/ c A

EE2-5

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2012

EEE/ISE PART II: MEng, BEng and ACGI

Corrected Copy

SIGNALS AND LINEAR SYSTEMS

Monday, 28 May 2:00 pm

Time allowed: 1:30 hours

There are THREE questions on this paper.

Answer ALL questions.

Question One carries 40% of the marks. The other 2 questions each carry 30%.

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : P.L. Dragotti
Second Marker(s) : P.T. Stathaki

© Imperial College London

Special Information for the Invigilators: none

Information for Candidates

Some Fourier Transforms

$$\mathrm{rect}(\tfrac{t}{\tau}) \iff \tau\,\mathrm{sinc}(\tfrac{\omega\tau}{2})$$

$$\tfrac{W}{\pi}\mathrm{sinc}(Wt) \iff \mathrm{rect}(\tfrac{\omega}{2W})$$

Time-integration property of the Fourier transform

$$\int_{-\infty}^{t} x(\tau)d\tau \iff \frac{X(\omega)}{j\omega} + \pi X(0)\delta(\omega)$$

Time-shifting property of the Fourier transform

$$x(t - t_d) \iff X(\omega)e^{-j\omega t_d}$$

The sum of the first M terms of a geometric series is:

$$\sum_{n=0}^{M-1} \rho^n = \frac{1-\rho^M}{1-\rho}$$

A useful Laplace transform

$$e^{\lambda t}u(t) \iff \frac{1}{s-\lambda}$$

The Questions

1. This question carries 40% of the mark.

   (a) Given the signal:

$$x(t) = \begin{cases} t & \text{for } \ 0 \le t \le 1 \\ 0 & \text{otherwise,} \end{cases}$$

   sketch and dimension each of the following signals:

   i. $x_1(t) = x(t-2)$

   [2]

   ii. $x_2(t) = x(-2t+3)$

   [2]

   (b) State with a brief explanation if the systems with the following input/output relationships are linear/non-linear, time-invariant/time-varying.

   i. $y(t) = 3 + 2x(t)$

   [2]

   ii. $y(t) = x(t)\sin(3t + \pi/4)$

   [2]

   (c) Given the following two signals:

$$f_1(t) = \begin{cases} t, & 0 \le t \le 1 \\ 2-t, & 1 < t \le 2 \\ 0, & \text{otherwise} \end{cases}$$

   and

$$f_2(t) = \delta(t+2) + \delta(t+1),$$

   sketch and dimension $c(t) = f_1(t) * f_2(t)$.

   [4]

Question 1 continues on next page

(d) Using the definition of the Laplace transform, compute the Laplace transform of $x(t) = u(t)$ where

$$u(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

and state clearly the Region of Convergence (ROC) of the transform.

[2]

(e) Using the time-integration property of the Fourier transform, determine the Fourier transform of $x(t) = u(t)$.

[2]

(f) Compare $X(\omega)$ with $X(s)$ for $s = j\omega$. Explain why they are different.

[2]

(g) Consider the electric circuit shown in Fig. 1.

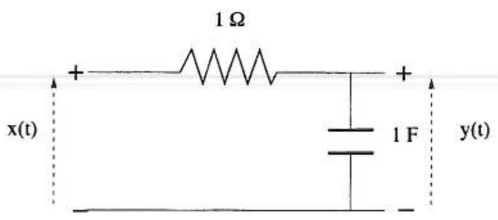


Figure 1: An *RC* circuit.

i. Find the linear differential equation that relates the input $x(t)$ to the output $y(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau$.

[4]

ii. Find the characteristic polynomial, characteristic roots and characteristic modes of this system.

[4]

iii. Find the zero-input component of the response $y(t)$ for $t \geq 0$, if the initial condition is $y(0) = 2$.

[4]

Question 1 continues on next page

(h) Consider the signal $x(t) = 4000\text{sinc}(4000\pi t)$

i. Sketch and dimension the Fourier transform of $x(t)$

[2]

ii. Determine the Nyquist sampling rate for $x(t)$

[2]

iii. Determine the Nyquist sampling rate for $x^2(t)$.

[2]

(i) Using the definition of the z-transform, compute the z-transform of

$$x[n] = a^n u[n] - a^n u[n-5],$$

where $a = 1/2$.

[4]

2. Consider the system connected in parallel as depicted in Fig. 2. Here the linear

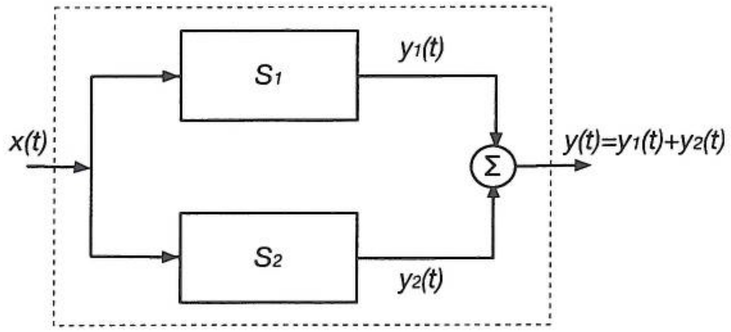


Figure 2: A parallel system.

system S_1 has the following input/output relationship:

$$\frac{d^2y_1}{dt^2} + 2\frac{dy_1}{dt} - 3y_1(t) = \frac{dx}{dt}$$

and the system S_2 has the following input/output relationship:

$$y_2(t) = 2x(t).$$

(a) Find the transfer function of S_1 and S_2.

[10]

(b) Find the transfer function of the parallel connected system.

[6]

(c) Assume the system was at rest when it was excited by the input $x(t) = e^{-2t}u(t)$, determine the exact expression of the output $y(t)$ for $t \geq 0$.

[14]

3. Consider the system $h(t)$ shown in Fig. 3. This system is obtained by cascading two 'zero-order hold' systems.

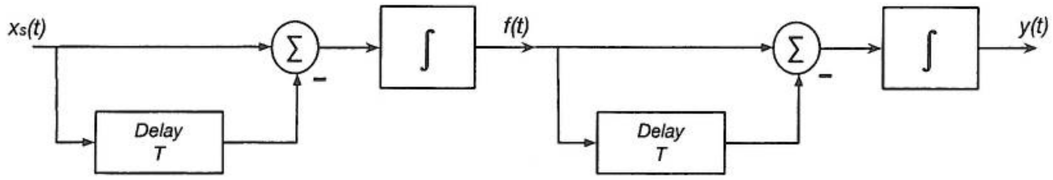


Figure 3: A cascade of two 'zero-order hold' systems .

(a) Find the unit impulse response $h(t)$ of the system.

[10]

(b) Find the frequency response $H(\omega)$ of the system.

[10]

(c) The system $h(t)$ is now used to reconstruct a sampled signal. Therefore the incoming signal is $x_s(t) = \sum_{n=-\infty}^{\infty} x_n \delta(t - nT)$, where $x_n = x(nT)$ are the samples. We assume the delay T is equal to the sampling period and $T = 1$. Assume

$$x(t) = \begin{cases} t^2 & \text{for } \; 0 \leq t \leq 3 \\ 0 & \text{otherwise.} \end{cases}$$

i. Derive the exact values of the samples $x_n = x(nT)$.

[5]

ii. Sketch and dimension the reconstructed signal $y(t)$.

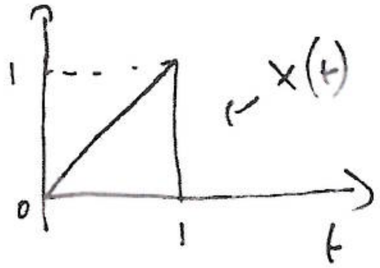
[5]

EE2-5 SIGNALS & LINEAR SYSTEMS
2012

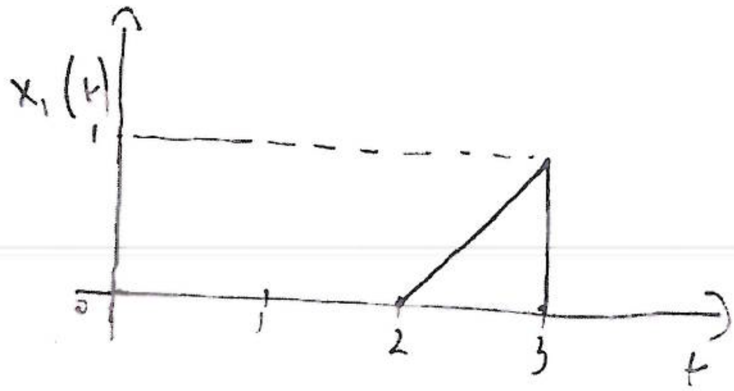
ANSWERS

QUESTION 1

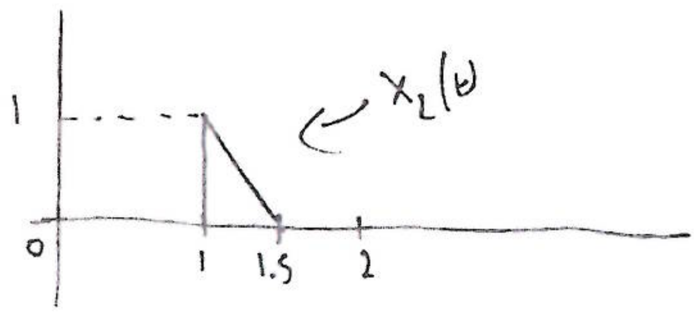
1.(a)



i.



$$x_2(t) = x(-2t+3) = x\left(-2\left(t-\tfrac{3}{2}\right)\right)$$



(b) LINEARITY MEANS THAT

2

IF

$x_1(t) \rightarrow y_1(t)$

AND

$x_2(t) \rightarrow y_2(t)$

THEN

$$\alpha x_1(t) + \beta x_2(t) \rightarrow \alpha y_1(t) + \beta y_2(t)$$

FEEDBACK: NOTE THAT MOST STUDENT GOT THIS WRONG BECAUSE THEY DID NOT REALIZE THAT A CONSTANT TERM LEADS TO NON-LINEAR SYSTEMS

THEREFORE

i. IS NON/LINEAR DUE TO THE DC TERM:

$$y(t) = 3 + 2(\alpha x_1(t) + \beta x_2(t)) \neq \alpha y_1(t) + \beta y_2(t) = (\alpha+\beta)3 + 2(\alpha x_1 + \beta x_2)$$

ii. IS INSTEAD LINEAR

$$y(t) = (\alpha x_1(t) + \beta x_2(t)) \sin(3t + \pi/4) = \alpha y_1(t) + \beta y_2(t)$$

TIME INVARIANCE MEANS THAT

IF

$x(t) \rightarrow y(t)$

THEN

$x(t-T) \rightarrow y(t-T)$

THEREFORE

i. IS TIME-INVARIANT $3 + x(t-T) = y(t-T)$

ii. IS TIME/VARING $x(t-T)\sin(3t + \pi/4) \neq y(t-T)$

3

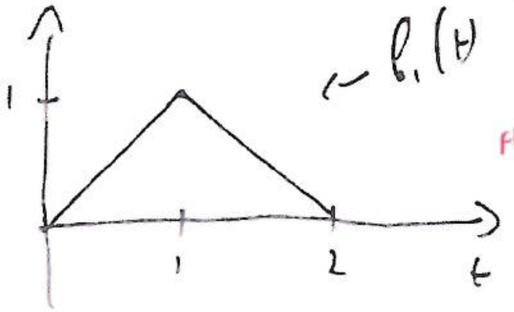
(c) FIRST NOTICE THAT

$$f(t) * \delta(t-2) = f(t-2)$$

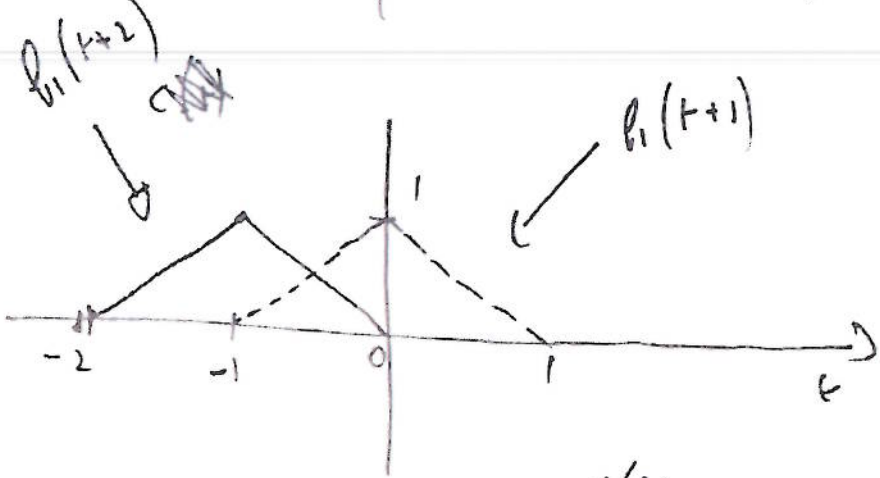
FEEDBACK: SOME STUDENTS DID NOT SEEM TO BE AWARE OF THIS BASIC FACT

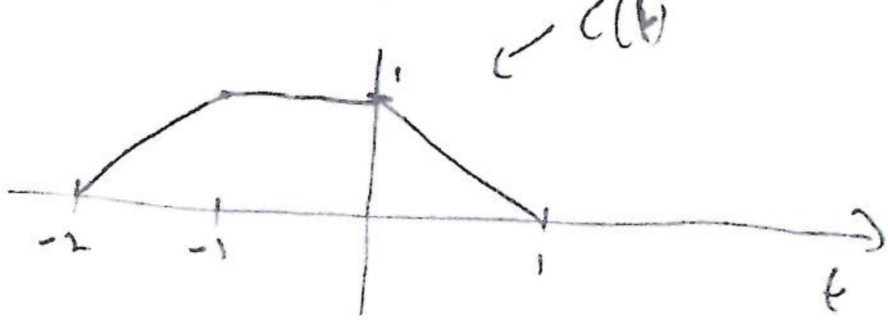
THEREFORE, BECAUSE OF LINEARITY OF CONVOLUTION, WE HAVE THAT

$$c(t) = f_1(t) + f_2(t) = f_1(t+2) + f_1(t+1)$$



FEEDBACK: MANY STUDENTS DID NOT SHIFTED SIGNAL PROPERLY TO THE 'LEFT' OF THE PLOT





(d)

$$X(s) = \int_{0^-}^{\infty} u(t)\, e^{-st}\, dt = \int_{0}^{\infty} e^{-st}\, dt = \left.\frac{e^{-st}}{-s}\right|_{0}^{\infty}$$

4

$$= \frac{1}{s}$$ ROC is $(\mathrm{Re}\, s) > 0$

FEEDBACK: s IS A COMPLEX NUMBER THUS ROC IS $(\mathrm{Re}\, s) > 0$. MOST STUDENT JUST WROTE $s > 0$ WHICH IS INCORRECT

(e)

NOTE THAT $\dfrac{du(t)}{dt} = \delta(t)$ AND $\displaystyle\int_{-\infty}^{t} \delta(\tau)\, d\tau = u(t)$

~~AND~~ MOREOVER

$$\delta(t) \Longleftrightarrow 1$$

~~USING~~ THEREFORE USING TIME-INTEGRATION PROPERTY

$$\int_{-\infty}^{t} x(\tau)\, d\tau \Longleftrightarrow \frac{X(\omega)}{j\omega} + \pi X(0)\,\delta(\omega)$$

WE OBTAIN

$$u(t) \Longleftrightarrow \frac{1}{j\omega} + \pi\,\delta(\omega)$$

(f) THEY ARE DIFFERENT SINCE 5

$$\int_{-\infty}^{\infty} u(t)\,dt = +\infty$$

OR ROC IN (d) DOES NOT CONTAIN THE AXIS $s = jw$

(g)

1. $$y(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)\,d\tau \qquad (1)$$

$$\frac{dy}{dt} = \frac{1}{C}\, i(t) \qquad (2)$$

$$y(t) + R\,i(t) = x(t) \qquad (3)$$

BY REPLACING (2) IN (3) WE OBTAIN

$$CR\frac{dy}{dt} + y(t) = x(t)$$

$$CR = 1 \Rightarrow \frac{dy}{dt} + y(t) = x(t)$$

6

CHARACTERISTIC POLYNOMIAL

$$(D+1)$$

CHARACTERISTIC ROOT $\lambda_1 = -1$

CHARACTERISTIC MODE $y(t) = c\, e^{\lambda_1 t} = c\, e^{-t}$

$y(0) = 2 \Rightarrow$ ZERO-INPUT RESPONSE:

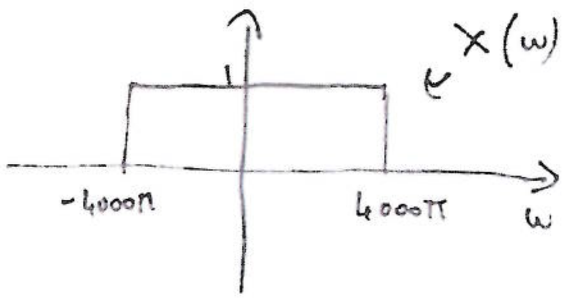
$$y(t) = 2e^{-t}$$

(h)

i $x(t) = 4000 \operatorname{sinc}(4000\pi t)$

USING THE TABLE

$$X(\omega) = \operatorname{rect}\left(\frac{\omega}{8000\pi}\right)$$



(ii) MAXIMUM NON-ZERO FREQUENCY OF $x(t)$

IS $f_m = \frac{4000\pi}{2\pi} = 2000\ Hz$. ~~thus fs~~

7

THE NYQUIST ~~FREQUENCY~~ SAMPLING RATE

IS $f_s = 2 \cdot f_m = 4000\,Hz = 4\,KHz$

(iii)

USING THE FACT THAT

$$x^2(t) = x(t) \cdot x(t) \Leftrightarrow \frac{1}{2\pi} X(\omega) * X(\omega)$$

WE IMMEDIATELY REALIZE THAT THE NEW SIGNAL HAS A BANDWIDTH WHICH IS TWICE THE ORIGINAL BANDWIDTH AND WHICH IS STILL CENTRED IN ZERO, THEREFORE

$$f_s = 8\,KHz$$

(i)

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n} = \sum_{n=0}^{\infty} a^n z^n (u[n] - u[n-5])$$

$$= \sum_{n=0}^{4} a^n z^n = \frac{(az)^5 - 1}{az - 1}$$

QUESTION 2

8

FEEDBACK: MOST STUDENT ANSWERED PROPERLY TO THIS QUESTION.

(a) TRANSFER FUNCTION:

$$H(s) = \frac{Y(s)}{X(s)}$$

THEREFORE, FOR S_1

$$\frac{d^2y}{dt^2} + 2\frac{dy_1}{dt} - 3y_1(t) = \frac{dx}{dt}$$

$$\Downarrow$$

$$(s^2 + 2s - 3)Y(s) = sX(s)$$

AND

$$H_1(s) = \frac{s}{(s^2+2s-3)} = \frac{s}{(s+3)(s-1)}$$

FOR S_2

$$y_2(t) = 2x(t)$$

THUS $H_2(s) = 2$

(b) $H(s) = H_1(s) + H_2(s) = 2 + \frac{s}{(s+3)(s-1)} = \frac{2s^2+5s-6}{(s+3)(s-1)}$

(c)

9

$$x(t) = e^{-2t}u(t) \Rightarrow X(s) = \frac{1}{s+2}$$

$$Y(s) = H(s)\cdot X(s) = \frac{2s^2+5s-6}{(s+3)(s-1)(s+2)}$$

USING PARTIAL FRACTION

$$Y(s) = \frac{A}{s+3} + \frac{B}{s-1} + \frac{C}{s+2} = \frac{2s^2+5s-6}{(s+3)(s-1)(s+2)}$$

WITH $A = -\frac{3}{4}$, $B = \frac{1}{12}$, $C = \frac{8}{3}$

SO

$$Y(s) = -\frac{3}{4}\frac{1}{s+3} + \frac{1}{12}\frac{1}{s-1} + \frac{8}{3}\frac{1}{s+2}$$

AND

$$y(t) = -\frac{3}{4}e^{-3t}u(t) + \frac{1}{12}e^{t}u(t) + \frac{8}{3}e^{-2t}u(t)$$

FEEDBACK: REMEMBER THE TERM 'u(t)'.

QUESTION 3

10

(a)

UNIT IMPULSE RESPONSE OBTAINED BY SETTING $x_S(t) = \delta(t)$

THEREFORE

$$f(t) = \int_{-\infty}^{t} \delta(\tau) - \delta(\tau - T)\, d\tau$$

$$= u(t) - u(t-T)$$

$$y(t) = h(t) = \int_{-\infty}^{t} f(\tau) - f(\tau - T)\, d\tau$$

$$= \int_{-\infty}^{t} u(\tau) - u(\tau - T) - u(\tau - T) + u(\tau - 2T)\, d\tau$$

SINCE

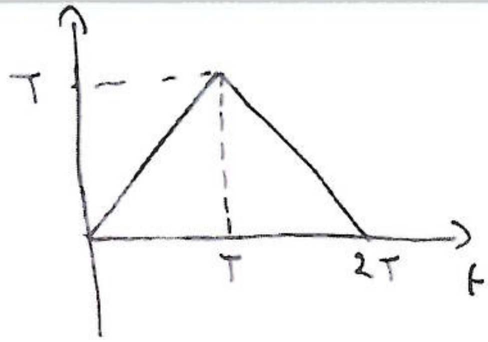
$$\int_{-\infty}^{t} u(t)\, dt = t\, u(t)$$

WE OBTAIN

$$h(t) = t\,u(t) - 2(t-T)\,u(t-T) + (t-2T)\,u(t-2T)$$

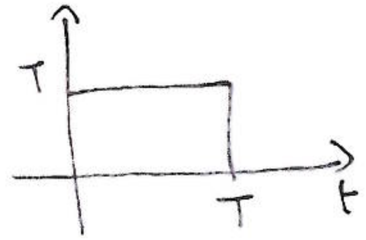
CONSEQUENTLY

$h(t) \to$



THE SAME RESULT
COULD HAVE BEEN OBTAINED
BY NOTICING THAT $h(t) = f(t) * f(t)$
AND

$f(t) \to$



(b)

$H(\omega)$ IS THE FOURIER TRANSFORM
OF $h(t)$.
SINCE $h(t) = f(t) * f(t)$ THEN
$H(\omega) = F(\omega) \cdot F(\omega)$ AND

USING TIME-SHIFTING PROPERTY
WE HAVE

$$f(t) = \mathrm{RECT}\left(\frac{t}{T} - 0.5\right) \Longleftrightarrow F(\omega) = T\,\mathrm{SINC}\left(\frac{\omega T}{2}\right) e^{-j\frac{\omega T}{2}}$$

AND

$$H(\omega) = F^2(\omega) = T^2 \mathrm{SINC}^2\left(\frac{\omega T}{2}\right) e^{-j\omega T}$$

FEEDBACK: MOST STUDENTS DID NOT REALIZE THAT $H(\omega) = F(\omega) \cdot F(\omega)$. THEY TRIED TO USE THE DEFINITION OF THE FOURIER TRANSFORM (FT) TO COMPUTE THE FT OF  THIS IS POSSIBLE BUT MUCH HARDER.

(c)

12

$$X_n = x(nT) = \begin{cases} 1 & \text{FOR } n=1 \\ 4 & \text{FOR } n=2 \\ 9 & \text{FOR } n=3 \\ 0 & \text{OTHERWISE} \end{cases}$$