# Maximum stack sizes in parallel diffusion on paths and cycles


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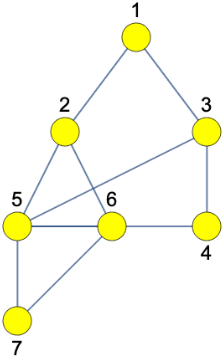
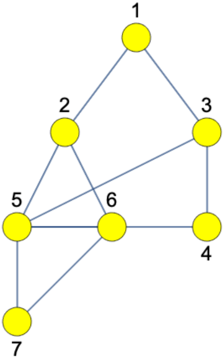
19 May 2023

Integers Conference 2023

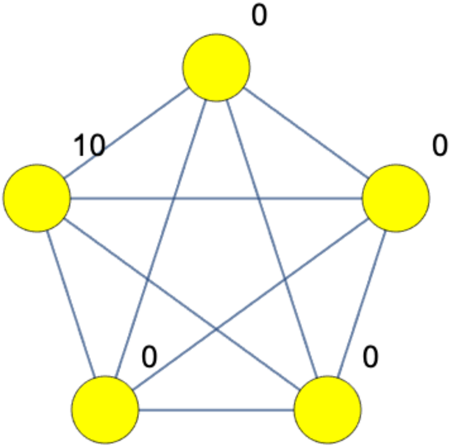
- Diffusion is a chip-firing game on a graph.
- Every vertex has a stack of chips.
- When the graph fires, each vertex sends one chip to each of its poorer neighbors.
- A graph together with an assignment of chips to the vertices is called a *configuration*.



- All vertices fire simultaneously.
- A vertex can have a negative number of chips.
- The set of all configurations on a graph forms a dynamical system.
- Total chipcount is preserved, so repeated firing will ultimately lead to periodic behavior.



# Example: $K_5$



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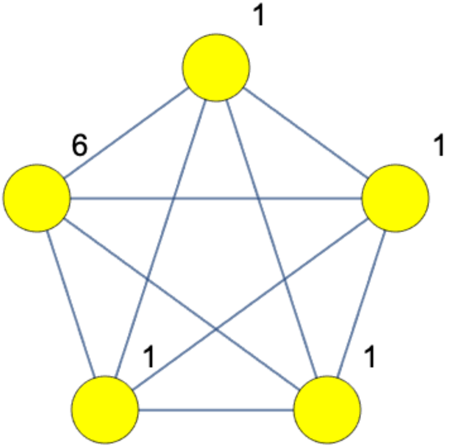

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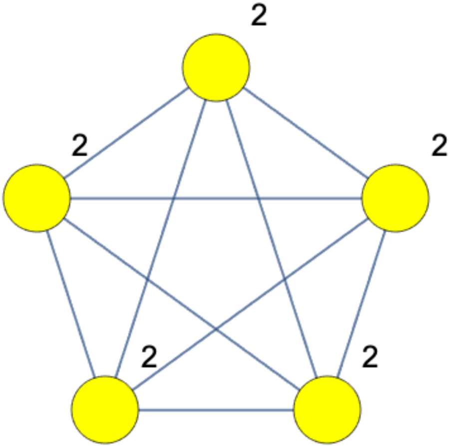
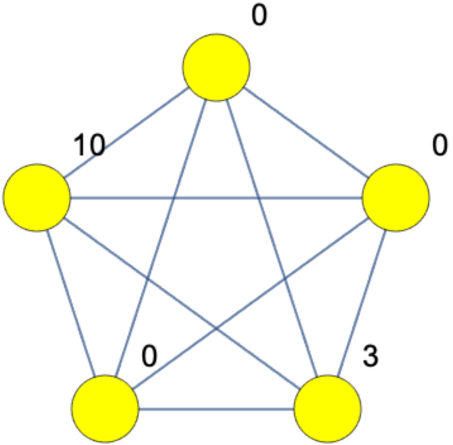



Stable Configuation

# Example: another configuration on $K_5$



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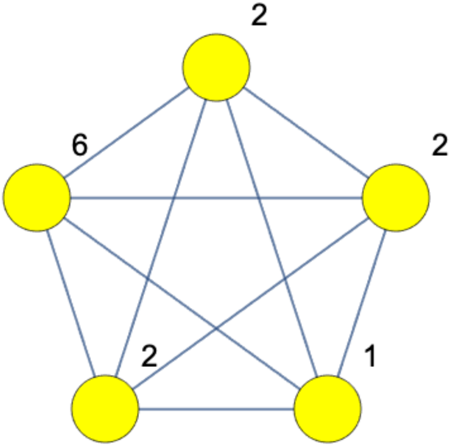

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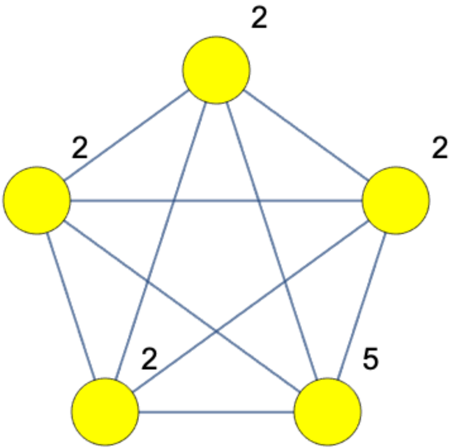

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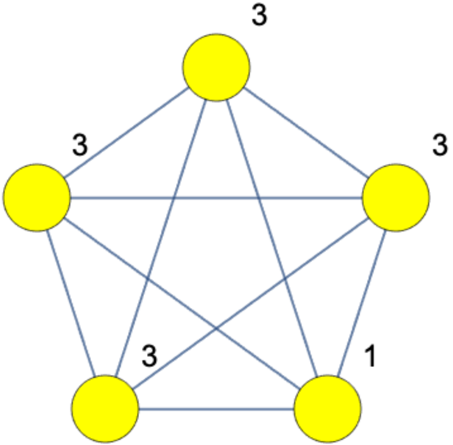


Period 2

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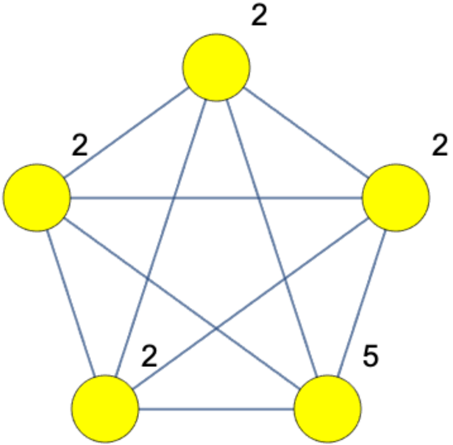
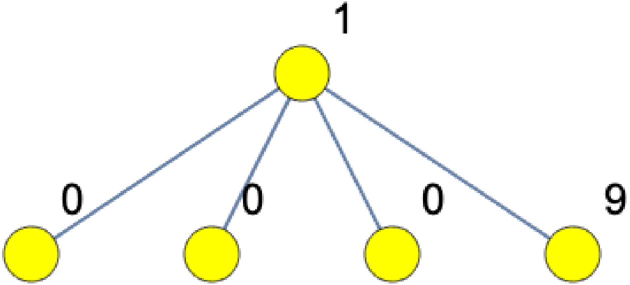



Period 2

# Example: Star $K_{1,4}$



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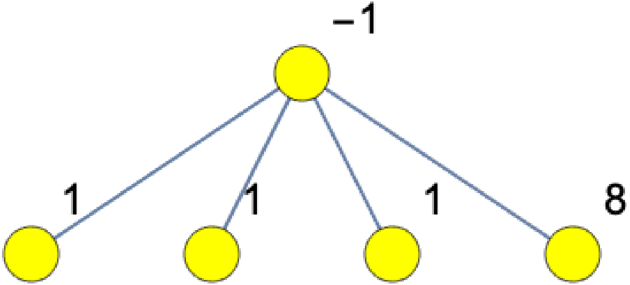

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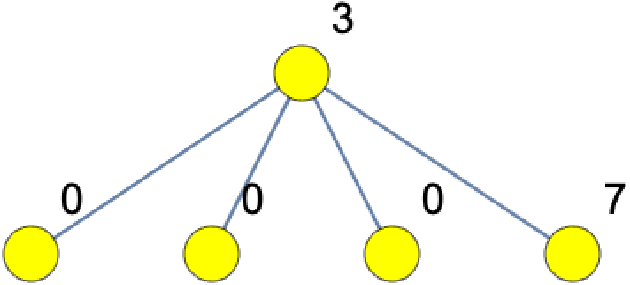

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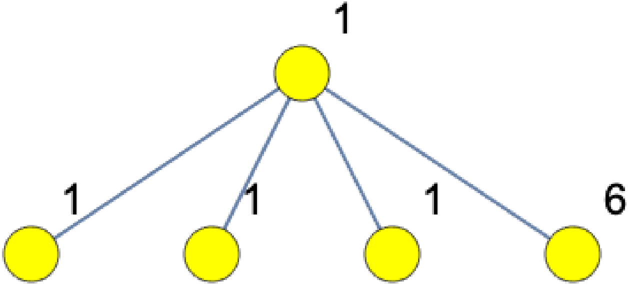

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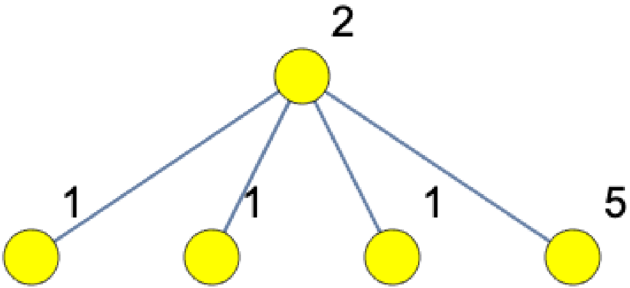

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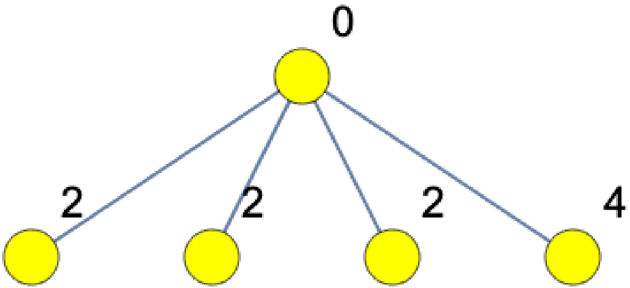


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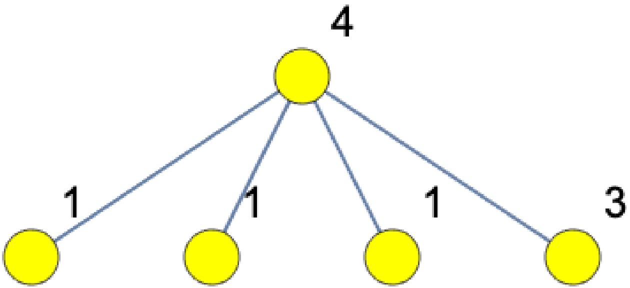


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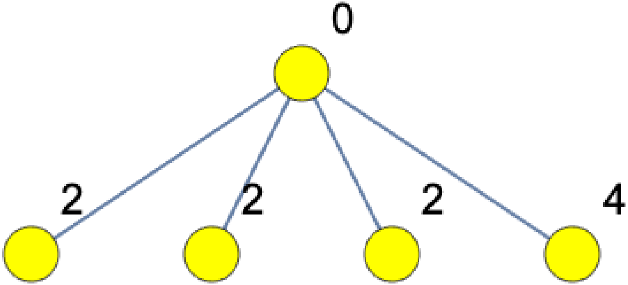


Period 2

# History

- Introduced by C.Duffey, T.F. Lidbetter, M.E. Messinger, R.J. Nowakowski, 2017
  - *A Variation on Chip-firing: the Diffusion Game.*
  - Conjecture: ultimately periodic with period 1 or 2.
- Long and Narayanan proved this conjecture, 2019.
  - *Diffusion on graphs is eventually periodic.*
- Cox, Mullen, and Nowakowski, 2020.
  - *Counting Path Configurations in Parallel Diffusion*

# Definitions

**$p_2$-configuration**

A configuration that is inside of a period of length 2.

**ground configuration**

A configuration in which the poorest vertex has exactly 0 chips.

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A configuration that is inside of a period of length 2.

### ground configuration

A configuration in which the poorest vertex has exactly 0 chips.

### P2G-configuration

For brevity, we will call a configuration that is both a $p_2$ configuration and a ground configuration a *P2G-configuration*.

# Main Question

Given a graph $G$, what is the largest possible stack size in a P2G-configuration on $G$?

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We will look at

- Paths
- Cycles

# Paths: small cases

## P2G-configurations

$P_1$: None. The only ground configuration is $\{0\}$, which is fixed.

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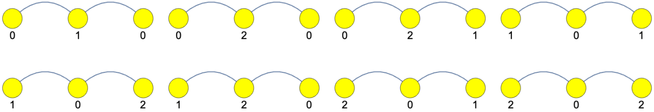
$P_3$: Eight




Largest Stack Size $= 2$

# Paths

**Theorem. CDL 2023**

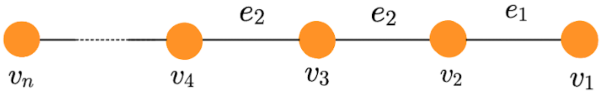
Let $C$ be a P2G-configuration on $P_n$, $n \geq 2$. Then the largest possible stack size in $C$ is $n-1$.

We show this in two parts:

1. Construct a P2GC on $P_n$ with a stack of size $n-1$.
2. Show that it's impossible to construct one with a stack size of $n$.

# Paths

Paths as start with $v_1$ and $e_1$ at the right and indices increase leftward, and assume that $v_1$ has 0 chips.

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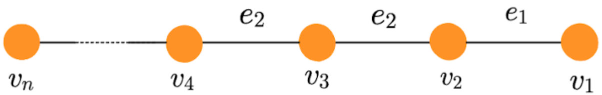


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Configuration $\Rightarrow$ Partial Orientation

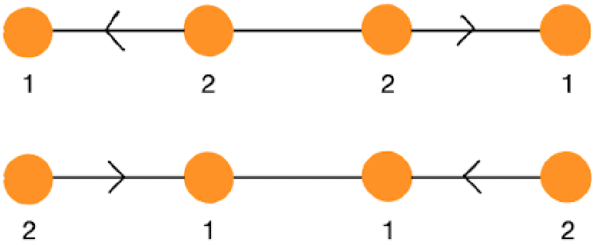
For each edge in the graph,

- leave it undirected if its incident vertices have the same stack size;
- direct the edge toward the vertex with smaller stack size.

# Paths: induced orientations

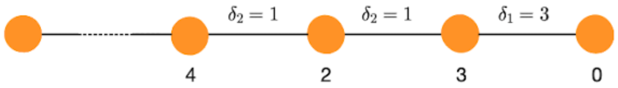
**Theorem LN**

Let $C$ be a $p_2$ configuration on a graph, and let $C'$ be the other $p_2$ configuration in the period. Then every undirected edge in $C$ is also undirected in $C'$, and every directed edge in $C$ has the opposite orientation in $C'$.



# Paths: $\delta$ values

For the edge $e_i$ between $v_i$ and$v_{i+1}$ let $\delta_i = |v_{i+1}| - |v_i|$



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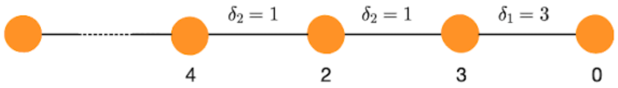


Observation: The $\delta$-values in a $p_2$-configuration are restricted.

- They can't be very large.
- Consecutive $\delta$-values can't differ by much.

# Paths: Lemma-tations on $\delta$-values

## Lemma 1

For any edge $e_i$ in a $p_2$ configuration on a path,

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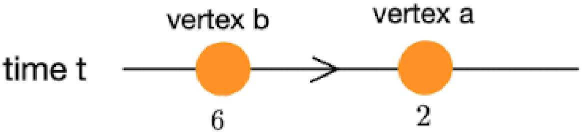

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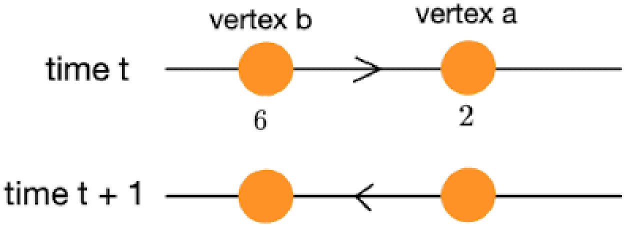

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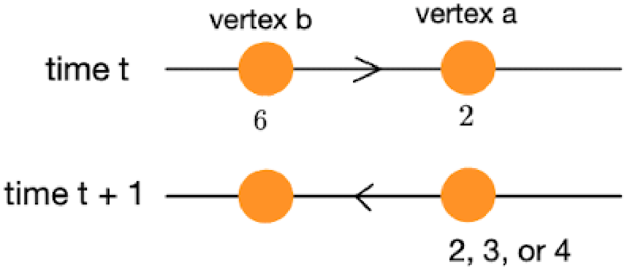

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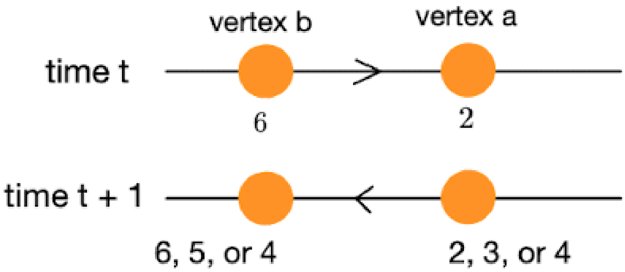

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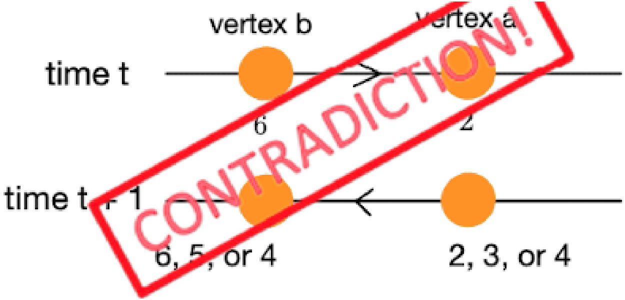

# Paths: Limitations on $\delta$-values

**Lemma 2**

Let $C$ be a $p_2$ configuration on a path. Then for any two consecutive edges $e_i$ and $e_{i+1}$, the ordered pair $(\delta_i, \delta_{i+1})$ cannot be any of the following:

$$\pm(3,3),\ \pm(2,3),\ \pm(3,2),\ \pm(2,2),\ \pm(3,1),\ \pm(1,3),$$

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Proof outline:

- $(0,0)$ is excluded by CMN.
- Others are excluded case-by-case using arguments similar to the previous proof.

# Paths: Usable $(\delta_i, \delta_{i+1})$ pairs

The table below shows all $(\delta_i, \delta_{i+1})$ pairs, with those eliminated by the lemmas grayed out.

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## Corollary

*Let $C$ be a $p_2$ configuration on a path. Then for any two consecutive edges $e_i$ and $e_{i+1}$, $-2 \leq \delta_i + \delta_{i+1} \leq 2$.*

# Paths: Limitations on $\delta$-values

## Lemmas from CMN *

Let $C$ be a $p_2$ configuration on $P_n$. Then ...

- No two consecutive $\delta$ values are 0.
- $\delta_1 \neq 0$ and $\delta_{n-1} \neq 0$.
- No three consecutive deltas have the same sign.
- If $\delta_i = 0$, then $\delta_{i-1}$ and $\delta_{i+1}$ cannot have the same sign.
- $\delta_1$ and $\delta_2$ cannot have the same sign, and
- $\delta_{n-2}$ and $\delta_{n-1}$ cannot have the same sign.
- $\delta_1 \neq \pm 3$ and $\delta_{n-1} \neq \pm 3$. (CDL)
- If $\delta_1 = \pm 2$, then $\delta_2 \neq 0$ (CDL)

*The lemmas in CMN are more general. They are restated here specifically for paths.

# $n-1$ chip construction

If we start with $|v_1| = 0$, then

$$|v_k| = \sum_{i=1}^{k-1} \delta_i.$$

Strategy: Pick the largest $\delta$-values that the lemmas will allow.

Because $\delta_i \leq 3$ and $\delta_i + \delta_{i+1} \leq 2$, the best we can hope for is to increase by roughly one chip per vertex.

# $n-1$ chip construction $n$ even $n=8$

$$\delta_i = \begin{cases} -2 & \text{if } i = 1 \\ 3 & \text{if } i \text{ is even} \\ -1 & \text{if } i > 1 \text{ and } i \text{ is odd.} \end{cases} \tag{1}$$

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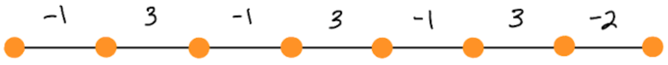

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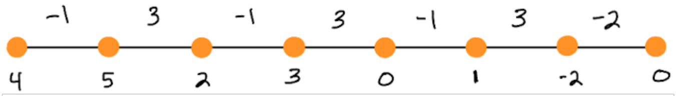

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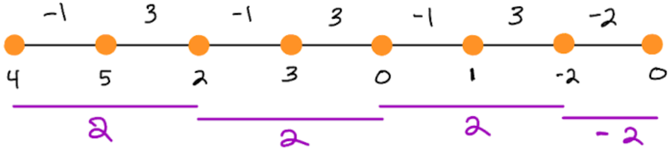

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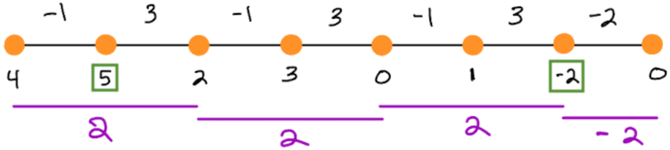

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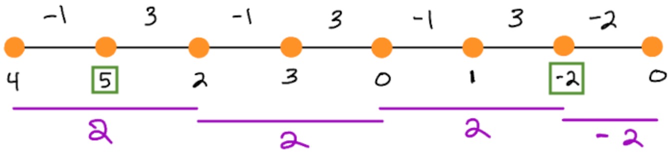


$|v_2| = -2$, and $|v_n| = \frac{n-1}{2} \cdot 2 = n-1$.
Adding two chips to each vertex results in a P2G-configuration with maximum stack size $n-1$.

# $n-1$ chip construction $n$ odd $\qquad n=9$

$$\delta_i = \begin{cases} -2 & \text{if } i = 1 \\ 3 & \text{if } i \text{ is even and } i < n-1 \\ -1 & \text{if } i > 1 \text{ and } i \text{ is odd.} \\ 2 & \text{if } i = n-1. \end{cases} \tag{2}$$

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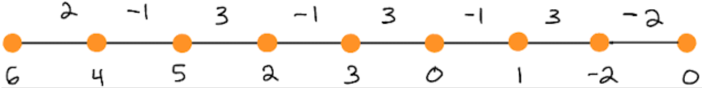

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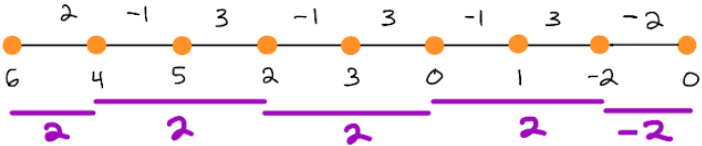
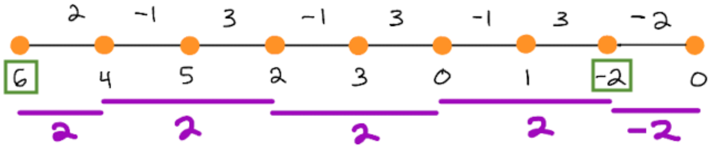


# $n-1$ chip construction $n$ odd $\qquad n=9$

$$\delta_i = \begin{cases} -2 & \text{if } i = 1 \\ 3 & \text{if } i \text{ is even and } i < n-1 \\ -1 & \text{if } i > 1 \text{ and } i \text{ is odd.} \\ 2 & \text{if } i = n-1. \end{cases} \tag{2}$$




$|v_2| = -2$, and $|v_{n-1}| = \frac{n-2}{2} \cdot 2 - 2 = n-3$.
Adding two chips to each vertex results in a P2F-configuration with maximum stack size $n-1$.

# Stack size of $n$ is not possible

## Theorem

*Let $n$ be a positive integer. There is no P2G-configuration on $P_n$ with a stack size of $n$ or greater.*

# Stack size of $n$ is not possible

## Theorem

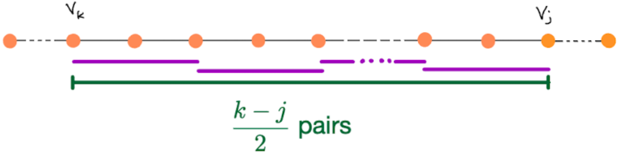
*Let $n$ be a positive integer. There is no P2G-configuration on $P_n$ with a stack size of $n$ or greater.*

Proof sketch:

Suppose that for some $|v_k| - |v_j| \geq n$, where $n \geq k > j \geq 1$

# $k - j$ even

If $k - j$ is even, then partition the $\delta$-values into $\frac{k-j}{2}$ pairs, and consider the pair-sums.




The average pair-sum is

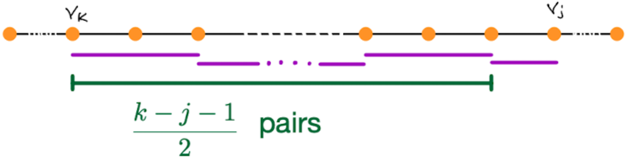
$$\frac{|v_k| - |v_j|}{\frac{k-j}{2}} = \frac{2(|v_k| - |v_j|)}{k-j} \geq \frac{2n}{k-j} > \frac{2n}{n-1} > 2,$$

meaning that at least one pair has $\delta_i + \delta_{i+1} > 2$, a contradiction.

# $k - j$ odd

If $k - j$ is odd, then partition the $\delta$-values into

$$\{\delta_j\}, \{\delta_{j+1}, \delta_{j+2}\}, \ldots, \{\delta_{k-2}, \delta_{k-1}\}$$




The sum of the $\delta$-pairs from $v_{j+1}$ to $v_k$ is $|v_k| - |v_j| + \delta_j$, making the average pair-sum

$$\frac{|v_k| - |v_j| + \delta_j}{\frac{k-j-1}{2}} = \frac{2(|v_k| - |v_j| + \delta_j)}{k-j-1} \geq \frac{2(n-\delta_j)}{k-j-1} \geq \frac{2(n-\delta_j)}{n-2}.$$

# $k - j$ odd

The average $\frac{2(n-\delta_j)}{n-2} > 2$ when $\delta_j < 3$.

If $\delta_j = 3$, then $j \geq 2$, $k - j - 1 \leq n - 3$, so the average is

$$\frac{2(n-\delta_j)}{k-j-1} = \frac{2(n-3)}{k-j-1} \geq \frac{2(n-3)}{n-3} = 2.$$

In all cases we are forced to have a pair of adjacent $\delta$-values that sum to 3 or greater, a contradiction.

# Cycles

From a computer search, the maximum stack size on a P2G-configuration on $C_3$ and $C_4$ are 2 and 3. For $n \geq 5$, the strategy is similar to that for paths.

- start with vertex $v_0$ having zero chips
- choose delta values that will make the stack sizes increase

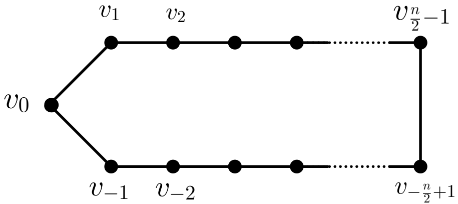
But we can't go all the way around because we'd get a large $\delta$-value when we return to $v_0$. So we need to

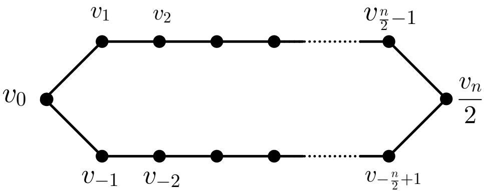
- start with vertex $v_0$ having zero chips
- go in both directions
- meet in the middle with a stack size that is roughly $\frac{n}{2}$

Since cycles don't have endpoints, we have a bit more freedom.

# Cycles:

Label the vertices as shown here

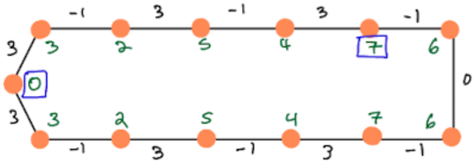



The construction depends on $n \pmod 4$.

# Cycles: $n$ odd $n = 4k + 1$

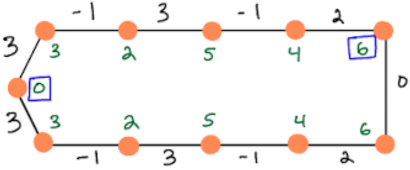
$$\delta_i = \begin{cases} 3 & \text{if } |i| \text{ is odd and } i \leq \frac{n-1}{2} \\ -1 & \text{if } |i| \text{ is even} \\ 0 & \text{for the middle edge} \end{cases} \tag{3}$$




$n = 13$

$$3\left(\frac{n-1}{4}\right) - 1\left(\frac{n-1}{4} - 1\right) = \frac{n+1}{2}$$

$$\delta_i = \begin{cases} 3 & \text{if } |i| \text{ is odd and } i < \frac{n-1}{2} \\ -1 & \text{if } |i| \text{ is even and } i < \frac{n-3}{2} \\ 2 & \text{if } |i| = \frac{n-1}{2} \\ 0 & \text{if } |i| = \frac{n+1}{2} \end{cases} \tag{4}$$

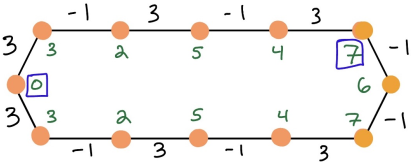



$n = 11$

$$2\left(\frac{n-3}{4}\right) + 2 = \frac{n+1}{2}$$

# Cycles: $n$ even $n = 4k$

$$\delta_i = \begin{cases} 3 & \text{if } |i| \text{ is odd} \\ -1 & \text{if } |i| \text{ is even} \end{cases} \tag{5}$$

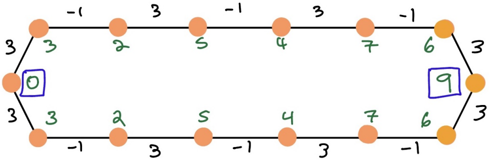



$$n = 12$$

$$3\left(\frac{n}{4}\right) - 1\left(\frac{n}{4} - 1\right) = \frac{n+2}{2}$$

# Cycles: $n$ even $n = 4k + 2$

$$\delta_i = \begin{cases} 3 & \text{if } |i| \text{ is odd} \\ -1 & \text{if } |i| \text{ is even} \end{cases} \tag{6}$$




$n = 14$

$$3\left(\frac{n+2}{4}\right) - 1\left(\frac{n-2}{4}\right) = \frac{n+4}{2}$$

**Theorem**

*Let $C$ be a ground configuration on $C_n$. Then the maximum possible stack size for a vertex in $C$ is*

$$\begin{cases} \frac{n+1}{2} & \text{if } n \text{ is odd} \\ \frac{n+2}{2} & \text{if } n = 4k \\ \frac{n+4}{2} & \text{if } n = 4k+2 \end{cases}$$

# References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Thank you.