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IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2010

EEE/ISE PART III/IV: MEng, BEng and ACGI

Corrected Copy

CONTROL ENGINEERING

Monday, 10 May 10:00 am

Time allowed: 3:00 hours

There are SIX questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

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CONTROL ENGINEERING

1. Consider a linear, single-output, discrete-time, system described by the equations

$$x(k+1) = Ax(k) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} x(k), \qquad y(k) = Cx(k) = \begin{bmatrix} 1 & -1 & 2 \end{bmatrix} x(k).$$

a) Study the observability and detectability properties of the system. Determine a basis for the unobservable subspace and the unobservable modes. [4 marks]

b) Consider the output sequence

$$y(0) = 0, \qquad y(1) = 0, \qquad y(2) = 0.$$

Determine all initial states $x(0)$ which are consistent with this output sequence. [4 marks]

c) Consider the output sequence

$$y(0) = 0, \qquad y(1) = 0, \qquad y(2) = 0.$$

Determine all final states $x(2)$ which are consistent with this output sequence. [6 marks]

d) Discuss, briefly, the results in parts b) and c). [2 marks]

e) Design an observer for the system with the property that for any initial value $e(0)$ of the estimation error one has

$$e(k) = 0$$

for all $k \geq \bar{k}$, for some $\bar{k} > 0$. Determine $\bar{k}$. [4 marks]

2. Consider a nonlinear, continuous-time, system described by the equations

$$\begin{aligned} \dot{x}_1 &= -3x_1 + x_2^2 + x_2, \\ \dot{x}_2 &= x_1 x_3^2 - 2x_2, \\ \dot{x}_3 &= -3x_2 u + \alpha x_3 + u, \end{aligned}$$

where $x(t) = [x_1(t), x_2(t), x_3(t)]'$, $u(t) \in \mathbb{R}$, and α is a constant.

a) Assume $u = 0$. Compute the equilibrium points of the system as a function of α. [4 marks]

b) Write the equations describing the linearized system around the equilibrium $x = 0$. [4 marks]

c) Study, using the principle of stability in the first approximation, the stability properties of the equilibrium $x = 0$ as a function of α. [4 marks]

d) Show that for all values of α the linearized system determined in part b) is not controllable but it is stabilizable. [4 marks]

e) Design a feedback control law $u = kx_3$ which renders the equilibrium $x = 0$ of the nonlinear system locally asymptotically stable for all values of α.
(Hint: note that k may be a function of α.) [4 marks]

3. Consider a nonlinear, continuous-time, system without input described by equations of the form

$$\begin{aligned} \dot{x} &= Ax + f(y), \\ y &= Cx, \end{aligned} \qquad (\star)$$

with $x(t) \in \mathbb{R}^n$ and $y(t) \in \mathbb{R}$. The matrices A and C have appropriate dimensions and

$$f(y) = \begin{bmatrix} f_1(y) \\ \vdots \\ f_n(y) \end{bmatrix}.$$

Assume that the pair $\{A, C\}$ is detectable and consider the problem of designing an observer for the nonlinear system. This problem can be solved as described below.

a) Let $\hat{x}$ be the estimate of the state x and define the estimation error $e = x - \hat{x}$. Let

$$\dot{\hat{x}} = \psi(\hat{x}, y).$$

Determine the function $\psi(\hat{x}, y)$ such that the differential equation for e is of the form

$$\dot{e} = (A + LC)e.$$

Discuss how the matrix L has to be selected to guarantee that the estimation error converges exponentially to zero. Discuss if, under the stated assumptions, it is possible to select L to achieve the above goal. [6 marks]

b) Consider the system

$$\dot{\xi}_1 = \xi_2, \qquad \dot{\xi}_2 = -3\xi_1^2\xi_2, \qquad y = \xi_1. \qquad (\star\star)$$

To design an observer for this system following the procedure in part a) we need to write the equations of the system in the form of equation $(\star)$ and verify the detectability assumption.

i) Consider the variables

$$x_1 = \xi_1, \qquad x_2 = \xi_2 + \xi_1^3.$$

Show that they define a change of coordinates, that is that there is a one to one relation between the variables (ξ_1, ξ_2) and the variables (x_1, x_2). [2 marks]

ii) Write the differential equation of the system in the (x_1, x_2) coordinates. Show that in these coordinates the system is described by equations of the form $(\star)$. Determine explicitly the matrices A, C and the function $f(y)$. [4 marks]

iii) Show that the matrices A and C determined in part b.ii) are such that the pair $\{A, C\}$ is detectable. Determine a matrix L such that the matrix $A + LC$ has all its eigenvalues equal to -1. [4 marks]

iv) Write explicitly the equations describing the observer and show how the observer can be used to estimate the state ξ_2 of the system $(\star\star)$. [4 marks]

4. The dynamic behaviour of a hot air balloon can be described by the equations

$$\ddot{h} = -\frac{1}{\tau_1}\dot{h} + \sigma\theta + w, \qquad \dot{\theta} = -\frac{1}{\tau_2}\theta + r,$$

where h indicates the height of the balloon, θ the temperature variation (with respect to some reference value) of the air in the balloon, w the vertical component of the wind, r the heat supplied to the air in the balloon, and τ_1, τ_2 and σ are positive constants, such that $\tau_1 \neq \tau_2$.

a) Let $x_1 = h$, $x_2 = \dot{h}$, $x_3 = \theta$, $u = [u_1, u_2]' = [r, w]'$, $y = [y_1, y_2]' = [h, \theta]'$ and $x = [x_1\ x_2\ x_3]'$. Write a state space realization of the considered system, i.e. determine matrices A, B_1, B_2, C_1 and C_2 such that

$$\dot{x} = Ax + B_1u_1 + B_2u_2, \qquad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} x.$$

[4 marks]

b) Study the controllability and observability properties of the system. [4 marks]

c) Assume that $u_1 = 0$ and that the only measurable output is y_2, that is consider the single-input, single output system described by the equations

$$\dot{x} = Ax + B_2u_2, \qquad y_2 = C_2x,$$

where the matrices A, B_2 and C_2 are as determined in part a).

i) Study the controllability and observability property of the system and determine the uncontrollable and unobservable modes. [4 marks]

ii) Write the equations of the system in the canonical form for non-reachable systems and identify the reachable subsystem and the unreachable subsystem. Show that the reachable subsystem is not observable. [4 marks]

iii) Exploiting the result in part c.ii) argue that

$$C_2e^{At}B_2 = 0$$

for all t.
(Hint: do not compute e^{At}!). [4 marks]

5. Consider two linear, single-input, single-output systems described by the equations

$$\begin{aligned} \sigma x_1 &= A_1 x_1 + B_1 u_1, & \sigma x_2 &= A_2 x_2 + B_2 u_2, \\ y_1 &= C_1 x_1, & y_2 &= C_2 x_2, \end{aligned}$$

where

$$A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad C_1 = \begin{bmatrix} 1 & 1 \end{bmatrix},$$

$$A_2 = [\lambda], \qquad B_2 = [1], \qquad C_2 = [1],$$

and λ is a constant parameter.

a) Study the reachability and observability properties of the two subsystems. [4 marks]

b) Consider the *interconnection equations*

$$u_2 = y_1 \qquad u = u_1 \qquad y = y_2.$$

i) Write a state space realization for the interconnected system with state $x = [x_1', x_2']'$, input u and output y. [4 marks]

ii) Study the reachability property of the interconnected system determined in part b.i). [4 marks]

iii) Study the observability property of the interconnected system determined in part b.i). [4 marks]

iv) Recall that the zeros of a single-input, single-output system

$$\sigma x = Ax + Bu, \qquad y = Cx,$$

with state $x(t) \in \mathbb{R}^n$, are the complex numbers s such that

$$\text{rank} \begin{bmatrix} sI - A & B \\ C & 0 \end{bmatrix} < n + 1.$$

Show that the interconnected system is non-reachable when a zero of the subsystem with state x_1 coincides with an eigenvalue of the matrix A_2.
(Comment: this provides a state-space description of the occurrence of pole-zero cancellations in the interconnection of systems described by transfer functions.) [4 marks]

6. Consider a linear, discrete-time, system described by the equation

$$x(k+1) = Ax(k) + Bu(k),$$

where $x(k) \in I\!R^n$ and $u(k) \in I\!R$ and the problem of designing a state feedback control law $u = Kx$ such that all eigenvalues of the closed-loop system are inside the unity disc.

Assume the system is reachable.

If the matrix A has only one eigenvalue with modulo larger or equal to 1, the problem can be solved *moving* only this eigenvalue and leaving all other eigenvalues unchanged. This goal can be achieved by means of the so-called Mitter procedure, described in what follows. (For simplicity we assume that the eigenvalue to be moved is *moved* to zero).

Mitter Procedure

(1) Let λ be the eigenvalue to be *moved* and compute v such that

$$v'A = \lambda v' \qquad\qquad v'B \neq 0.$$

(2) Select the feedback gain K as

$$K = \alpha v',$$

with

$$\alpha = -\frac{\lambda}{v'B}.$$

Assume now that

$$A = \begin{bmatrix} 0 & 1/2 & 0 \\ -1/2 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \qquad\qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

a) Compute the eigenvalues of A and verify that two have modulo strictly smaller than 1, while one has modulo larger or equal to 1. [2 marks]

b) Show that the system is reachable. [4 marks]

c) Apply Mitter procedure to determine a state feedback control law $u = Kx$ which *moves* the eigenvalue with modulo larger or equal to one to zero. [12 marks]

d) Verify that the matrix $A+BK$, with K as determined in part c), has two eigenvalues which coincide with the eigenvalues of A inside the unity disc, and one eigenvalue at zero. [2 marks]

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Question 1

a) The observability matrix of the system is

$$\mathcal{O} = \begin{bmatrix} 1 & -1 & 2 \\ -1 & -1 & -2 \\ -1 & 1 & -2 \end{bmatrix}.$$

Note that the first and third rows are linearly dependant, while the first and second are independent. Hence the observability matrix has rank 2, and the system is not observable. Note now that $\lambda(A) = \{0, +j, -j\}$ and since the unobservable subspace has dimension one, the unobservable mode has to be $s = 0$, which implies that the system is detectable (recall that it is a discrete-time system).
(The same conclusions can be obtained using the PBH observability test.)

b) All initial conditions $x(0)$ yielding the given output sequence are the solutions of the equations

$$y(0) = Cx(0) = 0, \qquad y(1) = CAx(0) = 0, \qquad y(2) = CA^2x(0) = 0,$$

or, equivalently, of the equation $\mathcal{O}x(0) = 0$. Since $\mathcal{O}$ has rank 2, the above equation have infinitely many solutions, given by

$$x(0) = \begin{bmatrix} -2\alpha & 0 & \alpha \end{bmatrix}'$$

for $\alpha \in \mathbb{R}$.

c) All final conditions $x(2)$ yielding the given output sequence can be obtained considering the initial conditions determined in part b) and *propagating* the initial conditions for two steps, that is

$$x(2) = A^2 \begin{bmatrix} -2\alpha \\ 0 \\ \alpha \end{bmatrix} = 0.$$

d) The main difference between the results of part b) and of part c) is in the fact that the set of initial conditions consistent with the given output sequence is a subspace, whereas the set of final conditions consistent with the given output sequence is only a point, *i.e.* the system is not observable but it is reconstructable.

e) The observer is described by the equations

$$\xi(k+1) = (A + LC)\xi(k) - Ly(k), \qquad \hat{x}(k) = \xi(k),$$

and the matrix $L = [L_1\ L_2\ L_3]'$ has to be selected so that $A + LC$ has all eigenvalues at 0. To achieve this goal note that

$$A + LC = \begin{bmatrix} 1 + L_1 & -1 - L_1 & 2L_1 \\ 1 + L_2 & -L_2 & 2 + 2L_2 \\ L_3 & -L_3 & 2L_3 \end{bmatrix}.$$

Selecting (note that other selections are possible)

$$L_1 = -1 \qquad L_2 = 0 \qquad L_3 = 0$$

yields

$$A + LC = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix},$$

which has all eigenvalues at 0. Note, finally, that

$$(A + LC)^2 \neq 0, \qquad\qquad (A + LC)^3 = 0,$$

hence, for the proposed design, $\bar{k} = 3$.

Question 2

a) The equilibrium points for $u = 0$ are the solutions of the equations

$$0 = -3x_1 + x_2^2 + x_2, \qquad 0 = x_1 x_3^2 - 2x_2, \qquad 0 = \alpha x_3.$$

($\alpha \neq 0$) The only solution is $x_1 = x_2 = x_3 = 0$.

($\alpha = 0$) Consider two sub-cases.

($x_3 = 0$) The second equation gives $x_2 = 0$ and the first equation $x_1 = 0$, hence the equilibrium $(0, 0, 0)$.

($x_3 \neq 0$) The second equation gives $x_1 = 2x_2/x_3^2$ which, replaced in the first equation gives $-6x_2/x_3^2 + x_2^2 + x_2 = 0$. This equation has two solutions

$$x_2 = 0 \qquad\qquad x_2 = \frac{6}{x_3^2} - 1.$$

The first solution yields the family of equilibria

$$(0, 0, \bar{x}_3)$$

with $\bar{x}_3 \neq 0$, while the second solution yields the family of equilibria

$$\left(\frac{2}{\bar{x}_3^2}(\frac{6}{\bar{x}_3^2} - 1), \frac{6}{\bar{x}_3^2} - 1, \bar{x}_3\right),$$

with $\bar{x}_3 \neq 0$.

b) The system linearized around the zero equilibrium is described by the equation

$$\dot{\delta}_x = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & \alpha \end{bmatrix} \delta_x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \delta_u.$$

c) The matrix A of the linearized system has eigenvalues -3, -2 and α. Hence, the zero equilibrium is locally asymptotically stable for $\alpha < 0$, it is unstable for $\alpha > 0$, while it is not possible to deduce any property, using the principle of stability in the first approximation, for $\alpha = 0$.

d) The reachability pencil of the linearized system is

$$\begin{bmatrix} sI - A & B \end{bmatrix} = \begin{bmatrix} s+3 & -1 & 0 & 0 \\ 0 & s+2 & 0 & 0 \\ 0 & 0 & s+\alpha & 1 \end{bmatrix}.$$

This matrix has rank 2 for $s = -3$ and $s = -2$, hence the system is not reachable, but stabilizable.

e) Selecting

$$u = kx_3 = -\alpha x_3 - x_3$$

yields a closed-loop linearized system with a matrix A having all eigenvalues with negative real part. Hence, the zero equilibrium of the closed-loop nonlinear system is locally asymptotically stable.

Question 3

a) Note that

$$\dot{e} = \dot{x} - \dot{\hat{x}} = Ax + f(y) - \psi(\hat{x}, y),$$

hence selecting

$$\psi(\hat{x}, y) = A\hat{x} + f(y) - L(y - C\hat{x})$$

yields

$$\dot{e} = (A + LC)e.$$

The matrix L has to be selected to render the error system $\dot{e} = (A+LC)e$ asymptotically stable. This goal can be achieved since by assumption the pair $\{A, C\}$ is detectable.

b) i) The relation

$$x_1 = \xi_1 \qquad\qquad x_2 = \xi_2 + \xi_1^3$$

can be inverted to give

$$\xi_1 = x_1 \qquad\qquad \xi_2 = x_2 - x_1^3,$$

hence the indicated transformation is a valid change of coordinates.

ii) The differential equations in the (x_1, x_2) coordinates are obtained as follows

$$\dot{x}_1 = \dot{\xi}_1 = \xi_2 = x_2 - x_1^3$$

and

$$\dot{x}_2 = \dot{\xi}_2 + 3\xi_1^2\dot{\xi}_1 = -3\xi_1^2\xi_2 + 3\xi_1^2\xi_2 = 0.$$

The system can thus be written as

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} -y^3 \\ 0 \end{bmatrix} \qquad\qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x.$$

iii) The observability matrix of the pair $\{A, C\}$, with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad\qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

as determined above, is

$$\mathcal{O} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

hence the pair $\{A, C\}$ is observable, and thus detectable. Finally, note that ($L = [L_1, L_2]'$)

$$A + LC = \begin{bmatrix} L_1 & 1 \\ L_2 & 0 \end{bmatrix},$$

hence the selection

$$L_1 = -2 \qquad\qquad L_2 = -1$$

is such that both eigenvalues of $A + LC$ are equal to -1.

iv) The equations of the observer are

$$\dot{\hat{x}} = \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix} \hat{x} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} y + \begin{bmatrix} -y^3 \\ 0 \end{bmatrix}.$$

Finally, from the estimates $\hat{x}_1$ and $\hat{x}_2$ of x_1 and x_2 an estimate of ξ_2 is given by

$$\hat{\xi}_2 = \hat{x}_2 - \hat{x}_1^3.$$

Question 4

a) The matrices A, B_1, B_2, C_1 and C_2 of the state space representation are

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1/\tau_1 & \sigma \\ 0 & 0 & -1/\tau_2 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \qquad C_2 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}.$$

b) The reachability matrix is

$$\mathcal{R} = \begin{bmatrix} 0 & 0 & 0 & 1 & \star & \star \\ 0 & 1 & \star & \star & \star & \star \\ 1 & 0 & \star & \star & \star & \star \end{bmatrix},$$

where the $\star$'s are elements that do not have to be computed explicitly. Note that $\text{rank}\mathcal{R} = 3$, hence the system is controllable.

The observability matrix is

$$\mathcal{O} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ \star & \star & \star \\ \star & \star & \star \\ \star & \star & \star \end{bmatrix}.$$

Note that $\text{rank}\mathcal{O} = 3$, hence the system is observable.

c) i) The reachability pencil is

$$[sI - A \;\; B_2] = \begin{bmatrix} s & -1 & 0 & 0 \\ 0 & s + 1/\tau_1 & -\sigma & 1 \\ 0 & 0 & s + 1/\tau_2 & 0 \end{bmatrix}.$$

The reachability pencil has rank 3 for all $s \neq -1/\tau_2$. Hence, the system is not controllable, and the uncontrollable mode is $s = -1/\tau_2$.

The observability pencil is

$$\begin{bmatrix} sI - A \\ C_2 \end{bmatrix} = \begin{bmatrix} s & -1 & 0 \\ 0 & s + 1/\tau_1 & -\sigma \\ 0 & 0 & s + 1/\tau_2 \\ 0 & 0 & 1 \end{bmatrix}.$$

The observability pencil has rank 3 for all $s \neq -1/\tau_1$ and $s \neq 0$. Hence, the system is not observable, and the unobservable modes are $s = -1/\tau_1$ and $s = 0$.

ii) The system is already in the canonical form. In fact, setting $x_a = [x_1 \; x_2]'$ and $x_b = x_3$ yields

$$\begin{aligned} \dot{x}_a &= A_{11}x_a + A_{12}x_b + B_{21}u, \\ \dot{x}_b &= A_{22}x_b, \\ y &= C_{22}x_b, \end{aligned}$$

where

$$A_{11} = \begin{bmatrix} 0 & 1 \\ 0 & -1/\tau_1 \end{bmatrix}, \; A_{12} = \begin{bmatrix} 0 \\ \sigma \end{bmatrix}, \; A_{22} = \begin{bmatrix} -\frac{1}{\tau_1} \end{bmatrix}, \; B_{21} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; C_{22} = \begin{bmatrix} 1 \end{bmatrix}.$$

Note that the output depends only upon the state x_b, hence the reachable subsystem does not contribute to the output, *i.e.* it is unobservable.

iii) From part c.ii) one concludes that the input signal affects only the states x_a, which does not contribute to the output, hence for $x_a(0) = 0$, $x_b(0) = 0$ and for any $u_2(t)$, $y(t) = 0$ for all t, which implies that $C_2 e^{At} B_2 = 0$ for all t.

Question 5

a) The reachability and observability matrices for the first system are

$$\mathcal{R}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}, \qquad \mathcal{O}_1 = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}.$$

Both matrices have rank equal to two, hence the system is reachable and observable.

The second system is trivially reachable and observable, since it is a scalar system and $B_2 \neq 0$ and $C_2 \neq 0$.

b) i) A state space realization for the interconnected system is

$$\dot{x} = Ax + Bu, \qquad y = Cx,$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & 1 & \lambda \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}.$$

ii) The reachability matrix of the interconnected system is

$$\mathcal{R} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 4 \\ 0 & 1 & 3+\lambda \end{bmatrix}.$$

Note that $\det\mathcal{R} = -1-\lambda$, hence the interconnected system is reachable provided $\lambda \neq -1$.

iii) The observability matrix of the interconnected system is

$$\mathcal{O} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & \lambda \\ \lambda & 3+\lambda & \lambda^2 \end{bmatrix}.$$

Note that $\det\mathcal{O} = 3$, hence the interconnected system is observable for all λ.

iv) Note that

$$\det \begin{bmatrix} sI - A_1 & B_1 \\ C_1 & 0 \end{bmatrix} = -s-1,$$

hence the first subsystem has a zero for $s = -1$. When $\lambda = -1$, the zero of the first subsystem coincides with the eigenvalue (that is the pole) of the second subsystem, and this is the reason why the interconnected system is unobservable for $\lambda = -1$.

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Question 6

a) Note that

$$\det(sI - A) = (s-2)(s^2 + 1/4).$$

The eigenvalues of A are $\lambda(A) = \{\frac{j}{2}, -\frac{j}{2}, 2\}$, hence two eigenvalues are inside the unity disk, and one it is outside.

b) The reachability matrix of the system is

$$\mathcal{R} = \begin{bmatrix} 0 & 0 & 1/2 \\ 0 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix},$$

hence the system is reachable.

c) The eigenvalue to move is $\lambda = 2$. Let $v = \begin{bmatrix} 0 & 0 & v_3 \end{bmatrix}'$, with $v_3 \neq 0$, and note that $v'A = 2v'$ and $v'B = v_3 \neq 0$. Hence, the feedback gain K is

$$K = -\frac{2}{v'B} v' = \begin{bmatrix} 0 & 0 & -2 \end{bmatrix}.$$

d) Note that

$$A + BK = \begin{bmatrix} 0 & 1/2 & 0 \\ -1/2 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The matrix $A + BK$ has eigenvalues $\lambda(A) = \{\frac{j}{2}, -\frac{j}{2}, 0\}$, *i.e.* two eigenvalues coincide with the eigenvalues of A inside the unity disk, and the third eigenvalue is at 0.