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Paper Number(s): **E1.5**

IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE
UNIVERSITY OF LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2002

EEE PART I: M.Eng., B.Eng. and ACGI

ENGINEERING MATERIALS

Friday, 7 June 10:00 am

There are FIVE questions on this paper.

Answer THREE questions.

Time allowed: 2:00 hours

Examiners responsible:

First Marker(s): Pike,W.T.

Second Marker(s): Tate,T.J.

E1.5

Fundamental constants:

Permittivity of free space, $\varepsilon_0 = 8.85 \times 10^{-12}$ F/m
Permeability of free space, $\mu_0 = 4\pi \times 10^{-7}$ H/m
Planck's constant, $h = 6.62 \times 10^{-34}$ Js
Boltzmann's constant, $k = 1.38 \times 10^{-23}$ J/K
Electron charge, $e = 1.6 \times 10^{-19}$ C
Electron mass, $m = 9.1 \times 10^{-31}$ kg
Speed of light, $c = 3 \times 10^{8}$ m/s

E1.5 Exam questions

1. (a) An electron is trapped in a infinitely deep one-dimensional potential well:

$$\begin{aligned} V(x) &= \infty, \quad x < 0 \\ &= 0, \quad 0 \le x \le L \\ &= \infty, \quad x > L \end{aligned} \qquad \text{(equation 1)}$$

Show that the wavefunction

$$\begin{aligned} \psi(x) &= 0, \quad x < 0 \\ &= A \sin kx, \quad 0 \le x \le L \\ &= 0, \quad x > L \end{aligned} \qquad \text{(equation 2)}$$

is a solution of the one-dimensional Schrödinger equation

$$-\frac{h^2}{8\pi^2 m}\frac{d^2\psi}{dx^2} + V(x)\psi(x) = E\psi(x)$$

By using the appropriate boundary conditions derive expressions for all possible k and E in terms of quantum number, n. Show that the energy difference between the two lowest states is

$$\Delta E = \frac{3h^2}{8mL^2}.$$

[9 marks]

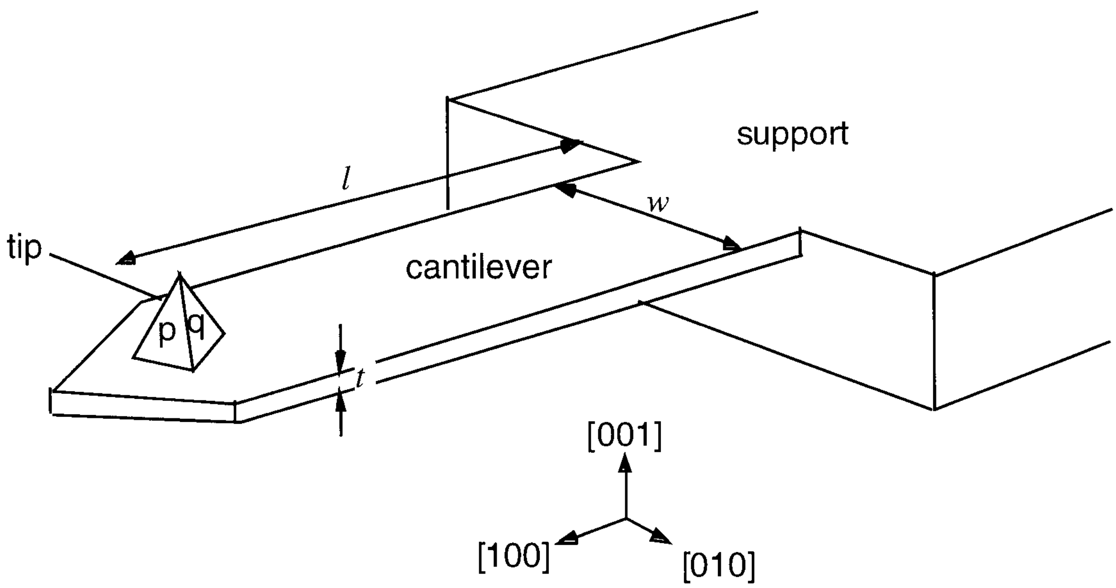
(b) A quantum-well infrared photodetector (QWIP) has been designed to detect infrared radiation of wavelength less than 10 μm. Assuming the quantum well of the QWIP can be approximated by the potential $V(x)$ in equation 1 and the infrared absorption occurs for a transition between the two lowest states, what should be the width of the well? [5 marks]

(c) The probability of finding a single electron confined to one dimension between x and x+dx is given by $|\psi(x)|^2 dx$. Explain why the electron wavefunction should obey the normalization condition:

$$\int_{-\infty}^{+\infty} |\psi(x)|^2 dx = 1$$

Hence, using the identity $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$, find the constant A in equation 2 for the ground state of the electron. [6 marks]

2. (a) The diagram below shows the cantilever and tip of an atomic-force microscope together with the crystal axes of the silicon used in its fabrication. The tip, in the shape of a pyramid, has been fabricated by anisotropic etching of the silicon, with the {111} planes resistant to the chosen etchant. Give the Miller indices for the crystal faces (p) and (q) and determine the angle of these faces to the vertical [001] direction. [5 marks]



(b) The tip is fabricated a distance l from the cantilever support. The cantilever has width w and thickness t. Show that the relevant second moment of inertia, I, for calculating the deflection of the cantilever is given by

$$I = \frac{wt^3}{12}$$

Using the expression

$$v(x) = -\frac{P}{EI}\left(l\frac{x^2}{2} - \frac{x^3}{6}\right)$$

for the small vertical deflections, v, of the cantilever, where x is the distance from the cantilever support, P is the force at the end of the cantilever, and E is Young's modulus for the cantilever material, derive an expression for the spring constant, k, of the cantilever for a force on the tip. [11 marks]

(c) The cantilever has a width 50 μm, length 1 mm and the Young's modulus for silicon is 1.1×10^{11} Nm^{-2}. Determine the thickness of the cantilever required to produce a force of 1 nN for a 1-nm tip deflection. [4 marks]

3. (a) Name and describe the three types of polarization observed in dielectrics and their relative associated frequencies. What types would you expect to see exhibited by (i) helium (ii) sodium chloride (iii) water? [6 marks]

(b) The frequency response of the dielectric constant of a material due to its orientational polarization is given by:

$$\varepsilon_r = \varepsilon_{rs} + \frac{\Delta\varepsilon}{1 + j\omega\tau}$$

What is denoted by ε_{rs}, $\Delta\varepsilon$, and τ? By consideration of the imaginary component of ε_r, determine the frequency of maximum absorption for an electromagnetic wave passing through the material. [6 marks]

(c) Hence show that the ratio of the real, k', to imaginary, k'', wavenumber for an electromagnetic wave at the peak absorption frequency is given by

$$\frac{k'}{k''} = \frac{4\varepsilon_{rs}}{\Delta\varepsilon} + 2$$

[8 marks]

4. (a) With reference to the capacitance of a parallel-plate capacitor, describe how polarization increases the dielectric constant of a material. [4 marks]

(b) A parallel-plate capacitor is produced by evaporating a thin layer of metal on the two largest sides of a thin slab of dielectric. The dielectric has a dielectric constant ε_r, and a thickness d. A voltage V is applied between the plates. By consideration of the energy stored in a capacitor show that the dielectric will feel a normal stress, σ, given by

$$\sigma = \frac{\varepsilon_0 \varepsilon_r V^2}{2d^2}$$

Is this stress compressive or extensive? [5 marks]

(c) The dielectric has a Young's modulus E and a Poisson's ratio of v. Show that the fractional change in the capacitance due to the stress on charging the capacitor is given by

$$\frac{\Delta C}{C} = (1 + 2v)\frac{\sigma}{E}$$

You may assume that the induced strains are all much less than 1.

For a particular parallel-plate capacitor, the dielectric has a Young's modulus of 10^9 Nm^{-2} a Poisson's ratio of 0.5, a dielectric constant of 2 and a thickness of 100 μm. What is the relative change, $\Delta C/C$, in the value of the capacitance when the originally uncharged capacitor is charged to 100 V? [11 marks]

5. (a) In the Weiss model of ferromagnetism, the magnetization, M, as a result of a magnetic field H at a temperature T is given by

$$M = n\mu_m \, \mathrm{L}\left(\frac{\mu_0\mu_m[H + \lambda M]}{kT}\right) \qquad \text{(equation 3)}$$

where L is the Langevin function. Explain the meanings of the parameters n, μ_m and λ? By suitable substitution, this relationship can be rewritten

$$\alpha x - \beta H = \mathrm{L}(x)$$

Find expressions for α and β. Demonstrate using graphical methods that the Weiss model predicts permanent magnetization for no external field if the temperature is below the Curie temperature, T_c. Derive an expression for the Curie temperature. [10 marks]

(b) Show graphically that above the Curie temperature magnetization will only occur with an external field, H. Show, by solving equation 3, that when x is small,

$$M = \frac{n\mu_m^2 \, \mu_0 H}{3k(T - T_c)}$$

[6 marks]

(c) What are the two limitations of the Weiss model in explaining observed ferromagnetic behaviour? [4 marks]

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E1.5 Exam answers

1.(a) Substitute $\psi(x)$ into the Schrödinger equation

$$-\frac{h^2}{8\pi^2 m}\frac{d^2\psi}{dx^2}+V(x)\psi(x)=E\psi(x)$$

to give

$$\frac{h^2}{8\pi^2 m}Ak^2\sin kx+V(x)A\sin kx=EA\sin kx$$ [1]

As $\psi(x)$ is only nonzero for $0 < x < L$, where V is zero,

$$\frac{h^2}{8\pi^2 m}k^2=E$$ [1]

From the boundary conditions, $\psi(x)=0$, when x = L, which implies

$$\sin kL=0$$ [1]

and therefore

$$kL=n\pi$$ [1]

where n is the quantum number and is any integer not equal to zero. [1]

Hence the general solutions for k and E are

$$k=\frac{n\pi}{L},$$

$$E=\frac{n^2h^2}{8mL^2}$$ [2]

The energy difference between the two lowest states will be

$$\Delta E=\frac{(2^2-1^2)h^2}{8mL^2}=\frac{3h^2}{8mL^2}$$ [2]

Total [9]

(b) For the QWIP,

$$\Delta E=h\upsilon$$

$$=\frac{hc}{\lambda}$$ [1]

$$=\frac{3\pi h^2}{8mL^2}$$

and so [1]

$$L=\sqrt{\frac{3h\lambda}{8mc}}$$

$$=\sqrt{\frac{3\times 6.63\times 10^{-34}\times 10^{-5}}{8\times 9.11\times 10^{-31}\times 3\times 10^{8}}}$$ [2]

$$=3.0\,\text{nm}$$

[1]

Total [5]

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(c) The probability of finding an electron in the region from x to x+dx is given by $|\psi(x)|^2dx$. Hence as the probability of finding the electron somewhere must be 1

$$\int_{-\infty}^{+\infty} |\psi(x)|^2 dx = 1$$ [2]

The ground state is given by

$$\psi(x) = A\sin\frac{\pi x}{L}$$ [1]

and so the normalization condition becomes

$$\int_0^{+L} A^2 \sin^2\left(\frac{\pi x}{L}\right) dx = 1$$ [1]

Using $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

$$\int_0^{+L} \frac{A^2}{2}\left[1-\cos\left(\frac{2\pi x}{L}\right)\right] dx = 1$$

So

$$\frac{A^2}{2}\left[x - \frac{2\pi}{L}\sin\left(\frac{2\pi x}{L}\right)\right]_0^L = 1$$ [1]

and $A=\pm\sqrt{(2/L)}$ [1]

Total [6]

2. (a) p: (111); q ($\bar{1}11$) [2]

The angle of the planes to the vertical is given by:

$$\cos^{-1}\theta = \frac{\mathbf{a.b}}{|\mathbf{a}||\mathbf{b}|}$$ [1]

where **a** is (111) and **b** is (001)

$$\cos^{-1}\theta = \frac{1}{1.\sqrt{3}}$$ [1]

and so the angle is 54.74° [1]

Total [5]

(b) The second moment of inertia is given by

$$I = \iint_A y^2 da$$ [2]

with $da = wdy$, and so

$$I = w\int_{-t/2}^{+t/2} y^2 dy$$ [1]

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$$= w\left[\frac{y^3}{3}\right]_{-t/2}^{+t/2}$$ [2]

$$= \frac{wt^3}{12}$$

From

$$v = -\frac{P}{EI}\left(l\frac{x^2}{2} - \frac{x^3}{6}\right)$$

when $x = l$, [1]

$$v = -\frac{Pl^3}{3EI}$$ [2]

The spring constant is defined by

$$P = -kv$$ [1]

and so

$$k = \frac{3EI}{l^3}$$ [2]

Total [11]

(c) Hence $k = \frac{Et^3w}{4l^3}$ [1]

To determine a thickness t we use

$$t = \left(\frac{4k}{Ew}\right)^{\frac{1}{3}} l$$ [2]

with k = 1 N/m,

$$t = \left(\frac{4}{1.1\times10^{11}\times 50\times10^{-6}}\right)^{\frac{1}{3}} 1\times10^{-3}$$ [2]

$$= 8.99\,\mu\text{m}$$

Total [4]

3. (a) Electronic – due to the linear motion of the electron cloud about the nucleus induced by an external electric field. Highest frequency (UV to visible). [1]

Molecular – due to relative motion of atoms in a molecule with different charges induced by an external electric field. Lower frequency than electronic polarization (IR). [1]

Orientational – due to the alignment of molecules with a permanent dipole with the external electric field. Lowest frequency (RF) [1].

(i) Helium – electronic
(ii) Sodium chloride – electronic and molecular
(iii) Water – electronic, molecular and orientational [3]

Total [6]

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(b) ε_{rs} is the static, or DC, or zero-frequency value of the dielectric constant.
$\Delta\varepsilon$ is the change in the dielectric constant from zero frequency to frequencies above the orientational-polarization frequencies. τ is the relaxation time constant of the orientational polarization. [3]

Given

$$\varepsilon_r = \varepsilon_{rs} + \frac{\Delta\varepsilon}{1 + j\omega\tau}$$

$$\varepsilon_r = \varepsilon'_r - j\varepsilon''_r$$

$$= \varepsilon_{rs} + \frac{\Delta\varepsilon(1 - j\omega\tau)}{1 + \omega^2\tau^2}$$

and so

$$\varepsilon''_r = \frac{\omega\tau\Delta\varepsilon}{1 + \omega^2\tau^2} \qquad [1]$$

To find the maximum value,

$$\frac{d\varepsilon''_r}{d\omega} = 0$$

and so

$$\frac{\tau\Delta\varepsilon\left(1 + \omega^2\tau^2\right) - \omega\tau\Delta\varepsilon.2\omega\tau^2}{\left(1 + \omega^2\tau^2\right)^2} = 0 \qquad [1]$$

$$\left(1 + \omega^2\tau^2\right) - 2\omega^2\tau^2 = 0$$

which gives $\omega = 1/\tau$. [1]

Total [6]

(c) From $k = \dfrac{\omega}{c} n$

$$k' = \frac{\omega}{c} n'$$

$$k'' = \frac{\omega}{c} n'' = \frac{\omega\varepsilon''_r}{c.2n'} \qquad [2]$$

and so

$$\frac{k'}{k''} = \frac{2n'^2}{\varepsilon''_r} \qquad [1]$$

which gives

$$\frac{k'}{k''} = \frac{2\varepsilon'_r}{\varepsilon'_r} \qquad [1]$$

The real part of the dielectric constant is given by

$$\varepsilon'_r = \varepsilon_{rs} + \frac{\Delta\varepsilon}{1 + \omega^2\tau^2} \qquad [1]$$

and so at $\omega = 1/\tau$,

$$\varepsilon'_r = \varepsilon_{rs} + \frac{\Delta\varepsilon}{2} \qquad [1]$$

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with

$$\varepsilon_r'' = \frac{\Delta\varepsilon}{2}$$ [1]

which gives

$$\frac{k'}{k''} = \frac{4\varepsilon_{rs}}{\Delta\varepsilon} + 2$$ [1]

Total [8]

4. (a) A dielectric is **polarized** in the presence of an electric field, and **surface charges** will be induced on each surface of the dielectric. These surface charges are of **opposite** sign to charges on the capacitor plates and so will reduce the electric flux in the material. Additional charges will accumulate on the plates of the capacitor to **compensate** for the charge cancellation. Hence the **capacitance** of the capacitor **increases**, implying an increase in the dielectric constant.

The key parts of the explanation are

1. Induced surface charges [1]
2. Charge cancellation [1]
3. Additional charging of plates [1]
4. Capacitance increase [1]

Total [4]

(b) Energy of a capacitor is given by

$$E = \frac{1}{2}CV^2$$

$$= \frac{\varepsilon_0\varepsilon_r AV^2}{2x}$$ [1]

and the force is given by

$$F = -\frac{dE}{dx}$$

$$= -\frac{d}{dx}\left(\frac{\varepsilon_0\varepsilon_r AV^2}{2x}\right)$$ [2]

$$= \frac{\varepsilon_0\varepsilon_r AV^2}{2x^2}$$

which for $x = d$ gives

$$\sigma = \frac{F}{A}$$

$$= \frac{\varepsilon_0\varepsilon_r V^2}{2d^2}$$ [1]

This force is compressive. [1]

Total [5]

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Originally

$$C = \frac{\varepsilon_0 \varepsilon_r A}{d}$$

Finally,

$$C' = \frac{\varepsilon_0 \varepsilon_r A'}{d'}$$

Due to the compressive stress,

$$\begin{aligned} d' &= d - \Delta d \\ &= d(1-\varepsilon) \end{aligned} \qquad [1]$$

where

$$\frac{\Delta d}{d} = \varepsilon = \frac{\sigma}{E} \qquad [1]$$

and E is Young's modulus and

$$\begin{aligned} A' &= A(1+\nu\varepsilon)^2 \\ &\approx A(1+2\nu\varepsilon) \end{aligned} \qquad [2]$$

for $\varepsilon << 1$. The final capacitance is given by

$$C' = \frac{\varepsilon_0 \varepsilon_r A(1+2\nu\varepsilon)}{d(1-\varepsilon)} \qquad [1]$$

which for $\varepsilon << 1$ gives

$$\begin{aligned} C' &= \frac{\varepsilon_0 \varepsilon_r A(1+2\nu\varepsilon)(1+\varepsilon)}{d} \\ &= \frac{\varepsilon_0 \varepsilon_r A[1+(1+2\nu)\varepsilon]}{d} \end{aligned} \qquad [2]$$

The difference in capacitance is therefore given by

$$\Delta C = \frac{\varepsilon_0 \varepsilon_r A(1+2\nu)\varepsilon}{d} \qquad [1]$$

and

$$\frac{\Delta C}{C} = (1+2\nu)\frac{\sigma}{E} \qquad [1]$$

Substituting

$$\sigma = \frac{\varepsilon_0 \varepsilon_r V^2}{2d^2}$$

and solving for the given parameters

$$\frac{\Delta C}{C} = (1+1)\frac{8.85\times10^{-12}\times2\times100^2}{2\times\left(10^{-4}\right)^2\times10^9} \qquad [1]$$

which is $1.77\text{x}10^{-8}$. [1]

Total [11]

5. In

$$M = n\mu_m \mathrm{L}\left[\frac{\mu_0\mu_m(H+\lambda M)}{kT}\right]$$

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n is the number of magnetic dipoles per unit volume, [1]
μ_m is the magnetic dipole moment of each dipole, [1]
λ is the Weiss constant representing the strength of the internal magnetic field due to the overall magnetization (the mean field). [1]

Putting

$$x = \frac{\mu_0 \mu_m (H + \lambda M)}{kT}$$

we get

$$\alpha x - \beta H = \mathrm{L}(x)$$

with

$$\alpha = \frac{kT}{\lambda n \mu_0 \mu_m^2} \quad [1]$$

and

$$\beta = \frac{1}{\lambda n \mu_m} \quad [1]$$

With H = 0, we have

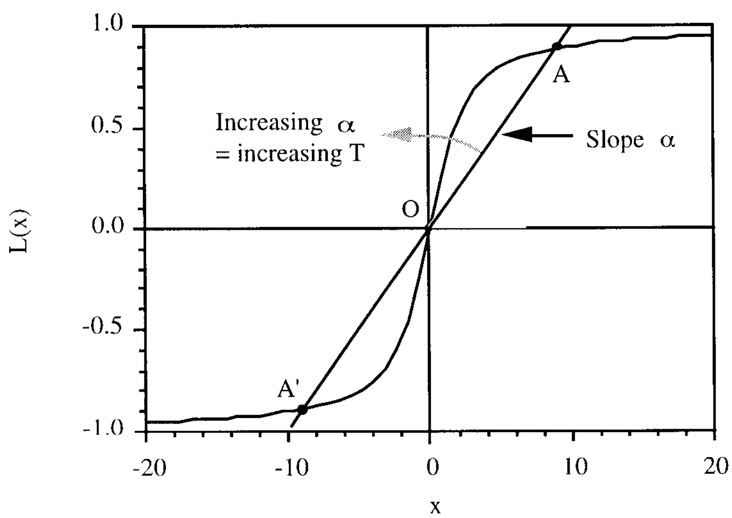
$$\alpha x = \mathrm{L}(x)$$

and the solution is given by the intersection of the two functions

$y_1 = \alpha x$, and

$y_2 = \mathrm{L}(x)$

which is plotted below:



[2]

The solutions are at the intersections A, A' and the origin. Only the solutions A and A' are stable. [1]

The slope of the Langevin function is 1/3. Therefore there will be no permament magnetization, with only a solution at the origin if α is more than x/3. Hence the transition will occur when

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$$\frac{kT}{\lambda n \mu_0 \mu_m^2} = \frac{1}{3}$$ [1]

and the Curie temperature is given by

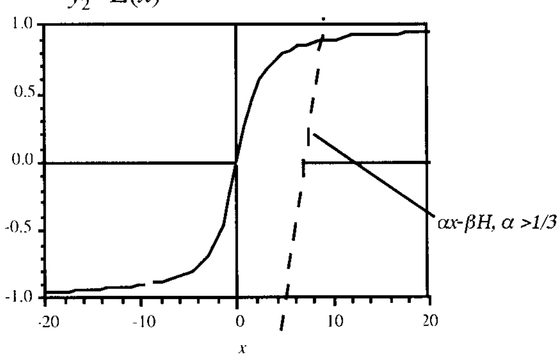
$$T_C = \frac{\lambda n \mu_0 \mu_m^2}{3k}$$ [1]

Total [10]

(b) Above the Curie temperature there will only be a solution at a non-zero magnetic field. The solution is now the intersection of

$y_1 = \alpha x - \beta H$ and

$y_2 = \mathrm{L}(x)$ [1]



[2]

From

$$M = n\mu_m \mathrm{L}\left[\frac{\mu_0 \mu_m (H + \lambda M)}{kT}\right]$$

When the argument of the Langevin function is small,

$$M = \frac{n\mu_0 \mu_m^2 \,(H + \lambda M)}{3kT}$$ [1]

and so

$$M = \frac{Hn\mu_0\mu_m^2}{3kT - \lambda n \mu_0 \mu_m^2}$$ [1]

giving the required expression

$$M = \frac{Hn\mu_0\mu_m^2}{3k(T - T_c)}$$ [1]

Total [6]

(c) Limitations of the Weiss model of ferromagnetism

(i) Unphysically high value for λ. The correlation of the dipole moments is not due to a mean field but is caused by quantum effects. [2]

(ii) Does not consider the effect of magnetic domains. Sudden change in magnetization predicted by the Weiss model is only seen in the smallest, single-domain particles. Similarly, unpoled ferromagnets often cancel out their external magnetic fields by aligning the poles of their magnetic domains in opposition. [2]

Total [4]