1- sided
master

IMPERIAL COLLEGE LONDON

E3.07
AU1
ISE3.11

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2007

EEE/ISE PART III/IV: MEng, BEng and ACGI

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DIGITAL SIGNAL PROCESSING

Tuesday, 15 May 2:30 pm

Time allowed: 3:00 hours

There are SIX questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s) :	P.A. Naylor
	Second Marker(s) :	D.P. Mandic

DIGITAL SIGNAL PROCESSING

1. a) Briefly describe the operation of the M-fold decimator and the L-fold expander shown in Figure 1.1 and give analytical expressions for $y_D(n)$ and $y_E(n)$ in terms of $x(n)$.

M-fold decimator L-fold expander

Figure 1.1

[6]

b) The filter

$$H(z) = \sum_{k=0}^{N} h(k) z^{-k}$$

is to be employed in a 2-fold decimation scheme using type 1 polyphase filters obtained from $H(z)$. The symmetry condition $h(k) = h(N-k)$ is satisfied.

i) Write down the polyphase components of $H(z)$ and derive their symmetry conditions for both N even and N odd.

ii) Design one simple example for the case $N = 4$ and one simple example for the case $N = 5$ that illustrate your solution.

[9]

c) Now consider upsampling. Draw the signal flow graph of a 2-fold upsampling system employing type 1 polyphase filters. State why it may be desirable to employ the Noble Identities in this context and draw a further signal flow graph illustrating their use. [5]

2. Consider a complex discrete-time signal $x(n)$ from which two new signals are derived as

$$x_1(n) = \frac{1}{2}(x(n) + x^*(-n))$$
$$x_2(n) = \frac{1}{2}(x(n) - x^*(-n))$$

where $x^*(n)$ denotes the complex conjugate of $x(n)$.

a) When $x(n)$ is periodic with one period having sample values $[1,\ 2,\ 3,\ -1]$, find $x_1(n)$ and $x_2(n)$. Hence state any symmetry properties of $x_1(n)$ and $x_2(n)$.
[5]

b) Show that if the Fourier transform of $x(n)$ is $X(e^{j\omega})$ then the Fourier transform of $x^*(-n)$ is $X^*(e^{j\omega})$.
[5]

c) Find the Fourier transform of $x_1(n)$ in terms of the Fourier transform of $x(n)$.
[5]

d) The 6-point sequence given by

$$V(k) = [1,\ 2,\ -3,\ -3,\ 2,\ 1], \quad k = 0,\ 1,\ \ldots,\ 5$$

has inverse discrete Fourier transform

$$v(n), \quad n = 0,\ 1,\ \ldots,\ 5.$$

Determine whether $v(n)$ is real or complex and give the reasoning to support your answer.
[5]

3. a) Define the following underlined terms.

i) A system is stable.

ii) A system is non-causal.

iii) A system is minimum phase.

[3]

b) Given the signal representation in the z-domain

$$X(z) = z^2 + 3z + 5 + \frac{2}{z^2 + 5z + 4}, \quad |z| > 4$$

find the corresponding signal representation in the time domain. [6]

c) Consider a linear time-invariant system described by the z-domain system function

$$H(z) = \frac{(z - \frac{1}{2})(z + 2)(z^2 + \frac{1}{9})}{(z^2 + 2z + 5)(z^2 - 4z + 13)}.$$

Plot and appropriately label the roots of $H(z)$. [6]

d) Determine any regions of convergence of $H(z)$ and state whether the inverse z-transform associated with each region of convergence is left-sided, right-sided or two-sided. Comment on the stability of $H(z)$. [5]

4. Consider a multirate system comprising an analysis filter bank followed by a synthesis filter bank. The analysis filter bank contains filters $H_k(z)$ and the synthesis filter bank contains filters $G_k(z)$ for $k = 0, 1, \ldots, L-1$.

a) Sketch and fully label the block diagram of this system in its basic form. [4]

b) State and explain the advantages of polyphase implementation of filter banks. Draw a block diagram of the multirate system of a) implemented in polyphase form. [4]

c) The filter $H_k(z)$ can be implemented in Type 1 polyphase form as

$$H_k(z) = \sum_{l=0}^{L-1} z^{-l} E_{kl}(z^L), \quad k = 0, 1, \ldots, L-1$$

Find the relationship between the coefficients of the filter $E_{kl}(z)$ and the coefficients of the filter $H_k(z)$. [3]

d) Show that the expression for $H_k(z)$ can be written in matrix form as

$$\mathbf{h}(z) = \mathbf{E}(z^L)\mathbf{e}(z)$$

and give the definitions of $\mathbf{h}(z)$, $\mathbf{E}(z)$ and $\mathbf{e}(z)$, where $\mathbf{E}(z)$ is called the polyphase component matrix of the analysis filter bank. [5]

e) Consider the matrix $\mathbf{R}(z)$ which is the polyphase component matrix of the synthesis filter bank. Define what is meant by *perfect reconstruction* in this context and deduce the condition on $\mathbf{R}(z)$ for perfect reconstruction.

[4]

5. Consider the digital filter

$$H(z) = 0.5(1-\alpha)\frac{1+z^{-1}}{1-\alpha z^{-1}}$$

with $|\alpha| < 1$.

a) Show that the magnitude squared frequency response can be written as

$$|H(e^{j\omega})|^2 = \frac{(1-\alpha)^2(1+\cos\omega)}{2(1+\alpha^2-2\alpha\cos\omega)}$$

[6]

b) Find the derivative of $|H(e^{j\omega})|^2$ with respect to ω and hence show that $|H(e^{j\omega})|$ is a monotonically decreasing function of ω for $0 < \omega < \pi$. [6]

c) Draw a labelled sketch of the magnitude of the frequency response $H(e^{j\omega})$ and mark on your sketch the -3 dB cut-off frequency ω_c. [5]

d) Determine an expression for α in terms of ω_c. [3]

6. A causal digital filter with input $x(n)$ has output $y(n)$ such that

$$y(n) = x(n) + x(n-2) + y(n-1) - 0.5y(n-2)$$

where n is the discrete time index.

a) Find the poles and zeros associated with this filter and sketch a plot of them on the z-plane. [6]

b) Determine an expression for the impulse response of the filter and show that the impulse response is real valued. [8]

c) Draw a labelled signal flow diagram for this system

i) in Direct Form I, [3]

ii) using the minimum number of memory elements. [3]

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1. a) A brief textual explanation is expected in addition to the equations:

$$y_D(n) = x(Mn)$$
$$y_E(n) = \begin{cases} x(n/L) & n = kL, \quad k \text{ integer} \\ 0 & \text{otherwise.} \end{cases}$$

[6]

b) The filter $H(z)$ is symmetric giving it a linear phase response.

The type 1 polyphase decompositions are given by

$$H(z) = E_0(z^2) + z^{-1}E_1(z^2)$$
$$E_0(z) = \sum_{n=0}^{N/2} h(2n)z^{-n}$$
$$E_1(z) = \sum_{n=0}^{N/2} h(2n+1)z^{-n}$$

For an even order example with order 4

$$H(z) = 1 + 2z^{-1} + 4z^{-2} + 2z^{-3} + z^{-4}$$
$$E_0 = 1 + 4z^{-1} + z^{-2}$$
$$E_1 = 2 + 2z^{-1}$$

which gives $E_0(z)$ and $E_1(z)$ to be symmetric.

For an odd order example

$$H(z) = 1 + 2z^{-1} + 3z^{-2} + 3z^{-3} + 2z^{-4} + z^{-5}$$
$$E_0 = 1 + 3z^{-1} + 2z^{-2}$$
$$E_1 = 2 + 3z^{-1} + 1$$

which gives $E_0(z)$ and $E_1(z)$ as an order-reversed pair.

[9]

c) The upsampler is shown in 3 forms in Figure 1.1, starting with the direct implementation, then the polyphase filter and finally the Noble identities are applied to make the filters functions of z not z^2. This improves computational efficiency by a factor of 2.

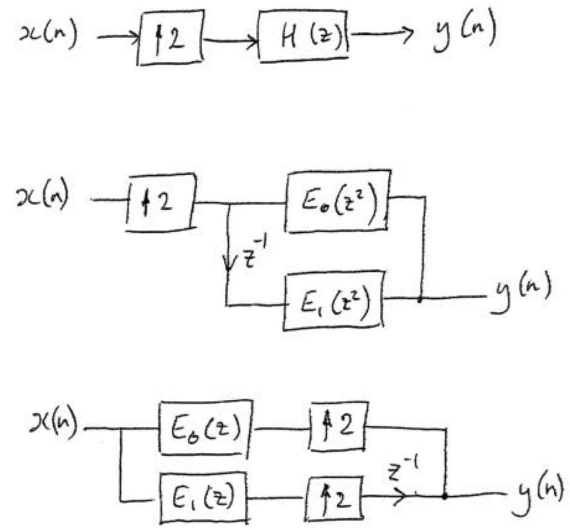


Figure 1.1

2. a)

$$\begin{aligned} x(n) &= [1 \quad 2 \quad 3 \quad -1] \\ x_1(n) &= [1 \quad 0.5 \quad 3 \quad 0.5] \\ x_2(n) &= [0 \quad 1.5 \quad 0 \quad -1.5] \end{aligned}$$

x_1 is even symmetric and x_2 is odd symmetric.

b) Consider

$$\mathscr{F}\{x(n)\} = X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-jn\omega}$$

Write

$$\begin{aligned} x(n)e^{-jn\omega} &= (x_r + jx_i)(\cos n\omega - j\sin n\omega) \\ &= x_r\cos(n\omega) + x_i\sin(n\omega) + j(x_i\cos(n\omega) - x_r\sin(n\omega)) \end{aligned}$$

Now see that the effect of conjugating $x(n)$ on the LHS is to replace ω with $-\omega$ on the RHS:

$$\begin{aligned} x^*(n)e^{-jn\omega} &= (x_r - jx_i)(\cos n\omega - j\sin n\omega) \\ &= x_r\cos(-n\omega) + x_i\sin(-n\omega) - j(x_i\cos(-n\omega) - x_r\sin(-n\omega)) \end{aligned}$$

since $\cos(-n\omega) = \cos(n\omega)$ and $\sin(-n\omega) = -\sin(n\omega)$. So the former expression is the complex conjugate of the latter with ω replaced by $-\omega$, and therefore

$$x^*(n) \Longleftrightarrow X^*(e^{-j\omega})$$

Hence, by replacing n by $-n$, the former and latter expressions become identical except for the conjugation so that

$$x^*(-n) \Longleftrightarrow X^*(e^{j\omega})$$

c)

$$\begin{aligned} \mathscr{F}\{x_1^*(n)\} &= \mathscr{F}\left\{\frac{1}{2}(x(n) + x^*(-n))\right\} \\ &= \frac{1}{2}\left(X(e^{j\omega}) + X^*(e^{j\omega})\right) \\ &= \frac{1}{2}\left(X_r(e^{j\omega}) + jX_i(e^{j\omega}) + X_r(e^{j\omega}) - jX_i(e^{j\omega})\right) \\ &= X_r(e^{j\omega}) \end{aligned}$$

where $X_r(e^{j\omega})$ represents the real part of $X(e^{j\omega})$ and $X_i(e^{j\omega})$ represents the imaginary part.

d) We require

$$X(k) = X(-k_{modN})$$

to obtain a real $x(n)$ which is not satisfied there. Hence IDFT is complex.

3. a) Bookwork

b)

$$X(z) = z^2 + 3z + 5 + \frac{2}{z^2 + 5z + 4}$$
$$= z^2 + 3z + 5 + \frac{2/3}{z+1} - \frac{2/3}{z+4}$$
$$x(n) = \delta(n+2) + 3\delta(n+1) + 5\delta(n) + \frac{2}{3}(-1)^{n-1}u(n-1) - \frac{2}{3}(-4)^{n-1}u(n-1)$$

c)

$$H(z) = \frac{(z-0.5)(z+2)(z^2+1/9)}{(z^2+2z+5)(z^2-4z+13)}.$$

has zeros at $z = 1/2, \quad -2, \quad \pm j/3$
and poles at $z = -1 \pm j2, \quad 2 \pm j3$

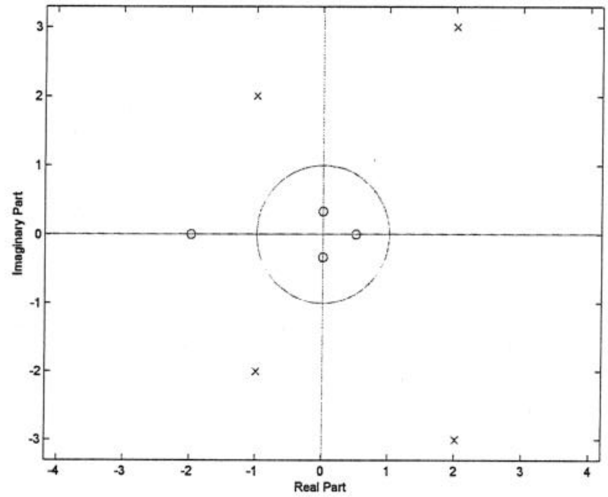


Figure 3.1

d) The moduli of the pole pairs are 2.24 and 3.61. Hence the possible ROCs are

i) $|z| < 2.24$
ii) $2.24 < |z| < 3.61$
iii) $|z| > 3.61$

For 1, the ROC is interior to all poles, hence inverse z-transform is left-sided.
For 3, the ROC is exterior to all poles, hence inverse z-transform is right-sided
For 2, the ROC is bounded by the two complex pole-pairs, hence inverse z-transform is two-sided.

For stability, the ROC must included the unit circle, hence only 1 is stable.

4. a)

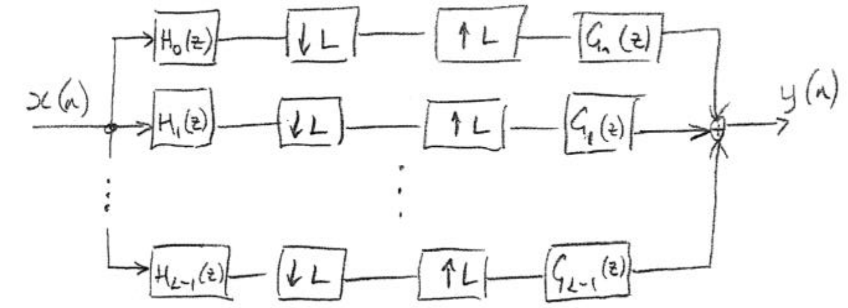


Figure 4.1

b) Polyphase filter banks benefit from efficiency in that all filter computations can be performed at the lowest sampling rate in the system.

c) For the kth filter, the coefficients of $E_{kl}(z)$ are a subset of the coefficients of $H(z)$ obtained by subsampling such that

$$e_{kl} = h_{kp} \quad p = lL$$

d)

$$\mathbf{h}(z) = \mathbf{E}(z^L)\mathbf{e}(z)$$

$$\mathbf{h}(z) = [H_0(z)\ H_1(z)\ \ldots\ H_{L-1}(z)]^T$$
$$\mathbf{e}(z) = [1\ z^{-1}\ \ldots\ z^{-(L-1)}]^T$$
$$\mathbf{E}(z) = \begin{bmatrix} E_{00}(z) & E_{01}(z) & \ldots & E_{0,L-1}(z) \\ E_{10}(z) & E_{11}(z) & \ldots & E_{1,L-1}(z) \\ \vdots & \vdots & \ddots & \vdots \\ E_{L-1,0}(z) & E_{L-1,1}(z) & \ldots & E_{L-1,L-1}(z) \end{bmatrix}$$

e)

$$\mathbf{R}(z)\mathbf{E}(z) = \mathbf{I}$$

5. a) For frequency response, z is replaced with $e^{j\omega}$ and for the modulus squared we compute HH^*. The modulus of the numerator is easily seen from geometry on the z-plane by considering the unit circle. The denominator terms need to be multiplied out.

b) By using the quotient rule

$$\frac{d|H(e^{j\omega})|^2}{d\omega} = \frac{-(1-\alpha)^2(1+2\alpha+\alpha^2)\sin\omega}{2(1+\alpha^2-2\alpha\cos\omega)^2}$$

The numerator is always negative since the sin runs only from $0 < \omega < \pi$. The denominator is always positive since it is a square. Hence the derivative is never positive.

c) The sketch is expected to show a maximum at $|H(e^{j0})| = 1$ and a minimum at $|H(e^{j\pi})| = 0$. A small number (two or three) of intermediate points will help to indicate the general shape. Marks will not be lost for lack of accuracy in the sketch providing the general trend is clear.

d) We require $|H(e^{j\omega})|^2 = 1/2$ which leads to

$$(1-\alpha)^2(1+\cos\omega_c) = 1+\alpha^2-2\alpha\cos\omega_c$$

so that

$$\cos\omega_c = \frac{2\alpha}{1+\alpha^2}.$$

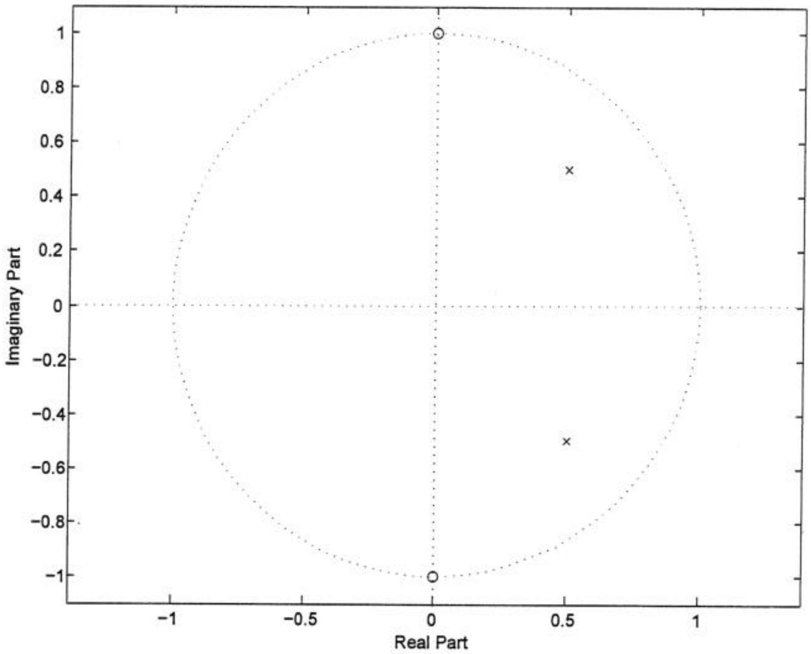
This has two possible solutions from which we pick the stable solution

$$\alpha = \frac{1-\sin\omega_c}{\cos\omega_c}.$$

6. a) The system function can be written

$$\frac{Y(z)}{X(z)} = \frac{1+z^{-2}}{1-z^{-1}+0.5z^{-2}}$$

This has zeros at $z = \pm j$ and poles at $z = 0.5 \pm j0.5$



b)

$$\begin{aligned} H(z) &= \frac{1+z^{-2}}{1-z^{-1}+0.5z^{-2}} \\ &= \frac{1+z^{-2}}{(1-pz^{-1})(1-p^*z^{-1})} \text{where} p = 0.5+j0.5 \\ &= 1 + \frac{z^{-1}+0.5z^{-2}}{(1-pz^{-1})(1-p^*z^{-1})} \end{aligned}$$

The inverse z-transform of 1 is $\delta(n)$. The inverse z-transform of the fraction is found by partial fraction expansion.

$$\frac{z^{-1}+0.5z^{-2}}{1+z^{-1}+0.5z^{-2}} = \frac{A}{z-p} + \frac{A^*}{z-p^*} \text{ where } p = -0.5+j0.5 \text{ and } A = 0.5-j.$$

leading to

$$h(n) = \delta(n) + Ap^n u(n-1) + A^*(p^*)^n u(n-1)$$

To show that his is real-valued, note that $\delta(n)$ is real by definition and show that the 2nd and 3rd terms on the RHS are real by writing

$$A = |A|\, e^{j\alpha} \text{ and } p = |p|\, e^{j\beta}$$

so that

$$\begin{aligned} Ap^n u(n-1) + A^*(p^*)^n u(n-1) &= |A|\,|p|^n \left(e^{j(\alpha+\beta)} + e^{-j(\alpha+\beta)}\right)\, u(n-1) \\ &= 2\,|A|\,|p|^n\, \cos(\alpha+\beta)\, u(n-1) \end{aligned}$$

c) The signal flow graphs are drawn as