Imperial College London

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2012

This paper is also taken for the relevant examination for the Associateship.

M3S8/M4S8

Time Series

Date: Someday, May or June 2012 Time: 2 – 4 pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

Note: Throughout this paper $\{\epsilon_t\}$ is a sequence of uncorrelated random variables (white noise) having zero mean and variance σ_ϵ^2, unless stated otherwise. The unqualified term "stationary" will always be taken to mean second-order stationary. All processes are real-valued unless stated otherwise. The sample interval is unity unless stated otherwise.

1. (a) (i) What is meant by saying that a stochastic process is stationary?

(ii) Let $\{Y_t\}$ be a stationary process with mean zero, and define

$$X_t = \nu_t + Y_t$$

where ν_t is a deterministic seasonal component with period 12. Let B be the backward shift operator. Express the autocovariance sequence $\{s_{W,\tau}\}$ for $\{W_t\}$, where $W_t = (1 - B^{12})X_t$, in terms of the autocovariance sequence $\{s_{Y,\tau}\}$ for $\{Y_t\}$.

(iii) Show that an oscillation at frequency $f = 1/3$ in a stationary process $\{X_t\}$ is eliminated when the filter with impulse response $\{g_{-2}, g_{-1}, g_0, g_1, g_2\} = \{-1, 4, 3, 4, -1\}/9$ is applied to $\{X_t\}$.

(b) A continuous-time process $\{X(t)\}$, with t in seconds, has spectral density function

$$S_{X(t)}(f) = \begin{cases} 1 - 2|f|, & |f| \leq 1/2, \\ 0, & \text{otherwise,} \end{cases}$$

with f in cycles/second. It is sampled with a sample interval $\Delta t = 2$ seconds to produce the discrete-time process $\{X_t\}$. What is the spectral density function of $\{X_t\}$?

(c) Consider the stochastic process $\{X_t\}$ defined by

$$X_t - 0.6X_{t-1} = \epsilon_t - 1.2\epsilon_{t-1},$$

with $\sigma_\epsilon^2 = 1$.

(i) Demonstrate that $\{X_t\}$ is stationary but not invertible.

(ii) Derive the spectral density function $S(f)$ of $\{X_t\}$ and evaluate it at $f = 1/4$, expressing the answer as a fraction of the form $x/34$ and give the value of x.

2. (a) Let Y_1 and Y_2 be independent and identically distributed random variables, each with mean zero and variance σ^2, and let c be a constant.

(i) Find the mean and covariance of $\{X_t\}$ defined by

$$X_t = Y_1 \cos(ct) + Y_2 \sin(ct),$$

and hence show that the process is stationary.

(ii) For the case $c = \pi/4$ show that $\{X_t\}$ is *strictly* stationary if and only if Y_1 and Y_2 each have the Gaussian (normal) distribution.
[*You will need to use Bernstein's theorem which states that if U and V are IID random variables, and $(U+V)/\sqrt{2}$ has the same distribution as U and V, then U and V are Gaussian (normal). Also recall that* $\cos(\pi/4) = 1/\sqrt{2}$.]

(b) Let Y_1 be a random variable with mean zero and variance unity, and let c be a constant.

(i) Find the mean and covariance of $\{X_t\}$ defined by

$$X_t = Y_1 \cos(ct),$$

and hence determine when the process is stationary.

(ii) Show that when the process in (b)(i) is stationary that its autocorrelation sequence must be of the form

$$\rho_\tau = (-1)^{|\ell\tau|}, \qquad \tau \in \mathbb{Z},$$

for some $\ell \in \mathbb{Z}$.

(iii) The sequence $\{\rho_\tau\}$ is positive semidefinite, if, for all $n \geq 1$, for any $t_1, t_2, \ldots, t_n$ contained in $\mathbb{Z}$, and for any set of nonzero real numbers $a_1, a_2, \ldots, a_n$

$$\sum_{j=1}^{n} \sum_{k=1}^{n} \rho_{t_j - t_k} a_j a_k \geq 0.$$

Show that this inequality condition is satisfied by $\rho_\tau = (-1)^{|\ell\tau|}$ for $\ell, \tau \in \mathbb{Z}$.

(c) The autocovariance sequence $\{s_\tau\}$ of a stationary process is related to its non-negative spectral density function $S(f)$ (assuming it exists) via

$$s_\tau = \int_{-1/2}^{1/2} \mathrm{e}^{\mathrm{i}2\pi f\tau} S(f) \mathrm{d}f.$$

Use this result to show that $\{s_\tau\}$ is necessarily positive semidefinite, i.e., for all $n \geq 1$, for any $t_1, t_2, \ldots, t_n$ contained in $\mathbb{Z}$, and for any set of nonzero real numbers $a_1, a_2, \ldots, a_n$

$$\sum_{j=1}^{n} \sum_{k=1}^{n} s_{t_j - t_k} a_j a_k \geq 0.$$

[*Hint: The integral of a non-negative function is non-negative.*]

3. Let $X_1, \ldots, X_N$ be a sample of size N from a stationary process $\{X_t\}$ with a non-zero mean μ and spectral density function $S(f)$. At lag $\tau = 0$ both the unbiased and biased estimators of the autocovariance sequence reduce to

$$\widehat{s}_0 \equiv \frac{1}{N} \sum_{t=1}^{N} \left(X_t - \overline{X}\right)^2 .$$

(i) Show that $E\{\widehat{s}_0\} = s_0 - \mathrm{var}\,\{\overline{X}\}$, where $s_0 = \mathrm{var}\,\{X_t\}$.

(ii) Define the spectral estimator where the exact mean is known and subtracted as

$$\widehat{S}(f) = \frac{1}{N} \left| \sum_{t=1}^{N} (X_t - \mu) \mathrm{e}^{-\mathrm{i}2\pi f t} \right|^2 .$$

Use the spectral representation theorem to show that the mean of the spectral estimator $\widehat{S}(f)$ is given by

$$E\{\widehat{S}(f)\} = \int_{-1/2}^{1/2} \mathcal{F}(f - f') S(f') \mathrm{d}f' ,$$

where $\mathcal{F}(f)$ denotes Fejer's kernel given by

$$\mathcal{F}(f) = \frac{1}{N} \left| \sum_{t=1}^{N} \mathrm{e}^{-\mathrm{i}2\pi f t} \right|^2 .$$

(iii) Demonstrate that

$$\mathrm{var}\,\{\overline{X}\} = (1/N) E\{\widehat{S}(0)\},$$

and hence that

$$E\{\widehat{s}_0\} = \int_{-1/2}^{1/2} \left(1 - \frac{1}{N}\mathcal{F}(f)\right) S(f)\,\mathrm{d}f.$$

(iv) Sketch the form of $(1/N)\mathcal{F}(f)$ and hence describe the kind of spectrum $S(f)$ that would give rise to a large discrepancy between $E\{\widehat{s}_0\}$ and s_0.

4. (a) Let $\{X_t\}$ be a stationary AR(p) process:

$$X_t - \phi_{1,p}X_{t-1} - \ldots - \phi_{p,p}X_{t-p} = \epsilon_t.$$

(i) Derive the Yule-Walker equations

$$\boldsymbol{\gamma}_p = \boldsymbol{\Gamma}_p \boldsymbol{\phi}_p \quad \text{and} \quad \sigma_\epsilon^2 = s_0 - \sum_{j=1}^{p} \phi_{j,p} s_j,$$

where $\boldsymbol{\gamma}_p = [s_1, s_2, \ldots, s_p]^T$; $\boldsymbol{\phi}_p = [\phi_{1,p}, \phi_{2,p}, \ldots, \phi_{p,p}]^T$ and

$$\boldsymbol{\Gamma}_p = \begin{bmatrix} s_0 & s_1 & \ldots & s_{p-1} \\ s_1 & s_0 & \ldots & s_{p-2} \\ \vdots & \vdots & & \vdots \\ s_{p-1} & s_{p-2} & \ldots & s_0 \end{bmatrix}.$$

(ii) Let $p = 1$ and $\phi_{1,1} = \phi$ with $|\phi| < 1$. Use results from the formulation of part (a)(i) to show that

$$s_\tau = \frac{\sigma_\epsilon^2 \phi^{|\tau|}}{1-\phi^2}, \quad |\tau| \geq 0.$$

Hence show that the variance of the sample mean $\bar{X} \equiv (X_1 + X_2 + X_3 + X_4)/4$ is

$$\mathrm{var}\{\bar{X}\} = \frac{\sigma_\epsilon^2 (2 + \phi + \phi^2)}{8(1-\phi)}.$$

[*Recall,* $\mathrm{var}\{\sum_{j=1}^{N} X_j\} = \sum_{j=1}^{N} \sum_{k=1}^{N} \mathrm{cov}\{X_j, X_k\}$.]

(b) (i) What is meant by saying two discrete time stochastic processes $\{X_t\}$ and $\{Y_t\}$ are jointly stationary stochastic processes?

(ii) Suppose $\{X_t\}$ and $\{Y_t\}$ are zero mean jointly stationary processes given by

$$X_t = \epsilon_t - \theta\epsilon_{t-1}; \qquad Y_t = \epsilon_{t-2}.$$

with $|\theta| < 1$. Derive the cross-spectrum $S_{XY}(f)$, and hence find the value of the magnitude squared coherence, $\gamma_{XY}^2(f)$, and explain its value in terms of the forms of the processes $\{X_t\}$ and $\{Y_t\}$.

1. (a) (i) $\{X_t\}$ is second-order stationary if $E\{X_t\}$ is a finite constant for all t, var$\{X_t\}$ is a finite constant for all t, and cov$\{X_t, X_{t+\tau}\}$, is a finite quantity depending only on τ and not on t. seen ⇓

4

(ii) sim. seen ⇓

$$\begin{aligned} W_t &= (1-B^{12})X_t = (X_t - X_{t-12}) \\ &= \nu_t + Y_t - \nu_{t-12} - Y_{t-12} \\ &= Y_t - Y_{t-12}. \end{aligned}$$

So $E\{W_t\} = 0$. Then

$$\begin{aligned} E\{W_t W_{t+\tau}\} &= E\{(Y_t - Y_{t-12})(Y_{t+\tau} - Y_{t+\tau-12})\} \\ &= 2s_{Y,\tau} - s_{Y,\tau-12} - s_{Y,\tau+12} = s_{W,\tau}. \end{aligned}$$

2

(iii) The frequency response/transfer function of the filter is given by

$$\begin{aligned} G(f) &= \frac{1}{9}\sum_{j=-2}^{2} g_j \mathrm{e}^{-\mathrm{i}2\pi f j} \\ &= [-1\cdot \mathrm{e}^{-\mathrm{i}4\pi f} + 4\cdot \mathrm{e}^{\mathrm{i}2\pi f} + 3 + 4\cdot \mathrm{e}^{-\mathrm{i}2\pi f} - 1\cdot \mathrm{e}^{-\mathrm{i}4\pi f}]/9 \\ &= [3 - 2\cos(4\pi f) + 8\cos(2\pi f)]/9. \end{aligned}$$

Consider $f = 1/3$. Now $\cos(\pi/3) = 0.5$, and hence $\cos(2\pi/3) = -0.5$ (by skew-symmetry about $\pi/2$) and $\cos(4\pi/3) = -0.5$ also (by symmetry about π) and so $G(1/3) = 3 - 2\cdot(-0.5) + 8\cdot(-0.5) = 0$ and hence oscillations at $f = 1/3$ are eliminated since $|G(1/3)|^2 S_X(1/3) = 0$.

4

(b) The Nyquist frequency for the discrete process is $f_{\mathcal{N}} = 1/(2\Delta t) = 1/4$ cycle/second. This is the folding frequency; the aliased spectrum is got by folding $S_{X(t)}(f)$ about $\pm f_{\mathcal{N}} = \pm 1/4$. Since the spectrum is triangular and takes the value 1/2 at $\pm f_{\mathcal{N}}$, the folding addition produces a rectangle: unseen ⇓

$$S_{X_t}(f) = \begin{cases} 1, & |f| \le 1/4, \\ 0, & \text{otherwise.} \end{cases}$$

4

(c) (i) For this process $\Phi(z) = 1 - 0.6z$, which has a root $10/6$ which is outside the unit circle so the process is stationary. But $\Theta(z) = 1 - 1.2z$ which has a root $10/12$ which is inside the unit circle so the process is not invertible. sim. seen ⇓

2

(ii) To find the spectrum of the ARMA(1,1) process, write

$$X_t - \phi X_{t-1} = Y_t; \qquad Y_t = \epsilon_t - \theta\epsilon_{t-1},$$

then we have

$$|G_\phi(f)|^2 S_X(f) = S_Y(f),$$

where $G_\phi(f) = 1 - \phi \mathrm{e}^{-\mathrm{i}2\pi f}$, and

$$S_Y(f) = |G_\theta(f)|^2 S_\epsilon(f),$$

where $G_\theta(f) = 1 - \theta \mathrm{e}^{-\mathrm{i}2\pi f}$, so that with $\phi = 0.6$ and $\theta = 1.2$,

$$\begin{aligned} S_X(f) &= S_\epsilon(f)\frac{|G_\theta(f)|^2}{|G_\phi(f)|^2} = \sigma_\epsilon^2 \frac{|1-1.2\mathrm{e}^{-\mathrm{i}2\pi f}|^2}{|1-0.6\mathrm{e}^{-\mathrm{i}2\pi f}|^2} \\ &= \frac{2.44 + 2.4\cos(2\pi f)}{1.36 + 1.2\cos(2\pi f)}. \end{aligned}$$

When $f = 1/4$ we have $\cos(2\pi f) = \cos(\pi/2) = 0$, so

$$S_X(1/4) = 244/136 = 61/34,$$

so $x = 61$. $\boxed{4}$

2. (a) (i) $\boxed{\text{unseen} \Downarrow}$

$$E\{X_t\} = E\{Y_1\}\cos(ct) + E\{Y_2\}\sin(ct) = 0.$$

Also for the covariance (which for $\tau = 0$ gives the variance),

$$\begin{aligned}
E\{X_t X_{t+\tau}\} &= E\{[Y_1\cos(ct) + Y_2\sin(ct)][Y_1\cos(c[t+\tau]) + Y_2\sin(c[t+\tau])]\} \\
&= E\{Y_1^2\}\cos(ct)\cos(c[t+\tau]) + E\{Y_1Y_2\}\cos(ct)\sin(c[t+\tau]) \\
&+ E\{Y_2Y_1\}\sin(ct)\cos(c[t+\tau]) + E\{Y_2^2\}\sin(ct)\sin(c[t+\tau]) \\
&= \sigma^2\cos(ct)\cos(c[t+\tau]) + \sigma^2\sin(ct)\sin(c[t+\tau]).
\end{aligned}$$

$\boxed{2}$

But, since $\cos(a-b) = \cos a\cos b + \sin a \sin b$,

$$E\{Y_t Y_{t+\tau}\} = \sigma^2\cos(c\tau) = s_\tau.$$

Therefore the process is always stationary. $\boxed{2}$

(ii) Firstly suppose that $\{X_t\}$ is strictly stationary. Then the marginal distribution of X_t is independent of $t \in \mathbb{Z}$. With $c = \pi/4$ the cases $t = 0$ and 1 give $X_0 = Y_1$ and $X_1 = (Y_1 + Y_2)/\sqrt{2}$ so that Y_1 and $(Y_1+Y_2)/\sqrt{2}$ have the same distribution. We know that Y_1 and Y_2 are IID. From Bernstein's theorem we can conclude that Y_1 and Y_2 are Gaussian.
Now suppose that Y_1 and Y_2 are Gaussian, then $\{X_t\}$ is a Gaussian process, (all finite-dimensional marginal distributions are multivariate Gaussian). The process is (second-order) stationary by part (i), and we know that a stationary Gaussian process is strictly stationary. $\boxed{6}$

(b) (i) $E\{Y_t\} = E\{X_0\}\cos(ct) = 0$. Taking $X_2 \equiv 0$ in (a), gives

$$E\{Y_t Y_{t+\tau}\} = \sigma^2\cos(ct)\cos(c[t+\tau]).$$

Since t and τ are integers, the process is stationary for $c = \ell\pi, \ell \in \mathbb{Z}$ and non-stationary otherwise, i.e.,

$$s_\tau = \sigma^2\cos(\ell\pi t)\cos(\ell\pi[t+\tau]).$$

$\boxed{2}$

(ii) Now $\cos(\ell\pi t) = (-1)^{\ell t}$ and $\cos(\ell\pi[t+\tau]) = (-1)^{\ell(t+\tau)}$ so that $\boxed{\text{unseen} \Downarrow}$

$$s_\tau = \sigma^2(-1)^{\ell t}(-1)^{\ell(t+\tau)} = \sigma^2(-1)^{\ell\tau},$$

for some choice $\ell \in \mathbb{Z}$. Hence $s_0 = \sigma^2$ and by symmetry $\rho_\tau = s_\tau/s_0 = (-1)^{|\ell\tau|}$, $\tau \in \mathbb{Z}$. $\boxed{2}$

(iii) $\boxed{\text{unseen} \Downarrow}$

$$\begin{aligned}
\sum_{j=1}^n\sum_{k=1}^n \rho_{t_j - t_k} a_j a_k &= \sum_{j=1}^n\sum_{k=1}^n (-1)^{\ell(t_j - t_k)} a_j a_k \\
&= \sum_{j=1}^n (-1)^{\ell t_j} a_j \sum_{k=1}^n (-1)^{\ell t_k} a_k = [\sum_{j=1}^n (-1)^{\ell t_j} a_j]^2 \geq 0.
\end{aligned}$$

$\boxed{2}$

(c) Replacing the autocovariance by its Fourier representation,

$$\begin{aligned}\sum_{j=1}^{n}\sum_{k=1}^{n} s_{t_j-t_k} a_j a_k &= \int_{-1/2}^{1/2} \sum_{j=1}^{n}\sum_{k=1}^{n} a_j a_k \mathrm{e}^{\mathrm{i}2\pi f(t_j-t_k)} S(f)\mathrm{d}f \\ &= \int_{-1/2}^{1/2} \left| \sum_{j=1}^{n} \mathrm{e}^{\mathrm{i}2\pi f t_j} \right|^2 S(f)\mathrm{d}f \geq 0,\end{aligned}$$

since the integral of a non-negative function is necessarily non-negative. $\boxed{4}$

3. (i) Here $\{X_t\}$ is a stationary process with mean value $\mu = E\{X_t\}$, and variance s_0. By definition, $\boxed{\text{sim. seen} \Downarrow}$

$$\begin{aligned}
\widehat{s}_0 &= \frac{1}{N}\sum_{t=1}^{N}(X_t - \overline{X})^2 = \frac{1}{N}\sum_{t=1}^{N}([X_t-\mu]-[\overline{X}-\mu])^2 \\
&= \frac{1}{N}\sum_{t=1}^{N}\left([X_t-\mu]^2 - 2[X_t-\mu][\overline{X}-\mu] + [\overline{X}-\mu]^2\right) \\
&= \frac{1}{N}\sum_{t=1}^{N}[X_t-\mu]^2 - 2[\overline{X}-\mu][\overline{X}-\mu] + [\overline{X}-\mu]^2 \\
&= \frac{1}{N}\sum_{t=1}^{N}[X_t-\mu]^2 - [\overline{X}-\mu]^2.
\end{aligned}$$

Taking the expectation of both sides and noting that $E\{\overline{X}\} = \mu$ yields

$$E\{\widehat{s}_0\} = \frac{1}{N}\sum_{t=1}^{N} E\{[X_t-\mu]^2\} - E\{[\overline{X}-\mu]^2\} = \mathrm{var}\,\{X_t\} - \mathrm{var}\,\{\overline{X}\} = s_0 - \mathrm{var}\,\{\overline{X}\},$$

the desired result. $\boxed{5}$

(ii) Let $\boxed{\text{sim. seen} \Downarrow}$

$$J(f) \equiv (1/\sqrt{N})\sum_{t=1}^{N}(X_t-\mu)\mathrm{e}^{-\mathrm{i}2\pi ft}.$$

By the spectral representation theorem $X_t - \mu = \int_{-1/2}^{1/2}\mathrm{e}^{\mathrm{i}2\pi f't}\,\mathrm{d}Z(f')$, where $\{Z(\cdot)\}$ is a process with orthogonal increments, and $E\{dZ(f)\} = 0$. Thus

$$\begin{aligned}
J(f) &= (1/\sqrt{N})\sum_{t=1}^{N}\left(\int_{-1/2}^{1/2}\mathrm{e}^{\mathrm{i}2\pi f't}\,\mathrm{d}Z(f')\right)\mathrm{e}^{-\mathrm{i}2\pi ft} \\
&= (1/\sqrt{N})\int_{-1/2}^{1/2}\sum_{t=1}^{N}\mathrm{e}^{-\mathrm{i}2\pi(f-f')t}\,\mathrm{d}Z(f') \\
&= \int_{-1/2}^{1/2} F(f-f')\,\mathrm{d}Z(f'),
\end{aligned}$$

where $F(f) = (1/\sqrt{N})\sum_{t=1}^{N}\mathrm{e}^{-\mathrm{i}2\pi ft}$. Now it is given that,

$$\widehat{S}(f) \equiv |J(f)|^2 = (1/N)\left|\sum_{t=1}^{N}(X_t-\mu)\mathrm{e}^{-\mathrm{i}2\pi ft}\right|^2.$$

Because $\{Z(\cdot)\}$ has orthogonal increments, we therefore have

$$E\{\widehat{S}(f)\} = \int_{-1/2}^{1/2}\mathcal{F}(f-f')S(f')\,\mathrm{d}f',$$

where

$$\mathcal{F}(f) \equiv |F(f)|^2 = (1/N)\left|\sum_{t=1}^{N}\mathrm{e}^{-\mathrm{i}2\pi ft}\right|^2.$$

$\boxed{5}$

(iii) Now $\boxed{\text{unseen} \Downarrow}$

$$\begin{aligned}\operatorname{var}\{\overline{X}\} &= E\{(\overline{X}-\mu)^2\} = (1/N^2)E\left\{\left(\sum_{t=1}^{N}(X_t-\mu)\right)^2\right\}\\ &= (1/N)E\{\widehat{S}(0)\},\end{aligned}$$

and from (ii), $E\{\widehat{S}(0)\} = \int_{-1/2}^{1/2}\mathcal{F}(f)S(f)\,\mathrm{d}f$ (by symmetry of spectral density function), and of course $s_0 = \int_{-1/2}^{1/2} S(f)\,\mathrm{d}f$, so that the result follows from part (i), i.e.,

$$E\{\widehat{s}_0\} = s_0 - \operatorname{var}\{\overline{X}\} = \int_{-1/2}^{1/2}\left(1-\frac{1}{N}\mathcal{F}(f)\right)S(f)\,\mathrm{d}f.$$

$\boxed{5}$

(iv) The function $(1/N)\mathcal{F}(f)$ is non-negative and symmetric about $f=0$ where it takes the max value 1. It is multi-lobed. The main lobe decreases from 1 to zero at $\pm 1/N$. Other much smaller side-lobes are to be found between the zeros at $k/N, k\in\mathbb{Z}\setminus\{0\}$ and the sidelobes decrease with increasing $|f|$. So $\left(1-\frac{1}{N}\mathcal{F}(f)\right)$ is close to 1 for $(1/N)\le|f|\le 1/2$, but decreases to zero as $|f|$ decreases from $1/N$ to zero. Since $s_0=\int_{-1/2}^{1/2}S(f)\mathrm{d}f$, we can expect a large discrepancy between s_0 and $E\{\widehat{s}_0\}$ if the SDF for frequencies $0\le|f|\le(1/N)$ largely determines the value of s_0, i.e., if most power in $\{X_t\}$ is concentrated in frequencies $0\le|f|\le(1/N)$. $\boxed{5}$

4. (a) (i) We start by multiplying the defining equation by $X_{t-\tau}$: seen ⇓

$$X_t X_{t-\tau} = \sum_{j=1}^{p} \phi_{j,p} X_{t-j} X_{t-\tau} + \epsilon_t X_{t-\tau}.$$

Taking expectations, for $\tau > 0$ and using the fact that for $\tau > 0$, $X_{t-\tau}$ only involves ϵ's for times earlier than t, so is uncorrelated with ϵ_t,

$$s_\tau = \sum_{j=1}^{p} \phi_{j,p} s_{\tau-j}. \tag{1}$$

Let $\tau = 1, 2, \ldots, p$, and recall that $s_{-\tau} = s_\tau$, to obtain

$$\begin{aligned} s_1 &= \phi_{1,p} s_0 + \phi_{2,p} s_1 + \ldots + \phi_{p,p} s_{p-1} \\ s_2 &= \phi_{1,p} s_1 + \phi_{2,p} s_0 + \ldots + \phi_{p,p} s_{p-2} \\ \vdots \quad & \quad \vdots \\ s_p &= \phi_{1,p} s_{p-1} + \phi_{2,p} s_{p-2} + \ldots + \phi_{p,p} s_0 \end{aligned}$$

or in matrix notation,

$$\boldsymbol{\gamma}_p = \Gamma_p \boldsymbol{\phi}_p,$$

where $\boldsymbol{\gamma}_p = [s_1, s_2, \ldots, s_p]^T$; $\boldsymbol{\phi}_p = [\phi_{1,p}, \phi_{2,p}, \ldots, \phi_{p,p}]^T$ and

$$\Gamma_p = \begin{bmatrix} s_0 & s_1 & \ldots & s_{p-1} \\ s_1 & s_0 & \ldots & s_{p-2} \\ \vdots & \vdots & & \vdots \\ s_{p-1} & s_{p-2} & \ldots & s_0 \end{bmatrix}$$

For σ_ϵ^2 we multiply the defining equation by X_t and take expectations to obtain

$$s_0 = \sum_{j=1}^{p} \phi_{j,p} s_j + E\{\epsilon_t X_t\} = \sum_{j=1}^{p} \phi_{j,p} s_j + \sigma_\epsilon^2. \tag{2}$$

4

(ii) When $p = 1$ the Yule-Walker equations (1) and (2) give

unseen ⇓

$$s_\tau = \phi s_{\tau-1} \quad \text{and} \quad \sigma_\epsilon^2 = s_0 - \phi s_1 = s_0 - \phi^2 s_0 = (1 - \phi^2) s_0$$

so $s_0 = \sigma_\epsilon^2 / [1 - \phi^2]$ and by iteration and symmetry, $s_\tau = \phi_{1,1}^{|\tau|} s_0$, giving

$$s_\tau = \frac{\sigma_\epsilon^2 \phi^{|\tau|}}{1 - \phi^2} \quad |\tau| \geq 0.$$

2

For the variance of the mean:

$$\begin{aligned} \text{var}\{\bar{X}\} &= \text{var}\left\{\frac{1}{4}(X_1 + X_2 + X_3 + X_4)\right\} \\ &= \frac{1}{16} [\text{var}\{X_1\} + \text{var}\{X_2\} + \text{var}\{X_3\} + \text{var}\{X_4\} \\ &+ 2\,\text{cov}\{X_1, X_2\} + 2\,\text{cov}\{X_2, X_3\} + 2\,\text{cov}\{X_3, X_4\} \\ &+ 2\,\text{cov}\{X_1, X_3\} + 2\,\text{cov}\{X_2, X_4\} + 2\,\text{cov}\{X_1, X_4\}]. \end{aligned}$$

So,

$$\begin{aligned}
\mathrm{var}\{\bar{X}\} &= \frac{1}{16}[4s_0 + 6s_1 + 4s_2 + 2s_3] \\
&= \frac{1}{16}\left[\frac{4\sigma_\epsilon^2}{1-\phi^2} + \frac{6\sigma_\epsilon^2\phi}{1-\phi^2} + \frac{4\sigma_\epsilon^2\phi^2}{1-\phi^2} + \frac{2\sigma_\epsilon^2\phi^3}{1-\phi^2}\right] \\
&= \frac{\sigma_\epsilon^2}{8(1-\phi^2)}[2 + 3\phi + 2\phi^2 + \phi^3] \\
&= \frac{\sigma_\epsilon^2}{8(1-\phi)}[2 + \phi + \phi^2].
\end{aligned}$$

$\boxed{4}$

(b) (i) Two real-valued discrete time stochastic processes $\{X_t\}$ and $\{Y_t\}$ are said to be jointly stationary stochastic processes if $\{X_t\}$ and $\{Y_t\}$ are each, separately, second-order stationary processes, and cov$\{X_t, Y_{t+\tau}\}$ is a function of τ only. $\boxed{\text{unseen} \Downarrow}$ $\boxed{2}$

(ii)

$$\begin{aligned}
s_{XY,\tau} = E\{X_t Y_{t+\tau}\} &= E\{(\epsilon_t - \theta\epsilon_{t-1})\epsilon_{t+\tau-2}\} \\
&= E\{\epsilon_t\epsilon_{t+\tau-2}\} - \theta E\{\epsilon_{t-1}\epsilon_{t+\tau-2}\} \\
&= \begin{cases} 0, & \text{if } \tau = 0, \\ -\theta\sigma_\epsilon^2, & \text{if } \tau = 1, \\ 0, & \text{if } \tau = -1, \\ \sigma_\epsilon^2, & \text{if } \tau = 2 \\ 0, & \text{if } \tau = -2 \\ 0, & \text{if } |\tau| \geq 3. \end{cases}
\end{aligned}$$

$\boxed{2}$

So

$$\begin{aligned}
S_{XY}(f) &= \sum_{\tau=0}^{2} s_{XY,\tau}\mathrm{e}^{-\mathrm{i}2\pi f\tau} = \sigma_\epsilon^2\left[-\theta\mathrm{e}^{-\mathrm{i}2\pi f} + \mathrm{e}^{-\mathrm{i}4\pi f}\right] \\
&= \sigma_\epsilon^2\mathrm{e}^{-\mathrm{i}2\pi f}[-\theta + \mathrm{e}^{-\mathrm{i}2\pi f}] \\
\Rightarrow |S_{XY}(f)|^2 &= \sigma_\epsilon^4[-\theta + \mathrm{e}^{-\mathrm{i}2\pi f}][-\theta + \mathrm{e}^{\mathrm{i}2\pi f}] \\
&= \sigma_\epsilon^4[1 + \theta^2 - 2\theta\cos(2\pi f)].
\end{aligned}$$

Also $S_X(f) = \sigma_\epsilon^2[1 - \theta\mathrm{e}^{-\mathrm{i}2\pi f}][1 - \theta\mathrm{e}^{\mathrm{i}2\pi f}]$ (for example from Q1(c)(ii)), i.e., $S_X(f) = \sigma_\epsilon^2[1+\theta^2 - 2\theta\cos(2\pi f)]$, and $S_Y(f) = \sigma_\epsilon^2$ as it is (shifted) white noise. So the magnitude squared coherence is given by

$$\gamma_{XY}^2(f) = \frac{|S_{XY}(f)|^2}{S_X(f)S_Y(f)} = 1.$$

$\boxed{4}$

The magnitude squared coherence is unity because $\{X_t\}$ and $\{Y_t\}$ are perfectly linearly related since we can write $X_t = Y_{t+2} - \theta Y_{t+1}$. $\boxed{2}$