

## 7.1 Integration by Parts

### 1 Intro to IBP

- IBP allows us to replace a difficult integral with an expression involving an easier integral.
- based on the product rule

Memorize this:

$$\int u \, dv = uv - \int v \, du$$

#### 1.0.1 Example:

Suppose we want to do this integral:

$$\int x \sin x \, dx \quad \text{this becomes } u + dv$$

1. first sort the integrand (including the  $dx$ ) into  $u$  and  $dv$  (The goal is to choose  $u$  and  $dv$  so that  $\int v \, du$  is easier than  $\int u \, dv$ .)
2. Find  $du$  by taking the derivative of  $u$ , and  $v$  by integrating  $dv$
3. Write out  $uv - \int v \, du$
4. Integrate!

$$\boxed{\begin{array}{ll} u = x & dv = \sin x \, dx \\ du = 1 \, dx & v = -\cos x \end{array}} \quad uv - \int v \, du$$

$$x(-\cos x) - \int (-\cos x) \, dx$$

$$-x \cos x + \int \cos x \, dx = \boxed{-x \cos x + \sin x + C}$$

$$\begin{aligned} \text{check: } \frac{d}{dx} (-x \cos x + \sin x) &= -x(-\sin x) + \cos x(-1) + \cos x \\ &= \underline{x \sin x} \end{aligned}$$

### 1.0.2 Example:

Use IBP to integrate

$$\int 3xe^{5x} dx \rightarrow 3x \cdot \frac{1}{5}e^{5x} - \int \frac{1}{5}e^{5x} \cdot 3 dx$$

$u = 3x \quad dv = e^{5x} dx$   
 $du = 3dx \quad v = \frac{1}{5}e^{5x}$

$$\frac{3}{5}xe^{5x} - \frac{3}{5} \int e^{5x} dx$$

$\frac{1}{5}e^{5x}$

$$= \boxed{\frac{3}{5}xe^{5x} - \frac{3}{25}e^{5x} + C}$$

Trig  
Exponential

### 1.0.3 Example:

Use IBP to integrate

$$\int \ln x dx = x \ln x - \int x \cdot \frac{1}{x} dx$$

$u = \ln x \quad dv = dx$   
 $du = \frac{1}{x} dx \quad v = x$

$$= x \ln x - \int dx$$

$$= \boxed{x \ln x - x + C}$$

## 2 Derivation of the IBP Formula

### 3 IBP with Definite Integrals

#### 3.0.1 Example:

$$\begin{aligned}
 \int_1^4 t \ln t dt &= \left[ \frac{1}{2} t^2 \ln t \right]_1^4 - \int_1^4 \frac{1}{t} \cdot \frac{1}{2} t^2 dt \\
 &\quad \boxed{u = \ln t \quad dv = t dt} \quad \boxed{du = \frac{1}{t} dt \quad v = \frac{1}{2} t^2} \\
 &= \frac{1}{2} t^2 \ln t - \frac{1}{2} \int_1^4 t dt \\
 &= \frac{1}{2} t^2 \ln t - \frac{1}{2} \cdot \frac{1}{2} t^2 \Big|_1^4 \\
 &= \frac{1}{2} t^2 \ln t - \frac{1}{4} t^2 \Big|_1^4 \\
 &= \left[ \frac{1}{2} (16) \ln 4 - \frac{1}{4} (16) \right] - \left[ \frac{1}{2} \cdot \ln 1 - \frac{1}{4} \right] \\
 &= 8 \ln 4 - 4 + \frac{1}{4} = \boxed{8 \ln 4 - \frac{15}{4}}
 \end{aligned}$$

## 4 Using IBP more than once

#### 4.0.1 Example:

$$\begin{aligned}
 \int x^2 \cos(3x) dx &= \frac{1}{3} x^2 \sin(3x) - \frac{2}{3} \int x \sin(3x) dx \\
 &\quad \boxed{u = x^2 \quad dv = \cos(3x) dx} \quad \boxed{du = 2x dx \quad v = \frac{1}{3} \sin(3x)} \\
 &\quad \text{Now use IBP again} \\
 &= \frac{1}{3} x^2 \sin(3x) - \frac{2}{3} \left[ -\frac{1}{3} x \cos(3x) - \int -\frac{1}{3} \cos(3x) dx \right] \\
 &= \frac{1}{3} x^2 \sin(3x) + \frac{2}{9} x \cos(3x) - \frac{2}{9} \int \cos(3x) dx \\
 &\quad \boxed{\int x^4 \sin x dx} \\
 &= \frac{1}{3} x^2 \sin(3x) + \frac{2}{9} x \cos(3x) - \frac{2}{27} \sin(3x) + C \\
 &\quad \text{would require} \\
 &\quad \text{4 rounds of} \\
 &\quad \text{IBP}
 \end{aligned}$$

## 5 Tabular IBP

The tabular method for integration by parts is useful for integrating the following types of integrals:

$$\int (\text{polynomial}) \sin(ax) dx \quad \int (\text{polynomial}) \cos(ax) dx \quad \int (\text{polynomial}) e^{ax} dx$$

$$\int x^2 \sin x dx$$

derivatives  $\downarrow +x^2$        $\sin x \downarrow \text{integrate}$

$-2x$        $-\cos x$

$+2$        $-\sin x$

$-0$        $\cos x$

$-x^2 \cos x + 2x \sin x + 2 \cos x + C$

### 5.0.1 Example:

$$\int (x^2 + x) e^{\frac{1}{2}x} dx$$

$+x^2 + x \quad e^{\frac{1}{2}x}$

$-2x + 1 \quad 2e^{\frac{1}{2}x}$

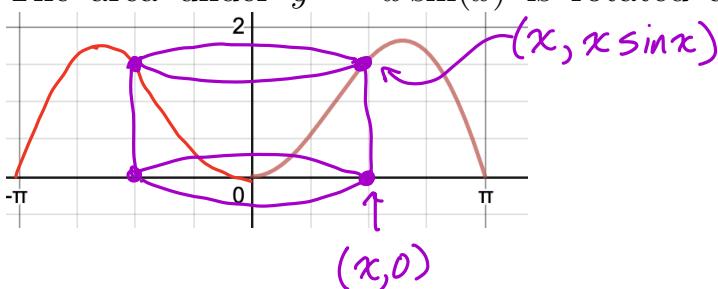
$+2 \quad 4e^{\frac{1}{2}x}$

$-0 \quad 8e^{\frac{1}{2}x}$

$2(x^2 + x)e^{\frac{1}{2}x} - 4(2x + 1)e^{\frac{1}{2}x} + 16e^{\frac{1}{2}x} + C$

### 5.0.2 Example:

The area under  $y = x \sin(x)$  is rotated around the  $y$ -axis. Find the volume.



$$\text{rad} = x$$

$$\text{height} = x \sin x$$

$$V = 2\pi \int_0^\pi x^2 \sin x dx = \left[ -x^2 \cos x + 2x \sin x + 2 \cos x \right]_0^\pi =$$