Master

UNIVERSITY OF LONDON

E2.8 Mathematics 3

B.ENG. AND M.ENG. EXAMINATIONS 2006

For Internal Students of Imperial College London

This paper is also taken for the relevant examination for the Associateship of the City & Guilds of London Institute

PART II : MATHEMATICS 3 (ELECTRICAL ENGINEERING)

Tuesday 30th May 2006 2.00 - 5.00 pm

Answer EIGHT questions.

Please answer questions from Section A and Section B in separate answerbooks.

A statistics data sheet is provided.

Corrected Copy

[*Before starting, please make sure that the paper is complete; there should be 10 pages, with a total of 12 questions. Ask the invigilator for a replacement if your copy is faulty.*]

1. Consider the mapping

$$w = \frac{1+i}{z} + 1$$

from the z-plane to the w-plane where $w = u + iv$.

(i) Show that the circle $x^2 + y^2 = R^2$ in the xy-plane maps to the circle

$$(u - 1)^2 + v^2 = 2/R^2$$

in the uv-plane.

(ii) Show that if the point (x, y) traverses the circle in the clockwise direction, then the point (u, v) traverses the circle in the counter-clockwise direction.

(iii) Show that the family of straight lines $y = \alpha x$ (α real) in the xy-plane maps to a family of straight lines in the uv-plane, all passing through a single point, which is to be determined.

Determine also the slope of the line in the uv-plane that corresponds to the straight line $y = \alpha x$ in the xy-plane.

2. Use the Residue Theorem to show that

(i)

$$\oint_C \frac{z\,dz}{(z-1)^2(z-i)} = 0 ,$$

where the contour C is the circle of radius 3 centred at the origin. What is the answer when C is changed to be a circle, radius $\frac{1}{2}$, centred at the origin?

(ii)

$$\oint_C \frac{z^2 dz}{(z-i)^3} = 2\pi i ,$$

where the contour C is the square with vertices at $\pm 2 + 2i$ and $\pm 2 - 2i$.

The residue of a complex function $f(z)$ at a pole $z = a$ of multiplicity m is given by the expression

$$\lim_{z \to a} \frac{1}{(m-1)!} \left[\frac{d^{m-1}}{dz^{m-1}} \{(z-a)^m f(z)\} \right] .$$

PLEASE TURN OVER

3. The complex function

$$\frac{e^{iz}}{z(z^2+1)(z^2+9)}$$

has a simple pole at $z = 0$, two simple poles in the upper half-plane at $z = i$ and $z = 3i$, and two more in the lower half-plane at $z = -i$ and $z = -3i$.

Show that

(i) the residue at $z = 0$ is $1/9$,

(ii) the residue at $z = i$ is $-e^{-1}/16$,

(iii) the residue at $z = 3i$ is $e^{-3}/144$.

Now consider the contour integral

$$\oint_C \frac{e^{iz}\,dz}{z(z^2+1)(z^2+9)},$$

where C is taken to be a large semi-circle in the upper half of the complex plane of radius R, with an additional small semi-circle taken below the pole at $z = 0$.

(iv) Show that the contribution to the above integral from this small semi-circle, in the limit when its radius goes to zero, is $\pi i/9$.

(v) Why is there no contribution to the above integral from the large semi-circle of radius R in the upper half-plane, in the limit when its radius goes to infinity?

(vi) Hence, show that

$$\int_{-\infty}^{\infty} \frac{\sin x\,dx}{x(x^2+1)(x^2+9)} = \frac{\pi(8e^3 - 9e^2 + 1)}{72e^3}.$$

The residue of a complex function $f(z)$ at a pole $z = a$ of multiplicity m is given by the expression

$$\lim_{z\to a} \frac{1}{(m-1)!}\left[\frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m f(z)\}\right].$$

4. (i) Use Fourier transforms to show that the Dirac delta-function has an integral representation of the form

$$\int_{-\infty}^{\infty} e^{\pm i a \tau}\, d\tau \;=\; 2\pi\delta(a)\,,$$

or with τ and a reversed.

(ii) Hence prove the integral relation between the two functions $f(t)$ and $g(t)$ and their Fourier transforms $\overline{f}(\omega)$ and $\overline{g}(\omega)$

$$\int_{-\infty}^{\infty} f(t) g^*(t)\, dt \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{f}(\omega)\overline{g}^*(\omega)\, d\omega\,,$$

where $*$ represents the complex conjugate.

(iii) If $f(t) = e^{-\omega_0|t|}$ and $g(t) = \cos(\Omega_0 t)$, where Ω_0 and $\omega_0 > 0$ are constant frequencies, show that

$$\int_{-\infty}^{\infty} e^{-\omega_0|t|}\cos(\Omega_0 t)\, dt \;=\; \frac{2\omega_0}{\omega_0^2+\Omega_0^2}\,.$$

5. You are given the result

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx \;=\; \pi\,.$$

(i) Use this to show that

$$\int_{-\infty}^{\infty} \frac{\sin pt}{t}\, dt \;=\; \begin{cases} +\pi, & p > 0 \\ -\pi, & p < 0, \end{cases}$$

where p is an arbitrary real number of either sign.

(ii) Use the result of the integral in (i) to show that the Fourier transform $\overline{f}(\omega)$ of the function

$$f(t) \;=\; \frac{\sin \frac{1}{2}t}{\frac{1}{2}t}$$

is the rectangle function

$$\overline{f}(\omega) \;=\; \begin{cases} 2\pi\,, & -\frac{1}{2} < \omega < \frac{1}{2}\,, \\ 0\,, & \omega < -\frac{1}{2},\;\; \frac{1}{2} < \omega\,. \end{cases}$$

Sketch $\overline{f}(\omega)$ against ω.

PLEASE TURN OVER

6. Two functions of time, $f(t)$ and $g(t)$, have Laplace transforms $\overline{f}(s) = \mathcal{L}\{f(t)\}$ and $\overline{g}(s) = \mathcal{L}\{g(t)\}$ respectively. The convolution product between these functions $f * g$ is defined by

$$f * g \;=\; \int_0^t f(u)g(t-u)\,du\ .$$

(i) Prove the Laplace convolution theorem

$$\mathcal{L}\{f * g\} \;=\; \overline{f}(s)\,\overline{g}(s)\,,$$

and hence show that

$$\mathcal{L}\left(\int_0^t f(u)\,du\right) \;=\; \frac{\overline{f}(s)}{s}.$$

(ii) Show also that if

$$\overline{G}(s) \;=\; \frac{1}{(1+s^2)^2}$$

then

$$G(t) \;=\; \tfrac{1}{2}\,(\sin t - t\cos t)\,.$$

Hence show that for $s > 0$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(1+s^2)^2}\right\} = 1 - \cos t - \tfrac{1}{2}t\sin t.$$

You may assume that $\mathcal{L}\{\sin\omega t\} \;=\; \frac{\omega}{s^2+\omega^2}$.

7. Consider a two-dimensional region R bounded by a closed piecewise smooth curve C. Using Green's Theorem in a plane, choose the components of a vector field $\boldsymbol{v}(x, y)$ in terms of two differentiable arbitrary scalar functions $P(x, y)$ and $Q(x, y)$ to prove the two-dimensional form of Stokes' Theorem

$$\int\int_R \boldsymbol{k} \cdot (\operatorname{curl} \boldsymbol{v})\, dxdy \;=\; \oint_C \boldsymbol{v} \cdot d\boldsymbol{r}\,. \qquad (*)$$

R is now designated as the region bounded by the parabola $y^2 = x$ and the line $y = x$.
Sketch this region.

If $\boldsymbol{v} = \frac{1}{2}(x^2\boldsymbol{i} + y^2\boldsymbol{j})$, show by direct calculation that

$$\oint_C \boldsymbol{v} \cdot d\boldsymbol{r} \;=\; 0\,.$$

By evaluating $\operatorname{curl} \boldsymbol{v}$, verify that

$$\int\int_R \boldsymbol{k} \cdot (\operatorname{curl} \boldsymbol{v})\, dxdy \;= 0\,.$$

Green's Theorem in a plane states that, for a two-dimensional region R bounded by a closed, piecewise smooth curve C,

$$\oint_C \{P(x,y)dx + Q(x,y)dy\} \;=\; \int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy\,.$$

PLEASE TURN OVER

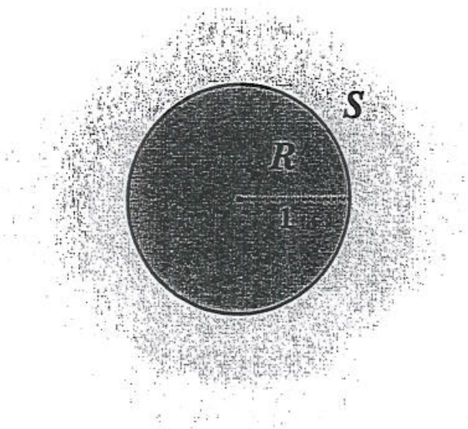
8. A change of variables from (u, v) to (x, y) is of the form $x = G_1(u, v)$ and $y = G_2(u, v)$. Define the Jacobian $\mathcal{J}$ of the change of variables in terms of a 2 x 2 determinant.

Suppose that

$$G_1(u, v) = \frac{u}{u^2 + v^2} \quad \text{and} \quad G_2(u, v) = \frac{v}{u^2 + v^2} .$$

Calculate $\mathcal{J}$.

Now consider the unit circle $u^2 + v^2 = 1$. Show that this change of variables maps the region R corresponding to the inside of the unit circle to the region S corresponding to the outside of the same circle.



Show also that the same change of variables maps the outside region S to the inside region R.

Write down an expression for the integral

$$\iint_R f(x, y)\, dx\, dy$$

in terms of an integral with respect to u and v over a region in the uv-plane which should be carefully identified. Hence compute the following integral

$$\iint_S \frac{4}{(u^2 + v^2)^2}\, du\, dv .$$

You may assume without calculation that the area of R is π.

9. Let $\boldsymbol{F} = (F_1, F_2, F_3)$ be a vector field and φ a scalar field in three dimensions. Define $\operatorname{grad}\varphi$, $\operatorname{div}\boldsymbol{F}$ and $\operatorname{curl}\boldsymbol{F}$.

(i) Show that

$$\operatorname{curl}(\varphi\boldsymbol{F}) = \varphi\operatorname{curl}\boldsymbol{F} + (\operatorname{grad}\varphi) \times \boldsymbol{F}$$

Hint: evaluate $\dfrac{\partial(\varphi F_3)}{\partial y} - \dfrac{\partial(\varphi F_2)}{\partial z}$ *and show that this is the first component of* $\varphi\operatorname{curl}\boldsymbol{F} + (\operatorname{grad}\varphi) \times \boldsymbol{F}$.

(ii) Hence show that if $\boldsymbol{v} = (v_1, v_2, v_3)$ is a vector such that

$$\boldsymbol{v} \cdot \operatorname{curl}\boldsymbol{F} = 0$$

then

$$\boldsymbol{v} \cdot \operatorname{curl}(\varphi\boldsymbol{F}) = \det\begin{vmatrix} v_1 & v_2 & v_3 \\ \dfrac{\partial\varphi}{\partial x} & \dfrac{\partial\varphi}{\partial y} & \dfrac{\partial\varphi}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

(iii) Let $\boldsymbol{k} = (0, 0, 1)$ be the unit vector in the z direction. Show that

$$\boldsymbol{k} \cdot \operatorname{curl}\boldsymbol{F} = 0$$

if and only if

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$$

(iv) Suppose that φ is independent of both x and y, that F_1 is independent of y and F_2 is independent of x.

Use parts (ii) and (iii) to show that

$$\boldsymbol{k} \cdot \operatorname{curl}(\varphi\boldsymbol{F}) = 0$$

PLEASE TURN OVER

10. Suppose that a vector field $\boldsymbol{F}(x, y)$ satisfies

$$\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = 0$$

for any closed loop C in the plane. Explain how to define a scalar potential $\varphi(x, y)$ for $\boldsymbol{F}$ and verify that your definition satisfies $\boldsymbol{F} = -\operatorname{grad} \varphi$.

Show that if $\boldsymbol{F} = -\operatorname{grad} \varphi$ then

$$\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y} .$$

Verify that this condition is satisfied for the following two-dimensional vector fields and derive the corresponding potential $\varphi(x, y)$ in each case:

(i) $\boldsymbol{F}(x, y) = (2x + y,\ x + 2y)$

(ii) $\boldsymbol{F}(x, y) = (3x^2y^2,\ 2x^3y)$

(iii) $\boldsymbol{F}(x, y) = (y \sin xy,\ x \sin xy)$

11. (i) Distributor A supplies 87% of the memory chips used by a computer firm and 5% of them are defective. Distributor B supplies the remaining 13%, of which 8% is defective.

    (a) Write down the expression for the probability that a randomly selected chip is defective.

    (b) Assuming that two randomly selected chips are both defective, obtain the probability that both are from Distributor A.

    (ii) Suppose a manufacturer of memory chips observes that the probability of chip failure is $p = 0.05$. A new procedure is introduced to improve the design of chips. 200 chips are produced using this new procedure and each is tested. We assume that these 200 tests are independent and each chip has the same probability of failure. We set the rule that we would accept the new procedure if 3 or fewer chips fail. Let

$$H_0: \ p = 0.05 \quad \text{and} \quad H_1: \ p < 0.05.$$

    Carefully define the Type I error and compute its probability.

    [*Hint: use the Poisson approximation to the Binomial distribution.*]

12. A computer network performance indicator X is modelled as a random variable with probability density function

$$f_X(x) = \begin{cases} x/8 & \text{if } \ 0 \leq x \leq 2, \\ k & \text{if } \ 2 < x < 4, \\ (6-x)/8 & \text{if } \ 4 \leq x \leq 6. \end{cases}$$

    (i) Find the value of the constant k.

    (ii) Sketch the pdf.

    (iii) Calculate the expected value $E(X)$.

    (iv) Calculate the variance $\text{var}\,(X)$.

    (v) Calculate the probability that the network performance indicator takes a value that does not exceed 5.

END OF PAPER

Pt 2 - Maths J

MATHEMATICS DEPARTMENT

MATHEMATICAL FORMULAE

1. VECTOR ALGEBRA

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1,\, a_2,\, a_3)$$

Scalar (dot) product: $\mathbf{a}\,.\,\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

Vector (cross) product:

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Scalar triple product:

$$[\mathbf{a},\,\mathbf{b},\,\mathbf{c}] = \mathbf{a}\,.\,\mathbf{b}\times\mathbf{c} = \mathbf{b}\,.\,\mathbf{c}\times\mathbf{a} = \mathbf{c}\,.\,\mathbf{a}\times\mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product: $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{c}\,.\,\mathbf{a})\mathbf{b} - (\mathbf{b}\,.\,\mathbf{a})\mathbf{c}$

2. SERIES

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \dots \quad (\alpha \text{ arbitrary, } |x|<1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots ,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^n\frac{x^{2n}}{(2n)!} + \dots ,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + \dots ,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^n\frac{x^{n+1}}{(n+1)} + \dots (-1 < x \le 1)$$

3. TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin(a+b) = \sin a\,\cos b + \cos a\,\sin b\,;$$

$$\cos(a+b) = \cos a\,\cos b - \sin a\,\sin b\,.$$

$$\cos iz = \cosh z\,;\quad \cosh iz = \cos z\,;\quad \sin iz = i\sinh z\,;\quad \sinh iz = i\sin z\,.$$

4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f\,D^n g + \tbinom{n}{1} Df\,D^{n-1}g + \dots + \tbinom{n}{r} D^r f\,D^{n-r}g + \dots + D^n f\;g\,.$$

(b) Taylor's expansion of $f(x)$ about $x = a$:

$$f(a+h) = f(a) + hf'(a) + h^2 f''(a)/2! + \dots + h^n f^{(n)}(a)/n! + \epsilon_n(h)\,,$$

where $\epsilon_n(h) = h^{n+1}f^{(n+1)}(a+\theta h)/(n+1)!, \quad 0<\theta<1\,.$

(c) Taylor's expansion of $f(x,\,y)$ about $(a,\,b)$:

$$f(a+h,\,b+k) = f(a,\,b) + [hf_x + kf_y]_{a.b} + 1/2!\left[h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}\right]_{a.b} + \dots$$

(d) Partial differentiation of $f(x,\,y)$:

i. If $y = y(x)$, then $f = F(x)$, and $\dfrac{dF}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dx}$.

ii. If $x = x(t)$, $y = y(t)$, then $f = F(t)$, and $\dfrac{dF}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$.

iii. If $x = x(u,\,v)$, $y = y(u,\,v)$, then $f = F(u,\,v)$, and

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}\,,\quad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}\,.$$

(e) Stationary points of $f(x,\,y)$ occur where $f_x = 0,\ f_y = 0$ simultaneously.
Let $(a,\,b)$ be a stationary point: examine $D = [f_{xx}f_{yy} - (f_{xy})^2]_{a.b}$.
If $D>0$ and $f_{xx}(a,\,b) < 0$, then $(a,\,b)$ is a maximum;
If $D>0$ and $f_{xx}(a,\,b) > 0$, then $(a,\,b)$ is a minimum;
If $D<0$ then $(a,\,b)$ is a saddle-point.

(f) Differential equations:

i. The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating factor $I(x) = \exp[\int P(x)(dx]$, so that $\dfrac{d}{dx}(Iy) = IQ$.

ii. $P(x,\,y)dx + Q(x,\,y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y$.

5. INTEGRAL CALCULUS

(a) An important substitution: $\tan(\theta/2) = t$:
$\sin\theta = 2t/(1+t^2)$, $\cos\theta = (1-t^2)/(1+t^2)$, $d\theta = 2\,dt/(1+t^2)$.

(b) Some indefinite integrals:

$$\int (a^2 - x^2)^{-1/2} dx = \sin^{-1}\left(\frac{x}{a}\right), \quad |x| < a.$$

$$\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1}\left(\frac{x}{a}\right) = \ln\left\{\frac{x}{a} + \left(1 + \frac{x^2}{a^2}\right)^{1/2}\right\}.$$

$$\int (x^2 - a^2)^{-1/2} dx = \cosh^{-1}\left(\frac{x}{a}\right) = \ln\left|\frac{x}{a} + \left(\frac{x^2}{a^2} - 1\right)^{1/2}\right|.$$

$$\int (a^2 + x^2)^{-1} dx = \left(\frac{1}{a}\right)\tan^{-1}\left(\frac{x}{a}\right).$$

6. NUMERICAL METHODS

(a) Approximate solution of an algebraic equation:

If a root of $f(x) = 0$ occurs near $x = a$, take $x_0 = a$ and
$x_{n+1} = x_n - [f(x_n)/f'(x_n)]$, $n = 0, 1, 2 \ldots$

(Newton Raphson method).

(b) Formulae for numerical integration: Write $x_n = x_0 + nh$, $y_n = y(x_n)$.

i. Trapezium rule (1-strip): $\int_{x_0}^{x_1} y(x)dx \approx (h/2)[y_0 + y_1]$.

ii. Simpson's rule (2-strip): $\int_{x_0}^{x_2} y(x)dx \approx (h/3)[y_0 + 4y_1 + y_2]$.

(c) Richardson's extrapolation method: Let $I = \int_a^b f(x)dx$ and let I_1, I_2 be two estimates of I obtained by using Simpson's rule with intervals h and $h/2$.

Then, provided h is small enough,

$$I_2 + (I_2 - I_1)/15,$$

is a better estimate of I.

7. LAPLACE TRANSFORMS

Function	Transform	Function	Transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)dt$	$af(t) + bg(t)$	$aF(s) + bG(s)$
df/dt	$sF(s) - f(0)$	d^2f/dt^2	$s^2F(s) - sf(0) - f'(0)$
$e^{at}f(t)$	$F(s-a)$	$tf(t)$	$-dF(s)/ds$
$(\partial/\partial\alpha)f(t,\alpha)$	$(\partial/\partial\alpha)F(s,\alpha)$	$\int_0^t f(t)dt$	$F(s)/s$
$\int_0^t f(u)g(t-u)du$	$F(s)G(s)$		
1	$1/s$	$t^n (n = 1, 2 \ldots)$	$n!/s^{n+1}$, $(s > 0)$
e^{at}	$1/(s-a)$, $(s > a)$	$\sin\omega t$	$\omega/(s^2+\omega^2)$, $(s > 0)$
$\cos\omega t$	$s/(s^2+\omega^2)$, $(s > 0)$	$H(t-T) = \begin{cases} 0, & t < T \\ 1, & t > T \end{cases}$	e^{-sT}/s, $(s, T > 0)$

8. FOURIER SERIES

If $f(x)$ is periodic of period $2L$, then $f(x+2L) = f(x)$, and

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \text{ where}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad n = 0, 1, 2, \ldots, \text{ and}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1, 2, 3, \ldots$$

Parseval's theorem

$$\frac{1}{L}\int_{-L}^{L}[f(x)]^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right).$$

Part 2 - Maths J

1. Probabilities for events

For events A, B, and C $\quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$

More generally $\quad P(\bigcup A_i) = \sum P(A_i) - \sum P(A_i \cap A_j) + \sum P(A_i \cap A_j \cap A_k) - \cdots$

The odds in favour of A $\quad P(A)\,/\,P(\overline{A})$

Conditional probability $\quad P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$ provided that $P(B) > 0$

Chain rule $\quad P(A \cap B \cap C) = P(A)\,P(B \mid A)\,P(C \mid A \cap B)$

Bayes' rule $\quad P(A \mid B) = \dfrac{P(A)\,P(B \mid A)}{P(A)\,P(B \mid A) + P(\overline{A})\,P(B \mid \overline{A})}$

A and B are independent if $\quad P(B \mid A) = P(B)$

A, B, and C are independent if $\quad P(A \cap B \cap C) = P(A)P(B)P(C)$, and

$$P(A \cap B) = P(A)P(B), \quad P(B \cap C) = P(B)P(C), \quad P(C \cap A) = P(C)P(A)$$

2. Probability distribution, expectation and variance

The probability distribution for a discrete random variable X is the complete set of probabilities $\{p_x\} = \{P(X = x)\}$

Expectation $\quad E(X) = \mu = \sum_x x p_x$

Sample mean $\quad \overline{x} = \dfrac{1}{n} \sum_k x_k$ estimates μ from random sample $x_1, x_2, \ldots, x_n$

Variance $\quad \operatorname{var}(X) = \sigma^2 = E\{(X - \mu)^2\} = E(X^2) - \mu^2$, where $E(X^2) = \sum_x x^2 p_x$

Sample variance $\quad s^2 = \dfrac{1}{n-1} \left\{ \sum_k x_k^2 - \dfrac{1}{n} \left(\sum_j x_j \right)^2 \right\}$ estimates σ^2

Standard deviation $\quad \operatorname{sd}(X) = \sigma$

If value y is observed with frequency n_y

$$n = \sum_y n_y, \quad \sum_k x_k = \sum_y y n_y, \quad \sum_k x_k^2 = \sum_y y^2 n_y$$

For function $g(x)$ of x, $\quad E\{g(X)\} = \sum_x g(x) p_x$

Skewness $\quad \beta_1 = E\left(\dfrac{X - \mu}{\sigma}\right)^3$ is estimated by $\dfrac{1}{n-1} \sum \left(\dfrac{x_i - \overline{x}}{s}\right)^3$

Kurtosis $\quad \beta_2 = E\left(\dfrac{X - \mu}{\sigma}\right)^4 - 3$ is estimated by $\dfrac{1}{n-1} \sum \left(\dfrac{x_i - \overline{x}}{s}\right)^4 - 3$

Sample median $\tilde{x}$. If the sample values $x_1, \ldots, x_n$ are ordered $x_{(1)} \le x_{(2)} \le \cdots \le x_{(n)}$

$$\tilde{x} = x_{(\frac{n+1}{2})} \text{ if } n \text{ is odd, and } \tilde{x} = \tfrac{1}{2}\left(x_{(\frac{n}{2})} + x_{(\frac{n+2}{2})}\right) \text{ if } n \text{ is even.}$$

α-quantile $Q(\alpha)$ is such that $P(X \le Q(\alpha)) = \alpha$

Sample α-quantile $\widehat{Q}(\alpha)$ is the sample value for which the proportion of values $\le \widehat{Q}(\alpha)$ is α (using linear interpolation between values on either side)

The sample median $\tilde{x}$ estimates the population median $Q(0.5)$.

3. Probability distribution for a continuous random variable

The cumulative distribution function (cdf) $F(x) = P(X \le x) = \int_{x_0=-\infty}^{x} f(x_0)\mathrm{d}x_0$

The probability density function (pdf) $f(x) = \dfrac{\mathrm{d}F(x)}{\mathrm{d}x}$

$$E(X) = \mu = \int_{-\infty}^{\infty} x\, f(x)\mathrm{d}x, \quad \mathrm{var}\,(X) = \sigma^2 = E(X^2) - \mu^2,$$

where $E(X^2) = \int_{-\infty}^{\infty} x^2\, f(x)\mathrm{d}x$

4. Discrete probability distributions

Discrete Uniform *Uniform* (n)

$$p_x = \frac{1}{n} \quad (x = 1, 2, \ldots, n) \qquad \mu = \tfrac{1}{2}(n+1), \quad \sigma^2 = \tfrac{1}{12}(n^2 - 1)$$

Binomial distribution *Binomial* (n, θ)

$$p_x = \binom{n}{x} \theta^x (1-\theta)^{n-x} \quad (x = 0, 1, 2, \ldots, n) \qquad \mu = n\theta, \quad \sigma^2 = n\theta(1-\theta)$$

Poisson distribution *Poisson* (λ)

$$p_x = \frac{\lambda^x e^{-\lambda}}{x!} \quad (x = 0, 1, 2, \ldots) \quad (\text{with } \lambda > 0) \qquad \mu = \lambda, \quad \sigma^2 = \lambda$$

Geometric distribution *Geometric* (θ)

$$p_x = (1-\theta)^{x-1}\theta \quad (x = 1, 2, 3, \ldots) \qquad \mu = \frac{1}{\theta}, \quad \sigma^2 = \frac{1-\theta}{\theta^2}$$

5. Continuous probability distributions

Uniform distribution *Uniform* (α, β)

$$f(x) = \begin{cases} \dfrac{1}{\beta - \alpha} & (\alpha < x < \beta), \\ 0 & (\text{otherwise}). \end{cases} \qquad \mu = (\alpha + \beta)/2, \quad \sigma^2 = (\beta - \alpha)^2/12.$$

Exponential distribution *Exponential* (λ)

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & (0 < x < \infty), \\ 0 & (-\infty < x \leq 0). \end{cases} \qquad \begin{array}{l} \mu = 1/\lambda\,, \\ \sigma^2 = 1/\lambda^2\,. \end{array}$$

Normal distribution $N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2 \right\} \quad (-\infty < x < \infty)$$

$$E(X) = \mu\,, \quad \mathrm{var}\,(X) = \sigma^2$$

Standard normal distribution $N(0,1)$

If X is $N(\mu, \sigma^2)$, then $Y = \dfrac{X-\mu}{\sigma}$ is $N(0,1)$

6. Reliability

For a device in continuous operation with failure time random variable T having pdf $f(t)$ $(t > 0)$

The reliability function at time t	$R(t)$	$=$	$P(T > t)$		
The failure rate or hazard function	$h(t)$	$=$	$f(t)/R(t)$		
The cumulative hazard	$H(t)$	$=$	$\int_0^t h(t_0)\,\mathrm{d}t_0$	$=$	$-\ln\{R(t)\}$
The Weibull(α, β) distribution has	$H(t)$	$=$	βt^α		

7. System reliability

For a system of k devices, which operate independently, let

$R_i = P(D_i) = P($"device i operates"$)$

The system reliability, R, is the probability of a path of operating devices

A system of devices in series operates only if every device operates

$$R = P(D_1 \cap D_2 \cap \cdots \cap D_k) = R_1 R_2 \cdots R_k$$

A system of devices in parallel operates if any device operates

$$R = P(D_1 \cup D_2 \cup \cdots \cup D_k) = 1 - (1-R_1)(1-R_2)\cdots(1-R_k)$$

8. Covariance and correlation

The covariance of X and Y $\quad \mathrm{cov}\,(X,Y) = E(XY) - \{E(X)\}\{E(Y)\}$

From pairs of observations $(x_1, y_1), \ldots, (x_n, y_n)$ $\quad S_{xy} = \sum_k x_k y_k - \frac{1}{n}(\sum_i x_i)(\sum_j y_j)$

$$S_{xx} = \sum_k x_k^2 - \frac{1}{n}(\sum_i x_i)^2, \qquad S_{yy} = \sum_k y_k^2 - \frac{1}{n}(\sum_j y_j)^2$$

Sample covariance $\quad s_{xy} = \frac{1}{n-1} S_{xy}$ estimates $\mathrm{cov}\,(X, Y)$

Correlation coefficient $\quad \rho = \mathrm{corr}\,(X, Y) = \dfrac{\mathrm{cov}\,(X, Y)}{\mathrm{sd}\,(X) \cdot \mathrm{sd}\,(Y)}$

Sample correlation coefficient $\quad r = \dfrac{S_{xy}}{\sqrt{S_{xx} S_{yy}}}$ estimates ρ

9. Sums of random variables

$$E(X+Y) = E(X) + E(Y)$$
$$\mathrm{var}\,(X+Y) = \mathrm{var}\,(X) + \mathrm{var}\,(Y) + 2\,\mathrm{cov}\,(X, Y)$$
$$\mathrm{cov}\,(aX + bY,\ cX + dY) = (ac)\,\mathrm{var}\,(X) + (bd)\,\mathrm{var}\,(Y) + (ad + bc)\,\mathrm{cov}\,(X, Y)$$

If X is $N(\mu_1, \sigma_1^2)$, Y is $N(\mu_2, \sigma_2^2)$, and $\mathrm{cov}\,(X, Y) = c$,
then $X + Y$ is $N(\mu_1 + \mu_2,\ \sigma_1^2 + \sigma_2^2 + 2c)$

10. Bias, standard error, mean square error

If t estimates θ (with random variable T giving t)

Bias of t $\quad \mathrm{bias}(t) = E(T) - \theta$

Standard error of t $\quad \mathrm{se}\,(t) = \mathrm{sd}\,(T)$

Mean square error of t $\quad$ MSE(t) $= E\{(T-\theta)^2\} = \{\mathrm{se}\,(t)\}^2 + \{\mathrm{bias}(t)\}^2$

If $\overline{x}$ estimates μ, then $\mathrm{bias}(\overline{x}) = 0$, $\mathrm{se}\,(\overline{x}) = \sigma/\sqrt{n}$, MSE$(\overline{x}) = \sigma^2/n$, $\widehat{\mathrm{se}}\,(\overline{x}) = s/\sqrt{n}$

Central limit property $\quad$ if n is fairly large, $\overline{x}$ is from $N(\mu,\ \sigma^2/n)$ approximately

11. Likelihood

The likelihood is the joint probability as a function of the unknown parameter θ.
For a random sample $x_1, x_2, \ldots, x_n$

$$\ell(\theta;\, x_1, x_2, \ldots, x_n) = P(X_1 = x_1 \mid \theta) \cdots P(X_n = x_n \mid \theta) \quad \text{(discrete distribution)}$$

$$\ell(\theta;\, x_1, x_2, \ldots, x_n) = f(x_1 \mid \theta) f(x_2 \mid \theta) \cdots f(x_n \mid \theta) \quad \text{(continuous distribution)}$$

The maximum likelihood estimator (MLE) is $\widehat{\theta}$ for which the likelihood is a maximum.

12. Confidence intervals

If $x_1, x_2, \ldots, x_n$ are a random sample from $N(\mu, \sigma^2)$ and σ^2 is known, then the 95% confidence interval for μ is $(\overline{x} - 1.96\frac{\sigma}{\sqrt{n}},\ \overline{x} + 1.96\frac{\sigma}{\sqrt{n}})$

If σ^2 is estimated, then from the Student t table for t_{n-1} we find $t_0 = t_{n-1,0.05}$

The 95% confidence interval for μ is $(\overline{x} - t_0\frac{s}{\sqrt{n}},\ \overline{x} + t_0\frac{s}{\sqrt{n}})$

13. Standard normal table

Values of pdf $\phi(y) = f(y)$ and cdf $\Phi(y) = F(y)$

y	$\phi(y)$	$\Phi(y)$	y	$\phi(y)$	$\Phi(y)$	y	$\phi(y)$	$\Phi(y)$	y	$\Phi(y)$
0	.399	.5	.9	.266	.816	1.8	.079	.964	2.8	.997
.1	.397	.540	1.0	.242	.841	1.9	.066	.971	3.0	.999
.2	.391	.579	1.1	.218	.864	2.0	.054	.977	0.841	.8
.3	.381	.618	1.2	.194	.885	2.1	.044	.982	1.282	.9
.4	.368	.655	1.3	.171	.903	2.2	.035	.986	1.645	.95
.5	.352	.691	1.4	.150	.919	2.3	.028	.989	1.96	.975
.6	.333	.726	1.5	.130	.933	2.4	.022	.992	2.326	.99
.7	.312	.758	1.6	.111	.945	2.5	.018	.994	2.576	.995
.8	.290	.788	1.7	.094	.955	2.6	.014	.995	3.09	.999

14. Student t table

Values $t_{m,p}$ of x for which $P(|X| > x) = p$, when X is t_m

m	p= 0.10	0.05	0.02	0.01	m	p= 0.10	0.05	0.02	0.01
1	6.31	12.71	31.82	63.66	9	1.83	2.26	2.82	3.25
2	2.92	4.30	6.96	9.92	10	1.81	2.23	2.76	3.17
3	2.35	3.18	4.54	5.84	12	1.78	2.18	2.68	3.05
4	2.13	2.78	3.75	4.60	15	1.75	2.13	2.60	2.95
5	2.02	2.57	3.36	4.03	20	1.72	2.09	2.53	2.85
6	1.94	2.45	3.14	3.71	25	1.71	2.06	2.48	2.78
7	1.89	2.36	3.00	3.50	40	1.68	2.02	2.42	2.70
8	1.86	2.31	2.90	3.36	∞	1.645	1.96	2.326	2.576

15. Chi-squared table

Values $\chi^2_{k,p}$ of x for which $P(X > x) = p$, when X is χ^2_k and $p = .995, .975,$ *etc*

k	.995	.975	.05	.025	.01	.005	k	.995	.975	.05	.025	.01	.005
1	.000	.001	3.84	5.02	6.63	7.88	18	6.26	8.23	28.87	31.53	34.81	37.16
2	.010	.051	5.99	7.38	9.21	10.60	20	7.43	9.59	31.42	34.17	37.57	40.00
3	.072	.216	7.81	9.35	11.34	12.84	22	8.64	10.98	33.92	36.78	40.29	42.80
4	.207	.484	9.49	11.14	13.28	14.86	24	9.89	12.40	36.42	39.36	42.98	45.56
5	.412	.831	11.07	12.83	15.09	16.75	26	11.16	13.84	38.89	41.92	45.64	48.29
6	.676	1.24	12.59	14.45	16.81	18.55	28	12.46	15.31	41.34	44.46	48.28	50.99
7	.990	1.69	14.07	16.01	18.48	20.28	30	13.79	16.79	43.77	46.98	50.89	53.67
8	1.34	2.18	15.51	17.53	20.09	21.95	40	20.71	24.43	55.76	59.34	63.69	66.77
9	1.73	2.70	16.92	19.02	21.67	23.59	50	27.99	32.36	67.50	71.41	76.15	79.49
10	2.16	3.25	13.31	20.48	23.21	25.19	60	35.53	40.48	79.08	83.30	88.38	91.95
12	3.07	4.40	21.03	23.34	26.22	28.30	70	43.28	48.76	90.53	95.02	100.4	104.2
14	4.07	5.63	23.68	26.12	29.14	31.32	80	51.17	57.15	101.9	106.6	112.3	116.3
16	5.14	6.91	26.30	28.85	32.00	34.27	100	67.33	74.22	124.3	129.6	135.8	140.2

16. The chi-squared goodness-of-fit test

The frequencies n_y are grouped so that the fitted frequency $\widehat{n}_y$ for every group exceeds about 5.

$X^2 = \sum_y \frac{(n_y - \widehat{n}_y)^2}{\widehat{n}_y}$ is referred to the table of χ^2_k with significance point p,

where k is the number of terms summed, less one for each constraint, *eg* matching total frequency, and matching $\overline{x}$ with μ.

17. Joint probability distributions

Discrete distribution $\{p_{xy}\}$, where $p_{xy} = P(\{X = x\} \cap \{Y = y\})$.

Let $p_{x\bullet} = P(X = x)$, and $p_{\bullet y} = P(Y = y)$, then

$$p_{x\bullet} = \sum_y p_{xy}, \quad \text{and} \quad P(X = x \,|\, Y = y) = \frac{p_{xy}}{p_{\bullet y}}$$

Continuous distribution

Joint cdf $\quad F(x,y) = P(\{X \le x\} \cap \{Y \le y\}) = \int_{x_0=-\infty}^{x} \int_{y_0=-\infty}^{y} f(x_0\,, y_0)\,\mathrm{d}x_0\,\mathrm{d}y_0$

Joint pdf $\quad f(x,y) = \dfrac{\mathrm{d}^2 F(x,y)}{\mathrm{d}x\,\mathrm{d}y}$

Marginal pdf of X $\quad f_X(x) = \int_{-\infty}^{\infty} f(x,y_0)\,\mathrm{d}y_0$

Conditional pdf of X given $Y=y$ $\quad f_{X|Y}(x|y) = \dfrac{f(x,y)}{f_Y(y)}$ (provided $f_Y(y) > 0$)

18. Linear regression

To fit the linear regression model $y = \alpha + \beta x$ by $\widehat{y}_x = \widehat{\alpha} + \widehat{\beta} x$ from observations $(x_1, y_1), \ldots, (x_n, y_n)$, the least squares fit is

$$\widehat{\alpha} = \overline{y} - \overline{x}\widehat{\beta}, \quad \widehat{\beta} = S_{xy}/S_{xx}$$

The residual sum of squares $\quad \mathrm{RSS} = S_{yy} - \dfrac{S_{xy}^2}{S_{xx}}$

$$\widehat{\sigma^2} = \frac{\mathrm{RSS}}{n-2}, \qquad \frac{n-2}{\sigma^2}\,\widehat{\sigma^2} \text{ is from } \chi^2_{n-2}$$

$$E(\widehat{\alpha}) = \alpha, \quad E(\widehat{\beta}) = \beta,$$

$$\mathrm{var}(\widehat{\alpha}) = \frac{\sum x_i^2}{n\,S_{xx}}\sigma^2, \quad \mathrm{var}(\widehat{\beta}) = \frac{\sigma^2}{S_{xx}}, \quad \mathrm{cov}(\widehat{\alpha}, \widehat{\beta}) = -\frac{\overline{x}}{S_{xx}}\sigma^2$$

$$\widehat{y}_x = \widehat{\alpha} + \widehat{\beta} x, \qquad E(\widehat{y}_x) = \alpha + \beta x, \qquad \mathrm{var}(\widehat{y}_x) = \left\{ \frac{1}{n} + \frac{(x-\overline{x})^2}{S_{xx}} \right\} \sigma^2$$

$$\frac{\widehat{\alpha} - \alpha}{\widehat{\mathrm{se}}\,(\widehat{\alpha})}, \quad \frac{\widehat{\beta} - \beta}{\widehat{\mathrm{se}}\,(\widehat{\beta})}, \quad \frac{\widehat{y}_x - \alpha - \beta x}{\widehat{\mathrm{se}}\,(\widehat{y}_x)} \quad \text{are each from } t_{n-2}$$

19. Design matrix for factorial experiments $\quad$ With 3 factors each at 2 levels

$$\mathbf{X} = \begin{pmatrix} 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

A5 E1

E1

	EXAMINATION QUESTIONS/SOLUTIONS 2005-06	Course II(3) Core Q8
Question 5(c)		Marks & seen/unseen
Parts		
(a)	if $x^2+y^2=R^2$, then $z = Re^{i\theta}$	2
	Also $1+i = \sqrt{2}e^{i\pi/4}$	1
	Thus $w = (\sqrt{2}/R)e^{-i(\theta-\frac{\pi}{4})} + 1$	
	$\Rightarrow u-1 = \frac{\sqrt{2}}{R}\cos\left(\theta-\frac{\pi}{4}\right)$	
	& $v = -\frac{\sqrt{2}}{R}\sin\left(\theta-\frac{\pi}{4}\right)$	
	Finally $(u-1)^2+v^2 = \frac{2}{R^2}$	5
(b)	The directions of traverse are opposite because of the sign change in front of $\sin\left(\theta-\frac{\pi}{4}\right)$	2
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A5 E1

E1

	EXAMINATION QUESTIONS/SOLUTIONS 2005-06	Course II(3) Core Q8
Question SOL		Marks & seen/unseen

Parts

c) If $y = \alpha x$, $w = \dfrac{1+i}{(1+i\alpha)x} + 1$ — 2

because $z = x + iy = (1+i\alpha)x$

Thus $w = \dfrac{(1+i)(1-i\alpha)}{(1+\alpha^2)x} + 1$

$$= \frac{(1+\alpha) + i(1-\alpha)}{(1+\alpha^2)x} + 1$$

$$\Rightarrow u - 1 = \frac{1+\alpha}{(1+\alpha^2)x}$$

$$v = \frac{1-\alpha}{(1+\alpha^2)x}$$ — 4

Hence $\dfrac{u-1}{v} = \dfrac{1+\alpha}{1-\alpha}$ — 2

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A S E1

E1

	EXAMINATION QUESTIONS/SOLUTIONS 2005-06	Course II(3) Core Q8
Question SOL		Marks & seen/unseen
Parts	or $v = \left(\frac{1-\alpha}{1+\alpha}\right)(u-1)$ This is a family of straight lines passing through the point $(1,0)$ The slope of the line corresponding to $y=\alpha x$ is $\left(\frac{1-\alpha}{1+\alpha}\right)$	 1 1 $\boxed{20}$
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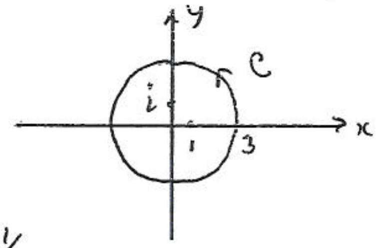
	EXAMINATION QUESTIONS/SOLUTIONS 2005-06	Course Paper 3
Question EE2		Marks & seen/unseen

Parts

a) $F(z) = \dfrac{z}{(z-1)^2(z-i)}$ has a simple pole at $z=i$ and a double pole at $z=1$ — Unseen, 2



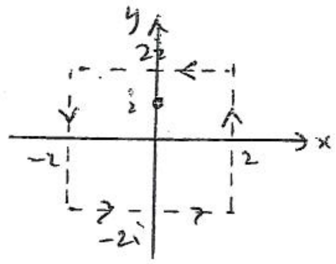
1) Residue at $z=i$ is $\dfrac{i}{(i-1)^2} = -\frac{1}{2}$ — 2

2) " " $z=1$ is $\lim_{z\to 1} \dfrac{d}{dz}\left(\dfrac{z}{z-i}\right) = \dfrac{-i}{(1-i)^2} = \frac{1}{2}$ — 3

Sum of residues $= -\frac{1}{2}+\frac{1}{2} = 0 \Rightarrow \oint_C F(z)\,dz = 0$ Res. Thm. — 2

If C is changed to a circle radius ½ centred at O the answer remains zero because no poles lie in C. — 1

b) $F(z) = \dfrac{z^2}{(z-i)^3}$ has a triple pole at $z=i$.



Residue at $z=i$ is

$$\lim_{z\to i} \frac{1}{2!}\frac{d^2}{dz^2}\left[(z-i)^3 F(z)\right]$$

$$= \frac{1}{2}\lim_{z\to i}\frac{d^2}{dz^2} z^2 = \frac{1}{2}\cdot 2 = 1$$ — 6

No other poles lie in C.

By Residue Thm $\oint_C F(z)\,dz = 2\pi i \times 1 = 2\pi i$ — 2

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E3

EXAMINATION QUESTIONS/SOLUTIONS 2005-06

Course: Paper 3

Question: EE3

Marks & seen/unseen: Unseen

Parts

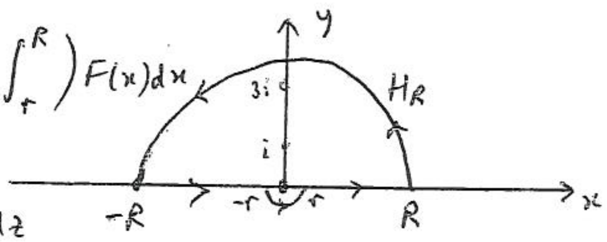
$$F(z) = \frac{e^{iz}}{z(z^2+1)(z^2+9)}$$ has poles (simple) at $z=0$, $z=i$ & $z=3i$ in upper ½-plane

a) Residue of $F(z)$ at $z=0$ is: $\frac{e^{-0}}{1\times 9} = \frac{1}{9}$ — 2

b) " " " at $z=i$ is: $\frac{e^{-1}}{i\times 2i\times 8} = -e^{-1}/16$ — 2

c) " " " at $z=3i$ is $\frac{e^{-3}}{3i\times(-8)\times 6i} = e^{-3}/144$ — 2

$$\oint_C F(z)\,dz = \left(\int_{-R}^{-r} + \int_r^R\right) F(x)\,dx + \int_{C_r} F(z)\,dz + \int_{H_R} F(z)\,dz$$ — 3



$C_r = re^{i\theta}$

$\theta: \pi \to 2\pi$

d) Now $\lim_{r\to 0}\int_{C_r} F(z)\,dz$

$$= \lim_{r\to 0}\int_\pi^{2\pi} \frac{e^{ire^{i\theta}}\, r i e^{i\theta}\, d\theta}{re^{i\theta}(r^2e^{2i\theta}+1)(r^2e^{2i\theta}+9)}$$

$$= i\int_\pi^{2\pi} \frac{1}{9}\,d\theta = \pi i/9.$$ — 5

e) Residue Theorem says:

$$\left(\frac{1}{9} - \frac{e^{-1}}{16} + \frac{e^{-3}}{144}\right)2\pi i = \lim_{R\to\infty}\left(\int_{-R}^{R} \frac{e^{ix}\,dx}{x(x^2+1)(x^2+9)} + \int_{H_R} F(z)\,dz\right) + \pi i/9$$ — 3

zero by Jordan's Lemma

Thus $$\int_{-\infty}^{\infty} \frac{\sin x\,dx}{x(x^2+1)(x^2+9)} = \pi\left(\frac{1}{9} - \frac{e^{-1}}{8} + \frac{e^{-3}}{72}\right)$$ — 2

$$= \frac{\pi}{72e^3}\left(8e^3 - 9e^2 + 1\right)$$ — 1

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E4

	EXAMINATION QUESTIONS/SOLUTIONS 2005-06	Course Paper 3
Question EE4		Marks & seen/unseen

Parts

a) $f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(\omega) e^{i\omega t} d\omega$; $\bar{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$ — Seen

$\therefore 2\pi \bar{f}(\omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \bar{f}(\omega') e^{i\omega' t - i\omega t} dt\, d\omega'$ (Reverse integ)

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{i(\omega'-\omega)t} dt \right) \bar{f}(\omega') d\omega'$$

According to the given δ-fn. relation we have

$$\delta(\omega'-\omega) = 2\pi \int_{-\infty}^{\infty} e^{i(\omega'-\omega)t} dt$$ — 6

If one performs this in reverse order one gets a "$-i$" with ω & t reversed etc – various options here.

b) $4\pi^2 \int_{-\infty}^{\infty} f(t) g^*(t) dt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \bar{f}(\omega) e^{i\omega t} d\omega \int_{-\infty}^{\infty} \bar{g}^*(\omega') e^{-i\omega' t} d\omega' dt$ — seen

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \underbrace{\left(\int_{-\infty}^{\infty} e^{i(\omega-\omega')t} dt \right)}_{2\pi\delta(\omega-\omega')} \bar{f}(\omega) \bar{g}^*(\omega') d\omega'$$

$$= 2\pi \int_{-\infty}^{\infty} \bar{f}(\omega)\delta(\omega-\omega')\bar{g}^*(\omega') d\omega\, d\omega' = 2\pi \int_{-\infty}^{\infty} \bar{f}(\omega)\bar{g}^*(\omega) d\omega$$ — 6

Thus $\int_{-\infty}^{\infty} f(t)\bar{g}^*(t) dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(\omega)\bar{g}^*(\omega) d\omega$

c) $f(t) = e^{-\omega_0 |t|}$; $g(t) = \cos \Omega_0 t$; $|t| = \begin{cases} t & t>0 \\ -t & t<0 \end{cases}$ — Unseen

$\bar{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t - \omega_0 |t|} dt = \int_{-\infty}^{0} e^{t(\omega_0 - i\omega)} dt + \int_{0}^{\infty} e^{-t(\omega_0 + i\omega)} dt$

$\bar{f}(\omega) = \frac{1}{\omega_0 - i\omega} + \frac{1}{\omega_0 + i\omega} = \frac{2\omega_0}{\omega_0^2 + \omega^2}$ $(\omega_0 > 0)$ — 8

$\bar{g}(\omega) = \pi[\delta(\omega - \Omega_0) + \delta(\omega + \Omega_0)]$.

$$\int_{-\infty}^{\infty} e^{-\omega_0|t|} \cos(\Omega_0 t) dt = \frac{\pi 2\omega_0}{2\pi} \int_{-\infty}^{\infty} \frac{\delta(\omega-\Omega_0)+\delta(\omega+\Omega_0)}{\omega_0 + \omega^2} d\omega$$

$$= \frac{2\omega_0}{\omega_0^2 + \Omega_0^2}$$

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E5

	EXAMINATION QUESTIONS/SOLUTIONS 2005-06	Course Paper 3
Question EE5		Marks & seen/unseen

Parts

$$\int_{-\infty}^{\infty} \frac{\sin pt}{t}\,dt \overset{(x=pt)}{=} \int_{-p\infty}^{p\infty} \frac{\sin x}{x}\,dx = \begin{cases} +\pi & p>0 \\ -\pi & p<0 \end{cases}$$ unseen 4

$$\bar{f}(\omega) = 2\int_{-\infty}^{\infty} \left(\frac{\sin\frac{1}{2}t}{t}\right) e^{-i\omega t}\,dt$$

$$i\bar{f}(\omega) = \int_{-\infty}^{\infty} \left[e^{it(\frac{1}{2}-\omega)} - e^{-it(\frac{1}{2}+\omega)}\right] dt/t$$ 4

$$= \int_{-\infty}^{\infty} \frac{e^{ip_1 t}}{t}\,dt - \int_{-\infty}^{\infty} \frac{e^{ip_2}}{t}\,dt \qquad p_1 = \tfrac{1}{2}-\omega, \quad p_2 = -\tfrac{1}{2}-\omega$$

$$I_1 = \begin{cases} \pi & \omega < \frac{1}{2} \\ -\pi & \omega > \frac{1}{2} \end{cases} \qquad I_2 = \begin{cases} \pi & \omega < -\frac{1}{2} \\ -\pi & \omega > \frac{1}{2} \end{cases}$$ 6

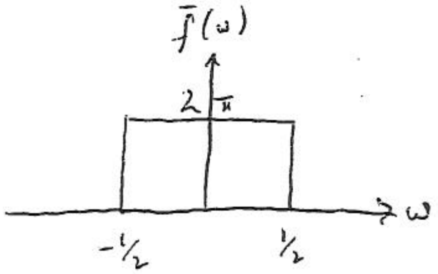
with $i\bar{f}(\omega) = (I_1 - I_2)\,i$ $\qquad I_n = \int_{-\infty}^{\infty} \frac{\sin p_n t}{t}\,dt$

Combining, we have (cosine part of integrals are 0)

$$\bar{f}(\omega) = \begin{cases} 2\pi, & -\frac{1}{2} < \omega < \frac{1}{2}, \\ 0, & \omega < -\frac{1}{2};\ \omega > \frac{1}{2}. \end{cases}$$ 4

The integrals combine s.t. $I_1 = \pi$ & $I_2 = -\pi$ in central interval but cancel in the outer parts.

Rectangle function.



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Setter's initials JD	Checker's initials AOC	Page number

EE 6

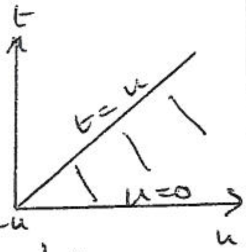
EXAMINATION QUESTIONS/SOLUTIONS 2005-06

Course: Paper 3

Question: EE6

Parts | Marks & seen/unseen

a) $\mathcal{L}(f*g) = \int_0^\infty e^{-st}\left(\int_0^t f(u)g(t-u)\,du\right)dt$ — Bookwork



$= \int_0^\infty f(u)\left(\int_u^\infty e^{-st}g(t-u)\,dt\right)du$

$v=t-u$

$= \int_0^\infty f(u)\left(\int_0^\infty e^{-s(u+v)}g(v)\,dv\right)du$

$= \left(\int_0^\infty f(u)e^{-su}\,du\right)\left(\int_0^\infty e^{-sv}g(v)\,dv\right) = \bar{f}(s)\bar{g}(s)$ — 4

Take $g(t)=1 \Rightarrow \bar{g}(s) = 1/s$ and $f*g = \int_0^t f(u)\,du$ — Non-bookwork 3

$\therefore \mathcal{L}\left(\int_0^t f(u)\,du\right) = \bar{f}(s)/s$

b) Take $\bar{f}(s) = \dfrac{1}{(1+s^2)}$ then $f(t) = \sin t$ — Seen similar

$\mathcal{L}(f*f) = [\bar{f}(s)]^2 = \dfrac{1}{(1+s^2)^2} = \bar{G}(s)$

$f*f = \mathcal{L}^{-1}\bar{G}(s) = G(t)$

Thus $G(t) = f*f = \int_0^t \sin u \sin(t-u)\,du$ — 7

$= \frac{1}{2}\int_0^t [\cos(2u-t) - \cos t]\,du = \frac{1}{2}\left[\frac{\sin(2u-t)}{2} - u\cos t\right]_0^t$

$G(t) = \frac{1}{2}[\sin t - t\cos t]$.

c) $\mathcal{L}^{-1}[\bar{F}(s)\,\bar{G}(s)] = F(t)*G(t)$ with $\bar{G}(s) = \dfrac{1}{(1+s^2)^2}$, $\bar{F}(s) = 1/s$, $F(t) = 1$ — Unseen

$= \int_0^t \frac{1}{2}[\sin u - u\cos u]\,du$ — 6

$= -\frac{1}{2}[\cos u]_0^t - \frac{1}{2}[u\sin u + \cos u]_0^t$

$= -\frac{1}{2}\{\cos t - 1 + t\sin t + \cos t - 1\}$

$= 1 - \cos t - \frac{1}{2}t\sin t$ (as advertised)

Setter's initials: JDG | Checker's initials: AOG | Page number

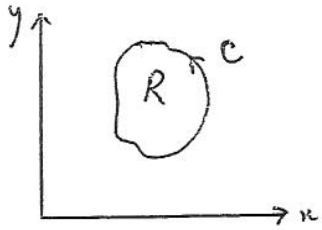
E7

	EXAMINATION QUESTIONS/SOLUTIONS 2005-06	Course Paper 3
Question EE7		Marks & seen/unseen

Parts

Choose $\underline{v} = \underline{i}P + \underline{j}Q$

$d\underline{r} = \hat{\underline{i}}dx + \hat{\underline{j}}dy$ Seen



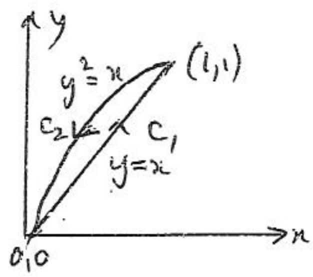
$\oint_c Pdx + Qdy = \oint_c \underline{v}\cdot d\underline{r}$

$$\text{curl}\,\underline{v} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ P & Q & 0 \end{vmatrix} = \hat{\underline{k}}(Q_x - P_y)$$ 8

Thus $\oint_c \underline{v}\cdot d\underline{r} = \oint_c Pdx + Qdy = \iint_R (Q_x - P_y)\,dxdy$

$= \iint_R \underline{k}\cdot\text{curl}\,\underline{v}\,dxdy$

$\oint_c \underline{v}\cdot d\underline{r} = \frac{1}{2}\oint_c x^2dx + y^2dy$ Unseen



$c_1: \; y = x$ 2

$c_2: \; y^2 = x$

$\int_{c_2} = -\frac{1}{2}\int_0^1 (2y^5 + y^2)\,dy = -\frac{1}{2}\left(\frac{1}{3}+\frac{1}{3}\right) = -1/3$ 4

$\int_{c_1} = \int_0^1 x^2dx = 1/3$ Thus $\oint_c = 1/3 - 1/3 = 0$ 4

$$\text{Finally} \quad \text{curl}\,\underline{v} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ \frac{1}{2}x^2 & \frac{1}{2}y^2 & 0 \end{vmatrix} = 0$$ 2

Thus $\iint_R \underline{k}\cdot\text{curl}\,\underline{v}\,dxdy = 0$.

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JDG	AOG	

Aero. II ...

II (3) E 8

Solution

$$J = \det \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$ 2

$$\frac{\partial x}{\partial u} = \frac{v^2-u^2}{(u^2+v^2)^2}$$

$$\frac{\partial x}{\partial v} = -\frac{2vu}{(u^2+v^2)^2}$$

$$\frac{\partial y}{\partial u} = -\frac{2uv}{(u^2+v^2)^2}$$ 4

$$\frac{\partial y}{\partial v} = \frac{u^2-v^2}{(u^2+v^2)^2}$$

$$\therefore J = \det \begin{vmatrix} \frac{v^2-u^2}{(u^2+v^2)^2} & -\frac{2vu}{(u^2+v^2)^2} \\ -\frac{2uv}{(u^2+v^2)^2} & \frac{u^2-v^2}{(u^2+v^2)^2} \end{vmatrix}$$

$$= -\frac{1}{(u^2+v^2)^2}$$ 2

A point in R satisfies $u^2+v^2 < 1$

Its image is $(x, y) = \left(\frac{u}{u^2+v^2}, \frac{v}{u^2+v^2}\right)$

and $x^2+y^2 = \frac{1}{(u^2+v^2)^2}(u^2+v^2) = \frac{1}{u^2+v^2}$

Thus $x^2+y^2 > 1$ i.e. the image is in S 3

Conversely a point in S satisfies $u^2+v^2>1$

$$x^2+y^2 = \frac{1}{u^2+v^2} < 1$$

i.e. the image is in R 2

Change the variables for the integral to get

$$\iint_R f(x,y)\,dx\,dy = \iint_{R^*} f(G_1(u,v), G_2(u,v))\,|J|\,du\,dv$$ 2

where $|J| = \dfrac{1}{(u^2+v^2)^2}$

and R^* = image of R under transformation

$= S$ from above 1

Identify r.h.s. with given integral

so $f = 4$

Thus r.h.s. = l.h.s.

$$= \iint_R 4\,dx\,dy$$

$= 4 \times$ area of unit circle 4

$= 4\pi$

Total $\boxed{20}$

A10 CH2 CE10 EE9 II(3)

C4

$$\text{grad } \varphi = \left(\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z}\right)$$

1 Mark

$$\text{div } \boldsymbol{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

1 Mark

$$\text{curl } \boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

where $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are unit vectors in the x, y and z directions respectively.

2 Marks

Only one or the other form of curl needs to be given

a) The first component of curl $\varphi\boldsymbol{F}$ is by above

$$\frac{\partial(\varphi F_3)}{\partial y} - \frac{\partial(\varphi F_2)}{\partial z} = \varphi\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \frac{\partial\varphi}{\partial y}F_3 - \frac{\partial\varphi}{\partial z}F_2$$

3 Marks

Now $\varphi\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)$ is the first component of φ curl $\boldsymbol{F}$ whilst

$$(\text{grad } \varphi) \times \boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial\varphi}{\partial x} & \frac{\partial\varphi}{\partial y} & \frac{\partial\varphi}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

so that $\frac{\partial\varphi}{\partial y}F_3 - \frac{\partial\varphi}{\partial z}F_2$ is the first component of $(\text{grad } \varphi) \times \boldsymbol{F}$.

2 Marks

We can either repeat the same calculation for the second and third components, or argue that the same relationship must hold by permuting F_1, F_2 and F_3.

1 Mark

b) We have

$$\begin{aligned} \boldsymbol{v}.(\text{curl } \varphi\boldsymbol{F}) &= \boldsymbol{v}.(\varphi \text{ curl } \boldsymbol{F}) + \boldsymbol{v}.((\text{grad } \varphi) \times \boldsymbol{F}) \\ &= \varphi\, \boldsymbol{v}.\text{curl } \boldsymbol{F} + \boldsymbol{v}.((\text{grad } \varphi) \times \boldsymbol{F}) \\ &= \boldsymbol{v}.((\text{grad } \varphi) \times \boldsymbol{F}) \end{aligned}$$

since we are given $\boldsymbol{v}.\text{curl } \boldsymbol{F} = 0$.

1 Mark

The standard formula for the triple product is

$$\boldsymbol{a}.(\boldsymbol{b}\times\boldsymbol{c}) = (a_1,a_2,a_3).\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

1 Mark

so that

$$\boldsymbol{v}.(\text{curl } \varphi\boldsymbol{F}) = \boldsymbol{v}.((\text{grad } \varphi)\times\boldsymbol{F}) = \begin{vmatrix} v_1 & v_2 & v_3 \\ \frac{\partial\varphi}{\partial x} & \frac{\partial\varphi}{\partial y} & \frac{\partial\varphi}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

as required.

2 Marks

c) We have

$$\begin{aligned} \boldsymbol{k}.(\text{curl } \boldsymbol{F}) &= (0,0,1)\left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z},\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x},\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right) \\ &= \frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y} \end{aligned}$$

Hence $\boldsymbol{k}.(\text{curl } \boldsymbol{F}) = 0$ if and only if $\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y} = 0$

2 Marks

d) If φ is independent of both x and y then $\frac{\partial\varphi}{\partial x} = \frac{\partial\varphi}{\partial y} = 0$.

1 Mark

If F_1 is independent of y and F_2 is independent of x then $\frac{\partial F_1}{\partial y} = 0$, $\frac{\partial F_2}{\partial x} = 0$, so that

$$\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y} = 0$$

and hence $\boldsymbol{k}.(\text{curl } \boldsymbol{F}) = 0$ by c)

1 Mark

Thus by b)

$$\boldsymbol{k}.(\text{curl } \varphi\boldsymbol{F}) = \begin{vmatrix} 0 & 0 & 1 \\ 0 & 0 & \frac{\partial\varphi}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = 0$$

2 Marks

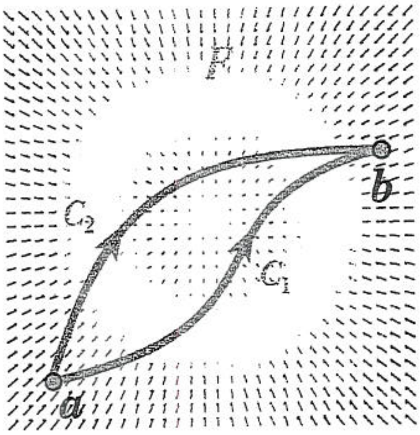
DB

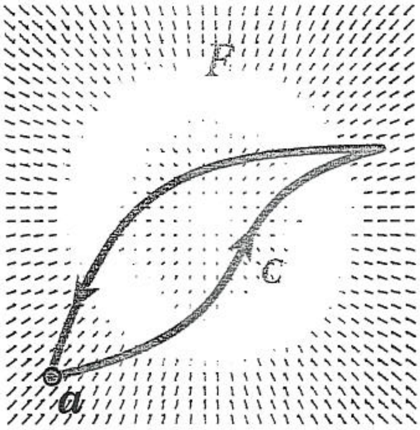
A II CH 3 EE 10 E10 II(3)

C5 Suppose that for any closed curve C we have

$$\oint_C \boldsymbol{F}.d\boldsymbol{r} \quad = \quad 0$$

Suppose that C_1 and C_2 are any two paths with the same end points $\boldsymbol{a}$ and $\boldsymbol{b}$:





Let C be the path consisting of traversing C_1 from $\boldsymbol{a}$ to $\boldsymbol{b}$ and then returning via C_2, as in the above figure on the right. Thus

$$0 \quad = \quad \oint_C \boldsymbol{F}.d\boldsymbol{r} \quad = \quad \int_{C_1} \boldsymbol{F}.d\boldsymbol{r} - \int_{C_2} \boldsymbol{F}.d\boldsymbol{r}$$

and hence

$$\int_{C_1} \boldsymbol{F}.d\boldsymbol{r} \quad = \quad \int_{C_2} \boldsymbol{F}.d\boldsymbol{r}$$

for any two paths C_1 and C_2 with the same end points. Hence $\int_C \boldsymbol{F}.d\boldsymbol{r}$ is independent of the path C, and only depends on its endpoints.

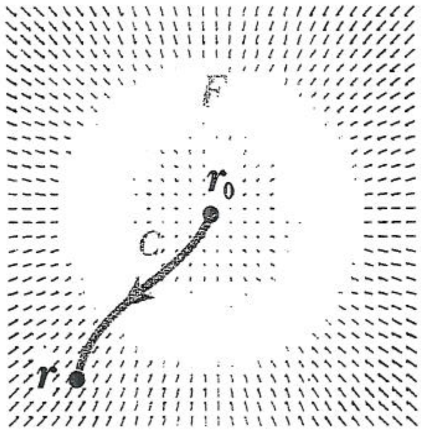
3 Marks

The diagrams are not required.

To construct the potential φ, fix an arbitrary base point $\boldsymbol{r}_0$ and set $\varphi(\boldsymbol{r}_0) = 0$. Then, given any other point $\boldsymbol{r} = (x,y)$, choose any path C from $\boldsymbol{r}_0$ to $\boldsymbol{r}$ and define

$$\varphi(\boldsymbol{r}) \quad = \quad -\int_C \boldsymbol{F}.d\boldsymbol{r}$$

By above, the right hand side is independent of the choice of path.



2 Marks

The diagram is not required.

To show that grad $\varphi = -\boldsymbol{F}$ choose $\boldsymbol{r}_0 = (0,0)$ and a path from $\boldsymbol{r}_0$ to $\boldsymbol{r}$ made up of vertical and horizontal segments C_y and C_x (see figure below). Parametrize each of these by t. Along C_y, we have $x = 0$ and $\boldsymbol{F}.d\boldsymbol{r} = F_2(0,t)\ dt$. Thus

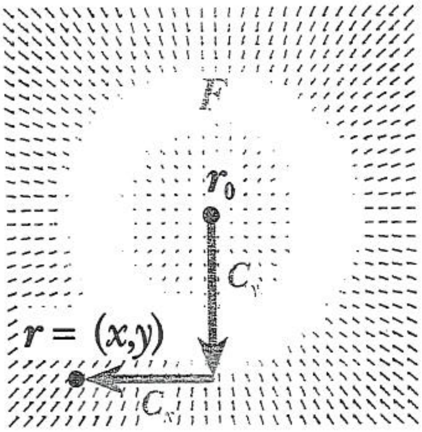
$$\int_{C_y} \boldsymbol{F}.d\boldsymbol{r} \quad = \quad \int_0^y F_2(0,t)\ dt$$

Along C_x we have $F.dr = F_2(t,y)\ dt$ and so

$$\int_{C_x} F.dr = \int_0^x F_1(t,y)\ dt$$

Thus

$$\varphi(x,y) = -\int_0^y F_2(0,t)\ dt - \int_0^x F_1(t,y)\ dt \qquad (1)$$



Differentiating with respect to x we note that the first term is independent of x, so that

$$\frac{\partial\varphi}{\partial x}(x,y) = -\frac{\partial}{\partial x}\int_0^x F_1(t,y)\ dt = -F_1(x,y)$$

2 Marks

The diagram is not required.

Similarly, taking a path consisting first of a horizontal segment and then a vertical one gives

$$\varphi(x,y) = -\int_0^x F_1(t,0)\ dt - \int_0^y F_2(x,t)\ dt \qquad (2)$$

Differentiating with respect to y

$$\frac{\partial\varphi}{\partial y}(x,y) = -\frac{\partial}{\partial y}\int_0^y F_2(x,Y)\ dY = -F_2(x,y)$$

1 Mark

Suppose that $F = -\operatorname{grad}\varphi$. By the equivalence of the two mixed second partial derivatives of φ we have

$$\frac{\partial F_2}{\partial x} = -\frac{\partial}{\partial x}\left(\frac{\partial\varphi}{\partial y}\right) = -\frac{\partial^2\varphi}{\partial x\partial y} = -\frac{\partial^2\varphi}{\partial y\partial x}$$
$$= -\frac{\partial}{\partial y}\left(\frac{\partial\varphi}{\partial x}\right) = \frac{\partial F_1}{\partial y}$$

as required

3 Marks

a) Suppose $F = (2x+y, x+2y)$. Then

$$\frac{\partial F_2}{\partial x} = 1 = \frac{\partial F_1}{\partial y}$$

as required. To compute φ we can use either Eq. 1 or Eq. 2. Here we choose to use Eq. 1:

$$\varphi(x,y) = -\int_0^y 2t\ dt - \int_0^x 2t+y\ dt$$
$$= -y^2 - xy - x^2$$

3 Marks

Any other method which gives a solution that differs from the one above by a constant is acceptable.

b) Suppose $F = (3x^2y^2, 2x^3y)$. Then

$$\frac{\partial F_2}{\partial x} = 6x^2y = \frac{\partial F_1}{\partial y}$$

as required. Again, we compute φ using Eq. 1:

$$\begin{aligned}\varphi(x,y) &= -\int_0^y 0\,dt - \int_0^x 3t^2y^2\,dt \\ &= x^3y^2\end{aligned}$$

3 Marks

Any other method which gives a solution that differs from the one above by a constant is acceptable.

c) Suppose $F = (y\sin xy, x\sin xy)$. Then

$$\frac{\partial F_2}{\partial x} = \sin xy + xy\cos xy = \frac{\partial F_1}{\partial y}$$

as required. Again, we compute φ using Eq. 1:

$$\begin{aligned}\varphi(x,y) &= -\int_0^y 0\,dt - \int_0^x y\sin ty\,dt \\ &= -1 + \cos xy\end{aligned}$$

Since the potential is only defined up to constant of integration, the -1 can be dropped.

3 Marks

Any other method which gives a solution that differs from the one above by a constant is acceptable.

GN/RC

II(3)
E 11 & 12

Solutions

Note: U means UNSEEN

11. (i)

(a) Define the events A = {chip is from Distributor A}, B = {chip is from Distributor B} and D = {chip is defective}, then

$$\mathrm{P}(D) = \mathrm{P}(D|A)\mathrm{P}(A) + \mathrm{P}(D|B)\mathrm{P}(B) = 0.87 \times 0.05 + 0.13 \times 0.08 = 0.054$$

$\boxed{5\ \mathrm{U}}$

(b) Define the events $E =$ {both chips are from Distributor A} and F ={both chips are defective}, then

$$\mathrm{P}(E|F) = \frac{\mathrm{P}(E \cap F)}{\mathrm{P}(F)} = \frac{(0.87 \times 0.05)^2}{(0.87 \times 0.05 + 0.13 \times 0.08)^2} = 0.651$$

$\boxed{5\ \mathrm{U}}$

(ii) A Type I error occurs when the null hypothesis H_0 is rejected but H_0 is true. From the assumption that the tests are independent and all have the same failure probability, we have that

$$\mathrm{P}(\text{ reject } H_0 \mid H_0 \text{ is true }) = \mathrm{P}(X \leq 3; p = 0.05) = \sum_{k=0}^{3} \binom{200}{k} (0.05)^k (0.95)^{200-k}$$

Since n is large and p is small, we can use the Poisson approximation with $\lambda = np = 200 \times 0.05 = 10$. Therefore the required probability is

$$\sum_{k=0}^{3} e^{-10} \frac{10^k}{k!} = 0.01$$

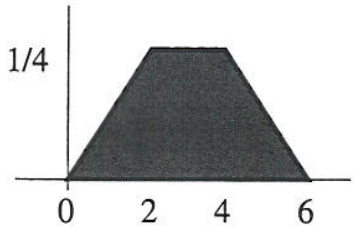
$\boxed{10\ \mathrm{U}}$ 20

12. (a) In order to find k we solve the following equation with respect to k:

$$1 = \int f_X(x)dx = \frac{1}{8}\int_0^2 xdx + k\int_2^4 dx + \frac{1}{8}\int_4^6 (6-x)dx = \frac{1}{4} + 2k + \frac{1}{4} \rightarrow k = \frac{1}{4}$$

$\boxed{4\ \mathrm{U}}$

(b) The pdf looks like a trapezium



$\boxed{1\ \text{U}}$

(c) The expected value is

$$\mathrm{E}(X) = \int x f_X(x)dx = \frac{1}{8}\int_0^2 x^2 dx + \frac{1}{4}\int_2^4 x dx + \frac{1}{8}\int_4^6 x(6-x)dx = \frac{1}{3} + \frac{3}{2} + \frac{7}{6} = 3$$

$\boxed{4\ \text{U}}$

(d) We need to compute $\mathrm{Var}(X) = \mathrm{E}(X^2) - \{\mathrm{E}(X)^2\}$,

$$E(X^2) = \int x^2 dx = \frac{1}{8}\int_0^2 x^3 dx + \frac{1}{4}\int_2^4 x^2 dx + \frac{1}{8}\int_4^6 x^2(6-x)dx = \frac{1}{2} + \frac{14}{3} + \frac{11}{2} = \frac{32}{3}$$

$$\mathrm{Var}(X) = \frac{32}{3} - 9 = \frac{5}{3}$$

$\boxed{6\ \text{U}}$

(e) The required probability is

$$\mathrm{P}(X \leq 5) = \int_0^5 f_X(x)dx = 1 - \int_5^6 f_X(x)dx = 1 - \frac{1}{8}\int_5^6 (6-x)dx = 1 - \frac{1}{16} = \frac{15}{16}$$

$\boxed{5\ \text{U}}$

20