

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2006

EEE/ISE PART I: MEng, BEng and ACGI

COMMUNICATIONS 1

Corrected Copy

Friday, 26 May 10:00 am

Time allowed: 2:00 hours

There are FOUR questions on this paper.

Q1 is compulsory.

Answer Q1 and any two of questions 2-4.

Q1 carries 40% of the marks. Questions 2 to 4 carry equal marks (30% each).

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible

First Marker(s) : P.L. Dragotti,

Second Marker(s) : M.K. Gurcan,

Special Information for the Invigilators: none

Information for Candidates

Some Fourier Transforms

$$\cos(\omega_0 t + \theta) \iff \pi[\delta(\omega - \omega_0)e^{-j\theta} + \delta(\omega + \omega_0)e^{j\theta}]$$

$$\text{rect}\left(\frac{t}{\tau}\right) \iff \tau \text{sinc}\left(\frac{\omega\tau}{2}\right)$$

$$\frac{W}{\pi} \text{sinc}(Wt) \iff \text{rect}\left(\frac{\omega}{2W}\right)$$

$$\frac{\alpha^2}{2\pi} \text{sinc}^2\left(\frac{\alpha t}{2}\right) \iff \Delta\left(\frac{\omega}{\alpha}\right)$$

where

$$\Delta(\omega) = \begin{cases} 1 - |\omega|, & |\omega| \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

Some useful trigonometric identities

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

$$\sin x \cos y = \frac{1}{2} \sin(x - y) + \frac{1}{2} \sin(x + y)$$

$$\cos x \cos y = \frac{1}{2} \cos(x - y) + \frac{1}{2} \cos(x + y).$$

Euler's formula

$$e^{jx} = \cos x + j \sin x.$$

Frequency modulation by a sinusoidal signal

$$\varphi_{FM}(t) = A \sum_{n=-\infty}^{\infty} J_n(\beta) \cos(\omega_c + n\omega_m)t.$$

where $\beta = \Delta f/B$.

Table of Bessel Function values (recall that $|J_n(\beta)| = |J_{-n}(\beta)|$):

n	$\beta = 1$	$\beta = 2$	$\beta = 5$	$\beta = 10$
0	0.765	0.224	-0.178	-0.246
1	0.440	0.577	-0.328	0.043
2	0.115	0.353	0.047	0.255

Roots of $J_0(x)$:

x	2.4048	5.5201	8.6537	11.7915
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The Questions

1. This question is compulsory.

(a) Consider the two signals $x_1(t)$ and $x_2(t)$ shown in Figure 1.1.

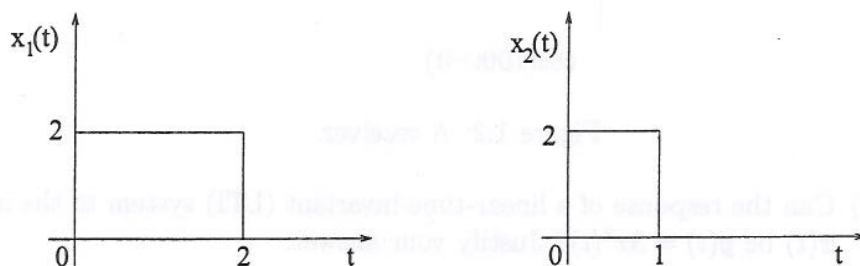


Figure 1.1: The two energy signals $x_1(t)$ and $x_2(t)$.

i. Determine the correlation coefficient between $x_1(t)$ and $x_2(t)$.

[4]

ii. Determine the energy of $y(t) = x_1(t) + x_2(t)$.

[4]

(b) Compute the Fourier transform of $x(t) = e^{-2t}u(t)$, where $u(t)$ is the unit step function defined by

$$u(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

[4]

(c) The received signal $s(t) = (\cos 10t) \cos 100t$ is multiplied by the local carrier $\cos(100t + \theta)$ and the result $x(t)$ is fed to a filter that has the frequency response

$$H(\omega) = \begin{cases} 1 & \text{for } |\omega| \leq 30 \text{ rad/s} \\ 0 & \text{otherwise} \end{cases}$$

giving the output $y(t)$, as shown in Figure 1.2.

i. For $\theta = 0$,

A. Sketch and dimension the Fourier transform of $x(t)$.

[4]

B. Sketch and dimension the Fourier transform of $y(t)$.

[4]

ii. for $\theta = \pi/4$ and $\theta = \pi/2$, write the exact expression for the output $y(t)$.

[4]

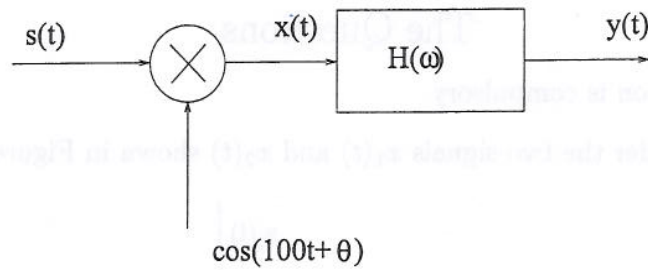


Figure 1.2: A receiver.

- (d) Can the response of a linear-time-invariant (LTI) system to the input $x(t)$ be $y(t) = 3x^2(t)$? Justify your answer.

[4]

- (e) Consider the FM signal

$$\varphi(t) = A \cos[2\pi f_c t + k_f \int_{-\infty}^t m(\alpha) d\alpha],$$

where $m(t) = 20 \cos(200t)$, $k_f = 10\pi$ and $f_c = 1000$ Hz.

- i. Compute the minimum and maximum instantaneous frequency of $\varphi(t)$.

[4]

- ii. If the power of $\varphi(t)$ is 8, find the value A .

[4]

- (f) A sinusoidal source $v(t) = 10 \sin(500\pi t)$ Volts with internal resistance $R = 50 \Omega$ is connected to a transmission line with characteristic impedance $Z_0 = 50 \Omega$. The transmission line is connected to a load Z_L (see Figure 1.3).

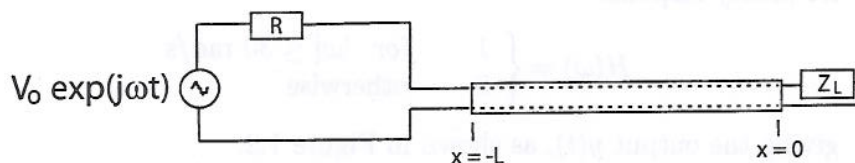


Figure 1.3: A transmission line connected to a sinusoidal source.

- i. Choose Z_L so that there is no reflection in the line.

[2]

- ii. For the value Z_L you found in part (i), find the exact expression for the current flowing in the circuit.

[2]

2. Consider the FM signal

$$\varphi(t) = 10 \cos[2\pi f_c t + k_f \int_{-\infty}^t x(\alpha) d\alpha],$$

where $f_c = 2000$ Hz, $k_f = 16\pi$ and $x(t) = A \cos 16\pi t$.

- (a) Using Carson's rule, the bandwidth of $\varphi(t)$ is $B_{FM} = 96$ Hz. Compute the amplitude A of $x(t)$.

[6]

- (b) Sketch and dimension the Fourier transform of $x(t)$.

[6]

- (c) Compute the power of the modulated signal $\varphi(t)$.

[6]

- (d) The modulated signal is now passed through an ideal band-pass filter $H(\omega)$ centered at $\omega_c = 2\pi f_c$ rad/s with a bandwidth of 40π rad/s (see Figure 2.1). Determine the power of the output signal $y(t)$.

[6]

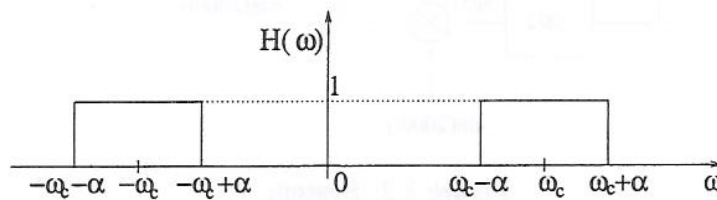


Figure 2.1: Ideal bandpass filter. In this case, $\omega_c = 2\pi f_c$ and $\alpha = 20\pi$ rad/s.

- (e) Find the smallest value of k_f that guarantees no power is transmitted at the carrier frequency.

[6]

3. A lowpass signal $x(t)$ has the Fourier transform shown in Figure 3.1. This

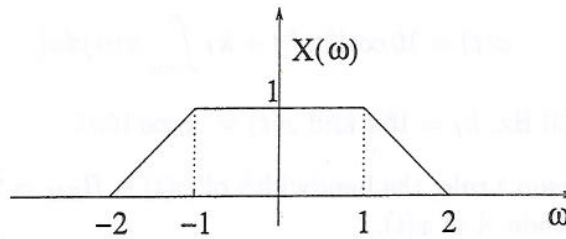


Figure 3.1: Fourier Transform of $x(t)$.

signal $x(t)$ is applied to the system shown in Figure 3.2. The block marked by $-\pi/2$ represents a block performing the Hilbert transform. The filter with transfer function $H(\omega)$ is an ideal lowpass filter with cut-off frequency $\omega_c = 1$ rad/s. That is, $H(\omega) = 1$ for $|\omega| \leq 1$ rad/s and is zero otherwise.

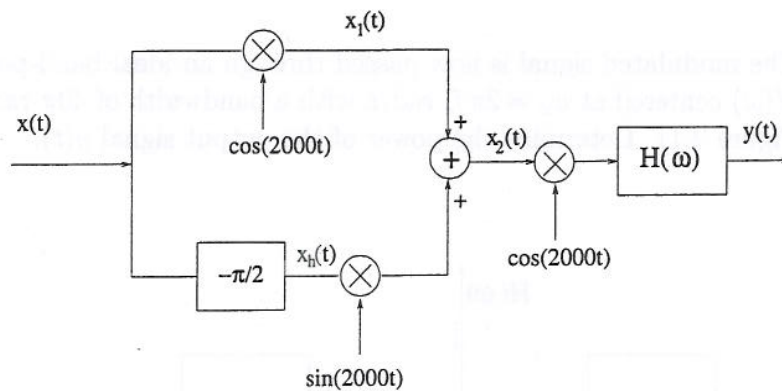


Figure 3.2: System

(a) Sketch and dimension the Fourier transform of $x_1(t)$.

[6]

(b) Sketch and dimension the Fourier transform of $x_2(t)$.

[6]

(c) Find an exact expression for $y(t)$.

[6]

(d) Find an exact expression for $x(t)$.

[12]

4. Consider a linear time-invariant system $h(t)$ where the input $x(t)$ and output $y(t)$ are related by the following linear differential equation:

$$\frac{d^2y(t)}{dt^2} + \frac{dy(t)}{dt} + \beta y(t) = \frac{dx(t)}{dt} + \alpha x(t).$$

- (a) Find the transfer function of $h(t)$. Recall that the transfer function is defined as $Y(\omega) = H(\omega)X(\omega)$.

[6]

- (b) Find the value of α such that $H(\omega) = 0$ for $\omega = 0$. Then choose β so that $|H(\omega)|^2$ has its maximum when $\omega = 2$ rad/s. (Assume β is real and $\beta > 0$).

[6]

- (c) Assume that $x(t) = e^{-t}u(t)$. Compute the Energy Spectral Density (ESD) of $x(t)$.

[6]

- (d) Compute the autocorrelation function $\psi(\tau) = \int_{-\infty}^{\infty} x(t)x(t+\tau)dt$. [Hint: Recall that in this case $\psi(\tau) = \psi(-\tau)$].

[6]

- (e) Verify that the Fourier transform of $\psi(\tau)$ is equal to the ESD of $x(t)$. [Hint: Use the fact that $x(-t) \Leftrightarrow X(-\omega)$].

[6]

QUESTION 1

(a)

$$i.) E_{x_1} = 4 \cdot 2 = 8$$

Basic

$$E_{x_2} = 4$$

$$C_{x_1, x_2} = \frac{1}{\sqrt{E_{x_1} E_{x_2}}} \int_{-\infty}^{\infty} x_1(t) x_2(t) dt =$$

$$= \frac{1}{4\sqrt{2}} \int_0^1 4 dt = \frac{1}{\sqrt{2}}$$

$$ii.) y(t) = x_1(t) + x_2(t)$$

Basic

$$E_y = E_{x_1} + E_{x_2} + 2\sqrt{E_{x_1} E_{x_2}} \cdot C_{x_1, x_2} =$$

$$= 12 + 8 = 20$$

(b) NO. SUCH A SYSTEM IS NOT LINEAR.

FOR INSTANCE

$$x_1(t) \rightarrow y(t) = x_1^2(t)$$

$$x_2(t) \rightarrow y(t) = x_2^2(t)$$

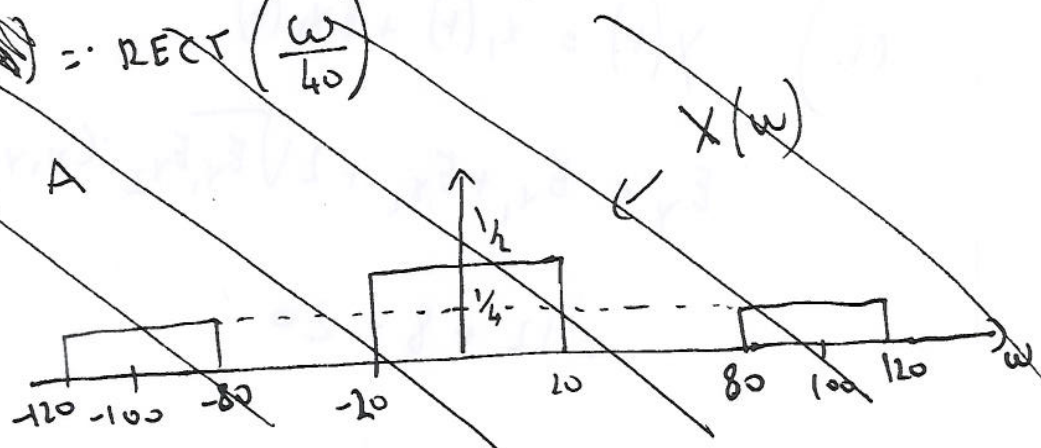
BUT

$$2x_1(t) + \beta x_2(t) \rightarrow y(t) = (2x_1(t) + \beta x_2(t))^2 \neq 2y_1(t) + \beta y_2(t)$$

$$c) X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_0^{\infty} e^{-2t} e^{-j\omega t} dt =$$

$$= \frac{1}{2+j\omega}$$

(d) $H(\omega) = \text{RECT}\left(\frac{\omega}{40}\right)$



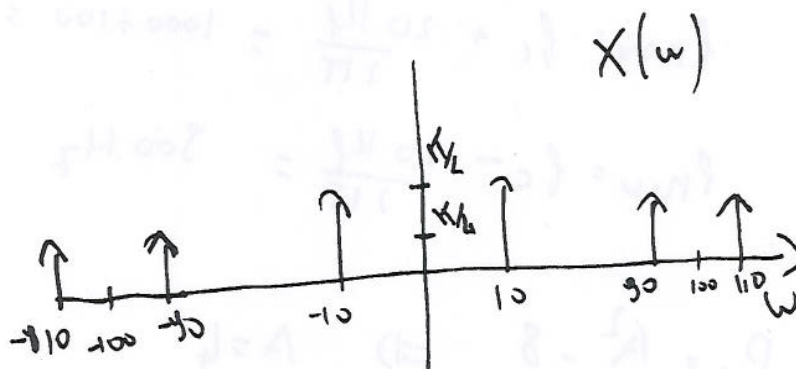
(u)

$$\cos 10t \Rightarrow \pi [\delta(\omega - 10) + \delta(\omega + 10)]$$

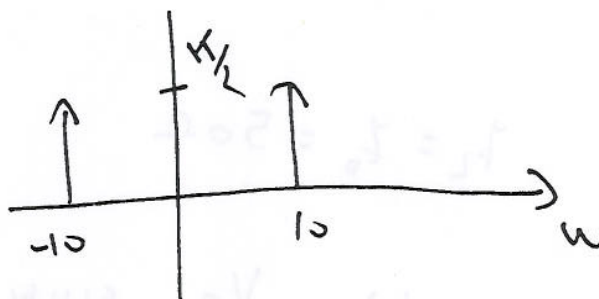
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i.

A.



B.

 $Y(\omega)$ 

ii

NEW

 $\theta \neq 0$

$$y(t) = \cos 10t \cdot \cos \theta$$

$$\theta = \pi/2$$

$$y(t) = 0$$

$$\theta = \pi/4$$

$$y(t) = \frac{\sqrt{2}}{2} \cos 10t$$

(2)

i. $\omega_i = 2\pi f_c + k_f m(t)$

$$f_i = f_c + \frac{k_f m(t)}{2\pi}$$

$$f_{max} = f_c + 20 \frac{k_f}{2\pi} = 1000 + 100 = 1100 \text{ Hz}$$

$$f_{min} = f_c - 20 \frac{k_f}{2\pi} = 900 \text{ Hz}$$

ii. $P_y = \frac{A^2}{2} = 8 \Rightarrow A = 4$

(8)

i. $Z_L = Z_0 = 50 \Omega$

ii. $i(t) = \frac{V_0}{R + Z_0} \sin \omega_c t = 0.1 \sin(500\pi t) \text{ A}$

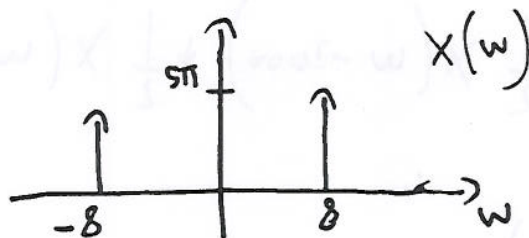
QUESTION 2

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$$(a) B_{FH} = 2(\Delta f + B) = 2\left(\frac{k_f m_p}{2\pi} + B\right) =$$

$$= 2(8 \cdot m_p + 8) = 96 \Rightarrow 16 m_p = 80 \Rightarrow m_p = 5.$$

(b)



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$$(c) P_{\psi} = \frac{100}{2} = 50$$

$$(d) \beta = \frac{\Delta f}{B} = 5. \quad \text{THE MODULATED SIGNAL CAN BE}$$

WRITTEN AS FOLLOWS:

$$\psi_{FH}(t) = 10 \sum_{n=-\infty}^{\infty} J_n(\beta) \cos(\omega_c + n\omega_m)t$$

$$\omega_m = 16\pi \text{ RAD/s}, \quad \omega_c = 2\pi f_c \text{ WITH } f_c = 2000 \text{ Hz}.$$

AFTER FILTERING

$$\psi_{FH}(t) = 10 \sum_{n=-1}^1 J_n(\beta) \cos(\omega_c + n\omega_m)t$$

THUS THE POWER IS

$$P_{\psi} = \frac{10^2}{2} J_0^2(\beta) + 2 \cdot \frac{10^2}{2} J_1^2(\beta) \approx 14.34$$

$$= 50(-0.178)^2 + 100 \cdot (-0.328)^2 = 12.34$$

$$(e) \text{ THIS HAPPENS WHEN } \beta \approx 2.4 \Rightarrow$$

$$\beta = \frac{k_f \cdot m_p}{2\pi B} = 2.4 \Rightarrow k_f \approx 7.68\pi$$

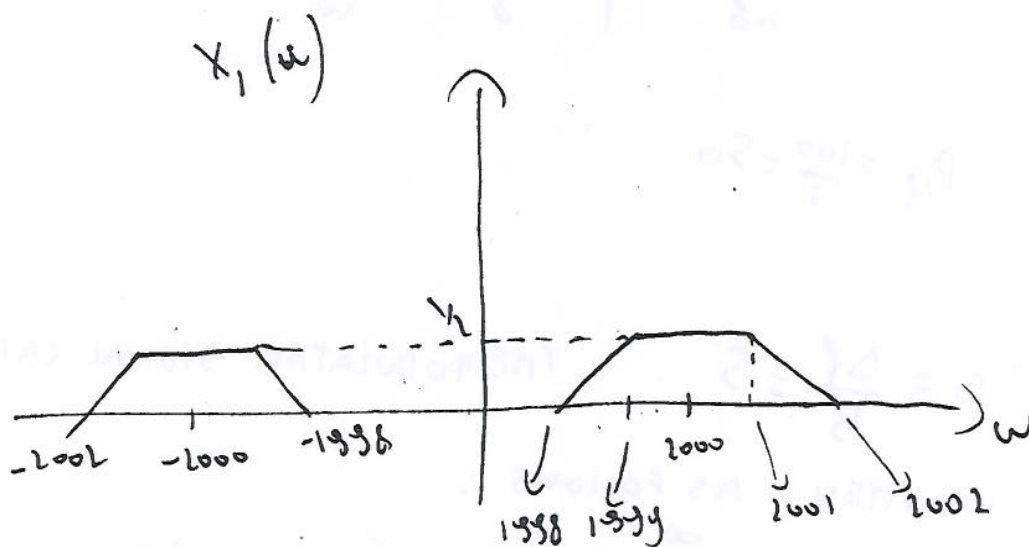
NEW

QUESTION 3

6

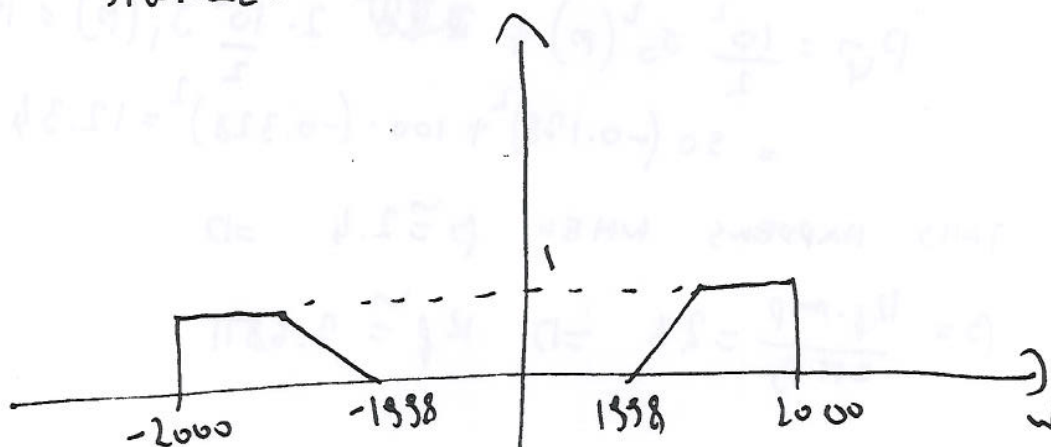
(a) $x_1(t) = x(t) \cos 2000t$

$$X_1(\omega) = \frac{1}{2} X(\omega - 2000) + \frac{1}{2} X(\omega + 2000)$$

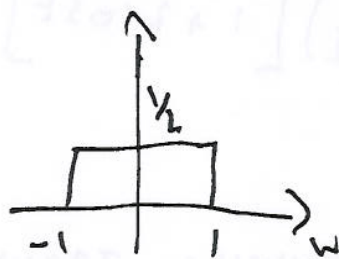


(b) $x_2(t) = x(t) \cos 2000t + x_H(t) \sin 2000t$

THUS, $x_2(t)$ IS AN SSB-LSB MODULATED SIGNAL.



(c) $Y(\omega) = \frac{1}{2} \text{RECT}\left(\frac{\omega}{2}\right)$



$$Y(t) = \frac{1}{2\pi} \text{sinc}\left(\frac{t}{2}\right)$$

(d) $X(\omega)$ CAN BE WRITTEN AS FOLLOWS

VER

~~$$X(\omega) = \Delta\left(\frac{\omega+1}{2}\right) + \Delta\left(\frac{\omega}{2}\right) + \Delta\left(\frac{\omega-1}{2}\right)$$~~

$$X(\omega) = \Delta(\omega+1) + \Delta(\omega) + \Delta(\omega-1)$$

WHERE

$$\Delta(\omega) = \begin{cases} 1 - |\omega| & |\omega| \leq 1 \\ 0 & \text{OTHERWISE} \end{cases}$$

$$\frac{\omega^2}{2\pi} \text{sinc}^2\left(\frac{\omega}{2}\right) \Leftrightarrow \Delta\left(\frac{\omega}{2}\right)$$

THUS

$$X(t) = \frac{1}{2\pi} \text{sinc}^2\left(\frac{t}{2}\right) + \frac{1}{2\pi} \text{sinc}^2\left(\frac{t}{2}\right) \left[e^{jt} + e^{-jt} \right] =$$

8

$$= \frac{1}{2\pi} \text{sinc}^2\left(\frac{t}{2}\right) + \frac{1}{\pi} \text{sinc}^2\left(\frac{t}{2}\right) \cos t =$$

$$= \frac{1}{2\pi} \text{sinc}^2\left(\frac{t}{2}\right) [1 + 2 \cos t].$$

QUESTION 4

(a) TAKING THE FOURIER TRANSFORM ON BOTH SIDES

$$- \omega^2 Y(\omega) + j\omega Y(\omega) + \beta Y(\omega) = j\omega X(\omega) + dX(\omega)$$

$$\Rightarrow H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{d + j\omega}{-\omega^2 + j\omega + \beta}$$

VER

(b)

$$H(0) = \frac{d}{\beta} = 0 \Rightarrow d = 0$$

$$|H(\omega)|^2 = \frac{\omega^2}{(\beta - \omega^2)^2 + \omega^2} = \frac{\omega^2}{\beta^2 + \omega^4 - 2\beta\omega^2 + \omega^2}$$

$$\frac{d(|H(\omega)|^2)}{d\omega} = \frac{2\omega(\omega^4 + (1-2\beta)\omega^2 + \beta^2) - \omega^2(4\omega^3 + 2(1-2\beta)\omega)}{\omega^4 + (1-2\beta)\omega^2 + \beta^2} =$$

$$= \frac{2\omega^5 + 2\omega\beta^2 - 4\omega^5}{(\quad)} = 0 \Rightarrow 2\omega(\beta^2 - \omega^4) = 0$$

$$\Rightarrow \omega_0 = \sqrt{\beta} = 2 \Rightarrow \beta = 4$$

$$(c) \quad x(t) = e^{-t} u(t) \quad (\Rightarrow) \quad X(\omega) = \frac{1}{1+j\omega}$$

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$$ESD = |X(\omega)|^2 = \frac{1}{1+\omega^2}$$

$$(d) \quad x(t) = e^{-t} u(t) \quad \text{AND} \quad x(t+\tau) = e^{-(t+\tau)} u(t+\tau)$$

THUS

$$\psi(\tau) = \int_{-\infty}^{\infty} x(t) x(t+\tau) dt =$$

NEW

$$= \int_0^{\infty} e^{-t} e^{-(t+\tau)} dt = e^{-\tau} \int_0^{\infty} e^{-2t} dt =$$

FOR $\tau > 0$

$$= \frac{1}{2} e^{-\tau}$$

SINCE $\psi(\tau) = \psi(-\tau) \Rightarrow \psi(\tau) = \frac{1}{2} e^{-|\tau|}$

$$(e) \quad \cancel{\psi(\tau)} \quad \frac{1}{2} e^{-\tau} u(\tau) \quad (\Rightarrow) \quad \frac{1}{2(1+j\omega)}$$

NEW

$$\frac{1}{2} e^{\tau} u(-\tau) \quad (\Rightarrow) \quad \frac{1}{2(1-j\omega)}$$

$$\text{THUS} \quad \frac{1}{2} e^{-|\tau|} = \frac{1}{2} e^{-\tau} u(\tau) + \frac{1}{2} e^{\tau} u(-\tau) \quad (\Rightarrow) \quad \frac{1}{(1+\omega^2)}$$

