EXAM SOLUTIONS

1. a) i) [2]

By using channel coding, digital signals are more immune to channel noise. Coding in digital communications can make the effect of noise arbitrarily small.

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The Fourier transform of the signal is $\frac{1}{3}\Lambda(\frac{f}{3})$, which is a baseband signal from -3 to 3 Hz. From Nyquist theorem, the minimum sampling rate for an exact reconstruction is $2 \times 3 = 6$ samples per second.

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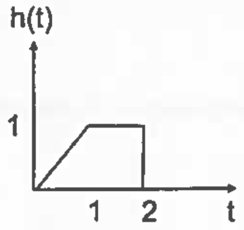
The optimal receive filter is a *matched filter*, given by $h(t) = g(2-t)$:



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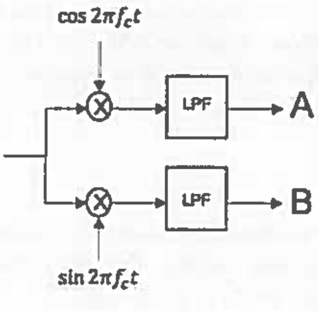

Each branch is a traditional coherent receiver, where A, B have the signs of $\cos\phi$ and $-\sin\phi$, respectively. Therefore, the decision is made as follows:

- $A > 0, B = 0 \rightarrow \phi = 0$,
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b) i) True. An SSS process has a shift-invariant joint distribution which makes the mean constant and the autocorrelation only a function of time difference. [2]

ii) False. [2]

iii) False. Being white is about the spectrum of the process, it does not specify its distribution. [2]

iv) False. Since nothing is known about the cross correlation of these two processes, the statement is not true in general. [3]

c) We have $R_X(\tau) = 3000\text{sinc}(3000\tau)$.

i) [4]

$X(t)$ and $X(t+\tau)$ are uncorrelated when

$$\mathbb{E}[X(t)X(t+\tau)] = \mathbb{E}[X(t)]\mathbb{E}[X(t+\tau)].$$

Since the process is zero-mean, we have $R_X(\tau) = 0$. This holds at $\tau = \frac{n}{3000}$ for non-zero integer n. Therefore, the sampling rates 3K, $\frac{3}{2}$K, $\frac{3}{4}$K, ... result in uncorrelated samples. Obviously, the maximum sampling rate is 3KHz.

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5 sec. $\times$ 1K samples/sec. $\times$ 10 bits/sample = 50K bits.

d) i) [6]

1. True. The output of an LTI system to a Gaussian process is Gaussian.
2. True. $S_Y(f) = S_X(f)|H(f)|^2$, and since $H(f)$ is bandlimited, so is $S_Y(f)$.
3. False. For Gaussian distributions WSS is equivalent to SSS. This follows from the nature of Gaussian distribution that only the mean and the second moment are sufficient to characterize the distribution. Since $Y(t)$ is WSS and Gaussian, it is also SSS.

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Since $X(t)$ is zero-mean, so is $Y(t)$. Therefore, $Y(1)$ is a zero-mean Gaussian random variable. Its pdf is symmetric with respect to the origin. Hence, $\Pr\{Y(1) \geq 0\} = 0.5$.

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$Y(t)$ is SSS, since it is both Gaussian and WSS. Therefore, the joint distribution of the pair $(Y(-1), Y(-2))$ is the same as that of $(Y(1), Y(2))$. Hence,

$$\begin{aligned}\Pr\{Y(-1)+Y(-2)>2\} &= 1-\Pr\{Y(-1)+Y(-2)\le 2\}\\ &= 1-\Pr\{Y(1)+Y(2)\le 2\}\\ &= 1-0.3=0.7\end{aligned}$$

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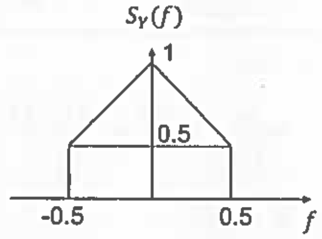

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$Y(t)$ is Gaussian. We found its mean to be 0.5. What remains to solve this problem is to find the variance of $Y(t)$. We have

$$\mathrm{E}[Y^2(t)] = R_Y(0) = \int_{-\infty}^{+\infty} S_Y(f)df = \frac{3}{4},$$

which can be verified from the figure of $S_Y(f)$. The variance of $Y(t)$ is $3/4-\mu_Y^2 = \frac{1}{2}$. Therefore,

$$\Pr\{Y(0)\ge 2\} = Q\left(\frac{2-0.5}{\sqrt{0.5}}\right).$$

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Since $Y(t)$ is Gaussian, $Y(1)+Y(2)+Y(3)$ is a Gaussian random variable with mean $3/2$. Due to symmetry, the probability is $1/2$.

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Due to the symmetry in the system, the optimal threshold in this case is trivial, and is given by 0.

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Probability of error when a "0" is transmitted is given by

$$P_{e0} = \int_{1+T}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2},$$

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The overall probability of error can be written as

$$P_e = p_0 \int_{1+T}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2} + p_1 \int_{1-T}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2}.$$

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From here we obtain

$$\ln \frac{p_0}{p_1} = \frac{(1+T)^2 - (1-T)^2}{2\sigma^2} = \frac{2T}{\sigma^2} = 4T.$$

We find $T = \frac{\ln 4}{4} \approx 0.35$.

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The error probability can be written as

$$P_e = p_0 \cdot Q(1+T) + p1 \cdot Q(1-T).$$

The receiver uses the T from above, given by $T = 0.35$. Then we have

$$\begin{aligned} P_e &= 0.3 \cdot Q(1.35) + 0.7 \cdot Q(0.65) \\ &\approx 0.3 \times 9 \times 10^{-2} + 0.7 \times 3 \times 10^{-1} \qquad \approx 0.237. \end{aligned}$$

This is almost double the probability of error the receiver was expecting to achieve, which is found as

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The diagram is given below, where we have $n(t) = n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t)$.

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From $m(t) = \sqrt{6}\cos(4\pi t)$, we have $W = 2$, and $P = 3$. The power of noise is

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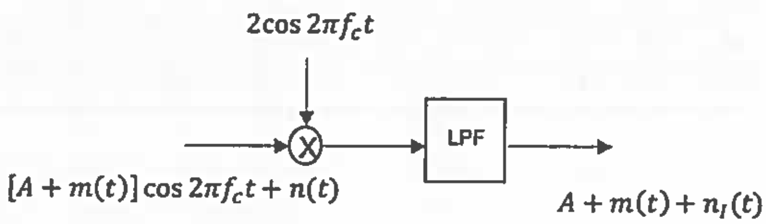
$$\mathrm{SNR_O} = \frac{3}{4}$$



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The performance of the above system is worse than that of a baseband system with the same transmitted power, because the transmitted power is $\frac{A^2+3}{2}$, and we have

$$\mathrm{SNR}_{AM} = \frac{3}{A^2+3}\mathrm{SNR}_{baseband},$$

where $\frac{3}{A^2+3} < 1$.

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The Huffman code results in $A = 0$, $B = 01$ and $C = 11$.

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The average codeword length is

$$0.5 \times 1 + 0.3 \times 2 + 0.2 \times 2 = 1.5 \text{ bits/symbol}.$$

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The Shannon bound on the minimal codeword length is given by the entropy of the source, which is $H(0.5, 0.3, 0.2) = 1.4855$ bits/symbol.

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The Huffman code is given as in the table below. The average codeword length is 3 bits/word, which is equivalent to 1.5 bits/symbol. This is the same as coding the symbols separately. Although there seems to be no advantage of joint coding for this particular example, combining multiple symbols generally increases the efficiency of the Huffman code even if the symbols are independent.

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101	AC	.1	.1	.15	.15	.15	.2			
110	CA	.1	.1	.1	.15	.15				
1000	BB	.09	.1	.1	.1					
1001	BC	.06	.09	.1						
1110	CB	.06	.06							
1111	CC	.04								

Examination Paper Submission document for 2017-2018 academic year.

For this exam, please write the main course code and the course title below.

Code:

Title:

We, the exam setter and the second marker, confirm that the following points have been discussed and agreed between us.

1. There is no full or partial reuse of questions.
2. This examination yields an appropriate range of marks that is well balanced, reflecting the quality of student (with weak students failing, capable students getting at least 40% and bright industrious students obtaining more than 70%)
3. The model answers give a fair indication of the amount of work needed to answer the questions. Each part has a comment indicating to the external examiners the nature of the question; i.e. whether it is bookwork, new theory, a new theoretical application, a calculation for a new example, etc.
4. The exam paper does not contain any grammar and spelling mistakes.
5. The marking schedule is shown in the answers document and the resolution of each allocated mark is better than 3/20 for each question.
6. The examination paper can be completed by the students within time allowed.

Signed (Setter):

Date: 15/3/2018

Signed (Second Marker):

Date: 19/03/2018

Please submit this form with exam paper and model answers, and associated coursework to the Undergraduate Office on Level 6 by the required submission date.

Checklist for Exam Setter and Second Marker

Our external examiners require ever higher standards regarding intellectual content and presentation.

In order to help meet these requirements in a uniform way, **a checklist is given for the Setter and Second Marker to print, complete, sign and submit to the UG Office with the exam paper and model answers, in separate envelopes.**

Guidelines for exam paper preparation are given on a separate page together with the link to download a templates for question and answer documents.

The completed, signed, checklists will be copied to the external examiners.

It is important that examiners keep to the deadline for their exam papers.

The 2017-18 deadlines can be seen on the internal webpages of Staff Information

http://www.imperial.ac.uk/electrical-engineering/internal/for-staff/assessment-and-examinations/

Enough time must be allowed for the second marker to review the paper and provide feedback. The Setter and Second-marker must discuss and agree a final version of the paper before it is handed in to UG office. Where applicable, please enclose a copy of course work and marking scheme.

With regards

Christos Papavassiliou

Guidelines for Setting Exam Papers:

Format and structure

- Exam papers must adhere to the prescribed format, style and layout. Template (Word and latex version) can be downloaded from http://www3.imperial.ac.uk/electricalengineering/teaching/staff/examsetting
- The college guidelines suggest about 70% of students in a course should get 1^{st}/2:1. The questions must be set with a target raw exam average of 67.5%. Each question should have a fair balance between easy and hard elements so that the majority of the students can score a decent mark in each question.
- Papers should have a good balance between theory and problems.
- All 1^{st} and 2^{nd} Year papers will have three questions. Question 1 must have 40% weight of total marks. It must have wide coverage. Question 2 and 3 must be detailed and each with 30% weight.
- Second marker must make an independent judgment on the difficulty, length and anticipated average score. In case of disagreement, both the markers should work together to produce an acceptable version for the UG office

Clarity

- Sentences should be as short, simple and direct as possible with data (whether numerical or textual explanation) separated as much as possible from the statements of the actions required.
- Use standard convention for units (The Electrical Engineering Handbook, 2nd Ed, 1997)
- Figures must be drawn with due care for clarity and should be labeled and numbered properly. The first figure related to question 'N' should be numbered as Fig N.1.
- Any special instructions must be on the first page after the title page. The department will produce the title page and give it to you to check. Please check that the data on this page is correct.
- Answers should be typed in separate sections showing marks at the end of each section. There should be as many sections as there are parts of a question each with an allocated mark.

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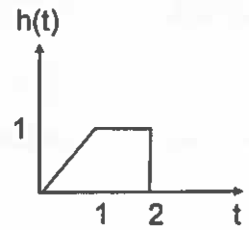

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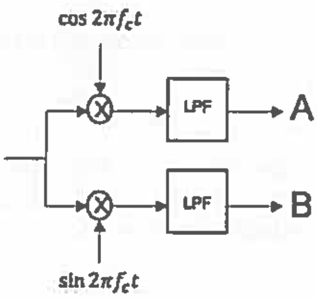

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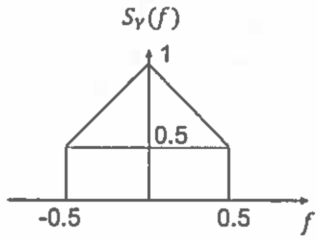

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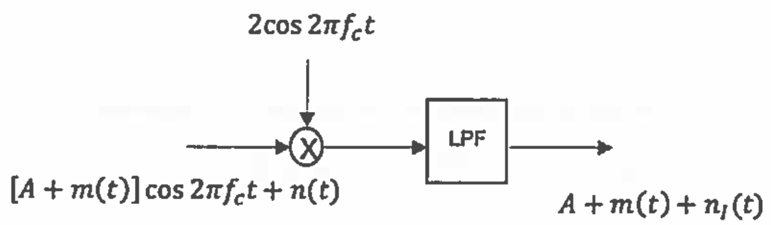

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