Imperial College
London

M4/5M3

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May-June 2017

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Introduction to Partial Differential Equations

Date: Friday 12 May 2017

Time: 14:00 - 16:00

Time Allowed: 2.5 Hours

This paper has 5 Questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the relevant main book(s) are full.

All required additional material will be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers to each question as per the table below.

Raw Mark	Up to 12	13	14	15	16	17	18	19	20
Extra Credit	0	½	1	1 ½	2	2 ½	3	3 ½	4

- Each question carries equal weight.
- Calculators may not be used.

1. The following two problems are related to the method of characteristics.

(a) Find the solution $u = u(x, y)$ to the problem

$$\begin{cases} x^2\partial_x u + xy\partial_y u - u^2 = 0 & \text{for } (x, y) \in \mathbb{R}^2, \\ u(x, y) = 1 & \text{for } x = y^2,\ y \in \mathbb{R}. \end{cases}$$

Determine where this solution becomes singular.

(b) Find the entropy solution to Burgers' equation

$$\partial_t u + u\partial_x u = 0, \quad t > 0,\ x \in \mathbb{R},$$

with the initial data

$$u(x, 0) = \begin{cases} x & \text{for } x < 1, \\ 0 & \text{for } x \geq 1. \end{cases}$$

2. Consider the initial-boundary value problem (IBVP) for the heat equation

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} & \text{in } t > 0,\ 0 < x < L, \\ u(t,0) = u_x(t,L) = 0 & \text{for } t \geq 0, \\ u(0,x) = f(x) & \text{for } 0 \leq x \leq L, \end{cases}$$

with $f \in C^1([0,L])$.

(a) Give a notion of a classical solution including necessary compatibility conditions on the data.

(b) Find the energy associated with the (IBVP) and deduce the uniqueness of the solution.

(c) Find the candidate solution to the (IBVP) using the separation of variables technique.

3. Consider the general wave equation with the source term on the line

$$\begin{cases} \partial_t^2 u - c^2 \partial_x^2 u = F(x,t) & \text{in } t > 0,\ x \in \mathbb{R}, \\ u(x,0) = \phi(x) & \text{for } x \in \mathbb{R}, \\ \partial_t u(x,0) = \psi(x) & \text{for } x \in \mathbb{R}. \end{cases}$$

Knowing that the classical solution of this problem is given by the D'Alambert formula

$$u(x,t) = \frac{\phi(x+ct)+\phi(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\,dy + \frac{1}{2c}\int_0^t \int_{x-ct+cs}^{x+ct-cs} F(y,s)\,dy\,ds,$$

find the solution to the problem on the half-line

$$\begin{cases} \partial_t^2 u - c^2 \partial_x^2 u = F(x,t) & \text{in } t > 0,\ x > 0 \\ u(x,0) = \phi(x) & \text{for } x \geq 0 \\ \partial_t u(x,0) = \psi(x) & \text{for } x \geq 0 \\ u(0,t) = h(t) & \text{for } t \geq 0 \end{cases}$$

$u = u_1 + u_2$ by following the steps:

(a) Find the solution u_1 to the IBVP with zero Dirichlet data (i.e. $h(t) = 0$),

(b) Recall that the general form of the classical solution to the one-dimensional wave equation:

$$\partial_t^2 u - c^2 \partial_x^2 u = 0$$

is of the form

$$u(x,t) = f(x+ct) + g(x-ct),$$

where $f, g \in C^2(\mathbb{R})$ are two arbitrary functions of a single variable. Deduce from this the form of solution u_2 to the complementary IBVP problem (i.e. $F(x,t) = \phi(x) = \psi(x) = 0$).

4. Let Ω be a bounded domain in $\mathbb{R}^3$, $u \in C^4(\Omega) \cap C^2(\overline{\Omega})$.

(a) Consider the following boundary value problem (BP):

$$\begin{cases} \Delta^2 u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \\ \Delta u = g & \text{on } \partial\Omega. \end{cases}$$

Show that the solution to the (BP) is unique.
Hint: Use the maximum principle for the harmonic functions.

(b) Show that if $u, v \in C^4(\Omega) \cap C^2(\overline{\Omega})$ are such that $\Delta u = \Delta v = u = v = 0$ on $\partial\Omega$, then

$$\int_\Omega (u\Delta^2 v - v\Delta^2 u)dx = 0.$$

(c) Define the averages:

$$S(r) = \frac{1}{4\pi r^2}\int_{\partial B(0,r)} u(x)\,d\sigma,$$
$$V(r) = \frac{1}{\frac{4}{3}\pi r^3}\int_{B(0,r)} \Delta u(x)dx.$$

Show that

$$\frac{d}{dr}S(r) = \frac{r}{3}V(r).$$

(d) Use the result of (c) to show that if u is biharmonic, i.e. $\Delta^2 u = 0$, then

$$S(r) = u(0) + \frac{r^2}{6}\Delta u(0).$$

Hint: Use the mean value theorem for the harmonic functions.

5. The following Cauchy problem for the traffic flow equation:

$$\begin{cases} \frac{\partial \rho}{\partial t} + \frac{\partial q(\rho)}{\partial x} = 0, & \text{for } t > 0 \ , \ x \in \mathbb{R}, \\ \rho(x,0) = \begin{cases} \rho_m & \text{for} \quad x < 0 \\ 0 & \text{for} \quad x > 0 \end{cases} \end{cases}$$

with $v(\rho) = v_m \left(1 - \frac{\rho}{\rho_m}\right)$, $v_m > 0$, and $q(\rho) = \rho v(\rho)$, models the dynamics of the cars after the traffic light situated at $x = 0$ changes from red to green.

(a) Compute the entropy solution of the problem.

(b) How long does it take for a car initially located at $x_0 < 0$ to pass the traffic lights?

(c) What is the trajectory of the car once it starts to move?

(d) How many cars will get through the traffic light if the green light lasts for t^* time units?

Solutions to the exam problems M345/M3 2017

1. (a) The characteristic system reads as

$$\begin{cases} \dfrac{dx}{ds} = x^2 \quad , \quad x(0,r) = r^2 \\ \dfrac{dy}{ds} = xy \quad , \quad y(0,r) = r \\ \dfrac{dz}{ds} = z^2 \quad , \quad z(0,r) = 1 \end{cases}$$

from where $x = \frac{r^2}{1-sr^2}$ and $\frac{dy}{ds} = \frac{r^2 y}{1-sr^2}$, so $y = \frac{r}{1-sr^2}$. Immediately from the equation on z, we obtain $z = \frac{1}{1-s}$.

Inverting, we get $r = \frac{x}{y}$, and from the formula for y: $sr^2 = 1 - \frac{r}{y}$, and so $s = \frac{y^2}{x^2} - \frac{1}{x}$. Thus

$$u(x,y) = z(s(x,y), r(x,y)) = \frac{1}{1 - \frac{y^2}{x^2} + \frac{1}{x}} = \frac{x^2}{x^2 + x - y^2}.$$

Therefore, the solution u is singular for $y^2 = x^2 + x$.

[8] SEEN SIMILAR

(b) The characteristic system reads as

$$\begin{cases} \dfrac{dt}{ds} = 1 \quad , \quad t(0,r) = 0 \\ \dfrac{dx}{ds} = z \quad , \quad x(0,r) = r \\ \dfrac{dz}{ds} = 0 \quad , \quad z(0,r) = u(0,r) := u_0(r) \end{cases}$$

The solution is thus constant on the characteristic lines $x = u_0(r)s + r$, $s = t$:

$$u(t,x) = u_0(r) = u_0(x - ut).$$

The characteristics are thus given by:

$$x = \begin{cases} rt + r & \text{for} \quad r < 1, \\ r & \text{for} \quad r \geq 1. \end{cases}$$

Since the initial condition is decreasing function, the characteristics will intersect and the shock will appear. Due to discontinuity of initial data, this shock will start from $t = 0$ at $x = 1$. Therefore, the entropy solution to Burgers' equation for $t > 0$ is given by

$$u(x,t) = \begin{cases} \frac{x}{1+t} & \text{for} \quad x < \sigma(t) \\ 0 & \text{for} \quad x > \sigma(t), \end{cases}$$

and $\sigma(t)$ can be determined from the Rankine-Hugoniot condition:

$$\sigma'(t) = \frac{1}{2}\frac{\sigma(t)}{t+1}, \quad \sigma(0) = 1,$$

from which $\sigma(t) = \sqrt{t+1}$.

The entropy condition $\frac{x}{1+t} > \sigma'(t) > 0$ is also satisfied as $\frac{\sqrt{t+1}}{t+1} > \frac{1}{2\sqrt{t+1}}$ and $\frac{1}{2\sqrt{t+1}} > 0$ for $t > 0$.

[12] SEEN SIMILAR

2. (a) We define a classical solution as a function $u \in C^{2,1}_{x,t}((0,\infty)\times(0,L))$ such that u and $\frac{\partial u}{\partial x}$ are continuous up to the boundary in x and t, i.e., $u \in C([0,\infty)\times[0,L))$ and $\frac{\partial u}{\partial x} \in C([0,\infty)\times(0,L])$. As a consequence, $f(x)$ has to satisfy the compatibility conditions $f(0)=0$ and $f'(L)=0$.

[4] SEEN SIMILAR

(b) We introduce the energy

$$E(u(t)) = \frac{1}{2}\int_0^L u^2 dx.$$

We can show that $E(u(t))$ is non increasing function of t, we compute

$$\frac{dE(u(t))}{dt} = \int_0^L u\partial_t u dx = \int_0^L u\partial_x^2 u dx = \dots$$

integrating by parts

$$\dots = -\int_0^L (\partial_x u)^2 dx + u\partial_x u\Big|_{x=0}^{x=L}.$$

The integral on the right hand side is thus nonpositive, the boundary term vanishes due to the assumptions on the boundary data. Therefore

$$\frac{dE(t)}{dt} \le 0,$$

and so

$$E(u(t)) \le E(u(0)), \quad \text{for } t>0.$$

Taking two solutions to the IBVP u and v, we obtain that the difference $w=u-v$ satisfies

$$\begin{cases} \partial_t w - \partial_x^2 w = 0 & \text{in } t>0,\ 0<x<L, \\ w(t,0) = w_x(t,L) = 0 & \text{for } t \ge 0, \\ w(0,x) = 0 & \text{for } 0 \le x \le L, \end{cases}$$

and so

$$\frac{1}{2}\int_0^L (w(x,t))^2 dx = E(w(t)) \le E(w(0)) = \frac{1}{2}\int_0^L (w(x,0))^2 dx = 0.$$

This means that $w(x,t)\equiv 0$, for $(x,t)\in[0,L]\times[0,\infty)$ meaning that $u=v$.

[8] SEEN SIMILAR

(c) We look for nonzero solutions of the form $u(t,x)=T(t)X(x)$ with $T(t)\in C^1(0,\infty)$ and $X(x)\in C^2(0,L)$, satisfying

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda \in \mathbb{R}$$

for any open set in (t,x) where $T(t)$ and $X(x)$ are nonzero. Therefore, we get that $T'(t) = -\lambda T(t)$ and $X''(x)+\lambda X(x)=0$ for $t>0$ and $0<x<L$ with $X(0)=X'(L)=0$. Now, we distinguish cases:

$\underline{\lambda = 0}$: Since $X''=0$, then X' is a constant, but since $X'(L)=0$ thus $X'(x)=0$. This in turn means that X is constant, but since $X(0)=0$, thus $X(x)=0$. Therefore $u(x,t)$ is equal to 0 in this case.

$\underline{\lambda > 0}$: The solutions of the ODEs are

$$X(x) = C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x).$$

The condition $X(0) = 0$ is equivalent to $C_1 = 0$ and the condition $X'(L) = 0$ is equivalent to $C_2 \cos(\sqrt{\lambda}L) = 0$. Therefore, either $\cos(\sqrt{\lambda}L) = 0$ or $C_1 = C_2 = 0$. The second case leads to trivial solutions, the first one leads to $\lambda = \frac{(n+1/2)^2\pi^2}{L^2}$ with n integer such that $n \geq 0$. We can now solve the equation for $T(t)$ given by $T'(t) = -\frac{(n+1/2)^2\pi^2}{L^2}T(t)$ to get $T(t) = Ce^{-\frac{(n+1/2)^2\pi^2}{L^2}t}$, with $C \in \mathbb{R}$. Summarizing, the solutions for this case are given by

$$u_n(t,x) = C_n e^{-\frac{(n+1/2)^2\pi^2}{L^2}t} \sin\left(\frac{(n+1/2)\pi}{L}x\right), \quad n \geq 0.$$

$\underline{\lambda < 0}$: The solutions of the ODEs are

$$X(x) = C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x}.$$

The condition $X(0) = 0$ is equivalent to $C_1 + C_2 = 0$ and the condition $X'(L) = 0$ is equivalent to $C_1 - C_2 = 0$. Therefore, $C_1 = C_2 = 0$ and no nontrivial solution is obtained in this case.

The candidate solution is thus of the form:

$$u(t,x) := \sum_{n=0}^{\infty} C_n e^{-\frac{(n+1/2)^2\pi^2}{L^2}t} \sin\left(\frac{(n+1/2)\pi}{L}x\right),$$

where the coefficients C_n have to match the expansion of the initial data $f(x)$, i.e.

$$f(x) = u(0,x) = \sum_{n=0}^{\infty} C_n \sin\left(\frac{(n+1/2)\pi}{L}x\right).$$

[8] SEEN SIMILAR

3. (a) When $h(t) = 0$ the solution to the IBVP:

$$\begin{cases} \partial_t^2 u_1 - c^2 \partial_x^2 u_1 = F(x,t) & \text{in } t > 0,\ x > 0 \\ u_1(x,0) = \phi(x) & \text{for } x \geq 0 \\ \partial_t u_1(x,0) = \psi(x) & \text{for } x \geq 0 \\ u_1(0,t) = 0 & \text{for } t \geq 0 \end{cases}$$

can be obtained by the method of odd extension for the data:

$\phi_{odd}(x) = \begin{cases} \phi(x) & \text{for } x \geq 0 \\ -\phi(-x) & \text{for } x < 0 \end{cases}$, $\psi_{odd}(x) = \begin{cases} \psi(x) & \text{for } x \geq 0 \\ -\psi(-x) & \text{for } x < 0 \end{cases}$, and $F_{odd}(x,t) = \begin{cases} F(x,t) & \text{for } x \geq 0 \\ -F(-x,t) & \text{for } x < 0 \end{cases}$.

From the D'Alambert formula, we deduce that a perturbation of initial condition at point $(0,0)$ changes the solution $u_1(x,t)$ in the interval $(x - ct, x + ct)$. Therefore for $x \geq ct$ the solution to the IBVP is the same, namely

$$u_1(x,t) = \frac{\phi(x+ct) + \phi(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\,dy + \frac{1}{2c}\int_0^t \int_{x-ct+cs}^{x+ct-cs} F(y,s)\,dyds.$$

For $0 < x < ct$ we compute using the d'Alambert formula and the odd extension that

$$\begin{aligned} u_1(x,t) =& \frac{\phi_{odd}(x+ct) + \phi_{odd}(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{odd}(y)\,dy + \frac{1}{2c}\int_0^t \int_{x-ct+cs}^{x+ct-cs} F_{odd}(y,s)\,dyds \\ =& \frac{\phi(x+ct) - \phi(x-ct)}{2} + \frac{1}{2c}\int_0^{x+ct} \psi(y)\,dy - \frac{1}{2c}\int_{x-ct}^{0} \psi(-y)\,dy \\ &+ \frac{1}{2c}\int_{t-\frac{x}{c}}^t \int_{x-c(t-s)}^{x+c(t-s)} F(y,s)\,dyds - \frac{1}{2c}\int_0^{t-\frac{x}{c}} \int_{x-c(t-s)}^{0} F(-y,s)\,dyds \\ &+ \frac{1}{2c}\int_0^{t-\frac{x}{c}} \int_0^{x+c(t-s)} F(y,s)\,dyds. \end{aligned}$$

Changing the variables in the second and the fourth integral $y \to -y$, we obtain

$$\begin{aligned} u_1(x,t) =& \frac{\phi(x+ct) - \phi(x-ct)}{2} + \frac{1}{2c}\int_{ct-x}^{x+ct} \psi(y)\,dy \\ &+ \frac{1}{2c}\int_{t-\frac{x}{c}}^t \int_{x-c(t-s)}^{x+c(t-s)} F(y,s)\,dyds + \frac{1}{2c}\int_0^{t-\frac{x}{c}} \int_{c(t-s)-x}^{x+c(t-s)} F(y,s)\,dyds \\ =& \frac{\phi(x+ct) - \phi(x-ct)}{2} + \frac{1}{2c}\int_{ct-x}^{x+ct} \psi(y)\,dy + \int_0^t \int_{|x-c(t-s)|}^{x+c(t-s)} F(y,s)\,dyds. \end{aligned}$$

[10] SEEN SIMILAR

(b) To find the solution u_2 to the IBVP:

$$\begin{cases} \partial_t^2 u_2 - c^2 \partial_x^2 u_2 = 0 & \text{in } t > 0,\ x > 0 \\ u_2(x,0) = 0 & \text{for } x \geq 0 \\ \partial_t u_2(x,0) = 0 & \text{for } x \geq 0 \\ u_2(0,t) = h(t) & \text{for } t \geq 0 \end{cases}$$

we use the general form of the solution

$$u_2(x,t) = f(x+ct) + g(x-ct). \tag{1}$$

We check the boundary conditions to determine functions f and g:
$u_2(x,0) = f(x) + g(x) = 0$, thus

$$f(x) = -g(x) \tag{2}$$

$\partial_t u_2(x,0) = cf'(x) - cg'(x) = 0$, thus

$$f'(x) = g'(x). \tag{3}$$

Differentiating (2) and comparing with (3), we obtain for $x > 0$ that:

$$f(x) = -g(x) = a,$$

for some constant a. On the other hand using the boundary condition $\tilde{u}_2(0,t) = f(ct) + g(-ct) = h(t)$, therefore since $ct > 0$

$$g(-ct) = h(t) - a, \quad \text{or} \quad g(s) = h(-s/c) - a,$$

for $s < 0$. Coming back to (1) we obtain that for $0 < x < ct$

$$u_2(x,t) = \begin{cases} a - a = 0 & \text{for} \quad x \geq ct \\ a + h((ct-x)/c) - a = h(t - x/c) & \text{for} \quad 0 < x < ct. \end{cases}$$

[10] UNSEEN

4. (a) Let us take $w = v - u$, where v, u are two solutions to the boundary value problem. Then w satisfies

$$\begin{cases} \Delta^2 w = 0 & \text{in } \Omega, \\ w = \Delta w = 0 & \text{on } \partial\Omega. \end{cases}$$

So, Δw is harmonic function and so, from the maximum principle we know that $\min w$ and $\max w$ are attained at the boundary $\partial\Omega$. Thus $\Delta w(x) \equiv 0$ for all $x \in \overline{\Omega}$. This implies that w itself is harmonic, but again $w = 0$ on $\partial\Omega$, therefore $w(x) \equiv 0$ for all $x \in \overline{\Omega}$.

Since $0 = w = u - v$, thus $u = v$ which means that the solution is unique.

[5] SEEN SIMILAR

(b) Using Green's identity

$$\int_\Omega (u\Delta^2 v - v\Delta^2 u)dx = \int_{\partial\Omega} u\frac{\partial \Delta v}{\partial n} d\sigma - \int_{\partial\Omega} v\frac{\partial \Delta u}{\partial n} d\sigma = 0$$

due to the boundary conditions for u and v. Alternatively, one can integrate by parts twice.

[3] SEEN SIMILAR

(c) Before differentiating, we need to rewrite $S(r)$ as an integral over the unit sphere. We take $x = rx'$ in $\mathbb{R}^3$, thus

$$S(r) = \frac{1}{4\pi r^2}\int_{\partial B(0,r)} u(x)\, d\sigma_x = \frac{1}{4\pi}\int_{\partial B(0,1)} u(rx')\, d\sigma_{x'},$$

and so, the derivative of S is just a derivative in the integrant

$$\begin{aligned} S'(r) =& \frac{1}{4\pi}\int_{\partial B(0,1)} \nabla u(rx')\cdot x'\, d\sigma_{x'} = \frac{1}{4\pi}\int_{\partial B(0,1)} \frac{\partial u}{\partial n}(rx')\, d\sigma_{x'} \\ =& \frac{1}{4\pi r^2}\int_{\partial B(0,r)} \frac{\partial u}{\partial n}(x)\, d\sigma_x = \frac{1}{4\pi r^2}\int_{B(0,r)} \Delta u(x)\, dx, \end{aligned}$$

where the last passage is obtained using Green's identity. The r.h.s. is then equal to $\frac{r}{3}V(r)$.

[6] SEEN SIMILAR

(d) If u is biharmonic, then Δu is harmonic and it has a mean value property. We therefore have

$$S'(r) = \frac{r}{3}V(r) = \frac{r}{3}\frac{1}{\frac{4}{3}\pi r^3}\int_{B(0,r)} \Delta u(x)\, dx = \frac{r}{3}\Delta u(0).$$

Solving the ODE, we obtain

$$S(r) = \frac{r^2}{6}\Delta u(0) + S(0) = u(0) + \frac{r^2}{6}\Delta u(0).$$

[6] UNSEEN

5. (Mastery question) (a)

(a) The characteristic system reads as

$$\begin{cases} \dfrac{dt}{ds} = 1 & , \qquad t(0,r) = 0 \\ \dfrac{dx}{ds} = v_m(1 - 2z/\rho_m) & , \qquad x(0,r) = r \\ \dfrac{dz}{ds} = 0 & , \quad z(0,r) = \rho(r,0) := \rho_0(r) \end{cases}$$

From the third equation it follows that $z(s,r) = \rho_0(r)$, so the solution is constant on characteristics. We may thus solve the first equation to get $s = t$ and the second equation to get the equations of characteristics:

$$x = v_m\left(1 - \frac{2\rho_0(r)}{\rho_m}\right)t + r.$$

Using the initial data we therefore have

$$x = \begin{cases} -v_m t + r & \text{for} \quad r < 0 \\ v_m t + r & \text{for} \quad r \geq 0, \end{cases}$$

and so, the solution reads

$$\rho(x,t) = \begin{cases} \rho_m & \text{for} \quad x < -v_m t \\ 0 & \text{for} \quad x \geq v_m t. \end{cases}$$

The empty space for $x \in (-v_m t, v_m t)$ should be filled with rarefaction wave. We look for its form by considering the "self-similar" solution of the form: $\rho\left(\frac{x}{t}\right)$, for which

$$\begin{aligned} & -\frac{x}{t^2}\rho'\left(\frac{x}{t}\right) + v_m(1 - 2\rho\left(\frac{x}{t}\right)/\rho_m)\rho'\left(\frac{x}{t}\right)\frac{1}{t} = 0, \\ \iff & -\xi\rho'(\xi) + v_m(1 - 2\rho(\xi)/\rho_m)\rho'(\xi) = 0, \quad \forall \xi = \frac{x}{t}, \\ \iff & \rho'(\xi)\left(v_m(1 - 2\rho(\xi)/\rho_m) - \xi\right) = 0. \end{aligned}$$

If ρ is not a constant, then $v_m(1 - 2\rho(\xi)/\rho_m) = \xi$ and thus

$$\rho(x,t) = \rho\left(\frac{x}{t}\right) = \frac{\rho_m}{2}\left(1 - \frac{x}{v_m t}\right), \quad \text{for} - v_m t < x < v_m t.$$

[8] SEEN SIMILAR

(b) The left radial of the rarefaction wave moves with the velocity $-v_m$. When this radial arrives to the point $x_0 < 0$ the car located at this point will start to move. The time it takes is thus equal to $t = \frac{x_0}{-v_m}$.

[4] SEEN SIMILAR

(c) From the previous point we know that the car starts to move at time $t = \frac{x_0}{-v_m}$. The velocity of the car is given by $v = v_m(1 - \rho/\rho_m)$, where ρ is the density in the rarefaction wave computed in point (a), we thus have

$$x'(t) = v(x,t) = v_m(1 - \rho(x,t)/\rho_m) = v_m\left(1 - \frac{1}{2}\left(1 - \frac{x}{v_m t}\right)\right) = \frac{v_m}{2}\left(1 + \frac{x}{v_m t}\right).$$

Therefore, we need to solve

$$x'(t) = \frac{x}{2t} + \frac{v_m}{2}, \quad x\left(\frac{x_0}{-v_m}\right) = x_0.$$

Solving the ODE we obtain

$$x = v_m t + C\sqrt{t}, \quad \text{and} \quad x_0 = v_m \frac{x_0}{-v_m} + C\sqrt{\frac{x_0}{-v_m}}.$$

Therefore, finally the trajectory of this car is:

$$x(t) = v_m t - 2\sqrt{-x_0 v_m t}.$$

[4] UNSEEN

(d) From the previous point we know that the car that will pass through the traffic light as the last one will have to have a position 0 at $t = t^*$, therefore

$$x(t^*) = v_m t^* - 2\sqrt{-x_0 v_m t^*} = 0 \iff x_0 = \frac{-v_m t^*}{4}.$$

So, during the green light only the cars ahead of this one will pass, which means that there will be $|x_0|\rho_m = \frac{\rho_m v_m t^*}{4}$ passing through the light in time $t = t^*$.

[4] UNSEEN

Imperial College London
Department of Mathematics

Examiner's Comments

Exam: M3 M3/ M45 /M3 **Session: 2016-2107**

Question 1

Please use the space below to comment on the candidates' overall performance in the exam. A brief paragraph highlighting common mistakes and parts of questions done badly (or well) is sufficient. Do not refer to individual candidates. The purpose of this exercise is to provide guidance to the external examiners, and to the candidates themselves, on how you feel the cohort faired. Your comments will be available to students online.

It is incredible that some of our ~~m~~ master students cannot solve $x' = x^2$ or $y' = y$ correctly. For the rest, the cohort perform pretty well.

Marker: José A. Carrillo

Signature: [signature] **Date:** 13/05/2017

Please return with exam marks (one report per marker)

Imperial College London
Department of Mathematics

Examiner's Comments

Exam: M3M3/M45/M3 **Session: 2016-2107**

Question 2

Please use the space below to comment on the candidates' overall performance in the exam. A brief paragraph highlighting common mistakes and parts of questions done badly (or well) is sufficient. Do not refer to individual candidates. The purpose of this exercise is to provide guidance to the external examiners, and to the candidates themselves, on how you feel the cohort faired. Your comments will be available to students online.

A not negligible amount of students and even master students donot know how to solve $\cos(\sqrt{\lambda}\, L) = 0$ in terms of λ ... the typical answer is $\lambda_n = \frac{n^2\pi^2}{L^2}$ ...

Marker: JOSE A. CARRILLO

Signature: [signature] **Date:** 13/05/2017

Please return with exam marks (one report per marker)

Imperial College London
Department of Mathematics

Examiner's Comments

Exam: M3M3 / M4-5 M3 **Session: 2016-2107**

Question 3

Please use the space below to comment on the candidates' overall performance in the exam. A brief paragraph highlighting common mistakes and parts of questions done badly (or well) is sufficient. Do not refer to individual candidates. The purpose of this exercise is to provide guidance to the external examiners, and to the candidates themselves, on how you feel the cohort faired. Your comments will be available to students online.

The cohort did quite poorly in the second part that has not been solved by none of the master students. It seems that the domain of dependence of the wave eq'n is not well understood by the students.

Marker: José A. Carrillo

Signature: [signature] **Date:** 13/05/2017

Please return with exam marks (one report per marker)

Imperial College London
Department of Mathematics

Examiner's Comments

Exam: M3M3 / M4P5/M3 **Session: 2016-2107**

Question 4

Please use the space below to comment on the candidates' overall performance in the exam. A brief paragraph highlighting common mistakes and parts of questions done badly (or well) is sufficient. Do not refer to individual candidates. The purpose of this exercise is to provide guidance to the external examiners, and to the candidates themselves, on how you feel the cohort faired. Your comments will be available to students online.

A typical mistake is to apply the maximum principle without checking first that the function is harmonic. Taking derivatives of functions where the domains depend on the variable seems to be complicated for some students.

Marker: JOSE A. CARRILLO

Signature: [signature] **Date:** 13/05/2017

Please return with exam marks (one report per marker)