EE4-27

SOLUTIONS: SYSTEMS IDENTIFICATION

1. Solution

a) Moving-average models of order n with $n \geq 1$ have a correlation function γ such that $\gamma(\tau) = 0, \forall\tau\colon |\tau| > n$. By inspecting Fig. 1.1 in the text of the exam paper, we notice that $\gamma(\tau) \neq 0$, $\tau \in [0,5]$. Therefore, it is immediate to conclude that moving-average stochastic models $MA(n)$, $n = 1,2$ have not a correlation function consistent with $\gamma(\tau)$ plotted in Fig. 1.1 in the text of the exam paper.

[4 Marks]

b) First, consider the auto-regressive stochastic model $AR(1)$:

$$AR(1): \quad v(t) = av(t-1) + \eta(t), \quad \eta(\cdot) \sim WN(0, \lambda_1^2)$$

and calculate the correlation function as follows (the process $v(\cdot)$ is zero-mean):

$$v(t) = av(t-1) + \eta(t) \Longrightarrow v(t)v(t-\tau) = av(t-1)v(t-\tau) + \eta(t)v(t-\tau).$$

Hence:

$$\mathbb{E}[v(t)v(t-\tau)] = a\mathbb{E}[v(t-1)v(t-\tau)] + \mathbb{E}[\eta(t)v(t-\tau)].$$

For $\tau = 0$, one has:

$$\gamma(0) = a\gamma(-1) + \lambda_1^2 \Longrightarrow \gamma(0) = a\gamma(1) + \lambda_1^2,$$

whereas, for $\tau \geq 1$, one has:

$$\gamma(\tau) = a\gamma(\tau - 1).$$

Referring to Fig. 1.1 in the text of the exam paper:

$$\gamma(0) = \frac{42}{5}, \; \gamma(1) = 6, \; \gamma(2) = \frac{18}{5}.$$

Then:

$$a = \frac{\gamma(1)}{\gamma(0)} = \frac{5}{7} \Longrightarrow \gamma(2) = a\gamma(1) = \frac{5}{7}6 = \frac{30}{7} \neq \frac{18}{5}$$

and thus the values of the correlation function of the model $AR(1)$ are not consistent with the values plotted in Fig. 1.1 in the text of the exam paper.

Now, consider the auto-regressive stochastic model $AR(2)$:

$$AR(2): \quad v(t) = a_1 v(t-1) + a_2 v(t-2) + \eta(t), \quad \eta(\cdot) \sim WN(0, \lambda_2^2)$$

and calculate the correlation function as follows (the process $v(\cdot)$ is zero-mean):

$$\begin{aligned} &v(t) = a_1 v(t-1) + a_2 v(t-2) + \eta(t) \\ &\Longrightarrow v(t)v(t-\tau) = a_1 v(t-1)v(t-\tau) + a_2 v(t-2)v(t-\tau) + \eta(t)v(t-\tau). \end{aligned}$$

Applying the expected value operator, one gets:

$$\mathbb{E}[v(t)v(t-\tau)] = a_1\mathbb{E}[v(t-1)v(t-\tau)] + a_2\mathbb{E}[v(t-2)v(t-\tau)] + \mathbb{E}[\eta(t)v(t-\tau)],$$

which gives:

$$\gamma(0) = a_1\gamma(1) + a_2\gamma(2) + \lambda_2^2,$$
$$\gamma(1) = a_1\gamma(0) + a_2\gamma(1)$$

and

$$\gamma(\tau) = a_1\gamma(\tau-1) + a_2\gamma(\tau-2),\ \forall\tau \geq 2.$$

Now, check that an auto-regressive stochastic model $AR(2)$ is consistent with the values of $\gamma(\tau)$ plotted in Fig. 1.1 in the text of the exam paper.

Clearly, one can write:

$$\begin{bmatrix} \gamma(1) & \gamma(2) & 1 \\ \gamma(0) & \gamma(1) & 0 \\ \gamma(1) & \gamma(0) & 0 \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \\ \lambda_2^2 \end{bmatrix} = \begin{bmatrix} \gamma(0) \\ \gamma(1) \\ \gamma(2) \end{bmatrix}$$

which gives

$$\begin{bmatrix} a_1 \\ a_2 \\ \lambda_2^2 \end{bmatrix} = \begin{bmatrix} \frac{5}{6} \\ -\frac{1}{6} \\ 4 \end{bmatrix}$$

Using the above relationship

$$\gamma(\tau) = a_1\gamma(\tau-1) + a_2\gamma(\tau-2)$$

for $\tau = 3,4,5$, it follows that

$$\gamma(3) = 2,\ \gamma(4) = \frac{16}{15},\ \gamma(5) = \frac{5}{9}.$$

These values of the correlation function $\gamma(\tau)$ are consistent with the respective ones reported in Fig. 1.1 in the text of the exam paper.

Moreover:

$$\left(1 - \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}\right)\mathscr{Z}[v(t)] = \mathscr{Z}[\eta(t)] \Longrightarrow \mathscr{Z}[v(t)] = \frac{z^2}{z^2 - \frac{5}{6}z + \frac{1}{6}}\mathscr{Z}[\eta(t)]$$

The roots of the polynomial $z^2 - \frac{5}{6}z + \frac{1}{6}$ are $z_1 = \frac{1}{3}$ and $z_2 = \frac{1}{2}$, both lying strictly inside the unit circle. Hence the process associated to the just-obtained $AR(2)$ model is stationary thus concluding that the correlation function of this process is consistent with the correlation function reported in Fig. 1.1 in the text of the exam paper.

[8 Marks]

c) From the answer to Question 1b), one gets:

$$\mathscr{Z}[v(t)] = G(z)\mathscr{Z}[\eta(t)] \text{ with } G(z) = \frac{z^2}{(z-\frac{1}{3})(z-\frac{1}{2})}.$$

Then, as $\lambda_2^2 = 4$ (see the answer to Question 1b)), the spectrum of the stationary stochastic process $v(\cdot)$ is given by:

$$\Gamma(\omega) = \left|G(e^{j\omega})\right|^2 \lambda_2^2 = \frac{4}{\left|e^{j\omega} - \frac{1}{3}\right|^2 \left|e^{j\omega} - \frac{1}{2}\right|^2}.$$

After a little algebra, we obtain

$$\Gamma(\omega) = \frac{4}{(\frac{5}{4} - \cos\omega)(\frac{10}{9} - \frac{2}{3}\cos\omega)}.$$

[4 Marks]

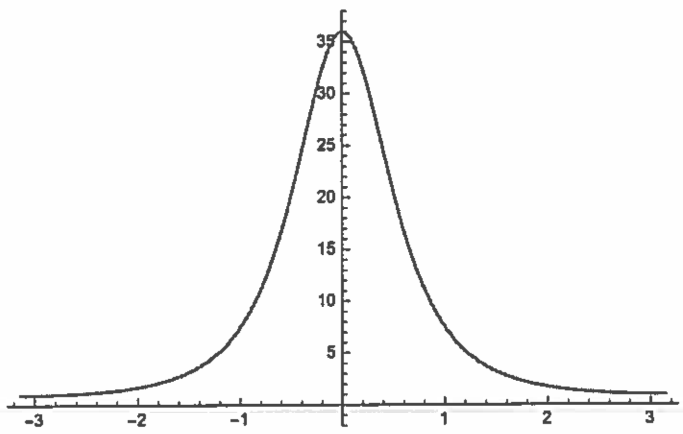

Figure 1.1 Plot of the spectrum $\Gamma(\omega) = \frac{4}{(\frac{5}{4} - \cos\omega)(\frac{10}{9} - \frac{2}{3}\cos\omega)}$.

d) The behaviour of $\Gamma(\omega)$ in the interval $\omega \in [-\pi, \pi]$ is shown in Figs. 1.1.

To sketch the plot in Fig. 1.1, a few values of the spectrum can be computed directly from the analytical expression of $\Gamma(\omega)$ obtained in the answer to Question 1c).

Alternatively, a few values of $\Gamma(\omega)$ can be directly computed by geometric considerations:

$$\Gamma(0) = \frac{4}{\left(1 - \frac{1}{3}\right)^2 \left(1 - \frac{1}{2}\right)^2} = 36$$

$$\Gamma(\pi/2) = \frac{4}{\left(1 + (\frac{1}{3})^2\right)\left(1 + (\frac{1}{2})^2\right)} = \frac{72}{25} = 2.88$$

$$\Gamma(\pi) = \frac{4}{\left(1 + \frac{1}{3}\right)^2 \left(1 + \frac{1}{2}\right)^2} = 1$$

| 4 Marks |

2. Solution

a) Consider the set

$$\mathcal{M} = \{(x(i), \hat{y}(i)),\, i = 0, 1, \ldots, N-1\}.$$

and the linear function

$$g(x,\theta) = ax + b, \quad \text{with } \theta = [a,b]^\top .$$

Introduce the error variable given by

$$e(i) = \hat{y}(i) - g(x(i),\theta) = \hat{y}(i) - ax(i) - b,\; i = 0, 1, \ldots, N-1 .$$

Moreover, consider the following cost function:

$$J(\theta) = \frac{1}{N}\sum_{i=0}^{N-1}[e(i)]^2 = \frac{1}{N}\sum_{i=0}^{N-1}[\hat{y}(i) - ax(i) - b]^2 .$$

The minimisation of $J(\theta)$ with respect to the unknown vector θ yields a linear approximation of the unknown function $f(x)$ in the least-squares sense. More precisely:

$$\theta^\circ = [a^\circ, b^\circ]^\top = \arg\min_{\theta} J(\theta)$$

provided that a unique minimum of $J(\theta)$ does exist.

[6 Marks]

b) To compute the optimal solution $\theta^\circ = [a^\circ, b^\circ]^\top$, we consider the gradient of $J(\theta)$ with respect to the vector θ:

$$\begin{aligned}\frac{\partial}{\partial\theta}J(\theta) &= \frac{1}{N}\sum_{i=0}^{N-1}\frac{\partial}{\partial\theta}\left\{[\hat{y}(i) - ax(i) - b]^2\right\} = \\ &= \frac{2}{N}\sum_{i=0}^{N-1}[\hat{y}(i) - ax(i) - b]\frac{\partial}{\partial\theta}[\hat{y}(i) - ax(i) - b] = \\ &= -\frac{2}{N}\sum_{i=0}^{N-1}[\hat{y}(i) - ax(i) - b]\,[x(i)\,,\,1]\end{aligned}$$

where $[x(i)\,,\,1]$ is a 2-dimensional row vector. Now impose

$$\frac{\partial}{\partial\theta}J(\theta) = [0,0],$$

that is,

$$\left\{\begin{aligned}&\sum_{i=0}^{N-1}[\hat{y}(i) - ax(i) - b]x(i) = 0\\ &\sum_{i=0}^{N-1}[\hat{y}(i) - ax(i) - b] = 0\end{aligned}\right. \implies \left\{\begin{aligned}&\sum_{i=0}^{N-1}\hat{y}(i)x(i) - a\sum_{i=0}^{N-1}x(i)^2 - b\sum_{i=0}^{N-1}x(i) = 0\\ &\sum_{i=0}^{N-1}\hat{y}(i) - a\sum_{i=0}^{N-1}x(i) - bN = 0\end{aligned}\right.$$

Therefore, in compact form, we state the following linear problem:

$$\begin{bmatrix}\sum_{i=0}^{N-1}x(i)^2 & \sum_{i=0}^{N-1}x(i)\\ \sum_{i=0}^{N-1}x(i) & N\end{bmatrix}\cdot\begin{bmatrix}a\\ b\end{bmatrix} = \begin{bmatrix}\sum_{i=0}^{N-1}\hat{y}(i)x(i)\\ \sum_{i=0}^{N-1}\hat{y}(i)\end{bmatrix}$$

thus obtaining (in case of existence of a single minimum solution)

$$\theta^\circ = \begin{bmatrix} a^\circ \\ b^\circ \end{bmatrix} = \begin{bmatrix} \sum_{i=0}^{N-1} x(i)^2 & \sum_{i=0}^{N-1} x(i) \\ \sum_{i=0}^{N-1} x(i) & N \end{bmatrix}^{-1} \begin{bmatrix} \sum_{i=0}^{N-1} \hat{y}(i) x(i) \\ \sum_{i=0}^{N-1} \hat{y}(i) \end{bmatrix}$$

[8 Marks]

c) Consider the set $\mathscr{M}$ of 5 measurements given in the text of Question 2c) of the exam paper:

$$\mathscr{M} = \{(0,-0.34), (0.5, 0.54), (1, 0.81), (1.5, 2.28), (2, 3.66)\}$$

and substitute the numerical values of $x(i)$ and $\hat{y}(i)$ into the general solution obtained in the answer to Question 2b) thus obtaining the following linear problem:

$$\begin{bmatrix} 15/2 & 5 \\ 5 & 5 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 11.82 \\ 6.95 \end{bmatrix} \Longrightarrow \theta^\circ = \begin{bmatrix} a^\circ \\ b^\circ \end{bmatrix} = \begin{bmatrix} 1.948 \\ -0.558 \end{bmatrix}$$

The plot of the linear approximation $g(x, \theta^\circ)$ of $f(x)$ it thus shown in Fig. 2.1 below (continuous line), in which the noisy samples are shown, too (circles):

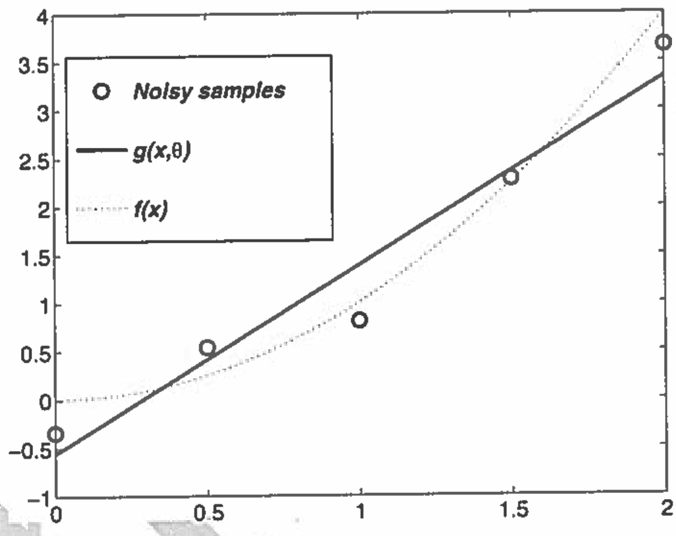

Figure 2.1 Plot of $g(x, \theta^\circ)$ (continuous line) and of the the noisy samples of $f(x)$ (circles). The dotted line depicts the behaviour of the the unknown function $f(x)$.

As can be noticed from Fig. 2.1, the straight line is such that the sum of the squares of the distances from the samples is minimised. Clearly, for a different realisation of the noise $\varepsilon(i)$, a different numerical solution of the least squares problem would have been obtained.

Remark for the students (not part of the solution of the exam paper). The plot of the unknown function $f(x)$ has been reported in Fig. 2.1 for the sake of completeness to compare the linear approximation $g(x, \theta^\circ)$ with the true function $f(x)$. However, the behaviour of $f(x)$ cannot be deduced from the information provided in the text of the exam paper. Hence plotting the behaviour of $f(x)$ is not part of the solution of this exam paper.

[6 Marks]

3. Solution

a) Consider the linear estimator

$$\widehat{T}(\alpha,\beta,\gamma) = \alpha T_1 + \beta T_2 + \gamma T_3$$

and apply the expected value operator:

$$\mathbb{E}[\widehat{T}(\alpha,\beta,\gamma)] = \alpha\mathbb{E}(T_1) + \beta\mathbb{E}(T_2) + \gamma\mathbb{E}(T_3) = (\alpha+\beta+\gamma)\overline{T}.$$

Then, the estimator $\widehat{T}(\alpha,\beta,\gamma)$ is unbiased if $\alpha+\beta+\gamma = 1$.

[4 Marks]

b) First compute $\mathrm{var}[\widehat{T}(\alpha,\beta,\gamma)]$. One gets:

$$\begin{aligned}\mathrm{var}[\widehat{T}(\alpha,\beta,\gamma)] &= \mathbb{E}\left\{\left[\alpha(T_1-\overline{T}) + \beta(T_2-\overline{T}) + \gamma(T_3-\overline{T})\right]^2\right\} \\ &= \alpha^2\mathrm{var}(T_1) + \beta^2\mathrm{var}(T_2) + \gamma^2\mathrm{var}(T_3) + 2\alpha\beta\mathbb{E}[(T_1-\overline{T})(T_2-\overline{T})] \\ &\quad + 2\alpha\gamma\mathbb{E}[(T_1-\overline{T})(T_3-\overline{T})] + 2\beta\gamma\mathbb{E}[(T_2-\overline{T})(T_3-\overline{T})]\end{aligned}$$

As data T_1, T_2, and T_3 are mutually independent by assumption, it follows that:

$$\mathbb{E}[(T_1-\overline{T})(T_2-\overline{T})] = 0;\ \mathbb{E}[(T_1-\overline{T})(T_3-\overline{T})] = 0;\ \mathbb{E}[(T_2-\overline{T})(T_3-\overline{T})] = 0.$$

Then:

$$\mathrm{var}[\widehat{T}(\alpha,\beta,\gamma)] = \alpha^2\mathrm{var}(T_1) + \beta^2\mathrm{var}(T_2) + \gamma^2\mathrm{var}(T_3)$$

Since

$$T_1 \sim \mathscr{G}(\overline{T},2), \quad T_2 \sim \mathscr{G}(\overline{T},4), \quad T_3 \sim \mathscr{G}(\overline{T},9/4),$$

one gets

$$\mathrm{var}[\widehat{T}(\alpha,\beta,\gamma)] = 2\alpha^2 + 4\beta^2 + \frac{9}{4}\gamma^2.$$

Moreover, since $\alpha+\beta+\gamma = 1$ in order to ensure that the estimator is unbiased, it follows that $\gamma = 1-\alpha-\beta$, thus obtaining

$$\mathrm{var}(\widehat{T}) = 2\alpha^2 + 4\beta^2 + \frac{9}{4}(1-\alpha-\beta)^2 = \frac{17}{4}\alpha^2 + \frac{25}{4}\beta^2 + \frac{9}{2}\alpha\beta - \frac{9}{2}\alpha - \frac{9}{2}\beta + \frac{9}{4}.$$

To minimise $\mathrm{var}(\widehat{T})$ with respect to α, β, we first compute:

$$\frac{\partial}{\partial\alpha}\mathrm{var}(\widehat{T}) = \frac{17}{2}\alpha + \frac{9}{2}\beta - \frac{9}{2}; \quad \frac{\partial}{\partial\beta}\mathrm{var}(\widehat{T}) = \frac{9}{2}\alpha + \frac{25}{2}\beta - \frac{9}{2}$$

Now, we impose

$$\frac{\partial}{\partial\alpha}\mathrm{var}(\widehat{T}) = 0; \quad \frac{\partial}{\partial\beta}\mathrm{var}(\widehat{T}) = 0$$

and we solve for α and β, that is:

$$\begin{cases} \frac{17}{2}\alpha + \frac{9}{2}\beta - \frac{9}{2} = 0 \\ \frac{9}{2}\alpha + \frac{25}{2}\beta - \frac{9}{2} = 0 \end{cases} \Longrightarrow \begin{cases} \alpha^\circ = \frac{18}{43} \simeq 0.42 \\ \beta^\circ = \frac{9}{43} \simeq 0.21 \end{cases}$$

Moreover $\gamma^\circ = 1 - \alpha^\circ - \beta^\circ = \frac{16}{43} \simeq 0.37$.

[7 Marks]

c) From the answer to Question 2b) it follows that

$$\mathrm{var}[\widehat{T}(\alpha,\beta,\gamma)] = 2\alpha^2 + 4\beta^2 + \frac{9}{4}\gamma^2$$

Replacing into this formula the values of $\alpha^\circ, \beta^\circ, \gamma^\circ$, one finally gets:

$$\mathrm{var}[\widehat{T}(\alpha^\circ,\beta^\circ,\gamma^\circ)] = \frac{36}{43} \simeq 0.84$$

[3 Marks]

d) Consider the empirical mean estimator defined in the text of the exam paper (Question 3d)):

$$\widetilde{T} = \frac{1}{3}(T_1 + T_2 + T_3)$$

and apply the expected value operator:

$$\mathbb{E}[\widetilde{T}] = \frac{1}{3}\mathbb{E}(T_1) + \frac{1}{3}\mathbb{E}(T_2) + \frac{1}{3}\mathbb{E}(T_3) = \overline{T}.$$

Then, the estimator $\widetilde{T}$ is unbiased.

Now, compute $\mathrm{var}(\widetilde{T})$:

$$\mathrm{var}(\widetilde{T}) = \mathbb{E}\left\{\left[\frac{1}{3}(T_1 - \overline{T}) + \frac{1}{3}(T_2 - \overline{T}) + \frac{1}{3}(T_3 - \overline{T})\right]^2\right\}$$

$$= \frac{1}{9}\mathbb{E}\left\{\left[(T_1 - \overline{T}) + (T_2 - \overline{T}) + (T_3 - \overline{T})\right]^2\right\} = \frac{1}{9}[\mathrm{var}(T_1) + \mathrm{var}(T_2) + \mathrm{var}(T_3),]$$

where the above result has been obtained again thanks to the assumption on mutual independence of the data T_1, T_2, and T_3 (see the answer to Question 3b)) that implies

$$\mathbb{E}[(T_1 - \overline{T})(T_2 - \overline{T})] = 0;\ \mathbb{E}[(T_1 - \overline{T})(T_3 - \overline{T})] = 0;\ \mathbb{E}[(T_2 - \overline{T})(T_3 - \overline{T})] = 0.$$

Since

$$T_1 \sim \mathcal{G}(\overline{T},2), \quad T_2 \sim \mathcal{G}(\overline{T},4), \quad T_3 \sim \mathcal{G}(\overline{T},9/4),$$

one gets $\mathrm{var}(\widetilde{T}) = \frac{11}{12} \simeq 0.92$.

Notice that

$$\mathrm{var}[\widehat{T}(\alpha^\circ,\beta^\circ,\gamma^\circ)] = \frac{36}{43} < \mathrm{var}(\widetilde{T}) = \frac{11}{12}$$

This result is consistent with the fact that the estimator $\widehat{T}(\alpha^\circ,\beta^\circ,\gamma^\circ)$ has been optimised exploiting the a-priori information about the *different* variances of the data. Instead, the empirical mean estimator $\widetilde{T}$ does not make use of this a-priori information and hence it should be sub-optimal. Clearly, in case of data having the same variance, the two estimators would coincide.

[6 Marks]

4. Solution

a) From the difference equation (see (4.1) in the exam paper)

$$v(t) = \frac{1}{5}v(t-1) + \frac{1}{3}\eta(t) + \frac{5}{3}\eta(t-1),$$

we obtain (with the usual interpretation of z^{-1} as a one-step backward shift operator)

$$\left(1 - \frac{1}{5}z^{-1}\right)v(t) = \frac{1}{3}(1+5z^{-1})\eta(t).$$

Then, the transfer function from the input process $\eta(\cdot)$ to the output process $v(\cdot)$ is

$$H_{\eta v}(z) = \frac{1}{3}\cdot\frac{z+5}{z-\frac{1}{5}}.$$

Since the input process $\eta(\cdot)$ is stationary and the pole of the transfer function $H_{\eta v}(z)$ lies strictly inside the unit circle, it can be concluded that the steady-state process $v(\cdot)$ is stationary.

The transfer function $H_{\eta v}(z)$ is not in canonical form. To determine the canonical form, let's consider

$$H_{\eta v}(z) = \frac{1}{3}\cdot\frac{z+5}{z-\frac{1}{5}}\cdot\frac{z+\frac{1}{5}}{z+\frac{1}{5}} = \frac{1}{3}\cdot\frac{z+\frac{1}{5}}{z-\frac{1}{5}}\cdot\frac{z+5}{z+\frac{1}{5}}$$

where the term $\frac{z+\frac{1}{5}}{z-\frac{1}{5}}$ is expressed in canonical form and the term $\frac{1}{3}\frac{z+5}{z+\frac{1}{5}}$ does not change the spectrum of the process except for a multiplication by a constant.

Then, the process $v(\cdot)$ can be equivalently generated by the model

$$v(t) = \frac{1}{5}v(t-1) + e(t) + \frac{1}{5}e(t-1) \qquad (\star)$$

where $e(\cdot) \sim WN(0, \lambda^2)$ and λ^2 is a suitable value of the variance of $e(\cdot)$ to be determined as follows. We first compute the variance of the process $v(\cdot)$. As $\mathbb{E}[v(t)] = 0$, it follows that:

$$\begin{aligned}\text{var}[v(t)] = \mathbb{E}[v(t)^2] = \gamma_v(0) &= \mathbb{E}\left\{\left[\frac{1}{5}v(t-1) + \frac{1}{3}\eta(t) + \frac{5}{3}\eta(t-1)\right]^2\right\} \\ &= \frac{1}{25}\mathbb{E}[v(t-1)^2] + \frac{1}{9}\mathbb{E}[\eta(t)^2] + \frac{25}{9}\mathbb{E}[\eta(t-1)^2] + \\ &+ \frac{2}{15}\mathbb{E}[v(t-1)\eta(t)] + \frac{2}{3}\mathbb{E}[v(t-1)\eta(t-1)] + \frac{10}{9}\mathbb{E}[\eta(t)\eta(t-1)]\end{aligned}$$

and thus

$$\gamma_v(0) = \frac{1}{25}\gamma_v(0) + 28 \Longrightarrow \gamma_v(0) = \frac{175}{6} \simeq 29.2$$

Now, by a similar algebra the variance can be computed using the above equivalent model $(\star)$:

$$\gamma_v(0) = \frac{1}{25}\gamma_v(0) + \frac{28}{25}\lambda^2 \Longrightarrow \lambda^2 = \frac{6}{7}\gamma_v(0) = 25$$

[5 Marks]

b) Consider the model of the process $v(\cdot)$ in canonical form determined in the answer to Question 4a), that is:

$$v(t) = \frac{1}{5}v(t-1) + e(t) + \frac{1}{5}e(t-1), \quad \text{with } e(\cdot) \sim WN(0,25).$$

Thus:

$$A(z)v(t) = C(z)e(t)$$

where

$$A(z) = 1 - \frac{1}{5}z^{-1}, \quad C(z) = 1 + \frac{1}{5}z^{-1}.$$

By carrying out one iteration of polynomial division of $C(z)$ by $A(z)$ one gets:

$$\begin{array}{cccc} 1 & \frac{1}{5}z^{-1} & 1 & -\frac{1}{5}z^{-1} \\ -1 & \frac{1}{5}z^{-1} & 1 & \\ // & \frac{2}{5}z^{-1} & & \end{array}$$

Then:

$$\widehat{W}(z) = \frac{C(z)}{A(z)} = 1 + z^{-1}\frac{\frac{2}{5}}{1 - \frac{1}{5}z^{-1}}.$$

and hence the transfer function of the one-step ahead predictor of $v(t+1)$ from the white noise process $e(t)$ is given by

$$\widehat{W}_1(z) = \frac{\frac{2}{5}}{1 - \frac{1}{5}z^{-1}}$$

Moreover, the transfer function of the one-step ahead predictor of $v(t+1)$ from the past data $v(t)$ is

$$W_1(z) = \frac{\frac{2}{5}}{1 + \frac{1}{5}z^{-1}}.$$

and, accordingly, the difference equation implementing the one-step ahead predictor of $v(t+1)$ from the data $v(t)$ is

$$\hat{v}(t+1|t) = -\frac{1}{5}\hat{v}(t|t-1) + \frac{2}{5}v(t).$$

To compute $\text{var}[\varepsilon_1(t)]$, from

$$\widehat{W}(z) = 1 + z^{-1}\frac{\frac{2}{5}}{1 - \frac{1}{5}z^{-1}}$$

it follows that

$$\text{var}[\varepsilon_1(t)] = \text{var}[v(t+1) - \hat{v}(t+1|t)] = 1 \cdot \text{var}[e(t+1)] = 25.$$

[4 Marks]

c) Using the expression

$$A(z)v(t) = C(z)e(t)$$

with

$$A(z) = 1 - \frac{1}{5}z^{-1}, \quad C(z) = 1 + \frac{1}{5}z^{-1}$$

that has been used in the answer to Question 4b), another iteration of polynomial division of $C(z)$ by $A(z)$ is carried out in the following:

$$
\begin{array}{llll}
1 & \frac{1}{5}z^{-1} & & 1 \quad\quad -\frac{1}{5}z^{-1} \\
-1 & \frac{1}{5}z^{-1} & & 1+\frac{2}{5}z^{-1} \\
// & \frac{2}{5}z^{-1} & & \\
// & -\frac{2}{5}z^{-1} & \frac{2}{25}z^{-2} & \\
// & // & \frac{2}{25}z^{-2} &
\end{array}
$$

and thus

$$\widehat{W}(z) = \frac{C(z)}{A(z)} = 1 + \frac{2}{5}z^{-1} + z^{-2}\frac{\frac{2}{25}}{1-\frac{1}{5}z^{-1}}$$

Therefore, the transfer function of the two-steps ahead predictor of $v(t+2)$ from the white noise process $\xi(t)$ is given by

$$\widehat{W}_2(z) = \frac{\frac{2}{25}}{1-\frac{1}{5}z^{-1}}$$

and thus the transfer function of the two-steps ahead predictor of $v(t+2)$ from the past data $v(t)$ is

$$W_2(z) = \frac{\frac{2}{25}}{1+\frac{1}{5}z^{-1}}$$

Finally, the difference equation implementing the two-step ahead predictor of $v(t+2)$ from the data $v(t)$ is

$$\hat{v}(t+2|t) = -\frac{1}{5}\hat{v}(t+1|t-1) + \frac{2}{25}v(t)\,.$$

To compute $\mathrm{var}[\varepsilon_2(t)]$, from

$$\widehat{W}(z) = \frac{C(z)}{A(z)} = 1 + \frac{2}{5}z^{-1} + z^{-2}\frac{\frac{2}{25}}{1-\frac{1}{5}z^{-1}}$$

it follows that

$$
\begin{aligned}
\mathrm{var}[\varepsilon_2(t)] &= \mathrm{var}[v(t+2) - \hat{v}(t+2|t)] = \\
&= 1\cdot\mathrm{var}[e(t+2)] + \left(\frac{2}{5}\right)^2 \cdot \mathrm{var}[e(t+1)] = \frac{29}{25}\cdot 25 = 29.
\end{aligned}
$$

The comparison between $\mathrm{var}[\varepsilon_1(t)]$ and $\mathrm{var}[\varepsilon_2(t)]$ gives

$$\mathrm{var}[\varepsilon_2(t)] = 29 > 25 = \mathrm{var}[\varepsilon_1(t)]$$

This confirms that the variance of the prediction error $\mathrm{var}[\varepsilon_r(t)]$ increases with the number r of steps-ahead of the prediction that is computed.

[6 Marks]

d) Proceeding as in the answer to Question 4a), one gets the transfer function from the input process $\xi(\cdot)$ to the output process $v(\cdot)$:

$$H_{\xi v}(z) = \frac{1}{3}\frac{z-5}{z-\frac{1}{5}}.$$

The computation of the complex spectrum of the process $v(\cdot)$ gives:

$$\Phi_v(z) = H_{\xi v}(z)H_{\xi v}(z^{-1})\cdot 9 = \frac{(z-5)(z-1/5)}{(z^{-1}-5)(z^{-1}-1/5)} = \cdots = 25.$$

Thus, the process $v(\cdot)$ is white:

$$v(t) = \frac{1}{3}\frac{z-5}{z-\frac{1}{5}}\xi(t) = \rho(t)$$

where $\rho \sim WN(0,25)$. Then, the r-step ahead predictions coincide with the expected value of the process for any r:

$$\hat{v}(t+1|t) = 0, \quad \text{var}[\varepsilon_1(t)] = 25$$

and

$$\hat{v}(t+2|t) = 0, \quad \text{var}[\varepsilon_2(t)] = 25.$$

[5 Marks]