UNIVERSITY OF LONDON [ISE 2.6 2000]

B.ENG. AND M.ENG. EXAMINATIONS 2000

For Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examination for the Associateship of the City and Guilds of London Institute.

INFORMATION SYSTEMS ENGINEERING 2.6

MATHEMATICS

Date Wednesday 3rd May 2000 2.00 - 4.00 pm

Answer FOUR questions, to include at least one from Section B

[Before starting, please make sure that the paper is complete. There should be SIX pages, with a total of SIX questions. Ask the invigilator for a replacement if this copy is faulty.]

[ISE 2.6 2000]

Section A

1. The Fourier transform of $f(t)$ is given by

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt .$$

Show that, if $f(t)$ is a real-valued and even function of t, then the Fourier transform $\hat{f}(\omega)$ is real-valued for all real ω.

Now let

$$g(t) = e^{-|t|} \quad \text{for all } t,$$

$$\text{and} \quad h(t) = \begin{cases} 1 & \text{for } -1 \le t \le 1, \\ 0 & \text{for all other values of } t. \end{cases}$$

Show that $\hat{h}(\omega) = 2\dfrac{\sin\omega}{\omega}$.

Assume that the Fourier transform of $g(t)$ is $\hat{g}(\omega) = \dfrac{2}{1+\omega^2}$.

Find the Fourier transforms of the functions $\dfrac{1}{1+t^2}$ and $\dfrac{\sin t}{t}$.

By considering a suitable convolution, show that

$$\int_{-\infty}^{\infty} e^{-|u|}\,\frac{\sin u}{u}\,du = \frac{\pi}{2}.$$

Find $\displaystyle\int_{-\infty}^{\infty}\left(\frac{\sin t}{t}\right)^2 dt$.

You may assume also

(a) if $f(t)$ has Fourier transform $\hat{f}(\omega)$, then $\hat{f}(t)$ has Fourier transform $2\pi f(-\omega)$,

(b) $f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{+i\omega t}\, d\omega$,

(c) if $f(t)$ and $g(t)$ have Fourier transforms $\hat{f}(\omega)$ and $\hat{g}(\omega)$ respectively, then the convolution $\int_{-\infty}^{\infty} f(t-u)\,g(u)\,du$ has Fourier transform $\hat{f}(\omega)\,\hat{g}(\omega)$,

(d) Parseval's Formula $\int_{-\infty}^{\infty} |f(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2\,d\omega$.

2. (i) By using Laplace transforms, find the function $g(t)$ which satisfies the equation

$$g(t) = \sin t + 2 \int_0^t g(t-u) \cos u \, du \qquad \text{for } t > 0 .$$

(ii) (a) By choosing a suitable parameterisation $x(t), y(t)$ for the straight line segment C leading from $(0, 0)$ to $(2, 1)$ in the xy plane, evaluate the path integral

$$I = \int_C (2x + ye^{xy}) \, dx + (xe^{xy} - 3y^2) \, dy.$$

(b) Assuming that a suitable potential function exists, check your answer to part (a) by finding this potential function.

3. (i) Make a careful sketch of the region of the xy plane over which the integral

$$\int_0^1 dy \int_1^{1/y} x^{-1} e^{-xy} \, dx$$

is taken. Change the order of integration, using your sketch as necessary, and hence evaluate the integral.

(ii) Sketch carefully the region of the xy plane over which the integral

$$\int_0^1 dx \int_{1-\sqrt{1-x^2}}^{x} \frac{x}{\sqrt{x^2+y^2}} \, dy$$

is taken. Use polar coordinates to show that the value of this integral is $\dfrac{1}{3\sqrt{2}}$.

4. Let

$$f(z) = \frac{1}{z^4+1}.$$

Show that the residues of $f(z)$ at its poles at $z = e^{i\pi/4}$ and $z = e^{i3\pi/4}$ are

$$\frac{1}{4}\left(-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) \text{ and } \frac{1}{4}\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) \text{ respectively.}$$

Suppose that $R > 1$ and consider the path $C = C_1 + C_2$ in the complex plane, where C_1 leads from $-R$ to $+R$ along the real axis and C_2 is the upper semi-circular path from $+R$ to $-R$.

Prove that

$$\int_C f(z)\,dz = \frac{\pi}{\sqrt{2}}.$$

Prove that

$$\left|\int_{C_2} f(z)\,dz\right| \leq \frac{\pi R}{R^4 - 1}.$$

Hence, by considering also the limits as $R \to +\infty$ of $\int_{C_1} f(z)\,dz$ and $\int_{C_2} f(z)\,dz$, deduce that

$$\int_{-\infty}^{\infty} \frac{1}{x^4+1}\,dx = \frac{\pi}{\sqrt{2}}.$$

Section B **[ISE 2.6 2000]**

1. Following the installation of a particular software package, an initial diagnostic installation test is performed; this test fails if it discovers installation errors and passes otherwise. If the test is failed the installation is rejected. If there are no installation errors the test is always passed. If there are installation errors the test discovers them 80% of the time. It is known that 1 in 10 installations will have installation errors.

   Determine the probability that

   (i) an installation is rejected;

   (II) there are errors present in an installation which has passed the test.

   If the initial test is passed a further independent test is performed; this test has exactly the same properties as the initial test, except that if there are installation errors it discovers them 99% of the time. If this test is failed the installation is rejected, otherwise it is accepted.

   (iii) Determine the probability that an installation with errors is accepted.

   The times, T_1 and T_2 (in minutes) taken to run the first and second tests are normally distributed with means, 10 and 50, and variances, 2 and 10, respectively.

   (iv) Determine the distribution of the total testing times for the two cases of the first test being passed or failed.

   (v) Determine the probability that the overall testing time is greater than 1 hour.

2. The lifetime, T, of a component, in hours, has probability density function

$$f(t) = \begin{cases} \lambda^2 t e^{-\lambda t} & t > 0 \\ 0 & \text{otherwise} . \end{cases}$$

(i) Determine the reliability function, and show that the hazard function is given by

$$z(t) = \frac{\lambda^2 t}{1 + \lambda t} .$$

(ii) A system is made up of n such components, operating independently, which operates if at least r components are operating.

(a) Determine an expression for the system reliability at 1 hour.

(b) If $\lambda = 0.4$ and $r = n$, what is the largest number of components that may be installed if the system reliability at 1 hour is required to be at least 0.5.

(c) If $\lambda = 0.4$ and $r = n - 1$, determine the system reliability at 1 hour when the system is made up of the number of components calculated in part (b).

Comment briefly on the reliability change under these less stringent requirements.

END OF PAPER

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PAPER I.S.E. 2.6

QUESTION

SOLUTION 1A

$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)\cos\omega t \, dt \; - i \int_{-\infty}^{\infty} f(t)\sin\omega t \, dt$

where the second $\int$ is 0 since $f(t)\sin\omega t$ is an odd function of t. 3

$\hat{h}(\omega) = \int_{-\infty}^{\infty} h(t)e^{-i\omega t}\,dt = \int_{-1}^{1} e^{-i\omega t}\,dt$

$= \left[\frac{e^{-i\omega t}}{-i\omega}\right]_{t=-1}^{1} = \frac{e^{-i\omega}}{-i\omega} - \frac{e^{i\omega}}{-i\omega} = 2\,\frac{\sin\omega}{\omega}$. 3

$\frac{1}{2}g(t)$ has F.T. $\frac{1}{1+\omega^2}$ so by Symmetry formula

$\frac{1}{1+t^2}$ has F.T $2\pi\,\frac{1}{2}g(-\omega) = \pi e^{-|\omega|}$. 3

Similarly $\frac{1}{2}h(t)$ has F.T. $\frac{\sin\omega}{\omega}$, so

$\frac{\sin t}{t}$ has F.T. $2\pi\,\frac{1}{2}h(-\omega) = \pi h(\omega)$. 2 Seen

$\int_{-\infty}^{\infty} e^{-|t-u|}\,\frac{\sin u}{u}\,du$ has F.T. $\hat{g}(\omega).\left(\text{F.T. of } \frac{\sin t}{t}\right)$ Similar Seen

$= \frac{2}{1+\omega^2}\,\pi h(\omega)$.

$\therefore$ Inversion gives

last integral $= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2}{1+\omega^2}\,\pi h(\omega)\,e^{+i\omega t}\,d\omega$,

and putting $t=0$ gives

$\int_{-\infty}^{\infty} e^{-|u|}\,\frac{\sin u}{u}\,du = \int_{-\infty}^{\infty}\frac{1}{1+\omega^2}h(\omega)\,d\omega$

$= \int_{-1}^{1}\frac{d\omega}{1+\omega^2} = \left[\tan^{-1}\omega\right]_{-1}^{1} = \frac{\pi}{2}$. 6

From Parseval's formula,

$\int_{-\infty}^{\infty}\left(\frac{\sin t}{t}\right)^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\text{F.T. of }\left(\frac{\sin t}{t}\right)^2\right|^2 d\omega$

$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |\pi h(\omega)|^2 d\omega = \frac{1}{2\pi}\int_{-1}^{1}\pi^2 d\omega = \pi$. 3

Setter : RIDLER-ROWE

Setter's signature : CJRR

Checker : HERBERT

Checker's signature : D M Hobert

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QUESTION

SOLUTION 2A

Unseen

(i) Taking Laplace transforms in the given equation (using tables for sine and cosine),

$$\bar{g}(p) = \frac{1}{p^2+1} + 2\bar{g}(p)\frac{p}{p^2+1}. \quad 3$$

$$\bar{g}(p)\frac{p^2+1-2p}{p^2+1} = \frac{1}{p^2+1}. \quad \therefore \bar{g}(p) = \frac{1}{(p-1)^2}. \quad 2$$

$$\therefore \text{ (from tables) } \quad g(t) = te^t. \quad 3$$

(8)

(ii) (a) Put $x = 2t$, $y = t$ for $0 \le t \le 1$. 1

$$I = \int_{t=0}^{1}\{(4t + te^{2t^2})2 + (2te^{2t^2} - 3t^2)\}\,dt \quad 2$$

$$= \int_0^1 (8t + 4te^{2t^2} - 3t^2)\,dt$$

$$= \left[4t^2 + e^{2t^2} - t^3\right]_0^1$$

$$= 4 + e^2 - 1 - 1 = 2 + e^2. \quad 2$$

(5)

(b) $P = \frac{\partial F}{\partial x} = 2x + ye^{xy}$, $Q = \frac{\partial F}{\partial y} = xe^{xy} - 3y^2$.

$$\therefore F = \int P\,dx = x^2 + e^{xy} + h(y). \quad 2$$

$$Q = \frac{\partial F}{\partial y} = xe^{xy} + \frac{dh}{dy}. \quad \therefore \frac{dh}{dy} = -3y^2.$$

$\therefore$ Take $h = -y^3$ and potential $F = x^2 + e^{xy} - y^3$. 3

$$\therefore I = F(\text{at finish }(2,1)) - F(\text{at start }(0,0))$$

$$= 4 + e^2 - 1 - 1$$

$$= 2 + e^2. \quad 2$$

(7)

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Checker : HERBERT Checker's signature : JM Herbert

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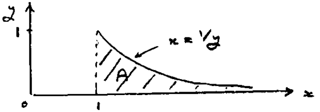
PAPER I.S.E. 2.6

QUESTION

SOLUTION 3A

All Unseen

(i)



2

$$\int_0^1 dy \int_1^{1/y} x^{-1} e^{-xy}\, dx = \iint_A x^{-1} e^{-xy}\, dx\, dy$$

$$= \int_{x=1}^{\infty} dx \int_{y=0}^{1/x} x^{-1} e^{-xy}\, dy$$ 2

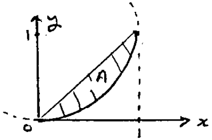
$$= \int_1^{\infty} dx \left[x^{-1}.(-x^{-1}) e^{-xy} \right]_{y=0}^{1/x}$$ 2

$$= \int_1^{\infty} x^{-2}(1-e^{-1})\, dx$$ 1

$$= (1-e^{-1})\left[-\tfrac{1}{x}\right]_1^{\infty} = 1-e^{-1}.$$ 1

(8)

(ii) $y = 1-\sqrt{1-x^2}$ gives $x^2+(y-1)^2 = 1$ i.e. circle centre $(0,1)$ radius 1.



3 for diagram

For $x = r\cos\theta$, $y = r\sin\theta$,

$x^2+(y-1)^2=1$ gives $r^2-2r\sin\theta = 0$ so $r = 2\sin\theta$.

$$\therefore \text{Integral} = \iint_A \frac{r\cos\theta}{r}\, r\,dr\,d\theta$$ 2

$$= \int_0^{\pi/4} d\theta \int_0^{2\sin\theta} r\cos\theta\, dr$$ 3 (limits)

$$= \int_0^{\pi/4} \left[\tfrac{1}{2} r^2 \cos\theta\right]_{r=0}^{2\sin\theta}$$ 2

$$= \int_0^{\pi/4} 2\sin^2\theta\cos\theta\, d\theta$$

$$= \left[\tfrac{2}{3}\sin^3\theta\right]_0^{\pi/4} = 1/3\sqrt{2}.$$ 2

(12)

Setter : RIDLER-ROWE Setter's signature : CRR

Checker : HERBERT Checker's signature : DM Hobert.

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PAPER ISE 2.6

QUESTION

SOLUTION 4A

Unseen

$z^4+1=0$ for $z = e^{i\pi/4 + in\pi/2}$, $n = 0,1,2,3$.

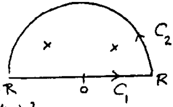
Simple zeros.

$\therefore \mathrm{Res}(f, e^{i\pi/4}) = \left.\dfrac{1}{\frac{d}{dz}(z^4+1)}\right|_{e^{i\pi/4}} = \dfrac{e^{-i3\pi/4}}{4} = \frac{1}{4}\left(\frac{-1}{\sqrt2} - \frac{i}{\sqrt2}\right)$, 3

$\mathrm{Res}(f, e^{3i\pi/4}) = \left.\dfrac{1}{\frac{d}{dz}(z^4+1)}\right|_{e^{3\pi i/4}} = \dfrac{e^{-i9\pi/4}}{4} = \frac{1}{4}\left(\frac{1}{\sqrt2} - \frac{i}{\sqrt2}\right)$. 2

By the Residue Theorem,

$\int_C f(z)\,dz = 2\pi i \times \{$sum of residues of f at poles inside $C\}$



$= 2\pi i\{\mathrm{Res}(f, e^{i\pi/4}) + \mathrm{Res}(f, e^{i3\pi/4})\}$

$= 2\pi i \, \frac{1}{4}\left(-\frac{i}{\sqrt2}\right) 2 = \pi/\sqrt2$. 5

For z on C_2, $|z^4+1| \geq |z^4| - 1 = R^4 - 1$

so $|f(z)| \leq \dfrac{1}{R^4-1}$.

$\therefore \left|\int_{C_2} f(z)\,dz\right| \leq \dfrac{1}{R^4-1} \times (\text{length } C_2) = \dfrac{\pi R}{R^4-1}$ $\quad\cdots(*)$ 5

$\int_C f = \int_{C_1} f + \int_{C_2} f$.

$\therefore \int_{C_1} f = \int_{-R}^{R} \dfrac{dx}{x^4+1} = \int_C f - \int_{C_2} f$

$\to \dfrac{\pi}{\sqrt2}$ as $R \to \infty$

since $\int_{C_2} f \to 0$ from $(*)$, giving result. 5

Setter : RIDLER-ROWE — Setter's signature : CJRR.

Checker : DR R BEAN(?) — Checker's signature : [signature]

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PAPER ISE 2.6

QUESTION

SOLUTION B1 (1 of 2

Let TF_1 = event initial test fails.

E = event installation errors present

$P(TF_1' | E') = 1$ $\quad P(TF_1 | E) = 0.8$

$P(TF_1 | E') = 0$ $\quad P(TF_1' | E) = 0.2$ 2

$P(E) = 0.1$ $\quad P(E') = 0.9$

(i) $P(\text{rejected}) = P(TF_1) = P(TF_1|E)P(E) + P(TF_1|E')P(E')$

$= 0.8 \times 0.1 + 0 \times 0.9$ 4

$= \underline{\underline{0.08}}$

(ii) $P(E | TF_1') = \dfrac{P(TF_1'|E)P(E)}{P(TF_1')} = \dfrac{0.2 \times 0.1}{0.92}$ 3

$= \underline{\underline{0.0217}}$

(iii) $P(TF_2' | E') = 1$ $\quad P(TF_2 | E) = 0.99$

$P(TF_2 | E') = 0$ $\quad P(TF_2' | E) = 0.01$

$P(\text{accepted} | E) = P(TF_1' \cap TF_2' | E)$

$= P(TF_1' | E) P(TF_2' | E)$

$= 0.2 \times 0.01$ 4

$= \underline{\underline{0.002}}$

Setter : E. J. McCoy

Checker : STEPHENS

Setter's signature : Emma McCoy

Checker's signature :

iv) If the first test fails, total testing time $T = T_1$

So $\underline{T \sim N(10,2)}$

If the first test is passed, $T = T_1 + T_2$

$E(T) = E(T_1) + E(T_2) = 10 + 50 = 60$

$Var(T) = var(T_1) + var(T_2) = 12$

So $\underline{T \sim N(60,12)}$

v) $P(T>60) = P(T>60|TF_1)P(TF_1) + P(T>60|TF_1')P(TF_1')$

$= P\left(\frac{T-10}{\sqrt{2}} > \frac{60-10}{\sqrt{2}}\right) \times 0.08 +$

$P\left(\frac{T-60}{\sqrt{12}} > \frac{60-60}{\sqrt{12}}\right) \times 0.99$

$= \left[1 - \Phi\left(\frac{50}{\sqrt{2}}\right)\right] \times 0.08 + \frac{1}{2} \times 0.99$

$= \underline{0.495}$

Setter : E.J. McCOY
Checker : STEPHENS.

Setter's signature : Emma McCoy
Checker's signature :

i) Reliability $R(t) = \int_t^\infty \lambda^2 t e^{-\lambda t} dt$ $\quad u = t \quad \frac{dv}{dt} = \lambda e^{-\lambda t}$

$\frac{du}{dt} = 1 \quad v = -e^{-\lambda t}$

$$R(t) = \lambda \left\{ [-te^{-\lambda t}]_t^\infty + \int_t^\infty e^{-\lambda t} dt \right\}$$

$$= \lambda t e^{-\lambda t} + [-e^{-\lambda t}]_t^\infty$$

$$= \lambda t e^{-\lambda t} + e^{-\lambda t}$$

$$= \underline{\underline{e^{-\lambda t}(1+\lambda t)}}$$ 4

Hazard $z(t) = \frac{f(t)}{R(t)} = \frac{\lambda^2 t e^{-\lambda t}}{e^{-\lambda t}(1+\lambda t)} = \underline{\underline{\frac{\lambda^2 t}{1+\lambda t}}}$ 2

ii) Let $R_s(t)$ = system reliability

Z = number of components operating at time t.

Then $Z \sim Bin(n,p)$ where $p = P(\text{comp. operating at time } t)$
$= P(T > t) = R(t)$ 2

(a)
$$R_s(t) = P(Z \geq r) = \sum_{j=r}^{n} \binom{n}{j} p^j (1-p)^{n-j}$$

$$= \underline{\underline{\sum_{j=r}^{n} \binom{n}{j} e^{-\lambda t j}(1+\lambda t)^j (1 - e^{-\lambda t}(1+\lambda t))^{n-j}}}$$ 3

$$R_s(1) = \underline{\underline{\sum_{j=r}^{n} \binom{n}{j} e^{-\lambda j}(1+\lambda)^j (1 - e^{-\lambda}(1+\lambda))^{n-j}}}$$

Setter : E.J. McCoy Setter's signature : Emma McCoy
Checker : Stephens Checker's signature : [signature]

(b) $\lambda = 0.4$, $r = n$, $t = 1$

$$R_S(t) = e^{-0.4n}(1+0.4)^n$$
$$= e^{-0.4n}(1.4)^n$$

want $R_S(t) \geq 0.5 \Rightarrow e^{-0.4n}(1.4)^n \geq 0.5$

$$\Rightarrow -0.4n + n\ln(1.4) \geq \ln(0.5)$$

$$\Rightarrow n \leq \frac{\ln(0.5)}{\ln(1.4) - 0.4} = 10.91$$

$\therefore$ largest number of components is $\underline{10}$. 5

(c) $\lambda = 0.4$, $r = n-1$, $t = 1$

$$R_S(t) = \sum_{j=9}^{10} \binom{10}{j} e^{-0.4j}(1.4)^j \left(1 - e^{-0.4}(1.4)\right)^{10-j}$$

$$= 10e^{-3.6}(1.4)^9(1 - 1.4e^{-0.4}) + e^{-4}(1.4)^{10}$$ 3

$$= \underline{\underline{0.877}}$$

Large increase in reliability by allowing 1 of the 10 components to fail. 1