

1. a) (i) $P(S_1) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}.$

E2.17

~~E2.20~~

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(ii) $S_1 \cup S_2 = \{a,b,c\}$

(iii) $S_1 \cap S_2 = \{a,b\}$

(iv) $S_2 - S_1 = \{\emptyset\}$

(v) $|S_1 \cup S_2| = 3$

[6 MARKS]

b) Finite: S_1 from (a)

Infinite - Countable: \mathbb{N}

Infinite - Uncountable: \mathbb{R}

[3 MARKS]

c) (i) No, e.g. $(2,1) \in R$ but $(1,2) \notin R.$

(ii) No, e.g. $(2,1) \in R$ and $(4,2) \in R$ but $(4,1) \notin R.$

(iii) No, e.g. $(1,1) \notin R.$

(iv) No, e.g. there is no element $(1,x)$ for any $x.$

(v) No - (ii) implies this. [6 MARKS]

d) (i) $\forall x (J(x) \rightarrow L(x))$

(ii) $\exists x (L(x) \wedge \neg J(x))$

(iii) $\forall x (J(x) \rightarrow A(x, \text{Jones}))$

(iv) $\forall x \forall y (A(x,y) \wedge J(x) \rightarrow J(y))$. [5 MARKS]

e) Simplify (i): b

Modus ~~Tollens~~ ^{Tollens} w/ (ii): $\neg C$

[2 MARKS]

1. f) (i) From the theorem, $f(x)$ is $O(x^3)$.

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$$|f(x)| = |x^3 + 2x^2 + 1| \geq |x^3| + |2x^2| + 1 \quad \text{for } x > 0$$

$$> |x^3|$$

So with any κ , e.g. $\kappa = 1$ Δ with $c = 1$,

$$\forall x ((x > \kappa) \rightarrow |f(x)| > c|x^3|) \quad \square$$

(ii) `proc1 (int x) {`
 for $i = 1$ to $x * x * x + 2 * x * x + 1 - 4$
 $avar = avar * 2$;
`}`

(iii) `proc2 (int x) {`
 if $x = 1$.
 return $2 * x * x$;
 else
 return $proc2(x-1) * proc2(x-1)$; [9 MARKS]
`}`

g) let $a > 1$ be a real number, $b > 1$ be an integer,
 $c > 0$ be a real number and $d > 0$ be a real number

let f be an increasing function s.t.
 $f(n) = a f(n/b) + cn^d$ whenever $n = b^k$
 for positive integer k .

- (i) If $a < b^d$, $f(n)$ is $O(n^d)$
- (ii) If $a = b^d$, $f(n)$ is $O(n^d \log n)$
- (iii) If $a > b^d$, $f(n)$ is $O(n^{\log_b a})$.

[9 MARKS]

3. a) (i) Let $x \in B = \mathbb{R}$.

Then $2^x \in \mathbb{R}$ and $2^x > 0 \Rightarrow 2^x \in \mathbb{R}_+$.

So $f(2^x, 0) = x \quad \square$. [3 MARKS]

(ii) Choose $A = \mathbb{R}_+ \times \{0\}$

Proof above holds for surjectivity.

For injectivity,

$$f(x_1, y_1) = f(x_2, y_2)$$

But $y_1 = y_2 = 0$.

$$\text{Also } \log_2(x_1 + 0) = \log_2(x_2 + 0)$$

$$\Rightarrow x_1 = x_2 \quad \square. \quad [6 \text{ MARKS}]$$

(iii) It is possible to obtain any ~~positive~~ non-negative value $v \in \mathbb{R}_+ \cup \{0\}$ from

$$f(2^v, 0) = v \quad \& \quad 2^v \geq 1$$

~~Also zero is possible in f~~

Negative values are not possible - we

would require $f(x, y) < 0$

$$\Rightarrow \log_2(x + y) < 0$$

$$\text{So either } \log_2 x < 0 \Rightarrow x < 1 \quad x$$

$$\text{or } \log_2(x+1) < 0 \Rightarrow x+1 < 1$$

$$\text{i.e. } x < 0 \quad x.$$

So image is $\mathbb{R}_+ \cup \{0\}$

[6 MARKS]

3. (b)(i) $R^n \subseteq R \Rightarrow R$ is transitive:

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$R^n \subseteq R \Rightarrow R^2 \subseteq R$. $(a,b) \in R$ and $(b,c) \in R$
then $(a,c) \in R^2$. But $R^2 \subseteq R \Rightarrow (a,c) \in R$.
So R is transitive.

R is transitive $\Rightarrow R^n \subseteq R$

True for $n=1$. Use induction to show $R^{n+1} \subseteq R$
assuming $R^n \subseteq R$.

Consider $(a,b) \in R^{n+1} = R \cdot R^n$

$\Rightarrow \exists x ((a,x) \in R \wedge (x,b) \in R^n)$

$R^n \subseteq R \Rightarrow (x,b) \in R$. R is transitive

$\Rightarrow (a,b) \in R$. So $R^{n+1} \subseteq R$. [6 MARKS]

(ii) $\forall a \in A \exists b \in B ((a,b) \in R)$

$\wedge \forall a \in A \forall b \in B \forall c \in B ((a,b) \in R \wedge (a,c) \in R$
 $\rightarrow b=c)$. [6 MARKS]

(iii) Trivially, $f: \{0\} \rightarrow \{0\}$
 $f(0)=0$ is transitive.

[3 MARKS]

4. a) (i) $T(\text{Steven}) \wedge I(\text{Steven})$

3. (ii) $G(\text{Steven}) \wedge \forall x (I(x) \rightarrow G(x))$

(iii) $\exists x (T(x) \wedge I(x))$

$\wedge \forall x \forall y (T(x) \wedge I(x) \wedge T(y) \wedge I(y) \rightarrow x=y)$

(iv) $\forall x (T(x) \wedge I(x) \rightarrow x = \text{Steven})$

(v) $T(\text{Amanda}) \wedge \neg T(\text{James})$

(vi) $\neg I(\text{Amanda})$

[12 MARKS]

b) Simplify (iii) $\Rightarrow \forall x \forall y (T(x) \wedge I(x) \wedge T(y) \wedge I(y) \rightarrow x=y)$

Universal Instantiation

$\forall x (T(x) \wedge I(x) \wedge T(\text{Steven}) \wedge I(\text{Steven}) \rightarrow x = \text{Steven})$

Hypothesis $\Rightarrow \forall x (T(x) \wedge I(x) \rightarrow x = \text{Steven})$

[9 MARKS]

c) Universal Instantiation on (iv)

$T(\text{Amanda}) \wedge I(\text{Amanda}) \rightarrow \text{Amanda} = \text{Steven}$

Modus Ponens

$\neg (T(\text{Amanda}) \wedge I(\text{Amanda}))$

$\equiv \neg T(\text{Amanda}) \vee \neg I(\text{Amanda})$ (*)

Simplify (v) $T(\text{Amanda})$ (+)

Disjunctive Syllogism (*) \wedge (+)

$\Rightarrow \neg I(\text{Amanda})$

[9 MARKS]

8. 4. a) $f(x)$ is $O(g(x)) \equiv \exists c \in \mathbb{R}^+ \exists \kappa \in \mathbb{R}^+ \forall x (x > \kappa) \rightarrow (|f(x)| \leq c |g(x)|)$
 $f(x)$ is $\Omega(g(x)) \equiv \exists c \in \mathbb{R}^+ \exists \kappa \in \mathbb{R}^+ \forall x (x > \kappa) \rightarrow (|f(x)| \geq c |g(x)|)$
 $f(x)$ is $\Theta(g(x)) \equiv [f(x) \text{ is } O(g(x))] \wedge [f(x) \text{ is } \Omega(g(x))]$.

[6 MARKS]

b) $\exists \kappa_1, \kappa_2, c_1, c_2$ s.t.

$$|f_1(x)| \leq c_1 |g_1(x)|, \quad x > \kappa_1$$

$$\& \quad |f_2(x)| \leq c_2 |g_2(x)|, \quad x > \kappa_2$$

By the triangle inequality,

$$|f_1(x) + f_2(x)| \leq |f_1(x)| + |f_2(x)|$$

$$\leq c_1 |g_1(x)| + c_2 |g_2(x)|, \quad x > \max(\kappa_1, \kappa_2)$$

$$\leq c_1 \max(|g_1(x)|, |g_2(x)|) +$$

$$c_2 \max(|g_1(x)|, |g_2(x)|)$$

$$= (c_1 + c_2) \max(|g_1(x)|, |g_2(x)|)$$

So with $c = c_1 + c_2$, $\kappa = \max(\kappa_1, \kappa_2)$,

$$f_1(x) + f_2(x) \text{ is } O(\max(|g_1(x)|, |g_2(x)|)).$$

[6 MARKS]

c)

```
proc1(int n) {
    total := 1;
    for i = 1 to 2*n
        total := total * n;
}
```

(Assuming loop
term n is
evaluated once)
[6 MARKS]

d)

```
proc2(int n) {
    if n = 1
        return 3*n;
    else if n = 2
        return 6*n*n*n;
}
```

else
return $n * \text{proc2}(n \text{ div } 3)$;
}

[6 MARKS]

8. (e) Proc1 has $O(n)$ exec time (#mults)

4. Proc2 has $O(\log n)$ exec time (#mults).

Proc therefore has $O(\max(n, \log n))$

$$= O(n)$$

$k \geq 1$

[6 MARKS]

H/H
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