IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2018

EEE PART I: MEng, BEng and ACGI

MATHEMATICS 1B (E-STREAM AND I-STREAM)

Corrected copy

Friday, 25 May 10:00 am

Time allowed: 2:00 hours

There are FOUR questions on this paper.

Answer ALL questions. All questions carry equal marks (25% each)

NO CALCULATORS ALLOWED.
Mathematical Formulae sheet provided

Any special instructions for invigilators and information for candidates are on page 1.

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EE1-10B MATHEMATICS II

1. a) Obtain the Fourier Transform of the function [7]

$$f(t) = \begin{cases} \cos(\pi t) & -1 \le t \le 1 \\ 0 & \text{otherwise} \end{cases} .$$

b) (i) Let $\mathcal{F}[f(t)]$ be the Fourier transform of $f(t)$ and $\mathcal{F}^{-1}[F(\omega)]$ the inverse Fourier transform of $F(\omega)$. Use the inverse Fourier Transform to show that

$$\mathcal{F}[\cos(at)] = \pi \left[\delta(\omega + a) + \delta(\omega - a) \right] .$$ [3]

(ii) Similarly, obtain $\mathcal{F}[\sin(at)]$. [4]

c) A cube has edges of length 2, with one corner at the origin and another corner at $P(2,2,2)$. A line L_1 has equation $\underline{\mathbf{r}} = (2,1,4) + \lambda(1,2,1)$. Find:

(i) the vector equation of the line L_2 through the origin and P; [2]

(ii) the cartesian equation of the plane Π through the centre of the cube and equidistant from the origin and P; [3]

(iii) the point Q, the intersection of Π and L_1; [3]

(iv) the distance between line L_2 and point Q. [3]

2. a) The vector $\underline{\mathbf{a}}$ satisfies

$$\underline{\mathbf{a}} \times \underline{\mathbf{v}} = \underline{\mathbf{w}}, \quad \text{and} \quad \underline{\mathbf{a}} \cdot \underline{\mathbf{v}} = 2,$$

where $\underline{\mathbf{v}} = (-1, 2, 1)$ and $\underline{\mathbf{w}} = (3, -10, 23)$. By taking the vector product of the first equation with one of the given vectors, or otherwise, find $\underline{\mathbf{a}}$. [5]

b) Consider the matrices

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -2 & 3 \\ -1 & 1 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & -1 & 3 \\ -2 & 1 & 0 \\ 3 & -1 & 1 \end{pmatrix}.$$

Obtain (i) $\det(A)$, (ii) $\det(B)$, (iii) $\det(AB)$, and (iv) $\det(BA)$. [4]

c) You are given a system of linear equations

$$\begin{array}{rcrcrcl} -x & + & 2y & - & z & = & 4 \\ 2x & - & 3y & + & z & = & \alpha \\ -3x & + & 2y & + & \beta z & = & 2 \end{array}$$

where α and β are constants.

(i) Write the system in the form $A\underline{\mathbf{x}} = \underline{\mathbf{b}}$ and find $\det(A)$. Hence find the conditions on α, β required so that the sytem has (I) no solution, (II) a unique solution, and (III) infinitely many solutions. In case (III) find these solutions. [6]

(ii) For $\beta = 0$, use Gaussian elimination to find the inverse matrix A^{-1}. [4]

d) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2 matrix, with distinct eigenvalues λ_1 and λ_2.

Considering $\det(A - \lambda I)$ for both eigenvalues, or otherwise, prove that

(i) $\text{Trace}(A) = \lambda_1 + \lambda_2$, and (ii) $\det(A) = \lambda_1 \lambda_2$. [6]

[Recall that the trace of a matrix is the sum of its diagonal elements.]

3. a) Given the matrix

$$A = \begin{pmatrix} 4 & -2 & -1 \\ 0 & 2 & -1 \\ 3 & -3 & 1 \end{pmatrix},$$

show that $\lambda = 2$ is one of the eigenvalues of A, find the other two eigenvalues, and obtain an eigenvector for $\lambda = 2$. [5]

b) Solve the first order ODE

$$(x^2+1)\frac{dy}{dx} + xy = \sqrt{x^2+1},$$

satisfying the initial condition $y(\sqrt{3}) = \sqrt{3}$. [6]

c) A solution of the second order differential equation

$$(1-x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 0,$$

can be found in the form of a series, using the Leibnitz-Maclaurin method.

(i) Differentiate the ODE n times and evaluate at $x = 0$ to obtain the recurrence relation

$$y^{(n+2)}(0) = n(n-3)y^{(n)}(0), \quad (n \geq 0).$$

where $y^{(k)}(0)$ is the k^{th} derivative of y, evaluated at zero, and we take $y^{(0)}(0) = y(0)$. [4]

(ii) If initial conditions are $y(0) = 1$ and $y'(0) = 1$, show that $y^{(n)}(0) = 0$ for all $n > 3$ and hence find the solution $y(x)$. [4]

(iii) The substitution $z = y'$ reduces the second-order ODE to a separable first-order ODE for $z(x)$. Solve this for z, and hence obtain $y(x)$, confirming your result from (ii). [6]

4. a) Find the solution of the differential equation

$$t^2\frac{dx}{dt} = t^2 + x^2 + tx.$$

satisfying the condition $x(1) = 1$. [6]

b) We would like to estimate the speed of light using Einstein's celebrated formula

$$E = mc^2.$$

To do so we carry out a controlled and very small (i.e. safe) nuclear detonation, where we measure the amount of energy (E) generated, and the amount of mass (m) consumed in the process. If the energy is measured to within 0.01% accuracy and the mass to within 0.03%, use the total differential to estimate the maximum error in the calculated speed of light, c. [6]

c) A function of two variables is given as

$$f(x,y) = x^3 + xy^2 - x.$$

i) Find the stationary points of $f(x,y)$ and determine their nature using the Hessian determinant. [7]

ii) Sketch the contours of the surface $z = f(x,y)$, clearly showing the stationary points. [6]