Non-disconnecting disentanglements of amalgamated 2-factorizations of complete multipartite graphs

C. D. Leach and C. A. Rodger*
Department of Discrete and Statistical Sciences
120 Math Annex
Auburn University, AL 36849-5307

Abstract

In this paper necessary and sufficient conditions are found for an edge-colored graph H to be the homomorphic image of a 2-factorization of a complete multipartite graph G in which each 2-factor of G has the same number of components as its corresponding color class in H. This result is used to completely solve the problem of finding hamilton decompositions of $K_{a,b}-E(U)$ for any 2-factor U of $K_{a,b}$.

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1 Introduction

A *G-decomposition* of a graph H is a partition of the edges of H such that each element of the partition induces a subgraph isomorphic to G. An *m-cycle system* of *order* n is a C_m-decomposition of K_n, where C_m is a cycle of length m (an m-cycle). The *leave* of an m-cycle decomposition of H is the set of edges in $K_n - E(H)$, where n is the number of vertices in H.

The history of cycle decompositions dates back to at least 1847, when Kirkman [8] solved the existence problem for 3-cycle systems of K_n. Throughout the 20th century, many results have appeared involving various m-cycle systems of K_n. Recently the spectrum problem for m-cycle sytems of K_n was completely solved – the case with m odd by Alspach and Gavlas [1], and the case with m even by Sajner [11]. In 1981, Sotteau [12] found necessary and sufficient conditions for the existence of m-cycle systems of complete bipartite graphs.

The existence of m-cycle decompositions with specified families of leaves have also been studied extensively. This problem was solved for all m in the case where the leave is a 1-factor by Alspach, Gavlas and Sajner [1, 11]. Colbourn and Rosa [5] found necessary and sufficient conditions for the existence of 3-cycle decompositions with leaves being the 2-regular (possibly non-spanning) graphs. Rodger and Fu have found necessary and sufficient conditions for the existence of partial 4-cycle systems of K_n with forest leaves [6] and 2-regular leaves [7]. Ashe, Fu and Rodger have extended these results to partial 6-cycle systems [2].

Hamilton decompositions date back to at least 1892, when Walecki solved the spectrum problem for hamilton decompositions of K_n. In 1976, Laskar and Auerbach [3] determined which complete multipartite graphs have hamilton decompositions. In 1997, Buchanan [4] used the technique of amalgamations to prove the existence of a hamilton decomposition of $K_n - E(G)$ for n odd with any 2-factor leave G.

In this paper, we obtain a general result concerning amalgamations of complete multipartite graphs in Section 2, then in Section 3 we apply this result to find necessary and sufficient conditions for the existence of hamilton decompositions of $K_{n,n} - E(U)$ with U being any 2-factor.

Throughout this paper, $m_G(u, v)$ denotes the number of edges in the graph G joining vertices u and v. The number of components of the graph G is denoted by $\omega(G)$. This paper makes extensive use of edge-colorings; $G(i)$ is used to denote the subgraph of G induced by the edges colored i. An edge-coloring is said to be *equitable* if $|d_{G(i)}(v) - d_{G(j)}(v)| \leq 1$ for all pairs of colors i, j and all vertices $v \in V(G)$. It has been proved by de Werra that for any $k \geq 1$ and for any bipartite multigraph B, there exists an equitable k-edge-coloring of B [13]. Notation and definitions not listed here can be found in [14].

2 Amalgamations and Outline 2-factorizations

Suppose G and H are graphs, with $|V(G)| \geq |V(H)|$ and $|E(G)| = |E(H)|$. A function $\psi : V(G) \to V(H)$ is said to be an *amalgamating function* of G if it is surjective and there exists a bijection $\phi : E(G) \to E(H)$ such that

(i) if $x, y \in V(G)$, $\psi(x) \neq \psi(y)$, and e joins x and y, then $\phi(e)$ joins $\psi(x)$ and $\psi(y)$ in H;

(ii) if e is a loop on $x \in V(G)$ in G, then $\phi(e)$ is a loop on $\psi(x)$ in H; and

(iii) if e joins $x, y \in V(G)$ in G, where $x \neq y$, but $\psi(x) = \psi(y)$, then $\phi(e)$ is a loop on $\psi(x)$ in H.

If these three conditions are satisfied, then H is said to be an *amalgamation* of G (or a *homomorphic image* of G), and G is said to be a *disentanglement* of H. If $w \in V(H)$, then $\psi^{-1}(w) = \{v \in V(G) \mid \psi(v) = w\}$, and $\eta(w) = |\psi^{-1}(w)|$ is said to be the *amalgamation number* of w. We denote the graph $H = (\psi(V(G)), \phi(E(G)))$ by $\psi(G)$, where for any set $S, f(S) = \{f(s) \mid s \in S\}$.

Let $K(n_1, \ldots, n_p)$ denote the complete multipartite graph (V, E) with partition $\{A_1, \ldots, A_p\}$ of V, where $|A_i| = n_i$ for $1 \leq i \leq p$; so two vertices are adjacent in this simple graph if and only if they are in different parts. Suppose further that $K(n_1, \ldots, n_p)$ has an edge-coloring $C : E(K(n_1, \ldots, n_p)) \to (1, 2, \ldots, \ell)$ for which the subgraph induced by each color class is a 2-factor. This coloring is a 2-factorization of $K(n_1, \ldots, n_p)$, and its existence implies that every vertex has degree 2ℓ, and hence every part must be the same size, say m. Thus we get $\ell = (p-1)m/2$. For brevity, we denote the p-partite graph $K(m, m, \ldots, m)$ by $K_m^{(p)}$.

Lemma 2.1 *Let ψ be an amalgamating function of $K_m^{(p)} = (V, E)$, with the additional property that for each $w \in \psi(V)$, $\psi^{-1}(w) \subseteq A_i$ for some i. Let $G = \psi(K_m^{(p)})$, with parts $A'_i = \{\psi(x) \mid x \in A_i\}$ for $1 \leq i \leq p$. Let $C : E(G) \to \{1, \ldots, \ell\}$ be an ℓ-edge-coloring for which each color class induces a 2-factor of G, and let $v, w \in V(G)$. Then*

1) $d_G(w) = \eta(w)(p-1)m$,

2) $m_G(v,w) = \begin{cases} \eta(v)\eta(w) & \textit{if } w \textit{ and } v \textit{ are in different parts,} \\ 0 & \textit{if } w \textit{ and } v \textit{ are in the same part,} \end{cases}$

3) $d_{G(i)}(w) = 2\eta(w)$, *and*

4) $\sum_{v\in A'_i} \eta(v) = m$.

Proof:

1) Each vertex of $K_m^{(p)}$ has degree $(p-1)m$, and $|\psi^{-1}(w)| = \eta(w)$, so $d_G(w) = \eta(w)(p-1)m$.

2) Let $v \in A'_i$ and $w \in A'_j$; so $\psi^{-1}(v) \subseteq A_i$ and $\psi^{-1}(w) \subseteq A_j$. If $i \neq j$ then each vertex of $\psi^{-1}(v)$ is joined to each vertex of $\psi^{-1}(w)$ by exactly one edge, and hence $m_G(v,w) = \eta(v)\eta(w)$. If $i = j$, then each vertex in $\psi^{-1}(v)$ is adjacent to no vertices in $\psi^{-1}(w)$, so $m_G(v,w) = 0$.

3) Each of the $\eta(w)$ vertices of $\psi^{-1}(w)$ is adjacent to 2 edges of $G(i)$ for $1 \leq i \leq \ell$, thus $d_{G(i)}(w) = 2\eta(w)$ for $1 \leq i \leq \ell$.

4) This follows since $|A_i| = m$ and $\bigcup_{v\in A'_i} \psi^{-1}(v) = A_i$.

□

An edge-colored graph G with partition $\{A'_1, \ldots, A'_p\}$ of its vertex set that satisfies conditions 1-4 of Lemma 1 for some function $\eta : V(G) \to \mathbb{N}$ and integers m and p, is said to be an *outline 2-factorization* of $K_m^{(p)}$ with parts $A'_1, \ldots, A'_p$.

3 Non-disconnecting disentanglements

By Lemma 2.1, any amalgamated 2-factorization of $K_m^{(p)}$ is an outline 2-factorization of $K_m^{(p)}$. We aim to show that the converse is also true. We will show this by proving that any outline 2-factorization G of $K_m^{(p)}$ has a disentanglement G' that is also a 2-factorization of $K_m^{(p)}$, a result that was first proved by Wantland and Rodger [10]. However, here we will do even more. Taking connectivity into account, we will show that there exists such a G' with $\omega(G'(i)) = \omega(G(i))$ for each color class $i \in \{1, \ldots, \ell\}$. This result leads to the interesting construction considered in Section 4.

Theorem 3.1 *If G is an outline 2-factorization of $K_m^{(p)}$ with parts $A'_1 \ldots, A'_p$, then G is the amalgamation of some edge-colored $H \cong K_m^{(p)}$, with $\omega(G(i)) = \omega(H(i))$ for each color class $i \in \{1, \ldots, \ell\}$.*

Proof: **Case 1:** Suppose $\eta(w) = 1$ for all $w \in V(G)$.

By Lemma 2.1, we have that

(1) $d_G(w) = (p-1)m$,

(2) $m_G(v,w) = \begin{cases} 1 & \text{if } w \text{ and } v \text{ are in different parts,} \\ 0 & \text{if } w \text{ and } v \text{ are in the same part,} \end{cases}$

(3) $d_{G(i)}(w) = 2$ for all $i \in \{1, \dots, \ell\}$, and

(4) $\sum_{v \in A'_i} 1 = |A'_i| = m$.

Then by (2) $G \cong K_m^{(p)}$ by (3) each color class is a 2-factor, and clearly $\omega(G(i)) = \omega(H(i))$ for all $i \in \{1, \dots, \ell\}$.

Case 2: Suppose $\eta(w) \geq 2$ for at least one vertex $w \in V(G)$. We first show that G has some disentanglement G' with $|V(G')| = |V(G)| + 1$, where G' is also an outline 2-factorization of $K_m^{(p)}$, and $\omega(G'(i)) = \omega(G(i))$ for every color class $i \in \{1, \dots, \ell\}$. For $1 \leq i \leq p$ let $A'_i = \{w_{i,j} | 1 \leq j \leq |A'_i|\}$. We can assume that $\eta(w_{p,1}) \geq 2$, and think of constructing G' by "splitting" $w_{p,1}$ into two vertices, say $w_{p,1}$ and $w_{p,0}$, and carefully "moving" one of the ends of some of the edges from $w_{p,1}$ to $w_{p,0}$. More formally, we let $V(G') = V(G) \cup \{w_{p,0}\}$, and define $\eta' : V(G') \to \mathbb{N}$ by

$$\eta'(w) = \begin{cases} \eta(w) - 1 & \text{if } w = w_{p,1}, \\ 1 & \text{if } w = w_{p,0}, \\ \eta(w) & \text{otherwise.} \end{cases}$$

The bijective relation between $E(G')$ and $E(G)$ is useful to recall throughout this proof. To determine how to redistribute the $\eta(w_{p,1})(p-1)m$ edges incident with $w_{p,1}$ in G between $w_{p,1}$ and $w_{p,0}$ in G', we form the associated bipartite multigraph B_1 with bipartition $\{A' = A'_1 \cup A'_2 \cup \dots \cup A'_{p-1}, \{c_1, \dots, c_\ell\}\}$ of $V(B_1)$ as follows : for each $i \in \{1, \dots, \ell\}$ and each $u \in A'$, an edge is placed between c_i and u for each edge of G joining $w_{p,1}$ and u colored i. B_1 has the following properties:

1) $d_{B_1}(c_i) = 2\eta(w_{p,1})$ for each $i \in \{1, \dots, \ell\}$, and

2) $d_{B_1}(u) = \eta(w_{p,1})\eta(u)$ for each $u \in A'$.

Recall that $\eta'(w_{p,0}) = 1$. Thus if G' is to be an outline 2-factorization of $K_m^{(p)}$, then we need $d_{G'}(w_{p,0}) = (p-1)m$, $d_{G'(i)}(w_{p,0}) = 2$ for $1 \leq i \leq l$, and $m_{G'}(w_{p,0}, v) = \eta'(v)$ for each $v \in A'$. We determine which edge ends are "moved" to $w_{p,0}$ by giving B_1 an equitable $\eta(w_{p,1})$-edge-coloring $C_1 : E(B_1) \to \{1, 2, \ldots, \eta(w_{p,1})\}$ [13]. Then each color class of C_1 contains 2 edges incident with each c_i and $\eta(u)$ edges incident with u for each $u \in A'$. Thus any color class corresponds to a set of edges of G, each incident with $w_{p,1}$, that can be detached from $w_{p,1}$ and rejoined to $w_{p,0}$ to make G' an outline 2-factorization of $K_m^{(p)}$ with amalgamation numbers given by η'. We now turn to the issue of connectivity. We want to disentangle G in such a way that $\omega(G(i)) = \omega(G'(i))$ for $i \in \{1, \ldots, \ell\}$. The procedure described above does not guarantee this, as the following indicates.

Let $G_i(v)$ be the subgraph of $G \setminus \{v\}$ induced by color class i, and let K be a component of $G_i(w_{p,1})$. The problem can arise if $\omega(G_i(w_{p,1})) \geq 2$, and if there are exactly two edges of color class i connecting K to $w_{p,1}$ in G. If both of these edges are detached from $w_{p,1}$ and rejoined to $w_{p,0}$, then the subgraph of $G'(i)$ induced by $V(K) \cup \{w_{p,0}\}$ is not connected to the rest of $G'(i)$, forming a new component. Thus $\omega(G'(i)) = \omega(G(i)) + 1$.

However, we can prevent this from happening by modifying the above procedure using two more associated bipartite multigraphs B_2 and B_3. Let B_2 be the subgraph of B_1 induced by color classes 1 and 2 of the coloring C_1. (We are guaranteed that there are at least 2 color classes, since $\eta(w_{p,1}) \geq 2$). B_2 satisfies:

1) $d_{B_2}(u) = 2\eta(u)$ for each $u \in A'$, and

2) $d_{B_2}(c_i) = 4$ for $i \in \{1, \ldots, \ell\}$.

Form a third bipartite graph, B_3, with $V(B_3) = V(B_2) \cup \{c_i' \mid i \in \{1, \ldots, \ell\}\}$ as follows. $E(B_3)$ is constructed from $E(B_2)$ by dividing the four edges incident with each c_i in B_2 into 2 pairs - one pair incident with c_i in B_3, the other pair incident with c_i' in B_3. If any two of the edges incident with c_i in B_1 correspond to the *only* two edges joining a component K of $G_i(w_{p,1})$ to $w_{p,1}$, then they are paired together (so possibly the four edges incident with c_i in B_1 will form two such pairs; clearly no edge can itself be in two such pairs). Any remaining edges incident with c_i in B_2 are arbitrarily paired. B_3 satisfies:

1) $d_{B_3}(u) = 2\eta(u)$ for $u \in A'$, and

2) $d_{B_3}(c_i) = d_{B_3}(c_i') = 2$ for $i \in \{1, \ldots, \ell\}$.

B_3 has an equitable 2-edge-coloring for which either color class corresponds to a subset of $E(G)$, the edges in which can be detached from $w_{p,1}$ and rejoined to $w_{p,0}$ to result in a graph G' which is also an outline 2-factorization of $K_m^{(p)}$. Furthermore, $\omega(G'(i)) = \omega(G(i))$ for each color class $i \in \{1, \ldots, \ell\}$.

This process of disentangling G, an outline 2-factorization of $K_m^{(p)}$ produces a new outline 2-factorization with one new vertex that has an amalgamation number of 1. It also reduces the amalgamation number of some other vertex by 1. If the process is applied repeatedly, eventually it will produce an outline 2-factorization of $K_m^{(p)}$, say H, in which all vertices have an amalgamation number of 1, and in which $\omega(H(i)) = \omega(G(i))$ for all $i \in \{1, \ldots, \ell\}$. By case 1, $H \cong K_m^{(p)}$. $\square$

4 Hamilton Decompositions of $K_{n,n} - E(U)$

It is easy to show that $K_{n,n}$ has a hamilton decomposition if and only if n is even (see below). Using this result, and the following corollary of Theorem 3.1, we find necessary and sufficient conditions for the existence of a hamilton decomposition of $K_{n,n} - E(U)$ for any 2-factor U of $K_{n,n}$. The proof provides a general method for constructing such a decomposition.

Corollary 4.1 *Let G be an outline 2-factorization of $K_{n,n}$, where n is an even integer. Then G is the amalgamation of some edge-colored $K_{n,n}$ with $\omega(G(i)) = \omega(K_{n,n}(i))$ for each color class i.*

Proof: This is the special case of Theorem 3.1 when $p = 2$. $\square$

We are now prepared to prove the following theorem:

Theorem 4.1 *Let U be any 2-factor of $K_{a,b}$. There exists a hamilton decomposition of $K_{a,b} - E(U)$ if and only if*

1. *$a = b = n$ and n is even, and*
2. *if $n = 4$ then U is not a pair of 4-cycles.*

Proof: First suppose that there exists a hamilton decomposition of $G = K_{a,b} - E(U)$. Since the edges are partitioned into 2-factors, clearly G is regular of even degree, so (1) follows. Also, removing the edges of a 2-factor consisting of two 4-cycles from $K_{4,4}$ results in a disconnected graph, so (2) follows. Therefore the necessity follows.

Now suppose that conditions (1-2) hold. In particular, throughout this proof n is assumed to be even. Let U be an arbitrary 2-factor of $K_{n,n}$. U is defined up to isomorphism by the length of its cycles, so we can assume that U consists of q cycles with lengths $s_1, s_2, \ldots, s_q$. We have that $\sum_{i=1}^{q} s_i = 2n$ and $4 \leq s_i \leq 2n$ for each $i \in \{1, \ldots, q\}$. It is also important to note that each cycle s_i necessarily contains $s_i/2$ vertices in each of the two parts of the graph $K_{n,n}$.

To exhibit a hamilton decomposition of $K_{n,n} - E(U)$, it suffices to construct an outline 2-factorization G with only one disconnected color class, such that this disconnected color class has q components $\kappa_1, \kappa_2, \ldots, \kappa_q$ satisfying $\sum_{v \in V(\kappa_i)} \eta(v) = s_i$ for each $i \in \{1, \ldots, q\}$. By applying Corollary 4.1, G can be disentangled into a copy H of $K_{n,n}$ without disconnecting any component of any color class. So after disentangling G, its connected color classes correspond to hamilton cycles in H. Furthermore, the remaining, disconnected color class of G corresponds to a 2-factor of $K_{n,n}$ consisting of q components, each of which must be a cycle; since no component of this color class was disconnected, the ith cycle will have length $\sum_{v \in \kappa_i} \eta(v) = s_i$. Hence the subgraph of $K_{n,n}$ induced by this color class is isomorphic to U. Thus the connected color classes form a hamilton decomposition of $K_{n,n} - E(U)$. Therefore it remains to show how to construct the initial outline 2-factorization.

First, we will construct a hamilton decomposition of $K_{n,n}$, which consists of $\frac{n}{2}$ hamilton cycles. Let $V(K_{n,n}) = \{v_{i,j} \mid i \in \{1, 2, \ldots, n\}, j \in \{L, R\}\}$ (L and R are used to denote the left and right parts of the graph). Each edge has one end in each part, so we can define the *difference* of an edge $\{v_{i,L}, v_{j,R}\}$, denoted $diff(\{v_{i,L}, v_{j,R}\})$, to be $j - i \pmod{n}$. Each color class in the edge-coloring $k : E(K_{n,n}) \to \{1, \ldots, \frac{n}{2}\}$, defined by

$$k(e) = \begin{cases} \frac{diff(e)}{2} + 1 (\bmod n) & \text{if } diff(e) \text{ is even, and} \\ \frac{diff(e)+1}{2} + 1 (\bmod n) & \text{if } diff(e) \text{ is odd} \end{cases}$$

is a hamilton cycle. We now amalgamate this edge-colored graph to form G, an outline 2-factorization of $K_{n,n}$. Let $V(G) = \{w_{i,j} : i \in \{1, 2, \ldots, q\},\ j \in \{L, R\}\}$. In both parts, for each i, we amalgamate $s_i/2$ vertices of $K_{n,n}$ to form one vertex of G. This is done by letting $\psi : V(K_{n,n}) \to V(G)$ be defined by $\psi(v_{i,j}) = w_{z,j}$ if and only if $\sum_{x=1}^{z-1} s_x/2 < i \leq \sum_{x=1}^{z} s_x/2$ and $j \in \{L, R\}$.

Now G is clearly an outline 2-factorization of $K_{n,n}$ with $\eta(w_{i,L}) = \eta(w_{i,R}) = s_i/2$ for $i \in \{1, \ldots, q\}$, and every color class of G is connected. We will modify the edge-coloring of G to disconnect color class 1 into q components. This is mainly accomplished by swapping edges between color classes 1 and 2.

Any vertex $w_{i,L}$ is incident with $2\eta(w_{i,L}) = s_i$ edges of color 1. By the definition of the edge-coloring k, exactly one of these edges joins $w_{i,L}$ to $w_{i-1,R}$, and the rest join $w_{i,L}$ to $w_{i,R}$. Since $\eta(w_{i,L}) \geq 2$, $w_{i,L}$ is incident with at least 4 edges of color 1, and thus there are at least 3 edges of color class 1 joining $w_{i,L}$ to $w_{i,R}$. Let S_1 be the set of all q edges of color 1 joining $w_{i,L}$ to $w_{i-1,R}$ for $i \in \{1, \ldots, q\}$. Similarly, any vertex $w_{i,L}$ is incident with $2\eta(w_{i,L}) = s_i \geq 4$ edges of color 2. Exactly three of these edges join $w_{i,L}$ to $w_{i+1,R}$, and the remaining $2\eta(w_{i,L}) - 3 \geq 1$ join $w_{i,L}$ to $w_{i,R}$. (If $s_i = 4$, then there is *exactly* one edge colored 2 joining $w_{i,L}$ to $w_{i,R}$.) For each $i \in \{1, \ldots, q\}$, pick an edge colored 2 joining $w_{i,L}$ to $w_{i,R}$ and put it in the set S_2.

Recolor the graph with $k' : E(G) \to \{1, \ldots, \frac{n}{2}\}$, defined by

$$k'(e) = \begin{cases} 1 & \text{if } e \in S_2 \\ 2 & \text{if } e \in S_1 \\ k(e) & \text{otherwise,} \end{cases}$$

and denote this edge-colored version of G by G'.

Now color class 1 of k' has q components, the ith component being induced by the two vertices $w_{i,L}$ and $w_{i,R}$. Furthermore, $\eta(w_{i,L}) + \eta(w_{i,R}) = s_i$. This recoloring has not affected any of the conditions from Lemma 2.1, so G' is an outline 2-factorization of $K_{n,n}$. However, color class 2 of k' may be disconnected.

Recall that in the original coloring k, if $\eta(w_{i,L}) = 2$, then there is exactly one edge colored 2 linking $w_{i,L}$ to $w_{i,R}$; if $\eta(w_{i,L}) = 2$ for some $i \in \{1, \ldots, q\}$, then this edge is placed in the set S_2 and so is recolored 1 by k', leaving $w_{i,L}$ adjacent in $G'(2)$ to only $w_{i+1,R}$ and $w_{i-1,R}$. Even so, if either q is odd or if $\eta(\omega_{i,L}) > 2$ for some $i \in \{1, \ldots, q\}$, then $G'(2)$ is connected; otherwise q is even and $s_i = 2$ for all $i \in \{1, \ldots, q\}$, in which case $G'(2)$ has exactly two components.

If color class 2 of k' is disconnected, then it can be reconnected without affecting the connectivity of any other color class providing that a third color class exists, as is shown below. Such a third color class must exist, for otherwise $n/2 = 2$, so $n = 4$ and U is a pair of 4-cycles, contradicting condition (2).

If $G'(2)$ is disconnected then some edges can be "swapped" between color classes 2 and 3 to reconnect color class 2 and while keeping color class 3 connected. There are exactly three edges of $G'(2)$ joining each $w_{i,L}$ to $w_{i+1,R}$. Pick one edge of $G'(2)$ that joins $w_{i,L}$ to $w_{i+1,R}$ for each $i = \{1, \ldots, q\}$ and put it in the set T_2. There are exactly three edges of $G'(3)$ linking $w_{i,L}$ to $w_{i+2,R}$. Since there are three such edges, we can recolor one of them without disconnecting the resulting graph induced by the edges colored 3. Pick one such edge colored 3 for each $i \in \{1, \ldots, q\}$ and place it in T_3. Now recoloring the graph with $k'' : E(G) \to \{1, \ldots, \frac{n}{2}\}$, defined by

$$k''(e) = \begin{cases} 2 & \text{if } e \in T_3, \\ 3 & \text{if } e \in T_2, \text{ and} \\ k'(e) & \text{otherwise} \end{cases}$$

will produce a new outline 2-factorization. Color class 2 of k'' is connected since $w_{i,L}$ is adjacent in $G(2)$ to both $w_{i+1,R}$ and $w_{i+2,R}$. Of course, color classes 4 through $n/2$ and color class 1 are unaffected by this recoloring.

By Corollary 4.1, we can disentangle G into a copy H of $K_{n,n}$. Color class 1 of k'' corresponds to a subgraph of H that is isomorphic to U. Color classes 2 through $n/2$ of k'' correspond to disjoint hamilton cycles of $H - E(U)$. Thus color classes 2 through $n/2$ correspond to a hamilton decomposition of $H - E(U) \cong K_{n,n} - E(U)$. □

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