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MODEL ANSWER and MARKING SCHEME

First Examiner	J. BARRIA	Paper Code	EE4.05+EE9-S07
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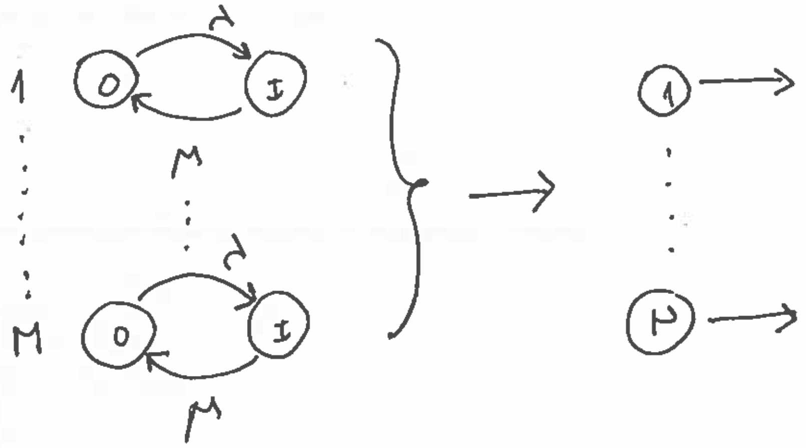
1a)



for $N \geq M$

In equilibrium one source is busy with probability:

$$P(\text{source busy}) = \frac{\lambda}{\lambda+\mu} \quad \text{(derive)}$$ 4

Hence we have M non-interacting sources each with busy probability p.

Then the number of busy sources is binomial (M, p)

$$\pi_i = \frac{\binom{M}{i} p^i (1-p)^{M-i}}{\sum_{j=0}^{M} \binom{M}{j} p^j (1-p)^{M-j}} \qquad i = 0, 1, \ldots, M$$

$$\sum_{j=0}^{M} \binom{M}{j} p^j (1-p)^{M-j} = 1$$ 6

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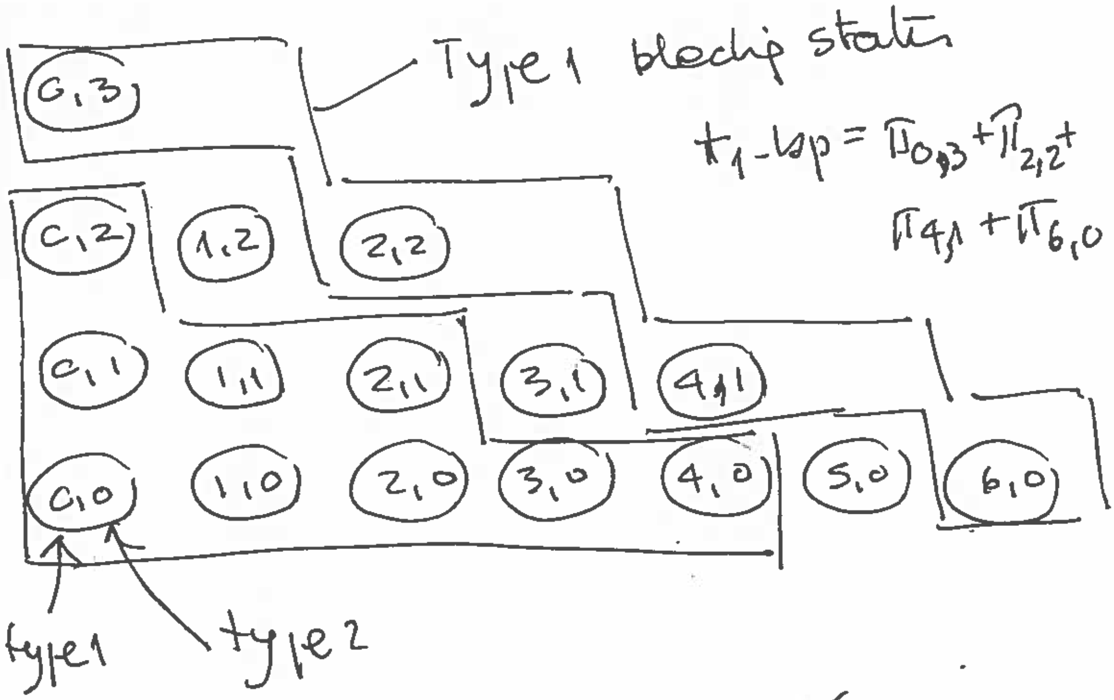
1b) State space (t_1, t_2) t_i = type i cells — 3

From the note the capacity of the system is 6 type 1 cells or 3 type 2 cells. — 1



$t_1\text{-bp} = \pi_{0,3} + \pi_{2,2} + \pi_{4,1} + \pi_{6,0}$ — 3

$$t_2\text{-bp} = 1 - \pi_{0,2} - \sum_{0}^{2} \pi_{i,1} - \sum_{0}^{4} \pi_{i,0}$$ — 3

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2a) 27000 message / minute = 450 messages / second

Uniform distribution

1 $= a$ 100 $= b = X_{max}$

$$E[x] = \frac{a+b}{2}$$

$$Var[x] = \frac{1}{12}(b-a)^2 = E[x^2] - [E(x)]^2$$

$$\frac{1}{12}(b-a)^2 = E[x^2] - \frac{1}{4}(a+b)^2$$

$$\therefore E[x^2] = \frac{4}{12}(a^2+b^2) - \frac{8}{12}ab$$

as $a \to 0$ $$E[x^2] = \frac{4}{12}b^2 = \frac{1}{3}X_{max}^2$$

$$\text{mean message time} = \frac{\lambda E[S^2]}{2(1-\rho)}$$

λ = 450 messages / second

$$E[S] = \frac{80}{2.048} \times 50.5\,\mu s = 1.97\,\mu s$$ 3

$\rho = \lambda E[S] = 0.88$

$E[S^2] \sim \frac{1}{3}[S_{max}^2]$

$S_{max} \sim 100$ packets × 80 bits

~ 8000 bits

$E[S^2] \sim \frac{1}{3}\left(\frac{8000}{2.048}\right)^2 \sim 5.08 \times 10^{-6}\ s^2$ — 3

mean waiting time $= \frac{\lambda E[S^2]}{2(1-\rho)} \sim 10.2$ ms — 1

For 1-packet message (top priority)

mean waiting time $= E[R]$ — 1

For 100-packet message (bottom priority)

mean waiting time $= \frac{E[R]}{(1-\rho)^2}$ — 2

2b. i

let N_t = no of items in the system at time t

Q_t = queue length at time t.

Then

$$P[Q_t = i \mid \text{Delay}] = P[Q_t = i \mid N_t \geq K]$$

$$= \frac{P[N_t = K+i]}{P[N_t \geq K]} = \frac{\pi_K \rho^i}{\sum_{j=0}^{\infty} \pi_K \rho^j}$$ 2

From the equilibrium equations

$$\pi_{K+i} = \pi_K \rho^i \qquad \rho = \frac{\text{offered traffic}}{\text{channel}}.$$

$$\sum_{j=0}^{\infty} \pi_{K+j} = \pi_K \sum_j \rho^j = \pi_K (1-\rho)^{-1}$$

$$\Rightarrow P[Q_t = i \mid \text{Delay}] = (1-\rho)\rho^i \qquad i \geq 0$$

$$\left(\frac{\pi_K \rho^i}{\sum_{j=0}^{\infty} \pi_K \rho^j} = \frac{\cancel{\pi_K} \rho^i}{\cancel{\pi_K} \sum_{j=0}^{\infty} \rho^j} = (1-\rho)\rho^i \right)$$ 2

2(b)(iii) For delayed arrivals:

$$P[W > z \mid Q_t = i] = P[<(i+1) \text{ departure in } (0,z)]$$

$$= \sum_{j=0}^{i} \frac{(K\mu z)^j}{j!} e^{-K\mu z}$$ 2

in equilibrium $P(Q_t = i) = (1-\rho)\rho^i$

Therefore for delayed arrivals

$$P(W > z) =$$

$$= \sum_{i=0}^{\infty} P(Q_t = i) P(W > z \mid Q_t = i)$$

$$= \sum_{i=0}^{\infty} (1-\rho)\rho^i \sum_{j=0}^{i} \frac{(K\mu z)^j}{j!} e^{-K\mu z}$$ 2

using $\lambda = K\mu\rho$

$$P(W \le z \mid W > 0) = 1 - e^{-K\mu(1-\rho)z} \qquad z \geq 0$$ 2

3a System M|M|K|K+Q $K = Q = 2$

$\lambda = 40\ s^{-1}$ $\mu = 40\ s^{-1}$

$\rho = \frac{\lambda}{K\mu} = 0.5$ (offered traffic | channel)

Mean queue length $E(Q_t) = P(\text{Delay})\, E(Q_t | \text{Delay})$

$P[\text{Loss}] = 0.043 = \pi_K \rho^2$

$\pi_K = 0.174$

$P[\text{Delay}] = \pi_K \left(\frac{1-\rho^2}{1-\rho} \right) = \pi_K (1+\rho)$

$P[\text{Delay}] = 0.261$

$E(Q_t) = 0.261\, \underbrace{E(Q_t | \text{Delay})}$

$\frac{\rho}{1-\rho} - \frac{2\rho^2}{1-\rho^2} \ldots = \frac{\rho}{1+\rho} = 0.333$

$E(Q_t) = 0.087$ 4

3a ii)

$$E[W] = \frac{1}{\lambda_A} E[Q_t]$$

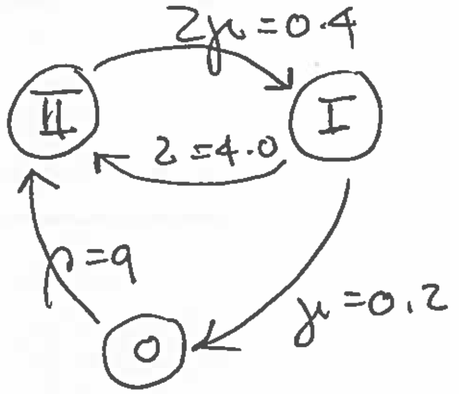
$$\lambda_A = \lambda [1 - P(\text{loss})]$$ (arrival rates for accepted messages)

$$= 38.3 \ 1/s$$

$$E[W] = \frac{1}{38.3} * 0.087 = 0.002271$$

$$= 2.27 \ ms$$ 5

3b



$$Q = \begin{bmatrix} -0.4 & 0.4 & \phi \\ 4.0 & -4.2 & 0.2 \\ 9.0 & 0 & -9.0 \end{bmatrix}$$ 3

solve

$$\begin{bmatrix} -0.4 & 4.0 & 9.0 \\ 0.4 & -4.2 & 0 \\ 0 & 0.2 & -9.0 \end{bmatrix} \begin{bmatrix} X_{II} \\ X_I \\ X_0 \end{bmatrix} = \begin{bmatrix} \phi \end{bmatrix}$$

$$X_{II} + X_I + X_0 = 1$$

<u>1st + 3rd</u>

$$-0.4 X_{II} + 4.2 X_I = 0$$

$$X_{II} = \frac{4.2}{0.4} X_I = 10.5 X_I$$

$$X_0 = 1 - X_I - 10.5 X_I$$

$$= 1 - 11.5 X_I$$

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$$-0.4X_{II} + 4.0X_I + 9.0X_0 = \phi$$

$$-0.4(10.5X_I) + 4X_I + 9(1 - 11.5X_I) = \phi$$

$$-4.2X_I + 4X_I + 9 - 103.5X_I = \phi$$

$$-0.2X_I + 9 - 103.5X_I = \phi$$

$$9 = 103.7X_I$$

$$\boxed{X_I = 0.08679}$$

$$X_{II} = 10.5X_I$$

$$\boxed{X_{II} = 0.911295}$$

$$X_{II} + X_I = 0.998085$$ 4

Average rate at which the system completes the work

$$E[Z(t)] = \sum r_i \pi_i$$

$$= 16 \times 0.911295 + 10 \times 0.08679$$

$$= 15.44862$$ 4

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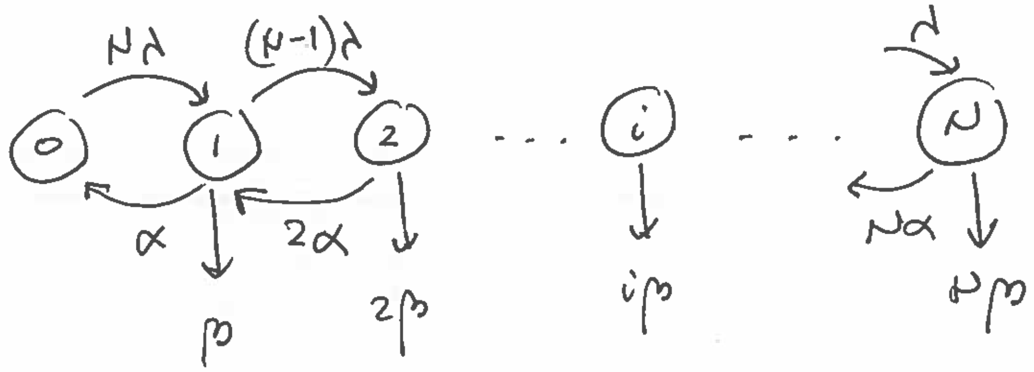
4a)



4

each source offers β cells/s

Average arrival rate when N active sources: $N\beta \frac{\lambda}{\alpha+\lambda}$ cells/s (slow steps) 2

The system is stable if:

$$N\beta \frac{\lambda}{\alpha+\lambda} < \nu$$

(mean arrival rate < mean service rate) 4

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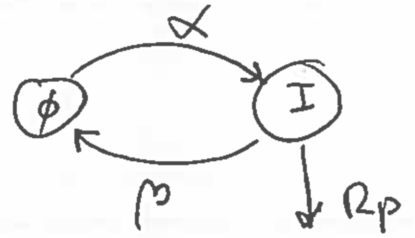
4b)



equivalent ON-OFF traffic source representation

probability one source is on $p = \frac{\alpha}{\alpha+\beta}$

Average rate of transmission for N sources $N \frac{\alpha}{\alpha+\beta} R_p = m R_p$ 2

where R_p is the peak rate in cells/s.

The variance σ^2 of a binomial distribution with parameter p is: $N p(1-p) = \sigma^2$ 4 derive

Define equivalent capacity

$$C_L = (m + k\sigma) R_p$$

$$C = \frac{C_L}{R_p} = m + k\sigma = Np + k\sqrt{Np(1-p)}$$ 4