Imperial College London

BSc/MSci EXAMINATION May 2012

This paper is also taken for the relevant Examination for the Associateship

ADVANCED CLASSICAL PHYSICS

For 3rd and 4th Year Physics Students

Monday, 21st May 2012: 14:00 to 16:00

Answer ALL parts of Section A and TWO questions from Section B.
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the THREE answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

SECTION A

1. (i) The equation of motion for a particle of mass m in a frame rotating with angular velocity vector $\boldsymbol{\omega}$ is

$$m\left.\frac{d^2\boldsymbol{r}}{dt^2}\right|_{\mathrm{R}} = \boldsymbol{F} - 2m\boldsymbol{\omega}\times\left.\frac{d\boldsymbol{r}}{dt}\right|_{\mathrm{R}} - m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r})\,,$$

where $\boldsymbol{F}$ is the force acting on the particle. What are the two other terms called? Which way do they point, if the particle is moving towards north on the northern hemisphere? [4 marks]

(ii) Consider a system of N particles with masses m_i and positions $\boldsymbol{r}_i$, $i = 1,\ldots,N$, in which all forces are central, so that the force $\boldsymbol{F}_{ij}$ on particle i due to its interaction with particle j is parallel to $\boldsymbol{r}_j - \boldsymbol{r}_i$. Show that if there are no external forces, the total angular momentum

$$\boldsymbol{L} = \sum_{i=1}^{N} m_i \boldsymbol{r}_i \times \dot{\boldsymbol{r}}_i$$

is conserved. According to Noether's theorem, what is the underlying cause of this conservation law? [5 marks]

(iii) A one-dimensional system has the Lagrangian function

$$L = \frac{1}{2}m\dot{x}^2 + \dot{x}f(x),$$

where $f(x)$ is a function of x.

(a) Find the Euler-Lagrange equation and solve it. [2 marks]

(b) Find the canonical momentum, and write down the Hamiltonian function. [2 marks]

(iv) Charge density ρ and current density $\boldsymbol{J}$ form a four vector $j^\mu = (\rho c, J_x, J_y, J_z)$. Write the continuity equation

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\boldsymbol{J} = 0$$

in the four-vector notation using the Einstein summation convention, showing explicitly that it agrees with the above expression. [4 marks]

(v) In terms of vector potential $\boldsymbol{A}$ and scalar potential ϕ, their electric and magnetic fields are

$$\begin{aligned}\boldsymbol{B} &= \nabla\times\boldsymbol{A},\\ \boldsymbol{E} &= -\frac{\partial\boldsymbol{A}}{\partial t} - \nabla\phi.\end{aligned}$$

Explain what is meant by gauge invariance and gauge transformations, showing what form they can have. [3 marks]

[Total 20 marks]

SECTION B

2. (i) From conservation of angular momentum $\boldsymbol{L}$ in the inertial frame, derive the Euler equations for free rotation,

$$
\begin{aligned}
I_1\dot{\omega}_1 + (I_3 - I_2)\,\omega_2\omega_3 &= 0, \\
I_2\dot{\omega}_2 + (I_1 - I_3)\,\omega_3\omega_1 &= 0, \\
I_3\dot{\omega}_3 + (I_2 - I_1)\,\omega_1\omega_2 &= 0,
\end{aligned}
$$

where I_1, I_2 and I_3 are the principal moments, and ω_1, ω_2 and ω_3 are the components of the angular velocity vector in the directions of the corresponding principal axes $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$ and $\hat{\mathbf{e}}_3$. [4 marks]

Consider now an object that has inertia tensor

$$
\bar{\bar{\boldsymbol{I}}} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} I_0,
$$

where I_0 is a constant with the dimensions of kgm^2.

(ii) Find the principal moments and the principal axes of inertia for this object. [6 marks]

(iii) Show that in the current case the speed of the rotation is constant

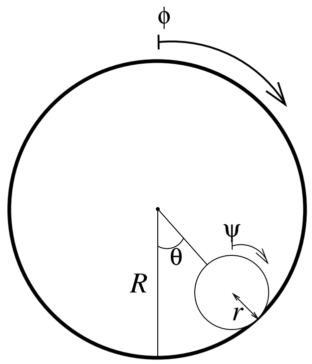
$$
\frac{d\boldsymbol{\omega}^2}{dt} = 0.
$$

What property of the inertia tensor is responsible for that? [2 marks]

(iv) Write down the Euler equations in the current case, and find the time evolution of the components ω_1, ω_2 and ω_3. [3 marks]

[Total 15 marks]

3. Consider a cylinder with mass m, radius r and moment of inertia $I_1 = mr^2/2$, placed on the inside of a larger hollow cylinder with internal radius R and moment of inertia $I_2 = mR^2$ (see figure). The smaller cylinder can roll without slipping on the inside of the larger cylinder, and the larger cylinder can rotate freely around its axis.



Let ϕ be the rotation angle of the larger cylinder, θ the angle describing the position of the smaller cylinder, and ψ the rotation angle of the smaller cylinder. Because the smaller cylinder is assumed not to slip, they satisfy the constraint $r(\dot{\psi}+\dot{\theta}) = R(\dot{\phi}+\dot{\theta})$.

(i) Show that the Lagrangian of the system is

$$L = \frac{3}{4}mR^2\dot{\phi}^2 + \frac{3}{4}m(R-r)^2\dot{\theta}^2 + \frac{1}{2}mR(R-r)\dot{\phi}\dot{\theta} + mg(R-r)\cos\theta.$$

[6 marks]

(ii) Derive the Euler-Lagrange equations and identify a conserved quantity.
[3 marks]

(iii) Show that the smaller cylinder's position angle θ satisfies the equation

$$\ddot{\theta} = -\frac{3g}{4(R-r)}\sin\theta.$$

[3 marks]

(iv) Assume that initially both cylinders are at rest, with the smaller cylinder displaced by its equilibrium position by angle $\theta_0 \ll 1$. Find the time evolution of the rotation angle ϕ. [3 marks]

[Total 15 marks]

4. (i) A charged particle with charge q travels in the positive direction along the x axis at the speed of light, in such a way that it passes through the origin at time $t = 0$. Use the retarded Lorenz-gauge potentials

$$\begin{aligned}\phi(\boldsymbol{r}, t) &= \frac{1}{4\pi\epsilon_0}\int dt'\, d^3\boldsymbol{r}'\delta\left(t' - t + \frac{|\boldsymbol{r}-\boldsymbol{r}'|}{c}\right)\frac{\rho(\boldsymbol{r}', t')}{|\boldsymbol{r}-\boldsymbol{r}'|}, \\ \boldsymbol{A}(\boldsymbol{r}, t) &= \frac{\mu_0}{4\pi}\int dt'\, d^3\boldsymbol{r}'\delta\left(t' - t + \frac{|\boldsymbol{r}-\boldsymbol{r}'|}{c}\right)\frac{\boldsymbol{J}(\boldsymbol{r}', t')}{|\boldsymbol{r}-\boldsymbol{r}'|},\end{aligned}$$

where ρ is the charge density and $\boldsymbol{J}$ is the current density, to show that

$$\phi(\boldsymbol{r}, t) = \begin{cases} \dfrac{q}{4\pi\epsilon_0}\dfrac{1}{ct - x}, & \text{if } ct > x, \\ 0, & \text{if } ct < x. \end{cases}$$

[10 marks]

(ii) Show that

$$\boldsymbol{A}(\boldsymbol{r}, t) = \begin{cases} \dfrac{q\mu_0}{4\pi}\dfrac{c\hat{\boldsymbol{\imath}}}{ct - x}, & \text{if } ct > x, \\ 0, & \text{if } ct < x. \end{cases}$$

[2 marks]

(iii) Calculate the electric and magnetic fields due to the moving charge at $x \neq ct$. [3 marks]

Hint: Remember that if function $f(x)$ has one zero at $x = x_0$, i.e., $f(x_0) = 0$, then

$$\delta(f(x)) = \frac{\delta(x - x_0)}{|f'(x_0)|}.$$

[Total 15 marks]

5. In terms of the electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$, the Faraday tensor is

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}.$$

(i) How does it transform under Lorentz transformation $x^{\mu'} = \Lambda^{\mu'}{}_{\nu} x^{\nu}$?

[2 marks]

(ii) Show that the quantity $F_{\mu\nu}F^{\mu\nu}$ is Lorentz invariant. Express it in terms of $\boldsymbol{E}$ and $\boldsymbol{B}$. [4 marks]

(iii) Write down the Faraday tensor that corresponds to a linearly polarised electromagnetic plane wave travelling in the z direction, given by

$$\begin{aligned} \boldsymbol{E}(t,\boldsymbol{r}) &= E\hat{\boldsymbol{\imath}} e^{ik(z-ct)}, \\ \boldsymbol{B}(t,\boldsymbol{r}) &= \frac{E}{c}\hat{\boldsymbol{\jmath}} e^{ik(z-ct)}. \end{aligned}$$

Show that it satisfies the Maxwell equations

$$\begin{aligned} \partial_\mu F^{\mu\nu} &= 0, \\ \partial^\mu F^{\nu\rho} + \partial^\nu F^{\rho\mu} + \partial^\rho F^{\mu\nu} &= 0. \end{aligned}$$

[5 marks]

(iv) Find the Faraday tensor in a frame boosted in the z direction in terms of the boosted coordinates $x^{\mu'}$? [4 marks]

[Total 15 marks]