

QUESTION 1

WAVELETS + APPLICATIONS
SOLUTIONS 2008

E445 1
IEE 4.47
CS-21
S022

$$a) C(\tau) = \frac{1}{2} \left[H(\tau^{\frac{1}{2}}) X(\tau^{\frac{1}{2}}) + H(-\tau^{\frac{1}{2}}) X(-\tau^{\frac{1}{2}}) \right]$$

$$Y(\tau) = \frac{1}{2} G(\tau) \left[H(\tau) X(\tau) + H(-\tau) X(-\tau) \right]$$

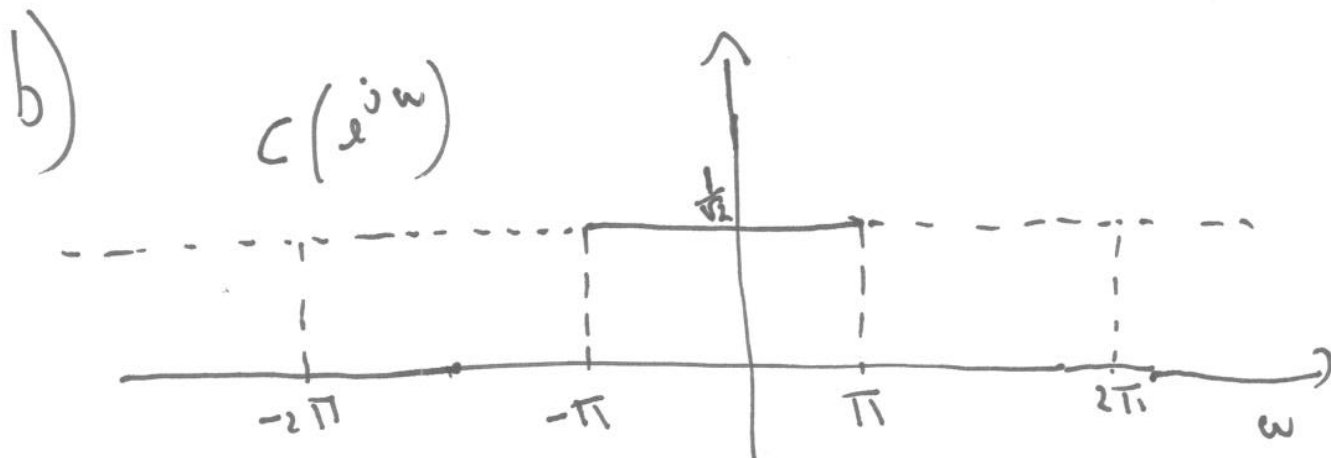
$$D(\tau) = X(\tau) - Y(\tau)$$

$$\hat{X}(\tau) = X(\tau) - \frac{1}{2} G(\tau) \left[H(\tau) X(\tau) + H(-\tau) X(-\tau) \right] + \frac{1}{2} F(\tau) \left[H(\tau) X(\tau) + H(-\tau) X(-\tau) \right]$$

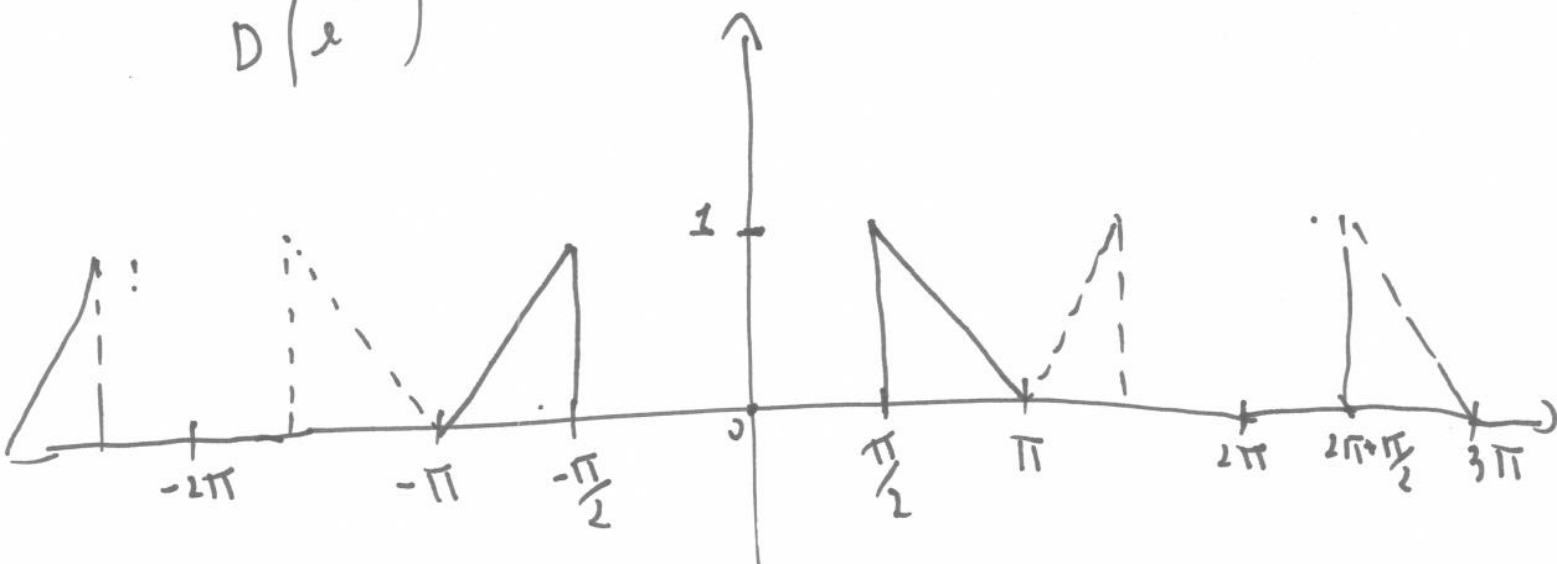
\Downarrow

$$PR: F(\tau) = G(\tau)$$

NOTICE THAT PR CONDITION DOES NOT DEPEND ON $H(\tau)$.

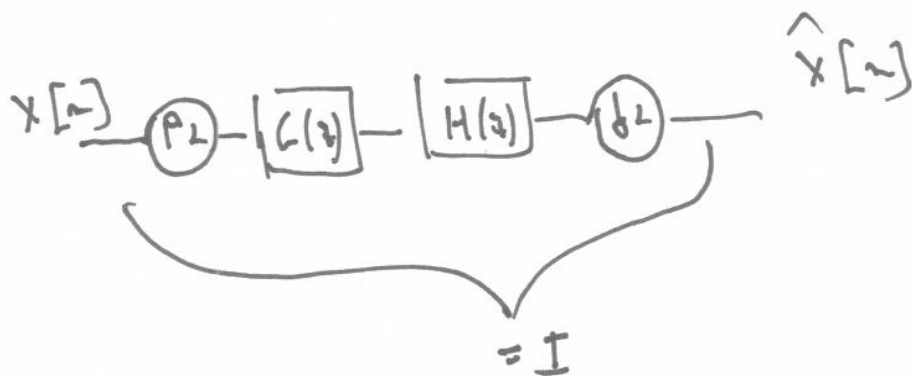


$$D(e^{j\omega})$$



c) THE SYSTEM IS NOT IDEMPOTENT : $p^2 \neq p$

IN FACT THE SYSTEM IS IDEMPOTENT IF



THAT IS $\hat{x}[n] = x[n]$

CHECK:

$$\begin{aligned} \hat{X}(z) &= \downarrow H(z) L(z) X(z) \Big|_z = \\ &= X\left(\frac{z}{2}\right) \left(H\left(z^{1/2}\right) L\left(z^{1/2}\right) + H\left(-z^{1/2}\right) L\left(-z^{1/2}\right) \right) \end{aligned}$$

$$= \frac{x(z)}{2} \underbrace{(z^{-1} + 6 + z)}_{\neq I}$$

d) THE SYSTEM IS IDEMPOTENT
IF AND ONLY IF

$$H(z) \cdot G(z) + H(-z) G(-z) = 2 \quad (1)$$

$$H(z) = (z+1)(z^{-1}+1)(a+bz+bz^{-1})$$

THE COEFFICIENTS a AND b MUST
BE SUCH THAT CONDITION (1) IS
SATISFIED.

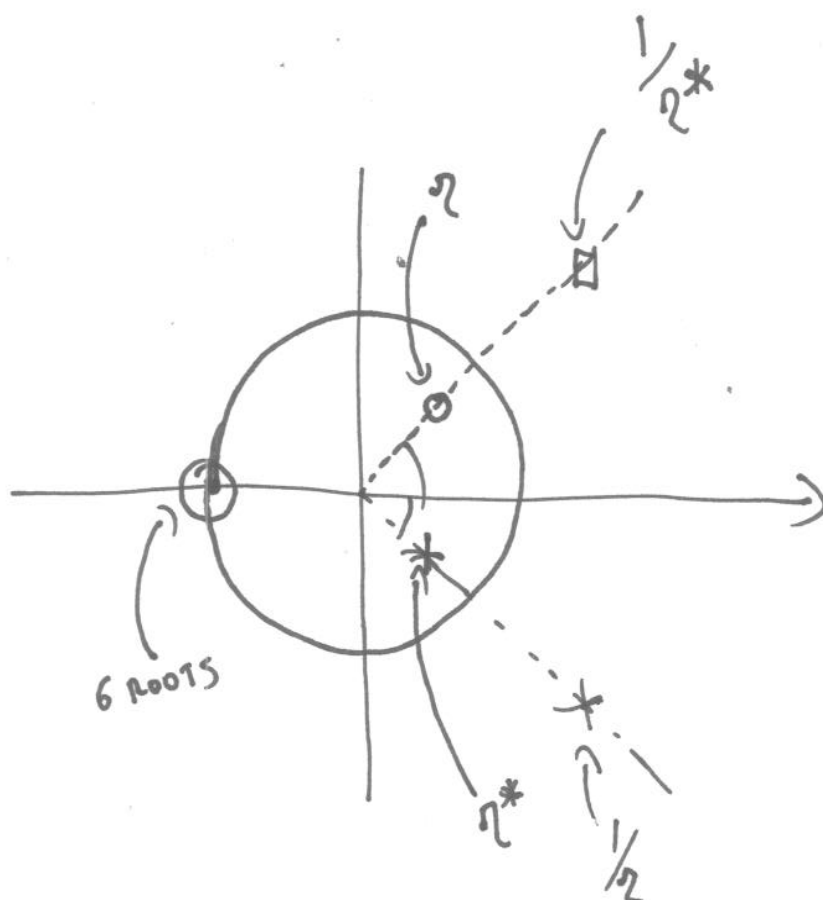
$$\text{THUS } a = \frac{1}{4} \text{ AND } b = -\frac{1}{16}.$$

NOTICE THAT THE UPPER-BRANCH
IS NOW PERFORMING AN OBLIQUE
PROJECTION WHICH IS GOOD NEWS.
HOWEVER, THE PROJECTION IS NOT
OPTIMAL IN THAT IT IS NOT
ORTHOGONAL.

QUESTION 2

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a)



$$b) \quad P(z) = (1+z)^3 (1+z^{-1})^3 (z-2) (z-2^{-1}) (z-2^*) (z-\frac{1}{2^*})$$

$$G_0(z) = (1+z^{-1})^3 (z-2) (z-2^*)$$

$$H_0(z) = (1+z)^3 (z-\frac{1}{2}) (z-\frac{1}{2^*})$$

$$G_1(z) = -z^{-1} G_0(-z^{-1})$$

$$H_1(z) = G_1(z^{-1})$$

$$c) \quad H_1(z) = -z \cancel{H_0(z)} \quad -z (1-z^{-1})^3 (z^*+2) (z+2^*)$$

THEREFORE $H_1(z)$ HAS A ZERO OF ORDER 3 ~~POLE~~ AT $w=0$:

$$\left. H_1(z^{jw}) \right|_{w=0} = 0 \quad \left. H_1^{(1)}(z^{jw}) \right|_{w=0} = 0 \quad \left. H_1^{(11)}(z^{jw}) \right|_{w=0} = 0$$

NOW

$$\sum_{12} (m-12)^2 h_1[12] = m^2 \sum_{12} h_1[12] - 2m \sum_{12} 12 h_1[12] + \sum_{12} 12^2 h_1[12]$$

$$= 0 \quad \text{SINCE}$$

$$\sum_{12} h_1[12] = \left. H_1(z^{jw}) \right|_{w=0} = 0$$

$$\sum_{12} 12 h_1[12] = j \left. \frac{\partial H_1}{\partial w} \right|_{w=0} = 0$$

$$\sum_{12} 12^2 h_1[12] = - \left. \frac{\partial^2 H_1(z^{jw})}{\partial w^2} \right|_{w=0} = 0$$

o) POSSIBLE FACTORIZATION!

$$G_0(z) = (1+z)^2 (1+z^{-1})^2$$

$$H_0(z) = (1+z) (1+z^{-1}) \Phi(z)$$

a)

TWO-SCALE EQUATIONS:

$$\psi(t) = \sqrt{2} \sum_n g_0[n] \psi(2t-n)$$

$$\tilde{\psi}(t) = \sqrt{2} \sum_n h_0[n] \tilde{\psi}(2t-n)$$

Moreover:

$$\langle \tilde{\psi}(t), \psi(t-n) \rangle = \delta[n] \quad (1)$$

USING THE TWO-SCALE EQUATIONS AND THE LINEARITY OF INNER PRODUCT, WE OBTAIN:

$$\begin{aligned} \langle \tilde{\psi}(t), \psi(t-n) \rangle &= \\ &= 2 \sum_k h_0[k] \sum_l g_0[l] \langle \tilde{\psi}(2t-k), \psi(2t-2n-l) \rangle \\ &= 2 \sum_k \sum_l h_0[k] g_0[l] \cdot \frac{1}{2} \langle \tilde{\psi}(x-k), \psi(x-2n-l) \rangle \end{aligned} \quad (2)$$

WHERE IN THE LAST EQUALITY WE HAVE REPLACED $2t$ WITH x .

BECAUSE OF (1) WE HAVE THAT

$$\langle \tilde{\psi}(x-k), \psi(x-2n-l) \rangle = \delta[k-(2n+l)]$$

EQ. 2, THEREFORE BECOMES:

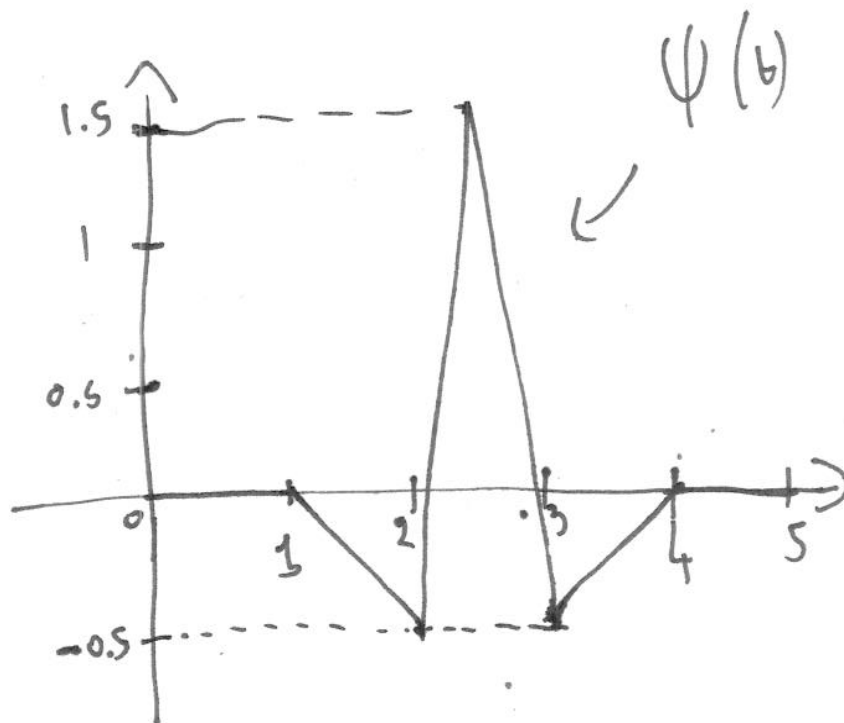
$$\langle \psi(t), \psi(t-n) \rangle = \sum_k \sum_l h_0[k] g_0[l] \delta[l - (2n + k)]$$

$$= \sum_l g_0[l] h_0[l + 2n] = \langle g_0[l], h_0[l + 2n] \rangle$$

BY COMPARING THIS LAST TERM ABOVE
WITH EQ. (1) WE OBTAIN THE
DESIRED RESULT:

$$\langle g_0[l], h_0[l + 2n] \rangle = \delta[n]$$

b)



c) $\tilde{\psi}(t)$ HAS TWO VANISHING MOMENTS

W/ FACT

$$\tilde{\psi}(t) = \sum_n \sqrt{2} h_1[n] \tilde{\psi}(2t-n)$$

AND

$$h_1[n] = (-1)^{n-1} g_0[1-n]$$

SINCE $g_0(t)$ HAS TWO ZEROS AT π ,
THIS MEANS THAT $h_1(z)$ HAS TWO
ZEROS AT $\omega=0$.

NOW,

$$\hat{\tilde{\psi}}(\omega) = \frac{1}{\sqrt{2}} H_1\left(e^{j\frac{\omega}{2}}\right) \hat{\tilde{\psi}}\left(\frac{\omega}{2}\right)$$

THIS IMPLIES THAT

$$\hat{\tilde{\psi}}(\omega) \Big|_{\omega=0} = 0 \quad \text{AND THAT} \quad \frac{d\hat{\tilde{\psi}}(\omega)}{d\omega} \Big|_{\omega=0} = 0$$

THUS $\tilde{\psi}(t)$ HAS TWO VANISHING
MOMENTS.

d) $f(t)$ IS 2-LIPSCHITS ($\alpha = 2$)

$$f(t) = p_{t_0}(t) + \varepsilon(t)$$

WHERE $p_{t_0}(t)$ IS A POLYNOMIAL OF
DEGREE $m = \lfloor 2 \rfloor$

$$\text{AND } |\varepsilon(t)| \leq K |t - t_0|^2 \quad (1)$$

THE WAVELET COEFFICIENTS IN THE CONE
OF INFLUENCE OF t_0 ARE THEN
GIVEN BY:

$$\langle f, \tilde{\psi}_{m,n} \rangle = \langle p_{t_0}(t), \tilde{\psi}_{m,n}(t) \rangle + \langle \varepsilon(t), \tilde{\psi}_{m,n}(t) \rangle$$

SINCE $2 = 1.8$ THEN $m = 1$ AND

SINCE $\tilde{\psi}(t)$ HAS TWO VANISHING MOMENTS

$$\langle p_{t_0}(t), \tilde{\psi}_{m,n}(t) \rangle = 0$$

USING EQ. 1 WE HAVE:

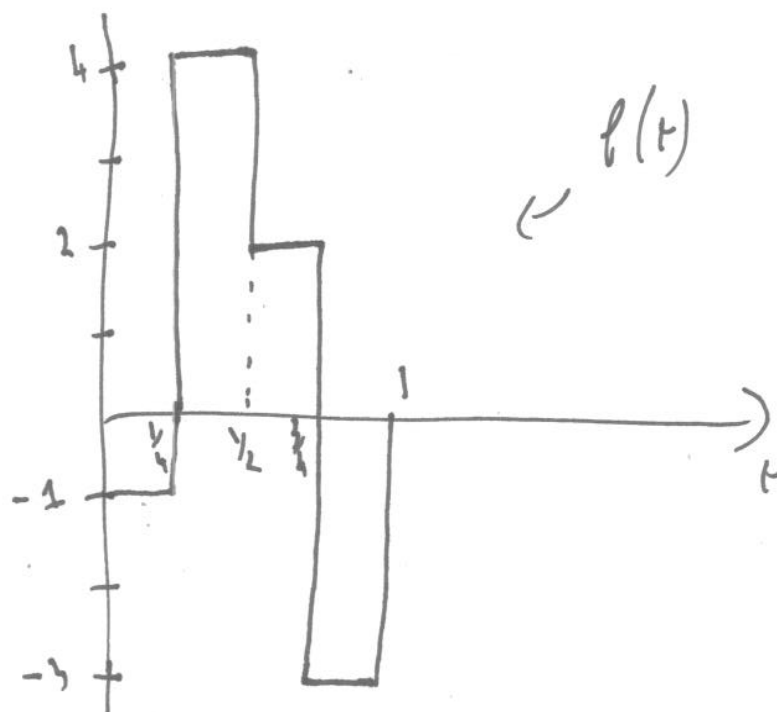
$$\begin{aligned} \langle f, \tilde{\psi}_{m,n} \rangle &= \langle \varepsilon(t), \tilde{\psi}_{m,n}(t) \rangle \leq K 2^{-m/2} \int_{-\infty}^{\infty} |t - t_0| |\tilde{\psi}(2^{-m}t - n)| dt \\ &= K 2^{m/2} \int_{-\infty}^{\infty} |x 2^m + n 2^m - t_0| |\tilde{\psi}(x)| dx \end{aligned}$$

$$\begin{aligned}
 & \stackrel{(a)}{\leq} KC 2^{m(2+\frac{1}{2})} \underbrace{\int_{-\infty}^{\infty} (|x| + |C|)^2 \psi(x) dx}_{=A} = \\
 & = C_1 2^{m(2+\frac{1}{2})}
 \end{aligned}$$

WHERE (a) FOLLOWS FROM THE FACT THAT
 WE ARE IN THE CONE OF INFLUENCE OF t_0 ,
 THEREFORE $|m 2^m - t_0| \leq C 2^m$. HERE 'C' IS
 THE COMPACT SUPPORT OF $\tilde{\psi}(t)$.

QUESTION 4

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a) $\varphi_{-2,m}(t) = 2\varphi(4t-m)$, $c_{j,m} = \langle f(t), \varphi_{j,m}(t) \rangle$.

~~6-2, 0~~ $c_{-2,0} = -\frac{1}{2}$

$$c_{-2,1} = 2$$

$$c_{-2,2} = 1$$

$$c_{-2,3} = -\frac{3}{2}$$

$$c_{-2,m} = 0 \quad m \neq 0, 1, 2, 3$$

b) SINCE THE BASIS IS ORTHO-NORMAL, WE HAVE THAT

$$c_{j,m} = \langle f(t), \varphi_{j,m}(t) \rangle \quad \text{AND}$$

$$d_{j,m} = \langle f(t), \psi_{j,m}(t) \rangle$$

a)

$$c_{0,0} = \frac{-2+4+2-5}{4} = \frac{1}{2}$$

b)

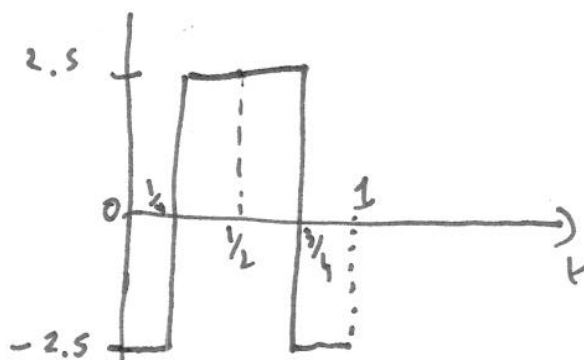
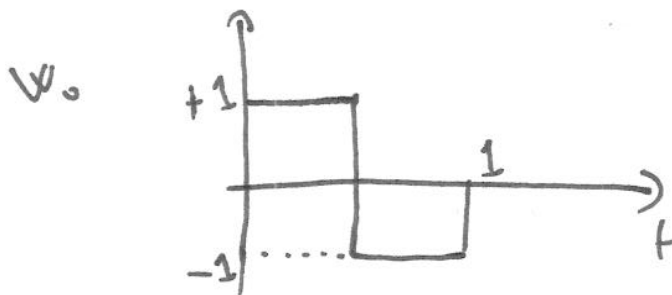
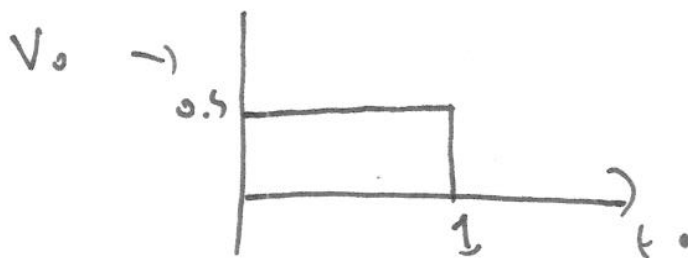
$$c_{0,m} = 0 \quad m \neq 0$$

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$$\begin{cases} d_{0,0} = 1 \\ d_{0,m} = 0 \quad m \neq 0 \end{cases}$$

$$\begin{cases} d_{-1,0} = -\frac{5}{4} \cdot \sqrt{2} \\ d_{-1,1} = \frac{5}{6} \cdot \sqrt{2} \\ d_{-1,m} = 0 \quad m \neq 0, 1 \end{cases}$$

c)



6)

(13)

$$\text{PARSEVAL} \Rightarrow \|f\|^2 = \sum_n |c_{n,0}|^2 + \sum_{j=1}^{\infty} \sum_n |d_{j,n}|^2$$

$$\|f\|^2 = \frac{1}{4} + \frac{16}{4} + \frac{4}{4} + \frac{9}{4} = \frac{15}{2}$$

$$|c_{0,0}|^2 + |d_{0,0}|^2 + |d_{-1,0}|^2 + |d_{-1,1}|^2 =$$

$$= \frac{1}{4} + 3 + \frac{25}{16} \cdot 2 + \frac{25}{16} \cdot 2 = \frac{15}{2} \quad \checkmark$$