Paper Number(s): **ISE2.9**

IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE
UNIVERSITY OF LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2001

ISE PART II: M.Eng. and B.Eng.

CONTROL SYSTEMS

Friday, 11 May 2:00 pm

There are FIVE questions on this paper.

Answer THREE questions.

Time allowed: 2:00 hours

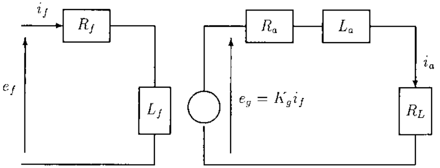
Corrected Copy

Examiners: Jaimoukha,I.M.

1. Consider the following circuit diagram representing a DC generator. Here

$$\begin{aligned} e_f &: \text{ applied field voltage} \\ i_f &: \text{ field current} \\ R_f &: \text{ field coil resistance} \\ L_f &: \text{ field coil inductance} \\ e_g &: \text{ generated voltage} \\ i_a &: \text{ armature current} \\ R_a &: \text{ armature resistance} \\ L_a &: \text{ armature inductance} \\ R_L &: \text{ load resistance.} \end{aligned}$$

It is assumed that the angular velocity of the armature is constant so that the generated voltage e_g is proportional to the field current, $e_g = K_g i_f$.



(a) Write down the two differential equations relating the field and armature currents to the applied field voltage. [5]

(b) Derive a state-variable model in the standard form:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t). \end{aligned}$$

Take the states to be the field and armature currents, the input to be the applied field voltage and the output to be the load voltage $e_L = i_a R_L$.

[10]

(c) Derive the transfer function between the applied field voltage e_f and the load voltage e_L. [5]

2. The figure below depicts a feedback control system with

$$G(s) = \frac{k}{(s+1)(s+2)}$$

where k is a design parameter. Design a stabilising compensator of the form

$$K(s) = \frac{1}{s-p}$$

as follows:

(a) Choose p so that when $r(t)$ is a unit step,

$$r(t) = 1, \quad t \geq 0,$$

applied at $t = 0$, the steady-state error must satisfy

$$\lim_{t\to\infty} e(t) = 0.$$

[4]

(b) Find the range of values of k such that the closed-loop is stable. [5]

(c) Find the value of $k > 0$ such that the closed-loop system is marginally stable. For this value of k, show that -3 is a closed-loop pole. When $r(t) = 0$, the closed-loop system is observed to oscillate. Find the frequency of this oscillation. [4]

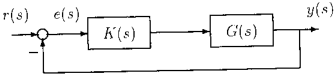
(d) Find the minimum value of k such that when $r(t)$ is a unit ramp,

$$r(t) = t, \quad t \geq 0,$$

applied at $t = 0$, the steady-state error must satisfy

$$\lim_{t\to\infty} e(t) \leq 1.$$

[7]



3. Consider the feedback control system shown in the figure below. Here,

$$G(s) = \frac{1}{s(s+2)}$$

and $K(s)$ is the transfer function of the compensator.

(a) For $K(s) = k$, a constant compensator, draw the root locus accurately as k varies in the range $0 \leq k \leq \infty$. **[5]**

(b) Design a phase lead compensator $K(s)$ to satisfy the following design specifications:

i. The closed-loop is stable.

ii. The two dominant poles have damping ratio $\zeta = .707$ and settling time of $2s$.

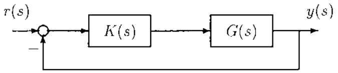
iii. The response due to the third pole decays at least as fast as e^{-3t}.

Draw a rough sketch of the root locus of the compensated system. **[12]**

(c) For the compensated system in Part (b), evaluate the steady state error

$$e_{ss} = \lim_{t \to \infty} [r(t) - y(t)]$$

when $r(t)$ is a unit step reference signal applied at $t = 0$. **[3]**



4. Consider the feedback control system in the figure below. Here,

$$G(s) = \frac{1}{(s+1)^3}$$

and $K(s)$ is the transfer function of a feedforward compensator.

(a) Sketch the Nyquist diagram of $G(s)$, clearly indicating the low and high frequency portions, as well as the real-axis intercepts. [7]

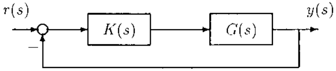
(b) Set $K(s) = K$, a constant compensator. Give the number of unstable closed-loop poles for all (positive and negative) K.

(c) Take $K = 1$. Determine the gain margin. [6]

(d) Without doing any actual design, briefly describe how a phase-lag compensator,

$$K(s) = k\frac{1 + s/\omega_0}{1 + s/\omega_p}, \qquad 0 > \omega_0 > \omega_p >$$

would improve the steady-state tracking properties without deteriorating the stability margins. [7]



5. The figure below depicts a position control system with velocity feedback. Here,

$$G(s) = \frac{1}{s}$$

and K_v and K_p are design parameters.

(a) Evaluate the transfer function from $r(t)$ to $y(t)$ and from $r(t)$ to $e(t)$.

[4]

(b) Find the values of K_p and K_v so that the following design specifications are satisfied:

i. The closed-loop system must be stable.

ii. Both closed-loop poles are placed at -2.

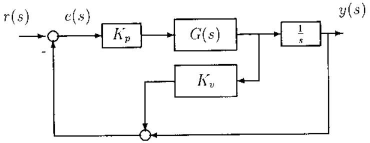
Draw the resulting root locus.

[12]

(c) Suppose that $r(s) = \frac{1}{s^3}$. Find the steady state error

$$e_{ss} = \lim_{t \to \infty} e(t)$$

[4]



SOLUTIONS (ISE2.9, Control Systems, 2001)

1. (a) The equation for the field circuit is

$$e_f(t) = R_f i_f(t) + L_f \frac{di_f(t)}{dt}$$

and the equation for the armature circuit is

$$K_g i_f(t) = (R_a + R_L) i_a(t) + L_a \frac{di_a(t)}{dt}$$

since $e_g(t) = K_g i_f(t)$.

(b) Let $x_1(t) = i_f(t), x_2(t) = i_a(t), u(t) = e_f(t)$ and $y(t) = R_L i_a(t)$. Then the above equations can be written as

$$\begin{aligned} \dot{x}_1(t) &= -\frac{R_f}{L_f} x_1(t) + \frac{1}{L_f} u(t) \\ \dot{x}_2(t) &= \frac{K_g}{L_a} x_1(t) - \frac{R_a + R_L}{L_a} x_2(t) \end{aligned}$$

and $y(t) = R_L x_2(t)$. In matrix form, this becomes

$$\begin{aligned} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} &= \begin{bmatrix} -\frac{R_f}{L_f} & 0 \\ \frac{K_g}{L_a} & -\frac{R_a+R_L}{L_a} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{L_f} \\ 0 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 0 & R_L \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}. \end{aligned}$$

(c) Taking the Laplace transform of the first equation in part (a):

$$i_f(s) = \frac{e_f(s)}{R_f + sL_f}.$$

The Laplace transform of the second equation then gives the required transfer function as

$$\frac{R_L i_a(s)}{e_f(s)} = \frac{R_L K_g}{(R_f + sL_f)(R_a + R_L + sL_a)}$$

2. (a) After some block diagram manipulations,

$$\frac{e(s)}{r(s)} = \frac{1}{1 + K(s)\dfrac{k}{(s+1)(s+2)}}.$$

Using the final value theorem of the Laplace transform,

$$\begin{aligned} e_{ss} &= \lim_{t\to\infty} e(t) = \lim_{s\to 0} se(s) = \lim_{s\to 0} \frac{sr(s)}{1 + K(s)\dfrac{k}{(s+1)(s+2)}} \\ &= \lim_{s\to 0} \frac{1}{1 + K(s)\dfrac{k}{(s+1)(s+2)}} \end{aligned}$$

since $r(s) = 1/s$. For zero steady-state error, we set $K(s) = 1/s$, resulting in a type 1 system, provided k is chosen so that the closed-loop system is stable.

(b) Taking $K(s) = 1/s$, gives the characteristic equation as

$$1 + \frac{k}{s(s+1)(s+2)} = 0 \Longrightarrow s^3 + 3s^2 + 2s + k = 0.$$

The Routh array is then

$$\begin{array}{c|cc} s^3 & 1 & 2 \\ s^2 & 3 & k \\ s & 2 - k/3 & \\ 1 & k & \end{array}$$

For stability, we require no sign changes in the first column. Thus the closed-loop will be stable for $0 < k < 6$.

(c) From Part (b), the closed-loop system is marginally stable when $k = 6$. The characteristic equation becomes,

$$s^3 + 3s^2 + 2s + 6 = 0.$$

It is then easy to confirm that -3 is a root of this equation. To find the frequency of the oscillations, we find the other roots by dividing the characteristic polynomial by $s + 3$:

$$s^3 + 3s^2 + 2s + 6 = (s+3)(s^2+2).$$

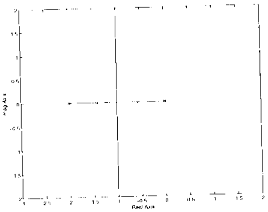
Thus the other poles are at $\pm j\sqrt{2}$ indicating that the frequency of the oscillations is $w = \sqrt{2}$ rad/second.

(d) When $r(s) = 1/s^2$ (unit ramp), we have

$$e_{ss} = \lim_{s\to 0} \frac{1}{s + \dfrac{k}{(s+1)(s+2)}} = 2/k.$$

Therefore, the minimum value of k so that $e_{ss} \le 1$ is $k = 2$.

3. (a) The plot is shown below.

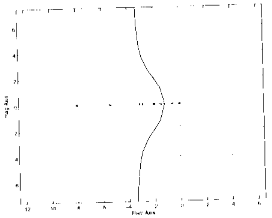


(b) Since $\zeta = 0.707$ and the settling time $T_s = 4/\zeta\omega_n = 2s$, it follows that $\zeta\omega_n = \sqrt{1-\zeta^2}\omega_n = 2$ so the required closed-loop dominant pole locations are $-2 + j2, -2 - j2$. Let the compensator transfer function be $K(s) = k(s-z)/(s-p)$. To ensure the response due to the third pole decays at least as fast as e^{-3t}, we place the compensator zero at $z = -3$. The location of the compensator pole p can be obtained as follows. Let the angle between the required pole $(-2+j2)$ and p be θ. Applying the angle criterion:

$$63.4^\circ - (90^\circ + 135^\circ + \theta) = \pm 180^\circ$$

or $\theta = 18.3^\circ$. Thus $p = -8$. Finally, k can be found from the magnitude criterion

$$k = |s(s+2)(s+8)/(s+3)|_{s=-2+j2} = 16.$$



(c) Let $e(t) = r(t) - y(t)$. Then $e(s) = r(s)/[1 + K(s)G(s)]$ and since the closed-loop system is stable, the final value theorem gives

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} se(s) = 0$$

since $r(s) = 1/s$ and $\lim_{s\to 0} G(s)K(s) = \infty$.

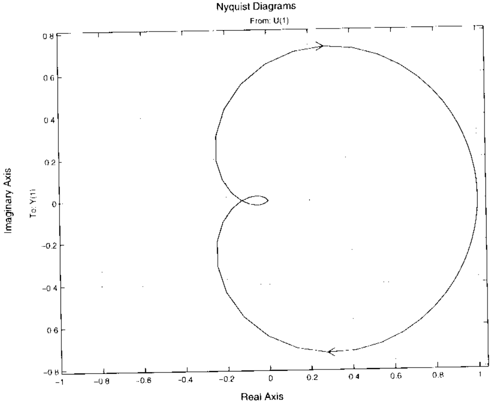
4. (a) The Nyquist plot, together with the unit circle centred on the origin is shown below. The real-axis intercepts can be found by setting the imaginary part of $G(j\omega)$ to zero. This gives intercepts at $\omega_i = 0, \pm\sqrt{3}, \infty$ and so $G(j\omega_i) = 1, -0.125, -0.125, 0$.

(b) The number of unstable closed-loop poles associated with gain K can be determined by the number of encirclements by $G(s)$ of the point $-\frac{1}{K}$. Thus

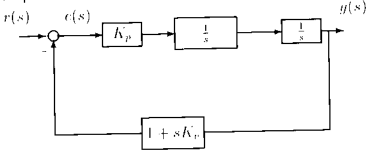
$$\begin{aligned} 0 < k < 8 &\implies \text{no unstable poles} \\ k > 8 &\implies \text{2 unstable poles} \\ -1 < k < 0 &\implies \text{no unstable poles} \\ k < -1 &\implies \text{1 unstable pole} \end{aligned}$$

(c) Since the intercept with the negative real axis is at -0.125, the gain margin is 8.

(d) The phase-lag compensator (with $k = 1$) has gain close to unity for frequencies below ω_p and gain close to $\frac{\omega_p}{\omega_0} < 1$ for frequencies beyond ω_0. The phase is negative and large between these two frequencies but insignificant below and above. It follows that we can use phase-lag compensation to increase low frequency gain by setting $k > 1$ (hence improving tracking properties) without introducing phase lag at high frequency (which would reduce the phase margin) by placing w_p and w_0 in the 'middle' frequency range.



5. (a) The feedback loop can be redrawn as follows:



$$e(s) = r(s) - (1 + K_v s)(K_p/s^2)e(s) \Rightarrow e(s) = \frac{s^2}{s^2 + K_p K_v s + K_p}$$

$$y(s) = \frac{K_p/s^2}{1 + K_p(1 + sK_v s)/s^2} r(s) = \frac{K_p}{s^2 + K_p K_v s + K_p} r(s)$$

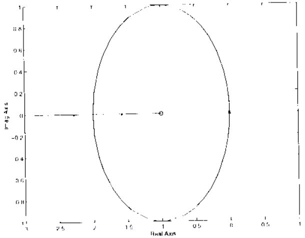
(b) The characteristic equation can be written as

$$1 + K_p K_v \frac{s + 1/K_v}{s^2} = 0.$$

The root locus of $\hat{G}(s) = \frac{s+1/K_v}{s^2}$ must therefore have a break point at -2 to satisfy the design specifications. Thus $\frac{\mathrm{d}}{\mathrm{d}s}\hat{G}(s)|_{s=-2} = 0$ so $K_v = 1$. To place the closed loop poles at -2, we use the gain criterion:

$$1 + K_p K_v \frac{s + 1/K_v}{s^2}|_{s=-2} = 0$$

This gives $K_p = 4$. The root locus of the compensated system is shown below.



(c) Since the closed-loop system is stable, we use the final value theorem:

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} se(s) = \frac{1}{K_p} = 0.25$$

from parts (a) and (b).