THE ANSWERS

Notations:

(a) B - Bookwork

(b) E - New example

(c) A - New application

1. a) This question checks your understanding of basic concepts of probability. It is very similar to many problems we have seen in the lectures/classes. Most students answered i) and ii) correctly. Most students could not answer iii) correctly! This is an A level question with a simple application of L'hospital rule to compute the limit. d) was answered correctly by about half of the students and results from a direct application of Chebyshev's inequality.

The CDF is given by $F_X(x) = \int_{-\infty}^{x} f_X(x)dx$ which leads to

$$F_X(x) = \begin{cases} 0, & x \leq 0, \\ 2x - x^2, & 0 < x < 1, \\ 1, & x \geq 1. \end{cases}$$

[2 - A]

ii) $E(X) = \int_{-\infty}^{\infty} x f_X(x)dx$. We get $E(X) = \int_0^1 x2(1-x)dx = 1/3$.

[2 - A]

$\mathrm{Var}(X) = E(X^2) - E(X)^2$. $E(X^2) = 1/6$. So $\mathrm{Var}(X) = 1/6 - 1/9 = 1/18$.

[2 - A]

iii) We write $m_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f_X(x)dx$. By integration by part,

$$m_X(t) = \begin{cases} \frac{2e^t}{t^2} - \frac{2}{t^2} - \frac{2}{t}, & t \neq 0, \\ 1, & t = 0. \end{cases}$$

[1 - A]

We can compute $E(X) = m'_X(0)$ and $E(X^2) = m''_X(0)$.

[1 - A]

We get $m'_X(t) = \frac{2e^t t - 4e^t + 4 + 2t}{t^3}$. Applying L'hospital rule 3 times, we get $E(X) = 1/3$.

[1 - A]

Similarly $m'_X(t) = \frac{2e^t t^2 - 8te^t + 12e^t - 12 - 4t}{t^4}$. Applying L'hospital rule 4 times, we get $E(X) = 1/6$ such that $\mathrm{Var}(X) = 1/6 - 1/9 = 1/18$.

[1 - A]

iv) By Chebyshev's inequality $P\left(\left|X-\frac{1}{3}\right| \geq \frac{1}{4}\right) \leq \frac{1}{1/16}E\left[(X-1/3)^2\right] = 16\mathrm{Var}(X) = 8/9$.

[2 - A]

The exact value can be computed as follows

$$
\begin{aligned}
P\left(\left|X-\frac{1}{3}\right| \geq \frac{1}{4}\right) &= 1-P\left(\left|X-\frac{1}{3}\right| \leq \frac{1}{4}\right) \\
&= 1-P\left(-\frac{1}{4} \leq X-\frac{1}{3} \leq \frac{1}{4}\right) \\
&= 1-P\left(\frac{1}{12} \leq X \leq \frac{7}{12}\right) \\
&= 1-F_X(\frac{7}{12})+F_X(\frac{1}{12}) \\
&= \frac{1}{3}
\end{aligned}
$$

[2 - A]

b) This question checks your understanding of basic concepts of statistics. It is very similar to many problems we have seen in the lectures/classes. Part iv) requires you to make the connection with CLT.

By the method of moments, we aim at finding the estimator by equating the sample mean with corresponding population mean, i.e. $E(X) = \overline{X}$.

[2 - A]

We can compute

$$E(X) = \int_0^\theta \frac{2x^2}{\theta^2}dx = \frac{2}{3}\theta.$$

We choose $\tilde{\theta}$ as $\frac{2}{3}\tilde{\theta} = \overline{X}$, i.e. $\tilde{\theta} = \frac{3}{2}\overline{X}$.

[2 - A]

ii) Expectation

$$E(\tilde{\theta}) = E(\frac{3}{2}\overline{X}) = \frac{3}{2}E(\overline{X}) = \frac{3}{2}E(X) = \theta.$$

[2 - A]

Variance

$$\mathrm{Var}(\tilde{\theta}) = \mathrm{Var}(\frac{3}{2}\overline{X}) = \frac{9}{4}\mathrm{Var}(\overline{X}) = \frac{9}{4}\frac{\mathrm{Var}(X)}{n}.$$

We can compute $E(X^2) = \int_0^\theta \frac{2x^3}{\theta^2}dx = \frac{1}{2}\theta^2$ such that $\mathrm{Var}(X) = \frac{1}{2}\theta^2 - \frac{4}{9}\theta^2 = \frac{1}{18}\theta^2$. This leads to $\mathrm{Var}(\tilde{\theta}) = \frac{\theta^2}{8n}$.

[2 - A]

iii) Since $E(\tilde{\theta}) = \theta$, the estimator is unbiased.

[4 - A]

iv) For large n, we can make use of the Central Limit Theorem and write

$$
\begin{aligned}
P\left(\tilde{\theta} \geq \theta\right) &= P(\frac{3}{2}\overline{X} \geq \theta) = P(\overline{X} \geq \frac{2}{3}\theta) \\
&= P(Z \geq \frac{\frac{2}{3}\theta - E(\overline{X})}{\sqrt{\mathrm{Var}(\overline{X})}}) = P(Z \geq 0) = 1/2
\end{aligned}
$$

where Z is a standard normal random variable.

[4 - A]

2. This question checks your understanding of basic concepts of probability. It is very similar to many problems we have seen in the lectures/classes. One difficulty is in part b) that requires to make a change of variable and use the Jacobian. Part g) is also a subquestion that was not correctly answered by most students. Always pay attention to the domain of the RVs.

We need to compute $E(X)$. We can first compute the marginal of X as

$$f_X(x) = \begin{cases} 2e^{-x}\int_x^{\infty} e^{-y}dy = 2e^{-2x}, & x > 0, \\ 0, & otherwise. \end{cases}$$

[2 - A]

This is an EXPO(2). Hence $E(X) = \frac{1}{2}$.

[2 - A]

b) We can first compute the Jacobian and write

$$\begin{vmatrix} \frac{\partial X}{\partial U} & \frac{\partial X}{\partial V} \\ \frac{\partial Y}{\partial U} & \frac{\partial Y}{\partial V} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = 1$$

[2 - A]

We then write

$$f_{U,V}(u,v) = \begin{cases} 2e^{-(u+2v)}, & u > 0, v > 0, \\ 0, & otherwise. \end{cases}$$

[2 - A]

c) The marginal are obtained by integration of the joint pdf as follows

$$f_U(u) = \begin{cases} e^{-u}, & u > 0, \\ 0, & otherwise. \end{cases}$$
$$f_V(v) = \begin{cases} 2e^{-2v}, & v > 0, \\ 0, & otherwise. \end{cases}$$

U is EXPO(1) and V is EXPO(2).

[2 - A]

d) Since $f_{U,V}(u,v) = f_U(u)f_V(v)$, U and V are two independent random variables.

[2 - A]

e) The conditional pdf $f_{U|V}(u|v)$ is given as

$$f_{U|V}(u|v) = f_U(u) = \begin{cases} e^{-u}, & u > 0, \\ 0, & otherwise. \end{cases}$$

[2 - A]

f) $E(U|V) = E(U) = 1$.

[2 - A]

g) $1 = E(U|V) = E(Y-X|X) = E(Y|X) - E(X|X) = E(Y|X) - X$. Hence $E(Y|X) = 1 + X$.

[2 - A]

h) $E(Y) = E_X E(Y|X) = E(1+X) = 1 + E(X) = 1 + \frac{1}{2} = \frac{3}{2}$.

[2 - A]