EE1-10B MATHEMATICS II

The below comments apply to common errors. Where there is no comment, the question was done well by most students.

1. a) Given the function

$$f(t) = e^{at} H(-t)\,,$$

where H is the Heaviside function, obtain $F(\omega)$, the Fourier transform of $f(t)$. State the condition on the constant a which is necessary for the existence of $F(\omega)$. [5]

SOLUTION

$$\begin{aligned}
\mathscr{F}[f(t)] &= \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt = \int_{-\infty}^{0} e^{at}e^{-i\omega t}\,dt \\
&= \left[\frac{1}{a-i\omega}e^{(a-i\omega)t}\right]_{-\infty}^{0} = \frac{1}{a-i\omega} - \lim_{K\to-\infty}\frac{e^{(a-i\omega)K}}{a-i\omega} \\
&= \frac{1}{a-i\omega}
\end{aligned}$$

as the limit is zero, provided $a > 0$. Answer $\mathrm{Re}[a-i\omega] > 0$ is also good.

Many did not recognize the condition $a > 0$, which makes the limit equal to 0.

b) Hence, or otherwise, obtain the inverse Fourier Transform of [5]

$$G(\omega) = \frac{1}{4-2i\omega-3i}\,.$$

SOLUTION

Begin by rewriting

$$G(\omega) = \frac{1}{4-2i\left(\omega+\frac{3}{2}\right)} = H\left(\omega+\frac{3}{2}\right)$$

where

$$H(\omega) = \frac{1}{4-2i} = \frac{1}{2}\left(\frac{1}{2-i\omega}\right),$$

and using (a) we have

$$h(t) = \mathscr{F}^{-1}[H(\omega)] = \frac{1}{2}e^{2t}H(-t)$$

and using the frequency shift from G to H, we obtain

$$g(t) = e^{-\frac{3}{2}it} h(t) = \frac{1}{2} e^{(2-3i/2)t} H(-t)\,.$$

Very few people recognized the frequency shift property. Very few recognized that part (a) could be used directly with the value of the constant $a = 2 - \frac{3}{2}i$. Lots of people started to compute the inverse Fourier transform directly from the definition, failing to recognize that they could use the results of part (a).

c) Given the plane with equation $\Pi : 2x - 3y + 5z = -4$,

i) Find the minimum distance from the point $P(1,-1,2)$ to Π; obtain the point on Π nearest to P. [4]

SOLUTION

The intersection of the line through P and normal to the plane will be the desired point, say A; hence the direction vector of the line is the normal vector of the plane, $(2,-3,5)$. The line has equation

$$\underline{\mathbf{r}} = (x,y,z) = (1,-1,2) + \lambda(2,-3,5)$$

and substituting into the equation of the plane:

$$2(1+2\lambda) - 3(-1-3\lambda) + 5(2+5\lambda) = 15 + 38\lambda = -4 \Rightarrow \lambda = -\frac{1}{2},$$

so the interection of line and plane, the point A on Π nearest to P is

$$A : (1,-1,2) - \frac{1}{2}(2,-3,5) = \left(0, \frac{1}{2}, -\frac{1}{2}\right)$$

and the minimum distance from P to Π is

$$|\overrightarrow{AP}| = \left|\left(1, -\frac{3}{2}, \frac{5}{2}\right)\right| = \frac{\sqrt{38}}{2}\,.$$

Most people got the procedure to find the point A. Some people, instead of computing $|PA|$, computed $|OA|$.

ii) Another plane has equation $\Phi : x + \alpha y + \beta z = 0$. Give all values of α and β that make Π and Φ orthogonal. [3]

SOLUTION For the planes to be orthogonal, the normal vectors need to be orthogonal:

$$(2,-3,5)\cdot(1,\alpha,\beta) = 0 \Rightarrow 2 - 3\alpha + 5\beta = 0$$

with solutions
$$(\alpha,\beta) = (2/3,0) + t(5/3,1)$$
where t is any real number.

Many people left the solution as $3\alpha + 5\beta = 2$, which is not complete, failing to recognize that infinite solutions (depending on some parameter t) solve such a condition. A few people erroneously made the cross product - instead of the dot product - equal to zero.

d) Given the vectors $\underline{\mathbf{u}} = (1,2,a)$, $\underline{\mathbf{v}} = (3,-4,b)$ and $\underline{\mathbf{w}} = (-5,6,c)$, find a condition on the scalars a,b,c so that $\underline{\mathbf{u}} \times \underline{\mathbf{v}} \cdot \underline{\mathbf{w}} = 0$.

Let this condition be satisfied. The vectors now form what kind of set? What is the determinant of the matrix whose columns are $\underline{\mathbf{u}}, \underline{\mathbf{v}}, \underline{\mathbf{w}}$? Finally, obtain scalars p,q such that $\underline{\mathbf{u}} = p\underline{\mathbf{v}} + q\underline{\mathbf{w}}$. [7]

SOLUTION

Rewrite as
$$\underline{\mathbf{w}} \cdot (\underline{\mathbf{u}} \times \underline{\mathbf{v}}) = \begin{vmatrix} -5 & 6 & c \\ 1 & 2 & a \\ 3 & -4 & b \end{vmatrix} = -5(2b+4a) - 6(b-3a) + c(-4-6),$$
so the condition is

$$a + 8b + 5c = 0, \qquad (*)$$

Begin by finding p,q: $\underline{\mathbf{u}} = p\underline{\mathbf{v}} + q\underline{\mathbf{w}} \Rightarrow (1,2,a) = p(3,-4,b) + q(-5,6,c)$ which is a set of three equations:
$$3p - 5q = 1, \quad -4p + 6q = 2, \quad \Rightarrow \quad p = -8, q = -5$$
and the third equation:
$$bp + cq = a \Rightarrow -8p - 5c = a,$$
is satisfied due to $(*)$.

Thus, the vectors form a linearly dependent set, and a matrix with linearly dependent columns has determinant zero.

Alternatively, from $\underline{\mathbf{w}} \cdot (\underline{\mathbf{u}} \times \underline{\mathbf{v}}) = 0$ we have that $\underline{\mathbf{w}}$ is orthogonal to $\underline{\mathbf{u}} \times \underline{\mathbf{v}}$, but as this is orthogonal to $\underline{\mathbf{u}}$ and $\underline{\mathbf{v}}$, we deduce that $\underline{\mathbf{w}}$ is in the plane defined by $\underline{\mathbf{u}}$ and $\underline{\mathbf{v}}$ and hence the vectors form a linearly dependent set.

Lots of people stated that the condition $\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ implies that the vectors $\mathbf{w}, \mathbf{u}, \mathbf{v}$ form an orthogonal set: this is incorrect. They failed to recognize that, since $\mathbf{w}$ is orthogonal to the cross product $\mathbf{u}, \mathbf{v}$ and since such cross product is orthogonal to both $\mathbf{u}, \mathbf{v}$, it must hold that all three vectors lie on the same plane, thus forming a linearly dependent set.

2. a) Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{pmatrix}.$$

i) Calculate A^2 and A^3 and find scalars ϕ and ψ such that [4]

$$A^3 + \phi A^2 + \psi A + I = \underline{\mathbf{0}},$$

where I is the identity matrix.

SOLUTION

$$A^2 = \begin{pmatrix} 1 & 4 & 6 \\ 0 & 8 & 13 \\ -2 & 3 & 6 \end{pmatrix}, \quad A^3 = \begin{pmatrix} -1 & 15 & 25 \\ -5 & 29 & 50 \\ -5 & 10 & 19 \end{pmatrix}.$$

and hence

$A^3 + \phi A^2 + \psi A + I$

$$= \begin{pmatrix} -1+\phi+\psi+1 & 15+4\phi+\psi & 25+6\phi+\psi \\ -5+\psi & 29+8\phi+2\psi+1 & 50+13\phi+3\psi \\ -5-2\phi-\psi & 10+3\phi+\psi & 19+6\phi+2\psi+1 \end{pmatrix}$$

$= \underline{\mathbf{0}}$. The $(2,1)$ entry gives $-5+\psi = 0 \Rightarrow \psi = 5$; the $(1,1)$ entry gives $\phi + \psi = 1 \Rightarrow \phi = -5$. Check with any other entry to confirm these.

Almost all correct answers for A^2 and A^3, but lots of people gave up here, seeming to think that $A^3 + \phi A^2 + \psi A + I = 0$ is too complicated. Of those that tried, nearly all got it right.

ii) Use the result from (i) to find the inverse of A. [4]

SOLUTION

To use (i), we observe that, given $A^3 - 5A^2 + 5A + I = \underline{\mathbf{0}}$ we can multiply on the left:

$$A^{-1}(A^3 - 5A^2 + 5A + I) = A^2 - 5A + 5I + A^{-1} = A^{-1}\underline{\mathbf{0}} = \underline{\mathbf{0}}$$

so that $A^{-1} = -A^2 + 5A - 5I =$

$$= -\begin{pmatrix} 1 & 4 & 6 \\ 0 & 8 & 13 \\ -2 & 3 & 6 \end{pmatrix} + 5\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{pmatrix} - 5\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 1 & -1 \\ 5 & -3 & 2 \\ -3 & 2 & -1 \end{pmatrix}$$

A large minority of students didn't know where to start and skipped this. Of those that realized multiplication by the inverse would do it, nearly all got it.

iii) Confirm your result in (ii) by calculating A^{-1} using Gaussian elimination. [4]

SOLUTION

Set up the augmented matrix and use row operations:

$$(A:I) = \begin{pmatrix} 1 & 1 & 1 & | & 1 & 0 & 0 \\ 1 & 2 & 3 & | & 0 & 1 & 0 \\ -1 & 1 & 2 & | & 0 & 0 & 1 \end{pmatrix} \underset{\substack{R_2-R_1 \\ R_3+R_1}}{\sim} \begin{pmatrix} 1 & 1 & 1 & | & 1 & 0 & 0 \\ 0 & 1 & 2 & | & -1 & 1 & 0 \\ 0 & 2 & 3 & | & 1 & 0 & 1 \end{pmatrix}$$

$$\underset{\substack{R_1-R_2 \\ R_3-2R_2}}{\sim} \begin{pmatrix} 1 & 0 & -1 & | & 2 & -1 & 0 \\ 0 & 1 & 2 & | & -1 & 1 & 0 \\ 0 & 0 & -1 & | & 3 & -2 & 1 \end{pmatrix} \underset{-R_3}{\sim} \begin{pmatrix} 1 & 0 & -1 & | & 2 & -1 & 0 \\ 0 & 1 & 2 & | & -1 & 1 & 0 \\ 0 & 0 & 1 & | & -3 & 2 & -1 \end{pmatrix}$$

$$\underset{\substack{R_1+R_3 \\ R_2-2R_3}}{\sim} \begin{pmatrix} 1 & 0 & 0 & | & -1 & 1 & -1 \\ 0 & 1 & 0 & | & 5 & -3 & 2 \\ 0 & 0 & 1 & | & -3 & 2 & -1 \end{pmatrix}$$

confirming the result from (ii).

Mostly Gaussian, less cofactor method.

b) Given a matrix

$$A = \begin{pmatrix} 1 & -4 & 2 \\ -4 & 1 & -2 \\ 2 & -2 & -2 \end{pmatrix}$$

i) Show that $\lambda = -3$ is one of the eigenvalues of A and find the other two. [4]

SOLUTION

Begin with $\det(A - \lambda I) = 0$ and use row/column operations to simplify:

$$\begin{vmatrix} 1-\lambda & -4 & 2 \\ -4 & 1-\lambda & -2 \\ 2 & -2 & -2-\lambda \end{vmatrix} \underset{R_1+R_2}{=} \begin{vmatrix} -3-\lambda & -3-\lambda & 0 \\ -4 & 1-\lambda & -2 \\ 2 & -2 & -2-\lambda \end{vmatrix}$$

$$\underset{C_2-C_1}{=} \begin{vmatrix} -3-\lambda & 0 & 0 \\ -4 & 5-\lambda & -2 \\ 2 & -4 & -2-\lambda \end{vmatrix} = 0$$

showing $\lambda = -3$ is an eigenvalue. Now expand by the first row:

$-(3+\lambda)[(5-\lambda)(-2-\lambda)+8] = -(3+\lambda)(\lambda^2 - 3\lambda - 18) = 0$

$\Rightarrow\ -(\lambda+3)^2(\lambda-6) = 0$, so the other eigenvalues are $\lambda = -3$ and $\lambda = 6$.

A lot of arithmetic errors. An impressive number of people multiplied out the cubic polynomial and factorized correctly, but most of the errors were in multiplying out or in factorizing. Of those that couldn't factorize, a few smart ones substituted the given eigenvalue and showed that $A+3I$ is singular, so -3 is an eigenvalue.

ii) Find eigenvectors corresponding to the three eigenvalues of A. [4]

SOLUTION

For $\lambda = 6$ we have $(A-6I)\underline{\mathbf{x}} = \underline{\mathbf{0}}$ giving

$$\begin{pmatrix} -5 & -4 & 2 \\ -4 & -5 & -2 \\ 2 & -2 & -8 \end{pmatrix} \underline{\mathbf{x}} = \underline{\mathbf{0}} \Rightarrow \begin{pmatrix} 1 & -1 & -4 \\ 0 & -9 & -18 \\ 0 & -9 & -18 \end{pmatrix} \underline{\mathbf{x}} = \underline{\mathbf{0}} \text{ using Row operations.}$$

Hence $y+2z = 0$ and $x = y+4z$. Choosing $z = 1$ we get the eigenvector $\underline{\mathbf{x}}_1 = (2,-2,1)$

For $\lambda = -3$ we have $(A+3I)\underline{\mathbf{x}} = \underline{\mathbf{0}}$ giving $\begin{pmatrix} 4 & -4 & 2 \\ -4 & 4 & -2 \\ 2 & -2 & 1 \end{pmatrix} \underline{\mathbf{x}} = \underline{\mathbf{0}}$ and all three rows give $2x-2y+z = 0$, two free variables. To ensure linear independence, we choose $y = 2, z = 0$ for one eigenvector and $y = 0, z = 2$ for the other, giving $\underline{\mathbf{x}}_2 = (2,2,0)$ and $\underline{\mathbf{x}}_3 = (-1,0,2)$.

Mostly arithmetic error or using the wrong eigenvalue. A few people gave zero eigenvectors due to incorrect eigenvalue. This should have been an indication that something was wrong! Some people found a single eigenvector - instead of two - for $\lambda = -3$.

iii) Using projection, or otherwise, find a set of orthonormal eigenvectors for A, and hence obtain the orthogonal diagonalization of A. [5]

SOLUTION

We note that $\underline{\mathbf{x}}_2$ and $\underline{\mathbf{x}}_3$ are both orthogonal to $\underline{\mathbf{x}}_1$, but not to each other. Given that a linear combination of eigenvectors corresponding to the same eigenvalue is also an eigenvector for that eigenvalue, we can use projection to get two orthogonal eigenvectors from $\underline{\mathbf{x}}_1$ and $\underline{\mathbf{x}}_2$.

If $\mu\underline{\mathbf{x}}_2$ is the projection of $\underline{\mathbf{x}}_3$ onto $\underline{\mathbf{x}}_2$, then $\underline{\mathbf{x}}_3 - \mu\underline{\mathbf{x}}_2$ is an eigenvector for $\lambda = -3$ and orthogonal to $\underline{\mathbf{x}}_2$. Hence

$$\mu = \frac{\underline{\mathbf{x}}_3 \cdot \underline{\mathbf{x}}_2}{\underline{\mathbf{x}}_2 \cdot \underline{\mathbf{x}}_2} = \frac{(-1,0,2)\cdot(2,2,0)}{(2,2,0)\cdot(2,2,0)} = -\frac{1}{4}$$

and so

$$\underline{\mathbf{x}}_3 - \mu \underline{\mathbf{x}}_2 = (-1,0,2) + \frac{1}{4}(2,2,0) = \frac{1}{2}(-1,1,4)$$

and we take $\underline{\mathbf{x}}_4 = (-1,1,4)$, for convenience, noting it is orthogonal to $\underline{\mathbf{x}}_1$ and $\underline{\mathbf{x}}_2$, as expected. Hence the set or orthonormal eigenvectors is

$$\hat{\underline{\mathbf{x}}}_1 = \frac{1}{3}(2,-2,1)\,, \quad \hat{\underline{\mathbf{x}}}_2 = \frac{1}{\sqrt{2}}(1,1,0)\,, \quad \hat{\underline{\mathbf{x}}}_4 = \frac{1}{\sqrt{18}}(-1,1,4)\,.$$

Given a symmetric matrix and a set of orthonormal eigenvectors, the orthogonal diagonalization is $A = PDP^T$ where

$$P = \begin{pmatrix} \frac{2}{3} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{18}} \\ -\frac{2}{3} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{18}} \\ \frac{1}{3} & 0 & \frac{4}{\sqrt{18}} \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 6 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

Most of the errors here were due to the fact that a random pair of independent eigenvectors for the repeated eigenvalue $\lambda = -3$ are not going to be orthogonal: many assumed this and skipped the last part which uses projection. A lot of people put the eigenvectors unnormalized into a matrix P. Many calculated the inverse of P by Gaussian elimination.

3. a) Find the general solution of the differential equation

$$(3t\cos x - 2x)\frac{dx}{dt} = 4t - 3\sin x\,.$$

Find also the particular solution satisfying the condition $x(1) = 0$. [6]

SOLUTION

Rewrite as

$$(3t\cos x - 2x)\frac{dx}{dt} - 4t + 3\sin x = 0\,.$$

with $P(x,t) = 3t\cos x - 2x$ and $Q(x,t) = -4t + 3\sin x$. Check for exactness:

$$\frac{\partial P}{\partial t} = 3\cos x = \frac{\partial Q}{\partial x}\,,$$

so the ODE is exact in the form $dF = 0$, where F is found by integrating P, Q:

$$\frac{\partial F}{\partial x} = P \Rightarrow F = \int P\,dx = \int 3t\cos x - 2x\,dx = 3t\sin x - x^2 + g(t)$$

where g is an arbitrary function. Similarly

$$\frac{\partial F}{\partial t} = Q \Rightarrow F = \int Q\,dt = \int -4t + 3\sin x\,dt = -2t^2 + 3t\sin x + h(x)$$

where h is another arbitrary function. Equating the two, we have $g(t) = -2t^2$ and $h(x) = -x^2$ so that

$$F(x,t) = 3t\sin x - x^2 - 2t^2$$

with $dF = 0 \Rightarrow F =$constant giving the solution:

$$3t\sin x - x^2 - 2t^2 = C\,.$$

and implementing the initial condition $x(1) = 0$ gives $C = -2$, so the particular solution is

$$3t\sin x - x^2 - 2t^2 = -2\,.$$

Some mixed up the partial derivatives for checking exactness and concluded the equation is not exact. Others integrated P resp. Q with respect to the wrong independent variables. Many, many left the answer in the form $F(x,t) = 3t\sin x - x^2 - 2t^2$ or $F(x,t) = 3t\sin x - x^2 - 2t^2 = C$ or $= -2$, both incorrect: $F(x,t)$ is just an aide to solving the ODE, it has nothing to do with the original equation which only has t and x. Some left the answer as $3t\sin x - x^2 - 2t^2 = C$.

b) Given the Bernoulli equation

$$x\frac{dy}{dx} + y = x^2y^2\,,$$

use the substitution $v = y^{-1}$ to obtain a first order linear equation in v, and hence solve for y. [6]

SOLUTION

Following the suggestion, the substitution

$$v = \frac{1}{y} \Rightarrow \frac{dv}{dx} = -\frac{1}{y^2}\frac{dy}{dx}$$

and dividing the original equation by $-xy^2$ we obtain

$$-\frac{1}{y^2}\frac{dy}{dx} - \frac{1}{x}\frac{1}{y} = -x\,,$$

and substituting, we have a linear equation:

$$\frac{dv}{dx} - \frac{1}{x}v = -x$$

with $P(x) = -1/x$ and the integrating factor

$$\mu(x) = e^{-\int \frac{1}{x}\,dx} = \frac{1}{x}\,.$$

Multiplying through by the integrating factor we get

$$\frac{1}{x}\frac{dv}{dx}-\frac{1}{x^2}v=\frac{d}{dx}\left(\frac{1}{x}v\right)=-1\Rightarrow\frac{1}{x}v=-x+C$$

and the solution for v is

$$v=\frac{1}{y}=-x^2+Cx\Rightarrow y=\frac{1}{Cx-x^2}\,.$$

Lots of errors in transforming to a linear equation, many people had problems obtaining the relation between derivatives $\frac{dv}{dx}$ and $\frac{dy}{dx}$. Often, this made the resulting ODE intractable. Some got it right, but then couldn't solve the linear ODE. A few made the classic error of adding the integration constant too late, and proceeded as follows

$$\frac{1}{x}v=-x\Rightarrow v=-x^2+C\Rightarrow\quad etc.$$

c) Solve the following second order differential equation: [8]

$$\frac{d^2y}{dx^2}-2\frac{dy}{dx}+5y=26\cos(3x)\,.$$

SOLUTION

The auxiliary equation is $\lambda^2-2\lambda+5=0\Rightarrow\lambda=1\pm2i$ and the complementary function is

$$y_c=e^x(c_1\cos2x+c_2\sin2x)\,,$$

where $c_{1,2}$ are arbitrary constants. For a particula integral, try

$$\begin{aligned}y_p&=A\cos3x+B\sin3x\\ y_p'&=-3A\sin3x+3B\cos3x\\ y_p''&=-9A\cos3x-9B\sin3x\end{aligned}$$

and substitute into the ODE to get

$$-9A\cos3x-9B\sin3x-2(-3A\sin3x+3B\cos3x)+5(A\cos3x+B\sin3x)=$$

$$(-9A-6B+5A)\cos3x+(-9B+6A+5B)\sin3x=26\cos3x$$

so equating coefficients we have $-4A-6B=26\,,\quad 6A-4B=0$ so that $A=-2$ and $B=-3$, and the particular integral together with the complementary function gives the general solution

$$y=y_c+y_p=e^x(c_1\cos2x+c_2\sin2x)-2\cos3x-3\sin3x\,.$$

Mostly fine with the PI, but a lot of people wrote the CF in terms of complex exponentials.

d) The height h of a regular cone, with volume V and radius of the circular base r, is found using
$$V = \frac{1}{3}\pi r^2 h\,.$$
Given that the percentage errors in the measurements of r and V are at most 0.5% and 0.2%, respectively, give an estimate for the maximum percentage error in the calculation of h. [5]

SOLUTION

Begin by writing
$$h = \frac{3V}{\pi r^2} \Rightarrow \frac{\partial h}{\partial V} = \frac{3}{\pi r^2}\,, \quad \frac{\partial h}{\partial r} = -\frac{6V}{\pi r^3}\,.$$
The total differential

$$dh = \frac{\partial h}{\partial V}dV + \frac{\partial h}{\partial r}dr$$
is used to estimate small differences $\Delta h\,, \Delta V\,, \Delta r$:
$$\Delta h \approx \frac{\partial h}{\partial V}\Delta V + \frac{\partial h}{\partial r}\Delta r = \frac{3}{\pi r^2}\Delta V - \frac{6V}{\pi r^3}\Delta r$$

and dividing by h on the left, and by $3V/(\pi r^2)$ on the right, we obtain

$$\frac{\Delta h}{h} \approx \frac{\frac{3}{\pi r^2}}{\frac{3V}{\pi r^2}}\Delta V - \frac{\frac{6V}{\pi r^3}}{\frac{3V}{\pi r^2}}\Delta r = \frac{\Delta V}{V} - 2\frac{\Delta r}{r}\,.$$
The relative error could be positive or negative, so we use the triangle inequality to estimate
$$\left|\frac{\Delta h}{h}\right| \leq \left|\frac{\Delta V}{V}\right| + 2\left|\frac{\Delta r}{r}\right| = 0.002 + 2(0.005) = 0.012\,,$$
so the maximum error in calculating h is 1.2%.

Mostly fine, the most common error lay in not using the triangle inequality, or in trying to solve for ΔV instead of Δh

4. a) A solution of the second order differential equation

$$\frac{d^2y}{dx^2} - 2xy = 0\,,$$

can be found in the form of a series with the Leibnitz-Maclaurin method. Given the initial conditions $y(0) = 1$ and $y'(0) = 0$, differentiate the ODE n times to obtain the recurrence relation

$$y^{(n+2)}(0) = 2ny^{(n-1)}(0)\,, \quad (n \geq 1)\,,$$

where $y^{(k)}(0)$ is the k^{th} derivative of y, evaluated at zero.

Obtain the first three non-zero terms of the series. [8]

SOLUTION

Using Leibnitz' Theorem we differentiate the ODE n times and get

$$y^{(n+2)} - 2\left[xy^{(n)} + n \cdot 1y^{(n-1)}\right] = 0$$

and setting $x = 0$ we have

$$y^{(n+2)}(0) - 2ny^{(n-1)}(0) = 0 \Rightarrow y^{(n+2)}(0) = 2ny^{(n-1)}(0)$$

as required. The initial conditions give
$y(0) = y^{(0)}(0) = 1$ and $y'(0) = y^{(1)}(0) = 0$.
From the ODE, setting $x = 0$ we obtain $y''(0) = y^{(2)}(0) = 0$. Hence the recurrence gives that
$n = 1 : y^{(3)}(0) = 2(1)y^{(0)}(0) = 2y(0) = 2$
$n = 2 : y^{(4)}(0) = 2(2)y^{(1)}(0) = 0$
$n = 3 : y^{(5)}(0) = 2(3)y^{(2)}(0) = 0$
$n = 4 : y^{(6)}(0) = 2(4)y^{(3)}(0) = 2^4 = 16$
Clearly, we will only get non-zero every third term, so
$n = 5,6 : y^{(7)}(0) = y^{(8)}(0) = 0$ and
$n = 7 : y^{(9)}(0) = 2(7)y^{(6)}(0) = 7 \cdot 2^5 = 224$
and so on. The Maclaurin series

$$y = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \cdots + \frac{y^{(n)}(0)}{n!}x^n + \cdots$$

gives the solution for this ODE:

$$y = 2 + \frac{16}{6!}x^6 + \frac{7 \cdot 2^5}{9!}x^9 + \cdots = 2 + \frac{1}{45}x^6 + \frac{1}{36 \cdot 45}x^9 + \cdots.$$

Quite a few people didn't manage to set up the recurrence. Of these, a few did solve the recurrence correctly. A lot of people didn't say why $y''(0) = 0$ but used it anyway. Some just skipped it and got odd results for higher terms.

b) If $u = f(\phi)$ where f is not specified, and $\phi = \dfrac{2x-y}{3xy}$, show that

$$y^2\frac{\partial u}{\partial y} + 2x^2\frac{\partial u}{\partial x} = 0\,,$$

SOLUTION

Begin with the chain rule:

$$\frac{\partial u}{\partial x} = \frac{du}{d\phi}\frac{\partial \phi}{\partial x} = \frac{du}{d\phi}\frac{3y^2}{(3xy)^2}\,,$$

and

$$\frac{\partial u}{\partial y} = \frac{du}{d\phi}\frac{\partial \phi}{\partial y} = \frac{du}{d\phi}\frac{(-6x^2)}{(3xy)^2}\,.$$

Hence

$$y^2\frac{\partial u}{\partial y} + 2x^2\frac{\partial u}{\partial x} = \frac{du}{d\phi}\left[y^2\frac{(-6x^2)}{(3xy)^2} + 2x^2\frac{3y^2}{(3xy)^2}\right] = 0\,,$$

as required.

Problem in original question led to many errors, not penalized. Many didn't see there was a problem and proceeded anyway. Mostly fine. Some did not see the simplification in the derivative: $\dfrac{\partial u}{\partial x} = \dfrac{du}{d\phi}\dfrac{1}{3x^2}$*, and similarly for the other derivative, leading to complications.*

c) A function of two variables is given as

$$f(x,y) = x(y+1)^2 - x^2 - x\,.$$

i) Find the stationary points of $f(x,y)$ and determine their nature using the Hessian determinant. [7]

SOLUTION

Begin with the partial derivatives, set equal to zero to find the stationary points:

$$\frac{\partial f}{\partial x} = (y+1)^2 - 2x - 1 = 0\,, \quad \frac{\partial f}{\partial y} = 2x(y+1) = 0\,,$$

The second of these gives immediate solutions $x = 0$ or $y = -1$. Substitute into the first equation:

$$x = 0 \Rightarrow (y+1)^2 - 1 = 0 \Rightarrow y+1 = \pm 1 \Rightarrow y = 0, -2$$

giving stationary points at $P_1 : (0,0)$ and $P_2 : (0,-2)$ and

$$y = -1 \Rightarrow -2x - 1 = 0 \Rightarrow x = -\frac{1}{2}$$

giving a third stationary point at $P_3 : (-\frac{1}{2}, -1)$.

To classify the stationary points, first obtain the second partial derivatives

$$f_{xx} = -2\,, \quad f_{yy} = 2x\,, \quad f_{xy} = f_{yx} = 2(y+1)\,,$$

Hence the Hessian determinant is

$$D = \begin{vmatrix} -2 & 2(y+1) \\ 2(y+1) & 2x \end{vmatrix} = -4x - 4(y+1)^2\,,$$

Evaluating the determinant at the stationary points we find:

$P_1 :\ H(0,0) = -4 < 0$ so P_1 is a saddle point;
$P_2 :\ H(0,-2) = -4 < 0$ so P_2 is a saddle point;
$P_3 :\ H(-\frac{1}{2},-1) = 2 > 0$, and $f_{xx}(-\frac{1}{2},-1) = -2 < 0$ so P_3 is a maximum.

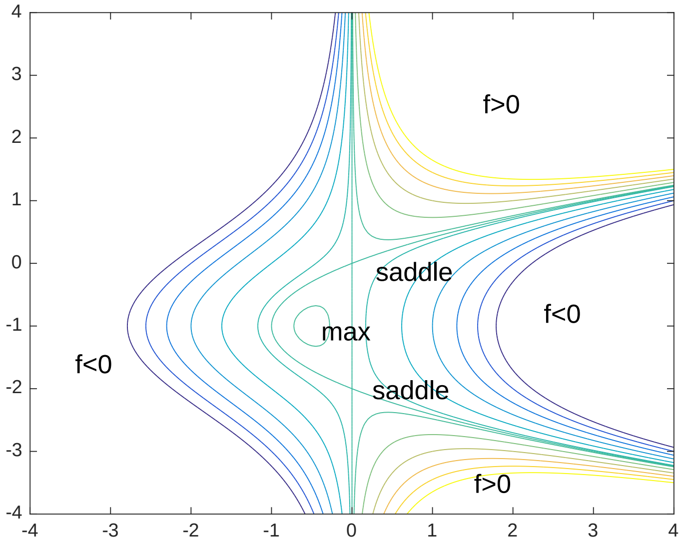
Many people took the equation $\frac{\partial f}{\partial y} = 2x(y+1) = 0$ *and concluded that* $(0,-1)$ *is a stationary point. Others solved the first equation and substituted into the second, leading to very complicated, and often incorrect, answers. Plugging these into the Hessian determinant then led to further problems.*

ii) Sketch the contours of the surface $z = f(x,y)$. [5]

SOLUTION

First set $f(x,y) = x(y+1)^2 - x^2 - x = x[(y+1)^2 - x - 1] = 0$, so contours with $f = 0$ when $x = 0$ or $x = (y+1)^2 - 1$: the $y-$axis and a horizontal parabola on $[-1,\infty]$. Check that, as expected, the saddle-points are the points of intersection of parabola and $y-$axis.

The maximum at $(-\frac{1}{2},-1)$ is in the area between the two zero contours, so $f > 0$ here, descending to $f = 0$ and then to $f < 0$ in adjacent areas. Crossing the positive $y-$axis from (say) $(-1,2)$ to $(1,2)$ we move from a region with $f < 0$ to $f = 0$ on the axis, and therefore continue with f increasing into a region where $f > 0$. Thus, the two areas in the first quadrant above the parabola and in the second quadrant below the parabola show contours with $f > 0$. This should all appear in the sketch:



Very variable answers. Many people ignored $x = 0$ and had contour lines crossing the $y-$axis. Other people did notice the contour $x = 0$ but still had contour lines following the parabola, crossing the $y-$axis. A number of people couldn't sketch a horizontally oriented parabola. The idea that contours meet at saddle points has bypassed many people.