EXAM SOLUTIONS

1. a) i) [2]

We first sample the signal. Sampling rate must be greater than twice the highest frequency component of the signal (i.e., Nyquist rate). Then the sample values are quantized, and each quantization interval is mapped to a distinct sequence of bits.

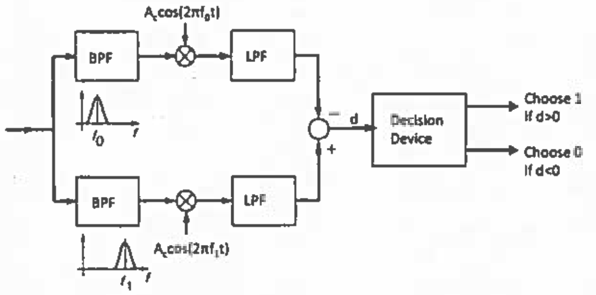
ii) [2]

A signal whose energy is concentrated in a frequency band around DC is called a baseband signal, whereas if the signal energy is concentrated away from DC, it is called a passband signal.

iii) [3]

Only the signal $z(t)$ can be a real baseband signal, since it is the only baseband signal that has conjugate symmetry.

iv) [5]



Each branch is a traditional coherent receiver for demodulation at frequencies f_0 and f_1, respectively. Whichever branch has higher energy, we make a decision in favour of that. That is, if a bit zero is transmitted, and the upper branch detects the transmitted waveform together with noise, while the lower branch only detects noise. The difference between the two should be negative as the upper branch will have more energy, and the decision device outputs 0.

b) i) True. For any autocorrelation function, we have $R_X(t,t) = E[X(t)X(t)] >= 0\,\forall t$. However, for the given function we have $R_X(t,t) = \sin(2t)$, which might take negative values. [2]

ii) True. This was proven in the lecture. [2]

iii) True. $\Phi(x) = P(X \leq x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}dt$. [2]

c) For a Gaussian random variable $X \sim \mathcal{N}(\mu, \sigma^2)$, we have shown in the lecture that $\mathrm{P}\{X > c\} = Q\left(\frac{c-\mu}{\sigma}\right)$.

i) $\mathrm{P}\{-5 < X < 2\} = \mathrm{P}\{-5 < X\} - \mathrm{P}\{2 < X\} = Q\left(\frac{-5+2}{\sqrt{2}}\right) - Q\left(\frac{2+2}{\sqrt{2}}\right) = Q\left(\frac{-3}{\sqrt{2}}\right) - Q\left(\frac{4}{\sqrt{2}}\right)$. [2]

ii) $\mathrm{P}\{X^2 - 2X > 3\} = \mathrm{P}\{(X-3)(X+1) > 0\} = \mathrm{P}\{(X < -1\} + \mathrm{P}\{(X > 3\} = \left[1 - Q\left(\frac{-1+2}{\sqrt{2}}\right)\right] + Q\left(\frac{3+2}{\sqrt{2}}\right) = 1 + Q\left(\frac{5}{\sqrt{2}}\right) - Q\left(\frac{1}{\sqrt{2}}\right)$. [3]

iii) $\mathrm{P}\{X \cdot Y - Y + 3X < 3\} = \mathrm{P}\{(X-1)(Y+3) < 0\} = \mathrm{P}\{X - 1 < 0, Y + 3 > 0\} + \mathrm{P}\{X - 1 > 0, Y + 3 < 0\} = \mathrm{P}\{X < 1\} \cdot \mathrm{P}\{Y > -3\} + \mathrm{P}\{X > 1\} \cdot \mathrm{P}\{Y < -3\} = \left[1 - Q\left(\frac{3}{\sqrt{2}}\right)\right] Q\left(\frac{-4}{\sqrt{3}}\right) + Q\left(\frac{3}{\sqrt{2}}\right)\left[1 - Q\left(\frac{-4}{\sqrt{3}}\right)\right]$. [4]

iv) Note that the random variables on the left and right of the inequality are both standard random variables, i.e., zero-mean unit-variance. Due to symmetry and their independence, the probability of one being greater than the other is 1/2. This is also equal to $Q(0)$. [3]

d) The autocorrelation of the given WSS random process can be found by taking the inverse Fourier transform of the given PSD. We have

$$R_X(\tau) = e^{-\alpha|\tau|} + \cos\left(\frac{\pi}{2}\tau\right) + 3.$$

For a real WSS process, we have

$$\begin{aligned} E\left[\left(X(t+\tau) - X(t)\right)^2\right] &= E[X^2(t+\tau) - 2X(t+\tau)X(t) + X^2(t)] \\ &= 2R_X(0) - 2R_X(\tau) \\ &= 2[R_X(0) - R_X(\tau)]. \end{aligned}$$

i) [5]

For $\tau = 2$, we have

$$2[R_X(0) - R_X(\tau)] = 2[3 - e^{-2\alpha}].$$

ii) [5]

$$\begin{aligned} E\left[\left(X(t+\tau) + X(t) - X(t-\tau)\right)^2\right] &= E[X^2(t+\tau) + X^2(t) + X^2(t-\tau) + 2X(t+\tau)X(t) \\ &\quad - 2X(t+\tau)X(t-\tau) - 2X(t)X(t-\tau)] \\ &= 3R_X(0) - 2R_X(2\tau) \\ &= 7 - 2e^{-4\alpha}. \end{aligned}$$

2. a) i) [4]

We have

$$E[X(t)] = E[Y(t)]\cos(2\pi f_c t) - E[Z(t)]\sin(2\pi f_c t)$$

Since cosine and sine waves are linearly independent, the mean value of $X(t)$ above can be constant if and only if $E[Y(t)] = E[Z(t)] = 0$, or $E[Y(t)] = k\cos(2\pi f_c t)$ and $E[Z(t)] = k\sin(2\pi f_c t)$, for some $k \in \mathscr{R}$.

ii) [6]

We know from above that $E[X(t)] = 0$.

We have

$$\begin{aligned}
R_X(t_1,t_2) =& E[X(t_1)X(t_2)],\\
=& E[Y(t_1)Y(t_2)]\cos(2\pi f_c t_1)\cos(2\pi f_c t_2)\\
&+ E[Z(t_1)Z(t_2)]\sin(2\pi f_c t_1)\sin(2\pi f_c t_2)\\
&- E[Y(t_1)Z(t_2)]\cos(2\pi f_c t_1)\sin(2\pi f_c t_2)\\
&- E[Y(t_2)Z(t_1)]\sin(2\pi f_c t_1)\cos(2\pi f_c t_2)\\
=& R_Y(t_1,t_2)\cos(2\pi f_c t_1)\cos(2\pi f_c t_2) + R_Z(t_1,t_2)\sin(2\pi f_c t_1)\sin(2\pi f_c t_2)\\
=& R_Y(t_1,t_2)\left[\frac{1}{2}\cos 2\pi f_c(t_1 - t_2) + \frac{1}{2}\cos 2\pi f_c(t_1 + t_2)\right]\\
&+ R_Z(t_1,t_2)\left[\frac{1}{2}\cos 2\pi f_c(t_1 - t_2) - \frac{1}{2}\cos 2\pi f_c(t_1 + t_2)\right]\\
=& \left[\frac{R_Y(t_1,t_2) + R_Z(t_1,t_2)}{2}\right]\cos 2\pi f_c(t_1 - t_2)\\
&+ \left[\frac{R_Y(t_1,t_2) - R_Z(t_1,t_2)}{2}\right]\cos 2\pi f_c(t_1 + t_2).
\end{aligned}$$

For this to depend only on the time difference, we need to have $R_Y(t_1,t_2) = R_Z(t_1,t_2)$, and both $R_Y(t_1,t_2)$ and $R_Z(t_1,t_2)$ must depend only on the time difference $t_1 - t_2$, i.e., $Y(t)$ and $Z(t)$ must be WSS. Then we have $R_X(\tau) = R_Y(\tau)\cos(2\pi f_c \tau)$.

iii) [3]

We have $S_X(f) = \frac{1}{2}(S_Y(f - f_c) + S_Y(f + f_c))$.

iv) [3]

Note that $X(t)$ is the model of a passband white Gaussian noise signal we have seen in the lecture. We have $S_X(f) = N_o$.

Since $Y(t)$ and $Z(t)$ are Gaussian and independent, $X(t)$ is also Gaussian. We also have shown that $X(t)$ is WSS under these conditions. Then it is also SSS.

b) i) [3]

Let X denote the input bit, and Y denote the decoded bit.

$$P(X = 0|Y = 1) = \frac{P(X = 0, Y = 1)}{P(Y = 1)} = \frac{q_0 p_0}{q_0 p_0 + (1 - q_0)(1 - p_1)}$$

ii) [6]

For an error that cannot be corrected, at least 2 out of 7 transmitted bits should be received in error. When all-zero sequence is transmitted, the error probability is given by

$$\sum_{i=2}^{7}\binom{7}{i}(1-p_0)^{7-i}p_0^i$$

Since $1-p_0 \simeq 1$, we can approximate the above sum with the first dominant term as $\binom{7}{2}10^{-6} = 0.21 \times 10^{-8}$.

iii) [5]

Since $p_0 = p_1$, all channel codewords are protected equally; and therefore, independent of the probabilities the messages, the average probability will be the same as in the previous question.

3. a) i) [3]

When a 0 is transmitted, the receiver makes an error if the received signal is above the threshold T. Then we can write the error probability as

$$\begin{aligned} P_{e0} &= \int_T^\infty \frac{1}{2b}e^{-|x|/b}dx \\ &= \int_T^\infty \frac{1}{2b}e^{-x/b}dx \\ &= \frac{1}{2}e^{-T/b}. \end{aligned}$$

ii) [3]

When a 1 is transmitted, the receiver makes an error if the received signal is below the threshold T. Then the corresponding error probability is

$$\begin{aligned} P_{e1} &= P\{A+W<T\} = P\{W<T-A\} = P\{W>A-T\} \\ &= \int_{A-T}^\infty \frac{1}{2b}e^{-x/b}dx \\ &= \frac{1}{2}e^{-(A-T)/b}. \end{aligned}$$

iii) [2]

$$P_e = p_0 \cdot P_{e0} + p_1 \cdot P_{e1}.$$

iv) [10]

$$P_e = p_0 \cdot \frac{1}{2}e^{-T/b} + p_1 \cdot \frac{1}{2}e^{-(A-T)/b}$$

To find the optimal threshold, we find T which satisfies $\frac{dP_e}{dT} = 0$. We have

$$\frac{dP_e}{dT} = -\frac{p_0}{2b}e^{-T/b} + \frac{p_1}{2b}e^{(T-A)/b} = 0.$$

This is equivalent to

$$\frac{p_1}{1-p_1} = e^{(A-2T)/b},$$

from which we get the optimal threshold value as

$$T = \frac{1}{2}\left(A - b\ln\frac{p_1}{1-p_1}\right).$$

For $p_1 = 2/3, A = 2$, and $b = 1/\ln(2)$, we have $T = (A-1)/2 = 1/2$. The error probability is then given by

$$\begin{aligned} P_e =& \frac{1}{6}e^{-T/b} + \frac{2}{3}\cdot\frac{1}{2}e^{-(A-T)/b} \\ =& 1/3\sqrt{2}. \end{aligned}$$

b) i) [2]

The word entropy is given by

$$\begin{aligned} -0.3\log_2 0.3 - 0.25\log_2 0.25 - 0.2\log_2 0.2 - 0.15\log_2 0.15 \\ -0.05\log_2 0. - 0.05\log_2 0.05 = 2.328 \text{ bits per word} \end{aligned}$$

ii) [3]

The letters have the following probabilities $\Pr(x) = 0.45$, $\Pr(y) = 0.35$ and $\Pr(z) = 0.2$. Therefore, the letter entropy is given by

$$-0.45\log_2 0.45 - 0.35\log_2 0.35 - 0.2\log_2 0.2 = 1.513 \text{ bits per letter}$$

iii) [3]

The entropy of the word distribution is equivalent to $2.328/3 = 0.776$ bits per letter; therefore, it is much less than the letter entropy. The reason is the following: When we evaluate the letter entropy, we assume that each letter appears with its corresponding probability in each word. However, we know that only certain letter combinations are possible. This reduces the uncertainty of the language, leading to a smaller entropy.

iv) [4]

Shortening the words has no effect on the total word entropy since the probabilities remain the same. The language becomes more efficient by using 33.3% less letters; however, it also becomes more prone to errors. In the new language, an error in a single letter may lead to another word, leading to confusion; whereas, in the original language, all single-letter errors can be detected (even though they may not be corrected).