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EE2-10B

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2012

EEE PART II: MEng, BEng and ACGI

Corrected Copy

FIELDS

Monday, 18 June 2:00 pm

Time allowed: 1:30 hours

There are TWO questions on this paper.

Answer BOTH questions. Question One carries 20 marks. Question Two carries 30 marks.

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : R.R.A. Syms

Second Marker(s) : S. Lucyszyn

2E Electromagnetic Fields 2012 – Formula sheet

- Vector calculus (Cartesian co-ordinates)

$\nabla = \underline{i}\,\partial/\partial x + \underline{j}\,\partial/\partial y + \underline{k}\,\partial/\partial z$

$\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$

$\mathrm{grad}(\phi) = \nabla\phi = \underline{i}\,\partial\phi/\partial x + \underline{j}\,\partial\phi/\partial y + \underline{k}\,\partial\phi/\partial z$

$\mathrm{div}(\underline{\mathbf{F}}) = \nabla\,.\,\underline{\mathbf{F}} = \partial \mathbf{F}_x/\partial x + \partial \mathbf{F}_y/\partial y + \partial \mathbf{F}_z/\partial z$

$\mathrm{curl}(\underline{\mathbf{F}}) = \nabla \times \underline{\mathbf{F}} = \underline{i}\,\{\partial \mathbf{F}_z/\partial y - \partial \mathbf{F}_y/\partial z\} + \underline{j}\,\{\partial \mathbf{F}_x/\partial z - \partial \mathbf{F}_z/\partial x\} + \underline{k}\,\{\partial \mathbf{F}_y/\partial x - \partial \mathbf{F}_x/\partial y\}$

Where ϕ is a scalar field and $\underline{\mathbf{F}}$ is a vector field

- Maxwell's equations – integral form

$\iint_A \underline{\mathbf{D}}\,.\,d\underline{a} = \iiint_V \rho\,dv$

$\iint_A \underline{\mathbf{B}}\,.\,d\underline{a} = 0$

$\int_L \underline{\mathbf{E}}\,.\,d\underline{L} = -\iint_A \partial\underline{\mathbf{B}}/\partial t\,.\,d\underline{a}$

$\int_L \underline{\mathbf{H}}\,.\,d\underline{L} = \iint_A [\underline{\mathbf{J}} + \partial\underline{\mathbf{D}}/\partial t]\,.\,d\underline{a}$

Where $\underline{\mathbf{D}}$, $\underline{\mathbf{B}}$, $\underline{\mathbf{E}}$, $\underline{\mathbf{H}}$, $\underline{\mathbf{J}}$ are time-varying vector fields

- Maxwell's equations – differential form

$\mathrm{div}(\underline{\mathbf{D}}) = \rho$

$\mathrm{div}(\underline{\mathbf{B}}) = 0$

$\mathrm{curl}(\underline{\mathbf{E}}) = -\partial\underline{\mathbf{B}}/\partial t$

$\mathrm{curl}(\underline{\mathbf{H}}) = \underline{\mathbf{J}} + \partial\underline{\mathbf{D}}/\partial t$

- Material equations

$\underline{\mathbf{J}} = \sigma\,\underline{\mathbf{E}}$

$\underline{\mathbf{D}} = \varepsilon\,\underline{\mathbf{E}}$

$\underline{\mathbf{B}} = \mu\,\underline{\mathbf{H}}$

- Theorems

$\iint_A \underline{\mathbf{F}}\,.\,d\underline{a} = \iiint_V \mathrm{div}(\underline{\mathbf{F}})\,dv$ – Gauss' theorem

$\int_L \underline{\mathbf{F}}\,.\,d\underline{L} = \iint_A \mathrm{curl}(\underline{\mathbf{F}})\,.\,d\underline{a}$ – Stokes' theorem

$\mathrm{curl}\,\{\mathrm{curl}(\underline{\mathbf{F}})\} = \mathrm{grad}\,\{\mathrm{div}(\underline{\mathbf{F}})\} - \nabla^2\underline{\mathbf{F}}$

2E Electromagnetic Fields 2011 – Formula sheet (continued)

- Electromagnetic waves (pure dielectric media)

Time dependent vector wave equation $\nabla^2\underline{\mathbf{E}} = \mu_0\varepsilon\, \partial^2\underline{\mathbf{E}}/\partial t^2$

Time independent scalar wave equation $\nabla^2\underline{E} = -\omega^2\mu_0\varepsilon_0\varepsilon_r\, \underline{E}$

For z-going, x-polarized waves $d^2E_x/dz^2 + \omega^2\mu_0\varepsilon_0\varepsilon_r\, E_x = 0$

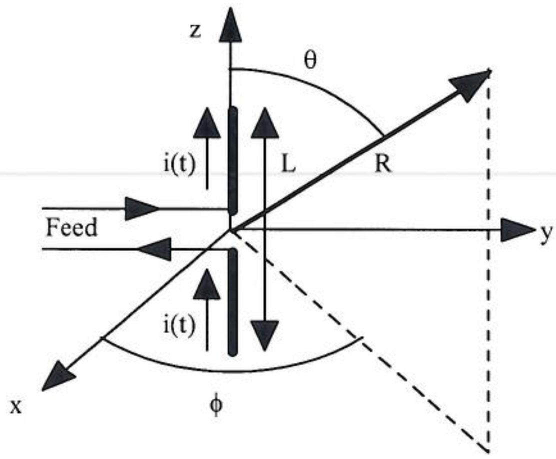
Where $\underline{E}$ is a time-independent vector field

- Antenna formulae

Far-field pattern of half-wave dipole

$E_\theta = j\, 60I_0\, \{\cos[(\pi/2)\cos(\theta)] / \sin(\theta)\}\, \exp(-jkR)/R$; $H_\phi = E_\theta/Z_0$

Here I_0 is peak current, R is range and $k = 2\pi/\lambda$



Power density $\underline{S} = 1/2\ \mathrm{Re}\,(\underline{E} \times \underline{H}^*) = S(R, \theta)$

Normalised radiation pattern $F(\theta, \phi) = S(R, \theta, \phi) / S_{max}$

Directivity $D = 1/\ \{1/4\pi \iint_{4\pi} F(\theta, \phi) \sin(\theta)\, d\theta\, d\phi\}$

Gain $G = \eta D$ where η is antenna efficiency

Effective area $A_e = \lambda^2 D/4\pi$

Friis transmission formula $P_r = P_t\,(\eta_t\eta_r A_t A_r / R^2\lambda^2)$

Fields 2012 – Questions

1. a) Discuss the implications for electromagnetic waves of **no more than two** of the following, illustrating your answer with diagrams, graphs or formulae where appropriate.

   i) Matching of electromagnetic fields at boundaries

   ii) The displacement current

   iii) Transmission via the ionosphere

[2 x 5]

b) Sketch the lumped-element circuit model of a transmission line. Referring to a co-axial cable, briefly explain the physical origin of any components that you have included.

[5]

c) Assuming that the sections are sufficiently short, derive a pair of first-order coupled differential equations relating voltages to currents. Uncouple the equations to obtain a new pair of second-order differential equations in terms of voltages and currents alone.

[5]

2. a) Figure 1a shows a dielectric slab of thickness d and refractive index n. Write down its optical transfer function τ. How is τ modified for N slabs in series, if the i^{th} slab has thickness d_i and refractive index n_I as shown in Figure 1b? Figure 1c shows a lens formed from glass of refractive index n polished into spherical surfaces of radii r_1 and r_2. At a distance R from the axis, the glass thickness is $d_g = d - (R^2/2)(1/r_1 + 1/r_2)$. What is the transfer function?

[8]

b) The electric field of a diverging spherical electromagnetic wave can be written in the scalar approximation as $E(r) = (E_0/r) \exp(-jk_0r)$, where E_0 is a constant and $k_0 = 2\pi/\lambda$. Derive a paraxial approximation for the field. How would the approximation modify if the wave were instead converging to a point? If the lens is used for imaging, show that the object position u and image position v are related by $1/u + 1/v = 1/f$, and find the focal length f.

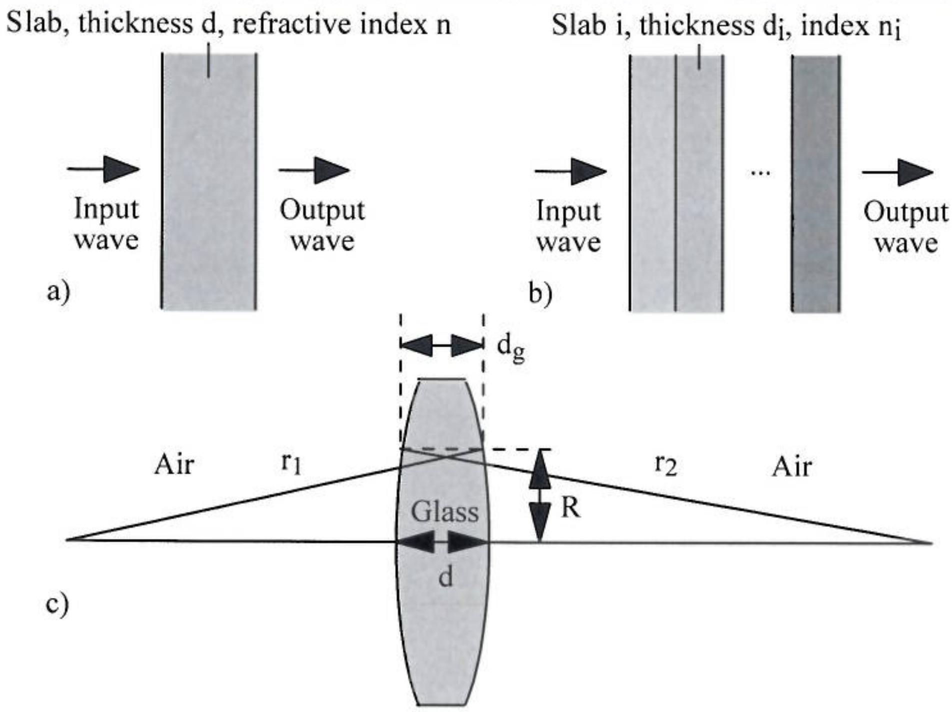
[10]

c) A plano-concave lens formed in glass of refractive index 1.5 has a focal length of -1m. Sketch the lens shape, indicating the image position for an object at infinity.

[6]

d) Explain why it is difficult to construct either a lens or a mirror for use in an imaging system at X-ray frequencies. Describe how imaging may still be carried out.

[6]



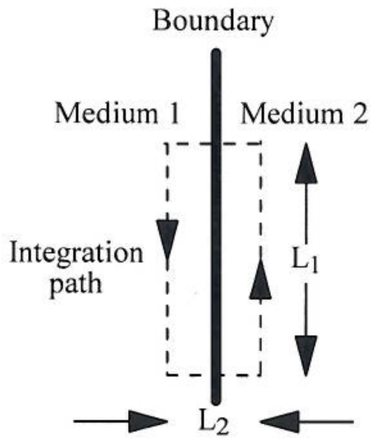
Slab, thickness d, refractive index n
Input wave
Output wave
a)
Slab i, thickness d_i, index n_i
...
Input wave
Output wave
b)
d_g
Air
r_1
r_2
Air
Glass
R
d
c)

Figure 1.

Fields 2012 – Solutions

1a)

i) Solutions for finite media can be constructed by matching solutions for infinite space at boundaries, using boundary conditions derived directly from Maxwell's equations. The figure below shows a boundary between two media. Medium 1 has time-dependent electric field $\underline{\mathbf{E}}_1$, and Medium 2 has a corresponding field $\underline{\mathbf{E}}_2$.



[2]

Faraday's law states that:

$\int_L \underline{\mathbf{E}} \cdot d\underline{L} = - \iint_A \partial\underline{\mathbf{B}}/\partial t \cdot d\underline{a}$

The integration path for the line integral is the rectangle shown. Two of the sides are parallel to the boundary and the rectangle is long and thin so that $L_2 << L_1$. If L_2 tends to zero, the area must tend to zero, so the surface integral must be zero. If L_1 is small, the fields can be considered uniform. Integration then gives:

$\int_L \underline{\mathbf{E}} \cdot d\underline{L} = (\mathbf{E}_{t1} - \mathbf{E}_{t2}) L_1$

Here $\mathbf{E}_{t1}$ and $\mathbf{E}_{t2}$ are the components of the two fields parallel to the long line elements. These components are tangential to the boundary. Applying Faraday's Law, we then get:

$\mathbf{E}_{t1} - \mathbf{E}_{t2} = 0$

Consequently the boundary condition for the electric field is that tangential components of $\underline{\mathbf{E}}$ must match.

Similar calculations can be done for all field quantities. The full set of conditions is:

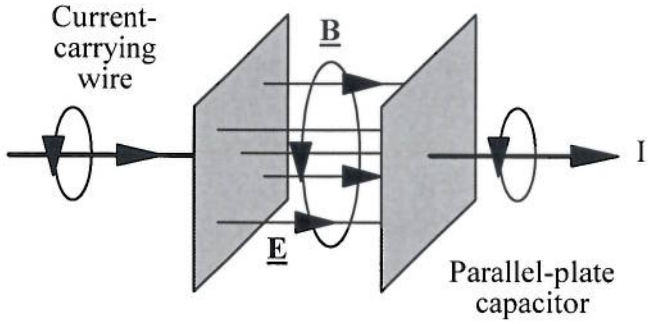
- Tangential components of $\underline{\mathbf{E}}$ and $\underline{\mathbf{H}}$ must match on a boundary

- Normal components of **D** and **B** must match on a boundary (apart from surface effects)

[3]

COMMON ERRORS: FAILURE TO INCLUDES EXAMPLE DIAGRAM OR PROOF; FAILURE TO GENERALISE RESULT BEYOND E TO H, D AND B.

ii) Maxwell noticed that the existing theory could not really describe the continuity of current flow through a simple circuit containing wires connecting to a capacitor.



[2]

By Gauss' law, the electric field between plates is Flux p.u.a = charge p.u.a. so $\varepsilon E = Q/A$

Maxwell noticed that $\varepsilon\, dE/dt = dD/dt = I/A$ is really a current density. He called this new term $\underline{\mathbf{J}}_D$, the displacement current, and added it to Ampere's law, which then became:

$\int_L \underline{\mathbf{H}} \cdot d\underline{L} = \iint_A [\underline{\mathbf{J}} + \partial\underline{\mathbf{D}}/\partial t] \cdot d\underline{a}$

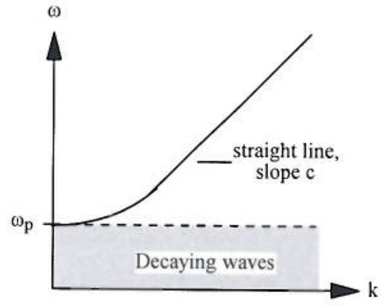
Here $\underline{\mathbf{J}}$ describes the flow of normal current (moving electrons) while $\underline{\mathbf{J}}_D = \partial\underline{\mathbf{D}}/\partial t$ describes the effect of time-varying fields.

$\underline{\mathbf{J}}_D$ will become larger as the rate of change of field (i.e. the frequency) rises. As a result, $\underline{\mathbf{J}}$ dominates at low frequency, while $\partial\underline{\mathbf{D}}/\partial t$ dominates at high frequency. It is then possible to have a flow of energy without any moving electrons at all. At a stroke, Maxwell unified the theory, and made the theoretical analysis of high frequency electromagnetic fields possible.

[3]

COMMON ERRORS: FAILURE TO INCLUDE DIAGRAM; LACK OF DISCUSSION OF EFFECT OF FREQUENCY.

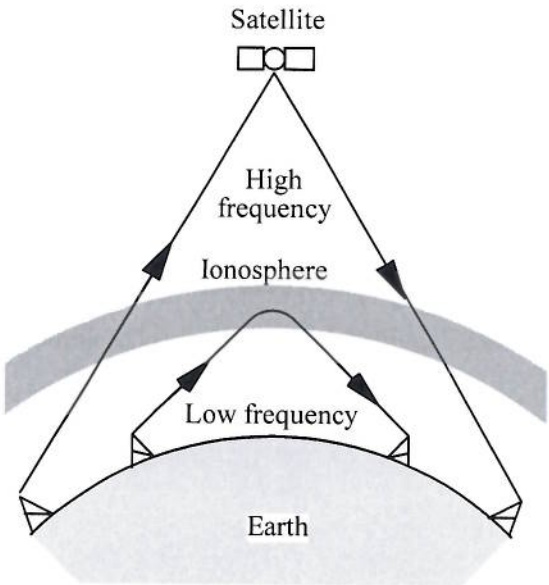
iii) The ionosphere is a region of the upper atmosphere containing layers of charged particles arranged as approximately spherical shells. It has the dispersion relation $\omega^2 = \omega_p{}^2 + c^2k^2$, giving the dispersion diagram shown in the LH figure below.



[2]

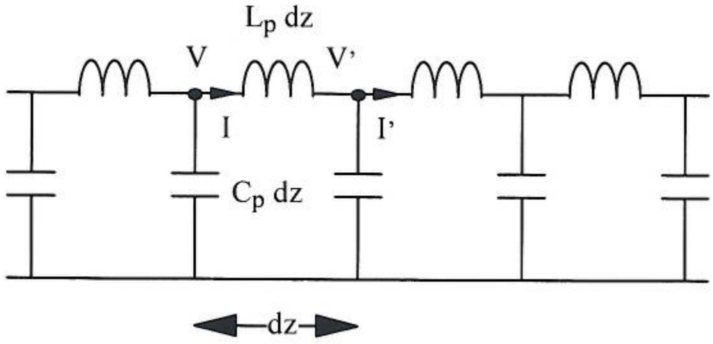
At angular frequency ω, the propagation constant is $k = (1/c)\ \sqrt{(\omega^2 - \omega_p{}^2)}$. There is no real solution for k for $\omega < \omega_p$, so the ionosphere reflects low frequencies. This allows over-the horizon radio communication by multiple reflection between the ionosphere and the oceans (which also contain ions) as shown in the RH figure, the method used by Marconi. However, transmission is unreliable because the weather disrupts the ionosphere. Modern communications use higher frequencies, to which the atmosphere is transparent. The signal is transmitted to a geostationary satellite, regenerated and retransmitted.

[3]



COMMON ERRORS: OMISSION OF DISPERSION EQUATION FOR WAVES IN THE IONOSPHERE; INCORRECT DRAWING OR OMISSION OF THE DISPERSION DIAGRAM.

xb) The lumped-element model of a transmission line is as shown below:

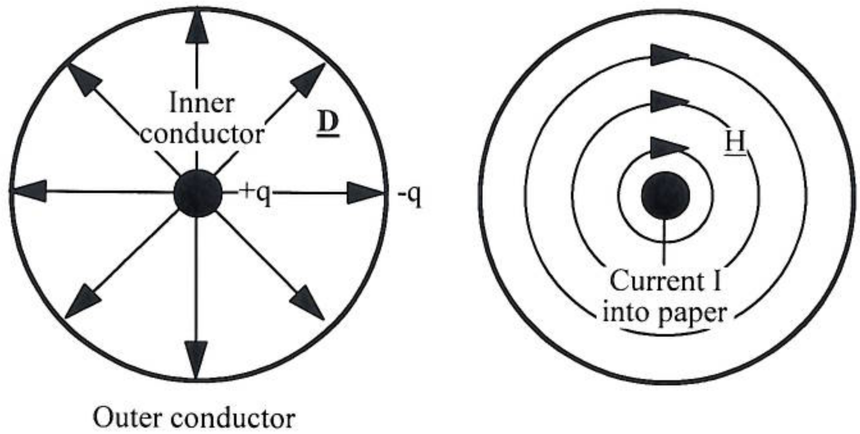


Here L_p is the per-unit length inductance and C_p is the per-unit length capacitance of the line, and the sections are dz in length.

[2]

GENERALLY ANSWERED CORRECTLY.

For a co-axial cable, the capacitance arises from the equal and opposite charges stored on the inner and outer conductors. The inductance arises from the magnetic flux stored between the inner and outer conductors.



[3]

COMMON ERRORS: FAILURE TO INCLUDE DIAGRAMS OF THE COAXIAL CABLE OR DISCUSS THE PHYSICAL ORIGINS OF L AND C.

c) Assigning nodal voltages V and V' and currents I and I' as shown we obtain at angular frequency ω:

$V' = V - j\omega L_p I\, dz$ $\quad$ $I' = I - j\omega C_p V\, dz$

If we now write:

$V' = V + (dV/dz)\, dz$ $\quad$ $I' = I + (dI/dz)\, dz$

Then by comparison:

$dV/dz = -j\omega L_p I$ $\quad$ $dI/dz = -j\omega C_p V$

[3]

GENERALLY ANSWERED CORRECTLY.

To uncouple the equations, differentiate to get:

$d^2V/dz^2 = -\, j\omega L_p dI/dz = -\omega^2 L_p C_p V$

$d^2I/dz^2 = -\omega^2 L_p C_p I$

[2]

COMMON ERRORS: ONLY DERIVING 1 UNCOUPLED EQUATION.

2a) In a medium of refractive index n, the propagation constant is $k_0 n$, where $k_0 = 2\pi/\lambda$. The time-independent scalar electric field of a travelling wave is therefore $E(z) = E_0 \exp(-jk_0 nz)$.

For a slab of thickness d, as in a) below, the transfer function is therefore $\tau_S = \exp(-jk_0 nd)$

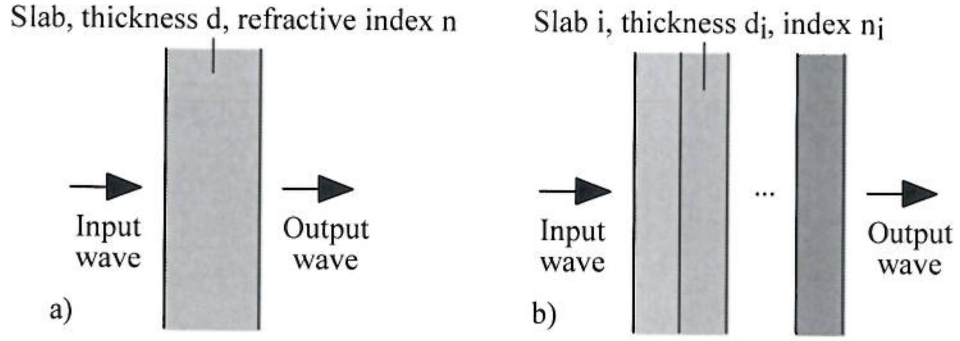
[2]

GENERALLY ANSWERED CORRECTLY.

For cascaded slabs as in b) below, the transfer function is therefore:

$\tau_S' = \exp(-jk_0 n_1 d_1) \exp(-jk_0 n_2 d_2) \ \ldots \exp(-jk_0 n_N d_N)$, or

$\tau_S' = \exp(-jk_0 \delta)$ where $\delta = {}_{i=1}\Sigma^N n_i d_i$ is the optical thickness



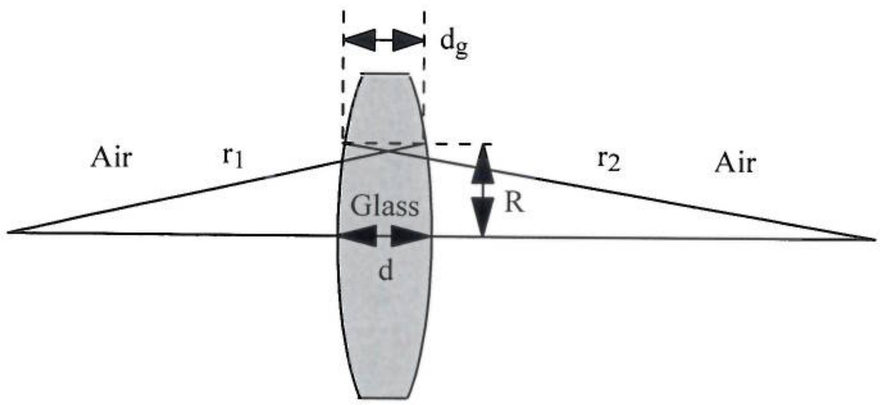
[2]

For the lens shown below, the glass thickness at a distance R is $d_g = d - (R^2/2)(1/r_1 + 1/r_2)$. The optical thickness is therefore:

$\delta = n \times d_g + 1 \times (d - d_g)$ or $\delta = nd - (n - 1)(R^2/2)(1/r_1 + 1/r_2)$

The transfer function is then $\tau = \exp(-jk_0\delta)$, or $\tau = \tau_S \tau_L$ where τ_S is as above and:

$\tau_L = \exp\{+jk_0(n - 1)(R^2/2)(1/r_1 + 1/r_2)\}$



[4]

COMMON ERRORS: FAILURE TO UNDERSTAND THE CONCEPT OF OPTICAL THICKNESS.

b) The time-independent scalar expression for a spherical wave is $E(r) = E_0/r \exp(-jk_0r)$

At distance z, $r^2 = z^2 + R^2$ where $R^2 = x^2 + y^2$

Consequently $r = z (1 + R^2/z^2)^{1/2}$

For small R, $r \approx z (1 + R^2/2z^2) = z + R^2/2z$

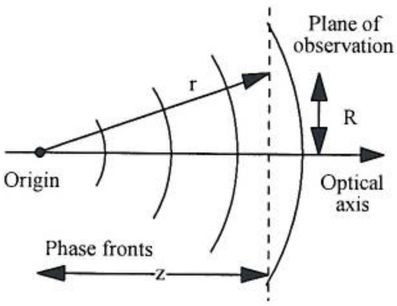
We can then approximate the phase variation as $\exp(-jk_0r) \approx \exp(-jk_0z) \exp(-jk_0R^2/2z)$

For the amplitude, which is slowly varying, we can instead put $E_0/r \approx E_0/z$

The full expression for a paraxial spherical wave is then:

$E \approx A(z) \exp(-jk_0R^2/2z)$ where $A(z) = E_0/z \exp(-jk_0z)$

[4]



COMMON ERRORS: FAILURE TO USE THE BINOMIAL APPROXIMATION.

If the wave is converging to a point z away, the paraxial wave expression modifies to:

$E \approx A(z) \exp(+jk_0R^2/2z)$

[2]

When the lens is used for imaging with object and image distances u and v as below, the input field in the plane of the lens is $E_{in} \approx A(z) \exp(-jk_0R^2/2u)$

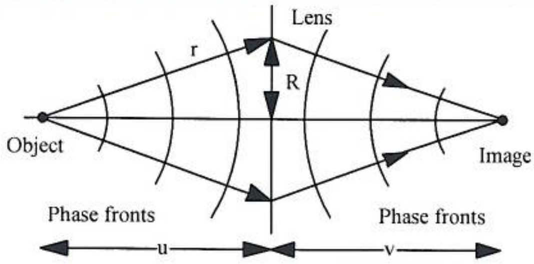
Similarly, the output field is $E_{out} \approx A(z) \exp(+jk_0R^2/2v)$

Ignoring the unimportant slab term, we should have $E_{out} = E_{in}\, \tau_L$

Hence, $\exp(+jk_0R^2/2v) = \exp(-jk_0R^2/2u) \exp\{+jk_0(n - 1) (R^2/2) (1/r_1 + 1/r_2)\}$

Consequently $1/v = -1/u + (n - 1)(1/r_1 + 1/r_2)$ and hence:

$1/u + 1/v = 1/f$, where $1/f = (n - 1)(1/r_1 + 1/r_2)$ defines the focal length of the lens



[4]

COMMON ERRORS: FAILURE TO EQUATE THE OUTPUT FIELD FROM THE LENS TO THAT OF A CONVERGING SPERICAL WAVE.

c) The focal length is $1/f = (n - 1)(1/r_1 + 1/r_2)$

For a plano-concave lens we make take r_1 to be infinite.

COMMON ERRORS: FAILURE TO UNDERSTAND THAT r IS INFINITE FOR A FLAT SURFACE.

If $n = 1.5$, and f is -1m, then $-1 = 0.5/r_2$ so that $r_2 = -0.5$ m.

[2]

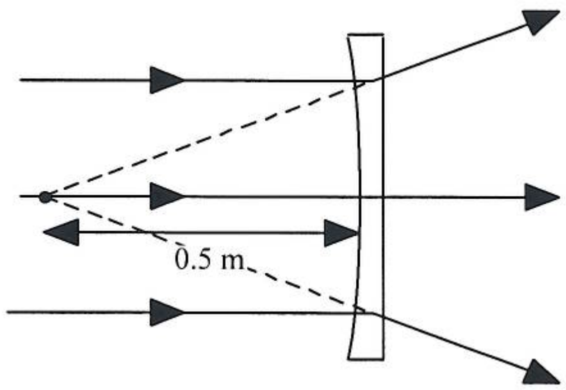
COMMON ERRORS: FAILURE TO REARRANGE THE EQUATION 1/u + 1/v = 1/f TO YIELD v CORRECTLY.

The imaging equation is $1/u + 1/v = 1/f$

If u is infinite, then $1/v = 1/f$, so that $v = -0.5$ m.

[2]

The lens shape and image position are then as shown below.



[2]

COMMON ERRORS: FAILURE TO UNDERSTAND THE CONCEPT OF A VIRTUAL IMAGE; FAILURE TO DRAW THE CORRECT RAY DIAGRAM WHEN v IS NEGATIVE; FAILURE TO DRAWTHE CORRECT LENS SHAPE WHEN r_2 IS NEGATIVE.

d) It is difficult to construct a lens or mirror for use at X-ray frequencies because the refractive index tends to unity and the reflectivity to zero. Imaging may still be carried out as shadowgraph (a conventional X-ray photograph) or by computer aided tomography.

[2]

GENERALLY ANSWERED CORRECTLY.

A CAT scanner is an imaging system that operates without the need for a lens. It consists of an X-ray source and a detector array, placed on either side of the patient. The source and detector array can be rotated around the patient, and detected signals are measured at many different angular positions. Each signal is determined by the integrated absorption between source and detector. Simultaneous equations are then solved to find the local absorber values and hence the overall distribution of absorption (the X-ray image).

[4]

COMMON ERRORS: FAILURE TO UNDERTAND THE NEED FOR A COMPUTATION TO FIND THE IMAGE IN A CAT SCANNER.