SOLUTIONS: COMPLEX CALCULUS
EE2L

1. Exercise

a) (similar to examples done in class) The curve $\partial^+ B$ can be parameterized as γ given below:

$$\gamma(t) = e^{it}, \qquad t \in [0, 2\pi].$$

By definition of integral of complex variable we have:

$$\int_{\partial^+ B} \frac{e^{iz}}{z}\, dz = \int_0^{2\pi} \frac{e^{i\gamma(t)}}{\gamma(t)} \dot{\gamma}(t)\, dt.$$

[2]

Since $\dot{\gamma}(t) = ie^{it}$ we see that:

$$\int_{\partial^+ B} \frac{e^{iz}}{z}\, dz = \int_0^{2\pi} ie^{i(\cos(t)+i\sin(t))}\, dt =$$

$$= \int_0^{2\pi} ie^{-\sin(t)} e^{i\cos(t)}\, dt =$$

$$= \int_0^{2\pi} ie^{-\sin(t)} [\cos(\cos(t)) + i\sin(\cos(t))]\, dt.$$

This proves that:

$$I = \mathrm{Im}\left[\int_{\partial^+ B} \frac{e^{iz}}{z}\, dz\right].$$

[2]

b) (similar to results derived in class) The function e^{iz} is a composition of holomorphic functions (multiplication times complex scalar and complex exponential). It is therefore holomorphic in $\mathbb{C}$. The function z is also holomorphic in $\mathbb{C}$, therefore their ratio is holomorphic whenever the denominator is different from 0. This is also the domain of definition of the function. The point $z = 0$ is the only singularity, and it is a pole of multiplicity 1, as it is easily verified by noticing that $e^{i0} = 1 \neq 0$ and realizing that $z \cdot \frac{e^{iz}}{z} = e^{iz}$ for all $z \neq 0$, which, as previously claimed is holomorphic. [3]

c) (similar to examples done in class) The residue of the function at 0 is given by:

$$\lim_{z \to 0} z \frac{e^{iz}}{z} = e^{i0} = 1.$$

Hence, by the Residue's theorem:

$$\int_{\partial^+ B} \frac{e^{iz}}{z}\, dz = 2\pi i \cdot 1 = 2\pi i.$$

It follows that $I = 2\pi$. [3]

d) (similar to problems solved in class) Differentiating u with respect to x and y yields:

$$\frac{\partial u}{\partial x} = \mathrm{e}^x \cos(y) + \mathrm{e}^{-y} \cos(x),$$

$$\frac{\partial u}{\partial y} = -\mathrm{e}^x \sin(y) - \mathrm{e}^{-y} \sin(x).$$

[1]

Differentiating again, yields:

$$\frac{\partial^2 u}{\partial x^2} = \mathrm{e}^x \cos(y) - \mathrm{e}^{-y} \sin(x),$$

$$\frac{\partial^2 u}{\partial y^2} = \mathrm{e}^{-y} \sin(x) - \mathrm{e}^x \cos(y).$$

[2]

It is easy to see that:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

which proves that u is a solution of the Laplace's equation in $\mathbb{R}^2$ and hence harmonic. [1]

e) (similar to problems solved in class) To find the conjugate v we exploit the Cauchy-Riemann's equations. In particular:

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = \mathrm{e}^x \cos(y) + \mathrm{e}^{-y} \cos(x).$$

This implies:

$$v(x,y) = \int \mathrm{e}^x \cos(y) + \mathrm{e}^{-y} \cos(x)\, dy =$$

$$= \mathrm{e}^x \sin(y) - \mathrm{e}^{-y} \cos(x) + c_1(x),$$

where $c_1(x)$ is an arbitrary function of x. [1]

Similarly:

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = \mathrm{e}^x \sin(y) + \mathrm{e}^{-y} \sin(x).$$

This implies:

$$v(x,y) = \int \mathrm{e}^x \sin(y) + \mathrm{e}^{-y} \sin(x)\, dx =$$

$$= \mathrm{e}^x \sin(y) - \mathrm{e}^{-y} \cos(x) + c_2(y),$$

where c_2 is an arbitrary function of y. [1]

Equating the previous expressions for v yields:

$$c_1(x) = c_2(y) = c$$

for some scalar $c \in \mathbb{R}$. Hence:

$$v(x,y) = \mathrm{e}^x \sin(y) - \mathrm{e}^{-y} \cos(x) + c.$$

[2]

f) (similar to examples done in class) To find the function g as requested we evaluate $u+iv$ on the real axis ($y=0$). This gives:

$$u(x,0)+iv(x,0)=e^x+\sin(x)+i[-\cos(x)+c].$$

Hence we may let:

$$g(z)=e^z+\sin(z)-i\cos(z)+ic=e^z-ie^{iz}+ic.$$

The function $g(z)$ equals $u(x,y)+iv(x,y)$ on the real axis, and is clearly holomorphic. Hence it coincides with $u(x,y)+iv(x,y)$ for all $(x,y)\in\mathbb{R}^2$. [3]

g) (unseen) A holomorphic function g is conformal at every point where $g'(z)\neq 0$. Hence, provided z is such that:

$$g'(z)=e^z+e^{-iz}\neq 0$$

[1]

Notice that, to have

$$e^z=-e^{-iz}=e^{-i(z+\pi)}$$

we need:

$$\mathrm{Re}(z)=\mathrm{Re}(-i(z+\pi))=\mathrm{Im}(z)$$

(which implies the two sides of the previous equation have the same modulus), and

$$\mathrm{Im}(z)+2k\pi=\mathrm{Im}(-i(z+\pi))$$

(which implies the two sides of the previous equation have the same argument). [2]

We have a (linear) system of two equations in two real unknowns, $\mathrm{Re}(z)$, $\mathrm{Im}(z)$, whose solution is:

$$\mathrm{Re}(z)=\mathrm{Im}(z)=\frac{2k+1}{2}\pi,$$

with k an arbitrary integer. Hence there are infinitely many point in $\mathbb{C}$ where g is not conformal, these are:

$$z=\frac{2k+1}{2}\pi+i\frac{2k+1}{2}\pi,\qquad k\in\mathbb{Z}.$$

[1]

2. Exercise (Similar to equations solved in class)

a) Let $X(s)$ denote $\mathscr{L}[x]$. Then $\mathscr{L}[dx/dt]=sX(s)$ and $\mathscr{L}[d^3x/dt^3]=s^3X(s)$. Applying Laplace's transforms to both sides of the differential equation yields:

$$(s^3+3s-4)X(s)=\frac{1}{s}.$$

[6]

Solving with respect to $X(s)$ yields:

$$X(s)=\frac{1}{s(s^3+3s-4)}.$$

[4]

Notice that: $(s^3+3s-4)=(s-1)(s^2+s+4)$, therefore:

$$X(s)=\frac{A}{s-1}+\frac{Bs+C}{s^2+s+4}+\frac{D}{s}$$

with:

$$A = \lim_{s \to 1} (s-1)X(s) = \frac{1}{6}$$

$$D = \lim_{s \to 0} sX(s) = -\frac{1}{4}$$

$$C = -\frac{1}{12}, \qquad B = \frac{1}{12}.$$

[5]

Taking inverse Laplace's transforms yields:

$$x(t) = \frac{e^t}{6} + \frac{e^{-\frac{t}{2}} \left(\cos\left(\frac{\sqrt{15}t}{2}\right) - \frac{\sqrt{15}\sin\left(\frac{\sqrt{15}t}{2}\right)}{5} \right)}{12} - \frac{1}{4}.$$

[5]

b) Since $x(t)$ needs to be constant we do have $\dot{x}(t) \equiv 0$, $\ddot{x}(t)$ and $x^{(3)}(t) \equiv 0$ for all $t > 0$. In particular, then, $-4x(t) = 1$ for all $t > 0$. Since $\dot{x}$, $\ddot{x}$ and x are continuous in t we may pick:

$$x(0) = -\frac{1}{4}, \dot{x}(0) = 0, \ddot{x}(0) = 0.$$

[5]