4.3 What Derivatives Tell About the Shape of a Graph

1 Increasing and Decreasing: Using $f'(x)$

Recall that for a function $f(x)$, we can evaluate $f'(c)$ at a number to determine the slope of the function at $x = c$.

Fact: The only places where a function can change from increasing to decreasing or vice-versa is at a critical number.

To find the intervals where $f(x)$ is increasing or decreasing, do the following:

1. Find the critical numbers of $f(x)$.
2. Cut the real number line into intervals at the critical numbers.
3. For each interval, pick a test value and check to see if $f'(x)$ is positive or negative.
    - if $f'(x)$ is positive at a test value, then then $f(x)$ is increasing on that test value's entire interval.
    - if $f'(x)$ is negative at a test value, then then $f(x)$ is decreasing on that test value's entire interval.

1.0.1 Example:

Find the intervals where $f(x) = 2x^3 - 2x^2 - \frac{7}{2}x + 1$ is increasing or decreasing.

The First Derivative Test: Suppose that c is a critical number of a continuous function f.

1. If f' changes from positive to negative at c, then f has a local maximum at $x = c$.
2. If f' changes from negative to positive at c, then f has a local minimum at $x = c$.
3. If the sign of f' does not change at c, then f has no local max or min at $x = c$.

1.0.2 Example:

Find the intervals where $y = x^3 + x$ is increasing or decreasing, and find any local max or mins.

1.0.3 Example:

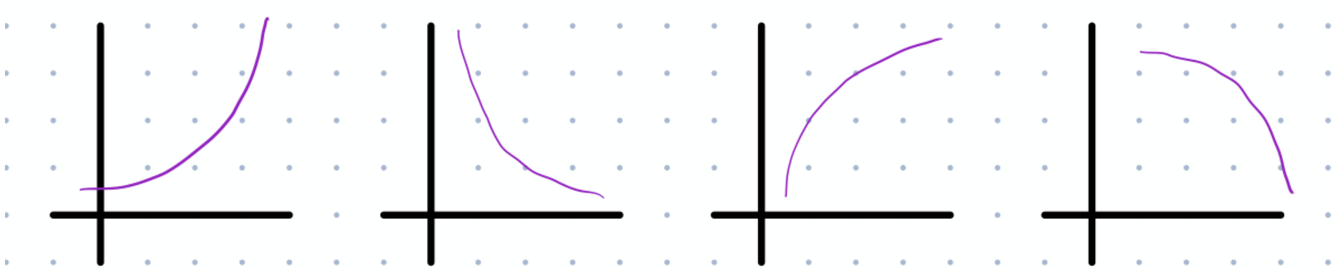
Find the intervals where $y = x^2 e^{\left(\frac{x}{2}\right)}$ is increasing or decreasing, and find any local max or mins.

1.0.4 Example:

Find the intervals where $y = x^{\frac{4}{3}} - 2x^{\frac{2}{3}}$ is increasing or decreasing, and find any local max or mins.

2 Concavity: Using $f''(x)$

Definition: A graph is **concave up** at a point if the tangent at that point lies below the graph, and **concave down** at a point if the tangent at that point lies above the graph.



Concave up, shaped like a cup,
Concave down, shaped like a frown

https://www.desmos.com/calculator/fs3lx8gptb
Consider the graphs of $y = x^2$ and $y = -x^2$

Fact: The second derivative, $f''(x)$ tell the **concavity** of $f(x)$.

- if $f''(c)$ is positive, then the graph is concave up at $x = c$.
- if $f''(c)$ is negative, then the graph is concave down at $x = c$.

To find the intervals where $f(x)$ is concave up or concave down, do the following:

1. Find the hypercritical numbers of $f(x)$. (that's my nonstandard term for numbers that make the **second derivative** zero or undefined.)
2. Cut the real number line into intervals at the hypercritical numbers.
3. For each interval, pick a test value and check to see if $f''(x)$ is positive or negative.
   - if $f''(x)$ is positive at a test value, then then $f(x)$ is concave up on that test value's entire interval.
   - if $f''(x)$ is negative at a test value, then then $f(x)$ is concave down on that test value's entire interval.

Definition: A point on a graph where the concavity changes sign is called an **inflection point**.

2.0.1 Example:

Find the intervals where $f(x) = \frac{4}{3}x^3 + x^2 - 5x$ is concave up or concave down, and find any inflection points.

2.0.2 Example:

Find the intervals where $f(x) = 3x^4 - 4x^3 + x$ is concave up or concave down, and find any inflection points.

2.0.3 Example:

Find the intervals where $y = 4xe^{x/2}$ is concave up or concave down, and find any inflection points.

2.1 Second Derivative Test

The Second Derivative Test: Suppose that the second derivative, f'' is continusou near c.

1. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$
2. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$
3. If $f'(c) = 0$ and $f''(c) = 0$, then the test does not tell you anything. (There may be a max, min, or neither at $x = c$).

2.1.1 Example:

Use the second derivative test to find any relative extrema of $f(x) = 4x^3 - 3x^2$

2.1.2 Example:

What does the second derivative test tell about the graphs $f(x) = (x+3)^4$ and $g(x) = -5x^5$?