

B.ENG. and M.ENG. EXAMINATIONS 2013

PART I : MATHEMATICS 1 (ELECTRICAL ENGINEERING)

Date Thursday 30th May 2013 10.00 - 12.00

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.

Answer Question 1 and THREE of the remaining five

Answer Section A and Section B in different answerbooks.

Question 1 carries twice the marks of each of the other questions.

CALCULATORS MAY NOT BE USED.

A mathematical formulae sheet is provided.

[Before starting, please make sure that the paper is complete; there should be 6 pages, with a total of SIX questions. Ask the invigilator for a replacement if your copy is faulty.]

1. (i) Find all roots of

$$z^4 + 2z^2 + 2 = 0$$

in polar form, i.e. $z = r \exp(i\varphi)$ with $r, \varphi \in \mathbb{R}$.

- (ii) Express

$$\exp(i\pi/3) + \exp(-i2\pi/3)$$

in Cartesian form, i.e. $z = x + iy$ with $x, y \in \mathbb{R}$.

- (iii) Find
- q
- so that the limit

$$\lim_{x \rightarrow \infty} x^q (\sqrt{x} - \sqrt{x-1})$$

is finite and non-zero.

*Do **not** use L'Hôpital's rule.*

- (iv) Find the limit

$$\lim_{x \rightarrow 0} \frac{\sin(x) - x}{x^3}$$

*You **can** use L'Hôpital's rule.*

You can use $\lim_{x \rightarrow 0} \sin(x)/x = 1$.

- (v) Differentiate

$$(\tan(x))^{\exp(x)}$$

- (vi) Integrate

$$\int \tan(x) \, dx$$

- (vii) Integrate

$$\int_0^1 \tan(x)^2 \, dx$$

Q1 CONTINUES ON THE NEXT PAGE

(viii) Find the Taylor expansion of

$$\frac{1}{\exp(x) + 1}$$

about $x = 0$ to first order (up to and including the term linear in x) and state the remainder term $R_2(x)$.

(ix) Find the general solution of the following first order ODE:

$$2y'(x) = \frac{y(x)^2}{x^2} + 1$$

(x) Find the general solutions of the following second order ODE:

$$y''(x) + y(x) = \sin(2x)$$

2. Find $\frac{dy}{dx}$ as a function of x in each of the following cases :

(i) $y(x) = \frac{\exp(2x)}{\sin(3x^2)} ;$

(ii) $y(x) = (1 + x^2) \tan^{-1}(x) ;$

Note: $\tan^{-1}(x)$ denotes the inverse tan function

(iii) $y(x) = \frac{x + 3}{(x + 1)(x - 2)} ;$

(iv) $y(x) = x^{(x^2)} .$

Note: This is x raised to the power x^2 .

(v) Find the following n th derivative

$$\frac{d^n}{dx^n} (x \ln(x))$$

for $n \geq 2$.

Note: Simplify the result as much as possible.

PLEASE TURN OVER

3. Evaluate the following limits *without using L'Hôpital's rule unless specified*:

(i) $\lim_{x \rightarrow 1} \frac{x}{\sin(x)} ;$

Do not use L'Hôpital's rule.

(ii) $\lim_{x \rightarrow 1} \frac{x-1}{\sin(x\pi)} ;$

Do not use L'Hôpital's rule.

(iii) $\lim_{x \rightarrow \infty} x^{2/3} \left((1+x)^{1/3} + (1-x)^{1/3} \right) ;$

Consider only real roots. Do not use L'Hôpital's rule.

(iv) $\lim_{x \rightarrow 1} \frac{x^n - 1}{x^m - 1}$ for integer $n, m > 0$;

*You **may** use L'Hôpital's rule.*

(v) $\lim_{x \rightarrow 1} \frac{\exp(x) - \exp(1)(1+x^2)/2}{(x-1)^3} ;$

Do use L'Hôpital's rule.

4. (i) Integrate $\int_0^{2\pi} \sin(mx) \sin(nx) dx$ for positive integers $0 < n, m \in \mathbb{N}$.

Note: The case $n = m$ requires special attention.

(ii) Show that $\int_0^\infty x^n \exp(-x^2) dx = \frac{n-1}{2} \int_0^\infty x^{n-2} \exp(-x^2) dx$ for any positive integer n , i.e. $0 < n \in \mathbb{N}$.

(iii) Integrate $\int \frac{(x-1)(x^2 + \frac{5}{2}x + 1)}{2x^2 - 3x + 1} dx$.

(iv) Integrate $\int \frac{dx}{2 + \cos(x)}$.

5. (i) Express in polar form

$$(1 - i)^3 \quad \text{and} \quad (\sqrt{3}i + 1)^{1/2} .$$

- (ii) Find the real part of $\left(\frac{1}{\sin(\theta) + i \cos(\theta)} \right)^4$ in terms of $\cos(2\theta)$.

- (iii) Find all solutions of the equation

$$\cos(z) = \frac{5}{3}$$

in the Cartesian form, i.e. $z = x + iy$ with $x, y \in \mathbb{R}$.

6. (i) Find the solution $y(x)$ of the differential equation

$$\frac{dy}{dx} = -xy(x) .$$

- (ii) Find the general solution $y(x)$ of the differential equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{1}{2}y(x) = \sin(x) .$$

- (iii) Find the solution $y(x)$ of the differential equation

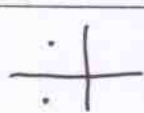
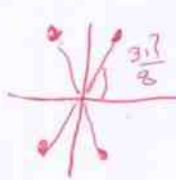
$$\frac{d^2y}{dx^2} = -xy(x)$$



by expressing $y(x)$ in a MacLaurin series, $y(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$,

and determining the recurrence relation for a_n . Determine the value of a_2 .

*Note: Do **not** attempt to solve the recurrence relation.*

END OF PAPER

	EXAMINATION QUESTIONS/SOLUTIONS 2012-13	Course
Question 1	TOPIC <i>General</i>	Marks & seen/unseen
Parts	<p>(i) $z^4 + 2z^2 + 2 = 0$ </p> <p>$\Rightarrow (z^2 + 1)^2 = -1 \Rightarrow z^2 = \pm 2^{0} - \frac{1}{2} i (\pi \pm \frac{\pi}{4})$</p> <p>$= \sqrt{2} e^{i(\frac{\pi}{2} \pm \frac{\pi}{4})}$</p> <p>$\Rightarrow z = 2^{\frac{1}{4}} e^{i\varphi}$ with $\varphi = \frac{3}{8}\pi, \frac{5}{8}\pi, \frac{11}{8}\pi, \frac{13}{8}\pi$</p> <p>or $z = \pm 2^{\frac{1}{4}} e^{i(\frac{\pi}{2} \pm \frac{\pi}{8})}$ </p> <p>(ii) $e^{i\frac{2\pi}{3}} + e^{-i\frac{2\pi}{3}} = \frac{1}{2} (\cos \frac{\pi}{3} + \cos \frac{2\pi}{3})$</p> <p>$+ \frac{1}{2} i (\sin \frac{\pi}{3} - \sin \frac{2\pi}{3})$</p> <p>$= \frac{1}{2} (\frac{1}{2} - \frac{1}{2})$</p> <p>$+ \frac{1}{2} i (\frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4}) = 0$</p> <p>or observe $-e^{i\frac{2\pi}{3}} = e^{i\frac{2\pi}{3} - 2\pi} = e^{-i\frac{2\pi}{3}}$</p> <p>(iii) $\sqrt{x} - \sqrt{x-1} = \frac{x - (x-1)}{\sqrt{x} + \sqrt{x-1}} = \frac{1}{\sqrt{x} + \sqrt{x-1}}$</p> <p>$\Rightarrow \lim_{x \rightarrow \infty} x^{\frac{1}{2}} (\sqrt{x} - \sqrt{x-1}) = \frac{1}{2}$ for $q = \frac{1}{2}$</p>	<p>4</p> <p>4</p> <p>4</p>
	Setter's initials <i>HYJ</i> Checker's initials <i>GP</i>	Page number 51

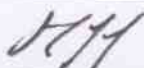
	EXAMINATION QUESTIONS/SOLUTIONS 2012-13	Course
Question 1	TOPIC General	Marks & seen/unseen
Parts	<p>(iv) $\lim_{x \rightarrow 0} \frac{\sin(x) - x}{x^3} = \lim_{x \rightarrow 0} \frac{\cos(x) - 1}{3x^2}$</p> <p>$= \lim_{x \rightarrow 0} \frac{-\sin(x)}{6x} = -\frac{1}{6}$</p> <p>(v) $\frac{d}{dx} (\tan(x))^{\exp(x)} = \frac{d}{dx} \exp(\ln(\tan(x)) \exp(x))$</p> <p>$= \left\{ \frac{1}{\tan(x)} \frac{1}{\cos^2(x)} \exp(x) + \ln(\tan(x)) \exp(x) \right\}$</p> <p>$\tan(x)^{\exp(x)}$</p> <p>$= \left\{ \frac{1}{\sin(x)} \frac{1}{\cos(x)} \exp(x) + \ln(\tan(x)) \exp(x) \right\}$</p> <p>$\tan(x)^{\exp(x)}$</p> <p>(vi) $\int \tan(x) dx = -\int \frac{d}{dx} \ln(\cos(x)) dx$</p> <p>$= -\ln(\cos(x)) + C$</p>	<p>4</p> <p>4</p> <p>4</p>
	Setter's initials 	Checker's initials 
		Page number S2

	EXAMINATION QUESTIONS/SOLUTIONS 2012-13	Course
Question 1	TOPIC <i>General</i>	Marks & seen/unseen
Parts	<p>(vii) $\int_0^1 \tan^2(x) dx = -\int \left(1 - \frac{1}{\cos^2(x)}\right) dx$</p> <p>$= -x + C + \int \frac{d}{dx} \tan x \, dx = \tan x - x + C$</p> <p>$= \tan(1) - 1$</p> <p>(viii) $f(x) = (\exp(x) + 1)^{-1}$</p> <p>$f'(x) = -(\exp(x) + 1)^{-2} \exp(x)$</p> <p>$f''(x) = 2(\exp(x) + 1)^{-3} \exp(2x) + f'(x)$</p> <p>$f(0) = 1/2$</p> <p>$f'(0) = -1/4$</p> <p>$f''(0) = 0$ (not needed)</p> <p>$\Rightarrow f(x) = \frac{1}{2} - \frac{1}{4}x + R_2(x)$</p> <p>with $R_2(x) = \frac{x^2}{2} f''(\xi) \quad \xi \in [0, x]$</p>	<p>4</p> <p>4</p>
	Setter's initials <i>KJ</i> Checker's initials <i>GP</i>	Page number 53

	EXAMINATION QUESTIONS/SOLUTIONS 2012-13	Course
Question 1	TOPIC <i>General</i>	Marks & seen/unseen
Parts (ix)	$2y' = \frac{y^2}{x^2} + 1 \quad \text{homogeneous}$ $= 2f(v) \quad v = \frac{y}{x}, \quad f(v) = \frac{1}{2}(v^2 + 1)$ $\Rightarrow \ln x + C' = \int \frac{dv}{f(v)-v} = 2 \int \frac{dv}{(v-1)^2}$ $= \frac{-2}{v-1} = \frac{2}{1-v}$ $\Rightarrow \frac{2}{\ln x + C'} = 1 - \frac{y}{x} \Rightarrow \underline{y = x - \frac{2x}{\ln x + C'}}$	4
	Setter's initials <i>HY1</i> Checker's initials <i>GP</i>	Page number 55

	EXAMINATION QUESTIONS/SOLUTIONS 2012-13		Course
Question	TOPIC	General	Marks & seen/unseen
Parts (x)	$y''(x) + y(x) = \sin(2x)$ <p>Complementary function, oscillatory can</p> $y_{CF}(x) = A \sin(x) + B \cos(x)$ <p>Particular integral, try $y_{PI}(x) = C \sin(2x) + D \cos(2x)$</p> $\Rightarrow (C - 4C) \sin(2x) + (D - 4D) \cos(2x) = \sin(2x)$ $\Rightarrow D = 0, \quad C = -\frac{1}{3}$ <p>General solution</p> $y(x) = A \sin(x) + B \cos(x) - \frac{1}{3} \sin(2x)$		4
	Setter's initials	Checker's initials	Page number
	11/11		56

40

	EXAMINATION QUESTIONS/SOLUTIONS 2012-13	Course
Question 2	TOPIC Differentiation	Marks & seen/unseen
Parts		
(i)	$y(x) = \frac{e^{2x}}{\sin(3x^2)} \Rightarrow y'(x) = \frac{2e^{2x}}{\sin(3x^2)} - 6x \frac{\cos(3x^2)e^{2x}}{\sin^2(3x^2)}$	4
(ii)	$y(x) = (1+x^2) \tan^{-1}(x) = \frac{2e^{2x}}{\sin(3x^2)} (1 - 3x \cot(3x^2))$ $\Rightarrow y'(x) = (1+x^2) \frac{1}{1+x^2} + 2x \tan^{-1}(x)$ $= 1 + 2x \tan^{-1}(x)$	4
(iii)	$y(x) = \frac{2x^2 - 2x - 4}{(x+1)(x-2)}$ <p>Either note that $y(x) = 2 \Rightarrow y'(x) = 0$ or break force:</p> $y'(x) = \frac{4x-2}{(x+1)(x-2)} - \frac{2x^2-2x-4}{(x+1)^2(x-2)^2} \underbrace{(2x-1)}_{(x+1)+(x-2)}$ $= 2 \frac{2x-1}{(x+1)(x-2)} \left\{ 1 - \frac{x^2-x-2}{(x+1)(x-2)} \right\} = 0$ <p style="text-align: center;">= 0</p>	<p>Separate sheet</p> <p>4</p>
(iv)	$y(x) = x^{(x^2)} = \exp(x^2 \ln(x))$ $\frac{d}{dx} y(x) = (2x \ln(x) + x) x^{(x^2)}$	4
(v)	$\frac{d^n}{dx^n} x \ln x = \sum_{k=0}^n \binom{n}{k} x^{(k)} (\ln x)^{(n-k)} = x \frac{(n-1)!}{x^n} (-1)^{n-1}$ $+ n \frac{(n-2)!}{x^{n-1}} (-1)^{n-2} = \frac{(n-2)!}{x^{n-1}} (-1)^{n-1} \{n-1-n\}$ $= \frac{(n-2)!}{x^{n-1}} (-1)^n$ <p>using $\frac{d^n}{dx^n} \ln x = \frac{(n-1)!}{x^n} (-1)^{n-1}$</p>	4
Setter's initials		<div style="border: 1px solid black; border-radius: 50%; width: 30px; height: 30px; display: flex; align-items: center; justify-content: center; margin: 0 auto;">20</div>
	Checker's initials GP	Page number 52

	EXAMINATION QUESTIONS/SOLUTIONS 2012-13	Course
Question 2		Marks & seen/unseen
Parts (iii)	$y(x) = \frac{x+3}{(x+1)(x-2)}$ $\frac{dy}{dx} = \frac{1}{(x+1)(x-2)} - \frac{(x+3)}{(x+1)^2(x-2)} - \frac{(x+3)}{(x+1)(x-2)^2}$ $= \frac{1}{(x+1)^2(x-2)^2} \left\{ (x+1)(x-2) - (x+3)(x-2) - (x+3)(x+1) \right\}$ $= \frac{1}{(x+1)^2(x-2)^2} \left\{ -2x+4 -x^2-4x-3 \right\} = -\frac{x^2+6x-1}{(x+1)^2(x-2)^2}$ <p><u>or</u></p> $y(x) = \frac{x+3}{(x+1)(x-2)} = \frac{1}{x-2} + \frac{2}{(x+1)(x-2)}$ $\frac{dy}{dx} = \frac{-1}{(x-2)^2} - \frac{2(x-2+x+1)}{(x+1)^2(x-2)^2}$ $= -\frac{x^2+2x+1+4x-2}{(x+1)^2(x-2)^2} = -\frac{x^2+6x-1}{(x+1)^2(x-2)^2}$	4
	Setter's initials	Page number
	Checker's initials	57

	EXAMINATION QUESTIONS/SOLUTIONS 2012-13		Course
Question 3	TOPIC Limits		Marks & seen/unseen
Parts			
(i)	$\lim_{x \rightarrow 1} \frac{x}{\sin(x)} = \frac{1}{\sin(1)}$		3 seen similar for all
(ii)	$\lim_{x \rightarrow 1} \frac{x-1}{\sin(x\pi)} = \lim_{y \rightarrow 0} \frac{y}{\sin(\pi+y\pi)}$ $= \lim_{u \rightarrow 0} \frac{u/\pi}{-\sin(u)} = -\frac{1}{\pi}$		5
(iii)	$\lim_{x \rightarrow \infty} x^{2/3} \left((1+x)^{1/3} + (1-x)^{1/3} \right)$ $= \lim_{x \rightarrow \infty} x^{2/3} \frac{(1+x) + (1-x)}{(1+x)^{2/3} - (1+x)^{1/3}(1-x)^{1/3} + (1-x)^{2/3}}$ $= 2 \lim_{x \rightarrow \infty} \frac{1}{\left(1+\frac{1}{x}\right)^{2/3} + \left(1+\frac{1}{x}\right)^{1/3}\left(1-\frac{1}{x}\right)^{1/3} + \left(1-\frac{1}{x}\right)^{2/3}}$ $= \frac{2}{3}$		5
(iv)	$\lim_{x \rightarrow 1} \frac{x^n - 1}{x^m - 1} = \lim_{x \rightarrow 1} \frac{n x^{n-1}}{m x^{m-1}} = \frac{n}{m} \text{ with L'Hopital}$ <p>or</p> $\lim_{x \rightarrow 1} \frac{x^n - 1}{x^m - 1} = \lim_{x \rightarrow 1} \frac{\overbrace{(x-1)(x^{n-1} + x^{n-2} + \dots + 1)}^{n \text{ terms}}}{\underbrace{(x-1)(x^{m-1} + x^{m-2} + \dots + 1)}_{m \text{ terms}}}$ $= \frac{n}{m}$		3
Setter's initials HJJ		Checker's initials GP	Page number 58

	EXAMINATION QUESTIONS/SOLUTIONS 2012-13	Course
Question 3	TOPIC <i>Limits</i>	Marks & seen/unseen
Parts (v)	$\lim_{x \rightarrow 1} \frac{e^x - e(1+x^2)/2}{(x-1)^3}$ $= \lim_{x \rightarrow 1} \frac{e^x - ex}{3(x-1)^2} = \lim_{x \rightarrow 1} \frac{e^x - e}{6(x-1)}$ $= \lim_{x \rightarrow 1} \frac{e^x}{6} = \frac{e}{6}$	<p>4</p> <p>(20)</p>
	Setter's initials <i>HPJ</i> Checker's initials <i>GP</i>	Page number 59

	EXAMINATION QUESTIONS/SOLUTIONS 2012-13		Course
Question 4	TOPIC Integration		Marks & seen/unseen
Parts i	$\int_0^{2\pi} \sin(nx) \sin(mx) dx = \frac{1}{2} \int_0^{2\pi} \cos((n-m)x) - \cos((n+m)x) dx$ <p>using</p> $\cos((n+m)x) = \cos(nx) \cos(mx) - \sin(nx) \sin(mx)$ $\cos((n-m)x) = \cos(nx) \cos(mx) + \sin(nx) \sin(mx)$ $\begin{cases} = 0 & \text{for } n \neq m \text{ (note } n, m > 0) \\ = \pi & \text{for } n = m \end{cases}$		5
	Setter's initials HJ1	Checker's initials GSP	Page number S10

EXAMINATION QUESTIONS/SOLUTIONS 2012-13

Course

Question

4

TOPIC

Integration

Marks &

seen/unseen

Parts

ii

$$\begin{aligned} \int_0^{\infty} x^n \exp(-x^2) dx &= \lim_{b \rightarrow \infty} \int_0^b x^n \exp(-x^2) dx \\ \int_0^b x^n e^{-x^2} dx &= \frac{1}{2} \int_0^{b^2} y^{\frac{n-1}{2}} e^{-y} dy \\ &= - \left[\frac{1}{2} y^{\frac{n-1}{2}} e^{-y} \right]_0^{b^2} + \frac{1}{2} \int_0^{b^2} \frac{n-1}{2} y^{\frac{n-3}{2}} e^{-y} dy \\ &= - [\dots] + \frac{n-1}{2} \int_0^b x^{n-2} e^{-x^2} dx \end{aligned}$$

Take $b \rightarrow \infty$ to remove $[-]$, so that

$$\int_0^{\infty} x^n \exp(-x^2) dx = \frac{n-1}{2} \int_0^{\infty} x^{n-2} e^{-x^2} dx$$

for $n > 1$

Seen
Similar
for all

5

Setter's initials

HY

Checker's initials

GP

Page number

S11

EXAMINATION QUESTIONS/SOLUTIONS 2012-13

Course

Question

4

TOPIC

Integration

Marks &

seen/unseen

Parts

(iii)

$$I = \int \frac{(x-1)(x^2 + \frac{5}{2}x + 1)}{2x^2 - 3x + 1} dx$$

Denominator $2x^2 - 3x + 1 = (x-1)(2x-1)$

$$\Rightarrow I = \int \frac{x^2 + \frac{5}{2}x + 1}{2x-1} dx = \int \frac{\frac{1}{2}x(2x-1) + 3x + 1}{2x-1} dx$$

$$= \int \frac{1}{2}x + \frac{3}{2} + \frac{\frac{5}{2}}{2x-1} dx = \frac{1}{4}x^2 + \frac{3}{2}x + \frac{5}{4}\ln(2x-1) + C$$

5

(iv)

$$\int \frac{dx}{2 + \cos(x)} \quad \text{use } t = \tan\left(\frac{x}{2}\right), \cos(x) = \frac{1-t^2}{1+t^2}$$

$$dx = \frac{2}{1+t^2} dt$$

$$\Rightarrow \int \frac{dx}{2 + \cos(x)} = \int \frac{1}{2 + \frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2} dt = \int \frac{2}{3+t^2} dt$$

$$= \frac{2}{\sqrt{3}} \int \frac{\sqrt{3}}{3+t^2} dt = \frac{2}{\sqrt{3}} \arctan\left(\frac{t}{\sqrt{3}}\right) + C$$

$$= \frac{2}{\sqrt{3}} \arctan\left(\frac{\tan\left(\frac{x}{2}\right)}{\sqrt{3}}\right) + C$$

5

20

Setter's initials

HLJ

Checker's initials

GP

Page number

512

EXAMINATION QUESTIONS/SOLUTIONS 2012-13

Course

Question

5

TOPIC

Complex numbers

Marks &

seen/unseen

Parts

(i)

$$(1-i)^3 = 2^{3/2} e^{-i\frac{\pi}{4}3} \text{ also allow } e^{i\frac{5}{4}\pi}$$

$$(1+\sqrt{3}i)^{1/2} = \sqrt{2} e^{i\frac{\pi}{3} \frac{1}{2}}, \sqrt{2} e^{i\frac{7}{6}\pi}$$

4

(ii)

$$\left(\frac{1}{\sin(\theta) + i\cos(\theta)} \right)^4 = \left(\frac{-i}{e^{-i\theta}} \right)^4$$

$$= e^{4i\theta} (= \cos(4\theta) + i\sin(4\theta))$$

$$= (e^{2i\theta})^2 = (\cos(2\theta) + i\sin(2\theta))^2$$

$$= (\cos(2\theta))^2 - (\sin(2\theta))^2 + 2i\sin(2\theta)\cos(2\theta)$$

$$= 2(\cos(2\theta))^2 - 1 + 2i\sin(2\theta)\cos(2\theta)$$

$$\Rightarrow \operatorname{Re} \left(\left(\frac{1}{\sin\theta + i\cos\theta} \right)^4 \right) = 2\cos^2 2\theta - 1$$

6

Setter's initials

AMY

Checker's initials

GP

Page number

513

20

EXAMINATION QUESTIONS/SOLUTIONS 2012-13

Course

Question

5

TOPIC

Complex numbers

Marks &

seen/unseen

Parts

(iii)

$$\cos(z) = \frac{1}{2} (e^{iz} + e^{-iz})$$

$$z = y - iz$$

$$\frac{5}{3} = \frac{1}{2} (e^x (\cos(y) + i \sin(y)) + e^{-x} (\cos(y) - i \sin(y)))$$

$$\Rightarrow \frac{5}{3} = \frac{1}{2} \cos(y) (e^x + e^{-x})$$

$$0 = \frac{1}{2} \sin(y) (e^x - e^{-x})$$

from the second the $e^x = e^{-x} = 1$ seems a solution but then there is no $y \in \mathbb{R}$ with $\cos(y) = \frac{5}{3} \Rightarrow y = n\pi, n \in \mathbb{Z}$
 $\Rightarrow \cos(y) = (-1)^n \Rightarrow$

$$\frac{5}{3} = \frac{1}{2} (-1)^n (e^x + e^{-x}) \rightarrow n \text{ even}$$

$$\Rightarrow e^{2x} - \frac{10}{3} e^x + 1 = 0$$

$$\Rightarrow e^x = \frac{\frac{10}{3} \pm \sqrt{\frac{100}{9} - 4}}{2} = \frac{5}{3} \pm \frac{4}{3}$$

$$\Rightarrow x = \pm \ln(3)$$

$$\Rightarrow z = \pm i \ln(3) + 2k\pi, k \in \mathbb{Z}$$

10

Setter's initials

HHH

Checker's initials

GP

Page number

S14

	EXAMINATION QUESTIONS/SOLUTIONS 2012-13		Course
Question 6	TOPIC ODEs		Marks & seen/unseen
Parts			
(i)	$y' = -xy$ separable $\int \frac{dy}{y} = -\int x dx \Rightarrow \ln y + C' = -\frac{1}{2}x^2$ $y(x) = A e^{-\frac{1}{2}x^2}$		4
(ii)	$y = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$		
↑			
<u>(iii)</u>	$y' = \sum_{n=1}^{\infty} a_n n \frac{x^{n-1}}{n!} = \sum_{n=0}^{\infty} a_{n+1} \frac{x^n}{n!}$		
(iv) next page	$y'' = \sum_{n=0}^{\infty} a_{n+2} \frac{x^n}{n!}$		
	$y'' = -xy \Rightarrow \sum_{n=0}^{\infty} a_{n+2} \frac{x^n}{n!} = -\sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n!}$ $= -\sum_{n=1}^{\infty} a_{n-1} \frac{x^n}{(n-1)!}$		
	$\Rightarrow \frac{a_{n+2}}{n!} = -\frac{a_{n-1}}{(n-1)!} \Rightarrow a_{n+2} = -n a_{n-1}, n \geq 1$		
	$a_2 = 0$ and $a_n = -a_{n-3} (n-2)$		
	$a_0 = ?$ $a_1 = ?$ $a_3 = -a_0$ $a_4 = -2a_1$ \dots		8
	Setter's initials H11	Checker's initials GP	Page number S15

	EXAMINATION QUESTIONS/SOLUTIONS 2012-2013		Course
Question 6	TOPIC ODEs		Marks & seen/unseen
Parts ii	$y'' + y' + \frac{1}{2}y = \sin(x)$ $a=1, b=1, c=\frac{1}{2} \quad b^2 - 4ac = -1 < 0 \quad \text{oscil}$ $y_{CF} = \left(A \sin \frac{x}{2} + B \cos \frac{x}{2} \right) e^{-\frac{x}{2}} \quad \begin{matrix} \Omega = \frac{1}{2} \\ -\frac{b}{2a} = -\frac{1}{2} \end{matrix}$ $y_{p1} = C \sin x + D \cos x$ $y_{p1}' = C \cos x - D \sin x$ $y_{p1}'' = -y_{p1}$ $\Rightarrow y_{p1}' - \frac{1}{2} y_{p1} = \left(C - \frac{D}{2} \right) \cos x - \left(D + \frac{C}{2} \right) \sin x = \sin x$ $\Rightarrow C = \frac{D}{2} \Rightarrow -D \frac{5}{4} = 1 \Rightarrow D = -\frac{4}{5}, C = -\frac{2}{5}$ $y_{p1} = -\frac{2}{5} \sin x - \frac{4}{5} \cos x$ <p>General soln:</p> $y(x) = \left(A \sin \frac{x}{2} + B \cos \frac{x}{2} \right) e^{-\frac{x}{2}} - \frac{2}{5} \sin x - \frac{4}{5} \cos x$		8
	Setter's initials <i>H.F.</i>	Checker's initials <i>GP</i>	Page number S16