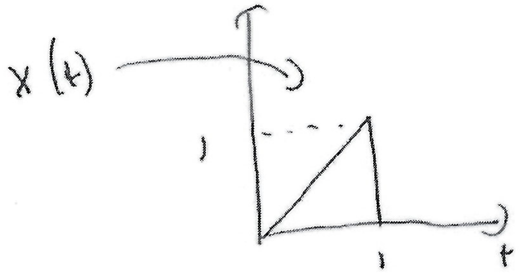
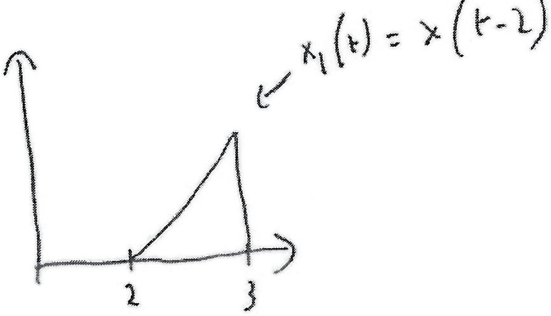
ANSWERS

QUESTION 1

(a)

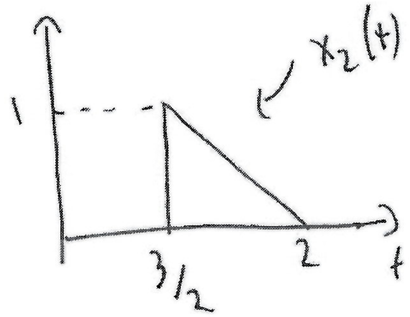


i.



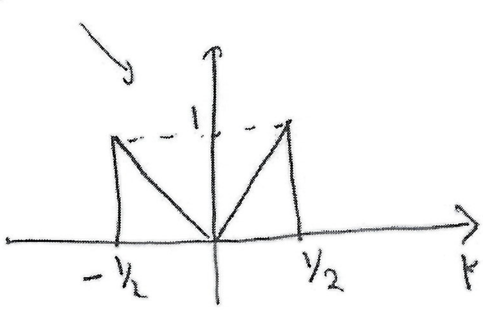
ii.

$$x_2(t) = x(-2(t-2))$$



iii.

$$x_3(t) = x(2t) + x(-2t)$$



(b)

i. $y(t) = x(t-2) + x(2-t)$

COMMENT: THIS IS THE CORRECT WAY TO CHECK LINEARITY

LET US CHECK LINEARITY

$$x_1(t) \to y_1(t) = x_1(t-2) + x_1(2-t)$$

$$x_2(t) \to y_2(t) = x_2(t-2) + x_2(2-t)$$

$$x_3(t) = a x_1(t) + b x_2(t) \to y_3(t) = a x_1(t-2) + b x_2(t-2) + a x_1(2-t) + b x_2(2-t)$$

$$= a y_1(t) + b y_2(t)$$

HENCE LINEAR

LET US CHECK TIME-INVARIANCE

$$x_1(t) \to y_1(t) = x_1(t-2) + x_1(2-t)$$

$$x_1(t-t_0) = x_2(t) \to y_2(t) = x_2(t-2) + x_2(2-t)$$

$$= x_1(t-t_0-2) + x_2(2-t-t_0)$$

$$\neq y_1(t-t_0)$$

NOTE THAT $y_1(t-t_0) = x_1(t-t_0-2) + x_1(2-t+t_0)$

HENCE TIME-VARIANT

~~[illegible]~~

~~[illegible]~~

COMMENT: MOST STUDENTS THOUGHT THIS WAS TIME-INVARIANT. THE TERM $x(2-t)$ MAKES IT TIME VARIANT. THINK OF THE SYSTEM $y(t) = x(-t)$ AND SKETCH THE OUTPUT WHEN THE INPUT IS DELAYED BY t_0.

ii.

$y(t) = x(t)\cos 3t$

THE SYSTEM IS LINEAR

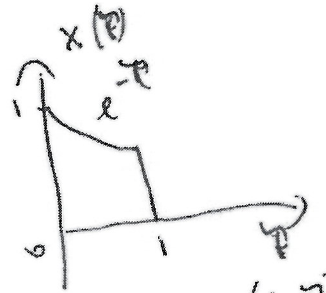
LET US CHECK FOR TIME-INVARIANCE

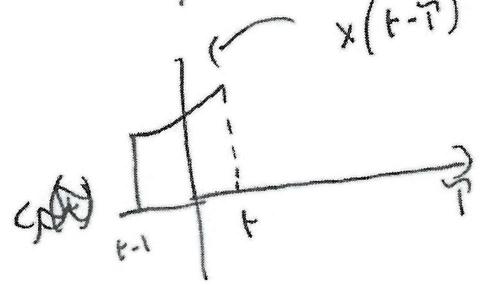
$$x_1(t) \rightarrow y_1(t) = x_1(t)\cos 3t$$

$$x_2(t) = x_1(t-t_0) \rightarrow y_2(t) = x_1(t-t_0)\cos 3t \neq y_1(t-t_0)$$

HENCE TIME-VARIANT

(c)





HENCE WE HAVE

$$x(t) * x(t) = \begin{cases} 0 & t \le 0 \\ \int_0^t x(\tau)\,x(t-\tau) & 0 < t \le 1 \\ \int_{t-1}^{1} x(\tau)\,x(t-\tau) & 1 < t \le 2 \\ 0 & t > 2 \end{cases}$$

$$\int_0^t x(\tau)\,x(t-\tau)\,d\tau = \int_0^t e^{-\tau} e^{-(t-\tau)}\,d\tau = t\,e^{-t} \qquad 0 \le t \le 1$$

$$\int_{t-1}^{1} x(\tau)\,x(t-\tau)\,d\tau = \int_{t-1}^{1} e^{-\tau} e^{-(t-\tau)}\,d\tau = (2-t)\,e^{-t} \qquad 1 < t \le 2$$

HENCE

$$x(t) * x(t) = \begin{cases} 0 & t \le 0 \\ t\,e^{-t} & 0 < t \le 1 \\ (2-t)\,e^{-t} & 1 < t \le 2 \\ 0 & t > 2 \end{cases}$$

COMMENTS: MOST STUDENTS COULD NOT COMPUTE THIS CONVOLUTION. ~~[illegible]~~ WHEN THE TWO SIGNALS HAVE COMPACT SUPPORT IT IS HELPFUL TO USE THE GRAPHICAL METHOD TO SEE THE INTERVALS IN WHICH THE INTEGRAL SHOULD BE SPLIT

(d)

$$y(t) = x(t-1) + x(-(t+1))$$

USING SHIFTING AND SCALING PROPERTY OF THE FT YEALDS:

$$Y(w) = X(w)\,e^{-jw} + X(-w)\,e^{jw}$$

$$= \frac{1}{w^2}\left(1 - jw - e^{-jw} + 1 + jw - e^{jw}\right)$$

$$= \frac{1}{w^2}\left(2 - e^{jw} - e^{-jw}\right) = 4\,\frac{\left(e^{j\frac{w}{2}} - e^{-j\frac{w}{2}}\right)^2}{\left(j\frac{w}{2}\right)^2}$$

$$= \left(\text{SINC}\,\frac{w}{2}\right)^2$$

(e)

i. CHARACTERISTIC POLYNOMIAL

$$s^2 + 4s + 3$$

CHARACTERISTIC ROOTS

$$s_{1,2} = -2 \pm \sqrt{4-3} = \begin{cases} -1 \\ -3 \end{cases}$$

CHARACTERISTIC MODES

$$c_1 e^{-t} + c_2 e^{-3t}$$

ii.

$$y(t) = c_1 e^{-t} + c_2 e^{-3t} \qquad t \geq 0$$

$$y(0) = c_1 + c_2 = 0$$
$$\Rightarrow \quad c_1 = 1$$
$$\dot{y}(0) = -c_1 - 3c_2 = 2 \qquad c_2 = -1$$

HENCE

$$y(t) = \left(e^{-t} - e^{-3t}\right) u(t)$$

iii.

$$\left(s^2 + 4s + 3\right) Y(s) = X(s)$$

SINCE $\quad X(s) = \frac{1}{s}$

WE HAVE THAT

$$Y(s) = \frac{1}{s(s+1)(s+3)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+3}$$

USING COVERING METHOD WE FIND THAT

$$Y(s) = \frac{1}{3s} - \frac{1}{2(s+1)} + \frac{1}{6(s+3)}$$

$$\Updownarrow$$

$$y(t) = \left(\frac{1}{3} - \frac{1}{2} e^{-t} + \frac{1}{6} e^{-3t}\right) u(t)$$

iv.

BECAUSE OF LINEARITY THE TOTAL ANSWER IS THE SUM OF THE RESPONSES IN PART ii. AND iii. THAT IS

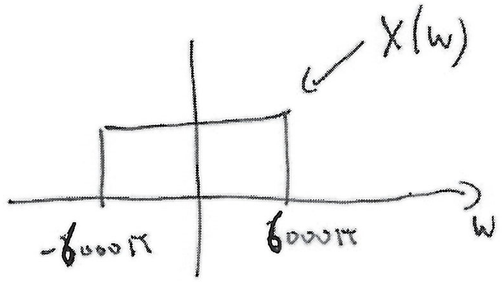
$$y_{TOT}(t) = \left(e^{-t} - e^{-3t}\right)u(t) + \left(\frac{1}{3} - \frac{1}{2}e^{-t} + \frac{1}{6}e^{-3t}\right)u(t)$$

$$= \left(\frac{1}{3} + \frac{1}{2}e^{-t} - \frac{5}{6}e^{-3t}\right)u(t)$$

(8) USING FOURIER TABLES

i.

$$6000\,\mathrm{SINC}(6000\pi t) \Leftrightarrow \mathrm{RECT}\left(\frac{w}{12000\pi}\right)$$



COMMENT: NORMALLY THE BANDWIDTH IS MEASURED IN Hz

ii)

NYQUIST RATE FOR $x(t)$: $f_s = 2\cdot\frac{6000\pi}{2\pi} = 6\,kHz$

(9)

$$\frac{X(z)}{z} = \frac{z+1}{z^2-5z+4} = \frac{z+1}{(z-4)(z-1)} = \frac{A}{z-4} + \frac{B}{z-1}$$

$$= \frac{5}{3}\frac{1}{z-4} - \frac{2}{3}\frac{1}{z-1}$$

HENCE

$$X(z) = \frac{5}{3}\frac{z}{z-4} - \frac{2}{3}\frac{z}{z-1}$$

USING Z-TRANSFORM TABLE WE GET

$$x[n] = \left(\frac{5}{3}(4)^n - \frac{2}{3}\right)u[n]$$

QUESTION 2

(a)

IN STEADY STATE THE INDUCTOR IS A SHORT CIRCUIT AND THE CAPACITOR IS AN OPEN CIRCUIT THEREFORE

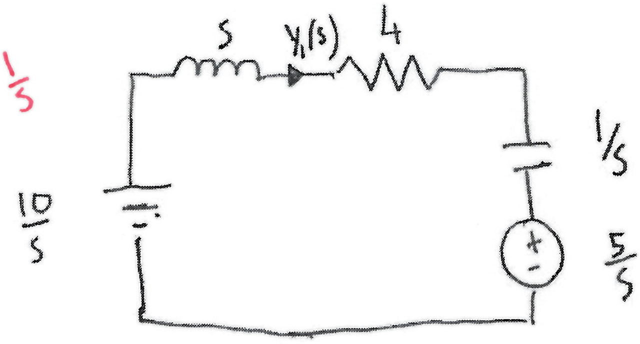
$$y_1(0^-) = 0$$

$$V_c(0^-) = 5V$$

COMMENTS : MOST STUDENTS DID THIS QUESTION CORRECTLY HOWEVER SOME HAD $y_1(0^-) \neq 0$ AND OTHERS ASSUMED $V_c(0^-) = 0$

(b) THE EQUIVALENT CIRCUIT IN THE LAPLACE DOMAIN IS:

REMEMBER THE $\frac{1}{s}$ TERM HERE.



THUS THE LOOP EQUATION IN THE LAPLACE DOMAIN IS:

$$\frac{10}{s} = sY_1(s) + 4Y_1(s) + \frac{Y_1(s)}{s} + \frac{5}{s} ;$$

$$Y_1(s)\left(s^2 + 4s + 1\right) = 5 ;$$

$$Y_1(s) = \frac{5}{s^2 + 4s + 1}$$

10

(c) $$Y_1(s) = \frac{5}{s^2+4s+1} = \frac{5}{(s+2-\sqrt{3})(s+2+\sqrt{3})} = \frac{5}{2\sqrt{3}}\left(\frac{1}{s+2-\sqrt{3}} - \frac{1}{s+2+\sqrt{3}}\right)$$

USING THE FACT THAT

$$e^{-\lambda t}u(t) \Longleftrightarrow \frac{1}{s+\lambda}$$

WE OBTAIN

$$y_1(t) = \frac{5}{2\sqrt{3}}\left(e^{-(2-\sqrt{3})t} - e^{-(2+\sqrt{3})t}\right)u(t)$$

COMMENTS: THIS IS A PHYSICALLY REALI~~ZA~~BLE CIRCUIT SO $y_1(t)$ SHOULD BE REAL VALUED. THE EXPONENTS SHOULD BE NEGATIVE AND THE SOLUTION SHOULD BE CONSISTENT WITH THE INITIAL CONDITIONS, ~~THAT~~ SPECIFICALLY IF $y_1(0^-)=0$ YOU EXPECT $y_1(0^+)=0$ AS WELL. YOU CAN HAVE A DISCONTINUITY IN ~~0~~ ZERO ONLY WHEN THE GENERATOR IS A DIRAC.

QUESTION 3

(a)

IF WE DENOTE WITH $F(s)$ THE OUTPUT OF THE FEEDBACK SYSTEM WE HAVE THAT

$$Y(s) = (s+2)\,F(s)$$

$$F(s) = K\left(X(s) - (s+a)^2 F(s)\right)$$

COMMENTS: MANY STUDENTS DID NOT GET THIS RIGHT.

THEREFORE

$$F(s) = \frac{K}{1 + K(s+a)^2}\,X(s)$$

AND

$$Y(s) = \frac{K(s+2)}{1 + K(s+a)^2}\,X(s)$$

THE TRANSFER FUNCTION IS:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{K(s+2)}{1 + K(s+a)^2}$$

(b) USING FINAL VALUE THEOREM WE HAVE THAT

$$y(\infty) = \lim_{t \to \infty} y(t) = \lim_{s \to 0} sY(s)$$

SINCE $X(s) = 1/s$ AND $K=1$

WE HAVE THAT

$$y(\infty) = \lim_{s \to 0} \frac{s+2}{1+(s+a)^2} = \frac{2}{1+a^2}$$

WE WANT $y(\infty) = \frac{2}{5}$ WITH $a > 0 \Rightarrow a = 2$.

(c) i.

$$Y(s) = \frac{s+2}{1+(s+1)^2} \cdot \frac{1}{s}$$

$$= \frac{C}{s} + \frac{As+B}{s^2+2s+2}$$

MULTIPLY BOTH SIDES BY s AND SET $s=0$

$$1 = C$$

THEN MULTIPLY BOTH SIDES BY s AND LET $s \to \infty$

$$0 = 1 + A \Rightarrow A = -1$$

FINALLY SET $s = -1$ TO FIND

$$B = -1$$

WE HAVE

$$Y(s) = \frac{1}{s} - \frac{s+1}{(s+1)^2+1}$$

USING LAPLACE TABLES WE OBTAIN

$$y(t) = u(t) - e^{-t}\cos t \; u(t)$$

ii.

FOR $t \geq 0$

$$y(t) = 1 - e^{-t}\cos t$$

$$\frac{dy}{dt} = e^{-t}\cos t + e^{-t}\sin t = 0$$

$$\Rightarrow e^{-t}(\cos t + \sin t) = 0$$

$$\Rightarrow \cos t = -\sin t \quad \Rightarrow \quad \tan t = -1$$

THIS HAPPENS AT $t = \pi/4 + \pi/2$.

THE OTHER POINT ARE EITHER LOCAL MINIMA OR LOCAL MAXIMA BECAUSE OF THE TERM e^{-t} IN THE EXPRESSION OF $y(t)$.