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E3.20

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2007

EEE PART III/IV: MEng, BEng and ACGI

DISCRETE MATHEMATICS AND COMPUTATIONAL COMPLEXITY

Monday, 21 May 2:00 pm

Time allowed: 3:00 hours

There are FIVE questions on this paper.

Answer Question One (29%), Question TWO (29%) and TWO other questions.

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : G.A. Constantinides

Second Marker(s) : T.J.W. Clarke

Notation

The following notation is used throughout this paper:

$\mathbb{R}$: The set of real numbers.

$\mathbb{Z}$: The set of integers.

$\mathbb{Z}_+$: The set of positive integers.

$\mathbb{C}$: The set of complex numbers.

$\mathbb{N}$: The set of natural numbers.

$\mathscr{P}(S)$: The power set of set S.

The Questions

1. **[Compulsory]**

a) For the sets $S_1 = \{1,2\}$, $S_2 = \{2,3\}$, list the elements of

i) $S_1 \cup S_2$,

ii) $S_1 \cap S_2$,

iii) $S_1 - S_2$,

iv) $S_1 \times S_2$,

v) $\mathscr{P}(S_1)$.

[7]

b) Consider the relation $R = \{(1,2),(2,3),(3,4)\}$ on the set $\mathbb{R}$.

i) Let R be a function from A to B. Find the smallest cardinality A and B for this to be possible.

ii) List the elements of $R \cdot R$.

iii) List the elements of the transitive closure of R.

iv) Draw the digraph of R.

v) How many functions are there from the set R to itself?

[9]

c) Express each of the following statements using appropriate logical syntax. You should take the set of complex numbers as the universe of discourse, and make use of a predicate $P(x)$, meaning 'x is a real number'.

i) Every real number, when squared, gives a non-negative real number.

ii) Every quadratic equation with complex coefficients has two complex roots.

iii) There are two distinct real numbers that are solutions to $x^2 - 1 = 0$.

[9]

d) i) Consider two functions $f(x)$ and $g(x)$. $f(x)$ is known to be $\Theta(x^2)$ and $g(x)$ is known to be $\Theta(x)$. Find functions $p(x)$ and $q(x)$ such that $f(x)+g(x)$ is $O(p(x))$ and $f(x)g(x)$ is $O(q(x))$.

ii) Write some pseudo-code for a function `fun1(x)` that has worst-case execution time $O(g(x))$ and for a function `fun2(x)` that has worst-case execution time $O(f(x))$.

iii) Show that if $r(x)$ is $\Theta(s(x))$ then $s(x)$ is $\Theta(r(x))$.

[9]

e) Write some pseudo-code for a function whose worst-case execution time satisfies $f(n) = 3f(n/2) + n$ whenever n is an even number. Find a big-O expression for the worst-case execution time of this function.

[6]

2. **[Compulsory]**

This question is concerned with the *reachability* decision problem, which can be expressed as follows. Given a relation R on a set A and two elements $a_1 \in A$ and $a_2 \in A$, is the following logical condition true? $\exists n((a_1, a_2) \in R^n)$, where universe of discourse is $\mathbb{Z}_+$. Figure 2.1 illustrates some pseudo-code for solving this decision problem.

a) Consider the two digraphs shown in Figure 2.2(a) and (b). For each case, answer the corresponding reachability instance, and calculate how many times the function shown in Figure 2.1 is called (including the initial call).

[14]

b) Consider $|R|$ as the size of a reachability instance, and assume that every atomic operation in the algorithm shown in Figure 2.1 takes $\Theta(1)$ time. Is this algorithm exponential time or polynomial time? Justify your answer.

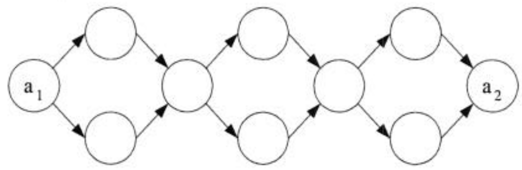
[13]

c) Suggest an improved version of this algorithm, write and explain the pseudo-code, and discuss whether this version is polynomial time.

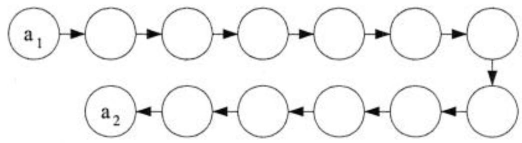
[13]

```
reach( R, a1, a2 )
begin
    if (a1,a2) ∈ R then
        result := true
    else begin
        result := false
        for every a such that (a1,a) ∈ R
        begin
            if reach( R, a, a2 ) then
                result := true
        end
    end
end
```

Figure 2.1 An algorithm for solving the reachability decision problem



(a) Digraph 1



(b) Digraph 2

Figure 2.2 Two digraphs

3. a) Consider the function $f : A \to \mathbb{C}$ defined by $f(x) = \frac{e^{jx}}{1-x}$, where $j = \sqrt{-1}$.

i) If $A = \mathbb{R} - K$, what is the smallest K, in the sense that if $A = \mathbb{R} - K'$ then $K \subseteq K'$?

ii) Show that for this choice of K, the function f is an injection.

iii) Show that f is not a surjection for this choice of K.

[10]

b) i) Show that the transitive closure of a relation R is equal to its connectivity relation R^*. You may assume that for an arbitrary relation Q, (i) Q is transitive iff Q^n is transitive for all positive integers n, (ii) Q is transitive iff $Q^n \subseteq Q$ for all positive integers n.

ii) Consider the function $g : S \to S$ defined by $g(x) = \lfloor \sqrt{x} \rfloor$. For $S = \{1, 2, ..., 16\}$, list the elements of $g \cdot g$ and the transitive closure g^* of g.

[20]

4. This question uses predicate logic to describe the behaviour of the simple circuit shown in Figure 4.1. Let the universe of discourse, corresponding to the set of clock periods, be $\mathbb{N}$. Each wire $i \in \{1,2\}$ is associated with a predicate $P_i(t)$. A logic-0 is present on a wire at a particular cycle t if the corresponding proposition $P_i(t)$ is false, and a logic-1 is present if the corresponding proposition $P_i(t)$ is true.

An axiom describing the function of the D-type flip-flop is given in equation (4.1).

$$\neg P_1(0) \wedge \forall t (P_1(t+1) \leftrightarrow P_2(t)). \tag{4.1}$$

a) Write a corresponding axiom for the inverter.

[4]

b) Write a proposition corresponding to the English sentence 'the inverter output at cycle 1 will have the opposite logical value to the inverter output at cycle 0'.

[5]

c) From the inverter and the flip-flop axioms, formally derive your proposition as conclusion. State the rule of inference used at each step in your working.

[15]

d) Show further that the conclusion $P_1(1)$ can be reached. State the rule of rule of inference used at each step in your working.

[6]

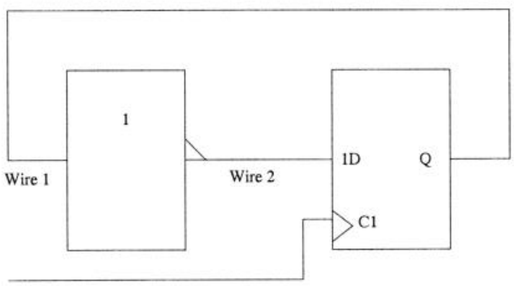


Figure 4.1 A circuit

5. This question is about multiplying two n-bit numbers, where n is a power of two, using only addition and shift arithmetic operations, each of which takes $\Theta(n)$ time for n bits. We shall use the operator '$<<$' to denote left-shift, *i.e.* $x << y$ means 'x left-shifted by y bits'. We shall also represent n-bit numbers by binary arrays of length n, where the most-sigificant bit is element $n-1$ and the least significant bit is element 0.

a) A possible multiplication algorithm is shown in Figure 5.1. Derive a big-Θ expression for the execution time of this algorithm.

[7]

b) Let us denote the least-significant $n/2$ bits of A by A_L and the most-significant $n/2$ bits by A_H, and similarly for B, so that $A = 2^{n/2}A_H + A_L$ and $B = 2^{n/2}B_H + B_L$. Notice that $AB = 2^n(A_H B_H) + 2^{n/2}\{(A_L + A_H)(B_L + B_H) - A_H B_H - A_L B_L\} + A_L B_L$. Use this observation to propose a recursive multiplication algorithm.

[8]

c) State the Master Theorem.

[8]

d) Derive a big-O expression for the recursive execution time, and comment on the result.

[7]

```
multiply( binary A[n], binary B[n] )
begin
   result := 0
   for i := n − 1 downto 0
      if( A[i] = 1 ) then
         result := (result << 1) + B
      else
         result := (result << 1)
end
```

Figure 5.1 An algorithm for multiplying two numbers

Discrete maths & Computational Complexity 1

SOLUTIONS

E2.17/E3.20

master - 16/4/07.

Q1

(NEW COMPUTED EXAMPLE)

a) (i) $S_1 \cup S_2 = \{1, 2, 3\}$

(ii) $S_1 \cap S_2 = \{2\}$

(iii) $S_1 - S_2 = \{1\}$

(iv) $S_1 \times S_2 = \{(1,2), (1,3), (2,2), (2,3)\}$

(v) $P(S_1) = \{\phi, \{1\}, \{2\}, \{1,2\}\}$.

[7]

b) (i) $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$

(ii) $R \circ R = \{(1,3), (2,4)\}$

(iii) $R^* = \{(1,2), (1,3), (1,4), (2,3), (2,4), (3,4)\}$.

(iv)

(v) $|R| = 3$ $\quad |R|^{|R|} = 3^3 = 27$

[9]

(NEW COMPUTED EXAMPLE)

c) i) $\forall x \left(P(x) \rightarrow (x^2 \geq 0) \right)$

ii) $\forall a \forall b \forall c \exists x_1 \exists x_2 \left(a x_1^2 + b x_1 + c = 0 \wedge \right.$
$\left. a x_2^2 + b x_2 + c = 0 \right)$

iii) $\exists x_1 \exists x_2 \left(x_1 \neq x_2 \wedge P(x_1) \wedge P(x_2) \right.$
$\left. \wedge\ x_1^2 - 1 = 0 \wedge x_2^2 - 1 = 0 \right)$

[9]

d) i) $p(x) = x^2$ (say) (NEW COMPUTED EXAMPLE)

$q(x) = x^3$ (say)

ii)

```
fun1 (x)
{ total := 0
  for i = 1 to x
      for j = 1 to x
          total := total + 1
}
```

```
fun2 (x)
{
  total := 0
  for i = 1 to x
      total := total + 1
}
```

(iii) $r(x)$ is $\Theta(s(x))$

$\exists c_1, c_2, K$ s.t.

$\Rightarrow c_1|s(x)| \leq |r(x)| \leq c_2|s(x)|$

when $x \geq K$

$c_1|s(x)| \leq |r(x)|$

$\Rightarrow |s(x)| \leq \frac{1}{c_1}|r(x)|$ (as c_1 +ve)

$\Rightarrow$ $s(x)$ is $O(r(x))$ ($c = \frac{1}{c_1}$, K as before)

$c_2|s(x)| \geq |r(x)|$

$\Rightarrow |s(x)| \geq \frac{1}{c_2}|r(x)|$ ($c = \frac{1}{c_2}$, K as before)

$\Rightarrow$ $s(x)$ is $\Omega(r(x))$

$\therefore$ $s(x)$ is $\Theta(r(x))$.

[9]

(NEW COMPUTED EXAMPLE)

e)

```
fun(n: integer)
{  total := 0;
   for i=1 to 3
      fun(⌊n/2⌋)

   for i=1 to n
      total := total+1;
}
```

(NEW COMPUTED EXAMPLE)

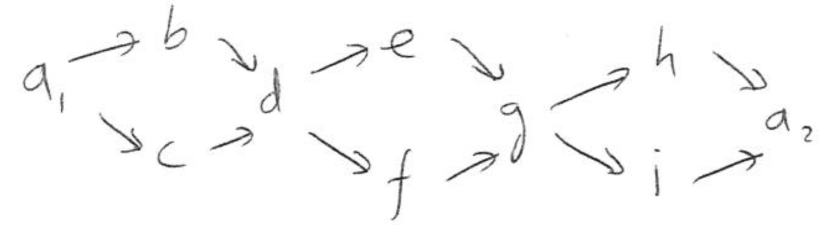
[6]

c.f. $f(n) = af(n/b) + cn^d$ $a > b^d$

$a=3,\ b=2,\ c=1,\ d=1$ $\Rightarrow \underline{\underline{O(n^{\log_2 3})}}$

Q2

a)(i)



$reach(R, a_1, a_2)$

calls $reach(R, b, a_2)$

& $reach(R, c, a_2)$

$reach(R, b, a_2)$ calls $reach(R, d, a_2)$

$reach(R, c, a_2)$ calls $reach(R, d, a_2)$

Total of 4 subroutine calls

Thus total $= 1 + 4 + 2 \times 4 + 2^2 \times 4 - 2^2 \times 2$

~~= 29 times~~ ~~= 25 time~~

$= \underline{\underline{21}}$ times

(ii) $a_1 \rightarrow b \rightarrow \cdots \rightarrow a_2$

$reach(R, a_1, a_2)$

calls $reach(R, b, a_2)$

etc.

Total of 11 subroutine calls + original call

$= \underline{\underline{12 \text{ calls}}}$.

[14]

(NEW COMPUTED EXAMPLE)

b) Extending the instance from Fig 1.2(a) gives a digraph with $1+3k$ nodes and $|R| = 4k$ edges.

Total # ~~call~~ subroutine calls is

$$4 + 2\times 4 + \dots + 2^{k-1}\times 4 - 2^{k-1}\times 2$$

$$= 4\left(1+2+\dots+2^{k-1}\right) - 2^{k}$$

$$= 4\left(\frac{1-2^k}{1-2}\right) - 2^k$$

$$= 4\left(2^k - 1\right) - 2^k$$

$$= 4\left(2^{|R|/4} - 1\right) - 2^{|R|/4}$$

is $\Omega\left(2^{|R|}\right)$

$\Rightarrow$ Exponential time.

[13]

(NEW COMPUTED EXAMPLE)

c) Two main improvements

(i) early exit

(ii) mark visited nodes

```
visited[ ] → init to false
reach(R, a1, a2)
begin
    if (a1, a2) ∈ R then
        result := true
    else begin
        result := false
        while( result = false ) do
            select next a s.t. (a1, a) ∈ R
            if reach(R, a, a2) then
                result := true
            else
            if NOT visited[a] then
                if reach(R, a, a2) then
                    result := true
        end
    end
end
```

[initials]

c) [continued]

Total #subroutine calls is at most #nodes.

~~⇒~~ ⇒ poly time. $(R \subseteq V \times V)$.

[13]

Q3

a) (i) Every real x has a corresponding $f(x) \in \mathbb{C}$ except 1. So $K = \{1\}$.

(ii) $f(x) = f(y)$

$$\frac{e^{jx}}{1-x} = \frac{e^{jy}}{1-y} \Rightarrow \left|\frac{e^{jx}}{1-x}\right| = \left|\frac{e^{jy}}{1-y}\right|$$

$$\Rightarrow \frac{1}{1-x} = \frac{1}{1-y} \Rightarrow 1-y = 1-x$$

$$\Rightarrow \underline{\underline{x = y}}$$

(iii) $\frac{1}{1-\pi} \in \mathbb{C}$

for $f(x) = \frac{1}{1-\pi}$

we would require $\frac{e^{jx}}{1-x} = \frac{1}{1-\pi}$

$$\Rightarrow \frac{1}{1-x} = \frac{1}{1-\pi} \Rightarrow x = \pi.$$

But at $x = \pi$, $f(x) = \frac{j}{1-\pi} \neq \frac{1}{1-\pi}$

CONTRADICTION.

[10]

b) (i) $R^* = R \cup R^2 \cup \ldots$

We want to show R^* is the smallest transitive relation containing R.

Proof: a) $R \subseteq R^*$ directly

We must show b) R^* is transitive

c) $R^* \subseteq S$ for any transitive S s.t. $R \subseteq S$.

b) $(a,b) \in R^*$, $(b,c) \in R^*$.

R^* is set of all paths $\Rightarrow$ $(a,c) \in R^*$;

follow path $a \to b$ & then $b \to c$.

c) Since S is transitive, S^n is transitive and $S^n \subseteq S$.

$S^* = S \cup S^2 \cup \ldots$, $S^n \subseteq S$

$\Rightarrow S^* \subseteq S$

$R \subseteq S \Rightarrow R^* \subseteq S^*$

$\therefore R^* \subseteq S^* \subseteq S$

(BOOKWORK)

(ii) $g = \{(1,1), (2,1), (3,1), (4,2), (5,2), (6,2), (7,2), (8,2), (9,3), (10,3), (11,3), (12,3), (13,3), (14,3), (15,3), (16,4)\}$

$g \cdot g = \{(1,1), (2,1), (3,1), (4,1), (5,1), (6,1), (7,1), (8,1), (9,1), (10,1), (11,1), (12,1), (13,1), (14,1), (15,1), (16,2)\}$

$g^* = \{(1,1), (2,1), (3,1), (4,1), (5,1), (6,1), (7,1), (8,1), (9,1), (10,1), (11,1) (12,1), (13,1), (14,1), (15,1), (16,1), (4,2), (5,2), (6,2), (7,2), (8,2), (16,2), (9,3), (10,3), (11,3), (12,3), (13,3), (14,3), (15,3), (16,4)\}.$

[illegible]

[20]

(NEW COMPUTED EXAMPLE)

4. a) $\forall t \, (P_2(t) \Leftrightarrow \neg P_1(t))$ [4]

b) $P_2(1) \Leftrightarrow \neg P_2(0)$ [5]

c) $\forall t \, (P_1(t+1) \Leftrightarrow P_2(t))$

(Simplification from 4.1)

$P_1(1) \Leftrightarrow P_2(0)$ (*)

(Universal instantiation)

$P_2(1) \Leftrightarrow \neg P_1(1)$ (+)

(Universal instantiation of (b))

$P_2(1) \rightarrow \neg P_1(1)$

(Simplification)

$P_2(0) \Rightarrow P_1(1)$

(Simplification of *)

$\neg P_1(1) \Rightarrow \neg P_2(0)$ [Contrapositive]

$\underline{P_2(1) \Rightarrow \neg P_2(0)}$

(Hypothetical Syllogism)

Also $P_1(1) \Rightarrow P_2(0)$ (Simplification of *)

$\neg P_2(0) \Rightarrow \neg P_1(1)$ [Contrapositive]

$\neg P_1(1) \Rightarrow P_2(1)$ (Simplification of +)

$\underline{\neg P_2(0) \rightarrow P_2(1)}$ (Hypothetical syllogism)

[15]

d) We have

$$P_2(1) \leftrightarrow \neg P_2(0)$$

and $\neg P_1(0)$ (simplification from 4.1)

The latter gives $P_2(0)$

(Universal instant + modus ponens)

Thus from (c) we get $\neg P_2(1)$

(modus tollens)

Finally, we obtain $P_1(1)$

(Universal instant + modus tollens)

~~[4]~~

[6]

Q5 a) "for" loop executes $\Theta(n)$ times.

Assuming tests take $\Theta(1)$ time, each iteration is $\Theta(n)$.

Total is $\Theta(n) \times \Theta(n) + \Theta(1)$

$$= \underline{\underline{\Theta(n^2)}}.$$

~~[8]~~ [7]

(NEW COMPUTED EXAMPLE)

b) newmult(binary A[n], binary B[n])

begin

~~[illegible]~~

if (n = 1)

if (A[1] = 0)

result := 0

else result := B

else begin

$A_H = A[n-1 \ldots n/2]$

$A_L = A[n/2 - 1 \ldots 0]$

$B_H = B[n-1 \ldots n/2]$

$B_L = B[n/2 - 1 \ldots 0]$

T1 = ~~A_H~~ newmult(A_H, B_H)

T2 = newmult($A_L + A_H$, $B_L + B_H$)

T3 = newmult(A_L, B_L)

result := (T1 << n) + (T2 − T1 − T3) << (n/2)

+ (T3).

end

end

~~[8]~~ [8]

c) Let $a \geq 1$ be real, $b > 1$ be integer, $c > 0$ be real, $d \geq 0$ be real

Let $f(n)$ be increasing s.t.

$$f(n) = a f(n/b) + cn^d \quad \text{when}$$

$$n = b^k, \text{ for +ve integer } k.$$

(i) If $a < b^d$, $f(n)$ is $O(n^d)$

(ii) If $a = b^d$, $f(n)$ is $O(n^d \log n)$

(iii) If $a > b^d$, $f(n)$ is $O(n^{\log_b a})$

~~[8]~~ [8]

(BOOKWORK)

d) We have

$$f(n) = 3f(n/2) + \Theta(n)$$

$a > b^d$

so $f(n)$ is $O\left(n^{\log_2 3}\right)$

and so is run time. This is an improvement on the $\Theta(n^2)$ algorithm.

~~[8]~~ [7]

(NEW COMPUTED EXAMPLE)