[E 2.9 (Maths 4) 2012]

B.ENG. AND M.ENG. EXAMINATIONS 2012

PART II Paper 4: MATHEMATICS (ELECTRICAL AND INFORMATION SYSTEMS ENGINEERING)

Date Friday 8th June 2012 2.00 - 4.00 pm

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.

Answer FOUR questions.

Please answer questions from Section A and Section B in separate answer-books.

A mathematical formulae sheet is provided.

Statistical data sheets are provided.

[*Before starting, please make sure that the paper is complete; there should be 5 pages, with a total of 6 questions. Ask the invigilator for a replacement if your copy is faulty.*]

SECTION A

1. Evaluate the following integral along the two given paths:

$$\int_{C_i} \left(y^2 \, \mathrm{d}x \, + \, 3xy \, \mathrm{d}y \right) \, ,$$

   (i) where C_1 is the parabolic arc $y = x^2$, from $x = 0$ to $x = 1$.

   (ii) where C_2 is the line segment $y = x$, from $x = 0$ to $x = 1$.

   Explain why these results are different, and hence evaluate the double integral

$$\iint_\Omega y \, \mathrm{d}x \, \mathrm{d}y,$$

   where Ω is the region bounded by the paths C_1 and C_2.

2. (i) State the two-dimensional divergence theorem.

   (ii) Evaluate the divergence of the vector field

$$\mathbf{u} \; = \; \sqrt{x^2 + y^2} \; (x \, \mathbf{i} + y \, \mathbf{j}) \; .$$

   (iii) Verify the 2-dimensional divergence theorem directly, by evaluating the two integrals

$$\oint_C \mathbf{u} \, . \, \mathbf{n} \; \mathrm{d}s \; ,$$

   and

$$\iint_R \nabla \, . \, \mathbf{u} \, \mathrm{d}x \, \mathrm{d}y,$$

   where R is the disc $x^2 + y^2 < a^2$, C is its boundary and $\mathbf{n}$ is the unit normal vector to this boundary.

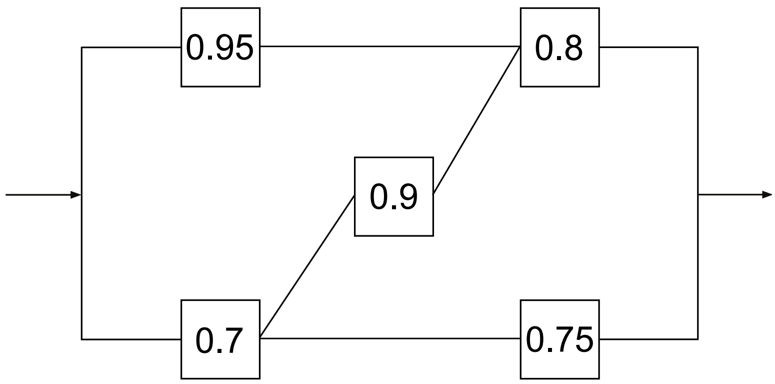
PLEASE TURN OVER

SECTION B

3. (i) Consider a machine which is used to make widgets. When the machine is adjusted correctly, 60% of the widgets it produces are of high quality, and 40% are of medium quality. When it is adjusted incorrectly, 30% of the widgets it produces are of high quality, and 70% of medium quality. In each case the levels of quality of different widgets can be assumed to be independent of each other. The machine is adjusted once a day. On any given day, the probability is 0.9 that it will be adjusted correctly and 0.1 that it will be adjusted incorrectly.

   Four widgets produced on the same day are selected at random and inspected. One of them is found to be of high quality, and three of medium quality. Given this evidence, what is the probability that the machine has been adjusted correctly that day?

   (ii) An electronic system consists of five components connected as illustrated in the accompanying diagram. The components work independently, with reliabilities as shown.



   Find the reliability of this system.

   Hint: Look at cases where the bottom-left fails, and where it functions, and combine the results.

4. In a computer network X and Y are two performance measures that vary randomly with bivariate density

$$f_{X,Y}(x, y = \begin{cases} kx^2 y & \text{if } -1 < x < 1 \text{ and } 0 < y < x^2 \,, \\ 0 & \text{otherwise.} \end{cases}$$

(i) Sketch the support of this density function and find the value of k.

(ii) Working directly with the joint density, find $\mathrm{E}(XY)$.

(iii) Find the marginal density $f_X(x)$ and hence evaluate $\mathrm{E}(X)$.

(iv) Find the conditional density $f_{Y|X}(y|x)$. Are X and Y independent? Compute $\mathrm{E}(Y|X = x)$ and write down the conditional expectation $\mathrm{E}(Y|X)$.

(v) Compute $\mathrm{Corr}(X, Y)$. Are X and Y uncorrelated?

5. (i) The random variable X follows a $\Gamma(5, \theta)$ distribution, that is,

$$f_X(x) = \begin{cases} \theta^5 x^4 e^{-\theta x}/4! & \text{if } x > 0 \\ 0 & \text{otherwise,} \end{cases}$$

where $\theta > 0$ is a parameter.

(a) Show that the moment generating function (MGF) of X is

$$m_X(t) = \left(1 - \frac{t}{\theta}\right)^{-5} \,, \quad \text{for } t < \theta \,.$$

Hint: Use the fact that the density integrates to 1 *for all positive values of* θ*.*

(b) Using the MGF, find the mean and variance of X.

(ii) Suppose that we filter a time series process $\{y_t\}$ by applying the five-point moving average

$$x_t = y_{t-2} - 6y_{t-1} + 11y_t - 6y_{t+1} + y_{t+2} \,.$$

(a) Show that the transfer function of this linear filter is $A(\omega) = (2\cos\omega - 3)^2$.

(b) Is this a low- or high-pass filter?

PLEASE TURN OVER

6. (i) What conditions does a time series process need to satisfy in order to be weakly (or second-order) stationary?

(ii) Let $\{\varepsilon_t\}$ be a white noise process with mean zero and variance σ^2. For each of the following time series processes, determine if it is weakly stationary and, if so, specify its autocorrelation function (ACF) and find its spectrum $f(\omega)$.

(a) $y_t = \varepsilon_t + 2\varepsilon_{t-1}$.

(b) $y_t = y_{t-1} + 0.5\varepsilon_t$.

(c) y_t independent $\mathrm{N}(2,4)$ for t odd, y_t independent $\mathrm{Exp}(0.5)$ for t even, and y_t independent of y_s for all $t \neq s$.

(d) $y_t = \varepsilon_0 + t\,\varepsilon_t$.

END OF PAPER

EE II (4) 2012

A1 (a) To find

$$\int_{C_1} y^2\,\mathrm{d}x + 3xy\,\mathrm{d}y,$$

set $y = x^2$, $\mathrm{d}y = 2x\mathrm{d}x$, giving

$$\int_{x=0}^{1} x^4 + 6x^4 dx = \frac{7}{5}\left[x^5\right]_0^1 = \frac{7}{5}.$$

(6 marks)

(b) Similarly, on C_2, setting $y = x$, $\mathrm{d}y = \mathrm{d}x$, we find

$$\int_{C_2} y^2\,\mathrm{d}x + 3xy\,\mathrm{d}y =$$

$$\int_{x=0}^{1} x^2\,\mathrm{d}x + 3x^2\,\mathrm{d}x =$$

$$\int_0^1 4x^2\,\mathrm{d}x = \left[4\frac{x^3}{3}\right]_0^1 = \frac{4}{3}.$$

(6 marks)

These results are different because $y^2\,\mathrm{d}x + 3xy\,\mathrm{d}y$ is not an exact differential. Indeed

$$\frac{\partial}{\partial x}(3xy) - \frac{\partial}{\partial y}y^2 = y \neq 0.$$

(4 marks)

Thus, by Green's Theorem,

$$\int\int_\Omega y\,\mathrm{d}x\mathrm{d}y = \int_{C_1-C_2} y^2\,\mathrm{d}x + 3xy\,\mathrm{d}y = \frac{7}{5} - \frac{4}{3} = \frac{1}{15}.$$

(4 marks)

EE II (4)

A2 (a) The two-dimensional divergence theorem states that for a differentiable vector field $\mathbf{u}(x, y)$, and a region R with boundary C, then:

$$\int\int_R \nabla.\mathbf{u}\mathrm{d}x\mathrm{d}y = \oint_C \mathbf{u}.\mathbf{n}ds.$$

Here C is taken to enclose R anticlockwise, and $\mathbf{n}$ is the outward normal.

(4 marks)

(b) The divergence of the vector field

$$\mathbf{u} = (x\mathbf{i} + y\mathbf{j})\sqrt{x^2 + y^2}$$

can be found using

$$\nabla.(\mathbf{v}\phi) = (\nabla.\mathbf{v})\phi + \mathbf{v}.\nabla(\phi),$$

with $\mathbf{v} = (x\mathbf{i} + y\mathbf{j})$, and $\phi = \sqrt{x^2 + y^2}$. We get

$$\nabla.\mathbf{u} = (\nabla.(x\mathbf{i} + y\mathbf{j}))\sqrt{x^2 + y^2} + (x\mathbf{i} + y\mathbf{j}).\nabla(\sqrt{x^2 + y^2})$$

$$= 2\sqrt{x^2 + y^2} + \frac{x^2 + y^2}{\sqrt{x^2 + y^2}} = 3\sqrt{x^2 + y^2}.$$

(4 marks)

(c) If R is the disc $x^2 + y^2 < a^2$, and C is its boundary, to evaluate the integral

$$\oint_C \mathbf{u}.\mathbf{n}ds,$$

we parametrise the circular boundary by $x = a\cos(\theta)$, $y = a\sin(\theta)$. We have $\mathbf{u}.\mathbf{n} = a^2$, $\mathrm{d}s = a\mathrm{d}\theta$. Thus

$$\oint_C \mathbf{u}.\mathbf{n}ds = \int_0^{2\pi} a^2 a\mathrm{d}\theta = 2\pi a^3.$$

(4 marks)

To evaluate

$$\int\int_R \nabla.\mathbf{u}\mathrm{d}x\mathrm{d}y,$$

Checked

R.L.J.

E2E II (4)

we change to polar coordinates, $x = r\cos(\theta)$, $y = r\sin(\theta)$, so that $\mathrm{d}x\mathrm{d}y = r\mathrm{d}r\mathrm{d}\theta$; thus

$$\int\int_R \nabla.\mathbf{u}\mathrm{d}x\mathrm{d}y = \int_0^a [\int_0^{2\pi} 3r^2\mathrm{d}\theta]\mathrm{d}r =$$

$$2\pi\int_0^a 3r^2\mathrm{d}r = 2\pi a^3.$$

The 2-dimensional divergence theorem is thus verified in this case. (4 marks)

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R.L.J.

E 2.8 (Maths 4) 2012 — Solutions

1. (a) Let C denote the event that the machine is adjusted correctly. Define X to be the number of high-quality widgets in a sample of 4. Then $X \sim \text{Bin}(4, p)$, where $p = 0.6$ if C occurs, and $p = 0.3$ if $\overline{C}$ occurs. Thus,

$$\mathrm{P}(X = 1|\, C\,) = \binom{4}{1} 0.6^1 (1 - 0.6)^{4-1} = 0.1536\,,$$
$$\mathrm{P}(X = 1|\, \overline{C}\,) = \binom{4}{1} 0.3^1 (1 - 0.3)^{4-1} = 0.4116\,.$$

The probability of observing this particular sample is

$$\begin{aligned}\mathrm{P}(X = 1) &= \mathrm{P}(X = 1|\, C)\mathrm{P}(C) + \mathrm{P}(X = 1|\, \overline{C}\,)P(\,\overline{C}\,)\\ &= 0.1536 \times 0.9 + 0.4116 \times 0.1 = 0.1794\,.\end{aligned}$$

Using Bayes' theorem, the required probability is

$$\mathrm{P}(C|X = 1) = \frac{\mathrm{P}(X = 1|\, C)\mathrm{P}(C)}{\mathrm{P}(X = 1)} = \frac{0.1536 \times 0.9}{0.1794} \approx \underline{0.7706}\,.$$

Unseen — 10 MARKS

(b) Label the components 1-5 as in the following diagram. Let R_i, $i = 1, \ldots, 5$, denote the event that component i functions, and R the event that the system functions.

```
   |---1---2---|
   |        /  |
---|     3     |---
   |    /      |
   |---4---5---|
```

We begin by conditioning on the status of component 4. If it fails, we remove it from the system and are left with components 1 and 2 only. This gives

$$\mathrm{P}(R|\, \overline{R_4}\,) = \mathrm{P}(R_1 \cap R_2) = \mathrm{P}(R_1)\mathrm{P}(R_2) = 0.95 \times 0.8 = 0.76\,.$$

If component 4 functions, the system reduces to

```
       |--1--|
   |--|       |--2--|
   |  |--3--|       |
---|                |---
   |                |
   |-------5------|
```

Combining components 1 and 3 gives

$$\mathrm{P}(R_{1,3}) = \mathrm{P}(R_1 \cup R_3) = 1 - \mathrm{P}(\overline{R}_1 \cap \overline{R}_3) = 1 - (1 - 0.95)(1 - 0.9) = 0.995\,.$$

These are connected in series with component 2, so the reliability of the top branch is

$$\mathrm{P}(R_{1,3} \cap R_2) = \mathrm{P}(R_{1,3})P(R_2) = 0.995 \times 0.8 = 0.796\,.$$

Thus, the reliability of the system (given that component 4 functions) is

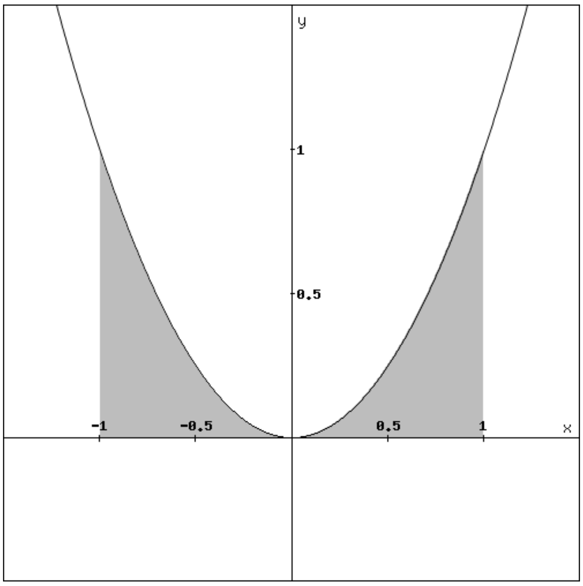
$$\begin{aligned}\mathrm{P}(R|R_4) &= \mathrm{P}\left[(R_{1,3} \cap R_2) \cup R_5\right] = 1 - \mathrm{P}[\overline{(R_{1,3} \cap R_2)} \cap \overline{R}_5] \\ &= 1 - [1 - \mathrm{P}(R_{1,3} \cap R_2)][1 - P(R_5)] \\ &= 1 - (1 - 0.796)(1 - 0.75) = 0.949\end{aligned}$$

Finally, we put the two parts together to obtain the reliability of the system

$$\begin{aligned}\mathrm{P}(R) &= \mathrm{P}(R|R_4)\mathrm{P}(R_4) + \mathrm{P}(R|\,\overline{R}_4\,)\mathrm{P}(\overline{R}) \\ &= 0.949 \times 0.7 + 0.76 \times (1 - 0.7) = \underline{0.8923}\,.\end{aligned}$$

Seen similar — 10 MARKS

2. (a) Sketch of the support:



The density must integrate to 1, so

$$\begin{aligned}1 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dydx = \int_{-1}^{1}\int_{0}^{x^2} kx^2y\,dydx \\ &= \int_{-1}^{1}\left[\frac{kx^2y^2}{2}\right]_0^{x^2} dx = \int_{-1}^{1}\frac{kx^6}{2}dx = \left[\frac{kx^7}{14}\right]_{-1}^{1} = \frac{k}{7}\,,\end{aligned}$$

hence $\underline{k = 7}$.

Unseen — 3 MARKS

(b)

$$\begin{aligned}
\mathrm{E}(XY) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{X,Y}(x,y)dydx = \int_{-1}^{1}\int_{0}^{x^2} 7x^3y^2\,dydx \\
&= \int_{-1}^{1}\left[\frac{7x^3y^3}{3}\right]_0^{x^2} dx = \int_{-1}^{1}\frac{7x^9}{3}dx = \left[\frac{7x^{10}}{30}\right]_{-1}^{1} \Rightarrow \\
&\underline{\mathrm{E}(XY) = 0}
\end{aligned}$$

Unseen — 3 MARKS

(c) From the first step of the integration in part (a), we can see that

$$f_X(x) = \begin{cases} 7x^6/2 & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

This is symmetric about the y axis, so $\underline{\mathrm{E}(X) = 0}$.

Unseen — 4 MARKS

(d) The required conditional density is

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \begin{cases} \frac{7x^2y}{7x^6/2} = \frac{2y}{x^4} & \text{if } -1 < x < 1 \text{ and } 0 < y < x^2 \\ 0 & \text{otherwise.} \end{cases}$$

It depends on x, thus $\underline{X \text{ and } Y \text{ are not independent}}$. We then have

$$\begin{aligned}
\mathrm{E}(Y|X=x) &= \int_{-\infty}^{\infty} y f_{Y|X}(y|x)dy = \int_0^{x^2}\frac{2y^2}{x^4}dy \\
&= \left[\frac{2y^3}{3x^4}\right]_0^{x^2} = \frac{2x^2}{3},
\end{aligned}$$

hence $\mathrm{E}(Y|X) = 2X^2/3$.

Unseen — 8 MARKS

(e) Using the results from previous calculations,

$$\mathrm{Cov}(X,Y) = \mathrm{E}(XY) - \mathrm{E}(X)\mathrm{E}(Y) = 0 - 0 \times \mathrm{E}(Y) = 0 \Rightarrow \underline{\mathrm{Corr}(X,Y) = 0},$$

thus, $\underline{X \text{ and } Y \text{ are uncorrelated}}$.

Unseen — 2 MARKS

3. (a) i.

$$\begin{aligned}
m_X(t) =& \mathrm{E}(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f_X(x)dx = \int_0^{\infty} e^{tx}\frac{\theta^5 x^4 e^{-\theta x}}{4!}dx \\
=& \int_0^{\infty}\frac{\theta^5 x^4 e^{-(\theta-t)x}}{4!}dx = \frac{\theta^5}{(\theta-t)^5}\int_0^{\infty}\frac{(\theta-t)^5 x^4 e^{-(\theta-t)x}}{4!}dx \Rightarrow
\end{aligned}$$

$$\underline{m_X(t) = (1 - t/\theta)^{-5}},$$

because the final integrand is a $\Gamma(5, \theta - t)$ density function, as long as $\theta - t > 0 \Leftrightarrow t < \theta$.

Unseen — 5 MARKS

ii. Differentiating the MGF gives

$$m_X'(t) = \frac{5}{\theta}\left(1 - \frac{t}{\theta}\right)^{-6}, \quad m_X''(t) = \frac{30}{\theta^2}\left(1 - \frac{t}{\theta}\right)^{-7},$$

so we have

$$\begin{aligned} \mathrm{E}(X) &= m_X'(0) = \frac{5}{\theta} \\ \mathrm{Var}(X) &= \mathrm{E}(X^2) - \mathrm{E}(X)^2 = m_X''(0) - (m_X'(0))^2 = \frac{30}{\theta^2} - \left(\frac{5}{\theta}\right)^2 = \frac{5}{\theta^2}. \end{aligned}$$

Alternatively, we can write down the power series expansion of $m_X(t)$ (a negative binomial series):

$$m_X(t) = \left(1 - \frac{t}{\theta}\right)^{-5} = 1 + 5\frac{t}{\theta} + \frac{5 \times (5+1)}{2}\left(\frac{t}{\theta}\right)^2 + \dots$$

and identify the first two moments as $5/\theta$ and $30/\theta^2$, giving the same result.

Unseen — 5 MARKS

(b) i. We can write the filter in the form

$$x_t = \sum_{k=-\infty}^{\infty} a_k y_{t-k},$$

where $a_0 = 11$, $a_1 = a_{-1} = -6$, $a_2 = a_{-2} = 1$, and $a_k = 0$ for all other k. The transfer function is

$$\begin{aligned} A(\omega) &= \sum_{k=-\infty}^{\infty} a_k e^{ik\omega} = e^{-2i\omega} - 6e^{-i\omega} + 11 - 6e^{i\omega} + e^{2i\omega} \\ &= (e^{-2i\omega} + e^{2i\omega}) - 6(e^{-i\omega} + e^{i\omega}) + 11 = 2\cos(2\omega) - 12\cos\omega + 11 \\ &= 2(2\cos^2\omega - 1) - 12\cos\omega + 11 = 4\cos^2\omega - 12\cos\omega + 9 \\ &= (2\cos\omega - 3)^2, \end{aligned}$$

as required.

Unseen — 8 MARKS

ii. The spectra of $\{x_t\}$ and $\{y_t\}$ are related by

$$f^{(x)}(\omega) = |A(\omega)|^2 f^{(y)}(\omega).$$

Since $|A(\omega)|^2 = (2\cos\omega - 3)^4$ is an increasing function on the interval $(0, \pi)$, the lower frequencies are damped down. Thus, this is a high-pass filter.

Seen similar — 2 MARKS

4. (a) The expectation $\mathrm{E}(y_t)$ must be constant, and the autocovariance $\mathrm{Cov}(y_t, y_{t+s})$ must be independent of t (i.e. a function of the lag s only).

Seen — 3 MARKS

(b) i. This is an MA(1) process. Its expectation is

$$\mathrm{E}(y_t) = \mathrm{E}(\varepsilon_t) + 2\mathrm{E}(\varepsilon_{t-1}) = 0\,,$$

and its autocovariance is

$$\mathrm{Cov}(y_t, y_{t+s}) = \begin{cases} 5\sigma^2 & \text{if } s = 0 \\ 2\sigma^2 & \text{if } s = 1 \\ 0 & \text{if } s \geq 2, \end{cases}$$

hence it is weakly stationary. The ACF is

$$\rho_s = \begin{cases} 1 & \text{if } s = 0 \\ 2/5 & \text{if } s = 1 \\ 0 & \text{if } s \geq 2, \end{cases}$$

and the spectrum is

$$f(\omega) = \gamma_0 + 2\sum_{k=1}^{\infty} \gamma_k \cos(k\omega) = \sigma^2(5 + 4\cos\omega)\,.$$

Seen similar — 6 MARKS

ii. This is a random walk. The variance is

$$\mathrm{Var}(y_t) = \mathrm{Var}(y_{t-1}) + 0.5^2\mathrm{Var}(\varepsilon_t) = \mathrm{Var}(y_{t-1}) + 0.25\sigma^2\,,$$

which is not constant, so the process is not weakly stationary.

Seen — 3 MARKS

iii. The distributions N(2, 4) and Exp(0.5) both have mean 2 and variance 4. Also, y_t and y_{t+s} are independent for all $s \neq 0$. Hence, the process is weakly stationary. The ACF is

$$\rho_s = \begin{cases} 1 & \text{if } s = 0 \\ 0 & \text{if } s \geq 1, \end{cases}$$

and the spectrum is

$$f(\omega) = \gamma_0 + 2\sum_{k=1}^{\infty} \gamma_k \cos(k\omega) = 4\,.$$

Unseen — 5 MARKS

iv. For $t > 0$, the variance is

$$\mathrm{Var}(y_t) = \mathrm{Var}(\varepsilon_0) + t^2\mathrm{Var}(\varepsilon_t) = \sigma^2 + t^2\sigma^2\,,$$

which is not constant, so the process is not weakly stationary.

Unseen — 3 MARKS