IMPERIAL COLLEGE LONDON

E4.24
C1.3
ISE4.21

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2008

MSc and EEE/ISE PART IV: MEng and ACGI

Corrected Copy

Q3 ✓ - check

DISCRETE-TIME SYSTEMS AND COMPUTER CONTROL

Monday, 28 April 2:30 pm

Time allowed: 3:00 hours

There are SIX questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : J.C. Allwright
Second Marker(s) : R.B. Vinter

Special information for Invigilators: None

Information for candidates:

Some useful transforms

f_k	$f^Z(z)$	$f^D(\gamma)$	
$i_k = 0^k$	1	T	
1^k	$\frac{z}{z-1}$	$\frac{1+\gamma T}{\gamma}$	
t_k	$\frac{Tz}{(z-1)^2}$	$\frac{1+\gamma T}{\gamma^2}$	
α^k	$\frac{z}{z-\alpha}$	$\frac{1+\gamma T}{\gamma-\overline{\alpha}}$	where $\overline{\alpha} = \frac{\alpha-1}{T}$
$k\alpha^k$	$\frac{z\alpha}{(z-\alpha)^2}$	$\frac{(1+\gamma T)(1+\overline{\alpha}T)}{T(\gamma-\overline{\alpha})^2}$	

$f(t)$	$f^L(s)$
$1(t)$	$\frac{1}{s}$
t	$\frac{1}{s^2}$
$\exp(\alpha T)$	$\frac{1}{s-\alpha}$

$$G^W(w) = G^Z\left(\frac{\mu+w}{\mu-w}\right) \text{ where } \mu = \frac{2}{T}$$

Some notation

Symbol(s)	Meaning
$'$	transposition of a vector or of a matrix
q	the forward shift operator
$f^Z(z), f^D(\gamma), f^F(j\omega), f^W(w)$	the Z-, Delta-, discrete-time Fourier and W-transforms, respectively, of the sequence $\{f_k\}$
$g^L(s)$	the Laplace transform of $g(t)$
t_k	kT

The Routh Test

Every root of $\alpha_0 w^n + \alpha_1 w^{n-1} + \ldots + \alpha_n = 0$ has strictly negative real part if and only if all $n+1$ entries in the first column of the following Routh-table are non-zero and have the same sign:

$1:$	α_0	α_2	α_4	$\ldots$
$2:$	α_1	α_3	α_5	$\ldots$
$3:$	$\frac{\alpha_1\alpha_2-\alpha_0\alpha_3}{\alpha_1}$	$\frac{\alpha_1\alpha_4-\alpha_0\alpha_5}{\alpha_1}$	$\frac{\alpha_1\alpha_6-\alpha_0\alpha_7}{\alpha_1}$	$\ldots$
$\ldots:$	$\ldots$	$\ldots$	$\ldots$	
$n+1:$	$\ldots$.			

The Jury Test

If $\alpha_n > 0$: every root of $d(z) := \alpha_n z^n + \alpha_{n-1} z^{n-1} + \ldots + \alpha_0 = 0$ has modulus strictly less than one if and only if

$$\text{(i)} \qquad d(1) > 0$$

and

$$\text{(ii)} \qquad d(-1) \begin{cases} > 0, & \text{if } n \text{ is even;} \\ < 0, & \text{if } n \text{ is odd} \end{cases}$$

and

$$\text{(iii)} \qquad |a_0| < a_n, |b_0| > |b_{n-1}|, |c_0| > |c_{n-2}|, \ldots,$$

where the b_i, c_i etc., are determined using the following Jury table

$1:$	a_0	a_1	a_2	$\ldots$	$\ldots$	a_n
$2:$	a_n	a_{n-1}	a_{n-2}	$\ldots$	$\ldots$	a_0
$3:$	b_0	b_1	b_2	$\ldots$	b_{n-1}	
	where $b_i = a_0 a_i - a_n a_{n-i}$					
$4:$	b_{n-1}	b_{n-2}	$\ldots$	b_1	b_0	
$5:$	c_0	c_1	$\ldots$	c_{n-2}		
	where $c_i = b_0 b_i - b_{n-1} b_{n-1-i}$					
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$		
$2n-3:$	$\ldots$.					

If $a_n < 0$: every root of $d(z) := a_n z^n + a_{n-1} z^{n-1} + \ldots a_0 = 0$ has modulus strictly less than one if and only if every root of $\overline{d}(z)$, defined by $\overline{d}(z) = -d(z)$, has modulus strictly less than one and this may be checked by applying the above methodology for the case $a_n > 0$ to $\overline{d}(z)$.

The Questions

In this examination: u_k, e_k, ε_k and y_k are all scalars.

1. (a) By starting with its q-formulation, derive a formula for $y^Z(z)$ for the following non-causal system, where $x_k \in \mathbb{R}^n$:

$$x_{k+1} = Ax_k + bu_{k+2} \colon x_0 = x_o$$
$$y_k = c' x_k.$$ [3]

(b) Consider the standard causal system

$$x_{k+1} = Ax_k + bu_k \colon x_0 = x_o \qquad (1.1)$$
$$y_k = c' x_k.$$

(i) State a formula for $y^Z(z)$ that involves $u^Z(z)$ and x_o. [1]

Suppose

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & \alpha \end{bmatrix}.$$

(ii) By calculating A^2, A^3 and A^4, show that, for all $k \geq 2$, we can write A^k as

$A^k = \alpha^{p(k)} A^2$ for an appropriate function $p(k)$, which should be specified. [5]

(iii) Suppose $u_k = 0$ for all k. Use the result of part (b-i) above to determine a formula for y_k that involves c', A, k and x_o. Hence write $y^Z(z)$ as an infinite series and sum the series to give $y^Z(z) = H^Z(z)x_0$ for an appropriate $H^Z(z)$ which should be specified as a formula that involves only c', powers of α and z together with A and A^2. [5]

(iv) Use the result of part (b-i) to determine the pulse Z-transfer function of system (1.1) from b, c' and $H^Z(z)$ of part (b-iii). [3]

(c) Consider the system of Figure 1.1 where SH denotes the special hold that, for all k, sets $u(t) = (t - t_k)u_k$ for all $t \in [t_k, t_{k+1})$. Here the sample period is T. Derive the corresponding difference equation that relates $x(t_{k+1})$ to $x(t_k)$ and u_k from the *Variation of Constants* formula. Do not evaluate any exponentials or integrals that are involved. [3]

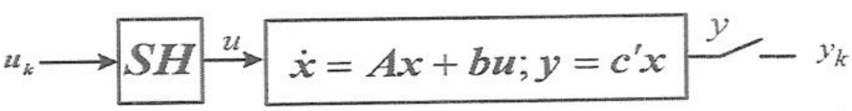

Figure 1.1

2. (a) For a linear discrete-time system, use the relationship between its pulse Z-transfer function $G^Z(z)$ and its pulse D-transfer function $G^D(\gamma)$ to derive the poles of $G^D(\gamma)$ from the poles of $G^Z(z)$. **[3]**

Derive the BIBO-stability condition for the poles of $G^D(\gamma)$ from the BIBO-stability condition for the poles of $G^Z(z)$. Represent graphically the condition for the poles of $G^D(\gamma)$ in terms of an appopriate set in the complex plane. **[2]**

(b) Consider the system of Figure 2.1 below with sample period T, where S is the system

$$\dot{x}(t) = Ax(t) + bu(t) : x(0) = x_o; \quad y(t) = c'x(t) \tag{2.1}$$

with $x(t) \in \mathbb{R}^n$. H ere A has distinct eigenvalues.

For $x_k \triangleq x(t_k)$, one way to approximate the solution of (2.1) is to use the approximating system (2.2) below and another way is to use the approximating system (2.3), where I_n denotes the $n \times n$ identity matrix :

$$x_{k+1} = [I_n + AT]x_k + Tbu_k\text{: } x_0 = x_o; \qquad y_k = c'x_k \tag{2.2}$$

$$x_{k+1} = (I_n+\tfrac{1}{2}AT)(I_n-\tfrac{1}{2}AT)^{-1}x_k + (I_n+\tfrac{1}{2}AT)Tbu_k : x_0 = x_o; \quad y_k = c'x_k. \tag{2.3}$$

(i) For approximation (2.2), determine a formula for $x^D(\gamma)$ in terms of $A, b, T, \gamma, u^D, x_o$.

Hint: start with $x^Z(z)$ and apply the general relationship between $x^D(\gamma)$ and $x^Z(z)$.

Hence determine the pulse D-transfer function of that system, called here $G^D(\gamma)$.

Relate your $G^D(\gamma)$ to the transfer function $G^L(s)$ for (2.1) between $y^L(s)$ and $u^L(s)$. **[5]**

(ii) Show that approximation (2.2) is not necessarily discrete-time BIBO-stable if system (2.1) is continuous-time BIBO-stable. **[4]**

(iii) Derive approximation (2.3) from the exact relationship between x_{k+1}, x_k and u_k.

Show that approximation (2.3) is discrete-time BIBO-stable if the continuous-time system (2.1) is continuous-time BIBO-stable. **[6]**

Hints for parts (b-ii,iii): Use spectral form and eigenvalues where appropriate.
If $M = V\text{diag}\{\mu_1, \mu_2, \ldots, \mu_n\}V^{-1}$ for some $V \in \mathbb{C}^{n\times n}$ then the μ_i are the eigenvalues of M.
You might find it useful to write, when appropriate, eigenvalues λ_i as $\Re(\lambda_i)+j\Im(\lambda_i)$.

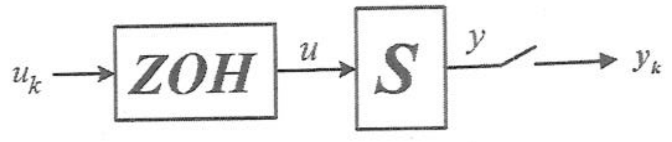

Figure 2.1

3. A controller calculates its output u_k from its input e_k using

$$u_k = h_0 e_k + h_1 e_{k-1} + h_2 e_{k-2},\ \forall k \geq 0$$

and let $h_k = 0$ for all $k > 2$.

Suppose $e_k = \cos(\omega t_k)$ for all $k \geq 0$ and $t_k = kT$.

(a) Without altering the order of summations, show that

$$u_k = \frac{e^{j\omega kT} h^F(\omega T)\ +\ e^{-j\omega kT} h^F(-\omega T)}{2},\ \forall k \geq 3.$$ **[5]**

Hence show that

$$u_k = |h^F(\omega T)| \cos(\omega t_k + \angle h^F(\omega T)),\ \forall k \geq 3$$

where $h^F(\Omega) = F\{h_0, h_1, h_2,\ 0, 0, 0, \dots\}(\Omega)$ and Ω is a digital frequency, **[4]**

(b) Choose $h_0,\ h_1$ and h_2 so that the controller has the pulse Z- transfer function

$$C^Z(z) = \frac{(z-j)(z+j)}{z^2}.$$ **[2]**

Suppose $T = 1$ s and the input to the controller is $u_k = \cos(\omega kT)$ for $\omega = \frac{\pi}{2}$ rad/s. Sketch the pole-zero configuration for the controller and use the sketch to predict the behaviour of its output u_k as $k \to \infty$. **[3]**

Determine $u^Z(z)$ from $C^Z(z)$ and $e^Z(z)$ and then, by considering $u^Z(z)$ without using a partial fraction expansion or residues, determine the outputs $u_0,\ u_1,\ u_2,\ \dots$.
Are these outputs consistent with the predicted outputs requested above?
Hint: $Z\{\cos(\omega t_k)\}(z) = \frac{z(z-\cos(\omega T))}{z^2-2z\cos(\omega T)+1}$. **[3]**

(c) Determine the canonical direct realization of the controller for which

$$C^Z(z) = \frac{(z-j)(z+j)}{(z-1)(z+2)}.$$ **[3]**

4. Consider the system of Figure 4.1 where $H^Z(z)$ is the pulse Z-transfer function of the process

$$x_{k+1} = Ax_k + bu_k; \quad y_k = c'x_k$$

where

$$A = \begin{bmatrix} 2.5 & -0.5 \\ 4.5 & -0.5 \end{bmatrix}, \; b = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}, \; c' = [0.25 \quad 0.25]$$

and $K \geq 0$.

(a) Show that the closed-loop pulse Z-transfer function is

$$\frac{0.25K(z+1)}{(z-1)^2+Kz-0.5K}.$$ [4]

(b) Use the Jury stability test to determine the maximum value of K_{max} such that the closed-loop system is BIBO-stable if $K \in (0, \, K_{max})$. [5]

(c) Determine the root-locus for the system and use it to compute the value of K which places at least one closed-loop pole at the origin. [5]

(d) For $K = 2$ and $r_k = k$ for all k, it can be shown that

$$e^Z(z) = \frac{(z+0.5)}{z(z-1)}.$$

Use the Inital Value Theorem to determine e_0. Use an inversion integral to determine e_k for $k \geq 2$. [3]

(e) Consider the calculations of part (a). State without proof, in one or two sentences, the reason that the above control system is unsatisfactory for any value of K. [3]

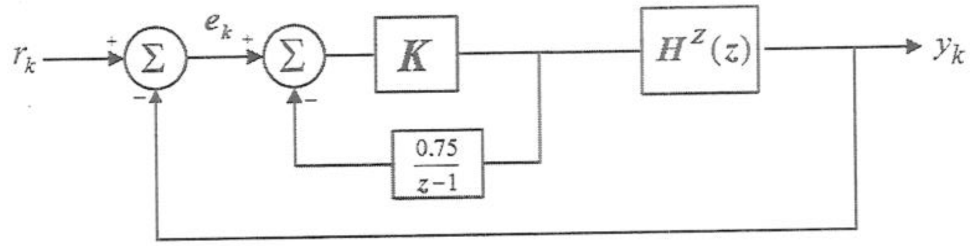

Figure 4.1

5. (a) Consider $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$ where these form a reachable pair.

Show that $VAV^{-1} \in \mathbb{R}^{n\times n}$ is a companion matrix where, for an appropriate $p \in \mathbb{R}^n$ which should be specified,

$$V = \begin{bmatrix} p' \\ p'A \\ \vdots \\ p'A^{n-1} \end{bmatrix}.$$

[8]

(b) Consider the system of Figure 5.1, where $K \geq 0$ and

$$A = \begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix},\ b = \begin{bmatrix} 4 \\ -2 \end{bmatrix},\ c' = [\,-0.5 \quad -1.5\,].$$

The pulse Z-transfer function of this discrete-time system in the forward path is

$$\tfrac{z-2}{z^2-1}.$$

(i) Sketch the root-locus for this system and state whether it is possible to choose K to locate both closed-loop poles at 0.2. **[2]**

(ii) Determine the corresponding companion matrix using the method of part (a) above and hence choose the feedback gain vector f so that the poles of the more general closed-loop control system

$$\begin{aligned} x_{k+1} &= Ax_k + bu_k : x_0 = x_o \\ y_k &= c'x_k \\ u_k &= r_k - f'x_k \end{aligned}$$

are at 0.2. **[10]**

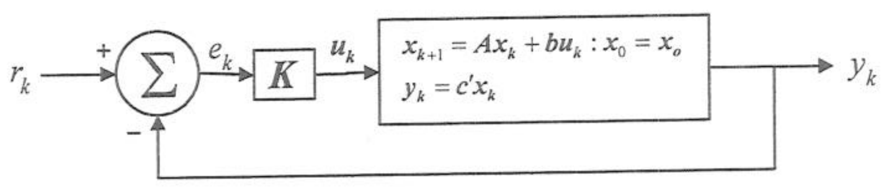

Figure 5.1

6. (a) Matrices A and M from $\mathbb{R}^{n\times n}$ are called similar if there is a non-singular matrix $P \in \mathbb{R}^{n\times n}$ such that $A = PMP^{-1}$.

Use the definition of an eigenvalue that involves a determinant to show that similar matrices have the same eigenvalues. **[3]**

(b) Consider the following system that consists of a plant, with state $x_k \in \mathbb{R}^n$ and scalar output y_k, controlled by feedback from the state $\widehat{x}_k$ of an observer :

$$\begin{aligned} x_{k+1} &= Ax_k + bu_k : x_0 = x_o \\ u_k &= r_k - f'\widehat{x}_k \\ y_k &= c'x_k \\ \widehat{x}_{k+1} &= (A-lc')\widehat{x}_k + ly_k + bu_k : \widehat{x}_0 = \widehat{x}_o. \end{aligned} \tag{6.1}$$

Here $b, f, l \in \mathbb{R}^n$.

(i) Let

$$\overline{x}_k \triangleq \begin{bmatrix} x_k \\ \widehat{x}_k \end{bmatrix}, \quad P = \begin{bmatrix} I_n & 0_n \\ I_n & -I_n \end{bmatrix} = P^{-1}$$

where I_n denotes the $n \times n$ identity matrix.

Determine $\overline{A} \in \mathbb{R}^{2n\times 2n}$ and $\overline{b} \in \mathbb{R}^{2n}$ such that (6.1) above can be represented as

$$\overline{x}_{k+1} = \overline{A}\overline{x}_k + \overline{b}r_k : \, x_0 = x_o.$$

By considering $P\overline{A}P^{-1}$, show that $\overline{A}$ is similar to a block upper-triangular matrix with diagonal blocks $A^f \triangleq A - bf'$ and $A^{ob} \triangleq A - lc'$.

Discuss briefly the practical significance of this. **[9]**

(ii) Now suppose that $r_k = 0$ for all k and the aim of the control system is to reduce x_k to zero.

Suppose $n = 2$ and f and l have been chosen to locate the eigenvalues of A^f and A^{ob} at the origin.

What are the eigenvalues of $\overline{A}$? Justify your answer.

Comment briefly on the significance of this.

Every matrix from $\mathbb{R}^{n\times n}$ is similar to a companion matrix. What are the special features of the companion matrix to which $\overline{A}$ is similar?

Explain the fact that the system reduces any non-zero x_0 to zero by the discrete-time t_4.

Comment briefly on this. **[8]**