Imperial College
London

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2015

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Integration Theory & Applications

Date: Thursday, 21 May 2015. Time: 2.00pm – 4.00pm. Time allowed: 2 hours.

This paper has FOUR questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the main book is full.

Statistical tables will not be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers to each question as per the table below.

Raw mark	up to 12	13	14	15	16	17	18	19	20
Extra credit	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4

- Each question carries equal weight.
- Calculators may not be used.

1. i. Consider sets of the plane which are subsets of $E = \{(x,y) : 0 \leq x \leq 1, 0 \leq y \leq 1\} \subset \mathbb{R}^2$. Give the definition of the outer measure, a measurable set (in the Lebesgue sense) and its measure.
ii. Let X be a set with a complete σ-additive measure μ on it. Let $f_n : X \to \mathbb{R}$, $n = 1, 2, \ldots$ be measurable functions on X converging everywhere on X as $n \to \infty$ to a function f. Show that f is measurable.

2. i. State the definition of a measurable simple function.
ii. State the definition of an integral of a simple function over a set of finite measure.
iii. Let $f : [0, \infty) \to \mathbb{R}$ be such that f is integrable with respect to the Lebesgue measure μ on $[0, \infty)$. Given $\epsilon > 0$, show that there exists a simple function φ vanishing outside a set of finite measure and such that

$$\int_0^\infty |f - \varphi| d\mu < \epsilon.$$

3. i. State the Levi theorem.
ii. Let A be a set with a complete σ-additive measure μ on it, such that $\mu(A) < \infty$. Let $f : A \to \mathbb{R}$ be integrable with respect to μ on A. Prove that if $\int_A |f| d\mu = 0$ then $f = 0$ a.e. on A.

4. i. Prove that any absolutely continuous function on a finite interval can be represented as a difference of two nondecreasing absolutely continuous functions.
ii. Let $f : [a, b] \to \mathbb{R}$ be a nondecreasing absolutely continuous function. Show that if a set $A \subset [a, b]$ has Lebesgue measure zero then $f(A)$ has Lebesgue measure zero.

	EXAMINATION SOLUTIONS 2014-15	Course M34PM19
Question 1		Marks & seen/unseen
Parts i.	The outer measure of a set A is $\mu^*(A) = \inf\limits_{A \subset \cup P_k} \sum_k m(P_k)$ over all coverings of A by finite or countable number of rectangles P_k; $m(P_k)$ is the area (measure) of the rectangle P_k. Let $\mathcal{R}(\mathcal{L})$ be the minimal ring generated by the semiring $\mathcal{L}$ of rectangles. A set A is called measurable if $\forall \varepsilon > 0 \ \exists B \in \mathcal{R}(\mathcal{L})$ such that $\mu^*(A \triangle B) < \varepsilon$. μ^* restricted to measurable sets is called the measure μ.	5 seen
	Setter's initials Checker's initials	Page number 1

	EXAMINATION SOLUTIONS 2014-15	Course
Question 1		Marks & seen/unseen
Parts ii.	Let $f_n(x) \to f(x)\ \forall x \in X$ as $n \to \infty$. Then $\{x : f(x) < c\} =$ $= \bigcup_k \bigcup_n \bigcap_{m>n} \{x : f_m(x) < c - \frac{1}{k}\}$ Indeed, if $f(x) < c$ then $\exists k$ s.t. $f(x) < c - \frac{2}{k}$. For this k $\exists n$ s.t. $\forall m \geq n : f_m(x) < c - \frac{1}{k}$. On the other hand, if for any fixed k, n we have for all $m > n$: $f_m(x) < c - \frac{1}{k}$ then $f(x) < c$, by taking $m \to \infty$. Since all $\{x : f_m(x) < c - \frac{1}{k}\}$ are measurable and measurable sets form a σ-algebra, $\{x : f(x) < c\}$ is measurable. As c is arbitrary, f is measurable.	15 seen
	Setter's initials Ch	Page number 2

	EXAMINATION SOLUTIONS 2014-15	Course
Question 2		Marks & seen/unseen
Parts i	A measurable function is called simple if it assumes at most countable number of values.	2 seen
ii	Let $f: X \to R$ be a simple function with values $y_1, y_2, \dots$ $y_j \neq y_k$ if $j \neq k$. Let $A \subset X$ be a set of finite measure. The integral of f over A, $\int_A f \, d\mu$, is the sum of the series $\sum_n y_n \mu(A_n)$ $A_n = \{x \in A : f(x) = y_n\}$ if the series converge absolutely.	3 seen
	Setter's initials	Checker's initials	Page number 3

	EXAMINATION SOLUTIONS 2014-15	Course
Question 2		Marks & seen/unseen
Parts iii	Since f is integrable over $[0,\infty)$, for given $\varepsilon>0$ $\exists\, x_0>0$ s.t. $\int_{x_0}^{\infty}\lvert f\rvert\,d\mu<\varepsilon/2$. Let $\varphi_n(x)=\frac{j}{n}$ for x such that $\frac{j}{n}\le f(x)<\frac{j+1}{n}$, $x\in[0,x_0]$, $j\in\mathbb{Z}$, $n\in\mathbb{Z}^+$. Then $\lvert\varphi_n(x)-f(x)\rvert\le 1/n$, $x\in[0,x_0]$. Let $\varphi_n(x)=0$ for $x\in[x_0,\infty)$ Choose $n>2x_0/\varepsilon$. Then $\int_0^{\infty}\lvert f-\varphi_n\rvert\,d\mu=\int_0^{x_0}\lvert f-\varphi_n\rvert\,d\mu+\int_{x_0}^{\infty}\lvert f\rvert\,d\mu\le\frac{x_0}{n}+\frac{\varepsilon}{2}<\varepsilon$. We have that $\varphi=\varphi_n$ since φ_n is simple by a theorem in lectures	15 unseen
	Setter's initials	Checker's initials	Page number 4

Course

Question 3

Marks & seen/unseen

Parts

i

Let $f_1(x) \leq f_2(x) \leq \cdots$ on A, f_n, $n = 1, 2 \ldots$ are integrable and $\int_A f_n \, d\mu \leq K$ for all n, where K is constant.

Then there exist a.e. a finite limit

$$f(x) = \lim_{n\to\infty} f_n(x),$$

f is integrable and

$$\int_A f_n \, d\mu \to \int_A f \, d\mu.$$

5 seen

	EXAMINATION SOLUTIONS 2014-15	Course
Question 3		Marks & seen/unseen
Parts		
ii	Let $A_k = \{x \in A : \lvert f \rvert \geq 1/k\}$. By Chebyshev inequality, $\mu(A_k) \leq k \int_A \lvert f \rvert d\mu = 0$, i.e. for any $k = 1, 2, \ldots$, $\mu(A_k) = 0$ Note that the set $B =$ $= \{x \in A : \lvert f(x) \rvert \neq 0\} =$ $= \bigcup_{k=1}^{\infty} A_k$ Since $A_1 \subset A_2 \subset \ldots$, continuity of the measure implies that $\mu(B) = \lim_{n \to \infty} \mu(A_n) = 0$. Thus $\mu\{x \in A : f(x) = 0\} =$ $= \mu(A) - \mu(B) = \mu(A)$, i.e. $f = 0$ a.e.	15 ~~unseen~~ seen
	Setter's initials / Checker's initials	Page number 6

	EXAMINATION SOLUTIONS 2014-15	Course
Question 4		Marks & seen/unseen
Parts i	Any a.c. function on an interval $[a,b]$ can be written as an indefinite integral, $$f(x)=\int_a^x f'(t)\,dt+f(a).$$ Let $$f_+'(t)=\begin{cases} f'(t) & \text{if } f'(t)\geq 0\\ 0, & \text{otherwise},\end{cases}$$ $$f_-'(t)=\begin{cases} -f'(t) & \text{if } f'(t)<0\\ 0, & \text{otherwise}.\end{cases}$$ Thus $f'(t)=f_+'(t)-f_-'(t)$, $f_\pm'(t)\geq 0$; and $f=f_1-f_2$, $$f_1=\int_a^x f_+'(t)\,dt+f(a);$$ $$f_2=+\int_a^x f_-'(t)\,dt;$$ $f_{1,2}$ are nondecreasing; and they are a.c. since they are indefinite integrals.	8 unseen
	Setter's initials	Checker's initials
		Page number 7

Parts: ii | 12 unseen

Fix any $\varepsilon > 0$. Since f is a.c. $\exists\, \delta > 0$ s.t. for any finite system of subintervals (a_j, b_j), $j = 1, \dots, n$ satisfying $\sum_1^n (b_j - a_j) < \delta$ and nonintersecting, we have $\sum_1^n |f(b_j) - f(a_j)| < \varepsilon$.

Let $A \subset [a,b]$, $\mu(A) = 0$.
Then for $B = A \setminus (\{a\} \cup \{b\})$ we have: $B \in (a,b)$, $\mu(B) = 0$

By a problem in exercises (or a known lemma), we can find an open set G such that $B \subset G \subset (a,b)$ and $\mu(G \setminus B) < \delta$. Therefore,

$$\mu(G) = \mu(B) + \mu(B \setminus G) < \delta$$

Parts ii

Since G is an open set,

$G = \bigcup_{j=1}^{\infty} (a_j, b_j)$, where

(a_j, b_j) - nonintersecting open intervals. We have

$\mu(G) = \sum_1^{\infty} (b_j - a_j) < \delta$.

Therefore (f is a.c.),

$\sum_1^n |f(b_j) - f(a_j)| < \varepsilon \quad \forall n$, so

$\sum_1^{\infty} |f(b_j) - f(a_j)| \leq \varepsilon$.

Since f is nondecreasing, it is obvious that

$f(A) \subset \bigcup_1^{\infty} [f(a_j), f(b_j)] \cup f(a) \cup f(b)$. Therefore, by subadditivity

$\mu^*(f(A)) \leq \sum_1^{\infty} |f(b_j) - f(a_j)| \leq \varepsilon$

$\Rightarrow \mu^*(f(A)) = 0 \Rightarrow \mu(f(A)) = 0$.