Final

Imperial College
London

[E2.8 (Maths 3) 2010]

B.ENG. AND M.ENG. EXAMINATIONS 2010

PART II Paper 3 : MATHEMATICS (ELECTRICAL ENGINEERING)

Date Wednesday 2nd June 2010 2.00 - 5.00 pm

*DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.*

**Answer EIGHT questions.**

**Please answer questions from Section A and Section B in separate answer-books.**

*A mathematical formulae sheet is provided.*

*Statistical data sheets are provided.*

[*Before starting, please make sure that the paper is complete; there should be SEVEN pages, with a total of TWELVE questions. Ask the invigilator for a replacement if your copy is faulty.*]

**SECTION A** **[E2.8 (Maths 3) 2010]**

1. Consider the mapping

$$w = \frac{1}{(z-i)^2}$$

from the $z$-plane to the $w$-plane, where $z = x + iy$ and $w = u + iv$.

   (i) What corresponds in the $w$-plane to the family of circles $x^2 + (y-1)^2 = c^2$ in the $z$-plane?
   Sketch both families.

   (ii) Sketch the lines $x = 0$ and $y = 1$ in the $z$-plane. To what does this correspond in the $w$-plane?
   Sketch the result.

   (iii) $y = ax + 1$ represents a family of straight lines of gradient $a$. What is the corresponding family of lines in the $w$-plane?
   Sketch both families.

   (iv) In the two special cases when $a = 1$ & $a = -1$, what corresponds to this in the $w$-plane?
   Sketch the result.

2. The complex function

$$\frac{e^{iz}}{z(z^2+1)(z^2+4)}$$

has simple poles in the upper half-plane at $z = i$ and $z = 2i$, with another at $z = 0$ and two simple poles in the lower half-plane at $z = -i$ and $z = -2i$. Show that the residues at the poles lying at $z = 0$, $z = i$ and $z = 2i$ are respectively $1/4$, $-e^{-1}/6$ and $e^{-2}/24$.

Now consider the contour integral

$$\oint_C \frac{e^{iz}\,dz}{z(z^2+1)(z^2+4)}$$

where $C$ is taken to be a semi-circle in the upper half of the complex plane, with an additional small semi-circular indentation below the pole at $z = 0$. Show that the contribution to the above integral from this indentation, in the limit when its radius goes to zero, is $\pi i/4$.

Using all this information, show that

$$\int_{-\infty}^{\infty} \frac{\sin x\,dx}{x(x^2+1)(x^2+4)} = \frac{\pi(3e-1)(e-1)}{12e^2}.$$

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3. Using the unit circle $z = e^{i\theta}$ as your contour $C$, convert the integral

$$I = \int_0^{2\pi} \frac{d\theta}{4 + \sin\theta}$$

to a complex integral over $C$, and hence show that

$$I = \frac{2\pi}{\sqrt{15}}\,.$$

4. If $\overline{f}(\omega)$ is the Fourier transform of $f(t)$, prove Parseval's equality

$$\int_{-\infty}^{\infty} |f(t)|^2\,dt \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} |\overline{f}(\omega)|^2\,d\omega\,.$$

The sinc-function $\mathrm{sinc}\,(t)$ and the tent function $\Lambda(t)$ are defined respectively by

$$\mathrm{sinc}\,(t) \;=\; \frac{\sin(t/2)}{(t/2)}\,,$$

$$\Lambda(t) \;=\; \begin{cases} 1+t, & -1 \le t \le 0\,, \\ 1-t, & 0 \le t \le 1\,. \end{cases}$$

Show that

(i) $$\overline{\Lambda}(\omega) \;=\; \mathrm{sinc}^2\,(\omega)\,,$$

(ii) $$\int_{-\infty}^{\infty} \mathrm{sinc}^4\,(\omega)\,d\omega \;=\; \frac{4\pi}{3}\,.$$

5. A second order ordinary differential equation, with initial values, takes the form

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = f(t)\,. \tag{1}$$

$x(t)$ has an initial value $x = x_0$ at $t = 0$, where $x_0$ is an unspecified non-zero constant, while $dx/dt = -x_0$ at $t = 0$.

By taking a Laplace transform of (1)and using the convolution theorem, show that

$$x(t) = x_0 e^{-t}\cos 2t + \frac{1}{2}\int_0^t e^{-t'}\sin(2t')\,f(t-t')\,dt'$$

is a solution of (1)and its initial conditions.

6. $\overline{f}(s) = \mathcal{L}\{f(t)\}$ and $\overline{g}(s) = \mathcal{L}\{g(t)\}$ are the Laplace transforms of two functions $f(t)$ and $g(t)$ respectively. The convolution of $f(t)$ with $g(t)$ is defined as

$$f * g = \int_0^t f(t')g(t-t')\,dt'\,.$$

Use double integration to prove the Laplace convolution theorem

$$\mathcal{L}\{f * g\} = \overline{f}(s)\overline{g}(s)$$

providing a sketch of the region over which the integration takes place.

Hence, or otherwise, show that

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \frac{1}{2}t\sin t\,.$$

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7. A closed boundary curve $C$ is formed from two parabolae which enclose a region $R$. The lower of the two is $y = x^2$ while the upper is $y^2 = x$. Sketch these curves, marking where they intersect. Evaluate the double integral

$$\int\int_R dxdy\,,$$

to show the area of the region $R$ between the two parabolae is $1/3$.

Use Green's theorem in a plane to show that

$$\oint_C \frac{2y^4\,dx - xy^3\,dy}{2x^2} \;=\; -\frac{3}{4}\,.$$

Evaluate the line integral directly, showing that the contribution from the lower parabola is zero.

*Green's Theorem in a plane states that for a two-dimensional region R bounded by a closed, piecewise smooth curve C:*

$$\oint_C \{P(x,y)dx \;+\; Q(x,y)dy\} \;=\; \int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy\,.$$

8. Consider the transformation of variables

$$u \;=\; \frac{x}{x^2+y^2}\,, \qquad v \;=\; \frac{y}{x^2+y^2}\,.$$

Show that

$$u^2 \;+\; v^2 \;=\; \frac{1}{x^2+y^2}$$

and hence show that the circle $x^2 + y^2 = a^2$ in the $x - y$ plane transforms to the circles $u^2 + v^2 = \frac{1}{a^2}$ in the $u - v$ plane.

Show that the line $y = mx$ in the $x - y$ plane transforms to the line $v = mu$ in the $u - v$ plane.

Show also that the Jacobian $J'$ for the transformation from $(u,\, v)$ to $(x,\, y)$ satisfies

$$|J'| \;=\; \frac{1}{(x^2+y^2)^2}\,.$$

Hence deduce $|J|$ where $J$ is the Jacobian for the transformation from $(x,\, y)$ to $(u,\, v)$.

Finally evaluate the integral

$$\int\int_R \frac{1}{(x^2+y^2)^2} \exp\left(\frac{1}{x^2+y^2}\right) dx\,dy$$

where $R$ is the region in the first quadrant between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ and the lines $y = \frac{1}{\sqrt{3}}x$ and $y = \sqrt{3}x$.

9. Let $\boldsymbol{F} = (F_1,\, F_2,\, F_3)$ and $\boldsymbol{A} = (A_1,\, A_2,\, A_3)$ be differentiable vector fields and $\phi$ and $\psi$ be differentiable scalar fields in three dimensions. Define grad $\phi$, div $\boldsymbol{F}$ and curl $\boldsymbol{F}$.

   Show that

   (i) div curl $\boldsymbol{A} = 0$ and curl grad $\psi = 0$ ,

   (ii) div $(\phi \boldsymbol{F}) = \boldsymbol{F} \,.\, \mathrm{grad}\, \phi + \phi\, \mathrm{div}\, \boldsymbol{F}$ ,

   (iii) curl $(\phi \boldsymbol{F}) = \phi\, \mathrm{curl}\, \boldsymbol{F} + \mathrm{grad}\, \phi \times \boldsymbol{F}$ .

   *In case* (iii) *it is only necessary to prove the result for one component.*

   When $\boldsymbol{F} = \mathrm{curl}\, \boldsymbol{A}$, write down div $(\phi \boldsymbol{F})$ in terms of $\phi$ and the components of $\boldsymbol{F}$.

   When $\boldsymbol{F} = \mathrm{grad}\, \psi$ write down curl $(\phi \boldsymbol{F})$ in terms of $\phi$ and $\psi$.

   If $\boldsymbol{F} = (x{+}yz,\; y{+}zx,\; z{+}xy)$, show that curl $\boldsymbol{F} = 0$ and find $\psi$ such that grad $\psi = \boldsymbol{F}$.

   (*Note alternative notation for* $\mathrm{grad}\phi = \nabla\phi$, $\mathrm{curl}\boldsymbol{F} = \nabla \times \boldsymbol{F}$ *and* $\mathrm{div}\boldsymbol{F} = \nabla \,.\, \boldsymbol{F}$.)

10. Write down the condition for the line integral

$$\int_{P,A}^{B} [\, f(x,\, y)\, dx + g(x,\, y)\, dy \,]$$

   to be independent of the path $P$ joining the initial point $A$ to the final point $B$.

   Find the value of $\alpha$ that makes the line integral

$$\int_{P,A}^{B} (x^3\, y^2\, dx + \alpha x^4\, y\, dy)$$

   independent of the path.

   Evaluate the line integral (which depends on $\beta$)

$$\int_{P,A}^{B} (x^3\, y^2\, dx + \beta\, x^4\, y\, dy)$$

   for two paths *of your own choice* joining $A(0,\, 0)$ to $B(1,\, 1)$ and verify that the answers are the same if the value of $\beta$ is equal to the value of $\alpha$ obtained above.

   Find the potential function $\psi(x,\, y)$ defined by the line integral for the value of $\alpha$ obtained above.

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11. The random variable $T$ has a truncated exponential distribution with probability density function

$$f(t) = \begin{cases} Ce^{-\lambda t}, & t > 1, \\ 0, & t \leq 1, \end{cases}$$

where $\lambda > 0$.

(i) Show that $C = \lambda e^{\lambda}$, derive the cumulative distribution function of $T$ and hence find the median of $T$ in terms of $\lambda$.

(ii) Find $P(T > t + s | T > s)$, where $s > 1$ and $t > 0$.

(iii) Suppose we have a random sample of size $n$, $t_1, \ldots, t_n$, from this truncated exponential distribution. Show that the maximum likelihood estimator of $\lambda$ is

$$\widehat{\lambda} = \frac{1}{\bar{t} - 1},$$

where $\bar{t}$ is the sample mean.

12. Consider the time series

$$y_t = t + 0.4e_t + 0.3e_{t-1} + 0.3e_{t-2},$$

where $\{e_t\}$ is white noise with $\text{Var}(e_t) = 1$.

(i) Define "white noise".

(ii) Find $\text{cov}(y_t, y_{t+s})$ for all $t$ and $s = 0, 1, 2, 3, \ldots$.

(iii) Find the autocorrelation $\rho_k$, $k = 1, 2, \ldots$ of $\{y_t\}$.

(iv) Is $\{y_t\}$ (weakly) stationary? Justify your answer.

(v) Consider the time series

$$x_t = y_t - t.$$

Is $\{x_t\}$ (weakly) stationary? Justify your answer.

(vi) Find the spectrum $f(w)$ of $\{x_t\}$.

**END OF PAPER**

MATHEMATICS DEPARTMENT

# MATHEMATICAL FORMULAE

## 1. VECTOR ALGEBRA

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1, a_2, a_3)$$

Scalar (dot) product: $\mathbf{a}\,.\,\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

Vector (cross) product:

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Scalar triple product:

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a}\,.\,\mathbf{b}\times\mathbf{c} = \mathbf{b}\,.\,\mathbf{c}\times\mathbf{a} = \mathbf{c}\,.\,\mathbf{a}\times\mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product: $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{c}\,.\,\mathbf{a})\mathbf{b} - (\mathbf{b}\,.\,\mathbf{a})\mathbf{c}$

## 2. SERIES

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \ldots \quad (\alpha \text{ arbitrary}, |x| < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n\frac{x^{2n}}{(2n)!} + \ldots,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + \ldots,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^n\frac{x^{n+1}}{(n+1)} + \ldots (-1 < x \le 1)$$



## 3. TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin(a+b) = \sin a \cos b + \cos a \sin b\,;$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b\,.$$

$$\cos iz = \cosh z\,; \quad \cosh iz = \cos z\,; \quad \sin iz = i\sinh z\,; \quad \sinh iz = i\sin z\,.$$

## 4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f\,D^n g + \binom{n}{1} Df\,D^{n-1}g + \ldots + \binom{n}{r} D^r f\,D^{n-r}g + \ldots + D^n f\,g\,.$$

(b) Taylor's expansion of $f(x)$ about $x = a$:

$$f(a+h) = f(a) + hf'(a) + h^2 f''(a)/2! + \ldots + h^n f^{(n)}(a)/n! + \epsilon_n(h)\,,$$

where $\epsilon_n(h) = h^{n+1} f^{(n+1)}(a+\theta h)/(n+1)!, \quad 0 < \theta < 1\,.$

(c) Taylor's expansion of $f(x, y)$ about $(a, b)$:

$$f(a+h, b+k) = f(a,b) + [hf_x + kf_y]_{a,b} + 1/2!\left[h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}\right]_{a,b} + \ldots$$

(d) Partial differentiation of $f(x, y)$:

i. If $y = y(x)$, then $f = F(x)$, and $\dfrac{dF}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dx}$.

ii. If $x = x(t)$, $y = y(t)$, then $f = F(t)$, and $\dfrac{dF}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$.

iii. If $x = x(u, v)$, $y = y(u, v)$, then $f = F(u, v)$, and

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}, \quad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$

(e) Stationary points of $f(x, y)$ occur where $f_x = 0$, $f_y = 0$ simultaneously.
Let $(a, b)$ be a stationary point: examine $D = [f_{xx}f_{yy} - (f_{xy})^2]_{a,b}$.
If $D > 0$ and $f_{xx}(a, b) < 0$, then $(a, b)$ is a maximum;
If $D > 0$ and $f_{xx}(a, b) > 0$, then $(a, b)$ is a minimum;
If $D < 0$ then $(a, b)$ is a saddle-point.

(f) Differential equations:

i. The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating factor $I(x) = \exp[\int P(x)(dx]$, so that $\dfrac{d}{dx}(Iy) = IQ$.

ii. $P(x, y)dx + Q(x, y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y$.

## 5. INTEGRAL CALCULUS

(a) An important substitution: $\tan(\theta/2) = t$ :
$\sin\theta = 2t/(1+t^2)$, $\cos\theta = (1-t^2)/(1+t^2)$, $d\theta = 2\,dt/(1+t^2)$.

(b) Some indefinite integrals:

$$\int (a^2 - x^2)^{-1/2} dx = \sin^{-1}\left(\frac{x}{a}\right), \quad |x| < a.$$

$$\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1}\left(\frac{x}{a}\right) = \ln\left\{\frac{x}{a} + \left(1 + \frac{x^2}{a^2}\right)^{1/2}\right\}.$$

$$\int (x^2 - a^2)^{-1/2} dx = \cosh^{-1}\left(\frac{x}{a}\right) = \ln\left|\frac{x}{a} + \left(\frac{x^2}{a^2} - 1\right)^{1/2}\right|.$$

$$\int (a^2 + x^2)^{-1} dx = \left(\frac{1}{a}\right)\tan^{-1}\left(\frac{x}{a}\right).$$

## 6. NUMERICAL METHODS

(a) Approximate solution of an algebraic equation:

If a root of $f(x) = 0$ occurs near $x = a$, take $x_0 = a$ and
$x_{n+1} = x_n - [f(x_n)/f'(x_n)]$, $n = 0, 1, 2 \ldots$

(Newton Raphson method).

(b) Formulae for numerical integration: Write $x_n = x_0 + nh$, $y_n = y(x_n)$.

i. Trapezium rule (1-strip): $\int_{x_0}^{x_1} y(x)dx \approx (h/2)[y_0 + y_1]$.

ii. Simpson's rule (2-strip): $\int_{x_0}^{x_2} y(x)dx \approx (h/3)[y_0 + 4y_1 + y_2]$.

(c) Richardson's extrapolation method: Let $I = \int_a^b f(x)dx$ and let $I_1$, $I_2$ be two estimates of $I$ obtained by using Simpson's rule with intervals $h$ and $h/2$. Then, provided $h$ is small enough,

$$I_2 + (I_2 - I_1)/15,$$

is a better estimate of $I$.



## 7. LAPLACE TRANSFORMS

| Function | Transform | Function | Transform |
|---|---|---|---|
| $f(t)$ | $F(s) = \int_0^\infty e^{-st} f(t)dt$ | $af(t) + bg(t)$ | $aF(s) + bG(s)$ |
| $df/dt$ | $sF(s) - f(0)$ | $d^2f/dt^2$ | $s^2F(s) - sf(0) - f'(0)$ |
| $e^{at}f(t)$ | $F(s-a)$ | $tf(t)$ | $-dF(s)/ds$ |
| $(\partial/\partial\alpha)f(t,\alpha)$ | $(\partial/\partial\alpha)F(s,\alpha)$ | $\int_0^t f(t)dt$ | $F(s)/s$ |
| $\int_0^t f(u)g(t-u)du$ | $F(s)G(s)$ | | |
| $1$ | $1/s$ | $t^n (n = 1, 2 \ldots)$ | $n!/s^{n+1}$, $(s > 0)$ |
| $e^{at}$ | $1/(s-a)$, $(s > a)$ | $\sin\omega t$ | $\omega/(s^2+\omega^2)$, $(s > 0)$ |
| $\cos\omega t$ | $s/(s^2+\omega^2)$, $(s > 0)$ | $H(t-T) = \begin{cases} 0, & t < T \\ 1, & t > T \end{cases}$ | $e^{-sT}/s$, $(s, T > 0)$ |

## 8. FOURIER SERIES

If $f(x)$ is periodic of period $2L$, then $f(x+2L) = f(x)$, and

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \text{ where}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad n = 0, 1, 2, \ldots, \text{ and}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1, 2, 3, \ldots$$

Parseval's theorem

$$\frac{1}{L}\int_{-L}^{L} [f(x)]^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right).$$

## 1. Probabilities for events

For events $A$, $B$, and $C$ $\quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$

More generally $\quad P(\bigcup A_i) = \sum P(A_i) - \sum P(A_i \cap A_j) + \sum P(A_i \cap A_j \cap A_k) - \cdots$

The odds in favour of $A$ $\quad P(A) \,/\, P(\overline{A})$

Conditional probability $\quad P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$ provided that $P(B) > 0$

Chain rule $\quad P(A \cap B \cap C) = P(A)\, P(B \mid A)\, P(C \mid A \cap B)$

Bayes' rule $\quad P(A \mid B) = \dfrac{P(A)\, P(B \mid A)}{P(A)\, P(B \mid A) + P(\overline{A})\, P(B \mid \overline{A})}$

$A$ and $B$ are independent if $\quad P(B \mid A) = P(B)$

$A$, $B$, and $C$ are independent if $\quad P(A \cap B \cap C) = P(A)P(B)P(C)$, and

$$P(A \cap B) = P(A)P(B), \quad P(B \cap C) = P(B)P(C), \quad P(C \cap A) = P(C)P(A)$$

## 2. Probability distribution, expectation and variance

The probability distribution for a discrete random variable $X$ is called the probability mass function (pmf) and is the complete set of probabilities $\{p_x\} = \{P(X = x)\}$

Expectation $\quad E(X) = \mu = \sum_x x p_x$

For function $g(x)$ of $x$, $\quad E\{g(X)\} = \sum_x g(x) p_x$, so $E(X^2) = \sum_x x^2 p_x$

Sample mean $\quad \overline{x} = \dfrac{1}{n} \sum_k x_k$ estimates $\mu$ from random sample $x_1, x_2, \ldots, x_n$

Variance $\quad \mathrm{var}\,(X) = \sigma^2 = E\{(X - \mu)^2\} = E(X^2) - \mu^2$

Sample variance $\quad s^2 = \dfrac{1}{n-1} \left\{ \sum_k x_k^2 - \dfrac{1}{n} \left( \sum_j x_j \right)^2 \right\}$ estimates $\sigma^2$

Standard deviation $\quad \mathrm{sd}\,(X) = \sigma$

If value $y$ is observed with frequency $n_y$

$$n = \sum_y n_y, \quad \sum_k x_k = \sum_y y n_y, \quad \sum_k x_k^2 = \sum_y y^2 n_y$$

Skewness $\quad \beta_1 = E\left( \dfrac{X - \mu}{\sigma} \right)^3$ is estimated by $\dfrac{1}{n-1} \sum \left( \dfrac{x_i - \overline{x}}{s} \right)^3$

Kurtosis $\quad \beta_2 = E\left( \dfrac{X - \mu}{\sigma} \right)^4 - 3$ is estimated by $\dfrac{1}{n-1} \sum \left( \dfrac{x_i - \overline{x}}{s} \right)^4 - 3$

Sample median $\quad \tilde{x}$ or $x_{\text{med}}$. Half the sample values are smaller and half larger

If the sample values $x_1, \ldots, x_n$ are ordered as $x_{(1)} \le x_{(2)} \le \cdots \le x_{(n)}$,
then $\tilde{x} = x_{(\frac{n+1}{2})}$ if $n$ is odd, and $\tilde{x} = \frac{1}{2}\left(x_{(\frac{n}{2})} + x_{(\frac{n+2}{2})}\right)$ if $n$ is even

$\alpha$-quantile $Q(\alpha)$ is such that $P(X \leq Q(\alpha)) = \alpha$

Sample $\alpha$-quantile $\widehat{Q}(\alpha)$ Proportion $\alpha$ of the data values are smaller

Lower quartile Q1 = $\widehat{Q}(0.25)$ one quarter are smaller

Upper quartile Q3 = $\widehat{Q}(0.75)$ three quarters are smaller

Sample median $\tilde{x} = \widehat{Q}(0.5)$ estimates the population median $Q(0.5)$

3. Probability distribution for a continuous random variable

The cumulative distribution function (cdf) $F(x) = P(X \leq x) = \int_{x_0=-\infty}^{x} f(x_0)\mathrm{d}x_0$

The probability density function (pdf) $f(x) = \dfrac{\mathrm{d}F(x)}{\mathrm{d}x}$

$E(X) = \mu = \int_{-\infty}^{\infty} x\, f(x)\mathrm{d}x$, $\mathrm{var}\,(X) = \sigma^2 = E(X^2) - \mu^2$, where $E(X^2) = \int_{-\infty}^{\infty} x^2\, f(x)\mathrm{d}x$

4. Discrete probability distributions

Discrete Uniform *Uniform* $(n)$

$$p_x = \frac{1}{n} \quad (x = 1, 2, \ldots, n) \qquad \mu = (n+1)/2,\ \sigma^2 = (n^2-1)/12$$

Binomial distribution *Binomial* $(n, \theta)$

$$p_x = \binom{n}{x} \theta^x (1-\theta)^{n-x} \quad (x = 0, 1, 2, \ldots, n) \qquad \mu = n\theta,\ \sigma^2 = n\theta(1-\theta)$$

Poisson distribution *Poisson* $(\lambda)$

$$p_x = \frac{\lambda^x e^{-\lambda}}{x!} \quad (x = 0, 1, 2, \ldots) \quad (\text{with } \lambda > 0) \qquad \mu = \lambda,\ \sigma^2 = \lambda$$

Geometric distribution *Geometric* $(\theta)$

$$p_x = (1-\theta)^{x-1}\theta \quad (x = 1, 2, 3, \ldots) \qquad \mu = \frac{1}{\theta},\ \sigma^2 = \frac{1-\theta}{\theta^2}$$

5. Continuous probability distributions

Uniform distribution *Uniform* $(\alpha, \beta)$

$$f(x) = \begin{cases} \dfrac{1}{\beta - \alpha} & (\alpha < x < \beta), \\ 0 & (\text{otherwise}). \end{cases} \qquad \mu = (\alpha+\beta)/2,\ \sigma^2 = (\beta-\alpha)^2/12$$

Exponential distribution *Exponential* $(\lambda)$

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & (0 < x < \infty), \\ 0 & (-\infty < x \leq 0). \end{cases} \qquad \mu = 1/\lambda,\ \sigma^2 = 1/\lambda^2$$

Normal distribution $N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\} \quad (-\infty < x < \infty), \quad E(X) = \mu, \quad \mathrm{var}(X) = \sigma^2$$

Standard normal distribution $N(0,1)$

If $X$ is $N(\mu, \sigma^2)$, then $Y = \dfrac{X-\mu}{\sigma}$ is $N(0,1)$

## 6. Reliability

For a device in continuous operation with failure time random variable $T$ having pdf $f(t)$ $(t > 0)$

| | | |
|---|---|---|
| The reliability function at time $t$ | $R(t)$ | $= P(T > t)$ |
| The failure rate or hazard function | $h(t)$ | $= f(t)/R(t)$ |
| The cumulative hazard function | $H(t)$ | $= \int_0^t h(t_0)\,\mathrm{d}t_0 = -\ln\{R(t)\}$ |
| The Weibull$(\alpha, \beta)$ distribution has | $H(t)$ | $= \beta t^\alpha$ |

## 7. System reliability

For a system of $k$ devices, which operate independently, let

$R_i = P(D_i) = P($"device $i$ operates"$)$

The system reliability, $R$, is the probability of a path of operating devices

A system of devices in series operates only if every device operates

$$R = P(D_1 \cap D_2 \cap \cdots \cap D_k) = R_1 R_2 \cdots R_k$$

A system of devices in parallel operates if any device operates

$$R = P(D_1 \cup D_2 \cup \cdots \cup D_k) = 1 - (1 - R_1)(1 - R_2) \cdots (1 - R_k)$$

## 8. Covariance and correlation

The covariance of $X$ and $Y$ $\quad \mathrm{cov}(X,Y) = E(XY) - \{E(X)\}\{E(Y)\}$

From pairs of observations $(x_1, y_1), \ldots, (x_n, y_n)$ $\quad S_{xy} = \sum_k x_k y_k - \frac{1}{n}(\sum_i x_i)(\sum_j y_j)$

$$S_{xx} = \sum_k x_k^2 - \frac{1}{n}(\sum_i x_i)^2, \qquad S_{yy} = \sum_k y_k^2 - \frac{1}{n}(\sum_j y_j)^2$$

| | | |
|---|---|---|
| Sample covariance | $s_{xy} = \dfrac{1}{n-1} S_{xy}$ | estimates $\mathrm{cov}(X,Y)$ |
| Correlation coefficient | $\rho = \mathrm{corr}(X,Y)$ | $= \dfrac{\mathrm{cov}(X,Y)}{\mathrm{sd}(X) \cdot \mathrm{sd}(Y)}$ |
| Sample correlation coefficient | $r = \dfrac{S_{xy}}{\sqrt{S_{xx} S_{yy}}}$ | estimates $\rho$ |

9. Sums of random variables

$$E(X+Y) = E(X)+E(Y)$$
$$\operatorname{var}(X+Y) = \operatorname{var}(X)+\operatorname{var}(Y)+2\operatorname{cov}(X,Y)$$
$$\operatorname{cov}(aX+bY,\ cX+dY) = (ac)\operatorname{var}(X)+(bd)\operatorname{var}(Y)+(ad+bc)\operatorname{cov}(X,Y)$$

If $X$ is $N(\mu_1,\sigma_1^2)$, $Y$ is $N(\mu_2,\sigma_2^2)$, and $\operatorname{cov}(X,Y)=c$, then $X+Y$ is $N(\mu_1+\mu_2,\ \sigma_1^2+\sigma_2^2+2c)$

10. Bias, standard error, mean square error

If $t$ estimates $\theta$ (with random variable $T$ giving $t$)

| | | |
|---|---|---|
| Bias of $t$ | $\operatorname{bias}(t)$ | $= E(T)-\theta$ |
| Standard error of $t$ | $\operatorname{se}(t)$ | $= \operatorname{sd}(T)$ |
| Mean square error of $t$ | $\operatorname{MSE}(t)$ | $= E\{(T-\theta)^2\} = \{\operatorname{se}(t)\}^2+\{\operatorname{bias}(t)\}^2$ |

If $\overline{x}$ estimates $\mu$, then $\operatorname{bias}(\overline{x})=0$, $\operatorname{se}(\overline{x})=\sigma/\sqrt{n}$, $\operatorname{MSE}(\overline{x})=\sigma^2/n$, $\widehat{\operatorname{se}}(\overline{x})=s/\sqrt{n}$

Central limit property  If $n$ is fairly large, $\overline{x}$ is from $N(\mu,\ \sigma^2/n)$ approximately

11. Likelihood

The likelihood is the joint probability as a function of the unknown parameter $\theta$.

For a random sample $x_1,x_2,\ldots,x_n$

$$\ell(\theta;\ x_1,x_2,\ldots,x_n) = P(X_1=x_1\,|\,\theta)\cdots P(X_n=x_n\,|\,\theta) \qquad \text{(discrete distribution)}$$
$$\ell(\theta;\ x_1,x_2,\ldots,x_n) = f(x_1\,|\,\theta)\,f(x_2\,|\,\theta)\cdots f(x_n\,|\,\theta) \qquad \text{(continuous distribution)}$$

The maximum likelihood estimator (MLE) is $\widehat{\theta}$ for which the likelihood is a maximum

12. Confidence intervals

If $x_1,x_2,\ldots,x_n$ are a random sample from $N(\mu,\sigma^2)$ and $\sigma^2$ is known, then the 95% confidence interval for $\mu$ is $(\overline{x}-1.96\frac{\sigma}{\sqrt{n}},\ \overline{x}+1.96\frac{\sigma}{\sqrt{n}})$

If $\sigma^2$ is estimated, then from the Student t table for $t_{n-1}$ we find $t_0=t_{n-1,0.05}$

The 95% confidence interval for $\mu$ is $(\overline{x}-t_0\frac{s}{\sqrt{n}},\ \overline{x}+t_0\frac{s}{\sqrt{n}})$

13. Standard normal table — Values of pdf $\phi(y) = f(y)$ and cdf $\Phi(y) = F(y)$

| $y$ | $\phi(y)$ | $\Phi(y)$ | $y$ | $\phi(y)$ | $\Phi(y)$ | $y$ | $\phi(y)$ | $\Phi(y)$ | $y$ | $\Phi(y)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | .399 | .5 | .9 | .266 | .816 | 1.8 | .079 | .964 | 2.8 | .997 |
| .1 | .397 | .540 | 1.0 | .242 | .841 | 1.9 | .066 | .971 | 3.0 | .999 |
| .2 | .391 | .579 | 1.1 | .218 | .864 | 2.0 | .054 | .977 | 0.841 | .8 |
| .3 | .381 | .618 | 1.2 | .194 | .885 | 2.1 | .044 | .982 | 1.282 | .9 |
| .4 | .368 | .655 | 1.3 | .171 | .903 | 2.2 | .035 | .986 | 1.645 | .95 |
| .5 | .352 | .691 | 1.4 | .150 | .919 | 2.3 | .028 | .989 | 1.96 | .975 |
| .6 | .333 | .726 | 1.5 | .130 | .933 | 2.4 | .022 | .992 | 2.326 | .99 |
| .7 | .312 | .758 | 1.6 | .111 | .945 | 2.5 | .018 | .994 | 2.576 | .995 |
| .8 | .290 | .788 | 1.7 | .094 | .955 | 2.6 | .014 | .995 | 3.09 | .999 |

14. Student t table — Values $t_{m,p}$ of $x$ for which $P(|X| > x) = p$, when $X$ is $t_m$

| $m$ | $p$= 0.10 | 0.05 | 0.02 | 0.01 | $m$ | $p$= 0.10 | 0.05 | 0.02 | 0.01 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 6.31 | 12.71 | 31.82 | 63.66 | 9 | 1.83 | 2.26 | 2.82 | 3.25 |
| 2 | 2.92 | 4.30 | 6.96 | 9.92 | 10 | 1.81 | 2.23 | 2.76 | 3.17 |
| 3 | 2.35 | 3.18 | 4.54 | 5.84 | 12 | 1.78 | 2.18 | 2.68 | 3.05 |
| 4 | 2.13 | 2.78 | 3.75 | 4.60 | 15 | 1.75 | 2.13 | 2.60 | 2.95 |
| 5 | 2.02 | 2.57 | 3.36 | 4.03 | 20 | 1.72 | 2.09 | 2.53 | 2.85 |
| 6 | 1.94 | 2.45 | 3.14 | 3.71 | 25 | 1.71 | 2.06 | 2.48 | 2.78 |
| 7 | 1.89 | 2.36 | 3.00 | 3.50 | 40 | 1.68 | 2.02 | 2.42 | 2.70 |
| 8 | 1.86 | 2.31 | 2.90 | 3.36 | $\infty$ | 1.645 | 1.96 | 2.326 | 2.576 |

15. Chi-squared table — Values $\chi^2_{k,p}$ of $x$ for which $P(X > x) = p$, when $X$ is $\chi^2_k$ and $p$ = .995, .975, *etc*

| $k$ | .995 | .975 | .05 | .025 | .01 | .005 | $k$ | .995 | .975 | .05 | .025 | .01 | .005 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | .000 | .001 | 3.84 | 5.02 | 6.63 | 7.88 | 18 | 6.26 | 8.23 | 28.87 | 31.53 | 34.81 | 37.16 |
| 2 | .010 | .051 | 5.99 | 7.38 | 9.21 | 10.60 | 20 | 7.43 | 9.59 | 31.42 | 34.17 | 37.57 | 40.00 |
| 3 | .072 | .216 | 7.81 | 9.35 | 11.34 | 12.84 | 22 | 8.64 | 10.98 | 33.92 | 36.78 | 40.29 | 42.80 |
| 4 | .207 | .484 | 9.49 | 11.14 | 13.28 | 14.86 | 24 | 9.89 | 12.40 | 36.42 | 39.36 | 42.98 | 45.56 |
| 5 | .412 | .831 | 11.07 | 12.83 | 15.09 | 16.75 | 26 | 11.16 | 13.84 | 38.89 | 41.92 | 45.64 | 48.29 |
| 6 | .676 | 1.24 | 12.59 | 14.45 | 16.81 | 18.55 | 28 | 12.46 | 15.31 | 41.34 | 44.46 | 48.28 | 50.99 |
| 7 | .990 | 1.69 | 14.07 | 16.01 | 18.48 | 20.28 | 30 | 13.79 | 16.79 | 43.77 | 46.98 | 50.89 | 53.67 |
| 8 | 1.34 | 2.18 | 15.51 | 17.53 | 20.09 | 21.95 | 40 | 20.71 | 24.43 | 55.76 | 59.34 | 63.69 | 66.77 |
| 9 | 1.73 | 2.70 | 16.92 | 19.02 | 21.67 | 23.59 | 50 | 27.99 | 32.36 | 67.50 | 71.41 | 76.15 | 79.49 |
| 10 | 2.16 | 3.25 | 13.31 | 20.48 | 23.21 | 25.19 | 60 | 35.53 | 40.48 | 79.08 | 83.30 | 88.38 | 91.95 |
| 12 | 3.07 | 4.40 | 21.03 | 23.34 | 26.22 | 28.30 | 70 | 43.28 | 48.76 | 90.53 | 95.02 | 100.4 | 104.2 |
| 14 | 4.07 | 5.63 | 23.68 | 26.12 | 29.14 | 31.32 | 80 | 51.17 | 57.15 | 101.9 | 106.6 | 112.3 | 116.3 |
| 16 | 5.14 | 6.91 | 26.30 | 28.85 | 32.00 | 34.27 | 100 | 67.33 | 74.22 | 124.3 | 129.6 | 135.8 | 140.2 |

16. **The chi-squared goodness-of-fit test**

The frequencies $n_y$ are grouped so that the fitted frequency $\widehat{n}_y$ for every group exceeds about 5.

$$X^2 = \sum_y \frac{(n_y - \widehat{n}_y)^2}{\widehat{n}_y}$$ is referred to the table of $\chi_k^2$ with significance point $p$,

where $k$ is the number of terms summed, less one for each constraint, *eg* matching total frequency, and matching $\overline{x}$ with $\mu$

17. **Joint probability distributions**

Discrete distribution $\{p_{xy}\}$, where $p_{xy} = P(\{X=x\} \cap \{Y=y\})$.

Let $p_{x\bullet} = P(X=x)$, and $p_{\bullet y} = P(Y=y)$, then

$$p_{x\bullet} = \sum_y p_{xy} \quad \text{and} \quad P(X=x \,|\, Y=y) = \frac{p_{xy}}{p_{\bullet y}}$$

**Continuous distribution**

Joint cdf $\quad F(x,y) = P(\{X \le x\} \cap \{Y \le y\}) = \int_{x_0=-\infty}^{x} \int_{y_0=-\infty}^{y} f(x_0, y_0)\, dx_0\, dy_0$

Joint pdf $\quad f(x,y) = \dfrac{d^2 F(x,y)}{dx\, dy}$

Marginal pdf of $X$ $\quad f_X(x) = \int_{-\infty}^{\infty} f(x, y_0)\, dy_0$

Conditional pdf of $X$ given $Y=y$ $\quad f_{X|Y}(x|y) = \dfrac{f(x,y)}{f_Y(y)}$ (provided $f_Y(y) > 0$)

18. **Linear regression**

To fit the linear regression model $y = \alpha + \beta x$ by $\widehat{y}_x = \widehat{\alpha} + \widehat{\beta} x$ from observations $(x_1, y_1), \ldots, (x_n, y_n)$, the least squares fit is $\quad \widehat{\alpha} = \overline{y} - \overline{x}\widehat{\beta}, \quad \widehat{\beta} = \dfrac{S_{xy}}{S_{xx}}$

The residual sum of squares $\quad \text{RSS} = S_{yy} - \dfrac{S_{xy}^2}{S_{xx}}$

$\widehat{\sigma^2} = \dfrac{\text{RSS}}{n-2} \qquad \dfrac{n-2}{\sigma^2}\widehat{\sigma^2}$ is from $\chi_{n-2}^2$

$$E(\widehat{\alpha}) = \alpha, \quad E(\widehat{\beta}) = \beta,$$

$$\text{var}(\widehat{\alpha}) = \frac{\sum x_i^2}{n S_{xx}}\sigma^2, \quad \text{var}(\widehat{\beta}) = \frac{\sigma^2}{S_{xx}}, \quad \text{cov}(\widehat{\alpha}, \widehat{\beta}) = -\frac{\overline{x}}{S_{xx}}\sigma^2$$

$$\widehat{y}_x = \widehat{\alpha} + \widehat{\beta} x, \qquad E(\widehat{y}_x) = \alpha + \beta x, \qquad \text{var}(\widehat{y}_x) = \left\{\frac{1}{n} + \frac{(x-\overline{x})^2}{S_{xx}}\right\}\sigma^2$$

$\dfrac{\widehat{\alpha} - \alpha}{\widehat{\text{se}}(\widehat{\alpha})}, \quad \dfrac{\widehat{\beta} - \beta}{\widehat{\text{se}}(\widehat{\beta})}, \quad \dfrac{\widehat{y}_x - \alpha - \beta x}{\widehat{\text{se}}(\widehat{y}_x)}$ are each from $t_{n-2}$

EE2 - Maths Paper 3 - (E2.8) EE 1

# EXAMINATION QUESTIONS/SOLUTIONS 2009-2010

Course: EE2 + AE2 paper 3

Question

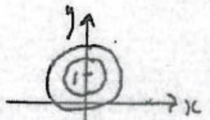
Parts | Marks & seen/unseen

$$w = \frac{1}{(z-i)^2} = \frac{1}{[x+i(y-1)]^2} = \frac{[x-i(y-1)]^2}{[x^2+(y-1)^2]^2} = u+iv$$

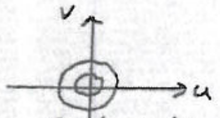
$$u = \frac{x^2-(y-1)^2}{[x^2+(y-1)^2]^2} \;;\quad v = \frac{-2x(y-1)}{[x^2+(y-1)^2]^2} \quad (*)$$ 4

(i) $u^2+v^2 = \dfrac{1}{[x^2+(y-1)^2]^2}$

$\therefore$ $x^2+(y-1)^2 = c^2$ corresponds to $u^2+v^2 = 1/c^4$ 6



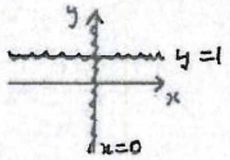

Circles radius c centred (0,1)

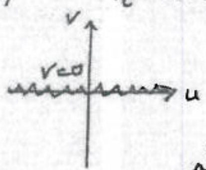



Circles radius $1/c^2$ centred (0,0)

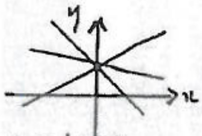
(ii) $x=0 \rightarrow v=0$

$y=1 \rightarrow$ from (*) 4




(iii)

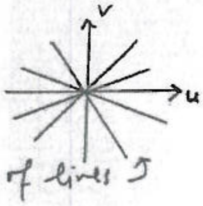



$y = 1+ax$

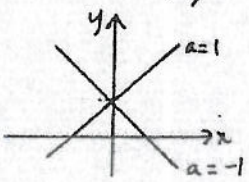
Family of lines.

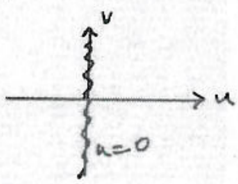
$u = \dfrac{(1-a^2)}{x^2(1+a^2)^2}$, $v = \dfrac{-2a}{x^2(1+a^2)^2}$ 4

$(1-a^2)v = -2au$ Family of lines




(iv) When $a=\pm 1$, $a^2=1$ so $(1-a^2)v = -2au \Rightarrow u=0$




2

Note to markers:
It is possible to do this question without explicitly writing down (*) [e.g. for part (i) we have $|z-i| = c$] - in this case allocate the 4 marks for (*) equally to parts (i)-(iv).

Setter's initials: JDG.

Checker's initials: AGW

Page number

EE 2

| | EXAMINATION QUESTIONS/SOLUTIONS 2009-2010 | Course<br>EE2<br>paper 3 |
|---|---|---|
| Question<br>EE2 | | Marks &<br>seen/unseen |

Parts

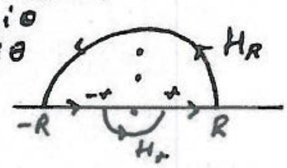
$\dfrac{e^{iz}}{z(z^2+1)(z^2+4)}$ has poles in the upper-$z$-plane at $0, i, 2i$

Residue at $z=0$ is $\left[\dfrac{e^{iz}}{(z^2+1)(z^2+4)}\right]_{z=0} = \frac{1}{4}$

Residue at $z=i$ is $\left[e^{iz}z^{-1}(z^2+4)^{-1}(z+i)^{-1}\right]_{z=i} = -e^{-1}/6$ 3

Residue at $z=2i$ is $\left[e^{iz}z^{-1}(z^2+1)^{-1}(z+2i)^{-1}\right]_{z=2i} = e^{-2}/24$

Residue Theorem: $H_R = Re^{i\theta}$, $H_r = re^{i\theta}$ 2 (pic)




$(*)\begin{cases} \oint_C F(z)\,dz = 2\pi i\left\{\frac{1}{4} - \frac{1}{6e} + \frac{1}{24e^2}\right\} \\ \oint_C F(z)\,dz = \left(\int_{-R}^{-r} + \int_{r}^{R}\right)\dfrac{e^{ix}\,dx}{x(x^2+1)(x^2+4)} + \int_{H_r} F(z)\,dz + \int_{H_R} F(z)\,dz \end{cases}$ 4

$\therefore \lim_{r\to 0} \int_{H_r} F(z)\,dz = \lim_{r\to 0} \int_{\pi}^{2\pi} \dfrac{\exp(ire^{i\theta})\, ire^{i\theta}\,d\theta}{re^{i\theta}(1+r^2e^{2i\theta})(4+r^2e^{2i\theta})}$

$= i\int_{\pi}^{2\pi} \frac{1}{4}\,d\theta = \pi i/4$ 4

$\lim_{R\to\infty} \int_{H_R} F(z)\,dz = 0$ by Jordan's Lemma i) $m = 1 > 0$ ii) Only poles iii) $\|F(z)\| \to 0$ as $R\to\infty$. 4

Thus in the limits $r\to 0$ & $R\to\infty$ $(*)$ becomes

$2\pi i\left(\frac{1}{4} - \frac{1}{6e} + \frac{1}{24e^2}\right) = \frac{\pi i}{4} + \int_{-\infty}^{\infty} \dfrac{e^{ix}\,dx}{x(x^2+1)(x^2+4)}$

Now $\int_{-\infty}^{\infty} \dfrac{\cos x\,dx}{x(x^2+1)(x^2+4)} = 0$ as the integrand is odd. 3

$\therefore \int_{-\infty}^{\infty} \dfrac{\sin x\,dx}{x(x^2+1)(x^2+4)} = 2\pi\left(\frac{1}{8} - \frac{1}{6e} + \frac{1}{24e^2}\right)$

$= \dfrac{\pi}{12e^2}(3e^2 - 4e + 1)$

$= \dfrac{\pi}{12e^2}(3e-1)(e-1)$

Unseen.

Setter's initials JDG | Checker's initials ADG | Page number

EE 3

| | EXAMINATION QUESTIONS/SOLUTIONS 2009-2010 | Course EE2 paper 3 |
|---|---|---|
| Question EE3 | | Marks & seen/unseen |

Parts

$\sin\theta = \frac{1}{2i}(z - z^{-1})$ $\qquad z = e^{i\theta}$

$I = \int_0^{2\pi} \frac{d\theta}{4+\sin\theta} = \frac{1}{i}\oint_c \frac{dz}{z[4+(z-z^{-1})/2i]}$ because $dz = iz\,d\theta$ — 6

$= 2\oint_c \frac{dz}{z^2+8iz-1}$

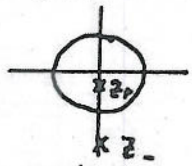
Now $z^2+8iz-1=0$ has roots at $z=z_+$ & $z=z_-$

where $2z_\pm = -8i \pm [-64+4]^{1/2}$ — 4

or $z_\pm = -4i \pm i\sqrt{15}$ $\quad z_+$ inside C

$z_-$ outside C

$\therefore \frac{1}{2}I = \oint_c \frac{dz}{(z-z_+)(z-z_-)}$



$\overset{\text{R.T.}}{=} 2\pi i \times$ (Res. of integrand at $z_+$) — 6

Residue of $(z-z_+)^{-1}(z-z_-)^{-1}$ at $z=z_+$

$= (z_+-z_-)^{-1}$

$= (2i\sqrt{15})^{-1}$ — 4

$\therefore \quad I = 2\pi/\sqrt{15}.$

Seen Similar

| Setter's initials | Checker's initials | Page number |
|---|---|---|
| JDG | AOG | |

EE 4

| | EXAMINATION QUESTIONS/SOLUTIONS 2009-2010 | Course EE2 paper 3 |
|---|---|---|
| Question EE4 | | Marks & seen/unseen |

Parts

a) $$\int_{-\infty}^{\infty}|f(t)|^2dt=\int_{-\infty}^{\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\omega t}\bar{f}(\omega)d\omega\right)\left(\int_{-\infty}^{\infty}\frac{1}{2\pi}e^{i\omega' t}\bar{f}(\omega')d\omega'\right)^{*}dt$$

$$=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}e^{i\omega t}\bar{f}(\omega)d\omega\right)\left(\int_{-\infty}^{\infty}e^{-i\omega' t}\bar{f}^{*}(\omega')d\omega'\right)dt$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\bar{f}(\omega)\left(\int_{-\infty}^{\infty}\bar{f}^{*}(\omega')\underbrace{\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i(\omega-\omega')t}dt\right\}}_{\delta(\omega-\omega')}d\omega'\right)d\omega$$ 8

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\bar{f}(\omega)\left(\int_{-\infty}^{\infty}\bar{f}^{*}(\omega')\delta(\omega-\omega')d\omega'\right)d\omega$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\bar{f}(\omega)\bar{f}^{*}(\omega)d\omega=\frac{1}{2\pi}\int_{-\infty}^{\infty}|\bar{f}(\omega)|^2d\omega \quad \text{Q.E.D.}$$ seen

b) i) $$\bar{\Lambda}(\omega)=\int_{-1}^{0}e^{-i\omega t}(1+t)dt+\int_{0}^{1}e^{-i\omega t}(1-t)dt$$

$$=-\frac{1}{i\omega}\left\{\int_{-1}^{0}(1+t)d(e^{-i\omega t})+\int_{0}^{1}(1-t)d(e^{-i\omega t})\right\}$$

$$\overset{\text{parts}}{=}-\frac{1}{i\omega}\int_{0}^{1}(e^{-i\omega t}-e^{i\omega t})dt=\frac{2}{\omega}\int_{0}^{1}\sin\omega t\,dt$$ 6

$$=\frac{2}{\omega^2}(1-\cos\omega)=\frac{4\sin^2\omega}{\omega^2}=\operatorname{sinc}^2\omega$$ seen

ii) $$\int_{-\infty}^{\infty}\operatorname{sinc}^4\omega\,d\omega=\int_{-\infty}^{\infty}|\bar{\Lambda}(\omega)|^2d\omega\overset{\text{Parseval}}{=}2\pi\int_{-\infty}^{\infty}|\Lambda(t)|^2dt$$

$$=2\pi\left\{\int_{-1}^{0}(1+t)^2dt+\int_{0}^{1}(1-t)^2dt\right\}$$ 6

$$=2\pi\left\{\left[t+t^2+t^3/3\right]_{-1}^{0}+\left[t-t^2+t^3/3\right]_{0}^{1}\right\}$$ unseen

$$=4\pi/3.$$

Setter's initials | Checker's initials ADG | Page number

EE5

EXAMINATION QUESTIONS/SOLUTIONS 2009-2010

Course: EE2 paper 3

Question: EE5

Marks & seen/unseen

Parts

$$\ddot{x} + 2\dot{x} + 5x = f(t) \; ; \; x(0) = x_0 ; \; \dot{x}(0) = -x_0$$

$$\therefore \left(s^2\bar{x}(s) - s x_0 + x_0\right) + 2\left(s\bar{x}(s) - x_0\right) + 5\bar{x}(s) = \bar{f}(s) \qquad 4$$

$$\therefore \left(s^2 + 2s + 5\right)\bar{x}(s) = (s+2)x_0 - x_0 + \bar{f}(s)$$

$$\therefore \bar{x}(s) = \underbrace{\frac{(s+1)x_0}{(s+1)^2 + 4}}_{\text{C.F.}} + \underbrace{\frac{\bar{f}(s)}{(s+1)^2+4}}_{\text{P.I.}} \qquad \text{completing the square} \qquad 3$$

$$\therefore \bar{x}(s) = x_0\,\bar{g}_1(s) + \bar{f}(s)\,\bar{g}_2(s)$$

$$\bar{g}_1(s) = \frac{s+1}{(s+1)^2+2^2} \Rightarrow g_1(t) = e^{-t}\mathcal{L}^{-1}\left(\frac{s}{s^2+2^2}\right)$$

$$= e^{-t}\cos 2t \qquad \text{Shift Thm} \qquad 4$$

$$\bar{g}_2(s) = \frac{1}{2}\cdot\frac{2}{(s+1)^2+2^2} = \frac{1}{2}e^{-t}\mathcal{L}^{-1}\left(\frac{2}{s^2+2^2}\right)$$

$$= \frac{1}{2}e^{-t}\sin 2t \qquad 4$$

$$\therefore x(t) = x_0\,g_1(t) + \mathcal{L}^{-1}\left\{\bar{f}(s)\,\bar{g}_2(s)\right\}$$

$$= x_0\,g_1(t) + f(t) * g_2(t) \qquad \text{Convolution Thm} \qquad 3$$

$$= x_0 e^{-t}\cos 2t + \frac{1}{2}\int_0^t e^{-t'}\sin(2t')f(t-t')\,dt' \qquad 2$$

Unseen but similar

Setter's initials: J.D.G.

Checker's initials: AOG

Page number

EE 6

EXAMINATION QUESTIONS/SOLUTIONS 2009-2010

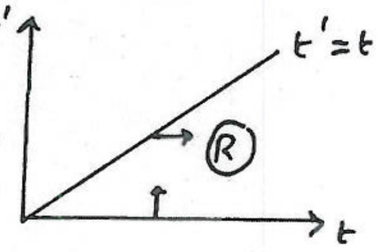
Course: EE2 paper 3

Question: EE6

Parts | Marks & seen/unseen

a) $\mathcal{L}(f*g) = \int_0^\infty e^{-st}\left(\int_0^t f(t')g(t-t')dt'\right)dt$

$= \int_0^\infty f(t')\left\{\int_{t'}^\infty e^{-st} g(t-t')dt\right\}dt'$




The change of limits occurs in order to cover the wedge-like region R after a change of order of integration.

4 picture

2 change of limits

Now let $\tau = t - t'$ to get

$\mathcal{L}(f*g) = \int_0^\infty f(t')\left(e^{-st'}\int_0^\infty e^{-s\tau}g(\tau)d\tau\right)dt'$ 4

$= \underline{\int_0^\infty f(t')e^{-st'}dt'}\int_0^\infty e^{-s\tau}g(\tau)d\tau = \bar{f}(s)\bar{g}(s)$ (seen)

b) Let $\frac{s}{(s^2+1)^2} = \frac{s}{s^2+1}\cdot\frac{1}{s^2+1} \equiv \bar{f}(s)\bar{g}(s)$ 2

$\therefore \bar{f}(s) = \frac{s}{s^2+1} \rightarrow f(t) = \cos t$ 2

$\bar{g}(s) = \frac{1}{s^2+1} \rightarrow g(t) = \sin t$ 2

$\therefore \mathcal{L}^{-1}\left(\frac{s}{(s^2+1)^2}\right) = \cos t * \sin t$ Convolution Thm

$= \int_0^t \sin(t-t')\cos t'\,dt'$

$= \frac{1}{2}\int_0^t \{\sin t + \sin(t-2t')\}dt'$

$= \frac{1}{2}t\sin t + \frac{1}{4}\left[\cos(t-2t')\right]_0^t$ 4

$= \frac{1}{2}t\sin t$

(unseen) but similar

Setter's initials: JDG

Checker's initials: AOG

Page number

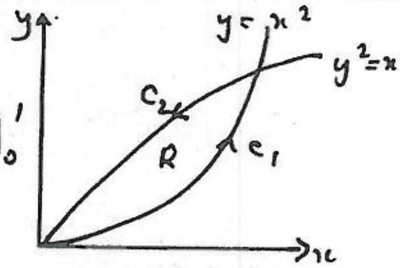
EE 7

| | EXAMINATION QUESTIONS/SOLUTIONS 2009-2010 | Course EE 2 paper 3 |
|---|---|---|
| Question EE7 | | Marks & seen/unseen |
| Parts | | |

a) $\iint dx\,dy = \int_0^1 \left( \int_{x^2}^{\sqrt{x}} dy \right) dx$

$= \int_0^1 (x^{1/2} - x^2)\,dx = \left[ \frac{2}{3} x^{3/2} - \frac{1}{3} x^3 \right]_0^1$ 2+2

$= 1/3$




b) $\oint_C P\,dx + Q\,dy \rightarrow P = y^4/x^2 \ \& \ Q = -y^3/2x$

$Q_x - P_y = -7/2 \cdot y^3/x^2$ 2

G.T.

$= \iint_R (Q_x - P_y)\,dx\,dy = -7/2 \iint (y^3/x^2)\,dx\,dy$ 4

$= -7/2 \int_0^1 x^{-2} \left( \int_{x^2}^{x^{1/2}} y^3\,dy \right) dx = -\frac{7}{8} \int_0^1 x^{-2}(x^2 - x^8)\,dx$

$= -7/8 \int_0^1 (1 - x^6)\,dx = -7/8 \cdot 6/7 = -3/4$ 4

c) $\int_{C_1} \frac{2y^4\,dx - xy^3\,dy}{2x^2}$ on $y = x^2 \Rightarrow dy = 2x\,dx$.

$= \int_0^1 \frac{1}{2x^2} (2x^8 - 2x^8)\,dx = 0$ 3

$\int_{C_2} \left( \frac{\quad}{\quad} \right) = \int_1^0 \frac{1}{2y^4} (4y^5 - y^5)\,dy$ on $x = y^2$

$= -\frac{3}{2} \int_0^1 y\,dy = -3/4.$ 3

Unseen

| Setter's initials | Checker's initials | Page number |
|---|---|---|
| J.D.G. | ADG | |

EE8

EXAMINATION QUESTIONS/SOLUTIONS 2009-2010

Course: CORE

Question: C3 p2

Parts

Marks & ~~seen~~/unseen

To evaluate the integral we first note that $R \to R^*$ where $R^*$ is the region between the two circles $u^2+v^2 = \frac{1}{4}$ and $u^2+v^2=1$ and the two lines $v = \frac{1}{\sqrt{3}}u$ and $v = \sqrt{3}\,u$. 3

for specification of $R^*$ including diagrams below

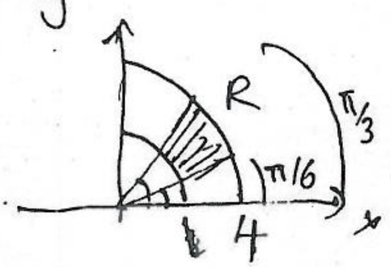
Then $\iint_R \frac{1}{(x^2+y^2)^2} \exp\left(\frac{1}{x^2+y^2}\right) dx\,dy$ 3

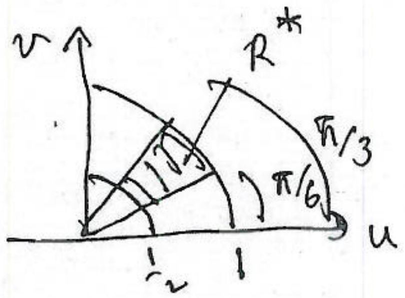
$= \iint_{R^*} \exp(u^2+v^2)\, du\, dv$

Now change to polar coordinates $\rho$ and $\theta$ where $u = \rho\cos\theta$, $v = \rho\sin\theta$ $|J| = \rho$

Thus integral $= \int_{\theta=\pi/6}^{\pi/3} \int_{\rho=\frac{1}{2}}^{1} \rho e^{\rho^2}\, d\rho\, d\theta$ 4

$= \frac{\pi}{6}\left[\frac{1}{2} e^{\rho^2}\right]_{1/2}^{1} = \frac{\pi}{12}\left[e - e^{1/4}\right]$




20

Setter's initials: RCJ.

Checker's initials: AOG

Page number: 2/2

EE8

EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2009-2010

Course: CORE

Question: C3 (i)

Marks & ~~seen~~/unseen

Parts

$$u^2+v^2 = \frac{x^2}{(x^2+y^2)^2} + \frac{y^2}{(x^2+y^2)^2} = \frac{1}{x^2+y^2}$$

Thus if $x^2+y^2=a^2$, $u^2+v^2 = \frac{1}{a^2}$ 2

If $y=mx$ then $u=\frac{x}{x^2+y^2}$ and $v=\frac{mx}{x^2+y^2}$ 2

and $\therefore$ $v = mu$.

$$\frac{\partial u}{\partial x} = \frac{1}{x^2+y^2} - \frac{x\,2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = -\frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial x} = -\frac{2xy}{(x^2+y^2)^2}, \quad \frac{\partial v}{\partial y} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

Thus $J' = \begin{vmatrix} \frac{y^2-x^2}{(x^2+y^2)^2} & -\frac{2xy}{(x^2+y^2)^2} \\ \frac{-2xy}{(x^2+y^2)^2} & \frac{x^2-y^2}{(x^2+y^2)^2} \end{vmatrix} = +\frac{1}{(x^2+y^2)^2}$ 4

Hence $|J'| = \frac{1}{(x^2+y^2)^2}$ 2

$J$ and $J'$ satisfy $|J| = \frac{1}{|J'|} = (x^2+y^2)^2$

Setter's initials: RLJ

Checker's initials: AOG

EE9

EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2009-2010

Course: COR5

Question: C4 p1

Marks & seen/unseen

Parts

$$\text{grad}\,\varphi = \underline{i}\frac{\partial\varphi}{\partial x} + \underline{j}\frac{\partial\varphi}{\partial y} + \underline{k}\frac{\partial\varphi}{\partial z}$$

$$\text{div}\,\underset{\sim}{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\text{curl}\,\underset{\sim}{F} = \begin{pmatrix} \hat{\underline{i}} & \hat{\underline{j}} & \underline{k} \\ \partial_x & \partial_y & \partial_z \\ F_1 & F_2 & F_3 \end{pmatrix} = \hat{\underline{i}}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \hat{\underset{\sim}{j}}(\dots) + \underline{k}(\dots)$$ 3

(i) $$\text{div curl}\,\underline{F} = \frac{\partial}{\partial x}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) = 0$$ 2

$$[\text{curl grad}\,\psi]_1 = \frac{\partial}{\partial y}\left(\frac{\partial\psi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial\psi}{\partial y}\right) = 0$$

similarly for other components

(ii) $$\text{div}(\varphi\underset{\sim}{F}) = \frac{\partial}{\partial x}(\varphi F_1) + \frac{\partial}{\partial y}(\varphi F_2) + \frac{\partial}{\partial z}(\varphi F_3)$$

$$= F_1\frac{\partial\varphi}{\partial x} + F_2\frac{\partial\varphi}{\partial y} + F_3\frac{\partial\varphi}{\partial z} + \varphi\frac{\partial F_1}{\partial x} + \varphi\frac{\partial F_2}{\partial y} + \varphi\frac{\partial F_3}{\partial z}$$ 2

$$= \underline{F}\cdot\text{grad}\,\varphi + \varphi\,\text{div}\,\underset{\sim}{F}$$

(iii) $$[\text{curl}(\varphi\underset{\sim}{F})]_1 = \frac{\partial}{\partial y}(\varphi F_3) - \frac{\partial}{\partial z}(\varphi F_2)$$

$$= \varphi\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \frac{\partial\varphi}{\partial y}F_3 - \frac{\partial\varphi}{\partial z}F_2$$ 2

$$= \varphi[\text{curl}\,\underset{\sim}{F}]_1 + [\text{grad}\,\varphi \times \underset{\sim}{F}]_1$$

similarly for other components

ALL SEEN

Setter's initials: RLi

Checker's initials: AOG

Page number: 1/2

EE9

EXAMINATION QUESTIONS/SOLUTIONS 2009-2010

Course: CORE

Question: C4 p2

Marks & seen/unseen

Parts

When $\underset{\sim}{F} = \text{curl}\,\underset{\sim}{A}$, $\text{div}\,\underset{\sim}{F} = \text{div curl}\,\underset{\sim}{A} = 0$ by (i). Hence by (ii)

$$\text{div}(\varphi \underset{\sim}{F}) = \underset{\sim}{F} \cdot \text{grad}\,\varphi = F_1 \frac{\partial \varphi}{\partial x} + F_2 \frac{\partial \varphi}{\partial y} + F_3 \frac{\partial \varphi}{\partial z}$$ 3

---

When $\underset{\sim}{F} = \text{grad}\,\psi$, $\text{curl}\,\underset{\sim}{F} = \text{curl grad}\,\psi = 0$ by (i). Hence by (iii)

$$\text{curl}(\varphi \underset{\sim}{F}) = \text{grad}\,\varphi \times \underset{\sim}{F} = \text{grad}\,\varphi \times \text{grad}\,\psi$$ 3

---

$$\text{curl}\,\underset{\sim}{F} = \begin{vmatrix} \underset{-}{i} & \underset{-}{j} & \underset{-}{k} \\ \partial_x & \partial_y & \partial_z \\ x+yz & y+zx & z+xy \end{vmatrix} = \underset{-}{i}(x-x) - \underset{-}{j}(y-y) + \underset{-}{k}(z-z) = 0$$

$$\frac{\partial \psi}{\partial x} = x + yz, \quad \frac{\partial \psi}{\partial y} = y + zx, \quad \frac{\partial \psi}{\partial z} = z + xy$$

$$\frac{\partial \psi}{\partial x} = x + yz \Rightarrow \psi(x,y,z) = \tfrac{1}{2}x^2 + xyz + f(y,z)$$

Substitute into eqn. for $\frac{\partial \psi}{\partial y}$ to get

$$xz + \frac{\partial f}{\partial y} = y + zx \Rightarrow f(y,z) = \tfrac{1}{2}y^2 + g(z)$$

Substitute $\psi$ into eqn. for $\frac{\partial \psi}{\partial z}$ to get

$$xy + \frac{dg}{dz} = z + xy \Rightarrow g = \tfrac{1}{2}z^2 + c$$ 5

$$\Rightarrow \psi(x,y,z) = \tfrac{1}{2}(x^2 + y^2 + z^2) + xyz + c.$$ 20

ALL UNSEEN

Setter's initials: RLJ

Checker's initials: AOG

Core Q5 Solution to C5 EE10

Line integral is independent of path if $\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y}$. 2

In this case $f = x^3y^2$, $g = \alpha x^4 y$

so l.i. is independent of path if

$$4\alpha x^3 y = 2x^3 y$$ 2

i.e. $\alpha = 1/2$.

Many paths are possible but two simple paths are $P_1$, a straight line of equation $y=x$ joining A to B, and $P_2$, a path of two parts with the first being a horizontal straight line $y=0$ running from A to C(1,0) and the second a vertical straight line $x=1$ running from C to B.

Then $$\int_{P_1 A}^{B} = \int_0^1 (x^5\,dx + \beta x^5)\,dx \quad (1)$$

$$= \frac{1}{6}(1+\beta)$$

and $$\int_{P_2, A}^{B} = \int_A^C + \int_C^B$$

$$= 0 + \beta\int_0^1 y\,dy$$

The first term is zero because y and dy are both zero on $P_1$. The second term uses the fact that $x=1$ and $dx=0$ on $P_2$ 8

thus $\int_{P_2,A}^{B} = \frac{1}{2}\beta \quad (2)$

RLJ. ADG (1)+(2) $\Rightarrow \beta = \frac{1}{2}$ same as $\alpha$

CONTINUED

Core Q 5 EE 10

Solution to CS (Continued)

$$\frac{\partial \varphi}{\partial x} = f = x^3 y^2, \quad \frac{\partial \varphi}{\partial y} = g = \frac{1}{2} x^4 y$$

$$\frac{\partial \varphi}{\partial x} = x^3 y^2 \Rightarrow \varphi(x,y) = \frac{x^4 y^2}{4} + \cancel{g}\, h(y)$$

Substitute into $\frac{\partial \varphi}{\partial y}$ to get

$$\frac{2x^4 y}{2} + \frac{dh}{dy} = \frac{1}{2} x^4 y$$

$$\therefore \frac{dh}{dy} = 0 \Rightarrow h = c$$

$$\varphi(x,y) = \frac{x^4 y^2}{4} + c.$$

8

Total 20

RLJ.

AOG

EE II (3) 11

i. We require

$$\begin{aligned}\int_1^\infty Ce^{-\lambda t}\,\mathrm{d}t &= 1\\ C\left[-\frac{e^{-\lambda t}}{\lambda}\right]_1^\infty &= 1\\ \frac{C}{\lambda}(0+(e^{-\lambda})) &= 1\\ C &= \lambda e^{\lambda}\end{aligned}$$

[4]

For $x \geq 1$, the cdf is

$$\begin{aligned}F(x) &= \lambda e^{\lambda}\int_1^x e^{-\lambda t}\,\mathrm{d}t\\ &= \lambda e^{\lambda}\left[-\frac{e^{-\lambda t}}{\lambda}\right]_1^x\\ &= \lambda e^{\lambda}\left(-\frac{e^{-\lambda x}}{\lambda}+\frac{e^{-\lambda}}{\lambda}\right)\\ &= 1-e^{\lambda(1-x)}\end{aligned}$$

[2]

So $F(x)=0$ for $x \leq 1$, $F(x) = 1-e^{\lambda(1-x)}$ for $x > 1$.

[1]

The median is the value $x$ such that $F(x) = 0.5$

$$\begin{aligned}1-e^{\lambda(1-x)} &= 0.5\\ e^{\lambda(1-x)} &= 0.5\\ \lambda(1-x) &= \log(0.5)\\ x &= 1-\frac{\log(0.5)}{\lambda}\end{aligned}$$

[3]

NA

EE II (3) 11

ii.

$$\begin{aligned}
P(T > t+s | T > s) &= \frac{P(\{T > t+s\} \cap \{T > s\})}{P(T > s)} \\
&= \frac{P(T > t+s)}{P(T > s)} = \frac{1 - F(t+s)}{1 - F(s)} \\
&= \frac{e^{\lambda(1-(t+s))}}{e^{\lambda(1-s)}} = \underline{e^{-\lambda t}}
\end{aligned}$$

[3]

iii.

$$\begin{aligned}
L(\lambda) &= \prod_{i=1}^{n} \lambda e^{\lambda} e^{-\lambda t_i} \\
&= \lambda^n e^{n\lambda} e^{-\lambda \sum_{i=1}^{n} t_i}
\end{aligned}$$

taking logs

$$\log(L(\lambda)) = n\log(\lambda) + n\lambda - \lambda \sum_{i=1}^{n} t_i$$

we seek a maximum

$$\frac{d\log(L(\lambda))}{d\lambda} = \frac{n}{\lambda} + n - \sum_{i=1}^{n} t_i$$

turning points occur when the derivative is 0, so

$$0 = \frac{n}{\hat{\lambda}} + n - \sum_{i=1}^{n} t_i = n + n\hat{\lambda} - \hat{\lambda} \sum_{i=1}^{n} t_i$$

9

agreed

so the maximum likelihood estimator of $\lambda$ is

$$\hat{\lambda} = \frac{-n}{n - \sum_{i=1}^{n} t_i} = \frac{1}{\frac{1}{n}\sum_{i=1}^{n} t_i - 1} = \frac{1}{\bar{t} - 1}$$

and examine the second derivative

$$\frac{d^2\log(L(\lambda))}{d\lambda^2} = -\frac{n}{\lambda^2}, < 0 \ \forall \ \lambda \text{ including } \hat{\lambda}$$

to verify that this solution is a maximum.

[7]

TOTAL [20]

# EXAMINATION SOLUTIONS 2009-10

| Question | | Course |
|---|---|---|
| 12 | | EE2(3) |

| Parts | | Marks & seen/unseen |
|---|---|---|
| (i) | A time series $\{e_t\}$ is called white noise if<br>$\mathrm{E}(e_t) = 0$ for all $t$,<br>$\mathrm{cov}(e_t, e_s) = 0$ for all $t \neq s$<br>$\mathrm{Var}(e_t)$ does not depend on $t$. | 3 |
| (ii) | $\mathrm{cov}(y_t, y_t) = \mathrm{Var}(y_t) = 0.4^2\,\mathrm{Var}(e_t) + 0.3^2\,\mathrm{Var}(e_{t-1}) + 0.3^2\,\mathrm{Var}(e_{t-2})$<br>$= 0.4^2 + 0.3^2 + 0.3^2 = 0.34$<br>$\mathrm{cov}(y_t, y_{t+1}) = 0.4 \cdot 0.3\,\mathrm{cov}(e_t, e_t) + 0.3 \cdot 0.3\,\mathrm{cov}(e_{t-1}, e_{t-1})$<br>$= 0.4 \cdot 0.3 + 0.3 \cdot 0.3 = 0.21$<br>$\mathrm{cov}(y_t, y_{t+2}) = 0.4 \cdot 0.3 = 0.12$<br>$\mathrm{cov}(y_t, y_{t+k}) = 0$ for $k = 3, 4, \ldots$ | 6 |
| (iii) | $\rho_1 = 0.21/0.34 \approx 0.618$<br>$\rho_2 = 0.12/0.34 \approx 0.353$<br>$\rho_k = 0$ for $k = 3, 4, \ldots$. | 3 |
| (iv) | $\mathrm{E}(y_t) = t + 0.3\,\mathrm{E}(e_t) + 0.5\,\mathrm{E}(e_{t-1}) + 0.2\,\mathrm{E}(e_{t-2}) = t$<br>$\{y_t\}$ is not stationary because $\mathrm{E}(y_t)$ depends on $t$. | 2 |
| (v) | $\mathrm{E}(x_t) = \mathrm{E}(y_t) - t = 0$, $\mathrm{cov}(x_t, x_{t+s}) = \mathrm{cov}(y_t, y_{t+s})$.<br>Since both $\mathrm{E}(x_t)$ and $\mathrm{cov}(x_t, x_{t+s})$ do not depend on $t$, the time series $\{x_t\}$ is stationary. | 3 |
| (vi) | The spectrum is given by<br>$$f(\omega) = \mathrm{cov}(x_t, x_t) + 2\sum_{k=1}^{\infty} \mathrm{cov}(x_t, x_{t+k}) \cos(k\omega) = 0.34 + 0.42\cos(\omega) + 0.24\cos(2\omega).$$ | 3 |

| Setter's initials | Checker's initials | Page number |
|---|---|---|
| | | |

EE II (3) 11

i. We require

$$\int_1^\infty Ce^{-\lambda t}\,\mathrm{d}t = 1$$
$$C\left[-\frac{e^{-\lambda t}}{\lambda}\right]_1^\infty = 1$$
$$\frac{C}{\lambda}(0 + (e^{-\lambda})) = 1$$
$$C = \lambda e^{\lambda}$$

[4]

For $x \geq 1$, the cdf is

$$\begin{aligned} F(x) &= \lambda e^{\lambda}\int_1^x e^{-\lambda t}\,\mathrm{d}t \\ &= \lambda e^{\lambda}\left[-\frac{e^{-\lambda t}}{\lambda}\right]_1^x \\ &= \lambda e^{\lambda}\left(-\frac{e^{-\lambda x}}{\lambda} + \frac{e^{-\lambda}}{\lambda}\right) \\ &= 1 - e^{\lambda(1-x)} \end{aligned}$$

[2]

So $F(x) = 0$ for $x \leq 1$, $F(x) = 1 - e^{\lambda(1-x)}$ for $x > 1$.

[1]

The median is the value $x$ such that $F(x) = 0.5$

$$\begin{aligned} 1 - e^{\lambda(1-x)} &= 0.5 \\ e^{\lambda(1-x)} &= 0.5 \\ \lambda(1-x) &= \log(0.5) \\ x &= 1 - \frac{\log(0.5)}{\lambda} \end{aligned}$$

[3]

NA

EE II (3) 11

ii.

$$\begin{aligned} P(T > t+s | T > s) &= \frac{P(\{T > t+s\} \cap \{T > s\})}{P(T > s)} \\ &= \frac{P(T > t+s)}{P(T > s)} = \frac{1 - F(t+s)}{1 - F(s)} \\ &= \frac{e^{\lambda(1-(t+s))}}{e^{\lambda(1-s)}} = \underline{e^{-\lambda t}} \end{aligned}$$

[3]

iii.

$$\begin{aligned} L(\lambda) &= \prod_{i=1}^{n} \lambda e^{\lambda} e^{-\lambda t_i} \\ &= \lambda^n e^{n\lambda} e^{-\lambda \sum_{i=1}^{n} t_i} \end{aligned}$$

taking logs

$$\log(L(\lambda)) = n\log(\lambda) + n\lambda - \lambda \sum_{i=1}^{n} t_i$$

we seek a maximum

$$\frac{d\log(L(\lambda))}{d\lambda} = \frac{n}{\lambda} + n - \sum_{i=1}^{n} t_i$$

turning points occur when the derivative is 0, so

$$0 = \frac{n}{\hat{\lambda}} + n - \sum_{i=1}^{n} t_i = n + n\hat{\lambda} - \hat{\lambda} \sum_{i=1}^{n} t_i$$

9

agreed

so the maximum likelihood estimator of $\lambda$ is

$$\hat{\lambda} = \frac{-n}{n - \sum_{i=1}^{n} t_i} = \frac{1}{\frac{1}{n}\sum_{i=1}^{n} t_i - 1} = \frac{1}{\bar{t} - 1}$$

and examine the second derivative

$$\frac{d^2\log(L(\lambda))}{d\lambda^2} = -\frac{n}{\lambda^2}, < 0 \ \forall \ \lambda \text{ including } \hat{\lambda}$$

to verify that this solution is a maximum.

[7]

**TOTAL [20]**

| | EXAMINATION SOLUTIONS 2009-10 | Course EE2(3) |
|---|---|---|
| Question 12 | | Marks & seen/unseen |
| Parts | | |
| (i) | A time series $\{e_t\}$ is called white noise if<br>$\mathrm{E}(e_t) = 0$ for all $t$,<br>$\mathrm{cov}(e_t, e_s) = 0$ for all $t \neq s$<br>$\mathrm{Var}(e_t)$ does not depend on $t$. | 3 |
| (ii) | $\mathrm{cov}(y_t, y_t) = \mathrm{Var}(y_t) = 0.4^2\,\mathrm{Var}(e_t) + 0.3^2\,\mathrm{Var}(e_{t-1}) + 0.3^2\,\mathrm{Var}(e_{t-2})$<br>$= 0.4^2 + 0.3^2 + 0.3^2 = 0.34$<br>$\mathrm{cov}(y_t, y_{t+1}) = 0.4 \cdot 0.3\,\mathrm{cov}(e_t, e_t) + 0.3 \cdot 0.3\,\mathrm{cov}(e_{t-1}, e_{t-1})$<br>$= 0.4 \cdot 0.3 + 0.3 \cdot 0.3 = 0.21$<br>$\mathrm{cov}(y_t, y_{t+2}) = 0.4 \cdot 0.3 = 0.12$<br>$\mathrm{cov}(y_t, y_{t+k}) = 0$ for $k = 3, 4, \ldots$ | 6 |
| (iii) | $\rho_1 = 0.21/0.34 \approx 0.618$<br>$\rho_2 = 0.12/0.34 \approx 0.353$<br>$\rho_k = 0$ for $k = 3, 4, \ldots$. | 3 |
| (iv) | $\mathrm{E}(y_t) = t + 0.3\,\mathrm{E}(e_t) + 0.5\,\mathrm{E}(e_{t-1}) + 0.2\,\mathrm{E}(e_{t-2}) = t$<br>$\{y_t\}$ is not stationary because $\mathrm{E}(y_t)$ depends on $t$. | 2 |
| (v) | $\mathrm{E}(x_t) = \mathrm{E}(y_t) - t = 0$, $\mathrm{cov}(x_t, x_{t+s}) = \mathrm{cov}(y_t, y_{t+s})$.<br>Since both $\mathrm{E}(x_t)$ and $\mathrm{cov}(x_t, x_{t+s})$ do not depend on $t$, the time series $\{x_t\}$ is stationary. | 3 |
| (vi) | The spectrum is given by<br>$$f(\omega) = \mathrm{cov}(x_t, x_t) + 2\sum_{k=1}^{\infty} \mathrm{cov}(x_t, x_{t+k}) \cos(k\omega) = 0.34 + 0.42\cos(\omega) + 0.24\cos(2\omega).$$ | 3 |
| | Setter's initials | Checker's initials | Page number |