IMPERIAL COLLEGE LONDON

EE4-29
EE9-CS3-2
EE9-FPN1-04

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2018

MSc and EEE/EIE PART IV: MEng and ACGI

Corrected copy

OPTIMIZATION

Thursday, 3 May 10:00 am

Time allowed: 3:00 hours

There are FOUR questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : A. Astolfi
Second Marker(s) : P.L. Dragotti

1. The Levenberg-Marquardt algorithm is a modification of Newton's method for the solution of nonlinear equations. In the case of the scalar equation

$$f(x) = 0,$$

with $x \in I\!R$ and f differentiable, the Levenberg-Marquardt algorithm can be written as (note that f' denotes the first derivative of f with respect to x)

$$x_{k+1} = x_k - 2\frac{f(x_k)}{f'(x_k) + f'(\bar{x})}, \qquad \bar{x} = x_k - \frac{f(x_k)}{f'(x_k)}.$$

Consider now the problem of minimizing the function

$$q(x) = \frac{x^4}{4} + \frac{4}{3}x^3 - 10x$$

(note that the global minimizer of the function q is $x = 1.365230013$).

a) Re-cast the considered minimization problem as the problem of finding the solution of a scalar equation. [2 marks]

b) Write Newton's iteration for the solution of the equation determined in part a). [2 marks]

c) Run four iterations of the Newton's iteration in part b) with $x_0 = 3$ and evaluate the first four values of the sequence of the relative errors

$$RE^N_{k+1} = \frac{x_{k+1} - x^\star}{(x_k - x^\star)^3},$$

that is evaluate RE^N_1, RE^N_2, RE^N_3 and RE^N_4. Hence argue that Newton's method does not have speed of convergence of order three. Explain why this is not un-expected. [6 marks]

d) Write now the Levenberg-Marquardt algorithm for the solution of the equation determined in part a).
(Hint: write the algorithm as two equations, that is do not substitute $\bar{x}$ into the first equation of the algorithm.) [2 marks]

e) Run four iterations of the Levenberg-Marquardt algorithm in part d) with $x_0 = 3$ and evaluate the first four values of the sequence of the relative errors

$$RE^{LM}_{k+1} = \frac{x_{k+1} - x^\star}{(x_k - x^\star)^3}.$$

Hence argue that the Levenberg-Marquardt algorithm is faster than Newton's algorithm. [8 marks]

(Comment: it is well-known that the Levenberg-Marquardt algorithm has, under similar assumptions to those required by Newton's method, speed of convergence of order three.)

2. Consider the optimization problem

$$\min_{x_1,x_2} x_1^2 + x_2^2 - 3,$$
$$x_1 - 3x_2 + 2 \leq 0.$$

a) Write the necessary conditions of optimality for this problem and determine a candidate optimal solution. [4 marks]

b) Show that the candidate optimal solution determined in part a) is a strict minimizer. [4 marks]

c) The considered problem can be transformed introducing the *slack* variable $s \in \mathbb{R}$ and re-writing the problem as

$$\begin{aligned} &\min_{x_1,x_2,s} x_1^2 + x_2^2 - 3, \\ &x_1 - 3x_2 + 2 + s = 0, \qquad\qquad (\star) \\ &-s \leq 0. \end{aligned}$$

Write necessary conditions of optimality for the transformed problem $(\star)$. [2 marks]

d) Consider now the solution of the transformed problem $(\star)$ using the so-called log penalty function method described by

$$\begin{aligned} &\min_{x_1,x_2,s} x_1^2 + x_2^2 - 3 - \varepsilon \log(s), \\ &x_1 - 3x_2 + 2 + s = 0, \end{aligned} \qquad (\star\star)$$

with $\varepsilon > 0$.

i) Write necessary conditions of optimality for the log penalty function problem $(\star\star)$. [2 marks]

ii) Show that one of the necessary conditions of optimality determined in part d.i) provides an *approximation* of the complementarity condition determined in part c). [4 marks]

iii) Determine the unique candidate optimal solution for the log penalty function problem $(\star\star)$ and show that this coincides with the optimal solution of the original problem as ε tends to zero. [4 marks]

3. Consider the optimization problem

$$\min_x f(x),$$
$$h(x) \leq 0,$$

with $x \in I\!R^n$ and $h : I\!R^n \to R$, that is there is only one inequality constraint.

The basic idea of the so-called "Lagrange relaxation" is to replace the constraint with a price in the objective function to penalize the constraint violation. Thus, the considered constrained optimization problem is transformed into the family of unconstrained optimization problems

$$\min_x f(x) + \mu h(x), \qquad (\star)$$

with $\mu \geq 0$. Note that the solution of the Lagrange relaxation problem is a function of μ. We denote this function with $q(\mu)$. The function $q(\mu)$ is such that for all $\mu \geq 0$ and all feasible x the inequality

$$q(\mu) \leq f(x)$$

holds. This inequality is called the weak duality condition and motivates the following Lagrangian dual problem

$$\max_\mu q(\mu),$$
$$\mu \geq 0.$$

The optimal solution of the Lagrangian dual problem provides a lower bound for the optimal solution of the original problem. If the largest lower bound of the Lagrangian dual problem coincides with the optimal solution of the original problem than we say that the duality gap is zero.

Consider now the optimization problem

$$\min_x x^2,$$
$$1 - x \leq 0.$$

a) Write the Lagrangian of the problem and the associated necessary conditions of optimality. [2 marks]

b) Determine, using the necessary conditions derived in part a), the optimal solution of the problem. [2 marks]

c) Write the Lagrange relaxation of the problem, that is write the family of optimization problems defined in $(\star)$. Hence, solve the problem to determine the function $q(\mu)$. [6 marks]

d) Write the Lagrangian dual problem for the considered optimization problem and determine its solution. Show that for the considered problem the duality gap is zero, that is the optimal cost $q^\star$ of the Lagrangian dual problem coincides with the optimal cost $f^\star$ of the original problem. Thus show that the optimal solution of the considered problem can be obtained by solving the equation $f(x) = f^\star = q^\star$. [6 marks]

e) Write the sequential penalty function F_ε for the considered optimization problem and determine its global minimizer. Show that as ε tends to zero the global minimum of F_ε converges to the value $f^\star$ determined in part d). [4 marks]

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4. Consider the optimization problem

This is correct

$$
\begin{aligned}
&\min_{x_1,x_2} x_1^2 - x_1\\
&2 - x_1 \le 0,\\
&(x_1-3)^2 - x_2 - 2 \le 0,\\
&1 - x_1 + x_2 \le 0.
\end{aligned}
$$

a) Plot the admissible set and show that all points are regular points. [2 marks]

b) Solve the problem using only graphical considerations. [2 marks]

c) State first order necessary conditions of optimality for this constrained optimisation problem. [4 marks]

d) Using the conditions derived in part c) compute candidate optimal solutions. [8 marks]

e) Determine if the conditions of strict complementarity hold at the candidate optimal points computed in part d). Hence, explain why second order sufficient conditions of optimality cannot be used to study the optimality properties of the candidate optimal points. [4 marks]