

B.ENG. AND M.ENG. EXAMINATIONS 2008

PART II : MATHEMATICS (ELECTRICAL ENGINEERING)

Date Thursday 5th June 2008 2.00 - 4.00 pm

Answer FOUR questions.

Please answer question from Section A and Section B in separate answerbooks.

*A mathematical formulae sheet is provided.*

*Statistical data sheets are provided.*

*[Before starting, please make sure that the paper is complete; there should be 4 pages, with a total of 6 questions. Ask the invigilator for a replacement if your copy is faulty.]*



## SECTION A

1. Consider a real  $n \times n$  symmetric matrix  $A$  with distinct eigenvalues  $\lambda_i$  and corresponding normalised eigenvectors  $e_i$  for  $i = 1, \dots, n$ .

- (i) Show that all the  $\lambda_i$  are real.  
 (ii) Show that the eigenvectors  $e_i$  obey the orthogonality relation

$$e_i^T e_j = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

- (iii) Show that the  $n \times n$  matrix

$$P = \{e_1 \ e_2 \ \dots \ e_n\}$$

satisfies the relation

$$P^T P = I,$$

where  $I$  is the  $n \times n$  unit matrix.

2. Find the eigenvalues and normalised eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & \sqrt{2} & 0 \\ \sqrt{2} & 1 & \sqrt{2} \\ 0 & \sqrt{2} & 1 \end{pmatrix}.$$

Now consider the quadratic form

$$Q = x_1^2 + 2\sqrt{2}x_1x_2 + x_2^2 + 2\sqrt{2}x_2x_3 + x_3^2$$

written as

$$Q = x^T A x,$$

where  $x = (x_1, x_2, x_3)^T$ . Show that  $Q$  can be written as

$$Q = 3y_1^2 + y_2^2 - y_3^2,$$

by finding a matrix  $P$  which satisfies  $x = Py$  where  $y = (y_1, y_2, y_3)^T$ .

Find  $y_1$ ,  $y_2$  and  $y_3$  in terms of  $x_1$ ,  $x_2$  and  $x_3$  from the matrix  $P$ .

PLEASE TURN OVER

## SECTION B

3. (i) Suppose that one observes the following sample :

$$-0.2, 0.3, 0.5, 0.2, -0.1, -0.1, 0.5$$

Assume that these are independent observations from an  $N(\mu, \sigma^2)$  distribution.

- (i) Sketch the empirical cumulative distribution function.
- (ii) Find the sample mean  $\bar{x}$  and the median.
- (iii) Find the sample variance  $s^2$  and the sample standard deviation  $s$ .
- (iv) Find a 90% confidence interval for the mean  $\mu$ .
- (v) Perform a t-test at the 5% level for the hypotheses

$$H_0 : \mu \leq 0 \text{ against } H_1 : \mu > 0.$$

4. Suppose that the random variables  $X$  and  $Y$  have the joint pdf

$$f(x, y) = \begin{cases} kxy^2 & \text{if } 0 \leq x \text{ and } 0 \leq y \text{ and } x + y \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

- (i) Show that  $k = 60$ .
- (ii) Find the marginal pdf of  $X$ .
- (iii) Show that  $E(X) = 1/3$ .
- (iv) Find  $\text{cov}(X, Y)$ . [You may use that  $E(Y) = 1/2$ .]
- (v) Are  $X$  and  $Y$  uncorrelated? Justify your answer.
- (vi) Are  $X$  and  $Y$  independent? Justify your answer.

5. Suppose one wants to estimate the proportion  $\theta$  of students cheating in exams. Instead of asking directly, the following setup is used: a group of  $n$  students is being interviewed. Every student is given a spinner which, with known probability  $p \neq 1/2$ , points to 1 and to 0 otherwise. Only the student observes the outcome of his/her spinner. If the spinner points to 1 and the student has cheated or the spinner points to 0 and the student has not cheated, he/she is asked to report a 1. Otherwise the student is asked to report a 0.

- (i) Show that the probability that a given student reports a 1 is

$$\theta(2p - 1) + 1 - p.$$

- (ii) What is the probability that he/she reports a 0?

- (iii) Show that the maximum likelihood estimator is

$$\hat{\theta} = \frac{1}{2p - 1} \left( \frac{1}{n} \sum_{i=1}^n X_i + p - 1 \right),$$

where  $X_i$  denotes the 0 or 1 the  $i$ th student reports.

- (iv) Is  $\hat{\theta}$  an unbiased estimator for  $\theta$ ?

6. (i) Let  $X$  be *Uniform*(0, 1).

Find the pdf of the random variable  $X^5$ .

- (ii) Let  $X$  be  $N(0, 1)$  and  $Y$  be  $N(5, 3)$ . Assuming that  $X$  and  $Y$  are independent, what is the distribution of  $X + Y$ ?

- (iii) Suppose that  $X$  and  $Y$  are independent, where  $X$  is *Uniform*(0, 1) and  $Y$  is *Exponential*( $\lambda$ ) for some  $\lambda > 0$ .

What is the pdf of the random variable  $X + Y$ ?

END OF PAPER



MATHEMATICAL FORMULAE

1. VECTOR ALGEBRA

$$a = a_1i + a_2j + a_3k = (a_1, a_2, a_3)$$

Scalar (dot) product:  $a \cdot b = a_1b_1 + a_2b_2 + a_3b_3$

Vector (cross) product: 
$$a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Scalar triple product: 
$$[a, b, c] = a \cdot b \times c = b \cdot c \times a = c \cdot a \times b = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product:  $a \times (b \times c) = (c \cdot a)b - (b \cdot a)c$

2. SERIES

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \dots \quad (\alpha \text{ arbitrary, } |x| < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^n \frac{x^{2n}}{(2n)!} + \dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \dots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^n \frac{x^{n+1}}{(n+1)} + \dots \quad (-1 < x \leq 1)$$

3. TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin(a+b) = \sin a \cos b + \cos a \sin b;$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b.$$

$$\cos iz = \cosh z; \quad \cosh iz = \cos z; \quad \sin iz = i \sinh z; \quad \sinh iz = i \sin z.$$

4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f D^n g + \binom{n}{1} Df D^{n-1} g + \dots + \binom{n}{n} D^n f D^0 g + \dots + D^n f g.$$

(b) Taylor's expansion of  $f(x)$  about  $x = a$ :

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \dots + \frac{h^n}{n!}f^{(n)}(a) + \epsilon_n(h),$$

where  $\epsilon_n(h) = h^{n+1}f^{(n+1)}(a + \theta h)/(n+1)!$ ,  $0 < \theta < 1$ .

(c) Taylor's expansion of  $f(x, y)$  about  $(a, b)$ :

$$f(a+h, b+k) = f(a, b) + [hf_x + kf_y]_{a,b} + 1/2! [h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}]_{a,b} + \dots$$

(d) Partial differentiation of  $f(x, y)$ :

i. If  $y = y(x)$ , then  $f = F(x)$ , and  $\frac{dF}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx}$ .

ii. If  $x = x(t)$ ,  $y = y(t)$ , then  $f = F(t)$ , and  $\frac{dF}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$ .

iii. If  $x = x(u, v)$ ,  $y = y(u, v)$ , then  $f = F(u, v)$ , and

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u}, \quad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial v}.$$

(e) Stationary points of  $f(x, y)$  occur where  $f_x = 0$ ,  $f_y = 0$  simultaneously. Let  $(a, b)$  be a stationary point: examine  $D = [f_{xx}f_{yy} - (f_{xy})^2]_{a,b}$ .

If  $D > 0$  and  $f_{xx}(a, b) < 0$ , then  $(a, b)$  is a maximum;

If  $D > 0$  and  $f_{xx}(a, b) > 0$ , then  $(a, b)$  is a minimum;

If  $D < 0$  then  $(a, b)$  is a saddle-point.

(f) Differential equations:

i. The first order linear equation  $dy/dx + P(x)y = Q(x)$  has an integrating factor  $I(x) = \exp[\int P(x)(dx)]$ , so that  $\frac{d}{dx}(Iy) = IQ$ .

ii.  $P(x, y)dx + Q(x, y)dy = 0$  is exact if  $\partial Q/\partial x = \partial P/\partial y$ .

## 5. INTEGRAL CALCULUS

- (a) An important substitution:  $\tan(\theta/2) = t$  :  
 $\sin \theta = 2t/(1+t^2)$ ,  $\cos \theta = (1-t^2)/(1+t^2)$ ,  $d\theta = 2dt/(1+t^2)$ .
- (b) Some indefinite integrals:

$$\int (a^2 - x^2)^{-1/2} dx = \sin^{-1} \left( \frac{x}{a} \right), \quad |x| < a.$$

$$\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1} \left( \frac{x}{a} \right) = \ln \left\{ \frac{x}{a} + \left( 1 + \frac{x^2}{a^2} \right)^{1/2} \right\}.$$

$$\int (x^2 - a^2)^{-1/2} dx = \cosh^{-1} \left( \frac{x}{a} \right) = \ln \left| \frac{x}{a} + \left( \frac{x^2}{a^2} - 1 \right)^{1/2} \right|.$$

$$\int (a^2 + x^2)^{-1} dx = \left( \frac{1}{a} \right) \tan^{-1} \left( \frac{x}{a} \right).$$

## 6. NUMERICAL METHODS

- (a) Approximate solution of an algebraic equation:

If a root of  $f(x) = 0$  occurs near  $x = a$ , take  $x_0 = a$  and  $x_{n+1} = x_n - [f(x_n)/f'(x_n)]$ ,  $n = 0, 1, 2, \dots$

(Newton Raphson method).

- (b) Formulae for numerical integration: Write  $x_n = x_0 + nh$ ,  $y_n = y(x_n)$ .

i. Trapezium rule (1-strip):  $\int_{x_0}^{x_1} y(x) dx \approx (h/2) [y_0 + y_1]$ .

ii. Simpson's rule (2-strip):  $\int_{x_0}^{x_2} y(x) dx \approx (h/3) [y_0 + 4y_1 + y_2]$ .

- (c) Richardson's extrapolation method: Let  $I = \int_a^b f(x) dx$  and let  $I_1, I_2$  be two

estimates of  $I$  obtained by using Simpson's rule with intervals  $h$  and  $h/2$ .

Then, provided  $h$  is small enough,

$$I_2 + (I_2 - I_1)/15,$$

is a better estimate of  $I$ .

## 7. LAPLACE TRANSFORMS

Function	Transform	Function	Transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t) dt$	$a f(t) + b g(t)$	$a F(s) + b G(s)$
$df/dt$	$s F(s) - f(0)$	$d^2 f/dt^2$	$s^2 F(s) - s f(0) - f'(0)$
$e^{at} f(t)$	$F(s-a)$	$t f(t)$	$-dF(s)/ds$
$(\partial/\partial \alpha) f(t, \alpha)$	$(\partial/\partial \alpha) F(s, \alpha)$	$\int_0^t f(t) dt$	$F(s)/s$
$\int_0^t f(u) g(t-u) du$	$F(s) G(s)$		

$$1 \quad 1/s \quad t^n (n = 1, 2, \dots)$$

$$e^{at} \quad 1/(s-a), (s > a) \quad \sin \omega t$$

$$\cos \omega t \quad s/(s^2 + \omega^2), (s > 0) \quad H(t-T) = \begin{cases} 0, & t < T \\ 1, & t > T \end{cases} \quad e^{-st}/s, (s, T > 0)$$

## 8. FOURIER SERIES

If  $f(x)$  is periodic of period  $2L$ , then  $f(x+2L) = f(x)$ , and

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \text{ where}$$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx, \quad n = 0, 1, 2, \dots, \text{ and}$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx, \quad n = 1, 2, 3, \dots$$

Parseval's theorem

$$\frac{1}{L} \int_{-L}^L [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$



## 1. Probabilities for events

For events  $A$ ,  $B$ , and  $C$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

More generally  $P(\cup A_i) = \sum P(A_i) - \sum P(A_i \cap A_j) + \sum P(A_i \cap A_j \cap A_k) - \dots$

The odds in favour of  $A$

$$P(A) / P(\bar{A})$$

Conditional probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad \text{provided that } P(B) > 0$$

Chain rule

$$P(A \cap B \cap C) = P(A)P(B|A)P(C|A \cap B)$$

Bayes' rule

$$P(A|B) = \frac{P(A)P(B|A)}{P(A)P(B|A) + P(\bar{A})P(B|\bar{A})}$$

$A$  and  $B$  are independent if

$$P(B|A) = P(B)$$

$A$ ,  $B$ , and  $C$  are independent if

$$P(A \cap B \cap C) = P(A)P(B)P(C), \text{ and}$$

$$P(A \cap B) = P(A)P(B), \quad P(B \cap C) = P(B)P(C), \quad P(C \cap A) = P(C)P(A)$$

## 2. Probability distribution, expectation and variance

The probability distribution for a discrete random variable  $X$  is called the probability mass function (pmf) and is the complete set of probabilities  $\{p_x\} = \{P(X = x)\}$

Expectation  $E(X) = \mu = \sum_x x p_x$

For function  $g(x)$  of  $x$ ,  $E\{g(X)\} = \sum_x g(x)p_x$ , so  $E(X^2) = \sum_x x^2 p_x$

Sample mean  $\bar{x} = \frac{1}{n} \sum_k x_k$  estimates  $\mu$  from random sample  $x_1, x_2, \dots, x_n$

Variance  $\text{var}(X) = \sigma^2 = E\{(X - \mu)^2\} = E(X^2) - \mu^2$

Sample variance  $s^2 = \frac{1}{n-1} \left\{ \sum_k x_k^2 - \frac{1}{n} \left( \sum_j x_j \right)^2 \right\}$  estimates  $\sigma^2$

Standard deviation  $\text{sd}(X) = \sigma$

If value  $y$  is observed with frequency  $n_y$

$$n = \sum_y n_y, \quad \sum_k x_k = \sum_y y n_y, \quad \sum_k x_k^2 = \sum_y y^2 n_y$$

Skewness  $\beta_1 = E\left(\frac{X - \mu}{\sigma}\right)^3$  is estimated by  $\frac{1}{n-1} \sum \left(\frac{x_i - \bar{x}}{s}\right)^3$

Kurtosis  $\beta_2 = E\left(\frac{X - \mu}{\sigma}\right)^4 - 3$  is estimated by  $\frac{1}{n-1} \sum \left(\frac{x_i - \bar{x}}{s}\right)^4 - 3$

Sample median  $\tilde{x}$  or  $x_{\text{med}}$ . Half the sample values are smaller and half larger

If the sample values  $x_1, \dots, x_n$  are ordered as  $x_{(1)} \leq x_{(2)} \leq \dots \leq x_{(n)}$ ,

then  $\tilde{x} = x_{(\frac{n+1}{2})}$  if  $n$  is odd, and  $\tilde{x} = \frac{1}{2}(x_{(\frac{n}{2})} + x_{(\frac{n+2}{2})})$  if  $n$  is even

$\alpha$ -quantile  $Q(\alpha)$  is such that  $P(X \leq Q(\alpha)) = \alpha$

Sample  $\alpha$ -quantile  $\widehat{Q}(\alpha)$  Proportion  $\alpha$  of the data values are smaller

Lower quartile  $Q1 = \widehat{Q}(0.25)$  one quarter are smaller

Upper quartile  $Q3 = \widehat{Q}(0.75)$  three quarters are smaller

Sample median  $\tilde{x} = \widehat{Q}(0.5)$  estimates the population median  $Q(0.5)$

### 3. Probability distribution for a continuous random variable

The cumulative distribution function (cdf)  $F(x) = P(X \leq x) = \int_{x_0=-\infty}^x f(x_0)dx_0$

The probability density function (pdf)  $f(x) = \frac{dF(x)}{dx}$

$E(X) = \mu = \int_{-\infty}^{\infty} x f(x)dx$ ,  $\text{var}(X) = \sigma^2 = E(X^2) - \mu^2$ , where  $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx$

### 4. Discrete probability distributions

Discrete Uniform *Uniform* ( $n$ )

$$p_x = \frac{1}{n} \quad (x = 1, 2, \dots, n) \quad \mu = (n+1)/2, \quad \sigma^2 = (n^2 - 1)/12$$

Binomial distribution *Binomial* ( $n, \theta$ )

$$p_x = \binom{n}{x} \theta^x (1-\theta)^{n-x} \quad (x = 0, 1, 2, \dots, n) \quad \mu = n\theta, \quad \sigma^2 = n\theta(1-\theta)$$

Poisson distribution *Poisson* ( $\lambda$ )

$$p_x = \frac{\lambda^x e^{-\lambda}}{x!} \quad (x = 0, 1, 2, \dots) \quad (\text{with } \lambda > 0) \quad \mu = \lambda, \quad \sigma^2 = \lambda$$

Geometric distribution *Geometric* ( $\theta$ )

$$p_x = (1-\theta)^{x-1} \theta \quad (x = 1, 2, 3, \dots) \quad \mu = \frac{1}{\theta}, \quad \sigma^2 = \frac{1-\theta}{\theta^2}$$

### 5. Continuous probability distributions

Uniform distribution *Uniform* ( $\alpha, \beta$ )

$$f(x) = \begin{cases} \frac{1}{\beta - \alpha} & (\alpha < x < \beta), \\ 0 & (\text{otherwise}). \end{cases} \quad \mu = (\alpha + \beta)/2, \quad \sigma^2 = (\beta - \alpha)^2/12$$

Exponential distribution *Exponential* ( $\lambda$ )

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & (0 < x < \infty), \\ 0 & (-\infty < x \leq 0). \end{cases} \quad \mu = 1/\lambda, \quad \sigma^2 = 1/\lambda^2$$

Normal distribution  $N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left\{ -\frac{1}{2} \left( \frac{x-\mu}{\sigma} \right)^2 \right\} \quad (-\infty < x < \infty), \quad E(X) = \mu, \quad \text{var}(X) = \sigma^2$$

Standard normal distribution  $N(0,1)$

If  $X$  is  $N(\mu, \sigma^2)$ , then  $Y = \frac{X - \mu}{\sigma}$  is  $N(0,1)$

## 6. Reliability

For a device in continuous operation with failure time random variable  $T$  having pdf  $f(t)$  ( $t > 0$ )

The reliability function at time  $t$   $R(t) = P(T > t)$

The failure rate or hazard function  $h(t) = f(t)/R(t)$

The cumulative hazard function  $H(t) = \int_0^t h(t_0) dt_0 = -\ln\{R(t)\}$

The Weibull( $\alpha, \beta$ ) distribution has  $H(t) = \beta t^\alpha$

## 7. System reliability

For a system of  $k$  devices, which operate independently, let

$$R_i = P(D_i) = P(\text{"device } i \text{ operates"})$$

The system reliability,  $R$ , is the probability of a path of operating devices

A system of devices in series operates only if every device operates

$$R = P(D_1 \cap D_2 \cap \dots \cap D_k) = R_1 R_2 \dots R_k$$

A system of devices in parallel operates if any device operates

$$R = P(D_1 \cup D_2 \cup \dots \cup D_k) = 1 - (1 - R_1)(1 - R_2) \dots (1 - R_k)$$

## 8. Covariance and correlation

The covariance of  $X$  and  $Y$   $\text{cov}(X, Y) = E(XY) - \{E(X)\}\{E(Y)\}$

From pairs of observations  $(x_1, y_1), \dots, (x_n, y_n)$   $S_{xy} = \sum_k x_k y_k - \frac{1}{n} (\sum_i x_i) (\sum_j y_j)$

$$S_{xx} = \sum_k x_k^2 - \frac{1}{n} (\sum_i x_i)^2, \quad S_{yy} = \sum_k y_k^2 - \frac{1}{n} (\sum_j y_j)^2$$

Sample covariance  $s_{xy} = \frac{1}{n-1} S_{xy}$  estimates  $\text{cov}(X, Y)$

Correlation coefficient  $\rho = \text{corr}(X, Y) = \frac{\text{cov}(X, Y)}{\text{sd}(X) \cdot \text{sd}(Y)}$

Sample correlation coefficient  $r = \frac{S_{xy}}{\sqrt{S_{xx} S_{yy}}}$  estimates  $\rho$

9. Sums of random variables

$$E(X + Y) = E(X) + E(Y)$$

$$\text{var}(X + Y) = \text{var}(X) + \text{var}(Y) + 2 \text{cov}(X, Y)$$

$$\text{cov}(aX + bY, cX + dY) = (ac) \text{var}(X) + (bd) \text{var}(Y) + (ad + bc) \text{cov}(X, Y)$$

If  $X$  is  $N(\mu_1, \sigma_1^2)$ ,  $Y$  is  $N(\mu_2, \sigma_2^2)$ , and  $\text{cov}(X, Y) = c$ , then  $X + Y$  is  $N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2 + 2c)$

10. Bias, standard error, mean square error

If  $t$  estimates  $\theta$  (with random variable  $T$  giving  $t$ )

$$\text{Bias of } t \quad \text{bias}(t) = E(T) - \theta$$

$$\text{Standard error of } t \quad \text{se}(t) = \text{sd}(T)$$

$$\text{Mean square error of } t \quad \text{MSE}(t) = E\{(T - \theta)^2\} = \{\text{se}(t)\}^2 + \{\text{bias}(t)\}^2$$

If  $\bar{x}$  estimates  $\mu$ , then  $\text{bias}(\bar{x}) = 0$ ,  $\text{se}(\bar{x}) = \sigma/\sqrt{n}$ ,  $\text{MSE}(\bar{x}) = \sigma^2/n$ ,  $\widehat{\text{se}}(\bar{x}) = s/\sqrt{n}$

Central limit property If  $n$  is fairly large,  $\bar{x}$  is from  $N(\mu, \sigma^2/n)$  approximately

11. Likelihood

The likelihood is the joint probability as a function of the unknown parameter  $\theta$ .

For a random sample  $x_1, x_2, \dots, x_n$

$$\ell(\theta; x_1, x_2, \dots, x_n) = P(X_1 = x_1 | \theta) \cdots P(X_n = x_n | \theta) \quad (\text{discrete distribution})$$

$$\ell(\theta; x_1, x_2, \dots, x_n) = f(x_1 | \theta) f(x_2 | \theta) \cdots f(x_n | \theta) \quad (\text{continuous distribution})$$

The maximum likelihood estimator (MLE) is  $\hat{\theta}$  for which the likelihood is a maximum

12. Confidence intervals

If  $x_1, x_2, \dots, x_n$  are a random sample from  $N(\mu, \sigma^2)$  and  $\sigma^2$  is known, then

the 95% confidence interval for  $\mu$  is  $(\bar{x} - 1.96 \frac{\sigma}{\sqrt{n}}, \bar{x} + 1.96 \frac{\sigma}{\sqrt{n}})$

If  $\sigma^2$  is estimated, then from the Student t table for  $t_{n-1}$  we find  $t_0 = t_{n-1, 0.05}$

The 95% confidence interval for  $\mu$  is  $(\bar{x} - t_0 \frac{s}{\sqrt{n}}, \bar{x} + t_0 \frac{s}{\sqrt{n}})$

13. Standard normal table Values of pdf  $\phi(y) = f(y)$  and cdf  $\Phi(y) = F(y)$

$y$	$\phi(y)$	$\Phi(y)$	$y$	$\phi(y)$	$\Phi(y)$	$y$	$\phi(y)$	$\Phi(y)$	$y$	$\Phi(y)$
0	.399	.5	.9	.266	.816	1.8	.079	.964	2.8	.997
.1	.397	.540	1.0	.242	.841	1.9	.066	.971	3.0	.999
.2	.391	.579	1.1	.218	.864	2.0	.054	.977	0.841	.8
.3	.381	.618	1.2	.194	.885	2.1	.044	.982	1.282	.9
.4	.368	.655	1.3	.171	.903	2.2	.035	.986	1.645	.95
.5	.352	.691	1.4	.150	.919	2.3	.028	.989	1.96	.975
.6	.333	.726	1.5	.130	.933	2.4	.022	.992	2.326	.99
.7	.312	.758	1.6	.111	.945	2.5	.018	.994	2.576	.995
.8	.290	.788	1.7	.094	.955	2.6	.014	.995	3.09	.999

14. Student t table Values  $t_{m,p}$  of  $x$  for which  $P(|X| > x) = p$ , when  $X$  is  $t_m$

$m$	$p=0.10$	0.05	0.02	0.01	$m$	$p=0.10$	0.05	0.02	0.01
1	6.31	12.71	31.82	63.66	9	1.83	2.26	2.82	3.25
2	2.92	4.30	6.96	9.92	10	1.81	2.23	2.76	3.17
3	2.35	3.18	4.54	5.84	12	1.78	2.18	2.68	3.05
4	2.13	2.78	3.75	4.60	15	1.75	2.13	2.60	2.95
5	2.02	2.57	3.36	4.03	20	1.72	2.09	2.53	2.85
6	1.94	2.45	3.14	3.71	25	1.71	2.06	2.48	2.78
7	1.89	2.36	3.00	3.50	40	1.68	2.02	2.42	2.70
8	1.86	2.31	2.90	3.36	$\infty$	1.645	1.96	2.326	2.576

15. Chi-squared table Values  $\chi_{k,p}^2$  of  $x$  for which  $P(X > x) = p$ , when  $X$  is  $\chi_k^2$  and  $p = .995, .975, \text{ etc}$

$k$	.995	.975	.05	.025	.01	.005	$k$	.995	.975	.05	.025	.01	.005
1	.000	.001	3.84	5.02	6.63	7.88	18	6.26	8.23	28.87	31.53	34.81	37.16
2	.010	.051	5.99	7.38	9.21	10.60	20	7.43	9.59	31.42	34.17	37.57	40.00
3	.072	.216	7.81	9.35	11.34	12.84	22	8.64	10.98	33.92	36.78	40.29	42.80
4	.207	.484	9.49	11.14	13.28	14.86	24	9.89	12.40	36.42	39.36	42.98	45.56
5	.412	.831	11.07	12.83	15.09	16.75	26	11.16	13.84	38.89	41.92	45.64	48.29
6	.676	1.24	12.59	14.45	16.81	18.55	28	12.46	15.31	41.34	44.46	48.28	50.99
7	.990	1.69	14.07	16.01	18.48	20.28	30	13.79	16.79	43.77	46.98	50.89	53.67
8	1.34	2.18	15.51	17.53	20.09	21.95	40	20.71	24.43	55.76	59.34	63.69	66.77
9	1.73	2.70	16.92	19.02	21.67	23.59	50	27.99	32.36	67.50	71.41	76.15	79.49
10	2.16	3.25	18.31	20.48	23.21	25.19	60	35.53	40.48	79.08	83.30	88.38	91.95
12	3.07	4.40	21.03	23.34	26.22	28.30	70	43.28	48.76	90.53	95.02	100.4	104.2
14	4.07	5.63	23.68	26.12	29.14	31.32	80	51.17	57.15	101.9	106.6	112.3	116.3
16	5.14	6.91	26.30	28.85	32.00	34.27	100	67.33	74.22	124.3	129.6	135.8	140.2

16. The chi-squared goodness-of-fit test

The frequencies  $n_y$  are grouped so that the fitted frequency  $\hat{n}_y$  for every group exceeds about 5.

$$X^2 = \sum_y \frac{(n_y - \hat{n}_y)^2}{\hat{n}_y} \text{ is referred to the table of } \chi_k^2 \text{ with significance point } p,$$

where  $k$  is the number of terms summed, less one for each constraint, eg matching total frequency, and matching  $\bar{x}$  with  $\mu$

17. Joint probability distributions

Discrete distribution  $\{p_{xy}\}$ , where  $p_{xy} = P(\{X = x\} \cap \{Y = y\})$ .

Let  $p_{x\bullet} = P(X = x)$ , and  $p_{\bullet y} = P(Y = y)$ , then

$$p_{x\bullet} = \sum_y p_{xy} \quad \text{and} \quad P(X = x | Y = y) = \frac{p_{xy}}{p_{\bullet y}}$$

Continuous distribution

Joint cdf  $F(x, y) = P(\{X \leq x\} \cap \{Y \leq y\}) = \int_{x_0=-\infty}^x \int_{y_0=-\infty}^y f(x_0, y_0) dx_0 dy_0$

Joint pdf  $f(x, y) = \frac{d^2 F(x, y)}{dx dy}$

Marginal pdf of  $X$   $f_X(x) = \int_{-\infty}^{\infty} f(x, y_0) dy_0$

Conditional pdf of  $X$  given  $Y = y$   $f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(y)} \quad (\text{provided } f_Y(y) > 0)$

18. Linear regression

To fit the linear regression model  $y = \alpha + \beta x$  by  $\hat{y}_x = \hat{\alpha} + \hat{\beta}x$  from observations

$(x_1, y_1), \dots, (x_n, y_n)$ , the least squares fit is  $\hat{\alpha} = \bar{y} - \bar{x}\hat{\beta}$ ,  $\hat{\beta} = \frac{S_{xy}}{S_{xx}}$

The residual sum of squares  $RSS = S_{yy} - \frac{S_{xy}^2}{S_{xx}}$

$$\widehat{\sigma^2} = \frac{RSS}{n-2} \quad \frac{n-2}{\sigma^2} \widehat{\sigma^2} \text{ is from } \chi_{n-2}^2$$

$$E(\hat{\alpha}) = \alpha, \quad E(\hat{\beta}) = \beta,$$

$$\text{var}(\hat{\alpha}) = \frac{\sum x_i^2}{n S_{xx}} \sigma^2, \quad \text{var}(\hat{\beta}) = \frac{\sigma^2}{S_{xx}}, \quad \text{cov}(\hat{\alpha}, \hat{\beta}) = -\frac{\bar{x}}{S_{xx}} \sigma^2$$

$$\hat{y}_x = \hat{\alpha} + \hat{\beta}x, \quad E(\hat{y}_x) = \alpha + \beta x, \quad \text{var}(\hat{y}_x) = \left\{ \frac{1}{n} + \frac{(x - \bar{x})^2}{S_{xx}} \right\} \sigma^2$$

$$\frac{\hat{\alpha} - \alpha}{\widehat{\text{se}}(\hat{\alpha})}, \quad \frac{\hat{\beta} - \beta}{\widehat{\text{se}}(\hat{\beta})}, \quad \frac{\hat{y}_x - \alpha - \beta x}{\widehat{\text{se}}(\hat{y}_x)} \text{ are each from } t_{n-2}$$