Wednesday

1. Test is Monday Feb 2
    A. Practice Test is on courseden
2. HW 6.2 6.3 due tonight - questions?
3. HW 6.4 Questions?
4. Finish 6.4 (Water pumping)
5. Cover 6.5 Average value of function

Friday

1. Review day. Go over practice test w/ Dr. Robinson (I won't be here because of a funeral)

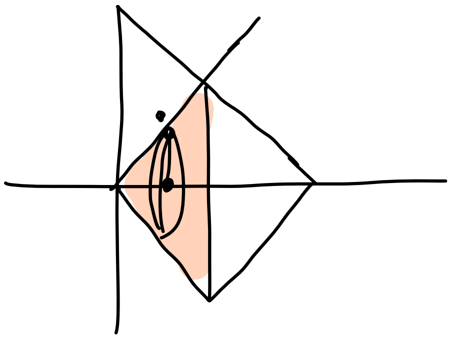
Monday

1. test 1

6.3 #4 $\quad y = x, \quad y = 1 - x, \quad x\text{-axis}$ $\qquad V = \pi \int_a^b (\text{radius})^2 \, dx$

thick = dx

radius = x

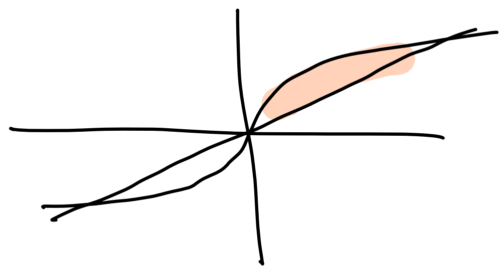


$$V = 2\pi \int_0^{\frac{1}{2}} x^2 \, dx = 2\pi \left. \frac{x^3}{3} \right]_0^{1/2}$$

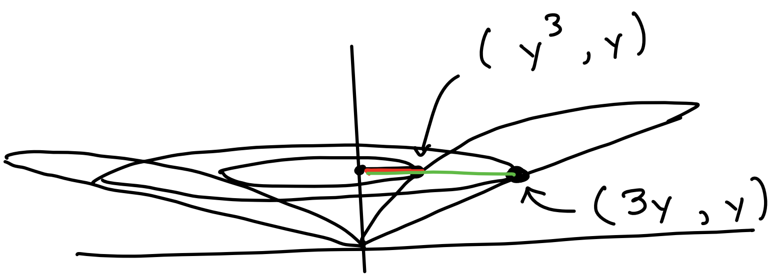
$$= \frac{2\pi}{3}\left(\frac{1}{2}\right)^3 = \frac{2\pi}{3} \cdot \frac{1}{8} = \frac{2\pi}{24}$$

$$= \boxed{\frac{\pi}{12}}$$

$$2\pi \int_{\frac{1}{2}}^{1} (1-x)^2 \, dx$$



6.2 #4 $\quad x = 3y \qquad x = y^3$



thickness = dy

$R = 3y$

$r = y^3$

$$\pi \int R^2 - r^2 \, dy = \pi \int_0^{\sqrt{3}} (3y)^2 - (y^3)^2 \, dy$$

$3y = y^3$

$3 = y^2$

$\sqrt{3} = y$

$$= \pi \int_0^{\sqrt{3}} 9y^2 - y^6 \, dy = \pi \left[3y^3 - \frac{1}{7}y^7 \right]_0^{\sqrt{3}}$$

$$= \pi \left[3(\sqrt{3})^3 - \frac{1}{7}(\sqrt{3})^7 \right] \approx 27.9843\ldots$$

$$\int 4y^3 - y^6 \, dy$$

6.2

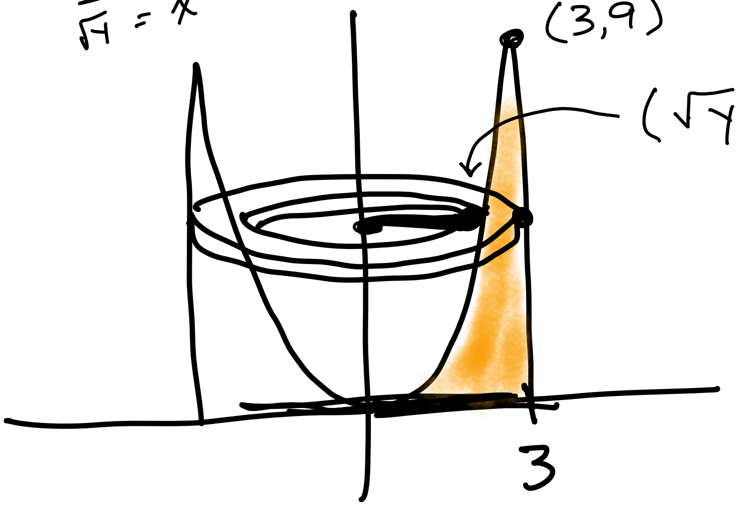
3. $y = x^2$, $y = 0$, $x = 3$ ↻ y-axis

$\sqrt{y} = x$

thick = dy

$R = 3$

$r = \sqrt{y}$



$$V = \pi \int_0^9 3^2 - (\sqrt{y})^2 \, dy$$

$$= \pi \int_0^9 9 - y \, dy$$

$$= \pi \left[9y - \frac{1}{2}y^2 \right]_0^9 = \pi \left[81 - \frac{1}{2}(81) \right] = \frac{81\pi}{2} \approx \boxed{127.23}$$

2.0.4 Example:

A spring with a natural length of 50 cm is stretched to a length of 90 cm. If the spring constant is 2 newtons per meter, how much energy is stored in the spring?

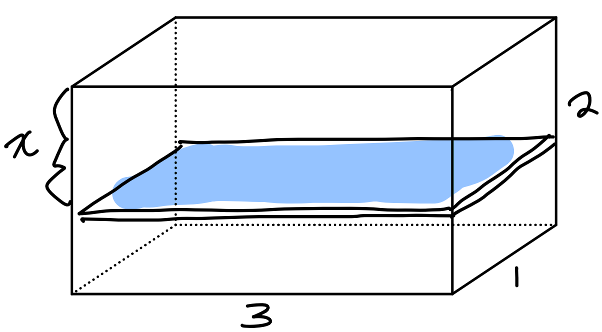
3 Water-pumping problems

The problems in this subsection involving pumping water out of various tanks by lifting the water to the top. The tricky bits are that

- the water at the bottom has to be lifted further than the water at the top, and
- the area of each "sheet" of water can vary if the sides of the tank are sloped.

3.0.1 Example:

Suppose you have a tank of water that is 1 meter wide, 3 meters long, and 2 meters tall. How much work is required to pump the water out of the tank?



Weight of Sheet × distance lifted.

Vol × density

Slice

Area = $3 \ m^2$

Volume = $3dx \ m^3$

Weight = $9800 \cdot 3dx$ N

density of water in N/m^3

Raise the slice x meters.

Work slice = $9800 \cdot 3x\,dx$ Joules.

$$\text{Work} = \int_0^2 9800 \cdot 3x\,dx$$

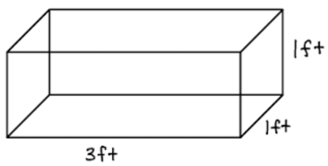
$$= 29400 \int_0^2 x\,dx$$

$$= 29400 \cdot \frac{1}{2} x^2 \Big]_0^2 = \boxed{58{,}800 \text{ J}}$$

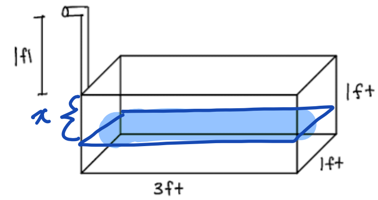
Worksheet 6.4: Work to Pump Water

Each tank shown here is full of water. Set up an integral for the work required to pump all of the water to the top of the tank (or top of the spout, if there is one). Round to the nearest whole number.

1. (a)



(b)



Volume $= 3 \cdot 1 \cdot dx = 3dx$ ft^3

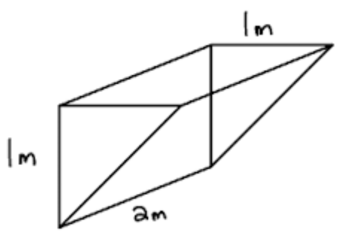
(Force) Weight $= 62.5 \cdot 3dx = 187.5\,dx$ lbs

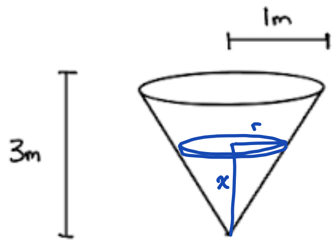
dist $= x + 1$ ft

Work $= 187.5(x+1)dx$ J

$$\boxed{\text{Total Work} = 187.5\int_0^1 x+1\,dx}$$

2.





3.

Sheet

$Vol = \pi r^2 dx$

$\frac{r}{x} = \frac{1}{3} \rightarrow 3r = x$

$r = \frac{x}{3}$

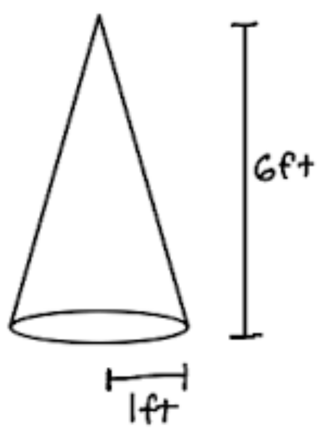
$Vol = \pi \left(\frac{x}{3}\right)^2 dx$

$Weight = 9800\pi \left(\frac{x^2}{9}\right) dx$

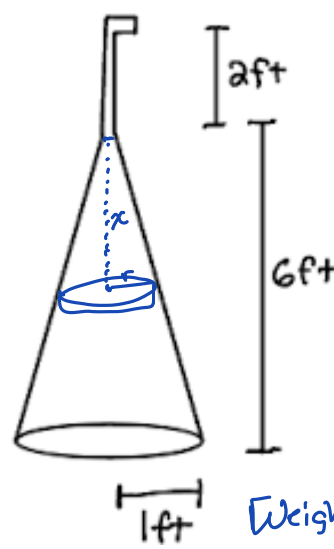
$= \frac{9800\pi}{9} x^2 dx$

distance lifted $= 3 - x$

$$\text{Total Work} = \frac{9800\pi}{9} \int_0^3 (3-x)\, x^2 dx \approx \boxed{23091\,J}$$



4. (a)



(b)

$V = \pi r^2 dx$

$\frac{x}{r} = \frac{6}{1}$

$6r = x$

$r = \frac{1}{6}x$

$V = \frac{\pi}{36} x^2 dx$

$Weight = \frac{62.5\pi\, x^2 dx}{36}$

$distance = x + 2$

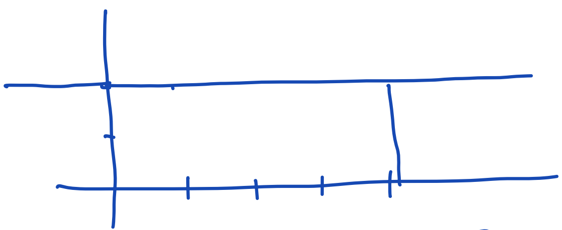
$$Work = \frac{62.5\pi}{36} \int_0^6 x^2 (x+2)\, dx$$

6.5 Average Value of a Function

The goal of this section is to find the average value of $f(x)$ over some interval $[a, b]$. This is slightly complicated by the fact that we're taking the average of an infinite number of values.
Let's start with some easy ones.

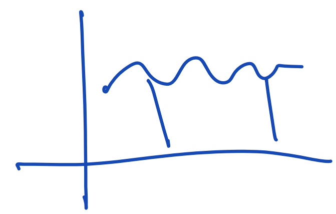
0.0.1 Example:

Find the average value of $f(x) = 2$ over the interval $[0, 4]$



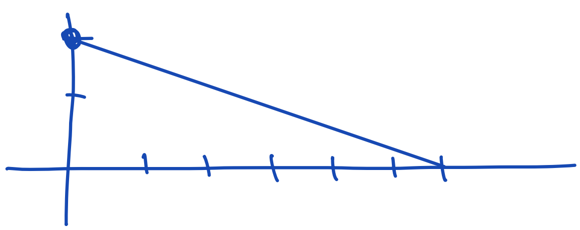
avg = 2.

Area = $4 \cdot 2$



0.0.2 Example:

Find the average value of $f(x) = 2 - \frac{1}{3}x$ over the interval $[0, 6]$



Avg = 1

Area = $\frac{1}{2} \cdot 6 \cdot 2$
$= 6$

Observation: Area = (length of interval) × (average value of the function)

Rearranging this gives

$$\text{(average value)} = \frac{\text{Area}}{\text{length of interval}}$$

Thus we have that

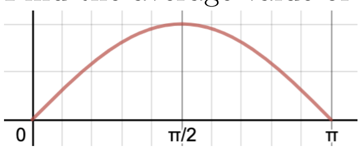
Average value of a function
The average value of the function $f(x)$ on the interval $[a, b]$ is given by

$$\frac{1}{b-a} \int_a^b f(x)\, dx$$

1 Examples

1.0.1 Example:

Find the average value of $f(x) = \sin x$ on $[0, \pi]$



$$\text{avg} = \frac{1}{\pi - 0}\int_0^\pi \sin x \, dx = \frac{1}{\pi}\Big[-\cos x\Big]_0^\pi$$

$$= \frac{1}{\pi}\Big[-\cos(\pi) + \cos(0)\Big] = \frac{1}{\pi}\Big[-(-1) + 1\Big]$$

$$= \frac{1}{\pi}\big[2\big] = \boxed{\frac{2}{\pi}} \approx .637$$

1.0.2 Example:

Find the average value of $g(x) = \sqrt{x}$ over $[4, 9]$.

$$\text{avg} = \frac{1}{9-4}\int_4^9 x^{1/2}\, dx = \frac{1}{5}\,\frac{2}{3}\,x^{3/2}\Big]_4^9 = \frac{2}{15}\left((9)^{3/2} - (4)^{3/2}\right)$$

$$= \frac{2}{15}\left(27 - 8\right) = \frac{2}{15}\cdot 19 = \boxed{\frac{38}{15}}$$

1.0.3 Example:

Find the average value of $f(x) = \frac{1}{x}$ on the interval $[1, 10]$

$$\text{avg} = \frac{1}{10-1}\int_1^{10} \frac{1}{x}\, dx = \frac{1}{9}\ln x\Big]_1^{10} = \frac{1}{9}\ln(10) - \frac{1}{9}\underbrace{\ln 1}_{0}$$

$$= \boxed{\frac{1}{9}\ln 10}$$