Final

E3.20

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2010

EEE PART III/IV: MEng, BEng and ACGI

DISCRETE MATHEMATICS AND COMPUTATIONAL COMPLEXITY

Monday, 24 May 2:30 pm

Time allowed: 3:00 hours

There are FIVE questions on this paper.

Answer Question One (40 marks), Question Two (40 marks), and TWO of Questions Three to Five (30 marks each). Note that this paper is marked out of 140.

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : G.A. Constantinides

Second Marker(s) : M.M. Draief

NOTATION

The following notation may be used throughout this paper:

$\mathbb{R}$: The set of real numbers.

$\mathbb{Z}$: The set of integers.

$\mathbb{N}$: The set of natural numbers.

$\mathscr{P}(S)$: The power set of set S.

The Questions

1. **[Compulsory]**

a) For the sets $S_1 = \{\emptyset, a\}$, and $S_2 = \{2, 3\}$, list the elements of:

i) $S_1 \cup S_2$,

ii) $S_1 \cap S_2$,

iii) $S_1 - S_2$,

iv) $S_1 \times S_2$,

v) $\mathscr{P}(S_1)$.

[9]

b) Provide one example of each of the following functions from $\mathbb{N}$ to $\mathbb{N}$:

i) An injection but not a surjection,

ii) A surjection but not an injection,

iii) A bijection,

iv) Neither a surjection nor an injection.

[6]

c) Express each of these statements using predicate logic syntax. You may assume the existence of an addition operation "+" over integers, and an equality predicate "=", both with the usual meaning, *e.g.* 4 + 1 = 3 + 2. The universe of discourse should be the set of integers.

i) "No matter which two integers a and b I choose, $a + b$ has the same value as $b + a$".

ii) "If I want to sum any three integers, it doesn't matter whether I add the first two integers first, and then add the third, or whether I add the last two, and then add the first".

iii) "For every integer a, there is another integer b such that no matter which integer c I choose, if I add a to c and then add b to the result, I get back to c".

iv) "There is an integer to which I can add any integer a, and I get the result a".

[6]

d) Solve the following recurrence relations, in each case stating whether the resulting sequence a_n is $O(n)$.

i) $a_n = a_{n-1}$ for $n > 1$ with $a_1 = 1$,

ii) $a_n = 2a_{n-1} + 1$ for $n > 1$ with $a_1 = 1$,

iii) $a_n = a_{n-1} + a_{n-2} + 1$ for $n > 1$ with $a_0 = 0$, $a_1 = 0$.

iv) $a_n = \frac{3}{4}a_{n-1} - \frac{1}{8}a_{n-2}$ for $n > 1$ with $a_0 = 1$, $a_1 = 1$.

[9]

e) Provide one example each of a relation on $A = \{1, 2, 3\}$ that has each of the following properties. The cardinality of the relation should be at least 1 in all cases.

i) reflexive but not symmetric or transitive,

ii) transitive but not reflexive or symmetric,

iii) symmetric but not reflexive or transitive,

iv) reflexive and transitive but not symmetric,

v) reflexive and symmetric by not transitive,

vi) transitive and symmetric but not reflexive,

vii) reflexive, symmetric, and transitive.

[10]

2. **[Compulsory]**

For two sets A and B, let $A \to B$ denote the set of all functions from A to B. This question will repeatedly refer to the set $M = (\mathbb{N} \to \{0,1\})$.

a) We can define a function $q : M \to \mathscr{P}(\mathbb{N})$ by $q(g) = \{i | g(i) = 1\}$. Show that q is a bijection.

[12]

b) Hence comment on the relationship between the cardinality of M and the cardinality of $\mathscr{P}(\mathbb{N})$.

[2]

Let us assume that there exists a bijection $h : \mathbb{N} \to M$. Consider a set $S = \{i | h(i)(i) = 0\}$.

c) Show that $S \in \mathscr{P}(\mathbb{N})$.

[8]

d) Show that $\neg \exists n (q(h(n)) = S)$.

[12]

e) Draw the appropriate conclusion about the assumption on the existence of h, and hence on the cardinality of $\mathscr{P}(\mathbb{N})$ and the countability of $\mathscr{P}(\mathbb{N})$, explaining your answers carefully.

[6]

3. Let A and B be finite sets.

a) State a formula for the number of functions from A to B in terms of the cardinalities of A and B.

[3]

b) Derive a formula for the number of injections from A to B in terms of the cardinalities of A and B.

[9]

c) Derive a formula for the number of surjections from A to B in terms of the cardinalities of A and B.

[9]

d) Derive a formula for the number of bijections from A to B in terms of the cardinalities of A and B.

[9]

4. a) Write a predicate logic expression for each of these statements, given that R is a relation on a set A. Use A as the universe of discourse.

i) "R is a reflexive relation".

ii) "R is a transitive relation".

[4]

R is said to be *antisymmetric* iff $\forall a \forall b (aRb \wedge bRa \rightarrow (a = b))$. A relation $\preceq$ is a *partial order* iff it is reflexive, antisymmetric, and transitive. A relation $\preceq$ is a *total order* if it is both a partial order and also $\forall a \forall b ((a \preceq b) \vee (b \preceq a))$.

Consider the relation $\preceq_1 = \{(a,b) | \exists k \in \mathbb{N}(b = ka)\}$ on the set $\mathbb{N}$ and $\preceq_2 = \{(a,b) | \exists k \in \mathbb{N}(b = ka)\}$ on the set $B = \{1,2,3,4,6,8,12\}$.

b) Show that $\preceq_1$ and $\preceq_2$ are both partial orders.

[12]

c) Show that neither $\preceq_1$ nor $\preceq_2$ are total orders.

[6]

d) Draw the digraph of $R_1 = \{(1,2),(1,3),(2,4),(2,6),(3,6),(4,8),(4,12),(6,12)\}$.

[4]

e) Draw the digraph of $\{(b,b) | b \in B\} \cup R_1^*$, where R_1^* denotes the transitive closure of R_1, and comment on its relationship to $\preceq_2$.

[4]

5. a) State the Master Theorem.

[8]

b) Write pseudo-code for four procedures, each operating on an array a of integers of length n, and respectively having execution time:

i) that is a $\Omega(n)$ function,

ii) that is a $\Omega(2^n)$ function,

iii) that the Master Theorem shows to be a $O(n)$ function,

iv) that the Master Theorem shows to be a $O(n^2 \log n)$ function.

You may assume that no compiler optimizations would be performed on your code.

[14]

c) Let Π denote the set of all problems. Let A denote the set of all algorithms. Let $Q(x, y, z)$ be the predicate "Algorithm x solves problem y in worst-case time $O(z)$", where z is a function of the size, n, of the problem instance. For example, $Q(\texttt{myalg}, \texttt{myprob}, n^2)$ states that `myalg` solves `myprob` in worst-case quadratic time. Let P be the set of all tractable problems. Define P in terms of Q using predicate logic syntax.

[4]

d) Give one example each of: a tractable problem, an unsolvable problem, and a solvable problem not known to be tractable.

[4]

Discrete Mathematics and Computational Complexity

SOLUTIONS 2010

1. a) (i) $S_1 \cup S_2 = \{\phi, a, 2, 3\}$

(ii) $S_1 \cap S_2 = \phi$

(iii) $S_1 - S_2 = \{\phi, a\}$

(iv) $S_1 \times S_2 = \{(\phi, 2), (\phi, 3), (a, 2), (a, 3)\}$

(v) $P(S_1) = \{\phi, \{\phi\}, \{a\}, \{\phi, a\}\}$

[9]

b) (i) $f(n) = n + 1$

(ii) $f(n) = \lfloor n/2 \rfloor$

(iii) $f(n) = n$

(iv) $f(n) = \lfloor n/2 \rfloor + 1$

[6]

c) (i) $\forall a \forall b (a + b = b + a)$

(ii) $\forall a \forall b \forall c ((a + b) + c = a + (b + c))$

(iii) $\forall a \exists b \forall c ((a + c) + b = c)$

(iv) $\exists b \forall a (b + a = a)$

[6]

d) (i) $a_n = 1 \quad (n \geq 1)$ is $O(n)$

(ii) $a_n = \alpha 2^n - 1 \quad (n \geq 1)$

$a_1 = 2\alpha - 1 = 1$

$\Rightarrow \alpha = 1$ so $a_n = 2^n - 1$. Not $O(n)$

(iii) Form $r^2 - r - 1 = 0$

$1^2 - 4 \cdot 1 \cdot (-1) \neq 0$, so distinct roots.

Roots are $r_1 = \dfrac{1 - \sqrt{1+4}}{2} = \frac{1}{2}(1 - \sqrt{5})$

$r_2 = \frac{1}{2}(1 + \sqrt{5})$.

$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n - \dfrac{1}{\cancel{1}+1-1}$

$= \alpha_1 r_1^n + \alpha_2 r_2^n - 1$

$\alpha_1 + \alpha_2 - 1 = 0 \quad \Rightarrow \quad \alpha_1 r_1 + \alpha_2 r_1 - r_1 = 0$

$\alpha_1 r_1 + \alpha_2 r_2 - 1 = 0$

So $\alpha_2 = \dfrac{r_1 - 1}{r_1 - r_2}$ $\quad\quad$ $\alpha_1 = \dfrac{1 - r_2}{r_1 - r_2}$

a_n is <u>not</u> $O(n)$. [9]

(iv) $(r - \frac{1}{2})(r - \frac{1}{4}) = 0$ $\rightarrow$ distinct roots.

$$\alpha_n = \alpha_1 \left(\tfrac{1}{2}\right)^n + \alpha_2 \left(\tfrac{1}{4}\right)^n$$

$$\left.\begin{array}{l} \alpha_1 + \alpha_2 = 1 \\ 2\alpha_1 + \alpha_2 = 4 \end{array}\right\} \Rightarrow \alpha_1 = 3, \quad \alpha_2 = -2$$

$a_n = 3\left(\frac{1}{2}\right)^n - 2\left(\frac{1}{4}\right)^n \quad n \geq 0$ <u>is</u> $O(n)$.

e) (i) $R = \{(1,1), (2,2), (3,3), (1,2), (2,3)\}$

(ii) $R = \{(1,2), (2,3)\cancel{\}}, (1,3)\}$

(iii) $R = \{(1,2), (2,3), (2,1), (3,2)\}$

(iv) $R = \{(1,1), (2,2), (3,3), (1,2)\}$

(v) $R = \{(1,1), (2,2), (3,3), (1,2), (2,3), (2,1), (3,2)\}$

(vi) $R = \cancel{\{\emptyset}\ \{(1,1), (1,2), (2,1), (2,2)\}$

(vii) $R = \{1,2,3\} \times \{1,2,3\}$.

[10]

2. (a) (i) Injection:

$$q(g_1) = q(g_2)$$

$$\Rightarrow \{i_1 \mid g_1(i_1) = 1\} = \{i_2 \mid g_2(i_2) = 1\}$$

i.e. $g_1(i_1) = 1 \iff g_2(i_1) = 1$

Since the co-domain of g is $\{0,1\}$,

we have $g_1(i_1) = g_2(i_1)$ for all i_1, and so

$$g_1 = g_2.$$

(ii) Surjection.

Let the elements of $S \in P(\mathbb{N})$ be given by

$$S = \{s_1, s_2, \dots\}.$$

We can construct g as

$$g(i) = \begin{cases} 0, & \text{if } i \notin S \\ 1, & \text{otherwise.} \end{cases}$$

Then $q(g) = S$. [12]

(b) They must therefore have the same cardinality. [2]

(c) $S = \{i \mid h(i)(i) = 0\}$.

Since the domain of h and the domain of the domain of $h(i)$ are both $\mathbb{N}$, $S \subseteq \mathbb{N}$. Hence $S \in P(\mathbb{N})$. [8]

(d) Consider $h(n)$. If $h(n)(n) = 0$, then $n \in S$.

But also $n \notin q(h(n))$, from the definition of q.

~~Then~~ If $h(n)(n) = 1$ then $n \notin S$.

But also $n \cancel{\notin} \in q(h(n))$, from the definition of q.

Hence $S \neq q(h(n))$.

Since this is true for all n, $\neg \exists n \mid q(h(n)) = S$. [12]

e) Hence there is no such bijection h. As a result, $|P(\mathbb{N})| \neq |\mathbb{N}|$. $P(\mathbb{N})$ is uncountable. [6]

3. a) $|B|^{|A|}$ (bookwork) [3]

b) $\dfrac{|B|!}{(|B|-|A|)!}$ ($|B| \times (|B|-1) \times \dots \times (|B|-|A|+1)$ – each time one fewer co-domain element to select) [9] [6]

c) $\dfrac{|A|!}{(|A|-|B|)!}$ ($|A| \times (|A|-1) \times \dots \times (|A|-|B|+1)$ – each time one fewer domain element to select) [9] [6]

d) $|A|!$ (note that this is also $|B|!$, since $|A| = |B|$) [9]

4. a) (i) $\forall a\,(aRa)$

(ii) $\forall a \forall b \forall c\,(aRb \wedge bRc \rightarrow aRc)$

b) (i) Reflexivity of $\preceq_1$

Consider $a \in \mathbb{Z}$. $(a,a) \in \preceq_1$ because $\exists k$ $(k=1)$ s.t. $a = k \cdot a$.

(ii) Transitivity of $\preceq_1$

Consider $(a,b) \in \preceq_1$ and $(b,c) \in \preceq_1$

Then $\exists k_1, k_2$ s.t.

$b = k_1 a$ and $c = k_2 b$.

Since $c = k_2 k_1 a$, $\exists k$ ~~s.t.~~ $(k = k_1 k_2)$

s.t. $c = ka$

$\therefore$ $(a,c) \in \preceq_1 \Rightarrow \preceq_1$ is transitive.

(iii) Antisymmetry.

$a \preceq b$ and $b \preceq a$.

Then $b = k_1 a$ and $a = k_2 b$

So $a = k_2 k_1 a \Rightarrow k_2 k_1 = 1$

Since $k_1, k_2 \in \mathbb{Z}$, $k_1 = 1$ and $k_2 = 1$.

Thus $a = b$.

$\therefore \preceq_1$ is a partial order.

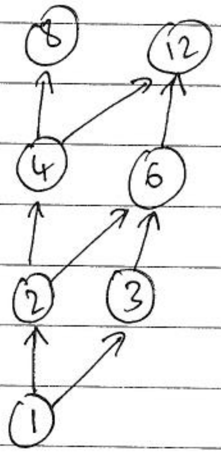
It follows that $\preceq_2$ is a partial order, since $\{1, 2, 3, 4, 6, 8, 12\} \subset \mathbb{N}$

c) $(4,6) \notin \preceq_1$, as ~~6 ≠ k s.t.~~ $4 = 6k$ is not satisfiable for k integer.

Equally, $(6,4) \notin \preceq_1$ as $6 = 4k$ is not satisfiable for k integer.

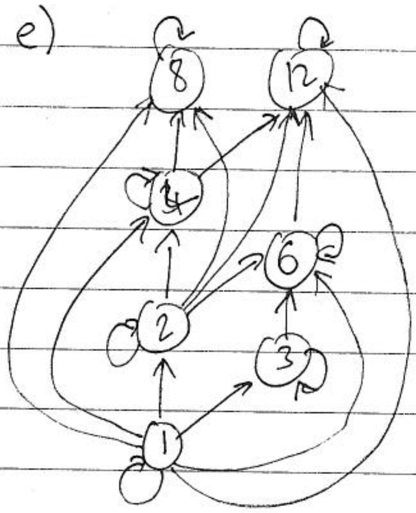
Since 4 & 6 are in the sets for both $\preceq_1$ & $\preceq_2$, the proof holds for both.

4. d)



[4]

e)



They are equal.

[4]

5. a) Let $a \geq 1$ be a real number, $b > 1$ be an integer,
$c > 0$ be a real number, and $d \geq 0$ be a real number.
Let f be an increasing function satisfying $f(n) = a f(n/b) \pm c n^d$
whenever $n = b^k$ for $k \in \mathbb{Z}^+$.
Then:

If $a < b^d$, $f(n)$ is $O(n^d)$
If $a = b^d$, $f(n)$ is $O(n^d \log n)$??
If $a > b^d$, $f(n)$ is $O(n^{\log_b a})$

(bookwork)

[8]

b) (i)

```
proc p1(a[n]: integer)
  t := 0
  for i = 1 to n
      t := t + a[i]
  end
  result := t
```

(ii)

```
proc p2(a[n]: integer)
if n > 1
result := p2(a[1 to n-1]) + p2(a[2 to n])
else
  result := 1
end
```

(iii)

```
proc p3(a[n]: integer)
if n > 1
    result := p3(a[1 to ⌊n/2⌋])
    for i = 1 to n
        result := result + 1
    end
else
    result := 1
end
```

(iv) proc p4(a[n]: integer)
if n > 1
  for i = 1 to 4
    result := result + p4(a[1 to ⌊n/2⌋])
  end
  for i = 1 to n·n
    result := result + 1
  end
else
  result := 1   [4]
end

~~c) P = { y | ∃x ∃ $Q(x, y, n$~~

c) $P = \{ y \mid \exists x \in A \ \exists b \in \mathbb{Z}^+ \ Q(x, y, n^b) \}$ [4]

d) tractable: matrix multiplication
unsolvable: halting problem
intractable (?): k-colouring of a graph.   [4]