IMPERIAL COLLEGE LONDON

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SC4

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2008

MSc and EEE PART IV: MEng and ACGI

PROBABILITY AND STOCHASTIC PROCESSES

Corrected Copy

Q3.

Thursday, 1 May 2:30 pm

Time allowed: 3:00 hours

There are SIX questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : K.K. Leung

Second Marker(s) : R.B. Vinter

Special Instructions for Invigilator: **None**

Information for Students: **Complementary Normal Distribution**

$$Q(x) = 1 - \Phi(x) = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dx$$

If needed, for any y different from all the x values given below, $Q(y)$ can be approximated by linear interpolation of the values of $Q(x)$ at the two x values closest to y.

x	$Q(x)$	x	$Q(x)$
0	5.00e-01	2.7	3.47e-03
0.1	4.60e-01	2.8	2.56e-03
0.2	4.21e-01	2.9	1.87e-03
0.3	3.82e-01	3.0	1.35e-03
0.4	3.45e-01	3.1	9.68e-04
0.5	3.09e-01	3.2	6.87e-04
0.6	2.74e-01	3.3	4.83e-04
0.7	2.42e-01	3.4	3.37e-04
0.8	2.12e-01	3.5	2.33e-04
0.9	1.84e-01	3.6	1.59e-04
1.0	1.59e-01	3.7	1.08e-04
1.1	1.36e-01	3.8	7.24e-05
1.2	1.15e-01	3.9	4.81e-05
1.3	9.68e-02	4.0	3.17e-05
1.3	8.08e-02	4.5	3.40e-06
1.5	6.68e-02	5.0	2.87e-07
1.6	5.48e-02	5.5	1.90e-08
1.7	4.46e-02	6.0	9.87e-10
1.8	3.59e-02	6.5	4.02e-11
1.9	2.87e-02	7.0	1.28e-12
2.0	2.28e-02	7.5	3.19e-14
2.1	1.79e-02	8.0	6.22e-16
2.2	1.39e-02	8.5	9.48e-19
2.3	1.07e-02	9.0	1.13e-19
2.4	8.20e-03	9.5	1.05e-21
2.5	6.21e-03	10.0	7.62e-24
2.6	4.66e-03		

1. a. A factory produces a mix of "good" and "bad" communication devices. The lifetime in seconds of "good" and "bad" devices is characterized by probability distribution functions (PDF's), $F_g(t)$ and $F_b(t)$, respectively. The respective probability of a randomly selected device being "good" and "bad" is p and $1-p$.

   i. Find the probability that a randomly selected device still functions after t seconds. [4]

   ii. To "weed out" the bad devices, every device is tested for t seconds. The devices that fail the test are discarded and only the "surviving" devices are sent to customers. For a target of 99% of the devices sent out are "good", find an expression that relates t to p, $F_g(t)$ and $F_b(t)$. (Hint: Use Bayes' rule.) [9]

   iii. If both PDF's, $F_g(t)$ and $F_b(t)$, are exponential distributions with rate λ and 1000λ, respectively, solve for the required testing time t from part ii. (Your formula for t will depend on λ and p.) [4]

b. Let $Y \equiv kX$ where k is a constant and X is a scalar random variable with probability density function (pdf) $f_X(x)$ and Laplace transform (L.T.) $F_X^*(s)$.

   i. Find the pdf $f_Y(y)$ for Y. [4]

   ii. Let $F_Y^*(s)$ be the L.T. of $f_Y(y)$. Express $F_Y^*(s)$ in terms of $F_X^*(s)$. [4]

2. a. Consider a random variable X that takes on an integer value k with probability

$$P_k = \frac{\alpha^k}{k!} e^{-\alpha} \qquad k = 0, 1, 2, \dots.$$

where $\alpha > 0$ is a constant. Derive the probability generating function $G^*(z)$ for X. Hence, or otherwise, obtain the mean and variance of X.

[9]

b. Consider K independent Poisson (arrival) processes. For each $i = 1$ to K, the i^{th} process has n arrivals during a time duration t with probability

$$P_n = \frac{(\lambda_i t)^n}{n!} e^{-\lambda_i t} \qquad n = 0, 1, 2, \dots.$$

where λ_i is the arrival rate for the i^{th} process. Now let us merge all K processes into a single process, i.e., a new process for which the number of arrivals within a given time period is the sum of the arrivals from all the K processes in the same period. Show that the merged process is also Poisson with rate $\lambda = \lambda_1 + \lambda_2 + \dots + \lambda_K$.

[9]

(Hint: To begin, consider the merging of two Poisson processes.)

c. Let X be a scalar random variable and $\Phi(s) = E[e^{sX}]$ where s can be real or complex.

i. Show that for any $\alpha > 0$ and for any real s,

$$P[e^{sX} \geq \alpha] \leq \frac{\Phi(s)}{\alpha}.$$

[4]

ii. For any real value x and $s > 0$, show that

$$P[X \geq x] \leq e^{-sx}\Phi(s).$$

[3]

3. a. Consider four scalar random variables, X, Y, U and V, which are related by the following relationships:

$$U = XY$$
$$V = X$$

Let $f_{XY}(x,y)$ be the joint probability density function (pdf) for X and Y. Furthermore, let $f_{UV}(u,v)$ denote the joint pdf for U and V, and $f_U(u)$ be the marginal pdf for U.

i. Express $f_{UV}(u,v)$ in terms of $f_{XY}(.,.)$. [7]

ii. Find an expression for $f_U(u)$. [3]

iii. Now assume that X and Y are independent and each is uniformly distributed between 0 and 1. Find a closed-form expression for $f_U(u)$ in this case. [5]

3.40 pm

b. Let $X(n)$ for n being an integer from $-\infty$ to $+\infty$ be a wide-sense stationary (WSS) process with an autocorrelation function $R(\tau) = e^{-\tau^2}$. We seek an estimate $\widetilde{X}(n)$ of $X(n)$ given $X(n-1)$ and $X(n-2)$ of the following form:

$$\widetilde{X}(n) = aX(n-2) + bX(n-1).$$

i. Find the constants a and b that minimize the mean square error. [6]

ii. Find the mean square error for such an estimate. [4]

4. Let $X(t)$ be a scalar wide-sense stationary (WSS) process with a normal distribution, zero mean $E[X(t)]=0$ for all t and an autocorrelation function $R(\tau)=e^{-2|\tau|}$. We further assume that the joint process of $\{X(t), X(t+\tau)\}$ has a jointly normal distribution for all t and τ.

a. Find the variance of $X(t)$. [2]

b. Express the probability $P[X(t)\le 2]$ as a function of $F(x)$ where

$$F(x)=\int_{-\infty}^{x}\frac{1}{\sqrt{2\pi}}e^{-y^2/2}dy.$$ [4]

c. Find $E[X(t+\tau)+X(t)]$ for any real τ. [2]

d. Find $E\{[X(t+\tau)+X(t)]^2\}$ for any real τ. [4]

e. Prove that the random variable $X(t+\tau)+X(t)$ has a normal distribution. [7]

Hint: The jointly normal density function for two random variables, *X* and *Y*, each of which has a marginal zero mean and unit variance, and their correlation coefficient denoted by ρ is:

$$f(x,y)=\frac{1}{2\pi\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[x^2-2\rho xy+y^2\right]\right\}$$

f. Derive the probability $P[|X(t+\tau)+X(t)|\le 1]$ as a function of $F(x)$ defined in part b.

[6]

Hint: Apply results in parts c to e.

5. a. The times between two consecutive events in a random experiment are independently, identically distributed (iid) with mean m and variance m^2. Let S_n denote the time when the n^{th} event occurs.

   i. Find the mean and variance of S_n, denoted by $E[S_n]$ and $VAR[S_n]$, respectively. [3]

   ii. Recall that S_{1000} denotes the time when the 1000th event occurs. Using the complementary normal distribution given at the beginning of this examination paper, determine the probability for $950\,m < S_{1000} \leq 1050\,m$. [5]

b. Consider a renewal process S_n for $n = 1, 2, 3, \ldots$ Let $\{X_i, i = 1, 2, 3, \ldots\}$ be a sequence of non-negative independent random variables with a common probability distribution function (PDF) $F(t)$, where X_i represents the time duration between the *(n-1)*st and n^{th} arrival (renewal) points. Let $\mu = E[X_i]$ for all $i = 1, 2, 3, \ldots$, $S_0 \equiv 0$ and

$S_n = \sum_{i=1}^{n} X_i$ for $n = 1, 2, 3, \ldots$ That is, S_n is the time when the n^{th} arrival occurs. Furthermore, let $N(t)$ denote the number of arrivals in the time duration $(0, t]$.

   i. Show that $P[N(t) = n] = F_n(t) - F_{n+1}(t)$ where $F_n(t)$ is the n-fold convolution of $F(t)$.

   [5]

   ii. Show that $E[N(t)] = \sum_{n=1}^{\infty} F_n(t)$. [5]

   iii. Show that with probability 1,

$$\frac{N(t)}{t} \to \frac{1}{\mu} \quad \text{as } t \to \infty.$$ [7]

6. Consider a discrete-time Markov chain with transition probabilities from state i to state j:

$$p_{ij} = e^{-\lambda} \sum_{n=0}^{j} \binom{i}{n} p^n q^{i-n} \frac{\lambda^{j-n}}{(j-n)!}$$

where i and $j = 0, 1, 2, \ldots$, $0 < p < 1$ and $p + q = 1$.

(Note: Don't be scared by the apparent complexity of the expression for p_{ij}. The results are in a simple, closed form. In addition, it is worth noting that $\binom{i}{n}$ represents the number of combinations in choosing n out of i, and is zero for $n > i$.)

a. Let π_i denote the equilibrium probability of state i. Write a set of equations expressing each π_i in terms of p_{ij} and π_j for $j = 0, 1, 2, \ldots$. [4]

b. Let $Q(z)$ be the probability generating function (PGF) for π_i for $i = 0, 1, 2, \ldots$ That is,

$$Q(z) = \sum_{i=0}^{\infty} \pi_i z^i .$$

From result in part a, find an expression relating $Q(z)$ to $Q(1 + p(z-1))$. [8]

c. Recursively (i.e., repeatedly) apply the result in part b to itself and show that the nth recursion gives

$$Q(z) = e^{\lambda(z-1)(1+p+p^2+\ldots+p^{n-1})} Q(1 + p^n(z-1)).$$

(Hint: Use mathematical induction.) [7]

d. By considering the result in part c in the limit of $n \to \infty$, find $Q(z)$ and then π_i for $i = 0, 1, 2, \ldots$ by expansion and inspection.

(Hint: Use the fact that $0 < p < 1$.) [6]