EE3-10
EE9CS3-1

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2011

MSc and EEE/ISE PART III/IV: MEng, BEng and ACGI

**MATHEMATICS FOR SIGNALS AND SYSTEMS**

Tuesday, 24 May 10:00 am

Time allowed: 3:00 hours

**There are THREE questions on this paper.**

**Answer ALL questions. All questions carry equal marks.**

**Any special instructions for invigilators and information for candidates are on page 1.**

Examiners responsible First Marker(s) : M.M. Draief
Second Marker(s) : D. Angeli

## MATHEMATICS FOR SIGNAL AND SYSTEMS

1. For a matrix $A$ in $\mathbb{R}^{n\times n}$, we define the $k$-th power of $A$ as $A^k = A^{k-1}\times A = A\times A^{k-1}$, for $k\geq 1$ and $A^0 = I$ the identity matrix. We denote by $\mathrm{Im}(A)$ and $\mathscr{N}(A)$ the range (or image) and the nullspace (or kernel) of $A$, respectively.

We say that two subspaces $V$ and $W$ of $\mathbb{R}^n$ are complementary, denoted by $V\oplus W = \mathbb{R}^n$, if (i) $V\cap W = \{\mathbf{0}\}$, where $\mathbf{0}$ is the zero vector in $\mathbb{R}^n$, and (ii) any vector $x\in\mathbb{R}^n$ can be written as $x = v+w$ where $v\in V$ and $w\in W$.

a) We let $n=3$ and define the matrix $A$ by

$$A = \begin{pmatrix} 4 & -1 & 5 \\ -2 & -1 & -1 \\ -4 & 1 & -5 \end{pmatrix}.$$

i) Derive $\mathrm{Im}(A)$ and $\mathscr{N}(A)$ and determine a basis for each of them. [ 3 ]

ii) Do we have $\mathrm{Im}(A)\oplus\mathscr{N}(A) = \mathbb{R}^3$? Justify your answer. [ 2 ]

iii) Let $A^2 = A\times A$. Derive $\mathrm{Im}(A^2)$ and $\mathscr{N}(A^2)$ and determine a basis for each of them. [ 3 ]

iv) Show that $\mathrm{Im}(A^2)\oplus\mathscr{N}(A^2) = \mathbb{R}^3$. [ 3 ]

b) We now let $n=4$ and define the matrix $A_m$ as follows

$$A_m = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & m & 0 & 0 \\ 1 & 0 & -m & -1 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

where $m\in\mathbb{R}$ is a parameter.

i) Derive bases for $\mathscr{N}(A_m)$ and $\mathrm{Im}(A_m)$. [ 3 ]

ii) For $m\neq 0$, show that $\mathrm{Im}(A_m)\oplus\mathscr{N}(A_m) = \mathbb{R}^4$. [ 2 ]

iii) We now fix $m=0$. Compute $A_0^3$.

Do we have $\mathrm{Im}(A_0^3)\oplus\mathscr{N}(A_0^3) = \mathbb{R}^4$?

Justify your answer. [ 2 ]

c) We define the following property

For $A\in\mathbb{R}^{n\times n}$, there exists an integer $p\geq 1$ such that $\mathrm{Im}(A^p)\oplus\mathscr{N}(A^p) = \mathbb{R}^n$, $(\star)$

i) Let $A$ be a non-singular (invertible) matrix. Find $p$ such that the property $(\star)$ is satisfied for $A$. Justify your answer. [ 1 ]

ii) Let $A$ be a projection. Find a value $p$ such that $(\star)$ is satisfied.

Explain your answer. A formal proof is not required. [ 1 ]

*In fact, the property $(\star)$ is satisfied for any matrix $A$.*

2. For $x, y$ two vectors in $\mathbb{R}^m$, we define the inner product $(x \mid y) = x^T y = \sum_{i=1}^m x_i y_i$ where $x_i$ and $y_i$ are the $i$-th coordinates of $x$ and $y$, respectively, and $^T$ is the operation of transposing a vector or a matrix. We also let the norm of $x$ be $||x|| = \sqrt{x^T x} = \sqrt{\sum_{i=1}^m x_i^2}$.

Let $A \in \mathbb{R}^{m \times n}$. For $v \in \mathbb{R}^m$, we define $v_0 \in \mathrm{Im}(A)$, the orthogonal projection of $v$ on $\mathrm{Im}(A)$, i.e.,

$$(v - v_0 \mid Ax) = 0, \text{ for all } x \in \mathbb{R}^n. \qquad (\star\star)$$

a) Let $x_0 \in \mathbb{R}^n$ such that $Ax_0 = v_0$.

i) Show that for all $x \in \mathbb{R}^n$, we have

$$||Ax - v||^2 = ||v - v_0||^2 + ||Ax - v_0||^2.$$

[ 3 ]

ii) Prove that $||Ax_0 - v|| = \min_{x \in \mathbb{R}^n} ||Ax - v||$. We will refer to $x_0$ as a *pseudo-solution* of the equation $Ax = v$. [ 2 ]

iii) Suppose that $A$ has zero-nullspace and let $x_1$ be a vector such that

$$||Ax_1 - v|| = ||v_0 - v||.$$

Show that $x_1 = x_0$. [ 3 ]

iv) By rewriting $(\star\star)$ in matrix form show that $x_0$ is a pseudo-solution of $Ax = v$ **if and only if** $x_0$ is a solution of the *normal equation*

$$A^T A x_0 = A^T v.$$

[ 2 ]

v) Assume that $A$ has zero-null space. Describe an algorithm for solving the normal equation using the Cholesky decomposition (the description of the Cholesky decomposition is not required). [ 2 ]

vi) Ignoring the cost of the Cholesky decomposition, how many additional operations does the previous algorithm (Question 2.a)v)) perform? [ 2 ]

b) Let $n = 3$,

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & 2 & 1 \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Compute the pseudo-solutions of $Ax = v$. [ 2 ]

c) Let $n$ be an integer greater or equal to 2 and define the following three real-valued vectors $(a_1, a_2, \cdots, a_n)$, $(b_1, b_2, \cdots, b_n)$ and $(c_1, c_2, \cdots, c_n)$. We would like to find two real numbers $\lambda$ and $\mu$ that minimise the following sum

$$\sum_{k=1}^{n} (\lambda a_k + \mu b_k - c_k)^2.$$

i) Restate the above minimisation problem in terms of finding the pseudo-inverse of a linear equation $Ax = v$. [ 1 ]

ii) Derive a condition on $(a_1, a_2, \cdots, a_n)$ and $(b_1, b_2, \cdots, b_n)$ so that the matrix $A$, defined in Question 2.c)i), has zero null-space. [ 1 ]

iii) Under the condition of Question 2.c) ii), solve the minimisation problem. Express $\lambda$ and $\mu$ in terms of inner products. [ 2 ]

3. We consider the set $\mathbb{R}_n[X]$ of polynomials with real coefficients and degrees less or equal to $n$ endowed with the inner product $\langle P,Q\rangle = \int_{-1}^{1} P(t)Q(t)dt$.

a) Show that $\langle P,Q\rangle = \int_{-1}^{1} P(t)Q(t)dt$ is indeed an inner product on $\mathbb{R}_n[X]$. [ 1 ]

b) Give the expression of $\langle P,Q\rangle$ when $P$ and $Q$ are polynomials in $\mathbb{R}_2[X]$ in terms of the coefficients of both $P$ and $Q$. [ 1 ]

c) Let $L$ be the application on $\mathbb{R}_n[X]$ such that

$$L(P) = \frac{d}{dX}\left[(X^2-1)\frac{dP}{dX}\right].$$

i) Show that if $P \in \mathbb{R}_n[X]$ then $L(P) \in \mathbb{R}_n[X]$ and that $L$ is a linear transformation on $\mathbb{R}_n[X]$. [ 2 ]

ii) Prove that, for all $P,Q$ in $\mathbb{R}_n[X]$, we have

$$\langle L(P),Q\rangle = \langle P,L(Q)\rangle.$$

[ 3 ]

*Hint: Perform integrations by parts.*

d) Let $P_0 = 1$ and for $k = 1,\cdots,n$, define the polynomial $P_k$ of degree $k$ as follows

$$P_k = \frac{d^k}{dX^k}\left((X^2-1)^k\right),$$

the $k$-th derivative of $(X^2-1)^k$.

i) Compute $P_1$ and $P_2$. [ 1 ]

ii) Derive an expression of $L(P_k)$ in terms of $P_k'$ and $P_k''$ the first and second derivatives of $P_k$, respectively. [ 1 ]

iii) Prove the following identity

$$(X^2-1)\frac{d[(X^2-1)^k]}{dX} - 2kX(X^2-1)^k = 0.$$

[ 1 ]

iv) By differentiating $(k+1)$ times the above expression, establish that

$$(X^2-1)P_k''(X) + 2XP_k'(X) = k(k+1)P_k(X).$$

[ 4 ]

*Hint: Use Leibniz's formula*

$$(fg)^{(k+1)} = \sum_{i=1}^{k+1}\binom{k+1}{i} f^{(i)}g^{(k+1-i)},$$

*where $f^{(i)}$ is the i-th derivative of $f$.*

v) Find the eigenvalues and eigenvectors of the transformation $L$. [ 2 ]

e) Let $k,l$ two integers between 0 and $n$.

i) Express $\langle L(P_k),P_l\rangle$ and $\langle L(P_l),P_k\rangle$ in terms of $\langle P_k,P_l\rangle$. [ 2 ]

ii) Prove that $(P_0,P_1,\cdots,P_n)$ is an orthogonal basis of $\mathbb{R}_n[X]$ when endowed with the inner product $\int_{-1}^{1} P(t)Q(t)dt$. [ 2 ]

*These polynomials are known as Legendre polynomials.*

EE3-10
EE9 CS3-1

# SOLUTIONS - 2011

MATHEMATICS FOR SIGNAL & SYSTEMS (1210-2011).

Q1

1 | 10

1

a)
$$A = \begin{bmatrix} 4 & -1 & 5 \\ -2 & -1 & -1 \\ -4 & 1 & -5 \end{bmatrix}$$

i)
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathcal{N}(A) \Rightarrow \begin{cases} 4x - y + 5z = 0 \\ -2x - y - z = 0 \\ -4x + y - 5z = 0 \end{cases} \Leftrightarrow \begin{cases} y = 4x + 5z \\ z = -2x - y \end{cases}$$

$$\Leftrightarrow \begin{cases} y = -x \\ z = -x \end{cases}$$

$$\mathcal{N}(A) = \{ x e_1 - x e_2 - x e_3 \; ; \; x \in \mathbb{R} \} = \text{Span}\left\{ \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \right\}.$$

From lecture $\text{Im}(A) = \text{Span}\{\text{column vectors}\}$.

Hence $\text{Im}(A) = \text{Span}\left\{ \begin{pmatrix} 4 \\ -2 \\ -4 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ -1 \\ -5 \end{pmatrix} \right\}$

Since $\begin{pmatrix} 5 \\ -1 \\ -5 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ -4 \end{pmatrix} - \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$, then

$$\text{Im}(A) = \text{Span}\left\{ \begin{pmatrix} 4 \\ -2 \\ -4 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

ii)
It is not difficult to see that

$$\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = \frac{1}{3}\left[ \begin{pmatrix} 4 \\ -2 \\ -4 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right] \Rightarrow$$

$$\text{Im}(A) \cap \mathcal{N}(A) \neq \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

So the answer is $\text{Im}(A) \oplus \mathcal{N}(A) \neq \mathbb{R}^3$.

1) a) iii) $A^2 = \begin{bmatrix} -2 & 2 & -4 \\ -2 & 2 & -4 \\ 2 & -2 & 4 \end{bmatrix}$.

Q.1 (+2)

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathcal{N}(A^2) \Leftrightarrow 2x - 2y + 4z = 0 \Leftrightarrow x = y - 2z.$

$\mathcal{N}(A^2) = \text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}.$

$\text{Im}(A^2) = \text{Span}\left\{ \begin{pmatrix} -2 \\ -2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix}, \begin{pmatrix} -4 \\ -4 \\ 4 \end{pmatrix} \right\}$

$= \text{Span}\left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\}.$

iv) $u \in \mathcal{N}(A^2) \cap \text{Im}(A^2) \Leftrightarrow \exists\ \alpha, \beta, \gamma \in \mathbb{R} \mid$

$u = \alpha(e_1 + e_2) + \beta(-2e_1 + e_3) = \gamma(-e_1 - e_2 + e_3)$

$\Leftrightarrow \begin{cases} \alpha - 2\beta + \gamma = 0 \\ \alpha - \gamma = 0 \\ \beta - \gamma = 0 \end{cases} \Leftrightarrow \alpha = \beta = \gamma = 0.$

Hence $\mathcal{N}(A^2) \cap \text{Im}(A^2) = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}.$

Moreover $\dim(\mathcal{N}(A^2)) + \dim(\text{Im}(A^2)) = 3.$

Note: One can use the fact that $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$ forms a basis of $\mathbb{R}^3$.

Q1

1/b)

$$\begin{pmatrix}x\\y\\z\\t\end{pmatrix} = u \in \mathcal{N}(A_m) \iff \begin{cases} -y=0\\ my=0\\ x-mz-t=0\\ y=0\end{cases} \Rightarrow \begin{cases} y=0\\ t=x-mz\end{cases}$$

$$\mathcal{N}(A_m) = \cancel{\text{Vect}}\,\text{Span}\left\{ \begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\-m\end{pmatrix} \right\}.$$

$$\text{Im}(A_m) = \text{Span}\left\{ \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\m\\0\\1\end{pmatrix} \right\}.$$

ii) $m \neq 0$.

$$u \in \mathcal{N}(A_m) \cap \text{Im}(A_m) \iff \exists\, \alpha, \beta, \gamma, \delta.$$

$$u = \alpha(e_1+e_4) + \beta(e_3 - me_4) = \gamma e_3 + \delta(-e_1 + me_2 + e_4)$$

$$\iff \begin{cases} \alpha + \delta = 0\\ m\delta = 0\\ \beta - \gamma = 0\\ \alpha - m\beta - \delta = 0 \end{cases} \Rightarrow \begin{array}{c} \alpha = \beta = \gamma = \delta = 0\\ (m \neq 0) \end{array}$$

Moreover $\dim(\text{Im}(A_m)) + \dim(\mathcal{N}(A_m)) = 4$.

iii) $m=0$ $\quad A_0^3 = \begin{bmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{bmatrix}$, Hence

trivially $\mathcal{N}(A_0) = \mathbb{R}^4$ & $\text{Im}(A_0) = \left\{\begin{pmatrix}0\\0\\0\end{pmatrix}\right\}$

$\Rightarrow \mathbb{R}^4 = \text{Im}(A_0) \oplus \mathcal{N}(A_0)$.

Q1 4

1/c)

i) A invertible $\Rightarrow$ $\mathcal{N}(A) = \left\{ \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \right\}$

$\mathrm{Im}(A) = \mathbb{R}^n$

$\Rightarrow$ $\mathcal{N}(A) \oplus \mathrm{Im}(A) = \mathbb{R}^n$

~~ii)~~ if A is a projection then

$\mathcal{N}(A) \oplus \mathrm{Im}(A) = \mathbb{R}^n$.

Q2

5

2/ a/ $(v - v_0 \mid Ax) = 0 \quad \forall x \in \mathbb{R}^3 \quad (*)$

i) $\|Ax - v\|^2 = \|Ax - v_0 + v_0 - v\|^2$

$= \|Ax - v_0\|^2 + \|v_0 - v\|^2 + 2(Ax - v_0 \mid v_0 - v)$

Remark that $(Ax - v_0 \mid v_0 - v) = (A(x - y_0) \mid v_0 - v)$

$= 0$ by $(*)$.

$\Rightarrow \|Ax - v\|^2 = \|Ax - v_0\|^2 + \|v_0 - v\|^2$

ii) $\|Ax - v\| \geq \|Ax - v_0\|$ unless $v = v_0$.

iii) $\|Ax_1 - v\|^2 = \|Ax_1 - Ax_0 + Ax_0 - v\|^2$

$= \|Ax_1 - Ax_0\|^2 + \|Ax_0 - v\|^2$

$+ 2\underbrace{(A(x_1 - x_0) \mid Ax_0 - v)}_{=0 \text{ by } (*)}$.

However $\|Ax_1 - v\| = \|Ax_2 - v\|$

$\Rightarrow \|Ax_1 - Ax_2\| = 0 \Rightarrow A(x_1 - x_2) = 0$

$\Rightarrow x_1 = x_2$ since $A$ has zero null-space.

2/ a/ $\delta 2$ 6

iv) $(**) \Leftrightarrow (v-v_0)^T Ax = 0 \quad \forall x$

$$\Leftrightarrow (A^T v - Ax_0)^T x = 0 \quad \forall x$$

$$\Leftrightarrow A^T(v - Ax_0) = 0$$

$$\Rightarrow A^T A x_0 = A^T v .$$

v) • Cholesky decomposition of $A^T A = L^T L \in \mathbb{R}^{n \times n}$.

$L \in$

$v \in \mathbb{R}^m$

$A^T \in \mathbb{R}^{n \times m}$

* $A^T v : (2m-1)\,n$.

* $A^T A : (2m-1)\,n^2$

* $L^T \underbrace{L x_0}_{} = A^T v$.

$L^T w = A^T v \quad : \quad n^2$

$L x_0 = w \quad : \quad n^2$

vi) In total ~~$2mn + 2mn^2 + 2n^2$ flops~~.

$$2mn - n + 2mn^2 - n^2 + 2n^2$$

$$= 2mn^2 + 2mn + n^2 - n \left(= \Theta(mn^2)\right),$$

Q2 7

2/c) iii) $A^T \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{matrix} \sum a_i c_i \\ \sum b_i c_i \end{matrix}$

$$\begin{bmatrix} \langle a, a\rangle & \langle a, b\rangle \\ \langle a, b\rangle & \langle b, b\rangle \end{bmatrix} \begin{pmatrix} \lambda_1 \\ \mu_2 \end{pmatrix} = \begin{matrix} \langle a, c\rangle \\ \langle b, c\rangle . \end{matrix}$$

$$\langle a, a\rangle \cancel{\lambda_1} + \langle a, b\rangle \cancel{\mu_2} = \langle a, c\rangle$$
$$\langle a, b\rangle \cancel{\lambda} + \langle b, b\rangle \cancel{\mu} = \langle b, c\rangle .$$

$$\lambda = \frac{\langle a, c\rangle - \langle a, b\rangle \mu}{\langle a, a\rangle}$$

$$\frac{\langle a, b\rangle}{\langle a, a\rangle}\left[\langle a, c\rangle - \langle a, b\rangle \mu\right] + \langle b, b\rangle \mu = \langle b, c\rangle$$

$$\langle b, b\rangle - \frac{\langle a, b\rangle^2}{\langle a, a\rangle} \mu = \langle b, c\rangle - \frac{\langle a, b\rangle \langle a, c\rangle}{\langle a, a\rangle}$$

$$\left\{ \begin{aligned} \mu &= \frac{\langle a, a\rangle \langle b, c\rangle - \langle a, b\rangle \langle a, c\rangle}{\langle a, a\rangle \langle b, b\rangle - \langle a, b\rangle^2} \\ \lambda &= \frac{\langle b, b\rangle \langle a, c\rangle - \langle a, b\rangle \langle b, c\rangle}{\langle a, b\rangle \langle b, b\rangle - \langle a, b\rangle^2} \end{aligned} \right.$$

8 Q3

3/

a) $\langle P, Q \rangle = \langle Q, P \rangle$

$\langle P, P \rangle = \int_{-1}^{1} P^2(t)\, dt \geq 0$

$\langle \lambda P + \mu R, Q \rangle = \lambda \langle P, Q \rangle + \mu \langle R, Q \rangle.$

$\langle P, P \rangle = 0 \Rightarrow P^2 = 0$ on $(-1, 1)$

$\Rightarrow P = 0.$

b) $P(x) = a_0 + a_1 x + a_2 x^2$

$Q(x) = b_0 + b_1 x + b_2 x^2$

$P(x) Q(x) = b_0 a_0 + (a_1 b_0 + a_0 b_1) x$

$+ (a_1 b_1 + a_2 b_0 + b_2 a_0) x^2 + (a_1 b_2 + a_2 b_1) x^3 + a_2 b_2 x^4$

$\langle P, Q \rangle = 2 b_0 a_0 + \frac{2 a_0 b_1 + a_2 b_0 + a_0 b_2}{3} + \frac{2 a_2 b_2}{5}.$

c) i) High degree in $L(P)$ comes

from $\frac{d}{dX} (X^2 - 1) \frac{dX^n}{dX}$

i.e $\frac{d}{dX} (X^2 - 1)\, n X^{n-1}$ which has

at most degree $n$.

Q3

(3)

Trivially we have

$$L(P + \lambda Q) = L(P) + \lambda L(Q).$$

ii) $$\langle L(P)\, Q\rangle = \int_{-1}^{1} \frac{d}{dt}(t^2-1)\frac{dP(t)}{dt}\, Q(t)\, dt.$$

Integration by part.

$$= \underbrace{\left[(t^2-1)\frac{dP}{dt}\, Q\right]_{-1}^{1}}_{=0} - \int_{-1}^{1} (t^2-1)\frac{dP}{dt}\frac{dQ}{dt}(t)\, dt$$

Integration by part,

$$= -\underbrace{\left[(t^2-1)\frac{dQ(t)}{dt}\, P(t)\right]_{-1}^{1}}_{=0} + \int_{-1}^{1} \frac{d}{dt}\left[(t^2-1)\frac{dQ}{dt}\right] P(t)\, dt$$

$$= \langle P, L(Q)\rangle.$$

d) i) $P_1 = 2X$, $P_2 = 12X^2 - 4$.

ii) $L(P_k) = \left((X^2-1)P_k'\right)' = 2X P_k' + (X^2-1)P_k''$.

iii) $$(X^2-1)\left((X^2-1)^k\right)' = (X^2-1)\cdot 2k\, X(X^2-1)^{k-1}$$
$$= 2k\, X(X^2-1)^k.$$

Q3 10

iv) Leibniz formula $\frac{d^{k+1}}{dx^{k+1}} f(x)g(x) = \sum_{i=0}^{k+1} \binom{k+1}{i} f^{(i)}(x) g^{(k+1-i)}(x)$

Applying this to d) iii).

$$(x^2-1) P_k''(x) + 2x(k+1) P_k'(x) + (k+1)k P_k(x)$$
$$- 2kx P_k'(x) - 2k(k+1) P_k(x) = 0.$$

$$\boxed{(x^2-1) P_k''(x) + 2x P_k'(x) = k(k+1) P_k(x)}$$

v) Eigenvalues $k(k+1)$ eigenvectors $P_k$.

e)

i) $\langle L(P_k), P_l \rangle = k(k+1) \langle P_k, P_l \rangle.$

$\langle L(P_l), P_k \rangle = l(l+1) \langle P_l, P_k \rangle$

ii) i) $\Rightarrow$ $k(k+1) \langle P_k, P_l \rangle = l(l+1) \langle P_k, P_l \rangle$

together with d) ii) $k \neq l \Rightarrow \langle P_k, P_l \rangle = 0.$

ii) Since $P_0 \ldots P_n$ is orthogonal of cardinality $n$ then it is an orthogonal basis of $\mathbb{R}_n[x]$.