UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 1996

MEng Honours Degrees in Computing Part IV
MSc Degree in Foundations of Advanced Information Technology
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the
Diploma of Membership of Imperial College
Associateship of the City and Guilds of London Institute

PAPER 4.73

DOMAIN THEORY AND FRACTALS
Monday, April 29th 1996, 2.00 - 4.00

Answer THREE questions

For admin. only: paper contains
4 questions
4 pages (excluding cover page)

1a Show that every continuous function on a cpo has a least fixed point.

b Let $f : D \to E$ be a monotone map between cpo's D and E. Prove that if f is onto (i.e. $\forall y \in E \exists x \in D.\ f(x) = y$) and if it reflects the order (i.e. $\forall x, x' \in D.\ f(x) \sqsubseteq f(x') \Rightarrow x \sqsubseteq x'$), then f is continuous.

c Let D be a poset and $\mathbf{2}$ the two-element poset $\{\bot, \top\}$ with $\bot \sqsubseteq \top$. Show that the poset of monotone maps from D to $\mathbf{2}$ ordered pointwise (i.e., $f \sqsubseteq g \overset{\text{def}}{\Longleftrightarrow} \forall x \in D.\ f(x) \sqsubseteq g(x)$) is isomorphic with the poset of upper subsets of D ordered by subset inclusion. (A subset $A \subseteq D$ is an *upper subset* by definition if $\forall x, x' \in D.\ x \in A \;\&\; x \sqsubseteq x' \;\Rightarrow\; x' \in D$.)

The three parts carry, respectively, 35%, 25%, 40% of the marks.

2a Define the function space $A \to B$ of two cpo's A and B. Let A_1, A_2, B_1, B_2 be cpo's. Given embedding-projection pairs $(e_1, p_1) : A_1 \lhd B_1$ and $(e_2, p_2) : A_2 \lhd B_2$, construct an embedding-projection pair $(e, p) : (A_1 \to A_2) \lhd (B_1 \to B_2)$ verifying that it is indeed an embedding-projection pair.

b Prove that projections preserve greatest lower bounds of arbitrary subsets when they exist. Show by a counter-example that projections need not preserve finite elements.

c Consider the domain equation

$$D \cong F(D) = (\mathbb{N}_\perp \otimes D)_\perp$$

in the category **CPO**.

i) Find, up to isomorphism, the iterates $D_n = F^n(\{\perp\})$ for $n = 0, 1, 2, 3$ and the corresponding embedding-projection pairs $(e_n, p_n) : D_n \lhd D_{n+1}$ for $n = 0, 1, 2$.

ii) Obtain the solution of the domain equation up to isomorphism.

The three parts carry, respectively, 20%, 20%, 60% of the marks.

Turn over...

3a Given an iterated function system in the plane $\mathbb{R}^2$, describe how one can obtain the attractor of the system as the least fixed point of a continuous function on a cpo of a class of subsets of $\mathbb{R}^2$ ordered by reverse inclusion.

b An iterated function system is given by the two maps $f_1, f_2 : \mathbb{R}^2 \to \mathbb{R}^2$ with

$$f_1((x,y)) = (x/2, y/2) \qquad f_2((x,y)) = (1 + (x/2), 1 + (y/2)).$$

Find the attractor of the system and compute its similarity dimension. Compute the Hausdorff distance between the attractor and the unit circle $\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$.

Find the attractor of the iterated function system $f_1, f_2, f_3, f_4 : \mathbb{R}^2 \to \mathbb{R}^2$ where f_1 and f_2 are as above and $f_3((x,y)) = (2,0)$ and $f_4((x,y)) = (0,2)$ for all $(x,y) \in \mathbb{R}^2$.

c The function fib : $\mathrm{N} \to \mathrm{N}$ is recursively defined by

$$\mathrm{fib}(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ \mathrm{fib}(n-2) + \mathrm{fib}(n-1) & \text{if } n > 1. \end{cases}$$

Construct a function $F : (\mathrm{N}_\perp \to_s \mathrm{N}_\perp) \to (\mathrm{N}_\perp \to_s \mathrm{N}_\perp)$, where $\mathrm{N}_\perp \to_s \mathrm{N}_\perp$ is the strict function space of N, such that its least fixed point gives the function fib. Prove that F is continuous and has a unique fixed point.

The three parts carry, respectively, 25%, 40%, 35% of the marks.

4a Let D and E be Scott domains, and let $a, c \in \mathcal{K}_D$ and $b, d \in \mathcal{K}_E$ be two pairs of finite elements of D and E respectively. Show that $(a \searrow b) \sqcup (c \searrow d)$ exists in the function space $D \to E$ iff either $a \sqcup c$ does not exist or $a \sqcup c$ and $b \sqcup d$ both exist. State clearly any results that you may wish to use in your proof.

b i) Let $(A, \vdash, \Delta)$ be an information system. Define the the *lower*, the *upper* and the *convex* power information systems, $\mathcal{P}_l(A)$, $\mathcal{P}_u(A)$ and $\mathcal{P}_c(A)$ respectively; carefully define the orders that you use.

ii) Let $R : A \to B$ be an approximable mapping between two information systems $A = (A, \vdash_A, \Delta_A)$ and $B = (B, \vdash_B, \Delta_B)$. Construct the the approximable mapping $\mathcal{P}_u(R) : \mathcal{P}_c(A) \to \mathcal{P}_c(A)$ and check that it satisfies the axioms for an approximable mapping.

The two parts carry, respectively, 40%, 60% of the marks.

End of paper