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DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING  
EXAMINATIONS 2012

MSc and EEE/ISE PART III/IV: MEng, BEng and ACGI

Corrected Copy

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**MATHEMATICS FOR SIGNALS AND SYSTEMS**

Friday, 4 May 2:30 pm

Time allowed: 3:00 hours

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**There are THREE questions on this paper.**

**Answer ALL questions. All questions carry equal marks.**

15:20 → correction (page 4/3 question 1c)

**Any special instructions for invigilators and information for candidates are on page 1.**

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# MATHEMATICS FOR SIGNAL AND SYSTEMS

1. We consider the following systems of linear equations

$$\begin{aligned} 2x_1 - x_2 &= 3 \\ -x_1 + 2x_2 - x_3 &= -5 \\ -x_2 + 2x_3 &= 5 \end{aligned} \quad (1.1)$$

- a) i) Write the system (1.1) in matrix form, i.e.  $Ax = y$  where  $A \in \mathbb{R}^{3 \times 3}$  and  $x, y \in \mathbb{R}^3$ . [ 1 ]
- ii) Compute the determinant of  $A$ . [ 1 ]
- iii) Determine  $x^*$  the solution of the system (1.1) and justify that it is the unique solution of the system (1.1). [ 1 ]

b) We now study the matrix  $J = \begin{pmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1/2 & 0 \end{pmatrix}$ .

- i) Write the system (1.1) in the form  $x = Jx + z$  where  $x, z \in \mathbb{R}^3$ . [ 2 ]
- ii) Find an orthogonal matrix  $P$ , i.e.  $P^T P = I$  where  $I$  is the identity matrix, such that  $J = PDP^T$  where

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & -1/\sqrt{2} \end{pmatrix}.$$

- iii) Compute  $J^k$  the  $k$ th power of  $J$  for all non-negative integers  $k$ . [ 2 ]
- Hint: Distinguish odd and even values of  $k$ .* [ 2 ]

c) Let  $x^{(0)} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$  and define the sequence of vectors  $x^{(0)}, x^{(1)}, \dots$  as follows. For  $k \geq 0$ ,

$$x^{(k+1)} = Jx^{(k)} + z,$$

where  $z$  is defined in b) i) and let  $\delta^{(k)} = x^{(k)} - x^*$ , where  $x^*$  is defined in a) iii). [ 1 ]

- i) Compute  $x^{(1)}$  and  $\delta^{(1)}$ . [ 1 ]
- ii) Show that  $\delta^{(k)} = PD^k P^T \delta^{(0)}$ ,  $P$  and  $D$  defined in b) ii). [ 1 ]
- iii) Show that  $\|D^k x\| \leq \frac{1}{2^{k/2}} \|x\|$  for all  $x \in \mathbb{R}^3$ . [ 3 ]
- iv) Show that for  $U$  an orthogonal matrix  $\|Ux\| = \|x\|$ . [ 1 ]
- v) Show that  $\|\delta^{(k)}\| \leq \sqrt{\frac{13}{2^k}}$ . [ 3 ]
- vi) Show that, for  $k \geq k_0$  where  $k_0 = \frac{\log(13) + 6\log(10)}{\log(2)}$ , we have

$$\|x^{(k)} - x^*\| \leq 10^{-3}.$$

[ 2 ]

2. The aim of this problem is to derive an algorithm for performing the  $QR$  decomposition using orthogonal matrices known as *Givens rotators*.

a) We start by considering rotators in  $\mathbb{R}^2$  given by  $Q = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$ , where  $\theta \in [0, 2\pi)$ .

i) Show that  $Q$  is an orthogonal matrix, i.e.  $Q^T Q = I$ . [ 1 ]

ii) Let  $x \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ . Find a rotator  $Q$  such that  $Q^T x = \begin{pmatrix} \|x\| \\ 0 \end{pmatrix}$ . [ 2 ]

iii) For  $A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$ . Find a rotator  $Q$  such that  $Q^T A = \frac{1}{\sqrt{2}} \begin{pmatrix} 2 & 5 \\ 0 & 1 \end{pmatrix}$ . [ 1 ]

iv) For  $A \in \mathbb{R}^{2 \times 2}$  non-singular, find a rotator  $Q$  such that  $Q^T A = R$ ,  $R$  upper triangular. [ 2 ]

b) We now examine the general case. Let  $A = (a_{ij})_{i,j=1,\dots,n} \in \mathbb{R}^{n \times n}$  be a non-singular matrix and define **Givens rotators** as follows.

$$Q^{(ij)} = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & c & 0 & \dots & 0 & -s \\ & & & 0 & 1 & \ddots & & 0 \\ & & & \vdots & \ddots & \ddots & \ddots & \vdots \\ & & & 0 & & \ddots & 1 & 0 \\ & & & s & 0 & \dots & 0 & c \\ & & & & & & & 1 & \\ & & & & & & & & \ddots & \\ & & & & & & & & & 1 \end{pmatrix} \quad (2.1)$$

The matrix  $Q^{(ij)}$  is such that all the entries are equal to 0 but the diagonal entries that are equal to 1 except entries  $(i, i)$  and  $(j, j)$  both equal to  $c = \cos(\theta)$ , and entry  $(i, j)$  equals  $-s$  and entry  $(j, i)$  equals  $s$  where  $s = \sin(\theta)$ .

i) Find  $Q$  a Givens rotator such that  $Q^T$  transforms  $x = (x_1, \dots, x_n)^T$  into a vector whose  $j$ th coordinate is equal to 0. [ 2 ]

ii) Show that  $Q^{(ij)} A$  and  $Q^{(ij)T} A$  only alter the  $i$ th and  $j$ th rows of  $A$ . [ 1 ]

iii) Show that  $A Q^{(ij)}$  and  $A Q^{(ij)T}$  only alter  $i$ th and  $j$ th column of  $A$ . [ 1 ]

c) We now describe how to perform the  $QR$ -decomposition using Givens rotators.

i) Find a Givens rotator  $Q^{(21)}$  such that  $Q^{(21)}(a_{11} \dots a_{n1})^T = (\star, 0, a_{31}, \dots, a_{n1})^T$  where  $\star$  is some real number. [ 2 ]

ii) Show that there are rotators of the form  $Q^{(21)}, \dots, Q^{(n1)}$  such that  $(Q^{(n1)})^T \dots (Q^{(21)})^T A$  has its first column of the form  $(\bullet, 0, \dots, 0)^T$  where  $\bullet$  is some real number. [ 3 ]

iii) Describe a method for deriving the  $QR$  decomposition using Givens rotators and derive its complexity. [ 5 ]

3. We define the family of *Hermite polynomials*  $(H_n(x))_{n \geq 0}$  by

$$H_0(x) = 1 \quad \text{and} \quad H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} (e^{-x^2}),$$

where  $\frac{d^n}{dx^n} (e^{-x^2})$  is the  $n$ th derivative of  $e^{-x^2}$ .

- a) i) Compute  $H_1, H_2, H_3$ . [ 2 ]  
 ii) Show that for all non-negative integer  $n$  we have

$$\frac{dH_n}{dx}(x) = 2xH_n(x) - H_{n+1}(x).$$

[ 2 ]

- iii) Show that for  $k < n$ ,  $\int_{-\infty}^{+\infty} x^k H_n(x) e^{-x^2} dx = 0$ ,

*Hint: Perform successive integrations by part and use the fact that for all non-negative integers  $k$  and  $l$ ,*

$$\lim_{x \rightarrow \infty} x^k \frac{d^l}{dx^l} (e^{-x^2}) = \lim_{x \rightarrow -\infty} x^k \frac{d^l}{dx^l} (e^{-x^2}) = 0$$

[ 2 ]

- iv) Show that  $\int_{-\infty}^{+\infty} x^n H_n(x) e^{-x^2} dx = n! \sqrt{\pi}$ .

*Hint: Use, without justification, the identity  $\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}$ .* [ 2 ]

- v) Show that the family of polynomials  $(H_n(x))_{n \geq 0}$  forms a family of orthogonal polynomials for the inner product  $\langle f, g \rangle = \int_{-\infty}^{+\infty} e^{-x^2} f(x) g(x) dx$ . [ 2 ]

- b) We now study the solutions of the differential equation

$$-\frac{d^2 f}{dx^2}(x) + x^2 f(x) = \lambda f(x), \quad (3.1)$$

for  $\lambda \in \mathbb{R}$  a parameter.

- i) By decomposing the polynomial  $xH_n$  in the basis  $(H_0, \dots, H_{n+1})$  of  $\mathbb{R}_{n+1}[X]$ , the space of polynomials of degree less or equal to  $n+1$ , and using a) iii) and v), show that

$$xH_n(x) = \frac{\langle xH_n, H_{n+1} \rangle}{\langle H_{n+1}, H_{n+1} \rangle} H_{n+1}(x) + \frac{\langle xH_n, H_n \rangle}{\langle H_n, H_n \rangle} H_n(x) + \frac{\langle xH_n, H_{n-1} \rangle}{\langle H_{n-1}, H_{n-1} \rangle} H_{n-1}(x).$$

[ 2 ]

- ii) Using a) ii), iii) and v), show that  $\langle xH_n, H_n \rangle = 0$ . [ 2 ]

In fact, one can show that

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0. \quad (3.2)$$

In the remainder, we assume that this holds and no justification is required

- iii) Using question a) ii) and b) i) and the identity (3.2), prove that

$$\frac{d^2 H_n}{dx^2}(x) - 2x \frac{dH_n}{dx}(x) + 2nH_n(x) = 0. \quad (3.3)$$

[ 3 ]

- iv) Using identity (3.3), show that the function  $f_n(x) = e^{-x^2/2} H_n(x)$  is solution of the differential equation (3.1) for  $\lambda = 2n + 1$ . [ 3 ]



Q1.

Q1 1/9

1/3

$$\begin{cases} 2x_1 - x_2 = 3 \\ -x_1 + 2x_2 - x_3 = -5 \\ -x_2 + 2x_3 = 5 \end{cases}$$

a)

$$i) \underbrace{\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}}_A \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_x = \underbrace{\begin{bmatrix} 3 \\ -5 \\ 5 \end{bmatrix}}_y. \quad [1]$$

$$ii) \det(A) = 8 - 2 - 2 = 4. \quad [1]$$

$$iii) A^{-1} = \frac{1}{4} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix} \quad [1]$$

$$x^* = y = A^{-1}x = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

A is non-singular,  $\det(A) \neq 0$ , by (i), so  $x^*$  is the unique solution of (1.1).

$$b) J = \begin{pmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1/2 & 0 \end{pmatrix}.$$

$$i) Jx = \begin{bmatrix} x_2/2 \\ x_1/2 + x_3/2 \\ x_2/2 \end{bmatrix}; \quad [2]$$

$$A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = y \Rightarrow x = Jx + \underbrace{\frac{1}{2} \begin{pmatrix} 3 \\ -5 \\ 5 \end{pmatrix}}_n$$

ii)  $P = \begin{bmatrix} 1/\sqrt{2} & 1/2 & 1/2 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{2} & 1/2 & 1/2 \end{bmatrix}$  Q1  $\frac{2}{9}$   
 $\frac{2}{3}$

eigenvectors of  $J$  associated  
to eigenvalue  $0, 1/\sqrt{2}, -1/\sqrt{2}$  respectively.

$P^{-1} = P^T$  since  $P$  orthogonal

①  $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & -1/\sqrt{2} \end{bmatrix}$  [2]

iii)  $\forall k \quad J^k = P D^k P^T$  [2]

$k=0 \quad J^{2k} = I$

$k \geq 1 \quad J^{2k} = \begin{bmatrix} 1/2^{k+1} & 0 & 1/2^{k+1} \\ 0 & 1/2^k & 0 \\ 1/2^{k+1} & 0 & 1/2^{k+1} \end{bmatrix}$

$k \geq 0 \quad J^{2k+1} = \begin{bmatrix} 0 & 1/2^{k+1} & 0 \\ 1/2^{k+1} & 0 & 1/2^{k+1} \\ 0 & 1/2^{k+1} & 0 \end{bmatrix}$

c) i)  $x^{(1)} = \frac{1}{2} \begin{pmatrix} 5 \\ -4 \\ 7 \end{pmatrix}; \quad \delta^{(1)} = \frac{1}{2} \begin{pmatrix} 3 \\ -2 \\ 3 \end{pmatrix}$  [1]

ii)  $\delta^{(k+1)} = x^{(k+1)} - x^* = (Jx^{(k)} + 2) - (Jx^* + 2)$   
 $= J(x^{(k)} - x^*) = J \delta^{(k)}$

By induction  $\delta^{(k)} = J^k \delta^{(0)} = P D^k P^T \delta^{(0)}$  [1]

iii)

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$D^k x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2^{-k/2} & 0 \\ 0 & 0 & (-1)^k 2^{-k/2} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\|D^k x\|^2 = \|(0, 2^{-k/2} x_2, 2^{-k/2} (-1)^k x_3)^T\|^2$$

$$= \frac{1}{2^k} (x_2^2 + x_3^2) \leq \left( \frac{1}{2^{k/2}} \|x\| \right)^2 \quad [3]$$

$$i) \quad \|Ux\|^2, \quad (U_n)^T U_n = x^T (\underbrace{U^T U}_I) x = x^T x = \|x\|^2 \quad [1]$$

$$\begin{aligned} ii) \quad \|\delta^{(k)}\| &= \|P D^k P^T \delta^{(0)}\| \\ &= \|D^k P^T \delta^{(0)}\| \quad \text{P orthogonal \& by (i).} \\ &\leq \frac{1}{2^{k/2}} \|P^T \delta^{(0)}\| \quad \text{by (ri)} \\ &= \frac{1}{2^{k/2}} \|\delta^{(0)}\| \quad \text{by (i); } P^T \text{ ortho.} \end{aligned}$$

It remains to compute  $\|\delta^{(0)}\|$

$$\|\delta^{(0)}\| = \left\| \begin{pmatrix} 0 \\ 3 \\ -2 \end{pmatrix} \right\| = \sqrt{13}. \quad [3]$$

$$\text{Hence} \quad \|\delta^{(k)}\| \leq \sqrt{\frac{13}{2^k}}.$$

$$ii) \quad \|\delta^{(k)}\| = \|x^{(k)} - x^*\| \leq \sqrt{\frac{13}{2^k}}. \quad \text{let } k_0 \text{ the smallest integer such that } \sqrt{\frac{13}{2^{k_0}}} \leq 10^{-3}.$$

$$\Rightarrow k_0 = \frac{\log(13) + 6 \log(10)}{\log(2)} \approx 24. \quad [2]$$

7/9

Q1

3/3

Q2

Q2 1/3

a)  $Q = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$ .

i)  $Q^T Q = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$   
 $= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  Using  $\cos^2 \theta + \sin^2 \theta = 1$  [1]

ii).

$$Q^T x = \begin{pmatrix} \cos \theta x_1 + \sin \theta x_2 \\ -\sin \theta x_1 + \cos \theta x_2 \end{pmatrix}$$

1st condition  $-\sin \theta x_1 + \cos \theta x_2 = 0$ . [2]  
 $\tan \theta = x_2/x_1$  (\*)

Let  $\theta \in [0, 2\pi)$  such that  $\cos \theta = \frac{x_1}{\sqrt{x_1^2 + x_2^2}}$   
 &  $\sin \theta = \frac{x_2}{\sqrt{x_1^2 + x_2^2}}$  then condition (\*) satisfied

and  $Q^T x = \begin{pmatrix} \sqrt{x_1^2 + x_2^2} / \sqrt{x_1^2 + x_2^2} \\ 0 \end{pmatrix} = \begin{pmatrix} \|x\| \\ 0 \end{pmatrix}$

iii) Let such  $\theta$  such that.

$\cos \theta = 1/\sqrt{2}$   $\sin \theta = 1/\sqrt{2}$  ;  $\theta = \pi/4$ .

Hence, for  $Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$   $Q^T \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix}$ .

and  $Q^T A = \frac{1}{\sqrt{2}} \begin{pmatrix} 2 & 5 \\ 0 & 1 \end{pmatrix}$  [1]



iv) For general  $A$ ;

5/9 [2]

Q2 2/3 Let  $Q$  such that  $\cos \theta = \frac{a_{11}}{\sqrt{a_{11}^2 + a_{21}^2}}$  ;  $\sin \theta = \frac{a_{21}}{\sqrt{a_{11}^2 + a_{21}^2}}$

$$b) i) (Q^{(ij)})^T \begin{pmatrix} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_j \\ \vdots \\ x_n \end{pmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_{i-1} \\ c x_i + s x_j \\ x_{i+2} \\ \vdots \\ x_{j-1} \\ -s x_i + c x_j \\ x_{j+2} \\ \vdots \\ x_n \end{bmatrix} \begin{matrix} \leftarrow i \\ \leftarrow j \end{matrix} \quad (**).$$

we want  $- \sin \theta x_i + \cos \theta x_j = 0$ .

[2]

Let  $\theta \in (0, 2\pi)$  such that  $\cos \theta = \frac{x_i}{\sqrt{x_i^2 + x_j^2}}$

$\sin \theta = \frac{x_j}{\sqrt{x_i^2 + x_j^2}}$ . In this case the  $j$ th coordinate is equal to 0 whereas the  $i$ th coordinate is given by  $\sqrt{x_i^2 + x_j^2}$ .

where the same transform  
ii) see  $(**)$  applies to each column of  $A$ . [1]

iii) by the transpose of ii) which is altered in row & row only columns become rows and vice-versa. So the only alterations take place in the columns of  $A$ . [1]

c) i)  $Q^{(21)}$  gives rotator of the form (2.1) [2]

where  $\cos \theta = \frac{a_{11}}{\sqrt{a_{11}^2 + a_{21}^2}}$  ;  $\sin \theta = \frac{a_{21}}{\sqrt{a_{11}^2 + a_{21}^2}}$ .

Let  $* = a'_{11} (= \sqrt{a_{11}^2 + a_{21}^2})$ .

ii)  $Q^{(31)}$  such that  $\cos \theta = \frac{a'_{11}}{\sqrt{a_{11}'^2 + a_{31}^2}}$  ;  $\sin \theta = \frac{a_{31}}{\sqrt{a_{11}'^2 + a_{31}^2}}$

More generally  $Q_{(k+1),1}^T$  such that if  $a_{1,1}^{(k)}$  the value  $\frac{6}{9}$   
 in entry  $(1,1)$  after application of  $Q_{k,1}^T \dots Q_{2,1}^T$   
 then  $\cos(\theta_{(k+1),1}) = \frac{(a_{1,1}^{(k)})^2}{\sqrt{(a_{1,1}^{(k)})^2 + a_{(k+1),1}^2}}$

$$\& \sin(\theta_{(k+1),1}) = \frac{a_{(k+1),1}}{\sqrt{(a_{1,1}^{(k)})^2 + a_{(k+1),1}^2}} \quad \cdot \quad \underline{Q^2} \quad \frac{3}{3}$$

Once we apply  $R^{(1)} = (Q_{n,1}^T \dots Q_{2,1}^T)A$  as above  
 the first column of  $R^{(1)}$  is of the form  
 $\begin{pmatrix} * \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ . [3]

iii) Once we reduced the 1st column following  
 ii). We can easily construct Givens rotators.

$Q_{n,2} \dots Q_{3,2}$  such that [5]

$R^{(2)} = Q_{n,2}^T \dots Q_{3,2}^T Q_{n,1}^T \dots Q_{2,1}^T A$  has  
 zeros in columns 1 & 2 below the main diagonal

In a similar fashion we can find rotators,  
 of the form  $Q_{2,1} \dots Q_{n,n-1}$  such that

A rough calculation gives  $R = Q_{n,n-1}^T \dots Q_{2,1}^T A$  is upper triangular  
 For the first column. (more precisely  $2n^2 \log(n)$ ).  
 for each of them we have 5 operations for  
 in  $\theta$  & one addition for  $\sin \theta$  - justification  
 and then. 2 BONUS POINTS For proper justification of complexity.

# Mathematics for signals & systems do 11/2, 12.

Q3

Q3 1/3 2/9

a) i)  $H_1(n) = (-1)^n e^{n^2} \frac{d}{dn} e^{-n^2} = 2n$

$H_2(n) = e^{n^2} \frac{d}{dn} (-2n e^{-n^2}) = 4n^2 - 2$  [2]

$H_3(n) = - e^{n^2} \frac{d}{dn} ((4n^2 - 2) e^{-n^2}) = 8n^3 - 12n$

ii)  $\frac{d}{dx} H_n(n) = (-1)^n \frac{d}{dn} e^{n^2} \frac{d^n}{dn^n} e^{-n^2}$   
 $= (-1)^n \left[ 2n e^{n^2} \frac{d^n}{dn^n} e^{-n^2} + e^{n^2} \frac{d^{n+1}}{dn^{n+1}} e^{-n^2} \right]$

$H'_n(x) = 2n H_n(n) - H_{n+1}(n)$

iii)  $\int_{-\infty}^{+\infty} e^{-n^2} x^k H_n(n) dn = (-1)^n \int_{-\infty}^{+\infty} x^k \frac{d^n}{dn^n} e^{-n^2} dn$

IP  $= \underbrace{\left[ (-1)^n x^k \frac{d^{n-1}}{dn^{n-1}} e^{-n^2} \right]_{-\infty}^{+\infty}}_{=0 \text{ by hint}} - (-1)^n \int_{-\infty}^{+\infty} k x^{k-1} \frac{d^{n-1}}{dn^{n-1}} e^{-n^2} dn$

Repeating step  $\dots = (-1)^{n+k} k! \int_{-\infty}^{+\infty} \frac{d^{n-k}}{dn^{n-k}} e^{-n^2} dn$   
 above k-ths using hint  $= (-1)^{n+k} k! \int_{-\infty}^{+\infty} \frac{d^{n-k+1}}{dn^{n-k+1}} e^{-n^2} dn$  by hint [2]

iv)  $\int_{-\infty}^{+\infty} x^n H_n(n) e^{-n^2} dn \stackrel{\text{as above}}{=} n! \int_{-\infty}^{+\infty} e^{-n^2} dn = \sqrt{\pi} \text{ by hint}$  [2]

v)  $\int_{-\infty}^{+\infty} H_k(n) H_n(n) e^{-n^2} dn = \int_{-\infty}^{+\infty} \sum_{l=0}^k h_l e^{n^2} H_l(n) e^{-n^2} dn$   
 $= 0$  by linearity of integral & 3) a) iii) [2]



b)  
i)

$\mathbb{Q}^{3^{2/3}}$

$$x H_n(x) = \sum_{k=0}^{n+1} \frac{\langle x H_n, H_k \rangle}{\langle H_k, H_k \rangle} H_k. \quad (*) \quad \frac{8}{9}$$

note that  $\langle x H_n, H_k \rangle = \langle H_n, x H_k \rangle$

and by a) iii)  $\langle H_n, x H_k \rangle$  is equal 0

whenever the degree of  $x H_k$  is smaller than  $n$ .  
that is to say  $k+1 < n$ ;  $k \leq n-2$ .

Hence  $(*) \Rightarrow$

$$x H_n = \frac{\langle x H_n, H_{n+1} \rangle}{\langle H_{n+1}, H_{n+1} \rangle} H_{n+1} + \frac{\langle x H_n, H_n \rangle}{\langle H_n, H_n \rangle} H_n + \frac{\langle x H_n, H_{n-1} \rangle}{\langle H_{n-1}, H_{n-1} \rangle} H_{n-1} \quad [2]$$

~~$$\langle x H_n, H_n \rangle = \int_{-\infty}^{+\infty} e^{-x^2} \left[ \frac{d}{dx} e^{-x^2} \right]^2 dx$$~~

by ii)  $\langle x H_n, H_n \rangle = \left\langle \frac{1}{2} \frac{dH_n}{dx} + \frac{1}{2} H_{n+1}, H_n \right\rangle$

$$= \frac{1}{2} \underbrace{\left\langle \frac{dH_n}{dx}, H_n \right\rangle}_{=0 \text{ by a) iii)}} + \frac{1}{2} \underbrace{\langle H_{n+1}, H_n \rangle}_{=0 \text{ by a) v)}}.$$

$$\langle x H_n, H_n \rangle = 0.$$

[2].

iii)

Q3 (3/3)

9/9

$$H_{n+1} - 2x H_n + 2n H_{n-1} = 0.$$

$$-H_n + 2n H_{n-1} = 0 \text{ by a) ii) } \Rightarrow H'_n = 2n H_{n-1} \quad (**)$$

$$\Rightarrow H_{n+1} - 2x H_n + H'_n = 0.$$

differentiating

$$H'_{n+1} - 2H_n - 2x H'_n + H''_n = 0.$$

$$\text{By } (***) \quad H'_{n+1} = (2n+2) H_n.$$

$$\text{Hence } (2n+2) H_n - 2H_n - 2x H'_n + H''_n = 0.$$

$$\Rightarrow H''_n - 2x H'_n + 2n H_n = 0.$$

[3]

$$12) \quad f'_n(x) = -x f_n(x) + e^{-x^2/2} H'_n.$$

$$f''_n(x) = -f_n(x) - x f'_n(x) - x e^{-x^2/2} H'_n + e^{-x^2/2} H''_n.$$

$$-f''_n + x^2 f_n = f_n + (x f'_n) + x e^{-x^2/2} H'_n - e^{-x^2/2} H''_n + x^2 f_n.$$

$$= f_n - \frac{x^2 f_n}{x^2 f_n} + x e^{-x^2/2} H'_n + x e^{-x^2/2} H'_n - e^{-x^2/2} H''_n + x^2 f_n$$

$$= f_n + \left[ \underbrace{2x H'_n - H''_n}_{2n H_n \text{ by b) iii)}} \right] e^{-x^2/2}.$$

$$= f_n + 2n f_n = (2n+1) f_n. \quad [3],$$