IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2014

EEE/EIE PART I: MEng, Beng and ACGI

ANALYSIS OF CIRCUITS

Friday, 6 June 10:00 am

Time allowed: 2:00 hours

There are THREE questions on this paper.

Answer ALL questions.
Q1 carries 40% of the marks. Questions 2 and 3 carry equal marks (30% each).

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s) :	D.M. Brookes
	Second Marker(s) :	P. Georgiou

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ANALYSIS OF CIRCUITS

Information for Candidates:

The following notation is used in this paper:

1. The voltage waveform at node X in a circuit is denoted by $x(t)$, the phasor voltage by X and the root-mean-square (or RMS) phasor voltage by $\widetilde{X} = \frac{X}{\sqrt{2}}$. The complex conjugate of X is X^*.
2. Component and source values in a circuit are normally given in Ohms, Farads, Henrys, Volts or Amps with the unit symbol omitted. Where an imaginary number is specified, it represents the complex impedance or phasor value.
3. Times are given in seconds unless otherwise stated.
4. Unless otherwise indicated, frequency response graphs should use a linear axis for phase and logarithmic axes for frequency and magnitude.

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1. a) Using nodal analysis, calculate the voltages at nodes X and Y of Figure 1.1. [5]

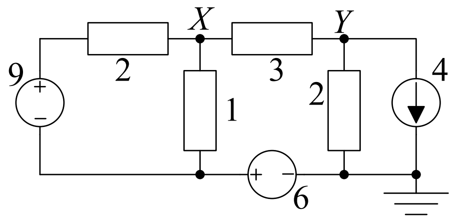

Figure 1.1

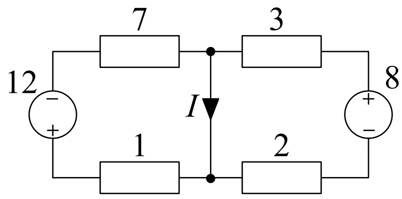

Figure 1.2

b) Use the principle of superposition to find the current I in Figure 1.2. [5]

c) Draw the Thévenin equivalent circuit of the network in Figure 1.3 and find the value of its components. [5]

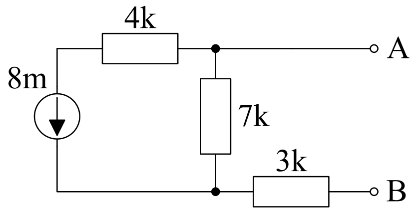

Figure 1.3

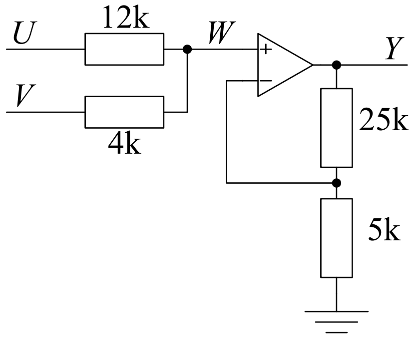

Figure 1.4

d) Assuming the opamp in the circuit of Figure 1.4 is ideal, give an expression for Y in terms of U and V. [5]

e) The graph of Figure 1.5 plots the output voltage, Y, against the input voltage, X, for the circuit shown in Figure 1.6. The graph consists of two straight lines that intersect at the point $(10, 10)$ and that pass through the origin and the point $(20, 12)$ respectively. Assuming that the forward voltage drop of the diode is $0.7\,\text{V}$, determine the values of the resistor, R, and the voltage source, V. [5]

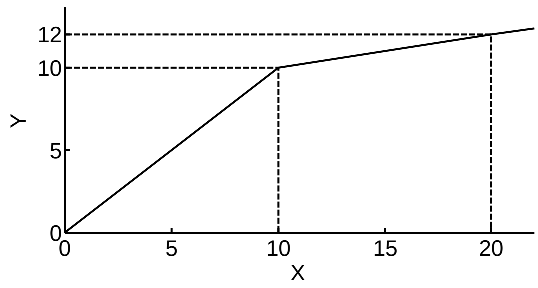

Figure 1.5

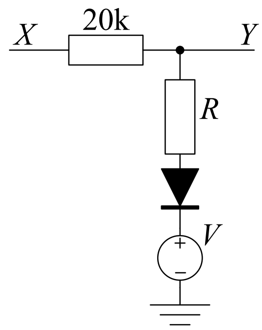

Figure 1.6

f) Determine the gain, $\frac{Y}{X}$, for the block diagram shown in Figure 1.7. The rectangular blocks are drawn with inputs at the left and outputs at the right and have gains of F and G respectively. The open circles represent adder/subtractors; their inputs have the signs indicated on the diagram and their outputs are W and Y respectively. [5]

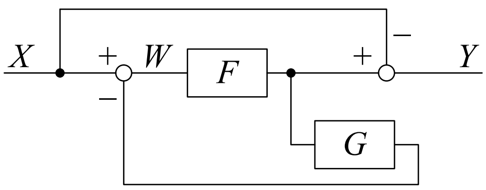

Figure 1.7

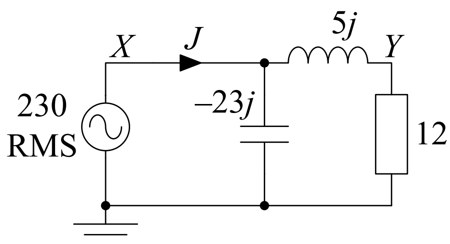

Figure 1.8

g) In the circuit of Figure 1.8, the RMS phasor $\widetilde{X} = 230$ and the component values shown indicate complex impedances. Determine the value of the RMS current phasor $\tilde{J}$ and of the complex power, $\widetilde{V} \times \tilde{I}^*$, absorbed by each of the four components.

[5]

h) Figure 1.10 shows a transmission line of length 100 m that is terminated in a resistive load, R, with reflection coefficient $\rho = +0.6$. The line has a propagation velocity of $u = 2 \times 10^8\,\mathrm{m/s}$. At time $t = 0$, a forward-travelling (i.e. left-to-right) pulse arrives at X with amplitude 4 V and duration $1.5\,\mu$s, as shown in Figure 1.9.

Draw a dimensioned sketch of the waveform at Y, a point 60 m from the end of the line, for $0 \le t \le 3\,\mu$s. Assume that no reflections occur at point X. [5]

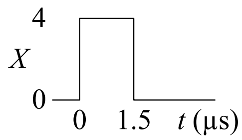

Figure 1.9

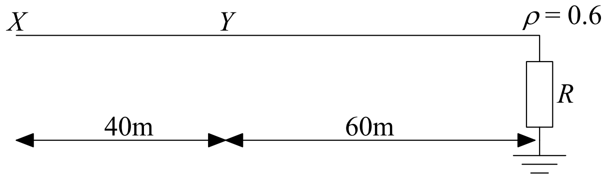

Figure 1.10

2. a) Show that the transfer function of the circuit of Figure 2.1 can be written in the form

$$H(j\omega)=\frac{Y(j\omega)}{X(j\omega)}=\frac{1}{\left(\frac{j\omega}{\omega_0}\right)^2+2\zeta\frac{j\omega}{\omega_0}+1}$$

and express the values of ω_0 and ζ in terms of the component values L, C and R. [5]

b) Give expressions for the low and high frequency asymptotes of $H(j\omega)$ and the angular frequency at which they have the same magnitude. [3]

c) Determine the magnitude and phase of $H(j\omega)$ at $\omega=\omega_0$. [2]

d) Show that $|H(j\omega)|^{-2}$ may be written as a polynomial with real coefficients in x where $x=\left(\frac{\omega}{\omega_0}\right)^2$. By differentiating this polynomial, or otherwise, show that the maximum value of $|H(j\omega)|$ occurs at $\omega=\omega_0\sqrt{1-2\zeta^2}$. [6]

e) Determine values of C and R so that $\omega_0=5000\,\mathrm{rad/s}$ and $\zeta=0.1$ given that $L=100\,\mathrm{mH}$. [2]

i) Sketch a dimensioned graph of $|H(j\omega)|$ in decibels using a logarithmic frequency axis. Your graph should include both the high and low frequency asymptotes in addition to a sketch of the true magnitude response. [3]

ii) If $x(t)=3\cos\omega_0 t$, determine the average power dissipation of the circuit and the peak value of the energy, $\frac{1}{2}Cy^2(t)$, stored in the capacitor.

[3]

iii) Determine the values of ω for which $\angle H(j\omega)=-45^\circ$ and -135°.

Hence sketch a dimensioned graph of $\angle H(j\omega)$ using a straight-line approximation with three segments. Your graph should use a logarithmic frequency axis and a linear phase axis. [6]

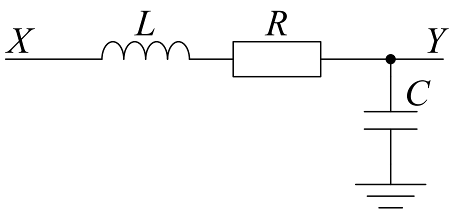

Figure 2.1

3. In the circuit of Fig. 3.1, the input, X, is driven by a voltage source as shown.

a) Derive an expression for the transfer function, $\frac{Y(j\omega)}{X(j\omega)}$ and determine the corner frequencies in its magnitude response. [4]

b) With the capacitor temporarily removed from the circuit, determine the Thévenin equivalent voltage and resistance of the remainder of the circuit at the terminals of the capacitor. [4]

c) Derive the time constant of the circuit, τ, in two ways: (i) from the Thévenin resistance found in part b) and (ii) from the denominator corner frequency found in part a). [2]

d) If the input voltage, $x(t)$, is given by

$$x(t) = \begin{cases} -2 & \text{for } t < 0 \\ +3 & \text{for } t \geq 0 \end{cases},$$

determine an expression for the output waveform, $y(t)$. Sketch its waveform over approximately the range $-\tau \leq t \leq 4\tau$. [7]

e) Assuming that the opamp in Fig. 3.2 is ideal, determine the transfer function, $\frac{V(j\omega)}{U(j\omega)}$. [4]

f) By considering the voltage across the capacitor, explain why an input voltage discontinuity of Δu will result in an output voltage discontinuity of the same amplitude. [2]

g) If $R = 20\,\text{k}\Omega$, $C = 20\,\text{nF}$ and the input voltage, $u(t)$, is given by

$$u(t) = \begin{cases} \sin 1000t & \text{for } t < 0 \\ 2\cos 2000t & \text{for } t \geq 0 \end{cases},$$

determine expressions for the output $v(t)$ for both positive and negative t. [7]

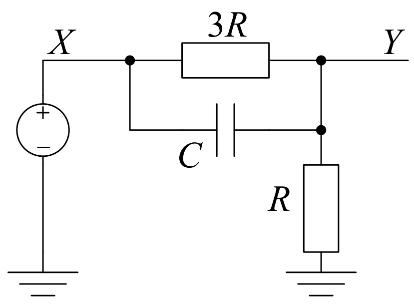

Figure 3.1

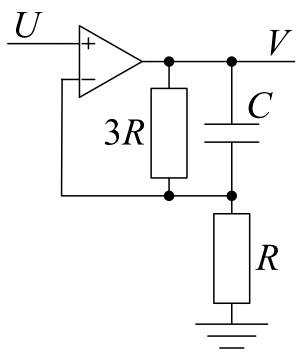

Figure 3.2

ANALYSIS OF CIRCUITS

**** Solutions 2014 ****

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The following notation is used in this paper:

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3. Times are given in seconds unless otherwise stated.
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1. a) Using nodal analysis, calculate the voltages at nodes X and Y of Figure 1.1.

[5]

We can immediately label the voltages on the bottom left and top left nodes as 6 *and* $6+9=15$ *respectively. We now write down KCL equation at node* X *to obtain*

$$\begin{aligned} \frac{X-15}{2}+X-6+\frac{X-Y}{3} &= 0 \\ \Rightarrow 3X-45+6X-36+2X-2Y &= 0 \\ \Rightarrow 11X-2Y &= 81 \end{aligned}$$

KCL at Y *gives*

$$\begin{aligned} \frac{Y-X}{3}+\frac{Y}{2}+4 &= 0 \\ \Rightarrow 2Y-2X+3Y+24 &= 0 \\ \Rightarrow -2X+5Y &= -24 \end{aligned}$$

Combining these gives $55X-4X=405-48 \quad \Rightarrow \quad X=\frac{357}{51}=7$

from which $5Y=-24+14=-10 \quad \Rightarrow \quad Y=\frac{-10}{5}=-2$

Several people wrote $\frac{X-9}{3}+\cdots$ *instead of* $\frac{X-15}{3}+\cdots$ *for KCL at* X *and a few wrote* $\cdots+\frac{Y-(-6)}{2}+\cdots$ *instead of* $\cdots+\frac{Y}{2}+\cdots$ *for KCL at* Y. *I advise labeling the nodes explicitly with their voltages on the diagram; a voltage source fixes the difference between two node voltages rather than the voltage at a particular node. Thus, the node with a ground symbol has, by definition, a voltage of* 0, *the node at the + side of the* $6\,\mathrm{V}$ *source has a voltage of* 6 *and the node at the + end of the* $9\,\mathrm{V}$ *source, therefore has a voltage of* $6+9=15$.

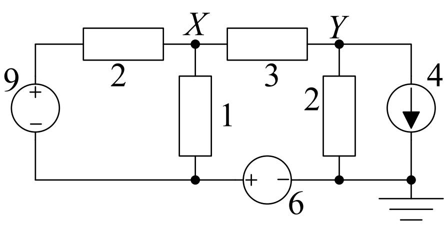


Figure 1.1

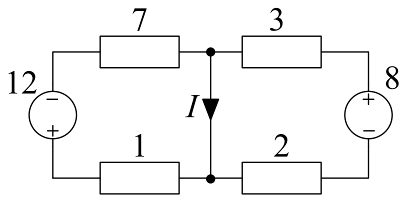


Figure 1.2

b) Use the principle of superposition to find the current I in Figure 1.2.

[5]

If we short circuit the $12\,\mathrm{V}$ *voltage source, the* $7\,\Omega$ *and* $1\,\Omega$ *are shorted out by the central link that carries* I *and so we have a current of* $I_A=\frac{8}{3+2}=+1.6\,\mathrm{A}$.

If we short circuit the $8\,\mathrm{V}$ *voltage source, the* $3\,\Omega$ *and* $2\,\Omega$ *are shorted out and so we have a current of* $I_B=\frac{-12}{7+1}=-1.5\,\mathrm{A}$.

By superposition, the total current is therefore $1.6-1.5=0.1\,\mathrm{A}$.

Some people treated the two ends of the wire carrying I as separate nodes. Each of the sub-circuits is simple enough to write the current down directly. A few people however used nodal analysis to solve the sub-circuits and often made mistakes. The most common mistake by far was not to define which node was ground (and therefore at $0\,\mathrm{V}$*); unless you do this, nodal analysis is hopeless since all the node voltages are undefined.*

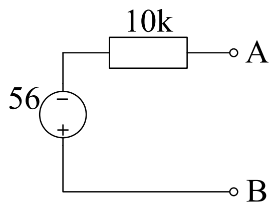
---

c) Draw the Thévenin equivalent circuit of the network in Figure 1.3 and find the value of its components. [5]

We can find the open circuit voltage by ignoring the $3\,\mathrm{k}$ *resistor (since there is no current flowing through it). The* $8\,mA$ *will therefore flow upwards through the* $7\,\mathrm{k}$ *resistor resulting in an open-circuit voltage of* $V_{AB} = -8 \times 7 = -56\,\mathrm{V}$.

To find the Thévenin resistance, we treat the current source as an open circuit. The $4\,\mathrm{k}$ *resistor now plays no part and the Thévenin resistance is therefore* $7 + 3 = 10\,\mathrm{k}$.

So the complete Thévenin equivalent is:



A more complicated approach is to do a full nodal analysis. If we define ground to be terminal B and label the top and bottom of the current source as X and Y respectively, then we want to find the voltage at A when a current I is flowing into it. KCL at A, X and Y gives

$$\begin{aligned}
\frac{A-X}{4} + \frac{A-Y}{7} - I &= 0 \Rightarrow 11A - 7X - 4Y = 28I \\
\frac{X-A}{4} + 8 &= 0 \Rightarrow A - X = 32 \\
\frac{Y-A}{7} + \frac{Y}{3} - 8 &= 0 \Rightarrow 3A - 10Y = -168
\end{aligned}$$

from which the solution is

$$\begin{pmatrix} A \\ X \\ Y \end{pmatrix} = \begin{pmatrix} 11 & -7 & -4 \\ 1 & -1 & 0 \\ 3 & 1 & -10 \end{pmatrix}^{-1} \begin{pmatrix} 28I \\ 32 \\ -168 \end{pmatrix} = \begin{pmatrix} -56 \\ -88 \\ 0 \end{pmatrix} + \begin{pmatrix} 10 \\ 10 \\ 3 \end{pmatrix} I$$

easily found using the simultaneous equation solver on the calculator by solving first with $I = 0$ *and then with* $I = 1$.

From the top row, $A = -56 + 10I$*, which gives the Thévenin component values directly.*

Several people thought the $7\,\mathrm{k}$ *and* $3\,\mathrm{k}$ *resistors were in parallel rather than in series (presumably because they shorted A and B together). When calculating* *the open-circuit voltage, some people thought that the* 8 mA *current would divide between the* 7 k *and* 3 k *resistors; in fact no current flows through the* 3 k *resistor when A and B are open-circuit. Some people calculated the component values but lost marks because they did not draw the diagram as the question asked. Many people got the sign of the voltage source wrong; often they calculated the correct answer of* -56 V *but then drew the wrong polarity on the diagram.*

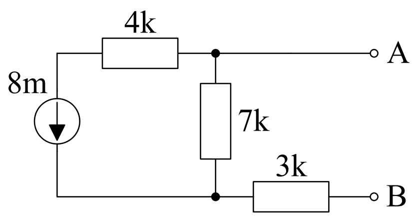


Figure 1.3

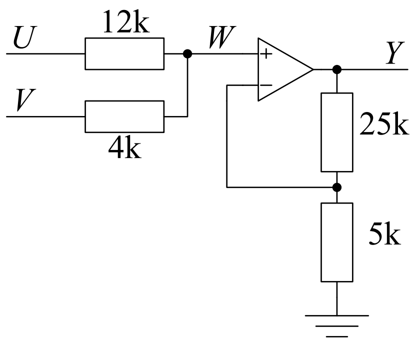


Figure 1.4

d) Assuming the opamp in the circuit of Figure 1.4 is ideal, give an expression for Y in terms of U and V. [5]

This is an non-inverting op-amp circuit and so we can write down $Y = W\left(1+\frac{25}{5}\right) = 6W$.

Applying KCL at node W *gives* $\frac{W-U}{12}+\frac{W-V}{4}=0$ *from which* $W-U+3W-3V=0$ *and hence* $W=\frac{U+3V}{4}$.

Putting these together gives $Y = 6W = 6\times\frac{U+3V}{4} = 1.5U+4.5V$.

Although the two inputs of an ideal opamp with negative feedback are at the same voltage, they are two distinct nodes nevertheless and hence give rise to two separate KCL equations (because current does not flow into one input and out of the other). A few people incorrectly wrote down a single KCL equation that included both nodes.A surprising number of people had sign errors in their algebra (in other questions as well) and wrote $Y=-1.5U+4.5V$*; if an input is connected only to the* $+$ *terminal of an opamp, then its gain will be positive and if it is connected only to the* $-$ *terminal, its gain will be negative.*

In this question, as in others, quite a few people gave answers that were equivalent to the correct answer, but that did not quite answer the question, e.g. $2Y=3U+9V$*; if the question asks for an expression for* Y*, then the last line of your answer should be "*$Y=\ldots$*".*

e) The graph of Figure 1.5 plots the output voltage, Y, against the input voltage, X, for the circuit shown in Figure 1.6. The graph consists of two straight lines that intersect at the point $(10, 10)$ and that pass through the origin and the point $(20, 12)$ respectively. Assuming that the forward voltage drop of the diode is $0.7\,\text{V}$, determine the values of the resistor, R, and the voltage source, V. [5]

The diode turns on when $Y = V + 0.7$. For this to be when $Y = 10\,\text{V}$ we must have $V = 9.3$.

To determine R, we can apply KCL at node Y when $X = 20$ and $Y = 12$:

$$\frac{12-20}{20} + \frac{12-10}{R} = 0$$

from which $-8R + 40 = 0$ which gives $R = 5\,\text{k}$.

Alternatively, we can say that, since the slope of the second part of the characteristic is $\frac{2}{10} = 0.2$, this must be the gain of the potential divider formed by $20\,\text{k}$ and R. To get a gain of $\frac{1}{5}$ you need $20\,\text{k} = 4R$ from which $R = 5\,\text{k}$ as before.

Yet another way to solve the problem is to write the KCL equation when the diode is on:

$$\begin{aligned} \frac{Y-X}{20} + \frac{Y-0.7-V}{R} &= 0 \\ \Rightarrow (Y-X)R - 20V &= 14 - 20Y \end{aligned}$$

Now we can substitute the two points $(10, 10)$ and $(20, 12)$ (both of which lie on the "diode on" part of the graph) to get a pair of simultaneous equations

$$\begin{aligned} 0R - 20V &= 14 - 200 = -186 \\ -8R - 20V &= 14 - 240 = -226 \end{aligned}$$

which we can solve for R and V.

Several wrote down incorrect equations for either the "diode on" or "diode off" situations. The diode acts either as an open circuit ("off") or else a $0.7\,\text{V}$ voltage source ("on"). When the diode is off, there is no current through R, so KCL at Y gives $\frac{Y-X}{20} = 0$. Several people assumed the "diode on" equation, $\frac{Y-X}{20} + \frac{Y-0.7-V}{R} = 0$, was valid all the time including at $X = Y = 0$.

Some people treated the circuit as a potential divider; this is possible but, since neither resistor is connected to ground, it is easy to make errors and most of those that tried this method got it wrong. The commonest error was to write $Y = X\frac{R}{20+R}$ which implicitly assumes the top of the diode is at $0\,\text{V}$. The correct potential divider expressions involve voltage differences across resistors (i.e. the voltage at one end minus the voltage at the other) giving $Y - (V + 0.7) = (X - (V + 0.7))\frac{R}{20+R}$ or alternatively $X - Y = (X - (V + 0.7))\frac{20}{20+R}$. Using nodal analysis is much less error-prone for circuits that are even a little bit complicated. Several people took the voltage across the diode to be $-0.7\,\text{V}$ instead of $+0.7\,\text{V}$; when the diode is on, the voltage and current polarities are the same as for a resistor (since they both absorb power).

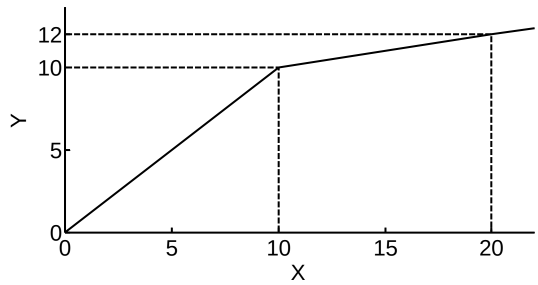


Figure 1.5

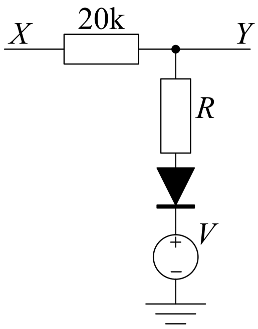


Figure 1.6

f) Determine the gain, $\frac{Y}{X}$, for the block diagram shown in Figure 1.7. The rectangular blocks are drawn with inputs at the left and outputs at the right and have gains of F and G respectively. The open circles represent adder/subtractors; their inputs have the signs indicated on the diagram and their outputs are W and Y respectively. [5]

For node W, we can write the following equation: $W = X - GFW$ from which we get $W = \frac{1}{1+FG}X$.

For node Y,we can write $Y = FW - X = \left(\frac{F}{1+FG} - 1\right)X = \frac{F-FG-1}{1+FG}X$ so the gain is $\frac{Y}{X} = \frac{F-FG-1}{1+FG}$.

Some wrote down the correct equations but were not sure what to do next; if you want to get Y in terms of X, you need to eliminate W. Some people wrote down dimensionally inconsistent equations such as $W = X - GF$; when writing down the equations for a block diagram, gains (such as F or G) must always multiply signals (such as X, W or Y). Although the question asked for $\frac{Y}{x}$, many people instead gave an expression for Y in terms of X (e.g. $Y = \frac{FX}{1+FG} - X$). This is, of course, equivalent but it is always a good idea to answer the question that was asked rather than a different question of your own creation.

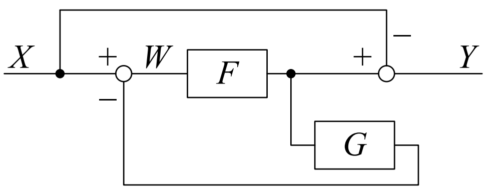


Figure 1.7

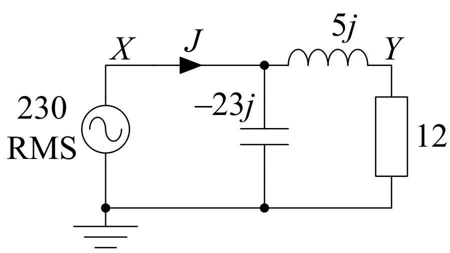


Figure 1.8

g) In the circuit of Figure 1.8, the RMS phasor $\widetilde{X} = 230$ and the component values shown indicate complex impedances. Determine the value of the RMS current phasor $\widetilde{J}$ and of the complex power, $\widetilde{V} \times \widetilde{I}^*$, absorbed by each of the four components.

[5]

The RMS current through he capacitor is $\frac{\widetilde{X}}{-23j} = 10j$.

The RMS current through the inductor is $\frac{\widetilde{X}}{12+5j} = \frac{2760-1150j}{169} = 16.3 - 6.8j$.

The total RMS current is therefore $\tilde{J} = 16.3 - 6.8j + 10j = \frac{2760-540j}{169} = 16.3 + 3.2j$.

Power absorbed by source is $-\widetilde{X} \times \tilde{J}^* = -230\,(16.3-3.2j) = -3.76+0.73j\,\text{kVA}$.

Power absorbed by capacitor is $\widetilde{X} \times (10j)^* = -2.3j\,\text{kVA}$.

Power absorbed by resistor is $\left|\tilde{I}\right|^2 R = |16.3-6.8j|^2 \times 12 = 313 \times 12 = 3.76\,\text{kW}$.

Power absorbed by inductor is $\left|\tilde{I}\right|^2 Z_L = |16.3-6.8j|^2 \times 5j = 313 \times 5j = 1.57j\,\text{kVA}$.

As expected the powers sum to zero and the complex power absorbed by a passive component (C, R or L) has the same phase as the component's impedance.

Many people, presumably following their training from school, gave "exact" answers like $\tilde{J} = \frac{2760-540j}{169}$. *Answers in this form are useful in engineering only in very rare cases since physical quantities such as resistance and current never have exact rational values except when they are* 0*; in questions based on component values, you should always give answers in decimal form. Component values are typically only accurate to 2 or at most 3 significant digits.*

Many people made errors doing complex arithmetic manually; note that the calculators supplied in the exam are able to do complex arithmetic but practising beforehand is advisable. Also, if you use the calculator, it is still advisable to write down intermediate values so that you will get some marks even if the final answer is wrong. A common error was to say $\frac{230}{-23j} = -10j$ *instead of* $+10j$. *Some took* $S = |\tilde{I}|Z$ *instead of* $S = |\tilde{I}|^2 Z$. *Others stated the correct expression but actually used* $S = \tilde{I}^2 Z$ *instead. In order to determine J, some people began by calculating the total impedance of the three passive components* $(= 13.6 - 2.65j\,\Omega)$*; although this is entirely correct, it is easier to calculate the capacitor current separately and add it to the current through the inductor/resistor combination (as is done above).*

Calculating the current through the capacitor is very easy as $\tilde{I}_C = \frac{\tilde{V}}{-23j}$. *Quite a few people used the current divider formula to calculate it as* $\tilde{I}_C = \frac{12+5j}{12+5j-23j}\tilde{J}$ *which is correct but much more effort and usually gave an answer that was not purely imaginary.*

Quite a few people gave the power <u>supplied by</u>, rather than <u>absorbed by</u> the source; indeed several gave this as their only answer which was not what the question asked for. One or two were troubled by the idea of a voltage source "absorbing" power at all; if there is only a single source, the complex power it <u>absorbs</u> will always have a negative real part (meaning that it is actually supplying average power to the rest of the circuit) but the imaginary part can have either sign.

The correct formulae for the power absorbed by a component with complex impedance Z are $S = \widetilde{V} \times \tilde{I}^* = |\tilde{I}|^2 Z = \frac{|\widetilde{V}|^2}{Z^*}$. *Several people omitted the squares, the conjugation and/or the modulus signs from the expressions even though the conjugation was given in the question); a few sprinkled their equations with* $\sqrt{2}$ *factors or else multiplied the expressions by* $\frac{1}{2}$ *which is not necessary if you are using RMS quantities like* $\widetilde{V}$ *and* $\tilde{I}$. *From the formula* $S = |\tilde{I}|^2 Z$ *it is clear that S and Z must have the same phase (i.e. complex argument) so the powers absorbed by R, C or L must respectively be real, negative-imaginary and* *positive-imaginary; many people gave answers that violated this constraint.*

Several people used the correct formula $S = |\tilde{I}|^2 Z$ but used the total current $\tilde{J}$ when calculating the power absorbed by the passive components. In fact, $\tilde{J}$ splits and flows partly thought the capacitor and partly through the inductor+resistor;in the formula, $\tilde{I}$ needs to be the current that actually flows through the component in question.

Several people talked about $j\omega L$ or $\frac{1}{j\omega C}$ and some said the question was impossible without knowing ω; however, in this case $j\omega L = 5j$ and $\frac{1}{j\omega C} = -23j$ so you do not need to know ω explicitly.

h) Figure 1.10 shows a transmission line of length 100 m that is terminated in a resistive load, R, with reflection coefficient $\rho = +0.6$. The line has a propagation velocity of $u = 2 \times 10^8$ m/s. At time $t = 0$, a forward-travelling (i.e. left-to-right) pulse arrives at X with amplitude 4 V and duration 1.5 μs, as shown in Figure 1.9.

Draw a dimensioned sketch of the waveform at Y, a point 60 m from the end of the line, for $0 \leq t \leq 3\,\mu$s. Assume that no reflections occur at point X. [5]

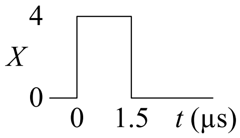


Figure 1.9

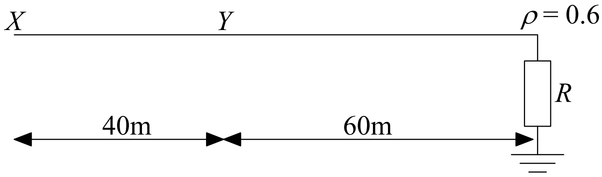
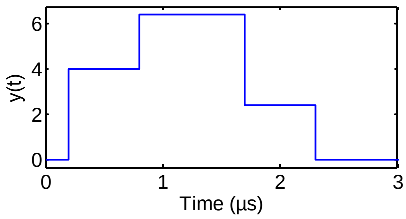


Figure 1.10

The velocity, u, is 200 m per µs. The forward wave takes 0.2 µs to reach Y and a further 0.6 µs to reflect from the end and return to Y. Therefore the waveform at Y is the sum of two overlapping waves: (i) a pulse of amplitude 4 V beginning at t = 0.2 µs (ending at t = 1.7 µs) and a pulse of $4 \times \rho = 2.4$ V beginning at t = 0.8 µs (ending at t = 2.3 µs). Where the pulses overlap, their combined voltage is $4 + 2.4 = 6.4$ V.



Most people who tried this got it right but quite a few did not attempt this question at all. A few people used a value of $\rho = -0.6$; in general, ρ can have either sign but in this question you are told that it is positive. Several assumed a value of $\rho_0 = +1$ at X even though the question explicitly said there were no reflections at X; this made the question somewhat harder. On a "dimensioned sketch" you should mark the values on the X and Y axes where interesting things happen: in this case, this means marking "0", "2.4", "4" and "6.4" on the vertical axis and "0.2", "0.8", "1.7" and "2.3" on the horizontal axis (unlike on the graph above).

---

2. a) Show that the transfer function of the circuit of Figure 2.1 can be written in the form

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{1}{\left(\frac{j\omega}{\omega_0}\right)^2 + 2\zeta\frac{j\omega}{\omega_0} + 1}$$

and express the values of ω_0 and ζ in terms of the component values L, C and R. [5]

Viewing the circuit as a potential divider, the transfer function is

$$\begin{aligned} H(j\omega) &= \frac{\frac{1}{j\omega C}}{j\omega L + R + \frac{1}{j\omega C}} \\ &= \frac{1}{(j\omega)^2 LC + j\omega RC + 1} \\ &= \frac{1}{\left(\frac{j\omega}{\omega_0}\right)^2 + 2\zeta\frac{j\omega}{\omega_0} + 1} \end{aligned}$$

where, by identifying coefficients, $\frac{1}{\omega_0^2} = LC$ *and* $\frac{2\zeta}{\omega_0} = RC$ *from which* $\omega_0 = \sqrt{\frac{1}{LC}}$ *and* $\zeta = \frac{\omega_0 RC}{2} = \frac{RC}{2\sqrt{LC}} = \frac{R}{2}\sqrt{\frac{C}{L}}$.

One person said they could not do this question because they had not revised the topic of "resonance". Actually, the question is pretty much self-contained, so it should be possible to do it even so (revision is still a good idea though). Many people find ζ *hard to write (or at least write clearly). Rather than matching the coefficients between the transfer function and the formula given in the question, some just remembered the formulae from the lectures:* $\omega_0 = \sqrt{\frac{c}{a}} = \sqrt{\frac{1}{LC}}$ *and* $\zeta = \frac{b}{\sqrt{4ac}} = \frac{RC}{\sqrt{4LC}}$*; this method works but relies on a good memory and also on the notation in the question exactly matching that used in the notes (luckily this was true). Many people did not precisely answer the question that was asked: if the question asks for an expression for* ζ*, then your answer should end with a line of the form* $\zeta = \ldots$ *rather than with some vaguely equivalent equation such as* $2\zeta = \frac{RC}{\sqrt{LC}}$ *or even* $4\zeta^2 = \frac{R^2C}{L}$.

---

b) Give expressions for the low and high frequency asymptotes of $H(j\omega)$ and the angular frequency at which they have the same magnitude. [3]

LF asymptote: $H_{LF}(j\omega) = 1$. *HF asymptote:* $H(j\omega) = \frac{1}{LC}(j\omega)^{-2} = \left(\frac{j\omega}{\omega_0}\right)^{-2}$. *The asymptotes have the same magnitude at* $\omega = \omega_0$.

The asymptotes are complex-valued functions of ω *that specify both the magnitude* <u>*and*</u> *the phase at low or high frequencies respectively. Some people just gave the magnitude of the asymptotes; this is incorrect.*

Some people confused "HF asymptote" with "value at $\omega = \infty$*". In this example, the HF asymptote is* $\left(\frac{j\omega}{\omega_0}\right)^{-2}$ *but the value at* $\omega = \infty$ *is* $H(j\infty) = 0$*. Notice that, because the asymptote is a function of* ω*, it tells you how the gain varies as* ω *approaches infinity not just when it actually equals infinity. Another way of* *looking at the distinction is that the asymptote is an entire line whereas $H(j\infty)$ is just a single point on the line.*

Note that at $\omega = \omega_0$ the values of the asymptotes are 1 and -1 respectively; these have the same magnitude (as required by the question) but not the same phase. Some people tried to find a value of ω at which the two asymptotes had exactly the same value (i.e. both magnitude and phase); this is not possible for a real-valued ω. Many people did not distinguish clearly between an asymptote and its absolute value e.g. writing false equations like $H(j\omega) = \left(\frac{j\omega}{\omega_0}\right)^{-2} = \left(\frac{\omega}{\omega_0}\right)^{-2}$.

---

c) Determine the magnitude and phase of $H(j\omega)$ at $\omega = \omega_0$. [2]

At $\omega = \omega_0$, $\left(\frac{j\omega}{\omega_0}\right)^2 = -1$ so $H(j\omega_0) = \frac{1}{-1+2\zeta j+1} = \frac{-j}{2\zeta}$. This has a magnitude of $|H(j\omega_0)| = \frac{1}{2\zeta}$ and a phase $\angle H(j\omega_0) = -\frac{\pi}{2}$. Note that $\zeta = \frac{\omega_0 RC}{2}$ is always positive.

Several people said that $H(j\omega_0) = 1$ since the high and low frequency asymptotes both have magnitude 1 at $\omega = \omega_0$ (albeit with different phases). However, at a resonance, the true gain is often not well apprixmated by the asymptotes; that is the main point of this question.

Some ignored the j^2 factor in the first term and set $\left(\frac{j\omega}{\omega_0}\right)^2 = +1$ instead of -1. Ignoring j in a complex number is like ignoring the difference between "North" and "East" when navigating using a map. Pretty much the entire behaviour of a quadratic resonance arises from the fact that at ω_0 the impedances of the inductor and capacitor cancel out.

Several people had surprising difficulty giving the magnitude of $\frac{-j}{2\zeta}$; some said $\frac{-1}{2\zeta}$ (which is negative) and others said $\frac{j}{2\zeta}$ (which is complex). The magnitude of a fraction is the magnitude of the numerator divided by the magnitude of the denominator and is always real and positive. Note that ζ is always real and, in our circuit, is bound to be positive (although in other circuits it might be negative).

Others had difficulty with the phase of $\frac{-j}{2\zeta}$; many tried to work it out using $\arctan()$ which is valid but definitely overkill. If in doubt about the phase of a simple complex number, plot it on an Argand diagram.

Although the question asked for "the magnitude and phase" of $H(j\omega_0)$, many people just wrote down its value as a complex number (and therefore lost marks).

---

d) Show that $|H(j\omega)|^{-2}$ may be written as a polynomial with real coefficients in x where $x = \left(\frac{\omega}{\omega_0}\right)^2$. By differentiating this polynomial, or otherwise, show that the maximum value of $|H(j\omega)|$ occurs at $\omega = \omega_0\sqrt{1-2\zeta^2}$. [6]

To find the magnitude squared, we take the sum of the squares of the real and imaginary parts; the middle term in the denominator of $H(j\omega)$ is imaginary while the other two terms are real (since $j^2 = -1$):

$$
\begin{aligned}
|H(j\omega)|^{-2} = \left|H(j\omega)^{-1}\right|^{2} &= \left|\left(\frac{j\omega}{\omega_0}\right)^2 + 2\zeta\frac{j\omega}{\omega_0} + 1\right|^2 \\
&= \left|\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right) + j\frac{2\zeta\omega}{\omega_0}\right|^2 \\
&= \left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \left(\frac{2\zeta\omega}{\omega_0}\right)^2 \\
&= (1-x)^2 + \left(2\zeta\sqrt{x}\right)^2 \\
&= (1-x)^2 + 4\zeta^2 x \\
&= x^2 + \left(4\zeta^2-2\right)x + 1
\end{aligned}
$$

Setting the derivative of this polynomial to zero to find its minimum (it must be a minimum rather than a maximum because the coefficient of x^2 is positive) gives

$$x_p = -\frac{4\zeta^2-2}{2} = 1-2\zeta^2$$

Hence

$$
\begin{aligned}
\left(\frac{\omega_p}{\omega_0}\right)^2 &= 1-2\zeta^2 \\
\Rightarrow \quad \omega_p &= \omega_0\sqrt{1-2\zeta^2}
\end{aligned}
$$

This is the minimum of $|H(j\omega)|^{-2}$ and so must be the maximum of $|H(j\omega)|$.

Many people found this quite hard because they were not completely familiar with the facts that if $z = a + jb$ is a complex number, then $|z|^2 = zz^* = a^2+b^2$ and also $\left|z^k\right| = |z|^k$ for any integer k (we use this above with $k = -1$). Many people either ignored the j or ignored the modulus signs; both of these mistakes make the algebra much harder as well as giving the wrong answer. Actually $|z|^2$ is much easier to work with than $|z|$ which involves square root signs. Some multiplied the numerator and denominator of $H(j\omega) = \frac{1}{\left(\frac{j\omega}{\omega_0}\right)^2 + 2\zeta\frac{j\omega}{\omega_0}+1}$ by its complex conjugate; doing this is almost always a bad idea in algebra because it converts quadratic expressions into quartic expressions.

Quite frequently, people assumed that if $H(j\omega) = \frac{1}{\left(\frac{j\omega}{\omega_0}\right)^2 + 2\zeta\frac{j\omega}{\omega_0}+1}$ then $|H(j\omega)| = \frac{1}{\left(\frac{\omega}{\omega_0}\right)^2 + 2\zeta\frac{\omega}{\omega_0}+1}$ formed by deleting all the j factors; this is not a valid way of calculating the magnitude of a complex number or expression. Alternatively, several assumed that $|H(j\omega)|^{-2} = H(j\omega)^{-2}$ which is true for real numbers but not for complex numbers. An approach that used even more algebra was to calculate $\left|H(j\omega)^2\right|$, i.e. to square it before taking the magnitude. Others just squared all the terms to get the (incorrect) squared magnitude $|H(j\omega)|^{-2} = \left(\frac{\omega}{\omega_0}\right)^4 + \left(2\zeta\frac{\omega}{\omega_0}\right)^2 + 1^2$; you need to add together all the real and all the imaginary terms and then square the two resultant sums.

If $z = a + jb$, then $|z|^2 = z \times z^* = (a+jb)(a-jb) = a^2+b^2$, i.e. $|z|^2$ equals the sum of the squares of its real and imaginary parts (this can also be seen by applying Pythagoras' theorem to the Argand diagram). Some people used the first expression rather than the second which results in much worse algebra: $|H(j\omega)|^{-2} = \left|\left(\frac{j\omega}{\omega_0}\right)^2 + 2\zeta\frac{j\omega}{\omega_0} + 1\right|^2 = \left(\left(\frac{j\omega}{\omega_0}\right)^2 + 2\zeta\frac{j\omega}{\omega_0} + 1\right)\left(\left(\frac{j\omega}{\omega_0}\right)^2 - 2\zeta\frac{j\omega}{\omega_0} + 1\right) =$ *.... A few ignored the j or else did not take the complex conjugate; either error results in the wrong answer entirely.*

Despite the instructions in the question, some substituted $x = \frac{\omega}{\omega_0}$ or even $x = j\omega$ instead of $x = \left(\frac{\omega}{\omega_0}\right)^2$. Others didn't make any substitution at all and worked entirely in ω which is fine but messier.

Even though the questions told you to use $|H(j\omega)|^{-2}$, a few brave people used $|H(j\omega)|^2$ instead. It is, of course, true that the maximum of $|H(j\omega)|^2$ is at the same value of ω as the minimum of $|H(j\omega)|^{-2}$ but differentiating $|H(j\omega)|^2$ is much more effort.

Many people who could not get the right answer "adjusted" their algebra so that the last line was a triumphant $\Rightarrow \quad \omega_p = \omega_0\sqrt{1-2\zeta^2}$ but the algebra adjustments counted as additional errors and so lost them additional marks. Better to confess that the answer is wrong but you cannot find your mistake.

---

e) Determine values of C and R so that $\omega_0 = 5000\,\text{rad/s}$ and $\zeta = 0.1$ given that $L = 100\,\text{mH}$. [2]

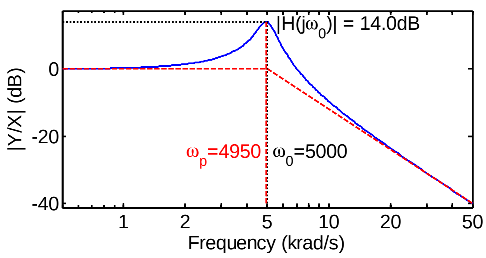
From $\omega_0 = \sqrt{\frac{1}{LC}}$, $C = \frac{1}{L\omega_0^2} = \frac{1}{0.1\times 25\times 10^6} = 0.4\,\mu\text{F} = 400\,\text{nF}$.

From $\zeta = \frac{\omega_0 RC}{2}$, $R = \frac{2\zeta}{C\omega_0} = \frac{0.2}{0.4\times 5\times 10^{3-6}} = 0.1\,\text{k}\Omega = 100\,\Omega$.

Most people got this right. However it sometimes involved a great deal of algebra. On the whole it is easiest to manipulate symbolic algebra into its simplest form first and only then substitute numerical values. This avoids wasted work such as taking square roots and then later on squaring the result (as many people did in this question).

---

i) Sketch a dimensioned graph of $|H(j\omega)|$ in decibels using a logarithmic frequency axis. Your graph should include both the high and low frequency asymptotes in addition to a sketch of the true magnitude response. [3]



From part d), the peak is at $\omega_p = \omega_0\sqrt{1-2\zeta^2} = 5000\times\sqrt{0.98} = 5000\times 0.9899 = 4950\,\text{rad/s}$.

From part c), the gain at $\omega_0 = 5000$ is $\frac{1}{2\zeta} = 5 = 14\,\text{dB}$.

The gradient of the HF asymptote is -2 or, equivalently $-40\,\text{dB}$ per decade, meaning that at $\omega = 10\omega_0$ the gain has fallen to approximately $-40\,\text{dB}$.

Several people derived the correct asymptote expressions but drew them with the wrong gradient. If the asymptote is $A\omega^k$ then it has a gradient of k on the log-log axes that we use for magnitude responses. This the LF asymptote has a gradient of 0, the HF asymptote has a gradient of -2 and they cross at $\omega_0 = 5000\,\text{rad/s}$. Several people drew the LF asymptote with a gradient of $+1$ (presumably remembering an example from the notes).

Several people showed the gradient of the HF asymptote as only $-20\,\text{dB}$ per decade.

---

ii) If $x(t) = 3\cos\omega_0 t$, determine the average power dissipation of the circuit and the peak value of the energy, $\frac{1}{2}Cy^2(t)$, stored in the capacitor.

[3]

At resonance, the total impedance is $j\omega_0 L + R + \frac{1}{j\omega_0 C} = R$ since the impedances of the L and C cancel out (we can see this either by substituting numerical values for L and C or algebraically because $\omega_0^2 LC = 1$ implies $\frac{1}{j\omega_0 C} = \frac{\omega_0^2 LC}{j\omega_0 C} = -j\omega_0 L$). So the average power dissipation is $\frac{\langle x^2(t)\rangle}{R} = \frac{1}{2}3^2 \times \frac{1}{R} = 45\,\text{mW}$.

Using phasors and the result from part c), $Y = \frac{Y}{X}(j\omega) \times X = \frac{1}{2\zeta j}X = \frac{1}{0.2j} \times 3 = -15j$. Hence the peak capacitor voltage is 15 and the peak energy stored is $\frac{1}{2}C \times 15^2 = 45\,\mu\text{J}$.

Several people said the average power dissipation was $\frac{X^2}{R}$ (which applies to DC levels) rather than $\frac{\langle x^2(t)\rangle}{R} = \frac{\frac{1}{2}\hat{X}^2}{R}$ which applies to a sine wave with peak voltage $\hat{X}$. Many said the peak energy was measured in Watts rather than Joules. The formula $P = \frac{V^2}{R}$ requires that V is the voltage across the resistor R. Several thought the power dissipation in R was $\frac{(x-y)^2}{R}$ even thought he voltage $x-y$ is not the voltage across the resistor but includes the inductor as well.

---

iii) Determine the values of ω for which $\angle H(j\omega) = -45^\circ$ and -135°.

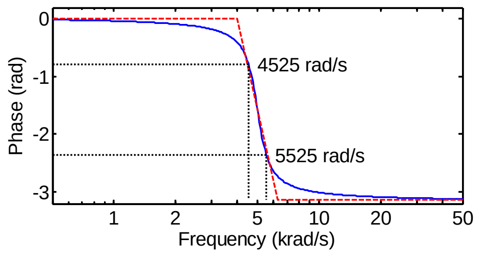
Hence sketch a dimensioned graph of $\angle H(j\omega)$ using a straight-line approximation with three segments. Your graph should use a logarithmic frequency axis and a linear phase axis.

[6]

$H(j\omega) = \frac{1}{\left(\frac{j\omega}{\omega_0}\right)^2 + 2\zeta\frac{j\omega}{\omega_0} + 1}$ so that $\angle H(j\omega) = -\arctan\left(\frac{2\zeta\frac{\omega}{\omega_0}}{1-\left(\frac{\omega}{\omega_0}\right)^2}\right)$. Since $\tan(-45^\circ) = -1$ and $\tan(-135^\circ) = \tan(+45^\circ) = +1$, we need the argument of $\arctan(\,)$ in the expression for $\angle H(j\omega)$ to equal ± 1. An equivalent geometrical approach from the Argand diagram is that

the real and imaginary parts of a complex number with an argument of $\pm 45°$ *must have equal magnitude. So we need* $1-\left(\frac{\omega}{\omega_0}\right)^2 = \pm 2\zeta\frac{\omega}{\omega_0}$ *or, equivalently,* $\left(\frac{\omega}{\omega_0}\right)^2 \pm 2\zeta\frac{\omega}{\omega_0} - 1 = 0$*. The roots of this equation are* $\frac{\omega}{\omega_0} = \mp\zeta \pm \sqrt{\zeta^2+1}$.

For $\zeta = 0.1$*, this gives* $\frac{\omega}{\omega_0} = \mp 0.1 \pm \sqrt{1.01} = \pm 0.905, \pm 1.105$*. Taking the positive frequencies,* $\angle H(j\omega) = -45°$ *at* $\omega = 4525$ *and* $\angle\frac{Y}{X}(j\omega) = -135°$ *at* $\omega = 5525$*.*



The central segment of the straight-line that passes through $(4525, \frac{-\pi}{4})$ *and* $(5525, \frac{-3\pi}{4})$ *passes through* $(\omega_a, 0)$ *and* $(\omega_b, -\pi)$ *where* $\omega_a = 5000 \times \left(\frac{4525}{5000}\right)^2 = 4095$ *and* $\omega_b = 5000 \times \left(\frac{5525}{5000}\right)^2 = 6105$*. The squared frequency ratios arise because we wish to double the phase shift relative to that at* ω_0*. Alternatively, for those with a good memory, the formula given in the lecture notes gives slightly different values of* $\omega_a = 10^{-\zeta}\omega_0 = 3972$ *and* $\omega_b = 10^{+\zeta}\omega_0 = 6295$*; this is the straight line approximation plotted above as a dashed red line. The solid blue curve shows the true phase (not requested).*

Most people found this difficult. It is a lot easier if you recognize straight off that if a complex number has an argument of $\pm 45°$ *then its real and imaginary parts have equal magnitude.*

Quite a few people assumed that the corners in the phase response were at $0.1\omega_0$ *and* $10\omega_0$ *which is true for linear factors. For quadratic factors however, the corresponding formulae are* $10^{-|\zeta|}\omega_0$ *and* $10^{+|\zeta|}\omega_0$ *which comes to the same thing if* $|\zeta| = 1$ *(its maximum value) but not otherwise.*

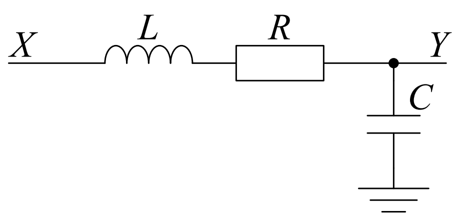


Figure 2.1

3. In the circuit of Fig. 3.1, the input, X, is driven by a voltage source as shown.

a) Derive an expression for the transfer function, $\frac{Y(j\omega)}{X(j\omega)}$ and determine the corner frequencies in its magnitude response. [4]

The circuit is a potential divider, and the impedance of $3R||C$ is $\left(\frac{1}{3R}+j\omega C\right)^{-1} = \frac{3R}{1+3j\omega RC}$ so the transfer function is

$$\frac{Y(j\omega)}{X(j\omega)} = \frac{R}{R+\frac{3R}{1+3j\omega RC}} = \frac{1+3j\omega RC}{4+3j\omega RC}.$$

The numerator corner frequency is $\omega_n = \frac{1}{3RC}$ and the denominator corner frequency is $\omega_d = \frac{4}{3RC}$.

Most people did this correctly but quite a few people made algebraic errors. It is helpful to write gains in terms of dimensionless terms (like ωRC or $\omega^2 LC$) because it is them easy to spot dimensional incompatibilities resulting for algebra errors. Expressions like $R+3$ or $R+j\omega C$ should never occur because they are not dimensionally consistent.

Several people implicitly took the impedance of a capacitor to be $j\omega C$ instead of $\frac{1}{j\omega C}$.

A few people wrote down KCL equations at Y (correct) and/or at X (incorrect). You cannot apply KCL at X because you do not know what current flows through the voltage source.

A good final check is to verify that the transfer function gives the correct gain at $\omega = 0$ (i.e. with the capacitor an open circuit) and $\omega = \infty$ (i.e. with the capacitor a short circuit). This simple test will detect most algebra errors.

---

b) With the capacitor temporarily removed from the circuit, determine the Thévenin equivalent voltage and resistance of the remainder of the circuit at the terminals of the capacitor. [4]

To determine the Thévenin equivalent voltage, we assume no current flows through the capacitor and determine the voltage across it as $V_{th} = 0.75X$.

To determine the Thévenin equivalent resistance, we connect the two grounds together, short-circuit the voltage source and measure the resistance of the resultant network, which consists of R and $3R$ in parallel. The resistance is therefore $R_{th} = 0.75R$.

Most got this right although quite a few gave the Thévenin resistance as $4R$.A few gave a dimensionally incorrect value such as $\frac{1}{4}$ which cannot be right since all the resistors in the circuit are proportional to R.

---

c) Derive the time constant of the circuit, τ, in two ways: (i) from the Thévenin resistance found in part b) and (ii) from the denominator corner frequency found in part a). [2]

The time constant is (i) $R_{th}C = 0.75RC$ or alternatively (ii) $\frac{1}{\omega_d} = 0.75RC$.

Mostly correct. Again several answers were dimensionally wrong, e.g. $\tau = \frac{4C}{3R}$ does not have dimensions of "time".

d) If the input voltage, $x(t)$, is given by

$$x(t) = \begin{cases} -2 & \text{for } t < 0 \\ +3 & \text{for } t \geq 0 \end{cases},$$

determine an expression for the output waveform, $y(t)$. Sketch its waveform over approximately the range $-\tau \leq t \leq 4\tau$. [7]

From part a), the DC gain of the circuit is $\frac{Y}{X}(0) = 0.25$. For $t < 0$, $y(t) = -2 \times 0.25 = -0.5$.

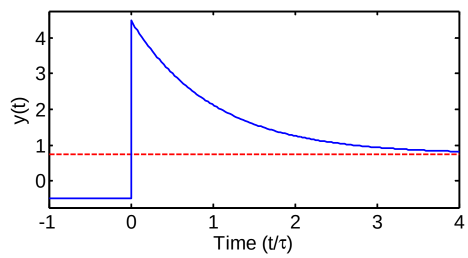
For $t \geq 0$, the steady state solution is $y_{SS}(t) = +3 \times 0.25 = 0.75$.

We can calculate $y(0+)$ in two ways:
(i) by ensuring the capacitor voltage, $y(t) - x(t)$, does not change instantly and
(ii) by noting that $y(0+) = y(0-) + \frac{Y}{X}(\infty) \times (x(0+) - x(0-))$.

Using method (i), at $t = 0-$ the capacitor voltage is $y(0-) - x(0-) = -0.5 - (-2) = 1.5$. At $t = 0+$, we therefore still have $1.5 = y(0+) - x(0+) = y(0+) - 3$. From this we get $y(0+) = 1.5 + 3 = 4.5$.

Alternatively, using method (ii) we have $y(0+) = y(0-) + \frac{Y}{X}(\infty) \times (x(0+) - x(0-)) = -0.5 + 1 \times (3 - (-2)) = 4.5$.

For $t \geq 0$, the output is therefore given by $y(t) = y_{SS}(t) + (y(0+) - y_{SS}(0+))\, e^{-\frac{t}{\tau}} = 0.75 + 3.75e^{-\frac{t}{\tau}}$ where $\tau = 0.75RC$ from part c). The dashed line in the plot below shows the asymptote $y(+\infty) = 0.75$.



Many people assumed that the output was continuous (i.e. had no discontinuity at $t = 0$). The most common reason given for this was that the capacitor voltage could not change instantly; although this true, it would only force Y to be continuous if the capacitor were connected between Y and ground which it isn't.

A few people gave complex values for the steady-state value of $y(t)$ and/or the transient amplitude; these quantities are always real-valued.

Several people gave the formula for $y(t)$ for $t \geq 0$ but did not say explicitly what it was for $t < 0$.

e) Assuming that the opamp in Fig. 3.2 is ideal, determine the transfer function, $\frac{V(j\omega)}{U(j\omega)}$. [4]

From the standard gain of a non-inverting amplifier, the gain is $\frac{V}{U} = 1 + \frac{Z}{R} = \frac{R+Z}{R}$ *where* Z *is the impedance of the* $3R||C$ *combination. Note that this is just the reciprocal of the gain* $\frac{Y}{X}$*. Making use of the previous result, we therefore have* $\frac{V(j\omega)}{U(j\omega)} = \frac{4+3j\omega RC}{1+3j\omega RC}$*. Alternatively, we can regard the circuit as a potential divider with* V *as the input and* U *as the output (since the inverting input of the opamp is constrained by negative feedback to equal* U*). Its transfer function is thus the inverse of Fig. 3.1. Yet another way is to use nodal analysis by doing KCL at the junction between the* R *and* $3R$*resistors (noting that the voltage at this node is equal to* U*) to get:* $\frac{U}{R} + \frac{U-V}{3R} + (U-V)j\omega C = 0$ *from which* $3U + U - V + (U-V)j3\omega RC = 0 \Rightarrow U(4+j3\omega RC) = V(1+j3\omega RC)$ *and the result follows.*

Some gave the transfer function as $\frac{V}{U}(j\omega) = 1 + \frac{3}{1+3j\omega RC}$ *which is correct (and got full marks) but it is more conventional to write transfer functions as a rational polynomial in* $j\omega$ *(i.e. one polynomial divided by another).*

Many people included an extra factor of R *in both numerator and denominator:* $\frac{V(j\omega)}{U(j\omega)} = \frac{4R+3j\omega R^2C}{R+3j\omega R^2C}$*. This is, of course, still correct but is a bit lazy and makes for a lot more calculation in part g).*

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f) By considering the voltage across the capacitor, explain why an input voltage discontinuity of Δu will result in an output voltage discontinuity of the same amplitude. [2]

If $u(t)$ *suddenly changes by* Δu*, then negative feedback will ensure that the inverting input of the opamp changes by the same amount and, since the voltage across the capacitor cannot change instantly,* V*, must jump by the same amount.*

Many people said (correctly) that this followed from the gain at $\omega = \infty$*:* $\frac{V}{U}(j\infty) = 1$*. However the question asked you to consider the voltage across the capacitor. Relatively few people got the chain of cause and effect right: the input discontinuity causes a change at* V_+ *and the negative feedback then adjusts* V *so that* V_- *becomes equal to* V_+*.*

In many cases, the answer given was rather vague and did not form a logical explanation; it is not sufficient to say "voltage across capacitor cannot change instantly" without relating the voltage across the capacitor explicitly to the input and output voltages.

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g) If $R = 20\,\text{k}\Omega$, $C = 20\,\text{nF}$ and the input voltage, $u(t)$, is given by

$$u(t) = \begin{cases} \sin 1000t & \text{for } t < 0 \\ 2\cos 2000t & \text{for } t \geq 0 \end{cases},$$

determine expressions for the output $v(t)$ for both positive and negative t. [7]

From part e), the transfer function is $\frac{V(j\omega)}{U(j\omega)} = \frac{4+3j\omega RC}{1+3j\omega RC}$ *with* $RC = 4\times 10^{-4}$.

At $\omega_1 = 1000$*,* $\omega_1 RC = 0.4$ *and so* $\frac{V}{U}(j\omega_1) = \frac{4+1.2j}{1+1.2j} = \frac{544}{244} - \frac{360}{244}j = 2.23 - 1.475j$*. Using phasors, for* $t < 0$*,* $U_1 = -j$ *and so* $V_1 = -j \times (2.23 - 1.475j) = -1.475 - 2.23j$*. Hence, for* $t < 0$*, we have* $v(t) = -1.475\cos 1000t + 2.23\sin 1000t$.

At $\omega_2 = 2000$, $\omega_2 RC = 0.8$ and so $\frac{V}{U}(j\omega_2) = \frac{4+2.4j}{1+2.4j} = \frac{976}{676} - \frac{720}{676}j = 1.44 - 1.065j$. Using phasors, for $t \ge 0$, $U_2 = 2$ and so $V_2 = 2 \times (1.44 - 1.065j) = 2.89 - 2.13j$. Hence $v_{2,SS}(t) = 2.89\cos 2000t + 2.13\sin 2000t$.

To determine the transient amplitude, we note that $v(0-) = -1.475$, $v_{SS}(0+) = 2.89$ and that $\Delta v = 1 \times \Delta u = 2$. Thus $v(0+) = v(0-) + \Delta v = -1.475 + 2 = 0.525$ and the transient amplitude is $0.525 - 2.89 = -2.365$.

Thus, for $t \ge 0$, we have $v_{2,SS}(t) = 2.89\cos 2000t + 2.13\sin 2000t - 2.365e^{-\frac{t}{\tau}}$ where $\tau = 3RC = 1.2\,\text{ms}$.

Several people got the wrong answer when evaluating $\frac{4+1.2j}{1+1.2j}$; it is well worth learning to use the complex arithmetic capabilities of the provided calculator to save time and reduce errors.

A few people wrote things like $v(t) = \frac{4+1.2j}{1+1.2j}\sin 1000t$ which does not make sense: never mix j and t in the same expression. Phasors are complex but do not involve t while waveforms are real and do not involve j.

Note that $u(0+)$ is the value of the waveform $u(t)$ at the specific time $t = 0+$ (where the $+$ means just a tiny bit after $t = 0$). Its value must therefore be a specific voltage and must be a real number that does not depend on t. Likewise, the transient amplitude is given by $A = v(0+) - v_{SS}(0+)$ is also a real number that does not depend on t.

Rather than using phasors, some people took the gain to be 4 (i.e. the DC gain) even thought he input signal was a sine wave. This may arise from confusion between "steady state" and "DC"; for $t > 0$ the "steady state" input is a continuous cosine wave: $u(t) = 2\cos 2000t$.

Giving an "exact" answer like $v(t) = -\frac{360}{244}\cos 1000t + \frac{544}{244}\sin 1000t$ makes no sense at all in an engineering problem where the component values and frequencies are only ever approximate. Much better to give the answer in decimal to, say, 3 significant figures.

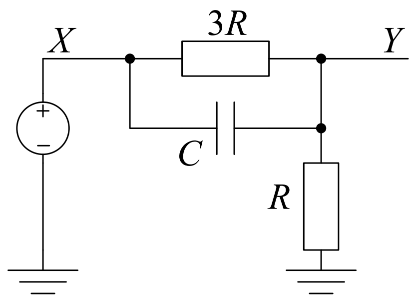

Figure 3.1

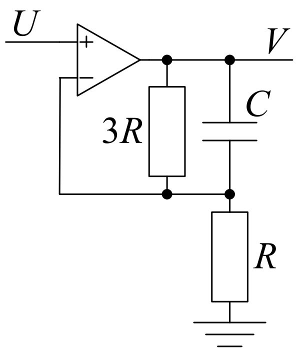

Figure 3.2