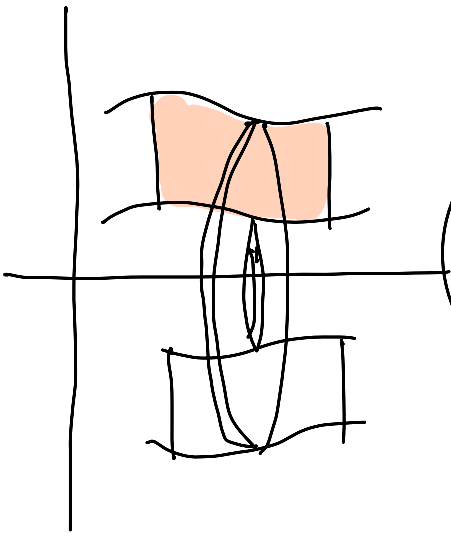
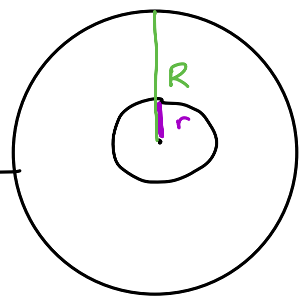
3 Washers

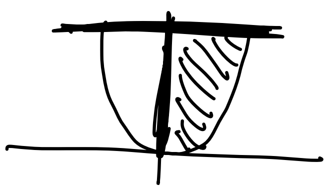
If the area does not touch the axis of rotation, the cross-section is a washer instead of a disk.



Area of Washer $= \pi R^2 - \pi r^2 = \pi(R^2 - r^2)$

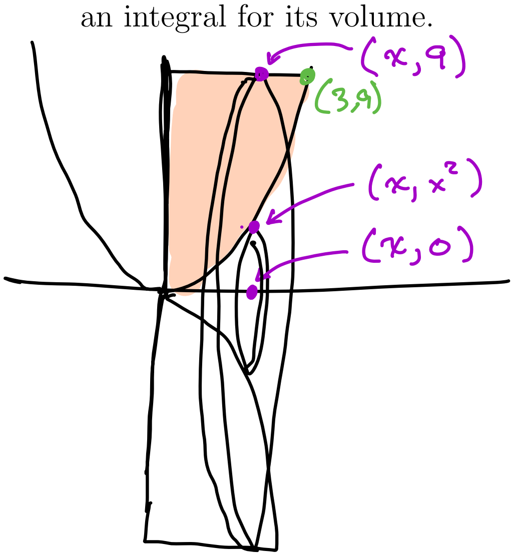


$$\boxed{\text{Vol} = \pi \int_a^b R^2 - r^2 \, dx}$$



3.0.1 Example:

Take the area bounded by $y = x^2$, the y-axis, and $y = 9$. Rotate it around the x-axis, and set up an integral for its volume.



$R = 9 \qquad r = x^2$

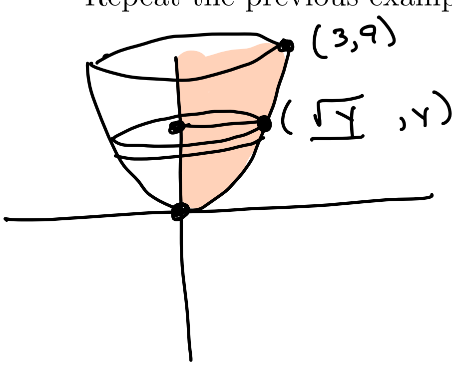
$$\boxed{V = \pi \int_0^3 9^2 - (x^2)^2 \, dx}$$

3.0.2 Example:

$y = x^2$

$\sqrt{y} = x$

Repeat the previous example, but this time rotate around the y-axis.



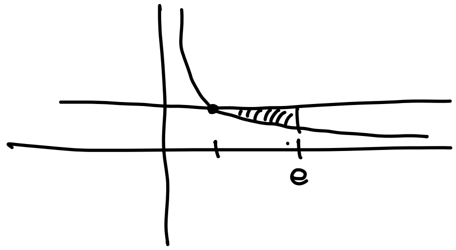
thickness $= dy$

Rad $= \sqrt{y}$

$\pi \int \text{Rad}^2 \, dy$

$$\pi \int_0^9 (\sqrt{y})^2 \, dy = \pi \int_0^9 y \, dy = \pi \cdot \frac{1}{2} y^2 \Big]_0^9$$

$$= \pi \cdot \frac{9^2}{2} = \boxed{\frac{81\pi}{2}}$$



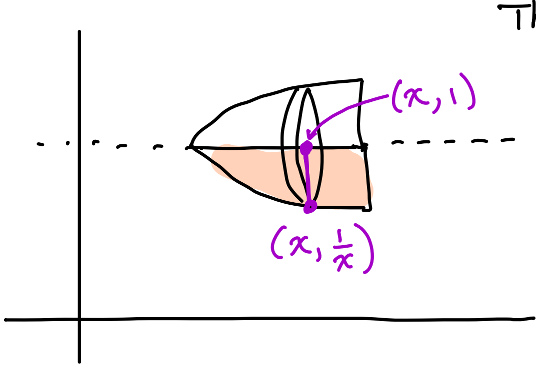
4 More Examples

4.0.1 Examples:

Take the area bounded by $y = 1$, $y = \frac{1}{x}$ $1 \leq x \leq e$, and set up an integral for the volume when it is rotated around (a) the line $y = 1$, (b) the x-axis, and (c) the y-axis.

Thick = dx

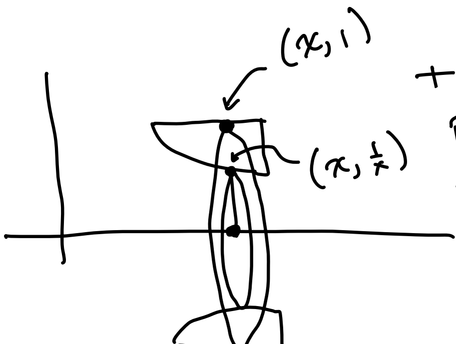
a)



Red = $1 - \frac{1}{x}$

$$V = \pi \int_1^e \left(1 - \frac{1}{x}\right)^2 dx$$

b)



thick = dx

$R = 1$

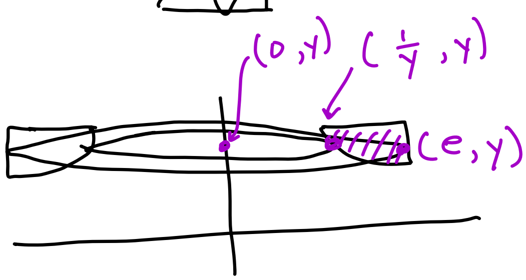
$r = \frac{1}{x}$

$$V = \pi \int_1^e 1^2 - \left(\frac{1}{x}\right)^2 dx$$

$y = \frac{1}{x}$

$x = \frac{1}{y}$

c)



Thick = dy

$R = e$

$r = \frac{1}{y}$

$y = \frac{1}{x}$

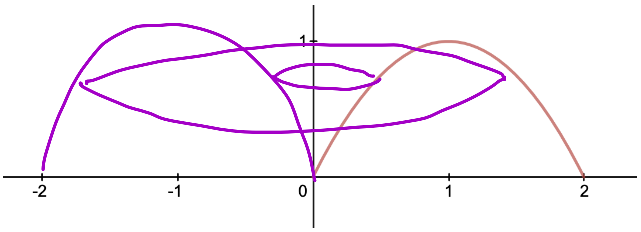
$$V = \pi \int_{\frac{1}{e}}^1 e^2 - \frac{1}{y^2} dy$$

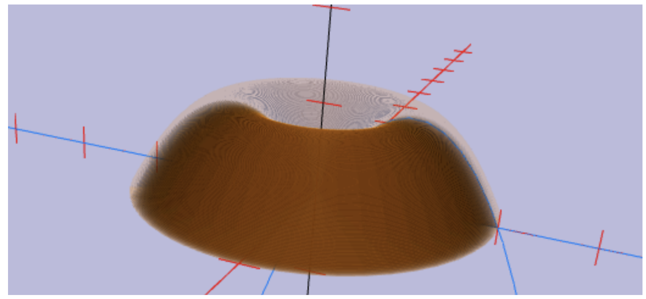
6.3 Cylindrical Shells

Some volumes are very difficult to find using disks and washers.

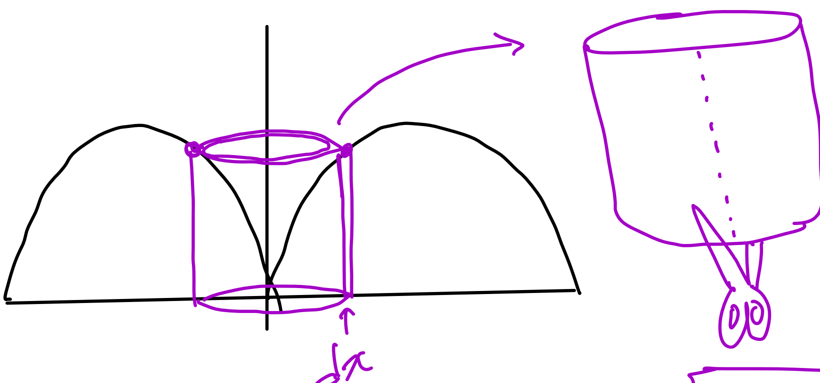
0.0.1 Example:

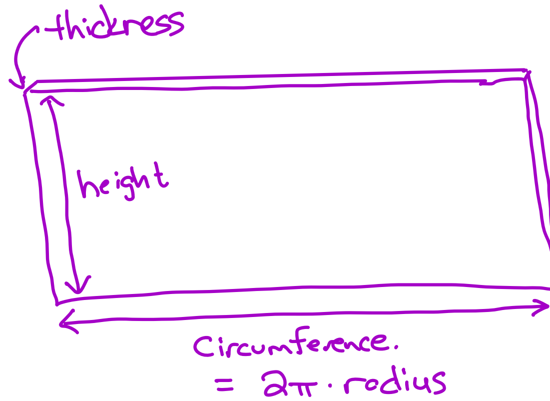
Area below $y = 2x - x^2$ rotated around the y-axis.





Better Method: Break it into cylindrical shells.





$$\boxed{V_{shell} = 2\pi\,(\text{radius})(\text{height})(\text{thickness})}$$

$$V_{solid} = 2\pi \int_a^b (\text{radius})(\text{height})(\text{thickness})$$

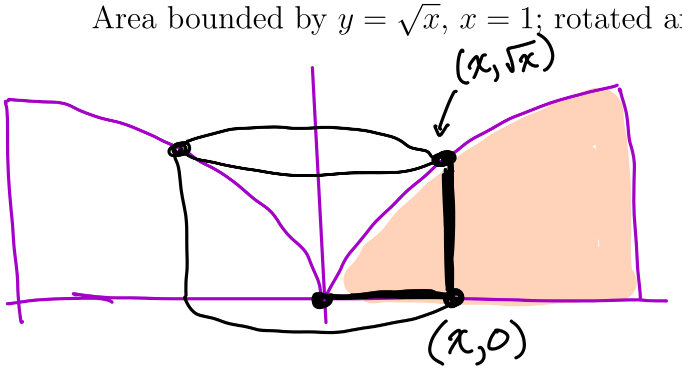
Cylindrical Shell Method: $\quad$ Volume $= 2\pi \displaystyle\int_a^b (\text{radius})(\text{height})(\text{thickness})$

$$\text{Thickness} = \begin{cases} dx & \text{if rotated around a vertical line} \\ dy & \text{if rotated around a horizontal line.} \end{cases}$$

← opposite from disks | washers

0.0.2 Example:

Area bounded by $y = \sqrt{x}$, $x = 1$; rotated around y-axis.



thickness $= dx$

radius $= x$

height $= \sqrt{x}$

$$V = 2\pi \int_0^1 x\sqrt{x}\,dx$$

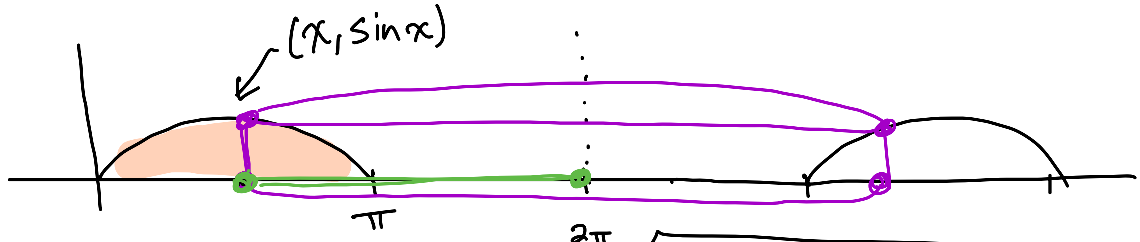
$$= 2\pi \int_0^1 x^{3/2}\,dx$$

$$= 2\pi \frac{2}{5} x^{5/2} \Big]_0^1 = \frac{4\pi}{5}(1-0) = \boxed{\frac{4\pi}{5}}$$

0.0.3 Example:

positive

Area bounded by $y = \sin x$, x-axis; rotated around the line $x = 2\pi$.



thick $= dx$

height $= \sin x$

radius $= 2\pi - x$

$$\boxed{V = 2\pi \int_0^{\pi} (2\pi - x)\sin x\,dx}$$