Multidecomposition of the complete graph into graph pairs of order 4 with various leaves

Atif Abueida [*] Sally Clark [†] David Leach [‡]

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Abstract

A *graph-pair of order* t is two non-isomorphic graphs G and H on t non-isolated vertices for which $G \cup H \cong K_t$ for some integer $t \geq 4$. Given a graph-pair (G, H), we say (G, H) divides some graph K if the edges of K can be partitioned into copies of G and H with at least one copy of G and at least one copy of H. We will refer to this partition as a (G, H)-*multidecomposition* of K.

In this paper, we consider the existence of multidecompositions of $K_n - F$ for the graph-pair of order 4 where F is a Hamiltonian cycle, a 1-factor or almost 1-factor.

1 Introduction

The authors in [1, 3] defined a *graph-pair of order* t as two non-isomorphic graphs G and H on t non-isolated vertices for which $G \cup H \cong K_t$ for some integer $t \geq 4$. For a given graph-pair (G, H), we say (G, H) divides K_n if the edges of K_n can be partitioned into copies of G and H with at least one copy of G and at least one copy of H. We will refer to this partition as a (G, H)-*multidecomposition*. The authors in [1, 2, 3] settled the problem of multidecomposition of K_n and λK_n into graph pairs of order 4 and 5. We will consider simple graphs of the form $K_n - F$, where $K_n - F$ denotes the graph K_n from which a Hamiltonian Cycle, a 1-factor or almost 1-factor has been removed. In this paper we will consider multidecomposition into the graph pair G and H whose union is the complete graph K_4.

Let G be a 4-cycle. Let H be a pair of K_2's. A 4-cycle on vertices $w, x, y,$ and z is denoted as $\{w, x, y, z\}$.

The *difference* of the edge joining vertices u and v is defined to be $\min\{u - v, v - u\}$, reducing modulo n.

2 Hamilton Cycle Leave

Theorem 1 *Let F be a Hamilton cycle on K_n. $K_n - F$ has a (G, H)-multidecomposition if and only if $n \equiv 0$ or 3 (mod 4) and $n \geq 7$.*

[*]**Corresponding author:** atif.abueida@notes.udayton.edu, Department of Mathematics, University of Dayton, Dayton, OH 45469-2316. Acknowledges support from Research Council, UD.

[†]sclark@bsc.edu, Division of Science and Mathematics, Birmingham-Southern College, 900 Arkadelphia Road, Birmingham , AL 35254

[‡]cleach@westga.edu, Department of Mathematics, University of West Georgia, Carrollton, GA 30118

Proof: Let $E = |E(K_n - F)|$. Then $E = \dfrac{n(n-1)}{2} - n = \dfrac{n^2-n}{2} - \dfrac{2n}{2} = \dfrac{n^2-3n}{2}$.

Since G and H both have an even number of edges, E must also be even. Thus $n^2 - 3n \equiv 0$(mod 4) $\Rightarrow n(n-3) \equiv 0$ (mod 4) $\Rightarrow n \equiv 0$ or 3 (mod 4).

Case $n \equiv 0$ (mod 4):

Let $V(K_n) = \mathbb{Z}_n$. The edges of K_n have differences $\{1, 2, \ldots, \frac{n}{2}\}$. Note that there are exactly n edges of each difference, except for the difference $\frac{n}{2}$ which has $\frac{n}{2}$ edges, and that $\frac{n}{2}$ is even because $n \equiv 0$ (mod 4). We will begin by constructing a 4-cycle decomposition of K_n with a 1-factor leave. We will do this by exploiting the fact that a 4-cycle can be constructed by using four edges whose differences sum to n. We break the 4-cycles into Type I and Type II.

Type I: For each $\alpha \in \{1, \ldots, \frac{n}{4} - 1\}$ there is a unique $\beta \in \{\frac{n}{4} + 1, \ldots, \frac{n}{2} - 1\}$ such that $\alpha + \beta = \frac{n}{2}$. The vertices $\{i, i+\alpha, i+\frac{n}{2}, i+\alpha+\frac{n}{2}\}$ form a 4-cycle that uses exactly two edges of difference α and two of difference β. This 4-cycle is a base block in the decomposition, and generates $\frac{n}{2}$ 4-cycles, one for each value of $i \in \{0, \ldots, \frac{n}{2} - 1\}$. This uses all the edges of K_n except for those with difference $\frac{n}{2}$ or $\frac{n}{4}$.

Type II: The edges of difference $\frac{n}{4}$ can be partitioned into 4-cycles by using the base block $\{i, i+\frac{n}{4}, i+\frac{n}{2}, i+\frac{3n}{4}\}$. This base block generates a 4-cycle for each value of $i \in \{0, \ldots, \frac{n}{4} - 1\}$, using all n edges of difference $\frac{n}{4}$.

The only edges that remain are the $\frac{n}{2}$ edges of difference $\frac{n}{2}$. These form a 1-factor leave. This decomposition can easily be modified to give a (G, H)-decomposition of K_n with a Hamilton cycle leave. This is accomplished by breaking up the Type I 4-cycles generated by the base block $\{i, i+1, i+\frac{n}{2}, i+\frac{n}{2}+1\}$. These 4-cycles consist of edges of difference 1 and $\frac{n}{2} - 1$. The edges of difference 1 form a Hamilton cycle, and the edges of differences $\frac{n}{2} - 1$ can be trivially partitioned into copies of H. Similarly, the $\frac{n}{2}$ edges of the 1-factor leave can be partitioned into $\frac{n}{4}$ copies of H.

Note that in this case the construction gives $\frac{n^2-6n}{8}$ copies of G. Since each copy of G can be decomposed into two copies of H, one can decompose all but one copy of G into copies of H.

Case $n \equiv 3$(mod 4):

If $n \equiv 3$(mod 8), then the number of differences is is $\frac{n-1}{2} \equiv 1$(mod 4). The difference 1 edges form the Hamilton cycle and we group the remaining differences into sets of four, each of which will generate copies of G. The groups of differences are defined by $\{4i+2, 4i+3, 4i+4, 4i+5\}$ for $0 \le i \le \frac{n-11}{8}$. Each group is used to build the base block $\{j, j+(4i+2), j+(8i+7), j+(4i+4)\}$. For $0 \le j \le n-1$, each base block will give n different copies of G.

The construction gives $\frac{n(n-3)}{8}$ copies of G. Since any copy of G can be decomposed into two copies of H, one can decompose any number (between 1 and $\frac{n^2-3n-8}{8}$) copies of G into copies of H.

If $n \equiv 7$(mod 8), then the number of differences is $\frac{n-1}{2} \equiv 3$(mod 4). The difference 1 edges form the Hamilton cycle leave. The differences 2 and $\frac{n-1}{2}$ edges will each form a Hamilton cycle. Each of these Hamilton cycles can be decomposed into $\frac{n-1}{2}$ copies of H with one edge leftover. Without loss of generality we can assume that the two leftover edges have distinct vertices. Hence they can be paired to form another copy of H. We group the remaining differences into sets of four, each of which will generate copies of G. The groups of differences are defined by $\{4i+3, 4i+4, 4i+5, 4i+6\}$ for $0 \le i \le \frac{n-15}{8}$. Each group is used to build the base block $\{j, j+(4i+3), j+(8i+9), j+(4i+5)\}$. For $0 \le j \le n-1$, each base block will give n different copies of G.

The construction gives $\frac{n(n-7)}{8}$ copies of G. Since any copy of G can be decomposed into two copies of H, one can decompose any number (between 0 and $\frac{n^2-7n-8}{8}$) copies of G into copies of H.■

3 1-Factor Leave

Theorem 2 *Let F be a 1-factor subgraph of K_n. $K_n - F$ has a (G,H)-multidecomposition if and only if $n \equiv 0$ or 2 $(mod\ 4)$ and $n \geq 6$. Moreover, if x is an integer between 1 and $\frac{1}{8}(n^2 - 2n) - 1$, then there exists a (G,H)-multidecomposition of $K_n - F$ having exactly x copies of G.*

Proof: (Necessity) In order for K_n to have a 1-factor, n must be even. $K_2 - F$ has no edges, and $K_4 - F$ consists of only a single copy of G.

(Sufficiency) In [4], Alspach and Gavlas show that $K_n - F$ has a G decomposition if and only if $|E(K_n - F)|$ is divisible by 4.

The number of edges in $K_n - F$ is

$$\frac{n(n-1)}{2} - \frac{n}{2} = \frac{n^2 - 2n}{2} = \frac{1}{2}(n^2 - 2n)$$

If $n = 2j$, then

$$|E(K_n - F)| = \frac{1}{2}((2j)^2 - 2(2j)) = \frac{1}{2}(4j^2 - 4j) = 2(j^2 - j)$$

$j^2 - j$ must be even, so $|E(K_n - F)| \equiv 0(\text{mod } 4)$ and hence there is a G-decomposition of $K_n - F$ consisting of exactly $\frac{n^2-2n}{8}$ copies of G. Since any copy of G can be decomposed into two copies of H, one can decompose any number (between 1 and $\frac{n^2-2n}{8} - 1$) of copies of G into two copies of H. ■

4 Almost 1-Factor Leave

For odd n, we define an *almost* 1*-factor* of K_n to be a 1-factor on K_{n-1}.

Theorem 3 *Let F be an almost 1-factor of K_n. Then $K_n - F$ has a (G,H)-multidecomposition if and only if n is odd and $n \geq 7$. Moreover, if x is an integer between 1 and $\frac{n^2-4n+3}{8}$ then there exists a (G,H)-multidecomposition of $K_n - F$ having exactly x copies of G.*

Proof:

(Necessity) $K_3 - F$ is a path on three vertices, which contains no copy of G or H as a subgraph. It can be shown that $K_5 - F$ has no (G,H)-multidecomposition.

(Sufficiency) Let $V(K_n - F) = \{0, 1, \ldots, n-2\} \cup \{\infty\}$. The edges of $K_n \setminus \{\infty\}$ have differences $\{1, \ldots, \frac{n-1}{2}\}$. In both cases the half difference $h = \frac{n-1}{2}$ makes up the almost 1-factor.

Case $n \equiv 3(\text{mod } 4)$: In this case there is no quarter difference, so h is odd and the remaining $h-1$ differences can be paired up as $(i, h-i)$ for $2 \leq i \leq \frac{h-1}{2}$. Each such pair gives a base block $(0, i, h, i+h)$ that can be rotated to generate $\frac{n-1}{2}$ copies of G.

We now partition the edges of differences 1 and $h-1$, and the edges incident with ∞ into copies of G and H.

The base block $(\infty, 0, 1, h)$ is rotated halfway around to generate $\frac{n-1}{2}$ copies of G.

The remaining edges form a cycle of length h, with pendant edge at each vertex on the cycle. Those edges can be easily decomposed into copies of H.

Case $n \equiv 1(\text{mod } 4)$: In this case h is even, so there is a quarter difference. Take the quarter difference $\frac{h}{2}$ to create the base block $(0, \frac{h}{2}, h, \frac{3h}{2})$, which is rotated one quarter of the way around to generate $\frac{n-1}{4}$ copies of G. The remaining edges are partitioned the same way as in case $n \equiv 3(\text{mod } 4)$.

Note that the construction gives $\frac{n^2-4n+3}{8}$ copies of G. Since each copy of G can be decomposed into two copies of H, one can decompose all but one of the copies of G into copies of H.■

In this paper we settled the existence of the multidecomposition of the complete graph into graph pairs of order 4 with a leave consisting of a Hamiltonian cycle, a 1-factor or almost 1-factor. A nice consequent result will be to show that there exists a multidecomposition of the complete graph into graph pairs of order 4 with a general 2-factor leave.

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