EE2-08B

THE ANSWERS (Mathematics 2B)

Notations:

(a) B - Bookwork

(b) E - New example

(c) A - New application

1. a) i) $\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y)$

[1 - B]

$E(XY) = \int_0^1 \int_0^1 2x^2 y\,dx\,dy = \frac{1}{3}$

[1 - A]

$E(X) = \int_0^1 2x^2 dx = \frac{2}{3}$

[1 - A]

$E(Y) = \int_0^1 y\,dy = \frac{1}{2}$

[1 - A]

$\mathrm{Cov}(X,Y) = \frac{1}{3} - \frac{2}{3}\frac{1}{2} = 0$. Hence $\mathrm{Corr}(X,Y) = 0$.

ii) $P(X \leq 0.25 \mid Y \geq 1/3) = \frac{P(X \leq 0.25 \cap Y \geq 1/3)}{P(Y \geq 1/3)}$

[1 - A]

$P(Y \geq 1/3) = \int_{1/3}^1 dy = \frac{2}{3}$

[1 - A]

$P(X \leq 0.25 \cap Y \geq 1/3) = \int_0^{0.25} \int_{1/3}^1 2x\,dy\,dx = \frac{1}{24}$

[1 - A]

$P(X \leq 0.25 \mid Y \geq 1/3) = \frac{1}{16}$

[1 - A]

iii) $\mathrm{Var}(3X - 2Y + 5) = 9\mathrm{Var}(X) + 4\mathrm{Var}(Y)$.

[1 - A]

$\mathrm{Var}(X) = E(X^2) - E(X)^2 = \frac{1}{18}$.

[1 - A]

$\mathrm{Var}(Y) = E(Y^2) - E(Y)^2 = \frac{1}{12}$.

[1 - A]

$\mathrm{Var}(3X - 2Y + 5) = \frac{5}{6}$.

[1 - A]

iv) independent if $f_{X,Y}(x,y) = f_X(x) f_Y(y),\ \forall x,y$.

[1 - A]

$$f_X(x) = \begin{cases} 2x, & 0 < x < 1, \\ 0, & \textit{otherwise.} \end{cases}$$

[1 - A]

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$$f_Y(y) = \begin{cases} 1, & 0 < y < 1 \\ 0, & otherwise. \end{cases}$$

[1 - A]

Yes, they are independent.

[1 - A]

b) i) Consider the random sample $X_1, \ldots, X_n$. Likelihood function $L(\theta) = f_{X_1,\ldots,X_n}(x_1,\ldots,x_n|\theta) = \prod_{i=1}^{n} f_{X_i}(x_i|\theta)$.

[1 - B]

$L(\theta) = \prod_{i=1}^{n} \theta\,(1-x_i)^{\theta-1} = \theta^n \left(\prod_{i=1}^{n}(1-x_i)\right)^{\theta-1}$.

[1 - A]

Log-likelihood function becomes $\log L(\theta) = n\log\theta + (\theta-1)\sum_{i=1}^{n}\log(1-x_i)$.

[1 - A]

Derivative to zero $\frac{d}{d\theta}\log L(\theta) = \frac{n}{\theta} + \sum_{i=1}^{n}\log(1-x_i) = 0$.

[1 - A]

Estimator $\hat{\theta} = \frac{-n}{\sum_{i=1}^{n}\log(1-X_i)}$.

[1 - A]

Concavity $\frac{d^2}{d\theta^2}\log L(\theta) = -\frac{n}{\theta^2} < 0$.

[1 - A]

ii) Evaluate the estimator using data such that the estimate of θ is given by $\hat{\theta} = \frac{-20}{\sum_{i=1}^{20}\log(1-x_i)} = 4.59$. [3 - A]

2. a) $P(X_1+X_2 \leq 1) = \int_0^1 \int_0^{1-x_1} f_{X_1,X_2}(x_1,x_2)dx_2dx_1.$

[1 - A]

Using independence,

$$f_{X_1,X_2}(x_1,x_2) = \begin{cases} 4(1-x_1)(1-x_2), & 0 < x_1 < 1, 0 < x_2 < 1, \\ 0, & \textit{otherwise.} \end{cases}$$

[1 - A]

$P(X_1+X_2 \leq 1) = \int_0^1 \int_0^{1-x_1} 4(1-x_1)(1-x_2)dx_2dx_1 = \frac{5}{6}.$

[1 - A]

$P(X_1+X_2 \leq 1) = 1 - P(X_1+X_2 \leq 1) = \frac{1}{6}.$

[1 - A]

b) i) $f_{X,Y}(x,y) = f_{Y|X}(y|x)f_X(x)$

[1 - B]

$$f_{X,Y}(x,y) = \begin{cases} \frac{2(1-x)}{x}\exp\left(-\frac{y}{x}\right), & 0 < x < 1, 0 < y < \infty, \\ 0, & \textit{otherwise.} \end{cases}$$

[1 - A]

ii) $E(Y|X=x) = \int_{-\infty}^{+\infty} y f_{Y|X}(y|x)dy$ for $0 < x < 1$.

[1 - B]

$E(Y|X=x) = \int_0^{+\infty} \frac{y}{x}\exp\left(-\frac{y}{x}\right)dy = x$. Hence $E(Y|X) = X$.

[2 - A]

iii) $E_X E(Y|X) = E(Y).$

[2 - A]

$E(Y) = \int_0^1 x2(x-1)dx = \frac{1}{3}.$ [2 - A]

iv) $\text{Var}(Y) = E(\text{Var}(Y|X)) + \text{Var}(E(Y|X)).$

[2 - A]

$\text{Var}(Y|X=x) = \int_0^{+\infty}(y - E(Y|X=x))^2 f_{Y|X}(y|x)dy = \int_0^{+\infty}(y-x)^2\frac{1}{x}e^{-\frac{y}{x}}dy = x^2$

[2 - A]

$E(\text{Var}(Y|X)) = \int_0^1 x^2 2(1-x)dx = \frac{1}{6}.$

[1 - A]

$\text{Var}(E(Y|X)) = \int_0^1 (x-\frac{1}{3})^2 2(1-x)dx = \frac{1}{18}.$

[1 - A]

$\text{Var}(Y) = \frac{1}{6} + \frac{1}{18} = \frac{2}{9}.$

c) Write $W = 1 - \sqrt{1-U}$.
$F_W(w) = P(W \leq w) = P\left(1 - \sqrt{1-U} \leq w\right) = P\left(1 - w \leq \sqrt{1-U}\right).$

[2 - A]

$$F_W(w) = \begin{cases} P\left((1-w)^2 \leq 1-U\right), & 0 \leq w \leq 1, \\ 1, & w > 1. \end{cases}$$

[2 - A]

$$F_W(w) = \begin{cases} -w^2 + 2w, & 0 \le w \le 1, \\ 0, & w < 0, \\ 1, & w > 1. \end{cases}$$

[1 - A]

$$f_W(w) = \frac{d}{dw} F_W(w) = \begin{cases} 2(1-w), & 0 \le w \le 1, \\ 0, & \textit{otherwise}. \end{cases}$$

[1 - A]