

## SOLUTIONS: Feedback Systems

1. a) i) The transfer function for the circuit in Figure 1.2 is given by

$$\frac{Z_f(s)}{Z_i(s)} = -\frac{C_i(s+1/R_i C_i)}{C_f(s+1/R_f C_f)} = -\frac{C_i s + 1/R_i}{C_f s + 1/R_f}$$

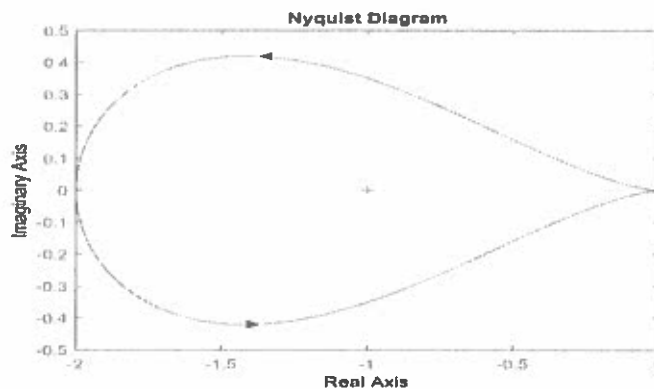
Putting in the values:  $G(s) = G_1(s)G_2(s) = \frac{1}{(s+1)(s+2)}$

- ii) The DC gain of  $G(s)$  is therefore  $G(0) = 0.5$ .

iii)  $y_{ss} := \lim_{t \rightarrow \infty} y(t) = \lim_{s \rightarrow 0} s y(s) = \lim_{s \rightarrow 0} s G(s) u(s) = G(0) = 0.5$ .

iv)  $\lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} s e(s) = \lim_{s \rightarrow 0} \frac{s r(s)}{1 + K_p G(s)} = \frac{1}{1 + K_p G(0)} \leq .01 \Rightarrow K_p \geq 198$ .

- b) i) The Nyquist diagram is shown below.



- ii) The Nyquist criterion:  $N = Z - P$  where  $N$  is the number of clockwise encirclements of  $-1/K_p$ ,  $Z$  is the number of unstable closed loop poles and  $P$  is the number of unstable open loop poles ( $= 1$ ). So for:

- $0 < K_p < 0.5$ ,  $N = 0$  and so  $Z = 1$ ,
- $0.5 < K_p < \infty$ ,  $N = -1$  and so  $Z = 0$
- $-\infty < K_p < 0$ ,  $N = 0$  and so  $Z = 1$ .
- when  $K_p = 0.5$  the closed loop is marginally stable.

- iii) For  $K_p = 1$ , the closed loop is stable. The loop gain can be increased without bound and reduced by up to 50% without losing stability.

- iv) The closed loop transfer function, DC gain and damping ratio are

$$H(s) = \frac{4K_p}{s^2 + s + 2(2K_p - 1)}, \quad H(0) = \frac{2K_p}{2K_p - 1}, \quad \zeta = \frac{1}{2\sqrt{2(2K_p - 1)}}$$

When  $r(t)$  is a unit step, good steady-state response requires  $H(0) \sim 1$ , or equivalently, large  $K_p$ . However, large  $K_p$  will result in small  $\zeta$ , which results in large overshoot and an oscillatory response.

2. a) The Nyquist diagram is shown in the next page. The real-axis intercepts can be found from the Routh array. The characteristic equation is

$$1 + K_p G(s) = 0 \Rightarrow s^2 + (3 - K_p)s + 2 + 3K_p = 0$$

The Routh array

$$\begin{array}{c|cc} s^2 & 1 & 2 + 3K_p \\ s^1 & 3 - K_p & \\ 1 & 2 + 3K_p & \end{array}$$

The values  $K_p = 3$  and  $K_p = -2/3$  result in a zero row, and so the real-axis intercepts are obtained as  $-1/K_p$  and the corresponding frequencies are obtained from the auxiliary polynomials and are  $-1/3, \omega = \sqrt{11}$  and  $3/2, \omega = 0$ .

- b) The gain margin = 3 from the Routh array. For the phase margin, we first find the real frequency  $\omega$  such that  $|G(j\omega)| = 1$ , or

$$9 + \omega^2 = (1 + \omega^2)(4 + \omega^2) \Leftrightarrow \omega^4 + 4\omega^2 - 5 = 0 \Leftrightarrow (\omega^2 - 1)(\omega^2 + 5) = 0 \Rightarrow \omega = \pm 1.$$

Therefore the cross-over frequency  $\omega_c = 1$  and the angle of  $G(j\omega_c) = -j$  is  $-90^\circ$  and so the phase margin =  $90^\circ$ . The stability margins are adequate so the design is robust against uncertainties in the gain and phase of  $G(s)$ . Furthermore, we expect the transient response to be acceptable since the Nyquist plot is not too close to the point  $-1$ .

- c) When  $K(s) = 1$ , the closed loop transfer function is

$$H(s) = \frac{G(s)}{1 + G(s)} = \frac{3 - s}{s^2 + 2s + 5} \Rightarrow H(0) = 0.6.$$

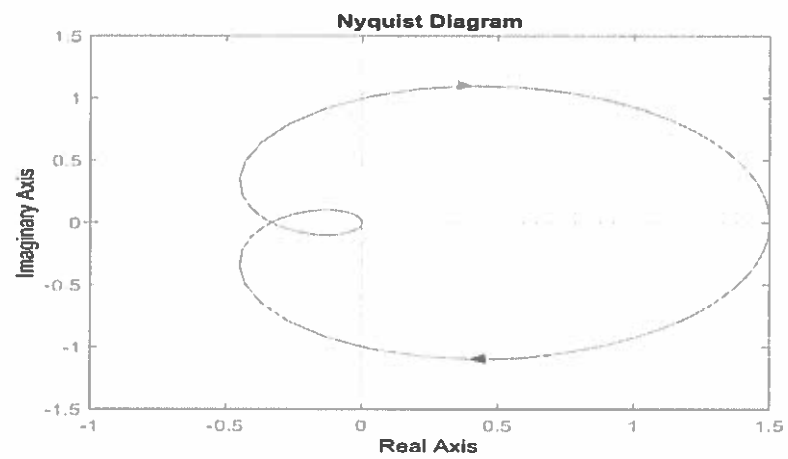
Therefore, the steady-state output is 0.6. Since the reference signal is 1,

the steady-state response of the system is not adequate.

- d) The Nyquist criterion:  $N = Z - P$  where  $N$  is the number of clockwise encirclements of  $-1/K_p$ ,  $Z$  is the number of unstable closed loop poles and  $P$  is the number of open loop poles ( $= 0$ ). So for

- $0 < K_p < 3$ ,  $N = 0$  and so  $Z = 0$ ,
- $3 < K_p < \infty$ ,  $N = 2$  and so  $Z = 2$ ,
- $-\infty < K_p < -2/3$ ,  $N = 1$  and so  $Z = 1$ ,
- $-2/3 < K_p < 0$ ,  $N = 0$  and so  $Z = 0$ ,
- when  $K_p = 3$  and  $K_p = -2/3$  the closed loop is marginally stable.

- e) Note from Parts b and c that the transient response is adequate while the steady-state response is not. Hence we need to improve the steady-state performance. Since phase-lag compensation increases low frequency gain, and so improves steady-state tracking it follows that the system requires phase-lag compensation.



3. a) For a critically damped response with a settling time of approximately 1 s, the poles must be repeated and located at  $p_1 = p_2 = -4$ .

- b) The closed-loop characteristic equation is  $1 + K_p G(s) = 0$  or

$$s^2 + 2s + K_p(s + 8/3) = s^2 + (2 + K_p)s + 8K_p/3 = 0.$$

- c) The Routh array is given by

$$\begin{array}{c|cc} s^2 & 1 & 8K_p/3 \\ s & 2 + K_p & \\ 1 & 8K_p/3 & \end{array}$$

Thus for closed-loop stability,  $K_p > 0$  and  $K_p > -2$ , or  $K_p > 0$ .

- d) To achieve the design specifications,  $s_1 = s_2 = -4$  must be roots of the characteristic equation. It follows that we must have

$$s^2 + (2 + K_p)s + 8K_p/3 = (s + 4)^2 = s^2 + 8s + 16.$$

Therefore  $K_p = 6$ .

- e) Here  $r(s) = 1/s^2$  and  $e(s) = r(s)/(1 + 6G(s))$ . Using the final value theorem of the Laplace transform:

$$\lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} s e(s) = 1/8.$$