IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2017

EEE PART I: MEng, BEng and ACGI

MATHEMATICS 1A (E-STREAM AND I-STREAM)

Thursday, 25 May 10:00 am

Time allowed: 2:00 hours

There are FOUR questions on this paper.

Answer ALL questions. All questions carry equal marks (25% each).

NO CALCULATORS ALLOWED.
Mathematical Formulae sheet provided

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s) :	D. Nucinkis, D. Nucinkis
	Second Marker(s) :	D.M. Brookes, D.M. Brookes

EE1-10A MATHEMATICS I

Information for Candidates:

Calculators are not permitted in this exam.

1. a) Express in the form $x+iy$: [4]

$$(i)\quad \frac{1-2i}{i-2}, \qquad (ii)\quad \left(\frac{1-\sqrt{3}i}{2}\right)^{2017}.$$

b) Sketch the locus of the complex number z satisfying [4]

$$z-\bar{z}=\frac{1}{\bar{z}}-\frac{1}{z}.$$

c) Obtain all complex solutions z, when [7]

$$(i)\quad \sinh z=-i, \qquad (ii)\quad \sin^2(iz)=1.$$

d) Obtain the limits [10]

$$(i)\ \lim_{x\to 0} x\cos(\cot x), \quad (ii)\ \lim_{x\to 0}\frac{x^2}{\ln(\cos x)}, \quad (iii)\ \lim_{x\to\pi/6}\frac{1-\sin(3x)}{\cot x-\sqrt{3}}.$$

2. a) Obtain the value of q for which the following limit exists and is non-zero, and state the value of the limit: [4]

$$\lim_{x\to\infty} x^q\left[(x+1)^{2/3}-(x-1)^{2/3}\right].$$

b) Differentiate to obtain $\dfrac{dy}{dx}$: [6]

$$(i)\quad y=(\sin x)^{\cos x}, \qquad (ii)\quad \cos(x)=\sin(y), \qquad (iii)\quad y^2=\cos(xy).$$

c) Given the function

$$f(x)=\frac{2x^2-5x+1}{x+1},$$

find all stationary points and their nature, obtain any asymptotes and give a sketch showing these and any other relevant features. [10]

d) Obtain the n^{th} derivative $\dfrac{d^ny}{dx^n}$ for [5]

$$y=x^2e^{-x}.$$

3. a) Evaluate the indefinite integrals: [8]

$$(i) \int \frac{4x-6}{x^2-3x+4}\,dx \quad (ii) \int \frac{1}{x\ln x}\,dx, \quad (iii) \int \frac{1}{4\sin x - 3\cos x - 5}\,dx.$$

b) Use a substitution to integrate $\dfrac{1}{\sqrt{x^2-1}}$ and hence show that [5]

$$\cosh^{-1} x = \ln(x + \sqrt{x^2-1}).$$

c) Obtain the Maclaurin series of $\dfrac{1}{e^{-x}+1}$ to first order with a remainder term. Explain how the error estimate from the remainder term can be improved without any more terms in the series. Obtain the improved error estimate. [8]

d) A convergent series can be bounded by two constants A, B:

$$A < \sum_{n=1}^{\infty} \frac{1}{n^3} < B.$$

Use the integral test to find one of the constants, and give a possible value for the other constant. [4]

4. a) Find the radius and interval of convergence of the infinite series [5]

$$\sum_{n=2}^{\infty} \frac{(3x)^n}{n(n-1)},$$

b) Without obtaining the Fourier Series of the function

$$f(x) = \begin{cases} x+2, & 0 \le x < 1.5 \\ 4-x, & 1.5 \le x < 3 \end{cases} \quad \text{and} \quad f(x+3) = f(x), \forall x,$$

find the values of the Fourier Series at $x = 0$ and $x = 1.5$. [3]

c) A function is defined as

$$f(x) = \begin{cases} 1-x & 0 \le x < 1 \\ 0 & 1 \le x < 2 \end{cases}$$

i) Obtain $g(x)$, the even extension of $f(x)$, with period $T = 4$ and sketch $g(x)$ for $-6 \le x \le 6$. [3]

ii) Obtain the Fourier cosine series of $g(x)$. [10]

[You may assume that $\cos(n\pi/2) = (-1)^{n/2}$ for even n.]

iii) By careful choice of a value of x in the results of (ii), or otherwise, calculate the infinite series [4]

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$$

Mathematical Formulae

MATHEMATICAL FORMULAE

1 Vector Algebra

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1, a_2, a_3)$$

Scalar(Dot) Product $\quad \mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

Vector (Cross) Product

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}.$$

Triple vector product $\quad \mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$

Triple scalar product

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1$$

2 Series

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\ldots+\frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}x^r+\ldots \ (n \text{ arbitrary}, |x|<1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^r}{r!} + \ldots \qquad \text{for all } x$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{r+1}\frac{x^r}{r} + \ldots \qquad \text{for } -1 < x \le 1$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + \frac{(-1)^r x^{2r}}{(2r)!} + \ldots \qquad \text{for all } x$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + \frac{(-1)^r x^{2r+1}}{(2r+1)!} + \ldots \qquad \text{for all } x$$

3 Trigonometric Identities and Hyperbolic Functions

$$\sin(A+B) = \sin A\cos B + \sin B\cos A$$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$

$$\cos(iz) = \cosh z\,, \quad \cosh(iz) = \cos z\,, \quad \sin(iz) = i\,\sinh(z)\,, \quad \sinh(iz) = i\,\sin z$$

5 Integral Calculus

1. An important substitution: $\tan(\theta/2) = t$; then

$$\sin\theta = \frac{2t}{1+t^2}, \quad \cos\theta = \frac{1-t^2}{1+t^2}, \quad d\theta = \frac{2\,dt}{1+t^2}.$$

2.

$$\int \frac{dx}{(a^2-x^2)^{1/2}} = \sin^{-1}\left(\frac{x}{a}\right), \quad |x| < a.$$

3.

$$\int \frac{dx}{(a^2+x^2)^{1/2}} = \sinh^{-1}\left(\frac{x}{a}\right) = \ln\left[\frac{x}{a} + \left(\frac{x^2}{a^2}+1\right)^{1/2}\right].$$

4.

$$\int \frac{dx}{(x^2-a^2)^{1/2}} = \cosh^{-1}\left(\frac{x}{a}\right) = \ln\left[\frac{x}{a} + \left(\frac{x^2}{a^2}-1\right)^{1/2}\right].$$

5.

$$\int \frac{dx}{a^2+x^2} = \left(\frac{1}{a}\right)\tan^{-1}\left(\frac{x}{a}\right).$$

6 Fourier Series

The following formulae assume that $f(x)$ satisfies the Dirichlet conditions and is periodic with period T, i.e. $f(x+T) = f(x)$. The general Fourier Series for $f(x)$ is

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2n\pi x}{T}\right) + b_n \sin\left(\frac{2n\pi x}{T}\right)\right]$$

where

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(x)\,\cos\left(\frac{2n\pi x}{T}\right)dx, \; b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(x)\,\sin\left(\frac{2n\pi x}{T}\right)dx, \; n = 0,1,2,3,\ldots$$

The series converges to $f(x)$ at points of continuity and to the mean value $\frac{1}{2}(f(x_+)+f(x_-))$ at points where $f(x)$ is discontinuous.

The complex form of the Fourier series is

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i2\pi nx/T}, \text{ where } c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(x)e^{-i2n\pi x/T}\,dx \text{ for every integer } n.$$

Half-range series: If $f(t)$ is an even (resp. odd) function, all sine (resp. cosine) terms vanish, and we have a half-range cosine (resp. sine) series. Let $L = T/2$. Then the coefficients are, respectively:

$$f(t) \text{ even: } b_n = 0,\; a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx. \qquad f(t) \text{ odd: } a_n = 0,\; b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

Parseval's Theorem: If the complex Fourier series of the T-periodic function $f(x)$ has coefficients c_n, and the real Fourier series of $f(x)$ has coefficients a_n and b_n, then

$$\frac{1}{T}\int_{-T/2}^{T/2} f^2(x)\,dx = \sum_{n=-\infty}^{\infty}|c_n|^2. \qquad \text{and} \qquad \frac{2}{T}\int_{-T/2}^{T/2} f^2(x)\,dx = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty} a_n^2 + b_n^2.$$