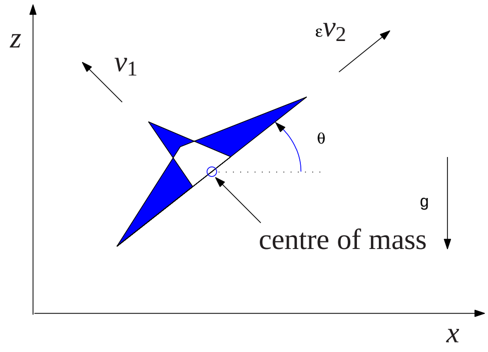
Modelling and control of multibody mechanical systems
Sample exam paper

1. Consider the (simplified and normalised) equations describing the motion of a vertical take-off and landing aircraft moving in a horizontal plane, namely

$$\begin{aligned}\ddot{x} &= -(\sin\theta)v_1 + \varepsilon(\cos\theta)v_2 \\ \ddot{z} &= (\cos\theta)v_1 + \varepsilon(\sin\theta)v_2 - g \\ \ddot{\theta} &= v_2,\end{aligned}$$

where (x,z) describes the position of the centre of mass of the aircraft in a vertical plane, θ the roll angle, g the gravity acceleration, v_1 and v_2 the control actions and $\varepsilon > 0$ a parameter that captures the effect of the “slopped” wings and induces a coupling between the vertical and the roll dynamics.



a) Show that the system, with $v_1 = v_2 = 0$, can be written as an Hamiltonian system with $M = I$. In particular, write the internal Hamiltonian $H_0(q,p)$, where $q = (x,y,\theta)$ and p are the corresponding momenta. [4 marks]

b) Verify that if $v_1 = v_2 = 0$ then $\dot{H}_0(q,p) = 0$. [2 marks]

c) Show that the system is not a simple Hamiltonian system, *i.e.* it is not possible to define a Hamiltonian function $H(q,p,u) = H_0(q,p) - H_1(q)v_1 - H_2(q)v_2$, with $H_0(q,p)$ as in part a), such that

$$\dot{q} = \left(\frac{\partial H}{\partial p}\right)' \qquad \dot{p} = -\left(\frac{\partial H}{\partial q}\right)'.$$

[4 marks]

d) Let v_1 and v_2 be constant. Compute the equilibrium points of the system. Give a physical interpretation for the obtained result. [4 marks]

e) Show that the point $(q,p) = (0,0)$ is an equilibrium of the system. What is the value of the input signals associated to this equilibrium? [2 marks]

f) Compute the linearization of the system around the equilibrium $(q,p) = (0,0)$. Show that this equilibrium can be locally asymptotically stabilised by a state feedback control law exploiting Lyapunov first method.
Finally, suppose that it is possible to measure only $y = q$. Is the equilibrium $(q,p) = (0,0)$ locally asymptotically stabilizable by a dynamic output feedback control law? [4 marks]

2. Consider a simple Hamiltonian system with

$$H_0(q,p) \quad = \frac{1}{2}p^2(1+\alpha q^2) + \frac{1}{2}q^2 - \frac{1}{n}q^n,$$

$\alpha > 0$, $n > 2$ and even, and $H_1(q) = q$.

a) Write the Hamiltonian equations of motion. [4 marks]

b) Find the equilibria of the system for $u = 0$. [2 marks]

c) Study the local stability properties of each equilibrium. [2 marks]

d) Asymptotically stabilise the system around the equilibrium $(0,0)$ using Lyapunov first method. [4 marks]

e) Using the shaping function method find a control law which globally asymptotic stabilises the equilibrium $(0,0)$. [8 marks]

3. A circular disk of mass m and radius a rolls without sliding on a horizontal plane as shown in Figure 3.1 (the disk in the figure has non-zero width for illustration purposes). The plane of the disk remains always vertical. The moment of inertia of the disk about its spin axis is I_{yy} and about a diameter is I_{zz}.

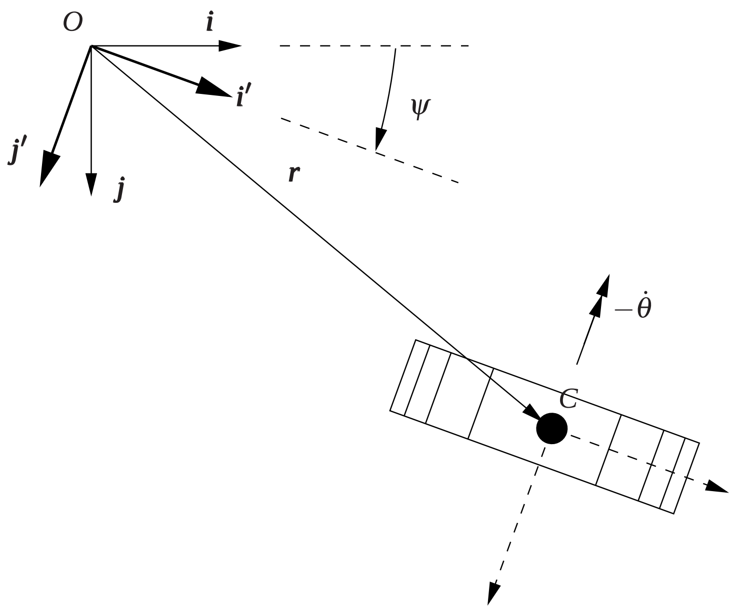


Figure 3.1 A disk rolling on a horizontal plane.

A moving Cartesian coordinate system with unit vectors $\boldsymbol{i'}$ and $\boldsymbol{j'}$ is used to analyse the motion of the object. This coordinate system has a fixed origin O but it rotates by an angle ψ so that it has the same orientation as the body fixed axes (shown with dashed lines on the object).

a) The coordinates of the centre of mass, C, in the moving reference frame are (x', y'). Give the kinetic energy of the disk in terms of the four generalised coordinates x', y', ψ, θ. [5]

b) Write the equations of the rolling constraint. [3]

c) Hence derive the equations of motion of the object. [8]

d) What are the forces that maintain the rolling constraint? [4]

4. a) A helicopter blade of mass m is attached onto a massless rotor that rotates with a fixed angular speed ω. The blade has a lagging freedom relative to the rotor described by the angle γ as shown in Figure 4.1.

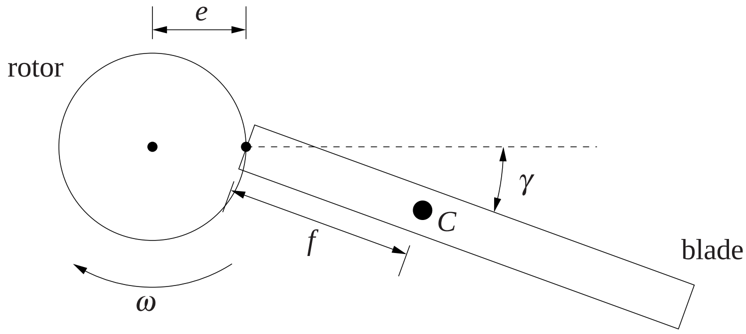


Figure 4.1 Plan view of a helicopter rotor with one blade.

The radius of the rotor is e, and the distance along the blade from the blade attachment point to the centre of mass of the blade, C, is f. The moment of inertia of the blade about the axis passing through the centre of mass and which is normal to the plane of the diagram is I_{zz}. A damping moment of magnitude $-D\dot{\gamma}$ opposes the motion of the blade relative to the rotor, where D is the damping coefficient.

i) Write an expression for the kinetic energy of the system. [4]

ii) Show that the lagging equation of motion is

$$(mf^2 + I_{zz})\ddot{\gamma} + D\dot{\gamma} + m\omega^2 ef \sin\gamma = 0.$$

[6]

b) Prove that for a general rigid body motion about a fixed point the rate of change of the kinetic energy T is given by

$$\frac{dT}{dt} = \boldsymbol{\Omega} \cdot \boldsymbol{N},$$

where $\boldsymbol{\Omega}$ is the angular velocity vector of the body and $\boldsymbol{N}$ is the external torque.
[10]

5. A cylindrical bar is made to rotate uniformly with an angular speed of ω about an axis passing through the centre of the bar and making an angle θ with the bar as shown in Figure 5.1.

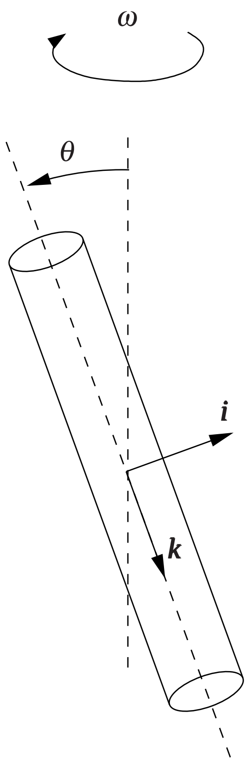


Figure 5.1 Cylindrical bar.

a) Find the angular velocity vector of the bar expressed in a body fixed reference frame. [2]

b) Find the angular momentum vector of the bar expressed in the same body fixed reference frame. [2]

c) Find the magnitude and direction of the torque driving the bar. [6]

d) If in addition to ω the bar rotates about its axis of symmetry with a speed of $\dot{\phi}$ then

   i) what are the new angular velocity and angular momentum vectors? [5]

   ii) what is the extra torque that is needed to drive the bar? [5]

6. A particle of mass m slides without friction on a wedge of angle α and mass M that can move without friction on a smooth horizontal surface, as shown in Figure 6.1.

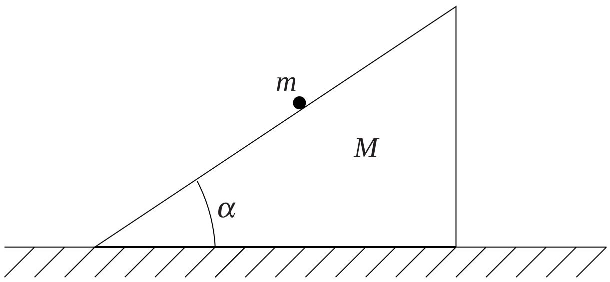


Figure 6.1 A particle slides on a wedge. The wedge slides on the horizontal surface.

a) Treating the constraint of the particle on the wedge by the method of Lagrange multipliers, find the equations of motion for the particle and wedge. [15]

b) Also obtain an expression for the forces of constraint between the particle and the wedge. [5]