UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 2003

BEng Honours Degree in Computing Part I
MEng Honours Degrees in Computing Part I
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the Associateship of the City and Guilds of London Institute

PAPER C142

DISCRETE MATHEMATICS

Monday 28 April 2003, 16:00
Duration: 90 minutes
(Reading time 5 minutes)

Answer THREE questions

Paper contains 4 questions
Calculators not required

1 Let A and B be arbitrary sets. Recall that the symmetric difference of A and B, denoted $A \triangle B$, is defined by

$$A \triangle B = \{x : (x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)\}$$

a For $A = \{3, 5, 7, 9\}$ and $B = \{2, 3, 4, 5, 6\}$, give $A \triangle B$, $A \cup B$, $A \cap B$ and $B - A$.

b Let A, B and C be arbitrary sets.

i) Draw the Venn diagrams of $(A \triangle B) \cup (C \triangle B)$ and $(A \triangle C) \triangle B$. Give a simple example to show that the two sets are not equal. Explain under what conditions they are equal.

ii) *Prove* that the equality $A \triangle B = (A \cup B) - (A \cap B)$ is true.

c Assume that A and B are finite sets. Define the cardinality of $A \triangle B$ in terms of the cardinality of $A \cap B$, A and B. Explain your answer.

d Show that $A \triangle C = B \triangle C$ implies $A = B$.

The four parts carry, respectively, 20%, 40%, 15% and 25% of the marks.

2a Let $A = \{a, b, c\}$ and $B = \{2, 4\}$. If possible, in each case, give a function from A to B which is

i) a bijection;

ii) one-to-one and not onto;

iii) onto and not one-to-one;

iv) neither one-to-one nor onto.

If not possible, explain why.

b Repeat part 2a when A is the set of natural numbers and B the set of integers.

c Given two arbitrary sets A and B, recall that A has the same cardinality as B if and only if there is a bijection from A to B.

i) Prove that the cardinality relation is reflexive, symmetric and transitive. You may state, rather than prove, any properties you use about bijections.

ii) Explain why the set of integers and the set of odd natural numbers have the same cardinality.

The three parts carry, respectively, 25%, 30% and 45% of the marks.

3a i) What does it mean for a graph to be *simple*?

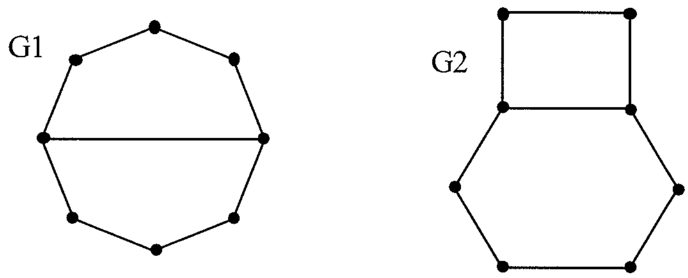
ii) What is the greatest number of arcs possible for a simple graph with n nodes (any $n \geq 1$)? Justify your answer.
Give an example for $n = 5$ to show that this greatest number can be achieved.

iii) What is the least number of arcs possible for a simple graph with n nodes, where each node has degree ≥ 3? Justify your answer.
Give an example for $n = 5$ to show that this least number can be achieved.

iv) Show that a simple connected graph with n nodes must have at least n-1 arcs (any $n \geq 1$).

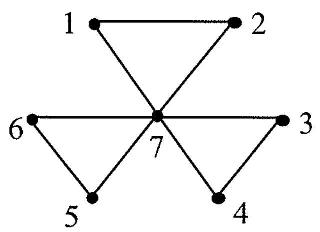
b i) What does it mean for two graphs to be *isomorphic*?

ii) The diagram shows graphs G1 and G2:



Is G1 isomorphic to G2?
Explain your answer.

iii) An *automorphism* is an isomorphism from a graph to itself.
How many automorphisms (including the identity) does the following graph possess?
Explain your answer.



The two parts carry, respectively, 55%, 45% of the marks.

4a Binary Search is applied to searching for an integer x in an ordered list L of integers with 9 distinct entries indexed from 0 to 8.

i) Draw the decision tree.
Include the possible ways that x fails to belong to L.

ii) What is the worst-case number W of comparisons?
In exactly what position or positions must x lie relative to the elements of L for the worst case to arise?
Why is this worst-case number W of comparisons optimal among all algorithms for searching a list of length 9 by comparisons?

iii) Now assume that x is in L (but the algorithm does not take advantage of this fact), and that all positions for x in L are equally likely.
Calculate the average number of comparisons.

b i) Write down the recurrence relation for the worst case number of comparisons needed by MergeSort on a list of length n (not necessarily a power of 2).
Briefly justify your answer.
Do not solve the recurrence relation.

ii) Let n be a power of 2, that is, $n=2^k$. Write down the recurrence relation for MergeSort when applied to a list of n distinct numbers which are already sorted.
Briefly justify your answer.

iii) Solve your recurrence relation from (ii).

iv) Again let $n=2^k$. How many comparisons does MergeSort take on the list $[1,2,\ldots,n-1,n,n,n-1,\ldots,2,1]$?

The two parts carry, respectively, 40%, 60% of the marks.