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EE9-CS4-1

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
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MSc and EEE/EIE PART IV: MEng and ACGI

Corrected Copy

PREDICTIVE CONTROL

Tuesday, 13 May 10:00 am

Time allowed: 3:00 hours

There are FOUR questions on this paper.

Answer FOUR questions.

All questions carry equal marks.

Any special instructions for invigilators and information for candidates are on page 1.

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PREDICTIVE CONTROL

1. Consider the nonlinear system

$$\frac{d^2q(t)}{dt^2} = -\frac{dq(t)}{dt}u(t), \; y(t) = q(t),$$

with initial conditions

$$q(0) = 1, \; \frac{dq(0)}{dt} = 1,$$

where $t \in \mathbb{R}$ denotes time, $q(t) \in \mathbb{R}$ is a scalar variable, $u(t) \in \mathbb{R}$ is the input and $y(t) \in \mathbb{R}$ is the output.

a) Show that the above system can be written in the form

$$\frac{dx(t)}{dt} = \begin{bmatrix} 0 & 1 \\ 0 & -u(t) \end{bmatrix} x(t), \quad y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t), \quad x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

by defining an appropriate state variable $x(t) \in \mathbb{R}^2$. [2]

b) Give the definition of the exponential $\exp(M)$ of a square matrix M. [1]

c) Use the definition of the matrix exponential to derive an expression for $\exp(M)$ if

$$M := \begin{bmatrix} 0 & h \\ 0 & 0 \end{bmatrix},$$

where $h \in \mathbb{R}$ is a given scalar. [2]

d) Use the definition of the matrix exponential to show that

$$F(\alpha) := \exp \begin{bmatrix} 0 & h \\ 0 & -\alpha h \end{bmatrix} = \begin{bmatrix} 1 & \left(1 - e^{-\alpha h}\right)/\alpha \\ 0 & e^{-\alpha h} \end{bmatrix}$$

where $h \in \mathbb{R}$ is a given scalar and $\alpha \in \mathbb{R}$ is non-zero. [5]

e) Suppose now that the state of the system above is sampled every h seconds and that the input to the system has a zero-order-hold, i.e. $u(t) = u_k := u(kh)$ for all $t \in [kh, (k+1)h)$, where k denotes the sample instant. Let $x_k := x(kh)$ and show that

$$x_{k+1} = A(u_k)x_k, \quad k = 0, 1, 2,$$

where $A(u_k) := F(u_k)$ if $u_k \neq 0$ and $A(u_k) := \exp(M)$ if $u_k = 0$. [4]

f) Suppose that $h = 1$ and the input sequence is given by $u_0 = 1$, $u_1 = 0$ and $u_2 = 1$. Compute the output sequence $y_k := y(kh)$ for $k = 0, 1, 2, 3$. [6]

2. We are interested in solving the following optimal control problem:

$$\min_{u_0,u_1,\ldots,u_{N-1}} \sum_{k=0}^{N-1} \left[\|Qx_{k+1}\|_2^2 + \|Ru_k\|_2^2 + \rho\|d(Fx_{k+1}-g)\|_1\right]$$

subject to

$$x_0 = \hat{x},\ x_{k+1} = Ax_k + Bu_k, \quad k = 0,1,\ldots,N-1,$$
$$u_\ell \leq u_k \leq u_h, \quad k = 0,1,\ldots,N-1,$$

where the state $x_k \in \mathbb{R}^n$, input $u_k \in \mathbb{R}^m$, the scalar $\rho > 0$, the matrices Q and R are square and the function $d : \mathbb{R}^p \to \mathbb{R}^p$ is defined as

$$d(z) := \begin{bmatrix} \max\{0,z_1\} \\ \vdots \\ \max\{0,z_p\}. \end{bmatrix}$$

Assume an estimate of the current state $\hat{x}$ is given.

You are required to convert the above problem into an equivalent QP of the form

$$\min_\theta \frac{1}{2}\theta' H\theta + c'\theta$$

subject to

$$M\theta \leq f$$

by introducing slack variables to handle the 1-norm stage cost.

a) Give a definition for θ, which should include all slack variables. Remember to define the size of your slack variables. [3]

b) How many rows should there be in H? Why? Note you do not need an expression for H in order to answer this. [2]

c) How many rows should there be in M? Why? Note you do not need an expression for M in order to answer this. [3]

d) Derive an expression for M in terms of the problem data. [4]

e) Derive an expression for f in terms of the problem data. [2]

f) Derive an expression for H in terms of the problem data. [2]

g) Derive an expression for c in terms of the problem data. [4]

3. A computationally efficient predictive control technique for handling constraints while tracking a constant reference, is to use a so-called 'reference governor'. The idea behind a reference governor is outlined below.

Suppose a single-input, open-loop asymptotically stable system is given by

$$x_{k+1} = Ax_k + Bu_k,$$

where the state $x_k \in \mathbb{R}^n$ is measured at each sample instant k. A dynamic reference governor is given by the scaled output of a first-order low-pass filter of the form

$$\begin{aligned} v_k &= v_{k-1} + \lambda(x_k, v_{k-1}, r)(r - v_{k-1}), \\ u_k &= F v_k, \end{aligned}$$

where the gain $F \in \mathbb{R}$ is computed off-line so as to ensure that the regulated variable

$$z_k := Hx_k \in \mathbb{R}$$

converges to any reference $r \in \mathbb{R}$ when no constraints are violated. The scalar $\lambda(x_k, v_{k-1}, r) \in [0, 1]$ is computed on-line at each sample instant so as to ensure constraint satisfaction. Note that if $v_{k-1} = r$ or $\lambda(x_k, v_{k-1}, r) = 1$, then $v_k = r$ and $u_k = Fr$.

a) Why does the inverse of $I - A$ exist? [2]

b) Suppose $u_k = Fr$ for all $k \geq K$ for some $K \in \mathbb{N}$. Show that

$$\lim_{k\to\infty} z_k = r$$

if

$$F = \frac{1}{H(I-A)^{-1}B}$$

and $H(I-A)^{-1}B \neq 0$. [8]

c) Suppose now that we would like to ensure that the constraints

$$Cx_k \leq d$$

are satisfied at all sample instants, where $C \in \mathbb{R}^{q\times n}$ and $d > 0$. In a reference governor this is often done by solving, at each sample instant, the scalar optimisation problem

$$\lambda(x_k, v_{k-1}, r) := \max_{\rho} \rho$$

subject to

$$\begin{aligned} s_{j+1} &= As_j + BFw_j, \quad j = 0, 1, \ldots, N-1, \quad s_0 = x_k, \\ w_j &= w_{j-1} + \rho(r - v_{k-1}), \quad j = 0, 1, \ldots, N-1, \quad w_{-1} = v_{k-1}, \\ Cs_j &\leq d, \quad j = 1, \ldots, N, \\ 0 &\leq \rho \leq 1. \end{aligned}$$

One can derive expressions for $G \in \mathbb{R}^M$ and $h \in \mathbb{R}^M$ such that

$$\lambda(x_k, v_{k-1}, r) = \max_{\rho}\{\rho \mid G\rho \leq h\}.$$

Derive the expression for M. [2]

d) Derive expressions for G and h, paying particular attention to the detail in the subscripts of the above optimisation problem. [7]

e) Which of G or h is a function of r? Give a reason why. [1]

4. In the following, please keep your answers clear and concise, using no more than 20 words per point, e.g. if the answer is worth 3 points, as in part a) below, then you should use no more than 60 words, etc. State the number of words you used for each answer. Marks will be deducted if you are found to have gone over the limit.

a) Explain why receding horizon control is a feedback control method. [3]

b) Why is feedback used in control system design? [1]

c) Why can feedback potentially have a negative effect on system behaviour? [2]

d) What is the main difference between 'receding horizon control' and 'variable horizon control'? [2]

e) Give an example of an application where it is preferable to implement a variable horizon controller, rather than a receding horizon controller. Give a reason for your answer. [2]

f) Give an example of an application of predictive control where it is preferable to use a 1-norm stage cost on the input, rather than a quadratic cost. Give a reason for your answer. [2]

g) Give an example of an application where you have hard constraints on the state. Explain why the constraints are hard and not soft. [2]

h) Give an example of an application where you have soft constraints on the state. Explain why the constraints are soft and not hard. [2]

i) Outline the technique and main ideas used in the majority of stability proofs in the predictive control literature. [4]