On the Minimum Relay Node Cover Problem

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Overview

Introduction

- ▷ Wireless Sensor Networks
- ▷ Two-Tired Relay Node Placement Problem
- ▷ Minimum Geographic Disk Cover Problem

Minimum Relay Node Cover Problem

- ▷ MRNC is Born!
- ▷ MGDC vs MRNC
- ▷ Appying The Shifting Strategy
- ▷ Inapproximability Analysis
- ▷ Straightforward Method

Conclusion and Future Works

- ▷ Conclusion
- ▷ Future Works

References

Two-Tired Wireless Sensor Network Example

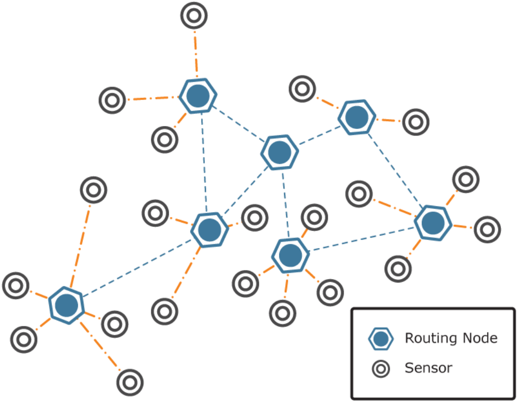

Figure 1: Wireless Sensor Network

[1]

2tRNP Problem Formulation [2]

Given:

1. A set $\mathfrak{X} = \{\mathfrak{x}_1, \mathfrak{x}_2, \ldots, \mathfrak{x}_\mathfrak{n}\}$ of sensor nodes with known locations.
2. $r > 0$ and $R > r$, the communication ranges for sensor nodes and relay nodes respectively.

Seek:

A placement of a set $\mathfrak{Y} = \{\mathfrak{y}_1, \mathfrak{y}_2, \ldots, \mathfrak{y}_\mathfrak{m}\}$ of relay nodes, such that

1. $\forall \mathfrak{x}_\mathfrak{i} \in \mathfrak{X}, \exists \mathfrak{y}_\mathfrak{j} \in \mathfrak{Y}$ such that $\|\mathfrak{x}_\mathfrak{i}\mathfrak{y}_\mathfrak{j}\| \leq r$.
2. The undirected graph $\mathfrak{G}\langle\mathfrak{V}, \mathfrak{E}\rangle$ is **connected**, where $\mathfrak{V} = \mathfrak{Y}$, and $\mathfrak{E} = \{(\mathfrak{y}_\mathfrak{i}, \mathfrak{y}_\mathfrak{j}) | \mathfrak{y}_\mathfrak{i}, \mathfrak{y}_\mathfrak{j} \in \mathfrak{Y}, \|\mathfrak{y}_\mathfrak{i}\mathfrak{y}_\mathfrak{j}\| \leq R\}$. By **connected**, it means every relay node in this graph is reachable from all other relay nodes.
3. $|\mathfrak{Y}|$ is minimum.

A General Framework for 2tRNP [2]

INPUT: A set $\mathfrak{X} = \{\mathfrak{x}_1, \mathfrak{x}_2, \ldots, \mathfrak{x}_\mathfrak{n}\}$ of sensor nodes in the Euclidean plane. $r > 0$ and $R > r$, the communication ranges for sensor nodes and relay nodes respectively. An approximation algorithm $\mathscr{A}$ for the *MGDC* problem, an approximation algorithm $\mathscr{B}$ for the *SMT-MSPBEL* problem.

OUTPUT: A set $\mathfrak{R} = \{\mathfrak{r}_1, \mathfrak{r}_2, \ldots, \mathfrak{r}_\mathfrak{k}\}$ of relay nodes.

Step 1: Apply $\mathscr{A}$ to $\mathfrak{X}$ to obtain a set of points $\mathfrak{C} = \{\mathfrak{c}_1, \ldots, \mathfrak{c}_\mathfrak{n}\}$. Without loss of generality, we assume that $\mathfrak{C}$ is minimal.

Step 2: Construct a set $\mathfrak{D} \subseteq \mathfrak{X}$ such that $\forall \mathfrak{c}_\mathfrak{i} \in \mathfrak{C}, \exists! \mathfrak{d}_\mathfrak{j} \in \mathfrak{D}$ such that $\|\mathfrak{d}_\mathfrak{j}\mathfrak{c}_\mathfrak{i}\| \leq r$, and $\forall \mathfrak{d}_\mathfrak{j} \in \mathfrak{D}, \exists \mathfrak{c}_\mathfrak{i} \in \mathfrak{C}$ such that $\|\mathfrak{d}_\mathfrak{j}\mathfrak{c}_\mathfrak{i}\| \leq r$.

Step 3: Apply $\mathscr{B}$ to obtain a set of relay nodes $\mathfrak{Y}$ that is an feasible solution for *SMT-MSPBEL*$(\mathfrak{D}, \mathfrak{R}, \mathfrak{R})$.

Step 4: Output $\mathfrak{R} = \mathfrak{C} \cup \mathfrak{D} \cup \mathfrak{Y}$.

The Core is ...

INPUT: A set $\mathfrak{X} = \{\mathfrak{x}_1, \mathfrak{x}_2, \ldots, \mathfrak{x}_\mathfrak{n}\}$ of sensor nodes in the Euclidean plane. $r > 0$ and $R > r$, the communication ranges for sensor nodes and relay nodes respectively. An approximation algorithm $\mathscr{A}$ for the *MGDC* problem, an approximation algorithm $\mathscr{B}$ for the *SMT-MSPBEL* problem.

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Step 4: Output $\mathfrak{R} = \mathfrak{C} \cup \mathfrak{D} \cup \mathfrak{Y}$.

MGDC Problem Formulation [2]

Given:

1. A set $\mathfrak{X} = \{\mathfrak{x}_1, \mathfrak{x}_2, \ldots, \mathfrak{x}_\mathfrak{n}\}$ of points in the Euclidean plane $\mathbb{R}^2$.
2. A constant value $r > 0$ as the radius of disks.

Seek:

A set $\mathfrak{B} = \{\mathfrak{b}_1, \mathfrak{b}_2, \ldots, \mathfrak{b}_\mathfrak{m}\}$ of disks such that

1. $\forall \mathfrak{x}_\mathfrak{i} \in \mathfrak{X}, \exists \mathfrak{b}_\mathfrak{j} \in \mathfrak{B}$ such that $\|\mathfrak{x}_\mathfrak{i}\mathfrak{b}_\mathfrak{j}\| \leq r$.
2. $|\mathfrak{B}|$ is minimum.

Two Things About MGDC

- In 1981, Masuyama,Shigeru, et al. and Fowler, Robert J. et al. proved that MGDC is *NP-hard*.[7][8]
- The state-of-art approach delivering a PTAS is the *Shifting Strategy* proposed by Hochbaum, Dorit S. et al. in 1985.[3]

The Shifting Strategy [3]

1. Given a set of points enclosed in an area $\mathcal{I}$ in a specified space, the goal is to find a minimal number of disks of diameter D to cover all the points.
2. Given a constant $l \geq 1$ called the **shifting parameter**, we partition $\mathcal{I}$ into vertical strips, left closed and right open, of width D.
3. Make each l consecutive strips into a group. We call the set of strips of width $l \cdot D$ a **shift partition**. There are l shift partitions.
4. Make partitions similarly in each dimension.
5. In each cell, e.g. a square, in the Euclidean plane, we search for a local optimal solution.
6. Combine all local solutions to obtain the final approximate solution for one shift partition.
7. Output the best one among all shift partitions.

A Shifting Strategy Example

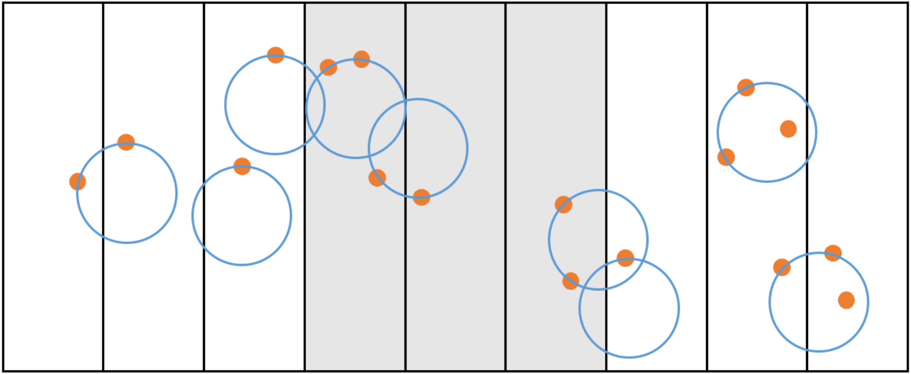

Figure 2: The shifting paramter $l = 3$. In the first shift, 9 disks are needed.

A Shifting Strategy Example (Cont.)

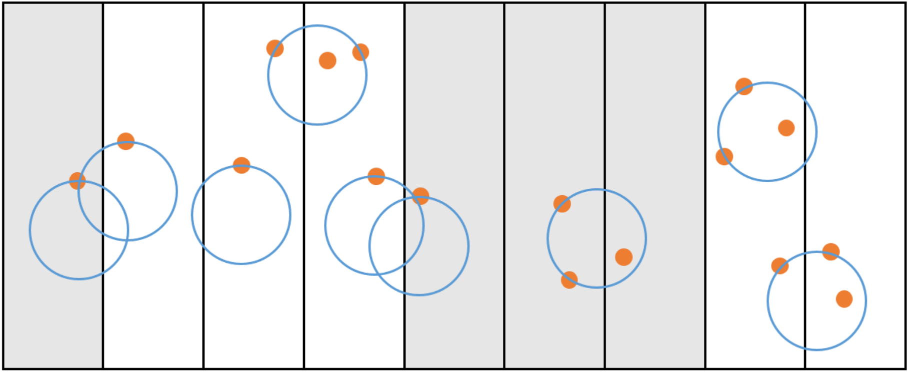

Figure 3: In the second shift, 9 disks are needed.

A Shifting Strategy Example (Cont.)

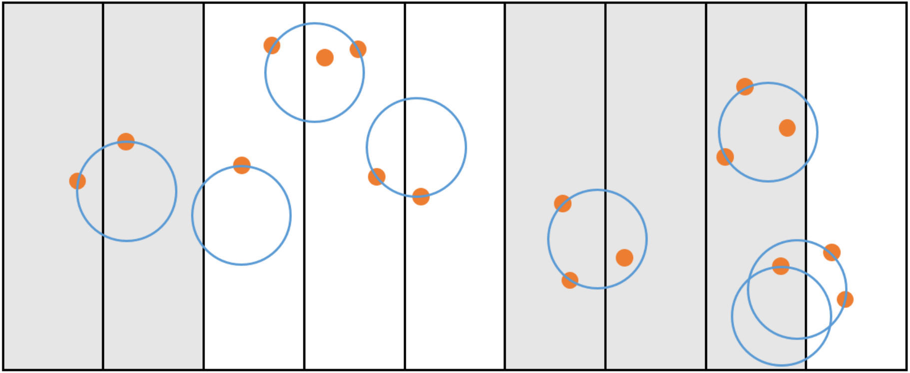

Figure 4: In the third shift, 8 disks are needed, and it is the best solution.

The Shifting Lemma [3]

- Let $\mathscr{A}$ be a local algorithm taking a given $l \cdot D$ width strip as input and delivering a set of disks fully covering all points in the strip.
- Let $\mathscr{A}(\mathcal{S}_i)$, where $\mathcal{S}_i \in \{\mathcal{S}_1, \ldots, \mathcal{S}_l\}$ is a given shift partition, represent a solution for all points applying $\mathscr{A}$ to each $l \cdot D$ strip.
- Let $\mathscr{S}_{\mathscr{A}}$ be the shift algorithm defined for $\mathscr{A}$ delivering the set of disks of the minimum cardinality among $\{\mathscr{A}(\mathcal{S}_1), \ldots, \mathscr{A}(\mathcal{S}_l)\}$.
- Let *OPT* denote the globally optimal solution, $|OPT|$ denote its cardinality.
- Let $\mathfrak{r}_{\mathscr{B}}$ denote the approximation ratio.

$$\mathfrak{r}_{\mathscr{S}_{\mathscr{A}}} \leq \mathfrak{r}_{\mathscr{A}} \cdot (1 + \frac{1}{l})$$

Rephrase MGDC

Given:

1. A set $\mathfrak{X} = \{\mathfrak{x}_1, \mathfrak{x}_2, \ldots, \mathfrak{x}_n\}$ of sensor nodes in the Euclidean plane $\mathbb{R}^2$.
2. A constant value $r > 0$ as the communication range of sensor nodes.
3. A constant value $R > r$ as the communication range of relay nodes.

Seek:

A set $\mathfrak{B} = \{\mathfrak{b}_1, \mathfrak{b}_2, \ldots, \mathfrak{b}_m\}$ of relay nodes such that

1. $\forall \mathfrak{x}_i \in \mathfrak{X}, \exists \mathfrak{b}_j \in \mathfrak{B}$ such that $\|\mathfrak{x}_i \mathfrak{b}_j\| \leq r$.
2. $|\mathfrak{B}|$ is minimum.

MRNC is Born!

What if...

<u>Given:</u>

1. A set $\mathfrak{X} = \{\mathfrak{x}_1, \mathfrak{x}_2, \ldots, \mathfrak{x}_n\}$ of sensor nodes in the Euclidean plane $\mathbb{R}^2$.
2. ~~A constant value $r > 0$ as the communication range of sensor nodes.~~ Each sensor node has a constant value $r_i > 0$ as its communication range.
3. A constant value $R > r$ as the communication range of relay nodes.

<u>Seek:</u>

A set $\mathfrak{B} = \{\mathfrak{b}_1, \mathfrak{b}_2, \ldots, \mathfrak{b}_m\}$ of relay nodes such that

1. $\forall \mathfrak{x}_i \in \mathfrak{X}, \exists \mathfrak{b}_j \in \mathfrak{B}$ such that $\|\mathfrak{x}_i \mathfrak{b}_j\| \leq r_i$.
2. $|\mathfrak{B}|$ is minimum.

MRNC is Born! (Cont.)

Obvious to see that...

- MRNC is a generalized version MGDC.
- MRNC is NP-Hard.

Question

Can the Shifting Strategy give MRNC a PTAS?

MRNC Problem Formulation

Given:

1. A set $\mathfrak{S} = \{\mathfrak{s}_1, \mathfrak{s}_2, \ldots, \mathfrak{s}_\mathfrak{n}\}$ of sensor nodes in the Euclidean plane $\mathbb{R}^2$.
2. Every sensor node has an individual communication range $r_i > 0$, and for any two different sensor nodes, $\mathfrak{s}_\mathfrak{i}, \mathfrak{s}_\mathfrak{j}$, such that $\mathfrak{i} \neq \mathfrak{j}$, it is possible that $r_i \neq r_j$.
3. A constant value $R > r_i$ as the communication range of relay nodes.

Seek:

A set $\mathfrak{C} = \{\mathfrak{c}_1, \mathfrak{c}_2, \ldots, \mathfrak{c}_\mathfrak{m}\}$ of relay nodes such that

1. $\forall \mathfrak{s}_\mathfrak{i} \in \mathfrak{C}, \exists \mathfrak{c}_\mathfrak{j} \in \mathfrak{C}$ such that $\|\mathfrak{s}_\mathfrak{i}\mathfrak{c}_\mathfrak{j}\| \leq r_i$.
2. $|\mathfrak{C}|$ is minimum.

NP Optimization Problem (NPO)

Definition 1 [4]

An **NP Optimization (NPO)** problem A is a fourtuple $(I, sol, m, goal)$ such that

1. I is the set of instances of A and it is recognizable in polynomial time.
2. Given an instance x of I, $sol(x)$ denotes the set of feasible solutions of x. A polynomial p exists such that, $\forall x \forall y, y \in sol(x) \wedge |y| \leq p(|x|)$. Moreover, $\forall x \forall y$ such that $|y| \leq p(|x|)$, it is decidable in polynomial time whether $y \in sol(x)$.
3. Given an instance x and a feasible solution y of x, $m(x, y)$ denotes the positive integer measure of y. The function m is computable in polynomial time and is also called the objective function.
4. $goal \in \{max, min\}$

L-Reduction ($\leq_L^p$)

Definition 2 [5]

Given two NPO problems F and G, a **L-reduction (linear reduction)** from F to G is a tuple $\langle t_1, t_2, \alpha, \beta \rangle$ such that

1. t_1, t_2 are polynomial time computable funtions, and α, β are positive constants.
2. $t_1 : \mathcal{I}_F \rightarrow \mathcal{I}_G$ and $\forall x \in \mathcal{I}_F$ and $\forall y \in \mathcal{S}_G(t_1(x)), t_2(x, y) \in \mathcal{S}_F(x)$.
3. $opt_G(t_1(x)) \leq \alpha \cdot opt_F(x)$.
4. $\forall x \in \mathcal{I}_F$ and $\forall y \in \mathcal{S}_G(t_1(x))$,
   $|opt_F(x) - m_F(x, t_2(x, y))| \leq \beta \cdot |opt_G(t_1(x)) - m_G(t_1(x), y)|$.

If F L-reduces to G, we write $F \leq_L^p G$.

$MGDC \leq_L^p MRNC$

Proposition 1: $MGDC \leq_L^p MRNC$

Since the MRNC problem is directly a generalized version of the MGDC problem, it is trivial to prove this proposition.

Theorem 1 [6]

Assume that X, Y are optimization problems, and there is an L-reduction from X to Y. If there is a PTAS for Y, then there is a PTAS for X. Equivalently, if X does not have a PTAS, then Y does not have a PTAS.

Trouble Encountered...

Proposition 1 cannot tell whether or not MRNC has any PTAS, and it is hard to directly prove that MRNC could L-reduce to MGDC or not. Therefore, we have to examine how the Shifting Strategy would perform on MRNC.

The Proof of the Shifting Lemma [3]

The Shifting Lemma

$$\mathfrak{r}_{\mathscr{S}_{\mathscr{A}}} \leq \mathfrak{r}_{\mathscr{A}} \cdot (1 + \frac{1}{l})$$

Proof

Let $Z^{\mathscr{A}}$ denote the value of the solution delivered by algorithm $\mathscr{A}$. Since $\mathfrak{r}_{\mathscr{A}}$ is defined as the supremum of $\frac{Z^{\mathscr{A}}}{|OPT|}$ over all problem instances, then

$$\mathfrak{r}_{\mathscr{A}} \geq \frac{Z^{\mathscr{A}(J)}}{|OPT_J|}$$

where J denotes the Jth $l \cdot D$ strip, and $|OPT_J|$ is the local optimal solution for this strip. Then it is obvious to see that

$$Z^{\mathscr{A}(S_i)} = \sum_{J in S_i} Z^{\mathscr{A}(J)} \leq \mathfrak{r}_{\mathscr{A}} \cdot \sum_{J in S_i} |OPT_J|$$

The Proof of the Shifting Lemma (Cont.) [3]

Our Goal

Find the relation between $Z^{S_{\mathscr{A}}} = \min_{i=1,\ldots,l}(Z^{\mathscr{A}(S_i)})$ and $|OPT|$.

Core Problem

When and where do the errors happen?

Proof of the Shifting Lemma. (Cont.)

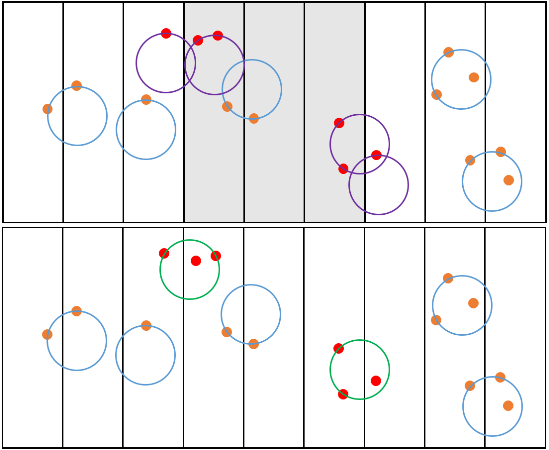

Figure 5: The optimal solution only needs 7 disks.

The Proof of the Shifting Lemma (Cont.) [3]

Errors

- Errors may happen when points in two adjacent strips could be covered by one disk.
- In an OPT, there might be some disks covering points across two adjacent strips. These disks could become errors in local optimal solutions.
- Let $OPT^{(i)}$ be the set of such disks in an OPT for the ith shift. Since for each of them, it could only cover points across at most two adjacent strips, then it is obvious to see that

$$\sum_{i=1}^{l} |OPT^{(i)}| \leq |OPT| \tag{1}$$

The Proof of the Shifting Lemma (Cont.) [3]

Proof (Cont.)

For a local optimal solution, it follows that

$$\sum_{JinS_i} |OPT_J| \leq |OPT| + |OPT^{(i)}| \tag{2}$$

From (1) and (2), it is easy to have

$$\sum_{i=1}^{l}(|OPT| + |OPT^{i}|) \leq (1+l) \cdot |OPT| \tag{3}$$

Therefore, it holds that

$$\min_{i=1,\dots,l}(\sum_{JinS_i} |OPT_J|) \leq \frac{1}{l} \cdot \sum_{i=1}^{l}(\sum_{JinS_i} |OPT_J|) \leq \frac{1+l}{l} \cdot |OPT| \tag{4}$$

The Proof of the Shifting Lemma (Cont.) [3]

Proof (Cont.)

Finally, we have

$$Z^{S_{\mathscr{A}}} \leq \min_{i=1,\ldots,l}(\mathfrak{r}_{\mathscr{A}} \cdot \sum_{JinS_i} |OPT_J|) \leq \mathfrak{r}_{\mathscr{A}} \cdot \frac{1+l}{l} \cdot |OPT| \tag{5}$$

The Proof of the Shifting Lemma (Cont.)

Time Complexity

If applying the Shifting Strategy twice, then the area $\mathcal{I}$ is divided into a grid, and for each cell, it is an $l \cdot D \times l \cdot D$ square. To fully cover such a square, it needs at most $4 \cdot l^2$ disks of diameter D.

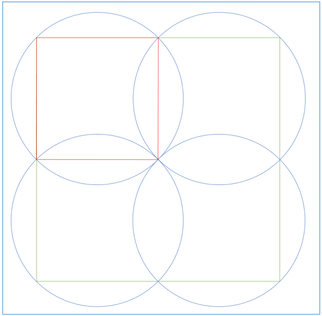

Figure 6: $l \cdot D \times l \cdot D$ square is covered by 4 unit disks.

The Proof of the Shifting Lemma (Cont.)

Time Complexity (Cont.)

There will be l^2 squares. If using two points to determine the position of a disk, then for each disk, it has at most 2 possible positions to place. We have n points in the plane. In each dimension, there will be l shifts. Therefore, the running time is $O(l^2 \cdot n^{2 \cdot 4 \cdot l^2})$.

Concepts Mapping

MGDC	**MRNC**
Point	Sensor Node
Disk	Relay Node (Overlap)
Points covered by a disk	Sensors overlapping together

Apply the Shifting Strategy to MRNC

CASE 1 Only a constant number of sensor nodes are of the different communication range from others.

- CASE 1-1 The special communication range is equal or smaller than the regular one.
- CASE 1-2 The special communication range is greater than the regular one.

CASE 2 m sensor nodes are of the special communication range from others where m is not constant.

CASE 3 No certain number of sensor nodes for either range.

CASE 1-1

Algorithm 1

INPUT: A set of sensor nodes $\mathfrak{S} = \mathfrak{S}_M \cup \mathfrak{S}_N$ in the Eulidean plane; r_M, r_N are communication ranges for $\mathfrak{S}_M, \mathfrak{S}_N$ respectively, and $r_M \geq r_N > 0$; $|\mathfrak{S}_M| = M$, $|\mathfrak{S}_N| = N$, and N is a constant; $l \geq 1$ is the Shifting Parameter.

OUTPUT: A set of relay nodes fully cover the sensor nodes.

STEP 1: Divide the area into $l \cdot 2r_M \times l \cdot 2r_M$ squares.

STEP 2: For each square, find a local optimal solution.

STEP 3: Go through all shifts, and find the best solution with the minimum cardinality.

CASE 1-1 (Cont.)

Time Complexity

This case is "large squares with some small sensors". For $\mathfrak{S}_M$, it is a typical application of the Shifting Strategy. However, because of the existence of $\mathfrak{S}_N$, $4 \cdot l^2$ relay nodes might not be fair enough to cover all sensors in a square. In the worst case, it needs at most $(4 \cdot l^2 + N)$, and it is still a constant, so the time complexity will not rise significantly.

CASE 1-1 (Cont.)

Approximation Performance

If thinking carefully, it is easy to find that the circumstance when errors would occur is still the same, i.e. sensors could only overlap with others across at most two adjacent strips. Therefore, the approximation performance will be still the same.

CASE 1-1

Algorithm 2

INPUT: A set of sensor nodes $\mathfrak{S} = \mathfrak{S}_M \cup \mathfrak{S}_N$ in the Eulidean plane; r_M, r_N are communication ranges for $\mathfrak{S}_M, \mathfrak{S}_N$ respectively, and $r_M \geq r_N > 0$; $|\mathfrak{S}_M| = M$, $|\mathfrak{S}_N| = N$, and N is a constant; $l \geq 1$ is the Shifting Parameter.

OUTPUT: A set of relay nodes fully cover the sensor nodes.

STEP 1: Divide the area into $l \cdot 2r_N \times l \cdot 2r_N$ squares.

STEP 2: For each square, find a local optimal solution.

STEP 3: Go through all shifts, and find the best solution with the minimum cardinality.

CASE 1-1 (Cont.)

Time Complexity

This case is "small squares with a lot of big sensors". For each square, even though there would be "big sensors", the procedure searching for the local optimal solution will not change. Therefore, the time complexity is still the same.

CASE 1-1 (Cont.)

Approximation Performance

In this case, sensors could overlap with others across more than 2 adjacent strips. Hence, (1) may not hold here, thereby the Shifting Lemma may not hold. Here are steps to compute how many errors could occur for one overlap corresponding to a relay node in an optimal solution.

STEP 1: Find the leftmost and the rightmost sensors of this overlap, s^j_{lm} and s^j_{rm} respectively.

STEP 2: Count the number of $l \cdot D$ strips to cover in each shift. Let the set of numbers be $\mathcal{E}_j = \{e^1_j, e^2_j, \ldots,\}$, where e^i_j is for the ith shift.

STEP 3: The total number of errors for one overlap over all shifts will be $\mathbb{E}_j = \sum_{e^i_j \in \mathcal{E}_j} (e^i_j - 1)$.

CASE 1-1 (Cont.)

Approximation Performance (Cont.)

It can be easily seen that

$$e_j^i \leq \lceil \frac{\|s_{lm}^j s_{rm}^j\|}{l \cdot D} \rceil \tag{6}$$

The upper bound of $\mathbb{E}_j$ will be

$$\mathbb{E}_j \leq |\mathcal{E}_j| \cdot \max_j(\lceil \frac{\|s_{lm}^j s_{rm}^j\|}{l \cdot D} \rceil) \leq l \cdot \max_j(\lceil \frac{\|s_{lm}^j s_{rm}^j\|}{l \cdot D} \rceil) \tag{7}$$

CASE 1-1 (Cont.)

Approximation Performance (Cont.)

From (2), $|OPT|^{(i)}$ can be seen as errors, thus

$$\begin{aligned}\sum_{i=1}^{l}(|OPT| + |ERR|^{(i)}) &\leq l \cdot |OPT| + \sum_{j=1}^{|OPT|} \mathbb{E}_j \\ &\leq [l + l \cdot \max_j(\lceil \frac{\|s_{lm}^j s_{rm}^j\|}{l \cdot D} \rceil)] \cdot |OPT|\end{aligned} \tag{8}$$

CASE 1-1 (Cont.)

Approximation Performance (Cont.)

From (4) and (8), it holds that

$$
\begin{aligned}
\min_{i=1,\ldots,l}\Big(\sum_{JinS_i} |OPT_J|\Big) &\leq \frac{1}{l}\cdot\sum_{i=1}^{l}\Big(\sum_{JinS_i} |OPT_J|\Big) \\
&\leq |OPT| + \frac{1}{l}\cdot\sum_{j=1}^{|OPT|}\mathbb{E}_j \\
&\leq [1+\max_j(\lceil\frac{\|s_{lm}^j s_{rm}^j\|}{l\cdot D}\rceil)]\cdot|OPT|
\end{aligned}
\tag{9}
$$

CASE 1-1 (Cont.)

Approximation Performance (Cont.)

If $\|s^j_{lm}s^j_{rm}\| \leq (l-x)\cdot D$, where $1 \geq x \geq 0$, then the number of shifts where errors may occur will be bounded by $l-x-1$.

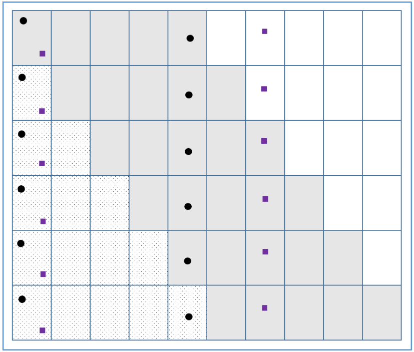

Figure 7: Example for errors over shifts.

CASE 1-1 (Cont.)

Approximation Performance (Cont.)

If thinking about (7) carefully, it is easy to find that the max number of errors over all shifts for a given overlap is

$$\max_j\left(\left\lceil\frac{\|s_{lm}^j s_{rm}^j\|}{D}\right\rceil\right) \tag{10}$$

Therefore, the average number of errors in one shift will be

$$\frac{\max_j\left(\left\lceil\frac{\|s_{lm}^j s_{rm}^j\|}{D}\right\rceil\right)}{l} \tag{11}$$

CASE 1-1 (Cont.)

Approximation Performance (Cont.)

Based on (11), (9) can be modified to

$$\min_{i=1,\ldots,l}\Big(\sum_{JinS_i}|OPT_J|\Big) \leq \Big[1+\frac{\max_j\big(\lceil\frac{\|s_{lm}^j s_{rm}^j\|}{D}\rceil\big)}{l}\Big]\cdot|OPT| \qquad (12)$$

If $r_N = r_M$, then (12) is the same as (4). In the CASE 1-1, the result would worsen if $\|s_{lm}^j s_{rm}^j\|$ increases. Moreover, since $\|s_{lm}^j s_{rm}^j\| \leq 2r_M$, if $\frac{r_M}{r_N}$ getting larger, then the approximation performance could change from a PTAS to a worse result which depends on the result of (11).

CASE 1-2

Algorithm 3

INPUT: A set of sensor nodes $\mathfrak{S} = \mathfrak{S}_M \cup \mathfrak{S}_N$ in the Eulidean plane; r_M, r_N are communication ranges for $\mathfrak{S}_M, \mathfrak{S}_N$ respectively, and $r_M < r_N$; $|\mathfrak{S}_M| = M$, $|\mathfrak{S}_N| = N$, and N is a constant; $l \geq 1$ is the Shifting Parameter.

OUTPUT: A set of relay nodes fully cover the sensor nodes.

STEP 1: Divide the area into $l \cdot 2r_M \times l \cdot 2r_M$ squares.

STEP 2: For each square, find a local optimal solution.

STEP 3: Go through all shifts, and find the best solution with the minimum cardinality.

CASE 1-2 (Cont.)

Time Complexity

This case is "small squares with some big sensors". It is a simpler version of the Algorithm 2. Therefore, the time complexity is still the same.

CASE 1-2 (Cont.)

Approximation Performance

Since there are only a constant number of big sensors, then the errors could only be in a constant amount.

$$\min_{i=1,\ldots,l}\left(\sum_{JinS_i} |OPT_J|\right) \leq \left(1+\frac{1}{l}\right)\cdot |OPT| + O(N) \qquad (13)$$

CASE 1-2

Algorithm 4

INPUT: A set of sensor nodes $\mathfrak{S} = \mathfrak{S}_M \cup \mathfrak{S}_N$ in the Eulidean plane; r_M, r_N are communication ranges for $\mathfrak{S}_M, \mathfrak{S}_N$ respectively, and $r_M < r_N$; $|\mathfrak{S}_M| = M$, $|\mathfrak{S}_N| = N$, and N is a constant; $l_M, l_N \geq 1$ is the Shifting Parameter.

OUTPUT: A set of relay nodes fully cover the sensor nodes.

STEP 1: Divide the area into $l_N \cdot 2r_N \times l_N \cdot 2r_N$ squares.

STEP 2: For each $l_N \cdot 2r_N \times l_N \cdot 2r_N$ square, we divide it into $l_M \cdot 2r_M \times l_M \cdot 2r_M$ squares. For each small square, we search for a local optimal solution. Then based on the local optimal solutions for small squares, we find the local optimal solution for the large squares.

STEP 3: Go through all shifts, and find the best solution with the minimum cardinality.

CASE 1-2 (Cont.)

Time Complexity

This case is "large squares with a lot of small sensors". In this case, we apply the Shifting Strategy twice actually. For each large square, there could be $O(M)$ amount of sensors, then it is a typical application of the Shifting Strategy. We have ${l_N}^2$ large squares, then the running time will rise with a coefficient ${l_N}^2$.

CASE 1-2 (Cont.)

Approximation Performance

Let SOL_M^i be a local optimal solution for the set of small sensors in a specific large strip. Let OPT_M^i be the corresponding optimal solution. Then,

$$\min(\sum_i |SOL_M^i|) \leq \min(\sum_i (1 + \frac{1}{l_M}) \cdot |OPT_M^i|) \qquad (14)$$

W.r.t. $\mathcal{S}_M$, OPT_M^i is only a local optimal solution. Hence, it holds that

$$\min(\sum_i |OPT_M^i|) \leq (1 + \frac{1}{l_N}) \cdot |OPT_M| \qquad (15)$$

From (14) and (15),

$$\min(\sum_i |SOL_M^i|) \leq (1 + \frac{1}{l_M}) \cdot (1 + \frac{1}{l_N}) \cdot |OPT_M| \qquad (16)$$

CASE 1-2 (Cont.)

Approximation Performance

Since there are only a constant number of big sensors, then

$$\min\left(\sum_{j} |SOL^{j}|\right) \leq \left(1 + \frac{1}{l_M}\right) \cdot \left(1 + \frac{1}{l_N}\right) \cdot |OPT| + O(N) \qquad (17)$$

CASE 1 Conclusion

Using the dominant type of sensors to apply the Shifting Strategy first, then fitting the rest in, could be better than the opposite ways, and specifically for the CASE 1, the approximation performance could still be PTAS.

The time complexity and the approximation performance is a dilemma.

CASE 2

Algorithm 5

INPUT: A set of sensor nodes $\mathfrak{S} = \mathfrak{S}_M \cup \mathfrak{S}_N$ in the Eulidean plane; r_M, r_N are communication ranges for $\mathfrak{S}_M, \mathfrak{S}_N$ respectively, and $r_M < r_N$; $|\mathfrak{S}_M| = M$, $|\mathfrak{S}_N| = N$, and N is a constant; $l_M, l_N \geq 1$ is the Shifting Parameter.

OUTPUT: A set of relay nodes fully cover the sensor nodes.

STEP 1: Apply the Shifting Strategy to $\mathfrak{S}_M$. The resultant set of overlaps is $\mathfrak{D}_M$.

STEP 2: Apply the Shifting Strategy to $\mathfrak{S}_N$. The resultant set of overlaps is $\mathfrak{D}_N$.

STEP 3: Merge $\mathfrak{D}_M$ and $\mathfrak{D}_N$.

STEP 4: Output the corresponding set of relay nodes.

CASE 2 (Cont.)

Algorithm 5 (STEP 3)

for each $\mathcal{O}_i^{\mathcal{A}_i} \in \mathfrak{D}_M$, **do**
  for each $c_j \in \mathcal{A}_i$, **do**
    for each $\mathcal{O}_l^{\mathcal{B}_l} \in \mathfrak{D}_N$, **do**
      if $Overlapping\_s(c_j, \mathcal{O}_l^{\mathcal{B}_l}) = True$, **then**
        $\mathcal{B}_l = \mathcal{B}_l \cup \{c_j\}$
        $\mathcal{A}_i = \mathcal{A}_i \setminus \{c_j\}$
        Stop going through the rest overlaps in $\mathfrak{D}_N$
      endif
    endfor
  endfor
  if $\mathcal{A}_i = \emptyset$, **then**
    $\mathfrak{D}_M = \mathfrak{D}_M \setminus \{\mathcal{O}_i^{\mathcal{A}_i}\}$
  endif
endfor

CASE 2 (Cont.)

Time Complexity

The first and the second steps are typical application of the Shifting Strategy. In the third step, *Overlapping*$_s(.,.)$ costs $O(2 \cdot N^2)$. Hence, the third step will cost $O(2 \cdot N^2 \cdot M)$. Therefore, the running time will still be dominated by the Shifting Strategy parts.

CASE 2 (Cont.)

Approximation Performance

Based on the proof of the Shifting Lemma, $SOL_M = OPT_M \cup R_M$ and $SOL_N = OPT_N \cup R_N$, where SOL means the final solution, and R means the errors. Therefore, it holds that

$$|SOL| \leq (1 + \frac{1}{l_M} + \frac{1}{l_M}) \cdot |OPT| \tag{18}$$

CASE 2 Conclusion

The general idea to tackle this case is the Divide-and-Conquer yet without recursion. The approximation performance is still a PTAS.

CASE 3

Algorithm 6

INPUT: A set of sensor nodes $\mathfrak{S} = \mathfrak{S}_1 \cup \ldots \cup \mathfrak{S}_n$ in the Eulidean plane; a set of communication ranges $\mathcal{R} = \{r_1, \ldots, r_n\}$, and they could be different from each other; $|\mathcal{S}_i| = N_i$; $l_1, \ldots, l_n \geq 1$ is the Shifting Parameter.

OUTPUT: A set of relay nodes fully cover the sensor nodes.

STEP 1: Apply the Shifting Strategy to each $\mathfrak{S}_i$. The resultant set of overlaps is $\mathfrak{D}_i$.

STEP 2: Sort the set of $\mathfrak{D}_i$ by r_i in the descending order. Let the result be $\mathbb{O}$.

STEP 3: Go through $\mathbb{O}$ and try to merge them as much as possible.

STEP 4: Output the corresponding set of relay nodes.

CASE 3 (Cont.)

Algorithm 6 (STEP 3)

for each $\mathfrak{D}_i \in \mathbb{O}$, **do**
    Merge $\mathfrak{D}_i$ to all $\mathfrak{D}_j, j \leq i$.
    if $\mathfrak{D}_i = \emptyset$ after merging, **then**
        $\mathbb{O} = \mathbb{O} \backslash \{\mathfrak{D}_i\}$
    endif
endfor

CASE 3 (Cont.)

Time Complexity

The first step is still a typical application of the Shifting Strategy. The second step will cost $O(n \log n)$. In the third step, the merging will happen at most $\frac{n \cdot (n-1)}{2}$ times, thus its running time is $O(n^2 \cdot (\max_i N_i)^3)$.

CASE 3 (Cont.)

Approximation Performance

According to CASE 2 and the Algorithm 5, it is easy to have

$$|SOL| \leq (1 + \frac{1}{l_1} + \frac{1}{l_2} + \ldots + \frac{1}{l_n}) \cdot |OPT| \tag{19}$$

This is an interesting result. The performance will depend on the properties of the series $1 + \frac{1}{l_1} + \frac{1}{l_2} + \ldots + \frac{1}{l_n}$. It could not be a PTAS, and could be any worse.

CASE 3 Conclusion

Even though there might be better algorithms applying the Shifting Strategy, the CASE 3 has still demonstrated it is hard to obtain a PTAS for MRNC by using the Shifting Strategy. The approximability comes from the partitioning of the plane; however, so does the errors. It is the hardest part when trying to address this problem, i.e. the complexity-and-approximability dilemma we mentioned before. If there is a way could consider the distribution of sensors more carefully, a better algorithm may exist.

Inapproximability Analysis

Conjecture 1

MRNC does not have any PTAS.

Conjecture 2

MRNC is not in the APX class.

General Idea

According to the L-Reduction and the Theorem 1, as long as we could find a problem $\mathcal{X}$ such that $\mathcal{X}$ is L-reducible to MRNC and $\mathcal{X}$ does not have any PTAS, then MRNC does not have any PTAS.

Inapproximability Analysis (Cont.)

Minimum Clique Partition (MCP) Problem [9]

Given:

1. A graph $\mathfrak{G} = \langle \mathfrak{V}, \mathfrak{E} \rangle$.

Seek:

A clique partition for $\mathfrak{G}$, i.e. a partition of $\mathfrak{V}$ into disjoint subsets $\mathfrak{V}_1, \ldots, \mathfrak{V}_k$ such that, for $1 \leq i \leq k$, the subgraph induced by $\mathfrak{V}_i$ is a complete graph, and the cardinality of the clique partition is minimum.

Inapproximability Analysis (Cont.)

MRNC is as hard as MCP

Consider each overlap in the solution of MRNC, we could map it to the clique in MCP. Then searching for minimum number of overlaps in MRNC is equivalent to searching for cliques in MCP. However, the opposite way is not valid, because in a given clique, it only tells every pair of vertices are connected together, which is interpreted as two-two sensors overlapping in MRNC yet not sufficient to tell if all corresponding sensors are overlapping together.

Based on this point, it is apparent to see that MRNC is as hard as MCP.

Inapproximability Analysis (Cont.)

Inapproximability of MCP

This problem is not approximable within $\mathfrak{V}^{\frac{1}{7-\epsilon}}$ for any $\epsilon > 0$. [9]

Question

Could we say MRNC at least has obtained a "lower bound" of the inapproximability?

Straightforward Method

Let us go back to my paper.

Conclusion

- MRNC with only two different communication ranges still has PTAS.
- The Shifting Strategy may hardly lead MRNC towards PTAS.
- General version MRNC is questionable to have PTAS or constant ratios.
- The inapproximability of MRNC is still unknown.

Future Works

- The complexity-and-approximability dilemma is very interesting.
- The inapproximability of MRNC should be determined.

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