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IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE
UNIVERSITY OF LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2000

MSc and EEE/ISE PART IV: M.Eng. and ACGI

SPECTRAL ESTIMATION AND ADAPTIVE SIGNAL PROCESSING

Friday, 19 May 2000, 10:00 am

Corrected Copy

There are FOUR questions on this paper.

Answer Question ONE and TWO others.

Question 1 carries 40% of the total mark and each other question carries 30%.

Question 1 (a) (iii)

Time allowed: 3:00 hours

Examiners: Dr J.A. Chambers, Dr J.M.C. Clark

Special instructions for invigilators: None

Information for candidates: None

1. The coefficient update equation for the signed-regressor (LMS) algorithm is given by

$$\underline{w}[n+1] = \underline{w}[n] + 2\mu e[n]\text{sign}(\underline{x}[n])$$

where $e[n] = d[n] - \underline{w}^T[n]\underline{x}[n]$; $d[n]$, $\underline{w}[n]$ and $\underline{x}[n]$ are respectively the desired response, coefficient vector and input vector at the n-th sample; and $(.)^T$ denotes vector transpose.

(a) Given Price's theorem, which states that for a pair, α and β, of zero mean jointly Gaussian random variables

$$E\{\alpha\,\text{sign}(\beta)\} = \frac{1}{\sigma_\beta}\sqrt{\frac{2}{\pi}}E\{\alpha\beta\}$$

and any other necessary assumptions, which should be stated, show that

(i) $E\{\text{sign}(\underline{x}[n])\underline{x}^T[n]\} = \dfrac{1}{\sigma_x}\sqrt{\dfrac{2}{\pi}}R$ where $R = E\{\underline{x}[n]\underline{x}^T[n]\}$;

(ii) $E\{\underline{v}[n+1]\} = (I - 2\mu\dfrac{1}{\sigma_x}\sqrt{\dfrac{2}{\pi}}R)E\{\underline{v}[n]\}$ where $\underline{v}[n] = \underline{w}[n] - \underline{w}_{opt}$

and $\underline{w}_{opt}$ is the Wiener solution;

(iii) the mean of the $\underline{w}[n]$ of length N, generated by the signed-regressor LMS algorithm converges if

$$0 < \mu < \frac{1}{N\sigma_x\sqrt{\frac{2}{\pi}}}$$

~~where N is the length of filter~~ and comment upon the learning trajectory of the signed-regressor LMS algorithm as compared to that of the conventional LMS algorithm.

(b)

(i) Define and discuss the term misadjustment as used to quantify the performance of an adaptive algorithm.

(ii) Given that $\|\underline{v}[n+1]\|^2 = E\{\underline{v}^T[n+1]\underline{v}[n+1]\}$ and the identity $E\{\underline{v}^T[n]\underline{x}[n]\underline{x}^T[n]\underline{v}[n]\} = E\{\underline{v}^T[n]E\{\underline{x}[n]\underline{x}^T[n]\}\underline{v}[n]\}$ verify that

$$\|\underline{v}[n+1]\|^2 = \|\underline{v}[n]\|^2 - 4\left(\frac{1}{\sigma_x}\sqrt{\frac{2}{\pi}}\mu - \mu^2 N\right)E\{\underline{v}^T[n]R\underline{v}[n]\} + 4\mu^2 N J_{min}$$

where N is the number of coefficients in the adaptive filter.

(iii) With the assumption that the signed-regressor LMS algorithm is convergent in the mean square, use the result in (ii) to calculate the misadjustment of the algorithm. Show how this expression simplifies for small μ.

2. The extended normal equations are represented in matrix form as

$$R \begin{bmatrix} 1 \\ \underline{a}_p \end{bmatrix} = \begin{bmatrix} E_p \\ \underline{0} \end{bmatrix}$$

where $\underline{a}_p = [a_p(1), a_p(2), \ldots, a_p(p)]^T$ is the parameter vector of a p-th order forward linear predictor and E_p is the corresponding prediction error power.

(a) Develop the Levinson-Durbin recursion for the solution of the extended normal equations and show explicitly the reflection coefficients necessary for the realisation of a forward linear prediction error filter in a lattice structure.

(b) A discrete time random signal, x[n], has autocorrelation sequence

$$r_x(\tau) = (0.2)^{|\tau|}$$

(i) Calculate the reflection coefficients of a second order forward linear prediction error filter and sketch the corresponding lattice realisation.

(ii) Show how the reflection coefficients change if uncorrelated zero mean white noise with variance $\sigma^2 = 0.1$ is added to x[n]. Comment upon the general effect of noise added to a discrete time random signal upon its corresponding reflection coefficients.

(c)

(i) Suggest and justify a procedure based upon reflection coefficients which can be used to test the extendibility of a finite length sequence

$$r_x(0), r_x(1), r_x(2), \ldots, r_x(p)$$

into a valid infinite support autocorrelation sequence.

(ii) Evaluate the constraints upon α and β so that the sequence

$$r_x(0) = 1, \;\; r_x(1) = \alpha, \;\; r_x(2) = \beta$$

is extendible.

(iii) When the sequence in (ii) is extendible, provide two different valid extensions.

3.

(a) Show in block diagram form how a sub-band adaptive filter can be used in a hands-free mobile phone for acoustic echo control and discuss the effect double talk has on the operation of the adaptive filter.

The coefficient update equation for the normalised least mean square (NLMS) algorithm is

$$\underline{w}[n+1] = \underline{w}[n] + \frac{\beta e[n]\underline{x}[n]}{\varepsilon + \|\underline{x}[n]\|^2}$$

where $e[n] = d[n] - \underline{w}^T[n]\underline{x}[n]$, and $d[n]$, $\underline{w}[n]$ and $\underline{x}[n]$ are respectively the desired response, coefficient vector and input vector at the n-th sample.

(b) Verify that the update equation $\underline{w}[n+1] = \underline{w}[n] + \mu e^p[n]\underline{x}[n]$ based upon the a posteriori error, $e^p[n] = d[n] - \underline{w}^T[n+1]\underline{x}[n]$, is equivalent to the NLMS algorithm. Comment upon the relationship between β, ε and μ.

(c) Derive the coefficient update equation for the affine projection (AP) algorithm which is a generalisation of NLMS based upon forcing the following L a posteriori errors to be zero

$$\begin{bmatrix} e^p[n] \\ e^p[n-1] \\ \vdots \\ e^p[n-L] \end{bmatrix} = \begin{bmatrix} d[n] \\ d[n-1] \\ \vdots \\ d[n-L] \end{bmatrix} - \begin{bmatrix} \underline{w}^T[n+1]\underline{x}[n] \\ \underline{w}^T[n+1]\underline{x}[n-1] \\ \vdots \\ \underline{w}^T[n+1]\underline{x}[n-L] \end{bmatrix} = \underline{0}$$

(d) Compare the computational complexity of the AP and the recursive least squares algorithms and state the condition for which the two algorithms are identical.

4.

(a) Explain the challenges in estimation of the power spectral density from a measurement of a discrete time random signal.

(b) An industrial spectrum analyser is based upon a continuous update of the estimate of the power spectral density of a discrete time random signal of the form

$$\hat{P}_i(f) = \alpha \hat{P}_{i-1}(f) + \frac{(1-\alpha)}{N}\left|\sum_{n=0}^{N-1} x_i[n]\exp(-j2\pi fn)\right|^2, \quad f \in (-\tfrac{1}{2},\tfrac{1}{2}].$$

where $x_i[n] = x[n + iN]$ is the i-th block of N data samples. The initialization of the update equation is $\hat{P}_i(f) = 0, \forall f$.

(i) Discuss in detail the philosophy which underlies this approach to spectrum estimation and comment upon the basis on which the parameter α should be chosen.

(ii) With the assumption that the blocks of $x[n]$ are from uncorrelated Gaussian discrete time random signals, and that $0 < \alpha < 1$, calculate the mean and variance of $\hat{P}_i(f)$, given that the variance of the periodogram estimator is given approximately by $\mathrm{var}\{\hat{P}_{PER}(f)\} \approx P_x^2(f)$.

(c) Suggest a model-based approach that could be used for on-line spectral estimation.

1. (i) a) Assume elements of $x[n]$ are zero mean Gaussian r.v.s from a wide sense stationary process

$$\text{sign}(\underline{x}[n]) = \begin{bmatrix} \text{sign}(x[n]) \\ \text{sign}(x[n-1]) \\ \vdots \\ \text{sign}(x[n-N+1]) \end{bmatrix}$$

$$\text{sign}(\underline{x}[n])\underline{x}^T[n] = \begin{bmatrix} \text{sign}(x[n])x[n] & \text{sign}(x[n])x[n-1] & \cdots & \text{sign}(x[n])x[n-N+1] \\ \vdots & \vdots & & \vdots \\ \text{sign}(x[n-N+1])x[n] & \cdots & \cdots & \text{sign}(x[n-N+1])x[n-N+1] \end{bmatrix}$$

From Price's theorem, $\text{sign}(x[n])x[n] = \frac{1}{\sigma_x}\sqrt{\frac{2}{\pi}}E\{x[n]x[n]\} = \frac{1}{\sigma_x}\sqrt{\frac{2}{\pi}}r_x(0)$

Thus,

$$E\{\text{sign}(\underline{x}[n])\underline{x}^T[n]\} = \begin{bmatrix} \frac{1}{\sigma_x}\sqrt{\frac{2}{\pi}}r_x(0) & \frac{1}{\sigma_x}\sqrt{\frac{2}{\pi}}r_x(1) & \cdots & \frac{1}{\sigma_x}\sqrt{\frac{2}{\pi}}r_x(N-1) \\ \vdots & \vdots & & \vdots \\ \frac{1}{\sigma_x}\sqrt{\frac{2}{\pi}}r_x(N-1) & \frac{1}{\sigma_x}\sqrt{\frac{2}{\pi}}r_x(N-2) & \cdots & \frac{1}{\sigma_x}\sqrt{\frac{2}{\pi}}r_x(0) \end{bmatrix}$$

$$= \frac{1}{\sigma_x}\sqrt{\frac{2}{\pi}}\,R \quad \blacksquare$$

(4)

b)

$$\begin{aligned}\underline{w}[n+1] &= \underline{w}[n] + 2\mu\,\text{sign}(\underline{x}[n])\left(d[n] - \underline{x}^T[n]\underline{w}[n]\right) \\ &= \underline{w}[n] - 2\mu\,\text{sign}(\underline{x}[n])\underline{x}^T[n]\underline{w}[n] + 2\mu d[n]\,\text{sign}(\underline{x}[n]) \\ &= \left(I - 2\mu\,\text{sign}(\underline{x}[n])\underline{x}^T[n]\right)\underline{w}[n] + 2\mu d[n]\,\text{sign}(\underline{x}[n]) \quad \text{— (I)}\end{aligned}$$

From the Wiener solution $R\underline{w}_{opt} = \underline{p}$, and exploiting the statistical independence assumption between $\underline{x}[n]$ and $\underline{w}[n]$, subtract $\underline{w}_{opt}$ from (I) and taking expectations

$$\begin{aligned}E\{\underline{w}[n+1] - \underline{w}_{opt}\} &= \left(I - 2\mu\frac{1}{\sigma_x}\sqrt{\frac{2}{\pi}}R\right)E\{\underline{w}[n]\} + 2\mu\frac{1}{\sigma_x}\sqrt{\frac{2}{\pi}}\underline{p} - \underline{w}_{opt} \\ &= \left(I - 2\mu\frac{1}{\sigma_x}\sqrt{\frac{2}{\pi}}R\right)E\{\underline{w}[n]\} - \left(I - 2\mu\frac{1}{\sigma_x}\sqrt{\frac{2}{\pi}}R\right)\underline{w}_{opt} \\ &= \left(I - 2\mu\frac{1}{\sigma_x}\sqrt{\frac{2}{\pi}}R\right)\left(E\{\underline{w}[n]\} - \underline{w}_{opt}\right)\end{aligned}$$

$$E\{\underline{v}[n+1]\} = \left(I - 2\mu\frac{1}{\sigma_x}\sqrt{\frac{2}{\pi}}R\right)E\{\underline{v}[n]\} \quad \blacksquare$$

(8)

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1\. Cont. c) R – positive definite

$R = Q\Lambda Q^T$, $\underline{v}'[n] \triangleq Q^T E\{\underline{v}[n]\}$, hence $\underline{v}'[n+1] = \left(I - 2\mu \frac{1}{\sigma_x}\sqrt{\frac{2}{\pi}}\Lambda\right)\underline{v}'[n]$

which is decoupled, thus $v_j'[n+1] = \left(1 - 2\mu\frac{1}{\sigma_x}\sqrt{\frac{2}{\pi}}\lambda_j\right)v_j'[n]$ $j = 1, 2, \ldots, N$,

for convergence to zero, $\left|1 - 2\mu\frac{1}{\sigma_x}\sqrt{\frac{2}{\pi}}\lambda_j\right| < 1$,

$$\Rightarrow -1 < 1 - 2\mu\frac{1}{\sigma_x}\sqrt{\frac{2}{\pi}}\lambda_j < 1$$

$$\Rightarrow 1 > \mu\frac{1}{\sigma_x}\sqrt{\frac{2}{\pi}}\lambda_j > 0, \quad 0 < \mu < \frac{1}{\left(\frac{1}{\sigma_x}\sqrt{\frac{2}{\pi}}\lambda_j\right)} \quad \forall j$$

Worst case, $0 < \mu < \dfrac{1}{\frac{1}{\sigma_x}\sqrt{\frac{2}{\pi}}\lambda_{max}}$, but $\lambda_{max} < tr(R) = N\sigma_x^2$,

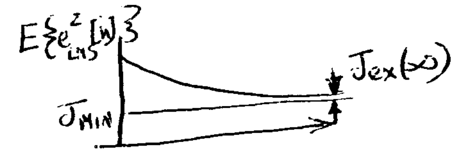
finally $0 < \mu < \dfrac{1}{\sigma_x\sqrt{\frac{2}{\pi}}N}$

Learning trajectory essentially identical to that of LMS, i.e.

$$E\{\underline{v}[n+1]\} = (I - 2\mu R)E\{\underline{v}[n]\}$$

Scale factor difference (8)

(ii) a) $\mathcal{M} \triangleq \dfrac{J_{ex}(\infty)}{J_{min}}$ Dimensionless (4)



b) From I (in (i) b)

$$\underline{v}[n+1] = \underline{v}[n] + 2\mu\, \text{sign}(\underline{x}[n])e[n]$$

but $e[n] = d[n] - \underline{w}^T[n]\underline{x}[n] = d[n] - (\underline{v}[n] + \underline{w}_{opt}[n])^T\underline{x}[n]$

$$= d[n] - \underline{w}_{opt}^T[n]\underline{x}[n] - \underline{v}^T[n]\underline{x}[n]$$

$$= e_o[n] - \underline{v}^T[n]\underline{x}[n] = e_o[n] - \underline{x}^T[n]\underline{v}[n]$$

so $\underline{v}[n+1] = \left(I - 2\mu\, \text{sign}(\underline{x}[n])\underline{x}^T[n]\right)\underline{v}[n] + 2\mu\, \text{sign}(\underline{x}[n])e_o[n]$

$$\|\underline{v}^T[n+1]\underline{v}[n+1]\|^2 = E\Big\{\left(\underline{v}^T[n](I - 2\mu\, \underline{x}[n]\,\text{sign}(\underline{x}^T[n])) + 2\mu e_o[n]\,\text{sign}(\underline{x}^T[n])\right).$$
$$\left((I - 2\mu\, \text{sign}(\underline{x}[n])\underline{x}^T[n])\underline{v}[n] + 2\mu\, \text{sign}(\underline{x}[n])e_o[n]\right)\Big\}$$

$$= E\{\underline{v}^T[n]\underline{v}[n]\} - 4\mu E\{\underline{v}^T[n]\underline{x}[n]\,\text{sign}(\underline{x}^T[n])\underline{v}[n]\} + 4\mu^2 N E\{\underline{v}^T[n]\underline{x}[n]\,\underline{x}^T[n]\,\underline{v}[n]\}$$
$$+ 4\mu^2 N E\{e_o^2[n]\} + 4\mu E\{\underline{v}^T[n]\,\text{sign}(\underline{x}[n])e_o[n]\} - 8\mu^2 N E\{\underline{v}^T[n]\underline{x}[n]e_o[n]\}$$

J.C. [signature]

(ii) b) Cont

From identity

$$E\{\underline{v}^T[n]\underline{x}[n]\,\mathrm{sign}(\underline{x}^T[n])\underline{v}[n]\} = \frac{1}{\sigma_x}\sqrt{\frac{2}{\pi}}\,E\{\underline{v}^T[n]R\underline{v}[n]\}$$

similarly

$$E\{\underline{v}^T[n]\underline{x}[n]\underline{x}^T[n]\underline{v}[n]\} = E\{\underline{v}^T[n]R\underline{v}[n]\}$$

and from independence assumption

$$E\{\underline{v}^T[n]\underline{x}[n]e_o[n]\} = E\{\underline{v}^T[n]\}\underbrace{E\{\underline{x}[n]e_o[n]\}}_{0} = 0$$

and

$$E\{\underline{v}^T[n]\mathrm{sign}(\underline{x}[n])e_o[n]\} = 0.$$

Thus

$\underline{\mathrm{II}}$ — $$\|\underline{v}[n+1]\|^2 = \|\underline{v}[n]\|^2 + 4\left(\mu^2N - \frac{1}{\sigma_x}\sqrt{\frac{2}{\pi}}\mu\right)E\{\underline{v}^T[n]R\underline{v}[n]\} + 4\mu^2NJ_{MIN}$$

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c) In convergence

$$\lim_{n\to\infty}\{\|\underline{v}[n+1]\|^2 = \|\underline{v}[n]\|^2\},$$

knowing $$\mathcal{M} = \frac{J_{ex}(\infty)}{J_{MIN}} = \frac{E\{\underline{v}^T[\infty]R\underline{v}[\infty]\}}{J_{MIN}}.$$

But from $\underline{\mathrm{II}}$ $$E\{\underline{v}^T[\infty]R\underline{v}[\infty]\} = \frac{\mu^2NJ_{MIN}}{\left(\frac{1}{\sigma_x}\sqrt{\frac{2}{\pi}}\mu - \mu^2N\right)}$$

$$\Rightarrow \mathcal{M} = \frac{\mu N}{\left(\frac{1}{\sigma_x}\sqrt{\frac{2}{\pi}} - \mu N\right)}$$

When $\mu \approx 0$, $\mathcal{M} = \mu N\sigma_x\sqrt{\pi/2}$

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J.C. [signature]

2 (i) Given the exchange matrix $J = J^{-1} = \begin{bmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{bmatrix}$

and R_{xx} which is Toeplitz, symmetric. Denote $R_{xx_{p+1\times p+1}} = R_{xx}^{p}$

$$J R_{xx}^{p} J^{-1} = R_{xx}$$

therefore $J R_{xx} J^{-1} \begin{bmatrix} 1 \\ \underline{a}_p \end{bmatrix} = \begin{bmatrix} E_p \\ \underline{0}_p \end{bmatrix} \Rightarrow R_{xx} \begin{bmatrix} a_p(p) \\ \vdots \\ a_p(1) \end{bmatrix} = R_{xx} \begin{bmatrix} \underline{a}_p^R \\ 1 \end{bmatrix}$

$$= J^{-1} \begin{bmatrix} E_p \\ \underline{0}_p \end{bmatrix} = \begin{bmatrix} \underline{0}_p \\ E_p \end{bmatrix}$$

Now,

$$R_{xx}^{p+1} \begin{bmatrix} 1 \\ \underline{a}_{p+1} \end{bmatrix} = \begin{bmatrix} R_{xx}^{p} & r_{xx}(p+1) \\ & \vdots \\ r_{xx}(-[p+1]) \dots & r_{xx}(0) \end{bmatrix} \begin{bmatrix} 1 \\ \underline{a}_{p+1} \end{bmatrix}$$

$$= \begin{bmatrix} r_{xx}(0) \dots & r_{xx}(p+1) \\ r_{xx}(-[p+1]) & R_{xx}^{p} \end{bmatrix} \begin{bmatrix} 1 \\ \underline{a}_{p+1} \end{bmatrix} = \begin{bmatrix} E_{p+1} \\ \underline{0}_{p+1} \end{bmatrix}$$

Finally, form

$$R_{xx}^{p+1} \begin{bmatrix} 1 \\ \underline{a}_p \\ 0 \end{bmatrix} = \begin{bmatrix} E_p \\ \underline{0}_p \\ \gamma_p \end{bmatrix} \quad \text{and} \quad R_{xx}^{p+1} \begin{bmatrix} 0 \\ \underline{a}_p^R \\ 1 \end{bmatrix} = \begin{bmatrix} \gamma_p \\ \underline{0}_p \\ E_p \end{bmatrix}$$

(where $\gamma_p = r_{xx}(-[p+1]) + \sum_{l=1}^{p} a_p(l) r_{xx}(-[p+1]+l)$)

Set $\Gamma_{p+1} = -\gamma_p / E_p$ (** Reflection Coefficient **)

Thus $R_{xx}^{p+1} \begin{bmatrix} 1 \\ \underline{a}_{p+1} \end{bmatrix} = R_{xx}^{p+1} \left(\begin{bmatrix} 1 \\ \underline{a}_p \\ 0 \end{bmatrix} + \Gamma_{p+1} \begin{bmatrix} 0 \\ \underline{a}_p^R \\ 1 \end{bmatrix} \right) = \begin{bmatrix} E_{p+1} \\ \underline{0}_{p+1} \end{bmatrix} = \begin{bmatrix} E_p(1-\Gamma_{p+1}^2) \\ \underline{0}_{p+1} \end{bmatrix}$

Levinson-Durbin recursion

$$\begin{bmatrix} 1 \\ \underline{a}_{p+1} \end{bmatrix} = \begin{bmatrix} 1 \\ \underline{a}_p \\ 0 \end{bmatrix} + \Gamma_{p+1} \begin{bmatrix} 0 \\ \underline{a}_p^R \\ 1 \end{bmatrix} \quad \text{and} \quad E_{p+1} = E_p(1-\Gamma_{p+1}^2)$$

Initialisation

$a_0(0) = 1, \quad E_0 = r_{xx}(0).$ 10

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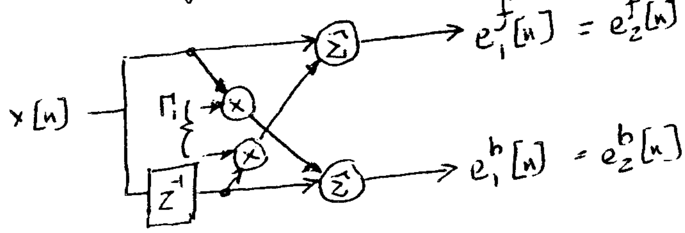
2 (ii) a) $\Gamma_1 = -\dfrac{r_0}{\epsilon_0} = -\dfrac{0.2}{1.0} = -0.2$

$\epsilon_1 = 1.0(1-(0.2)^2) = 0.96$

$\Gamma_2 = -\dfrac{r_1}{\epsilon_1} = -\dfrac{(r_{xx}(2) + a_1(1)\, r_x(1))}{0.96} = -\dfrac{(0.04 - 0.2 \times 0.2)}{0.96} = 0$

NB. Process, first order AR, hence



(7)

b) White noise will only effect $r_{xx}(0)$, hence $r_{xx}(0) = 1.1$, and (3)

$\Gamma_1 = \dfrac{-0.2}{1.1} = 0.\dot{1}\dot{8}$

General effect, to reduce magnitude of reflection coefficients

(iii) a) A necessary and sufficient condition is for the reflection coefficients to be bounded in magnitude by unity.

If $r_x(0), r_x(1) \cdots r_x(p)$ is extendible, R_{xx} must be positive definite which implies that $|\Gamma_j| < 1$. Or, if $|\Gamma_j| < 1$ for $j = 1, \cdots, p$, then $\Gamma_j^e = \begin{cases} \Gamma_j \,; & j = 1, \cdots, p \\ 0 & j > p \end{cases}$ represents a valid extension $-AR(p)$. (5)

b) $|\Gamma_1| = \alpha < 1$ and $|\Gamma_2| = \left| \dfrac{r_{xx}(2) + a_1(1) r_{xx}(1)}{\epsilon_1} \right| = \left| \dfrac{r_{xx}(2) + \Gamma_1 r_x(1)}{r_x(0)(1 - \Gamma_1^2)} \right|$

$= \left| \dfrac{\beta - \alpha^2}{1 - \alpha^2} \right| < 1$ (3)

or $2\alpha^2 - 1 < \beta < 1$

c) $\Gamma = [\Gamma_1, \Gamma_2, 0, 0, \cdots]^T$ (2)

$\Gamma = [\Gamma_1, \Gamma_2, 0.5, 0, \cdots]^T$

(30/30)

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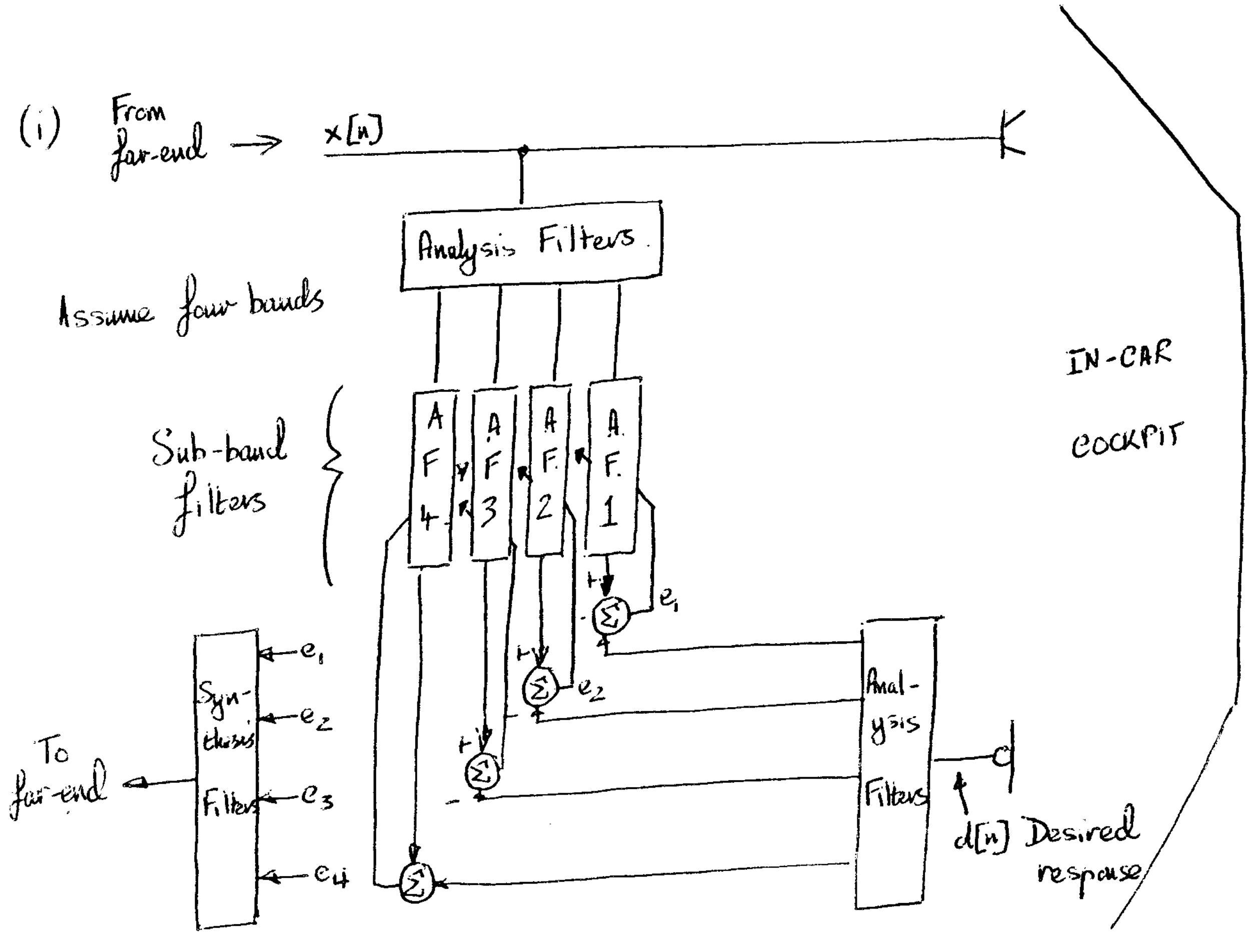
3 (i)



Double talk, SNR in desired response drops dramatically, typical to use a double talk detector and "freeze" the adaptive filters. (6)

(ii)
$$\underline{w}[n+1] = \underline{w}[n] + \mu\left[d[n] - \underline{w}^T[n+1]\underline{x}[n]\right]\underline{x}[n]$$
$$= \underline{w}[n] + \mu d[n]\underline{x}[n] - \mu\left[\underline{x}[n]\underline{x}^T[n]\right]\underline{w}[n+1]$$

$$\Rightarrow \left[I + \mu\underline{x}[n]\underline{x}^T[n]\right]\underline{w}[n+1] = \underline{w}[n] + \mu d[n]\underline{x}[n]$$
$$= \underline{w}[n] + \mu\left[d[n] - \underline{w}^T[n]\underline{x}[n]\right]\underline{x}[n] + \mu\underline{w}^T[n]\underline{x}[n]\underline{x}[n]$$
$$= \underline{w}[n] + \mu d[n]\underline{x}[n] + \mu\left[\underline{x}[n]\underline{x}^T[n]\right]\underline{w}[n] - \mu\underline{w}^T[n]\underline{x}[n]\underline{x}[n]$$
$$= \left[I + \mu\underline{x}[n]\underline{x}^T[n]\right]\underline{w}[n] + \mu d[n]\underline{x}[n] - \mu\underline{w}^T[n]\underline{x}[n]\underline{x}[n]$$

$$\Rightarrow \underline{w}[n+1] = \underline{w}[n] + \mu e[n]\left[I + \mu\underline{x}[n]\underline{x}^T[n]\right]^{-1}\underline{x}[n]$$

With Woodbury's identity
$$\left(I + \mu\underline{x}[n]\underline{x}^T[n]\right)^{-1} = I - \frac{\mu\,\underline{x}[n]\underline{x}^T[n]}{1 + \mu\|\underline{x}[n]\|^2}$$

Thus
$$\underline{w}[n+1] = \underline{w}[n] + \mu e[n]\underline{x}[n] - \mu^2\frac{e[n]\,\underline{x}[n]\underline{x}^T[n]\underline{x}[n]}{(1 + \mu\|\underline{x}[n]\|^2)}$$
and $\underline{x}[n]\underline{x}^T[n]\underline{x}[n] = \underline{x}[n]\|\underline{x}[n]\|^2$
$$= \underline{w}[n] + \mu e[n]\underline{x}[n] - \mu^2\frac{e[n]\|\underline{x}[n]\|^2\underline{x}[n]}{(1 + \mu\|\underline{x}[n]\|^2)}$$

J C

3) (ii) Cont.

$$\underline{w}[n+1] = \underline{w}[n] + \mu e[n]\left[1 - \frac{\mu\|\underline{x}[n]\|^2}{1+\mu\|\underline{x}[n]\|^2}\right]\underline{x}[n]$$ 7/9

$$= \underline{w}[n] + \frac{1}{\frac{1}{\mu} + \|\underline{x}[n]\|^2} e[n]\,\underline{x}[n]$$

c.f. NLMS $\underline{w}[n+1] = \underline{w}[n] + \dfrac{\beta e[n]\underline{x}[n]}{\epsilon + \|\underline{x}[n]\|^2}$

Thus equivalence when $\beta = 1$, $\epsilon = 1/\mu$. 14

(iii) Update $\underline{w}[n+1] = \underline{w}[n] + \Delta\underline{w}[n]$

where $\Delta\underline{w}[n]$ is such that $\underset{L\times 1}{\underline{e}^P} = \underset{L\times 1}{\underline{d}} - \underset{L\times N}{\begin{bmatrix}\underline{x}^T[n]\\ \vdots \\ \underline{x}^T[n-L]\end{bmatrix}} \underset{N\times 1}{\underline{w}[n+1]} = \underline{0}$

$$\Rightarrow \quad X^T\underline{w}[n+1] = \underline{d}$$

$$\Rightarrow \quad X^T(\underline{w}[n] + \Delta\underline{w}[n]) = \underline{d}$$

$$\Rightarrow \quad X^T\underline{w}[n] + X^T\Delta\underline{w}[n] = \underline{d}$$

$$\Rightarrow \quad X^T\Delta\underline{w}[n] = \underline{d} - X^T\underline{w}[n] = \underline{e}^{priori}$$

Undetermined set of equations, use pseudo inverse $\Delta\underline{w}[n] = (X^T)^{\#^{-1}}\underline{e}^{priori}$

Thus AP coefficient update $\underline{w}[n+1] = \underline{w}[n] + X(X^TX)^{-1}\underline{e}^{priori}$ 8

(iv) RLS $\sim O(N^2)$ Two algorithms identical when $N = L$.

AP $\sim O(L^3 + NL)$ 2

30/30

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4 (i) Amount of data is limited – may be very small due to application or requirement of statistical stationarity over the observation.

Data is often contaminated by noise or interference.

Any prior knowledge of the process should be exploited.

(ii) a) The goal with the spectrum analyser is to continuously refine the spectrum estimate as new data is read. With the arrival of each new data block, the periodogram is calculated and averaged with the previous spectrum estimate. Note, a simple running average is not used to circumvent the need for a stationarity assumption. The choice of $0 < \alpha < 1$ forgets the past value of $\hat{P}_i(e^{j\omega})$ as new data is measured. When $\alpha = 0$, only the periodogram of the most recent data values is used. The estimator uses an exponentially weighted average of previous periodograms.

b) If $Q_i(f) \triangleq \frac{1}{N}\left|\sum_{n=0}^{N-1} x_i[n] e^{-j2\pi f n}\right|^2$

$$\hat{P}_i(f) = \alpha \hat{P}_{i-1}(f) + (1-\alpha) Q_i(f)$$

As $\hat{P}_{-1}(f) = 0$, $\hat{P}_i(f) = \sum_{k=0}^{i} (1-\alpha)\alpha^k Q_{i-k}(f)$.

Bartlett Window

Since $Q_{i-k}(f)$ is the periodogram, $E\{Q_{i-k}(f)\} = P_X(f) * W_B(f)$

Thus, $E\{\hat{P}_i(f)\} = \sum_{k=0}^{i} (1-\alpha)\alpha^k \left[P_X(f) * W_B(f)\right]$

$$= \left[P_X(f) * W_B(f)\right](1-\alpha) \underbrace{\sum_{k=0}^{i} \alpha^k}_{\frac{1-\alpha^{i+1}}{1-\alpha}}$$

$$= (1-\alpha^{i+1})\left[P_X(f) * W_B(f)\right]$$

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4 (ii) b) Cont.

As the blocks are uncorrelated and Gaussian

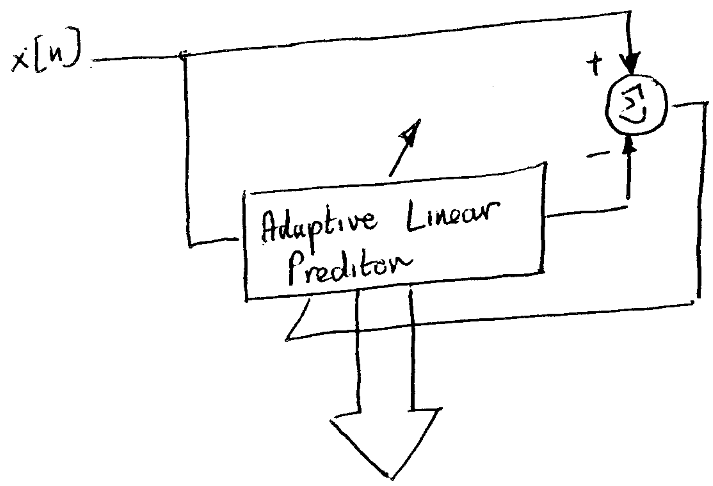
$$\text{var}\{\hat{P}_l(f)\} = \sum_{k=0}^{l} (1-\alpha)\alpha^k \, \text{var}\{Q_{l-k}(f)\}$$

from the question, $\text{var}\{Q_{l-k}(f)\} \approx P_x^2(f)$,

thus

$$\text{var}\{\hat{P}_l(f)\} \approx (1-\alpha^{l+1}) P_x^2(f)$$

(iii) Assume AR model – with order p.



$$\hat{P}_l(f) = \frac{1}{\left|1 + \hat{a}_{p,l}(1)e^{-j2\pi f} + \dots + \hat{a}_{p,l}(p)e^{-j2\pi p f}\right|^2}$$

Jonathan Chambers.

21-11-99.

JC
MLL