UNIVERSITY OF LONDON [II(3)E 2001]

B.ENG. AND M.ENG. EXAMINATIONS 2001

For Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examination for the Associateship.

PART II : MATHEMATICS 3 (ELECTRICAL ENGINEERING)

Wednesday 6th June 2001 2.00 - 5.00 pm

Answer EIGHT questions.

[*Before starting, please make sure that the paper is complete; there should be 7 pages, with a total of 12 questions. Ask the invigilator for a replacement if your copy is faulty.*]

1. Consider the mapping

$$w = \frac{1}{z - i}$$

from the z-plane to the w-plane, where $z = x + iy$ and $w = u + iv$.

(i) Show that, in the z-plane, the family of circles centred at $(0, 1)$ with radius a

$$x^2 + (y - 1)^2 = a^2$$

maps to another family of circles in the w-plane. What is the radius of this family and where is its centre?

(ii) What is the image in the w-plane of the x-axis ($y = 0$) in the z-plane? Show that the curve that represents this image passes through the origin in the w-plane.

(iii) Show that the family of straight lines $y = cx$ in the z-plane with $c =$ constant have the image in the w-plane represented by

$$\left(u - \frac{c}{2}\right)^2 + \left(v - \frac{1}{2}\right)^2 = \frac{1}{4}(1 + c^2)\,.$$

2. By choosing a suitable contour C in the upper half of the complex plane, use the contour integral

$$\oint_C \frac{e^{iz}dz}{(z^2 + 4)(z^2 + 1)}$$

to show that

$$\int_{-\infty}^{\infty} \frac{\cos x\, dx}{(x^2 + 4)(x^2 + 1)} = \frac{\pi}{6}\left(\frac{2e - 1}{e^2}\right)\,.$$

PLEASE TURN OVER

3. (i) Show that if C is a circle of arbitrary radius r centred at the origin, then the value of the complex integral

$$\oint_C \frac{dz}{z}$$

is independent of r. What is this value?

(ii) Use the Residue Theorem to show that

$$\oint_C \frac{z\,dz}{(z-1)^2\,(z-i)} \;=\; 0\,,$$

where the contour C is the circle of radius 2 centred at the origin. What is the answer when C is changed to be the rectangle with vertices at $\pm\frac{1}{2}+2i$ and $\pm\frac{1}{2}-2i$?

Recall that the residue of a complex function $f(z)$ at a pole $z=a$ of multiplicity m is given by the expression

$$\lim_{z\to a}\frac{1}{(m-1)!}\left[\frac{d^{m-1}}{dz^{m-1}}\;\{(z-a)^m\,f(z)\}\right]\;.$$

4. The Fourier convolution of the functions $f(t)$ and $g(t)$ is defined by

$$f*g \;=\; \int_{-\infty}^{\infty} f(u)g^*(t-u)\,du$$

where g^* is the complex conjugate of g. If $\overline{f}(\omega)$ and $\overline{g}(\omega)$ are the Fourier transforms of $f(t)$ and $g(t)$ respectively, prove the Fourier convolution theorem

$$\int_{-\infty}^{\infty} e^{-i\omega t}(f*g)\,dt \;=\; \overline{f}(\omega)\,\overline{g}\,(\omega).$$

For a function $f(t)$, if $\gamma(t)$ is defined by

$$\gamma(t) \;=\; \frac{f*f}{\int_{-\infty}^{\infty}|f(t)|^2\,dt}$$

show that

$$\int_{-\infty}^{\infty}\overline{\gamma}(\omega)\,d\omega \;=\; 2\pi.$$

5. (i) A second order ordinary differential equation takes the form

$$\frac{d^2x}{dt^2} + \omega^2 x = f(t)\,,$$

where $f(t)$ is an arbitrary piecewise smooth function. It has initial conditions

$$x = \frac{dx}{dt} = 0 \text{ when } t = 0\,.$$

Use the Laplace convolution theorem to show that

$$x(t) = \frac{1}{\omega}\int_0^t \sin(\omega u) f(t-u)\,du.$$

(ii) A third order ordinary differential equation takes the form

$$\frac{d^3x}{dt^3} + 3\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + x = f(t)$$

where $f(t)$ is an arbitrary piecewise smooth function. $x(t)$ and its first two derivatives satisfy the conditions

$$x = \frac{dx}{dt} = \frac{d^2x}{dt^2} = 0 \quad \text{when } t = 0.$$

Use the shift and convolution theorems to show that

$$x(t) = \frac{1}{2}\int_0^t e^{-u}u^2 f(t-u)\,du.$$

6. Given that

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x}dx = i\pi,$$

show that

$$\int_{-\infty}^{\infty} \frac{e^{ipx}}{x}dx = \begin{cases} +i\pi, & p > 0 \\ -i\pi, & p < 0, \end{cases}$$

where p is an arbitrary real number. Hence show that the Fourier transform $\overline{f}(\omega)$ of the function

$$f(t) = \frac{\sin t/2}{t/2}$$

is given by

$$\overline{f}(\omega) = \begin{cases} 2\pi\,, & -\frac{1}{2} < \omega < \frac{1}{2}\,, \\ 0\,, & \omega < -\frac{1}{2}, \quad \omega > \frac{1}{2}\,. \end{cases}$$

PLEASE TURN OVER

7. (i) The double integral I_1 is given by

$$I_1 = \iint_{R_1} (x+y)^2 \cos\,(x^2 - y^2)\, dxdy\,,$$

where R_1 is the finite region in the x-y plane enclosed by the lines $x = 0,\ y = 0$ and $y = 1 - x$.

Show that, by using the transformation,

$$u \;=\; x \;-\; y, \quad v \;=\; x \;+\; y\,,$$

the integral can be written as

$$I_1 \;=\; \frac{1}{2}\int_0^1 v^2 \left(\int_{-v}^{v} \cos(uv)\, du\right) dv\,.$$

Hence evaluate I_1.

(ii) Use the same transformation to evaluate

$$I_2 \;=\; \iint_{R_2} (x^2 + y^2)\, dxdy\,,$$

where R_2 is the interior of the square bounded by $y \;=\; \pm\, x, \quad y \;=\; \pm\,(x-1)$.

8. A vector field $\mathbf{F}$ is defined as

$$\mathbf{F} \;=\; 2xye^z\mathbf{i} \;+\; x^2e^z\mathbf{j} \;+\; (x^2ye^z \;+\; z^2 \;+\; 3z)\mathbf{k}\,.$$

(i) Find div $\mathbf{F}$ and curl $\mathbf{F}$.

(ii) Find a function $\phi\,(x,\, y,\, z)$ such that $\mathbf{F} \;=\; \nabla\phi$.

(iii) Evaluate

$$\frac{\partial^2}{\partial z^2}\,(x\,\mathbf{F}\,.\,\mathbf{i} \;-\; 2\phi)\,.$$

9. (i) The vector field $\mathbf{F}$ is defined by

$$\mathbf{F} \;=\; (y^2 \cos x)\,\mathbf{i} \;+\; (\alpha y \sin x)\,\mathbf{j}\,,$$

where α is a constant. Find the value of α such that $\operatorname{curl}\mathbf{F} = \mathbf{0}$.

(ii) Consider the integral

$$I \;=\; \int_C (y^2 \cos x\, dx \;+\; \beta y \sin x\, dy),\ \ (\beta \text{ constant})\,,$$

where C is a curve joining the points (0, 0) and ($\pi/2$, 1).

Evaluate I in the following cases:

(a) C is the line $y = (2/\pi)x$;

(b) C is the curve $y = \sin x$.

Show that the answers to (a) and (b) are equal for one particular value of β and find that value.

Explain why the value of α found in part (i) is the same as this value of β.

10. P and Q are continuous functions of x and y with continuous first partial derivatives in a simply connected region R with a piecewise smooth boundary C. Green's Theorem in a plane says that

$$\oint_C (Pdx + Qdy) = \int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\, dxdy.$$

Find a two-dimensional vector $\boldsymbol{u}$, defined in terms of P and Q, to show that Green's Theorem can be re-expressed as the two-dimensional version of the Divergence Theorem

$$\oint_C \boldsymbol{u}.\boldsymbol{n}\, ds = \int\int_R \operatorname{div}\boldsymbol{u}\, dxdy$$

where $\boldsymbol{n}$ is the unit normal to the curve C.

If $\boldsymbol{u}$ is given by $\boldsymbol{u} = x^2\boldsymbol{i} + y^2\boldsymbol{j}$ and R is the first quadrant of the circle of unit radius, evaluate the right hand side of the Divergence Theorem to show that

$$\int\int_R \operatorname{div}\boldsymbol{u}\, dxdy = 4/3.$$

PLEASE TURN OVER

11. Let $A_1, \ldots, A_k$ form a partition of a sample space and B be some event. Use the definition of conditional probability and the theorem of total probabilities to derive Bayes's formula for $P(A_i \mid B)$.

    It is estimated that 5% of optical disks produced by a manufacturer are faulty. A disk may be subjected to an initial diagnostic test. If there is a fault, the test gives a diagnosis of 'faulty' with probability 0.8; if there is no fault the test gives a diagnosis of 'OK' with probability 0.95. If the test gives a diagnosis of 'faulty', the disk is rejected. A disk is chosen at random and tested. What is the probability that

    (i) the test gives a diagnosis of 'OK'?

    (ii) a disk is faulty which has been given a diagnosis of 'OK'?

    If the initial test gives the diagnosis 'OK', a further independent test is performed; this test has exactly the same properties as the initial test, except that if there is a fault, the test gives a diagnosis of 'faulty' 99% of the time. If this test gives a diagnosis of 'OK' the disk is accepted for use, otherwise it is rejected.

    (iii) Determine the probability that a faulty disk is accepted for use.

12. Let X and Y be two random variables. The coefficient of correlation between X and Y is given by

$$\rho_{X,Y} = \frac{\operatorname{cov}\{X, Y\}}{[\operatorname{var}\{X\} \operatorname{var}\{Y\}]^{1/2}} = \frac{E\{XY\} - E\{X\} E\{Y\}}{[\operatorname{var}\{X\} \operatorname{var}\{Y\}]^{1/2}} .$$

    (i) What does it measure? How should values of $\rho_{X,Y}$ of -1, 0 and 1 be interpreted?

    (ii) If X and Y are independent, what is $\rho_{X,Y}$?

    Let X and Y have the joint probability density function given by

$$f_{X,Y}(x, y) = \begin{cases} x^{-1}, & 0 \le y \le x \le 1, \\ 0, & \text{otherwise}. \end{cases}$$

    (iii) Calculate $E\{XY\}$ and $E\{X\}$ and $E\{Y\}$, and hence find the value of $\operatorname{cov}\{X, Y\}$.

    (iv) Are X and Y independent?

END OF PAPER

MATHS 3
2001

E1

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EXAMINATION QUESTION / SOLUTION

SESSION : 2000 - 2001

PAPER 3

QUESTION

SOLUTION 14

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$$w = \frac{1}{z-i} = \frac{1}{x+i(y-1)} = \frac{x-i(y-1)}{x^2+(y-1)^2} = u+iv$$

$$u = \frac{x}{x^2+(y-1)^2} \; ; \quad v = \frac{-(y-1)}{x^2+(y-1)^2}$$

$$u^2+v^2 = \frac{1}{x^2+(y-1)^2} \qquad (*)$$ 3

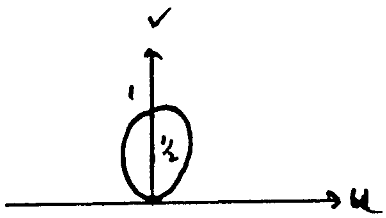
a) The family of circles $x^2+(y-1)^2 = a^2$ map to $u^2+v^2 = 1/a^2$ – a family centred at $(0,0)$ radius a^{-1}. 3

b) $y=0 \Rightarrow u = \frac{x}{x^2+1}, \quad v = \frac{1}{x^2+1}$

From $(*)$ $\therefore u^2+v^2 = \frac{1}{x^2+1} = v$

$$\therefore u^2+(v-1/2)^2 = (1/2)^2$$ 4

A circle centred $(0, 1/2)$ radius $\frac{1}{2}$



c) $y = cx \quad u = \frac{x}{x^2+(cx-1)^2}, \quad v = \frac{1-cx}{x^2+(cx-1)^2}$

$$u^2+v^2 = \frac{1}{x^2+(cx-1)^2} \qquad v = \frac{1}{x^2+(cx-1)^2} - cu$$ 5

$$\therefore u^2+v^2 = v+cu$$

$$\therefore \left(u-\tfrac{c}{2}\right)^2+\left(v-\tfrac{1}{2}\right)^2 = \tfrac{1}{4}(1+c^2)$$

Family of circles centred at $\left(\frac{c}{2}, \frac{1}{2}\right)$ radius $\frac{1}{2}\sqrt{1+c^2}$.

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EXAMINATION QUESTION / SOLUTION

SESSION : 2000 - 2001

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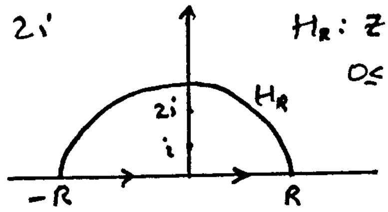
PAPER 3

QUESTION

SOLUTION 15

Two simple poles in C at $z = i, 2i$

$H_R: z = Re^{i\theta}$ $0 \le \theta \le \pi$



Cauchy's Thm $\Rightarrow$

$$\oint \frac{e^{iz}\,dz}{(z^2+4)(z^2+1)} = 2\pi i \{\text{Sum of Residues}\}$$

Residue at $z = i$ $= -\frac{i}{6}e^{-1}$

" " $z = 2i$ $= \frac{e^{-2}}{(1-4)4i} = \frac{i e^{-2}}{12}$

$$\therefore \oint_c = 2\pi i^2\left(\frac{e^{-2}}{12} - \frac{e^{-1}}{6}\right) = \frac{\pi}{6}\left(\frac{2e-1}{e^2}\right)$$ 6

Now $$\oint_c \frac{e^{iz}\,dz}{(z^2+4)(z^2+1)} = \int_{-R}^{R}\frac{e^{ix}\,dx}{(x^2+4)(x^2+1)} + \int_{H_R}\frac{e^{iz}\,dz}{(z^2+4)(z^2+1)}$$

Now take the limit $R \to \infty$

$$\lim_{R\to\infty}\oint_c = \int_{-\infty}^{\infty}\frac{e^{ix}\,dx}{(x^2+4)(x^2+1)} + \lim_{R\to\infty}\int_{H_R}$$ 4

Now by Jordan's Lemma $\lim_{R\to\infty}\int_{H_R} = 0$ provided

(i) Only singularities in upper ½-plane are poles ✓ 3

(ii) $f(z) \to 0$ as $R \to \infty$ ✓ $\int_{H_R} e^{imz}f(z)\,dz = \int_{H_R}$

(iii) $m > 0$ ✓

Moreover
$$\int_{-\infty}^{\infty} = \int_{-\infty}^{\infty}\frac{(\cos x + i\sin x)\,dx}{(x^2+4)(x^2+1)}$$

$$= \int_{-\infty}^{\infty}\frac{\cos x\,dx}{(x^2+4)(x^2+1)} + i.0 \text{ by symmetry}$$ 2

$$\therefore \int_{-\infty}^{\infty}\frac{\cos x\,dx}{(x^2+4)(x^2+1)} = \frac{\pi}{6}\left(\frac{2e-1}{e^2}\right)$$

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EXAMINATION QUESTION / SOLUTION

SESSION : 2000 - 2001

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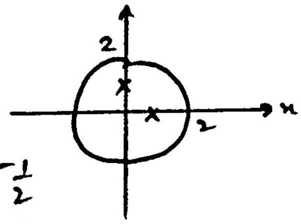
PAPER 3

QUESTION

SOLUTION 16

(ii) $\oint_c F(z)\,dz = 2\pi i \times \{\text{sum of Residues of } F(z) \text{ in } C\}$

Residues at $z = i$ (simple)
$z = 1$ (double)



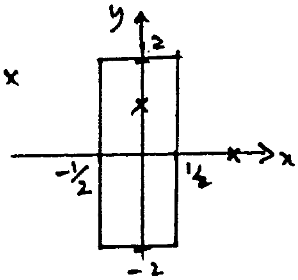
calculated: Res. at $z=i$ is $\frac{i}{(i-1)^2} = \frac{-1}{2}$ 2

" " $z=1$ is $\left[\frac{d}{dz}\left(\frac{z}{z-i}\right)\right]_{z=1}$

$= \left[\frac{(z-i)-z}{(z-i)^2}\right]_{z=1} = \frac{-i}{-2i} = \frac{1}{2}$ 6

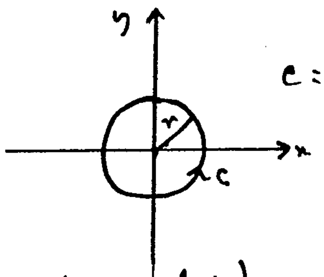
Sum of Residues $= -\tfrac{1}{2} + \tfrac{1}{2} = 0$

If the contour is changed to the box then $z=1$ is excluded and we have $2\pi i \times (-\tfrac{1}{2}) = -\pi i$ 2



(i) $\oint_c \frac{dz}{z} = \int_0^{2\pi} \frac{i r e^{i\theta}\,d\theta}{r e^{i\theta}}$

$= i\int_0^{2\pi} d\theta$



$c: z = re^{i\theta}$
$0 \le \theta \le 2\pi$

$= 2\pi i$ (independent of r) 5

Alternatively, by the Residue Theorem $F(z) = \frac{1}{z}$ has one simple pole at $z=0$ in C

$\therefore \oint_c \frac{dz}{z} = 2\pi i \times (\text{Res at } z=0)$

Res. of $1/z$ at $z=0$ $= 1$

$\therefore \oint_c \frac{dz}{z} = 2\pi i$ regardless of size of circle.

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MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

SESSION : 2000 - 2001

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PAPER 3

QUESTION 17

SOLUTION

$$f*g = \int_{-\infty}^{\infty} f(t')g(t-t')\,dt'$$ ($t' \equiv u$ in question)

$$F.T(f*g) = \int_{-\infty}^{\infty} e^{-i\omega t}(f*g)\,dt$$

$$= \int_{-\infty}^{\infty} e^{-i\omega t}\left(\int_{-\infty}^{\infty} f(t')g(t-t')\,dt'\right)dt$$

Let $\tau = t - t'$: exchanging the order of integration causes no problem as the domain is doubly infinite

$$\therefore FT(f*g) = \int_{-\infty}^{\infty} f(t')\left(\int_{-\infty}^{\infty} e^{-i\omega t} g(t-t')\,dt\right)dt'$$

$$= \int_{-\infty}^{\infty} f(t')\left(e^{-i\omega t'}\int_{-\infty}^{\infty} e^{-i\omega\tau} g(\tau)\,d\tau\right)dt'$$

$$= \left(\int_{-\infty}^{\infty} f(t')e^{-i\omega t'}\,dt'\right)\left(\int_{-\infty}^{\infty} e^{-i\omega\tau} g(\tau)\,d\tau\right)$$ $\boxed{10}$

$$= \quad \bar{f}(\omega) \qquad\qquad \bar{g}(\omega)$$

Now $$\gamma(t) = \int_{-\infty}^{\infty} f(t')f(t-t')\,dt' \Big/ \int_{-\infty}^{\infty} |f(t)|^2\,dt$$

Numerator is a number so we have

$$\bar{\gamma}(\omega) = F.T.(\gamma(t)) = \left(\int_{-\infty}^{\infty} |f|^2\,dt\right)^{-1}\left[\bar{f}(\omega)\bar{f}(\omega)\right]$$ $\boxed{5}$ 2

from the Convolution Thm. 2

Now $$\int_{-\infty}^{\infty} |f(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\bar{f}(\omega)|^2\,d\omega \quad \text{(Parseval)}$$ 1

$$\int_{-\infty}^{\infty} \bar{\gamma}(\omega)\,d\omega = 2\pi$$

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E5

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EXAMINATION QUESTION / SOLUTION

SESSION : 2000 - 2001

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PAPER 3

QUESTION 18

SOLUTION 18

a) $\ddot{x} + \omega^2 x = f(t)$ ($t' \equiv u$ in question)

Using the Math. formulae $\mathcal{L}(\ddot{x}) = s^2\bar{x}(s) - s x(0) - \dot{x}(0)$

$= s^2 \bar{x}(s)$ in this case

$\therefore (s^2+\omega^2)\bar{x}(s) = \bar{f}(s)$ 2

$\therefore \bar{x}(s) = \bar{f}(s)\bar{g}(s)$ where $\bar{g}(s) = \dfrac{1}{s^2+\omega^2}$ 2

Hence $g(t) = \frac{1}{\omega}\sin\omega t$ $(s>0)$

and $x(t) = \mathcal{L}^{-1}[\bar{f}(s)\bar{g}(s)] = \int_0^t g(t')f(t-t')dt'$

$= \frac{1}{\omega}\int_0^t \sin(\omega t')f(t-t')dt'$ Convolution Thm $\boxed{7}$ 3

b) $\dddot{x} + 3\ddot{x} + 3\dot{x} + x = f(t)$

Now $\int_0^\infty e^{-st}\dddot{x}\,dt = \int_0^\infty e^{-st}d(\ddot{x})$

$= [\ddot{x}e^{-st}]_0^\infty + s\int_0^\infty \ddot{x}e^{-st}dt$ 2

For $s>0$, we have, with ICs $\ddot{x}(0) = \dot{x}(0) = x(0)$

$\mathcal{L}(\dddot{x}) = s\mathcal{L}(\ddot{x}) = s^3\bar{x}(s)$

$\therefore (s^3 + 3s^2 + 3s + 1)\bar{x}(s) = \bar{f}(s)$ 3

or $(s+1)^3\bar{x}(s) = \bar{f}(s) \Rightarrow \bar{x}(s) = \underbrace{\bar{g}(s)}_{(s+1)^{-3}}\bar{f}(s)$

Now $\mathcal{L}(s^{-3}) = \frac{1}{2}t^2$, $\boxed{8}$

plus the shift theorem, gives

$g(t) = \frac{1}{2}t^2e^{-t}$

$\therefore x(t) = \frac{1}{2}\int_0^t e^{-t'}t'^2 f(t-t')dt'$ 3

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EXAMINATION QUESTION / SOLUTION

SESSION : 2000 – 2001

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PAPER 3

QUESTION

SOLUTION 19

In $\int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx$, put $x = pt$ $dx = p\,dt$ $x = \infty \Rightarrow$

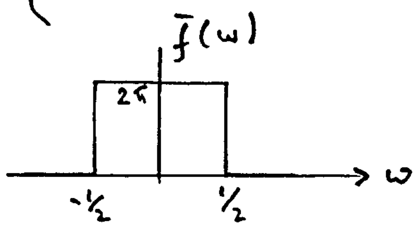
$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx = \int_{-\infty/p}^{\infty/p} \frac{e^{ipt}}{t} dt = \begin{cases} i\pi & p>0 \\ -i\pi & p<0 \end{cases}$$ 3

$$\text{F.T.}\left(\frac{\sin t/2}{t/2}\right) = \int_{-\infty}^{\infty} e^{-i\omega t}\left(\frac{e^{it/2} - e^{-it/2}}{2i\,t/2}\right) dt$$ 6

$$= -i \int_{-\infty}^{\infty} \frac{e^{i(\frac{1}{2}-\omega)t}}{t} dt + i \int_{-\infty}^{\infty} \frac{e^{i(-\omega-\frac{1}{2})t}}{t} dt$$

$$= \begin{cases} -i\,(i\pi) + i\,(-i\pi) & \omega < \frac{1}{2} \\ -i\,(i\pi) + i\,(-i\pi) & -\frac{1}{2} < \omega \\ -i\,(-i\pi) + i\,(-i\pi) & \omega > \frac{1}{2} \\ -i\,(i\pi) + i\,(i\pi) & \omega < -\frac{1}{2} \end{cases}$$ 6

$$= \begin{cases} 2\pi & -\frac{1}{2} < \omega < \frac{1}{2} \\ 0 & \omega > \frac{1}{2},\ \omega < -\frac{1}{2} \end{cases}$$



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EXAMINATION QUESTION / SOLUTION

SESSION : 2000 - 2001

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PAPER 3

QUESTION

SOLUTION 20

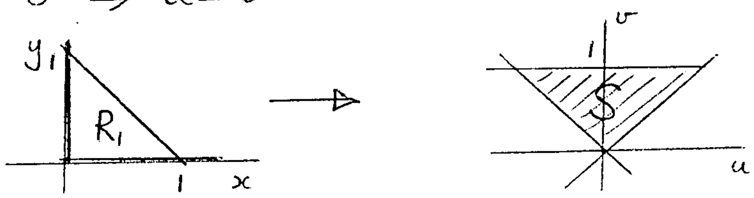
(i) Jacobian is given by $J = \begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix}^{-1}$

$= \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix}^{-1} = (1-(-1))^{-1} = \underline{\underline{\tfrac{1}{2}}}$. 2

We have $u = x-y,\ v = x+y \Rightarrow x = \frac{1}{2}(u+v);\ y = \frac{1}{2}(v-u)$

Thus $x=0 \Rightarrow u=-v$ and $y=1-x \Rightarrow v=1$. 3

$y=0 \Rightarrow u=v$

i.e. 

So: $I_1 = \int_{v=0}^{v=1}\int_{u=-v}^{u=v} v^2\cos(uv)\,du\,dv\cdot\left(\tfrac{1}{2}\right)$ ← from J 2

$= \int_{v=0}^{v=1} \frac{1}{2}v^2\left[\frac{\sin(uv)}{v}\right]_{u=-v}^{u=v} dv$

$= \int_0^1 v\sin(v^2)\,dv$ Subst $t=v^2$.

$= \frac{1}{2}\int_0^1 \sin t\,dt = \underline{\underline{\tfrac{1}{2}(1-\cos(1))}}$ 2

(ii) As above, $J = \frac{1}{2}$.

$x^2+y^2 = \left(\frac{1}{2}(u+v)\right)^2 + \left(\frac{1}{2}(u-v)\right)^2 = \frac{1}{2}(u^2+v^2)$. 2

$y = \pm x \Rightarrow \underline{u=0},\ \underline{v=0}$

$y = x-1 \Rightarrow \underline{u=1},\quad y=1-x \Rightarrow \underline{v=1}$ 2

and $I_2 = \int_{v=0}^{v=1}\int_{u=0}^{u=1} \frac{1}{2}(u^2+v^2)\cdot\frac{1}{2}\,du\,dv$.

$= \frac{1}{4}\int_0^1 \left[\frac{u^3}{3}+v^2u\right]_{u=0}^{u=1} dv$

$= \frac{1}{4}\int_0^1 \left(\frac{1}{3}+v^2\right)dv = \frac{1}{4}\left[\frac{1}{3}v+\frac{v^3}{3}\right]_0^1 = \underline{\underline{\tfrac{1}{6}}}$. 2

Total 15

Setter : WALTON Setter's signature : Andrew Walton

Checker: JACOBS Checker's signature : R. L. Jacobs

E8

MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

SESSION : 2000 - 2001

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PAPER 3
BEng / MEng
PART 2

QUESTION

SOLUTION 22

$$\operatorname{div} \underline{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \qquad \underline{F} = (F_1, F_2, F_3)$$

(i)

so $\operatorname{div} \underline{F} = 2ye^z + x^2ye^z + 2z + 3$ (3)

$$\text{Curl } \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xye^z & x^2e^z & x^2ye^z+z^2+3z \end{vmatrix}$$

$$= \underline{i}\left(x^2e^z - x^2e^z\right) - \underline{j}\left(2xye^z - 2xye^z\right) + \underline{k}\left(2xe^z - 2xe^z\right)$$ (3)

$= 0$

(ii) $\frac{\partial \varphi}{\partial x} = 2xye^z \Rightarrow \varphi = x^2ye^z + g_1(y,z)$ (2)

$\frac{\partial \varphi}{\partial y} = x^2e^z \Rightarrow \varphi = x^2ye^z + g_2(y,z)$

$\frac{\partial \varphi}{\partial z} = x^2ye^z + z^2 + 3z \Rightarrow \varphi = x^2ye^z + \frac{z^3}{3} + \frac{3z^2}{2} + g_3(x,y)$ (4)

Hence $\boxed{\varphi = x^2ye^z + \frac{z^3}{3} + \frac{3z^2}{2} + \text{constant}}$

(iii) $\underline{F}\cdot\underline{i} = 2xye^z$

$x\underline{F}\cdot\underline{i} = 2x^2ye^z$

so $x\underline{F}\cdot\underline{i} - 2\varphi = -\frac{2z^3}{3} - 3z^2 + \text{const.}$ (3)

$\frac{\partial^2}{\partial z^2}\left(x\underline{F}\cdot\underline{i} - 2\varphi\right) = -4z - 6$

$\boxed{15}$

Setter : ATKINSON   Setter's signature : C. Atkinson

Checker : J. D. GIBBON   Checker's signature : J. D. Gibbon

E9

MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

SESSION : 2000 - 2001

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PAPER 3

QUESTION

SOLUTION 23

(i) $\underline{F} = \alpha y \sin x \,\hat{j} + y^2 \cos x \,\hat{i}$

$$\Rightarrow \text{Curl}\,\underline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & 0 \\ y^2\cos x & \alpha y \sin x & 0 \end{vmatrix} = \hat{k}\left(\alpha y \cos x - 2y\cos x\right)$$

$$= (\alpha - 2) y \cos x \,\hat{k}$$

$$= 0 \text{ iff } \alpha = 2$$ 3

(ii) $$\underline{I} = \int_{(0,0)}^{(\frac{\pi}{2},1)} (y^2 \cos x)\,dx + (\beta y \sin x)\,dy$$

(a) Consider C to be $y = \frac{2}{\pi}x \quad (0 \leq x \leq \frac{\pi}{2})$

Then subst for y to get:

$$I = \int_0^{\pi/2} \left(\left(\frac{2}{\pi}\right)^2 x^2 \cos x + \beta\left(\frac{2}{\pi}\right) x \sin x \left(\frac{2}{\pi}\right)\right) dx$$

$$= \left(\frac{2}{\pi}\right)^2 \int_0^{\pi/2} (x^2 \cos x + \beta x \sin x)\,dx$$ 2

by parts

$$= \left(\frac{2}{\pi}\right)^2 \left\{ \underbrace{\left[x^2 \sin x - \beta x \cos x \right]_0^{\pi/2}}_{(\pi/2)^2} - \int_0^{\pi/2} (2x \sin x - \beta \cos x)\,dx \right\}$$

$$= \left(\frac{2}{\pi}\right)^2 \left\{ \left(\frac{\pi}{2}\right)^2 - 2\left(\underbrace{\left[-x\cos x\right]_0^{\pi/2}}_{\text{zero}} + \int_0^{\pi/2} \cos x\,dx\right) + \beta \int_0^{\pi/2} \cos x\,dx \right\}$$

$$= \left(\frac{2}{\pi}\right)^2 \left\{ \left(\frac{\pi}{2}\right)^2 + (\beta - 2) \right\} = 1 + \frac{4}{\pi^2}(\beta - 2)$$ 3

(b) Let $y = \sin x$ and subst. for y: $(0 \leq x \leq \frac{\pi}{2})$

$$I = \int_0^{\pi/2} (\sin^2 x \cos x + \beta \sin^2 x \cos x)\,dx$$

$$= (1+\beta)\left[\frac{\sin^3 x}{3}\right]_0^{\pi/2} = \frac{1}{3}(1+\beta)$$ 3

Equating answers to (a) & (b): $1 + \frac{4}{\pi^2}(\beta - 2) = \frac{1}{3} + \frac{\beta}{3}$

$$\Rightarrow \beta\left(\frac{4}{\pi^2} - \frac{1}{3}\right) = \frac{8}{\pi^2} - \frac{2}{3} = 2\left(\frac{4}{\pi^2} - \frac{1}{3}\right)$$ 2

$\Rightarrow \beta = 2$ FINAL PART: Curl $\underline{F} = 0 \Leftrightarrow \int_C \underline{F}.d\underline{r}$ is path independent 2

So for $\beta = 2$ the answers to (a) & (b) must agree.

Setter : WALTON Setter's signature : Andrew Walton

Checker : JACOBS Checker's signature : R. L. Jacobs

Total 15

MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

SESSION : 2000 - 2001

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PAPER 3

QUESTION

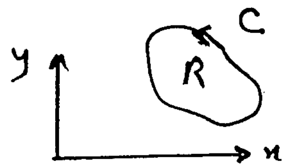
SOLUTION 29

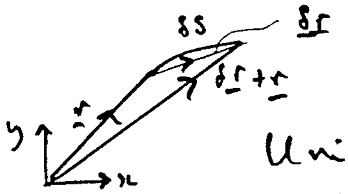
Bookwork

$$\oint_C (P\,dx + Q\,dy) = \iint_R (Q_x - P_y)\,dx\,dy$$

Define a vector $\underline{u} = \hat{\underline{i}}Q - \underline{j}P$

$\therefore$ $\operatorname{div}\underline{u} = Q_x - P_y$ 2





Unit tangent vector $\hat{\underline{T}}$ defined as

$$\hat{\underline{T}} = \frac{d\underline{r}}{ds} = \hat{\underline{i}}\frac{dx}{ds} + \hat{\underline{j}}\frac{dy}{ds}$$

Unit normal $\hat{\underline{n}}$ satisfies $\hat{\underline{n}}\cdot\hat{\underline{T}} = 0$ 4

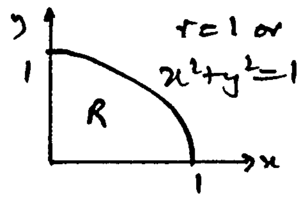
so $\hat{\underline{n}} = \pm\left(\hat{\underline{j}}\frac{dx}{ds} + \hat{\underline{i}}\frac{dy}{ds}\right)$

$\therefore$ $\underline{u}\cdot\hat{\underline{n}} = Q\frac{dy}{ds} + P\frac{dx}{ds}$ 2

$\therefore$ G.T. can be re-expressed as

$$\oint_C (\underline{u}\cdot\hat{\underline{n}})\,ds = \iint_R (\operatorname{div}\underline{u})\,dx\,dy$$

$\underline{u} = \hat{\underline{i}}x^2 + \hat{\underline{j}}y^2 \Rightarrow \operatorname{div}\underline{u} = 2(x+y)$



$\therefore \iint_R \operatorname{div}\underline{u}\,dx\,dy = 2\iint_R (x+y)\,dx\,dy$ 7

$= 2\iint r(\cos\theta+\sin\theta)\,r\,dr\,d\theta$

$= 2\int_0^1 r^2\,dr \int_0^{\pi/2}(\cos\theta+\sin\theta)\,d\theta$

$dx\,dy = r\,dr\,d\theta$

Because

$dx\,dy = \begin{vmatrix} c & s \\ -rs & rc \end{vmatrix} dr\,d\theta$

$= r\,dr\,d\theta$

Setter : J.D. GIBBON

Checker : HERBERT

Setter's signature : J.D. Gibbon

Checker's signature : [signature]

E11

MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

SESSION : 2000 - 2001

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PAPER EE II (3)

QUESTION

SOLUTION Stats 31

$P(A_i|B) = P(A_i \cap B)/P(B) = P(B|A_i)P(A_i)/P(B)$ and

by law of total probabilities $P(B) = \sum_{j=1}^{k} P(B|A_j)P(A_j)$.

Hence $P(A_i|B) = \dfrac{P(B|A_i)P(A_i)}{\sum_{j=1}^{k} P(B|A_j)P(A_j)}$ 4

Let ok = test says 'ok' ; F = faulty disk. Then

$P(ok|\bar{F}) = 0.95$ $\quad P(\overline{ok}|F) = 0.8$ $\quad P(F) = 0.05$ 2

so $P(ok|F) = 0.2$ $\quad P(\bar{F}) = 0.95$

(i) $P(ok) = P(ok|F)P(F) + P(ok|\bar{F})P(\bar{F})$

$= (0.2 \times 0.05) + (0.95 \times .95) = \underline{0.9125}$ 3

(ii) $P(F|ok) = \dfrac{P(ok|F)P(F)}{P(ok|F)P(F) + P(ok|\bar{F})P(\bar{F})} = 0.2 \times 0.05 / .9125$ 3

$= \underline{0.01096}$ to 5 dp

(iii) $P(\overline{ok}_2|F) = 0.99$

$P(\text{accepted}|F) = P(ok_1 \cap ok_2|F)$ $\quad (ok_1 \equiv ok)$

$= P(ok_1|F)P(ok_2|F)$

$= 0.2 \times 0.01$

$= \underline{0.002}$. 3

15

Setter : ATWalden Setter's signature : ATW

Checker : D JHAND Checker's signature : DJH

E12

MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

SESSION : 2000 - 2001

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PAPER EEII(3)

QUESTION

SOLUTION Stats 32

i) The coefficient of correlation is a measure of the strength of the linear relationship between two random variables. If $|\rho_{X,Y}| = 1$ they are perfectly linearly related, if $\rho_{X,Y} = 0$, they are not linearly related. 4

ii) If X and Y are independent $\rho_{X,Y} = 0$. 2

(iii) $f_{X,Y}(x,y) = \begin{cases} x^{-1} & 0 \le y \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$

$$E\{XY\} = \int_{x=0}^{1}\int_{y=0}^{x} xyx^{-1}\,dx\,dy = \int_{x=0}^{1}\int_{y=0}^{x} y\,dx\,dy = \int_{x=0}^{1} \frac{y^2}{2}\Big|_0^x = \int_{x=0}^{1}\frac{x^2}{2}dx = \frac{x^3}{6}\Big|_0^1 = \frac{1}{6}.$$ 2

$$E\{X\} = \int_{x=0}^{1}\int_{y=0}^{x} 1\,dx\,dy = \int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = 1/2.$$ 2

$$E\{Y\} = \int_{x=0}^{1}\int_{y=0}^{x} \frac{y}{x}\,dx\,dy = \int_{x=0}^{1} \frac{1}{x}\cdot\frac{y^2}{2}\Big|_0^x dx = \int_{x=0}^{1}\frac{1}{x}\frac{x^2}{2}dx = \int_0^1 \frac{x}{2}dx = \frac{x^2}{4}\Big|_0^1 = 1/4.$$ 2

So

$$\mathrm{Cov}\{X,Y\} = \frac{1}{6} - \frac{1}{4}\cdot\frac{1}{2} = \frac{1}{24}.$$ 7 1

(iv) $\rho_{X,Y} \neq 0$ since $\mathrm{cov}\{X,Y\} = 1/24$. Hence X and Y are not independent. 2

15

Setter : AT Walden   Setter's signature : ATW

Checker : DJ HAND   Checker's signature : DJH