EE 4-24

SOLUTIONS: DISCRETE-TIME SYSTEMS AND COMPUTER CONTROL

1. Solution

a) The differential equation describing the dynamics of the accelerometer can be rewritten in terms of the measured acceleration as follows:

$$\frac{1}{K}\frac{d^2}{dt^2}a_m(t)+\frac{K_f}{KM}\frac{d}{dt}a_m(t)+\frac{K_e}{KM}a_m(t)=a(t)$$

Taking the Laplace transform on both sides of the above differential equation (setting to zero the initial conditions as it should be always done when computing transfer functions) and re-arranging, we finally get

$$G(s)=\frac{A_m(s)}{A(s)}=\frac{K}{s^2+\frac{K_f}{M}s+\frac{K_e}{M}}$$

where $A_m(s)=\mathscr{L}[a_m(t)]$ and $A(s)=\mathscr{L}[a(t)]$.

[3 marks]

b) The function expressing the time-profile of the acceleration $a(t)$ depicted in Fig. 1.2 of the text of the exam paper is given by

$$a(t)=5\cdot 1(t)-8\cdot 1(t-5)+3\cdot 1(t-10)$$

where $1(t)$ denotes the continuous-time unit step function. Thus, taking the Laplace transform, we obtain

$$A(s)=\mathscr{L}[5\cdot 1(t)-8\cdot 1(t-5)+3\cdot 1(t-10)]=\frac{5-8e^{-5s}+3e^{-10s}}{s}$$

Using the values of the constants in the expression of the transfer function computed in the answer to Question 1a), we obtain

$$A_m(s)=\frac{6}{s(s^2+5s+6)}(5-8e^{-5s}+3e^{-10s})$$

[5 marks]

c) Let us first compute

$$\begin{aligned}\mathscr{Z}\left[\frac{6}{s(s^2+5s+6)}\right] &= \mathscr{Z}\left[6\left(\frac{1}{6}\frac{1}{s}-\frac{1}{2}\frac{1}{s+2}+\frac{1}{3}\frac{1}{s+3}\right)\right]\\ &= \frac{1}{1-z^{-1}}-3\frac{1}{1-\frac{1}{e^2}z^{-1}}+2\frac{1}{1-\frac{1}{e^3}z^{-1}}\\ &= \frac{z}{e^5}\frac{(e-1)^2(1+2e+(2e^2+e^3)z)}{(z-1)(z-\frac{1}{e^2})(z-\frac{1}{e^3})}\end{aligned}$$

Now, by noticing that the factors e^{-5s} and e^{-10s} in the s-domain translate in the factors z^{-5} and z^{-10} in the z-domain, respectively, because $T = 1$s, we immediately get

$$A_m(z) = \frac{z}{e^5} \frac{(e-1)^2(1+2e+(2e^2+e^3)z)}{(z-1)(z-\frac{1}{e^2})(z-\frac{1}{e^3})}(5-8z^{-5}+3z^{-10})$$

[8 marks]

d) The discrete-time equivalent model for the accelerometer cannot be determined because the absence of a sampling device at the input makes it impossible to construct a transfer function in the z-domain.

[4 marks]

2. Solution

a) The transfer function $G(z)$ can be written as

$$G(z) = \frac{10z^2 - 8z}{5z^2 - 8z + 3} = 2\frac{z(z-\frac{4}{5})}{(z-1)(z-\frac{3}{5})}$$

As $G(z)$ has a pole in $z = 1$, the position constant does not take on a finite value, that is,

$$k_p = \lim_{z\to 1} G(z) = \infty$$

Concerning the velocity constant, we have

$$k_v = \lim_{z\to 1} \frac{z-1}{z} G(z) = \lim_{z\to 1} \frac{z-1}{z} 2\frac{z(z-\frac{4}{5})}{(z-1)(z-\frac{3}{5})} = 1$$

[3 marks]

b) We proceed by using the "computational method". Noticing that $\mathscr{Z}[u(k)] = 1$ and denoting $H(z) = \mathscr{Z}[h(k)]$, we have

$$H(z) = 2\frac{z(z-\frac{4}{5})}{(z-1)(z-\frac{3}{5})} \cdot 1 = \frac{2z^2 - \frac{8}{5}z}{z^2 - \frac{8}{5}z + \frac{3}{5}}$$

and hence the following difference equation relating the sequences h and u at different time steps can be written as follows:

$$h(k+2) - \frac{8}{5}h(k+1) + \frac{3}{5}h(k) = 2u(k+2) - \frac{8}{5}u(k+1)$$

which allows to compute $h(k)$, $k \geq 2$ provided that the initial samples $h(0)$ and $h(1)$ are available. Let us compute these two initial samples. First, we consider the difference equation for $k = -2$:

$$h(0) - \frac{8}{5}h(-1) + \frac{3}{5}h(-2) = 2u(0) - \frac{8}{5}u(-1)$$

which immediately gives $h(0) = 2u(0) = 2$. Next, consider the difference equation for $k = -1$:

$$h(1) - \frac{8}{5}h(0) + \frac{3}{5}h(-1) = 2u(1) - \frac{8}{5}u(0)$$

which gives $h(1) = \frac{8}{5}$. Finally, we consider the difference equation for $k = 0$:

$$h(2) - \frac{8}{5}h(1) + \frac{3}{5}h(0) = 2u(2) - \frac{8}{5}u(1)$$

from which we get $h(2) = \frac{34}{25}$.

[5 marks]

c) Let us write the input sequence as

$$u(k) = u_1(k) + u_2(k)$$

where, denoting by $1(k)$ the discrete-time unit step function, we let

$$u_1(k) = 4 \cdot 1(k) \quad \text{and} \quad u_2(k) = 2\sin(2k) \cdot 1(k)$$

Linearity implies that we can write

$$Y(z) = Y_1(z) + Y_2(z)$$

where $Y(z) = \mathscr{Z}[y(k)]$, $Y_1(z) = \mathscr{Z}[y_1(k)]$, and $Y_2(z) = \mathscr{Z}[y_2(k)]$, with $y_1(k)$ and $y_2(k)$ being the output responses to the input sequences $u_1(k)$ and $u_2(k)$, respectively. We have:

$$Y_1(z) = \frac{2z^2 - \frac{8}{5}z}{z^2 - \frac{8}{5}z + \frac{3}{5}} \cdot \frac{4z}{z-1}$$

Expanding $Y_1(z)/z$ in partial fractions, we get

$$\frac{Y_1(z)}{z} = \frac{A}{(z-1)^2} + \frac{B}{z-1} + \frac{C}{z-\frac{3}{5}}$$

where A, B, C are suitable constants. Since, we only have to compute the "steady-state" output sequence $y_{ss}(k)$, there is no need compute the constant C because the corresponding response mode vanishes for $k \to \infty$ (the pole $\frac{3}{5}$ lies strictly inside the unit circle). Then, by denoting with $y_{ss}^{(1)}(k)$ the "steady-state" output sequence generated by $u_1(k)$ and with $Y_{ss}^{(1)}(z) = \mathscr{Z}[y_{ss}^{(1)}(k)]$ its $\mathscr{Z}$ transform, we have

$$\frac{Y_{ss}^{(1)}(z)}{z} = \frac{A}{(z-1)^2} + \frac{B}{z-1}$$

with

$$A = \lim_{z\to 1}\left[(z-1)^2 \frac{8z^2 - \frac{32}{5}z}{(z-1)^2(z-\frac{3}{5})}\right] = 4$$

$$B = \lim_{z\to 1}\left\{\frac{d}{dz}\left[(z-1)^2 \frac{8z^2 - \frac{32}{5}z}{(z-1)^2(z-\frac{3}{5})}\right]\right\} = 14$$

Then

$$y_{ss}^{(1)}(k) = (4k+14)\cdot 1(k)$$

Now, we consider the input $u_2(k)$. We have:

$$Y_2(z) = \frac{2z^2 - \frac{8}{5}z}{z^2 - \frac{8}{5}z + \frac{3}{5}} \cdot \frac{2z\sin(2)}{z^2 - 2z\cos(2) + 1}$$

Expanding $Y_2(z)/z$ in partial fractions, we get

$$\frac{Y_2(z)}{z} = \frac{A}{(z-1)} + \frac{B}{z-\frac{3}{5}} + \frac{C}{z-e^{-2j}} + \frac{C^*}{z-e^{2j}}$$

As before, we denote with $y_{ss}^{(2)}(k)$ the "steady-state" output sequence generated by $u_2(k)$ and with $Y_{ss}^{(2)}(z) = \mathscr{Z}[y_{ss}^{(2)}(k)]$ its $\mathscr{Z}$ transform. Hence (again, the pole $\frac{3}{5}$ lies strictly inside the unit circle)

$$\frac{Y_{ss}^{(2)}(z)}{z} = \frac{A}{(z-1)} + \frac{C}{z-e^{-2j}} + \frac{C^*}{z-e^{2j}}$$

with

$$A = \lim_{z\to 1}\left[(z-1)\frac{4z^2 - \frac{16}{5}z}{(z-1)(z-\frac{3}{5})} \cdot \frac{\sin(2)}{z^2 - 2z\cos(2) + 1}\right] = \frac{2\sin(2)}{2 - 2\cos(2)} \simeq 0.64$$

$$C = \lim_{z \to e^{-2j}} \left[(z-e^{-2j}) \frac{(4z^2 - \frac{16}{5}z)\sin(2)}{(z-1)(z-\frac{3}{5})(z-e^{-2j})(z-e^{2j})} \right] \simeq -0.61 - 1.17j$$

Hence, after some algebra, we get

$$y_{ss}^{(2)}(k) = 0.64 \cdot 1(k) + (C \cdot e^{-2jk} + C^* \cdot e^{2jk}) \cdot 1(k) \simeq [0.64 + 2.65\sin(2k - 0.48)] \cdot 1(k)$$

Finally

$$y_{ss}(k) = y_{ss}^{(1)}(k) + y_{ss}^{(2)}(k) = [4k + 14.64 + 2.65\sin(2k - 0.48)] \cdot 1(k)$$

[9 marks]

d) The frequency discrete-time frequency response theorem can be used to determine

$$\mathscr{Z}^{-1}\left(\frac{Cz}{z-e^{-2j}} + \frac{C^* z}{z-e^{2j}}\right)$$

without the need to compute constants C and C^*. In fact:

$$\mathscr{Z}^{-1}\left(\frac{Cz}{z-e^{-2j}} + \frac{C^* z}{z-e^{2j}}\right) = 2 \cdot \left|G(e^{j2})\right| \sin\left(2k + \operatorname{Arg} G(e^{j2})\right)$$

where

$$G(z) = \frac{10z^2 - 8z}{5z^2 - 8z + 3}$$

[3 marks]

3. Solution

a) Since $T = 1/10$, we can write (in the Laplace domain)

$$H(s)G(s) = (1 - e^{-s/10})\frac{1}{s(s+1)^2}$$

Now, using the usual procedure to calculate $\mathscr{Z}$ transforms of terms involving the ZOH, we obtain

$$\begin{aligned}
HG(z) &= (1-z^{-1})\,\mathscr{Z}\left[\frac{1}{s(s+1)^2}\right] = (1-z^{-1})\,\mathscr{Z}\left[\frac{1}{s} - \frac{1}{(s+1)^2} - \frac{1}{s+1}\right] \\
&= (1-z^{-1})\left[\frac{1}{1-z^{-1}} - \frac{e^{-1/10}}{10}\frac{z^{-1}}{(1-e^{-1/10}z^{-1})^2} - \frac{1}{1-e^{-1/10}z^{-1}}\right] \\
&= 1 - \frac{e^{-1/10}}{10}\frac{z-1}{(z-e^{-1/10})^2} - \frac{z-1}{z-e^{-1/10}}
\end{aligned}$$

After some algebra, we finally get

$$HG(z) = \frac{(1-\frac{11}{10}e^{-1/10})z + e^{-1/5} - \frac{9}{10}e^{-1/10}}{(z-e^{-1/10})^2}$$

[4 marks]

b) To compute the discrete-time approximation with the Tustin transformation, we let $s = \frac{2}{T}\frac{z-1}{z+1}$ (with $T = 1/10$) in the expression of $C(s)$. Then, we obtain

$$C_{Tu}(z) = C(s)|_{s=20\frac{z-1}{z+1}} = \frac{K}{20\frac{z-1}{z+1} + K} = \frac{K}{20+K}\frac{z+1}{z + \frac{K-20}{K+20}}$$

Instead, to compute the discrete-time approximation through the pole-zero correspondence, the poles and zeros in continuous-time have to be matched in discrete-time. Hence, we have

$$C_{pz}(z) = \alpha\frac{K(1+z^{-1})}{1-e^{-K/10}z^{-1}}$$

where α should be computed so as to match the gain at $s=0$ which gives

$$\alpha = \frac{1-e^{-K/10}}{2K} \Longrightarrow C_{pz}(z) = \frac{1-e^{-K/10}}{2}\frac{z+1}{z-e^{-K/10}}$$

Comparing the two approximations $C_{Tu}(z)$ and $C_{pz}(z)$, we notice that both have (by construction) the same static gain. Moreover, they both have a zero in $z=-1$ which corresponds to the zero at $s=-\infty$ in continuous-time. Finally, comparing the poles, we notice that for small values of K ($K \leq 5$) they are approximately the same, whereas, for larger values of K ($K > 20$), the pole of $C_{Tu}(z)$ becomes negative, whereas the pole of $C_{pz}(z)$ always remains positive.

[4 marks]

c) Thanks to the location of the samplers that can be ascertained from Fig. 3.1 in the text of the exam paper, we have

$$G_{cl}(z) = \frac{Y(z)}{R(z)} = \frac{C_{pz}(z)HG(z)}{1+M(z)C_{pz}(z)HG(z)}$$

In the answer to Question 3a) we got

$$HG(z) = \frac{(1-\frac{11}{10}e^{-1/10})z + e^{-1/5} - \frac{9}{10}e^{-1/10}}{(z-e^{-1/10})^2}$$

whereas from the answer to Question 3b) we have

$$C_{Tu}(z) = \frac{K}{20+K}\frac{z+1}{z+\dfrac{K-20}{K+20}}$$

Some algebra finally yields

$$G_{cl}(z) = \frac{2K(z+1)[(10e^{1/5} - 11e^{1/10})z + 10 - 9e^{1/10}]}{400(z-1)(e^{1/10}z-1)^2 + K(z+1)[30 + 10e^{1/5}z(2z+1) - 3e^{1/10}(3+17z)]}.$$

[6 marks]

d) To get a second-order closed-loop characteristic polynomial, we set a specific value $\tilde{K}$ so as to cancel the open-loop zero of the term $(1-\frac{11}{10}e^{-1/10})z + e^{-1/5} - \frac{9}{10}e^{-1/10}$ in the numerator of $HG(z)$, thus getting, after some algebra,

$$\tilde{K} = \frac{100(e^{1/10}-1)^2}{5e^{1/5} - 5 - e^{1/10}} \simeq 600$$

Using $\tilde{K}$, we obtain the following open-loop transfer function:

$$\frac{\tilde{K}}{20+\tilde{K}} \cdot \frac{z+1}{z+\dfrac{\tilde{K}-20}{\tilde{K}+20}} \cdot \frac{(1-\frac{11}{10}e^{-1/10})z + e^{-1/5} - \frac{9}{10}e^{-1/10}}{(z-e^{-1/10})^2}$$

$$= \frac{\tilde{K}}{20+\tilde{K}} \cdot (1-\frac{11}{10}e^{-1/10}) \cdot \frac{z+1}{(z-e^{-1/10})^2}$$

We denote by $\tilde{G}_{cl}(z)$ the discrete-time closed-loop transfer function obtained with the above selection $K = \tilde{K}$. After some simple algebra, we get:

$$\tilde{G}_{cl}(z) = \frac{\beta(z+1)}{z^2 + (\beta/2 - 2e^{-1/10})z + e^{1/5} + \beta/2}$$

with

$$\beta = \frac{\tilde{K}}{20+\tilde{K}} \cdot (1-\frac{11}{10}e^{-1/10}) = \frac{(e^{1/10}-1)^2}{2e^{1/5}}$$

and hence

$$\tilde{G}_{cl}(z) = \frac{\frac{(e^{1/10}-1)^2}{2e^{1/5}}(z+1)}{z^2 + \frac{1}{4}(1+e^{-1/5} - 10e^{-1/10})z + e^{1/5} + \frac{1}{4}e^{-1/5}(e^{1/10}-1)^2}$$

Since

$$\frac{1}{4}e^{-1/5}(e^{1/10}-1)^2 > 1$$

it can be concluded that the discrete-time closed-loop system described by the transfer function $\tilde{G}_{cl}(z)$ is unstable.

[6 marks]

4. Solution

a) To compute the transfer function $G_{cl}^{(ry)}(z)$, we first set $D(z) = N(z) = 0$. By inspection of the location of the samplers in the block diagram shown in Fig. 4.1 of the text of the exam paper, we immediately get the following relationships:

$$Y(z) = HG(z)C(z)E(z) \quad \text{and} \quad E(z) = R(z) - M(z)Y(z)$$

from which, after some algebra, we immediately obtain

$$G_{cl}^{(ry)}(z) = \frac{C(z)HG(z)}{1 + M(z)C(z)HG(z)}$$

[4 marks]

b) To compute the transfer function $G_{cl}^{(dy)}(z)$, we first set $R(z) = N(z) = 0$. By inspection of the location of the samplers in the block diagram shown in Fig. 4.1 of the text of the exam paper, we immediately get the following relationships:

$$Y(z) = HG(z)[-D(z) + C(z)E(z)] \quad \text{and} \quad E(z) = -M(z)Y(z)$$

from which, after some algebra, we get

$$G_{cl}^{(dy)}(z) = -\frac{HG(z)}{1 + M(z)C(z)HG(z)}$$

Finally, To compute the transfer function $G_{cl}^{(ny)}(z)$, we first set $R(z) = D(z) = 0$. Again, by inspection of the location of the samplers in the block diagram shown in Fig. 4.1 of the text of the exam paper, we immediately get the following relationships:

$$Y(z) = HG(z)C(z)E(z) \quad \text{and} \quad E(z) = -M(z)[N(z) + Y(z)]$$

from which, after some algebra, it follows that

$$G_{cl}^{(ny)}(z) = -\frac{M(z)C(z)HG(z)}{1 + M(z)C(z)HG(z)}$$

[6 marks]

c) The velocity error can be computed as follows:

$$e_v = \lim_{z \to 1} \frac{z-1}{z} E(z) = \lim_{z \to 1} \frac{z-1}{z} G_{cl}^{(re)}(z) R(z)$$

where $G_{cl}^{(re)}(z)$ denotes the closed-loop discrete-time transfer function from the reference input variable $R(z)$ and the error variable $E(z)$ and where

$$R(z) = \mathscr{Z}[k \cdot 1(k)] = \frac{Tz^{-1}}{(1 - z^{-1})^2} = \frac{1}{10} \frac{z}{(z-1)^2}$$

Clearly:

$$E(z) = R(z) - Y(z) \quad \text{and} \quad Y(z) = HG(z)C(z)E(z)$$

from which, we immediately obtain ($M(z) = 1$)

$$G_{cl}^{(re)}(z) = \frac{1}{1 + C(z)HG(z)} = \frac{1}{1 + \left(K_P + K_I \dfrac{z}{z-1}\right) \dfrac{1 - e^{-T}}{z - e^{-T}}}$$

Then

$$e_v = \lim_{z\to 1} \frac{z-1}{z} \frac{1}{1+\left(K_P + K_I \frac{z}{z-1}\right) \frac{1-e^{-1/10}}{z-e^{-1/10}}} \frac{1}{10} \frac{z}{(z-1)^2}$$

After a few simple algebraic manipulations, we get

$$e_v = \lim_{z\to 1} \frac{1}{10} \frac{z-e^{-1/10}}{(z-1)(z-e^{-1/10}) + [K_P(z-1)+K_I z](1-e^{-1/10})} = \frac{1}{10K_I}$$

Hence any choice of K_I such that $K_I \geq 10$ implies $|e_v| \leq 0.01$ (provided that the discrete-time closed-loop control system is asymptotically stable). Then, let us set $K_I = 10$ and check whether there exist a selection of K_P ensuring that the closed-loop asymptotic stability is attained.

With the above choice $K_I = 10$, the closed-loop poles are given by solutions of the polynomial equation

$$(z-1)(z-e^{-1/10}) + [k_P(z-1)+10z](1-e^{-1/10}) = 0$$

which, after some algebra, can be rewritten as

$$z^2 + Az + B = 0$$

where

$$A = K_P(1-e^{-1/10}) + 9 - 11e^{-1/10}$$
$$B = K_P(e^{-1/10} - 1)$$

The two roots of the above polynomial equation are located strictly inside the unit circle (hence guaranteeing the closed-loop asymptotic stability) if and only if

$$B > -1 - A; \quad B < 1; \quad B > A - 1$$

After some simple algebra, we obtain that the above inequalities are satisfied for any selection of K_P such that

$$0 < K_P < \frac{11e^{-1/10} - 9}{1 - e^{-1/10}} \simeq 10$$

[10 marks]