IMPERIAL COLLEGE LONDON

E3.09
C1.0
ISE3.9

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2008

EEE/ISE PART III/IV: MEng, BEng and ACGI

CONTROL ENGINEERING

Corrected Copy

Monday, 12 May 10:00 am

Time allowed: 3:00 hours

There are SIX questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

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CONTROL ENGINEERING

1. Consider a linear, single-input, single-output, continuous-time system described by the equations

$$\dot{x} = Ax + Bu \qquad y = Cx \tag{1.1}$$

where $x(t) \in I\!R^n$, $n \geq 2$, $u(t) \in I\!R$, $y(t) \in I\!R$, and A, B, and C are matrices of appropriate dimensions.

Consider another linear, single-input, single-output, continuous-time system described by the equations

$$\dot{\xi} = F\xi + Gv \qquad \eta = H\xi \tag{1.2}$$

where $\xi(t) \in I\!R^2$, $v(t) \in I\!R$, $\eta(t) \in I\!R$, and F, G, and H are matrices of appropriate dimensions.

System (1.2) is said to *match* system (1.1) at the points s_1 and s_2, with $s_1 \neq s_2$, if (I denotes the identity matrix of appropriate dimension)

$$H(s_1 I - F)^{-1} G = C(s_1 I - A)^{-1} B \qquad H(s_2 I - F)^{-1} G = C(s_2 I - A)^{-1} B.$$

Let

$$F = \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix} - \begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} \qquad G = \begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix}$$

$$H = \begin{bmatrix} C(s_1 I - A)^{-1} B & C(s_2 I - A)^{-1} B \end{bmatrix}.$$

a) Show that, with the above selection of F, G and H, system (1.2) matches system (1.1) at s_1 and s_2 for all Δ_1 and Δ_2. [8 marks]

b) Show that, with the above selection of F, G and H, system (1.2) is reachable for any Δ_1 and Δ_2 such that $\Delta_1 \Delta_2 \neq 0$. (Recall that $s_1 \neq s_2$.) [4 marks]

c) Assume that $(C(s_1 I - A)^{-1} B) = (C(s_2 I - A)^{-1} B) = \kappa$. Show that, with the above selection of F, G and H, system (1.2) is observable if and only if $\kappa \neq 0$. (Recall that $s_1 \neq s_2$.) [4 marks]

d) Assume that $s_1 = 0$ and $s_2 = 1$. Consider the above selection of F. Select Δ_1 and Δ_2 such that system (1.2) has two eigenvalues at -1. [4 marks]

2. Consider a linear, discrete-time system described by the equations

$$x_1(k+1) = x_2(k), \qquad x_2(k+1) = Gx_1(k) + Bu(k), \tag{2.1}$$

where $x(k) = [x_1'(k)\ x_2'(k)]'$, with $x_1(k) \in I\!R^n$ and $x_2(k) \in I\!R^n$ for some $n \geq 1$, is the state, $u(k) \in I\!R^m$, for some $m \leq n$, is the input and G and B are matrices of appropriate dimensions.

a) Show that the system (2.1) is reachable if and only if the system

$$\xi(k+1) = G\xi(k) + Bv(k),$$

with $\xi(k) \in I\!R^n$ and $v(k) \in I\!R^m$ is reachable. [6 marks]

b) Assume $m = n$ and $B = I$. Show, using the result of part a), that the system (2.1) is reachable. [2 marks]

c) Assume $m = n$ and $B = I$. To design a state-feedback control law which asymptotically stabilizes system (2.1) one could proceed in steps, as detailed below.

i) Consider the system

$$x_1(k+1) = v(k).$$

Let $v(k) = Kx_1(k)$, and determine one $K = K'$ such that the system

$$x_1(k+1) = Kx_1(k)$$

is asymptotically stable. [2 marks]

ii) Consider the signal

$$e(k) = x_2(k) - Kx_1(k),$$

with K as selected in part c.i), and the system (2.1). Write an expression for $e(k+1)$ in terms of $x_1(k)$, $x_2(k)$ and $u(k)$. [4 marks]

iii) Determine $u(k)$ such that $e(k) = 0$ for all $k \geq 1$. [2 marks]

iv) Argue that the state feedback control law determined in part c.iii) asymptotically stabilizes the discrete-time system (2.1). [4 marks]

3. Consider the linear, single-input, continuous-time system described by the equations

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = Ax + Bu = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u.$$

The input $u(t)$ of the system is connected to a zero-order-hold and the state $x(t)$ is measured with a sampler. Let $T > 0$ be the sampling time.

We wish to design a state feedback control law to asymptotically stabilize the system.

a) Write the approximate Euler model of the considered system. [2 marks]

b) Show that the approximate Euler model determined in part a) is reachable for any $T > 0$. [2 marks]

c) Consider the approximate Euler model determined in part a) and design a state feedback control law $u = Kx = K_1x_1 + K_2x_2$ that assigns both eigenvalues of the resulting closed-loop system at $s = 0$. (Recall that, since the Euler approximate model is a discrete-time system, the state feedback control law is such that the closed-loop system is asymptotically stable.) [4 marks]

d) To assess the efficacy of the state feedback control law designed in part c) on the system, consider the exact model of the continuous-time system in the presence of the hold and the sampler. This model is given by

$$x(k+1) = A_d x(k) + B_d u(k),$$

where $A_d = e^{AT}$ and $B_d = \int_0^T e^{A(T-\tau)} B \, d\tau$.

i) Compute explicitly the matrices A_d and B_d for the considered system. [6 marks]

ii) Consider the closed-loop system

$$x(k+1) = A_d x(k) + B_d K x(k),$$

where K is as in part c), and A_d and B_d are as in part d.i). Discuss the stability properties of the resulting closed-loop system and discuss if the design based on the Euler model is effective. [6 marks]

4. Consider a nonlinear, single-input, continuous-time system described by the equations

$$\dot{x}_1 = x_1^2 + x_2 \qquad \dot{x}_2 = x_1 x_2 + u$$

where $x(t) = [x_1(t) \;\; x_2(t)]' \in I\!R^2$ is the state and $u(t) \in I\!R$ is the control input, and the problem of designing a state feedback control law which asymptotically stabilizes the equilibrium $x = 0$ of the system.

This problem can be solved using two different approaches, detailed in parts a) and b) below.

a) i) Write the linearization of the system at $x = 0$. [2 marks]

ii) Verify that the linearized system is reachable. [2 marks]

iii) Design a linear state feedback control law $u = K_a x$ which assigns both eigenvalues of the closed-loop linearized system at -1. [2 marks]

iv) Argue that the zero equilibrium of the nonlinear system in closed loop with the control law $u = K_a x$ is locally asymptotically stable. [2 marks]

b) i) Let $y(t) = x_1(t)$. Show that $\ddot{y} + \alpha(x_1, x_2) = u$, for some function $\alpha(\cdot)$, which should be specified. [4 marks]

ii) Design a nonlinear state feedback control law $u = K_b(x)$ such that $\ddot{y} + 2\dot{y} + y = 0$. [2 marks]

iii) Argue that the zero equilibrium of the nonlinear system in closed loop with the control law $u = K_b(x)$ is globally asymptotically stable. [4 marks]

c) Discuss briefly advantages and disadvantages of the control laws $u = K_a x$ and $u = K_b(x)$, designed in parts a) and b), respectively, in terms of the stability properties of the zero equilibrium of the associated closed-loop system and in terms of their complexity. [2 marks]

5. Consider a linear, single-input, single-output, discrete-time system described by the equations

$$x(k+1) = Ax(k) + Bu(k) \qquad y(k) = Cx(k)$$

where $x(k) \in I\!R^3$ is the state, $u(k) \in I\!R$ is the input, $y(k) \in I\!R$ is the output,

$$A = \begin{bmatrix} 0 & \alpha & 0 \\ -\alpha & 0 & 1 \\ 0 & 0 & -1/2 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$$

and $\alpha \in I\!R$ is a constant parameter.

Recall that a discrete-time system is said to be reconstructable if all unobservable modes are at $s = 0$.

a) Study the observability, detectability and reconstructability properties of the system as a function of α. [8 marks]

b) Determine for which values of α it is possible to design an observer such that the state estimation error $e(k) = x(k) - \hat{x}(k)$, where $\hat{x}(k)$ is the estimate of $x(k)$, is identically equal to zero for all $k \geq 3$.
(Do not design the observer.) [6 marks]

c) Assume $\alpha = 0$. Determine the unobservable subspace and write the system in the canonical decomposition for unobservable systems. [6 marks]

6. Consider a linear, single-input, single-output, continuous-time system described by the equations

$$\dot{x} = Ax + Bu + Pd \qquad\qquad y = Cx$$

where $x(t) \in I\!R^n$ is the state, $u(t) \in I\!R$ is the input, $d(t) \in I\!R$ is the disturbance, and $y(t) \in I\!R$ is the output.

Consider the problem of *decoupling* the effect of the disturbance from the output by means of a suitably designed state feedback control law. This problem is solvable if the condition (C) below holds.

(C) There exists a non-negative integer $\kappa \leq n$ such that

$$CB = 0 \quad CAB = 0 \quad \cdots \quad CA^{\kappa-2}B = 0 \quad CA^{\kappa-1}B \neq 0,$$

$$CP = 0 \quad CAP = 0 \quad \cdots \quad CA^{\kappa-2}P = 0 \quad CA^{\kappa-1}P = 0,$$

and

$$\text{rank} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{\kappa-1} \end{bmatrix} = \kappa.$$

Let

$$A = \begin{bmatrix} a_{11} & 1 & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & 0 & 1 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \qquad B = \begin{bmatrix} b_1 \\ b_2 \\ 0 \\ 1 \end{bmatrix} \qquad P = \begin{bmatrix} p_1 \\ p_2 \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}.$$

a) Show that condition (C) holds for some κ. [4 marks]

b) Show that the system can be written in the form

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & S \end{bmatrix} x + B(u + Lx) + \begin{bmatrix} P_1 \\ 0 \end{bmatrix} d,$$

$$y = \begin{bmatrix} 0 & C_2 \end{bmatrix} x$$

with $x_2(t) \in I\!R^{\kappa}$,

$$S = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

and A_{11}, A_{12} L, P_1 and C_2 matrices of appropriate dimensions, which should be specified. [10 marks]

c) Consider the state feedback control law

$$u = -Lx + K_2 x_2$$

with $K_2 \in I\!R^{1\times\kappa}$.
Write equations for the closed-loop system and argue that the control law solves the considered disturbance decoupling problem.
Finally, show that for the closed-loop system one has

$$y(t) = C_2 e^{Ft} x_2(0),$$

for some matrix $F \in I\!R^{\kappa\times\kappa}$. [6 marks]