EE4-23
EE9-CS2-1
EE9-FPN2-05

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2017

MSc and EEE/EIE PART IV: MEng and ACGI

**Corrected copy**

**STABILITY AND CONTROL OF NON-LINEAR SYSTEMS**

Friday, 12 May 10:00 am

Time allowed: 3:00 hours

**There are FOUR questions on this paper.**

**Answer ALL questions.**

*All questions carry equal marks*

**Any special instructions for invigilators and information for candidates are on page 1.**

Examiners responsible First Marker(s) : D. Angeli
Second Marker(s) : E.C. Kerrigan

1. Consider the following two-dimensional time-invariant nonlinear system:

$$\begin{aligned}\dot{x}_1 &= -\sin(x_1 - x_2)\\ \dot{x}_2 &= x_2 - x_2^2 + x_1\end{aligned}$$

a) Discuss existence and uniqueness of solutions. [ 2 ]

b) Find the nullclines of the system and sketch them graphically. [ 3 ]

c) Identify the regions of the phase-plane in which the nullclines partition $\mathbb{R}^2$ and the orientation of the vector-field in each one of them. [ 3 ]

d) Find, if any, those regions that are forward invariant. [ 3 ]

e) Find all equilibria of the system. [ 2 ]

f) Linearize the system around each equilibrium and discuss the local phase-portrait. [ 4 ]

g) Sketch the global phase portrait of the system. [ 3 ]

2. Consider the following 3-dimensional, nonlinear system of differential equations:

$$\begin{aligned} \dot{x}_1 &= -\mathrm{atan}(x_1) + x_2 \\ \dot{x}_2 &= -2x_1 + x_3^3 \\ \dot{x}_3 &= -x_3 - x_2 \end{aligned}$$

along with the candidate Lyapunov function given below:

$$V(x) = \frac{x_1^2}{2} + \alpha x_2^2 + \beta x_3^4.$$

a) Prove that for all $\alpha$ and $\beta$ positive, $V(x)$ is a positive definite and radially unbounded function. [ 5 ]

b) Choose $\alpha$ and $\beta$ positive, so that the function $\dot{V}(x)$ is (at least) negative semi-definite. [ 5 ]

c) Show that the origin is a globally asymptotically stable equilibrium. [ 5 ]

d) Consider next the system obtained by replacing the equation for $\dot{x}_3$ by the following one:

$$\dot{x}_3 = x_3 + x_2.$$

Show, by using suitable instability criteria, that the origin is unstable.
[ *Hint: start out with a Lyapunov function of the following type:* $V(x) = -\frac{x_1^2}{2} + \alpha x_2^2 + \beta x_3^4$ *where* $\alpha$ *and* $\beta$ *are not constrained in sign. Does* $V$ *meet all assumptions of the instability criterion?* ] [ 5 ]

3. Consider the following 3-dimensional nonlinear system with state variable $x = [x_1, x_2, x_3]'$ and scalar input $d$:

$$\begin{aligned} \dot{x}_1 &= \operatorname{atan}(d - x_1) \\ \dot{x}_2 &= -x_2^5 + x_1^2 x_3^2 \\ \dot{x}_3 &= x_1^3 - x_3. \end{aligned}$$

a) Decompose the system as a cascade interconnection of subsystems (this may involve two or more subsystems). [ 4 ]

b) Show that each of the subsystems is Input-to-State Stable. [ 6 ]

c) Compute a suitable upper-bound of class $\mathcal{K}$ to the nonlinear gain from input $d$ to state $x$. [ 6 ]

d) Show that Input-to-State Stability is a form of dissipativity. What kind of estimate on solutions of an ISS system can be achieved by regarding the corresponding ISS Lyapunov function as a storage function of a dissipative system? [ 4 ]

4. Consider the following affine nonlinear control system:

$$\begin{aligned}
\dot{x}_1 &= x_1 + x_3 \\
\dot{x}_2 &= -x_2^2 + u \\
\dot{x}_3 &= -x_3 + x_2 + \sin(x_1)
\end{aligned}$$

$$y = x_1.$$

a) Compute the relative degree from input $u$ to output $y$ and emphasize whether the notion adopted is local or global. [ 4 ]

b) Design an input-output linearizing state feedback. [ 3 ]

c) By employing pole placement, find a suitable stabilizing output feedback controller for the linear system resulting from the previous question. [ 3 ]

d) Modify the output equation as follows: $y = x_3$. Compute the associated relative degree, from input $u$ to the newly defined output variable and rewrite the system in normal form. [ 6 ]

e) Compute the zero-dynamics and comment on their stability. [ 4 ]