EE2-08A MATHEMATICS

1. Given the complex mapping from $z = x + iy$ to $w = u + iv$:

$$w = \frac{1}{z+i}$$

a) Show that circles $x^2 + (y+1)^2 = a^2$ in the $z-$plane map to circles in the $w-$plane, and give the equation of the circles in terms of u, v. [4]

b) Show that the axes in the $z-$plane map to an axis and a circle in the $w-$plane. Obtain the axis and circle. [3]

c) Obtain the images in w of the lines $y = x - 1$ and $y = -1$. [3]

2. Given the real integral

$$I = \int_0^{2\pi} \frac{d\theta}{(5 + 3\cos\theta)^2},$$

a) Use the substitution $z = e^{i\theta}$ to show that

$$I = -i \oint_C \frac{4z\, dz}{(3z+1)^2(z+3)^2},$$

where C is the unit circle in the complex plane. [6]

b) Using Cauchy's residue theorem, or otherwise, calculate I. [4]

Recall that the residue of a complex function $F(z)$ at a pole $z = a$ of multiplicity m is given by the expression

$$\lim_{z \to a} \left\{ \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m F(z) \right] \right\}.$$

3. a) The complex function

$$F(z) = \frac{e^{imz}}{(z^2+4)^2}$$

has two double poles. Find the residue at the pole lying in the upper half of the complex plane. [5]

b) Consider the contour integral $I = \oint_C \frac{e^{imz}}{(z^2+4)^2}\, dz\,,$

where the closed contour C consists of a semi-circle in the complex upper half-plane, taken in the anti-clockwise sense, and $m > 0$.

Using the result from (a), Cauchy's Residue Theorem and Jordan's lemma, show that

$$\int_{-\infty}^{\infty} \frac{\cos mx}{(x^2+4)^2}\, dx = \frac{(2m+1)\pi}{16} e^{-2m}\,.$$

[10]

4. a) Two functions $f(t)$ and $g(t)$ have Laplace transforms $\overline{f}(s) = \mathscr{L}[f(t)]$ and $\overline{g}(s) = \mathscr{L}[g(t)]$, respectively. If the convolution of $f(t)$ with $g(t)$ is defined as

$$f \star g = \int_0^t f(u)g(t-u)\, du\,,$$

prove the Laplace Convolution theorem: $\mathscr{L}[f \star g] = \overline{f}(s)\overline{g}(s)\,.$ [5]

b) Use the Laplace Convolution theorem to solve the second order ordinary differential equation

$$\frac{d^2x}{dt^2} + 9x = \sin 3t\,,$$

with initial conditions $x(0) = x'(0) = 0\,.$ [10]

[Recall the identity $2\sin A \sin B = \cos(A-B) - \cos(A+B)$.]