

Master

IMPERIAL COLLEGE LONDON

E4.10
C2.1
SC4

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2004

MSc and EEE PART IV: M.Eng. and ACGI

PROBABILITY AND STOCHASTIC PROCESSES

Time allowed: 3:00 hours

There are SIX questions on this paper.
Answer FOUR questions.

**Any special instructions for invigilators and information for
candidates are on page 1.**

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Information for Candidates

The Kalman Filter

Consider signal and observation processes $\{x_k\}$ and $\{y_k\}$, respectively, that satisfy:

$$\begin{aligned}x_k &= Ax_{k-1} + e_{k-1} \\y_k &= Cx_k + v_k, \quad k = 1, 2, \dots\end{aligned}$$

Here, $\{e_k\}$ and $\{v_k\}$ are zero mean Gaussian processes with covariances

$$\text{cov}\{e_k\} = Q^{(s)} \quad \text{and} \quad \text{cov}\{v_k\} = Q^{(o)} \quad \text{for all } k.$$

The initial state x_0 is a Gaussian random variable with specified mean and covariance:

$$E[x_0] = \hat{x}_0 \quad \text{and} \quad \text{cov}\{x_0\} = P_0.$$

Assume that

x_0 , $\{e_k\}$ and $\{v_k\}$ are independent random variables and $Q^{(o)}$ is invertible.

The conditional mean \hat{x}_k and conditional variance P_k of x_k given y_1, \dots, y_k are related to the conditional mean \hat{x}_{k-1} and conditional variance P_{k-1} of x_{k-1} given y_1, \dots, y_{k-1} , via the intermediate variable $P_{k|k-1}$, by the following equations:

$$\begin{aligned}P_{k|k-1} &= AP_{k-1}A^T + Q^{(s)} \\P_k &= P_{k|k-1} - P_{k|k-1}C^T(CP_{k|k-1}C^T + Q^{(o)})^{-1}CP_{k|k-1} \\K(k) &= P_{k|k-1}C^T(CP_{k|k-1}C^T + Q^{(o)})^{-1} \\ \hat{x}_k &= A\hat{x}_{k-1} + K(k)(y_k - CA\hat{x}_{k-1}).\end{aligned}$$

1. (a) Possible failures in a communication link joining points a and b are represented by the state ('open' or 'closed') of switches S_1, \dots, S_5 in Figure 1(a). Assume that the switches fail (are open) independently and

$$P[S_i \text{ is closed}] = p \quad \text{for } i = 1, 2, \dots, 5$$

for some constant p , $0 < p < 1$. What is the probability that the path between a and b will be closed? [10]

Hint: Consider separately the cases ' S_3 is closed' and S_3 is open, i.e. use the formula

$$P[E] = P[E|S_3]P[S_3] + P[E|\bar{S}_3]P[\bar{S}_3]$$

where $E = \{\text{'there is a closed path from } a \text{ to } b'\}$ and, for $i = 1, \dots, 5$, $S_i = \{\text{' } S_i \text{ is closed'}\}$.

- (b) A discrete random signal S , that takes values $S = 1$ or $S = 2$, is transmitted at point A in Fig. 1(b). The signal is received at point B , after passage through a channel that is modelled as an amplifier with gain K . The amplifier fails randomly:

$$K = \begin{cases} 2 & \text{if amplifier is functioning} \\ 1 & \text{if amplifier fails (no amplification).} \end{cases}$$

Assume that amplifier failure is independent of the value of the signal and

$$\begin{aligned} P[S = 1] &= \alpha, & P[S = 2] &= (1 - \alpha) \\ P[K = 2] &= \beta, & P[K = 1] &= (1 - \beta) \end{aligned}$$

for some constants α , $0 < \alpha < 1$, and β , $0 < \beta < 1$.

The received signal at B is $R = 2$. What is the probability that the amplifier has failed? [10]

Hint: Calculate $P[R = 2|K = 1]$ and $P[R = 2|K = 2]$ and use Bayes Rule.

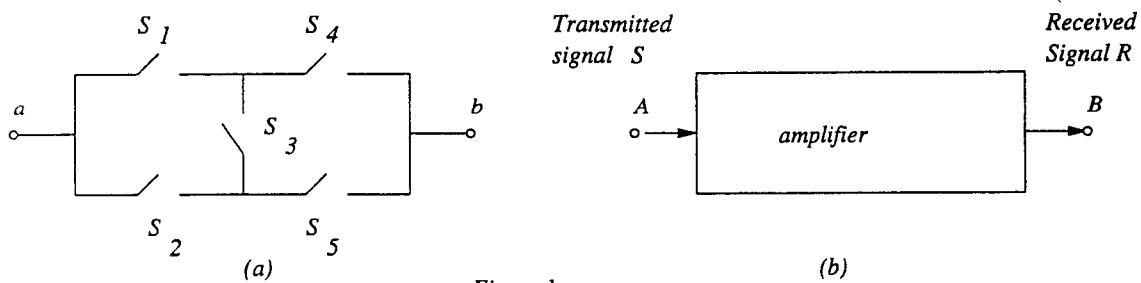


Figure 1

2. (a) A random variable $T(\omega)$ has the exponential probability density

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t}, & \text{for } t \geq 0 \\ 0, & \text{for } t < 0. \end{cases}$$

Here $\lambda (> 0)$ is a parameter.

Calculate the characteristic function of $T(\omega)$. Hence determine the mean m_T and the variance σ_T^2 of $T(\omega)$. [8]

2. (b) The lifetime $T(\omega)$ of an electronic device is modelled as a random variable with exponential distribution. For fixed $S > 0$ and $t > S$ calculate the conditional probability

$$P[T(\omega) \leq t | T(\omega) > S].$$

Hence calculate the conditional density of $T(\omega)$, given the event ' $T(\omega) \geq S$ '. [4]

Show that the conditional mean and variance of $T(\omega)$ given ' $T(\omega) \geq S$ ' (i.e. the expected life time and its variance, given the the device has not failed before time S) are

$$m_{T|T \geq S} = S + m_T \quad \text{and} \quad \sigma_{T|T \geq S}^2 = \sigma_T^2,$$

where m_T and σ_T^2 are the unconditional mean and variance of $T(\omega)$, calculated in (a). [6]

Comment on the suitability of modelling the lifetime of a device using the exponential distribution, in the light of your calculation. [2]

Hint: When evaluating integrals of the form

$$I = \int_S^\infty g(t) dt$$

use a change of variables ' $t' = t - S$ ', i.e. express the integral

$$I = \int_0^\infty g(t' + S) dt'.$$

Note also that you have formulae for the moments $\int_0^\infty t^k f_T(t) dt$, $k = 1, 2$, from part (a).

3. A noisy measurement $Y(\omega)$ is made of the position $X(\omega)$ of an object along a line. Assume that the measurement noise is additive, i.e.

$$Y(\omega) = X(\omega) + N(\omega).$$

Assume also that the noise $N(\omega)$ and the signal $X(\omega)$ are independent, that $X(\omega)$ is uniformly distributed on $[-a, a]$:

$$f_X(x) = \begin{cases} (1/2)a & \text{for } -a \leq x \leq a \\ 0 & \text{otherwise} \end{cases}$$

for some parameter $a > 0$, and that $N(\omega)$ is normally distributed, with zero mean and unit variance:

$$f_N(n) = \frac{1}{(2\pi)^{\frac{1}{2}}} e^{-\frac{1}{2}n^2}.$$

Derive expressions for the conditional density of $X(\omega)$ given $Y(\omega)$, $f_{X|Y}(x|y)$. [4]

Hence derive expressions for the (nonlinear) least squares estimate $\hat{X}(y)$ of $X(\omega)$ given $Y(\omega) = y$. [6]

(In these expressions, you do not have to evaluate the integrals involved).

Show that, as $a \rightarrow \infty$,

$$\hat{X}(y) \rightarrow \hat{x}_{ML}$$

where, for fixed y , \hat{x}_{ML} maximizes the ‘likelihood function’

$$x \rightarrow f_{Y|X}(y|x)$$

[4]

Hint: Obtain a formula for the joint probability density $f_{XY}(x, y)$ by using the formula $f_{XY}(x, y) = f_{Y|X}(y|x)f_X(x)$.

4. The position $X(\omega)$ of an object in one dimensional space needs to be estimated from noisy measurements. For this purpose, K cheap, identical sensors are to be used.

Assume that the i^{th} sensor measurement $Y_i(\omega)$ is related to $X(\omega)$ according to

$$Y_i(\omega) = X(\omega) + N_i(\omega) \quad \text{for } i = 1, 2, \dots, K,$$

for random variables $N_1(\omega), \dots, N_K(\omega)$. Assume further that $X(\omega), N_1(\omega), \dots, N_K(\omega)$ are zero mean, independent random variables and

$$\text{var}\{X\} = \sigma_X^2 \quad \text{and} \quad \text{var}\{N_1\} = \dots = \text{var}\{N_K\} = \sigma_N^2.$$

Determine

- (i): the linear least squares estimate \hat{X} of X given Y_1, \dots, Y_n [12]
 (ii): the mean square estimation error of \hat{X} . [6]

Now assume that $\sigma_N^2 = 1 \text{ m}^2$ and $\sigma_X^2 = 0.5 \text{ m}^2$. Suppose that we require

$$E|\hat{X} - X|^2 \leq 0.1 \text{ m}^2.$$

Determine the minimum number of sensors K required to achieve this specification. [2]

Hint: Determine the linear least squares estimate \hat{X} from first principles, *not* by using the general formula for multi-dimensional linear least squares estimation. Use the fact that, by symmetry, all the weights in the linear least squares estimator are the same.

5. (a) Consider a stationary scalar output process $\{y_k\}$ and vector state process $\{x_k\}$ governed by the equations

$$\begin{cases} x_{k+1} = Ax_k + be_k \\ y_k = c^T x_k. \end{cases} \quad (1)$$

Here, A is a given $n \times n$ matrix and b and c are given n -vectors. $\{e_k\}$ is a sequence of zero mean, uncorrelated random variables, each with unit variance. Develop formulae for the covariance matrix of x_k and the variance of y_k :

$$R_x(0) = E[x_k x_k^T] \quad \text{and} \quad R_y(0) = E[y_k^2].$$

[8]

- (b) Now suppose the matrices in (5.1) are as follows:

$$A = \begin{bmatrix} -k & -\alpha \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad c^T = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Here, α is a system constant. k is a design parameter (the gain in some inner feedback loop) that is to be chosen to reduce the variance of $\{y_k\}$.

- (i): For fixed α and k , obtain a formula for the variance of y_k . [8]

- (ii): Assume $\alpha = 0.25$. Determine the value of k that minimizes $E[y_k^2]$. [4]

You should assume that all the values of α and k considered are such that (1) is a stable dynamical system.

6. The position of a stationary object along a line is modelled as the scalar Gaussian random variable x_0 . Measurements y_k of the position are taken at times $k = 1, 2, \dots$. It is assumed that

$$y_k = x_0 + v_k$$

where $\{v_k\}$ is a sequence of zero mean, independent Gaussian random variables, each with variance σ_0^2 . Assume also that x_0 has zero mean and variance P_0 .

Let \hat{x}_k and P_k be the conditional mean and variance of x_0 , given y_1, \dots, y_k , for $k = 1, 2, \dots$. Use the Kalman filter equations to derive the following recursive equations for P_k^{-1}

$$P_{k+1}^{-1} = \sigma_0^{-2} + P_k^{-1} \quad (2)$$

and

$$\hat{x}_{k+1} = (1 - \sigma_0^{-2} P_{k+1}) \hat{x}_k + \sigma_0^{-2} P_{k+1} y_{k+1}.$$

[16]

By using (2), or otherwise, show that

$$P_k \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

Hint: Introduce the ‘state equation’ $x_{k+1} = Ax_k + Be_k$ with $A = 1$ and $B = 0$.

[4]

E4.10, C21, SC4 Prob. + Stochastic Processes Exam 2004. Answers

(a) Write S_i = "switch S_i is closed", for $i=1, \dots, 5$. Note that, for any i, j , $i \neq j$,

$$P(S_i \cup S_j) = 1 - P(\bar{S}_i \cap \bar{S}_j) \quad (\text{by de Morgan's rule})$$

$$= 1 - P(\bar{S}_i) P(\bar{S}_j) \quad (\text{by independence})$$

$$= 1 - (1-p)^2$$

The event E = "there is a closed path from a to b " has probability

$$P[E|S_3] P[S_3] + P[E|\bar{S}_3] P[\bar{S}_3] \quad (\bar{S}_3 = \text{complement of } S_3)$$

$$= P[(S_1 \cup S_2) \cap (S_4 \cup S_5)] p + P[(S_1 \cap S_4) \cup (S_2 \cap S_5)] (1-p)$$

$$= P[S_1 \cup S_2] \cdot P[S_4 \cup S_5] p + (1 - P[\bar{S}_1 \cap \bar{S}_4]) (1 - P[\bar{S}_2 \cap \bar{S}_5]) (1-p)$$

$$= (1 - (1-p)^2)^2 p + (1 - (1-p^2)^2) (1-p)$$

$$= (2p - p^2)^2 p + (2p^2 - p^4) (1-p) = (2-p)^2 p^3 + (2-p^2) (1-p) p^2$$

$$= 4p^3 - 4p^4 + p^5 + 2p^2 - 2p^3 - p^4 + p^5 = 2p^2 + 2p^3 - 5p^4 + 2p^5$$

$$= p^2 (2 + 2p - 5p^2 + 2p^3)$$

(b) $P[R=2|K=1] = P[S=2] = (1-\alpha)$

Also, $P[R=2|K=2] = P[S=1] = \alpha$, $P[K=1] = (1-\beta)$, $P[K=2] = \beta$

By Bayes Rule

$$P[\text{'amplifier fails'} | R=2] = P[K=1 | R=2]$$

$$= \frac{P[R=2|K=1] P[K=1]}{P[R=2|K=1] P[K=1] + P[R=2|K=2] P[K=2]}$$

$$= \frac{(1-\alpha)(1-\beta)}{(1-\alpha)(1-\beta) + \alpha\beta}$$

Prob. + Stoch. Processes Exam 2004, Answers

2/6

$$2(a) \quad \Phi_T(\theta) = E\{e^{j\theta T(\omega)}\} = \int_0^\infty e^{j\theta t} \lambda e^{-\lambda t} dt = \lambda \int_0^\infty e^{-(\lambda - j\theta)t} dt$$

$$= -\lambda (\lambda - j\theta)^{-1} \exp\{(\lambda - j\theta)t\} \Big|_{t=0}^\infty = \lambda (\lambda - j\theta)^{-1}$$

$$\frac{d}{d\theta} \Phi_T(\theta) \Big|_{\theta=0} = \frac{+j\lambda}{(\lambda - j\theta)^2} \Big|_{\theta=0} = \frac{j}{\lambda} = j m_T \Rightarrow m_T = \frac{1}{\lambda}$$

$$\frac{d^2}{d\theta^2} \Phi_T(\theta) \Big|_{\theta=0} = \frac{2\lambda j^2}{(\lambda - j\theta)^3} \Big|_{\theta=0} = \frac{2j^2}{\lambda^2} = j^2 E\{T^2\} \Rightarrow E\{T^2\} = \frac{2}{\lambda^2}$$

$$\text{So } \sigma_T^2 = E\{T^2\} - m_T^2 = \frac{1}{\lambda^2}$$

$$(b) \quad P[T(\omega) \leq t | T(\omega) > s] = \frac{P[s < T(\omega) \leq t]}{P[T(\omega) > s]} = \frac{\int_s^t \lambda e^{-\lambda t} dt}{\int_s^\infty \lambda e^{-\lambda t} dt}$$

$$= -e^{-\lambda t} \Big|_s^t / e^{-\lambda s} = (e^{-\lambda s} - e^{-\lambda t}) / e^{-\lambda s} = 1 - e^{-\lambda(t-s)} \quad (\text{for } t > s)$$

Hence

conditional density of $T(\omega)$, given $T(\omega) > s$ is

$$[4] \quad f_{T|T>s}(t) = \frac{d}{dt} \{ \dots \} = \begin{cases} \lambda e^{-\lambda(t-s)} & t \geq s \\ 0 & \text{otherwise} \end{cases}$$

We have

$$m_{T|T>s} = \int_s^\infty \lambda t e^{-\lambda(t-s)} dt = \int_0^\infty \lambda (t+s) e^{-\lambda t} ds$$

$$= \int_0^\infty \lambda t e^{-\lambda t} ds + s \lambda \int_0^\infty e^{-\lambda t} ds$$

$$= m_T + s e^{-\lambda t} \Big|_0^\infty = s + \frac{1}{\lambda} \quad (\text{by part (a)})$$

$$= s + m_T$$

Also,

$$\sigma_{T|T>s}^2 = \int_s^\infty \lambda t^2 e^{-\lambda(t-s)} dt - m_{T|T>s}^2$$

$$= \int_0^\infty \lambda (t+s)^2 e^{-\lambda t} dt - m_{T|T>s}^2 = \int_0^\infty \lambda t^2 e^{-\lambda t} dt + 2s \int_0^\infty \lambda t e^{-\lambda t} dt + s^2 \int_0^\infty \lambda e^{-\lambda t} dt - m_{T|T>s}^2$$

$$= \frac{2}{\lambda^2} + 2s \times \frac{1}{\lambda} + s^2 - (s + \frac{1}{\lambda})^2$$

$$[6] \quad = \frac{1}{\lambda^2} + (\frac{1}{\lambda^2} + 2s \times \frac{1}{\lambda} + s^2) - (\frac{1}{\lambda^2} + 2s \times \frac{1}{\lambda} + s^2) = \frac{1}{\lambda^2}$$

$$= \sigma_T^2$$

Notice that, if the device has not failed up to time s ,

then its remaining expected lifetime $m_{T|T>s} - s$ is the same as when it was new. This is a reasonable model, over the medium term, for many electronic devices. However it can be expected to be a poor model in the long term.

[2]

Prob + Stoch. Processes Exam 2004. Answers

3. $Y = X + N$. Since X and N are independent, it follows that

$$f_{Y|X}(y|x) = \frac{1}{(2\pi)^{1/2}} \exp\left\{-\frac{1}{2}(y-x)^2\right\}$$

Since $f_X(x) = \begin{cases} (2a)^{-1} & \text{if } -a \leq x \leq +a \\ 0 & \text{otherwise} \end{cases}$

It follows $f_{XY}(x,y) = (2a)^{-1} (2\pi)^{-1/2} \exp\left\{-\frac{1}{2}(y-x)^2\right\}$

But that the conditional density of X given Y is (for $-a \leq x \leq a$)

$$f_{X|Y}(x|y) = \frac{f_{XY}}{f_Y} = \frac{(2a)^{-1} (2\pi)^{-1/2} \exp\left\{-\frac{1}{2}(y-x)^2\right\}}{(2a)^{-1} (2\pi)^{-1/2} \int_{-a}^{+a} \exp\left\{-\frac{1}{2}(y-x)^2\right\} dx}$$

We have

$$f_{X|Y}(x|y) = \begin{cases} \frac{\frac{1}{(2\pi)^{1/2}} \exp\left\{-\frac{1}{2}(y-x)^2\right\}}{\frac{1}{(2\pi)^{1/2}} \int_{-a}^{+a} \exp\left\{-\frac{1}{2}(y-x)^2\right\} dx} & -a \leq x \leq +a \\ 0 & \text{otherwise} \end{cases}$$

The conditional mean of X given Y is therefore

$$\hat{X} (= \int x f_{X|Y} dx) = \frac{(a)}{(b)}$$

where $(a) = \frac{1}{(2\pi)^{1/2}} \int_{-a}^{+a} x \exp\left\{-\frac{1}{2}(y-x)^2\right\} dx$

and

$$(b) = \frac{1}{(2\pi)^{1/2}} \int_{-a}^{+a} \exp\left\{-\frac{1}{2}(y-x)^2\right\} dx$$

As $a \rightarrow \infty$

$$(a) \rightarrow \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} x \exp\left\{-\frac{1}{2}(y-x)^2\right\} dx = y$$

$$(b) \rightarrow \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \exp\left\{-\frac{1}{2}(y-x)^2\right\} dx = 1$$

(We have used here the facts that $\frac{1}{(2\pi)^{1/2}} \exp\left\{-\frac{1}{2}(y-x)^2\right\}$ is a probability density with mean y .) We see that $\hat{X} \rightarrow y$.

But \hat{X}_{ml} maximizes $x \rightarrow f_{Y|X}(y|x) = \frac{1}{(2\pi)^{1/2}} \exp\left\{-\frac{1}{2}(y-x)^2\right\}$.

Clearly $\hat{X}_{\text{ml}} = y$, so we have shown

$$\hat{X} \rightarrow \hat{X}_{\text{ml}} \quad \text{as} \quad a \rightarrow \infty$$

Prob + Stochastic Processes Exam 2004. Answers

4 Since all random variables involved are zero mean, the 'constant' component in the linear least squares estimator is zero. By symmetry,

$$\hat{x} = \alpha \sum_{i=1}^n Y_i$$

The mean square error is

$$\begin{aligned} J(\alpha) &= E\left[\left(X - \alpha \sum_{i=1}^n Y_i\right)^2\right] = E\left[\left(X - \alpha \sum_{i=1}^n (X + N_i)\right)^2\right] \\ &= E\left[\left((1 - \alpha n)X - \alpha \sum_{i=1}^n N_i\right)^2\right] \\ &= (1 - \alpha n)^2 E[X^2] + \alpha^2 n E[N_i^2] \\ &= (\alpha n - 1)^2 \sigma_x^2 + \alpha^2 n \sigma_N^2 \end{aligned}$$

Minimizing parameter, α^* , satisfies

$$\frac{d}{d\alpha} J(\alpha^*) = 0, \quad \text{i.e.} \quad 2(\alpha^* n - 1)n\sigma_x^2 + 2\alpha^* n \sigma_N^2 = 0$$

$$\text{whence } \alpha^* = \sigma_x^2 / (n\sigma_x^2 + \sigma_N^2).$$

[12] Linear least squares estimate is $\hat{x} = \frac{\sigma_x^2}{n\sigma_x^2 + \sigma_N^2} \sum_{i=1}^n Y_i$

The mean square error is

$$\begin{aligned} J(\alpha^*) &= \left(\frac{n\sigma_x^2}{n\sigma_x^2 + \sigma_N^2} - 1\right)^2 \sigma_x^2 + \frac{n\sigma_x^4 \sigma_N^2}{(n\sigma_x^2 + \sigma_N^2)^2} \\ &= \frac{\sigma_N^4 \sigma_x^2 + n\sigma_x^4 \sigma_N^2}{(n\sigma_x^2 + \sigma_N^2)^2} = \frac{\sigma_x^2 \sigma_N^2 (\sigma_N^2 + n\sigma_x^2)}{(n\sigma_x^2 + \sigma_N^2)^2} = \frac{\sigma_x^2 \sigma_N^2}{n\sigma_x^2 + \sigma_N^2} \end{aligned}$$

For $\sigma_N^2 = 1 \text{ m}^2$, $\sigma_x = 0.5 \text{ m}^2$, $J(\alpha^*) = \frac{0.5}{1 + 0.5n}$

We require $J(\alpha^*) \leq 0.1 \text{ m}^2$, i.e.

$$0.5 \leq 0.1 \times (1 + 0.5n)$$

$$\text{or } 5 \leq 1 + 0.5n \quad \text{or } 8 \leq n$$

[2] The least number of sensors required then is 8

(a) Prob. + Stochastic Processes. Exam 2004. Answers

5 (i) We have $E\{x_k, x_{k+1}^T\} = E\{(Ax_k + be_k)(Ax_k + be_k)^T\}$ (from state equations)

But x_k is zero mean and a linear function of e_{k-1}, e_{k-2}, \dots . Since the e_k 's are uncorrelated and zero mean, $E\{e_k x_k^T\} = 0$. Expanding, we have

$$R_x(0) = E\{x_{k+1} x_{k+1}^T\} = A E\{x_k x_k^T\} A^T + 0 + b E\{e_k e_k^T\} b^T = A R_x(0) A^T + b b^T.$$

[8] But then $R_y(0) = E\{C^T x_k x_k^T C\} = C^T R_x(0) C$. Lyapunov equation

(b) (i) For $A = \begin{bmatrix} -k & -\alpha \\ 1 & 0 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, the Lyapunov equation is

$$\begin{aligned} (i) P &= \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} = \begin{bmatrix} -k & -\alpha \\ 1 & 0 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} \begin{bmatrix} -k & 1 \\ -\alpha & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} k^2 P_{11} + 2\alpha k P_{12} + \alpha^2 P_{22} & -k P_{11} - \alpha P_{12} \\ -k P_{11} - \alpha P_{12} & P_{11} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

Equating entries in this matrix equation gives.

$$P_{11} = k^2 P_{11} + 2\alpha k P_{12} + \alpha^2 P_{22} + 1$$

$$P_{12} = -k P_{11} - \alpha P_{12}$$

$$P_{22} = P_{11}$$

We have $P_{12} = -\frac{k P_{11}}{1+\alpha}$

Hence $P_{11} = (k^2 - \frac{2\alpha k^2}{1+\alpha} + \alpha^2) P_{11} + 1$

and

$$P_{11} = \frac{1}{1 - (1 - \frac{2\alpha}{1+\alpha})k^2 - \alpha^2}$$

[8] Then $R_y(0) = C^T R_x(0) C = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = P_{11}$

$$= \frac{1}{1 - (1 - \frac{2\alpha}{1+\alpha})k^2 - \alpha^2}$$

for $\alpha = 0.25$

[4] (i) Since, $1 - \frac{2\alpha}{1+\alpha} > 0$, $R_y(0)$ is maximized by $\underline{k=0}$.

Prob. + Stochastic Processes, Exam 2004, Answers

6 Let $\{x_k\}$ be the sequence $\{x_0, x_1, \dots\}$. We model $\{x_k\}$ and $\{y_k\}$ as

$$\begin{cases} x_{k+1} = a x_k + b e_k \\ y_k = c x_k + v_k \end{cases}$$

with $a=1$, $b=0$ and $c=1$. Here $\{v_k\}$ is white noise with $\text{cov}\{v_k\} = \sigma_0^2$.

The Kalman filter equations give update equations for the conditional mean and covariance of x_k , \hat{x}_k and P_k respectively:

$$P_{k+1|k} = A P_k A^T + Q = P_k$$

$$P_{k+1} = P_k - P_k C^T (C P_k C^T + \sigma_0^2)^{-1} C P_k = P_k - \frac{P_k^2}{P_k + \sigma_0^2} = \frac{\sigma_0^2 P_k}{P_k + \sigma_0^2} \quad (1)$$

$$K_{k+1} = P_k C (C P_k C^T + \sigma_0^2)^{-1} = \frac{P_k}{P_k + \sigma_0^2}$$

and

$$\hat{x}_{k+1} = \hat{x}_k + \left(\frac{P_k}{P_k + \sigma_0^2} \right) (y_{k+1} - \hat{x}_k). \quad (2)$$

We can write (2) as

$$\hat{x}_{k+1} = \frac{\sigma_0^2}{P_k + \sigma_0^2} \hat{x}_k + \frac{P_k}{P_k + \sigma_0^2} y_{k+1} = \underbrace{\left(1 - \frac{\sigma_0^2}{P_k + \sigma_0^2} \right)}_{\frac{P_k}{P_k + \sigma_0^2}} \hat{x}_k + \frac{\sigma_0^2}{P_k + \sigma_0^2} y_{k+1}$$

(we have used (1)).

Also

$$[16] \quad \underline{P_{k+1}^{-1} = \sigma_0^{-2} + P_k^{-1}}, \quad k = 0, 1, \dots \quad (3)$$

(3) implies

$$P_k^{-1} = \sigma_0^{-2} + k P_0^{-1}$$

It follows from this equation that

$$P_k^{-1} \rightarrow \infty \quad \text{as } k \rightarrow \infty$$

Hence

$$[4] \quad P_k \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$