IMPERIAL COLLEGE LONDON

E4.08
SO14
ISE4.33

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2004

MSc and EEE/ISE PART IV: MEng and ACGI

DIGITAL IMAGE PROCESSING

Thursday, 13 May 10:00 am

Time allowed: 3:00 hours

There are FOUR questions on this paper.

Answer THREE questions.

Corrected Copy

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s) :	T. Stathaki
	Second Marker(s) :	T.J.W. Clarke

1. (a) (i) A real digital image $f(x,y)$ of size $M \times N$ has Discrete Fourier Transform $F(u,v)$. $G(u,v)$ is created by the algorithm $G(u,v) = F(u,v)$ followed by $G(u_0,v_0) = 0$ and $G(M-u_0, N-v_0) = 0$, where (u_0,v_0) is a specific two-dimensional frequency pair. A new image $g(x,y)$ is created by taking the Inverse Discrete Fourier Transform of $G(u,v)$. Find the difference $f(x,y) - g(x,y)$.

[3]

(ii) Find the energy in an image of size $M \times N$ in a given band of frequencies $u_1 \le u \le u_2$, $v_1 \le v \le v_2$, as a percentage of the total energy. **[3]**

(b) Let $f(x,y)$ denote an $M \times M$ -point two-dimensional (2-D) sequence (M is a positive power of 2). In implementing the 2-D Discrete Walsh Transform (DWT) of $f(x,y)$, we relate $f(x,y)$ to a new $M \times M$ -point sequence $W(u,v)$.

(i) Define the sequence $W(u,v)$ in terms of $f(x,y)$. **[1]**

(ii) Find the Walsh Transform of the following image at points $(0,0)$ and $(3,0)$.

$$\begin{bmatrix} 5 & 6 & 8 & 10 \\ 6 & 6 & 5 & 7 \\ 4 & 5 & 3 & 6 \\ 8 & 7 & 5 & 5 \end{bmatrix}$$

[4]

(c) Consider the population of vectors $\underline{f}$ of the form

$$\underline{f} = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix}.$$

Each component $f_i(x,y), i = 1,2$ represents an image. The population arises from the formation of the vectors across the entire collection of pixels.

Consider now a population of vectors $\underline{g}$ of the form

$$\underline{g} = \begin{bmatrix} g_1(x,y) \\ g_2(x,y) \end{bmatrix}$$

where the vectors $\underline{g}$ are the Karhunen-Loeve transforms of the vectors $\underline{f}$.

The mean of the population $\underline{f}$ calculated as part of the transform is

$$\underline{m}_{\underline{f}} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}.$$

The covariance matrix of the population $\underline{f}$ calculated as part of the transform is

$$\underline{C}_{\underline{f}} = \frac{1}{25}\begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$$

with eigenvalues $\lambda_1 = 0.2331$ and $\lambda_2 = 0.0069$.

(i) Find the vectors $\underline{g}$. **[5]**

(ii) Find the covariance matrix of the population $\underline{g}$. **[2]**

(iii) Suppose that a credible job could be done of reconstructing approximations to the two original images by using one principal component image. What would be the mean square error incurred in doing so? Express your answer as a percentage of the maximum possible error. **[2]**

2. (a) Consider a 64×64 image with 4 grey levels. The normalised grey levels are 0, $1/3$, $2/3$ and 1. The number of pixels with the corresponding grey levels, are shown in the following table.

Grey level	Number of pixels
0	1813
1/3	1506
2/3	574
1	203

(i) Draw the histogram of the image.

[1]

(ii) Determine the equalised histogram.

[2]

(iii) Draw the equalised histogram.

[1]

(b) What happens if you apply histogram equalisation twice to the same image? Justify your answer.

[4]

(c) After histogram equalisation will an image have more, the same or fewer distinct grey levels? Justify your answer.

[4]

(d) A mean filter is a linear filter but a median filter is not. Why? Justify your answer.

[4]

(e) Let $f(x,y)$ denote an $M \times N$ image. Suppose that the pixel intensities r are represented by 8 bits. Moreover the histogram $h(r)$ of the image is available. Find the value of $\sum_{u=0}^{M-1}\sum_{v=0}^{N-1}|F(u,v)|^2$, with $F(u,v)$ the Discrete Fourier Transform of the image.

[4]

3. We are given the blurred and noisy version $g(x,y)$ of an image $f(x,y)$ such that in lexicographic ordering

$$g = Hf + n$$

where H is the degradation matrix which is assumed to be block-circulant, and n is the noise term which is assumed to be zero mean, independent from the original image and white. Moreover, f and g are the lexicographically ordered original and degraded image respectively.

(a) (i) Consider the inverse filtering image restoration technique. Write down without proof the expressions for both the inverse filter estimator and the restored image both in the spatial domain and the frequency domain. Explain all symbols used.

[2]

(ii) Discuss briefly the disadvantages of the inverse filtering image restoration technique.

[2]

(iii) Consider the pseudo-inverse filtering image restoration technique. Write down the expression for the pseudo-inverse filter estimator in the frequency domain. Discuss briefly the advantage of the pseudo-inverse filtering image restoration technique.

[2]

(b) (i) Consider the iterative constrained least squares image restoration technique. Write down without proof the expressions for both the iterative constrained least squares estimator and the restored image both in the spatial domain. Explain all symbols used.

[3]

(ii) Discuss the method of spatially adaptive iterative constrained least squares image restoration technique.

[5]

(c) One class of filters considered for reducing background noise in images has a frequency response $W(u,v)$ given by

$$W(u,v) = \left[\frac{S_{ff}(u,v)}{S_{ff}(u,v) + S_{nn}(u,v)}\right]^{\beta}$$

where $S_{ff}(u,v)$ is the original image power spectrum and $S_{nn}(u,v)$ is the noise power spectrum. If $\beta = 1$, the filter is a Wiener filter. If $\beta = \frac{1}{2}$, the filter is called a power spectrum filter. Suppose $S_{ff}(u,v)$ has a lowpass character and its amplitude decreases as u and v increase, while $S_{nn}(u,v)$ is approximately constant independent of u and v.

(i) For a given $S_{ff}(u,v)$ and $S_{nn}(u,v)$, which filter reduces more the noise, the Wiener filter or the power spectrum filter? Justify your answer.

[3]

(ii) For a given $S_{ff}(u,v)$ and $S_{nn}(u,v)$, which filter blurs the image more, the Wiener filter or the power spectrum filter? Justify your answer.

[3]

4. (a) Consider an image with intensity $f(x,y)$ that can take three possible values $A=0, B=128, C=255$ with the probabilities shown in Figure 4.1 below. We wish to compress the image using the lossless JPEG compression standard. Suppose that the prediction residual for pixel with intensity x in the following Figure 4.2 is defined as $r=x-y$ where y is the function $y=a$.

Find the residuals, their probabilities and their corresponding codewords. Suppose that the Huffman codewords for the following categories in the lossless JPEG are:

Category 0: 010
Category 7: 11110
Category 8: 111110

Intensity	**Probability**
$A=0$	$\frac{1}{6}$
$B=128$	$\frac{2}{6}$
$C=255$	$\frac{3}{6}$

Figure 4.1

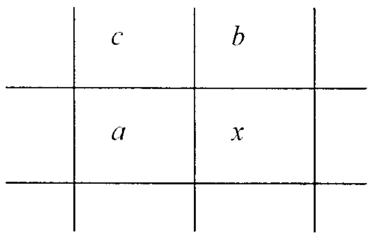


Figure 4.2

[8]

(b) Consider a Discrete Memoryless Source (DMS) that consists of two symbols s_1, s_2 with probabilities p_1, p_2 respectively with $p_1 = 0.8$ and $p_2 = 0.2$. Suppose that the two symbols are to be transmitted using extended by two Huffman coding.

(i) Find the Huffman codewords. Is Huffman code unique? Justify your answer.

[5]

(ii) Find the Huffman codewords if it is known that the probability of a 0 being transmitted as a 1 is higher than the probability of a 1 being transmitted as a 0.

[5]

(iii) Determine the redundancy of the Huffman code for this example.

[2]

1. (a) (i) $f(x,y)-g(x,y)=\sum_{u=0}^{M-1}\sum_{v=0}^{M-1}(F(u,v)-G(u,v))e^{j2\pi(ux/M+vy/N)}$

$$=F(u_0,v_0)e^{j2\pi(u_0x/M+v_0y/N)}+F(M-u_0,M-v_0)e^{j2\pi[(M-u_0)x/M+(N-v_0)y/N]}$$
$$=F(u_0,v_0)e^{j2\pi(u_0x/M+v_0y/N)}+F^*(u_0,v_0)e^{-j2\pi(u_0x/M+v_0y/N)}$$
$$=F(u_0,v_0)e^{j2\pi(u_0x/M+v_0y/N)}+F^*(u_0,v_0)\left(e^{j2\pi(u_0x/M+v_0y/N)}\right)^*$$
$$=F(u_0,v_0)e^{j2\pi(u_0x/M+v_0y/N)}+\left(F(u_0,v_0)e^{j2\pi(u_0x/M+v_0y/N)}\right)^*$$
$$=2\,\mathrm{Re}\{F(u_0,v_0)e^{j2\pi(u_0x/M+v_0y/N)}\}$$

The function $\mathrm{Re}\{\cdot\}$ denotes the real part of a complex number.
If we write $F(u_0,v_0)=A(u_0,v_0)+jB(u_0,v_0)$ we have
$f(x,y)-g(x,y)$
$=2\{A(u_0,v_0)\cos[2\pi(u_0x/M+v_0y/N)]-B(u_0,v_0)\sin[2\pi(u_0x/M+v_0y/N)]\}$
Thus, the difference $f(x,y)-g(x,y)$ is a 2-D sinewave.

[3]

(ii) The required energy is $\sum_{u=u_1}^{u_2}\sum_{v=v_1}^{v_2}|F(u,v)|^2$. The total energy is $\sum_{u=0}^{M-1}\sum_{v=0}^{N-1}|F(u,v)|^2$. As a percentage of the total energy the required is ($\sum_{u=u_1}^{u_2}\sum_{v=v_1}^{v_2}|F(u,v)|^2 / \sum_{u=0}^{M-1}\sum_{v=0}^{N-1}|F(u,v)|^2$)%100.

[3]

(b)

(i) $W(u,v)=\frac{1}{M}\sum_{x=0}^{M-1}\sum_{y=0}^{M-1}f(x,y)\left[\prod_{i=0}^{n-1}(-1)^{(b_i(x)b_{n-1-i}(u)+b_i(y)b_{n-1-i}(v))}\right]$

[1]

(ii) For the calculation of $W(0,0)$ we have:
$u=(0)_{10}=(00)_2 \Rightarrow b_0(u)=0, b_1(u)=0$
$v=(0)_{10}=(00)_2 \Rightarrow b_0(v)=0, b_1(v)=0$

$$W(0,0)=\frac{1}{M}\sum_{x=0}^{M-1}\sum_{y=0}^{M-1}f(x,y)\left[\prod_{i=0}^{n-1}(-1)^{(b_i(x)\cdot 0+b_i(y)\cdot 0)}\right]$$
$$=\frac{1}{M}\sum_{x=0}^{M-1}\sum_{y=0}^{M-1}f(x,y)\left[\prod_{i=0}^{n-1}(-1)^{0}\right]=\frac{1}{M}\sum_{x=0}^{M-1}\sum_{y=0}^{M-1}f(x,y)=\frac{96}{4}=24$$

For the calculation of $W(3,0)$ we have:
$u=(3)_{10}=(11)_2 \Rightarrow b_0(u)=1, b_1(u)=1$
$v=(0)_{10}=(00)_2 \Rightarrow b_0(v)=0, b_1(v)=0$

$$W(0,0)=\frac{1}{M}\sum_{x=0}^{M-1}\sum_{y=0}^{M-1}f(x,y)\left[\prod_{i=0}^{n-1}(-1)^{(b_i(x)\cdot 1+b_i(y)\cdot 0)}\right]$$
$$=\frac{1}{M}\sum_{x=0}^{M-1}\sum_{y=0}^{M-1}f(x,y)\left[\prod_{i=0}^{n-1}(-1)^{b_i(x)}\right]=\frac{1}{M}\sum_{x=0}^{M-1}\sum_{y=0}^{M-1}f(x,y)(-1)^{b_1(x)+b_0(x)}$$

$$=\frac{1}{M}\sum_{y=0}^{M-1}(f(0,y)-f(1,y)-f(2,y)+f(3,y))=\frac{12}{4}$$

[4]

(c)

(i) First we must find the eigenvectors of $\underline{C}_{\underline{f}}$. These are $\underline{e}_2 = \begin{bmatrix} 0.3827 \\ -0.9239 \end{bmatrix}$ and $\underline{e}_1 = \begin{bmatrix} -0.9239 \\ -0.3827 \end{bmatrix}$. The matrix A is $A = \begin{bmatrix} \underline{e}_1^T \\ \underline{e}_2^T \end{bmatrix}$.

Therefore $\underline{g} = \underline{A}(\underline{f} - \underline{m}_{\underline{f}})$ **[5]**

(ii) $\underline{C}_{\underline{g}} = \begin{bmatrix} 0.2331 & 0 \\ 0 & 0.0069 \end{bmatrix}$ **[2]**

(iii) 0.0069. 100% of the maximum error.

[2]

2. (a) Imagine a 64×64 image with 4 grey levels. The normalised grey levels are 0, $1/3$, $2/3$ and 1.

Grey level	Number of pixels
0	1813
1/3	1506
2/3	574
1	203

(i) $p(0) = 1813/4096$, $p(1/3) = 1506/4096$, $p(2/3) = 574/4096$, $p(1) = 203/4096$.
Drawing is obvious.

[1]

(ii) $T(0) = 1813/4096$, $T(1/3) = (1506 + 1813)/4096$, $T(2/3) = (574 + 1506 + 1813)/4096$, $T(1) = 1$

[2]

(iii) Obvious.

[1]

(b) $s = T(r) = \int_0^r p_r(w)dw$. Suppose we histogram equalise s to get

$z = T(s) = \int_0^s p_s(w)dw = \int_0^s 1 \cdot dw = s$. So after the first time nothing happens.

[4]

(c) The same as each grey level is mapped to a new distinct grey level

[4]

(d) Median filters are non linear filters because for two sequences $x(n)$ and $y(n)$

$$\text{median}\{x(n) + y(n)\} \neq \text{median}\{x(n)\} + \text{median}\{y(n)\}$$

For example consider two signals with three samples each, i.e.,
$x(n) = \{10,1,1\}$ and $y(n) = \{3,5,10\}$. Therefore, $x(n) + y(n) = \{13,6,11\}$
$\text{median}\{x(n)\} = 1$, $\text{median}\{y(n)\} = 5$, $\text{median}\{x(n) + y(n)\} = 11$. Thus,
$\text{median}\{x(n) + y(n)\} = 11 \neq 6 = \text{median}\{x(n)\} + \text{median}\{y(n)\}$

[4]

(d) $\sum_{u=0}^{M-1}\sum_{v=0}^{N-1}|F(u,v)|^2 = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)^2$

Suppose that $h(r) = \frac{n_r}{MN}, r = 0,\ldots,255 \Rightarrow n_r = MNh(r)$. Note that n_r is the number of pixels with intensity r. In that case $\sum_{u=0}^{M-1}\sum_{v=0}^{N-1}|F(u,v)|^2 = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)^2 = \sum_{i=0}^{255} r_i^{\,2} n_i$

[4]

3.

(a) (i) The objective is to minimize $J(\mathbf{f}) = \|\mathbf{n}(\mathbf{f})\|^2 = \|\mathbf{y} - \mathbf{Hf}\|^2$. We set the first derivative of the cost function equal to zero $\frac{\partial J(\mathbf{f})}{\partial \mathbf{f}} = 0 \Rightarrow -2\mathbf{H}^T(\mathbf{y} - \mathbf{Hf}) = \mathbf{0}$. If $\mathbf{H}^{-1}$ exists then $\mathbf{f} = (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\mathbf{y}$. If $M = N$ then $\mathbf{f} = \mathbf{H}^{-1}\mathbf{y}$. According to the previous analysis if $\mathbf{H}$ (and therefore $\mathbf{H}^{-1}$) is block circulant the above problem can be solved as a set of $M \times N$ scalar problems as follows $F(u,v) = \frac{H^*(u,v)Y(u,v)}{|H(u,v)|^2} \Rightarrow f(i,j) = \Im^{-1}\left[\frac{H^*(u,v)Y(u,v)}{|H(u,v)|^2}\right]$

[2]

(ii) Suppose first that the additive noise $n(i,j)$ is negligible. A problem arises if $H(u,v)$ becomes very small or zero for some point (u,v) or for a whole region in the (u,v) plane. In that region inverse filtering cannot be applied. Note that in most real applications $H(u,v)$ drops off rapidly as a function of distance from the origin. If these points are known they can be neglected in the computation of $F(u,v)$.

In the presence of external noise we have that $\hat{F}(u,v) = \frac{H^*(u,v)(Y(u,v) - N(u,v))}{|H(u,v)|^2} = \frac{H^*(u,v)Y(u,v)}{|H(u,v)|^2} - \frac{H^*(u,v)N(u,v)}{|H(u,v)|^2} \Rightarrow$

$\hat{F}(u,v) = F(u,v) - \frac{N(u,v)}{H(u,v)}$. If $H(u,v)$ becomes very small, the term $N(u,v)$ dominates the result.

[2]

(iii)

$$\hat{F}(u,v) = \begin{cases} \dfrac{H^*(u,v)Y(u,v)}{|H(u,v)|^2} & H(u,v) \neq 0 \\ \\ 0 & H(u,v) = 0 \end{cases}$$

[2]

(b) (i) In this method we attempt to solve the problem of constrained restoration iteratively. As already mentioned the following functional is minimized $M(\mathbf{f},\alpha) = \|\mathbf{y} - \mathbf{Hf}\|^2 + \alpha\|\mathbf{Cf}\|^2$. The necessary condition for a minimum is that the gradient of $M(\mathbf{f},\alpha)$ is equal to zero. That is $\mathbf{\Phi}(\mathbf{f}) = \nabla_{\mathbf{f}} M(\mathbf{f},\alpha) = (\mathbf{H}^T\mathbf{H} + \alpha\mathbf{C}^T\mathbf{C})\mathbf{f} - \mathbf{H}^T\mathbf{y}$. The initial estimate and the updating rule for obtaining the restored image are now given by

$\mathbf{f}_0 = \beta\mathbf{H}^T\mathbf{y}$

$\mathbf{f}_{k+1} = \mathbf{f}_k + \beta[\mathbf{H}^T\mathbf{y} - (\mathbf{H}^T\mathbf{H} + \alpha\mathbf{C}^T\mathbf{C})\mathbf{f}_k]$

It can be proved that the above iteration (known as **Iterative CLS** or **Tikhonov-Miller Method)** converges if $0 < \beta < \frac{2}{|\lambda_{\max}|}$ where $\lambda_{\max}$ is the maximum eigenvalue of the matrix $(\mathbf{H}^T\mathbf{H} + \alpha\mathbf{C}^T\mathbf{C})$.

[3]

(ii) The functional to be minimized takes the form $M(\mathbf{f},\alpha) = \|\mathbf{y} - \mathbf{Hf}\|^2_{\mathbf{W}_1} + \alpha\|\mathbf{Cf}\|^2_{\mathbf{W}_2}$ where $\|\mathbf{y} - \mathbf{Hf}\|^2_{\mathbf{W}_1} = (\mathbf{y} - \mathbf{Hf})^T\mathbf{W}_1(\mathbf{y} - \mathbf{Hf})$, $\|\mathbf{Cf}\|^2_{\mathbf{W}_2} = (\mathbf{Cf})^T\mathbf{W}_2(\mathbf{Cf})$, $\mathbf{W}_1, \mathbf{W}_2$ are diagonal matrices, the choice of which can be justified in various ways. The entries in both matrices are non-negative values and less than or equal to unity. In that case $\mathbf{\Phi(f)} = \nabla_{\mathbf{f}} M(\mathbf{f},\alpha) = (\mathbf{H}^{\mathrm{T}}\mathbf{W}_1^{\mathrm{T}}\mathbf{W}_1\mathbf{H} + \alpha\mathbf{C}^{\mathrm{T}}\mathbf{W}_2^{\mathrm{T}}\mathbf{W}_2\mathbf{C})\mathbf{f} - \mathbf{H}^{\mathrm{T}}\mathbf{W}_1\mathbf{y}$. A more specific case is $M(\mathbf{f},\alpha) = \|\mathbf{y} - \mathbf{Hf}\|^2 + \alpha\|\mathbf{Cf}\|^2{}_{\mathrm{w}}$ where the weighting matrix is incorporated only in the regularization term. This method is known as **weighted regularised image restoration**. The entries in matrix $\mathbf{W}$ will be chosen so that the high-pass filter is only effective in the areas of low activity and a very little smoothing takes place in the edge areas.

[5]

(c)

(i) The power $\beta = 1/2$ increases the value of the filter so the power spectrum filter removes less noise than the Wiener filter.

[3]

(ii) According to (i) the Wiener filter.

[3]

Question 4

(a)

Differentials	Probabilities
0 = A-A & B-B & C-C	(1 + 4 + 9)/36 = 14/36
A-B = -128 = 0111 1111	2/36
B-A = 128 = 10000000	2/36
A-C = -255 = 0000 0000	3/36
C-A = 255 = 1111 1111	3/36
B-C = -127 = 0000000	6/36
C-B = 127 = 1111111	6/36

Codewords

0	010
128 :	1000 0000 / 11111 0
-128 :	0111 1111 / 11111 0
127 :	1111111 / 1111 0
-127 :	000 0000 / 1111 0
255 :	1111 1111 / 11111 0
-255 :	0000 0000 / 11111 0

(b)

(i)

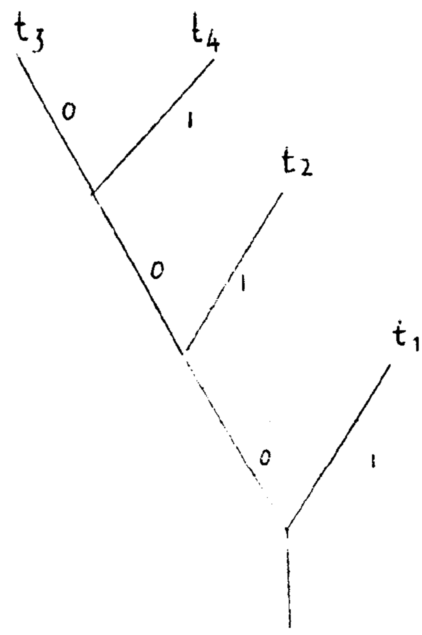
new symbol	probability
$s_1 s_1 = t_1$	$p_1^2 = 0.64$
$s_1 s_2 = t_2$	$p_1 p_2 = 0.16$
$s_2 s_1 = t_3$	$p_1 p_2 = 0.16$
$s_2 s_2 = t_4$	$p_2^2 = 0.04$

t_1	0.64
t_2	0.16
t_3	0.16
t_4	0.04

t_1	0.64
t_2	0.16
$\{t_3, t_4\}$	0.2

t_1	0.64
$\{t_3, t_4\}$	0.2
t_2	0.16

t_1	0.64
$\{\{t_3, t_4\}, t_2\}$	0.36



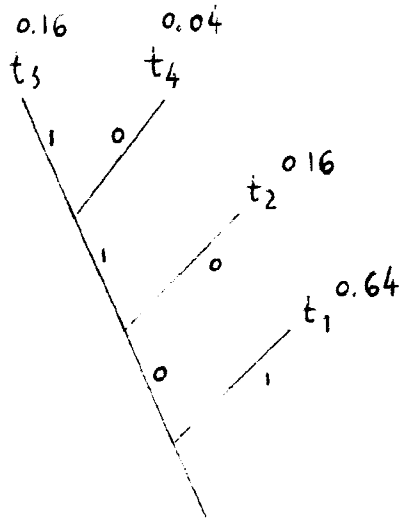
t_1 : 0

t_2 : 01

t_3 : 000

t_4 : 001

(ii)



1's are preferable; when two branches are merged the 1 goes to the branch with the higher probability

(iii)

$$\text{Entropy } H(S) = -\sum_{i=1}^{n} p_i \log_2 p_i \quad , \quad n=4$$

$$l_{avg} = 1 \cdot 0.64 + 2 \cdot 0.16 + 3 \cdot 0.04 + 3 \cdot 0.16$$

$$\text{Redundancy} = l_{avg} - H(S)$$