SOLUTIONS: SYSTEMS IDENTIFICATION

1. Solution

a) Consider the moving-average stationary stochastic process $w(\cdot)$ described in Question 1-a) of the text of the exam paper:

$$MA(1): \; w(t) = e(t) + ce(t-1)$$

with $c \in \Re$, $c \neq 0$, $|c| < 1$, and $e(\cdot) \sim WN(0, \sigma_e^2)$

One gets:

$$w(t) = (1 + cz^{-1})e(t)$$

and thus

$$\Gamma_w(\omega) = \left|e^{j\omega} + c\right|^2 \sigma_e^2 = \left[1 + c^2 + 2c\cos(\omega)\right] \sigma_e^2$$

By imposing

$$\Gamma_w(0) = \Gamma_v(0), \quad \Gamma_w(\pi/2) = \Gamma_v(\pi/2), \quad \Gamma_w(\pi) = \Gamma_v(\pi)$$

one gets:

$$\begin{cases} (1 + c^2 + 2c)\, \sigma_e^2 = \dfrac{9}{4} \\[2ex] (1 + c^2)\, \sigma_e^2 = \dfrac{5}{4} \\[2ex] (1 + c^2 - 2c)\, \sigma_e^2 = \dfrac{1}{4} \end{cases}$$

and hence, after some algebra, it follows that

$$\begin{cases} 2c\sigma_e^2 = 1 \\[2ex] \dfrac{5}{4}\sigma_e^4 - \sigma_e^2 - \dfrac{1}{4} = 0 \end{cases} \implies \begin{cases} c = \dfrac{1}{2} \\[2ex] \sigma_e^2 = 1 \end{cases}$$

which is the only feasible solution.

[7 Marks]

b) Setting $c = \frac{1}{2}$ and $\sigma_e^2 = 1$, the moving-average stationary stochastic process $w(\cdot)$ is described as

$$w(t) = e(t) + \frac{1}{2}e(t-1)$$

with $e(\cdot) \sim WN(0,1)$.

It turns out that:

$$\gamma_w(0) = \text{var}[w(t)] = \mathbb{E}[e(t)^2] + c^2\mathbb{E}[e(t-1)^2] + 2c\mathbb{E}[e(t)e(t-1)] = \frac{5}{4}$$

$$\gamma_w(1) = \gamma_w(-1) = \mathbb{E}[w(t)w(t-1)] = \mathbb{E}\{[e(t) + \frac{1}{2}e(t-1)] \cdot [e(t-1) + \frac{1}{2}e(t-2)]\} = \frac{1}{2}$$

and $\gamma_w(\tau) = 0$, $\forall \tau$ such that $|\tau| \geq 2$.

[5 Marks]

c) Consider the auto-regressive stationary stochastic process $y(\cdot)$ described in Question 1-c) of the text of the exam paper:

$$AR(1): \ y(t) = ay(t-1) + \eta(t)$$

where $a \in \Re$, $a \neq 0$, $|a| < 1$, and $\eta(\cdot) \sim WN(0, \sigma_\eta^2)$.

One gets:

$$y(t) = \frac{1}{1 - az^{-1}} \eta(t)$$

and thus

$$\Gamma_y(\omega) = \frac{1}{|e^{j\omega} - a|^2} \sigma_\eta^2 = \frac{1}{1 + a^2 - 2a\cos(\omega)} \sigma_\eta^2$$

By imposing

$$\Gamma_y(0) = \Gamma_v(0), \quad \Gamma_y(\pi/2) = \Gamma_v(\pi/2), \quad \Gamma_y(\pi) = \Gamma_v(\pi)$$

one gets:

$$\begin{cases} \dfrac{1}{1 + a^2 - 2a} \sigma_\eta^2 = \dfrac{9}{4} \\[2ex] \dfrac{1}{1 + a^2} \sigma_\eta^2 = \dfrac{5}{4} \\[2ex] \dfrac{1}{1 + a^2 + 2a} \sigma_\eta^2 = \dfrac{1}{4} \end{cases}$$

and hence, after some algebra, it follows that

$$\begin{cases} a = \dfrac{8}{9} \sigma_\eta^2 \\[2ex] \dfrac{45}{116} = \dfrac{1}{4} \quad \text{inconsistent: not an identity} \end{cases}$$

Therefore, there is no solution, that is, no AR(1) stationary stochastic process has a spectrum such that

$$\Gamma_y(0) = \Gamma_v(0), \quad \Gamma_y(\pi/2) = \Gamma_v(\pi/2), \quad \Gamma_y(\pi) = \Gamma_v(\pi)$$

[8 Marks]

2. a) Consider the complex spectrum $\Phi_v(z)$ given in Question 2 of the text of the exam paper:

$$\Phi_v(z) = \frac{(z+1/4)(z+4)}{(z-2)(1/2-z)} = \frac{(z+1/4)(z+4)}{(z-1/2)(2-z)}$$

First, one gets:

$$\Phi_v(z) = \sqrt{2}\frac{(z+1/4)}{(z-1/2)}\frac{1}{\sqrt{2}}\frac{(z+4)}{(2-z)}$$

Since

$$\Phi_v(z) = H(z)H(z^{-1})\mathrm{var}(\xi) = H(z)H(z^{-1})$$

Thus:

$$H(z) = \sqrt{2}\frac{(z+1/4)}{(z-1/2)}$$

[5 Marks]

b) Consider the transfer function $H(z)$ obtained in the answer to Question 2-a):

$$H(z) = \sqrt{2}\frac{(z+1/4)}{(z-1/2)}$$

To make $H(z)$ a spectral factor in canonical form, we consider $\eta(\cdot) \sim WN(0,2)$. Hence:

$$v(t) = H(z)\xi(t) = \frac{1}{\sqrt{2}}H(z)\eta(t) = \widehat{H}(z)\eta(t)$$

where

$$\widehat{H}(z) = \frac{(z+1/4)}{(z-1/2)}$$

is a spectral factor in canonical form.

Thus:

$$A(z)v(t) = C(z)\eta(t)$$

where

$$A(z) = 1 - \frac{1}{2}z^{-1}, \quad C(z) = 1 + \frac{1}{4}z^{-1}.$$

By carrying out one iteration of polynomial division of $C(z)$ by $A(z)$ one gets:

$$\begin{array}{cccc} 1 & \frac{1}{4}z^{-1} & 1 & -\frac{1}{2}z^{-1} \\ -1 & \frac{1}{2}z^{-1} & 1 & \\ // & \frac{3}{4}z^{-1} & & \end{array}$$

Then:

$$\widehat{H}(z) = \frac{C(z)}{A(z)} = 1 + z^{-1}\frac{\frac{3}{4}}{1-\frac{1}{2}z^{-1}}.$$

and hence the transfer function of the one-step ahead predictor of $v(t+1)$ from the white noise process $\eta(t)$ is given by

$$\widehat{H}_1(z) = \frac{\frac{3}{4}}{1-\frac{1}{2}z^{-1}}$$

Moreover, the transfer function of the one-step ahead predictor of $v(t+1)$ from the past data $v(t)$ is

$$H_1(z) = \frac{\frac{3}{4}}{1+\frac{1}{4}z^{-1}}.$$

and, accordingly, the difference equation implementing the one-step ahead predictor of $v(t+1)$ from the data $v(t)$ is

$$\hat{v}(t+1|t) = -\frac{1}{4}\hat{v}(t|t-1) + \frac{3}{4}v(t).$$

[3 Marks]

c) Consider the one-step ahead prediction error when using the predictor determined in the answer to Question 2-b):

$$\varepsilon_1(t) = v(t+1) - \hat{v}(t+1|t)$$

From

$$\widehat{H}(z) = \frac{C(z)}{A(z)} = 1 + z^{-1}\frac{\frac{3}{4}}{1-\frac{1}{2}z^{-1}}$$

one immediately gets

$$\varepsilon_1(t) = \eta(t+1)$$

Hence $\mathbb{E}[\varepsilon_1(t)] = \mathbb{E}[\eta(t+1)] = 0$ and

$$\mathrm{var}[\varepsilon_1(t)] = \mathrm{var}[v(t+1) - \hat{v}(t+1|t)] = 1 \cdot \mathrm{var}[\eta(t+1)] = 2.$$

[3 Marks]

d) Using the expression

$$A(z)v(t) = C(z)\eta(t)$$

with

$$A(z) = 1 - \frac{1}{2}z^{-1}, \quad C(z) = 1 + \frac{1}{4}z^{-1}.$$

that has been used in the answer to Question 2-b), two additional iterations of polynomial division of $C(z)$ by $A(z)$ are carried out in the following:

$$\begin{array}{llll}
1 & \frac{1}{4}z^{-1} & & \\
-1 & \frac{1}{2}z^{-1} & & \\
// & \frac{3}{4}z^{-1} & & \\
// & -\frac{3}{4}z^{-1} & \frac{3}{8}z^{-2} & \\
// & // & \frac{3}{8}z^{-2} & \\
// & // & -\frac{3}{8}z^{-2} & \frac{3}{16}z^{-3} \\
// & // & // & \frac{3}{16}z^{-3}
\end{array}
\qquad
\begin{array}{lll}
1 & -\frac{1}{2}z^{-1} & \\
1 & \frac{3}{4}z^{-1} & \frac{3}{8}z^{-2}
\end{array}$$

and thus

$$\widehat{H}(z) = \frac{C(z)}{A(z)} = 1 + \frac{3}{4}z^{-1} + \frac{3}{8}z^{-2} + z^{-3}\frac{\frac{3}{16}}{1-\frac{1}{2}z^{-1}}$$

Therefore, the transfer function of the two-steps ahead predictor of $v(t+3)$ from the white noise process $\eta(t)$ is given by

$$\widehat{H}_3(z) = \frac{\frac{3}{16}}{1-\frac{1}{2}z^{-1}}$$

and thus the transfer function of the two-steps ahead predictor of $v(t+3)$ from the past data $v(t)$ is

$$H_3(z) = \frac{\frac{3}{16}}{1+\frac{1}{4}z^{-1}}$$

Finally, the difference equation implementing the two-step ahead predictor of $v(t+3)$ from the data $v(t)$ is

$$\hat{v}(t+3|t) = -\frac{1}{4}\hat{v}(t+2|t-1) + \frac{3}{16}v(t).$$

[3 Marks]

e) Consider the three-step ahead prediction error when using the predictor determined in the answer to Question 2-d):

$$\varepsilon_3(t) = v(t+3) - \hat{v}(t+3|t)$$

From

$$\widehat{H}(z) = \frac{C(z)}{A(z)} = 1 + \frac{3}{4}z^{-1} + \frac{3}{8}z^{-2} + z^{-3}\frac{\frac{3}{16}}{1-\frac{1}{2}z^{-1}}$$

one gets

$$\varepsilon_3(t) = \eta(t+3) + \frac{3}{4}\eta(t+2) + \frac{3}{8}\eta(t+1)$$

Hence

$$\mathbb{E}[\varepsilon_3(t)] = \mathbb{E}[\eta(t+3)] + \frac{3}{4}\mathbb{E}[\eta(t+2)] + \frac{3}{8}\mathbb{E}[\eta(t+1)] = 0$$

and

$$\begin{aligned} \mathrm{var}[\varepsilon_3(t)] &= \mathrm{var}[v(t+3) - \hat{v}(t+3|t)] \\ &= 1\cdot\mathrm{var}[\eta(t+3)] + \frac{9}{16}\mathrm{var}[\eta(t+2)] + \frac{9}{64}\mathrm{var}[\eta(t+1)] = \frac{109}{32} \simeq 3.4 \end{aligned}$$

[3 Marks]

f) The comparison between $\mathrm{var}[\varepsilon_1(t)]$ and $\mathrm{var}[\varepsilon_3(t)]$ gives

$$\mathrm{var}[\varepsilon_3(t)] = \frac{109}{32} > 2 = \mathrm{var}[\varepsilon_1(t)]$$

This confirms that the variance of the prediction error $\mathrm{var}[\varepsilon_r(t)]$ increases with the number r of steps-ahead of the prediction that is computed.

[3 Marks]

3. a) Using a trigonometric well-known formula, one gets

$$\begin{aligned} c(j) &= \gamma + K\sin[100\pi t(j) + \varphi] \\ &= \gamma + K\sin[100\pi t(j)]\cos(\varphi) + K\cos[100\pi t(j)]\sin(\varphi) \end{aligned}$$

Moreover, according to the text of Question 3-a), K is supposed to be known. Thus, the parametric model $\mathscr{M}$ can be written as

$$\mathscr{M}:\ \widehat{c}(t,\alpha) = \sum_{i=1}^{3} \alpha_i \rho_i(t)$$

where

$$\rho_1(t) := 1; \quad \rho_2(t) := K\sin[100\pi t]; \quad \rho_3(t) := K\cos[100\pi t]$$

and where $\alpha := [\alpha_1, \alpha_2, \alpha_3]^\top$ is a three-dimensional vector collecting the parameters to be determined using the available measurements.

[7 Marks]

b) Model $\mathscr{M}$ can be rewritten as follows:

$$\mathscr{M}:\ \widehat{c}(t,\alpha) = \Psi(t)^\top \alpha$$

where the vector $\Psi(t)$ is defined as follows:

$$\Psi(t) := \begin{bmatrix} 1 \\ K\sin[100\pi t] \\ K\cos[100\pi t] \end{bmatrix}$$

and where the vector α is defined in the answer to Question 3-a).

Consider the set of time-instants and current measurements

$$\Theta = \{(t(j), c_m(j)),\ j = 1, \ldots, M\}.$$

and the above-defined function

$$\widehat{c}(t,\alpha) = \Psi(t)^\top \alpha$$

Introduce the error variable given by

$$e(j) = c_m(j) - \widehat{c}[t(j),\alpha] = c_m(j) - \Psi[t(j)]^\top \alpha,\ j = 1, \ldots, M.$$

Moreover, consider the following cost function:

$$J(\alpha) = \frac{1}{M}\sum_{j=1}^{M}[e(j)]^2 = \frac{1}{M}\sum_{j=1}^{M}\{c_m(j) - \Psi[t(j)]^\top \alpha\}^2.$$

The minimisation of $J(\theta)$ with respect to the unknown vector α yields the optimal (in the least-squares sense) set of parameters $\alpha_1^\circ, \alpha_2^\circ, \alpha_3^\circ$ of model $\mathscr{M}$. Specifically:

$$\alpha^\circ = [\alpha_1^\circ, \alpha_2^\circ, \alpha_3^\circ]^\top = \arg\min_{\alpha} J(\alpha)$$

provided that a unique minimum of $J(\alpha)$ does exist.

To compute the optimal solution $\alpha^\circ = [\alpha_1^\circ, \alpha_2^\circ, \alpha_3^\circ]^\top$, the gradient of $J(\alpha)$ with respect to the vector α is determined:

$$\begin{aligned}
\frac{\partial}{\partial\alpha}J(\alpha) &= \frac{1}{M}\sum_{j=1}^{M}\frac{\partial}{\partial\alpha}\left\{[c_m(j)-\Psi[t(j)]^\top\alpha]^2\right\} = \\
&= \frac{2}{M}\sum_{j=1}^{M}[c_m(j)-\Psi[t(j)]^\top\alpha]\frac{\partial}{\partial\alpha}\left[c_m(j)-\Psi[t(j)]^\top\alpha\right] = \\
&= -\frac{2}{M}\sum_{j=1}^{M}[c_m(j)-\Psi[t(j)]^\top\alpha]\Psi[t(j)]^\top
\end{aligned}$$

where, according to the answer to Question 3-b), $\Psi(t)^\top$ is a 3-dimensional row vector. Now, by imposing

$$\frac{\partial}{\partial\alpha}J(\alpha) = [0,0,0],$$

after some algebra, the following linear problem is obtained:

$$\left\{\sum_{j=1}^{M}\Psi[t(j)]\Psi[t(j)]^\top\right\}\alpha = \sum_{j=1}^{M}c_m(j)\Psi[t(j)]$$

that has a unique solution α° if the matrix

$$\sum_{j=1}^{M}\Psi[t(j)]\Psi[t(j)]^\top$$

is non-singular.

Finally, using the specific form of vector $\Psi(t)$ obtained in the answer to Question 3-b), one has:

$$\sum_{j=1}^{M}\Psi[t(j)]\Psi[t(j)]^\top =$$

$$\begin{bmatrix}
M & \sum_{j=1}^{M}K\sin[100\pi t(j)] & \sum_{j=1}^{M}K\cos[100\pi t(j)] \\
\sum_{j=1}^{M}K\sin[100\pi t(j)] & \sum_{j=1}^{M}K^2\sin[100\pi t(j)]^2 & \sum_{j=1}^{M}K^2\sin[100\pi t(j)]\cos[100\pi t(j)] \\
\sum_{j=1}^{M}K\cos[100\pi t(j)] & \sum_{j=1}^{M}K^2\sin[100\pi t(j)]\cos[100\pi t(j)] & \sum_{j=1}^{M}K^2\cos[100\pi t(j)]^2
\end{bmatrix}$$

and

$$\sum_{j=1}^{M}c_m(j)\Psi[t(j)] = \begin{bmatrix}
\sum_{j=1}^{M}c_m(j) \\
K\sum_{j=1}^{M}c_m(j)\sin[100\pi t(j)] \\
K\sum_{j=1}^{M}c_m(j)\cos[100\pi t(j)]
\end{bmatrix}$$

[8 Marks]

c) Consider the set Θ given in the text of Question 3-c) of the exam paper:

$$\Theta = \{(0, 0.65), (0.005, 1.16), (0.01, -0.08), (0.015, -0.33), (0.02, 1.07)\}$$

and substitute the numerical values of $t(j)$ and $c_m(j)$ into the general solution obtained in the answer to Question 3-b).

After some algebra, one gets:

$$\begin{bmatrix} 5 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 2.47 \\ 1.49 \\ 1.8 \end{bmatrix} \Longrightarrow \alpha^\circ = \begin{bmatrix} \alpha_1^\circ \\ \alpha_2^\circ \\ \alpha_3^\circ \end{bmatrix} = \begin{bmatrix} 0.4 \\ 0.745 \\ 0.466 \end{bmatrix}$$

and thus the optimal (in the least-squares sense) parametric model $\mathcal{M}$ of the unknown current $c(t)$ takes on the form

$$\mathcal{M} : \widehat{c}(t, \alpha) = 0.4 + 0.745 \cdot \sin(100\pi t) + 0.466 * \cos(100\pi t)$$

Fig. 3.1 shows the diagram of the approximate function $\widehat{c}(t, \alpha)$ (dashed line) and the five noisy measurements c_m of the current belonging to the set Θ.

It can be noticed that the approximate model fits quite well the measurements. The availability of a larger measurement set would be beneficial to devise an improved parametric model.

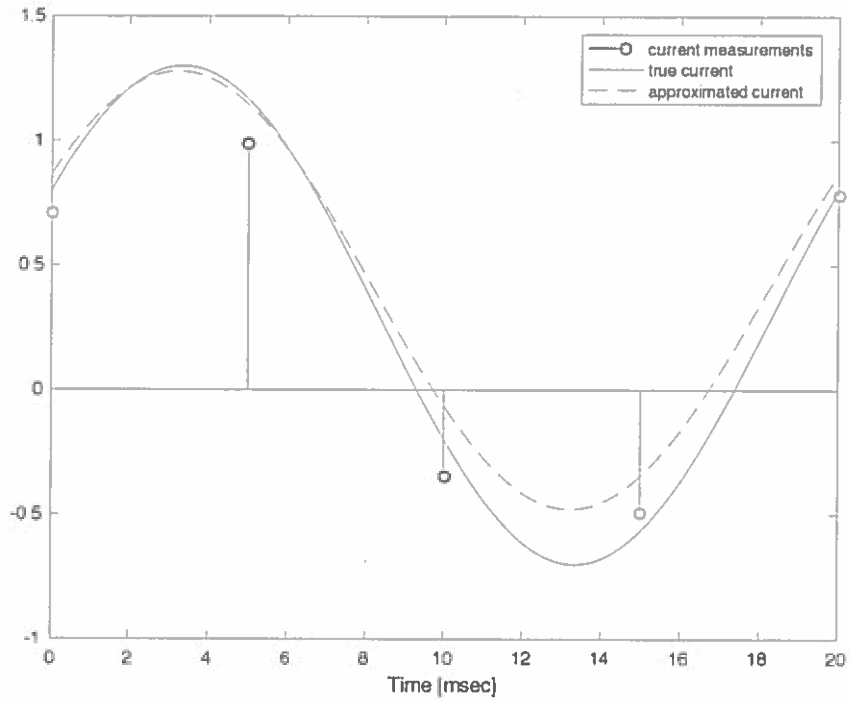

Figure 3.1

Remark for the students (not part of the solution of the exam paper).

In Fig. 3.1, the time-behaviour of the true current is also shown (continuous line) for completeness and for the sake of comparison.

[5 Marks]

4. a) One refers to Case 1 in the text of the exam paper where $y(\cdot) = v_1(\cdot)$. The model in prediction form is

$$\widehat{\mathscr{M}}(\theta): \quad \hat{y}(t|t-1) = ay(t-1)$$

The estimate $\hat{a}(N)$ converges almost surely to the minima of

$$\begin{aligned}\bar{J}(a) &= \mathbb{E}\left\{[y(t)-\hat{y}(t|t-1)]^2\right\} = \mathbb{E}\left\{[y(t)-ay(t-1)]^2\right\} \\ &= \mathbb{E}[y(t)^2] - 2a\mathbb{E}[y(t)y(t-1)] + a^2\mathbb{E}[y(t-1)^2] = (1+a^2)\gamma_y(0) - 2a\gamma_y(1)\end{aligned}$$

where $\gamma_y(\tau)$ denotes the correlation function of the process $y(\cdot)$.

Therefore, $\bar{J}(a)$ has a single minimum attained for $\bar{a} = \gamma_y(1)/\gamma_y(0)$. One computes $\gamma_y(1)$ as follows:

$$\gamma_y(1) = \mathbb{E}[y(t)y(t-1)] = \mathbb{E}\left\{\left[-\frac{1}{2}y(t-1)+\xi(t)\right]y(t-1)\right\} = -\frac{1}{2}\gamma_y(0)$$

Thus $\bar{a}_1 = \gamma_y(1)/\gamma_y(0) = -\frac{1}{2}$.

[6 Marks]

b) One refers to Case 2 in the text of the exam paper, where $y(\cdot) = v_2(\cdot)$, and computes $\gamma_y(0)$ and $\gamma_y(1)$:

$$\begin{aligned}\gamma_y(0) &= \mathbb{E}[y(t)^2] = \mathbb{E}\left\{\left[-\tfrac{1}{2}y(t-1)+\xi(t)+\tfrac{1}{4}\xi(t-1)\right]^2\right\} \\ &= \tfrac{1}{4}\gamma_y(0) + \mathrm{var}[\xi(t)] + \tfrac{1}{16}\mathrm{var}[\xi(t)] - \tfrac{1}{4}\mathbb{E}[y(t-1)\xi(t-1)]\end{aligned}$$

Hence $\gamma_y(0) = \frac{13}{12}\mathrm{var}[\xi(t)] = \frac{13}{12}$.

$$\begin{aligned}\gamma_y(1) &= \mathbb{E}[y(t)y(t-1)] = \mathbb{E}\left\{\left[-\tfrac{1}{2}y(t-1)+\xi(t)+\tfrac{1}{4}\xi(t-1)\right]y(t-1)\right\} \\ &= -\tfrac{1}{2}\gamma_y(0) + \tfrac{1}{4}\mathrm{var}[\xi(t)] = -\tfrac{7}{24}\mathrm{var}[\xi(t)] = -\tfrac{7}{24}\end{aligned}$$

Thus $\bar{a}_2 = \gamma_y(1)/\gamma_y(0) = -\frac{7}{26}$.

[7 Marks]

c) We have:

$$\mathrm{var}[v_1(t)-\bar{a}_1v_1(t-1)] = \mathbb{E}\left\{\left[-\frac{1}{2}v_1(t-1)+\xi(t)+\frac{1}{2}v_1(t-1)\right]^2\right\} = \mathrm{var}[\xi(t)]$$

$$\begin{aligned}\mathrm{var}[v_2(t)-\bar{a}_2v_2(t-1)] &= \mathbb{E}\left\{\left[-\tfrac{3}{13}v_2(t-1)+\xi(t)+\tfrac{1}{4}\xi(t-1)\right]^2\right\} \\ &= \tfrac{2705}{2704}\mathrm{var}[\xi(t)]\end{aligned}$$

As should be expected because of the presence of the coloured noise in model generating $v_2(\cdot)$, the prediction error $v_2(t)-\bar{a}_2v_2(t-1)$ is not white and

$$\mathrm{var}[v_2(t)-\bar{a}_2v_2(t-1)] = \frac{2705}{2704}\mathrm{var}[\xi(t)] > \mathrm{var}[\xi(t)] = \mathrm{var}[v_1(t)-\bar{a}_1v_1(t-1)]$$

Finally, let us address the case $\xi(\cdot) \sim WN(0,3)$. As we have seen in the previous answers, $\gamma_y(0)$ and $\gamma_y(1)$ are always proportional to $\mathrm{var}[\xi(t)]$ and hence the values $\bar{a}_1 = \gamma_y(1)/\gamma_y(0)$ when $y(\cdot) = v_1(\cdot)$ and $\bar{a}_2 = \gamma_y(1)/\gamma_y(0)$ when $y(\cdot) = v_2(\cdot)$ do not depend on $\mathrm{var}[\xi(t)]$.

[7 Marks]