IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2017

MSc Control Systems

GAME THEORY

Corrected copy

Thursday, 11 May 10:00 am

Time allowed: 3:00 hours

There are FIVE questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : R.B. Vinter

Second Marker(s) : D. Angeli

Information for candidates:

No special instructions for candidates.

1. A zero-sum matrix game for players A and B has pay-off matrix S, given by

$$S \quad = \quad \begin{array}{c|cccc} A\backslash B & & & & \\ \hline & 6 & 5 & 7 & 0 \\ & 5 & 7 & 0 & 8 \end{array} \quad .$$

Player A seeks to minimize the pay-off.

(i): Using graphical means, derive a mixed saddlepoint equilibrium $(\bar{y}, \bar{z})$ for the game. [12]

Calculate the equilibrium pay-off V for $(\bar{y}, \bar{z})$. [2]

(ii): Let $y^* := (1/V)\bar{y}$. Show that y^* is a solution to the linear program

$$(LP) \quad \begin{cases} \text{Maximize } y^T 1_2 \\ \text{over 2-vectors } y \text{ satisfying} \\ S^T y \leq 1_4 \\ y \geq 0 \end{cases}$$

Here, for an given integer n, 1_n is the n-vector $1_n := [1, \ldots, 1]^T$. [6]
Hint: Compute, and sketch, the region R in 2D space of vectors y satisfying the constraints of (LP), and hence show that the cost function $y^T 1_2$ cannot be increased, if we replace y^* by any other point in R.

2. Consider the multistage zero sum game, involving players A and B.

1st stage: A chooses L, M or R.(This is his first action.)

2rd stage: B observes whether the event 'A has chosen either L or M for his first action' has occured, or has not occured. B then chooses L or R.

3rd stage: A remembers his first action in the first stage and chooses L or R for his second action.

Figure 2 summarizes the information sets for each of the players, and also specifies the 'zero sum' pay-offs, which A seeks to minimize and B seeks to maximize.

(Note that, if A's first action is R, then the value of the pay-off does not depend on A's second action.)

(i): Reformulate the game as a 2-person game for A and B, in normal form. [13]

(ii): Identify any pure, weakly dominated strategies. [2]

(iii): Identify all Nash equilibria for the pair of players A and B in pure strategies, and state which of them are admissible. [5]

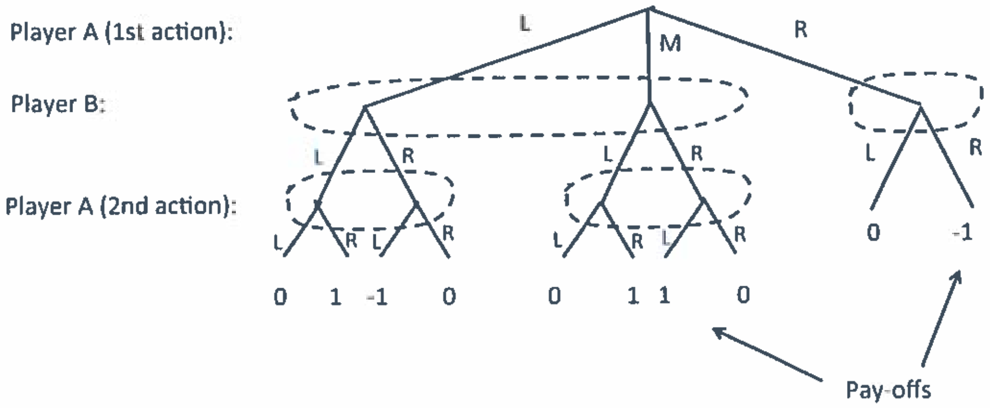

Figure 2

3: A network has N subscribers, each of whom can choose to dial up for a connection. However, because of capacity constraints, at most $N-1$ subscribers who dial up can be connected.

* If N subscribers dial up, none of the subscribers are connected.
* If $N-1$, or fewer, subscribers dial up, all of the subscribers are connected.

The satisfaction level of an individual subscriber is quantified as follows:

Satisfaction Level	Scenario
3	The subscriber dials up and is connected
0	The subscriber does not dial up
-1	The subscriber dials up and is not connected

Consider an N-person game, in which each subscriber is regarded as a player. For each i, the i'th subscriber chooses a probability to dial up. The pay-off for each subscriber is the expected value of that subscriber's satisfaction level, when all the subscribers dial up randomly and independently, according to their chosen probabilities. All subscribers wish to maximize their pay-offs.

(a): Calculate the probability $\bar{\beta}(N)$, $0 < \bar{\beta}(N) < 1$, such that $(\bar{\beta}(N), \ldots, \bar{\beta}(N))$ is a (symmetric) Nash equilibrium for the game. [16]

(b): Assume all subscribers are implementing their Nash equilibrium strategy. Determine the probability $p(N)$ that there is no communication to the network, due to overloading. Sketch $p(N)$, as $N \to \infty$. [4]

4(A): A: Consider a 3-person continuous game, involving the leader X and the 2 followers A and B. Player X chooses a real number $x > 0$. Players A and B observe x, then Player A chooses a number a and Player B chooses a number b simultaneously. (Players A and B have no knowledge of each other's choices.) The pay-offs for players X, A and B, which they each seek to minimize, are

$$L^X(x,a,b) = -a-b+x^2,\ L^A(x,a,b) = ab-xa+a^2-\frac{1}{2}b^2, L^B(x,a,b) = ab-2ax+b^2-\frac{1}{4}a^2 .$$

What value of x should X choose in each of the following two cases?

(i): Each player A and B chooses a safety strategy $a(x)$ and $b(x)$, respectively. (The safety strategies will depend on x). [9]

(ii): Players A and B choose a Nash equilibrium $(a'(x), b'(x))$. (The Nash equlibrium will depend on x.) [6]

B: A continuous game has 2 players, A and B. Player A chooses a number a and player B chooses a number b. Player A's pay-off is $L^A(a,b)$.

Define the inverted response curve (for Player B) to be the graph in 2D space of the function

$$\phi^+(a) := \arg\max_{b'} L^A(a,b') .$$

(Equivalently stated, 'for given a, $b = \phi^+(a)$ maximizes $L^A(a,b')$) over b''.)

Show that, if $\bar{a}$ is a safety strategy for the A player, then $(\bar{a}, \phi(\bar{a}))$ is a point at which the inverted response curve is tangent to a contour of points (a,b) on which $L^A(.,.)$ is a constant. [5]

(The above configuration is represented graphically in Figure 4. You should assume that $L^A(a,b)$ achieves a minimum *inside* the region R.)

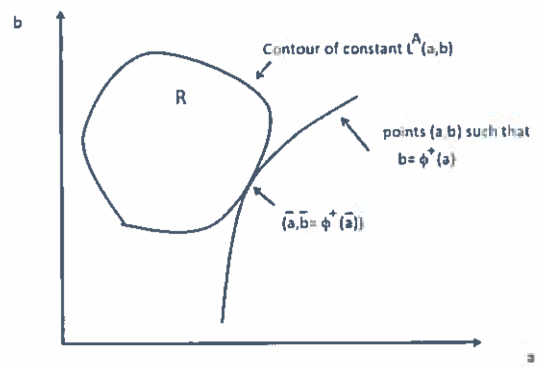

Figure 4

5(A): Consider the zero-sum dynamic game, with a scalar state variable:

$$\begin{cases} \underset{\{u_0(x)\ldots,u_{N-1}(x)\}}{\text{Minimize}} \quad \underset{\{v_0(x)\ldots,v_{N-1}(x)\}}{\text{Maximize}} \quad \sum_{t=0}^{N-1} L(x_t, u_t, v_t) + L(x_N) \\ \text{subject to} \\ x_{t+1} = f(x_t, u_t, v_t) \;\; \text{for } t = 0, \ldots, N-1 \\ u_t \in \Lambda \text{ and } v_t \in \Lambda \;\; \text{for } t = 0, \ldots, N-1 \\ x_0 \text{ a given number}. \end{cases}$$

involving two players, who choose state feedback strategies $\{u_0(x) \ldots, u_{N-1}(x)\}$ and $\{v_0(x) \ldots, v_{N-1}(x)\}$.

(i): Briefly summarise the dynamic programming approach to find a saddlepoint equilibrium, with respect to state feedback strategies, for the two players. [5]

(ii): Now assume that

$L(x) = x$

$f(x, u, v) = x(1 + u^T S(x) v)$

$S(x)$ is the state dependent 2×2 matrix $S(x) = \begin{bmatrix} 2 & 1 \\ 1 & x \end{bmatrix}$

and Λ is the simplex set in 2D space

$$\Lambda = \{(w_1, w_2) \mid w_1 \geq 0, w_2 \geq 0, w_1 + w_2 = 1\}.$$

Obtain formulae for:

(a): the saddlepoint feedback strategies of the two players [8]

(b): the pay-off for these strategies and also the corresponding pay-off. [7]

You should assume

Positivity condition: $\quad x > 1$,

for all values of the state x of interest.

Hint: Consider value functions $V_0(x), \ldots, V_N(x)$ that have the special structure

$$V_t(x) = k_t x + c_t, t = 0, \ldots, N,$$

for constants $k_0 > 0, \ldots, k_N > 0$ and constants $c_0, \ldots, c_N$.