Final

Imperial College
London

[E 2.9 (Maths 4) 2010]

B.ENG. AND M.ENG. EXAMINATIONS 2010

PART II Paper 4 : MATHEMATICS (ELECTRICAL ENGINEERING)

Date Thursday 3rd June 2010 2.00 - 4.00 pm

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.

Answer FOUR questions.

Please answer questions from Section A and Section B in separate answer-books.

A mathematical formulae sheet is provided.

Statistical data sheets are provided.

[Before starting, please make sure that the paper is complete; there should be 6 pages, with a total of 6 questions. Ask the invigilator for a replacement if your copy is faulty.]

SECTION A

1. (i) You are given the result that the distinct eigenvalues of a real symmetric $n \times n$ matrix A are real. Show that the corresponding normalized eigenvectors are orthogonal; that is

$$\boldsymbol{a}_i^T \boldsymbol{a}_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Thus show that the matrix $P = \{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n\}$ is orthogonal; that is, P satisfies the relation $P^T P = I$.

(ii) Consider the 3×3 matrix

$$A = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 128 \end{pmatrix}.$$

Find the eigenvalues λ_i and their corresponding normalized eigenvectors $\boldsymbol{a}_i$ of A.

Using these, or otherwise, show that the matrix $P = \{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3\}$ can be written as

$$P = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \\ \sqrt{2} & 0 & 0 \end{pmatrix}.$$

If $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$, show that $P^{-1} A P = \Lambda$.

2. Find the eigenvalues and normalised eigenvectors of the matrix

$$A = \begin{pmatrix} 13 & \sqrt{13} & 0 \\ \sqrt{13} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

By writing the quadratic form

$$Q = 13x_1^2 + 2\sqrt{13}\,x_1x_2 + x_2^2 + x_3^2$$

as

$$Q = \boldsymbol{x}^T A\boldsymbol{x},$$

where $\boldsymbol{x} = (x_1, x_2, x_3)^T$, show that Q can be written in the diagonal form

$$Q = y_1^2 + 14y_3^2\,,$$

by finding a matrix P which satisfies $\mathbf{x} = P\boldsymbol{y}$ where $\boldsymbol{y} = (y_1, y_2, y_3)^T$.

Find y_1, y_2 and y_3 in terms of x_1, x_2 and x_3 from the matrix P.

SECTION B

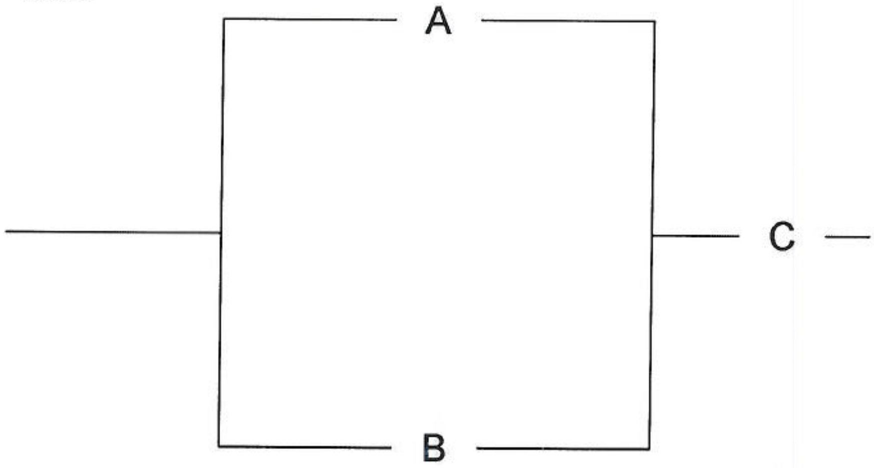
3. (i) For events A and B, where $P(A) > 0$ and $P(B) > 0$, show that

   (a) $P(A|B) = P(B|A)$ if $P(A) = P(B)$.

   (b) $P(A) = P(A \cap B) + P(A \cap B')$.

   (c) Using the result in (b), and assuming that A and B are independent, show that A and B' are also independent.

   (ii) Consider the following network, where components A, B and C each fail independently with probability 0.05. Derive the probability that the system functions.



   (iii) A component has a random failure time, T, with hazard function

$$z(t) = \begin{cases} \frac{t}{1+t} & t > 0\ , \\ 0 & \text{otherwise}\ . \end{cases}$$

   Compute the cumulative hazard function and the reliability function.

PLEASE TURN OVER

4. The daily electrical consumption in a town has a normal distribution with mean 500 megawatts and standard deviation 150 megawatts. The power station, which provides power for this town alone, has an output of $Y = 650$ megawatts with probability 0.7 and an output of $Y = 800$ megawatts with probability 0.3, on any particular day.

   (i) If the output is 650 megawatts on a particular day, calculate the probability that the demand cannot be met from the power station.

   (ii) Show that the probability that the demand cannot be met on a particular day is 0.118.

   (iii) If the demand cannot be met on a particular day, there is a power cut (but a maximum of only one power cut per day). Assuming that outputs and demands on different days are independent, what is the distribution of the number of power cuts in any seven day period?

   (iv) What is the probability that there are no more than 2 power cuts caused by unsatisfied demand in a seven day period?

5. The random variable X has density function

$$f(x) = \begin{cases} (x/\theta^2)\, e^{-(x/\theta)} & x > 0\ , \\ 0 & \text{otherwise}\ . \end{cases}$$

for some $\theta > 0$. For this random variable $E[X] = 2\theta$ and $E[X^2] = 6\theta^2$.

   (i) Compute the variance of X.

   (ii) Suppose a random sample of n observations is drawn from a population with density given by X. Obtain an expression for the maximum likelihood estimator, $\hat{\theta}$, of θ.

   (iii) Obtain an expression for $E[\hat{\theta}]$, and state whether this estimator is biased.

   (iv) Obtain the method of moments estimator for θ based on $E[X]$.

6. Suppose that we observe the following sample:

$$2.5, \quad 2.2, \quad 0.4, \quad 4.0, \quad 1.8, \quad -0.2, \quad -3.4.$$

Assume that these are independent observations from a $N(\mu, \sigma^2)$ distribution.

(i) Sketch the empirical cumulative distribution function.

(ii) Find the sample mean and sample median.

(iii) Find the sample variance and the sample standard deviation.

(iv) Suppose we believe that $\mu = \frac{1}{2}$, but a third party claims that $\mu < \frac{1}{2}$. Perform an appropriate test, at the 5% significance level, to test their claim.

END OF PAPER

M A T H E M A T I C S   D E P A R T M E N T

MATHEMATICAL FORMULAE

1. VECTOR ALGEBRA

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1, a_2, a_3)$$

Scalar (dot) product: $\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

Vector (cross) product:

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} .$$

Scalar triple product:

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a}.\mathbf{b}\times\mathbf{c} = \mathbf{b}.\mathbf{c}\times\mathbf{a} = \mathbf{c}.\mathbf{a}\times\mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product: $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{c}.\mathbf{a})\mathbf{b} - (\mathbf{b}.\mathbf{a})\mathbf{c}$

2. SERIES

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \dots \quad (\alpha \text{ arbitrary, } |x| < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots ,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^n\frac{x^{2n}}{(2n)!} + \dots ,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + \dots ,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^n\frac{x^{n+1}}{(n+1)} + \dots (-1 < x \le 1)$$

3. TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin(a+b) = \sin a\cos b + \cos a\sin b\ ;$$

$$\cos(a+b) = \cos a\cos b - \sin a\sin b\ .$$

$$\cos iz = \cosh z\ ; \quad \cosh iz = \cos z\ ; \quad \sin iz = i\sinh z\ ; \quad \sinh iz = i\sin z\ .$$

4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f\,D^n g + \binom{n}{1} Df\,D^{n-1}g + \dots + \binom{n}{r} D^r f\,D^{n-r}g + \dots + D^n f\,g\ .$$

(b) Taylor's expansion of $f(x)$ about $x = a$:

$$f(a+h) = f(a) + hf'(a) + h^2 f''(a)/2! + \dots + h^n f^{(n)}(a)/n! + \epsilon_n(h)\ ,$$

where $\epsilon_n(h) = h^{n+1}f^{(n+1)}(a+\theta h)/(n+1)!, \quad 0 < \theta < 1\ .$

(c) Taylor's expansion of $f(x, y)$ about (a, b):

$$f(a+h, b+k) = f(a, b) + [hf_x + kf_y]_{a,b} + 1/2!\left[h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}\right]_{a,b} + \dots$$

(d) Partial differentiation of $f(x, y)$:

i. If $y = y(x)$, then $f = F(x)$, and $\dfrac{dF}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dx}$.

ii. If $x = x(t)$, $y = y(t)$, then $f = F(t)$, and $\dfrac{dF}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$.

iii. If $x = x(u, v)$, $y = y(u, v)$, then $f = F(u, v)$, and

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}, \quad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$

(e) Stationary points of $f(x, y)$ occur where $f_x = 0$, $f_y = 0$ simultaneously.
Let (a, b) be a stationary point: examine $D = [f_{xx}f_{yy} - (f_{xy})^2]_{a,b}$.
If $D > 0$ and $f_{xx}(a, b) < 0$, then (a, b) is a maximum;
If $D > 0$ and $f_{xx}(a, b) > 0$, then (a, b) is a minimum;
If $D < 0$ then (a, b) is a saddle-point.

(f) Differential equations:

i. The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating factor $I(x) = \exp[\int P(x)(dx)]$, so that $\dfrac{d}{dx}(Iy) = IQ$.

ii. $P(x, y)dx + Q(x, y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y$.

5. INTEGRAL CALCULUS

(a) An important substitution: $\tan(\theta/2) = t$:
$\sin\theta = 2t/(1+t^2)$, $\cos\theta = (1-t^2)/(1+t^2)$, $d\theta = 2\,dt/(1+t^2)$.

(b) Some indefinite integrals:

$$\int (a^2 - x^2)^{-1/2} dx = \sin^{-1}\left(\frac{x}{a}\right), \quad |x| < a.$$

$$\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1}\left(\frac{x}{a}\right) = \ln\left\{\frac{x}{a} + \left(1 + \frac{x^2}{a^2}\right)^{1/2}\right\}.$$

$$\int (x^2 - a^2)^{-1/2} dx = \cosh^{-1}\left(\frac{x}{a}\right) = \ln\left|\frac{x}{a} + \left(\frac{x^2}{a^2} - 1\right)^{1/2}\right|.$$

$$\int (a^2 + x^2)^{-1} dx = \left(\frac{1}{a}\right)\tan^{-1}\left(\frac{x}{a}\right).$$

6. NUMERICAL METHODS

(a) Approximate solution of an algebraic equation:

If a root of $f(x) = 0$ occurs near $x = a$, take $x_0 = a$ and
$x_{n+1} = x_n - [f(x_n)/f'(x_n)]$, $n = 0, 1, 2 \ldots$

(Newton Raphson method).

(b) Formulae for numerical integration: Write $x_n = x_0 + nh$, $y_n = y(x_n)$.

i. Trapezium rule (1-strip): $\int_{x_0}^{x_1} y(x)dx \approx (h/2)[y_0 + y_1]$.

ii. Simpson's rule (2-strip): $\int_{x_0}^{x_2} y(x)dx \approx (h/3)[y_0 + 4y_1 + y_2]$.

(c) Richardson's extrapolation method: Let $I = \int_a^b f(x)dx$ and let I_1, I_2 be two estimates of I obtained by using Simpson's rule with intervals h and $h/2$. Then, provided h is small enough,

$$I_2 + (I_2 - I_1)/15,$$

is a better estimate of I.

7. LAPLACE TRANSFORMS

Function	Transform	Function	Transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)dt$	$af(t) + bg(t)$	$aF(s) + bG(s)$
df/dt	$sF(s) - f(0)$	d^2f/dt^2	$s^2F(s) - sf(0) - f'(0)$
$e^{at}f(t)$	$F(s-a)$	$tf(t)$	$-dF(s)/ds$
$(\partial/\partial\alpha)f(t,\alpha)$	$(\partial/\partial\alpha)F(s,\alpha)$	$\int_0^t f(t)dt$	$F(s)/s$
$\int_0^t f(u)g(t-u)du$	$F(s)G(s)$		
1	$1/s$	$t^n (n = 1, 2 \ldots)$	$n!/s^{n+1}$, $(s > 0)$
e^{at}	$1/(s-a)$, $(s > a)$	$\sin\omega t$	$\omega/(s^2+\omega^2)$, $(s > 0)$
$\cos\omega t$	$s/(s^2+\omega^2)$, $(s > 0)$	$H(t-T) = \begin{cases} 0, & t < T \\ 1, & t > T \end{cases}$	e^{-sT}/s, $(s, T > 0)$

8. FOURIER SERIES

If $f(x)$ is periodic of period $2L$, then $f(x + 2L) = f(x)$, and

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \text{ where}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad n = 0, 1, 2, \ldots, \text{ and}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1, 2, 3, \ldots$$

Parseval's theorem

$$\frac{1}{L}\int_{-L}^{L} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right).$$

1. Probabilities for events

For events A, B, and C $\quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$

More generally $\quad P(\bigcup A_i) = \sum P(A_i) - \sum P(A_i \cap A_j) + \sum P(A_i \cap A_j \cap A_k) - \cdots$

The odds in favour of A $\quad P(A) \,/\, P(\overline{A})$

Conditional probability $\quad P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$ provided that $P(B) > 0$

Chain rule $\quad P(A \cap B \cap C) = P(A)\,P(B \mid A)\,P(C \mid A \cap B)$

Bayes' rule $\quad P(A \mid B) = \dfrac{P(A)\,P(B \mid A)}{P(A)\,P(B \mid A) + P(\overline{A})\,P(B \mid \overline{A})}$

A and B are independent if $\quad P(B \mid A) = P(B)$

A, B, and C are independent if $\quad P(A \cap B \cap C) = P(A)P(B)P(C)$, and

$$P(A \cap B) = P(A)P(B), \quad P(B \cap C) = P(B)P(C), \quad P(C \cap A) = P(C)P(A)$$

2. Probability distribution, expectation and variance

The probability distribution for a discrete random variable X is called the probability mass function (pmf) and is the complete set of probabilities $\{p_x\} = \{P(X = x)\}$

Expectation $\quad E(X) = \mu = \sum_x x p_x$

For function $g(x)$ of x, $\quad E\{g(X)\} = \sum_x g(x) p_x$, so $E(X^2) = \sum_x x^2 p_x$

Sample mean $\quad \overline{x} = \dfrac{1}{n} \sum_k x_k$ estimates μ from random sample $x_1, x_2, \ldots, x_n$

Variance $\quad \mathrm{var}(X) = \sigma^2 = E\{(X-\mu)^2\} = E(X^2) - \mu^2$

Sample variance $\quad s^2 = \dfrac{1}{n-1} \left\{ \sum_k x_k^2 - \dfrac{1}{n} \left(\sum_j x_j \right)^2 \right\}$ estimates σ^2

Standard deviation $\quad \mathrm{sd}(X) = \sigma$

If value y is observed with frequency n_y

$$n = \sum_y n_y, \quad \sum_k x_k = \sum_y y n_y, \quad \sum_k x_k^2 = \sum_y y^2 n_y$$

Skewness $\quad \beta_1 = E\left(\dfrac{X-\mu}{\sigma}\right)^3$ is estimated by $\dfrac{1}{n-1} \sum \left(\dfrac{x_i - \overline{x}}{s}\right)^3$

Kurtosis $\quad \beta_2 = E\left(\dfrac{X-\mu}{\sigma}\right)^4 - 3$ is estimated by $\dfrac{1}{n-1} \sum \left(\dfrac{x_i - \overline{x}}{s}\right)^4 - 3$

Sample median $\quad \tilde{x}$ or x_{med}. Half the sample values are smaller and half larger

If the sample values $x_1, \ldots, x_n$ are ordered as $x_{(1)} \le x_{(2)} \le \cdots \le x_{(n)}$,
then $\tilde{x} = x_{(\frac{n+1}{2})}$ if n is odd, and $\tilde{x} = \frac{1}{2}\left(x_{(\frac{n}{2})} + x_{(\frac{n+2}{2})}\right)$ if n is even

α-quantile $Q(\alpha)$ is such that $P(X \leq Q(\alpha)) = \alpha$

Sample α-quantile $\widehat{Q}(\alpha)$ Proportion α of the data values are smaller

Lower quartile Q1 $= \widehat{Q}(0.25)$ one quarter are smaller

Upper quartile Q3 $= \widehat{Q}(0.75)$ three quarters are smaller

Sample median $\tilde{x} = \widehat{Q}(0.5)$ estimates the population median $Q(0.5)$

3. Probability distribution for a continuous random variable

The cumulative distribution function (cdf) $F(x) = P(X \leq x) = \int_{x_0=-\infty}^{x} f(x_0)\mathrm{d}x_0$

The probability density function (pdf) $f(x) = \dfrac{\mathrm{d}F(x)}{\mathrm{d}x}$

$E(X) = \mu = \int_{-\infty}^{\infty} x\, f(x)\mathrm{d}x$, $\operatorname{var}(X) = \sigma^2 = E(X^2) - \mu^2$, where $E(X^2) = \int_{-\infty}^{\infty} x^2\, f(x)\mathrm{d}x$

4. Discrete probability distributions

Discrete Uniform *Uniform* (n)

$p_x = \dfrac{1}{n} \quad (x = 1, 2, \ldots, n)$ $\qquad \mu = (n+1)/2,\ \sigma^2 = (n^2-1)/12$

Binomial distribution *Binomial* (n, θ)

$p_x = \binom{n}{x} \theta^x (1-\theta)^{n-x} \quad (x = 0, 1, 2, \ldots, n)$ $\qquad \mu = n\theta,\ \sigma^2 = n\theta(1-\theta)$

Poisson distribution *Poisson* (λ)

$p_x = \dfrac{\lambda^x e^{-\lambda}}{x!} \quad (x = 0, 1, 2, \ldots)$ (with $\lambda > 0$) $\qquad \mu = \lambda,\ \sigma^2 = \lambda$

Geometric distribution *Geometric* (θ)

$p_x = (1-\theta)^{x-1}\theta \quad (x = 1, 2, 3, \ldots)$ $\qquad \mu = \dfrac{1}{\theta},\ \sigma^2 = \dfrac{1-\theta}{\theta^2}$

5. Continuous probability distributions

Uniform distribution *Uniform* (α, β)

$$f(x) = \begin{cases} \dfrac{1}{\beta-\alpha} & (\alpha < x < \beta), \\ 0 & (\text{otherwise}). \end{cases}$$

$\mu = (\alpha+\beta)/2,\ \sigma^2 = (\beta-\alpha)^2/12$

Exponential distribution *Exponential* (λ)

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & (0 < x < \infty), \\ 0 & (-\infty < x \leq 0). \end{cases}$$

$\mu = 1/\lambda,\ \sigma^2 = 1/\lambda^2$

Normal distribution $N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2 \right\} \quad (-\infty < x < \infty), \quad E(X) = \mu, \quad \mathrm{var}(X) = \sigma^2$$

Standard normal distribution $N(0,1)$

If X is $N(\mu, \sigma^2)$, then $Y = \frac{X-\mu}{\sigma}$ is $N(0,1)$

6. Reliability

For a device in continuous operation with failure time random variable T having pdf $f(t)$ $(t > 0)$

The reliability function at time t $\quad R(t) = P(T > t)$

The failure rate or hazard function $\quad h(t) = f(t)/R(t)$

The cumulative hazard function $\quad H(t) = \int_0^t h(t_0)\, dt_0 = -\ln\{R(t)\}$

The Weibull(α, β) distribution has $\quad H(t) = \beta t^{\alpha}$

7. System reliability

For a system of k devices, which operate independently, let

$R_i = P(D_i) = P($"device i operates"$)$

The system reliability, R, is the probability of a path of operating devices

A system of devices in series operates only if every device operates

$R = P(D_1 \cap D_2 \cap \cdots \cap D_k) = R_1 R_2 \cdots R_k$

A system of devices in parallel operates if any device operates

$R = P(D_1 \cup D_2 \cup \cdots \cup D_k) = 1 - (1 - R_1)(1 - R_2) \cdots (1 - R_k)$

8. Covariance and correlation

The covariance of X and Y $\quad \mathrm{cov}(X, Y) = E(XY) - \{E(X)\}\{E(Y)\}$

From pairs of observations $(x_1, y_1), \ldots, (x_n, y_n)$ $\quad S_{xy} = \sum_k x_k y_k - \frac{1}{n}(\sum_i x_i)(\sum_j y_j)$

$$S_{xx} = \sum_k x_k^2 - \frac{1}{n}(\sum_i x_i)^2, \qquad S_{yy} = \sum_k y_k^2 - \frac{1}{n}(\sum_j y_j)^2$$

Sample covariance $\quad s_{xy} = \frac{1}{n-1} S_{xy}$ estimates $\mathrm{cov}(X, Y)$

Correlation coefficient $\quad \rho = \mathrm{corr}(X, Y) = \frac{\mathrm{cov}(X, Y)}{\mathrm{sd}(X) \cdot \mathrm{sd}(Y)}$

Sample correlation coefficient $\quad r = \frac{S_{xy}}{\sqrt{S_{xx} S_{yy}}}$ estimates ρ

9. Sums of random variables

$$E(X+Y) = E(X)+E(Y)$$
$$\text{var}(X+Y) = \text{var}(X)+\text{var}(Y)+2\,\text{cov}(X,Y)$$
$$\text{cov}(aX+bY,\ cX+dY) = (ac)\,\text{var}(X)+(bd)\,\text{var}(Y)+(ad+bc)\,\text{cov}(X,Y)$$

If X is $N(\mu_1,\sigma_1^2)$, Y is $N(\mu_2,\sigma_2^2)$, and $\text{cov}(X,Y)=c$, then $X+Y$ is $N(\mu_1+\mu_2,\ \sigma_1^2+\sigma_2^2+2c)$

10. Bias, standard error, mean square error

If t estimates θ (with random variable T giving t)

Bias of t	$\text{bias}(t)$	$= E(T)-\theta$
Standard error of t	$\text{se}(t)$	$= \text{sd}(T)$
Mean square error of t	MSE(t)	$= E\{(T-\theta)^2\} = \{\text{se}(t)\}^2+\{\text{bias}(t)\}^2$

If $\overline{x}$ estimates μ, then $\text{bias}(\overline{x})=0$, $\text{se}(\overline{x})=\sigma/\sqrt{n}$, $\text{MSE}(\overline{x})=\sigma^2/n$, $\widehat{\text{se}}(\overline{x})=s/\sqrt{n}$

Central limit property If n is fairly large, $\overline{x}$ is from $N(\mu,\ \sigma^2/n)$ approximately

11. Likelihood

The likelihood is the joint probability as a function of the unknown parameter θ.

For a random sample $x_1,x_2,\ldots,x_n$

$$\ell(\theta;\ x_1,x_2,\ldots,x_n) = P(X_1=x_1\,|\,\theta)\ \cdots\ P(X_n=x_n\,|\,\theta) \qquad \text{(discrete distribution)}$$
$$\ell(\theta;\ x_1,x_2,\ldots,x_n) = f(x_1\,|\,\theta)\,f(x_2\,|\,\theta)\ \cdots\ f(x_n\,|\,\theta) \qquad \text{(continuous distribution)}$$

The maximum likelihood estimator (MLE) is $\widehat{\theta}$ for which the likelihood is a maximum

12. Confidence intervals

If $x_1,x_2,\ldots,x_n$ are a random sample from $N(\mu,\sigma^2)$ and σ^2 is known, then the 95% confidence interval for μ is $(\overline{x}-1.96\frac{\sigma}{\sqrt{n}},\ \overline{x}+1.96\frac{\sigma}{\sqrt{n}})$

If σ^2 is estimated, then from the Student t table for t_{n-1} we find $t_0=t_{n-1,0.05}$

The 95% confidence interval for μ is $(\overline{x}-t_0\frac{s}{\sqrt{n}},\ \overline{x}+t_0\frac{s}{\sqrt{n}})$

13. Standard normal table   Values of pdf $\phi(y) = f(y)$ and cdf $\Phi(y) = F(y)$

y	$\phi(y)$	$\Phi(y)$	y	$\phi(y)$	$\Phi(y)$	y	$\phi(y)$	$\Phi(y)$	y	$\Phi(y)$
0	.399	.5	.9	.266	.816	1.8	.079	.964	2.8	.997
.1	.397	.540	1.0	.242	.841	1.9	.066	.971	3.0	.999
.2	.391	.579	1.1	.218	.864	2.0	.054	.977	0.841	.8
.3	.381	.618	1.2	.194	.885	2.1	.044	.982	1.282	.9
.4	.368	.655	1.3	.171	.903	2.2	.035	.986	1.645	.95
.5	.352	.691	1.4	.150	.919	2.3	.028	.989	1.96	.975
.6	.333	.726	1.5	.130	.933	2.4	.022	.992	2.326	.99
.7	.312	.758	1.6	.111	.945	2.5	.018	.994	2.576	.995
.8	.290	.788	1.7	.094	.955	2.6	.014	.995	3.09	.999

14. Student t table   Values $t_{m,p}$ of x for which $P(|X| > x) = p$, when X is t_m

m	p= 0.10	0.05	0.02	0.01	m	p= 0.10	0.05	0.02	0.01
1	6.31	12.71	31.82	63.66	9	1.83	2.26	2.82	3.25
2	2.92	4.30	6.96	9.92	10	1.81	2.23	2.76	3.17
3	2.35	3.18	4.54	5.84	12	1.78	2.18	2.68	3.05
4	2.13	2.78	3.75	4.60	15	1.75	2.13	2.60	2.95
5	2.02	2.57	3.36	4.03	20	1.72	2.09	2.53	2.85
6	1.94	2.45	3.14	3.71	25	1.71	2.06	2.48	2.78
7	1.89	2.36	3.00	3.50	40	1.68	2.02	2.42	2.70
8	1.86	2.31	2.90	3.36	∞	1.645	1.96	2.326	2.576

15. Chi-squared table   Values $\chi^2_{k,p}$ of x for which $P(X > x) = p$, when X is χ^2_k and p = .995, .975, *etc*

k	.995	.975	.05	.025	.01	.005	k	.995	.975	.05	.025	.01	.005
1	.000	.001	3.84	5.02	6.63	7.88	18	6.26	8.23	28.87	31.53	34.81	37.16
2	.010	.051	5.99	7.38	9.21	10.60	20	7.43	9.59	31.42	34.17	37.57	40.00
3	.072	.216	7.81	9.35	11.34	12.84	22	8.64	10.98	33.92	36.78	40.29	42.80
4	.207	.484	9.49	11.14	13.28	14.86	24	9.89	12.40	36.42	39.36	42.98	45.56
5	.412	.831	11.07	12.83	15.09	16.75	26	11.16	13.84	38.89	41.92	45.64	48.29
6	.676	1.24	12.59	14.45	16.81	18.55	28	12.46	15.31	41.34	44.46	48.28	50.99
7	.990	1.69	14.07	16.01	18.48	20.28	30	13.79	16.79	43.77	46.98	50.89	53.67
8	1.34	2.18	15.51	17.53	20.09	21.95	40	20.71	24.43	55.76	59.34	63.69	66.77
9	1.73	2.70	16.92	19.02	21.67	23.59	50	27.99	32.36	67.50	71.41	76.15	79.49
10	2.16	3.25	13.31	20.48	23.21	25.19	60	35.53	40.48	79.08	83.30	88.38	91.95
12	3.07	4.40	21.03	23.34	26.22	28.30	70	43.28	48.76	90.53	95.02	100.4	104.2
14	4.07	5.63	23.68	26.12	29.14	31.32	80	51.17	57.15	101.9	106.6	112.3	116.3
16	5.14	6.91	26.30	28.85	32.00	34.27	100	67.33	74.22	124.3	129.6	135.8	140.2

16. The chi-squared goodness-of-fit test

The frequencies n_y are grouped so that the fitted frequency $\widehat{n}_y$ for every group exceeds about 5.

$X^2 = \sum_y \frac{(n_y - \widehat{n}_y)^2}{\widehat{n}_y}$ is referred to the table of χ^2_k with significance point p,

where k is the number of terms summed, less one for each constraint, *eg* matching total frequency, and matching $\overline{x}$ with μ

17. Joint probability distributions

Discrete distribution $\{p_{xy}\}$, where $p_{xy} = P(\{X = x\} \cap \{Y = y\})$.

Let $p_{x\bullet} = P(X = x)$, and $p_{\bullet y} = P(Y = y)$, then

$p_{x\bullet} = \sum_y p_{xy}$ and $P(X = x \,|\, Y = y) = \frac{p_{xy}}{p_{\bullet y}}$

Continuous distribution

Joint cdf $F(x,y) = P(\{X \le x\} \cap \{Y \le y\}) = \int_{x_0=-\infty}^{x} \int_{y_0=-\infty}^{y} f(x_0\,, y_0)\, \mathrm{d}x_0\, \mathrm{d}y_0$

Joint pdf $f(x,y) = \frac{\mathrm{d}^2 F(x,y)}{\mathrm{d}x\, \mathrm{d}y}$

Marginal pdf of X $f_X(x) = \int_{-\infty}^{\infty} f(x, y_0)\, \mathrm{d}y_0$

Conditional pdf of X given $Y = y$ $f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$ (provided $f_Y(y) > 0$)

18. Linear regression

To fit the linear regression model $y = \alpha + \beta x$ by $\widehat{y}_x = \widehat{\alpha} + \widehat{\beta} x$ from observations $(x_1, y_1), \ldots, (x_n, y_n)$, the least squares fit is $\widehat{\alpha} = \overline{y} - \overline{x}\widehat{\beta}$, $\widehat{\beta} = \frac{S_{xy}}{S_{xx}}$

The residual sum of squares $\mathrm{RSS} = S_{yy} - \frac{S_{xy}^2}{S_{xx}}$

$\widehat{\sigma^2} = \frac{\mathrm{RSS}}{n-2}$ $\frac{n-2}{\sigma^2}\widehat{\sigma^2}$ is from χ^2_{n-2}

$E(\widehat{\alpha}) = \alpha$, $E(\widehat{\beta}) = \beta$,

$\mathrm{var}\,(\widehat{\alpha}) = \frac{\sum x_i^2}{n\, S_{xx}}\sigma^2$, $\mathrm{var}\,(\widehat{\beta}) = \frac{\sigma^2}{S_{xx}}$, $\mathrm{cov}\,(\widehat{\alpha}, \widehat{\beta}) = -\frac{\overline{x}}{S_{xx}}\sigma^2$

$\widehat{y}_x = \widehat{\alpha} + \widehat{\beta} x$, $E(\widehat{y}_x) = \alpha + \beta x$, $\mathrm{var}\,(\widehat{y}_x) = \left\{ \frac{1}{n} + \frac{(x - \overline{x})^2}{S_{xx}} \right\} \sigma^2$

$\frac{\widehat{\alpha} - \alpha}{\widehat{\mathrm{se}}\,(\widehat{\alpha})}$, $\frac{\widehat{\beta} - \beta}{\widehat{\mathrm{se}}\,(\widehat{\beta})}$, $\frac{\widehat{y}_x - \alpha - \beta x}{\widehat{\mathrm{se}}\,(\widehat{y}_x)}$ are each from t_{n-2}

EXAMINATION QUESTIONS/SOLUTIONS 2009-2010

EE2 - Maths Paper 4 (E2.9)

Course: EE2 paper 4

Question: EE4 1

Marks & seen/unseen

Parts

i) Write $A\underline{a}_i = \lambda_i \underline{a}_i$; $A\underline{a}_j = \lambda_j \underline{a}_j$ $\lambda_i \neq \lambda_j$

$\underline{a}_i^T A^T \underline{a}_j = \lambda_i \underline{a}_i^T \underline{a}_j$ $\qquad$ $\underline{a}_i^T A \underline{a}_j = \lambda_j \underline{a}_i^T \underline{a}_j$ $\qquad$ 2

When $A^T = A \Rightarrow (\lambda_i - \lambda_j)\underline{a}_i^T \underline{a}_j = 0$

Because $\lambda_i \neq \lambda_j$, we have $\underline{a}_i^T \underline{a}_j = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$ (*) $\qquad$ 2

Form $P^T P$ from $P = (\underline{a}_1 \underline{a}_2 \cdots \underline{a}_n)$

$$P^T P = \begin{pmatrix} \underline{a}_1^T \\ \underline{a}_2^T \\ \vdots \\ \underline{a}_n^T \end{pmatrix} (\underline{a}_1 \underline{a}_2 \cdots \underline{a}_n) = \begin{pmatrix} \underline{a}_1^T\underline{a}_1 & \underline{a}_1^T\underline{a}_2 & \cdots & \underline{a}_1^T\underline{a}_n \\ \underline{a}_2^T\underline{a}_1 & \underline{a}_2^T\underline{a}_2 & \cdots & \underline{a}_2^T\underline{a}_n \\ \underline{a}_n^T\underline{a}_1 & \cdots & & \underline{a}_n^T\underline{a}_n \end{pmatrix} \xleftarrow{\text{Using (*)}} = I$$ 4

ii) $A = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 128 \end{pmatrix}$ $\quad \lambda_1 = 128$ & $(\lambda-3)^2 = 1$

$\lambda_2 = 4 \quad \lambda_3 = 2$ $\qquad$ 3

$\lambda_1 = 128$; $\underline{a}_1 = (0,0,1)^T$ $\qquad$ 1

$\lambda_2 = 4$ $\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 124 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \Rightarrow \begin{matrix} b = a \\ c = 0 \end{matrix}$ $\quad \underline{a}_2 = \frac{1}{\sqrt{2}}(1,1,0)^T$ $\qquad$ 2

$\lambda_3 = 2$ $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 126 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \Rightarrow \begin{matrix} b = -a \\ c = 0 \end{matrix}$ $\quad \underline{a}_3 = \frac{1}{\sqrt{2}}(1,-1,0)^T$ $\qquad$ 2

$\therefore P = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \\ \sqrt{2} & 0 & 0 \end{pmatrix}$

$$AP = \frac{1}{\sqrt{2}}\begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 128 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \\ \sqrt{2} & 0 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 4 & 2 \\ 0 & 4 & -2 \\ 128\sqrt{2} & 0 & 0 \end{pmatrix}$$ 4

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \\ \sqrt{2} & 0 & 0 \end{pmatrix}\begin{pmatrix} 128 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix} = P\Lambda,$$

with $\Lambda = \text{diag}(128, 4, 2)$.

Seen similar

Setter's initials: JDG $\qquad$ Checker's initials $\qquad$ Page number

EXAMINATION QUESTIONS/SOLUTIONS 2009-2010

Course: EE2 paper 4

Question: EE4 Q2

Parts | Marks & seen/unseen

$$A = \begin{pmatrix} 13 & \sqrt{13} & 0 \\ \sqrt{13} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \lambda_1 = 1 \text{ with } \lambda(\lambda-14)=0$$
$$\lambda_2 = 0 \quad \lambda_3 = 14.$$ 3

$\lambda_1 = 1$, $\underline{a}_1 = (0,0,1)^T$ 1

$$\lambda_2 = 0 \quad \begin{pmatrix} 13 & \sqrt{13} & 0 \\ \sqrt{13} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \quad \begin{matrix} c=0 \\ b=-a\sqrt{13} \end{matrix} \quad \underline{a}_2 = \frac{1}{\sqrt{14}}(1,-\sqrt{13},0)^T$$ 2

$$\lambda_3 = 14 \quad \begin{pmatrix} -1 & \sqrt{13} & 0 \\ \sqrt{13} & -13 & 0 \\ 0 & 0 & -13 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \quad \begin{matrix} c=0 \\ a=b\sqrt{13} \end{matrix} \quad \underline{a}_3 = \frac{1}{\sqrt{14}}(\sqrt{13},1,0)^T$$ 2

$$P = (\underline{a}_1\ \underline{a}_2\ \underline{a}_3) = \frac{1}{\sqrt{14}}\begin{pmatrix} 0 & 1 & \sqrt{13} \\ 0 & -\sqrt{13} & 1 \\ \sqrt{14} & 0 & 0 \end{pmatrix}; \Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 14 \end{pmatrix}$$ 2

$$Q = 13x_1^2 + 2\sqrt{13}\,x_1x_2 + x_2^2 + x_3^2 = \underline{x}^T A \underline{x}$$

With $\underline{x} = P\underline{y}$ or $\underline{y} = P^T\underline{x}$ where P is given above as an orthogonal mx ($P^T = P^{-1}$), then

$$Q = (P\underline{y})^T A (P\underline{y}) = \underline{y}^T (P^T A P)\underline{y}$$ 3

But $AP = P\Lambda \Rightarrow P^T A P = \Lambda$

$$\therefore \quad Q = \underline{y}^T \Lambda \underline{y} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2$$ 3

$$\therefore \quad Q = y_1^2 + 14 y_3^2$$ 1

$$\underline{y} = P^T\underline{x} \Rightarrow \underline{y} = \frac{1}{\sqrt{14}}\begin{pmatrix} 0 & 0 & \sqrt{14} \\ 1 & -\sqrt{13} & 0 \\ \sqrt{13} & 1 & 0 \end{pmatrix}\underline{x}$$

so
$$y_1 = x_3,$$
$$\sqrt{14}\,y_2 = x_1 - \sqrt{13}\,x_2,$$
$$\sqrt{14}\,y_3 = \sqrt{13}\,x_1 + x_2.$$ 3

seen similar

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EXAMINATION QUESTIONS/SOLUTIONS 2009-2010

Course: EE2 paper 4

Question: EE4 Q2

Marks & seen/unseen

Parts

$$A = \begin{pmatrix} 13 & \sqrt{13} & 0 \\ \sqrt{13} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \lambda_1 = 1 \text{ with } \lambda(\lambda - 14) = 0$$
$$\lambda_2 = 0 \quad \lambda_3 = 14.$$ 3

$\lambda_1 = 1, \quad \underline{a}_1 = (0,0,1)^T$ 1

$$\lambda_2 = 0 \quad \begin{pmatrix} 13 & \sqrt{13} & 0 \\ \sqrt{13} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \quad \begin{matrix} c = 0 \\ b = -a\sqrt{13} \end{matrix} \quad \underline{a}_2 = \frac{1}{\sqrt{14}}(1, -\sqrt{13}, 0)^T$$ 2

$$\lambda_3 = 14 \quad \begin{pmatrix} -1 & \sqrt{13} & 0 \\ \sqrt{13} & -13 & 0 \\ 0 & 0 & -13 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \quad \begin{matrix} c = 0 \\ a = b\sqrt{13} \end{matrix} \quad \underline{a}_3 = \frac{1}{\sqrt{14}}(\sqrt{13}, 1, 0)^T$$ 2

$$P = (\underline{a}_1\ \underline{a}_2\ \underline{a}_3) = \frac{1}{\sqrt{14}}\begin{pmatrix} 0 & 1 & \sqrt{13} \\ 0 & -\sqrt{13} & 1 \\ \sqrt{14} & 0 & 0 \end{pmatrix}; \Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 14 \end{pmatrix}$$ 2

$$Q = 13x_1^2 + 2\sqrt{13}\,x_1x_2 + x_2^2 + x_3^2 = \underline{x}^T A \underline{x}$$

With $\underline{x} = P\underline{y}$ or $\underline{y} = P^T\underline{x}$ where P is given above as an orthogonal mx $(P^T = P^{-1})$, then 3

$$Q = (P\underline{y})^T A (P\underline{y}) = \underline{y}^T(P^TAP)\underline{y}$$

But $\quad AP = P\Lambda \Rightarrow P^TAP = \Lambda$

$$\therefore \quad Q = \underline{y}^T \Lambda \underline{y} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2$$ 3

$$\therefore \quad Q = y_1^2 + 14y_3^2$$ 1

$$\underline{y} = P^T\underline{x} \Rightarrow \underline{y} = \frac{1}{\sqrt{14}}\begin{pmatrix} 0 & 0 & \sqrt{14} \\ 1 & -\sqrt{13} & 0 \\ \sqrt{13} & 1 & 0 \end{pmatrix}\underline{x}$$

so $\quad y_1 = x_3,$

$\sqrt{14}\,y_2 = x_1 - \sqrt{13}\,x_2,$ 3

$\sqrt{14}\,y_3 = \sqrt{13}\,x_1 + x_2.$

seen similar

Setter's initials: JDG.

Checker's initials

Page number

NA EE II (4) (3)

3
Solution 1a

i. **a.** From the definition of conditional probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A \cap B)}{P(B)}\frac{P(A)}{P(A)} = P(B|A)\frac{P(A)}{P(B)}.$$

so $P(A|B) = P(B|A)$ if and only if $P(A) = P(B)$.

[2]

b.

$$\begin{aligned} A &= A \cap \Omega \\ &= A \cap (B \cup B') \\ &= (A \cap B) \cup (A \cap B') \quad \text{by the distributive rule} \end{aligned}$$

This is a disjoint union, so Axiom III says

$$P(A) = P(A \cap B) + P(A \cap B')$$

[2]

c. From the previous result

$$\begin{aligned} P(A \cap B') &= P(A) - P(A \cap B) \\ &= P(A) - P(A)P(B) \\ &= P(A)(1 - P(B)) \\ &= P(A)P(B') \end{aligned}$$

[4]

ii. Let A denote that component A functions (and similarly for B and C) Then

$$P(A) = P(B) = P(C) = 1 - 0.05 = 0.95$$

For the system to function, S say,

$$\begin{aligned} P(S) &= P((A \cup B) \cap C) \\ &= P(A \cup B)P(C) \\ &= [P(A) + P(B) - P(A \cap B)]\, P(C) \\ &= [P(A) + P(B) - P(A)P(B)]\, P(C) \\ &= \left[0.95 + 0.95 - 0.95^2\right] 0.95 \approx \underline{0.948} \end{aligned}$$

[5]

NA

EE II (3)

3

Solution 3b

iii. Obtain the cumulative hazard for $u > 0$

$$\begin{aligned} H(u) &= \int_0^u \frac{t}{1+t}\, dt = \int_0^u 1 - \frac{1}{1+t}\, dt \\ &= [t - \log(1+t)]_0^u \\ &= u - \log(1+u) \end{aligned}$$

[5]

For the reliability function

$$H(t) = -\log(R(t)) \implies R(t) = \exp(-H(t))$$

So

$$\begin{aligned} R(t) &= \exp(-H(t)) = \exp(\log(1+t) - t) \\ &= (1+t)\exp(-t) \end{aligned}$$

[2]

TOTAL [20]

NA EEII (4) 4

Solution 4a

i. Let D = {demand cannot be met}, and let X = daily consumption, and Y=output.

$$P(D|Y = 650) = P(X > 650)$$

where $X \sim N(500, (150)^2)$. So,

$$\begin{aligned}
P(D|Y = 650) &= P\left(\frac{X - 500}{150} > \frac{650 - 500}{150}\right) \\
&= P\left(Z > \frac{150}{150}\right) \qquad \text{where } Z \sim N(0, 1) \\
&= 1 - P\left(Z \leq \frac{150}{150}\right) \\
&= 1 - \Phi(1.0) \approx 1 - 0.841 = \underline{0.159}
\end{aligned}$$

[5]

ii. Using total probability

$$P(D) = P(D|Y = 650)P(Y = 650) + P(D|Y = 800)P(Y = 800)$$

So we also require $P(D|Y = 800) = P(X > 800)$

$$\begin{aligned}
P(D|Y = 800) &= P\left(\frac{X - 500}{150} > \frac{800 - 500}{150}\right) \\
&= P\left(Z > \frac{300}{150}\right) \\
&= 1 - P(Z \leq 2) \\
&= 1 - \Phi(2.0) \approx 1 - 0.977 = \underline{0.023}
\end{aligned}$$

[5]

Hence the probability that the demand cannot be met on a particular day is

$$P(D) = (0.159 \times 0.7) + (0.023 \times 0.3) = \underline{0.1182}.$$

[2]

NA EE II (4) 4

Solution 4 2b

iii. In any week, we have a sequence of Bernoulli trials that are independent and have a common probability of success. Since we are interested in the number of successes in a week, a Binomial distribution is the appropriate model.

[2]

Thus, let C =number of power cuts in a week. Then $C \sim \text{Bin}(n, \theta)$, where $n = 7$ and $\theta = P(D) = 0.1182$.

[2]

iv. Require $P(C \le 2)$:

$$
\begin{aligned}
P(C \le 2) &= P(C=0) + P(C=1) + P(C=2) \\
&= \sum_{j=0}^{2} \binom{7}{j} \theta^j (1-\theta)^{7-j} \\
&= \binom{7}{0}(0.1182)^0(0.8818)^7 + \binom{7}{1}(0.1182)^1(0.8818)^6 + \binom{7}{2}(0.1182)^2(0.8818)^5 \\
&= (0.8818)^7 + 7(0.1182)(0.8818)^6 + 21(0.1182)^2(0.8818)^5 \approx \underline{0.960}.
\end{aligned}
$$

[4]

TOTAL [20]

NA

EEII(4) 5

Solution ~~3~~ 5a

i.

$$var[X] = E[X^2] - E[X]^2 = 6\theta^2 - (2\theta)^2 = 2\theta^2$$

[2]

ii. To obtain the maximum likelihood estimator

$$L(\theta) = \prod_{i=1}^{n} f(x_i|\theta) = \prod_{i=1}^{n} \frac{x_i}{\theta^2} e^{-\frac{x_i}{\theta}}$$
$$= \frac{1}{\theta^{2n}} e^{-\frac{1}{\theta}\sum_{i=1}^{n} x_i} \prod_{i=1}^{n} x_i$$

Taking logs

$$\log L = -2n \log\theta - \frac{1}{\theta}\sum_{i=1}^{n} x_i + \log\prod_{i=1}^{n} x_i$$

Differentiate with respect to θ

$$\frac{d\log L}{d\theta} = -\frac{2n}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^{n} x_i$$

Equate with zero and solve

$$-\frac{2n}{\hat\theta} + \frac{1}{\hat\theta^2}\sum_{i=1}^{n} x_i = 0$$
$$-2n\hat\theta + \sum_{i=1}^{n} x_i = 0$$
$$\hat\theta = \frac{1}{2n}\sum_{i=1}^{n} x_i$$

[7]

Verify this is a maximum

$$\frac{d^2\log L}{d\theta^2} = \frac{2n}{\theta^2} - \frac{2}{\theta^3}\sum_{i=1}^{n} x_i$$

NA EE II (4) 5

Solution ~~3~~ 5b

evaluate at the estimator

$$\left.\frac{2n}{\theta^2} - \frac{2}{\theta^3}\sum_{i=1}^{n} x_i\right|_{\theta=\frac{1}{2n}\sum_{i=1}^{n} x_i} = \frac{8n^3}{\left(\sum_{i=1}^{n} x_i\right)^2} - \frac{16n^3}{\left(\sum_{i=1}^{n} x_i\right)^3}\sum_{i=1}^{n} x_i$$
$$= \frac{8n^3}{\left(\sum_{i=1}^{n} x_i\right)^2} - \frac{16n^3}{\left(\sum_{i=1}^{n} x_i\right)^2} = -\frac{8n^3}{\left(\sum_{i=1}^{n} x_i\right)^2}$$

9

This is always negative, and hence the likelihood estimator is a maximum.

[3]

iii. Require

$$E[\hat{\theta}] = E[\frac{1}{2n}\sum_{i=1}^{n} X_i] = \frac{1}{2n}E[\sum_{i=1}^{n} X_i]$$
$$= \frac{1}{2n}\sum_{i=1}^{n} E[X_i]$$
$$= \frac{n}{2n}E[X_i]$$

Given that $E[X_i] = 2\theta\,\forall i$, $E[\hat{\theta}] = \theta$.

[3]

The bias is $E[\hat{\theta}] - \theta = \theta - \theta = 0$. The estimator is unbiased for θ.

[2]

iv. Now

$$E[X] = 2\theta \quad \text{and} \quad \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

The method of moments estimator of θ is then

$$2\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} X_i$$
$$\hat{\theta} = \frac{1}{2n}\sum_{i=1}^{n} X_i$$

[3]

TOTAL [20]

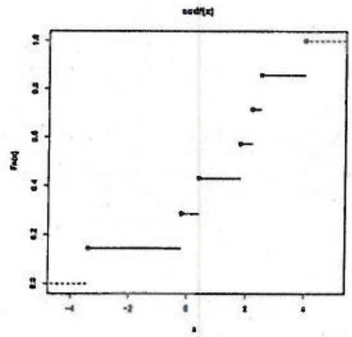
NA EEII(4) 6

Solution 6a

i. Require

$x_{(i)}$	-3.4	-0.2	0.4	1.8	2.2	2.5	4.0
$\hat{F}(x_{(i)})$	0.14	0.29	0.43	0.57	0.71	0.86	1.00

yielding



[4]

ii.

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{1}{7} \times 7.3 \approx \underline{1.04}$$

[2]

Median is $\underline{1.8}$.

[1]

iii.

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2 = \frac{1}{6} \times 34.477 \approx \underline{5.75}$$

[2]

the sample standard deviation is then $\sqrt{5.75} \approx \underline{2.40}$.

[1]

NA ~~LE~~ II (4) 6

Solution ~~4~~6b

iv. Require a one-tail test, with null and alternative hypotheses

$$H_0 : \mu = \frac{1}{2} \quad \text{vs} \quad H_1 : \mu < \frac{1}{2}$$

[2]

A t-test is appropriate here. The test statistic is

$$t = \frac{\bar{x} - 1/2}{\frac{s}{\sqrt{n}}} = \frac{1.04 - 0.5}{\frac{2.40}{2.65}}$$
$$= \frac{0.504}{0.906} \approx \underline{0.599}$$

[2]

Under the null hypothesis, the test statistics has a t distribution with $n - 1 = 6$ degrees of freedom. Noting the ~~the~~ formula sheet only gives two sided values, the critical value is given by $-t_{6,0.1} = -1.94$.

[4]

Since $t > -1.94$ we have no evidence to reject the null hypothesis, and hence there is no evidence to support the claim.

[2]

TOTAL [20]

9