EE4-27
EE9-CS5-2
EE9-FPN2-08

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2018

MSc and EEE/EIE PART IV: MEng and ACGI

SYSTEMS IDENTIFICATION

Corrected copy

Thursday, 17 May 10:00 am

Time allowed: 3:00 hours

There are FOUR questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : T. Parisini
Second Marker(s) : S.A. Evangelou

1. Given a stationary stochastic process $v(\cdot)$ with $\mathbb{E}(v) = 0$, consider its spectrum $\Gamma_v(\omega)$ which takes on the following values:

$$\Gamma_v(0) = \frac{9}{4}, \quad \Gamma_v(\pi/2) = \frac{5}{4}, \quad \Gamma_v(\pi) = \frac{1}{4}, \tag{1.1}$$

a) Consider the moving-average stationary stochastic process $w(\cdot)$ given by

$$MA(1): \; w(t) = e(t) + ce(t-1)$$

where $c \in \Re$, $c \neq 0$, $|c| < 1$, and $e(\cdot) \sim WN(0, \sigma_e^2)$.

Compute all the pairs of parameters (c, σ_e^2) such that

$$\Gamma_w(0) = \Gamma_v(0), \quad \Gamma_w(\pi/2) = \Gamma_v(\pi/2), \quad \Gamma_w(\pi) = \Gamma_v(\pi).$$

where $\Gamma_w(\omega)$ denotes the spectrum of the stochastic process $w(\cdot)$ and where $\Gamma_v(0), \Gamma_v(\pi/2), \Gamma_v(\pi)$ take on the values given in (1.1).

[7 Marks]

b) Determine the analytical expression of the correlation function $\gamma_w(\tau), \forall \tau$ of the stationary stochastic process $w(\cdot)$ defined in Question 1-a) using the pairs of parameters (c, σ_e^2) determined in your answer to Question 1-a).

[5 Marks]

c) Consider the auto-regressive stationary stochastic process $y(\cdot)$ given by

$$AR(1): \; y(t) = ay(t-1) + \eta(t)$$

where $a \in \Re$, $a \neq 0$, $|a| < 1$, and $\eta(\cdot) \sim WN(0, \sigma_\eta^2)$.

Compute, if possible, values of the parameters a and σ_η^2 such that

$$\Gamma_y(0) = \Gamma_v(0), \quad \Gamma_y(\pi/2) = \Gamma_v(\pi/2), \quad \Gamma_y(\pi) = \Gamma_v(\pi).$$

where $\Gamma_y(\omega)$ denotes the spectrum of the stochastic process $y(\cdot)$ and where $\Gamma_v(0), \Gamma_v(\pi/2), \Gamma_v(\pi)$ take on the values given in (1.1). Discuss your findings.

[8 Marks]

2. Consider a stationary stochastic process $v(\cdot)$ with $\mathbb{E}(v) = 0$ and characterized by the complex spectrum

$$\Phi_v(z) = \frac{(z+1/4)(z+4)}{(z-2)(1/2-z)}$$

a) Determine a transfer function $H(z)$ such that the *stationary* stochastic process $v(\cdot)$ can be modelled as

$$v(t) = H(z)\xi(t)$$

where $\xi(\cdot)$ is a white stochastic process $\xi(\cdot) \sim WN(0,1)$.

[5 Marks]

b) Determine the difference equation yielding the optimal one-step ahead prediction $\hat{v}(t+1|t)$ of $v(t+1)$ on the basis of past values $v(t), v(t-1), v(t-2), \ldots$ of the process $v(\cdot)$.

[3 Marks]

c) Denoting by $\varepsilon_1(t) = v(t+1) - \hat{v}(t+1|t)$ the one-step ahead prediction error when using the predictor determined in your answer to Question 2-b), determine its model as a stochastic process; moreover compute $\mathbb{E}(\varepsilon_1)$ and $\text{var}[\varepsilon_1]$.

[3 Marks]

d) Determine the difference equation yielding the optimal three-step ahead prediction $\hat{v}(t+3|t)$ of $v(t+3)$ on the basis of past values $v(t), v(t-1), v(t-2), \ldots$ of the process $v(\cdot)$.

[3 Marks]

e) Denoting by $\varepsilon_3(t) = v(t+3) - \hat{v}(t+3|t)$ the three-step ahead prediction error when using the predictor determined in your answer to Question 2-d), determine its model as a stochastic process; moreover compute $\mathbb{E}(\varepsilon_3)$ and $\text{var}[\varepsilon_3]$.

[3 Marks]

f) Compare $\text{var}[\varepsilon_1(t)]$ computed in your answer to Question 2-c) with $\text{var}[\varepsilon_3(t)]$ computed in your answer to Question 2-e). Comment on your findings.

[3 Marks]

3. Consider the electrical circuit depicted in Fig. 3.1.

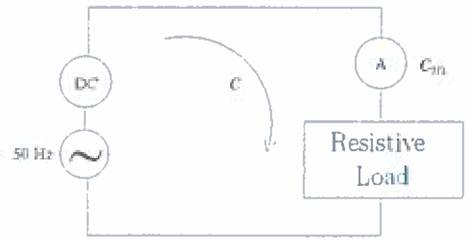

Figure 3.1 Electrical circuit with a resistive load and a DC combined with 50Hz AC power supply.

A current meter is available providing a measurement c_m of the current c flowing in the circuit.

An experiment to identify the DC bias and the phase of the current c is set up. A set of M given values of time-samples $t(j)$, $j = 1, \ldots, M$ are selected. Correspondingly, a set of M measurements of the current are recorded by the current meter:

$$c_m(j) = c(j) + \xi(j),\ j = 1, \ldots, M$$

where $c(j) = \gamma + K\sin[100\pi t(j) + \varphi]$ and ξ is a zero-mean white process $\xi(\cdot) \sim WN(0, \lambda^2)$.

a) Assuming the amplitude K to be known, devise a *parametric* model $\mathcal{M}$ of the unknown current $c(t)$ of the form

$$\mathcal{M}:\ \hat{c}(t, \alpha) = \sum_{i=1}^{N} \alpha_i \rho_i(t)$$

for a suitable choice of N and of the functions $\rho_i(t), i = 1, \ldots, N$, where $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_N]^T$ is a vector of parameters to be determined using the available measurements. [*Hint: exploit the known form of the current $c(t)$ and the trigonometric properties on the* $\sin(\cdot)$ *function*].

[7 Marks]

b) Consider the parametric model $\mathcal{M}$ designed in your answer to Question 3-a). The parameters $\alpha_i, i = 1, \ldots, N$ have to be determined using a set of time-instants and current measurements

$$\Theta = \{(t(j), c_m(j)),\ j = 1, \ldots, M\}.$$

Devise a least-squares method the solution of which provides the optimal (in the least-squares sense) sets of parameters α_i^o, $i = 1, \ldots, N$ and determine its general solution in analytical form.

[8 Marks]

c) Consider the following set Θ of 5 time-instants and corresponding measurements of current, where $K = 1$:

$$\Theta = \{(0, 0.65), (0.005, 1.16), (0.01, -0.08), (0.015, -0.33), (0.02, 1.07)\}$$

Using the general solution obtained in your answer to Question 3-b), compute the optimal parameters α_i^o, $i = 1, \ldots, N$. Sketch the plot of the time-behaviour of the obtained approximate model and also plot on the same diagram the 5 points of the set Θ. Discuss your findings.

[5 Marks]

4. Consider two different stochastic processes $v_1(\cdot)$ and $v_2(\cdot)$ generated as illustrated in Fig. 4.1.

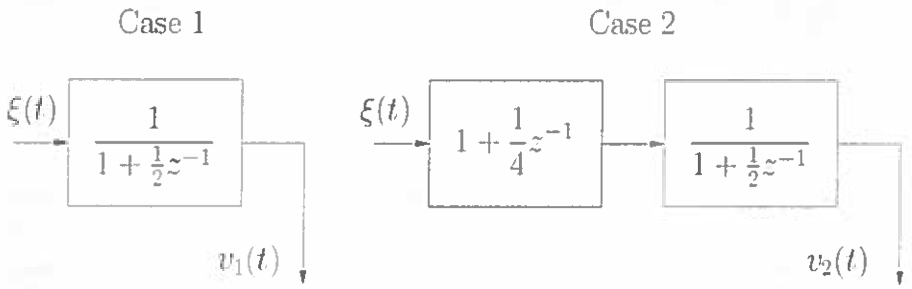

Figure 4.1 Two stochastic data-generation mechanisms.

Moreover, consider the family of AR(1) stochastic models

$$\mathcal{M}(\theta): \quad y(t) = ay(t-1) + \eta(t), \quad \theta := a$$

where, with reference to Case 1 in Fig. 4.1, $y(t) = v_1(t)$, whereas with reference to Case 2 in Fig. 4.1, $y(t) = v_2(t)$.

In both Case 1 and Case 2, consider an identification procedure in which $\widehat{a}(N)$ is the least squares estimate of a based on N measurements $\{y(1), y(2), \dots, y(N)\}$.

a) **Case 1.** The stochastic process is $y(\cdot) = v_1(\cdot)$ with $\xi(\cdot) \sim WN(0,1)$. Determine the value $\bar{a}_1$ the estimate $\widehat{a}(N)$ approaches for large values of N (that is, $\bar{a}_1 = \lim_{N\to\infty} \widehat{a}(N)$, a.s.).

[6 Marks]

b) **Case 2.** The stochastic process is $y(\cdot) = v_2(\cdot)$ with $\xi(\cdot) \sim WN(0,1)$. Determine the value $\bar{a}_2$ the estimate $\widehat{a}(N)$ approaches for large values of N (that is, $\bar{a}_2 = \lim_{N\to\infty} \widehat{a}(N)$, a.s.).

[7 Marks]

c) With reference to Case 1 of Question 4-a) and Case 2 of Question 4-b), compute and compare the variances of the prediction errors

$$\text{var}[v_1(t) - \bar{a}_1 v_1(t-1)] \quad \text{and} \quad \text{var}[v_2(t) - \bar{a}_2 v_2(t-1)]$$

Comment on your findings.

Moreover, consider a different variance for the noise ξ, namely $\xi(\cdot) \sim WN(0,3)$. State whether or not the values of $\bar{a}_1$ and $\bar{a}_2$ computed in the answers to Questions 4-a) and 4-b) are different because of the different value taken on by the variance of ξ. Justify your answer.

[7 Marks]