EE - stream

(E2.8)

UNIVERSITY OF LONDON

[II(3)E 2002]

B.ENG. AND M.ENG. EXAMINATIONS 2002

For Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examination for the Associateship.

PART II : MATHEMATICS 3 (ELECTRICAL ENGINEERING)

Wednesday 29th May 2002 2.00 - 5.00 pm

Answer EIGHT questions.
Answers to Section A *questions must be written in a different answer book from answers to* Section B *questions.*

[Before starting, please make sure that the paper is complete; there should be 7 pages, with a total of 12 questions. Ask the invigilator for a replacement if your copy is faulty.]

1. Show that the function

$$u(x,y) \;=\; \sin x \cosh y \;+\; 2\cos x \sinh y$$

satisfies Laplace's equation.

By integrating the Cauchy-Riemann equations directly, find the conjugate function $v(x,y)$ and hence show that $w = u + iv$ can be expressed as

$$w \;=\; (1-2i)\sin z \;+\; ic$$

where $z = x + iy$ and c is an arbitrary real constant.

2. (i) Consider a complex function $f(z)$ which can be written in the form

$$f(z) \;=\; \frac{1}{g(z)}\,,$$

where $g(z)$ has a simple zero at $z = a$.

Show that the residue of $f(z)$ at $z = a$ is $1/g'(a)$.

If, instead, $g(z)$ has a double zero at $z = a$, show that the residue at $z = a$ is

$$-\frac{2}{3}\,\frac{g'''(a)}{[g''(a)]^2}.$$

(ii) If

$$f(z) \;=\; \frac{1}{z^4+1}\,,$$

show that of the four simple poles lying on the unit circle, two are located in the upper half-plane at

$$z = e^{i\pi/4} \text{ and } z = e^{3i\pi/4}.$$

By considering a semi-circular contour in the upper half-plane, show that

$$\int_{-\infty}^{\infty} \frac{dx}{x^4+1} \;=\; \frac{\pi}{\sqrt{2}}.$$

Recall that the residue of a complex function $f(z)$ at a pole $z = a$ of multiplicity m is given by the expression

$$\lim_{z\to a} \frac{1}{(m-1)!}\left[\frac{d^{m-1}}{dz^{m-1}}\left\{(z-a)^m f(z)\right\}\right].$$

PLEASE TURN OVER

3. Consider the contour integral

$$\oint_C \frac{e^{iz}\,dz}{z(z^2+1)}$$

where the contour C is a semi-circle in the upper half of the complex plane, with an additional small semi-circular deformation below the pole at $z = 0$.

Use the Residue Theorem to show that

$$\int_0^\infty \frac{\sin x\,dx}{x(x^2+1)} = \frac{\pi(e-1)}{2e}\,.$$

4. The tent function $\Lambda(t)$, the sinc-function $\operatorname{sinc} t$, and the square wave $\Pi(t)$, are defined respectively by

$$\Lambda(t) = \begin{cases} 1+t, & -1 \le t \le 0, \\ 1-t, & 0 \le t \le 1, \\ 0, & \text{otherwise,} \end{cases}$$

$$\operatorname{sinc} t = \frac{\sin(t/2)}{(t/2)},$$

and

$$\Pi(t) = \begin{cases} 1, & -1/2 \le t \le 1/2, \\ 0, & \text{otherwise.} \end{cases}$$

Show that the Fourier transform of $\Pi(t)$ is given by

$$\overline{\Pi}(\omega) = \operatorname{sinc}\omega$$

and the Fourier transform of $\Lambda(t)$ is given by

$$\overline{\Lambda}(\omega) = \operatorname{sinc}^2\omega.$$

5. (i) The Laplace transform $\mathcal{L}\{y(t)\}$ of a function $y(t)$ is denoted as

$$\overline{y}(s) = \mathcal{L}\{y(t)\} = \int_0^\infty y(t)e^{-st}\,dt\,.$$

Show that, for $Re(s) > 0$,

$$\begin{aligned}\mathcal{L}\{\dot{y}(t)\} &= s\overline{y}(s) - y(0),\\ \mathcal{L}\{\ddot{y}(t)\} &= s^2\overline{y}(s) - sy(0) - \dot{y}(0),\end{aligned}$$

provided $y(t)$ and $\dot{y}(t)$ vanish sufficiently fast as $t \to \infty$.

(ii) A function $y(t)$ satisfies the differential equation

$$\ddot{y} + 3\dot{y} + 2y = f(t),$$

subject to the initial conditions $\dot{y}(0) = 0$ and $y(0) = \alpha$, a constant. $f(t)$ is a given function that is not specified here. Using a Laplace transform and the Laplace Convolution Theorem, obtain the solution of this differential equation in the form

$$y(t) = \alpha\left(2e^{-t} - e^{-2t}\right) + \int_0^t \left\{e^{-(t-u)} - e^{-2(t-u)}\right\} f(u)\,du.$$

6. If a function $f(t)$ is periodic in time t with fixed period T such that $f(t) = f(t-T)$ with $T > 0$, show that for $s > 0$ its Laplace transform $\overline{f}(s)$ is given by

$$\overline{f}(s) = \frac{1}{1-\exp(-sT)}\int_0^T f(t)\exp(-st)\,dt.$$

If $f(t)$ is the periodic square wave of period T

$$f(t) = \begin{cases} 1, & 0 \le t \le T/2, \\ 0, & T/2 < t \le T, \end{cases}$$

show that its Laplace transform $\overline{f}(s)$ is given by

$$\overline{f}(s) = \frac{1}{s}\left(\frac{1-\exp(-sT/2)}{1-\exp(-sT)}\right).$$

Explain what happens in the limit $T \to \infty$.

PLEASE TURN OVER

7. The double integral I_1 is given by

$$I_1 = \int\int_A e^{-\alpha(x^2+y^2)} dxdy ,$$

where A is the finite region enclosed by the curve $x^2 + y^2 = R^2$.

Sketch the region of integration A and, upon using the substitution $x = r\cos\theta$, $y = r\sin\theta$ (polar coordinates), show that

$$I_1 = \frac{\pi}{\alpha}(1 - e^{-\alpha R^2}).$$

Calculate $\lim_{R\to\infty} I_1$ and hence deduce the value of the integral

$$I_2 = \int_{-\infty}^{+\infty} e^{-\alpha x^2} dx .$$

Hint: relate I_2^2 to I_1.

8. (i) Show that

$$\operatorname{div}(\operatorname{curl}\mathbf{A}) = 0, \quad \operatorname{curl}(\nabla\phi) = \mathbf{0}$$

for a general three-dimensional vector field $\mathbf{A}$ and scalar field ϕ.

(ii) Suppose that $\mathbf{A}$ satisfies the relation

$$\operatorname{curl}\mathbf{A} = \lambda\mathbf{A}$$

for some scalar λ.

Prove the following results:

(a) $\operatorname{div}\mathbf{A} = 0$;

(b) $\nabla^2\mathbf{A} + \lambda^2\mathbf{A} = \mathbf{0}$;

(c) $\lambda = \dfrac{\operatorname{curl}\mathbf{A}\cdot\operatorname{curl}(\operatorname{curl}\mathbf{A})}{(\operatorname{curl}\mathbf{A})\cdot(\operatorname{curl}\mathbf{A})}$.

You may assume the vector identity: $\operatorname{curl}(\operatorname{curl}\mathbf{A}) = \nabla(\operatorname{div}\mathbf{A}) - \nabla^2\mathbf{A}$.

9. (i) Show that

$$\int_C \left\{(x^2 + 2xy + 3y^2)dx + (x^2 + 6xy + 2y^2)dy\right\}$$

is independent of the path C, joining the initial point to the final point.

Evaluate the integral for a path C from $(0, 0)$ to $(1, 2)$.

(ii) Let C be a circle in the $x - y$ plane with centre at the origin and radius 1, described in the counter-clockwise direction. Let R be the region inside the circle.

Evaluate

$$\oint_C \{(2x - y)dx + (x - 2y)dy\}$$

(a) directly, and

(b) using Green's Theorem.

Green's Theorem in the plane states that:

$$\oint_C (Pdx + Qdy) = \int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy\,.$$

10. Consider a two-dimensional region R bounded by a closed piecewise smooth curve C. Using Green's Theorem in a plane (see below), choose the components of a vector field $\mathbf{v}(x, y)$ in terms of $P(x, y)$ and $Q(x, y)$ to prove the two-dimensional form of Stokes's Theorem

$$\int\int_R \mathbf{k}.(\mathrm{curl}\,\mathbf{v})\,dxdy = \oint_C \mathbf{v}.d\mathbf{r} \tag{1}$$

where $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$.

If $\mathbf{v} = y^2\mathbf{i} + x^2\mathbf{j}$, and R is the finite region bounded by the hyperbola $y = \frac{1}{4x}$, and the lines $x = 1$ and $y = x$, sketch the region R in the $x - y$ plane and, by evaluating the line integral on the right hand side of (1), or otherwise, show that

$$\int\int_R \mathbf{k}.(\mathrm{curl}\,\mathbf{v})\,dxdy = \frac{5}{48}.$$

Green's Theorem in a plane states that for a two-dimensional region R bounded by a closed, piecewise smooth curve C, then

$$\oint_C \{P(x,y)dx + Q(x,y)dy\} = \int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy.$$

PLEASE TURN OVER

11. (i) Events E_1 and E_2 are exclusive and E_2 and E_3 are independent. Draw a Venn diagram to illustrate the most general structure for these events. Suppose that $\text{pr}(E_1) = p_1$, $\text{pr}(E_2) = p_2$, $\text{pr}(E_3) = p_3$ and $\text{pr}(E_1 \mid E_3) = p_{13}$. Express $\text{pr}(E_2 \mid E_3)$, $\text{pr}(E_1 \cap E_3)$, $\text{pr}(E_3 \mid E_1)$ and $\text{pr}(E_1 \mid E_2)$ in terms of p_1, p_2, p_3 and p_{13}.

(ii) The insulating material in a transformer is subject to three types of deterioration, D_1, D_2 and D_3. From extensive historical data, the probabilities are well estimated as $\text{pr}(D_1) = 0.1$, $\text{pr}(D_2) = 0.05$, $\text{pr}(D_3) = 0.01$, and $\text{pr}(D_1 \cup D_2) = 0.12$. It is also known that D_1 and D_3 occur independently, and that D_2 and D_3 are exclusive. Compute the probabilities of the following states of deterioration:

(a) D_1 and D_3 both present;
(b) D_1 and D_2 both present;
(c) D_1, D_2 and D_3 all present;
(d) D_2 present, given that D_1 is present;
(e) D_1 present, given that D_3 is present.

12. (i) The discrete random variables X_1 and X_2 have joint probability function $p(x_1, x_2)$ and marginal probability functions $p_1(x_1)$ and $p_2(x_2)$, respectively. Write down an expression that defines the expected value $E(X_1)$ and prove that $E(X_1 + X_2) = E(X_1) + E(X_2)$.

(ii) The random variables X_1 and X_2 are independent with distributions $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$, respectively.
What are the distributions of $X_1 + X_2$ and $2X_1 - 3X_2$?

(iii) The random variable X_3 has mean μ_3 and variance σ_3^2, X_4 has mean μ_4 and variance σ_4^2 and X_3 and X_4 are correlated with correlation coefficient ρ. Calculate the mean and variance of $2X_3 - 3X_4$.

(iv) The energy requirements of two systems, which have to satisfy varying loads, are random variables with distributions well-approximated by $N(1.41, 0.11)$ and $N(1.11, 0.07)$. Because of other demands, which themselves vary randomly, the available supply is distributed as $N(3.66, 0.18)$. The two system requirements and the supply are independent random variables.
What is the probability that the supply can meet the demand from the two systems?

For part (iv) you may use the following table of values of the standard normal distribution function.

x	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$\Phi(x)$	0.5000	0.5398	0.5793	0.6179	0.6554	0.6915	0.7257	0.7580	0.7881	0.8159
x	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
$\Phi(x)$	0.8413	0.8643	0.8849	0.9032	0.9192	0.9332	0.9452	0.9554	0.9641	0.9713
x	2.0	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9
$\Phi(x)$	0.9772	0.9821	0.9861	0.9893	0.9918	0.9938	0.9953	0.9965	0.9974	0.9981

END OF PAPER

2nd yr data-sheet

MATHEMATICS DEPARTMENT

MATHEMATICAL FORMULAE

1. VECTOR ALGEBRA

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1, a_2, a_3)$$

Scalar (dot) product: $\mathbf{a}\,.\,\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

Vector (cross) product:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Scalar triple product:

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a}\,.\,\mathbf{b}\times\mathbf{c} = \mathbf{b}\,.\,\mathbf{c}\times\mathbf{a} = \mathbf{c}\,.\,\mathbf{a}\times\mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product: $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c}\,.\,\mathbf{a})\mathbf{b} - (\mathbf{b}\,.\,\mathbf{a})\mathbf{c}$

2. SERIES

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \ldots \quad (\alpha \text{ arbitrary, } |x| < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \ldots,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^n \frac{x^{n+1}}{(n+1)} + \ldots (-1 < x \le 1)$$

Data sheet

3. TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin(a+b) = \sin a \cos b + \cos a \sin b\,;$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b\,.$$

$$\cos iz = \cosh z\,; \quad \cosh iz = \cos z\,; \quad \sin iz = i \sinh z\,; \quad \sinh iz = i \sin z\,.$$

4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f\,D^n g + \binom{n}{1} Df\,D^{n-1}g + \ldots + \binom{n}{r} D^r f\,D^{n-r}g + \ldots + D^n f\,g\,.$$

(b) Taylor's expansion of $f(x)$ about $x = a$:

$$f(a+h) = f(a) + hf'(a) + h^2 f''(a)/2! + \ldots + h^n f^{(n)}(a)/n! + \epsilon_n(h),$$

where $\epsilon_n(h) = h^{n+1} f^{(n+1)}(a + \theta h)/(n+1)!, \quad 0 < \theta < 1\,.$

(c) Taylor's expansion of $f(x, y)$ about (a, b):

$$f(a+h, b+k) = f(a,b) + [hf_x + kf_y]_{a,b} + 1/2!\left[h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}\right]_{a,b} + \ldots$$

(d) Partial differentiation of $f(x, y)$:

i. If $y = y(x)$, then $f = F(x)$, and $\dfrac{dF}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dx}$.

ii. If $x = x(t)$, $y = y(t)$, then $f = F(t)$, and $\dfrac{dF}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$.

iii. If $x = x(u, v)$, $y = y(u, v)$, then $f = F(u, v)$, and

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}, \quad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$

(e) Stationary points of $f(x, y)$ occur where $f_x = 0$, $f_y = 0$ simultaneously.
Let (a, b) be a stationary point: examine $D = [f_{xx}f_{yy} - (f_{xy})^2]_{a,b}$.
If $D > 0$ and $f_{xx}(a, b) < 0$, then (a, b) is a maximum;
If $D > 0$ and $f_{xx}(a, b) > 0$, then (a, b) is a minimum;
If $D < 0$ then (a, b) is a saddle-point.

(f) Differential equations:

i. The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating factor $I(x) = \exp[\int P(x)(dx]$, so that $\dfrac{d}{dx}(Iy) = IQ$.

ii. $P(x, y)dx + Q(x, y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y$.

5. INTEGRAL CALCULUS

(a) An important substitution: $\tan(\theta/2) = t$:
$\sin\theta = 2t/(1+t^2)$, $\cos\theta = (1-t^2)/(1+t^2)$, $d\theta = 2\,dt/(1+t^2)$.

(b) Some indefinite integrals:

$$\int (a^2 - x^2)^{-1/2} dx = \sin^{-1}\left(\frac{x}{a}\right), \quad |x| < a.$$

$$\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1}\left(\frac{x}{a}\right) = \ln\left\{\frac{x}{a} + \left(1 + \frac{x^2}{a^2}\right)^{1/2}\right\}.$$

$$\int (x^2 - a^2)^{-1/2} dx = \cosh^{-1}\left(\frac{x}{a}\right) = \ln\left|\frac{x}{a} + \left(\frac{x^2}{a^2} - 1\right)^{1/2}\right|.$$

$$\int (a^2 + x^2)^{-1} dx = \left(\frac{1}{a}\right)\tan^{-1}\left(\frac{x}{a}\right).$$

6. NUMERICAL METHODS

(a) Approximate solution of an algebraic equation:

If a root of $f(x) = 0$ occurs near $x = a$, take $x_0 = a$ and
$x_{n+1} = x_n - [f(x_n)/f'(x_n)]$, $n = 0, 1, 2 \ldots$

(Newton Raphson method).

(b) Formulae for numerical integration: Write $x_n = x_0 + nh$, $y_n = y(x_n)$.

i. Trapezium rule (1-strip): $\int_{x_0}^{x_1} y(x)dx \approx (h/2)[y_0 + y_1]$.

ii. Simpson's rule (2-strip): $\int_{x_0}^{x_2} y(x)dx \approx (h/3)[y_0 + 4y_1 + y_2]$.

(c) Richardson's extrapolation method: Let $I = \int_a^b f(x)dx$ and let I_1, I_2 be two estimates of I obtained by using Simpson's rule with intervals h and $h/2$.

Then, provided h is small enough,

$$I_2 + (I_2 - I_1)/15,$$

is a better estimate of I.

7. LAPLACE TRANSFORMS

Function	Transform	Function	Transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)dt$	$af(t) + bg(t)$	$aF(s) + bG(s)$
df/dt	$sF(s) - f(0)$	d^2f/dt^2	$s^2F(s) - sf(0) - f'(0)$
$e^{at}f(t)$	$F(s-a)$	$tf(t)$	$-dF(s)/ds$
$(\partial/\partial\alpha)f(t,\alpha)$	$(\partial/\partial\alpha)F(s,\alpha)$	$\int_0^t f(t)dt$	$F(s)/s$
$\int_0^t f(u)g(t-u)du$	$F(s)G(s)$		
1	$1/s$	$t^n (n = 1, 2 \ldots)$	$n!/s^{n+1}$, $(s > 0)$
e^{at}	$1/(s-a)$, $(s > a)$	$\sin\omega t$	$\omega/(s^2+\omega^2)$, $(s > 0)$
$\cos\omega t$	$s/(s^2+\omega^2)$, $(s > 0)$	$H(t-T) = \begin{cases} 0, & t < T \\ 1, & t > T \end{cases}$	e^{-sT}/s, $(s, T > 0)$

8. FOURIER SERIES

If $f(x)$ is periodic of period $2L$, then $f(x + 2L) = f(x)$, and

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \text{ where}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad n = 0, 1, 2, \ldots, \text{ and}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1, 2, 3, \ldots$$

Parseval's theorem

$$\frac{1}{L}\int_{-L}^{L} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right).$$

September 2000

SOLUTIONS - PART 2 - MATHS 3 - 2002

MATHEMATICS FOR ENGINEERING STUDENTS

(EE Stream)

EXAMINATION QUESTION / SOLUTION

SESSION : 2001-2002

E 1

PAPER 3

QUESTION

SOLUTION 15

Please write on this side only, legibly and neatly, between the margins

$u = \sin x \cosh y + 2\cos x \sinh y$

$u_x = \cos x \cosh y - 2\sin x \sinh y$, $u_y = \sin x \sinh y + 2\cos x \cosh y$

$u_{xx} = -\sin x \cosh y - 2\cos x \sinh y$; $u_{yy} = \sin x \cosh y + 2\cos x \sinh y$

$\therefore \underline{u_{xx} + u_{yy} = 0}$ 5

Hence $\exists$ a conjugate function v such that (CR equations)

$v_y = u_x = \cos x \cosh y - 2\sin x \sinh y$ 1)

$v_x = -u_y = -\sin x \sinh y - 2\cos x \cosh y$ 2)

Integrate : 1) $\rightarrow$ $v = \cos x \sinh y - 2\sin x \cosh y + A(x)$

2) $\rightarrow$ $v = \cos x \sinh y - 2\sin x \cosh y + B(y)$

$\therefore A = B = \text{const} = c$ 5

$\therefore \underline{v = \cos x \sinh y - 2\sin x \cosh y + c}$

$\therefore W = u + iv = \sin x \cosh y + 2\cos x \sinh y + i\left(\cos x \sinh y - 2\sin x \cosh y\right) + ic$

$= (1-2i)\sin x \cosh y + (2+i)\cos x \sinh y + c_1$ 2

$= (1-2i)\{\sin x \cosh y + i\cos x \sinh y\} + c_1$

Now $\cos(iy) = \frac{1}{2}\left(e^{i(iy)} + e^{-i(iy)}\right) = \frac{1}{2}\left(e^{y} + e^{-y}\right) = \cosh y$ 2

$\sin(iy) = \frac{1}{2i}\left(e^{i(iy)} - e^{-i(iy)}\right) = \frac{1}{2i}\left(e^{-y} - e^{y}\right) = i\sinh y$

$\therefore W = (1-2i)\{\sin x \cos(iy) + \cos x \sin(iy)\} + c_1$

$= (1-2i)\sin(x+iy) + c_1$ 1

$\underline{= (1-2i)\sin z + c_1}$

Setter : J. D. GIBBON Setter's signature : J. D. Gibbon

Checker : HERBERT Checker's signature : [signature] i)

$f = \frac{1}{g(z)}$; Expand $g(z)$ as a Taylor series about $z=a$

$$g(z) = g(a) + (z-a)g'(a) + \tfrac{1}{2}(z-a)^2 g''(a) + \tfrac{1}{6}(z-a)^3 g'''(a) \ldots$$

i) If g has a simple zero at $z=a$ then $g(a)=0$ ~~but~~

ii) if g has a double zero at $z=a$ then $g'(a)=0$ also.

$\therefore$ i) $f(z) = [(z-a)g'(a) + \ldots]^{-1}$

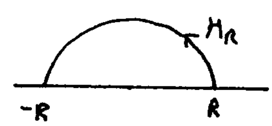
so Res of $f(z)$ at $z=a$ is $\frac{1}{g'(a)}$ 3

ii) Res. of $f(z)$ at $z=a$ with a double pole is $\lim_{z\to a} \frac{d}{dz}\left\{\frac{(z-a)^2}{\frac{1}{2}(z-a)^2 g''(a) + \frac{(z-a)^3}{6} g'''(a) \ldots}\right\}$

$$= \lim_{z\to a} \frac{d}{dz}\left[\tfrac{1}{2}g''(a) + \tfrac{1}{6}(z-a)g'''(a) + \ldots\right]^{-1}$$ 4

$$= -\tfrac{1}{6}g'''(a)\left[\tfrac{1}{2}g''(a)\right]^{-2} = -\tfrac{2}{3}\cdot\frac{g'''(a)}{(g''(a))^2}$$

ii) $\oint_C \frac{dz}{z^4+1} = \lim_{R\to\infty}\left(\int_{-R}^{R}\frac{dx}{x^4+1} + \int_{H_R}\frac{dz}{z^4+1}\right)$



$1+z^4$ has simple zeros at $\begin{cases} e^{\pi i/4},\ e^{3\pi i/4} & (\text{upper } \tfrac{1}{2}) \\ e^{5\pi i/4},\ e^{7\pi i/4} & (\text{lower } \tfrac{1}{2}) \end{cases}$

Now $\lim_{R\to\infty}\int_{H_R}\frac{dz}{z^4+1} = 0$ because degree of den. >1 & only sings are poles.

From formula above:

Res. at $e^{\pi i/4}$ is $\frac{1}{4e^{3\pi i/4}}$ 4

Res. at $e^{3\pi i/4}$ is $1/4e^{9\pi i/4}$

$\therefore \int_{-\infty}^{\infty}\frac{dx}{x^4+1} = \frac{2\pi i}{4}\left\{e^{-\frac{3\pi i}{4}} + e^{-9\pi i/4}\right\}$

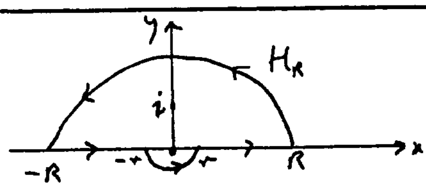
Now $\{e^{-3\pi i/4} + e^{-9\pi i/4}\} = \{-e^{\pi i/4} + e^{-\pi i/4}\} = -2i\sin\pi/4 = -2i/\sqrt{2}$ 4

$\therefore \int_{-\infty}^{\infty}\frac{dx}{x^4+1} = \frac{2\pi}{2\sqrt{2}} = \frac{\pi}{\sqrt{2}}$

Setter : J. D. GIBBON   Setter's signature : J.D. Gibbon

Checker : HERBERT   Checker's signature : Dr C Herbert

$$\oint_C \frac{e^{iz}}{z(z^2+1)}dz$$



Within C there are two simple poles : $z=0$ and $z=i$

i) Res. at $z=0$ is 1

ii) Res. at $z=i$ is $\dfrac{e^{-1}}{i \times 2i} = -\frac{1}{2}e^{-1}$] Sum is $1-\frac{1}{2e}$ 3

$$\oint_C = \int_{-R}^{-r} + \int_{\gamma} + \int_r^R + \int_{H_R}$$ Take $R\to\infty$, $r\to 0$

a) By Jordan's Lemma $\int_{H_R} \to 0$ as $R\to 0$ because, for integrals of the type $\int_{H_R} e^{iaz}f(z)dz$ 3

i) $a=1>0$

ii) $f(z)=\dfrac{1}{z(z^2+1)} \to 0$ as $z\to\infty$ with the degree of denominator >1.

iii) $f(z)$ has only poles in upper half-plane.

b) Small semi-circle is $z=re^{i\theta}$, $\pi\le\theta\le 2\pi$

$$\therefore \lim_{r\to 0}\int_{\gamma}\frac{e^{iz}dz}{z(z^2+1)} = \lim_{r\to 0}\int_{\pi}^{2\pi}\frac{\exp(ire^{i\theta})ie^{i\theta}d\theta}{e^{i\theta}(r^2e^{2i\theta}+1)}$$

$$= i\int_{\pi}^{2\pi}d\theta = \pi i$$ 3

c) $$\lim_{\substack{R\to\infty \\ r\to 0}}\left(\int_{-R}^{-r}+\int_r^R\right)\frac{e^{ix}dx}{x(x^2+1)} = \int_{-\infty}^{\infty}\frac{e^{ix}dx}{x(x^2+1)}$$ 3

$$= \int_0^{\infty}\frac{(e^{ix} \cancel{+} - e^{-ix})dx}{x(x^2+1)} = 2i\int_0^{\infty}\frac{\sin x\,dx}{x(x^2+1)}$$

Finally, using the Residue Theorem we have

$$2\pi i\left(1-\frac{1}{2e}\right) = \pi i + 2i\int_0^{\infty}\frac{\sin x\,dx}{x(x^2+1)}$$ 3

so $$\int_0^{\infty}\frac{\sin x\,dx}{x(x^2+1)} = \frac{\pi}{2}\left(1-\frac{1}{e}\right)$$

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Checker : HERBERT

Checker's signature : Dr P Herbert.

MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

SESSION : 2001-2002 E4

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PAPER 3

QUESTION

SOLUTION 18

$$\bar{\Pi}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t}\,\Pi(t)\,dt$$

$$= \int_{-1/2}^{1/2} 1.\,e^{-i\omega t}\,dt + 0$$

$$= -\frac{1}{i\omega}\left[e^{-i\omega/2} - e^{i\omega/2}\right] = \frac{\sin \omega/2}{\omega/2}$$

$$= \operatorname{sinc} \omega$$ 5

$$\bar{\Lambda}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t}\Lambda(t)\,dt$$

$$= \int_{-1}^{0}(1+t)\,e^{-i\omega t}dt + \int_{0}^{1}(1-t)e^{-i\omega t}dt$$

Now $\int e^{-i\omega t}dt = \frac{i}{\omega}\left[e^{-i\omega t}\right]$

$$\int t\,e^{-i\omega t}dt = \frac{i}{\omega}\int t\,d(e^{-i\omega t}) = \frac{i}{\omega}\left[t\,e^{-i\omega t}\right] - \frac{i}{\omega}\int e^{-i\omega t}dt$$

$$= \frac{i}{\omega}\left[t\,e^{-i\omega t}\right] + \frac{1}{\omega^2}\left[e^{-i\omega t}\right]$$ 5

$$\therefore\ \bar{\Lambda}(\omega) = \frac{i}{\omega}\left[\cancel{e^{-i\omega}} - \cancel{e^{i\omega}}\right] + \frac{i}{\omega}\left[0 + \cancel{e^{i\omega}}\right] + \frac{1}{\omega^2}\left[1 - e^{i\omega}\right]$$

$$-\frac{i}{\omega}\left[\cancel{e^{-i\omega}} - 0\right] - \frac{1}{\omega^2}\left[e^{-i\omega} - 1\right]$$

$$= \frac{1}{\omega^2}\left[2 - e^{i\omega} - e^{-i\omega}\right] = \frac{2}{\omega^2}(1 - \cos\omega)$$

But $\cos\omega = 1 - 2\sin^2\tfrac{1}{2}\omega$

$$\therefore\ \bar{\Lambda}(\omega) = 4\,\frac{\sin^2\frac{1}{2}\omega}{\omega^2} = \operatorname{sinc}^2\omega$$ 5

Setter : J. D. GIBBON Setter's signature : J.D. Gibbon

Checker : HERBERT Checker's signature : [signature]

MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

SESSION : 2001-2002

E5

PAPER 3

QUESTION

SOLUTION 19

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$$\mathcal{L}(\dot{y}) = \int_0^\infty e^{-st}\dot{y}\,dt = \int_0^\infty e^{-st}\,dy$$

$$= \left[y e^{-st}\right]_0^\infty + s\int_0^\infty y e^{-st}dt \qquad s>0$$

$$= s\bar{y}(s) - y(0)$$ 2

$$\mathcal{L}(\ddot{y}) = \int_0^\infty e^{-st}\ddot{y}\,dt = \int_0^\infty e^{-st}\,d(\dot{y})$$

$$= \left[\dot{y} e^{-st}\right]_0^\infty + s\int_0^\infty e^{-st}\dot{y}\,dt \qquad s>0$$

$$= -\dot{y}(0) + s\left(s\bar{y}(s) - y(0)\right)$$

$$= s^2\bar{y}(s) - \dot{y}(0) - s y(0)$$ 3

(ii) Now apply above formulae to $\ddot{y} + 3\dot{y} + 2y = f(t)$

$$\therefore\ s^2\bar{y}(s) - \cancel{\dot{y}(0)} - s y(0) + 3s\bar{y}(s) - 3y(0) + 2\bar{y}(s) = \bar{f}(s)$$

$$\therefore\ (s^2+3s+2)\bar{y}(s) = (s+3)y(0) + \bar{f}(s)$$

$$\therefore\ \bar{y}(s) = \frac{(s+3)\,y(0)}{(s+1)(s+2)} + \frac{\bar{f}(s)}{(s+1)(s+2)} = \left(\frac{2}{s+1} - \frac{1}{s+2}\right)y(0) + \frac{\bar{f}(s)}{s+1} - \frac{\bar{f}(s)}{s+2}$$ 5

Invert:

$$y(t) = \left(2e^{-t} - e^{-2t}\right)y(0)$$

$$+ \mathcal{L}^{-1}\left(\frac{\bar{f}(s)}{(s+1)}\right) - \mathcal{L}^{-1}\left(\frac{\bar{f}(s)}{s+2}\right)$$

Convolution Thm: $\mathcal{L}(f*g) = \bar{f}(s)\bar{g}(s)$

(Formula sheet)

$$\therefore\ y(t) = \left(2e^{-t} - e^{-2t}\right)y(0) +$$ 5

$$+ \int_0^t \left(e^{-(t-u)} - e^{-2(t-u)}\right)f(u)\,du.$$

Setter : J. D. GIBBON

Setter's signature : J. D. Gibbon

Checker : HERBERT

Checker's signature : J. C. Herbert

MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

SESSION : 2001-2002

E6

PAPER 3

QUESTION

SOLUTION 20

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$$\int_0^\infty f(t)\,e^{-st}\,dt = \int_0^T f(t)\,e^{-st}\,dt + \int_T^{2T} f(t)\,e^{-st}\,dt + \cdots$$

Consider $\displaystyle\int_T^{2T} f(t)\,e^{-st}\,dt$

and define $\tau = t - T$, so

$$\int_T^{2T} f(t)\,e^{-st}\,dt = \int_0^T e^{-s(\tau+T)} f(\tau+T)\,d\tau$$

invoking periodicity

$$= e^{-sT}\int_0^T e^{-s\tau} f(\tau)\,d\tau$$

Seen on a sheet.

$\therefore$ clearly $\displaystyle\int_{nT}^{(n+1)T} = e^{-snT}\int_0^T e^{-st} f(t)\,dt$

$$\therefore \int_0^\infty f(t)\,e^{-st}\,dt = \left(1 + e^{-sT} + e^{-2sT} + \cdots\right)\int_0^T$$

For $s>0$ $\qquad = \left(1 - e^{-sT}\right)^{-1}\displaystyle\int_0^T e^{-st} f(t)\,dt$ 10

$$\int_0^T f(t)\,e^{-st}\,dT = \int_0^{T/2} 1.\,e^{-st}\,dt + 0 = -\frac{1}{s}\left[e^{-st}\right]_0^{T/2}$$

$$= \frac{1}{s}\left(1 - e^{-sT/2}\right)$$

$$\therefore \bar{f}(s) = \frac{1}{s}\left(\frac{1 - e^{-sT/2}}{1 - e^{-sT}}\right)$$ 3

When $T \to \infty$ $(s>0)$, $\bar{f}(s) \to \frac{1}{s}$. This is the L.T. of $f(t) = 1$: the square wave period $\to \infty$ and $f(t) \to 1$ uniformly in $0 \le t \le \infty$. 2

Setter : J.D. GIBBON

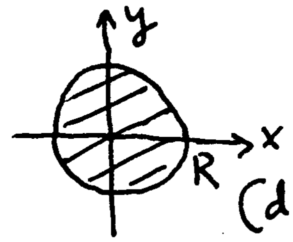
Setter's signature : J.D. Gibbon

Checker : HERBERT

Checker's signature : AJC Herbert

$x^2+y^2=R^2$ - circle of radius R

Hence the area of integration



(disk) 2

Jacobian: $\frac{\partial x}{\partial r}\frac{\partial y}{\partial \theta}-\frac{\partial x}{\partial \theta}\frac{\partial y}{\partial r}=r(\cos^2\theta+\sin^2\theta)=r.$

Region $x^2+y^2<R^2 \Rightarrow 0<\theta<2\pi,\ 0<r<R,$ 4

also $x^2+y^2=r^2$. Therefore

$$I_1=\int_0^R r\,dr\int_0^{2\pi} e^{-\alpha r^2}d\theta$$

To calculate I_1 substitute $r^2=u$: 4

$$I_1=\pi\int_0^{R^2} du\,e^{-\alpha u}=-\frac{\pi}{\alpha}e^{-\alpha u}\Big|_0^{R^2}=\underline{\underline{\frac{\pi}{\alpha}\left(1-e^{-\alpha R^2}\right)}}$$

Therefore the limit $I_1(\infty)=\lim_{R\to\infty} I_1 = \frac{\pi}{\alpha}$. 1

Observe that

$$I_2^2=\left[\int_{-\infty}^{+\infty}dx\,e^{-\alpha x^2}\right]^2=\int_{-\infty}^{+\infty}dx\int_{-\infty}^{+\infty}dy\,e^{-\alpha(x^2+y^2)}=I_1(\infty)$$ 4

Hence $I_2=\underline{\underline{\sqrt{\frac{\pi}{\alpha}}}}$

Total 15

Setter : Gogolin   Setter's signature : A. Gogolin

Checker : WALTON   Checker's signature : Andrew Walton (a)(i) Let $\underline{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$

Then $\text{Curl}\,\underline{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ A_1 & A_2 & A_3 \end{vmatrix}$

$= \hat{i}(A_{3y} - A_{2z}) - \hat{j}(A_{3x} - A_{1z}) + \hat{k}(A_{2x} - A_{1y})$

So $\text{div}(\text{Curl}\,\underline{A}) = \frac{\partial}{\partial x}(A_{3y} - A_{2z}) + \frac{\partial}{\partial y}(A_{1z} - A_{3x}) + \frac{\partial}{\partial z}(A_{2x} - A_{1y})$

$= 0$, as required.

(a)(i) 4 MARKS

(a)(ii)

$\text{Curl}(\underline{\nabla}\varphi) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ \varphi_x & \varphi_y & \varphi_z \end{vmatrix}$

$= \hat{i}(\varphi_{zy} - \varphi_{yz}) - \hat{j}(\varphi_{zx} - \varphi_{xz}) + \hat{k}(\varphi_{yx} - \varphi_{xy})$

$= \underline{0}$, as required.

(a)(ii) 3 MARKS

(b) $\text{Curl}\,\underline{A} = \lambda\underline{A} \quad (*)$

(i) Taking div of both sides:

$\underbrace{\text{div}(\text{Curl}\,\underline{A})}_{=0 \text{ by (a)}} = \lambda\,\text{div}\,\underline{A} \Rightarrow \text{div}\,\underline{A} = 0$

(b)(i) 2 MARKS

(ii) Take curl of both sides:

$\underbrace{\text{Curl}(\text{Curl}\,\underline{A})}_{\underline{\nabla}(\text{div}\,\underline{A}) - \nabla^2\underline{A}} = \lambda\,\text{Curl}\,\underline{A}$

(given in question)

But $\text{div}\,\underline{A} = 0 \therefore \nabla^2\underline{A} = -\lambda\,\text{Curl}\,\underline{A} = -\lambda^2\underline{A}$ (using $(*)$)

Hence $\nabla^2\underline{A} + \lambda^2\underline{A} = 0$

(ii) 3 MARKS

(iii) Using (ii) we have $\text{Curl}(\text{Curl}\,\underline{A}) = \lambda\,\text{Curl}\,\underline{A}$

So RHS of (iii) becomes

$$\frac{(\text{Curl}\,\underline{A})\cdot(\lambda\,\text{Curl}\,\underline{A})}{(\text{Curl}\,\underline{A})\cdot(\text{Curl}\,\underline{A})} = \lambda, \text{ as required.}$$

(iii) 3 MARKS

$\boxed{\text{Total } 15}$

Setter: A. WALTON Setter's signature: Graham Walton
Checker: A. GOGOLIN Checker's signature: A. Gogolin

MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

SESSION : 2001-2002

E9

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PAPER 3

QUESTION

SOLUTION 24

(i) We know that $\int_C (f\,dx + g\,dy)$ is independent of the path joining the initial point to the final point if $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$. Here, $f = (x^2+2xy+3y^2)$, $g = (x^2+6xy+2y^2)$, so $\frac{\partial f}{\partial y} = 2x+6y = \frac{\partial g}{\partial x}$, as required. 4

Let the path be $y = 2x$. Then

$$\int_C = \int_0^1 \{(x^2+4x^2+12x^2)\,dx + (x^2+12x^2+8x^2).2\,dx\}$$

$$= \int_0^1 59x^2\,dx$$

$$= \frac{1}{3}.59.$$ 4

(ii) (a) Put $x = \cos\theta$, $y = \sin\theta$

Then $dx = -\sin\theta\,d\theta$ $dy = \cos\theta\,d\theta$,

and the integral $= \int_0^{2\pi} \{-(2\cos\theta - \sin\theta)\sin\theta + (\cos\theta - 2\sin\theta)\cos\theta\}\,d\theta$

$$= \int_0^{2\pi} [(\cos^2\theta + \sin^2\theta) - 4\sin\theta\cos\theta]\,d\theta$$

$$= \int_0^{2\pi} (1 - 2\sin 2\theta)\,d\theta$$

$$= [\theta + \cos 2\theta]_0^{2\pi}$$

$$= 2\pi$$ 4

(b) Apply Green's Theorem with

$f = 2x - y$ $g = x - 2y$.

We get

$\frac{\partial g}{\partial x} = 1$ $\frac{\partial f}{\partial y} = -1$ 3

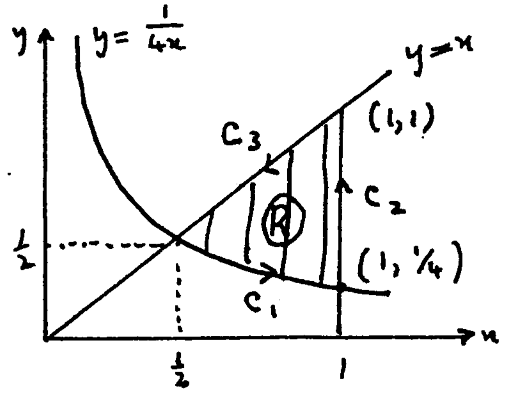
$\therefore$ The integral $= \iint_R 2\,dx\,dy = 2A$ where A = area of the circle $= 2\pi$.

Setter : G.D. JAMES Setter's signature : G James

Checker : A.J. MESTEL Checker's signature : J. [signature]

i) Chose $\underline{v} = \underline{\hat{i}}P + \underline{\hat{j}}Q$; $d\underline{r} = \underline{\hat{i}}dx + \underline{\hat{j}}dy \Rightarrow \underline{v}\cdot d\underline{r} = P\,dx + Q\,dy$

$\underline{k}\cdot\text{curl}\,\underline{v} = \underline{k}\cdot\begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ P & Q & 0 \end{vmatrix} = Q_x - P_y$. Hence result 5



ii) $\oint_c = \int_{c_1} + \int_{c_2} + \int_{c_3}$

$\int_{c_1}(y^2dx + x^2dy)$ 2

$= \frac{1}{4}\int_{\frac{1}{2}}^{1}\left(\frac{1}{4x^2} - 1\right)dx$

$= -\frac{1}{4}\left[\frac{1}{4x} + x\right]_{1/2}^{1}$

$= -\frac{1}{4}\left\{\frac{1}{4} + 1 - \frac{1}{2} - \frac{1}{2}\right\} = -\frac{1}{16}$ 2

$\int_{c_2}(y^2dx + x^2dy) = \int_{\frac{1}{4}}^{1} dy = 3/4$ 2

$C_2: x=1,\ y: \frac{1}{4}\to 1$

$\int_{c_3}(y^2dx + x^2dy) = 2\int_{1}^{\frac{1}{2}} x^2 dx = \frac{2}{3}\left[x^3\right]_1^{1/2} = -\frac{7}{8}\cdot\frac{2}{3} = -\frac{7}{12}$ 2

$C_3: y=x \quad x: 1\to\frac{1}{2}$

Total: $\oint_c(y^2dx + x^2dy) = -\frac{1}{16} + \frac{3}{4} - \frac{7}{12} = \frac{11}{16} - \frac{7}{12}$

$= \frac{1}{4}\left(\frac{11}{4} - \frac{7}{3}\right) = 5/48$ 2

[I have checked the answer by doing the area integral over R two different ways:

$\text{LHS} = 2\iint_R (x-y)\,dx\,dy = 2\int_{1/2}^{1}\left(\int_{\frac{1}{4x}}^{x}(x-y)\,dy\right)dx = 2\int_{1/2}^{1}\left[xy - \frac{y^2}{2}\right]_{\frac{1}{4x}}^{x}dx$

$= \int_{1/2}^{1}\left(x^2 - \frac{1}{2} + \frac{1}{16x^2}\right)dx = \left[\frac{x^3}{3} - \frac{x}{2} - \frac{x^{-1}}{16}\right]_{1/2}^{1} = 5/48$

Setter : J. D. GIBBON   Setter's signature : J. D. Gibbon
Checker : A. WALTON   Checker's signature : Andrew Walton

MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

SESSION : 2001-2002

E 11

PAPER EE2 13

QUESTION 1

SOLUTION 33

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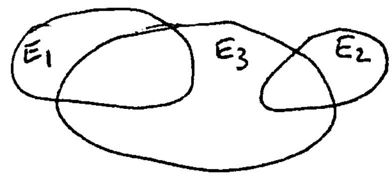
(i) Venn diagram - as seen 2



$\mathrm{pr}(E_2 \mid E_3) = p_2$ 1

$\mathrm{pr}(E_1 \cap E_3) = p_{13}p_3$ 1

$\mathrm{pr}(E_3 \mid E_1) = p_{13}p_3/p_1$ 1

$\mathrm{pr}(E_1 \mid E_2) = 0$ 1

(ii) (a) prob$=$ $\mathrm{pr}(D_1) \times \mathrm{pr}(D_3) = 0.001$; 1

(b) prob$=$ $\mathrm{pr}(D1 \cap D2) = \mathrm{pr}(D_1) + \mathrm{pr}(D_2) - \mathrm{pr}(D1 \cup D2)$

$= 0.1 + 0.05 - 0.12 = 0.03$; 2

(c) $\mathrm{pr}(D_1 \cap D_2 \cap D_3) = 0$ since D_2 and D_3 exclusive; 2

(d) $\mathrm{pr}(D_2 \mid D_1) = \mathrm{pr}(D_2 \cap D_1)/\mathrm{pr}(D_1) = 0.03/0.1 = 0.30$; 2

(e) $\mathrm{pr}(D_1 \mid D_3) = \mathrm{pr}(D_1) = 0.1$, since independent 2

15

Setter : MJ CROWDER

Setter's signature : MJ Crowder

Checker : AT WALDEN

Checker's signature : AWalden

MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

SESSION : 2001-2002

E 12

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PAPER EE2 (3)

QUESTION 2

SOLUTION 34

(i) Expression: $\mathrm{E}(X_1) = \sum_{x_1} x_1 p_1(x_1)$ 1

Proof: $\mathrm{E}(X_1 + X_2) = \sum_{x_1,x_2}(x_1 + x_2)p(x_1, x_2)$

$= \sum_{x_1,x_2} x_1 p(x_1, x_2) + \sum_{x_1,x_2} x_2 p(x_1, x_2)$

$= \sum_{x_1} x_1 p_1(x_1) + \sum_{x_2} x_2 p_2(x_2) = \mathrm{E}(X_1) + \mathrm{E}(X_2)$ 3

(ii) $\mathrm{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$ 1

$\mathrm{N}(2\mu_1 - 3\mu_2, 4\sigma_1^2 + 9\sigma_2^2)$ 2

(iii) mean=$2\mu_3 - 3\mu_4$ 1

variance= $4\sigma_3^2 - 12\rho\sigma_3\sigma_4 + 9\sigma_4^2$ 3

(iv) prob = pr(Supply$\geq$Dem1+Dem2),

where Supply–(Dem1+Dem2) is N(1.14,0.36), 2

so prob $= 1 - \Phi(1.14/0.6) = 1 - \Phi(1.9) = 0.0287$ 2

15

Setter : MJ CROWDER Setter's signature : MJ Crowder

Checker : AT WALDEN Checker's signature : AT Walden