UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 2003

BEng Honours Degree in Computing Part I
MEng Honours Degrees in Computing Part I
BSc Honours Degree in Mathematics and Computer Science Part I
MSci Honours Degree in Mathematics and Computer Science Part I
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the Associateship of the City and Guilds of London Institute
This paper is also taken for the relevant examinations for the Associateship of the Royal College of Science

PAPER C140=MC140

LOGIC

Friday 2 May 2003, 16:00
Duration: 90 minutes
(Reading time 5 minutes)

Answer THREE questions

Paper contains 4 questions
Calculators not required

1a Let A and B be propositional formulas. Explain what it means to say that

i A logically implies B;

ii A and B are logically equivalent.

b The formulas $(p \rightarrow q) \rightarrow r$ and $p \rightarrow (q \rightarrow r)$ are not logically equivalent but one does logically imply the other.

i State the direction in which the logical implication does hold, and give a proof of it using natural deduction.
Do not rewrite any formula using equivalences.

ii By describing an appropriate situation show that the logical implication in the other direction does not hold.

c Use natural deduction to show that $\vdash p \vee \neg p$.
Do not rewrite any formula using equivalences, and do not use any lemmas.
You may use the derived rule PC if you wish.

The three parts carry, respectively 20%, 50% and 30% of the marks.

2a Using only these equivalences to rewrite formulas:

1. $P \equiv \neg\neg P$
2. $\neg\exists x P \equiv \forall x \neg P$
3. $P \rightarrow Q \equiv \neg P \vee Q$
4. $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$
5. $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$
6. $\forall x(P \wedge Q) \equiv \forall x P \wedge \forall x Q$

establish the new equivalence

$$\exists x(P \rightarrow Q) \equiv \exists x \neg P \vee \exists x Q$$

At each step make clear which of the equivalences 1-6 is being used.

b Without using equivalences to rewrite any formula, use natural deduction alone (a lemma of the form $A \vee \neg A$ may be used if desired) to show the following, making clear at each step the inference rule being used:

i $\exists x(P(x) \rightarrow Q(x)) \vdash \exists x \neg P(x) \vee \exists x Q(x)$

ii $\forall x(Q(x) \rightarrow P) \vdash \forall x Q(x) \rightarrow P$

The two parts carry, respectively 40% and 60% of the marks.

3. L is the 2-sorted signature with sorts Nat and [Nat], containing constants 0, 1, 2, ... :Nat and [ ]:[Nat], function symbols +, –, x, :, ++, !!, and relation symbols <, ≤ and merge, of appropriate sorts. The intended semantics is a structure whose domain consists of the natural numbers 0, 1, 2, ... (sort Nat) and all lists of natural numbers (sort [Nat]). The symbols have the usual meanings. For example, merge(ys, zs, xs) holds iff xs is a permutation of ys++zs and the relative order of entries in ys and zs is retained in xs. You are also given an L-formula in(n, xs) expressing that the natural number n occurs in the list xs.

a Below is an English description of an operation together with an incorrect attempt to formalise the operation logically in the language L:

removesquares: [Nat] → [Nat]
pre: none
post: zs is the result of removing from xs all the square numbers (and no others), where zs = removesquares xs
∃us:[Nat](merge(us, zs, xs) ∧ ∀k:Nat(in(k, us) → ∃n:Nat(k=nxn)))

i Give an example input and output that satisfies the given logical expression of the post-condition but not the given English one. Translate the logical expression of the post-condition into English. Explain what is wrong with the logical expression of the post-condition.

ii Amend the logical expression of the post-condition so that it correctly formalises the intention expressed by the given English one.

b Formalise in the language L the pre-condition and post-condition expressed in English in the following operation (if you consider the given English to be ambiguous then write down in clear English any additional assumptions you make):

middle: [Nat] → Nat
pre: the input list contains at least three different natural numbers
post: the output natural number is in the input list but is neither the largest nor the smallest entry in it.

c Formalise in the language L the post-condition expressed in English in the following operation:

sort_no_dups: [Nat] → [Nat]
pre: none
post: the output list contains the same elements as the input list but they are sorted in ascending order and have no duplicates.

The three parts carry, respectively 30%, 35% and 35% of the marks.

4a What is the signature (L) of the following sentence?

S: $\forall$ x,y (P(x,f(y)) $\rightarrow$ Q(f(x),y)) $\wedge$ P(a,f(b)) $\wedge$ P(b,b)

Enumerate all terms occurring in S.

b Give an L-structure M such that the sentence S in part a is true in M.

c Determine whether it is the case that S $\models$ Q(f(b),b), by considering appropriate structures.

d Given a structure N, an assignment is a function allocating one object in the domain of N to each variable. Justify the need for introducing assignments when evaluating formulas in structures.

e Is the L-formula $\exists$x (P(x,x) $\wedge \neg$ Q(x,y)) valid? Briefly justify your answer.

The five parts carry, respectively, 20%, 25%, 25%, 10%, 20% of the marks.