Imperial College
London

M4/5A2

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May-June 2017

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Fluid Dynamics I

Date: Wednesday 24 May 2017

Time: 14:00 - 16:30

Time Allowed: 2.5 Hours

This paper has 5 Questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the relevant main book(s) are full.

All required additional material will be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers to each question as per the table below.

Raw Mark	Up to 12	13	14	15	16	17	18	19	20
Extra Credit	0	½	1	1 ½	2	2 ½	3	3 ½	4

- Each question carries equal weight.
- Calculators may not be used.

1. It is known that the Earth is not a perfect sphere. It is also known that the pressure in the ocean increases with depth much faster than it decreases in the atmosphere. Keeping this in mind, find the shape of the Earth by assuming that it may be thought of as a rotating volume of fluid surrounded by vacuum. The fluid is kept together through the action of the gravitational force. Assume that this force has only a radial component, which is proportional to the distance from the Earth's centre, namely

$$f_r = -\alpha r.$$

You need to perform the following tasks:

(a): Thanks to the fact that the fluid motion is symmetric with respect to the Earth's axis, it is convenient to use spherical polar coordinates (see Figure 1), where the Navier–Stokes equations are given by (2). Place the x-axis along the axis of the Earth's rotation.

Assuming that the fluid rotates as a solid body, what are the velocity components V_r, V_ϑ and V_ϕ?

(b): Given that the angular velocity of the Earth's rotation is Ω, and the Earth's radius at the North Pole is R_0, show that at any other meridional angle ϑ (measured from the North Pole), the distance R from the Earth's surface to the centre is given by

$$R = \frac{R_0}{\sqrt{1 - \dfrac{\Omega^2}{\alpha}\sin^2\vartheta}}. \qquad (1)$$

(c): Choose the plain surface S in Figure 1 to coincide with the (x, y) plane, and show that equation (1) may be written as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Hence, conclude that the Earth is an ellipsoid. When are its principle axes, a and b?

(d): For the Earth, Ω^2/α is small. If there is a planet for which Ω^2/α is not small, then which of the assumptions you have used will fail?

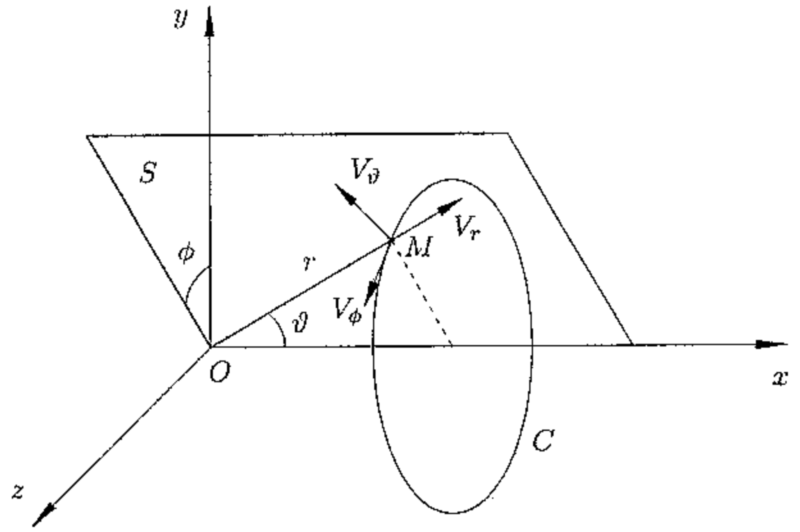


Figure 1: Spherical polar coordinates.

$$\frac{\partial V_r}{\partial t} + V_r\frac{\partial V_r}{\partial r} + \frac{V_\vartheta}{r}\frac{\partial V_r}{\partial \vartheta} + \frac{V_\phi}{r\sin\vartheta}\frac{\partial V_r}{\partial \phi} - \frac{V_\vartheta^2 + V_\phi^2}{r} = f_r - \frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\bigg(\frac{\partial^2 V_r}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 V_r}{\partial \vartheta^2} + \frac{1}{r^2\sin^2\vartheta}\frac{\partial^2 V_r}{\partial \phi^2} + \frac{2}{r}\frac{\partial V_r}{\partial r} + \frac{1}{r^2\tan\vartheta}\frac{\partial V_r}{\partial \vartheta} - \frac{2}{r^2}\frac{\partial V_\vartheta}{\partial \vartheta} - \frac{2}{r^2\sin\vartheta}\frac{\partial V_\phi}{\partial \phi} - \frac{2V_r}{r^2} - \frac{2V_\vartheta}{r^2\tan\vartheta}\bigg), \tag{2a}$$

$$\frac{\partial V_\vartheta}{\partial t} + V_r\frac{\partial V_\vartheta}{\partial r} + \frac{V_\vartheta}{r}\frac{\partial V_\vartheta}{\partial \vartheta} + \frac{V_\phi}{r\sin\vartheta}\frac{\partial V_\vartheta}{\partial \phi} + \frac{V_r V_\vartheta}{r} - \frac{V_\phi^2}{r\tan\vartheta} = f_\vartheta - \frac{1}{\rho r}\frac{\partial p}{\partial \vartheta} + \nu\bigg(\frac{\partial^2 V_\vartheta}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 V_\vartheta}{\partial \vartheta^2} + \frac{1}{r^2\sin^2\vartheta}\frac{\partial^2 V_\vartheta}{\partial \phi^2} + \frac{2}{r}\frac{\partial V_\vartheta}{\partial r} + \frac{1}{r^2\tan\vartheta}\frac{\partial V_\vartheta}{\partial \vartheta} - \frac{2\cos\vartheta}{r^2\sin^2\vartheta}\frac{\partial V_\phi}{\partial \phi} + \frac{2}{r^2}\frac{\partial V_r}{\partial \vartheta} - \frac{V_\vartheta}{r^2\sin^2\vartheta}\bigg), \tag{2b}$$

$$\frac{\partial V_\phi}{\partial t} + V_r\frac{\partial V_\phi}{\partial r} + \frac{V_\vartheta}{r}\frac{\partial V_\phi}{\partial \vartheta} + \frac{V_\phi}{r\sin\vartheta}\frac{\partial V_\phi}{\partial \phi} + \frac{V_r V_\phi}{r} + \frac{V_\vartheta V_\phi}{r\tan\vartheta} = f_\phi - \frac{1}{\rho r\sin\vartheta}\frac{\partial p}{\partial \phi} + \nu\bigg(\frac{\partial^2 V_\phi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 V_\phi}{\partial \vartheta^2} + \frac{1}{r^2\sin^2\vartheta}\frac{\partial^2 V_\phi}{\partial \phi^2} + \frac{2}{r}\frac{\partial V_\phi}{\partial r} + \frac{1}{r^2\tan\vartheta}\frac{\partial V_\phi}{\partial \vartheta} + \frac{2}{r^2\sin\vartheta}\frac{\partial V_r}{\partial \phi} + \frac{2\cos\vartheta}{r^2\sin^2\vartheta}\frac{\partial V_\vartheta}{\partial \phi} - \frac{V_\phi}{r^2\sin^2\vartheta}\bigg), \tag{2c}$$

$$\frac{\partial V_r}{\partial r} + \frac{1}{r}\frac{\partial V_\vartheta}{\partial \vartheta} + \frac{1}{r\sin\vartheta}\frac{\partial V_\phi}{\partial \phi} + \frac{2V_r}{r} + \frac{V_\vartheta}{r\tan\vartheta} = 0. \tag{2d}$$

2. Analyse the following two viscous two-dimensional flows of incompressible fluid:

(a): The first one is the flow down an infinite flat slope under the action of the gravitational field g. The angle between the slope and horizontal is α; see Figure 2. Assume that the fluid forms a layer of constant thickness h. Assume also that the flow is steady and none of the fluid-dynamic functions depends on the coordinate x measured down the slope. Your task is to find the velocity distribution across the layer.

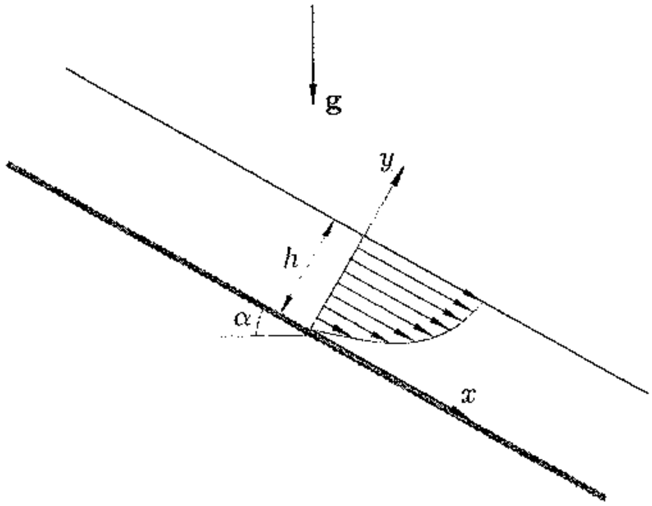


Figure 2: Fluid layer on the downslope.

Hint: Use the Navier–Stokes equations written in Cartesian coordinates with x directed down the slope (see Figure 2), and recall that the tangential stress

$$\tau_{yx} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

is zero at the upper edge of the fluid layer.

(b): Now consider an incompressible viscous fluid that occupies a semi-infinite region on one side of an infinite flat plate (as shown in Figure 3). The plate performs oscillatory motion

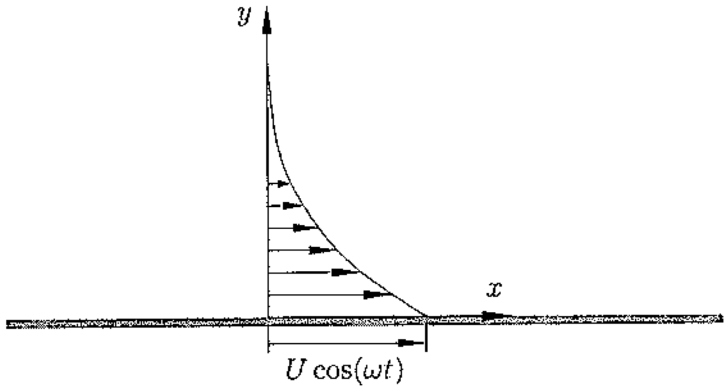


Figure 3: Flow above an oscillating plate.

in its plane with velocity given by

$$u = U\cos(\omega t). \tag{3}$$

continued on next page ...

Here U is the amplitude of the oscillations and ω is the frequency. Find the velocity distribution in the fluid above the surface.

Suggestion: Use Cartesian coordinates with the x-axis along the oscillating plate. Deduce that in the flow considered the Navier–Stokes equations reduce to

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}.$$

Seek the solution to this equation in the form

$$u = f(y)e^{i\omega t} + \overline{f}(y)e^{-i\omega t},$$

where $f(y)$ is a complex-valued function, and $\overline{f}(y)$ is the complex conjugate of $f(y)$.

3. A log of semicircular cross-section is placed on the ground as shown in Figure 4. Given that the radius of the log is a and the mass per unit length in the spanwise direction is m, find the critical wind speed V_∞ capable of lifting the log.

When solving this problem, start with analysis of the pressure force acting on a small element of the contour of the upper surface of the body. Project the force on the vertical

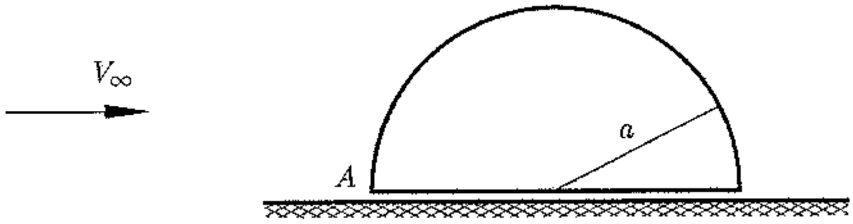


Figure 4: Log on the ground.

axis, and show that the integral force

$$L = \int_{-a}^{a} (p_0 - p)\, dx,$$

where x is measured along the ground, p is the pressure on the upper surface of the log and p_0 is the pressure in the gap between the log and the ground. You may assume that p_0 is constant and coincides with the pressure at the front stagnation point A.

Using the Bernoulli equation, determine the pressure distribution on the upper surface of the log and the pressure in the gap between the log and the ground.

Finally, balancing the lift force with the log weight, find the critical value of the free-stream velocity.

4. Consider a two-dimensional source of strength q placed at a point $z = b$ outside a circular cylinder as shown in Figure 5. The cylinder is centred at the coordinate origin and has radius a.

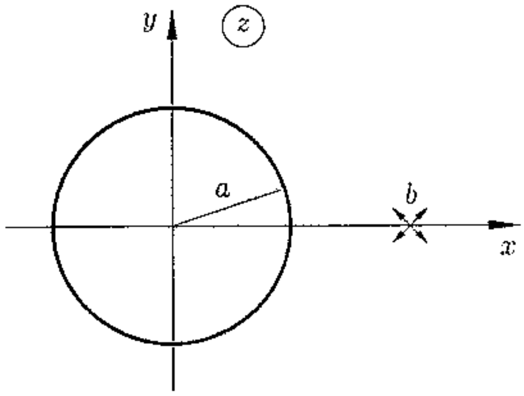


Figure 5: Flow over a circular cylinder due to a source.

Using the Joukovskii transformation

$$\zeta = \frac{1}{2}\left(z + \frac{a^2}{z}\right),$$

map the cylinder onto an infinitely thin flat plate occupying the interval $[-a, a]$ on the real axis in the auxiliary ζ-plane. Find the position ζ_0 of the source in the ζ-plane.

Now, notice that an infinitely thin flat plate aligned with the source does not affect the flow from the source. Hence, write the complex potential in the auxiliary plane as

$$W(\zeta) = \frac{q}{2\pi}\ln(\zeta - \zeta_0).$$

Finally, return to the physical z-plane, and show that the flow may be treated as being composed of the source situated at $z = b$, an additional source situated inside the cylinder at the point $z = a^2/b$, and a sink at $z = 0$.

Calculate the components (X, Y) of the integral pressure force acting on the cylinder. When performing this task you may use without proof the Blasius–Chaplygin formula

$$X - iY = i\frac{\rho}{2}\oint\limits_C \left(\frac{dw}{dz}\right)^2 dz, \tag{4}$$

where C is the cylinder surface that should be traced in the counter-clockwise direction.

Does the source blow the cylinder away or attract it?

Suggestion: When calculating the integral in (4), show that near $z = b$

$$\frac{dw}{dz} = \frac{q}{2\pi}\left(\frac{1}{z-b} + \frac{a^2/b}{b^2 - a^2}\right),$$

and apply the residue theorem.

5. Consider an inviscid incompressible fluid flow above flat ground on which a thin fence of height h is installed; in Figure 6, the ground coincides with the x-axis and the fence occupies an interval $[0, h]$ of the y-axis. The flow velocity far upstream of the fence is V_∞.

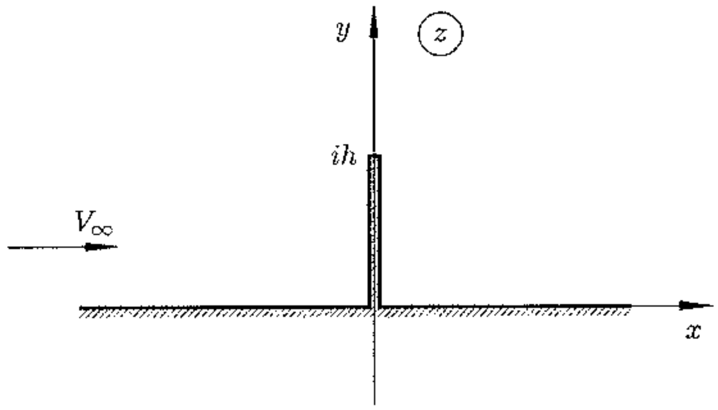


Figure 6: Flow over a fence.

Deduce that the conformal mapping of the physical z-plane shown in Figure 6 onto the upper half-plane in the auxiliary ζ-plane is given by

$$\zeta = \sqrt{z^2 + h^2}. \tag{5}$$

Assume that in the auxiliary plane the complex potential may be written as

$$W(\zeta) = \widetilde{V}_\infty \zeta.$$

and find the value of the real constant $\widetilde{V}_\infty$, taking into account that in the physical plane the free-stream velocity is V_∞.

Find the pressure distribution along the ground ($z = x$) by using the Bernoulli equation.

Calculate the integral pressure force

$$F = \int_{-\infty}^{\infty} (p - p_\infty)\, dx$$

acting upon the ground; here p_∞ is the unperturbed pressure in the oncoming flow.

Finally show that near the tip of the fence, the streamlines assume a parabolic shape. To perform this task take into account that in the auxiliary ζ-plane the streamlines are parallel to the real ξ-axis. Hence, write equation (5) as

$$\xi + id = \sqrt{z^2 + h^2},$$

where d is a constant. Simplify the right-hand side of this equation assuming that z is close to ih.

Solutions

Problem 1. [Unseen]

(a): Using Figure 1 the students should be able to see that the velocity components are given by

$$V_\phi = \Omega r \sin\vartheta, \qquad V_r = 0, \qquad V_\vartheta = 0, \tag{s1.1}$$

[2 marks]

(b): Substituting (s1.1) into equations (2a) and (2b) results in

$$\frac{1}{\rho}\frac{\partial p}{\partial r} = \Omega^2 r \sin^2\vartheta - \alpha r, \tag{s1.2a}$$

$$\frac{1}{\rho}\frac{\partial p}{\partial \vartheta} = \Omega^2 r^2 \sin\vartheta\cos\vartheta. \tag{s1.2b}$$

[4 marks]

Integrating the first of these we have

$$\frac{p}{\rho} = \tfrac{1}{2}\Omega^2 r^2 \sin^2\vartheta - \tfrac{1}{2}\alpha r^2 + \Phi(\vartheta). \tag{s1.3}$$

In order find function $\Phi(\vartheta)$, one needs to substitute (s1.3) into (s1.2b). This results in

$$\Phi'(\vartheta) = 0,$$

and we can conclude that $\Phi(\vartheta)$ is a constant, i.e.

$$\Phi(\vartheta) = C.$$

Thus the pressure everywhere inside the Earth is given by

$$p = \rho\Big(\tfrac{1}{2}\Omega^2 r^2 \sin^2\vartheta - \tfrac{1}{2}\alpha r^2 + C\Big). \tag{s1.4}$$

The equation for the Earth's surface is obtained by setting the pressure, p, to zero,

$$\tfrac{1}{2}\Omega^2 R^2 \sin^2\vartheta - \tfrac{1}{2}\alpha R^2 + C = 0. \tag{s1.5}$$

[4 marks]

Now the constant, C, can be found by taking into account that at the North pole, where $\vartheta = 0$, the radius $R = R_0$. We see that

$$C = \tfrac{1}{2}\alpha R_0^2,$$

which allows us to write the equation (s1.5) for the Earth surface as

$$\tfrac{1}{2}\Omega^2 R^2 \sin^2\vartheta - \tfrac{1}{2}\alpha R^2 + \tfrac{1}{2}\alpha R_0^2 = 0.$$ **[2 marks]**

It remains to solve this equation for R, and we will see that the Earth radius depends on the meridional angle ϑ as

$$R = \frac{R_0}{\sqrt{1 - \frac{\Omega^2}{\alpha}\sin^2\vartheta}}. \tag{s1.6}$$

[2 marks]

(c): Placing plane S into the (x, y)-plane, we can write

$$R = \sqrt{x^2 + y^2}, \qquad \sin\vartheta = \frac{y}{R}. \tag{s1.7}$$

Substituting the second equation on (s1.7) into (s1.6) and taking squaring on both sides, we have

$$R^2 = \frac{R_0^2}{1 - \frac{\Omega^2}{\alpha}\frac{y^2}{R^2}},$$

or equivalently,

$$R^2 - \frac{\Omega^2}{\alpha}y^2 = R_0^2.$$

Now we use the first equation in (s1.7). We have

$$x^2 + \left(1 - \frac{\Omega^2}{\alpha}\right)y^2 = R_0^2.$$

This may be written as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

with

$$a = R_0, \qquad b = \frac{R_0}{\sqrt{1 - \frac{\Omega^2}{\alpha}}}.$$

[4 marks]

(d): As the planet become more 'pancake-like', the gravitational force deviates from the radial direction.

[2 marks]

Problem 2. (a) [unseen], (b) [seen]

The students are expected to remember that for a two-dimensional flow the Navier–Stokes equations are written in Cartesian coordinates as

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = f_x - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{s2.1a}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = f_y - \frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right), \tag{s2.1b}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{s2.1c}$$

[2 marks]

(a). With the x-axis directed downslope, the components of the body force are given by

$$f_x = g\sin\alpha, \qquad f_y = -g\cos\alpha.$$

Taking this into account, the x- and y-momentum equations (s2.1a), (s2.1b) are written as

$$0 = g\sin\alpha - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}, \tag{s2.2}$$

$$0 = -g\cos\alpha - \frac{1}{\rho}\frac{\partial p}{\partial y}. \tag{s2.3}$$

[2 marks]

Integrating (s2.3), we have

$$p = -\rho g\cos\alpha\, y + \Phi(x). \tag{s2.4}$$

To find $\Phi(x)$, we note that at the upper edge of the fluid layer

$$p\Big|_{y=h} = p_a.$$

We have

$$\Phi(x) = p_a + \rho g\cos\alpha\, h. \tag{s2.5}$$

Substituting (s2.5) back into (s2.4)

$$p = p_a + \rho g\cos\alpha(h-y),$$ **[2 marks]**

we see that the pressure does not depend on x, and therefore, equation (s2.2) reduces to

$$\frac{\partial^2 u}{\partial y^2} = -\frac{g}{\nu}\sin\alpha.$$ **[2 marks]**

The general solution of this equation is

$$u = -\frac{g}{2\nu}\sin\alpha y^2 + C_1 y + C_2. \tag{s2.6}$$

Constants C_1 and C_2 may be found from the boundary conditions

$$u\Big|_{y=0} = 0, \qquad \frac{\partial u}{\partial y}\Big|_{y=h} = 0. \tag{s2.7}$$

The first of (s2.7) shows that

$$C_2 = 0. \tag{s2.8}$$

The second condition gives

$$C_1 = \frac{gh}{\nu}\sin\alpha. \tag{s2.9}$$

Substituting (s2.8) and (s2.9) back into (s2.6), we find

$$u = \frac{gh}{2\nu}\sin\alpha(2h-y)y.$$

[2 marks]

(b). Obviously, in this flow all the derivatives with respect to x are zeros. Therefore, it follows from the continuity equation (s2.1c) that

$$\frac{\partial v}{\partial y} = 0. \tag{s2.10}$$

Integrating (s2.10) with the impermeability condition on the plate surface

$$v\Big|_{y=0} = 0,$$

we have

$$v = 0$$

everywhere in the flow field.

[1 mark]

It is now easily seen that the x-momentum equation (s2.1a) reduces to

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial y^2}. \tag{s2.11}$$

[2 marks]

It has to be solved with the boundary conditions

$$u\Big|_{y=0} = \tfrac{1}{2}Ue^{i\omega t} + \tfrac{1}{2}Ue^{-i\omega t}, \qquad u\Big|_{y=\infty} = 0. \tag{s2.12}$$

We represent the sought solution in the form

$$u = f(y)e^{i\omega t} + \overline{f}(y)e^{-i\omega t}. \tag{s2.13}$$

Substitution of (s2.13) into (s2.11) results in

$$i\omega f(y)e^{i\omega t} - i\omega\overline{f}(y)e^{-i\omega t} = \nu f''(y)e^{i\omega t} + \nu\overline{f}''(y)e^{-i\omega t}.$$

Separating the terms with $e^{i\omega t}$ from the terms with $e^{-i\omega t}$ we obtain the two equations

$$i\omega f = \nu f'', \qquad -i\omega \overline{f} = \nu \overline{f}''.$$

However, the second of these is simply the complex conjugate of the first equation. This means that in order to find function $f(y)$ we need to solve the equation

$$i\omega f = \nu f''. \tag{s2.14}$$

while the boundary conditions (s2.12) reduce to

$$f(0) = U, \qquad f(\infty) = 0. \tag{s2.15}$$

[3 marks]

The complementary solution to (s2.14) are sought in the form

$$f(y) = Ce^{\lambda y},$$

leading to the following characteristic equation

$$i\omega = \nu\lambda^2.$$

We see that

$$\lambda_{1,2} = \pm\sqrt{i\frac{\omega}{\nu}} = \pm e^{i\pi/4}\sqrt{\frac{\omega}{\nu}} = \pm(1+i)\sqrt{\frac{\omega}{2\nu}}.$$ **[2 marks]**

The general solution of equation (s2.14) is written as

$$f(y) = C_1 e^{\lambda_1 y} + C_2 e^{\lambda_2 y}. \tag{s2.16}$$

Since the real part of λ_1 is positive, $f(y)$ becomes infinitely large as $y \to \infty$ unless

$$C_1 = 0. \tag{s2.17}$$

In order to find C_2 we used the first of conditions (s2.15). We have

$$C_2 = \frac{1}{2}U. \tag{s2.18}$$

Substituting (s2.17) and (s2.18) back into (s2.16), we have

$$f(y) = \frac{1}{2}Ue^{-(1+i)\sqrt{\frac{\omega}{2\nu}}y}. \tag{s2.19}$$

It remains to substitute (s2.19) into (s2.13), and we arrive at the conclusion that

$$u = \Re\left\{Ue^{-\sqrt{\frac{\omega}{2\nu}}y}e^{i\left(\omega t - \sqrt{\frac{\omega}{2\nu}}y\right)}\right\} = Ue^{-\sqrt{\frac{\omega}{2\nu}}y}\cos\left(\omega t - \sqrt{\frac{\omega}{2\nu}}y\right).$$ **[2 marks]**

Problem 3. [unseen, equation (s3.4) was used by the students for other flows]

In order to calculated the lift force acting on the half-log one needs to calculate the integral pressure forces on the upper and lower sides of the log. We shall start with the upper surface; see Figure 1(a). If we consider a small element of the log contour of length dl then the pressure force acting on this element (per unit length in the spanwise direction) is perpendicular to the surface and its modulus

$$|d\,\mathbf{P}| = pdl.$$

The projection of the force on the y-axis is

$$dL_u = -pdl\cos\alpha. \tag{s3.1}$$

Here minus is used to indicate that the force is directed downwards, i.e. against the y-axis.

[2 marks]

It is easily seen from Figure 1(b) that

$$dl\cos\alpha = dx,$$

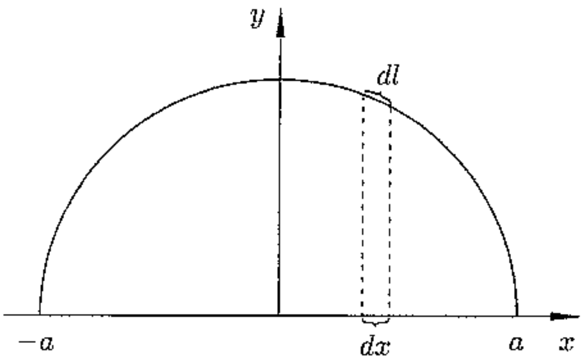
which allows to write (s3.1) as

$$dL_u = -pdx.$$

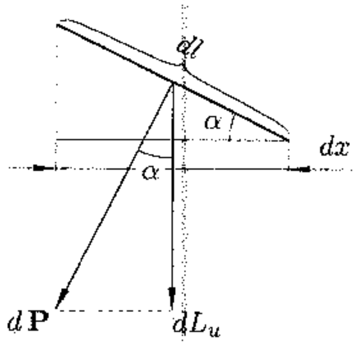
Integrating over the entire upper surface we have

$$L_u = -\int_{-a}^{a} pdx. \tag{s3.2}$$

[3 marks]



(a) Element of the surface.



(b) Zoomed surface element.

Figure 1: Calculation of the pressure force acting on the upper surface of the log.

Now we turn to calculation of the pressure force acting on the lower side of the log. Since the gap between the log and the ground is small, the air there should be almost stagnant. This means that the pressure in the gap is constant and coincides with the pressure p_0 at the front stagnation point A. Let us consider a small element dx of the lower surface of the log; see Figure 1(a). The pressure force acting on this element is directed vertically upwards and equals

$$dL_l = p_0 dx. \qquad \textbf{[2 marks]}$$

Integration over the entire lower surface gives

$$L_l = \int_{-a}^{a} p_0 dx. \tag{s3.3}$$

Adding (s3.3) to (s3.2) we find that the resultant lift force is

$$L = \int_{-a}^{a} (p_0 - p) dx. \tag{s3.4}$$

[2 marks]

In order to determine the pressure p on the log surface we notice that the flow over the log is identical to the upper half of the flow past a circular cylinder. This means that the velocity V on the upper surface of the log

$$V = 2V_\infty \sin\vartheta. \qquad \textbf{[3 marks]}$$

Here ϑ is the angle between the position vector and the x-axis; see Figure 2.

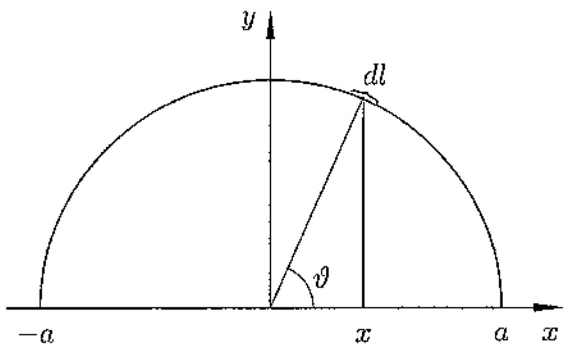


Figure 2: Coordinate x and position angle ϑ.

Using the Bernoulli equation

$$\frac{p}{\rho} + \frac{V^2}{2} = \frac{p_\infty}{\rho} + \frac{V_\infty^2}{2}, \tag{s3.5}$$

[1 mark]

we can find the pressure distribution on the log surface,

$$p = p_\infty + \frac{\rho}{2} V_\infty^2 - 2\rho V_\infty^2 \sin^2\vartheta. \tag{s3.6}$$

The pressure p_0 at the stagnation point may be calculated by setting $V = 0$ in (s3.5),

$$p_0 = p_\infty + \frac{\rho}{2} V_\infty^2. \tag{s3.7}$$

Substituting (s3.6) and (s3.7) into (s3.4), we have

$$L = 2\rho V_\infty^2 \int\limits_{-a}^{a} \sin^2\vartheta \; dx. \qquad \textbf{[2 marks]}$$

Since $x = r\cos\vartheta$, this formula may be rewritten as

$$L = -2\rho V_\infty^2 a \int\limits_{\pi}^{0} \sin^2\vartheta \cdot \sin\vartheta d\vartheta = 2\rho V_\infty^2 a \int\limits_{0}^{\pi} (1 - \cos^2\vartheta)\sin\vartheta d\vartheta =$$

$$= 2\rho V_\infty^2 a \Big(-\cos\vartheta + \frac{1}{3}\cos^3\vartheta\Big)\Big|_0^\pi = \frac{8}{3}\rho V_\infty^2 r.$$

[3 marks]

The critical wind speed is reached when the lift force becomes equal to the weight mg per unit length in the spanwise direction:

$$\frac{8}{3}\rho V_\infty^2 a = mg.$$

We have

$$V_\infty = \sqrt{\frac{3}{8}\frac{mg}{\rho a}}. \qquad \textbf{[2 marks]}$$

Problem 4. **[unseen]**

The Joukovskii transformation

$$\zeta = \frac{1}{2}\left(z + \frac{a^2}{z}\right) \tag{s4.1}$$

maps the cylinder in the physical z-plane onto a flat plate in the auxiliary ζ-plane occupying an interval $[-a, a]$ on the real axis. The source is mapped into the point

$$\xi_0 = \frac{1}{2}\left(b + \frac{a^2}{b}\right). \tag{s4.2}$$

[3 marks]

Since the source produces velocity parallel to the plate, we can write the complex potential in the auxiliary plane as

$$W(\zeta) = \frac{q}{2\pi}\ln(\zeta - \xi_0). \tag{s4.3}$$

Substitution of (s4.1) and (s4.2) into (s4.3) yields

$$w(z) = \frac{q}{2\pi}\ln\left[\frac{1}{2}\left(z + \frac{a^2}{z}\right) - \frac{1}{2}\left(b + \frac{a^2}{b}\right)\right]. \tag{s4.4}$$

[2 marks]

The argument of the logarithm may be rearranged as

$$\frac{1}{2}\left(z + \frac{a^2}{z}\right) - \frac{1}{2}\left(b + \frac{a^2}{b}\right) = \frac{1}{2}(z-b)\frac{1}{z}\left(z - \frac{a^2}{b}\right),$$

and we have

$$w(z) = \frac{q}{2\pi}\ln(z-b) + \frac{q}{2\pi}\ln\left(z - \frac{a^2}{b}\right) - \frac{q}{2\pi}\ln z - \frac{q}{2\pi}\ln(2).$$

Since the complex potential is defined to within an arbitrary constant we can write

$$w(z) = \frac{q}{2\pi}\ln(z-b) + \frac{q}{2\pi}\ln\left(z - \frac{a^2}{b}\right) - \frac{q}{2\pi}\ln z. \tag{s4.5}$$

We see that the flow is composed of the source at $z = b$, a source at the inverse point $z = a^2/b$ inside the circle and a sink at the centre of the circle $z = 0$.

[3 marks]

Differentiation of (s4.5) results in

$$\frac{dw}{dz} = \frac{q}{2\pi}\left(\frac{1}{z-b} + \frac{1}{z - a^2/b} - \frac{1}{z}\right).$$

Near $z = b$ we have

$$\frac{dw}{dz} = \frac{q}{2\pi}\left(\frac{1}{z-b} + \frac{1}{b - a^2/b} - \frac{1}{b}\right) = \frac{q}{2\pi}\left(\frac{1}{z-b} + \frac{a^2/b}{b^2 - a^2}\right), \qquad \textbf{[2 marks]}$$

and

$$\left(\frac{dw}{dz}\right)^2 = \frac{q^2}{4\pi^2}\left[\frac{1}{(z-b)^2} + \frac{2a^2/b}{b^2-a^2}\frac{1}{z-b} + \cdots\right]. \tag{s4.6}$$

[3 marks]

Since the point $z = b$ is external to the contour C, the residue theorem reads

$$X - iY = -i\frac{\rho}{2}\mathrm{Res}\left\{\left(\frac{dw}{dz}\right)^2\right\}\Bigg|_{z=b} \tag{s4.7}$$

[3 marks]

It follows from (s4.6) that

$$\mathrm{Res}\left\{\left(\frac{dw}{dz}\right)^2\right\}\Bigg|_{z=b} = 2\pi i\frac{q^2}{4\pi^2}\frac{2a^2/b}{b^2-a^2}. \tag{s4.8}$$

Substitution of (s4.8) into (s4.7) yields

$$X - iY = -\frac{\rho}{2}2\pi i\frac{q^2}{4\pi^2}\frac{2a^2/b}{b^2-a^2}. \tag{s4.9}$$

Separating the real and imaginary parts in (s4.9) we have

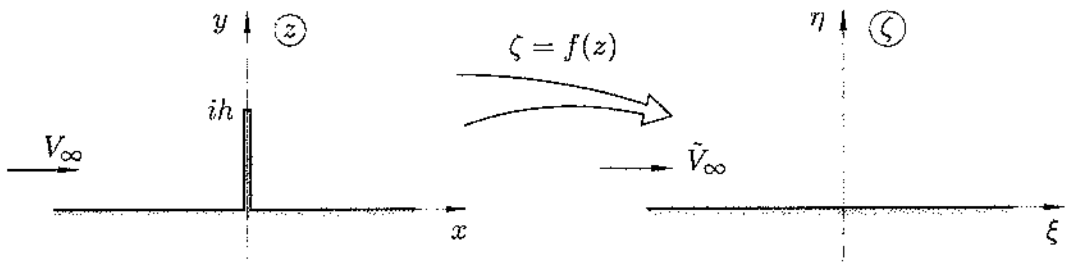
$$X = \rho\frac{q^2}{2\pi}\frac{a^2/b}{b^2-a^2}, \qquad Y = 0.$$ **[2 marks]**

We see that the force is directed towards the source. This, of course, is not what one would expect. Perhaps, this is because the above theory does not take into account flow separation from the cylinder surface.

[2 marks]

Problem 5. [Seen up to equation (s5.5). The rest is new for the students]

To describe the flow we perform the mapping of the physical z-plane onto the auxiliary ζ-plane where the rigid body surface coincides with the real ξ-axis as shown in Figure 3, and the flow is uniform such that the complex potential



(a) Physical plane. (b) Auxiliary plane.

Figure 3: Required mapping.

$$W(\zeta) = \tilde{V}_\infty \zeta. \tag{s5.1}$$

To find the function $f(z)$ which perform the required conformal transformation, let us first consider the mapping

$$\zeta_1 = z^2. \qquad \textbf{[2 marks]}$$

It transforms the rigid body surface (the flat wall together with the plate) onto a branch cut in the ζ_1-plane along the real ξ-axis with the tip $z = ih$ being mapped into the point (see Figure 4)

$$\zeta_1 = (ih)^2 = -h^2. \qquad \textbf{[1 mark]}$$

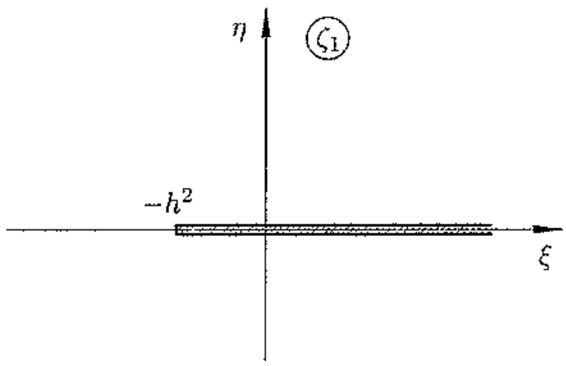


Figure 4: ζ_1-plane.

With the transformation

$$\zeta_2 = \zeta_1 + h^2 \qquad \textbf{[1 mark]}$$

which performs parallel translation of the complex plane along the real axis we move the tip of the branch cut into the coordinate origin $\zeta_2 = 0$ as shown in Figure 5. It remains to "open" the cut with the help of the function

$$\zeta = \sqrt{\zeta_2}. \qquad \textbf{[2 marks]}$$

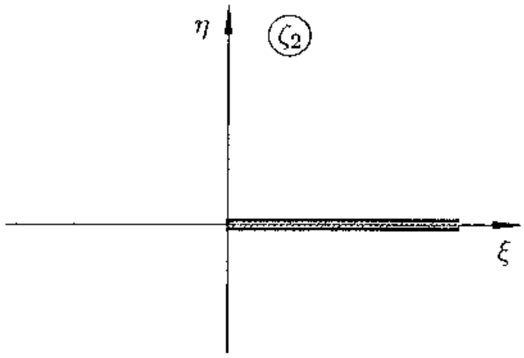


Figure 5: ζ_2-plane.

We see that the sought mapping function is

$$\zeta = \sqrt{\zeta_2} = \sqrt{\zeta_1 + h^2} = \sqrt{z^2 + h^2}. \tag{s5.2}$$

Substituting (s5.2) into (s5.1) we arrive at the conclusion that the sought complex potential in the $z-$plane is written as

$$w(z) = \tilde{V}_\infty \sqrt{z^2 + h^2}. \tag{s5.3}$$

The complex-conjugate velocity at any point in the z-plane is

$$\overline{V} = \frac{dw}{dz} = \tilde{V}_\infty \frac{z}{\sqrt{z^2 + h^2}}. \tag{s5.4}$$

[2 marks]

In particular, in the oncoming flow ($z \to \infty$)

$$\overline{V} \to \tilde{V}_\infty,$$

which means that $\tilde{V}_\infty = V_\infty$, and we can write equations (s5.3) and (s5.4) as

$$w(z) = V_\infty \sqrt{z^2 + h^2}, \qquad \overline{V} = V_\infty \frac{z}{\sqrt{z^2 + h^2}}. \tag{s5.5}$$

[2 marks]

On the wall surface, where $z = x$ we have

$$\overline{V} = V_\infty \frac{x}{\sqrt{x^2 + h^2}}.$$

Therefore, using the Bernoulli equation

$$\frac{p}{\rho} + \frac{V^2}{2} = \frac{p_\infty}{\rho} + \frac{V_\infty^2}{2},$$ **[2 marks]**

we can find that

$$p = p_\infty + \frac{\rho}{2}\left(V_\infty^2 - V^2\right) = p_\infty + \frac{\rho}{2}\left(V_\infty^2 - V_\infty^2 \frac{x^2}{x^2 + h^2}\right) = p_\infty + \frac{1}{2}\rho V_\infty^2 \frac{h^2}{x^2 + h^2},$$

and the integral for the pressure force may be written as

$$F = 2\int_0^\infty (p - p_\infty)dx = \rho V_\infty^2 h^2 \int_0^\infty \frac{dx}{x^2 + h^2}.$$

Introducing a new integration variable t such that

$$x = ht, \qquad dx = h\,dt,$$

we have

$$F = \rho V_\infty^2 h \int_0^\infty \frac{dt}{t^2+1} = \rho V_\infty^2 h \arctan t\Big|_0^\infty = \rho V_\infty^2 h\,\frac{\pi}{2}. \qquad \textbf{[2 marks]}$$

Equation (s5.2) may be expressed as

$$\xi + id = \sqrt{z^2 + h^2} = \sqrt{(z - ih)(z + ih)}. \tag{s5.6}$$

For z close to ih the right-hand side of (s5.6) may be approximated as

$$\sqrt{(z - ih)(z + ih)} = \sqrt{2ih(z - ih)}$$

Consequently, we can write

$$\xi + id = \sqrt{2ih(z - ih)}. \tag{s5.7}$$

[3 marks]

Representing z in the form $z = x + iy$, and taking squares on both sides of (s5.7), we have

$$\xi^2 + 2id\xi - d^2 = 2ih\big[x + i(y - h)\big]. \tag{s5.8}$$

Separation of the real and imaginary parts in (s5.8) yields

$$\xi^2 - d^2 = -2h(y - h), \qquad 2d\xi = 2hx. \tag{s5.9}$$

Eliminating ξ from (s5.9) results in

$$\frac{h^2}{d^2}x^2 - d^2 = -2h(y - h), \qquad \textbf{[3 marks]}$$

which shows that the streamlines are parabolas.

Imperial College London
Department of Mathematics

Examiner's Comments

Exam: Fluid Dynamics I, A2 Session: 2016-2107

Question 1

Please use the space below to comment on the candidates' overall performance in the exam. A brief paragraph highlighting common mistakes and parts of questions done badly (or well) is sufficient. Do not refer to individual candidates. The purpose of this exercise is to provide guidance to the external examiners, and to the candidates themselves, on how you feel the cohort faired. Your comments will be available to students online.

This question requires careful examination of the terms of the Navier-Stokes equations in spherical coordinates. A significant simplification would to to notice that the viscous terms disappear in a "solid motion". Only few students used this observation. Also, only few students knew that the gravitational force is radial when the mass of fluid is close to spherical shape.

Marker: A. Ruban

Signature: Anatoly I Ruban Date: 31/05/2017

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Imperial College London
Department of Mathematics

Examiner's Comments

Exam: ______________________ **Session: 2016-2107**

Question 2

Please use the space below to comment on the candidates' overall performance in the exam. A brief paragraph highlighting common mistakes and parts of questions done badly (or well) is sufficient. Do not refer to individual candidates. The purpose of this exercise is to provide guidance to the external examiners, and to the candidates themselves, on how you feel the cohort faired. Your comments will be available to students online.

Question 2 consists of two Problems. These are standard exact solutions of the Navier-Stokes equations. The students found these rather easy. Still a few students could not write down the Navier-Stokes equations in cartesian coordinates.

Marker: A. Ruban

Signature: [signature] Date: 31/05/2017

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Examiner's Comments

Exam: ______________________ **Session: 2016-2107**

Question 3

Please use the space below to comment on the candidates' overall performance in the exam. A brief paragraph highlighting common mistakes and parts of questions done badly (or well) is sufficient. Do not refer to individual candidates. The purpose of this exercise is to provide guidance to the external examiners, and to the candidates themselves, on how you feel the cohort faired. Your comments will be available to students online.

This question proved to be rather difficult for a number of students, despite the projection of the pressure forces was discussed in detail in lectures and a number of other problems. I guess, next year more attention to development of "physical intuition" is needed.

Marker: A. Kuban

Signature: Anatoly Kuban Date: 31/05/2017

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Department of Mathematics

Examiner's Comments

Exam: ______________________ **Session: 2016-2107**

Question 4

Please use the space below to comment on the candidates' overall performance in the exam. A brief paragraph highlighting common mistakes and parts of questions done badly (or well) is sufficient. Do not refer to individual candidates. The purpose of this exercise is to provide guidance to the external examiners, and to the candidates themselves, on how you feel the cohort faired. Your comments will be available to students online.

This is a straightforward question, and the majority of students were able to deal with it.

Marker: A. Kuba

Signature: [signature] Date: 31/05/2017

Please return with exam marks (one report per marker)