SOLUTIONS: DISCRETE-TIME SYSTEMS AND COMPUTER CONTROL

1. Solution

a) In the Laplace domain, owing to the presence of the impulse sampling shown in Fig. 1.1, one gets:

$$Y_1(s) = G(s)X^*(s) \Longrightarrow Y_1^*(s) = G^*(s)X^*(s)$$

Then, in the z-domain, it follows that:

$$Y_1(z) = G(z)X(z) = \mathscr{Z}\left(\frac{1}{s(s+1)}\right) \cdot \mathscr{Z}\left[e^{-t} \cdot u(t)\right]$$

Thus:

$$\begin{aligned} Y_1(z) &= \mathscr{Z}\left(\frac{1}{s} - \frac{1}{s+1}\right) \cdot \mathscr{Z}\left[e^{-t} \cdot u(t)\right] = \\ &= \left(\frac{1}{1-z^{-1}} - \frac{1}{1-e^{-T}z^{-1}}\right) \cdot \frac{1}{1-e^{-T}z^{-1}} = \frac{z^2(1-e^{-T})}{(z-1)(z-e^{-T})^2} \end{aligned}$$

Now, to determine $y_1(kT)$, $k = 0, 1, \ldots$, the partial fraction expansion of $\dfrac{Y_1(z)}{z}$ has to be carried out. Thus:

$$\frac{Y_1(z)}{z} = \frac{z(1-e^{-T})}{(z-1)(z-e^{-T})^2} = \frac{c_1}{z-1} + \frac{c_2}{(z-e^{-T})^2} + \frac{c_3}{(z-e^{-T})}$$

where:

$$c_1 = \left.\frac{z(1-e^{-T})}{(z-e^{-T})^2}\right|_{z=1} = \frac{1}{1-e^{-T}}$$

$$c_2 = \left.\frac{z(1-e^{-T})}{(z-1)}\right|_{z=e^{-T}} = -e^{-T}$$

$$c_3 = \left.\frac{(1-e^{-T})(z-1) - z(1-e^{-T})}{(z-1)^2}\right|_{z=e^{-T}} = -\frac{1}{1-e^{-T}}$$

Then:

$$Y_1(z) = \frac{1}{1-e^{-T}}\frac{z}{z-1} - e^{-T}\frac{z}{(z-e^{-T})^2} - \frac{1}{1-e^{-T}}\frac{z}{(z-e^{-T})}$$

and hence

$$y_1(kT) = \mathscr{Z}^{-1}[Y_1(z)] = \frac{1}{1-e^{-T}}(1-e^{-kT}) - ke^{-kT}, \quad k = 0, 1, \ldots$$

[8 marks]

b) In the Laplace domain, owing to the presence of the impulse sampling shown in Fig. 1.2, one gets:

$$Y_2(s) = \frac{1-e^{-sT}}{s}G(s)X^*(s) \Longrightarrow Y_2^*(s) = \left[(1-e^{-sT})\frac{G(s)}{s}\right]^* X^*(s)$$

Then, in the z-domain, it follows that:

$$Y_2(z) = (1-z^{-1})\mathcal{Z}\left[\frac{G(s)}{s}\right]\cdot\mathcal{Z}\left[e^{-t}\cdot u(t)\right]$$

Thus:

$$\begin{aligned}
Y_2(z) &= (1-z^{-1})\mathcal{Z}\left(\frac{1}{s^2(s+1)}\right)\cdot\mathcal{Z}\left[e^{-t}\cdot u(t)\right] = \\
&= (1-z^{-1})\left(\frac{Tz^{-1}}{(1-z^{-1})^2}-\frac{1}{1-z^{-1}}+\frac{1}{1-e^{-T}z^{-1}}\right)\cdot\frac{1}{1-e^{-T}z^{-1}} = \\
&\left(\frac{T}{z-1}-1+\frac{z-1}{z-e^{-T}}\right)\cdot\frac{z}{z-e^{-T}} = \\
&= \frac{z[z(T-1+e^{-T})+1-(T+1)e^{-T}]}{(z-1)(z-e^{-T})^2}
\end{aligned}$$

Now, to determine $y_2(kT)$, $k=0,1,\ldots$, the same approach used in the answer to Question 1a) can be followed, that is, the partial fraction expansion of $\frac{Y_2(z)}{z}$ has to be carried out. Thus:

$$\frac{Y_2(z)}{z} = \frac{z(T-1+e^{-T})+1-(T+1)e^{-T}}{(z-1)(z-e^{-T})^2} = \frac{c_1}{z-1}+\frac{c_2}{(z-e^{-T})^2}+\frac{c_3}{(z-e^{-T})}$$

where:

$$c_1 = \left.\frac{z(T-1+e^{-T})+1-(T+1)e^{-T}}{(z-e^{-T})^2}\right|_{z=1} = \frac{T}{1-e^{-T}}$$

$$c_2 = \left.\frac{z(T-1+e^{-T})+1-(T+1)e^{-T}}{(z-1)}\right|_{z=e^{-T}} = -(1-e^{-T})$$

$$c_3 = \left.\frac{(T-1+e^{-T})(z-1)-(z(T-1+e^{-T})+1-(T+1)e^{-T})}{(z-1)^2}\right|_{z=e^{-T}} = -\frac{T}{1-e^{-T}}$$

Then:

$$Y_2(z) = \frac{T}{1-e^{-T}}\frac{z}{z-1}-(1-e^{-T})\frac{z}{(z-e^{-T})^2}-\frac{T}{1-e^{-T}}\frac{z}{(z-e^{-T})}$$

and hence

$$y_2(kT) = \mathcal{Z}^{-1}[Y_2(z)] = \frac{T}{1-e^{-T}}(1-e^{-kT})+(1-e^{T})ke^{-kT}, \quad k=0,1,\ldots$$

[8 marks]

c) From the answers to Question 1a) and 1b), setting $T = 1\,\text{sec}$, it follows that:

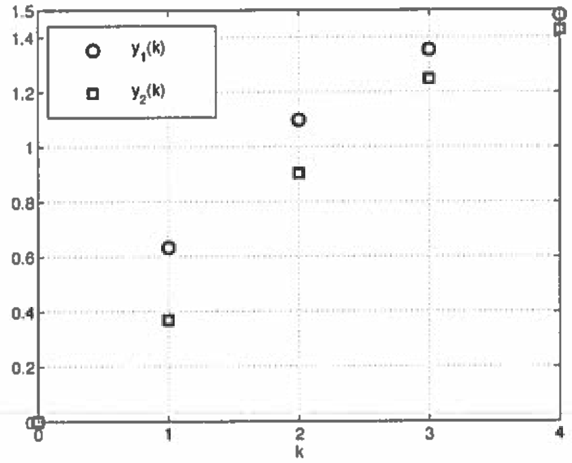
$$y_1(0)=0;\, y_1(1)\simeq 0.63;\, y_1(2)\simeq 1.1;\, y_1(3)\simeq 1.35;\, y_1(4)\simeq 1.48$$

and

$$y_2(0)=0;\, y_2(1)\simeq 0.37;\, y_2(2)\simeq 0.9;\, y_2(3)\simeq 1.25;\, y_2(4)\simeq 1.43$$

The first five values of the sequences $y_1(k)$ and $y_2(k)$ are plotted in the following figure.

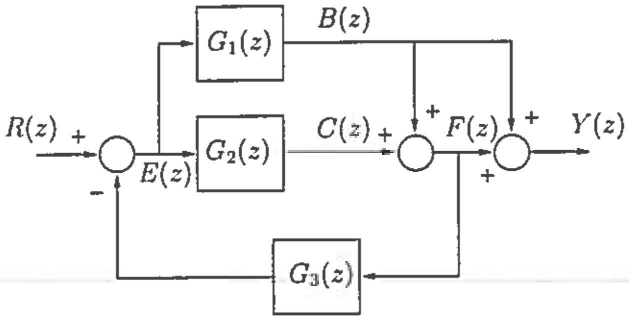
Comparing the two sequences, one can notice that they are slightly different in the initial transient. Moreover, it turns out that the sequence $y_2(k)$ is delayed with respect the sequence $y_1(k)$. This is not surprising and it is caused by the presence of the ZOH. When $k\to\infty$, the two sequences approach the same limit value because $\lim_{k\to\infty} ke^{-kT}=0$.



[4 marks]

2. Solution

a) To answer this question, one sets $D(z) = 0$ and considers the following block scheme



in which some additional variables have been added for the sake of the subsequent algebraic derivations. The following relationships hold (the dependence on z has been dropped for notational simplicity):

$$Y = F + B\,;\; E = R - G_3F\,;\; F = B + C\,;\; B = G_1E\,;\; C = G_2E$$

After some algebra, it follows that

$$Y(z) = \frac{2G_1(z) + G_2(z)}{1 + G_3(z)[G_1(z) + G_2(z)]}R(z)$$

Replacing the expressions of $G_1(z)$, $G_2(z)$, and $G_3(z)$, one gets:

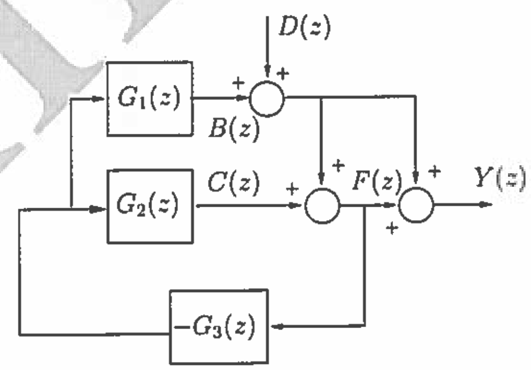
$$Y(z) = \frac{2\frac{z-1}{3z+2} + \frac{K}{z}}{1 + \frac{z}{z-1}\left(\frac{z-1}{3z+2} + \frac{K}{z}\right)}R(z)$$

and finally

$$Y(z) = H_{ry}(z)R(z) = \frac{2z^3 + (3K-4)z^2 + (2-K)z - 2K}{4z^3 + (3K-2)z^2 + (2K-2)z}R(z)$$

[5 marks]

b) A procedure analogous to the one addressing Question 2a) is carried out setting $R(z) = 0$ and considering the following block scheme:



In this case, the following relationships hold (the dependence on z has been dropped again for notational simplicity):

$$Y = F + B + D\,;\; F = B + C + D\,;\; B = -G_1G_3F\,;\; C = -G_2G_3F$$

After some algebra, it follows that

$$Y(z) = \frac{2+G_2(z)G_3(z)}{1+G_3(z)[G_1(z)+G_2(z)]}D(z)$$

Replacing the expressions of $G_1(z)$, $G_2(z)$, and $G_3(z)$, one gets:

$$Y(z) = \frac{2+\frac{K}{z}\frac{z}{z-1}}{1+\frac{z}{z-1}\left(\frac{z-1}{3z+2}+\frac{K}{z}\right)}D(z)$$

and finally

$$Y(z) = H_{dy}(z)D(z) = \frac{2z+K-2}{4z^2+(3K-2)z+2K-2}D(z)$$

[5 marks]

c) To analyse the stability of the overall dynamic system as a function of the parameter $K \in \Re$, consider the denominator of the closed-loop transfer function $H_{ry}(z)$:

$$p(z) = 4z^3+(3K-2)z^2+(2K-2)z = z(4z^2+(3K-2)z+2K-2)$$

Thus, one needs to check for which values of $K \in \Re$ the roots of $p(z)$ (that is, the closed-loop poles) lie strictly inside the unit circle. It is possible to focus only on the roots of the polynomial $4z^2+(3K-2)z+2K-2$ as the remaining root is $z=0$.

Introducing the bilinear transformation $z = \frac{1+w}{1-w}$, it follows that one has to check whether the roots of the following polynomial $\widetilde{p}(w)$ have negative real part:

$$\widetilde{p}(w) = (1+w)^2+\frac{3K-2}{4}(1-w^2)+\frac{K-1}{2}(1-w)^2 = \frac{4-K}{4}w^2+(3-K)w+\frac{5K}{4}$$

Then, all the roots of $\widetilde{p}(w)$ have strictly negative real part if $K \in (0,3)$ thus concluding that the overall discrete-time system is asymptotically stable if $K \in (0,3)$.

[5 marks]

d) Consider the transfer function $H_{dy}(z) = \frac{Y(z)}{D(z)}$ determined in the answer to Question 2b). One has:

$$Y(z) = H_{dy}(z)D(z) = \frac{2z+K-2}{4z^2+(3K-2)z+2K-2}$$

since $D(z) = \mathscr{Z}[\delta(k)] = 1$. Substituting $K=1$, after some algebra one gets:

$$Y(z) = \frac{1}{2}\frac{z-1/2}{z(z+1/4)}$$

Now, to determine $y(k)$, $k=0,1,\ldots$, the partial fraction expansion of $\dfrac{Y(z)}{z}$ has to be carried out. Thus:

$$\frac{Y(z)}{z}=\frac{1}{2}\frac{z-1/2}{z^2(z+1/4)}=\frac{c_1}{z^2}+\frac{c_2}{z}+\frac{c_3}{z+1/4}$$

where:

$$c_1=\frac{1}{2}\frac{z-1/2}{z+1/4}\bigg|_{z=0}=-1$$

$$c_2=\frac{1}{2}\frac{3/4}{(z+1/4)^2}\bigg|_{z=0}=24$$

$$c_3=\frac{1}{2}\frac{z-1/2}{z^2}\bigg|_{z=-1/4}=-24$$

Then:

$$Y(z)=-\frac{1}{z}+24-24\frac{z}{z+1/4}$$

and hence

$$y(k)=-\delta(k-1)+24\cdot\delta(k)-24(-1/4)^k\cdot u(k),\quad k=0,1,\ldots$$

where $u(k)$ is the discrete-time unit step sequence.

[5 marks]

3. Solution

a) Consider

$$G(s) = \frac{1/2}{(s^2+1/4)(s^2+9/4)}$$

The poles are the roots of the denominator, that is:

$$p_1 = j\frac{1}{2},\ p_2 = -j\frac{1}{2},\ p_3 = j\frac{3}{2},\ p_4 = -j\frac{3}{2}$$

Since $Re(p_i) = 0, i = 1,2,3,4$ and the poles are distinct, it can be concluded that the continuous-time system described by $G(s)$ is marginally stable.

[3 marks]

b) For the generic sampling time T, we can write (in the Laplace domain)

$$H(s)G(s) = (1-e^{-sT})\frac{G(s)}{s} = (1-e^{-sT})\frac{1/2}{s(s^2+1/4)(s^2+9/4)}$$

Now, by the usual procedure to calculate $\mathscr{Z}$ transforms of terms involving the ZOH, we obtain

$$\begin{aligned} HG(z) &= (1-z^{-1})\,\mathscr{Z}\left[\frac{1/2}{s(s^2+1/4)(s^2+9/4)}\right] = \\ &= (1-z^{-1})\,\mathscr{Z}\left[\frac{c_1}{s} + \frac{c_2}{s-j/2} + \frac{c_2^*}{s+j/2} + \frac{c_3}{s-j3/2} + \frac{c_3^*}{s+j3/2}\right] \end{aligned}$$

where c_2^* and c_3^* are the complex-conjugates of c_2 and c_3, respectively. After some algebra, one gets:

$$c_1 = \frac{8}{9},\ c_2 = c_2^* = -\frac{1}{2},\ c_3 = c_3^* = \frac{1}{18}$$

Thus:

$$HG(z) = \frac{z-1}{z}\left[\frac{8}{9}\frac{z}{z-1} - \frac{1}{2}\left(\frac{z}{z-e^{-jT/2}} + \frac{z}{z-e^{jT/2}}\right) + \frac{1}{18}\left(\frac{z}{z-e^{-j3T/2}} + \frac{z}{z-e^{j3T/2}}\right)\right]$$

After some algebra, one finally gets

$$HG(z) = \frac{8}{9} + (z-1)\left(\frac{\cos(T/2)-z}{z^2-2z\cos(T/2)+1} + \frac{1}{9}\frac{z-\cos(3T/2)}{z^2-2z\cos(3T/2)+1}\right)$$

[8 marks]

c) Setting $T = \pi$ sec and substituting this value of T into the the expression of $HG(z)$ computed in the Answer to Question 3b), after some easy algebra it follows immediately that

$$HG(z) = \frac{8}{9}\frac{z+1}{z^2+1}$$

[3 marks]

d) For a given value of the sampling time T, a point s in the Laplace complex plane is mapped into a point z in the $\mathscr{Z}$-plane through the following relationship:

$$z = e^{sT}$$

Applying this relationship to the poles of $G(s)$ computed in the Answer to Question 3a) with $T = \pi$ sec, one immediately gets:

$$z_1 = e^{p_1 T} = e^{j\frac{1}{2}\pi} = j,\, z_2 = e^{p_2 T} = e^{-j\frac{1}{2}\pi} = -j$$

and

$$z_3 = e^{p_3 T} = e^{j\frac{3}{2}\pi} = -j,\, z_4 = e^{p_4 T} = e^{-j\frac{3}{2}\pi} = j$$

Thus the four continuous-time poles are mapped in only two distinct points in the $\mathscr{Z}$-plane that coincide with the poles of $HG(z)$ determined in the Answer to Question 3c). This is caused by the specific choice of T, for which only the poles p_1 and p_2 fall in the primary strip $\left[-\frac{\omega_s}{2}, \frac{\omega_s}{2}\right]$ with $\omega_s = \frac{2\pi}{T} = 2\text{rad/sec}$ and for the fact that the pair of poles p_3 and p_4 lie in the complementary strip at a distance from the pair p_1 and p_2 that is exactly equal to the sampling time. Hence, this choice of the sampling time causes an *aliasing* phenomenon.

[6 marks]

4. Solution

a) Considering the block-diagram in Fig. 4.1(b), it is immediate to compute the open-loop transfer function $G(s)$ from the input u to the pointing angle θ, that is:

$$G(s) = \frac{\Theta(s)}{U(s)} = \frac{B}{s(Js+B)} = \frac{1}{s(1+sJ/B)} = \frac{1}{s(1+10s)}$$

where the value $J/B = 10$ has been used, as stated in Question 4a).

[3 marks]

b) Again, with reference to the block-diagram in Fig. 4.1(b), one first determines the closed-loop transfer function from the reference input θ_s to the error variable e. It follows that:

$$E(s) = \frac{1}{1+C(s)G(s)}\Theta_s(s) = \frac{s(1+as)}{as^2+s+K}\Theta_s(s)$$

As $K > 0, a > 0$, the closed-loop system is asymptotically stable. Now:

$$\Theta_s(s) = \mathscr{L}(0.01 \cdot t) = \frac{1}{100}\frac{1}{s^2}$$

and hence

$$E(s) = \frac{1}{100}\frac{1+as}{s(as^2+s+K)}$$

Since the closed-loop system is asymptotically stable, the final-value theorem can be used thus getting:

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{1}{100}\frac{1+as}{as^2+s+K} = \frac{1}{100K}$$

Then, any choice of K such that $K \geq 1$ ensures that $\lim_{t\to\infty} |e(t)| \leq 0.01$.

As can be noticed by inspecting the closed-loop transfer function computed above, the closed-loop poles are the roots of the second-order polynomial

$$p(s) = s^2 + \frac{1}{a}s + \frac{K}{a}$$

Comparing the polynomial $p(s)$ with the generic second-order polynomial expressed in terms of the damping-ration ξ and the natural angular frequency ω_n:

$$s^2 + 2\xi\omega_n s + \omega_n^2$$

it is immediate to conclude that the choice $a = 1$, $K = 1$ implies that $\xi = 0.5$ and $\omega_n = 1\text{rad/sec}$. Then, the controller

$$C(s) = \frac{1+10s}{1+s}$$

ensures that all the specifications set in Question 4b) are satisfied thus concluding the controller-design in continuous-time.

[5 marks]

c) The continuous-time controller $C(s)$ obtained in the answer to Question 3b) is:

$$C(s) = \frac{1+10s}{1+s}$$

and hence its transfer function has one zero and one pole.

Thus, the generic form for the "pole-zero correspondence" discrete-time approximation $C_d(z)$ of the controller $C(s)$ is:

$$C_d(z) = K_d \frac{z-z_1}{z-p_1}$$

Taking into account of the value of the sampling time $T = 0.2$sec, the discrete-time zero z_1 corresponding to the continuous-time zero is:

$$z_1 = e^{-1/10 \cdot 1/5} = e^{-1/50} \simeq 0.98$$

whereas the discrete-time pole p_1 corresponding to the continuous-time pole is:

$$p_1 = e^{-1 \cdot 1/5} = e^{-1/5} \simeq 0.82$$

As the static gain has to be preserved in order to keep the tracking steady-state requirement, one has to match the gain at 0, that is:

$$\lim_{z \to 1} C_d(z) = \lim_{s \to 0} C(s) \Longrightarrow K_d \frac{1-e^{-1/50}}{1-e^{-1/5}} = 1 \Longrightarrow K_d = \frac{1-e^{-1/5}}{1-e^{-1/50}} \simeq 9.154$$

Then, the discrete-time approximation $C_d(z)$ of the controller $C(s)$ is:

$$C_d(z) = \frac{1-e^{-1/5}}{1-e^{-1/50}} \cdot \frac{z-e^{-1/50}}{z-e^{-1/5}}$$

The equivalent discrete-time model $HG(z)$ for the plant $G(s)$ connected to the ZOH and the sampler is determined as follows:

$$H(s)G(s) = (1-e^{-sT})\frac{G(s)}{s} = (1-e^{-sT})\frac{1}{s^2(1+10s)}$$

Now, by the usual procedure to calculate $\mathscr{Z}$ transforms of terms involving the ZOH, we obtain

$$HG(z) = (1-z^{-1})\,\mathscr{Z}\left[\frac{1}{s^2(1+10s)}\right] =$$

$$= (1-z^{-1})\,\mathscr{Z}\left[\frac{1}{s^2} - \frac{10}{s} + \frac{10}{s+1/10}\right]$$

Hence:

$$HG(z) = \frac{z-1}{z}\left(\frac{1}{5}\frac{z}{(z-1)^2} - 10\frac{z}{z-1} + 10\frac{z}{z-e^{-1/50}}\right)$$

which finally gives:

$$(10e^{-1/50} - 49/5) \cdot \frac{z + \frac{10-51/5 \cdot e^{-1/50}}{10e^{-1/50}-49/5}}{(z-1)(z-e^{-1/50})} \simeq 0.002\frac{z+0.99}{(z-1)(z-0.98)}$$

[6 marks]

d) The closed-loop transfer function $G_{cl}(z) = \Theta(z)/\Theta_s(z)$ is given by:

$$G_{cl}(z) = \frac{C_d(z)HG(z)}{1 + C_d(z)HG(z)}$$

First, it is convenient to compute the open-loop transfer function $C_d(z)HG(z)$. One has:

$$\begin{aligned} C_d(z)HG(z) &= \frac{1 - e^{-1/5}}{1 - e^{-1/50}} \cdot (10e^{-1/50} - 49/5) \cdot \frac{z + \frac{10 - 51/5 \cdot e^{-1/50}}{10e^{-1/50} - 49/5}}{(z-1)(z - e^{-1/5})} \\ &\simeq 0.018 \frac{z + 0.99}{(z-1)(z-0.82)} \end{aligned}$$

where the "exact cancellation" between the zero of $C_d(z)$ and one of the poles of $HG(z)$ has been carried out.

The closed-loop transfer function $G_{cl}(z)$ is:

$$G_{cl}(z) \simeq 0.018 \cdot \frac{z + 0.99}{z^2 - 1.8z + 0.837}$$

The closed-loop poles are $p_1^{cl} \simeq 0.9 + j0.16$ and $p_2^{cl} \simeq 0.9 - j0.16$. Then, the closed-loop discrete-time system is asymptotically stable.

The corresponding points in the s-plane can be computed using the following inverse relationship:

$$z = e^{sT} \Longrightarrow s = \frac{1}{T}\log(z)$$

computed for $z = p_1^{cl}$ and $z = p_2^{cl}$ which, after some algebra, gives

$$\tilde{s} \simeq -0.437 + j0.89, \; \tilde{s}^* \simeq -0.437 - j0.89$$

The second-order polynomial having $\tilde{s}$ and $\tilde{s}^*$ as roots is:

$$s^2 + 0.874s + 1.153$$

and the associated damping ratio and natural angular frequency are:

$$\tilde{\xi} \simeq 0.4, \; \tilde{\omega}_n \simeq 1.07$$

The comparison with the values of ξ and ω_n obtained with the continuous-time controller $C(s)$ shows that the digital implementation of the control system causes a slight degradation in the damping ratio whereas the closed-loop bandwidth is approximately the same. A re-design of $C(s)$ or a choice of a smaller sampling-time may be needed.

[6 marks]