IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2007

EEE/ISE PART II: MEng, BEng and ACGI

CORRECTED COPY

SIGNALS AND LINEAR SYSTEMS

Tuesday, 22 May 2:00 pm

Time allowed: 2:00 hours

There are FOUR questions on this paper.

Q1 is compulsory.
Answer Q1 and any two of questions 2-4.
Q1 carries 40% of the marks. Questions 2 to 4 carry equal marks (30% each).

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : P.T. Stathaki, P.T. Stathaki
Second Marker(s) : A.G. Constantinides, A.G. Constantinides

1.

Consider the discrete-time system with the following input-output relationship

$$y[n] = 2x[n] - x[n+1] - x[n-1]. \qquad (1)$$

(i) Is this system linear and time invariant (LTI)? Justify your answer.

[5]

(ii) Find the impulse response $h[n]$ of the system and express it compactly in a mathematical form. Sketch the impulse response.

[5]

(iii) Find the step response $s[n]$ of the system and express it compactly in a mathematical form. Sketch the step response.

[5]

(iv) By performing the discrete time convolution $y[n] = x[n] * h[n]$ find the output $y[n]$ of the system defined in equation (1), when the input is the following signal

$$x[n] = \begin{cases} 1, & n = 0,1 \\ 0, & \text{otherwise.} \end{cases}$$

[5]

(v) Consider a discrete signal $x[n]$ with Discrete Time Fourier Transform (DTFT) $X(e^{j\omega})$. Find the time signal with Discrete Time Fourier Transform $j\dfrac{dX(e^{j\omega})}{d\omega}$.

[5]

(vi) Find the frequency response of the system defined in equation (1). Find the amplitude and the phase of the frequency response.

[5]

(vii) Consider a discrete signal $x[n]$ with z-transform $X(z)$. Find the z-Transform of the signal $x[n-n_0]$ with n_0 any integer.

[5]

(viii) Find the z-Transform of the output $y[n]$ of the system defined in equation (1) above, when the input is the function $x[n]$ defined in (iv).

[5]

2.

(a) Consider a discrete signal $x[n]$ that is periodic with fundamental period N and Fourier Series coefficients c_k. Find the Fourier Series coefficients of the signal $y[n] = x[n] - x[n-1]$ as functions of the Fourier Series coefficients c_k of the signal $x[n]$.

[7]

(b) Let $x[n]$ be a discrete periodic signal with fundamental period $N = 10$ and Fourier Series coefficients c_k. Let

$$x[n] = \begin{cases} 1, & 0 \le n \le 7 \\ 0, & 8 \le n \le 9 \end{cases}$$

Also, let

$$y[n] = x[n] - x[n-1].$$

(i) Show that $y[n]$ has a fundamental period of 10.

[4]

(ii) Determine the Fourier Series coefficients of $y[n]$ from its samples.

[8]

(iii) Using the Fourier Series coefficients of $y[n]$ and the result of part (a) above, determine the Fourier Series coefficients c_k of $x[n]$, for $k \neq 0$. Determine c_0 separately from the samples of $x[n]$.

[11]

3.

(a) Consider a continuous time signal $x(t)$ which is sampled uniformly with sampling period T_s to obtain the signal $x_s(t) = x(t)\sum_{k=-\infty}^{+\infty}\delta(t-kT_s)$, where $\delta(t)$ is the continuous time impulse function. Find the Fourier transform of the sampled signal $x_s(t)$.

[Hint: Use the Fourier Series representation of the function $\sum_{k=-\infty}^{+\infty}\delta(t-kT_s)$.]

[10]

(b) Consider a continuous time signal $x(t)$ with Fourier transform $X(\omega) = \Pi(\frac{\omega}{4\pi \times 10^3})$ where ω is the angular frequency and $\Pi(\omega)$ is defined as:

$$\Pi(\omega) = \begin{cases} 1 & |\omega| \le 0.5 \\ 0 & \text{otherwise} \end{cases}$$

We sample $x(t)$ uniformly with sampling period T_s to obtain the signal $x_s(t) = x(t)\sum_{k=-\infty}^{+\infty}\delta(t-kT_s)$.

(i) Sketch the Fourier transform of $x_s(t)$, $X_s(\omega)$, assuming $T_s = 0.1$ ms.

[10]

(ii) Sketch the Fourier transform of $x_s(t)$, $X_s(\omega)$, assuming $T_s = \frac{4}{7}$ ms. Comment on the result.

[10]

4.

(a) (i) Find the analytical expression and the region of convergence (ROC) of the z-Transform of the discrete signal $x[n] = a^n u[n+1]$, with a real and $u[n]$ the discrete unit step function defined as

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

[6]

(ii) Find the analytical expression and the region of convergence (ROC) of the z-Transform of the discrete signal $x[n] = -a^n u[-n-2]$, with a real and $u[n]$ the discrete unit step function.

[6]

For parts (a) (i)-(a) (ii) you may wish to use the relationship $\sum_{n=0}^{+\infty} x^n = \frac{1}{1-x}$, if $|x| < 1$.

(b) Consider a LTI system with input $x[n]$ and output $y[n]$ related by the difference equation

$$y[n] - \frac{9}{2}y[n-1] + 2y[n-2] = -7x[n]$$

Determine the impulse response and its z-Transform in the following three cases:

(i) The system is causal.

(ii) The system is stable.

(iii) The system is neither stable nor causal.

Find the ROC of the z-Transform in each of the above cases.

[18]

E2.5 SIGNALS + LINEAR SYSTEMS - SOLUTIONS - 2007

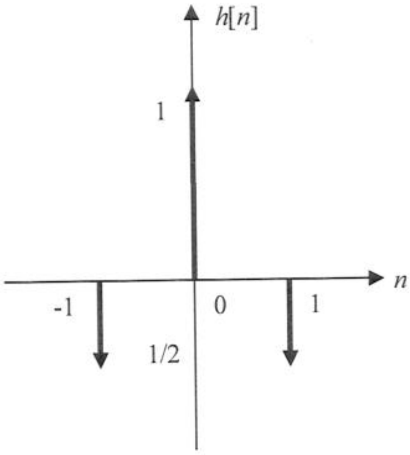
1.
Consider the discrete-time system with the following input-output relationship

$$y[n] = 2x[n] - x[n+1] - x[n-1]. \qquad (1)$$

(i) Yes, since if the inputs $x_1[n]$ and $x_2[n]$ produce the outputs $y_1[n]$ and $y_2[n]$ respectively, the input $a_1x_1[n]+a_2x_2[n]$ will produce the output $a_1y_1[n]+a_2y_2[n]$. Furthermore, if the input $x_1[n]$ produces the output $y_1[n]$, the input $x_1[n-n_0]$ will produce the output $y_1[n-n_0]$.

[5]

(ii) The impulse response of the system $h[n]$ is defined as the output of the system when the input is the impulse function $\delta[n]$. Therefore, $h[n]=\dfrac{2\delta[n]-\delta[n+1]-\delta[n-1]}{2}$. This function is shown below:



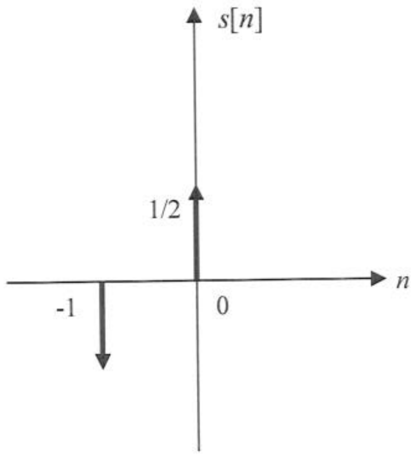
[5]

(iii) Find the step response $s[n]$ of the system and express it compactly in a mathematical form. Sketch the step response. The step response of the system $s[n]$ is defined as the output of the system when the input is the unit step function $u[n]$. Therefore, $s[n]=\dfrac{2u[n]-u[n+1]-u[n-1]}{2}$. This function is shown below:

$$s[-1]=\frac{2u[-1]-u[0]-u[-2]}{2}=-\frac{1}{2}$$

$$s[0]=\frac{2u[0]-u[1]-u[-1]}{2}=\frac{1}{2}$$

$s[n]=0, n\geq 1$



[5]

(iv) $y[n]=x[n]*h[n]=x[n]*\frac{2\delta[n]-\delta[n+1]-\delta[n-1]}{2}=\frac{2x[n]-x[n+1]-x[n-1]}{2}$

$$y[-1]=\frac{2x[-1]-x[0]-x[-2]}{2}=-\frac{1}{2}$$

$$y[0]=\frac{2x[0]-x[1]-x[-1]}{2}=\frac{1}{2}$$

$$y[2]=\frac{2x[2]-x[3]-x[1]}{2}=-\frac{1}{2}$$

[5]

(v) $X(e^{j\omega})=\sum_{n=-\infty}^{\infty}x[n]e^{-j\omega n}\Rightarrow j\frac{dX(e^{j\omega})}{d\omega}=j\sum_{n=-\infty}^{\infty}(-jn)x[n]e^{-j\omega n}\Rightarrow j\frac{dX(e^{j\omega})}{d\omega}=\sum_{n=-\infty}^{\infty}nx[n]e^{-j\omega n}$.

Therefore, the signal with Discrete Time Fourier Transform $j\frac{dX(e^{j\omega})}{d\omega}$ is the signal $nx[n]$.

[5]

(vi) $Y(e^{j\omega})=2X(e^{j\omega})-e^{j\omega}X(e^{j\omega})-e^{-j\omega}X(e^{j\omega})$. The frequency response is $H(e^{j\omega})=2-e^{j\omega}-e^{-j\omega}=2-2\cos\omega=2(1-\cos\omega)$. The amplitude response is the same and the phase response is zero

[5]

(vii) $\sum_{n=-\infty}^{\infty}x[n-n_0]z^{-n}=\sum_{n=-\infty}^{\infty}x[n-n_0]z^{-(n-n_0)}z^{-n_0}=z^{-n_0}X(z)$

[5]

(viii) $Y(z)=(1+z^{-1})\frac{2-z-z^{-1}}{2}$

[5]

2.

(a) $x[n] = \sum_{k=\langle N\rangle} c_k e^{jk\omega_0 n}$ and $x[n-1] = \sum_{k=\langle N\rangle} c_k e^{jk\omega_0 (n-1)} = \sum_{k=\langle N\rangle} e^{-jk\omega_0} c_k e^{jk\omega_0 n}$.

Therefore, $x[n]-x[n-1] = \sum_{k=\langle N\rangle} (1-e^{-jk\omega_0}) c_k e^{jk\omega_0 n}$ and thus, the FS coefficients of the signal $x[n]-x[n-1]$ are $(1-e^{-jk\omega_0})c_k$.

[7]

(b)

(i) $x[n]$ is periodic with period 10 and therefore $x[n-1]$ is periodic with period 10. Hence, $y[n] = x[n]-x[n-1]$ will be periodic with period 10.

[4]

(ii)
$$y[0] = x[0]-x[-1] = x[0]-x[9] = 1$$
$$y[n] = x[n]-x[n-1] = 0,\ 1 \le n \le 7$$
$$y[8] = x[8]-x[7] = -1$$
$$y[9] = x[9]-x[8] = 0$$
$$c_k^y = \frac{1}{N}\sum_{n=\langle N\rangle} y[n] e^{-jk\omega_0 n} = \frac{1}{10}(1\cdot e^{-jk\omega_0 0} - e^{-jk\omega_0 8}) = \frac{1}{10}(1-e^{-jk\frac{2\pi}{10}8})$$

[8]

(iii)
$$c_k^y = \frac{1}{10}(1-e^{-jk\frac{2\pi}{10}8}) = (1-e^{-jk\frac{2\pi}{10}})c_k^x \Rightarrow c_k^x = \frac{1-e^{-jk\frac{2\pi}{10}8}}{1-e^{-jk\frac{2\pi}{10}}}$$
$$c_0^x = \frac{1}{N}\sum_{n=\langle N\rangle} x[n] = \frac{8}{10}$$

[11]

3.

(a) The function $\sum_{k=-\infty}^{+\infty}\delta(t-kT)$ is periodic with period T and therefore, it can be written using Fourier Series representation as $\sum_{k=-\infty}^{+\infty}\delta(t-kT)=\sum_{k=-\infty}^{+\infty}c_k e^{jk\omega_s t}$, $\omega_s=\frac{2\pi}{T}$ with $c_k=\frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}}\delta(t)e^{-jk\omega_s t}dt=\frac{1}{T}$. Therefore, $\sum_{k=-\infty}^{+\infty}\delta(t-kT)=\sum_{k=-\infty}^{+\infty}\frac{1}{T}e^{jk\omega_s t}=\frac{1}{T}\sum_{k=-\infty}^{+\infty}e^{-jk\omega_s t}$, $\omega_s=\frac{2\pi}{T}$.

Hence, $x_s(t)=x(t)\sum_{k=-\infty}^{+\infty}\delta(t-kT_s)=\frac{1}{T}x(t)\sum_{k=-\infty}^{+\infty}e^{-jk\omega_s t}$. The Fourier transform of $x(t)e^{-jk\omega_s t}$ is $X(j\omega+jk\omega_s)$ and therefore, $X_s(j\omega)=\frac{1}{T}\sum_{k=-\infty}^{k=+\infty}X(j\omega+jk\omega_s)$

[10]

(b)

(i) In that case $\omega_s=2\pi f_s=2\pi\frac{1}{T_s}=10\cdot 2\pi 10^3$ and the Fourier transform $X_s(\omega)$ is given in Figure 2 below. **[10]**

(ii) In that case $\omega_s=2\pi f_s=2\pi\frac{1}{T_s}=2\pi\frac{7}{4}10^3=2\cdot 2\pi 10^3-\frac{1}{4}2\pi 10^3$ and the Fourier transform $X_s(\omega)$ is given in Figure 3 below. As we see there is aliasing since the sampling frequency does not satisfy the Nyquist criterion according to which $\omega_s=2\pi f_s\geq 2(2\pi\times 10^3)\Rightarrow f_s\geq 2\times 10^3\Rightarrow T_s=\frac{1}{f_s}\leq 0.5\times 10^{-3}$ **[10]**

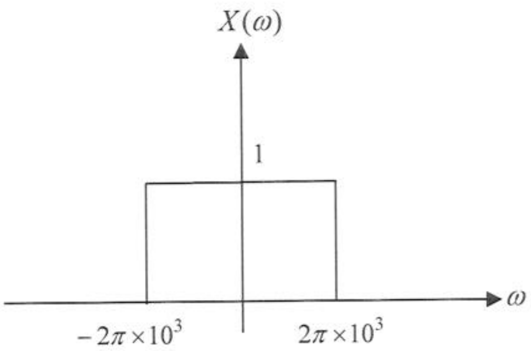


Figure 1: Fourier transform $X(\omega)$

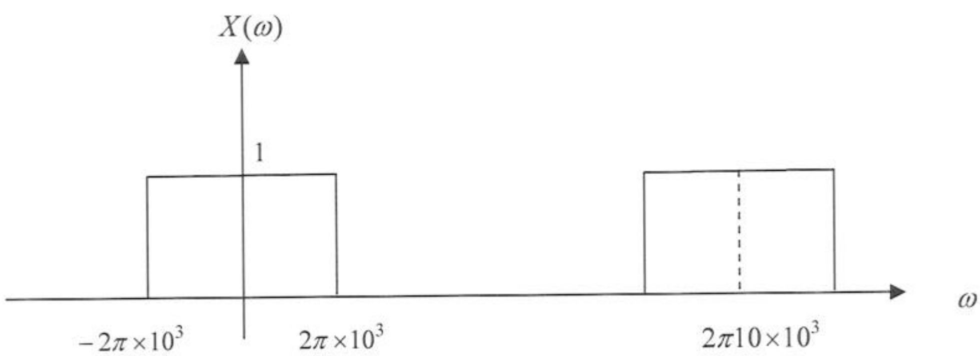


Figure 2: Fourier transform $X_s(\omega)$ for $T_s=0.1$ ms

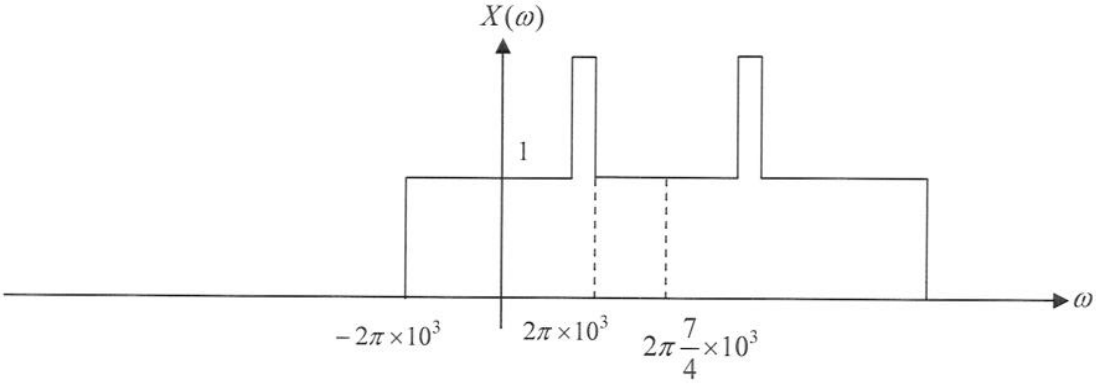


Figure 3: Fourier transform $X_s(\omega)$ for $T_s = \frac{4}{7}$ ms

[5]

4.

(a)

(i)

$$u[n+1]=\begin{cases}1, & n+1\geq 0 \Rightarrow n\geq -1\\ 0, & \text{otherwise.}\end{cases}$$

$$X(z)=\sum_{n=-1}^{\infty}a^n z^{-n}=a^{-1}z+\sum_{n=0}^{\infty}a^n z^{-n}=a^{-1}z+\sum_{n=0}^{\infty}(az^{-1})^n=a^{-1}z+\frac{1}{1-az^{-1}}=\frac{z}{a}\frac{z}{z-a},|a|<|z|$$

[6]

(ii)

$$u[-n-2]=\begin{cases}1, & -n-2\geq 0 \Rightarrow n\leq -2\\ 0, & \text{otherwise.}\end{cases}$$

$$X(z)=-\sum_{n=-\infty}^{-2}a^n z^{-n}=-\sum_{n=2}^{\infty}a^{-n}z^n=-\sum_{n=0}^{\infty}a^{-n}z^n+1+a^{-1}z=-\sum_{n=0}^{\infty}(a^{-1}z)^n+1+a^{-1}z$$

$$=\frac{-1}{1-a^{-1}z}+1+a^{-1}z=\frac{z}{a}\frac{z}{z-a},|z|<|a|$$

[6]

(b)

$$Y(z)[1-\frac{9}{2}z^{-1}+2z^{-2}]=-7X(z)\Rightarrow H(z)=\frac{-7}{(1-\frac{1}{2}z^{-1})(1-4z^{-1})}=\frac{1}{(1-\frac{1}{2}z^{-1})}-\frac{8}{(1-4z^{-1})}$$

$$=\frac{z}{(z-\frac{1}{2})}-\frac{8z}{(z-4)}=\frac{z}{(z-\frac{1}{2})}-8\frac{z}{(z-4)}$$

(i) The system is causal.

$$h[n]=(\frac{1}{2})^n u[n]-8(4)^n u[n],|z|>4$$

(ii) The system is stable.

$$h[n]=(\frac{1}{2})^n u[n]+8(4)^n u[-n-1],|z|>\frac{1}{2}\cap|z|<4$$

(iii) The system is neither stable nor causal.

$$h[n]=-(\frac{1}{2})^n u[-n-1]+8(4)^n u[-n-1],|z|<\frac{1}{2}$$

[18]