EE1-10B Solutions 2016

Marks

(1)

a) $\mathcal{F}\{f(t)\}$

$$= \int_{-T/2}^{T/2} 1\, e^{-i\omega t}\, dt = \left[-\frac{e^{-i\omega t}}{i\omega} \right]_{-T/2}^{T/2} \qquad (2)$$

$$= \frac{1}{i\omega}\left(e^{i\omega T/2} - e^{-i\omega T/2} \right)$$

$$= \frac{2\sin(\omega T/2)}{\omega} = T\,\mathrm{sinc}(\omega T/2) \qquad (3)$$

b) $\mathcal{F}\left[e^{i\omega_0 t} g(t)\right] = \int_{-\infty}^{+\infty} e^{i\omega_0 t} g(t) e^{-i\omega t}\, dt \qquad (2)$

$$= \int_{-\infty}^{+\infty} g(t)\, e^{-i(\omega - \omega_0)t}\, dt$$

$$= G(\omega - \omega_0) \qquad (3)$$

c) Solve $\begin{pmatrix} 1 & -2 & 1 & 3 \\ -2 & 1 & 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 1 & 3 \\ 0 & -3 & 4 & 7 \end{pmatrix}$ $(R_2 + 2R_1)$

$R_2 \Rightarrow -3y + 4z = 7$, free variable, let $z = \lambda \in \mathbb{R}$ (2)

Then $y = -\frac{7}{3} + \frac{4}{3}\lambda$, and $R_1 \Rightarrow x = 2y - z + 3$

$$= 2\left(-\tfrac{7}{3} + \tfrac{4}{3}\lambda\right) - \lambda + 3$$

$$= -\tfrac{5}{3} + \tfrac{5}{3}\lambda$$

$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -5/3 \\ 7/3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 5/3 \\ 4/3 \\ 1 \end{pmatrix}$ $\Rightarrow$ equation of a line in $\mathbb{R}^3$ (3)

d) i) $A^2 = \begin{pmatrix} \alpha^2-1 & -\alpha \\ \alpha & -1 \end{pmatrix}$

$A^3 = \begin{pmatrix} \alpha^3-2\alpha & 1-\alpha^2 \\ \alpha^2-1 & -\alpha \end{pmatrix} = I_2$, if $\alpha = -1$ (2)

$A^4 = \begin{pmatrix} \alpha^4-3\alpha^2+1 & 2\alpha-\alpha^3 \\ \alpha^3-2\alpha & 1-\alpha^2 \end{pmatrix} = I_2$, if $\alpha = 0$ (3)

ii) $\det A = 1 \Rightarrow A^{-1}$ exists $\forall \alpha$. 2

If $A = A^{-1}$ would have $A^2 = I$

but $(A^2)_{22} = -1$, hence impossible $\forall \alpha$. (3)

1: a, c: seen similar

b: Bookwork

d: new

② a) Following the hint, Marks

$$\underline{b}\cdot\underline{y} = \underline{b}\cdot\underline{a} + \underline{b}\cdot(\underline{b}\times\underline{y})$$
$$= \underline{b}\cdot\underline{a} \quad \text{as } \underline{b},\ \underline{b}\times\underline{y} \text{ orthogonal}$$

and

$$\underline{b}\times\underline{y} = \underline{b}\times\underline{a} + \underline{b}\times(\underline{b}\times\underline{y})$$
$$= \underline{b}\times\underline{a} + (\underline{b}\cdot\underline{y})\underline{b} - (\underline{b}\cdot\underline{b})\underline{y} \qquad (3)$$

Now substitute
$$\underline{b}\times\underline{y} = \underline{y} - \underline{a}$$
$$\underline{b}\cdot\underline{y} = \underline{b}\cdot\underline{a}$$
in last eq^n to get

$$\underline{y} - \underline{a} = \underline{b}\times\underline{a} + (\underline{b}\cdot\underline{a})\underline{b} - (\underline{b}\cdot\underline{b})\underline{y}$$

and solve $\underline{y} = \dfrac{1}{1+\underline{b}\cdot\underline{b}}\left(\underline{b}\times\underline{a} + (\underline{b}\cdot\underline{a})\underline{b} + \underline{a}\right)$ (3)

b) i) $\det A = 2\alpha - 3 - (\alpha+3) + (1+2) = \alpha - 3$, hence (1)
unique sol^n when $\alpha \neq 3$. Let $\alpha = 3$, then do row ops: (1)

$$\left(\begin{array}{ccc|c} 1&1&1&1\\ 1&2&3&\beta\\ -1&1&3&2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1&1&1&1\\ 0&1&2&\beta-1\\ 0&2&4&3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1&1&1&1\\ 0&1&2&\beta-1\\ 0&0&0&5-2\beta \end{array}\right) \qquad (1)$$

For $\alpha = 3$, $\begin{cases} \beta = 5/2 & \text{gives infinitely many sol}^{n}\text{s} \\ \beta \neq 5/2 & \text{gives no sol}^{n}. \end{cases}$ (2)

ii)
$$\left(\begin{array}{ccc|ccc} 1&1&1&1&0&0\\ 1&2&3&0&1&0\\ -1&1&2&0&0&1 \end{array}\right) \underset{\substack{R_2-R_1\\ R_3+R_1}}{\sim} \left(\begin{array}{ccc|ccc} 1&1&1&1&0&0\\ 0&1&2&-1&1&0\\ 0&2&3&1&0&1 \end{array}\right) \underset{\substack{R_1-R_2\\ R_3-2R_2}}{\sim} \left(\begin{array}{ccc|ccc} 1&0&-1&2&-1&0\\ 0&1&2&-1&1&0\\ 0&0&-1&3&-2&1 \end{array}\right) \qquad (2)$$

$$\underset{-1R_3}{\sim} \left(\begin{array}{ccc|ccc} 1&0&-1&2&-1&0\\ 0&1&2&-1&1&0\\ 0&0&1&-3&2&-1 \end{array}\right) \underset{\substack{R_1+R_3\\ R_2-2R_3}}{\sim} \left(\begin{array}{ccc|ccc} 1&0&0&-1&1&-1\\ 0&1&0&5&-3&2\\ 0&0&1&-3&2&-1 \end{array}\right) \Rightarrow A^{-1} = \begin{pmatrix} -1&1&-1\\ 5&-3&2\\ -3&2&-1 \end{pmatrix} \quad (2)$$

iii) Rewrite as
$$\begin{aligned} x+y+z&=2\\ x+2y+3z&=5\\ -x+y+2z&=1 \end{aligned} \Rightarrow A\underline{x}=\underline{b},\quad \underline{x} = A^{-1}\begin{pmatrix}2\\5\\1\end{pmatrix}$$

$$= \begin{pmatrix} -2+5-1\\ 10-15+2\\ -6+10-1 \end{pmatrix} = \begin{pmatrix} 2\\ -3\\ 3 \end{pmatrix} \quad (2)$$

2) i) $|B-\lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = (1-\lambda)(1-\lambda)-4 = \lambda^2-2\lambda-3=0$

$\Rightarrow \lambda = 3, -1$ are eigenvalues. (2)

For $\lambda = -1 \Rightarrow$

$(B+I)\underline{x} = \underline{0} \Rightarrow \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow x = -y$

eigenvector: $k\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. $(k \in \mathbb{R})$

For $\lambda = 3 \Rightarrow$

$(B-3I)\underline{x} = \underline{0} \Rightarrow \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x = y$

eigenvector $k\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. (2)

ii) Normalize: $\hat{\underline{x}}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\hat{\underline{x}}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(distinct eigenvalues $\Rightarrow$ eigenvectors orthogonal)

$P = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ so that $D = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$ (3)

$P^{-1} = P^T = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, as orthogonal matrix. (1)

③ a) Rewrite ODEs

marks

$$\underbrace{\left(2xte^{x^2} + t^2\sin x + \sec^2 x\right)}_{P(x,t)}\frac{dx}{dt} + \underbrace{e^{x^2} - 2t\cos x - \cos t}_{Q(x,t)} = 0$$

This is exact if $\frac{\partial P}{\partial t} = \frac{\partial Q}{\partial x}$

check: $\left.\begin{aligned}\frac{\partial P}{\partial t} &= 2xe^{x^2} + 2t\sin x \\ \frac{\partial Q}{\partial x} &= 2xe^{x^2} + 2t\sin x\end{aligned}\right\}$ ✓ (2)

Hence $H(x,t) = 0$ with

$$P = \frac{\partial H}{\partial x}, \quad Q = \frac{\partial H}{\partial t}$$

$\Rightarrow H = \int P\,dx = te^{x^2} - t^2\cos x + \tan x + f(t)$

and $H = \int Q\,dt = te^{x^2} - t^2\cos x - \sin t + g(x)$ (3)

where f, g are arbitrary functions. Equate the two $\Rightarrow f = \sin t$, $g = \tan x$ and the solution is

$$te^{x^2} - t^2\cos x + \tan x - \sin t = C \quad (1)$$

where C is a constant.

b) i) C.F:

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0 \quad \text{with}$$

auxiliary equation $m^2 - 2m + 2 = 0$

$$\Rightarrow m = 1 \pm i$$

giving the complementary function

$$y_c = e^x(A\cos x + B\sin x) \quad (2)$$

with A, B arbitrary constants

ii) P.I.

As $\cos(3x)$ is not in C.F., sufficient to try $y_p = C\cos(3x) + D\sin(3x)$

$$\Rightarrow y_p' = -3C\sin(3x) + 3D\cos(3x)$$

$$y_p'' = -9C\cos(3x) - 9D\sin(3x) \quad (2)$$

substitute these into ODE $\Rightarrow$ Marks

$-9C\cos 3x - 9D\sin 3x - 2[-3C\sin 3x + 3D\cos 3x] + 2[C\cos 3x + D\sin 3x] = 85\cos 3x$

collect like terms $\Rightarrow$

$-7C - 6D = 85$
$6C - 7D = 0$ $\Rightarrow$ solve to obtain:
$C = -7$
$D = -6$

$Y_p = -7\cos 3x - 6\sin 3x$ (2)

iii) Hence General Solution is $Y_p + Y_c$

$Y = e^x(A\cos x + B\sin x) - 7\cos 3x - 6\sin 3x$ (1)

c) Homogeneous equation, rewrite as

$$\frac{dx}{dt} = \left(\frac{x}{t}\right)^2 + \frac{x}{t} + 1$$

Let $v = \frac{x}{t} \Rightarrow x = vt$, $\frac{dx}{dt} = v + t\frac{dv}{dt}$ (2)

and ODE becomes

$$v + t\frac{dv}{dt} = v^2 + v + 1$$

or $t\frac{dv}{dt} = v^2 + 1$ which is separable

$$\int \frac{1}{v^2+1}\,dv = \int \frac{1}{t}\,dt \Rightarrow$$ (2)

$$\tan^{-1}(v) = \ln|t| + C$$

$\Rightarrow$ back to t/x:

$$\tan^{-1}\left(\frac{x}{t}\right) = \ln|t| + C$$

I.C $x(1) = 1$ $\Rightarrow$

$\tan^{-1}(1) = \ln 1 + C \Rightarrow C = \frac{\pi}{4}$

Sol^n is $\tan^{-1}\left(\frac{x}{t}\right) = \ln|t| + \frac{\pi}{4}$ (2)

d) $$\cos y \frac{dy}{dx} = 3x + \sin y$$ Marks

i) $u = 3x + \sin y$ so take $\frac{d}{dx}$:

$$\frac{du}{dx} = 3 + \cos y \frac{dy}{dx}$$

$$= 3 + u$$ (2)

ii) Integrate $u' = 3+u \Rightarrow$

$$\int \frac{1}{3+u}\, du = \int 1\, dx$$

$\ln|3+u| = x + C$ (take exp:)

$3 + u = Ke^x \Rightarrow \underline{u = Ke^x - 3}$ (2)

ii) $\Rightarrow$ back to x/y:

$$3 + 3x + \sin y = Ke^x$$

iii) I.C.: $y(0) = 0$ gives

$3 = K \Rightarrow$

$\sin y = 3(e^x - x - 1)$ or

$$\underline{y = \sin^{-1}(3e^x - 3x - 3)}$$ (2)

Q2: a) unseen
b) seen similar
c) " "

Q3: a, b, c : seen similar
d : unseen

Marks

4) a) i) $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial \rho}\frac{\partial \rho}{\partial x} + \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial x} = \frac{\partial f}{\partial \rho} + \frac{\partial f}{\partial \phi} = \left(\frac{\partial}{\partial \rho} + \frac{\partial}{\partial \phi}\right) f$

$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial \rho}\frac{\partial \rho}{\partial t} + \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial t} = -c\frac{\partial f}{\partial \rho} + c\frac{\partial f}{\partial \phi} = c\left(\frac{\partial}{\partial \phi} - \frac{\partial}{\partial \rho}\right) f$ (2)

Hence:

$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial \rho} + \frac{\partial f}{\partial \phi}\right) = \left(\frac{\partial}{\partial \rho} + \frac{\partial}{\partial \phi}\right)\left(\frac{\partial f}{\partial \rho} + \frac{\partial f}{\partial \phi}\right) =$

$= \frac{\partial^2 f}{\partial \rho^2} + 2\frac{\partial^2 f}{\partial \rho \partial \phi} + \frac{\partial^2 f}{\partial \phi^2}$ (2)

Similarly:

$\frac{\partial^2 f}{\partial t^2} = \frac{\partial}{\partial t}\left(c\left(\frac{\partial f}{\partial \phi} - \frac{\partial f}{\partial \rho}\right)\right) = c\left(\frac{\partial}{\partial \phi} - \frac{\partial}{\partial \rho}\right)\left(c\left(\frac{\partial f}{\partial \phi} - \frac{\partial f}{\partial \rho}\right)\right)$

$= c^2\left(\frac{\partial^2 f}{\partial \rho^2} - 2\frac{\partial^2 f}{\partial \rho \partial \phi} + \frac{\partial^2 f}{\partial \rho^2}\right)$ (2)

ii) equate the two expressions

$\frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 f}{\partial \rho^2} - 2\frac{\partial^2 f}{\partial \rho \partial \phi} + \frac{\partial^2 f}{\partial \phi^2} = \frac{\partial^2 f}{\partial \rho^2} + 2\frac{\partial^2 f}{\partial \rho \partial \phi} + \frac{\partial^2 f}{\partial \phi^2} = \frac{\partial^2 f}{\partial x^2}$

$\Rightarrow \quad 4\frac{\partial^2 f}{\partial \rho \partial \phi} = 0$ as req^d. (2)

iii) $\frac{\partial}{\partial \rho}\left(\frac{\partial f}{\partial \phi}\right) = 0 \Rightarrow \frac{\partial f}{\partial \phi} = h(\phi)$, arbitrary function

$\Rightarrow f = \int h(\phi)\, d\phi$

$= g_1(\phi) + g_2(\rho)$, another arbitrary f^n. (2)

hence $f = g_1(x+ct) + g_2(x-ct)$ (2)

b) i) $g(x,y) = x^2y - xy^2 + y - x$

$\Rightarrow g_x = 2xy - y^2 - 1$
$g_y = x^2 - 2xy + 1$

To obtain stationary pts, set $= 0 \Rightarrow$ (2)

① $2xy = y^2+1$
② $2xy = x^2+1$

$\Rightarrow x^2 = y^2, \ x = \pm y$

marks

Case $x=y$, substitute into ② to get

$2x^2 = x^2+1 \Rightarrow x = \pm 1,\ y = x = \pm 1$

$\Rightarrow$ stationary pts at $(1,1)$, $(-1,-1)$

Case $x=-y$, substitute into ② to get

$-2x^2 = x^2+1 \Rightarrow$ No solⁿ.

The stationary points are $(1,1)$ and $(-1,-1)$ (2)

(ii) Calculate the Hessian matrix

$$H(g) = \begin{pmatrix} g_{xx} & g_{yx} \\ g_{xy} & g_{yy} \end{pmatrix} = \begin{pmatrix} 2y & 2x-2y \\ 2x-2y & -2x \end{pmatrix}$$ (2)

$H(g)(1,1) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \Rightarrow \det Hg(1,1) = -4$

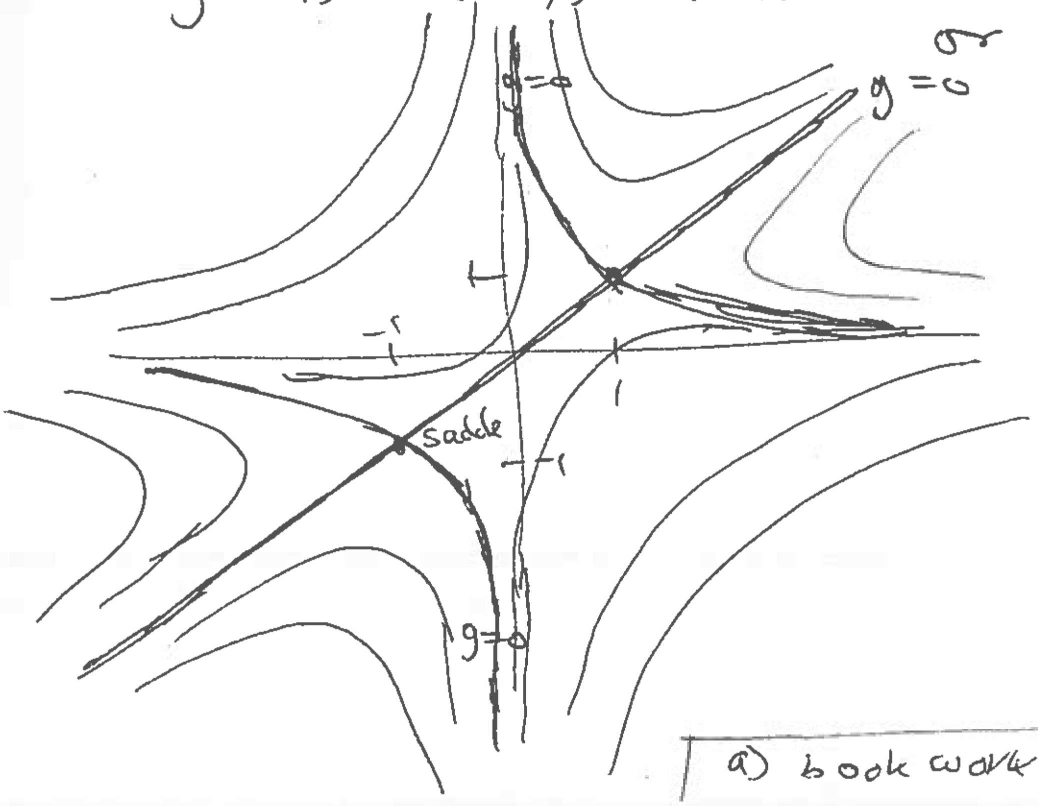
$\Rightarrow$ $(1,1)$ is a saddle point

$H(g)(-1,-1) = \begin{pmatrix} -2 & 0 \\ 0 & +2 \end{pmatrix} \Rightarrow \det = -4$

$(-1,-1)$ is a saddle point (2)

($\det H < 0 \Rightarrow$ saddle)

(iii) $g(x,y) = (x-y)(xy-1) = 0 \Rightarrow x=y$ or $xy=1 \Rightarrow y=\frac{1}{x}$



$g=0$ on $x=y$ and $y=\frac{1}{x}$. (2)

(3)

a) book work b) seen similar