Paper Number(s): E4.22
C1.2

IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE
UNIVERSITY OF LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2001

MSc and EEE PART IV: M.Eng. and ACGI

LINEAR OPTIMAL CONTROL

Monday, 30 April 10:00 am

There are SIX questions on this paper.

Answer FOUR questions.

Time allowed: 3:00 hours

Corrected Copy

Examiners: Astolfi,A. and Weiss,G.

Special instructions for invigilators: None

Information for candidates:

System:	$\dot{x}(t) = A(t) + Bu(t),\ x(0) = x_0.$
Quadratic cost function:	$J(x_0, u) = \int_0^\infty [x(t)'Qx(t) + u(t)'Ru(t)]\, dt,$ $Q = Q' \geq 0,\ R = R' > 0.$
Riccati equation:	$A'P + PA + Q - PBR^{-1}B'P = 0.$
Optimal control law:	$u(t) = -R^{-1}B'Px(t) = -Kx(t).$
Minimum cost:	$x_0'Px_0.$
Return difference inequality for scalar u:	$\lvert 1 + K(j\omega I - A)^{-1}B \rvert \geq 1,$
Minimum principle:	$\dot{x} = f(x, u),$

$$J(x_0, u) = \int_0^\infty L(x(t), u(t))dt,$$

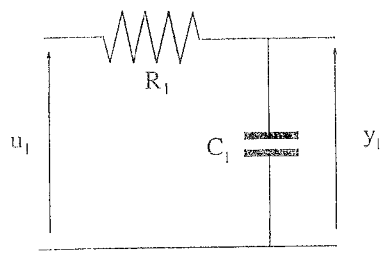
$$H(x, u, \lambda_0, \lambda) = \lambda_0 L(x, u) + \lambda^T f(x, u),$$

$$\dot{\lambda}^* = -\left.\frac{\partial H}{\partial x}\right|^T_{(x^*, u^*, t_f^*)},$$

$$H(x^*, \omega, \lambda_0^*, \lambda^*) \geq H(x^*, u^*, \lambda_0^*, \lambda^*),\ \forall \omega,$$

$$H(x^*, u^*, \lambda_0^*, \lambda^*) = 0.$$

1. Consider the linear electric networks in Figure 1, with $R_1 > 0$, $R_2 > 0$, $R_3 > 0$, $C_1 > 0$ and $C_2 > 0$. Let u_1 be the driving voltage, x_1 be the voltage across the capacitor C_1, and y_1 be the output voltage for the first circuit. Let u_2 be the driving voltage, x_2 be the voltage across the capacitor C_2, and y_2 be the output voltage for the second circuit.



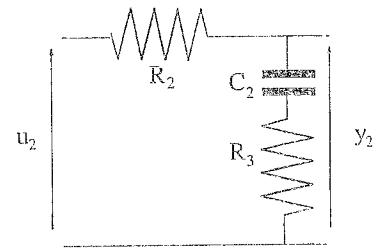


Figure 1.

(a) Using Kirchhoff's laws, or otherwise, express the dynamics of both circuits in the standard state-space form.

(b) Study the controllability and the observability of the two dynamical systems determined in (a).

Consider the interconnected network in Figure 2, obtained setting $u_2 = y_1$ via an ideal op-amp and regarding u_1 and y_2 as input and output of the resulting network. Suppose that no current flows into the input terminals of the op-amp.

(c) Study the controllability and observability of the system as a function of the parameters R_1, R_2, R_3, C_1 and C_2.

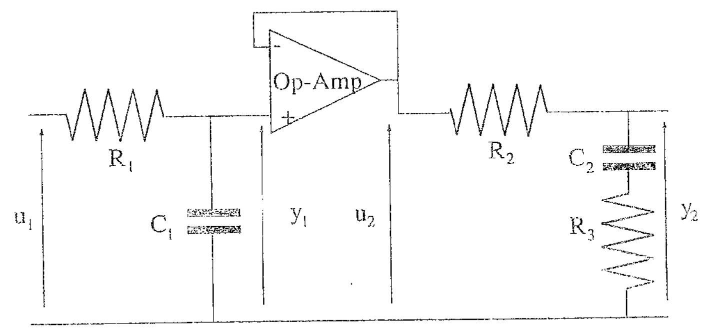


Figure 2.

2. The linearised model of a system composed of a tractor pulling a trailer along a straight path is described by the equations

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= u\\ \dot{x}_3 &= -x_3 - \lambda u\\ y &= x_1\end{aligned}$$

where $u \in I\!R$ is the input, $y \in I\!R$ is the output and λ is a constant parameter.

(a) Discuss the properties of controllability and stabilisability as a function of λ.

(b) Discuss the properties of observability and detectability.

(c) Using the results established in part (b) design an observer to reconstruct asymptotically the state x_2.

(d) Let

$$u = k_1 x_1 + k_2 x_2,$$

and compute values for k_1 and k_2 such that the closed-loop system has eigenvalues at $\{-1/2, -1/2, -1\}$.

(e) Using the results in parts (c) and (d) discuss why on-line measurements of the variable x_1 are sufficient to design a stabilizing (dynamic) output feedback controller.

3. Consider a system described by the following transfer function from u to y

$$W(s) = \frac{1}{s^2},$$

and the quadratic cost (to be minimised)

$$J(x_0, u) = \int_0^{\infty} \left(q\, y^2(t) + \frac{1}{q}\, u^2(t) \right) dt,$$

with $q > 0$.

(a) Write the minimal state space realization of the system with the pair $\{A, B\}$ in controllable canonical form.

(b) Write the Algebraic Riccati Equation (ARE) associated with the optimal control problem and verify that the hypotheses to solve the optimal control problem are verified.

(c) Compute the solutions of the ARE derived in (b) and verify that one solution is always positive for any $q > 0$.

(d) Compute the optimal control law as a function of q and plot on the complex plane the position of the eigenvalues of the closed-loop system as a function of $q > 0$.

4. Consider the system

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} x + \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix} u$$

and the quadratic cost (to be minimised)

$$J(x_0, u) = \int_0^\infty \left(x_1^2(t) + 2x_2^2(t) + u_1^2(t) + u_2^2(t) \right) dt.$$

(a) Show that the system is controllable if and only if $b_1 b_2 \neq 0$ and that the system is stabilizable if and only if $b_2 \neq 0$.

(b) Write the ARE associated with the considered optimal control problem.

(c) Assume $b_1 b_2 \neq 0$. Find the positive definite solution P of the ARE derived in part (b) as a function of b_1 and b_2 and compute the optimal state feedback control law and the optimal closed loop system. (Hint: consider a diagonal P.)

(d) Let $b_2 = 1$. Compute b_1 such that the optimal closed loop system computed in part (c) has two coincident eigenvalues.

5. Consider the system

$$\dot{x} = -x + 2u.$$

(a) Write all stabilizing state feedback control laws $u = -kx$.

(b) Let $u = -kx$ and compute all k such that the feedback law is stabilizing and *optimal in some sense*. (Hint: use the return difference inequality.)

(c) Consider the cost function

$$J(x_0, u) = \int_0^\infty \left(qx^2(t) + u^2(t)\right) dt.$$

Compute q such that the control law $u = -3x$ is optimal with respect to the considered quadratic cost.

6. Consider the nonlinear system

$$\dot{x} = x + u^3$$

and the problem of finding a bounded control law $|u(t)| \leq 1$ that drives the state of the system from $x(0) = x_i$ to $x(t_f) = 0$ in minimum time.

(a) Compute explicitly all the initial states $x(0) = x_i$ that can be steered to $x(t_f) = 0$ by the bounded control.

(b) Write the necessary conditions for optimality in the case of normal extremals.

(c) Write the optimal control as a function of the optimal co-state $\lambda^*(t)$.

(d) Write the optimal control as a function of the optimal state $x^*(t)$.

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(a) $\dfrac{u_1 - x_1}{R_1} = i = C_1 \dot{x}_1 \qquad y_1 = x_1$

$$\Sigma_1 \begin{cases} \dot{x}_1 = -\dfrac{1}{R_1 C_1} x_1 + \dfrac{1}{R_1 C_1} u_1 \\ y_1 = x_1 \end{cases}$$

(1)

$$\frac{u_2 - x_2 - R_3 C_2 \dot{x}_1}{R_2} = C \dot{x}_2 \qquad y_2 = \frac{R_2 x_2 + R_3 u_2}{R_2 + R_3}$$

$$\Sigma_2 \begin{cases} \dot{x}_2 = -\dfrac{1}{(R_2+R_3)C_2} x_2 + \dfrac{1}{(R_2+R_3)C_2} u_2 \\ y_2 = \dfrac{R_2}{R_2+R_3} x_2 + \dfrac{R_3}{R_2+R_3} u_2 \end{cases}$$

(2)

(b) Σ_1 is controllable and observable $\forall R_i > 0,\ C_i > 0$
Σ_2 is " " " "

(1)

$$\Sigma_{in} \begin{cases} \dot{x}_1 = -\dfrac{1}{R_1 C_1} x_1 + \dfrac{1}{R_1 C_1} u_1 \\ \dot{x}_2 = -\dfrac{1}{(R_2+R_3)C_2} x_2 + \dfrac{1}{(R_2+R_3)C_2} x_1 \\ y_2 = \dfrac{R_3}{R_2+R_3} x_1 + \dfrac{R_2}{R_2+R_3} x_2 \end{cases}$$

(2)

(c)

$$\mathcal{C} = \begin{bmatrix} \dfrac{1}{R_1 C_1} & * \\ 0 & \dfrac{1}{(R_1+R_3)C_2 R_1 C_1} \end{bmatrix} \Rightarrow \det \mathcal{C} \neq 0 \quad \forall R_i > 0 \ \forall C_i > 0$$

$\Uparrow$ controllable $\forall R_i > 0\ \forall C_i > 0$

(2)

$$\mathcal{O} = \begin{bmatrix} \dfrac{R_3}{R_2+R_3} & \dfrac{R_2}{R_2+R_3} \\ \dfrac{R_2}{(R_2+R_3)^2 C_2} - \dfrac{R_3}{(R_2+R_3)R_1 C_1} & -\dfrac{R_2}{(R_2+R_3)^2 C_2} \end{bmatrix} \Rightarrow \det \mathcal{O} = \frac{(R_3 C_2 - R_1 C_1) R_2}{(R_2+R_3)^2 R_1 C_1 C_2}$$

$\Downarrow$

Obs. $\forall R_i > 0,\ \forall C_i > 0$ if $R_3 C_2 \neq R_1 C_1$

(2)

2)

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \\ -\lambda \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

(a) $\mathcal{C} = [B, AB, A^2B] = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -\lambda & \lambda & -\lambda \end{bmatrix}$ $\quad \det \mathcal{C} = \lambda$

If $\lambda \neq 0 \Rightarrow$ Controllable

If $\lambda = 0 \Rightarrow$ No contr., but stabilizable (2)

(b) $\mathcal{O} = \begin{bmatrix} C \\ CA^1 \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad \det \mathcal{O} = 0$

Not observable, however the "unobservable state" is x_3, and $\dot{x}_3 = -x_3 \Rightarrow$ Detectable. (2)

(c) Observer $\quad \dot{\xi} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \xi + \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} [y - \xi_1] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$

with $l_1 < 0$, $l_2 < 0$.

Note: only the **subsystem** (x_1, x_2) is used for the design.

ξ_i is an estimation of x_i. (3)

(d) $A_{cl} = A + BK$, with $K = [k_1, k_2, 0]$, is $A_{cl} = \begin{bmatrix} 0 & 1 & 0 \\ k_1 & k_2 & 0 \\ -\lambda k_1 & -\lambda k_2 & -1 \end{bmatrix}$

ch. pol $= (s+1)(s^2 - k_2 s - k_1) \Rightarrow k_1 = -1 \quad k_2 = -1/4$ (1)

(e) Using the "separation **principle**" use

$$u = k_1 \xi_1 + k_2 \xi_2$$

there is no need to estimate x_3! (2)

— Sol 2 —

3)

(a) $\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$

2

(b) $ARE = \begin{bmatrix} (p_{12}^2 - 1)q & p_{11} - q p_{12} p_{22} \\ p_{11} - q p_{12} p_{22} & 2p_{12} - q p_{22}^2 \end{bmatrix} \qquad P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$

HP: (A, B) contr ✓

$Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \geq 0$ ✓

$R = \frac{1}{q} > 0$ ✓

$(A, Q^{1/2})$ obs ✓

3

(c) From ARE_{11} $p_{12} = \pm 1$

$p_{12} = 1 \longrightarrow p_{22} = \pm\sqrt{\frac{2}{q}}$ only $+\sqrt{\frac{2}{q}}$ is ok

$\Downarrow$

$p_{11} = \sqrt{2q}$

$P = \begin{bmatrix} \sqrt{2q} & 1 \\ 1 & \sqrt{\frac{2}{q}} \end{bmatrix} \Longleftarrow$

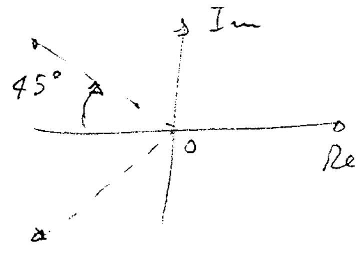
This is > 0 $\forall q > 0$

$p_{12} = -1 \longrightarrow p_{22}^2 < 0$ not good

3

(d) $k^* = [-q, -\sqrt{2q}] \qquad A_{cl} = \begin{bmatrix} 0 & 1 \\ -q & -\sqrt{2q} \end{bmatrix}$

$\lambda_{1,2} = -\frac{1}{2}\sqrt{2q} \pm \frac{1}{2} i \sqrt{2q}$



2

— Sol 3 —

4)

(a)

$$\mathcal{C} = [B, AB] = \left[\begin{array}{cc|cc} b_1 & 0 & -b_1 & 0 \\ 0 & b_2 & 0 & 4b_2 \end{array}\right]$$

rank $\mathcal{C} = 2$ if $b_1 \cdot b_2 \neq 0 \Rightarrow$ Contr

If $b_2 \neq 0$ and $b_1 = 0$

$\dot{x}_1 = -x_1$ ← Not contr

$\dot{x}_2 = 4x_2 + b_2 u_2$ ← Contr

$\Downarrow$

Stabilizable

3

If $b_2 = 0$ $\quad \dot{x}_2 = 4x_2$ $\quad$ Not stable $\Rightarrow$ Not stabilizable

(b) $$ARE = \begin{bmatrix} 1 - 2p_{11} - p_{11}^2 b_1^2 & 0 \\ 0 & 2 + 8p_{22} - p_{22}^2 b_2^2 \end{bmatrix} \qquad P = \begin{bmatrix} P_{11} & 0 \\ 0 & P_{22} \end{bmatrix}$$

2

(c) $$p_{11} = \frac{-1 + \sqrt{1+b_1^2}}{b_1^2} > 0 \qquad p_2 = \frac{4 + \sqrt{16+2b_2^2}}{b_2^2} > 0$$

$$K^* = \begin{bmatrix} -\frac{-1+\sqrt{1+b_1^2}}{b_1} & 0 \\ 0 & -\frac{4+\sqrt{16+2b_2^2}}{b_2} \end{bmatrix} \qquad A_{cl} = \begin{bmatrix} -\sqrt{1+b_1^2} & 0 \\ 0 & -\sqrt{16+2b_2^2} \end{bmatrix}$$

3

(d) $\lambda_1 = -\sqrt{1+b_1^2}$ $\qquad \lambda_2 = -\sqrt{18}$ $\qquad$ ($b_2 = 1$!)

$\lambda_1 = \lambda_2 \Rightarrow b_1 = \pm\sqrt{17}$.

2

5)

(a) $\dot{x} = -x + 2u$

$u = -kx$

$-1 - 2k < 0$ for stability

$\Downarrow$

$k > -1/2$ (2)

(b) Return difference inequality

$$\left|1 + \frac{2k}{j\omega + 1}\right| \geq 1 \Rightarrow 1 + \frac{4k(k+1)}{\omega^2+1} \geq 1$$

$\Downarrow$

$k \leq -1 \quad k \geq 0 \Leftarrow k(k+1) \geq 0$

optimal in some sense + stability

$\Downarrow$

$k \geq 0$ (4)

(c) $u^* = -R^{-1}B'Px = -3x \qquad R = 1 \qquad B = 2$

$\hookrightarrow P = 3/2 > 0$

ARE $\Rightarrow A'P + PA - PBB'P + Q = 0 \Rightarrow Q = 12$

(4)

6)

(a) $\dot{x} = x + u^3$

If $|u(t)| \le 1$ only $x_i(0)$: $|x_i(0)| < 1$

can be steered to the origin. (2)

(b) $H = 1 + \lambda(x + u^3)$

$\dot{x}^* = x^* + u^{*3}$

$\dot{\lambda}^* = -\lambda^*$ (3)

$H(x^*, w, \lambda^*) \ge H(x^*, u^*, \lambda^*) \quad \forall w \in [-1, 1]$

$\lambda^* w^3 \ge \lambda^* u^{*3} \quad \forall w \in [-1, 1]$

(c) $\lambda^*(t) = \lambda^*(0)\, e^{-t}$ $\quad \lambda^*(0) > 0 \Rightarrow \lambda^*(t) > 0$ and vice versa.

$u^*(\lambda^*) = -\text{sign}(\lambda^*) \Rightarrow u^*$ is ± 1. (2)

(d) if $u^* = +1$ $\quad \dot{x} = x + 1$ $\quad$ x goes to zero only if $x(0) < 0$.

if $u^* = -1$ $\quad \dot{x} = x - 1$ $\quad$ x goes to zero only if $x(0) > 0$.

$u^*(x) = -\text{sign}(x)$ (3)