Imperial College
London

M4/5P19

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May-June 2017

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Measure and Integration

Date: Wednesday 17 May 2017

Time: 14:00 - 16:00

Time Allowed: 2.5 Hours

This paper has 5 Questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the relevant main book(s) are full.

All required additional material will be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers to each question as per the table below.

Raw Mark	Up to 12	13	14	15	16	17	18	19	20
Extra Credit	0	½	1	1 ½	2	2 ½	3	3 ½	4

- Each question carries equal weight.
- Calculators may not be used.

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1. (a) Let $E \subset \mathbb{R}^d$ be a Lebesgue measurable set. Define what it means for a function $f : E \to \mathbb{R}$ to be Lebesgue measurable. In the following, measurable always means Lebesgue measurable.

(b) Let $f : \mathbb{R}^d \to \mathbb{R}$ be measurable. Show that the following functions are measurable:

(i) $g : \mathbb{R}^d \to \mathbb{R}$ defined by $g(x) = f(x+v)$ for fixed $v \in \mathbb{R}^d$,

(ii) $h : \mathbb{R}^d \to \mathbb{R}$ defined by $h(x) = \sin |f(x)|$.

(c) Let $\mathbb{R}^d = A \cup B$ with A and B measurable sets. Prove that $f : \mathbb{R}^d \to \mathbb{R}$ is measurable if and only if the restrictions $f|_A$ and $f|_B$ are measurable.

(d) TRUE or FALSE (Please give a proof or a counterexample.)

(i) There is a non-measurable function whose square is measurable.

(ii) The square of a measurable function is always measurable.

(iii) The square of a Lebesgue integrable function is always Lebesgue integrable.

2. (a) State but do not prove the Montone Convergence Theorem.

(b) Prove Fatou's Lemma from the Montone Convergence Theorem.
HINT: Start from $\inf_{n \geq k} f_n \leq f_j$ for all $j \geq k$.

(c) Let (f_n) be a sequence of measurable functions on $\mathbb{R}^d$. Assume

$$f_n \to f \text{ pointwise} \quad , \quad f_n \geq 0 \quad , \quad \int f_n \to \int f < \infty .$$

Prove that $\int |f - f_n| \to 0$.
HINT: Verify and use $|f - f_n| = 2(f - f_n)^+ - (f - f_n)$, then apply the Dominated Convergence Theorem.

(d) Show by an example that the conclusion in (c) fails without the assumption that $f_n \geq 0$.

3. (a) State but do not prove Fubini's theorem for the Lebesgue integral.

   (b) Use Fubini's theorem to prove that

$$\lim_{n\to\infty}\int_0^n \frac{\sin x}{x}dx = \frac{\pi}{2}.$$

   Hint: Use that $\int_0^\infty e^{-tx}dt = \frac{1}{x}$.

4. (a) Define what it means for a function $g : [a,b] \to \mathbb{R}$ to be absolutely continuous.

   (b) Prove that if $g : [a,b] \to \mathbb{R}$ is absolutely continuous and $|g'| \leq C$ a.e. on $[a,b]$ for some constant C, then g is Lipschitz.

   For parts (c) and (d) let $f : [0,1] \to [0,1]$ be the Cantor-Lebesgue function. You can use without proof all properties of the Cantor-Lebesgue function established in class or on the example sheets.

   (c) Define $h : [0,1] \to [0,1]$ by $h(x) = \frac{1}{2}x + \frac{1}{2}f(x)$. Prove that $h : [0,1] \to [0,1]$ is a bijection. Prove that h is of bounded variation but not absolutely continuous.

   (d) Define the curve $\gamma : [0,1] \to \mathbb{R}^2$ by $\gamma(t) = \begin{bmatrix} x(t) = t \\ y(t) = f(t) \end{bmatrix}$, which is the graph of f. Explain why γ is rectifiable and compute its length.

5. Mastery Question. Let $(X, \mathcal{M}, \mu)$ be a measure space and $p \in [1, \infty)$.

(a) Define the space $L^p(X, \mu)$.

(b) Prove the Chebychev inequality: If $f \in L^p(X, \mu)$, then for $\lambda \in (0, \infty)$ one has the estimate

$$\mu(\{x \in X \mid |f(x)| > \lambda\}) \leq \lambda^{-p} \|f\|_{L^p}^p .$$

(c) Let $f_1, f_2, ... f_n$ be real-valued measurable functions on X. Prove that for $p_1, p_2, ..., p_n$ real numbers in $(1, \infty)$ satisfying $\frac{1}{p_1} + \frac{1}{p_2} + ... + \frac{1}{p_n} = 1$ one has the estimate

$$\|f_1 \cdots f_n\|_{L^1} \leq \|f_1\|_{L^{p_1}} \cdots \|f_n\|_{L^{p_n}} .$$

(d) Let (f_n) be a sequence in $L^p(X, \mu)$ and $f \in L^p(X, \mu)$ such that $\lim_{n\to\infty} \|f_n - f\|_{L^p} = 0$. Let (g_n) be a sequence of real-valued measurable functions on X with $g_n \to g$ a.e. on X and $|g_n| \leq M$ for every $n \in \mathbb{N}$. Prove that

$$\lim_{n\to\infty} \|f_n g_n - fg\|_{L^p} = 0 .$$

Solutions Measure and Integration Exam 2017

March 6, 2017

1 Question 1

1.1 Part (a) (SEEN: 2 points)

The function $f : E \to \mathbb{R}$ is Lebesgue measurable if the set $f^{-1}(U) = \{x \in E \mid f(x) \in U\}$ is measurable for any $U \subset \mathbb{R}$ open. [Various equivalent definitions also receive full marks, of course.]

1.2 Part (b) (SEEN SIMILAR: 3+3points)

For (i), since f is measurable, the sets

$$f^{-1}(U) = \{x \mid f(x) \in U\} = E_U$$

are measurable for any $U \subset \mathbb{R}$ open. Then the v-translated sets $E'_U = \{x \mid x + v \in E_U\}$ are also measurable. But

$$g^{-1}(U) = \{x \mid g(x) \in U\} = \{x \mid f(x+v) \in U\} = E'_U .$$

For (ii), note that the function $k : x \mapsto \sin|x|$ is continuous and that $h(x) = k \circ f(x)$. In lectures we observed that the composition of a continuous function with a measurable function is again measurable.

1.3 Part (c) (UNSEEN: 6 points)

Note first that the restriction $f|_A$ is measurable means

$$(f|_A)^{-1}(U) = f^{-1}(U) \cap A \quad \text{is measurable for any } U \subset \mathbb{R} \text{ open}$$

and similarly for $f|_B$. If f is measurable on $\mathbb{R}^d = A \cup B$, then $f^{-1}(U)$ is measurable for any $U \subset \mathbb{R}$ open, which since A and B are measurable implies that $f^{-1}(U) \cap A$ and $f^{-1}(U) \cap B$ are measurable for any $U \subset \mathbb{R}$ open, which says that f is measurable on A and B.

Conversely, assuming that both $f^{-1}(U) \cap A$ and $f^{-1}(U) \cap B$ are measurable for any $U \subset \mathbb{R}$ open, we have

$$f^{-1}(U) = \left(f^{-1}(U) \cap A\right) \bigcup \left(f^{-1}(U) \cap B\right)$$

which is measurable.

1.4 Part (d) (SEEN SIMILAR: 2+2+2 points)

1. TRUE. Let $\mathcal{N}$ be a non-measurable set of $[0,1]$ as defined in class and let f be equal to 1 on $\mathcal{N}$ and -1 on the complement. The function f is not measurable but $f^2 \equiv 1$ the constant function is.
2. TRUE. The map $\Phi : \mathbb{R} \to \mathbb{R}$ given by $x \mapsto x^2$ is continuous. We have $f^2(x) = \Phi \circ f(x)$ which is the composition of a continuous function with a measurable function. Hence f^2 is measurable (as in part (ii) of (b)).
3. FALSE: Take $f(x) = x^{-\frac{3}{4}}$ on $(0,1)$.

2 Question 2

2.1 Part (a) (SEEN: 6 points)

Theorem 2.1 (Monotone Convergence Theorem, MCT). *Let (f_n) be a sequence of (extended real-valued) measurable functions with $f_n \geq 0$ and $f_n \to f$ a.e. Suppose in addition $f_n(x) \leq f_{n+1}(x)$ holds a.e. in x for any n. Then*

$$\lim_{n\to\infty} \int f_n = \int f . \tag{1}$$

2.2 Part (b) (UNSEEN: 6 points)

We let (f_n) be a sequence of non-negative measurable functions with $f_n \to f$ a.e. Starting from the hint

$$\begin{aligned} \inf_{n\geq k} f_n &\leq f_j \quad \text{for all } j \geq k \\ \int \inf_{n\geq k} f_n &\leq \int f_j \quad \text{for all } j \geq k \quad \text{by monotonicity} \\ \int \inf_{n\geq k} f_n &\leq \inf_{j\geq k} \int f_j \end{aligned} \tag{2}$$

Now since $(\tilde{f}_k) = (\inf_{n\geq k} f_n)$ is an increasing sequence of non-negative measurable functions, the MCT applies on the left hand side when taking the limit $k \to \infty$, while on the right hand side we obtain the $\liminf \int f_j$. Hence $\int f \leq \liminf_{j\to\infty} \int f_j$, which is the conclusion of Fatou's Lemma.

2.3 Part (c) (UNSEEN: 6 points)

We first verify the hint (2 points):

$$|f - f_n| = (f-f_n)^+ + (f-f_n)^- = 2(f-f_n)^+ - \Big((f-f_n)^+ - (f-f_n)^-\Big) = 2(f-f_n)^+ - (f-f_n) .$$

Note that $2(f-f_n)^+ \leq 2f$ (since $f_n \geq 0$). Since $\int f < \infty$ we can therefore apply the dominant convergence theorem to deduce

$$\lim_{n\to\infty} \int (f-f_n)^+ = \int \lim_{n\to\infty} (f-f_n)^+ = 0 .$$

Since by the third assumption also

$$\lim_{n\to\infty} \int (f-f_n) = \int f - \lim_{n\to\infty} \int f_n = 0$$

we deduce that $\lim_{n\to\infty} \int |f - f_n| = 0$ by plugging in the hint and using the limits above (4 points).

2.4 Part (d) (UNSEEN: 2 points)

Take a sequence (f_n) as follows $f_n = \frac{1}{n}$ on $[0, n]$ and $f_n = -\frac{1}{n}$ on $[-n, 0)$ and zero otherwise. Then $f_n \to 0$ pointwise and $\int f_n = 0$ for all n. But $\int |f - f_n| = \int |f_n| = 2$ does not go to zero.

3 Question 3

3.1 Part (a) (SEEN: 8 points)

Theorem 3.1 (Fubini). *Let $f : \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \to \mathbb{R}$ be integrable on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$. Then for almost every $y \in \mathbb{R}^{d_2}$*

1. *The slice f^y is integrable in $\mathbb{R}^{d_1}$*
2. *The function defined by $y \mapsto \int_{\mathbb{R}^{d_1}} f^y(x)\,dx$ is integrable in $\mathbb{R}^{d_2}$.*

Moreover, the integral of f can be computed iteratively

3.
$$\int_{\mathbb{R}^{d_2}} dy \int_{\mathbb{R}^{d_1}} f(x,y)\,dx = \int_{\mathbb{R}^{d_2}} dy \int_{\mathbb{R}^{d_1}} f^y(x)\,dx = \int_{\mathbb{R}^d} f\,.$$

3.2 Part (b) (UNSEEN: 12 points)

Using the hint we first note

$$\int_0^n \frac{\sin x}{x} dx = \int_0^n dx \sin x \int_0^\infty e^{-tx} dt\,. \quad \text{(2 points)}$$

We would like to apply Fubini to change the order of integration. To justify this we need that $\sin x \cdot e^{-tx}$ is integrable over $[0,n] \times [0,\infty)$. To compute that the integral of $|\sin x| e^{-tx}$ over $[0,n] \times [0,\infty)$ is finite we apply Tonelli's theorem which allows us to compute $\int\int |\sin x| e^{-tx}$ in any order, so

$$\int_0^n |\sin x| \int_0^\infty e^{-tx} dt = \int_0^n \frac{|\sin x|}{x} dx < \infty\,.$$

where the inequality follows since $\frac{\sin x}{x}$ is bounded by 1 in $[0,\pi]$ and trivially by $\frac{1}{x}$ in $[\pi, n]$. (3 points) By Fubini we can now interchange the order of integration and compute

$$\int_0^\infty dt \int_0^n \sin x\; e^{-tx} dx\,.$$

An integration by parts yields

$$\int_0^n \sin x e^{-tx} dx = \frac{(1 - \cos n e^{-tn}) + \sin n e^{-tn}(-t)}{1+t^2} \qquad \text{(4 points)}$$

To compute

$$\lim_{n\to\infty} \int_0^n dt \frac{(1 - \cos n e^{-tn}) + \sin n e^{-tn}(-t)}{1+t^2} = \lim_{n\to\infty} \int_0^\infty dt \chi_{[0,n]} \frac{(1 - \cos n e^{-tn}) + \sin n e^{-tn}(-t)}{1+t^2}$$

we apply the dominant convergence theorem noting that the integrand is dominated for $n \geq 1$ by $\frac{1+te^{-t}}{1+t^2}$ which is integrable, which reduces the problem to computing $\int_0^\infty dt \frac{1}{1+t^2} = \frac{\pi}{2}$ (3 points).

4 Question 4

4.1 Part (a) (SEEN: 4 points)

The function $g : [a,b] \to \mathbb{R}$ is absolutely continuous if for any $\epsilon > 0$ there exists a $\delta > 0$ such that for any collection of disjoint intervals (a_k, b_k), $k = 1, ..., N$ of $[a,b]$ we have

$$\sum_{k=1}^{N}(b_k - a_k) < \delta \qquad \Longrightarrow \qquad \sum_{k=1}^{N}|g(b_k) - g(a_k)| < \epsilon\,.$$

4.2 Part (b) (SEEN SIMILAR: 4 points)

Since g is absolutely continuous on $[a,b]$ the fundamental theorem of calculus for the Lebesgue integral holds on any interval $[x,y]$ with $a \leq x \leq y \leq b$. Therefore

$$|g(y) - g(x)| = |\int_x^y g'(t)\,dt| \leq C|y-x|\,,$$

where the assumption has been used in the last step. This shows that g is Lipschitz on $[a,b]$.

4.3 Part (c) (SEEN SIMILAR: 3+3 points)

(i) We first note that h is continuous being the sum of two continuous functions. It is also of bounded variation since again both f and x are. By a result from lectures it is differentiable a.e. and clearly $h'(x) = \frac{1}{2}$ a.e. (recall $f'(x) = 0$ a.e.). It follows from a result in lectures that for $x \leq y$

$$h(y) - h(x) \geq \int_x^y h'(t)dt = \frac{1}{2}(y-x) \geq 0 \tag{3}$$

and hence that h is injective. Since $h(0) = 0$ and $h(1) = 1$ and h is continuous we conclude by the Intermediate Value Theorem that h is also surjective.

(ii) The inequality (3) establishes that h is monotone. Since it is also bounded, by a result from lectures h is BV. It cannot be AC because then the FTC would hold. However $1 - 0 = h(1) - h(0) \neq \int_0^1 \frac{1}{2}dt = \frac{1}{2}$.

4.4 Part (d) (UNSEEN: 6 points)

Let $0 = t_0 < t_1 < ... < t_N = 1$ be any partition of $[0,1]$. We compute

$$\sum_{j=1}^{N}|\gamma(t_j) - \gamma(t_{j-1})| = \sum_{j=1}^{N}\sqrt{(t_j - t_{j-1})^2 + (f(t_j) - f(t_{j-1}))^2} \leq \sum_{j=1}^{N}|t_j - t_{j-1}| + \sum_{j=1}^{N}|f(t_j) - f(t_{j-1})| = 2$$

using that f is monotone. This shows that γ is rectifiable and $L(\gamma) \leq 2$. To show the length is equal to 2 we find for $\epsilon > 0$ a partition whose length is bigger than $2 - \epsilon$. This is done as follows. We partition $[0,1]$ into 3^n intervals of length 3^{-n} and call this partition $\mathcal{P}_n$. We recall that 2^n of these intervals, say I_i, belong to C_n (the sets used in the definition of the Cantor-Set). By construction of the Cantor Lebesgue function we know that f increases by $\frac{1}{2^n}$ on each interval I_i (in the sense that $f(t_i) - f(t_{i-1}) = \frac{1}{2^n}$ for the endpoints of I_i) while f is constant on the other $3^n - 2^n$ intervals. Therefore, we have

$$\sum_{\mathcal{P}_n}|\gamma(t_j) - \gamma(t_{j-1})| = \sum_{i=1}^{2^n}\sqrt{\left(\frac{1}{3^n}\right)^2 + \left(\frac{1}{2^n}\right)^2} + \sum_{i=1}^{3^n-2^n}\sqrt{\left(\frac{1}{3^n}\right)^2 + 0} = \sqrt{(2/3)^{2n}+1} + 1 - (2/3)^n\,.$$

Choosing n sufficiently large, we find that the partition has length arbitrarily close to 2.

5 Solution to the Mastery Question

5.1 Part (a) (SEEN, 3 points)

For f a measurable (say real-valued) measurable function on X we define

$$\|f\|_{L^p} := \left(\int_X |f|^p d\mu(x)\right)^{\frac{1}{p}}$$

and

$$L^p(X,\mu) := \{f : X \to \mathbb{R} : \ f \text{ is measurable and } \ \|f\|_{L^p} < \infty\}.$$

5.2 Part (b) (SEEN SIMILAR, 3 points)

This is proven just as the L^1-case treated on one of the example sheets:

$$\int_X |f|^p d\mu(x) \geq \int_{\{x\in X \ | \ |f(x)|>\lambda\}} |f|^p d\mu(x) \geq \lambda^p \cdot \mu\left(\{x \in X \mid |f(x)| > \lambda\}\right)$$

which is already the claim.

5.3 Part (c) (SEEN SIMILAR, 6 points)

This is proven by induction. The case $n = 2$ is Hoelder's inequality. Assume now the inequality holds for $n-1$ and let $p_1, ..., p_n$ in $(1,\infty)$ be given with $\frac{1}{p_1} + ... + \frac{1}{p_n} = 1$. We have

$$\|f_1 \cdot ... \cdot f_n\|_{L^1} \leq \|f_1 \cdot ... f_{n-1}\|_{L^q} \|f_n\|_{L^{p_n}}$$

for $\frac{1}{q} + \frac{1}{p_n} = 1$ by Hoelder's inequality. Furthermore,

$$\|f_1 \cdot ... f_{n-1}\|_{L^q} = (\||f_1|^q ... |f_{n-1}|^q\|_{L^1})^{\frac{1}{q}} = \|f_1\|_{L^{qr_1}} ... \|f_{n-1}\|_{L^{qr_{n-1}}}$$

for any $r_1, ..., r_{n-1}$ in $(1,\infty)$ with $\frac{1}{r_1} + \frac{1}{r_2} + ... + \frac{1}{r_{n-1}} = 1$ by the induction assumption. If we choose $r_i = \frac{p_i}{q}$ for $i = 1, ... n-1$, the desired inequality follows – hence all we need to check is whether this choice satisfies $r_i > 1$ and $\frac{1}{r_1} + \frac{1}{r_2} + ... + \frac{1}{r_{n-1}} = 1$. Now from $\frac{1}{q} + \frac{1}{p_n} = 1$ and $\frac{1}{p_1} + ... + \frac{1}{p_{n-1}} + \frac{1}{p_n} = 1$ we deduce

$$\frac{1}{q} = \frac{1}{p_1} + ... + \frac{1}{p_{n-1}}$$

from which both inequalities are immediate.

5.4 Part (d) (UNSEEN, 8 points)

We clearly have $|g| \leq M$ a.e. Write

$$\|f_n g_n - fg\|_{L^p} \leq \|f(g_n - g) + g_n(f_n - f)\|_{L^p} \leq \|f(g_n - g)\|_{L^p} + \|g_n(f_n - f)\|_{L^p}.$$

Note that the second term on the right hand side goes to zero as $n \to \infty$ in view of

$$\|g_n(f_n - f)\|_{L^p} \leq M\|f_n - f\|_{L^p},$$

while for the first term on the right hand side we observe that the integrand is dominated by

$$\int_X |f|^p |g_n - g|^p d\mu(x) \leq \int_X |f|^p \left(|g_n| + |g|\right)^p d\mu(x) \leq \int_X 2^p M^p |f|^p d\mu(x) < \infty,$$

which allows to apply the dominant convergence theorem:

$$\lim_{n\to\infty} \int_X |f|^p |g_n - g|^p d\mu(x) = \int_X |f|^p \lim_{n\to\infty} |g_n - g|^p d\mu(x) = 0$$

using the pointwise convergence $g_n \to g$ a.e.