1. Consider the following two-dimensional time-invariant nonlinear system:

$$\begin{aligned}\dot{x}_1 &= -x_1^3 + \tfrac{7}{2}x_1 + x_2 \\ \dot{x}_2 &= -x_2 + \max\{|x_1|, 1\}\mathrm{sign}(x_1)\end{aligned}$$

where $\mathrm{sign}(r)$ denotes,

$$\mathrm{sign}(r) = \begin{cases} 1 & \text{for } r > 0 \\ 0 & \text{for } r = 0 \\ -1 & \text{for } r < 0 \end{cases}$$

a) Is the function $\max\{|x_1|, 1\}\mathrm{sign}(x_1)$ Lipschitz continuous? (justify your answer) [2]

b) Discuss existence and uniqueness of solutions. [2]

c) Find and sketch the nullclines of the vector-field. [3]

d) Identify the regions of the phase-plane in which the nullclines partition $\mathbb{R}^2$ and the orientation of the vector-field in each one of them. [3]

e) Find out, if any, those regions that are forward invariant. [3]

f) Find all equilibria of the system. [2]

g) Linearize the system around each equilibrium and discuss the local phase-portrait. [3]

h) Sketch the global phase portrait of the system, emphasizing equilibria and how their local portraits merge into each other. [2]

2. Consider the following nonlinear autonomous system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\alpha & \beta \\ \gamma & -\delta \end{bmatrix} \begin{bmatrix} x_1^3 \\ x_2^3 \end{bmatrix}$$

where $x = [x_1, x_2]'$ is the two-dimensional state vector and $\alpha, \beta, \gamma, \delta$ are positive parameters.

a) Decompose the system as the feedback interconnections of two scalar subsystems, corresponding to the variables x_1 and x_2, respectively. [3]

b) Show that each of the subsystems can be shown to be Input to State Stable with respect to its own input. [2]

c) Compute an ε-parameterized upper-bound of the Input-to-State gain of each subsystem, so as to be as tight as possible. [4]

d) Show, by applying the small-gain theorem, that $\alpha\delta > \beta\gamma$ implies Global Asymptotic Stability of the interconnected system. [3]

e) Compare this stability condition with the set of parameters values for which matrix A defined as

$$A := \begin{bmatrix} -\alpha & \beta \\ \gamma & -\delta \end{bmatrix},$$

is Hurwitz (viz. all its eigenvalues have negative real part). [3]

f) Set $\alpha = \delta = 1$ and $\beta = \gamma = 2$. Show that the functions $V_1(x) = (x_1 - x_2)^2$ and $V_2(x) = (x_1 + x_2)^2$ have, respectively, negative semi-definite and positive semi-definite derivatives along solutions of the system. [3]

g) Show that, for the parameters values selected in question f), the origin is an unstable equilibrium. (Hint: combine $V_1(x)$ and $V_2(x)$ to find a function with positive definite derivative and apply the instability criterion) [2]

3. Consider the Single-Input Single-Output nonlinear system described by the following set of equations

$$\begin{aligned}
\dot{x}_1 &= x_1 + 2x_2 - \sin(x_1 + x_2) + 2x_2x_3 + [2 + \cos(x_2)]u \\
\dot{x}_2 &= -x_2 + \sin(x_1 + x_2) \\
\dot{x}_3 &= -x_3^3 + x_2x_3 \\
\\
y &= x_1 + x_2
\end{aligned}$$

with state $x = [x_1, x_2, x_3]' \in \mathbb{R}^3$, input u and output y, respectively.

a) Find the relative degree of the system, for $x = 0$, and justify whether this is locally or globally defined. [4]

b) Find an Input-Output linearizing feedback. [4]

c) Apply the Input-Output linearizing feedback computed previously and write the system in normal form. [4]

d) Highlight the internal dynamics and discuss their stability or instability properties (ISS, GAS, local Asyptotic Stability). [4]

e) Design a state-feedback to globally asymptotically stabilize the origin of the original system. [4]

4. Consider the following control affine nonlinear system:

$$\begin{aligned} \dot{x}_1 &= g(x_2) \\ \dot{x}_2 &= -g(x_1) + u \end{aligned}$$

where $g : \mathbb{R} \to \mathbb{R}$ is any locally Lipschitz real-valued scalar function, $x \in \mathbb{R}^2$ is the state variable and $u \in \mathbb{R}$ is the input.

a) Show that for a suitable choice of the output $y = h(x)$, the system is loss-less and passive (possibly with a non lower bounded storage function). [5]

b) Let $g(r) = e^r - e^{-r}$. Consider the static feedback $u = -x_2$. Show that the origin is Globally Asymptotically Stable. [5]

c) Argue that Global Asymptotic Stability may also be achieved by means of bounded controls. For instance fulfilling the constraint $|u| \leq 1$. (Hint: modify the feedback by including some form of saturation nonlinearity). [5]

d) Consider next the situation arising with $g(r) = r^2$. Let $u = 0$ and show that the origin is unstable. (Hint: use Lyapunov instability criteria). [5]