EE4-05
EE9CS7-22
EE9SO7

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2012

MSc and EEE PART IV: MEng and ACGI

TRAFFIC THEORY & QUEUEING SYSTEMS

Monday, 21 May 10:00 am

Time allowed: 3:00 hours

There are FOUR questions on this paper.

Answer ALL questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s) :	J.A. Barria
	Second Marker(s) :	D.P. Mandic

Special instructions for students

1. Erlang Loss formula recursive evaluation:

$$E_N(\rho) = \frac{\rho E_{N-1}(\rho)}{N + \rho E_{N-1}(\rho)}$$
$$E_0(\rho) = 1$$

2. Engset Loss formula recursive evaluation (for a fixed M and $p = \alpha/1+\alpha$):

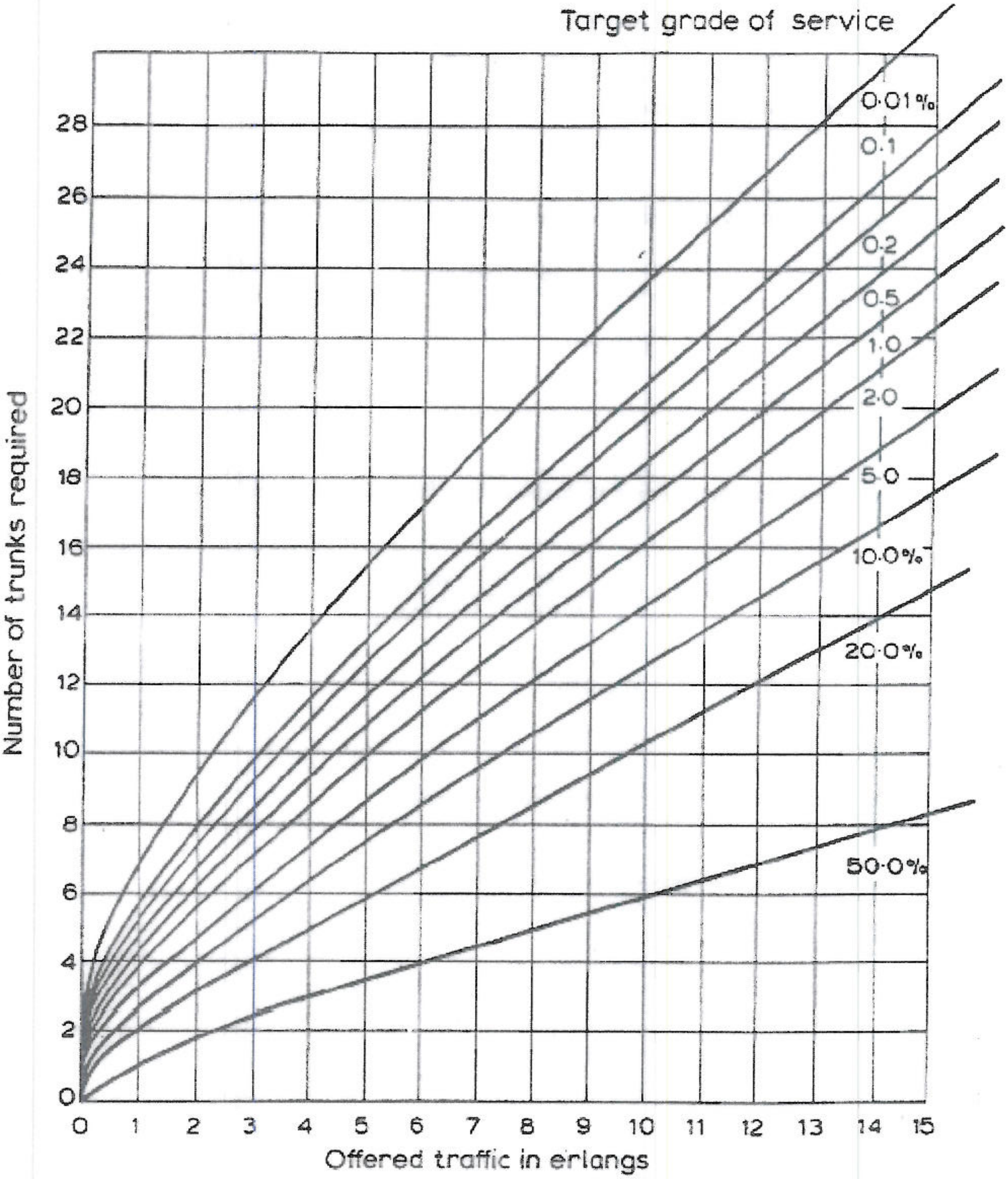
$$e_N = \frac{(M-N+1)\alpha e_{N-1}}{N+(M-N+1)\alpha e_{N-1}}$$
$$e_0 = 1$$
$$\alpha = \lambda/\mu$$

3. Traffic capacity on basis of Erlang B formula (next page).

Note: for large ρ, N is approximately linear: $N \approx 1.33\rho + 5$

4. Expected residual time

$$E[R] = \frac{1}{2}\lambda E[S^2]$$



Traffic capacity on basis of Erlang B formula.

The Questions

1.

a) Consider the simple re-attempt model represented in Figure 1.1.

Let N_t denote the number of busy channels at time t.

i) Describe and discuss the characteristics of the re-attempt model. Discuss relevant assumptions to make the model simpler and tractable within the well known Markov chain framework. [5]

ii) For the simplified model defined in i) find the probability distribution that a demand is submitted j times in total. [4]

iii) Derive the total offered traffic to the system including re-attempts. [3]

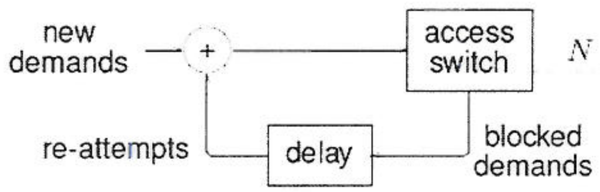


Figure 1.1

b) Explain what is meant by, (i) the global balance equations, and (ii) the local balance equations, for a continuous-time stationary Markov chain.

Explain the relation between the two sets of equations.

Hint: you can use as an example the Erlang or Engset models. [4]

c) Define the terms "call congestion" and "time congestion". Derive an expression for these two measures of congestion. Explain why these two measures have the same value in the Erlang traffic model but different values in the Engset model. [4]

2.

a) Figure 2.1 depicts an M/M/1 closed queuing network model approximation of the token pool buffer representation (shown in Figure 2.2) of the Leaky bucket algorithm.

i) With the help of Fig. 2.2, explain the operations of the Leaky bucket algorithm using the variables identified in Fig. 2.1. (M, D, λ, λ^*).

[6]

ii) Derive the average throughput for the model represented in Figure 2.1. Clearly show all your calculations.

[6]

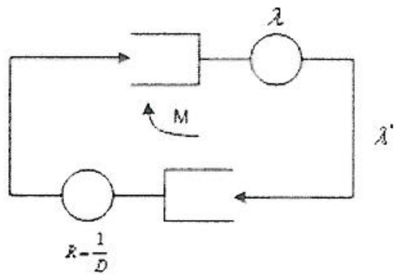


Figure. 2.1.

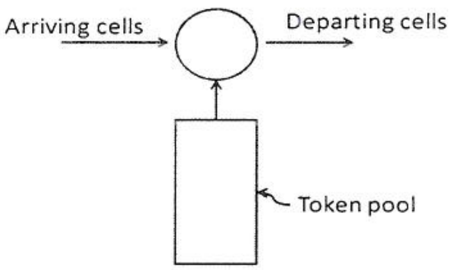


Figure. 2.2.

b) For the system represented by the Markov chain in Fig 2.3. derive the probability, (π_0), that the system is empty.

[8]

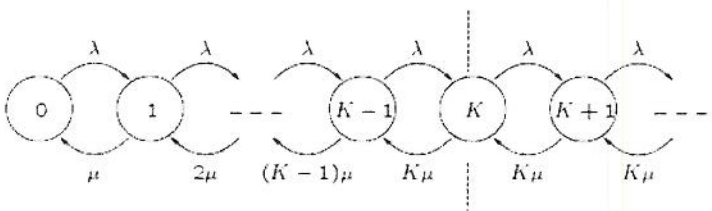


Figure. 2.3.

3.

a) A call centre accepts incoming requests on a first in first out basis and all requests join a waiting calls queue.

The incoming traffic to this call centre has been measured and is 8 Erlangs.

Each enquiry has an average holding time of 75 seconds.

At all times there are 10 positions available to answer incoming requests.

i) Derive the call blocking probability for the system described.

[6]

ii) Determine the probability that an incoming request will be blocked if the waiting call queue size is five (5) calls.

[2]

iii) Determine the probability that an incoming request will be blocked if the waiting queue size is ten (10) calls.

[2]

b) Show that for an N-channel link fed by M sources ($M > N$) the mean carried traffic can be expressed by:

$$\rho_c = \left[\frac{(1 - B_c)\alpha M}{1 + (1 - B_c)\alpha} \right]$$

where, B_c is the call congestion, and α is the offered traffic per free source.

Hint: the total offered traffic is the product of the offered traffic per free source and the mean number of sources.

[10]

4.

a) A random mixture of two Poisson streams is offered to a buffered single server link with capacity 32 Mbps.

The requests of the aggregate Poisson stream is an aggregate of two streams with the following characteristics:

- Stream 1 requests arrivals at a rate of 3000 messages/s and all messages consist of 10 packets;

-Stream 2 requests arrivals at a rate of 12000 messages/minute and all messages consists of 40 packets.

All packet sizes are 50 bytes.

If the link is now operating at half of its capacity:

i) Determine the overall mean message waiting time when the buffer protocol is FIFO, and [7]

ii) Determine the overall mean message waiting time when all Stream 1 messages are given non-pre-emptive priority. [7]

b) Consider a multiprocessor system consisting of ***n*** processors. At least one (1) processor is needed for the system to be up.

Assume:

- Each processor fails at a rate γ,

- Each processor is repaired at a rate τ,

- The coverage probability is c,

- The average reconfiguration delay after a covered failure is $1/\delta$,

- The average re-boot delay after a non-recoverable failure is $1/\beta$.

i) Define the state space of the system, and derive all transition rates. [3]

ii) Derive an expression for the system un-availability. [3]

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EE9CS7-22
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Q1 a) The effect of re-attempts is important under heavy-traffic conditions but difficult to analyse. as it result in a non-Poisson arrival stream.

2

Simple re-attempt model:

Assumptions

- A blocked demand is re-submitted with probability p
- Re-submission occur after a long interval

Then, the re-attempts can be treated as additional new demands to the system, and the feedback effect simply increases the offered load

3

If B_c = call congestion (w the re-attempts)

then P(demand is blocked and re-submitted) $= B_c p$

so P(demand is submitted j times in total) $=$

$$(B_c p)^{j-1}(1-B_c p)$$

4

This is a geometric distribution

From the geometric distribution the mean number of attempts $= \dfrac{1}{1-B_c p} = \bar{N}$

Then if offered traffic without reattempts $= \rho_0$

Total traffic including re-attempts $=$

$$\bar{N}\rho_0 = \frac{\rho_0}{1-B_c p}$$

3

Q1 b) If $\{X_t\}$ is an irreducible Markov chain the equilibrium distribution is unique and can be shown that this distribution (π_i; $i = 1, 2, \dots$) satisfy (in scalar form) the following Equation

$$\sum_{i \in E} \pi_i q_{ij} = 0 \quad \text{for each } i \in E$$

where q_{ij} are the coefficients of the transition matrix Q. If we separate out the self-transition term q_{jj}

$$\sum_{i \neq j} \pi_i q_{ij} = -\pi_j q_{jj} \quad \left(\text{and by definition } -q_{jj} = \sum_{i \neq j} \pi_j q_{ji}\right)$$

$$\sum_{i \neq j} \pi_i q_{ij} = \sum_{i \neq j} \pi_j q_{ji}$$ ← Global balance Equations

In certain types of CTMC a stronger set of conditions might apply: there must be flux balance between each pair of states. In such case we must have:

$$\pi_i q_{ij} = \pi_j q_{ji} \quad \text{for each } (i, j \neq i)$$ ← Local balance Equations

Example (Erlang or Engset)

Engset:

$\lambda_0 = M\lambda$; $\lambda_i = (1 - i/M)\lambda_0$; $\mu_i = i\mu$

$$i\mu\pi_i = \left(M - (i-1))\lambda_0\right)\pi_{i-1}$$ ← Local BE

$$\left(M - (i-1)\lambda_0\right)\pi_{i-1} + (i+1)\mu\pi_{i+1} = (M-i)\lambda_0\pi_i + i\mu\pi_i$$ ← Global BE

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Second Examiner:

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Q1 c

Call congestion: fraction of arriving calls / demands that will meet saturation (and hence be blocked)

Time congestion: Portion of time for which the link is saturated

Erlang Model

When the arrival stream is a Poisson stream the arrival rate (λi) is independent of the link state. Hence

P (call meets saturation) = P (system is saturated)

That is $B_C = B_T = \Pi_N = E_N(\rho)$

$\rho = E(N_t) = \frac{\lambda}{\mu}$

N = Number of links.

Engset Model: Any given idle source will see the link occupancy pattern generated by the remaining $(M-1)$ sources and the state distribution seen by the idle source will be $\hat{\pi}_i(M, p) = \pi_i(M-1, p)$

So an arrival demand will find the link in a state i with probability $\hat{\pi}_i(M, p)$ hence

$B_C = \Pi_N(M-1, p)$ and $B_C < B_T$ in this case

M = Number of sources

p = probability that one source is busy.

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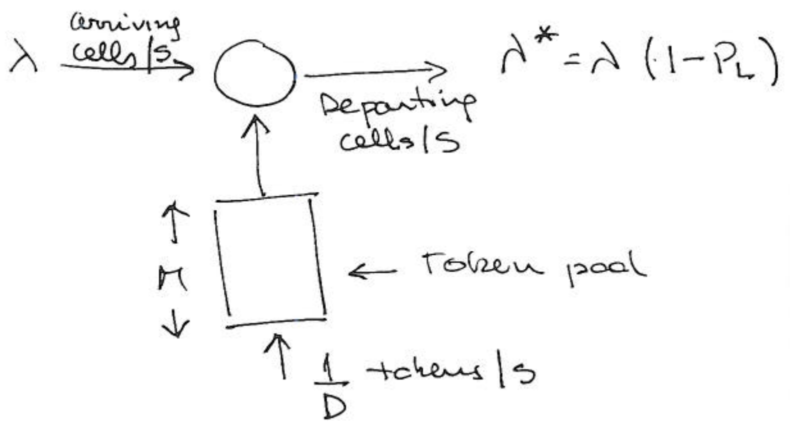
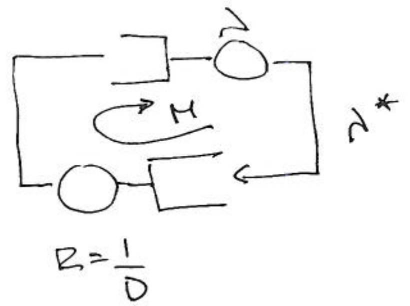
Q2 (b)) i)



Tokens are generated one per D seconds.
λ^* = average throughput. This is different from offered load (λ) because of possible cell loss.
Departing cells will be enabled only if there are tokens in token pool.

ii)



- Cells generated at Poisson rate λ (only if upper queue has token). This queue increases at a rate $R = D^{-1}$ [s^{-1}]
- M tokens circulating in it (i.e. at most M cells served in succession)

$\lambda^* = \lambda(1-P_L) = \lambda(1-P_0)$

P_L = Probability upper queue empty = Probability lower queue full (using M/M/1/M)

$P_L = \dfrac{\rho^M(1-\rho)}{1-\rho^{M+1}}$, $\rho = \dfrac{\lambda}{R} = \lambda D$, $\lambda^* = \lambda\left[\dfrac{1-\rho^M}{1-\rho^{M+1}}\right]$

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Paper Code: | Second Examiner:

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Q2 Equilibrium distribution

b) The local balance equation holds and the balance equations are

$$\pi_i = \left(\frac{\lambda}{i\mu}\right)\pi_{i-1} = \left(\frac{A}{i}\right)\pi_{i-1} \qquad i \le K$$

$$\pi_i = \left(\frac{\lambda}{K\mu}\right)\pi_{i-1} = \rho\,\pi_{i-1} \qquad i \ge K$$

$$\pi_i = \begin{cases} \left(\frac{A^i}{i!}\right)\pi_0 & \text{if } i \le K \\ \left(\frac{A^K}{K!}\right)\rho^{i-K}\pi_0 & \text{if } i \ge K \end{cases}$$

$$\pi_K = \left(\frac{A^K}{K!}\right)\pi_0$$

for $i \ge K$ we can write $i = K + j$ $(j \ge 0)$

$$\pi_{K+j} = \left(\frac{A^K}{K!}\right)\rho^j\pi_0 = \rho^j\pi_K$$

Normalisation gives $\pi_0 = \frac{1}{S}$

$$S = \sum_{i=0}^{K}\left(\frac{A^i}{i!}\right) + \left(\frac{A^K}{K!}\right)\frac{\rho}{1-\rho}$$

$$\pi_0 = \frac{1}{(A^K/K!)}\left[\frac{(1-\rho)E_K(A)}{(1-\rho)+\rho E_K(A)}\right]$$

where $E_K(A)$ is the Erlang loss for A erlangs

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Q3 a) M|M|K|N system

system size $= N = K + Q$

$K = 10$ (Nr of positions)

$Q =$ queue size

Local balance equations

$$\pi_i = \left(\frac{A^i}{i!}\right)\pi_0 \qquad 0 \le i \le K$$

$$= \left(\frac{A^K}{K!}\right)\rho^{i-K}\pi_0 \qquad K \le i \le K+Q$$

$$S = \frac{A^K}{K!}\left[E_K^{-1}(A) + \frac{\rho(1-\rho^Q)}{1-\rho}\right], \quad \rho \ne 1$$

$$\pi_0 = \frac{1}{(A^K/K!)}\left[\frac{(1-\rho)E_K(A)}{(1-\rho)+\rho(1-\rho^Q)E_K(A)}\right], \quad \rho \ne 1$$

$$P(\text{Loss}) = P(\text{Buffer full}) = P(N_t = K+Q)$$

$$= \pi_K\rho^Q = \left(\frac{A^K}{K!}\right)\rho^Q = \left[\frac{(1-\rho)\rho^Q E_K(K\rho)}{(1-\rho)+\rho(1-\rho^Q)E_K(K\rho)}\right]$$

$\rho = 0.8$

$E_K(K\rho) = 0.122 \quad (E_{10}(8))$

$P(\text{Loss}) = 0.030 \qquad Q = 5$

$= 0.009 \qquad Q = 10$

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Paper Code: | Second Examiner:

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Q3 i) Engset model ($N < M$) : In this case there is congestion therefore some of this offered loads is not carried.

offered traffic ρ_o = [offered traffic / free source] × E (Nr of free sources)

$$\rho_o = \alpha \times E(M - N_t)$$

$$= \alpha \times [M - E(N_t)]$$

$$= \alpha M - \alpha [(1 - B_c)\rho_o]$$

$$\rho_o = \frac{\alpha M}{1 + (1 - B_c)\alpha}$$

and the total carried traffic is

$$\rho_c = (1 - B_c)\rho_o$$

and

B_c = call congestion

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Paper Code: | Second Examiner:

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Q4 a)

$$\left.\begin{aligned}\lambda_1 &= 3000 \text{ 1/s}\\ S_1 &= \frac{10\times 400}{16\times 10^6} = 0.25\times 10^{-3}\text{s}\end{aligned}\right\} \rho_1 = \lambda_1 S_1 = 0.75$$

$$\left.\begin{aligned}\lambda_2 &= 200 \text{ 1/s}\\ S_2 &= \frac{40\times 400}{16\times 10^6} = 10^{-3}\text{s}\end{aligned}\right\} \rho_2 = \lambda_2 S_2 = 0.20$$

% type 1 messages $= \dfrac{\lambda_1}{\lambda_1+\lambda_2} = \dfrac{30}{32}$

% type 2 messages $= \dfrac{\lambda_2}{\lambda_1+\lambda_2} = \dfrac{2}{32}$

FIFO

$S_1^2 = 0.0625\times 10^{-6}$ $\quad S_2^2 = 1\times 10^{-6}$

$$E(S^2) = \left[\frac{30}{32}\times 0.625 + \frac{2}{32}\times 1\right]10^{-6} = 0.121\times 10^{-6}$$

$$E(R) = \frac{1}{2}\lambda_1 S_1^2 + \frac{1}{2}\lambda_2 S_2^2 = 0.19375 \text{ ms}$$

$$E(W) = \frac{E(R)}{1-\rho_1-\rho_2} = 3.875 \text{ ms}$$

ii) Non-pre-emptive priority for stream 1

$$E(W_1) = \left[\frac{E(R)}{1-\rho_1}\right] = \frac{0.19375}{0.25} = 0.775 \text{ ms}$$

$$E(W_2) = \left[\frac{E(W_1)}{1-\rho_1-\rho_2}\right] = \frac{0.775}{0.05} = 15.5 \text{ ms}$$

$$E(W) = \frac{\lambda_1}{\lambda_1+\lambda_2}E(W_1) + \frac{\lambda_2}{\lambda_1+\lambda_2}E(W_2) = 1.695 \text{ ms}$$

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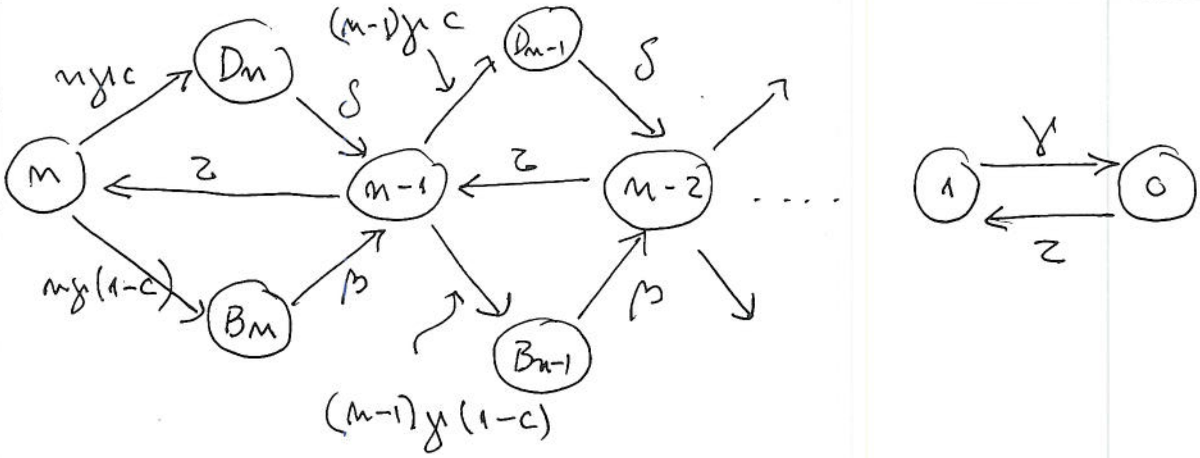
Q4 b)



D_n = HW failure; Reconfigures to state $n-1$

B_n = SW failure; Reboots to state $n-1$

$$\text{System unavailability} = 1 - \sum_{j=1}^{n} \pi_j$$

$$= \pi_0 + \sum_{j=2}^{n} \pi_{D_j} + \sum_{j=2}^{n} \pi_{B_j}$$