EE3-10
EE9-CS3-1

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2016

MSc and EEE/EIE PART III/IV: MEng, BEng and ACGI

MATHEMATICS FOR SIGNALS AND SYSTEMS

Corrected copy

Monday, 12 December 9:00 am

Time allowed: 3:00 hours

10-53 am. Examiner
Put on notice board.
Question 3C - Delete
last Sentence.

There are THREE questions on this paper.

Answer ALL questions. All questions carry equal marks.

NO Calculators allowed

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : P.T. Stathaki
Second Marker(s) : A. Gyorgy

1. a) By using row reduction on the augmented matrix choose a, b such that the system below

$$x + ay = 2$$
$$4x + 8y = b$$

has
(i) a unique solution; **[1]**
(ii) multiple solutions; **[1]**
(iii) no solution. **[1]**

b) Three linear systems $Ax = \underline{d}$, $Bx = \underline{e}$, $Cx = \underline{f}$ have the augmented matrices

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

respectively, given in reduced row echelon form.
(i) Which of the matrices A, B, C are invertible? **[2]**
(ii) How many solutions does each system have? **[3]**
(iii) Find the rank of each matrix A, B, C. **[2]**

c) Let V be the subspace of $\Re^4$ consisting of all solutions to the system of equations written in matrix form as

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Let W be the orthogonal complement of V. Find row vectors $\underline{v} \in V$ and $\underline{w} \in W$ so that $\underline{v} + \underline{w} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$. (Hint: Use the relationship $\underline{v} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} - \underline{w}$.) **[6]**

d) If A is an invertible matrix of dimension 4×4, describe all vectors in the null space of the 4×8 matrix $B = \begin{bmatrix} A & A \end{bmatrix}$. **[4]**

2. a) Consider a matrix Q which is symmetric and orthogonal. Show that each eigenvalue is equal to 1 or -1. **[2]**

b) Consider a 3×3 matrix Q which is symmetric and orthogonal. Mark each statement (i)-(v) **True** or **False**. Justify your answer.

(i) Q does not have repeated eigenvalues. **[2]**

(ii) Q is always positive definite. **[1]**

(iii) Q is diagonalizable. **[1]**

(iv) Q is non-singular. **[1]**

(v) The matrix $P=\frac{1}{2}(Q+I)$ is a projection matrix, with I being the identity matrix.

[3]

c) Find the projection matrix onto the subspace that is formed by the vectors $\begin{bmatrix}1 & 2 & 1 & 1\end{bmatrix}^T$ and $\begin{bmatrix}-1 & 1 & 0 & 1\end{bmatrix}^T$. Use it to compute the projection of the vector $\begin{bmatrix}1 & 1 & 2 & 1\end{bmatrix}^T$ onto that subspace. **[3]**

d) Consider the given matrix

$$A=\begin{bmatrix}4 & -3 & 3\\ 3 & -2 & 3\\ 3 & -3 & 4\end{bmatrix}$$

with eigenvectors

$$v_1=\begin{bmatrix}\frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{3}}\end{bmatrix},\ v_2=\begin{bmatrix}\frac{1}{\sqrt{2}}\\ 0\\ -\frac{1}{\sqrt{2}}\end{bmatrix},\ v_3=\begin{bmatrix}0\\ \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix}.$$

The three eigenvectors are ordered in descending order of magnitude of their associated eigenvalues, i.e. the first eigenvector corresponds to the largest eigenvalue.
Give the diagonalized form of two distinct matrices B,C such that $B^2=A=C^2$. You do not need to calculate the inverse matrix involved in the diagonalization. **[4]**

e) Suppose that the matrices A,B of dimension $m\times n$ have the same four subspaces. If they are both in reduced row echelon form,

$$A=\begin{bmatrix}I & F\\ 0 & 0\end{bmatrix},\ B=\begin{bmatrix}I & G\\ 0 & 0\end{bmatrix}$$

prove that F must equal G.

[3]

3. a) Consider a matrix A of dimension 3×3 with eigenvalues $\lambda_1 = 0, \lambda_2 = c, \lambda_3 = 2$ where c is a real scalar, and associated eigenvectors

$$v_1 = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix},\ v_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0\end{bmatrix},\ v_3 = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\1\\-2\end{bmatrix}$$

(i) For which values of c is matrix A diagonalizable? [2]
(ii) For which values of c is matrix A symmetric? [2]
(iii) For which values of c is matrix A positive definite? [1]
(iv) For which values of c is matrix A positive semi-definite? [1]
(v) For which values of c is A a Markov matrix? [2]
(vi) Could $P = \frac{1}{2}A$ be a projection matrix? Justify your answer. [2]

b) Consider the Singular Value Decomposition of the matrix $A = U\Sigma V^T$. Suppose $\Sigma = \begin{bmatrix}3 & 0\\0 & 2\end{bmatrix}$ and U and V each have two columns.

(i) What can we say about the dimension and rank of matrix A? [2]
(ii) Show that if $\Sigma = \begin{bmatrix}3 & 0\\0 & 0\end{bmatrix}$, a basis for the null space of A is the second column of V. [2]

c) Consider the matrix A

$$A = \begin{bmatrix}3 & 2 & 2\\2 & 3 & -2\end{bmatrix}$$

Find the Singular Value Decomposition of matrix $A = U\Sigma V^T$. It is given that a set of orthogonal but not orthonormal eigenvectors of A^TA is $\begin{bmatrix}1\\-1\\4\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\-1\\-1/2\end{bmatrix}$. The given order corresponds to the order of the eigenvalues of A^TA according to their magnitude, with the first being the eigenvector that corresponds to the largest eigenvalue. [6]