EVERY NON-TRIVIALLY $(2, r)$-REGULAR GRAPH IS REGULAR

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Abstract. Let G be a simple graph. We say G is $(2, r)$-regular if for any pair of distinct vertices u and w, $|N(u) \cup N(w)| = r$. The graph G, which is neither complete nor empty, is strongly regular if it is regular and the number of vertices adjacent to two vertices u and w depends only on whether u and w are adjacent or not. In this note we prove that every non-trivially $(2, r)$-regular graph is strongly regular. As a result of this we show that if a non-trivially $(2, r)$-regular graph of order n exists then $4(n-1)(n-r)+1$ is a perfect square.

1. Introduction

Let $G = (V, E)$ be a simple graph of order n and let $t \leq n$. We say G is a (t, r)-*regular* graph (see [3]) if for every set $S \subset V(G)$ with $|S| = t$ we have $|\bigcup_{v \in S} N(v)| = r$.
In this note we deal with non-trivially $(2, r)$-regular graphs. A $(2, r)$-regular graph is non-trivial if $r > 1$ and the graph is not complete. In [3], $(2, r)$-regular graphs were classified for $r = 0, 1, 2$. But characterizing $(2, r)$-regular graphs for $r \geq 3$ has proven to be a much more difficult problem (see [2, 4, 5, 6]). In [4] it is stated that the existence of non-regular $(2, r)$-regular graphs is open. In the next section we will answer this question.

Let $G = (V, E)$ be a simple graph of order n which is neither complete nor empty. We say G is a *strongly regular* (n, d, μ_1, μ_2) graph (see [1]) if G is regular ($|N(u)| = d$ for all $u \in V$) and for every distinct pair of vertices u and w:

(1) if $\{u, w\} \in E$ then $|N(u) \cap N(w)| = \mu_1$;
(2) if $\{u, w\} \notin E$ then $|N(u) \cap N(w)| = \mu_2$.

Since $|N(u) \cup N(w)| = |N(u)| + |N(w)| - |N(u) \cap N(w)|$ for every distinct pair of vertices u and w, it follows that a strongly regular (n, d, μ, μ) graph is a $(2, 2d - \mu)$-regular graph. In this note we prove that a non-trivially $(2, r)$-regular graph of order n is a strongly regular $(n, d, 2d - r, 2d - r)$ graph for some d dependent on n and r. As an immediate result of this we show that if a non-trivially $(2, r)$-regular graph of order n exists then $4(n-1)(n-r)+1$ is a perfect square.

2. $(2, r)$-regular graphs and their associated PBDs

An (n, K, λ) *pairwise balanced design* (PBD) consists of an order pair $(V, \mathcal{B})$, where V is a set of n elements and $\mathcal{B}$ is a collection of subsets of V (called blocks), each of size $k \in K$, such that every distinct pair of V appears in precisely λ blocks (see [8]). The following result shows that $(2, r)$-regular graphs are related to particular PBDs. The proof of this theorem is straightforward and can be found in [6].

Theorem 1. *Let G be a $(2, r)$-regular graph of order n with vertex set $V = \{1, 2, 3 \ldots, n\}$. Define $\mathcal{B} = \{B_1, B_2, \ldots, B_n\}$ where $B_i = V \setminus N(i)$ for $i = 1, 2, \ldots, n$. Then $(V, \mathcal{B})$ is a PBD with $\lambda = n - r$. Moreover,*

(1) *$i \in B_i$ for every $i \in V$;*
(2) *$i \in B_j$ if and only if $j \in B_i$ for $i, j \in V$;*

(3) $|B_i \cap B_j| = \lambda$ *for* $i, j \in V$ *and* $i \neq j$.
(4) G *is an* $(n-k)$*-regular graph if and only if* $|B_i| = k$ *for all* $i \in \{1, 2, \ldots, n\}$.

We now define an *incidence* matrix $A = [a_{ij}]$, of order n, for the PBD given in Theorem 1 by $a_{ij} = 1$ if $i \in B_j$ and $a_{ij} = 0$ otherwise. Then:
(1) $a_{ii} = 1$ for all $i \in V$;
(2) $A = A^t$ (that is $a_{ij} = a_{ji}$ for all $i, j \in V$);
(3) $A^2 = D + \lambda J$, where D is a diagonal matrix of order n with diagonal elements $d_i = |B_i| - \lambda$ and J is the matrix of order n with every entry equal 1.

The following conjecture was stated in [6].

Conjecture 2. *Let* $A = [a_{ij}]$ *be a* $(0,1)$ *symmetric matrix of order* n *and* $a_{ii} = 1$ *for each* i*. If the off-diagonal elements of* A^2 *are all equal then the diagonal elements of* A^2 *are all equal.*

The authors recently discovered that Ryser in 1970 has proved (see [7]) a stronger theorem. In order to state this theorem we need the following definitions. A *permutation* matrix is a matrix obtained by permuting the rows and columns of the identity matrix. Two matrices A and B are *equivalent* if there exists a permutation matrix P such that $A = P^T BP$, where P^T is the transpose of P. Define

$$B = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ 1 & & & \\ \cdot & & & \\ \cdot & & \mathbf{0} & \\ \cdot & & & \\ 1 & & & \end{bmatrix} \quad (n \geq 2) \qquad C = \begin{bmatrix} 0 & 1 & \ldots & 1 \\ 1 & & & \\ \cdot & & & \\ \cdot & & \mathbf{Q} & \\ \cdot & & & \\ 1 & & & \end{bmatrix} \quad (n \geq 4)$$

where $\mathbf{0}$ is the zero matrix of order $n-1$ and $\mathbf{Q}$ is a symmetric permutation matrix of order $n-1$. A *line* in a matrix is a row or a column of that matrix.

Theorem 3. *(See* [7]*) Let* A *be a* $(0,1)$ *matrix of order* $n > 1$ *that satisfies the matrix equation* $A^2 = D + \lambda J$*, where* D *is a diagonal matrix and* λ *is a positive integer. Then* A *has constant line sums* c *except for the* $(0,1)$ *matrices* A *of order* n *with* $\lambda = 1$ *equivalent to* B *or* C *and the* $(0,1)$ *matrix* A *of order* 5 *with* $\lambda = 2$ *equivalent to*

$$F = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Furthermore, if A *has constant line sums* c*, then* $A^2 = dI + \lambda J$*, where* $c^2 = d + \lambda n$ *and* $-\lambda < d \leq c - \lambda$.

Theorem 4. *Every non-trivially* $(2, r)$*-regular graph of order* n *is a strongly regular* $(n, d, 2d - r, 2d - r)$ *graph, where* $d = ((2n-1) - \sqrt{4(n-1)(n-r)+1})/2$.

Proof. Let $G = (V, E)$ be a non-trivially $(2, r)$-regular graph of order n and let $(V, \mathcal{B})$ be the PBD given in Theorem 1. Assume A is an incidence matrix for this PBD. Then A satisfies the assumptions of Conjecture 2. If off-diagonal elements of A are all zero then A is the identity matrix. It is easy to see that G is complete in this case. This is a contradiction. Now assume that the off-diagonal entries of A are non-zero. Since $a_{ii} = 1$, for all i, it follows that A is not equivalent to matrices B, C or F. Therefore A has constant line sums by

Theorem 3. (That is to say that Conjecture 2 is true.) So $|B_i| = k$ for all $i \in V$. Therefore by Part 4 of Theorem 1, G is an $(n-k)$-regular graph. Thus, for every distinct pair of vertices u and w we have

$$|N(u) \cap N(w)| = |N(u)| + |N(w)| - |N(u) \cup N(w)| = (n-k) + (n-k) - r.$$

Hence G is a strongly regular $(n, d, 2d-r, 2d-r)$ graph, where $d = n-k$. By Theorem 3 we also have $k^2 = (k-\lambda) + \lambda n$ or $k(k-1) = \lambda(n-1)$, where $\lambda = n-r$. Now since $k = n-d$ it follows that $(n-d)(n-d-1) = (n-r)(n-1)$ and $d = ((2n-1) - \sqrt{4(n-1)(n-r)+1})/2$ since $d < n$. □

Corollary 5. *If a non-trivially $(2, r)$-regular graph of order n exists then $4(n-1)(n-r)+1$ is a perfect square.*

Example 6. *The graph $K_4 \times K_4$ is a non-trivial $(2, 10)$-regular graph on 16 vertices (see for example [6]). By Theorem 4 this graph is also a strongly regular $(16, 6, 2, 2)$ graph.*

We conclude this paper with the following interesting question.

Question 7. Classify all simple graphs which have the following properties. For every distinct pair of vertices u and w:
(1) if u and w are joined then $|N(u) \cup N(w)| = r_1$;
(2) if u and w are not joined then $|N(u) \cup N(w)| = r_2$.

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