Imperial College London

BSc/MSci EXAMINATION May 2012

This paper is also taken for the relevant Examination for the Associateship

STATISTICAL MECHANICS

For Third-Year and Fourth-Year Physics Students

Thursday, May 24, 2012: 10.00 to 12.00

Answer ALL questions in Section A. This section carries 20 marks.
Answer TWO questions in Section B. Each question carries 15 marks.

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the FOUR answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in FOUR answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

SECTION A

1. Consider site percolation in one dimension with site occupation probability p. The cluster distribution $n(s,p)$ gives the density of clusters with s adjacent occupied sites.

(i) Show that $n(s,p)$ can be written in the form

$$n(s,p) = (1-p)^2 e^{-s/s_\xi}$$

where $s_\xi(p) = +1/|\ln p|$. [3 marks]

(ii) Give a physical interpretation for $s_\xi(p)$, and explain the behaviour of $s_\xi(p)$ in the two limits of $p \to 0^+$ (approaching zero from above) and $p \to 1^-$ (approaching unity from below). [2 marks]

(iii) Show that, near the percolation threshold p_c, the form for $s_\xi(p)$ is consistent with the power-law form:

$$s_\xi \sim \frac{1}{(p_c - p)^{1/\sigma}}\,.$$

Find the values for p_c and σ. [2 marks]

[Total 7 marks]

2. Consider an Ising ferromagnet with spins on a three-dimensional cubic lattice. The energy E of the system in an applied magnetic field H is given by

$$E = -J \sum_{\langle ij \rangle} s_i s_j - H \sum_i s_i \,. \tag{1}$$

(i) Explain the variable s_i of the Ising model. Give and explain the sign of J suitable for a ferromagnet. [2 marks]

For an Ising ferromagnet at temperature T in an applied field H, the Landau free energy density (per site) is given by

$$f_{\mathrm{L}}(m, T, H) \simeq f_0(T) + \frac{a}{2}(T - T_c)m^2 + \frac{b}{4}m^4 - mH$$

where $m = \langle s_i \rangle$ is the magnetisation, and T_c, a and b (all positive) are phenomenological parameters.

Consider first the ferromagnet in the absence of an applied magnetic field ($H = 0$).

(ii) Sketch the magnetisation (per site) of the Ising ferromagnet as a function of temperature. Indicate clearly the behaviour at low temperatures ($T \simeq 0$), near $T = T_c$, and at high temperatures ($T \gg T_c$). [2 marks]

(iii) Sketch the Landau free energy density f_{L} at $H = 0$ as a function of m for the two temperatures: $T > T_c$ and $T < T_c$. Hence, deduce the equilibrium magnetisation $m_0(T)$ as a function of temperature T, as predicted by Landau theory. [4 marks]

A magnetic field $H \neq 0$ is applied to the ferromagnet parallel to the spin axis.

(iv) Sketch the magnetisation per site, m, as a function of the applied field H in the range $-\infty < H < +\infty$ for *two* temperatures: $T > T_c$ and $T < T_c$. Indicate clearly the behaviour near zero field and at high fields.
Hint: make sure your answer here is consistent with your answer to part (ii).
[3 marks]

(v) Write down the self-consistent equation given by mean field theory for the magnetisation per site, m, at temperature T and in non-zero field H in a three-dimensional cubic lattice.
Give a *physical* picture to explain your equation and its relationship with the magnetisation of a *single spin* in a field H:

$$m = \tanh\left(\frac{H}{k_{\mathrm{B}}T}\right)$$

where H is expressed in the same units as in Eq. (1). [2 marks]

[Total 13 marks]

SECTION B

3. In the site percolation problem, the strength of the percolating cluster P_∞ is defined as the *fraction* of lattice sites belonging to the percolating cluster.

(i) Consider site percolation on a Bethe lattice with coordination number $z = 4$ and site occupation probability p. Define $Q(p)$ as the *probability* that a branch of the tree-like lattice is *not* connected to infinity *via* its sub-branches. It can be shown that:

$$P_\infty = p(1 - Q^4) \quad \text{with} \quad (Q-1)\left(Q^2 + Q + 1 - \frac{1}{p}\right) = 0. \qquad [\text{ DO NOT PROVE }]$$

Find possible solutions for Q and give a *physical* interpretation of these solutions in terms of percolation. Hence, show that the percolation threshold for a $z = 4$ Bethe lattice is $p_c = 1/3$. [5 marks]

(ii) Consider now site percolation on a cubic lattice of linear size L in three dimensions. For the infinite lattice, the percolation transition is a continuous phase transition. As $p \to p_c^+$, P_∞ has the power-law form:

$$P_\infty(p, L = \infty) \sim (p - p_c)^\beta \qquad \text{with } \beta = 0.418\,,$$

and the typical cluster length scale $\xi(p)$ diverges as

$$\xi(p) \sim |p - p_c|^{-\nu} \qquad \text{with } \nu = 0.877\,.$$

(a) Sketch $P_\infty(p, L)$ as a function of p in the range $0 \le p \le 1$ for three cases: $L = \infty$, L_1 and L_2, where L_1 and L_2 are finite and $L_1 > L_2$. Comment on the behaviour near p_c. [3 marks]

(b) Explain why we expect

$$P_\infty(p, L) = P_\infty(p, \infty) f(\xi/L)\,.$$

[3 marks]

(c) Hence show that, for large L ($\gg$ lattice spacing),

$$P_\infty(p = p_c, L) \propto \frac{1}{L^y}\,.$$

Give the numerical value for the exponent y. [4 marks]

[Total 15 marks]

4. In Ginzburg-Landau theory, the equation of state for a three-dimensional Ising ferromagnet describes the magnetisation $m(\mathbf{r})$ in response to a spatially varying external field $H(\mathbf{r})$:

$$-c\nabla^2 m + a(T - T_c)m + bm^3 = H(\mathbf{r}) \tag{1}$$

where a, b, c and T_c are phenomenological parameters.

(i) A *weak* periodic field with wavevector $\mathbf{q}$ is applied: $H(\mathbf{r}) = h_\mathbf{q} \cos(\mathbf{q}\cdot\mathbf{r})$ at a temperature T above T_c. This results in a periodic magnetisation: $m(\mathbf{r}) = \chi_\mathbf{q} h_\mathbf{q} \cos(\mathbf{q}\cdot\mathbf{r})$ to first order in $h_\mathbf{q}$.

(a) Show that

$$\chi_\mathbf{q} = \frac{1}{c|\mathbf{q}|^2 + a(T - T_c)} \qquad \text{for } T > T_c\,.$$

Give a criterion for the validity of this weak-field result in terms of the magnitude $h_\mathbf{q}$ of the applied field. [4 marks]

(b) Extract a temperature-dependent characteristic length scale $\xi(T)$ from the above expression for $\chi_\mathbf{q}$ and explain its physical significance. [3 marks]

In the absence of an applied field ($H = 0$), there are thermal fluctuations in the magnetisation $m(\mathbf{r})$ of the system at all wavelengths. The Landau free energy for the magnetisation $m(\mathbf{r})$ in zero field is given by:

$$F = \int d^3\mathbf{r} \left[\frac{c}{2}|\nabla m|^2 + \frac{a}{2}(T - T_c)m^2 + \frac{b}{4}m^4\right]. \tag{2}$$

(ii) Find the Ginzburg-Landau free energy $F(m_\mathbf{q})$ for a weak periodic magnetisation $m(\mathbf{r}) = m_\mathbf{q}\cos(\mathbf{q}\cdot\mathbf{r})$ in a system of volume V in zero field ($H = 0$). Give your answer to up to second order in $m_\mathbf{q}$.
You may assume that, ignoring boundary effects, $\int_V \cos^2(\mathbf{q}\cdot\mathbf{r})d^3\mathbf{r} = V/2$.
[3 marks]

A beam of spin-polarised neutrons can be scattered by the thermal fluctuations in the magnetisation. For a wave $m(\mathbf{r}) = m_\mathbf{q}\cos\mathbf{q}\cdot\mathbf{r}$, a neutron gains momentum $\hbar\mathbf{q}$ in the scattering process. The intensity $S(\mathbf{q})$ of the scattered beam is proportional to the mean square average $\langle m_\mathbf{q}^2\rangle$ of the wave magnitude.

(iii) Argue that the mean square average of the wave magnitude at wavevector $\mathbf{q}$ and in thermal equilibrium at temperature T is given by

$$\langle m_\mathbf{q}^2\rangle \propto \frac{k_\mathrm{B}T}{V}\chi_\mathbf{q}\,.$$

[2 marks]

(iv) Sketch the scattered intensity $S(\mathbf{q})$ of neutrons passing through the ferromagnet as a function of $|\mathbf{q}|$ at two temperatures: $T = 1.1T_c$ and $1.3T_c$. [3 marks]

[Total 15 marks]

5. Consider a ferromagnet near the Curie point at temperature T and in an external field H. The scaling hypothesis postulates that the singular part of the free energy density f_s obeys the scaling form

$$f_s(t,h) = b^{-d} f_s(b^{y_t} t, b^{y_h} h) \tag{1}$$

where d is the number of dimensions, $t \equiv (T - T_c)/T_c$ is the reduced temperature, $h \equiv H/(k_\mathrm{B} T_c)$ is a reduced field and T_c is the Curie temperature.

(i) In a renormalisation group (RG) transformation with scale factor b, the original system with parameters t and h becomes a coarse-grained system with new parameters t' and h'. According to the above scaling relation, how should t' and h' be related to t and h? [2 marks]

(ii) Consider the ferromagnet at precisely $T = T_c$. How does the magnetisation of this system respond to an external field H? Hence, describe qualitatively how the parameter h should behave under successive RG transformations for $h \neq 0$. Comment on the sign of the exponent y_h. [3 marks]

The heat capacity of the system is given by: $c_V = -T\partial^2 f/\partial T^2$, where f is the free energy density. This diverges as the system is cooled down in zero field from a temperature above T_c towards the Curie temperature T_c:

$$c_V(H=0) \sim t^{-\alpha} \qquad \text{as } t \to 0^+.$$

(iii) Assume that the total free energy is dominated by its singular part f_s. Use the scaling relation (1) to derive an analogous scaling relation between $c_V(t,h)$ and $c_V(b^{y_t} t, b^{y_h} h)$. [3 marks]

(iv) Hence, show that the exponent α is related to the exponents y_t by:

$$\alpha = 2 - \frac{d}{y_t}.$$

[3 marks]

(v) How does the critical behaviour of c_V differ if the system is heated up from a temperature T below T_c to $T = T_c$? [1 mark]

(vi) A small external field H is applied to this system near T_c. Sketch how the heat capacity c_V behaves as a function of temperature T near T_c at zero applied field and at a non-zero applied field H. How does your sketch vary with increasing H? [3 marks]

[Total 15 marks]

6. Consider the site percolation problem on a two-dimensional triangular lattice, with site occupation probability p. A real-space renormalisation group (RSRG) scheme for this model with linear scale factor $b = \sqrt{3}$ transforms p to p' as follows

$$p' \equiv R_b(p) = p^2(3-2p). \qquad \text{[DO NOT PROVE]}$$

(i) Identify the three fixed points of the transformation $R_b(p)$. [3 marks]

(ii) By considering how $R_b(p)$ behaves near $p = 0$ and $p = 1$, explain how the parameter p flows under successive RSRG steps in the regimes $p \simeq 0$ and $p \simeq 1$. [4 marks]

(iii) Sketch on the p-axis (in the range $0 \leq p \leq 1$) how the parameter p flows under successive RSRG steps for any initial value of p. What does this renormalisation flow mean in terms of the percolation behaviour of this problem? You should identify the percolation threshold as given by this RSRG scheme. [3 marks]

The cluster length scale $\xi(p)$ diverges as the system approaches a critical point:

$$\xi(p) \sim |p - p_c|^{-\nu} \quad \text{as } p \to p_c\,. \tag{1}$$

(iv) Explain why $\xi(p)$ obeys the relation: $\xi(R_b(p)) = \xi(p)/b$. [1 mark]

(v) Assuming that the power law (1) holds, show that the exponent ν in the divergence of the cluster length scale can be related to the properties of the RSRG transformation by

$$\nu = \frac{\ln b}{\ln|R'_b(p_c)|}$$

where $R'_b(p) \equiv dR_b/dp$. You may assume that $R_b(p)$ is smooth and differentiable near p_c. [3 marks]

(vi) Find the estimate of ν given by this RSRG scheme. [1 mark]

[Total 15 marks]