3.1 Derivatives of Polynomials and Exponential Functions

1 Derivatives of Power Functions

1.1 The Power Rule:

For any real number n,

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

2 Proof of the Power Rule

Before we prove the power rule, let's look at powers of $(x+h)$. (This follows from the binomial theorem.)

$(x+h)^0 = 1$

$(x+h)^1 = x+h$

$(x+h)^2 = x^2 + 2xh + h^2$

$(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$

$(x+h)^n = x^n + (n-1)x^{n-1}h +$ more terms, each with h raised to a larger power than 1

Let $f(x) = x^n$

Then $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

There are a few special cases worth mentioning:

- $\frac{d}{dx}(x) = 1$
- $\frac{d}{dx}(1) = 0$

2.0.1 Examples:

$$\frac{d}{dx}\left(x^5\right)$$

$$\frac{d}{dx}\left(x^3\right)$$

$$\frac{d}{dx}\left(\sqrt{x}\right)$$

$$\frac{d}{dx}\left(\sqrt[3]{x}\right)$$

$$\frac{d}{dx}\left(\frac{1}{x}\right)$$

$$\frac{d}{dx}\left(\frac{1}{x^4}\right)$$

2.1 Addition, Subtraction, and Constant Multiples

Differentiation (taking the derivative) is a *linear operation*. That means that

- $\dfrac{d}{dx}\left[f(x) + g(x)\right] = \dfrac{d}{dx}f(x) + \dfrac{d}{dx}g(x)$
- $\dfrac{d}{dx}\left[f(x) - g(x)\right] = \dfrac{d}{dx}f(x) - \dfrac{d}{dx}g(x)$
- $\dfrac{d}{dx}\left[cf(x)\right] = c\dfrac{d}{dx}f(x)$

2.1.1 Examples:

$$\frac{d}{dx}(4x^2)$$

$$\frac{d}{dx}(3x^9)$$

$$\frac{d}{dx}(x^3 + x^5)$$

$$\frac{d}{dx}(3\sqrt{x})$$

2.2 More Special Cases to Memorize:

- $\frac{d}{dx}(cx) = c$ for any constant c.

- $\frac{d}{dx}(c) = 0$ for any constant c.

2.2.1 Examples:

Find the derivatives of the following functions:

$f(x) = 3x^3 + 2x^2 - 5x + 9$

$A(r) = \pi r^2$

$s(t) = 10 + 33t - 16t^2$

$V(r) = \dfrac{4}{3}\pi r^3$

2.3 More complicated examples:

2.3.1 Example:

$f(x) = (5x^3 - 4)^2$

2.3.2 Example:

$$f(x) = \frac{5x^2 - 15x + 10}{5x}$$

2.3.3 Example:

$f(x) = (2x^2 + 3)(x + 5)$

2.3.4 Example:

$f(x) = \dfrac{6x^2 + 3x + 9}{3x + 1}$ We can't do this one yet!

3 Tangent lines

3.0.1 Example:

Find the equation of the line that is tangent to $y = 4 + 3x - x^2$ at the point $(3, 4)$.

3.0.2 Example:

Find all places where the line tangent to $f(x) = 4x^3 - 21x^2 - 90x$ has a horizontal tangent.

3.0.3 Example:

Find all places where the line tangent to $g(x) = 1/x$ has a tangent with slope $-\frac{1}{16}$.

4 Velocity and Acceleration:

Recall that if $s(t)$ gives the position of an object at time t, then

- $s'(t) = v(t)$ gives the velocity at time t and
- $s''(t) = v'(t) = a(t)$ gives the acceleration at time t.

Suppose a particle moves along a straight line, and its position is given by $s(t) = 2 + 3t - \frac{1}{2}x^2$.

4.0.1 Example:

On another planet, a ball is thrown straight up. Its height in feet is given by the function $h(t) = 6 + 20t - 4.4t^2$. Find the velocity and acceleration functions, and then determine the speed at which the ball impacts the ground.

5 Derivatives of Exponential Functions

5.1 Derivative of e^x

$$\frac{d}{dx}\left(e^x\right) = e^x$$

5.1.1 Examples:

Find the derivatives of the following functions:

$f(x) = 3e^x$

$g(x) = 4e^x - x^3$

5.2 Derivative of a^x

$$\frac{d}{dx}\left(a^x\right) = e^x \ln(a)$$

5.2.1 Examples:

Find the derivatives of the following functions:

$f(x) = 2^x + x^2$

$g(x) = e^x + 2^x$