Imperial College London

Course: M3S8/M4S8/M5S8
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External:
Date: January 19, 2015

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2015

M3S8/M4S8/M5S8

Time Series

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Note: Throughout this paper $\{\epsilon_t\}$ is a sequence of uncorrelated random variables (white noise) having zero mean and variance σ_ϵ^2, unless stated otherwise. The unqualified term "stationary" will always be taken to mean second-order stationary. All processes are real-valued unless stated otherwise. The sample interval is unity unless stated otherwise.

1. (a) (i) What is meant by saying that a stochastic process is stationary?

   (ii) Consider the following process of ARMA(2,1) form:

$$X_t = \frac{31}{20}X_{t-1} - \frac{3}{5}X_{t-2} + \epsilon_t - \frac{4}{3}\epsilon_{t-1}.$$

   Is this process stationary? Is it invertible?

   (b) Are the following statements true or false?

   (i) the random variables in a white noise sequence are always identically distributed;

   (ii) the spectral density function $S(f)$ of a stationary time series is always real-valued;

   (iii) as more tapering is performed with direct spectral estimators, the resolution of the estimator increases;

   (iv) if $\{\boldsymbol{\epsilon}_t\}$ is bivariate white noise then $\epsilon_{1,t}$ and $\epsilon_{2,t}$ are always uncorrelated.

   (c) Let $\{X_t\}$ be defined by

$$X_t - \phi X_{t-1} = \epsilon_t - \theta\epsilon_{t-1}, \quad t = 1, 2, 3, 4, \ldots$$

   with $X_0 = 0$ and $\epsilon_0 = 0$. ϕ and θ are constants and $|\phi|, |\theta| < 1$.

   (i) Find the 3×3 covariance matrix of X_1, X_2, X_3.

   (ii) Explain the form of the covariance matrix when $\phi = \theta$.

2. (a) Suppose $\{X_t\}$ is an MA(q) process with zero mean, i.e., X_t can be expressed in the form

$$X_t = -\theta_{0,q}\epsilon_t - \theta_{1,q}\epsilon_{t-1} - \ldots - \theta_{q,q}\epsilon_{t-q},$$

where the $\theta_{j,q}$'s are constants ($\theta_{0,q} \equiv -1, \theta_{q,q} \neq 0$). Show that its autocovariance sequence is given by

$$s_\tau = \begin{cases} \sigma_\epsilon^2 \sum_{j=0}^{q-|\tau|} \theta_{j,q}\theta_{j+|\tau|,q}, & \text{if } |\tau| \leq q, \\ 0, & \text{if } |\tau| > q. \end{cases}$$

(b) Let $\{X_t\}$ be the stationary zero mean MA(1) process $X_t = \epsilon_t - \theta\epsilon_{t-1}$.

(i) Show that its spectral density function takes the form

$$S(f) = \sigma_\epsilon^2[1 + \theta^2 - 2\theta\cos(2\pi f)].$$

(ii) Use this spectral density function to find the lag-1 autocovariance s_1.

(c) Let $\{X_t\}$ be the normal (Gaussian) stationary zero mean MA(1) process $X_t = \epsilon_t - \theta\epsilon_{t-1}$.

(i) Express $\operatorname{cov}\{X_t, X_{t+\tau}^3\}$, $\tau \in \mathbb{Z}$, in terms of θ and σ_ϵ^2, where cov denotes covariance.
You will need to use the following version of the Isserlis Theorem: If X_j, X_k, X_l, X_m are any four real-valued Gaussian/normal random variables with zero mean then

$$E\{X_jX_kX_lX_m\} = E\{X_jX_k\}E\{X_lX_m\} + E\{X_jX_l\}E\{X_kX_m\} + E\{X_jX_m\}E\{X_kX_l\}.$$

(ii) Hence show that $\operatorname{corr}\{X_t, X_t^3\} = \sqrt{(3/5)}$, where corr denotes correlation.
You will need to use the following result: If X is a real-valued Gaussian/normal random variable with mean μ and variance σ^2 then

$$E\{(X-\mu)^r\} = \begin{cases} \frac{r!}{(r/2)!}\frac{\sigma^r}{2^{r/2}}, & r \text{ even} \\ 0, & r \text{ odd.} \end{cases}$$

3. (a) If the X's are real-valued random variables and the c's are real-valued constants, use the definition of covariance to show that

$$\operatorname{cov}\left\{\sum_j c_{0,j}X_{0,j}, \sum_k c_{1,k}X_{1,k}\right\} = \sum_j \sum_k c_{0,j}c_{1,k}\operatorname{cov}\{X_{0,j}, X_{1,k}\},$$

where cov denotes covariance and the summations are over finite sets of integers.

(b) Let $\{X_t\}$ be a normal (Gaussian) stationary zero mean process having autocovariance sequence $\{s_{X,\tau}\}$ and spectral density function $S_X(f)$. Denote the periodogram estimator of $S_X(f)$, based on a sample of size N from the process, by $\widehat{S}_X^{(p)}(f)$. Assume that, for $0 < |f| < 1/2$, the ratio $2\widehat{S}_X^{(p)}(f)/S_X(f)$ may be taken to be distributed as a χ_2^2 random variable, i.e., a chi-squared random variable with two degrees of freedom. [If χ_ν^2 denotes a chi-squared random variable with ν degrees of freedom then $E\{\chi_\nu^2\} = \nu$ and $\operatorname{var}\{\chi_\nu^2\} = 2\nu$.]

Let $\{X_{m,t}\}$, for $m = 1, \ldots, M$, represent $M \geq 2$ zero mean, normal (Gaussian) stationary processes, independent of each other, each also having the autocovariance sequence $\{s_{X,\tau}\}$ and spectral density function $S_X(f)$. Since the different realizations are independent it follows that $\operatorname{cov}\{X_{m,t}, X_{n,u}\} = 0$ for any t and u when $m \neq n$. Define the stationary process $\{\overline{X}_t\}$ by $\overline{X}_t = \frac{1}{\sqrt{M}}\sum_{m=1}^{M} X_{m,t}$.

(i) By first computing $\operatorname{cov}\{\overline{X}_t, \overline{X}_{t+\tau}\}$, find the spectrum of $\{\overline{X}_t\}$.

(ii) Giving your arguments clearly, determine the variance, for $0 < |f| < 1/2$, of the periodogram based on $\overline{X}_t, t = 1, \ldots, N$.

(iii) Now suppose we form the periodogram $\widehat{S}_m^{(p)}(f)$, based on $X_{m,t}, t = 1, \ldots, N$, for each of the M time series, and use their average $\frac{1}{M}\sum_{m=1}^{M}\widehat{S}_m^{(p)}(f)$ as an estimator of $S_X(f)$. Giving your arguments clearly, find the variance of this estimator for $0 < |f| < 1/2$.

(c) Suppose a stationary continuous-time process $\{X(t)\}$ with parameter θ, has spectral density function given by

$$S_{X(t)}(f) = \begin{cases} |1 - \theta\,\mathrm{e}^{-\mathrm{i}4\pi f}|^2, & |f| \leq 1/2, \\ 0, & \text{otherwise.} \end{cases}$$

This continuous-time process is sampled with sampling interval $\Delta t = 2$ to produce a discrete-time stationary process $\{X_t\}$.

Derive, with full reasoning, the (aliased) spectral density function $S_{X_t}(f)$ of $\{X_t\}$ in the interval $|f| \leq f_{\mathcal{N}}$, where $f_{\mathcal{N}}$ is the Nyquist frequency for the process $\{X_t\}$.

4. (a) Let $\{X_t\}$ be a zero mean stationary AR(p) process: $X_t - \phi_{1,p}X_{t-1} - \ldots - \phi_{p,p}X_{t-p} = \epsilon_t$.

(i) Derive the Yule-Walker equations $\boldsymbol{\gamma}_p = \boldsymbol{\Gamma}_p\boldsymbol{\phi}_p$ and $\sigma_\epsilon^2 = s_0 - \sum_{j=1}^p \phi_{j,p}s_j$, for estimation of the parameter vector $\boldsymbol{\phi}_p = [\phi_{1,p}, \phi_{2,p}, \ldots, \phi_{p,p}]^T$ and white noise variance σ_ϵ^2, where $\boldsymbol{\gamma}_p = [s_1, s_2, \ldots, s_p]^T$ and

$$\boldsymbol{\Gamma}_p = \begin{bmatrix} s_0 & s_1 & \ldots & s_{p-1} \\ s_1 & s_0 & \ldots & s_{p-2} \\ \vdots & \vdots & & \vdots \\ s_{p-1} & s_{p-2} & \ldots & s_0 \end{bmatrix}$$

and s_τ is the lag-τ autocovariance.

(ii) To obtain estimates of the $\{\phi_{j,p}\}$ and σ_ϵ^2 in practice we replace the $\{s_\tau\}$ in $\boldsymbol{\Gamma}_p, \boldsymbol{\gamma}_p$ and σ_ϵ^2, above, by their estimates. Suppose for an AR(2) process we obtain $\widehat{s}_0 = 5, \widehat{s}_1 = 4$, and $\widehat{s}_2 = 2$. Use the Yule-Walker equations to obtain estimates $\widehat{\boldsymbol{\phi}}_p$ and $\widehat{\sigma}_\epsilon^2$ and hence give the estimated form of the AR(2) model.

(iii) If a real-valued AR(2) process has a characteristic equation having complex roots these come in the form of a complex-conjugate pair which can be written $z_1 = (1/r)\,\mathrm{e}^{-\mathrm{i}2\pi f'}, z_2 = (1/r)\,\mathrm{e}^{\mathrm{i}2\pi f'}$. The spectral density function is then

$$S_X(f) = \frac{\sigma_\epsilon^2}{[1 - 2r\cos(2\pi(f' + f)) + r^2][1 - 2r\cos(2\pi(f' - f) + r^2]}.$$

For the estimated model of part (a)(ii), find r and f', giving f' in the form $f' = [1/(2\pi)]\cos^{-1}(x)$ where x needs to be determined.
Hence describe a major feature of the estimated spectrum, $\widehat{S}_X(f)$.

(b) (i) What is meant by saying two discrete time stochastic processes $\{X_t\}$ and $\{Y_t\}$ are jointly second-order stationary stochastic processes?

(ii) Suppose $\{X_t\}$ and $\{Y_t\}$ are zero mean jointly second-order stationary processes given by

$$X_t = \phi X_{t-1} + \epsilon_t, \qquad Y_t = \epsilon_t - \theta\epsilon_{t-1}, \qquad \text{with } |\phi|, |\theta| < 1.$$

Find the cross-covariance sequence $\{s_{XY,\tau}\}$, where $s_{XY,\tau} = E\{X_tY_{t+\tau}\}$.
[Hint: Express the autoregressive process in moving average form.]

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May-June 2015

M3S8/M4S8/M5S8

Time Series (SOLUTIONS)

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1. (a) (i) $\{X_t\}$ is second-order stationary if $E\{X_t\}$ is a finite constant for all t, var$\{X_t\}$ is a finite constant for all t, and cov$\{X_t, X_{t+\tau}\}$, is a finite quantity depending only on τ and not on t. seen ⇓ 4

(ii) In characteristic polynomial form we have sim. seen ⇓

$$\Phi(z) = 1 - \frac{31}{20}z + \frac{3}{5}z^2; \qquad \Theta(z) = 1 - \frac{4}{3}z.$$

The roots of $\Phi(z)$ are $\frac{31\pm1}{24} = \frac{4}{3}, \frac{5}{4}$. These are both outside the unit circle so the process is stationary. 3

The root of $\Theta(z)$ is 3/4 which is inside the unit circle so the process is not invertible (i.e., not representable as a well-defined autoregression). 1

(b) (i) FALSE: white noise variables need not be identically distributed; unseen ⇓

(ii) TRUE: a spectral density function is always real-valued;

(iii) FALSE: as more tapering is performed the resolution *decreases*;

(iv) FALSE: the components $\epsilon_{1,t}$ and $\epsilon_{2,t}$ may be correlated. 4

(c) (i) sim. seen ⇓

$$\begin{aligned}
X_1 &= \phi X_0 + \epsilon_1 - \theta\epsilon_0 = \epsilon_1 \Rightarrow \mu_{X_1} = E\{X_1\} = 0 \\
X_2 &= \phi X_1 + \epsilon_2 - \theta\epsilon_1 = (\phi-\theta)\epsilon_1 + \epsilon_2 \Rightarrow \mu_{X_2} = E\{X_2\} = 0 \\
X_3 &= \phi X_2 + \epsilon_3 - \theta\epsilon_2 = \phi(\phi-\theta)\epsilon_1 + \phi\epsilon_2 + \epsilon_3 - \theta\epsilon_2 \\
&= \phi(\phi-\theta)\epsilon_1 + (\phi-\theta)\epsilon_2 + \epsilon_3 \Rightarrow \mu_{X_3} = E\{X_3\} = 0.
\end{aligned}$$

Then 2

$$\begin{aligned}
\mathrm{var}\{X_1\} &= E\{\epsilon_1^2\} = \sigma_\epsilon^2 \\
\mathrm{var}\{X_2\} &= E\{[(\phi-\theta)\epsilon_1 + \epsilon_2]^2\} \\
&= (\phi-\theta)^2\sigma_\epsilon^2 + \sigma_\epsilon^2 = [1 + (\phi-\theta)^2]\sigma_\epsilon^2 \\
\mathrm{var}\{X_3\} &= E\{[\phi(\phi-\theta)\epsilon_1 + (\phi-\theta)\epsilon_2 + \epsilon_3]^2\} \\
&= [1 + (1+\phi^2)(\phi-\theta)^2]\sigma_\epsilon^2 \\
\mathrm{cov}\{X_2, X_3\} &= E\{[(\phi-\theta)\epsilon_1 + \epsilon_2][\phi(\phi-\theta)\epsilon_1 + (\phi-\theta)\epsilon_2 + \epsilon_3]\} \\
&= \phi(\phi-\theta)^2\sigma_\epsilon^2 + (\phi-\theta)\sigma_\epsilon^2 = (\phi-\theta)[1 + \phi(\phi-\theta)]\sigma_\epsilon^2 \\
\mathrm{cov}\{X_1, X_2\} &= E\{\epsilon_1[(\phi-\theta)\epsilon_1 + \epsilon_2]\} = (\phi-\theta)\sigma_\epsilon^2 \\
\mathrm{cov}\{X_1, X_3\} &= E\{\epsilon_1[\phi(\phi-\theta)\epsilon_1 + (\phi-\theta)\epsilon_2 + \epsilon_3]\} = \phi(\phi-\theta)\sigma_\epsilon^2.
\end{aligned}$$

So covariance matrix is 4

$$\sigma_\epsilon^2 \begin{bmatrix} 1 & (\phi-\theta) & \phi(\phi-\theta) \\ (\phi-\theta) & 1 + (\phi-\theta)^2 & (\phi-\theta)[1+\phi(\phi-\theta)] \\ \phi(\phi-\theta) & (\phi-\theta)[1+\phi(\phi-\theta)] & 1 + (1+\phi^2)(\phi-\theta)^2 \end{bmatrix}.$$

(ii) When $\phi = \theta$ we just have $X_t = \epsilon_t$ for $t = 1, 2, 3$ and so the covariance matrix should be $\sigma_\epsilon^2 \boldsymbol{I}_3$, which is what we get. unseen ⇓ 2

2. (a) Since $E\{\epsilon_t\epsilon_{t+\tau}\} = 0 \ \ \forall\, \tau \neq 0$ we have for $\tau \geq 0$. $\boxed{\text{seen} \Downarrow}$

$$s_\tau = \text{cov}\{X_t, X_{t+\tau}\} \;=\; \sum_{j=0}^{q}\sum_{k=0}^{q} \theta_{j,q}\theta_{k,q} E\{\epsilon_{t-j}\epsilon_{t+\tau-k}\}.$$

This is always identically zero if $\tau > q$. For $q \geq \tau \geq 0$, the double sum is only non-zero along the diagonal specified by $k = j+\tau$ so $s_\tau = \sigma_\epsilon^2 \sum_{j=0}^{q-\tau} \theta_{j,q}\theta_{j+\tau,q}$. Now, $s_\tau = s_{-\tau}$, and so the autocovariance sequence is given by

$$s_\tau = \begin{cases} \sigma_\epsilon^2 \sum_{j=0}^{q-|\tau|} \theta_{j,q}\theta_{j+|\tau|,q}, & \text{if } |\tau| \leq q, \\ 0, & \text{if } |\tau| > q. \end{cases}$$

$\boxed{4}$

(b) (i) From linear filtering, input $\mathrm{e}^{\mathrm{i}2\pi ft}$ to the filter $L(\epsilon_t) = \epsilon_t - \theta\epsilon_{t-1} = X_t$ to obtain the frequency response function $G(f)$:

$$\begin{aligned} L\{\mathrm{e}^{\mathrm{i}2\pi ft}\} &= \mathrm{e}^{\mathrm{i}2\pi ft}(1 - \theta\mathrm{e}^{-\mathrm{i}2\pi f}) \Rightarrow G(f) = 1 - \theta\mathrm{e}^{-\mathrm{i}2\pi f} \\ \Rightarrow |G(f)|^2 &= 1 + \theta^2 - 2\theta\cos(2\pi f). \end{aligned}$$

Then use the fact that the output spectrum is the input spectrum times $|G(f)|^2$:

$$S(f) = |G(f)|^2 S_\epsilon(f) = \sigma_\epsilon^2[1 + \theta^2 - 2\theta\cos(2\pi f)].$$

$\boxed{3}$

(ii) For s_1, using the inverse Fourier transform, $\boxed{\text{unseen} \Downarrow}$

$$\begin{aligned} s_1 &= \int_{-1/2}^{1/2} S(f)\mathrm{e}^{\mathrm{i}2\pi f\cdot 1}\mathrm{d}f = \sigma_\epsilon^2\int_{-1/2}^{1/2}[1+\theta^2 - 2\theta\cos(2\pi f)]\cos(2\pi f)\mathrm{d}f \text{ (since real)} \\ &= \sigma_\epsilon^2[1+\theta^2]\int_{-1/2}^{1/2}\cos(2\pi f)\mathrm{d}f - 2\theta\sigma_\epsilon^2\int_{-1/2}^{1/2}\cos^2(2\pi f)\mathrm{d}f = -\sigma_\epsilon^2\theta. \end{aligned}$$

(Here we used $\cos^2(2\pi f) = (1/2)[1 + \cos(4\pi f)]$. Also $\cos(2\pi f)$ and $\cos(4\pi f)$ integrate to zero over $(-1/2, 1/2]$). $\boxed{5}$

$\boxed{\text{unseen} \Downarrow}$

(c) (i) From part (a) we have for the MA(1):

$$s_\tau = \begin{cases} \sigma_\epsilon^2(1+\theta^2), & \tau = 0, \\ -\sigma_\epsilon^2\theta, & |\tau| = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Since $E\{X_t\} = 0$

$$\begin{aligned}
\operatorname{cov}\{X_t, X_{t+\tau}^3\} &= E\{X_t X_{t+\tau}^3\} - E\{X_t\}E\{X_{t+\tau}^3\} \\
&= E\{X_t X_{t+\tau}^3\} \\
&= E\{X_t X_{t+\tau}\}E\{X_{t+\tau}^2\} + E\{X_t X_{t+\tau}\}E\{X_{t+\tau}^2\} \\
&+ E\{X_t X_{t+\tau}\}E\{X_{t+\tau}^2\} \\
&= 3 s_\tau s_0 \\
&= \begin{cases} 3\sigma_\epsilon^4(1+\theta^2)^2, & \tau = 0, \\ -3\sigma_\epsilon^4\theta(1+\theta^2), & |\tau| = 1, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

$\boxed{4}$

(ii) Now

$\boxed{\text{unseen} \Downarrow}$

$$\operatorname{corr}\{X_t, X_t^3\} = \frac{\operatorname{cov}\{X_t, X_t^3\}}{[\operatorname{var}\{X_t\}\operatorname{var}\{X_t^3\}]^{1/2}}.$$

We know from part (a) that $\operatorname{var}\{X_t\} = \sigma^2 = s_0 = \sigma_\epsilon^2(1+\theta^2)$. For $\operatorname{var}\{X_t^3\}$ we need to use the hint with $\mu = 0, \sigma^2 = s_0$, and for $r = 3$ and $r = 6$:

$$\begin{aligned}
\operatorname{var}\{X_t^3\} &= E\{X_t^6\} - E^2\{X_t^3\} = E\{X_t^6\} \\
&= 15 s_0^3,
\end{aligned}$$

so

$$\operatorname{corr}\{X_t, X_t^3\} = \frac{\operatorname{cov}\{X_t, X_t^3\}}{[s_0 \cdot 15 s_0^3]^{1/2}} = \frac{\operatorname{cov}\{X_t, X_t^3\}}{s_0^2\sqrt{15}} = \frac{3s_0^2}{s_0^2\sqrt{15}} = \sqrt{(3/5)}.$$

$\boxed{4}$

3. (a) Noting that the means are not stated to be zero: $\boxed{\text{unseen} \Downarrow}$

$$
\begin{aligned}
& \operatorname{cov}\Big\{\sum_j c_{0,j}X_{0,j}, \sum_k c_{1,k}X_{1,k}\Big\} \\
= \ & E\bigg\{\Big(\sum_j c_{0,j}X_{0,j} - E\Big\{\sum_j c_{0,j}X_{0,j}\Big\}\Big)\Big(\sum_k c_{1,k}X_{1,k} - E\Big\{\sum_k c_{1,k}X_{1,k}\Big\}\Big)\bigg\} \\
= \ & E\bigg\{\Big[\sum_j c_{0,j}\big(X_{0,j} - E\{X_{0,j}\}\big)\Big]\Big[\sum_k c_{1,k}\big(X_{1,k} - E\{X_{1,k}\}\big)\Big]\bigg\} \\
= \ & \sum_j\sum_k c_{0,j}c_{1,k}E\left\{\big(X_{0,j} - E\{X_{0,j}\}\big)\big(X_{1,k} - E\{X_{1,k}\}\big)\right\} \\
= \ & \sum_j\sum_k c_{0,j}c_{1,k}\operatorname{cov}\{X_{0,j}, X_{1,k}\},
\end{aligned}
$$

as required. $\boxed{4}$

(b) (i) Using the result of part (a) and the independence of the series we have $\boxed{\text{unseen} \Downarrow}$

$$
\begin{aligned}
\operatorname{cov}\{\overline{X}_t, \overline{X}_{t+\tau}\} &= \operatorname{cov}\left\{\frac{1}{\sqrt{M}}\sum_{m=1}^{M} X_{m,t}, \frac{1}{\sqrt{M}}\sum_{n=1}^{M} X_{n,t+\tau}\right\} \\
&= \sum_{m=1}^{M}\sum_{n=1}^{M}\frac{1}{\sqrt{M}}\frac{1}{\sqrt{M}}\operatorname{cov}\{X_{m.t}, X_{n,t+\tau}\} \\
&= \frac{1}{M}\sum_{m=1}^{M}\operatorname{cov}\{X_{m,t}, X_{m,t+\tau}\} = \frac{1}{M}\sum_{m=1}^{M} s_{X,\tau} = s_{X,\tau}.
\end{aligned}
$$

$\boxed{2}$

Its spectrum is $S_X(f)$ too since there is a 1:1 relationship between autocovariances sequences and spectral density functions. $\boxed{1}$

(ii) $\{\overline{X}_t\}$ is zero mean, and has the same distribution (normal/Gaussian) [it is a linear combinationof normals], autocovariance and spectrum as $\{X_t\}$ in the preamble, so we can use the result that the periodogram of $\{\overline{X}_t\}$ can be taken to have the same distribution as the random variable $[S_X(f)/2]\cdot\chi_2^2$, for which

$$
\operatorname{var}\{[S_X(f)/2]\cdot\chi_2^2\} = \frac{S_X^2(f)}{4}\cdot 4 = S_X^2(f), \quad 0 < |f| < 1/2.
$$

$\boxed{4}$

(iii) The time series on which they are based are independent so we know that the $\widehat{S}_m^{(p)}(f), m = 1,\ldots,M$ are independent. Also the variance of each $S_m^{(p)}(f)$ is $S_X^2(f)$ from the logic in part (ii). So for $0 < |f| < 1/2$,

$$
\operatorname{var}\left\{\frac{1}{M}\sum_{m=1}^{M}\widehat{S}_m^{(p)}(f)\right\} = \frac{1}{M^2}\sum_{m=1}^{M}\operatorname{var}\left\{\widehat{S}_m^{(p)}(f)\right\} = \frac{1}{M^2}\sum_{m=1}^{M} S_X^2(f) = \frac{S_X^2(f)}{M}.
$$

OR, alternatively,

$$\operatorname{var}\left\{\frac{1}{M}\frac{2}{S_X(f)}\left[S_1^{(p)}(f)+\cdots+S_M^{(p)}(f)\right]\right\}=\operatorname{var}\left\{\frac{1}{M}\chi^2_{2M}\right\}=\frac{4}{M}$$

so

$$\operatorname{var}\left\{\frac{1}{M}\sum_{m=1}^{M}\widehat{S}_m^{(p)}(f)\right\}=\frac{4}{M}\frac{S_X^2(f)}{4}=\frac{S_X^2(f)}{M}.$$

$\boxed{4}$

(c) The sdf of the continuous-time process is $\boxed{\text{unseen} \Downarrow}$

$$S_{X(t)}(f)=\begin{cases}|1-\theta\,\mathrm{e}^{-\mathrm{i}4\pi f}|^2, & |f|\le 1/2,\\ 0, & \text{otherwise.}\end{cases}$$

This can be rewritten as

$$S_{X(t)}(f)=\begin{cases}(1+\theta^2)-2\theta\cos(4\pi f), & |f|\le 1/2,\\ 0, & \text{otherwise.}\end{cases}$$

The Nyquist frequency is $f_{\mathcal{N}}=1/(2\Delta_t)=1/4$. $\boxed{1}$

Now $\cos(4\pi f)=1$ at $f=0,\pm 1/2$ and -1 at $f=\pm 1/4$. So in the interval $[0,1/2]$ the spectrum is symmetric about $f=1/4$ with a minimum at this value. Likewise in the interval $[-1/2,0]$ the spectrum is symmetric about $f=-1/4$ with a minimum at this value. [Of course it is also symmetric overall about $f=0$]. Since it is symmetric about $f=1/4$ folding it about this Nyquist frequency to obtain the aliased result for $S_{X_t}(f)$ is equivalent to doubling all values, i.e.,

$$S_{X_t}(f)=2S_{X(t)}(f)=2|1-\theta\,\mathrm{e}^{-\mathrm{i}4\pi f}|^2,\qquad |f|\le 1/4.$$

[A sketch is not required for full marks but a proper explanation about the symmetries should be given.] $\boxed{4}$

4. (a) (i) We start by multiplying the defining equation by X_{t-k}: $\boxed{\text{seen} \Downarrow}$

$$X_t X_{t-k} = \sum_{j=1}^{p} \phi_{j,p} X_{t-j} X_{t-k} + \epsilon_t X_{t-k}.$$

Taking expectations, for $k > 0$:

$$s_k = \sum_{j=1}^{p} \phi_{j,p} s_{k-j}.$$

Let $k = 1, 2, \ldots, p$ and recall that $s_{-\tau} = s_\tau$ to obtain

$$\begin{aligned} s_1 &= \phi_{1,p} s_0 + \phi_{2,p} s_1 + \ldots + \phi_{p,p} s_{p-1} \\ s_2 &= \phi_{1,p} s_1 + \phi_{2,p} s_0 + \ldots + \phi_{p,p} s_{p-2} \\ \vdots &\quad \vdots \\ s_p &= \phi_{1,p} s_{p-1} + \phi_{2,p} s_{p-2} + \ldots + \phi_{p,p} s_0 \end{aligned}$$

or in matrix notation,

$$\boldsymbol{\gamma}_p = \boldsymbol{\Gamma}_p \boldsymbol{\phi}_p.$$

Finally, we need to estimate σ_ϵ^2. To do so, we multiply the defining equation by X_t and take expectations to obtain

$$s_0 = \sum_{j=1}^{p} \phi_{j,p} s_j + \mathsf{E}\{\epsilon_t X_t\} = \sum_{j=1}^{p} \phi_{j,p} s_j + \sigma_\epsilon^2,$$

so that

$$\sigma_\epsilon^2 = s_o - \sum_{j=1}^{p} \phi_{j,p} s_j.$$

$\boxed{5}$

(ii) We use $\boxed{\text{sim. seen} \Downarrow}$

$$\begin{bmatrix} \phi_{1,2} \\ \phi_{2,2} \end{bmatrix} = \frac{1}{s_0^2 - s_1^2} \begin{bmatrix} s_0 & -s_1 \\ -s_1 & s_0 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} \frac{s_0 s_1 - s_1 s_2}{s_0^2 - s_1^2} \\ \frac{-s_1^2 + s_0 s_2}{s_0^2 - s_1^2} \end{bmatrix}$$

and

$$\sigma_\epsilon^2 = s_0 - \phi_{1,2} s_1 - \phi_{2,2} s_2.$$

Substituting the given values we obtain

$$\phi_{1,2} = 4/3,\ \phi_{2,2} = -2/3,\ \sigma_\epsilon^2 = 1.$$

$\boxed{3}$

(iii) The roots are the solution of $\boxed{\text{sim. seen} \Downarrow}$

$$(1 - \frac{4}{3} z + \frac{2}{3} z^2) = 0$$

We obtain

$$z_1, z_2 = [\frac{4}{3} \pm \sqrt{[-72/81]}]/(4/3) = 1 \pm \frac{1}{\sqrt{2}} \mathrm{i}.$$

So

$$|z_1|^2, |z_2|^2 = (1 + (1/2)) = 3/2 \Rightarrow r = \sqrt{(2/3)}.$$

Also,

$$\Re\{\frac{1}{r}\mathrm{e}^{-\mathrm{i}2\pi f'}\} = \frac{1}{r}\cos(2\pi f') = \Re\{1 + \frac{1}{\sqrt{2}}\mathrm{i}\} = 1,$$

so $f' = [1/(2\pi)]\cos^{-1}(\sqrt{(2/3)})$ and $x = r = \sqrt{(2/3)}$. $\boxed{2}$

Generally speaking complex roots will induce a peak in the spectrum, indicating an oscillatory tendency for frequencies about f'. Also, the closer the value of r to unity, the more dominant the oscillation. Since here the value of $r = \sqrt{(2/3)} > 0.8$, the root is quite close to the unit circle and we would expect a noticeable peak in the spectrum around f'. $\boxed{2}$

(b) (i) Two real-valued discrete time stochastic processes $\{X_t\}$ and $\{Y_t\}$ are said to be jointly stationary stochastic processes if $\{X_t\}$ and $\{Y_t\}$ are each, separately, second-order stationary processes, and cov$\{X_t, Y_{t+\tau}\}$ is a function of τ only. $\boxed{\text{seen} \Downarrow}$ $\boxed{2}$

(ii) Firstly write X_t in the MA(∞) form $X_t = \sum_{k=0}^{\infty}\phi^k\epsilon_{t-k}$. Then $\boxed{\text{unseen} \Downarrow}$

$$\begin{aligned} E\{X_tY_{t+\tau}\} &= E\left\{\sum_{k=0}^{\infty}\phi^k\epsilon_{t-k}[\epsilon_{t+\tau} - \theta\epsilon_{t+\tau-1}]\right\} \\ &= \sum_{k=0}^{\infty}\phi^k E\{\epsilon_{t-k}\epsilon_{t+\tau}\} - \theta\sum_{k=0}^{\infty}\phi^k E\left\{\epsilon_{t-k}\epsilon_{t+\tau-1}\right\} \end{aligned}$$

The first term gives $\phi^k\sigma_\epsilon^2$ when $k = -\tau$ and the second gives $-\theta\phi^k\sigma_\epsilon^2$ when $k = 1 - \tau$. So

$$s_{XY,\tau} = \begin{cases} 0, & \tau \geq 2; \\ -\theta\sigma_\epsilon^2, & \tau = 1; \\ \sigma_\epsilon^2\phi^{|\tau|}(1 - \theta\phi), & \tau \leq 0. \end{cases}$$

$\boxed{6}$