UNIVERSITY OF LONDON **[E2.11 2006]**

B.ENG. AND M.ENG. EXAMINATIONS 2006

For Internal Students of Imperial College London

This paper is also taken for the relevant examination for the Associateship of the City & Guilds of London Institute

INFORMATION SYSTEMS ENGINEERING E2.11

MATHEMATICS

Date Wednesday 31st May 2006 2.00 - 4.00 pm

Answer FOUR questions, to include at least one from Section B

Answers to Section A *questions must be written in a different answer book from answers to* Section B *questions.*

A statistics formula sheet is provided

[Before starting, please make sure that the paper is complete. There should be SIX pages, with a total of SIX questions. Ask the invigilator for a replacement if this copy is faulty.]

[E2.11 2006]

Section A

1. We define the Fourier transform $\widehat{f}(\omega)$ of a function $f(t)$ as

$$\widehat{f}(\omega) \;=\; \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt\;.$$

If f is smooth and $|\, f(t)\,| \to 0$ sufficiently fast as $t \to \pm\infty$, show that one can evaluate the Fourier transforms of $f'(t)$, $f''(t)$, $tf(t)$ and $tf'(t)$ in the following forms :

(i) $\widehat{f'(t)}\;(\omega) \;=\; a_1(\omega)\, \widehat{f}(\omega)\;;$

(ii) $\widehat{f''(t)}(\omega) \;=\; a_2(\omega)\, \widehat{f}(\omega)\;;$

(iii) $t\widehat{f(t)}(\omega) \;=\; a_3(\omega)\, \dfrac{d\widehat{f}(\omega)}{d\omega}\;;$

(iv) $t\widehat{f'(t)}(\omega) \;=\; -\,\widehat{f}(\omega) \;+\; a_4(\omega)\, \dfrac{d\widehat{f}(\omega)}{d\omega}\;.$

Find a_1, a_2, a_3, a_4, which may be constants or functions of w.

(v) Find the Fourier transform of

$$e^{-a|t|}\;\cos bt\;,$$

where $a > 0$.

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[E2.11 2006]

2. (i) Take the Laplace transform of

$$y(t) \;=\; \sin t \;+\; \int_0^t y(u)\,du\,, \qquad t \geq 0$$

and hence find the solution $y(t)$.

What is $y(0)$?

(ii) Show that

$$\int_{(0,0)}^{(u,v)} \left[(x^2 y \;+\; \cos x)\,dx \;+\; \frac{x^3}{3}\,dy \right]$$

is independent of integration path from $(0,\,0)$ to $(u,\,v)$.

Choose a suitable path and integrate along it to evaluate the integral.

(iii) Determine a function $\Phi\,(x,\,y)$ such that

$$\frac{\partial \Phi}{\partial x} \;=\; x^2 y \;+\; \cos x\,,$$

$$\frac{\partial \Phi}{\partial y} \;=\; \frac{x^3}{3}\,.$$

(iv) What is the value of the integral in (ii) in terms of Φ ?

3. (i) By switching to polar coordinates, show that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \operatorname{sech}^2\,(x^2 \;+\; y^2)\; dx\, dy \;=\; \pi \;.$$

You may use

$$\frac{d}{du}\,\tanh u \;=\; \frac{1}{\cosh^2 u} \;=\; \operatorname{sech}^2 u \;.$$

(ii) Sketch the region in the $x - y$ plane, over which the integral

$$\int_0^1 \int_{y^2}^{y^{2/3}} \frac{\cos x}{\sqrt{x}}\; dx\, dy$$

is taken. Hence, give the correct form of the integral if the order of integration is reversed.

(iii) Evaluate the resulting integral in (ii).

4. The Fourier transform of $f(x) = \dfrac{x}{x^2 + 2x + 2}$ is

$$\widehat{f}(\omega) \;=\; \int_{-\infty}^{\infty} \frac{x}{x^2 + 2x + 2}\, e^{-i\omega x}\, dx \;, \qquad \omega > 0 \;.$$

Evaluate this integral using the semicircle method. The semicircle includes the real axis and is closed in the lower-half of the complex plane.

Justify why the integral along the semicircular part may be neglected in the limit of infinite radius.

5. A particular application accesses two computer files which are downloaded sequentially. The downloading times, T_1 and T_2, of the two files are modelled as independent normal random variables with means μ_1 and μ_2 and *known* standard deviations $\sigma_1 = 0.3$ and $\sigma_2 = 0.4$ respectively. The total downloading time is given by $T = T_1 + T_2$.

   (i) Initially it is assumed that $\mu_1 = 2$ seconds and $\mu_2 = 3$ seconds. Find:

      (a) $P(T_1 > 2.6)$;

      (b) $P(T_2 > 4)$.

   (ii) To assess the accuracy of the reported means, a sample of 50 downloading times is measured for each file giving sample means of 2.01 and 3.97 for T_1 and T_2 respectively.

      (a) Find 95% confidence intervals for μ_1 and μ_2 .

      (b) Comment on these confidence intervals in light of the initial assumptions in part (i).

      (c) What is the distribution of the total downloading time T?

      (d) Determine a 95% confidence interval for the mean total downloading time.

      (e) Under the assumptions in part (i) determine the probability that the sample mean total downloading time is greater than the value obtained for the upper bound of the confidence interval calculated in part (ii)(d).
      Comment on your result.

6. A particular component is obtained from sources A or B, with 90% chance of being obtained from source A and the remaining 10% chance from source B. The lifetimes, T_A and T_B of components of type A and B in hours, have probability density function

$$f(t) \;=\; \lambda e^{-\lambda t} \qquad t > 0\,,$$

with $\lambda = 0.2$ and $\lambda = 0.5$ for components from sources A and B respectively.

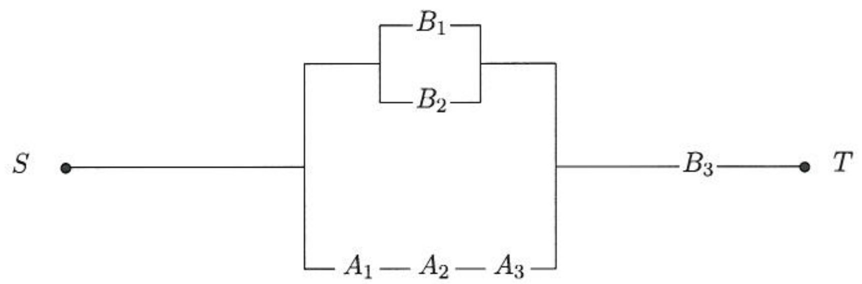
(i) (a) Determine the reliability functions and hazard rates associated with T_A and T_B.

(b) Determine the reliability of each type of component at 2 hours.

(c) Determine the reliability of a randomly selected component at 2 hours.

(d) Given that a component is still operating at 2 hours, what is the probability that it was obtained from A?

(ii) A system is made up using components, A_1, A_2, A_3 from source A and B_1, B_2, B_3 from source B. The system functions if there is a path of non-defective components between S and T.



(a) Assuming that the lifetimes of all components are independent, determine the reliability of the system at 2 hours.

(b) Suppose the parallel components B_1 and B_2 are replaced by a single component obtained from source A. Show that this change does not decrease the reliability of the system at 2 hours.

END OF PAPER

Solution ~~Question~~ 1	EXAMINATION QUESTIONS/SOLUTIONS 2005-06	Course ISE 2
		Marks & seen/unseen

Parts

(i) $\widehat{f'(t)}(\omega) = \int_{-\infty}^{\infty} f'(t)\, e^{-i\omega t}\, dt$

$= \left[f(t)\, e^{-i\omega t}\right]_{-\infty}^{\infty} - (-i\omega)\int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$

3 $= 0 + i\omega \hat{f}(\omega) \Rightarrow a_1(\omega) = i\omega.$

(ii) $\widehat{f''(t)}(\omega) = i\omega \widehat{f'(t)}(\omega)$ by (i)

$= (i\omega)(i\omega)\hat{f}(\omega)$ by (i) again

3 $= -\omega^2 \hat{f}(\omega) \Rightarrow a_2(\omega) = -\omega^2.$

(iii) $\widehat{tf(t)}(\omega) = \int_{-\infty}^{\infty} t f(t)\, e^{-i\omega t}\, dt$

$= \frac{1}{-i} \frac{d}{d\omega} \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$

3 $= i \frac{d\hat{f}(\omega)}{d\omega} \Rightarrow a_3(\omega) = i.$

(iv) $\widehat{tf'(t)}(\omega) = i \frac{d}{d\omega} \widehat{f'(t)}(\omega)$ by (iii)

$= i \frac{d}{d\omega}\left[i\omega \hat{f}(\omega)\right]$ by (i)

3 $= -\hat{f}(\omega) - \omega \frac{d\hat{f}(\omega)}{d\omega} \Rightarrow a_4(\omega) = -\omega.$

SEEN

(v) $I = \int_{-\infty}^{\infty} e^{-a|t|} \cos bt\, e^{-i\omega t}\, dt = \int_{-\infty}^{0} e^{at} \cos bt\, e^{-i\omega t}\, dt + \int_0^{\infty} e^{-at} \cos bt\, e^{-i\omega t}\, dt$

$= \int_0^{\infty} e^{-at} \cos bt\, e^{i\omega t}\, dt + \int_0^{\infty} e^{-at} \cos bt\, e^{-i\omega t}\, dt$

$= A + \bar{A} \leftarrow$ complex conjugate.

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EXAMINATION QUESTIONS/SOLUTIONS 2005-06

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~~Question~~ Solution 1

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Parts

$$\cos bt = \frac{1}{2}\left(e^{ibt} + e^{-ibt}\right).$$

$$A = \int_0^\infty e^{-at} \cos bt \, e^{i\omega t} dt = \int_0^\infty \frac{e^{-at}}{2}\left(e^{ibt} + e^{-ibt}\right) e^{i\omega t} dt$$

$$= \frac{1}{2} \int_0^\infty \left[e^{[-a+i(b+\omega)]t} + e^{[-a+i(\omega-b)]t} \right] dt$$

$$= \frac{1}{2}\left[\frac{1}{a - i(\omega+b)} + \frac{1}{a - i(\omega-b)} \right]$$

$$I = A + A^* = \frac{1}{2}\left[\frac{1}{a-i(\omega+b)} + \frac{1}{a+i(\omega-b)} + \frac{1}{a+i(\omega+b)} + \frac{1}{a-i(\omega-b)} \right]$$

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$$= \frac{1}{2}\left[\frac{2a}{a^2 + (\omega+b)^2} + \frac{2a}{a^2 + (\omega-b)^2} \right]$$

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Solution	EXAMINATION QUESTIONS/SOLUTIONS 2005-06	Course ISE2
~~Question~~ 2		Marks & seen/unseen

Parts

4 (i) Laplace transform: $\tilde{y}(p) = \dfrac{1}{1+p^2} + \dfrac{\tilde{y}(p)}{p}$ UNSEEN

$$\tilde{y}(p) = \frac{p}{(p-1)(p^2+1)} = \frac{A}{p-1} + \frac{Bp+C}{p^2+1}$$ ← solve for A, B, C

$$= \frac{1}{2}\left[\frac{1}{p-1} - \frac{p}{p^2+1} + \frac{1}{p^2+1}\right]$$

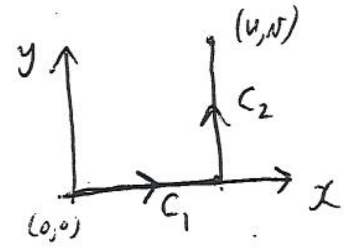
4 Using tables: $y(t) = \frac{1}{2}\left[e^t - \cos t + \sin t\right]$. SEEN SIMILAR

1 $y(0) = \frac{1}{2}[1-1+0] = 0$, which agrees with original equation.

(ii) $I = \int_{(0,0)}^{(u,v)} P\,dx + Q\,dy$ is independent of path SEEN

2 if $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ .. Both equal x^2, so true here.

Choose path:



$$I = \int_{C_1} P\,dx + Q\,\underbrace{dy}_{0} + \int_{C_2} P\,\underbrace{dx}_{0} + Q\,dy$$

$$= \int_0^u (x^2 \cdot 0 + \cos x)\,dx + \int_0^v \left(u^3/3 \right) dy$$ SEEN SIMILAR

4 $= +\sin u + \frac{1}{3}u^3 v$

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EXAMINATION QUESTIONS/SOLUTIONS 2005-06

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Parts

(iii) $\Phi = \frac{x^3}{3} y + \sin x + f(y)$

$\frac{\partial \Phi}{\partial y} = \frac{x^3}{3} + f'(y) = \frac{x^3}{3}$ so $f'(y) = 0$

Chose $f(y) = 0$.

3 $\Phi = \frac{1}{3} x^3 y + \sin x.$ SEEN SIMILAR

2 (iv) $I = \Phi(u, v) - \Phi(0,0)$ SEEN

$= \frac{1}{3} u^3 v + \sin u.$

EXAMINATION QUESTIONS/SOLUTIONS 2005-06

Course: ISE 2

Solution

Question 3

Marks & seen/unseen

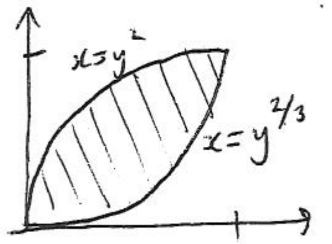
Parts

(i) $x^2+y^2=r^2$, $dx\,dy = r\,dr\,d\theta$.

4 $$I = \int_0^\infty dr \int_0^{2\pi} d\theta \; r \operatorname{sech}^2 r^2 \qquad \text{Let } u=r^2,\; du = 2r\,dr$$ SEEN SIMILAR

4 $$= 2\pi \, \frac{1}{2} \int_0^\infty \operatorname{sech}^2 u \, du = \pi .$$

(ii)



6 $$I = \int_0^1 \int_{x^{1/2}}^{x^{3/2}} \frac{\cos x}{\sqrt{x}} \, dy \, dx$$ SEEN SIMILAR

(iii) $$I = \int_0^1 \left[\frac{y \cos x}{\sqrt{x}} \right]_{x^{3/2}}^{x^{1/2}} dx$$

$$= \int_0^1 (-x\cos x + \cos x)\, dx$$

$$= -[x \sin x]_0^1 + \int_0^1 \sin x \, dx + [\sin x]_0^1$$

6 $$= -\sin 1 + [-\cos x]_0^1 + \sin 1$$

$$= -\cos 1 + 1$$

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Solution ~~Question~~ 4	EXAMINATION QUESTIONS/SOLUTIONS 2005-06	Course ISE 2
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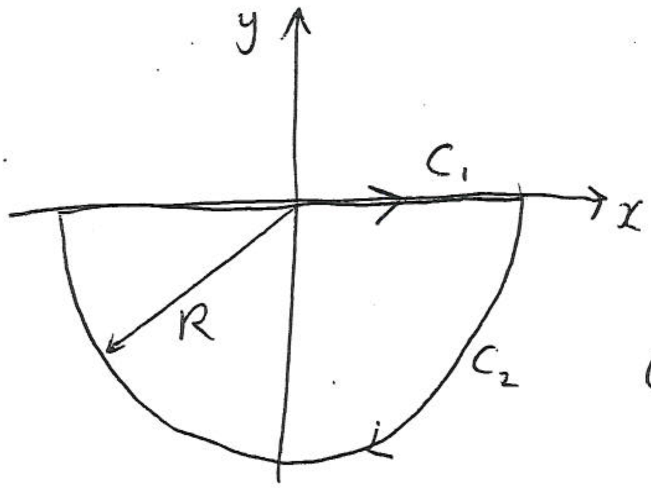
Parts

Consider the semicircular path:



3 (picture) $C = C_1 + C_2$

As $R \to \infty$, the integral

2 $$I = I_1 + I_2 = \int_{C_1+C_2} \frac{z e^{-i\omega z}}{z^2+2z+2} dz \quad , \quad z = x+iy$$

is equal to $\hat{f}(\omega)$, as long as we can neglect the integral over C_2. To show that this is the case, let $z = Re^{i\theta}$, $\theta : 0 \mapsto -\pi$.

$$I_2 = \int_0^{-\pi} \frac{Re^{i\theta} \exp(-i\omega Re^{i\theta})}{z^2+2z+2} \, iRe^{i\theta} d\theta$$

We have $|\exp(-i\omega Re^{i\theta})| = |\exp(\omega R \sin\theta - i\omega R\cos\theta)|$
$= \exp(\omega R \sin\theta)$.

But $\sin\theta < 0$ on C_2, so the exponential $\to 0$ — SEEN SIMILAR

4 as $R \to \infty$, faster than any other growing terms in the integrand. (Note that $\omega > 0$)

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Parts

Hence, $I_2 = 0$.

3 I_1 will equal $-2\pi i \times$ (sum of residues in C) (there is a minus sign since it is a clockwise path).

2 There are simple poles at $z = \dfrac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$.

2 Only $-1-i$ is inside C.

2 The residue is $\left.\dfrac{z e^{-i\omega z}}{2z+2}\right|_{z=-1-i} = \dfrac{(-1-i)\, e^{+i\omega(1+i)}}{2(-1-i)+2}$.

Hence,

$$I = I_1 = +2\pi i \frac{(-1-i)}{2i} e^{+\omega(-1+i)}$$

2

$$\boxed{I = -\pi\,(1+i)\, e^{+\omega(-1+i)}}$$

Course ISE Sec B

Question

Marks & seen/unseen

Parts

5. (i) (a)

$$\begin{aligned} T_1 &\sim N(2, 0.3^2) \ \Rightarrow \ Z = \frac{T_1 - 2}{0.3} \sim N(0,1) \\ \mathrm{P}(T_1 > 2.6) &= \mathrm{P}\left(Z > \frac{2.6-2}{0.3}\right) \\ &= \mathrm{P}(Z > 2) = 1 - \Phi(2) = 1 - 0.977 = 0.023. \end{aligned}$$

$\boxed{2}$

(b)

$$\begin{aligned} T_2 &\sim N(3, 0.4^2) \ \Rightarrow \ Z = \frac{T_2 - 3}{0.4} \sim N(0,1) \\ \mathrm{P}(T_2 > 4) &= \mathrm{P}\left(Z > \frac{4-3}{0.4}\right) \\ &= \mathrm{P}(Z > 2.5) = 1 - \Phi(2.5) = 1 - 0.994 = 0.006. \end{aligned}$$

$\boxed{2}$

(ii) (a) Let $\overline{t_1}$ be the sample mean for file 1. The 95% confidence interval for μ_1 is

$$\begin{aligned} \left(\overline{t_1} - 1.96\frac{\sigma_1}{\sqrt{n}}, \overline{t_1} + 1.96\frac{\sigma_1}{\sqrt{n}}\right) &= \left(2.01 \pm 1.96\frac{0.3}{\sqrt{50}}\right) \\ &= (2.01 - 0.083, 2.01 - 0.083) \\ &= (1.927, 2.093) \end{aligned}$$

$\boxed{3}$

Let $\overline{t_2}$ be the sample mean for file 2. The 95% confidence interval for μ_2 is

$$\begin{aligned} \left(\overline{t_2} - 1.96\frac{\sigma_2}{\sqrt{n}}, \overline{t_2} + 1.96\frac{\sigma_2}{\sqrt{n}}\right) &= \left(3.97 \pm 1.96\frac{0.4}{\sqrt{50}}\right) \\ &= (3.97 - 0.111, 3.97 + 0.111) \\ &= (3.859, 4.081) \end{aligned}$$

$\boxed{3}$

(b) The CI for μ_1 agrees with the assumed value in part (i), while the CI for μ_2 does not contain the value 3 - the data do not support this value. $\boxed{1}$

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(c)

$$\begin{aligned} T &= T_1 + T_2 \\ \mathrm{E}(T) &= \mu = \mu_1 + \mu_2 \\ \mathrm{var}(T) &= 0.3^2 + 0.4^2 = 0.5^2 \\ \Rightarrow T &\sim N(\mu_1 + \mu_2, 0.5^2) \end{aligned}$$

$\boxed{2}$

(d) The 95% confidence interval for μ is

$$\begin{aligned} &\left(2.01 + 3.97 - 1.96\frac{0.5}{\sqrt{50}}, \qquad 2.01 + 3.97 + 1.96\frac{0.5}{\sqrt{50}}\right) \\ &= (5.98 - 0.1386, 5.98 + 0.1386) \\ &= (5.841, 6.119) \end{aligned}$$

$\boxed{4}$

(e) Under the assumption in part (i)

$$\begin{aligned} \overline{T} &\sim N\left(2 + 3, \frac{0.5^2}{50}\right) \\ \Rightarrow \mathrm{P}(\overline{T} > 6.119) &= \mathrm{P}\left(Z > \frac{6.119 - 5}{0.5/\sqrt{50}}\right) \\ &= 1 - \Phi(15.8) = 1 \end{aligned}$$

this does not agree with the CI in part (ii)(d), the recorded downloading times cast doubt on the assumed values.

$\boxed{3}$

Question

Marks & seen/unseen

Parts

6\. (i) (a)

$$R(t) = \int_t^\infty \lambda e^{-\lambda s}\,ds = \left[-e^{-\lambda s}\right]_0^\infty = e^{-\lambda t}$$

$$h(t) = \frac{f(t)}{R(t)} = \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}} = \lambda.$$ 4

For T_A:

$$R_A(t) = e^{-0.2t}; \quad h_A(t) = 0.2.$$ 1

For T_B:

$$R_B(t) = e^{-0.5t}; \quad h_B(t) = 0.5.$$ 1

(b)

$$R_A(2) = e^{-0.4} = 0.670; \quad R_B(2) = e^{-1} = 0.368.$$ 2

(c)

$$\begin{aligned} \mathrm{P}(T>2) &= \mathrm{P}(T>2\,|\,A)\mathrm{P}(A) + \mathrm{P}(T>2\,|\,B)\mathrm{P}(B) \\ &= 0.670 \times 0.9 + 0.368 \times 0.1 = 0.640. \end{aligned}$$ 2

(d)

$$\begin{aligned} \mathrm{P}(A\,|\,T>2) &= \frac{\mathrm{P}(T>2\,|\,A)\mathrm{P}(A)}{\mathrm{P}(T>2)} \\ &= \frac{0.670 \times 0.9}{0.640} = 0.942. \end{aligned}$$ 3

(ii) (a) Let T be the lifetime of the system

$$\begin{aligned} \mathrm{P}(T>2) &= R_B(2)\left(1-(1-R_B(2))^2(1-R_A(2)^3)\right) \\ &= 0.368(1-(1-0.368)^2(1-0.670^3)) = 0.265. \end{aligned}$$ 4

(b) If B_1 and B_2 were replaced with a single component of type A then

$$\begin{aligned} \mathrm{P}(T>2) &= R_B(2)\left(1-(1-R_A(2))(1-R_A(2)^3)\right) \\ &= 0.368(1-(1-0.670)(1-0.670^3)) = 0.283. \end{aligned}$$

So, B_1 and B_2 can be replaced without decreasing the reliability. 3

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