4.4 L'Hopital's Rule & Indeterminate Forms

1 Indeterminate Quotients: $\frac{0}{0}$ and $\frac{\infty}{\infty}$

The forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$ are called **indeterminant**. They are undefined, and *limits* that have either form can take on any value from $-\infty$ to ∞.

To evaluate limits of these two forms, we can use L'Hopital's Rule:

L'Hopital's Rule

Let f and g are differentiable functions, and $g'(x) \neq 0$ near a (except possibly at a).

$$\text{If } \lim_{x \to a} f(x) = 0 \text{ and } \lim_{x \to a} g(x) = 0$$

or

$$\text{If } \lim_{x \to a} f(x) = \pm\infty \text{ and } \lim_{x \to a} g(x) = \pm\infty,$$

$$\text{Then } \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

1.1 Some Basic Examples

1.1.1 Example:

$$\lim_{x \to 3} \frac{x^2 + x - 12}{x^2 + 2x - 15}$$

1.1.2 Example:

$$\lim_{x \to 0} \frac{\cos x - 1}{x}$$

1.1.3 Example:

$$\lim_{x \to 0} \frac{\sin 6x}{x^3 + 2x}$$

1.1.4 Example:

$$\lim_{x\to\infty} \frac{3x^2+x+1}{4x^2+9x-5}$$

Warning: Don't use L'Hopital's Rule if you don't have an indeterminate form!

1.1.5 Example:

$$\lim_{x\to 2} \frac{x^2+2x}{x+2}$$

Note: Sometimes you have to use L'Hopital's Rule more than once.

1.1.6 Example:

$$\lim_{x\to\infty} \frac{x^3+x}{e^{2x}+x}$$

1.1.7 Example:

$$\lim_{x\to\infty} \frac{x\ln x}{x^2+x+1}$$

1.1.8 Example:

$$\lim_{x\to\infty} \frac{\ln x}{x+2}$$

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2 Growth rates of different types of functions

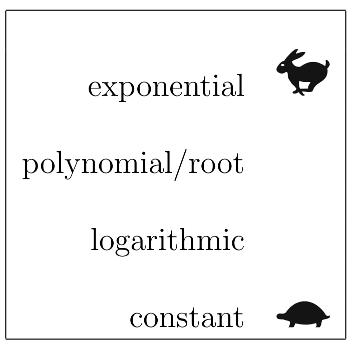
Suppose we have a limit of the form

$$\lim_{x\to\infty} \frac{T(x)}{B(x)} \text{ that goes to } \tfrac{\infty}{\infty}.$$

This is a race:

- $T(x)$ tries to make the limit ∞ or $-\infty$
- $B(x)$ tries to make the limit 0.
- The faster function wins.

Certain classes of functions grow faster than others, summarized in this chart:



2.0.1 Examples where $T(x)$ and $B(x)$ are from different classes.

$$\lim_{x\to\infty} \frac{x^9+4x^6-9x+4}{e^x}$$

$$\lim_{x\to\infty} \frac{e^x}{x^9+4x^6-9x+4}$$

$$\lim_{x\to\infty} \frac{5x}{\ln x}$$

$$\lim_{x\to\infty} \frac{\sqrt{x}}{\ln x}$$

2.0.2 More examples.

If $T(x)$ or $B(x)$ can be split into multiple terms, then we only need to look at the fastest term of each. (terms are things that are added or subtracted together)

$$\lim_{x \to \infty} \frac{x^2 + \ln x}{\ln x + e^x}$$

$$\lim_{x \to \infty} \frac{x^4 + e^{3x}}{9x^9 + 100}$$

2.0.3 Examples where $T(x)$ and $B(x)$ are from the same class.

In these cases, it depends on which class.

$$\lim_{x \to \infty} \frac{x^2 + x + 1}{3x^3 + x}$$

$$\lim_{x \to \infty} \frac{e^{3x}}{e^{2x}}$$

$$\lim_{x \to \infty} \frac{2^x}{3^x}$$

$$\lim_{x \to \infty} \frac{\sqrt{x^2 + 1}}{\sqrt{x^2 + x}}$$

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3 Indeterminate Products: $0 \cdot \infty$

For these, we rewrite the product as a quotient.

3.0.1 Example:

$$\lim_{x \to 0^+} x \ln x$$

3.0.2 Example:

$\lim_{x\to-\infty} x^2e^x$

3.0.3 Example:

$\lim_{x\to0^+} \sin x \ln x$

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4 Indeterminate Differences: $\infty - \infty$

Suppose we have a limit of the form

$$\lim_{x\to\infty} F(x) - G(x) \quad \text{that goes} \quad \text{to } \infty - \infty.$$

This is a race:

- $F(x)$ tries to make the limit ∞.
- $B(x)$ tries to make the limit $-\infty$.
- The faster function wins.

Some of these are relatively easy:

4.0.1 Example:

a. $\lim_{x\to\infty} 3x^4 - 4x^2$

b. $\lim_{x\to\infty} \ln x - 2x$

c. $\lim_{x\to\infty} e^x - x^5$

Some are tougher:

4.0.2 Example:

$\lim_{x \to \frac{\pi}{2}^-} \sec x - \tan x$

4.0.3 Example:

$\lim_{x \to \infty} \sqrt{x^2 + 12x} - \sqrt{x^2 + 5x}$

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5 Indeterminate Powers: 0^0, 1^∞, and ∞^0