$$A = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix} \qquad \begin{vmatrix} 3-\lambda & 1 & 0 \\ 1 & 3-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{vmatrix} = 0 \quad \Rightarrow \lambda_1 = 5 \text{ and } (\lambda-3)^2 - 1 = 0$$

$$\lambda^2 - 6\lambda + 8 = 0$$

$$(\lambda-2)(\lambda-4) = 0 \qquad \therefore \ \lambda_1 = 5, \quad \lambda_2 = 4, \quad \lambda_3 = 2.$$ 4

Eigenvectors: $\lambda_1 = 5 \quad \underline{a}_1 = (0,0,1)^T$ 2

$\lambda_2 = 4 \quad \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0 \quad b = a \quad \underline{a}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \to \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ 2

$\lambda_3 = 2 \quad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0 \quad b = -a \quad \underline{a}_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \to \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ 2

Form the mx $P = (\underline{a}_1\ \underline{a}_2\ \underline{a}_3)$, to give

$$AP = P\Lambda \qquad \Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_3 \end{pmatrix}$$

$$\therefore \ \Lambda = P^{-1}AP$$

Check: $AP = (A\underline{a}_1, A\underline{a}_2, A\underline{a}_3)$

$= (\lambda_1\underline{a}_1\ \lambda_2\underline{a}_2\ \lambda_3\underline{a}_3)$ 5

and

$P\Lambda = (\underline{a}_1\ \underline{a}_2\ \underline{a}_3)\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$

$= (\lambda_1\underline{a}_1\ \lambda_2\underline{a}_2\ \lambda_3\underline{a}_3)$ ✓

To form P we write

$$P = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \\ \sqrt{2} & 0 & 0 \end{pmatrix}$$ 5

$$\Lambda = \begin{pmatrix} 5 & & 0 \\ & 4 & \\ 0 & & 2 \end{pmatrix}.$$

Note that P is orthogonal so $P^{-1} = P^T$. Hence $\Lambda = P^TAP$ can be evaluated directly. If a student takes this route he/she should be given credit.

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$$A = \begin{pmatrix} 11 & \sqrt{11} & 0 \\ \sqrt{11} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \lambda = 1 \text{ and } (\lambda-1)(\lambda-11) - 11 = 0$$

so $\lambda^2 - 12\lambda = 0 \Rightarrow \lambda = 0, 12.$

$\lambda_1 = 12 \quad \lambda_2 = 1 \quad \lambda_3 = 0$ L

evecs: $\lambda_1 = 12$: $\underline{a}_1 = \begin{pmatrix} \sqrt{11} \\ 1 \\ 0 \end{pmatrix} \rightarrow \frac{1}{\sqrt{12}}\begin{pmatrix} \sqrt{11} \\ 1 \\ 0 \end{pmatrix}$ 4

$\lambda_3 = 0 \qquad \underline{a}_3 = \begin{pmatrix} 1 \\ -\sqrt{11} \\ 0 \end{pmatrix} \rightarrow \frac{1}{\sqrt{12}}\begin{pmatrix} 1 \\ -\sqrt{11} \\ 0 \end{pmatrix}$ L

$\lambda_2 = 1 \qquad \underline{a}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$]

Form the mx from normalized $\underline{a}_i$: $P = (\underline{a}_1\ \underline{a}_2\ \underline{a}_3)$

$$AP = P\Lambda \qquad = \frac{1}{\sqrt{12}}\begin{pmatrix} \sqrt{11} & 0 & 1 \\ 1 & 0 & -\sqrt{11} \\ 0 & \sqrt{12} & 0 \end{pmatrix}$$

Moreover $P^{-1} = P^T$ (bookwork)

$$\therefore \quad \Lambda = P^T A P \qquad \Lambda = \mathrm{diag}(12, 1, 0)$$

$Q = \underline{x}^T A \underline{x} \Rightarrow A = \begin{pmatrix} 11 & \sqrt{11} & 0 \\ \sqrt{11} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and, with $\underline{x} = P\underline{y}$ 6

$$\therefore Q = \underline{y}^T(P^T A P)\underline{y} = \underline{y}^T(P^T A P)\underline{y}$$
$$= \underline{y}^T \Lambda \underline{y}$$
$$= 12y_1^2 + y_2^2 + 0y_3^2$$

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EXAMINATION QUESTION / SOLUTION

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EE2

QUESTION

3

SOLUTION

3.

(a) (i) $(1-p)^{10} = 0.9044$ 2

(ii) $10p(1-p)^9 = 0.09135$ 2

(iii) 1-P(0 or 1) $= 1 - 0.9044 - 0.09135 = 0.0042$ 2

last part: need n so that $1-(1-p)^n > \frac{1}{2}$, i.e. $n\log(1-p) < \log\frac{1}{2}$,

i.e. $n > \log\frac{1}{2}/\log(1-p) = 68.97$ 3

(b) $\mathrm{P}(mf \mid A\cap B) = \mathrm{P}(A\cap B \mid mf)\mathrm{P}(mf)/\mathrm{P}(A\cap B)$

but, $\mathrm{P}(A\cap B) = \mathrm{P}(A\cap B \mid mf)\mathrm{P}(mf) + \mathrm{P}(A\cap B \mid \overline{mf})\mathrm{P}(\overline{mf})$ 6

$= p_A p_B q + (1-p_A)(1-p_B)(1-q)$

so, $\mathrm{P}(mf \mid A\cap B) = p_A p_B q/\{p_A p_B q + (1-p_A)(1-p_B)(1-q)\}$

$\mathrm{P}(mf \mid A\cap \bar{B}) = \mathrm{P}(A\cap \bar{B} \mid mf)\mathrm{P}(mf)/\mathrm{P}(A\cap \bar{B})$

but $\mathrm{P}(A\cap \bar{B} \mid mf) = \mathrm{P}(A \mid mf) - \mathrm{P}(A\cap B \mid mf) = p_A - p_A p_B$

and $\mathrm{P}(A\cap \bar{B}) = \mathrm{P}(A\cap \bar{B} \mid mf)\mathrm{P}(mf) + \mathrm{P}(A\cap \bar{B} \mid \overline{mf})\mathrm{P}(\overline{mf})$ 5

$= \{p_A - p_A p_B\} + \{(1-p_A) - (1-p_A)(1-p_B)\}$

$= p_A(1-p_B) + (1-p_A)p_B,$

so $\mathrm{P}(mf \mid A\cap \bar{B}) = p_A(1-p_B)q/\{p_A(1-p_B)q + (1-p_A)p_B(1-q)\}$

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QUESTION
4

SOLUTION

4.

distn fn: $F(v) = \mathrm{P}(V \le v) = \int_0^v f(v)dv = [-(1+v/\xi)^{-1}]_0^v$

$= 1-(1+v/\xi)^{-1} = v/(\xi+v)$ 4

median: $\frac{1}{2} = F(m) = m/(\xi+m) \Rightarrow m = \xi$ 2

$\mathrm{P}(V > a+b \mid V > a) = \mathrm{P}(V > a+b)/\mathrm{P}(V > a) = (1+\frac{a+b}{\xi})^{-1}/(1+\frac{a}{\xi})^{-1}$

$= (\xi+a)/(\xi+a+b)$ 4

$\mathrm{P}(\text{all } v_i \text{ in range}) = \prod_{i=1}^n \mathrm{P}(a < v_i < b) = \{F(b)-F(a)\}^n$

$= \{(1+\frac{a}{\xi})^{-1} - (1+\frac{b}{\xi})^{-1}\}^n = \{\frac{\xi(b-a)}{(\xi+a)(\xi+b)}\}^n$ 4

P(at most 2 below a): first, $F(a) = 1/(3+1) = 1/4$. 2

Then, prob= P(none below)+P(1 below)+P(2 below)

$= (\frac{3}{4})^4 + 4(\frac{3}{4})^3(\frac{1}{4}) + 6(\frac{1}{4})^2(\frac{3}{4})^2 = 3^5/4^4 = 243/256 = 0.9492$ 4

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QUESTION
5

SOLUTION

5.

$\mathrm{E}(X^{-1}) = \frac{1}{2}\xi^3 \int_0^\infty x e^{-\xi x} dx = \frac{1}{2}\xi^3 (1/\xi^2) = \frac{1}{2}\xi$ 3

$\mathrm{var}(X^{-1}) = \mathrm{E}(X^{-2}) - \mathrm{E}(X^{-1})^2 = \frac{1}{2}\xi^3 \int_0^\infty e^{-\xi x} dx - (\frac{1}{2}\xi)^2 = \frac{1}{2}\xi^2 - \frac{1}{4}\xi^2 = \frac{1}{4}\xi^2$ 4

$\mathrm{E}(t) = 2n^{-1} \sum_{i=1}^n \mathrm{E}(x_i^{-1}) = 2n^{-1} \sum_{i=1}^n (\frac{1}{2}\xi) = \xi$ (unbiased) 4

$$mse(t) = \mathrm{var}(t) + \mathrm{bias}(t)^2 = \mathrm{var}(2n^{-1} \sum_{i=1}^n x_i^{-1}) + 0$$
$$= 4n^{-2} \sum_{i=1}^n \mathrm{var}(x_i^{-1}) = 4n^{-2} \sum_{i=1}^n (\tfrac{1}{4}\xi^2) = \xi^2/n$$ 6

consistent: yes, because $mse(t) \to 0$ as $n \to \infty$ 3

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QUESTION
6

SOLUTION

6.

$\mathrm{E}(y_t) = 0$, $\mathrm{var}(y_t) = (1 + \frac{1}{4} + \frac{1}{16})\sigma_e^2 = \frac{21}{16}\sigma_e^2$ 4

$$\mathrm{cov}(y_t, y_{t-s}) = \mathrm{cov}(e_t + \tfrac{1}{2}e_{t-1} + \tfrac{1}{4}e_{t-2}, e_{t-s} + \tfrac{1}{2}e_{t-s-1} + \tfrac{1}{4}e_{t-s-2})$$

$$= \begin{cases} (\frac{1}{2} + \frac{1}{8})\sigma_e^2 = \frac{5}{8}\sigma_e^2 & \text{for } s = 1 \\ \frac{1}{4}\sigma_e^2 & \text{for } s = 2 \\ 0 & \text{for } s > 2 \end{cases}$$ 5

stationary: yes, since mean and covar fn independent of t 2

spectrum: $f(\omega) = \Re\{\sum_{k=-\infty}^{\infty} \gamma_k e^{ik\omega}\}$, where $\gamma_k = \mathrm{cov}(y_t, y_{t+k})$

$$f(\omega) = \Re\{\gamma_0 + 2\gamma_1 e^{i\omega} + 2\gamma_2 e^{2i\omega}\} = \tfrac{21}{16}\sigma_e^2 + \tfrac{10}{8}\sigma_e^2\cos\omega + \tfrac{2}{4}\sigma_e^2\cos(2\omega)$$

$$= \tfrac{13}{16}\sigma_e^2 + \tfrac{10}{8}\sigma_e^2\cos\omega + \sigma_e^2\cos^2\omega = \tfrac{\sigma_e^2}{16}(13 + 20\cos\omega + 16\cos^2\omega)$$ 6

low-pass: since $f(\omega)$ decreases from $f(0) = \frac{49}{16}\sigma_e^2$ to $f(\pi) = \frac{9}{16}\sigma_e^2$

as ω increases from 0 to π (though not monotonically) 3

(In fact, $\frac{d}{d\omega}f(\omega) = \frac{\sigma_e^2}{16}(-20\sin\omega - 32\sin\omega\cos\omega) = -\frac{\sigma_e^2}{4}\sin\omega(5 + 8\cos\omega)$

so $f(\omega)$ takes minimum value $\frac{27}{64}\sigma_e^2$ at $\cos^{-1}(-5/8)$.)

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