EE4-23
EE9CS2-1

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2011

MSc and EEE/ISE PART IV: MEng and ACGI

STABILITY AND CONTROL OF NON-LINEAR SYSTEMS

Monday, 16 May 10:00 am

Time allowed: 3:00 hours

There are SIX questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s) :	D. Angeli
	Second Marker(s) :	E.C. Kerrigan

1. Consider the following autonomous system with state variable $x = [x_1\, x_2]' \in \mathbb{R}^2$:

$$\dot{x}(t) = \begin{cases} A_1 x(t) & \text{if } x_1(t)x_2(t) \leq 0 \\ A_2 x(t) & \text{if } x_1(t)x_2(t) > 0 \end{cases} \tag{1.1}$$

where A_1 and A_2 are matrices defined according to:

$$A_1 = \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 3 \\ -3 & -1 \end{bmatrix}.$$

a) Is the system linear or nonlinear ? Does it fulfill the standard conditions for existence and/or unicity of solutions ? (justify your answers). [2]

b) Compute the equilibria of the system. Is the system linearizable around equilibria ? [2]

c) Next we proceed to a detailed study of the phase portrait. Compute the eigenvalues and eigenvectors of A_1 and sketch the phase portrait of the linear system $\dot{x} = A_1 x$. [2]

d) Sketch the phase portrait of the linear system $\dot{x} = A_2 x$. [2]

e) Merge the two phase-portraits previously sketched in order to obtain the phase-portrait of system (1.1). [3]

f) Exploiting the previous graphical analysis, infer whether the system's equilibria are asymptotically stable or not (justify your response). [4]

g) Write the system in polar coordinates; can you exploit this representation to carry out a stability analysis of the model ? [5]

2. Consider the following differential equation:

$$\ddot{y} = -y^3 + y - \dot{y}(y^2 - 2)$$

a) Find a state-space description of the system. [3]

b) Find the equilibria of the system. [2]

c) Linearize the system around each equilibrium and sketch the local phase-portrait around each equilibrium, (Hint: exploit eigenvector information) [5]

d) Consider the following subset of state-space:

$$\mathscr{S} = \{(y\dot{y}) : y^4/4 - y^2/2 + \dot{y}^2/2 < -1/8\}$$

Sketch its shape on the phase-plane (Hint: study the function $y^4/4 - y^2/2$; find the intersection of $\mathscr{S}$ with the y coordinate axis). [3]

e) Show that the set $\mathbb{R}^2 \backslash \mathscr{S}$ is forward invariant. [3]

f) Knowing that there exists a ball of large radius which is forward invariant, merge the previous local information into a consistent global phase-portrait of the system. [4]

3. Consider the following two-dimensional nonlinear system:

$$\begin{aligned}\dot{x}_1 &= -2x_1 - x_2^2 \\ \dot{x}_2 &= -2x_1x_2 - 4x_2^3\end{aligned}$$

Let $f(x)$ denote the function $(x_1, x_2)' \mapsto (-2x_1 - x_2^2, -2x_1x_2 - 4x_2^3)'$.

a) Show that there exists a function $V(x) : \mathbb{R}^2 \to \mathbb{R}$ such that $-\frac{\partial V}{\partial x}' = f(x)$, (this is sometimes called a *gradient* system); [4]

b) Show that $V(\cdot)$ can be chosen so that $V(0) = 0$, $V(\cdot)$ is positive definite and radially unbounded; [4]

c) Use the $V(\cdot)$ previously constructed to prove global asymptotic stability of the origin; [4]

d) Consider now a perturbed system $\dot{x} = f(x) + \varepsilon g(x)$ where

$$g(x) = \begin{bmatrix} -2x_1x_2 - 4x_2^3 \\ 2x_1 + x_2^2 \end{bmatrix}$$

and $\varepsilon \in \mathbb{R}$ is arbitrary. Show that the origin is still globally asymptotically stable. [4]

e) Consider next the following nonlinear system:

$$\dot{x} = g(x) + (x_1 + x_2)^2 f(x)$$

Explain why the origin is its only equilibrium and discuss its stability (Hint: notice that f and g are orthogonal). [4]

4. Consider the nonlinear control system with state $x = [x_1\, x_2\, x_3]' \in \mathbb{R}^3$ and scalar input u given below:

$$\begin{aligned}\dot{x}_1 &= \cos(x_2 - x_1)x_3 + u \\ \dot{x}_2 &= x_1 + \cos(x_2 - x_1)x_3 + u \\ \dot{x}_3 &= x_2 - x_1.\end{aligned}$$

a) Compute the relative degree with respect to the output $y = x_3$; [3]

b) Find a state feedback and a change of coordinates which globally linearizes the system from input to state; [3]

c) Design a feedback to achieve global asymptotic output tracking of any $\mathscr{C}^3$ reference signal $r(t)$; [4]

d) Consider next the output $y = x_2 + x_3 - x_1$. Compute the corresponding relative degree; [3]

e) Design a feedback which results in linear Input-Output dynamics with respect to the newly defined output signal y; [2]

f) Write the equations of the system in canonical form, highlighting the zero-dynamics (relative to the newly defined output signal y); [3]

g) Design a feedback to asymptotically track constant set-points r with the newly defined output signal y (justify your design); [2]

5. Consider the following nonlinear control system:

$$\begin{aligned}\dot{x}_1 &= \alpha_1 x_2 x_3 + u_1 \\ \dot{x}_2 &= \alpha_2 x_1 x_3 + u_2 \\ \dot{x}_3 &= \alpha_3 x_1 x_2 + u_3\end{aligned}$$

with state $x = [x_1\, x_2\, x_3]' \in \mathbb{R}^3$ and input $u = [u_1\, u_2\, u_3]' \in \mathbb{R}^3$. The parameters $\alpha_1, \alpha_2, \alpha_3$ are uncertain and fulfill the following equation: $\alpha_1 + \alpha_2 + \alpha_3 = 0$.

a) Show that this defines a passive (and lossless) system with respect to the output $y = x$; [5]

b) Consider next the following controller (basically a nonlinear PI controller):

$$\dot{z} = v, \qquad w = K_I z + K_P v^3$$

with state $z \in \mathbb{R}^3$, input $v \in \mathbb{R}^3$ and output $w \in \mathbb{R}^3$. The matrix K_I is any symmetric and positive definite matrix, whereas K_P is diagonal and positive definite. The symbol v^3 denotes $[v_1^3\, v_2^3\, v_3^3]'$, viz. the componentwise cubic power of vector v. Prove that this system is passive from input v to output w; [5]

c) Consider next the following feedback interconnection:

$$u = -w, \qquad v = y - r$$

where $r \in \mathbb{R}^3$ is a constant set-point. Compute the equilibria of the interconnected system. [5]

d) Show that for $r = 0$ the origin is a globally asymptotically stable equilibrium of the closed-loop system. [5]

6. Consider the following nonlinear system:

$$\begin{aligned} \dot{x}_1 &= -\text{sat}(x_1) + x_2 \\ \dot{x}_2 &= -x_2^3 - x_1^k + d \end{aligned}$$

with state $x = [x_1\, x_2] \in \mathbb{R}^2$, and input $d \in \mathbb{R}$. The variable k is a positive integer to be defined later on. The function sat denotes the standard piecewise affine saturation function

$$\text{sat}(x) = \begin{cases} x & \text{if } |x| \leq 1 \\ 1 & \text{if } x \geq 1 \\ -1 & \text{if } x \leq -1 \end{cases}$$

a) Set $k = 1$; prove that the system is Input-to-State stable (Hint: try to use $x'x$ as a candidate Lyapunov function); [5]

b) Let k be any odd integer; show that the system is Input-to-State stable; [5]

c) Set $k = 2$; compute the equilibria of the system; [4]

d) Show that the equilibrium 0 is unstable for $k = 2$; (Hint: using nullclines, find a forward invariant set next to it; study solutions initialized in this set; where do they converge ?). [6]

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SOLUTIONS: STABILITY AND CONTROL OF NONLINEAR SYSTEMS
MASTER IN CONTROL

1. Exercise

   a) The system is not linear; indeed for strictly positive x_1 we have, $f(x_1,0)+f(-x_1,0)=[-1,-3]'x_1-[0,-3]'x_1=[-x_1,0]\neq 0$, which contradicts linearity. The function $f(x_1,x_2)$ is not continuous, so existence and unicity of solutions are not guaranteed by the standard Caratheodory theorem for existence and unicity of solutions. To verify discontinuity just notice that for positive x_2:

$$\lim_{x_1\to 0^+} f(x_1,x_2)=[3x_2,-x_2]'\neq[3x_2,0]'=\lim_{x_1\to 0^-} f(x_1,x_2).$$

   b) Notice that $x\neq 0$ implies $A_1x\neq 0$ and $A_2x\neq 0$. Hence the only equilibrium is the origin. The system is not linearizable around the origin since $f(x_1,x_2)$ is not $\mathscr{C}^1$ (notice that $\partial f/\partial x$ is discontinuous at $x_1=0$).

   c) The eigenvalues of A_1 are $\lambda=-1\pm 3j$. Hence the phase portrait corresponding to A_1 is that of a stable focus. Notice that the focus is in canonical form, hence $x_1^2+x_2^2$ is strictly decreasing along solutions of $\dot{x}=A_1x$. See Fig. 1.1.

   d) The eigenvalues of A_2 are $\lambda=\pm 3j$. Hence, the phase portrait corresponding to A_2 is that of a center. Notice that the center is in canonical form, hence $x_1^2+x_2^2$ is constant along solutions of $\dot{x}=A_2x$. See Fig. 1.2.

   e) Merging the two phase portraits results in the qualitative portrait shown in Fig. 1.3.

   f) The equilibrium is indeed asymptotically stable. Lyapunov stability is trivial since:

$$x_1^2(t)+x_2^2(t)\leq x_1^2(0)+x_2^2(0)$$

   Attractivity follows since each time the solution transits in the first or third quadrant, the modulus of $x_1^2+x_2^2$ reduces by a (constant) factor $\alpha<1$.

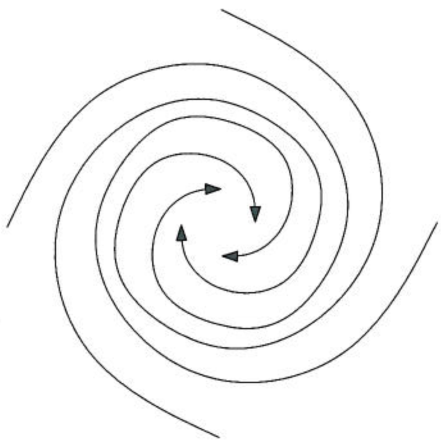

Figure 1.1 Phase portrait of $\dot{x}=A_1x$

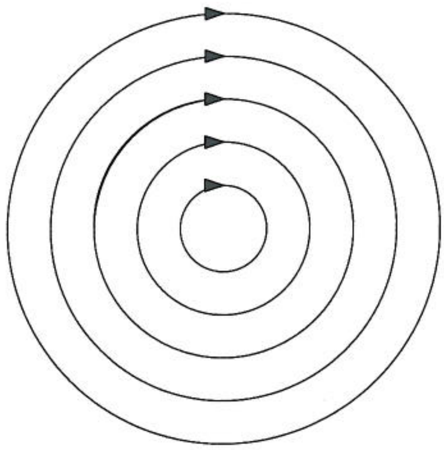

Figure 1.2 Phase portrait of $\dot{x} = A_2 x$

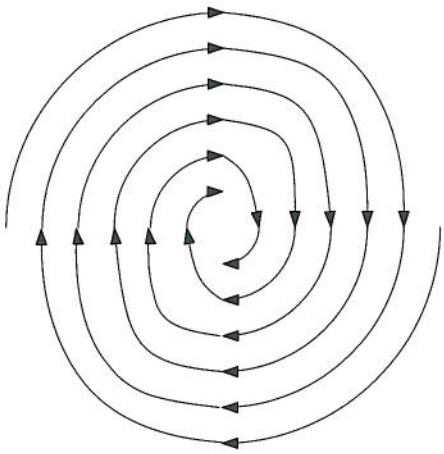

Figure 1.3 Phase portrait of the nonlinear system

g) In polar coordinates,

$$r = \sqrt{x_1^2 + x_2^2} \qquad x_1 = r\cos(\theta) \qquad x_2 = r\sin(\theta)$$

For $x_1 x_2 \geq 0$ we have:

$$\dot{r} = \frac{1}{\sqrt{x_1^2 + x_2^2}}(\dot{x}_1 x_1 + \dot{x}_2 x_2) = -\sqrt{x_1^2 + x_2^2} = -r$$

For $x_1 x_2 < 0$ instead $\dot{r} = 0$. Computing the derivative of θ yields (regardless of the quadrant):

$$\dot{\theta} = \frac{\dot{x}_2 x_1 - \dot{x}_1 x_2}{x_1^2 + x_2^2} = -3.$$

Hence, $\theta(t) = \theta(0) - 3t$, while:

$$r(t) = e^{-\int_0^t [\text{sign}(\sin(\theta(t))\cos(\theta(t)))+1]/2 dt} r(0).$$

Hence, $r(t) \to 0$ exponentially which shows asymptotic stability.

2. Exercise

a) We let $x = [y, \dot{y}]' = [x_1, x_2]'$ so that a state space description can be obtained by letting:

$$\dot{x} = [\dot{x}_1, \dot{x}_2]' = [\dot{y}, \ddot{y}]' = [x_2, -x_1^3 + x_1 - x_2(x_1^2 - 2)]'.$$

Hence, $\dot{x} = f(x)$ with:

$$f(x) = \begin{bmatrix} x_2 \\ -x_1^3 + x_1 - x_2(x_1^2 - 2) \end{bmatrix}.$$

b) The equilibria of the system are found for $f(x) = 0$. This implies $x_2 = 0$ and therefore $-x_1^3 + x_1 = 0$. Consequently there are 3 equilibria:

$$x_{e1} = [0, 0]', \quad x_{e2} = [-1, 0]', \quad x_{e3} = [1, 0]'.$$

c) Computing the Jacobian of $f(x)$ yields:

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -3x_1^2 + 1 - 2x_1 x_2 & 2 - x_1^2 \end{bmatrix}.$$

Linearizing around x_{e1} yields:

$$\dot{\delta x} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \delta x$$

The associated characteristic polynomial is $s^2 - 2s - 1$ and eigenvalues are $\lambda_{1,2} = 1 \pm \sqrt{2}$. As eigenvalues have opposite sign and are real x_{e1} is a saddle point. The associated eigenvectors are:

$$v_+ = [1, 1+\sqrt{2}]' \qquad v_- = [1, 1-\sqrt{2}]'$$

Linearizing around x_{e2} and x_{e3} yields:

$$\dot{\delta x} = \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix} \delta x$$

The associated characteristic polynomial is $s^2 - s + 2$ with roots: $\lambda_{1,2} = \frac{1 \pm \sqrt{7}j}{2}$. Hence the equilibria x_{e2}, x_{e3} are unstable foci.

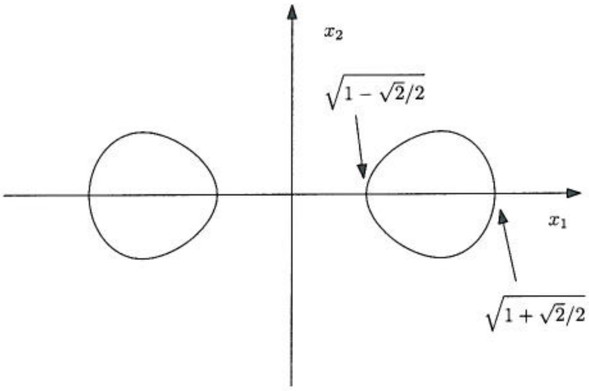

Figure 2.1 A sketch of the set $\mathscr{S}$

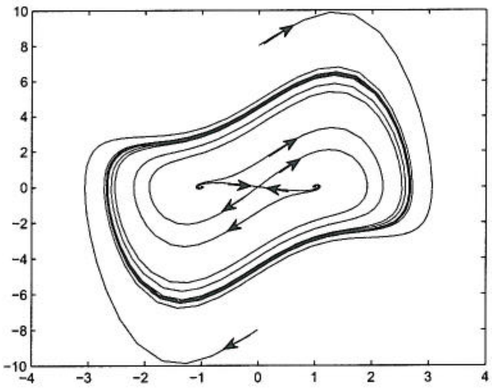

Figure 2.2 Phase-portrait of the system

d) In order to understand the shape of $\mathscr{S}$ notice that the function $y^4/4 - y^2/2$ has two global minima in $y = \pm 1$ respectively and a local maximum in $y = 0$. The set $\mathscr{S}$ meets the $x_2 = 0$ axis for $x_1^4/4 - x_1^2/2 = -\frac{1}{8}$, that is for $x_1^2 = 1 \pm \sqrt{2}/2$. Hence, the set $\mathscr{S}$ has two connected components which approximately look like ovals. As the definition of $\mathscr{S}$ only involves even powers of y and $\dot{y}$ the set is symmetric with respect to both cartesian axes. See Fig. 2.1.

e) Take any point in the boundary of $\mathscr{S}$. Clearly, $y^2 \in [1-\sqrt{2}/2, 1+\sqrt{2}/2]$. Hence $y^2 - 2 < 0$ for all such points. Derive now the function $y^4/4 - y^2/2 + \dot{y}^2/2$ along solutions of the system. We get:

$$\frac{d}{dt}\frac{y^4}{4} - \frac{y^2}{2} + \frac{\dot{y}^2}{2} = y^3\dot{y} - y\dot{y} + \dot{y}\ddot{y} = \dot{y}[\ddot{y} + y^3 - y] = -\dot{y}^2(y^2 - 2) \geq 0$$

This means that the function $\frac{y^4}{4} - \frac{y^2}{2} + \frac{\dot{y}^2}{2} + 1/8$ is non-decreasing along solutions of the system and that initial conditions in $\mathbb{R}^2 \setminus \mathscr{S}$ give rise to solutions contained in $\mathbb{R}^2 \setminus \mathscr{S}$.

f) The phase portrait of the system is shown in Fig. 2.2. It is not the only global portrait compatible with the collected local information.

3. Exercise

a) The function $V(x) = x_1^2 + x_1 x_2^2 + x_2^4$ is such that $f(x) = -\frac{\partial V}{\partial x}$;

b) Notice that $V(x)$ is continuously differentiable (it is actually a polynomial), $V(0) = 0$, moreover:

$$V(x) = (x_1^2 + x_1 x_2^2 + x_2^4/4) + 3x_2^4/4 = (x_1 + x_2^2/2)^2 + 3x_2^4/4 > 0$$

hence $V(x)$ is positive definite. We show next that it is also radially unbounded. To this end notice that $|x| \to +\infty$ implies either $|x_1| \to +\infty$ or $|x_2| \to +\infty$. In the latter case, clearly $V(x) \geq 3/4x_2^4 \to +\infty$. Assume instead $|x_2|$ is bounded, then:

$$V(x) \geq (x_1 + x_2^2/2)^2 \to +\infty$$

as $|x_1| \to +\infty$ (regardless of the bound on $|x_2|$).

c) Notice that $\frac{\partial V}{\partial x} = 0$ iff $x = 0$. Moreover:

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x) = -|f(x)|^2$$

As $x = 0$ is the only equilibrium $\dot{V}$ is negative definite. Hence, by the Lyapunov stability theorem we can conclude global asymptotic stability of the equilibrium at the origin.

d) Since $\frac{\partial V}{\partial x} g(x) = 0$ we have that along solutions of the perturbed system:

$$\dot{V}(x) = \frac{\partial V}{\partial x}[f(x) + \varepsilon g(x)] = \frac{\partial V}{\partial x} f(x) = -|f(x)|^2 < 0$$

Hence the equilibrium in 0 is globally asymptotically stable also for the perturbed system.

e) Notice that $f(x)$ and $g(x)$ are orthogonal for all $x \neq 0$, whereas they are both 0 only for $x = 0$. Hence, the only equilibrium is for $x = 0$. We may again use $V(x)$ as a candidate Lyapunov function:

$$\dot{V}(x) = \frac{\partial V}{\partial x}[g(x) + (x_1 + x_2)^2 f(x)] = -(x_1 + x_2)^2 |f(x)|^2 \leq 0$$

Notice that $\dot{V}(x)$ is negative semi-definite. In fact, $\dot{V}$ vanishes on the line $L := \{[x_1, x_2] : x_1 + x_2 = 0\}$. To see the largest invariant set contained in L we may compute:

$$\dot{x}_1 + \dot{x}_2|_{x_1 + x_2 = 0} = -2x_1 - x_2^2 + 2x_1 x_2 + 4x_2^3 = 4x_2^3 - 3x_2^2 + 2x_2$$

Notice that this is zero iff $x_2 = 0$ (two additional roots are complex conjugate). Hence $x_1 = 0$. Thus the origin is the largest invariant set contained in L and the equilibrium is globally asymptotically stable by the Lasalle's criterion.

4. Exercise

a) Taking derivatives of y yields:

$$\dot{y} = x_2 - x_1 \qquad \ddot{y} = x_1 \qquad y^{(3)} = \cos(x_2 - x_1)x_3 + u$$

Hence the (globally defined) relative degree is 3.

b) One may let $u = -\cos(x_2 - x_1)x_3 + v$. Indeed, taking $\xi = [y, \dot{y}, \ddot{y}]'$ yields the following linear dynamics:

$$\dot{\xi} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xi + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} v$$

c) One may get asymptotic tracking of time-varying smooth reference trajectories just by letting:

$$v(t) = -3(\ddot{y}(t) - \ddot{r}(t)) - 3(\dot{y}(t) - \dot{r}(t)) - (y(t) - r(t)) + r^{(3)}(t)$$

Letting e denote the tracking error $e(t) = y(t) - r(t)$, we have the following closed-loop dynamics:

$$e^{(3)} + 3\ddot{e} + 3\dot{e} + e = 0$$

which corresponds to an asymptotically stable linear system.

d) Given the new output $y = x_2 + x_3 - x_1$, we have:

$$\dot{y} = x_2 - x_1 + \dot{x}_2 - \dot{x}_1 = x_2 \qquad \ddot{y} = x_1 + \cos(x_2 - x_1)x_3 + u.$$

Hence, the relative degree with respect to this new output is 2 (globally defined).

e) The feedback $u = -x_1 - \cos(x_2 - x_1)x_3 + v$ gives linear Input-Output dynamics.

f) Let $z = x_2 - x_1$; we may consider the following set of coordinates $[y\dot{y}z]'$. Taking derivatives yields:

$$\ddot{y} = v \qquad \dot{z} = \dot{x}_2 - \dot{x}_1 = x_1 = \dot{y} - z$$

Notice that the zero-dynamics are Input-to-State stable (they are linear and asymptotically stable).

g) One may define $v = -2\dot{y} - (y - r)$ Accordingly the closed-loop equations read: $\ddot{y} = -2\dot{y} - (y - r)$ and $\dot{z} = -z + \dot{y}$. The first one implies that $y(t)$ asymptotically tracks any constant reference r and $\dot{y} \to 0$. By virtue of the second equation it follows that $z(t)$ is bounded and asymptotically tends to 0, thus showing asymptotic stability of the equilibrium $[r, 0, 0]'$.

5. Exercise

a) Let $V(x) = \frac{1}{2}x'x$ be the candidate storage function. We have:

$$\dot{V}(x) = x'\dot{x} = (\alpha_1 + \alpha_2 + \alpha_3)x_1x_2x_3 + y'u = y'u$$

Hence the system is lossless.

b) For the PI controller we have:

$$w'v = z'K_I v + v'K_P v^3 = z'K_I \dot{z} + v'K_P v^3 \geq \frac{d}{dt}\frac{1}{2}z'K_I z$$

Hence the system is passive from input v to output w.

c) To find equilibria we have to solve the equations $\dot{z} = 0$ and $\dot{x} = 0$ simultaneously.

$$\dot{z} = 0 \Rightarrow v = 0 \Rightarrow r = y = x.$$

Hence $x = r$. From the equation $\dot{x} = 0$ we have:

$$z = K_I^{-1}\begin{bmatrix} \alpha_1 r_2 r_3 \\ \alpha_2 r_1 r_3 \\ \alpha_3 r_1 r_2 \end{bmatrix}.$$

d) Consider the candidate Lyapunov function: $W(x,z) = \frac{1}{2}x'x + \frac{1}{2}z'K_I z$. Clearly W is positive definite and radially unbounded (with respect to the state $[x,z]$ of the closed-loop system). Moreover, taking derivatives of W along solutions yields:

$$\dot{W} = y'u + z'K_I\dot{z} = y'u + w'v - v'K_P v^3 = -x'K_P x^3 \leq 0$$

Hence $\dot{W}$ is negative semidefinite. In particular $\dot{W}$ vanishes on the set $K = \{[x,z] : x = 0\}$. In order to prove global asymptotic stability of the origin we need to find the larges invariant set contained in K. Notice that:

$$x = 0 \text{ and } \dot{x} = 0 \Rightarrow K_I z = 0 \Rightarrow z = 0$$

Hence the origin is the largest invariant set contained in K and by virtue of the Lasalle's invariance principle, 0 is a globally asymptotically stable equilibrium.

6. Exercise

a) Take the candidate Lyapunov function $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$. Deriving it along solutions of the system yields:

$$\begin{aligned} \dot{V}(x,d) &= x_1\dot{x}_1 + x_2\dot{x}_2 = -\text{sat}(x_1)x_1 + x_1x_2 - x_2^4 - x_1x_2 + x_2 d \\ &= -\text{sat}(x_1)x_1 - x_2^4 + x_2 d \leq -\text{sat}(x_1)x_1 - x_2^4 + |x_2||d| \end{aligned}$$

We consider separately the two cases:

i) $-x_2^4/2 + |x_2||d| \leq 0$; in this case:

$$\dot{V} \leq -\text{sat}(x_1)x_1 - x_2^4/2$$

ii) $-x_2^4/2 + |x_2||d| \geq 0$; in this case $|x_2|^3 \leq 2|d|$; hence:

$$\dot{V} \leq -\text{sat}(x_1)x_1 - x_2^4 + |d|\sqrt{3}2|d|.$$

Combining the 2 inequalities we have:

$$\dot{V} \leq -- \operatorname{sat}(x_1)x_1 - x_2^4/2 + |d|\sqrt{3}2|d|.$$

Notice that $\operatorname{sat}(x_1)x_1 + x_2^4/2$ is positive definite and radially unbounded. Hence there exists a class $\mathscr{K}_\infty$ function α such that

$$\alpha(|x|) \leq \operatorname{sat}(x_1)x_1 + x_2^4/2.$$

Exploiting this last consideration we obtain:

$$\dot{V} \leq -\alpha(|x|) + |d|\sqrt{3}2|d|$$

which shows that V is an ISS-Lyapunov function and therefore the system is Input-to-State Stable.

b) The function $V(x)$ used previously does not give the desired inequality for an arbitrary odd k. Hence we modify $V(x)$ as follows:

$$\tilde{V}(x) = \frac{x_1^{k+1}}{k+1} + \frac{x_2^2}{2}.$$

As k is an odd positive integer $\tilde{V}$ is positive definite and radially unbounded. Deriving it along solutions of the system yields:

$$\begin{aligned}\dot{\tilde{V}} &= x_1^k\dot{x}_1 + x_2\dot{x}_2 = -\operatorname{sat}(x_1)x_1^k + x_1^k x_2 - x_2^4 - x_2x_1^k + x_2 d \\ &= -\operatorname{sat}(x_1)x_1^k - x_2^4 + x_2 d \leq -\operatorname{sat}(x_1)x_1^k - x_2^4 + |x_2||d|\end{aligned}$$

We consider separately the two cases:

i) $-x_2^4/2 + |x_2||d| \leq 0$; in this case:

$$\dot{\tilde{V}} \leq -\operatorname{sat}(x_1)x_1^k - x_2^4/2$$

ii) $-x_2^4/2 + |x_2||d| \geq 0$; in this case $|x_2|^3 \leq 2|d|$; hence:

$$\dot{\tilde{V}} \leq -\operatorname{sat}(x_1)x_1^k - x_2^4 + |d|\sqrt{3}2|d|.$$

Notice that $\operatorname{sat}(x_1)x_1^k + x_2^4/2$ is positive definite and radially unbounded. Hence there exists a class $\mathscr{K}_\infty$ function $\tilde{\alpha}$ such that

$$\tilde{\alpha}(|x|) \leq \operatorname{sat}(x_1)x_1^k + x_2^4/2.$$

Exploiting this last consideration we obtain:

$$\dot{\tilde{V}} \leq -\tilde{\alpha}(|x|) + |d|\sqrt{3}2|d|$$

which shows that $\tilde{V}$ is an ISS-Lyapunov function and therefore the system is Input-to-State Stable.

c) Let us look at the nullclines of the system. These are shown in Fig. 6.1. Notice that they only have 2 intersections, at the equilibria $x_{e1} = [0,0]'$ and $x_{e2} = [-1,-1]'$.

d) Consider the small lens shaped region in between x_{e1} and x_{e2} and the two nullclines. We highlighted the direction of the vector-field on the boundary of such a region. Clearly, this is a forward invariant set. Any solution initialized in its interior stays there for ever. Moreover it fulfills $\dot{x}_1 < 0$ and $\dot{x}_2 < 0$. Which means that both x_1 and x_2 are monotone decreasing and since they are also bounded they admit a limit which is necessarily an equilibrium. Hence, $x(t) \to x_{e2}$ for all initial conditions in the considered region. This shows that the equilibrium x_{e1} is unstable as we may pick $x(0)$ arbitrarily close to x_{e1} and still get a solution which moves out of a fixed neighborhood of x_{e1} (for instance a ball of radius $1/2$ centered at the origin).

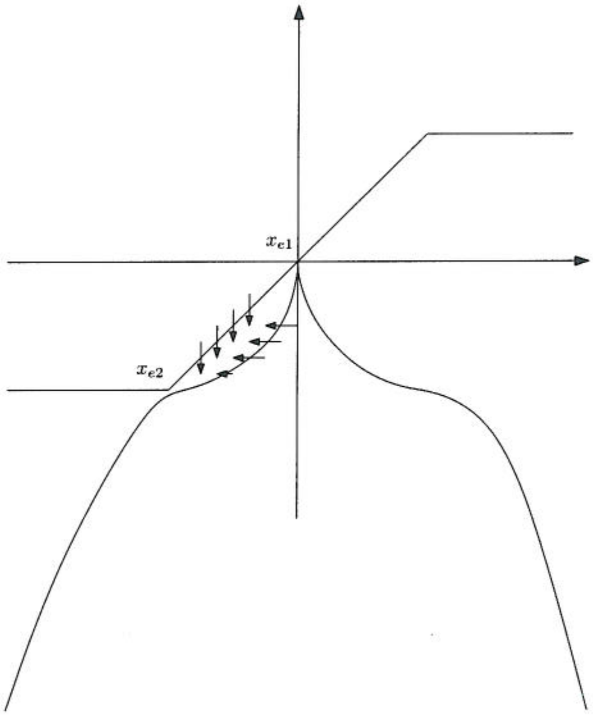

Figure 6.1 Nullclines and equilibria