

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2007

EEE/ISE PART II: MEng, BEng and ACGI

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SIGNALS AND LINEAR SYSTEMS

Tuesday, 22 May 2:00 pm

Time allowed: 2:00 hours

There are **FOUR** questions on this paper.

Q1 is compulsory.

Answer Q1 and any two of questions 2-4.

Q1 carries 40% of the marks. Questions 2 to 4 carry equal marks (30% each).

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible

First Marker(s) : P.T. Stathaki, P.T. Stathaki

Second Marker(s) : A.G. Constantinides, A.G. Constantinides

1.

Consider the discrete-time system with the following input-output relationship

$$y[n] = 2x[n] - x[n+1] - x[n-1]. \quad (1)$$

- (i) Is this system linear and time invariant (LTI)? Justify your answer. [5]
- (ii) Find the impulse response $h[n]$ of the system and express it compactly in a mathematical form. Sketch the impulse response. [5]
- (iii) Find the step response $s[n]$ of the system and express it compactly in a mathematical form. Sketch the step response. [5]
- (iv) By performing the discrete time convolution $y[n] = x[n] * h[n]$ find the output $y[n]$ of the system defined in equation (1), when the input is the following signal [5]
- $$x[n] = \begin{cases} 1, & n = 0, 1 \\ 0, & \text{otherwise.} \end{cases}$$
- (v) Consider a discrete signal $x[n]$ with Discrete Time Fourier Transform (DTFT) $X(e^{j\omega})$. Find the time signal with Discrete Time Fourier Transform $j \frac{dX(e^{j\omega})}{d\omega}$. [5]
- (vi) Find the frequency response of the system defined in equation (1). Find the amplitude and the phase of the frequency response. [5]
- (vii) Consider a discrete signal $x[n]$ with z-transform $X(z)$. Find the z-Transform of the signal $x[n - n_0]$ with n_0 any integer. [5]
- (viii) Find the z-Transform of the output $y[n]$ of the system defined in equation (1) above, when the input is the function $x[n]$ defined in (iv). [5]

2.

- (a) Consider a discrete signal $x[n]$ that is periodic with fundamental period N and Fourier Series coefficients c_k . Find the Fourier Series coefficients of the signal $y[n] = x[n] - x[n-1]$ as functions of the Fourier Series coefficients c_k of the signal $x[n]$.

[7]

- (b) Let $x[n]$ be a discrete periodic signal with fundamental period $N=10$ and Fourier Series coefficients c_k . Let

$$x[n] = \begin{cases} 1, & 0 \leq n \leq 7 \\ 0, & 8 \leq n \leq 9 \end{cases}$$

Also, let

$$y[n] = x[n] - x[n-1].$$

- (i) Show that $y[n]$ has a fundamental period of 10.

[4]

- (ii) Determine the Fourier Series coefficients of $y[n]$ from its samples.

[8]

- (iii) Using the Fourier Series coefficients of $y[n]$ and the result of part (a) above, determine the Fourier Series coefficients c_k of $x[n]$, for $k \neq 0$. Determine c_0 separately from the samples of $x[n]$.

[11]

3.

- (a) Consider a continuous time signal $x(t)$ which is sampled uniformly with sampling period T_s to obtain the signal $x_s(t) = x(t) \sum_{k=-\infty}^{+\infty} \delta(t - kT_s)$, where $\delta(t)$ is the continuous time impulse function. Find the Fourier transform of the sampled signal $x_s(t)$.

[Hint: Use the Fourier Series representation of the function $\sum_{k=-\infty}^{+\infty} \delta(t - kT_s)$.]

[10]

- (b) Consider a continuous time signal $x(t)$ with Fourier transform $X(\omega) = \Pi\left(\frac{\omega}{4\pi \times 10^3}\right)$ where ω is the angular frequency and $\Pi(\omega)$ is defined as:

$$\Pi(\omega) = \begin{cases} 1 & |\omega| \leq 0.5 \\ 0 & \text{otherwise} \end{cases}$$

We sample $x(t)$ uniformly with sampling period T_s to obtain the signal $x_s(t) = x(t) \sum_{k=-\infty}^{+\infty} \delta(t - kT_s)$.

- (i) Sketch the Fourier transform of $x_s(t)$, $X_s(\omega)$, assuming $T_s = 0.1$ ms.

[10]

- (ii) Sketch the Fourier transform of $x_s(t)$, $X_s(\omega)$, assuming $T_s = \frac{4}{7}$ ms. Comment on the result.

[10]

4.

- (a) (i) Find the analytical expression and the region of convergence (ROC) of the z-Transform of the discrete signal $x[n] = a^n u[n+1]$, with a real and $u[n]$ the discrete unit step function defined as

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

[6]

- (ii) Find the analytical expression and the region of convergence (ROC) of the z-Transform of the discrete signal $x[n] = -a^n u[-n-2]$, with a real and $u[n]$ the discrete unit step function.

[6]

For parts (a) (i)-(a) (ii) you may wish to use the relationship $\sum_{n=0}^{+\infty} x^n = \frac{1}{1-x}$, if $|x| < 1$.

- (b) Consider a LTI system with input $x[n]$ and output $y[n]$ related by the difference equation

$$y[n] - \frac{9}{2}y[n-1] + 2y[n-2] = -7x[n]$$

Determine the impulse response and its z-Transform in the following three cases:

- (i) The system is causal.
- (ii) The system is stable.
- (iii) The system is neither stable nor causal.

Find the ROC of the z-Transform in each of the above cases.

[18]

E2.5 SIGNALS + LINEAR SYSTEMS -

SOLUTIONS - 2007

1.

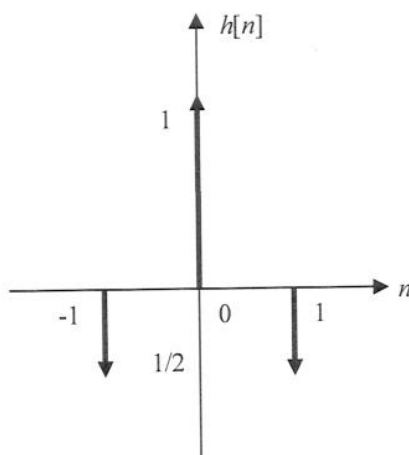
Consider the discrete-time system with the following input-output relationship

$$y[n] = 2x[n] - x[n+1] - x[n-1]. \quad (1)$$

- (i) Yes, since if the inputs $x_1[n]$ and $x_2[n]$ produce the outputs $y_1[n]$ and $y_2[n]$ respectively, the input $a_1x_1[n] + a_2x_2[n]$ will produce the output $a_1y_1[n] + a_2y_2[n]$. Furthermore, if the input $x_1[n]$ produces the output $y_1[n]$, the input $x_1[n - n_0]$ will produce the output $y_1[n - n_0]$.

[5]

- (ii) The impulse response of the system $h[n]$ is defined as the output of the system when the input is the impulse function $\delta[n]$. Therefore, $h[n] = \frac{2\delta[n] - \delta[n+1] - \delta[n-1]}{2}$. This function is shown below:



[5]

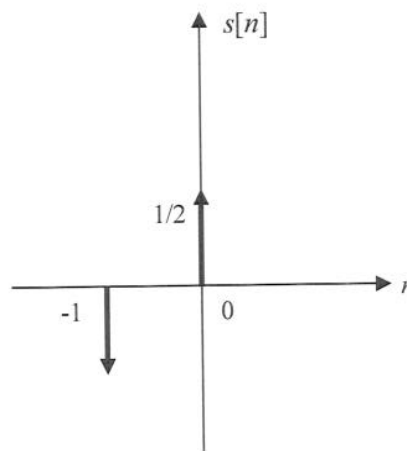
- (iii) Find the step response $s[n]$ of the system and express it compactly in a mathematical form. Sketch the step response. The step response of the system $s[n]$ is defined as the output of the system when the input is the unit step function $u[n]$. Therefore,

$$s[n] = \frac{2u[n] - u[n+1] - u[n-1]}{2}. \text{ This function is shown below:}$$

$$s[-1] = \frac{2u[-1] - u[0] - u[-2]}{2} = -\frac{1}{2}$$

$$s[0] = \frac{2u[0] - u[1] - u[-1]}{2} = \frac{1}{2}$$

$$s[n] = 0, n \geq 1$$



[5]

$$\begin{aligned}
 \text{(iv)} \quad y[n] &= x[n] * h[n] = x[n] * \frac{2\delta[n] - \delta[n+1] - \delta[n-1]}{2} = \frac{2x[n] - x[n+1] - x[n-1]}{2} \\
 y[-1] &= \frac{2x[-1] - x[0] - x[-2]}{2} = -\frac{1}{2} \\
 y[0] &= \frac{2x[0] - x[1] - x[-1]}{2} = \frac{1}{2} \\
 y[2] &= \frac{2x[2] - x[3] - x[1]}{2} = -\frac{1}{2}
 \end{aligned}
 \tag{5}$$

$$\text{(v)} \quad X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \Rightarrow j \frac{dX(e^{j\omega})}{d\omega} = j \sum_{n=-\infty}^{\infty} (-jn)x[n]e^{-j\omega n} \Rightarrow j \frac{dX(e^{j\omega})}{d\omega} = \sum_{n=-\infty}^{\infty} nx[n]e^{-j\omega n} .$$

Therefore, the signal with Discrete Time Fourier Transform $j \frac{dX(e^{j\omega})}{d\omega}$ is the signal $nx[n]$.

$$\begin{aligned}
 \text{(vi)} \quad Y(e^{j\omega}) &= 2X(e^{j\omega}) - e^{j\omega}X(e^{j\omega}) - e^{-j\omega}X(e^{j\omega}) . \quad \text{The frequency response is} \\
 H(e^{j\omega}) &= 2 - e^{j\omega} - e^{-j\omega} = 2 - 2\cos\omega = 2(1 - \cos\omega) . \quad \text{The amplitude response is the same and the} \\
 &\quad \text{phase response is zero}
 \end{aligned}
 \tag{5}$$

$$\text{(vii)} \quad \sum_{n=-\infty}^{\infty} x[n-n_0]z^{-n} = \sum_{n=-\infty}^{\infty} x[n-n_0]z^{-(n-n_0)}z^{-n_0} = z^{-n_0}X(z)$$

$$\text{(viii)} \quad Y(z) = (1+z^{-1}) \frac{2-z-z^{-1}}{2}$$

2.

$$(a) \quad x[n] = \sum_{k \in \langle N \rangle} c_k e^{jk\omega_0 n} \text{ and } x[n-1] = \sum_{k \in \langle N \rangle} c_k e^{jk\omega_0 (n-1)} = \sum_{k \in \langle N \rangle} e^{-jk\omega_0} c_k e^{jk\omega_0 n}.$$

Therefore, $x[n] - x[n-1] = \sum_{k \in \langle N \rangle} (1 - e^{-jk\omega_0}) c_k e^{jk\omega_0 n}$ and thus, the FS coefficients of the signal

$$x[n] - x[n-1] \text{ are } (1 - e^{-jk\omega_0}) c_k.$$

[7]

(b)

(i) $x[n]$ is periodic with period 10 and therefore $x[n-1]$ is periodic with period 10. Hence, $y[n] = x[n] - x[n-1]$ will be periodic with period 10.

[4]

$$(ii) \quad y[0] = x[0] - x[-1] = x[0] - x[9] = 1$$

$$y[n] = x[n] - x[n-1] = 0, \quad 1 \leq n \leq 7$$

$$y[8] = x[8] - x[7] = -1$$

$$y[9] = x[9] - x[8] = 0$$

$$c_k^y = \frac{1}{N} \sum_{n \in \langle N \rangle} y[n] e^{-jk\omega_0 n} = \frac{1}{10} (1 \cdot e^{-jk\omega_0 0} - e^{-jk\omega_0 8}) = \frac{1}{10} (1 - e^{-jk \frac{2\pi}{10} 8})$$

[8]

$$(iii) \quad c_k^y = \frac{1}{10} (1 - e^{-jk \frac{2\pi}{10} 8}) = (1 - e^{-jk \frac{2\pi}{10}}) c_k^x \Rightarrow c_k^x = \frac{1 - e^{-jk \frac{2\pi}{10} 8}}{1 - e^{-jk \frac{2\pi}{10}}}$$

$$c_0^x = \frac{1}{N} \sum_{n \in \langle N \rangle} x[n] = \frac{8}{10}$$

[11]

3.

(a) The function $\sum_{k=-\infty}^{+\infty} \delta(t - kT)$ is periodic with period T and therefore, it can be written using

Fourier Series representation as $\sum_{k=-\infty}^{+\infty} \delta(t - kT) = \sum_{k=-\infty}^{+\infty} c_k e^{jk\omega_s t}$, $\omega_s = \frac{2\pi}{T}$ with

$$c_k = \frac{1}{T} \int_{-\frac{T}{2}}^{+\frac{T}{2}} \delta(t) e^{-jk\omega_s t} dt = \frac{1}{T}. \text{ Therefore, } \sum_{k=-\infty}^{+\infty} \delta(t - kT) = \sum_{k=-\infty}^{+\infty} \frac{1}{T} e^{jk\omega_s t} = \frac{1}{T} \sum_{k=-\infty}^{+\infty} e^{-jk\omega_s t}, \omega_s = \frac{2\pi}{T}.$$

Hence, $x_s(t) = x(t) \sum_{k=-\infty}^{+\infty} \delta(t - kT_s) = \frac{1}{T} x(t) \sum_{k=-\infty}^{+\infty} e^{-jk\omega_s t}$. The Fourier transform of $x(t)e^{-jk\omega_s t}$ is

$$X(j\omega + jk\omega_s) \text{ and therefore, } X_s(j\omega) = \frac{1}{T} \sum_{k=-\infty}^{+\infty} X(j\omega + jk\omega_s)$$

[10]

(b)

(i) In that case $\omega_s = 2\pi f_s = 2\pi \frac{1}{T_s} = 10 \cdot 2\pi 10^3$ and the Fourier transform $X_s(\omega)$ is given in Figure 2 below.

[10]

(ii) In that case $\omega_s = 2\pi f_s = 2\pi \frac{1}{T_s} = 2\pi \frac{7}{4} 10^3 = 2 \cdot 2\pi 10^3 - \frac{1}{4} 2\pi 10^3$ and the Fourier transform $X_s(\omega)$ is given in Figure 3 below. As we see there is aliasing since the sampling frequency does not satisfy the Nyquist criterion according to which

$$\omega_s = 2\pi f_s \geq 2(2\pi \times 10^3) \Rightarrow f_s \geq 2 \times 10^3 \Rightarrow T_s = \frac{1}{f_s} \leq 0.5 \times 10^{-3}$$

[10]

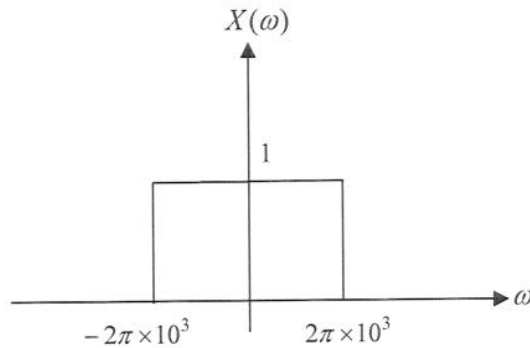


Figure 1: Fourier transform $X(\omega)$

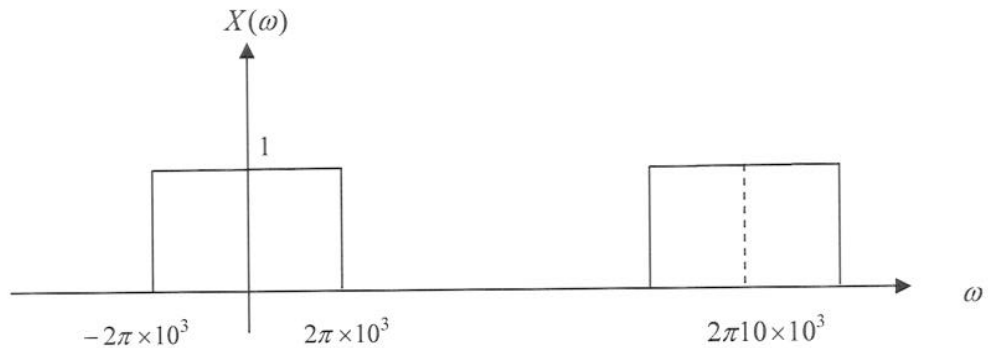


Figure 2: Fourier transform $X_s(\omega)$ for $T_s = 0.1$ ms

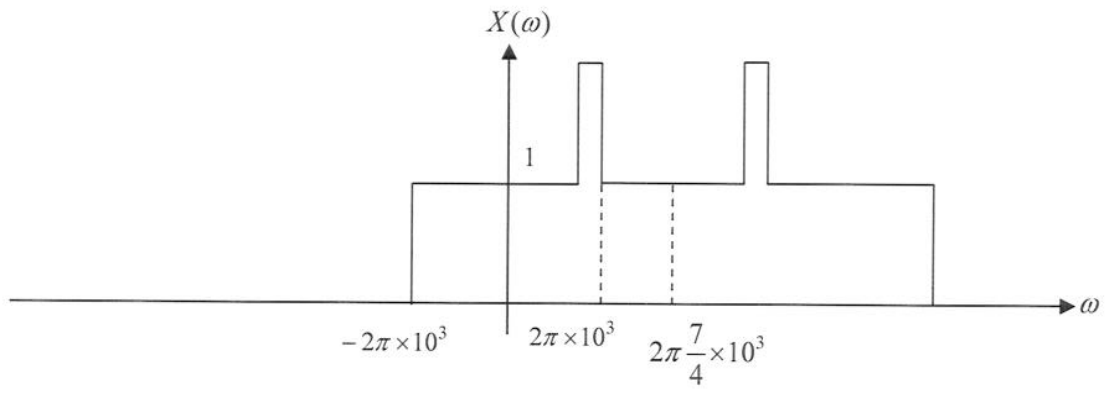


Figure 3: Fourier transform $X_s(\omega)$ for $T_s = \frac{4}{7}$ ms

[5]

4.

(a)

(i)

$$u[n+1] = \begin{cases} 1, & n+1 \geq 0 \Rightarrow n \geq -1 \\ 0, & \text{otherwise.} \end{cases}$$

$$X(z) = \sum_{n=-1}^{\infty} a^n z^{-n} = a^{-1}z + \sum_{n=0}^{\infty} a^n z^{-n} = a^{-1}z + \sum_{n=0}^{\infty} (az^{-1})^n = a^{-1}z + \frac{1}{1-az^{-1}} = \frac{z}{a} \frac{z}{z-a}, |a| < |z|$$

[6]

(ii)

$$u[-n-2] = \begin{cases} 1, & -n-2 \geq 0 \Rightarrow n \leq -2 \\ 0, & \text{otherwise.} \end{cases}$$

$$\begin{aligned} X(z) &= -\sum_{n=-\infty}^{-2} a^n z^{-n} = -\sum_{n=2}^{\infty} a^{-n} z^n = -\sum_{n=0}^{\infty} a^{-n} z^n + 1 + a^{-1}z = -\sum_{n=0}^{\infty} (a^{-1}z)^n + 1 + a^{-1}z \\ &= \frac{-1}{1-a^{-1}z} + 1 + a^{-1}z = \frac{z}{a} \frac{z}{z-a}, |z| < |a| \end{aligned}$$

[6]

(b)

$$Y(z)[1 - \frac{9}{2}z^{-1} + 2z^{-2}] = -7X(z) \Rightarrow H(z) = \frac{-7}{(1 - \frac{1}{2}z^{-1})(1 - 4z^{-1})} = \frac{1}{(1 - \frac{1}{2}z^{-1})} - \frac{8}{(1 - 4z^{-1})}$$

$$= \frac{z}{(z - \frac{1}{2})} - \frac{8z}{(z - 4)} = \frac{z}{(z - \frac{1}{2})} - 8 \frac{z}{(z - 4)}$$

(i) The system is causal.

$$h[n] = (\frac{1}{2})^n u[n] - 8(4)^n u[n], |z| > 4$$

(ii) The system is stable.

$$h[n] = (\frac{1}{2})^n u[n] + 8(4)^n u[-n-1], |z| > \frac{1}{2} \cap |z| < 4$$

(iii) The system is neither stable nor causal.

$$h[n] = -(\frac{1}{2})^n u[-n-1] + 8(4)^n u[-n-1], |z| < \frac{1}{2}$$

[18]