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IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2005

EEE/ISE PART III/IV: MEng, BEng and ACGI

DIGITAL SIGNAL PROCESSING

Thursday, 28 April 10:00 am

Time allowed: 3:00 hours

There are SIX questions on this paper.

Answer FOUR questions.

Corrected Copy

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : P.A. Naylor
Second Marker(s) : D.P. Mandic

Special Instructions for Invigilators: None

Information for Candidates:

Sequence	z-transform
$\delta(n)$	1
$a^n u(n)$	$\dfrac{1}{1-az^{-1}}$
$\left(r^n \cos \omega_0 n\right)u(n)$	$\dfrac{1-\left(r\cos\omega_0\right)z^{-1}}{1-\left(2r\cos\omega_0\right)z^{-1}+r^2z^{-2}}$

Table 1 : z-transform pairs

$\delta(n)$ is defined to be the unit impulse function.
$u(n)$ is defined to be the unit step function.

Numbers in square brackets against the right margin of the following pages are a guide to the marking scheme.

1 (a) Write down the Type 1 polyphase form of a filter $H(z)$. [4]

(b) Consider the system shown in Figure 1 in which P and Q represent linear operators with L inputs and L outputs.

(i) What function would this system typically perform? [2]

(ii) For this typical function, describe in words the relationship of $v_i(n) \quad i=0,1,\ldots,L-1$ to $x(n)$. [3]

(iii) Determine expressions for the transfer functions from the input signal $x(n)$ to the signals $v_i(n) \quad i=0,1,\ldots,L-1$. [5]

(c) Consider a lowpass filter $H_0(z)$ with impulse response $h_0(n)$ and normalized cut-off frequency $\pi/2$. Derive a highpass filter $H_1(z)$ and give its impulse response $h_1(n)$ such that $H_0(z)$ and $H_1(z)$ are Quadrature Mirror Filters. Sketch an example of the magnitude frequency responses of the filters for the frequency range -2π to 2π. [6]

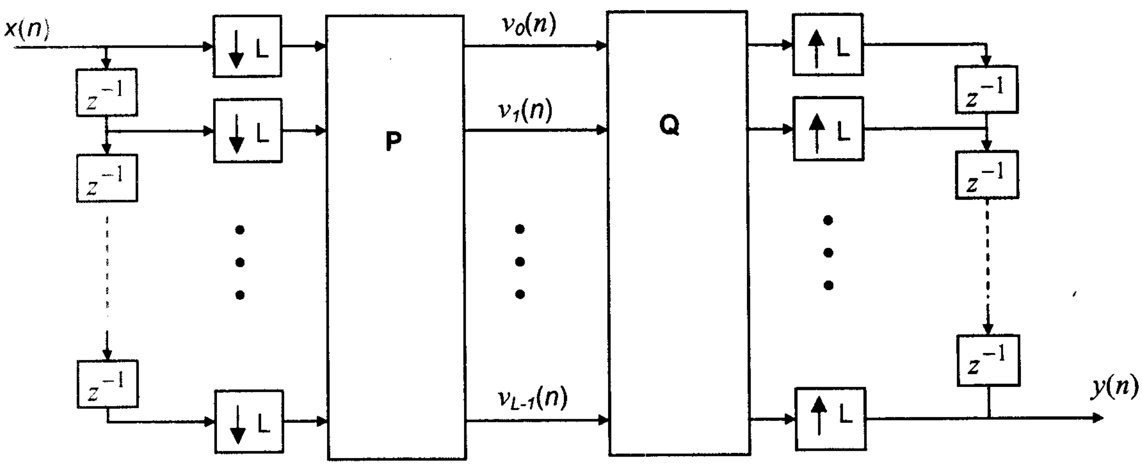


Figure 1.

2. Consider a discrete-time sequence $\{x(n)\}$.

(a) Give the definition of the z-transform $X(z)$. [2]

(b) Explain what is meant by the *Region of Convergence* for z-transforms. [3]

(c) State briefly any significant similarities and differences between the z-transform and the Laplace transform. Illustrate your answer using sketches in the z and s domains. Comment on the way in which the spectrum of discrete-time signals is represented in the z-domain. [5]

(d) Find the inverse z-transform of $H(z)=\dfrac{z^2+z+1}{z^2+3z+2}$, $1<|z|<2$. [10]

3. Let $X(k)$ be the DFT of a discrete-time signal $x(n)$ of length *N* samples.

(a) Consider vectors $\mathbf{x}$ and $\mathbf{X}$ representing the time domain and frequency domain data respectively, and a matrix $\mathbf{D}_N$ known as the DFT matrix. The DFT operation can be expressed in the following matrix form [7]

$$\mathbf{X}=\mathbf{D}_N\mathbf{x}.$$

Write out in full the vectors $\mathbf{X}$ and $\mathbf{x}$, and the DFT matrix $\mathbf{D}_N$, showing their elements in terms of $x(n)$, $X(k)$ and the term $W_N=e^{-j2\pi/N}$.

(b) When $x(n)$ is complex it can be written [8]

$$x(n)=g(n)+j\,h(n).$$

Show that

$$G(k)=\frac{1}{2}\left(X(k_N)+X^*(-k_N)\right) \qquad \text{and}$$

$$H(k)=\frac{1}{2j}\left(X(k_N)-X^*(-k_N)\right)$$

where $G(k)$ is the DFT of $g(n)$, $H(k)$ is the DFT of $h(n)$, X^* is the complex conjugate of X and the subscript *N* indicates modulo *N* indexing.

(c) Given two 4-point real sequences $p(n)=\{1, 2, 0, 1\}$ and $q(n)=\{2, 2, 1, 1\}$, use the method of part (b) to formulate, and write out in full, the matrix form for this particular case including the matrix elements. Hence find the DFTs of $p(n)$ and $q(n)$. [5]

4

(a) Draw the block diagram of a two-point DFT process and hence show with the aid of appropriate diagrams how a four-point decimation-in-time FFT operation can be implemented efficiently. [6]

(b) Discuss briefly the features of a DSP microprocessor which would facilitate efficient computation of an FFT algorithm. [4]

(c) The system shown in Figure 2 represents a filter with input $X(z)$ and output $Y(z)$.

Deduce the important characteristics of this filter and find expressions for any zeros and/or poles. [10]

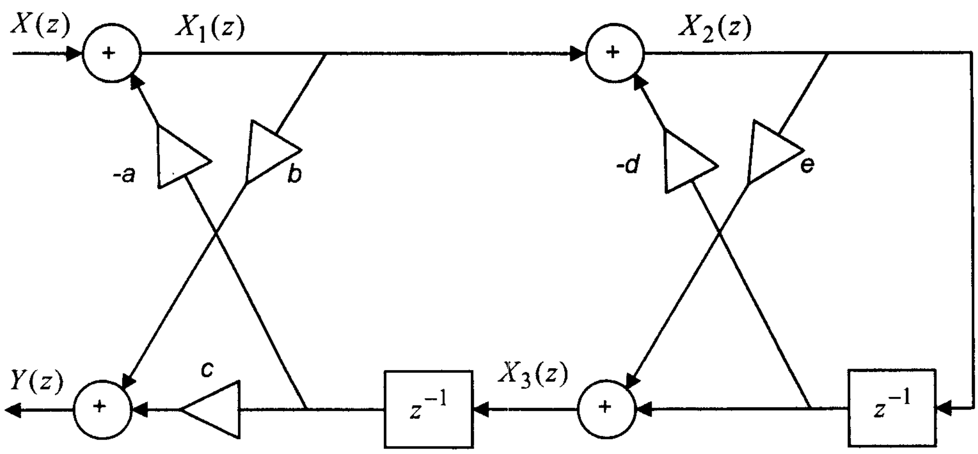


Figure 2.

5 A finite impulse response digital filter has an output $y(n)$ for an input $x(n)$

$$y(n) = x(n) + x(n-3)$$

where n is the discrete time index.

(a) Find the poles and zeros associated with this filter and sketch a plot of them on the z-plane. [4]

(b) Write an expression and sketch a plot (in dB) for the filter's magnitude response and hence determine the gain of the filter at a frequency of $\pi/8$. Mark this frequency and corresponding gain on your plot. [6]

(c) Define the group delay of a digital filter and state the units in which group delay is measured. [3]

(d) If the digital filter above operates with a sampling period of T seconds, what is the group delay of the filter at a frequency of $\frac{0.1}{T}$? [7]

6

(a) What is the relationship between system function and frequency response of a discrete-time system? [3]

(b) A communications channel can be represented by a linear shift invariant system defined by the difference equation [6]

$$y(n) = x(n) - 0.8x(n-1) + 0.8x(n-2).$$

Write down the system function, $H(z) = \frac{Y(z)}{X(z)}$, for this system and plot its poles, zeros and region of convergence on the z-plane.

(c) It is desired to equalise the channel in (b) by passing the signal $y(n)$ through a filter $G(z)$ such that the signal $x(n)$ is recovered exactly. Determine the required system function $G(z)$ for the equaliser and plot the poles and zeros on the z-plane. [6]

(d) Describe with reference to the z-plane what is meant by a non-minimum phase channel. State what problem occurs when attempting to equalize such a channel. Suggest possible approaches to this problem. [5]

1.

a) $H(z)=\sum_{l=0}^{L-1} z^{-l}E_l(z^L)$ with $E_l(z)=\sum_n e_l(n)z^{-n}$ and $e_l(n)=h(nL+l)$

b) Subband analysis and synthesis.

The signals vi(n) represent the subband division (or analysis) of x(n). They are formed from the effective bandpass filtering of x(n).

$$\begin{bmatrix} H_0(z) \\ \vdots \\ H_{L-1}(z) \end{bmatrix} = \begin{bmatrix} P_{00}(z^L) & P_{01}(z^L) & \cdots & P_{0,L-1}(z^M) \\ \vdots & & \ddots & \\ P_{L-1,0}(z^L) & & \cdots & P_{L-1,L-1}(z^L) \end{bmatrix} \begin{bmatrix} 1 \\ z^{-1} \\ \vdots \\ z^{-(M-1)} \end{bmatrix}$$

c) $h_1(n)=h_0(n)(-1)^n$

Sketch should show symmetry points at +/- pi/2. The effective frequency translation of H0 to form H1 must be made clear for full marks.

2.

a) $X(z)=\sum_{n=-\infty}^{\infty} x(n)z^{-n}$

b) ROC encloses the range of z for which the z-transform expression converges. Outside the ROC the z-transform does not exist in any meaningful way.

c) The z-transform in the discrete-time case corresponds to the Laplace transform in the continuous time case. Sketches should show the significance of the unit circle vs. the jw axis. The frequency response of discrete-time signals is periodic. These are compactly represented by multiple rotations around the unit circle in z.

d) $h(n)=0.5\delta(n)-(-1)^n u(n)-1.5(-2)^n u(-n-1)$

3.
(a)

$$\mathbf{x} = [x(0)\ x(1) \ldots x(N-1)]^T$$
$$\mathbf{X} = [X(0)\ X(1) \ldots X(N-1)]^T$$
$$D_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & & \\ \vdots & \vdots & & \ddots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}$$

(b)

$$x(n) = g(n) + jh(n)$$
$$X(k) = \sum_{n=0}^{N-1} (g(n) + jh(n)) e^{-j2\pi kn/N}$$
$$X^*(k) = \sum_{n=0}^{N-1} (g(n) - jh(n)) e^{j2\pi kn/N}$$
$$X^*(-k) = \sum_{n=0}^{N-1} (g(n) - jh(n)) e^{-j2\pi kn/N}$$
$$0.5(X(k) + X^*(-k)) = 0.5\left(\sum_n g(n) W_N^{kn} + j\sum_n h(n) W_N^{kn} + \sum_n g(n) W_N^{kn} - j\sum_n h(n) W_N^{kn} \right)$$
$$= \sum_n g(n) W_N^{kn}$$
$$= G(k)$$

Similar expressions follow for $H(k)$.

(c)

Form the complex sequence from the 2 real sequences as $x(n) = p(n) + jq(n)$ then

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 1+2j \\ 2+2j \\ j \\ 1+j \end{bmatrix} = \begin{bmatrix} 4+6j \\ 2 \\ -2 \\ 2j \end{bmatrix}$$

$$X^*(k) = \{4 - j6 \quad 2 \quad -2 \quad -j2\}$$
$$X^*(-k) = \{4 - j6 \quad -j2 \quad -2 \quad 2\}$$

$$P(k) = \{4 \quad 1-j \quad -2 \quad 1+j\}$$
$$Q(k) = \{6 \quad 1-j \quad 0 \quad 1+j\}$$

4.
(a)
[Bookwork]

(b)
bit-reversed addressing
fast MAC

(c)

$$X_1 = X - az^{-1}X_3$$
$$X_2 = X_1 - dz^{-1}X_2$$
$$X_3 = X_2(e + z^{-1})$$
$$Y = bX_1 + cz^{-1}X_3$$

$$X_2 = \frac{X_1}{1 + dz^{-1}}$$
$$X_3 = \frac{e + z^{-1}}{1 + dz^{-1}}X_1$$
$$X_1 = X - \frac{az^{-1}(e + z^{-1})}{1 + dz^{-1}}X_1$$
$$= \frac{(1 + dz^{-1})}{1 + (d + ae)z^{-1} + az^{-2}}X$$
$$\frac{Y}{X} = \frac{b + (bd + ce)z^{-1} + cz^{-2}}{1 + (d + ae)z^{-1} + az^{-2}} = \frac{c + (bd + ce)z^{1} + bz^{2}}{a + (d + ae)z^{1} + z^{2}}$$

The filter is therefore 2nd order IIR.

The zeros are given by $\dfrac{-(bd + ce) \pm \sqrt{(bd + ce)^2 - 4bc}}{2c}$ and

the poles are given by $\dfrac{-(d + ae) \pm \sqrt{(d + ae)^2 - 4a}}{2a}$.

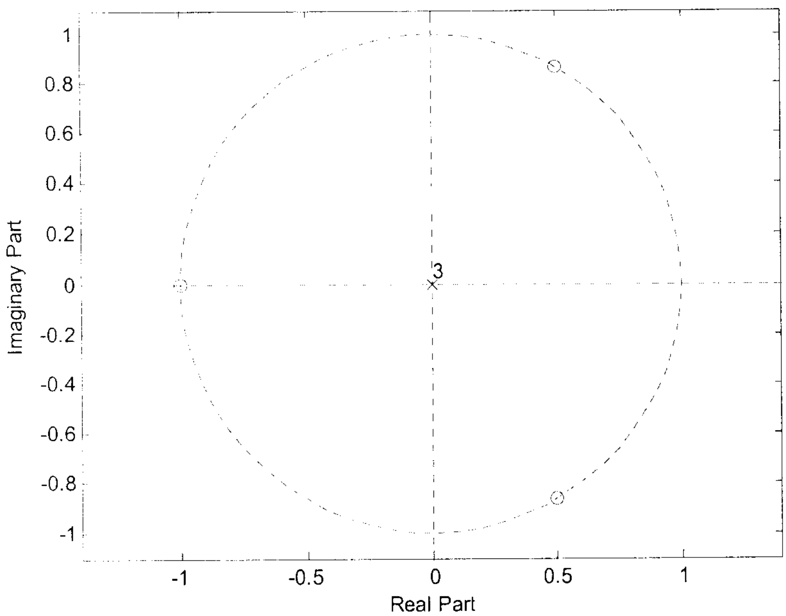
5.

(a)

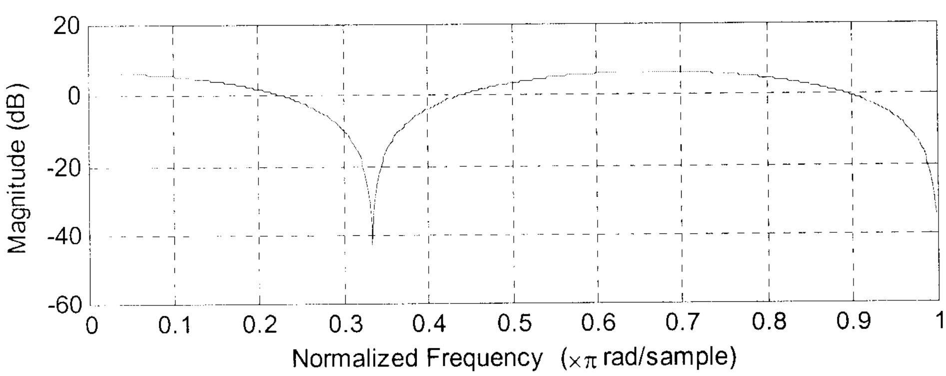
$$H(z) = 1 + z^{-3} = \frac{z^3 + 1}{z^3}$$

Considering the numerator, the solutions of $z^3 = -1$ give zeros at -1, 0.5 + j0.866, 0.5000 - j0.866.

Consider the denominator, the solutions of $z^3 = 0$ give 3 poles at the origin.



(b)



Frequency response is given by

$$H(e^{j\omega}) = 1 + e^{-j3\omega} = 1 + \cos 3\omega - j\sin 3\omega$$

$$\left|H(e^{j\omega})\right| = \sqrt{(1+\cos 3\omega)^2 + (\sin 3\omega)^2} = \sqrt{2 + 2\cos 3\omega}$$

For $\omega = \pi/8$, $\left|H(e^{j\omega})\right| = \sqrt{2 + 2\cos\left(3\pi/8\right)} = 1.66 \simeq 4.4\text{ dB}$

(c) Group delay is the negative derivative of phase wrt frequency. Units of seconds.

(d) For FIR filters such as this, the phase response is linear – only need to find the gradient of the linear function. Two frequency points are sufficient.

$$\angle H(e^{j\omega}) = \tan^{-1}\left(\frac{\sin 3\omega}{1+\cos 3\omega}\right)$$

For $\omega = 0, \quad \angle = 0$.

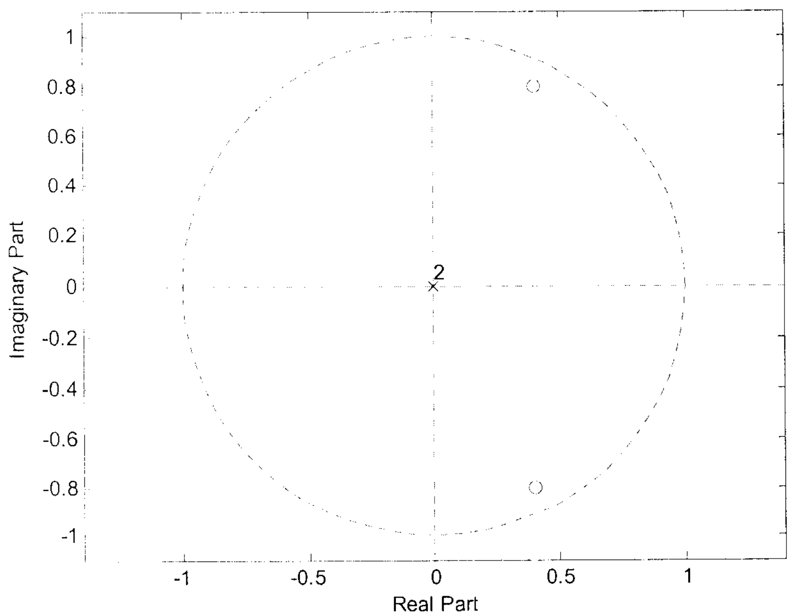
For $\omega = \pi/6, \quad \angle = \tan^{-1}\left(\frac{\sin \pi/2}{1+\cos \pi/2}\right) = \tan^{-1}(1) = \pi/4$

Therefore the group delay = 1.5T seconds at all frequencies.

6.

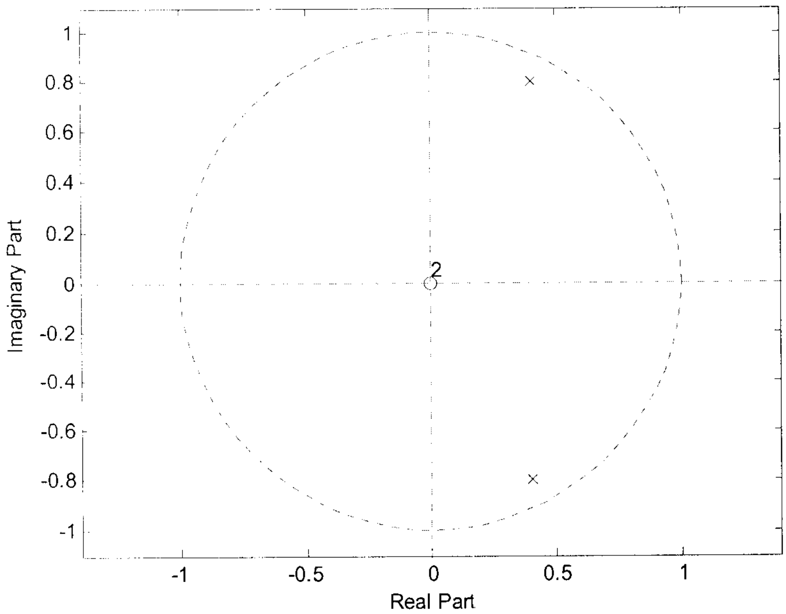
(a) Frequency response is the z-transform evaluated on the unit circle in the z-plane.

(b) $H(z) = 1 - 0.8z^{-1} + 0.8z^{-2}$ with roots at 0.4 +/-j0.8.



(c) Equalizer must have poles to cancel the zeros giving

$$G(z) = \frac{z^{-2}}{1 - 0.8z^{-1} + 0.8}$$



(d) In a non-minimum phase channel, the zeros are outside the unit circle and therefore give rise to unstable poles in the equalizer. A possible approach is to reflect the 'unstable' poles inside the unit circle which does not affect the magnitude response of the equalizer.