

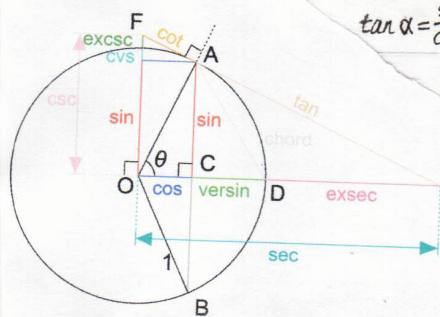
$\sqrt{x^2} = |x|$ 加绝对值是算数平方根，真平方根可以是负数

三角函数

No. 基数
Date 三角

$$\operatorname{Im}[e^{i(\alpha+\beta)}] = \operatorname{Im}[(c_\alpha + i s_\alpha)(c_\beta + i s_\beta)]$$

特殊值	0°	15°	18°	22.5°	30°	36°	45°	奇变偶不变，符号看象限
\sin	0	$\frac{\sqrt{6}-\sqrt{2}}{4}$	$\frac{\sqrt{5}-1}{4}$	$\frac{\sqrt{2}-\sqrt{2}}{2}$	$\frac{1}{2}$	$\frac{\sqrt{10}-2\sqrt{5}}{4}$	$\frac{\sqrt{2}}{2}$	$\sin(90^\circ-\alpha) = \cos\alpha$
\cos	1	$\frac{\sqrt{6}+\sqrt{2}}{4}$	$\frac{\sqrt{10}+2\sqrt{5}}{4}$	$\frac{\sqrt{2}+\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{5}+1}{4}$	$\frac{\sqrt{2}}{2}$	$\cos(90^\circ-\alpha) = \sin\alpha$
\tan	0	$2-\sqrt{3}$			$\frac{\sqrt{3}}{3}$		1	$\tan(90^\circ-\alpha) = \cot\alpha$



$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha}, \sin^2\alpha + \cos^2\alpha = 1, \tan^2\alpha + 1 = \sec^2\alpha, \cot^2\alpha + 1 = \csc^2\alpha$$

二倍角

$$\sin 2\alpha = 2 \sin\alpha \cos\alpha$$

$$\sin^2\alpha = \frac{1 - \cos 2\alpha}{2}$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$

$$\cos^2\alpha = \frac{1 + \cos 2\alpha}{2}$$

$$\tan 2\alpha = \frac{2 \tan\alpha}{1 - \tan^2\alpha}$$

$$\tan^2\alpha = \frac{1 - \cos 2\alpha}{1 + \cos 2\alpha}$$

三倍角

$$\sin 3\alpha = 3 \sin\alpha - 4 \sin^3\alpha$$

$$\sin \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos\alpha}{2}}$$

$$\cos 3\alpha = 4 \cos^3\alpha - 3 \cos\alpha$$

$$\cos \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos\alpha}{2}}$$

$$\tan 3\alpha = \frac{3 \tan\alpha - \tan^3\alpha}{1 - 3 \tan^2\alpha}$$

$$\tan \frac{\alpha}{2} = \frac{\sin\alpha}{1 + \cos\alpha} = \frac{1 - \cos\alpha}{\sin\alpha}$$

基本公式

$$\sin(\alpha + \beta) = \sin\alpha \cos\beta + \sin\beta \cos\alpha$$

$$\sin 3\alpha = 3 \sin\alpha - 4 \sin^3\alpha$$

$$\sin(\alpha - \beta) = \sin\alpha \cos\beta - \sin\beta \cos\alpha$$

$$\cos 3\alpha = 4 \cos^3\alpha - 3 \cos\alpha$$

$$\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$$

$$\tan 3\alpha = \frac{3 \tan\alpha - \tan^3\alpha}{1 - 3 \tan^2\alpha}$$

$$\cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta$$

$$\sin 4\alpha = 4 \sin\alpha \cos\alpha - 8 \sin^3\alpha \cos\alpha$$

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha \tan\beta}$$

$$= 4 \cos^3\alpha \sin\alpha - 4 \sin^3\alpha \cos\alpha$$

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha \tan\beta}$$

$$\cos 4\alpha = 8 \cos^4\alpha - 8 \cos^2\alpha + 1$$

和差化积

$$\tan 4\alpha = \frac{4 \tan\alpha - 4 \tan^3\alpha}{1 - 6 \tan^2\alpha + \tan^4\alpha}$$

$$\sin\alpha + \sin\beta = 2 \sin \frac{\alpha+\beta}{2} \cos \frac{\alpha-\beta}{2}$$

$$\tan(\alpha + \frac{\pi}{4}) = \frac{1 + \tan\alpha}{1 - \tan\alpha} = \frac{\sin\alpha + \cos\alpha}{\sin\alpha - \cos\alpha}$$

$$\sin\alpha - \sin\beta = 2 \sin \frac{\alpha-\beta}{2} \cos \frac{\alpha+\beta}{2}$$

$$\sin 5\alpha = 16 \sin^5\alpha - 20 \sin^3\alpha + 5 \sin\alpha$$

$$\cos\alpha + \cos\beta = 2 \cos \frac{\alpha+\beta}{2} \cos \frac{\alpha-\beta}{2}$$

$$\tan^2\alpha \sin^2\alpha = \tan^2\alpha - \sin^2\alpha$$

$$\cos\alpha - \cos\beta = -2 \sin \frac{\alpha+\beta}{2} \sin \frac{\alpha-\beta}{2}$$

$$\cos 5\alpha = 16 \cos^5\alpha - 20 \cos^3\alpha + 5 \cos\alpha$$

$$\sin(\alpha + \beta) \sin(\alpha - \beta) = \frac{1}{2} [\cos(\alpha + \beta) - \cos(\alpha - \beta)]$$

$$(\sin\alpha \pm \cos\alpha)^2 = 1 \pm \sin 2\alpha$$

$$\cos(\alpha + \beta) \cos(\alpha - \beta) = \frac{1}{2} [\cos(\alpha + \beta) + \cos(\alpha - \beta)]$$

$$\tan 5\alpha = \frac{5 \tan\alpha - 10 \tan^3\alpha + \tan^5\alpha}{1 - 10 \tan^2\alpha + 5 \tan^4\alpha}$$

$$\sin\alpha \cos\beta = \frac{1}{2} [\sin(\alpha + \beta) + \sin(\alpha - \beta)]$$

$$\sin 3\alpha = 4 \sin\alpha \sin(\frac{\pi}{3} - \alpha) \sin(\frac{\pi}{3} + \alpha)$$

$$\cos\alpha \sin\beta = \frac{1}{2} [\sin(\alpha + \beta) - \sin(\alpha - \beta)]$$

$$\tan(\alpha + \beta) \tan(\alpha - \beta) = \frac{\tan^2\alpha - \tan^2\beta}{1 - \tan^2\alpha \tan^2\beta}$$

$$\tan\alpha + \cot\alpha = \frac{1}{\sin\alpha \cos\alpha}$$

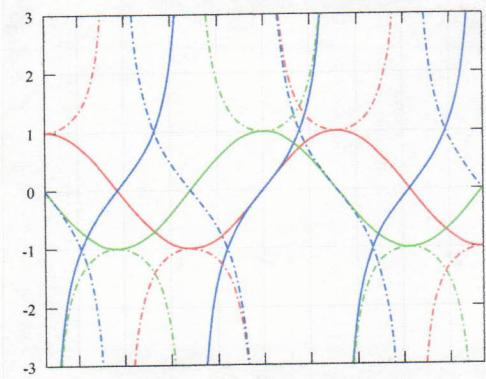
$$\tan\alpha - \cot\alpha = -\frac{2}{\tan 2\alpha}$$

降幂

半角

万能

其它



$$\sin x < x < \tan x$$



[向量] $a \sin x + b \cos x = \sqrt{a^2 + b^2} \sin(x + \phi)$, $\tan \phi = \frac{b}{a}$

$$\sin(\arccos x) = \sqrt{1-x^2} \quad \sin(\arctan x) = \frac{x}{\sqrt{1+x^2}}$$

$$\arctan \frac{1}{2} + \arctan \frac{1}{3} = \arctan 1$$

正弦定理 $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$ (外接圆半径)

余弦定理 $c^2 = a^2 + b^2 - 2ab \cos C$

正切定理 $\frac{a+b}{a-b} = \frac{\tan \frac{A+B}{2}}{\tan \frac{A-B}{2}}$

射影定理 $a \cos B + b \cos A = c$

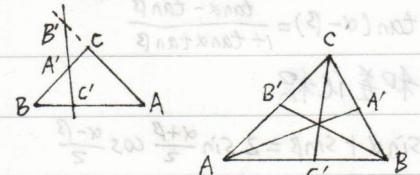
$S_{\triangle ABC} = \frac{1}{2} ab \sin C = 2R^2 \sin A \sin B \sin C = \frac{abc}{4R} = \frac{r}{2}(a+b+c) = \sqrt{p(p-a)(p-b)(p-c)}$, $p = \frac{1}{2}(a+b+c)$



$$\text{中线长 } AB^2 + AC^2 = \frac{1}{2} BC^2 + 2AD^2$$

$$\text{角平分线 } \frac{AB}{AC} = \frac{BP}{PC}$$

$$\text{直角三角形射影定理 } AH^2 = BH \cdot HC$$



梅涅劳斯定理 A', B', C' 共线 $\Leftrightarrow \frac{AB'}{B'C} \cdot \frac{CA'}{A'B} \cdot \frac{BC'}{C'A} = 1$

塞瓦定理 AA', BB', CC' 共点或平行 $\Leftrightarrow \frac{AC'}{CB} \cdot \frac{BA'}{AC} \cdot \frac{CB'}{BA} = 1$

托勒密定理 $AB \cdot CD + AD \cdot BC \geq AC \cdot BD$, 当且仅当 A, B, C, D 共圆时取等号

西姆松定理 过三角形外接圆上一点作三边垂线, 三垂足共线 (西姆松线)

张角定理 A, C, B 共线 $\Leftrightarrow \frac{\sin(\alpha+\beta)}{PC} = \frac{\sin \alpha}{PB} + \frac{\sin \beta}{PA}$ ($\alpha+\beta < \pi$)

欧拉公式 内心外心距离满足 $d^2 = R^2 - 2Rr$ (I_r 内心, O 外心, G 重心, H 垂心, I_A 旁心)

解析法常用三角恒等式 [和差化积, 降幂]

$$\sin A + \sin B + \sin C = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} \quad \sin^2 A + \sin^2 B + \sin^2 C = 2 + 2 \cos A \cos B \cos C$$

$$\cos A + \cos B + \cos C = 1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \quad \cos^2 A + \cos^2 B + \cos^2 C = 1 - 2 \cos A \cos B \cos C$$

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C \quad \tan \frac{A}{2} \tan \frac{B}{2} + \tan \frac{B}{2} \tan \frac{C}{2} + \tan \frac{C}{2} \tan \frac{A}{2} = 1$$

勾股定理 Pythagorean Theorem 直角三角形

斜截式	$y = kx + b$	$k \neq \infty$	点到直线距离 $d = \frac{ Ax_0 + By_0 + C }{\sqrt{A^2 + B^2}}$
点斜式	$y - y_1 = k(x - x_1)$	$k \neq \infty$	
两点式	$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$	$k \neq 0, k \neq \infty$	线关于点对称 $A(2x_0 - x) + B(2y_0 - y) + C = 0$
截距式	$\frac{x}{a} + \frac{y}{b} = 1$	$a \neq 0, b \neq 0$	点关于线对称 ① 中点在线上 ② $B\Delta x = A\Delta y$
一般式	$Ax + By + C = 0$	A, B 不同时为零	线关于线对称 ① 取一点 ② 到角 $\tan\theta = \frac{k_2 - k_1}{1 + k_1 k_2}$
法线式	$x \cos\theta + y \sin\theta = r$	圆切线方程	
(原点圆) 切点弦方程	$x_0 x + y_0 y = r^2$	圆切线长 $\sqrt{x_0^2 + y_0^2 + Dx_0 + Ey_0 + F}$	
两交点连线: 交点圆系方程	$\lambda = -1$	由直译写圆方程 $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$	

	椭圆 $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	双曲线 $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	抛物线 $y^2 = 2px$
半焦距 c	$c^2 = a^2 - b^2$	$c^2 = a^2 + b^2$	
离心率 $e = \frac{c}{a}$	$(0, 1)$	$(1, +\infty)$	
准线 $\rightarrow +c$	$x = \pm \frac{a^2}{c}$	$x = \pm \frac{a^2}{c}$	$x = -\frac{P}{2}$
焦参数 焦点准线距离	$P = \frac{b^2}{c}$	$P = \frac{b^2}{c}$	P
半通径长 $\leftarrow x_0$	$a \pm ex_0$	$e x_0 \pm a = \frac{3dn}{2s} = 1, 3dn \cdot \frac{1}{s} = V$	$x_0 + \frac{P}{2}$
焦半径长	$a \pm ex_0$	$e x_0 \pm a = \frac{3dn}{2s} = 1, 3dn \cdot \frac{1}{s} = V$	$x_A + x_B + P$
焦点 直角系	$2a \pm e(x_A + x_B)$		
弦长 极系	[锥曲线极系方程 $P = \frac{ep}{1 - e \cos\theta}$]	$\frac{2ep}{1 - e^2 \cos^2\theta}$ (θ 为焦射线与半径夹角)	
点差法结论	$(\frac{x^2}{X} + \frac{y^2}{Y} = 1) k_{弦} k_{中} = -\frac{Y}{X}$ (双曲线中点不在 $(0, 1)$ 区)	$[k_{弦} = y'_x] k_{弦} \cdot y'_\phi = P$	
切点弦方程	$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1$	$\frac{x_0 x}{a^2} - \frac{y_0 y}{b^2} = 1$	$y_0 y = P(x + x_0)$
焦点△面积 $\theta = \angle FPF_2$	$b^2 \tan \frac{\theta}{2}$	$b^2 \cot \frac{\theta}{2}$	
	到 $C_1 C_2$ 和为 $2a$	到 $C_1 C_2$ 差为 $2a$	直角时互余

统计分布

$$S_{\text{斜棱柱侧}} = C_{\text{直截面}} \cdot l_{\text{母线}}$$

$$S_{\text{正棱台侧}} = \frac{1}{2}(c+c')h', c, c' \text{ 底面边长}, h' \text{ 斜高}$$

$$S_{\text{圆台侧}} = \pi(r+r')l \left[= \frac{1}{2}(c+c')l \right]$$

$$S_{\text{球}} = 4\pi R^2, V_{\text{球}} = \frac{4}{3}\pi R^3, V_{\text{锥}} = \frac{1}{3}V_{\text{柱}}$$

$$V_{\text{台体}} = \frac{1}{3}h(S + \sqrt{SS'} + S')$$

$$\text{APB三点共线 } \overrightarrow{OP} = \lambda_1 \overrightarrow{OA} + \lambda_2 \overrightarrow{OB} \text{ 且 } \lambda_1 + \lambda_2 = 1$$

$$\text{ABCP四点共面 } \overrightarrow{OP} = \lambda_1 \overrightarrow{OA} + \lambda_2 \overrightarrow{OB} + \lambda_3 \overrightarrow{OC} \text{ 且 } \lambda_1 + \lambda_2 + \lambda_3 = 1$$

$$d_{\text{点面}} = |\overrightarrow{\text{斜线}} \cdot \overrightarrow{\text{法线}}|$$

欧拉公式 无洞多面体顶点V、面数F、棱数E 满足 $V+F-E=2$ 有洞变零

正四面体 六条棱相等 (作为对角线外接正方体)

$$\text{记边长为 } a, \text{ 表面积 } S = \sqrt{3}a^2, V = \frac{a^3}{6\sqrt{2}}, r = \frac{a}{2\sqrt{3}}, h = 4r, R = 3r$$

直角四面体 有一个三面角是直三面角 (长方体的一角)

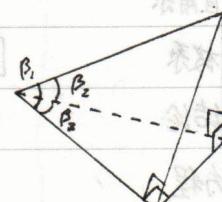
$$S_p = S_A^2 + S_B^2 + S_C^2, V = \frac{1}{6}abc, r = \frac{abc}{2S} = \frac{S_A + S_B + S_C - S_p}{a+b+c}, R = \frac{1}{2}\sqrt{a^2 + b^2 + c^2}, \frac{1}{h^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$$

等面四面体 三组对棱分别相等 (作为对角线外接长方体)

$$R = \frac{1}{4}\sqrt{2(a^2 + b^2 + c^2)} = \frac{k}{2}, V = \frac{1}{3}\sqrt{(k^2 - a^2)(k^2 - b^2)(k^2 - c^2)} = \frac{4}{3}S_x \cdot r$$

直棱四面体 三条相连棱形成三边直角折线

$$\cos \beta_1 = \cos \beta_2 = \cos \beta_3$$



重心四面体 四条高线交于一点 (作为对角线外接平行六面体各面是菱形)

对棱互相垂直

莱布尼茨公式 三角形 $AP^2 + BP^2 + CP^2 = 3PG^2 + \frac{1}{3}(a^2 + b^2 + c^2)$

四面体 (P到4顶点距离平方和) $= 4PG^2 + \frac{1}{4}(6\text{棱长平方和})$

$$\text{四面体的立体角 } \tan^2\left(\frac{\alpha}{4}\right) = \tan\left(\frac{\pi}{2}\right) \tan\left(\frac{s-\alpha}{2}\right) \tan\left(\frac{s-\beta}{2}\right) \tan\left(\frac{s-\gamma}{2}\right) \quad S = \frac{1}{2}(\alpha + \beta + \gamma)$$

微分表

等价无穷小 $x \rightarrow 0$ 的比值的极限为 $\lim_{x \rightarrow 0^+} x^x = 1$

$$\sin x \sim x \quad 1 - \cos x \sim \frac{x^2}{2} \quad \arcsin x \sim x \quad \tan x \sim x \quad \arctan x \sim x$$

$$e^x - 1 \sim x \quad a^x - 1 \sim x \ln a \quad (1+x)^\alpha \sim 1 + \alpha x \quad (\alpha \in \mathbb{R}) \quad \ln(1+x) \sim x \quad \log_a(1+x) \sim \frac{x}{\ln a}$$

$$\text{无穷大的速度 } (\alpha, \beta, \gamma > 0) \quad \lim_{x \rightarrow \infty} \sqrt[2]{x} = 1 \quad \lim_{x \rightarrow \infty} \sqrt[3]{x!} = \infty \quad (\text{数理斯特林}) \quad \sqrt{x} \sim \frac{x}{2}$$

$$\lim_{x \rightarrow 0^+} x^\alpha (\ln x)^\beta = 0 \quad \lim_{x \rightarrow \infty} (1 + \frac{1}{x})^x = e \quad x \rightarrow \infty \quad \ln^\alpha x \ll x^\beta \ll e^{rx} \ll x! \ll x^x \ll (x!)^2$$

$$\text{导函数} \quad (x^\alpha)' = \alpha x^{\alpha-1} \quad (a^x)' = a^x \ln a \quad (\log_a x)' = \frac{1}{x \ln a}$$

$$(\sin x)' = \cos x \quad (\cos x)' = -\sin x \quad (\tan x)' = \sec^2 x \quad (\cosh x)' = \sinh x \quad (\sinh x)' = \cosh x \quad (\tanh x)' = \operatorname{sech}^2 x$$

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}} \quad (\arccos x)' = \frac{-1}{\sqrt{1-x^2}} \quad (\arctan x)' = \frac{1}{1+x^2} \quad (\operatorname{arsinh} x)' = \frac{1}{\sqrt{1+x^2}} \quad (\operatorname{arch} x)' = \frac{1}{\sqrt{x^2-1}} \quad (\operatorname{artanh} x)' = \frac{1}{1-x^2}$$

$$(\csc x)' = -\csc x \cot x \quad (\sec x)' = \sec x \tan x \quad (\cot x)' = -\csc^2 x$$

$$(\operatorname{arcsec} x)' = \frac{-1}{|x|\sqrt{x^2-1}} \quad (\operatorname{arccsc} x)' = \frac{1}{|x|\sqrt{x^2-1}} \quad (\operatorname{arccot} x)' = -\frac{1}{1+x^2}$$

$$\text{法则 } (f \pm g)' = f' \pm g' \quad (fg)' = f'g + fg' \quad \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2} \quad (\text{可推广到有限个})$$

$$d(f \pm g) = df \pm dg \quad d(cf) = cdf \quad d(fg) = gdf + fdg \quad d\left(\frac{f}{g}\right) = \frac{gdf - f dg}{g^2} \quad (g \neq 0) \quad \frac{d^n f}{dx^n} = \frac{d^n f}{du^n} \left(\frac{du}{dx}\right)^n$$

$$\text{反函数求导 } [f^{-1}(y)]'_y = \frac{1}{f'(x)} \quad \text{微分形式不变性 } df = f'(u)du, \text{无论 } u \text{ 是自变量还是中间变量}$$

$$\text{高阶导数 } (a^x)^{(n)} = a^x \ln^n a \quad (a > 0) \quad (e^{kx})^{(n)} = k^n e^{kx} \quad \left(\frac{1}{x \pm 1}\right)^{(n)} = (-1)^n \frac{n!}{(x \pm 1)^{n+1}}$$

$$(\sin kx)^{(n)} = k^n \sin(kx + n \cdot \frac{\pi}{2}) \quad (\cos kx)^{(n)} = k^n \cos(kx + n \cdot \frac{\pi}{2})$$

$$\text{莱布尼茨公式 } (fg)^{(n)} = \sum_{k=0}^n C_n^k f^{(n-k)} g^{(k)}$$

$$\text{欧拉齐次函数定理 } k \text{ 次齐次 } n \text{ 元函数 } f(\lambda x_1, \dots, x_n) = \lambda^k f(x_1, \dots, x_n), \text{ 有 } \sum_{i=1}^n \frac{\partial f}{\partial x_i} x_i = kf$$

行列式求导 一行/列求导其它行/列不动, 所有行/列的单独求导的求和

$$\begin{aligned} \text{Jacobian matrix} \quad J &= \frac{\partial (f_1, \dots, f_m)}{\partial (x_1, \dots, x_n)} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \dots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} & \text{Wronskian determinant} \quad W(x) = W(f_1, \dots, f_n) = \begin{vmatrix} f_1(x) & \dots & f_n(x) \\ f_1'(x) & \dots & f_n'(x) \\ \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & \dots & f_n^{(n-1)}(x) \end{vmatrix} \\ \text{雅可比矩阵} \quad J &= m \times n & \text{雅可比行列式 } (f_1, \dots, f_n \text{ 有公共定义域, } n-1 \text{ 次可微}) \end{aligned}$$

隐函数组求导

微分表

积分表

$$\int \tan x \, dx = -\ln |\cos x| + C \quad \int \cot x \, dx = \ln |\sin x| + C \quad \int \operatorname{th} x \, dx = \ln(\operatorname{ch} x) + C$$

$$\int \sec x \, dx = \ln |\sec x + \tan x| + C \quad \int \csc x \, dx = \ln |\csc x - \cot x| + C = \ln |\tan \frac{x}{2}|$$

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a} \arctan \frac{x}{a} + C \quad \int \frac{dx}{x^2-a^2} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C \quad \int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin \frac{x}{a} + C$$

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \ln(x+\sqrt{x^2+a^2}) + C = \operatorname{arsh} \frac{x}{a} + C \quad \int \frac{dx}{\sqrt{x^2-a^2}} = \ln|x+\sqrt{x^2-a^2}| + C = \frac{x}{|a|} \operatorname{arsh} \frac{|x|}{a} + C$$

有理函数分解因式 $\frac{Ax+B}{(x-a)^n}$, 三角函数拆奇次项多项式, 三角换元画辅助三角形换回

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx = \int_0^{\frac{\pi}{2}} \cos^n x \, dx = \frac{n-1}{n} I_{n-2} = \frac{(n-1)!!}{n!!} \rightarrow \text{奇 } I_1 = 1, \text{ 偶 } I_0 = \frac{\pi}{2}$$

$$\int_0^{\frac{\pi}{2}} \tan^n x \, dx = \left[\int_0^{\frac{\pi}{2}} \tan^{n-2} x (\sec^2 x - 1) \, dx \right] = \frac{1}{n-2} - \int_0^{\frac{\pi}{2}} \tan^{n-2} x \, dx$$

$$\text{凑微分 } \int_0^1 f(y) dy \int_0^y f(t) dt = \int_0^1 d \left(\int_0^y f(x) dx \right) \int_0^y f(t) dt = \frac{1}{2} \left(\int_0^y f(t) dt \right)^2 \Big|_0^1$$

$$\text{凑全微分 } y \, dx + x \, dy = d(xy) \quad x \, dx + y \, dy = d\left(\frac{x^2+y^2}{2}\right) \quad \frac{1}{y^2} (y \, dx - x \, dy) = d\left(\frac{x}{y}\right)$$

$$\frac{1}{x^2+y^2} (x \, dx + y \, dy) = d\sqrt{x^2+y^2} \quad \frac{1}{x^2+y^2} (x \, dx + y \, dy) = d\left[\frac{1}{2} \ln(x^2+y^2)\right] \quad \frac{1}{(x-y)^2} (y \, dx - x \, dy) = d\left(\frac{xy}{x-y}\right)$$

$$\frac{1}{xy} (y \, dx - x \, dy) = d\left(\ln\left|\frac{x}{y}\right|\right) \quad \frac{1}{x^2+y^2} (y \, dx - x \, dy) = d\left(\arctan\frac{x}{y}\right) \quad \frac{1}{x^2+y^2} (y \, dx - x \, dy) = \frac{1}{2} d\left(\ln\left|\frac{x-y}{x+y}\right|\right)$$

$$\text{分部积分 } \int_a^b f(x) g(x) \, dx = f(x)g(x) \Big|_a^b - \int_a^b f'(x) g(x) \, dx$$

$$\text{配对积分 } \int [f'(x) + f(x) g'(x)] e^{g(x)} \, dx = e^{g(x)} f(x) + C$$

$$\text{变量积分 } \frac{d}{dt} \left(\int_{v(t)}^{u(t)} f(x) \, dx \right) = f(u(t)) u'(t) - f(v(t)) v'(t) \quad (\text{常数限求导为零})$$

$$\text{换元必换限 } \int_a^b f(x) \, dx \stackrel{x=u(t)}{=} \int_{u(a)}^{u(b)} f(u(t)) u'(t) \, dt$$

$$\int_a^b f(u(t)) u'(t) \, dt \stackrel{\text{配元不换限}}{=} \int_a^b f(u(t)) \, du(t) \stackrel{\text{换元必换限}}{=} \int_{u(a)}^{u(b)} f(x) \, dx$$

$$\text{Improper Integral} \quad \int_a^\infty \frac{dx}{x^p} = \frac{a^{1-p}}{p-1} \quad (a>0, p>1)$$

$$\int_0^\infty \frac{dx}{e^x+1} = \frac{\pi i}{2} \quad [(2I)^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} \, dx \, dy = \int_0^{\infty} r \, dr \int_0^{\infty} e^{-r^2} \, dr = \pi]$$

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} \, dx = (n-1) \Gamma(n-1) = (n-1)! \quad \Gamma(\frac{1}{2}) = [2 \int_0^\infty e^{-t^2} \, dt] = \sqrt{\pi}$$

$$\int_0^\infty \frac{x \, dx}{e^x+1} = \int_0^\infty \left(\sum_{k=1}^{\infty} (-1)^{k-1} e^{-kx} \right) x \, dx = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k^2} = \frac{\pi^2}{12}$$

$$I_n = \int_0^\infty \frac{x^{n-1} \, dx}{e^x-1} = \sum_{k=1}^{\infty} \frac{1}{k^n} \Gamma(n) \quad I_2 = \frac{\pi^2}{6} \approx 1.645 \quad I_{\frac{3}{2}} \approx 2.315 \quad I_3 \approx 2.404 \quad I_{\frac{5}{2}} \approx 1.783 \quad I_4 = \frac{\pi^4}{15} \approx 6.494$$

$$I_n = \int_0^\infty e^{-\alpha x^2} x^n \, dx = -\frac{\partial}{\partial \alpha} \int_0^\infty e^{-\alpha x^2} x^{n-2} \, dx \quad I_0 = \frac{\sqrt{\pi}}{2} \alpha^{-\frac{1}{2}} \quad I_1 = \frac{1}{2} \alpha^{-1} \quad I_2 = \frac{\sqrt{\pi}}{4} \alpha^{-\frac{3}{2}} \quad I_3 = \frac{1}{2} \alpha^{-2} \quad I_4 = \frac{3\sqrt{\pi}}{8} \alpha^{-\frac{5}{2}}$$

积分表

级数表

傅里叶<信号>

$$\int_0^1 f(x) dx = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n f\left(\frac{i}{n}\right)$$

$$F\left(e^{(x-x_0)\frac{d}{dx}}\right) f(x) |_{x_0}$$

Taylor formula

$$泰勒公式 f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \begin{cases} \text{佩亚诺余项 } O((x-x_0)^n) \\ \text{拉格朗日余项 } \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1} (x_0 \leq \xi \leq x) \\ \text{柯西余项 } \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)(x-\xi)^n (x_0 \leq \xi \leq x) \end{cases}$$

麦克劳林公式 ($0 < \xi < x$)

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + o(x^n)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^n \frac{x^n}{n} + o(x^n)$$

求导

$$\sin x = x - \frac{x^3}{3!} - \dots + (-1)^{\frac{n-1}{2}} \frac{x^{2n-1}}{(2n-1)!} + o(x^{2n}) \quad x \in \mathbb{R}$$

$$\cos x = 1 - \frac{x^2}{2!} - \dots + (-1)^n \frac{x^{2n}}{(2n)!} + o(x^{2n+1}) \quad x \in \mathbb{R}$$

$$\text{牛顿二项式公式 } (x+y)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^{\alpha-k} y^k \quad \binom{\alpha}{k} = \frac{\alpha(\alpha-1)\dots(\alpha-k+1)}{k!} \quad (\alpha \in \mathbb{R})$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!} x^3 + \dots + \binom{\alpha}{n} x^n + \dots$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots + (-x)^n + \dots \quad x \in (-1, 1)$$

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{2 \cdot 4} x^2 + \frac{1 \cdot 3}{2 \cdot 4 \cdot 6} x^3 + \dots + (-1)^{\frac{n-1}{2}} \frac{(2n-3)!!}{(2n)!!} x^n + \dots \quad x \in [-1, 1]$$

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{1}{2}x + \frac{1 \cdot 3}{2 \cdot 4} x^2 - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} x^3 + \dots + (-1)^n \frac{(2n-1)!!}{(2n)!!} x^n + \dots \quad x \in (-1, 1)$$

$$[\text{积 } \frac{1}{1+x^2}] \arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots + (-1)^n \frac{x^{2n+1}}{2n+1} \quad x \in [-1, 1] \rightarrow \pi = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

$$\operatorname{sh} x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \quad \operatorname{ch} x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \quad x \in \mathbb{R} \quad \operatorname{arth} x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} \quad x \in (-1, 1)$$

级数表

常微表

初等积分法 将微分方程的解用初等函数或其积分表示

$$\textcircled{1} \frac{dy}{dx} = f(x)g(y) \quad \text{分离变量 } \frac{dy}{g(y)} = f(x)dx \quad \text{若 } g(y_0) = 0 \text{ 的 } y = y_0 \text{ 不在通解里, 要补上这个特解}$$

$$\textcircled{2} \frac{dy}{dx} = f\left(\frac{x}{y}\right) \quad \text{令 } \frac{y}{x} = u, y' = (ux)' = u'x + u = f(u) \rightarrow \textcircled{1} \quad \text{若 } f(u) = u_0 \text{ 要补上特解 } u = u_0$$

$$\textcircled{3} \quad \text{令 } C_1 = C_2 = 0 \rightarrow \textcircled{2}$$

$$\frac{dy}{dx} = f\left(\frac{a_1x+b_1y+C_1}{a_2x+b_2y+C_2}\right) \quad \begin{cases} |a_1 b_1| = 0 \rightarrow f\left(\frac{u+C_1}{\lambda u+C_2}\right) \rightarrow \textcircled{1} \\ |a_1 b_1| \neq 0 \rightarrow f\left(\frac{a_1x+b_1y}{a_2x+b_2y}\right) \rightarrow \textcircled{3} \end{cases} \quad \textcircled{4} \frac{dy}{dx} = xf\left(\frac{y}{x^2}\right) \quad \text{令 } \frac{y}{x^2} = u$$

$$\textcircled{5} \frac{dy}{dx} = f(ax+by+c) \quad \text{令 } ax+by+c = u$$

$$\text{first-order homogeneous } \frac{dy}{dx} = P(x)y \quad \textcircled{1} \quad y = C e^{\int P(x)dx}$$

$$\text{nonhomogeneous } \frac{dy}{dx} = P(x)y + Q(x) \quad [\text{常数变易法}] \quad y = e^{\int P(x)dx} \left(\int Q(x)e^{-\int P(x)dx} dx + C \right)$$

$$\text{Bernoulli differential equation} \quad \frac{dy}{dx} = P(x)y + Q(x)y^n \quad (P, Q \text{ 连续}, n \neq 0, 1) \quad \frac{1-n}{1-n} \frac{dy^{1-n}}{dx} = P(x)y^{1-n} + Q(x) \quad (n > 0 \text{ 时补上 } y=0)$$

$$\text{total differential equation} \quad \text{全微分方程 } P(x, y)dx + Q(x, y)dy = 0 = du(x, y) = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \iff \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

$$(P, Q \text{ 有连续偏导}) \quad \text{通解 } u(x, y) = C \quad \langle \text{常微分方程与因子 分步组合法} \rangle$$

$$\text{积分与路径无关法 } u = \int_{x_0}^x P(t, y_0)dt + \int_{y_0}^y Q(x, t)dt = \int_{y_0}^y Q(x_0, t)dt + \int_{x_0}^x P(t, y)dt$$

$$\text{不定积分法 } \frac{\partial u}{\partial x} = P \quad \text{两边对 } x \text{ 求导} \rightarrow u = \int P dx + \varphi(y) \rightarrow \frac{\partial u}{\partial y} = Q \rightarrow \text{对比得 } \varphi$$

$$\text{integrating factor } \mu(x, y) \neq 0 \text{ 连续可微, 使 } \mu(Pdx + Qdy) = du$$

-阶隐式微分方程 有连续偏导, 不能显式地解出 $\frac{dy}{dx}$ 或 $\frac{dx}{dy}$ 的表达式

$$\textcircled{1} \quad P = \varphi(x, c), y = f(x, \varphi(x, c))$$

$$\textcircled{1} \quad y = f(x, y') \quad \frac{y' = P}{\text{两边对 } x \text{ 求导}} \rightarrow P = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial p} \cdot \frac{dp}{dx} \quad \text{通解} \quad \begin{cases} \textcircled{2} \quad x = \psi(p, c), y = f(\psi(p, c), p) \quad (\text{参数形式的通解}) \\ \textcircled{3} \quad \psi(x, p, c) = 0, y = f(x, p) \quad (p \text{ 为参数, } c \text{ 为常数}) \end{cases}$$

$$\textcircled{2} \quad x = f(y, y') \quad \frac{y' = P}{\text{两边对 } y \text{ 求导}} \rightarrow \frac{1}{P} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial p} \cdot \frac{dp}{dy} \quad \text{通解} \quad \begin{cases} \psi(y, p, c) = 0, x = f(y, p) \end{cases}$$

$$\textcircled{3} \quad F(x, y') = 0 \quad \frac{dy = Pdx}{F(x, p) \text{ 化参数式}} \quad \begin{cases} x = \varphi(t) \\ P = \psi(t) \end{cases} \rightarrow \begin{cases} x = \varphi(t) \\ y = \int \psi(t) \varphi'(t) dt + C \end{cases} \quad \rightarrow \text{然后验证分母=0是否是奇解}$$

$$\textcircled{4} \quad F(y, y') = 0 \quad \frac{dx = \frac{1}{P} dy}{F(y, p) \text{ 化参数式}} \quad \begin{cases} y = \varphi(t) \\ P = \psi(t) \end{cases} \rightarrow \begin{cases} y = \varphi(t) \\ x = \int \frac{\psi(t)}{\varphi'(t)} dt + C \end{cases} \quad \begin{array}{l} \text{若 } F(y, 0) = 0 \text{ 有实根 } y = k \\ \text{则 } y = k \text{ 是奇解} \end{array}$$

\langle 商化? $\rangle F(\lambda) = 0$ 有 k 重根 λ_i
 [消去 $(\lambda - \lambda_i)^k$] $\Rightarrow F^{(i)}(\lambda_i) = 0$ (i 从 0 到 $k-1$) $\text{No. 且 } F^{(k)}(\lambda_i) \neq 0$

Date

合集

给定单参数曲线族 $\Phi(x, y; c) = 0$ 包络不属于曲线族, 过曲线每一点有曲线族中一曲线和它在这点相切

C-判别曲线 $\Phi = 0$ 和 $\Phi'_c = 0$ 消去 C 得到的曲线 二包络 (需画草图检验)

Th. 微分方程奇解上每一点都至少还有一个解存在

P-判别曲线 $F(x, y, p) = 0$ 连续可微, 奇解 $\subseteq F = 0$ 和 $F_p = 0$ 消去 p 得到的曲线 (需化回原方程检验)

Th. 一阶 (其包络无几何意义) 微分方程通解的包络 \Leftrightarrow 微分方程的奇解

克莱罗微分方程 $y = xp + f(p)$, $p = y'$, f 连续可微 [两边对 x 求导 $0 = y''(x + f')$] 通解是 P 换成 C 的直线族

n 阶齐次线性微分方程 $y^{(n)} + \sum_{i=1}^n a_i(x)y^{(n-i)} = 0$ 有 n 个解 $y_{1:n}(x)$, 则 $W[y_{1:n}(x_0)] = W[y_{1:n}(x_0)] e^{-\int_{x_0}^x a_i(t)dt}$

Liouville formula 刘维尔公式 \rightarrow 若 $y_1(x)$ 是 $y'' + a_1(x)y' + a_2(x)y = 0$ 的解, 则 $y_2(x) = y_1(x) \left(C_1 \int \frac{1}{y_1^2(x)} e^{-\int_{x_0}^x a_1(t)dt} dx + C_2 \right)$ 也是方程的解

n 阶常系数齐次线性微分方程 $y^{(n)} + \sum_{i=1}^n a_i y^{(n-i)} = 0$ [设 $y = e^{\lambda x}$, 方程化为 $(\lambda^n + \sum_{i=1}^n a_i \lambda^{n-i}) e^{\lambda x} = 0$] 解 特征方程

k 重根 (实重根入对应 k 个 $\rightarrow y = x^j e^{\lambda x}$, $j = 0, \dots, (k-1)$) [范德蒙] 各基本解线性无关

复重根 $\alpha \pm \beta i$ 对应 $2k$ 个 $\rightarrow y = x^j e^{\alpha x} \sin \beta x$, $y = x^j e^{\alpha x} \cos \beta x$ [复值解性质变换] 乘 C_i 线性组合即通解

$y^{(n)} + \sum_{i=1}^n a_i y^{(n-i)} = f(x)$, f 为连续函数 (非齐次可用常数变易法, 但有两类特殊形式用比较系数法更简便)

$f(x) = P_m(x) e^{\lambda x} \rightarrow y^* = x^k Q_m(x) e^{\lambda x}$ (k 为 λ 的重根数, 不是根取 $k=0$, 下标 m 表示 m 次多项式) 复系数

$f(x) = (A_{m1}(x) \cos \beta x + B_{m2}(x) \sin \beta x) e^{\alpha x} \rightarrow y^* = x^k (P_{n1}(x) \cos \beta x + Q_{n2}(x) \sin \beta x) e^{\alpha x}$ ($n_1, n_2 \leq \max\{m_1, m_2\}$)

欧拉方程 $x^n y^{(n)} + \sum_{i=1}^n a_i x^{n-i} y^{(n-i)} = 0$ $\xrightarrow{t = \ln|x|} x^i y^{(i)} = \frac{d}{dt} (\frac{d}{dt} - 1) \cdots (\frac{d}{dt} - i+1) y$

[解出 $y = e^{\lambda t} \rightarrow y = x^\lambda$] 特征方程为 $\lambda(\lambda-1)\cdots(\lambda-n+1) + \sum_{i=1}^n a_i \lambda^{i-n+1} = 0$

实重根 $\rightarrow y = \ln^j |x| x^\lambda$, 复重根 $\rightarrow \ln^j |x| x^\alpha \sin(\beta \ln |x|)$, $\ln^j |x| x^\alpha \cos(\beta \ln |x|)$

无阻尼自由振动 $\frac{d^2x}{dt^2} + \omega^2 x = 0$ (微小振动 $\theta \approx 0$) 通解 $x = A \sin(\omega t + \theta)$, $A = \sqrt{C_1^2 + C_2^2}$, $\theta = \arctan \frac{C_1}{C_2}$

有阻尼自由振动 $\frac{d^2x}{dt^2} + 2\zeta \frac{dx}{dt} + \omega^2 x = 0$ (+阻尼) $x = A e^{-\zeta t} \sin(\omega_d t + \theta)$, $\omega_d^2 = \omega^2 - \zeta^2$ ($\zeta > 1$ 则至少过一次平衡位置)

小阻尼强迫振动 $\frac{d^2x}{dt^2} + 2\zeta \frac{dx}{dt} + \omega^2 x = H \sin(pt + \theta_0)$ $x = \text{齐通} + \frac{H \sin(pt + \theta_0)}{\sqrt{(p^2 - \omega^2)^2 + 4\zeta^2 p^2}}$ ($p = \sqrt{\omega^2 - 2\zeta^2}$ 时共振)

参考文献

$$s = \sigma + j\omega$$

傅里叶变换

$$\mathcal{F}[f(t)] = F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \xleftarrow[\sigma=0]{\text{双边拉普拉斯}} \mathcal{L}[f(t)] = F(s) = \int_0^{\infty} f(t) e^{-st} dt$$

傅里叶逆变换

$$\mathcal{F}^{-1}[F(\omega)] = f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega \quad (\text{绝对收敛})$$

$$\mathcal{L}^{-1}[F(s)] = f(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} F(s) e^{st} ds \quad (s>0)$$

对称

$$\mathcal{F}[\mathcal{F}[f(t)]] = 2\pi f(-t)$$

线性

$$\mathcal{F}[\sum a f_i(t)] = \sum a F_i(\omega)$$

$$\mathcal{L}[\sum a f_i(t)] = \sum a F_i(s)$$

延时

$$\mathcal{F}[f(a(t-t_0))] = \frac{1}{|a|} F\left(\frac{\omega}{a}\right) e^{-j\omega \frac{t_0}{a}} \quad (a \neq 0)$$

$$\mathcal{L}[f(a(t-t_0)) u(t-t_0)] = \frac{1}{a} F\left(\frac{s}{a}\right) e^{-s \frac{t_0}{a}}$$

频移

$$\mathcal{F}^{-1}[F(\omega+\omega_0)] = f(t) e^{-j\omega_0 t}$$

$$\mathcal{L}^{-1}[F(s+a)] = f(t) e^{-at}$$

微分

$$\mathcal{F}[f^{(n)}(t)] = (j\omega)^n F(\omega)$$

$$\mathcal{L}[f^{(n)}(t)] = s^n F(s) - s^{n-1} f(0_-) - \dots - s^0 f^{(n-1)}(0_-)$$

$$\mathcal{F}^{-1}[F^{(n)}(\omega)] = (-j\omega)^n f(t)$$

$$\mathcal{L}^{-1}[F^{(n)}(s)] = (-t)^n f(t)$$

卷积

$$\mathcal{F}[f_1 * f_2] = F_1 \cdot F_2 \quad \text{可推广到n个}$$

$$\mathcal{L}[f_1 * f_2] = F_1 \cdot F_2$$

$$\mathcal{F}[f_1 \cdot f_2] = \frac{1}{2\pi} F_1 * F_2$$

$$\mathcal{L}[f_1 \cdot f_2] = \frac{1}{2\pi j} F_1 * F_2$$

积分

$$\mathcal{F}\left[\int_{-\infty}^t f(\tau) d\tau\right] = \frac{1}{j\omega} F(\omega) + \pi F(0) \delta(\omega)$$

$$\mathcal{L}\left[\int_0^t f(\tau) d\tau\right] = \frac{1}{s} [F(s) + f'(0_-)]$$

$$\mathcal{L}\left[\int_s^{\infty} F(\sigma) d\sigma\right] = \frac{1}{s} f(t)$$

$$\mathcal{F}[1] = 2\pi \delta(\omega)$$

$$\mathcal{F}[\delta(t)] = 1 \quad \mathcal{F}[u(t)] = \frac{1}{j\omega} + \pi \delta(\omega)$$

$$\mathcal{L}[\delta^{(n)}(t)] = s^n$$

$$\mathcal{F}[e^{j\omega_0 t}] = 2\pi \delta(\omega - \omega_0)$$

$$\mathcal{F}[e^{-at} u(t)] = \frac{1}{j\omega + a} \quad \mathcal{L}[t^n e^{-at}] = \frac{n!}{(s+a)^{n+1}} \quad (s > -a)$$

$$\mathcal{F}[\cos(\omega_0 t)] = \frac{1}{2} [\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$$

$$\mathcal{L}[\cos t + j \sin t = e^{j\omega t}] \rightarrow \frac{s + j\omega}{s^2 + \omega^2} = \frac{1}{s - j\omega}$$

$$\mathcal{F}[\sin(\omega_0 t)] = j\pi [\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]$$

$$\mathcal{F}[\Pi_{\omega}] = \pi \operatorname{Sa}\left(\frac{\omega}{2}\right)$$