Generalized Leech trees

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Abstract

In 1975, Leech introduced the problem of labelling the edges of a tree with distinct positive integers so that the sums along distinct paths in the tree were distinct, and the set of such path-sums were consecutive starting with 1. We generalize this problem to labellings from arbitrary finite Abelian groups, with a particular focus on direct products of the additive group of $\mathbb{Z}_2$.

1 Definitions and history

Let T be a tree on n vertices and $n-1$ edges; since every pair of vertices in T is joined by exactly one path, there are $\binom{n}{2}$ distinct paths in T. Let P be this set of paths. Given a labelling $w : E(T) \to \mathbb{Z}^+$ of the edges of T, for a path $p \in P$ we define

$$w(p) = \sum_{e \in p} w(e)$$

Such a labelling is a *Leech labelling* if

$$\{w(p) | p \in P\} = \left[\binom{n}{2}\right]$$

In other words, each path in T has a distinct sum of weights along its edges, and those sums are the consecutive integers 1 through $\binom{n}{2}$. We say that T is a *Leech tree* if it admits a Leech labelling.

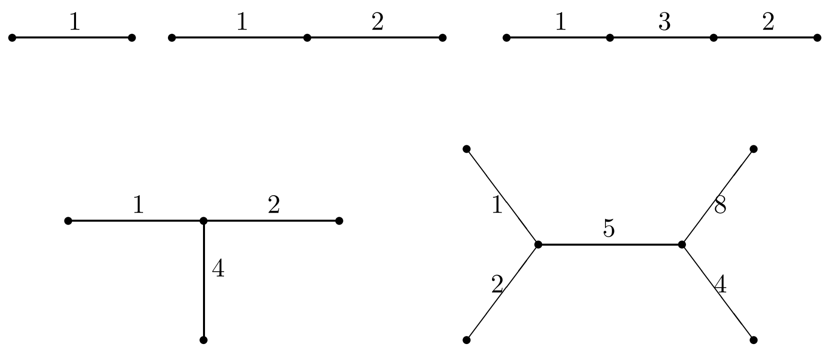

Figure 1: Leech labellings of all known Leech trees.

Only five Leech trees are known, presented in Figure 1; they were listed along with their labellings in the original 1975 paper [3] by Leech. The motivation there was an application in electrical engineering: such a labelling would give a way of constructing a universal resistor with possible impedance of 1 up to $\binom{n}{2}$ units from $n-1$ simple resistors.

In 1977, Taylor [8] used a parity argument to prove the following key result:

Theorem 1.1 (Taylor, 1977)**.** *If T is a Leech tree on n vertices, then n is either a square or two more than a square.*

Yang, Zhang, and Ding [9] consider a weaker version of the problem: given a positive integer n, what is the largest integer N such that there is a tree on n vertices and a labelling on its edges with distinct integers such that $1, \ldots, N$ appear as path-sums? They performed computer searches for values of n up to 11, distinguishing between branched and unbranched trees.

More recently, Székely, Wang, and Zhang [6, 7] performed computer searches to show the non-existence of Leech trees of orders 9 and 11. They also provided upper bounds on the diameter and maximum degree of Leech trees, and conjecture that only finitely many Leech trees exist.

Leech labellings have also been studied under the name "perfect distance labellings", by Calhoun *et al* [1] and Calhoun and Polhill [2]. The former paper broadens the search to "distinct distance labellings", where the condition that the path-sums be consecutive is removed; the question of interest is the minimum largest path-sum required for a tree on n vertices. (In the course of this, they show that there cannot be a Leech tree on 16 vertices.) It also extends the initial problem to forests (and graphs in general), an investigation that is taken up in the latter paper.

2 G-Leech trees

Let $(G, +)$ be a finite Abelian group with identity element 0. A *G-Leech labelling* of a tree T with path-set P is a function $w : E(T) \to G$ such that the weight-sums $w(p)$ for $p \in P$ are all distinct and $\{w(p)|p \in P\} = G - \{0\}$. Clearly if T has n vertices then $|G| = \binom{n}{2} + 1$. Naturally, we call a tree that admits a G-Leech labelling a *G-Leech tree*.

We want G to be Abelian because we assume our trees to be undirected; one could allow for non-Abelian groups by considering oriented paths (that is, distinguishing between the path from u to v and the path from v to u), though of course we must then require that G contains no elements of order 2. We exclude the group's identity element from the set of path-sums by analogy with the original problem: with this constraint, Leech labellings with path-sums of 1 up to m directly translate to $\mathbb{Z}_{m+1}$-Leech labellings. However, one could easily consider the alternative problem of requiring *every* group element, including the identity, to appear as a path-sum; in this case, one must ensure that the path containing 0 is maximal.

Two cases of particular interest to use are the *binary* Leech labellings, where $G = \mathbb{Z}_2^r$ for some r, and the *cyclic* Leech labellings, where $G = \mathbb{Z}_m$ for some m. In this paper, we solve the existence problem for the former case completely.

Taylor's result on Leech trees extends naturally to G-Leech trees, as long as G allows for a notion of parity. The proof we give of the following result is essentially identical to the proof of Theorem 1.1 as reproduced in [6].

Theorem 2.1. *If G is a finite Abelian group of even order and T is a G-Leech tree on n vertices then $n = k^2 + 2$ for some positive integer k.*

Proof. Let $\varphi : G \to \mathbb{Z}_2$ be the natural homomorphism. Given a G-Leech labelling $w : E(T) \to G$ take the composition φw which maps each edge to either 0 or 1; define a bipartition of the vertices of T into sets A, B such that if $e = uv$ then $\varphi w(e) = 1$ iff u and v are in different sets. (This defines a unique bipartition up to renaming the sets.) Any odd path (that is, path with a sum that maps to 1 under φ) must have one end-point in each of A and B. Therefore, there are exactly $|A| \cdot |B|$ such paths.

Since $|G|$ is even, we must have $|G| - 1 = \binom{n}{2}$ odd since every group element but one appears on a path of T. Of these, slightly more than half – $\frac{1}{2}\left[\binom{n}{2} + 1\right]$ – are odd paths. Therefore:

$$\begin{aligned}(|A|-|B|)^2 &= (|A|+|B|)^2 - 4|A|\cdot|B| \\ &= n^2 - 2\left[\binom{n}{2}+1\right] \\ &= n-2\end{aligned}$$

□

3 Binary Leech trees

Here we are labelling the edges of trees with binary strings, with addition defined componentwise in $\mathbb{Z}_2$, and we want the set of path sums to be precisely the set on non-zero binary strings. To do this with strings of length r on a tree with n vertices, we must have that $2^r - 1 = \binom{n}{2}$. There are precisely four pairs of positive values that satisfy this constraint, derived from the Ramanujan-Nagell numbers [4]:

n	r
2	1
3	2
6	4
91	12

Of these four cases, the first two are easily dispatched; there is exactly one labelling of K_2 with a non-zero one-bit string, which (miraculously!) is a Leech labelling, while any pair of two distinct non-zero two-bit strings will give a binary Leech labelling of P_3. The last pair is inadmissible by Theorem 2.1. This leaves us only the case of $n = 6$ to contend with.

There are six non-isomorphic trees on six vertices; however, if we can find a solution for one tree then we can find a solution for all trees (of that order). Given a edge-labelling f of T and $u, v \in V(T)$ let $f(uv_T)$ denote the sum of the weights along the u, v-path in T.

Lemma 3.1. *Let T be a tree on $n \geq 2$ vertices with binary edge-labelling $w : E(T) \to \mathbb{Z}_2^r$. Let v be a leaf in T with neighbour u, and let $x \neq v$ be a neighbour of u. Construct a new tree $T' = T + vx - uv$, and define $w' : E(T') \to \mathbb{Z}_2^r$ by*

$$w'(e) = \begin{cases} w(e) & e \in E(T) \cap E(T') \\ w(uv) + w(ux) & e = vx \end{cases}$$

Then for any vertices $y, z \in T$, $w(yz_T) = w'(yz_{T'})$.

Proof. If neither y nor z is equal to v, then the path between them does not include v and hence the values of w and w' coincide for every edge in the y, z-path.

If $y = v$, then either the v, z-path in T goes through x or it does not. If it does, then the corresponding path in T' shares with the path in T all edges in the x, z-path and therefore the path-sum, and by construction $w'(vx) = w(uv) + w(ux)$, as required. Otherwise, the v, z-path in T and in T' coincide on the edges of the u, z-path, and $w'(vx) + w'(ux) = w(uv)$ as required. □

Corrolary 3.2. *For any positive integer n, either all trees on n vertices are binary Leech trees or none are.*

Proof. By performing successive "leaf-shifts" of the type used in the previous lemma, any tree can be transformed into a star; since the operation is reversible, any tree can therefore be transformed into any other tree of the same order. □

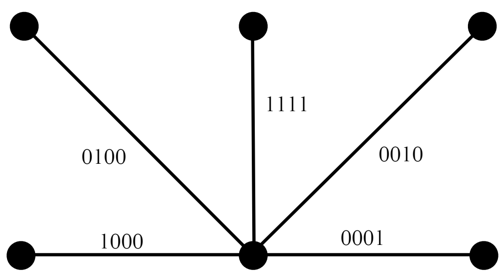

Figure 2: A binary Leech labelling of $K_{1,5}$.

We demonstrate that the star $K_{1,5}$ is a binary Leech tree, and therefore that all trees on six vertices are binary Leech trees, in Figure 2. There are many more such labellings; a computer search revealed $20160 = 120 \times 168$ Leech labellings of a labelled $K_{1,5}$, yielding 168 Leech labellings of an unlabelled 5-star. This is evocative in that it is the number of ovoids in the projective 3-space over GF(2). (Recall that an *ovoid* in the space $PG(3, q)$, with q a prime power, is a set of $q^2 + 1$ points with no three collinear. It is well-known that every subplane intersects an ovoid in either 1 or $q + 1$ points, a fact that we have some use for. See [5] for further details about ovoids in projective 3-space.) This is no mere numerical coincidence:

Theorem 3.3. *Let S be a set of five binary strings of length 4. Then S gives a Leech labelling of $K_{1,5}$ iff S is an ovoid in $PG(3, 2)$.*

Proof. Let S be the labels of a binary Leech labelling of $K_{1,5}$. If S does not comprise an ovoid in $PG(3,2)$ then some $a, b, c \in S$ must be collinear: $a + b = c$. But this gives two paths in the star that have the same sum, which is impossible. Therefore S must be an ovoid.

Now let S be the points of an ovoid, say $S = \{a, b, c, d, e\}$. We know that no sum of two elements of S can be a third, since that would give three collinear points; we need therefore only check that no two pair-sums are equal. So suppose that $x = a + b = c + d$; then the subplane that contains a, b, c must also contain x since $x = a + b$, and also d since $d = x + c$. But this gives a subplane of the space containing four points of the ovoid, which is impossible. Therefore all of the pair-sums must be distinct both from the members of S and from each other, meaning that S is a label set for a binary Leech labelling of the 5-star. □

One further note on this fact is that the leaf-shifting operation from Lemma 3.1 has the property that the sum on any "named" path is invariant: that is, if we name two vertices u, v in T and perform a sequence of leaf-shifts on T then the weight of the u, v path in any tree in the resulting sequence is equal to that in T. Using leaf-shifts on the star gives six labellings that are equivalent in this sense (because there are six choices for the name of the central vertex); between this equivalency and the graph automorphisms we end up with twenty-eight families of labellings of $K_{1,5}$, each with $6!$ instantiations. This holds true for the other trees on six vertices as well, as the $6!$ can be derived directly as the number of ways to label the vertices of the tree in question.

4 Some notes on cyclic Leech trees

Our investigations on the cyclic Leech trees are much less developed. To give the flavour of the work we shall describe some results concerning the first few admissible cyclic groups: $\mathbb{Z}_2, \mathbb{Z}_4, \mathbb{Z}_7, \mathbb{Z}_{11}$, and $\mathbb{Z}_{16}$.

Notice that, for all of these except $\mathbb{Z}_{11}$, we get examples of cyclic Leech trees for free: as we remarked earlier, any (integer) Leech labelling can also be interpreted as a cyclic Leech labelling. Thus, the trees in Figure 1 are cyclic Leech trees. We can find one more example for $\mathbb{Z}_{16}$ quite easily: the leaf-shifting operation described in Lemma 3.1 works in a more general setting, as long as the supporting edge of the shift (ux in the terminology of the lemma) has a label of order 2 in the group. Applying this to the known Leech tree on six vertices gives us another cyclic Leech tree, depicted in Figure 3.

For $\mathbb{Z}_{11}$, there are three possible candidates for cyclic Leech labellings.

Lemma 4.1. *The star $K_{1,4}$ is not a $\mathbb{Z}_{11}$-Leech tree.*

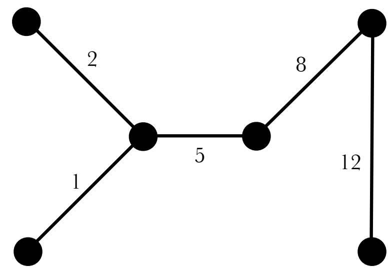

Figure 3: A $\mathbb{Z}_{16}$-Leech labelling of a tree.

Proof. Suppose that a, b, c, d comprise a cyclic Leech labelling of the 4-star. Clearly no two of these are additive inverses (and in fact, for any $x \in G$ at most one of x and $-x$ can appear in a G-Leech labelling of a graph), and hence we must have the element $-a$ appearing as a sum of two others.

We cannot have, say, $-a = b + c$ for the following reason: suppose we took the sum of all of the path-sums under this labelling. Since each edge occurs in four paths, this sum must be $4(a + b + c + d)$; since this is the sum of all non-zero group elements in $\mathbb{Z}_{11}$, it must also be 0. Therefore, $a+b+c+d = 0$; if it were the case that $-a = b+c$ then we have $a+b+c = 0$ and hence $d = 0$, which is impossible.

Therefore, without loss of generality let $-a = a + b$, and hence $b = -2a = 9a$. By the same line of reasoning, however, $-b$ must be expressed as a sum of b with another edge label, and so forth; at some point this chain must lead us back around to a, meaning that $a = 9^k a$ for some $k \leq 4$. This is also impossible, however; therefore $-a$ cannot be a path-sum under this labelling and so the labelling is not a cyclic Leech labelling. □

Note that in the above proof we exploited the properties of $\mathbb{Z}_{11}$ as a ring, a useful set of tools in these analyses.

Lemma 4.2. *All trees on five vertices except for $K_{1,4}$ are $\mathbb{Z}_{11}$-Leech trees.*

Proof. For P_5, label the edges 1, 9, 7, 8 in sequence along the path. (A computer search has shown that this is unique up to group and graph automorphisms.)

For the other tree with degree sequence (3,2,1,1,1), label as follows: 8 on the central edge, 1 and 2 on the edges pendant to the vertex of degree 3, and 7 on the edge pendant to the vertex of degree 2. □

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