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IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2017

MSc and EEE PART IV: MEng and ACGI

PROBABILITY AND STOCHASTIC PROCESSES

Corrected copy

Monday, 15 May 10:00 am

Time allowed: 3:00 hours

There are FOUR questions on this paper.

Answer ALL questions. All questions carry equal marks.

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : C. Ling

Second Marker(s) : D. Angeli

Information for students

Each of the four questions has 25 marks.

The Questions

1. Random variables.

a) Calculate the probability that a hand of 13 cards dealt from a normal shuffled pack of 52 contains exactly two kings and one ace. What is the probability that it contains exactly one ace given that it contains exactly two kings?

X

You may use Stirling's formula $n! \approx \sqrt{2\pi}\, n^{n+1/2} e^{-n}$.

[10]

b) Given the characteristic function of a Gaussian random variable $X \sim N(0, \sigma^2)$:

$$\Phi_X(\omega) = e^{-\sigma^2\omega^2/2},$$

derive its fourth moment $E[X^4]$. You should give the step-by-step derivation, not just the answer.

[10]

c) Suppose X_1 and X_2 are independent $N(0, \sigma^2)$ random variables. Find the characteristic function of $Y = X_1 X_2$.

[5]

X Tutor came to explain / write on the board 1 a) — no joker in the pack of cards

2. Estimation.

a) A random variable X has exponential distribution with parameter $\lambda = 10$:

$$f_X(x) = \frac{1}{\lambda} e^{-x/\lambda}$$

We wish to estimate parameter λ from $n = 100$ i.i.d. samples of X. Compute the Cramer-Rao lower bound $\mathrm{Var}[\hat{\lambda}] = \frac{1}{I(\lambda)}$ where the Fisher information

$$I(\lambda) = -E\left[\frac{\partial^2}{\partial \lambda^2} \ln f_X(x_1, \cdots, x_n; \lambda)\right].$$

[10]

b) Consider tossing a fair coin for n times. The variable X_i describes the outcome of the i-th toss: $X_i = 1$ if heads shows and $X_i = 0$ if tail shows. Let $X = \sum_{i=1}^{n} X_i$.

i) State the distribution of X, and then compute its expectation and its variance.

[5]

ii) Using Chebyshev's inequality, derive an upper bound on $P(X \geq \frac{5n}{8})$.

[10]

3. Random processes.

a) Assume U is uniformly distributed on $[-\pi, \pi]$, and V is independent of U with probability density function $f_V(x)$. Show that the random process

$$X(n) = e^{j(U-Vn)}$$

is wide-sense stationary and that its power spectral density is $f_V(\omega)$.

[10]

b) Consider the auto-regressive process

$$Y(n) = \alpha Y(n-1) + Z(n)$$

where α is a real number satisfying $|\alpha| < 1$, and $Z(n)$ is an i.i.d. sequence with zero mean and unit variance.

i) Derive the autocorrelation function $R_Y(n)$ of $Y(n)$.

[7]

ii) Suppose we wish to predict $Y(n+2)$ from $Y(n)$, $Y(n-1)$, ..., $Y(1)$. Note that this is a **two**-step ahead predictor. The coefficients of the linear MMSE estimator

$$Y(n+2) = \sum_{i=1}^{n} c_i Y(i)$$

are given by the Wiener-Hopf equation

$$\boldsymbol{Rc} = \boldsymbol{r}$$

where $\boldsymbol{c} = [c_1, c_2, \dots, c_n]^T$, $\boldsymbol{r} = [R_Y(n+1), R_Y(n), \dots, R_Y(2)]^T$, and $\boldsymbol{R}$ is an n-by-n matrix whose (i, j)th entry is $R_Y(i-j)$. Find the best coefficients and the associated mean-square error.

[8]

4. Martingale and Markov chains.

a) Let $X_1, X_2, \cdots, X_n, \cdots$ be a sequence of random variables with finite means satisfying

$$E[X_{n+1}|X_1, X_2, \cdots, X_n] = aX_n + bX_{n-1}$$

where $0 < a, b < 1$ and $a + b = 1$. Find a value of β such that

$$S_n = \beta X_n + X_{n-1}$$

forms a martingale with respect to sequence X.

[10]

b) Consider a symmetric random walk on the two-dimensional grid $\{(x,y): x, y \in 0, \pm1, \pm2, \ldots\}$. This Markov chain is a sequence $\{X_n\}$ where $P\{X_{n+1} = X_n + \epsilon\} = \frac{1}{4}$ and the vector $\epsilon \in \{(\pm1,0), (0,\pm1)\}$. Suppose the chain starts at the origin $X_0 = (0,0)$.

i) Derive the probability $P\{X_{2n} = (0,0)\}$. [5]

ii) Show that the origin is a recurrent state. [10]

Hint: Being recurrent requires $\sum_n P\{X_{2n} = (0,0)\} = \infty$.