IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2011

EEE PART II: MEng, BEng and ACGI

Corrected Copy

FIELDS AND DEVICES

Monday, 20 June 2:00 pm

Time allowed: 2:00 hours

There are FOUR questions on this paper.

Answer ALL questions. Questions 1 and 3 each carry 20% of the marks.
Questions 2 and 4 each carry 30% of the marks.

This paper has 2 sections. Use a separate answer book for each section.

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : K. Fobelets, R.R.A. Syms
Second Marker(s) : W.T. Pike, Z. Durrani

© Imperial College London

Special instructions for invigilators

This exam consists of **2 sections**. Section A: **Devices** and section B: **Fields**. Each section has to be solved in their respective answer books. Check that 2 different answer books are available for the students.

Special instructions for students

Use different answers books for each section:

Devices: answer book **A**

Fields: answer book **B**

Constants and Formulae for section A: Devices

permittivity of free space:	$\varepsilon_o = 8.85 \times 10^{-12}$ F/m
permeability of free space:	$\mu_o = 4\pi \times 10^{-7}$ H/m
intrinsic carrier concentration in Si:	$n_i = 1.45 \times 10^{10}$ cm^{-3} at $T = 300$K
dielectric constant of Si:	$\varepsilon_{Si} = 11$
dielectric constant of SiO_2:	$\varepsilon_{ox} = 4$
thermal voltage:	$kT/e = 0.026$V at $T = 300$K
charge of an electron:	$e = 1.6 \times 10^{-19}$ C

$$\left.\begin{aligned} J_n(x) &= e\mu_n n(x)E(x) + eD_n\frac{dn(x)}{dx} \\ J_p(x) &= e\mu_p p(x)E(x) - eD_p\frac{dp(x)}{dx} \end{aligned}\right\} \quad \text{Drift-diffusion current equations}$$

$$I_{DS} = \frac{\mu C_{ox}W}{L}\left((V_{GS} - V_T)V_{DS} - \frac{V_{DS}^2}{2}\right) \quad \text{MOSFET current}$$

$$\left.\begin{aligned} J_n &= \frac{eD_n n_p}{L_n}\left(e^{\frac{eV}{kT}} - 1\right) \\ J_p &= \frac{eD_p p_n}{L_p}\left(e^{\frac{eV}{kT}} - 1\right) \end{aligned}\right\} \quad \text{Diode diffusion currents}$$

$$V_0 = \frac{kT}{e}\ln\left(\frac{N_A N_D}{n_i^2}\right) \quad \text{Built-in voltage}$$

$$c = c_0 \exp\left(\frac{eV}{kT}\right) \text{ with } \begin{cases} c = p_n \text{ or } n_p \\ c_0 \text{ bulk minority carrier concentration} \end{cases} \quad \text{Minority carrier injection under bias } V$$

$$\delta c = \Delta c \exp\left(\frac{-x}{L}\right) \text{ with } \begin{cases} \delta c = \delta p_n \text{ or } \delta n_p \\ \Delta c \text{ the excess carrier concentration} \\ \text{at the edge of the depletion region} \end{cases} \quad \text{Excess carrier concentration as a function of distance}$$

$$L = \sqrt{D\tau} \quad \text{Diffusion length}$$

$$D = \frac{kT}{e}\mu \quad \text{Einstein relation}$$

$$W_{depl} = \left[\frac{2\varepsilon V_0}{e}\frac{N_A + N_D}{N_A N_D}\right]^{1/2} \quad \text{Depletion width in pn diode}$$

$$C_{diff} = \frac{e}{kT}I\tau \quad \text{Diffusion capacitance}$$

Constants and Formulae for section B: Fields

Vector calculus (Cartesian co-ordinates)

$\nabla = \underline{i}\,\partial/\partial x + \underline{j}\,\partial/\partial y + \underline{k}\,\partial/\partial z$

$\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$

$\text{grad}(\phi) = \nabla\phi = \underline{i}\,\partial\phi/\partial x + \underline{j}\,\partial\phi/\partial y + \underline{k}\,\partial\phi/\partial z$

$\text{div}(\underline{\mathbf{F}}) = \nabla \cdot \underline{\mathbf{F}} = \partial \mathbf{F}_x/\partial x + \partial \mathbf{F}_y/\partial y + \partial \mathbf{F}_z/\partial z$

$\text{curl}(\underline{\mathbf{F}}) = \nabla \times \underline{\mathbf{F}} = \underline{i}\,\{\partial \mathbf{F}_z/\partial y - \partial \mathbf{F}_y/\partial z\} + \underline{j}\,\{\partial \mathbf{F}_x/\partial z - \partial \mathbf{F}_z/\partial x\} + \underline{k}\,\{\partial \mathbf{F}_y/\partial x - \partial \mathbf{F}_x/\partial y\}$

Where ϕ is a scalar field and $\underline{\mathbf{F}}$ is a vector field

Maxwell's equations – integral form

$\iint_A \underline{\mathbf{D}} \cdot d\underline{a} = \iiint_V \rho \, dv$

$\iint_A \underline{\mathbf{B}} \cdot d\underline{a} = 0$

$\int_L \underline{\mathbf{E}} \cdot d\underline{L} = -\iint_A \partial\underline{\mathbf{B}}/\partial t \cdot d\underline{a}$

$\int_L \underline{\mathbf{H}} \cdot d\underline{L} = \iint_A [\underline{\mathbf{J}} + \partial\underline{\mathbf{D}}/\partial t] \cdot d\underline{a}$

Where $\underline{\mathbf{D}}$, $\underline{\mathbf{B}}$, $\underline{\mathbf{E}}$, $\underline{\mathbf{H}}$, $\underline{\mathbf{J}}$ are time-varying vector fields

Maxwell's equations – differential form

$\text{div}(\underline{\mathbf{D}}) = \rho$

$\text{div}(\underline{\mathbf{B}}) = 0$

$\text{curl}(\underline{\mathbf{E}}) = -\partial\underline{\mathbf{B}}/\partial t$

$\text{curl}(\underline{\mathbf{H}}) = \underline{\mathbf{J}} + \partial\underline{\mathbf{D}}/\partial t$

Material equations

$\underline{\mathbf{J}} = \sigma \, \underline{\mathbf{E}}$

$\underline{\mathbf{D}} = \varepsilon \, \underline{\mathbf{E}}$

$\underline{\mathbf{B}} = \mu \, \underline{\mathbf{H}}$

Theorems

$\iint_A \underline{\mathbf{F}} \cdot d\underline{a} = \iiint_V \text{div}(\underline{\mathbf{F}}) \, dv$ – Gauss' theorem

$\int_L \underline{\mathbf{F}} \cdot d\underline{L} = \iint_A \text{curl}(\underline{\mathbf{F}}) \cdot d\underline{a}$ – Stokes' theorem

$\text{curl}\,\{\text{curl}(\underline{\mathbf{F}})\} = \text{grad}\,\{\text{div}(\underline{\mathbf{F}})\} - \nabla^2\underline{\mathbf{F}}$

© Imperial College London

Electromagnetic waves (pure dielectric media)

Time dependent vector wave equation $\nabla^2\underline{\mathbf{E}} = \mu_0\varepsilon\, \partial^2\underline{\mathbf{E}}/\partial t^2$

Time independent scalar wave equation $\nabla^2\underline{E} = -\omega^2\mu_0\varepsilon_0\varepsilon_r\, \underline{E}$

For z-going, x-polarized waves $d^2E_x/dz^2 + \omega^2\mu_0\varepsilon_0\varepsilon_r\, E_x = 0$

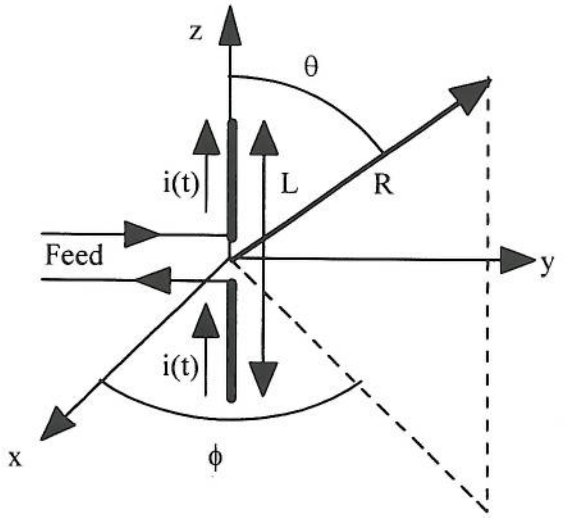
Where $\underline{E}$ is a time-independent vector field

Antenna formulae

Far-field pattern of half-wave dipole

$E_\theta = j\, 60I_0\, \{\cos[(\pi/2)\cos(\theta)] / \sin(\theta)\}\, \exp(-jkR)/R$; $H_\phi = E_\theta/Z_0$

Here I_0 is peak current, R is range and $k = 2\pi/\lambda$



Power density $\underline{S} = 1/2\ \mathrm{Re}\,(\underline{E} \times \underline{H}^*) = S(R, \theta)$

Normalised radiation pattern $F(\theta, \phi) = S(R, \theta, \phi) / S_{max}$

Directivity $D = 1/\ \{1/4\pi \iint_{4\pi} F(\theta, \phi) \sin(\theta)\, d\theta\, d\phi\}$

Gain $G = \eta D$ where η is antenna efficiency

Effective area $A_e = \lambda^2 D/4\pi$

Friis transmission formula $P_r = P_t\,(\eta_t\eta_r A_t A_r/R^2\lambda^2)$

© Imperial College London

SECTION A: SEMICONDUCTOR DEVICES

1.

a) What is the value of the minority carrier concentration at both Ohmic contacts of a pn diode with $N_A = 10^{16}$ cm^{-3} and $N_D = 10^{18}$ cm^{-3}?

[4]

b) Prove that the current in a long forward biased n$^+$p diode is determined by the electron diffusion current. Note that the diffusion constant and diffusion length are the same order of magnitude for electrons and holes.

[4]

c) The steady-state diffusion equation for holes is given by: $\frac{d^2\delta p(x)}{dx^2} = \frac{\delta p(x)}{\tau_p}$ (1).

Which function (sine, cosine hyperbolic, exponential, linear) describes $\delta p(x)$ for:

i) the SHORT diode approximation?

ii) the LONG diode approximation?

[4]

d) When a pn diode is switched with a step voltage $e(t)$ from $+E$ to $-E$, the current through the diode changes as given in fig. 1. What is the physical process that causes the delay time t_{sd}?

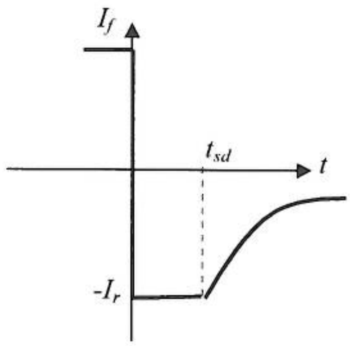


Figure 1. The current through a pn diode when switching from forward to reverse bias.

[4]

e) What is the relationship (=,>,<) between the current gain, β in the BJT without recombination in the base ($\beta_{no\text{-}recom}$) and a BJT with recombination in the base (β_{recom}). Explain your answer in one short sentence.

[4]

SECTION A: SEMICONDUCTOR DEVICES

2.

a) Sketch the energy band diagram (E_c, E_v, E_G, E_F) of a pn$^+$ junction under reverse bias conditions. Ensure that the relative distances between E_c, E_v, and E_F as well as the widths of the depletion regions are consistent with the given doping densities.

[6]

b) Sketch the variation of the minority carrier concentration $p_n(x)$ and $n_p(x)$ in each region of the diode under reverse bias. You may assume that the lengths of the neutral layers are much larger than the minority carrier diffusion lengths. Ensure that the relative values of the minority carrier concentration are consistent with the doping density given in a) and take the depletion widths into account in your drawing. Indicate clearly junction and contact 'values'.

[6]

c) Give the relation (=,>,<) between the carrier generation rate G and carrier recombination rate R in the neutral regions where $| x | < | L |$, with L the minority carrier diffusion length. Explain your answer in one short sentence.

[4]

d) The time dependent variation of the excess charge in the neutral region of the pn$^+$ diode is given by:

$$i(t) = \frac{Q(t)}{\tau} + \frac{dQ(t)}{dt}$$

Derive the expression for the time variation of the excess charge when the uncharged diode is switched on at $t = 0$ to a DC value of I_f.

[10]

e) If you were to use a pn$^+$ diode for the fabrication of a BJT, would it be as the emitter-base or base-collector junction? Explain your answer in one short sentence.

[4]

© Imperial College London

SECTION B: FIELDS

3. Discuss the implications for electromagnetic waves of **any two** of the following, illustrating your answer with diagrams, graphs or formulae where appropriate.

   a) Transmission lines, characteristic impedance and input impedance

   b) The effect of atmospheric properties on radio, microwave and optical links

   c) Antenna directivity, antenna arrays and early-warning radar

   d) Optical imaging systems, night vision systems and X-ray CAT scanners.

[20]

SECTION B: FIELDS

4.

a) Give a physical interpretation of the integral form of Gauss' law. Show how Gauss' theorem may be used to transform Gauss' law into the alternative differential form.

[6]

b) Starting with the integral form of Faraday's law, and assuming a rectangular integration path, derive the boundary condition that must be satisfied by electric fields at an interface between two media.

[6]

c) An electromagnetic wave is incident on a boundary between two dielectric media of refractive index n_1 and n_2 as shown in Figure 2 below. Assuming that $n_1 < n_2$, sketch the directions of the outgoing waves. Assuming that the incident electric field is polarized in the y-direction, write down expressions for the electric fields in each medium. Show how application of the boundary matching condition gives rise to Alhazen's and Snell's laws.

[8]

d) Using Snell's law, show that total internal reflection may occur, and identify the necessary conditions. What kind of wave exists in Medium 2 under these circumstances? Sketch its direction and the transverse variation of its field.

[10]

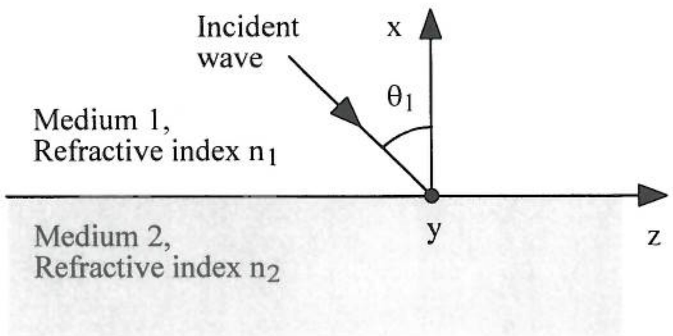


Figure 2. An electromagnetic wave incident on a boundary between two dielectric media of refractive index n_1 and n_2.

SECTION A: SEMICONDUCTOR DEVICES - SOLUTIONS 2011

1.

a) at the p-contact: $n_{p_o} = \frac{n_i^2}{N_A} = 21\,10^3\ cm^{-3}$

at the n-contact: $p_{n_o} = \frac{n_i^2}{N_D} = 210\ cm^{-3}$

[4]

b) In an n$^+$p diode N_D>>N_A. From formulae sheet:

$$J_n = \frac{eD_n n_p}{L_n}\left(e^{\frac{eV}{kT}} - 1\right)$$
$$J_p = \frac{eD_p p_n}{L_p}\left(e^{\frac{eV}{kT}} - 1\right)$$
$$\frac{J_n}{J_p} = \frac{L_p D_n n_p}{D_p L_n p_n} \approx \frac{n_i^2 N_D}{N_A n_i^2} = \frac{N_D}{N_A} >> 1$$
$$\Rightarrow J_n >> J_p$$

[4]

c) i) linear

ii) exponential

[4]

d) Discharging of the minority carrier concentration that was stored in the neutral regions under forward bias. At t=tsd the excess charge in the neutral regions is zero. (This time delay is related to the diffusion capacitance.)

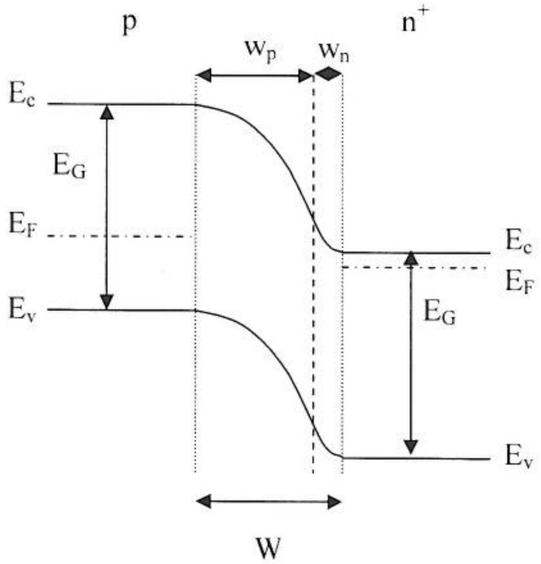
[4]

e) $\beta_{no\text{-}recom} > \beta_{recom}$. Without recombination: $\beta = \frac{I_{E\rightarrow B}}{I_{B\rightarrow E}}$ and with recombination: $\beta = \frac{I_{E\rightarrow B} - I''_B}{I_{B\rightarrow E} + I''_B}$. "$\rightarrow$" direction of carrier injection across EB junction, I''_B is the base current due to recombination.

[4]

2.

a)



[4]

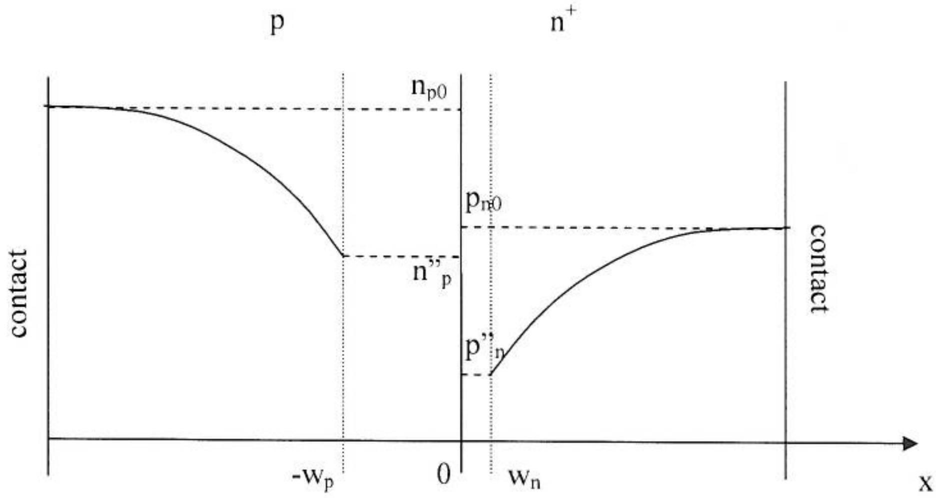
E_F @ p side higher in energy than @ n side (reverse bias)

$w_p >> w_n$ (due to heavy n-type doping)

$E_c - E_F |_{@p} >> E_c - E_F |_{@n}$

E_G constant [6]

b)



[6]

c) $G < R$. Recombination rate is larger since there are excess minority carriers in the regions: $| x | < | L |$. The excess will recombine.

[4]

d) The hole current can be neglected and thus the total diode current is electron current, thus $I_f = I_n$. The stored charge $Q(t)$ is thus $Q_n(t)$. Thus the differential equation becomes

$$i_n(t) = \frac{Q_n(t)}{\tau_n} + \frac{dQ_n(t)}{dt}$$

 © Imperial College London

Turn ON: from 0 bias to ON
Switching happens at t=0s.
Boundary conditions:
For $t<0$ $i_n(t=0)=0$, $Q_n(t=0)=0$.
For $t\geq 0$ $i_n(t)=I_f$
$t \geq 0$

$$I_f = \frac{Q_n(t)}{\tau_n} + \frac{dQ_n(t)}{dt}$$

$$\frac{dQ_n(t)}{Q_n(t) - I_f\tau_n} = \frac{-dt}{\tau_n}$$

$$\int_{Q_n(t=0)}^{Q_n(t)} \frac{dQ_n(t)}{Q_n(t) - I_f\tau_n} = \int_0^t \frac{-dt}{\tau_n}$$

$$\ln\left(Q_n(t) - I_f\tau_n\right)\Big|_{Q_n(t)} - \ln\left(Q_n(t) - I_f\tau_n\right)\Big|_{Q_n(0)} = \frac{-t}{\tau_n}$$

$$\ln\left(\frac{Q_n(t) - I_f\tau_n}{-I_f\tau_n}\right) = \frac{-t}{\tau_n}$$

$$Q_n(t) - I_f\tau_n = -I_f\tau_n \exp\left(\frac{-t}{\tau_n}\right)$$

$$Q_n(t) = I_f\tau_n\left(1 - \exp\left(\frac{-t}{\tau_n}\right)\right)$$

[10]

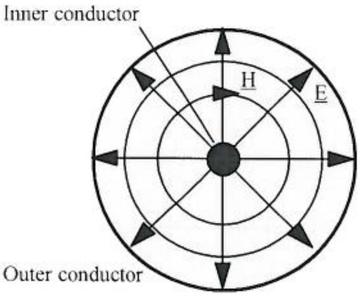
e) emitter-base. The n+ layer should be the emitter and the p layer the base for an npn BJT. A heavily doped emitter will give large current gain.

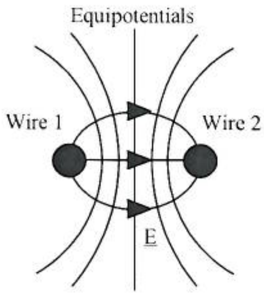
[4]

SECTION A: FIELDS - SOLUTIONS

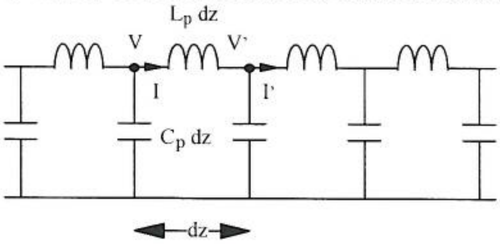
3.

a) A transmission line is a high frequency electronic link, which has a constant cross-section formed by a pair of parallel conductors spaced apart by dielectric. Examples include two-wire and coaxial cables, and the coplanar waveguides and microstrip used in PCBs and MMICs. Important properties include the distribution of electric and magnetic fields near the conductors, and equi-potential lines. The conductors will typically have distributed capacitance and inductance, which can be found in the static approximation by first solving for the field distributions and then applying Gauss' law (in the former case) and Ampere's law (in the latter).

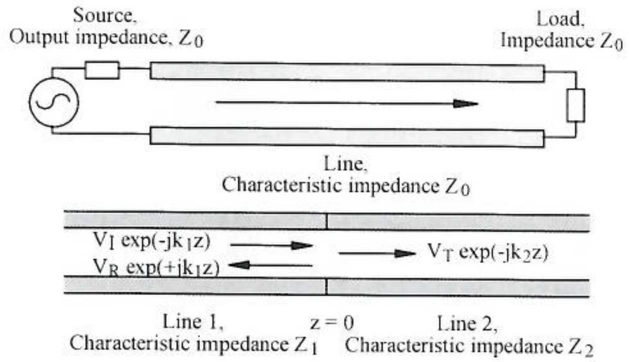




Propagation can be modelled in terms of the transmission line equations, which are derived by applying Kirchhoff's laws to a ladder network containing distributed inductance and capacitance. Current and voltages satisfy the second order differential equations $d^2V/dz^2 = -\omega^2 L_p C_p V$ and $d^2I/dz^2 = -\omega^2 L_p C_p I$. The effect of the distributed reactance is to give a unique propagation velocity $v_{ph} = 1/(L_P C_P)^{1/2}$ to waves on the line and real characteristic impedance $Z_0 = (L_P/C_P)^{1/2}$ to an infinite length of line.



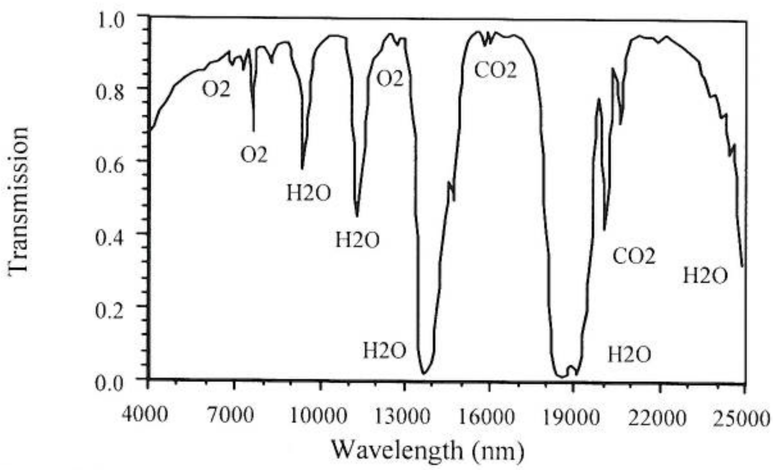
Discontinuities in the characteristic impedance generate reflection, so an optimised transmission line system will match the output impedance of the source, the load impedance and the line impedance. The input impedance of a finite length of line terminated by an arbitrary load will differ from the characteristic impedance, and may be a strong function of frequency. Techniques for matching different impedances (for example, quarter-wave transformers) exist, but are typically narrow band.



[10]

b) The main effects of atmospheric properties on electromagnetic waves are absorption, scattering, diffraction and reflection.

Absorption is a loss of energy that occurs in a number of specific regions of the electromagnetic spectrum, due to excitation of transitions between different vibrational configurations in the molecules comprising the atmosphere. The visible and infrared absorption spectrum is as shown below (NB students will probably be unable to reproduce these details). Water absorption has a strong effect on microwaves at 2.45 GHz frequency.



Rayleigh scattering is a random redirection of energy that is caused by inhomogeneities such as water droplets and dust particles in the atmosphere. Since its effects rise as $1/\lambda^4$, it predominantly affects short wavelengths. Rayleigh scattering has a strong effect at optical wavelengths, where it is responsible for the colour of the sky, and the opacity of mist, fog and seawater.

Diffraction is the spreading of energy that follows when a beam is passed through a finite aperture or is emitted from a finite source. Diffraction effects rise if the width of the source is small when measured in wavelengths. Consequently, radio waves are strongly affected by diffraction.

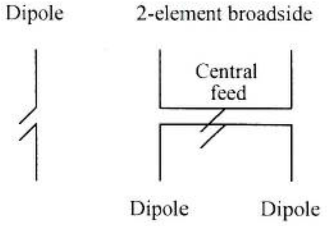
Reflection of electromagnetic waves occurs at the ionosphere, a set of concentric shells surrounding the atmosphere that contain ions formed by bombardment with energetic particles from space. The ionosphere forms a conducting 'surface' that can reflect waves at frequencies below the plasma frequency. The ionosphere and the earth's surface (especially seawater)

© Imperial College London

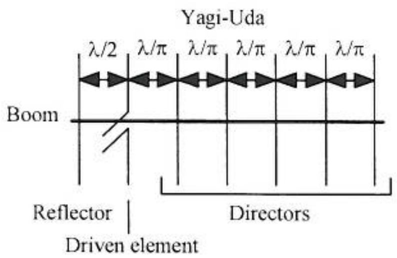
provide a pair of conducting surfaces that allow radio communication beyond the line-of-sight limit. However, such a link is relatively unstable and strongly affected by weather variations. The ionosphere is transparent above the plasma frequency, allowing over-the-horizon communication via geostationary satellites.

[10]

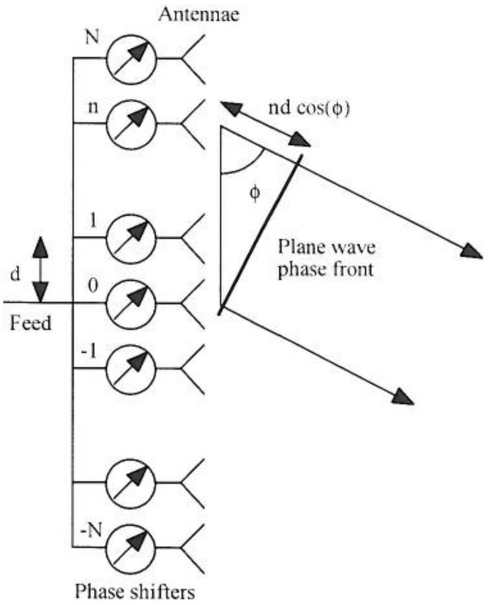
c) Because radio waves are strongly affected by diffraction, the energy emitted from a simple dipole antenna diverges strongly, as a spherical wave. Such an element has very low directivity. However, the energy can be concentrated in a particular direction by using multiple dipoles. The elements are placed at well-defined separations, and the amplitude and phase of the signals fed to each dipole are carefully controlled. The far-field contributions from all the elements may then be arranged to sum in-phase in a particular direction, thus increasing the directivity by many orders of magnitude.



The beneficial effects may be exploited in signal transmission and in reception. Applications include boosting the received signal in a radio or TV system, or increasing the range of a radar system. Arrays with individual feeds are too expensive for domestic use. However, the Yagi array (the universal terrestrial TV reception aerial) allows a similar effect to be achieved using a feed to just one dipole, with the other dipoles being used as passive reflectors or directors.

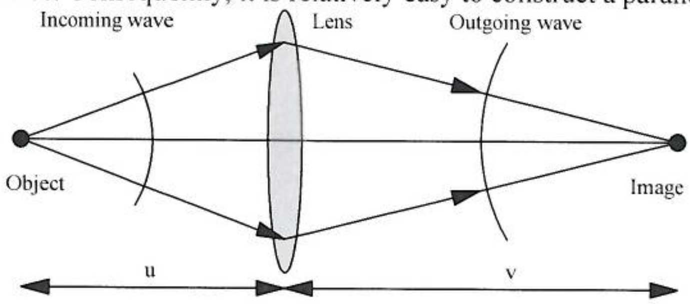


Military radars use multiple feeds. To maximise directivity (and hence the range of target detection, and the warning time) they may be constructed in extremely large sizes. At this point the antenna is normally too bulky to be steered mechanically. Instead, the beam direction is scanned electronically, by varying the phases of the signals sent to each element using microwave phase shifters.

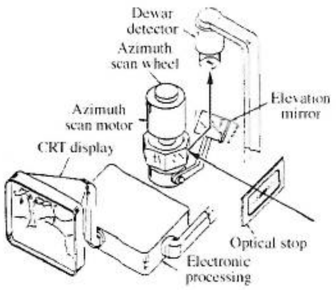


[10]

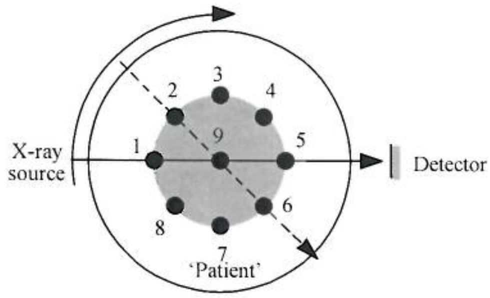
d) Optical imaging systems detect moderately energetic radiation that is scattered from an object. The original radiation may come from a natural source such as the sun, or an artificial source such as a light bulb, light emitting diode or laser. Image formation may be considered as a process in which radiation diverging from a point object is redirected by a lens into radiation converging onto a point image, with more complex images being built up from a set of points. Image formation is governed by the imaging formula $1/u + 1/v = 1/f$, where u and v are object and image distances and f is the focal length. The wavelengths are such that transparent materials for image forming components such as lenses may easily be provided, and background absorption from the atmosphere is low. The photons involved have sufficient energy to promote electronic transitions leading to chemical changes in the eye or in photographic film, or band-to-band transitions in semiconductors. Because photodiodes and other semiconductor devices can use a large energy gap such detectors may be relatively immune to thermal noise. Consequently, it is relatively easy to construct a parallel imaging system.



Night vision systems operate when normal sources of illumination are absent. Consequently, such systems must detect thermal radiation emitted by the object itself. Although suitable transparent materials exist for constructing lenses, and the atmosphere is transparent to long wavelength IR near 4 μm and in the range 8-12 μm, the photons involved have so little energy that band-to-band transitions can only operate in semiconductors with small energy gaps. As a result, detectors must be constructed using complex compound semiconductors (e.g. CdHgTe), suffer from thermal noise, and often require cooling. It is therefore harder to construct a parallel imaging system. Some night vision systems operate by image intensification and wavelength conversion; others by scanning a beam across a single cooled detector.



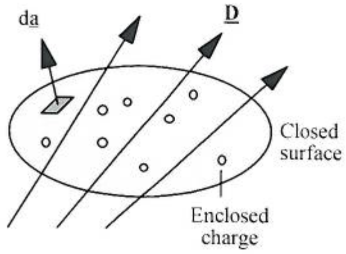
X-ray imaging systems operate at very short wavelength, using highly energetic photons that are strongly penetrating. The frequencies are so high that most materials have a refractive index approaching unity, so it is very difficult to form lenses, and mirrors can only operate at grazing incidence. Generally image contrast is therefore obtained from differences in absorption during transmission, and simple X-ray images are shadowgraphs. Computer-aided tomography (CAT) operates by measuring shadowgraphs obtained for multiple directions of incidence, and using the combined information to deduce the 3D arrangement of absorbers that must have given rise to them.



[10]

4

a) Consider electric flux out of a closed surface enclosing charges



In integral form, Gauss' Law states that $\iint_A \underline{\mathbf{D}} \cdot d\underline{a} = \iiint_V \rho \, dv$

Here $d\underline{a}$ is a vector representing a small area of the surface

The modulus of $d\underline{a}$ is the size of the area

The direction of $d\underline{a}$ is the normal vector to the area

$\underline{\mathbf{D}} \cdot d\underline{a}$ is therefore the amount of electric flux normal to the small area

$\iint_A \underline{\mathbf{D}} \cdot d\underline{a}$ is the sum over the surface, i.e. the total flux out

If ρ is the charge density then $\iiint_V \rho \, dv$ is the integral of the charge over the volume

Consequently the integral equation also states that Flux out = charge enclosed

[4]

Gauss' Theorem states that:

$\iint_A \underline{\mathbf{F}} \cdot d\underline{a} = \iiint_V \text{div}(\underline{\mathbf{F}}) \, dv$

Applying Gauss' Theorem to Gauss'Law:

$\iint_A \underline{\mathbf{D}} \cdot d\underline{a} = \iiint_V \text{div}(\underline{\mathbf{D}}) \, dv = \iiint_V \rho \, dv$

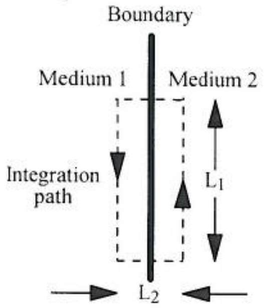
The second and third integrals are now both volume integrals. Since their integration region is undefined, the integrands must be equal. Hence $\text{div}(\underline{\mathbf{D}}) = \rho$, the differential form of Gauss' Law.

[2]

b) Faraday's law states that

$\int_L \underline{\mathbf{E}} \cdot d\underline{L} = - \iint_A \partial\underline{\mathbf{B}}/\partial t \cdot d\underline{a}$

To apply Faraday's law, we define the integration path as a rectangle straddling the interface between two media as shown. Two of the sides are parallel to the boundary and the rectangle is long and thin so that $L_2 << L_1$.



If we let L_2 tend to zero, the area must tend to zero, so the RHS of Faraday's equation must be zero. If L_1 is small but non-zero, the fields can be considered uniform. Integration gives:

$\int_L \underline{\mathbf{E}} \cdot d\underline{L} = (\mathbf{E}_{t1} - \mathbf{E}_{t2}) L_1$

Here $\mathbf{E}_{t1}$ and $\mathbf{E}_{t2}$ are the components of the two fields parallel to the line elements parallel to the boundary. These components are tangential to the boundary and are called tangential components.

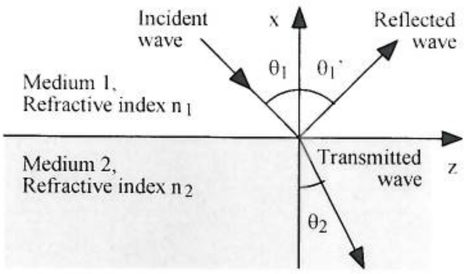
Applying Faraday's Law, we then get:

$\mathbf{E}_{t1} - \mathbf{E}_{t2} = 0$

Consequently the boundary condition for the electric field is that tangential components of $\underline{\mathbf{E}}$ must match across boundary.

 © Imperial College London

c) The directions of the reflected and refracted waves are as shown below. **[6]**



[2]

Solutions can be written down as incident and reflected waves in Medium 1, and a transmitted wave in Medium 2, so that:

$E_{y1} = E_I \exp[-jk_0n_1(z \sin(\theta_1) - x \cos(\theta_1))] + E_R \exp[-jk_0n_1(z \sin(\theta_1') + x \cos(\theta_1'))]$

$E_{y2} = E_T \exp[-jk_0n_2(z \sin(\theta_2) - x \cos(\theta_2))]$

Here E_I, E_R and E_T are the amplitudes of the incident, reflected and transmitted waves and $k_0 = 2\pi/\lambda$ is the propagation constant in free space.

[2]

The boundary condition that must be satisfied is that tangential components of $\underline{E}$ must match across a boundary. Since $\underline{E}$ is wholly tangential, we must have $E_{y1} = E_{y2}$ on $x = 0$, or:

$E_I \exp[-jk_0n_1z \sin(\theta_1)] + E_R \exp[-jk_0n_1z \sin(\theta_1')] = E_T \exp[-jk_0n_2z \sin(\theta_2)]$

[2]

The only way this equation can be satisfied for all z is if:

$n_1 \sin(\theta_1) = n_1\sin(\theta_1') = n_2\sin(\theta_2)$

Hence $\theta_1 = \theta_1'$ (Alhazen's law)

$n_1 \sin(\theta_1) = n_2\sin(\theta_2)$ (Snell's law)

[2]

d) From Snell's law:

$\theta_2 = \sin^{-1}\{(n_1/n_2) \sin(\theta_1)\}$

Real solutions for θ_2 do not exist if $(n_1/n_2) \sin(\theta_1) > 1$.

Since $\sin(\theta_1) < 1$, we require $n_1 > n_2$, so that the wave must be incident from the high-index side.

Assuming that this is the case, the critical condition will just be reached when $(n_1/n_2) \sin(\theta_1) = 1$. The critical angle is $\theta_C = \sin^{-1}(n_2/n_1)$.

[4]

The electric field in medium 2 can be written as $E_{y2} = E_T \exp[-jk_0n_2z \sin(\theta_2)] \exp[+jk_0n_2x \cos(\theta_2)]$. From Snell's law we know that $n_2\sin(\theta_2) = n_1 \sin(\theta_1)$

Consequently $n_2\cos(\theta_2) = n_2\{1 - (n_1/n_2)^2\sin^2(\theta_1)\}^{1/2}$

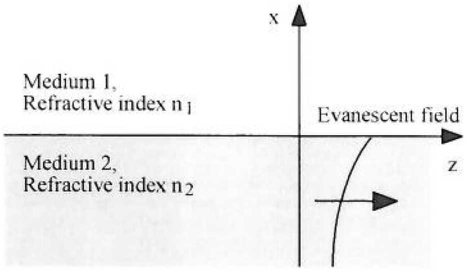
For incidence beyond the critical angle, the contents of the curly bracket must be negative.

Hence $n_2\cos(\theta_2) = \pm j\alpha$, where $\alpha = n_2\{(n_1/n_2)^2\sin^2(\theta_1) - 1\}^{1/2}$

Consequently the wave in Medium 2 can be written as $E_{y2} = E_T \exp[-jk_0n_1z \sin(\theta_1)] \exp[+k_0\alpha x]$.

[4]

This expression describes a wave travelling parallel to the interface, decaying exponentially below. It can be represented thus:


x
Medium 1,
Refractive index n_1
Evanescent field
z
Medium 2,
Refractive index n_2

[2]