Master

E3.09
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ISE3.9

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2007

EEE/ISE PART III/IV: MEng, BEng and ACGI

CONTROL ENGINEERING

Friday, 27 April 2:30 pm

Time allowed: 3:00 hours

There are SIX questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : A. Astolfi
Second Marker(s) : C. Ling

CONTROL ENGINEERING

1. Consider a linear, single-input, single-output, continuous-time system described by the equations

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ \beta \end{bmatrix} u \qquad y = \begin{bmatrix} 1 & \alpha \end{bmatrix} x.$$

a) Study the reachability and stabilizability properties of the system as a function of β. [4 marks]

b) Study the observability and detectability properties of the system as a function of α. [4 marks]

c) Design, using the separation principle, an output feedback control law such that all eigenvalues of the closed-loop system are at -2. Discuss for which values of α and β it is possible to design such a control law. [10 marks]

d) Consider a static output feedback control law

$$u = Ky.$$

Assume $\beta = 0$ and $\alpha \geq 0$. Determine for which values of K the closed-loop system is asymptotically stable. [2 marks]

2. Consider an inverted pendulum described by the equation

$$Ml^2\ddot{\theta} = Mgl\sin\theta + u,$$

where θ describes the angle of the pendulum with respect to a vertical axis directed upward, M is the mass of the pendulum, l is the length of the pendulum, g is the gravitational acceleration, and u is an external torque. (Obviously $M > 0$, $l > 0$ and $g > 0$!)

a) Write the system in state space form. [2 marks]

b) Assume u is constant and compute all equilibrium points of the system. [4 marks]

c) Compute the linearized system around the equilibrium point corresponding to $u = 0$ and $\theta = 0$. [4 marks]

d) Show that the equilibrium point in part c) is unstable. [2 marks]

e) Assume $M = 1$, $l = 1$ and $g = 10$. Design a state feedback control law $u = Kx$ which asymptotically stabilizes the linearized system determined in part c). [4 marks]

f) Assume $l = 1$, $g = 10$ and $u = Kx$ as determined in part e). Determine for which values of M the closed-loop linearized system is asymptotically stable. [4 marks]

3. Consider a linear, single-input, single output, discrete-time system described by the equations

$$x^+ = Ax + Bu + Pd \qquad\qquad y = Cx,$$

where $x \in X = \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}$ is the control input, $y(t) \in \mathbb{R}$ is the output and $d(t) \in \mathbb{R}$ is a disturbance. The effect of the disturbance on the output $y(t)$ has to be cancelled by means of a suitably designed control action.

Assume that the disturbance $d(t)$ is such that

$$d^+ = Sd.$$

The problem of cancelling the effect of the disturbance d on the output y can be solved selecting a control law of the form

$$u = Kx + Ld,$$

where K is such that the system

$$x^+ = (A + BK)x$$

is asymptotically stable, and

$$L = \Gamma - K\Pi,$$

with $\Pi \in \mathbb{R}^{n\times 1}$ and $\Gamma \in \mathbb{R}$ solutions of the equations (known as the FBI equations)

$$\Pi S = A\Pi + B\Gamma + P \qquad\qquad 0 = C\Pi.$$

Assume

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad P = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} C_1 & C_2 & C_3 \end{bmatrix} \qquad\qquad S = [1],$$

with $C_i \in \mathbb{R}$, for $i = 1,2,3$.

a) Find K such that the matrix $A + BK$ has all eigenvaues at zero. [4 marks]

b) Show that, for the selected matrices, the FBI equations have solutions Π and Γ if and only if

$$C_3 = 0$$

or

$$C_1 - C_2 \neq 0.$$

[8 marks]

c) Using the results in parts a) and b) write a control law which solves the considered disturbance cancellation problem. [2 marks]

d) The control law determined in part c) requires the knowledge of the state x of the system and of the disturbance d. It is possible to circumvent this problem by constructing an observer for the system with state x and d and output y. Assume that $C_1 \neq 0$, $C_2 = C_3 = 0$, and show that it is possible to design an asymptotic observer for such a system. (Do not design the observer!) [6 marks]

4. Consider a nonlinear, continuous-time system described by the equation

$$\dot{x} = f(x),$$

with $x \in X = I\!R^n$. Suppose $x = 0$ is an equilibrium.

To study global asymptotic stability of the equilibrium $x = 0$ of the system the following condition is often used.

Krasowsky condition. *The equilibrium $x = 0$ of the system $\dot{x} = f(x)$ is globally asymptotically stable if the matrix*

$$\frac{\partial f(x)}{\partial x} + \left(\frac{\partial f(x)}{\partial x}\right)'$$

has all its eigenvalues with negative real part for all $x \in I\!R^n$.

Consider the system

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -x_1 \\ -x_2 - x_2^3 + hx_3 \\ kx_2 - x_3 - x_3^3 \end{bmatrix},$$

with h and k constant.

a) Show that the system has an equilibrium for $x = 0$ and compute the linearized model of the system around the equilibrium $x = 0$. [4 marks]

b) Study the stability properties of the linearized system as a function of h and k. In particular show that the linearized system is asymptotically stable if $1 - kh > 0$, it is stable if $1 - kh = 0$ and it is unstable if $1 - kh < 0$ [5 marks]

c) Using the principle of stability in the first approximation discuss the stability properties of the zero equilibrium of the nonlinear system as a function of h and k. [5 marks]

d) Using Krasowsky condition for global asymptotic stability show the following. If $h + k = 0$ then the equilibrium $x = 0$ of the nonlinear system is globally asymptotically stable. Hence argue that $x = 0$ is the only equilibrium of the system. [6 marks]

5. Consider a linear, single-input, single-output, discrete-time system described by the equations

$$x(k+1) = Ax(k) + Bu(k) \qquad y(k) = Cx(k),$$

where $x \in X = I\!R^n$ is the state, $u(k) \in I\!R$ is the input and $y(k) \in I\!R$ is the output.

Suppose that the initial state is $x(0) = 0$. The matrices A, B and C are unknown and also the dimension n of the state space is unknown.

The unknown n, A, B and C can be determined by performing the following experiment. Set

$$u(k) = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{if } k \geq 1, \end{cases}$$

and let $y(0), y(1), y(2), \cdots$, be the corresponding output sequence. The Hankel matrix associated with this output sequence is defined as

$$H = \begin{bmatrix} y(1) & y(2) & y(3) & \cdots \\ y(2) & y(3) & y(4) & \cdots \\ y(3) & y(4) & y(5) & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

The dimension of the state space is equal to the rank of H, *i.e.*

$$n = \text{rank} H.$$

The matrices A, B and C can be selected with the following form

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & \cdots & -\alpha_{n-1} \end{bmatrix} \qquad B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \end{bmatrix}.$$

a) Let H_n be the matrix composed of the first n rows and the first $n+1$ columns of H. This matrix has rank n. Show that

$$H_n = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \begin{bmatrix} B & AB & \ldots & A^{n-1}B & A^nB \end{bmatrix}. \tag{5.1}$$

[4 marks]

b) Show that the observability matrix

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

coincides with the identity matrix. [2 marks]

c) Using equation (5.1) show that

$$B = \begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(n) \end{bmatrix}$$

[4 marks]

d) Assume $n = 2$. Using equation (5.1) show that the coefficients α_0 and α_1 are the solutions of a linear system of equations. [6 marks]

e) Assume that $y(1) = 0$, $y(2) = 1$ and $y(k) = 0$ for all $k \geq 3$. Construct the Hankel matrix associated with this output sequence and compute its rank. Hence compute matrices A, B and C such that equation (5.1) holds. [4 marks]

6. Consider a linear, single-input, single-output, continuous-time system described by the equation

$$\dot{x} = Ax + Bu = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u,$$

with initial state $x(0)$, and the problem of determining a state feedback control law which stabilizes the system and minimises the cost

$$J(x_0, u) = \int_0^\infty (x'Qx + R^2u^2(t))dt$$

with

$$Q = C'C,$$
$$C = \begin{bmatrix} C_1 & 0 \end{bmatrix},$$

$C_1 > 0$ and $R > 0$. The sought after control law can be determined by means of the following steps.

a) Verify that the system with output $y = Cx$ is reachable and observable. [4 marks]

b) Consider the Hamiltonian matrix

$$H = \begin{bmatrix} A & -\frac{BB'}{R^2} \\ -Q & -A'. \end{bmatrix}$$

Show that the characteristic polynomial of H is $p(s) = s^4 + \frac{C_1^2}{R^2}$ and compute the eigenvalues of H as a function of C_1 and R. [8 marks]

c) Let $u = Kx = K_1x_1 + K_2x_2$. Find K_1 and K_2 such that the eigenvalues of the resulting closed-loop system coincide with the eigenvalues of H having negative real part. Such a feedback control law solves the considered problem. [4 marks]

d) Finally, the optimal cost associated with the initial state $x(0)$ is

$$x'(0)Px(0),$$

where P is a symmetric matrix such that

$$K = -\frac{B'P}{R^2}.$$

Assume $x(0) = [0, x_2(0)]'$ and determine the optimal cost associated to this initial state. [4 marks]

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Question 1

a) The reachability matrix is

$$R = \begin{bmatrix} 1 & -1 \\ \beta & 1-2\beta \end{bmatrix},$$

and $\det(R) = 1-\beta$. Hence the system is reachable for all $\beta \neq 1$. For $\beta = 1$, consider the reachability pencil

$$\left[\begin{array}{c|c} sI - A & B \end{array}\right] = \left[\begin{array}{cc|c} s+1 & 0 & 1 \\ -1 & s+2 & 1 \end{array}\right],$$

and note that it loses rank for $s = -2$. Hence, the unreachable mode is $s = -2$, and the system is stabilizable.

b) The observability matrix is

$$O = \begin{bmatrix} 1 & \alpha \\ \alpha - 1 & -2\alpha \end{bmatrix},$$

and $\det(O) = -\alpha - \alpha^2$. Hence the system is observable for all $\alpha \neq 0$ and $\alpha \neq -1$. For $\alpha = 0$, consider the observability pencil

$$\left[\begin{array}{c} sI - A \\ \hline C \end{array}\right] = \left[\begin{array}{cc} s+1 & 0 \\ -1 & s+2 \\ \hline 1 & 0 \end{array}\right],$$

and note that it loses rank for $s = -2$. Hence, the unobservable mode is $s = -2$, and the system is detectable. For $\alpha = -1$, the observability pencil is

$$\left[\begin{array}{c} sI - A \\ \hline C \end{array}\right] = \left[\begin{array}{cc} s+1 & 0 \\ -1 & s+2 \\ \hline 1 & -1 \end{array}\right],$$

and note that it loses rank for $s = -1$. Hence, the unobservable mode is $s = -1$, and the system is detectable.

c) To design an output feedback control law with the separation principle and with the given requirement on the eigenvalues we have to find matrices $K = [K_1\ K_2]$ and $L = [L_1\ L_2]$ such that the eigenvalues of $A + BK$ and $A + LC$ are all equal to -2.
Note that

$$A + BK = \begin{bmatrix} -1 + K_1 & K_2 \\ 1 + \beta K_1 & -2 + \beta K_2 \end{bmatrix}.$$

Its characteristic polynomial is

$$s^2 + (3 - K_1 - \beta K_2)s + (-\beta K_2 - 2K_1 + 2 - K_2)$$

and this should be equal to $(s+2)^2$. As a result we have to solve the equations

$$3 - K_1 - \beta K_2 = 4 \qquad\qquad -\beta K_2 - 2K_1 + 2 - K_2 = 4,$$

yielding

$$K_1 = -1 \qquad\qquad K_2 = 0.$$

This implies that the state feedback problem is solvable for any β. In fact, it is solvable for $\beta \neq 1$, by reachability of the system, and for $\beta = 1$ because the unreachable mode coincides with one of the desired closed-loop eigenvalues.
Note now that

$$A + LC = \begin{bmatrix} -1 + L_1 & \alpha L_1 \\ 1 + L_2 & -2 + \alpha L_2 \end{bmatrix}.$$

Its characteristic polynomial is

$$s^2 + (3 - L_1 - L_2\alpha)s + (-L_2\alpha - 2L_1 + 2 - L_1\alpha)$$

and this should be equal to $(s+2)^2$. As a result we have to solve the equations

$$3 - L_1 - L_2\alpha = 4 \qquad\qquad -L_2\alpha - 2L_1 + 2 - L_1\alpha = 4,$$

yielding

$$L_1 = L_2 = -\frac{1}{1+\alpha}.$$

This implies that the output injection problem is solvable for any $\alpha \neq -1$. In fact, it is solvable for $\alpha \neq -1$ and $\alpha \neq 0$, by observability of the system, and for $\alpha = 0$ because the unobservable mode coincides with one of the desired closed-loop eigenvalues. For $\alpha = -1$ it is not solvable because the unobservable mode is $s = -1$.
Finally, the output feedback control law is described by

$$\dot{\xi} = (A + BK + LC)\xi - Ly \qquad\qquad u = K\xi.$$

d) If a static output feedback control law is used then the closed-loop system is described by

$$\dot{x} = (A + BKC)x = \begin{bmatrix} -1 + K & K\alpha \\ 1 + \beta K & K\beta\alpha - 2. \end{bmatrix}$$

If $\beta = 0$ the characteristic polynomial of $A + BKC$ is

$$s^2 + (3 - K)s + (2 - 2K - \alpha K).$$

To have asymptotic stability all coefficients of this polynomial have to be positive (by Routh test), hence

$$K < \frac{2}{2+\alpha} \qquad\qquad K < 3.$$

Note that the first inequality implies the second (by positivity of α), and the admissible values of K include $K = 0$.

Question 2

a) To write the system in state space form define the state variables $x_1 = \theta$ and $x_2 = \dot{\theta}$. As a result we have the equations

$$\dot{x}_1 = x_2 \qquad \dot{x}_2 = \frac{g}{l}\sin x_1 + \frac{1}{Ml^2}u.$$

b) The equilibrium points of the system are the solutions of the equations $\dot{x}_1 = \dot{x}_2 = 0$. This implies $x_2 = 0$ and

$$0 = g\sin x_1 + \frac{1}{Ml}u.$$

Therefore if

$$|u| \le gMl$$

we have infinitely many equilibria, i.e. all solutions of the equation

$$\sin x_1 = -\frac{1}{gMl}u.$$

Note that, from a physical point of view, only two of these solutions are distinct, i.e. describe different positions of the pendulum.
If

$$|u| > gMl$$

the system does not have any equilibrium.
(The above result has a very simple physical interpretation. If the input torque u is constant and smaller, in absolute value, than the torque generated by the gravity then the pendulum can be in equilibrium, otherwise the pendulum will rotate indefinitely.)

c) The linearized system around the equilibrium point $x_1 = 0$, $x_2 = 0$, $u = 0$ is

$$\dot{\delta}_x = A\delta_x + B\delta_u = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & 0 \end{bmatrix}\delta_x + \begin{bmatrix} 0 \\ \frac{1}{Ml^2} \end{bmatrix}\delta_u.$$

d) The characteristic polynomial of the matrix A of the linearized system is

$$s^2 - \frac{g}{l},$$

which has the roots

$$s_1 = -\sqrt{\frac{g}{l}} < 0 \qquad s_2 = \sqrt{\frac{g}{l}} > 0.$$

Therefore, by the principle of stability in the first approximation, the equilibrium is unstable.

e) Setting $M = 1$, $l = 1$, $g = 10$ and $K = [K_1\ K_2]$ yields

$$A + BK = \begin{bmatrix} 0 & 1 \\ 10 + K_1 & K_2 \end{bmatrix}.$$

By Routh test, this matrix has all eigenvalues with negative real part if $K_2 < 0$ and $10 + K_1 < 0$. We can, for example, select $K_1 = -11$ and $K_2 = -1$.

f) If M varies we have

$$A + BK = \begin{bmatrix} 0 & 1 \\ 10 - \frac{11}{M} & -\frac{1}{M} \end{bmatrix},$$

and this matrix has all eigenvalues with negative real part if

$$10 - \frac{11}{M} < 0$$

or equivalently if

$$M < \frac{11}{10}.$$

(It is interesting to note that

- the selection of $K_2 < 0$ does not affect the values of M for which we have asymptotic stability;
- a reduction in M does not yield an unstable closed-loop system;
- to cope with large values of M it is necessary to select a large (in absolute value) and negative K_1.)

Question 3

a) Note that (set $K = [K_1\ K_2\ K_3]$)

$$A + BK = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ K_1 & K_2 & K_3 + 1 \end{bmatrix}$$

and

$\det(sI - (A + BK)) = s^3 + (-K_3 - 4)s^2 + (3K_3 - K_2 + 5)s + (2K_2 - K_1 - 2 - 2K_3)$.

This polynomial should be equal to s^3, hence

$$K_1 = -8 \qquad K_2 = -7 \qquad K_3 = -4.$$

b) Note that

$$\begin{bmatrix} \Pi_1 \\ \Pi_2 \\ \Pi_3 \end{bmatrix} = \Pi S = A\Pi + B\Gamma + P = \begin{bmatrix} 2\Pi_1 + \Pi_2 \\ \Pi_2 + \Pi_3 + 1 \\ \Pi_3 + \Gamma + 1 \end{bmatrix}$$

and

$$0 = C\Pi = C_1\Pi_1 + C_2\Pi_2 + C_3\Pi_3.$$

From the first equation we obtain

$$\Pi_2 = -\Pi_1 \qquad \Pi_3 = -1 \qquad \Gamma = -1,$$

which, replaced in the second equation, yields

$$0 = (C_1 - C_2)\Pi_1 - C_3.$$

This equation, in the unknown Π_1, has a solution Π_1 if $C_3 = 0$, yielding $\Pi_1 = 0$, or if $C_1 - C_2 \neq 0$, yielding $\Pi_1 = \frac{C_3}{C_1 - C_2}$.

c) The required control law is

$$u = Kx + Ld = Kx + (\Gamma - K\Pi)d = \begin{bmatrix} -8 & -7 & -4 \end{bmatrix} x + (-5 + \Pi_1)d,$$

where Π_1 is as computed above.

d) The extended system, with state d and x, and $C_2 = C_3 = 0$ is described by the equations

$$\begin{bmatrix} d^+ \\ x^+ \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} d \\ x \end{bmatrix}$$

$$y = \begin{bmatrix} 0 & C_1 & 0 & 0 \end{bmatrix} \begin{bmatrix} d \\ x \end{bmatrix}.$$

The observability matrix of this system is

$$O = \begin{bmatrix} 0 & C_1 & 0 & 0 \\ 0 & 2C_1 & C_1 & 0 \\ C_1 & 4C_1 & 3C_1 & C_1 \\ 5C_1 & 8C_1 & 7C_1 & 4C_1 \end{bmatrix},$$

which has rank four for all $C_1 \neq 0$. Hence, the system is observable and it is possible to reconstruct the states x and d from measurements of y (and u).

Question 4

a) Replacing $x_1 = x_2 = x_3 = 0$ in the differential equations yields $\dot{x}_1 = \dot{x}_2 = \dot{x}_3 = 0$. Hence $x = 0$ is an equilibrium. The linearized system around this equilibrium is described by

$$\dot{\delta}_x = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & h \\ 0 & k & -1 \end{bmatrix} \delta_x.$$

b) The characteristic polynomial of the linearized system is

$$(s+1)(s^2 + 2s + (1 - kh)).$$

Hence, by Routh test, if $1 - kh > 0$ the system is asymptotically stable, if $1 - kh < 0$ the system is unstable, if $1 - hk = 0$ the system is stable (not asymptotically).

c) The zero equilibrium of the nonlinear system is locally asymptotically stable if $1 - kh > 0$, and unstable if $1 - kh < 0$. If $1 - hk = 0$ the principle of stability in the first approximation does not allow to draw any conclusion on the stability properties of such equilibrium.

d) Note that

$$\frac{\partial f(x)}{\partial x} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 - 3x_2^2 & h \\ 0 & k & -1 - 3x_3^2 \end{bmatrix}$$

hence

$$\frac{\partial f(x)}{\partial x} + \left(\frac{\partial f(x)}{\partial x} \right)' = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 - 6x_2^2 & h + k \\ 0 & h + k & -2 - 6x_3^2 \end{bmatrix}.$$

If $h + k = 0$ the matrix

$$\frac{\partial f(x)}{\partial x} + \left(\frac{\partial f(x)}{\partial x} \right)'$$

is diagonal and has all eigenvalues negative. Therefore, by Krasowsky condition, the zero equilibrium of the nonlinear system is globally asymptotically stable. This implies that, for any initial condition $x(0)$, we have $\lim_{t \to \infty} x(t) = 0$, hence the system cannot have any other equilibrium point.

Question 5

a) Note that

$$y(1) = CB \qquad y(2) = CAB \quad \cdots \quad y(i) = CA^{i-1}B \quad \cdots,$$

hence

$$H_n = \begin{bmatrix} CB & CAB & \cdots & CA^nB \\ CAB & CA^2B & \cdots & CA^{n+1}B \\ \vdots & \vdots & \vdots & \vdots \\ CA^{n-1}B & CA^nB & \cdots & CA^{2n-1}B \end{bmatrix}$$

and this coincides with

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \begin{bmatrix} B & AB & \ldots & A^nB \end{bmatrix}.$$

b) By a direct computation we have

$$CA = [0\ 1\ 0\ 0\ \cdots\ 0], \qquad CA^2 = [0\ 0\ 1\ 0\ \cdots\ 0], \qquad \cdots$$

which proves the claim.

c) Since the observability matrix is the identity, equation (5.1) is

$$H_n = \begin{bmatrix} B & AB & \ldots & A^nB \end{bmatrix}.$$

Therefore, B is equal to the first column of H_n, i.e.

$$B = \begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(n) \end{bmatrix}.$$

d) Note that (recall that $n = 2$)

$$H_2 = \begin{bmatrix} y(1) & y(2) & y(3) \\ y(2) & y(3) & y(4) \end{bmatrix} = \begin{bmatrix} B & AB & A^2B \end{bmatrix}$$

where

$$AB = \begin{bmatrix} B_2 \\ -\alpha_0 B_1 - \alpha_1 B_2 \end{bmatrix} \qquad A^2B = \begin{bmatrix} -\alpha_0 B_1 - \alpha_1 B_2 \\ -\alpha_0 B_2 - \alpha_1(-\alpha_0 B_1 - \alpha_1 B_2) \end{bmatrix}.$$

Therefore,

$$B_1 = y(1) \qquad B_2 = y(2),$$
$$-\alpha_0 y(1) - \alpha_1 y(2) = y(3)$$

and

$$-\alpha_0 y(2) - \alpha_1(-\alpha_0 B_1 - \alpha_1 B_2) = -\alpha_0 y(2) - \alpha_1(y(3)) = y(4).$$

Therefore, to determine α_0 and α_1 we have to solve the last two linear equations.

e) The Hankel matrix associated to the given output sequence is

$$H = \begin{bmatrix} 0 & 1 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix},$$

which has rank equal to two.
Exploiting the results of the previous points we have

$$B_1 = 0 \qquad B_2 = 1$$

and the equations

$$-\alpha_1 y(2) = 0 \qquad -\alpha_0 y(2) = 0,$$

yielding $\alpha_0 = \alpha_1 = 0$. Therefore,

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Question 6

a) The system is in reachability canonical form, hence it is reachable. The observability matrix is

$$O = C_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

hence the system is observable.

b) The Hamiltonian matrix is

$$H = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{R^2} \\ \hline -C_1^2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right].$$

To determine the characteristic polynomial $p(s)$ of H compute the determinant of $sI-H$ using the 'expansion by minors method' starting from the first row. This yields $p(s) = s(s^3) + 1(\frac{C_1^2}{R^2})$. Note now that

$$p(s) = s^4 + \frac{C_1^2}{R^2} = \left(s^2 + \sqrt{2}\sqrt{\frac{C_1}{R}}s + \frac{C_1}{R}\right)\left(s^2 - \sqrt{2}\sqrt{\frac{C_1}{R}}s + \frac{C_1}{R}\right)$$

The eigenvalues of H are the roots of $p(s)$, namely

$$\sqrt{\frac{C_1}{R}}\left(\pm\frac{\sqrt{2}}{2} \pm i\frac{\sqrt{2}}{2}\right).$$

c) K_1 and K_2 have to be such that the characteristic polynomial of

$$A + BK = \begin{bmatrix} 0 & 1 \\ K_1 & K_2 \end{bmatrix},$$

namely

$$s^2 - K_2 s - K_1,$$

equals

$$s^2 + \sqrt{2}\sqrt{\frac{C_1}{R}}s + \frac{C_1}{R}.$$

As a result,

$$K_1 = -\frac{C_1}{R} \qquad K_2 = -\sqrt{2}\sqrt{\frac{C_1}{R}}$$

d) Let

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix},$$

and note that

$$x(0)'Px(0) = [0 \; x_2(0)]P\begin{bmatrix} 0 \\ x_2(0) \end{bmatrix} = x_2(0)^2 P_{22}.$$

Finally,

$$K = [K_1 \ K_2] = -\frac{B'P}{R^2} = -\frac{1}{R^2}[P_{12} \ P_{22}],$$

hence

$$P_{22} = \sqrt{2}\sqrt{\frac{C_1}{R}}R^2,$$

and the optimal cost is $x_2(0)^2\sqrt{2}\sqrt{\frac{C_1}{R}}R^2$.