EE4-65

THE ANSWERS

Notations:

(a) B - Bookwork

(b) E - New example

(c) A - New application

1. a) i) The diversity gain is the slope at high SNR of the error curve vs. the SNR on a log-log scale, i.e. $-\lim_{\rho\to\infty}\frac{\log(P_e(\rho))}{\log(\rho)}$ with ρ being the SNR. For (a), the diversity gain is 1 as the error rate decreases by 10^{-1} when the SNR is increased from 50dB to 60dB.

[1 - E]

For (b), the diversity gain is 2 as the error rate decreases by 10^{-2} when the SNR is increased from 30dB to 40dB.

[1 - E]

For (c), the diversity gain is 3 as the error rate decreases by 10^{-3} when the SNR is increased from roughly 26dB to 36dB.

[1 - E]

For (d), the diversity gain is 4 as the error rate decreases by 10^{-4} when the SNR is increased from 10dB to 20dB.

[1 - E]

ii) The simplest approach is to perform transmit diversity visa matched beamforming

for (a), matched beamforming with $n_r = 1$, $n_t = 1$

[1 - E]

for (b), matched beamforming with $n_r = 1$, $n_t = 2$

[1 - E]

for (c), matched beamforming with $n_r = 1$, $n_t = 3$

[1 - E]

for (d), matched beamforming with $n_r = 1$, $n_t = 4$

[1 - E]

b) i) The multiplexing gain is the pre-log factor of the ergodic capacity at high SNR, i.e. $g_s = \lim_{\rho\to\infty}\frac{C_{CDIT}}{\log_2(\rho)}$. Hence by increasing the SNR by 3dB (e.g. from 27dB to 30dB), the ergodic capacity increases by g_s bits/s/Hz.

(a) $g_s = 1$.

[1 - E]

(b) $g_s = 2$.

[1 - E]

(c) $g_s = 2$.

[1 - E]

(d) $g_s = 3$.

[1 - E]

(e) $g_s = 4$.

[1 - E]

ii) There are several possible configurations that satisfy to $n_r + n_t = 9$, namely 5×4, 4×5, 6×3, 3×6, 7×2, 2×7, 1×8 and 8×1. The matching between curves and antenna configurations is easily identified by using the following two arguments: 1) The multiplexing gain with CDIT at high SNR is given by $\min\{n_t, n_r\}$. 2) With CDIT only, the input covariance matrix in i.i.d. channel is $\mathbf{Q} = 1/n_t \mathbf{I}_{n_t}$. This implies that 5×4, 6×3, 7×2 and 8×1 outperform 4×5, 3×6, 2×7 and 1×8, respectively.

(a) $n_r \times n_t = 8 \times 1$ or 1×8

[1 - E]

(b) $n_r \times n_t = 2 \times 7$

[1 - E]

(c) $n_r \times n_t = 7 \times 2$

[1 - E]

(d) $n_r \times n_t = 3 \times 6$ or 6×3

[1 - E]

(e) $n_r \times n_t = 4 \times 5$ or 5×4

[1 - E]

c) The two columns are orthogonal. We can simply apply

$$\mathbf{g}_1 = \begin{bmatrix} 1 & 1-j & 1 & 1+j \end{bmatrix} / \sqrt{6},$$

to match with stream 1 and null out the interference from the second stream.

[2 - E]

Similarly, for the second stream, we can apply

$$\mathbf{g}_2 = \begin{bmatrix} 1 & 0 & -1 & 0 \end{bmatrix} / \sqrt{2},$$

The combiner is given by

$$\mathbf{G} = \begin{bmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \end{bmatrix}$$

[2 - E]

This is a (normalized) received matched filter. Given that the columns are orthogonal the matched filter maximizes the SNR, nulls out the inter-stream interference.

[2 - E]

d) In order to maximize the multiplexing gain, we can schedule both users on the same resource and separate them using linear precoding. In order to maximize the SNR and not experience any multi-user interference, we would need the transmit precoder $\mathbf{w}_1$ and $\mathbf{w}_2$ to be such that $\mathbf{h}_2\mathbf{w}_1 = 0$ and $\mathbf{w}_1$ being matched to $\mathbf{h}_1$, i.e. $\mathbf{w}_1 = \mathbf{h}_1^H$. Similarly $\mathbf{h}_1\mathbf{w}_2 = 0$ and $\mathbf{w}_2$ has to be matched to $\mathbf{h}_2$, i.e.

$\mathbf{w}_2 = \mathbf{h}_2^H$.

[3 - E]

This leads to the conditions $1 + e^{j2\pi d/\lambda(\cos\theta_1 - \cos\theta_2)} = 0$ and $1 + e^{j2\pi d/\lambda(\cos\theta_2 - \cos\theta_1)} = 0$, which is achieved if $2\pi d/\lambda(\cos\theta_1 - \cos\theta_2) = k\pi$ with $k = \pm 1, 2, 3, \ldots$. We would choose the location of the users such that $\cos\theta_1 - \cos\theta_2 = \frac{k\lambda}{2d}$.

[3 - E]

e) i) The system model can be written as

$$y = \sqrt{E_s}\mathbf{h}\mathbf{w}c + n,$$

where $\mathbf{h} \triangleq [h_1, \ldots, h_{n_t}]$ represents the MISO channel vector, $\mathbf{w}$ is the transmit precoder, n is the AWGN noise with variance σ_n^2, E_s is the average transmit power, c is a symbol chosen in a unit average constellation. The choice that maximizes the receive SNR is given by

$$\mathbf{w} = \frac{\mathbf{h}^H}{\|\mathbf{h}\|}.$$

The ML detection rule is written as

$$\arg\min_c \left| y - \sqrt{E_s}\mathbf{h}\mathbf{w}c \right|^2$$

[3 - B]

ii) The output SNR can be written as $\rho_{out} = \rho u$ with $u = \|\mathbf{h}\|^2$. The error probability can approximated as For QAM constellations, the error probability is given, assuming ML detection, by

$$\bar{P} \approx \int_0^\infty \bar{N}_e Q\left(d_{min}\sqrt{\frac{\rho u}{2}}\right) p_u(u)\, du,$$

$$\leq \bar{N}_e \mathscr{E}\left\{ e^{-\frac{d_{min}^2 \rho u}{4}} \right\} \qquad \text{(using Chernoff bound } Q(x) \leq \exp\left(-\frac{x^2}{2}\right)\text{)}$$

where $\bar{N}_e$ and d_{min} are respectively the number of nearest neighbors and minimum distance of separation of the underlying constellation. Since u is a χ^2 random variable with $2n_t$ degrees of freedom, using the MGF of a χ^2 random variable, the above average upper-bound is given by

$$\bar{P} \leq \bar{N}_e \left(\frac{1}{1 + \rho d_{min}^2/4}\right)^{n_t}$$

[4 - B]

iii) At high SNR,

$$\bar{P} \overset{\rho\nearrow}{\leq} \bar{N}_e \left(\frac{\rho d_{min}^2}{4}\right)^{-n_t}.$$

The diversity gain is equal to n_t. [3 - B]

2. a) The matrix is clearly rank 1. [3 - A]

This highlights that the maximum number of independent streams to be transmitted over this channel is equal to 1. [3 - A]

b) We find that

$$\mathbf{H}^H\mathbf{H} = \begin{bmatrix} |a|^2 & 0 \\ 0 & |b|^2 \end{bmatrix}.$$

The capacity over the deterministic channel writes as

$$C(\mathbf{H}) = \max_{P_1,P_2} \left(\log_2 \left(1 + \frac{P_1}{\sigma_n^2}|a|^2 \right) + \log_2 \left(1 + \frac{P_2}{\sigma_n^2}|b|^2 \right) \right)$$

with $P_1 + P_2 = P$. The optimal power allocation is given by the water-filling solution

$$P_1^\star = \left(\mu - \frac{\sigma_n^2}{|a|^2} \right)^+, \quad P_2^\star = \left(\mu - \frac{\sigma_n^2}{|b|^2} \right)^+$$

with μ computed such that $P_1^\star + P_2^\star = P$.

[2 - A]

Assuming $P_1^\star$ and $P_2^\star$ are positive, $\mu = \frac{P}{2} + \frac{1}{2}\left(\frac{\sigma_n^2}{|a|^2} + \frac{\sigma_n^2}{|b|^2} \right)$. If $\mu - \frac{\sigma_n^2}{|b|^2} \leq 0$, $P_2^\star = 0$ and $P_1^\star = P$. The capacity writes as

$$C(\mathbf{H}) = \log_2 \left(1 + \frac{P}{\sigma_n^2}|a|^2 \right).$$

[2 - A]

If $\mu - \frac{\sigma_n^2}{|b|^2} > 0$, $P_1^\star = \frac{P}{2} - \frac{\sigma_n^2}{2|a|^2} + \frac{\sigma_n^2}{2|b|^2}$ and $P_2^\star = \frac{P}{2} + \frac{\sigma_n^2}{2|a|^2} - \frac{\sigma_n^2}{2|b|^2}$. The capacity writes as

$$C(\mathbf{H}) = \log_2 \left(1 + \frac{P_1^\star}{\sigma_n^2}|a|^2 \right) + \log_2 \left(1 + \frac{P_2^\star}{\sigma_n^2}|b|^2 \right).$$

[2 - A]

c) By the distance-product criterion, the diversity gain is given by $n_r L_{min} = n_r \min l_{\mathbf{C},\mathbf{E}}$ where $\min l_{\mathbf{C},\mathbf{E}}$ refers to the minimum effective length over all possible non-zero error matrices.

[3 - A]

We note that $d^* = a$, $c^* = b$, $b^* = c$ and $a^* = d$. Given the three codewords, the possible non-zero error matrices are given by

$$\mathbf{a} - \mathbf{b} = \begin{bmatrix} 0 & a-d & b-a & c-b \end{bmatrix},$$

$$\mathbf{a} - \mathbf{c} = \begin{bmatrix} a-b & 0 & b-a & c-b \end{bmatrix},$$

$$\mathbf{b} - \mathbf{c} = \begin{bmatrix} a-b & d-a & 0 & 0 \end{bmatrix}.$$

Hence, $l_{\mathbf{a},\mathbf{b}} = 3$, $l_{\mathbf{a},\mathbf{c}} = 3$ and $l_{\mathbf{b},\mathbf{c}} = 2$, leading to $L_{min} = 2$ and a total diversity gain of 2.

[3 - A]

d) i) The diversity gain of n_t can be obtained with and without CSIT. Without CSIT, such a diversity gain can for instance be achieved with the use of delay-diversity.

[6 - A]

ii) To serve a total of $2n_t$ streams without interference, the transmitter would need $2n_t$ transmit antennas. Alternatively, with n_t antennas, the transmitter could serve $n_t/2$ terminals with 2 streams each. [6 - A]

3. a) i) Consider the two-user Gaussian SISO Multiple Access Channel over the deterministic channels h_1 and h_2. The system model is written as $y = h_1 c_1 + h_2 c_2 + n$ where the transmit power constraint at transmitter i is given by $\mathcal{E}\{|c_i|^2\} \leq P$ for $i = 1, 2$. The noise is AWGN with variance σ_n^2.

[3-B]

ii) The sum-rate is given by $R_1 + R_2 = \log_2\left(1 + \frac{P}{\sigma_n^2}\left(|h_1|^2 + |h_2|^2\right)\right)$. The sum-rate can be achieved by decoding message of user 1 first by treating user 2's message as noise and then perform SIC, i.e. cancel user 1 message and decode user 2's message subject to noise only.

[4-B]

iii) The strategy is not unique since both ordering (user 1 decoded first or user 2 decoded first) could be performed to achieve the same sum-rate. Moreover any time sharing between the two ordering strategies would also achieve the same sum-rate.

[4-B]

iv) Since transmitter 1 is far away, it is subject to a larger path loss and therefore lower average SNR. It would be more fair to operate the system such that user 1 gets the same rate as if it is alone in the system. That would imply user 2 message would have to be decoded first subject to interference from user 1 message. Cancelling user 2 message would leave user 1 message left with the Gaussian noise. This would incur a loss in user 2 rate compared to the case where it was alone in the system but since it is close to the receiver, its rate is already relatively large. Specifically the rates allocated to user 1 is $R_1 = \log_2\left(1 + \frac{P}{\sigma_n^2}|h_1|^2\right)$ and to user 2 is $R_2 = \log_2\left(1 + \frac{P|h_2|^2}{\sigma_n^2 + P|h_1|^2}\right)$.

[4-B]

b) i) The received signal of terminal 1 writes as

$$y_1 = \mathbf{h}_1\mathbf{p}_1 c_1 + \mathbf{h}_1\mathbf{p}_2 c_2 + n_1$$

where y_1 is the $[1 \times 1]$ received signal at user 1, $\mathbf{h}_1$ is the channel between the transmitter and user 1, $\mathbf{P} = [\mathbf{p}_1, \mathbf{p}_2]$ is the $[2 \times 2]$ precoder at the transmitter made of two sub-precoders $\mathbf{p}_1$ and $\mathbf{p}_2$ of size $[2 \times 1]$ (precoding data for terminal 1 and 2 respectively), and $\mathbf{c}_1 = [c_1, c_2]^T$ is the $[2 \times 1]$ transmit symbol vector at the transmitter whose entries are unit-average energy independent symbols. The received signal aof terminal 2 is similarly expressed as

$$y_2 = \mathbf{h}_2\mathbf{p}_1 c_1 + \mathbf{h}_2\mathbf{p}_2 c_2 + n_2$$

. The transmit power writes as $\mathrm{Tr}\left\{\mathcal{E}\left\{\mathbf{Pc}(\mathbf{Pc})^H\right\}\right\} = \mathrm{Tr}\left\{\mathbf{PP}^H\right\} = P$. Hence the power allocation to each stream is naturally accounted for in the precoder.

[5-B]

ii) The maximum multiplexing gain is equal to 2 and can be obtained by designing the precoders such that $\mathbf{h}_1\mathbf{p}_2 = 0$ and $\mathbf{h}_2\mathbf{p}_1 = 0$. This can be simply done by choosing $\mathbf{P}$ as the inverse of the concatenated channel matrix and is knowns as ZFBF. The sum-rate achieved under sum a precoding scheme would be $R_s = \log_2(1 + \frac{|\mathbf{h}_1\mathbf{p}_1|^2}{\sigma_n^2}) + \log_2(1 + \frac{|\mathbf{h}_2\mathbf{p}_2|^2}{\sigma_n^2})$. Assuming for instance a uniform power allocation, both log function would scale linearly with the transmit power (in dB) at high SNr, therefore leading to a multiplexing gain of 2.

[5-B]

iii) If the CSIT is imperfect, ZFBF with uniform power allocation would lead to a saturating sumrate as the SNR increases due to the presence of the multi-user interference originating from $\mathbf{h}_1\mathbf{p}_2 \neq 0$ and $\mathbf{h}_2\mathbf{p}_1 \neq 0$ that would scale with the transmit power. The system becomes interference limited and the sum-rate saturates therefore leading to a zero multiplexing gain at high SNR. By allocating all the transmit power to a single user, the system is converted to a dynamic TDMA, does not experience sum-rate saturation anymore and would achieve a multiplexing gain of 1 at high SNR.

[5-B]