Probability and Stochastic Processes

Exam 2017 solutions

B—bookwork, E—new example, T—new theory

1.

a) We denote by

students did well

A: exactly one Ace, and KK: exactly two Kings

We want to calculate: $P(A|KK) = \dfrac{P(A \cap KK)}{P(KK)}$,

[2B]

where $P(A \cap KK) = \dfrac{\binom{4}{1}\binom{4}{2}\binom{44}{10}}{\binom{52}{13}} \approx 0.094$,

[3B]

and $P(KK) = \dfrac{\binom{4}{2}\binom{48}{11}}{\binom{52}{13}} \approx 0.214$.

[3B]

[2B]

So, $P(A|KK) \approx 0.44$

In general, students did well

b) Recall the definition of the characteristic function

$$\Phi_X(\omega) = E\left(e^{jX\omega}\right) = E\left[\sum_{k=0}^{\infty} \frac{(j\omega X)^k}{k!}\right] = \sum_{k=0}^{\infty} j^k \frac{E(X^k)}{k!}\omega^k$$

$$= 1 + jE(X)\omega + j^2 \frac{E(X^2)}{2!}\omega^2 + \cdots + j^k \frac{E(X^k)}{k!}\omega^k + \cdots.$$ [2E]

It is readily verified that

$$E(X^k) = \frac{1}{j^k} \frac{\partial^k \Phi_X(\omega)}{\partial \omega^k}\bigg|_{\omega=0}, \quad k \geq 1.$$

In particular, [2E]

$$E(X^4) = \frac{\partial^4 \Phi_X(\omega)}{\partial \omega^4}\bigg|_{\omega=0}$$

Now, let’s go step by step.

$$\frac{\partial \Phi_X(\omega)}{\partial \omega} = -\sigma^2 \omega e^{-\sigma^2\omega^2/2}$$ [1E]

$$\frac{\partial^2 \Phi_X(\omega)}{\partial \omega^2} = -\sigma^2 e^{-\sigma^2\omega^2/2} + \sigma^4\omega^2 e^{-\sigma^2\omega^2/2}$$ [1E]

$$\frac{\partial^3 \Phi_X(\omega)}{\partial \omega^3} = \sigma^4 \omega e^{-\sigma^2\omega^2/2} + 2\sigma^4 \omega e^{-\sigma^2\omega^2/2} - \sigma^6\omega^3 e^{-\sigma^2\omega^2/2}$$ [1E]

$$= 3\sigma^4 \omega e^{-\sigma^2\omega^2/2} - \sigma^6\omega^3 e^{-\sigma^2\omega^2/2}$$

$$\frac{\partial^4 \Phi_X(\omega)}{\partial \omega^4} = 3\sigma^4 e^{-\sigma^2\omega^2/2} - 3\sigma^6\omega^2 e^{-\sigma^2\omega^2/2} + \text{terms containing } \omega$$ [1E]

We only care the first term, since all others = 0 if ω = 0. Thus

$$E(X^4) = \left.\frac{\partial^4 \Phi_X(\omega)}{\partial \omega^4}\right|_{\omega=0} = 3\sigma^4$$ [2E]

c) The derivation is not unique. One approach is the following:

$$\Phi_Y(\omega) = E\{e^{j\omega X_1 X_2}\} = E\left\{E\{e^{j\omega X_1 X_2}|X_2\}\right\}$$ [1T]

Note that the inner expection is

$$\Phi_{X_1}(\omega X_2) = e^{-\sigma^2\omega^2 X_2^2/2}$$ [1T]

Now we average over X_2 to obtain

$$\Phi_Y(\omega) = E\left\{e^{-\frac{\sigma^2\omega^2 X_2^2}{2}}\right\} = \int e^{-\frac{\sigma^2\omega^2 x_2^2}{2}} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x_2^2}{2\sigma^2}}\ dx_2$$ [1T]

$$= \frac{1}{\sqrt{2\pi}\sigma}\int e^{-\frac{(1+\sigma^4\omega^2)x_2^2}{2\sigma^2}} dx_2$$

$$= \frac{\frac{\sigma}{\sqrt{1+\sigma^4\omega^2}}}{\sigma} \frac{1}{\sqrt{2\pi}\frac{\sigma}{\sqrt{1+\sigma^4\omega^2}}} \int e^{-\frac{(1+\sigma^4\omega^2)x_2^2}{2\sigma^2}} dx_2$$ [1T]

Notice that the integrand is a density, we have

$$= \frac{1}{\sqrt{1+\sigma^4\omega^2}}$$ [1T]

This is the tricky part; very few students did it right. Derivation is not unique; students got marks as long as the approach is valid.

2. Students did well.

a) The joint distribution of the samples is given by

$$f_X(x_1, \dots, x_n; \lambda) = \prod_{i=1}^{n} \frac{1}{\lambda} e^{-x_i/\lambda} = \frac{1}{\lambda^n} e^{-\sum x_i/\lambda}$$

Then the log-likelihood function is [3E]

$$\ln f_X(x_1, \dots, x_n; \lambda) = -n \ln \lambda - \sum x_i/\lambda$$

Now take the derivatives:

$$\frac{\partial}{\partial \lambda} \ln f_X(x_1, \dots, x_n; \lambda) = -\frac{n}{\lambda} + \frac{\sum x_i}{\lambda^2}$$

[3E]

$$\frac{\partial^2}{\partial \lambda^2} \ln f_X(x_1, \dots, x_n; \lambda) = \frac{n}{\lambda^2} - 2\frac{\sum x_i}{\lambda^3}$$

Then the Fisher information is given by

$$I = -E\left[\frac{n}{\lambda^2} - 2\frac{\sum x_i}{\lambda^3}\right] = -\frac{n}{\lambda^2} + \frac{2n\lambda}{\lambda^3} = \frac{n}{\lambda^2}$$ [2E]

So the Cramer-Rao bound is

$$\mathrm{Var}[\hat{\lambda}] = \frac{1}{I(\lambda)} = \frac{\lambda^2}{n} = \frac{100}{100} = 1$$ [2E]

b) i) Since $X_i, i = 1, \cdots, n$ are independent Bernoulli variables, $X = \sum_{i=1}^{n} X_i$ is Binomial with

$$X \sim B\left(n, \frac{1}{2}\right)$$

[1B]

$$E(X) = E\left(\sum_{i=1}^{n} X_i\right) = nE(X_1) = n/2$$ [2B]

$$\mathrm{Var}(X) = \mathrm{Var}\left(\sum_{i=1}^{n} X_i\right)$$

$= n\mathrm{Var}(X_1)$ (Independence)

[2B]

$= n/4$

Part i) is easy

Students did less well in Part ii). Some didn't use Chebyshev inequality, which lost mark.

ii) If we note that

$$\left\{\left|X-\frac{n}{2}\right|\geq\frac{n}{8}\right\}=\left\{X\geq\frac{5n}{8}\right\}\cup\left\{X\leq\frac{3n}{8}\right\}$$ [2E]

then

$$P\left(\left|X-\frac{n}{2}\right|\geq\frac{n}{8}\right)\geq P\left(X\geq\frac{5n}{8}\right)$$ [3E]

Thus

$$P\left(X\geq\frac{5n}{8}\right)\leq P\left(\left|X-\frac{n}{2}\right|\geq\frac{n}{8}\right)$$ [2E]

$$\leq\frac{\operatorname{var}(X)}{(n/8)^2}=\frac{16}{n}$$

[3E]

The answer is not unique. Due to symmetry, the following bound also holds:

$$P\left(X\geq\frac{5n}{8}\right)=\frac{1}{2}P\left(\left|X-\frac{n}{2}\right|\geq\frac{n}{8}\right)$$

$$\leq\frac{1}{2}\frac{\operatorname{var}(X)}{(n/8)^2}=\frac{8}{n}$$

3. It's a proof problem, which requires some mathematical maturity.

a) Firstly consider the mean

$E[X(n)] = E\left[e^{j(U-Vn)}\right] = E\left[e^{j(U)}\right]E\left[e^{j(-Vn)}\right] = 0$ [3E]

because U is uniform and independent of V.

Secondly consider the autocorrelation function

$$R_X(m) = E[X(n+m)X(n)^*] = E\left[e^{j(U-V(m+n))}e^{-j(U-Vn)}\right] \quad [3E]$$

$$= E\left[e^{j(-Vm)}\right] = \Phi_V(-m)$$

where Φ_V denotes the characteristic function of V:

$$\Phi_V(-m) = E[e^{-jVm}] = \int e^{-jVm} f_V(x)dx \quad [2E]$$

Hence it is wide-sense stationary.

Recall the Wiener-Khinchin relation

Power spectral density ⇔ autocorrelation function [2E]

via the Fourier transform. Since $f_V(\omega)$ is the inverse Fourier transform of $R_X(m)$, it is exactly the power spectral density.

b)

The system transfer function is given by

$$H(z) = \frac{1}{1-az^{-1}} = \sum_{n=0}^{\infty} a^n z^{-n} \quad [2B]$$

provided $|a| < 1$. Thus $h(n) = a^n, \quad |a| < 1$ [2B]

represents the impulse response of an AR(1) stable system.

We get the output autocorrelation sequence of an AR(1) process to be

$$R_y(n) = \delta(n) * \{a^{-n}\} * \{a^n\} = \sum_{k=0}^{\infty} a^{|n|+k} a^k = \frac{a^{|n|}}{1-a^2} \quad [3B]$$

ii) Recall the Wiender-Hopf equation

$$\mathbf{c} = \mathbf{R}^{-1}\mathbf{r}$$

Be carefully it's a 2-step ahead predictor

$$\begin{pmatrix} 1 & \alpha & \cdots & \alpha^{n-1} \\ \alpha & 1 & & \alpha^{n-2} \\ \vdots & & \ddots & \vdots \\ \alpha^{n-1} & & \cdots & \alpha \quad 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} \alpha^{n+1} \\ \alpha^n \\ \vdots \\ \alpha^2 \end{pmatrix}$$ [2E]

Whose solution is given by

[2E]

$$c_n = \alpha^2, \qquad c_k = 0, \qquad 1 \le k < n$$

The MMSE predictor turns out to be a one-tap predictor

$$\hat{Y}(n+2) = \alpha^2 Y(n)$$ [2E]

The mean-square error is

$$MSE = E\left[\left(Y(n+2) - \alpha^2 Y(n)\right)^2\right]$$

[2E]

$$= R_Y(0) - 2\alpha^2 R_Y(2) + \alpha^4 R_Y(0) = \frac{1-\alpha^4}{1-\alpha^2} = 1 + \alpha^2$$

4.

a) Just use the definition of martingale. However, some students forgot.

$$E[S_{n+1}|X_1, \dots, X_n] = E[\beta X_{n+1} + X_n|X_1, \dots, X_n]$$ [2E]

$$= \beta(aX_n + bX_{n-1}) + X_n = (\beta a + 1)X_n + \beta b X_{n-1}$$ [3E]

If it's a martingale, then

$$(\beta a + 1)X_n + \beta b X_{n-1} = S_n = \beta X_n + X_{n-1}$$ [3E]

Hence

$$\beta = \frac{1}{b} = \frac{1}{1-a}$$ [2E]

Part b is quite hard. Few students worked it out.

b)

i) Of the $2n$ steps, suppose the chain goes upward for i steps, leftward for j steps. It returns to the origin if and only it also goes downward for i steps, rightward for j steps. Here we must have $i + j = n$. [2E]

Therefore,

$$P\{X_{2n} = (0,0)\} = \left(\frac{1}{4}\right)^{2n} \sum_{i+j=n} \frac{(2n)!}{(i!\,j!)^2}$$ [3E]

ii) The above formula may be rewritten as

$$P\{X_{2n} = (0,0)\} = \left(\frac{1}{2}\right)^{2n} \binom{2n}{n} \sum_{i+j=n} \frac{(n)!}{2^n(i!\,j!)} \frac{(n)!}{2^n(i!\,j!)}$$

$$\leq \left(\frac{1}{2}\right)^{2n} \binom{2n}{n} M \sum_{i+j=n} \frac{(n)!}{2^n(i!\,j!)}$$ [3T] (*)

where

$$M = \max\{\sum_{i+j=n} \frac{(n)!}{2^n(i!\,j!)}\} \approx \frac{(n)!}{2^n\left(\left(\frac{n}{2}\right)!\right)^2}$$

Further, the sum in (*) equals 1, since the summands form a probability distribution. It follows that [2T]

$$P\{X_{2n} = (0,0)\} \leq \left(\frac{1}{8}\right)^{n} \frac{(2n)!}{n!\left(\left(\frac{n}{2}\right)!\right)^2}$$

Using Stirling's formula, we obtain

$$P\{X_{2n} = (0,0)\} \leq \left(\frac{1}{8}\right)^{n} \frac{\sqrt{2\pi(2n)}\left(\frac{2n}{e}\right)^{2n}}{\sqrt{2\pi(n)}\left(\frac{n}{e}\right)^{n}\left(\sqrt{2\pi\left(\frac{n}{2}\right)}\left(\frac{\frac{n}{2}}{e}\right)^{\frac{n}{2}}\right)^2} = Cn^{-1}$$ [2T]

for some constant C.

The key point is to use Stirling's formula

Finally, we find that

$$\sum_{n} P\{X_{2n} = (0,0)\} = C \sum_{n>1} n^{-1} = \infty \qquad \text{[3T]}$$

because the sum diverges. We therefore conclude that the origin is a recurrent state.