Fair Hamilton Decompositions of Complete Multipartite Graphs

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Abstract

A fair hamilton decomposition of the complete multipartite graph G is a set of hamilton cycles in G whose edges partition the edges of G in such a way that, for each pair of parts and for each pair of hamilton cycles H_1 and H_2, the difference in the number of edges in H_1 and H_2 joining vertices in these two parts is at most one. In this paper we completely settle the existence of such decompositions. The proof is constructive, using the method of amalgamations (graph homomorphisms).

1   Introduction

The *complete multipartite graph* $K_{a_1,\ldots,a_p}$ is the simple graph whose vertex set can be partitioned into p parts $A_1,\ldots,A_p$, with $|A_i| = a_i$ for $1 \le i \le p$, in such a way that two vertices are joined if and only if they occur in different parts. If $m = a_1 = \cdots = a_p$ then we denote this graph by $K_m^{(p)}$.

A hamilton decomposition of a graph G is a partition of $E(G)$ in such a way that each element of the partition induces a hamilton cycle in G. Walecki [10] proved a classic result in graph theory in 1892, showing that there exists a hamilton decomposition of K_n if and only if n is odd. In 1976, Laskar and Auerbach [8] settled a companion problem, showing that there exists a hamilton decomposition of $K_{a_1,\ldots,a_p}$ if and only if $m = a_1 = \cdots = a_p$ and $(p-1)m$ is even.

Over the past 20 years Hilton, Nash-Williams and Rodger along with various other mathematicians have pioneered the use of graph homomorphisms (under the name of amalgamations) to obtain results in the area of graph decompositions (see [5, 6, 7, 12]). In particular, amalgamations provide a different proof of those results of Walecki, Laskar and Auerbach; in so doing, they also

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obtain a method to successfully attack the problem of embedding a given edge-coloring of K_n or $K_n^{(q)}$ into an edge-coloring of K_m or $K_m^{(p)}$ respectively in which each color class is a hamilton cycle [5, 6].

Amalgamations have also been used by Buchanan [2] for cracking another difficult problem that has stumped people for years, namely to show that regardless of which 2-factor F is removed from K_{2n+1}, the resulting graph $K_{2n+1} - E(F)$ has a hamilton decomposition. This result was recently extended to complete bipartite graphs [9], and although progress has been made, the corresponding result for $K_m^{(p)}$ still remains open.

It is a common objective in mathematics when dealing with partitioning problems to ask for certain characteristics to be evenly shared among the parts on the partition. For example, edge-colorings can be considered to be a partition of the edges into color classes. One might then also require that: the sizes of each pair of color classes be within one of each other (equalized edge-colorings); or, at each vertex the edges are shared as evenly as possible among the color classes (equitable edge-colorings); or for multigraphs, the edges between each pair of vertices be shared as evenly as possible among the color classes (balanced edge-colorings). DeWerra and McDiarmid [3, 11] first showed that every bipartite graph B has an equitable k-edge-coloring, for all $k \geq 1$; in fact B has a k-edge-coloring that is simultaneously equalized, balanced and equitable (see Lemma 5.1 of [7], for example). Such results for graphs in general are very difficult to come by [4].

Similarly, the *n-cube* Q_n has vertex set consisting of all binary n-tuples, and two vertices are joined by an edge if and only if they disagree in precisely one component, say component i, then the edge is said to have *direction i*. Then the set of edges of Q_n have a natural partition $\{E_1, \ldots, E_n\}$, where E_i is the set of edges of direction i. While studying Gray codes, Bhat and Savage [1] managed to show that for all $n \geq 3$, there exists a hamilton cycle H in which the edges are selected as evenly as possible from the sets $E_1, \ldots, E_n$; so H contains $2\lfloor 2^{n-1}/n \rfloor$ or $2\lceil 2^{n-1}/n \rceil$ edges in E_i for $1 \leq i \leq n$.

In this spirit, the purpose of this paper is to solve a related difficult problem while simultaneously demonstrating the flexibility and power of using graph homomorphisms. A hamilton decomposition $\{H_1, \ldots, H_k\}$ of $K_{a_1, \ldots, a_p}$ is said to be *fair* if for $1 \leq i < j \leq p$ and for $1 \leq k_1 \leq k_2 \leq k$

$$| \, |E_{k_1}(A_i, A_j)| - |E_{k_2}(A_i, A_j)| \, | \leq 1,$$

where $E_\ell(A_i, A_j)$ is the set of edges in H_ℓ joining vertices in A_i to vertices in A_j (we also write

$E_\ell(v_i, v_j)$ if $A_i = \{v_i\}$ and $A_j = \{v_j\}$). So in such a hamilton decomposition, the edges between each pair of parts are shared as evenly as possible among the hamilton cycles. In this paper we settle the existence of fair hamilton decompositions of $K_{a_1, \ldots, a_p}$ (see Theorem 2.2). This result can be obtained neatly using amalgamations, yet it is not clear that any other known technique could solve such a problem.

In this paper, we allow graphs to contain multiple edges.

2 The Main Result

For the purposes of this paper, it is unnecessary to delve into the method of amalgamations, since the results concerning this technique used here have already been proved. Briefly, the method starts with a set of graphs G_1 that satisfy a set of properties $\mathcal{P}_1$. It then finds a set of properties $\mathcal{P}_2$ such that *any* graph satisfying the properties in $\mathcal{P}_2$ is the homomorphic image of *some* graph in G_1. So, if G_2 is the set of graphs satisfying the properties in $\mathcal{P}_2$, then to prove that G_1 is non-empty, it suffices to find a graph in G_2. Finding such a graph is much simpler than finding a graph in G_1 directly.

The result on amalgamations needed here is the following theorem. It follows from each of two different more general results which were proved originally by Hilton and Rodger (see Theorem 1 of [6]) and by Leach and Rodger (see Theorem 3.1 of [9]).

Theorem 2.1 *Let $\ell, m \geq 1$, and let G be an ℓ-edge-colored graph with $V(G) = \{v_1, \ldots, v_p\}$. Let $G(k)$ denote the subgraph of G induced by the set of edges colored k.*

If in G

(1) for $1 \leq i < j \leq p$, the number of edges joining v_i to v_j is m^2,

(2) for $1 \leq k \leq \ell$ and for $1 \leq i \leq p$, $d_{G(k)}(v_i) = 2m$,

(3) for $1 \leq k_1 < k_2 \leq \ell$ and for $1 \leq i < j \leq p$, $| |E_{k_1}(v_i, v_j)| - |E_{k_2}(v_i, v_j)| | \leq 1$, and

(4) for $1 \leq k \leq \ell$, $G(k)$ is connected,

then there exists a fair hamilton decomposition of $K_m^{(p)}$.

Remark If H is a fair hamilton decomposition of $K_m^{(p)}$ and if G is a graph homomorphism formed from H by mapping each vertex in A_i to v_i for $1 \leq i \leq p$, then clearly G would satisfy properties (1-4). Theorem 2.1 actually shows that the reverse is true, namely that *any* graph satisfying (1-4) is the homomorphic image of *some* fair hamilton decomposition of $K_m^{(p)}$.

We are now ready to prove the main result.

Theorem 2.2 *There exists a fair hamilton decomposition of $K_{a_1,\dots,a_p}$ if and only if $a_1 = \cdots = a_p = m$ and $(p-1)m$ is even.*

Proof: First suppose there exists a hamilton decomposition of $K_{a_1,\dots,a_p}$. Then $K_{a_1,\dots,a_p}$ must be regular, so $a_1 = \cdots = a_p = m$ for some integer m. Also, $|E(K_m^{(p)})| = p(p-1)m^2/2$, and each hamilton cycle has mp edges, so there are $\ell = (p-1)m/2$ hamilton cycles in the decomposition. This must be an integer, so $(p-1)m$ must be even.

Next, suppose that $a_1 = \cdots = a_p = m$, so $K_{a_1,\dots,a_p} = K_m^{(p)}$, and that $(p-1)m$ is even. By Theorem 2.1, it suffices to find a graph satisfying conditions (1-4). So let G be a graph with $V(G) = \{v_i \mid 0 \leq i \leq p-1\}$ that satisfies condition (1). It remains to edge-color G so that conditions (2-4) are met. To do so, we assign a *color vector* $c(u,v)$ of length ℓ to each pair $\{u,v\} \subseteq V(G)$, where the ith component of $c(u,v)$ represents the number of edges colored i that join u to v in G.

To meet conditions 1, 2, and 3, it is equivalent to require that the assignment of vectors must satisfy the following properties:

i) for each vector $c(u,v) = (x_0, \dots, x_{\ell-1})$, $\sum_{i=0}^{\ell-1} x_i = m^2$,

ii) $\sum_{w \in N(v)} c(v,w) = (2m, 2m, \dots, 2m)$ for all $v \in V(G)$, and

iii) for each vector $c(u,v) = (x_0, \dots, x_{\ell-1})$, $|x_i - x_j| \leq 1$ for $0 \leq i < j < \ell$,

respectively. Clearly conditions (i) and (iii) are met if and only if

(a) $\alpha = m^2 (\bmod \ \ell)$ entries in each vector are $a = \lfloor m^2/\ell \rfloor + 1$, and the remaining $\ell - (m^2 (\bmod \ \ell))$ entries of each vector are $b = \lfloor m^2/\ell \rfloor$.

Here, the way these vectors are obtained differs according to whether p is even or odd. In each case, having defined the color vectors so that they satisfy conditions (i-iii), the issue of ensuring each color class is connected is then addresssed.

4

Case 1: p is even

When p is even, K_p has a 1-factorization consisting of $p-1$ 1-factors $F_0 \ldots F_{p-2}$. These are used to find $p-1$ color vectors $c_0, \ldots, c_{p-2}$, and we define $c(u,v) = c_i$ for each $\{u,v\} \in F_i$. The vectors $c_0, \ldots, c_{p-2}$ can be described by the rows of a $(p-1) \times \ell$ array $\mathcal{C}$, called a *coloring array* (so row r in the coloring array corresponds to the color vector assignment to each edge in the one 1-factor F_r, and each column corresponds to a color). The coloring array $\mathcal{C}$ can be constructed by defining its $(i,j)^{th}$ entry to be

$$
c_{i,j} = \begin{cases} a & \text{if } j \in \{i\alpha, i\alpha+1, \ldots, (i+1)\alpha-1\} \text{ (reducing each calculation modulo } \ell), \text{ and} \\ b & \text{otherwise} \end{cases}
$$

(the rows and columns of $\mathcal{C}$ are indexed by 0 to $p-2$ and 0 to $\ell-1$ respectively).

Then clearly $\mathcal{C}$ meets condition (iii). Also, each row contains α a's as required by condition (a), so (i) is satisfied. For $0 \le r \le p-2$, the subarray $\mathcal{C}(r)$ of $\mathcal{C}$ formed by rows 0 to r is easily seen to have the property that each of columns 0 to $(r+1)\alpha - 1 \pmod{\ell}$ has exactly one more a than each of the columns $(r+1)\alpha \pmod{\ell}$ to ℓ. Therefore, since $(p-1)\alpha \equiv (p-1)m^2 \pmod{\ell} \equiv 0 \pmod{\ell}$, each column in $\mathcal{C} = \mathcal{C}(p-2)$ has the same number of a's as each other column. So, as each row sum of $\mathcal{C}$ is m^2 (by (i)), the sum of all entries in $\mathcal{C}$ is $m^2(p-1)$, so each column sum is $m^2(p-1)/\ell = 2m$. Therefore (ii) is also satisfied. So $\mathcal{C}$ satisfies conditions (1-3).

It remains to address condition (4). If $m^2 \ge \ell$ then $b = \lfloor m^2/\ell \rfloor \ge 1$ and so by (a) each entry in $\mathcal{C}$ is non-zero. Therefore each color appears on at least one edge between each pair of vertices in G, so obviously each color class is connected.

If $m^2 < \ell$, or equivalently $p > 2m+1$, then $b = 0$ and so each color appears on no edges between some pairs of vertices. In this case, we specifically require that the 1-factorization of K_p be defined by

$$
F_r' = \{\{\infty, r\}, \{r+j(\text{mod } p-1), r-j(\text{mod } p-1)\} \mid j \in \mathbb{Z}_{p/2}\} \text{ for each } r \in \mathbb{Z}_{p-1},
$$

then rename r with v_r for $r \in \mathbb{Z}_{p-1}$ and ∞ with v_{p-1} to form F_r. Then for $0 \le r \le p-2$, $F_r' \cup F_{r+1}'$ is the hamilton cycle $(r, r+2, r-2, r+4, \ldots, r+p-2, r-p+2)$, where all calculations are done modulo $p-1$. Therefore, if $\mathcal{C}$ has the additional property that

for each column j (i.e. for each color j) there exists an $r \in \mathbb{Z}_{p-1}$ such that $c_{r,j} = c_{r+1,j} = 1$ (†)

then color class j in G contains a hamilton cycle, and so is connected. We consider two cases in turn, the second of which requires redefining $\mathcal{C}$.

Case 1.1: $\ell > m^2 > \ell/2$

Since $m^2 > \ell/2$, we know $2m > (p-1)/2$. So by (ii) and since $a = 1$ and $b = 0$, more than half of the $p-1$ entries in each column of $\mathcal{C}$ are 1's. So by the Pigeonhole Principle, $\mathcal{C}$ has the additional property (†), and each color class in G is connected.

Case 1.2: $m^2 \leq \ell/2$

First note that since p is even in Case 1, $\ell/2 = (p-1)m/4 \neq m^2$. Similarly $\ell/m^2 = (p-1)/2m$ and so m^2 does not divide ℓ. Instead of the previous definition, in this case we define $\mathcal{C}$ with three types of rows.

Type I: for $0 \leq s \leq \lfloor \ell/m^2 \rfloor - 1$, let

$$c_{2s,j} = c_{2s+1,j} = \begin{cases} 1 & \text{for } sm^2 \leq j < (s+1)m^2, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Type II: for $s = \lfloor \ell/m^2 \rfloor$, let

$$c_{2s,j} = c_{2s+1,j} = 1 \text{ for } sm^2 \leq j \leq \ell - 1,$$

$$c_{2s,j} = 1 \text{ for } 0 \leq j \leq (s+1)m^2 - \ell - 1,$$

$$c_{2s+1,j} = 1 \text{ for } (s+1)m^2 - \ell \leq j \leq 2(s+1)m^2 - 2\ell - 1, \text{ and}$$

$c_{2s,j}$ and $c_{2s+1,j}$ are 0 for each remaining value of $j \in \mathbb{Z}_\ell$.

Type III: for $2\lfloor \ell/m^2 \rfloor + 2 \leq r \leq p - 2$, let

$$c_{r,j(\mathrm{mod}\ \ell)} = \begin{cases} 1 & \text{if } rm^2 - 2\ell \leq j \leq (r+1)m^2 - 2\ell - 1, \text{ and} \\ 0, \end{cases}$$

where the last range on j is determined by noting that

$$(2(\lfloor \ell/m^2 \rfloor + 1)m^2 - 2\ell) + (r - (2\lfloor \ell/m^2 \rfloor + 2))m^2 = rm^2 - 2\ell. \tag{$*$}$$

Since each row of $\mathcal{C}$ clearly contains m^2 1's, (i) is satisfied, and obviously (iii) is satisfied since each entry in $\mathcal{C}$ is 0 or 1. The first two types of rows ensure that in each column of $\mathcal{C}$, the first 1 is immediately followed by another 1, so each color class is connected (by †). Types II and III

are then defined so that the remaining 1's in $\mathcal{C}$ sweep across its columns in a cyclic fashion, so again the number of 1's in each pair of columns of $\mathcal{C}$ clearly differs by at most one. Since the sum of the entries in $\mathcal{C}$ is $(p-1)m^2 = 2\ell m$ (by (i)), it follows that each column sum is $2m$, so (ii) is satisfied. So giving G the edge-coloring determined by $\mathcal{C}$ in each case ensures that properties (1-4) are satisfied as required.

Case 2: p is odd

When p is odd there exists a hamilton decomposition $\{H_0, \ldots, H_{(p-3)/2}\}$ of K_p. In this case we define color vectors $c_0, \ldots, c_{(p-3)/2}$, then for each $\{u,v\} \in E(H_i)$ we define $c(u,v) = c_i$ for $0 \le i \le (p-3)/2$. As in the previous case, we construct a coloring array $\mathcal{C}$, the rows of which are the color vectors. However, since each hamilton cycle has *two* edges that meet each vertex, condition (ii) is met by ensuring that each column sum is m. We still require each row sum to be m^2 to meet condition (i), and each entry in $\mathcal{C}$ must be within one of each other entry in $\mathcal{C}$ to meet (iii).

Define $\mathcal{C}$ by letting

$$
c_{i,j} = \begin{cases} a & \text{if } j \in \{i\alpha, i\alpha+1, \ldots, (i+1)\alpha - 1\}, \text{ (reducing each calculation modulo } \ell), \text{ and} \\ b & \text{otherwise,} \end{cases}
$$

for $0 \le i \le (p-3)/2$, where $a = \lfloor m^2/\ell \rfloor + 1, b = \lfloor m^2/\ell \rfloor$, and $\alpha = m^2 \pmod{\ell}$. Then each row sum is m^2 since $\alpha a + (\ell - \alpha)b = m^2$, so (i) is satisfied. Again, the a's are placed in $\mathcal{C}$ so that the number of a's in each pair of columns is within one. So since the sum of all entries in $\mathcal{C}$ is $m^2(p-1)/2 = \ell m$, each column sum is exactly m; thus (ii) is satisfied. Clearly (iii) is satisfied. Finally, since each column sum in $\mathcal{C}$ is $m \ge 1$, each color appears on the edges of at least one hamilton cycle in G, so each color class in G is connected. So G satisfies properties (1-4) as required.

Therefore, in each case the result follows by applying 2.1 to G. $\qquad\square$

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