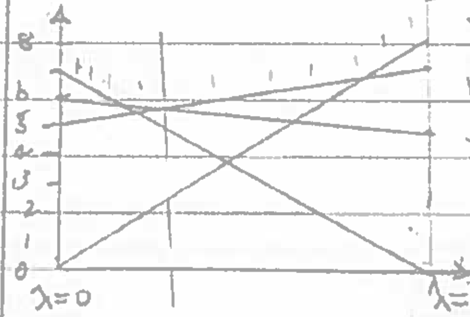
Model Answers to 2017 Game Theory Exam

1 (i) The pay-off matrix is $\begin{array}{c|c} A \backslash B & \\ \hline & \begin{bmatrix} 6 & 5 & 7 & 0 \\ 5 & 7 & 0 & 8 \end{bmatrix} \end{array}$

To calculate a safety strategy for A "=" $(1-\lambda) \times$ row (1) + $\lambda \times$ row (2) we look at the diagram:



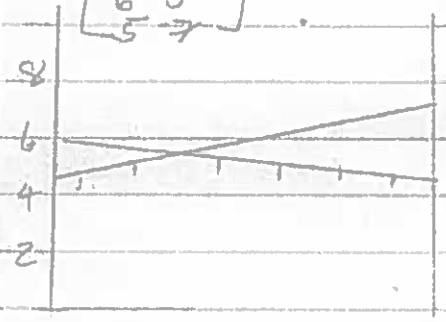
The worst case pay-off (for A) is minimized when A chooses λ to satisfy

$$6(1-\lambda) + 5\lambda = 5(1-\lambda) + 7\lambda$$

with solution $\lambda = \frac{1}{3}$.

So A's mixed strategy is $\frac{2}{3} \times$ row (1) $+ \frac{1}{3} \times$ row 2

Because, when A chooses a safety strategy, it will not be favorable for B to choose columns (3) or (4), we can ignore these. The pay-off matrix then becomes $\begin{bmatrix} 6 & 5 \\ 5 & 7 \end{bmatrix}$.



The worst case pay-off (for B) is maximized when B chooses μ to satisfy

$$6(1-\mu) + 5\mu = 5(1-\mu) + 7\mu$$

with solution $\mu = \frac{1}{3}$.

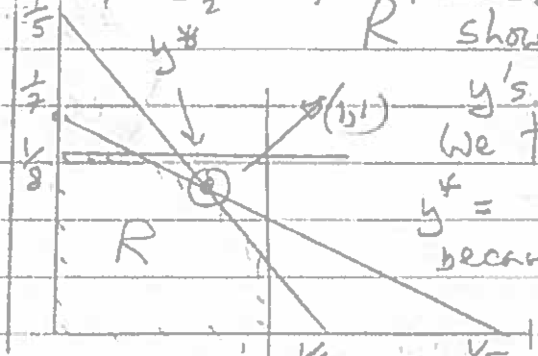
So A's mixed strategy is $\frac{2}{3} \times$ row (1) $+ \frac{1}{3} \times$ row (2) $+ 0 \times$ row (3) $(+ 0 \times$ row (4)$)$

The pay-off is $\left(\frac{2}{3}, \frac{1}{3}\right) \begin{bmatrix} 6 & 5 & 7 & 0 \\ 5 & 7 & 0 & 8 \end{bmatrix} \begin{pmatrix} 2/3 \\ 1/3 \\ 0 \\ 0 \end{pmatrix} = \frac{17}{3} = V$

The equivalent LP giving A's safety strategy is: $\left\{ \begin{array}{l} \text{Max } y_1 + y_2 \\ \begin{bmatrix} 6 & 5 \\ 5 & 7 \\ 7 & 0 \\ 0 & 8 \end{bmatrix} \le 1 \\ y_1 \ge 0, \; y_2 \ge 0 \end{array} \right.$

The region in 2-space of y's satisfying the constraints is defined by the inequalities:

$6y_1 + 5y_2 \le 1$, $5y_1 + 7y_2 \le 1$, $7y_1 \le 1$, $8y_2 \le 1$, $y_1 \ge 0$, $y_2 \ge 0$.



R shows the 'feasible' region for y's.

We find (by solving $6y_1 + 5y_2 = 1$, $5y_1 + 7y_2 = 1$), $y^* = \left(\frac{2}{17}, \frac{1}{17}\right)$. This solves the (LP) because it is furthest feasible point in the (1,1) direction. We see

$$y^* = \frac{1}{V} \times \left(\frac{2}{3}, \frac{1}{3}\right)$$

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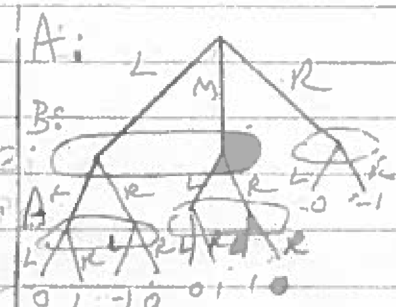
2. 

Because Player A remembers his first action the strategies for Player A are
$A1 = \binom{L}{L}$, $A2 = \binom{L}{R}$, $A3 = \binom{M}{L}$, $A4 = \binom{M}{R}$, $(A5) = \binom{R}{L}$, $A6 = \binom{R}{R}$

For the given B-player information sets, the strategies for Player B are

$B1 = \begin{cases} L & \text{if } A = L \text{ or } M \\ L & \text{if } A = R \end{cases}$, $B2 = \begin{cases} L & \text{if } A = L \text{ or } M \\ R & \text{if } A = R \end{cases}$

$B3 = \begin{cases} R & \text{if } A = L \text{ or } M \\ R & \text{if } A = R \end{cases}$ $B4 = \begin{cases} R & \text{if } A = L \text{ or } M \\ L & \text{if } A = R \end{cases}$

The pay-offs for the different choices are

A \ B	B1	B2	B3	B4
A1	LLL = 0	LLL = 0	LRL = -1	LRL = -1
A2	LLR = 1	LLR = 1	LRR = 0	LRR = 0
A3	MLL = 0	MLL = 0	MRL = 1	MRL = 1
A4	MLR = 1	MLR = 1	MRR = 0	MRR = 0
A5	RLL = 0	RRL = -1	RRL = -1	RLL = 0
A6	RLR = 0	RRR = -1	RRR = -1	RLR = 0

A \ B					
	$\boxed{0}$	0	-1	-1	(weakly)
⇒	+1	+1	0	0	← dominated
	0	0	+1	+1	← dominated
	+1	+1	0	0	← dominated
	$\boxed{0}$	-1	-1	0	
	$\boxed{0}$	-1	-1	0	
			↑ (weakly) dominated		

rows 2, 3 and 4 are weakly dominated (by row 1)
column 3 is weakly dominated (by column 4)

There are 3 Nash equilibria (in pure strategies):
(A1, B1) = LLL, (A5, B1) = RLL, (A6, B1) = RLR
All these Nash equilibria are admissible.

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3 Consider the following events

(a) A_1 : {Player 1 dials up and Players 2, ..., N dials up}

A_2 : {Player 1 dials up and at least one of the players 2, .., N does not dial up}

A_3 : {Player 1 does not dial up}

The satisfaction levels for player 1 are:

satisfaction level $= -1$ if A_1 occurs

" $= +3$ if A_2 occurs

" $= 0$ if A_3 occurs

Let $(\bar\beta, \ldots, \bar\beta)$ be a symmetric collection of probabilities for the N-players. We assume that $0 < \bar\beta < 1$. Take another, arbitrary, probability β for the 1st player. Then, for the first player

$$\text{Pay off}(\beta, \bar\beta, \ldots, \bar\beta) \le \text{Pay off}(\bar\beta, \ldots, \bar\beta), \quad (*)$$

since $(\bar\beta, \ldots, \bar\beta)$ is a Nash equilibrium. But

Pay off $(\beta, \bar\beta, \ldots, \bar\beta) = (-1)\,\text{Prob}\{A_1\} + 3\,\text{Prob}\{A_2\} + 0\,\text{Prob}\{A_3\}$

Prob{A_1} = Prob{Player 1 dials up} × Prob{Player 2..N did all dial up} (by independence)

$= \beta \times \bar\beta^{N-1}$

Prob{A_2} = Prob{Player 1 dials up} × (1 − Prob{Prob 2..N all dial up})

$= \beta \times (1 - \bar\beta^{N-1})$

Prob{A_3} = Prob{Play 1 does not dial up} $= (1-\beta)$

So pay-off is $-1 \times \beta \times \bar\beta^{N-1} + 3 \times \beta(1 - \bar\beta^{N-1}) + 0 \times \beta$

$= (-\bar\beta^{N-1} + 3(1 - \bar\beta^{N-1}))\,\beta$

If this is maximized over β at $\bar\beta$ (see (*)), the slope

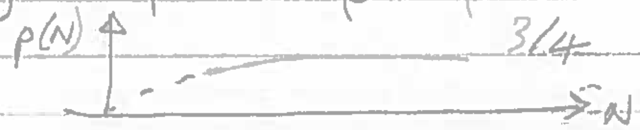
$$-\bar\beta^{N-1} + 3(1 - \bar\beta^{N-1}) = 0$$

$\Rightarrow \quad 4\bar\beta^{N-1} = 3$, So $\bar\beta = (3/4)^{1/(N-1)}$

(b) Communication fails when all subscribers dial in.

The probability $= \bar\beta^N = \bar\beta \times \bar\beta^{N-1} = 3/4 \left(\frac{3}{4}\right)^{1/N-1}$.

p(N)



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4A(i) $L^X(x,a,b) = -a - b + x^2$, $L^A(x,a,b) = ab - xa + a^2 - \frac{1}{2}b^2$
and $L^B(x,a,b) = ab - 2xb + b^2 - \frac{1}{4}a^2$
Fix x. Then for any a, $L^A(x,a,\cdot)$ is maximized
when $\partial L^A/\partial b \; L^A(x,a,b) = 0 \Rightarrow a - b = 0 \Rightarrow b = a$
A will choose a to satisfy $\partial/\partial a \; L^A(x,a,b=a) = 0$
$\Rightarrow \frac{\partial}{\partial a}\{a^2 - xa + a^2 - \frac{1}{2}a^2\} = 0 \Rightarrow 3a - x = 0$
This is A's safety strategy. } or $a = \frac{1}{3}x$.
Likewise, B's safety strategy is obtained by solving
$\partial/\partial a \; L^B(x,a,b) = 0 \Rightarrow b - \frac{1}{2}a \Rightarrow a = 2b$
and $\partial/\partial b \; L^B(x, a=2b, b) = \frac{\partial}{\partial b}\{2b^2 - 2xb + b^2 - b^2\} = 0$
$\Rightarrow 4b - 2x = 0$ or $b = x/2$.
X's pay-off is $L^X(x, a=\frac{1}{3}x, b=\frac{x}{2}) = -\frac{1}{3}x - \frac{x}{2} + x^2$
This is minimized when $2x = 5/6 \Rightarrow x = \frac{5}{12}$
The combined strategies of X, A, B are
$\left(x = \frac{5}{12},\; a = \frac{5}{36},\; b = \frac{5}{24}\right)$

A(ii) Now assume A and B choose a and b to give a Nash equi
Then $\frac{\partial}{\partial a} L^A(x,a,b) = 0$, $\partial/\partial b \; L^B(x,a,b) = 0$
$\Rightarrow b - x + 2a = 0$ and $a - 2x + 2b = 0$
These can be solved to give $a = 0$, $b = x$
X's pay-off is $L^X(x, a=0, b=x) = -x + x^2$
This is minimized when $2x = 1 \Rightarrow x = \frac{1}{2}$
Now the strategies are: $\left(x = \frac{1}{2},\; a = 0,\; b = \frac{1}{2}\right)$

B The safety strategy $\bar{a}$ of A minimizes $\max_{b\in B}\{L^A(a,b')\}$
By definition of $\phi(\cdot)$, $\max_{b'\in B} L^A(\bar{a},b') = L^A(\bar{a},\bar{b})$

Now take any $a \in A$. Then $\max_{b'\in B} L^A(a,b') = L^A(a,\phi(a))$
But, by the tangency condition, $(a,\phi(a))$, which is on the inverted response curve, lies outside the region R.
So $L^A(\bar{a},\bar{b}) < L^A(a,\phi(a))$. Hence $\bar{a}$ is a safety strategy.

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5 (i) The dynamic programming approach is to solve for $V_0(x), \ldots V_N(x)$

$$\begin{cases} V_t(x) = \min_u \max_v \{ V_{t+1}(f(x,u,v)) + L(x)\} & t=0,\ldots,N-1 \quad (1) \\ V_N(x) = L(x) \end{cases}$$

We assume a "saddle point" $(\bar{u}_t(x), \bar{v}_t(x))$ exists for each $t=0,\ldots,N-1$ and relevant values of x.

Then $\{(\bar{u}_t(x), \bar{v}_t(x))\}$ is a saddle point for the dynamic game, and $V_0(x_0)$ is the value of the game.

(ii) For the data as given, $V_N(x) = k_N x + c_N$ with $k_N = 1$, $c_N = 0$

To confirm $V_t(x)$ has 'linear structure' for all t, assume $V_{t+1}(x) = k_{t+1}x + c_{t+1}$ (for some $k_{t+1} > 0$, some c_{t+1})

The right side of (1) is

$$(k_{t+1} + 1)x + c_{t+1} + k_{t+1} x \left(\min_u \max_v u^T S(x) v\right) \quad (2)$$

$\min_u \max_v u^T S(x) v = \min_u \max_v (u_1, u_2) \begin{bmatrix} 2 & 1 \\ 1 & x \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$,

i.e. the pay-off of a zero sum game in mixed strategies.

The saddle strategies $(\bar{u}, \bar{v})$ are $((1-\mu, \mu), (1-\nu, \nu))$ where

$(1-\mu)2 + \mu = (1-\mu)1 + x\mu \Rightarrow \bar{u} = \left(\frac{x-1}{x}, \frac{1}{x}\right)$

$(1-\nu)2 + \nu = (1-\nu)1 + x\nu \Rightarrow \bar{v} = \left(\frac{x-1}{x}, \frac{1}{x}\right)$

The pay-off is

$$\bar{u}^T S(x) \bar{v} = \frac{1}{x^2}\left(2(x-1)^2 + 2(x-1) + x\right) = \frac{(2x-1)\cancel{x}}{x^{\cancel{2}}}$$

So (2) becomes

$$(k_{t+1} + 1)x + c_{t+1} + k_{t+1} \cancel{x} \times \frac{2x-1}{\cancel{x}}$$

$$= (3k_{t+1} + 1)x + c_{t+1} - k_{t+1}$$

So $V_t(x) = k_t x + c_t$, where $k_t = 3k_{t+1} + 1$, $c_t = c_{t+1} - k_{t+1}$

Summary: Calculate k_t, c_t, $t=0,\ldots,N-1$ from

$$\begin{cases} k_t = 3k_{t+1} + 1 & \text{with final values } k_N = 1,\ c_N = 0 \\ c_t = c_{t+1} - k_{t+1} \end{cases}$$

Then the value of the game is $k_0 x_0 + c_0$ $(= V_0(x))$

The saddle point strategies are

$$(\bar{u}(x), \bar{v}(x)) = \left(\left(\frac{x-1}{x}, \frac{1}{x}\right), \left(\frac{x-1}{x}, \frac{1}{x}\right)\right) \quad (x > 1)$$