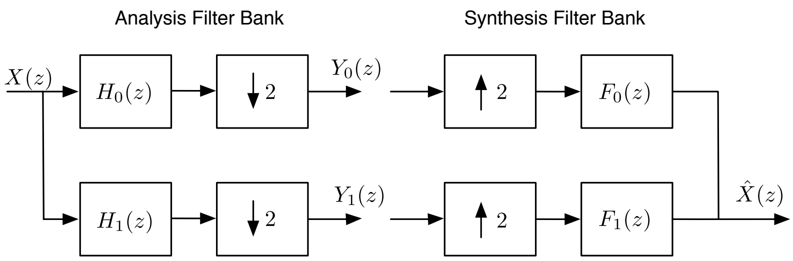
Digital Signal Processing

1. a) Sketch the block diagram of a 2-band maximally decimated filter bank structure showing the analysis and synthesis filter banks connected directly in cascade. [3]

Discuss the effects of aliasing in a QMF maximally decimated 2-band filter bank. Include an explanation for aliasing in terms of the aliasing component matrix $\mathbf{H}(z)$. Include supporting analysis. Include a comparison to the case of oversampled filter banks. [5]

Derive the aliasing cancelling conditions. [2]

Solution:



If the signals $X_{0,1}(z)$ at the output of filters $H_{0,1}(z)$ are not band-limited to

$$|X_k(e^{j\omega})| = 0, \quad \begin{cases} |\omega| < \frac{\pi}{2}, & k = 0 \\ \frac{\pi}{2} < |\omega| < \pi & k = 1 \end{cases}$$

then aliasing will result from the processes of decimation. This can be explained in terms of the expression for the reconstruction, $\hat{X}(z)$, of the input signal, $X(z)$, in the form

$$2\hat{X}(z) = \begin{bmatrix} X(z) & X(-z) \end{bmatrix} \mathbf{H}(z) \begin{bmatrix} F_0(z) \\ F_1(z) \end{bmatrix}$$

where $X(-z)$ is the term describing the aliasing.

In oversampled filter banks, the decimation factor is chosen so that the transition band of the analysis filters is accommodated within the Nyquist frequency after downsampling, hence aliasing is substantially avoided at the cost of reduced computational efficiency.

In the case of QMF filter banks such as this, alias cancellation can be achieved if the synthesis bank is designed satisfying

$$H_0(-z)F_0(z) + H_1(-z)F_1(z) = 0$$

so for example

$$F_0(z) = H_1(-z) \quad \text{and} \quad F_1(z) = -H_0(-z).$$

b) Consider a linear time-invariant system output signal $y(n)$, input signal $x(n)$ and impulse response $\{h_k\}$ for $k = 0, \ldots, N$ given by

k	h_k
0	-0.0087
1	0.0000
2	0.2518
3	0.5138
4	0.2518
5	0.0000
6	-0.0087

and

$$y(n) = \sum_{k=0}^{N} h_k x(n-k).$$

The frequency response of this system is shown in Fig. 1.1.

Use this system to design aliasing cancelling 2-band maximally decimated QMF analysis and synthesis filter banks. Your design should include the coefficients of all filters. [7]

Describe fully the reconstruction properties of this filter bank and state the group delay of the analysis and synthesis filter banks. [3]

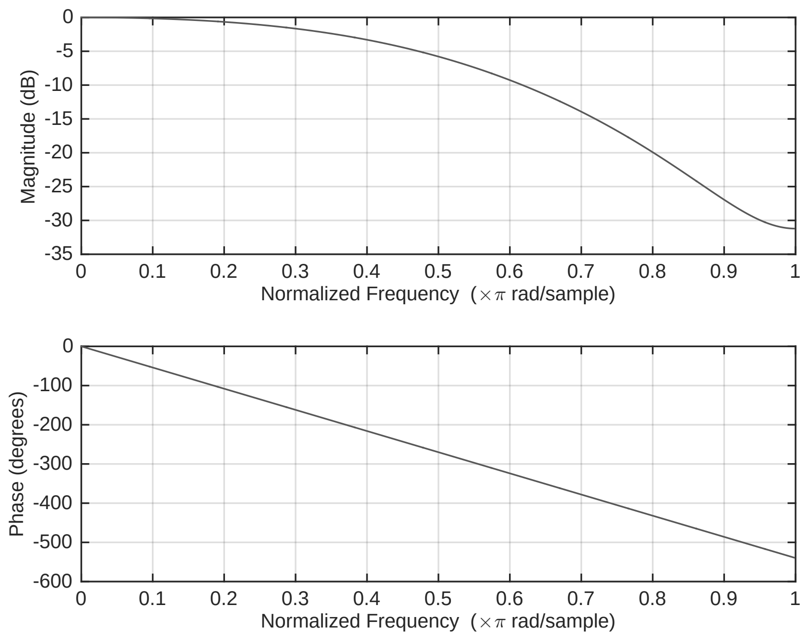


Figure 1.1 Frequency Response

Solution: Start by noting that the given system is a lowpass prototype, and denote it $H(z)$. Then for alias cancellation we require

$$\begin{aligned} H_1(z) &= H_0(-z) \\ F_0(z) &= H_1(-z) = H_0(z) \\ F_1(z) &= -H_0(-z) \end{aligned}$$

so that the filters become

$$\begin{aligned} h_0 &= \begin{bmatrix} -0.0087 & 0.0000 & 0.2518 & 0.5138 & 0.2518 & 0.0000 & -0.0087 \end{bmatrix} \\ h_1 &= \begin{bmatrix} -0.0087 & 0.0000 & 0.2518 & -0.5138 & 0.2518 & 0.0000 & -0.0087 \end{bmatrix} \\ f_0 &= h_0 \\ f_1 &= \begin{bmatrix} 0.0087 & 0.0000 & -0.2518 & 0.5138 & -0.2518 & 0.0000 & 0.0087 \end{bmatrix}. \end{aligned}$$

For reconstruction, the reconstruction transfer function is

$$T(z) = \frac{1}{2}\left[H_0(z)H_1(-z) - H_1(z)H_0(-z)\right].$$

The key properties to mention are:

- perfect amplitude reconstruction cannot be obtained if $H_0(z)$ and $H_1(Z)$ are FIR under the above alias cancellation constraint;
- linear phase response is obtained; no phase distortion;
- group delay of 3 samples.

2. a) Show that a 4-point DFT can be expressed in terms of two 2-point DFTs using the Decimation-in-Time FFT approach and write down the recombination equations. [6]

Illustrate your solution using a signal flow graph. [4]

Solution:

$$X(k) = \sum_{n=0}^{3} x(n) W_4^{nk} \quad k = 0, 1, 2, 3$$
$$= x(0)W_4^0 + x(1)W_4^k + x(2)W_4^{2k} + x(3)W_4^{3k}$$

Next, using $W_4^{0k} = W_2^{0k}$ and $W_4^{2k} = W_2^{1k}$, and since we are using decimation-in-time we define $x_e(0) = x(0)$ and $x_e(1) = x(2)$, we can write that

$$x(0)W_4^0 + x(2)W_4^{2k} = x_e(0)W_2^0 + x_e(1)W_2^{1k}$$

which is a 2-point DFT of the even-indexed sequence $x_e(n)$.

Next, since $W_4^{2k} = W_2^k$ we can write that

$$x(1)W_4^k + x(3)W_4^{3k} = W_4^k \left(x(1)W_4^{0k} + x(3)W_4^{2k} \right)$$
$$= W_4^k \left(x_o(0)W_2^{0k} + x_o(1)W_2^k \right)$$

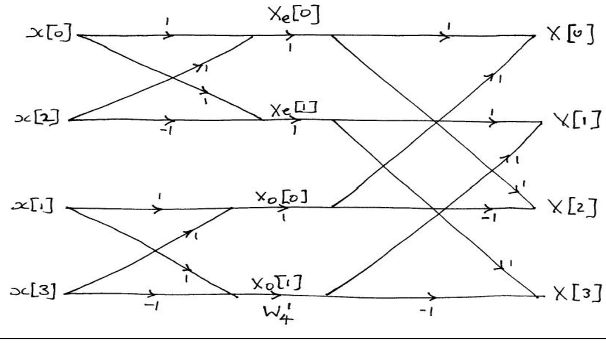
which is a 2-point DFT of the odd-indexed sequence $x_o(n)$. Finally we can write

$$X(k) = X_e(k) + W_4^k X_o(k) \quad k = 0, 1, 2, 3.$$

The recombination equations are given by

$$X(0) = X_e(0) + X_o(0)$$
$$X(1) = X_e(1) + W_4^1 X_o(1)$$
$$X(2) = X_e(0) - X_o(0)$$
$$X(3) = X_e(1) - W_4^1 X_o(1).$$

The signal flow graphs is as follows.



b) Determine the percentage reduction in the number of real multiply operations when a 1024-point DFT is computed using the Decimation-in-Time FFT algorithm instead of a direct implementation of the DFT. [4]

Solution: The DFT requires $4N^2$ real multiplications whereas for the DIT-FFT $2N\log_2(N)$ real multiplications are needed (assuming the trigonometric functions are available). For $N = 1024$, the DFT requires 4194304 and the DIT-FFT requires 20480. The percentage reduction is therefore 99.51%.

c) Find the magnitude and phase spectra of the discrete-time signal

$$p(n) = \begin{bmatrix} 0.1 & -0.1 & 0.0 & -0.1 \end{bmatrix}.$$

[6]

Solution: The magnitude spectrum is given by

$$|P(e^{j\omega})| = \begin{bmatrix} 0.1000 & 0.1000 & 0.3000 & 0.1000 \end{bmatrix}$$

and the phase spectrum is quick to determine knowing symmetry properties of the DFT and is given by

$$\angle P(e^{j\omega}) = \begin{bmatrix} \pi & 0 & 0 & 0 \end{bmatrix}.$$

3. Consider a discrete-time sequence $x(n)$ having z-transform $X(z)$.

a) State the expression for $X(z)$ in terms of $x(n)$ and explain what is meant by the *Region of Convergence* in this context. [4]

Solution:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)x^{-n}$$

ROC defines the range of z for which the z-transform expression converges. Outside the ROC the z-transform does not exist in a meaningful way.

b) Describe the similarities and differences between the z-transform and the Laplace transform. Illustrate your answer using relevant sketches. Comment on the way in which the spectrum of discrete-time signals is represented in the z-domain. [5]

Solution: The z-transform in the discrete-time case is analagous to the Laplace transform in the continuous time case. Sketches should show the significance of the unit circle in z vs. the $j\omega$ axis on the s-plane. The frequency response of discrete-time signals is periodic with period 2π. The periodic spectrum is compactly represented by multiple rotations around the unit circle in z.

c) i) A particular linear time-invariant discrete-time system is described by the difference equation

$$y(n) = 0.2y(n-1) + 2x(n).$$

Using z-transform relationships, find expressions for the system function $H(z)$ of this system, and the unit impulse response $h(n)$ of this system. [4]

Solution:

$$Y(z) = 0.2Y(z)z^{-1} + 2X(z)$$
$$\Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{2}{1-0.2z^{-1}}.$$

The inverse z-transform of $H(z)$ gives

$$h(n) = 2(0.2)^n u(n).$$

ii) Consider the signal

$$x(n) = \begin{cases} a^n, & 0 \leq n \leq M-1 \\ 0 & \text{otherwise} \end{cases}$$

where M is an integer and $0 < a < 1$.

Determine the z-transform $X(z)$ of the signal $x(n)$ for any integer value of M.

Sketch in the z-plane a representation of $X(z)$ for the particular case of $M = 8$. [4]

Solution:

$$X(z) = \sum_{n=0}^{M-1} \left(az^{-1}\right)^n = \frac{1-(az^{-1})^M}{1-az^{-1}} = \frac{z^M - a^M}{z^{M-1}(z-a)}$$

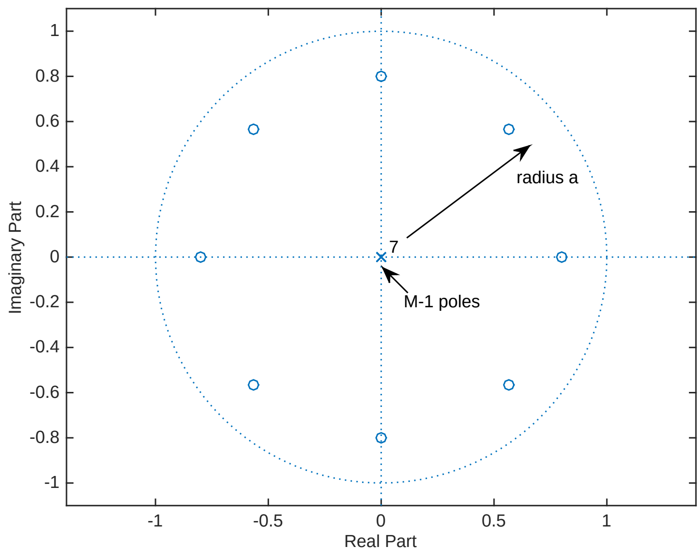
The roots of the numerator are given by $z^M = a^M$ such that the zeros of the transfer function are at

$$z_k = ae^{j2\pi k/M} \quad k = 0, \ldots, M-1.$$

The roots of the numerator give one pole at $z = a$ and $M-1$ poles at $z = 0$. The pole at $z = a$ and the zero for $k = 0$ cancel so that

$$X(z) = \frac{(z-z_1)(z-z_2)\cdots(z-z_{M-1})}{z^{M-1}}.$$

The sketch should show the roots as zeros and poles.



iii) Find the inverse z-transform of

$$G(z) = \frac{1}{1-1.5z^{-1}+0.5z^{-2}}$$

for the 3 cases of the ROC:

$$|z| > 1,$$
$$|z| < 0.5,$$
$$0.5 < |z| < 1.$$

[3]

Solution: By partial fraction expansion

$$G(z) = \frac{2}{1 - z^{-1}} - \frac{1}{1 - 0.5z^{-1}}.$$

For the case of ROC $|z| > 1$, the position of the poles indicates both partial fractions contribute causal terms and hence

$$g(n) = 2u(n) - (0.5)^n u(n).$$

For the case of ROC $|z| < 0.5$, the position of the poles indicates both partial fractions contribute anticausal terms and hence

$$g(n) = -2u(-n-1) + (0.5)^n u(-n-1).$$

For the case of ROC $0.5 < |z| < 1$, the position of the poles indicates the signal is two-sided and hence

$$g(n) = -2u(-n-1) - (0.5)^n u(n).$$

4. A single stage of a lattice structure is shown in Fig. 4.1.

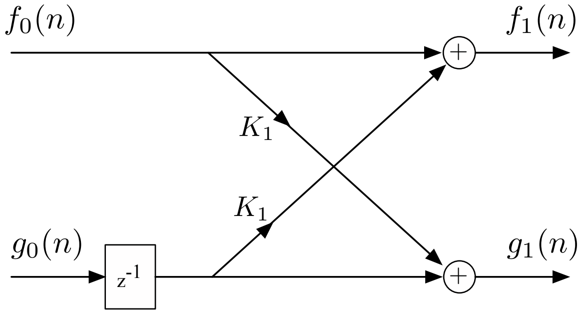


Figure 4.1 Single Stage Lattice Structure

a) Write expressions for $f_1(n)$ and $g_1(n)$ in terms of $f_0(n)$ and $g_0(n)$. [2]

Solutions:

$$f_1(n) = f_0(n) + K_1 g_0(n-1)$$
$$g_1(n) = K_1 f_0(n) + g_0(n-1)$$

b) Defining $x(n)$ as an input signal that is connected to both $f_0(n)$ and $g_0(n)$, and taking the output $y(n)$ as the signal $f_1(n)$, show that the structure implements a first order FIR filter and write down the difference equation for the filter in terms of $x(n)$, $y(n)$ and the filter coefficients denoted a_k, for integer k. Include formulae for a_k. [4]

Solutions:

$$y(n) = x(n) + a_1 x(n-1)$$
$$a_1 = K_1$$

c) Sketch a block diagram of two single stage lattice structures in cascade. The first stage is as given in Fig. 4.1 with $x(n)$ as an input signal that is connected to both $f_0(n)$ and $g_0(n)$, as in part b). The second stage uses K_2 in place of K_1 in the first stage. Denote the f and g outputs of the second stage $f_2(n)$ and $g_2(n)$ respectively.

Give expressions for $f_2(n)$ and $g_2(n)$ in terms of $f_1(n)$ and $g_1(n)$. [3]

Solutions:

$$f_2(n) = f_1(n) + K_2 g_1(n-1)$$
$$g_2(n) = K_2 f_1(n) + g_1(n-1)$$

d) Hence show that the cascaded structure implements a second order FIR filter and write down the difference equation for the filter in terms of $x(n)$, $y(n)$, K_1 and K_2.

Give expressions for the corresponding filter coefficients a_1 and a_2 in terms of K_1 and K_2. [5]

Solutions:

$$\begin{aligned} f_2(n) &= x(n) + K_1 x(n-1) + K_2\left(K_1 x(n-1) + x(n-2)\right) \\ &= x(n) + K_1(1+K_2)x(n-1) + K_2 x(n-2) \\ a_1 &= K_1(1+K_2) \\ a_2 &= K_2 \end{aligned}$$

e) Now consider a cascade of m single stage lattice filters for any integer m. Denote this filter's transfer function

$$A_m(z) = \frac{Y(z)}{X(z)} = \frac{F_m(z)}{F_0(z)}.$$

The filter with coefficients b_k and transfer function $B_m(z)$ is defined as

$$B_m(z) = \frac{G_m(z)}{X(z)}.$$

Write down a general expression for b_k in terms of a_k and determine $B_m(z)$ in terms of $A_m(z)$. [6]

Solutions:

$$\begin{aligned} b_k &= a(m-k) \quad \text{for} \quad k = 0, 1, \ldots, m \\ B(z) &= z^{-m} A(z^{-1}) \end{aligned}$$