

Estimation + Fault Detection, 2008 Exam: Model Answers

1. (a) The discrete time process satisfies

$$x_k = x(kh) = e^{\tilde{F}h} x[(k-1)h] + \int_{(k-1)h}^{kh} e^{\tilde{F}(kh-\sigma)} \tilde{b} w(\sigma) d\sigma$$

where $\tilde{F} = \begin{bmatrix} -\alpha_1 & 0 \\ 0 & -\alpha_2 \end{bmatrix}$, $\tilde{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

By properties of the stochastic integral, the v_k 's are zero mean + indep.

Also, $\text{cov}\{v_k\} = \int_0^h (e^{\sigma \tilde{F}} \tilde{b}) (e^{\sigma \tilde{F}} \tilde{b})^T d\sigma$ ($\sigma = kh - \sigma'$)

We have

$$F = \begin{bmatrix} e^{-\alpha_1 h} & 0 \\ 0 & e^{-\alpha_2 h} \end{bmatrix} \text{ and } Q = \int_0^h \begin{bmatrix} e^{-\alpha_1 \sigma} & 0 \\ 0 & e^{-\alpha_2 \sigma} \end{bmatrix} \begin{bmatrix} e^{-\alpha_1 \sigma} & e^{-\alpha_2 \sigma} \end{bmatrix} d\sigma = \begin{bmatrix} \frac{1}{2\alpha_1} (1 - e^{-2\alpha_1 h}) & \frac{1}{\alpha_1 + \alpha_2} (1 - e^{-(\alpha_1 + \alpha_2)h}) \\ \frac{1}{\alpha_1 + \alpha_2} (1 - e^{-(\alpha_1 + \alpha_2)h}) & \frac{1}{2\alpha_2} (1 - e^{-2\alpha_2 h}) \end{bmatrix}$$

[8] and Q is as given by (*)

(b) $x_k = F x_{k-1} + v_k$, so $E\{x_k x_k^T\} = E\{F x_{k-1} + v_k\} E\{F x_{k-1} + v_k\}^T$

Since x_{k-1} depends on past v_k 's, x_{k-1} and v_k are indep., so

$$R = E\{x_k x_k^T\} = F E\{x_{k-1} x_{k-1}^T\} F^T + 0 + Q = F R F^T + Q$$

[4] So $R = F R F^T + Q$

Inserting calculated values for F and Q gives

$$\begin{bmatrix} r_{11} & r_{12} \\ r_{12} & r_{22} \end{bmatrix} = \begin{bmatrix} e^{-\alpha_1 h} & 0 \\ 0 & e^{-\alpha_2 h} \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} \\ r_{12} & r_{22} \end{bmatrix} \begin{bmatrix} e^{-\alpha_1 h} & 0 \\ 0 & e^{-\alpha_2 h} \end{bmatrix} + \begin{bmatrix} \frac{1}{2\alpha_1} (1 - e^{-2\alpha_1 h}) & \frac{1}{\alpha_1 + \alpha_2} (1 - e^{-(\alpha_1 + \alpha_2)h}) \\ \frac{1}{\alpha_1 + \alpha_2} (1 - e^{-(\alpha_1 + \alpha_2)h}) & \frac{1}{2\alpha_2} (1 - e^{-2\alpha_2 h}) \end{bmatrix}$$

So

$$\begin{aligned} (1 - e^{-2\alpha_1 h}) r_{11} &= \frac{1}{2\alpha_1} (1 - e^{-2\alpha_1 h}) & \Rightarrow r_{11} &= \frac{1}{2\alpha_1} \\ (1 - e^{-(\alpha_1 + \alpha_2)h}) r_{12} &= \frac{1}{\alpha_1 + \alpha_2} (1 - e^{-(\alpha_1 + \alpha_2)h}) & r_{12} &= \frac{1}{\alpha_1 + \alpha_2} \\ (1 - e^{-2\alpha_2 h}) r_{22} &= \frac{1}{2\alpha_2} (1 - e^{-2\alpha_2 h}) & r_{22} &= \frac{1}{2\alpha_2} \end{aligned}$$

So $R = \begin{bmatrix} \frac{1}{2\alpha_1} & \frac{1}{\alpha_1 + \alpha_2} \\ \frac{1}{\alpha_1 + \alpha_2} & \frac{1}{2\alpha_2} \end{bmatrix}$

[4]

covariance of unsampled process

[2] R is independent of h because $E\{x_k x_k^T\} = E\{x(t) x(t)^T\}$

(c) The square of the corr. $f^h = \rho(x_k^1, x_k^2)^2 = \frac{r_{12}^2}{r_{11} \times r_{22}}$

$$[2] = \frac{\left(\frac{1}{\alpha_1 + \alpha_2}\right)^2}{\frac{1}{2\alpha_1} \times \frac{1}{2\alpha_2}} = \frac{\alpha_2 / \alpha_1}{(1 + \alpha_2 / \alpha_1)^2} = \frac{\delta}{(1 + \delta)^2} = \frac{100}{(101)^2} < 0.01$$

So $|\rho(x_k^1, x_k^2)| \leq 0.1$

2(a) $m_x = 0$ and $m_y = E[x + n + b] = 0 + 0 + P \cdot 1 = P$

$$E[x(y - m_y)] = E[x^2 + xn + x(b - P)] = \sigma_x^2 + 0 + 0$$

$$E[(y - m_y)^2] = E[(x + n + (b - P))^2] = \sigma_x^2 + \sigma_n^2 + (1 - P)P$$

It follows

$$\hat{x}_{LS} = K(y - \alpha) \quad \text{where } K = \frac{\sigma_x^2}{\sigma_x^2 + \sigma_n^2 + (1 - P)P}, \alpha = P.$$

and

$$[8] \quad E[\|x - \hat{x}_{LS}\|^2] = \sigma_x^2 \left[1 - \frac{\sigma_x^2}{\sigma_x^2 + \sigma_n^2 + (1 - P)P} \right]$$

(ii) By Bayes Rule, $p(b=1|y) = \frac{p(y|b=1)p(b=1)}{p(y|b=0)p(b=0) + p(y|b=1)p(b=1)}$

But $p(y|b=1) = N(1, \sigma_x^2 + \sigma_n^2)(y)$, $p(y|b=0) = N(0, \sigma_x^2 + \sigma_n^2)$,
(since n and b are independent).

$$\text{So } p(b=1|y) = \frac{PN(1, \sigma_x^2 + \sigma_n^2)(y)}{(1 - P)N(0, \sigma_x^2 + \sigma_n^2)(y) + PN(1, \sigma_x^2 + \sigma_n^2)(y)} = \alpha(y)$$

$$\text{Note: } \alpha(y) = \frac{P}{P + \exp\left\{-\frac{1}{2} \frac{(y - 1)^2}{\sigma_x^2 + \sigma_n^2}\right\}(1 - P)} \quad (1)$$

Then

$$p(x|y) = p(x|y, b=0)p(b=0|y) + p(x|y, b=1)p(b=1|y)$$

But $p(x|y, b=0)$ and $p(x|y, b=1)$ are obtained simply by solving the standard linear least squares problem with b a constant ($b=0$ or 1) i.e.

$$p(x|y, b=0) = N(K_1 y, \sigma_x^2 \left(1 - \frac{\sigma_x^2}{\sigma_x^2 + \sigma_n^2}\right))$$

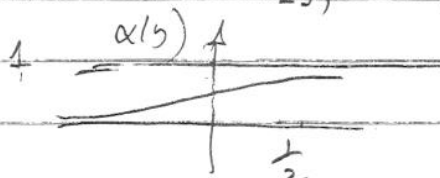
and

$$p(x|y, b=1) = N(K_1(y - 1), \sigma_x^2 \left(1 - \frac{\sigma_x^2}{\sigma_x^2 + \sigma_n^2}\right)) \quad \text{where } K = \frac{\sigma_x^2}{\sigma_x^2 + \sigma_n^2}$$

It follows from (1)

$$[8] \quad \hat{x}_{NLS} = (1 - \alpha(y)) K_1 y + \alpha(y) K_1 (y - 1) \\ = \underline{K_1 (y - \alpha(y))}$$

Since $K_1 > K_{LS}$, $\hat{x}_{NLS} \begin{cases} > x_{LS} & \text{for } y \text{ large (+ve)} \\ < x_{LS} & \text{for } y \text{ large (-ve)} \end{cases}$



The offset monotonically increases

[2] for 0 to 1, and is $\frac{P}{1+P}$ when $y = \frac{1}{2}$.

3.

Denote by $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}$ the covariance of $\begin{bmatrix} x \\ y \end{bmatrix}$.

We calculate, using $y = x \cos \theta + n$ (with $\cos \theta = c$)

$E[x] = E[y] = 0$, and $P_{22} = c^2 \text{Var}\{x\} + \text{Var}\{n\} = 1 + c^2$. Also

$$P_{12} = c \quad \text{and} \quad P_{11} = 1. \quad \left(P = \begin{bmatrix} 1 & c \\ c & 1+c^2 \end{bmatrix} \right)$$

$$[8] \quad \text{So, } p(x, y) = \frac{1}{1 \times (2\pi)} \times \exp \left\{ -\frac{1}{2} \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & c \\ c & 1+c^2 \end{bmatrix}^{-1} \begin{bmatrix} x \\ y \end{bmatrix} \right\}$$

$$\text{But } \begin{bmatrix} 1 & c \\ c & 1+c^2 \end{bmatrix}^{-1} = \begin{bmatrix} 1+c^2 & -c \\ -c & 1 \end{bmatrix} \quad (\text{Note: } \det[\dots] = 1)$$

Also

$$p(y) = \frac{1}{\sqrt{2\pi}(1+c^2)^{1/2}} \exp \left\{ -\frac{1}{2} \frac{y^2}{1+c^2} \right\}$$

So

$$p(x|y) = \frac{\sqrt{1+c^2}}{(2\pi)^{1/2}} \exp \left\{ -\frac{1}{2} \left[(1+c^2)x^2 - 2cx y + y^2 - \frac{y^2}{(1+c^2)} \right] \right\}$$

Factorize:

$$(1+c^2) \left[x^2 - \frac{2c}{(1+c^2)} x y + \frac{c^2}{(1+c^2)^2} y^2 \right]$$

$$= (1+c^2) \left[x - \frac{c}{1+c^2} y \right]^2$$

$$[6] \quad \text{So, } p(x|y) = \frac{1}{(2\pi\sigma^2)^{1/2}} \exp \left\{ -\frac{1}{2} \frac{|x - m|^2}{\sigma^2} \right\}$$

$$\text{where } m = \frac{\cos \alpha}{1 + \cos^2 \alpha} y \quad \text{and} \quad \sigma^2 = \frac{1}{1 + \cos^2 \alpha}$$

$$[4] \quad \text{So } E[x|y] = \frac{\cos \alpha}{1 + \cos^2 \alpha} y \quad \text{and} \quad \text{var}\{x|y\} = \frac{1}{1 + \cos^2 \alpha} \quad (=:\sigma^2)$$

We require that $P[|x - E[x|y]| > 0.8] \leq 0.2$

Since (from data) $P[|x - E[x|y]| > 0.9\sigma] = 0.2$

We must have $0.9\sigma \leq 0.8$,

$$\text{i.e. } \sqrt{\frac{1}{1 + \cos^2 \alpha}} \leq \frac{0.8}{0.9} \quad \text{or} \quad \cos^2 \alpha > 0.2656$$

$$[2] \quad \text{or} \quad \alpha \leq \cos^{-1} \sqrt{0.2656}$$

4(a) We know $x_k = F^k x_0$, so $y_k = x_k + v_k = F^k x_0 + v_k$. Since x_0 and v_0 are independent, and v_k is independent of $y_{1:k-1}$, and finally $v_k \sim N(0, Q)$, we have

$$[3] \quad p(y_k | y_{1:k-1}, x_0) = p(y_k | x_0) = N(F^k x_0, Q)(y_k)$$

(b) Insert $p(x_0 | y_{1:k}) = N(\hat{x}_{0|k}, P_{0|k})$ etc into Bayes rule gives

$$-\frac{1}{2} (x_0 - \hat{x}_{0|k})^T P_{0|k}^{-1} (x_0 - \hat{x}_{0|k}) = -\frac{1}{2} (y_k - F^k x_0)^T Q^{-1} (y_k - F^k x_0) - \frac{1}{2} (x_0 - \hat{x}_{0|k-1})^T P_{0|k-1}^{-1} (x_0 - \hat{x}_{0|k-1}) + (\text{term that depends only on } y_{1:k})$$

Expand this equation (and multiply by $-\frac{1}{2}$) \Rightarrow

$$x_0^T \left[P_{0|k}^{-1} - (F^k)^T Q^{-1} F^k - P_{0|k-1}^{-1} \right] x_0 - 2 x_0^T \left[P_{0|k}^{-1} \hat{x}_{0|k} - (F^k)^T Q^{-1} y_k - P_{0|k-1}^{-1} \hat{x}_{0|k-1} \right] + (\text{term independent of } x_0) = 0$$

Since this equation is valid for all x_0 ,

$$[6] \quad P_{0|k}^{-1} = P_{0|k-1}^{-1} + (F^k)^T Q^{-1} F^k \quad (*)$$

and

$$P_{0|k}^{-1} \hat{x}_{0|k} = P_{0|k-1}^{-1} \hat{x}_{0|k-1} + (F^k)^T Q^{-1} y_k$$

[7] [8]

(c) Specialize to scalar case, and set $Q=1$, $F=\sqrt{0.5}$. Then, by (*),

$$P_{0|k}^{-1} = P_{0|0}^{-1} + \sum_{j=1}^k a^j, \quad \text{where } a = 0.5$$

$$= P_{0|0}^{-1} + a \left[\sum_{j=0}^k a^j \right] \rightarrow P_{0|0}^{-1} + \frac{a}{1-a} \quad (\text{as } k \rightarrow \infty)$$

$$[4] \quad \text{So } P_{0|\infty}^{-1} = 1 + 0.5/0.5 = 2$$

$$\text{whence } P_{0|\infty} = \frac{1}{2}$$

5(a) Assume $X \sim \tilde{p}(x) = \alpha \delta(x+\sigma) + (1-2\alpha)\delta(x) + \alpha \delta(x-\sigma)$
 then $E[X] = \alpha(-\sigma) + (1-2\alpha) \times 0 + \alpha(\sigma) = 0$ $m_x = 0$ (\checkmark)
 and $\text{var}[X] = \sigma^2 \alpha + (1-2\alpha) \times 0^2 + \alpha \times \sigma^2 = 2\alpha \sigma^2$

[4] We require then $\alpha = \frac{1}{2}$ for $E[X^2] = \sigma^2$.

(b) $Y = X^3 + V$. Clearly Y has zero mean. So, to construct the least squares estimate we must calculate

$$E[XY] = E[X^4 + XV] = E[X^4] + E[X]E[V],$$

$$= \sigma^4 \times \frac{1}{2} + 0^2 \times 0 + \sigma^4 \times \frac{1}{2} = \sigma^4$$

$$\text{and } E[Y^2] = E[(X^3 + V)(X^3 + V)] = E[X^6] + 2E[V]E[X^3] + E[V^2]$$

$$= \sigma^6 \times \frac{1}{2} + \sigma^6 \times \frac{1}{2} + 0 + \sigma_n^2 = \sigma^6 + \sigma_n^2$$

It follows that the linear least squares estimate is

$$\hat{x} = ky$$

where

[10] $k = \frac{E[XY] E[Y^2]^{-1}}{1} = \frac{\sigma^4}{\sigma^6 + \sigma_n^2}$

(c) The approximate linear least squares estimate based on linearizing $h(x)$ about m_x is

$$\hat{x}' = k'y, \text{ when } k' = \frac{\sigma^2 H}{H^2 H + \sigma_n^2}$$

in which $H = h_x(m_x)$.

[4] But $h_x(m_x) = \frac{d}{dx} x^3 \Big|_{x=m_x=0} = 3x^2 \Big|_{x=0} = 0$.

So the approximate linear least squares estimate is simply

$$\hat{x}' = 0 \quad (= m_x)$$

[2] It is a bad estimate because it does not make use of the measurement

$$b(a) \quad y_k + 0.5 y_{k-1} = e_k + d e_k \quad \text{--- (1)}$$

So $E[(y_k + 0.5 y_{k-1})^2] = E[(e_k + d e_{k-1})^2]$. This implies

$$r_d(0) + r_d(1) + 0.25 r_d(0) = 1 + d^2 \quad \text{--- (2)}$$

Multiply across (1) by y_{k-1} and take expectations. This gives

$$E[y_k y_{k-1} + 0.5 y_{k-1}^2] = E[e_k y_{k-1} + d y_{k-1} e_{k-1}]$$

whence $r_d(1) + 0.5 r_d(0) = 0 + d E[y_k e_k]$.

Also, multiply across (1) by e_k and take $E[\dots]$. This gives

$$E[y_k e_k] + 0 = 1 + 0. \quad \text{So}$$

$$r_d(1) + 0.5 r_d(0) = d \quad \text{--- (3)}$$

From (2) and (3), $0.75 r_d(0) = 1 - d + d^2$. Hence

$$r_d(0) = \frac{4}{3} (1 - d + d^2) \quad [10]$$

$$(b) \quad \text{The likelihood function is } LR(z) = \frac{p(z|d=2)}{p(z|d=0)}$$

where $p(z|d=2) = N(0, 4)$ and $p(z|d=0) = N(0, 4/3)$

The log-likelihood function is

$$LLF(z) = \frac{1}{2} \left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2} \right) z^2 + \text{constant} \quad (\sigma_0^2 = 4/3, \sigma_1^2 = 4)$$

$$\text{So, } P_0[LLF(z) > c] = P_0[z^2 > c' := \left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right)^{-1} (c - \text{const.})]$$

We must choose c' (and hence c) such that

$$\text{Prob}[z^2 > c'] = 0.05 \quad (z \sim N(0, 3.4))$$

$$= \text{Prob}[z'^2 > \frac{3}{4} c'] = 0.05 \quad (z' \sim N(0, 1))$$

$$\text{From tables, } 3/4 c' = 3.84 \Rightarrow c' = \frac{4 \times 3.84}{3} = 5.12$$

$$\text{N-P test: } \begin{cases} \text{If } z^2 \leq 5.12 & \text{choose } H_0 \\ \text{If } z^2 > 5.12 & \text{choose } H_1 \end{cases} \quad [6]$$

Also

$$P_1[z^2 > 5.12] \quad (z \sim N(0, 4))$$

$$= \text{Prob}[(z')^2 > 5.12/4 = 1.28] = 0.26$$

We conclude that the power of the test is 0.26. (Poor test - will fail to detect many 'faults'!)

[4]