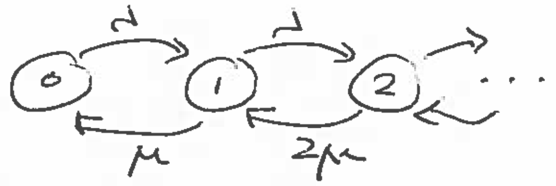
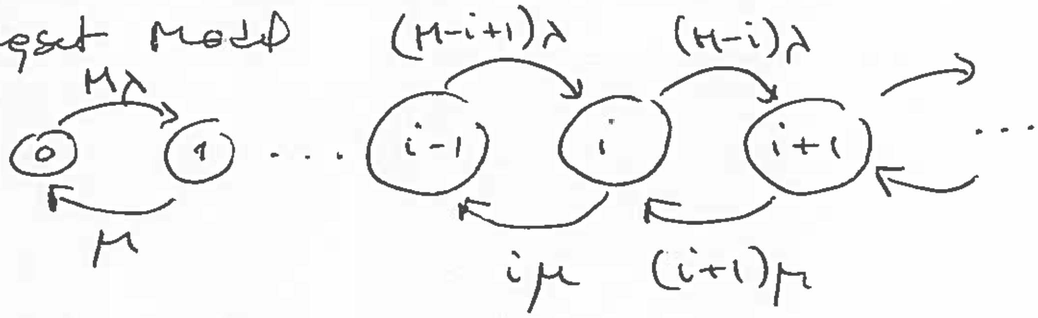
Q1 a) i) Erlang Model



Arrival Poisson stream (λ)
Exponential channel holding time (μ)

Engset Model



ii) Engset Model

Global (GBE)

$$[(M-i)\lambda + i\mu]\pi_i = (M-i+1)\lambda\pi_{i-1} + (i+1)\mu\pi_{i+1}$$

(LBE)

$$(M-i+1)\lambda\pi_{i-1} = i\mu\pi_i$$

iii)

$$\left.\begin{aligned}(M-i+1)\lambda\pi_{i-1} &= i\mu\pi_i\\ (M-i)\lambda\pi_i &= (i+1)\mu\pi_{i+1}\end{aligned}\right\}\ \text{LBE} \times 2$$

$$(M-i)\lambda\pi_i + i\mu\pi_i = (M-i+1)\lambda\pi_{i-1} + (i+1)\mu\pi_{i+1}$$

$$[(M-i)\lambda + i\mu]\pi_i = (M-i+1)\lambda\pi_{i-1} + (i+1)\mu\pi_{i+1}$$

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Q1 b) i)

$N \sim \Delta\rho + c$

from the provided graph $C = 4$

Also from the graph $\Delta = \dfrac{20-14}{13-8}$ ← ΔN / ← $\Delta\rho$

$= \dfrac{7}{5}$

ii)

1800 calls/h → 30 calls/m

$30 \frac{\text{calls}}{\text{m}} \times 3\,\text{m} = 90$ Erlangs

$N = \frac{7}{5} \times 90 + 4 = 7 \times 18 + 4 = 130$

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Q2

a) i) R_i = residual service time = time until first departure seen by ith arrival.

For a FIFO queue discipline and for the ith arrival:

S_i = service time

W_i = waiting time

Q_i = queue length found on arrival.

Then we have:

$$W_i = R_i + \sum_{j=1}^{Q_i} S_{i-j}$$

which is the total time to serve all items ahead of the ith arrival.

ii) take expectation of W_i and taking expectation on a sum of Q_i iid RVs:

$$E[W_i] = E[R_i] + E[Q_i]E[S]$$

since Poisson see an unbiased sample of queue behaviour

$$E[W] = E[R] + E[Q]E[S]$$

and $E[Q] = \lambda E[W]$

$$E[W] = E[R] + \rho E[W]$$

$$E[W] = \frac{E[R]}{1-\rho}$$

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Q2a iii) assuming that $\{R_t\}$ is ergodic

$$E[R_t] = \lim_{T\to\infty} \frac{1}{T}\int_0^T R_t\,dt = \lim_{T\to\infty} \frac{1}{2}\left(\frac{M_T}{T}\right)\left[\frac{1}{M_T}\sum_{i=1}^{M_t} s_i^2\right]$$

$\frac{M_T}{T}$ = service completion rate = mean arrival rate (λ)

$$E[R_t] = \frac{1}{2}\lambda E[S^2]$$

$$E[W] = \frac{\lambda E[S^2]}{2(1-\rho)}$$

Q2b i)

$$P(Q_t = i \mid \text{all servers busy}) = P(Q_t = i \mid N_t \geq K)$$

$$= \frac{P(N_t = K+i)}{\sum_{j=0}^{\infty} P(N_t = K+j)}$$

$$= \frac{\pi_K \rho^i}{\sum_{j=0}^{\infty} \pi_K \rho^j}$$

$$P(Q_t = i \mid \text{delay}) = (1-\rho)\rho^i \quad i=0,1,\ldots$$

Q2b) ii)

$$\sum_{i=0}^{\infty} i(1-\rho)\rho^i = (1-\rho)[\rho + 2\rho^2 + 3\rho^3 + 4\rho^4 + \dots]$$
$$= \rho + 2\rho^2 + 3\rho^3 + 4\rho^4 + \dots - \rho^2 - 2\rho^3 - 3\rho^4 - \dots]$$
$$= \rho + \rho^2 + \rho^3 + \rho^4 \dots = \rho\underbrace{(1+\rho+\rho^2+\dots)}_{\sum \rho^i = \frac{1}{1-\rho}} = \rho$$
$$= \rho/(1-\rho)$$

iii)

$$\underbrace{\sum_{i=0}^{\infty} i^2(1-\rho)\rho^i}\ - \frac{\rho^2}{(1-\rho)^2} = \frac{\rho+\rho^2}{(1-\rho)^2} - \frac{\rho^2}{(1-\rho)^2} = \frac{\rho}{(1-\rho)^2}$$

$$= (1-\rho)(1\rho + 4\rho^2 + 9\rho^3 + 16\rho^4 + \dots)$$
$$= \frac{(1-\rho)^3}{(1-\rho)^2}(\rho + 4\rho^2 + 9\rho^3 + 16\rho^4 + \dots) = \rho - 3\rho^2 + 3\rho^3 - \rho^4 + 4\rho^2$$
$$- 12\rho^3 + 12\rho^4 - 4\rho^5 + 19\rho^3$$
$$- 27\rho^4 + 16\rho^4 + \dots$$
$$= \frac{\rho+\rho^2}{(1-\rho)^2}$$

iv) Unconditional queue-length distribution
- Delayed arrival see a geometric queue length
- Non delayed arrivals see zero queue length.

$$P(Q_t=i) = P(\text{delay}) \cdot P(Q_t=i \mid \text{delay}) +$$
$$P(\text{no delay})\, P(Q_t = i \mid \text{no delay})$$
$$= \begin{cases} 1, & \text{if } i=0 \\ 0, & \text{if } i>0 \end{cases}$$

$$P(\text{Delay}) = D_K(A)$$

$$P(Q_t=i) = \begin{cases} D_K(A)(1-\rho)\rho^i & \text{if } i>0 \\ 1-\rho D_K(A) & \text{if } i=0 \end{cases}$$

Q3 a) 250 m/s — 60% – single 160 bits log
— 40% – 2 × 160 bits log

Channel transmission rate 64 Kbits/s

$$E[S] = \frac{6}{10}B + \frac{4}{10}2B = \frac{6B+8B}{10} = \frac{14B}{10}$$

$$E[S^2] = \frac{6}{10}B^2 + \frac{4}{10}(2B)^2 = \frac{6}{10}B^2 + \frac{4\cdot4}{10}B^2 = \frac{22}{10}B^2$$

$B = 160$ bits $\quad B = \frac{160}{64K} = 2.5\,ms$

$$\lambda_1 = 0.6 \times 250 = 150$$

$$\lambda_2 = 0.4 \times 250 = 100$$

$$E[R] = \frac{1}{2}\lambda E[S^2] = \frac{1}{2} \times 250 \times \frac{22}{10}(2.5)^2 = 1.718\ ms$$

Priority 1 $\quad E[W_1] = \frac{E[R]}{1-\rho_1} =$

$$E[T_1] = E[W_1] + E[S_1] =$$

Priority 2 $\quad E[W_2] = \frac{E[W_1]}{1-\rho_1-\rho_2} =$

$$E[T_2] = E[W_2] + E[S_2] =$$

$$E[S_1] = \frac{160}{64K} \longrightarrow \rho_1 = \lambda_1 E[S_1] = \frac{160}{64K} \cdot 150 = 0.375$$

$$E[S_2] = \frac{320}{64K} \longrightarrow \rho_2 = \lambda_2 E[S_2] = \frac{320}{64K} \times 100 = 0.5$$

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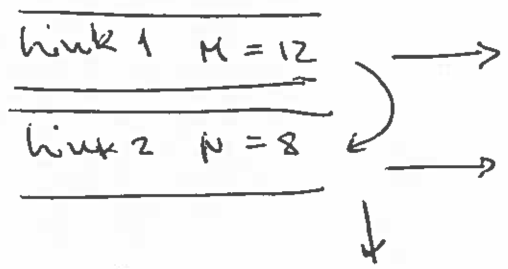
Q3 4)



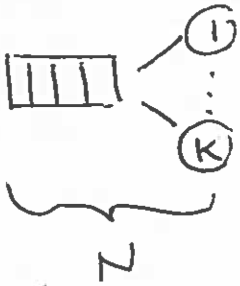
$B_1 = E_{12}(10)$

$B_2 = E_{20}(10)$

carried traffic link 1 $= \rho(1 - B_1)$

carried traffic link 2 $= B_1\rho - B_2\rho = (B_1 - B_2)\rho$

Q4 a)



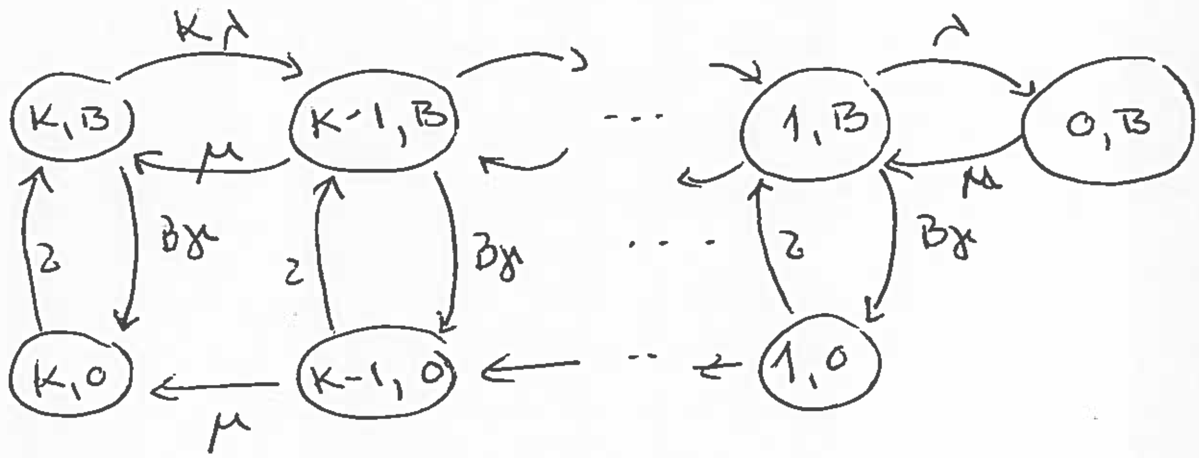
Buffer stages = $N - K = B$

i) State space (K, B)

K = # of servers

B = Buffer

ii)



iii) faulty state = $(x, 0) + (0, B)$

Q4 b) i) The underlying assumption of a FFM model is that the number of packets during an active period is very large and appears like a continuous flow of fluid.

In the case of a statistical multiplexer this approach is valid if the number of sources and the capacity of the server are very large. In this case the buffer occupancy can be approximated by a continuous random variable x.

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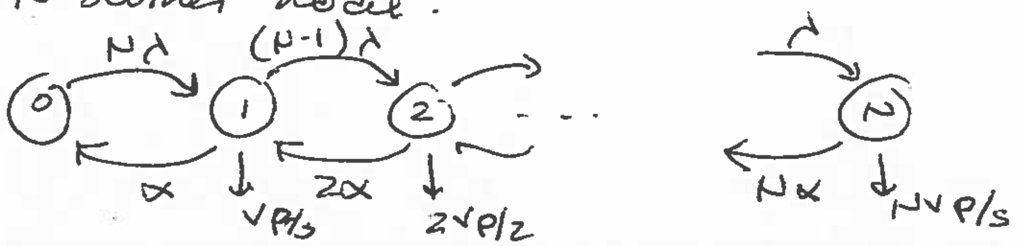
Q4 ii)

N sources model.



let $F_i(t,x)$ = the probability that the buffer occupancy is less or equal to x with i sources on

Set first a generating equation for $F_i(t+\Delta t, x)$ at an incremental time Δt later:

$$F_i(t+\Delta t, x) = [N-(i-1)]\lambda \Delta t\, F_{i-1}(t,x) + (i+1)\alpha \Delta t\, F_{i+1}(t,x) + \{1-[(N-i)\lambda + i\alpha]\Delta t\} F_i[t, x-\underbrace{(i-c)\alpha\Delta t}_{\Delta x}] + o(\Delta t)$$

$i\alpha - \alpha c = h$: Rate of filling the buffer. Hence it should start at : $x - h\Delta t$.

let

$\Delta x = (i-c)\alpha\Delta t$, $h_1 = [N-(i-1)]\lambda$, $h_2 = (i+1)\alpha$,

$h_3 = [(N-i)\lambda + i\alpha]$

$$F_i(t+\Delta t, x) = h_1 \Delta t\, F_{i-1}(t,x) + h_2 \Delta t\, F_{i+1}(t,x) + [1+h_3\Delta t] F_i(t, x-\Delta x)$$

$$-F_i(t,x) = -F_i(t,x)$$

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Q4b) iii)

$$\frac{F_i(t+\Delta t, x) - F_i(t,x)}{\Delta t} = h_1 F_{i-1}(t,x) + h_2 F_{i+1}(t,x) + h_3 F_i(t, x-\Delta x) + \frac{F_i(t, x-\Delta x) - F_i(t,x)}{\Delta x}\frac{\Delta x}{\Delta t}$$

$$\frac{\partial F_i(t,x)}{\partial t} = h_1 F_{i-1}(t,x) + h_2 F_{i+1}(t,x) + h_3 F_i(t,x-\Delta x) - \left(\frac{F_i(t,x) - F_i(t,x-\Delta x)}{\Delta x}\right)(i-c)\alpha$$

lim $\Delta x = 0$

$$\frac{\partial F_i(t,x)}{\partial t} = h_1 F_{i-1}(t,x) + h_2 F_{i+1}(t,x) + h_3 F_i(t,x) - (i-c)\alpha \frac{\partial F_i(t,x)}{\partial x}$$

statistical equilibrium:

$$\frac{\partial F_i(x,t)}{\partial t} = 0 \quad \text{and} \quad F_i(t,x) \to F_i(x)$$

$$(i-c)\alpha \frac{d F_i(x)}{\partial x} = [N-(i-1)]\lambda F_{i-1}(x) - [(N-i)\lambda + i\alpha] F_i(x) + (i+1)\alpha F_{i+1}(x)$$

$$G_i(x) = 1 - F_i(x)$$