UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 2001

MSc in Computing Science
BSc Honours Degree in Mathematics and Computer Science Part III
MSci Honours Degree in Mathematics and Computer Science Part III
MSc in Advanced Computing
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the Associateship of the Royal College of Science

PAPER C336=I3.6

PERFORMANCE ANALYSIS

Wednesday 2 May 2001, 14:00
Duration: 120 minutes

Answer THREE questions

Paper contains 4 questions
Calculators not required

1 a Show that, in a steady state M/M/1 queue with constant service rate μ and constant arrival rate λ, the probability that the queue length is n is $(1-\rho)\rho^n$ where $\rho=\lambda/\mu$. What is a necessary and sufficient condition for a steady state to exist?

b What is an N-phase *Markov modulated Poisson process* (MMPP)? If the arrival rate in phase k is λ_k and the equilibrium probability k-vector of the Markov phase-process is π, in the steady state,

i) What is the average arrival rate?

ii) In a long observation period of length T, how many arrivals would you expect to see when the phase is k?

iii) Estimate the probability that a random arrival comes in phase k.

c The IPP/M/1 queue has interrupted Poisson (IPP) arrivals and exponential service times with parameter μ. An IPP is a 2-phase MMPP in which the arrival rate in one phase (phase 0, say) is zero. Suppose the arrival rate in the other phase (phase 1, say) is 1. If the instantaneous transition rates from phase 0 to 1 and phase 1 to 0 are respectively α and β, derive a condition for the queue to have a steady state.

The three parts carry, respectively, 30%, 50% *and* 20% *of the marks.*

2 a i) State the conditions for a discrete-time stochastic process $X = \{X_n \mid n=0,1,2,...\}$ to be a *Markov Chain* and define its *one-step transition probability matrix* Q.

ii) Give an example of a *reducible* discrete time Markov chain.

iii) Give an example of a *periodic* discrete time Markov chain.

iv) State the *steady state theorem* for discrete time Markov chains, specifying clearly the conditions under which it holds. Give a plausible argument to derive it, *assuming* the steady state exists.

b In a merging switch, there are two input channels and one output channel, with buffers at each. In every basic time unit, a cell arrives in each input buffer with probability p and a cell departs from the output buffer, if non-empty, with probability q. In addition, n cells are taken from the input buffers, processed and placed into the output buffer, providing there were at least n in the input buffers at the start of the time unit; if not, all cells are transferred. Assuming that all tasks are stochastically identical and that buffers are unbounded, describe a *two-dimensional* discrete time Markov chain model for this system by:

i) defining a suitable state space;

ii) defining the one-step transition probability matrix;

iii) writing down a set of linear equations for the equilibrium state probabilities. What constraints would you expect on p, q and n for equilibrium to exist?

3 a Write down the balance equations for a closed queueing network of M servers and population N, where node i has constant service rate μ_i and a task that completes service at node i proceeds immediately to node j with constant probability p_{ij} $(1 \le i,j \le M)$. State Jackson's Theorem for closed networks.

b Consider a *cyclic* queueing network ($p_{12} = p_{21} = 1$) with two tasks, A and B, and two servers, 1 and 2, each with rate μ. Let $E(\lambda)$ denote an exponential random variable with parameter λ, so that the service time random variable at each server is $E(\mu)$. *Assuming* the Job Observer Property, on arrival of task A at queue 1,

i) What is the probability that task B is at server 2?

ii) If B is at server 2, what is the probability distribution of the time to the next service completion (of either task)?

iii) Show that the random variable for the remaining cycle time of task A (i.e. the time left before A next joins queue 1) is then a sum of two $E(\mu)$ random variables.

iv) Conversely, if B is at server 1 on A's arrival there, now what are the random variables for the time to the next service completion and the remaining cycle time after that?

v) Given that the sum of an $E(2\mu)$ random variable and a random variable that is 0 with probability 0.5 and $E(\mu)$ with probability 0.5 is $E(\mu)$, show that cycle time is an Erlang-3 random variable with parameter μ.

The two parts carry, respectively, 40% *and* 60% *of the marks.*

4 a State Little's result, paying particular attention to the conditions under which it holds and explaining precisely the terms involved in the formula.

b An M/G/1 queue at equilibrium has arrival rate λ and mean service time $1/\mu$.

i) Apply Little's result to the queue of tasks waiting to commence service.

ii) What is the probability that the server is busy just before an arrival instant? What properties do you need for this to hold?

iii) Show that the mean number of tasks waiting to commence service is $R\lambda^2/(\mu-\lambda)$ where R is the mean of the remaining service time of a task observed in service at a random instant.

c In a closed queueing network of M nodes with K customers, let the throughput along some given arc be T and the average number of visits made by a task to server i between successive traversals of that arc be v_i $(1 \le i \le M)$.

i) Show that $L_i = Tv_iW_i$ and that $K = T\sum_{i=1}^{M} v_iW_i$ where L_i is the mean queue length and W_i is the mean waiting time at server i $(1 \le i \le M)$.

ii) Explain how to calculate the v_i from the network's routing probabilities.

iii) If all service nodes are infinite servers, show that throughput is proportional to population.

The three parts carry, respectively, 25%, 35% *and* 40% *of the marks.*

End of paper