

E2.5 Signals and Linear Systems Solutions 2008

All questions are unseen.

Question 1 is compulsory.

Answer to Question 1

a)

If $x_1 \rightarrow y_1$ and $x_2 \rightarrow y_2$, for a linear system,

$$k_1 x_1 + k_2 x_2 \rightarrow k_1 y_1 + k_2 y_2 \text{ where } k_1 \text{ and } k_2 \text{ are constants.}$$

[2]

b)

$$x(t) = e^{j\theta} = \cos \theta + j \sin \theta$$

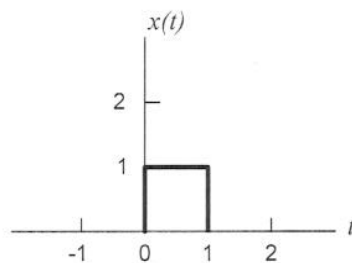
Therefore,

Even: $\cos \theta$

Odd: $j \sin \theta$.

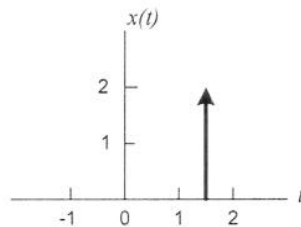
[2]

c) i)



[3]

ii)



[3]

d) i)

$$v_s(t) = Ri(t) + v_c(t)$$

$$v_c(t) = \frac{1}{C} \int_{-\infty}^t i(\tau) d\tau$$

$$x(t) = v_s(t), \quad y(t) = i(t).$$

$$\therefore Ry(t) + \frac{1}{C} \int_{-\infty}^t y(\tau) d\tau = x(t).$$

Differentiate both sides wrt t :

$$\begin{aligned} R \frac{dy}{dt} + \frac{1}{C} y(t) &= \frac{dx}{dt} \\ \Rightarrow \frac{dy}{dt} + \frac{1}{RC} y(t) &= \frac{1}{R} \frac{dx}{dt} \end{aligned}$$

[3]

ii) Take Laplace transform on both sides:

$$\left(s + \frac{1}{RC}\right) Y(s) = \frac{1}{R} sX(s).$$

$$\therefore H(s) = \frac{Y(s)}{X(s)} = C \times \frac{s}{RCs + 1}.$$

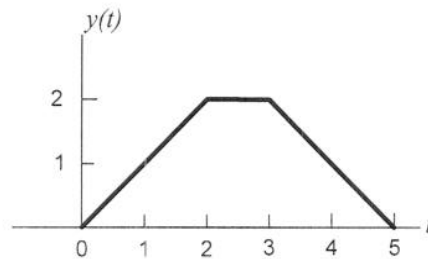
[3]

e)

$$\begin{aligned} y(t) &= h(t) * x(t) \\ &= (2e^{-3t} - e^{-2t})u(t) * e^{-t}u(t) \\ &= 2e^{-3t}u(t) * e^{-t}u(t) - e^{-2t}u(t) * e^{-t}u(t) \\ &= \left[\frac{2(e^{-t} - e^{-3t})}{2} - \frac{(e^{-t} - e^{-2t})}{1} \right] u(t) \\ &= (e^{-2t} - e^{-3t})u(t) \end{aligned}$$

[4]

f)



[4]

g) The complex zeros are given by:

$$z^2 - z + \frac{5}{2} = 0.$$

Therefore the zeros are at:

$$z = \frac{1 \pm \sqrt{1-10}}{2} = \frac{1}{2} \pm j\frac{3}{2}.$$

The poles are given by:

$$p^2 + 5p + 5 = (p+4)(p+1) = 0.$$

Therefore the poles are at:

$$p = -1 \text{ and } p = -4.$$

[4]

h) By definition of Fourier transform,

$$\text{FT of } x(t-t_0) = \int_{-\infty}^{\infty} x(t-t_0)e^{-j\omega t} dt.$$

Let $\tau = t - t_0$,

$$\begin{aligned} \int_{-\infty}^{\infty} x(t-t_0)e^{-j\omega t} dt &= \int_{-\infty}^{\infty} x(\tau)e^{-j\omega(\tau+t_0)} d\tau \\ &= e^{-j\omega t_0} \int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau} d\tau \\ &= e^{-j\omega t_0} X(\omega) \end{aligned}$$

[4]

i) Divide $F[z]$ by z , and perform partial fraction:

$$\frac{F[z]}{z} = \frac{z-7}{z^2-5z+4} = \frac{z-7}{(z-1)(z-4)} = \frac{2}{z-1} - \frac{1}{z-4}.$$

$$F[z] = 2\frac{z}{z-1} - \frac{z}{z-4}$$

$$\therefore f[k] = [2 - 4^k]u[k].$$

[4]

j)

i) Nyquist rate is $2 \times 4.5 \times 10^6 = 9$ MHz.

Therefore the actual sampling rate = $9 \text{ MHz} \times 1.2 = 10.8 \text{ MHz}$.

[2]

ii) 1024 levels require 10 bits per sample. Therefore bit-rate is:

$$10.8 \times 10^6 \times 10 = 108 \text{ Mbits/sec.}$$

[2]

Answer to Question 2

a) Express the differential equation in terms of D operators:

$$(D^2 + 6D + 9)y(t) = (2D + 9)x(t) \Rightarrow Q(D)y(t) = P(D)x(t)$$

$$Q(D) = (D^2 + 6D + 9), \quad P(D) = (2D + 9)$$

The characteristic equation is therefore:

$$(\lambda^2 + 6\lambda + 9) = 0 \Rightarrow (\lambda + 3)^2 = 0.$$

$$\therefore y_0(t) = (c_1 + c_2 t)e^{-3t} \quad \text{and} \quad \dot{y}_0(t) = [-3(c_1 + c_2 t) + c_2]e^{-3t}$$

Setting $t = 0$, and substituting $e^{-3t} y_0(0) = 0$ and $\dot{y}_0(0) = 1$, gives

$$\left. \begin{array}{l} 0 = c_1 \\ 1 = -3c_1 + c_2 \end{array} \right\} \Rightarrow \begin{array}{l} c_1 = 0 \\ c_2 = 1 \end{array}$$

$$\therefore y_0(t) = te^{-3t} \quad \text{and} \quad \dot{y}_0(t) = (-3t + 1)e^{-3t}$$

Now the impulse response can be calculated:

$$\begin{aligned} h(t) &= [P(D)y_0(t)]u(t) \\ &= [2y_0(t) + 9\dot{y}_0(t)]u(t) \\ &= (-6te^{-3t} + 2e^{-3t} + 9te^{-3t})u(t) \\ &= (2 + 3t)e^{-3t}u(t) \end{aligned}$$

[15]

b) The system response to $u(t)$ is $g(t)$, and the response to the step $u(t - \tau)$ is $g(t - \tau)$ (time-invariant property).

It is given that $\Delta f = \frac{\Delta f}{\Delta \tau} \Delta \tau = \dot{f}(\tau) \Delta \tau$. The step component at $t = n\Delta \tau$ therefore has a height of $\dot{f}(n\Delta \tau) \Delta \tau$, and can be expressed as $[\dot{f}(n\Delta \tau) \Delta \tau] u(t - n\Delta \tau)$. This gives a response $\Delta y(t)$ at the output, where

$$\Delta y(t) = [\dot{f}(n\Delta \tau) \Delta \tau] g(t - n\Delta \tau).$$

Therefore, the total response due to ALL step components is:

$$\begin{aligned} y(t) &= \lim_{\Delta \tau \rightarrow 0} \sum_{n=-\infty}^{\infty} \dot{f}(n\Delta \tau) g(t - n\Delta \tau) \Delta \tau \\ &= \int_{-\infty}^{\infty} \dot{f}(\tau) g(t - \tau) d\tau \\ &= \dot{f}(\tau) * g(\tau) \\ &= \dot{f}(t) * g(t). \end{aligned}$$

[15]

Answer to Question 3

a) i) From definition of Fourier transform,

$$\begin{aligned}
 F(\omega) &= \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \\
 &= \int_{-\tau}^0 e^{-j\omega t} dt - \int_0^{\tau} e^{-j\omega t} dt \\
 &= -\frac{1}{j\omega} e^{-j\omega t} \Big|_{-\tau}^0 - \frac{1}{j\omega} e^{-j\omega t} \Big|_0^{\tau} \\
 &= -\frac{1}{j\omega} + \frac{1}{j\omega} e^{j\omega\tau} + \frac{1}{j\omega} e^{-j\omega\tau} - \frac{1}{j\omega} \\
 &= -\frac{2}{j\omega} + \frac{2}{j\omega} \cos \omega\tau \\
 &= j \frac{4}{\omega} \sin^2 \left(\frac{\omega\tau}{2} \right)
 \end{aligned}$$

[10]

ii) Express $f(t)$ as sum of two rectangular functions:

$$f(t) = \text{rect}\left(\frac{t+\tau/2}{\tau}\right) - \text{rect}\left(\frac{t-\tau/2}{\tau}\right)$$

Given that

$$\text{rect}\left(\frac{t}{\tau}\right) \Leftrightarrow \tau \text{sinc}\left(\frac{\omega\tau}{2}\right),$$

apply time-shifting property gives

$$\text{rect}\left(\frac{t \pm \tau/2}{\tau}\right) \Leftrightarrow \tau \text{sinc}\left(\frac{\omega\tau}{2}\right) e^{\pm j\omega\tau/2}.$$

Therefore

$$\begin{aligned}
 F(\omega) &= \tau \text{sinc}\left(\frac{\omega\tau}{2}\right) e^{+j\omega\tau/2} - \tau \text{sinc}\left(\frac{\omega\tau}{2}\right) e^{-j\omega\tau/2} \\
 &= 2j \tau \text{sinc}\left(\frac{\omega\tau}{2}\right) \sin\left(\frac{\omega\tau}{2}\right) \\
 &= j \frac{4}{\omega} \sin^2 \left(\frac{\omega\tau}{2} \right)
 \end{aligned}$$

[10]

b)

$$f(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma^2}}$$

and $\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma^2}} \Leftrightarrow e^{-\sigma^2\omega^2/2}.$

Parseval's Theorem states:

$$E_f = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega.$$

Given

$$F(\omega) = e^{-\sigma^2\omega^2/2}$$

we obtain:

$$E_f = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\sigma^2\omega^2} d\omega.$$

Let $\sigma\omega = \frac{x}{\sqrt{2}}$, then $\sigma^2\omega^2 = \frac{x^2}{2}$ and $d\omega = \frac{1}{\sigma\sqrt{2}} dx$.

Therefore

$$E_f = \frac{1}{2\pi} \frac{1}{\sigma\sqrt{2}} \int_{-\infty}^{\infty} e^{-x^2/2} dx = \frac{1}{2\sigma\sqrt{\pi}}.$$

[10]

Answer to Question 4

a) Taking z -transform of both sides:

$$zY[z] - 0.5Y[z] = zF[z] + 0.8F[z].$$

Therefore

$$H[z] = \frac{Y[z]}{F[z]} = \frac{z + 0.8}{z - 0.5}.$$

b) The frequency response is given by:

$$H[e^{j\Omega}] = \frac{e^{j\Omega} + 0.8}{e^{j\Omega} - 0.5} = \frac{(\cos \Omega + 0.8) + j \sin \Omega}{(\cos \Omega - 0.5) + j \sin \Omega}.$$

Therefore, the amplitude response is

$$\begin{aligned} |H[e^{j\Omega}]|^2 &= H[e^{j\Omega}]H[e^{-j\Omega}]. \\ &= \frac{(e^{j\Omega} + 0.8)(e^{-j\Omega} + 0.8)}{(e^{j\Omega} - 0.5)(e^{-j\Omega} - 0.5)} \\ &= \frac{1.64 + 1.6 \cos \Omega}{1.25 - \cos \Omega} \end{aligned}$$

The phase response is

$$\angle H[e^{j\Omega}] = \tan^{-1} \left(\frac{\sin \Omega}{\cos \Omega + 0.8} \right) - \tan^{-1} \left(\frac{\sin \Omega}{\cos \Omega - 0.5} \right).$$

c) Since $f[k] = \cos(0.5k - \frac{\pi}{3})$, $\Omega = 0.5$.

Therefore

$$|H[e^{j\Omega}]|^2 = \frac{1.64 + 1.6 \cos 0.5}{1.25 - \cos 0.5} = 8.174$$

$$|H[e^{j\Omega}]| = 2.86$$

$$\begin{aligned} \angle H[e^{j\Omega}] &= \tan^{-1} \left(\frac{\sin 0.5}{\cos 0.5 + 0.8} \right) - \tan^{-1} \left(\frac{\sin 0.5}{\cos 0.5 - 0.5} \right) \\ &= 0.2784 - 0.9037 \\ &= -0.6253 \text{ radian or } 35.83^\circ \end{aligned}$$

Therefore, the system response is

$$y[k] = 2.86 \cos(0.5k - \frac{\pi}{3} - 0.6253) = 2.86 \cos(0.5k - 1.6725).$$