The chromatic number of $K^2(9,4)$ is 11

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Abstract

In 2004, Kim and Nakprasit showed that the choromatic number of $K^2(9,4)$ is at least 11. In this note we present an 11-coloring for $K^2(9,4)$. This proves that the chromatic number of $K^2(9,4)$ is 11.

Keywords: Kneser graph, square, graph coloring

1 Introduction

For a simple graph G, let G^2 be the *square* of G, obtained from G as follows: the vertex set $V(G^2)$ is $V(G)$, and two distinct vertices $u, v \in V(G^2)$ are adjacent if and only if the distance between u and v in G is at most 2. We use **[4]** for terminology and notation which are not defined here.

Let $[n] = \{1, 2, \ldots, n\}$. We denote the family of k-element subsets of $[n]$ by $\binom{[n]}{k}$. For $n \geq 2k$, the vertex set of the *Kneser graph* $K(n,k)$ is $\binom{[n]}{k}$, and two vertices A and B are adjacent in $K(n,k)$ if and only if $A \cap B = \emptyset$. When $n = 2k+1$, the graph $K(2k+1,k)$ is also called an *odd* graph. Two vertices A and B of $K^2(2k+1,k)$ are adjacent if and only if $A \cap B = \emptyset$ or $|A \cap B| = k-1$. The *chromatic number* of a graph G, written $\chi(G)$, is the minimum number of colors needed to color the vertices so that adjacent vertices receive different colors. An *independent set* in a graph is a set of pairwise nonadjacent vertices. The *independent number* of a graph G, written $\alpha(G)$, is the maximum size of an independent set of vertices. In **[1]** it was proved that $\alpha(K^2(9,4)) = 12$. Note that two vertices A and B of $K^2(9,4)$ are nonadjacent if and only if $|A \cap B| = 1$ or 2. Using the fact that $\chi(G) \geq \dfrac{|V(G)|}{\alpha(G)}$, we have $\chi(K^2(9,4)) \geq 11$, as it was noted in **[1]**.

2 The main result

Since $K(5,2)$ is the Petersen graph, $K^2(5,2)$ is a complete graph on 10 vertices. Hence, $\chi(K^2(5,2)) = 10$. In **[1]** it was shown that $\chi(K^2(7,3)) = 6$. Figure 1 displays a partition of $V(K^2(9,4))$ into eleven independent sets ($\mathcal{I}_j,\ j \in [11]$). This proves that:

Theorem 1 $\chi(K^2(9,4)) = 11$.

Using *nauty* **[3]**, we see that $\mathcal{I}_i \cong \mathcal{I}_j$ for $i, j \in [7]$. In particular,

$$
\begin{aligned}
\mathcal{I}_2 &= (1,3,2)(4,9,6)(7,5,8)\mathcal{I}_1 \\
\mathcal{I}_3 &= (5,8,6)(1,7,9,4,2)\mathcal{I}_1 \\
\mathcal{I}_4 &= (3,8,4,7,5,9)\mathcal{I}_1 \\
\mathcal{I}_5 &= (6,7)(1,9,8,3,4,2)\mathcal{I}_1 \\
\mathcal{I}_6 &= (5,8)(1,4,3,9,7,6,2)\mathcal{I}_1 \\
\mathcal{I}_7 &= (1,2)(4,9,7,8,5,6)\mathcal{I}_1
\end{aligned}
$$

1 2 3 9	1 2 3 6	1 2 5 7	1 2 3 8	1 2 5 6	1 2 3 4	1 2 3 7
1 2 4 6	1 2 4 5	1 2 8 9	1 2 6 7	1 2 7 9	1 2 6 9	1 2 4 9
1 3 4 5	1 3 4 9	1 3 4 7	1 3 5 6	1 3 5 8	1 3 6 8	1 3 4 8
1 3 7 8	1 5 8 9	1 3 5 9	1 4 5 8	1 4 6 7	1 4 7 9	1 5 6 7
1 5 6 8	1 6 7 8	1 4 6 8	1 4 6 9	1 4 8 9	1 5 7 8	1 6 8 9
1 6 7 9	2 3 5 7	2 3 5 6	1 7 8 9	2 3 4 7	2 3 5 9	2 3 5 8
2 3 6 7	2 3 8 9	2 3 7 8	2 3 4 9	2 3 6 8	2 4 5 8	2 3 6 9
2 4 5 7	2 4 6 8	2 4 6 9	2 5 6 8	2 4 5 9	2 4 6 7	2 4 5 6
2 5 8 9	2 4 7 9	3 4 5 8	2 5 7 9	3 4 6 9	2 7 8 9	2 4 7 8
3 4 6 8	3 4 5 6	3 6 7 9	3 4 5 7	3 5 7 9	3 4 8 9	3 4 5 9
3 5 6 9	3 4 7 8	4 5 7 9	3 6 8 9	4 5 7 8	3 5 6 7	3 4 6 7
4 7 8 9	5 6 7 9	5 6 7 8	4 6 7 8	6 7 8 9	4 5 6 9	5 7 8 9
$\mathcal{I}_1$	$\mathcal{I}_2$	$\mathcal{I}_3$	$\mathcal{I}_4$	$\mathcal{I}_5$	$\mathcal{I}_6$	$\mathcal{I}_7$
1 2 3 5	1 2 4 7	1 2 6 8	1 2 4 8			
1 2 7 8	1 2 5 8	1 3 5 7	1 2 5 9			
1 3 4 6	1 3 6 7	1 3 6 9	1 3 7 9			
1 4 5 7	1 3 8 9	1 4 5 9	1 4 5 6			
1 5 6 9	1 5 7 9	1 4 7 8	2 3 4 6			
2 4 8 9	2 3 4 5	2 3 4 8	2 5 7 8			
2 5 6 7	2 3 7 9	2 6 7 9	2 6 8 9			
3 4 7 9	2 5 6 9	3 7 8 9	3 5 6 8			
3 5 8 9	2 6 7 8	4 5 6 7	4 5 8 9			
3 6 7 8	3 5 7 8	5 6 8 9	4 6 7 9			
4 5 6 8	4 6 8 9					
$\mathcal{I}_8$	$\mathcal{I}_9$	$\mathcal{I}_{10}$	$\mathcal{I}_{11}$			

Figure 1: A partition of $\binom{[9]}{4}$ into eleven independent sets.

Sketch of our search.

1. Start with the following seven disjoint Steiner triple systems of order 9 (see **[2]**).

$$
\begin{aligned}
\mathcal{S}_1 &= \{124, 139, 158, 167, 236, 257, 289, 345, 378, 468, 479, 569\}\\
\mathcal{S}_2 &= \{129, 134, 156, 178, 235, 248, 267, 368, 379, 457, 469, 589\}\\
\mathcal{S}_3 &= \{127, 135, 146, 189, 238, 249, 256, 347, 369, 458, 579, 678\}\\
\mathcal{S}_4 &= \{126, 138, 147, 159, 239, 245, 278, 346, 357, 489, 568, 679\}\\
\mathcal{S}_5 &= \{125, 137, 148, 169, 234, 268, 279, 356, 389, 459, 467, 578\}\\
\mathcal{S}_6 &= \{123, 149, 157, 168, 247, 258, 269, 348, 359, 367, 456, 789\}\\
\mathcal{S}_7 &= \{128, 136, 145, 179, 237, 246, 259, 349, 358, 478, 567, 689\}
\end{aligned}
$$

2. For each $i \in [7]$ and $j \in [12]$, add x_{ij} to block $B_j \in \mathcal{S}_i$ in such a way that the resulting set of twelve 4-subsets of [9] is an independent set in $K^2(9,4)$. Computer searches show that there are precisely 288 independent sets in $K^2(9,4)$ containing $\mathcal{S}_i$ for each $i \in [7]$. All 2016($= 7 \times 288$) of these are isomorphic.

3. Find seven disjoint independent sets among these 2016 independent sets. Let R be the remaining 42 4-subsets of [9] which do not appear in these seven independent sets.

4. Use computer searches to partition R into four independent sets in $K^2(9,4)$. If such a partition exists, then we have 11 independent sets in $K^2(9,4)$ which partition $V(K^2(9,4))$. Otherwise, go to Step (3) and generate a new set of seven disjoint independent sets.

References

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