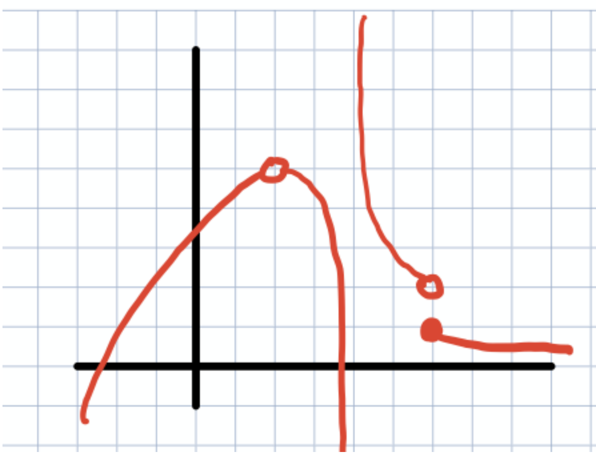
Continuity

1 Informal Concept

Informally, a function is **continuous** if you can trace along its graph without picking up your pencil (i.e. it has no breaks or missing points along the graph).

A function is discontinuous at a point c if you have to pick up your pencil at $x = c$ while tracing the graph.



Three types of discontinuities:

- removable (aka a hole in the graph)
- jump (an abrupt jump)
- infinite (graph has a vertical asymptote)

2 Formal Definition

The function $f(x)$ is **continuous at the number** a if

$$\lim_{x \to a} f(x) = f(a).$$

Note that there are three things that this entails:

- $f(a)$ is defined (or equivalently, a is in the domain of $f(x)$),
- the limit $\lim\limits_{x \to a} f(x)$ exists, and
- they are equal.

Facts:

- Polynomials and $\sin(x)$ and $\cos(x)$ are continuous on $(-\infty, \infty)$.
- Rational functions are discontinuous at the numbers that make their denominator zero (i.e. holes and VA), but continuous elsewhere.
- $\sqrt{x}$ is continuous on $[0, \infty)$

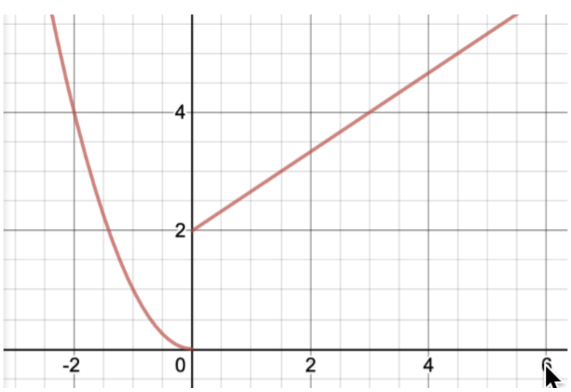
2.1 One-sided continuity

The function $f(x)$ is **continuous from the left at** a if $\lim\limits_{x \to a^-} f(x) = f(a)$.

The function $f(x)$ is **continuous from the right at** a if $\lim\limits_{x \to a^+} f(x) = f(a)$.

2.1.1 Example

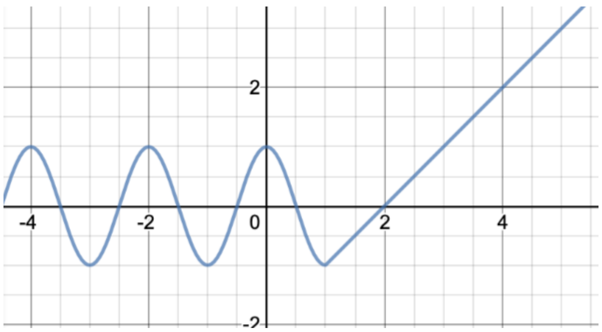
$$f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ \frac{2}{3}x + 2 & \text{if } x \geq 0 \end{cases}$$



2.1.2 Example

Show that this function is continuous on $(-\infty, \infty)$

$$f(x) = \begin{cases} \cos(\pi x) & \text{if } x \leq 1 \\ x - 2 & \text{if } x > 1 \end{cases}$$



2.1.3 Example

Is this function continuous on $(-\infty, \infty)$?

$$f(x) = \begin{cases} \cos(\pi x) & \text{if } x < 1 \\ x - 2 & \text{if } x > 1 \end{cases}$$

2.1.4 Example

Find the value of c that makes this function continuous.

$$f(x) = \begin{cases} 5\sqrt{x} & \text{if } x \leq 1 \\ 3x + c & \text{if } x > 1 \end{cases}$$

2.1.5 Example

Find the value of b that makes this function continuous.

$$f(x) = \begin{cases} -\frac{2}{3}x + 4 & \text{if } x < 2 \\ x^2 + b & \text{if } x \geq 2 \end{cases}$$