E4.25
C4.1
ISE4.23

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2006

MSc and EEE/ISE PART IV: MEng and ACGI

DESIGN OF LINEAR MULTIVARIABLE CONTROL SYSTEMS

Thursday, 4 May 2:30 pm

Time allowed: 3:00 hours

Corrected Copy

There are SIX questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s) :	I.M. Jaimoukha
	Second Marker(s) :	D.J.N. Limebeer

Special Information for Invigilators :	None
Information for Candidates :	None

1. (a) Let the transfer matrix $G(s)$ have a state space realization,

$$G(s) \stackrel{s}{=} \left[\begin{array}{ccc|cc} 1 & 0 & 1 & 1 & 3 \\ 3 & -2 & 2 & 2 & 4 \\ 0 & 0 & 4 & 0 & 0 \\ \hline 1 & 0 & 2 & 0 & 1 \\ 4 & 0 & 3 & 1 & 0 \end{array}\right].$$

Find the uncontrollable and/or unobservable modes and determine whether the realization is detectable and stabilizable. **[6]**

(b) Consider a state–variable model described by the dynamics

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t), \end{aligned}$$

and denote the corresponding transfer matrix by $H(s)$. Suppose that there exists $P = P' > 0$ such that

$$\begin{bmatrix} A'P + PA + C'C & PB \\ B'P & -I \end{bmatrix} < 0.$$

(i) Prove that A is stable. **[6]**

(ii) By defining the Lyapunov function

$$V(t) = x(t)'Px(t),$$

the cost function

$$J := \int_0^\infty [y(t)'y(t) - u(t)'u(t)]dt,$$

and using a property of the integral $\int_0^\infty \dot{V}(t)dt$, or otherwise, prove that

$$\|H\|_\infty < 1.$$

State clearly the assumptions required on $u(t)$, $x(0)$ and $x(\infty)$. **[8]**

2. Consider the nominal and actual loops shown in Figure 2 below. Here

$$L(s) = G(s)K(s)$$

is the loop gain, where $G(s)$ is a given system model and $K(s)$ is a compensator. The transfer matrix $\Delta(s)$ represents a perturbation and it is assumed that $\Delta(s)$ is stable.

(a) Define internal stability for the nominal loop, and derive necessary and sufficient conditions for which this feedback loop is internally stable.

[6]

(b) Suppose that the transfer matrix $G(s)$ in the nominal loop in Figure 2 is stable. Derive a parameterization of all internally stabilizing controllers for the feedback loop.

[6]

(c) Suppose that

$$G(s) = \frac{1}{s+1} G_o(s)$$

where $G_o(s)$ is a stable and minimum–phase transfer matrix (that is, $G_o(s)^{-1}$ is stable). By using the answer to Part (b) above and the small gain theorem, or otherwise, find the maximum $\mathcal{H}_\infty$ norm of Δ for which there always exists a stabilizing controller for the actual loop in Figure 2.

[8]

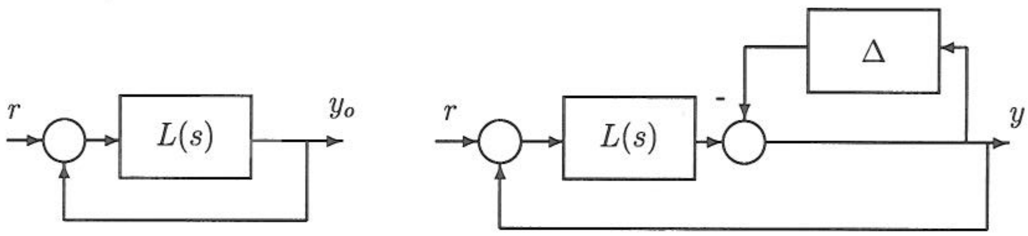

Figure 2

3. Consider the regulator shown in Figure 3 for which it is assumed that the triple (A, B, C) is minimal and $x(0)=x_0$.

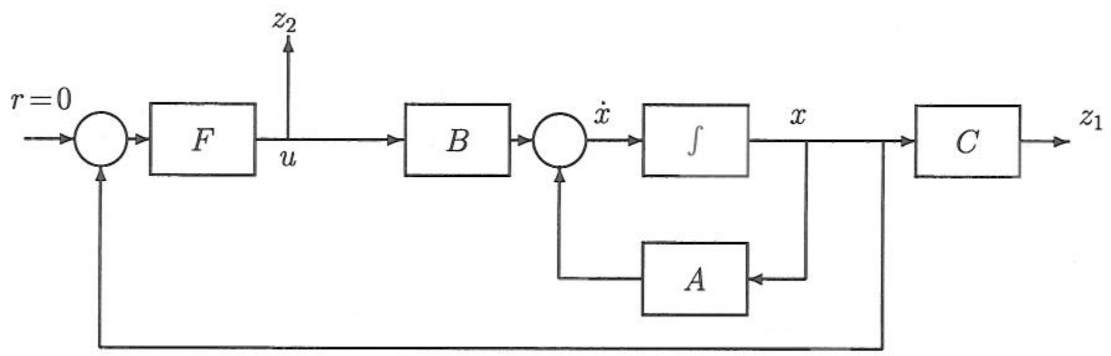

Figure 3

Let $z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$. A stabilizing state-feedback gain matrix F is to be designed such that the cost function $J := \int_0^\infty z(t)^T z(t) dt$ is minimized.

Let $V(t) = x(t)^T P x(t)$ where $P = P^T$ is the unique positive definite solution of the algebraic Riccati equation

$$A^T P + PA + C^T C - PBB^T P = 0$$

(a) Assuming the closed loop is asymptotically stable, obtain an expression for $\int_0^\infty \dot{V}(t) dt$ in terms of x_0. [5]

(b) Evaluate an expression for J using

(i) $u(t) = Fx(t)$,

(ii) Part (a),

(iii) the algebraic Riccati equation,

(iv) an appropriate completion of a square.

Using this expression, find F that minimizes J. Give also the minimum value of J. [5]

(c) Prove that the closed loop is stable by showing that $\dot{V}(t) < 0$ along closed-loop trajectories. [5]

(d) Suppose that $A = B = C = x_0 = 1$. Find P, F and the minimum value of J. Verify that the closed loop is stable. [5]

4. Consider the feedback loop shown in Figure 4 where $G(s)$ represents a plant model and $K(s)$ represents an internally stabilizing compensator. Suppose that

$$G(s) \stackrel{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cc|cc} -1 & -1 & 1 & 1 \\ -1 & -1.25 & 0.6 & 0.8 \\ \hline 1 & 0.6 & 0 & 0 \\ 1 & 0.8 & 0 & 0 \end{array}\right] \in \mathcal{RH}_\infty.$$

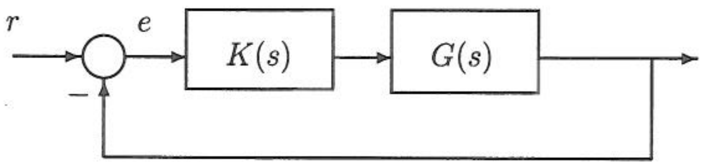

Figure 4

(a) Show that the given realization for $G(s)$ is balanced and evaluate the Hankel singular values of $G(s)$.

[6]

(b) By using:

- the answer to Part (a),
- the small gain theorem (which should be stated),
- and a parameterization of the set of all internally stabilizing controllers,

derive a technique to design a first order internally stabilizing controller $K(s)$ for $G(s)$.

[8]

(c) Since there are many controllers which satisfy the design specifications in Part (b), explain how to choose the controller so that the loop DC gain is acceptable.

[6]

5. Consider the feedback configuration in Figure 5. Here, $G(s)$ is a nominal plant model and $K(s)$ is a compensator. The signal $w(s)$ represents a disturbance on the input of the plant. The design specifications are to synthesize a compensator $K(s)$ such that the feedback loop is internally stable and:

- For good disturbance rejection, it is required that,

$$\|e(j\omega)\| < |w_1(j\omega)^{-1}|\,\|w(j\omega)\|\,\forall\,\omega.$$

- To limit the control effort, it is required that,

$$\|u(j\omega)\| < |w_2(j\omega)^{-1}|\,\|w(j\omega)\|, \forall\,\omega.$$

- For good regulation it is required that,

$$\|y(j\omega)\| < |w_3(j\omega)^{-1}|\,\|w(j\omega)\|, \forall\,\omega,$$

where $w_1(s)$, $w_2(s)$ and $w_3(s)$ are suitable filters and where $\|\cdot\|$ denotes the Euclidean norm.

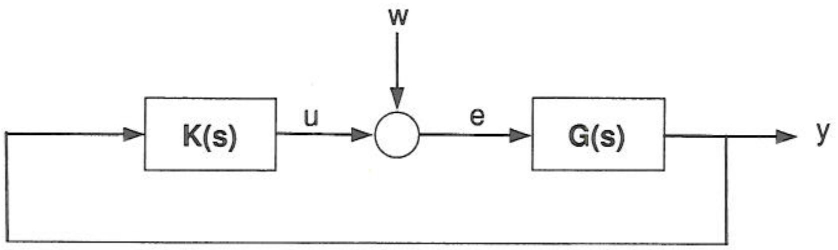

Figure 5

(a) Derive $\mathcal{H}_\infty$-norm bounds, in terms of $G(s), K(s), w_1(s)$, $w_2(s)$ and $w_3(s)$ that are sufficient to achieve the design specifications.

[6]

(b) Derive a generalized regulator formulation of the design problem that captures the sufficient conditions in Part (a).

[7]

(c) Assume that $K(s)$ achieves the design specifications in Part (a). Suppose that an input multiplicative uncertainty $\Delta(s)$ is introduced so that the actual plant is $G(s)\,[I + \Delta(s)]$ where $\Delta(s)$ is a stable transfer matrix. Derive an upper bound on $\|\Delta(j\omega)\|$, for all ω, for which closed loop stability is guaranteed.

[7]

6. Consider the regulator shown in Figure 6 for which it is assumed that the triple (A, B, C) is minimal and $x(0)=0$.

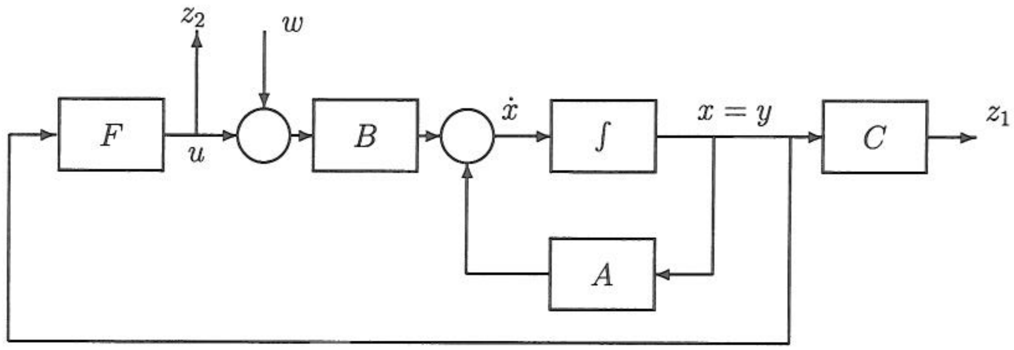

Figure 6

Let $z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ and let H denote the transfer matrix from w to z. A stabilizing state-feedback gain matrix F is to be designed such that, for given $\gamma > 0$, $\|H\|_\infty < \gamma$.

(a) Write down the generalized regulator system for this design problem. **[5]**

(b) By using the Lyapunov function $V(t) = x(t)^T X x(t)$, where X is to be determined, derive sufficient conditions for the solution of the design problem. Your conditions should be in the form of the existence of a certain solution to an algebraic Riccati equation. It should also include an expression for F and an expression for the worst-case disturbance w. **[10]**

(c) Suppose that $A = B = C = 1$. Find the smallest γ_{opt} such that for all $\gamma > \gamma_{opt}$, there exists a solution to the design problem in Part (b) above. **[5]**

[END]

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Design of Linear Multivariable Control Systems

Solutions 2006

1. (a) Since $[A - sI \;\; B]$ loses rank for $s = 4$ it is an uncontrollable mode, and since $[A^T - sI \;\; C^T]$ loses rank for $s = -2$, it is an unobservable mode. Since the uncontrollable mode is unstable, the realization is not stabilizable, and since the unobservable mode is stable, the realization is detectable.

   (b) (i) The $(1,1)$ block of the inequality gives the inequality $A'P+PA+C'C < 0$. Let $z \neq 0$ be a right eigenvector of A and let λ be the corresponding eigenvalue. Then multiplying the inequality from the left by z' and from the right by z gives $(\lambda + \bar{\lambda})z'Pz + z'C'Cz < 0$. Since $P > 0$ it follows that $z'Pz > 0$ and since $z'C'Cz \geq 0$ it follows that $\lambda+\bar{\lambda} < 0$ so that A is stable.

   (ii) Since A is stable, $\|H\|_\infty < 1$ if and only if, with $x(0) = 0$,

$$J := \int_0^\infty [y'y - u'u]dt < 0,$$

for all $u(t)$ such that $\|u\|_2 < \infty$. If $\|u\|_2$ is bounded, then $\lim_{t\to\infty} x(t) = 0$. Now,

$$\int_0^\infty \frac{\mathrm{d}}{\mathrm{dt}}[x'Px]dt = x(\infty)'Px(\infty) - x(0)'Px(0) = 0.$$

So,

$$\begin{aligned} 0 &= \int_0^\infty \dot{x}'Px + x'P\dot{x}dt = \int_0^\infty [(Ax+Bu)'Px + x'P(Ax+Bu)]dt \\ &= \int_0^\infty [x'(A'P+PA)x + x'PBu + u'B'Px]dt \end{aligned}$$

Use $y = Cx$ and add the last expression to J

$$\begin{aligned} J &= \int_0^\infty [x'(A'P+PA+C'C)x + x'PBu + u'B'Px - u'u]dt \\ &= \int_0^\infty \begin{bmatrix} x' & u' \end{bmatrix} \begin{bmatrix} A'P+PA+C'C & PB \\ B'P & -I \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} dt \\ &< 0 \end{aligned}$$

from the inequality in the question. This proves the result.

2. (a) Inject a signal d in between $G(s)$ and $K(s)$ and call the input to $G(s)$, u and the input to $K(s)$, e. The loop is internally stable if and only if the transfer matrix from $\begin{bmatrix} d \\ r \end{bmatrix}$ to $\begin{bmatrix} u \\ e \end{bmatrix}$ is stable (no poles in the closed right half plane). Since

$$\begin{bmatrix} d \\ r \end{bmatrix} = \begin{bmatrix} I & -K \\ -G & I \end{bmatrix} \begin{bmatrix} u \\ e \end{bmatrix} =: T(s) \begin{bmatrix} u \\ e \end{bmatrix}$$

the loop is internally stable if and only if $T^{-1}(s)$ is stable.

(b) Since $G(s)$ is stable, we proceed as follows. Note that

$$\begin{bmatrix} I & -K \\ -G & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ -G & I \end{bmatrix} \begin{bmatrix} I & -K \\ 0 & I-GK \end{bmatrix}.$$

Hence

$$\begin{bmatrix} I & -K \\ -G & I \end{bmatrix}^{-1} = \begin{bmatrix} I & -K \\ 0 & I-GK \end{bmatrix}^{-1} \begin{bmatrix} I & 0 \\ -G & I \end{bmatrix}^{-1} = \begin{bmatrix} I & K(I-GK)^{-1} \\ 0 & (I-GK)^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ G & I \end{bmatrix}.$$

Finally, since $(I-GK)^{-1} = I + GK(I-GK)^{-1}$, it follows that if G is stable, then the loop is internally stable if and only if $Q := K(I-GK)^{-1}$ is stable. Rearranging terms shows that K is internally stabilizing if and only if $K = Q(I+GQ)^{-1}$ for some stable Q.

(c) Let ϵ be the input to Δ and δ be the output of Δ. Then $\epsilon = -(I-GK)^{-1}\delta$. Substituting the expression for stabilizing K from Part (b), and the expression for $G(s)$,

$$[I - G(s)K(s)]^{-1} = I + G(s)Q(s) = I + \frac{1}{s+1} G_o(s)Q(s).$$

Since $G_o(s)^{-1}$ is stable, we can set $Q(s) = G_o(s)^{-1}\hat{Q}(s)$ for some stable $\hat{Q}(s)$. It follows that

$$[I - G(s)K(s)]^{-1} = I + \frac{1}{s+1}\hat{Q}(s).$$

The small gain theorem implies that to guarantee internal stability we require $\|\Delta\|_\infty \left\| I + \frac{1}{s+1}\hat{Q}(s) \right\|_\infty < 1$. However, $\left\| I + \frac{1}{s+1}\hat{Q}(s) \right\|_\infty \geq 1$ for any $\hat{Q}(s)$. It follows we can guarantee internal stability only if $\|\Delta\|_\infty < 1$.

3. (a) Let $V=x^TPx$ and set $u=Fx$. Provided that $P=P^T>0$ and $\dot{V}<0$ along closed-loop trajectories, we can assume $\lim_{t\to\infty} x(t)=0$. Then

$$\dot{V}=\dot{x}^TPx+x^TP\dot{x}=x^T\left(A^TP+PA+F^TB^TP+PBF\right)x.$$

Integrating from 0 to ∞ and using $x(\infty)=0$,

$$\int_0^\infty x^T\left(A^TP+PA+F^TB^TP+PBF\right)xdt=-x_0^TPx_0.$$

(b) Using the definition of J and adding the last equation,

$$J=x_0^TPx_0+\int_0^\infty x^T[A^TP+PA+C^TC+F^TF+F^TB^TP+PBF]xdt.$$

Completing the squares by using

$$(F+B^TP)^T(F+B^TP)=F^TF+F^TB^TP+PBF+PBB^TP,$$

$$J=x_0^TPx_0+\int_0^\infty\{x^T[A^TP+PA+C^TC-PBB^TP]x+\left\|(F+B^TP)x\right\|^2\}dt.$$

Since the last term is always nonnegative, it follows that the minimizing value of F is given by $F=-B^TP$. Since the term in square brackets is zero from the Riccati equation, it follows that the minimum value of J is $x_0^TPx_0$.

(c) It remains to prove $\dot{V}<0$ along the state-trajectory with $u=Fx$. But using the expression for $\dot{V}(t)$ in Part (a), the Riccati equation and the expression for F, we get

$$\dot{V}=x^T\left(A^TP+PA+F^TB^TP+PBF\right)x=-x^T(C^TC+PBB^TP)x<0$$

for all $x\neq 0$ (since (A,B,C) is assumed minimal) proving closed-loop stability.

(d) Putting in the numbers in the Riccati equation and the expression for F, we get $P=1+\sqrt{2}$, $F=-1-\sqrt{2}$ and the minimum value of J is $1+\sqrt{2}$. The closed loop A–matrix is given by $A+BF=-\sqrt{2}$ demonstrating closed-loop stability.

4. (a) The realization of $G(s)$ is balanced if

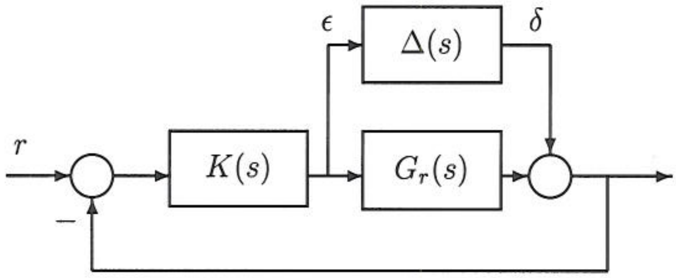
$$A\Sigma + \Sigma A' + BB' = A'\Sigma + \Sigma A + C'C = 0$$

for $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2) > 0$ and where the $\sigma_i's$ are the Hankel singular values of $G(s)$. A simple calculation gives $\Sigma = \mathrm{diag}(1, 0.4)$.

(b) Let $G_r(s)$ denote a first-order balanced truncation of $G(s)$. Then $G_r(s) = G(s) + \Delta(s)$ where

$$\|\Delta\|_\infty \leq 2\sum_{i=2}^{2} \sigma_i = 0.8.$$

Then replacing $G(s)$ by $G_r(s)$ in the loop of Figure 4 is equivalent to:



Now

$$\epsilon = -K(I + G_r K)^{-1}\delta$$

and so the loop is stable if $\|K(I + G_r K)^{-1}\|_\infty < \frac{1}{\|\Delta\|_\infty} \leq 1.25$ from the small gain theorem. However, the set of all internally stabilizing controllers for $G_r(s)$ is given by:

$$K = Q(I - G_r Q)^{-1}$$

for stable Q. Furthermore,

$$K(I + G_r K)^{-1} = Q.$$

Thus we can take $Q = qI_2$ where q is constant (to guarantee a first order controller) and $|q| < 1.25$ (to guarantee stabilization of G).

(c) The DC loop gain is given by

$$G(0)K(0) = G(0)q\left[I - G_r(0)q\right]^{-1} = G(0)\left[q^{-1}I - G_r(0)\right]^{-1}.$$

A high DC loop gain ensures good tracking for DC signals. Now,

$$q^{-1}I - G_r(0) = \begin{bmatrix} q^{-1} - 1 & -1 \\ -1 & q^{-1} - 1 \end{bmatrix}.$$

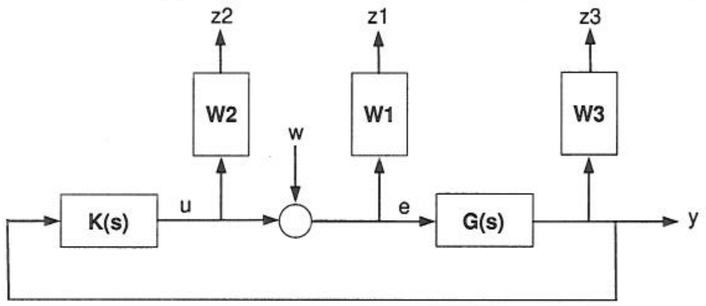
A little calculation shows that this is singular for $q = 0.5$ (which is allowed by Part (b) above), thus ensuring infinite loop DC gain.

5. (a) It is clear that we require K to be internally stabilizing.

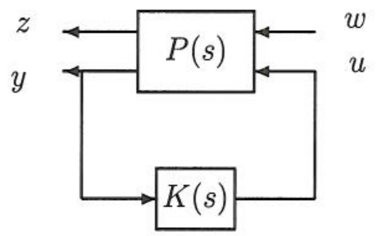
- A simple calculation shows that, $e(s) = S(s)w(s)$ where $S(s) = [I - K(s)G(s)]^{-1}$. Thus $\|e(j\omega)\| \leq \|S(j\omega)\|\|w(j\omega)\|$. It follows that a sufficient condition to achieve the first design specification is $\|S(j\omega)\| < |w_1^{-1}(j\omega)|, \forall\omega$ or equivalently $\|W_1 S\|_\infty < 1$, where $W_1 = w_1 I$.
- A similar calculation shows that, $u(s) = K(s)G(s)S(s)w(s)$. Thus $\|u(j\omega)\| \leq \|K(j\omega)G(j\omega)S(j\omega)\|\|w(j\omega)\|$. It follows that a sufficient condition to achieve the second design specification is $\|K(j\omega)G(j\omega)S(j\omega)\| < |w_2^{-1}(j\omega)|$, $\forall\omega$ or equivalently $\|W_2 KGS\|_\infty < 1$, where $W_2 = w_2 I$.
- Another calculation shows that $y(s) = G(s)S(s)w(s)$. Thus $\|y(j\omega)\| \leq \|G(j\omega)S(j\omega)\|\|w(j\omega)\|$. It follows that a sufficient condition to achieve the third design specification is $\|G(j\omega)S(j\omega)\| < |w_3^{-1}(j\omega)|, \forall\omega$ or equivalently $\|W_3 GS\|_\infty < 1$, where $W_3 = w_3 I$.

To satisfy all design requirements, it is sufficient that $\left\| \begin{bmatrix} W_1 S \\ W_2 KGS \\ W_3 GS \end{bmatrix} \right\|_\infty < 1.$

(b) The design specifications reduce to the requirement that the transfer matrix from w to $z = [z_1^T \ z_2^T \ z_3^T]^T$ in the following diagram has $\mathcal{H}_\infty$-norm less than 1.



The corresponding generalized regulator formulation is to find an internally stabilizing K such that $\|\mathcal{F}_l(P, K)\|_\infty < 1$:



$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \left[\begin{array}{c|c} W_1 & W_1 \\ 0 & W_2 \\ W_3 G & W_3 G \\ \hline G & G \end{array}\right].$$

(c) Let the input to Δ be ϵ and the output from Δ be δ. Then $\epsilon = KGS\delta$ and since KGS is stable, the small gain theorem implies closed-loop stability if $\|\Delta(j\omega)K(j\omega)G(j\omega)S(j\omega)\| < 1, \forall\omega$. Since K achieves the design specifications of Part (a), $\|\Delta(j\omega)\| < |w_2(j\omega)|, \forall\,\omega$ is the maximal stability radius.

27/[illegible]/06

6. (a) The generalized regulator formulation is given by

$$\begin{bmatrix} z(s) \\ y(s) \end{bmatrix} = P(s) \begin{bmatrix} w(s) \\ u(s) \end{bmatrix},\ u(s) = Fy(s),\ P(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{c||c|c} A & B & B \\ \hline\hline C & 0 & 0 \\ 0 & 0 & I \\ \hline I & 0 & 0 \end{array}\right].$$

(b) The requirement $\|H\|_\infty < \gamma$ is equivalent to $J := \|z\|_2^2 - \gamma^2 \|w\|_2^2 < 0$, with $\|v\|_2^2 := \int_0^\infty \|v(t)\|^2 dt$. Let $V = x^T X x$ and set $u = Fx$. Provided that $X = X^T > 0$ and $\dot{V} < 0$ along the closed-loop trajectory, we can assume $\lim_{t\to\infty} x(t) = 0$. Then

$$\dot{V} = \dot{x}^T X x + x^T X \dot{x} = x^T \left(A^T X + XA + F^T B^T X + XBF\right) x + x^T XBw + w^T B^T X x.$$

Integrating from 0 to ∞ and using $x(0) = x(\infty) = 0$,

$$0 = \int_0^\infty [x^T \left(A^T X + XA + F^T B^T X + XBF\right) x + x^T XBw + w^T B^T X x] dt.$$

Using the definition of J and adding the last equation, $J =$

$$\int_0^\infty \{x^T [A^T X + XA + C^T C + F^T F + F^T B^T X + XBF] x - [\gamma^2 w^T w - x^T XBw - w^T B^T X x]\} dt.$$

Let $Z = F + B^T X$. Completing the squares by using

$$Z^T Z = F^T F + F^T B^T X + XBF + XBB^T X$$

$$\|(\gamma w - \gamma^{-1} B^T X x)\|^2 = \gamma^2 w^T w - w^T B^T X x - x^T XBw + \gamma^{-2} x^T XBB^T X x,$$

$$J = \int_0^\infty \{x^T [A^T X + XA + C^T C - (1-\gamma^{-2}) XBB^T X] x + \|Zx\|^2 - \left\|\gamma w - \gamma^{-1} B^T X x\right\|^2\} dt.$$

Thus two sufficient conditions for $J < 0$ are the existence of X such that

$$A^T X + XA + C^T C - (1-\gamma^{-2}) XBB^T X = 0, \qquad X = X^T > 0.$$

The state feedback gain is $F = -B^T X$ (ensuring $Z = 0$) and the worst case disturbance is $w^* = \gamma^{-2} B^T X x$. The closed-loop with these feedback laws is $\dot{x} = [A - (1-\gamma^{-2}) BB^T X] x$ and a third condition is therefore $Re\ \lambda_i [A - (1 - \gamma^{-2}) BB^T X] < 0,\ \forall\, i$.
It remains to prove $\dot{V} < 0$ along state-trajectory with $u = Fx$ and $w = 0$. But

$$\dot{V} = x^T \left(A^T X + XA + F^T B^T X + XBF\right) x = -x^T (C^T C + (1 + \gamma^{-2}) XBB^T X) x < 0$$

for all $x \neq 0$ (since (A, B, C) is assumed minimal) proving closed-loop stability.

(c) Putting in the numbers in the Riccati equation, we get $X = \frac{1 \pm \sqrt{2-\gamma^{-2}}}{1-\gamma^{-2}}$. For the stability condition, we need to choose the positive square root, so $X = \frac{1+\sqrt{2-\gamma^{-2}}}{1-\gamma^{-2}}$. It follows that the optimal γ is the infimum value of γ for which $X > 0$, so $\gamma_{opt} = 1$.