Imperial College London

[E2.8 (Maths 3) 2013]

B.ENG. AND M.ENG. EXAMINATIONS 2013

PART II Paper 3 : MATHEMATICS (ELECTRICAL ENGINEERING)

Date Thursday 30th May 2013 2.00 - 4.00 pm

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.

Answer FOUR questions.

A mathematical formulae sheet is provided.

[*Before starting, please make sure that the paper is complete; there should be 6 pages, with a total of SIX questions. Ask the invigilator for a replacement if your copy is faulty.*]

1. (i) Find where the function below is stationary, characterize such points and sketch it.

$$F(x, y) = x^2 + 4x + y^2$$

(ii) Define the gradient vector of a function $f(x, y)$ and explain its direction and magnitude.

A bead is a circular ellipsoid with volume $V = \frac{4}{3}\pi ab^2$ and $a = 10$, $b = 20$. I make a small error in fabricating the shape. I first measure a and find $\Delta a = 0.01$ (a small error in a). I then measure the volume of the ellipsoid and find it is unchanged. What can I conclude about my error in b, Δb ?

I construct a chain of beads out of alternating ellipsoidal and spherical beads (the chain has an even number of beads, $2N$). Each spherical bead has radius $r = 10$. Write out an expression for the volume of the chain in terms of N, a, b, r.

I make a small change $\Delta a, \Delta b, \Delta r$ which maximizes the change in volume of my chain (which still has $2N$ beads).
What ratios $\dfrac{\Delta b}{\Delta a}$ and $\dfrac{\Delta r}{\Delta a}$ did I use?

2. (i) Write out the Cauchy-Riemann equations. Provide a one or two line explanation of their meaning.

(ii) Consider the map

$$w = \frac{1}{z - (1 + i)}$$

Show that it is analytic everywhere except at $z = 1 + i$.

Is this conformal?

Two straight lines intersect in the z-plane at $z = 10 + 10i$. What two things can be said, briefly, about this intersection under the map w?

(iii) What do the lines $y = 0$ and $x = 0$ in the z-plane become under the map w above? (It might help to find expressions for u, v and $u^2 + v^2$ which connect points $x + iy$ in the z-plane to points $u + iv$ in the w-plane.)

Provide a sketch showing the transformed versions of the locuses $x = 0$ and $y = 0$ in the w-plane.

Identify any points of intersection of these two locuses in the w-plane. What is the x, y co-ordinate in the z-plane in the limit as these intersection points are approached in the w-plane?

PLEASE TURN OVER

3. (i) Evaluate

$$\oint_C \frac{z^3}{(z-2)^3}\, dz\ ,$$

where the closed contour C is a counter clockwise unit circle in the z-plane with centre at $z = 2$.

What is this integral when the contour C is instead specified as the boundary of a square with the same centre?

(ii) By considering a unit circle contour in the z-plane, and a suitable substitution for $\sin\theta$, evaluate the integral below:

$$\int_0^{2\pi} \frac{1}{\sin\theta + i}\, d\theta.$$

(iii) Write down the residue theorem for

(a) a contour containing a single simple pole,
(b) a contour containing two simple poles.

Explain very briefly, through a diagram of an appropriate contour or otherwise, how result (b) can be obtained from result (a).

Note: The residue of a complex function $f(z)$ at a pole $z = a$ of multiplicity m is given by

$$\lim_{z\to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left\{(z-a)^m f(z)\right\}\ .$$

4. (i) Provide a simple sketch of

$$\frac{1}{(a^2+t^2)^2}$$

as a function of t with a being a positive constant.

Write out an expression for its Fourier Transform (using the frequency space variable ω).

Write out Jordan's Lemma.

Explain how this can be used to express integrals of the form

$$\int_{-\infty}^{\infty} e^{imx} F(x)\, dx \quad (m \geq 0)$$

in terms of a contour integral.

(ii) Using the above and assuming $\omega < 0$ find the Fourier Transform of $\dfrac{1}{(a^2+t^2)^2}$, $a > 0$.

Briefly explain how you would solve this for $\omega > 0$.

Note: The residue of a complex function $f(z)$ at a pole $z = a$ of multiplicity m is given by

$$\lim_{z \to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left\{ (z-a)^m f(z) \right\} .$$

5. (i) If $\overline{f}(\omega)$ is the Fourier Transform of $f(t)$ prove Parseval's equality

$$\int_{-\infty}^{\infty} |f(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\overline{f}(\omega)|^2 \, d\omega$$

You might need the identity

$$\int_{-\infty}^{\infty} e^{\pm i\Omega t} dt = 2\pi\delta(\Omega)$$

(ii) Find the Fourier transform of $e^{-|t|}$

(iii) Find the Fourier Transform of $H(t)e^{-bt}$ where $H(t)$ is the Heaviside step function $H(t) = 1$ for $t > 0$ and $H(t) = 0$ for $t < 0$ and where $b > 0$.

(iv) Using the Fourier Convolution theorem write down the Fourier Transform of

$$\int_{-\infty}^{\infty} e^{-|t'|} H(t-t') e^{-b(t-t')} \, dt' \qquad (1)$$

where $b > 0$.

(v) By performing the integral (1) directly, while carefully considering the cases $t > 0$ and $t < 0$, show that

$$\int_{-\infty}^{\infty} e^{-|t'|} H(t-t') e^{-b(t-t')} \, dt' \;=\; \begin{cases} \dfrac{(b+1)e^{-t} - 2e^{-bt}}{b^2 - 1} & t > 0 \\ \dfrac{e^{t}}{1+b} & t < 0 \end{cases} \qquad (2)$$

(vi) Find the Fourier Transform of the right hand side of equation (2). (You might like to use this to check your answer to (iv)).

PLEASE TURN OVER

6. (i) Given that $\overline{f}(s) = \mathcal{L}\{f(t)\}$ is the Laplace transform of $f(t)$, prove that when a is a real constant

$$\mathcal{L}\{e^{at}f(t)\} = \overline{f}(s-a) \quad Re(s) > a.$$

(ii) Find the inverse Laplace Transform of

$$\overline{f}(s) = \frac{4}{s(s-4)}$$

(iii) Show that the Laplace Transforms of $\sin\omega t$ and $\cos\omega t$ are $\dfrac{\omega}{s^2+\omega^2}$, $s > 0$ and $\dfrac{s}{s^2+\omega^2}$, $s > 0$ respectively.

Use the Laplace Convolution theorem to solve the equation below for $x(t)$:

$$\frac{d^2x}{dt^2} + 4x = \cos 2t \quad \text{when} \quad x(t=0) = 0 \quad \text{and} \quad \frac{dx}{dt}(t=0) = 0$$

Recall the identity $\sin(A+B) + \sin(A-B) = 2\sin A\cos B$

END OF PAPER

EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2012-13

Course: EE3

Question: 1

TOPIC: Functions of multiple variables

Marks & seen/unseen

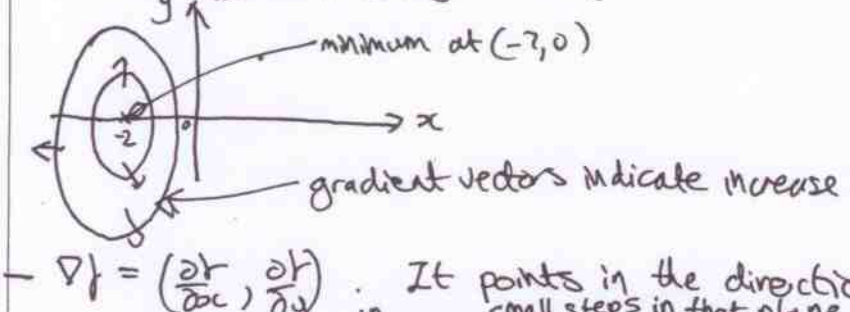
Parts

i) $F(x,y) = x^2+4x+y^2$ stationary at $x=-2$ $y=0$

$\frac{\partial f}{\partial x} = 2x+4$ $\frac{\partial^2 f}{\partial x^2} = 2$ $\frac{\partial^2 f}{\partial x \partial y} = 0$

$\frac{\partial f}{\partial y} = 2y$ $\frac{\partial^2 f}{\partial y^2} = 2$

at $(-2,0)$ $\frac{\partial^2 f}{\partial x^2} > 0$; $\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 - f_{xx} f_{yy} = 0 - 4 < 0$ $\Rightarrow$ minimum — 4 s



ii) – $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$. It points in the direction in the (x,y)-plane in which small steps in that plane ~~which~~ yield the greatest positive change in f. [or accept variants]. — 4

- it's magnitude is the rate of change of f in the direction specified above. — seen

– $V = \frac{4}{3}ab^2\pi \Rightarrow \Delta V = 4b^2\Delta a + 8ab\Delta b$ $a=10,\ b=20$

if $\Delta V = 0$, $\Delta a = 0.01 \Rightarrow 0 = [4 \times 20^2 \times 0.01 + 8 \times 10 \times 20 \times \Delta b]\frac{\pi}{3}$ — 4

$\Delta b =$ ~~-0.005~~ -0.01

– $V_{chain} = N\left(\frac{4}{3}ab^2\pi + \frac{4}{3}\pi r^3\right)$. — 1

$\Delta V = N[\frac{\pi}{3}4b^2\Delta a + 8ab\Delta b\frac{\pi}{3} + 4\pi r^2\Delta r]$ — 6.

$\nabla V = \begin{pmatrix} N4b^2\pi/3 \\ N8ab\pi/3 \\ N4\pi r^2 \end{pmatrix}$ so $\frac{\Delta b}{\Delta a} = \frac{2a}{b} = 1$

$\frac{\Delta r}{\Delta a} = \frac{3r^2}{b^2} = \frac{3 \times 100}{400} = \frac{3}{4}$

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EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2012-13

Course: EE 3

Question: 2

TOPIC: Complex Variables I

Marks & seen/unseen

Parts

- $u_x = v_y$; $u_y = -v_x$ — 2
- They represent the constraint that $\lim_{\Delta z \to 0} \frac{\Delta f(z)}{\Delta z}$ should be independent of the direction in which $\Delta z \to 0$ — 2

  Accept $\frac{\partial f(z)}{\partial z^*} = 0$ or any reasonable alternative.

- Check satisfies CR-equ^{n}s. $w = \frac{1}{(x-1)+i(y-1)} = \frac{(x-1)-i(y-1)}{(x-1)^2+(y-1)^2} = u+iv$ — ~~3~~ 4

$u_x = \frac{1}{(x-1)^2+(y-1)^2} + \frac{-2(x-1)^2}{((x-1)^2+(y-1)^2)^2}$; $u_y = \frac{-2(y-1)(x-1)}{((x-1)^2+(y-1)^2)^2}$

$v_y = \frac{-1}{(x-1)^2+(y-1)^2} + \frac{2(y-1)^2}{((x-1)^2+(y-1)^2)^2}$; $v_x = \frac{(y-1).2.(x-1)}{((x-1)^2+(y-1)^2)^2}$

$u_y = -v_x$ clearly.

$u_x = [(x-1)^2+(y-1)^2 - 2(x-1)^2]/g(x,y)$ $\quad v_y = [-(x-1)^2-(y-1)^2+2(y-1)^2]/g(x,y)$

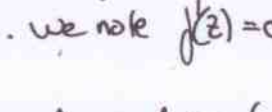
$g(x,y) = ((x-1)^2+(y-1)^2)^2$.

So analytic everywhere save at $z = 1+i$.

$\Rightarrow$ conformal where it is analytic. we note $f'(z) = 0$ occurs nowhere.

- The angle of intersection is preserved and so too is the ordering of the lines — 2

(accept equivalent ordering remarks)



This works because the map is conformal at this point.

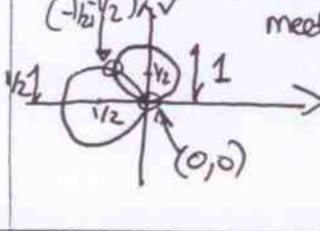
- $u^2+v^2 = \frac{1}{(x-1)^2+(y-1)^2}$. When $y=0$ $v = \frac{1}{(x-1)^2+1}$

$u^2+v^2 = v$

$\Rightarrow u^2+(v-1/2)^2 = 1/4 \Rightarrow$ circle centre $(0, 1/2)$ radius $1/2$. — 5

when $x=0$ $u = \frac{-1}{(y-1)^2+1}$; $u^2+v^2 = -u \Rightarrow (u+1/2)^2+v^2 = 1/4$

$\Rightarrow$ circle centre $(-1/2, 0)$ radius $1/2$



meet at right angles at both points.

$(-1/2, 1/2) \to (0,0)$ in z-plane. — 5

$(0,0) \to$ any mention of infinity scores.

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Page number

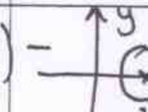
EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2012-13 | Course: EE 3

Question: 3

TOPIC: Complex Variables II

Marks & seen/unseen

Parts

i) 

Pole of multiplicity 3 at $z = 2$.
Pole is contained in contour

Residue at pole is $\lim_{z\to 2} \frac{1}{2!}\frac{d^2}{dz^2}\frac{(z-2)^3 z^3}{(z-2)^3}$ **5**

$= \frac{1}{2} \times 3 \times 2 \times 2 = 6.$

By the residue thrm the value of integral is $12\pi i$.

– Pole still enclosed and no other poles ⇒ answer unchanged **2**

ii) Consider contour $z = e^{i\theta}$ $dz = ie^{i\theta}d\theta = iz\,d\theta$

noting that $\sin\theta = (z - z^{-1})/2i$ **10**

consider integral

$$\oint \frac{1}{\left(\left(\frac{z-z^{-1}}{2i}\right)+i\right)iz}dz = \oint \frac{2}{z^2-2z-1}dz = \cancel{\oint \frac{2}{(z-(1+\sqrt{2}))}}$$

$$= \oint \frac{2}{(z-(1+\sqrt{2}))(z-(1-\sqrt{2}))}dz$$

Note that only pole at $z = 1-\sqrt{2}$ is contained in contour and is a simple pole.

⇒ residue at $z = 1-\sqrt{2}$

$$\lim_{z\to 1-\sqrt{2}}\left[\frac{2}{z-(1+\sqrt{2})}\right] = \frac{2}{1-\sqrt{2}-1-\sqrt{2}} = -\frac{1}{\sqrt{2}}.$$

By the residue theorem, which is $2\pi i \times$ (sum of residues enclosed) integral is $\sqrt{2}\cdot\pi i$. Because of the equivalence of the contour integral and the original use this is the desired answer.

iii) a) $2\pi i \times$ value of residue at the simple pole
b) $2\pi i \times$ the sum of the two residues at each of the simple poles. **1**

– Accept any contour/schematic or argument using cuts e.g. **2**

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EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2012-13 | Course: EE 3

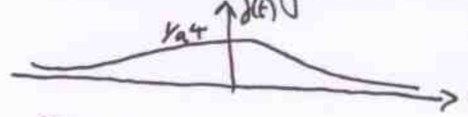
Question: 4

TOPIC

Fourier Transforms 1 – A

Marks & seen/unseen

Parts

- 

symmetrical and decaying away at $\pm\infty$. — 1

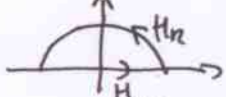
- $\int_{-\infty}^{\infty} e^{-i\omega t} \dfrac{1}{(a^2+t^2)^2}\, dt \; (*)$ — 1

- If the only singularities of $F(z)$ are poles then on a semicircular contour of radius R as drawn below — 3.

then $\lim_{R\to\infty} \int_{H_R} e^{imz} F(z)\, dz = 0$

provided $m > 0$ and $|F(z)| \to 0$ as $R\to\infty$. If $m=0$ faster convergence to zero is required for $F(z)$.

Or equivalent.

- consider the contour C_R 

 — 4.

then $\lim_{R\to\infty} \oint_{C_R} e^{imz} F(z)\, dz = 2\pi i \times$(sum of residues enclosed by above contour)

$$= \lim_{R\to\infty} \Big[\underbrace{\int_H e^{imz} F(z)\, dz}_{①} + \underbrace{\int_{H_R} e^{imz} F(z)\, dz}_{②} \Big]$$

where if J's Lemma applies ② $\to 0$

and we note that ① becomes $\int_{-\infty}^{\infty} e^{imx} F(x)\, dx$ since $z = x$ along H.

$\Rightarrow \int_{-\infty}^{\infty} e^{imx} F(x)\, dx = 2\pi i \times$ (sum of residues enclosed).

- Suppose $F(z) = \frac{1}{(a^2+z^2)^2}$. We note that $m > 0$ since $-\omega > 0$ for $(*)$ above and that $F(z)$ decays away sufficiently fast. So can use J's Lemma. — 1 seen

so $\lim_{R\to\infty} \oint_{C_R} e^{-i\omega z} \dfrac{1}{(a^2+z^2)^2}\, dz$ using the contour above

$= \int_{-\infty}^{\infty} e^{-i\omega t} \dfrac{1}{(a^2+t^2)^2}\, dt + \lim_{R\to\infty} \cancel{\int_{H_R}}^{\,0} = 2\pi i \times$ sum of residues enclosed

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	EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2012-13	Course EE 3
Question 4	TOPIC Fourier Transforms 1-B	Marks & seen/unseen

Parts

Residues $z^2=-a^2$ $\quad z=\pm ai$

Two double poles at ai and $-ai$ $(a>0)$

$\Rightarrow$ one pole at ai in upper half plane so

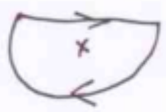
$$\lim_{z\to ai}\frac{1}{1!}\frac{d}{dz}\left[\frac{e^{-i\omega z}(z-ai)^2}{(z-ai)^2(z+ai)^2}\right]$$

$$=\lim_{z\to ai}\left[\frac{-i\omega e^{-i\omega z}}{(z+ai)^2}+\frac{e^{-i\omega z}\cdot -2}{(z+ai)^3}\right]$$

$$= ie^{+a\omega}(a\omega-1)/4a^3$$

so $\displaystyle\int_{-\infty}^{\infty} e^{-i\omega t}\frac{1}{(a^2+t^2)^2}\,dt = \frac{-\pi e^{+a\omega}}{2a^3}(a\omega-1)$ $\quad \omega<0,\ a>0$

- For $\omega>0$ consider a corresponding semicircular contour in the lower half plane: ~~but no~~



some indication of clockwise direction.

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EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2012-13

Course

EE 3

Question 5

TOPIC

Fourier Transforms 2

Marks & seen/unseen

Parts

i) $\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\bar{f}(\omega')e^{i\omega' t}d\omega'\right]\left[\int_{-\infty}^{\infty}\bar{f}^*(\omega'')e^{-i\omega'' t}d\omega''\right]dt = \int_{-\infty}^{\infty}|f(t)|^2 dt$ 3

$= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\bar{f}(\omega')\int_{-\infty}^{\infty}\bar{f}^*(\omega'')\underbrace{\int_{-\infty}^{\infty}e^{i(\omega'-\omega'')t}dt}_{2\pi\,\delta(\omega'-\omega'')}\,d\omega''\,d\omega'$

$= \frac{1}{2\pi}\int_{-\infty}^{\infty}|\bar{f}(\omega')|^2 d\omega'$

ii) $\int_0^{\infty}e^{-(i\omega+1)t}dt + \int_{-\infty}^{0}e^{-(i\omega-1)t}dt = \left[\frac{e^{-(i\omega+1)t}}{-(i\omega+1)}\right]_0^{\infty} + \left[\frac{e^{-(i\omega-1)t}}{-(i\omega-1)}\right]_{-\infty}^{0}$ 3

$= \frac{1}{i\omega+1} + \frac{1}{1-i\omega} = \frac{2}{1+\omega^2}$

iii) $\int_0^{\infty}e^{-bt}e^{-i\omega t}dt = \left[\frac{e^{-(i\omega+b)t}}{-(b+i\omega)}\right]_0^{\infty} = \frac{1}{b+i\omega}$ 3

iv) $FT[f(t)*g(t)] = \bar{f}(\omega)\bar{g}(\omega)$ so $Eq(1)$ becomes 2

$\frac{2}{(1+\omega^2)}\frac{1}{(b+i\omega)}$ under transformation.

v) consider first $t>0$ $H(t-t')=0$ $t-t'<0$, $t<t'$

Integral becomes

$\int_{-\infty}^{0}e^{-|t'|}e^{-b(t-t')}dt' + \int_0^{t}e^{-|t'|}e^{-b(t-t')}dt'$ 6

$= e^{-bt}\left(\left[\frac{e^{(1+b)t'}}{1+b}\right]_{-\infty}^{0} + \left[\frac{e^{(b-1)t'}}{(b-1)}\right]_0^{t}\right)$

$= \frac{1}{b^2-1}\left((1+b)e^{-t} - 2e^{-bt}\right)$.

now consider $t<0$.

$\int_{-\infty}^{t}e^{t'}e^{-b(t-t')}dt' = e^{-bt}\left[\frac{e^{t'(1+b)}}{1+b}\right]_{-\infty}^{t} = \frac{e^{t}}{1+b}$.

vi) consider FT of two parts. $t<0$.

$\int_{-\infty}^{0}e^{-i\omega t}\frac{e^{t}}{1+b}dt = \frac{1}{(1+b)(1-i\omega)}$ (1)

$t>0$. $\frac{1}{b^2-1}\int_0^{\infty}e^{-i\omega t}\left[(b+1)e^{-t} - 2e^{-bt}\right]dt = \frac{1}{(b^2-1)}\left(\left[\frac{e^{-(i\omega+1)t}(b+1)}{-(i\omega+1)}\right]_0^{\infty} + \left[\frac{-2e^{-(i\omega+b)t}}{-(i\omega+b)}\right]_0^{\infty}\right) = \frac{1}{b^2-1}\left[\frac{b+1}{i\omega+1} + \frac{-2}{i\omega+b}\right]$ (2) 3

FT of (2) is (1)+(2) Accept this as an answer

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EXAMINATION QUESTIONS/SOLUTIONS 2012-13

Course

Question

TOPIC

Marks & seen/unseen

Parts

Check step (not required)

$$① + ② = \frac{1}{(b^2-1)(\omega^2+1)} \times \overbrace{\left[(b-1)(1+i\omega) + (b+1)(1-i\omega)\right]}^{2b-2i\omega} - \frac{2}{(i\omega+b)(b^2-1)}$$

$$= \frac{(2b-2i\omega)(b+i\omega) - 2(\omega^2+1)}{(i\omega+b)(b^2-1)(\omega^2+1)} = \frac{2\cancel{(b^2-1)}}{(i\omega+b)(\omega^2+1)\cancel{(b^2-1)}}.$$

This checking step is not required for full marks.

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EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2012-13

Course: EE 3

Question: 6

TOPIC: Laplace Transforms

Parts | | Marks & seen/unseen

ii) $\bar{f}(s) = \frac{A}{s} + \frac{B}{s-4}$ $\quad A=-B$, $A=-1$. — 5

$= \frac{1}{s-4} - \frac{1}{s}$

By shift theorem $e^{at}f(t) \Rightarrow F(s-a)$ and also $1 \Rightarrow 1/s$

so $f(t) = e^{4t} - 1$.

iii) $-\int_0^\infty e^{-st} e^{i\omega t}\,dt = \left[\frac{e^{(i\omega - s)t}}{i\omega - s}\right]_0^\infty = \frac{1}{s-i\omega} = \frac{s+i\omega}{s^2+\omega^2}$ — 5

since $e^{i\omega t} = \cos\omega t + i\sin\omega t$.

it follows that $\cos\omega t \Rightarrow \frac{s}{s^2+\omega^2}, s>0$

$\sin\omega t \Rightarrow \frac{\omega}{s^2+\omega^2}, s>0$

$- s^2\bar{x}(s) - 1 + 4\bar{x}(s) = s/s^2+4$ — ~~10~~ 8

$\bar{x}(s) = \frac{1}{s^2+4} + \frac{s}{(s^2+4)^2}$

$\mathcal{L}$ convolution theorem $f * g \Rightarrow F(s)G(s)$

$x(t) = \frac{1}{2}\sin 2\omega t + \int_0^t \frac{1}{2}\sin 2\omega t' \cos 2\omega(t-t')\,dt'$

using double angle formula

$\sin 2\omega t' \cos 2\omega(t-t') = \frac{1}{2}[\sin 2\omega t + \sin(4\omega t' - 2t)]$

$x(t) = \frac{1}{2}\sin 2\omega t + \frac{t}{2}\sin 2\omega t + 0$.

2 i) $\mathcal{L}[\exp(at) f(t)] = \underbrace{\int_0^\infty e^{-(s-a)t} f(t)\,dt}_{\text{only valid if } Re(s)>a} = \bar{f}(s-a)$ — 2

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