Imperial College London

MEng (Engineering) Examination 2016

Year 1

AE1-107 Mathematics Term 1

Monday 18th January 2016: 10.00 to 12.00
[2 hour]

There are ***FOUR*** questions.
Full marks may be obtained for
complete answers to ***ALL FOUR*** questions.

A data sheet is attached

The use of lecture notes is NOT allowed.

Question 1

(a) Consider the function $f(x)$ defined as $f(x) = \sin(4x)$.

i. Using the definition of derivative as a limit, find the derivative $f'(x)$ of the function $f(x)$.
You may assume that $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$ and $\lim_{x\to 0} \frac{1-\cos(x)}{x} = 0$. [20%]

ii. Sketch the curve $f(x)$ and identify extrema (maxima and/or minima) and inflexion points if any (it is not necessary to use the definition of derivative as a limit to find higher order derivatives). [15%]

(b) Evaluate the following limits:

i. $\lim_{x\to 0} \frac{\ln(1-x)}{\sin(2x)}$ [15%]

ii. $\lim_{x\to 0} x^3 \sin\left(\frac{1}{x}\right)$ [15%]

iii. $\lim_{x\to +\infty} \left(\frac{x+3}{x}\right)^x$. [15%]

(c) The displacement, x, of a mass in a vibrating system is given by,

$$x(t) = (1+t)e^{-wt}$$

where w is a constant corresponding to the natural frequency of vibration.

i. Compute $\dot{x} = \frac{dx}{dt}$ and $\ddot{x} = \frac{d\dot{x}}{dt}$. [10%]

ii. Show that $\ddot{x} + 2w\dot{x} + w^2 x = 0$. [10%]

Question 2

(a) Determine $\frac{dy}{dx}$ in each of the following cases:

i. $x(t) = \frac{1}{2}t^2 + 2$ and $y(t) = \sin(3t+1)$ [15%]

ii. $x^2 \sin(y) + xy = 2.$ [15%]

(b) Determine the following integrals:

i. $\int \frac{1}{x^2 + x - 6} dx$ [15%]

ii. $\int \frac{1}{1 + \cos(x) + \sin(x)} dx$ (you may use $t = \tan(x/2)$) [15%]

iii. $\int \frac{3x^8 + 5x^6 - 4x^5 + 15x^4 - x}{x^5} dx.$ [15%]

(c) Given that

$$I_n = \int (\sin(x))^n dx, \ n = 0, 1, 2, \ldots,$$

and using integration by parts, show that for $n \geq 2$ [15%]

$$I_n = -\frac{1}{n}\cos(x)(\sin(x))^{n-1} + \frac{n-1}{n} I_{n-2}.$$

Use this formula to evaluate I_4. [10%]

Question 3

(a) Determine if the following series converge:

i. $\sum_{n=1}^{\infty} \frac{4n+5}{100n}$ [12.5%]

ii. $\sum_{n=1}^{\infty} \frac{x^{4n}}{n!}$. [12.5%]

(b) The maximum tension, T, in a suspended cable is given by

$$T = \frac{Wl}{2}\sqrt{1+\left(\frac{l}{4s}\right)^2}$$

where W is the load per unit length, l is the total span and s is the sag. Using an appropriate Maclaurin series show that

$$T = \frac{Wl}{2} + \frac{Wl^3}{64s^2} - \frac{Wl^5}{4096s^4} + \frac{Wl^7}{131072s^6} \cdots$$

[25%]

(c) Find all stationary points of the function

$$f(x,y) = x^2 + xy + x - 2x\sqrt{x^2+y^2}$$

where the square root is assumed to be positive. [25%]

(d) Find and classify the stationary points of the function

$$g(x,y) = x(y^2 - 2y + 1).$$

[25%]

Question 4

(a) Give the definition of:

i. A periodic function, which has period $L > 0$; [5%]

ii. An odd function; and an even function. [5%]

(b) Consider the function

$$f(x) = x(\pi - x)$$

defined on the interval $0 \le x \le \pi$.

i. Sketch the odd extension of f to the real line and find its Fourier series. [30%]

ii. Using Parseval's theorem, or otherwise, find the value of

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^6}.$$

[20%]

(c) i. Show that $z\bar{z} = |z|^2$ for any complex number $z \in \mathbb{C}$. [5%]

ii. Show that

$$\frac{|w|^2 - |z|^2}{|w-z|^2} = \mathrm{Re}\left(\frac{w+z}{w-z}\right),$$

for any $w, z \in \mathbb{C}$. [20%]

iii. Determine the locus of points in the complex plane which satisfy

$$(\mathrm{Im} z)^2 + 1 \le z(1-i) + \bar{z}(1+i).$$

[15%]

Question 1

(a) i. From the defintion of derivatives as a limit, we have

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
f'(x) &= \lim_{h\to 0} \frac{1}{h}\left[\sin(4x+4h)-\sin(4x)\right] \\
f'(x) &= \lim_{h\to 0} \frac{1}{h}\left[\sin(4x)\cos(4h)+\cos(4x)\sin(4h)-\sin(4x)\right] \\
f'(x) &= \lim_{h\to 0} \frac{1}{h}\left[\sin(4x)[\cos(4h)-1]+\cos(4x)\sin(4h)\right] \\
f'(x) &= \sin(4x)\lim_{h\to 0}\frac{\cos(4h)-1}{h}+\cos(4x)\lim_{h\to 0}\frac{\sin(4h)}{h} \\
f'(x) &= 4\sin(4x)\lim_{h\to 0}\frac{\cos(4h)-1}{4h}+4\cos(4x)\lim_{h\to 0}\frac{\sin(4h)}{4h}
\end{aligned}
$$

Using the limits given in the exam ($\lim_{x\to 0}\frac{\sin(x)}{x}=1$ and $\lim_{x\to 0}\frac{1-\cos(x)}{x}=0$), we get

$$f'(x) = 4\cos(4x)$$

Note that the following identity was used:

$$\sin(A+B)=\sin(A)\cos(B)+\cos(A)\sin(B)$$

[20%]

ii. Extreme and inflexion points can be obtained from,

$$f'(x)=0=4\cos(4x)$$

or

$$4x=(2n-1)\frac{\pi}{2} \qquad \to \qquad x=\frac{(2n-1)\pi}{8}$$

for any integer n. The second derivative can clarify the nature of the extreme points (maximum or minimum):

$$f''(x)=-16\sin(4x)$$

Therefore the infinite number of extreme values alternates between local maxima and minima. When $x=\frac{(4k-3)\pi}{8}$, we obtain maxima ($f''(x)=-16<0$), while for $x=\frac{(4k-1)\pi}{8}$ we obtain minima ($f''(x)=16>0$). See figure 1 for the sketch of the curve.
Finally, when $x=\pm n\frac{\pi}{4}$, we have $f''(x)=0$, corresponding to inflection points.

[15%]

(b) i. The limit is of the form $\frac{0}{0}$ therefore it is possible to apply directly l'Hôpital's rule

$$\lim_{x\to 0}\frac{\ln(1-x)}{\sin(2x)}=\lim_{x\to 0}\frac{\frac{-1}{1-x}}{2\cos(2x)}=\lim_{x\to 0}\frac{-1}{2(1-x)\cos(2x)}=-\frac{1}{2}$$

[15%]

2

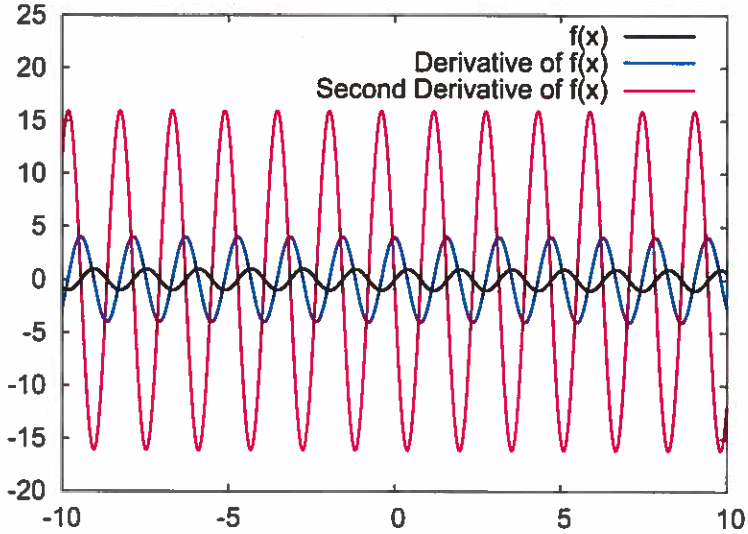

Figure 1: Plot of the function $\sin(4x)$ and its derivatives.

ii. The first thing to notice is that

$$-1 \le \sin(x) \le 1$$

As we are dealing with a limit we can avoid $x = 0$ and then we can write

$$-1 \le \sin\left(\frac{1}{x}\right) \le 1$$

Multiplying the previous relation by x^3 gives

$$-x^3 \le x^3 \sin\left(\frac{1}{x}\right) \le x^3$$

We have $\lim_{x\to 0} x^3 = 0$ and $\lim_{x\to 0} -x^3 = 0$. Therefore by using the Squeeze theorem we have

$$\lim_{x\to 0} x^3 \sin\left(\frac{1}{x}\right) = 0$$

[15%]

iii. Let us define y as $\left(\frac{x+3}{x}\right)^x$. Then we can write

$$\ln(y) = x \ln\left(\frac{x+3}{x}\right) = x\ln\left(1+\frac{3}{x}\right)$$

Using the following relation (from datasheet)

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^n \frac{x^{n+1}}{(n+1)} + \ldots (-1 < x \le 1)$$

we can write for large x

$$\ln(y) = x\ln\left(\frac{3}{x} - \frac{9}{2x^2} + \frac{27}{3x^3} + \ldots\right) = 3 - \frac{9}{2x} + \frac{27}{3x^2} + \ldots$$

So $\ln(y) = 3$. Hence $\lim_{x\to +\infty} y = e^3$ [15%]

(c) i. Derivatives are given as

$$\dot{x} = (1)\mathrm{e}^{-wt} + (1+t)(-w)\mathrm{e}^{-wt} = \mathrm{e}^{-wt}(1-w-wt)$$

and

$$\ddot{x} = -w\mathrm{e}^{-wt}(1-w-wt) + (-w)\mathrm{e}^{-wt} = -w\mathrm{e}^{-wt}(2-w-wt)$$

[10%]

ii. Substituting the values for $\ddot{x}, \dot{x}$ and x gives:

$$\mathrm{e}^{-wt}(-2w + w^2 + w^2 t) + 2w\mathrm{e}^{-wt}(1-w-wt) + w^2(1+t)\mathrm{e}^{-wt} = 0$$

$$\mathrm{e}^{-wt}\left(-2w + w^2 + w^2 t + 2w - 2w^2 - 2w^2 t + w^2 + w^2 t\right) = 0$$

$$0 = 0$$

[10%]

4

Question 2

(a) i.

$$\frac{dx}{dt} = t \qquad \frac{dy}{dt} = 3\cos(3t+1)$$
$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3\cos(3t+1)}{t}$$

[15%]

ii. Using the product rule we obtain

$$2x\sin(y) + y + x^2\frac{dy}{dx}\cos(y) + x\frac{dy}{dx} = 0,$$

Rearranging

$$\frac{dy}{dx} = -\frac{2x\sin(y)+y}{x^2\cos(y)+x}.$$

[15%]

(b) i. We have

$$\frac{1}{x^2-x+6} = \frac{1}{(x+3)(x-2)} = \frac{-1/5}{x+3} + \frac{1/5}{x-2}$$

so that

$$\begin{aligned}\int \frac{1}{x^2-x+6}dx &= -\frac{1}{5}\int\frac{1}{x+3} + \frac{1}{5}\int\frac{1}{x-2}\\ &= -\frac{1}{5}\ln|x+3| + \frac{1}{5}\ln|x-2| + C\\ &= \frac{1}{5}\ln\left|\frac{x-2}{x+3}\right| + C\end{aligned}$$

[15%]

ii. Using $t = \tan(x/2))$ and the datasheet, we have

$$\sin(x) = \frac{2t}{1+t^2}, \qquad \cos(x) = \frac{1-t^2}{1+t^2} \qquad \text{and} \qquad \frac{dx}{dt} = \frac{2}{1+t^2}$$

Therefore we obtain

$$\begin{aligned}\int\frac{1}{1+\cos(x)+\sin(x)}dx &= \int\frac{1}{1+\frac{1-t^2}{1+t^2}+\frac{2t}{1+t^2}}\frac{2}{1+t^2}dt\\ &= \int\frac{2}{1+t^2+2t+1-t^2}dt = \int\frac{2}{2+2t}dt\\ &= \int\frac{1}{1+t}dt = \ln|1+t| + C = \ln|1+\tan(x/2)| + C\end{aligned}$$

[15%]

iii.

$$\begin{aligned}\int\frac{3x^8+5x^6-4x^5+15x^4-x}{x^5}dx &= 3\int\frac{x^8}{x^5}dx + 5\int\frac{x^6}{x^5}dx + 15\int\frac{x^4}{x^5} - 4\int\frac{x^5}{x^5}dx - \int\frac{x}{x^5}dx\\ &= 3\int x^3dx + 5\int xdx + 15\int\frac{1}{x} - 4\int dx - \int\frac{1}{x^4}dx\\ &= \frac{3}{4}x^4 + \frac{5}{2}x^2 + 15\ln|x| - 4x + \frac{1}{3x^3} + C\end{aligned}$$

[15%]

(c) For $n \geq 2$ we write $(\sin(x))^n = \sin(x)(\sin(x))^{n-1}$ and we use integration by parts

$$\int udv = [uv] - \int vdu$$

with

$$u = (\sin(x))^{n-1}; \qquad v = -\cos(x)$$

and

$$du = (n-1)(\sin(x))^{n-2}(\cos(x))dx; \qquad dv = \sin(x)dx$$

Therefore

$$\begin{aligned} I_n = \int(\sin(x))^n dx &= -\cos(x)(\sin(x))^{n-1} + (n-1)\int(\sin(x))^{n-2}(\cos(x))^2 dx \\ &= -\cos(x)(\sin(x))^{n-1} + (n-1)\int(1-(sin(x))^2)(\sin(x))^{n-2}dx \\ &= -\cos(x)(\sin(x))^{n-1} + (n-1)\int(\sin(x))^{n-2} - (\sin(x))^n dx \\ &= -\cos(x)(\sin(x))^{n-1} + (n-1)I_{n-2} - (n-1)I_n \end{aligned}$$

Rearraging the terms leads to

$$nI_n = -\frac{1}{n}\cos(x)(\sin(x))^{n-1} + \frac{n-1}{n}I_{n-2}.$$

In order to compute I_4 we need I_0 and I_2 [15%]

$$I_0 = \int(\sin(x))^0 dx = \int 1dx = x + C$$
$$I_2 = -\frac{1}{2}\cos(x)\sin(x) + \frac{1}{2}I_0 = -\frac{1}{2}\cos(x)\sin(x) + \frac{x}{2} + D$$
$$I_4 = -\frac{1}{4}\cos(x)(\sin(x))^3 + \frac{3}{4}I_2 = -\frac{1}{4}\cos(x)(\sin(x))^3 + \frac{3}{8}(-\cos(x)\sin(x) + x) + E$$

[10%]

Question 3

(a) i. Since

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{4n+5}{100n} = \frac{4}{100} \neq 0$$

the series diverge. Note that the ratio test is not conclusive and that the integral test can also be used for this series. [12.5%]

ii. By the ratio test,

$$\lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} = \lim_{n\to\infty} \frac{n!}{(n+1)!} \frac{|x|^{4(n+1)}}{|x|^{4n}} = \lim_{n\to\infty} \frac{1}{n+1}|x|^4 = 0 < 1$$

for any x. The radius of convergence is ∞. [12.5%]

(b) In the specific case of $(1+x^2)^{\frac{1}{2}}$ and using the data sheet, we have

$$\begin{aligned}(1+x^2)^{\frac{1}{2}} &= 1 + \frac{1}{2}x^2 + \frac{\frac{1}{2}(-\frac{1}{2})}{2!}(x^2)^2 + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{3!}(x^2)^3 + \dots \\ &= 1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \frac{1}{16}x^6 + \dots\end{aligned}$$

With $x = \frac{l}{4s}$ we obtain:

$$\begin{aligned}\left[1+\left(\frac{l}{4s}\right)^2\right]^{\frac{1}{2}} &= 1 + \frac{1}{2}\left(\frac{l}{4s}\right)^2 - \frac{1}{8}\left(\frac{l}{4s}\right)^4 + \frac{1}{16}\left(\frac{l}{4s}\right)^6 \dots \\ &= 1 + \frac{l^2}{(2\times 4^2)\,s^2} - \frac{l^4}{(8\times 4^4)\,s^4} + \frac{l^6}{(16\times 4^6)\,s^6} \dots \\ &= 1 + \frac{l^2}{32s^2} - \frac{l^4}{2048s^4} + \frac{l^6}{65536s^6} \dots\end{aligned}$$

Introducing this last result in the original expression for T gives

$$\begin{aligned}T &= \frac{Wl}{2}\left[1 + \frac{l^2}{32s^2} - \frac{l^4}{2048s^4} + \frac{l^6}{65536s^6} \dots\right] \\ &= \frac{Wl}{2} + \frac{Wl^3}{64s^2} - \frac{Wl^5}{4096s^4} + \frac{Wl^7}{131072s^6} \dots\end{aligned}$$

[25%]

3. (c) (i) A stationary point (x, y) of f is one which satisfies

$$f_x(x, y) = 0 = f_y(x, y)$$

This implies that

$$f_x = 2x + y + 1 - 2\sqrt{x^2+y^2} - \frac{2x^2}{\sqrt{x^2+y^2}} = 0, \quad \text{and} \quad f_y = x - \frac{2xy}{\sqrt{x^2+y^2}} = 0.$$

[10%]

The equation $f_y = 0$ implies that either $x = 0$ or $2y = \sqrt{x^2+y^2}$.
In the former case, the equation $f_x = 0$ reduces to

$$f_x(0, y) = y + 1 - 2\sqrt{y^2} = 0 \Rightarrow 1 = \pm 2y - y \Rightarrow y = -\frac{1}{3}, 1.$$

In the latter, note that $y \geq 0$ and $4y^2 = x^2 + y^2 \Rightarrow x = \pm\sqrt{3}y$. Hence, the equation for $f_x = 0$ reduces to

$$\pm 2\sqrt{3}y + y - 4y - \frac{6y^2}{2y} + 1 = 0 \Rightarrow y = \frac{1}{6 \mp 2\sqrt{3}}.$$

Hence, the stationary points are

$$(x, y) = \left\{ \left(0, -\frac{1}{3}\right), (0, 1), \left(\frac{\sqrt{3}}{6-2\sqrt{3}}, \frac{1}{6-2\sqrt{3}}\right), \left(-\frac{\sqrt{3}}{6+2\sqrt{3}}, \frac{1}{6+2\sqrt{3}}\right) \right\}.$$

[15%]

(ii) To find the stationary points, first calculate $g_x = 0 = g_y$,

$$g_x = (y-1)^2 = 0 \quad \text{and} \quad g_y = 2(y-1)x = 0.$$

Hence, $(x, 1)$ is a stationary point for any value of x. [10%]
To classify the stationary points consider a perturbation (s,t) to a given stationary point $(x, 1)$, i.e. $g(x+s, 1+t) = (x+s)t^2$.

Case $x < 0$**:** For $s \in (-|x|, |x|)$ and $t \in \mathbb{R}$ we have that

$$g(x+s, 1+t) = (x+s)t^2 \leq 0$$

Since $g(x, 1) = 0$, it follows that $(x, 1)$ is a maximum if $x < 0$. [5%]

Case $x > 0$**:** Similarly, for $s \in (-x, x)$ and $t \in \mathbb{R}$,

$$g(x+s, 1+t) = (x+s)t^2 \geq 0.$$

Hence, $(x, 1)$ is a minimum if $x > 0$. [5%]

Case $x = 0$**:** In this case $g(t, 1+t) = t^3$ which is strictly negative for $t < 0$ and strictly positive for $t > 0$. Hence, $(0, 1)$ is neither a minimum nor a maximum. Therefore $(0, 1)$ is a saddle point. [5%]

[*Note that* $\Delta(x, 1) = 0$ *meaning that the discriminant test is inconclusive.* [5%] *will be given for this comment if full marks are not otherwise obtained for part (c)(ii).*]

(a) (i) A function f is periodic with period L is $f(x+L) = f(x)$ for any $x \in \mathbb{R}$. [5%]

(ii) A function f is: odd if $f(x) = -f(-x)$ for any $x \in \mathbb{R}$; and even if $f(x) = f(-x)$ for any $x \in \mathbb{R}$. [5%]

(b) The odd extension is

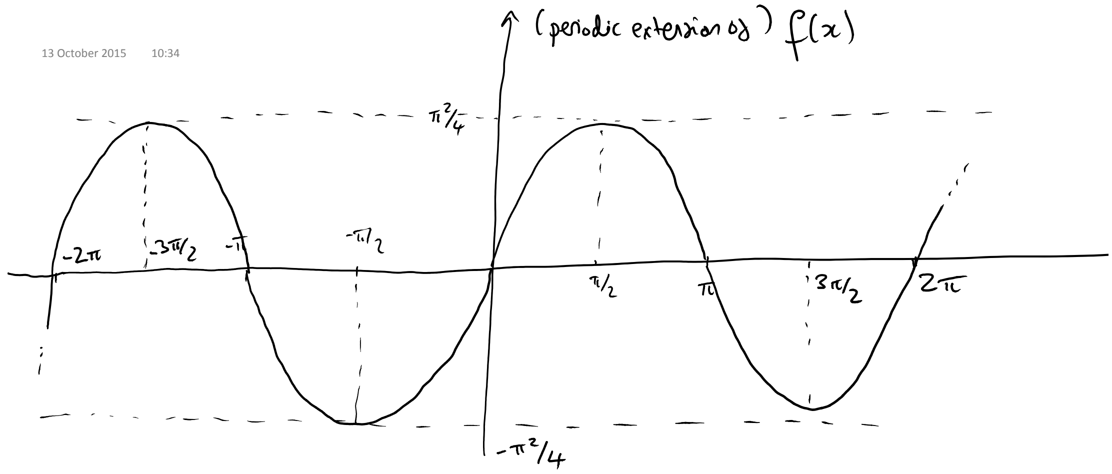

Figure 1: Periodic Extension of $f(x)$. [10%]

To find the Fourier series, note that since the chosen extension is odd, $a_n = 0, n \geq 0$. [5%]

The remaining coefficients are given by

$$\begin{aligned}
b_n = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\sin(nx)dx &= \frac{2}{\pi}\left(x(\pi-x)\cdot -\frac{1}{n}\cos(nx)\right)_0^{\pi} + \frac{2}{n\pi}\int_0^{\pi}(\pi - 2x)\cos(nx)dx \\
&= \frac{2}{n\pi}\left((\pi - 2x)\cdot\frac{1}{n}\sin(nx)\right)_0^{\pi} + \frac{4}{n^2\pi}\int_0^{\pi}\sin(nx)dx \\
&= \frac{4}{n^3\pi}\left(-\cos(nx)\right)_0^{\pi} \\
&= \frac{4}{n^3\pi}\left(1 - (-1)^n\right) \\
&= \begin{cases} 0, & n \text{ even} \\ \frac{8}{n^3\pi}, & n \text{ odd} \end{cases}
\end{aligned}$$

Hence, the Fourier series of $F(x)$ is

$$F(x) = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\sin((2n-1)x).$$

[15%]

(iii) Parseval's theorem applied to f implies that

$$\frac{2}{\pi}\int_0^{\pi} f(x)^2 dx = \sum_{n=1}^{\infty} b_n^2 = \frac{64}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^6}.$$

[10%]

A simple calculation implies that $\int_0^{\pi} f(x)^2 dx = \frac{\pi^5}{30}$ and, hence,

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^6} = \frac{\pi^6}{960}.$$

[10%]

(c) (i) Let $z = x + iy$. Then $z\bar{z} = (x+iy)(x-iy) = x^2 + y^2 = |z|^2$ [5%]

(ii)

$$\frac{w+z}{w-z} = \frac{(w+z)(\bar{w}-\bar{z})}{|w-z|^2} = \frac{|w|^2 - |z|^2}{|w-z|^2} + \frac{z\bar{w} - w\bar{z}}{|w-z|^2}.$$

Now, $\overline{[z\bar{w} - w\bar{z}]} = \bar{z}w - \bar{w}z = -[z\bar{w} - w\bar{z}]$. Hence, $z\bar{w} - w\bar{z}$ is purely imaginary. Since $|w-z|^2$ is real, it follows that

$$\frac{|w|^2 - |z|^2}{|w-z|^2} = \mathrm{Re}\left(\frac{w+z}{w-z}\right).$$

[20%]

(iii) Let $z = x + iy$. Then the inequality can be written

$$\begin{aligned} y^2 + 1 &\le z + \bar{z} + i(\bar{z} - z) \\ &= 2\mathrm{Re}(z) + 2\mathrm{Im}(z) \\ &= 2x + 2y \end{aligned}$$

Therefore, $(y-1)^2 \le 2x$. Thus, the locus contains all points lying on and within the parabola in the right-half complex plane which touches the imaginary axis at i and passes through the points $\frac{1}{2}$ and $\frac{1}{2} + 2i$. [15%]