master

UNIVERSITY OF LONDON [E2.11 (Maths) ISE 2007]

B.ENG. AND M.ENG. EXAMINATIONS 2007

For Internal Students of Imperial College London

This paper is also taken for the relevant examination for the Associateship of the City & Guilds of London Institute

INFORMATION SYSTEMS ENGINEERING E2.11

MATHEMATICS

Date Thursday 31st May 2007 2.00 - 4.00 pm

Answer FOUR questions, to include at least one from Section B

Answers to Section A *questions must be written in a different answer book from answers to* Section B *questions.*

A statistics formula sheet is provided

[Before starting, please make sure that the paper is complete. There should be FIVE pages, with a total of SIX questions. Ask the invigilator for a replacement if this copy is faulty.]

Section A

1. **Fourier Transform**

Let f, g and h be functions of time t possessing Fourier transforms and related by

$$g(t) = f(t-d) - f(t+d) \quad \text{and} \quad h'(t) = g(t)\,.$$

Let

$$f(t) = \begin{cases} 1 & \text{for} \quad -d \le t \le d\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

(i) Write the formula defining the Fourier transform $\mathrm{FT}f(t) = \hat{f}(\omega)$ of the function $f(t)$ as an integral over time t and evaluate the integral.

(ii) Express the Fourier transform $\mathrm{FT}h(t) = \hat{h}(\omega)$ in terms of $\mathrm{FT}f(t) = \hat{f}(\omega)$.

(iii) The integral formula defining the convolution of the function $f(t)$ with the function $g(t)$ is

$$(f \times g)\,(t) \;=\; \int_{-\infty}^{\infty} f(u)\; g(t-u)\, du\,.$$

Write your formula for $\mathrm{FT}h(t) = \hat{h}(\omega)$ as the FT of the convolution of two functions, identify the functions and evaluate the integral.

2. **Laplace Transform**

(i) Write the formula for the Laplace transform $L[f] = \overline{f}(p)$ of a function $f(t),\ t > 0$.
Write expressions for $L[tf]$, $L[f']$ and $L[f'']$ in terms of $L[f] = \overline{f}(p)$.
Recall the definition $(f*g)(t) = \int_0^t f(u)g(t-u)du$ for the convolution of two functions f and g.
Write $\overline{f*g}$ in terms of $\overline{f}(p)$ and $\overline{g}(p)$.

Solve the following differential equations by using the Laplace transform.

(ii) $y'' - 3y' + 2y = a(t), \quad t > 0, \quad y(0) = 0 = y'(0)$
where $a(t)$ has Laplace transform $\bar{a}(p)$. Express the answer in terms of $a(t)$ and a function which you should determine.

(iii) $x' + x - y = e^t\,, \qquad -x + y' + y = e^t\,, \qquad x(0) = 0,\ y(0) = 2\,.$

PLEASE TURN OVER

3. **Volume Integration**

A horizontal plane slices through a sphere of radius a at height $z = a\cos\theta > 0$.
Calculate the volumes of the two pieces by using the two different methods which follow.

(i) Compute the smaller volume by integrating in cylindrical coordinates.
Find the larger volume by subtraction.

(ii) Compute the smaller volume by integrating in spherical polar coordinates. For this you may use the formula for the volume of a cone, $V_{cone} = BaseArea \times Height/3$.
Find the larger volume by subtraction.

4. **Contour Integration**

Use contour integration to compute the inverse Fourier transform $f(t)$ for $t > 0$ of

$$\hat{f}(\omega) = \frac{1}{\omega(\omega^2 + a^2)}$$

for a positive real constant $a > 0$ by performing the following steps:

(i) Write the Cauchy integral formula for an analytic function $f(z)$.

(ii) Write the formula for the inverse Fourier transform $f(t)$ of this $\hat{f}(\omega)$ for $t > 0$.

(iii) (c1) Write the formula for the inverse Fourier transform in (i) as an integral around a contour C in the complex plane.
(c2) Draw a sketch that identifies the contour C and the directions of integration on each of its components.
(c3) Explain why you chose this contour to evaluate the inverse Fourier transform $f(t)$ for $t > 0$ of $\hat{f}(\omega)$ above.

(iv) Identify locations of poles enclosed by C, compute their residues and evaluate the integral around the contour using the Cauchy integral formula.

(v) Evaluate integrals on appropriate pieces of the contour.

(vi) Determine the function $f(t)$ whose Fourier transform is $\hat{f}(\omega)$ above.

PLEASE TURN OVER

5. The signal strength of a wireless router from a laptop is classified into three categories: *excellent, good* and *weak*, depending on the distance, X, of the laptop from the router,

Classification	Distance
Excellent	$X < 10\,\mathrm{m}$
Good	$10\,\mathrm{m} < X < 30\,\mathrm{m}$
Weak	$X > 30\,\mathrm{m}$

The distance, X, from the router follows an exponential distribution with parameter $\lambda = 0.1$,

$$f(x) = \begin{cases} \lambda \exp(-\lambda x) & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

(i) Find an expression for $\mathrm{P}(X < x)$.

(ii) Determine the probabilities that the signal strength will be classified as excellent, good and weak.

The observed probabilities of successfully downloading a file if the signal strength is classified as excellent, good or weak are 1, 0.9 and 0.1 respectively.

(iii) Find the unconditional probability that a file is downloaded successfully.

(iv) If a file is downloaded successfully, determine the probability that the signal strength was classified as excellent.

Assuming that files are downloaded sequentially and independently, with the probability of successful download given by the solution to part (iii),

(v) determine the maximum number of files that can be downloaded, such that the probability that they are all downloaded successfully is greater than 0.5.

(vi) find an expression for the probability that the first unsuccessful download occurs after n download attempts.

PLEASE TURN OVER

6. The lifetimes, T_A and T_B of components of type A and B, in hours, are modelled by normal distributions with variances 4 and 9 respectively.

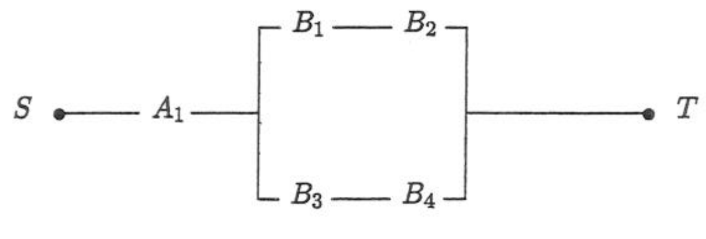
   (i) The lifetimes of a sample of size $n_A = 16$ components of type A have a sample mean of $\bar{x}_A = 26$, and the lifetimes of a sample of size $n_B = 16$ components of type B have a sample mean of $\bar{x}_B = 30$. Calculate 95% confidence intervals for the mean lifetimes of components A and B.

   (ii) Find expressions for the reliability and hazard function of a component with lifetime $T \sim N(\mu, \sigma^2)$.

   (iii) Assuming that the mean lifetimes of components A and B are 26 and 30 hours respectively, find the reliabilities of components of type A and of type B at one day.

   (iv) Comment on a potential problem of modelling lifetimes using a normal distribution.

The following network is constructed using one component, A_1, of type A and four components, B_1, B_2, B_3 and B_4, of type B, all operating independently. The network functions if there is a path of functioning components between S and T.



   (v) Determine the reliability of the network at one day.

END OF PAPER

MATHEMATICS DEPARTMENT

MATHEMATICAL FORMULAE

1. VECTOR ALGEBRA

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1,\, a_2,\, a_3)$$

Scalar (dot) product: $\mathbf{a}\,.\,\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

Vector (cross) product:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Scalar triple product:

$$[\mathbf{a},\, \mathbf{b},\, \mathbf{c}] = \mathbf{a}\,.\,\mathbf{b}\times\mathbf{c} = \mathbf{b}\,.\,\mathbf{c}\times\mathbf{a} = \mathbf{c}\,.\,\mathbf{a}\times\mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product: $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c}\,.\,\mathbf{a})\mathbf{b} - (\mathbf{b}\,.\,\mathbf{a})\mathbf{c}$

2. SERIES

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \ldots \quad (\alpha \text{ arbitrary, } |x| < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots\,,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n\frac{x^{2n}}{(2n)!} + \ldots\,,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + \ldots\,,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^n\frac{x^{n+1}}{(n+1)} + \ldots (-1 < x \le 1)$$

3. TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin(a+b) = \sin a\,\cos b + \cos a\,\sin b\,;$$

$$\cos(a+b) = \cos a\,\cos b - \sin a\,\sin b\,.$$

$$\cos iz = \cosh z\,; \quad \cosh iz = \cos z\,; \quad \sin iz = i\sinh z\,; \quad \sinh iz = i\sin z\,.$$

4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f\,D^n g + \tbinom{n}{1}\,Df\,D^{n-1}\,g + \ldots + \tbinom{n}{r}\,D^r f\,D^{n-r}g + \ldots + D^n f\,g\,.$$

(b) Taylor's expansion of $f(x)$ about $x = a$:

$$f(a+h) = f(a) + hf'(a) + h^2f''(a)/2! + \ldots + h^nf^{(n)}(a)/n! + \epsilon_n(h)\,,$$

where $\epsilon_n(h) = h^{n+1}f^{(n+1)}(a+\theta h)/(n+1)!, \quad 0 < \theta < 1\,.$

(c) Taylor's expansion of $f(x,\, y)$ about $(a,\, b)$:

$$f(a+h,\, b+k) = f(a,\, b) + [hf_x + kf_y]_{a.b} + 1/2!\,\left[h^2f_{xx} + 2hkf_{xy} + k^2f_{yy}\right]_{a.b} + \ldots$$

(d) Partial differentiation of $f(x,\, y)$:

i. If $y = y(x)$, then $f = F(x)$, and $\dfrac{dF}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dx}$.

ii. If $x = x(t)$, $y = y(t)$, then $f = F(t)$, and $\dfrac{dF}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$.

iii. If $x = x(u,\, v)$, $y = y(u,\, v)$, then $f = F(u,\, v)$, and

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}\,, \quad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}\,.$$

(e) Stationary points of $f(x,\, y)$ occur where $f_x = 0$, $f_y = 0$ simultaneously.
Let $(a,\, b)$ be a stationary point: examine $D = [f_{xx}f_{yy} - (f_{xy})^2]_{a.b}$.
If $D > 0$ and $f_{xx}(a,\, b) < 0$, then $(a,\, b)$ is a maximum;
If $D > 0$ and $f_{xx}(a,\, b) > 0$, then $(a,\, b)$ is a minimum;
If $D < 0$ then $(a,\, b)$ is a saddle-point.

(f) Differential equations:

i. The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating factor $I(x) = \exp[\int P(x)(dx]$, so that $\dfrac{d}{dx}(Iy) = IQ$.

ii. $P(x,\, y)dx + Q(x,\, y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y$.

5. INTEGRAL CALCULUS

(a) An important substitution: $\tan(\theta/2) = t$:
$\sin\theta = 2t/(1+t^2)$, $\cos\theta = (1-t^2)/(1+t^2)$, $d\theta = 2\,dt/(1+t^2)$.

(b) Some indefinite integrals:

$$\int (a^2 - x^2)^{-1/2} dx = \sin^{-1}\left(\frac{x}{a}\right), \quad |x| < a .$$

$$\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1}\left(\frac{x}{a}\right) = \ln\left\{\frac{x}{a} + \left(1 + \frac{x^2}{a^2}\right)^{1/2}\right\} .$$

$$\int (x^2 - a^2)^{-1/2} dx = \cosh^{-1}\left(\frac{x}{a}\right) = \ln\left|\frac{x}{a} + \left(\frac{x^2}{a^2} - 1\right)^{1/2}\right| .$$

$$\int (a^2 + x^2)^{-1} dx = \left(\frac{1}{a}\right)\tan^{-1}\left(\frac{x}{a}\right) .$$

6. NUMERICAL METHODS

(a) Approximate solution of an algebraic equation:

If a root of $f(x) = 0$ occurs near $x = a$, take $x_0 = a$ and
$x_{n+1} = x_n - [f(x_n)/f'(x_n)]$, $n = 0, 1, 2 \ldots$

(Newton Raphson method).

(b) Formulae for numerical integration: Write $x_n = x_0 + nh$, $y_n = y(x_n)$.

i. Trapezium rule (1-strip): $\int_{x_0}^{x_1} y(x)dx \approx (h/2)\,[y_0 + y_1]$.

ii. Simpson's rule (2-strip): $\int_{x_0}^{x_2} y(x)dx \approx (h/3)\,[y_0 + 4y_1 + y_2]$.

(c) Richardson's extrapolation method: Let $I = \int_a^b f(x)dx$ and let I_1, I_2 be two estimates of I obtained by using Simpson's rule with intervals h and $h/2$.

Then, provided h is small enough,

$$I_2 + (I_2 - I_1)/15 ,$$

is a better estimate of I .

7. LAPLACE TRANSFORMS

Function	Transform	Function	Transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)dt$	$af(t) + bg(t)$	$aF(s) + bG(s)$
df/dt	$sF(s) - f(0)$	d^2f/dt^2	$s^2F(s) - sf(0) - f'(0)$
$e^{at}f(t)$	$F(s-a)$	$tf(t)$	$-dF(s)/ds$
$(\partial/\partial\alpha)f(t,\alpha)$	$(\partial/\partial\alpha)F(s,\alpha)$	$\int_0^t f(t)dt$	$F(s)/s$
$\int_0^t f(u)g(t-u)du$	$F(s)G(s)$		
1	$1/s$	$t^n (n = 1, 2 \ldots)$	$n!/s^{n+1}$, $(s > 0)$
e^{at}	$1/(s-a)$, $(s > a)$	$\sin\omega t$	$\omega/(s^2+\omega^2)$, $(s > 0)$
$\cos\omega t$	$s/(s^2+\omega^2)$, $(s > 0)$	$H(t-T) = \begin{cases} 0, & t < T \\ 1, & t > T \end{cases}$	e^{-sT}/s, $(s, T > 0)$

8. FOURIER SERIES

If $f(x)$ is periodic of period $2L$, then $f(x+2L) = f(x)$, and

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \text{ where}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad n = 0, 1, 2, \ldots, \text{ and}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1, 2, 3, \ldots$$

Parseval's theorem

$$\frac{1}{L}\int_{-L}^{L} [f(x)]^2\, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right) .$$

1. Probabilities for events

For events A, B, and C $\quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$

More generally $\quad P(\bigcup A_i) = \sum P(A_i) - \sum P(A_i \cap A_j) + \sum P(A_i \cap A_j \cap A_k) - \cdots$

The odds in favour of A $\quad P(A) \,/\, P(\overline{A})$

Conditional probability $\quad P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$ provided that $P(B) > 0$

Chain rule $\quad P(A \cap B \cap C) = P(A)\, P(B \mid A)\, P(C \mid A \cap B)$

Bayes' rule $\quad P(A \mid B) = \dfrac{P(A)\, P(B \mid A)}{P(A)\, P(B \mid A) + P(\overline{A})\, P(B \mid \overline{A})}$

A and B are independent if $\quad P(B \mid A) = P(B)$

A, B, and C are independent if $\quad P(A \cap B \cap C) = P(A)P(B)P(C)$, and

$$P(A \cap B) = P(A)P(B), \quad P(B \cap C) = P(B)P(C), \quad P(C \cap A) = P(C)P(A)$$

2. Probability distribution, expectation and variance

The probability distribution for a discrete random variable X is called the probability mass function (pmf) and is the complete set of probabilities $\{p_x\} = \{P(X = x)\}$

Expectation $\quad E(X) = \mu = \sum_x x p_x$

For function $g(x)$ of x, $\quad E\{g(X)\} = \sum_x g(x) p_x$, so $E(X^2) = \sum_x x^2 p_x$

Sample mean $\quad \overline{x} = \dfrac{1}{n} \sum_k x_k$ estimates μ from random sample $x_1, x_2, \ldots, x_n$

Variance $\quad \mathrm{var}\,(X) = \sigma^2 = E\{(X-\mu)^2\} = E(X^2) - \mu^2$

Sample variance $\quad s^2 = \dfrac{1}{n-1} \left\{ \sum_k x_k^2 - \dfrac{1}{n} \left(\sum_j x_j \right)^2 \right\}$ estimates σ^2

Standard deviation $\quad \mathrm{sd}\,(X) = \sigma$

If value y is observed with frequency n_y

$$n = \sum_y n_y, \quad \sum_k x_k = \sum_y y n_y, \quad \sum_k x_k^2 = \sum_y y^2 n_y$$

Skewness $\quad \beta_1 = E\left(\dfrac{X-\mu}{\sigma}\right)^3$ is estimated by $\dfrac{1}{n-1} \sum \left(\dfrac{x_i - \overline{x}}{s}\right)^3$

Kurtosis $\quad \beta_2 = E\left(\dfrac{X-\mu}{\sigma}\right)^4 - 3$ is estimated by $\dfrac{1}{n-1} \sum \left(\dfrac{x_i - \overline{x}}{s}\right)^4 - 3$

Sample median $\quad \widetilde{x}$ or x_{med}. Half the sample values are smaller and half larger

If the sample values $x_1, \ldots, x_n$ are ordered as $x_{(1)} \le x_{(2)} \le \cdots \le x_{(n)}$, then $\widetilde{x} = x_{(\frac{n+1}{2})}$ if n is odd, and $\widetilde{x} = \frac{1}{2}\left(x_{(\frac{n}{2})} + x_{(\frac{n+2}{2})}\right)$ if n is even

α-quantile $Q(\alpha)$ is such that $P(X \le Q(\alpha)) = \alpha$

Sample α-quantile $\widehat{Q}(\alpha)$ Proportion α of the data values are smaller

Lower quartile Q1 $= \widehat{Q}(0.25)$ one quarter are smaller

Upper quartile Q3 $= \widehat{Q}(0.75)$ three quarters are smaller

Sample median $\widetilde{x} = \widehat{Q}(0.5)$ estimates the population median $Q(0.5)$

3. Probability distribution for a continuous random variable

The cumulative distribution function (cdf) $\quad F(x) = P(X \le x) = \int_{x_0=-\infty}^{x} f(x_0)\mathrm{d}x_0$

The probability density function (pdf) $\quad f(x) = \dfrac{\mathrm{d}F(x)}{\mathrm{d}x}$

$E(X) = \mu = \int_{-\infty}^{\infty} x\, f(x)\mathrm{d}x\,,\quad \mathrm{var}\,(X) = \sigma^2 = E(X^2) - \mu^2,$ where $E(X^2) = \int_{-\infty}^{\infty} x^2\, f(x)\mathrm{d}x$

4. Discrete probability distributions

Discrete Uniform *Uniform* (n)

$$p_x = \frac{1}{n} \quad (x = 1, 2, \ldots, n) \qquad \mu = (n+1)/2,\ \ \sigma^2 = (n^2-1)/12$$

Binomial distribution *Binomial* (n, θ)

$$p_x = \binom{n}{x} \theta^x (1-\theta)^{n-x} \quad (x = 0, 1, 2, \ldots, n) \qquad \mu = n\theta\,,\ \ \sigma^2 = n\theta(1-\theta)$$

Poisson distribution *Poisson* (λ)

$$p_x = \frac{\lambda^x e^{-\lambda}}{x!} \quad (x = 0, 1, 2, \ldots) \quad (\text{with } \lambda > 0) \qquad \mu = \lambda\,,\ \ \sigma^2 = \lambda$$

Geometric distribution *Geometric* (θ)

$$p_x = (1-\theta)^{x-1}\theta \quad (x = 1, 2, 3, \ldots) \qquad \mu = \frac{1}{\theta}\,,\ \ \sigma^2 = \frac{1-\theta}{\theta^2}$$

5. Continuous probability distributions

Uniform distribution *Uniform* (α, β)

$$f(x) = \begin{cases} \dfrac{1}{\beta-\alpha} & (\alpha < x < \beta), \\ 0 & (\text{otherwise}). \end{cases} \qquad \mu = (\alpha+\beta)/2\,,\ \ \sigma^2 = (\beta-\alpha)^2/12$$

Exponential distribution *Exponential* (λ)

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & (0 < x < \infty), \\ 0 & (-\infty < x \le 0). \end{cases} \qquad \mu = 1/\lambda\,,\ \ \sigma^2 = 1/\lambda^2$$

Normal distribution $N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2 \right\} \quad (-\infty < x < \infty), \quad E(X) = \mu, \quad \mathrm{var}(X) = \sigma^2$$

Standard normal distribution $N(0,1)$

If X is $N(\mu, \sigma^2)$, then $Y = \dfrac{X-\mu}{\sigma}$ is $N(0,1)$

6. Reliability

For a device in continuous operation with failure time random variable T having pdf $f(t)$ $(t > 0)$

The reliability function at time t $\quad R(t) = P(T > t)$

The failure rate or hazard function $\quad h(t) = f(t)/R(t)$

The cumulative hazard function $\quad H(t) = \int_0^t h(t_0)\,\mathrm{d}t_0 = -\ln\{R(t)\}$

The Weibull(α, β) distribution has $\quad H(t) = \beta t^\alpha$

7. System reliability

For a system of k devices, which operate independently, let

$R_i = P(D_i) = P($"device i operates"$)$

The system reliability, R, is the probability of a path of operating devices

A system of devices in series operates only if every device operates

$$R = P(D_1 \cap D_2 \cap \cdots \cap D_k) = R_1 R_2 \cdots R_k$$

A system of devices in parallel operates if any device operates

$$R = P(D_1 \cup D_2 \cup \cdots \cup D_k) = 1 - (1 - R_1)(1 - R_2) \cdots (1 - R_k)$$

8. Covariance and correlation

The covariance of X and Y $\quad \mathrm{cov}(X,Y) = E(XY) - \{E(X)\}\{E(Y)\}$

From pairs of observations $(x_1, y_1), \ldots, (x_n, y_n)$ $\quad S_{xy} = \sum_k x_k y_k - \frac{1}{n}(\sum_i x_i)(\sum_j y_j)$

$$S_{xx} = \sum_k x_k^2 - \frac{1}{n}(\sum_i x_i)^2, \qquad S_{yy} = \sum_k y_k^2 - \frac{1}{n}(\sum_j y_j)^2$$

Sample covariance $\quad s_{xy} = \dfrac{1}{n-1} S_{xy}$ estimates $\mathrm{cov}(X,Y)$

Correlation coefficient $\quad \rho = \mathrm{corr}(X,Y) = \dfrac{\mathrm{cov}(X,Y)}{\mathrm{sd}(X) \cdot \mathrm{sd}(Y)}$

Sample correlation coefficient $\quad r = \dfrac{S_{xy}}{\sqrt{S_{xx} S_{yy}}}$ estimates ρ

9. Sums of random variables

$$E(X+Y) = E(X)+E(Y)$$

$$\mathrm{var}\,(X+Y) = \mathrm{var}\,(X)+\mathrm{var}\,(Y)+2\,\mathrm{cov}\,(X,Y)$$

$$\mathrm{cov}\,(aX+bY,\ cX+dY) = (ac)\,\mathrm{var}\,(X)+(bd)\,\mathrm{var}\,(Y)+(ad+bc)\,\mathrm{cov}\,(X,Y)$$

If X is $N(\mu_1,\sigma_1^2)$, Y is $N(\mu_2,\sigma_2^2)$, and $\mathrm{cov}\,(X,Y)=c$, then $X+Y$ is $N(\mu_1+\mu_2,\ \sigma_1^2+\sigma_2^2+2c)$

10. Bias, standard error, mean square error

If t estimates θ (with random variable T giving t)

Bias of t — $\mathrm{bias}\,(t) = E(T)-\theta$

Standard error of t — $\mathrm{se}\,(t) = \mathrm{sd}\,(T)$

Mean square error of t — $\mathrm{MSE}(t) = E\{(T-\theta)^2\} = \{\mathrm{se}\,(t)\}^2+\{\mathrm{bias}\,(t)\}^2$

If $\overline{x}$ estimates μ, then $\mathrm{bias}\,(\overline{x})=0$, $\mathrm{se}\,(\overline{x})=\sigma/\sqrt{n}$, $\mathrm{MSE}(\overline{x})=\sigma^2/n$, $\widehat{\mathrm{se}}\,(\overline{x})=s/\sqrt{n}$

Central limit property — If n is fairly large, $\overline{x}$ is from $N(\mu,\ \sigma^2/n)$ approximately

11. Likelihood

The likelihood is the joint probability as a function of the unknown parameter θ.

For a random sample $x_1,x_2,\ldots,x_n$

$$\ell(\theta;\,x_1,x_2,\ldots,x_n) = P(X_1=x_1\,|\,\theta)\,\cdots\,P(X_n=x_n\,|\,\theta) \qquad \text{(discrete distribution)}$$

$$\ell(\theta;\,x_1,x_2,\ldots,x_n) = f(x_1\,|\,\theta)\,f(x_2\,|\,\theta)\,\cdots\,f(x_n\,|\,\theta) \qquad \text{(continuous distribution)}$$

The maximum likelihood estimator (MLE) is $\widehat{\theta}$ for which the likelihood is a maximum

12. Confidence intervals

If $x_1,x_2,\ldots,x_n$ are a random sample from $N(\mu,\sigma^2)$ and σ^2 is known, then the 95% confidence interval for μ is $(\overline{x}-1.96\frac{\sigma}{\sqrt{n}},\ \overline{x}+1.96\frac{\sigma}{\sqrt{n}})$

If σ^2 is estimated, then from the Student t table for t_{n-1} we find $t_0=t_{n-1,0.05}$

The 95% confidence interval for μ is $(\overline{x}-t_0\frac{s}{\sqrt{n}},\ \overline{x}+t_0\frac{s}{\sqrt{n}})$

13. Standard normal table — Values of pdf $\phi(y) = f(y)$ and cdf $\Phi(y) = F(y)$

y	$\phi(y)$	$\Phi(y)$	y	$\phi(y)$	$\Phi(y)$	y	$\phi(y)$	$\Phi(y)$	y	$\Phi(y)$
0	.399	.5	.9	.266	.816	1.8	.079	.964	2.8	.997
.1	.397	.540	1.0	.242	.841	1.9	.066	.971	3.0	.999
.2	.391	.579	1.1	.218	.864	2.0	.054	.977	0.841	.8
.3	.381	.618	1.2	.194	.885	2.1	.044	.982	1.282	.9
.4	.368	.655	1.3	.171	.903	2.2	.035	.986	1.645	.95
.5	.352	.691	1.4	.150	.919	2.3	.028	.989	1.96	.975
.6	.333	.726	1.5	.130	.933	2.4	.022	.992	2.326	.99
.7	.312	.758	1.6	.111	.945	2.5	.018	.994	2.576	.995
.8	.290	.788	1.7	.094	.955	2.6	.014	.995	3.09	.999

14. Student t table — Values $t_{m,p}$ of x for which $P(|X| > x) = p$, when X is t_m

m	$p=$ 0.10	0.05	0.02	0.01	m	$p=$ 0.10	0.05	0.02	0.01
1	6.31	12.71	31.82	63.66	9	1.83	2.26	2.82	3.25
2	2.92	4.30	6.96	9.92	10	1.81	2.23	2.76	3.17
3	2.35	3.18	4.54	5.84	12	1.78	2.18	2.68	3.05
4	2.13	2.78	3.75	4.60	15	1.75	2.13	2.60	2.95
5	2.02	2.57	3.36	4.03	20	1.72	2.09	2.53	2.85
6	1.94	2.45	3.14	3.71	25	1.71	2.06	2.48	2.78
7	1.89	2.36	3.00	3.50	40	1.68	2.02	2.42	2.70
8	1.86	2.31	2.90	3.36	∞	1.645	1.96	2.326	2.576

15. Chi-squared table — Values $\chi^2_{k,p}$ of x for which $P(X > x) = p$, when X is χ^2_k and $p = .995, .975$, *etc*

k	.995	.975	.05	.025	.01	.005	k	.995	.975	.05	.025	.01	.005
1	.000	.001	3.84	5.02	6.63	7.88	18	6.26	8.23	28.87	31.53	34.81	37.16
2	.010	.051	5.99	7.38	9.21	10.60	20	7.43	9.59	31.42	34.17	37.57	40.00
3	.072	.216	7.81	9.35	11.34	12.84	22	8.64	10.98	33.92	36.78	40.29	42.80
4	.207	.484	9.49	11.14	13.28	14.86	24	9.89	12.40	36.42	39.36	42.98	45.56
5	.412	.831	11.07	12.83	15.09	16.75	26	11.16	13.84	38.89	41.92	45.64	48.29
6	.676	1.24	12.59	14.45	16.81	18.55	28	12.46	15.31	41.34	44.46	48.28	50.99
7	.990	1.69	14.07	16.01	18.48	20.28	30	13.79	16.79	43.77	46.98	50.89	53.67
8	1.34	2.18	15.51	17.53	20.09	21.95	40	20.71	24.43	55.76	59.34	63.69	66.77
9	1.73	2.70	16.92	19.02	21.67	23.59	50	27.99	32.36	67.50	71.41	76.15	79.49
10	2.16	3.25	13.31	20.48	23.21	25.19	60	35.53	40.48	79.08	83.30	88.38	91.95
12	3.07	4.40	21.03	23.34	26.22	28.30	70	43.28	48.76	90.53	95.02	100.4	104.2
14	4.07	5.63	23.68	26.12	29.14	31.32	80	51.17	57.15	101.9	106.6	112.3	116.3
16	5.14	6.91	26.30	28.85	32.00	34.27	100	67.33	74.22	124.3	129.6	135.8	140.2

16. $\boxed{\text{The chi-squared goodness-of-fit test}}$

The frequencies n_y are grouped so that the fitted frequency $\widehat{n}_y$ for every group exceeds about 5.

$X^2 = \sum_y \frac{(n_y - \widehat{n}_y)^2}{\widehat{n}_y}$ is referred to the table of χ_k^2 with significance point p,

where k is the number of terms summed, less one for each constraint, *eg* matching total frequency, and matching $\overline{x}$ with μ

17. $\boxed{\text{Joint probability distributions}}$

Discrete distribution $\{p_{xy}\}$, where $p_{xy} = P(\{X = x\} \cap \{Y = y\})$.

Let $p_{x\bullet} = P(X = x)$, and $p_{\bullet y} = P(Y = y)$, then

$$p_{x\bullet} = \sum_y p_{xy} \quad \text{and} \quad P(X = x \,|\, Y = y) = \frac{p_{xy}}{p_{\bullet y}}$$

$\boxed{\text{Continuous distribution}}$

Joint cdf $\quad F(x,y) = P(\{X \le x\} \cap \{Y \le y\}) = \int_{x_0=-\infty}^{x} \int_{y_0=-\infty}^{y} f(x_0, y_0)\, \mathrm{d}x_0\, \mathrm{d}y_0$

Joint pdf $\quad f(x,y) = \frac{\mathrm{d}^2 F(x,y)}{\mathrm{d}x\, \mathrm{d}y}$

Marginal pdf of X $\quad f_X(x) = \int_{-\infty}^{\infty} f(x, y_0)\, \mathrm{d}y_0$

Conditional pdf of X given $Y = y$ $\quad f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$ (provided $f_Y(y) > 0$)

18. $\boxed{\text{Linear regression}}$

To fit the linear regression model $y = \alpha + \beta x$ by $\widehat{y}_x = \widehat{\alpha} + \widehat{\beta} x$ from observations $(x_1, y_1), \ldots, (x_n, y_n)$, the least squares fit is $\quad \widehat{\alpha} = \overline{y} - \overline{x}\widehat{\beta}, \quad \widehat{\beta} = \frac{S_{xy}}{S_{xx}}$

The residual sum of squares $\quad \text{RSS} = S_{yy} - \frac{S_{xy}^2}{S_{xx}}$

$\widehat{\sigma^2} = \frac{\text{RSS}}{n-2} \qquad \frac{n-2}{\sigma^2}\,\widehat{\sigma^2}$ is from χ^2_{n-2}

$$E(\widehat{\alpha}) = \alpha, \quad E(\widehat{\beta}) = \beta,$$

$$\operatorname{var}(\widehat{\alpha}) = \frac{\sum x_i^2}{n\, S_{xx}} \sigma^2, \quad \operatorname{var}(\widehat{\beta}) = \frac{\sigma^2}{S_{xx}}, \quad \operatorname{cov}(\widehat{\alpha}, \widehat{\beta}) = -\frac{\overline{x}}{S_{xx}} \sigma^2$$

$$\widehat{y}_x = \widehat{\alpha} + \widehat{\beta} x, \qquad E(\widehat{y}_x) = \alpha + \beta x, \qquad \operatorname{var}(\widehat{y}_x) = \left\{ \frac{1}{n} + \frac{(x - \overline{x})^2}{S_{xx}} \right\} \sigma^2$$

$\frac{\widehat{\alpha} - \alpha}{\widehat{\text{se}}\,(\widehat{\alpha})}, \quad \frac{\widehat{\beta} - \beta}{\widehat{\text{se}}\,(\widehat{\beta})}, \quad \frac{\widehat{y}_x - \alpha - \beta x}{\widehat{\text{se}}\,(\widehat{y}_x)}$ are each from t_{n-2}

E2.11 - ISE MATHS - 2nd yr.

	EXAMINATION SOLUTIONS 2006-07	Course ISE 2.6
Question 1	Fourier transform	Marks (ALL SEEN)

Parts

(a) The Fourier transform FT$f(t) = \hat{f}(\omega)$ of the function $f(t)$ — 6 marks

$$f(t) = \begin{cases} 1 & \text{for} \quad -d \le t \le d \\ 0 & \text{otherwise} \end{cases}$$

is given as an integral over time t which is easily evaluated as

$$\hat{f}(\omega) = \mathsf{FT}(f(t)) = \int_{-\infty}^{\infty} e^{-i\omega t} f(t)\, dt = \frac{2\sin(\omega d)}{\omega}$$

(b) Consequently, since $g(t) = h'(t)$ implies $\hat{g}(\omega) = i\omega\hat{h}(\omega)$ one finds — 7 marks

$$\begin{aligned} \hat{g}(\omega) &= \mathsf{FT}\Big(f(t-d) - f(t+d)\Big) = \Big(e^{i\omega d} - e^{-i\omega d}\Big)\hat{f}(\omega) \\ &= 2i\,\sin(\omega d)\hat{f}(\omega) = \widehat{(h')}(\omega) = i\omega\hat{h}(\omega) \end{aligned}$$

That is,

$$\hat{h}(\omega) = \frac{2\sin(\omega d)}{\omega}\hat{f}(\omega) = \frac{4\sin^2(\omega d)}{\omega^2}$$

(c) $f * g(t) = \int_{-\infty}^{\infty} f(t-s)g(s)\, ds$ ← Given

The product of Fourier transforms of two functions is the Fourier transform of their convolution, $\hat{f}(\omega)\hat{g}(\omega) = (\widehat{f * g})$, so — 7 marks

$$\hat{h}(\omega) = \frac{2\sin(\omega d)}{\omega}\hat{f}(\omega) = \hat{f}(\omega)^2 = (\widehat{f * f})$$

where $f * f$ is the convolution of f with itself, namely

$$(f * f)(t) = \int_{-\infty}^{\infty} f(t-s)f(s)\, ds = \begin{cases} t + 2d & \text{for} \quad -2d \le t \le 0 \\ 2d - t & \text{for} \quad 0 \le t \le 2d \\ 0 & \text{otherwise} \end{cases}$$

	Setter's initials	Checker's initials	Page number
	d2h	RLJ	1

	EXAMINATION SOLUTIONS 2006-07	Course ISE 2.6
Question 2	Laplace transform	Marks (ALL SEEN)

(a) The Laplace transform $\bar{f}(p)$ of a function $f(t)$, $t > 0$, is defined as $L[f] = \bar{f}(p) = \int_0^\infty e^{-pt} f(t)dt$. It satisfies $L[tf] = -\frac{d}{dp}\bar{f}(p)$, $L[f'] = \overline{f'}(p) = -f(0) + p\bar{f}(p)$, $L[(f')'] = \overline{f''}(p) = -f'(0) + pL[f']$, and $\overline{f * g} = \bar{f}\,\bar{g}$, where $(f * g)(t) = \int_0^t f(u)g(t-u)du$.

5 marks

Parts

(b) Laplace transform both sides of the equation, using the formulas for the transformation of $y'(t)$ and $y''(t)$ and initial conditions $y(0)$ and $y'(0)$:

$$p^2\bar{y} - 3p\,\bar{y} + 2\bar{y} = \bar{a} \quad \Longleftrightarrow \quad \bar{y} = \frac{\bar{a}}{p^2 - 3p + 2}$$

The solution $\bar{y}$ is a product of Laplace transforms, therefore $y(t)$ is a convolution of $a(t)$ and the inverse Laplace transform of $(p^2-3p+2)^{-1}$. To find the latter, use partial fractions to write

7 marks

$$\frac{1}{p^2 - 3p + 2} = \frac{1}{p-2} - \frac{1}{p-1}$$

which has inverse Laplace transform $e^{2t} - e^t$. Hence, we have

$$y(t) = a(t) * (e^{2t} - e^t) = \int_0^t a(t-u)(e^{2u} - e^u)\, du\,.$$

(c) Taking the Laplace transform of the two equations and using the initial conditions for x and y, we have

$$p\,\bar{x} + \bar{x} - \bar{y} = \frac{1}{p-1}, \qquad -\bar{x} - 2 + p\,\bar{y} + \bar{y} = \frac{1}{p-1}.$$

8 marks

Solving for $\bar{x}$ and $\bar{y}$ yields,

$$\bar{x} = \frac{3}{(p-1)(p+2)}, \qquad \bar{y} = \frac{2p+1}{(p-1)(p+2)}.$$

These are easily inverted using partial fractions as,

$$x(t) = e^t - e^{-2t}, \qquad y(t) = e^t + e^{-2t}.$$

	Setter's initials: dzh	Checker's initials: RLJ	Page number: 2

EXAMINATION SOLUTIONS 2006-07

Course ISE 2.6

Question 3 — Volume integrals

Marks (ALL SEEN)

Parts

(a) In cylindrical coordinates $(x, y, z) = (r\cos\phi, r\sin\phi, z)$ the sphere is given by $z^2 = a^2 - r^2$, so that $z = \sqrt{a^2 - r^2}$ for $z \geq 0$. The volume of the cap above $z = a\cos\theta > 0$ is given in cylindrical coordinates by

$$\begin{aligned} V_{cap} &= \int_0^{2\pi} d\phi \int_0^{a\sin\theta} r dr \int_{a\cos\theta}^{\sqrt{a^2-r^2}} dz \\ &= \pi \int_0^{(a\sin\theta)^2} \Big[\sqrt{a^2-\rho} - a\cos\theta\Big] d\rho \text{ with } \rho = r^2 \\ &= \pi\Big[-\frac{2}{3}(a^2-\rho)^{3/2} - a\rho\cos\theta\Big]_{\rho=0}^{\rho=a^2\sin^2\theta} \\ &= \frac{2\pi a^3}{3}\Big[1 - \cos\theta\Big(\cos^2\theta + \frac{3}{2}\sin^2\theta\Big)\Big] \\ &= \frac{2\pi a^3}{3}\Big[1 - \cos\theta\Big(1 + \frac{1}{2}\sin^2\theta\Big)\Big] \end{aligned}$$

3 for limits
3 integral

4 correct integration

This is the smaller volume. The larger volume is given by

$$\frac{4\pi a^3}{3} - V_{cap} = \Big[1 + \cos\theta\Big(1 + \frac{1}{2}\sin^2\theta\Big)\Big]$$

1 mark

(b) In spherical coordinates $(x, y, z) = (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$ the polar angle θ cuts out a volume consisting of a cap for $z > a\cos\theta$ plus a cone for $z < a\cos\theta$. The sum of the volumes of the cap and cone is given by

$$V_{cap} + V_{cone} = \int_0^{2\pi} d\phi \int_0^a r^2 dr \int_{\cos\theta}^1 d\cos\theta = \frac{2\pi a^3}{3}(1 - \cos\theta)$$

9 marks

Now

$$V_{cone} = \frac{1}{3}B \times H = \frac{1}{3}\pi a^2 \sin^2\theta \times a\cos\theta$$

So

$$V_{cap} = \frac{2\pi a^3}{3}(1 - \cos\theta) - \frac{\pi a^3}{3}\sin^2\theta\cos\theta$$

which agrees with Part (a) and the larger volume follows.

Setter's initials: dZh

Checker's initials: RLJ

Page number: 3

	EXAMINATION SOLUTIONS 2006-07	Course ISE 2.6
Question 4	Contour integration	Marks (ALL SEEN)

Parts

(a) The Cauchy integral formula is given by

1 mark

$$\frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z-\alpha} = \begin{cases} f(\alpha) & \text{if contour } C \text{ encloses } z=\alpha \\ 0 & \text{otherwise} \end{cases}$$

(b) The inverse Fourier transform of $\hat{f}(\omega) = [\omega(\omega^2+a^2)]^{-1}$ is given by

1 mark

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\omega t}d\omega}{\omega(\omega^2+a^2)}$$

(c) As a contour integral in the complex plane, this is

$$f(t) = \frac{1}{2\pi}\int_C \frac{e^{izt}dz}{z(z^2+a^2)} \quad \text{with} \quad t>0$$

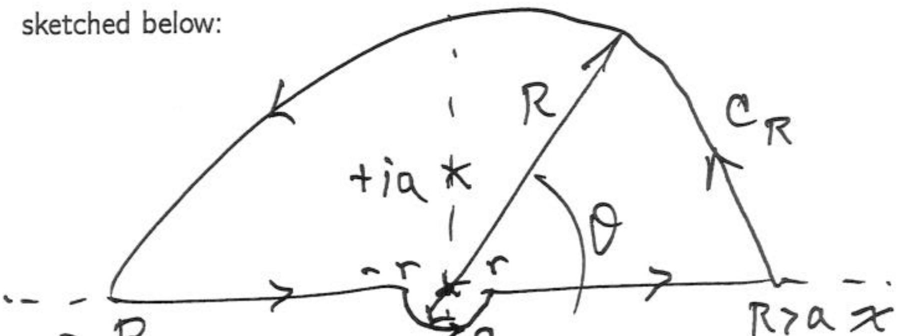
2 marks for poles

The integrand has first order poles at $z=0$ and $z=\pm ia$. An appropriate contour C consists of two real segments $x \in [-R,-r]$ and $x \in [r,R]$ with $0<r<R$ and $R>a$, plus the following two semicircles,

6 marks for contour (including sketch)

$$C_R: \quad z = Re^{i\theta} \text{ and } 0 \le \theta \le \pi, \quad C_r: \quad z = re^{i\theta} \text{ and } \pi \le \theta \le 2\pi$$

sketched below:



This contour C is chosen so that:
(1) the required integral along real axis emerges as $r \to 0$ and $R \to \infty$;
(2) the poles at $z=0$ and $z=+ia$ are enclosed; and
(3) the integral along the contour C_R vanishes in the limit $R \to \infty$ by the ML Theorem.
Alternatively, one may choose $C_r: \ z = re^{i\theta}, \ \pi \ge \theta \ge 0$.

	Setter's initials dzh	Checker's initials RLJ	Page number 4

Question 4 Contour integration (continued)

Marks (ALL SEEN)

Parts

(d) The pole at $z = -ia$ is not enclosed by C, so it does not contribute. The residues of $\frac{e^{izt}}{z(z^2+a^2)}$ at the poles $z = 0$ and $z = ia$ evaluate the Cauchy integral as

$$2\pi i \times \{\text{Sum of residues}\} = 2\pi i \times \left\{\frac{1}{a^2} - \frac{e^{-at}}{2a^2}\right\}$$

3 marks

(e) This integral is equal to the sum of its components

$$\begin{aligned}\frac{\pi i}{a^2}\left(2 - e^{-at}\right) &= \int_{-R}^{-r} \frac{e^{ixt}dx}{x(x^2+a^2)} + \int_{r}^{R} \frac{e^{ixt}dx}{x(x^2+a^2)} \\ &\quad + \underbrace{\int_{C_r} \frac{e^{izt}dz}{z(z^2+a^2)}}_{\lim r \to 0 \text{ is } \pi i/a^2} + \underbrace{\int_{C_R} \frac{e^{izt}dz}{z(z^2+a^2)}}_{\text{vanishes as } R \to \infty}\end{aligned}$$

2 marks

Closed in the upper half plane, the magnitude of the integral along C_R falls (exponentially) faster than its length grows. Hence, by the ML theorem, this integral vanishes in the limit $R \to \infty$ for $t > 0$.

In the limit $r \to 0$ the integral along C_r is given by

$$\lim_{r \to 0} \int_{\pi}^{2\pi} \frac{e^{rt(i\cos\theta - \sin\theta)} i d\theta}{(r^2 e^{2i\theta} + a^2)} = \int_{\pi}^{2\pi} \frac{1}{a^2}\, i d\theta = \frac{\pi i}{a^2}$$

2 marks

Thus,

$$\frac{\pi i}{a^2}\left(2 - e^{-at}\right) = \int_{-\infty}^{\infty} \frac{e^{ixt}dx}{x(x^2+a^2)} + \frac{\pi i}{a^2}$$

2 marks

(f) and the inverse Fourier transform we seek is

1 mark

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{ixt}dx}{x(x^2+a^2)} = \frac{i}{2a^2}\left(1 - e^{-at}\right)$$

for $t > 0$.

Setter's initials dZh

Checker's initials RLJ

EXAMINATION QUESTIONS/SOLUTIONS 2006-07

Course ISE Sec B

Question | Marks & seen/unseen

Parts

5. (i) part seen

$$P(X < x) = \int_0^x \lambda e^{-\lambda x} = \left[-e^{-\lambda x}\right]_0^x = 1 - e^{-\lambda x} = 1 - e^{-0.1x}.$$ 3

(ii)

$$P(\text{excellent}) = P(X < 10) = 1 - e^{-0.1\times 10} = 0.6321,$$ 1

$$\begin{aligned} P(\text{good}) &= P(10 < X < 30) = P(X < 30) - P(X < 10) \\ &= 1 - e^{-0.1\times 30} - (1 - e^{-0.1\times 10}) = 0.3181, \end{aligned}$$ 2

$$P(\text{weak}) = P(X > 30) = 1 - P(X < 30) = e^{-0.1\times 30} = 0.0498.$$ 1

(iii) Let S = event that the file is successfully downloaded.

$$\begin{aligned} P(S) &= P(S \mid \text{excellent})P(\text{excellent}) + P(S \mid \text{good})P(\text{good}) \\ &\quad + P(S \mid \text{weak})P(\text{weak}) \\ &= 1.0 \times 0.6321 + 0.9 \times 0.3181 + 0.1 \times 0.0498 = 0.9234. \end{aligned}$$ 4

(iv)

$$\begin{aligned} P(\text{excellent} \mid S) &= \frac{P(S \mid \text{excellent})P(\text{excellent})}{P(S)} \\ &= \frac{1.0 \times 0.6321}{0.9234} = 0.6845. \end{aligned}$$ 2

(v) Let Y be the number of successfully downloaded files out of n attempts. Then $Y \sim \text{Bin}(n, 0.9234)$ and $P(Y = n) = (0.9234)^n$.

$$\begin{aligned} P(Y = n) = (0.9234)^n &> 0.5 \Rightarrow n\log(0.9234) < \log(0.5) \\ \Rightarrow n &< \log(0.5)/\log(0.9234) = 8.6977 \end{aligned}$$

The maximum number of files is 8. 4

(vi) Let F = event that the first unsuccessful download occurs at the nth download

unseen

$$\begin{aligned} P(F) &= P((n-1)\text{successful}) \times P(n\text{th unsuccessful}) \\ &= (0.9234)^{n-1}(1 - 0.9234) = (0.9234)^{n-1}0.0766. \end{aligned}$$ 3

Setter's initials GM | Checker's initials MJC | Page number 6

6. (i) Let μ_A and μ_B be the mean lifetime of components of types A and B respectively. — part seen ↓

The 95% CI for μ_A is

$$\left(\bar{x}_A \pm 1.96\sigma_A/\sqrt{n}\right) = (26 \pm 1.96/2) = (25.02, 26.98).$$ $\boxed{3}$

The 95% CI for μ_B is

$$\left(\bar{x}_B \pm 1.96\sigma_B/\sqrt{n}\right) = (30 \pm 1.96 \times 3/4) = (28.53, 31.47).$$ $\boxed{3}$

(ii) Reliability:

$$R(t) = \mathrm{P}(T > t) = \mathrm{P}\left(\frac{T-\mu}{\sigma} > \frac{t-\mu}{\sigma}\right) = 1 - \Phi\left(\frac{t-\mu}{\sigma}\right)$$ $\boxed{2}$

Hazard:

$$\begin{aligned} h(t) = \frac{f(t)}{R(t)} &= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2\right\} \Big/ \left(1 - \Phi\left(\frac{t-\mu}{\sigma}\right)\right) \\ &= \frac{\sigma^{-1}\phi((t-\mu)/\sigma)}{1 - \Phi((t-\mu)/\sigma)} \end{aligned}$$ $\boxed{2}$

(iii)

$$R_A(24) = 1 - \Phi\left(\frac{24-26}{2}\right) = 1 - \Phi(-1) = \Phi(1) = 0.841.$$ $\boxed{2}$

$$R_B(24) = 1 - \Phi\left(\frac{24-30}{3}\right) = 1 - \Phi(-2) = \Phi(2) = 0.977.$$ $\boxed{2}$

(iv) Potentially gives negative lifetimes! $\boxed{1}$

(v) Let N = event that the network is functioning after 24 hours, A and B_1, B_2, B_3, B_4 be the events that individual components are functioning after 24 hours, and $R_N(24)$ = the reliability of the network at $t = 24$ hours:

$$\begin{aligned} N &= A \cap ((B_1 \cap B_2) \cup (B_3 \cap B_4)) \\ \Rightarrow R_N(24) &= R_A(24)\left(R_B^2(24) + R_B^2(24) - R_B^4(24)\right) \\ &= 0.841\left(2 \times 0.977^2 - 0.977^4\right) \\ &= 0.839. \end{aligned}$$ method seen $\boxed{5}$