

Mathematics for signal &amp; systems 2007/2008.

(1)

SOLUTIONS - 2008

(1)

$$a) \quad i) \quad A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$(ii) \quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(iii) We will show that  $\mathcal{M}$  is a subspace of  $M_3(\mathbb{C})$

First,  $\mathcal{M} \subset M_3(\mathbb{C})$  and  $\mathcal{M} \neq \emptyset$ . (as  $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{M}$ ).

Let  $\alpha \in \mathbb{C}$  and  $A, B \in \mathcal{M}$  i.e.

$$l_i(A) = c_j(A) \quad \text{and} \quad l_i(B) = c_j(B) \quad (*)$$

for  $i, j = 1, 2, 3$

It is not difficult to see that  $l_i$  and  $c_j$  are linear

$$\text{so that } l_i(\alpha A + B) = \alpha l_i(A) + l_i(B)$$

similarly for  $c_j$ . Hence  $l_i(\alpha A + B) = c_j(\alpha A + B)$

for  $i, j = 1, 2, 3$  by (\*) above.

If  $A, B \in \mathcal{M}$  and  $\alpha \in \mathbb{C}$ , then

$$\begin{aligned} \alpha(\alpha A + B) &= l_1(\alpha A + B) = \alpha l_1(A) + l_1(B) \\ &= \alpha \alpha(A) + \alpha(B) \end{aligned}$$

Hence  $\alpha$  is linear.

1/18

(2)

(ii) It is straightforward to check that  $J \in \mathcal{M}$

$$\text{as } l_i(J) = c_j(J) = 3.$$

Let  $A = (a_{ij})_{i,j=1,2,3}$

then

$$AJ = \begin{bmatrix} a_{11} + a_{12} + a_{13} & a_{11} + a_{12} + a_{13} & a_{11} + a_{12} + a_{13} \\ a_{21} + a_{22} + a_{23} & a_{21} + a_{22} + a_{23} & a_{21} + a_{22} + a_{23} \\ a_{31} + a_{32} + a_{33} & a_{31} + a_{32} + a_{33} & a_{31} + a_{32} + a_{33} \end{bmatrix}$$

$$JA = \begin{bmatrix} a_{11} + a_{21} + a_{31} & a_{12} + a_{22} + a_{32} & a_{13} + a_{23} + a_{33} \\ a_{11} + a_{21} + a_{31} & a_{12} + a_{22} + a_{32} & a_{13} + a_{23} + a_{33} \\ a_{11} + a_{21} + a_{31} & a_{12} + a_{22} + a_{32} & a_{13} + a_{23} + a_{33} \end{bmatrix}$$

so that  $AJ = \begin{pmatrix} l_1(a) & l_1(a) & l_1(a) \\ l_2(a) & l_2(a) & l_2(a) \\ l_3(a) & l_3(a) & l_3(a) \end{pmatrix}$

$$JA = \begin{pmatrix} c_1(a) & c_2(a) & c_3(a) \\ c_1(a) & c_2(a) & c_3(a) \\ c_1(a) & c_2(a) & c_3(a) \end{pmatrix}.$$

. If  $AJ = JA = \lambda A$

then from the previous computation  $l_i(A) = c_j(A) = \lambda$

$$\text{and } \lambda = \kappa(A).$$

. If  $A \in \mathcal{M}$  then  $AJ = JA = \kappa(A) J$ .

2/18

(3)

b) (i) Let  $a \in \mathbb{C}$ ,  $A, B \in \mathcal{M}^0$ .

We show that  $\mathcal{M}^0$  is a subspace of  $\mathcal{M}$ .

$\mathcal{M}^0 \subset \mathcal{M}$  and  $\mathcal{M}^0 \neq \emptyset$  as  $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{M}^0$ .

as  $\alpha$ ,  $k$  and  $\text{anti}$  are linear operators on  $\mathcal{M}$ , it is not difficult to check that

if  $A$  and  $B$  are such  $\alpha(A) = k(A) = \text{anti}(A)$   
 $\alpha(B) = k(B) = \text{anti}(B)$

then the same holds for  $aA + bB$ .

(ii)  $(G, G^T, J)$  independent.

$$aG + bG^T + cJ = 0$$

$$\Rightarrow \begin{aligned} a + b + c &= 0; & -2a + c &= 0, & a - b + c &= 0 \\ c &= 0, & 2b + c &= 0, & -a - b + c &= 0. \end{aligned}$$

$$\Rightarrow a = b = c = 0 \Rightarrow (G, G^T, J) \text{ independent.}$$

$$\alpha(J) = 3; \quad kJ = 3 \quad \& \quad \text{anti } J = 3 \Rightarrow J \in \mathcal{M}^0.$$

$$\alpha(G) = 0, \quad \text{anti } G = 0 \quad \& \quad kG = 0 \Rightarrow G \in \mathcal{M}^0$$

Similarly for  $G^T$ .

(4)

$$\text{If } A \in \mathcal{L}^0 \Rightarrow$$

$$a_{k1} + a_{k+2} + a_{k3} =$$

$$a_{1k} + a_{2k} + a_{3k} =$$

$$a_{11} + a_{22} + a_{33} =$$

$$a_{13} + a_{22} + a_{31}$$

$$a_{11} + a_{12} + a_{13} = a_{21} + a_{22} + a_{23} = a_{31} + a_{33} + a_{32} =$$

$$a_{11} + a_{21} + a_{31} = a_{12} + a_{22} + a_{32} = a_{13} + a_{23} + a_{33} =$$

$$a_{11} + a_{22} + a_{33} = a_{13} + a_{22} + a_{32}.$$

Solving the above system, it is not difficult to show that.

$$A = \frac{a_{22} - a_{12}}{2} G + \frac{a_{23} - a_{22}}{2} G^T + a_{22} J.$$

$$\text{Hence } \mathcal{L}^0 = \text{Span}(G, G^T, J) \text{ and } \dim \mathcal{L}^0 = 3.$$

6/18

②

Bookwork (created in ph sheet 3).

(1)

(2)

(a)

(i) The only difficult point is to show that if

$$\langle P, P \rangle = 0 \quad \text{then } P = 0$$

$$\int_{-1}^1 \frac{P^2(t)}{\sqrt{1-t^2}} dt = 0 \quad \Rightarrow \quad P(t) = 0 \quad \text{on } (-1, 1)$$

$P(t)$  is a polynomial with an infinite number of roots (zeros); then it is the zero polynomial.

(b) (ii)

(c) By induction.

$$\begin{aligned} T_0(\cos \theta) &= 1 & \text{for } T_0(x) &= 1 \\ T_1(\cos \theta) &= \cos \theta & \text{for } T_1(x) &= x. \end{aligned}$$

Suppose that for any  $k \leq n$  there exists a  $T_k$  such that  $\cos k\theta = T_k(\cos \theta)$ .

$$\begin{aligned} \cos(n+1)\theta &= \cos(n\theta)\cos\theta - \sin(n\theta)\sin\theta \\ &= T_n(\cos\theta)\cos\theta - \frac{1}{2}(\cos(n+1)\theta + \cos(n-1)\theta) \end{aligned}$$

$$\Rightarrow \cos(n+1)\theta = 2T_n(\cos\theta)\cos\theta - T_{n-1}(\cos\theta).$$

$$\text{if we let } T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

$$\text{then } \cos(n+1)\theta = T_{n+1}(\cos\theta).$$

The uniqueness stems from the fact that if  $P_n$  is such that  $P_n(\cos\theta) = \cos n\theta$  then the polynomial  $T_n - P_n$  is 0 on  $[-1, 1]$ ; hence  $T_n - P_n = 0$ .

where is solution for c)?

(d)  
(iii)

for  $m, n \in \mathbb{N}$ .

$$\int_{-1}^1 \frac{T_m(x) T_n(x)}{\sqrt{1-x^2}} dx = \int_{-\pi}^0 \frac{T_m(\cos \theta) T_n(\cos \theta)}{\sin \theta} (-\sin \theta) d\theta$$

with the change of variable  $t = \cos \theta$

$$\begin{aligned} \text{Therefore, } \langle T_m, T_n \rangle &= \int_0^\pi \cos(n\theta) \cos(m\theta) d\theta \\ &= \int_0^\pi \cos\left(\frac{n+m}{2}\theta\right) d\theta + \int_0^\pi \cos\left(\frac{n-m}{2}\theta\right) d\theta \\ &= 0 \quad \text{if } m \neq n. \end{aligned}$$

$$\begin{aligned} \bullet \quad m = n > 0 \quad \text{then} \quad \langle T_m, T_m \rangle &= \int_0^\pi \frac{1 - \cos^2 2m\theta}{2} d\theta \\ &= \pi/2 \end{aligned}$$

$$\bullet \quad m = n = 0 \quad \text{then} \quad \langle T_0, T_0 \rangle = \pi.$$

e) We can in addition show that  $T_n$  satisfies a differential equation.

On the hand

$$\frac{\partial}{\partial \theta} T_n(\cos \theta) = -\sin \theta T_n'(\cos \theta)$$

$$\frac{\partial^2}{\partial \theta^2} T_n(\cos \theta) = -\cos \theta T_n'(\cos \theta) + \sin^2 \theta T_n''(\cos \theta)$$

$$= (1 - \cos^2 \theta) T_n''(\cos \theta) - \cos \theta T_n'(\cos \theta).$$

$$\text{On the other hand, } \frac{\partial^2}{\partial x^2} T_n(\cos \theta) = \frac{\partial^2}{\partial \theta^2} \cos(n\theta) = -n^2 \cos n\theta$$

$$\Rightarrow -n^2 T_n(\cos \theta) = (1 - \cos^2 \theta) T_n''(\cos \theta) - \cos \theta T_n'(\cos \theta)$$

$T_n$  is the solution of  $\left| (1-x^2)y'' - xy' = -n^2 y \right|$  7/16

③

a) Bookwork.

b)

(i)

We will show that  $\mathcal{S}_n$  and  $\mathcal{A}_n$  are subspaces of  $M_n(\mathbb{R})$ .

$$* \mathcal{S}_n \subset M_n(\mathbb{R}) \quad \mathcal{S}_n \neq \emptyset \quad \text{as} \quad \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix} \in \mathcal{S}_n.$$

$$\alpha, A, B \quad ; \quad \alpha \in \mathbb{R} \text{ and } A, B \in \mathcal{S}_n.$$

$$\frac{1}{2} (\alpha A + B)^T = \alpha A^T + B^T = \alpha A + B$$

Hence  $\mathcal{S}_n$  is a vector space.

~~The~~

$$* \mathcal{A}_n \subset M_n(\mathbb{R}) \quad \mathcal{A}_n \neq \emptyset \quad \text{as} \quad \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix} \in \mathcal{A}_n.$$

$$\alpha \in \mathbb{R} \text{ and } A, B \in \mathcal{A}_n.$$

$$\begin{aligned} (\alpha A + B)^T &= \alpha A^T + B^T = -\alpha A - B \\ &= -(\alpha A + B) \end{aligned}$$

Hence  $\mathcal{A}_n$  is a vector space.

If  $A \in \mathcal{S}_n$  then.

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{12} & & & \vdots \\ \vdots & & & a_{nn} \\ a_{1n} & \dots & & \end{pmatrix}$$

The nly "degrees of freedom" are the upper triangular coefficients

Hence

$$\boxed{\dim(\mathcal{S}_n) = \frac{n(n+1)}{2}}$$



If  $A \in \mathcal{I}_n$  then 
$$\begin{pmatrix} 0 & a_{12} & \dots & a_{1n} \\ -a_{11} & & & \\ \vdots & & & \\ a_{1n} & \dots & & 0 \end{pmatrix}$$

Hence 
$$\dim(\mathcal{I}_n) = \frac{n(n-1)}{2}$$

(ii)

$(M + M^T)^T = M^T + M$  so that  $M + M^T \in \mathcal{I}_n$ .

$(M - M^T)^T = M^T - M = -(M - M^T) \Rightarrow M - M^T \in \mathcal{I}_n$ .

(iii)

Let  $A \in \mathcal{I}_n$  and  $B \in \mathcal{I}_n$ .

$(A, B) = \text{tr}(A^T B) = \text{tr}(AB)$

$(B, A) = \text{tr}(B^T A) = \text{tr}(BA)$

as  $(A, B) = (B, A)$  since  $(\cdot, \cdot)$  is an inner product

and  $\text{tr}(AB) = \text{tr}(BA)$ , this implies that

$\text{tr}(AB) = (A, B) = 0$  so that  $A \perp B$

This being true for any  $A \in \mathcal{I}_n$  and  $B \in \mathcal{I}_n$ , we have

$\mathcal{I}_n$  orthogonal to  $\mathcal{I}_n$ .

Let  $M \in \mathcal{M}_n(\mathbb{R})$  we can see that

$$M = \left( \frac{\pi + \pi^T}{2} \right) + \left( \frac{\pi - \pi^T}{2} \right).$$

by (ii)  $\frac{\pi + \pi^T}{2} \in \mathcal{J}_n$  and  $\frac{\pi - \pi^T}{2} \in \mathcal{A}_n$ .

This shows the existence of such decomposition.

Let us prove the uniqueness.

$$M = S + A \quad S \in \mathcal{J}_n \text{ and } A \in \mathcal{A}_n.$$

$$\text{then } M^T = S - A \quad \Rightarrow \quad \begin{cases} S = \frac{\pi + \pi^T}{2} \\ A = \frac{\pi - \pi^T}{2} \end{cases}$$

which concludes the proof.

(iv) by the previous question it is easily seen

that the orthogonal projection in  $\mathcal{J}_n$  is given by

$$p_{\mathcal{J}_n}(M) = \frac{\pi + \pi^T}{2}$$

$$\text{and } p_{\mathcal{A}_n}(M) = \frac{\pi - \pi^T}{2}.$$

(iv)

(iii)

Let  $M \in M_n(\mathbb{R})$  we can see that

$$M = \left( \frac{M+M^T}{2} \right) + \left( \frac{M-M^T}{2} \right)$$

ans! by the first part of this question.

$$\frac{M+M^T}{2} \in \mathcal{S}_n \text{ ans!}$$

④ Bookwork .

⑤ Bookwork .

## Solution Pb 4

(8)

The pathology of Banach spaces cannot occur in Hilbert spaces.

Theorem:

Let  $X$  be a Hilbert space and let  $M$  be a closed linear subspace of  $X$ .

For  $x_0 \notin M$ , consider

$$\delta = \inf \{ \|x_0 - y\| ; y \in M \}.$$

Then,

(i) There exist a unique  $y_0 \in M$  such that  $\|x_0 - y_0\| = \delta$

(ii)  $x_0 - y_0$  is orthogonal to  $M$  ( $\forall y \in M; \langle x_0 - y_0, y \rangle = 0$ )

Remark: This theorem says that the unique point in  $M$  closest to  $x_0$  is found by "dropping a perpendicular from  $x_0$  to  $M$ ". It is important to note that the theorem is not true for inner product spaces that are not complete.

Proof:

By the definition of the infimum of a set, there exists a sequence  $(y_m)_m$  in  $M$  such that

$$\delta_m = \|x - y_m\| \quad \text{and} \quad \delta_m \rightarrow \delta \quad \text{when } m \rightarrow +\infty.$$

We first show that  $(y_m)_m$  is a Cauchy sequence.

$$\|y_m - y_n\|^2 = \|(y_m - x) + (x - y_n)\|^2 = \|(y_m - x) - (y_n - x)\|^2.$$

By the parallelogram equality:  $(\|a+b\|^2 + \|a-b\|^2 = 2(\|a\|^2 + \|b\|^2))$

we have,

(9)

$$\begin{aligned}\|y_m - y_m\|^2 &\leq 2(\|y_m - x\|^2 + \|y_m - x\|^2) - \|y_m + y_m - 2x\|^2 \\ &= 2(\delta_m^2 + \delta_m^2) - 2\left\|\frac{y_m + y_m}{2} - x\right\|^2\end{aligned}$$

As  $\Pi$  is a subspace of  $X$  and  $y_m, y_m \in X$ ;  $\frac{y_m + y_m}{2} \in \Pi$  as well.

By the definition of  $\delta$  as the smallest  $\|x_0 - y\|$  for  $y \in \Pi$ .

$$\left\|\left(\frac{y_m + y_m}{2}\right) - x\right\| \geq \delta.$$

$$\text{Hence, } \|y_m - y_m\|^2 \leq 2(\delta_n^2 + \delta_m^2) - 2\delta^2.$$

as  $\delta_m \rightarrow \delta$  when  $m \rightarrow \infty$ , taking  $m \rightarrow +\infty$  and  $m \rightarrow +\infty$  in the previous inequality yields:

$$\lim_{m, m \rightarrow +\infty} \|y_m - y_m\| = 0.$$

So  $(y_m)_{m \rightarrow \infty}$  is indeed a Cauchy sequence.

$X$  is a Hilbert space, in particular a complete space, thus there exists a  $y_0 \in X$  such that  $\lim y_m = y_0$ . Moreover  $(y_m)_m$  is a sequence in  $\Pi$  which is closed, this implies that  $y_0 \in \Pi$ .

This shows that, there is a  $y_0 \in \Pi$  such that

$$\begin{aligned}\delta &= \inf \{ \|x_0 - y\|; y \in \Pi \} \\ &= \|x_0 - y_0\|\end{aligned}$$

To prove (i) of the theorem, we need to prove the uniqueness of  $y_0$ .

14/18

Assume that  $y_0 \in M$  and  $y_* \in \Pi$  both satisfy:

$$\|x - y_0\| = \delta \quad \text{and} \quad \|x - y_*\| = \delta.$$

We want to show that  $y_0 = y_*$ .

By the parallelogram equality,

$$\begin{aligned} \|y_* - y_0\|^2 &= \|(y_* - x) - (y_0 - x)\|^2 \\ &= 2\|y_0 - x\|^2 + 2\|y_* - x\|^2 - \|(y_* - x) + (y_0 - x)\|^2 \\ &= 2\delta^2 + 2\delta^2 - 4\left\|\frac{y_* + y_0}{2} - x\right\|^2 \end{aligned}$$

$$\frac{y_* + y_0}{2} \in \Pi; \quad \leq 4\delta^2 - 4\delta^2 = 0. \quad \Rightarrow y_* = y_0.$$

To conclude the proof of the Theorem, it remains to prove (ii)  $x_0 - y_0$  is orthogonal to  $M$ .

Let  $y \in M$ , as  $M$  is a vector space  $y_0 + \alpha y \in \Pi$  for any  $\alpha \in \mathbb{C}$ ;

From the definition of  $\delta$   $\|x_0 - (y_0 + \alpha y)\|^2 \geq \delta^2$ , so that

$$\delta^2 \leq \langle x_0 - y_0 - \alpha y, x_0 - y_0 - \alpha y \rangle = \|x_0 - y_0\|^2 + |\alpha|^2 \|y\|^2 - 2 \operatorname{Re}(\alpha \langle y, x_0 - y_0 \rangle).$$

$$\text{since } \|x_0 - y_0\| = \delta \quad = \delta^2 + |\alpha|^2 \|y\|^2 - 2 \operatorname{Re}(\alpha \langle y, x_0 - y_0 \rangle).$$

Which implies that  $|\alpha|^2 \|y\|^2 - 2 \operatorname{Re}(\alpha \langle y, x_0 - y_0 \rangle) \geq 0$

Let  $\alpha = \beta \langle x_0 - y_0, y \rangle$ ;  $\beta \in \mathbb{R}$ , we have.

$$\beta^2 |\langle x_0 - y_0, y \rangle|^2 \|y\|^2 - 2\beta |\langle x_0 - y_0, y \rangle|^2 \geq 0. \quad \text{15/18}$$

This inequality holds for all  $\beta \in \mathbb{R}$ . This cannot occur unless the coefficient of  $\beta$  is equal to 0, otherwise the left hand side of the inequality will change sign.

Hence,  $\langle x_0 - y_0, y \rangle = 0$

□

Example 5: SOLUTION P35

(i) Find the minimum over  $(a, b) \in \mathbb{R}^2$  of

$$I(a, b) = \int_0^\pi \left( \sin t - (at^2 + bt) \right)^2 dt.$$

Let  $L^2[0, \pi] = \{ \text{function } f \text{ such that } \int_0^\pi |f(t)|^2 dt < +\infty \}$

with the inner product  $\langle f, g \rangle = \int_0^\pi f(t) \bar{g}(t) dt$ .

$I(a, b) = \left( \text{the distance between } \sin(t) \text{ and the vector space spanned by } t^2 \text{ and } t \right)^2$

We can use the projection theorem to minimise this distance.

To this end we need the orthogonal projection of  $\sin t$  on  $\text{Span}(t, t^2) = \{ at^2 + bt, a, b \in \mathbb{R} \}$ .

$$\begin{cases} \langle \sin t - (\alpha t^2 + \beta t), t \rangle = 0 & (1) \end{cases}$$

$$\begin{cases} \langle \sin t - (\alpha t^2 + \beta t), t^2 \rangle = 0 & (2) \end{cases}$$

$$(1) \Rightarrow \int_0^\pi t \sin t dt - \alpha \int_0^\pi t^3 dt - \beta \int_0^\pi t^2 dt = 0$$

$$\Rightarrow \left[ t(-\cos t) \right]_0^\pi + \int_0^\pi \cos t dt - \alpha \frac{\pi^4}{4} - \beta \frac{\pi^3}{3} = 0$$

$$\Rightarrow \left[ \alpha \frac{\pi^4}{4} + \beta \frac{\pi^3}{3} = \pi \right] \quad (1')$$



$$\begin{aligned}
 (2) & \Rightarrow \int_0^{\pi} t^2 \sin t \, dt - \alpha \int_0^{\pi} t^4 \, dt - \beta \int_0^{\pi} t^3 \, dt = 0 \\
 & \Rightarrow \left[ t^2 (-\cos t) \right]_0^{\pi} + \int_0^{\pi} 2t \cos t \, dt - \alpha \frac{\pi^5}{5} - \beta \frac{\pi^4}{4} = 0 \\
 & \Rightarrow \pi^2 + 2 \left[ t (\sin t) \right]_0^{\pi} - 2 \int_0^{\pi} \sin t \, dt - \alpha \frac{\pi^5}{5} - \beta \frac{\pi^4}{4} = 0.
 \end{aligned}$$

$$\boxed{\alpha \frac{\pi^5}{5} + \beta \frac{\pi^4}{4} = \pi^2 + 2 \left[ \cos t \right]_0^{\pi} = \pi^2 - 4} \quad (2').$$

$$(1') \Rightarrow 3\pi\alpha + 4\beta = \frac{12}{\pi^2}$$

$$(2') \Rightarrow 4\pi\alpha + 5\beta = \frac{20}{\pi^2} - \frac{80}{\pi}$$

$$5 \times (1') - 4(2') \Rightarrow -\pi\alpha = \frac{60}{\pi^2} - \frac{80}{\pi^2} + \frac{320}{\pi}$$

$$\boxed{\alpha = \frac{20}{\pi^3} - \frac{320}{\pi^2}}$$

$$4(1') - 3(2') \Rightarrow \beta = \frac{48}{\pi^2} - \frac{60}{\pi^2} + \frac{240}{\pi}$$

$$\boxed{\beta = \frac{240}{\pi} - \frac{12}{\pi^2}}$$

The orthogonal projection of  $\sin t$  on  $\text{Span}\{t^2, t\}$  is given by

$$\left( \frac{20}{\pi^3} - \frac{320}{\pi^2} \right) t^2 + \left( \frac{240}{\pi} - \frac{12}{\pi^2} \right) t.$$

To compute the minimum of  $I(a,b)$  above, we only need to compute the following integral.

$$\int_0^\pi (\sin t - \alpha t^2 - \beta t)^2 dt \quad \alpha, \beta \text{ defined above.}$$

$$= \int_0^\pi \sin^2 t \, dt + \int_0^\pi \alpha^2 t^4 \, dt + \int_0^\pi \beta^2 t^2 \, dt \\ - 2\alpha \int_0^\pi t^2 \sin t \, dt - 2\beta \int_0^\pi t \sin t \, dt + 2\alpha\beta \int_0^\pi t^3 \, dt$$

$$= \int_0^\pi \left( \frac{1 - \cos 2t}{2} \right) dt + \alpha^2 \frac{\pi^5}{5} + \beta^2 \frac{\pi^3}{3} - 2\alpha (\pi^2 - 4) - 2\beta \pi + \frac{2\alpha\beta}{4} \pi^4$$

$$= \frac{\pi}{2} + \alpha^2 \frac{\pi^5}{5} + \beta^2 \frac{\pi^3}{3} - 2\alpha \pi^2 + 8\alpha - 2\beta \pi + \frac{\alpha\beta}{2} \pi^4$$

~~Replacing~~ Replacing  $\alpha$  and  $\beta$  by their computed values gives

the  $\min_{a,b \in \mathbb{R}} I(a,b) = I(\alpha, \beta)$ .

(ii) Find the minimum of  $\int_{-1}^1 (x^3 - ax^2 - bx - c)^2 dx = I(a,b,c)$   
 $a, b, c \in \mathbb{R}$ .

To minimise this integral that can be interpreted as the distance between  $x^3$  and the vector space  $\text{Span}\{x^2, x, 1\}$ .

We need to find the orthogonal projection of  $x^3$  on  $\text{Span}\{x^2, x, 1\}$

for the inner product  $\langle f, g \rangle = \int_{-1}^1 f(t) \overline{g(t)} \, dt$ .