UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE TECHNOLOGY AND MEDICINE

[E303/ISE3.3]

DEPARTMENT of ELECTRICAL and ELECTRONIC ENGINEERING
M.Eng, B.Eng and A.C.G.I. EXAMINATIONS 2001
PART III

Solutions 2001
COMMUNICATION SYSTEMS

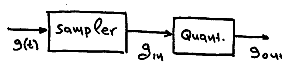
Examiner responsible: Dr A. Manikas

ANSWER to Q1

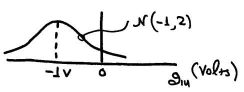
1)	A	B	C	D	E
2)	A	B	C	D	E
3)	A	B	C	D	E
4)	A	B	C	D	E
5)	A	B	C	D	E
6)	A	B	C	D	E
7)	A	B	C	D	E
8)	A	B	C	D	E
9)	A	B	C	D	E
10)	A	B	C	D	E
11)	A	B	C	D	E
12)	A	B	C	D	E
13)	A	B	C	D	E
14)	A	B	C	D	E
15)	A	B	C	D	E
16)	A	B	C	D	E

ANSWER to Q2

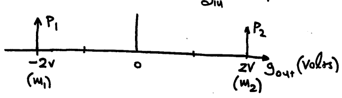
a>



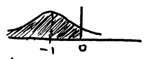
$pdf_{g_{in}} = pdf_{g(t)} =$



$pdf_{g_{out}} =$



$P_1 = \Pr(g_{out} = -2V) = \Pr(g_{in} < 0V) =$



$$= 1 - T\left(\frac{|0-(-1)|}{2}\right) = 1 - \underbrace{T\left(\frac{1}{2}\right)}_{0.3} = 0.7.$$

$P_2 = \Pr(g_{out} = +2V) = 1 - P_1 = 0.3$

b>

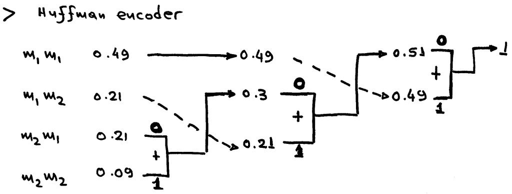
$$rms = \sqrt{P_{g_{out}}} = \sqrt{(-2)^2 P_1 + 2^2 P_2} = 2V$$

$$mean = (-2)\cdot \underset{0.7}{\underset{\uparrow}{P_1}} + 2\underset{0.3}{\underset{\uparrow}{P_2}} = -0.8V$$

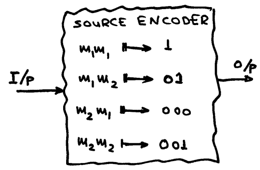
c>

$$(\underline{M}\times\underline{M}, \underline{q}) = \begin{pmatrix} m_1m_1 \rightarrow \Pr(m_1,m_1) = 0.7\times0.7 = 0.49 \\ m_1m_2 \rightarrow \Pr(m_1,m_2) = 0.7\times0.3 = 0.21 \\ m_2m_1 \rightarrow \Pr(m_2,m_1) = 0.3\times0.7 = 0.21 \\ m_2m_2 \rightarrow \Pr(m_2,m_2) = 0.3\times0.3 = 0.09 \end{pmatrix}$$

d> Huffman encoder



∴



e> $\overline{\ell}_2 = 1\times0.49 + 2\times0.21 + 3\times0.21 + 3\times0.09 = 1.81 \frac{\text{bits}}{\text{symbol}}.$

and

$$H(M) = \underbrace{-P_1\log_2 P_1}_{-0.36} - \underbrace{P_2\log_2 P_2}_{-0.521} = 0.8813 \frac{\text{bits}}{\text{level}}.$$

$$H(\underline{M}) \leq \frac{\overline{\ell}_2}{2} \leq H(\underline{M}) + \frac{1}{2} \Rightarrow 0.8813 \leq 0.905 \leq 1.3813$$

i.e. inequality is satisfied.

f> "single level" encoder : average length $= 1 \frac{\text{bit}}{\text{level}}$.
"double level" encoder : 1 symbol = 2 levels
therefore
average length $= 0.905 \frac{\text{bits}}{\text{level}}$
∴ better

g> $F_g = 4\,\text{kHz} \Rightarrow F_s = 8\,\text{kHz} \Rightarrow r_M = 8\text{K}\ \frac{\text{levels}}{\text{sec}}$.

and

$r_{MM} = 4\text{K}\ \frac{\text{double-levels}}{\text{sec}}$

information rates:

* $\underset{\text{(one-level approach)}}{r_{inf}} = \underset{8\text{K}}{r_M} \cdot \underset{0.8813}{H(\underline{M})} = 7.0503\ \frac{\text{kbits}}{\text{sec}}$

* $\underset{\text{(two-level approach)}}{r_{inf}} = \underset{4\text{K}}{r_{MM}} \cdot H(\underline{M}\times\underline{M})$

$H(\underline{M}\times\underline{M}) = 0.49\log_2 0.49 - 2\times 0.21\log_2 0.21 - 0.09\log_2 0.09 = 1.7626\ \frac{\text{bits}}{\underbrace{\text{double-level}}_{\text{symbol}}}$.

$\therefore \underset{\text{(two-level approach)}}{r_{inf}} = 4\text{K} \times 1.7626 = 7.0503\ \frac{\text{kbits}}{\text{sec}}$

ie $\underset{\text{(one-level)}}{r_{inf}} = \underset{\text{(two-level)}}{r_{inf}}$ (as it was expected)

data rates

$\underset{\text{(one-level)}}{r_{data}} = \underset{8\text{K}}{r_M} \cdot \underset{1}{\ell_1} = 8\text{K}\ \frac{\text{bits}}{\text{sec}}$

$\underset{\text{(double-level)}}{r_{data}} = \underset{4\text{K}}{r_{MM}} \cdot \underset{1.81}{\ell_2} = 7.24\text{K}\ \frac{\text{bits}}{\text{sec}}$

ie $\underset{\text{(one-level)}}{r_{data}} > \underset{\text{(double-level)}}{r_{data}}$

ANSWER to Q3

a>

* PCM using a 256-level uniform quantizer:

$SNR_q = 4.77 + 6\gamma - 20\log_{10} CF$ dB $\boxed{1}$

$Q = 256 \Rightarrow 2^\gamma = 256 \Rightarrow \gamma = \log_2 256 = 8\ \frac{\text{bits}}{\text{level}}$

CF = CREST FACTOR of the signal $g(t) \equiv \frac{\text{peak}}{\text{rms}}$

peak $= \hat{g} = 2$ Volts

rms $= \sigma_g = \sqrt{P_g}$ where →

$$P_g = 2\int_0^2 g^2 \cdot \underbrace{pdf_g(g)}_{\frac{1}{2}\Lambda\left(\frac{g}{2}\right)}\, dg = 2\int_0^2 g^2 \frac{1}{2}\,\frac{2-g}{2}\, dg = \cancel{\frac{2}{2}}\int_0^2 \left(g^2 - \frac{1}{2}g^3\right) dg = \left(\frac{g^3}{3} - \frac{1}{2}\frac{g^4}{4}\right)\Big|_0^2 = \frac{8}{3} - \frac{16}{8} = \frac{2}{3}$$

i.e. rms $= \sqrt{\frac{2}{3}}$

$\therefore \boxed{1} \Rightarrow SNR_q = 4.77 + 6\times 8 - \underbrace{20\log_{10}\frac{2}{\sqrt{\frac{2}{3}}}}_{20\log_{10}\sqrt{6} = 7.7815} = 44.9885$ dB

* PCM using an A-law/8-bit with A = 97.6:

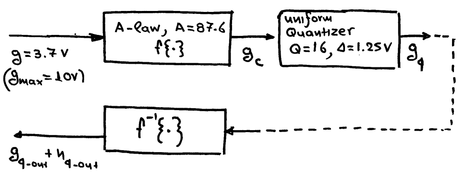
$SNR_q = 4.77 + 6\underset{8\text{bits}}{\gamma} - 20\log_{10}(1 + \ln A) = 38.0059$ dB

* $r_{cs} = r_b = \gamma F_s = 8 \times 2\times 10\text{K} \Rightarrow r_{cs} = 160\text{K}\ \frac{\text{channel symbols}}{\text{sec}}$

$B_{PCM} \geq \frac{r_{cs}}{2} \Rightarrow B_{PCM_{min}} = \frac{r_{cs}}{2} = 80\,\text{kHz}$

* if g(t) changes then $\begin{cases} SNR_{q,unif} = \uparrow \text{or} \downarrow \\ SNR_{q,A\text{-}law} = \text{constant} = 38.0058\ \text{dB} \\ \quad \text{i.e. it is independent of the statistics of the signal} \end{cases}$

b> *



* transmitter's part (top branch):

$\left.\begin{aligned} x &= \frac{g}{g_{max}} = \frac{3.7}{10} = 0.37 \\ \frac{1}{A} &= \frac{1}{87.6} = 0.0114 \end{aligned}\right\} \Rightarrow \frac{1}{A} \leq x < 1$

$\therefore g_c = \frac{1 + \ln(A|x|)}{1 + \ln A} \times \overset{10\text{V}}{g_{max}} = 8.1833\text{V}$

However $\underset{7.5\text{V}}{b_{14}} < g_c < \underset{8.75\text{V}}{b_{15}}$. Therefore $g_q = m_{15} = 8.125\text{V}$

* receiver's part (lower branch):

$g_{q\text{-out}} = \frac{1}{A}\exp\left(\left|\frac{m_{15}}{g_{max}}\right|(1 + \ln A) - 1\right) \times \overset{10\text{V}}{g_{max}} = 3.5839\text{V}$

$|n_{q\text{-out}}| = |g - g_{q\text{out}}| = |3.7 - 3.5839| = 0.1161$

* if the A-law coder is removed then

$\underset{2.5\text{V}}{b_{10}} < g < \underset{3.75\text{V}}{b_{11}} \Rightarrow g_q = m_{11} = 3.125\text{V}$

$\Rightarrow |n_{q\text{-out}}| = |g - g_q| = |3.7 - 3.125| = 0.575\text{V}$

ANSWER to Q4

a> $F_g = 4\times 10^3$ Hz

$F_s = 2\times F_g = 8\times 10^3$ Hz

$\overline{\ell}_3 = 1\times\frac{27}{64} + \left(3\times\frac{9}{64}\right)\times 3 + \left(5\times\frac{3}{64}\right)\times 3 + 5\times\frac{1}{64} = 2.4685\ \frac{\text{bits}}{\text{trip.}}$

$= 2.4685\ \frac{\text{bits}}{\text{triple-level}}$

Alphabet: $\underline{x} = \begin{pmatrix} x_1 = 1 \\ x_2 = 0 \end{pmatrix}$ $\boxed{M = 2 \text{ channel symbols}}$

Probabilities: $\underline{p} = \begin{bmatrix} p_1 = Pr(H_1) = 0.6344 \\ p_2 = Pr(H_0) = 0.3656 \end{bmatrix}$ ← to be proven

$H_x = -\sum_{m=1}^{2} p_m \log_2(p_m) = -\underline{p}^T \cdot \log_2(\underline{p}) = 0.9473\ \frac{\text{bits}}{\text{symbol}}$

$r_{inf} = H_x \cdot r_{cs}$

symbol rate $= r_b = F_s \frac{1}{3}\overline{\ell}_3 = 6583.3\ \frac{\text{symbols}}{\text{sec}}$

i.e. $r_{inf} = 0.9473 \times 6583.3 = 6236.4\ \frac{\text{bits}}{\text{symbol}}$

$r_d = \underset{1\text{bit}}{\ell} \cdot \underset{6583.3}{r_{cs}} = 6583.3\ \frac{\text{bits}}{\text{sec}}$

b> $P_e = 0.6344\times 0.05 + 0.3656\times 0.2 = 0.1048$

c> $1 = x_1 \longmapsto \overset{\sqrt{\frac{3}{8}}}{A_1} \cdot \Lambda\left(\frac{t}{0.5T_{cs}}\right)$ of Energy $= E_1 = ?$

$0 = x_2 \longmapsto 0$ Volts ie Energy $= E_2 = 0$

$E_1 = 2\int_0^{T_{cs}/2} A_1^2 \Lambda^2\left(\frac{t}{0.5T_{cs}}\right) dt$ (where $A_1 = \sqrt{\frac{3}{8}}$)

$= 2\int_0^{0.5T_{cs}} A_1^2 \left(\frac{-t + 0.5T_{cs}}{0.5T_{cs}}\right)^2 dt$

$= \ldots = \frac{1}{3}A_1^2 T_{cs} = \frac{1}{3}\,\frac{3}{8}\,T_{cs} = \frac{1}{8}T_{cs}$

$$E_b = E_1 \cdot P_1 + E_2 P_2 = \frac{1}{8} T_{cs} P_1 + 0 = 1.2046 \times 10^{-5}$$

$$EUE = \frac{E_b}{\underbrace{N_0}_{2\times 10^{-3}}} = 6.0228 \times 10^{-3} \quad \text{(data EUE)}$$

$$BUE = \frac{B}{r_b} = \frac{B}{r_{cs}} = \frac{B}{2B} = \frac{1}{2}$$

$$\boxed{\text{Note: } B = \frac{r_{cs}}{2} \Rightarrow r_{cs} = 2B = r_b}$$

$$\text{data point} = (EUE, BUE) = \left(6.0228\times 10^{-3}, \tfrac{1}{2}\right)$$

$$CS = \text{inf. point} = (EUE_{inf}, BUE_{inf}) = (\text{data point}) \times \frac{\log_2 \overset{2}{M}}{H_{Mut}}$$

therefore the mutual entropy H_{Mut} of the channel should be estimated

ie $H_{Mut} = H_Y - H_{Y|X}$ (or $H_{Mut} = H_X - H_{X|Y}$)

as follows:

* $\underline{p} = \begin{bmatrix} 0.6344 \\ 0.3656 \end{bmatrix}$; $\underline{\underline{F}} = \begin{bmatrix} 0.95, & 0.2 \\ 0.05, & 0.8 \end{bmatrix}$; $\underline{q} = \underline{\underline{F}} \cdot \underline{p} = \begin{bmatrix} 0.6758 \\ 0.3242 \end{bmatrix}$

  $\underline{\underline{J}} = \underline{\underline{F}} \cdot \text{diag}(\underline{p}) = \begin{bmatrix} 0.6027 & , 0.0731 \\ 0.0317 & , 0.2925 \end{bmatrix}$

* $H_Y = -\underline{q}^T \cdot \log_2(\underline{q}) = 0.9089 \frac{\text{bits}}{\text{symbol}}$

* $H_{Y|X} = -\sum_{m=1}^{2}\sum_{k=1}^{2} J_{km} \log_2\left(\frac{J_{km}}{p_m}\right) = -\|\underline{\underline{J}} \odot \log_2(\underline{\underline{F}})\|_{1*}$

  $= 0.4456 \frac{\text{bits}}{\text{symbol}}$

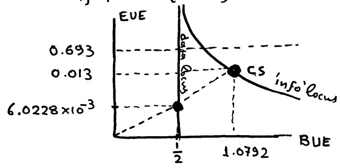
* $H_{Mut} = H_Y - H_{Y|X} = 0.4633 \frac{\text{bits}}{\text{symbol}}$

Note: a different approach is to use the following expression

$$H_{Mut} = -\left\|\underline{\underline{J}} \odot \log_2\left(\oslash \frac{\underline{\underline{F}} \cdot \underline{p} \cdot \underline{p}^T}{\underline{\underline{J}}}\right)\right\|_{1*} \frac{\text{bits}}{\text{symbol}} = 0.4633$$

where $\|\text{matrix}\|_{1*}$ = sum of the elements of the matrix-argument.

d> $\therefore$ CS = inf. point = (0.013, 1.0792)



e> $\therefore$ CS = inf. point = (0.013, 1.0792) $\Rightarrow$

CS is not realizable (since $EUE_{inf} = 0.013 < \underbrace{0.693}_{\text{theoretical limit}}$)

f> $$SNR_{in} = \frac{EUE_{inf}}{BUE_{inf}} = \frac{EUE_d}{BUE_d} = 0.012 \Rightarrow SNR_{in} = -19.2082\,\text{dB}$$