Imperial College London

MSci EXAMINATION May 2012

This paper is also taken for the relevant Examination for the Associateship

GENERAL RELATIVITY

For 4th-Year Physics Students

Tuesday, 22nd May 2012: 14:00 to 16:00

The paper consists of two sections: A and B
Section A contains one question, comprising small parts. [20 marks total]
Section B will contain four questions on selected parts of the course. [15 marks each]

Candidates are required to:
Answer **ALL** *parts of Section A and* **TWO** *questions from Section B.*

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the 3 answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in 3 answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

Conventions:

We use conventions as in lectures. In particular we take $(-,+,+,+)$ signature.

You may find the following formulae useful:

The Christoffel symbol is defined as,

$$\Gamma^{\mu}{}_{\alpha\beta} \equiv \frac{1}{2} g^{\mu\nu} \left(\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta} \right)$$

The covariant derivative is given as,

$$\nabla_\mu v^\nu \equiv \partial_\mu v^\nu + \Gamma^{\nu}{}_{\mu\alpha} v^\alpha$$

The Riemann tensor is defined as,

$$R_{\alpha\beta\mu}{}^{\delta} \quad = \quad \partial_\beta \Gamma^{\delta}{}_{\alpha\mu} - \partial_\alpha \Gamma^{\delta}{}_{\beta\mu} + \Gamma^{\nu}{}_{\alpha\mu} \Gamma^{\delta}{}_{\beta\nu} - \Gamma^{\nu}{}_{\beta\mu} \Gamma^{\delta}{}_{\alpha\nu}$$

For a Lagrangian of a curve $x^\mu(\lambda)$ of the form,

$$F = \int d\lambda \, \mathcal{L}(x^\mu, \frac{dx^\mu}{d\lambda})$$

the Euler-Lagrange equations are,

$$\frac{d}{d\lambda} \left(\frac{\partial \mathcal{L}}{\partial (\frac{dx^\mu}{d\lambda})} \right) = \frac{\partial \mathcal{L}}{\partial x^\mu}$$

Section A

Answer all of section A.

SECTION A

1. This question concerns timelike geodesics.

(i) State how the components of a vector w^μ and the metric $g_{\mu\nu}$ transform under a coordinate transformation $x \to x'$.

[4 marks]

(ii) Consider a timelike curve $x^\mu(\lambda)$ parameterized by λ. Recall the curve's tangent is $v^\mu = dx^\mu/d\lambda$. What condition must v^μ satisfy?

[1 mark]

(iii) Using the properties of derivatives show this does indeed transform correctly as a vector.

[2 marks]

(iv) Give the expression for the proper time τ along the timelike curve as an integral in λ from a starting point $x_0 = x(\lambda_0)$ to an endpoint $x_1 = x(\lambda_1)$.

[2 marks]

(v) Consider a change to another parameter μ so that $\lambda = \lambda(\mu)$;

(a) Show the proper time between the points x_0 and x_1 is invariant. (You may assume that $d\lambda/d\mu > 0$.)

[4 marks]

(b) Show we may always choose an *affine* parameterization where $d\tau/d\lambda = 1$.

[2 marks]

(vi) Use an affine parameterization and extremize the proper time of the curve using the Euler-Lagrange equations to show a timelike geodesic obeys,

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu{}_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0$$

[5 marks]

[Total 20 marks]

Section B

Answer 2 out of the 4 questions in the following section.

SECTION B

2. This question concerns curvature and the Bianchi identities.

(i) Consider the Riemannian geometry with line element,

$$ds^2 = dr^2 + \sinh^2 r \, d\phi^2$$

with coordinates $x^\mu = (r, \phi)$. Compute *all* the components of the Christoffel symbol for this metric.

[4 marks]

(ii) Compute *only* the component $R_{r\phi r}{}^{\phi}$ of the Riemann tensor. You should find,

$$R_{r\phi r}{}^{\phi} = -1$$

Using this result and the symmetries of the Riemann tensor, deduce *all* the components of $R_{\alpha\beta\mu\nu}$.

[4 marks]

(iii) Using these answers show the Ricci tensor obeys, $R_{\mu\nu} = \lambda g_{\mu\nu}$ for some constant λ which you should calculate.

[2 marks]

(iv) For a *general* geometry prove the Bianchi identity $\nabla_{[\alpha} R_{\beta\gamma]\mu}{}^{\nu} = 0$ - recall that using Riemann normal coordinates simplifies this calculation. From this derive the contracted Bianchi identity that involves the Ricci tensor and Ricci scalar, but not the Riemann tensor.

[5 marks]

[Total 15 marks]

3. This question concerns acceleration of a massive particle in general spacetimes.

(i) Given a timelike curve $x^\mu(\tau)$ with parameter given by proper time τ along the curve, and tangent $v^\mu = dx^\mu/d\tau$, recall the 4-acceleration a^μ is defined as,

$$a^\mu = v^\alpha \nabla_\alpha v^\mu$$

For the case of Minkowski spacetime, and taking Minkowski coordinates, show this is equivalent to the usual Special Relativity result,

$$a^\mu = \frac{d^2 x^\mu}{d\tau^2}$$

[3 marks]

(ii) For a general spacetime show that given a timelike vector w^μ and an orthogonal vector u^μ, then u^μ must be space-like.

[2 marks]

(iii) Recalling that the tangent obeys $v^\mu v_\mu = -1$, show that 4-acceleration, a^μ, is *orthogonal* to v^μ for particle motion in a general spacetime.

[4 marks]

(iv) Consider the 2-dimensional spacetime with line element,

$$ds^2 = -r^2 dt^2 + dr^2$$

and coordinates $x^\mu = (t, r)$. Show that this metric is related to the Minkowski metric,

$$ds^2 = -dT^2 + dZ^2$$

by the coordinate transform,

$$T = r \sinh t\,, \quad Z = r \cosh t$$

[3 marks]

(v) Use the fact that $\Gamma^t{}_{tt} = 0$ and $\Gamma^r{}_{tt} = r$, to show by direct calculation that a timelike observer following a curve r = constant experiences a *space-like* acceleration that is *orthogonal* to 4-velocity and obeys,

$$a^\mu a_\mu = \frac{1}{r^2} = \text{constant}$$

[3 marks]

[Total 15 marks]

4. This question concerns the FLRW spacetime.

(i) Recall the FLRW metric for a flat spatial geometry using a conformal time coordinate is given as,

$$ds^2 = a(\tau)^2 \left(-d\tau^2 + \delta_{ij} dx^i dx^j\right) \ , \quad i = 1, 2, 3$$

What are its symmetries? Write down a Killing vector field that generates spatial translation.

[3 marks]

(ii) Consider a timelike geodesic curve parameterized by its proper time t, so that $x^\mu(t) = \left(\tau(t), x^i(t)\right)$. Write down a Lagrangian for this curve that must be varied to obtain the geodesic equations. By varying this show that,

$$a^2 \frac{dx^i}{dt} = \text{constant}$$

[4 marks]

(iii) We see from this that an observer on a comoving trajectory - ie. constant x^i - follows a timelike geodesic. How is such a comoving observer's proper time t related to the conformal time τ?

[2 marks]

(iv) Follow a similar procedure as that above for timelike geodesics to explicitly show *null* geodesics are 45^o lines in the τ, x^i coordinates so that,

$$x^i = c^i \pm \frac{v^i}{|v|} \tau$$

where c^i and v^i are constants, and $|v| = \sqrt{\delta_{ij} v^i v^j}$.

[4 marks]

(v) A comoving source emits radiation at frequency ω_0 at time $\tau = \tau_0$, and a comoving observer receives it at time $\tau = \tau_1$. Compute the frequency of the received radiation in terms of the emitted frequency and the scale factor at these times.

[2 marks]

[Total 15 marks]

5. This question concerns the Newtonian limit of the Einstein equations and perfect fluids. Recall that for the conventions used in lectures the line element for the Newtonian spacetime is;

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \ , \quad \text{with} \quad g_{\mu\nu} = \eta_{\mu\nu} - 2\epsilon\, \Phi(x^i)\, \delta_{\mu\nu} + O(\epsilon^{3/2})$$

where $x^\mu = (t, x^i)$ with $i = 1, 2, 3$ and $\epsilon \to 0$ gives the Newtonian limit. Assume that the Newtonian potential $\Phi(x^i)$ is static so $\partial_t \Phi = 0$.

(i) Suppose a static particle at constant spatial position $x^i = x_0^i$ emits radiation at a constant frequency ω_0. What is the frequency of this radiation that a static observer at constant spatial position $x^i = x_1^i$ sees?

[4 marks]

(ii) Consider a particle, mass m, moving on a timelike geodesic, $x^\mu(\tau)$, with proper time τ. As in lectures we take it to be *slowly moving* so;

$$\frac{dx^\mu}{d\tau} = \left(1 + \epsilon\, f + O(\epsilon^{3/2}),\ \epsilon^{1/2}\, v^i + O(\epsilon^{3/2})\right)$$

Assume that $k^\mu = (1, 0, 0, 0)$ is a Killing vector field to all orders in ϵ. For this slowly moving particle compute the conserved energy E associated to this Killing vector. You should find,

$$E = m\left(1 + \epsilon\left(\frac{1}{2}\,\delta_{ij} v^i v^j + \Phi(x^k)\right) + O(\epsilon^{3/2})\right)$$

What is the physical interpretation of the various terms in this expression?

[4 marks]

(iii) The components of the Ricci tensor are;

$$\begin{aligned} R_{tt} &= \epsilon\, \delta_{ab} \partial_a \partial_b \Phi + O(\epsilon^{3/2}) \\ R_{ti} &= R_{it} = O(\epsilon^{3/2}) \\ R_{ij} &= \epsilon\, \delta_{ij}\ (\delta_{ab} \partial_a \partial_b \Phi) + O(\epsilon^{3/2}) \end{aligned}$$

Use these to compute (to leading order in ϵ) the components of the stress tensor that satisfies the Einstein equations for this spacetime.

[3 marks]

(iv) Show that pressureless dust perfect fluid in this spacetime may give rise to this stress tensor. How is the dust density related to the Newtonian potential?

[4 marks]

[Total 15 marks]