

UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE TECHNOLOGY AND MEDICINE

[E303/ISE3.3]



DEPARTMENT of ELECTRICAL and ELECTRONIC ENGINEERING
M.Eng, B.Eng and A.C.G.I. EXAMINATIONS 2001
PART III

Solutions 2001 COMMUNICATION SYSTEMS

Examiner responsible: Dr A. Manikas

Solutions

page-1

ANSWER to Q1

- 1) A B C D E
- 2) A B C D E
- 3) A B C D E
- 4) A B C D E
- 5) A B C D E
- 6) A B C D E
- 7) A B C D E
- 8) A B C D E
- 9) A B C D E
- 10) A B C D E
- 11) A B C D E
- 12) A B C D E
- 13) A B C D E
- 14) A B C D E
- 15) A B C D E
- 16) A B C D E

Solutions

page-2

ANSWER to Q2

a>

$$p.d.f_{g_m} = p.d.f_{g(t)} = \mathcal{N}(-1, 2)$$

$$p.d.f_{g_{o4t}} =$$

$$P_1 = \Pr(g_{o4t} = -2V) = \Pr(g_m < 0V) =$$

$$= 1 - \mathcal{T}\left(\frac{10 - (-1)}{2}\right) = 1 - \mathcal{T}\left(\frac{1}{2}\right) = 0.7$$

$$P_2 = \Pr(g_{o4t} = +2V) = 1 - P_1 = 0.3$$

b>

$$r.m.s = \sqrt{P_{g_{o4t}}} = \sqrt{(-2)^2 P_1 + 2^2 P_2} = 2V$$

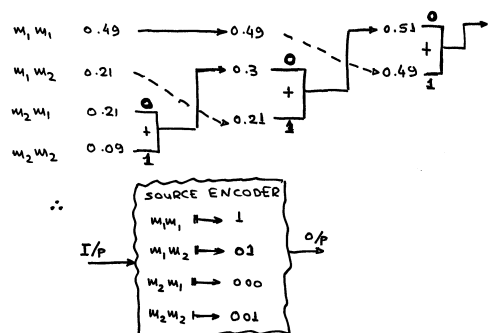
$$mean = (-2) \cdot P_1 + 2 P_2 = -0.8V$$

\uparrow 0.7 \uparrow 0.3

c>

$$(M \times M, \underline{q}) = \begin{pmatrix} w_1 w_1 \rightarrow \Pr(w_1, w_1) = 0.7 \times 0.7 = 0.49 \\ w_1 w_2 \rightarrow \Pr(w_1, w_2) = 0.7 \times 0.3 = 0.21 \\ w_2 w_1 \rightarrow \Pr(w_2, w_1) = 0.3 \times 0.7 = 0.21 \\ w_2 w_2 \rightarrow \Pr(w_2, w_2) = 0.3 \times 0.3 = 0.09 \end{pmatrix}$$

d> Huffman encoder



e>

$$\bar{L}_2 = 1 \times 0.49 + 2 \times 0.21 + 3 \times 0.21 + 3 \times 0.09 = 1.81 \frac{\text{bits}}{\text{symbol}}$$

and

$$H(M) = -P_1 \log_2 P_1 - P_2 \log_2 P_2 = 0.8813 \frac{\text{bits}}{\text{level}}$$

$\underbrace{-0.36}_{P_1} \quad \underbrace{-0.521}_{P_2}$

$$H(M) \leq \bar{L}_2 \leq H(M) + \frac{1}{2} \Rightarrow 0.8813 \leq 0.905 \leq 1.3813$$

i.e. inequality is satisfied.

f> "single level" encoder: average length = 1 $\frac{\text{bit}}{\text{level}}$
 "double level" encoder: 1 symbol = 2 levels
 therefore
 average length = 0.905 $\frac{\text{bits}}{\text{level}}$
 ∴ better

Solutions

page-3

Solutions

page-4

$$g) F_g = 4 \text{ kHz} \Rightarrow F_s = 8 \text{ kHz} \Rightarrow r_m = 8 \text{ k levels/sec.}$$

and

$$r_{mm} = 4 \text{ k double levels/sec}$$

information rates:

$$* r_{mf} = r_m \cdot H(M) = 7.0503 \frac{\text{kbits}}{\text{sec}}$$

(one-level approach) $\frac{8 \text{ k}}{0.8813}$

$$* r_{mf} = r_{mm} \cdot H(M \times M)$$

(two-level approach) $\frac{4 \text{ k}}{\uparrow}$

$$\rightarrow 0.43 \log_2 0.43 - 2 \times 0.21 \log_2 0.21 - 0.09 \log_2 0.09$$

$$= 1.7626 \frac{\text{bits}}{\text{double-level symbol}}$$

$$\therefore r_{mf} = 4 \text{ k} \times 1.7626 = 7.0503 \frac{\text{kbits}}{\text{sec}}$$

(two-level approach)

$$\text{i.e. } r_{mf} = r_{mf} \quad (\text{as it was expected})$$

(one-level) $\frac{7.0503}{\text{sec}}$ (two-level) $\frac{7.0503}{\text{sec}}$

data rates

$$r_{\text{data}} = r_m \cdot \ell_1 = 8 \text{ k bits/sec}$$

(one-level) $\frac{8 \text{ k}}{1}$

$$r_{\text{data}} = r_{mm} \cdot \ell_2 = 7.24 \text{ k bits/sec}$$

(double-level) $\frac{4 \text{ k}}{1.81}$

$$\text{i.e. } r_{\text{data}} > r_{\text{data}}$$

(one-level) $\frac{8 \text{ k}}{\text{sec}}$ (double-level) $\frac{7.24 \text{ k}}{\text{sec}}$

ANSWER to Q3

* PCM using a 256-level uniform quantizer:

$$\text{SNR}_q = 4.77 + 6\gamma - 20 \log_{10} \text{CF dB} \quad [1]$$

$$Q = 256 \Rightarrow 2^\gamma = 256 \Rightarrow \gamma = \log_2 256 = 8 \frac{\text{bits}}{\text{level}}$$

$$\text{CF} = \text{crest factor of the signal } g(t) = \frac{\text{peak}}{\text{rms}}$$

$$\text{peak} = \hat{g} = 2 \text{ Volts}$$

$$\text{rms} = \sigma_g = \sqrt{P_g}$$

$$P_g = 2 \int_0^2 g^2 \cdot \text{pdf}_g(g) dg = \int_0^2 g^2 \cdot \frac{1}{2} \Lambda\left(\frac{g}{2}\right) dg$$

$$\text{where } \rightarrow \left\{ \begin{aligned} &= 2 \int_0^2 g^2 \cdot \frac{1}{2} \cdot \frac{2-g}{2} dg \\ &= \frac{2}{2} \int_0^2 \left(g^2 - \frac{1}{2} g^3 \right) dg \\ &= \left(\frac{g^3}{3} - \frac{1}{8} g^4 \right) \Big|_0^2 \\ &= \frac{8}{3} - \frac{16}{8} = \frac{2}{3} \end{aligned} \right.$$

$$\text{i.e. } \text{rms} = \sqrt{\frac{2}{3}}$$

$$\therefore [1] \Rightarrow \text{SNR}_q = 4.77 + 6 \times 8 - 20 \log_{10} \frac{2}{\sqrt{\frac{2}{3}}} = 44.9885 \text{ dB}$$

$20 \log_{10} \sqrt{6} = 7.7815$

* PCM using an A-law/8-bit with $A = 97.6$:

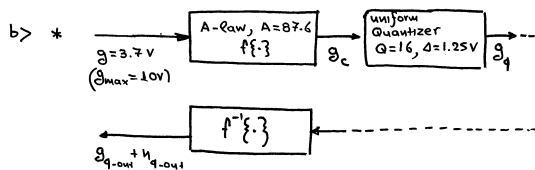
$$\text{SNR}_q = 4.77 + 6\gamma - 20 \log_{10} (1 + \ell_1 A) = 38.0059 \text{ dB}$$

(bits)

$$* r_{cs} = r_b = \gamma F_s = 8 \times 2 \times 10^3 \Rightarrow r_{cs} = 160 \text{ k channel symbols/sec}$$

$$B_{PCM} \geq \frac{r_{cs}}{2} \Rightarrow B_{PCM} = \frac{r_{cs}}{2} = 80 \text{ kHz}$$

* if $g(t)$ changes then $\left\{ \begin{aligned} &\text{SNR}_{q, \text{unif}} = \uparrow \text{ or } \downarrow \\ &\text{SNR}_{q, \text{A-law}} = \text{constant} = 38.0059 \text{ dB} \\ &\text{i.e. it is independent of the statistics of the signal} \end{aligned} \right.$



* transmitter's part (top branch):

$$\alpha = \frac{g}{g_{\max}} = \frac{3.7}{10} = 0.37 \Rightarrow \frac{1}{A} \leq \alpha < 1$$

$$\frac{1}{A} = \frac{1}{87.6} = 0.0114$$

$$\therefore g_c = \frac{1 + \ell_1(A-1)}{1 + \ell_1 A} \times g_{\max} = 8.1833 \text{ V}$$

$$\text{However } b_{14} < g_c < b_{15} \text{ . Therefore } g_q = u_{15} = 8.125 \text{ V}$$

$\frac{7.5 \text{ V}}{\downarrow} \quad \frac{8.75 \text{ V}}{\downarrow}$

* receiver's part (lower branch):

$$g_{1-out} = \frac{1}{A} \exp \left(\left| \frac{u_{15}}{g_{\max}} \right| (1 + \ell_1 A) \right) \times g_{\max} = 3.5839 \text{ V}$$

$$|u_{q-out}| = |g - g_{q-out}| = |3.7 - 3.5839| = 0.1161$$

* if the A-law coder is removed then

$$b_{10} < g < b_{11} \Rightarrow g_q = u_{11} = 3.125 \text{ V}$$

$\frac{2.5 \text{ V}}{\downarrow} \quad \frac{3.75 \text{ V}}{\downarrow}$

$$\Rightarrow |u_{q-out}| = |g - g_q| = |3.7 - 3.125| = 0.575 \text{ V}$$

ANSWER to Q4

$$\alpha> F_g = 4 \times 10^3 \text{ Hz}$$

$$F_s = 2 \times F_g = 8 \times 10^3 \text{ Hz}$$

$$\bar{\ell}_3 = 1 \times \frac{2^2}{64} + \left(3 \times \frac{9}{64} \right) \times 3 + \left(5 \times \frac{3}{64} \right) \times 3 + 5 \times \frac{1}{64} = 2.4685 \frac{\text{bits}}{\text{triple-level}}$$

$$\text{Alphabet: } \mathcal{X} = \left\{ \begin{aligned} &x_1 = 1 \\ &x_2 = 0 \end{aligned} \right\} \quad (M=2 \text{ channel symbols})$$

$$\text{Probabilities: } \mathcal{P} = \left\{ \begin{aligned} &P_1 = \Pr(H_1) = 0.6344 \\ &P_2 = \Pr(H_0) = 0.3656 \end{aligned} \right\} \quad \leftarrow \text{to be proven}$$

$$H_x = - \sum_{m=1}^2 p_m \log_2(p_m) = - \mathcal{P}^T \cdot \log_2(\mathcal{P}) = 0.9473 \frac{\text{bits}}{\text{symbol}}$$

$$r_{mf} = H_x \cdot r_{cs}$$

symbol rate = $r_b = F_s \frac{1}{3} \bar{\ell}_3 = 6583.3 \frac{\text{symbols}}{\text{sec}}$

$$\text{i.e. } r_{mf} = 0.9473 \times 6583.3 = 6236.4 \frac{\text{bits}}{\text{symbol}}$$

$$r_d = \frac{\ell}{\uparrow} \cdot r_{cs} = 6583.3 \frac{\text{bits}}{\text{sec}}$$

$\frac{1 \text{ bit}}{\downarrow} \quad \frac{6583.3}{\downarrow}$

$$b> P_e = 0.6344 \times 0.05 + 0.3656 \times 0.2 = 0.1048$$

$$c> 1 = x_1 \mapsto A_1 \cdot \Lambda\left(\frac{t}{0.5 T_{cs}}\right) \text{ of Energy} = E_1 = ?$$

$$0 = x_2 \mapsto 0 \text{ Volts i.e. Energy} = E_2 = 0$$

$$E_1 = 2 \int_{-T_{cs}/2}^{T_{cs}/2} A_1^2 \Lambda^2\left(\frac{t}{0.5 T_{cs}}\right) dt \quad (\text{where } A_1 = \sqrt{\frac{2}{3}})$$

$$= 2 \int_{-0.5 T_{cs}}^{0.5 T_{cs}} A_1^2 \left(\frac{-t + 0.5 T_{cs}}{0.5 T_{cs}} \right)^2 dt$$

$$= \dots = \frac{1}{3} A_1^2 T_{cs} = \frac{1}{3} \frac{2}{8} T_{cs} = \frac{1}{8} T_{cs}$$

$$E_b = E_1 \cdot P_1 + E_2 \cdot P_2 = \frac{1}{8} T_{cs} P_1 + 0 = 1.2046 \times 10^{-5}$$

$$EVE = \frac{E_b}{N_0 q} = \frac{6.0228 \times 10^{-3}}{2 \times 10^3} \quad (\text{data EVE})$$

$$BUE = \frac{B}{r_b} = \frac{B}{r_{cs}} = \frac{B}{2B} = \frac{1}{2}$$

Note: $B = \frac{r_{cs}}{2} \Rightarrow r_{cs} = 2B = r_b$

data point = $(EVE, BUE) = (6.0228 \times 10^{-3}, \frac{1}{2})$ ②

$$CS = \text{inf. point} = (EVE_{\text{inf}}, BUE_{\text{inf}}) = (\text{data point}) \times \frac{\log_2 M}{H_{\text{mut}}}$$

therefore the mutual entropy H_{mut} of the channel should be estimated

ie $H_{\text{mut}} = H_Y - H_{Y|X}$ (or $H_{\text{mut}} = H_X - H_{X|Y}$)

as follows:

$$* P = \begin{bmatrix} 0.6344 & \\ 0.3656 & \end{bmatrix}; \quad \mathbf{F} = \begin{bmatrix} 0.95 & 0.2 \\ 0.05 & 0.8 \end{bmatrix}; \quad \mathbf{q} = \mathbf{F} \cdot \mathbf{P} = \begin{bmatrix} 0.6758 & \\ 0.3242 & \end{bmatrix}$$

$$\mathbf{J} = \mathbf{F} \cdot \text{diag}(\mathbf{P}) = \begin{bmatrix} 0.6027 & 0.0731 \\ 0.0317 & 0.2925 \end{bmatrix}$$

$$* H_Y = -\mathbf{q}^T \cdot \log_2(\mathbf{q}) = 0.9089 \frac{\text{bits}}{\text{symbol}}$$

$$* H_{Y|X} = -\sum_{m=1}^2 \sum_{k=1}^2 J_{km} \log_2 \left(\frac{J_{km}}{P_m} \right) = -\|\mathbf{J} \odot \log_2(\mathbf{F})\|_{1A}$$

$$= 0.4456 \frac{\text{bits}}{\text{symbol}}$$

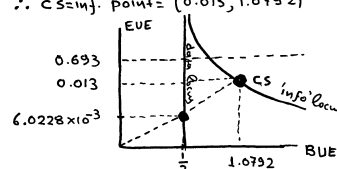
$$* H_{\text{mut}} = H_Y - H_{Y|X} = 0.4633 \frac{\text{bits}}{\text{symbol}}$$

Note: a different approach is to use the following expression

$$H_{\text{mut}} = -\|\mathbf{J} \odot \log_2 \left(\frac{\mathbf{F} \cdot \mathbf{P} \cdot \mathbf{P}^T}{\mathbf{J}} \right)\|_{1A} \frac{\text{bits}}{\text{symbol}} = 0.4633$$

where $\|\text{matrix}\|_{1A}$ = sum of the elements of the matrix-argument.

d> $\therefore CS = \text{inf. point} = (0.013, 1.0792)$



e> $\therefore CS = \text{inf. point} = (0.013, 1.0792) \Rightarrow$ CS is not realizable (since $EVE_{\text{inf}} = 0.013 < 0.693$) ^{theoretical H_{mut}}

f> $SNR_{1H} = \frac{EVE_{\text{inf}}}{BUE_{\text{inf}}} = \frac{EVE_d}{BUE_d} = 0.012 \Rightarrow SNR_{1H} = -19.2082 \text{ dB}$