Mathematics EE1-10A SOLUTIONS Marks

1 (a) i) $z = i^i = \left(e^{+i\frac{\pi}{2}}\right)^i = e^{-\pi/2}$ (2)

ii) $\frac{3-i}{2+i} \cdot \frac{2-i}{2-i} = \frac{6-5i+i^2}{4+1} = 1-i$ (2)

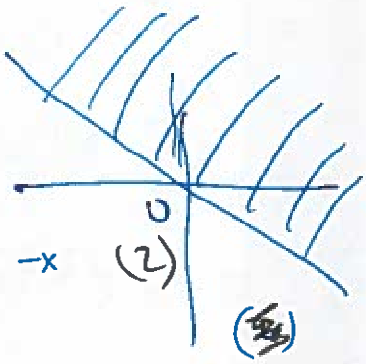
(b) All positive, so can square $\Rightarrow$

$|x+iy+i|^2 > |x+iy-1|^2 \Rightarrow$

$x^2+(y+1)^2 > (x-1)^2+y^2$

$2y+1 > -2x+1$ (2)

$y > -x$



$\Rightarrow$ ½ of plane $\mathbb{C}$, above diagonal $y=-x$ (2)

(4)

(c) $\lim_{x\to 0} \frac{x^2}{1-\left(1-\frac{1}{2}x^2+\frac{1}{4!}x^4-\dots\right)}$ (Maclaurin) (2)

$= \lim_{x\to 0} \frac{1}{+\frac{1}{2}-\frac{1}{4!}x^2+\dots} = \underline{\underline{2}}$ (2)

(d) Check: "$\frac{0}{0}$" $\to$ apply l'Hôpital

$\lim = \lim_{x\to\frac{\pi}{3}} \frac{-\sec^2 x}{-3\sin(3x)} = \lim_{x\to\frac{\pi}{3}} \frac{1}{3\sin(3x)\cos^2 x}$ (2)

$\Rightarrow \frac{1}{3\cdot 0\cdot\left(\frac{1}{2}\right)^2} \to \infty$ (2)

limit does not exist

(e) $y = (\ln x)^x \Rightarrow$

$\ln y = x\ln(\ln x)$

$\frac{1}{y}\frac{dy}{dx} = \ln(\ln x) + x\,\frac{1}{\ln x}\cdot\frac{1}{x}$ (2)

$\frac{dy}{dx} = y\left[\ln(\ln x) + \frac{1}{\ln x}\right] = \ln(\ln x)(\ln x)^x + (\ln x)^{x-1}$ (2)

or similar

(f) $\frac{d}{dx}(xy) + \frac{d}{dx}(\ln(xy)) = 0 \Rightarrow$ (1)

$1\cdot y + x\frac{dy}{dx} + \frac{1}{xy}\cdot\left[y + x\frac{dy}{dx}\right] = 0 \Rightarrow \left(x+\frac{1}{y}\right)\frac{dy}{dx} = -\left(y+\frac{1}{x}\right)$

$\frac{dy}{dx} = -\frac{\left(y+\frac{1}{x}\right)}{x+\frac{1}{y}} = -\frac{y}{x}$

(3)

Marks

(g) use $x = \sin u$ then $\begin{cases} x=0 \to u=0 \\ x=1 \to u=\frac{\pi}{2} \\ dx = \cos u\, du \end{cases}$

$$\Rightarrow \int_0^1 \sqrt{1-x^2}\,dx$$

$$= \int_0^{\pi/2} \sqrt{1-\sin^2 u}\,\cos u\, du$$

$$= \int_0^{\pi/2} \cos^2 u\, du = \frac{1}{2}\int_0^{\pi/2} \cos 2u + 1\, du \qquad (2)$$

$$= \frac{1}{2}\left[\frac{\sin 2u}{2} + u\right]_0^{\pi/2} = \pi/4 \qquad (2)$$

(h) substitute $u = \ln x$

$du = \frac{1}{x}dx$, $\quad x=1 \to u=0$, $x=e \to u=1$ (2)

$$\int_1^e \frac{(\ln x)^2}{x}dx = \int_0^1 u^2\,du = \frac{u^3}{3}\Big]_0^1 = \underline{\underline{\frac{1}{3}}} \qquad (2)$$

(i) Ratio test

(I) $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{n+1}{3^{n+1}}\cdot\frac{3^n}{n}\right| = \frac{1}{3}\left|\frac{n+1}{n}\right| \to \frac{1}{3}$ as $n\to\infty$ (2)

$<1 \to$ converges

II $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)!}{3^{n+1}}\cdot\frac{3^n}{n!}\right| = \left|\frac{n+1}{3}\right| \to \infty$, diverges (2)

(j) odd $\to a_n = 0$

$\frac{1}{2}$ range: $\quad b_n = \frac{2}{\pi}\int_0^{\pi} 1\cdot\sin(nt)\,dt$

$$= \frac{2}{\pi}\left[-\frac{\cos(nt)}{n}\right]_0^{\pi}$$

$$= \frac{-2}{n\pi}\left[(-1)^n - 1\right] = \begin{cases} \frac{4}{n\pi}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases} \qquad (3)$$

$$f(t) = \frac{4}{\pi}\sum_{\substack{n=1 \\ (n\text{ odd})}}^{\infty} \frac{1}{n}\sin(nt) \qquad \Rightarrow \text{ as req}^d \quad (n = 2m-1) \qquad (4)$$

Q1: seen similar

② a) write $x^q\left(\sqrt{x-1}-\sqrt{x}\right)$

$$= x^q\sqrt{x}\left(\sqrt{1-\tfrac{1}{x}}-1\right)$$

$$= x^{q+\frac{1}{2}}\left[\left(1-\tfrac{1}{2}\left(\tfrac{1}{x}\right)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2}\left(\tfrac{1}{x}\right)^2+\text{h.o.t}\right)-1\right]$$

(Bin. expansion, given $\left|\frac{1}{x}\right|<1$ as $x\to\infty$)

$$= x^{q+\frac{1}{2}}\left[-\frac{1}{2x}-\frac{1}{8x^2}\cdots\right] \quad (2)$$

To have finite and non zero limit require $q=\frac{1}{2}$

$$= -\frac{1}{2}-\frac{1}{8x} + \text{terms in } \frac{1}{x^3}, \frac{1}{x^4}\cdots$$

and $\lim\limits_{x\to\infty}$ is $-\frac{1}{2}$ (2)

b) Formula sheet (or otherwise)

$\Rightarrow \frac{d}{dx}\left(\cosh^{-1}x\right) = \frac{1}{\sqrt{x^2-1}}$

$\Rightarrow$ Let $u=\cosh^{-1}x$

$du = \frac{1}{\sqrt{x^2-1}}\,dx$

and integral $\Rightarrow \int u\,du = \frac{u^2}{2}+C$

$$= \frac{1}{2}\left[\cosh^{-1}x\right]^2+C$$

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a) seen similar b) new

~~b) unseen~~

c) i) $I_n = \int_0^{\pi} \cos^n x \, dx = \int_0^{\pi} \cos x \left(\cos^{n-1} x\right) dx$

$$= \sin x \cos^{n-1} x \Big]_0^{\pi} + (n-1)\int_0^{\pi} \cos^{n-2} x \, \sin^2 x \, dx \quad (2)$$

$$= (n-1)\int_0^{\pi} \cos^{n-2} x \, (1-\cos^2 x)\, dx$$

$$= (n-1)\left[I_{n-2} - I_n\right] \Rightarrow \quad (2)$$

$$I_n\left[1+n-1\right] = (n-1) I_{n-2} \Rightarrow$$

$$I_n = \frac{n-1}{n} I_{n-2} \quad (2)$$

ii) $I_0 = \int_0^{\pi} \cos^0 x \, dx = \int_0^{\pi} 1 \, dx = \pi$

$$\Rightarrow I_2 = \frac{2-1}{2} I_0 = \frac{\pi}{2}$$

$$I_4 = \frac{4-1}{4} I_2 = \frac{3\pi}{8}$$

$$I_6 = \frac{6-1}{6} I_4 = \frac{5\pi}{16} \quad (2)$$

d) $\frac{dy}{dx} = \frac{1}{x+\sqrt{x^2+1}}\left(1 + \frac{1}{2}(x^2+1)^{-1/2} \cdot 2x\right) \quad (2)$

$$= \frac{1}{x+\sqrt{x^2+1}}\left(1 + \frac{x}{\sqrt{x^2+1}}\right)$$

$$= \frac{1}{x+\sqrt{x^2+1}}\left(\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}\right) = \frac{1}{\sqrt{x^2+1}} \quad (2)$$

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both: similar to examples seen in class

3 a) i) Use substitution $t = \tan\left(\frac{x}{2}\right)$

$$\rightarrow dx = \frac{2\,dt}{1+t^2}$$

$$\sin x = \frac{2t}{1+t^2}$$

(Formula sheet)

$$\Rightarrow \int \frac{1}{1+\sin x}\,dx = \int \frac{1}{1+\frac{2t}{1+t^2}} \frac{2\,dt}{1+t^2} \qquad (2)$$

$$= \int \frac{2\,dt}{1+t^2+2t} = \int \frac{2\,dt}{(1+t)^2}$$

$$= -\frac{2}{1+t} + C$$

$$= -\frac{2}{1+\tan\left(\frac{x}{2}\right)} + C \qquad (2)$$

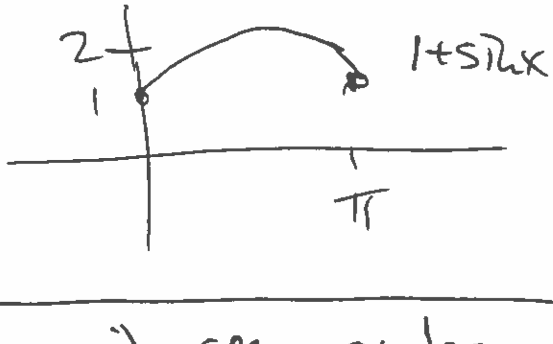
ii) Area : improper integral

requires $\displaystyle\lim_{k\to\pi} \int_0^k f(x)\,dx$

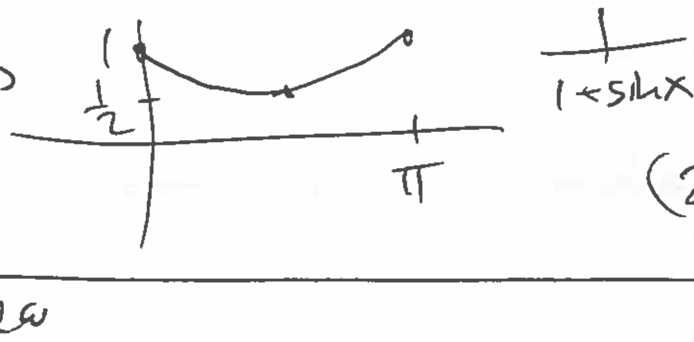
$$= \lim_{k\to\pi}\left[-\frac{2}{1+\tan\left(\frac{k}{2}\right)} + \frac{2}{1+\tan 0}\right]$$

as $k \to \pi$, $\tan\left(\frac{k}{2}\right) \to \infty$, so $\frac{2}{1+\tan\left(\frac{k}{2}\right)} \to 0$

leaving Area $= \frac{2}{1+0} = 2$ (2)



$\rightarrow$



(2)

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i) seen similar ii) new

Marks

b) de Moivre:

$$\cos 5\theta + i\sin 5\theta = (\cos\theta + i\sin\theta)^5$$

$$= \cos^5\theta + 5\cos^4\theta(i\sin\theta) + 10\cos^3\theta(i\sin\theta)^2 + 10\cos^2\theta(i\sin\theta)^3 + 5\cos\theta(i\sin\theta)^4 + (i\sin\theta)^5$$

$$= \text{Real Part} + i(5\sin\theta\cos^4\theta - 10\sin^3\theta\cos^2\theta + \sin^5\theta) \quad (2)$$

Hence

$$\sin 5\theta = \sin^5\theta + 5\sin\theta(1-\sin^2\theta)^2 - 10\sin^3\theta(1-\sin^2\theta)$$

$$= \sin^5\theta + 5\sin\theta(1 - 2\sin^2\theta + \sin^4\theta) - 10\sin^3\theta + 10\sin^5\theta$$

$$= \underset{C}{16}\sin^5\theta - \underset{B}{20}\sin^3\theta + \underset{A}{5}\sin\theta \quad (2)$$

c) i) $f(x) = \ln(1+x) \Rightarrow f(0) = 0$

$f'(x) = \dfrac{1}{1+x}$ $\qquad f'(0) = 1$

$f''(x) = -\dfrac{1}{(1+x)^2}$ $\qquad f''(0) = -1$

$f'''(x) = +\dfrac{2}{(1+x)^3}$ $\qquad f'''(0) = 2!$

$f^{(4)}(x) = -\dfrac{2\cdot 3}{(1+x)^4}$ $\qquad f^{(4)}(0) = -3!$

$f^{(5)}(x) = +\dfrac{2\cdot 3\cdot 4}{(1+x)^5}$

Maclaurin series is $f(0) + f'(0)x + \dfrac{f''(0)}{2!}x^2 + \dfrac{f'''(0)}{3!}x^3 + \dfrac{f^{(4)}(0)}{4!}x^4 \cdots$

$$= x - \frac{1}{2}x^2 + \frac{2!}{3!}x^3 - \frac{3!}{4!}x^4 + R_4$$

$$= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + R_4 \quad (3)$$

as reqd

where $R_4 = \dfrac{f^{(5)}(c)}{5!}x^5$

$$= \frac{1}{5(1+c)^5}x^5 \qquad (0 < |c| < |x|) \quad (2)$$

Look: Similar to examples seen in lectures

(i) $\left|\frac{a_{n+1}}{a_n}\right| = \left| - \frac{\frac{1}{n+1} x^{n+1}}{\frac{1}{n} x^n} \right|$

$= \left|\frac{n}{n+1}\right| |x| < 1$ (as $n \to \infty$)

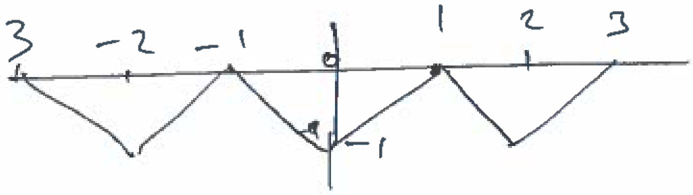
for convergence

as $\lim_{n\to\infty} \frac{n}{n+1} = 1 \Rightarrow |x| < 1$

as required. (3)

{similar to examples seen}

Q4 Mark

a) i)  (2)

ii) Even so $b_n = 0$. For a_n use ½-range series:

$$a_n = 2\int_0^1 (x-1)\cos(n\pi x)\,dx$$

$$a_0 = 2\int_0^1 x-1\,dx = 2\left[\frac{x^2}{2} - x\right]_0^1 = 2\left[\frac{1}{2}-1\right] = -1 \qquad (1)$$

$$a_n = 2\left\{\left[(x-1)\frac{\sin(n\pi x)}{n\pi}\right]_0^1 - \int_0^1 \frac{\sin(n\pi x)}{n\pi}\,dx\right\}$$

$$= 2\left[(x-1)\frac{\sin(n\pi x)}{n\pi} + \frac{\cos(n\pi x)}{(n\pi)^2}\right]_0^1 \qquad (2)$$

$$= 2\left\{0 - 0 + \frac{\cos(n\pi)}{(n\pi)^2} - \frac{1}{(n\pi)^2}\right]$$

$$= \begin{cases} -\frac{4}{(n\pi)^2}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases} \qquad (2)$$

$$\Rightarrow f(x) = -\frac{1}{2} - \frac{4}{\pi^2}\sum_{n \text{ odd}}^{\infty} \frac{1}{n^2}\cos(n\pi x)$$

$$= -\frac{1}{2} - \frac{4}{\pi^2}\sum_{m=1}^{\infty} \frac{1}{(2m-1)^2}\cos[(2m-1)\pi x] \qquad (1)$$

iii) Parseval →

$$2\int_0^1 (x-1)^2\,dx = \frac{1}{2}a_0^2 + \sum a_n^2 \Rightarrow$$

$$\frac{2}{3} = \frac{1}{2}(-1)^2 + \sum_{n \text{ odd}}\left(-\frac{4}{n^2\pi^2}\right)^2 \Rightarrow$$

$$\frac{1}{6} = \frac{16}{\pi^4}\sum_{\substack{n=1 \\ (n \text{ odd})}}^{\infty} \frac{1}{n^4} \rightarrow \frac{\pi^4}{96} = \sum_{n \text{ odd}} \frac{1}{n^4} \qquad (3)$$

Need $\sum_{n=1}^{\infty} \frac{1}{n^4} = \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} \frac{1}{n^4} + \sum_{\substack{n=1 \\ \text{even}}}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{96} + \sum_{m=1}^{\infty} \frac{1}{(2m)^4}$

$$= \frac{\pi^4}{96} + \frac{1}{16}\sum_{n=1}^{\infty} \frac{1}{n^4} \quad \text{(reindex)}$$

so finally $\left(1-\frac{1}{16}\right)\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{96} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{16}{15}\left(\frac{\pi^4}{96}\right) = \frac{\pi^4}{90}$ (3)

(b) i) $y = e^{-2x}$

$y' = -2e^{-2x}$

$y'' = 4e^{-2x}$

$\vdots$

$y^{(n)} = (-2)^n e^{-2x}$ (2)

ii) Need Leibnitz' Formula:

$$y^{(n)}(x) = x\frac{d^n}{dx^n}(\ln x) + n(1)\frac{d^{n-1}}{dx^{n-1}}(\ln x) + 0 + 0 \ldots \quad (*)$$

Have seen $\ln(x+1)$ in Q3

$\rightarrow \frac{d^n}{dx^n}(\ln x) = (-1)^{n+1}\frac{(n-1)!}{x^n}$ $\rightarrow$ into $*$.

$\rightarrow \frac{d^{n-1}}{dx^{n-1}}(\ln x) = (-1)^{n-2}\left(\frac{(n-2)!}{x^{n-1}}\right)$ (2)

to give $y^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{x^{n-1}} - (-1)^{n-1}\frac{n(n-2)!}{x^{n-1}}$

$= \frac{(-1)^{n-1}}{x^{n-1}}(n-2)!$

(as $(n-1)! - n(n-2)! = (n-2)!\{n-1-n\}$)

(valid for $n \geq 2$) (2)

(a) seen similar

(b) seen similar