SOLUTIONS

EE4-47 Modelling and control of multibody mechanical systems

Model answers

Question 1

a) $r\,dr\,d\theta\,dz$ [3 marks]

b) i) The moment of inertia about the axis of symmetry (Z axis) is:

$$I_{zz} = \int (x^2 + y^2)dm = \rho \int_V (x^2 + y^2)dV = \rho \int_V r^2 dV,$$

where r is the radial distance from the Z axis. If we consider an infinitesimal volume element given in cylindrical polar coordinates then I_{zz} becomes

$$I_{zz} = \rho \int_0^h \int_0^{2\pi} \int_0^{\frac{Rz}{h}} r^3 dr d\theta dz$$

in which the upper limit for the integration with respect to r is a variable given by Rz/h, which varies from the minimum radius of 0 when $z = 0$ to the maximum radius of R when $z = h$. Hence, after some effort,

$$I_{zz} = \frac{\rho \pi h R^4}{10}.$$

Note that the integration which involves the variable limit, which is a function of z, should be performed before the integration with respect to z. Finally, the mass of the cone is given by

$$m = \rho V = \frac{1}{3}\rho \pi R^2 h,$$

and therefore I_{zz} becomes

$$I_{zz} = \frac{3}{10} m R^2.$$

[8 marks]

ii) The moment of inertia about the X axis can be found similarly via the equation

$$I_{xx} = \int (y^2 + z^2)dm = \rho \int_V (y^2 + z^2)dV = \rho \int_V (r^2 \cos^2\theta + z^2)dV.$$

By using a similar cylindrical polar volume element as above the volume integral becomes

$$I_{xx} = \rho \int_0^h \int_0^{2\pi} \int_0^{\frac{Rz}{h}} (r^2 \cos^2\theta + z^2) r dr d\theta dz = \frac{1}{20}\rho \pi R^2 h \left(R^2 + 4h^2\right),$$

and by making use of the mass expression of the cone

$$I_{xx} = \frac{3}{20} m \left(R^2 + 4h^2\right).$$

[7 marks]

iii) Due to the symmetry of the cone I_{yy} is the same as I_{xx}, i.e.

$$I_{yy} = I_{xx}.$$

[2 marks]

Question 2

a) The kinetic energy is $T = \frac{1}{2}m\left(\dot{x}^2 + \dot{z}^2\right)$.
The potential energy is $V = -mgz$, with the level of point O corresponding to zero gravitational potential energy.
The Lagrangian is $L = T - V = \frac{1}{2}m\left(\dot{x}^2 + \dot{z}^2\right) + mgz$. [5 marks]

b) The constraint equation is

$$z = -\frac{x^2}{4a} + a, \tag{1}$$

by differentiating

$$\dot{z} = -\frac{x}{2a}\dot{x}, \tag{2}$$

and by differentiating once again

$$\ddot{z} = -\frac{1}{2a}\left(\dot{x}^2 + x\ddot{x}\right). \tag{3}$$

The Lagrangian equation with respect to the generalised coordinate x is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} + \lambda\frac{x}{2a} = 0,$$

which yields

$$m\ddot{x} + \lambda\frac{x}{2a} = 0,$$

and therefore

$$\lambda = -\frac{2ma\ddot{x}}{x}. \tag{4}$$

The Lagrangian equation with respect to the generalised coordinate z is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} + \lambda = 0,$$

which yields

$$m\ddot{z} - mg + \lambda = 0,$$

and by using Equations (3) and (4)

$$\left(\frac{x}{2a} + \frac{2a}{x}\right)\ddot{x} + \frac{\dot{x}^2}{2a} + g = 0,$$

and finally by multiplying by x

$$\left(\frac{x^2}{2a} + 2a\right)\ddot{x} + \frac{x\dot{x}^2}{2a} + gx = 0, \tag{5}$$

[8 marks]

c) The force applied by the wire on the mass has two components, one in the X direction, which can be seen from the x Lagrangian equation to be $-\lambda\frac{x}{2a}$ (or equivalently $m\ddot{x}$), and one in the Z direction which can be seen from the z Lagrangian equation to be $-\lambda$ (or equivalently $m\ddot{z} - mg$). Therefore the magnitude of the total force is

$$F_{wire} = \sqrt{\left(-\lambda\frac{x}{2a}\right)^2 + (-\lambda)^2},$$

and by using Equation (4)

$$F_{wire} = m\ddot{x}\sqrt{1+\frac{4a^2}{x^2}}.$$

Equivalently

$$F_{wire} = \sqrt{(m\ddot{x})^2 + (m\ddot{z} - mg)^2},$$

and by using Equation (3)

$$F_{wire} = \sqrt{(m\ddot{x})^2 + \left(-\frac{m}{2a}\left(\dot{x}^2 + x\ddot{x}\right) - mg\right)^2}.$$

The direction of the wire force is given by its slope which is

$$F_{wire_{slope}} = \frac{-\lambda}{-\lambda\frac{x}{2a}} = \frac{2a}{x}.$$

The slope of a tangent to the wire can be found by differentiating Equation (1) and is given by $-\frac{x}{2a}$. Therefore the wire force is always perpendicular to the wire as expected due to the absence of friction. [4 marks]

d) For small x the equation of motion is

$$2a\ddot{x} + gx = 0,$$

and therefore the mass executes simple harmonic motion with angular frequency

$$\sqrt{\frac{g}{2a}}.$$

[3 marks]

Question 3

a) 3 degrees of freedom, generalised coordinates are r, θ and ϕ. [2 marks]

b) The position vector is $\boldsymbol{r_C} = r\boldsymbol{e_r} + a\boldsymbol{e_\theta}$. By differentiating the position vector we obtain the velocity vector $\dot{\boldsymbol{r}}_C = (\dot{r} - a\dot{\theta})\boldsymbol{e_r} + r\dot{\theta}\boldsymbol{e_\theta}$. Finally, by differentiating the velocity vector we obtain the acceleration vector

$$\ddot{\boldsymbol{r}}_C = \left(\ddot{r} - a\ddot{\theta} - r\dot{\theta}^2\right)\boldsymbol{e_r} + \left(2\dot{r}\dot{\theta} - a\dot{\theta}^2 + r\ddot{\theta}\right)\boldsymbol{e_\theta}. \tag{6}$$

[3 marks]

c) The viscous damping force is given by $-\mu(a(\dot{\phi} - \dot{\theta}) + \dot{r})\boldsymbol{e_r}$. [3 marks]

d) The system has two bodies, therefore we consider the motion of each one in turn. The equation of motion of the centre of mass of the ball in vector form is

$$(-\mu(a(\dot{\phi} - \dot{\theta}) + \dot{r}) - mg\sin\theta)\boldsymbol{e_r} + (F - mg\cos\theta)\boldsymbol{e_\theta} = m\ddot{\boldsymbol{r}}_C,$$

where F is the perpendicular force applied by the beam on the ball to keep the ball touching the beam. By using Equation (6)

$$(-\mu(a(\dot{\phi}-\dot{\theta})+\dot{r})-mg\sin\theta)\boldsymbol{e_r}+(F-mg\cos\theta)\boldsymbol{e_\theta} = m\left(\ddot{r} - a\ddot{\theta} - r\dot{\theta}^2\right)\boldsymbol{e_r}+m\left(2\dot{r}\dot{\theta} - a\dot{\theta}^2 + r\ddot{\theta}\right)\boldsymbol{e_\theta}.$$

The first equation of motion (of the system) is found by collecting the $\boldsymbol{e_r}$ terms

$$m\left(\ddot{r} - a\ddot{\theta} - r\dot{\theta}^2 + g\sin\theta\right) + \mu(a(\dot{\phi} - \dot{\theta}) + \dot{r}) = 0. \tag{7}$$

The normal force exerted by the beam on the ball is found by collecting the $\boldsymbol{e_\theta}$ terms

$$F = m\left(2\dot{r}\dot{\theta} - a\dot{\theta}^2 + r\ddot{\theta} + g\cos\theta\right). \tag{8}$$

The motion of the ball about its centre of mass is given by

$$\boldsymbol{N_{ba}} = \frac{d\boldsymbol{H_{ba}}}{dt},$$

therefore

$$I_{ba}\ddot{\phi} = F_{viscous}a,$$

or by using Equation (7), the second equation of motion is

$$I_{ba}\ddot{\phi} - ma\left(\ddot{r} - a\ddot{\theta} - r\dot{\theta}^2 + g\sin\theta\right) = 0.$$

The third equation of motion can be found by considering the motion about the centre of the beam, given by

$$\boldsymbol{N_{be}} = \frac{d\boldsymbol{H_{be}}}{dt}.$$

A reaction force equal and opposite to F is acting on the beam from the ball, therefore

$$I_{be}\ddot{\theta} = -Fr,$$

which, upon substitution of F from Equation (8), gives the third equation of motion

$$\left(I_{be} + mr^2\right)\ddot{\theta} - mar\dot{\theta}^2 + 2mr\dot{r}\dot{\theta} + mgr\cos\theta = 0.$$

[9 marks]

e) This is F found earlier and shown in Equation (8). [3 marks]

Question 4

a) i) $\boldsymbol{r_1} = l_1\boldsymbol{e_{r1}}$, $\boldsymbol{r_2} = l_1\boldsymbol{e_{r1}} + l_2\boldsymbol{e_{r2}}$. [2 marks]

ii) By differentiating the position vector of the first mass we obtain $\dot{\boldsymbol{r}}_1 = l_1\dot{\theta}_1\boldsymbol{e_{\theta1}}$. To differentiate the second position vector we require $\dot{\boldsymbol{e}}_{r2}$ given by

$$\dot{\boldsymbol{e}}_{r2} = \boldsymbol{\Omega} \times \boldsymbol{e_{r2}} = \left(\dot{\theta}_1\boldsymbol{i} + \dot{\theta}_2\boldsymbol{e_{r1}}\right) \times \boldsymbol{e_{r2}} = -\dot{\theta}_1 \sin\theta_2\boldsymbol{e_{r1}} + \dot{\theta}_2\boldsymbol{e_{\theta2}}.$$

Therefore

$$\dot{\boldsymbol{r}}_2 = -l_2\dot{\theta}_1 \sin\theta_2\boldsymbol{e_{r1}} + l_1\dot{\theta}_1\boldsymbol{e_{\theta1}} + l_2\dot{\theta}_2\boldsymbol{e_{\theta2}},$$

which can also be expressed in terms of the basis vectors of only one of the moving Cartesian coordinate systems, $\boldsymbol{e_{r1}}$, $\boldsymbol{e_{\theta1}}$ and $\boldsymbol{i}$, as

$$\dot{\boldsymbol{r}}_2 = -l_2\dot{\theta}_1 \sin\theta_2\boldsymbol{e_{r1}} + (l_1\dot{\theta}_1 + l_2\dot{\theta}_2\cos\theta_2)\boldsymbol{e_{\theta1}} + l_2\dot{\theta}_2\sin\theta_2\boldsymbol{i}.$$

[4 marks]

b) i) The total kinetic energy of the system is

$$T = \frac{1}{2}m_1l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2\left(l_2^2\dot{\theta}_1^2\sin^2\theta_2 + (l_1\dot{\theta}_1 + l_2\dot{\theta}_2\cos\theta_2)^2 + l_2^2\dot{\theta}_2^2\sin^2\theta_2\right),$$

which can be rewritten as

$$T = \frac{1}{2}(m_1 + m_2)l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2\left(l_2^2\dot{\theta}_1^2\sin^2\theta_2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos\theta_2 + l_2^2\dot{\theta}_2^2\right).$$

[3 marks]

ii) The potential energy of the system, with zero potential energy at the level of point O, is

$$V = -(m_1 + m_2)gl_1\cos\theta_1 + m_2gl_2\sin\theta_1\sin\theta_2.$$

[4 marks]

iii) The Lagrangian function of the system is

$$L = T - V = \frac{1}{2}(m_1 + m_2)l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2\left(l_2^2\dot{\theta}_1^2\sin^2\theta_2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos\theta_2 + l_2^2\dot{\theta}_2^2\right)$$
$$+(m_1 + m_2)gl_1\cos\theta_1 - m_2gl_2\sin\theta_1\sin\theta_2.$$

[1 mark]

c) The Lagrangian equation with respect to the generalised coordinate θ_1 is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\theta}_1}\right) - \frac{\partial L}{\partial\theta_1} = 0,$$

which yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left((m_1 + m_2)l_1^2\dot{\theta}_1 + m_2l_2^2\dot{\theta}_1\sin^2\theta_2 + m_2l_1l_2\dot{\theta}_2\cos\theta_2\right) + (m_1+m_2)gl_1\sin\theta_1 + m_2gl_2\cos\theta_1\sin\theta_2 = 0,$$

which gives the first equation of motion as

$$(m_1 + m_2)l_1^2\ddot{\theta}_1 + m_2l_2^2\ddot{\theta}_1\sin^2\theta_2 + 2m_2l_2^2\dot{\theta}_1^2\sin\theta_2\cos\theta_2 + m_2l_1l_2\ddot{\theta}_2\cos\theta_2 - m_2l_1l_2\dot{\theta}_2^2\sin\theta_2$$

$$+(m_1 + m_2)gl_1 \sin\theta_1 + m_2 g l_2 \cos\theta_1 \sin\theta_2 = 0.$$

The Lagrangian equation with respect to the generalised coordinate θ_2 is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} = 0,$$

which yields the second equation of motion

$$l_2\ddot{\theta}_2 - l_2\dot{\theta}_1^2 \sin\theta_2 \cos\theta_2 + l_1\dot{\theta}_1\dot{\theta}_2 \sin\theta_2 + g\sin\theta_1\cos\theta_2 = 0.$$

[6 marks]

Answers