Imperial College
London

M3S8

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May-June 2016

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Time Series

Date: **Monday 23rd May 2016**

Time: **09.30 – 11.30**

Time Allowed: **2 Hours**

This paper has Four Questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the relevant main book(s) are full.

Statistical tables will not be provided.

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Note: Throughout this paper $\{\epsilon_t\}$ is a sequence of uncorrelated random variables (white noise) having zero mean and variance σ_ϵ^2, unless stated otherwise. The unqualified term "stationary" will always be taken to mean second-order stationary. All processes are real-valued unless stated otherwise. The sample interval is unity unless stated otherwise.

1. (a) (i) What is meant by saying that a stochastic process is stationary?

(ii) Given that the autocovariance sequence $\{s_\tau\}$ of a stationary process is symmetric about $\tau = 0$, show that its spectral density function $S(f)$ is symmetric about $f = 0$.

(b) Write down the missing material marked $\boxed{?}$ to complete the following:

(i) A time series $\{X_t\}$ is called Gaussian/normal if, for all $n \geq 1$ and any $t_1, \ldots, t_n \in T$ (where T is the index set), $\boxed{?}$;

(ii) the MA(1) process $X_t = \epsilon_t - \theta\epsilon_{t-1}$ (with θ a finite constant) and the MA(1) process $\boxed{?}$ have identical autocorrelation sequences;

(iii) $S(f)\mathrm{d}f$ is the average contribution (over all realizations of the time series) to $\boxed{?}$ from components with frequencies in a small interval about f;

(iv) two real-valued discrete-time processes $\{X_t\}$ and $\{Y_t\}$ are called jointly stationary stochastic processes if each is separately a second-order stationary process, and $\boxed{?}$;

(v) the (magnitude squared) coherence $\gamma_{XY}^2(f)$ between two jointly stationary processes $\{X_t\}$ and $\{Y_t\}$ at frequency f can be written as

$$\gamma_{XY}^2(f) = \frac{|E\{\boxed{?}\}|^2}{E\{|\mathrm{d}Z_X(f)|^2\}E\{|\mathrm{d}Z_Y(f)|^2\}},$$

where $\mathrm{d}Z_X(f)$ and $\mathrm{d}Z_Y(f)$ are the orthogonal increments for $\{X_t\}$ and $\{Y_t\}$ respectively.

(c) Let the process $\{X_t\}$ be defined by

$$X_t = \phi X_{t-2} + \epsilon_t, \quad t = 2, 3, 4, \ldots$$

with $X_0 = 0, X_1 = 1$ and $\phi > 0$.

Find the 3×3 covariance matrix of X_2, X_3, X_4. Is it a symmetric Toeplitz matrix?

2. (a) Suppose the stationary process $\{X_t\}$ can be written as a one-sided linear process, $X_t = \sum_{k=0}^{\infty} \psi_k \epsilon_{t-k}$. We wish to construct the l-step ahead forecast

$$X_t(l) = \sum_{k=0}^{\infty} \delta_k \varepsilon_{t-k}.$$

Show that the l-step prediction variance $\sigma^2(l) = E\{(X_{t+l} - X_t(l))^2\}$ is minimized by setting $\delta_k = \psi_{k+l}$, $k \geq 0$.

(b) A continuous-time stationary process $\{X(t)\}$, with t in seconds (s), has spectral density function

$$S_{X(t)}(f) = \begin{cases} 1 - \frac{1}{4}(|f| - 6), & 6 < |f| \leq 10, \\ 0, & \text{otherwise}, \end{cases}$$

with f in cycles/s. It is sampled with a sample interval $\Delta t = 0.1$s to produce the discrete-time process $\{X_t\}$.

What is the spectral density function $S_{X_t}(f)$ of $\{X_t\}$ for $|f| < f_N$, where f_N is the Nyquist frequency? *A solution via a graphical argument is perfectly acceptable.*

(c) Let $\{X_t\}$ be a sequence of independent and identically-distributed Gaussian/normal random variables, each with mean zero and variance 1. Define

$$Y_t = \begin{cases} X_t, & t \text{ even}, \\ (X_{t-1}^2 - 1)/\sqrt{2}, & t \text{ odd}. \end{cases}$$

(i) Show that $\{Y_t\}$ is a white noise process and determine its mean and variance. *You may assume without proof that* $\text{cov}\{Y_t, Y_{t+\tau}\} = 0$ *for all t and $|\tau| \geq 2$. You will need to use the following result: If X is a real-valued Gaussian/normal random variable with mean μ and variance σ^2 then*

$$E\{(X-\mu)^r\} = \begin{cases} \frac{r!}{(r/2)!} \frac{\sigma^r}{2^{r/2}}, & r \text{ even} \\ 0, & r \text{ odd}. \end{cases}$$

(ii) Compute

$$E\{Y_{n+1} | Y_0, \ldots, Y_n\}$$

firstly for n odd, and secondly for n even.

Is $\{Y_t\}$ not just white but in fact an independent and identically distributed sequence?

3. (a) Let $\{X_t\}$ be a zero mean stationary AR(p) process: $X_t - \phi_{1,p}X_{t-1} - \ldots - \phi_{p,p}X_{t-p} = \epsilon_t$.

(i) Derive the Yule-Walker equations $\gamma_p = \Gamma_p \phi_p$ and $\sigma_\epsilon^2 = s_0 - \sum_{j=1}^{p} \phi_{j,p} s_j$, for estimation of the parameter vector $\phi_p = [\phi_{1,p}, \phi_{2,p}, \ldots, \phi_{p,p}]^T$ and white noise variance σ_ϵ^2, where $\gamma_p = [s_1, s_2, \ldots, s_p]^T$ and

$$\Gamma_p = \begin{bmatrix} s_0 & s_1 & \ldots & s_{p-1} \\ s_1 & s_0 & \ldots & s_{p-2} \\ \vdots & \vdots & & \vdots \\ s_{p-1} & s_{p-2} & \ldots & s_0 \end{bmatrix}$$

and s_τ is the lag-τ autocovariance.

(ii) Consider the AR(2) process $X_t - 0.5X_{t-2} = \epsilon_t$. Given $\text{var}\{X_t\} = 2$, use the Yule-Walker equations to find the value of σ_ϵ^2.

(b) Let $\{X_t\}$ be a zero mean stationary Gaussian/normal process and let $Y_t = g(X_t)$ where g is an instantaneous non-linear transform (for example, $g(X_t) = X_t^2$). The cross-covariance sequence $\{s_{XY,\tau}\}$ is defined in the usual way as $s_{XY,\tau} = \text{cov}\{X_t, Y_{t+\tau}\}$.

(i) Show that

$$\text{cov}\{X_t, Y_{t+\tau}\} = \text{cov}\{X_{t+\tau}, g(X_{t+\tau})\}\rho_{X,\tau},$$

where $\{\rho_{X,\tau}\}$ is the autocorrelation sequence for $\{X_t\}$.
[*Hints: (1) In computing the expectation for* $\text{cov}\{X_t, Y_{t+\tau}\}$ *condition on* $X_{t+\tau}$ *and use iterated expectation. (2) For two correlated zero-mean Gaussian/normal random variables* Z_1, Z_2 *with variances* $\sigma_{Z_1}^2$ *and* $\sigma_{Z_2}^2$*, respectively,* $E\{Z_1|Z_2\} = \left(\frac{\text{cov}\{Z_1,Z_2\}}{\sigma_{Z_2}^2}\right) Z_2$.]

(ii) Now define g as the sign function

$$g(a) = \begin{cases} 1, & a > 0 \\ 0, & a = 0 \\ -1, & a < 0, \end{cases}$$

and let $\{X_t\}$ additionally be white noise with variance σ_X^2.
Making use of the known distribution of $X_{t+\tau}$ show that

$$\text{cov}\{X_{t+\tau}, g(X_{t+\tau})\} = \left(\frac{2}{\pi}\right)^{1/2} \sigma_X,$$

and then use the result in (b)(i) to find $S_{XY}(f)$, the cross-spectral density function for $\{X_t\}$ and $\{Y_t\}$.

4. (a) (i) State the three defining properties of a linear time-invariant digital filter.

(ii) Let $\{X_t\}$ be a zero mean stationary process. Define the seasonal differencing of $\{X_t\}$ via $Y_t = (1 - B^{12})X_t$ where B is the backward shift operator. Derive the spectrum $S_Y(f)$ of $\{Y_t\}$ in terms of the spectrum of $\{X_t\}$ and the sine function.

(b) Let $\{W_t\}$ and $\{\eta_t\}$ be two zero mean stationary processes which are uncorrelated with each other at all times. Their autocovariance sequences are given by

$$s_{W,\tau} = \begin{cases} 4 & \text{if } \tau = 0, \\ -1 & \text{if } |\tau| = 1, \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad s_{\eta,\tau} = \begin{cases} 2 & \text{if } \tau = 0, \\ 1 & \text{if } |\tau| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Define the stationary process

$$X_t = \tfrac{1}{2}W_{t-1} + \eta_t.$$

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$$g_u = \begin{cases} b & \text{if } u = -1 \\ a & \text{if } u = 0 \\ 0 & \text{otherwise} \end{cases}$$

which minimize the mean square error $E\{(Y_t - W_t)^2\}$.

5. (a) The Fourier transforms referred to in (i) and (ii) below are as defined in the reading material which was made available previously.

(i) The functions $g(\tau) = \mathrm{e}^{-\pi\tau^2}$ and $G(f) = \mathrm{e}^{-\pi f^2}$ are a Fourier transform pair, i.e., $g(\tau) \longleftrightarrow G(f)$. Let $\widetilde{g}(\tau) = A\, g(C\tau)\, \mathrm{e}^{\mathrm{i}2\pi f_0 \tau}$, where $A, C > 0$ and f_0 are all real-valued constants.
Express the Fourier transform, $\widetilde{G}(f)$, of $\widetilde{g}(\tau)$ in terms of $G(\cdot)$.

(ii) Consider a *continuous-time* complex-valued stationary process $\{X(t)\}$ with 'Gaussian-shaped' autocovariance function: $s(\tau) = A\mathrm{e}^{-B\tau^2 + \mathrm{i}2\pi f_0 \tau}$, $|\tau| < \infty$, where $A, B > 0$ and f_0 are all real-valued constants. Use the result in (i) to show that the corresponding spectral density function for this process is given by

$$S(f) = A\left(\frac{\pi}{B}\right)^{1/2} \mathrm{e}^{-\pi^2 (f-f_0)^2/B}, \quad |f| < \infty.$$

(b) Let $\{X(t)\}$ be a *continuous-time* stationary process with spectral density function $S_X(f), f \in \mathbb{R}$. For $\Delta_{\mathrm{t}} > 0$ let

$$\overline{X}(t) = \frac{1}{\Delta_{\mathrm{t}}} \int_{t-\Delta_{\mathrm{t}}}^{t} X(u)\, \mathrm{d}u$$

represent the average value of the process over the given interval.

(i) Show that $\{\overline{X}(t)\}$ is a linear time-invariant filtering of $\{X(t)\}$.

(ii) Hence find the spectral density function $S_{\overline{X}}(\cdot)$ of the stationary process $\{\overline{X}(t)\}$.

In the remainder of the question assume $S_X(f) = C$ for $f \in \mathbb{R}$, where C is a positive constant. Since $S_X(\cdot)$ is flat, we can regard it as the spectral density function for a continuous-time version of white noise. (It is really a convenient fiction.)

(iii) Find $\mathrm{var}\{\overline{X}(t)\}$.
Hint:

$$\int_0^\infty \frac{\sin^2(ax)}{x^2} \mathrm{d}x = \frac{a\pi}{2}.$$

(iv) Let $Y_t = \overline{X}(t\,\Delta_{\mathrm{t}})$, $t \in \mathbb{Z}$, be a discrete-time process formed by taking samples Δ_{t} time units apart from $\{\overline{X}(t)\}$. Using the spectral aliasing formula show that the spectral density function $S_Y(\cdot)$ of $\{Y_t\}$ is constant over $|f| < 1/(2\Delta_{\mathrm{t}})$.
Hint:

$$\sum_{k=-\infty}^{\infty} \frac{1}{(a+k)^2} = \frac{\pi^2}{\sin^2(\pi a)}.$$

(v) Explain why $\{Y_t\}$ in (b)(iv) has the spectrum of a white noise process.

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Course: M3S8 [Q1-4]/M4S8/M5S8[Q1-5]
Setter:
Checker:
Editor:
External:
Date: January 8, 2016

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2016

M3S8 [Q1-4]/M4S8/M5S8[Q1-5]

Time Series (SOLUTIONS)

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1. (a) (i) $\{X_t\}$ is second-order stationary if $E\{X_t\}$ is a finite constant for all t, var$\{X_t\}$ is a finite constant for all t, and cov$\{X_t, X_{t+\tau}\}$, is a finite quantity depending only on τ and not on t. seen ⇓ 4

(ii) Now $S(f) = \sum_{\tau=-\infty}^{\infty} s_\tau e^{-i2\pi f\tau}$. So $S(-f) = \sum_{\tau=-\infty}^{\infty} s_\tau e^{i2\pi f\tau}$. Let $\tau' = -\tau$, then, using $s_{-\tau'} = s_{\tau'}$ (given), unseen ⇓

$$S(-f) = \sum_{\tau'=\infty}^{-\infty} s_{-\tau'} e^{-i2\pi f\tau'} = \sum_{\tau'=-\infty}^{\infty} s_{\tau'} e^{-i2\pi f\tau'} = S(f).$$

3

(b) The missing text is seen ⇓

(i) the joint distribution of $X_{t_1}, \ldots, X_{t_n}$ is multivariate Gaussian/normal;

(ii) $X_t = \epsilon_t - \frac{1}{\theta}\epsilon_{t-1}$;

(iii) the process variance or power;

(iv) cov$\{X_t, Y_{t+\tau}\}$ is a function of τ only;

(v) $dZ_X^*(f)dZ_Y(f)$. [*This is the form for complex variables used in the course, but conjugation of the second term instead would be acceptable as this really boils down to a definition.*] 5

(c) sim. seen ⇓

$$\begin{aligned}
X_2 &= \phi X_0 + \epsilon_2 = \epsilon_2 \Rightarrow \mu_{X_2} = E\{X_2\} = 0 \\
X_3 &= \phi X_1 + \epsilon_3 = \phi + \epsilon_3 \Rightarrow \mu_{X_3} = E\{X_3\} = \phi \\
X_4 &= \phi X_2 + \epsilon_4 = \phi\epsilon_2 + \epsilon_4 \Rightarrow \mu_{X_4} = E\{X_4\} = 0.
\end{aligned}$$

Then 2

$$\begin{aligned}
\text{var}\{X_2\} = E\{(X_2 - \mu_{X_2})^2\} &= E\{\epsilon_2^2\} = \sigma_\epsilon^2 \\
\text{var}\{X_3\} = E\{(X_3 - \mu_{X_3})^2\} &= E\{\epsilon_3^2\} = \sigma_\epsilon^2 \\
\text{var}\{X_4\} = E\{(X_4 - \mu_{X_4})^2\} &= E\{[\phi^2\epsilon_2^2 + 2\phi\epsilon_2\epsilon_4 + \epsilon_4^2\} \\
&= [1 + \phi^2]\sigma_\epsilon^2 \\
\text{cov}\{X_2, X_3\} = E\{(X_2 - \mu_{X_2})(X_3 - \mu_{X_3})\} &= E\{\epsilon_2\epsilon_3\} = 0 \\
\text{cov}\{X_2, X_4\} = E\{(X_2 - \mu_{X_2})(X_4 - \mu_{X_4})\} &= E\{\epsilon_2[\phi\epsilon_2 + \epsilon_4]\} = \phi\sigma_\epsilon^2 \\
\text{cov}\{X_3, X_4\} = E\{(X_3 - \mu_{X_3})(X_4 - \mu_{X_4})\} &= E\{\epsilon_3[\phi\epsilon_2 + \epsilon_4]\} = 0.
\end{aligned}$$

So covariance matrix is

$$\sigma_\epsilon^2 \begin{bmatrix} 1 & 0 & \phi \\ 0 & 1 & 0 \\ \phi & 0 & 1+\phi^2 \end{bmatrix}.$$

5

This matrix is non-Toeplitz because of varying terms on the main diagonal. 1

2. (a) We want to minimize, $\boxed{\text{seen} \Downarrow}$

$$
\begin{aligned}
E\{(X_{t+l} - X_t(l))^2\} &= E\left\{\left(\sum_{k=0}^{\infty}\psi_k\epsilon_{t+l-k} - \sum_{k=0}^{\infty}\delta_k\epsilon_{t-k}\right)^2\right\} \\
&= \mathrm{E}\left\{\left(\sum_{k=0}^{l-1}\psi_k\epsilon_{t+l-k} + \sum_{k=0}^{\infty}[\psi_{k+l} - \delta_k]\epsilon_{t-k}\right)^2\right\} \\
&= \sigma_\epsilon^2\left\{\left(\sum_{k=0}^{l-1}\psi_k^2\right) + \sum_{k=0}^{\infty}(\psi_{k+l} - \delta_k)^2\right\}.
\end{aligned}
$$

The first term is independent of the choice of the $\{\delta_k\}$ and the second term is clearly minimized by choosing $\delta_k = \psi_{k+l}, k = 0, 1, 2, \ldots$. $\boxed{4}$

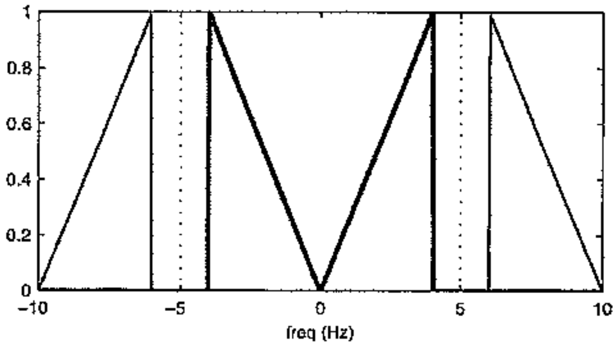

Figure 1: Aliasing

(b) See Fig. 1. The non-zero spectrum of the continuous-time process is delineated by the thin lines making up two triangles symmetric about zero. The aliasing (Nyquist) frequency is given by $f_{\mathcal{N}} = 1/(2\Delta_t) = 5$Hz when $\Delta_t = 0.1$s. The non-zero spectrum of the discretely-sampled process is found by reflecting about the Nyquist frequency, giving the two triangles delineated by the heavy lines. The spectrum for $|f| < f_{\mathcal{N}}$ is thus $\boxed{\text{sim. seen} \Downarrow}$

$$
S(f) = \begin{cases} \frac{1}{4}|f| & \text{for } |f| \le 4 \\ 0 & \text{for } 4 < |f| \le 5. \end{cases}
$$

$\boxed{4}$

(c) (i) We firstly show that $\{Y_t\}$ is white noise, mean 0 and variance 1. $\boxed{\text{unseen} \Downarrow}$

$$
E\{Y_t\} = \begin{cases} E\{X_t\} = 0, & t \text{ even} \\ \frac{1}{\sqrt{2}}[E\{X_{t-1}^2\} - 1] = \frac{1}{\sqrt{2}}[\mathrm{var}\{X_{t-1}\} - 1] = 0, & t \text{ odd.} \end{cases}
$$

$\boxed{2}$

Now find the variance $E\{Y_t^2\}$. It is given by

$$
\begin{cases} E\{X_t^2\} = 1, & t \text{ even} \\ E\{\frac{1}{2}(X_{t-1}^2 - 1)^2\} = \frac{1}{2}[E\{X_{t-1}^4\} - 2E\{X_{t-1}^2\} + 1] \overset{hint}{=} \frac{1}{2}[3 - 2 + 1] = 1, & t \text{ odd.} \end{cases}
$$

$\boxed{2}$

Now find the covariance at lag $\tau = 1$, $E\{Y_t Y_{t+1}\}$. It takes the form

$$\begin{cases} \frac{1}{\sqrt{2}} E\{X_t(X_t^2 - 1)\} = \frac{1}{\sqrt{2}} [E\{X_t^3\} - E\{X_t\}] \overset{hint}{=} 0, & t \text{ even} \\ \frac{1}{\sqrt{2}} E\{X_{t+1}(X_{t-1}^2 - 1)\} \overset{iid}{=} \frac{1}{\sqrt{2}} [E\{X_{t+1}\} E\{X_{t-1}^2 - 1\}] = 0, & t \text{ odd,} \end{cases}$$

Now find the covariance at lag $\tau = -1$, $E\{Y_t Y_{t-1}\}$. It takes the form

$$\begin{cases} \frac{1}{\sqrt{2}} E\{X_t(X_{t-2}^2 - 1)\} \overset{iid}{=} \frac{1}{\sqrt{2}} [E\{X_t\} E\{X_{t-2}^2 - 1\}] = 0, & t \text{ even,} \\ \frac{1}{\sqrt{2}} E\{X_{t-1}(X_{t-1}^2 - 1)\} = \frac{1}{\sqrt{2}} [E\{X_{t-1}^3\} - E\{X_{t-1}\}] \overset{hint}{=} 0, & t \text{ odd.} \end{cases}$$

$\boxed{4}$

The mean and variance are time-invariant. Since the $|\tau| = 1$ covariance is zero for any t and it is given that $\text{cov}\{Y_t, Y_{t+\tau}\} = 0$ for all t and $|\tau| \geq 2$ we conclude that $\{Y_t\}$ is a stationary and uncorrelated process, i.e. white noise, with mean zero and variance unity. $\boxed{1}$

(ii) For n odd ($n + 1$ even),

$$E\{Y_{n+1} | Y_0, \ldots, Y_n\} = E\{X_{n+1} | X_0, X_2, X_4, \ldots, X_{n-1}\} = E\{X_{n+1}\} = 0,$$

since the conditioning is on independent variables. $\boxed{1}$

For n even ($n + 1$ odd),

$$E\{Y_{n+1} | Y_0, \ldots, Y_n\} = E\left\{ \frac{X_n^2 - 1}{\sqrt{2}} \Big| X_0, X_2, X_4, \ldots, X_n \right\} = \frac{X_n^2 - 1}{\sqrt{2}} = \frac{Y_n^2 - 1}{\sqrt{2}}.$$

$\boxed{1}$

So we see that $\{Y_t\}$ cannot be an IID sequence. $\boxed{1}$

3. (a) (i) We start by multiplying the defining equation by X_{t-k}: $\boxed{\text{seen} \Downarrow}$

$$X_t X_{t-k} = \sum_{j=1}^{p} \phi_{j,p} X_{t-j} X_{t-k} + \epsilon_t X_{t-k}.$$

Taking expectations, for $k > 0$:

$$s_k = \sum_{j=1}^{p} \phi_{j,p} s_{k-j}.$$

Let $k = 1, 2, \ldots, p$ and recall that $s_{-\tau} = s_\tau$ to obtain

$$\begin{aligned}
s_1 &= \phi_{1,p}s_0 + \phi_{2,p}s_1 + \ldots + \phi_{p,p}s_{p-1} \\
s_2 &= \phi_{1,p}s_1 + \phi_{2,p}s_0 + \ldots + \phi_{p,p}s_{p-2} \\
\vdots & \quad \vdots \\
s_p &= \phi_{1,p}s_{p-1} + \phi_{2,p}s_{p-2} + \ldots + \phi_{p,p}s_0
\end{aligned}$$

or in matrix notation,

$$\boldsymbol{\gamma}_p = \boldsymbol{\Gamma}_p \boldsymbol{\phi}_p.$$

Finally, we need to estimate σ_ϵ^2. To do so, we multiply the defining equation by X_t and take expectations to obtain

$$s_0 = \sum_{j=1}^{p} \phi_{j,p} s_j + E\{\epsilon_t X_t\} = \sum_{j=1}^{p} \phi_{j,p} s_j + \sigma_\epsilon^2,$$

so that $\sigma_\epsilon^2 = s_o - \sum_{j=1}^{p} \phi_{j,p} s_j$. $\boxed{5}$

(ii) Here $\phi_{1,2} = 0, \phi_{2,2} = 0.5$ and $s_0 = \text{var}\{X_t\} = 2$. Then $\boxed{\text{sim. seen} \Downarrow}$

$$\sigma_\epsilon^2 = s_0 - \phi_{1,2}s_1 - \phi_{2,2}s_2 = 2 - 0.5s_2.$$

But

$$s_2 = \phi_{1,2}s_1 + \phi_{2,2}s_0 = 0 + 0.5 \cdot 2 = 1.$$

So $\sigma_\epsilon^2 = 2 - 0.5 \cdot 1 = 3/2$. $\boxed{3}$

(b) (i) Using the fact that $\{X_t\}$ has mean zero, then hints [1] and [2], $\boxed{\text{unseen} \Downarrow}$

$$\begin{aligned}
\text{cov}\{X_t, Y_{t+\tau}\} &= E\{X_t\, g(X_{t+\tau})\} - E\{X_t\}E\{g(X_{t+\tau})\} \\
&= E\{X_t\, g(X_{t+\tau})\} \stackrel{[1]}{=} E\{E\{X_t\, g(X_{t+\tau})|X_{t+\tau}\}\} \\
&= E\{g(X_{t+\tau})E\{X_t|X_{t+\tau}\}\} \\
&\stackrel{[2]}{=} E\left\{g(X_{t+\tau})\left(\frac{\text{cov}\{X_t, X_{t+\tau}\}}{\sigma_X^2}\right) X_{t+\tau}\right\} \\
&= E\left\{g(X_{t+\tau})X_{t+\tau}\right\}\left(\frac{\text{cov}\{X_t, X_{t+\tau}\}}{\sigma_X^2}\right) \\
&= \text{cov}\{X_{t+\tau}, g(X_{t+\tau})\}\rho_{X,\tau}.
\end{aligned}$$

$\boxed{7}$

unseen $\Downarrow$

(ii) Firstly we find the form of $\text{cov}\{X_{t+\tau}, g(X_{t+\tau})\}$.

$$\begin{aligned}
\text{cov}\{X_{t+\tau}, g(X_{t+\tau})\} &= E\{X_{t+\tau} \cdot \text{sign}\{X_{t+\tau}\}\} \\
&= E\{|X_{t+\tau}|\} \\
&= \int_{-\infty}^{\infty} |x| \frac{1}{\sigma_X \sqrt{(2\pi)}} \mathrm{e}^{-x^2/2\sigma_X^2} \mathrm{d}x \\
&= 2\int_0^{\infty} y\sigma_X \frac{1}{\sigma_X \sqrt{(2\pi)}} \mathrm{e}^{-y^2/2} \sigma_X \mathrm{d}y \\
&= -\frac{2\sigma_X}{\sqrt{(2\pi)}} \int_0^{\infty} -y\mathrm{e}^{-y^2/2} \mathrm{d}y \\
&= -\left(\frac{2}{\pi}\right)^{1/2} \sigma_X \left[\mathrm{e}^{-y^2/2}\right]_0^{\infty} \\
&= \left(\frac{2}{\pi}\right)^{1/2} \sigma_X.
\end{aligned}$$

So

$$\text{cov}\{X_{t+\tau}, g(X_{t+\tau})\}\rho_{X,\tau} = \left(\frac{2}{\pi\sigma_X^2}\right)^{1/2} s_{X,\tau}.$$

$\boxed{3}$

Since $\{X_t\}$ is now taken to be also white noise with variance σ_X^2, we know $S_X(f) = \sigma_X^2$. So Fourier transforming throughout the expression

$$\text{cov}\{X_t, Y_{t+\tau}\} = s_{XY,\tau} = \left(\frac{2}{\pi\sigma_X^2}\right)^{1/2} s_{X,\tau}$$

we get

$$S_{XY}(f) = \left(\frac{2\sigma_X^2}{\pi}\right)^{1/2}, \qquad |f| \leq 1/2.$$

$\boxed{2}$

It could be pointed out that both $\{X_t\}$ and $\{Y_t\}$ are stationary — they are both independent sequences under the assumption of white and Gaussian and hence independent X_t's — and $\text{cov}\{X_t, Y_{t+\tau}\}$ does not depend on t, so the two processes are jointly stationary, but this is not required here for full marks.

4. (a) (i) Let $\{x_t\}, \{y_t\}, \{x_{1,t}\}$ and $\{x_{2,t}\}$ be discrete-time sequences. $\boxed{\text{seen} \Downarrow}$

[1] Scale-preservation: Given a non-zero constant α,

$$L\{\alpha x_t\} = \alpha L\{x_t\}.$$

[2] Superposition:

$$L\{x_{1,t} + x_{2,t}\} = L\{x_{1,t}\} + L\{x_{2,t}\}.$$

[3] Time invariance: If $y_t = L\{x_t\}$ then

$$L\{x_{t+\tau}\} = y_{t+\tau}.$$

$\boxed{3}$

(ii) We have that $Y_t = X_t - X_{t-12}$ so $L\{X_t\} = X_t - X_{t-12}$ and if we input the complex exponential $e^{i2\pi ft}$ we get $\boxed{\text{sim. seen} \Downarrow}$

$$L\{e^{i2\pi ft}\} = e^{i2\pi ft} - e^{i2\pi f(t-12)} = e^{i2\pi ft}(1 - e^{-i24\pi f}) = e^{i2\pi ft}G(f).$$

So

$$\begin{aligned} |G(f)|^2 &= |1 - e^{-i24\pi f}|^2 = |e^{-i12\pi f}(e^{i12\pi f} - e^{-i12\pi f})|^2 \\ &= |e^{-i12\pi f} 2i \sin(12\pi f)|^2 = 4\sin^2(12\pi f). \end{aligned}$$

Therefore,

$$S_Y(f) = 4\sin^2(12\pi f)\, S_X(f).$$

$\boxed{4}$

(b) (i) $\boxed{\text{unseen} \Downarrow}$

$$\begin{aligned} E\{X_t X_{t+\tau}\} &= E\{(\tfrac{1}{2}W_{t-1} + \eta_t)(\tfrac{1}{2}W_{t+\tau-1} + \eta_{t+\tau})\} \\ \Rightarrow s_{X,\tau} &= \tfrac{1}{4}s_{W,\tau} + s_{\eta,\tau}, \end{aligned}$$

since expectations of other cross-products are zero due to $\{W_t\}$ and $\{\eta_t\}$ being uncorrelated. Fourier transforming then gives

$$S_X(f) = \tfrac{1}{4}S_W(f) + S_\eta(f).$$

Now

$$\begin{aligned} S_W(f) &= \sum_{\tau=-\infty}^{\infty} s_{W,\tau} e^{-i2\pi f\tau} = -e^{-i2\pi f\cdot(-1)} + 4 - e^{-i2\pi f\cdot 1} \\ &= 4 - (e^{-i2\pi f} + e^{i2\pi f\tau}) = 4 - 2\cos(2\pi f). \end{aligned}$$

Similarly,

$$S_\eta(f) = e^{-i2\pi f\cdot(-1)} + 2 + e^{-i2\pi f\cdot 1} = 2 + 2\cos(2\pi f)$$

So for $|f| \leq 1/2$,

$$S_X(f) = 1 - \tfrac{1}{2}\cos(2\pi f) + 2 + 2\cos(2\pi f) = 3(1 + \tfrac{1}{2}\cos(2\pi f)).$$

$\boxed{4}$

(ii) Now $Y_t = \sum_u g_u X_{t-u} = g_{-1}X_{t+1} + g_0 X_t = bX_{t+1} + aX_t$. So we want to find a, b to minimize $M = E\{(bX_{t+1} + aX_t - W_t)^2\}$ which is

$$\begin{aligned}
M &= E\{(\tfrac{1}{2}bW_t + b\eta_{t+1} + \tfrac{1}{2}aW_{t-1} + a\eta_t - W_t)^2\} \\
&= E\{([\tfrac{1}{2}b - 1]W_t + \tfrac{1}{2}aW_{t-1} + b\eta_{t+1} + a\eta_t)^2\} \\
&= [\tfrac{1}{2}b - 1]^2 E\{W_t^2\} + [\tfrac{1}{2}a]^2 E\{W_{t-1}^2\} + b^2 E\{\eta_{t+1}^2\} + a^2 E\{\eta_t^2\} \\
&+ 2[\tfrac{1}{2}b - 1][\tfrac{1}{2}a]E\{W_t W_{t-1}\} + 2abE\{\eta_{t+1}\eta_t\},
\end{aligned}$$

since the expectation of each of the other cross-products is zero because they involve different series which are uncorrelated. So

$$\begin{aligned}
M &= ([\tfrac{1}{2}b - 1]^2 + [\tfrac{1}{2}a]^2)s_{W,0} + (a^2 + b^2)s_{\eta,0} \\
&+ 2[\tfrac{1}{2}b - 1][\tfrac{1}{2}a]s_{W,1} + 2abs_{\eta,1} \\
&= 4(\tfrac{1}{4}b^2 - b + 1 + \tfrac{1}{4}a^2) + 2(a^2 + b^2) - 1(\tfrac{ab}{2} - a) + 2ab \\
&= 3a^2 + 3b^2 + \tfrac{3}{2}ab - 4b + a + 4.
\end{aligned}$$

$\boxed{5}$

So

$$\frac{\partial M}{\partial a} = 6a + \tfrac{3}{2}b + 1 \quad \text{and} \quad \frac{\partial M}{\partial b} = 6b + \tfrac{3}{2}a - 4.$$

Setting to zero and multiplying through the second equation by 4 gives

$$\begin{aligned}
6a + \tfrac{3}{2}b + 1 &= 0 \\
24b + 6a - 16 &= 0
\end{aligned}$$

So the second minus the first equation gives

$$\tfrac{45}{2}b - 17 = 0 \Rightarrow b = \tfrac{34}{45}.$$

Then the first equation gives

$$6a + \tfrac{3}{2} \cdot \tfrac{34}{45} = -1 \Rightarrow a = -\tfrac{48}{135}.$$

So $(a, b) = (-\frac{48}{135}, \frac{34}{45})$. $\boxed{2}$

We need to check this is a minimum. Now

$$\left|\begin{matrix} \frac{\partial^2 M}{\partial a^2} & \frac{\partial^2 M}{\partial a \partial b} \\ \frac{\partial^2 M}{\partial b \partial a} & \frac{\partial^2 M}{\partial b^2} \end{matrix}\right|_{a,b} = \left|\begin{matrix} 6 & \frac{3}{2} \\ \frac{3}{2} & 6 \end{matrix}\right| > 0$$

and $\frac{\partial^2 M}{\partial a^2}|_{a,b} = 6 > 0$, so that the solution is indeed a minimum point. $\boxed{2}$

5. (a) (i) Using the change of variable $t = C\tau$ $\boxed{\text{unseen} \Downarrow}$

$$\begin{aligned}
\widetilde{G}(f) &= \int_{-\infty}^{\infty} \widetilde{g}(\tau)\mathrm{e}^{-\mathrm{i}2\pi f\tau}\,\mathrm{d}\tau = \int_{-\infty}^{\infty} \left(Ag(C\tau)\mathrm{e}^{\mathrm{i}2\pi f_0\tau}\right)\mathrm{e}^{-\mathrm{i}2\pi f\tau}\,\mathrm{d}\tau \\
&= A\int_{-\infty}^{\infty} g(C\tau)\mathrm{e}^{-\mathrm{i}2\pi(f-f_0)\tau}\,\mathrm{d}\tau = \frac{A}{C}\int_{-\infty}^{\infty} g(t)\mathrm{e}^{-\mathrm{i}2\pi(f-f_0)t/C}\,\mathrm{d}t \\
&= \frac{A}{C}G([f-f_0]/C) \longleftrightarrow \widetilde{g}(\tau).
\end{aligned}$$

$\boxed{3}$

(ii) With $C = \sqrt{B/\pi}$, we have

$$\widetilde{g}(\tau) = Ag(C\tau)\mathrm{e}^{\mathrm{i}2\pi f_0\tau} = A\mathrm{e}^{-\pi C^2\tau^2}\mathrm{e}^{\mathrm{i}2\pi f_0\tau} = A\mathrm{e}^{-B\tau^2+\mathrm{i}2\pi f_0\tau} = s(\tau),$$

and from (i) the Fourier transform of $s(\cdot)$ is

$$\frac{A}{C}G([f-f_0]/C) = \frac{A}{\sqrt{B/\pi}}\mathrm{e}^{-\pi(f-f_0)^2/C^2} = A\left(\frac{\pi}{B}\right)^{1/2}\mathrm{e}^{-\pi^2(f-f_0)^2/B} = S(f).$$

Hence $s(\cdot) \longleftrightarrow S(\cdot)$, as required. $\boxed{3}$

(b) (i) let $L\{x(\cdot)\}$ be the filter whose output is the function $y(\cdot)$ given by

$$y(t) = \frac{1}{\Delta_t}\int_{t-\Delta_t}^{t} x(u)\,\mathrm{d}u.$$

Since the output of $L\{\alpha x(\cdot)\}$ is given by

$$\frac{1}{\Delta_t}\int_{t-\Delta_t}^{t} \alpha x(u)\,\mathrm{d}u = \alpha\frac{1}{\Delta_t}\int_{t-\Delta_t}^{t} x(u)\,\mathrm{d}u = \alpha y(t),$$

scale preservation holds. Assuming $L\{x_1(\cdot)\} = y_1(\cdot)$ and $L\{x_2(\cdot)\} = y_2(\cdot)$, since the output of $L\{x_1(\cdot) + x_2(\cdot)\}$ is given by

$$\begin{aligned}
\frac{1}{\Delta_t}\int_{t-\Delta_t}^{t} [x_1(u) + x_2(u)]\,\mathrm{d}u &= \frac{1}{\Delta_t}\int_{t-\Delta_t}^{t} x_1(u)\,\mathrm{d}u + \frac{1}{\Delta_t}\int_{t-\Delta_t}^{t} x_2(u)\,\mathrm{d}u \\
&= y_1(t) + y_2(t),
\end{aligned}$$

superposition holds. Finally, letting $u' = u + \tau$, since

$$\frac{1}{\Delta_t}\int_{t-\Delta_t}^{t} x(u+\tau)\,\mathrm{d}u = \frac{1}{\Delta_t}\int_{t+\tau-\Delta_t}^{t+\tau} x(u')\,\mathrm{d}u' = y(t+\tau),$$

time invariance holds. It follows that $\{\overline{X}(t)\}$ can be regarded as the output from an LTI *analog* filter. $\boxed{3}$

(ii) If $e^{i2\pi ft}$ is the input to the filter then the output is $e^{i2\pi ft}G(f)$ where $G(f)$ is the filter transfer function. So,

$$\begin{aligned} e^{i2\pi ft}G(f) &= \frac{1}{\Delta_t}\int_{t-\Delta_t}^{t} e^{i2\pi fu}\,du = \frac{1}{i2\pi f\,\Delta_t}\left(e^{i2\pi ft} - e^{i2\pi f(t-\Delta_t)}\right) \\ &= e^{i2\pi ft}\frac{e^{-i\pi f\,\Delta_t}}{i2\pi f\,\Delta_t}\left(e^{i\pi f\,\Delta_t} - e^{-i\pi f\,\Delta_t}\right) = e^{i2\pi ft}\frac{e^{-i\pi f\,\Delta_t}}{\pi f\,\Delta_t}\sin(\pi f\,\Delta_t). \end{aligned}$$

The process $\{\overline{X}(t)\}$ thus has a spectral density function given by

$$S_{\overline{X}}(f) = |G(f)|^2\,S_X(f) = \frac{\sin^2(\pi f\,\Delta_t)}{(\pi f\,\Delta_t)^2}S_X(f).$$

3

(iii)

$$\mathrm{var}\{\overline{X}(t)\} = \int_{-\infty}^{\infty} S_{\overline{X}}(f)\,df = \int_{-\infty}^{\infty} |G(f)|^2\,S_X(f)\,df = C\int_{-\infty}^{\infty}\frac{\sin^2(\pi f\,\Delta_t)}{(\pi f\,\Delta_t)^2}\,df.$$

Let $a = \pi\Delta_t$. Then

$$\mathrm{var}\{\overline{X}(t)\} = \frac{C}{a^2}\int_{-\infty}^{\infty}\frac{\sin^2(af)}{f^2}\,df = 2\frac{C}{a^2}\int_{0}^{\infty}\frac{\sin^2(af)}{f^2}\,df = 2\frac{C}{a^2}\cdot\frac{a\pi}{2} = \frac{C}{\Delta_t}.$$

3

(iv) We use $\sin^2(x+k\pi) = \sin^2(x)$. Then

$$\begin{aligned} S_Y(f) &= \sum_{k=-\infty}^{\infty} S_{\overline{X}}(f + \tfrac{k}{\Delta_t}) \\ &= \sum_{k=-\infty}^{\infty}\frac{\sin^2\left(\pi[f+\frac{k}{\Delta_t}]\,\Delta_t\right)}{\left(\pi[f+\frac{k}{\Delta_t}]\,\Delta_t\right)^2}S_X\left(f+\tfrac{k}{\Delta_t}\right) \\ &= C\sum_{k=-\infty}^{\infty}\frac{\sin^2\left(\pi[f+\frac{k}{\Delta_t}]\,\Delta_t\right)}{\left(\pi[f+\frac{k}{\Delta_t}]\,\Delta_t\right)^2} = C\sum_{k=-\infty}^{\infty}\frac{\sin^2(\pi f\,\Delta_t + k\pi)}{(\pi f\,\Delta_t + k\pi)^2} \\ &= \frac{C\sin^2(\pi f\,\Delta_t)}{\pi^2}\sum_{k=-\infty}^{\infty}\frac{1}{(f\,\Delta_t + k)^2} \\ &= \frac{C\sin^2(\pi f\,\Delta_t)}{\pi^2}\frac{\pi^2}{\sin^2(\pi f\,\Delta_t)} = C, \qquad |f| \le \frac{1}{2\,\Delta_t}. \end{aligned}$$

3

(v) Interpret the integral defining $\overline{X}(t)$ as being over the open interval $(t-\Delta_t, t)$, then Y_t and Y_u for $t \neq u$ are based on nonoverlapping portions of a continuous-time white noise process. Since distinct random variables from this process are uncorrelated, any random variable from $\{X(t)\}$ that was used to form Y_t is uncorrelated with those forming Y_u and hence Y_t and Y_u are uncorrelated, i.e., $\{Y_t\}$ is white noise.

2