EE3-10
EE9-CS3-1

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2014

MSc and EEE/EIE PART III/IV: MEng, Beng and ACGI

Corrected Copy

MATHEMATICS FOR SIGNALS AND SYSTEMS

Tuesday, 14 January 10:00 am

Time allowed: 3:00 hours

There are THREE questions on this paper.

Answer ALL questions. All questions carry equal marks.

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : M.M. Draief
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Mathematics for Signal and Systems

1. Two questions 1.a and 1.b below are independent.

We say that two subspaces V and W of $\mathbb{R}^n$ are complementary, denoted by $V \oplus W = \mathbb{R}^n$, if (i) $V \cap W = \{\mathbf{0}\}$, where $\mathbf{0}$ is the zero vector in $\mathbb{R}^n$, and (ii) any vector $x \in \mathbb{R}^n$ can be written as $x = v + w$ where $v \in V$ and $w \in W$.

a) Let P be the matrix defined as

$$P = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

i) Describe a basis of $\mathrm{Ker}(P)$ the null-space (kernel) of P, and $\mathrm{Ran}(P)$ the range of P. Justify your answer. [3]

ii) Show that $\mathbb{R}^4 = \mathrm{Ker}(P) \oplus \mathrm{Ran}(P)$. [2]

iii) Show that for $x \in \mathrm{Ker}(P)$ and $y \in \mathrm{Ran}(P)$, we have $x^T y = 0$. [2]

iv) Conclude that P is an orthogonal projection. [3]

b) Define the matrix A_m as follows

$$A_m = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & m & 0 & 0 \\ 1 & 0 & -m & -1 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

where $m \in \mathbb{R}$ is a parameter.

i) Derive bases for $\mathrm{Ker}(A_m)$ and $\mathrm{Ran}(A_m)$. [3]

ii) For $m \neq 0$, show that $\mathrm{Ran}(A_m) \oplus \mathrm{Ker}(A_m) = \mathbb{R}^4$. [2]

iii) We now fix $m = 0$. Compute A_0^3. [2]

iv) Do we have $\mathrm{Ran}(A_0^3) \oplus \mathrm{Ker}(A_0^3) = \mathbb{R}^4$?

Justify your answer. [3]

2. Let $A = (a_{ij})_{i,j=1,\dots,n} \in \mathbb{R}^{n\times n}$ be a symmetric matrix, i.e, $A^T = A$ such that for all $x \in \mathbb{R}^n$ with $x \neq 0$ we have

$$x^T A x > 0 .$$

Matrices satisfying the above properties are known as *positive-definite matrices*

a) Let $e_i \in \mathbb{R}^n$ with all its entries equal to 0 except the i-th entry which is equal to 1. Show that, for $i = 1, \dots, n$, we have $a_{ii} = e_i^T A e_i > 0$. [1]

b) Let C be the Schur complement of a_{11} in A, i.e.

$$C = A_{22} - \frac{1}{a_{11}} A_{21} A_{12} ,$$

where

$$A = \begin{pmatrix} a_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

with a_{11} is a scalar, $A_{21} \in \mathbb{R}^{n-1}$, and $A_{22} \in \mathbb{R}^{(n-1)\times(n-1)}$ and $A_{12} \in \mathbb{R}^{1\times(n-1)}$.

i) Justify the fact that $C = A_{22} - \frac{1}{a_{11}} A_{21} A_{21}^T$. [1]

ii) Let $v \in \mathbb{R}^{n-1}$ and define $x \in \mathbb{R}^n$ such that

$$x = \begin{pmatrix} -(1/a_{11}) A_{21}^T v \\ v \end{pmatrix} .$$

Show that $x^T A x = v^T C v$ and that C is a positive-definite matrix. [3]

c) In what follows we will show that there exists a lower-triangular matrix $L \in \mathbb{R}^{n\times n}$ such that $A = LL^T$. This factorisation is known as the *Cholesky decomposition.*

i) Let L be given by

$$L = \begin{pmatrix} l_{11} & 0^T \\ L_{21} & L_{22} \end{pmatrix}$$

with l_{11} is a scalar, $L_{21} \in \mathbb{R}^{n-1}$, and $L_{22} \in \mathbb{R}^{(n-1)\times(n-1)}$ and $0 \in \mathbb{R}^{n-1}$. Write the block structure of the matrix LL^T. [2]

ii) Let $A = LL^T$. Show that $l_{11} = \sqrt{a_{11}}$, $L_{21} = (1/l_{11}) A_{21}$, and $L_{22} L_{22}^T = A_{22} - L_{21} L_{21}^T$. [2]

iii) Describe a recursive procedure to construct the lower-triangular matrix L such that $A = LL^T$. [4]

iv) Describe how one would use the above procedure to solve the linear equation $Ax = y$ for $A \in \mathbb{R}^{n\times n}$ positive definite. [3]

d) Define the following matrix A

$$A = \begin{pmatrix} 25 & 15 & -5 \\ 15 & 18 & 0 \\ -5 & 0 & 11 \end{pmatrix}$$

i) Apply the Cholesky decomposition to the matrix A above. [2]

ii) Use it to solve the equation $Ax = y$ where $y = \begin{pmatrix} 30 \\ 15 \\ -16 \end{pmatrix}$. [2]

3. Let m and n be two positive integers with $m \leq n$. We consider $A \in \mathbb{R}^{(n+1)\times(m+1)}$ the matrix defined by

$$A = \begin{pmatrix} 1 & x_0 & \dots & x_0^m \\ 1 & x_1 & \dots & x_1^m \\ \vdots & \vdots & & \vdots \\ 1 & x_n & \dots & x_n^m \end{pmatrix},$$

where $x_0, \dots, x_n$ are n distinct real numbers.

Let $\mathbf{0}$ be the vector with all its entries equal to 0 (we will use the same notation for both the zero vector of $\mathbb{R}^{m+1}$ and the one of $\mathbb{R}^{n+1}$). In what followed we define the vector $v = \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_m \end{pmatrix} \in \mathbb{R}^{m+1}$.

a) i) Show that if $Av = \mathbf{0}$ then $v = \mathbf{0}$. [1]

Hint: Use the fact if the polynomial $P(x) = v_0 + v_1 x + \dots + v_m x^m$ has n distinct roots then $P(x) = 0$.

ii) Using the previous question, show that if $A^T A v = \mathbf{0}$ then $v = \mathbf{0}$. [2]

iii) Fix $y \in \mathbb{R}^{n+1}$. Justify the fact that the linear equation $A^T A x = A^T y$ admits a unique solution w. [2]

b) In the remainder of this problem, we will denote the solution in 2. a) iii) by w, i.e.

$$A^T A w = A^T y.$$

For $v \in \mathbb{R}^{m+1}$ and $y \in \mathbb{R}^{n+1}$, define $g(v) = (y - Av)^T (y - Av)$.

i) Show that $g(w) = y^T y - y^T A w$, with w defined in 2. a) iii). [2]

ii) Prove that $g(v) - g(w) = (w - v)^T A^T A (w - v)$. [2]

Hint: Use the fact that $||A(w-v)||^2 = ||(Aw - y) - (Av - y))||^2$.

iii) Show that for all $v \in \mathbb{R}^{m+1}$, we have $g(v) \geq g(w)$ and that $g(v) = g(w)$ if and only if $v = w$. [3]

c) Let P be a polynomial such that $P(x) = \sum_{k=0}^{m} v_k x^k$. We define the quantity

$$\Phi_m(P) = \sum_{i=0}^{n} (y_i - P(x_i))^2 .$$

Let $y = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^{n+1}$.

i) Show that $\Phi_m(P) = g(v)$. [2]

ii) Using question 3.b.iii), show that there exists a polynomial P_w such that $\Phi_m(P) \geq \Phi_m(P_w)$. [2]

d) Let $n = m = 3, x_0 = -1, x_1 = 0, x_2 = 1, x_3 = 2, y_0 = 1, y_1 = 2, ,y_2 = 1, y_3 = 0$.

i) Solve $A^T A v = A^T y$. [2]

ii) Derive the expression of the polynomial in $\mathbb{R}_3[X]$ that minimizes Φ_3 and give the minimum value of Φ_3 on $\mathbb{R}_3[X]$. Justify your answer. [2]