IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2014

EEE PART I: MEng, BEng and ACGI

Corrected Copy

MATHEMATICS 1A (E-STREAM AND I-STREAM)

Thursday, 29 May 10:00 am

Time allowed: 2:00 hours

There are THREE questions on this paper.

Answer ALL questions

Any special instructions for invigilators and information for candidates are on page 1.

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MATHEMATICS 1

Information for Candidates:

Calculators are not permitted in this exam.

1. a) i) A function $s(x)$ has 4 real roots and has a continuous derivative. If $g(x) = s'(x)$, what is the minimum number of real roots of $g(x)$? Justify your answer. [4 marks]

ii) If the order of a polynomial is known and its roots are known, when is this enough information to specify the polynomial uniquely – always, sometimes or never? Justify your answer. [4 marks]

iii) If $f(x)$ and $g(x)$ are 4^{th} and 3^{rd} order polynomials respectively, then solving $f(x) = g(x)$ is equivalent to finding the roots of a polynomial of what order? Justify your answer. [4 marks]

b) i) What is the name of the type of function which describes the locus of a point which is always equidistant from a given line and a given reference point? [2 marks]

ii) Find the function $y(x)$ which is equidistant from the line $y = 2$ and the point $(x, y) = (0, 4)$. [4 marks]

c) i) For a complex number X, where

$$X = \frac{a+ib}{a-ib}$$

and a and b are real, find expressions for the modulus and argument of X. [4 marks]

ii) For X as in (c)(i) above, if $a = b$, find the value of X in the form $u + iv$. [4 marks]

d) i) Euler's equation gives $e^{i\theta}$ in terms of trigonometric functions. Write Euler's equation. [2 marks]

ii) Using Euler's equation, show that $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$. [2 marks]

iii) Using (d)(i) and (d)(ii) above, derive trigonometric identities for $\sin 3\theta$ and $\cos 3\theta$ in terms of powers of $\sin\theta$ and $\cos\theta$. [4 marks]

2. Consider the function

$$y = f(x) = \frac{1}{4}x^2 - \frac{1}{2}\ln x,$$

defined for $x > 0$.

a) Compute the first and second derivatives of the function. Hence determine the only stationary point of the function. Show that $\frac{d^2 f}{dx^2} > 0$, for all $x > 0$, and hence that the stationary point is a minimum. [6 marks]

b) Plot the graph of the function for $x \in [0,3]$. Clearly indicate the stationary point and the value of the function for $x \to 0$. Note that the function f takes positive values for all $x > 0$. [4 marks]

c) Compute the indefinite integral

$$I = \int f(x)dx.$$

[5 marks]

d) Consider the region A in the (x,y)-plane defined as follows. The region is bounded from above by the graph of the function f and from below by the x-axis. The region is bounded from the left by the line $x = 1$ and from the right by the line $x = 2$.

i) Compute the area of the region A. [4 marks]

ii) Determine the length of the perimeter of the region A. Note that the perimeter is composed of four curves and that the length of each of these must be computed separately. [6 marks]

iii) Compute the volume of the solid of revolution obtained by rotating the region A around the x-axis. [8 marks]

3. The complex Fourier series for a periodic function, $u(t)$, with period $T = \frac{1}{F}$ is given by

$$u(t) = \sum_{n=-\infty}^{\infty} U_n e^{j2\pi nFt}$$

where j denotes $\sqrt{-1}$.

a) Show that, if m and n are integers, then

$$\int_0^T e^{j2\pi mFt} e^{j2\pi nFt} dt = \begin{cases} T & \text{if } m = -n \\ 0 & \text{otherwise} \end{cases}.$$

[6 marks]

b) Hence show that

$$\frac{1}{T}\int_0^T u(t) e^{-j2\pi mFt} dt = U_m.$$

State clearly any assumptions you make. [8 marks]

c) Suppose $u(t)$ has period $T_u = 4$ and $u(t) = e^{-0.3t}$ for $0 \le t < 4$.

By evaluating the integral in part b), determine an expression for the complex Fourier coefficients U_n. Simplify the expression where possible. [10 marks]

d) Suppose $v(t)$ has period $T_v = 8$ and $v(t) = \begin{cases} u(t) & \text{for } 0 \le t < 4 \\ u(-t) & \text{for } -4 \le t < 0 \end{cases}.$

i) Sketch a graph showing both $u(t)$ and $v(t)$ on the same set of axes over the range $-5 \le t \le 5$. [4 marks]

ii) The partial Fourier series of order N are defined by

$$u_N(t) = \sum_{n=-N}^{N} U_n e^{j2\pi nFt}$$

$$v_N(t) = \sum_{n=-N}^{N} V_n e^{j2\pi nFt}$$

where U_n and V_n are the complex Fourier coefficients of $u(t)$ and $v(t)$ respectively.

Explain why, for any fixed value of N, $v_N(t)$ will generally be a better approximation of $u(t)$ over the range $0 \le t \le 4$ than $u_N(t)$. You are not required to determine expressions for $u_N(t)$ and $v_N(t)$.

[5 marks]