CHARACTERIZATIONS FOR RATIONAL PAIRS Á LA SCHWEDE-TAKAGI AND KOLLÁR-KOVÁCS

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ABSTRACT. This work establishes triangulated characterizations for notions of rational pairs á la Schwede-Takagi and Kollár-Kovács. We use a concept of generation in triangulated categories, introduced by Bondal and Van den Bergh, to study these classes of singularities for pairs. One component of our work introduces rational pairs á la Kollár-Kovács for quasi-excellent schemes of characteristic zero, which gives a Kovács style splitting criterion and a Kovács-Schwede style cohomological vanishing result.

1. INTRODUCTION

This short note provides simple triangulated characterizations for notions of rational pairs á la Schwede-Takagi [ST08] and Kollár-Kovács [Kol13]. These notions are an analog of rational singularities in the minimal model program. Loosely speaking, such singularities tell us that the cohomology of particular line bundle behaves similarly to a line bundle on a smooth variety.

Recall a variety Y over a field of characteristic zero is said to have *rational singularities* if the natural map $\mathcal{O}_Y \to \mathbb{R}f_*\mathcal{O}_{\widetilde{Y}}$ is an isomorphism where $f\colon \widetilde{Y} \to Y$ is a resolution of singularities. It has been shown this is equivalent to the natural map $\mathcal{O}_Y \to \mathbb{R}f_*\mathcal{O}_{\widetilde{Y}}$ splitting [Kov00, Bha12, Mur21]. This is a weaker condition than required by the definition, and has motivated similar criterion's for detecting other singularities [SVV23, FL24, Sch07, GM23, McD23].

There are two approaches for defining an analog of rational singularities, via Schwede-Takagi [ST08] and Kollár-Kovács [Kol13], in the context of minimal model program when working with some notion of pairs. We explore both in our work. See Sections 1.1 and 1.2 for details. A splitting criterion for the Schwede-Takagi style of pairs exists, see [ST08, Theorem 3.11]. There is not a similar criterion known for Kollár-Kovács style pairs, which motivated early stages of our work. But later we show such an analog (see Proposition 3.5).

Our work takes steps towards an even weaker condition than splitting, but some terminology is needed. We briefly recall a notion of generation for a triangulated category $\mathcal{T}$, which was introduced in [BVdB03]. Let G be an object of $\mathcal{T}$. The smallest triangulated subcategory of $\mathcal{T}$ containing G and closed under direct summands is denoted $\langle \mathcal{S} \rangle$. Objects of $\langle G \rangle$ can be *finitely built* from G using only a finite number of shifts, cones and direct summands. If one wanted to count the number of cones, then $\langle \mathcal{S} \rangle_{n+1}$ denotes the subcategory of objects in $\mathcal{T}$ which can be finitely built from $\mathcal{S}$ using finite coproducts, direct summands, shifts, and *at most n cones*. See Section 2 for details.

The concept of generation in the bounded derived category of coherent sheaves, denoted $D^b_{\operatorname{coh}}(X)$, of a Noetherian scheme X is connected to various singularities arising in algebraic geometry and commututative algebra. Specifically, for characterizations of rational singularities, Du Bois singularities, and (derived) splinters [LV24]; as well as singularities of prime characteristic [BIL+23]. Our work extends such further to show singularities of pairs can be detected in a more relaxed fashion.

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1.1. **Schwede-Takagi pairs.** A 'pair' in the style of [ST08] consists of a quasi-compact separated normal irreducible scheme Y that is essentially of finite type over a field k of characteristic zero[1], an ideal sheaf $\mathscr{I}$ on Y, and a nonnegative real number c. The notion of rational singularities was extending to this setting of pairs, which is analogous to Kawamata log terminal (klt) pairs in the minimal model program. Any such data is typically denoted by $(Y, \mathscr{I}^c)$.

There is a notion of *log resolution* for such datum $(Y, \mathscr{I}^c)$. This a proper birational morphism $f: \widetilde{Y} \to Y$ from a regular scheme such that $\mathscr{I} \cdot \mathcal{O}_{\widetilde{Y}} = \mathcal{O}_{\widetilde{Y}}(-G)$ is a line bundle and $\operatorname{exc}(f) \cup \operatorname{supp}(G)$ is a simple normal crossings divisor[2]. See [ST08, KLM97, KM08] for details. We recall the definition for rational pairs in the sense of [ST08].

Definition A. *We say $(Y, \mathscr{I}^c)$ has* ***rational singularities*** *(á la Schwede-Takagi) if the natural map $\mathcal{O}_Y \to \mathbb{R}f_* \mathcal{O}_{\widetilde{Y}}(\lfloor c \cdot G \rfloor)$ is an isomorphism for $f: \widetilde{Y} \to Y$ a log resolution of $(X, \mathscr{I}^c)$ with $\mathscr{I} \cdot \mathcal{O}_{\widetilde{Y}} = \mathcal{O}_{\widetilde{Y}}(-G)$.*

This brings attention to our first result.

Theorem B. *(see Theorem 3.3) Suppose f is locally projective. Then $(Y, \mathscr{I}^c)$ has rational singularities á la Schwede-Takagi if, and only if, $\mathcal{O}_Y$ belongs to $\langle \mathbb{R}f_* \mathcal{O}_{\widetilde{Y}}(\lfloor c \cdot G \rfloor) \rangle_1$.*

Theorem B gives a simpler criterion to check whether (Y, D) is rational á la Schwede-Takagi as opposed to asking for the natural map $\mathcal{O}_Y \to \mathbb{R}f_* \mathcal{O}_{\widetilde{Y}}(\lfloor c \cdot G \rfloor)$ to be an isomorphism. Specifically, it says we only need to require that $\mathcal{O}_Y$ be a direct summand of object which is a finite direct sum of shifts of $\mathbb{R}f_* \mathcal{O}_{\widetilde{Y}}(\lfloor c \cdot G \rfloor)$ (e.g. an object of the form $\bigoplus_{n \in \mathbb{Z}} \mathbb{R}f_* \mathcal{O}_{\widetilde{Y}}(\lfloor c \cdot G \rfloor)^{\oplus r_n}[n]$). This gives us a weaker characterization than [ST08, Theorem 3.11] where Definition A was shown to be equivalent to the natural map $\mathcal{O}_Y \to \mathbb{R}f_* \mathcal{O}_{\widetilde{Y}}(\lfloor c \cdot G \rfloor)$ splitting, yet our statement doesn't directly work with said map.

1.2. **Kollár-Kovács pairs.** This component of our work is motivated by a notion of 'pairs' in the flavor of [Kol13], which is fairly new. But we work in slightly more general setting than loc. cit. Specifically, the class of schemes considered below are those which are quasi-compact separated quasi-excellent of characteristic zero and admit a dualizing complex. A 'pair' in this setting will be any such scheme Y which is normal and irreducible, with a choice of Weil divisor D on Y whose coefficients are all one. We fix such a datum in this subsection, and denote it by (Y, D).

There is a notion of *thrifty resolution*, related to that which appeared in Section 1.1, for the pair (Y, D). This is a proper birational morphism $f: \widetilde{Y} \to Y$ from a regular scheme $\widetilde{Y}$ such that the strict transform D_Y of D is a simple normal crossing (snc) divisor, $\operatorname{exc}(f)$ does not contain any stratum[3] of (Y, D_Y), and $f(\operatorname{exc}(f))$ does not contain any stratum of the snc locus[4] of (Y, D_Y). These morphisms always exist by [Tem18, Theorem 1.1.6], and see [Kol13, Definition 2.79] or [Tem18, §1.1.5] for details.

We now propose the following definition for rational pairs in this generality.

Definition C. *We say (Y, D) has* ***rational singularities*** *(á la Kollár-Kovács) if the natural map $\mathcal{O}_Y(-D) \to \mathbb{R}f_* \mathcal{O}_{\widetilde{Y}}(-D_Y)$ is an isomorphism for some thrifty resolution $f: \widetilde{Y} \to Y$ of (Y, D).*

A key difference between Definition C and Definition A is that simple normal crossing pairs are of the former, whereas not necessarily of the latter. This brings attention to our next result.

[1] We only impose Y to be normal and irreducible for our work, and this not required for other notions in [ST08].

[2] $\operatorname{exc}(f)$ denotes the exceptional locus of f, which is defined as the closed subset $\widetilde{Y} \setminus U$ for U the largest open subscheme for which f is an isomorphism over

[3] A *stratum*, in the sense of [Kol13, Definition 1.7], is an irreducible component of $\bigcap_{i \in I} D_i$ where I is a subset of J and $D_Y = \sum_{j \in J} D_j$.

[4] This is the largest open subscheme U of Y such that $(U, D|_U)$ is a snc pair in the sense of [Kol13, Definition 1.7].

Theorem D. *(see Theorem 3.6) The pair (Y, D) is a rational à la Kollár-Kovács if, and only if, $\mathcal{O}_Y(-D)$ is an object of $\langle \mathbb{R}f_*\mathcal{O}_Y(-D_Y)\rangle_1$ for some thrifty resolution $f\colon \widetilde{Y} \to Y$ of (Y, D) which is locally projective*

Theorem B gives a simpler criterion to check whether (Y, D) is rational á la Kollár-Kovács as opposed to asking for the natural map $\mathcal{O}_Y(-D) \to \mathbb{R}f_*\mathcal{O}_{\widetilde{Y}}(-D_Y)$ to be an isomorphism. Specifically, it says we only need to require that $\mathcal{O}_Y(-D)$ be a direct summand of object which is a finite direct sum of shifts of $\mathbb{R}f_*\mathcal{O}_{\widetilde{Y}}(-D_Y)$ (e.g. an object of the form $\bigoplus_{n\in\mathbb{Z}} \mathbb{R}f_*\mathcal{O}_{\widetilde{Y}}(-D_Y)^{\oplus r_n}[n]$).

There are a few steps to proving Theorem D that mimic classical results in the literature. One being a variation of [Kov00, Theorem 1], [Bha12, Theorem 2.12], and [Mur21, Theorem 9.5] in the context of Definition C (see Proposition 3.5). The other being a cohomological vanishing statement for thrifty resolutions in this general setting (see Lemma 3.4).

1.3. **Notation.** Let X be a scheme. The following triangulated categories are of interest to our work:

(1) $D(X) := D(\mathrm{Mod}(X))$ is the derived category of $\mathcal{O}_X$-modules.
(2) $D_{\mathrm{Qcoh}}(X)$ is the (strictly full) subcategory of $D(X)$ consisting of complexes with quasi-coherent cohomology.
(3) $D^b_{\mathrm{coh}}(X)$ is the (strictly full) subcategory of $D(X)$ consisting of complexes having bounded and coherent cohomology.

If X is affine, then we might at times abuse notation and write $D_{\mathrm{Qcoh}}(A)$ where $A := H^0(X, \mathcal{A})$ are the global sections; similar conventions will occur for the other categories. There is a triangulated equivalence of $D_{\mathrm{Qcoh}}(X)$ with $D(\mathrm{Qcoh}(X))$ [Sta24, Tag 09T4], and $D^b_{\mathrm{coh}}(X)$ with $D^b(\mathrm{coh}(X))$ [Sta24, Tag 0FDB]. We freely use this throughout.

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2. Preliminaries

This section is a very brisk recap on generation for triangulated categories. We freely pull content from [BVdB03, Rou08, ABIM10, Nee21]. Let $\mathcal{T}$ be a triangulated category with shift functor $[1]\colon \mathcal{T} \to \mathcal{T}$. Suppose $\mathcal{S}$ is a subcategory of $\mathcal{T}$.

Definition 2.1. A subcategory $\mathcal{S}$ is said to be **thick** if it is a triangulated subcategory of $\mathcal{T}$ which is closed under direct summands. The smallest thick subcategory containing $\mathcal{S}$ in $\mathcal{T}$ is denoted by $\langle \mathcal{S} \rangle$; if $\mathcal{S}$ consists of a single object G, then $\langle \mathcal{S} \rangle$ will be written as $\langle G \rangle$.

Definition 2.2.

(1) $\mathrm{add}(\mathcal{S})$ is the smallest strictly full[5] subcategory of $\mathcal{T}$ containing $\mathcal{S}$ which is closed under shifts, finite coproducts, and direct summands.
(2) $\langle \mathcal{S} \rangle_0$ consists of all objects in $\mathcal{T}$ isomorphic to the zero objects
(3) $\langle \mathcal{S} \rangle_1 := \mathrm{add}(\mathcal{S})$.
(4) $\langle \mathcal{S} \rangle_n := \mathrm{add}\{\mathrm{cone}(\phi) : \phi \in \mathrm{Hom}_{\mathcal{T}}(\langle \mathcal{S} \rangle_{n-1}, \langle \mathcal{S} \rangle_1)\}$; $\langle \mathcal{S} \rangle_n$ will be denoted by $\langle G \rangle_n$ if $\mathcal{S}$ consists of a single object G.

A rich study for generation has developed in the case $\mathcal{T} = D^b_{\mathrm{coh}}(X)$ where X is a Noetherian scheme. This includes, which is far from complete, the following: [Rou08, IT19, Aok21, Lan24, BIL+23, DLT23, Ola23, LO24, DL24a, DL24b]. Recent attention has been drawn to noncommutative flavors of loc. cit. [ELS20, DLM24a, DLM24b]. There are interesting notions and invariants which arise from Definition's 2.1 and 2.2 (see [Rou08, ABIM10]), but these lie outside the scope of our work.

[5]This means closed under isomorphisms in the ambient category.

Definition 2.3. ([Ati56, WW76]) Let $\mathcal{C}$ be an additive category. We say that $\mathcal{C}$ is a **Krull-Schmidt** category if every object of $\mathcal{C}$ is isomorphic to a finite coproduct of objects having local endomorphism rings. Moreover, an object of $\mathcal{C}$ is said to be **indecomposable** if it is not isomorphic to a coproduct of two nonzero objects.

If $\mathcal{C}$ is a Krull-Schmidt category, then every object is isomorphic to a finite coproduct of indecomposables, which is unique up to permutations [Kra15, Theorem 4.2]. The case where $\mathcal{C}$ satisfies a linearity condition over a Noetherian complete local ring yield instances of Krull-Schmidt categories (see [LC07, Corollary B]).

Example 2.4. (see [LV24, Lemma 2.7]) Let X be a proper scheme over a Noetherian complete local ring. Then $\operatorname{coh}(X)$ and $D^b_{\operatorname{coh}}(X)$ are Krull-Schmidt categories. Moreover, $\mathcal{O}_X$ is an indecomposable object in both categories whenever X is integral.

3. Results

This section proves our main results.

3.1. A key ingredient.

We need a few tools which allows us to leverage local algebra in the global setting. The following stems from a similar, yet mildly different, vein as [Kol13, Theorem 2.74]. A key difference is that we only impose f_*E belonging to $\langle \mathbb{R}f_*E\rangle_1$ as opposed requiring the natural map $f_*E \to \mathbb{R}f_*E$ split as in loc. cit.

Lemma 3.1. Let $f\colon Y \to X$ be a proper birational morphism of integral Noetherian schemes. Suppose X is proper over a Noetherian complete local ring. Consider a torsion free Cohen-Macaulay sheaf E on Y such that f_*E is an indecomposable object of $\operatorname{coh}(X)$ and $\mathbb{R}^j f_*\mathcal{H}^{-\dim X}(\mathbb{R}\mathcal{H}om(E,\omega_Y^\bullet)) = 0$ for $j \neq 0$. Then $\mathbb{R}^j f_*E = 0$ for $j \neq 0$ if f_*E belongs to $\langle \mathbb{R}f_*E\rangle_1$. Moreover, the natural map $f_*E \to \mathbb{R}f_*E$ is a quasi-isomorphism in $D^b_{\operatorname{coh}}(X)$.

Proof. Let $n = \dim X$. There is a chain of isomorphisms in $D^b_{\operatorname{coh}}(X)$:

$$\begin{aligned}\xi\colon \mathbb{R}\mathcal{H}om(\mathbb{R}f_*E,\omega_X^\bullet) &\xrightarrow{\cong} \mathbb{R}f_*\mathbb{R}\mathcal{H}om(E,\omega_Y^\bullet)\\ &\xrightarrow{\cong} \mathbb{R}f_*\big(\mathcal{H}^{-n}(\mathbb{R}\mathcal{H}om(E,\omega_Y^\bullet)[n])\big)\\ &\xrightarrow{\cong} f_*\big(\mathcal{H}^{-n}(\mathbb{R}\mathcal{H}om(E,\omega_Y^\bullet))\big)[n]\end{aligned}$$

which respectively arise from [Har66, § III.1.1, § VII.3.4], [Kol13, Corollary 2.70], and our hypothesis. This tells us that $\mathbb{R}\mathcal{H}om(\mathbb{R}f_*E,\omega_X^\bullet)$ is a complex concentrated in degree $-n$, which ensures $\mathcal{H}^{-n}(\xi)$ is an isomorphism.

It follows by [Kol13, Lemma 2.69] that $\mathcal{H}^{-n}(\mathbb{R}\mathcal{H}om(E,\omega_Y^\bullet))$ is isomorphic to $\mathcal{H}om(E,\omega_Y)$, and so, $\mathbb{R}^{-n}\mathcal{H}om(\mathbb{R}f_*E,\omega_X^\bullet)$ is isomorphic to $f_*\mathcal{H}om(E,\omega_Y)$. However, ω_Y is a torsion free coherent $\mathcal{O}_Y$-module by [Sta24, Tag 0AWK & Tag 0AXY], and so, [Sta24, Tag 0AXZ] tells us that $\mathcal{H}om(E,\omega_Y)$ is torsion free. Then, by [GD71, Proposition 7.4.5], $f_*\mathcal{H}om(E,\omega_Y)$ is torsion free because f is a proper birational morphism between integral Noetherian schemes. Hence, $\mathbb{R}^{-n}\mathcal{H}om(\mathbb{R}f_*E,\omega_X^\bullet)$ is torsion free.

Observe that f_*E is a direct summand of $\bigoplus_{n\in\mathbb{Z}}\mathbb{R}f_*E^{\oplus r_n}[n]$ from the hypothesis f_*E belonging to $\langle \mathbb{R}f_*E\rangle_1$. It follows from Example 2.4, coupled with our hypothesis f_*E is indecomposable, that f_*E is a direct summand of $\mathbb{R}f_*E$. This gives us maps $\alpha\colon f_*E \to \mathbb{R}f_*E$ and $\beta\colon \mathbb{R}f_*E \to E$ whose composition is the identity of f_*E in $D^b_{\operatorname{coh}}(X)$. Now apply the functor $\mathbb{R}\mathcal{H}om(-,\omega_X^\bullet)$ to get maps $\beta' := \mathbb{R}\mathcal{H}om(\beta,\omega_X^\bullet)$ and $\alpha' := \mathbb{R}\mathcal{H}om(\alpha,\omega_X^\bullet)$ for which $\alpha'\circ\beta'$ is the identity map on $\mathbb{R}\mathcal{H}om(f_*E,\omega_X^\bullet)$.

It suffices to show that β' is an isomorphism. Indeed, as this would give us a chain of isomorphisms in $D^b_{\mathrm{coh}}(X)$:

$$\begin{aligned} f_* E &\cong \mathbb{R}\mathcal{H}om(\mathbb{R}\mathcal{H}om(f_*E, \omega_X^\bullet), \omega_X^\bullet) \\ &\cong \mathbb{R}\mathcal{H}om(\mathbb{R}f_*(\mathcal{H}^{-n}(\mathbb{R}\mathcal{H}om(E, \omega_Y^\bullet)))[n], \omega_X^\bullet) \quad (\xi \circ \beta' \text{ is an isomorphism}) \\ &\cong \mathbb{R}f_*\mathbb{R}\mathcal{H}om(\mathcal{H}^{-n}(\mathbb{R}\mathcal{H}om(E, \omega_Y^\bullet))[n], \omega_Y^\bullet) \\ &\cong \mathbb{R}f_*\mathbb{R}\mathcal{H}om(\mathbb{R}\mathcal{H}om(E, \omega_Y^\bullet), \omega_X^\bullet) \\ &\cong \mathbb{R}f_* E. \end{aligned}$$

Consequently, we would know that $\mathbb{R}^j f_* E = 0$ for $j \neq 0$, which would furnish the desired claim.

We are left to verify that β' is an isomorphism. There are induced morphisms on cohomology sheaves $\mathcal{H}^j(\alpha') \circ \mathcal{H}^j(\beta')$, which must be the identity of $\mathbb{R}^j\mathcal{H}om(f_*E, \omega_X^\bullet)$ for each j. Hence, $\mathbb{R}^j\mathcal{H}om(f_*E, \omega_X^\bullet)$ is a direct summand of $\mathbb{R}^j\mathcal{H}om(\mathbb{R}f_*E, \omega_X^\bullet)$ for all j. It follows that $\mathbb{R}\mathcal{H}om(f_*E, \omega_X^\bullet)$ is concentrated in degree $-n$ because it is a direct summand of an object with the same property. This ensures we only need to check that the induced morphism $\mathcal{H}^{-n}(\beta')$ is an isomorphism. There is a Zariski dense open subset U in X for which f is an isomorphism over, which ensures $\mathcal{H}^{-n}(\beta')$ is generically an isomorphism. Hence, $\operatorname{coker}(\mathcal{H}^{-n}(\beta'))$ is a torsion sheaf in $\operatorname{coh}(Y)$. However, $\mathbb{R}^{-n}\mathcal{H}om(\mathbb{R}f_*E, \omega_X^\bullet)$ being torsion free tells us it cannot have a nonzero torsion sheaf as a direct summand, which tells us $\mathcal{H}^{-n}(\beta')$ is an isomorphism in $\operatorname{coh}(X)$, and completes the proof. □

The following is elementary, but will be useful for later proofs when used appropriately.

Lemma 3.2. Let X be a normal irreducible Noetherian scheme. Suppose $f\colon A \to B$ is a section of coherent torsion free $\mathcal{O}_X$-modules of equivalent rank in codimension one. Then $f_p\colon A_p \to B_p$ is an isomorphism for each p in X such that $\dim \mathcal{O}_{X,p} = 1$. Additionally, if A has property (S_2), then $f\colon A \to B$ is an isomorphism.

Proof. The last claim follows from [Sta24, Tag 0AVS] if we can show the first. Let p be in X such that $\dim \mathcal{O}_{X,p} = 1$. Then f_ξ is an isomorphism at the generic point ξ of X. This tells us $\operatorname{coker}(f_p)$ is a torsion $\mathcal{O}_{X,p}$-module. But B_p is a free $\mathcal{O}_{X,p}$-module of rank coinciding with that of A_p. Hence, $\operatorname{coker}(f_p) = 0$ as desired. □

3.2. **Schwede-Takagi pairs.** We use the notation of Section 1.1.

Theorem 3.3. With notation above. Suppose $f\colon \widetilde{Y} \to Y$ is a log resolution of $(Y, \mathcal{I}^c)$ which is locally projective. Then $(Y, \mathcal{I}^c)$ has rational singularities á la Schwede-Takagi if, and only if, $\mathcal{O}_Y$ belongs to $\langle \mathbb{R}f_*\mathcal{O}_{\widetilde{Y}}(\lfloor c \cdot G \rfloor)\rangle_1$.

Proof. The forward direction is obvious, and so, we only check the converse. It follows from f being locally projective that there is an affine open cover U_i of Y for which the natural morphisms $\widetilde{Y} \times_Y U_i \to U_i$ is projective. If we can show the cone of the natural map $\mathcal{O}_Y \to \mathbb{R}f_*\mathcal{O}_{\widetilde{Y}}(\lfloor c \cdot G \rfloor)$ has empty support when restricted to each U_i, then the desired claim follows. Hence, we can impose that Y be affine and f be projective.

Let $\mathcal{L} := \mathcal{O}_{\widetilde{Y}}(\lfloor cG \rfloor)$. Choose a point p in Y. Consider the fibered square:

$$\begin{tikzcd} Y' \arrow[r, "\phi"] \arrow[d, "s"'] & \operatorname{Spec}(\widehat{\mathcal{O}}_{Y,p}) \arrow[d, "t"] \\ \widetilde{Y} \arrow[r, "f"'] & Y \end{tikzcd}$$

where t is the canonical morphism. We know from [Liu02, Theorem 8.1.24] that f can be expressed as a sequence of blowups[6]. However, the blowup construction is compatible under flat base change

[6] f is a projective birational morphism between integral Noetherian schemes with affine target

(see [Sta24, Tag 0805]), and so, ϕ can also be expressed as sequence of blowups. Then ϕ is a proper birational morphism. We know that $\mathcal{O}_{Y,p}$ is a quasi-excellent normal domain as these properties are stable under localization. It follows from [Sta24, Tag 0C23] that $\widehat{\mathcal{O}}_{Y,p}$ is a normal domain as well. Moreover, by [Sta24, Tag 02ND], an induction argument on the length of the sequence of blowups for f will tells us Y' is a Noetherian integral scheme. We leverage these observations below to appeal to both Lemma's 3.1 and 3.2

It follows from [Sta24, Tag 08IB] that $t^*\mathbb{R}f_*\mathcal{L}$ is isomorphic to $\mathbb{R}\phi_* s^*\mathcal{L}$. If $\mathcal{O}_Y$ is in $\langle \mathbb{R}f_*\mathcal{L}\rangle_1$, then $\widehat{\mathcal{O}}_{Y,p}$ is in $\langle t^*\mathbb{R}f_*\mathcal{L}\rangle_1$ as $t^*\mathcal{O}_Y = \widehat{\mathcal{O}}_{Y,p}$, and so, $\widehat{\mathcal{O}}_{Y,p}$ is in $\langle \mathbb{R}\phi_* s^*\mathcal{L}\rangle_1$. This implies that $\widehat{\mathcal{O}}_{Y,p}$ is a direct summand of $\bigoplus_{n\in\mathbb{Z}} \mathbb{R}\phi_* s^*\mathcal{L}^{\oplus r_n}[n]$. It follows from Example 2.4 that $\widehat{\mathcal{O}}_{Y,p}$ is a direct summand of $\mathbb{R}\phi_* s^*\mathcal{L}$, giving us maps $\widehat{\mathcal{O}}_{Y,p} \to \mathbb{R}\phi_* s^*\mathcal{L} \to \widehat{\mathcal{O}}_{Y,p}$ whose composition is the identity. Taking 0-th cohomology sheaves gives us a sequence of maps $\widehat{\mathcal{O}}_{Y,p} \xrightarrow{\alpha} \phi_* s^*\mathcal{L} \xrightarrow{\beta} \widehat{\mathcal{O}}_{Y,p}$ whose composition is the identity in $\operatorname{coh}(\widehat{\mathcal{O}}_{Y,p})$. Note that $\phi_* s^*\mathcal{L}$ is torsion free by [GD71, Proposition 7.4.5] and $\widehat{\mathcal{O}}_{Y,p}$ is (S_2) as it is a normal domain [Sta24, Tag 0345]. Hence, Lemma 3.2 ensures the map $\alpha\colon \widehat{\mathcal{O}}_{Y,p} \to \phi_* s^*\mathcal{L}$ is an isomorphism, and so by Example 2.4, $\phi_* s^*\mathcal{L}$ is indecomposable in $\operatorname{coh}(\widehat{\mathcal{O}}_{Y,p})$.

We know, by [ST08, Lemma 3.5], that $\mathbb{R}\mathcal{H}om(\mathbb{R}f_*\mathcal{L}, \omega_Y^\bullet)$ is an object concentrated in one degree. Consider the following string of isomorphisms in $D_{\mathrm{Qcoh}}(\widehat{\mathcal{O}}_{Y,p})$:

$$\begin{aligned}
\mathbb{R}\phi_*(\mathcal{H}^{-\dim X}(\mathbb{R}\mathcal{H}om(s^*\mathcal{L}, \omega_{Y}^\bullet))) &\cong \mathbb{R}\phi_*(\mathbb{R}\mathcal{H}om(s^*\mathcal{L}, \omega_{Y'_n}^\bullet)[-n]) && (\text{[Kol13, Lemma 2.69]})\\
&\cong \mathbb{R}\mathcal{H}om(\mathbb{R}\phi_* s^*\mathcal{L}, \omega_{\widehat{\mathcal{O}}_{Y,p}}^\bullet)[-n] && (\text{Grothendieck duality})\\
&\cong \mathbb{R}\mathcal{H}om(t^*\mathbb{R}f_*\mathcal{L}, t^*\omega_{\mathcal{O}_{Y,p}}^\bullet)[-n] && (\text{[Sta24, Tag 0A7G, Tag 0A86]})\\
&\cong t^*\mathbb{R}\mathcal{H}om(\mathbb{R}f_*\mathcal{L}, \omega_Y^\bullet)[-n] && (\text{[GW23, Proposition 22.70]}).
\end{aligned}$$

Hence, from the exactness of t^*, it follows that $\mathbb{R}^b\phi_*(\mathcal{H}^{-\dim X}(\mathbb{R}\mathcal{H}om(s^*\mathcal{L}, \omega_Y^\bullet))) = 0$ for $b > 0$. It follows from Lemma 3.1, coupled with our work in prior paragraph, that $\mathbb{R}^j f_* s^*\mathcal{L} = 0$ for $j > 0$. Then the natural map $\phi_* s^*\mathcal{L} \to \mathbb{R}\phi_* s^*\mathcal{L}$ is an isomorphism, and so in particular, the natural map $t^* f_*\mathcal{L} \to t^*\mathbb{R}f_*\mathcal{L}$ is an isomorphism in $D^b_{\mathrm{coh}}(\widehat{\mathcal{O}}_{Y,p})$.

Let $q\colon \operatorname{Spec}(\mathcal{O}_{Y,p}) \to Y$ be the canonical morphism. We know that t factors through q. Consider the distinguished triangle in $D^b_{\mathrm{coh}}(Y)$:

$$f_*\mathcal{L} \xrightarrow{h} \mathbb{R}f_*\mathcal{L} \to C \to (f_*\mathcal{L})[1]$$

where h is the natural map. Now pulling back along t gives us a distinguished triangle in $D^b_{\mathrm{coh}}(\widehat{\mathcal{O}}_{Y,p})$:

$$t^* f_*\mathcal{L} \xrightarrow{t^*(h)} t^*\mathbb{R}f_*\mathcal{L} \to t^*C \to (t^* f_*\mathcal{L})[1].$$

It follows that t^*C is the zero object because we have checked that $t^*(h)$ is an isomorphism above. Consider the distinguished triangle in $D^b_{\mathrm{coh}}(\mathcal{O}_{Y,p})$:

$$q^* f_*\mathcal{L} \xrightarrow{q^*(h)} q^*\mathbb{R}f_*\mathcal{L} \to q^*C \to (q^* f_*\mathcal{L})[1].$$

If t^*C is the zero object, then q^*C is the zero object in $D^b_{\mathrm{coh}}(\mathcal{O}_{Y,p})$, see [Let21, Corollary 2.12]. However, tying all this together, we have shown that the natural map $h\colon f_*\mathcal{L} \to \mathbb{R}f_*\mathcal{L}$ must be an isomorphism in $D^b_{\mathrm{coh}}(Y)$. Our discussion above, coupled with Lemma 3.2, tells us the natural map $\mathcal{O}_Y \to f_*\mathcal{L}$ is an isomorphism, which completes the proof. □

3.3. **Kollár-Kovács pairs.** We use the notation of Section 1.2.

Lemma 3.4. Let $f\colon \widetilde{Y} \to Y$ be a thrifty resolution of (Y, D). Then $\mathbb{R}^j f_*(\omega_Y \otimes^{\mathbb{L}} \mathcal{O}_{\widetilde{Y}}(D_Y)) = 0$ for $j > 0$.

Proof. This is essentially the same argument to the alternative proof of [KS16, Lemma 2.5]. □

Any pair of thrifty resolutions for (Y, D) can be dominated by a third thrifty resolution of (Y, D) ♣ Pat: [todo:details]. This tells us Definition C is independent of the choice of thrifty resolution. The following is a variation of [Kov00, Theorem 1], [Bha12, Theorem 2.12] and [Mur21, Theorem 9.5] in our setting. Its proof follows in a similar vein, so we only give a sketch.

Proposition 3.5. (Kovács) The pair (Y, D) is a rational à la Kollár-Kovács if, and only if, the natural map $\mathcal{O}_Y(-D) \to \mathbb{R}f_*\mathcal{O}_{\widetilde{Y}}(-D_Y)$ splits for some thrifty resolution $f\colon \widetilde{Y} \to Y$ of (Y, D).

Proof. It is evident the forward direction holds, so we check the converse. Suppose there is a thrifty resolution $f\colon \widetilde{Y} \to Y$ of (Y, D) such that the natural map $\xi\colon \mathcal{O}_Y(-D) \to \mathbb{R}f_*\mathcal{O}_{\widetilde{Y}}(-D_Y)$ splits in $D^b_{\operatorname{coh}}(Y)$. This tells us that $\mathbb{R}\mathcal{H}om(\mathcal{O}_Y(-D), \omega^\bullet_Y)$ is a direct summand of $\mathbb{R}\mathcal{H}om(\mathbb{R}f_*\mathcal{O}_{\widetilde{Y}}(-D_Y), \omega^\bullet_Y)$. It follows from [Har66, § III.1.1, § VII.3.4] that $\mathbb{R}\mathcal{H}om(\mathbb{R}f_*\mathcal{O}_{\widetilde{Y}}(-D_Y), \omega^\bullet_Y)$ is isomorphic to the complex $\mathbb{R}f_*\big(\mathbb{R}\mathcal{H}om_{\widetilde{Y}}(\mathcal{O}_{\widetilde{Y}}(-D_Y), \omega_{\widetilde{Y}})\big)$. The following vanishing for $j > 0$ follows by Lemma 3.4:

$$\begin{aligned} 0 &= \mathbb{R}^j f_*(\omega_{\widetilde{Y}} \otimes^{\mathbb{L}} \mathcal{O}_{\widetilde{Y}}(D_Y)) \\ &\cong \mathbb{R}^j f_*\big(\mathbb{R}\mathcal{H}om_{\widetilde{Y}}(\mathcal{O}_{\widetilde{Y}}(-D_Y), \omega_{\widetilde{Y}})\big). \end{aligned}$$

Consequently, we see that $\mathbb{R}\mathcal{H}om(\mathcal{O}_Y(-D), \omega^\bullet_Y)$ is concentrated in one degree. It can be checked that $\mathbb{R}\mathcal{H}om(\xi, \omega^\bullet_X)$ is an isomorphism, and applying $\mathbb{R}\mathcal{H}om(-, \omega^\bullet_Y)$ once more implies ξ is an isomorphism as desired. □

It follows from [Tem18, Theorem 1.1.6] that we can find a thrifty resolution of (Y, D) which is locally projective.

Theorem 3.6. With notation above. Then (Y, D) is a rational à la Kollár-Kovács if, and only if, $\mathcal{O}_Y(-D)$ is an object of $\langle \mathbb{R}f_*\mathcal{O}_Y(-D_Y)\rangle_1$ for some thrifty resolution $f\colon \widetilde{Y} \to Y$ of (Y, D) which is locally projective.

Proof. The forward direction is obvious, whereas the converse follows from a similar argument for Theorem 3.3 if coupled with Proposition 3.5. □

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