Hamilton Decompositions of Complete Multipartite Graphs with any 2-Factor Leave

C. D. Leach and C. A. Rodger
Department of Discrete and Statistical Sciences
235 Allison Lab
Auburn University, AL 36849-5307

Abstract

For $m \geq 1$ and $p \geq 3$, given a set of integers $s_1, \ldots, s_q$ with $s_j \geq p+1$ for $1 \leq j \leq q$ and $\sum_{j=1}^{q} s_j = mp$, necessary and sufficient conditions are found for the existence of a hamilton decomposition of the complete p-partite graph $K_{m,\ldots,m} - E(U)$, where U is a 2-factor of $K_{m,\ldots,m}$ consisting of q cycles, the jth cycle having length s_j. This result is then used to completely solve the problem when $p = 3$, removing the condition that $s_j \geq p+1$.

1 Introduction

A hamilton decomposition of a graph is a partition of its edge set into spanning cycles. In 1982, Walecki [7] proved that K_n has a hamilton decomposition if and only if n is odd. Laskar and Auerbach [4] showed in 1976 that the complete p-partite graph $K_{a_0,\ldots,a_{p-1}}$ has a hamilton decomposition if and only if both $m(p-1)$ is even and $m = a_0 = \cdots = a_{p-1}$. Using the technique of vertex amalgamations (or graph homomorphisms), Hilton and Rodger [3] presented new proofs of both results. The methods developed by Hilton and Rodger proved to be very powerful in finding hamilton decompositions of other graphs.

In 1997, Buchanan [2] used amalgamations to show that for any odd n, and any 2-factor U of K_n, $K_n - E(U)$ has a hamilton decomposition. Recently, Rodger and Leach [5] solved the corresponding problem for the complete bipartite graph $K_{a,b} - E(U)$. In this paper, sufficient conditions are found for the existence of a hamilton decomposition of the complete p-partite graph $K_{m,\ldots,m} - E(U)$ where U is a 2-factor of $K_m^{(p)}$ that is specified by its cycle lengths. This general result is then used to completely solve this problem for complete tripartite graphs.

In this paper, graphs can have multiple edges, but no loops. Let $m_G(v,w)$ denote the number of edges joining v to w in G. $G(i)$ denotes the subgraph of G induced by the edges colored i, $G[E]$ denotes the subgraph of G induced by E, and the complete p-partite graph $K_{m,\ldots,m}$ is denoted by $K_m^{(p)}$. Let $y(\mathrm{mod}\ p)$ denote the integer x satisfying $x \equiv y(\mathrm{mod}\ p)$ and $x \in \{0,1,\ldots,p-1\} = \mathbb{Z}_p$.

2 Hamilton decompositions with a 2-regular leave.

The results concerning amalgamations that we need here have already been proved, so there is no need to explicitly describe the method here. We only need the following theorem, which is a corollary of a theorem by and Leach and Rodger [5].

Theorem 2.1 *Let $\ell, m \geq 1$, let $p \geq 2$, let G be an ℓ-edge-colored p-partite graph with vertex partition $\{A_0,\ldots,A_{p-1}\}$, and let $\eta : V(G) \to \mathbb{N}$. If*

1) $d_G(w) = \eta(w)(p-1)m$, for all $w \in V(G)$,

2) $m_G(v,w) = \begin{cases} \eta(v)\eta(w) & \text{if } v \text{ and } w \text{ are in different parts,} \\ 0 & \text{if } v \text{ and } w \text{ are in the same part,} \end{cases}$

3) $d_{G(i)}(w) = 2\eta(w)$, *for all* $w \in V(G)$,

4) $\sum_{v \in A_i} \eta(v) = m$, *for* $0 \le i \le p-1$, *and*

5) for $2 \le j \le \ell$, $G(j)$ *is connected; and* $G(1)$ *consists of* q *components* $C_1, \dots, C_q$ *with* $\sum_{v \in C_j} \eta(v) = s_j$ *for* $1 \le j \le q$,

then there exists a hamilton decomposition of $K_m^{(p)} - E(U)$ *for some* 2*-factor* U *consisting of* q *cycles, the* j*th cycle having length* s_j.

We also use the following result of Bermond, Favaron, and Maheo [1]:

Theorem 2.2 *If* G *is a* 4*-regular connected Cayley graph on* n *vertices, then* G *can be decomposed into two hamilton cycles.*

Let $|u-v|_d = \min\{(u-v) \mod d, (v-u) \mod d\} \in \mathbb{Z}_d$.

These results allow us to prove the following corollary.

Theorem 2.3 *Let* $m \ge 1$ *and* $p \ge 2$. *Let* $s_1, \dots, s_q$ *be integers such that* $s_j \ge p+1$ *for* $1 \le i \le q$ *and* $\sum_{j=1}^{q} s_j = mp$. *There exists a* 2*-factor* U *of* $K_m^{(p)}$ *with cycles of length* $s_1, \dots, s_q$ *such that* $K_m^{(p)} - E(U)$ *has a hamilton decomposition if and only if* $m(p-1)$ *is even.*

Proof: Suppose that such a hamilton decomposition exists. Since $K_m^{(p)} - E(U)$ is regular of degree $m(p-1)-2$, any hamilton decomposition consists of $(m(p-1)-2)/2$ hamilton cycles. Thus $m(p-1)$ must be even.

Now suppose that $m(p-1)$ is even. If $q=1$, then $s_1 = mp$ and the problem reduces to finding a hamilton decomposition of $K_m^{(p)}$; this has already been solved (see [3, 4]). For the remainder of the proof, we assume that $q \ge 2$. If $p=2$, then the result was proved in [5], so we can also assume that $p \ge 3$.

By Theorem 2.1, it suffices to construct an edge-colored graph G that satisfies conditions (1-5). G is constructed by first amalgamating K, an edge-colored graph isomorphic to $K_m^{(p)}$, and then recoloring the edges of two color classes in the resulting amalgamated graph.

Let $V(K) = \{0, 1, \dots, mp-1\}$, and let $\{A'_0, \dots, A'_{p-1}\}$ be a partition of $V(K)$ by defining $v \in A'_i$ if and only if $v \equiv i \pmod p$. We begin with the edge-coloring $c : E(K) \to \mathbb{Z}_{mp/2} \setminus \{jp \mid j \in \mathbb{Z}_{\lfloor m/2 \rfloor}\}$, defined by $c(\{u,v\}) = |u-v|_{mp}$. So each of the $\ell = (p-1)m/2$ color classes induces a 2-regular Cayley graph. Moreover, the union of any two color classes is a 4-regular Cayley graph. Also $G(a) \cup G(b)$ is connected if and only if $\gcd(\{a,b,mp\}) = 1$ (see Lemma 7.3.3 of [6], or [8] for example). The edges of $G(1) \cup G(2)$ will be used to construct one hamilton cycle and U. The remaining edges can be decomposed into hamilton cycles as shown below (although these may not be the hamilton cycles in our final decomposition).

Name the colors used with $c_1, \dots c_{m(p-1)/2} = c_\ell$ so that $c_i < c_{i+1}$ for $1 \le i < \ell$; so $C = \{c_1, \dots, c_\ell\} = \mathbb{Z}_{mp} \setminus \{jp | j \in \mathbb{Z}_{\lfloor m/2 \rfloor}\}$. It follows that

(a) if p is odd, then $c_{2i} = c_{2i} + 1$ for $1 \le i \le \lfloor m(p-1)/4 \rfloor$, and

(b) if p is even, then for $1 \le i \le m(p-1)/2$

  (i) $c_{i+1} = c_i + 1$ if $c_i \not\equiv -1 \pmod p$

  (ii) $c_{i+1} = c_i + 2$ if $c_i \equiv -1 \pmod p$.

Notice that if p is even and $c_i \equiv -1 \pmod p$, then both c_i and c_{i+1} are odd with $c_{i+1} - c_i = 2$, so since $\gcd(c_i, c_{i+1})$ divides $c_{i+1} - c_i$ and divides c_i, it follows that $\gcd(c_i, c_{i+1}) = 1$. Also, if $c_{i+1} = c_i + 1$, then clearly $\gcd(c_i, c_{i+1}) = 1$. Therefore in every case, for $1 \le i \le \lfloor m(p-1)/4 \rfloor$, $\gcd(c_{2i-1}, c_{2i}) = 1$. Then it follows from Theorem 2.2 that for $1 \le i \le \lfloor m(p-1)/4 \rfloor$, $G(c_{2i-1}) \cup G(c_{2i})$ has a hamilton decomposition. Furthermore, if $|C|$ is odd, then either m is odd and $p \equiv 3 \pmod 4$, or $m \equiv 2 \pmod 4$ and p is even. In either case, $c_{m(p-1)/2} = \lfloor (mp-1)/2 \rfloor$. So since in such cases $mp - \lfloor (mp-1)/2 \rfloor = \lfloor m(p-1)/2 \rfloor + 1$, it follows that $\gcd(\lfloor (mp-1)/2 \rfloor, mp) = 1$; therefore $G(c_{m(p-1)/2})$ is connected, so is itself a hamilton cycle.

Form a new edge-coloring of K with the colors $1, \dots, \ell = m(p-1)/2$ by letting the edges colored 1 or 2 retain their color, but then letting each other color class induce a hamilton cycle.

Let G be the graph with vertex set $\{w_{j,i} \mid 0 \le i \le p-1, 1 \le j \le q\}$ and edge-coloring $c : E(G) \to \{1, \dots, \ell\}$ formed by taking the homomorphic image of K with its new edge-coloring defined by the homomorphism $f : V(K) \to V(G)$ where $f(v) = w_{j,i}$ if $i \equiv v \pmod p$ and $\sum_{x=0}^{j-1} s_x \le v < \sum_{x=0}^{j} s_x$, where for convenience $s_0 = 0$. (The bijective relation between $E(K)$ and $E(G)$ allows G to inherit the ℓ-edge-coloring given to K.) So G is a p-partite graph with parts $A_i = \{w_{j,i} \mid 1 \le j \le q\}$, $0 \le i \le p-1$. By defining $\eta(w_{j,i}) = |f^{-1}(w_{j,i})|$, G is easily seen to satisfy conditions (1-4) of Theorem 2.1. The set $L_j = \{w_{j,i} | 0 \le i \le p-1\}$ is referred to as the jth *layer* of the graph G.

G certainly does not meet condition 5 of Theorem 2.1 with this inherited coloring because $G(1)$ is connected; also $G(2)$ may be disconnected. So we must recolor the edges colored 1 or 2. Note however that $G(i)$ is connected for $3 \leq i \leq \ell$, and this new coloring $\bar{c}$ will not affect such color classes. We define $\bar{c}$ such that $G(i)$ is connected for $2 \leq i \leq \ell$, and such that each layer induces a component of $G(1)$. Of course, care must be taken to ensure that G still satisfies condition 3 when colored with $\bar{c}$. The coloring $\bar{c}$ is defined by exchanging some edges between color classes 1 and 2.

We define multisets T_1 and T_2 of edges in G colored 1 and 2 respectively in such a way that for each $v \in V(G)$, $d_{G[T_1]}(v) = d_{G[T_2]}(v)$. Then recoloring the edges in T_1 with 2 and the edges in T_2 with 1 clearly produces a new edge-coloring $\bar{c}$ of G which satisfies conditions (1-4) of Theorem 2.1; to avoid confusion between the two edge-colorings, we let $\bar{G}(i)$ denote the subgraph of G induced by the ith color class of $\bar{c}$. The choices of T_1 and T_2 are also made so that condition (5) holds for $\bar{c}$; this is shown after T_1 and T_2 are defined. T_1 and T_2 are formed from the union of sets $T_{1,j}$ for $(0 \leq j \leq q)$ and $T_{2,j}$ for $(1 \leq j \leq q)$ of edges incident with vertices in L_j defined below.

For $1 \leq j \leq q$, exactly one edge of $G(1)$ connects layer j to layer $j+1$. Let $T_{1,0}$ be the set of all q such edges. Also for each layer L_j, let $w_{j,a}$ be the unique vertex in L_j that is adjacent in $G(1)$ to a vertex in L_{j-1}; we say that color 1 *enters* L_j at $w_{j,a}$. Let

$$\alpha = \begin{cases} s_j (\text{mod } p) - 1 & \text{if } s_j (\text{mod } p) \geq 3, \\ p + s_j (\text{mod } p) - 1 & \text{if } 0 \leq s_j (\text{mod } p) \leq 2. \end{cases}$$

Now color 1 enters layer j at $w_{j,a}$, so it leaves at $w_{j,a+\alpha}$ (since $s_j \equiv \alpha + 1 (\text{mod } p)$). For $1 \leq j \leq q$, let

$$T_{1,j} = \{\{w_{j,a+i}, w_{j,a+i+1}\} | 1 \leq i \leq \alpha - 2\},$$

and let

$$T_{2,j} = \{\{w_{j,a+i-1}, w_{j,a+i+1} | 1 \leq i \leq \alpha - 1\},$$

where all sums in the subscripts are reduced modulo p (see Figure 1). When $p = 4$ and $s_j \equiv 1$ or 2 (mod p), $T_{2,j}$ is a multiset. Two edges in G that join the same pair of vertices are said to be *twin* edges.

Since $T_{1,0}$ contains all the edges of $G(1)$ that bridge layers, and T_2 contains no edges that bridge layers, clearly $\bar{G}(1)$ consists of at least q components. To show that $\bar{G}(1)$ consists of exactly

q components, we must show that for $1 \leq j \leq q$, $\bar{G}[L_j](1)$ is connected. First note that for $1 \leq j \leq q$, $G[L_j](1)$ is connected since $s_j \geq p+1$, so $G[L_j](1)$ contains a spanning cycle.

G

L_{j-1} $a-1$ a

$a+\alpha$

L_j

L_{j+1}

edges colored 1

edges colored 2

$\bar{G}$

L_{j-1} $a-1$ a

$a+\alpha$

L_j

L_{j+1}

Suppose that $s_j (\text{mod } p) = 0$ or that $s_j (\text{mod } p) \geq 3$. Then $\alpha \leq p-1$, and so clearly no pair of twin edges are included in $T_{1,j}$ for any value of j. Also it is easy to check that $T_{1,j}$ includes at most one edge that has no twin in G. Therefore it follows that $\bar{G}[L_j](1)$ contains a hamilton path (that is also in $G[L_j](1)$) and so is clearly connected.

Otherwise $1 \leq s_j (\text{mod } p) \leq 2$ and in each of two cases a walk w in $\bar{G}[L_j](1)$ is given below that spans L_j:

1) if p is even then $w = (w_{j,a+2}, w_{j,a+4}, \ldots, w_{j,a-2}, w_{j,a}, w_{j,a+1}, w_{j,a+3}, \ldots, w_{j,a-1})$;

2) if p is odd then $w = (w_{j,a+1}, w_{j,a+3}, \ldots, w_{j,a-2}, w_{j,a}, w_{j,a+2}, w_{j,a+4}, \ldots, w_{j,a-1})$.

Therefore, in every case each layer of G induces a component of $\bar{G}(1)$.

Now we must show that $\bar{G}(2)$ is connected. In $G(2)$, for $1 \leq j \leq q$, there are two edges joining L_j to L_{j+1} (and in $\bar{G}(2)$ there are three such edges, one of which is in $G(1)$). We will show that $\bar{G}(2)$ consists of a single component by showing that for $1 \leq j \leq q$, $\bar{G}[L_j](2)$ is a subgraph of some connected subgraph of $\bar{G}(2)$.

First suppose that $s_j (\text{mod } p) = 0$ or $s_j (\text{mod } p) \geq 3$. Note that for $1 \leq j \leq q$, $G[L_j](2)$ consists of one component if p is odd and two components if p is even (induced by vertices with subscripts that have even or odd first components). Also every edge in $T_{2,j}$ has a twin in $G(2)$, so removing the edges in $T_{2,j}$ from $G[L_j](2)$ produces a subgraph of $\bar{G}[L_j](2)$ that has the same components as $G[L_j](2)$. Furthermore, in these cases $T_{1,j}$ is nonempty, so if $G[L_j](2)$ consists of two components then any edge of $T_{1,j}$ will connect them (since if p is even, then each edge of $T_{1,j}$ joins vertices whose first components have different parities). Thus each of these layers induces a single component of $\bar{G}(2)$.

In the remaining cases, the path $(w_{j,a}, w_{j-1,a-1}, w_{j,a+1}, w_{j,a+2}, \ldots, w_{j,a-1})$ is a path in $\bar{G}(2)$ that contains all vertices in $\bar{G}[L_j](2)$ (see figure 1). Thus each layer is a subgraph of a connected subgraph of $\bar{G}(2)$.

Since each layer is either connected, or is a subgraph of a connected subgraph of $\bar{G}(2)$, and since each layer is connected to the layer above and below, it follows that $\bar{G}(2)$ is connected.

Therefore the graph G colored with $\bar{c}$ satifies properties (1-5), so the result follows from Theorem 2.1. □

3 Hamilton Decompositions of $K_{m,m,m}$ with a 2-factor leave

The following corollary follows immediately from Theorem 2.2.

Corollary 3.1 *Let F be a set of q vertex-disjoint cycles with the length of the jth cycle being s_j, $\sum_{j=1}^{q} s_j = 3m$, and $s_j \geq 4$ for $1 \leq j \leq q$. Then for some 2-factor $U \cong F$ of $K_{m,m,m}$, $K_{m,m,m} - E(U)$ has a hamilton decomposition.*

In this result, the condition that $s_j \geq 4$ for $1 \leq j \leq q$ does not seem necessary, and indeed this is the case as the following theorem shows.

Theorem 3.1 *Let F be a set of q vertex-disjoint cycles with the length of the jth cycle being s_j. Then there exists a 2-factor $U \cong F$ of $K_{m,m,m}$, such that $K_{m,m,m} - E(U)$ has a hamilton decomposition if and only if $\sum_{j=1}^{q} s_j = 3m$.*

Proof: The proof of necessity is clear, so assume that $\sum_{j=1}^{q} s_j = 3m$. This proof closely follows the proof of Theorem 2.3. In Theorem 2.3, the condition that $s_j \geq p+1$ was only used to show that $\bar{G}(1)$ and $\bar{G}(2)$ are connected. So for this result, we need only show the two properties that when $s_j = 3$ then $\bar{G}[L_j](1)$ is connected, and that $\bar{G}(2)$ is connected when F contains cycles of length 3.

The first property is satisfied since for $1 \leq j \leq q$, if $s_j = 3$, then $\bar{G}[L_j](1) = K_3$. For $1 \leq j \leq q$, L_j is connected to L_{j+1} in $\bar{G}(2)$, so for the second property, we need only to show that if $s_j = 3$ then L_j is contained in a connected subgraph of $\bar{G}(2)$. Again we let $w_{j,a}$ be the unique vertex in L_j that is joined by an edge of $G(1)$ to a vertex on layer $j-1$. If $s_j = 3$, then $\{w_{j,a}, w_{j,a-1} = w_{j,a+2}\}$ is placed into $T_{2,j}$ and $\{w_{j,a}, w_{j-1,a+2}\}$ and $\{w_{j,a+2}, w_{j+1,a}\}$ are placed into $T_{1,j}$. Thus $\{w_{j,a}, w_{j-1,a-1}, w_{j,a+1}, w_{j+1,a}, w_{j,a+2}\}$ is a path within $\bar{G}(2)$ that spans L_j and is connected to L_{j-1}.

Thus $\bar{G}(2)$ is connected and the theorem is proved. □

As a final comment we remark that, following the proof of Theorem 3.1, one could probably show that the condition $s_j \geq p+1$ can be removed for other small values of p. The quest now though is to find a construction that removes it for all values of p.

References

[1] J. Bermond, O. Favaron, M. Maheo, Hamilton decomposition of Cayley graphs of degree 4, J. Combin. Th. Ser. B, 46 (1989), 142-253.

[2] H. Buchanan, Ph.D. Dissertation, University of West Virginia, 1996.

[3] A. J. W. Hilton and C. A. Rodger, Hamilton decompositions of complete regular s-partite graphs, Discrete Math., 58 (1986), 63-78.

[4] R. Laskar and B. Auerbach, On the decompositions of r-partite graphs into edge-disjoint hamilton circuits, Discrete Math., 14 (1976), 146-155.

[5] C. D. Leach and C. A. Rodger, Non-disconnecting disentanglements of amalgamated 2-factorizations of complete multipartite graphs, J. Combin. Designs, to appear.

[6] C. C. Lindner and C. A. Rodger, Design Theory, CRC Press, Boca Raton, 1997.

[7] D. E. Lucas, Recreations Mathématiques, 2 (1892) Gauthiers Villars, Paris.

[8] G. Stern and H. Lenz, Steiner triple systems with given sub-spaces; another proof of the Doyen-Wilson Theorem, Boll. Un. Mat. Ital. (5) 17-A (1980), 109-114.