UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 2002

MSci Honours Degree in Mathematics and Computer Science Part IV
MEng Honours Degrees in Computing Part IV
MSc in Advanced Computing
for Internal Students of the Imperial College of Science, Technology and Medicine

*This paper is also taken for the relevant examinations for the Associateship of the City and Guilds of London Institute*
*This paper is also taken for the relevant examinations for the Associateship of the Royal College of Science*

# PAPER C499

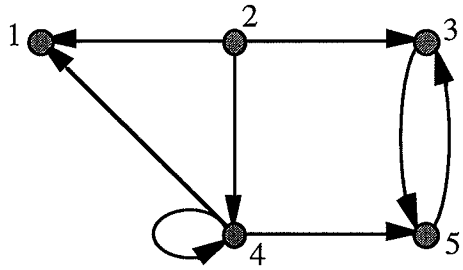
# MODAL AND TEMPORAL LOGIC

Wednesday 1 May 2002, 14:00
Duration: 120 minutes

*Answer THREE questions*

Paper contains 4 questions
Calculators not required

1a Consider the Kripke frame F = (W,R) shown below. The arrows denote the accessibility relation R — for example, R(2,3) holds, but R(1,2) does not.




i) Let g be the assignment into F given by g(p) = {1, 4, 5}. Evaluate the formula $A = \Box(p \to \Box p)$ at world 2 in the Kripke model (F,g). Show all working.

ii) Find an assignment h into F with h(p) = {1, 4, 5} and such that the formula $B = \Box\Box(\Diamond\Diamond p \to \Diamond\Diamond q) \wedge \Box(q \to \neg p)$ is true at world 2 in the Kripke model (F,h). Show all working.

b i) Explain what it means to say that a Kripke frame is a *p-morphic image* of another Kripke frame.

ii) Draw a diagram of a Kripke frame G = (X,S), *with at most three worlds,* that is a p-morphic image of the Kripke frame F of part a.

c i) Let H be a Kripke frame with the following property: for any worlds t, u, if u is accessible from t, then there exists a world w that is accessible from both t and u. Show directly from your definition in part b(i) that any p-morphic image of H also has this property.

ii) A *cycle* in a Kripke frame J = (W,R) is a sequence $w_1, w_2, \ldots, w_n$ of n distinct worlds in W (for some finite $n \geq 2$) such that $R(w_1,w_2)$, $R(w_2,w_3)$, $\ldots$, $R(w_{n-1},w_n)$, $R(w_n,w_1)$ all hold. (For example, 3,5 is a cycle in the frame F of part a.) J is said to be *acyclic* if there are *no* cycles in J.

Show that there is no modal formula C such that for any Kripke frame J, the formula C is valid in J if and only if J is acyclic. You may use standard facts about p-morphisms so long as you quote them clearly.

*The three parts carry, respectively, 30%, 30%, 40% of the marks.*

2a The normal modal logic $K$ is the smallest set of formulas including:

- Propositional Tautologies
- $\Box(p \to q) \to (\Box p \to \Box q)$
- The rules: MP, Uniform Substitution, and Necessitation.

i) Define the notion of Hilbert System.

ii) Show that if $K \vdash \phi \to \psi$, then $K \vdash \Box\phi \to \Box\psi$.

iii) Show that $K \vdash (\Box p \vee \Box q) \to \Box(p \vee q)$.

b Let D be the formula $\Box\Box p \to p$.

i) Using Salqvist's algorithm, find a first-order frame condition that is true in an arbitrary Kripke frame if and only if D is valid in that frame.

ii) What does your frame condition say in English?

c i) Define the truth conditions for a *normal* conditional $A \Rightarrow B$ in a standard conditional model $\mathcal{M} = \langle W, f, h \rangle$, $f : W \times Pow(W) \to Pow(W)$.
In what sense is a standard relational (Kripke) model a special case of a standard conditional model?

ii) Consider a form of strict implication $A \multimap B$ defined as $A \multimap B \stackrel{\text{def}}{=} \Box(A \to B)$ where $\Box$ is a *normal* modality.
$A \multimap B$ is a normal conditional. So therefore: suppose $\mathcal{M}' = \langle W, R, h \rangle$ is a standard relational (Kripke) model for $\Box$. Show that there is a corresponding standard conditional model $\mathcal{M} = \langle W, f, h \rangle$ for $\multimap$. In other words, show how $f$ in $\mathcal{M}$ can be defined in terms of the relation $R$ of $\mathcal{M}'$.
*Hint:* For $X$, $Y$, $Z$ any subsets of $W$, $X \subseteq (W - Y) \cup Z$ iff $X \cap Y \subseteq Z$.

iii) Show that the axiom schema MP

$$\text{MP.} \qquad (A \Rightarrow B) \to (A \to B)$$

is validated by the following condition on a standard conditional model $\mathcal{M}$:

$$\text{(mp)} \qquad \text{if } w \in X \text{ then } w \in f(w, X), \text{ for any } X \subseteq W \text{ and any } w \in W.$$

*The three parts carry, respectively, 30%, 35%, 35% of the marks.*

3a Consider the normal modal logic $S4 = K \cup \{\Box p \to p\} \cup \{\Box p \to \Box\Box p\}$, and a set of formulas $\Gamma \subseteq \mathcal{L}$, where $\mathcal{L}$ is the language of propositional modal logic.

i) Define the notion of maximal $S4$-consistency.

ii) Show that if $\Gamma = \{\Box\alpha_1, \ldots, \Box\alpha_n, \neg\Box\beta\}$ is $S4$-consistent then so is $\Gamma' = \{\Box\alpha_1, \ldots, \Box\alpha_n, \neg\beta\}$.

b Consider the normal logic $L = K \cup \{\Diamond\Box p \to \Diamond p\}$, i.e., the smallest set of formulas including:

- Propositional Tautologies
- $\Box(p \to q) \to (\Box p \to \Box q)$
- $\Diamond\Box p \to \Diamond p$
- The rules: MP, Uniform Substitution, and Necessitation.

i) Let $\mathcal{F}$ be the class of frames $\{F = (W, R)\}$ such that, for all $w, w' \in W$:

$$\text{if } wRw' \text{ then there exists a } w'' \in W \text{ such that } w'Rw'' \text{ and } wRw''.$$

Prove that L is sound and complete with respect to the class of frames $\mathcal{F}$.

ii) Prove frame correspondence for the axiom $\Diamond\Box p \to \Diamond p$ above, i.e., that given any frame $F$, $F \models \Diamond\Box p \to \Diamond p$ if and only if $F$ is such that, for all $w, w' \in W$:

$$\text{if } wRw' \text{ then there exists a } w'' \in W \text{ such that } w'Rw'' \text{ and } wRw''.$$

*The two parts carry, respectively, 40% and 60% of the marks.*

4a Let $\mathcal{M} = \langle W, \nu, h\rangle$ be a neighbourhood model: $W$ is a set of worlds, $h$ is a valuation function for the atoms, and $\nu : W \to Pow(Pow(W))$ is the neighbourhood function.

i) Define the truth set $\|A\|^{\mathcal{M}}$ of a formula $A$ in model $\mathcal{M}$, and state the truth conditions for $\Box A$ and $\Diamond A$ in model $\mathcal{M}$. Also express the truth conditions using the inverse function $f$ of $\nu$, defined as: $w \in f(X)$ iff $X \in \nu(w)$.

ii) Identify model conditions that validate the following schema B below. Justify your answer.

$$\text{B.} \qquad A \to \Diamond\Box A$$

You may find it easier to state the conditions using the function $f$ instead of $\nu$.

b Let $S$ be a (classical) system of modal logic.

i) Define the proof set $|A|_S$ of a formula $A$ in the system $S$.

ii) A model $\mathcal{M}^S = \langle W, \nu, h\rangle$ is a canonical model for the system $S$ when $W = |\top|_S$, $h(p) = |p|_S$, and for every $w \in \mathcal{M}$, $\Box A \in w$ iff $|A|_S \in \nu(w)$, i.e., $|\Box A|_S = f(|A|_S)$. Explain how such a canonical model can be used to prove completeness of the system $S$ with respect to a given class $C$ of neighbourhood models.

c Let $EG$ be the smallest classical system of modal logic containing the following schema G:

$$\text{G.} \qquad \Diamond\Box A \to \Box\Diamond A$$

Show that $EG$ is sound and complete with respect to neighbourhood models satisfying the following model condition (g), for all $X \subseteq W$:

$$\text{(g)} \qquad W - f(W - f(X)) \subseteq f(W - f(W - X))$$

($f$ is the inverse of $\nu$, as in part (a) above).

For the completeness proof, use the canonical model in which $f$ is defined as follows:

- $f(|A|_S) = |\Box A|_S$
- $f(X) = X$, for $X \neq |A|_S$ any $A$.

You may assume that if $X$ is a proof set, i.e., if $X = |A|_S$ for some formula $A$, then $W - X$ is also a proof set.

*The three parts carry, respectively, 25%, 35%, 40% of the marks.*