Imperial College
London

MATH97083

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2020

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Applied Probability

Date: 13th May 2020

Time: 13.00pm - 15.30pm (BST)

Time Allowed: 2 Hours 30 Minutes

Upload Time Allowed: 30 Minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS AS SEPARATE PDFs TO THE RELEVANT DROPBOXES ON BLACKBOARD (ONE FOR EACH QUESTION) WITH COMPLETED COVERSHEETS WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

We denote the natural numbers including 0 by $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

1. (a) Consider a discrete-time homogeneous Markov chain $(X_n)_{n\in\mathbb{N}_0}$ with state space $E = \{1,2,3,4,5,6,7,8\}$ and transition matrix given by

$$\mathbf{P} = \begin{pmatrix} \frac{1}{4} & 0 & \frac{3}{4} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

(i) Draw the transition diagram. (2 marks)

(ii) Specify the communicating classes and determine whether they are transient, null recurrent or positive recurrent. *Please note that you need to justify your answers.* (3 marks)

(iii) Find all stationary distributions. (5 marks)

(b) Suppose the influenza virus exists in K different strains, where $K \geq 2$. Each year, the virus either stays the same with probability $1-a$, for $a \in (0,1)$, or mutates to any of the other strains with equal probability. Suppose you can model the virus mutation by a discrete-time homogeneous Markov chain.

(i) We denote the state space by $E = \{1, \dots, K\}$. State the corresponding 1-step transition probabilities of the Markov chain. (3 marks)

(ii) You decide to group the states: You consider the modified state space $\widetilde{E} = \{I, O\}$ where I stands for the initial state and O for the collection of the other $K-1$ states.

(1.) State the corresponding 1-step transition probabilities of the Markov chain on $\widetilde{E}$. (2 marks)

(2.) Show that, for $n \in \mathbb{N}$,

$$p_{II}(n+1) = p_{II}(n)\left\{1 - a - \frac{a}{K-1}\right\} + \frac{a}{K-1},$$

and state all results from the lectures which you apply in your proof. (5 marks)

(Total: 20 marks)

2. (a) Let T be a nonnegative discrete random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $A \in \mathcal{F}$ be an event with $\mathbb{P}(A) > 0$. Show that

$$\mathbb{E}(T|A) = \sum_{n=1}^{\infty} \mathbb{P}(T \geq n|A).$$

(2 marks)

(b) Consider a discrete-time homogeneous Markov chain on a countable state space E. Suppose that the Markov chain is irreducible, has a stationary distribution denoted by $\boldsymbol{\pi}$ and all states are recurrent.

(i) Show that $\pi_i = \mu_i^{-1}$ for all $i \in E$, where μ_i denotes the mean recurrence time for state i. (5 marks)

(ii) Show that all states are positive recurrent. (3 marks)

(c) Consider a homogeneous Markov chain $(X_n)_{n\in\{0,1,2,\ldots,1000\}}$ with state space $E = \{1, 2, 3, 4\}$ and transition matrix given by

$$\mathbf{P} = \begin{pmatrix} 0.5 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0.5 \\ 0 & 0.5 & 0 & 0.5 \\ 0.5 & 0 & 0.5 & 0 \end{pmatrix}.$$

Answer the following questions about this Markov chain, justifying your answers.

(i) Is this Markov chain irreducible? (2 marks)

(ii) How many stationary distributions does this Markov chain have? Find all stationary distributions. (3 marks)

(iii) Is this Markov chain time-reversible? (5 marks)

(Total: 20 marks)

3. (a) Consider two independent and Poisson distributed random variables $X \sim \mathrm{Poi}(\lambda)$ and $Y \sim \mathrm{Poi}(\mu)$ with $\lambda, \mu > 0$. Show that

$$X \Big| X+Y = n \sim \mathrm{Bin}\left(n, \frac{\lambda}{\lambda+\mu}\right), \quad \text{for } n \in \mathbb{N}.$$

You may state and use without proof the distribution of $X+Y$. (4 marks)

(b) Let $(N_t)_{t\geq 0}$ denote a Poisson process with rate $\lambda > 0$. For $t_1 < t_2$, show that

$$N_{t_1} \Big| N_{t_2} = n \sim \mathrm{Bin}\left(n, \frac{t_1}{t_2}\right), \quad \text{for } n \in \mathbb{N}.$$

(5 marks)

(c) Let $(N_t)_{t\geq 0}$ denote a Poisson process with rate $\lambda > 0$. Let $(Z_i)_{i\in\mathbb{N}}$ denote independent and identically distributed random variables with Bernoulli distribution with parameter $p > 0$. Suppose that $(Z_i)_{i\in\mathbb{N}}$ and $(N_t)_{t\geq 0}$ are independent. For $t \geq 0$, define

$$X_t = \sum_{i=1}^{N_t} Z_i, \qquad Y_t = N_t - X_t.$$

(i) Show that $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ are Poisson processes with rates λp and $\lambda(1-p)$, respectively. (4 marks)

(ii) Also show that for any $t \geq 0$, X_t and Y_t are independent. (2 marks)

(d) Consider the Cramér-Lundberg model in insurance mathematics.

(i) State the model for the total claim amount. (3 marks)

(ii) Derive the cumulative distribution function of the total claim amount at a fixed point in time. (2 marks)

(Total: 20 marks)

4. (a) Let $N = (N_t)_{t\geq 0}$ and $M = (M_t)_{t\geq 0}$ denote independent Poisson processes with rates $\lambda > 0$ and $\mu > 0$, respectively. Let $T = \inf\{t \geq 0 : M_t = 1\}$ denote the random time when the first jump in M occurs. Determine $\mathbb{P}(N_{T/2} = k)$ for $k \in \mathbb{N}_0$ and name the distribution.

(7 marks)

(b) Consider a population of N individuals consisting at time 0 of one 'infective' and $N - 1$ 'susceptibles'. The process changes only by susceptibles becoming infective. If, at some time t, there are i infectives, then, for each susceptible, there is a probability of $i^2\lambda\delta + o(\delta)$ of becoming infective in $(t, t + \delta]$ for $\lambda, \delta > 0$.

(i) If we consider the event of becoming an infective as a birth, what is the birth rate λ_i of the process, when there are i infectives? (2 marks)

(ii) Let T denote the time to complete the epidemic, i.e. the first time when all N individuals are infective.

(1.) Derive $\mathbb{E}(T)$ (without using any type of generating functions). (2 marks)

(2.) Show that the Laplace transform of T is given by

$$\mathbb{E}[e^{-sT}] = \prod_{i=1}^{N-1} \left(\frac{\lambda_i}{\lambda_i + s}\right), \qquad \text{for } s \geq 0.$$

(2 marks)

(3.) Derive $\mathbb{E}(T)$ by using the Laplace transform given in (2.). (2 marks)

You may leave your solution in (1.) and (3.) as a sum.

(c) Give an example of two non-identical continuous-time Markov chains which have the same jump chain. (5 marks)

(Total: 20 marks)

Mastery question based on additional reading material.

5. Let $X = (X_t)_{t\geq 0}$ be a continuous-time Markov chain on a countable state space E with generator $\mathbf{G}$. We assume that the Markov chain is minimal.

(a) (i) Give a definition for the state $i \in E$ to be recurrent. (2 marks)

(ii) Give a definition for the state $i \in E$ to be transient. (1 mark)

(b) Suppose $E = \{1, 2, 3, 4\}$ and

$$\mathbf{G} = \begin{pmatrix} -1 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & -\frac{1}{2} & 0 & \frac{1}{4} \\ \frac{1}{6} & 0 & -\frac{1}{3} & \frac{1}{6} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

For each state in the state space, decide whether it is null/positive recurrent or transient and justify your answer.

(5 marks)

(c) Now let $E = \{1, 2\}$ and

$$\mathbf{G} = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}.$$

(i) Find the stationary distribution of X and justify your answer. (4 marks)

(ii) Find the stationary distribution of the jump chain associated with X. (2 marks)

(iii) Formulate a general result which describes the relationship between the stationary distributions of X and of its jump chain and show how the result applies in the context of the Markov chain considered in this question (part (c)). (4 marks)

(iv) Find the transition matrix $\mathbf{P}_t = (p_{ij}(t))_{i,j\in E}$ for all $t \geq 0$. (2 marks)
Hint: You may use without a proof that $\mathbf{G} = \mathbf{O}\mathbf{D}\mathbf{O}^{-1}$, where

$$\mathbf{O} = \begin{pmatrix} 1 & -\frac{1}{2} \\ 1 & 1 \end{pmatrix}, \qquad \mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & -3 \end{pmatrix}, \qquad \mathbf{O}^{-1} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{2}{3} \end{pmatrix}.$$

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2020

This paper is also taken for the relevant examination for the Associateship.

MATH96052/MATH97083

Applied Probability (Solutions)

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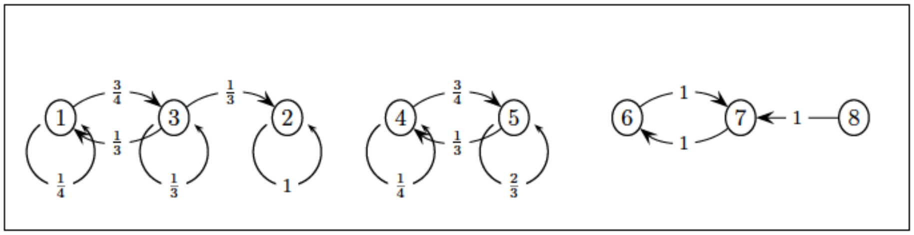
1. (a) (i) The transition diagram is given by meth seen $\Downarrow$



2,A

(ii) We have a finite state space which can be divided into five communicating classes: The classes $T_1 = \{1,3\}, T_2 = \{8\}$ are not closed and hence transient. 1,A
The classes $C_1 = \{2\}, C_2 = \{4,5\}, C_3 = \{6,7\}$ are finite and closed and hence positive recurrent. 2,A

(iii) This Markov chain does not have a unique stationary distribution $\boldsymbol{\pi}$ since we have three closed (essential) communicating classes. For the transient states we know from the lectures that $\pi_i = 0$ for $i = 1,3,8$. 1,A
For the positive recurrent states, we solve $\pi_2 \cdot 1 = \pi_2$, $(\pi_4, \pi_5) = (\pi_4, \pi_5)\begin{pmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$ and $(\pi_6, \pi_7) = (\pi_6, \pi_7)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, which leads to $\pi_2 = \pi_2$, $\pi_5 = \frac{9}{4}\pi_4$ and $\pi_6 = \pi_7$. 2,A

There are various ways of representing all possible stationary distributions (only one is needed!), e.g.:

* $\boldsymbol{\pi} = (0, \pi_2, 0, \pi_4, \frac{9}{4}\pi_4, \pi_6, \pi_6, 0)$ for all $\pi_2, \pi_4, \pi_6 \geq 0$ with $\pi_2 + \frac{13}{4}\pi_4 + 2\pi_6 = 1$,
* $\boldsymbol{\pi} = (0, \pi_2, 0, \frac{4}{9}\pi_5, \pi_5, \pi_6, \pi_6, 0)$ for all $\pi_2, \pi_5, \pi_6 \geq 0$ with $\pi_2 + \frac{13}{9}\pi_5 + 2\pi_6 = 1$,
* $\boldsymbol{\pi} = a(0,1,0,0,0,0,0,0) + b(0,0,0,\frac{4}{13},\frac{9}{13},0,0,0) + c(0,0,0,0,0,\frac{1}{2},\frac{1}{2},0)$ for all $a,b,c \geq 0$ with $a+b+c = 1$. 2,A

(b) (i) We have $p_{ii} = 1 - a$ for all $i \in E$ and we have that $p_{ij} = b$ for all $i \neq j$ for some $b \in (0,1)$, hence $1 = \sum_{j=1}^{K} p_{ij} = 1 - a + (K-1)b \Leftrightarrow b = \frac{a}{K-1}$, which implies that $p_{ij} = \frac{a}{K-1}$ for all $i,j \in E, i \neq j$. unseen $\Downarrow$ 3,B

(ii) (1.) Here the transition matrix corresponding to the Markov chain on $\widetilde{E} = \{I, O\}$ is given by

$$\mathbf{P} = \begin{pmatrix} 1-a & a \\ \frac{a}{K-1} & 1 - \frac{a}{K-1} \end{pmatrix}.$$

2,B

(2.) Let $n \in \mathbb{N}$. From the Chapman-Kolmogorov equations, we have that $\mathbf{P}^{n+1} = \mathbf{P}^n \mathbf{P}$, hence

$$\begin{aligned} p_{II}(n+1) &= p_{II}(n) \cdot p_{II} + p_{IO}(n) \cdot p_{OI}. \\ &= p_{II}(n) \cdot (1-a) + p_{IO}(n) \cdot \frac{a}{K-1}. \end{aligned}$$

2,C

Since $\mathbf{P}^n$ is a stochastic matrix, we have that $p_{II}(n) + p_{IO}(n) = 1 \Leftrightarrow p_{IO}(n) = 1 - p_{II}(n)$. 2,B

Hence

$$
\begin{aligned}
p_{II}(n+1) &= p_{II}(n)\cdot p_{II} + p_{IO}(n)\cdot p_{OI}\\
&= p_{II}(n)\cdot p_{II} + (1-p_{II}(n))\cdot p_{OI}\\
&= p_{II}(n)(p_{II}-p_{OI}) + p_{OI}\\
&= p_{II}(n)\left(1-a-\frac{a}{K-1}\right) + \frac{a}{K-1}.
\end{aligned}
$$

[2 marks for applying and mentioning the Chapman-Kolmogorov equations, 2 marks for using and mentioning that $\mathbf{P}^n$ is a stochastic matrix, 1 mark for deriving the final formula. If students made a mistake in the first part (1.) of the question and plugged in the wrong 1-step transition probabilities here, then they should not be penalised again, but can achieve full marks in this part of the question.] $\boxed{\text{1,B}}$

2. (a) We use the definition of the conditional expectation and the fact that $m = \sum_{n=0}^{m-1} 1$ to deduce that sim. seen ⇓

$$\mathbb{E}(T|A) = \sum_{m=0}^{\infty} m\mathbb{P}(T = m|A) = \sum_{m=0}^{\infty}\sum_{n=0}^{m-1} \mathbb{P}(T = m|A) = \sum_{n=0}^{\infty}\sum_{m=n+1}^{\infty} \mathbb{P}(T = m|A)$$
$$= \sum_{n=0}^{\infty} \mathbb{P}(T \geq n+1|A) = \sum_{n=1}^{\infty} \mathbb{P}(T \geq n|A).$$

2,A

(b) (i) Suppose that $X_0 \sim \boldsymbol{\pi}$ (i.e. $\mathbb{P}(X_0 = i) = \pi_i$ for each i). Let $T_i = \inf\{n \geq 1 : X_n = i\}$ denote the first hitting time for state $i \in E$, using part (a), we get $\pi_j \mu_j = \mathbb{P}(X_0 = j)\mathbb{E}(T_j|X_0 = j) = \sum_{n=1}^{\infty} \mathbb{P}(T_j \geq n|X_0 = j)\mathbb{P}(X_0 = j) = \sum_{n=1}^{\infty} \mathbb{P}(T_j \geq n, X_0 = j)$. seen ⇓ 2,D

Define $a_n := \mathbb{P}(X_m \neq j, 0 \leq m \leq n)$, for $n \in \mathbb{N}_0$.

Then $\mathbb{P}(T_j \geq 1, X_0 = j) = \mathbb{P}(X_0 = j)$ (since $T_j \geq 1$ by definition) and for $n \geq 2$

$$\begin{aligned}
&\mathbb{P}(T_j \geq n, X_0 = j) = \mathbb{P}(X_0 = j, X_m \neq j, 1 \leq m \leq n-1) \\
= \;& \mathbb{P}(X_m \neq j, 1 \leq m \leq n-1) - \mathbb{P}(X_m \neq j, 0 \leq m \leq n-1) \\
= \;& \mathbb{P}(X_m \neq j, 0 \leq m \leq n-2) - \mathbb{P}(X_m \neq j, 0 \leq m \leq n-1) \\
= \;& a_{n-2} - a_{n-1},
\end{aligned}$$

where we have used the homogeneity of the chain and the law of total probability. 2,D

Then, summing over n (noting that we are dealing with a telescoping sum) leads to

$$\begin{aligned}
\pi_j \mu_j &= \mathbb{P}(X_0 = j) + \sum_{n=2}^{\infty} (a_{n-2} - a_{n-1}) \\
&= \mathbb{P}(X_0 = j) + \mathbb{P}(X_0 \neq j) - \lim_{n\to\infty} a_n \\
&= 1 - \lim_{n\to\infty} a_n.
\end{aligned}$$

However, $\lim_{n\to\infty} a_n = \mathbb{P}(X_m \neq j, \ \forall m) = 0$ by the recurrence of j. That is, $\pi_j^{-1} = \mu_j$ if $\pi_j > 0$. 1,D

(ii) To see that $\pi_j > 0$ for all j, suppose the converse; then

$$0 = \pi_j = \sum_{i\in E} \pi_i p_{ij}(n) \geq \pi_i p_{ij}(n)$$

for all i, n, yielding that $\pi_i = 0$ whenever $i \to j$. However, the chain is irreducible, so that $\pi_i = 0$ for each i - a contradiction to the fact that $\boldsymbol{\pi}$ is a stationary vector. Thus $\mu_i < \infty$ and all states are positive. 3,C

(c) (i) Yes, this Markov chain is irreducible since all states communicate with each other. meth seen ⇓

2,A

(ii) Since the Markov chain is irreducible and the state space is finite, there is only one finite and closed communicating class, hence all states are positive recurrent. From a theorem in lectures we can then deduce that there is a unique stationary distribution. We observe that the transition matrix is doubly-stochastic and hence we know from lectures/problem class that the uniform distribution $\pi = (0.25, 0.25, 0.25, 0.25)$ is the unique stationary distribution of the Markov chain. [**Solving $\boldsymbol{\pi} = \pi \boldsymbol{P}$ for $\pi_i \geq 0, \sum_{i=1}^{4} \pi_i = 1$ to derive the stationary distribution is of course also a valid approach.**] [**1 mark for the correct answer and 2 marks for the justification.**] $\boxed{3,\mathrm{B}}$

(iii) From the lectures we know that the Markov chain is time-reversible if and only if the detailed-balance equations hold, i.e. $\pi_i p_{ij} = \pi_j p_{ji}$ for all $i, j \in \{1, 2, 3, 4\}$. Here we have that $\pi_i = \pi_j$ for all i, j, hence we need a symmetric transition matrix for the detailed-balance equations to hold. However, here we have e.g. $p_{14} = 0 \neq p_{41} = 0.5$, hence the detailed-balance equations do not hold and hence the Markov chain is not time-reversible. [**2 marks for "not time-reversible", 3 marks for justification**] $\boxed{5,\mathrm{B}}$

3. (a) We recall from lectures that $X + Y \sim \text{Poi}(\lambda + \mu)$. $\boxed{\text{unseen} \Downarrow}$

$\boxed{1,\text{A}}$

Let $n \in \mathbb{N}$ and $k \in \{0, 1, \ldots, n\}$, then, using Bayes' rule, we have

$$\mathbb{P}(X = k | X + Y = n) = \frac{\mathbb{P}(X + Y = n | X = k)\mathbb{P}(X = k)}{\mathbb{P}(X + Y = n)}$$
$$= \frac{\mathbb{P}(Y = n - k | X = k)\mathbb{P}(X = k)}{\mathbb{P}(X + Y = n)} \underset{\text{by independence of } X \text{ and } Y}{=} \frac{\mathbb{P}(Y = n - k)\mathbb{P}(X = k)}{\mathbb{P}(X + Y = n)}$$
$$= \frac{\mu^{n-k}}{(n-k)!} e^{-\mu} \frac{\lambda^k}{k!} e^{-\lambda} \left(\frac{(\lambda + \mu)^n}{n!} e^{-(\mu+\lambda)} \right)^{-1} = \binom{n}{k} \left(\frac{\lambda}{\lambda + \mu} \right)^k \left(\frac{\mu}{\lambda + \mu} \right)^{n-k},$$

which is indeed the probability mass function of a $\text{Bin}\left(n, \frac{\lambda}{\lambda+\mu}\right)$ random variable. $\boxed{3,\text{A}}$

(b) Since $(0, t_1], (t_1, t_2]$ are disjoint, the increments $N_{t_1} - N_0 = N_{t_1}, N_{t_2} - N_{t_1}$ are independent. Also, we have that $N_{t_1} \sim \text{Poi}(\lambda t_1), N_{t_2} \sim \text{Poi}(\lambda t_2), N_{t_2} - N_{t_1} \sim \text{Poi}(\lambda(t_2 - t_1))$. Hence we can apply part (a) with $X = N_{t_1}$ and $Y = N_{t_2} - N_{t_1}$ to conclude that $\boxed{\text{unseen} \Downarrow}$

$$N_{t_1} | N_{t_2} = n \sim \text{Bin}\left(n, \frac{\lambda t_1}{\lambda t_1 + \lambda(t_2 - t_1)}\right) \Leftrightarrow N_{t_1} | N_{t_2} = n \sim \text{Bin}\left(n, \frac{t_1}{t_2}\right).$$

$\boxed{5,\text{A}}$

(c) We answer (i) and (ii) jointly: Both $X = (X_t)_{t \geq 0}$ and $Y = (Y_t)_{t \geq 0}$ are stochastic processes. We need to check the defining properties of the Poisson process: $\boxed{\text{sim. seen} \Downarrow}$

* $X_0 = 0$ and $Y_0 = 0$ almost surely, since $N_0 = 0$ almost surely. $\boxed{1,\text{A}}$
* Independent increments: $(N_t)_{t \geq 0}$ has independent increments. By construction (the Z_is are all independent of each other and of (N_t)) $(X_t)_{t \geq 0}$ and $(Y_t)_{t \geq 0}$ inherit the independent increments property from $(N_t)_{t \geq 0}$.
* Stationary increments: $(N_t)_{t \geq 0}$ has stationary increments. By construction (the Z_is are all independent of each other and of (N_t) and identically distributed) $(X_t)_{t \geq 0}$ and $(Y_t)_{t \geq 0}$ inherit the stationary increments property from $(N_t)_{t \geq 0}$. $\boxed{1,\text{A}}$
* Let $n, m \in \mathbb{N}_0$, $t \geq 0$, then

$$\mathbb{P}(X_t = m, Y_t = n) = \mathbb{P}(X_t = m, N_t = m + n)$$
$$= \mathbb{P}(X_t = m | N_t = m + n)\mathbb{P}(N_t = m + n)$$
$$= \binom{m+n}{m} p^m (1-p)^n \frac{(\lambda t)^{m+n}}{(m+n)!} e^{-\lambda t}$$
$$= \frac{(\lambda t p)^m}{m!} e^{-\lambda t p} \frac{(\lambda t(1-p))^n}{n!} e^{-\lambda t(1-p)} = \mathbb{P}(X_t = m)\mathbb{P}(Y_t = n).$$

So, we found that X_t and Y_t are independent with $X_t \sim \text{Poi}(\lambda t p)$ and $Y_t \sim \text{Poi}(\lambda t(1-p))$ and X and Y are Poisson processes with rates λp and $\lambda(1-p)$, respectively. [**The final 4 marks are split as follows: 2 marks for the marginal distributions (possibly derived separately), 2 marks to show the independence.**] $\boxed{2,\text{A; } 2,\text{B}}$

(d) (i) The total claim amount in the Cramér-Lundberg model is defined as a compound Poisson process $(S_t)_{t\geq 0}$ satisfying $\boxed{\text{seen} \Downarrow}$

$$S_t = \begin{cases} \sum_{i=1}^{N_t} Y_i, & N_t > 0, \\ 0, & N_t = 0, \end{cases}$$

where $N = (N_t)_{t\geq 0}$ is a Poisson process of rate $\lambda > 0$ and the claim size process is denoted by $Y = (Y_i)_{i\in\mathbb{N}}$, where the Y_i denote positive i.i.d. random variables with finite mean and finite variance. Also, N and Y are assumed to be independent of each other. [**Any alternative correct definition of the (compound) Poisson process should be awarded full marks.**] $\boxed{3,\text{A}}$

(ii) Let $t \geq 0$. Then $F_{S_t}(x) = \mathbb{P}(S_t \leq x) = 0$ for $x < 0$. Let $x \geq 0$, then

$$\begin{aligned} F_{S_t}(x) = \mathbb{P}(S_t \leq x) &= \sum_{n=0}^{\infty} \mathbb{P}(S_t \leq x, N_t = n) = \sum_{n=0}^{\infty} \mathbb{P}\left(\sum_{i=1}^{n} Y_i \leq x, N_t = n\right) \\ &= \sum_{n=0}^{\infty} \mathbb{P}\left(\sum_{i=1}^{n} Y_i \leq x\right) \mathbb{P}(N_t = n) = \sum_{n=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!} \mathbb{P}\left(\sum_{i=1}^{n} Y_i \leq x\right). \end{aligned}$$

$\boxed{2,\text{A}}$

4. (a) Since N and M are independent, we deduce that N and T are independent. From lectures we know that $T \sim \text{Exp}(\mu)$. unseen $\Downarrow$

2,D

Then, using the continuous version of the law of total probability, we have for $k \in \mathbb{N}_0$:

$$\mathbb{P}(N_{T/2} = k) = \int_{-\infty}^{\infty} \mathbb{P}(N_{T/2} = k | T = t) f_T(t) dt = \int_0^{\infty} \mathbb{P}(N_{t/2} = k | T = t) \mu e^{-\mu t} dt$$
$$\overset{N,T \text{ independent}}{=} \int_0^{\infty} \mathbb{P}(N_{t/2} = k) \mu e^{-\mu t} dt = \int_0^{\infty} \frac{(\lambda t/2)^k}{k!} e^{-\lambda t/2} \mu e^{-\mu t} dt.$$

2,D

We change the variables $z = (\lambda/2 + \mu)t$ and use the fact that $\int_0^{\infty} z^k e^{-z} dz = \Gamma(k+1) = k!$ to deduce that

$$\mathbb{P}(N_{T/2} = k) = \frac{\mu}{k!} \left(\frac{\lambda}{2}\right)^k \left(\frac{\lambda}{2} + \mu\right)^{-(k+1)} \int_0^{\infty} z^k e^{-z} dz$$
$$= \mu \left(\frac{\lambda}{2}\right)^k \left(\frac{\lambda}{2} + \mu\right)^{-(k+1)} = \left(\frac{\lambda}{\lambda + 2\mu}\right)^k \left(\frac{2\mu}{\lambda + 2\mu}\right).$$

2,D

Hence $N_{T/2}$ has a geometric distribution with parameter $\frac{2\mu}{\lambda+2\mu}$. 1,D

(b) (i) If there are i infectives, then there are $N - i$ susceptibles and hence the birth rate is given by $\lambda_i = (N-i)i^2\lambda$ if $i = 1, \ldots, N-1$ and 0 otherwise. sim. seen $\Downarrow$

(ii) (1.) Let X_i be the time spent in state i (where i denotes the number of infectives), then we have that the time to complete the epidemic is 2,B

$$T = X_1 + \cdots + X_{N-1},$$

where the X_i are independent of each other with $X_i \sim \text{Exp}(\lambda_i)$. By the linearity of the expectation,

$$\mathbb{E}[T] = \sum_{i=1}^{N-1} \mathbb{E}(X_i) = \sum_{i=1}^{N-1} \frac{1}{\lambda_i} = \sum_{i=1}^{N-1} \frac{1}{(N-i)i^2\lambda}.$$

2,C

(2.) Let $s \geq 0$. Using the notation from (1.), the Laplace transform of X_i is given by

$$\mathbb{E}(e^{-sX_i}) = \int_0^{\infty} e^{-sx} \lambda_i e^{-\lambda_i x} dx = \lambda_i \int_0^{\infty} e^{-(s+\lambda_i)x} dx = \frac{\lambda_i}{\lambda_i + s}.$$

Hence

$$\mathbb{E}[e^{-sT}] = \mathbb{E}[e^{-s\sum_{i=1}^{N-1} X_i}] \overset{\text{independence of } X_i s}{=} \prod_{i=1}^{N-1} \mathbb{E}[e^{-sX_i}] = \prod_{i=1}^{N-1} \left(\frac{\lambda_i}{\lambda_i + s}\right).$$

2,C

(3.) To compute the expectation, we calculate the logarithm of the Laplace transform and use the fact that

$$\mathbb{E}[T] = -\frac{d}{ds}\left[\log\left\{\mathbb{E}[e^{-sT}]\right\}\right]\Bigg|_{s=0}.$$

Here $\log\left\{\mathbb{E}[e^{-sT}]\right\} = \sum_{i=1}^{N-1}\log\left(\frac{\lambda_i}{\lambda_i+s}\right)$ and 1,C

$$\frac{d}{ds}\log\left\{\mathbb{E}[e^{-sT}]\right\} = \sum_{i=1}^{N-1}\frac{(\lambda_i+s)}{\lambda_i}\cdot\frac{(\lambda_i+s)\cdot 0-\lambda_i\cdot 1}{(\lambda_i+s)^2} = -\sum_{i=1}^{N-1}\frac{1}{(\lambda_i+s)}.$$

Hence,

$$\mathbb{E}[T] = \sum_{i=1}^{N-1}\frac{1}{\lambda_i} = \sum_{i=1}^{N-1}\frac{1}{(N-i)i^2\lambda}.$$

1,C

(c) Many examples are possible: For instance, take a non-explosive birth process starting at 0 with rates $\lambda_i > 0$ for $i \in \mathbb{N}_0$ and a Poisson process of rate $\lambda > 0$. Both processes are continuous-time Markov chains on $\mathbb{N}_0$. meth seen $\Downarrow$ 3,D

Then the jump chain for both processes, denoted by $(Z_n)_{n\in\mathbb{N}_0}$, is given by $Z_n = n$ for $n \in \mathbb{N}_0$ and its transition probabilities are given by $p_{ij}^Z = 1$ when $j = i+1$ and 0 otherwise (for $i, j \in \mathbb{N}_0$), since both continuous-time Markov chains are non-decreasing processes which can only jump up by one step at a time. 1,D 1,C

Alternatively, one could consider Poisson processes of different rates etc.

Mastery question based on additional reading material. unseen $\Downarrow$

5. (a) (i) We say that state $i \in E$ is recurrent if $\mathbb{P}(\{t \geq 0 : X_t = i\}$ is unbounded $|X_0 = i) = 1$. 2

(ii) We say that state $i \in E$ is transient if $\mathbb{P}(\{t \geq 0 : X_t = i\}$ is unbounded $|X_0 = i) = 0$. 1

(b) We derive the transition matrix of the corresponding jump chain:

$$\mathbf{P} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We observe that the jump chain has two communicating classes: $T = \{1, 2, 3\}, C = \{4\}$. T is not closed, hence all states in T are transient. C is finite and closed, hence state 4 is (positive) recurrent. 3

We know that if a state is recurrent (transient) for the jump chain, then it is recurrent (transient) for the continuous-time Markov chain. So we conclude that states 1, 2, 3 are transient and state 4 is recurrent for the continuous-time Markov chain. Moreover, since $g_{44} = 0$, we have that state 4 is positive recurrent. 2

(c) (i) We denote the stationary distribution of X by $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$. We solve $\boldsymbol{\lambda}\mathbf{G} = \mathbf{0}$, for $\lambda_1, \lambda_2 \geq 0, \lambda_1 + \lambda_2 = 1$. Here we have $-\lambda_1 + 2\lambda_2 = 0 \Leftrightarrow \lambda_1 = 2\lambda_2$. Then $1 = \lambda_1 + \lambda_2 \Leftrightarrow 1 = 3\lambda_2 \Leftrightarrow \lambda_1 = \frac{2}{3}, \lambda_2 = \frac{1}{3}$. 2

Since $\mathbf{G}$ is irreducible and recurrent, we get that $\boldsymbol{\lambda}\mathbf{G} = \mathbf{0} \Leftrightarrow \boldsymbol{\lambda}\mathbf{P}_t = \boldsymbol{\lambda}$ for all $t \geq 0$, where $(\mathbf{P}_t)_{t\geq 0}$ denotes the matrix of transition probabilities associated with X. 2

(ii) The transition matrix of the associated jump chain is given by

$$\mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We denote the stationary distribution of the jump chain by $\boldsymbol{\pi} = (\pi_1, \pi_2)$. We solve $\boldsymbol{\pi}\mathbf{P} = \boldsymbol{\pi}$, for $\pi_1, \pi_2 \geq 0, \pi_1 + \pi_2 = 1$. Here we have $\pi_1 = \pi_2$. Then $1 = \pi_1 + \pi_2 \Leftrightarrow \pi_1 = \frac{1}{2}, \pi_2 = \frac{1}{2}$. 2

(iii) Let $\mathbf{G}$ be a generator and let $\mathbf{P}$ denote the transition matrix of the associated jump chain. Let $\boldsymbol{\lambda}$ be a measure. Then $\boldsymbol{\lambda}$ satisfies $\boldsymbol{\lambda}\mathbf{G} = \mathbf{0}$ if and only if $\boldsymbol{\mu}\mathbf{P} = \boldsymbol{\mu}$ where $\mu_i = \lambda_i \cdot (-g_{ii})$ for all $i \in E$. 3

In our case, we can obtain $\boldsymbol{\mu}$ as follows: $\mu_1 = \lambda_1 \cdot (-g_{11}) = \frac{2}{3} \cdot 1 = \frac{2}{3}, \mu_2 = \lambda_2 \cdot (-g_{22}) = \frac{1}{3} \cdot 2 = \frac{2}{3}$, i.e. $\mu_1 = \mu_2$. When we normalise, we obtain that $\boldsymbol{\pi} = (0.5, 0.5)$ is the stationary distribution of the jump chain. 1

(iv) From lectures, we know that, for $t \geq 0$, we have $\mathbf{P}_t = e^{t\mathbf{G}} = \sum_{n=0}^{\infty} \frac{t^n}{n!}\mathbf{O}\mathbf{D}^n\mathbf{O}^{-1} = \mathbf{O}\sum_{n=0}^{\infty} \frac{t^n}{n!}\mathbf{D}^n\mathbf{O}^{-1}$. Hence

$$\mathbf{P}_t = \mathbf{O}\begin{pmatrix} e^{t\cdot 0} & 0 \\ 0 & e^{-3t} \end{pmatrix}\mathbf{O}^{-1} = \begin{pmatrix} \frac{2}{3} + \frac{1}{3}e^{-3t} & \frac{1}{3} - \frac{1}{3}e^{-3t} \\ \frac{2}{3} - \frac{2}{3}e^{-3t} & \frac{1}{3} + \frac{2}{3}e^{-3t} \end{pmatrix}.$$

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