

The Solutions for E3.18, 2008

Model answer to Q 1(a): Derivations and Computed Example

For a half-height waveguide (i.e. its height dimension 'b' is half that of the width dimension 'a':

- i) Using (1.1), derive an expression for the length of the cavity in terms of 'a' and the various frequency terms.

An ideal air-filled rectangular waveguide has a guided-wavelength given by the following expression:

$$\lambda_g = \frac{\lambda_o}{\sqrt{1 - \left(\frac{f_c}{f_o}\right)^2}} \quad (1.1)$$

All variables have their usual meaning.

$$\text{for half-height : } b = \frac{a}{2}$$

$$\lambda_c = 2a \quad \therefore f_c = \frac{c}{2a} \quad \therefore a = \frac{\lambda_c}{2} = \frac{c}{2f_c} \quad \text{also} \quad \lambda_o = \frac{c}{f_o} \quad \therefore a = \frac{\lambda_o}{2} \left(\frac{f_o}{f_c}\right)$$

$$\text{for } TE_{101} \text{ mode : } l = \frac{\lambda_g}{2} = \frac{\lambda_o / 2}{\sqrt{1 - \left(\frac{f_c}{f_o}\right)^2}} = \frac{a}{\left(\frac{f_o}{f_c}\right) \sqrt{1 - \left(\frac{f_c}{f_o}\right)^2}} = \frac{a}{\sqrt{\left(\frac{f_o}{f_c}\right)^2 - 1}}$$

[4]

- ii) Using (i), derive an expression for the internal volume of the cavity.

$$\text{Volume} \equiv a b l = \frac{a^3}{2 \sqrt{\left(\frac{f_o}{f_c}\right)^2 - 1}}$$

[2]

- iii) Using (i), derive an expression for the internal area of the cavity.

$$\text{Area} \equiv 2(al + ab + bl) = 2a^2 \left[\frac{1}{2} + \frac{1}{\sqrt{\left(\frac{f_o}{f_c}\right)^2 - 1}} + \frac{1}{2\sqrt{\left(\frac{f_o}{f_c}\right)^2 - 1}} \right] = a^2 \left[1 + \frac{3}{\sqrt{\left(\frac{f_o}{f_c}\right)^2 - 1}} \right]$$

[3]

- iv) Using (1.2) and assuming that $f_o/f_c = \sqrt{2}$, derive an expression for the unloaded-Q-factor in terms of 'a' and classical skin depth.

For the TE₁₀₁ mode, the unloaded Q-factor for an air-filled rectangular waveguide resonant cavity is given by the following expression:

$$Q_u|_{TE_{101}} \cong \frac{2 \text{ Volume}}{\delta_o \text{ Area}} \quad (1.2)$$

All variables have their usual meaning.

$$\text{Volume} = \frac{a^3}{2\sqrt{\left(\frac{f_o}{f_c}\right)^2 - 1}} \rightarrow \frac{a^3}{2} \quad \text{and} \quad \text{Area} = a^2 \left[1 + \frac{3}{\sqrt{\left(\frac{f_o}{f_c}\right)^2 - 1}} \right] \rightarrow 4a^2$$

$$\therefore Q_u|_{TE_{101}} \cong \frac{2 \text{ Volume}}{\delta_o \text{ Area}} \rightarrow \frac{a}{4\delta_o}$$

[4]

v) Using (iv), calculate the unloaded-Q-factor for a 15.5 GHz resonant cavity made with copper walls having a DC bulk conductivity of 5.8×10^7 S/m.

$$\lambda_o = \frac{c}{f_o} = 19.355 \text{ mm} \quad \text{and} \quad a = \frac{\lambda_o}{2} \left(\frac{f_o}{f_c} \right) = 13.686 \text{ mm}$$

$$\delta_o = \sqrt{\frac{2}{\omega_o \mu_o \sigma_o}} = \sqrt{\frac{2}{2\pi 15.5 \times 10^9 \times 4\pi \times 10^{-7} \times 5.8 \times 10^7}} = 0.531 \mu\text{m}$$

$$\therefore Q_u|_{TE_{101}} \cong \frac{a}{4\delta_o} = 6,446$$

[4]

Model answer to Q 1(b): Derivation

For a cubical cavity (i.e. all internal dimensions are equal) derive an expression for the unloaded-Q-factor in terms of 'a' and classical skin depth and show that this has a 33.333% higher unloaded Q-factor.

$$\text{Volume} = a^3 \quad \text{and} \quad \text{Area} = 6a^2$$

$$\therefore Q_u|_{TE_{101}} \cong \frac{2 \text{ Volume}}{\delta_o \text{ Area}} \rightarrow \frac{a}{3\delta_o} \quad \text{which is } 4/3 = 1.3333333 \text{ higher than with half-height}$$

[3]

Model answer to Q 2(a): Computed Example

$$\epsilon_r' = 12.86 \quad \text{and} \quad \tan \delta = \frac{\epsilon_r''}{\epsilon_r'} = 0.0006 \quad \therefore \epsilon_r'' = 7.716 \times 10^{-3}$$

$$\sigma = \sigma' - j\sigma'' = j\omega\epsilon_o(\epsilon_r - 1) \quad \text{and} \quad \epsilon_r = \epsilon_r' - j\epsilon_r''$$

$$\therefore \sigma' = \omega\epsilon_o\epsilon_r'' \quad \text{and} \quad -\sigma'' = \omega\epsilon_o(\epsilon_r' - 1)$$

$$\therefore \sigma = 0.129 - j(-197.932) \text{ S/m}$$

[5]

Model answer to Q 2(b): Computed Example

$$\rho_o = 8 \text{ k}\Omega \cdot \text{cm} = 80 \Omega \cdot \text{m}$$

$$\therefore \sigma_o = \frac{1}{\rho_o} = 0.0125 \text{ S/m}$$

Therefore, it can be seen that at 300 GHz the conductivity is 10.32 times greater than the originally quoted value suggests.

[3]

Model answer to Q 2(c): Computed Example

$$\rho = \frac{\eta - \eta_o}{\eta + \eta_o} \quad \text{where} \quad \eta_o = \sqrt{\frac{\mu_o}{\epsilon_o}} \quad \text{and} \quad \eta = \sqrt{\frac{\mu_o\mu_r}{\epsilon_o\epsilon_r}} \rightarrow \sqrt{\frac{\mu_o}{\epsilon_o\epsilon_r'}}$$

$$\therefore \rho = \frac{1 - \sqrt{\epsilon_r'}}{1 + \sqrt{\epsilon_r'}} = -0.564$$

$$\Gamma = |\rho|^2 = 31.8\%$$

[5]

Model answer to Q 2(d): New Derivation

$$H(z) = H(0)e^{-\gamma z} + H(0)e^{+\gamma z}$$

$$P_{\text{ABSORBED}} = |H(z)|_{z=0}^2 R_s = 4|H(0)|^2 R_s$$

$$P_{\text{INCIDENCE}} = |H(0)|^2 \eta_o$$

$$\Gamma = \frac{P_{\text{REFLECTED}}}{P_{\text{INCIDENCE}}} \quad \text{where} \quad P_{\text{REFLECTED}} = P_{\text{INCIDENCE}} - P_{\text{ABSORBED}}$$

$$\therefore \Gamma = 1 - \frac{P_{\text{ABSORBED}}}{P_{\text{INCIDENCE}}} = 1 - 4 \frac{R_s}{\eta_o}$$

[5]

Model answer to Q 2(e): Computed Example

$$\Gamma = 1 - 4 \frac{R_s}{\eta_o} = 1 - 4 \frac{0.1}{120\pi} = 99.89\%$$

[2]

$$H(z) = H(0)e^{-\gamma z} + H(0)e^{+\gamma z}$$

$$P_{\text{ABSORBED}} = |H(z)|_{z=0}^2 R_s = 4|H(0)|^2 R_s$$

$$P_{\text{INCIDENCE}} = |H(0)|^2 \eta_o$$

$$\Gamma = \frac{P_{\text{REFLECTED}}}{P_{\text{INCIDENCE}}} \quad \text{where} \quad P_{\text{REFLECTED}} = P_{\text{INCIDENCE}} - P_{\text{ABSORBED}}$$

$$\therefore \Gamma = 1 - \frac{P_{\text{ABSORBED}}}{P_{\text{INCIDENCE}}} = 1 - 4 \frac{R_s}{\eta_o}$$

[7]

Model answer to Q 3(a): Bookwork and New Derivation

$$R_{DC} = \frac{1}{\sigma_o} \left(\frac{\text{length}, l}{\text{area}} \right) \rightarrow \frac{R_{DC}}{l} = \frac{1}{\sigma_o (\pi R^2)}$$

[2]

Model answer to Q 3(b): Bookwork and New Derivation

$$W_m = \frac{\mu_o}{2} \int_{\text{volume}} H^2 \cdot dv \quad \text{where} \quad v = (2\pi r)l \quad \text{and} \quad H = \frac{I}{2\pi R} \left(\frac{r}{R} \right) \quad \text{where} \quad r < R$$

$$\therefore W_m = \frac{\mu_o}{2} \int_{\text{volume}} \left(\frac{I}{2\pi R} \left(\frac{r}{R} \right) \right)^2 \cdot dv \quad \text{where} \quad v = (2\pi r)l$$

$$\therefore W_m = \frac{\mu_o}{2} \left(\frac{I}{2\pi R} \left(\frac{1}{R} \right) \right)^2 (2\pi r)l \int_0^R r^3 \cdot dr = \frac{\mu_o}{2} \left(\frac{I}{2\pi R} \left(\frac{1}{R} \right) \right)^2 (2\pi r)l \left[\frac{r^4}{4} \right]_0^R = \frac{\mu_o I^2 l}{16\pi}$$

$$W_m = \frac{1}{2} L_{DC} I^2 \rightarrow \frac{L_{DC}}{l} = \frac{\mu_o}{8\pi} \neq f(R)$$

[4]

Model answer to Q 3(c): Bookwork and New Derivation

$$\text{Im pedance, } Z = Z_s \left(\frac{\text{length}, l}{\text{width}, 2\pi R} \right) \quad \text{with} \quad Z_s = R_s(1+j) \quad \text{and} \quad R_s = \frac{1}{\sigma_o \delta_o}$$

[2]

Model answer to Q 3(d): Bookwork and New Derivation

$$\frac{Z}{l} = \frac{R_{HF}}{l} + j\omega \frac{L_{HF}}{l}$$

$$\therefore \frac{R_{HF}}{l} = \frac{1}{\sigma_o \delta_o 2\pi R} = \left(\frac{R}{2\delta_o} \right) \left(\frac{R_{DC}}{l} \right)$$

[2]

Model answer to Q 3(e): Bookwork and New Derivation

$$\therefore \frac{L_{HF}}{l} = \frac{1}{\sigma_o \delta_o 2\pi R \omega} = \left(\frac{2\delta_o}{R} \right) \left(\frac{L_{DC}}{l} \right)$$

[2]

Model answer to Q 3(f): Bookwork and New Derivation

No, since the resistance and inductive reactance are equal at all frequencies.

[2]

Model answer to Q 3(e): Bookwork and New Derivation

$$\frac{L_{HF}}{l} = \left(\frac{2\delta_o}{R} \right) \left(\frac{L_{DC}}{l} \right) = \left(\frac{2\delta_o}{R} \right) \left(\frac{\mu_o}{8\pi} \right) = \frac{\mu_o \delta_o}{2 \times (2\pi R)}$$

$$\therefore L_{HF} = \frac{\mu_o \delta_o}{2} \left(\frac{\text{length}, l}{\text{width}, 2\pi R} \right)$$

[2]

Model answer to Q 3(f): Bookwork and Computed example

$$\delta_o = \sqrt{\frac{2}{\omega \mu_o \sigma_o}}$$

$$\sigma_o = \frac{1}{22.14 \times 10^{-9} \Omega m}$$

$$\therefore \delta_o = 1.248 \mu m$$

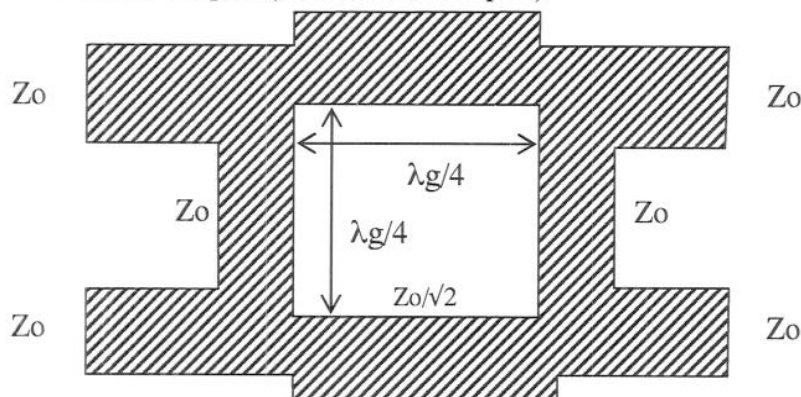
$$\therefore L_{HF} = \frac{\mu_o \delta_o}{2} \left(\frac{10^{-3}}{2\pi \times 25 \times 10^{-6} / 2} \right) = 0.01 nH$$

$$\therefore R_{HF} = \omega L_{HF} = 0.226 \Omega$$

[4]

Model answer to Q 4(a): Bookwork

90° 3dB Directional Coupler (Branch-line Coupler)

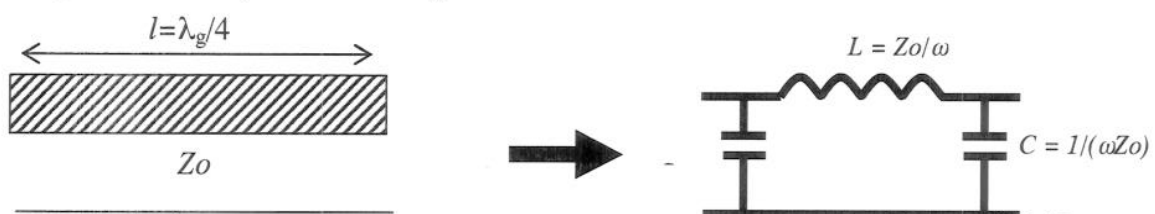


- Works on the interference principle, therefore, narrow fractional bandwidth (15% maximum)
- No bond-wires or isolation resistors required
- Wider tracks make it easier to fabricate and is, therefore, good for lower loss and higher power applications
- Simple design but large
- Meandered lines are possible for lower frequency applications

[5]

Model answer to Q 4(b): Bookwork and Computed Example

The lumped-element equivalent of a $\lambda_g/4$ transmission line is shown below.



All the previous distributed-element couplers can be transformed into equivalent lumped-element couplers by simply replacing all the $\lambda_g/4$ lengths of transmission lines with the above π -network. Since lumped-element components have a lower Q-factor, when compared to distributed-element components, there is an insertion loss penalty. Also, because this π -network is clearly a low-pass filter, having a cut-off frequency, $f_c = \frac{1}{2\pi\sqrt{LC}}$, there is also a bandwidth penalty.

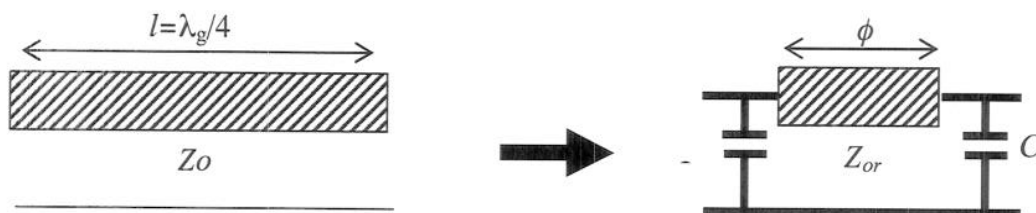
$L = 4.3 \text{ nH}$ and $C = 1.7 \text{ pF}$ for the $Z_o = 50 \Omega$ sections of line

$L = 3.0 \text{ nH}$ and $C = 2.4 \text{ pF}$ for the $Z_o = 35 \Omega$ sections of line

[5]

Model answer to Q 4(c): Bookwork and Computed Example

- Lumped-Distributed Couplers



In this 'reduced-size' technique, each $\lambda_g/4$ line is replaced with the above π -network.

$$Z_{or} = \frac{Z_o}{\sin \phi} \quad \text{and} \quad C = \frac{\cos \phi}{\omega Z_o}$$

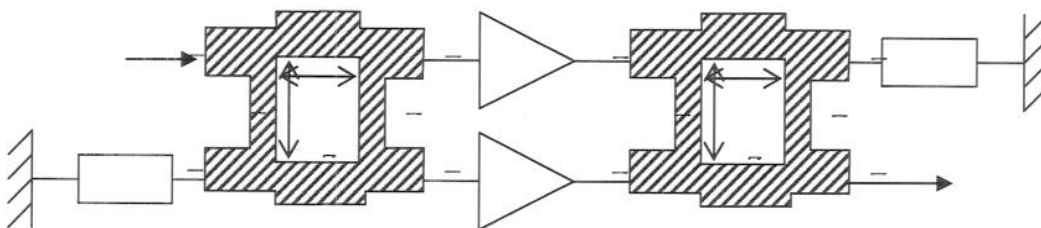
With $\phi = 45^\circ$,

$Z_{or} = 70.7 \Omega$ and $C = 1.22 \text{ pF}$ for the $Z_o = 50 \Omega$ sections of line

$Z_{or} = 50.0 \Omega$ and $C = 1.72 \text{ pF}$ for the $Z_o = 35 \Omega$ sections of line

[5]

Model answer to Q 4(d): Solution given in class



[5]

Model answer to Q5(a): New Derivation

$$z_{IN} = \frac{z + jz_{TX} \tan \vartheta}{z_{TX} + jz \tan \theta} \equiv z_o$$

$$\therefore Z_{TX} (Z + jZ_{TX} \tan \vartheta) = Z_o(Z_{TX} + jZ \tan \theta)$$

$$\text{Re}\{LHS\} \equiv \text{Re}\{RHS\}$$

$$\therefore \theta = \tan^{-1} \left\{ \frac{Z_{TX} (Z_o - R)}{XZ_o} \right\}$$

$$\text{Im}\{LHS\} \equiv \text{Im}\{RHS\}$$

$$\tan \theta = \frac{Z_{TX} X}{Z_o R - Z_{TX}^2} \equiv \frac{Z_{TX} (Z_o - R)}{XZ_o}$$

$$\therefore Z_{TX} = \sqrt{Z_o R - \frac{X^2 Z_o}{Z_o - R}}$$

[7]

Model answer to Q5(b): New Derivation

From the last expression in 4(a), the limits are:

$$R \neq Z_o \quad \text{and} \quad X < \sqrt{R(Z_o - R)}$$

[3]

Model answer to Q5(c): Computed Example

For a 2 nH inductance in series with a 3 Ω resistance at 900 MHz, the termination load impedance is $Z = 2 + j11.31 \Omega$.

Using the expressions from 5(b), R is not equal to 50 Ω and $X < 11.87 \Omega$, so both values are within the acceptable mathematical limits.

Using the expressions from 5(a), $Z_{TX} = 3.73 \Omega$ and $\vartheta = 16.5^\circ$.

[7]

Model answer to Q5(d): Bookwork

The value of Z_{TX} calculated in 5(c) would be considered very low in general. In practice, a conventional microstrip line could not be used to implement such a low impedance because the width of the signal line would be too wide. However, thin-film microstrip technology may be suitable as the widths of the lines are much narrower.

[3]

Model answer to Q 6(a): Bookwork and Derivation Exercise

The voltage and current on the line can be represented as :

$$V(z) = V_+ (e^{-\gamma z} + \rho(0)e^{+\gamma z})$$
$$I(z) = I_+ (e^{-\gamma z} - \rho(0)e^{+\gamma z})$$

It can be found that : $V_+ = 0.5(V(0) + Z_0 I(0))$ and $V_- = 0.5(V(0) - Z_0 I(0))$

\therefore incident wave power, $P_+ = \frac{|V_+|^2}{Z_0}$ and reflected wave power, $P_- = \frac{|V_-|^2}{Z_0}$

If Z_0 is taken to be purely real, the time-average power flow along the line is:

$$P(z) = \text{Re}\{V(z)I(z)^*\} = \text{Re}\left\{V_+ (e^{-\gamma z} + \rho(0)e^{+\gamma z}) I_+^* (e^{-\gamma z} - \rho(0)e^{+\gamma z})^*\right\}$$

where, $\rho(z) = \rho(0)e^{+2\gamma z} \equiv \rho(0)e^{+j2\beta z}$ for a lossless line

$$P(z) = \text{Re}\left\{\frac{|V_+|^2}{Z_0} (1 + \rho(z))(1 - \rho(z)^*)\right\} = \text{Re}\left\{\frac{|V_+|^2}{Z_0} (1 + \rho(z))(1 - \rho(z)^*)\right\} = \frac{|V_+|^2}{Z_0} (1 - |\rho(z)|^2)$$

but, $|\rho(z)| = |\rho(0)|$ for a lossless transmission line

$$\therefore P(z) = \frac{|V_+|^2}{Z_0} (1 - |\rho(0)|^2) = P_+ \left(1 - \frac{P_-}{P_+}\right) = \frac{|V_+|^2}{Z_0} (1 - |\rho(0)|^2) = P_+ \left(1 - \frac{P_-}{P_+}\right) = (P_+ - P_-)$$

This shows that, for a lossless transmission line, time-average power flow is independent of the line length and is equal to the incident wave power minus the reflected wave power.

[5]

Model answer to Q 6(b): Bookwork

The guided wavelength, λ_g , is defined as the distance between two successive points of equal phase on the wave at a fixed instance in time. The phase velocity of a wave is defined as the speed at which a constant phase point travels down the line. Frequency dispersion is said to occur when $\beta \neq \omega \cdot \text{constant}$. Dispersion can occur when $v_p = f(\omega)$, i.e. when $Dk = f(\omega)$. It can be shown that zero dispersion in a lossy line can also occur, but only when $RC = GL$:

$$\gamma^2 = (R + j\omega L)(G + j\omega C) \quad \text{and} \quad RC = GL$$

$$\therefore \alpha(\omega) = \alpha(0) = \sqrt{RG} \neq f(\omega) \quad \text{and} \quad \beta = \omega\sqrt{LC}$$

$$\text{also, Group Velocity, } V_g = \frac{\partial \omega}{\partial \beta} = \frac{1}{\sqrt{LC}} \equiv v_p \neq f(\omega)$$

[5]

Model answer to Q 6(c): Bookwork

$$Z_{in} = j\omega L + \frac{Z_o \frac{1}{j\omega C}}{Z_o + \frac{1}{j\omega C}} \equiv Z_o$$

$$\therefore Z_o = \frac{j\omega L}{2} \left(1 \mp \sqrt{1 - \frac{4}{\omega^2 LC}} \right)$$

$$\therefore \text{Cut-off frequency, } f_c = \frac{1}{\pi\sqrt{LC}} \text{ representing the bandwidth, i.e. when } \frac{4}{\omega^2 LC} = 1$$

$$Z_o = \begin{cases} \sqrt{\frac{L}{C}} & \text{when } \omega \ll \omega_c \text{ i.e. purely real} \\ \text{Complex} & \text{when } 0 < \omega < \omega_c \\ j\sqrt{\frac{L}{C}} & \text{when } \omega = \omega_c \text{ i.e. purely imaginary} \\ \text{Imajinary} & \text{when } \omega \geq \omega_c \end{cases}$$

[5]

Model answer to Q 6(d): Computed Example

$$Z_{in}(\omega_c) = jZ_o(\omega \ll \omega_c) \quad \text{and} \quad |\rho(\omega_c)|^2 = 1 \Rightarrow 0 \text{ dB} \quad \text{and} \quad |\tau(\omega_c)|^2 = 1 - |\rho(\omega_c)|^2 = 0 \Rightarrow -\infty \text{ dB}$$

[5]