Control engineering exam paper - Model answers

Question 1

a) The equilibria of the system are obtained solving the equations

$$0 = x_1\left(-x_1 + \frac{u}{1+x_2}\right), \qquad 0 = x_2(-x_2+u),$$

with $u > 0$ and constant. The first equation yields $x_1 = 0$ or $x_1 = \frac{u}{1+x_2}$. The second equation yields $x_2 = 0$ or $x_2 = u$. There are, therefore, four equilibrium points:

$$P_1 = (0,0) \qquad P_2 = (0,u) \qquad P_3 = (u,0) \qquad P_4 = (\frac{u}{1+u}, u).$$

b) The linearized models are described by equations of the form $\dot{\delta}_x = A_i\delta_x + B_i\delta_u$, where the matrices A_i's and B_i's are the Jacobian matrices of the generating function of the system, with respect to x and u, respectively, evaluated at the point P_i. Therefore

$$A_1 = \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} \frac{u}{1+u} & 0 \\ 0 & -u \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 \\ u \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -u & -u^2 \\ 0 & u \end{bmatrix}, \qquad B_3 = \begin{bmatrix} u \\ 0 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} -\frac{u}{1+u} & -\frac{u^2}{(1+u)^3} \\ 0 & -u \end{bmatrix}, \qquad B_4 = \begin{bmatrix} \frac{u}{(1+u)^2} \\ u \end{bmatrix}.$$

c) Recall that $u > 0$. Note that

- $\lambda(A_1) = \{u\}$, hence P_1 is unstable;
- $\lambda(A_2) = \{-u, \frac{u}{1+u}\}$, hence P_2 is unstable;
- $\lambda(A_3) = \{-u, u\}$, hence P_3 is unstable;
- $\lambda(A_4) = \{-u, -\frac{u}{1+u}\}$, hence P_4 is (locally) asymptotically stable.

d) The controllability matrices of the four linearized models are

$$\mathcal{C}_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \mathcal{C}_2 = \begin{bmatrix} 0 & 0 \\ u & -u^2 \end{bmatrix},$$

$$\mathcal{C}_3 = \begin{bmatrix} u & -u^2 \\ 0 & 0 \end{bmatrix}, \qquad \mathcal{C}_4 = \begin{bmatrix} \frac{u}{(1+u)^2} & -\frac{u^2}{(1+u)^2} \\ u & -u^2 \end{bmatrix}.$$

Note that

$$\det\mathcal{C}_1 = \det\mathcal{C}_2 = \det\mathcal{C}_3 = \det\mathcal{C}_4 = 0,$$

hence all linearized models are not controllable.

Question 2

a) With the given selection of state variables we have

$$\dot{x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a_1 & a_2 & 0 & 0 \\ a_3 & a_4 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ -1/l_1 \\ -1/l_2 \end{bmatrix} u,$$

where

$$a_1 = \frac{(m+1)g}{l_1} \qquad a_2 = \frac{mg}{l_1} \qquad a_3 = \frac{mg}{l_2} \qquad a_4 = \frac{(m+1)g}{l_2}.$$

b) The reachability matrix is

$$\mathcal{C} = \left[\begin{array}{cccc} B & AB & A^2B & A^3B \end{array}\right] = \begin{bmatrix} 0 & -\frac{1}{l_1} & 0 & -g\frac{m+1}{l_1^2} - g\frac{m}{l_1 l_2} \\ 0 & -\frac{1}{l_2} & 0 & -g\frac{m+1}{l_2^2} - g\frac{m}{l_1 l_2} \\ -\frac{1}{l_1} & 0 & -g\frac{m+1}{l_1^2} - g\frac{m}{l_1 l_2} & 0 \\ -\frac{1}{l_2} & 0 & -g\frac{m+1}{l_2^2} - g\frac{m}{l_1 l_2} & 0 \end{bmatrix},$$

and its determinant is

$$\det\mathcal{C} = -g^2\frac{(l_1 - l_2)^2}{l_1^4 l_2^4}.$$

As a result, the system is reachable (controllable) if and only if $l_1 \neq l_2$.

c) The observability matrix is

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ g\frac{m+1}{l_1} - g\frac{m}{l_2} & -g\frac{m+1}{l_2} + g\frac{m}{l_1} & 0 & 0 \\ 0 & 0 & g\frac{m+1}{l_1} - g\frac{m}{l_2} & -g\frac{m+1}{l_2} + g\frac{m}{l_1} \end{bmatrix},$$

and its determinant is

$$\det\mathcal{O} = -g^2(2m+1)^2\frac{(l_1 - l_2)^2}{l_1^2 l_2^2}.$$

As a result, the system is observable if and only if $l_1 \neq l_2$.

d) If $l_1 = l_2 = l$ then, subtracting the two equations describing the system yields

$$l(\ddot{\theta}_1 - \ddot{\theta}_2) = g(\theta_1 - \theta_2),$$

hence

$$l\ddot{\xi} = g\xi.$$

Note that this subsystem is not affected by the input u, and it has one positive and one negative eigenvalue, hence it is unstable. As a result, for $l_1 = l_2$ the system is not stabilizable.

Question 3

a) The equilibrium points are the (constant) solutions of the equation

$$x(t) = Ax(t)$$

hence the solutions of

$$(I - A)\bar{x} = \begin{bmatrix} \frac{1}{2} & -\frac{2}{5} \\ \frac{1-k}{2} & \frac{1}{5} \end{bmatrix} \bar{x} = 0.$$

Note that

$$\det(I - A) = \frac{2k - 1}{10},$$

hence for all $k \neq \frac{1}{2}$ the system has a unique equilibrium, whereas for $k = 1/2$ the system has infinitely many equilibria given by

$$\bar{x} = \alpha \begin{bmatrix} 4 \\ 5 \end{bmatrix},$$

for any $\alpha \in I\!R$.

b) The characteristic polynomial of the matrix A is

$$p(z) = z^2 - \frac{13}{10}z + \frac{k+1}{5},$$

and its roots are

$$z_{1,2} = \frac{13}{20} \pm \frac{\sqrt{89 - 80k}}{20}.$$

Note that the roots are real and positive for all $k \in [0, 1]$, and that the root with the "$-$" sign in front of the square root is always smaller than 1. The root with the "+" sign in front of the square root is larger than 1 for $k \in [0, 1/2)$, it is equal to 1 for $k = 1/2$, and it is smaller than 1 for $k \in (1/2, 1]$. In summary, the system is unstable for $k \in [0, 1/2)$, stable for $k = 1/2$, asymptotically stable for $k \in (1/2, 1]$.

c) Recall that $x(t) = A^t x(0)$, and note that since A has all non-negative entries for $k \in [0, 1]$, A^t has non-negative entries for all $t \geq 0$. Therefore if $x(0)$ has non-negative entry then $x(t)$ is the linear combination of the entries of $x(0)$ with non-negative coefficients, hence it has non-negative components.

d) i) Note that

$$z(t+1) = 5x_1(t+1) - 4x_2(t+1) = \frac{2}{5}z(t).$$

As a result, for any initial condition,

$$z(t) = \left(\frac{2}{5}\right)^t z(0),$$

which implies that $z(t)$ tends to zero as t goes to infinity, which proves the claim.

ii) Since all trajectories converge to the line $5x_1 - 4x_2 = 0$, the asymptotic revenue is

$$\lim_{t\to\infty} y(t) = (C_1 k - \frac{5}{4}C_2) \lim_{t\to\infty} x_1(t).$$

Hence the asymptotic revenue is non-negative provided

$$C_1 k - \frac{5}{4}C_2 \geq 0.$$

Question 4

a) The relation between the variables (x_1, x_2, x_3) and (H, O, W) can be written as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = T \begin{bmatrix} H \\ O \\ W \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} H \\ O \\ W \end{bmatrix}.$$

Note that the matrix T is invertible, hence there is a one-to-one relation between the two sets of variables. Finally

$$\begin{bmatrix} H \\ O \\ W \end{bmatrix} = T^{-1} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 - x_1 \\ \dfrac{x_2 - x_1}{2} \\ x_1 \end{bmatrix}.$$

b) Note that

$$\dot{x}_2 = \dot{W} + 2\dot{O} = 0 \qquad \dot{x}_3 = \dot{W} + \dot{H} = 0.$$

Hence

$$x_2(t) = x_2(0) \qquad x_3(t) = x_3(0),$$

which means that $x_2(t)$ and $x_3(t)$ are constant, i.e. $W(t) + 2O(t)$ and $W(t) + H(t)$ remain constant.

c) Note that

$$\dot{x}_1 = k_1 x_2 x_3^2 - (2k_2 + k_1 x_3^2 + 2k_1 x_2 x_3)x_1 + k_1(2x_3 + x_2)x_1^2 - k_1 x_1^3$$

and since $x_2(t) = x_2(0)$ and $x_3(t) = x_3(0)$

$$\dot{x}_1 = k_1 x_2(0) x_3^2(0) - (2k_2 + k_1 x_3^2(0) + 2k_1 x_2(0) x_3(0))x_1 + k_1(2x_3(0) + x_2(0))x_1^2 - k_1 x_1^3.$$

As a result (note that $x_2(0)$ and $x_3(0)$ are non-negative)

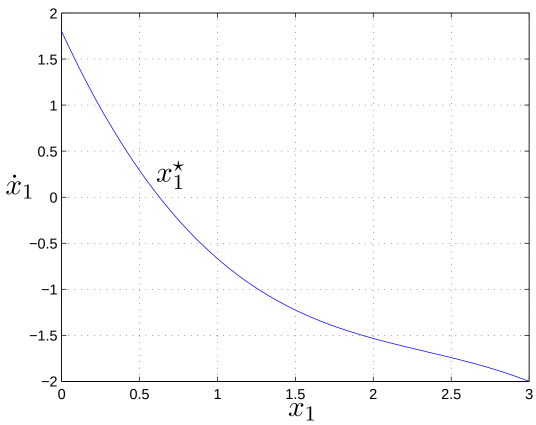
$$A = k_1 x_2(0) x_3^2(0) \geq 0, \qquad B = 2k_2 + k_1 x_3^2(0) + 2k_1 x_2(0) x_3(0) > 0,$$

$$C = 2x_3(0) + x_2(0) \geq 0, \qquad D = k_1 > 0.$$

d) i) Note that $\dot{x}_1$ is a cubic function of x_1 and that

$$\dot{x}_1|_{x_1=0} = A > 0 \qquad \lim_{x_1 \to \infty} \dot{x}_1(x_1) = -\infty.$$

As a result, $\dot{x}_1$ as a function of x_1 has the shape in the figure below.



Note that, since $\dot{x}_1 > 0$, for $x_1 < x_1^\star$, and $\dot{x}_1 < 0$, for $x_1 > x_1^\star$, the equilibrium $x_1^\star$ is globally asymptotically stable.

ii) In the (x_1, x_2, x_3) coordinates the system is described by the equations

$$\dot{x}_1 = A - Bx_1 + Cx_1^2 - Dx_1^3 \qquad \dot{x}_2 = 0 \qquad \dot{x}_3 = 0.$$

Hence, for any x_{2e} and x_{3e} there is a unique $x_{1e} = x_{1e}(x_{2e}, x_{3e})$ such that the point (x_{1e}, x_{2e}, x_{3e}) is an equilibrium. This means that the system has infinitely many equilibria, parameterized by x_{2e} and x_{3e}. The principle of stability in the first approximation cannot be used to assess stability of these equilibria. However, because of the structure of the $\dot{x}_2$ and $\dot{x}_3$ equation, and of what established in part d.i), these equilibria are stable, non-asymptotically.

Question 5

a) Since A is upper diagonal, its eigenvalues are the elements of the diagonal. As a result, the eigenvalues of A are both equal to -1, hence they are constant and with negative real part.

b) The system can be re-written as

$$\dot{x}_1 = -x_1 + e^{2t}x_2, \qquad \dot{x}_2 = -x_2,$$

hence (recall that $t_0 = 0$)

$$x_2(t) = e^{-t}x_2(0),$$

yielding

$$\dot{x}_1 = -x_1 + e^{t}x_2(0).$$

Using Lagrange formula for integrating this equation yields

$$x_1(t) = \left(x_1(0) - \frac{1}{2}x_2(0)\right)e^{-t} + \frac{1}{2}x_2(0)e^{t}.$$

Combining the expressions of $x_1(t)$ and $x_2(t)$ in matrix form yields

$$x(t) = \begin{bmatrix} e^{-t} & -\frac{1}{2}e^{-t} + \frac{1}{2}e^{t} \\ 0 & e^{-t} \end{bmatrix} x(0) = \Phi(t,0)x(0).$$

Note that $\Phi(0,0) = I$ and that

$$\frac{d\Phi(t,0)}{dt} = A(t)\Phi(t,0),$$

as requested.

c) By inspection, it is clear that, if $x_2(0) \neq 0$ then

$$\lim_{t\to\infty} \|x(t)\| = \infty.$$

Hence for almost all initial conditions the solutions are unbounded, whereas the solutions are bounded only if $x_2(0) = 0$.

d) The system is stable, if and only if, $\Phi(t,0)$ is bounded, hence the system is not stable.

e) Repeating the arguments in part a) we obtain

$$x_2(t) = e^{-t}x_2(0)$$

and

$$\begin{aligned} x_1(t) &= e^{-t}x_1(0) + \int_0^t e^{-(t-\tau)}e^{-\tau}b(\tau)d\tau\ x_2(0) \\ &= e^{-t}x_1(0) + e^{-t}\int_0^t b(\tau)d\tau\ x_2(0). \end{aligned}$$

Note now that since $b(t) \leq \bar{b}$ then

$$|\int_0^t b(\tau)d\tau| \leq \bar{b}t,$$

hence $x_1(t)$ is bounded and converges to zero. Therefore, the state transition matrix for this system is bounded and converges to zero, as $t \to \infty$, which implies that the system is asymptotically stable.

Question 6

a) The PBH reachability test states that a system is reachable if and only if

$$\text{rank}\,[sI - A \quad B] = n,$$

for all $s \in \lambda(A)$. Suppose now that there is a left eigenvector w of A which is orthogonal to B, i.e.

$$wA = \lambda w \qquad wB = 0.$$

This can be rewritten as

$$w\,[\lambda I - A \quad B] = 0,$$

which implies that the reachability pencil loses rank for $s = \lambda$. Hence, the system is reachable if and only if the reachability pencil has rank equal to n for all $s \in \lambda(A)$, which is equivalent to the fact that there is no left eigenvector of A which is orthogonal to B.
Note that we have used the fact that a matrix M is full rank if and only if $wM \neq 0$ for all vectors $w \neq 0$.

b) The PBH observability test states that a system is observable if and only if

$$\text{rank}\begin{bmatrix} sI - A \\ C \end{bmatrix} = n,$$

for all $s \in \lambda(A)$. Suppose now that there is a right eigenvector v of A which is orthogonal to C, i.e.

$$Av = \lambda v \qquad Cv = 0.$$

This can be rewritten as

$$\begin{bmatrix} \lambda I - A \\ C \end{bmatrix} v = 0,$$

which implies that the observability pencil loses rank for $s = \lambda$. Hence, the system is observable if and only if the observability pencil has rank equal to n for all $s \in \lambda(A)$, which is equivalent to the fact that there is no right eigenvector of A which is orthogonal to C.

c) For the considered system we have

$$A = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & \dots & & & \lambda_n \end{bmatrix} \qquad B = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ \vdots \\ B_n \end{bmatrix} \qquad C = \begin{bmatrix} C_1 & C_2 & C_3 & \dots & C_n \end{bmatrix}.$$

i) The left eigenvectors of A are

$$w_1 = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \qquad w_2 = \begin{bmatrix} 0 & 1 & \cdots & 0 \end{bmatrix} \quad \dots \quad w_n = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}.$$

There is a left eigenvector of A orthogonal to B if and only if there is a $B_i = 0$. Hence, the system is reachable if and only if

$$B_1 B_2 \dots B_n \neq 0.$$

ii) The right eigenvectors of A are

$$v_1 = w_1' \qquad v_2 = w_2' \quad \dots \quad v_n = w_n'.$$

There is a right eigenvector of A orthogonal to C if and only if there is a $C_i = 0$. Hence, the system is observable if and only if

$$C_1 C_2 \dots C_n \neq 0.$$

d) The left eigenvectors of the given A are

$$w_1 = \begin{bmatrix} \alpha & \beta & 0 \end{bmatrix} \qquad\qquad w_2 = \begin{bmatrix} \alpha & 0 & \gamma \end{bmatrix}$$

for any α, β and γ such that $|\alpha| + |\beta| > 0$ and $|\alpha| + |\gamma| > 0$. Note that, for example,

$$w_1 B = \alpha B_1 + \beta B_2,$$

and this can be rendered zero selecting $\alpha = B_2$ and $\beta = -B_1$, if $B_1 \neq 0$ or $B_2 \neq 0$, or selecting any nonzero α and β is $B_1 =$ and $B_2 = 0$. As a result, there is (always) a left eigenvector of A orthogonal to B, hence the system is not reachable.