CONTROL ENGINEERING – MODEL ANSWERS

1. a) The reachability matrix is

$$\mathscr{R} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

This has rank equal to two, hence the system is not reachable. To check controllability, note that

$$A^3 \in \mathrm{Im}\mathscr{R},$$

hence the system is controllable. Similarly, one could observe that the unreachable mode is $\lambda = 0$, hence controllability. [4 marks]

b) The set of states $\mathscr{R}_2$ that can be reached from $x(0) = 0$ in two steps is given by

$$x(2) = ABu(0) + Bu(1) = \begin{bmatrix} u(0) \\ u(1) \\ 0 \end{bmatrix}.$$

$\mathscr{R}_2$ is a two dimensional subspace. [4 marks]

c) The set of states $\mathscr{C}_2$ that can be controlled to zero in two steps is given by the states which satisfy the condition

$$0 = x(2) = A(Ax(0) + Bu(0)) + Bu(1) = \begin{bmatrix} x_3(0) - x_1(0) + u(0) \\ -x_2(0) + u(1) \\ 0 \end{bmatrix},$$

for some selection of $u(0)$ and $u(1)$. This shows that all states can be controlled to zero in two steps: $\mathscr{C}_2$ is a three dimensional subspace, that is spans the whole state space. The set $\mathscr{C}_2$ is larger than the set $\mathscr{R}_2$ because the system is controllable, but not reachable. [6 marks]

d) Note that ($K = [k_1\ k_2\ k_3]$)

$$A + BK = \begin{bmatrix} 0 & 1 & 0 \\ k_1 - 1 & k_2 & k_3 + 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The characteristic polynomial of $A + BK$ is

$$p(\lambda) = \lambda^3 - k_2\lambda^2 + (1 - k_1)\lambda.$$

Selecting $k_2 = 0$ and $k_1 = 1$ yields $p(\lambda) = \lambda^3$, as requested. Note that, since the unreachable mode is at $\lambda = 0$ there problem of assigning the eigenvalues is feasible, and has infinitely many solutions, namely

$$K = \begin{bmatrix} 1 & 0 & k_3 \end{bmatrix}.$$

Note that $KK' = 1 + k_3^2$, hence the *optimal* K is

$$K = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

[6 marks]

2. a) For this part $h = 1$, that is $T(t+1) = u(t)$.

i) Note that

$$T(t+1) = u(t) = u(t-1) - \alpha(T(t) - T_0)$$

and that

$$T(t) = u(t-1).$$

As a result

$$T(t+1) = T(t) - \alpha(T(t) - T_0) = (1-\alpha)T(t) + \alpha T_0.$$

[2 marks]

ii) The system in part a.i) is stable if and only if

$$-1 < 1 - \alpha < 1,$$

or, equivalently,

$$\alpha \in (0,2).$$

Since $\alpha \in (1,2)$ then the closed-loop system is asymptotically stable and for any initial condition the temperature converges to the desired value T_0. [2 marks]

b) For this part $h = 2$, that is $T(t+2) = u(t)$.

i) Note that

$$T(t+2) = u(t) = u(t-1) - \alpha(T(t) - T_0)$$

and that

$$T(t+1) = u(t-1).$$

As a result

$$T(t+2) = T(t+1) - \alpha(T(t) - T_0).$$

[4 marks]

ii) Let $y(t) = x_1(t) = T(t)$, $x_2(t) = T(t+1)$ and $u(t) = T_0$. Then

$$x_1^+ = x_2, \qquad x_2^+ = x_2 - \alpha(x_1 - u).$$

As a result

$$A = \begin{bmatrix} 0 & 1 \\ -\alpha & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ \alpha \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

[6 marks]

iii) The charcateristic polynomial of the matrix A is

$$p(z) = \det(zI - A) = z^2 - z + \alpha.$$

Using the transformation

$$z = \frac{1+s}{1-s}$$

one obtains a rational function the numerator of which is

$$\bar{p}(s) = (2+\alpha)s^2 + 2(1-\alpha)s + \alpha.$$

Note that for $\alpha \in (1,2)$ the coefficient of "s" is negative, whereas the coefficients of "s^2" and "s^0" are positive. The closed-loop system is therefore unstable. [4 marks]

c) A fast boiler let you control the temperature to the desired value, a slow one will not! [2 marks]

3. a) The three equations are

$$\dot{x}_1 - x_2 + x_3 = 0 \qquad \dot{x}_2 + x_1 - x_2 + u = 0, \qquad x_1 - x_3 + u = 0.$$

[2 marks]

b) From the third equation one obtains

$$x_3 = x_1 + u,$$

which replaced in the $\dot{x}_1$, $\dot{x}_2$ and y equations gives

$$\dot{x}_1 = -x_1 + x_2 - u, \qquad \dot{x}_2 = -x_1 + x_2 + u, \qquad y = 2x_1 + x_2 + u.$$

[2 marks]

c) From part b) one obtains

$$A_r = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}, \qquad B_r = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \qquad C_r = \begin{bmatrix} 2 & 1 \end{bmatrix}, \qquad D_r = \begin{bmatrix} 1 \end{bmatrix}.$$

[4 marks]

d) Note that

$$\det(\lambda E - A) = \lambda^2$$

and that

$$\det(\lambda I - A_r) = \lambda^2$$

hence the descriptor system and the reduced system have the same eigenvalues.

[4 marks]

e) Note that

$$\mathscr{O}_r = \begin{bmatrix} C_r \\ C_r A_r \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -3 & 3 \end{bmatrix}.$$

As a result, $\det \mathscr{O}_r \neq 0$ and the reduced system is observable. Note now that

$$\begin{bmatrix} \lambda E - A \\ C \end{bmatrix} = \begin{bmatrix} \lambda & -1 & 1 \\ 1 & \lambda - 1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

and this has rank equal to three for $\lambda = 0$, which is the only eigenvalue of the system. As a result, the descriptor system is observable. [4 marks]

f) Note that

$$\mathscr{R}_r = \begin{bmatrix} B_r & A_r B_r \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & 2 \end{bmatrix}.$$

As a result, $\det \mathscr{R}_r \neq 0$ and the reduced system is controllable. Controllability of the reduced system means that for any initial condition $x_1(0)$ and $x_2(0)$ there exists and interval $[0, T]$ and an input signal, defined in this interval, such that

$$x_1(t) = 0 \qquad x_2(t) = 0 \qquad u(t) = 0$$

for all $t \geq T$. Note now that $x_3(t)$ is such that, by what established in part a), $x_3(t) = x_1(t) + u(t)$, hence $x_3(t) = 0$, for all $t \geq T$, which implies controllability of the descriptor system. [4 marks]

4. Consider a nonlinear, continuous-time, system described by the equations

$$\dot{x} = f(x) = \begin{bmatrix} x_2^2 \\ x_3^2 \\ 0 \end{bmatrix},$$

with $x(t) = [x_1(t), x_2(t), x_3(t)]' \in I\!R^3$.

a) The equilibrium points are the solutions of the equations

$$x_2^2 = 0, \qquad x_3^2 = 0.$$

Hence, all points described by $\bar{x} = (\bar{x}_1, 0, 0)$, with $\bar{x}_1$ any real number, are equilibria of the system. [2 marks]

b) The linearization of the system around each of the equilibrium point is given by

$$\dot{\delta}_x = A\delta_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

[4 marks]

c) The characteristic polynomial of the matrix A is $\det(\lambda I - A) = \lambda^3$. All eigenvalues are at zero, and have geometric multiplicity one. Hence the linearized system is stable (non-asymptotically). [2 marks]

d) A direct integration yields

$$x_3(t) = x_3(0), \qquad x_2(t) = x_2(0) + x_3^2(0)t,$$

$$x_1(t) = x_1(0) + x_2^2(0)t + x_2(0)x_3^2(0)t^2 + \frac{1}{3}x_3^4(0)t^3.$$

Note that trajectories move away from the equilibria and become unbounded, hence the equilibria are unstable. [6 marks]

e) Note that

$$y = x_1, \qquad \dot{y} = x_2^2, \qquad \ddot{y} = 2x_2x_3^2, \qquad \dddot{y} = 2x_3^4, \qquad \ddddot{y} = 0,$$

as claimed. Using $x_1 = y$, $x_2 = \dot{y}$, $x_3 = \ddot{y}$ and $x_4 = \dddot{y}$ as state variables yields

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = x_3, \qquad \dot{x}_3 = x_4, \qquad \dot{x}_4 = 0,$$

or, equivalently,

$$\dot{x}_e = A_e x_e = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} x_e, \qquad y_e = C_e x_e = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x_e.$$

[6 marks]