EE2-08A MATHEMATICS

Where no comments are made, most students answered correctly and there were no remarkable or systematic errors.

1. a) Show that the function $u(x,y) = 2\cos x \cosh y + \sin x \sinh y$ satisfies Laplace's equation and integrate the Cauchy-Riemann equations to find its harmonic conjugate $v(x,y)$. [5]

SOLUTION

Obtain the derivatives

$$u_x = -2\sin x \cosh y + \cos x \sinh y \Rightarrow u_{xx} = -2\cos x \cosh y - \sin x \sinh y$$

and

$$u_y = 2\cos x \sinh y + \sin x \cosh y \Rightarrow u_{yy} = 2\cos x \cosh y + \sin x \sinh y$$

so that

$$u_{xx} + u_{yy} = -2\cos x \cosh y - \sin x \sinh + 2\cos x \cosh y + \sin x \sinh y = 0$$

and so $u(x,y)$ is a solution of Laplace's equation and a complex conjugate $v(x,y)$ exists.

To find v, we solve the Cauchy-Riemann equations:

$$u_x = v_y \Rightarrow v = \int u_x\, dy = \int -2\sin x \cosh y + \cos x \sinh y\, dy = -2\sin x \sinh y + \cos x \cosh y + A(x)$$

and

$$u_y = -v_x \Rightarrow v = -\int u_y\, dx = -\int 2\cos x \sinh y + \sin x \cosh y\, dx = -2\sin x \sinh y + \cos x \cosh y + B(y)$$

so the arbitrary functions $A(x)$ and $B(y)$ are constants and

$$v(x,y) = -2\sin x \sinh y + \cos x \cosh y + C\,.$$

b) Hence obtain $f(z) = u(x,y) + iv(x,y)$ where f is an analytic function of $z = x + iy$, simplifying as much as possible. [4]

SOLUTION

Write down $f = u + iv$:

$$f(z) = 2\cos x \cosh y + \sin x \sinh y - 2i \sin x \sinh y + i \cos x \cosh y + iC$$

and noting the symmetry, we rearrange:

$$f(z) = (2+i)\cos x \cosh y + (1-2i)\sin x \sinh y = (2+i)\cos x \cosh y - (2+i)i \sin x \sinh y + iC$$

and the next steps are clear:

$$f(z) = (2+i)[\cos x \cos(iy) - \sin x \sin(iy)] + iC = (2+i)\cos(x+iy) + iC\,,$$

so $f(z) = (2+i)\cos z + iC\,.$

A few people tried to avoid converting $i\sinh y$ *to* $\sin(iy)$*, and similarly for* $\cosh y$*, leading to problems. Use of exponential function not helpful here!*

2. a) The complex function

$$F(z) = \frac{1}{z(z^2+1)}$$

has three simple poles. Find the residues at the poles lying in the upper half of the complex plane and at the origin. [4]

SOLUTION Poles are at $z = 0, \pm i$.

Residue at $z = 0$:

$$\lim_{z \to 0} z \frac{1}{z(z^2+1)} = \lim_{z \to 0} \frac{1}{(z^2+1)} = 1$$

Residue at $z = i$:

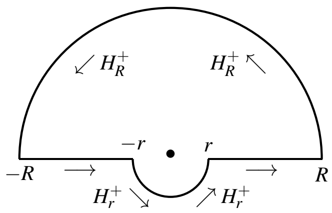
$$\lim_{z \to i} (z-i) \frac{1}{z(z^2+i)} = \lim_{z \to i} \frac{1}{z(z+i)} = \frac{1}{i(2i)} = -\frac{1}{2}$$

b) Consider the contour integral $I = \oint_C \frac{1}{z(z^2+1)}\, dz\,,$

where the closed contour C is taken to be the union of a semi-circle of radius R, lying in the upper half-plane, with a small semi-circle of radius r indented into the lower half-plane, both centred at $z = 0$ and the real intervals $[-R, -r]$ and $[r, R]$.

SOLUTION

Always useful to draw C:



i) Show that the contribution to I from the indented semi-circle of radius r, in the limit $r \to 0$, is $i\pi$.

SOLUTION

For the small semicircle we have

$$I_r = \int_{H_r} f(z)\, dz$$

Let $z = re^{i\theta}$, where $\pi \le \theta \le 2\pi$ - we are moving anticlockwise from $(-r,0)$ to $(r,0)$. Then $dz = ire^{i\theta}$. Substituting:

$$I_r = \int_{\pi}^{2\pi} \frac{i\, re^{i\theta}\, d\theta}{re^{i\theta}(1 + r^2 e^{i2\theta})}$$

$$\lim_{r \to 0} I_r = i \int_{\pi}^{2\pi} \frac{1}{1+0}\, d\theta = i\pi$$

The denominator simplifies given that $r^2 e^{i2\theta} \to 0$ as $r \to 0$. [3]

Argument sometimes sloppy, not always clear that the order of integration and limit can be varied. Similarly with direction of contour being anti-clockwise. Quite a few people tried to fit an incorrect calculation to the desired answer, resulting in strange logical steps.

ii) Use Jordan's lemma to show that the contribution to I from the arc of the larger semi-circle, in the limit $R \to \infty$, is zero.

SOLUTION Since:

- the only singularities of the function are poles
- $m = 0$ but
- $\lim_{R\to\infty} |F(z)| = \left|\frac{1}{z(z^2+1)}\right| \to 0$ faster than $|1/z|$;

$\lim_{R\to\infty} \int_{H_R} f(z)\, dz = 0$ as conditions hold for Jordan's lemma. [3]

Marked tendency to skip some of the detail here.

iii) Hence use your results from (a) and the Residue Theorem to obtain

$$\int_{-\infty}^{\infty} \frac{1}{x(x^2+1)}\, dx,$$

SOLUTION

Using the Residue Theorem, $I = 2\pi i$(sum of residues inside C),

$$2\pi i\left(1-\frac{1}{2}\right) = \oint_C f(z)\, dz = \int_{H_R} f(z)\, dz + \int_{-R}^{-r} f(z)\, dz + \int_{H_r} f(z)\, dz + \int_{r}^{R} f(z)\, dz$$

Taking the limits as $r \to 0$ and $R \to \infty$ does not affect the value of I obtained using the residue theorem, and with (i) and (ii), the RHS simplifies to

$$\pi i = \pi i + 0 + \int_{-\infty}^{\infty} \frac{1}{x(x^2+1)} dx$$

and so the answer is zero. This can be confirmed by observing that we are taking a symmetric integral of an odd function. [4]

Careful: limits must be taken at same time, can't leave one item with r, R while already presuming limits $0, \infty$ in another.

3. a) Given the integral of the real variable θ,

$$I = -\int_0^{2\pi} \sin[\cos(\theta) - \theta] e^{-\sin(\theta)}\, d\theta\,.$$

use the substitution $z = e^{i\theta}$ to show that I is equal to the real part of the complex contour integral

$$\oint_C \frac{e^{iz}}{z^2}\, dz,$$

where the contour C is the unit circle in the complex plane. [5]

SOLUTION

Using the substitution $z = e^{i\theta} \Rightarrow dz = ie^{i\theta}\, d\theta$ where $\theta = 0 \ldots 2\pi$ traces the unit circle, the contour C, so we can substitute as follows:

$$\oint_C \frac{e^{iz}}{z^2}\, dz = \int_0^{2\pi} \frac{e^{i(e^{i\theta})} ie^{i\theta}\, d\theta}{\left(e^{i\theta}\right)^2} = i\int_0^{2\pi} e^{i\left(e^{i\theta}-\theta\right)}\, d\theta = i\int_0^{2\pi} e^{i[\cos(\theta)+i\sin(\theta)-\theta]}\, d\theta \qquad (*)$$

$$= i\int_0^{2\pi} e^{i[\cos(\theta)-\theta]} e^{-\sin\theta}\, d\theta = i\int_0^{2\pi} e^{-\sin\theta} \left\{\cos[\cos(\theta)-\theta] + i\sin[\cos(\theta)-\theta]\right\}\, d\theta$$

and multiplying by i we take the real part to obtain the required result.

Quite a few sloppy answers here, especially in the first and third steps of the substitution (*) Some did not recover from this. Many tried substituting $\cos\theta = \frac{1}{2}(z+1/z)$ *and similarly for* $\sin\theta$ *into the real integral, which led nowhere due to (1) extreme complication, (2) the missing imaginary part.*

b) Using Cauchy's residue theorem, or otherwise, calculate I. [4]

SOLUTION The function $\dfrac{e^{iz}}{z^2}$ has a double pole at $z = 0$, so the residue is

$$\lim_{z\to 0} \frac{d}{dz}\left[(z-0)^2 f(z)\right] = \lim_{z\to 0} \frac{d}{dz}\left(e^{iz}\right) = \lim_{z\to 0} ie^{iz} = i\,.$$

So by Cauchy's Residue theorem,

$$\oint_C \frac{e^{iz}}{z^2}\, dz \; = 2\pi i(\text{Sum of Residues inside } C) = 2\pi i(i) = -2\pi\,.$$

Hence I is the real part of the above, which is real anyway, so $I = -2\pi$.

4. Consider the following second-order ODE

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 13y = f(x)$$

for some input function $f(x)$ and intial conditions $y(0) = y'(0) = 0$.

a) Take Laplace transforms to write the ODE in the form [3]

$$\bar{y}(s) = \bar{h}(s)\bar{f}(s)$$

SOLUTION Taking Laplace transforms of both sides, including the zero initial conditions we have

$$s^2\bar{y} + 6s\bar{y} + 13\bar{y} = \bar{f}(s) \Rightarrow \bar{y} = \frac{1}{s^2+6s+13}\bar{f}(s)$$

so that $\bar{h}(s) = \dfrac{1}{s^2+6s+13}$.

b) Hence use the Laplace convolution and shift theorems to write the solution in the form

$$y(x) = h(x) \star f(x)$$

,

where $h \star f$ is the convolution of $f(x)$ and $h(x)$, and $\mathscr{L}[h(x)] = \bar{h}(s)$. [5]

SOLUTION The convolution theorem states that $\mathscr{L}[h(x) * f(x)] = \bar{h}(s)\bar{f}(s)$, so we first find

$$h(x) = \mathscr{L}^{-1}[\bar{h}(s)] = \mathscr{L}^{-1}\left[\frac{1}{(s+3)^2+4}\right] = e^{-3x}\frac{1}{2}\mathscr{L}^{-1}\left[\frac{2}{s^2+2^2}\right] = \frac{1}{2}e^{-3x}\sin(2x)$$

using the table of transforms and the first shift-theorem.

Hence the solution $y(x)$ is the convolution of $h(x)$ and $f(x)$:

$$y(x) = \frac{1}{2}\int_0^x e^{-3u}\sin(2u)f(x-u)\,du.$$

[The other choice of convolution integral works equally well in (c).]

Many did not recall how to formulate the convolution integral, or left the answer in terms of $h \star f$.

c) If $f(x) = e^{-3x}$, obtain the solution $y(x)$ by solving the integral found in part (b). [4]

SOLUTION Taking f as given, the integral simplifies:

$$y(x) = \frac{1}{2}\int_0^x e^{-3u}\sin(2u)e^{-3(x-u)}\,du = \frac{1}{2}e^{-3x}\int_0^x \sin(2u)\,du = \frac{1}{2}e^{-3x}\left[-\frac{\cos 2u}{2}\right]_0^x = \frac{1}{4}e^{-3x}(1-\cos 2x).$$

Many of those that didn't have the correct convolution integral nevertheless used the result from (d), often correct, to somehow arrive at the same thing here.

d) With $f(x) = e^{-3x}$, take Laplace transforms of the ODE and use partial fractions and the shift theorem to take the inverse Laplace transform and find $y(x)$, and thus confirm the result obtained in (c). [6]

SOLUTION

Taking Laplace transforms:

$$s^2\bar{y} + 6s\bar{y} + 13\bar{y} = \frac{1}{s+3} \Rightarrow \bar{y} = \frac{1}{(s+3)(s^2+6s+13)} = \frac{As+B}{s^2+6s+13} + \frac{C}{s+3}$$

Partical fractions gives $A = -1/4\,, B = -3/4, C = 1/4$, so

$$4\bar{y} = \frac{-s-3}{s^2+6s+13} + \frac{1}{s+3} = -\frac{s+3}{(s+3)^2+4} + \frac{1}{s+3}$$

and taking inverse Laplace transforms we get

$$4y(x) = -\mathscr{L}^{-1}\left[\frac{s+3}{(s+3)^2+4}\right] + \mathscr{L}^{-1}\left[\frac{1}{s+3}\right] = -e^{-3x}\mathscr{L}^{-1}\left[\frac{s}{s^2+2^2}\right] + e^{-3x}$$

where we have used the first shift-theorem. Finally, the last transform is a cosine and

$$4y(x) = e^{-3x} - e^{-3x}\cos(2x)\,,$$

so $y = \dfrac{1}{4}e^{-3x}(1-\cos 2x)$, as before.

Arithmetic errors in the partial fractions, mostly, and incorrectly applied shift theorem. This then led to some students making more errors, trying to match up the solution to part (c). Or a correct procedure here, but trying to force it into an incorrect form obtained in (c).