E4.25
CS1.2

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2010

MSc and EEE/ISE PART IV: MEng and ACGI

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DESIGN OF LINEAR MULTIVARIABLE CONTROL SYSTEMS

Tuesday, 18 May 10:00 am

Time allowed: 3:00 hours

There are SIX questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : I.M. Jaimoukha
Second Marker(s) : E.C. Kerrigan

1. a) Let the transfer matrix $G(s)$ have a state space realisation

$$G(s) \stackrel{s}{=} \left[\begin{array}{cc|cc} -1 & 2 & 1 & 2 \\ 0 & 3 & 0 & 0 \\ \hline 0 & 1 & 1 & 2 \\ 0 & 2 & 3 & 4 \end{array}\right].$$

i) Find the uncontrollable and/or unobservable modes and determine whether the realisation is detectable and stabilisable. [4]

ii) Obtain a minimum realisation of $G(s)$. Comment on your answer. [4]

b) Consider a state–variable model described by the dynamics

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t). \end{aligned}$$

i) Suppose there exists $P = P' \succ 0$ such that

$$AP + PA' \prec 0.$$

Prove that A is stable. [4]

ii) Assume that $A + A' \prec 0$. Suppose there exists $P = P'$ such that

$$AP + PA' \prec 0.$$

Prove that $P \succ 0$. [4]

iii) Suppose there exist $P = P' \succ 0$ and Z such that

$$AP + PA' + BZ + Z'B' \prec 0.$$

Prove that the pair (A, B) is stabilisable. [4]

2. a) Define internal stability for the feedback loop shown in Figure 2 below and derive necessary and sufficient conditions for which this feedback loop is internally stable. [3]

b) Suppose that the transfer matrix $G(s)$ in the feedback loop in Figure 2 is stable. Derive a parameterization of all internally stabilizing controllers $K(s)$ for the feedback loop. [~~5~~] 4

c) Suppose that

$$G \stackrel{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] := \left[\begin{array}{cc|cc} -4 & 0 & 4 & 0 \\ 0 & -1 & 0 & 1 \\ \hline 4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right].$$

[4] no Hankels

Show that the given realisation for $G(s)$ is balanced and evaluate the Hankel singular values of $G(s)$.

d) Consider the feedback loop in Figure 2. Suppose that $G(s)$ is as defined in Part (c). Design an internally stabilising compensator $K(s)$ such that

i) $K(s)$ is diagonal. [~~4~~] 3

ii) $K(s)$ has McMillan degree 1. [~~4~~] 3

iii) The singular values of the DC loop gain are as large as possible. [~~4~~] 3

Hint: Obtain a first order balanced truncation $G_r(s)$ of $G(s)$, write $G(s) = G_r(s) + \Delta(s)$, use the fact that $\|\Delta\|_\infty$ is less than or equal to 'twice the sum of tail' and base your design on $G_r(s)$.

10:15

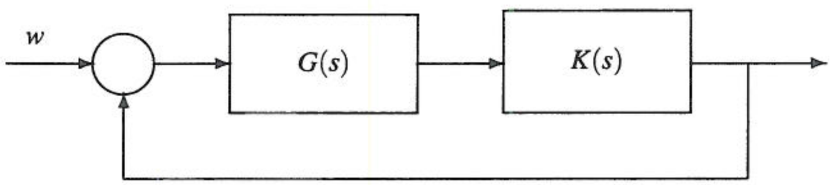

Figure 2

3. Consider the regulator in Figure 3.1 for which it is assumed that (A,B) is controllable and $x(0)=x_0$. A stabilizing state-feedback gain matrix F is to be designed such that the cost function $J := \int_0^\infty (u(t)'Ru(t) + x(t)'x(t))\,dt$ is minimized, where $R = R' \succ 0$.

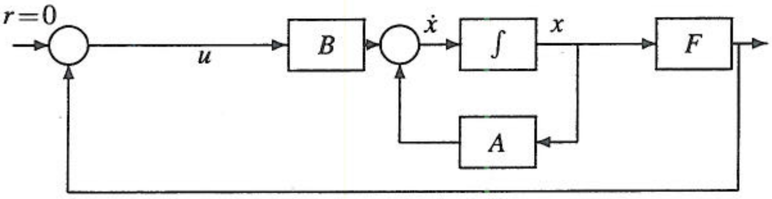

Figure 3.1

Let $V(t)=x(t)'Px(t)$ where $P=P'$ is the solution of an algebraic Riccati equation.

a) Assuming the closed loop is asymptotically stable, obtain an expression for $\int_0^\infty \dot{V}(t)dt$ in terms of x_0. [4]

b) Evaluate an expression for J using the following identity

$$(F+R^{-1}B'P)'R(F+R^{-1}B'P) = F'RF + F'B'P + PBF + PBR^{-1}B'P$$

Using this expression, find F that minimizes J. Give also the minimum value of J and the algebraic Riccati equation satisfied by P. [4]

c) Prove that, for the value of F chosen in part (b), the closed loop system in Figure 3.1 is stable. State clearly the assumption on P required to guarantee stability. [4]

d) Assume that $R = I$ and let $G(s) = (sI-A)^{-1}B$ and define $L(s) = I - FG(s)$. Using the algebraic Riccati equation show that

$$L(j\omega)'L(j\omega) = I + G(j\omega)'G(j\omega)$$

[4]

e) Using the answers to Parts (a)–(d) derive a robustness interpretation in terms of Figure 3.2. State clearly the assumptions on $\Delta(s)$. [4]

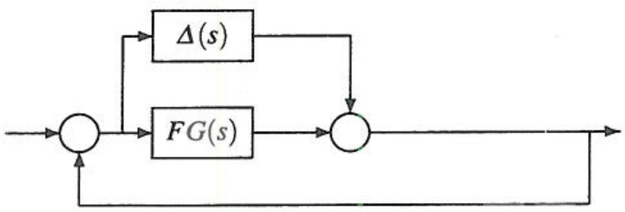

Figure 3.2

4. Consider the feedback configuration in Figure 4. Here, $G(s)$ is a nominal plant model and $K(s)$ is a compensator. The stable transfer matrices $\Delta_1(s)$ and $\Delta_2(s)$ represent uncertainties.

The design specification are to synthesize a compensator $K(s)$ such that the feedback loop is internally stable when:

- $\Delta_1 = 0$ and $\|\Delta_2(j\omega)\| \leq |w_2(j\omega)|, \forall\omega$, and,
- $\Delta_2 = 0$ and $\|\Delta_1(j\omega)\| \leq |w_1(j\omega)|, \forall\omega$,

where $w_1(s)$ and $w_2(s)$ are appropriate weighting functions.

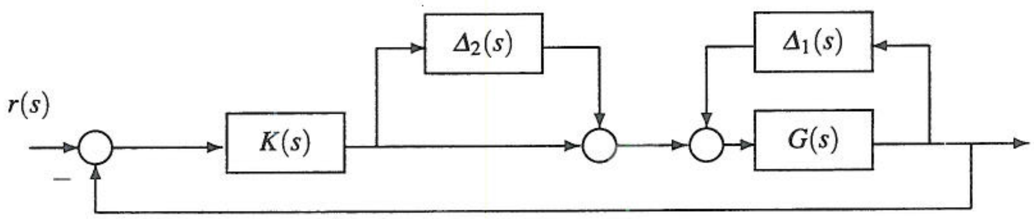

Figure 4

a) Derive $\mathcal{H}_\infty$–norm bounds, in terms of $G(s)$, $K(s)$, $w_1(s)$ and $w_2(s)$ that are sufficient to achieve the design specifications. [6]

b) Define suitable cost signals $z_1(s)$ and $z_2(s)$, external signal $w(s)$, measured signal $y(s)$ and control signal $u(s)$ and draw a block diagram, showing all these signals, as well as suitable weighting functions. [6]

c) Hence derive a generalised regulator formulation of the design problem that captures the sufficient conditions of Part (a). [8]

Hint: For Part (b), the external signal $w(s)$ may not be the same as the signal $r(s)$ shown in Figure 4.

5. Consider a state–variable model described by the dynamics

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t), \end{aligned}$$

and denote the corresponding transfer matrix by $H(s)$. Let $\gamma > 0$ be given and suppose that there exists $Q = Q' \succ 0$ such that

$$AQ + QA' + \gamma^{-2}BB' + QC'CQ \prec 0.$$

a) Prove that A is stable. [5]

b) By defining the Lyapunov function

$$V(t) = x(t)'Px(t),$$

where $P = Q^{-1}$, the cost function

$$J := \int_0^\infty [y(t)'y(t) - \gamma^2 u(t)'u(t)]dt,$$

and using a property of the integral $\int_0^\infty \dot{V}(t)dt$, or otherwise, prove that

$$\|H\|_\infty < \gamma.$$

State clearly the assumptions required on $u(t)$, $x(0)$ and $x(\infty)$. [10]

HINT: You may want to pre– and post–multiply the matrix inequality by Q^{-1} and complete a square.

c) Suppose that $A = -1$, $B = 1$ and $C = 1$. By using the answers to Parts (a) and (b), find $\|H\|_\infty$. [5]

6. Consider the regulator shown in Figure 6. Assume that

- The triple (A,B,C) is minimal
- $x(0)=0$
- The matrix C has full column rank.

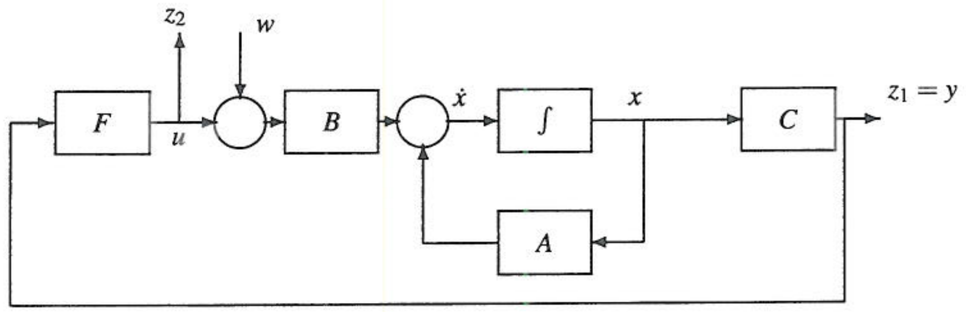

Figure 6

Let

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

and let $H(s)$ denote the transfer matrix from w to z.

A stabilizing static output feedback gain matrix F is to be designed such that, for $\gamma > 0$, $\|H\|_\infty < \gamma$.

a) Write down the generalized regulator system for this design problem. [6]

b) By using the Lyapunov function $V(t) = x(t)'Xx(t)$, where X is to be determined, derive sufficient conditions for the solution of the design problem.

Your conditions should be in the form of the existence of a certain solution to an algebraic Riccati equation.

It should also include an expression for F and an expression for the worst-case disturbance w. [10]

HINT: Consider using a left–inverse of C to obtain F.

c) Comment on your solution to Part (b) in the case that C does not have full column rank. [4]

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SOLUTIONS: DESIGN OF LINEAR MULTIVARIABLE CONTROL SYSTEMS 2010

1. a) i) Since $\begin{bmatrix} A - sI & B \end{bmatrix}$ loses rank for $s = 3$, 3 is an uncontrollable mode, and since $\begin{bmatrix} A' - sI & C' \end{bmatrix}$ loses rank for $s = -1$, -1 is an unobservable mode. Since the uncontrollable mode is unstable, the realisation is not stabilisable and since the unobservable mode is stable, the realisation is detectable.

ii) By removing the uncontrollable and unobservable parts we get the minimal realisation

$$G(s) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

That is, $G(s)$ is a static system and has McMillan degree 0.

b) i) Suppose that λ is an eigenvalue of A and let $z \neq 0$ be the corresponding left eigenvector. Then $z'A = \lambda z'$. Pre– and post–multiplying the matrix inequality by z' and z, respectively, we get

$$(\lambda + \bar{\lambda})z'Pz < 0.$$

Since $z \neq 0$ and $P \succ 0$, this implies that $z'Pz > 0$ so that $\lambda + \bar{\lambda} < 0$ and so A is stable.

ii) Assume that $A + A' \prec 0$. Suppose that λ is an eigenvalue of P and let $z \neq 0$ be the corresponding eigenvector. Then $Pz = \lambda z$. Pre– and post–multiplying the matrix inequality by z' and z, respectively, we get

$$\lambda z'(A + A')z < 0.$$

Since $z \neq 0$ and $A + A' \prec 0$, this implies that $z'(A + A')z < 0$ so that $\lambda > 0$ and so $P \succ 0$.

iii) The pair (A, B) is stabilisable if and only if there exists K such that $A + BK$ is stable. That is, the pair (A, B) is stabilisable if and only if there exist K and $P = P' \succ 0$ such that

$$(A + BK)P + P(A + BK)' \prec 0.$$

Comparing this with the inequality in the question, it follows that the pair (A, B) is stabilisable by identifying Z with KP.

2. a) Inject a signal $r(s)$ in between $G(s)$ and $K(s)$ and let $u(s)$ be the input to $G(s)$ and $y(s)$ be the input to $K(s)$. The loop is internally stable if and only if the transfer matrix from $\begin{bmatrix} w(s) \\ r(s) \end{bmatrix}$ to $\begin{bmatrix} u(s) \\ y(s) \end{bmatrix}$ is stable (no poles in the closed right half plane). Since

$$\begin{bmatrix} w(s) \\ r(s) \end{bmatrix} = \begin{bmatrix} I & -K(s) \\ -G(s) & I \end{bmatrix} \begin{bmatrix} u(s) \\ y(s) \end{bmatrix} =: T(s) \begin{bmatrix} u(s) \\ y(s) \end{bmatrix}$$

the loop is internally stable if and only if $T(s)^{-1}$ is stable.

b) Since $G(s)$ is stable, we proceed as follows. Note that

$$\begin{bmatrix} I & -K(s) \\ -G(s) & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ -G(s) & I \end{bmatrix} \begin{bmatrix} I & -K(s) \\ 0 & I - G(s)K(s) \end{bmatrix}.$$

Hence

$$\begin{aligned} \begin{bmatrix} I & -K(s) \\ -G(s) & I \end{bmatrix}^{-1} &= \begin{bmatrix} I & -K(s) \\ 0 & I - G(s)K(s) \end{bmatrix}^{-1} \begin{bmatrix} I & 0 \\ -G(s) & I \end{bmatrix}^{-1} \\ &= \begin{bmatrix} I & K(s)\,(I - G(s)K(s))^{-1} \\ 0 & (I - G(s)K(s))^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ G(s) & I \end{bmatrix}. \end{aligned}$$

Finally, since $(I - G(s)K(s))^{-1} = I + G(s)K(s)\,(I - G(s)K(s))^{-1}$, it follows that if $G(s)$ is stable, then the loop is internally stable if and only if $Q(s) := K(s)\,(I - G(s)K(s))^{-1}$ is stable. Rearranging terms shows that $K(s)$ is internally stabilizing if and only if $K(s) = Q(s)\,(I + G(s)Q(s))^{-1}$ for some stable $Q(s)$.

c) It can be easily verified that $A\Sigma + \Sigma A' + BB' = A'\Sigma + \Sigma A + C'C = 0$ where $\Sigma = \mathrm{diag}(2, 0.5)$. Thus the realisation for $G(s)$ is balanced and the Hankel singular values are 2 and 0.5.

d) Following the hint, we can write that $G(s) = G_r(s) + \Delta(s)$ where

$$G_r(s) \stackrel{s}{=} \left[\begin{array}{c|cc} -4 & 4 & 0 \\ \hline 4 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right], \qquad \|\Delta\|_\infty \le 1$$

and use $G_r(s)$ in the design. Since G_r is stable and K is required to be internally stabilising, $K = Q(I + G_r Q)^{-1}$ for some stable Q from Part (b). We search for a stable Q to satisfy the design requirements. For K to have a McMillan degree 1 we choose Q to be nondynamic. For K to be diagonal we choose a diagonal Q, say $Q = \mathrm{diag}(q_1, q_2)$. The small gain theorem implies that for K to stabilise the loop in Figure 2 for all Δ such that $\|\Delta\|_\infty \le 1$, we must have that $\|Q\| < 1$, or equivalently, $|q_1| < 1$ and $|q_2| < 1$. The DC loop gain is given by $G(0)K(0) = \mathrm{diag}(\frac{4q_1}{1+4q_1}, q_2)$ and so we choose $q_1 = -0.25$ and any q_2 such that $|q_2| < 1$.

3. a) Let $V = x'Px$ and set $u = Fx$. Provided that $P = P' \succ 0$ and $\dot{V} < 0$ along closed-loop trajectories, we can assume $\lim_{t \to \infty} x(t) = 0$. Then

$$\dot{V} = \dot{x}'Px + x'P\dot{x} = x'\left(A'P + PA + F'B'P + PBF\right)x.$$

Integrating from 0 to ∞ and using $x(\infty) = 0$,

$$\int_0^\infty x'\left(A'P + PA + F'B'P + PBF\right)x dt = -x_0'Px_0.$$

b) Using the definition of J and adding the last equation,

$$J = x_0'Px_0 + \int_0^\infty x'\left(A'P + PA + I + F'RF + F'B'P + PBF\right)x dt.$$

Completing the square using the given identity gives

$$J = x_0'Px_0 + \int_0^\infty \left(x'\left(A'P + PA + I - PBR^{-1}B'P\right)x + \left\|R^{\frac{1}{2}}(F + R^{-1}B'P)x\right\|^2\right) dt.$$

Since the last term is always nonnegative, it follows that the minimizing value of F is given by $F = -R^{-1}B'P$. We can set the term in square brackets to zero provided P satisfies the Riccati equation,

$$A'P + PA + I - PBR^{-1}B'P = 0.$$

It follows that the minimum value of J is $x_0'Px_0$.

c) For closed loop stability we need to prove that $A_c := A - BR^{-1}B'P$ is stable. The Riccati equation can be written as $A_c'P + PA_c + I + PBR^{-1}B'P = 0$. Let $\lambda \in \mathscr{C}$ be an eigenvalue of A_c and $z \neq 0$ be the corresponding eigenvector. Pre– and post–multiplying the Riccati equation be z' and z respectively gives $(\lambda + \bar{\lambda})z'Pz + z'z + z'PBR^{-1}B'Pz = 0$. Since $P \succ 0$ and $z \neq 0$, $z'Pz > 0$, $z'z > 0$ and $z'PBR^{-1}B'Pz \geq 0$. It follows that $\lambda + \bar{\lambda} < 0$ and the closed loop is stable.

d) Setting $R = I$ and by direct evaluation, $L(j\omega)'L(j\omega) =$

$$I - F(j\omega I - A)^{-1}B - B'(-j\omega I - A')^{-1}F' + B'(-j\omega I - A')^{-1}F'F(j\omega I - A)^{-1}B$$

But $F'F = A'P + PA + I = -(-j\omega I - A')P - P(j\omega I - A) + I$ from the Riccati equation. So, $L(j\omega)'L(j\omega)$

$$\begin{aligned}
&= I - F(j\omega I - A)^{-1}B - B'(-j\omega I - A')^{-1}F' \\
&\quad + B'(-j\omega I - A')^{-1}[-(-j\omega I - A')P - P(j\omega I - A) + I](j\omega I - A)^{-1}B \\
&= I - [F + B'P](j\omega I - A)^{-1}B - B'(-j\omega I - A')^{-1}[F' + PB] \\
&\quad + B'(-j\omega I - A')^{-1}(j\omega I - A)^{-1}B = I + G(j\omega)'G(j\omega)
\end{aligned}$$

e) Suppose that, due to uncertainties in the model, the actual system is given by Figure 3.2 where $G(s)$ is defined in part (d) and $\Delta(s)$ is a stable perturbation. Let ε be the input to Δ and δ be the output of Δ. Then $\varepsilon = \delta + FG\varepsilon = (I - FG)^{-1}\delta$. Using the small gain theorem (since the regulator is stable and the perturbation is assumed stable), the loop is stable if $\|\Delta(I - FG)^{-1}\|_\infty < 1$. But Part (d) implies that $\underline{\sigma}[I - FG(j\omega)] \geq 1 \forall \omega$ which implies $\|(I - FG)^{-1}\|_\infty \leq 1$. This shows that the loop will tolerate perturbations Δ of size $\|\Delta\|_\infty < 1$ without losing internal stability.

4. a) It is clear that we require $K(s)$ to be internally stabilising.

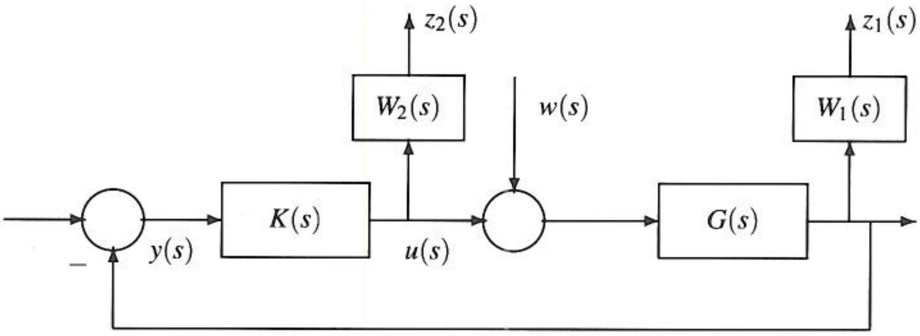
i) Suppose that $\Delta_1 = 0$ and let the input to Δ_2 be ε_2 while the output from Δ_2 be δ_2. Then a calculation shows that $\varepsilon_2 = -(I+KG)^{-1}KG\delta_2$. Using the small gain theorem, to satisfy the first design requirement, it is sufficient that $\left\|\Delta_2(j\omega)\left(I+K(j\omega)G(j\omega)\right)^{-1}K(j\omega)G(j\omega)\right\| < 1, \forall\omega$. This can be satisfied if $\|W_2(I+KG)^{-1}KG\|_\infty < 1$, where $W_2 = w_2 I$.

ii) An analogous procedure shows that to satisfy the second design requirement, it is sufficient that $\left\|\Delta_1(j\omega)G(j\omega)\left(I+K(j\omega)G(j\omega)\right)^{-1}\right\| < 1, \forall\omega$. This can be satisfied if $\left\|W_1 G(I+KG)^{-1}\right\|_\infty < 1$, where $W_1 = w_1 I$.

Thus, to satisfy both design requirements, it is sufficient that

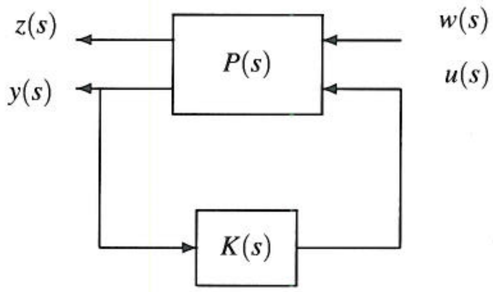
$$\left\| \begin{bmatrix} W_1 G(I+KG)^{-1} \\ W_2(I+KG)^{-1}KG \end{bmatrix} \right\|_\infty < 1.$$

b) All the requested signals are shown in the block diagram shown below.



c) The corresponding generalised regulator formulation is to find an internally stabilising $K(s)$ such that $\|\mathscr{F}_l(P,K)\|_\infty < 1$ where

$$z(s) = \begin{bmatrix} z_1(s) \\ z_2(s) \end{bmatrix},\ P(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} = \left[\begin{array}{c|c} W_1(s)G(s) & W_1(s)G(s) \\ 0 & W_2(s) \\ \hline -G(s) & -G(s) \end{array}\right].$$



5. By pre– and post–multiplying the matrix inequality by $P = Q^{-1}$ we get

$$A'P + PA + C'C + \gamma^{-2}PBB'P \prec 0.$$

a) The inequality implies that $A'P + PA \prec 0$. Let $z \neq 0$ be a right eigenvector of A and let λ be the corresponding eigenvalue. Then multiplying the inequality from the left by z' and from the right by z gives $(\lambda + \bar{\lambda})z'Pz < 0$. Since $P \succ 0$ and $z \neq 0$ then $z'Pz > 0$ and it follows that $\lambda + \bar{\lambda} < 0$ so that A is stable.

b) Since A is stable, $\|H\|_\infty < \gamma$ if and only if, with $x(0) = 0$,

$$J := \int_0^\infty [y'y - \gamma^2 u'u] dt < 0,$$

for all $u(t)$ such that $\|u\|_2 < \infty$. If $\|u\|_2$ is bounded, then $\lim_{t \to \infty} x(t) = 0$. Now,

$$\int_0^\infty \frac{\mathrm{d}}{\mathrm{dt}} (x'Px)\, dt = x(\infty)'Px(\infty) - x(0)'Px(0) = 0.$$

So,

$$\begin{aligned} 0 &= \int_0^\infty \dot{x}'Px + x'P\dot{x} dt = \int_0^\infty \big((Ax+Bu)'Px + x'P(Ax+Bu)\big)\, dt \\ &= \int_0^\infty \big(x'(A'P+PA)x + x'PBu + u'B'Px\big)\, dt. \end{aligned}$$

Using $y = Cx$, adding the last expression to J and using the identity

$$-(\gamma u - \gamma^{-1}B'Px)'(\gamma u - \gamma^{-1}B'Px) = x'PBu + u'B'Px - \gamma^2 u'u - \gamma^{-2}x'PBB'Px.$$

we get

$$\begin{aligned} J &= \int_0^\infty \big(x'\left(A'P + PA + C'C\right)x + x'PBu + u'B'Px - \gamma^2 u'u\big)\, dt \\ &= \int_0^\infty \big(x'(A'P + PA + C'C + \gamma^{-2}PBB'P)x - \|(\gamma u - \gamma^{-1}B'Px)\|^2\big)\, dt < 0 \end{aligned}$$

from the inequality. It follows that $\|H\|_\infty < \gamma$.

c) Using Parts (a) and (b), $\|H\|_\infty$ is the smallest γ for which there exists $P = P' \succ 0$ such that the inequality is satisfied. That is, it is the smallest γ such that

$$-2P + \gamma^{-2} + P^2 \prec 0$$

for some positive P, which is $\gamma = 1$, and so $\|H\|_\infty = 1$.

6. a) The generalized regulator formulation is given by

$$\begin{bmatrix} z(s) \\ y(s) \end{bmatrix} = P(s) \begin{bmatrix} w(s) \\ u(s) \end{bmatrix},\ u(s) = Fy(s),\ P(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{c||c|c} A & B & B \\ \hline\hline C & 0 & 0 \\ 0 & 0 & I \\ \hline C & 0 & 0 \end{array}\right].$$

b) The requirement $\|H\|_\infty < \gamma$ is equivalent to $J := \|z\|_2^2 - \gamma^2 \|w\|_2^2 < 0$. Let $V = x'Xx$ and set $u = FCx$. Provided that $X = X' \succ 0$ and $\dot{V} < 0$ along the closed-loop trajectory, we can assume $\lim_{t \to \infty} x(t) = 0$. Then

$$\dot{V} = \dot{x}'Xx + x'X\dot{x} = x'\left(A'X + XA + C'F'B'X + XBFF\right)x + x'XBw + w'B'Xx.$$

Integrating from 0 to ∞ and using $x(0) = x(\infty) = 0$,

$$0 = \int_0^\infty \left(x'\left(A'X + XA + C'F'B'X + XBFC\right)x + x'XBw + w'B'Xx\right) dt.$$

Using the definition of J and adding the last equation, $J =$

$$\int_0^\infty \left(x'\left(A'X + XA + C'C + C'F'FC + C'F'B'X + XBFC\right)x - \left(\gamma^2 w'w - x'XBw - w'B'Xx\right)\right) dt.$$

Let $Z = FC + B'X$. Completing the squares by using

$$\begin{aligned} Z'Z &= C'F'FC + C'F'B'X + XBFC + XBB'X \\ \|(\gamma w - \gamma^{-1}B'Xx)\|^2 &= \gamma^2 w'w - w'B'Xx - x'XBw + \gamma^{-2}x'XBB'Xx, \end{aligned}$$

$$J = \int_0^\infty \left(x'\left(A'X + XA + C'C - (1-\gamma^{-2})XBB'X\right)x + \|Zx\|^2 - \left\|\gamma w - \gamma^{-1}B'Xx\right\|^2\right) dt.$$

Thus two sufficient conditions for $J < 0$ are the existence of X such that

$$A'X + XA + C'C - (1-\gamma^{-2})XBB'X = 0, \qquad X = X' \succ 0.$$

A feedback gain is $F = -B'XC^\dagger$, where $C^\dagger$ denotes a left–inverse of C, and the worst case disturbance is $w^* = \gamma^{-2}B'Xx$. The closed-loop (using the optimal state–feedback and worst–case disturbance) is $\dot{x} = \left(A - (1-\gamma^{-2})BB'X\right)x$ and a third condition is therefore $Re\ \lambda_i\left(A - (1-\gamma^{-2})BB'X\right) < 0,\ \forall\, i$.

It remains to prove $\dot{V} < 0$ along state-trajectory with $u = FCx$ and $w = 0$. But

$$\dot{V} = x'\left(A'X + XA + C'F'B'X + XBFC\right)x = -x'(C'C + (1+\gamma^{-2})XBB'X)x < 0$$

for all $x \neq 0$ (since (A, B, C) is assumed minimal) proving closed-loop stability.

c) In the case that C does not have full column rank, then we cannot guarantee that the the equation $Z = FC + B'X = 0$ has a solution for F and the method will break down.