1. a) i) Let $z(t)$ be the position of the box. The force equations are

$$f = K_1 z + D\dot{z} + K_2(z - y), \qquad M\ddot{y} + K_2(y - z) = 0.$$

Taking Laplace transforms, substituting and eliminating z gives

$$G(s) = \frac{1}{s^3 + (1 + K_1)s^2 + s + K_1}.$$

ii) The Routh array is:

$$\begin{array}{c|cc} s^3 & 1 & 1 \\ s^2 & 1 + K_1 & K_1 \\ s & \frac{1}{1+K_1} & \\ 1 & K_1 & \end{array}$$

So $K_1 > 0$ for positive signs for the first column and therefore stability.

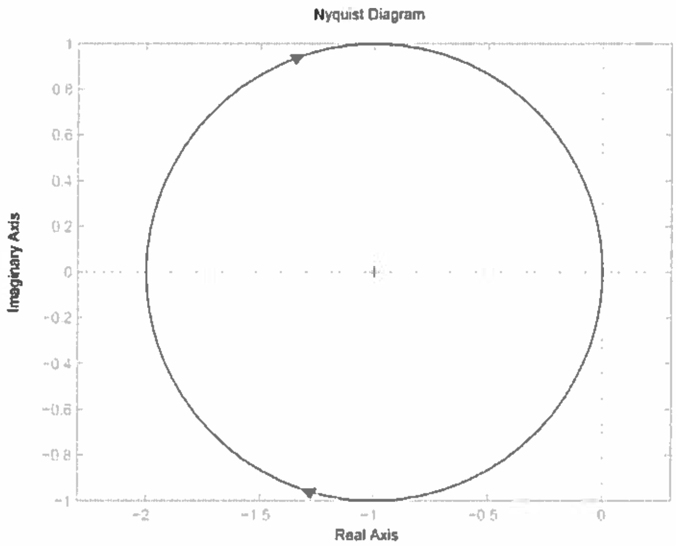
iii) When $K_1 = 0$ the array has a zero entry in the first column corresponding to marginal stability. Substituting $K_1 = 0$ into $G(s)$ gives the poles as the roots of $s(s^2 + s + 1)$ which are $0, -0.5 \pm j0.5\sqrt{3}$.

iv) Using the final value theorem and the fact that $f(s) = 1/s$,

$$y_{ss} := \lim_{t\to\infty} y(t) = \lim_{s\to 0} sy(s) = \lim_{s\to 0} sG(s)f(s) = \lim_{s\to 0} \frac{sG(s)}{s} = G(0) = \frac{1}{K_1}.$$

So for $y_{ss} = 2$, $K_1 = 0.5$.

b) i) The Nyquist diagram is shown below.



ii) The Nyquist criterion states that $N = Z - P$, where N is the number of clockwise encirclements by the Nyquist diagram of the point $-k^{-1}$, P is the number of unstable open-loop poles and Z is the number of unstable closed-loop poles. Since $G(s)$ has one unstable pole, $P = 1$.

- When $-\infty < k < 0.5$, $N = 0$ so $Z = 1$.
- When $0.5 < k < \infty$, $N = 1$ so $Z = 2$.
- When $k = 0.5$, the closed-loop is $\dfrac{0.5G(s)}{1+0.5G(s)} = \dfrac{1}{s^3}$ and so there are three closed-loop poles at the origin.

iii) A PD compensator has the form $K(s) = k(s+z)$. The characteristic equation for the closed-loop is

$$1 + K(s)G(s) = 1 + \frac{2k(s+z)}{s^3-1} = 0 \Rightarrow s^3 + 2ks + 2kz - 1 = 0$$

Since the coefficient of s^2 is zero, the closed-loop cannot be stabilised.

iv) Since $G(s) = \frac{2}{(s-1)(s^2+s+1)}$ then $G(s)K(s) = \frac{2k}{(s-1)(s^2+2s+3)}$. The characteristic equation for the closed-loop is

$$1 + K(s)G(s) = 1 + \frac{2k}{(s-1)(s^2+2s+3)} = 0 \Rightarrow s^3 + s^2 + s + 2k - 3 = 0$$

The Routh array:
$$\begin{array}{c|cc} s^3 & 1 & 1 \\ s^2 & 1 & 2k-3 \\ s & 2(2-k) & \\ 1 & 2k-3 & \end{array}$$

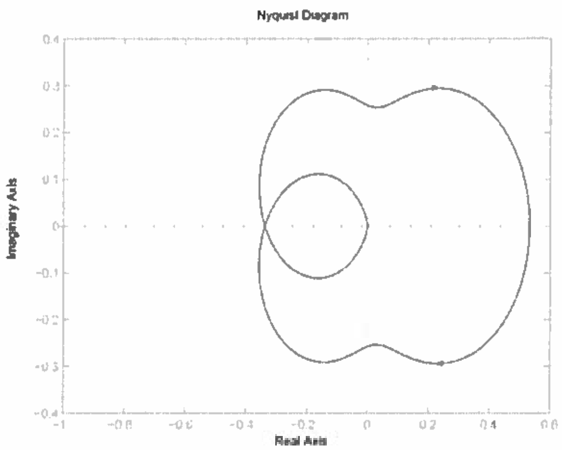
For stability, we need $1.5 < k < 2$ which is clearly possible.

2. a) The characteristic equation for the closed-loop is

$$1+KG(s) = 1+\frac{K}{s^3+as^2+bs+c} = 0 \Rightarrow s^3+as^2+bs+c+K = 0$$

The Routh array:
$$\begin{array}{c|cc} s^3 & 1 & b \\ s^2 & a & c+K \\ s & b-\frac{c+K}{a} & \\ 1 & c+K & \end{array}$$

The real-axis intercepts: $0, -\frac{1}{ab-c}, 1/c$. A typical Nyquist diagram is:



b) We have $N = Z - P$, where N is the number of clockwise encirclements by the Nyquist diagram of $-K^{-1}$, P is the number of unstable poles of G and Z is the number of unstable closed–loop poles. To find P, the Routh array for $G(s)$:

$$\begin{array}{c|cc} s^3 & 1 & b \\ s^2 & a & c \\ s & b-\frac{c}{a} & \\ 1 & c & \end{array}$$

Since $a > 0$, $c > 0$ and $ab > c$ then $G(s)$ is always stable and so $P = 0$.

- When $-\infty < K < -c$, $N = 1$ so $Z = 1$.
- When $-c < K < ab-c$, $N = 0$ so $Z = 0$.
- When $ab-c < K < \infty$, $N = 2$ so $Z = 2$.
- When $K = -c$, the closed–loop is marginally stable (a pole at 0)
- When $K = ab-c$: the closed–loop is marginally stable (poles at $\pm j\sqrt{b}$)

c) The gain margin is $ab-c$ since the real–axis intercept is at $\frac{1}{ab-c}$.

d) Since $ab-c \geq 2$, the gain margin is at least 2 for all parameter values.

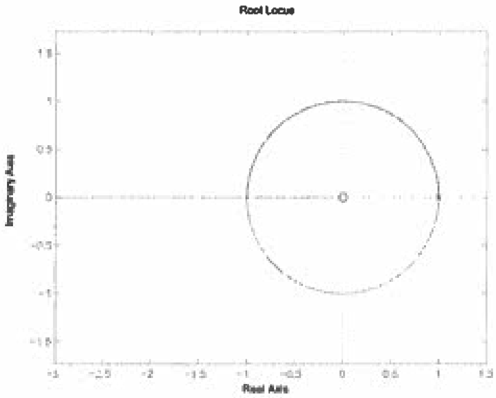
e) The gain and phase margins are adequate and we expect good transient responses. However, the DC gain $G(0) = 1/c$ is less than 1 and so we need to improve the steady state performance. Since phase–lag compensation increases low frequency gain, and hence improve steady–state tracking it follows that the system requires phase–lag compensation.

3. a) For a maximum overshoot of 5% and a settling time of 4 seconds the closed–loop poles must be placed at $s_0, \bar{s}_0 = -1 \pm j$.

b) For $z=0$, the closed–loop characteristic equation is

$$1+G(s)=0 \Rightarrow 1-\frac{2s}{s^2+ks+1}=0 \Rightarrow s^2-2s+1+ks=0 \Rightarrow 1+k\overbrace{\frac{s}{(s-1)^2}}^{\hat{G}(s)}=0$$

i) We plot the root locus of $\hat{G}$:



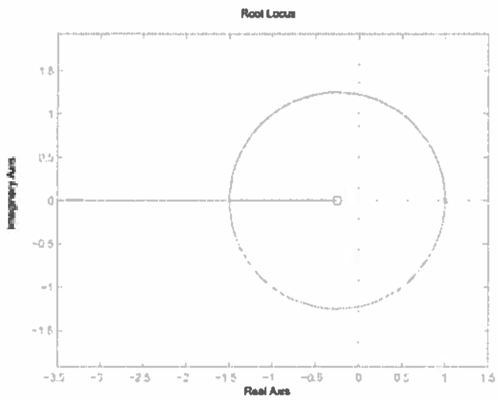
ii) Thus a settling time of 4s is only achievable with the closed–loop poles set at -1 and so the response is not oscillatory. The corresponding k is obtained from the gain criterion as $k=-1/\hat{G}(-1)=4$.

c) For general z, proceeding as before, the closed–loop characteristic equation is

$$1+G(s)=0 \Rightarrow 1-\frac{2s}{s^2+k(s+z)+1}=0 \Rightarrow s^2-2s+1+k(s+z)=0 \Rightarrow 1+k\overbrace{\frac{s+z}{(s-1)^2}}^{\hat{G}(s)}=0$$

i) $\hat{G}(s)$ has two poles at 1 and a zero at $-z$. Let the angle from $s_0=-1+j$ to $-z$ be θ and to 1 be θ_1. The angle criterion requires $\theta=2\theta_1-180°$ and after some trigonometry this gives $z=0.25$.

ii) For $z=0.25$, the root–locus of $\hat{G}(s)$ is shown below.



iii) The gain criterion requires $k=-1/\hat{G}(s_0)$ and so $k=4$.