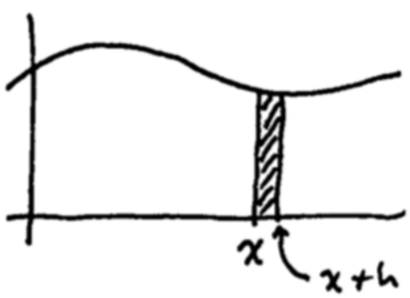
5.3 The Fundamental Theorem of Calculus

1 FToC Part I

Let $F(x) = \int_a^x f(t)\,dt$ (Note that $F(x)$ is a function of x, not t.)

$F(x)$ gives the area under the curve from a to x. Let's look at a thin slice right at x.



So $F(x+h) - F(x) \approx f(x) \cdot h$, which means $f(x) \approx \dfrac{F(x+h) - F(x)}{h}$

As $h- \to 0$, the approximation becomes exact, meaning that

$$f(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h},$$

which implies that $f(x) = F'(x)$.

Fundamental Theorem of Calculus Part I:

If f is continuous on the interval $[a, b]$, then the function

$$F(x) = \int_a^x f(t)\,dt, \quad a \leq x \leq b$$

is also continuous on $[a, b]$, differentiable on (a, b), and

$$F'(x) = f(x).$$

Note: Part 1 of the FToC essentially says that the processes of integration and differentiation "undo" each other. It can be written more compactly using Leibniz notation, as

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

Note that the constant a does not factor in at all.

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2 FToC Part II

Fundamental Theorem of Calculus Part II:

If f is continuous on the interval $[a, b]$, then

$$\int_a^b f(x)\,dx, = F(b) - F(a)$$

where F is an antiderivative of f.

Notation: We denote $F(b) - F(a)$ using a vertical bar or large brackets with b subscripted and a superscripted:

$$F(x)\Big|_a^b = F(b) - F(a),$$

$$\Big[F(x)\Big]_a^b = F(b) - F(a),$$

2.1 FToC Part II Examples:

In each example, evaluate the definite integral

2.1.1 Example:

$$\int_0^3 9 - x^2\,dx$$

2.1.2 Example:

$$\int_0^{\pi} \sin x \; dx$$

2.1.3 Example:

$$\int_0^{4} 1 + 3x \; dx$$

2.1.4 Example:

$$\int_1^{e^3} \frac{1}{x} \; dx$$

2.1.5 Example:

$$\int_1^{2} \frac{x^2 + 4x}{x} \; dx$$