ANALYSIS OF CIRCUITS

**** Solutions 2016 ****

Information for Candidates:

The following notation is used in this paper:

1. The voltage waveform at node X in a circuit is denoted by $x(t)$, the phasor voltage by X and the root-mean-square (or RMS) phasor voltage by $\widetilde{X} = \frac{X}{\sqrt{2}}$. The complex conjugate of X is X^*.
2. Component and source values in a circuit are normally given in Ohms, Farads, Henrys, Volts or Amps with the unit symbol omitted. Where an imaginary number is specified, it represents the complex impedance or phasor value.
3. Times are given in seconds unless otherwise stated.
4. Unless otherwise indicated, frequency response graphs should use a linear axis for phase and logarithmic axes for frequency and magnitude.
5. The real and imaginary parts of a complex number, X, are written $\Re(X)$ and $\Im(X)$ respectively.

Key: B=bookwork, U=unseen example

1. a) i) Using nodal analysis, calculate the voltage at node X of Figure 1.1. [3]

ii) Calculate the power absorbed by each of the three sources in the circuit. [3]

[U] KCL at node X gives

$$\begin{aligned} \frac{X-29}{4}+\frac{X-2}{3}-2 &= 0 \\ \Rightarrow 3X-87+4X-8-24 &= 0 \\ \Rightarrow 7X &= 119 \\ \Rightarrow X &= 17 \end{aligned}$$

The current throught the $4\,\Omega$ *resistor is* $\frac{29-X}{4}=\frac{12}{4}=3\,\mathrm{A}$ *from left to right. The power absorbed by the* $29\,\mathrm{V}$ *source is therefore* $29\times-3=-87\,\mathrm{W}$.

The current throught the $4\,\Omega$ *resistor is* $3+2=5\,\mathrm{A}$ *(by KCL). The power absorbed by the* $2\,\mathrm{V}$ *source is therefore* $2\times+5=10\,\mathrm{W}$.

The power absorbed by the $2A$ *source is* $17\times-2=-34\,\mathrm{W}$.

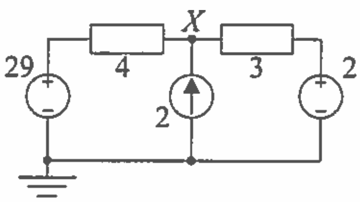

Figure 1.1

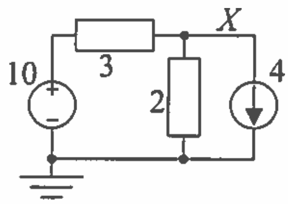

Figure 1.2

b) Use the principle of superposition to find the voltage X in Figure 1.2. [5]

[U] If we open circuit the $4\,\mathrm{A}$ *current source, the* $3\,\Omega$ *and* $2\,\Omega$ *resistors form a potential divider and* $X_1=\frac{2}{3+2}\times10=+4\,\mathrm{V}$.

If we short circuit the $10\,\mathrm{V}$ *voltage source, the* $3\,\Omega$ *and* $2\,\Omega$ *resistors are in parallel and equal* $\frac{3\times2}{3+2}=1.2\,\Omega$. *The voltage at X is then* $X_2=-4\times1.2=-4.8\,\mathrm{V}$.

By superposition, the total voltage is therefore $X=X_1+X_2=4-4.8=-0.8\,\mathrm{V}$.

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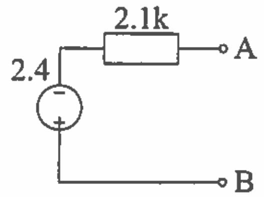
c) Draw the Thévenin equivalent circuit of the two-terminal network in Figure 1.3 and find the values of its components. [5]

[U] We can find the open circuit voltage by treating the circuit as a potential divider; this gives $V_{AB}=\frac{3}{5+3+2}\times8=2.4\,\mathrm{V}$.

We can find the Thévenin resistance by short-circuiting the voltage source. The remaining network has a resistance of $3||(5+2) = 3||7 = \frac{3\times 7}{3+7} = 2.1\,\mathrm{k}\Omega$.

So the complete Thévenin equivalent is:



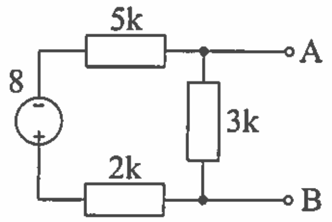


Figure 1.3

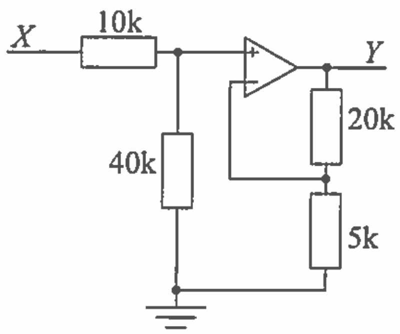


Figure 1.4

d) Assuming the opamp in the circuit of Figure 1.4 is ideal, give an expression for Y in terms of X. [4]

[U] This is a potential divider follower by a non-inverting op-amp circuit and so we can write down $Y = \left(1+\frac{20}{5}\right) \times \frac{40}{10+40} \times X = 5 \times 0.8 \times X = 4X$.

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e) Determine R_1 and R_2 in Figure 1.5 so that $Y = 0.1X$ and the parallel combination of R_1 and R_2 has an impedance of $50\,\Omega$. [5]

[U] The gain of the potential divider is $0.1 = \frac{R_2}{R_1+R_2}$ *which implies that* $0.1R_1 = (1-0.1)R_2 = 0.9R_2$ *from which* $R_1 = 9R_2$.

Substituting this relationship into the parallel impedance formula gives $50 = \frac{R_1R_2}{R_1+R_2} = \frac{9R_2^2}{10R_2} = 0.9R_2$ *from which* $R_2 = 55.6\,\Omega$ *and* $R_1 = 9R_2 = 500\,\Omega$.

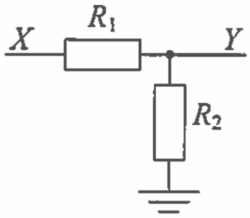

Figure 1.5

f) i) The diagram of Figure 1.6 shows a 50 Hz AC source with r.m.s. voltage $\widetilde{V} = 230\,\text{V}$ driving a load with impedance $3.5 + 2.5j\,\Omega$. Determine the complex power supplied to the load, $S = \widetilde{V} \times \widetilde{I}^*$, and also the power factor, $\lambda = \frac{\Re(S)}{|S|}$. [3]

[U] We can calculate the complex power directly as $S = \widetilde{V} \times \widetilde{I}^* = \frac{|\widetilde{V}|^2}{Z^*} = \frac{230^2}{3.5-2.5j} = 10 + 7.15\,\text{kVA}$. *Alternatively, and less directly, the load current is* $\widetilde{I} = \frac{\widetilde{V}}{Z} = \frac{230}{3.5+2.5j} = 43.5 - 31.1j$ *which gives the complex power as* $S = \widetilde{V} \times \widetilde{I}^* = 230\,(43.5 + 31.1j) = 10 + 7.15\,\text{kVA}$. *The power factor is therefore* $\lambda = \cos\angle S = \frac{\Re(S)}{|S|} = \frac{10}{12.3} = 0.814$.

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ii) A capacitor, C, is now connected across the load, as indicated in Figure 1.7. Calculate the value of C (in μF) required to increase the power factor to 0.95. [3]

[U] The complex power absorbed by the capacitor is purely imaginary and equals $S_C = \frac{|\widetilde{V}|^2}{Z_C^*} = -j\omega C|\widetilde{V}|^2 = -j100\pi C|\widetilde{V}|^2$. *A power factor of* 0.95 *for the total power absorbed,* $S + S_c$, *means that* $\angle\,(S + S_C) = \cos^{-1} 0.95 = 0.318$. *It follows that* $\frac{\Im(S+S_C)}{\Re(S+S_C)} = \tan\,(S + S_C) = \tan 0.318 = 0.329$. *So since* $\Re\,(S + S_C) = \Re\,(S) = 10\,\text{kW}$, *we can write* $|S_C| = |0.329 \times 10 - 7.15|\,\text{k} = 3.86\,\text{kVA}$. *So* $C = \frac{|S_C|}{100\pi|\widetilde{V}|^2} = 232\,\mu\text{F}$.

Figure 1.6

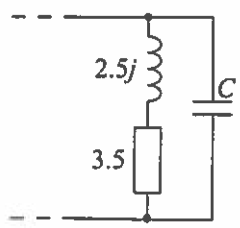


Figure 1.7

g) Determine the gain, $\frac{Y}{X}$, for the block diagram shown in Figure 1.8. The rectangular blocks are drawn with inputs at the left and outputs at the right and have gains of F and G respectively. The open circles represent adder/subtractors; their inputs have the signs indicated on the diagram and their outputs are V and Y respectively. [4]

[U] We can write down the following equations from the block diagram:

$$\begin{aligned} V &= FX - GV \\ Y &= GV - X \end{aligned}$$

We need to eliminate V from these equations:

$$\begin{aligned}
V &= FX - GV \\
\Rightarrow (1+G)V &= FX \\
\Rightarrow V &= \frac{F}{1+G}X \\
Y &= GV - X \\
Y &= \left(\frac{F}{1+G} - 1\right)X \\
\frac{Y}{X} &= \frac{F-G-1}{1+G}
\end{aligned}$$

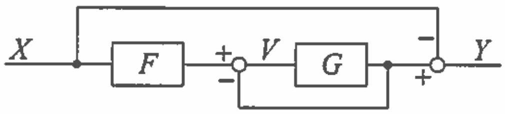


Figure 1.8

h) Figure 1.10 shows a transmission line of length 10 m that is terminated in a resistive load, R, with reflection coefficient $\rho = -0.4$. The line has a propagation velocity of $u = 2 \times 10^8\,\mathrm{m/s}$. At time $t = 0$, a forward-travelling (i.e. left-to-right) pulse arrives at X with amplitude 5 V and duration 100 ns, as shown in Figure 1.9.

Draw a dimensioned sketch of $y(t)$ for $0 \le t \le 250\,\mathrm{ns}$ where $y(t)$ is the waveform at Y, a point 7 m from the end of the line. Label the voltages taken by $y(t)$ and the times at which it changes. Assume that no reflections occur at point X. [5]

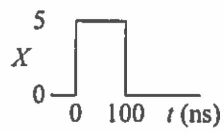


Figure 1.9

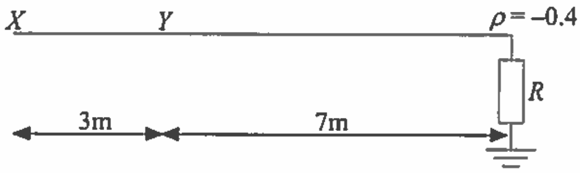
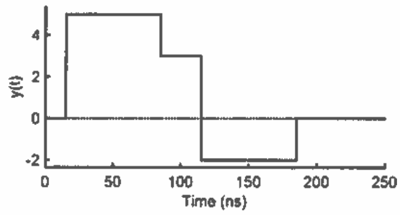


Figure 1.10

[U] The pulse will take $\frac{3}{u} = 15\,\mathrm{ns}$ *to reach Y and then a further* $\frac{2\times 7}{u} = 70\,ns$ *for its reflection to return to Y. So the waveform at Y is the sum of two components: (A) an incoming pulse of amplitude 5 lasting between 15 and 115 ns and (B) a reflected pulse of amplitude* $5 \times \rho = -2$ *lasting between 85 and 185 ns.*

Thus $y(t)$ *changes at* $\{15, 85, 115, 185\}$ ns *to voltages* $\{5, 3, -2, 0\}$ V. *A graph of* $y(t)$ *is shown below.*


y(t)
4
2
0
-2
0
50
100
150
200
250
Time (ns)

2. a) i) Assuming that the op-amp in Figure 2.1 is ideal, show that the output voltage at time t is given by [3]

$$y(t)=y(0)-\frac{1}{R_1C}\int_0^t x(\tau)d\tau.$$

[U] The V_- input of the op-amp is a virtual ground and so the voltage across the capacitor is $y-0=y$ and the current through it (from right to left according to the passive sign convention) is $i=-\frac{x}{R_1}$. Applying the capacitor equation, $i=C\frac{dv}{dt}$, we therefore obtain $-\frac{x}{R_1}=C\frac{dy}{dt}$ from which $\frac{dy}{dt}=-\frac{1}{R_1C}x$ and integratingg both sides from 0 to t gives the required integral equation.

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ii) If $x(t)$ has a constant value of 12 V and $y(0)=5\,\text{V}$, determine the value of R_1 such that $y(t)=-5\,\text{V}$ when $t=1\,\text{ms}$. [3]

[U] Substituting for $x(\tau)$ and $y(0)$ gives $y(t)=5-\frac{1}{R_1C}\int_0^t 12d\tau=5-\frac{12t}{R_1C}$. So, for $t=1\,\text{ms}$, $y(t)=-5=5-\frac{12t}{R_1C}$ gives $R_1=\frac{1.2t}{C}=12\,\text{k}\Omega$.

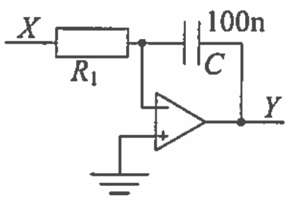

Figure 2.1

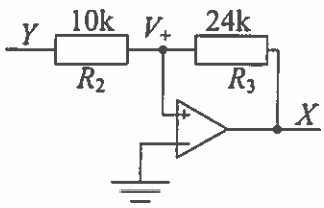

Figure 2.2

b) In the positive-feedback circuit of Figure 2.2, the op-amp has zero input current and its output voltage, x, is given by

$$x=\begin{cases}+12\,\text{V} & \text{if } v_+>v_-\\ -12\,\text{V} & \text{if } v_+<v_-\end{cases}$$

where v_+ and v_- are the voltages at the op-amp input terminals.

i) Show that $v_+=\frac{5}{17}x+\frac{12}{17}y$. [2]

[U] This is a weighted-average cicuit with $v_+=\frac{24y+10x}{34}$. Alternatively, applying KCL at node V_+ gives $\frac{v_+-y}{10}+\frac{v_+-x}{24}=0$ which leads to the same equation.

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ii) Hence show that the op-amp output, x, is given by

$$x=\begin{cases}+12\,\text{V} & \text{if } y>a\\ +12\,\text{V or }-12\,\text{V} & \text{if } -a<y<a\,.\\ -12\,\text{V} & \text{if } y<-a\end{cases}$$

Determine the value of a and explain what determines the value of x when $-a < y < a$. [4]

[U] If $x = +12$, then, from the previous part, $v_+ = \frac{24y+120}{34}$ whereas if $x = -12$, then $v_+ = \frac{24y-120}{34}$. In the first case $v_+ > 0$ provided $24y + 120 > 0 \Leftrightarrow y > -5$ and in the second case $v_+ > 0$ provided $24y - 120 > 0 \Leftrightarrow y > +5$. Thus, if $y > +5$, then in either case $v_+ > 0$ and $x = +12$. Similarly, if $y < -5$ then, regardless of the initial value of x, $v_+ < 0$ and $x = -12$.

If, however, $-5 < y < +5$ then v_+ will be greater or less than 0 according to whether $x = +12$ or -12. It follows that in this case, x will remain at its current value which is determined by whether y was most recently > 5 or < -5.

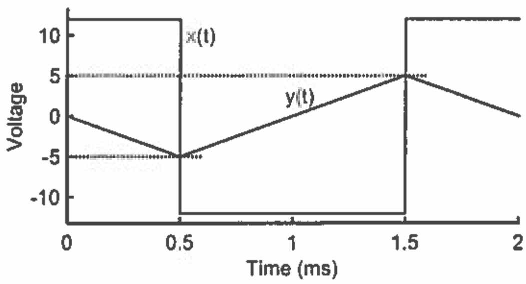
All this is expressed in the equation given in the question with $a = 5$.

[This question is continued on the next page]

c) The circuits of parts a) and b) are combined to form the oscillator circuit shown in Figure 2.3. The resistor R_1 has the value determined in part a).

Given the initial conditions $x(0) = +12V$ and $y(0) = 0V$, draw a dimensioned sketch of the waveforms $x(t)$ and $y(t)$ for $0 < t < 2\,\text{ms}$. Your sketch should label the peak value of $y(t)$ and the times when $x(t)$ switches. [6]

[U] From part b) we know that $x(t)$ will switch between ± 12 when $y(t)$ reaches ± 5. Also, from part a), we know that when $x(t) = \pm 12$, $y(t)$ has a gradient of $\mp 10\,\text{V/ms}$. So, $y(t)$ will ramp down to -5 in $0.5\,\text{ms}$, ramp up to $+5$ and then ramp down again as shown in the graph below. So the max and min values of $y(t)$ are ± 5 and the switching points of $x(t)$ occur at 0.5 and $1.5\,\text{ms}$.



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d) The output of the oscillator, y, forms the input to the circuit shown in Figure 2.4. You may assume that the diodes have a voltage drop of 0.7V when forward-biased and a current of zero when reverse-biased.

i) Give an expression for v in terms of y when both the diodes are reverse-biased. By considering the voltage across R_6, determine the range of input voltages, y, for whch both diodes will be reverse-biased. [4]

[U] If both diodes are off, the circuit will be an inverting amplifier and $v = -\frac{15.1}{38.6}y = -0.39y$. The voltage across the diodes (and across R_6) will be $v_6 = -\frac{9.6}{38.6}y$. The assumption that the diodes are both off is only true if this voltage lies in the range ± 0.7 or, in other words, that $|y| < \frac{0.7\times 38.6}{9.6} = 2.81$.

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ii) Give an expression for v in terms of y when diode D_1 is forward-biased. Determine the range of input voltages, y, for whch diode D_1 will be forward-biased. [4]

[U] If D_1 is forward-biased, current must be flowing from left to right in R_4 and so $y > 2.81$. The voltage v is equal to the sum of the votage across R5 and the voltage across D_1 and is therefore $v = -\frac{5.5}{38.6}y - 0.7 = -0.14y - 0.7$. For the specific input $y_0 = 2.81$, we have $-0.39y_0 = -0.14y_0 - 0.7 = -1.09$ so the two expressions are equal at the switching point,

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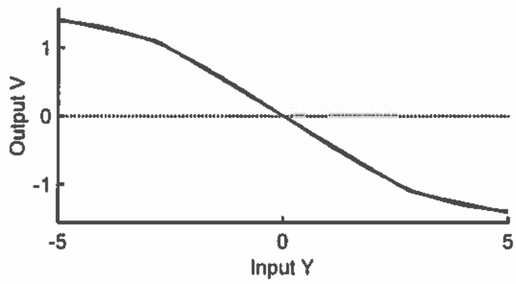
iii) Draw a dimensioned sketch showing the graph of v versus y for $-5 < y < +5$. [4]

[U] Combining the previous results, we have

$$v = \begin{cases} -0.14y + 0.7 & \text{for } y < -2.81 \\ -0.39y & \text{for } -2.81 < y < 2.81 \\ -0.14y - 0.7 & \text{for } y > 2.81 \end{cases}$$

Specific value of v are: $v(\pm 5) = \mp 1.4$ and $v(\pm 2.81) = \mp 1.09$.

Although not requested, the graph below also includes the target sine wave that the circuit is intended to approximate.



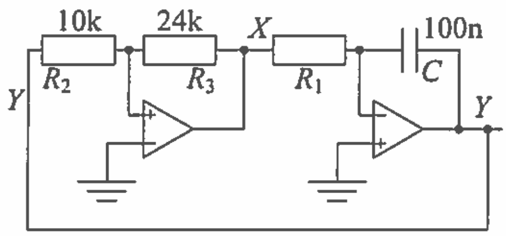


Figure 2.3

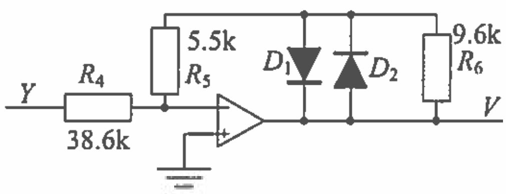


Figure 2.4

3. Figure 3.1 shows a shows a circuit whose input and output votages are $x(t)$ and $y(t)$ respectively.

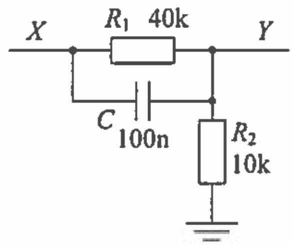


Figure 3.1

a) i) Determine the frequency response, $G(j\omega) = \frac{Y}{X}$. [4]

[U] Treating the circuit as a potential divider,

$$G(j\omega) = \frac{R_2}{R_2 + \frac{R_1 \times \frac{1}{j\omega C}}{R_1 + \frac{1}{j\omega C}}} = \frac{R_2}{R_2 + \frac{R_1}{j\omega CR_1 + 1}} = \frac{R_2(j\omega CR_1 + 1)}{j\omega CR_1R_2 + R_1 + R_2}$$

$$= = \frac{j\omega CR_1 + 1}{j\omega CR_1 + 5} = \frac{j0.004\omega + 1}{j0.004\omega + 5}$$

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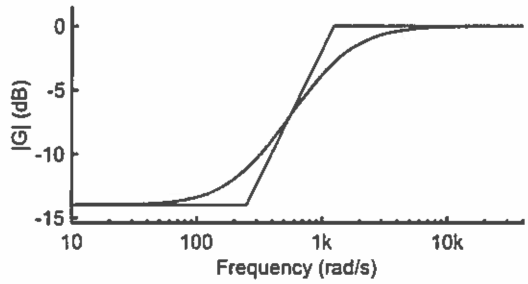
ii) Draw a dimensioned sketch of the straight-line approximation to the magnitude response, $|G(j\omega)|$, showing the values of any corner frequencies (in rad/s) and the gains of any horizontal portions of the response (in dB). [4]

[U] From the transfer function, the corner frequencies are $\omega_n = \frac{1}{R_1C} = 250$ in the numerator and $\omega_d = \frac{5}{R_1C} = 1250$ in the denominator. The low frequency asymptote is $0.2 = -14$dB and the high frequency gain is $1 = 0$dB.

In the graph below, the straight-line approximation is shown in red and the true response (not requested) in blue.



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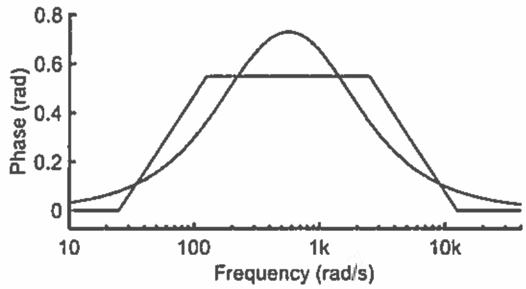
iii) Draw a dimensioned sketch of the straight-line approximation to the phase response, $\angle G(j\omega)$, showing the values of any corner frequencies (in rad/s) and the phase of any horizontal portions of the response (in rad). [4]

[U] The phase response has corner frequencies at: $0.1\omega_n = 25^+$, $10\omega_n = 2500^-$, $0.1\omega_d = 125^-$ *and* $10\omega_d = 12500^+$ *where the superscript indicates the sign of the gradient change at each corner. The frequency increment from* $\omega = 25$ *to* $\omega = 125$ *is* $\log_{10}\frac{125}{25} = 0.699$ *decades and the slope is* $\frac{\pi}{4}$ *per decade, so the phase increment is* $\frac{\pi}{4} \times 0.699 = 0.549$. *So the horizontal portions or the response have values* $\{0, 0.549, 0\}$ rad.

In the graph below, the straight-line approximation is shown in red and the true response (not requested) in blue.



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b) Determine the time constant of the circuit. [3]

[U] We can determine the time constant in two ways:

(A) If we short-circuit the input voltage source (i.e. connect X to ground), the Thévenin resistance at the teminals of the capacitor is $R_{Th} = 40||10 = 8$k. *Thus the time-constant of the circuit is* $\tau = R_{Th}C = 0.8$ ms.

(B) The time constant is also equal to the reciprocal of the denominator corner frequency: $\tau = \frac{1}{\omega_d} = \frac{1}{1250} = 0.8$ ms.

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c) If the input, shown in Figure 3.2, is given by

$$x(t) = \begin{cases} -5 & \text{for } t < 0 \\ +5 & \text{for } t \geq 0 \end{cases},$$

i) determine expressions for $y(t)$ both for $t < 0$ and for $t \geq 0$, [6]

[U] From the transfer function, the DC gain is 0.2. *Therefore the steady-state output is*

$$y_{SS}(t) = \begin{cases} -1 & \text{for } t < 0 \\ +1 & \text{for } t \geq 0 \end{cases}.$$

It follows that the complete output is

$$y(t) = \begin{cases} -1 & \text{for } t < 0 \\ 1 + Ae^{-\frac{t}{\tau}} & \text{for } t \geq 0 \end{cases}$$

where $\tau = 0.8ms$ from part b).

To determine A, we need to calculate $y(0)$. We do this in one of two ways.

(A) at $t = 0-$, the voltage across the capacitor is $y(0-) - x(0-) = (-1) - (-5) = 4$. Since the capacitor voltage cannot change instantaneously, $y(0+) - x(0+) = 4$ which means that $y(0+) = 4 + x(0+) = 4 + 5 = 9$.

(B) The output discontinuity will be $y(0+) - y(0-) = G(\infty) \times (x(0+) - x(0-)) = 1 \times 10 = 10$ from which $y(0+) = y(0-) + 10 = -1 + 10 = 9$.

So, now we have $y(0+) = 9 = 1 + A$ from which $A = 8$. The final expression for $y(t)$ is therefore

$$y(t) = \begin{cases} -1 & \text{for } t < 0 \\ 1 + 8e^{-\frac{t}{\tau}} & \text{for } t \geq 0 \end{cases}$$

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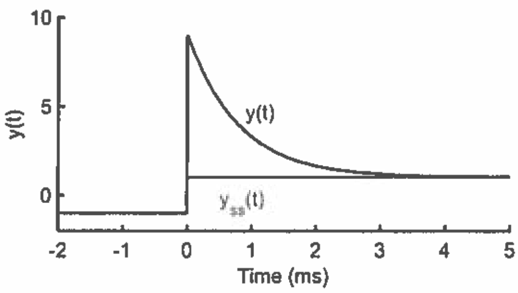
ii) draw a dimensioned sketch of the waveform of $y(t)$. [3]

[U] $y(t)$is plotted blow in blue. Although not requested, the graph below also shows $y_{SS}(t)$ in red.



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d) If the input, shown in Figure 3.3, is given by

$$x(t) = \begin{cases} \sin(500t) & \text{for } t < 0 \\ \sin(1000t) & \text{for } t \geq 0 \end{cases},$$

determine expressions for $y(t)$ both for $t < 0$ and for $t \geq 0$. [6]

[U] The steady-state gains are

$$G(j\omega) = \begin{cases} \frac{2j+1}{2j+5} = \frac{9+8j}{29} = 0.3103 + 0.2759j & \text{for } t < 0 \\ \frac{4j+1}{4j+5} = \frac{21+16j}{41} = 0.5122 + 0.3902j & \text{for } t \geq 0 \end{cases}.$$

So, since the phasor inputs are $X = -j$ for both segments, the steady state phasor outputs are

$$Y_{SS} = \begin{cases} 0.2759 - 0.3103j & \text{for } t < 0 \\ 0.3902 - 0.5122j & \text{for } t \geq 0 \end{cases}$$

and the corresponding waveforms are

$$y(t) = \begin{cases} 0.2759\cos 500t + 0.3103\sin 500t & \text{for } t < 0 \\ 0.3902\cos 1000t + 0.5122\sin 1000t + Be^{-\frac{t}{\tau}} & \text{for } t \geq 0 \end{cases}.$$

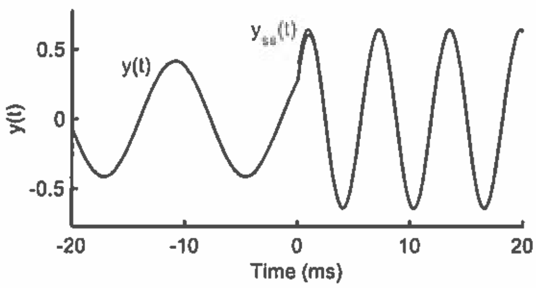
Since there is no input discontinuity, there will be no output discontinuity either so $y(0-) = y(0+)$*. So we can write*

$$\begin{aligned} y(0-) = 0.2759 &= y(0+) = 0.3902 + B \\ \Rightarrow B &= 0.2759 - 0.3902 = -0.1143 \end{aligned}$$

and the full expression for $y(t)$ *is therefore*

$$y(t) = \begin{cases} 0.2759\cos 500t + 0.3103\sin 500t & \text{for } t < 0 \\ 0.3902\cos 1000t + 0.5122\sin 1000t - 0.1143e^{-\frac{t}{\tau}} & \text{for } t \geq 0 \end{cases}.$$

Although not requested, the graphs of $y_{SS}(t)$ *and* $y(t)$ *are plotted below. The difference between them is negligible for* $t > 2$ ms.



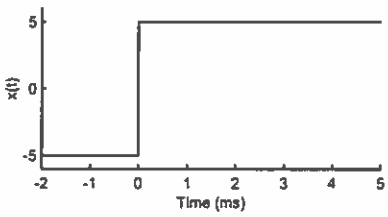


Figure 3.2

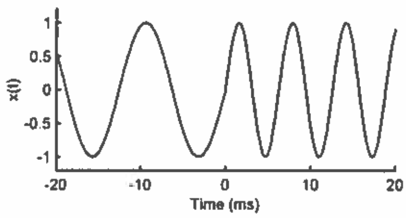


Figure 3.3