EE4-08
EE9-SO14

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2013

MSc and EEE/EIE PART IV: MEng and ACGI

DIGITAL IMAGE PROCESSING

Friday, 10 May 10:00 am

Time allowed: 3:00 hours

There are FOUR questions on this paper.

Answer THREE questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s) :	P.T. Stathaki
	Second Marker(s) :	C. Mehring

1. a) i) Explain why the two dimensional Discrete Fourier Transform is separable. **[2]**

   ii) Show that the two dimensional Discrete Fourier Transform can be implemented using one dimensional Discrete Fourier Transforms. **[2]**

b) Let $f(x,y)$ denote an $M \times N$-point 2-D sequence that is zero outside $0 \le x \le M-1$, $0 \le y \le N-1$, where M and N are integer powers of 2. In implementing the standard Discrete Cosine Transform of $f(x,y)$, we relate $f(x,y)$ to a new $M \times N$-point 2-D sequence $C(u,v)$.

   i) Define the sequence $C(u,v)$ in terms of $f(x,y)$. **[2]**

   ii) Define the concept of energy compaction in the Discrete Cosine Transform. **[2]**

   iii) Discuss how we can use the Discrete Cosine Transform for image compression. **[2]**

c) Consider the population of vectors $\underline{f}$ of the form

$$\underline{f} = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix}.$$

Each component $f_i(x,y), i = 1,2$ represents an image of size $M \times M$, where M is even. The population arises from the formation of vectors across the entire collection of pixels. The two images are defined as follows:

$$f_1(x,y) = \begin{cases} r_1 & 1 \le x \le \frac{M}{2}, 1 \le y \le M \\ s_1 & \frac{M}{2} \le x \le M, 1 \le y \le M \end{cases}$$

$$f_2(x,y) = \begin{cases} r_2 & 1 \le x \le \frac{M}{2}, 1 \le y \le M \\ s_2 & \frac{M}{2} \le x \le M, 1 \le y \le M \end{cases}$$

Consider now a population of random vectors of the form

$$\underline{g} = \begin{bmatrix} g_1(x,y) \\ g_2(x,y) \end{bmatrix}$$

where the vectors $\underline{g}$ are the Karhunen-Loeve (KL) transforms of the vectors $\underline{f}$.

   i) Find the images $g_1(x,y)$ and $g_2(x,y)$ using the Karhunen-Loeve (KL) transform. **[8]**

   ii) Comment on whether you could obtain the result of c)-i) above using intuition rather than by explicit calculation. **[2]**

2. a) Suppose we applied histogram equalization to a given image whose pixels all take a constant value $c \in [0,255]$. Let $h_{\text{out}}(s)$ denote the resulting (equalized) histogram of pixel values s taking values in [0, 255]. Sketch the plot of $h_{\text{out}}(s)$, and label the plot axes.

[5]

b) Suppose an image $f(x,y)$ is corrupted by additive Gaussian noise $n(x,y)$ with variance σ^2, where the corrupted image is $\tilde{f}(x,y) = f(x,y) + n(x,y)$. The Laplacian filter is applied to $\tilde{f}(x,y)$ yielding a new image $g(x,y)$. What is the variance of $g(x,y)$? Does the Laplacian filter reduce noise? Justify your answers.

[5]

c) The two images shown below in **Figure 2.1** are quite different, but their histograms are identical. Both images have size 80 × 80, with black and white pixels. Suppose that both images are blurred with a 3×3 smoothing mask. Would the resultant histograms still be the same? Draw the two histograms and explain your answer.
Note: the grey lines are used to signify the boundaries of the two images but not part of them.

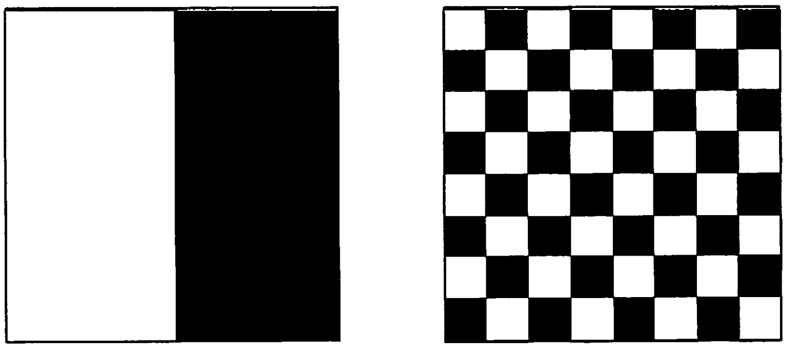


Figure 2.1

[5]

d) Discuss the advantages and disadvantages of global and local histogram equalization.

[5]

3. a) We are given the degraded version g of an image f such that in lexicographic ordering

$$g = Hf + n$$

where H is the degradation matrix which is assumed to be block-circulant, and n is the noise term which is assumed to be zero mean, independent and white. The images have size $N \times N$.

i) Consider the Inverse Filtering image restoration technique. Write down without proof the general expressions for both the Inverse filter estimator and the restored image in both spatial and frequency domains and explain all symbols used. **[4]**

ii) Discuss the disadvantages of the Inverse Filtering image restoration technique. **[4]**

b) In a particular scenario, the image under consideration is blurred due to relative motion between the image and the camera. The pixel of the image g at location (x, y) is related to the corresponding pixel of image f through the following relationship:

$$g(x, y) = f(x, y) + f(x, y-1) + f(x, y-2)$$

i) Consider the Inverse Filtering image restoration technique. Find the expressions for both the Inverse filter estimator and the restored image in the frequency domain. **[6]**

ii) Find the specific frequencies for which the restored image cannot be estimated. **[6]**

4. a) Consider a grey level image $f(x, y)$ with grey levels from 0 to 255. Assume that the image $f(x, y)$ has medium contrast. Furthermore, assume that the image $f(x, y)$ contains large areas of slowly varying intensity. Consider the image $g(x, y) = f(x, y) - f(x-1, y-1)$.

   i) Sketch a possible histogram of the image $f(x, y)$. **[2]**

   ii) Discuss the characteristics of the histogram of the image $g(x, y)$. **[2]**

   iii) Explain which of the two images is more amenable to compression using Huffman code. **[2]**

b) The following **Figure 4.1** shows a 10×10 image with 3 different grey levels (black, grey, white).

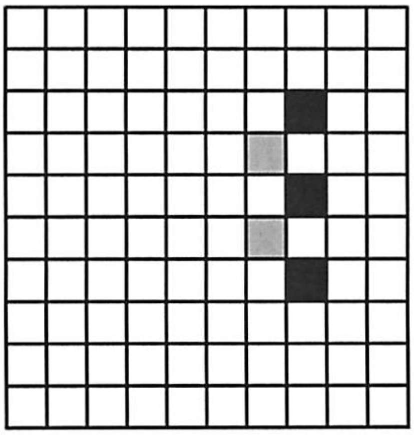


Figure 4.1

   i) Derive the probability of appearance (that forms the histogram) for each intensity (grey) level. Calculate the entropy of this image. **[2]**

   ii) Derive the Huffman code. **[2]**

   iii) Calculate the average length of the fixed length code and that of the derived Huffman coding. **[2]**

   iv) Calculate the ratio of image size (in bits) between using the fixed length coding and Huffman coding. Calculate the relative coding redundancy. **[2]**

   v) Derive the extended-by-two Huffman code and calculate the average length of it. **[4]**

   vi) Calculate the ratio of image size (in bits) between using the fixed length coding and extended-by-two Huffman coding. Calculate the relative coding redundancy. **[2]**

Question 1 (c)

$$f_1(x,y) = \left[\, r_1 \;\vdots\; s_1 \,\right] \qquad f_2(x,y) = \left[\, r_2 \;\vdots\; s_2 \,\right]$$

$$f_1 - m_1 = \left[\, \frac{r_1 - s_1}{2} \;\vdots\; -\frac{r_1 - s_1}{2} \,\right] \qquad f_2 - m_2 = \left[\, \frac{r_2 - s_2}{2} \;\vdots\; -\frac{r_2 - s_2}{2} \,\right]$$

$$E\{(f_1 - m_1)^2\} = \left(\frac{r_1 - s_1}{2}\right)^2 \text{ call this } A^2$$

$$E\{(f_2 - m_2)^2\} = \left(\frac{r_2 - s_2}{2}\right)^2 \text{ call this } B^2$$

$$E\{(f_1 - m_1)(f_2 - m_2)\} = AB$$

Covariance matrix : $\begin{bmatrix} A^2 & AB \\ AB & B^2 \end{bmatrix}$, Eigenvalues A^2+B^2, 0

Eigenvectors:

$$\begin{bmatrix} A^2 & AB \\ AB & B^2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = (A^2+B^2) \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \cancel{A^2x} + ABy = \cancel{A^2x} + B^2x$$

$$\Rightarrow ABy = B^2x \Rightarrow$$

$$Ay = Bx \Rightarrow$$

$$y = \frac{B}{A}x$$

Eigenvector 1 : $\begin{bmatrix} x \\ \frac{B}{A}x \end{bmatrix}$

Modulus 1 : $x^2 + \frac{B^2}{A^2}x^2 = 1 \Rightarrow \frac{A^2+B^2}{A^2} \cdot x^2 = 1 \Rightarrow$

$$x = \frac{A}{\sqrt{A^2+B^2}}, \text{ call } \sqrt{A^2+B^2} = D \Rightarrow$$

eigenvector 1 = $\begin{bmatrix} \frac{A}{C} \\ \frac{B}{C} \end{bmatrix}$

Eigenvector 2:

$$\begin{bmatrix} A^2 & AB \\ AB & B^2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0 \Rightarrow A^2x + ABy = 0 \Rightarrow Ax + By = 0 \Rightarrow y = \frac{-A}{B}x$$

Eigenvector 2: $\begin{bmatrix} x \\ -\frac{A}{B}x \end{bmatrix}$

Modulus 1: $x^2 + \frac{A^2}{B^2}x^2 = 1 \Rightarrow \frac{A^2+B^2}{B^2}x^2 = 1 \Rightarrow x = \frac{B}{C}$

Eigenvector 2: $\begin{bmatrix} \frac{B}{C} \\ -\frac{A}{C} \end{bmatrix}$

Matrix A of the transformation:

$$\begin{bmatrix} \frac{A}{C} & \frac{B}{C} \\ \frac{B}{C} & -\frac{A}{C} \end{bmatrix}$$

Image 1 (zero mean version) $[A \mid -A]$

Image 2 (zero mean version) $[B \mid -B]$

Original vectors $\begin{bmatrix} A \\ B \end{bmatrix}$ or $\begin{bmatrix} -A \\ -B \end{bmatrix}$

Tranformed vectors

$$\begin{bmatrix} \frac{A}{C} & \frac{B}{C} \\ \frac{B}{C} & -\frac{A}{C} \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} \frac{A^2+B^2}{C} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{A}{C} & \frac{B}{C} \\ \frac{B}{C} & -\frac{A}{C} \end{bmatrix}\begin{bmatrix} -A \\ -B \end{bmatrix} = \begin{bmatrix} \frac{-A^2-B^2}{C} \\ 0 \end{bmatrix}$$

Second Image is 0

First image is $\left[\sqrt{A^2+B^2} \;\vdots\; -\sqrt{A^2+B^2} \right]$ [8]

The images have same content (a vertical edge in the middle) and for that reason KL will only give 1 image [2]

Question 2

(a) New histogram

[5]

(b)

0	-1	0
-1	4	-1
0	-1	0

Laplacian

σ_f^2 variance of f

σ_n^2 variance of noise

Variance of Laplacian of $f+n$

$$16\sigma_f^2 + \sigma_f^2 + \sigma_f^2 + \sigma_f^2 + 16\sigma_n^2 + \sigma_n^2 + \sigma_n^2 + \sigma_n^2$$

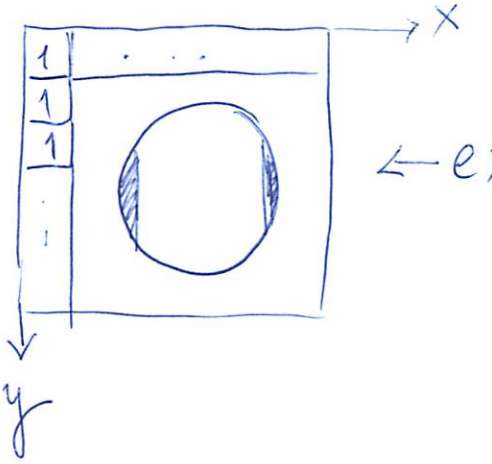
$$= 19\sigma_f^2 + 19\sigma_n^2$$

no Laplacian increases noise [5]

Question 3

b i) $H(u,v) = ?$

$h(m,n)$ degradation model



← extended degradation mask

Image size N, Degradation size 3 ⇒

$M = N+3-1 = N+2$

$$H(u,v) = \sum\sum h(\cancel{x},\cancel{y}) e^{-j\frac{2\pi}{M}(ux+vy)}$$

$$= 1 + e^{-j\frac{2\pi}{M}\cdot v} + e^{-j\frac{2\pi}{M}2v} = e^{-j\frac{2\pi}{M}v}\left(e^{-j\frac{2\pi}{M}v} + 1 + e^{+j\frac{2\pi}{M}v}\right)$$

$$= e^{-j\frac{2\pi}{M}v} \cdot 2\cos\frac{2\pi}{M}v$$

✓ Frequence of restored image

$$\tilde{F}(u,v) = \frac{G(u,v)}{e^{-j\frac{2\pi}{M}v}\left(2\cos\frac{2\pi}{M}v + 1\right)}$$

v cannot take values for which $\cos\frac{2\pi}{M}v = -1/2$ [6]

ii) $\cos\frac{2\pi}{M}v = -\frac{1}{2} \Rightarrow \frac{2\pi}{M}v = 2k\pi + \frac{2\pi}{3}$ [6]

$k=0 \Rightarrow \frac{2\pi}{M}v = \frac{2\pi}{3} \Rightarrow v = \frac{M}{3}$, $k=1$ $\frac{2\pi}{M}v = 2\pi + \frac{2\pi}{3} \Rightarrow \frac{v}{M} = 4/3$ not valid

Question 4(b)

Huffman code for the 3 symbol alphabet is shown on the first table.

$H = 0.335$ bits/symbol

$l_{avg} = 1.05$ bits/symbol

redundancy $= 0.715$ bits/symbol or 213% of entropy

Huffman code for extended alphabet

$l_{avg} = 1.222$ bits/new symbol <u>or</u>

0.611 bits/original symbol

redundancy $= 72\%$ of entropy