EE4-47

Modelling and control of multibody mechanical systems

Model answers

Question 1

a) Two single-axis-rotation transformation matrices are needed.

$$D_\psi = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which is the rotation matrix by angle ψ about a z axis.

$$B_\phi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix},$$

which is the rotation matrix by angle ϕ about an x axis.

The angular velocity vector of the axle in axle-fixed coordinates is

$$\Omega_a = B_\phi D_\psi \begin{bmatrix} 0 \\ 0 \\ \dot\psi \end{bmatrix} + B_\phi \begin{bmatrix} \dot\phi \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \dot\phi \\ \dot\psi \sin\phi \\ \dot\psi \cos\phi \end{bmatrix}.$$

[3 marks]

b) The inertia matrix of the wheel with respect to a set of axes parallel to the (unspun) axle-fixed axes and with origin the centre of mass of the wheel is

$$I_{COM} = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{xx} \end{bmatrix},$$

due to the symmetry of the wheel. We can shift the origin of this set of axes by a distance l along the axle, to the point A. We can then find the new inertia tensor with respect to the axle-fixed axes by adding to the inertia tensor about the centre of mass of the wheel a difference term as follows

$$I_{af} = I_{COM} + \begin{bmatrix} ml^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & ml^2 \end{bmatrix},$$

which amounts to

$$I_{af} = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{xx} \end{bmatrix} + \begin{bmatrix} ml^2 & 0 & 0 \\ 0 & & 0 \\ 0 & 0 & ml^2 \end{bmatrix} = \begin{bmatrix} I_{xx} + ml^2 & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{xx} + ml^2 \end{bmatrix}.$$

[3 marks]

c) In addition to the transformation matrices defined in part a) we also need

$$C_\theta = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}.$$

which is the rotation matrix by angle θ about a y axis.

The angular velocity vector of the wheel in axle-fixed coordinates is

$$\boldsymbol{\Omega}_w = B_\phi D_\psi \begin{bmatrix} 0 \\ 0 \\ \dot\psi \end{bmatrix} + B_\phi \begin{bmatrix} \dot\phi \\ 0 \\ 0 \end{bmatrix} + C_\theta^T \begin{bmatrix} 0 \\ \dot\theta \\ 0 \end{bmatrix} = \begin{bmatrix} \dot\phi \\ \dot\psi \sin\phi + \dot\theta \\ \dot\psi \cos\phi \end{bmatrix}.$$

[3 marks]

d) The angular momentum vector is $\boldsymbol{H} = I_{af}\boldsymbol{\Omega}_w$ and therefore

$$\boldsymbol{H} = \begin{bmatrix} (I_{xx} + ml^2)\dot\phi \\ I_{yy}(\dot\psi \sin\phi + \dot\theta) \\ (I_{xx} + ml^2)\dot\psi \cos\phi \end{bmatrix},$$

or in vector notation

$$\boldsymbol{H} = (I_{xx} + ml^2)\dot\phi \boldsymbol{i}' + I_{yy}(\dot\psi \sin\phi + \dot\theta)\boldsymbol{j}' + (I_{xx} + ml^2)\dot\psi \cos\phi \boldsymbol{k}'.$$

[3 marks]

e) The axle is horizontal, therefore $\phi = 0$. Also, $\dot\psi = -\omega_{yaw}$ and $\dot\theta = \omega_{spin}$.

i) By substituting ϕ, $\dot\psi$ and $\dot\theta$ into the expression for $\boldsymbol{H}$ found above we obtain

$$\boldsymbol{H} = I_{yy}\omega_{spin}\boldsymbol{j}' - (I_{xx} + ml^2)\omega_{yaw}\boldsymbol{k}'.$$

[2 marks]

ii) By substituting ϕ, $\dot\psi$ and $\dot\theta$ into the expression for $\boldsymbol{\Omega}_a$ found above we obtain $\boldsymbol{\Omega}_a = -\omega_{yaw}\boldsymbol{k}'$. The external torque vector can be found by considering the equation of motion about point A given by $\frac{d\boldsymbol{H}}{dt} = \boldsymbol{N}$. Therefore,

$$\boldsymbol{N} = -\omega_{yaw}\boldsymbol{k}' \times \left(I_{yy}\omega_{spin}\boldsymbol{j}' - (I_{xx} + ml^2)\omega_{yaw}\boldsymbol{k}'\right) = I_{yy}\omega_{spin}\omega_{yaw}\boldsymbol{i}'.$$

[4 marks]

iii) The moment of the couple provided by the weight of the wheel and the corresponding reaction force in the joint A is $mgl\boldsymbol{i}'$. Therefore,

$$l = \frac{I_{yy}\omega_{spin}\omega_{yaw}}{mg}.$$

[2 marks]

Question 2

a) i) The moment of inertia about the axis of symmetry (Z axis) is:

$$I_{zz} = \int (x^2 + y^2) dm = \rho \int_V (x^2 + y^2) dV = \rho \int_V r^2 dV,$$

where r is the radial distance from the Z axis. If we consider an infinitesimal volume element given in cylindrical polar coordinates then I_{zz} becomes

$$I_{zz} = \rho \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_0^{2\pi} \int_{R_1}^{R_2} r^3 dr d\theta dz = \frac{\rho \pi h \left(R_2^4 - R_1^4\right)}{2}.$$

But the mass of the cylindrical tube is given by

$$m = \rho V = \rho \pi \left(R_2^2 - R_1^2\right) h,$$

and therefore I_{zz} becomes

$$I_{zz} = \frac{1}{2} m \left(R_2^2 + R_1^2\right).$$

[6 marks]

ii) The moment of inertia about the X axis can be found similarly via the equation

$$I_{xx} = \int (y^2 + z^2) dm = \rho \int_V (y^2 + z^2) dV = \rho \int_V (r^2 \cos^2\theta + z^2) dV.$$

By using a similar cylindrical polar volume element as above the volume integral becomes

$$I_{xx} = \rho \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_0^{2\pi} \int_{R_1}^{R_2} (r^2 \cos^2\theta + z^2) r dr d\theta dz = \rho \left(\frac{\left(R_2^4 - R_1^4\right) \pi h}{4} + \frac{\left(R_2^2 - R_1^2\right) \pi h^3}{12} \right)$$

$$= \rho \pi h \left(R_2^2 - R_1^2\right) \left(\frac{R_2^2 + R_1^2}{4} + \frac{h^2}{12} \right),$$

and by making use of the mass expression of the tube

$$I_{xx} = \frac{1}{12} m \left(3(R_2^2 + R_1^2) + h^2\right).$$

[6 marks]

iii) Due to the symmetry of the cylindrical tube I_{yy} is the same as I_{xx}, i.e.

$$I_{yy} = I_{xx}.$$

[2 marks]

b) In the thick wall case $R_1 = \frac{1}{3} R_2$. In the thin wall case R_2 remains the same since the machining is done in the inside of the tube, but R_1 increases to $R_1 = \frac{2}{3} R_2$ since the wall is now only R_1 thick. Note that the mass of the tube in the two cases is different, therefore we make use of the I_{zz} expression derived above prior to the substitution of the mass. Hence, the fraction of reduction in the Z-axis moment of inertia is

$$\frac{I_{zz,thick} - I_{zz,thin}}{I_{zz,thick}} = 1 - \frac{I_{zz,thin}}{I_{zz,thick}} = 1 - \frac{\frac{\rho \pi h \left(R_2^4 - \left(\frac{2}{3} R_2\right)^4\right)}{2}}{\frac{\rho \pi h \left(R_2^4 - \left(\frac{1}{3} R_2\right)^4\right)}{2}} = 1 - \frac{1 - \frac{16}{81}}{1 - \frac{1}{81}} = 1 - \frac{65}{80} = \frac{3}{16}.$$

[6 marks]

Question 3

a) i) $\boldsymbol{r_M} = x\boldsymbol{i}$ and $\boldsymbol{r_m} = x\boldsymbol{i} + l\boldsymbol{e_r}$. [1 mark]

ii) By differentiating the position vector $\dot{\boldsymbol{r}}_{\boldsymbol{M}} = \dot{x}\boldsymbol{i}$ and

$$\dot{\boldsymbol{r}}_{\boldsymbol{m}} = \dot{x}\boldsymbol{i} + l\dot{\theta}\boldsymbol{e_\theta} = \dot{x}\cos\theta\boldsymbol{e_r} + (-\dot{x}\sin\theta + l\dot{\theta})\boldsymbol{e_\theta}.$$

[1 mark]

iii) By differentiating the velocity vector $\ddot{\boldsymbol{r}}_{\boldsymbol{M}} = \ddot{x}\boldsymbol{i}$ and

$$\ddot{\boldsymbol{r}}_{\boldsymbol{m}} = \ddot{x}\boldsymbol{i} + l\ddot{\theta}\boldsymbol{e_\theta} - l\dot{\theta}^2\boldsymbol{e_r} = (\ddot{x}\cos\theta - l\dot{\theta}^2)\boldsymbol{e_r} + (-\ddot{x}\sin\theta + l\ddot{\theta})\boldsymbol{e_\theta}.$$

[3 marks]

b)

$$(F_r - mg\sin\theta)\boldsymbol{e_r} - mg\cos\theta\boldsymbol{e_\theta} = m\left((\ddot{x}\cos\theta - l\dot{\theta}^2)\boldsymbol{e_r} + (-\ddot{x}\sin\theta + l\ddot{\theta})\boldsymbol{e_\theta}\right).$$

Hence:

i) Collecting the $\boldsymbol{e_\theta}$ terms,

$$g\cos\theta - \ddot{x}\sin\theta + l\ddot{\theta} = 0. \qquad (1)$$

[3 marks]

ii) Collecting the $\boldsymbol{e_r}$ terms, the force in the rod, F_r, is

$$F_r = m\left(g\sin\theta + \ddot{x}\cos\theta - l\dot{\theta}^2\right).$$

[3 marks]

c) $x + a\phi = 0$ or $\dot{x} + a\dot{\phi} = 0$. [1 mark]

d)

$$(F_x - F_r\cos\theta)\boldsymbol{i} + (F_r\sin\theta - R)\boldsymbol{k} = M\ddot{x}\boldsymbol{i},$$

where F_x is the horizontal friction force that maintains the rolling constraint and R is the normal reaction force on the wheel from the horizontal surface. Therefore, by collecting the $\boldsymbol{i}$ terms, substituting the expression for F_r found above and making use of Equation (1),

$$F_x = (M + m)\ddot{x} - ml\sin\theta\ddot{\theta} - ml\dot{\theta}^2\cos\theta.$$

[4 marks]

e) By considering the motion about the wheel centre of mass according to $\frac{d\boldsymbol{H}}{dt} = \boldsymbol{N}$, the second equation of motion is derived,

$$I\ddot{\phi} = T_d + F_x a,$$

and by substituting F_x from the equation above

$$I\ddot{\phi} - a\left((M + m)\ddot{x} - ml\sin\theta\ddot{\theta} - ml\dot{\theta}^2\cos\theta\right) = T_d.$$

Finally substituting $\ddot{x} = -a\ddot{\phi}$ from the constraint equation and $l\ddot{\theta} = \sin\theta\ddot{x} - g\cos\theta$ from Equation (1), yields

$$\left(I + Ma^2 + ma^2\cos^2\theta\right)\ddot{\phi} + mal\cos\theta\dot{\theta}^2 - mga\sin\theta\cos\theta = T_d.$$

[4 marks]

Question 4

a) $\boldsymbol{r}_C = r\boldsymbol{e}_r + a\boldsymbol{e}_\theta$. [2 marks]

b) By differentiating the position vector we obtain $\dot{\boldsymbol{r}}_C = (\dot{r} - a\dot{\theta})\boldsymbol{e}_r + r\dot{\theta}\boldsymbol{e}_\theta$. [2 marks]

c) The total kinetic energy of the ball is

$$T_{ba} = \frac{1}{2}m\left((\dot{r} - a\dot{\theta})^2 + r^2\dot{\theta}^2\right) + \frac{1}{2}I_{ba}\dot{\phi}^2.$$

The potential energy of the ball, with zero potential energy at the level of point O, is

$$V_{ba} = mg(r\sin\theta + a\cos\theta).$$

The kinetic energy of the beam is $T_{be} = \frac{1}{2}I_{be}\dot{\theta}^2$ and the potential of the beam is zero. Therefore, the Lagrangian function is

$$L = T_{ba} + T_{be} - V_{ba} = \frac{1}{2}m\left((\dot{r} - a\dot{\theta})^2 + r^2\dot{\theta}^2\right) + \frac{1}{2}I_{ba}\dot{\phi}^2 + \frac{1}{2}I_{be}\dot{\theta}^2 - mg(r\sin\theta + a\cos\theta).$$

[6 marks]

d) The rolling constraint in this case is holonomic and it is given by $r + a\phi = 0$.
[2 marks]

e) The Lagrangian equation with respect to the generalised coordinate r is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} + \lambda = 0,$$

in which λ is the Lagrange multiplier corresponding to the rolling constraint. The force that maintains the rolling constraint is $-\lambda$. Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m(\dot{r} - a\dot{\theta})\right) - mr\dot{\theta}^2 + mg\sin\theta = -\lambda,$$

which yields

$$-\lambda = m\left(\ddot{r} - a\ddot{\theta} - r\dot{\theta}^2 + g\sin\theta\right).$$

The Lagrangian equation with respect to the generalised coordinate θ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0,$$

which yields the first equation of motion

$$-ma\ddot{r} + \left(ma^2 + mr^2 + I_{be}\right)\ddot{\theta} + 2mr\dot{r}\dot{\theta} + mgr\cos\theta - mga\sin\theta = 0.$$

The Lagrangian equation with respect to the generalised coordinate ϕ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} + \lambda a = 0,$$

which gives $I_{ba}\ddot{\phi} + \lambda a = 0$, and upon substitution of λ it yields the second equation of motion

$$I_{ba}\ddot{\phi} - ma\left(\ddot{r} - a\ddot{\theta} - r\dot{\theta}^2 + g\sin\theta\right) = 0.$$

By using the rolling constraint equation $\ddot{\phi} = -\frac{\ddot{r}}{a}$ and substituting in the above equation, the second equation of motion can be obtained in terms of the generalised coordinates r and θ only as follows:

$$\left(\frac{I_{ba}}{a^2} + m\right)\ddot{r} + m\left(-a\ddot{\theta} - r\dot{\theta}^2 + g\sin\theta\right) = 0.$$

[8 marks]