FINAL PROJECT – MAT 3770

(INDIVIDUALIZED STUDY)

CUNY New York City College of Technology

MAT 3770 – OPTIMIZATION

Final Project – Optimization Methods

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Abstract

Deterministic optimization methods or techniques encountered throughout the course MAT 3770 – Optimization, plays a crucial role across various fields and applications. It involves finding the best solution to a problem with known and fixed parameters. Encountered in the course are three key areas where deterministic optimization is widely used. Linear Programming (LP) is a branch of deterministic optimization extensively used in operations research. It deals with optimizing a linear objective function subject to linear equality and inequality constraints. LP finds applications in resource allocation, production planning, logistics, and supply chain management. Integer programming (IP) extends LP to cases where decision variables are required to take on integer values. IP is particularly useful in problems involving discrete decision variables, such as scheduling and project management. Lagrange Multipliers (LM) is a mathematical technique widely used to solve constrained optimization problems. LM plays a significant role in various mathematical fields. LM is often applied to constrained optimization which includes linear programming, quadratic programming, and nonlinear programming, where constraints on the decision variables need to be considered.

Final Project – Optimization Methods

Optimization is a mathematical approach used to find the best possible solution for a given problem. It involves maximizing or minimizing a function, often referred to as an objective function, subject to certain constraints. Optimization problems are prevalent in various fields such as engineering, economics, finance, and operations research. Two main types of optimization problems are unconstrained optimization and constrained optimization.

Unconstrained optimization deals with finding the maximum or minimum of a function without any restrictions or constraints on the decision variables. Mathematically, it involves finding values for the variables that either maximize or minimize the objective function. In the MAT 3770 – Optimization course the method(s) of focus for unconstrained optimization include critical points, first derivative test, and the second derivative tests.

Integer programming (IP) is a specialized form of optimization where decision variables are restricted to integer values. This adds an additional layer of complexity to the optimization problem, as the solutions must not only satisfy the objective function and constraints but also involve discrete choices. IP problems often arise in real-world situations where decisions need to be made in whole units, such as in production planning or project scheduling. Solving integer programming problems requires specific algorithms, including branch-and-bound and cutting-plane methods, to efficiently explore the discrete solution space and find the optimal integer solution.

Constrained optimization, on the other hand, considers problems where the optimization is subject to certain constraints. Lagrange Multipliers (LM) is a powerful technique used to solve constrained optimization problems. The method involves introducing LM, which are coefficients associated with each constraint, and forming a Lagrangian function. The critical points of this

Lagrangian function, where its partial derivatives are zero, correspond to solutions that satisfy both the objective function and the constraints. Lagrange Multipliers are particularly useful when dealing with equality constraints. This method transforms the constrained problem into an unconstrained one, making it easier to find the optimal solution.

Optimization is a versatile and essential tool for decision-making in various disciplines. Unconstrained optimization addresses problems without restrictions on variables, while constrained optimization, especially with Lagrange Multipliers, tackles problems subject to constraints. Integer programming extends the scope of optimization to scenarios where decisions are discrete, offering solutions for problems with inherent integer constraints.

Duality and sensitivity analysis are important concepts in optimization that provide valuable insights into the relationships between primal and dual problems, as well as the impact of parameter changes on the optimal solution.

In this project various problems will be identified, analyzed, modeled, and solved based on the various prior mentioned categorizations and techniques or methods. Analytical development will be assisted by computer algebra systems (CAS). As well, a small summary of duality and sensitivity analysis will be provided with relevance to constrained optimization methods such as the Lagrange Multiplier method.

Constrained Optimization: Lagrange Multiplier Method

The first problem to be discussed concerns profit optimization application involving a personal computer manufacturer where its theory is that profit and marketing are positively correlated. A manufacturer of personal computers sells 10,000 units per month of a basic model. The cost of manufacturer is \$700 per unit, and the wholesale price is \$950. During the last quarter, the manufacturer lowered the price \$100 in a few test markets, and the result was a 50%

increase in sales. The company advertises the product at a cost of $50,000 per month. Its advertising agency claims that increasing the advertising budget by $10,000 per month would result in a sales increase of 200 units per month. Management has agreed to consider an increase in the advertising budget to no more than $100,000 per month. The directive is to determine the price and the advertising budget that will maximize profit; such to be done using the five-step method. Modeling as a constrained optimization problem and solving by the Lagrange multiplier method.

The Five-Step Method. The optimization five-step method is a general approach to solving optimization problems, From the course texts of Meerschaert, “a general procedure that can be used to solve problems using mathematical modeling” (pp.3). It provides a systematic way to tackle problems where the goal is to find the maximum or minimum of a given function, subject to certain constraints. Here are the five steps of the optimization method:

Step 1. Ask the question.

• Make a list of all the variables in the problem, including appropriate units.

• Be careful not to confuse variables and constants.

• State any assumptions you are making about these variables, including equations and inequalities.

• Check units to make sure that your assumptions make sense.

• State the objective of the problem in precise mathematical terms.

Step 2. Select the modeling approach.

• Choose a general solution procedure to be followed in solving this problem.

• Generally speaking, success in this step requires experience, skill, and familiarity with the relevant literature.

• In this book we will usually specify the modeling approach to be used.

Step 3. Formulate the model.

• Restate the question posed in step 1 in the terms of the modeling approach specified in step 2.

• You may need to relabel some of the variables specified in step 1 in order to agree with the notation used in step 2.

• Note any additional assumptions made in order to fit the problem described in step 1 into the mathematical structure specified in step 2.

Step 4. Solve the model.

• Apply the general solution procedure specified in step 2 to the specific problem formulated in step 3.

• Be careful in your mathematics. Check your work for math errors. Does your answer make sense?

• Use appropriate technology. Computer algebra systems, graphics, and numerical software will increase the range of problems within your grasp, and they also help reduce math errors.

Step 5. Answer the question.

• Rephrase the results of step 4 in nontechnical terms.

• Avoid mathematical symbols and jargon.

• Anyone who can understand the statement of the question as it was presented to you should be able to understand your answer.

The solution produced will have such a five-step method as the undercurrent. Defining the decisions variables, constants, and any assumptions you are making about these variables, including equations and inequalities.

P: the wholesale price per unit.

Q: the quantity of units sold per month.

A: the advertising budget per month.

R: revenue = $P \times Q$

C: cost = $700 \times Q$; $AE -$ being the advertising expense, $500 + A$.

P: profit, being the difference between revenue and cost.

The observed constraints:

The initial quantity sold is 10, 000 units per month.

After the price reduction, sales increased by 50%, so the new quantity sold is $1.5 \times 10,000$.

The advertising increase of $10, 000 results in an additional 200 units sold per month.

The advertising budget is limited toa maximum of $100, 000 per month.

The optimization problem:

$$\max_{P,Q,A}[(P-700)\cdot Q - 50{,}000]$$

Subject to the constraints:

The demand increase due to a price reduction: $Q = 10{,}000 + 0.5 \times 10,000 \rightarrow \lambda_1$

The advertising increase constraint: $Q \leq 10{,}000 + 200 \times \left(\frac{A-50{,}000}{10{,}000}\right) \rightarrow \lambda_2$

The advertising budget constraint: $A \leq 100{,}000 \rightarrow \lambda_3$

The Lagrangian function for the problem is:

$$L(P,Q,A,\lambda_1,\lambda_2,\lambda_3,\lambda_4) = (P-700)\cdot Q - 50{,}000 - \lambda_1 \cdot (Q - 10{,}000 - 0.5 \times 10{,}000) - \lambda_2 \cdot \left(Q - 10{,}000 - 200 \times \left(\frac{A-50{,}000}{10{,}000}\right)\right) + \lambda_3 \cdot (A - 100,000)$$

The first-order partial derivatives:

$$\frac{\partial L}{\partial P} = Q$$

$$\frac{\partial L}{\partial Q} = P - 700 - \lambda_1 - \lambda_2$$

$$\frac{\partial L}{\partial A} = \frac{-200\lambda_2}{10{,}000} + \lambda_3$$

$$\frac{\partial L}{\partial \lambda_1} = -(Q - 10{,}000 - 0.5\ \times 10{,}000)$$

$$\frac{\partial L}{\partial \lambda_2} = -\left(Q - 10{,}000 - 200\ \times \left(\frac{A - 50{,}000}{10{,}000}\right)\right)$$

$$\frac{\partial L}{\partial \lambda_3} = -(A - 100{,}000)$$

The first-order partial derivatives are then set to zero:

$$\frac{\partial L}{\partial P} = 0,\quad \frac{\partial L}{\partial Q} = 0,\quad \frac{\partial L}{\partial A} = 0,\quad \frac{\partial L}{\partial \lambda_1} = 0,\quad \frac{\partial L}{\partial \lambda_2} = 0,\quad \frac{\partial L}{\partial \lambda_3} = 0$$

Some observations:

$$Equation 1: Q = \ 0\ \ contradicts\ \ the\ demand\ constraint$$

$$A = 100{,}000$$

$$\lambda_2 = 50\lambda_3$$

$$Equation\ 4: Q = 10{,}000 + 0.5\ \times 10{,}000 = 15{,}000$$

There doesn't seem to be further rational information to advance with problem algebraically.

Fortunately, by reading carefully, the task of the problem is to model and solve; the latter said without any specification, where in the real-world people make the most of such or use what they have. Taking advantage of such:

```
In [2]: from scipy.optimize import minimize
# Define the profit function
def profit_function(variables):
P, Q, A, lambda1, lambda2, lambda3 = variables
```

```
    # Objective function (negative because we're maximizing)
    return -((P - 700) * Q - 50000 - lambda1 * (Q - 15000) - lambda2
* (Q - 10000 - 2 * (A - 50000) / 10000) + lambda3 * (A - 100000))
    # Constraints
    constraints = (
    {'type': 'eq', 'fun': lambda variables: variables[3] *
(variables[1] - 15000)}, # Constraint 1
    {'type': 'eq', 'fun': lambda variables: variables[4] *
(variables[1] - 10000 - 2 * (variables[2] - 50000) / 10000)}, #
Constraint 2
    {'type': 'ineq', 'fun': lambda variables: 100000 - variables[2]},
# Constraint 3
    )
    # Initial guess
    initial_guess = [950, 10000, 50000, 0, 0, 0]
    # Solve the optimization problem
    result = minimize(profit_function, initial_guess, method='SLSQP',
constraints=constraints)
    # Extract the optimal values
    optimal_price, optimal_quantity, optimal_advertising_budget, _,
_, _ = result.x
    # Display the results
    print(f"Optimal Price: ${optimal_price:.2f}")
    print(f"Optimal Quantity: {optimal_quantity:.2f} units")
    print(f"Optimal Advertising Budget:
${optimal_advertising_budget:.2f}")
    print(f"Maximum Profit: ${-result.fun:.2f}")
```

```
    Optimal Price: $950.00
    Optimal Quantity: 10000.00 units
    Optimal Advertising Budget: $50000.00
    Maximum Profit: $2450000.00
```

Constrained Optimization: Polynomial Multivariate Optimization Problem

Multivariable optimization involves finding the maximum or minimum of a function that depends on multiple variables. Particularly for a multivariate polynomial as the objective function there is no direct field application, rather to focus on the ability the competently find the optimal values. Consider the case of maximizing the following function:

$$f(x, y) = 2xy - x^2 - 2y^2 + 3x + 5$$

in the following domain:

$$\begin{Bmatrix} -10 \le x \le 10 \\ -10 \le y \le 10 \end{Bmatrix}$$

Such a problem can be solved with basic knowledge of calculus. One may start by determining the critical points by finding where the (first) derivatives with respect to x and y, then setting such partial derivatives equal to zero and solving for x and y. Then checking the function at the critical points and boundary points to determine at what point the maximum value resides.

Finding the partial derivatives with respect to x and y:

$$\frac{\partial f}{\partial x} = 2y - 2x + 3$$

$$\frac{\partial f}{\partial y} = 2x - 4y$$

Set these partial derivative equal to zero and solve for x and y:

$$2y - 2x + 3 = 0$$

$$2x - 4y = 0$$

$$x = 2y$$

$$2y - 4y + 3 = 0 \to y = \frac{3}{2} \to x = 3$$

So the critical point is $\left(3, \frac{3}{2}\right)$

Now, it comes down to evaluating $f(x, y)$:

$$f\left(3, \frac{3}{2}\right) = 2(3)\left(\frac{3}{2}\right) - (3)^2 - 2\left(\frac{3}{2}\right)^2 + 3(3) + 5 = \frac{19}{2}$$

$$f(-10, -10) = 2(-10)(-10) - (-10)^2 - 2(-10)^2 + 3(-10) + 5$$

$$= 200 - 100 - 200 - 30 + 5 = -125$$

$$f(-10,10) = 2(-10)(10) - (-10)^2 - 2(10)^2 + 3(-10) + 5$$

$$= -200 - 100 - 200 - 30 + 5 = -525$$

$$f(10, -10) = 2(10)(-10) - (10)^2 - 2(-10)^2 + 3(10) + 5 = -200 - 100 - 200 + 35$$

$$= -465$$

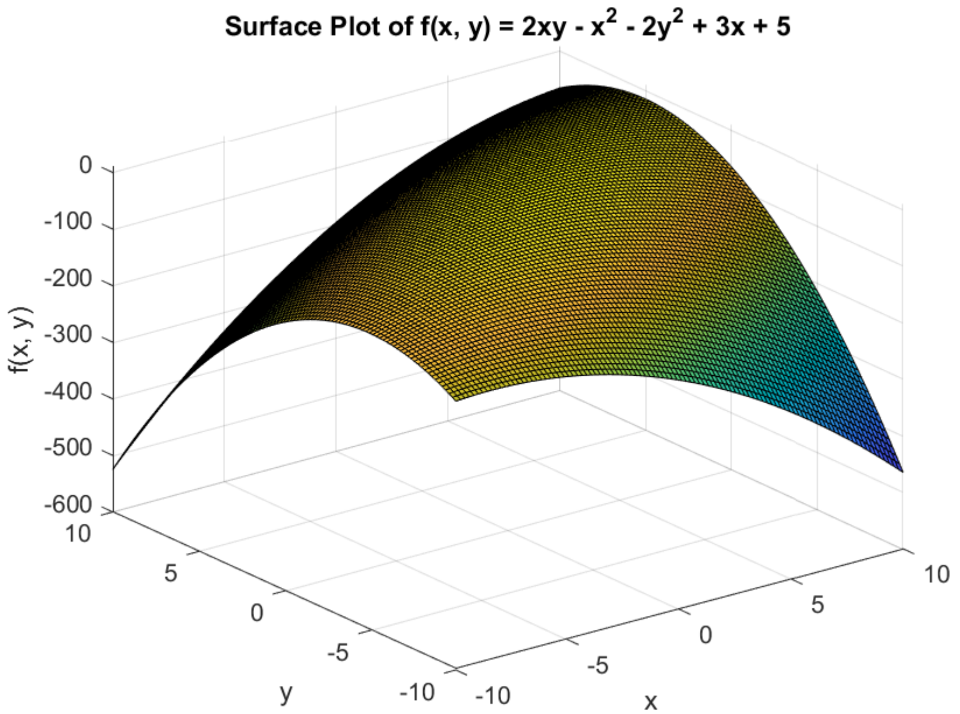
$$f(10,10) = 2(10)(10) - (10)^2 - 2(10)^2 + 3(10) + 5 = 200 - 100 - 200 + 35 = -65$$

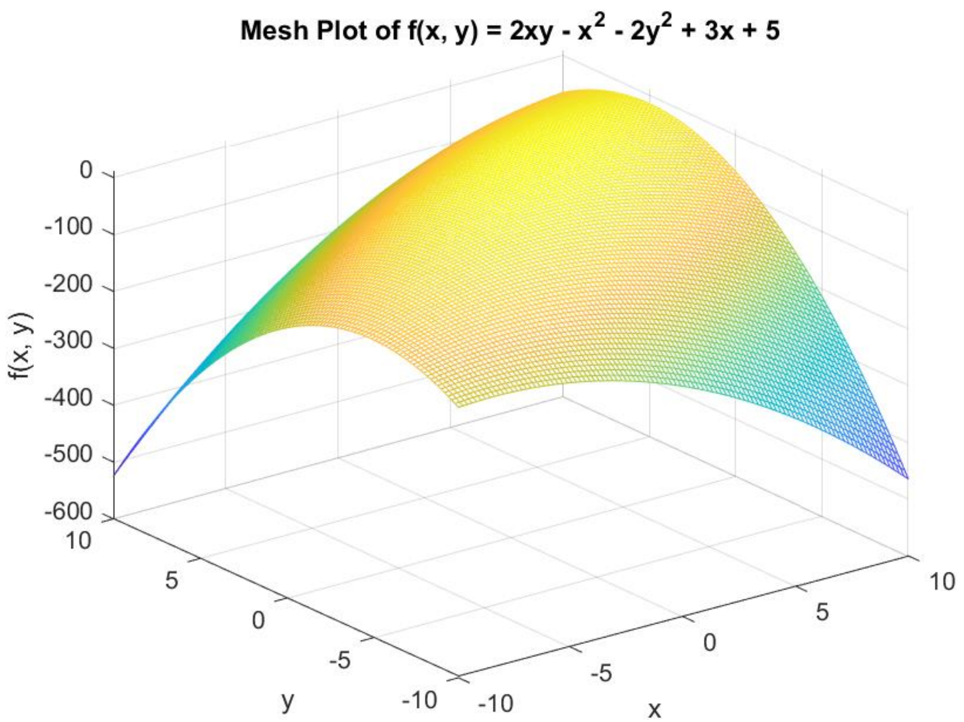
Therefore, a (global) maximum of $\frac{19}{2}$ at $\left(3, \frac{3}{2}\right)$.

Using MATLAB to generate a surface plot:

```
% Define the function

f = @(x, y) 2*x.*y - x.^2 - 2*y.^2 + 3*x + 5;

% Generate x and y values
x = linspace(-10, 10, 100); % adjust the number of points for smoother
plot
y = linspace(-10, 10, 100);
[X, Y] = meshgrid(x, y);

% Evaluate the function for each pair of (X, Y)
Z = f(X, Y);

% Create a surface plot
figure;
surf(X, Y, Z);
```

```
% Add labels and title
xlabel('x');
ylabel('y');
zlabel('f(x, y)');
title('Surface Plot of f(x, y) = 2xy - x^2 - 2y^2 + 3x + 5');


% Define the function
f = @(x, y) 2*x.*y - x.^2 - 2*y.^2 + 3*x + 5;

% Generate x and y values
x = linspace(-10, 10, 100); % adjust the number of points for smoother
plot
y = linspace(-10, 10, 100);
[X, Y] = meshgrid(x, y);

% Evaluate the function for each pair of (X, Y)
Z = f(X, Y);

% Create a mesh plot
figure;
mesh(X, Y, Z);

% Add labels and title
xlabel('x');
ylabel('y');
zlabel('f(x, y)');
title('Mesh Plot of f(x, y) = 2xy - x^2 - 2y^2 + 3x + 5');
```


Surface Plot of f(x, y) = 2xy - x2 - 2y2 + 3x + 5
f(x, y)
0
-100
-200
-300
-400
-500
-600
10
5
0
-5
-10
y
-10
-5
0
5
10
x


Mesh Plot of f(x, y) = 2xy - x2 - 2y2 + 3x + 5
f(x, y)
0
-100
-200
-300
-400
-500
-600
10
5
0
-5
-10
y
-10
-5
0
5
10
x

Integer Programming.

Integer programming (IP) is a specialized form of optimization where decision variables are restricted to integer values. This adds an additional layer of complexity to the optimization problem, as the solutions must not only satisfy the objective function and constraints but also involve discrete choices. IP problems often arise in real-world situations where decisions need to be made in whole units, such as in production planning or project scheduling. Solving integer programming problems requires specific algorithms, including branch-and-bound and cutting-plane methods, to efficiently explore the discrete solution space and find the optimal integer solution. A problem of interest:

$$\max_{x,y}[3x + y]$$

$$s.t: \quad \begin{Bmatrix} 3x + y \leq 3000 \\ x + y \leq 2000 \\ -10x + y \leq 1000 \\ x \geq 0, y \geq 0 \end{Bmatrix}$$

1. Solving $3x + y = 3000$ and $x + y = 2000$:

   Subtracting the second equation from the first,

$$\begin{Bmatrix} (3x + y) - (x + y) = 3000 - 2000 \\ 2x = 1000 \\ x = 500 \end{Bmatrix}$$

   Substituting $x = 500$ into $x + y = 2000$,

$$\begin{Bmatrix} 500 + y = 2000 \\ y = 1500 \end{Bmatrix}$$

   So, the first intersection point is $(500, 1500)$.

2. Solving $3x + y = 3000$ and $-10x + y = 1000$:

   Subtracting the latter equation from the first,

$$\begin{Bmatrix} (3x+y)-(-10x+y)=3000-1000 \\ 13x=2000 \\ x\approx 153.8 \end{Bmatrix}$$

Substituting such value for x into $-10x+y=1000$,

$$\begin{Bmatrix} -1538+y=1000 \\ y=2538 \end{Bmatrix}$$

So, the second intersection point is approximately $(153.8,\ 2538)$.

3. Solving $x+y=2000$ and $-10x+y=1000$:

   Add the first equation to the second:

$$\begin{Bmatrix} (x+y)+(-10x+y)=2000+1000 \\ -9x+2y=3000 \\ y=\dfrac{9x+3000}{2} \end{Bmatrix}$$

   Substitute this expression into $x+y=2000$,

$$\begin{Bmatrix} x+\dfrac{9x+3000}{2}=2000 \\ \dfrac{2x+9x+3000}{2}=2000 \\ 2x+9x+3000=4000 \\ 11x=1000 \\ x\approx 90.91 \end{Bmatrix}$$

   Substituting such value for x into $y=\frac{9x+3000}{2}$,

$$y=\frac{9(90.91)+3000}{2}=1909.095$$

   So, the third intersection point is approximately $(90.91,\ 1909.095)$

4. Now, to take the objective function $f(x,y)=3x+y$ at each of these points and check which one maximizes the function:

$$f(153.8, 2538)=2999.4$$

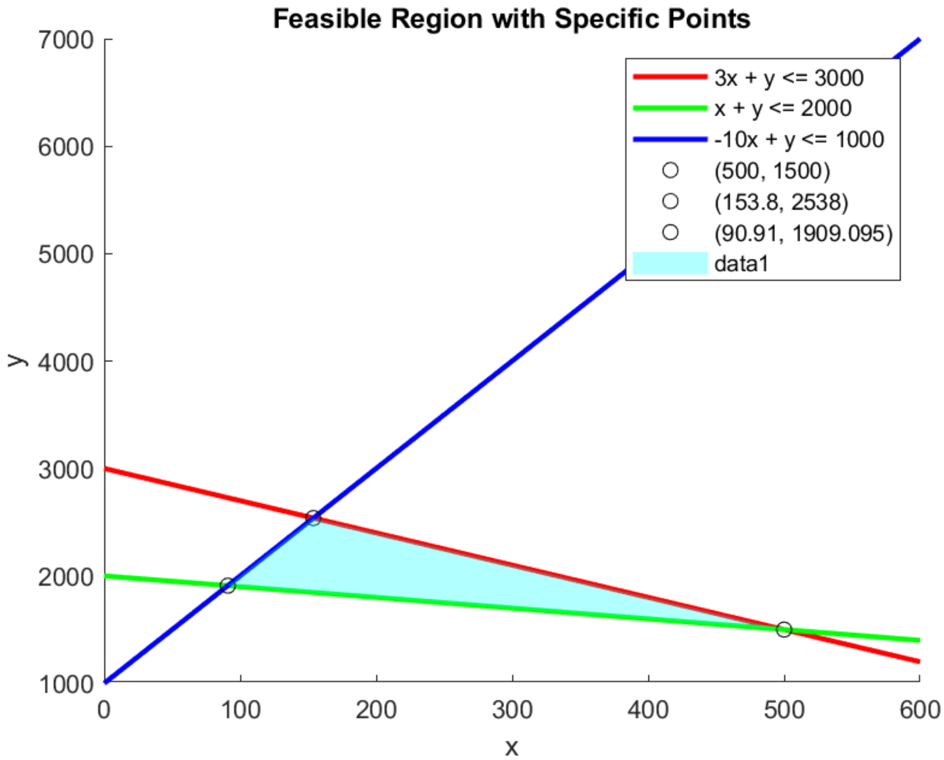
$$f(500, 1500)=3000$$

$$f(90.91, 1909.095\ )=2181.83$$

So, there's maximum of 3000 at point $(500, 1500)$.

As for a graphical demonstration in MATLAB:

```
% Define the constraint equations

x = linspace(0, 600, 1000); % Adjust the range as needed
y1 = 3000 - 3 * x;
y2 = 2000 - x;
y3 = 1000 + 10 * x;

% Plot the constraints
figure;
hold on;
plot(x, y1, 'r', 'LineWidth', 2, 'DisplayName', '3x + y <= 3000');
plot(x, y2, 'g', 'LineWidth', 2, 'DisplayName', 'x + y <= 2000');
plot(x, y3, 'b', 'LineWidth', 2, 'DisplayName', '-10x + y <= 1000');

% Plot specific points
plot(500, 1500, 'ko', 'DisplayName', '(500, 1500)');
plot(153.8, 2538, 'ko', 'DisplayName', '(153.8, 2538)');
plot(90.91, 1909.095, 'ko', 'DisplayName', '(90.91, 1909.095)');

% Add labels and legend
xlabel('x');
ylabel('y');
title('Feasible Region with Specific Points');
legend('show');

% Highlight the feasible region
fill([500, 153.8, 90.91, 500], [1500, 2538, 1909.095, 1500], 'c',
'FaceAlpha', 0.3, 'EdgeAlpha', 0);

hold off;
```



The convexity concern is accounted for as well. Convexity refers to the shape of the feasible region and the objective function. A set or a function is considered convex if, roughly speaking, any line segment connecting two points in the set or on the graph of the function lies entirely within the set or above the graph. Convexity is a desirable property because it ensures that optimization problems are well-behaved, and there is a higher likelihood of finding global optima.

As for the adjective "integer" in integer programming a further constraint must be placed on the decision variables, namely, $x, y \in \mathbb{Z}$. Using MATLAB:

```
% Objective function coefficients

f = [-3; -1];  % Coefficients for the objective function 3x + y
%In the MATLAB code provided, the objective function is defined as
      % f = [-3; -1]This corresponds to the coefficients of the
      % objective function where using the intlinprogin MATLAB
      % minimizes the objective function by default;
      % thus to maximize 3x + y, you need to minimize -3x – y
      % which is achieved by using the coefficients [-3; -1].
```

```
% Inequality constraints matrix and vector
A = [3, 1; 1, 1; -10, 1];  % Coefficients for the inequality
constraints
b = [3000; 2000; 1000];    % Right-hand side values for the inequality
constraints

% Lower bounds for variables
lb = [0; 0];

% Solve the integer linear programming problem
intcon = [1; 2];  % Variables x and y are integers
[x, fval, exitflag] = intlinprog(f, intcon, A, b, [], [], lb);

% Display the results
disp('Optimal values:');
disp(['x = ' num2str(x(1))]);
disp(['y = ' num2str(x(2))]);
disp(['Optimal objective value: ' num2str(-fval)]);  % Negate fval
because intlinprog minimizes by default
disp(['Exit flag: ' num2str(exitflag)]);
```

Such above code identifies the optimal value point to be $(500, 1500)$, with an optimum objective value of 3000.

Duality and Sensitivity.

Optimization is a fundamental process in various fields, from engineering to finance, where the goal is to find the best possible solution under certain constraints. Two critical concepts in optimization are duality and sensitivity analysis. Duality establishes a profound connection between primal and dual problems, while sensitivity analysis sheds light on the impact of parameter changes on the optimal solution.

Duality in Optimization. Duality is a fundamental concept in optimization theory, providing a powerful link between primal and dual problems. For a convex optimization problem, the primal problem (P) seeks to minimize an objective function subject to constraints, while the dual problem (D) aims to maximize a function derived from the Lagrangian of the primal problem. The duality theorem asserts that the optimal value of the dual problem is a lower bound on the optimal value of the primal problem. Strong duality occurs when these optimal values are equal.

The Lagrangian, formed by introducing Lagrange multipliers for the constraints, plays a significant role in establishing duality. The dual variables associated with the inequality constraints provide valuable information about the sensitivity of the primal problem to changes in the constraints.

Sensitivity Analysis in Optimization. Sensitivity analysis is a vital tool for assessing the robustness of optimal solutions to changes in problem parameters. Several aspects of sensitivity analysis provide insights into different facets of the optimization problem:

1. Sensitivity of Objective Function Coefficients – Changes in the coefficients of the objective function impact both primal and dual solutions. In the primal, it influences the optimal solution and value, while in the dual, it is reflected in the Lagrange multipliers.
2. Sensitivity of Right-Hand Side Values (RHS) – Alterations in the RHS of constraints affect the optimal solution in the primal, and the dual variables provide information on how sensitive the optimal value is to these changes.
3. Sensitivity of Constraint Coefficients – Variations in the coefficients of the constraints impact the primal solution, and the dual variables provide insights into the trade-offs between constraints.
4. Sensitivity of Constraint Boundaries – Changes in the constraint boundaries influence the optimal solution in the primal, and the dual variables convey information about the sensitivity of the constraints.

Duality and sensitivity analysis enhance our understanding of optimization problems. Duality provides a theoretical framework for establishing relationships between primal and dual problems, and sensitivity analysis offers a practical tool for decision-makers to evaluate the robustness of optimal solutions. Together, these concepts contribute to the development of efficient optimization algorithms, facilitate insightful interpretations of solutions, and guide decision-makers in making informed choices in dynamic environments. The marriage of duality and sensitivity analysis underscores the versatility and applicability of optimization theory in addressing real-world challenges. The journal article of Kornbluth makes extensive use of duality and sensitivity with optimization (Kornbluth 2017, pp. 599 – 614).

Demonstrated throughout this paper are some applications of constrained optimization, with implementation of the five-step method, and multiple techniques applied to solve such applications (the Lagrange Multiplier Method and basic calculus). Followed by a short notion of duality and sensitivity analysis in optimization. The trio of constrained optimization, duality, and sensitivity analysis forms a powerful and cohesive framework that significantly contributes to the understanding and practical application of optimization problems.

Constrained optimization lies at the core of addressing real-world challenges, where decision-makers are often faced with the task of finding optimal solutions under a myriad of constraints. From engineering designs to financial planning, the ability to navigate complex solution spaces while adhering to constraints is essential. Constrained optimization techniques provide the means to systematically explore these spaces and identify solutions that balance competing objectives.

The concept of duality establishes an elegant connection between primal and dual problems, revealing deep insights into the structure of optimization problems. Through Lagrangian formulation, duality not only provides a theoretical bridge between the primal and dual spaces but also offers a practical method for understanding trade-offs, assessing feasibility, and even deriving efficient algorithms for optimization. The duality theorem, with its implications of optimality and strong duality, adds a layer of elegance to the optimization landscape, enhancing our ability to interpret and solve complex problems.

Sensitivity analysis complements constrained optimization by offering a systematic approach to understanding how changes in problem parameters impact the optimal solution. Its utility extends across various dimensions, from assessing the impact of objective function coefficients to evaluating the robustness of solutions to changes in constraint boundaries. Sensitivity analysis provides decision-makers with valuable information about the stability and reliability of optimal solutions, enabling them to make informed decisions in the face of uncertainty and changing conditions.

Together, these three components create a holistic approach to optimization. Constrained optimization sets the stage by defining the problem structure, duality establishes a theoretical and practical link between primal and dual spaces, and sensitivity analysis guides decision-makers in navigating the complexities of dynamic environments. This integrated approach not only aids in solving optimization problems efficiently but also empowers individuals and organizations to make robust decisions that stand up to the challenges of the real world. In essence, the constructive interaction of constrained optimization, duality, and sensitivity analysis forms a cornerstone in the realm of decision-making and problem-solving.

References

1. Kornbluth, J. H. S. (1974). Duality, Indifference and Sensitivity Analysis inr Multiple Objective Linear Programming, *Journal of the Operational Research Society*, 25:4, 599-614
2. Meerschaert, M. M. (2013). *Mathematical Modeling*. Elsevier Inc. New York.