EE3-09

CONTROL ENGINEERING

1. a) The rechability matrix is

$$\mathscr{R} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

The rank of $\mathscr{R}$ is one, hence the system is not reachable. Note now that

$$A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

hence $\text{Im}A^3 \subset \text{Im}\mathscr{R}$, which implies that the system is controllable.

The observability matrix

$$\mathscr{O} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

is full rank: the system is observable. [8 marks]

b) Note that, since $u(0) = 0$,

$$x(1) = Ax(0) = \begin{bmatrix} x_2(0) \\ 0 \\ x_3(0) \end{bmatrix},$$

hence all initial conditions described by $x(0) = [\ \star \quad 0 \quad 0\ ]'$ with $\star$ any number, are such that $x(1) = 0$. [2 marks]

c) Note that, since $u(0) = u(1) = 0$,

$$x(2) = A^2x(0) = \begin{bmatrix} 0 \\ 0 \\ x_3(0) \end{bmatrix},$$

hence all initial conditions described by $x(0) = [\ \star \quad \star \quad 0\ ]'$ with $\star$ any number, are such that $x(2) = 0$. [2 marks]

d) Note that

$$y(0) = u(0) \qquad y(1) = u(0) + u(1) \qquad y(2) = u(0) + u(1) + u(2).$$

Hence, the conditions $y(0) = y(1) = 0$ and $y(2) = 1$ yield $u(0) = 0$, $u(1) = 0$ and $u(2) = 1$. The output sequence resulting from the above input sequence, extended with $u(t) = 0$, for all $t \geq 3$, is

$$y(3) = 1, \qquad y(4) = 1, \quad \cdots.$$

The reason why the output sequence is constant is that the state $x(3) = [0\ 0\ 1]'$ is an equilibrium of the system for $u = 0$. Hence, the state of the system is driven from $x(0) = 0$ to $x(3) = [0\ 0\ 1]'$ by the input $u(0) = 0$, $u(1) = 0$ and $u(2) = 1$ and remains therein for all $t \geq 3$. [8 marks]

2. a) To begin with note that

$$A = \begin{bmatrix} 1 & \alpha \\ 1 & 1-2\alpha \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

and that $\det A = 1 - 3\alpha$. If $1 - 3\alpha \neq 0$, the matrix A is invertible, hence the only equilibrium, for $u = 0$, is $x = 0$. If $\alpha = 1/3$, then to find the equilibrium points we need to solve the equations $\dot{x}_1 = \dot{x}_2 = 0$, that is

$$0 = x_1 + \frac{1}{3}x_2 \qquad 0 = x_1 + \frac{1}{3}x_2.$$

This means that all points described by

$$x = \delta \begin{bmatrix} 1 \\ -3 \end{bmatrix},$$

with δ any number, are equilibrium points, that is for $\alpha = 1/3$ and $u = 0$ the system has infinitely many equilibrium points on a straigh line. [4 marks]

b) As for part a), if $1 - 3\alpha \neq 0$, the matrix A is invertible, hence the only equilibrium is

$$x = \frac{\bar{u}}{3\alpha - 1} \begin{bmatrix} 2\alpha - 1 \\ 1 \end{bmatrix}.$$

If $\alpha = 1/3$ one has to solve the equations $\dot{x}_1 = \dot{x}_2 = 0$, that is

$$0 = x_1 + \frac{1}{3}x_2 + \bar{u} \qquad 0 = x_1 + \frac{1}{3}x_2.$$

These equations do not have any solution for $\bar{u} \neq 0$, that is the system does not have any equilibrium point. [4 marks]

c) If the matrix A is invertible, regardless of the value of the input signal, the system has one equilibrium point. If A is not invertible, the existence of equilibrium points depends upon the value of the input signal. If $\bar{u} = 0$ there are infinitely many equilibria, whereas if $\bar{u} \neq 0$ there are no equilibria. [2 marks]

d) The characteristic polynomial of the matrix A is

$$p(\lambda) = \lambda^2 + (2\alpha - 2)\lambda + (1 - 3\alpha).$$

For all values of α the coefficient of $p(\lambda)$ have different sign, hence the system is unstable. [6 marks]

e) The equations of the closed-loop system are

$$\dot{x} = (A - kBC)x, \qquad y = Cx,$$

with

$$A - kBC = \begin{bmatrix} 1-k & \alpha \\ 1 & 1-2\alpha \end{bmatrix}.$$

The characteristic polynomial of the matrix $A - kBC$ is

$$p_k(\lambda) = \lambda^2 + (2\alpha - 2 + k)\lambda + (1 - 3\alpha + 2k\alpha - k).$$

The closed-loop system is asymptotically stable for all k and α such that

$$2\alpha - 2 + k > 0 \qquad 1 - 3\alpha + 2k\alpha - k > 0.$$

Note that if $\alpha > 1/2$ there is always a sufficiently large positive k such that the above conditions are satisfied. [4 marks]

3. a) The reachability and observability matrices of the continuous-time system are

$$\mathscr{R} = \begin{bmatrix} 0 & 1 \\ 1 & \lambda \end{bmatrix} \qquad \mathscr{O} = \begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix},$$

hence the system is controllable (and reachable) and observable for all λ.

[4 marks]

b) A direct computation yields

$$A_d = \begin{bmatrix} 1+T\lambda+\frac{T^2\lambda^2}{2} & T+T^2\lambda \\ 0 & 1+T\lambda+\frac{T^2\lambda^2}{2} \end{bmatrix}, \quad B_d = \begin{bmatrix} 0 \\ T \end{bmatrix}, \quad C_d = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

[4 marks]

c) The reachability and observability matrices of the discrete-time system are

$$\mathscr{R}_d = \begin{bmatrix} 0 & T^2(1+T\lambda) \\ T & T\left(1+T\lambda+\frac{T^2\lambda^2}{2}\right) \end{bmatrix}, \quad \mathscr{O}_d = \begin{bmatrix} 1 & 0 \\ 1+T\lambda+\frac{T^2\lambda^2}{2} & T(1+T\lambda). \end{bmatrix}.$$

Both matrices lose rank for $T = -1/\lambda$. For this value of the sampling time (assuming $\lambda < 0$) the system is not reachable nor observable. [6 marks]

d) For $T = -1/\lambda$ one has

$$A_d = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \qquad B_d = \begin{bmatrix} 0 \\ -\frac{1}{\lambda} \end{bmatrix}, \qquad C_d = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Note that $C_d A_d^k B_d = 0$ for all $k \geq 0$. Hence, for $x(0) = 0$ one has

$$y(k) = 0,$$

for all $k \geq 0$, regardless of the input sequence. [6 marks]

4. a) The matrices A and B are given by

$$A = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix} \qquad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}.$$

[2 marks]

b) The reachability pencil is

$$[\lambda I - A \;\; B] = \left[\begin{array}{ccccc|c} \lambda - \lambda_1 & 0 & 0 & \cdots & 0 & b_1 \\ 0 & \lambda - \lambda_2 & 0 & \cdots & 0 & b_2 \\ 0 & 0 & \lambda - \lambda_3 & \cdots & 0 & b_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda - \lambda_n & b_n \end{array}\right].$$

The rank of the pencil has to be checked for each eigenvalue of A, that is for $\lambda = \lambda_i$, with $i = 1, \cdots, n$. Setting $\lambda = \lambda_1$ yields

$$[\lambda_1 I - A \;\; B] = \left[\begin{array}{ccccc|c} 0 & 0 & 0 & \cdots & 0 & b_1 \\ 0 & \lambda_1 - \lambda_2 & 0 & \cdots & 0 & b_2 \\ 0 & 0 & \lambda_1 - \lambda_3 & \cdots & 0 & b_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_1 - \lambda_n & b_n \end{array}\right].$$

The matrix $[\lambda_1 I - A \;\; B]$ has full rank if and only if $\lambda_i \neq \lambda_1$, for all $i = 2, \cdots, n$ and $b_1 \neq 0$. The same argument can be repeated for all λ_i. Hence the system is controllable if and only if $\lambda_i \neq \lambda_j$, for $i \neq j$, and $b_i \neq 0$, for all $i = 1, \cdots, n$.

[10 marks]

c) The system is now described by the matrices

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

i) Note that

$$A + BK = \begin{bmatrix} -1 + 2k_1 & 2k_2 \\ k_1 & 1 + k_2 \end{bmatrix}.$$

The characteristic polynomial of this matrix is

$$p(\lambda) = \lambda^2 + (-2k_1 - k_2)\lambda + (2k_1 - k_2 - 1).$$

Setting $k_1 = 0$ and $k_2 = -2$ yields the desired closed-loop eigenvalues.

[6 marks]

ii) The perturbed closed-loop matrix is

$$A_\delta = \begin{bmatrix} -1 & -4 + \delta \\ 0 & -1 \end{bmatrix},$$

and this is asymptotically stable for all δ. [2 marks]