Digital Signal Processing

1. a) A discrete-time signal $x(n)$ is given by

$$x(n) = -2 + 2\cos\frac{n\pi}{4} + \cos\frac{n\pi}{2} + \frac{1}{2}\cos\frac{3n\pi}{4}.$$

i) Determine the period in samples of $x(n)$. [3]

ii) Determine $|X(k)|$, the magnitude spectrum of $x(n)$. [5]

iii) Draw a labelled sketch of $|X(k)|$. [3]

iv) Verify Parseval's relation for this case by computing the power in both the time and frequency domains. [3]

Solution:
The period $N = 8$ samples.
The sample values are as tabulated:

n	$x(n)$
0	1.5000
1	-0.9393
2	-3.0000
3	-3.0607
4	-3.5000
5	-3.0607
6	-3.0000
7	-0.9393

Then using $X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}$ and by setting $k = 0, 1, \ldots N-1$, the values of $X(k)$ are found to be

$$\begin{aligned} X(0) &= -16 \\ X(1) &= X(7) = 8 \\ X(2) &= X(6) = 4 \\ X(3) &= X(5) = 2 \\ X(4) &= 0 \end{aligned}$$

and since these values are all real $|X(k)| = X(k)$.

The labelled sketch must include axis labels and clearly sketched discrete values for $|X(k)|$.

The energy in the sequence over one period is given by

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2$$

by Parseval's relation. This is verified in this case with both terms evaluating to $6.625N$.
Note that the normalization factor N in the above equation may appear on the other side of the equality depending on the choice on normalization in the DFT.

b) Given a discrete-time signal $x(n)$ having Fourier transform

$$F\{x(n)\} = \frac{1}{1 - ae^{-j\omega}}$$

find the Fourier transforms of

i) $x(n+2)$, [2]

ii) $x(n) \circledast x(n-2)$, [2]

iii) $x(n) \circledast x(-n)$, [2]

where $\circledast$ represents circular convolution.

Solution:

$$
\begin{aligned}
X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x(n+2)e^{-jn\omega} \\
&= \sum_{m=-\infty}^{\infty} x(m)e^{-jm\omega}e^{j2\omega}, \quad \text{where} \quad m = n+2 \\
&= X(\omega)e^{j2\omega}.
\end{aligned}
$$

$$
\begin{aligned}
X(e^{j\omega}) &= X(e^{j\omega})X(e^{j\omega})e^{-j2\omega} \\
&= X^2(e^{j\omega})e^{-j2\omega}.
\end{aligned}
$$

$$
\begin{aligned}
X(e^{j\omega}) &= X(e^{j\omega})X(e^{-j\omega}) \\
&= \frac{1}{1-ae^{-j\omega}}\frac{1}{1-ae^{j\omega}} \\
&= \frac{1}{1-2a\cos\omega+a^2}.
\end{aligned}
$$

2. The bilinear transform describing a mapping between the s-plane and the z-plane can be written

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right).$$

a) Let z and s be denoted

$$z = re^{j\omega}$$
$$s = \sigma + j\Omega.$$

Explain the result of the bilinear transform on $s = \sigma + j\Omega$ for the cases of $\sigma < 0$, $\sigma = 0$ and $\sigma > 0$. Include illustrative labelled sketches of the s-plane and z-plane.

[5]

Solution:
$\sigma < 0$ maps to the inside of the unit circle in the z-plane.
$\sigma = 0$ maps to the the unit circle in the z-plane.
$\sigma > 0$ maps to the outside of the unit circle in the z-plane.

b) Explain what is meant by frequency warping in the context of the bilinear transform and write an expression for the frequency ω in terms of Ω.

[3]

Solution:

$$\omega = 2\tan^{-1}\frac{\Omega T}{2}.$$

c) Consider a continuous-time bandpass filter with system function

$$H(s) = \frac{(\Omega_u - \Omega_l)s}{s^2 + (\Omega_u - \Omega_l)s + \Omega_l\Omega_u}$$

where Ω_u and Ω_l are the upper and lower band edge frequencies respectively.

i) Apply the bilinear transform to convert $H(s)$ to a discrete-time IIR filter $H(z)$ with sampling period T s. (*Hint: Do not consider frequency warping.*) [6]

ii) Write out the difference equation for the filter's output $y(n)$ given the input signal $x(n)$. [4]

iii) Draw an illustrative labelled sketch of the magnitude frequency response of $H(z)$. [2]

Solution:

$$H(s) = \frac{(\Omega_u - \Omega_l)s}{s^2 + (\Omega_u - \Omega_l)s + \Omega_l\Omega_u}$$

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$H(z) = (\Omega_u - \Omega_l)\frac{\frac{2}{T}(1-z^{-1})(1+z^{-1})}{(\frac{2}{T})^2(1-z^{-1})^2 + (\Omega_u - \Omega_l)(\frac{2}{T})(1-z^{-1})(1+z^{-1}) + \Omega_u\Omega_l(1+z^{-1})^2}$$

$$= \frac{2(\alpha-\beta) - 2(\alpha-\beta)z^{-2}}{4 + 2(\alpha-\beta) + \alpha\beta - 2(4-\alpha\beta)z^{-1} + [4 - 2(\alpha-\beta) + \alpha\beta]z^{-2}}$$

with $\alpha = \Omega_u T$ and $\beta = \Omega_l T$.
The difference equation is then given by

$$y(n) = \frac{1}{4 + 2(\alpha-\beta) + \alpha\beta} \times$$
$$[2(\alpha-\beta)x(n) - 2(\alpha-\beta)x(n-2) + 2(4-\alpha\beta)y(n-1) - [4 - 2(\alpha-\beta) + \alpha\beta]y(n-2)]$$

Key points of the labelled sketch include the overall spectral shape, the d.c. gain and the band edges.

3. a) Consider a maximally decimated 2-band analysis filter bank directly connected in cascade to a corresponding synthesis filter bank.

i) Draw a labelled sketch of this analysis-synthesis filter bank employing analysis filters $H_0(z)$ and $H_1(z)$ and synthesis filters $F_0(z)$ and $F_1(z)$. Denote the input signal as $x(n)$ with z-transform $X(z)$, the subband signals as $y_0(n)$ and $y_1(n)$ with z-transforms $Y_0(z)$ and $Y_1(z)$ respectively, and the output of the synthesis filter bank as $\hat{x}(n)$ with z-transform $\hat{X}(z)$. [4]

ii) Derive expressions for $Y_0(z)$ and $Y_1(z)$ in terms of $X(z)$, $H_0(z)$ and $H_1(z)$. [4]

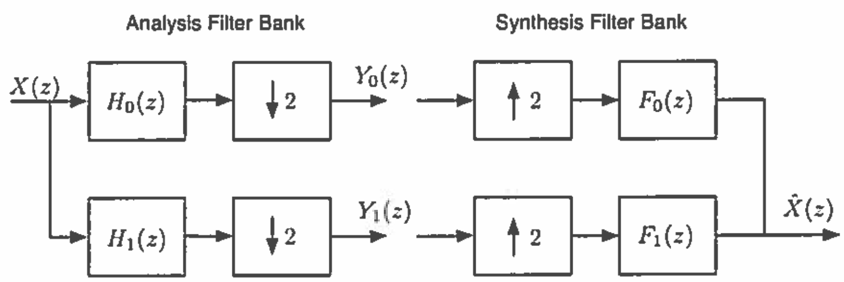
iii) Derive an expression for $\hat{X}(z)$ in terms of $X(z)$, $H_0(z)$, $H_1(z)$, $F_0(z)$ and $F_1(z)$. [4]

iv) Show that the expression for $\hat{X}(z)$ can be written in matrix form including the matrix term [2]

$$\mathbf{F} = \begin{bmatrix} F_0(z) \\ F_1(z) \end{bmatrix}.$$

Solution

The analysis filter bank has the form



The expressions follow as:

$$X_k(z) = H_k(z)X(z) \quad k = 0,1$$
$$Y_k(z) = \frac{1}{2}\left(X_k(z^{\frac{1}{2}}) + X_k(-z^{\frac{1}{2}})\right) \quad k = 1,2$$
$$\hat{X}(z) = \frac{1}{2}(H_0(z)F_0(z) + H_1(z)F_1(z))X(z) + \frac{1}{2}(H_0(-z)F_0(z) + H_1(-z)F_1(z))X(-z)$$
$$= \frac{1}{2}\begin{bmatrix} X(z) & X(-z) \end{bmatrix}\begin{bmatrix} H_0(z) & H_1(z) \\ H_0(-z) & H_1(-z) \end{bmatrix}\begin{bmatrix} F_0(z) \\ F_1(z) \end{bmatrix}$$
$$= \frac{1}{2}\begin{bmatrix} X(z) & X(-z) \end{bmatrix}\begin{bmatrix} H_0(z) & H_1(z) \\ H_0(-z) & H_1(-z) \end{bmatrix}\mathbf{F}(z).$$

b) Consider the system shown in Fig. 3.1 for which the input signal $x(n)$ has the spectrum shown in Fig. 3.2 and $H_B(z)$ is a bandpass filter with magnitude frequency response shown in Fig. 3.3. Draw a labelled sketch of the spectrum of the signal $y(m)$ and explain your answer. [6]

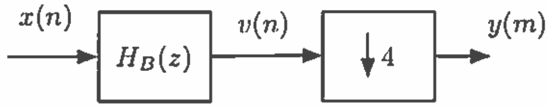


Figure 3.1 Multirate system

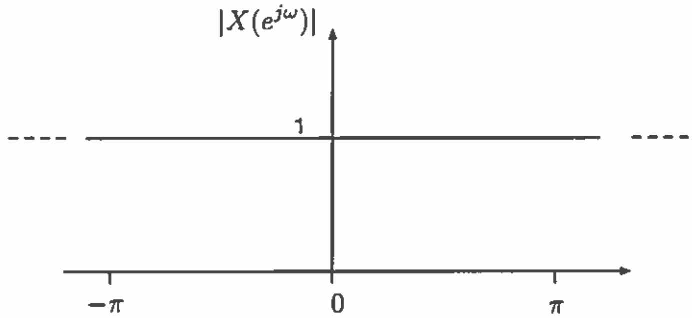


Figure 3.2 Input signal magnitude spectrum

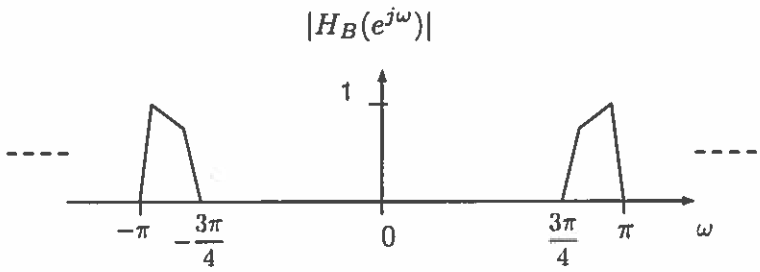
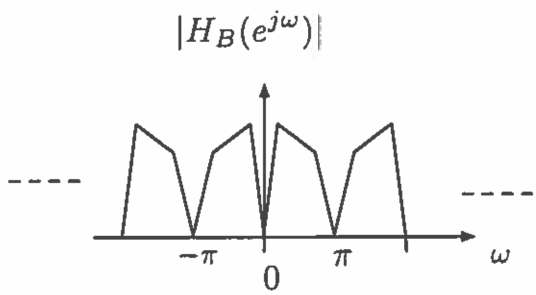


Figure 3.3 Filter magnitude frequency response

Solution

The resulting spectrum has the form of



4. The Discrete Fourier Transform can be written

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk} \quad k = 0, 1, \ldots, N-1.$$

a) Show that

i) $W_N^{k+N/2} = -W_N^k$

ii) $W_N^{k+N} = W_N^k$ [4]

Solution:
W_N is the complex exponential $e^{-j2\pi/N}$. The two properties can be shown by expanding the complex exponentials into trigonometric forms.

b) i) Derive the 4-point Radix-2 Decimation-in-Time FFT algorithm and draw the signal flow graph. [7]

ii) Write a clear explanation of the terms *Radix-2* and *Decimation-in-Time* in this context. [2]

iii) Determine the number of real multiply operations required to compute the 8-point Radix-2 Decimation-in-Time FFT. Ignore multiplications by $0, +1$ and -1. [3]

Solution:
Starting from the definition of the DFT we can then expand the DFT specifically for 2 points to obtain

$$X(0) = x(0) + x(1)$$
$$X(1) = x(0) - x(1).$$

For the case of $N = 4$ we obtain

$$X(k) = \sum_{n=0}^{3} x(n) W_4^{nk}.$$

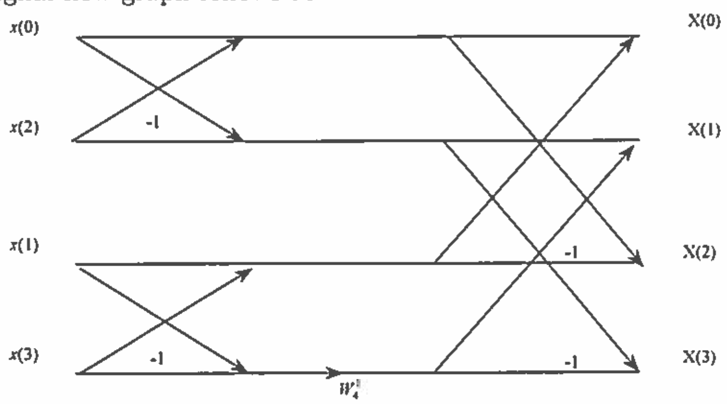
The derivation continues by performing decimation-in-time and employing symmetry properties of W (which should be shown explicitly) and leads to

$$X(k) = X_e(k) + W_4^k X_o(k) \quad k = 0, 1, 2, 3$$

where the subscripts e and o indicate even and odd indexed sub-sequences respectively. Hence the 4-point DFT can be written as two 2-point DFTs. The last stage of the derivation is to formulate the recombination equations as

$$X(0) = X_e(0) + X_o(0)$$
$$X(1) = X_e(1) + W_4^1 X_o(1)$$
$$X(2) = X_e(0) - X_o(0)$$
$$X(3) = X_e(1) - W_4^1 X_o(1)$$

The signal flow graph follows as



The terms *Radix-2* and *Decimation-in-Time* refer to decomposition of the FFT using structures based on 2-point DFTs, and decimation in time refers to this decomposition occurring in the time domain as opposed to the frequency domain, and involves also re-ordering of the input samples. The number of real multiplication operations is 4x (or 3x with Karatsuba method) the number of complex multiplies - in general it is right to assume complex operations in the DFT. The 8-point DIT FFT requires 4x 2-point DFTs + 2x recombination equations from 2-point to 4-point plus 1x recombination equations from 4-point to 8-point.

c) For a discrete-time signal $x(n)$ of length N samples with DFT $X(k)$, con-

sider a new sequence $y(n)$ of length $2N$ such that

$$y(n) = \begin{cases} x(n/2), & \text{for} \, n \, \text{even} \\ 0, & \text{for} \, n \, \text{odd.} \end{cases}$$

Find an expression for the DFT of $y(n)$ in terms of $X(k)$. [4]

Solution

$$\begin{aligned} Y(k) &= \sum_{n=0}^{2N-1} y(n) W_N^{nk} \quad k = 0, 1, \ldots 2N-1 \\ &= \sum_{n=0}^{2N-1} y(n) W_{2N}^{nk} \quad n \, \text{even} \quad \mathrm{k} = 0, 1, \ldots 2\mathrm{N}-1 \\ &= \sum_{m=0}^{N-1} y(2m) W_N^{mk} = \sum_{m=0}^{N-1} x(n) W_N^{mk} \\ &= \begin{cases} X(k) & k = 0, 1, \ldots N-1 \\ X(k-N) & k = N, N+1, \ldots 2N-1. \end{cases} \end{aligned}$$