Control engineering exam paper - Model answers

Question 1

- The controllability matrix is
$$\mathcal{R} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$
which has rank three. Hence the system is controllable. The observability matrix is
$$\mathcal{O} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$
which has rank three. Hence the system is observable. [4 marks]

- Note that $y = Cx = x_2$, hence $\dot{y} = \dot{x}_2 = x_3 = C_1 x$, with $C_1 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$. [2 marks]

- The feedback is given by
$$u = -k^2 x_2 - k x_3.$$
The closed-loop system is described by the equation
$$\dot{x} = A_{cl} x = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -k^2 & -k \end{bmatrix} x.$$
The characteristic polynomial of A_{cl} is
$$\det(\lambda I - A_{cl}) = \lambda^3 + (k+1)\lambda^2 + k(k+1)\lambda + (k^2 - 1).$$
Routh test shows that all roots of the polynomial have negative real part for all $k > 1 = k_\star$. [6 marks]

- Since $\dot{y} = x_3$, $\ddot{y} = \dot{x}_3 = x_1 + u = C_2 x + u$, with $C_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$. [2 marks]

- Note that
$$u = -k^3 y - k^2 \dot{y} - k\ddot{y} = -k^3 x_2 - k^2 x_3 - k(x_1 + u).$$
This can be rewritten as
$$u = -\frac{k^3}{1+k} x_2 - \frac{k^2}{1+k} x_3 - \frac{k}{1+k} x_1,$$
which is well-defined for all $k \neq -1$. The closed-loop system is described by the equation
$$\dot{x} = \tilde{A}_{cl} x = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 - \dfrac{k}{1+k} & -\dfrac{k^3}{1+k} & -\dfrac{k^2}{1+k} \end{bmatrix} x.$$
The characteristic polynomial of $\tilde{A}_{cl}$ is
$$\det(\lambda I - \tilde{A}_{cl}) = \lambda^3 + \frac{k^2 + k + 1}{k+1}\lambda^2 + k^2 \lambda + \frac{k^3 - 1}{k+1}.$$
Routh test shows that all roots of the polynomial have negative real part for all $k > 1 = k_\circ$. [6 marks]

Question 2

- The closed-loop system is described by the equation $\dot{x} = Ax$, with

$$A = \begin{bmatrix} -\alpha_1 & \alpha_1 & 0 \\ \alpha_2 & -\alpha_2 - \alpha_3 & \alpha_3 \\ 0 & \alpha_4 & -\alpha_4 \end{bmatrix}.$$

[4 marks]

- Note that

$$\det A = 0,$$

which shows that A has a zero eigenvalue. [2 marks]

- The characteristic polynomial of A is (recall that it has a zero eigenvalue)

$$\det(\lambda I - A) = \lambda(\lambda^2 + (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)\lambda + (\alpha_1\alpha_4 + \alpha_4\alpha_2 + \alpha_1\alpha_3)).$$

Selecting, for example,

$$\alpha_1 = 1, \qquad \alpha_2 = 1, \qquad \alpha_3 = 1, \qquad \alpha_4 = 1,$$

yields

$$\det(\lambda I - A) = \lambda(\lambda^2 + 4\lambda + 3) = \lambda(\lambda + 3)(\lambda + 1).$$

[6 marks]

- The differential equations are

$$\dot{z}_{12} = 3\,x_2 - 2\,x_1 - x_3, \qquad \dot{z}_{23} = x_1 - 3\,x_2 + 2\,x_3.$$

[2 marks]

These can be rewritten as

$$\dot{z}_{12} = -2\,z_{12} + z_{23}, \qquad \dot{z}_{23} = z_{12} - 2\,z_{23}.$$

As a result

$$F = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}.$$

The characteristic polynomial of F is

$$\det(\lambda I - F) = (\lambda + 1)(\lambda + 3),$$

which shows that the matrix F has eigenvalues equal to -1 and -3. [4 marks]

The above implies that

$$\lim_{t\to\infty} x_1(t) - x_2(t) = \lim_{t\to\infty} x_2(t) - x_3(t) = 0,$$

which is the same as condition $(\star)$. [2 marks]

Question 3

- The state equations are

$$x_1(k+1) = a_1 x_1(k) + x_2(k) + a_1 u(k), \qquad x_2(k+1) = a_0 x_1(k) + a_0 u(k),$$

$$y(k) = x_1(k) + b_2 u(k).$$

As a result

$$y(k+1) = x_1(k+1) + u(k+1) = a_1 x_1(k) + x_2(k) + a_1 u(k) + u(k+1),$$

and

$$\begin{aligned} y(k+2) &= a_1 x_1(k+1) + x_2(k+1) + a_1 u(k+1) + u(k+2) \\ &= a_1 y(k+1) + x_2(k+1) + u(k+2) \\ &= a_1 y(k+1) + a_0 y(k) + u(k+2), \end{aligned}$$

which shows that the state-space and the input-output descriptions are equivalent. [8 marks]

- The reachability matrix is

$$\mathcal{R} = \begin{bmatrix} a_1 & a_1^2 + a_0 \\ a_0 & a_0 a_1 \end{bmatrix},$$

and

$$\det\mathcal{R} = -a_0^2.$$

As a result the system is reachable if $a_0 \neq 0$. If $a_0 = 0$ the system is controllable (regardless of the values of a_1) . [6 marks]

- The observability matrix is

$$\mathcal{O} = \begin{bmatrix} 1 & 0 \\ a_1 & 1 \end{bmatrix},$$

hence the system is observable for any value of the constants a_0 and a_1. [2 marks]

- Selecting $a_0 = 0$ yields

$$y(k+2) + a_1 y(k+1) = u(k+2)$$

and replacing $k+1$ with k yields the given system. [4 marks]

Question 4

- The equilibria of the system satisfy the equations

$$x_1 = \alpha \sin x_2, \qquad x_2 = -\alpha \sin x_1.$$

  Eliminating x_2 yields

$$x_1 = \alpha \sin(-\alpha \sin x_1).$$

  This equation, for $\alpha \in (0, 1]$ has the unique solution $x_1 = 0$, hence $(0, 0)$ is the only equilibrium of the system. [4 marks]

- The linearization of the system around the zero equilibrium is given by

$$\delta_x^+ = A\delta_x + B\delta_u,$$

  with

$$A = \begin{bmatrix} 0 & \alpha \\ -\alpha & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

  [4 marks]

- The characteristic polynomial of the matrix A is

$$\det(\lambda I - A) = \lambda^2 + \alpha^2,$$

  which has roots inside the unity disk for all $|\alpha| < 1$. For $\alpha = 1$, the roots are $\pm j$. As a result, the linearized system is asymptotically stable for all $|\alpha| < 1$, and stable for $\alpha = 1$. [4 marks]

- Let $K = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$ and note that

$$A + BK = \begin{bmatrix} K_1 & \alpha + K_2 \\ \alpha & 0 \end{bmatrix}.$$

  Hence, selecting $K_1 = 0$ and $K_2 = -\alpha$, yields two eigenvalues at zero. [4 marks]

- The observability matrix is

$$\mathcal{O} = \begin{bmatrix} c_1 & c_2 \\ -c_2\alpha & c_1\alpha \end{bmatrix},$$

  and $\det\mathcal{O} = \alpha(c_1^2 + c_2^2)$. The system is therefore observable provided that $c_1^2 + c_2^2 \neq 0$.
  Let $L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$ and note that

$$A + LC = \begin{bmatrix} L_1c_1 & \alpha + L_1c_2 \\ -\alpha + L_2c_1 & L_2c_2 \end{bmatrix}.$$

  Hence, selecting $L_1 = -\alpha\dfrac{c_2}{c_1^2 + c_2^2}$ and $L_2 = \alpha\dfrac{c_1}{c_1^2 + c_2^2}$ yields two eigenvalues at zero.
  [4 marks]