Imperial College London

MEng (Engineering) Examination 2016

Year 1

AE1-101 Introduction to Aerodynamics

Friday 27th May 2016: 14.00 to 16.00
[2 hours]

The paper is divided into Section A and Section B
There are ***FOUR*** questions. All questions carry the same weight
Candidates may obtain full marks for complete answers to ***ALL*** questions.

You must answer each section in a separate answer booklet

The equations of motion for steady, two-dimensional, viscous flow are as follows:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$
$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

The use of lecture notes is NOT allowed.

Section A

1. (a) State in words and derive the equation of conservation of mass for one-dimensional flow through a rectangular control volume. [25%]

(b) Determine which of the velocity component sets given below satisfy the incompressible conservation of mass equation.

i.

$$\begin{aligned} u &= x + y, \\ v &= x - y. \end{aligned}$$

ii.

$$\begin{aligned} u &= A\sin(xy), \\ v &= -A\sin(xy). \end{aligned}$$

iii.

$$\begin{aligned} u &= 2x^2 + 3y, \\ v &= -2xy + 3y^2 + 3zy, \\ w &= -\frac{3}{2}z^2 - 2xz - 6yz. \end{aligned}$$

[30%]

(c) The streamfunctions are represented by

(i) $\Psi = x^2 - y^2$ (ii) $\Psi = x^2 + y^2$

determine the velocity and its direction at $(x = 2, y = 2)$. [25%]

(d) If the expression for a stream function is described by

$$\Psi = x^3 - 3xy^2$$

determine whether the flow is rotational or irrotational. [20%]

2. (a) Consider the flow along a circular section pipe.

   i. Write the Reynolds number in terms of mass flow rate, Q, pipe diameter, D, and the dynamic viscosity, μ.

   ii. For a fixed mass flow rate what effect does decreasing the diameter of a pipe have on the Reynolds number?

   iii. At what pipe diameter would you expect to start to see transition to turbulence if the fluid in the pipe is air, with a dynamic viscosity of 1.82×10^{-5} kg m^{-1} s^{-1}, and the mass flow rate is 1×10^{-3} kg s^{-1}? Justify your answer. [40%]

   (b) The fully developed velocity profile in a pipe of diameter D is

$$u(r) = u_{max}\left[1 - 4\left(\frac{r}{D}\right)^2\right]$$

   where u_{max} is the maximum velocity.

   i. Show that the average velocity, $\bar{u}$, is related to the peak velocity, u_{max}, by $\bar{u} = u_{max}/2$.

   ii. Show that the pressure pressure gradient in the pipe is

$$\frac{dp}{dx} = -\frac{16\mu u_{max}}{D^2}.$$

[60%]

Section B

3. (a) State, in words, the general control volume form for the equation of momentum conservation. [10%]

   (b) Using a rectangular control volume, derive the partial differential equation describing the unsteady two-dimensional, x-component of the momentum equation when the control volume is acted upon by both a pressure force and a viscous forces due to a laminar boundary layer. [45%]

   (c) A plane jet of water of thickness 25 mm is guided by a turning vane as shown in figure 1. The jet has a speed of 25 m/s and enters at an angle of 25 degrees to the horizontal and exits at an angle of 45 degrees to the horizontal. Assuming steady flow, determine the horizontal and vertical forces per unit depth experienced by the vane due to the momentum change of the jet of water. In which direction does the restraining force act? You may assume the density of water is 1000 kg/m^3. [45%]

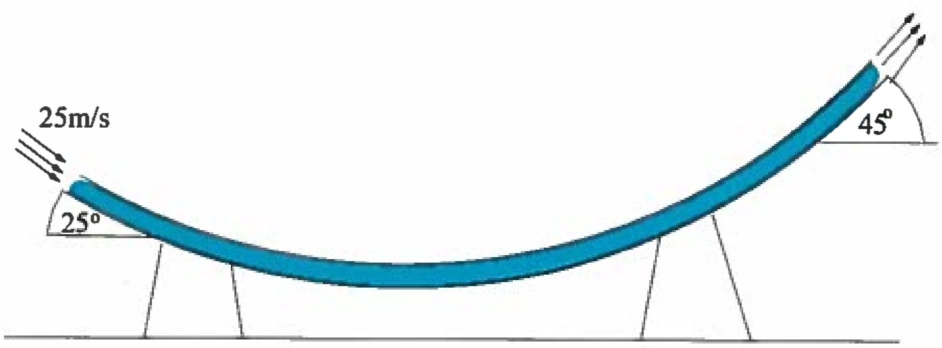

Figure 1

4. (a) State the mathematical conditions for a steady, laminar channel flow to be fully developed. [15%]

(b) A viscous flow between two long cylinders as shown in figure 2 is set into motion by counterclockwise rotation of the inner cylinder at angular velocity Ω_1. The inner cylinder has a radius of r_1 and the outer cylinder has a radius of r_2. Given that the equation of conservation of mass for a steady, incompressible flow in cylindrical coordinates is

$$\frac{\partial\, r v_r}{\partial r} + \frac{\partial v_\theta}{\partial \theta} = 0,$$

and the momentum equation for steady flow in the circumferential direction is

$$v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \nu\left(\frac{\partial^2 v_\theta}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right)$$

where v_r and v_θ are the velocities in the radial and circumferential directions:

(i) Determine the variation of v_θ as a function of r if the flow is steady and laminar. (Hint: $\partial p/\partial\theta = 0$ and note that $r\partial v_\theta/\partial r + v_\theta = \partial(r v_\theta)/\partial r$.) [70%]

(ii) If the inner cylinder were made stationary and the outer cylinder started to rotate, an analogous solution could be derived. If both cylinders are then allowed to move with different angular velocities, can the solutions simply be added together? Justify your answer. [15%]

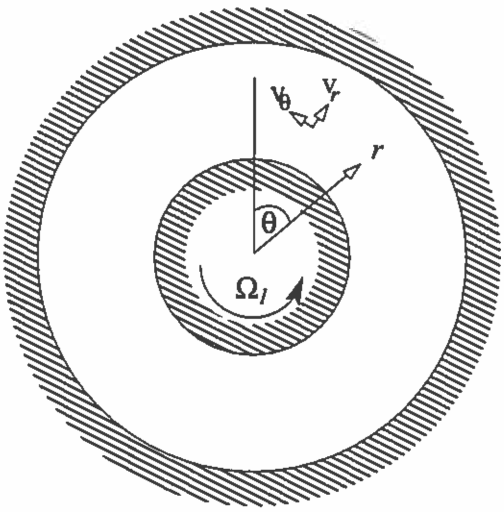

Figure 2

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Marks

a) The rate of change in time of mass within a control volume equals the net flux of mass through the control volume.

$$\dot{M} = \rho u A$$

For rectangular control volume $\delta A = 0$

incompressible flow $\frac{D\rho}{Dt} = 0 \ldots$

$\dot{M} =$ constant so we have

$$\nabla \underline{u} = 0 \quad \text{or} \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$ 25

→ see end.

b) i, $u = x + y$ $\quad \partial u/\partial x = 1$

$v = x - y$ $\quad \partial v/\partial y = -1$

$$\partial u/\partial x + \partial v/\partial y = 0.$$

ii) $u = A \sin xy$ $\quad \frac{\partial u}{\partial x} = Ay \cos xy$

$v = -A \sin xy$

$$\frac{\partial v}{\partial y} = -Ax \cos xy$$

$$\therefore \partial u/\partial x + \partial v/\partial y \neq 0$$

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iii) $u = 2x^2 + 3y$

$v = -2xy + 3y^2 + 3zy$

$w = -\frac{3}{2}z^2 - 2xz - 6yz$

$\frac{\partial u}{\partial x} = 4x \qquad \frac{\partial v}{\partial y} = -2x + 6y + 3z$

$\frac{\partial w}{\partial z} = -3z - 2x - 6y$

$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \ldots 0\,!$ 30

c) (i) $\psi = x^2 - y^2$ (ii) $\psi = x^2 + y^2$

$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}$

(i) $\frac{\partial \psi}{\partial x} = 2x \qquad \frac{\partial \psi}{\partial y} = -2y \qquad u = -2y$

$v = -2x$

$u(2,2) = -4$

(ii) $v(2,2) = -4.$

$\frac{\partial \psi}{\partial x} = 2x \qquad \frac{\partial \psi}{\partial y} = 2y \qquad u = 2y \quad v = -2x$

$u(2,2) = 4 \quad v(2,2) = -4$ 25

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d) $\psi = x^3 - 3xy^2$

need vorticity in (x,y) plane $\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$

$$= -\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2}$$

$\psi = x^3 - 3xy^2$

$\frac{\partial \psi}{\partial x} = 3x^2 - 3y^2 \qquad \frac{\partial^2 \psi}{\partial x^2} = 6x.$

$\frac{\partial \psi}{\partial y} = -6xy \qquad \frac{\partial^2 \psi}{\partial y^2} = -6x$ 20

irrotational $\omega_z = 0$.

(a) without assuming incompressibility

rate of change of mass $\frac{\partial \rho}{\partial t} \Delta x \Delta y \Delta z$

Net mass flux $\rho_2 u_2 \Delta y \Delta z - \rho_1 u_1 \Delta y \Delta z$

$\therefore \frac{\partial \rho}{\partial t} \Delta x \Delta y \Delta z + (\rho_2 u_2 - \rho_1 u_1) \Delta y \Delta z$

$$\frac{\partial \rho}{\partial t} + \frac{\Delta(\rho u)}{\Delta x} = 0.$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0.$$

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A2 (4)

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a)

i) $Re = \rho \frac{\bar{u} D}{\mu}$ $\quad Q = \rho \bar{u} A = \rho \bar{u} \pi \frac{D^2}{4}$

$$Re = \frac{4Q}{\pi \mu D}$$

ii) $Q = \rho \bar{u} A = \text{constant}$ so as A decreases $\bar{u}$ increases (if ρ = const)

iii) at $Re \simeq 2000-2500$ transition occurs

$$Q = 1 \times 10^{-3}$$

$$\mu = 1.82 \times 10^{-5}$$

$$D = \frac{4 \times 1 \times 10^{-3}}{\pi \, 1.82 \times 10^{-5} \, 2000}$$

$$D \simeq \frac{1}{5\pi \, 182} \text{ m} \approx 35 \text{ mm.}$$

40

b)

$$u(r) = u_{max}\left(1 - 4\left(\frac{r}{D}\right)^2\right)$$

$$\bar{u} = \frac{4}{\pi D^2} \int_0^{D/2} u(r) \, 2\pi r \, dr$$

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$$\bar{u} = \frac{4u_{max}}{\pi D^2}\int_0^{D/2}\left[1 - 4\left(\frac{r}{D}\right)^2\right]2\pi r\,dr$$

$$= \frac{8u_m}{D^2}\left[\frac{1}{2}r^2 - \frac{r^4}{D^2}\right]_0^{D/2}$$

$$= \frac{8u_m}{\cancel{D^2}}\left[\frac{1}{2}\frac{\cancel{D}^2}{4} - \frac{\cancel{D}^4}{16\cancel{D}^2}\right]$$

$$= 8u_m\left[\frac{1}{8} - \frac{1}{16}\right] = \frac{u_m}{2}$$

ii

$$\left.\frac{\partial u}{\partial r}\right|_{r=D/2} = -u_m\left(\frac{8r}{D^2}\right)\Bigg|_{r=D/2} = \frac{-4u_m}{D}.$$

$$\tau_w = -\frac{D}{4}\frac{dp}{dx} = \mu\left.\frac{\partial u}{\partial r}\right|_{r=D/2} = -\frac{4u_m}{D}\mu.$$

$$\frac{dp}{dx} = \frac{16\mu u_m}{D^2}$$

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0 for quoting from memory.

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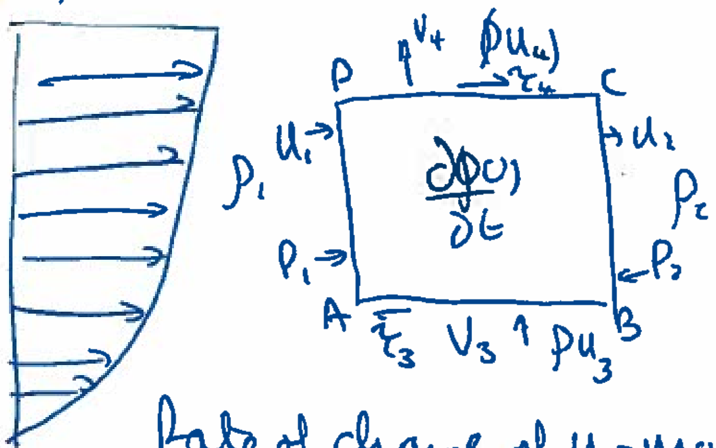
Question 3

1(a)

The rate of change of momentum within a Control Volume plus the net flux of momentum out of a Control Volume is equal to the applied forces acting on the Control Volume. — 10

1(b)



$\tau = \mu \frac{\partial u}{\partial y}$ — 5

Rate of change of u-momentum

$\frac{\partial \rho u}{\partial t} \Delta x \Delta y \Delta z$. — 5

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1(b)

Net flux:

$$(\rho u^2)_2 \Delta y \Delta z - (\rho u^2)_1 \Delta y \Delta z + (\rho u v)_4 \Delta x \Delta z - (\rho u v)_3 \Delta x \Delta z$$ 5

Pressure forces:

$$P_1 \Delta y \Delta z - P_2 \Delta y \Delta z.$$ 5

Shear stress forces.

$$\tau_4 \Delta x \Delta z - \tau_3 \Delta x \Delta z$$ 10

Full Balance

$$\frac{\partial \rho u}{\partial t} \Delta x \Delta y \Delta z + \Delta(\rho u^2) \Delta y \Delta z + \Delta(\rho u v) \Delta x \Delta z = \overset{(P_2 - P_1)}{-\Delta P} \Delta y \Delta z + \Delta \tau \Delta x \Delta z$$

$\div \Delta x \Delta y \Delta z.$

$$\frac{\partial \rho u}{\partial t} + \frac{\Delta \rho u^2}{\Delta x} + \frac{\Delta (\rho u v)}{\Delta y} = -\frac{\Delta P}{\Delta x} + \frac{\Delta \tau}{\Delta y}.$$

limit $\Delta x, \Delta y, \to 0$

$$\frac{\partial \rho u}{\partial t} + \frac{\partial \rho u^2}{\partial x} + \frac{\partial \rho u v}{\partial y} = -\frac{\partial P}{\partial x} + \frac{\partial \tau}{\partial y}.$$ 10

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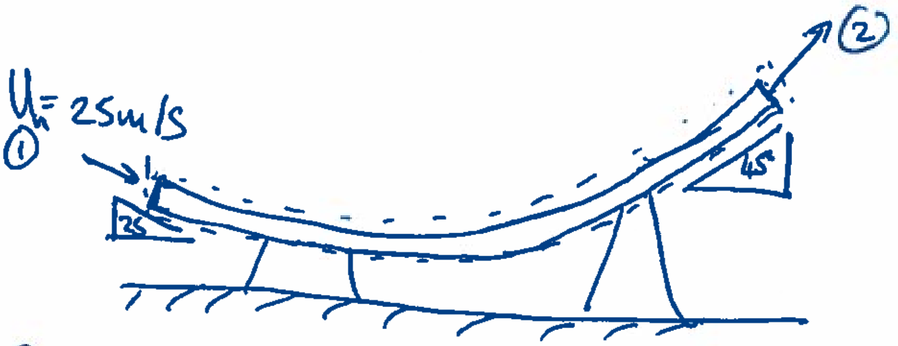
Finally since $\tau = \mu \frac{\partial u}{\partial y}$

$$\frac{\partial \rho u}{\partial t} + \frac{\partial \rho u^2}{\partial x} + \frac{\partial \rho u v}{\partial y} = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2}$$ 5.

1(c)



Consider momentum balance.

Δmom = Pressure Forces + restraing forces

We are only asked to evaluate the force due to momentum change

x-Force.

$$-\rho u_1 U_n a_1 + \rho u_2 U_n a_2 = F_x$$ 10

y-Force

$$-\rho v_1 U_n a_1 + \rho v_2 U_n a_2 = F_y$$ 10

where $U_n = 25$ m/s. $a_1 = a_2 = 25$ mm (per unit depth)

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Marks

1(c) $V_1 = -U_n \sin 25 = -10.57$.

$V_2 = U_2 \sin 45 = +17.68$

$U_1 = +U_n \cos 25 = +22.66$ 10

$U_2 = U_n \cos 45 = +17.68$

So $F_x = 1000 \times 25 \times 0.0025 \times (22.66 - 17.68)$

$= 311.25$ N/m. 5

$F_y = 1000 \times 25 \times 0.0025 (17.68 + 10.57)$

$= 1765.63$ N/m 5

This is the force acting on the fluid / C.V. and so the restraining force acts in the opposite direction $-F_x, -F_y$. 5.

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2 Question 4

(a) We assume that the velocity when fully developed does not change wrt respect to the flow direction

So flow in the x-direction.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = 0$$

15.

(b) (i) Starting with conservation of mass

$$\frac{\partial (r V_r)}{\partial r} + \frac{\partial V_\theta}{\partial \theta} = 0$$

If flow is fully developed $\frac{\partial}{\partial \theta} = 0$ since flow is in cylindrical direction.

$$\therefore \frac{\partial (r V_r)}{\partial r} = 0 \Rightarrow r V_r = \text{Const}$$

Since at $r = r_2$, $r = r_1$ $V_r = 0$ only solution that is possible is $V_r = 0$ i.e. Const $= 0$.

15.

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(11)

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(b) (i)

Now considering terms in the θ-momentum equation which do not include V_r terms or $\frac{\partial}{\partial\theta}$ terms since these are all zero, i.e.

$$\cancel{V_r}^{0}\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial \cancel{u_\theta}}{\cancel{\partial\theta}}^{0} + \frac{\cancel{V_r}^{0} u_\theta}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial\theta}$$

$$+ \nu\left(\frac{\partial^2 u_\theta}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r^2}\frac{\partial^2 \cancel{u_\theta}}{\cancel{\partial\theta^2}}^{0} + \frac{2}{r}\frac{\partial \cancel{V_r}}{\partial\theta}^{0}\right)$$

Since there is no pressure gradient in θ-direction $\frac{\partial p}{\partial\theta} = 0$ and we have

$$\nu\left(\frac{\partial^2 u_\theta}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)\right) = 0 \qquad (*)$$ 15.

Integrating w.r.t. r

$$\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} = \text{Const}$$

$$\Rightarrow r\frac{\partial u_\theta}{\partial r} + u_\theta = \text{Const} \times r$$

10

$$\Rightarrow \frac{\partial}{\partial r}(r u_\theta) = \text{Const} \times r \quad \left(\text{using hint } r\frac{\partial u_\theta}{\partial r} + u_\theta = \frac{\partial (r u_\theta)}{\partial r}\right)$$

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(b)(i)

integrating w.r.t r again

$$r u_\theta = \frac{C_1 r^2}{2} + C_2.$$

or $$u_\theta = \frac{C_1}{2} r + \frac{C_2}{r}.$$ 10

Now imposing B.C.'s

$$u_\theta(r_2) = 0 \Rightarrow C_1 r_2 + \frac{C_2}{r_2} = 0 \quad (a)$$

$$u_\theta(r_1) = \Omega r_1 \Rightarrow C_1 r_1 + \frac{C_2}{r_1} = \Omega r_1 \quad (b)$$ 10

From (a) $C_1 = -\frac{C_2}{r_2^2}$

Sub in (b) $$-\frac{C_2 r_1}{r_2^2} + \frac{C_2}{r_1} = \Omega r_1$$

$$C_2\left(\frac{r_2^2 - r_1^2}{r_2^2 r_1}\right) = \Omega r_1 \Rightarrow C_2 = \frac{\Omega r_2^2 r_1^2}{(r_2^2 - r_1^2)}.$$

$$C_1 = \frac{-\Omega r_1^2}{(r_2^2 - r_1^2)}$$

$$u_\theta(r) = \frac{-\Omega r_1^2}{(r_2^2 - r_1^2)} r + \frac{\Omega r_2^2 r_1^2}{(r_2^2 - r_1^2)}$$ 10

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b(ii)

Solution for different rotations of the inner and outer solutions can be added because equation (*) is linear 15