UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 2000

BEng Honours Degree in Computing Part III
MSc in Computing Science
BEng Honours Degree in Information Systems Engineering Part III
MEng Honours Degree in Information Systems Engineering Part III
BEng Honours Degree in Mathematics and Computer Science Part III
MEng Honours Degree in Mathematics and Computer Science Part III
MSc in Advanced Computing
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the Associateship of the City and Guilds of London Institute
This paper is also taken for the relevant examinations for the Associateship of the Royal College of Science

PAPER C336=I3.6

PERFORMANCE ANALYSIS

Thursday 11 May 2000, 14:30
Duration: 120 minutes

Answer THREE questions

Paper contains 4 questions

1 a State *Jackson's Theorem* for both *open* and *closed* queueing networks. Define the term *visitation rate* and show how this quantity can be calculated for each node from the network's *routing probabilities.*

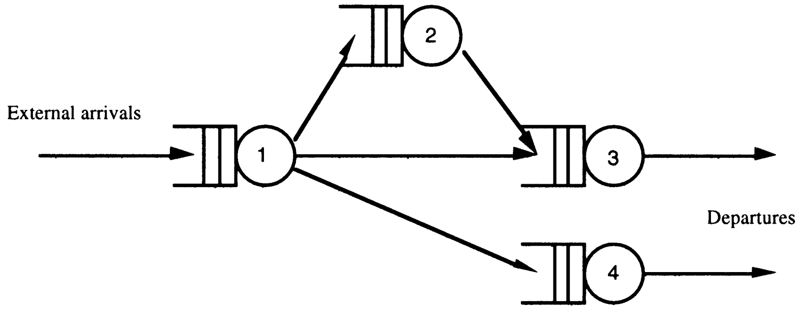
b In the open Jackson network shown below, external arrivals have rate Λ and the routing probabilities from node 1 to nodes 2, 3 and 4 are α, β and γ respectively. The service rate of node i is μ_i ($i = 1,2,3,4$).

i) What is the probability distribution of the response time at node 1?

ii) What is the probability that both nodes 1 and 2 are idle?

iii) How would you determine the response time distribution along the path containing nodes 1 and 4? What is the result when $\mu_4 = \mu_1 - (1-\gamma)\Lambda$?

iv) Why is it difficult to find the response time distribution along the path containing nodes 1, 2 and 3?



The two parts carry, respectively, 40% *and* 60% *of the marks.*

2 a Define the term *Poisson arrival process* and write down the mean interarrival time of such a process with *rate* λ. State and prove the *memoryless property* of Poisson processes.

b Given two independent Poisson processes with rates λ_1, λ_2 at an arbitrarily chosen time, derive the probability distribution function of the length of time to the next arrival from either process. Hence or otherwise, prove the *superposition property* of Poisson processes.

c A Poisson process P_1 with rate λ is *duplicated* forming the pair of *identical* Poisson processes P_1 and P_2; i.e. at every arrival instant of P_1, there is also an arrival of P_2, and *vice-versa.* P_1 passes through a delay unit that outputs every input exactly d seconds after its arrival.

i) Show that the output process, P_3 say, of the delay unit is Poisson with rate λ;

ii) The processes P_1 and P_3 are now merged. Show that their superposition is *not* Poisson. Why does the superposition property no longer hold?

3 a i) Define the *Markov property* in the context of continuous time stochastic processes.

ii) What is meant by the term *probability flux*?

iii) State the *Steady State Theorem* for continuous time Markov processes and justify it informally using a notion of *flux balance*.

b A transaction processing node, with Poisson arrivals at rate λ and negative exponential service times with mean $1/\mu$, operates as an M/M/1 queue except that whenever the node becomes idle immediately after processing a transaction, it performs maintenance operations. These last for a time that is negative exponentially distributed with mean $1/\theta$ and, whilst in progress, arriving transactions are lost.

i) Draw the state transition diagram for this queue.

ii) Write down a sufficient set of balance equations to determine the equilibrium probabilities (when they exist) of the system's states. (*Hint*: Use appropriate contours to simplify the equations.)

iii) What is the condition for equilibrium to exist?

iv) Show that, at equilibrium, the node's utilisation (proportion of time spent processing transactions) is $\dfrac{\theta\rho}{\theta+\lambda(1-\rho)}$ where $\rho = \lambda/\mu$.

The two parts carry, respectively, 40% *and* 60% *of the marks.*

4 Consider a closed Markovian queueing network of M nodes and k stochastically identical tasks, in which there is a path from any node to any other.

a i) Explain why the network must have a steady state.

ii) Let $T(k)$ be the throughput along one given arc connecting two nodes and let v_i be the average number of visits a task makes to node i between successive transits across that arc ($1 \le i \le M$). Show that the arrival rate to node i is Tv_i.

iii) Hence derive the equilibrium equations of *mean value analysis*:

$$L_i(k) = v_i T(k) W_i(k); \quad W_i(k) = (1+L_i(k-1))/\mu_i\,; \quad k= T(k) \sum_{i=1}^{M} v_i W_i(k)$$

where μ_i, $L_i(k)$ and $W_i(k)$ are respectively the service rate, mean queue length and mean waiting time at node i ($1 \le i \le M$), for population k.

b i) Write down an expression for the utilisation $u_i(k)$ of server i.

ii) As the population k increases, what happens to the utilisations of the servers? As $k \to \infty$, what is the maximum of the utilisations? *Use only intuitive arguments, not part a of the question.*

iii) Using part a, prove that $L_i(k) = u_i(k)(1 + L_i(k-1))$, ($1 \le i \le M$).

iv) As $k \to \infty$, suppose at server 1, the utilisation converges to the value $u<1$ and the mean queue length converges to L. Prove that $L = u/(1-u)$.

End of paper