5.5 u-Substitution

So far, the rules we've learned for taking integrals/antiderivatives have all been just using the derivative rules backward.

u-substitution is what lets us "undo the chain rule".

Suppose you're asked to do these two integrals:

$$\int 2x\cos(x^2)\,dx \qquad\qquad \int \frac{3x^2}{1+x^6}\,dx$$

If you look at these for a long time and really think about them, you might realize that the integrands (the things inside the integrals) look like derivatives where the chain rule was used.

But that's not a method. How do we do this if we don't see the derivative step?

u-Substitution

If u is some function of x, we can use u-substitution to do integrals of the form

$$\int f(u)u'\,dx.$$

1. Find u. (u is nested inside another function)
2. Take the differential of u to get $du = u'dx$.
3. Solve the differential for dx: $dx = \frac{du}{u'}$ OR replace $u'dx$ with du in the integrand.
4. Substitute into the integral
5. Cancel any remaining x's
6. Integrate with respect to u
7. Put the answer back in terms of x.

0.0.1 Example:

$$\int x^3(1-x^4)^2 \, dx$$

0.0.2 Example:

$$\int (2x+1)\cos(x^2+x+1) \, dx$$

0.0.3 Example:

$$\int \frac{\sqrt{\ln x}}{x} \, dx$$

0.0.4 Example:

$$\int \sin^2 x \cos x \; dx$$

0.0.5 Example:

$$\int \cos(7x) \; dx$$

0.0.6 Example:

$$\int x e^{x^2+3} \; dx$$

1 u-Substitution with Definite Integrals

When doing u-substitution with definite integrals, you have two options:

1. At the end of the problem, put the answer back in terms of x and plug in the original endpoints. **OR**

2. When you do the substitution, translate your endpoints from x-values to u-values.

Both are perfectly fine, but I'm almost always going to translate my endpoints.

1.0.1 Example:

$$\int_0^1 3x(x^2-1)^2\,dx$$

1.0.2 Example:

$$\int_{-2}^2 3x(4-x^2)\,dx$$

1.0.3 Example:

$$\int_0^1 \frac{8x}{(2x^2+1)^3}\,dx$$

1.0.4 Example:

$$\int_0^e \frac{3x^2}{x^3+1}\,dx$$

1.0.5 Example:

$$\int_0^{\frac{\pi}{4}} \tan(x)\,dx$$