

ON $(2, r)$ -REGULAR GRAPHS

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ABSTRACT. A graph G is $(2, r)$ -regular if for any pair of distinct vertices x and y , $|N(x) \cup N(y)| = r$. Here we use Hadamard matrices to construct an infinite class of $(2, r)$ -regular graphs with $r \geq 3$, none of which are complete. This is an interesting result since it was previously believed that for $r \geq 3$, the only regular $(2, r)$ -regular graph is K_r .

1. INTRODUCTION

In a simple graph G , the generalized degree of a set $S \subset V(G)$ is defined as $|\bigcup_{v \in S} N(v)|$. Over the past two decades, many results have been published concerning generalized degrees and other parameters concerning neighborhood unions. In particular, the concept of generalized regularity has received much attention.

Definition 1. (See [3]) Let $G = (V, E)$ be a simple graph of order n and let t be no greater than the vertex independence number of G . We say G is a $(t, r)^i$ -regular graph if for every independent set $S \subset V(G)$ with $|S| = t$ we have $|\bigcup_{v \in S} N(v)| = r$.

Definition 2. (See [4]) Let $G = (V, E)$ be a simple graph of order n and let $t \leq n$. We say G is a (t, r) -regular graph if for every set $S \subset V(G)$ with $|S| = t$ we have $|\bigcup_{v \in S} N(v)| = r$.

Note that a (t, r) -regular graph is necessarily $(t, r)^i$ -regular if t is not greater than the vertex independence number of the graph. It is worth mentioning that the definition and notation for (t, r) -regularity are not completely standardized across the literature. (t, r) -regular graphs are sometimes called u_t -regular graphs, and some articles (see for example [2, 3]) define (t, r) -regular as we have defined $(t, r)^i$ -regular here.

The specific case $t = 2$ has been studied in several articles including [2, 4, 6]. In [4], Haynes and Markus gave the following result characterizing $(2, r)$ -regular graphs for small values of r :

Theorem 3. For $r \leq 2$, the $(2, r)$ -regular graphs are characterized as follows:

- (1) The only $(2, 0)$ -regular graphs are \bar{K}_n .
- (2) No $(2, 1)$ -regular graphs exist.
- (3) A graph G is $(2, 2)$ -regular if and only if $G = mK_2$.

Characterizing $(2, r)$ -regular graphs for $r \geq 3$ has proven to be a much more difficult problem, and currently remains open. Faudree and Knisley proved [2] that if $r \geq 3$, then for n sufficiently large, depending on r , the only $(2, r)^i$ -regular graphs of order n are of the form $K_s + mK_p$, for some integers $s \geq 0$, $p \geq 1$, and $m \geq 2$ satisfying $s + mp = n$ and $s + 2(p-1) = r$. Haynes and Markus [4] applied this to prove that for $r \geq 3$, for n sufficiently large (depending on r) there are no $(2, r)$ -regular graphs of order n .

Haynes and Markus also asserted, in [4], that for $r \geq 3$ the only regular $(2, r)$ -regular graph is K_r . But there was a mistake in their proof, and this result is untrue. The line graph of $K_{4,4}$ and the Shrikhande graph noted by Cameron [1] are $(2, 10)$ -regular graphs which are not complete graphs. In Section 3, we develop a construction for $(2, r_n)$ -regular graphs of order 4^n for all integers $n \geq 2$, none of which are complete graphs (including $K_4 \times K_4$ when $n = 2$).

2. COMPUTATIONAL RESULTS

In this section we study $n \times n$ symmetric matrices $A = [a_{ij}]$ with elements zeros and ones and with $a_{ii} = 1$, $1 \leq i \leq n$, such that the off-diagonal elements of A^2 are all equal. These matrices are related to $(2, r)$ -regular graphs.

Theorem 4. Let $G = (V, E)$ be a $(2, r)$ -regular graph of order n . Define the block B_i by $B_i = V \setminus N(i)$ for every $i \in V = \{1, 2, 3, \dots, n\}$. Then

- (1) $i \in B_i$ for every $i \in V$;
- (2) $|B_1 \cup B_2 \cup B_3 \cup \dots \cup B_n| = n$;
- (3) $i \in B_j$ if and only if $j \in B_i$ for $i, j \in V$;
- (4) $|B_i| = s_i$ if and only if i appears in s_i blocks for each $i \in V$;
- (5) $|B_i \cap B_j| = \lambda = n - r$ for $i, j \in V$ and $i \neq j$;
- (6) every pair of distinct elements of V occurs in λ blocks;
- (7) if i occurs in blocks $B_{j_1}, B_{j_2}, \dots, B_{j_k}$, where $k = s_i$, then

$$\lambda(n-1) = \sum_{m=1}^k (|B_{j_m}| - 1).$$

Proof. 1. Since $i \notin N(i)$ and $B_i = V \setminus N(i)$ it follows that $i \in B_i$ for every $i \in V$.

2. Since $B_i \subseteq V$ and $i \in B_i$ for every $i \in V$ we have $\cup_{i=1}^n B_i = V$.

3. If the vertex i is not in the neighbourhood of the vertex j then j is not in the neighbourhood of i . So the result follows.
4. This is obvious by (3).
5. First note that $B_i \cap B_j = V \setminus (N(i) \cup N(j))$ for all $i, j \in V$. On the other hand, since G is a $(2, r)$ -regular graph we have $|N(i) \cup N(j)| = r$ if $i \neq j$. Now the result follows.
6. First note that if $i, j \in B_k$ for some $k \in V$ then by (3) we have $k \in B_i \cap B_j$. Now by (5) we have $|B_i \cap B_j| = \lambda$ if $i \neq j$. So the pair $\{i, j\}$ occurs in precisely λ blocks.
7. Each j in $\{1, \dots, n\}$, $j \neq i$, lies in exactly λ of the blocks B_{j_m} , $m = 1, \dots, k$, by (6). Therefore $\sum_{m=1}^k (|B_{j_m}| - 1) = \sum_{m=1}^k |B_{j_m} \setminus \{i\}|$ counts each of the $n - 1$ elements of $V \setminus \{i\}$, λ times.

□

Corollary 5. Let G be a $(2, r)$ -regular graph. Define $\mathcal{B} = \{B_1, B_2, \dots, B_n\}$ where $B_i = V \setminus N(i)$ for $i = 1, 2, \dots, n$. Then (V, \mathcal{B}) is a pairwise balanced design (PBD) on V with $\lambda = n - r$ (see [8] for the definition of a PBD).

Definition 6. We define an $n \times n$ *incidence* matrix $A = [a_{ij}]$ for the PBD given in Corollary 5 by

$$a_{ij} = \begin{cases} 1 & \text{if } i \in B_j; \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 7. Let $G = (V, E)$ be a $(2, r)$ -regular graph of order n . Define $\mathcal{B} = \{B_1, B_2, \dots, B_n\}$ where $B_i = V \setminus N(i)$ and $|B_i| = s_i$ for $i = 1, 2, \dots, n$. Suppose that A is an incidence matrix for the PBD (V, \mathcal{B}) . Then

- (1) $a_{ii} = 1$ for all $i \in V$;
- (2) $A = A^t$ (that is $a_{ij} = a_{ji}$ for all $i, j \in V$);
- (3) $A^2 = (c_{ij})$, where

$$c_{ij} = \begin{cases} s_i & \text{if } i = j \\ \lambda = n - r & \text{otherwise.} \end{cases}$$

Note that s_i is the number of ones in row i of matrix A .

Corollary 8. Let $G = (V, E)$ be a $(2, r)$ -regular graph of order n . Then G is an $(n - s)$ -regular graph if and only if $s_i = s$ for all $i \in \{1, 2, \dots, n\}$, where s_i is as in Corollary 7.

For $n = 2, 3, \dots, 10$, using a computer program, we examined every $n \times n$ zero-one matrix A which satisfies Conditions 1, 2 and 3 in Corollary 7. The result of this search is given in the following theorem.

Theorem 9. (1) If $n = 2, 4, 6$ or 8 then $s_i = 1, n - 1$ or n for all i . So a $(2, r)$ -regular graph with n vertices, $n \in \{2, 4, 6, 8\}$, is a complete graph

$(s_i = 1)$, a 1-factor $(s_i = n - 1)$ or the complement of a complete graph $(s_i = n)$.

(2) If $n = 3, 5, 7$ or 9 then $s_i = 1$ or n for all i . So a $(2, r)$ -regular graph with n vertices, $n \in \{3, 5, 7, 9\}$, is either a complete graph $(s_i = 1)$ or the complement of a complete graph $(s_i = n)$.

We end this section with the following conjecture.

Conjecture 10. Let $A = [a_{ij}]$ be an $n \times n$ symmetric matrix with elements zeros and ones and $a_{ii} = 1$ for each i . If the off-diagonal elements of A^2 are all equal then the diagonal elements of A^2 are all equal.

We note that by Theorem 9 the conjecture is true for $1 \leq n \leq 9$. The conjecture is also true if the off-diagonal elements of A^2 are all 2 or all 3 (see [5]).

3. CLASSES OF $(2, r)^i$ -REGULAR AND $(2, r)$ -REGULAR GRAPHS

In this section, using Hadamard matrices of order 4^n , $n \geq 2$, we construct infinite families of $(2, r_n)^i$ -regular and $(2, r_n)$ -regular graphs for $r_n \geq 3$, none of which are complete graphs.

Definition 11. ([7] Chapter 18) A *Hadamard matrix* of order n is an n by n matrix H with elements $+1$ and -1 , such that

$$HH^t = nI,$$

where I is the identity matrix of order n .

Definition 12. ([7] Chapter 18) Let A be an m by n matrix with elements a_{ij} and B another matrix. The matrix

$$\begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{bmatrix}$$

consisting of mn blocks with the size of B is called the *Kronecker product* $A \otimes B$ of the matrices A and B .

Definition 13. Let H_1 be the following Hadamard matrix.

$$H_1 = \begin{bmatrix} +1 & -1 & -1 & -1 \\ -1 & +1 & -1 & -1 \\ -1 & -1 & +1 & -1 \\ -1 & -1 & -1 & +1 \end{bmatrix}.$$

Define $H_n = H_1 \otimes H_{n-1}$, where \otimes is the *Kronecker product* of H_{n-1} and H_1 , for $n \geq 2$.

Theorem 14. ([7]) The matrix H_n is a Hadamard matrix of order 4^n for all positive integer n .

Definition 15. Let $A = [a_{ij}]$ be any matrix and let $i \neq k$. Then rows i and k of A are said to have λ_{ik} ones in common if $|\{j \mid a_{ij} = a_{kj} = 1\}| = \lambda_{ik}$ and λ'_{ik} negative ones in common if $|\{j \mid a_{ij} = a_{kj} = -1\}| = \lambda'_{ik}$.

Lemma 16. Let H be a Hadamard matrix of order m and let λ_{ik} and λ'_{ik} be as in Definition 15.

- (1) If $i \neq k$ then $2(\lambda_{ik} + \lambda'_{ik}) = m$.
- (2) Let r_i be the number of ones in row i . Then, for $i \neq k$, $r_i + r_k - 2\lambda_{ik} = m/2$.

Proof. 1. Consider the inner product of rows i and k . Since the inner product is 0, we must have $\lambda_{ik} + \lambda'_{ik} = m - (\lambda_{ik} + \lambda'_{ik})$. So the result follows.

2. Consider the inner product of rows i and k . $r_i - \lambda_{ik}$ is the number of ones in row i which multiply -1 in row k and $r_k - \lambda_{ik}$ is the number of ones in row k which multiply -1 in row i . But all the negative ones in the inner product must come from one of these two sources, so, since the inner product is 0, we must have $r_i - \lambda_{ik} + r_k - \lambda_{ik} = \lambda_{ik} + \lambda'_{ik}$. So $r_i + r_k - 2\lambda_{ik} = m/2$ by Part 1. \square

Corollary 17. Let H , λ_{ik} and r_i be as in Lemma 16. If $r_i = r_j = r$ is constant, then $\lambda_{ik} = r - m/4$ is also constant.

Theorem 18. Let $H_n = [h_{ij}]$ be as in Definition 13. Then

- (1) $h_{ii} = 1$ for all $1 \leq i \leq 4^n$;
- (2) $H_n = H_n^t$;
- (3) each row of H_n has precisely $s_n = 2^{n-1}(2^n + (-1)^n)$ ones;
- (4) the rows i and k , $i \neq k$, have precisely $s_n - 4^{n-1}$ ones in common.

Proof. A simple induction argument and Definition 13 show the correctness of Parts 1 and 2. For Part 3 first note that

$$\begin{aligned} \text{number of ones in row } i \text{ of } H_n &= \text{number of ones in row } i' \text{ of } H_{n-1} + \\ &\quad 3(\text{number of negative ones in row } i' \text{ of } H_{n-1}), \end{aligned}$$

where $i \equiv i' \pmod{4^{n-1}}$. Now one can apply an induction on n to prove that the number of ones in row i of H_n is the same for all $i \in \{1, 2, 3, \dots, 4^n\}$. Let s_n be the number of ones in each row of H_n . Then we have $s_n = s_{n-1} + 3(4^{n-1} - s_{n-1})$. That is $s_n = 3 \cdot 4^{n-1} - 2s_{n-1}$. Using this recursive relation for s_n the reader can verify $s_n = 2^{n-1}(2^n + (-1)^n)$.

Part 4 follows by Part 3 and Corollary 17. \square

Corollary 19. Let $H_n = [h_{ij}]$ be as in Definition 13 and $m = 4^n$. Let $A_n = [a_{ij}]$ be the $m \times m$ matrix with $a_{ii} = 0$ and $a_{ij} = (h_{ij} + 1)/2$ for $i \neq j$. (The 16×16 matrix A_2 is shown in Example 21.) Let $B_n = [b_{ij}]$ be the $m \times m$ matrix with $b_{ij} = -(h_{ij} - 1)/2$. (Thus the off-diagonal elements of A_n are 1 where the elements of H_n are 1 and 0 otherwise, and the elements of B_n are 1 where the elements of H_n are -1 and 0 elsewhere.) Then

- (1) every row of A_n contains $s_n - 1$ ones;
- (2) if $i \neq k$ and $a_{ik} = 0$ then rows i and k have $s_n - 4^{n-1}$ ones in common;
- (3) every row of B_n contains $4^n - s_n$ ones;
- (4) if $i \neq k$ then rows i and k of B_n have $3 \cdot 4^{n-1} - s_n$ ones in common.

Theorem 20. Let n be a positive integer and let s_n be as in Theorem 18.

1. There exists a graph H of order 4^n such that

- i. H is $(s_n - 1)$ -regular;
- ii. H is $(2, s_n + 4^{n-1} - 2)^i$ -regular;

2. There exists a graph G of order 4^n such that

- i. G is $(4^n - s_n)$ -regular;
- ii. G is $(2, 5 \cdot 4^{n-1} - s_n)$ -regular;
- iii. G is not a complete graph.

Proof. 1. Label the rows and columns of A_n (see Corollary 19) by the elements of $V = \{1, 2, 3, \dots, 4^n\}$. Define the graph H on V as follows: two vertices $i, j \in V$, $i \neq j$, are adjacent if and only if $a_{ij} = 1$. By Corollary 19 we see that H is an $(s_n - 1)$ -regular and $(2, s_n + 4^{n-1} - 2)^i$ -regular graph. 2. Label the rows and columns of B_n (see Corollary 19) by the elements of $V = \{1, 2, 3, \dots, 4^n\}$. Define the graph G on V as follows: two vertices $i, j \in V$, $i \neq j$, are adjacent if and only if $b_{ij} = 1$. (Note that $G = \bar{H}$.) By Corollary 19 we see that G is a $(4^n - s_n)$ -regular and $(2, 5 \cdot 4^{n-1} - s_n)$ -regular graph. Obviously, G is not a complete graph. \square

Note that the PBD associated with the graph G (see Corollary 5) is actually a symmetric $(4^n, 4^n - s_n, 3 \cdot 4^{n-1} - s_n)$ balanced incomplete block design (SBIBD). See [8] for the definition of a SBIBD.

The following example illustrates Corollary 19 and Theorem 20 when $n = 2$.

Example 21. Let $n = 2$. Then

$$A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Note that A_2 is symmetric and if $A_2^2 = [c_{ij}]$ then $c_{ii} = 10$ and $c_{ij} = 6$ for all $i, j \in \{1, 2, 3, \dots, 16\}$. The graph H associated with A_2 (see Theorem 20) is a 9-regular and $(2, 12)^i$ -regular graph. The graph G associated with B_2 (see Theorem 20) is a 6-regular and $(2, 10)$ -regular graph. Note that $G = K_4 \times K_4$ (see Figure 1).

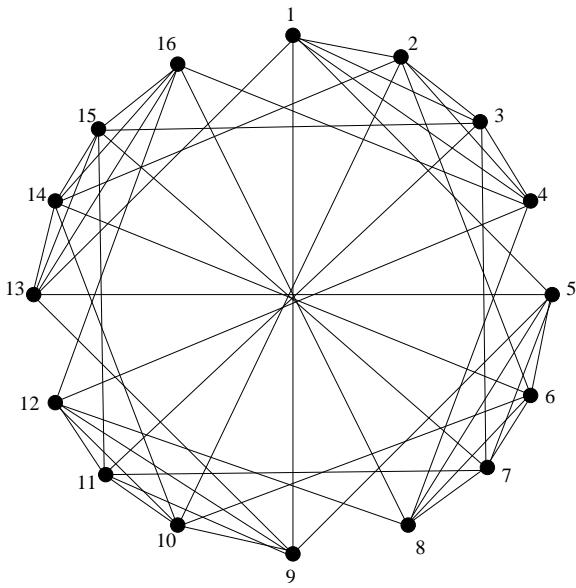


Figure 1: A 6-regular and (2,10)-regular graph

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