

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING  
EXAMINATIONS 2011

MSc and EEE/ISE PART IV: MEng and ACGI

**WAVELETS AND APPLICATIONS**

Thursday, 26 May 10:00 am

Time allowed: 3:00 hours

**There are FOUR questions on this paper.**

**Answer ALL questions.**

*All questions carry equal marks.*

**Any special instructions for invigilators and information for candidates are on page 1.**

Examiners responsible	First Marker(s) :	P.L. Dragotti
	Second Marker(s) :	K.D. Harris

**Special Information for the Invigilators: NONE**

**Information for Candidates:**

*Sub-sampling by an integer  $N$ :*

$$x_{\downarrow N}[n] \longleftrightarrow \frac{1}{N} \sum_{k=0}^{N-1} X(e^{j(\omega-2\pi k)/N}) = \frac{1}{N} \sum_{k=0}^{N-1} X(W_N^k z^{1/N}),$$

where

$$W_N^k = e^{-j2\pi k/N}.$$

*Poisson summation formula:*

$$\sum_{n=-\infty}^{\infty} f(t-n) = \sum_{k=-\infty}^{\infty} \hat{f}(2\pi k) e^{j2\pi kt}.$$

## The Questions

### 1. Multirate Signal Processing:

- (a) The deterministic autocorrelation function for a real-valued stable sequence  $x[n]$  is defined as

$$c_{xx}[n] = \sum_{k=-\infty}^{\infty} x[k]x[k+n].$$

- i. Show that the  $z$ -transform of  $c_{xx}[n]$  is  $C_{xx}(z) = X(z)X(z^{-1})$ .

[6]

- ii. Using the above result, express the orthogonality condition  $\langle g[n], g[n-2k] \rangle = \delta_k$  in the  $z$ -domain.

[6]

- (b) *Interpolation followed by decimation:* Given an input  $x[n]$ , consider upsampling by 2 followed by interpolation with a filter  $H(z)$ . Then to recover the original signal, apply filtering with a filter  $G(z)$  followed by downsampling by 2 in order to obtain a reconstruction  $\hat{x}[n]$  (see Fig. 1a). What does the product  $P(z) = H(z)G(z)$  have

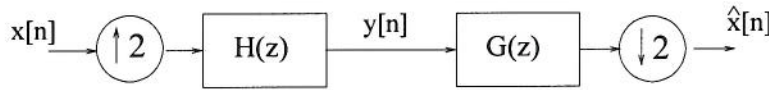


Figure 1a: Interpolation followed by decimation.

to satisfy in order for  $\hat{x}[n]$  to be equal to  $x[n]$ ?

[6]

- (c) *Successive interpolation:* Consider the system in Fig. 1b where the output  $y^{(1)}[n]$  is an interpolated version of the input sequence  $x[n]$ . We would like  $y^{(1)}[2n] = x[n]$ ,

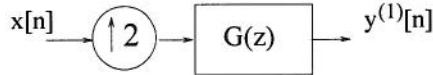


Figure 1b: Interpolation of  $x[n]$ .

while  $y^{(1)}[2n+1]$  is interpolated. What conditions does that impose on  $G(z)$ ?

[7]

2. You want to construct a wavelet transform that uses powers of three instead of powers of two. For this purpose, you will consider the centered box function  $\varphi(x)$  such that  $\varphi(x) = 1$  for  $x \in [-\frac{1}{2}, \frac{1}{2}]$  and  $\varphi(x) = 0$  otherwise.

(a) Start by showing that  $\varphi(x)$  is a valid scaling function. That is, show that

- i. it satisfies partition of unity:  $\sum_n \varphi(x - n) = 1$ .

[4]

- ii. it satisfies the Riesz basis criterion:

$$0 < A \leq \sum_{k=-\infty}^{\infty} |\hat{\varphi}(\omega + 2\pi k)|^2 \leq B < \infty.$$

[4]

- iii. it satisfies a 3-scale equation  $\varphi(x/3) = \sqrt{3} \sum_n h_0[n] \varphi(x - n)$ .

[4]

- (b) Construct two orthogonal and compactly supported wavelets  $\psi_a(t)$  and  $\psi_b(t)$ . We want  $\psi_a(t)$  to be symmetric and  $\psi_b(t)$  to be anti-symmetric. Moreover, we want the shortest possible solution. [Hint: recall that  $\psi_a(x/3) = \sqrt{3} \sum_n g_a[n] \varphi(x - n)$  and that  $\psi_b(x/3) = \sqrt{3} \sum_n g_b[n] \varphi(x - n)$ . Thus, you just need to find the coefficients  $g_a[n]$  and  $g_b[n]$  satisfying all the above requirements: shortest possible support, (anti-)symmetry and orthogonality].

[8]

- (c) Give the corresponding perfect reconstruction three-channel filter bank.

[5]

3. *Spectral factorization.* Following the footsteps of some great wavelet researchers, you

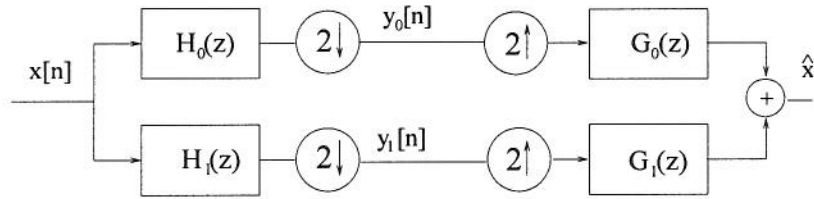


Figure 1c: Two-channel filter bank.

want to design your own family of orthogonal and biorthogonal filter banks using the spectral factorization method. You are an original person: instead of considering  $z = -1$ , you want to put a maximum number of zeros at  $z = 1$ .

- (a) Start by determining the shortest symmetric polynomials of the form  $P(z) = (1 - z)^L(1 - z^{-1})^L B(z)$  for  $L = 1$  and  $L = 2$  such that  $P(z) + P(-z) = 2$ .

[8]

- (b) Based on the polynomial  $P(z)$  of part (a), construct an orthogonal filter bank.

[8]

- (c) Based on the polynomial  $P(z)$  in part (a), construct all possible linear phase biorthogonal filter banks using the constraint  $H_0(-1) = 1$ .

[9]

4. *Meyer's Wavelet.* We are going to design a valid scaling function  $\varphi(t)$  and the corresponding wavelet in the frequency domain. We are going to follow a methodology first proposed by Meyer. Construct the scaling function  $\varphi(t)$  such that:

$$\hat{\varphi}(\omega) = \begin{cases} \sqrt{a(2 + \frac{3\omega}{2\pi})} & \omega \leq 0 \\ \sqrt{a(2 - \frac{3\omega}{2\pi})} & \omega \geq 0 \end{cases}$$

where  $\hat{\varphi}(\omega)$  is the Fourier transform of  $\varphi(t)$  and

$$a(\omega) = \begin{cases} 0 & \omega \leq 0 \\ 3\omega^2 - 2\omega^3 & 0 \leq \omega \leq 1 \\ 1 & \omega \geq 1 \end{cases}$$

Show that  $\varphi(t)$  satisfies the three criteria of a valid scaling function. More specifically:

- (a) Show that  $\varphi(t)$  satisfies partition of unity:

$$\sum_{n=-\infty}^{\infty} \varphi(t - n) = 1.$$

[6]

- (b) Show that  $\langle \varphi(t - n), \varphi(t - m) \rangle = \delta_{n,m}$ . This is equivalent to showing that  $\{\varphi(t - n)\}_{n \in \mathbb{Z}}$  is an orthonormal basis of the space  $V_0 = \text{span}\{\varphi(t - n)\}_{n \in \mathbb{Z}}$ . [Hint: operate in the frequency domain].

[7]

- (c) Finally, show pictorially and by operating in the frequency domain, that the two-scale equation

$$\varphi(t) = \sqrt{2} \sum_{n=-\infty}^{\infty} g_0[n] \varphi(2t - n)$$

is satisfied when  $G_0(e^{j\omega}) = \sqrt{2} \sum_{k=-\infty}^{\infty} \hat{\varphi}(2\omega + 4\pi k)$ , where  $G(e^{j\omega})$  is the discrete-time Fourier transform of  $g_0[n]$ .

[6]

- (d) Given  $\varphi(t)$ , the corresponding wavelet  $\psi(t)$  satisfies the following two-scale relation:

$$\psi(t) = \sqrt{2} \sum_{n=-\infty}^{\infty} g_1[n] \varphi(2t - n).$$

Derive the frequency domain expression of  $g_1[n]$ .

[6]

# SOLUTIONS

Wavelets and  
Applications  
2011

1/12

## QUESTION 1

EE 4-45  
EE 95022

$$\begin{aligned} (u) \\ (i) \quad c_{xx}(z) &= \sum_{m=-\infty}^{\infty} c_{xx}[m] z^{-m} \\ &= \sum_m \sum_{l} x[l] x[l+m] z^{-m} \end{aligned}$$

REPLACE  $m$  WITH  $l+m$ .

THIS LEADS TO

$$\begin{aligned} c_{xx}(z) &= \sum_l \sum_{l} x[l] x[l] z^{l-l} \\ &= \sum_l x[l] z^{-l} \cdot \sum_l x[l] z^l \\ &= X(z) \cdot X(z^{-1}) \end{aligned}$$

□

(ii)

$$\langle g[n], g[n-2N] \rangle = \sum_n g[n] \cdot g[n-2N]$$

WE DENOTE

$$P[k] = \sum_n g[n] g[n-k]$$

FOR BECAUSE OF (i)  $P(z) = G(z) \cdot G(z^{-1})$



2

$\langle y[n], y[n-2n] \rangle$  IS EQUIVALENT TO  
SUB-SAMPLING  $p[n]$  BY A FACTOR 2.  
THEREFORE

THE Z-TRANSFORM OF  $\langle y[n], y[n-2n] \rangle$  IS

$$\frac{1}{2} P(z^{1/2}) + \frac{1}{2} P(-z^{1/2}) = \frac{1}{2} G(z^{1/2}) G(z^{-1/2}) + \frac{1}{2} G(-z^{1/2}) G(-z^{-1/2})$$

PUTTING EVERYTHING TOGETHER, WE  
OBTAIN

$$\langle y[n], y[n-2n] \rangle = \delta_{11} \quad (\rightarrow) \quad \frac{1}{2} G(z^{1/2}) G(z^{-1/2}) + \frac{1}{2} G(-z^{1/2}) G(-z^{-1/2}) = 1$$

WHICH LEADS TO THE FOLLOWING ORTHOGONALITY  
CONDITION:

~~$$G(z^{1/2}) G(z^{-1/2}) +$$~~

$$G(z) G(z^{-1}) + G(-z) G(-z^{-1}) = 2$$



b)

$$Y(z) = H(z) X(z)$$

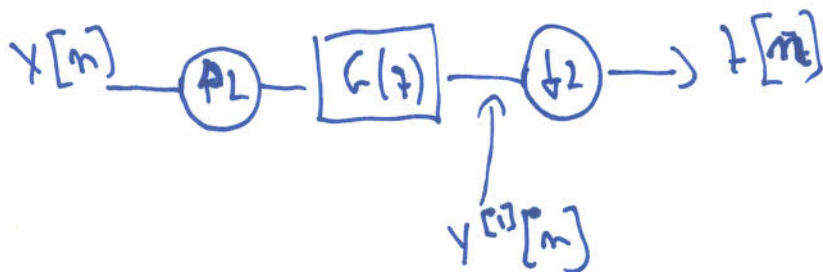
$$\hat{X}(z) = X(z) \left[ \frac{1}{2} \left[ G(z^{1/2}) H(z^{1/2}) + G(-z^{1/2}) H(-z^{1/2}) \right] \right]$$

THEREFORE

$$\hat{X}(z) = X(z) \Leftrightarrow G(z) H(z) + G(-z) H(-z) = 2$$

c)

CONSIDER THE FOLLOWING SYSTEM:

THE CONDITION  $y^{(1)}[n] = x[n]$ 

IS EQUIVALENT TO IMPOSING

THAT  $z[n] = x[n]$  WHICH IMPLIES THAT

$$X(z) = z(z) = \frac{1}{2} \left[ G(z^{1/2}) + G(-z^{1/2}) \right] X(z)$$



$$G(z) + G(-z) = 2$$

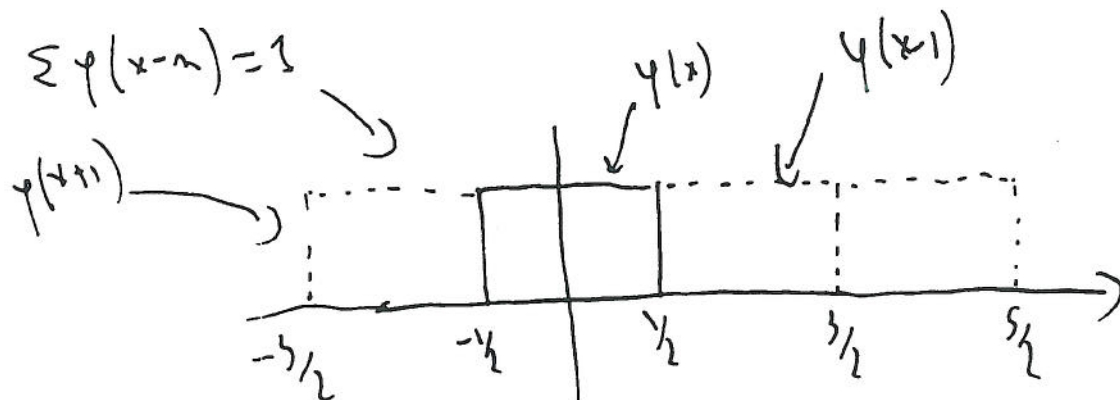
## QUESTION 2

4

2) (a)

i)  $\psi(x)$

CLEARLY SATISFIES PARTITION OF UNITY.



ii) MOREOVER

$$\langle \psi(x), \psi(x-n) \rangle = \delta_n$$

THUS  ~~$\psi(x)$~~   $\sum_n |\hat{\psi}(\omega + 2n\pi)|^2 = 1 \Rightarrow$

$\psi(x)$  IS AN ORTHONORMAL BASIS

iii) ONE CAN EASILY SEE GRAPHICALLY THAT

$$\psi\left(\frac{x}{3}\right) = \sum_n \sqrt{3} h_0[n] \psi(x-n)$$

WITH

$$\begin{cases} h_0[-3] = h_0[0] = h_0[1] = \frac{1}{\sqrt{3}} \\ \text{AND } h_0[n] = 0 \quad n \neq -3, 0, 1. \end{cases}$$

(b)

 ~~$\psi_a(x)$  MUST BE SYMMETRIC~~

$$\psi_a(x/3) = \sqrt{3} \sum_n g_a[n] \psi(x-n)$$

 $\psi_a(x)$  MUST BE SYMMETRIC. HOWEVER

$$(*) \langle \psi_a(x), \psi_a(x-3m) \rangle = \delta_m$$

$$(**) \langle \psi_a(x), \psi_a(x-3n) \rangle = 0$$

IF  $g_a[n] = 0$  FOR  $n \notin \{-1, 1, 0\}$ CONDITION  $(*)$  IS SATISFIED UP TO A CONSTANT FACTORFOR SYMMETRY  $g_a[1] = g_a[-1]$ 

$$\text{THUS} \begin{cases} \langle \psi_a(x), \psi_a(x) \rangle = 1 \\ \langle \psi_a(x), \psi(x) \rangle = 0 \end{cases} \text{ IMPLIES}$$

$$\text{THAT} \quad g_a[0] = \frac{2}{\sqrt{6}}$$

$$g_a[-1] = g_a[1] = -\frac{1}{\sqrt{6}}$$

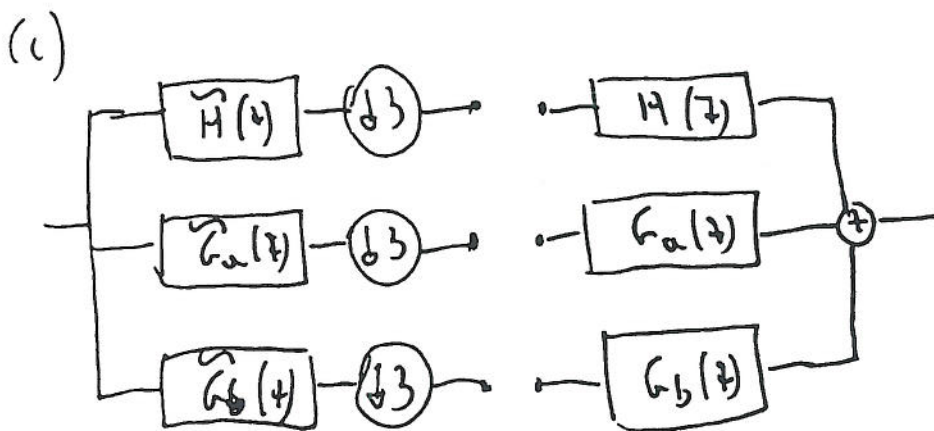
$$\psi_b(x/3) = \sqrt{3} \sum_n g_b[n] \psi(x-n)$$

 $\psi_b(x)$  MUST BE ANTI-SYMMETRIC

AND HAS TO SATISFY THE ORTHOGONALITY CONDITIONS. THIS LEADS TO

$$g_b[m] = 0 \quad m \neq \{-1, 1\}$$

$$\text{AND} \quad g_b[1] = -g_b[-1] = \frac{1}{\sqrt{3}}$$



WHERE THE ANALYSIS FILTERS  
ARE THE TIME REVERSED VERSIONS OF  
THE SYNTHESIS ONES.

# QUESTION 3

4

$$a) \quad B(t) = a + b(t + t^{-1})$$

$$\begin{aligned} P(t) + P(-t) &= (1-t)(1-t^{-1})(a + b(t + t^{-1})) + \\ &\quad (1+t)(1+t^{-1})(a - b(t + t^{-1})) = \\ &= (4a - 4b) - 2b(t^2 + t^{-2}) = 2 \Rightarrow \end{aligned}$$

$$\begin{cases} b = 0 \\ a = \frac{1}{2} \end{cases} \Rightarrow B(t) = \frac{1}{2}$$

$$L = 2$$

$$\begin{aligned} P(t) + P(-t) &= (1-t)^2(1-t^{-1})^2(a + b(t + t^{-1})) + \\ &\quad (1+t)^2(1+t^{-1})^2(a - b(t + t^{-1})) = \\ &= (t^{-2} + t^2)(2a - 8b) + (12a - 20b) = 2 \end{aligned}$$

$$\Downarrow$$

$$\begin{cases} 2a - 8b = 0 \\ 12a - 20b = 2 \end{cases}$$

$$\Downarrow$$

$$b = \frac{1}{8} \quad a = \frac{1}{2} \Rightarrow B(t) = \frac{1}{8}(4 + (t + t^{-1}))$$

DUE TO THE CONSTRAINT THIS IS THE  
SHORTEST SOLUTION

b) ORTHOGONALITY IMPLIES

$$P(z) = G_0(z) G_0(z^{-1}) \quad \text{AND} \quad H_0(z) = G_0(z^{-1})$$

THUS

$$L=1 \Rightarrow G_0(z) = \frac{1}{\sqrt{2}} (1 - z^{-1}) = H_0(z^{-1})$$

$$L=2 \Rightarrow G_0(z) = \frac{1}{2\sqrt{2}} (1 - z^{-1})^2 (2 + \sqrt{3} + z^{-1}) \frac{1}{\sqrt{2+\sqrt{3}}} = H_0(z^{-1})$$

THE FILTERS  $G_1(z)$  AND  $H_1(z)$  ARE IN BOTH CASES GIVEN BY:

$$G_1(z) = -z^{-1} G_0(-z^{-1})$$

$$H_1(z) = G_1(z^{-1})$$

c) BIORTHOGONAL CASE

$$L=1 \quad P(z) = (1-z)(1-z^{-1}) = \frac{1}{2} \quad \text{WITH} \quad H_0(-1) = 1$$

$H_0(z)$	$G_0(z)$
1	$\frac{1}{2} (1-z)(1-z^{-1})$
$\frac{1}{2} (1-z^{-1})$	$(1-z)$
$\frac{1}{4} (1-z)(1-z^{-1})$	2



$$L=2$$

$$p(t) = (1-t)^2 (1-t^{-1})^2 (t + 4 + t^{-1}) \frac{1}{8} \quad \text{with } H_0(-1)=1$$

$H_0(t)$	$G_0(t)$	
1	$(1-t)^2 (1-t^{-1})^2 (t + 4 + t^{-1}) \cdot \frac{1}{8}$	
$\frac{1}{2} (1-t)$	$(1-t)(1-t^{-1})^2$	$\frac{1}{4}$
$\frac{1}{4} (1-t)^2$	$(1-t^{-1})^2$	$\frac{1}{2}$
$\frac{1}{8} (1-t)^2 (1-t^{-1})$	$(1-t^{-1})$	
$\frac{1}{16} (1-t^2)(1-t^{-1})^2$		2

# QUESTION 4

10

(a) USING POISSON SUM FORMULA  
WE HAVE THAT

$$\sum_{n=-\infty}^{\infty} \varphi(t-n) = \sum_{k=-\infty}^{\infty} \hat{\varphi}(2\pi k) e^{j2\pi k t}$$

WE THUS NEED TO PROVE THAT

$$\sum_{k=-\infty}^{\infty} \hat{\varphi}(2\pi k) e^{j2\pi k t} = 1$$

OR EQUIVALENTLY

$$\hat{\varphi}(2\pi k) = \begin{cases} 1 & \text{FOR } k=0 \\ 0 & \text{FOR } k \neq 0 \end{cases}$$

BUT BY CONSTRUCTION

$$\hat{\varphi}(0) = \alpha(2) = 1$$

$$\hat{\varphi}(2\pi k) = \begin{cases} \alpha(2-3k) = 0 & k > 0 \\ \alpha(2+3k) = 0 & k < 0 \end{cases}$$

□

(b)

$\{\varphi(t-n)\}_{n \in \mathbb{Z}}$  is an orthonormal family from  $L_2(\mathcal{R})$ . To that end, we ~~use the Poisson formula and instead~~ show that ~~the following~~

$$\sum_{k \in \mathbb{Z}} |\Phi(\omega + 2k\pi)|^2 = 1. \quad (1)$$

From Figure 4.8 it is clear that for  $\omega \in [-(2\pi/3) - 2n\pi, (2\pi)/3 - 2n\pi]$

$$\sum_k |\Phi(\omega + 2k\pi)|^2 = |\Phi(\omega + 2n\pi)|^2 = 1.$$

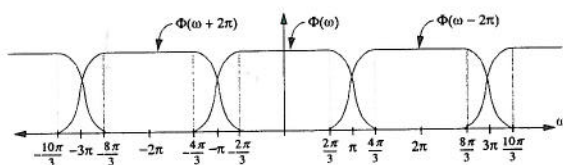


Figure 4.8 Pictorial proof that  $\{\varphi(t-n)\}_{n \in \mathbb{Z}}$  form an orthonormal family in  $L^2(\mathcal{R})$ .

The only thing left is to show (1) holds in overlapping regions. Thus, take for example,  $\omega \in [(2\pi)/3, (4\pi)/3]$ :

$$\begin{aligned} \Phi(\omega)^2 + \Phi(\omega - 2\pi)^2 &= a\left(2 - \frac{3\omega}{2\pi}\right) + a\left(2 + \frac{3(\omega - 2\pi)}{2\pi}\right) \\ &= a\left(2 - \frac{3\omega}{2\pi}\right) + a\left(-1 + \frac{3\omega}{2\pi}\right) \\ &= a\left(2 - \frac{3\omega}{2\pi}\right) + a\left(1 - \left(2 - \frac{3\omega}{2\pi}\right)\right) \\ &= 1. \end{aligned}$$

The last equation follows from the definition of ~~the function~~  $a(\omega)$ .

(c)

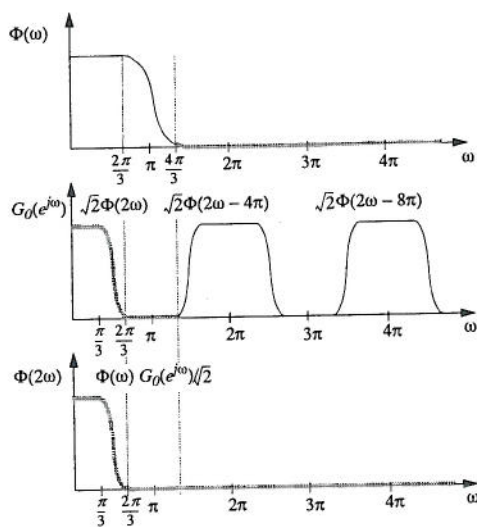
WE NEED TO SHOW FIND A

 $G(e^{j\omega})$  SUCH THAT THE FOLLOWING IS SATISFIED:

$$\hat{\psi}(2\omega) = \frac{G_0(e^{j\omega})}{\sqrt{2}} \hat{\psi}(\omega) \quad (2)$$

WE CAN SEE PICTORALLY FROM THE FIGURE BELOW THAT EQ. (2) IS SATISFIED WHEN

$$G_0(e^{j\omega}) = \sqrt{2} \sum_{k=-\infty}^{\infty} \hat{\psi}(2\omega + 4k\pi)$$



(d)

BECAUSE OF ORTHOGONALITY

$$G_1(e^{j\omega}) = -G_0^*(e^{j(\omega+\pi)}) e^{-j\omega}$$

THUS

$$\hat{\psi}(\omega) = -\frac{1}{\sqrt{2}} e^{-j\frac{\omega}{2}} \sum_{k=-\infty}^{\infty} \hat{\psi}(\omega + (k+1)\pi) \hat{\psi}\left(\frac{\omega}{2}\right)$$