Solutions: Stability and Control of Nonlinear Systems
Master in Control

1. Exercise

a) The first nullcline corresponds to the set:

$$\mathcal{N}_1 = \{(x_1,x_2) : \sin(\sqrt{x_1^2+x_2^2}) = 0\} = \{(x_1,x_2) : \exists k \in \mathbb{N} : \sqrt{x_1^2+x_2^2} = k\pi\}.$$

Therefore, $\mathcal{N}_1$ is the union of circles of radius $k\pi$ centered at the origin, for $k = 0,1,2,\ldots$. The second nullcline is defined by the equation $\mathcal{N}_2 = \{(x_1,x_2) : -x_2+x_1 = 0\}$ that is the diagonal bisecting positive and negative orthants. Equilibria are located at their intersection:

$$\begin{cases} \sqrt{x_1^2+x_2^2} & = k\pi \\ x_1 & = x_2 \end{cases}$$

Substituting the second equation into the first one yields:

$$\sqrt{2x_1^2} = k\pi \Rightarrow (x_1,x_2) = \pm\frac{k}{\sqrt{2}}(\pi,\pi).$$

b) We remark that the function $\sqrt{x_1^2+x_2^2} = |(x_1,x_2)|$ is globally Lipschitz, for all x_1,x_2; in fact by the triangular inequality:

$$\big||(x_1,x_2)| - |(z_1,z_2)|\big| \le |(x_1,x_2)-(z_1,z_2)|.$$

Since $\sin(\cdot)$ is globally Lipschitz, then the composition $\sin(\sqrt{x_1^2+x_2^2})$ is a globally Lipschitz continuous function. Overall $f(x)$ is Lipschitz continuous, hence solutions of the system exist and are unique.

c) The function $\sqrt{x_1^2+x_2^2}$ is differentiable everywhere, except for $(x_1,x_2) = 0$. Notice that partial derivatives converge to different numbers coming from positive xs or negative xs. A similar problem occurs for $\sin(\sqrt{x_1^2+x_2^2})$. Hence the system around the equilibrium at $(0,0)$ is not linearizable.The system is instead linearizable around all equilibria $\pm\frac{k}{\sqrt{2}}(\pi,\pi)$, with $k>0$.

d) Taking derivatives of the vector-field f with respect to x yields:

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \cos(\sqrt{x_1^2+x_2^2})\frac{x_1}{\sqrt{x_1^2+x_2^2}} & \cos(\sqrt{x_1^2+x_2^2})\frac{x_2}{\sqrt{x_1^2+x_2^2}} \\ 1 & -1 \end{bmatrix}.$$

Evaluating the Jacobian at equilibria $\pm\frac{k}{\sqrt{2}}(\pi,\pi)$, $k>0$, yields the following linearized systems:

$$\dot{\delta x} = \begin{bmatrix} \pm\frac{(-1)^k}{\sqrt{2}} & \pm\frac{(-1)^k}{\sqrt{2}} \\ 1 & -1 \end{bmatrix} \delta x.$$

For equilibria of type $\frac{k}{\sqrt{2}}(\pi,\pi)$, with $k>0$ even, or equilibria of type $-\frac{k}{\sqrt{2}}(\pi,\pi)$ for $k>0$ odd, the characteristic polynomial reads:

$$\chi(s)=s^2+(1-\frac{1}{\sqrt{2}})s-\sqrt{2}$$

which exhibits a permanence ad a variation of sign. Hence this has a positive and a negative real root. The local phase-portrait is that of a saddle-point:

$$\chi(s)=s^2+\left(1+\frac{1}{\sqrt{2}}\right)s+\sqrt{2}.$$

This polynomial has complext conjugate roots with negative real part; hence the local phase-portrait is that of a stable focus.

e) The global phase-portrait is shown in Fig. 1.1.

f) We take as a Lyapunov function $V(x)=x_1$. Clearly 0 is an accumulation point of $\{x: V(x)>0\}$. Moreover:

$$\dot{V}(x)=\sin(\sqrt{x_1^2+x_2^2})>0$$

for all $x\neq 0$ with $|x|<\pi$. By Lyapunov's instability criterion we can conclude that the equilibrium at the origin is unstable.

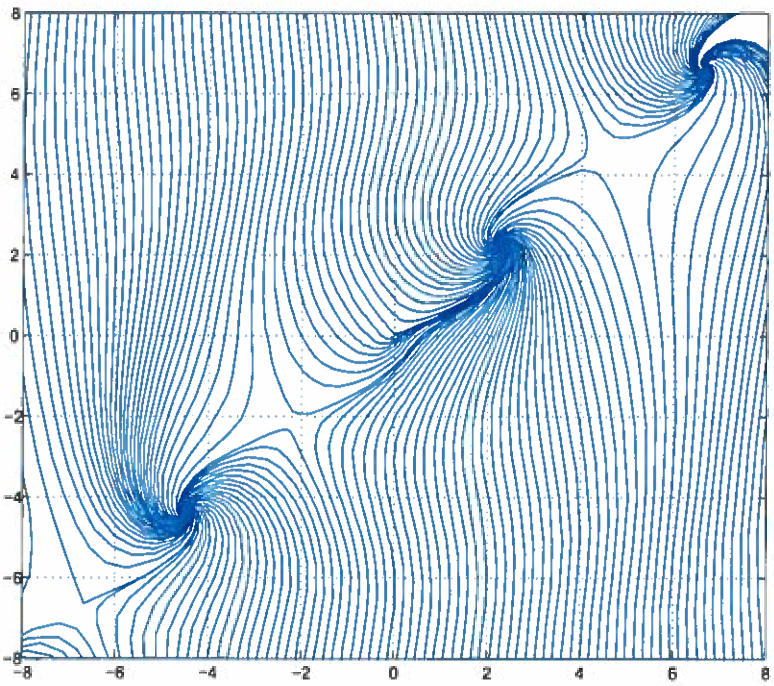

Figure 1.1 Phase portrait

2. Exercise

a) We choose as a candidate Lyapunov function $V(x) = \frac{x_1^2+x_2^2}{2}$. Deriving along solutions of the system yields:

$$\dot{V} = x_1\dot{x}_1 + x_2\dot{x}_2 = -x_1^2 + x_1x_2^3 - x_1x_2^3 - 2x_2^2 + x_2 d$$
$$\leq -x_1^2 - x_2^2 + \frac{d^2}{4}.$$

Hence V is an ISS-Lyapunov function and the system is ISS.

b) Notice that, from the previous inequality we see that:

$$|x|^2 \geq d^2/8 \Rightarrow \dot{V} \leq -|x|^2/2$$

This implies that $2\sqrt{2}|d|$ is an upper-bound to the gain from $|d|$ to $|x|$;

c) Consider that

$$z\dot{z} = -z^2(1+d_1^2+d_2^2) \leq 0$$

hence $z^2(t)/2$ is a non-increasing function of time. In particular: $|z(t)| \leq |z(0)|$.

d) Letting $W(z) = z^2/2$ we see that:

$$\dot{W} = -z^2(1+d_1^2+d_2^2) \leq -z^2$$

hence UGAS follows for all compact sets $D \subset \mathbb{R}^2$, provided $(d_1, d_2) \in D$.

e) Consider the feedback interconnection of equations:

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2^3 \\ \dot{x}_2 &= -x_1x_2^2 - 2x_2 + z \\ \dot{z} &= -z(1+x_1^2+x_2^2). \end{aligned}$$

A direct Lyapunov proof could be used to show Global Asymptotic Stability. We argue otherwise; from item c) we know that the z component of solutions is uniformly bounded (regardless of initial conditions), as long as solutions are defined. From ISS of the x subsystem then, the x-component of the solution is uniformly bounded. Therefore solutions are defined for all positive times. Moreover, the evolution takes place in a compact set. From UGAS we see that z tends to zero asymptotically. Hence, by virtue of the CICS property of ISS systems also $x(t)$ converges to zero.

3. Exercise

a) We pick as a state variable $x = [x_1, x_2, x_3, x_4] = [x_1, x_2, \dot{x}_1, \dot{x}_2]$; the corresponding equations read:

$$\begin{aligned} \dot{x}_1 &= x_3 \\ \dot{x}_2 &= x_4 \\ \dot{x}_3 &= -x_1 - k(x_1 - x_2) \\ \dot{x}_4 &= k(x_1 - x_2) + u \\ \\ y &= x_4 \end{aligned}$$

b) We consider the following storage function: $S(x) = \frac{x_3^2+x_4^2}{2} + \int_0^{x_1-x_2} k(x)\,dx + \frac{x_1^2}{2}$; taking derivatives of S yields:

$$\dot{S} = x_3\dot{x}_3 + x_4\dot{x}_4 + k(x_1 - x_2)(\dot{x}_1 - \dot{x}_2) + x_1\dot{x}_1 =$$

$$= -x_1x_3 - k(x_1 - x_2)x_3 + x_4k(x_1 - x_2) + ux_4 + k(x_1 - x_2)(x_3 - x_4) + x_1x_3 = uy.$$

Hence the system is passive and loss-less.

c) Taking $V(x) = S(x)$ as a candidate Lyapunov function we see that:

- $V(x)$ is positive definite. Each of its term is in fact non-negative. Moreover, $V(x) = 0$ if and only if $x_3 = 0$, $x_4 = 0$, $x_1 = 0$ and $\int_0^{x_1-x_2} k(x)dx = 0$. Notice that the latter equality is only possible if $x_1 = x_2$, thus $V(x) = 0 \Leftrightarrow x = 0$;
- $V(x)$ is radially unbounded; $V(x) \leq M$ clearly implies a bound on x_3, x_4 and x_1. Moreover, since k is monotone, the mechanical energy associated to the spring grows at least linearly with its elongation $x_1 - x_2$. Therefore, a bound on V also implies $|x_1 - x_2|$ bounded. Overall then sublevel sets of V are bounded.
- Differentiating along solutions yields:

$$\dot{V}(x) = uy = -x_4^2 \leq 0.$$

We consider the set $\{x : x_4 = 0\}$ and look for invariant sets inside it. If $x(t) \in \{x : x_4 = 0\}$ for all t, then $x(t) \in \{x : x_4 = 0 \& k(x_1 - x_2) - x = 0\} = \{x : x_4 = 0 \& x_1 - x_2 = 0\}$ for all t; differentiating once more we see that: $x(t) \in \{x : x_4 = 0, x_1 - x_2 = 0, x_3 - x_4 = 0\}$ for all t and finally $x(t) \in \{x : x_4 = 0, x_1 - x_2 = 0, x_3 - x_4 = 0, -x_1 - 2k(x_1 - x_2) - x_4 = 0\} = \{0\}$ for all t. Hence the largest invariant set in Ker$[\dot{V}] = \{0\}$ so that by Lasalle's criterion the origin is Globally Asymptotically Stable.

d) We realize the PI controller as a scalar system:

$$\dot{z} = v \qquad w = v + z$$

with state z, input$v$ and output w. Notice that

$$\frac{d}{dt}\frac{z^2}{2} = \dot{z}z = wv - v^2$$

which shows passivity of the device. Next, we may consider the Lyapunov function:

$$W(x, z) = V(x) + \frac{z^2}{2}.$$

This is positive definite and radially unbounded. Moreover:

$$\dot{W} = \dot{V} + z\dot{z} = uy + wv - v^2 - x_4^2 = -2x_4^2 \leq 0$$

Again by Lasalle's criterion it can be seen that the largest invariant set in $\text{Ker}[\dot{W}] = \{(x,z) : x = 0\}$. Hence, solutions approach asymptotically this set. Notice that z need not converge to 0 asymptotically.

4. Exercise

a) Equilibria for $u = 0$ are the solutions of the following system of equations:

$$\begin{cases} -2\mathrm{atan}(x_1) + x_2 = 0 \\ -\mathrm{atan}(x_1) - x_2 + x_3 = 0 \\ -\sin(x_3) = 0 \end{cases}$$

Hence, from the last equation we see that $x_3 = k\pi$ for any $k \in \mathbb{Z}$. Substituting $x_2/2$ in place of $\mathrm{atan}(x_1)$ in the second equation we see that $3x_2/2 = x_3$ and therefore x_2 need to fulfill $x_2 = 2k\pi/3$. Finally, $x_1 = \tan(k\pi/3)$. Hence, for all $k \in \mathbb{Z}$, $[\tan(k\pi/3), 2k\pi/3, k\pi]'$ is an equilibrium point.

b) Taking derivatives of y, yields:

$$\begin{aligned} \dot{y} &= -2\mathrm{atan}(x_1) + x_2 \\ \ddot{y} &= -2\tfrac{x_2 - 2\mathrm{atan}(x_1)}{1+x_1^2} - \mathrm{atan}(x_1) - x_2 + x_3 \\ y^{(3)} &= q(x) + ue^{-x_2^2} \end{aligned}$$

where $q(x)$ is given by:

$$q(x) = -2\frac{\left[-2\frac{-2\mathrm{atan}(x_1)+x_2}{1+x_1^2} - \mathrm{atan}(x_1) - x_2 + x_3\right](1+x_1^2) - 2x_1(x_2 - 2\mathrm{atan}(x_1))^2}{(1+x_1^2)^2}$$

$$-\frac{x_2 - 2\mathrm{atan}(x_1)}{1+x_1^2} - (x_3 - x_2 - \mathrm{atan}(x_1)) - \sin(x_3).$$

This means that the relative degree is equal to 3 and is globally defined, as $e^{-x_2^2} \neq 0$ for all $x \in \mathbb{R}^3$.

c) Letting $u = e^{x_2^2}[v - q(x)]$ and taking $\bar{x} = [y, \dot{y}, \ddot{y}]'$, yields the equation:

$$\dot{\bar{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \bar{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} v.$$

d) Applying the pole placement method, we can achieve closed loop poles in -1, by letting $v = -3\ddot{y} - 3\dot{y} - y$.

e) For $y = x_3$, we see that $\dot{y} = -\sin(x_3) + ue^{-x_2^2}$. Therefore the relative degree is again globally defined and equals 1. The input-output linearizing feedback is given as:

$$v = e^{x_2^2}(\sin(x_3) + v)$$

This results in the following normal form:

$$\begin{aligned} \dot{y} &= v \\ \dot{\xi}_1 &= \xi_2 - 2\mathrm{atan}(\xi_1) \\ \dot{\xi}_2 &= -\mathrm{atan}(\xi_1) - \xi_2 + y \end{aligned} ,$$

provided we define the internal dynamics state as $\xi = [x_1, x_2]'$.

f) We show next that the internal dynamics are ISS with respect to the input y. Take the following candidate Lyapunov function:

$$V(\xi) = \int_0^{\xi_1} \mathrm{atan}(z)dz + \xi_2^2/2.$$

Clearly V is positive definite and radially unbounded. Moreover, taking derivatives along solutions of the system yields:

$$\begin{aligned}\dot{V} &= \operatorname{atan}(\xi_1)[-2\operatorname{atan}(\xi_1)+\xi_2]+\xi_2[-\operatorname{atan}(\xi_1)-\xi_2+y] \\ &= -2\operatorname{atan}^2(\xi_1)-\xi_2^2+\xi_2 y \leq -2\operatorname{atan}^2(\xi_1)-\xi_2^2/2+y^2/2.\end{aligned}$$

This allows to prove ISS for sufficiently small values of the input. Hence, any feedback stabilizing the y equation, for instance $v=-y$, results in a converging signal $y(t)$ and in globally defined solutions. Moreover ξ asymptotically approaches zero thanks to the CICS property. Overall, the closed-loop system is GAS at the origin.