

Paper Number(s): E4.22
C1.2

IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE
UNIVERSITY OF LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2001

MSc and EEE PART IV: M.Eng. and ACGI

LINEAR OPTIMAL CONTROL

Monday, 30 April 10:00 am

There are SIX questions on this paper.

Answer FOUR questions.

Time allowed: 3:00 hours

Corrected Copy

Examiners: Astolfi, A. and Weiss, G.

Special instructions for invigilators:

None

Information for candidates:

System:

$$\dot{x}(t) = A(t) + Bu(t), \quad x(0) = x_0.$$

Quadratic cost function:

$$J(x_0, u) = \int_0^\infty [x(t)'Qx(t) + u(t)'Ru(t)] dt,$$
$$Q = Q' \geq 0, \quad R = R' > 0.$$

Riccati equation:

$$A'P + PA + Q - PBR^{-1}B'P = 0.$$

Optimal control law:

$$u(t) = -R^{-1}B'Px(t) = -Kx(t).$$

Minimum cost:

$$x_0'Px_0.$$

Return difference inequality for scalar u :

$$|1 + K(j\omega I - A)^{-1}B| \geq 1,$$

Minimum principle:

$$\dot{x} = f(x, u),$$

$$J(x_0, u) = \int_0^\infty L(x(t), u(t))dt,$$

$$H(x, u, \lambda_0, \lambda) = \lambda_0 L(x, u) + \lambda^T f(x, u),$$

$$\dot{\lambda}^* = -\frac{\partial H}{\partial x} \Big|_{(x^*, u^*, t_f^*)}^T,$$

$$H(x^*, \omega, \lambda_0^*, \lambda^*) \geq H(x^*, u^*, \lambda_0^*, \lambda^*), \quad \forall \omega,$$

$$H(x^*, u^*, \lambda_0^*, \lambda^*) = 0.$$

1. Consider the linear electric networks in Figure 1, with $R_1 > 0$, $R_2 > 0$, $R_3 > 0$, $C_1 > 0$ and $C_2 > 0$. Let u_1 be the driving voltage, x_1 be the voltage across the capacitor C_1 , and y_1 be the output voltage for the first circuit. Let u_2 be the driving voltage, x_2 be the voltage across the capacitor C_2 , and y_2 be the output voltage for the second circuit.



Figure 1.

- Using Kirchhoff's laws, or otherwise, express the dynamics of both circuits in the standard state-space form.
- Study the controllability and the observability of the two dynamical systems determined in (a).

Consider the interconnected network in Figure 2, obtained setting $u_2 = y_1$ via an ideal op-amp and regarding u_1 and y_2 as input and output of the resulting network. Suppose that no current flows into the input terminals of the op-amp.

- Study the controllability and observability of the system as a function of the parameters R_1 , R_2 , R_3 , C_1 and C_2 .

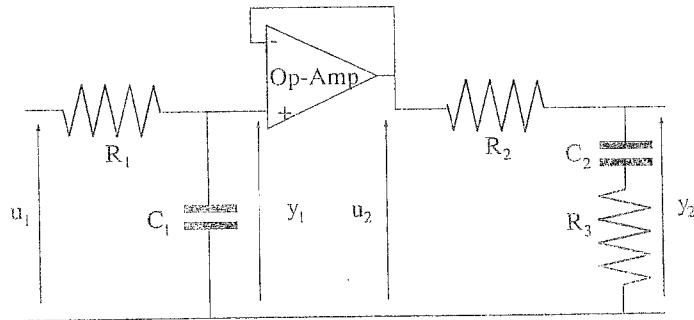


Figure 2.

2. The linearised model of a system composed of a tractor pulling a trailer along a straight path is described by the equations

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= u \\ \dot{x}_3 &= -x_3 - \lambda u \\ y &= x_1\end{aligned}$$

where $u \in \mathbb{R}$ is the input, $y \in \mathbb{R}$ is the output and λ is a constant parameter.

- (a) Discuss the properties of controllability and stabilisability as a function of λ .
- (b) Discuss the properties of observability and detectability.
- (c) Using the results established in part (b) design an observer to reconstruct asymptotically the state x_2 .
- (d) Let

$$u = k_1 x_1 + k_2 x_2,$$

and compute values for k_1 and k_2 such that the closed-loop system has eigenvalues at $\{-1/2, -1/2, -1\}$.

- (e) Using the results in parts (c) and (d) discuss why on-line measurements of the variable x_1 are sufficient to design a stabilizing (dynamic) output feedback controller.

3. Consider a system described by the following transfer function from u to y

$$W(s) = \frac{1}{s^2},$$

and the quadratic cost (to be minimised)

$$J(x_0, u) = \int_0^\infty \left(q y^2(t) + \frac{1}{q} u^2(t) \right) dt,$$

with $q > 0$.

- (a) Write the minimal state space realization of the system with the pair $\{A, B\}$ in controllable canonical form.
- (b) Write the Algebraic Riccati Equation (ARE) associated with the optimal control problem and verify that the hypotheses to solve the optimal control problem are verified.
- (c) Compute the solutions of the ARE derived in (b) and verify that one solution is always positive for any $q > 0$.
- (d) Compute the optimal control law as a function of q and plot on the complex plane the position of the eigenvalues of the closed-loop system as a function of $q > 0$.

4. Consider the system

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} x + \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix} u$$

and the quadratic cost (to be minimised)

$$J(x_0, u) = \int_0^\infty \left(x_1^2(t) + 2x_2^2(t) + u_1^2(t) + u_2^2(t) \right) dt.$$

- (a) Show that the system is controllable if and only if $b_1 b_2 \neq 0$ and that the system is stabilizable if and only if $b_2 \neq 0$.
- (b) Write the ARE associated with the considered optimal control problem.
- (c) Assume $b_1 b_2 \neq 0$. Find the positive definite solution P of the ARE derived in part (b) as a function of b_1 and b_2 and compute the optimal state feedback control law and the optimal closed loop system. (Hint: consider a diagonal P .)
- (d) Let $b_2 = 1$. Compute b_1 such that the optimal closed loop system computed in part (c) has two coincident eigenvalues.

5. Consider the system

$$\dot{x} = -x + 2u.$$

- (a) Write all stabilizing state feedback control laws $u = -kx$.
- (b) Let $u = -kx$ and compute all k such that the feedback law is stabilizing and *optimal in some sense*. (Hint: use the return difference inequality.)
- (c) Consider the cost function

$$J(x_0, u) = \int_0^\infty (qx^2(t) + u^2(t)) dt.$$

Compute q such that the control law $u = -3x$ is optimal with respect to the considered quadratic cost.

6. Consider the nonlinear system

$$\dot{x} = x + u^3$$

and the problem of finding a bounded control law $|u(t)| \leq 1$ that drives the state of the system from $x(0) = x_i$ to $x(t_f) = 0$ in minimum time.

- (a) Compute explicitly all the initial states $x(0) = x_i$ that can be steered to $x(t_f) = 0$ by the bounded control.
- (b) Write the necessary conditions for optimality in the case of normal extremals.
- (c) Write the optimal control as a function of the optimal co-state $\lambda^*(t)$.
- (d) Write the optimal control as a function of the optimal state $x^*(t)$.

1)

SOLUTIONS - LINEAR OPTIMAL CONTROL - 2001

E4.22

C.1.2

$$(a) \quad \frac{u_1 - x_1}{R_1} = \dot{x}_1 = C_1 \dot{x}_1, \quad y_1 = x_1,$$

$$\Sigma_1 \begin{cases} \dot{x}_1 = -\frac{1}{R_1 C_1} x_1 + \frac{1}{R_1 C_1} u_1 \\ y_1 = x_1 \end{cases}$$

(1)

$$\frac{u_1 - x_1 - R_3 C_2 \dot{x}_1}{R_2} = \dot{x}_2, \quad y_2 = \frac{R_2 x_1 + R_3 u_2}{R_2 + R_3}$$

$$\Sigma_2 \begin{cases} \dot{x}_2 = -\frac{1}{(R_2 + R_3) C_2} x_2 + \frac{1}{(R_2 + R_3) C_2} u_2 \\ y_2 = \frac{R_2}{R_2 + R_3} x_2 + \frac{R_3}{R_2 + R_3} u_2 \end{cases}$$

(2)

(b) Σ_1 is controllable and observable $\forall R_i > 0, C_i > 0$
 Σ_2 is " " " " " " " " (1)

$$\Sigma_{inv} \begin{cases} \dot{x}_1 = -\frac{1}{R_1 C_1} x_1 + \frac{1}{R_1 C_1} u_1 \\ \dot{x}_2 = -\frac{1}{(R_2 + R_3) C_2} x_2 + \frac{1}{(R_2 + R_3) C_2} x_1 \\ y_2 = \frac{R_3}{R_2 + R_3} x_1 + \frac{R_2}{R_2 + R_3} x_2 \end{cases}$$

(2)

controllable $\forall R_i > 0 \quad \forall C_i > 0$

(c)

$$G = \begin{bmatrix} \frac{1}{R_1 C_1} & * \\ 0 & \frac{1}{(R_2 + R_3) C_2 R_1 C_1} \end{bmatrix} \Rightarrow \det G \neq 0 \quad \forall R_i > 0 \quad \forall C_i > 0$$

(2)

$$Q = \begin{bmatrix} \frac{R_3}{R_2 + R_3} & \frac{R_2}{R_2 + R_3} \\ \frac{R_2}{(R_2 + R_3)^2 C_2} - \frac{R_3}{(R_2 + R_3) R_1 C_1} & -\frac{R_2}{(R_2 + R_3)^2 C_2} \end{bmatrix} \Rightarrow \det Q = \frac{(R_3 C_2 - R_1 C_1) R_2}{(R_2 + R_3)^2 R_1 C_1 C_2}$$

 \Downarrow

Obs. $\forall R_i > 0, \forall C_i > 0$
 if $R_3 C_2 \neq R_1 C_1$

(2)

- Sol 1 -

2)

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \\ -\lambda \end{bmatrix} \quad C = [1 \quad 0 \quad 0]$$

$$(a) \quad \mathcal{C} = [B, AB, A^2B] = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -\lambda & \lambda & -\lambda \end{bmatrix} \quad \det \mathcal{C} = \lambda$$

If $\lambda \neq 0 \Rightarrow$ controllable

If $\lambda = 0 \Rightarrow$ Not con, but stabilizable

(2)

$$(b) \quad \mathcal{O} = \begin{bmatrix} C \\ CA^1 \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \det \mathcal{O} = 0$$

Not observable, however the "unobservable state"

is x_3 , and $\dot{x}_3 = -x_3 \Rightarrow$ Detectable.

(2)

$$(c) \quad \text{Observer} \quad \dot{\xi} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \xi + \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} [y - \xi_1] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

with $l_1 < 0, l_2 < 0$.

Note: only the subsystem (x_1, x_2) is used for the design.

ξ_i is an estimation of x_i .

(3)

$$(d) \quad A_c = A + BK, \text{ with } K = [k_1, k_2, 0], \text{ is } A_c = \begin{bmatrix} 0 & 1 & 0 \\ k_1 & k_2 & 0 \\ -\lambda k_1 & -\lambda k_2 & -1 \end{bmatrix}$$

$$\text{ch. pol} = (s+1)(s^2 - k_2 s - k_1) \Rightarrow k_1 = -2 \quad k_2 = -1/4$$

(1)

(e) Using the "separation principle" use

$$u = k_1 \xi_1 + k_2 \xi_2$$

there is no need to estimate x_3 !

(2)

—Sol 2—

3)

$$(a) \quad \dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

(2)

$$(b) \quad ARE = \begin{bmatrix} (p_{12}^2 - 1)q & p_{11} - q p_{12} p_{22} \\ p_{11} - q p_{12} p_{22} & 2p_{12} - q p_{22}^2 \end{bmatrix} \quad P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$$

HP: (A, B) con/nc ✓

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \geq 0 \quad \checkmark$$

$$R = \frac{1}{q} > 0 \quad \checkmark$$

(A, Q^{1/2}) obs ✓

(3)

(c) From ARE, $p_{12} = \pm 1$

$$p_{12} = 1 \rightarrow p_{22} = \pm \sqrt{\frac{2}{q}} \quad \text{only } +\sqrt{\frac{2}{q}} \text{ is ok}$$

⇓

$$P = \begin{bmatrix} \sqrt{2}q & 1 \\ 1 & \sqrt{\frac{2}{q}} \end{bmatrix} \quad \leftarrow$$

$$p_{11} = \sqrt{2}q$$

This is > 0

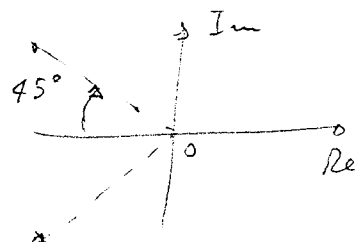
✓ q > 0

$$p_{12} = -1 \rightarrow p_{22}^2 < 0 \quad \text{not good}$$

(3)

$$(d) \quad K^* = [-q, -\sqrt{2}q] \quad A_{cl} = \begin{bmatrix} 0 & 1 \\ -q & -\sqrt{2}q \end{bmatrix}$$

$$\lambda_{1,2} = -\frac{1}{2}\sqrt{2}q \pm \frac{1}{2}i\sqrt{2}q$$



(2)

- Sol 3 -

4)

(a)

$$C = [B, AB] = \begin{bmatrix} b_1 & 0 & | & -b_1 & 0 \\ 0 & b_2 & | & 0 & 4b_2 \end{bmatrix}$$

$\text{rank } C = 2$ if $b_1 \cdot b_2 \neq 0 \Rightarrow$ Center

If $b_2 \neq 0$ and $b_1 = 0$ $\dot{x}_1 = -x_1$ \leftarrow Not center
 $\dot{x}_2 = 4x_2 + b_2 u_2$ \leftarrow Center

$$\Downarrow$$

Stabilizable

(3)

If $b_2 = 0$ $\dot{x}_2 = 4x_2$ Not stable \Rightarrow Not stabilizable

$$(b) \quad ALC = \begin{bmatrix} 1 - 2p_{11} - p_{11}^2 b_1^2 & 0 \\ 0 & 2 + 8p_{22} - p_{22}^2 b_2^2 \end{bmatrix} \quad P = \begin{bmatrix} p_{11} & 0 \\ 0 & p_{22} \end{bmatrix} \quad (2)$$

$$(c) \quad p_{11} = \frac{-1 + \sqrt{1+b_1^2}}{b_1^2} > 0 \quad p_{22} = \frac{4 + \sqrt{16+2b_2^2}}{b_2^2} > 0$$

$$K^* = \begin{bmatrix} -\frac{-1 + \sqrt{1+b_1^2}}{b_1} & 0 \\ 0 & -\frac{4 + \sqrt{16+2b_2^2}}{b_2} \end{bmatrix} \quad A_d = \begin{bmatrix} -\sqrt{1+b_1^2} & 0 \\ 0 & -\sqrt{16+2b_2^2} \end{bmatrix} \quad (3)$$

$$(d) \quad \lambda_1 = -\sqrt{1+b_1^2} \quad \lambda_2 = -\sqrt{16} \quad (b_2 = 1!)$$

$$\lambda_1 = \lambda_2 = 0 \quad b_1 = \pm \sqrt{7}$$

(2)

— Sol 4 —

5)

$$(a) \quad \dot{x} = -x + 2u$$

$$u = -kx$$

$$-1 - 2k < 0 \quad \text{for stability}$$

\Downarrow

$$k > -1/2 \quad (2)$$

(b) Return difference inequality

$$\left| 1 + \frac{2k}{j\omega + 1} \right| \geq 1 \Rightarrow 1 + \frac{4k(k+1)}{\omega^2 + 1} \geq 1$$

\Downarrow

$$k \leq -1 \quad k \geq 0 \quad \Leftarrow k(k+1) \geq 0$$

Optimal in some sense + stability

\Downarrow

$$k \geq 0$$

(4)

$$(c) \quad u^* = -R^{-1}B'Px = -3x \quad R=1 \quad B=3$$

$$\Downarrow P = 3/2 > 0$$

$$ARE \Rightarrow A'P + PA - PBB'P + Q = 0 \Rightarrow Q = 12$$

(4)

— Sol 5 —

6)

$$(a) \quad \dot{x} = x + u^3$$

$$\text{if } |u(t)| \leq 1 \quad \text{only } x_i(0); \quad |x_i(0)| < 1$$

can be steered to the origin. (2)

$$(b) \quad H = 1 + \lambda(x + u^3)$$

$$\dot{x}^* = x^* + u^{*3}$$

$$\dot{\lambda}^* = -\lambda^*$$

(3)

$$H(x^*, u, \lambda^*) \geq H(x^*, u^*, \lambda^*) \quad \forall u \in [-1, 1]$$

$$\lambda^* u^3 \geq \lambda^* u^{*3} \quad \forall u \in [-1, 1]$$

$$(c) \quad \lambda^*(t) = \lambda^*(0) e^{-t} \quad \lambda^*(0) > 0 \Rightarrow \lambda^*(t) > 0 \quad \text{and vice versa.}$$

$$u^*(\lambda^*) = -\text{sign}(\lambda^*) \Rightarrow u^* \text{ is } \pm 1. \quad (2)$$

$$(d) \quad \text{if } u^* = +1 \quad \dot{x} = x + 1 \quad x \text{ goes to zero only if } x(0) < 0.$$

$$\text{if } u^* = -1 \quad \dot{x} = x - 1 \quad x \text{ goes to zero only if } x(0) > 0.$$

$$u^*(x) = -\text{sign}(x)$$

(3)