3.6 Derivatives of Logarithms

1 Derivative of ln(x)

Facts: $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$ and $\dfrac{d}{dx}\big[\ln(f(x))\big] = \dfrac{f'(x)}{f(x)}$ (Memorize this)

1.0.1 Example:

Find the derivative of $y = \ln(\sin x)$

1.0.2 Example:

Find the derivative of $y = x^3 \ln(x)$

1.0.3 Example:

Find the derivative of $y = \ln(x^3 \sin x)$

1.0.4 Example:

Find the derivative of $y = \cos x \ln(x^2)$

2 Derivative of other logarithms

Facts: $\dfrac{d}{dx}(\log_b x) = \dfrac{1}{x \ln b}$ and $\dfrac{d}{dx}\left[\log_b(f(x))\right] = \dfrac{f'(x)}{\ln b \cdot f(x)}$

2.0.1 Example:

Find the derivative of $y = \log_2(x^2 + 5x + 1)$

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3 Review of log rules

1. $\log_a(M^N) = N \log_a(M)$
2. $\log_a(MN) = \log_a M + \log_a N$
3. $\log_a(M/N) = \log_a M - \log_a N$

We can use these rules to expand a complicated logarithm into several simpler logarithms.

3.0.1 Example:

$$\ln\left(\frac{xy^3}{z^4}\right) = \ln x + 3\ln y - 4\ln z$$

3.0.2 Examples:

Use the logarithm Rules to write as a sum or difference of two or more logaritms with no exponents:

$$\ln\left[\frac{x^4(x-7)^5}{\sqrt{2x-9}}\right] \qquad \log\left[x^6(x+7)^4\right]$$

3.0.3 Examples:

Use the logarithm Rules to write as a single logarithm:

$2\ln x^3 - 3\ln(2x+1) + \ln(x^2+1)$ $\qquad$ $\ln(2x+4) - 3\ln(x+3) - \frac{1}{2}\ln x$

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4 Using Log Rules to make derivatives easier

Suppose we are asked to find the derivative of $y = \ln\left(\dfrac{x^2}{\sin x}\right)$. It is HIGHLY preferable to use the log rules to expand it first, then take the derivative.

4.0.1 Example:

Find the derivative of $y = \ln(x^3 \cos x\sqrt{3x^2+4})$

4.0.2 Example:

Find the derivative of $y = \ln \sqrt{\dfrac{5x-9}{1-x^2}}$

4.0.3 Example:

Find the derivative of $y = \ln \dfrac{x^2 e^{2x}}{x^2+1}$

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5 Logarithmic Differentiation

5.1 Making hard derivatives easier

Taking the derivative of the function below would be painful. It would be easier if there were an ln in front of the fraction.

$$y = \frac{x^2(1-x)^9}{x^4 \tan x}$$

So let's take the ln of both sides.

$$\ln y = \ln \frac{x^2(1-x)^9}{x^4 \cos x}$$

Now we'll use implicit differentiation. Note that on the left, we have to use the chain rule. (The derivative of $\ln y$ is $\frac{y'}{y}$.

5.1.1 Example:

Use logarithmic differentiation to find the derivative of $y = (1 + x + x^2)^4(x^2 - 4)^7$

5.1.2 Example:

Differentiate the function $y = \dfrac{\sqrt{x}(x^3+9)}{(2x+4)^3}$

5.2 Functions with x both in the base and exponent

The simplest example of this is the function $y = x^x$. The only way we can find y' is by logarithmic differentiation.

5.2.1 Example:

Differentiate $y = (\tan x)^x$

5.2.2 Example:

Find the derivative of $y = (\ln x)^{\sin x}$