

**IMPERIAL COLLEGE LONDON**

**DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING**  
**EXAMINATIONS 2014**

**MSc and EEE/EIE PART III/IV: MEng, Beng and ACGI**

Corrected Copy

**MATHEMATICS FOR SIGNALS AND SYSTEMS**

**Tuesday, 14 January 10:00 am**

**Time allowed: 3:00 hours**

**There are THREE questions on this paper.**

**Answer ALL questions. All questions carry equal marks.**

**Any special instructions for invigilators and information for candidates are on page 1.**

**Examiners responsible**      **First Marker(s) :**      **M.M. Draief**  
   **Second Marker(s) :**      **D. Angeli**

## MATHEMATICS FOR SIGNAL AND SYSTEMS

1. Two questions 1.a and 1.b below are independent.

We say that two subspaces  $V$  and  $W$  of  $\mathbb{R}^n$  are complementary, denoted by  $V \oplus W = \mathbb{R}^n$ , if (i)  $V \cap W = \{0\}$ , where  $0$  is the zero vector in  $\mathbb{R}^n$ , and (ii) any vector  $x \in \mathbb{R}^n$  can be written as  $x = v + w$  where  $v \in V$  and  $w \in W$ .

- a) Let  $P$  be the matrix defined as

$$P = \frac{1}{2} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

- i) Describe a basis of  $\text{Ker}(P)$  the null-space (kernel) of  $P$ , and  $\text{Ran}(P)$  the range of  $P$ . Justify your answer. [3]
  - ii) Show that  $\mathbb{R}^4 = \text{Ker}(P) \oplus \text{Ran}(P)$ . [2]
  - iii) Show that for  $x \in \text{Ker}(P)$  and  $y \in \text{Ran}(P)$ , we have  $x^T y = 0$ . [2]
  - iv) Conclude that  $P$  is an orthogonal projection. [3]
- b) Define the matrix  $A_m$  as follows

$$A_m = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & m & 0 & 0 \\ 1 & 0 & -m & -1 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

where  $m \in \mathbb{R}$  is a parameter.

- i) Derive bases for  $\text{Ker}(A_m)$  and  $\text{Ran}(A_m)$ . [3]
- ii) For  $m \neq 0$ , show that  $\text{Ran}(A_m) \oplus \text{Ker}(A_m) = \mathbb{R}^4$ . [2]
- iii) We now fix  $m = 0$ . Compute  $A_0^3$ . [2]
- iv) Do we have  $\text{Ran}(A_0^3) \oplus \text{Ker}(A_0^3) = \mathbb{R}^4$ ?  
Justify your answer. [3]

2. Let  $A = (a_{ij})_{i,j=1,\dots,n} \in \mathbb{R}^{n \times n}$  be a symmetric matrix, i.e.  $A^T = A$  such that for all  $x \in \mathbb{R}^n$  with  $x \neq 0$  we have

$$x^T A x > 0.$$

Matrices satisfying the above properties are known as *positive-definite matrices*

- a) Let  $e_i \in \mathbb{R}^n$  with all its entries equal to 0 except the  $i$ -th entry which is equal to 1. Show that, for  $i = 1, \dots, n$ , we have  $a_{ii} = e_i^T A e_i > 0$ . [1]
- b) Let  $C$  be the Schur complement of  $a_{11}$  in  $A$ , i.e.

$$C = A_{22} - \frac{1}{a_{11}} A_{21} A_{12},$$

where

$$A = \begin{pmatrix} a_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

with  $a_{11}$  is a scalar,  $A_{21} \in \mathbb{R}^{n-1}$ , and  $A_{22} \in \mathbb{R}^{(n-1) \times (n-1)}$  and  $A_{12} \in \mathbb{R}^{1 \times (n-1)}$ .

- i) Justify the fact that  $C = A_{22} - \frac{1}{a_{11}} A_{21} A_{21}^T$ . [1]
- ii) Let  $v \in \mathbb{R}^{n-1}$  and define  $x \in \mathbb{R}^n$  such that

$$x = \begin{pmatrix} -(1/a_{11}) A_{21}^T v \\ v \end{pmatrix}.$$

Show that  $x^T A x = v^T C v$  and that  $C$  is a positive-definite matrix. [3]

- c) In what follows we will show that there exists a lower-triangular matrix  $L \in \mathbb{R}^{n \times n}$  such that  $A = LL^T$ . This factorisation is known as the *Cholesky decomposition*.

- i) Let  $L$  be given by

$$L = \begin{pmatrix} l_{11} & 0^T \\ L_{21} & L_{22} \end{pmatrix}$$

with  $l_{11}$  is a scalar,  $L_{21} \in \mathbb{R}^{n-1}$ , and  $L_{22} \in \mathbb{R}^{(n-1) \times (n-1)}$  and  $0 \in \mathbb{R}^{n-1}$ . Write the block structure of the matrix  $LL^T$ . [2]

- ii) Let  $A = LL^T$ . Show that  $l_{11} = \sqrt{a_{11}}$ ,  $L_{21} = (1/l_{11}) A_{21}$ , and  $L_{22} L_{22}^T = A_{22} - L_{21} L_{21}^T$ . [2]
- iii) Describe a recursive procedure to construct the lower-triangular matrix  $L$  such that  $A = LL^T$ . [4]
- iv) Describe how one would use the above procedure to solve the linear equation  $Ax = y$  for  $A \in \mathbb{R}^{n \times n}$  positive definite. [3]

- d) Define the following matrix  $A$

$$A = \begin{pmatrix} 25 & 15 & -5 \\ 15 & 18 & 0 \\ -5 & 0 & 11 \end{pmatrix}$$

- i) Apply the Cholesky decomposition to the matrix  $A$  above. [2]
- ii) Use it to solve the equation  $Ax = y$  where  $y = \begin{pmatrix} 30 \\ 15 \\ -16 \end{pmatrix}$ . [2]

3. Let  $m$  and  $n$  be two positive integers with  $m \leq n$ . We consider  $A \in \mathbb{R}^{(n+1) \times (m+1)}$  the matrix defined by

$$A = \begin{pmatrix} 1 & x_0 & \dots & x_0^m \\ 1 & x_1 & \dots & x_1^m \\ \vdots & \vdots & & \vdots \\ 1 & x_n & \dots & x_n^m \end{pmatrix},$$

where  $x_0, \dots, x_n$  are  $n$  distinct real numbers.

Let  $\mathbf{0}$  be the vector with all its entries equal to 0 (we will use the same notation for both the zero vector of  $\mathbb{R}^{m+1}$  and the one of  $\mathbb{R}^{n+1}$ ). In what followed we define the vector

$$v = \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_m \end{pmatrix} \in \mathbb{R}^{m+1}.$$

- a) i) Show that if  $Av = \mathbf{0}$  then  $v = \mathbf{0}$ . [ 1 ]

*Hint:* Use the fact if the polynomial  $P(x) = v_0 + v_1x + \dots + v_mx^m$  has  $n$  distinct roots then  $P(x) = 0$ .

- ii) Using the previous question, show that if  $A^T Av = \mathbf{0}$  then  $v = \mathbf{0}$ . [ 2 ]

- iii) Fix  $y \in \mathbb{R}^{n+1}$ . Justify the fact that the linear equation  $A^T Ax = A^T y$  admits a unique solution  $w$ . [ 2 ]

- b) In the remainder of this problem, we will denote the solution in 2. a) iii) by  $w$ , i.e.

$$A^T Aw = A^T y.$$

For  $v \in \mathbb{R}^{m+1}$  and  $y \in \mathbb{R}^{n+1}$ , define  $g(v) = (y - Av)^T (y - Av)$ .

- i) Show that  $g(w) = y^T y - y^T Aw$ , with  $w$  defined in 2. a) iii). [ 2 ]

- ii) Prove that  $g(v) - g(w) = (w - v)^T A^T A (w - v)$ . [ 2 ]

*Hint:* Use the fact that  $\|A(w - v)\|^2 = \|(Aw - y) - (Av - y)\|^2$ .

- iii) Show that for all  $v \in \mathbb{R}^{m+1}$ , we have  $g(v) \geq g(w)$  and that  $g(v) = g(w)$  if and only if  $v = w$ . [ 3 ]

- c) Let  $P$  be a polynomial such that  $P(x) = \sum_{k=0}^m v_k x^k$ . We define the quantity

$$\Phi_m(P) = \sum_{i=0}^n (y_i - P(x_i))^2.$$

$$\text{Let } y = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^{n+1}.$$

- i) Show that  $\Phi_m(P) = g(v)$ . [ 2 ]

- ii) Using question 3.b.iii), show that there exists a polynomial  $P_w$  such that  $\Phi_m(P) \geq \Phi_m(P_w)$ . [ 2 ]

- d) Let  $n = m = 3$ ,  $x_0 = -1$ ,  $x_1 = 0$ ,  $x_2 = 1$ ,  $x_3 = 2$ ,  $y_0 = 1$ ,  $y_1 = 2$ ,  $y_2 = 1$ ,  $y_3 = 0$ .

- i) Solve  $A^T Av = A^T y$ . [ 2 ]

- ii) Derive the expression of the polynomial in  $\mathbb{R}_3[X]$  that minimizes  $\Phi_3$  and give the minimum value of  $\Phi_3$  on  $\mathbb{R}_3[X]$ . Justify your answer. [ 2 ]