EE4-26
EE9-CS5-3
EE9-FPN2-07

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2016

MSc and EEE PART IV: MEng and ACGI

ESTIMATION AND FAULT DETECTION

Thursday, 19 May 10:00 am

Time allowed: 3:00 hours

Corrected Copy

There are FOUR questions on this paper.

Answer ALL questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : T. Parisini
Second Marker(s) : D. Angeli

Estimation and Fault Detection

Information for candidates:

- **One-step ahead Kalman predictor:**

$$\hat{x}(t+1|t) = F\hat{x}(t|t-1) + K(t)[y(t) - H\hat{x}(t|t-1)]$$

- **Kalman predictor gain**

$$K(t) = FP(t)H^\top \left(V_2 + HP(t)H^\top\right)^{-1}, \quad t = 1,2,\ldots$$

- **Riccati equation**

$$P(t+1) = F\left[P(t) - P(t)H^\top \left(V_2 + HP(t)H^\top\right)^{-1} HP(t)\right]F^\top + V_1, \quad t = 1,2,\ldots$$

- **Realization: observer canonical form**

  Given

$$Y(s)/U(s) = \frac{b_m s^m + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots a_1 s + a_0} \quad \text{with} \quad m < n$$

  then:

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & -a_{n-2} \\ 0 & \cdots & 0 & 1 & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_m \\ 0 \\ \vdots \\ 0 \end{bmatrix} u \\ \\ y = [0 \cdots 0\, 1] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} \end{cases}$$

1. Consider the following discrete-time dynamic system affected by state and output disturbances:

$$\begin{cases} x(t+1) = ax(t) + \xi(t) \\ \\ y(t) = bx(t) + \eta(t) \end{cases} \tag{1.1}$$

where $a, b \in \Re$; $\xi(\cdot) \sim WGN(0,4)$ and $\eta(\cdot) \sim WGN(0,1)$ are Gaussian zero-mean stochastic processes. Moreover, the stochastic processes $\xi(\cdot)$ and $\eta(\cdot)$ are supposed to be mutually independent. Finally, the initial state is a random variable $x(1) \sim WGN(1,4)$ which is assumed to be independent of $\xi(\cdot)$ and $\eta(\cdot)$.

a) For the system in (1.1), discuss the existence of the one-step ahead steady-state Kalman predictor, for any possible value of $a, b, \in \Re$.

[4 Marks]

b) Set $a = 1/3$, $b = 3$.

i) Consider the one-step ahead time-varying optimal Kalman predictor of the state x. Compute 3 values of the Riccati matrix $P(t)$, $t = 1,2,3$ and the corresponding values of the gain $K(t)$, $t = 1,2,3$. Establish whether the sequence of Riccati matrices $P(t)$ eventually converges to a positive-definite steady-state matrix $\overline{P}$ for increasing values of t.

[3 Marks]

ii) Write down the Algebraic Riccati Equation (ARE) and show that the ARE has an admissible solution $\overline{P}$. Compute the corresponding constant gain $\overline{K}$. Compare the time-behavior of the sequences $P(t)$, $t = 1,2,3$ and $K(t)$, $t = 1,2,3$, determined in your answer to Question 1b)-i), with the steady-state values $\overline{P}$ and $\overline{K}$. Comment on your findings.

[3 Marks]

iii) Let $e(t) = x(t) - \hat{x}(t|t-1)$ denote the state prediction error when the steady-state Kalman predictor (characterised by matrices $\overline{P}$ and $\overline{K}$ determined in your answer to Question 1b)ii)) is used. Show that the stochastic process $e(\cdot)$ is stationary and compute $\mathbb{E}(e)$ and $\mathrm{var}(e)$.

[3 Marks]

c) Set $a = 1/3$, $b = 0$.

i) Write down the ARE and show that it has an admissible solution $\widetilde{P}$. Compute $\widetilde{P}$, as well as the corresponding constant gain $\widetilde{K}$.

[3 Marks]

ii) Let $\widetilde{e}(t) = x(t) - \widetilde{x}(t|t-1)$ denote the optimal state prediction error of the one-step ahead steady-state predictor associated with the gain $\widetilde{K}$, which generates the state prediction $\widetilde{x}(t|t-1)$. Compute the variance $\mathrm{var}(\widetilde{e})$ and compare $\mathrm{var}(\widetilde{e})$ with $\mathrm{var}(e)$ calculated in your answer to Question 1b-iii). Comment on your findings.

[4 Marks]

2. Consider two electrical motors MS_1, MS_2 modelled as first-order transfer functions:

$$MS_1 : G_1(s) = \frac{1}{1+0.1s}; \quad MS_2 : G_2(s) = \frac{1}{1+0.2s}$$

Motors MS_1 and MS_2 are mechanically connected as shown in Fig. 2.1.

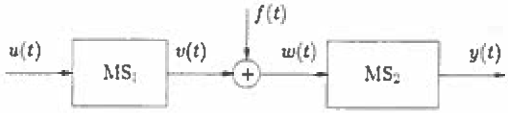

Figure 2.1 Block diagram for Question 2.

where $u(t)$ is a known input and $f(t)$ denotes a sinusoidal vibration due to a mechanical fault occurring at time T_0. Suppose that only one vibration fault may occur during the whole time-horizon $t \in (0, \infty)$. The vibration fault $f(t)$ takes the form:

$$f(t) = K\sin(t), \forall t \geq T_0, \tag{2.1}$$

where T_0 denotes the *unknown* time of fault occurrence and $K > 0$ is an *unknown* scalar.

a) Consider the system depicted in Fig. 2.1 for $t < T_0$.

i) Determine a state-space description of the whole dynamic system depicted in Fig. 2.1, where $u(t)$ is the input and $y(t)$ is the output.

[3 marks]

ii) Denote by $\hat{x}(t)$ the estimate of the state $x(t)$, within the state equations determined in your answer to Question 2a)-i), by $e(t) = x(t) - \hat{x}(t)$ the estimation error, and suppose that its dynamics obeys

$$\dot{e}(t) = Fe(t).$$

Design a full-order state observer such that the eigenvalues of F are $\lambda_1 = -5, \lambda_2 = -6$ and determine the expression of the residual

$$\varepsilon(t) = Ce(t), \forall t \in (0, T_0)$$

for a given value $\tilde{e}$ of the initial estimation error $e(0)$, where C is the output matrix determined in your answer to Question 2a)-i).

[6 marks]

b) Now, consider the presence of a vibration fault of the form given in Equation (2.1) for $t \geq T_0$.

i) Modify the state equations determined in your answer to Question 2a)-i) and the structure of the observer designed in your answer to Question 2a)-ii) so as to be able to construct a fault detection scheme based on the instantaneous estimate $\widehat{f}(t)$ of the vibration fault $f(t)$ for $t \geq T_0$. *Please do not attempt to design the fault estimator.*

[6 marks]

ii) Prove that the unknown amplitude K of the fault modelled in equation (2.1) can be determined from the knowledge of the output function $y(t), t \geq T_0$. Justify your answer.

[5 marks]

3. Consider a vehicle M moving on a straight line in the plane $x-y$ at a constant velocity denoted by the vector $v \neq 0$ (see Fig. 3.1). The symbols r_x and r_y denote the positions of the vehicle with respect to the x axis and the y axis, respectively, and ϑ denotes the constant angle formed by the direction of movement and the x-axis. Sensors are available to provide measurements of the positions $r_x(t)$ and $r_y(t)$ at every time-instant $t \geq 0$. It is assumed that $r_x(t) > 0$, $\forall t \geq 0$ and $r_y(t) > 0$, $\forall t \geq 0$.

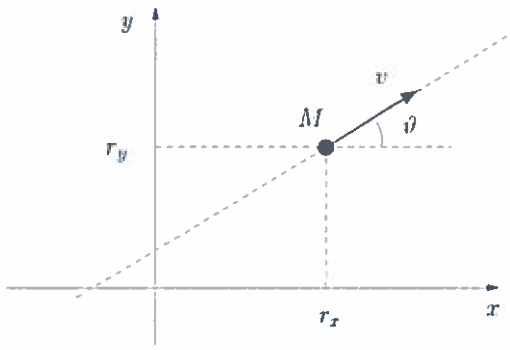

Figure 3.1 Scheme of vehicle movement for Question 3.

a) Determine a state-space description of the movement of the vehicle M on the $x-y$ plane taking into account that no inputs are acting on the vehicle and that two outputs, denoted by w_1 and w_2, have to be introduced that coincide with the respective position measurements, that is, $w_1(t) = r_x(t)$ and $w_2(t) = r_y(t)$.

[5 marks]

b) Devise an observer-based architecture to estimate the state of the system according to the state equations determined in your answer to Question 3a), and to also *simultaneously* estimate the angle ϑ. Verify that it is possible to design such an observer. *Please do not attempt to design the observer.*

[7 marks]

c) Denoting by $\hat{z}(t)$ the estimate of the state $z(t)$ defined in your answer to Question 3a), let $e_z(t) = z(t) - \hat{z}(t)$ denote the state estimation error. Suppose that the dynamics of the error e_z obeys

$$\dot{e}_z(t) = Fe_z(t).$$

In the context of the observer-based architecture determined in your answer to Question 3b), design a full-order observer such that the eigenvalues of F are:

$$\lambda_1 = -2,\ \lambda_2 = -2,\ \lambda_3 = -3,\ \lambda_4 = -3$$

[8 marks]

4. Consider the continuous-time dynamic system depicted in Fig. 4.1.

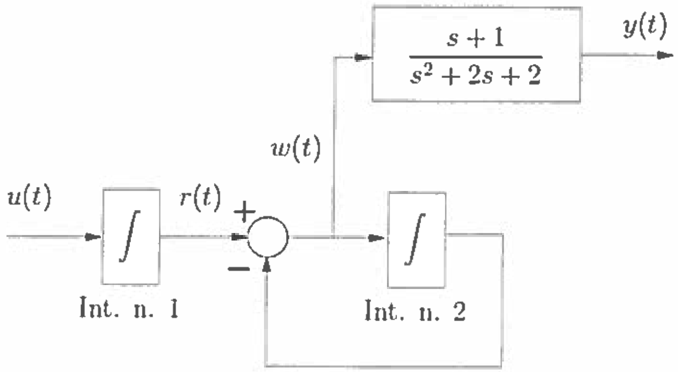

Figure 4.1 Block diagram for Question 4.

The system depicted in Fig. 4.1 is composed by the connection of three blocks: two integrators denoted by "Int. n. 1" and "Int. n. 2", and a third block with transfer function $\frac{s+1}{s^2+2s+2}$.

a) Define a state vector for each of the three blocks in Fig. 4.1 and devise a state-space description of the whole interconnected system where $u(t)$ is the input and $y(t)$ is the output.

[3 marks]

b) Determine the transfer function $G_{uw}(s)$ between the input $u(t)$ and the intermediate variable $w(t)$ and the transfer function $G_{uy}(s)$ between the input $u(t)$ and the output $y(t)$. Compare the order of the transfer function $G_{uy}(s)$ with the order of the state space realisation determined in your answer to Question 4a). Comment on your findings.

[3 marks]

c) Analyse the observability of the whole system in Fig. 4.1 from the output $y(t)$.

[3 marks]

d) Determine a state-space description of the dynamic system depicted in Fig. 4.1, which is equivalent to that determined in your answer to Question 4a) and that identifies the observable and the non-observable sub-systems (if any). Moreover, determine a basis for the non-observable vector subspace X_{no}.

[6 marks]

e) Denote by $x_1(0^-)$ the component of the initial state $x(0^-)$ of the state-space description, determined in your answer to Question 4a), corresponding to the block denoted by "Int. n. 1". Explain why it is not possible to determine $x_1(0^-)$ by using the output function $y(t)$, $t \geq 0$. Justify your answer.

[5 marks]

