Imperial College London

M4/5F22

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May-June 2017

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Mathematical Finance: Introduction to Option

Date: Tuesday 16 May 2017

Time: 14:00 - 16:00

Time Allowed: 2.5 Hours

This paper has 5 Questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the relevant main book(s) are full.

All required additional material will be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers to each question as per the table below.

Raw Mark	Up to 12	13	14	15	16	17	18	19	20
Extra Credit	0	½	1	1 ½	2	2 ½	3	3 ½	4

- Each question carries equal weight.
- Calculators may not be used.

1. Discuss the concept of limited liability, with reference to, e.g., bankruptcy and moral hazard.

2. In a two-period binary model, at each node the stock goes up by a factor of $5/4$ or down by a factor of $4/5$, each with positive probability. The payoff is $(S-8)_+$, with S the final stock-price; the initial stock-price is 8. Neglect interest.

   (i) Find the martingale probability p^* that the stock goes up.

   (ii) Working down the tree, find the value of the option at each node.

   (iii) Working up the tree, find the hedging portfolio at the time-0 and time-1 nodes.

3. (i) Define Brownian motion $B = (B_t)$.

   (ii) Find its covariance.

   (iii) For $c \in (0, \infty)$, show that the scaled process B_c, where

   $$B_c(t) := B(c^2 t)/c \qquad (t \geq 0),$$

   is again Brownian motion.

   (iv) Discuss the limitations this imposes on the suitability of Brownian motion as a model for driving noise (or uncertainty) in financial modelling.

4. (i) Define volatility.

   (ii) Comment briefly on: historic volatility; implied volatility; the volatility surface.

   (iii) How do option prices depend on volatility, and why?

   (iv) Discuss the effect of trading volume on volatility, and its implications for market stability.

5. Write rho (ρ) for the 'Greek' giving the sensitivity of option prices to the (riskless) interest rate r. You may quote the Black-Scholes formula for a call price,

$$C_t := S_t \Phi(d_1) - K e^{-r(T-t)} \Phi(d_2), \qquad (BS)$$

where

$$\phi(x) := e^{-\frac{1}{2}x^2}/\sqrt{2\pi}, \qquad \Phi(x) := \int_{-\infty}^{x} \phi(u) du,$$

$\tau := T - t$ the time to expiry,

$$d_1 := \frac{\log(S/K) + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}, \qquad d_2 := \frac{\log(S/K) + (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} = d_1 - \sigma\sqrt{\tau}.$$

(i) Show that for European calls, $\rho > 0$.

(ii) Give the financial interpretation of this.

(iii) Show that for European puts, $\rho < 0$.

(iv) Again, give the financial interpretation of this.

(v) Do these results extend to American options?

M3F22/M4F22/M5F22 EXAMINATION SOLUTIONS 2016-17

Q1: *Limited liability; bankruptcy; moral hazard.*
Limited liability.

All business transactions involve an exchange of goods or services between a willing buyer (aiming to 'buy at a minimum') and a willing seller (aiming to 'sell at a maximum'), each acting in (or with a degree of) good faith – *trusting the counter-party* to fulfill their obligations. When one party cannot do this, the transaction cannot take place as contracted, and both parties stand to lose: the defaulter as he may be forced into bankruptcy, the other as he may contract a financial loss (as he in turn may find himself unable to fulfil an obligation dependent on this deal).

Before limited liability (introduced in mid-Victorian times, in UK), a defaulter was liable to the *full financial loss* so incurred by his counter-party. This made trading, and setting up a business, very risky (especially as there was no Welfare State in those days to supply a safety net!). So limited liability (plc = public limited company) was introduced, and this enabled the growth of the modern business system, and modern capitalism. [7]

Bankruptcy.

When a firm goes bankrupt, the firm dies, leaving a loss; the firm's assets are then assessed by the liquidator, and divided up between the creditors. The net result is that *a debt is not met in full*, and so that the unfulfilled debt is *written off*. Thus bankruptcy is a mechanism whereby *debt can be written off.* [6]

Moral hazard.

The moral hazard inherent in this is that firms may be irresponsibly tempted to take excessive risks. If these pay off, the firm (and its board of directors, and shareholders, and employees) benefits. If they do not, the firm dies; the outstanding debt is written off. The directors may become undischarged bankrupts for a period, but are then (like murderers sentenced to life imprisonment on eventual release) able to re-enter the business world. The danger is that they may be tempted again to take unjustified risks, with *other people's money.* Meanwhile, the shareholders have no redress, and the employees have lost their jobs, through no fault of their own. [7]

[Largely seen – lectures]

Q2. *Two-period binary model.*

(i) *Martingale probability.*

We determine the risk-neutral probability p^* so as to make the option a fair game [martingale]: with S_0 the initial price,

$$S_0 = p^* S_0.5/4 + (1-p^*)S_0.4/5 : \ 1 = \frac{4}{5} + p^*(\frac{5}{4} - \frac{4}{5}) : \ \frac{1}{5} = p^*.\frac{9}{20} : \ p^* = \frac{4}{9}. \ [4]$$

(ii) *Pricing.* The time-2 stock prices S_2 are $S_0(5/4)^2$ (uu), S_0 (ud), $S_0.(4/5)^2$ (dd); payoffs (values) $V_2 = [S_2 - 8]_+$, which with $S_0 = 8$ are $9/2$ (uu), 0 (ud, dd). [2]

Work down the tree (as usual). The value V_1 at the two time-1 nodes are:

$$u - \text{node}: \qquad p^*.\frac{9}{2} + (1-p^*).0 = \frac{4}{9}.\frac{9}{2} = 2; \qquad d - \text{node}: \qquad 0. \qquad [3]$$

The value of the option at time 0 is

$$V_0 = p^*.V_1(u) + (1-p^*).V_1(d) = \frac{4}{9}.2 = \frac{8}{9}. \qquad [3]$$

(iii) *Hedging.*

Work up the tree (as given). From each node, the option is equivalent to ϕ_0 cash and ϕ_1 stock; the hedging portfolio is $H = (\phi_0, \phi_1)$.

Time 0.

$$u: \qquad \phi_0 + \phi_1.8.\frac{5}{4} = 2, \qquad d: \qquad \phi_0 + \phi_1.8.\frac{4}{5} = 0.$$

Subtract:

$$\phi_1.8.(\frac{5}{4} - \frac{4}{5}) = 2; \quad \phi_1.4.\frac{9}{20} = 1; \quad \phi_1 = \frac{5}{9};$$

$$\phi_0 = -\phi_1.8.\frac{4}{5} = -\frac{5}{9}.\frac{32}{5} = -\frac{32}{9} : \ H = (-\frac{32}{9}, \frac{5}{9}) : \text{ short } 32/9 \text{ cash, long } 5/9 \text{ stock.}$$

[4]

Time 1, d node: option worthless; $H = (0, 0)$.

Time 1, u node: stock up to 10, so (for the second time-period)

$$u: \qquad \phi_0 + \phi_1.10.\frac{5}{4} = \frac{9}{2}, \qquad d: \qquad \phi_0 + \phi_1.10.\frac{4}{5} = 0.$$

Subtract:

$$\phi_1.10.(\frac{5}{4} - \frac{4}{5}) = \frac{9}{2}; \quad \phi_1.10.\frac{9}{20} = \frac{9}{2}; \quad \phi_1 = 1;$$

$$\phi_0 = -\phi_1.8. = -8 : \quad H = (-8, 1) : \quad \text{short 8 cash, long 1 stock.} \quad [4]$$

[Similar seen, for the one-period case: Lectures and Problems]

Q3: *Brownian motion and scale.*
(i) *Brownian motion* (BM) $B = (B(t))$ is defined as the process with:
(a) $B(0) = 0$; [1]
(b) B has stationary independent Gaussian increments, with
$B(s+t) - B(s) \sim N(0,t)$ for all $s \geq 0$; [2]
(c) the paths $t \mapsto B(t)$ are continuous (in t, a.s. in ω). [1]
(ii) *Brownian covariance.* For $s \leq t$,

$$B_t = B_s + (B_t - B_s), \qquad B_s B_t = B_s^2 + B_s(B_t - B_s).$$

Take expectations: on the left we get $cov(B_s, B_t)$. The first term on the right is, as $E[B_s] = 0$, $var(B_s) = s$. As Brownian motion (BM) has independent increments, $B_t - B_s$ is independent of B_s, so

$$E[B_s(B_t - B_s)] = E[B_s].E[B_t - B_s] = 0.0 = 0.$$

Combining, $cov(B_s, B_t) = s$ for $s \leq t$. Similarly, for $t \leq s$ we get t. Combining, $cov(B_s, B_t) = \min(s,t)$. [5]
(iii) *Brownian scaling.* With $B_c(t) := B(c^2 t)/c$,

$$cov(B_c(s), B_c(t)) = E[B(c^2 s)/c.B(c^2 t)/c] = c^{-2}\min(c^2 s, c^2 t) = \min(s,t) = cov(B_s, B_t).$$

So B_c has the same mean 0 and covariance $\min(s,t)$ as BM. It is also (from its definition) continuous, Gaussian, stationary independent increments etc. So it has all the defining properties of BM. So it *is* BM. [4]
(iv) *BM in financial modelling.*

This limits the usefulness of BM as a model for the driving noise in a model of a financial market (modelling the effect of the unpredictable flow of new price-sensitive information). For, real financial markets are *scale-sensitive*: [1]
(a) small financial agents are price-takers not price-makers; with big financial agents, this is reversed; [3]
(b) utility functions U show *curvature*: for small amounts of money, the graph of U is (approximately) straight, so utility is effectively the same as cash; with large amounts, the Law of Diminishing Returns sets in, and this is *not* so. [3]

[Seen – lectures and problems]

Q4. (a) *Volatility.* The Black-Scholes formula involves the parameter σ (where σ^2 is the variance of the stock per unit time), called the *volatility* of the stock. In financial terms, this represents how sensitive the stock-price is to new information - how 'volatile' the market's assessment of the stock is. This volatility parameter is very important, *but* we do not know it; instead, we have to *estimate* the volatility for ourselves. There are two approaches: [2]
(b) *Historic volatility*: here we use Time Series methods to estimate σ from past price data. Clearly the more variability we observe in runs of past prices, the more volatile the stock price is; we can estimate σ like this given enough data. [3]
Implied volatility: match observed option prices to theoretical option prices. For, the price we see options traded at tells us what the *market* thinks the volatility is (estimating volatility this way works because the dependence is monotone). [3]
Volatility surface. If the Black-Scholes model were perfect, these two estimates would agree (to within sampling error). But discrepancies can be observed, which shows the imperfections of our model. Volatility graphed against price S, or strike K, typically shows a *volatility smile* (or even smirk). Graphed against S and K in 3 dimensions, we get the *volatility surface.* [3]
(c) Volatility dependence is given by *vega* $:= \partial c/\partial\sigma$ for calls, $\partial p/\partial\sigma$ for puts. From the Black-Scholes formula (which gives the price explicitly as a function of σ), one can check by calculus that $\partial c/\partial\sigma > 0$, and similarly for puts (or, use the result for calls and put-call parity). *Options like volatility.* The more uncertain things are (the higher the volatility), the more valuable protection against adversity becomes (the higher the option price). [3]
(d) The classical view of volatility is that it is caused by future uncertainty, and shows the market's reaction to the stream of new information. However, studies taking into account when different markets are open and closed (time-zones!) [there are only about 250 trading days in the year] have shown that the volatility is less when markets are closed than when they are open. This suggests that *trading itself is one of the main causes of volatility.* [3]

The introduction of a small transaction tax would have the effect of decreasing trading. This would increase market stability: trading is one of the causes of volatility; options like volatility. So trading tends to cause an increase in trading in options, and so on. Ultimately this tends to induce market instability. So conversely, market stability would benefit from a reduction in trading volumes caused by a transaction tax. [3]
[Mainly seen -- lectures]

Q5 (*Mastery question*): *Rho.*

(i) *Rho for calls.*

With $\phi(x) := e^{-\frac{1}{2}x^2}/\sqrt{2\pi}$, $\Phi(x) := \int_{-\infty}^{x} \phi(u)du$, $\tau := T - t$ the time to expiry, the Black-Scholes call price is, with d_1, d_2 as given,

$$C_t := S_t\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2). \qquad (BS)$$

So as $d_2 = d_1 - \sigma\sqrt{\tau}$,

$$\phi(d_2) = \frac{e^{-\frac{1}{2}(d_1-\sigma\sqrt{\tau})^2}}{\sqrt{2\pi}} = \frac{e^{-\frac{1}{2}d_1^2}}{\sqrt{2\pi}}.e^{d_1\sigma\sqrt{\tau}}.e^{-\frac{1}{2}\sigma^2\tau} = \phi(d_1).e^{d_1\sigma\sqrt{\tau}}.e^{-\frac{1}{2}\sigma^2\tau}.$$

Exponentiating the definition of d_1,

$$e^{d_1\sigma\sqrt{\tau}} = (S/K).e^{r\tau}.e^{\frac{1}{2}\sigma^2\tau}.$$

Combining,

$$\phi(d_2) = \phi(d_1).(S/K).e^{r\tau} : \qquad Ke^{-r\tau}\phi(d_2) = S\phi(d_1). \qquad (*)$$

Differentiating (BS) partially w.r.t. r gives, by $(*)$,

$$\begin{aligned} \rho := \partial C/\partial r &= S\phi(d_1)\partial d_1/\partial r - Ke^{-r\tau}\phi(d_2)\partial d_2/\partial r + K\tau e^{-r\tau}\Phi(d_2) \\ &= S\phi(d_1)\partial(d_1 - d_2)/\partial r + K\tau e^{-r\tau}\Phi(d_2) \\ &= S\phi(d_1)\partial(\sigma\sqrt{\tau})/\partial r + K\tau e^{-r\tau}\Phi(d_2) = K\tau e^{-r\tau}\Phi(d_2) : \end{aligned}$$

$$\rho > 0. \qquad [7]$$

(ii) *Financial interpretation.*

As r increases, cash becomes more attractive compared to stock. So stock *buyers* have a 'buyer's market', favouring them. So for *calls* (options to buy), $\rho > 0$. [3]

(iii) *Rho for puts.*

By put-call parity, $S + P - C = Ke^{-r\tau}$:

$$\partial P/\partial r = \partial C/\partial r - K\tau e^{-r\tau} = -K\tau e^{-r\tau}[1 - \Phi(d_2)] = -K\tau e^{-r\tau}\Phi(-d_2) < 0.$$

[3]

(iv) *Financial interpretation.*

As above: as r increases, stock *sellers* also operate in a buyer's market, but this is against them. So for *puts* (options to sell), $\rho < 0$. [3]

(v) *American options.*

All this extends to American options,via the *Snell envelope*, which is *order-preserving*. The discounted value of an American option is the Snell envelope $\tilde{U}_{n-1} = \max(\tilde{Z}_{n-1}, E^*[\tilde{U}_n|\mathcal{F}_{n-1}])$ of the discounted payoff $\tilde{Z}_n$ (exercised early at time $n < N$), with terminal condition $U_N = Z_N, \tilde{U}_N = \tilde{Z}_N$. As r increases, the Z-terms increase for calls (rho is positive for European calls). As the Zs increase, the Us increase (above: backward induction on n – dynamic programming, as usual for American options). Combining: as r increases, the U-terms increase. So rho is also positive for American calls. Similarly, rho is negative for American puts. **[4]**
[Similar to 'vega positive', done in Problems]

Imperial College London
Department of Mathematics

Examiner's Comments

Exam: M3F22 Math Finance: Introduction to Option Pricing **Session: 2016-2107**

Question 1

Please use the space below to comment on the candidates' overall performance in the exam. A brief paragraph highlighting common mistakes and parts of questions done badly (or well) is sufficient. Do not refer to individual candidates. The purpose of this exercise is to provide guidance to the external examiners, and to the candidates themselves, on how you feel the cohort faired. Your comments will be available to students online.

This was on limited liability, bankruptcy and moral hazard.

There is a lot one could say here. Solutions tended to be ~~too~~ brief. This may be a result of unfamiliarity with writing English prose — but this is a valuable transferable skill.

Some candidates did not have a clear understanding of what limited liability is.

Overall: quite well done; very variable.

Marker: N. H. BINGHAM

Signature: N. H. Bingham Date: 21 · 5 · 2017

Please return with exam marks (one report per marker)

Imperial College London
Department of Mathematics

Examiner's Comments

Exam: M3F22 Session: 2016-2107

Question 2

Please use the space below to comment on the candidates' overall performance in the exam. A brief paragraph highlighting common mistakes and parts of questions done badly (or well) is sufficient. Do not refer to individual candidates. The purpose of this exercise is to provide guidance to the external examiners, and to the candidates themselves, on how you feel the cohort faired. Your comments will be available to students online.

This was on the two-period binary model, pricing and hedging.

This question split the class. Those who knew what they were doing did it quickly and efficiently. Those who didn't tended to flounder at length.

The question boiled down to two calculations of the one-period binary model — very familiar.

Some of the arithmetic would not pass muster in primary school.

Marker: N. H. BINGHAM

Signature: N H. Bingham Date: 21.5.2017

Please return with exam marks (one report per marker)

Imperial College London
Department of Mathematics

Examiner's Comments

Exam: M3F22 Session: 2016-2107

Question 3

Please use the space below to comment on the candidates' overall performance in the exam. A brief paragraph highlighting common mistakes and parts of questions done badly (or well) is sufficient. Do not refer to individual candidates. The purpose of this exercise is to provide guidance to the external examiners, and to the candidates themselves, on how you feel the cohort faired. Your comments will be available to students online.

This was on Brownian motion (definition, covariance, scaling), and the suitability of BM as a model for driving noise in financial modelling.

The BM theory was reasonably well done (though a number of people thought that a covariance has only one argument!)

In the last part: BM is scale-invariant, but the economic/financial world obviously is not. Discrete v. continuous time is not the point here.

Marker: N.H. BINGHAM

Signature: N.H. Bingham Date: 21.5.2017

Please return with exam marks (one report per marker)

Imperial College London
Department of Mathematics

Examiner's Comments

Exam: M3F22 Session: 2016-2107

Question 4

Please use the space below to comment on the candidates' overall performance in the exam. A brief paragraph highlighting common mistakes and parts of questions done badly (or well) is sufficient. Do not refer to individual candidates. The purpose of this exercise is to provide guidance to the external examiners, and to the candidates themselves, on how you feel the cohort faired. Your comments will be available to students online.

This was on volatility.

Hardly anyone defined volatility, despite being explicitly asked to do so at the beginning of the question.

The middle parts were quite well done.

In the last part, trading volume — an aggregate quantity — tended to be confused with sizes of individual trades.

Marker: N.H. BINGHAM

Signature: N.H. Bingham Date: 21 · 5 · 2017

Please return with exam marks (one report per marker)