EE4-47
EE9CS6-1

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2011

MSc and EEE PART IV: MEng and ACGI

MODELLING AND CONTROL OF MULTI-BODY MECHANICAL SYSTEMS

Friday, 6 May 2:30 pm

Time allowed: 3:00 hours

Corrected Copy

There are SIX questions on this paper.

Answer FOUR questions.

All questions carry equal marks.

This is an OPEN BOOK examination.

Any special instructions for invigilators and information for candidates are on page 1.

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Second Marker(s) : S. Evangelou, A. Astolfi

MODELLING AND CONTROL OF MULTIBODY MECHANICAL SYSTEMS

1. Consider a simple Hamiltonian system with internal Hamiltonian

$$H_0(q,p) = \frac{1}{2}p^2(1+q^2) + 1 - \cos q,$$

and interaction Hamiltonian $H_1(q) = q^2$. Let u denote the input signal.

a) Write the Hamiltonian equations of motion. [2]

b) Write the potential energy $U(q)$ of the system and compute its stationary points. Hence determine the equilibrium points of the system for $u = 0$. [2]

c) Assume u is constant. Discuss the existence of equilibrium points as a function of u. [6]

d) Let $u = 0$. Consider the equilibrium point $(q, p) = (0, 0)$. Argue that the internal Hamiltonian is positive definite around this equilibrium point. Compute the time derivative of the internal Hamiltonian along the trajectories of the system, hence discuss the stability properties of the equilibrium point. [6]

e) Design a control law which *adds damping* , that is such that the time derivative of the internal Hamiltonian along the trajectories of the closed-loop system is negative semi-definite. [4]

2. An axisymmetric wheel is held by forks and is free to spin relative to the forks as shown in Figure 2.1; the wheel is assumed to have no spin inertia. The wheel has mechanical trail e and the moving assembly has mass offset f with respect to the vertical king-pin bearing. The king-pin moves forward with displacement x under the influence of a longitudinal force F_x. The moving assembly has mass m and z-moment of inertia I_{zz}, while the king-pin is assumed massless. The forks are free to steer by angle θ with respect to the king-pin. A lateral tyre force Y acts at the tyre-ground contact.

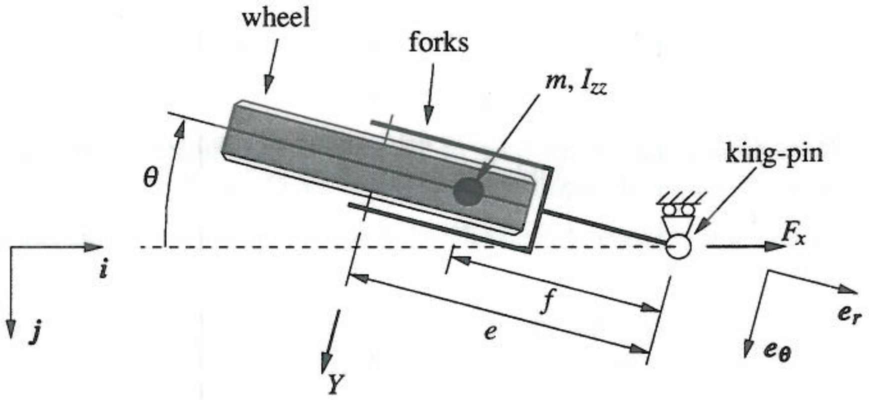

Figure 2.1 Plan view of a simple caster.

Make use of unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ (not shown) associated with an earth-fixed right-handed Cartesian coordinate system, and polar unit vectors $\boldsymbol{e_r}$ and $\boldsymbol{e_\theta}$ attached onto the wheel assembly with their origin coinciding with the king-pin bearing.

a) State the number of degrees of freedom of the system, and the associated generalised coordinates. [1]

b) Write the position vector of the centre of mass of the moving assembly. [2]

c) Determine the velocity vector of the same point. [2]

d) Compute the kinetic energy of the system. [3]

e) Use the Lagrangian approach to derive the equations of motion of the moving assembly. [10]

f) Ignore the tyre force and assume that the assembly is moving forward with constant acceleration a_f. Find whether the frequency of small steering oscillations increases or decreases as the acceleration increases. [2]

3. An axisymmetric wheel is held by forks and is free to spin relative to the forks as shown in Figure 3.1; the wheel is assumed to have no spin inertia. The wheel has mechanical trail e and the moving assembly has mass offset f with respect to the vertical king-pin bearing. The king-pin moves forward with displacement x under the influence of a longitudinal force F_x. The moving assembly has mass m and z-moment of inertia I_{zz}, while the king-pin is assumed massless. The forks are free to steer by angle θ with respect to the king-pin. A lateral tyre force Y acts at the tyre-ground contact.

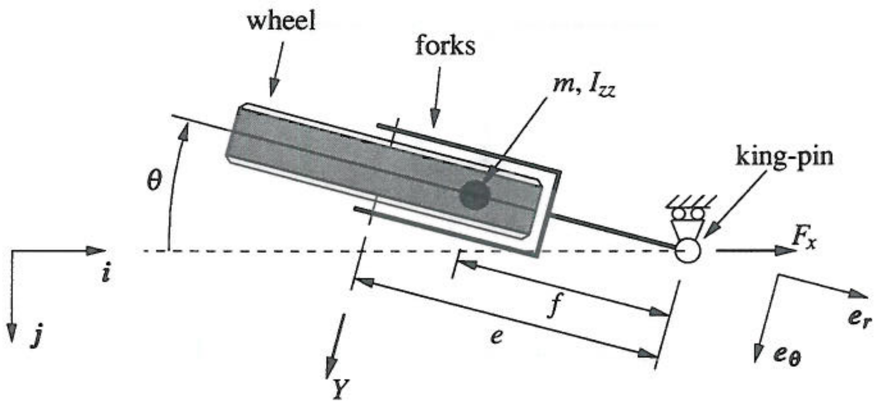

Figure 3.1 Plan view of a simple caster.

Make use of unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ (not shown) associated with an earth-fixed right-handed Cartesian coordinate system, and polar unit vectors $\boldsymbol{e_r}$ and $\boldsymbol{e_\theta}$ attached onto the wheel assembly with their origin coinciding with the king-pin bearing.

a) Determine the acceleration vector of the centre of mass of the moving assembly. [2]

b) Use the Newtonian approach to derive for the moving assembly:

i) The equations of motion. [8]

ii) The force of constraint which prevents the king-pin from moving in the $\boldsymbol{j}$ direction. [4]

c) The lateral force developed by the tyre is proportional to the sideslip and acts to oppose the slip. The following equation describes this situation:

$$Y = -C\frac{v_{lat}}{|v_{long}|},$$

where C is the constant of proportionality, v_{lat} is the lateral velocity of the tyre at the point of contact with the ground and v_{long} is the longitudinal velocity of the wheel. Consider that the king-pin is moving forward with a constant speed u and that θ is small.

i) Derive an expression for Y in terms of the generalised coordinates and velocities. [3]

ii) Determine the equation of motion of the moving assembly. [3]

4. A car kinetic energy recovery system (KERS), shown in Figure 4.1, consists of a flywheel that rotates about its spin axis with a fixed angular speed Ω_y. The spin axis is attached on the car chassis horizontally. The car is in cornering motion such that the spin axis rotates about the vertical direction with a fixed angular speed Ω_v.

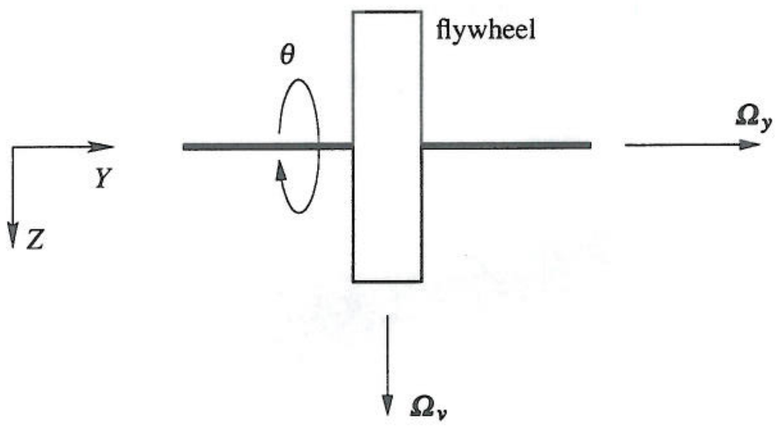

Figure 4.1 Kinetic energy recovery system (KERS).

The flywheel is axisymmetric with spin moment of inertia I_{yy} and radial moment of inertia, passing through the centre of mass, I_{xx}.

a) The body-fixed axes of the flywheel are initially aligned with the earth-fixed axes X, Y and Z, in which X is into the page. The rotation of this body is represented by three Euler angles ψ, ϕ and θ in the yaw-roll-pitch convention. By making use of the standard single-axis-rotation transformation matrices, show that the body angular velocity vector, $\boldsymbol{\Omega}$, in the body-fixed coordinate system is given by

$$\begin{bmatrix} -\dot{\psi}\sin\theta\cos\phi + \dot{\phi}\cos\theta \\ \dot{\psi}\sin\phi + \dot{\theta} \\ \dot{\psi}\cos\theta\cos\phi + \dot{\phi}\sin\theta \end{bmatrix}$$

[10]

b) Determine the kinetic energy of the flywheel. [4]

c) Assume that the spin axis remains horizontal such that $\phi = 0$. Use the Lagrangian approach to show that the magnitude of the "gyroscopic" moment that is exerted by the flywheel on the chassis, with direction perpendicular to both $\boldsymbol{\Omega}_y$ and $\boldsymbol{\Omega}_v$, is given by

$$|I_{yy}\Omega_y\Omega_v|$$

[6]

5. A helicopter blade of mass m is attached on a rotor of negligible mass that rotates with a fixed angular speed ω about its centre axis. The blade is free to flap relative to the rotor as shown in Figure 5.1.

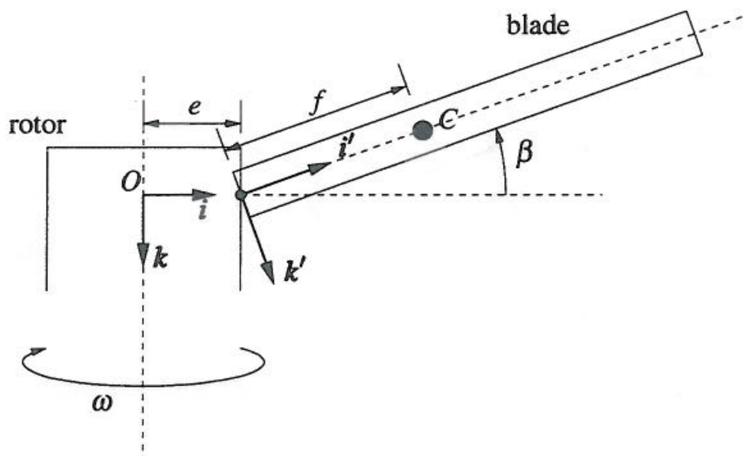

Figure 5.1 A helicopter rotor with one blade.

Two moving Cartesian coordinate systems, one attached on the rotor, with fixed origin O, and one on the blade, with unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$, and $\boldsymbol{i}'$, $\boldsymbol{j}'$ and $\boldsymbol{k}'$ respectively, are used to analyse the motion of the blade. The $\boldsymbol{j}$ and $\boldsymbol{j}'$ vectors are parallel to each other and have a direction out of the page at the instant shown. The point of attachment of the blade is at distance e from the rotor axis, and the centre of mass, C, of the blade is at distance f from the point of attachment, along the $\boldsymbol{i}'$ direction. The moments of inertia of the blade, I_{xx}, I_{yy} and I_{zz}, about axes in the $\boldsymbol{i}'$, $\boldsymbol{j}'$ and $\boldsymbol{k}'$ directions respectively, through the centre of mass, are principal. The effect of gravity is neglected.

It can be shown that the velocity vector of the centre of mass of the blade is

$$\dot{\boldsymbol{r}} = (e\omega + f\omega\cos\beta)\boldsymbol{j}' - f\dot{\beta}\boldsymbol{k}'.$$

a) Write an expression for the acceleration vector of the centre of mass of the blade. [4]

b) Determine the reaction force between the blade and the rotor. [3]

c) Write an expression for the angular velocity vector of the blade, $\boldsymbol{\Omega}'$, in terms of $\boldsymbol{i}'$, $\boldsymbol{j}'$ and $\boldsymbol{k}'$. [2]

d) Compute the angular momentum vector of the blade. [2]

e) In the following derivations use the <u>vectorial</u> approach:

i) Show that the flapping equation of motion is

$$(I_{yy} + mf^2)\ddot{\beta} + (I_{zz} - I_{xx})\omega^2 \sin\beta\cos\beta + mf(e + f\cos\beta)\omega^2 \sin\beta = 0.$$

[6]

ii) Derive expressions for the reaction moments between the blade and the rotor. [3]

6. A helicopter blade of mass m is attached onto a massless rotor that rotates with a fixed angular speed ω. The blade has a lagging freedom relative to the rotor described by the angle γ as shown in Figure 6.1.

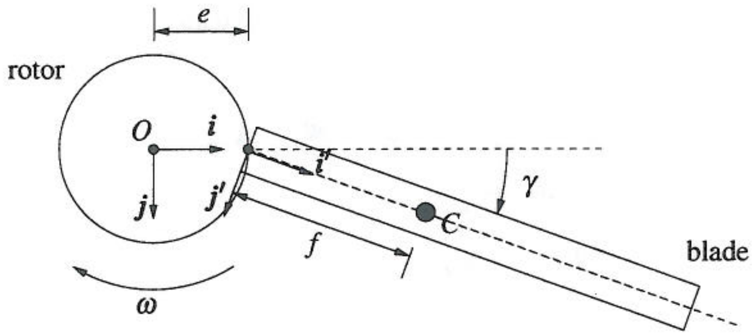

Figure 6.1 Plan view of a helicopter rotor with one blade.

Two moving Cartesian coordinate systems, one attached on the rotor, with fixed origin O, and one on the blade, with unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$, and $\boldsymbol{i}'$, $\boldsymbol{j}'$ and $\boldsymbol{k}'$ respectively, are used to analyse the motion of the blade. The $\boldsymbol{k}$ and $\boldsymbol{k}'$ vectors are parallel to each other and have a direction in the page. The radius of the rotor is e, and the distance along the $\boldsymbol{i}'$ direction from the blade attachment point to the centre of mass of the blade, C, is f. The moment of inertia of the blade about the axis passing through the centre of mass and which is normal to the plane of the diagram is I_{zz}. A damping moment of magnitude $-D\dot{\gamma}$ opposes the motion of the blade relative to the rotor, where D is the damping coefficient.

a) Find the velocity vector of the centre of mass of the blade in terms of $\boldsymbol{i}'$, $\boldsymbol{j}'$ and $\boldsymbol{k}'$. [2]

b) Hence compute the acceleration vector of the centre of mass of the blade. [5]

c) Determine the force of reaction between the blade and the rotor. [2]

d) Use the vectorial approach to show that the lagging equation of motion is

$$(mf^2 + I_{zz})\ddot{\gamma} + D\dot{\gamma} + m\omega^2 ef \sin\gamma = 0.$$

[8]

e) Assume that D is zero. Specify a simple mechanical system which has identical motion as the lagging motion of the blade. [3]

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Modelling and control of multibody mechanical systems

Model answers 2011

Question 1

a) The Hamiltonian of the system is

$$H(q,p,u) = H_0(q,p) - H_1(q)u.$$

The Hamiltonian equations of the system are

$$\dot{q} = (1+q^2)p \qquad \dot{p} = -p^2 q - \sin q + 2qu.$$

b) The potential energy is

$$U(q) = 1 - \cos q.$$

The stationary points are the solution of the equations

$$0 = \frac{\partial U}{\partial q} = \sin q$$

which are given by $q = k\pi$, with k integer. Hence, for $u = 0$, the equilibrium points are

$$(q,p) = (k\pi, 0),$$

with k integer.

c) The equilibrium points are given by $(q,p) = (q,0)$, where q is solution of the equation

$$0 = -\sin q + 2qu,$$

or equivalently

$$u = \frac{\sin q}{2q}.$$

This equation has, possibly multiple solutions, provided

$$-0.1083 \approx \min_q \frac{\sin q}{2q} \le u \le \max_q \frac{\sin q}{2q} = \frac{1}{2}.$$

d) Around the equilibrium $(q,p) = (0,0)$ the internal Hamiltonian can be approximated by

$$H_0(q,p) \approx \frac{1}{2}p^2 + \frac{1}{2}q^2,$$

hence the internal Hamiltonian is locally positive definite around the origin. By a general property of Hamiltonian systems

$$\dot{H}_0 = 2(1+q^2)qpu.$$

Thus, for $u = 0$ we have that $\dot{H}_0 = 0$. The function H_0 is positive definite around the origin and its time derivative is identically zero, hence the origin is a locally, non-asymptotically, stable equilibrium.

e) The damping injection control law is designed to render $\dot{H}_0$ negative semidefinite. In particular, selecting

$$u = -k\left(2(1+q^2)qp\right),$$

with $k > 0$ yields

$$\dot{H}_0 = -k\left(2(1+q^2)qp\right)^2 \le 0.$$

Question 2

a) 2 degrees of freedom. Generalised coordinates: x, θ.

b) $\boldsymbol{r} = x\boldsymbol{i} - f\boldsymbol{e_r}$

c) $\dot{\boldsymbol{r}} = \dot{x}\boldsymbol{i} - f\dot{\theta}\boldsymbol{e_\theta}$ or $\dot{\boldsymbol{r}} = (\dot{x} + f\dot{\theta}\sin\theta)\boldsymbol{i} - f\dot{\theta}\cos\theta\boldsymbol{j}$

d) $T = \frac{1}{2}m\dot{\boldsymbol{r}}\cdot\dot{\boldsymbol{r}} + \frac{1}{2}I_{zz}\dot{\theta}^2 = \frac{1}{2}m\left((\dot{x} + f\dot{\theta}\sin\theta)^2 + (f\dot{\theta}\cos\theta)^2\right) + \frac{1}{2}I_{zz}\dot{\theta}^2$ or

$$T = \frac{1}{2}m\left(\dot{x}^2 + 2f\dot{x}\dot{\theta}\sin\theta + f^2\dot{\theta}^2\right) + \frac{1}{2}I_{zz}\dot{\theta}^2.$$

e) The Lagrangian is $L = T - V = T$. The Lagrangian equation with respect to the generalised coordinate x is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = F_x - Y\sin\theta,$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m(\dot{x} + f\dot{\theta}\sin\theta)\right) = F_x - Y\sin\theta,$$

or

$$m\left(\ddot{x} + f\ddot{\theta}\sin\theta + f\dot{\theta}^2\cos\theta\right) = F_x - Y\sin\theta.$$

The Lagrangian equation with respect to the generalised coordinate θ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = -eY,$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m(f\dot{x}\sin\theta + f^2\dot{\theta}) + I_{zz}\dot{\theta}\right) - mf\dot{x}\dot{\theta}\cos\theta = -eY,$$

or

$$(mf^2 + I_{zz})\ddot{\theta} + mf\ddot{x}\sin\theta = -eY. \tag{1}$$

f) $\ddot{x} = a_f$ and $Y = 0$. From Equation (1)

$$(mf^2 + I_{zz})\ddot{\theta} + mfa_f\sin\theta = 0,$$

and since small perturbations are considered θ is small such that

$$(mf^2 + I_{zz})\ddot{\theta} + mfa_f\theta = 0.$$

It is easy to show that the frequency of oscillations is

$$\sqrt{\frac{mfa_f}{mf^2 + I_{zz}}},$$

and therefore the frequency increases with increasing acceleration.

Question 3

a) The velocity vector is $\dot{\boldsymbol{r}} = \dot{x}\boldsymbol{i} - f\dot{\theta}\boldsymbol{e_\theta}$ and by direct differentiation the acceleration vector is

$$\ddot{r} = \ddot{x}\boldsymbol{i} - f\ddot{\theta}\boldsymbol{e_\theta} + f\dot{\theta}^2\boldsymbol{e_r},$$

or

$$\ddot{r} = (\ddot{x} + f\ddot{\theta}\sin\theta + f\dot{\theta}^2\cos\theta)\boldsymbol{i} + (-f\ddot{\theta}\cos\theta + f\dot{\theta}^2\sin\theta)\boldsymbol{j}.$$

b) i) The motion of the centre of mass is given by

$$\sum \boldsymbol{F} = m\ddot{\boldsymbol{r}},$$

or

$$(F_x - Y\sin\theta)\boldsymbol{i} + (F_y + Y\cos\theta)\boldsymbol{j} = m\ddot{\boldsymbol{r}},$$

where F_y is the constraint force which prevents the king-pin from moving in the $\boldsymbol{j}$ direction. Therefore

$$F_x - Y\sin\theta = m(\ddot{x} + f\ddot{\theta}\sin\theta + f\dot{\theta}^2\cos\theta), \tag{2}$$

and

$$F_y + Y\cos\theta = m(-f\ddot{\theta}\cos\theta + f\dot{\theta}^2\sin\theta). \tag{3}$$

The motion about the centre of mass is given by

$$\frac{\mathrm{d}\boldsymbol{H}}{\mathrm{d}t} = \boldsymbol{N},$$

or

$$I_{zz}\ddot{\theta}\boldsymbol{k} = f\boldsymbol{e_r} \times (F_x\boldsymbol{i} + F_y\boldsymbol{j}) - (e-f)\boldsymbol{e_r} \times Y\boldsymbol{e_\theta},$$

or by substituting Equations (2) and (3),

$$I_{zz}\ddot{\theta} = -f\sin\theta\left(m(\ddot{x} + f\ddot{\theta}\sin\theta + f\dot{\theta}^2\cos\theta) + Y\sin\theta\right)$$

$$+f\cos\theta\left(m(-f\ddot{\theta}\cos\theta + f\dot{\theta}^2\sin\theta) - Y\cos\theta\right) - (e-f)Y,$$

or

$$I_{zz}\ddot{\theta} = m(-f\sin\theta\ddot{x} - f^2\ddot{\theta}) - eY,$$

or

$$(mf^2 + I_{zz})\ddot{\theta} + mf\ddot{x}\sin\theta = -eY. \tag{4}$$

ii) From Equation (3)

$$F_y = m(-f\ddot{\theta}\cos\theta + f\dot{\theta}^2\sin\theta) - Y\cos\theta.$$

c) i) The velocity vector of the contact point is

$$\dot{\boldsymbol{r}}_{\boldsymbol{c}} = \dot{x}\boldsymbol{i} - e\dot{\theta}\boldsymbol{e_\theta}.$$

Therefore

$$Y = -C\frac{\dot{\boldsymbol{r}}_{\boldsymbol{c}}\cdot\boldsymbol{e_\theta}}{\dot{\boldsymbol{r}}_{\boldsymbol{c}}\cdot\boldsymbol{e_r}},$$

or

$$Y = -C\frac{(\dot{x}\boldsymbol{i} - e\dot{\theta}\boldsymbol{e_\theta}) \cdot \boldsymbol{e_\theta}}{(\dot{x}\boldsymbol{i} - e\dot{\theta}\boldsymbol{e_\theta}) \cdot \boldsymbol{e_r}},$$

or

$$Y = -C\frac{-\dot{x}\sin\theta - e\dot{\theta}}{\dot{x}\cos\theta},$$

or due to small angles (also substituting $\dot{x}$ with u)

$$Y = C\theta + \frac{Ce}{u}\dot{\theta}. \tag{5}$$

ii) Since $\dot{x}$ is constrained to be constant we have only one degree of freedom. The motion is given by Equation (4) with Y substituted from Equation (5):

$$(mf^2 + I_{zz})\ddot{\theta} + \frac{Ce^2}{u}\dot{\theta} + Ce\theta = 0.$$

Question 4

a) Three single-axis-rotation transformation matrices are needed.

$$D_\psi = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which is the rotation matrix by angle ψ about a z axis.

$$B_\phi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix},$$

which is the rotation matrix by angle ϕ about an x axis.

$$C_\theta = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}.$$

which is the rotation matrix by angle θ about a y axis.

The angular velocity vector in body-fixed coordinates is

$$\begin{aligned}\boldsymbol{\Omega} &= C_\theta B_\phi \begin{bmatrix} 0 \\ 0 \\ \dot\psi \end{bmatrix} + C_\theta \begin{bmatrix} \dot\phi \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \dot\theta \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -\dot\psi\sin\theta\cos\phi + \dot\phi\cos\theta \\ \dot\psi\sin\phi + \dot\theta \\ \dot\psi\cos\theta\cos\phi + \dot\phi\sin\theta \end{bmatrix}\end{aligned}$$

b) Direct application of the kinetic energy formula given in the notes, with the products of inertia set to zero and with $I_{zz} = I_{xx}$, gives

$$T = \frac{1}{2}I_{xx}(-\dot\psi\sin\theta\cos\phi + \dot\phi\cos\theta)^2 + \frac{1}{2}I_{yy}(\dot\psi\sin\phi + \dot\theta)^2 + \frac{1}{2}I_{xx}(\dot\psi\cos\theta\cos\phi + \dot\phi\sin\theta)^2.$$

c) $\phi = 0$ implies that there is a constraint moment M_ϕ acting on the spin axis of the flywheel preventing it from tilting. The Lagrangian $L = T - V = T$. Therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot\phi}\right) - \frac{\partial L}{\partial\phi} = M_\phi,$$

or

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big(I_{xx}(-\dot\psi\sin\theta\cos\phi + \dot\phi\cos\theta)\cos\theta + I_{xx}(\dot\psi\cos\theta\cos\phi + \dot\phi\sin\theta)\sin\theta\Big) \\ -I_{xx}(-\dot\psi\sin\theta\cos\phi + \dot\phi\cos\theta)(\dot\psi\sin\theta\sin\phi) \\ -I_{yy}(\dot\psi\sin\phi + \dot\theta)\dot\psi\cos\phi - I_{xx}(\dot\psi\cos\theta\cos\phi + \dot\phi\sin\theta)(-\dot\psi\cos\theta\sin\phi) = M_\phi.\end{aligned}$$

By applying the constraint $\phi = 0$ we obtain

$$M_\phi = -I_{yy}\dot\theta\dot\psi.$$

But $\dot\psi = \Omega_v$ and $\dot\theta = \Omega_y$, and the moment exerted by the flywheel on the chassis is $-M_\phi$ (action-reaction) given by

$$I_{yy}\Omega_y\Omega_v.$$

Question 5

a) The angular velocity of the rotor fixed coordinate system is

$$\boldsymbol{\Omega} = \omega \boldsymbol{k},$$

and the angular velocity of the blade fixed coordinate system is

$$\boldsymbol{\Omega}' = \omega \boldsymbol{k} + \dot{\beta} \boldsymbol{j}'. \tag{6}$$

The acceleration vector of the centre of mass is found by differentiating the velocity vector. Therefore

$$\ddot{\boldsymbol{r}} = \boldsymbol{\Omega} \times (e\omega + f\omega\cos\beta)\boldsymbol{j}' - \boldsymbol{\Omega}' \times f\dot{\beta}\boldsymbol{k}' - f\omega\dot{\beta}\sin\beta\boldsymbol{j}' - f\ddot{\beta}\boldsymbol{k}',$$

or

$$\ddot{\boldsymbol{r}} = -(e + f\cos\beta)\omega^2\boldsymbol{i} - 2f\dot{\beta}\omega\sin\beta\boldsymbol{j} - f\dot{\beta}^2\boldsymbol{i}' - f\ddot{\beta}\boldsymbol{k}'.$$

b) The reaction force between the blade and the rotor is the only force acting on the blade. Therefore

$$\boldsymbol{F} = m\ddot{\boldsymbol{r}},$$

or

$$\boldsymbol{F} = m\left(-(e + f\cos\beta)\omega^2\boldsymbol{i} - 2f\dot{\beta}\omega\sin\beta\boldsymbol{j} - f\dot{\beta}^2\boldsymbol{i}' - f\ddot{\beta}\boldsymbol{k}'\right).$$

c) By finding the components along the $\boldsymbol{i}'$, $\boldsymbol{j}'$ and $\boldsymbol{k}'$ directions in Equation (6)

$$\boldsymbol{\Omega}' = -\omega\sin\beta\boldsymbol{i}' + \dot{\beta}\boldsymbol{j}' + \omega\cos\beta\boldsymbol{k}'.$$

d) Since I_{xx}, I_{yy} and I_{zz} are principal moments of inertia the angular momentum vector is given by

$$\boldsymbol{H} = -I_{xx}\omega\sin\beta\boldsymbol{i}' + I_{yy}\dot{\beta}\boldsymbol{j}' + I_{zz}\omega\cos\beta\boldsymbol{k}'.$$

e) The motion about the centre of mass is given by

$$\frac{\mathrm{d}'\boldsymbol{H}}{\mathrm{d}t} + \boldsymbol{\Omega}' \times \boldsymbol{H} = \boldsymbol{N},$$

or

$$-I_{xx}\omega\dot{\beta}\cos\beta\boldsymbol{i}' + I_{yy}\ddot{\beta}\boldsymbol{j}' - I_{zz}\omega\dot{\beta}\sin\beta\boldsymbol{k}' +$$
$$(I_{zz}-I_{yy})\omega\dot{\beta}\cos\beta\boldsymbol{i}' + (I_{zz}-I_{xx})\omega^2\sin\beta\cos\beta\boldsymbol{j}' + (I_{xx}-I_{yy})\omega\dot{\beta}\sin\beta\boldsymbol{k}' = -f\boldsymbol{i}'\times\boldsymbol{F} + N_x\boldsymbol{i}' + N_z\boldsymbol{k}',$$

in which N_x and N_z are reaction moments between the blade and the rotor in the $\boldsymbol{i}'$ and $\boldsymbol{k}'$ directions. Therefore

$$\begin{gathered}
-I_{xx}\omega\dot{\beta}\cos\beta\boldsymbol{i}' + I_{yy}\ddot{\beta}\boldsymbol{j}' - I_{zz}\omega\dot{\beta}\sin\beta\boldsymbol{k}' + \\
(I_{zz} - I_{yy})\omega\dot{\beta}\cos\beta\boldsymbol{i}' + (I_{zz} - I_{xx})\omega^2\sin\beta\cos\beta\boldsymbol{j}' + (I_{xx} - I_{yy})\omega\dot{\beta}\sin\beta\boldsymbol{k}' = \\
-mf\left((e + f\cos\beta)\omega^2\sin\beta\boldsymbol{j} - 2f\dot{\beta}\omega\sin\beta\boldsymbol{k}' + f\ddot{\beta}\boldsymbol{j}'\right) + N_x\boldsymbol{i}' + N_z\boldsymbol{k}'.
\end{gathered} \tag{7}$$

i) Collecting terms in $\boldsymbol{j}'$ we obtain

$$(I_{yy} + mf^2)\ddot{\beta} + (I_{zz} - I_{xx})\omega^2\sin\beta\cos\beta + mf(e + f\cos\beta)\omega^2\sin\beta = 0.$$

ii) Collecting the $\boldsymbol{i}'$ terms in Equation (7) we obtain

$$N_x = (I_{zz} - I_{yy} - I_{xx})\omega\dot{\beta}\cos\beta.$$

Collecting the $\boldsymbol{k}'$ terms in Equation (7) we obtain

$$N_z = -(I_{zz} + 2mf^2 + I_{yy} - I_{xx})\omega\dot{\beta}\sin\beta.$$

Question 6

a) The position vector of the blade centre of mass is

$$\boldsymbol{r} = e\boldsymbol{i} + f\boldsymbol{i}',$$

and therefore the velocity vector is

$$\dot{\boldsymbol{r}} = e\omega\boldsymbol{j} + f(\omega + \dot{\gamma})\boldsymbol{j}',$$

or

$$\dot{\boldsymbol{r}} = e\omega\sin\gamma\boldsymbol{i}' + (e\omega\cos\gamma + f(\omega + \dot{\gamma}))\boldsymbol{j}'.$$

b) The acceleration vector is found by differentiating the velocity vector, so that

$$\ddot{\boldsymbol{r}} = (e\omega\dot{\gamma}\cos\gamma - (\omega + \dot{\gamma})(e\omega\cos\gamma + f(\omega + \dot{\gamma})))\,\boldsymbol{i}' + (e\omega(\omega + \dot{\gamma})\sin\gamma - e\omega\dot{\gamma}\sin\gamma + f\ddot{\gamma})\,\boldsymbol{j}',$$

or

$$\ddot{\boldsymbol{r}} = \left(-e\omega^2\cos\gamma - f(\omega + \dot{\gamma})^2\right)\boldsymbol{i}' + \left(e\omega^2\sin\gamma + f\ddot{\gamma}\right)\boldsymbol{j}'.$$

c) Ignoring the effect of gravity, the only force acting on the blade is the reaction force between the blade and the rotor. Therefore

$$\boldsymbol{F} = m\ddot{\boldsymbol{r}},$$

or

$$\boldsymbol{F} = m\left(\left(-e\omega^2\cos\gamma - f(\omega + \dot{\gamma})^2\right)\boldsymbol{i}' + \left(e\omega^2\sin\gamma + f\ddot{\gamma}\right)\boldsymbol{j}'\right).$$

d) The motion about the centre of mass is given by

$$\frac{\mathrm{d}'\boldsymbol{H}}{\mathrm{d}t} + \boldsymbol{\Omega}' \times \boldsymbol{H} = \boldsymbol{N},$$

or

$$I_{zz}\ddot{\gamma}\boldsymbol{k}' = -f\boldsymbol{i}' \times \boldsymbol{F} - D\dot{\gamma}\boldsymbol{k}',$$

or

$$I_{zz}\ddot{\gamma} = -mf(e\omega^2\sin\gamma + f\ddot{\gamma}) - D\dot{\gamma},$$

or

$$(mf^2 + I_{zz})\ddot{\gamma} + D\dot{\gamma} + m\omega^2 ef\sin\gamma = 0.$$

e) When $D = 0$,

$$\ddot{\gamma} + \frac{m\omega^2 ef}{mf^2 + I_{zz}}\sin\gamma = 0.$$

The motion is equivalent to the motion of a simple pendulum under gravity with length

$$l = \frac{mf^2 + I_{zz}}{m\omega^2 ef}g.$$