UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 1997

BEng Honours Degree in Computing Part I
MEng Honours Degrees in Computing Part I
BSc Honours Degree in Mathematics and Computer Science Part I
MSci Honours Degree in Mathematics and Computer Science Part I
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the
Associateship of the Royal College of Science
Associateship of the City and Guilds of London Institute

PAPER 1.3 / MC1.3

DISCRETE MATHEMATICS

Thursday, April 24th 1997, 4.00 - 5.30

Answer THREE questions

For admin. only: paper contains 4 questions

1a Let W be the set $\{a, b, c\}$. Write out the elements of each of the following sets:

i) the powerset of W, $\text{Pow}(W)$;

ii) the Cartesian product $W \times W$;

iii) $\{\,|W|,\ |\text{Pow}(W)|,\ |W \times W|,\ |\text{Pow}(W \times W)|,\ |\text{Pow}(\text{Pow}(W \times W))|\,\}$.

b Let A and B be subsets of a set U.
Write out each of the following as expressions involving only unions and/or intersections of the sets A, B, $U - A$, $U - B$.

i) $A - B$;

ii) $U - (A \cap B)$;

iii) $U - (A \cup B)$.

c Let U be a set and f a function $f\colon U \to \text{Pow}(U)$.
Let d be a function $d\colon \text{Pow}(U) \to \text{Pow}(U)$, defined as follows, for all $A \subseteq U$:

$$d(A) = \{w \in U \mid f(w) \subseteq A\} \quad \text{i.e. } w \in d(A) \text{ iff } f(w) \subseteq A$$

i) Find an expression for $d(U)$ involving only U.

ii) Show that, for all $A, B \subseteq U$, $d(A \cap B) = d(A) \cap d(B)$.

iii) Let p be the following function $p\colon \text{Pow}(U) \to \text{Pow}(U)$:

$$p(A) = U - d(U - A)$$

Show that, for all $A, B \subseteq U$, $p(A \cup B) = p(A) \cup p(B)$.

The three parts carry, respectively, 30%, 20%, 50% of the marks.

2a Let N_6 be the set $\{1, 2, 3, 4, 5, 6\}$.
Let S be the following binary relation on N_6:

$$S = \{ (1,1), (1,2), (1,3), (1,4), (3,4), (4,5), (4,6) \}$$

i) Draw a directed graph representing S.

ii) The reflexive, transitive closure of a binary relation R on a set A is

$$R^* = R^+ \cup \mathrm{id}_A$$

where R^+ is the transitive closure of R and id_A is the identity relation on A. Calculate the reflexive, transitive closure S^* of S and present it as a (6x6) matrix.
(It is not necessary to define the term 'transitive closure'.)

b i) State and define the three properties which are required for a relation to be a (non-strict) partial ordering on a set.

ii) The relation S^* of part (a) is a (non-strict) partial ordering on the set N_6.
Is S^* a total (linear) ordering on N_6 ?
Either prove that it is total or give a counter-example to show it is not.

iii) What are the minimal and maximal elements of S^* (if any)?
(It is not necessary to define what 'minimal' and 'maximal' mean.)

c Let A be any set and f a function (not necessarily onto) $f: A \rightarrow N_6$.
Let R be a binary relation on A defined as follows, for all $x, y \in A$:

$$x \; R \; y \quad \text{iff} \quad f(x) \; S^* f(y)$$

i) Show that R is a *pre-order* (a reflexive and transitive relation) on A.
(You only need the fact that S^* is a partial ordering on N_6. You do not need to refer to specific elements of S^*.)

ii) Show that R is a partial ordering on A iff the function f is one-one.
(You need to show both halves of the 'if and only if'.)

iii) Suppose R as defined here *is* a partial ordering on A.
What can be said about the cardinality of the set A ?
(Justify your answer.)

The three parts carry, respectively, 25%, 25%, 50% of the marks.

Turn over ...

3a i) What is an Eulerian circuit in a graph? Give a necessary and sufficient condition for a graph G to be Eulerian, i.e. to have an Eulerian circuit.

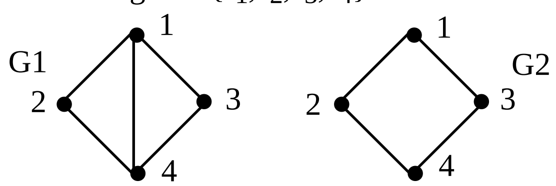
Let a graph G be *k-Eulerian* if G can be made Eulerian by the addition of no more than k arcs. (So an Eulerian graph is 0-Eulerian.)

ii) Find a value of k (depending on n) which guarantees that any graph with n nodes is k-Eulerian. Your value of k should be as low as possible. Justify your answer.

iii) For n=5, show by means of an example that your value of k is best possible.

b For the present purposes, an *interval* is a subset [a,b] of **R** (the real numbers), where $a \le b$, and [a,b] means $\{x \in \mathbf{R}: a \le x \le b\}$. Let S be a finite set of intervals I. Let the associated simple *interval graph* G(S) be defined as follows: The nodes of G(S) are the members of S. Join two different nodes I and I´ iff I and I´ overlap, i.e. $I \cap I' \neq \emptyset$.

i) Consider the following two graphs G1 and G2 on the nodes {1,2,3,4}. Say whether each is an interval graph, i.e. is $G(\{I_1,I_2,I_3,I_4\})$ for some set of intervals $\{I_1,I_2,I_3,I_4\}$. Justify your answers by giving an example $\{I_1,I_2,I_3,I_4\}$ or showing that $\{I_1,I_2,I_3,I_4\}$ cannot exist.

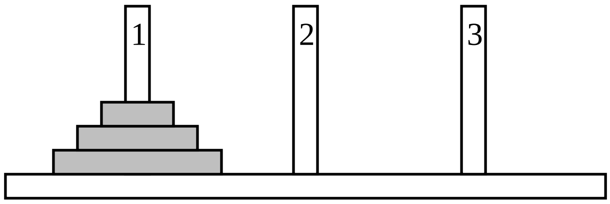


ii) Let K_n be the complete graph on n nodes (i.e. where every pair of distinct nodes is joined exactly once). Show by induction on n that if $G(\{I_1,\dots,I_n\})$ is K_n then $I_1 \cap \dots \cap I_n \neq \emptyset$.

4a i) Describe the Mergesort algorithm for sorting a list (call it L) of n integers.

ii) Obtain a recurrence relation for W(n), the worst-case number of comparisons used by Mergesort.

b In the Towers of Hanoi problem, there are three pegs and n discs, such that discs 1 to n are strictly increasing in size. Initially the n discs are all on peg 1 in order of size, with disc n at the bottom (illustrated below for n=3). The object is to move all the discs to peg 3, by moving one disc at a time from one peg to another, ensuring that at no time is a larger disc placed on top of a smaller disc.



The following is a recursive algorithm to solve the problem:

3-Hanoi
Move discs 1 to n-1 to peg 2;
Move disc n to peg 3;
Move discs 1 to n-1 to peg 3.

i) Obtain a recurrence relation for the number of steps S(n) taken by 3-Hanoi.

ii) Solve the recurrence relation.

iii) Is 3-Hanoi best possible (in terms of the number of steps taken)? Explain your answer briefly.

c Now suppose that we have the same problem as in part (b), except that there are four pegs and the problem is to move the discs from peg 1 to peg 4. For simplicity suppose n is a power of 2. We use a new algorithm as follows:

4-Hanoi
Move discs 1 to n/2 to peg 2;
Move discs n/2+1 to n to peg 4;
Move discs 1 to n/2 to peg 4.

i) Obtain a recurrence relation for the number of steps T(n) taken by 4-Hanoi. [Hint: it will use S from (b) above]

ii) Find the order of T(n) in terms of S and thereby show that T(n) has strictly lower order than S(n).

End of paper

