DIGITAL SIGNAL PROCESSING
SOLUTIONS 2003

Special Instructions for Invigilators: None

Information for Candidates:

Sequence	z-transform
$\delta(n)$	1
$u(n)$	$\frac{1}{1-z^{-1}}$
$a^n u(n)$	$\frac{1}{1-az^{-1}}$
$\left(r^n \cos\omega_0 n\right)u(n)$	$\frac{1-(r\cos\omega_0)z^{-1}}{1-(2r\cos\omega_0)z^{-1}+r^2z^{-2}}$

Table 1 : z-transform pairs

$\delta(n)$ is defined to be the unit impulse function.
$u(n)$ is defined to be the unit step function.

Numbers in square brackets against the right margin of the following pages are a guide to the marking scheme.

[4]

1. (a) Write down the expression for $x(n)$ as the inverse discrete Fourier transform of $X(k)$. Briefly explain the meaning of each term in the expression and give an illustrative example of a practical application of the inverse discrete Fourier transform.

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{j2\pi kn/N} \qquad n = 0, 1, 2, \ldots, N-1$$

Explanation:
k is an integer counter of points in the frequency domain
n is an integer counter of samples in the time domain
X(k) are the frequency domain points
x(n) are the time domain samples related to X(k) through the IDFT
N is the number of points in X(k) and also in x(n).

Practical application:
In FIR filter design, x(n) are the impulse response values and coefficients for a frequency sampled version of the desired frequency response X(k).

(b) Explain the meaning of the terms *minimum phase, maximum phase* and *mixed phase* in the context of causal stable FIR filters. [4]

In a minimum phase filter, the phase response is such that the zeros are all inside the unit circle, i.e. for the i[th] root $|z_i| < 1 \quad \forall i$.

In a maximum phase filter $|z_i| > 1$.
A mixed phase filter has zeros both inside and outside the unit circle.

(c) Given $X(k) = [1, 2, 1, 2]$, determine $x(n)$.

[6]

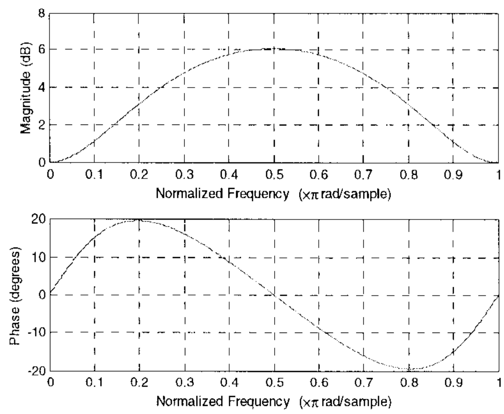
x(n) = [1.5, 0, -0.5, 0]

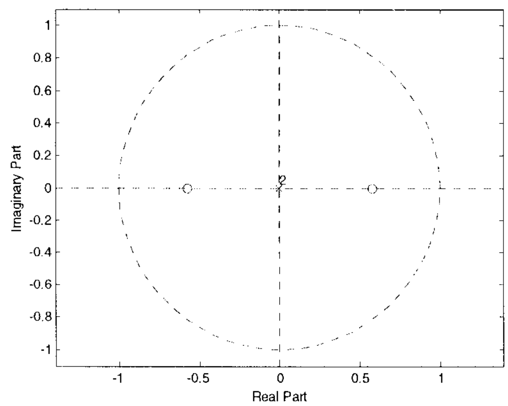
Consider that $x(n)$ is the impulse response of an FIR filter. Plot the roots of the filter's transfer function on the *z*-plane. Sketch graphs showing the main features of the filter's magnitude response, in dB, and phase response. Comment on whether this filter is minimum phase, maximum phase or mixed phase. [6]

$$H(z) = 1.5 - 0.5z^{-2} = \frac{1.5z^2 - 0.5}{z^2}$$

Numerator: A quadratic factorisation of $1.5z^2 - 0.5$ leads to roots at +/-0.5774.
Denominator: Yields 2 coincident roots at the origin.

The frequency response is given at frequencies of $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$. It is therefore only necessary to study the response between these points by evaluating the magnitude and phase of $H(e^{j\omega}) = 1.5 - 0.5e^{-j2\omega}$ for points such as $\omega = \frac{\pi}{4}$ and $\frac{3\pi}{4}$. At $\frac{\pi}{4}$, $H(z) = 1.5 + j0.5$ giving $|H(e^{j\omega})|$ = 3.98 dB and $\angle H(e^{j\omega})$ = 18 degrees. At $\frac{3\pi}{4}$, $H(z) = 1.5 - j0.5$ giving the same magnitude and phase of –18 degrees.





Since all the zeros are inside the unit circle, the filter is minimum phase.

2. (a) Define the autocorrelation function, $\gamma_{xx}(l)$, of a real-valued signal $x(n)$ and give a short description of autocorrelation including its method of computation and the significance of l. [4]

$$\gamma_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n)x(n+l)$$ (–ve sign is also correct due to symmetry)

l is the relative shift between the signal and a copy of the signal.
For any value of l, the autocorrelation is the sum of the point-by-point products of the samples in the signal and a version of the signal shifted by l.

(b) Write down the important properties of $\gamma_{xx}(l)$. [3]

$\gamma_{xx}(l) = \gamma_{xx}(-l)$

$\gamma_{xx}(0) = \sum x^2(n)$ equivalent to the energy in the signal

$|\gamma_{xx}(l)| \le \gamma_{xx}(0)$ i.e. the autocorrelation is a maximum for zero shift.

(c) The short-term autocorrelation function of a signal $x(n)$ can be defined as [13]

$$\gamma_{ST}(k,n) = \sum_{m=-\infty}^{\infty} x(m)w(n-m)x(m+k)w(n-m-k)$$

for which $w(n)$ is a window function.

Show that $\gamma_{ST}(k,n)$ can be computed using a structure of the form of Figure 2.1 and determine an expression for the digital filter $h_k(n)$.

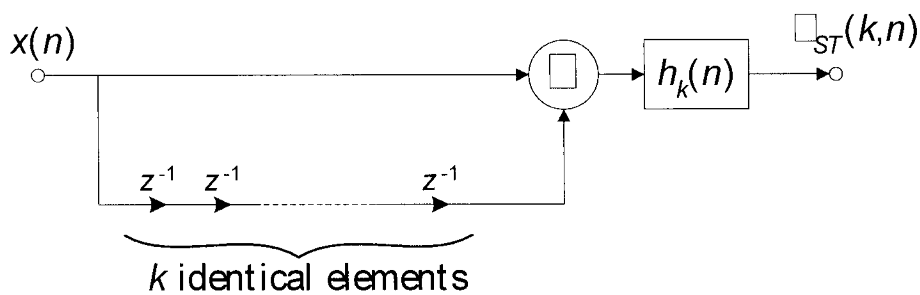


Figure 2.1

Let $m + k = s$. Then $\gamma_{ST}(k,n) = \sum_{s=-\infty}^{\infty} x(s-k)w(n-s+k)x(s)w(n-s)$. Thus γ_{ST} can be found as the convolution of $x(n)x(n-s)$ with $h_k(n) = w(n)w(n+k)$.

3. Write down the necessary conditions (i) in the frequency domain and (ii) in the z-domain for a filter $\mathcal{A}(z)$ to be a stable allpass filter and briefly describe a practical application of allpass filters. [10]

(i) $\left|\mathcal{A}(e^{jw})\right| = 1$

(ii) In the z domain, we require that if $z = re^{if}$ is a pole of the system function, then there must also exist a zero at $z = \frac{1}{r}e^{-if}$. For stable allpass filters, all the poles must lie within the unit circle and hence all the zeros must be outside.

Allpass filter can be used for phase equalisation.

Give the z domain system function for a second order allpass filter using the real filter coefficients ℓ_i for integer i.

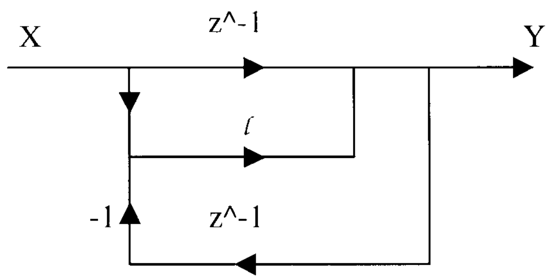
$$\mathcal{A}(z) = \frac{\ell_2 + \ell_1 z^{-1} + z^{-2}}{1 + \ell_1 z^{-1} + \ell_2 z^{-2}}$$

Derive and sketch the signal flow graph of a first-order allpass filter that employs only one non-trivial multiplication operation per output sample. [10]

Start by considering

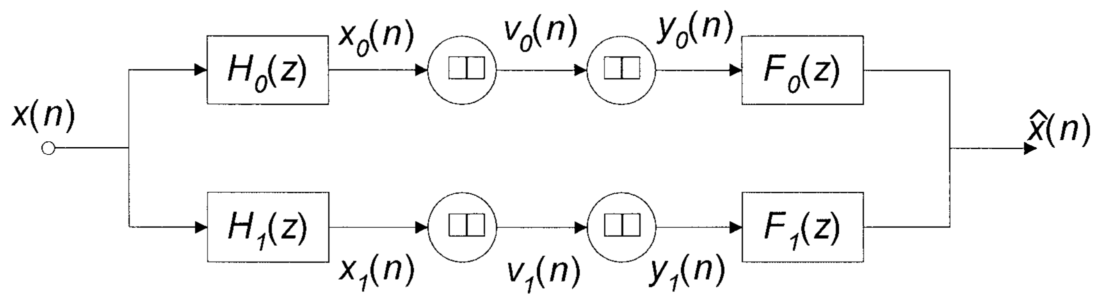
$$\mathcal{A}(z) = \frac{\mathcal{Y}(z)}{X(z)} = \frac{\ell + z^{-1}}{1 + \ell z^{-1}}$$

$$y(n) = (x(n) - y(n-1))\ell + x(n-1)$$



4. Consider a multirate system consisting of an analysis filter bank followed by a synthesis filter bank. The filter banks have two bands of equal bandwidth. The input signal to the system is $x(n)$ and the output is $\hat{x}(n)$.

(a) Draw a signal flow diagram of the system and label all signals in the system. Give an expression for each signal in terms of $X(z)$, the z-transform of $x(n)$. [5]



$$X_k(z) = H_k(z)X(z) \qquad k = 0,1$$

$$V_k(z) = \frac{1}{2}\left(X_k(z^{\frac{1}{2}}) + X_k(-z^{\frac{1}{2}}\right) \qquad k = 0,1$$

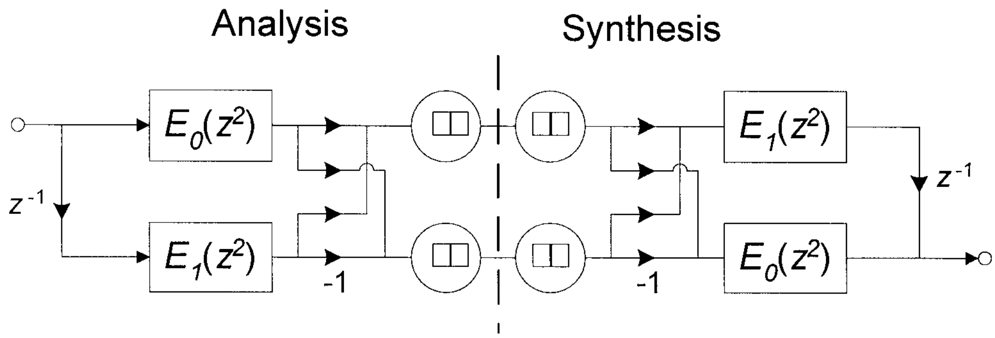
$$Y_k(z) = V_k(z^2) = \frac{1}{2}(X_k(z) + X_k(-z)) = \frac{1}{2}(H_k(z)X(z) + H_k(-z)X(-z)) \qquad k = 0,1$$

(b) Consider a half-band lowpass prototype filter $H_0(z)$. Write down expressions for the filters in your multirate system in terms of $H_0(z)$. Give reasons for your choices. [3]

Use QMF filters
Design for alias cancellation
$H_1(z) = H_0(-z)$
$F_0(z) = H_0(z)$
$F_1(z) = -H_1(z)$

(c) Show how each of the filters in the multirate system can be represented in a 2-phase, Type 1 polyphase form and, hence, draw and label the signal flow diagram of the analysis and synthesis filter banks using filters in Type 1 polyphase filters. [6]

$$H(z) = E_0(z^2) + z^{-1}E_1(z^2)$$



(d) State the Noble Identities and briefly describe how they can be used to improve the efficiency of multirate systems. [2]

> By interchanging the order of multirate blocks and filters through the use of the Noble Identities, it is possible to implement all filtering at the lowest sample rate, thereby reducing computational load.

(e) Using the Nobel Identities, redraw the two-band multirate system in an efficient Type 1 polyphase form. Calculate the ratio of the number of filtering computations required in your efficient implementation compared to the number required in the direct implementation of the system described in part (a). [4]

> The application of the Noble Identities brings a saving of a factor of 2. A further saving of a factor of 2 comes from the way in which the polyphase implementation gives both highpass and lowpass filters from the same computations – one be the sum and the other the difference of the two component filters $E_0(z)$ and $E_1(z)$.

5. (a) Consider the discrete-time signal $x(n)$ and the continuous-time signal $x_a(t)$, with $x_a(nT) = x(n)$ for sampling period T.

(i) Express $x(n)$ in terms of the superposition of complex exponentials. [2]

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j2\pi kn/N} \qquad n = 0,1,\ldots,N-1$$

(ii) State an expression for the amplitudes of the exponentials. [2]

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} \qquad k = 0,1,\ldots,N-1$$

(iii) Hence or otherwise, show that the Fourier Transform of $x(n)$ is periodic with period 2π. [3]

Write an expression for $x_a(nT)$ as the sum of integrals over intervals of $2\pi / T$:

$$x_a(t) = \frac{1}{2\pi}\sum_{r=-\infty}^{\infty}\int_{(2r-1)\pi/T}^{(2r+1)\pi/T} X_a(j\Omega)e^{j\Omega nT}d\Omega$$

Replace $\Omega \leftarrow (\Omega' + 2\pi r / T)$:

$$x(n) = \frac{1}{2\pi}\sum_{r=-\infty}^{\infty}\int_{-\pi/T}^{\pi/T} X_a\left(j\left(\Omega' + \frac{2\pi r}{T}\right)\right)e^{j\Omega' nT}e^{2\pi nr}d\Omega'$$

Use $e^{j2\pi rn} = 1 \quad \forall (r,n)$ integer, reverse order of sum and integration and use $\Omega' = \omega/T$:

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{1}{T}\sum_{r=-\infty}^{\infty} X_a\left(j\left(\frac{\omega}{T} + \frac{2\pi r}{T}\right)\right)\right]e^{j\omega n}d\omega$$

Note that this is in the form of an IDFT and hence

$X(e^{j\omega}) = \frac{1}{T}\sum_{r=-\infty}^{\infty} X_a\left(j\left(\frac{\omega}{T} + \frac{2\pi r}{T}\right)\right)$ which is periodic with period 2π.

(iv) Show how $q(t)$ can be reconstructed from $x(n)$ stating any necessary conditions. [4]

Write the inverse FT expression for $x_a(t)$ for the range $-\pi/T \le \Omega \le \pi/T$

$$x_a(t) = \frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} X_a(\Omega)e^{j\Omega t}d\Omega$$

As shown in (iii) for the range $-\pi \le \omega \le \pi$, $X(e^{j\omega}) = X(e^{j\Omega T}) = \frac{1}{T}X_a(\Omega)$

giving: $x_a(t) = \frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} TX(e^{j\Omega T})e^{j\Omega t}d\Omega.$

Using the DTFT we have: $X(e^{j\Omega T}) = \sum_{n=-\infty}^{\infty} x_a(nT)e^{-j\Omega nT}$ and then write:

$$x_a(t) = \frac{T}{2\pi}\int_{-\pi/T}^{\pi/T}\left[\sum_{n=-\infty}^{\infty} x_a(nT)e^{-j\Omega nT}\right]e^{j\Omega t}d\Omega.$$

Lastly, change the order of integration and summation:

$$x_a(t) = \sum_{n=-\infty}^{\infty} x_a(nT)\left[\frac{T}{2\pi}\int_{-\pi/T}^{\pi/T} e^{j\Omega(t-nT)}d\Omega\right]$$

$$= \sum_{n=-\infty}^{\infty} x_a(nT)\frac{\sin\left(\frac{\pi}{T}(t-nT)\right)}{\frac{\pi}{T}(t-nT)}$$

This can be recognized as the convolution of $x(n)$ with the sinc function $\frac{\sin\left(\pi t/T\right)}{\pi t/T}$ which is equivalent to an ideal lowpass filter with cutoff frequency of $\omega = \pi$.

(b) A particular LTI system has input $x(n)$ and output $y(n)$. An experiment was conducted which employed an input signal

$$x(n) = \begin{cases} 1, & n = 0,1 \\ 0, & \text{otherwise} \end{cases}$$

and the corresponding output was found to be

$$y(n) = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}.$$

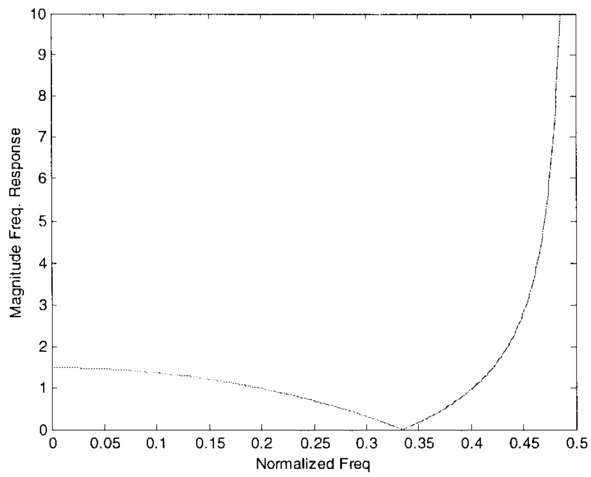
(i) Determine the frequency response of the system using the DTFT. [3]

$$\frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{n=0}^{N-1} e^{-jn\omega T}}{\sum_{n=0}^{1} e^{-jn\omega T}} = \frac{1-e^{-jN\omega T}}{1-e^{-j2\omega T}} = e^{-j(N-2)\omega T/2}\frac{\sin(N\omega T/2)}{\sin(\omega T)}$$

(ii) Sketch the magnitude response for $N = 3$. [6]

For N=3 $\left|\frac{Y(e^{j\omega})}{X(e^{j\omega})}\right| = \frac{\left|1+e^{-j\omega}+e^{-j2\omega}\right|}{\left|1+e^{-j\omega}\right|}$ for $T = f_s = 1$

The quadratic numerator can be factored to show zeros at $-0.5 \pm \frac{\sqrt{3}}{2}i$ corresponding to a normalized frequency of $2\pi/3$. The d.c. gain is 3/2 and the gain at π tends to $\left|\frac{-1}{0}\right|$. Hence we obtain the sketch plot as follows:



10
9
8
7
6
5
4
3
2
1
0
Magnitude Freq. Response
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
Normalized Freq

6. (a) Consider the two-sided function $x(n) = a^{n}$.

Write down the definition of the z-transform of $x(n)$. [6]

$$X(z) = \sum_{-\infty}^{\infty} x(n) z^{-n} = \sum_{-\infty}^{\infty} a^{n} z^{-n}$$

By considering $x(n)$ as the sum of two one-sided functions, show that $x(n) = a^{n}$ does not have a z-transform.

$$X(z) = \sum_{-\infty}^{-1} a^{n} z^{-n} + \sum_{0}^{\infty} a^{n} z^{-n}$$

The first term converges for $|z| < |a|$ and the second term converges for $|z| > |a|$. Hence it can be seen that the two ROCs have no common regions.

(b) Using long division, find the inverse z-transform of [6]

$$\mathcal{H}(z) = \frac{1 + 2z^{-1}}{1 + 0.4z^{-1} - 0.12z^{-2}}.$$

$$1 + 1.6z^{-1} + 0.52z^{-2} + 0.4z^{-3} - 0.22z^{-4} + \ldots$$

Hence determine the first 5 samples of the impulse response of $\mathcal{H}(z)$

$$h(n) = \{1.0, 1.6, -0.52, 0.4, -0.22\}$$

(c) Find a causal, stable, IIR equalizer $\mathcal{G}(z)$ such that $\left|\mathcal{H}(e^{jw})\mathcal{G}(e^{jw})\right| = 1$. [8]

The roots of H(z) can be found as:
zeros: 0, -2
poles: -0.6, 0.2

To create the equaliser, we could attempt to form $\mathcal{G}(z) = \mathcal{H}^{-1}(z)$ but the zero at −2 will then become an unstable pole. So, first reflect this root inside the unit circle and compensate for the gain:

$$\mathcal{G}(z) = 0.5 \times \frac{1 + 0.4z^{-1} - 0.12z^{-2}}{1 + 0.5z^{-1}}$$