EE4-29
EE9-CS3-2
EE9-FPN1-04

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2017

MSc and EEE/EIE PART IV: MEng and ACGI

OPTIMIZATION

Corrected copy

Thursday, 4 May 10:00 am

Time allowed: 3:00 hours

There are FOUR questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

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Second Marker(s) : P.L. Dragotti

OPTIMISATION

1. Consider the problem of minimizing the function

$$\min_x f(x),$$

with

$$f(x) = \frac{1}{2}x_1^2 - x_1 x_2 + \frac{1}{4}x_2^4 - \frac{1}{3}x_2^3.$$

a) Compute the stationary points of the function f. [2 marks]

b) Using second order sufficient conditions of optimality classify the stationary points determined in part a). Hence, determine the global minimum of f. [6 marks]

c) Consider the problem of minimizing the function using the so-called gradient method with extrapolation, that is the method defined by the iteration

$$X_{k+1} = X_k - \alpha_k \nabla f(X_k) + \beta_k (X_k - X_{k-1}),$$

with $\alpha_k > 0$ and $\beta_k \in [0, 1)$, for all $k \geq 0$, and $X_{-1} = X_0$. Let $X_0 = (1, 1)$.

i) Argue that the first step of the gradient method with extrapolation coincides with the first step of the gradient method. [2 marks]

ii) Run one iteration of the gradient method with extrapolation and determine the point X_1. Note that X_1 is a function of α_0 hence write a condition on α_0 such that the algorithm is a descent algorithm, that is $f(X_1) < f(X_0)$. Explain why β_0 does not appear in the descent condition $f(X_1) < f(X_0)$. [4 marks]

iii) Pick $\alpha_k = 1/2$ for all k. Run one more iteration of the gradient method with extrapolation (using as initial condition the point X_1 determined in part c.ii), that is compute the point X_2. Determine a condition on β_1 yielding a descent algorithm. Explain why $\beta_1 = 0$ is a feasible selection of β_1 and argue that it is not the best selection. [6 marks]

2. The proximal method is a descent method in which the problem

$$\min_x f(x) \qquad (1)$$

is replaced by the sequence of problems

$$\min_x \left(f(x) + \frac{1}{2\gamma_k}\|x - x_k\|^2 \right) \qquad (2)$$

where x_k is the current estimate of the solution of the problem (1), x_{k+1} is the solution of the minimization problem (2), and $\gamma_k > 0$.

Consider the quadratic function

$$f(x) = \frac{1}{2}x'Qx + c'x + d,$$

with $Q = Q' > 0$. Recall that the function has a global minimizer at $x^* = -Q^{-1}c$.

a) Write the optimization problem used in the proximal method and state under what conditions the problem has a unique solution. [2 marks]

b) Solve explicitly the optimization problem (2), that is determine x_{k+1} as a function of x_k. In particular, write the relation between x_{k+1} and x_k in the form

$$x_{k+1} = Ax_k + b, \qquad (3)$$

in which A is a matrix and b is a vector. (Note that A and b change as a function of k.) Write explicitly the matrix A and the vector b as a function of Q, c and γ_k. [6 marks]

c) Determine the fixed point $\bar{x}$ of equation (3), that is the point $\bar{x}$ such that

$$\bar{x} = A\bar{x} + b,$$

and show that the point is the global minimizer of the quadratic function. [2 marks]

d) Show that the iteration (3) is globally convergent for all $\gamma_k > 0$. This can be achieved using the following steps.

i) Show that

$$A = (\gamma_k Q + I)^{-1}$$

and that $A'A < I$. [2 marks]

ii) Write the iteration (3) in the form

$$x_{k+1} - x^* = A(x_k - x^*) + \tilde{b}$$

and show that $\tilde{b} = 0$. [4 marks]

iii) Exploit the results in parts d.i) and d.ii) to demonstrate the global convergence claim. Discuss also the effect of the parameter γ_k on the speed of convergence of the algorithm. [4 marks]

3. Consider the optimization problem

$$\min_{x_1,x_2,x_3} x_1+x_2+x_3^2,$$
$$x_1-1=0,$$
$$x_1^2+x_2^2-1=0.$$

a) Plot the admissible set and show that all points are non-regular points. [2 marks]

b) Solve the problem using only graphical considerations. [2 marks]

c) State first order necessary conditions of optimality for such a constrained optimization problem and show that the conditions do not give any candidate optimal solution. Discuss why this is the case. [6 marks]

d) The considered problem can be *regularized* as follows.

   i) Determine two constrains that could replace the constrains in the considered problem and which are such that all points are regular points. [2 marks]

   ii) State first order necessary conditions of optimality for the constrained optimization problem with the constraints determined in part d.i) and show that there is only one candidate optimal solution. [4 marks]

   iii) Using second order sufficient conditions of optimality show that the candidate optimal solution is indeed a solution of the considered optimization problem. [4 marks]

4. Consider the optimization problem

$$\min_{x_1,x_2} 9x_1^2 - 54x_1 + 13x_2^2 - 78x_2$$

$$x_1 - 4 \leq 0,$$

$$x_2 - 6 \leq 0,$$

$$3x_1 + 2x_2 - 18 \leq 0,$$

$$x_1 \geq 0, \quad x_2 \geq 0.$$

a) State first order necessary conditions of optimality for this constrained optimisation problem. [4 marks]

b) Using the conditions derived in part a) compute candidate optimal solutions. Show that there is only one candidate optimal solution and that this is in the interior of the admissible set.
(Hint: note that at least three of the Kuhn-Tucker multiplier have to be simultaneously zero.) [8 marks]

c) Replace now the *minimization sign* with the *maximization sign*. Note that this is equivalent to changing the sign of the objective function. The results in part b) can be used to determine candidate optimal solutions for the maximization problem provided one changes sign to the multipliers. Using the above argument, determine candidate optimal solutions for the maximization problem and the global solution of the problem. [8 marks]