UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 2001

MSci Honours Degree in Mathematics and Computer Science Part IV
MEng Honours Degrees in Computing Part IV
MSc in Advanced Computing
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the Associateship of the City and Guilds of London Institute
This paper is also taken for the relevant examinations for the Associateship of the Royal College of Science

PAPER C438

COMPLEXITY

Monday 14 May 2001, 10:00
Duration: 120 minutes

Answer THREE questions

Paper contains 4 questions
Calculators not required

1a i) Explain briefly the notions of NP, P-time reduction, and NP-complete.

ii) Let Q be a language in NP, and suppose that some NP-complete language C reduces to Q in P-time. Explain briefly why **(a)** Q must be NP-complete, and **(b)** if $Q \in P$ then $P = NP$. You may assume without proof any properties of P-time reduction that you need, provided you state them clearly.

b Explain clearly what the following problems are:

i) Hamiltonian path (HAM)

ii) Tripartite matching (TRI)

iii) KNAPSACK

c An undirected graph G is *complete* if for every two distinct nodes x, y of G, (x,y) is an edge of G.

The problem MONHC ("monochromatic Hamiltonian circuit") is the following: given a *complete* undirected graph G, each of whose *edges* is coloured either white or black, does G have a Hamiltonian circuit whose edges are all the same colour? Equivalently, can the nodes of G be enumerated as $v_1, v_2, \ldots, v_n$ such that the edges $(v_1,v_2), (v_2,v_3), \ldots, (v_{n-1},v_n)$, and (v_n,v_1) are either all black or all white?

Show that MONHC is NP-complete. You may assume the NP-completeness of standard problems, if you state your assumptions clearly.

2 In this question, NL denotes the class of languages NSPACE(log(n)).

a Explain the meaning of the following terms:

i) LOGSPACE reduction

ii) NL-complete

iii) the problem RCH (graph reachability)

iv) the problem 2SAT

v) the *complement* co-L of a language L.

Why is P-time reduction inappropriate for defining NL-completeness?

b i) State, without proof, the theorem of Immerman–Szelepscényi on space complexity classes. What is its consequence for NL?

ii) Sketch a proof to show that if L is an NL-complete language then so is co-L.

c i) Outline a proof that co-RCH reduces (in LOGSPACE) to 2SAT.
[It may help to view a directed graph edge (x,y) as a clause $\neg x \vee y$.]

ii) Hence, or otherwise, show that 2SAT is NL-complete. You may assume that $2SAT \in NL$, LOGSPACE reduction is transitive, and RCH is NL-complete.

3a i) What does it mean for a family of Boolean circuits C_n to be *uniform*?

ii) Why do we require uniformity when using Boolean circuits to define a complexity class?

iii) Define the class NC_j, for $j \geq 1$.

b A word $w \in \{0,1\}^*$ is said to be *alternating* if successive characters are always different (so, for instance, 010 is alternating, but 001 is not). Explain how to construct a uniform family of circuits C_n which recognises the language of alternating words. You should aim to keep the depth of your circuits as low as you can.

Also state the size and depth of C_n.

c The *concatenation* of languages L_1 and L_2, written L_1L_2, is defined as follows: $w \in L_1L_2$ iff $\exists w_1, w_2$ such that $w = w_1w_2$ and $w_1 \in L_1$, $w_2 \in L_2$.

Show carefully that if L_1 and L_2 are NC_j then so is L_1L_2 (any $j \geq 1$).

The three parts carry, respectively, 30%, 30%, 40% of the marks.

4a i) Define the function problems FSAT and FHAM.

ii) Define the classes FNP and FP.

b i) Show that if SAT is in P then FSAT is in FP.

ii) Show carefully that if HAM is in P then FHAM is in FP.

c A *clique* of size k in an undirected graph G is a subset of k nodes of G which are all adjacent to each other. The problem CLIQUE is: given a graph G and a number k, does G have a clique of size k? The associated functional problem FCLIQUE is: given a graph G, find a clique in G of maximal size.

Show that if CLIQUE is in P, then FCLIQUE is in FP.

The three parts carry, respectively, 25%, 40%, 35% of the marks.