IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2018

EEE PART II: MEng, BEng and ACGI

Corrected copy

MATHEMATICS 2A (E-STREAM AND I-STREAM)

Thursday, 24 May 2:00 pm

Time allowed: 1:30 hours

There are FOUR questions on this paper.

Answer ALL questions

NO CALCULATORS ALLOWED. Table of Laplace transforms included

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s) :	D. Nucinkis
	Second Marker(s) :	B. Clerckx

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EE2-08A MATHEMATICS

1. a) Show that the function $u(x,y) = 2\cos x \cosh y + \sin x \sinh y$ satisfies Laplace's equation and integrate the Cauchy-Riemann equations to find its harmonic conjugate $v(x,y)$. [5]

   b) Hence obtain $f(z) = u(x,y) + iv(x,y)$ where f is an analytic function of $z = x + iy$, simplifying as much as possible. [4]

2. a) The complex function
$$F(z) = \frac{1}{z(z^2+1)}$$
has three simple poles. Find the residues at the poles lying in the upper half of the complex plane and at the origin. [4]

   b) Consider the contour integral $I = \oint_C \frac{1}{z(z^2+1)}\, dz$,

   where the closed contour C is taken to be the union of a semi-circle of radius R, lying in the upper half-plane, with a small semi-circle of radius r indented into the lower half-plane, both centred at $z = 0$ and the real intervals $[-R, -r]$ and $[r, R]$.

   i) Show that the contribution to I from the indented semi-circle of radius r, in the limit $r \to 0$, is $i\pi$.

   ii) Use Jordan's lemma to show that the contribution to I from the arc of the larger semi-circle, in the limit $R \to \infty$, is zero.

   iii) Hence use your results from (a) and the Residue Theorem to obtain
$$\int_{-\infty}^{\infty} \frac{1}{x(x^2+1)}\, dx,$$

   Be sure to explain carefully your evaluation of I, the limiting behaviour of r and R, and your use of Cauchy's Residue Theorem and Jordan's lemma. [10]

Recall that the residue of a complex function $F(z)$ at a pole $z = a$ of multiplicity m is given by the expression
$$\lim_{z\to a}\left\{\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[(z-a)^m F(z)\right]\right\}.$$

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3. a) Given the integral of the real variable θ,

$$I = -\int_0^{2\pi} \sin[\cos(\theta) - \theta] e^{-\sin(\theta)}\, d\theta .$$

use the substitution $z = e^{i\theta}$ to show that I is equal to the real part of the complex contour integral

$$\oint_C \frac{e^{iz}}{z^2}\, dz ,$$

where the contour C is the unit circle in the complex plane. [5]

b) Using Cauchy's residue theorem, or otherwise, calculate I. [4]

4. Consider the following second-order ODE

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 13y = f(x)$$

for some input function $f(x)$ and initial conditions $y(0) = y'(0) = 0$.

a) Take Laplace transforms to write the ODE in the form [3]

$$\bar{y}(s) = \bar{h}(s)\bar{f}(s) ,$$

where $\mathscr{L}[y(x)] = \bar{y}(s)$ and $\mathscr{L}[f(x)] = \bar{f}(s)$.

b) Hence, use the Laplace convolution and shift theorems to write the solution in the form

$$y(x) = h(x) \star f(x) ,$$

where $h \star f$ is the convolution of the functions $f(x)$ and $h(x)$, and $\mathscr{L}[h(x)] = \bar{h}(s)$. [5]

c) If $f(x) = e^{-3x}$, obtain the solution $y(x)$ by solving the integral found in part (b). [4]

d) With $f(x) = e^{-3x}$, take Laplace transforms of the ODE and use partial fractions and the shift theorem to take the inverse Laplace transform and find $y(x)$, and thus confirm the result obtained in (c). [6]

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Table of Laplace transforms

$f(t)$	$\mathcal{L}\{f(t)\} \equiv F(s)$
A	$\frac{A}{s}$, $\quad \mathrm{Re}(s) > 0$
e^{at}	$\frac{1}{s-a}$, $\quad \mathrm{Re}(s) > a$
$t^n, \quad n = 1, 2 \ldots$	$\frac{n!}{s^{n+1}}$, $\quad \mathrm{Re}(s) > 0$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$, $\quad \mathrm{Re}(s) > 0$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$, $\quad \mathrm{Re}(s) > 0$
$e^{at} f(t)$	$F(s-a)$
$t^n f(t)$	$(-1)^n \frac{d^n F}{ds^n}$
$\frac{df}{dt}$	$sF(s) - f(0)$
$\frac{d^2 f}{dt^2}$	$s^2 F(s) - sf(0) - \frac{df}{dt}(0)$
$H(t-a)$	$\frac{e^{-as}}{s}$
$\delta(t-a)$	e^{-as}, $\quad a > 0$
$f(t-a)H(t-a)$	$e^{-as} F(s)$