1 The following predicates are used to represent arrays that have very few non-null elements (called sparse arrays). Note that in every array indices start at 0.

In(x, y, z) means that the value stored at index *y* of array *x* is *z*.
First(u, v) means that the first index in array *u* with a value is *v*.
End(u, v) means that the last possible index in array *u* is *v*.

a Give the representation, in terms of *In*, *First* and *End*, of the sparse array *A*.

	0	1	2	3	4	5	6	7	8	9
A =		19			25					

b Translate the following into **natural** English.

i) $\forall x \forall u \forall v[In(B, x, u) \wedge In(B, x, v) \rightarrow u = v]$
ii) $\neg \exists u \exists z \exists w[In(B, z, w) \wedge First(B, u) \wedge z < u]$
iii) $\forall x \forall u[First(B, x) \wedge In(B, x, u) \rightarrow u \geq 0]$

c Using the predicates *In*, *First*, *End* and the arithmetic relations $<$. $\leq$, $=$, translate the following into logic.

Define any other predicates you introduce.

i) For any indices i and j in array C, j is the next index to i (written next(C, i, j)) iff $i \leq j$ and there is no value stored at any indices between i and j.
ii) Every possible index in array A has a value.
iii) The values in array B appear in ascending order.
iv) There is no (common) index of arrays B and C with a value stored in both B and C.

The three parts carry, respectively, 10%, 30% , 60% of the marks.

2 a Give a natural deduction derivation of (4) from assumptions (1), (2) and (3). Do *not* rewrite any data using equivalences.

(1) $\forall x \forall y[R(x,y) \rightarrow R(y,x)]$
(2) $\forall x \forall y[R(x,y) \wedge R(y,x) \rightarrow x=y]$
(3) $\forall x[D(x) \rightarrow \neg R(x,x)]$
(4) $\neg \exists x \exists y[D(x) \wedge R(x,y)]$

b Show by natural deduction (without using equivalences):

$\vdash (p \rightarrow q) \leftrightarrow ((p \wedge q) \leftrightarrow p)$

c i) Show by a truth analysis that $((p \rightarrow q) \wedge p) \equiv (p \wedge q)$

ii) Show by equivalences: $(p \rightarrow q) \leftrightarrow ((p \wedge q) \leftrightarrow p) \equiv$ true
State clearly any equivalences you use.
(**Hint**: Use part ci) and the fact that $\leftrightarrow$ is associative and commutative.)

The three parts carry, respectively, 35%, 25%, 40% of the marks.

3 a i) Find a structure with domain = {0,1} to show that $\exists x.\ P(x) \nvDash P(a)$.
Explain your answer.

ii) Explain *carefully* why, if P(a) is true in a structure, then $\exists x.\ P(x)$ is also true in that structure.

iii) Find a structure with domain ={integers} that shows
$\exists y\ [\ f(y) = a]\ \nvDash\ \forall x \exists y\ [f(y) = x]$
Justify your answer *fully*.

b i) Translate into logic "There exist exactly two things".

ii) Explain why the sentence given in answer to part bi) cannot be true in any structure whose domain has either 3 elements or 1 element.

c Use natural deduction to show
$\forall y \forall z[\ (\exists x[f(y)=x \wedge f(z)=x]) \rightarrow y=z]\ \vdash \forall u \forall w[f(u)=f(w) \rightarrow u=w]$

The three parts carry, respectively, 45%, 25% , 30% of the marks.

4 a In this part you may use any of the standard rules of natural deduction (including the PC rule), **except** *$\rightarrow$E* or *$\vee$E*. You may also use the rules *$\rightarrow$E(alt)* and *$\vee$E(alt)* given below. You should not rewrite any data using equivalences.

$$\textbf{→E(alt)}\quad \frac{A \rightarrow B \qquad \neg B}{\neg A} \qquad \textbf{∨E(alt)}\quad \frac{A \vee B \qquad \neg B}{A} \qquad \left(\text{or } \frac{B \vee A \qquad \neg B}{A}\right)$$

Show by natural deduction: $(A \vee B) \wedge (A \rightarrow C) \wedge (B \rightarrow C) \vdash C$

b This part uses sentences (7) - (9) below.

(7) $(\exists x.\neg P(x)) \rightarrow \exists t[\neg P(t) \wedge \forall u[\neg P(u) \rightarrow u \geq t]]$

(8) $\forall t[\forall u[\neg P(u) \rightarrow u \geq t] \rightarrow \forall v[v<t \rightarrow P(v)]]$

(9) $\forall m[\forall v[v<m \rightarrow P(v)] \rightarrow P(m)]$

i) Translate (7) into natural English, where P(x) means "x belongs to set P".

ii) Complete the following natural deduction proof that (7), (8), (9) $\vdash \forall k.\ P(k)$ by filling in those parts marked by ??.

In the case that the part to be filled in is the reason for a step, include the line numbers used in its derivation.

1		$(\exists x.\neg P(x)) \rightarrow \exists t[\neg P(t) \wedge \forall u[\neg P(u) \rightarrow u \geq t]]$	(7)
2		$\forall t[\forall u[\neg P(u) \rightarrow u \geq t] \rightarrow \forall v[v<t \rightarrow P(v)]]$	(8)
3		$\forall m[\forall v[v<m \rightarrow P(v)] \rightarrow P(m)]$	(9)
4	$\forall$I K		
5		(??)	
6		(??)	(??)
7		$\exists t[\neg P(t) \wedge \forall u[\neg P(u) \rightarrow u \geq t]]$	($\rightarrow$E, 1,6)
8	T $\exists$E	(??)	
9		(??)	(??)
10		(??)	(??)
11		$\forall v[v < T \rightarrow P(v)]$	($\forall\rightarrow$E, 2,10)
12		P(T)	(??)
13		(??)	(??)
14		(??)	(??)
15		(??)	(PC)
16		$\forall k.P(k)$	($\forall$I)

iii) Using the translation of P(x) given in part bi) and the usual meanings of < and $\geq$, translate the completed proof of part bii) into natural English.

The two parts carry, respectively, 30%, 70% of the marks.

