Paper Number(s): **E2.5**
ISE2.7

IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE
UNIVERSITY OF LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2001

EEE/ISE PART II: M.Eng., B.Eng. and ACGI

SIGNALS AND LINEAR SYSTEMS

Wednesday, 13 June 2:00 pm

There are FIVE questions on this paper.

Answer THREE questions.

Time allowed: 2:00 hours

Examiners: Constantinides,A.G. and Barria,J.A.

1. Sketch the signal $x(t) = |\sin(\omega_0 t)|$ and show that its Fourier Series is given by

$$x(t) = \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{1}{3}\cos(2\omega_0 t) + \frac{1}{15}\cos(4\omega_0 t) + \frac{1}{35}\cos(6\omega_0 t) + \cdots\right)$$

[17]

It is given that the sum of two new signals $x_1(t)$ and $x_2(t)$ is

$$x_1(t) + x_2(t) = x(t)$$

Moreover it is known that

$$x_1(t) - x_2(t) = \sin(\omega_0 t)$$

Sketch the signals $x_1(t)$ and $x_2(t)$.

[6]

From the above relationships determine without further Fourier analysis the Fourier series for $x_1(t)$ and $x_2(t)$.

[10]

2. The transfer function of a real time-invariant system is given as

$$H(s) = \frac{1}{s-1}$$

a) The output $y(t)$ from this system is observed to be

$$\begin{aligned} y(t) &= e^{-at} - e^{-3at} & & for \quad t \geq 0 \\ &= 0 & & for \quad t < 0 \\ where \quad & a = 1 & & \end{aligned}$$

Determine the input $x(t)$.

[12]

b) What is the output when the input is $x_1(t) = 10e^{j3t}$ for all t?

[5]

c) Determine the output when the input is $x_2(t) = 10e^{-j3t}$ for all t.

[5]

d) Hence show that when the input is $x_3(t) = 6\cos(3t)$ for all t the output is given by

$$y_3(t) = \frac{6}{10}\left[\cos(3t) + 3\sin(3t)\right]$$

[11]

3. a) Determine the z-transform $X(z)$ of the signal

$$x(n) = (-0.8)^n - 0.2^n \qquad for \quad n = 0,1,2,3,...$$

Hence, determine the poles and zeros of $X(z)$ and place them on the z-plane.

[13]

b) A transfer function $H(z)$ is given as

$$H(z) = 1 + 3z^{-2} + 3z^{-4} + z^{-6}$$

What is the amplitude response at $\theta = \frac{\pi}{2}$?

[6]

The input signal to $H(z)$ is $x(n) = \cos(n\omega_0 T)$, where $\omega_0 = 2\pi.10^3 \, rad/s$ and $T = \frac{1}{4}10^{-3} \, s$. What is the output from $H(z)$?

[7]

Determine the zeros and poles of $H(z)$ on the z-plane and sketch their positions.

[7]

4. A causal digital filter has an impulse response $h(n)$ given by

$$h(0)=1 \quad h(1)=-4 \quad h(2)=6 \quad h(3)=-4 \quad h(4)=1 \quad , \quad h(n)=0 \quad for \quad n \geq 5$$

Determine its transfer function $H(z)$.

[9]

Sketch the amplitude response of $H(z)$ and indicate on your sketch the main features.

[9]

The input $x(n)$ has a z-transform $X(z)$ given by

$$X(z)=\frac{(a+bz^{-1})}{(1-z^{-1})}$$

where a, b are real coefficients.

Determine the output $y(n)$ for $n = 0,1,2,3,4$ and show that $y(n)=0$ for $n \geq 5$.

[15]

5. i) Show that the Laplace Transform of $\cos(\omega_0 t)$, $t \geq 0$ is given by $\dfrac{s}{s^2 + \omega_0^2}$

[5]

ii) The Laplace transform of a causal signal $x(t)$ is given as $X(s)$. Determine the Laplace Transform of $e^{-\alpha t}x(t)$, $t \geq 0$ and $\alpha > 0$.

[4]

iii) A causal linear time-invariant system has a transfer function

$$H(s) = 2\frac{s+\alpha}{(s+\alpha)^2 + \omega_0^2}$$

a) Determine the differential equation that relates the input $x(t)$ and the output $y(t)$.

[8]

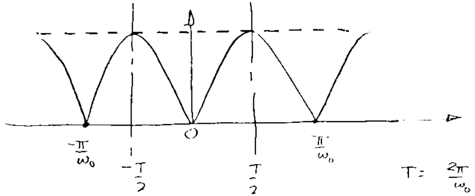
b) Use the above results, or otherwise, to determine the impulse response $h(t)$.

[12]

c) Sketch the amplitude frequency response of $H(s)$ for $\alpha > 0$.

[4]

1.



$$c_k = \frac{1}{T}\int_{-\frac{T}{2}}^{T/2} x(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T}\int_{-\frac{T}{2}}^{0} -\sin\omega_0 t\, e^{-jk\omega_0 t}\, dt$$

$$+ \frac{1}{T}\int_{0}^{T/2} \sin\omega_0 t\, e^{-jk\omega_0 t}\, dt$$

or

$$c_k = \frac{1}{T}\int_0^{T/2} \sin\omega_0 t\, \left(e^{jk\omega_0 t} + e^{-jk\omega_0 t}\right) dt$$

$$= \frac{1}{T}\int_0^{T/2} 2\sin\omega_0 t \cdot \cos(k\omega_0 t)\, dt$$

$$= \frac{1}{T}\int_0^{T/2} \left[\sin(k+1)\omega_0 t - \sin(k-1)\omega_0 t\right] dt$$

$$= \frac{1}{\omega_0 T}\left[-\frac{\cos(k+1)\omega_0 t}{k+1} + \frac{\cos(k-1)\omega_0 t}{k-1}\right]_0^{T/2}$$

$$= \frac{1}{\omega_0 T}\left[\frac{(-1)^k}{k+1} + \frac{(-1)^{k-1}}{k-1} + \frac{1}{k+1} - \frac{1}{k-1}\right]$$

for k even, $k+1$ and $k-1$ are odd

ie $C_k = \frac{1}{2\pi}\left[\frac{2}{k+1} - \frac{2}{k-1}\right] = -\frac{2}{\pi}\,\frac{1}{k^2-1}$

for k odd $k+1$, $k-1$ are even ie $c_k = 0$.

Hence

$$x(t) = \frac{2}{\pi} - \frac{2}{\pi}\sum_{l=1}^{\infty} \frac{e^{-j2l\omega_0 t} + e^{+j2l\omega_0 t}}{((2l)^2-1)}$$

or

$$x(t) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{l=1}^{\infty} \frac{\cos 2l\omega_0 t}{4l^2-1}$$

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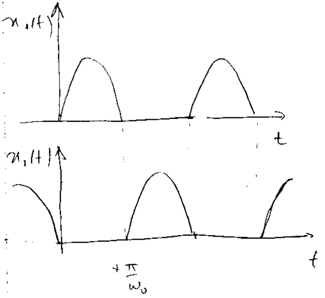
From $x_1(t) + x_2(t) = x(t)$

and $x_1(t) - x_2(t) = \sin w_0 t$

we have

$$x_1(t) = (x(t) + \sin w_0 t)/2$$

$$x_2(t) = (x(t) - \sin w_0 t)/2$$



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Hence

$$x_1(t) = \frac{1}{\pi} - \frac{2}{\pi} \sum_{l=1}^{\infty} \frac{\cos 2lw_0 t}{4l^2-1} + \frac{1}{2}\sin w_0 t$$

$$x_2(t) = \frac{1}{\pi} - \frac{2}{\pi} \sum_{l=1}^{\infty} \frac{\cos 2lw_0 t}{4l^2-1} - \frac{1}{2}\sin w_0 t.$$ 10

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2 a) From the Laplace transform of $y(t)$ we have

$$Y(s) = \frac{1}{s+a} - \frac{1}{s+3a} = H(s) \cdot X(s)$$

ie $$X(s) = (s+1)\cdot\left[\frac{1}{s+1} - \frac{1}{s+3}\right] = (s+1)\,\frac{s+3-s-1}{(s+1)(s+3)}$$

or $$X(s) = \frac{2}{s+3}.$$

Hence $$x(t) = 2e^{-3t}$$ 12

b) $$y_1(t) = \frac{10}{j3+1}\,e^{j3t} = 10\,\frac{1-j3}{10}\,e^{j3t}$$ 5

c) $$y_2(t) = \frac{10}{-j3+1}\,e^{-j3t} = 10\,\frac{1+j3}{10}\,e^{-j3t}$$ 5

d) $$y_3(t) = \frac{1}{2}\left[y_1(t) + y_2(t)\right]\cdot\frac{6}{10}$$

$$= \frac{1}{2}\cdot\frac{6}{10}\left[(1-j3)\,e^{j3t} + (1+j3)\,e^{-j3t}\right]$$

$$= \frac{6}{10}\left[\frac{e^{j3t}+e^{-j3t}}{2} + 3j\left(\frac{e^{-j3t}-e^{j3t}}{2}\right)\right]$$

$$= \frac{6}{10}\left[\cos 3t - 3\left(\frac{e^{j3t}-e^{-j3t}}{2j}\right)\right]$$

or $$y_3(t) = \frac{6}{10}\left[\cos 3t - 3\sin 3t\right]$$ 11

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HW

3.

a) $x(n) = (-0.8)^n - 0.2^n$

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} (-0.8)^n z^{-n} - \sum_{n=0}^{\infty} (0.2)^n . z^{-n}$$

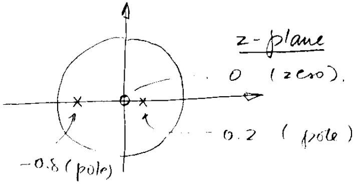
$$= \sum_{n=0}^{\infty} (-0.8z^{-1})^n - \sum_{n=0}^{\infty} (0.2z^{-1})^n$$

or $$X(z) = \frac{1}{1 + 0.8z^{-1}} - \frac{1}{1 - 0.2z^{-1}}$$ 8

Write
$$X(z) = \frac{1 - 0.2z^{-1} - 1 - 0.8z^{-1}}{(1+0.8z^{-1})(1-0.2z^{-1})}$$

$$= \frac{-z^{-1}}{(1+0.8z^{-1})(1-0.2z^{-1})}$$

$$\left.\begin{array}{ll} \text{poles @} & -0.8 \ , \ 0.2 \\ \text{zeros @} & 0 \end{array}\right\} \text{ on z-plane}$$



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b) $H(z) = 1 + 3z^{-2} + 3z^{-4} + z^{-6} = (1+z^{-2})^3$

when $\theta = \pi/2$ $z = j$ and hence

$H(j) = (1-1)^3 = 0$ 6

The normalised frequency of the input signal $x_1(n)$ is $\theta_0 = \omega_0 T$

i.e. $\theta_0 = 2\pi \times 10^3 \times \frac{1}{4} \times 10^{-3} = \pi/2$

i.e the input signal spectrum will be multiplied by H(z), and H(z) is zero at the input frequency. Therefore the output will be zero.

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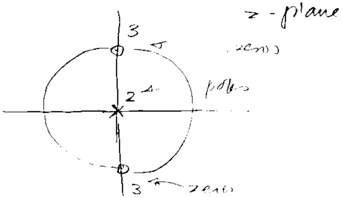
Poles of H(z) 2 at z = 0
zeros of H(z) 3 at each of +j and -j



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HW

4.

$$H(z) = h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4}$$

$$= 1 - 4z^{-1} + 6z^{-2} - 4z^{-3} + z^{-4}$$

$$H(z) = (1 - z^{-1})^4$$

9

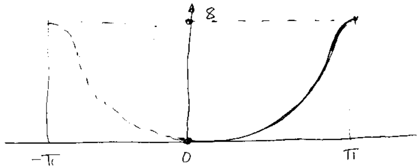
The amplitude response is obtained from $H(z)$ with $z = \exp(j\theta)$, θ - normalised frequency

i.e. $$A(\theta) = |H(e^{j\theta})| = |1 - e^{-j\theta}|^4$$

$$= \left| \left\{ e^{-j\theta/2} \left[e^{j\theta/2} - e^{-j\theta/2} \right] \right\}^4 \right|$$

$$= (2j)^4 \sin^4 \theta/2$$

or $$A(\theta) = 8 \cdot \sin^4 \theta/2$$



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$$Y(z) = H(z) \cdot (a + bz^{-1}) / (1 - z^{-1})$$

$$= (1 - z^{-1})^4 \cdot \frac{a + bz^{-1}}{(1 - z^{-1})} = (a + bz^{-1})(1 - z^{-1})^3$$

$(1 - z^{-1})^3 = 1 - 3z^{-1} + 3z^{-2} - z^{-3}$

$\times (a)$ $\quad a - 3az^{-1} + 3az^{-2} - az^{-3}$

$\times (bz^{-1})$ $\quad 0 + bz^{-1} - 3bz^{-2} + 3bz^{-3} - bz^{-4}$

or

$$Y(z) = a + (b - 3a)z^{-1} + 3(a - b)z^{-2} + (3b - a)z^{-3} - bz^{-4}$$

Hence we have

$$y(0) = a$$
$$y(1) = (b-3a)$$
$$y(2) = 3(a-b)$$
$$y(3) = (3b-a)$$
$$y(4) = -b$$

and $y(n) = 0$ for n more than 4. 15

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5.

(i) $\mathcal{L}\{\cos\omega_0 t\} = \int_0^\infty \cos\omega_0 t \cdot e^{-st}\,dt = \mathrm{Re}\left[\int_0^\infty e^{j\omega_0 t}\, e^{-st}\,dt\right]$

$= \mathrm{Re}\left[\int_0^\infty e^{-(s-j\omega_0)t}\,dt\right] = \mathrm{Re}\ \frac{1}{-(s-j\omega_0)}\, e^{-(s-j\omega_0)t}\Big|_0^\infty$

$= \mathrm{Re}\ \frac{1}{s-j\omega_0} = \frac{s}{s^2+\omega_0^2}$ 5

(ii) $\mathcal{L}\{e^{-\alpha t}x(t)\} = \int_0^\infty e^{-\alpha t}\,x(t)\,e^{-st}\,dt = \int_0^\infty e^{-(\alpha+s)t}\,x(t)\,dt$

$X(s+\alpha)$ 4

(iii) $H(s) = \frac{Y(s)}{X(s)} = 2\,\frac{s+\alpha}{(s+\alpha)^2+\omega_0^2} = \frac{2s+2\alpha}{s^2+2\alpha s+\alpha^2+\omega_0^2}$

(a)

or $Y(s)\left[s^2+2\alpha s+\alpha^2+\omega_0^2\right] = 2sX(s)+2\alpha X(s)$

i.e. $\frac{d^2y(t)}{dt^2} + 2\alpha\frac{dy(t)}{dt} + (\alpha^2+\omega_0^2)\,y(t) = 2\frac{dx(t)}{dt} + 2\alpha x(t)$ 8

(b)

$h(t) = \mathcal{L}^{-1}\left\{2\,\frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}\right\} = 2\,\mathcal{L}^{-1}\left\{\frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}\right\}$

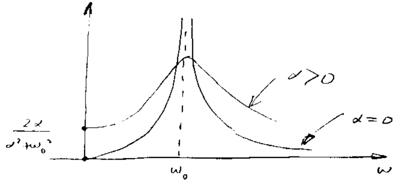
$= 2\,\mathcal{L}^{-1}\{V(s+\alpha)\}$

where $V(s) = \frac{s}{s^2+\alpha^2}$ i.e. $\mathcal{L}^{-1}V(s) = \cos\omega_0 t$

and hence

$h(t) = 2e^{-\alpha t}\cdot\cos\omega_0 t$ 12

(c)



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5.

(i) $\mathcal{L}\{\cos\omega_0 t\} = \int_0^{\infty} \cos\omega_0 t \cdot e^{-st}\,dt \quad \mathrm{Re}\left[\int_0^{\infty} e^{j\omega_0 t} e^{-st}\,dt\right]$

$= \mathrm{Re}\left[\int_0^{\infty} e^{-(s-j\omega_0)t}\,dt\right] = \mathrm{Re}\ \frac{1}{-(s-j\omega_0)} e^{-(s-j\omega_0)t}\Big|_0^{\infty}$

$= \mathrm{Re}\ \frac{1}{s-j\omega_0} = \frac{s}{s^2+\omega_0^2}$ 5

(ii) $\mathcal{L}\{e^{-\alpha t}x(t)\} = \int_0^{\infty} e^{-\alpha t} x(t) e^{-st}\,dt = \int_0^{\infty} e^{-(\alpha+s)t} x(t)\,dt$

$X(s+\alpha)$ 4

(iii) $H(s) = \frac{Y(s)}{X(s)} = 2\,\frac{s+\alpha}{(s+\alpha)^2+\omega_0^2} = \frac{2s+2\alpha}{s^2+2\alpha s+\alpha^2+\omega_0^2}$

(a)

or $Y(s)\left[s^2+2\alpha s+\alpha^2+\omega_0^2\right] = 2sX(s)+2\alpha X(s)$

i.e. $\frac{d^2y(t)}{dt^2} + 2\alpha\frac{dy(t)}{dt} + (\alpha^2+\omega_0^2)\,y(t) = 2\frac{dx(t)}{dt} + 2\alpha x(t)$ 8

(b) $h(t) = \mathcal{L}^{-1}\{\ 2\ \frac{s+\alpha}{\ }\ \} = 2\,\mathcal{L}^{-1}\left\{\frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}\right\}$

wh. $\quad\quad$ $\therefore\ \mathcal{L}^{-1}Y(s) = \cos\omega_0 t$

and

12

(c)

4

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