

EEE/EIE PART I: MEng, BEng and ACGI

Time allowed: 2:00 hours

**Q1 carries 40% of the marks. Questions 2 and 3 carry equal marks (30% each).**

Examiners responsible      First Marker(s) :      D.M. Brookes  
Second Marker(s) :      P. Georgiou

# ANALYSIS OF CIRCUITS

## Information for Candidates:

The following notation is used in this paper:

1. The voltage waveform at node  $X$  in a circuit is denoted by  $x(t)$ , the phasor voltage by  $X$  and the root-mean-square (or RMS) phasor voltage by  $\tilde{X} = \frac{X}{\sqrt{2}}$ . The complex conjugate of  $X$  is  $X^*$ .
2. Component and source values in a circuit are normally given in Ohms, Farads, Henrys, Volts or Amps with the unit symbol omitted. Where an imaginary number is specified, it represents the complex impedance or phasor value.
3. Times are given in seconds unless otherwise stated.
4. Unless otherwise indicated, frequency response graphs should use a linear axis for phase and logarithmic axes for frequency and magnitude.
5. The real and imaginary parts of a complex number,  $X$ , are written  $\Re(X)$  and  $\Im(X)$  respectively.

1. a) i) Using nodal analysis, calculate the voltage at node  $X$  of Figure 1.1. [ 3 ]
- ii) Calculate the power absorbed by each of the three sources in the circuit of Figure 1.1. [ 3 ]

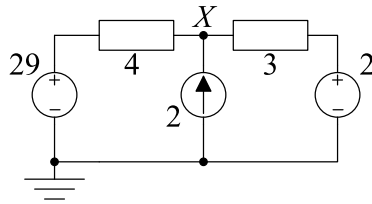


Figure 1.1

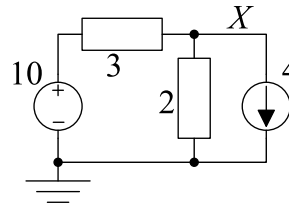


Figure 1.2

- b) Use the principle of superposition to find the voltage  $X$  in Figure 1.2. [ 5 ]
- c) Draw the Thévenin equivalent circuit of the two-terminal network in Figure 1.3 and find the values of its components. [ 5 ]

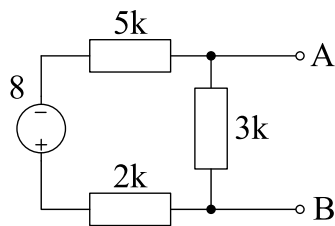


Figure 1.3

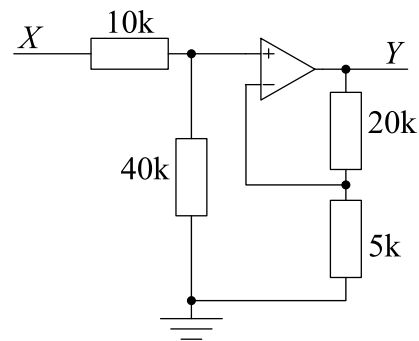


Figure 1.4

- d) Assuming the opamp in the circuit of Figure 1.4 is ideal, give an expression for  $Y$  in terms of  $X$ . [ 4 ]
- e) Determine  $R_1$  and  $R_2$  in Figure 1.5 so that  $Y = 0.1X$  and the parallel combination of  $R_1$  and  $R_2$  has an impedance of  $50\Omega$ . [ 5 ]

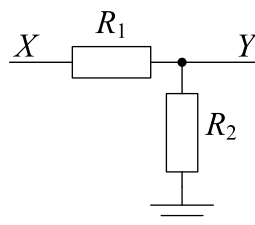


Figure 1.5

- f) i) The diagram of Figure 1.6 shows a 50 Hz AC source with r.m.s. voltage  $\tilde{V} = 230 \text{ V}$  driving a load with impedance  $3.5 + 2.5j \Omega$ . Determine the complex power supplied to the load,  $S = \tilde{V} \times \tilde{I}^*$ , and also the power factor,  $\lambda = \frac{\Re(S)}{|S|}$ . [ 3 ]
- ii) A capacitor,  $C$ , is now connected across the load, as indicated in Figure 1.7. Calculate the value of  $C$  (in  $\mu\text{F}$ ) required to increase the power factor to 0.95. [ 3 ]



Figure 1.6

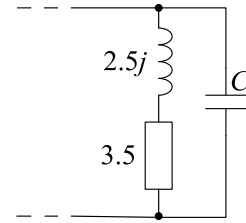


Figure 1.7

- g) Determine the gain,  $\frac{Y}{X}$ , for the block diagram shown in Figure 1.8. The rectangular blocks are drawn with inputs at the left and outputs at the right and have gains of  $F$  and  $G$  respectively. The open circles represent adder/subtractors; their inputs have the signs indicated on the diagram and their outputs are  $V$  and  $Y$  respectively. [ 4 ]

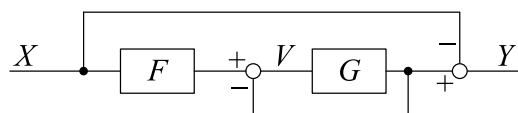


Figure 1.8

- h) Figure 1.10 shows a transmission line of length 10m that is terminated in a resistive load,  $R$ , with reflection coefficient  $\rho = -0.4$ . The line has a propagation velocity of  $u = 2 \times 10^8 \text{ m/s}$ . At time  $t = 0$ , a forward-travelling (i.e. left-to-right) pulse arrives at  $X$  with amplitude 5 V and duration 100ns, as shown in Figure 1.9.

Draw a dimensioned sketch of  $y(t)$  for  $0 \leq t \leq 250 \text{ ns}$  where  $y(t)$  is the waveform at  $Y$ , a point 7m from the end of the line. Label the voltages taken by  $y(t)$  and the times at which it changes. Assume that no reflections occur at point  $X$ . [ 5 ]

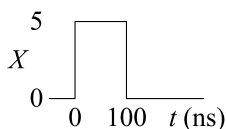


Figure 1.9

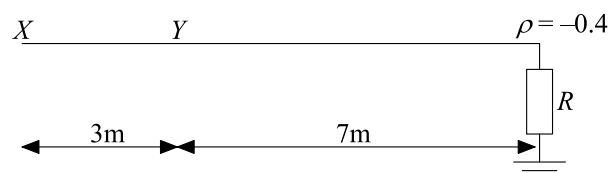


Figure 1.10

2. a) i) Assuming that the op-amp in Figure 2.1 is ideal, show that the output voltage at time  $t$  is given by [ 3 ]

$$y(t) = y(0) - \frac{1}{R_1 C} \int_0^t x(\tau) d\tau.$$

- ii) If  $x(t)$  has a constant value of 12 V and  $y(0) = 5$  V, determine the value of  $R_1$  such that  $y(t) = -5$  V when  $t = 1$  ms. [ 3 ]

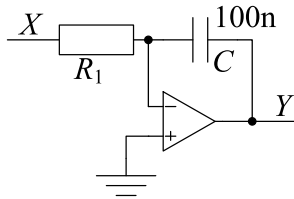


Figure 2.1

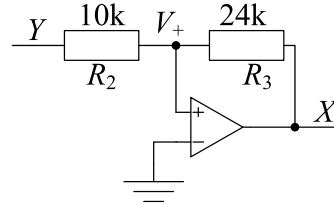


Figure 2.2

- b) In the positive-feedback circuit of Figure 2.2, the op-amp has zero input current and its output voltage,  $x$ , is given by

$$x = \begin{cases} +12 \text{ V} & \text{if } v_+ > v_- \\ -12 \text{ V} & \text{if } v_+ < v_- \end{cases}$$

where  $v_+$  and  $v_-$  are the voltages at the op-amp input terminals.

- i) Show that  $v_+ = \frac{5}{17}x + \frac{12}{17}y$ . [ 2 ]
- ii) Hence show that the op-amp output,  $x$ , is given by

$$x = \begin{cases} +12 \text{ V} & \text{if } y > a \\ +12 \text{ V or } -12 \text{ V} & \text{if } -a < y < a \\ -12 \text{ V} & \text{if } y < -a \end{cases}$$

Determine the value of  $a$  and explain what determines the value of  $x$  when  $-a < y < a$ . [ 4 ]

[This question is continued on the next page]

- c) The circuits of parts a) and b) are combined to form the oscillator circuit shown in Figure 2.3. The resistor  $R_1$  has the value determined in part a).  
Given the initial conditions  $x(0) = +12\text{ V}$  and  $y(0) = 0\text{ V}$ , draw a dimensioned sketch of the waveforms  $x(t)$  and  $y(t)$  for  $0 < t < 2\text{ ms}$ . Your sketch should label the peak value of  $y(t)$  and the times when  $x(t)$  switches. [ 6 ]
- d) The output of the oscillator,  $y$ , forms the input to the circuit shown in Figure 2.4. You may assume that the diodes have a voltage drop of  $0.7\text{ V}$  when conducting and a current of zero when their forward voltage is less than  $0.7\text{ V}$ .
- Give an expression for  $v$  in terms of  $y$  when neither diode is conducting. By considering the voltage across  $R_6$ , determine the range of input voltages,  $y$ , for which neither diode will be conducting. [ 4 ]
  - Give an expression for  $v$  in terms of  $y$  when diode  $D_1$  is conducting. Determine the range of input voltages,  $y$ , for which diode  $D_1$  will be conducting. [ 4 ]
  - Draw a dimensioned sketch showing the graph of  $v$  versus  $y$  for the range  $-5 < y < +5$ . [ 4 ]

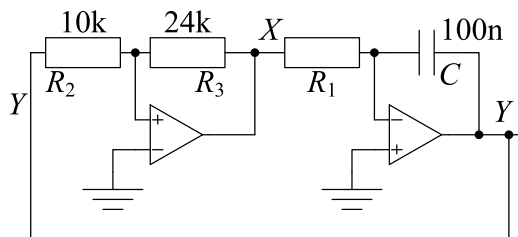


Figure 2.3

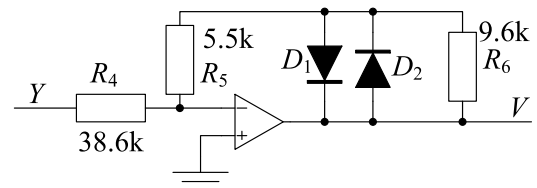


Figure 2.4

3. Figure 3.1 shows a circuit whose input and output voltages are  $x(t)$  and  $y(t)$  respectively.

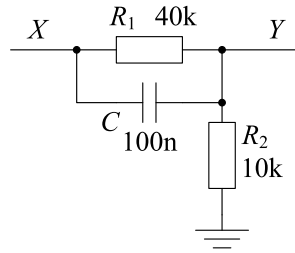


Figure 3.1

- a) i) Determine the frequency response,  $G(j\omega) = \frac{Y}{X}$ . [ 4 ]
- ii) Draw a dimensioned sketch of the straight-line approximation to the magnitude response,  $|G(j\omega)|$ , showing the values of any corner frequencies (in rad/s) and the gains of any horizontal portions of the response (in dB). [ 4 ]
- iii) Draw a dimensioned sketch of the straight-line approximation to the phase response,  $\angle G(j\omega)$ , showing the values of any corner frequencies (in rad/s) and the phase of any horizontal portions of the response (in rad). [ 4 ]
- b) Determine the time constant of the circuit. [ 3 ]
- c) If the input, shown in Figure 3.2, is given by

$$x(t) = \begin{cases} -5 & \text{for } t < 0 \\ +5 & \text{for } t \geq 0 \end{cases},$$

- i) determine expressions for  $y(t)$  both for  $t < 0$  and for  $t \geq 0$ , [ 6 ]
- ii) draw a dimensioned sketch of the waveform of  $y(t)$ . [ 3 ]
- d) If the input, shown in Figure 3.3, is given by

$$x(t) = \begin{cases} \sin(500t) & \text{for } t < 0 \\ \sin(1000t) & \text{for } t \geq 0 \end{cases},$$

determine expressions for  $y(t)$  both for  $t < 0$  and for  $t \geq 0$ . [ 6 ]

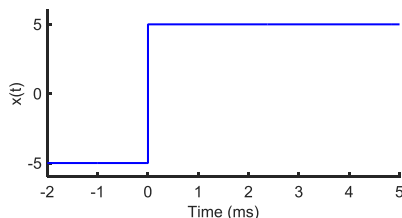


Figure 3.2

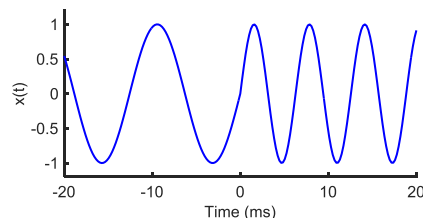


Figure 3.3

## ANALYSIS OF CIRCUITS

### \*\*\*\* Solutions 2016 \*\*\*\*

#### Information for Candidates:

The following notation is used in this paper:

1. The voltage waveform at node  $X$  in a circuit is denoted by  $x(t)$ , the phasor voltage by  $X$  and the root-mean-square (or RMS) phasor voltage by  $\hat{X} = \frac{X}{\sqrt{2}}$ . The complex conjugate of  $X$  is  $X^*$ .
2. Component and source values in a circuit are normally given in Ohms, Farads, Henrys, Volts or Amps with the unit symbol omitted. Where an imaginary number is specified, it represents the complex impedance or phasor value.
3. Times are given in seconds unless otherwise stated.
4. Unless otherwise indicated, frequency response graphs should use a linear axis for phase and logarithmic axes for frequency and magnitude.
5. The real and imaginary parts of a complex number,  $X$ , are written  $\Re(X)$  and  $\Im(X)$  respectively.

**Key: B=bookwork, U=unseen example**



1. a) i) Using nodal analysis, calculate the voltage at node X of Figure 1.1. [ 3 ]

---

[U] KCL at node X gives

$$\begin{aligned}\frac{X-29}{4} + \frac{X-2}{3} - 2 &= 0 \\ \Rightarrow 3X - 87 + 4X - 8 - 24 &= 0 \\ \Rightarrow 7X &= 119 \\ \Rightarrow X &= 17\end{aligned}$$

*None yet.*

---

- ii) Calculate the power absorbed by each of the three sources in the circuit of Figure 1.1. [ 3 ]
- 

[U] The current through the  $4\Omega$  resistor is  $\frac{29-X}{4} = \frac{12}{4} = 3\text{ A}$  from left to right. The power absorbed by the  $29\text{ V}$  source is therefore  $29 \times -3 = -87\text{ W}$ .

The current through the  $4\Omega$  resistor is  $3 + 2 = 5\text{ A}$  (by KCL). The power absorbed by the  $2\text{ V}$  source is therefore  $2 \times +5 = 10\text{ W}$ .

The power absorbed by the  $2\text{ A}$  source is  $17 \times -2 = -34\text{ W}$ .

Although not requested, the power absorbed by the  $4\Omega$  and  $3\Omega$  resistors is  $3^2 \times 4 = 36\text{ W}$  and  $5^2 \times 3 = 75\text{ W}$  respectively. Summing all the powers gives  $36 + 75 - 87 + 10 - 34 = 0$  as expected.

Several used superposition (i.e. calculated the power absorbed by one source when the other two sources were set to zero). However although superposition applies to currents and voltages, it doesn't apply to powers, so this gives the wrong answer. Many people had the wrong sign for one or more of the powers; the power absorbed is always  $V \times I$  where  $V$  and  $I$  are measured using the passive sign convention, i.e. their arrows are in opposite directions. Note that, in this circuit, the  $2\text{ V}$  source is absorbing power (as when charging a rechargeable battery); some people assumed that all sources always supply power. Surprisingly some tried to use the power expressions  $\frac{V^2}{R}$  or  $I^2 R$  to calculate the power absorbed by the sources (e.g.  $\frac{29^2}{4}$  for the  $29\text{ V}$  source); these expressions only apply to the power absorbed by a resistor. A few people didn't read the question and calculated the power absorbed by the resistors rather than the power absorbed by the sources. Several people calculated the current through the two voltage sources using the  $2\text{ A}$  source followed by a current divider; this is incorrect since, in a current divider, the resistors must be in parallel (i.e. connected at both ends).

---

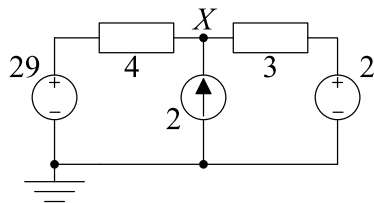


Figure 1.1

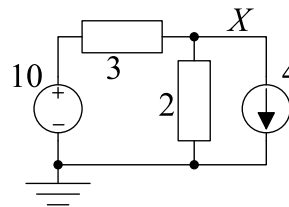


Figure 1.2

- b) Use the principle of superposition to find the voltage  $X$  in Figure 1.2. [ 5 ]

[U] If we open circuit the 4A current source, the 3Ω and 2Ω resistors form a potential divider and  $X_1 = \frac{2}{3+2} \times 10 = +4\text{V}$ .

If we short circuit the 10V voltage source, the 3Ω and 2Ω resistors are in parallel and equal  $\frac{3 \times 2}{3+2} = 1.2\Omega$ . The voltage at  $X$  is then  $X_2 = -4 \times 1.2 = -4.8\text{V}$ .

By superposition, the total voltage is therefore  $X = X_1 + X_2 = 4 - 4.8 = -0.8\text{V}$ .

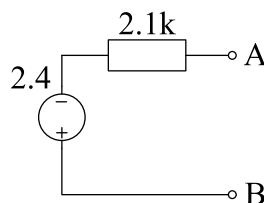
The most common mistake was to have  $X_2 = +4.8\text{V}$  which results in a final incorrect answer of  $X = X_1 + X_2 = 4 + 4.8 = 8.8\text{V}$ .

- c) Draw the Thévenin equivalent circuit of the two-terminal network in Figure 1.3 and find the values of its components. [ 5 ]

[U] We can find the open circuit voltage by treating the circuit as a potential divider; this gives  $V_{AB} = \frac{3}{5+3+2} \times 8 = 2.4\text{V}$ .

We can find the Thévenin resistance by short-circuiting the voltage source. The remaining network has a resistance of  $3 \parallel (5+2) = 3 \parallel 7 = \frac{3 \times 7}{3+7} = 2.1\text{k}\Omega$ .

So the complete Thévenin equivalent is:



Most people got this right. Some added all the resistors to get  $R_{th} = 10\text{k}\Omega$ ; we need the resistance between terminals A and B which consists of two resistors in parallel. Quite a few people calculated  $V_{th} = 4$  because they took the voltage across  $3\text{k}\Omega + 2\text{k}\Omega$  instead of just across the  $3\text{k}\Omega$  resistor. A few people calculated the correct component values but lost marks because they did not draw the Thévenin equivalent circuit even though the question asked for it. A few people had a double negative for  $V_{th}$ : they drew the voltage source with the “-” sign at the top and labelled it as “-2.4”. Several people said that the  $2\text{k}$  resistor had no current through it and could therefore be ignored; the logic behind this was unclear to me.

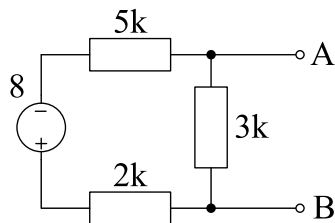


Figure 1.3

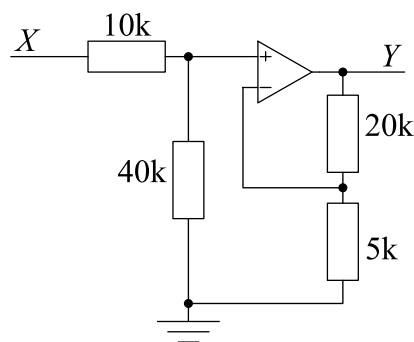


Figure 1.4

- d) Assuming the opamp in the circuit of Figure 1.4 is ideal, give an expression for  $Y$  in terms of  $X$ . [ 4 ]

[U] This is a potential divider followed by a non-inverting op-amp circuit and so we can write down  $Y = \left(1 + \frac{20}{5}\right) \times \frac{40}{10+40} \times X = 5 \times 0.8 \times X = 4X$ . Alternatively, we can denote the op-amp input voltages by  $V$  (since they are equal because of the negative feedback) and write down the following KCL equations:  $\frac{V-X}{10} + \frac{V}{40} = 0$  and  $\frac{V-Y}{20} + \frac{V}{5} = 0$  from which  $5V = 4X$  and  $5V = Y$  from which  $Y = 4X$ .

*Most people got this right.*

- e) Determine  $R_1$  and  $R_2$  in Figure 1.5 so that  $Y = 0.1X$  and the parallel combination of  $R_1$  and  $R_2$  has an impedance of  $50\Omega$ . [ 5 ]

[U] The gain of the potential divider is  $0.1 = \frac{R_2}{R_1+R_2}$  which implies that  $0.1R_1 = (1 - 0.1)R_2 = 0.9R_2$  from which  $R_1 = 9R_2$ .

Substituting this relationship into the parallel impedance formula gives  $50 = \frac{R_1 R_2}{R_1 + R_2} = \frac{9R_2^2}{10R_2} = 0.9R_2$  from which  $R_2 = 55.6\Omega$  and  $R_1 = 9R_2 = 500\Omega$ .

*Most people were able to do this OK. Many people gave the result as  $R_2 = \frac{500}{9}\Omega$  instead of its decimal equivalent; leaving answers as exact fractions maybe what is wanted in a mathematics exam but makes no sense at all in engineering since the component values are unlikely to be known more accurately than  $\pm 5\%$ . In engineering, all numerical answers should be given in decimal.*

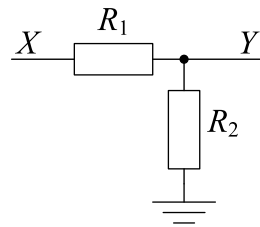


Figure 1.5

- f) i) The diagram of Figure 1.6 shows a 50 Hz AC source with r.m.s. voltage  $\tilde{V} = 230 \text{ V}$  driving a load with impedance  $3.5 + 2.5j \Omega$ . Determine the complex power supplied to the load,  $S = \tilde{V} \times \tilde{I}^*$ , and also the power factor,  $\lambda = \frac{\Re(S)}{|S|}$ . [ 3 ]

[U] We can calculate the complex power directly as  $S = \tilde{V} \times \tilde{I}^* = \frac{|\tilde{V}|^2}{Z^*} = \frac{230^2}{3.5 - 2.5j} = 10 + j7.15 \text{ kVA}$ . Alternatively, and less directly, the load current is  $\tilde{I} = \frac{\tilde{V}}{Z} = \frac{230}{3.5 + 2.5j} = 43.5 - 31.1j$  which gives the complex power as  $S = \tilde{V} \times \tilde{I}^* = 230(43.5 + 31.1j) = 10 + j7.15 \text{ kVA} = 1.23 \angle 35.5^\circ \text{ kVA}$ . The power factor is therefore  $\lambda = \cos \angle S = \frac{\Re(S)}{|S|} = \frac{10}{12.3} = 0.814$ .

Most people got this OK. One or two did not realise that  $\Re(s)$  meant the real part of  $S$ ; this was defined in the rubric on page 1 of the exam paper. Note that, since  $S = |\tilde{I}|^2 Z$ ,  $\angle S = \angle Z$  and so  $\lambda = \cos \angle Z = \frac{3.5}{\sqrt{3.5^2 + 2.5^2}} = 0.814$  is an alternative expression for  $\lambda$ . Some people wrongly inserted a factor of  $\frac{1}{2}$  when calculating the complex power;  $S = \tilde{V} \times \tilde{I}^* = \frac{1}{2} V \times I^*$  and, since the question specifies  $\tilde{V}$ , we do not need the factor of  $\frac{1}{2}$ . Other people calculated  $V = \sqrt{2} \times \tilde{V} = 325.3$  which is correct but unhelpful. Despite being given the formula in the question, several people calculated the complex power as  $S = \tilde{V} \times Z^*$  for reasons that were unclear to me. Surprisingly many people copied the component values down incorrectly (e.g. 3.2 instead of 3.5).

- ii) A capacitor,  $C$ , is now connected across the load, as indicated in Figure 1.7. Calculate the value of  $C$  (in  $\mu\text{F}$ ) required to increase the power factor to 0.95. [ 3 ]

[U] The complex power absorbed by the capacitor is purely imaginary and equals  $S_C = \frac{|\tilde{V}|^2}{Z_C^*} = -j\omega C |\tilde{V}|^2 = -j100\pi C |\tilde{V}|^2$  from which  $C = \frac{jS_C}{\omega |\tilde{V}|^2}$ .

A power factor of 0.95 for the total power absorbed,  $S + S_C$ , means that  $\angle(S + S_C) = \cos^{-1}(0.95) = 0.318 = 18.2^\circ$ . It follows that  $\frac{\Im(S + S_C)}{\Re(S + S_C)} = \tan(S + S_C) = \tan(0.318) = 0.329$ ; you can also derive this from  $\tan \theta = \frac{\sin \theta}{\cos \theta} = \sqrt{\sec^2 \theta - 1} = \sqrt{\frac{1}{0.95^2} - 1} = \sqrt{0.108} = 0.329$ .

So since  $\Re(S + S_C) = \Re(S) = 10 \text{ kW}$ , we can write  $S_C = j(0.329 \times 10 - 7.15) = -j3.86 \text{ kVA}$ . So  $C = \frac{jS_C}{100\pi |\tilde{V}|^2} = \frac{3860}{100\pi \times 52900} = 232 \mu\text{F}$ .

An alternative way to calculate  $S_C$  is  $\lambda = 0.95 = \frac{\Re(S + S_C)}{|S + S_C|} = \frac{10 \text{ kW}}{|S + S_C|}$  from which  $|S + S_C| = \frac{10}{0.95} = 10.526 = \sqrt{S_R^2 + |S_L + S_C|^2} = \sqrt{10^2 + |7.15j + S_C|^2} \text{ kVA}$ . Rearranging this gives  $S_C = j(\sqrt{10.526^2 - 10^2} - 7.15) = j(3.29 - 7.15) = -j3.86 \text{ kVA}$ .

Many people found this hard. In quite a few cases, people calculated a slightly messy expression for the total load impedance in terms of

*C but were not quite sure what to do next. Several people took  $\omega = 50$  instead of  $\omega = 2\pi \times 50 = 100\pi$ . Quite a few people calculated  $S_C = -j3.86\text{kVA}$  correctly but then said  $S_C = \frac{1}{j\omega C}$  instead of  $S_C = -j\omega C |\tilde{V}|^2$ .*

---

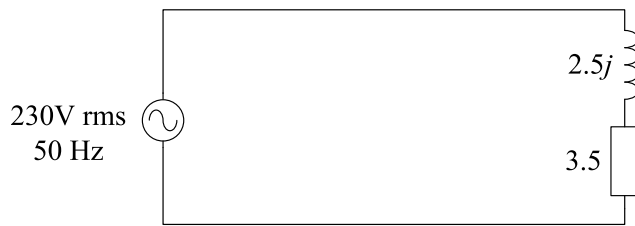


Figure 1.6

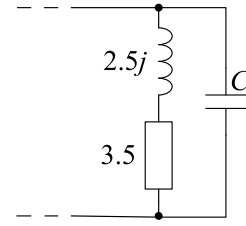


Figure 1.7

- g) Determine the gain,  $\frac{Y}{X}$ , for the block diagram shown in Figure 1.8. The rectangular blocks are drawn with inputs at the left and outputs at the right and have gains of  $F$  and  $G$  respectively. The open circles represent adder/subtractors; their inputs have the signs indicated on the diagram and their outputs are  $V$  and  $Y$  respectively. [ 4 ]

---

*[U] We can write down the following equations from the block diagram:*

$$\begin{aligned} V &= FX - GV \\ Y &= GV - X \end{aligned}$$

*We need to eliminate  $V$  from these equations:*

$$\begin{aligned} V &= FX - GV \\ \Rightarrow (1+G)V &= FX \\ \Rightarrow V &= \frac{F}{1+G}X \\ Y &= GV - X \\ Y &= \left( \frac{FG}{1+G} - 1 \right) X \\ \frac{Y}{X} &= \frac{FG - G - 1}{1+G} \end{aligned}$$

*Most people got this correct. However some wrote down dimensionally inconsistent equations like  $V = FX - G$ . Whenever you add or subtract two quantities, they must have the same dimension. If all the nodes in a block diagram have the same dimension (e.g. volts), then the gains (e.g.  $F$  and  $G$ ) are dimensionless. Several people wrote  $V = FX - Y$  for reasons that I didn't understand. It is worth making the obvious point that if the question asks for the value of  $\frac{Y}{X}$ , then the last line of your answer should be of the form  $\frac{Y}{X} = \dots$  and not something like  $Y = \left( \frac{XF}{1+G} \right) G - X$  which, although true, does not answer the question. There were a surprising number of algebra errors, usually involving minus signs. Many people did not have a systematic way of eliminating  $V$  from*

the pair of equations but instead seemed to do random algebraic manipulations until it disappeared.

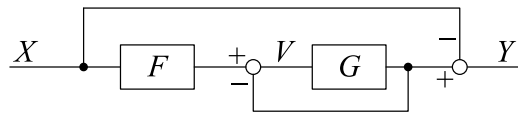


Figure 1.8

- h) Figure 1.10 shows a transmission line of length 10 m that is terminated in a resistive load,  $R$ , with reflection coefficient  $\rho = -0.4$ . The line has a propagation velocity of  $u = 2 \times 10^8$  m/s. At time  $t = 0$ , a forward-travelling (i.e. left-to-right) pulse arrives at  $X$  with amplitude 5 V and duration 100 ns, as shown in Figure 1.9.

Draw a dimensioned sketch of  $y(t)$  for  $0 \leq t \leq 250$  ns where  $y(t)$  is the waveform at  $Y$ , a point 7 m from the end of the line. Label the voltages taken by  $y(t)$  and the times at which it changes. Assume that no reflections occur at point  $X$ .

[ 5 ]

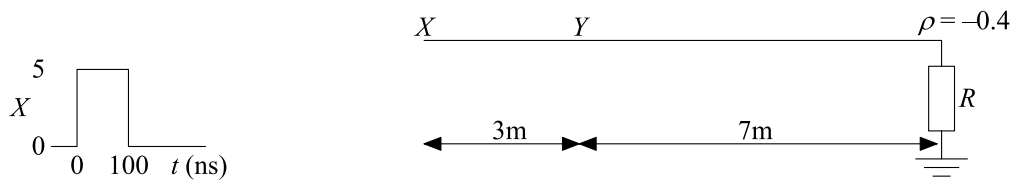
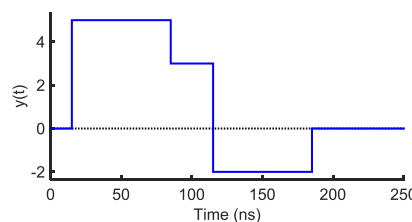


Figure 1.9

Figure 1.10

[U] The pulse will take  $\frac{3}{u} = 15$  ns to reach  $Y$  and then a further  $\frac{2 \times 7}{u} = 70$  ns for its reflection to return to  $Y$ . So the waveform at  $Y$  is the sum of two components: (A) an incoming pulse of amplitude 5 lasting between 15 and 115 ns and (B) a reflected pulse of amplitude  $5 \times \rho = -2$  lasting between 85 and 185 ns.

Thus  $y(t)$  changes at  $\{15, 85, 115, 185\}$  ns to voltages  $\{5, 3, -2, 0\}$  V. A graph of  $y(t)$  is shown below.



Quite a few people did not attempt this question, but almost everyone who did attempt it got it right. The most common mistake was to make the reflected pulse start at  $t = 70$  ns instead of  $t = 85$  ns.

2. a) i) Assuming that the op-amp in Figure 2.1 is ideal, show that the output voltage at time  $t$  is given by [ 3 ]

$$y(t) = y(0) - \frac{1}{R_1 C} \int_0^t x(\tau) d\tau.$$

[U] The  $V_-$  input of the op-amp is a virtual ground and so the voltage across the capacitor is  $y - 0 = y$  and the current through it (from right to left according to the passive sign convention) is  $i = -\frac{x}{R_1}$ . Applying the capacitor equation,  $i = C \frac{dy}{dt}$ , we therefore obtain  $-\frac{x}{R_1} = C \frac{dy}{dt}$  from which  $\frac{dy}{dt} = -\frac{1}{R_1 C} x$  and integrating both sides from 0 to  $t$  gives the required integral equation.

Many people calculated the frequency response,  $\frac{Y}{X} = -\frac{1}{j\omega R_1 C}$  but this only applies if  $y(t)$  is a sine wave; the integral expression given in the question is, in contrast, valid for any waveform  $x(t)$ , not just for a sine wave. There was some confusion over the sign of the capacitor voltage; from the passive sign convention, if the capacitor current flows from left to right, then the capacitor voltage must be  $v_- - Y = -Y$ . Note that the  $\tau$  inside the integral is a dummy variable and nothing to do with the value,  $R_1 C$ , which is not the time constant for this circuit and is, in any case, irrelevant to this problem.

- ii) If  $x(t)$  has a constant value of 12 V and  $y(0) = 5$  V, determine the value of  $R_1$  such that  $y(t) = -5$  V when  $t = 1$  ms. [ 3 ]

[U] Substituting for  $x(\tau)$  and  $y(0)$  gives  $y(t) = 5 - \frac{1}{R_1 C} \int_0^t 12 d\tau = 5 - \frac{12t}{R_1 C}$ . So, for  $t = 1$  ms,  $y(t) = -5 = 5 - \frac{12t}{R_1 C}$  gives  $R_1 = \frac{1.2t}{C} = 12$  k $\Omega$ .

Mostly done OK although quite a few people got the wrong power of 10 when converting the units into numerical values; the most common wrong answer was  $R_1 = 12$  M $\Omega$ . Several people incorrectly said that  $\int_0^t x(\tau) d\tau = [\frac{1}{2}x^2]_0^t$  because they integrated with respect to  $x$  rather than  $\tau$ .

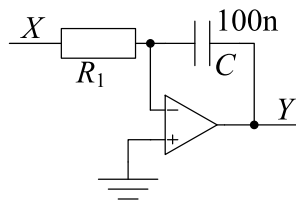


Figure 2.1

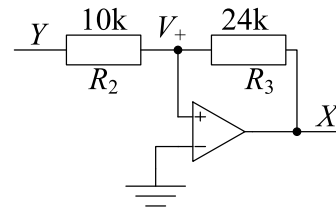


Figure 2.2

- b) In the positive-feedback circuit of Figure 2.2, the op-amp has zero input current and its output voltage,  $x$ , is given by

$$x = \begin{cases} +12 \text{ V} & \text{if } v_+ > v_- \\ -12 \text{ V} & \text{if } v_+ < v_- \end{cases}$$



where  $v_+$  and  $v_-$  are the voltages at the op-amp input terminals.

- i) Show that  $v_+ = \frac{5}{17}x + \frac{12}{17}y$ . [ 2 ]

---

*[U] This is a weighted-average circuit with  $v_+ = \frac{24y+10x}{34}$ . Alternatively, applying KCL at node  $V_+$  gives  $\frac{v_+-y}{10} + \frac{v_+-x}{24} = 0$  which leads to the same equation.*

*Almost everyone got this right. Most used nodal analysis but a few used superposition.*

---

- ii) Hence show that the op-amp output,  $x$ , is given by

$$x = \begin{cases} +12 \text{ V} & \text{if } y > a \\ +12 \text{ V or } -12 \text{ V} & \text{if } -a < y < a \\ -12 \text{ V} & \text{if } y < -a \end{cases}$$

Determine the value of  $a$  and explain what determines the value of  $x$  when  $-a < y < a$ . [ 4 ]

---

*[U] Since  $v_+ = \frac{5}{17}x + \frac{12}{17}y$ , we have  $v_+ > 0$  (which is the condition for  $x$  to become +12) if and only if  $y > -\frac{5}{12}x$ .*

*If  $x = +12$ , then  $v_+ > 0 \Leftrightarrow y > -5$  which means that  $x$  will remain at +12 unless  $y$  goes below  $-5$  in which case it will switch to  $x = -12$ .*

*If  $x = -12$ , then  $v_+ > 0 \Leftrightarrow y > +5$  which means that  $x$  will remain at  $-12$  unless  $y$  goes above  $+5$  in which case it will switch to  $x = +12$ .*

*All this is expressed in the equation given in the question with  $a = 5$ . If  $-5 < y < +5$  then  $x$  will remain at its current value.*

*Very few people gave a logically watertight proof; in particular few gave a convincing justification of the circuit's behaviour when  $-5 < y < +5$ . Most people correctly calculated the inequalities on  $y$  for the two possible values of  $x$ . The first case considered in the answer above shows that  $\{x = +12 \text{ and } y > -5\} \Rightarrow v_+ > 0 \rightarrow x = +12$ . Several people took this to be equivalent to  $\{y > -5\} \Rightarrow x = +12$  but you cannot ignore a logical condition like this; it also leads to a logical contradiction because the corresponding case when  $x = -12$  results in  $\{y < +5\} \Rightarrow x = -12$  which is contradictory for  $-5 < y < +5$ . Quite a lot of people correctly calculated that when  $x = +12$  then  $v_+ > 0 \Leftrightarrow y > -5$  but didn't believe the algebra and quietly changed the  $-5$  to  $+5$  to match the question or else said that this meant  $a = -5$  in the question (which doesn't really make sense); the correct interpretation of this result is that if  $x = +12$  then it will stay there for as long as  $y > -5$ .*

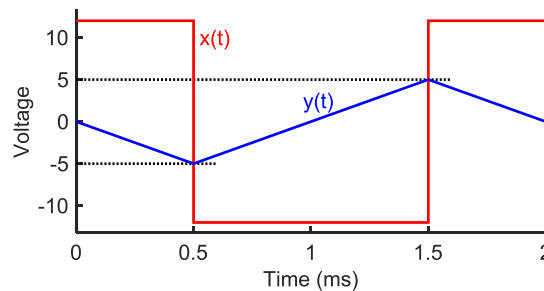
---

[This question is continued on the next page]

- c) The circuits of parts a) and b) are combined to form the oscillator circuit shown in Figure 2.3. The resistor  $R_1$  has the value determined in part a).

Given the initial conditions  $x(0) = +12\text{ V}$  and  $y(0) = 0\text{ V}$ , draw a dimensioned sketch of the waveforms  $x(t)$  and  $y(t)$  for  $0 < t < 2\text{ ms}$ . Your sketch should label the peak value of  $y(t)$  and the times when  $x(t)$  switches. [ 6 ]

[U] From part b) we know that  $x(t)$  will switch between  $\pm 12$  when  $y(t)$  reaches  $\pm 5$ . Also, from part a), we know that when  $x(t) = \pm 12$ ,  $y(t)$  has a gradient of  $\mp 10\text{ V/ms}$ . So,  $y(t)$  will ramp down to  $-5$  in  $0.5\text{ ms}$ , ramp up to  $+5$  and then ramp down again as shown in the graph below. So the max and min values of  $y(t)$  are  $\pm 5$  and the switching points of  $x(t)$  occur at  $0.5$  and  $1.5\text{ ms}$ .



*Quite a few people missed this out completely. Several people only plotted  $y(t)$  even though the question asked for  $x(t)$  as well. Note that because the capacitor current is proportional to  $x(t)$  which is piecewise constant,  $y(t)$ , consists of straight lines rather than the negative exponentials that several people assumed. Some people had the graph of  $y(t)$  inverted.*

- d) The output of the oscillator,  $y$ , forms the input to the circuit shown in Figure 2.4. You may assume that the diodes have a voltage drop of  $0.7\text{ V}$  when conducting and a current of zero when their forward voltage is less than  $0.7\text{ V}$ .

- i) Give an expression for  $v$  in terms of  $y$  when neither diode is conducting. By considering the voltage across  $R_6$ , determine the range of input voltages,  $y$ , for which neither diode will be conducting. [ 4 ]

[U] If both diodes are off, the circuit will be an inverting amplifier and  $v = -\frac{15.1}{38.6}y = -0.39y$ . The current through the feedback path is  $\frac{y}{38.6}$  from left to right, so since the diodes are non-conducting, this all flows through  $R_6$  and the voltage across  $R_6$  measured upwards (and hence also across the diodes) will be  $v_6 = \frac{9.6}{38.6}y$ . The assumption that the diodes are both off is only true if this voltage lies in the range  $\pm 0.7$  or, in other words, that  $|\frac{9.6}{38.6}y| < 0.7 \Rightarrow |y| < \frac{0.7 \times 38.6}{9.6} = 2.81$ .

*Most people got the correct expression for  $v$  but quite a few missed out the minus sign (which makes a big difference). Some people give the equivalent condition,  $|v| < \frac{0.7 \times 15.1}{9.6} = 1.101$  which is correct but not what the question asked. Many many people found difficulty in calculating the voltage across the diodes. Many assumed that the voltage across the diodes is equal to  $v$  whereas it actually equals the voltage across  $R_6$  as hinted in the question ( $v$  is the voltage across  $R_5 + R_6$  instead); others took the voltage across the diodes to equal*

*the voltage across  $R_5$ . It is possible to use a potential divider approach:  $v_6 = -\frac{R_6}{R_5+R_6}v = \frac{R_6}{R_4+R_5+R_6}(y-v)$ ; several people tried this but almost always used  $y$  in place of  $(y-v)$  when using the second form. Some showed that the diodes were non-conducting when  $-1.1 < v < +1.1$  which is correct but not what the question asked (which was for the range of  $y$  rather than  $v$ ). In a similar vein, the question asked for  $v$  in terms of  $y$  which means an equation with  $v$  on the left side (like  $v = -0.39y$ ) but many people gave an expression for  $y$  in terms of  $v$  instead (like  $y = -2.56v$ ) which is true but not what was asked.*

- ii) Give an expression for  $v$  in terms of  $y$  when diode  $D_1$  is conducting. Determine the range of input voltages,  $y$ , for which diode  $D_1$  will be conducting. [ 4 ]

*[U] If  $D_1$  is conducting, current must be flowing from left to right in  $R_4$  and so  $y > 2.81$ . The voltage  $v$  is equal to the sum of the voltage across  $R_5$  and the voltage across  $D_1$  and is therefore  $v = -\frac{5.5}{38.6}y - 0.7 = -0.14y - 0.7$ . For the specific input  $y_0 = 2.81$ , we have  $-0.39y_0 = -0.14y_0 - 0.7 = -1.09$  so the two expressions are equal at the switching point,*

*Many people got the sign of the diode voltage wrong: the voltage at the junction of  $R_5$  and  $R_6$  is equal to  $V + 0.7$ . Quite a few people assumed that  $v = -0.39y$  from part (i) was still true. Quite a few people assumed that the voltage across the diode was zero when it was conducting (even though the question said that it was 0.7 V).*

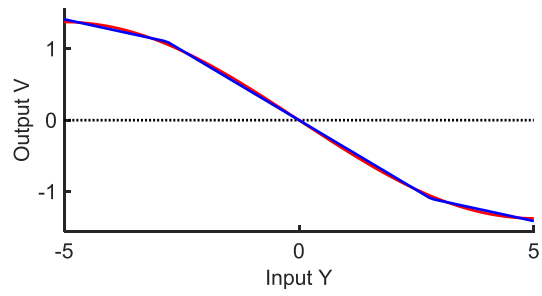
- iii) Draw a dimensioned sketch showing the graph of  $v$  versus  $y$  for the range  $-5 < y < +5$ . [ 4 ]

*[U] Combining the previous results, we have*

$$v = \begin{cases} -0.14y + 0.7 & \text{for } y < -2.81 \\ -0.39y & \text{for } -2.81 < y < 2.81 \\ -0.14y - 0.7 & \text{for } y > 2.81 \end{cases}$$

*Specific value of  $v$  are:  $v(\pm 5) = \mp 1.4$  and  $v(\pm 2.81) = \mp 1.09$ .*

*Although not requested, the graph below also includes the target sine wave that the circuit is intended to approximate; connecting this to the previous circuit makes a crude sine wave oscillator.*



*Several people had abrupt discontinuities in the graph. However, the circuit has negative feedback and all the components have continuous characteristics so there is nothing that can cause switching.*

---

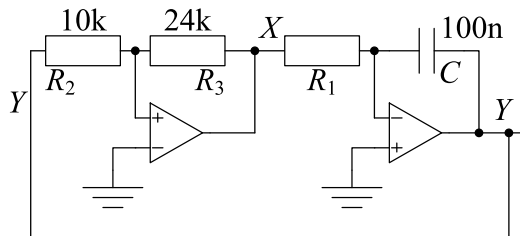


Figure 2.3

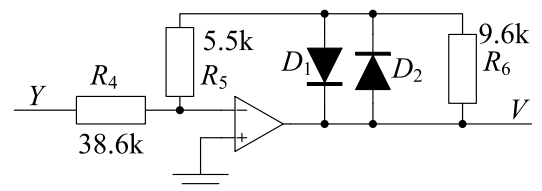


Figure 2.4

3. Figure 3.1 shows a circuit whose input and output voltages are  $x(t)$  and  $y(t)$  respectively.

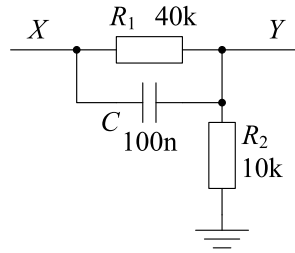


Figure 3.1

- a) i) Determine the frequency response,  $G(j\omega) = \frac{Y}{X}$ . [ 4 ]

[U] Treating the circuit as a potential divider,

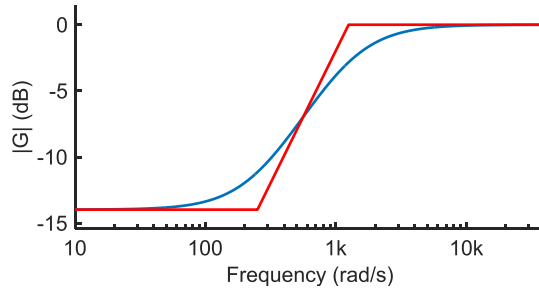
$$\begin{aligned} G(j\omega) &= \frac{R_2}{R_2 + \frac{R_1 \times \frac{1}{j\omega C}}{R_1 + \frac{1}{j\omega C}}} = \frac{R_2}{R_2 + \frac{R_1}{j\omega CR_1 + 1}} = \frac{R_2(j\omega CR_1 + 1)}{j\omega CR_1 R_2 + R_1 + R_2} \\ &= \frac{j\omega CR_1 + 1}{j\omega CR_1 + 5} = \frac{j0.004\omega + 1}{j0.004\omega + 5} = \frac{j\omega + 250}{j\omega + 1250} \end{aligned}$$

Mostly done OK. Quite a lot of people left the answer in symbolic form without substituting in the component values. Others did substitute in the component values but sometimes made mistakes with the powers of ten in the units such as writing 40 instead of 40k. One or two people said the impedance of a capacitor is  $j\omega C$  instead of  $\frac{1}{j\omega C}$ . This is an easily made error but you can tell that an expressions like  $R + j\omega C$  or  $R + 1$  are always wrong because they are dimensionally inconsistent; another warning sign is if, when you substitute in the numerical values, you end up adding quantities that have widely different orders of magnitude e.g.  $10^4 + 10^{-4}$ .

- ii) Draw a dimensioned sketch of the straight-line approximation to the magnitude response,  $|G(j\omega)|$ , showing the values of any corner frequencies (in rad/s) and the gains of any horizontal portions of the response (in dB). [ 4 ]

[U] From the transfer function, the corner frequencies are  $\omega_n = \frac{1}{R_1 C} = 250$  in the numerator and  $\omega_d = \frac{5}{R_1 C} = 1250$  in the denominator. The low frequency asymptote is  $0.2 = -14\text{dB}$  and the high frequency gain is  $1 = 0\text{dB}$ .

In the graph below, the straight-line approximation is shown in red and the true response (not requested) in blue.

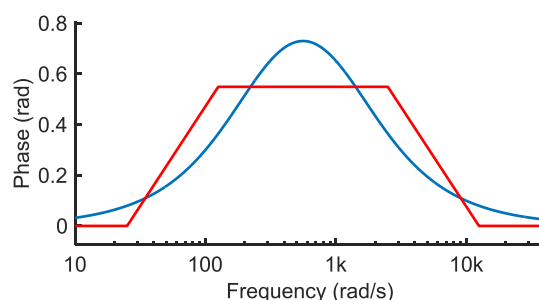


Mostly done OK. A few thought that a gain of 1 corresponds to 1 dB instead of 0 dB. Notice that, since the sloping segment has a gradient of 1 ( $= 20 \text{ dB per decade}$ ), the ratio of the corner frequencies is the same as the ratio of the gains at the corner frequencies:  $\frac{1250}{250} = \frac{1}{0.2}$ ; this was not true in everyone's answer. Quite a few people calculated the low and high frequency asymptotes correctly but then drew the graph flipped horizontally (i.e. with a low frequency gain of 1 and a high frequency gain of 0.2) for reasons that were not clear. When drawing a graph with a log frequency axis, there is no "correct" frequency at which to start the horizontal axis since  $0 \text{ rad/s}$  is never visible on the axis ( $\log 0 = -\infty$ ). A reasonable frequency range to plot is from 0.1 times the lowest corner frequency to 10 times the highest corner frequency although if you want to use the same range for a phase plot, it needs to be even larger. Some people began the plot at the lowest corner frequency and so did not include the horizontal low-frequency asymptote.

- iii) Draw a dimensioned sketch of the straight-line approximation to the phase response,  $\angle G(j\omega)$ , showing the values of any corner frequencies (in rad/s) and the phase of any horizontal portions of the response (in rad). [ 4 ]

[U] The phase response has corner frequencies at:  $0.1\omega_n = 25^+$ ,  $10\omega_n = 2500^-$ ,  $0.1\omega_d = 125^-$  and  $10\omega_d = 12500^+$  where the superscript indicates the sign of the gradient change at each corner. The frequency increment from  $\omega = 25$  to  $\omega = 125$  is  $\log_{10} \frac{125}{25} = \log_{10} 5 = 0.699$  decades and the slope is  $\frac{\pi}{4}$  per decade, so the phase increment is  $\frac{\pi}{4} \times 0.699 = 0.549$ . So the horizontal portions of the response have values  $\{0, 0.549, 0\} \text{ rad}$ .

In the graph below, the straight-line approximation is shown in red and the true response (not requested) in blue.



Most people got the general shape correct although a few inverted it.

*Note that each corner frequency in the magnitude plot corresponds to two gradient changes in the phase plot: one positive and one negative (some people made both changes in the same direction). If a linear factor is  $(a j\omega + b)$  then, provided that  $a$  and  $b$  have the same sign (the normal case), the first gradient change is in the same direction as in the magnitude plot (positive for numerator; negative for denominator). If  $a$  and  $b$  have opposite signs then the sign of the gradient changes are both reversed. The first and last segments of the plot have the phases corresponding to the low and high frequency asymptotes; in this case these are 0.2 and 1 which both have zero phase. Quite a lot of people said the peak phase shift was either  $\frac{\pi}{2}$  or  $\frac{\pi}{4}$  instead of  $\frac{\pi}{4} \times 0.699 = 0.549$ ; others just left the vertical axis of their graph completely unlabelled. Some people calculated the actual phase shift at  $\omega = 125$  but, as can be seen from the blue line above, this is much less than that of the straight line approximation. Some people thought that the phase at low frequencies would be  $\pi$  (or less logically  $0.5\pi$ ) because the gain in dB was the negative number  $-14$  dB; this is not true since  $-14$  dB corresponds to a gain of 0.2 which is a positive number.*

- 
- b) Determine the time constant of the circuit. [ 3 ]
- 

*[U] We can determine the time constant in two ways:*

*(A) If we short-circuit the input voltage source (i.e. connect X to ground), the Thévenin resistance at the terminals of the capacitor is  $R_{Th} = 40 || 10 = 8$  k. Thus the time-constant of the circuit is  $\tau = R_{Th}C = 0.8$  ms.*

*(B) The time constant is also equal to the reciprocal of the denominator corner frequency:  $\tau = \frac{1}{\omega_d} = \frac{1}{1250} = 0.8$  ms.*

*Almost everyone got this right. A very few said the resistors were in series rather than in parallel. A surprising number of people thought that  $8 \times 10^{-4}$  s was equal to 80 ms or 0.8  $\mu$ s. As with some of the other questions, several people left the answer as an unevaluated algebraic expression,  $\frac{R_1 R_2 C}{R_1 + R_2}$  or fraction  $\frac{1}{1250}$ ; you will normally lose marks if you do this.*

---

- c) If the input, shown in Figure 3.2, is given by

$$x(t) = \begin{cases} -5 & \text{for } t < 0 \\ +5 & \text{for } t \geq 0 \end{cases},$$

- i) determine expressions for  $y(t)$  both for  $t < 0$  and for  $t \geq 0$ , [ 6 ]
- 

*[U] From the transfer function, the DC gain is 0.2. Therefore the steady-state output is*

$$y_{ss}(t) = \begin{cases} -1 & \text{for } t < 0 \\ +1 & \text{for } t \geq 0 \end{cases}.$$

It follows that the complete output is

$$y(t) = \begin{cases} -1 & \text{for } t < 0 \\ 1 + Ae^{-\frac{t}{\tau}} & \text{for } t \geq 0 \end{cases}$$

where  $\tau = 0.8\text{ms}$  from part b).

To determine  $A$ , we need to calculate  $y(0)$ . We do this in one of two ways.

(A) at  $t = 0-$ , the voltage across the capacitor is  $y(0-) - x(0-) = (-1) - (-5) = 4$ . Since the capacitor voltage cannot change instantaneously,  $y(0+) - x(0+) = 4$  which means that  $y(0+) = 4 + x(0+) = 4 + 5 = 9$ .

(B) The output discontinuity will be  $y(0+) - y(0-) = G(\infty) \times (x(0+) - x(0-)) = 1 \times 10 = 10$  from which  $y(0+) = y(0-) + 10 = -1 + 10 = 9$ .

So, now we have  $y(0+) = 9 = 1 + A$  from which  $A = 8$ . The final expression for  $y(t)$  is therefore

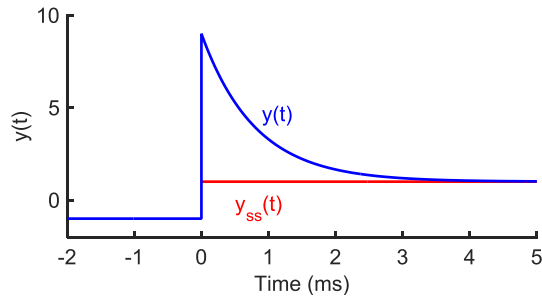
$$y(t) = \begin{cases} -1 & \text{for } t < 0 \\ 1 + 8e^{-\frac{t}{\tau}} & \text{for } t \geq 0 \end{cases}$$

Many people did not give an explicit formula for  $y(t)$  for  $t < 0$  as the question demanded; you need something like the final expression in the above model answer. Quite a lot of people were rather sloppy with their notation and, for example, wrote “ $y(0+) = 1$ ” when they actually meant “ $y_{SS}(t) = 1$  for  $t < 0$ ”; in fact  $y(0+) = 9$  and  $y_{SS}(0+) = 1$ . There are several ways to calculate  $y(0+)$ ; one of these is to calculate the capacitor voltage  $y - x$  at time  $t = 0-$  and say this must have the same value at  $t = 0+$ , i.e.  $y(0+) - x(0+) = y(0-) - x(0-) = -1 - (-5) = 4$ . Many people did this correctly but then applied it in the wrong way by saying things like  $y(0+) = y_{SS}(0+) + 4$  or  $y(0+) = y(0-) + 4$ . The correct deduction follows directly from the previous equation and is that  $y(0+) = x(0+) + 4 = 9$ ; mistakes like this are much less likely if you express the capacitor voltage explicitly as the difference between two node voltages (i.e. use  $y(t) - x(t)$ ) rather than talking about a vaguely defined capacitor voltage,  $v_C$ .

- 
- ii) draw a dimensioned sketch of the waveform of  $y(t)$ . [ 3 ]
- 

[U]  $y(t)$  is plotted below in blue. Although not requested, the graph below also shows  $y_{SS}(t)$  in red.





*Mostly correct. A sketched graph should have numerical scales on both axes; many people omitted the time-axis scale entirely.*

- d) If the input, shown in Figure 3.3, is given by

$$x(t) = \begin{cases} \sin(500t) & \text{for } t < 0 \\ \sin(1000t) & \text{for } t \geq 0 \end{cases}$$

determine expressions for  $y(t)$  both for  $t < 0$  and for  $t \geq 0$ .

[ 6 ]

*[U] The steady-state gains are*

$$G(j\omega) = \begin{cases} \frac{2j+1}{2j+5} = \frac{9+8j}{29} = 0.3103 + 0.2759j & \text{for } t < 0 \\ \frac{4j+1}{4j+5} = \frac{21+16j}{41} = 0.5122 + 0.3902j & \text{for } t \geq 0 \end{cases}$$

*So, since the phasor inputs are  $X = -j$  for both segments, the steady state phasor outputs are*

$$Y_{SS} = \begin{cases} 0.2759 - 0.3103j & \text{for } t < 0 \\ 0.3902 - 0.5122j & \text{for } t \geq 0 \end{cases}$$

*and the corresponding waveforms are*

$$y(t) = \begin{cases} 0.2759 \cos 500t + 0.3103 \sin 500t & \text{for } t < 0 \\ 0.3902 \cos 1000t + 0.5122 \sin 1000t + Be^{-\frac{t}{\tau}} & \text{for } t \geq 0 \end{cases}$$

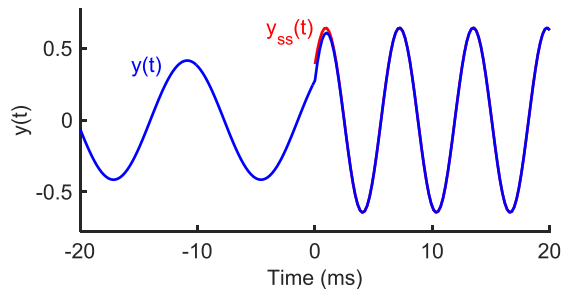
*Since there is no input discontinuity, there will be no output discontinuity either so  $y(0-) = y(0+)$ . So we can write*

$$\begin{aligned} y(0-) = 0.2759 &= y(0+) = 0.3902 + B \\ \Rightarrow B &= 0.2759 - 0.3902 = -0.1143 \end{aligned}$$

*and the full expression for  $y(t)$  is therefore*

$$\begin{aligned} y(t) &= \begin{cases} 0.2759 \cos 500t + 0.3103 \sin 500t & \text{for } t < 0 \\ 0.3902 \cos 1000t + 0.5122 \sin 1000t - 0.1143e^{-\frac{t}{\tau}} & \text{for } t \geq 0 \end{cases} \\ &= \begin{cases} 0.415 \cos(500t - 0.844) & \text{for } t < 0 \\ 0.644 \cos(1000t - 0.920) - 0.1143e^{-1250t} & \text{for } t \geq 0 \end{cases} \end{aligned}$$

*Although not requested, the graphs of  $y_{SS}(t)$  and  $y(t)$  are plotted below. The difference between them is negligible for  $t > 2$  ms.*



Mostly very well done. Note that the amplitude of the transient is always a real-valued constant (in this case  $-0.1143$ ); a few people either made it complex-valued or else a function of time. Similarly,  $y(t)$  is real-valued and so writing  $y(t) = 0.276 - 0.31j$  is meaningless and is bound to be incorrect. A few people assumed that the steady state gain was  $0.2$  possibly because they confused “steady state” with “DC”; if the input is a sine wave, the “steady state” gain is the gain at the sine wave’s frequency. One or two people thought that  $\sin(500t)$  meant that  $\omega = 2\pi \times 500$  rather than the correct value of  $\omega = 500$ . Expressing the waveforms as in the form  $y(t) = 0.644 \cos(1000t - 0.920)$  takes more effort and makes it harder to calculate  $y(0+)$ ; several people did not nevertheless. Quite a lot of people wrote “ $y(0-)$ ” when they actually meant “ $y(t)$  for  $t < 0$ ”; these are not the same thing at all as the first is a specific value while the second is a function of  $t$ .

---

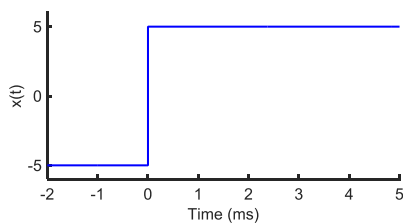


Figure 3.2

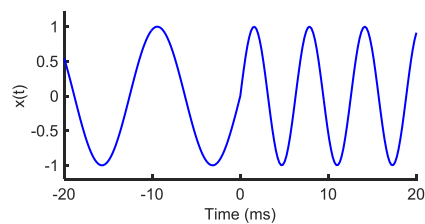


Figure 3.3