UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 1998

BEng Honours Degree in Computing Part III
MEng Honours Degrees in Computing Part IV
BEng Honours Degree in Information Systems Engineering Part III
MEng Honours Degree in Information Systems Engineering Part III
BSc Honours Degree in Mathematics and Computer Science Part III
MSci Honours Degree in Mathematics and Computer Science Part IV
MSci Honours Degree in Mathematics and Computer Science Part III
MSc Degree in Advanced Computing
MSc Degree in Computing Science
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the
Diploma of Membership of Imperial College
Associateship of the Royal College of Science
Associateship of the City and Guilds of London Institute

PAPER 3.36 / 4.36 / I3.6

PERFORMANCE ANALYSIS

Monday, April 27th 1998, 2.00 - 4.00

Answer THREE questions

For admin. only: paper contains 4 questions

1 a Define the terms *continuous time Markov chain* (or Markov process) and *probability flux.* State the steady state theorem for continuous time Markov chains, either formally or informally in terms of probability flux, giving conditions under which it is valid.

b The operation of an Ethernet with one backoff allowed is modelled as a Markov process with state space $\{(n, b) \mid n \geq 0, b \in \{0,1\}\}$. The arrival rate of new messages is λ and mean transmission time is $1/\mu$. When a new arrival finds the Ethernet busy, it backs off and retries, also at rate λ, but messages that find the Ethernet busy a second time are lost. In the state descriptor, n represents the number of messages waiting to retry, b=1 if the Ethernet is busy and b=0 if not.

i) Under what assumptions does the state evolve as a Markov process?

ii) Draw a state transition diagram and write down the balance equations.

iii) Prove that $\lambda(i+2)p_{i,1} = (i+1)(\mu + \lambda(i+2))p_{i+1,1}$ where $p_{n,b}$ is the equilibrium probability of state (n, b), and hence show that

$$p_{i,1} = \frac{\lambda(i+1)}{\prod_{j=2}^{i+1}(\mu+j\lambda)} \; p_{0,1}$$

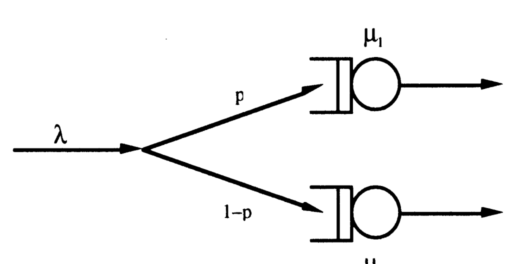
The two parts carry, respectively, 40% *and* 60% *of the marks.*

2 a In a closed, steady state Markovian queueing network of M nodes and k customers, with *throughput* $T(k)$ along a specially chosen arc, derive the following equations for the mean queue lengths $L_i(k)$ and mean waiting times $W_i(k)$ at node i ($1 \leq i \leq M$):

$L_i(k) = v_i T(k) W_i(k)$; $W_i(k) = (1+L_i(k-1))/\mu_i$; $k = T(k) \sum_{i=1}^{M} v_i W_i(k)$

where v_i is the *visitation rate* of server i and μ_i is its service rate ($1 \leq i \leq M$).

b Consider the steady state open network, with arrival rate λ, shown below and its closure, with a fixed population, formed by routing all departures back to the input marked λ and disallowing external arrivals.



i) Show that in the closure, response time (between successive departures from either server) is minimised when throughput is maximised.

ii) Assuming that mean waiting time at server 1 is $1/(\mu_1 - p\lambda)$, and at server 2 is $1/(\mu_2-(1-p)\lambda)$, find the value of p that minimises response time.

3 a In a Poisson arrival process with rate λ, the number of arrivals N in a time interval of length t is a Poisson random variable with probability mass function $P(N=i) = e^{-\lambda t}(\lambda t)^i/i!$.

i) Show that the interarrival time is an exponential random variable with parameter λ and that the density function of the time between an arrival and the nth successive arrival is $\frac{\lambda(\lambda t)^{n-1}}{(n-1)!} e^{-\lambda t}$.

ii) State and prove the *superposition property* of Poisson processes.

iii) State *Jackson's Theorem* for *open* queueing networks.

b In a client-server system, a set of m independent, homogeneous workstations are connected to a central computer (server) by two networks. Transactions are sent from the workstations over one network to the server which processes them and returns a file via the other network. Each workstation submits transactions as an independent Poisson process with rate λ and the networks and server are all modelled as M/M/1 queues with equal service rate μ.

i) *Assuming* that an M/M/1 queue at equilibrium has length $n \geq 0$ with probability $(1-\rho)\rho^n$, where ρ is the ratio of arrival rate to service rate, show that the waiting time random variable (queueing time *plus* time in service) is exponential with parameter equal to the product of the service rate and $(1-\rho)$. State clearly any other assumptions that you make.

ii) *Assuming* that the waiting times of a transaction at each network and at the server are independent, find the probability density function of response time.

4 a Explain why the M/G/1 queue is *not* a Markov process. At what instants does the Markov property hold and why? Briefly outline how this property may be used to obtain the equilibrium queue length probability distribution at these instants.

b State Little's result for queueing systems in equilibrium and sketch a proof of it using the notion of charging customers per unit time spent in the system, or otherwise.

c Apply Little's result to write down an equation for the mean number of customers waiting to commence service in an M/G/1 queue. Hence, stating any other assumptions you make, show that the *mean queueing time* in an M/G/1 queue, with arrival rate λ and service rate μ, is $Q = \frac{\lambda r}{\mu - \lambda}$ where r is the mean of the remaining service time of any customer being served just before an arrival instant. You may assume that there is a customer in service at an arrival instant with probability λ/μ. What is the mean time spent by a customer in the system?

End of paper