

**Imperial College
London****[E1.10 (Maths 1) 2010]****B.ENG. and M.ENG. EXAMINATIONS 2010****PART I : MATHEMATICS 1 (ELECTRICAL ENGINEERING)****Date Wednesday 2nd June 2010 10.00 am - 1.00 pm*****DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.*****Answer EIGHT questions.****CALCULATORS MAY NOT BE USED.***A mathematical formulae sheet is provided.*

[Before starting, please make sure that the paper is complete; there should be 6 pages, with a total of 10 questions. Ask the invigilator for a replacement if your copy is faulty.]

[E1.10 (Maths 1) 2010]

1. (i) Explain why the function

$$f(x) = \begin{cases} \frac{1}{2} - x^2 & \text{if } x \leq 0, \\ x^2 + C & \text{if } x > 0, \end{cases}$$

is differentiable at $x = 0$ only for the choice $C = 1/2$. For this value of C , prove that f is differentiable at $x = 0$ by computing the limits

$$\lim_{h \rightarrow 0^+} \frac{f(0+h) - f(0)}{h} \quad \text{and} \quad \lim_{h \rightarrow 0^-} \frac{f(0+h) - f(0)}{h}.$$

- (ii) Sketch the graph of $\cos^{-1}(x)$ and find the derivative of $\cos^{-1}(x)$.

- (iii) Differentiate

$$(\cos(x))^{\cos^{-1}(x)}.$$

2. (i) Calculate

$$\lim_{x \rightarrow 0} \frac{\ln(1+x)}{x}.$$

- (ii) By performing the substitution

$$y = \frac{1}{x},$$

or otherwise, calculate

$$\lim_{y \rightarrow \infty} \left(1 + \frac{1}{y}\right)^y.$$

- (iii) The power series

$$\sum_{n=1}^{\infty} u_n x^n$$

has coefficients given by

$$u_n = \frac{n^n}{n!}.$$

Show that the $(n+1)$ th term divided by the n th term is

$$\left(\frac{n+1}{n}\right)^n x,$$

and hence calculate the radius of convergence of this power series.

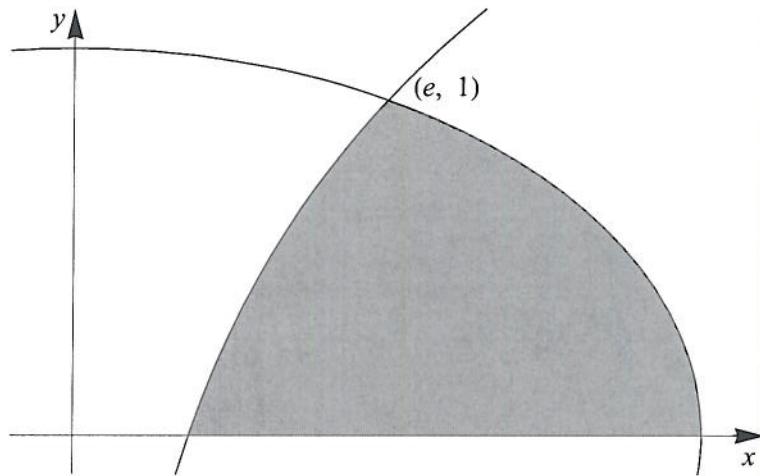
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[E1.10 (Maths 1) 2010]

3. The graph of the logarithm $y = \ln(x)$ meets the ellipse

$$\frac{x^2}{4e^2} + \frac{3y^2}{4} = 1$$

at $(e, 1)$. Find the area of the finite region enclosed in the first quadrant by these curves and the x -axis, as in the figure below



4. (i) If

$$z_1 = -1 + 2i,$$

$$z_2 = 3 + 3i,$$

find $z_1 z_2$ and $\frac{z_1}{z_2}$ in the form $x + iy$ for some real x and y and verify that:

$$|z_1 z_2| = |z_1| |z_2|.$$

- (ii) Find all complex solutions of $z^3 = 1$. Show that the sum of all these solutions is 0. Indicate the position of each solution in the complex plane.

- (iii) De Moivre's theorem states

$$(\cos \theta + i \sin \theta)^n = \cos(n\theta) + i \sin(n\theta).$$

Using this theorem, or otherwise, find an expression for $\sin(3\theta)$ in terms of powers of $\sin \theta$.

[E1.10 (Maths 1) 2010]

5. Consider the function

$$f(x) = \frac{e^{-x^2}}{2 - x^2}.$$

- (i) State, giving reasons, whether $f(x)$ is even, odd or neither.
- (ii) Find all vertical and horizontal asymptotes.
- (iii) Use (ii) to determine the sign of $f(x)$, for all x .
- (iv) Find all the points where $f'(x) = 0$.
- (v) Determine any local minima and maxima of $f(x)$.
- (vi) Sketch the graph of $f(x)$.

6. (i) Find a vector equation for the line of intersection of the planes $x + y + z = 1$ and $2x - y + 3z = 0$.

Find the equation of a plane through the origin which is perpendicular to the line of intersection found above.

- (ii) Given two vectors \mathbf{a} and \mathbf{b} in three dimensional space, show that the vector

$$\mathbf{a}_p = \frac{(\mathbf{a} \cdot \mathbf{b})}{|\mathbf{b}|^2} \mathbf{b}$$

is the projection of \mathbf{a} onto \mathbf{b} . Verify that $\mathbf{a} - \mathbf{a}_p$ is a vector perpendicular to \mathbf{b} .

Decompose the vector $(1, 2, 3)$ into a sum of two vectors, one of which is parallel to $(1, 1, 1)$ and the other perpendicular to $(1, 1, 1)$.

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[E1.10 (Maths 1) 2010]

7. Let

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ -1 & 2 & 3 \end{pmatrix}.$$

(i) Find A^2 and A^3 . Show that

$$A^3 - 5A^2 + 7A + I = 0.$$

(ii) Hence, or otherwise, show that $|A| \neq 0$, and hence that

$$A^{-1} = -A^2 + 5A - 7I.$$

Using this result, find A^{-1} .

(iii) Check your answer in (ii) by finding A^{-1} , using row operations on A .

8. (i) Find, in the form $y = f(x)$, the solution of the differential equation

$$\frac{dy}{dx} = \frac{4x - y}{x + 2y},$$

for which $y(0) = 1$.

(ii) Find the general solution of the linear differential equation

$$\frac{dy}{dx} \cos x + y \sin x = 1,$$

using an integrating factor, and hence prove that all solution curves pass through the point $(\pi/2, 1)$.

[E1.10 (Maths 1) 2010]

9. A forced linear oscillator with adjustable damping has equation of motion

$$\frac{d^2x}{dt^2} + k \frac{dx}{dt} + x = f(t).$$

Given that $x(0) = 0.24$ and $\dot{x}(0) = 0$, and that $f(t)$ is identically zero, find the solution of this equation in the following cases:-

- (i) $k = 2.5$;
- (ii) $k = 2.0$;
- (iii) $k = 1.2$.

For the case $k = 2.0$, find also the solution, with the same initial conditions, given that

$$f(t) = e^{-t}.$$

10. A function $y(x)$ satisfies the equation

$$y'' + (4x^2 + 1)y = 0$$

and the initial conditions $y(0) = 0$, $y'(0) = 1$.

Differentiate the equation m times using the Leibnitz formula to show that for $m \geq 2$

$$y^{(m+2)}(0) = - \left[y^{(m)}(0) + 4m(m-1)y^{(m-2)}(0) \right].$$

Deduce that

$$y(x) = \sum_{m=0}^{\infty} c_m x^{2m+1},$$

where c_m are constants.

Find the first three non-zero terms of this series.

END OF PAPER

MATHEMATICS DEPARTMENT

3. TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

MATHEMATICAL FORMULAE

$$\sin(a+b) = \sin a \cos b + \cos a \sin b;$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b.$$

$$\cos iz = \cosh z; \quad \cosh iz = \cos z; \quad \sin iz = i \sinh z; \quad \sinh iz = i \sin z.$$

1. VECTOR ALGEBRA

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1, a_2, a_3)$$

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

Vector (cross) product:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix},$$

Scalar triple product:

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = \mathbf{b} \cdot \mathbf{c} \times \mathbf{a} = \mathbf{c} \cdot \mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product: $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c} \cdot \mathbf{a})\mathbf{b} - (\mathbf{b} \cdot \mathbf{a})\mathbf{c}$

2. SERIES

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \dots \quad (\alpha \text{ arbitrary, } |x| < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^n \frac{x^{2n}}{(2n)!} + \dots,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \dots,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^n \frac{x^{n+1}}{(n+1)} + \dots \quad (-1 < x \leq 1)$$

4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f D^n g + (^n_1) Df D^{n-1} g + \dots + (^n_n) D^n f D^{n-r} g + \dots + D^n f g.$$

(b) Taylor's expansion of $f(x)$ about $x = a$:

$$f(a+h) = f(a) + hf'(a) + h^2 f''(a)/2! + \dots + h^n f^{(n)}(a)/n! + \epsilon_n(h),$$

where $\epsilon_n(h) = h^{n+1} f^{(n+1)}(a+\theta h)/(n+1)!$, $0 < \theta < 1$.

(c) Taylor's expansion of $f(x, y)$ about (a, b) :

$$f(a+h, b+k) = f(a, b) + [h f_x + k f_y]_{a,b} + 1/2! [h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}]_{a,b} + \dots$$

(d) Partial differentiation of $f(x, y)$:

i. If $y = y(x)$, then $f = F(x)$, and $\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx}$.

ii. If $x = x(t)$, $y = y(t)$, then $f = F(t)$, and $\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$.

iii. If $x = x(u, v)$, $y = y(u, v)$, then $f = F(u, v)$, and

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u}, \quad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial v}.$$

(e) Stationary points of $f(x, y)$ occur where $f_x = 0$, $f_y = 0$ simultaneously.
Let (a, b) be a stationary point: examine $D = [f_{xx} f_{xy} - (f_{xy})^2]_{a,b}$.

If $D > 0$ and $f_{xx}(a, b) < 0$, then (a, b) is a maximum;

If $D > 0$ and $f_{xx}(a, b) > 0$, then (a, b) is a minimum;

If $D < 0$ then (a, b) is a saddle-point.

(f) Differential equations:

i. The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating factor $I(x) = \exp[\int P(x)(dx)]$, so that $\frac{d}{dx}(Iy) = IQ$.

ii. $P(x, y)dx + Q(x, y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y$.

5. INTEGRAL CALCULUS

- (a) An important substitution: $\tan(\theta/2) = t$:
 $\sin \theta = 2t/(1+t^2)$, $\cos \theta = (1-t^2)/(1+t^2)$, $d\theta = 2dt/(1+t^2)$.

- (b) Some indefinite integrals:

$$\int (a^2 - x^2)^{-1/2} dx = \sin^{-1}\left(\frac{x}{a}\right), \quad |x| < a.$$

$$\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1}\left(\frac{x}{a}\right) = \ln \left\{ \frac{x}{a} + \left(1 + \frac{x^2}{a^2}\right)^{1/2} \right\}.$$

$$\int (x^2 - a^2)^{-1/2} dx = \cosh^{-1}\left(\frac{x}{a}\right) = \ln \left| \frac{x}{a} + \left(\frac{x^2}{a^2} - 1 \right)^{1/2} \right|.$$

$$\int (a^2 + x^2)^{-1} dx = \left(\frac{1}{a} \right) \tan^{-1}\left(\frac{x}{a}\right).$$

7. LAPLACE TRANSFORMS

Function	Transform	Function	Transform	Transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t) dt$	$a f(t) + b g(t)$	$a F(s) + b G(s)$	
df/dt	$sF(s) - f(0)$	$d^2 f/dt^2$	$s^2 F(s) - s f(0) - f'(0)$	
$e^{at} f(t)$	$F(s-a)$	$t f(t)$	$-dF(s)/ds$	
$(\partial/\partial a)f(t, a)$	$(\partial/\partial a)F(s, a)$	$f'_0 f(t) dt$	$F'(s)/s$	
$\int_0^t f(u)g(t-u) du$	$F(s)G(s)$			
1	$1/s$	$t^n (n=1, 2, \dots)$	$n!/s^{n+1}$, ($s > 0$)	
e^{at}	$1/(s-a)$, ($s > a$)	$\sin \omega t$	$\omega/(s^2 + \omega^2)$, ($s > 0$)	
$\cos \omega t$	$s/(s^2 + \omega^2)$, ($s > 0$)	$I(t-T) = \begin{cases} 0, & t < T \\ 1, & t > T \end{cases}$	e^{-sT}/s , ($s, T > 0$)	

6. NUMERICAL METHODS

- (a) Approximate solution of an algebraic equation:

If a root of $f(x) = 0$ occurs near $x = a$, take $x_0 = a$ and
 $x_{n+1} = x_n - [f(x_n)/f'(x_n)]$, $n = 0, 1, 2, \dots$

(Newton-Raphson method).

- (b) Formulae for numerical integration: Write $x_n = x_0 + nh$, $y_n = y(x_n)$.

- i. Trapezium rule (1-strip): $\int_{x_0}^{x_1} y(x) dx \approx (h/2)[y_0 + y_1]$.

- ii. Simpson's rule (2-strip): $\int_{x_0}^{x_2} y(x) dx \approx (h/3)[y_0 + 4y_1 + y_2]$.

- (c) Richardson's extrapolation method: Let $I = \int_a^b f(x) dx$ and let I_1 , I_2 be two estimates of I obtained by using Simpson's rule with intervals h and $h/2$.

Then, provided h is small enough,

$$I_2 + (I_2 - I_1)/15,$$

is a better estimate of I .

8. FOURIER SERIES

If $f(x)$ is periodic of period $2L$, then $f(x+2L) = f(x)$, and

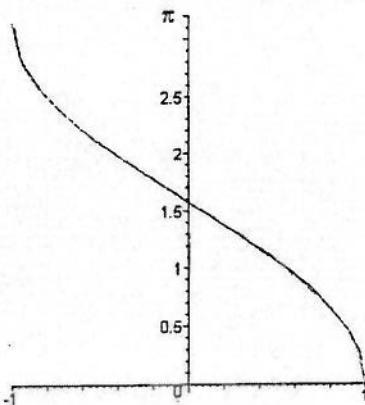
$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \text{ where}$$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx, \quad n = 0, 1, 2, \dots \text{ and}$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx, \quad n = 1, 2, 3, \dots$$

Parseval's theorem

$$\frac{1}{L} \int_{-L}^L [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

	EXAMINATION QUESTIONS/SOLUTIONS 2009-2010	Course EEI(1)
Question C1		Marks & seen/unseen
	(a) The function is not continuous at $x = 0$ if $C \neq 1/2$, so f can only be differentiable if $C = 1/2$. For $C = 1/2$, $\lim_{h \rightarrow 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0^+} \frac{\frac{1}{2}h^2 + \frac{1}{2} - \frac{1}{2}}{h} = \lim_{h \rightarrow 0^+} h = 0,$ $\lim_{h \rightarrow 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0^-} \frac{\frac{1}{2}h^2 + \frac{1}{2} - \frac{1}{2}}{h} = \lim_{h \rightarrow 0^-} h = 0.$ Since the lateral limits $\lim_{h \rightarrow 0^+} \frac{f(0+h) - f(0)}{h}$ and $\lim_{h \rightarrow 0^-} \frac{f(0+h) - f(0)}{h}$ coincide, f is differentiable at $x = 0$ and $f'(0) = 0$.	2 2 2 1
Parts	(b) The graph of $\cos^{-1}(x)$ is 	3
	Using the inverse function rule $(\cos^{-1})'(x) = \frac{1}{\cos'(\cos^{-1}(x))} = \frac{1}{-\sin(\cos^{-1}(x))} = -\frac{1}{\sqrt{1-\cos^2(\cos^{-1}(x))}} = -\frac{1}{\sqrt{1-x^2}}.$ [Other methods also acceptable].	4
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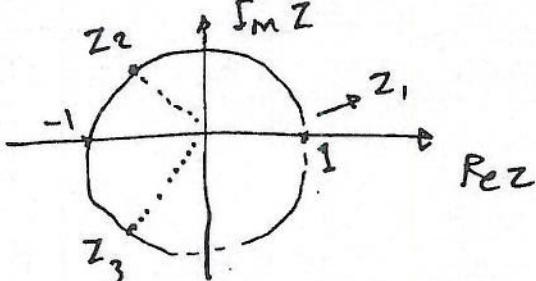
	EXAMINATION QUESTIONS/SOLUTIONS 2009-2010	Course EEI(1)
Question C1	Marks & seen/unseen	
(c) In order to differentiate $f(x) = \cos(x)^{\cos^{-1}(x)}$ we need first to take logarithms	2	
$\ln(f(x)) = \ln(\cos(x)^{\cos^{-1}(x)}) = \cos^{-1}(x) \ln(\cos(x))$ so that we can then differentiate	2	
$\frac{1}{f(x)} f'(x) = (\cos^{-1})' (x) \ln(\cos(x)) + \cos^{-1}(x) (\ln \circ \cos)'(x)$ $= -\frac{1}{\sqrt{1-x^2}} \ln(\cos(x)) - \cos^{-1}(x) \frac{\sin(x)}{\cos(x)}.$	2	
Therefore	2	
$f'(x) = -f(x) \left(\frac{1}{\sqrt{1-x^2}} \ln(\cos(x)) + \cos^{-1}(x) \tan(x) \right)$ $= -\cos(x)^{\cos^{-1}(x)} \left(\frac{1}{\sqrt{1-x^2}} \ln(\cos(x)) + \cos^{-1}(x) \tan(x) \right).$	2	
Parts	20	
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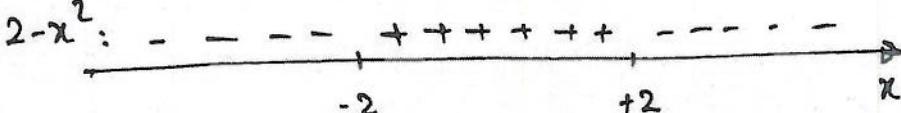
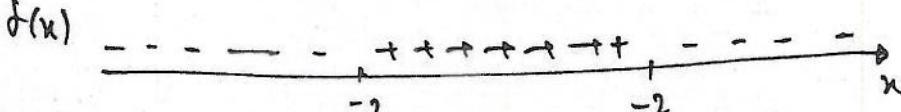
	EXAMINATION QUESTIONS/SOLUTIONS 2009-2010	Course EEG FIRST YR ENG CORE
Question C2		EE 1 (1) Marks & seen/unseen
Parts		
(i)	$\lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = \lim_{x \rightarrow 0} \frac{x - x^2/2 + x^3/3 - \dots}{x}$ $= \lim_{x \rightarrow 0} 1 - x/2 + x^2/3 - \dots$ $= \underline{1}.$ <p>OR L'Hopital's rule</p> $\lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = \lim_{x \rightarrow 0} \frac{1/(1+x)}{1}$ $= \underline{1}.$	
(ii)	$\lim_{y \rightarrow \infty} \left(1 + \frac{1}{y}\right)^y = \lim_{x \rightarrow 0^+} (1+x)^{1/x}$ $= \lim_{x \rightarrow 0^+} [e^{1/x \ln(1+x)}]$ $= e^1 \text{ from (i)} = \underline{e}.$	5
(iii)	$\frac{a_{n+1}}{a_n} = \frac{u_{n+1} x^{n+1}}{u_n x^n}$ $= \frac{(n+1)^{n+1}}{(n+1)!} \times \frac{n!}{n^n} \times x$	5
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		Page number 1 OF 2

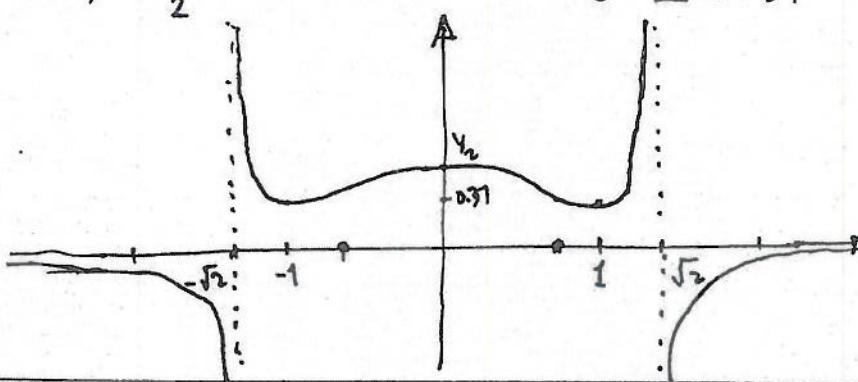
	EXAMINATION QUESTIONS/ <u>SOLUTIONS</u> 2009-2010	Course FIRST YR ENG CORE EE (1)
Question C2		Marks & seen/unseen
Parts	$= \frac{(n+1)^{n+1}}{(n+1)n^n} \times x$ $= \frac{(n+1)^n}{n^n} x = \left(\frac{n+1}{n}\right)^n x$	5
	Now, $\lim_{n \rightarrow \infty} \left(\frac{n+1}{n}\right)^n x = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n x$ $= e^x.$	
	For convergence, we require $ e^x < 1$ that is, $ x < \frac{1}{e}$.	
	Radius of convergence is $\frac{1}{e}$.	5
		20
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		Page number 2 OF 2

	EXAMINATION QUESTIONS/SOLUTIONS 2009-2010	Course EE I (1)
Question C3	Marks & seen/unseen	
	The area of the figure that we want to calculate is delimited by the horizontal x -axis, the graph of $y = \ln(x)$, and the graph of $y = \frac{2}{\sqrt{3}} \sqrt{1 - \frac{x^2}{4e^2}}$. Therefore, the area is given by	1
	$\int_1^e \ln(x) dx + \frac{2}{\sqrt{3}} \int_e^{2e} \sqrt{1 - \frac{x^2}{4e^2}} dx$ (Correct identification of integration limits)	2 2
	On the one hand, to integrate $\int_1^e \ln(x) dx$ we apply integration by parts $v' = 1 \quad v = x$ $u = \ln(x) \implies u' = 1/x$ so that $\int_1^e \ln(x) dx = [x \ln(x)]_1^e - \int_1^e x \frac{1}{x} dx = e - [x]_1^e = 1.$	3 3
Parts	On the other hand, to evaluate $\frac{2}{\sqrt{3}} \int_e^{2e} \sqrt{1 - \frac{x^2}{4e^2}} dx$ we apply the following change of variables $\frac{x}{2e} = \sin(u) \implies dx = 2e \cos(u) du$ which implies that the limits of integration change as follows: $x = e \implies \frac{1}{2} = \sin(u) \implies u = \frac{\pi}{6}$ $x = 2e \implies 1 = \sin(u) \implies u = \frac{\pi}{2}$ Therefore $\begin{aligned} \frac{2}{\sqrt{3}} \int_e^{2e} \sqrt{1 - \frac{x^2}{4e^2}} dx &= \frac{4e}{\sqrt{3}} \int_{\pi/6}^{\pi/2} \sqrt{1 - \sin^2(u)} \cos(u) du = \frac{4e}{\sqrt{3}} \int_{\pi/6}^{\pi/2} \cos^2(u) du \\ &= \frac{4e}{\sqrt{3}} \int_{\pi/6}^{\pi/2} \frac{1+\cos(2u)}{2} du = \frac{2e}{\sqrt{3}} \left[u + \frac{\sin(2u)}{2} \right]_{\pi/6}^{\pi/2} \\ &= \frac{2e}{\sqrt{3}} \left[\frac{\pi}{3} - \frac{\sqrt{3}}{4} \right] \end{aligned}$	2 2 3
	The final result is $\text{Area} = \frac{2e}{\sqrt{3}} \left[\frac{\pi}{3} - \frac{\sqrt{3}}{4} \right] + 1 \quad (\text{correct value})$	2 20
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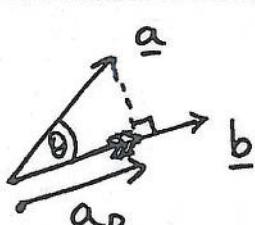
	EXAMINATION QUESTIONS/SOLUTIONS 2009-2010	Course
Question		EE I (1)
Parts		Marks & seen/unseen
(i)	$Z_1 Z_2 = (-1 + 2i)(3 + 3i)$ $= -3 + 6i^2 + 6i - 3i = -9 + 3i.$ $\frac{Z_1}{Z_2} = \frac{(-1 + 2i)}{(3 + 3i)} = \frac{(-1 + 2i)(3 - 3i)}{(3 + 3i)(3 - 3i)}$ $= \frac{-3 - 6i^2 + 3i + 6i}{18} = \frac{-3 + 9i}{18}$ $= \frac{1}{6} + \frac{1}{2}i.$ $ Z_1 = \sqrt{(-1+2i)(-1-2i)} = \sqrt{5}$ $ Z_2 = \sqrt{(3+3i)(3-3i)} = 3\sqrt{2}$ $ Z_1 Z_2 = \sqrt{(-9+3i)(-9-3i)} = \sqrt{90} = 3\sqrt{2} \cdot \sqrt{5} = Z_1 Z_2 $ $Z = r e^{i\theta} \Rightarrow z^3 = r^3 e^{i3\theta} = 1 e^{i2k\pi i} \Rightarrow$ $r = 1, \theta = 2k\pi, k = 0, 1, 2$ Solutions : $Z_1 = 1, Z_2 = e^{\frac{2i\pi}{3}}, Z_3 = e^{\frac{4i\pi}{3}}$ $Z_1 + Z_2 + Z_3 = 1 + e^{\frac{2i\pi}{3}} + e^{\frac{4i\pi}{3}} = (1 + \cos \frac{2\pi}{3} + \cos \frac{4\pi}{3}) +$ $+ i(\sin \frac{2\pi}{3} + \sin \frac{4\pi}{3}) = (1 - \cos \frac{\pi}{3} - \cos \frac{\pi}{3}) +$ $i(\sin \frac{\pi}{3} - \sin \frac{\pi}{3}) = (1 - \frac{1}{2} - \frac{1}{2}) = 0$	2 2 2 1 1 1 1
Setter's initials	V.S.	Checker's initials
	P.J.R	Page number

	EXAMINATION QUESTIONS/SOLUTIONS 2009-2010	Course
Question		Marks & seen/unseen
C4	Solutions ($\frac{2}{2}$)	EE I (1)
Parts (ii) Continued		2
(iii)	<p>For $-\theta$ De Moivre's theorem states</p> $(\cos \theta - i \sin \theta)^n = \cos(n\theta) - i \sin(n\theta) \quad (*)$ <p>For $n=3$ by subtracting (*) from the original De Moivre's theorem we get,</p> $\sin 3\theta = \frac{1}{2i} [(\cos \theta + i \sin \theta)^3 - (\cos \theta - i \sin \theta)^3] = 3$ $\frac{1}{2i} \left[(\cos^3 \theta + 3i \sin \theta \cos^2 \theta + 3i^2 \sin^2 \theta \cos \theta + i^3 \sin^3 \theta) \right.$ $\left. - (\cos^3 \theta - 3i \sin \theta \cos^2 \theta + 3i^2 \sin^2 \theta \cos \theta - i^3 \sin^3 \theta) \right] =$ $\frac{1}{2i} [6i \sin \theta \cos^2 \theta - 2i \sin^3 \theta] =$ $3 \sin \theta (1 - \sin^2 \theta) - \sin^3 \theta =$ $3 \sin \theta - 4 \sin^3 \theta.$	3 20
	Setter's initials V.S.	Checker's initials R.J.R.
		Page number 2

	EXAMINATION QUESTIONS/SOLUTIONS 2009-2010	Course
Question		EE I (1)
Parts	Solutions	Marks & seen/unseen
ii)	$f(-x) = f(x)$ even.	2
(iii)	$\lim_{x \rightarrow \infty} f(x) = 0 \Rightarrow y = 0$ horizontal asymptote.	1
	$\lim_{x \rightarrow \sqrt{2}} f(x) = \infty \Rightarrow x = \sqrt{2}$ vertical asymptote	1
	$\lim_{x \rightarrow -\sqrt{2}} f(x) = \infty \Rightarrow x = -\sqrt{2}$ vertical asymptote	1
(iv)	Since $f(x) = 0$ has no solutions and also e^{-x^2} is always positive we have: $2-x^2$:  Then $f(x)$: 	1
	$f'(x) = \frac{-2x e^{-(2-x^2)} - (-2x) e^{-(2-x^2)}}{(2-x^2)^2} =$ $\underline{\underline{\frac{2x e^{-(x^2-1)}}{(2-x^2)^2}}}$	2
	Setter's initials V.S.	Checker's initials DTP
		Page number 1

	EXAMINATION QUESTIONS/SOLUTIONS 2009-2010	Course
Question		EE 1(1)
C5	Solutions	Marks & seen/unseen
Parts		
(iv)	$f'(x) = 0 \Rightarrow x(x^2 - 1) = 0 \Rightarrow x = 0, \pm 1$	2
continued		
(v)	We should find the sign of $f''(x)$ at $x = 0, \pm 1$: $f''(x) = \frac{2(2-x^2)^2 e^{-x} (-2x(4x^3-x) + 3x^2 - 1) + 8x(2-x^2)(x^3-x)e^{-x}}{(2-x^2)^4}$ $= \frac{2e^{-x}(2x^6 - 5x^4 + 7x^2 - 2)}{(2-x^2)^3}$	2
	$f''(0) = \frac{2 \times -2}{8} = -\frac{1}{2} < 0 \Rightarrow x=0$ maximum	1
	$f''(1) = f''(-1) = \frac{2e^{-1}(2-5+7-2)}{1} = 4e^{-1} > 0 \Rightarrow$	1
(vi)	$f(0) = \frac{1}{2}$ $f(1) = f(-1) = e^{-1} \approx 0.37$	1
		3
	Setter's initials V.S.	Page number 2
	Checker's initials DTP	

	EXAMINATION QUESTIONS/SOLUTIONS 2009-2010	Course Core EE I (1)
Question C6	ANSWER	Marks & seen/unseen
Parts (a)	<p><u>Note</u> Any reasonable method accepted.</p> <p>Normal to $x+y+z=1$ is $\underline{n}_1 = (1, 1, 1)$</p> <p>" " $2x-y+3z=0$ is $\underline{n}_2 = (2, -1, 3)$</p> <p>Direction of line is $\underline{n}_1 \times \underline{n}_2 = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 1 \\ 2 & -1 & 3 \end{vmatrix}$</p> $= (4, -1, -3)$	1 1 2
	<p>Equation is $\underline{x} = \underline{x}_0 + t(4, -1, -3)$</p> <p>where \underline{x}_0 is any point on the line</p> <p>Find \underline{x}_0: Put $x=0$ in eqns of the planes</p> $\begin{cases} y+z=1 \\ -y+3z=0 \end{cases} \Rightarrow \underline{x}_0 = (0, \frac{3}{4}, \frac{1}{4})$	2 2 2
	<p>Plane perpendicular to the line has normal $\underline{n} = (4, -1, -3)$</p> <p>$\Rightarrow$ equation is $4x - y - 3z = \text{const.}$</p> <p>Since plane passes through the origin the equation is $4x - y - 3z = 0$</p>	4
	Setter's initials DTP	Checker's initials JWB
		Page number 1

	EXAMINATION QUESTIONS/SOLUTIONS 2009-2010	Course
Question	ANSWER	CORE SEI (1)
Parts	Marks & seen/unseen	
(b)	 $\cos \theta = \frac{(\underline{a} \cdot \underline{b})}{ \underline{a} \underline{b} }$ $\underline{a}_p = \underline{a} \cos \theta \frac{\underline{b}}{ \underline{b} } = \frac{(\underline{a} \cdot \underline{b})}{ \underline{b} ^2} \underline{b}$ <p style="text-align: right;">4</p>	
	<p>To show $\underline{a} - \underline{a}_p \perp \underline{b}$, compute</p> $(\underline{a} - \underline{a}_p) \cdot \underline{b} = \underline{a} \cdot \underline{b} - \frac{(\underline{a} \cdot \underline{b})}{ \underline{b} ^2} (\underline{b} \cdot \underline{b}) = 0$ <p style="text-align: right;">2</p>	
	<p>Here $\underline{a} = (1, 2, 3)$, $\underline{b} = (1, 1, 1)$</p> $\underline{a}_p = \frac{6}{3} (1, 1, 1) = (2, 2, 2)$ <p style="text-align: right;">1</p>	
	<p>other vector is $\underline{a} - \underline{a}_p = (-1, 0, 1)$</p> <p style="text-align: right;">1</p>	
	<p>(Check $(-1, 0, 1) \cdot (1, 1, 1) = 0$)</p> <p style="text-align: right;">2</p>	
	<p>\Rightarrow required decomposition is</p> $(1, 2, 3) = (2, 2, 2) + (-1, 0, 1)$	
	<p>Setter's initials</p> <p>DTP</p>	<p>Checker's initials</p> <p>DWB</p>
		<p>Page number</p> <p>2</p>

EXAMINATION QUESTIONS/SOLUTIONS 2009-2010		Course										
Question	Answer	Marks & seen/unseen										
7												
Parts	<p>(i) $A^2 = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ -1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ -1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} -3 & 12 & 20 \\ -2 & 5 & 8 \\ -4 & 6 & 9 \end{pmatrix}$</p> <p>$A^3 = \begin{pmatrix} -3 & 12 & 20 \\ -2 & 5 & 8 \\ -4 & 6 & 9 \end{pmatrix} \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ -1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} -23 & 46 & 72 \\ -10 & 17 & 26 \\ -13 & 16 & 23 \end{pmatrix}$</p> <p>$A^3 - 5A^2 + 7A + I = \begin{pmatrix} -23 + 15 + 7 + 1 & 46 - 60 + 4 & 72 - 100 + 28 \\ -10 + 10 + 0 & 17 - 25 + 7 + 1 & 26 - 40 + 14 \\ -13 + 20 - 7 & 16 - 30 + 4 & 23 - 45 + 21 + 1 \end{pmatrix} = 0.$</p> <p>(ii) $A = 1 \left \begin{matrix} 1 & 2 \\ 2 & 3 \end{matrix} \right - \left \begin{matrix} 2 & 4 \\ 1 & 2 \end{matrix} \right = -1 \neq 0 \Rightarrow A^{-1} \text{ exists}$</p> <p>$A^{-1}(A^3 - 5A^2 + 7A + I) = 0 \Rightarrow A^{-1} = -A^2 + SA - 7I$</p> <p>$A^{-1} = \begin{pmatrix} 3+5-7 & -12+10 & -20+20 \\ 2+0 & -5+5-7 & -8+10 \\ 4-5 & -6+10 & -9+15-7 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 0 \\ 2 & -7 & 2 \\ -1 & 4 & -1 \end{pmatrix}$</p> <p>(iii)</p> <table style="margin-left: auto; margin-right: auto;"> <tr> <td>$\begin{array}{ccc ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 1 \\ -1 & 2 & 3 & 1 & 0 & 0 \end{array}$</td> <td>$\rightarrow$</td> <td>$\begin{array}{ccc ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 4 & 7 & 1 & 0 & 1 \end{array}$</td> <td>$\rightarrow$</td> <td>$\begin{array}{ccc ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}$</td> </tr> </table> <table style="margin-left: auto; margin-right: auto;"> <tr> <td>$\begin{array}{ccc ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 4 & -1 \end{array}$</td> <td>$\rightarrow$</td> <td>$\begin{array}{ccc ccc} 1 & 2 & 0 & 1 & 5 & -16 \\ 0 & 1 & 0 & 2 & -7 & 2 \\ 0 & 0 & 1 & -1 & 4 & -1 \end{array}$</td> <td>$\rightarrow$</td> <td>$\begin{array}{ccc ccc} 1 & 0 & 0 & 1 & 1 & -2 \\ 0 & 1 & 0 & 2 & -7 & 2 \\ 0 & 0 & 1 & -1 & 4 & -1 \end{array}$</td> </tr> </table>	$\begin{array}{ccc ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 1 \\ -1 & 2 & 3 & 1 & 0 & 0 \end{array}$	\rightarrow	$\begin{array}{ccc ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 4 & 7 & 1 & 0 & 1 \end{array}$	\rightarrow	$\begin{array}{ccc ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}$	$\begin{array}{ccc ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 4 & -1 \end{array}$	\rightarrow	$\begin{array}{ccc ccc} 1 & 2 & 0 & 1 & 5 & -16 \\ 0 & 1 & 0 & 2 & -7 & 2 \\ 0 & 0 & 1 & -1 & 4 & -1 \end{array}$	\rightarrow	$\begin{array}{ccc ccc} 1 & 0 & 0 & 1 & 1 & -2 \\ 0 & 1 & 0 & 2 & -7 & 2 \\ 0 & 0 & 1 & -1 & 4 & -1 \end{array}$	2 2 2 1 2 2 4 5 20
$\begin{array}{ccc ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 1 \\ -1 & 2 & 3 & 1 & 0 & 0 \end{array}$	\rightarrow	$\begin{array}{ccc ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 4 & 7 & 1 & 0 & 1 \end{array}$	\rightarrow	$\begin{array}{ccc ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}$								
$\begin{array}{ccc ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 4 & -1 \end{array}$	\rightarrow	$\begin{array}{ccc ccc} 1 & 2 & 0 & 1 & 5 & -16 \\ 0 & 1 & 0 & 2 & -7 & 2 \\ 0 & 0 & 1 & -1 & 4 & -1 \end{array}$	\rightarrow	$\begin{array}{ccc ccc} 1 & 0 & 0 & 1 & 1 & -2 \\ 0 & 1 & 0 & 2 & -7 & 2 \\ 0 & 0 & 1 & -1 & 4 & -1 \end{array}$								
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	<u>EXAMINATION QUESTIONS/SOLUTIONS 2009-2010</u>	Course 1ST YR ENG CORE
Question C9		EE I(1) Marks & seen/unseen
Parts (i)	<p>Homogeneous. Let $y(x) = xu(x)$; then $y' = u + xu'$ by product rule.</p> <p>This gives $u + xu' = \frac{4-u}{1+2u}$</p> <p>and thus $x \frac{du}{dx} = \frac{(4-u) - (u+2u^2)}{1+2u}$ $= -\frac{2(u^2+u-2)}{2u+1}$</p> <p>Separating the variables,</p> $\int \frac{2u+1}{u^2+u-2} du = -\int \frac{2}{x} dx$ $\therefore \ln(u^2+u-2) = \ln(x^{-2}) + C$ <p>hence $u^2+u-2 = \frac{A}{x^2}$</p> <p>and $y^2 + xy - 2x^2 = A$.</p> <p>From quadratic formula (or otherwise)</p> $y = \frac{-x \pm \sqrt{x^2 + 8x^2 + 4A}}{2}$ $= \frac{-x \pm \sqrt{9x^2 + 4A}}{2}$ <p>$y(0) = 1$ implies +ve branch, $A = 1$ and we have</p> $y = \frac{-x + \sqrt{9x^2 + 4}}{2}$ <p>[Accept any valid method. Penalise selecting wrong branch or both branches: 2 marks]</p> <p>[continues...]</p>	4 4 4
	Setter's initials <u>AB</u>	Checker's initials <u>VS.</u>
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	EXAMINATION QUESTIONS/SOLUTIONS 2009-2010	Course 1ST YE ENG CORE
Question C9 (cont)		EE I(C) Marks & seen/unseen
Parts (ii)	<p>Lclear. Rewrite as</p> $\frac{dy}{dx} + y \tan x = \sec x.$ <p>$\int \sec x dx = \ln \sec x + C \therefore$ suitable integrating factor is $\sec x$</p> <p>Rewrite as</p> $\sec x \frac{dy}{dx} + y \sec x \tan x = \sec^2 x$ <p>Exact LHS:</p> $\frac{d}{dx} [y \sec x] = \sec^2 x$ $\therefore y \sec x = \tan x + C, C \text{ constant.}$ <p>giving $y = \underline{\sin x + C \cos x}$</p> <p>$x = \pi/2 \Rightarrow y = \sin \pi/2 + C \cos \pi/2$ $= 1 + 0C = 1. \checkmark$ $(\text{for all values of } C)$</p> <p>[Accept any valid method.]</p>	3 3 2 20
	Setter's initials PDR	Checker's initials VS.
		Page number 2 of 2

	<u>EXAMINATION QUESTIONS/SOLUTIONS 2009-2010</u>	Course 1ST YR ENG CORE
Question C10		<u>E I(1)</u> Marks & seen/unseen
Parts	<p>Aux eqn $\lambda^2 + k\lambda + 1 = 0$, roots $\lambda = -k \pm \sqrt{k^2 - 4}$</p> <p>(i) $k=2.5$: $\lambda = \frac{-2.5 \pm \sqrt{1.25}}{2} = -0.5, -2.0$</p> <p>gen'l soln: $x = Ae^{-0.5t} + Be^{-2.0t}$ $\therefore \dot{x} = -0.5Ae^{-0.5t} - 2.0Be^{-2.0t}$</p> <p>$x(0) = 0.24 \Rightarrow A + B = 0.24$</p> <p>$\dot{x}(0) = 0 \Rightarrow A = -4B$</p> <p>Hence $-3B = 0.24$, giving $B = -0.08$, $A = 0.32$ and $x = 0.32e^{-0.5t} - 0.08e^{-2.0t}$</p> <p>(ii) $k=2.0$: $\lambda = -1.0$, repeated gen'l soln: $x = e^{-t}(At + B)$ $\therefore \dot{x} = e^{-t}(-At - B + A)$</p> <p>$x(0) = 0.24 \Rightarrow B = 0.24$</p> <p>$\dot{x}(0) = 0 \Rightarrow A = B$ and $x = 0.24e^{-t}(t + 1)$</p> <p>(iii) $k=1.2$: $\lambda = \frac{-1.2 \pm \sqrt{-2.56}}{2} = -0.6 \pm 0.8i$</p> <p>gen'l soln: $x = e^{-0.6t}(A \cos 0.8t + B \sin 0.8t)$ $\therefore \dot{x} = e^{-0.6t}(-0.6A \cos 0.8t - 0.6B \sin 0.8t + 0.8A \sin 0.8t + 0.8B \cos 0.8t)$</p> <p>$x(0) = 0.24 \Rightarrow A = 0.24$</p> <p>$\dot{x}(0) = 0 \Rightarrow B = 3/4A = 0.18$ and $x = e^{-0.6t}(0.24 \cos 0.8t + 0.18 \sin 0.8t)$</p> <p>Last bit: comp. fn. is $x = e^{-t}(At + B)$ Seek partic. soln in the form $x = Mt^2 e^{-t}$ $\dot{x} = e^{-t}(-Mt^2 + 2Mt)$ $\ddot{x} = e^{-t}(Mt^2 - 4Mt + 2M)$ [continues...]</p>	2 4 4
	Setter's initials <u>PJR</u>	Checker's initials <u>DJR</u>
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	EXAMINATION QUESTIONS/SOLUTIONS 2009-2010	Course Core I EE I(1)	
Question	C 10	Marks & seen/unseen	
Parts	$\ddot{x} + 2\dot{x} + x = e^{-t}(Mt^2 - 4Mt + 2M - 2M\epsilon^2 + 4M\epsilon + M\epsilon^2)$ $\Rightarrow 2Me^{-t}$ and therefore $M = \frac{1}{2}$. <u>gen'l soln</u> $x = e^{-t}\left(\frac{1}{2}\epsilon^2 + A\epsilon + B\right)$ $\dot{x} = e^{-t}\left(-\frac{1}{2}\epsilon^2 - A\epsilon - B + \epsilon + A\right)$ $x(0) = 0.24 \Rightarrow B = 0.24$ $\dot{x}(0) = 0.48 \Rightarrow A = B$ and <u>$x = e^{-t}\left(\frac{1}{2}\epsilon^2 + 0.24\epsilon + 0.24\right)$</u>	3 3 3 20	
	Setter's initials	Checker's initials	
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	EXAMINATION QUESTIONS/SOLUTIONS 2009-2010	Course
Question	Answer	Marks & seen/unseen
Parts	$y''(x) = -(4x^2 + 1) y(x)$ Leibnitz for $m \geq 2 \Rightarrow$ $y^{(m+2)}(x) = -[(4x^2 + 1)y(x)]^{(m)}$ $= -[(4x^2 + 1)y^{(m)}(x) + m \cdot 8x \cdot y^{(m-1)}(x)$ $+ \frac{m(m-1)}{2} \cdot 8 \cdot y^{(m-2)}(x)]$ $\Rightarrow y^{(m+2)}(0) = -[y^{(m)}(0) + 4m(m-1)y^{(m-2)}(0)]$ for $m \geq 2$ Leibnitz for $m=1 \Rightarrow$ $y'''(x) = -[(4x^2 + 1)y'(x) + 8xy(x)]$ $\Rightarrow y'''(0) = -y'(0)$ $y(0)=0, y'(0)=1 \Rightarrow y''(0) = -y(0) = 0$ and $y'''(0) = -1.$ $y(0)=y''(0) \Rightarrow y^{(2m)}(0)=0 \quad m \geq 2$ $\Rightarrow y(x) = \sum_{m=0}^{\infty} c_m x^{2m+1}$ with $c_m = \frac{y^{(2m+1)}(0)}{(2m+1)!}$ $c_0 = y'(0) = 1, \quad c_1 = \frac{y^{(3)}(0)}{3!} = -\frac{1}{6}$ $c_2 = \frac{y^{(5)}(0)}{5!} = -\frac{1}{120} [y^{(3)}(0) + 4 \cdot 3 \cdot 2 \cdot y'(0)] = -\frac{23}{120}$ $y(x) = x - \frac{1}{6}x^3 - \frac{23}{120}x^5 \dots$	Seen Similar
	Setter's initials JWB	Checker's initials Df
		Page number