3.10 Linear Approximations & Differentials

1 The Main Idea behind both

Suppose we need to estimate the following things, possibly with no calculator available:

- $\sqrt{50}$
- $(3.95)^2$
- $\log 105$
- $\sin\left(\frac{11\pi}{60}\right)$

Linear Approximations and Differentials let us use derivatives to get **accurate** estimates to these sorts of questions.

2 Linear Approximations

You're given some function $f(x)$, and you're asked to approximate the value of $f(x)$ for some ramdomish x value, that happens to be near a "nice" value you could plug in. Call the nice value a.

Find the equation of the tangent line through a and call this tangent line $y = L(x)$. (L for "linearization").

When x is near a, we have that $f(x) \approx L(x)$.

Linearization of $f(x)$ near a: $f(x) \approx L(x) = f(a) + f'(a)(x - a)$

2.1 Examples

2.1.1 Example:

Find the linearization of $f(x) = 2x^3 - 3x + 4$ at $a = 1$, and use it to approximate $f(1.1)$.

2.1.2 Example:

Use a linear approximation to estimate $\sqrt[3]{8.07}$. Compare the approximation to the actual answer using a calculator.

2.1.3 Examples:

Use a linear approximation to estimate $\log(98)$

3 Differentials

Recall that $\frac{dy}{dx} = f'(x)$. If we multiply this out we get

$$dy = f'(x)dx.$$

The dx and dy are called **differentials**. The differentials can be thought of as approximate small changes in x and y, respectively.

3.0.1 Teaching Example:

Suppose a metal square initially has dimensions 3×3 inches. Now suppose the square is heated up, and expands to 3.08×3.08 inches. We will find the exact change in area, and we will use differentials to approximate the change in area.

Given a function $y = f(x)$, and a value of dx representing a small change in x, **the Actual Change in y is approximately** $dy = f'(x)dx$

Differentials can also be interpreted as a way to approximate error in y given error in x:

Given a function $y = f(x)$, and a value of dx representing the approximate error in x, **The error in y is approximately** $dy = f'(x)dx$

3.1 Examples:

3.1.1 Example:

A metal cube is $2 \times 2 \times 2$ inches. When heated up, its side length increases by .021 in. Use differentials to approximate the change in volume of the cube.

3.1.2 Example:

A circular disk is measured as having a radius of 21mm. The caliper used to measure it has a tolerance of .01mm, meaning that the error in measurement is at most .01mm. Find the approximate error, and the maximum and minimum possible areas of the disk.