Imperial College London

BSc/MSci EXAMINATION June 2012

This paper is also taken for the relevant Examination for the Associateship

ELECTRICITY & MAGNETISM

For First-Year Physics Students

Wednesday, 13th June 2012 14:00 to 16:00

Answer both questions in Section A and TWO questions from Section B.
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the FOUR answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in FOUR answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

SECTION A

1. (i) Consider a non-conducting sphere of radius R with non-homogeneous charge density $\rho = \rho(r) = r$, where r is the radial co-ordinate.

(a) Find the electric field inside and outside of the sphere [3 marks]

(b) Find and plot the electric potential inside and outside of the sphere [4 marks]

(c) How much energy does it take to assemble this configuration from individual charges? [4 marks]

(ii) A charge distribution produces an electric potential

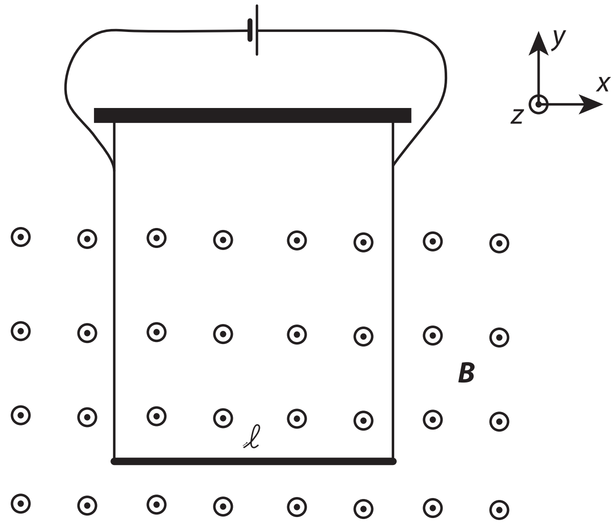
$$V(x) = \begin{cases} -10x\ V/m - 20\ V & x < -1\ m \\ 5x^2\ V/m^2 - 15\ V & -1\ m \leq x \leq 1\ m \\ 10x\ V/m - 20\ V & x > 1\ m \end{cases}$$

(a) Calculate and plot the electric field for $x < -1\ m$, $-1\ m \leq x \leq 1\ m$ and $x > 1\ m$ [2 marks]

(b) This potential is due a slab of charge of constant volume charge density ρ (C/m^3). What is the location of the slab? [2 marks]

(c) Determine the charge density ρ in (C/m^3) units not forgetting its sign. [5 marks]

(iii) A current carrying wireframe is hung by its upper horizontal section in the $x - y$ plane. The lower horizontal section of the rectangular wire frame, having a mass density of α (kg/m) and length ℓ, is immersed in a homogeneous and time stationary magnetic field $\boldsymbol{B} = B_0\hat{\boldsymbol{k}}$.

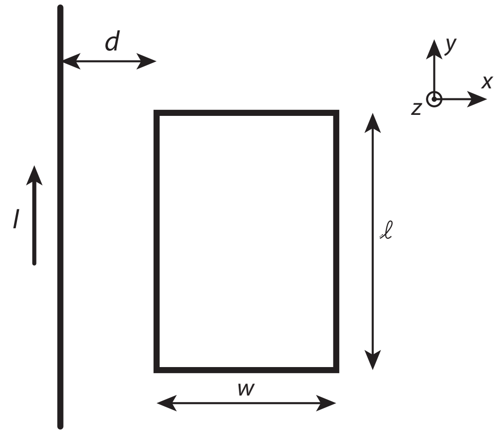


If the vertical wires of the frame are flexible, what is the minimum strength and direction of the current I in the wire such that there is a non-zero total upward force on the horizontal section? [4 marks]

(iv) A straight infinite wire carries an upward current I and is placed next to a rectangular wire loop with its long side parallel with the wire. The distance between

[This question continues on the next page . . .]

the wire and the loop is d and the loop has dimensions $\ell \times w$ as shown in the figure.



(a) Determine the magnetic flux through the wire loop due to the constant current I in the wire [3 marks]

(b) What is the induced emf and the current in the loop? [1 mark]

(c) Assume now that the current in the straight wire varies in time $I(t) = (1 + 2t)\ A$ where t is in seconds. What is the induced emf and the direction of the current in the loop? [2 marks]

[Total 30 marks]

2. (i) A student project involves constructing two 1 m lengths of wire with resistances as close as possible to 100Ω and 500Ω respectively. Nichrome wire is made available in diameters of 25, 50 and 120 μm. Nichrome has a resistivity of $1.1 \times 10^{-6}\,\Omega\,$m.

Which diameters should the students choose for each wire? [2 marks]

The students connect the two wires together into a 2 m length and then connect a 6V battery across the two loose ends. Using an oscilloscope they measure how the voltage changes along the 2 m length of the wires.

Sketch the result you would expect them to get [1 mark]

What is the current density in each of the wire segments? [1 mark]

Use the current densities and the conductivity to work out the electric field in each wire [1 mark]

Check your result for the electric field by integrating it along the length of each wire [1 mark]

(ii) Use complex analysis to show that in the following circuit the current I is related to the input voltage, V_{in} by

$$I = V_{in}\left(\frac{1}{2R} + j\frac{\omega C}{2}\right)$$

[2 marks]

What is the phase of I relative to V_{in} at low and high frequencies? [1 mark]

Hence work out the amplitude and phase of V_{out}. [1 mark]

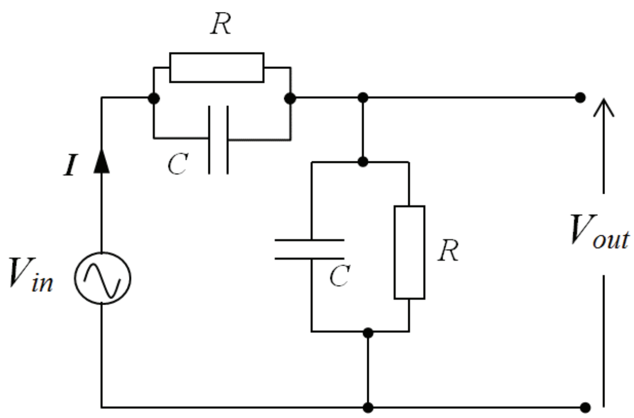


Figure 1: Circuit for Question (ii)

(iii) A current source (d.c.) with source resistance, R_S, is used to drive current through a load resistor, R_L.

Draw the circuit described. [1 mark]

Show that the power delivered to the load resistor is

$$P = I^2 R_S^2 \frac{R_L}{(R_S + R_L)^2}$$

[This question continues on the next page . . .]

[2 marks]

Show that the power is maximised when $R_L = R_S$ [1 mark]

(iv) For the operational amplifier circuit given below show that, for an ideal operational amplifier

$$\frac{V_{out}}{V_{in}} = -\frac{Z_F}{Z_{in}}$$

State all the assumptions you make about the operational amplifier. [2 marks]

If a single capacitor, C, is used for Z_F and a resistor, R, is used for Z_{in} show that the circuit performs an integration in terms of both amplitude and phase. [You may use complex analysis or otherwise] [2 marks]

If, instead, a single resistor, R, is used for Z_F and a capacitor, C, is used for Z_{in} show that the circuit performs a differentiation. [2 marks]

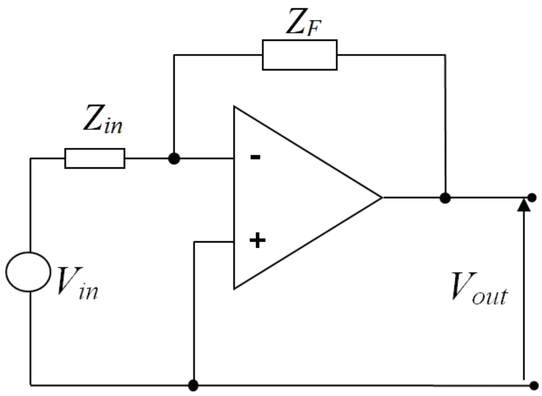


Figure 2: Circuit for Question (iv)

[Total 20 marks]

SECTION B

3. A uniformly charged non-conducting thin circular disk of radius R and surface charge density σ is located in the $x-y$ plane with its centre at the origin of a right handed Cartesian co-ordinate system.

(i) Calculate the electric potential due to the disk along the $+z$ axis [8 marks]

(ii) From the above result or otherwise show that the electric field at a point on the z axis is given by

$$\mathbf{E} = \frac{\sigma}{2\epsilon_0}(1 - \cos\alpha)\hat{\mathbf{k}}$$

where α is the angle at which the edge of the disk is seen from the observation point on axis [6 marks]

(iii) From the above result derive an expression for the electric field when the radius of the disk becomes much larger than the observation distance. Comment on your result. [2 marks]

(iv) From the result obtained in point (ii) above derive an expression for the electric field when the radius of the disk is much smaller than the observation distance. Comment on your result. [3 marks]

(v) Show that the electric potential at $(x, 0, 0)$ for $x > R$ is given by

$$V(x, 0, 0) = \frac{\sigma}{4\pi\epsilon_0}\int_0^R\int_0^{2\pi}\frac{\rho d\rho d\phi}{\sqrt{\rho^2 + x^2 - 2\rho x\cos\phi}}$$

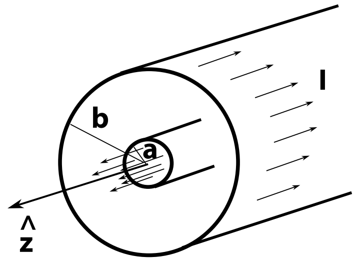
but DO NOT attempt to evaluate the integral. [3 marks]

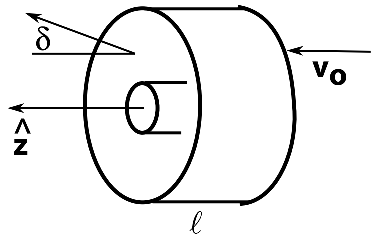
(vi) Show that $V(x, 0, 0)$ for $x \gg R$ approximates the potential due to a point charge [3 marks]

[Total 25 marks]

4. (i) State Ampere's Law and explain the meanings of the various quantities and expressions involved. [5 marks]

(ii) An infinite coaxial current system carries a current I distributed uniformly in a core of radius a. A thin cylindrical shell of radius $b > a$ carries an equal total current in the opposite direction as shown in the left figure.





(a) Find the magnetic field in the region $a < r < b$ and show that the magnetic field is zero outside the system. [5 marks]

(b) A short segment of length ℓ of such a configuration is used to deflect electrons (mass m, charge $-e$) initially travelling with $\boldsymbol{v} = v_o\hat{\boldsymbol{z}}$ at a radial position mid-way between the core and shell, i.e., at $r = (a + b)/2$. Assuming that $v_z \approx v_o$ throughout the subsequent motion, and neglecting any end effects due to the finite length of this segment, show that after traversing the length ℓ at a roughly constant radial distance $(a + b)/2$ each electron has received a positive radial impulse

$$\Delta p_r = \int F_r\, dt \approx F_r\, \Delta t = \frac{\mu_o e I \ell}{\pi(a + b)}$$

where p_r is the radial momentum of the electron and F_r is the radial component of the force. [5 marks]

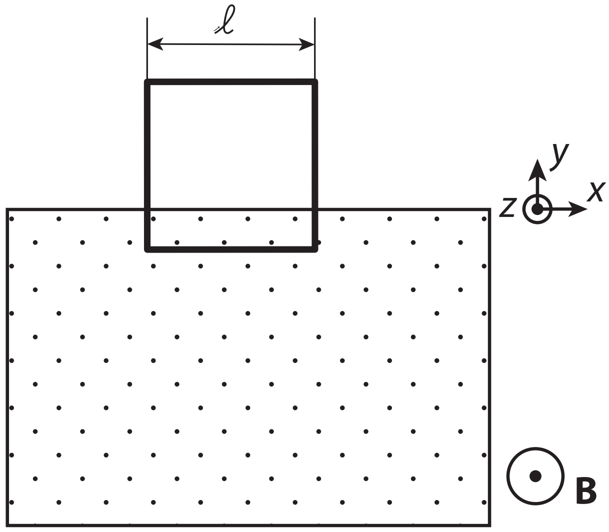
(c) Use this result to determine the angular deflection δ of the electrons. [5 marks]

(d) Hence estimate the current I required to use such a deflection system in a television requiring $\tan\delta = 0.1$ with the following parameters: $a = 5\,\text{mm}$, $b = 10\,\text{mm}$, $\ell = 50\,\text{mm}$, $v_0 = 2 \times 10^7 m/s$. [5 marks]

[Total 25 marks]

5. (i) State Faraday's induction law and discuss how Lenz's law is also expressed by it. [3 marks]

(ii) Consider the arrangement shown in the figure below.



A square wire loop of resistance R and mass m moves with constant velocity $\boldsymbol{v} = -\hat{\boldsymbol{j}}$. At $t = 0$ the bottom edge of the loop enters a time stationary, uniform magnetic field $\boldsymbol{B} = \hat{\boldsymbol{k}}$. The spatial extent of the magnetic field in the x direction is infinite and in the y direction is semi-infinite. Neglect self-inductance.

(a) Determine the magnitude and direction of the current induced in the wire loop while it is partly outside and then fully inside the magnetic field. [5 marks]

(b) Calculate the power dissipated in the wire loop while it is partly outside and then fully inside the magnetic field. [3 marks]

(iii) Consider now that the loop falls under gravity into the magnetic field with its bottom edge entering the field at $t = 0$.

(a) Determine the differential equation of motion [5 marks]

(b) What is the terminal velocity of the frame? [2 marks]

(c) Plot the velocity as a function of time and hence determine if the frame will reach terminal velocity while partly outside the magnetic field. [7 marks]

[Total 25 marks]

6. (i) The figure shows a common filter circuit. Without detailed calculation at this stage explain what type of filter this is. [4 marks]

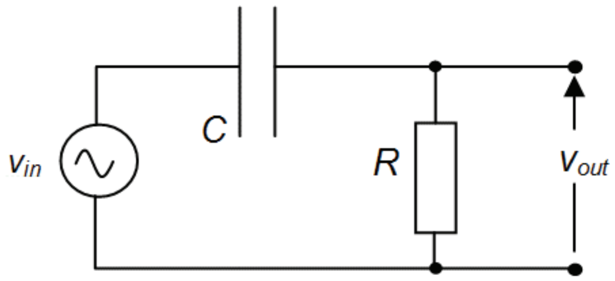


Figure 1: Circuit for Question (2.i)

(ii) Show that the transfer function for this circuit is

$$\frac{V_{out}}{V_{in}} = \frac{\omega^2 C^2 R^2}{1 + \omega^2 C^2 R^2} + j\frac{\omega CR}{1 + \omega^2 C^2 R^2}$$

[4 marks]

(iii) Use the result of (ii) to verify your explanation given in (i) above [4 marks]

(iv) Show that the -3dB point for the filter occurs when $\omega = 1/CR$ [4 marks]

(v) Sketch the Bode plots (amplitude and phase) for this circuit, properly annotating the key features [6 marks]

(vi) Without detailed analysis explain what would happen if a second identical filter were added before the output. Use a new Bode plot (sketch) for the amplitude to illustrate your answer. [3 marks]

[Total 25 marks]