EE4-24
EE9-CS1-3

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2018

MSc and EEE/EIE PART IV: MEng and ACGI

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DISCRETE-TIME SYSTEMS AND COMPUTER CONTROL

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Tuesday, 22 May 10:00 am

Time allowed: 3:00 hours

There are FOUR questions on this paper.

Answer ALL questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : G. Scarciotti
Second Marker(s) : A. Astolfi

DISCRETE-TIME SYSTEMS AND COMPUTER CONTROL

Information for candidates:

- $Z\left(\frac{1}{s}\right) = Z(1) = \frac{z}{z-1}$
- $Z\left(\frac{1}{s+a}\right) = Z\left(e^{-akT}\right) = \frac{z}{z-e^{-aT}}$
- $Z\left(\frac{b}{(s+a)^2+b^2}\right) = Z\left(e^{-akT}\sin bkT\right) = \frac{ze^{-aT}\sin bT}{z^2-2ze^{-aT}\cos bT+e^{-2aT}}$
- $Z\left(\frac{s+a}{(s+a)^2+b^2}\right) = Z\left(e^{-akT}\cos bkT\right) = \frac{z^2-ze^{-aT}\cos bT}{z^2-2ze^{-aT}\cos bT+e^{-2aT}}$
- $Z\left(\frac{1}{s^2}\right) = Z(kT) = \frac{zT}{(z-1)^2}$
- $Z\left(ka^{k-1}\right) = \frac{z}{(z-a)^2}$
- $Z(x(k+n)) = z^nX(z) - z^nx(0) - z^{n-1}x(1) - \cdots - zx(n-1)$
- Transfer function of the ZOH: $H_0(s) = \frac{1-e^{-sT}}{s}$
- Transfer function of the FOH: $H_1(s) = \frac{1+Ts}{T}\left(\frac{1-e^{-sT}}{s}\right)^2$
- Bilinear transformation: $z = \frac{1+w}{1-w}$
- Note that, for a given signal r, or $r(t)$, $R(z)$ denotes its Z-transform. All signals are assumed to be zero for negative times.

1. Consider the difference equation

$$x(k+2)+\alpha x(k+1)+\beta x(k)=u(k),$$

where α and β are constant. No assumption on the initial values of $x(k)$ is made at this point.

a) Determine the function $X(z)$.

[3 marks]

b) Let $\alpha=-2$ and $\beta=1$. Consider the input

$$U(z)=\frac{z(z-1)}{(z-\frac{1}{2})^2},$$

and assume that $x(1)=0$. No other assumption on the initial values of $x(k)$ is made at this point.

i) Compute the solution $x(k)$.

[3 marks]

ii) Assess the convergence properties of the solution.

[3 marks]

c) Let $\alpha=-\frac{3}{2}$, $\beta=\frac{1}{2}$, $x(0)=0$, $x(1)=0$ and $u(1)=0$. Assume that the input is given by

$$u(k)=\sin\left(\frac{\pi}{2}k\right).$$

i) The solution $x(k)$ can be decomposed in transient response $x_{tr}(k)$ and steady-state response $x_{ss}(k)$, as $x(k)=x_{tr}(k)+x_{ss}(k)$. The transient response is such that $\lim_{k\to+\infty}x_{tr}(k)=0$. Note that the steady-state response is properly defined only if $\lim_{k\to\infty}x(k)-x_{tr}(k)\neq\infty$. Compute the solution $x(k)$ and identify the transient response $x_{tr}(k)$ and the steady-state response $x_{ss}(k)$.

[3 marks]

ii) If the input of the system is changed to $u(k)=1+\sin\left(\frac{\pi}{2}k\right)$, how does the steady-state response change? Why?

[3 marks]

d) Consider the function

$$X(z)=\frac{1}{z-a}U(z)$$

and the input

$$U(z)=\frac{z}{z-b},$$

where a and b are constant. Generalize the previous observations. To this end solve the following parts.

i) Compute $x(k)$ assuming $a=b$. Study the convergence properties of $x(k)$ when $a=b=1$.

[2 marks]

ii) Compute $x(k)$ assuming $a\neq b$. Study the convergence properties of $x(k)$ as a function of b when $a=1$.

[3 marks]

2. Consider a unity feedback system with open-loop transfer function

$$P(s) = k\frac{1}{s+2},$$

with $k > 0$.

a) Show that the closed-loop system is stable for all $k > 0$.

[1 marks]

b) **Effect of ZOH:** Assume that the input of the system $P(s)$ is connected to a sampler and to a **zero-order hold** and that the output is connected to a sampler. Let $T > 0$ be the sampling period.

i) Determine the discrete-time equivalent transfer function of the open-loop system.

[2 marks]

ii) Show that the closed-loop system is asymptotically stable for all $k \in \left(0, \overline{K}\right)$. Compute $\overline{K}$.

[2 marks]

iii) Determine $\overline{K}$ for $T \to 0$ and for $T \to \infty$.

[2 marks]

c) **Effect of FOH:** Assume that the input of the system $P(s)$ is connected to a sampler and to a **first-order hold** and that the output is connected to a sampler. Let $T > 0$ be the sampling period.

i) Determine the discrete-time equivalent transfer function of the open-loop system.

[4 marks]

ii) Show that the closed-loop system is asymptotically stable for all $k \in \left(0, \overline{\overline{K}}\right)$. Compute $\overline{\overline{K}}$.

[5 marks]

iii) Determine $\overline{\overline{K}}$ for $T \to 0$ and for $T \to \infty$.

[3 marks]

d) Comment on the differences between the results obtained in part a), b) and c).

[1 marks]

3. Consider the digital control system in Figure 3.1.

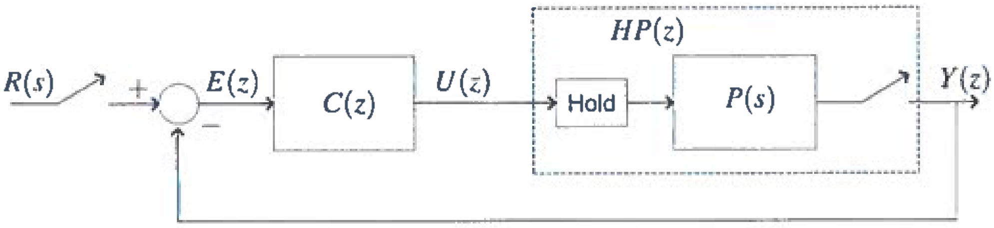

Figure 3.1 Block diagram for question 3.

The continuous-time system is defined by

$$
\begin{aligned}
\dot{x}(t) &= Fx(t) + Gu(t), \\
y(t) &= Hx(t),
\end{aligned}
\tag{3.1}
$$

with

$$
F = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \qquad G = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad H = \begin{bmatrix} 1 & 0 \end{bmatrix}.
$$

Let the sampling time be $T = 1$ sec.

a) Discretize the continuous-time state-space equations (3.1). To this end, determine the matrices A, B and C of the representation

$$
\begin{aligned}
x(k+1) &= Ax(k) + Bu(k), \\
y(k) &= Cx(k),
\end{aligned}
\tag{3.2}
$$

when the input $u(t)$ is sampled and fed to a zero-order hold.
Hint: recall that $e^{Ft} = \mathscr{L}^{-1}\left[(sI-F)^{-1}\right]$, *where* $\mathscr{L}^{-1}$ *indicates the inverse Laplace transform operator.*

[4 marks]

b) Compute the input-output transfer function of the discrete-time system (3.2).

[3 marks]

c) Compute the input-output transfer function $P(s)$ of the continuous-time state-space system (3.1).

[2 marks]

d) Compute the equivalent discrete-time model $HP(z)$ for the plant $P(s)$ interconnected to the zero-order hold and the sampler. Show that $HP(z)$ is the same transfer function as the one obtained in part b).

[3 marks]

e) Consider system 3.2 ~~(3.1)~~. Determine a state-feedback control $u(k) = -Kx(k)$ such that the closed-loop response is deadbeat (all closed-loop eigenvalues are zero). Note that Ackermann's formula is not your quickest option.

[4 marks]

f) Determine a controller described by an equation of the form

$$
C(z) = \frac{s_0 z^m + s_1 z^{m-1} + \cdots + s_{m-1} z + s_m}{z^m + v_1 z^{m-1} + \cdots + v_{m-1} z + v_m}
$$

such that the closed-loop response is deadbeat (all poles are at zero). Select m such that $C(z)$ has the smallest possible order. There is no need to give the numerical values of the coefficient s_i and v_i, just write the equations that relate one another these quantities (preferably in matrix form).

[4 marks]

4. Consider the discrete-time system

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix}\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} d \\ 0 \end{bmatrix} u(k),$$
$$y(k) = \begin{bmatrix} 0 & e \end{bmatrix}\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix},$$

with the initial state

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix},$$

where x_{10}, x_{20}, a, b, c, d and e are constant.

a) Determine for which values of

i) a, b, c and d the system is reachable.

ii) a, b, c and e the system is observable.

[4 marks]

b) For which values of a, b, c and d the system is controllable.

[8 marks]

c) Let $a = 0$ and $b = c = d = e = 1$.

i) Determine the sequence of inputs $\{u(k)\}$ to reach the state

$$\begin{bmatrix} x_1(2) \\ x_2(2) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

starting from any initial condition.

ii) Determine the initial state $x(0)$ and the states $x(1)$ and $x(2)$, which are consistent with the values $y(0) = 1$, $y(1) = 2$, $u(0) = 2$ and $u(1) = 1$.

[4 marks]

d) Let $b = 0$ and $a = d = e = 1$. Determine, if possible, the sequence of inputs $\{u(k)\}$ to control the state to zero for

i) $c = 0$;

ii) $c = 1$.

[4 marks]