THE THREE FACTOR PAPER

C. D. Leach
Department of Mathematics
State University of West Georgia
Carrollton, GA 30118

C. A. Rodger
Department of Discrete and Statistical Sciences
235 Allison Lab
Auburn University, AL 36849-5307

Abstract

We show that for any 2-factor U of K_n with n even, there exists a 3-factor T of K_n such that $E(U) \subset E(T)$ such that $K_n - E(T)$ admits a hamilton decomposition. This is proved with using a new result that concerns graph decompositions that are fairly divided, but not necessarily regular.

1 Introduction

Hamilton decompositions have been studied since 1892, when Walecki [8] proved that K_n has a hamilton decomposition if and only if n is odd. Laskar and Auerbach [5] proved in 1976 that the complete p-partite graph $K_{m,\dots,m}$ has a hamilton decomposition when $m(p-1)$ is even, and that when $m(p-1)$ is odd, that $K_{m,\dots,m}$ has a hamilton decomposition with a 1-factor leave.

The technique of vertex amalgamations (graph homomorphisms), which was developed in the 1980s by Rodger and Hilton, has proven very powerful in constructing hamilton decompositions of various classes of graphs. Hilton [3] used amalgamation to produce a new proof of Walecki's result, and Hilton and Rodger used amalgamation to produce new proofs [4] of Walecki's and Laskar and Auerbach's results; both of these papers extended the known results by allowing many of the hamilton cycles to be partially predetermined. In 1997, Buchanan used amalgamations to prove that for any 2-factor U of K_n, n odd, $K_n - E(U)$ admits a hamilton decomposition. Recently Rodger and Leach solved the corresponding existence problem for complete bipartite graphs [7], and obtained a partial solution for complete multipartite graphs [6].

The result of this paper is an extension of Buchanan's result. Buchanan constructed a hamilton decomposition of K_n with a 2-factor removed when n is necessarily odd. When n is even, if a hamilton decomposition is to exist once the edges of a k-factor are removed from K_n, then it is necessary that k be *odd*. Walecki's method solves the case where $k = 1$, and here we settle the existence of such a hamilton decomposition when $k = 3$. However, there are many possible 3-factors that one might remove. Our solution is sufficiently general that it allows the 3-factor to contain *any* specified 2-factor (see Theorem 3.2).

Another result of this paper is that in Section 2 we obtain a theorem that extends our knowledge of amalgamations, and is of interest in its own right. Most of the previous results on amalgamations of graph colorings have dealt with edge-colorings in which each color class induces a regular subgraph. The preliminary results in this paper are of note because they deal with amalgamations of graphs with non-regular colorings that are fairly divided in that they share the edges of each color class as evenly as possible among the vertices.

In this paper, graphs can contain multiple edges and loops. $\omega(G)$ is the number of components of the graph G, $m(u, v)$ is the number of edges between vertices u and v, and $G(i)$ is the subgraph of G induced by the edges colored i.

We also make extensive use of edge-colorings in this paper. In particular, equitable and balanced edge-colorings are used. An ℓ-edge-coloring of the graph G is said to be *equitable* if for each vertex $w \in V(G)$, the edges incident with w are distributed among the color classes as evenly as possible. An ℓ-edge-coloring of G is said to be *balanced* if for any pair of vertices $u, v \in V(G)$, the edges joining u and v are distributed among the color classes as evenly as possible.

2 Disentangling Amalgamated Graphs

Amalgamating a graph H can be thought of as taking a graph and squashing some of the vertices together to form "larger" vertices in the amalgamated graph G. Any edges incident with the original vertices in H are then incident with the new vertex in G, and any edge joining two of the squashed vertices in H becomes a loop on the new vertex. The number of vertices squashed together to form the new vertex w is the *amalgamation number* $\eta(w)$ of w. The resulting graph is the *amalgamation* of the original. Formally this is represented by a graph homomorphism $f : V(G) \to V(H)$; so for example if $w \in V(H)$, then $\eta(w) = |f^{-1}(w)|$. The reverse process, in which vertices of a graph are pulled apart to make "smaller" vertices is called *disentangling*, and hence produces a *disentanglement* of the original graph. We begin this section by looking at the properties of amalgamations of an edge-colored K_n.

Lemma 2.1 *Let $H \cong K_n$ be an ℓ-edge-colored graph, and let $f : V(H) \to V(G)$ be an amalgamating function with amalgamation numbers given by the function $\eta : V(G) \to \mathbb{N}$. Then G satisfies that following conditions for any vertices $w, v \in V(G)$:*

1) $d(w) = \eta(w)(n-1)$,

2) $m(w, v) = \eta(w)\eta(v)$ if $w \neq v$,

3) w has $\binom{\eta(w)}{2}$ loops, and

4) $d_{G(i)}(w) = \sum_{u \in f^{-1}(w)} d_{H(i)}(w)$ for $1 \leq i \leq \ell$.

Proof: The conditions follow from the properties of amalgamations, as described below.

1) This follows since w was formed by amalgamating $\eta(w)$ vertices of H, each of which had degree $(n-1)$.

2) w and v were formed by amalgamating $\eta(w)$ and $\eta(v)$ vertices of H. Since $f^{(-1)}(w)$ and $f^{(-1)}(v)$ are disjoint there are $\eta(w)\eta(v)$ edges joining w and v.

3) There are $\binom{\eta(w)}{2}$ pairs of vertices in $f^{-1}(w)$, each of which admits a loop in G.

4) This follows since each vertex $u \in f^{-1}(w)$ contributes $d_{H(i)}(u)$ to the degree of w.

$\square$

Lemma 2.1 describes the properties of an graph formed by taking the amalgamation of an edge-colored complete graph. The aim now is to show that any graph G that satisfies conditions $(1-4)$ for some function η is the amalgamation of an edge-colored graph $H \cong K_n$ in which for each color class i, the edges incident with v in H are shared as evenly as possible among the $\eta(v)$ vertices in $f^{-1}(v11)$ formed when disentangling v to form K_n. Furthermore, we will show that for any color i in which $\frac{d_{G(i)}(w)}{\eta(w)} \geq 2$ is an even integer for all $w \in V(G)$, we can disentangle G in such a way that $\omega(G(i)) = \omega(H(i))$.

Theorem 2.1 *Let G be an ℓ-edge-colored graph satisfying conditions 1-4 of Lemma 2.1 for the function $\eta : V(G) \to \mathbb{N}$. Then there exists a disentanglement H of G that satisfies*

i) $H \cong K_n$,

ii) for any $z \in V(G)$, $|d_{H(i)}(v) - d_{H(i)}(u)| \leq 1$ for all $i \in \{1, \ldots, \ell\}$ and all $u, v \in f^{-1}(z)$.

iii) if $\dfrac{d_{G(i)}(z)}{\eta(z)}$ is an even integer for all $z \in V(G)$, then $\omega(G(i)) = \omega(H(i))$. (*)

Proof:

Case 1: If $\eta(z) = 1$ for all $z \in V(G)$, then conditions (1-4) of Lemma 2.1 can easily be used to show that $H = G \cong K_n$, and (*i* - *iii*) trivially hold.

Case 2: Suppose $\eta(w) \geq 2$ for some vertex $w \in V(G)$. We will first show that G has a disentanglement G' with $|V(G')| = |V(G)| + 1$, where G' also satisfies condition $(1-4)$ of Lemma 2.1 for some new function $\eta' : V(G') \to \mathbb{N}$. We do the disentangling by constructing a new graph G' with $V(G') = V(G) \cup \{w'\}$, and define η' by

$$\eta'(v) = \begin{cases} \eta(v) - 1 & \text{if } v = w, \\ 1 & \text{if } v = w', \\ \eta(v) & \text{otherwise.} \end{cases}$$

To decide which edges to detach from w and "move" to w', we construct the bipartite graph B_1. $V(B_1)$ has the bipartition $(C, D) = (\{c_1, \ldots, c_\ell\}, \{V(G) - w) \cup \{\mathcal{L}\}\})$. For each color i and each vertex $u \in D - \{\mathcal{L}\}$, an edge is placed between c_i and u for each edge of $G(i)$ joining w and u. Additionally two edges are placed between $\mathcal{L}$ and c_i for each loop on w in $G(i)$. The vertices of B_1 are easily seen to have the following degrees:

$$d_{(B_1)}(v) = \begin{cases} d_{G(i)}(w) & \text{if } v = c_i, \\ \eta(w)(\eta(w) - 1) & \text{if } v = \mathcal{L}, \text{and} \\ \eta(w)\eta(v) & \text{otherwise.} \end{cases}$$

It is instructive to note that each edge of B_1 not incident with $\mathcal{L}$corresponds to a non-loop edge in G that is incident with w. Each edge of B_1 is incident with one of the vertices in $\{c_1, \ldots, c_\ell\}$, which tells the color of the corresponding edge in G. At this point, the edges of B_1 are uncolored.

By [2, 9] we can give B_1 an equitable and balanced $\eta(w)$-edge-coloring. (Note that each color class of $\mathcal{K}$ contains $\sum_{v \in D} d_{B_1}(v)/\eta(v) = \eta(v)\left((\sum_{v \in V(G)} \eta(v)) - 1\right)/\eta(v) = n - 1$ edges.) We give B_1 such a coloring $\mathcal{K}$.

We can nearly form G' by using the edges colored 1 in B_1. Each edge in $B_1(1)$ incident with $\mathcal{L}$ directly corresponds to an edge in G that is moved to w'. Also, if there are x edges in $B_1(1)$ incident with $\mathcal{L}$, then x loops on w in G are removed and replaced with x edges joining w to w' in G'. Note that since $\mathcal{K}$ is balanced, and $\eta(w) \geq 2$, at most half the edges in B_1 incident with $\mathcal{L}$ are are colored 1, so there are indeed x loops incident with w in G (recall each loop in G corresponds to *two* edges in B_1).

Let $\alpha_i = \lfloor \frac{d_{G(i)}(w)}{\eta(w)} \rfloor$ for $1 \leq i \leq \ell$. For each color i of $\mathcal{K}$,

$$d_{(B_1(i))}(v) = \begin{cases} \alpha_i \text{ or } \alpha_i + 1 & \text{if } v = c_i, \\ \eta(w) - 1 & \text{if } v = \mathcal{L}, \\ \eta(v) & \text{otherwise.} \end{cases}$$

We can use one color class of $\mathcal{K}$ to find a set of $n - 1$ edges that can be moved from w to a new vertex w' to form a new graph G' that satisfies properties $(1 - 4)$ for η', as the following argument shows.

1) Since $n-1$ edges are moved from w to w', $d_{G'}(w') = n-1 = \eta'(w')(n-1)$, and $d_{G'}(w) = \eta(w)(n-1)-(n-1) = \eta'(w)(n-1)$

2) The only edges of G affected by the process are the edges incident with w. The coloring ensures that for each $v \in V(G)-\{w\}$, exactly $\eta(v)$ edges joining w to v are moved to w'. This leaves $\eta(w)\eta(v)-\eta(v) = \eta'(w)\eta(v)$ edges joining w to v in G'.

3) No loops were placed on w', which is what is needed, since $\binom{\eta'(w')}{2} = \binom{1}{2} = 0$. Since w had $\binom{\eta(w)}{2}$ loops in G, $\eta(w)-1$ of which were replaced with edges joining w and w', there remain $\binom{\eta(w)}{2} - (\eta(w)-1) = \binom{\eta(w)-1}{2} = \binom{\eta'(w)}{2}$ loops on w in G'.

4) This clearly holds.

Repeating this process will eventually admit a graph G^* in which $\eta(w) = 1$ for all $w \in V(G^*)$. By case 1, G^* satisfies conditions $(1-4)$ of Lemma 2.1 and so satisfies conditions $(i-ii)$ of this theorem. However, we are not guaranteed that G^* satisfies condition (iii), so we need to modify the process a little.

Let $G_i(v)$ be the subgraph of $G \setminus \{v\}$ induced by color-class i. If for every $w \in V(G)$, $\frac{d_{G(i)}}{\eta(w)}$ is an even integer, then the number of edges joining each component of $G_i(v)$ to v must be even (indeed, every edge cut must be even since the sum of the degrees of the vertices in any graph must be even). The only way that $G'(i)$ will have more components than $G(i)$ is if every edge joining some component of $G_i(v)$ to w in G is moved to the new vertex w' in forming G'. We avoid this happening by guaranteeing that at most half the edges joining each component in $G_i(w)$ to w are moved to w'.

To do this we consider two color-classes of $\mathcal{K}$ (we know that there are at least 2 colors in the coloring $\mathcal{K}$ since we are assuming that $\eta(w) \geq 2$.) Let B_2 be the subgraph of B_1 induced by 2 color classes of $\mathcal{K}$. In B_2,

$$\begin{aligned}
&2\alpha_i \leq d_{B_2}(c_i) \leq 2\alpha_i + 2 && \text{for } 1 \leq i \leq \eta(w),\\
&d_{B_2}(c_i) = 2\alpha_i && \text{if condition (*) holds,}\\
&d_{B_2}(\mathcal{L}) = 2(\eta(w)-1), &&\\
&d_{B_2}(v) = 2\eta(v) && \text{if } v \in D \setminus \{\mathcal{L}\}.
\end{aligned}$$

We will use B_2 to form a third bipartite graph B_3. For each color i for which condition (*) holds, split c_i into α_i vertices $c_{i,1}, \ldots, c_{i,\alpha_i}$ of degree 2 in such a way that as many of those vertices as possible are incident with two edges that correspond to two edges in $G(i)$ that join w to the same component of $G_i(w)$. B_3 is constructed by taking a copy of B_2 and splitting each color vertex c_i into two vertices c_i and c_i', and moving $\lfloor \frac{d_{B_2}(c_i)}{2} \rfloor$ edges from c_i to c_i'. If a set of edges in B_2 corresponds to the only edges of $G(i)$ joining w to a component of $G_i(w)$ (this set contains at most $\alpha_i + 1$ edges), then they are grouped together either on c_i or c_i'. The remaining edges are grouped arbitrarily. The degrees of B_3 are given by

$$
\begin{array}{ll}
d_{B_3}(c_{i,j}) = 2 & \text{for } 1 \le i \le \mathcal{K}, 1 \le j \le \alpha_i \text{ if conditionn (*) holds,} \\
\alpha_i \le d_{B_3}(c_{i,j}) \le \alpha_i + 1 & \text{otherwise,} \\
d_{B_3}(\mathcal{L}) = 2(\eta(w) - 1), & \\
d_{B_3}(v) = 2\eta(v) & \text{if } v \in D \setminus \{\mathcal{L}\}.
\end{array}
$$

By [2, 9] we can give B_3 an equitable and balanced 2-edge-coloring. This gives two color classes, either of which can be used to determine a set of edges of G that can be moved to w' to make G' meets conditions $(1-4)$ of Lemma 2.1. Furthermore if condition (*) holds then exactly one of the two edges incident with $c_{i,j}$ will determine an edge of $G(i)$ to be moved to w'; so at most half the edges joining each component of $G_i(w)$ to w in G ill be moved to w' in G'. Therefore G' also has the property that $\omega(G(i)) = \omega(G'(i))$ for each i satisfying condition (*).

By repeatedly disentangling a single vertex, we get a series of graphs, each having one more vertex than the previous. Ultimately, this will give a graph G^* for which $\eta(v) = 1$ for all $v \in V(G^*)$. This graph satisfies the conditions $(i - iii)$. □

In the preceding theorem, we did not make any assumptions about the original ℓ-edge-coloring of G. If we assume that $d_{G(i)}(v)/\eta(v) = k$ for some integer k and for all $v \in V(G)$, then the edges in H of color $i$induce a k-factor of H.

Corollary 2.1 *Let G be an ℓ-edge-colored graph, let $\eta : V(G) \to \mathbb{N}$, and let $d_{G(i)}(w) = \eta(w) \cdot r_i$ for $1 \le i \le \ell$ and for some integer r_i. If G satisfies*

1) $d(w) = \eta(w)(n-1)$,

2) $m(w,v) = \eta(w)\eta(v)$,

3) w has $\binom{\eta(w)}{2}$ loops, and

4) $d_{G(i)}(w) = r_i \cdot \eta(w)$ for all $w \in V(G)$.

then G has a disentanglement H that satisfies

i) $H \cong K_n$,

ii) $d_{H(i)}(v)$ *is* r_i*-regular for* $1 \leq 1 \leq \ell$, *and*

iii) if r_i *is even, then* $\omega(G(i)) = \omega(H(i))$ *for* $1 \leq i \leq \ell$.

Proof: This follows immediately from Theorem 2.1. □

3 Hamilton Decompositions of K_n with a 3-factor removed.

We will now use Corollary 2.1 to show that for any 2-factor U of K_n (n even), there exists a 3-factor T containing U such that $K_n - E(T)$ admits a hamilton decomposition.

Let $V(K_n) = \{v_1, \ldots, v_{n-1}\}$. We define the difference of the edge joining v_i and v_j as $|j - i|$, and let E_i denote the set of edges having difference i. $E(K_n)$ can be partitioned into $\{E_1, \ldots, E_{\frac{n}{2}}\}$. For any sets E_i and E_{i+1}, $E_i \cup E_{i+1}$ induces a connected, 4-regular Cayley graph. This allows us to make use of the following result of Bermond, Favaron, and Maheo [1]:

Theorem 3.1 *If G is a 4-regular connected Cayley graph on n vertices, then G can be decomposed into two hamilton cycles.*

Theorem 3.2 *Let U be any 2-factor of K_n, n even. Then there exists a 3-factor T of K_n with $E(U) \subset E(T)$ such that $K_n - E(T)$ admits a hamilton decomposition.*

Proof: To show the existence of a hamilton decomposition of $K_n - E(T)$, we need only construct a graph that is the amalgamation of $K_n - E(T)$, and then apply Corollary 2.1.

Let U consist of q cycles, the i^{th} cycle having length s_i. Let $V(K_n) = \{v_0, \ldots, v_n\}$. Let $c : E(K_n) \to \{1, \ldots, \frac{n}{2}\}$ be the edge-coloring obtained by assigning color i to the edges of E_i. We will construct the amalgamation of $K_n - E(T)$ by first partitioning $E(K_n)$ such that all but two parts induce hamilton cycles (one part will induces pair of $\frac{n}{2}$-cycles, the other a 1-factor), amalgamating K_n, and then swapping some edges between some of the parts to form U. T will be the union of U and the 1-factor.

First, we need to construct the initial partition. This breaks into two cases:

Case 1: If $n \equiv 2 \pmod 4$, then by Theorem 3.1 $E_{2i-1} \cup E_{2i}$ can be decomposed into two hamilton cycles for $2 \leq i \leq \frac{n-2}{4}$.
Case 2: If $n \equiv 0 \pmod 4$, then by Theorem 3.1 $E_{2i-1} \cup E_{2i}$ can be decomposed into two hamilton cycles for $2 \leq i \leq \frac{n-4}{4}$, leaving $E_{\frac{n}{2}-1}$ unpaired. However, $\gcd(\frac{n}{2} - 1, n) = 1$, so $E_{\frac{n}{2}}$ is itself a hamilton cycle.
In both cases, $E_{\frac{n}{2}}$ remains unpaired, and is the 1-factor leave, E_1 is also a hamilton cycle, and E_2 is a pair of $\frac{n}{2}$-cycles.

We will now use the edges of the $E_1 \cup E_2$ to construct the 2-factor U. This is done by amalgamating K_n to form G, and then swapping some edges between E_1 and E_2. Let $V(G) = \{w_i : 1 \leq i \leq q\}$, let $\psi : V(K_n) \to V(G)$ be defined by

$$\psi(v_i) = w_j \text{ if and only if } \sum_{x=1}^{j-1} s_x < i \leq \sum_{x=1}^{j} s_x,$$

and let G inherit the edge-coloring c. For each pair of vertices w_i, w_{i+1}, there is exactly one edge of E_1 joining them. Since there are q vertices, there are exactly q such edges. Call this set of edges X. Also, since each cycle of U must have length at least 3, then each vertex w of G has at least one loop from E_2. We pick one such loop from each vertex and place it in the set Y. We now recolor G with the coloring $c' : V(G) \to \{1, \ldots, \frac{n}{2}\}$ defined by

$$c'(e) = \begin{cases} 1 & \text{if } e \in X \\ 2 & \text{if } e \in Y \\ c(e) & \text{otherwise.} \end{cases}$$

$G(1)$ consists of q components, but $G(2)$ consists of a single component since each vertex w_i is joined by 3 edges of color 2 to w_{i+1} after the recoloring. Also, $G(i)$ is connected for $3 \leq i \leq \frac{n}{2} - 1$. By Theorem 2.1, G can be disentangled without breaking any component of any color class. Doing so gives $G(1) \cong U$, and thus $G(1) \cup G(\frac{n}{2}) \cong T$. All other color classes are hamilton cycles, and thus form a hamilton decomposition of $K_n - E(T)$. □

References

[1] J. Bermond, O. Favaron, M. Maheo, Hamilton decomposition of Cayley graphs of degree 4, J. Combin. Th. Ser. B, 46 (1989), 142-253.

[2] D. DeWerra, Equitable colorations of graphs, Rev. Fran. Inf. Rech. Oper., 5 (1971), 3-8.

[3] A. J. W. Hilton, Hamilton decompositions of complete graphs, J. Combin. Theory (B), 36 (1984), 125-134.

[4] A. J. W. Hilton and C. A. Rodger, Hamilton decompositions of complete regular s-partite graphs, Discrete Math., 58 (1986), 63-78.

[5] R. Laskar and B. Auerbach, On the decompositions of r-partite graphs into edge-disjoint hamilton circuits, Discrete Math., 14 (1976), 146-155.

[6] C. D. Leach and C. A. Rodger, Hamilton Decompositions of Complete Multipartite Graphs with any 2-Factor Leave, to appear.

[7] C. D. Leach and C. A. Rodger, Non-disconnecting disentanglements of amalgamated 2-factorizations of complete multipartite graphs, J. Combin. Designs, 9 (2001) 460-467.

[8] D. E. Lucas, Recreations Mathématiques, 2 (1892) Gauthiers Villars, Paris.

[9] C. J. H. McDiarmid, The solution of a timetabling problem, J. Inst. Math. Appl., 9 (1972), 23-34.