UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 2001

MSci Honours Degree in Mathematics and Computer Science Part IV
MEng Honours Degrees in Computing Part IV
MSc in Advanced Computing
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the Associateship of the City and Guilds of London Institute
This paper is also taken for the relevant examinations for the Associateship of the Royal College of Science

PAPER C499

MODAL AND TEMPORAL LOGIC

Friday 11 May 2001, 14:00
Duration: 120 minutes

Answer THREE questions

Paper contains 4 questions
Calculators not required

1 All formulas in this question are built from atoms, $\top$, and $\bot$ using $\wedge$, $\vee$, $\neg$, $\Box$, $\Diamond$.

a Explain the following terms:

i) a Kripke frame,

ii) a positive formula,

iii) a negative formula,

iv) a strongly positive formula,

v) an untied formula,

vi) a Sahlqvist formula.

b Let $F = (W,R)$ be a Kripke frame, and let g, h be assignments of atoms to subsets of W, satisfying $g(q) \subseteq h(q)$ for all atoms q.

i) Let A be a positive formula. Using induction on A, prove that for all worlds t in W, if $(F,g),t \models A$ then $(F,h),t \models A$.

ii) State and justify an analogous statement for negative formulas.

c Let C be the formula $\Diamond\Box p \wedge \Diamond\Box q \rightarrow \Box\Diamond(p\wedge q)$. Using Sahlqvist's algorithm, find a first-order frame condition that is true in a Kripke frame F if and only if C is valid in F. What does your frame condition say in English?

2a Define the semantics of the temporal connectives F, P, Y, T, U, and S.

b Express the following statements in temporal logic. You may take the underlying flow of time to be the integers ($\mathbb{Z}$,<), the time points representing consecutive days. You may use any of the connectives from part a, and propositional atoms meaning "UK is in the Euro", "The pound is going down" and "The dollar is going down". For example, "UK joins the Euro" could be expressed by (UK is in the Euro) $\wedge \neg$Y(UK is in the Euro).

i) Once the UK joins the Euro it will never leave the Euro.

ii) The pound has always gone down when the dollar was going down, and sometimes even when it wasn't.

iii) The dollar will not fall until the UK joins the Euro. [Remember that of course, the UK may never join the Euro!]

iv) The UK will not join the Euro unless at that time the pound has been going down for a while.

v) Ever since the UK joined the Euro, the dollar has fallen, and it will continue to fall as the UK remains in the Euro.

c Let R be a new unary temporal connective, with semantics over the integers $\mathbb{Z}$ defined as follows: Rq is true at t if and only if q is true at t–1, t, or t+1. (Informally, Rq is true now just if q is true at some time around now.)

i) Consider a model M = ($\mathbb{Z}$,<,h) where h(q) = {...,-8, -4, 0, 4, 8, ...} for any atom q. Let A be any formula of the temporal logic FP — that is, it is made from atoms, $\top$, $\bot$ using the Boolean connectives, F, and P. Outline an argument by induction on A to show that A is equivalent in M to one of: $\top$, $\bot$, q, $\neg$q.

ii) Deduce from part c(i) that the connective R is not expressible in the temporal logic FP over the flow of time ($\mathbb{Z}$,<).

iii) Find a formula of the temporal logic US that is equivalent to Rq in the flow of time ($\mathbb{Z}$,<).

The three parts carry, respectively, 30%, 30%, 40% of the marks.

3a Consider a *normal* modal logic L, and a class of frames $\mathcal{F}$.

i) Define the notions of soundness and completeness of the logic L with respect to the class of frames $\mathcal{F}$.

ii) Define the canonical model M_C^L for L. State, without proof, the property of the canonical model that provides completeness proofs for the logic L.

iii) Let F_C^L be the canonical frame for L. Show that if $F_C^L \in \mathcal{F}$, then L is complete with respect to $\mathcal{F}$.

b Consider the normal logic S5 defined by:

Tautologies
axiom K: $\Box(p \to q) \to (\Box p \to \Box q)$
axiom T: $\Box p \to p$
axiom 4: $\Box p \to \Box\Box p$
axiom 5: $\Diamond p \to \Box\Diamond p$
Rules: MP, Uniform Substitution, and Necessitation.

Prove that S5 is sound and complete with respect to the class of frames $\mathcal{F} = \{(W, \sim) \mid \sim \text{ is an equivalence relation on } W\}$.

Give full details for the parts corresponding to axioms T and 4 and outline the rest of the proof. You may assume that a relation is an equivalence iff it is reflexive and euclidean.

c A model $M = (W, R, h)$ is *universal* if R is universal on W, i.e., if for any w, w' in W we have that wRw'.

Investigate whether the canonical model M_C^{S5} for S5 is universal.

The three parts carry, respectively, 30%, 40%, 30% of the marks.

4a i) Define what is meant by a *classical* system of modal logic.

ii) Show, by constructing a suitable derivation, that every *classical* system of modal logic containing theorems:

C. $(\Box A \wedge \Box B) \rightarrow \Box(A \wedge B)$

P. $\neg\Box\bot$

is also closed under the following rule:

$$\frac{\neg(A \wedge B)}{\neg(\Box A \wedge \Box B)}$$

b i) Define the structure of models and the truth conditions for formulas $\Box A$ in *neighbourhood semantics*. Include in your answer the definition of the *truth set* of a formula A in a model.

ii) Find suitable model conditions on a neighbourhood model to validate the schemas C and P of part (a). It is *not* necessary to prove completeness.

c i) Define the standard semantics of a *normal* conditional $A \Rightarrow B$ where the conditional is read as a form of relative necessity.

ii) Find suitable model conditions to validate the following schemas:

I. $A \Rightarrow A$

S. $((A \Rightarrow B) \wedge (B \Rightarrow C)) \rightarrow (A \Rightarrow C)$

It is *not* necessary to prove completeness.

iii) Show that any model satisfying the conditions in part c(ii) validates also the following schema:

Aug. $(A \Rightarrow B) \rightarrow ((A \wedge \mathrm{C}) \Rightarrow B)$

The three parts carry, respectively, 20%, 40%, 40% of the marks.