UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 1999

MEng Honours Degrees in Computing Part IV
MSci Honours Degree in Mathematics and Computer Science Part IV
MSc Degree in Advanced Computing
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the
Diploma of Membership of Imperial College
Associateship of the Royal College of Science
Associateship of the City and Guilds of London Institute

PAPER 4.38

COMPLEXITY

Thursday, May 13th 1999, 10.00 – 12.00

Answer THREE questions

For admin. only:
paper contains 4 questions

1a i) Explain what it means to say that a decision problem (or language) *reduces* to another *in polynomial time.*

ii) Show that a decision problem is NP-complete if and only if (a) it is in NP, and (b) some NP-complete problem reduces to it in polynomial time. You may assume without proof standard facts about reduction.

iii) What is the problem 3SAT?

b Let G be an undirected graph. An *independent set in G* is a set X of nodes of G that contains no edges of G (i.e., such that for no x, y $\in$ X is (x,y) an edge of G).

Consider the following decision problem:

IND *(independent set)*
Given: an undirected graph G, and an integer $k \geq 0$ written in binary.
Problem: is there an independent set of G containing exactly k nodes?

Show that IND is NP-complete. [You may assume that 3SAT is NP-complete.]

c Consider the following variants of IND:

i) IND_u: as for IND except that the number k is *written in unary.*

ii) IND_p:
Given: an undirected graph G, an integer $k \geq 0$ written in binary, and a node x of G.
Problem: does G have an independent set X with exactly k nodes and with x $\in$ X?

iii) IND_{17}:
Given: an undirected graph G.
Problem: is there an independent set X of G with exactly 17 nodes?

Are these problems NP-complete? Briefly justify your answer.

2a i) State, without proof, Savitch's theorem on space complexity classes.

ii) State, without proof, the Immerman–Szelepscényi theorem.

b Consider the following decision problem:

> CYC *(cycle)*
> *Given:* a directed graph G, and distinct nodes x, y of G.
> *Problem:* is there a cycle in G containing x and y (i.e., a path in G from x to y and back to x)?

Show that CYC $\in$ NLOGSPACE. Here, and below, NLOGSPACE denotes the class NSPACE(log n).

c i) What is the problem 2SAT?

ii) Briefly explain why 2SAT $\in$ P.

d Using parts a and b, refine part c to conclude that 2SAT $\in$ NLOGSPACE. You may assume without proof that NLOGSPACE is closed under logspace reduction.

The four parts carry, respectively, 20%, 20%, 35%, 25% of the marks.

Turn over...

3a Construct a Boolean circuit which when given an input word $x \in \{0,1\}^*$ decides whether x has two successive 1s. Your circuit must have depth $O(\log n)$. Explain your construction briefly. What is your circuit's size?

b i) Define the classes NC_j and NC.

ii) What is the relationship between NC and P? Justify your answer briefly.

c Explain why the reachability problem RCH for graphs belongs to NC_2.

d i) What does it mean for a language to be P-complete? Give an example of a P-complete problem.

ii) Why is it generally believed that no problem can be P-complete and NC? If this were not the case, what would be the consequence for P, NC and NC_j?

The four parts carry, respectively, 25%, 30%, 20%, 25% of the marks.

4a i) Define *Monte Carlo Turing machine* (for a language L). Also define the class RP.

ii) Let L_1 and L_2 be two RP languages. Let L be defined by

$$L = \{xy : x \in L_1, y \in L_2\}$$

(Here xy is the concatenation of x and y.) Show that $L \in RP$.

b i) Describe a zero knowledge proof that a graph G can be 3-coloured. Explain briefly why the proof works.

ii) Describe a zero knowledge proof that a graph G has a clique of size k. (A clique in a graph is a subset of the nodes where every node is connected to every other node). Explain briefly why the proof works. [Hint: consider permutations of the adjacency matrix of G.]

End of paper