

7.1 Integration by Parts

1 Intro to IBP

- IBP allows us to replace a difficult integral with an expression involving an easier integral.
- based on the product rule

Memorize this:

$$\int u dv = uv - \int v du$$

1.0.1 Example:

Suppose we want to do this integral:

$$\int x \sin x dx$$

this becomes $u + dv$

1. first sort the integrand (including the dx) into u and dv (The goal is to choose u and dv so that $\int v du$ is easier than $\int u dv$.)
2. Find du by taking the derivative of u , and v by integrating dv
3. Write out $uv - \int v du$
4. Integrate!

$$\begin{array}{ll} u = x & dv = \sin x dx \\ du = 1 dx & v = -\cos x \end{array} \quad uv - \int v du$$

$$\underbrace{x}_{u} \underbrace{(-\cos x)}_v - \int \underbrace{(-\cos x)}_v \underbrace{dx}_{du}$$

$$-x \cos x + \int \cos x dx = \boxed{-x \cos x + \sin x + C}$$

$$\begin{aligned} \text{check: } \frac{d}{dx} (-x \cos x + \sin x) &= -x(-\sin x) + \cos x(-1) + \cos x \\ &= \underline{x \sin x} \end{aligned}$$

How to pick u: Inverse trig
Logarithms
Algebraic (polynomials)
Trig
Exponential

1.0.2 Example:

Use IBP to integrate

$$\int 3xe^{5x} dx$$

$$\boxed{\begin{array}{ll} u = 3x & dv = e^{5x} dx \\ du = 3dx & v = \frac{1}{5}e^{5x} \end{array}}$$

$$3x \cdot \frac{1}{5}e^{5x} - \int \frac{1}{5}e^{5x} \cdot 3 dx$$

$$\frac{3}{5}xe^{5x} - \frac{3}{5} \int e^{5x} dx$$

$\underbrace{\int e^{5x} dx}_{\frac{1}{5}e^{5x}}$

$$= \boxed{\frac{3}{5}xe^{5x} - \frac{3}{25}e^{5x} + C}$$

1.0.3 Example:

Use IBP to integrate

$$\int \ln x dx$$

$$\boxed{\begin{array}{ll} u = \ln x & dv = dx \\ du = \frac{1}{x} dx & v = x \end{array}}$$

$$= x \ln x - \int x \cdot \frac{1}{x} dx$$

$$= x \ln x - \int dx$$

$$= \boxed{x \ln x - x + C}$$

2 Derivation of the IBP Formula

3 IBP with Definite Integrals

3.0.1 Example:

$$\begin{aligned}\int_1^4 t \ln t \, dt &= \left[\frac{1}{2} t^2 \ln t \right]_1^4 - \int_1^4 \frac{1}{t} \cdot \frac{1}{2} t^2 \, dt \\ \boxed{\begin{array}{l} u = \ln t \quad dv = t \, dt \\ du = \frac{1}{t} \, dt \quad v = \frac{1}{2} t^2 \end{array}} &= \frac{1}{2} t^2 \ln t - \frac{1}{2} \int_1^4 t \, dt \\ &= \left[\frac{1}{2} t^2 \ln t - \frac{1}{2} \cdot \frac{1}{2} t^2 \right]_1^4 \\ &= \left[\frac{1}{2} t^2 \ln t - \frac{1}{4} t^2 \right]_1^4 = \left[\frac{1}{2} (16) \ln 4 - \frac{1}{4} (16) \right] - \left[\frac{1}{2} \cdot \ln 1 - \frac{1}{4} \right] \\ &= 8 \ln 4 - 4 + \frac{1}{4} = \boxed{8 \ln 4 - \frac{15}{4}}\end{aligned}$$

4 Using IBP more than once

4.0.1 Example:

$$\begin{aligned}\int x^2 \cos(3x) \, dx &= \frac{1}{3} x^2 \sin(3x) - \frac{2}{3} \int x \sin(3x) \, dx \\ \boxed{\begin{array}{l} u = x^2 \quad dv = \cos(3x) \, dx \\ du = 2x \, dx \quad v = \frac{1}{3} \sin(3x) \end{array}} &\quad \text{Now use IBP again} \\ &\quad \boxed{\begin{array}{l} u = x \quad dv = \sin 3x \, dx \\ du = dx \quad v = -\frac{1}{3} \cos(3x) \end{array}} \\ &= \frac{1}{3} x^2 \sin(3x) - \frac{2}{3} \left[-\frac{1}{3} x \cos(3x) - \int -\frac{1}{3} \cos(3x) \, dx \right] \\ &= \frac{1}{3} x^2 \sin(3x) + \frac{2}{9} x \cos(3x) - \frac{2}{9} \int \cos(3x) \, dx \\ &= \frac{1}{3} x^2 \sin(3x) + \frac{2}{9} x \cos(3x) - \frac{2}{27} \sin(3x) + C \end{aligned}$$

$\int x^4 \sin x \, dx$
would require
4 rounds of
IBP

5 Tabular IBP

The tabular method for integration by parts is useful for integrating the following types of integrals:

$$\begin{array}{l}
 \int (\text{polynomial}) \sin(ax) dx \quad \int (\text{polynomial}) \cos(ax) dx \quad \int (\text{polynomial}) e^{ax} dx \\
 \int x^2 \sin x dx
 \end{array}$$

$\begin{array}{r} \text{derivatives} \\ \downarrow \\ +x^2 \\ -2x \\ +2 \\ -0 \end{array}$	$\begin{array}{r} \text{integrate} \\ \downarrow \\ \sin x \\ -\cos x \\ -\sin x \\ \cos x \end{array}$	$\boxed{-x^2 \cos x + 2x \sin x + 2 \cos x + C}$
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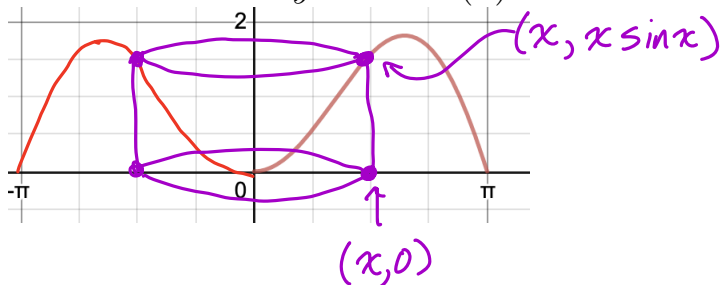
5.0.1 Example:

$$\int (x^2 + x) e^{\frac{1}{2}x} dx$$

$\begin{array}{r} +x^2 + x \\ -2x + 1 \\ +2 \\ -0 \end{array}$	$\begin{array}{r} e^{\frac{1}{2}x} \\ 2e^{\frac{1}{2}x} \\ 4e^{\frac{1}{2}x} \\ 8e^{\frac{1}{2}x} \end{array}$	$2(x^2 + x)e^{\frac{1}{2}x} - 4(2x + 1)e^{\frac{1}{2}x} + 16e^{\frac{1}{2}x} + C$
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5.0.2 Example:

The area under $y = x \sin(x)$ is rotated around the y -axis. Find the volume.



$$\text{rad} = x$$

$$\text{height} = x \sin x$$

$$V = 2\pi \int_0^\pi x^2 \sin x dx = 2\pi \left[-x^2 \cos x + 2x \sin x + 2 \cos x \right]_0^\pi =$$