Imperial College
London

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2015

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Probability & Statistics I

Date: Thursday, 14 May 2015. Time: 10.00am – 12.00noon. Time allowed: 2 hours.

This paper has FOUR questions.

Candidates should start their solutions to each question in a new main answer book

Supplementary books may only be used after the relevant main book(s) are full.

Statistical tables are provided on pages 5 & 6.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers to each question as per the table below.

Raw mark	up to 12	13	14	15	16	17	18	19	20
Extra credit	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4

- Each question carries equal weight.
- Calculators may not be used.

1. (a) State the three axioms of probability for events defined on a sample space Ω.

(b) Prove from the axioms that, for any event $E \subseteq \Omega$, $\mathsf{P}(E) \leq 1$.

(c) If E and F are independent events, prove that E and F^C are independent.

(d) Professor Nasty challenges you to the following game, which involves three, three-sided dice: one red, one green and one blue. Each of the dice have equal probability of landing on each of the three sides independently of the other dice .

The red die has the numbers $\{3,3,3\}$;
The green die has the numbers $\{4,4,1\}$;
The blue die has the numbers $\{2,2,5\}$.

Professor Nasty lets you choose one of the dice and then she chooses one of the remaining two dice. You both roll your chosen dice once and the winner is the person whose die shows the bigger number.

(i) What is the probability that Blue beats Green?

(ii) What is the probability that Green beats Red?

(iii) Given your calculations, you choose Blue. What colour should Professor Nasty choose to maximise her probability of winning?

(iv) You play again, but this time Professor Nasty chooses first. She rolls each of the dice and chooses the colour with the highest score. Once she has chosen, you choose one of the two remaining dice at random. If you chose Blue and won, what is the probability that Professor Nasty chose Green?

2. The random variables X and Y are independently distributed with $X \sim Poisson(\lambda)$ and $Y \sim Gamma(2, 1)$.

(a) Find an expression for $\alpha = \mathsf{P}(X \geq 2)$.

(b) Prove that

$$\mathsf{P}(Y \leq \lambda) = \alpha.$$

(c) (i) Show that the moment generating function of X is given by

$$M_X(t) = \exp\{\lambda(e^t - 1)\}.$$

(ii) Hence show that $\mathsf{E}_{f_X}(X) = \mathsf{var}_{f_X}(X) = \lambda$.

(d) Suppose X_1 and X_2 are independently distributed with the same distribution as X.

(i) Determine the probability mass function of $Z = X_1 + X_2$.

(ii) For what range of values of λ will $\mathsf{P}(Z > 0) > 0.5$?

3. Consider the continuous random variables $U \sim Uniform(0, 1)$ and $X = \sqrt{-2\log(1-U)}$.

(a) Determine $F_U(u)$, the cumulative distribution function of U.

(b) Find an expression for $F_X(x)$ in terms of $F_U(\cdot)$.

(c) Hence prove that $f_X(x)$, the probability density function of X, is given by

$$f_X(x) = xe^{-x^2/2}, \quad x > 0.$$

(d) If $Y \sim Uniform(-a, a), a > 0$, independent of X (where X is defined as above) and $Z = X + Y$, use the fact that

$$f_Z(z) = \int f_X(z-y) f_Y(y) \mathrm{d}y$$

to determine the form of $f_Z(z)$ when $z > a$.

(e) Show that

$$\mathsf{P}(Z > a) = \frac{1}{2a} \int_0^{2a} e^{-u^2/2} \mathrm{d}u.$$

4. Continuous random variables X and Y have joint probability density function (pdf), $f_{X,Y}(x,y)$.

(a) Prove that,

$$\mathsf{var}_{f_{X,Y}}(aX+bY) = a^2\mathsf{var}_{f_X}(X) + 2ab\mathsf{cov}_{f_{X,Y}}(X,Y) + b^2\mathsf{var}_{f_Y}(Y),$$

for constants a, b.

(b) Suppose the joint pdf of X and Y is given by,

$$f_{X,Y}(x,y) = \frac{2}{3}(x+2y), \quad 0 \le x \le 1; 0 \le y \le 1.$$

(i) Determine $f_X(x)$, the marginal pdf of X.

(ii) Determine $f_Y(y)$, the marginal pdf of Y.

(iii) Determine $\mathsf{E}_{f_X}(X)$ and $\mathsf{E}_{f_Y}(Y)$ and $\mathsf{E}_{f_{X,Y}}(XY)$.

(iv) Determine $\mathsf{E}_{f_{X,Y}}(X-Y)$.

(v) Find $P(X > Y)$. Comment on this result in light of your answer to part (iv).

DISCRETE DISTRIBUTIONS

	RANGE $\mathbb{X}$	PARAMETERS	MASS FUNCTION f_X	CDF F_X	$\mathrm{E}_{f_X}[X]$	$\mathrm{Var}_{f_X}[X]$	MGF M_X
$Bernoulli(\theta)$	$\{0,1\}$	$\theta \in (0,1)$	$\theta^x(1-\theta)^{1-x}$		θ	$\theta(1-\theta)$	$1-\theta+\theta e^t$
$Binomial(n,\theta)$	$\{0,1,...,n\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{n}{x}\theta^x(1-\theta)^{n-x}$		$n\theta$	$n\theta(1-\theta)$	$(1-\theta+\theta e^t)^n$
$Poisson(\lambda)$	$\{0,1,2,...\}$	$\lambda \in \mathbb{R}^+$	$\dfrac{e^{-\lambda}\lambda^x}{x!}$		λ	λ	$\exp\{\lambda(e^t-1)\}$
$Geometric(\theta)$	$\{1,2,...\}$	$\theta \in (0,1)$	$(1-\theta)^{x-1}\theta$	$1-(1-\theta)^x$	$\dfrac{1}{\theta}$	$\dfrac{(1-\theta)}{\theta^2}$	$\dfrac{\theta e^t}{1-e^t(1-\theta)}$
$NegBinomial(n,\theta)$	$\{n,n+1,...\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{x-1}{n-1}\theta^n(1-\theta)^{x-n}$		$\dfrac{n}{\theta}$	$\dfrac{n(1-\theta)}{\theta^2}$	$\left(\dfrac{\theta e^t}{1-e^t(1-\theta)}\right)^n$
or	$\{0,1,2,...\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{n+x-1}{x}\theta^n(1-\theta)^x$		$\dfrac{n(1-\theta)}{\theta}$	$\dfrac{n(1-\theta)}{\theta^2}$	$\left(\dfrac{\theta}{1-e^t(1-\theta)}\right)^n$

For **CONTINUOUS** distributions (see over), define the **GAMMA FUNCTION**

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx$$

and the LOCATION/SCALE transformation $Y = \mu + \sigma X$ gives

$$f_Y(y) = f_X\left(\frac{y-\mu}{\sigma}\right)\frac{1}{\sigma} \qquad F_Y(y) = F_X\left(\frac{y-\mu}{\sigma}\right) \qquad M_Y(t) = e^{\mu t}M_X(\sigma t) \qquad \mathrm{E}_{f_Y}[Y] = \mu + \sigma\mathrm{E}_{f_X}[X] \qquad \mathrm{Var}_{f_Y}[Y] = \sigma^2\mathrm{Var}_{f_X}[X]$$

CONTINUOUS DISTRIBUTIONS							
	$\mathbb{X}$	PARAMS.	PDF f_X	CDF F_X	$\mathrm{E}_{f_X}[X]$	$\mathrm{Var}_{f_X}[X]$	MGF M_X
$Uniform(\alpha,\beta)$ (stand. model $\alpha=0,\beta=1$)	(α,β)	$\alpha<\beta\in\mathbb{R}$	$\dfrac{1}{\beta-\alpha}$	$\dfrac{x-\alpha}{\beta-\alpha}$	$\dfrac{(\alpha+\beta)}{2}$	$\dfrac{(\beta-\alpha)^2}{12}$	$\dfrac{e^{\beta t}-e^{\alpha t}}{t(\beta-\alpha)}$
$Exponential(\lambda)$ (stand. model $\lambda=1$)	$\mathbb{R}^+$	$\lambda\in\mathbb{R}^+$	$\lambda e^{-\lambda x}$	$1-e^{-\lambda x}$	$\dfrac{1}{\lambda}$	$\dfrac{1}{\lambda^2}$	$\left(\dfrac{\lambda}{\lambda-t}\right)$
$Gamma(\alpha,\beta)$ (stand. model $\beta=1$)	$\mathbb{R}^+$	$\alpha,\beta\in\mathbb{R}^+$	$\dfrac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}$		$\dfrac{\alpha}{\beta}$	$\dfrac{\alpha}{\beta^2}$	$\left(\dfrac{\beta}{\beta-t}\right)^\alpha$
$Weibull(\alpha,\beta)$ (stand. model $\beta=1$)	$\mathbb{R}^+$	$\alpha,\beta\in\mathbb{R}^+$	$\alpha\beta x^{\alpha-1}e^{-\beta x^\alpha}$	$1-e^{-\beta x^\alpha}$	$\dfrac{\Gamma(1+1/\alpha)}{\beta^{1/\alpha}}$	$\dfrac{\Gamma\left(1+\frac{2}{\alpha}\right)-\Gamma\left(1+\frac{1}{\alpha}\right)^2}{\beta^{2/\alpha}}$	
$Normal(\mu,\sigma^2)$ (stand. model $\mu=0,\sigma=1$)	$\mathbb{R}$	$\mu\in\mathbb{R},\sigma\in\mathbb{R}^+$	$\dfrac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\dfrac{(x-\mu)^2}{2\sigma^2}\right\}$		μ	σ^2	$e^{\{\mu t+\sigma^2t^2/2\}}$
$Student(\nu)$	$\mathbb{R}$	$\nu\in\mathbb{R}^+$	$\dfrac{(\pi\nu)^{-\frac{1}{2}}\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\left\{1+\frac{x^2}{\nu}\right\}^{(\nu+1)/2}}$		0 (if $\nu>1$)	$\dfrac{\nu}{\nu-2}$ (if $\nu>2$)	
$Pareto(\theta,\alpha)$	$\mathbb{R}^+$	$\theta,\alpha\in\mathbb{R}^+$	$\dfrac{\alpha\theta^\alpha}{(\theta+x)^{\alpha+1}}$	$1-\left(\dfrac{\theta}{\theta+x}\right)^\alpha$	$\dfrac{\theta}{\alpha-1}$ (if $\alpha>1$)	$\dfrac{\alpha\theta^2}{(\alpha-1)(\alpha-2)}$ (if $\alpha>2$)	
$Beta(\alpha,\beta)$	$(0,1)$	$\alpha,\beta\in\mathbb{R}^+$	$\dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$		$\dfrac{\alpha}{\alpha+\beta}$	$\dfrac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	

Imperial College
London

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BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2015

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M1S

Probability & Statistics I (Solutions)

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1. (a) **Axioms of Probability**
Given a σ−field, $\mathcal{F}$ (a set of subsets of the sample space Ω.) For events $E, E_1, E_2, \ldots \in \mathcal{F}$, then the probability function, $\mathsf{P}(\cdot)$, must satisfy:

(I) $\mathsf{P}(E) \geq 0$.
(II) $\mathsf{P}(\Omega) = 1$.
(III) If $E_1, E_2, \ldots$, are pairwise disjoint then
$\mathsf{P}(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mathsf{P}(E_i)$ (Countable additivity). $\boxed{3}$

(Do not need to specify σ−field, could instead say: for events $E, E_1, \ldots \subseteq \Omega$. Lose 1 mark if finite rather than countable additivity specified)

unseen ⇓

(b)

$$E \cup E^C = \Omega \Rightarrow \mathsf{P}(E) + \mathsf{P}(E^C) = \mathsf{P}(\Omega), \text{ axiom III as } E \cap E^C = \phi$$
$$\Rightarrow \mathsf{P}(E) + \mathsf{P}(E^C) = 1, \text{ axiom II}$$
$$\Rightarrow \mathsf{P}(E) \leq 1, \text{ axiom I as } \mathsf{P}(E^C) \geq 0 \text{ and } \mathsf{P}(E) \geq 0.$$

$\boxed{2}$

seen ⇓

(c)

$$\mathsf{P}(E \cap F^C) = \mathsf{P}(F^C \cap E) = \mathsf{P}(F^C \mid E)\mathsf{P}(E) = (1 - \mathsf{P}(F \mid E))\mathsf{P}(E)$$
$$= (1 - \mathsf{P}(F))\mathsf{P}(E) \text{ (from } \mathsf{P}(F \mid E) = \mathsf{P}(F) \text{ as } E \text{ and } F \text{ indep)}$$
$$= \mathsf{P}(F^C)\mathsf{P}(E),$$

therefore E and F^C are independent. $\boxed{2}$

(d) Let B, R and G reprensent the number on the Blue, Red and Green die respectively. sim. seen ⇓
Let B_i, R_i, G_i = event Blue, Red or Green chosen on the ith choice, $i = 1, 2$.
Let W = event you win.

(i)

$$\mathsf{P}(\text{Blue beats Green}) = \mathsf{P}((B = 2 \cap G = 1) \cup (B = 5)) = \mathsf{P}(B = 2)\mathsf{P}(G = 1) + \mathsf{P}(B = 5)$$
$$= \frac{2}{3} \cdot \frac{1}{3} + \frac{1}{3} = \frac{5}{9}.$$

$\boxed{2}$

(ii)

$$\mathsf{P}(\text{Green beats Red}) = \mathsf{P}(G = 4) = \frac{2}{3}.$$

$\boxed{1}$

(iii) We know that Blue beats Green with probability 5/9, need to calculate:

$$\mathsf{P}(\text{Blue beats Red}) = \mathsf{P}(B = 5) = \frac{1}{3}.$$

Professor Nasty should should Red to maximise her probability of winning. $\boxed{2}$

(iv)

$$\begin{aligned}
\mathsf{P}(R_1) &= \mathsf{P}(\text{Red beats both Blue and Green}) = \mathsf{P}(G = 1 \cap B = 2) \\
&= \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9}. \\
\mathsf{P}(B_1) &= \mathsf{P}(\text{Blue beats both Red and Green}) = \mathsf{P}(B = 5) = \frac{1}{3}. \\
\mathsf{P}(G_1) &= \mathsf{P}(\text{Green beats both Red and Blue}) = \mathsf{P}(G = 4 \cap B = 2) \\
&= \frac{2}{3} \cdot \frac{2}{3} = \frac{4}{9}.
\end{aligned}$$

$\boxed{3}$

Want to calculate:

$$\begin{aligned}
\mathsf{P}(G_1 \mid W \cap B_2) &= \frac{\mathsf{P}(W \cap B_2 \mid G_1)\mathsf{P}(G_1)}{\mathsf{P}(W \cap B_2)} \\
&= \frac{\mathsf{P}(W \mid G_1 \cap B_2)\mathsf{P}(B_2 \mid G_1)\mathsf{P}(G_1)}{\mathsf{P}(W \cap B_2)}.
\end{aligned}$$

$\boxed{2}$

R_1 and G_1 partition B_2, so, from the theorem of total probability:

$$\begin{aligned}
\mathsf{P}(W \cap B_2) &= \mathsf{P}(W \cap B_2 \mid G_1)\mathsf{P}(G_1) + \mathsf{P}(W \cap B_2 \mid R_1)\mathsf{P}(R_1) \\
&= \mathsf{P}(W \mid G_1 \cap B_2)\mathsf{P}(B_2 \mid G_1)\mathsf{P}(G_1) + \mathsf{P}(W \mid R_1 \cap B_2)\mathsf{P}(B_2 \mid R_1)\mathsf{P}(R_1) \\
&= \frac{5}{9} \cdot \frac{1}{2} \cdot \frac{4}{9} + \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{2}{9} = \frac{10}{81} + \frac{3}{81} = \frac{13}{81}.
\end{aligned}$$

$\boxed{2}$

So,

$$\mathsf{P}(G_1 \mid W \cap B_2) = \frac{\frac{10}{81}}{\frac{13}{81}} = \frac{10}{13}.$$

$\boxed{1}$

unseen ⇓

2. (a)

$$X \sim Poisson(\lambda) \Rightarrow f_X(x) = \frac{e^{-\lambda}\lambda^x}{x!}, x = 0, 1, 2, \ldots$$

$$\begin{aligned}\alpha &= \mathsf{P}(X \geq 2) = 1 - \mathsf{P}(X \leq 1) \\ &= 1 - (\mathsf{P}(X = 0) + \mathsf{P}(X = 1)) = 1 - e^{-\lambda} - \lambda e^{-\lambda}.\end{aligned}$$

$\boxed{3}$

(b)

$$Y \sim \textit{Gamma}(2, 1) \Rightarrow f_Y(y) = xe^{-x},\ x \geq 0.$$

$$\begin{aligned}\mathsf{P}(Y \leq \lambda) &= \int_0^\lambda xe^{-x}\,dx \qquad \begin{matrix} u = x & \frac{dv}{dx} = e^{-x} \\ \frac{du}{dx} = 1 & v = -e^{-x}\end{matrix} \\ &= \left[-xe^{-x}\right]_0^\lambda + \int_0^\lambda e^{-x} = -\lambda e^{-\lambda} + \left[-e^{-x}\right]_0^\lambda \\ &= -\lambda e^{-\lambda} + (-e^{-\lambda} + 1) = 1 - e^{-\lambda} - \lambda e^{-\lambda} = \alpha,\end{aligned}$$

as required.

$\boxed{3}$

(c) (i)

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$$\begin{aligned}M_X(t) = \mathsf{E}_{f_X}(e^{tX}) &= \sum_{x=0}^{\infty} e^{tx}\frac{e^{-\lambda}\lambda^x}{x!} \\ &= e^{-\lambda}\sum_{x=0}^{\infty}\frac{(e^t\lambda)^x}{x!} = e^{-\lambda}\exp(e^t\lambda) = \exp(\lambda(e^t - 1)).\end{aligned}$$

as required.

$\boxed{3}$

(ii)

$$\begin{aligned}\mathsf{E}_{f_X}(X) = M^{(1)}(0) &= \exp(\lambda(e^t - 1))\lambda e^t|_{t=0} \\ &= \lambda\exp(\lambda(e^t - 1) + t)|_{t=0} = \lambda.\end{aligned}$$

$\boxed{2}$

$$\begin{aligned}\mathsf{var}_{f_X}(X) &= \mathsf{E}_{f_X}(X^2) - \mathsf{E}^2_{f_X}(X) = M^{(2)}(0) - \lambda^2 \\ &= \lambda(\exp(\lambda(e^t - 1) + t))(\lambda e^t - 1)|_{t=0} - \lambda^2 \\ &= \lambda(1 + \lambda) - \lambda^2 = \lambda.\end{aligned}$$

$\boxed{3}$

sim. seen ⇓

(d) $Z = X_1 + X_2$ where X_1 and X_2 are both independent *Poisson*(λ).

(i)

$$\begin{aligned} M_Z(t) &= M_{X_1}(t)M_{X_2}(t) = \exp(\lambda(e^t - 1))\exp(\lambda(e^t - 1)) \\ &= \exp(2\lambda(e^t - 1)) \end{aligned}$$

So $Z \sim$ *Poisson*(2λ) and

$$f_Z(z) = \frac{e^{-2\lambda}(2\lambda)^z}{z!}, z = 0, 1, 2, \ldots$$

$\boxed{4}$

unseen ⇓

(ii)

$$\mathsf{P}(Z > 0) = 1 - \mathsf{P}(Z = 0) = 1 - e^{-2\lambda}.$$

So,

$$\begin{aligned} \mathsf{P}(Z > 0) > 0.5 &\Rightarrow 1 - e^{-2\lambda} > 0.5 \Rightarrow e^{-2\lambda} < 0.5 \\ &\Rightarrow -2\lambda < \log(0.5) \Rightarrow \lambda > \frac{-\log(0.5)}{2} \\ &\Rightarrow \lambda > \frac{\log(2)}{2}. \end{aligned}$$

$\boxed{2}$

meth seen ↓

3. (a) $U \sim$ *Uniform*$(0,1)$, the range of U is $(0,1)$, so the range of $X = \sqrt{-2\log(1-U)}$ is $(0,\infty)$.
$F_U(u) = 0, u \le 0, F_U(u) = 1, u \ge 1$,

$$F_U(u) = \mathsf{P}(U \le u) = \int_0^u f_U(t)\mathrm{d}t = \int_0^u 1\mathrm{d}t = u$$
$$\Rightarrow F_U(u) = u, \quad u \in (0,1).$$

3

(b)

$$\begin{aligned}F_X(x) &= \mathsf{P}(X \le x) = \mathsf{P}(\sqrt{-2\log(1-U)} \le x) \\ &= \mathsf{P}(-2\log(1-U) \le x^2) = \mathsf{P}\left(\log(1-U) \ge -\frac{x^2}{2}\right) \\ &= \mathsf{P}\left(1-U \ge \mathrm{e}^{-x^2/2}\right) = \mathsf{P}(U \le 1-\mathrm{e}^{-x^2/2}) \\ &= F_U(1-\mathrm{e}^{-x^2/2}),\ x > 0.\end{aligned}$$

4

(c)

$$f_X(x) = \frac{\mathrm{d}}{\mathrm{d}x}F_X(x) = x\mathrm{e}^{-x^2/2},\ x > 0.$$

2

unseen ⇓

(d)

$$f_Y(y) = \frac{1}{2a}, y \in (-a,a), \quad f_X(x) = x\mathrm{e}^{-x^2/2}, x > 0.$$

$Z = X + Y$ so the range of Z is $(-a,\infty)$. 1

If $z > a$, then for $y \in (-a,a)$, we have that $z - y > 0$, so using convolution we have:

$$\begin{aligned}f_Z(z) &= \int f_X(z-y)f_Y(y)\,\mathrm{d}y = \int_{-a}^{a}(z-y)\mathrm{e}^{-(z-y)^2/2}\frac{1}{2a}\,\mathrm{d}y \\ &= \frac{1}{2a}\left[\mathrm{e}^{-(z-y)^2/2}\right]_{-a}^{a} \\ &= \frac{1}{2a}\left(\mathrm{e}^{-(z-a)^2/2} - \mathrm{e}^{-(z-a)^2/2}\right), z > a.\end{aligned}$$

4

(e)

$$\mathsf{P}(Z > a) = \int_a^\infty f_Z(z)\,\mathrm{d}z$$

$\boxed{2}$

$$\begin{aligned}
\mathsf{P}(Z > a) &= \frac{1}{2a}\int_a^\infty \mathrm{e}^{-(z-a)^2/2}\,\mathrm{d}z - \frac{1}{2a}\int_a^\infty \mathrm{e}^{-(z+a)^2/2}\,\mathrm{d}z \quad \text{let } u = z - a \text{ and } u = z + a \text{ resp.}\\
&= \frac{1}{2a}\int_0^\infty \mathrm{e}^{-u^2/2}\mathrm{d}u - \frac{1}{2a}\int_{2a}^\infty \mathrm{e}^{-u^2/2}\mathrm{d}u\\
&= \frac{\sqrt{2\pi}}{2a}\int_0^\infty \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-u^2/2}\mathrm{d}u - \frac{\sqrt{2\pi}}{2a}\int_{2a}^\infty \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-u^2/2}\mathrm{d}u\\
&= \frac{\sqrt{2\pi}}{2a}\left\{\left(\frac{1}{2} - \left(\frac{1}{2} - \int_0^{2a}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-u^2/2}\mathrm{d}u\right)\right)\right\}\\
&= \frac{1}{2a}\int_0^{2a}\mathrm{e}^{-u^2/2}\mathrm{d}u.
\end{aligned}$$

using properties of the standard normal pdf. $\boxed{4}$

unseen ⇓

4. (a)

$$\begin{aligned}
\mathsf{var}_{f_{X,Y}}(aX+bY) &= \mathsf{E}_{f_{X,Y}}((aX+bY)^2) - \mathsf{E}^2_{f_{X,Y}}(aX+bY)\\
&= \mathsf{E}_{f_{X,Y}}(a^2X^2+2abXY+b^2Y^2) - (a\mathsf{E}_{f_X}(X)+b\mathsf{E}_{f_Y}(Y))^2\\
&= a^2\mathsf{E}_{f_X}(X^2)+2ab\mathsf{E}_{f_{X,Y}}(XY)+b^2\mathsf{E}_{f_Y}(Y^2) - (a^2\mathsf{E}^2_{f_X}(X)+2ab\mathsf{E}_{f_X}(X)\mathsf{E}_{f_Y}(Y)+b^2\mathsf{E}^2_{f_Y}(Y))\\
&= a^2(\mathsf{E}_{f_X}(X^2)-\mathsf{E}^2_{f_X}(X)) + b^2(\mathsf{E}_{f_Y}(Y^2)-\mathsf{E}^2_{f_Y}(Y)) + 2ab(\mathsf{E}_{f_{X,Y}}(XY)-\mathsf{E}_{f_X}(X)\mathsf{E}_{f_Y}(Y))\\
&= a^2\mathsf{var}_{f_X}(X)+b^2\mathsf{var}_{f_Y}(Y)+2ab\mathsf{cov}_{f_{X,Y}}(X,Y).
\end{aligned}$$

$\boxed{3}$

sim. seen ⇓

(b) (i)

$$\begin{aligned}
f_X(x) &= \int_0^1 f_{X,Y}(x,y)\,\mathrm{d}y = \int_0^1 \frac{2}{3}(x+2y)\,\mathrm{d}y\\
&= \left[\frac{2xy}{3}+\frac{2y^2}{3}\right]_0^1 = \frac{2x}{3}+\frac{2}{3}\\
&= \frac{2}{3}(x+1), \quad 0 \le x \le 1.
\end{aligned}$$

$\boxed{2}$

(ii)

$$\begin{aligned}
f_Y(y) &= \int_0^1 f_{X,Y}(x,y)\,\mathrm{d}x = \int_0^1 \frac{2}{3}(x+2y)\,\mathrm{d}x\\
&= \left[\frac{x^2}{3}+\frac{4xy}{3}\right]_0^1 = \frac{1}{3}+\frac{4y}{3}\\
&= \frac{1}{3}(4y+1), \quad 0 \le y \le 1.
\end{aligned}$$

$\boxed{2}$

(iii)

$$\begin{aligned}
\mathsf{E}_{f_X}(X) &= \int_0^1 \frac{2}{3}(x^2+x)\,\mathrm{d}x = \frac{2}{3}\left[\frac{x^3}{3}+\frac{x^2}{2}\right]_0^1\\
&= \frac{2}{3}\left(\frac{1}{3}+\frac{1}{2}\right) = \frac{2}{3}\cdot\frac{5}{6} = \frac{5}{9}.
\end{aligned}$$

$\boxed{2}$

$$\begin{aligned}
\mathsf{E}_{f_Y}(y) &= \int_0^1 \frac{1}{3}(4y^2+y)\,\mathrm{d}x = \frac{1}{3}\left[\frac{4y^3}{3}+\frac{y^2}{2}\right]_0^1\\
&= \frac{1}{3}\left(\frac{4}{3}+\frac{1}{2}\right) = \frac{1}{3}\cdot\frac{11}{6} = \frac{11}{18}.
\end{aligned}$$

$\boxed{2}$

$$
\begin{aligned}
\mathsf{E}_{f_{X,Y}}(XY) &= \int_0^1 \int_0^1 \frac{2}{3} xy(x+2y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \left[\frac{2}{3}\left(\frac{x^3 y}{3} + x^2 y^2\right)\right]_0^1 \mathrm{d}y \\
&= \int_0^1 \frac{2}{3}\left(\frac{y}{3} + y^2\right)\,\mathrm{d}y = \left[\frac{2}{3}\left(\frac{y^2}{6} + \frac{y^3}{3}\right)\right]_0^1 \\
&= \frac{2}{3}\left(\frac{1}{6} + \frac{1}{3}\right) = \frac{2}{3} \cdot \frac{3}{6} = \frac{1}{3}.
\end{aligned}
$$

$\boxed{3}$

$\boxed{\text{unseen} \Downarrow}$

(iv)

$$
\mathsf{E}_{f_{X,Y}}(X-Y) = \mathsf{E}_{f_X}(X) - \mathsf{E}_{f_Y}(Y) = \frac{5}{9} - \frac{11}{18} = -\frac{1}{18}.
$$

$\boxed{2}$

(v)

$$
\begin{aligned}
\mathsf{P}(X>Y) &= \int_0^1 \int_0^x f_{X,Y}(x,y)\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 \int_0^x \frac{2}{3}(x+2y)\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_0^1 \left[\frac{2}{3}(xy+y^2)\right]_0^x \mathrm{d}x = \int_0^1 \frac{2}{3}(x^2+x^2)\,\mathrm{d}x \\
&= \left[\frac{2}{3} \cdot \frac{2x^3}{3}\right]_0^1 = \frac{4}{9}.
\end{aligned}
$$

$\boxed{3}$

Could also formulate as $\int_0^1 \int_y^1 f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y$.
$\mathsf{P}(X>Y) < 0.5$ which is consistent with $\mathsf{E}_{f_{X,Y}}(X-Y) < 0$. $\boxed{1}$