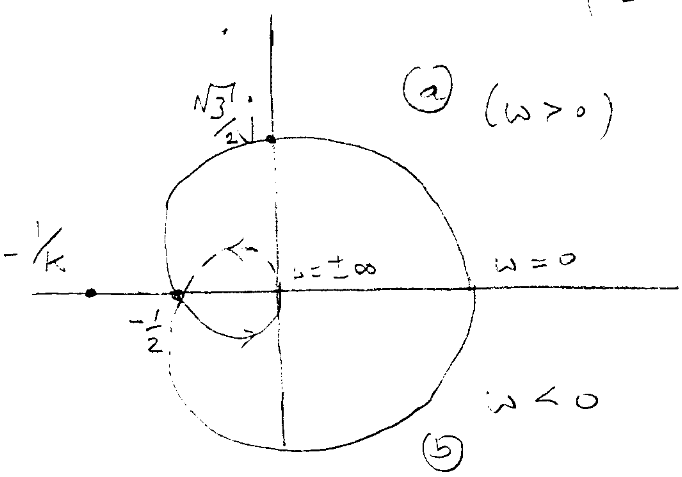
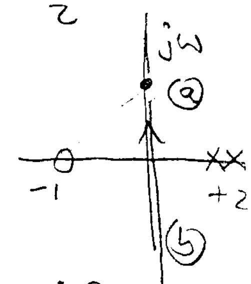
1 (a). Nyquist's encirclement theorem: $N = P_c - P_o$,
where N = # { clockwise encirclements of the $-1+0j$ point }
P_c = # { $\mathbb{C}$ poles strictly in right half-space of $\mathbb{C}$ }
[2] P_o = # { Φ poles strictly in right half-space of $\mathbb{C}$ }

Set $k=1$.
$$G(j\omega) = \frac{(1+j\omega)(-1-j\omega)^2}{(\omega^2+1)^2} = \frac{(1+j\omega)\{(1-\omega^2)+2j\omega\}}{(\omega^2+1)^2}$$
$$= \frac{(1-3\omega^2)+[(1-\omega^2)+2]j\omega}{(1+\omega^2)^2}$$

Intercept with real axis occurs when $\mathrm{Im}\{G(j\bar{\omega})\}=0$, i.e. when $\bar{\omega}=\sqrt{3}$. Then $G(j\bar{\omega}) = -8/16 = -\frac{1}{2}$
Intercept with imag. axis occurs when $\mathrm{Re}\{G(j\bar{\bar{\omega}})\}=0$ i.e. when $\bar{\bar{\omega}} = \sqrt{1/3}$. Then $G(j\bar{\bar{\omega}}) = \left(\frac{8}{3} / \frac{16}{9}\right)\frac{1}{\sqrt{3}}\, j = \frac{\sqrt{3}}{2}$



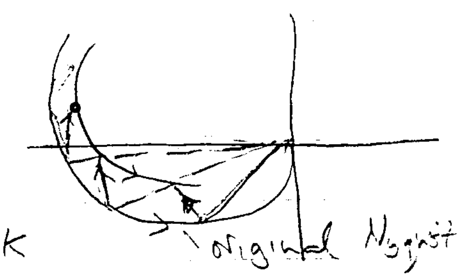


$G(j0) = 1e^{j0}$
$G(j\infty) = 0e^{-j\pi/2}$

[12]

For $0<K<2$, there are no encirclements. So
$0 = P_c - P_o = P_c - 2$ Hence $P_c = 2$ (2 unstable $\mathbb{C}$ poles)
For $2<K$, there are 2 anticlockwise encirclements, so
$-2 = P_c - 2$. Hence $P_c = 0$ (no unstable poles)
[2] Summary: unstable for $0 \le K < 2$ and stable for $K > 2$.

(b) The effect of the extra poles is to rotate the original Nyquist diagram clockwise. This change moves the intercept of Nyquist diagram with the real axis to the right. This increases the range of K values for which the system is unstable.
[4] Summary: the modified system is unstable



2 (i) We require (for phase margin ϕ_m and gain cross-over frequency $\bar{\omega}$)

$$G_c(j\bar{\omega})\, G(j\bar{\omega}) = 1 \measuredangle -180^\circ + \phi_m . \quad \text{— (1)}$$

But $|G_c(j\bar{\omega})| = \dfrac{\sqrt{(\bar{\omega}/\omega_0)^2+1}}{\sqrt{(\bar{\omega}/\omega_1)^2+1}} \cdot K = \dfrac{\cos\theta_1}{\cos\theta_0} \cdot K$ (when $\tan\theta_0 = \dfrac{\bar{\omega}}{\omega_0}$, $\tan\theta_1 = \bar{\omega}/\omega_1$).

and $\measuredangle G_c(j\omega) = \tan^{-1}\left(\dfrac{\bar{\omega}}{\omega_0}\right) - \tan^{-1}\left(\dfrac{\bar{\omega}}{\omega_1}\right) = \theta_0 - \theta_1$

From (1) then, $\cos\theta_1 K |G(j\bar{\omega})| = \cos\theta_0$

4 and $\theta_0 - \theta_1 + \measuredangle G(j\bar{\omega}) = -180^\circ + \phi_m$, whence $\theta_0 - \theta_1 = \theta$

(ii) Write $G_c G(s) = \frac{1}{s}\tilde{G}(s)$, where $\tilde{G}(s) = G_c(s) \cdot \dfrac{4}{(s+1)(s+2)}$

Then $E(s) = \dfrac{1}{1+G_cG(s)} \times \dfrac{1}{s^2} = \dfrac{s}{s+\tilde{G}(s)} \times \dfrac{1}{s^2}$

So $\lim\limits_{t\to\infty} e(t) = \lim\limits_{s\downarrow 0} s E(s) = \lim\limits_{s\downarrow 0} \dfrac{\cancel{s^2}}{s+\tilde{G}(s)} \times \dfrac{1}{\cancel{s^2}} = \dfrac{1}{\tilde{G}(0)} = \dfrac{1}{2K}$

We want $\lim\limits_{t\to\infty} e(t) = \frac{1}{2}\ rs^{-1}$. Hence $K = 1$

To achieve $\phi_m = 40^\circ$ with gain cross-over freq. $\bar{\omega} = 1.7\ rs^{-1}$

we must arrange that

$$\theta = -180^\circ + \phi_m - \measuredangle G(j\bar{\omega}) = -180^\circ + 40^\circ + 170.1^\circ = 49.9^\circ$$

According to the given formulae

$$\frac{\bar{\omega}}{\omega_0} = \frac{1 - |G(j\bar{\omega})|\cos\theta}{|G(j\bar{\omega})|\sin\theta} = \frac{1 - 0.4545 \cdot 0.6441}{0.4545 \cdot 0.7649} = 2.0344$$

$$\frac{\bar{\omega}}{\omega_1} = \frac{\cos\theta - |G(j\bar{\omega})|}{\sin\theta} = \frac{0.6441 - 0.4545}{0.7649} = 0.2479$$

It follows

13 $\omega_0 (= 1.7/2.0344) = 0.8356$, $\omega_1 (= 1.7/0.2479) = 6.8573$

The ratio of compensator break frequencies is

$$\omega_1/\omega_0 = 8.2$$

Large values of ω_1/ω_0 should be avoided because they give rise to large peak values of the control signal, in response to a step change of the reference signal, which may saturate or damage the control actuator. Here, ω_1/ω_0 is quite modest and these difficulties should not arise for a

3 sensibly chosen actuator.

3(a) Equating voltages across the three limbs of the circuit gives

$$V_c + \frac{dV_c}{dt} = i_L + \frac{di_L}{dt} = -\left(i_L + \frac{dV_c}{dt}\right) = V_{out} \quad — (*)$$

(we have used the fact that $i_c = dV_c/dt$).

From (*),

$$2\frac{dV_c}{dt} = -V_c - i_L$$

and

$$\frac{di_L}{dt} = -2i_L - \frac{dV_c}{dt} = -2i_L + \frac{1}{2}V_c + \frac{1}{2}i_L$$

Hence $di_L/dt = \frac{1}{2}V_c - 3/2\, i_L$

We also have $V_{out} = V_c + \frac{dV_c}{dt} = V_c - \frac{1}{2}V_c - \frac{1}{2}i_L = \frac{1}{2}V_c - \frac{1}{2}i_L$

Writing the equations in state space form, we obtain

[14] $$\frac{d}{dt}\begin{pmatrix} V_c \\ i_L \end{pmatrix} = \underbrace{\begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ +\frac{1}{2} & -3/2 \end{bmatrix}}_{A}\begin{pmatrix} V_c \\ i_L \end{pmatrix}, \quad V_{out} = \underbrace{\left[\tfrac{1}{2} \;\; -\tfrac{1}{2}\right]}_{c^T}\begin{pmatrix} V_c \\ i_L \end{pmatrix}$$

(b) The observability matrix is

$$M = \begin{bmatrix} \leftarrow c^T \rightarrow \\ \leftarrow c^T A \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{4} + \frac{3}{4} = +\frac{1}{2} \end{bmatrix}$$

[3] Since $\det M = 0$, the system is unobservable.

(c) We see that

$$\frac{d}{dt}V_{out}(t) = \frac{1}{2}\frac{dV_c}{dt} - \frac{1}{2}\frac{di_L}{dt}$$

$$= \frac{1}{2}\left(-\frac{1}{2}V_c - \frac{1}{2}i_L\right) - \frac{1}{2}\left(\frac{1}{2}V_c - \frac{3}{2}i_L\right)$$

$$= -\frac{1}{2}V_c + \frac{1}{2}i_L = -V_{out}(t)$$

But $V_{out}(0) = \frac{1}{2}V_c(0) - \frac{1}{2}i_L(0) = 0$

[3] It follows that $V_{out}(t) = e^{-t}V_{out}(0) = 0$ for all $t \geq 0$.

4(a). We have $\dddot{y} = -2\ddot{y} - \dot{y} + u$. Let $x_1 = y$, $x_2 = \dot{y}$, $x_3 = \ddot{y}$. Then $\dot{x}_1 = x_2$, $\dot{x}_2 = x_3$ and $\dot{x}_3 = -2x_3 - x_2 + u$. So a state-space realisation is:

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -2 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{b} u \quad \text{and} \quad y = \underbrace{[1 \; 0 \; 0]}_{c^T} x$$

We require a closed loop characteristic polynomial

$$\gamma(s) = (s+3)(s+2+j)(s+2-j) = (s+3)(s^2+4s+5)$$
$$= s^3 + 7s^2 + 15s + 15 \qquad (*)$$

The closed loop system matrix, for state feedback

$$u = -k^T x,$$

is

$$(A - bk^T) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -k_1 & -k_2 & -k_3 \end{bmatrix}$$

with characteristic polynomial

$$s^3 + k_3 s^2 + k_2 s + k_1 \qquad (**)$$

Matching coefficients in $(*)$ and $(**)$ gives

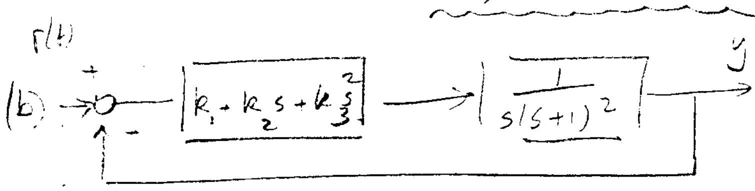
$$(k_1 \;\; k_2 \;\; k_3) = (15 \;\; 15 \;\; 7)$$

We have $u = -k_1 y - k_2 \dot{y} - k_3 \ddot{y}$. In the s-domain

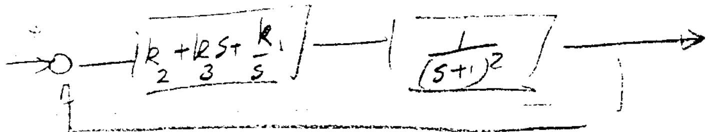
$$u(s) = -(k_1 + k_2 s + k_3 s^2)\, y(s)$$

So the compensator transfer function is

$$D(s) = 7 + 15s + 15s^2$$

(b) 

has the same closed loop poles as:



It follows from (a) that the PID controller will place closed loop poles at $-3, -2 \pm j$ if

$$k_2 + k_3 s + k_1 \frac{1}{s} = K\left(1 + T_D s + \frac{1}{T_I s}\right)$$

or

$$K = 15, \quad T_D = \frac{7}{15}, \quad T_I = 1$$

5. Take as state variables $x_1 = y$ and $x_2 = \dot{y}$. Then the problem can be reformulated as the 'general' quadratic cost control problem with

[4] $Q = \begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix}$, $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\underline{x}_0 = \begin{pmatrix} y(0) \\ \dot{y}(0) \end{pmatrix}$

Expanding the Riccati eqn. for this data, we obtain

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} + \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} = 0$$

Equating entries gives

$0 + 0 + \alpha - P_{12}^2 = 0$ — (1)

$0 + P_{11} + 0 - P_{12}P_{22} = 0$ — (2)

$P_{11} + 0 + 0 - P_{22}P_{12} = 0$ (duplicates previous equation)

$P_{12} + P_{12} + 0 - P_{22}^2 = 0$ — (3)

The first equation gives

$$P_{12} = \pm\sqrt{\alpha}$$

To solve (3), we must have $P_{12} = \alpha^{\frac{1}{2}}$. Then $P_{22} = \pm\sqrt{2}\,\alpha^{1/4}$

Then $P_{11} = \pm\sqrt{2}\,\alpha^{3/4}$. Possible solutions are

$$P = \begin{bmatrix} -\sqrt{2}\,\alpha^{3/4} & \alpha^{\frac{1}{2}} \\ \alpha^{\frac{1}{2}} & -\sqrt{2}\,\alpha^{1/4} \end{bmatrix} \text{ and } \begin{bmatrix} +\sqrt{2}\,\alpha^{3/4} & \alpha^{\frac{1}{2}} \\ \alpha^{\frac{1}{2}} & +\sqrt{2}\,\alpha^{\frac{1}{4}} \end{bmatrix}$$

We reject the first solution, because it is not positive definite. This leaves the second solution. The feedback law is

$$u = -b^T P x = -P_{12}x_1 - P_{22}x_2 = -P_{12}y - P_{22}\dot{y}$$

It follows that the optimal feedback law is

[14] $$u = -\alpha^{\frac{1}{2}}y - \sqrt{2}\,\alpha^{\frac{1}{4}}\dot{y}$$

The closed loop system equation is $\dot{x} = A_{cl}x$, where

$$A_{cl} = (A - bb^TP) = \begin{bmatrix} 0 & 1 \\ -P_{12} & -P_{22} \end{bmatrix}$$

This has characteristic polynomial $s^2 + 2\left(\frac{\alpha^{\frac{1}{4}}}{\sqrt{2}}\right)s + \alpha^{\frac{1}{2}}$

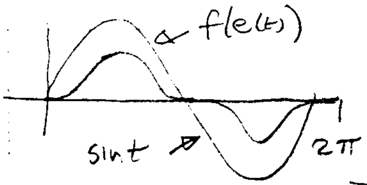
or $s^2 + 2\zeta\omega_n(\alpha)s + \omega_n^2(\alpha)$

with $\zeta = 1/\sqrt{2}$ and $\omega_n = \alpha^{\frac{1}{4}}$

It follows that, as α increases, the damping factor of the closed loop system remains the same ($\zeta = 1/\sqrt{2}$), but the 'states' $y, \dot{y}$

[2] converge to zero increasingly rapidly.

6 Take $e(t) = A\sin t$. Then $f(e(t)) = \begin{cases} A^2\sin^2 t & 0 \le t \le \pi \\ -A^2\sin^2 t & \pi < t \le 2\pi \end{cases}$



The first harmonic has amplitude

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin t\,dt.$$

By symmetry $a_1 = 2 \times \frac{1}{\pi}\int_0^{\pi} A^2 \sin^3 t\,dt$

Using 'hint', we have

$$a = 2A^2/\pi \int_0^{\pi} \sin^3 t\,dt = 2A^2/\pi\left[-\cos t + \tfrac{1}{3}\cos^3 t\right]_0^{\pi}$$

$$= 2A^2/\pi\left(+2 - \tfrac{1}{3}.2\right) = \frac{8A^2}{3\pi}$$

Hence, describing function is

$$N(A) \quad (= a_1/A) = \frac{8A}{3\pi}.$$

Limit cycle equation is $G(j\bar\omega)N(A) = -1 + 0j$

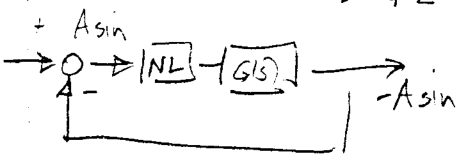
i.e. $\dfrac{(j\bar\omega+4)^2}{-\omega^2(j\bar\omega+1)} = -\dfrac{1}{N(A)}$ or $\dfrac{((16-\bar\omega^2) + 8j\bar\omega)(1-j\bar\omega)}{\bar\omega^2(1+\bar\omega^2)} = \dfrac{1}{N(A)}$.

whence $\dfrac{(16 + 7\bar\omega^2) + j\bar\omega(-8+\bar\omega^2)}{\bar\omega^2(1+\bar\omega^2)} = \dfrac{1}{N(A)}$

$\mathrm{Im}\{..\} = 0 \Rightarrow \bar\omega = 2\sqrt{2}$ rs^{-1} (frequency of limit cycle &c.)

$\mathrm{Re}\{..\} = 1/N(A) \Rightarrow$

$$72/72 = \frac{1}{N(A)} \text{ or } 1 = \frac{3\pi}{2A} \Rightarrow A = \frac{3\pi}{8}.$$

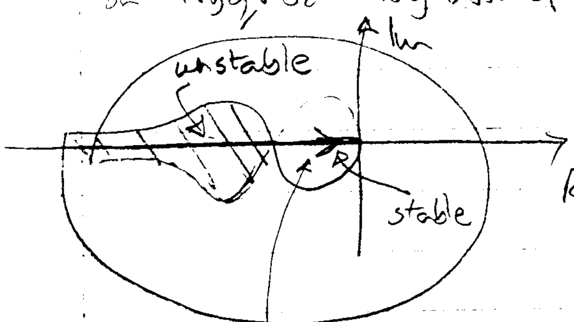


From block diagram, output is approx.

$$c(t) = -A\sin\bar\omega t$$

Hence amplitude at output $= \dfrac{3\pi}{8}$

To assess limit cycle stability, superpose locus of $-1/N(A)$ on Nyquist diagram of $G(s)$



locus of $-1/N(A)$ (arrow indicates increasing A)

As A increases, $-1/N(A)$ moves from an 'unstable' region to a 'stable' region.
It follows the limit cycle is stable.