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IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE
UNIVERSITY OF LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2001

MSc and EEE/ISE PART IV: M.Eng. and ACGI

DESIGN OF LINEAR MULTIVARIABLE CONTROL SYSTEMS

Thursday, 3 May 10:00 am

There are SIX questions on this paper.

Answer FOUR questions.

Time allowed: 3:00 hours

Examiners: Jaimoukha,I.M. and Weiss,G.

Corrected Copy

Special instructions for invigilators: None

Information for candidates: None

1. Let the transfer matrix $G(s)$ have a state space realisation

$$G(s) \stackrel{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] := \left[\begin{array}{cccc|cc} 1 & 2 & 0 & 0 & 1 & 2 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 3 & 4 \\ 0 & 0 & 0 & 5 & 0 & 0 \\ \hline 2 & 3 & 0 & 0 & 1 & 0 \\ 1 & 4 & 0 & 0 & 0 & 1 \end{array}\right].$$

(a) Find the uncontrollable and/or unobservable modes and determine whether the realisation is detectable and stabilisable. [4]

(b) Suppose that $K \in \mathcal{R}^{2\times 4}$ and $L \in \mathcal{R}^{4\times 2}$ are arbitrary matrices. Determine two of the eigenvalues of $A - BK$ and two of the eigenvalues of $A - LC$. Explain how you arrive at your answer. [4]

(c) Find a minimal realisation for $G(s)$. [4]

(d) Find the McMillan form of $G(s)$ and determine the pole and zero polynomials. What is the McMillan degree of $G(s)$? [9]

(e) Determine the system zeros, indicating the type of each zero. [4]

2. (a) Define internal stability for the feedback loop in Figure 1, and derive necessary and sufficient conditions for which this loop is internally stable. **[6]**

(b) Suppose that $G(s)$ is given by

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+2} \\ 0 & \frac{1}{s+1} \end{bmatrix}.$$

Give a parameterisation of all internally stabilising controllers for $G(s)$. **[7]**

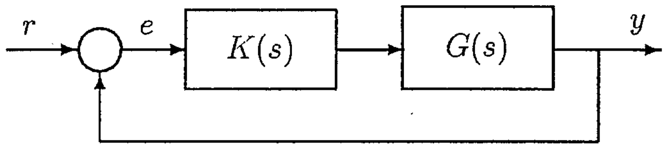

Figure 1

(c) Let $G(s)$ be as given in Part (b) and let $S(s)$ denote the transfer matrix from r to e in Figure 1. Suppose now that an uncertainty on $G(s)$ is introduced as shown in Figure 2, with $\Delta(s)$ a stable transfer matrix satisfying

$$\|\Delta(j\omega)\| \leq |1 + j\omega|^2, \ \forall \omega \in \mathcal{R},$$

Using the answer to Part (b) design a controller $K(s)$ which internally stabilises the feedback loop in Figure 2 for all $\Delta(s)$, and such that $\|S(0)\| \leq 0.1$. **[12]**

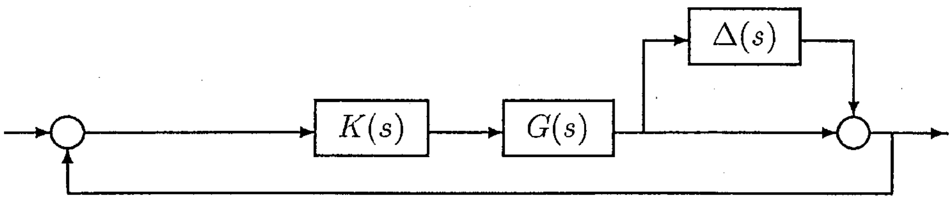

Figure 2

3. (a) State Gershgorin's Theorem concerning the location of the eigenvalues of a complex matrix. Let

$$A = \begin{bmatrix} -4 & 0 & 0 & \delta_1 \\ 0 & -3 & \delta_2 & 0 \\ 0 & \delta_2 & -2 & 0 \\ \delta_1 & 0 & 0 & -1 \end{bmatrix} \in \mathcal{R}^{4\times 4}.$$

Using Gershgorin's Theorem, give a range of values of δ_1 and δ_2 for which A is guaranteed to have all its eigenvalues in the left half of the complex plane.

[7]

(b) For the feedback loop in Figure 3, let K be a constant diagonal matrix. State a Nyquist type stability criterion in terms of the direct Nyquist array of $G(s)$.

[6]

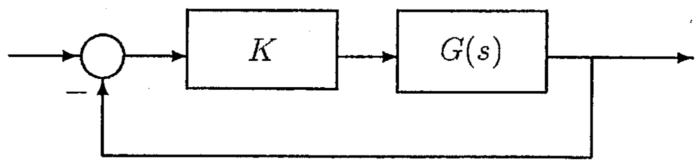

Figure 3

(c) Consider the feedback loop in Figure 4. Here

$$G(s) = \begin{bmatrix} \frac{5}{s-1} & 0 \\ 0 & \frac{5}{s+1} \end{bmatrix}, \qquad K = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix},$$

and $\Delta(s)$ is a transfer matrix representing a structured uncertainty of the form

$$\Delta(s) = \begin{bmatrix} 0 & \delta_{12}(s) \\ \delta_{21}(s) & 0 \end{bmatrix}$$

such that $\|\Delta\|_\infty < 1$. Use the answer to Part (b) to derive the range of values of k_1 and k_2 for which the closed-loop system is guaranteed to be internally stable.

[12]

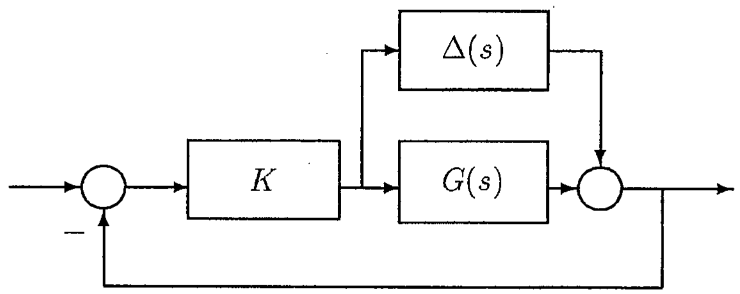

Figure 4

4. Figure 5 illustrates the implementation of the control law $u(t) = -Kx(t) + r(t)$ which (when $r(t) = 0$) minimises

$$J(x_0, u) = \int_0^\infty \|Cx(t)\|^2 + \|u(t)\|^2 \, dt$$

subject to $\dot{x}(t) = Ax(t) + Bu(t)$, $x(0) = x_0$ where $K = B'P$ and $P = P'$ is the unique stabilising solution of the Riccati equation $A'P + PA - PBB'P + C'C = 0$. Here, T' denotes the complex conjugate transpose of T. Assume that the triple (A, B, C) is minimal. Let $F(s) = (sI - A)^{-1}B$, $G(s) = C(sI - A)^{-1}B$ and $L(s) = I + KF(s)$.

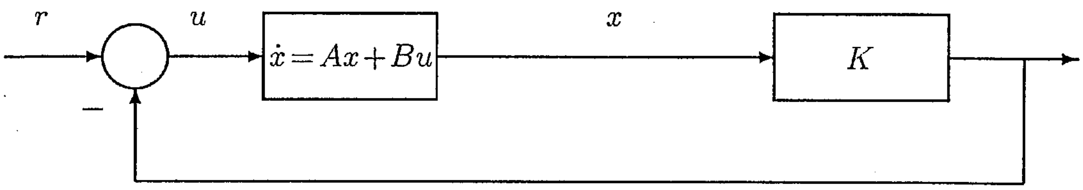

Figure 5

(a) Show that

$$L(j\omega)'L(j\omega) = I + G(j\omega)'G(j\omega).$$

[8]

(b) Suppose that $G(s) = \frac{4}{s+3}$. Derive a balanced, minimal state-space realisation $G(s) = C(sI - A)^{-1}B$ and evaluate K for this realisation. [5]

(c) Let $G(s)$ and K be as in Part (b). Suppose a stable uncertainty Δ is introduced as shown in Figure 6. Derive the maximal stability radius (using the $\mathcal{H}_\infty$-norm as a measure) for Δ that can be deduced from Part (a) and the small gain theorem.

[12]

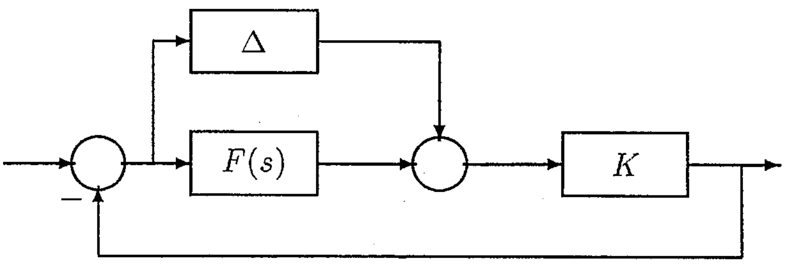

Figure 6

5. Consider the feedback loop shown in Figure 7 where $G(s)$ represents a plant model and $K(s)$ represents an internally stabilising compensator. Suppose that

$$G(s) \stackrel{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cc|cc} -4 & -4 & 2 & 2 \\ -4 & -5 & 1.2 & 1.6 \\ \hline 2 & 1.2 & 0 & 0 \\ 2 & 1.6 & 0 & 0 \end{array}\right] \in \mathcal{RH}_\infty.$$

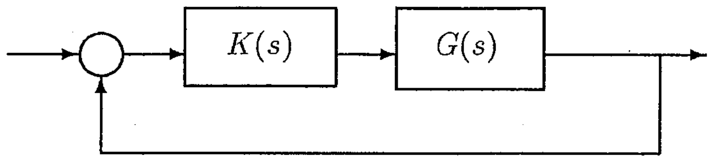

Figure 7

(a) Show that the given realisation for $G(s)$ is balanced and evaluate the Hankel singular values of $G(s)$.

[6]

(b) Design a family of first order internally stabilising controllers $K(s)$ for $G(s)$ using the following procedure:

i. Replace $G(s)$ in Figure 7 by a first order approximation $G_r(s)$ and give an upper bound on $\|G(s) - G_r(s)\|_\infty$.

ii. Find the set of all internally stabilising controllers for the new feedback loop.

iii. Using the small gain theorem and the bound on $\|G(s) - G_r(s)\|_\infty$, choose a family of first order internally stabilising controllers for the feedback loop of Figure 7. **[12]**

(c) Design a non-dynamic internally stabilising controller K for $G(s)$ such that $\|K\| \geq 1$. **[7]**

(Hint: Use the procedure outlined in Part (b) and the fact that $G_r(s)$ is rank deficient.)

6. Consider the feedback configuration in Figure 8. Here, $G(s)$ is a nominal plant model and $K(s)$ is a compensator. The signals $r(s)$ and $n(s)$ are the Laplace transforms of the reference and sensor noise, respectively. The design specifications are to synthesise a compensator $K(s)$ such that the feedback loop is internally stable and:

- For good tracking, it is required that, when $n(s) = 0$,
$$\frac{\|e(j\omega)\|}{\|r(j\omega)\|} < |w_1(j\omega)^{-1}|, \forall\omega.$$
- For good sensor noise attenuation it is required that, when $r(s) = 0$,
$$\frac{\|y(j\omega)\|}{\|n(j\omega)\|} < |w_2(j\omega)^{-1}|, \forall\omega$$

where $w_1(s)$ is a low pass and $w_2(s)$ is a high pass filter.

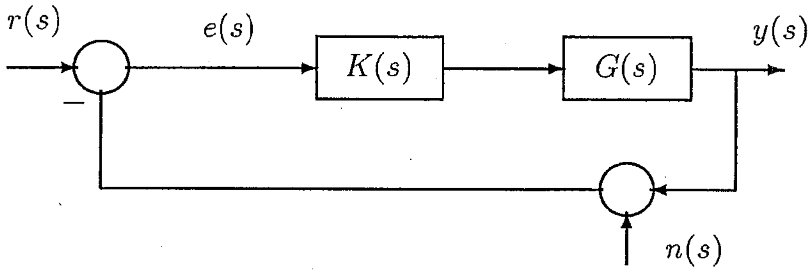

Figure 8

(a) Derive $\mathcal{H}_\infty$-norm bounds, in terms of $G(s), K(s), w_1(s)$ and $w_2(s)$ that are sufficient to achieve the design specifications. [8]

(b) Derive a generalised regulator formulation of the design problem that captures the sufficient conditions in Part (a). [8]

(c) Assume that a compensator $K(s)$ achieves the design specifications in Part (a). Suppose now that an uncertainty $\Delta(s)$ is introduced as shown in Figure 9. Assume that $\Delta(s)$ is a stable transfer matrix. Derive the maximal stability radius for $\|\Delta(j\omega)\|, \forall\omega$. [9]

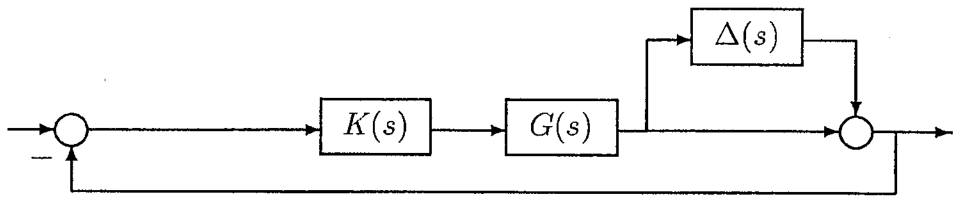

Figure 9

[END]

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1. (a) Since $[A-sI \;\; B]$ loses rank for $s=3$ and $s=5$, they are uncontrollable modes, and since $[A^T - sI \;\; C^T]$ loses rank for $s=4$ and $s=5$, they are unobservable modes. Since the uncontrollable mode are unstable, the realisation is not stabilisable, and since the unobservable modes are unstable, the realisation is not detectable. [4]

(b) Since the modes $\lambda = 3$ and $\lambda = 5$ are uncontrollable, they cannot be assigned via state feedback and so they are eigenvalues of $A - BK$. Similarly, since $\lambda = 4$ and $\lambda = 5$ are unobservable modes, they cannot be assigned via output injection and so they are eigenvalues of $A - LC$. [4]

(c) By removing the uncontrollable and/or unobservable modes we get the minimal realisation

$$G(s) \stackrel{s}{=} \left[\begin{array}{c|cc} 1 & 1 & 2 \\ \hline 2 & 1 & 0 \\ 1 & 0 & 1 \end{array}\right] = \begin{bmatrix} \frac{s+1}{s-1} & \frac{4}{s-1} \\ \frac{1}{s-1} & \frac{s+1}{s-1} \end{bmatrix} = \frac{1}{s-1}\begin{bmatrix} s+1 & 4 \\ 1 & s+1 \end{bmatrix}.$$

[4]

(d) By performing the following elementary operations: (1) $r_1 \leftrightarrow r_2$, (2) $r_2 := r_2 - (s+1)r_1$, (3) $c_2 := c_2 - (s+1)c_1$, (4) $c_2 = -c_2$, the McMillan form of $G(s)$ is given by,

$$G(s) = \begin{bmatrix} s+1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{s-1} & 0 \\ 0 & s+3 \end{bmatrix} \begin{bmatrix} 1 & s+1 \\ 0 & -1 \end{bmatrix} =: L(s)M(s)R(s),$$

where $L(s)$ and $R(s)$ are unimodular.
The pole and zero polynomials are given by

$$p(s) = s-1, \qquad z(s) = s+3$$

respectively. The McMillan degree is the degree of the pole polynomial and is therefore equal to 1. [9]

(e) Since $s=3$ and $s=5$ are uncontrollable modes, they are input decoupling zeros. Since $s=4$ and $s=5$ are unobservable modes, they are output decoupling zeros. So, $s=5$ is an input/output decoupling zero. It follows from Part (d) that the system has a transmission zero at $s=-3$. [4]

2. (a) Inject a signal d in between $G(s)$ and $K(s)$ and call the input to $G(s)$ u. The loop is internally stable if and only if the transfer matrix from $\begin{bmatrix} d \\ r \end{bmatrix}$ to $\begin{bmatrix} u \\ e \end{bmatrix}$ is stable (no poles in the closed right half plane). Since

$$\begin{bmatrix} d \\ r \end{bmatrix} = \begin{bmatrix} I & -K \\ -G & I \end{bmatrix} \begin{bmatrix} u \\ e \end{bmatrix} =: T(s) \begin{bmatrix} u \\ e \end{bmatrix}$$

the loop is internally stable if and only if $T^{-1}(s)$ is stable.

[6]

(b) Since $G(s)$ is stable, we proceed as follows. Note that

$$\begin{bmatrix} I & -K \\ -G & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ -G & I \end{bmatrix} \begin{bmatrix} I & -K \\ 0 & I-GK \end{bmatrix}.$$

Hence

$$\begin{bmatrix} I & -K \\ -G & I \end{bmatrix}^{-1} = \begin{bmatrix} I & -K \\ 0 & I-GK \end{bmatrix}^{-1} \begin{bmatrix} I & 0 \\ -G & I \end{bmatrix}^{-1} = \begin{bmatrix} I & K(I-GK)^{-1} \\ 0 & (I-GK)^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ G & I \end{bmatrix}.$$

Finally, since $(I-GK)^{-1} = I + GK(I-GK)^{-1}$, it follows that if G is stable, then the loop is internally stable if and only if $Q := K(I-GK)^{-1}$ is stable. Rearranging terms shows that K is internally stabilising if and only if $K = Q(I+GQ)^{-1}$ for some stable Q.

[7]

(c) Since K is required to be internally stabilising, $K = Q(I+GQ)^{-1}$ for some stable Q from Part (b). We search for a stable Q to satisfy the design requirements. Let the input to Δ be ϵ while the output from Δ be δ. Then a simple calculation shows that $\epsilon = C\delta$ where $C = (I-GK)^{-1}GK$ is the complementary sensitivity which is stable. Now

$$\begin{aligned} S &= (I-GK)^{-1} = I + GK(I-GK)^{-1} = I + GQ, \\ C &= GK(I-GK)^{-1} = GQ. \end{aligned}$$

The small gain theorem implies that for K to stabilise the loop in Figure 2 for all Δ, we must have

$$\|G(j\omega)Q(j\omega)\| < \frac{1}{|1+j\omega|^2}$$

so we choose

$$Q(s) = h\frac{1}{(s+1)^2}\, G^{-1}(s) = h \begin{bmatrix} \frac{1}{s+1} & \frac{-1}{s+2} \\ 0 & \frac{1}{s+1} \end{bmatrix}$$

where $-1 < h < 1$ is to be determined. Since

$$S(0) = I + G(0)Q(0) = (1+h)I_2$$

it follows that any $-1 < h \leq -0.9$ will satisfy the design specifications. **[12]**

3. (a) Gershgorin's Theorem: Let A be any $n \times n$ complex matrix. The eigenvalues of A lie in $\mathcal{D}_1$, the union of the discs,

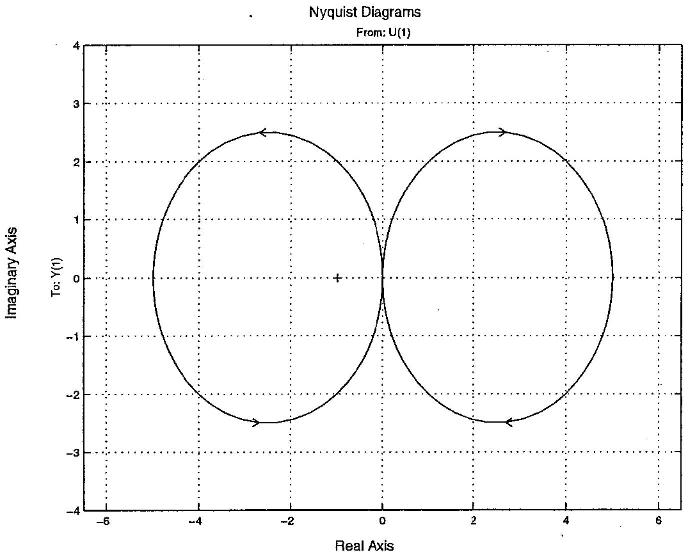
$$|l - m_{ii}| \leq \sum_{j \neq i} |m_{ij}|, \quad i = 1, \ldots, n,$$

and they also lie in $\mathcal{D}_2$, the union of the discs,

$$|l - m_{ii}| \leq \sum_{j \neq i} |m_{ji}|, \quad i = 1, \ldots, n.$$

For the given A, taking column sums, the Gershgorin discs are centred on $-4, -3, -2$ and -1 and have radii $|\delta_1|, |\delta_2|, |\delta_2|$ and $|\delta_1|$, respectively. Thus $\mathcal{D}_2$ lies in the closed left half plane if $-1 < \delta_1 < 1$ and $-2 < \delta_2 < 2$. **[7]**

(b) Let $G(s)$ have P closed right half plane poles. Assume that $K^{-1} + G(s)$ is diagonally dominant, that is, $|\frac{1}{k_i} + G_{ii}(s)| \geq \sum_{j \neq i} |G_{ji}(s)|$, for all i and for all s on the Nyquist contour. Let the ith Gershgorin band of $G(s)$ encircle the point $-\frac{1}{k_i}$ a total of N_i times anticlockwise. Then the loop is internally stable if and only if $\sum_i N_i = P$. **[6]**



(c) For the given $G(s)$, $P = 1$. The Nyquist plots for G_{11} (left circle) and G_{22} (right circle) are shown above. Since $||\Delta||_\infty < 1$, it follows that $||\delta_{12}||_\infty < 1$ and $||\delta_{21}||_\infty < 1$. Thus the Gershgorin circles all have radius 1 at most. To guarantee stability, it is sufficient that the number of anticlockwise encirclements by the Gershgorin bands of G_{11} of $-\frac{1}{k_1}$ and the number of anticlockwise encirclements by the Gershgorin bands of G_{22} of $-\frac{1}{k_2}$ is 1. Thus the closed-loop system is stable if $-4 < -\frac{1}{k_1} < -1$ (equivalently, if $0.25 < k_1 < 1$) and if $-\frac{1}{k_2} > 6$ (equivalently, if $k_2 > -\frac{1}{6}$) or $-\frac{1}{k_2} < -1$ (equivalently, if $k_2 < 1$). **[12]**

4. (a) By direct evaluation, $L(j\omega)'L(j\omega) =$

$$I+K(j\omega I-A)^{-1}B+B'(-j\omega I-A')^{-1}K'+B'(-j\omega I-A')^{-1}K'K(j\omega I-A)^{-1}B.$$

But

$$K'K = A'P + PA + C'C = -(-j\omega I - A')P - P(j\omega I - A) + C'C$$

from the Riccati equation. So, $L(j\omega)'L(j\omega)$

$$\begin{aligned} &= I + K(j\omega I - A)^{-1}B + B'(-j\omega I - A')^{-1}K' \\ &\quad +B'(-j\omega I - A')^{-1}[-(-j\omega I - A')P - P(j\omega I - A) + C'C](j\omega I - A)^{-1}B \\ &= I + [K - B'P](j\omega I - A)^{-1}B + B'(-j\omega I - A')^{-1}[K' - PB] \\ &\quad +B'(-j\omega I - A')^{-1}C'C(j\omega I - A)^{-1}B \\ &= I + G(j\omega)'C'CG(j\omega). \end{aligned}$$

[8]

(b) A minimal, balanced state-space realisation of $G(s)$ is given by

$$G(s) \stackrel{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} -3 & 2 \\ \hline 2 & 0 \end{array}\right].$$

The Riccati equation becomes

$$-3P - 3P - 4P^2 + 4 = 0$$

which has a stabilising solution $P = 0.5$. Hence $K = B'P = 1$. [5]

(c) Let ϵ be the input to Δ and δ be the output of Δ. Then

$$\epsilon = -K(\delta + G\epsilon) = -(I + KG)^{-1}K\delta.$$

Using the small gain theorem (since the regulator is stable and the perturbation is assumed stable), the loop is stable if $\|\Delta(I + KG)^{-1}K\|_\infty < 1$. But Part (a) implies that $\underline{\sigma}[I + KG(j\omega)] \geq 1$ which implies $\|(I + KG)^{-1}\|_\infty \leq 1$. Furthermore, $K = 1$ from Part (b). Hence the loop will tolerate perturbations of size (measured in the $\mathcal{H}_\infty$ norm) at least 1 without losing internal stability, since $\|\Delta\|_\infty < 1$ implies that

$$\|\Delta(I + KG)^{-1}K\|_\infty < 1.$$

[12]

5. (a) The realisation of $G(s)$ is balanced if

$$A\Sigma + \Sigma A' + BB' = A'\Sigma + \Sigma A + C'C = 0$$

for $\Sigma = \text{diag}(\sigma_1, \sigma_2) > 0$ and where the $\sigma_i's$ are the Hankel singular values of $G(s)$. A simple calculation gives $\Sigma = \text{diag}(1, 0.4)$. **[6]**

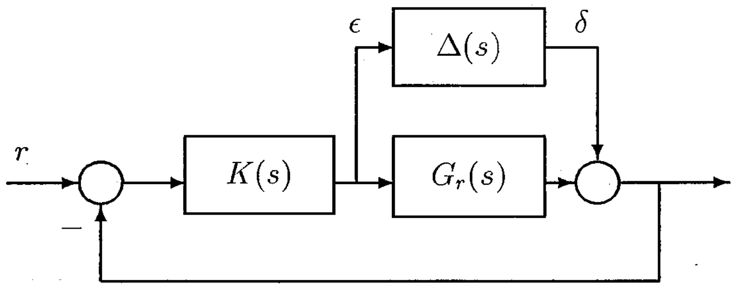
(b) Let

$$G_r(s) \stackrel{s}{=} \left[\begin{array}{c|cc} -4 & 2 & 2 \\ \hline 2 & 0 & 0 \\ 2 & 0 & 0 \end{array}\right] = \frac{4}{s+4}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

denote a first order balanced truncation of $G(s)$. Then $G_r(s) = G(s) + \Delta(s)$ where

$$\|\Delta\|_\infty \leq 2\sum_{i=2}^{2}\sigma_i = 0.8.$$

Then replacing $G(s)$ by $G_r(s)$ in the loop of Figure 7 is equivalent to:



Now $\epsilon = -K(I + G_rK)^{-1}\delta$ and so the small gain theorem implies that the loop is stable if $\|\Delta K(I - G_rK)^{-1}\|_\infty < 1$ which is guaranteed if $\|K(I - G_rK)^{-1}\|_\infty < 1.25$ since $\|\Delta\|_\infty \leq 0.8$. However, the set of all internally stabilising controllers for $G_r(s)$ is given by $K = Q(I + G_rQ)^{-1}$ for stable Q. Furthermore, $K(I - G_rK)^{-1} = Q$. Thus we can take $Q = qI_2$ where q is constant (to guarantee a first order controller) and $|q| < 1.25$ (to guarantee stabilisation of G). For example, taking $q = -1$ gives

$$K(s) \stackrel{s}{=} \left[\begin{array}{c|cc} -3 & 1 & 1 \\ \hline 1 & -1 & 0 \\ 1 & 0 & -1 \end{array}\right].$$

[12]

(c) Arguing as in Part (b), the set of all internally stabilising controllers for $G_r(s)$ is given by $K = Q(I + G_rQ)^{-1}$ for stable Q. Since G_r has rank 1, we can ensure that K is non-dynamic by choosing non-dynamic Q such that $G_rQ = 0$. A possible choice is

$$Q = q\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},$$

giving $K = Q$. Finally, to guarantee stabilisation of G and to ensure that $\|K\| \geq 1$ we choose q such that $1 \leq \|Q\| = \sqrt{2}|q| < 1.25$, or $\frac{1}{\sqrt{2}} \leq |q| < \frac{1.25}{\sqrt{2}}$.

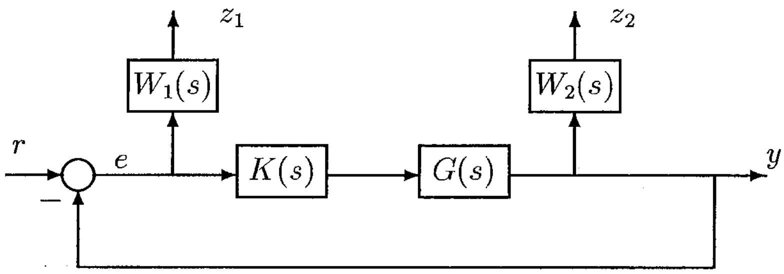
[7]

6. (a) It is clear that we require K to be internally stabilising.

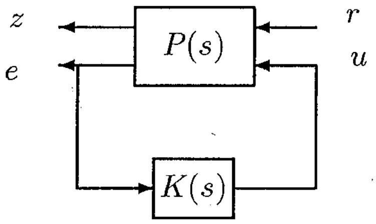
- A simple calculation shows that, when $n(s) = 0$, $e(s) = -S(s)r(s)$ where $S(s) = [I+G(s)K(s)]^{-1}$ is the sensitivity. Thus $\|e(j\omega)\| \leq \|S(j\omega)\|\|r(j\omega)\|$. It follows that a sufficient condition to achieve the first design specification is $\|S(j\omega)\| < |w_1^{-1}(j\omega)|, \forall\omega$ or equivalently $\|W_1S\|_\infty < 1$, where $W_1 = w_1 I$.
- When $r(s) = 0$, a similar calculation shows that $y(s) = -C(s)n(s)$ where $C(s) = G(s)K(s)[I+G(s)K(s)]^{-1}$ is the complementary sensitivity. Thus $\|y(j\omega)\| \leq \|C(j\omega)\|\|n(j\omega)\|$. It follows that a sufficient condition to achieve the second design specification is $\|C(j\omega)\| < |w_2^{-1}(j\omega)|, \forall\omega$ or equivalently $\|W_2C\|_\infty < 1$, where $W_2 = w_2 I$.

Thus, to satisfy both design requirements, it is sufficient (but not necessary) that $\left\| \begin{bmatrix} W_1 S \\ W_2 C \end{bmatrix} \right\|_\infty < 1$. [8]

(b) The design specifications reduce to the requirement that the transfer matrix from r to $z = [z_1^T \ z_2^T]^T$ in the following diagram has $\mathcal{H}_\infty$-norm less than 1.



The corresponding generalised regulator formulation is to find an internally stabilising K such that $\|\mathcal{F}_l(P,K)\|_\infty < 1$:



$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \left[\begin{array}{c|c} W_1 & -W_1G \\ 0 & W_2G \\ \hline I & -G \end{array}\right].$$

[8]

(c) Let the input to Δ be ϵ while the output from Δ be δ. Then $\epsilon = -C\delta$ where $C = (I+GK)^{-1}GK$ is the complementary sensitivity which is stable. Using the small gain theorem, closed-loop stability is assured provided that $\|\Delta(j\omega)C(j\omega)\| < 1, \forall\omega$. Since $K(s)$ achieves the design specifications of Part (a), $\|\Delta(j\omega)\| < |w_2^{-1}(j\omega)|, \forall\omega$ gives the maximal stability radius. [9]