Imperial College London

BSc/MSci EXAMINATION May 2012

This paper is also taken for the relevant Examination for the Associateship

DYNAMICAL SYSTEMS AND CHAOS

For 3rd and 4th Year Physics Students

Wednesday, 30th May 2012: 14:00 to 16:00

Answer three questions
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the THREE answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

1. (i) Weed with population density $x(t)$ and sea urchins with population density $y(t)$ live in a pool at the edge of the open sea.

The dynamics of this closed ecosystem are governed by the equations

$$\frac{dx}{dt} = x(1 - \tfrac{1}{2}y - 2x)$$

$$\frac{dy}{dt} = y(2x - \tfrac{1}{2} - \tfrac{1}{2}y)$$

Show that there are four critical points, of which only one corresponds to coexistence of the two species. [4 marks]

(ii) By considering the eigenvalue/eigenvector problems for the local linear approximations near the critical points, classify the local behaviours and use them to sketch the full phase plane portrait for the system in all quadrants.

[12 marks]

(iii) State carefully what happens as time $t \to \infty$ for the following possible initial states:

(a) $x(0) = 0, \quad y(0) = 10.$

(b) $x(0) = 1, \quad y(0) = 0.$

(c) $x(0) = 5, \quad y(0) = 10.$

[4 marks]

[Total 20 marks]

2. (i) In a simple model of the dynamics of malaria (due to Ross and Macdonald) the equations are:

$$\begin{aligned}\frac{dx}{dt} &= \frac{abM}{N}\,y(1-x) - rx \\ \frac{dy}{dt} &= ax(1-y) - \mu y,\end{aligned}$$

where $x(t), y(t)$ are respectively the proportions of the human population, N, and the female mosquito population, M, which are infected and a, b, r, μ are positive constants characterising biting, infection, recovery and mortality rates.

(a) Show that the critical point at the origin (0,0) is asymptotically stable when $R < 1$, but unstable when $R > 1$, where

$$R = \frac{Ma^2b}{N\mu r}.$$

[7 marks]

(b) Explain *briefly* and illustrate why the disease can sustain itself within these populations if and only if $R > 1$. [3 marks]

(ii) Malkus and Howard ($\sim$ 1970) considered a Lorenz waterwheel with the equations:

$$\begin{aligned}\frac{dx}{dt} &= \sigma(y-x) \\ \frac{dy}{dt} &= \rho x - y - xz \\ \frac{dz}{dt} &= -z + xy\end{aligned}$$

where σ, ρ are positive constants.

(a) Show that the stability of the critical point, P_1, at the origin, then depends on eigenvalues $\lambda_1 = -1$, and λ_2, λ_3 satisfying

$$\lambda^2 + (\sigma+1)\lambda + \sigma(1-\rho) = 0.$$

[6 marks]

(b) Show that the eigenvectors corresponding to λ_2, λ_3 in part (ii)(a) are orthogonal if $\sigma = \rho$.

[4 marks]

[Total 20 marks]

3. (i) For the two-dimensional map:

$$\begin{aligned} x_{n+1} &= F(x_n, y_n) \\ y_{n+1} &= G(x_n, y_n) \end{aligned}$$

show that the asymptotic stability of a fixed point (X, Y) requires the eigenvalues λ of the matrix

$$M = \begin{pmatrix} \frac{\partial F}{\partial X} & \frac{\partial F}{\partial Y} \\ \frac{\partial G}{\partial X} & \frac{\partial G}{\partial Y} \end{pmatrix}$$

to be such that $|\lambda| < 1$. [5 marks]

(ii) State *briefly* what is meant by:

(a) a limit cycle [3 marks]

(b) fractal dimension [3 marks]

and for an integrable dynamical system of two degrees of freedom:

(c) an invariant two-torus [3 marks]

(d) orbital closure [3 marks]

(e) torus breakdown under non-integrable perturbation leading to chaos. [3 marks]

[Total 20 marks]

4. The discrete variable x_n satisfies the quadratic map

$$x_{n+1} = x_n^2 + a,$$

where a is a real parameter.

(i) Show that:

(a) there is an asymptotically stable fixed point attractor only when $-\frac{3}{4} < a < \frac{1}{4}$, [5 marks]

(b) there is an asymptotically stable two-cycle attractor only when $a_0 < a < -\frac{3}{4}$ where a_0 is to be found. [7 marks]

(ii) (a) show that the linear transformation

$$x_n = Ay_n + \frac{1}{2}B$$

leads to a general quadratic map for y_n

$$y_{n+1} = Ay_n^2 + By_n + f(a, A, B).$$

Hence find a as a function of r for the logistic map $y_{n+1} = ry_n(1 - y_n)$. [4 marks]

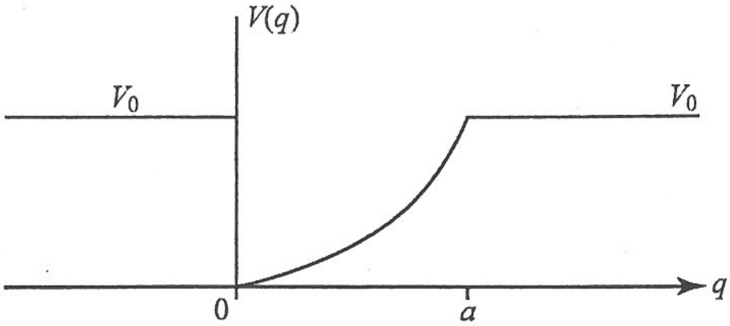
(b) Comment briefly on the consequences for the logistic map of the results of (i)(a) and (i)(b) above. [4 marks]

[Total 20 marks]

5. A particle of mass m moves along the q axis in the potential

$$V(q) = \left| \begin{array}{lll} V_0 & \text{for} & q < 0 \\ V_0 \dfrac{q^2}{a^2} & \text{for} & 0 \leqslant q \leqslant a \\ V_0 & \text{for} & q > a \end{array} \right.$$

where V_0, a are positive constants.



(i) Use the Hamiltonian to sketch the phase plane portrait for this system. Explain briefly why the flow of this system preserves area in the phase plane. [6 marks]

(ii) find the action I for the bounded oscillations in terms of energy E. [5 marks]

(iii) Show that the period τ of the oscillations is given by

$$\tau = \pi a \left(\frac{m}{2V_0} \right)^{1/2} .$$

[4 marks]

(iv) If the potential well becomes narrower, but with the same depth, on a time scale very long compared with τ, find how the energy and period of oscillation of the particle vary with a. Show that the particle must then eventually leave the well. [5 marks]

[Total 20 marks]