IMPERIAL COLLEGE OF SCIENCE TECHNOLOGY AND MEDICINE
UNIVERSITY OF LONDON

[E303/ISE3.3]

DEPARTMENT of ELECTRICAL and ELECTRONIC ENGINEERING
EXAMINATIONS 2002

EEE/ISE PART III/IV: M.Eng., B.Eng. and ACGI

SOLUTIONS 2002
COMMUNICATION SYSTEMS

There are FOUR questions (Q1 to Q4)

Answer question ONE (in separate booklet) and TWO other questions.

Question 1 has 20 multiple choice questions numbered 1 to 20, all carrying equal marks.
There is only one correct answer per question.

Distribution of marks

Question-1: 40 marks
Question-2: 30 marks
Question-3: 30 marks
Question-4: 30 marks

The following are provided:

- ***A table of Fourier Transforms***
- ***A "Gaussian Tail Function" graph***

Examiner: Dr A. Manikas

ANSWER to Q1

1)	**A**	**B**	**C**	**D**	**E**
2)	**A**	**B**	**C**	**D**	**E**
3)	**A**	**B**	**C**	**D**	**E**
4)	**A**	**B**	**C**	**D**	**E**
5)	**A**	**B**	**C**	**D**	**E**
6)	**A**	**B**	**C**	**D**	**E**
7)	**A**	**B**	**C**	**D**	**E**
8)	**A**	**B**	**C**	**D**	**E**
9)	**A**	**B**	**C**	**D**	**E**
10)	**A**	**B**	**C**	**D**	**E**
11)	**A**	**B**	**C**	**D**	**E**
12)	**A**	**B**	**C**	**D**	**E**
13)	**A**	**B**	**C**	**D**	**E**
14)	**A**	**B**	**C**	**D**	**E**
15)	**A**	**B**	**C**	**D**	**E**
16)	**A**	**B**	**C**	**D**	**E**
17)	**A**	**B**	**C**	**D**	**E**
18)	**A**	**B**	**C**	**D**	**E**
19)	**A**	**B**	**C**	**D**	**E**
20)	**A**	**B**	**C**	**D**	**E**

ANSWER to Q2

a)

$$p_e = \mathbb{T}\left\{\sqrt{(1-\rho)\,EUE}\right\}$$

$$E_b = \frac{1}{2}\int_0^{T_{cs}} \left(s_0^2(t)+s_1^2(t)\right)dt = \frac{1}{2}\int_0^{8\mu}\left((1m)^2+(1m)^2\right)dt = \underbrace{8\times10^{-12}}$$

$$\rho = \frac{1}{E_b}\int_0^{8\mu} s_0(t)\,s_1(t)\,dt =$$

$$= \frac{1}{8\times10^{-12}}\left(\int_0^{2\mu}10^{-6}\,dt + \int_{2\mu}^{4\mu}(-10^{-6})\,dt + \int_{4\mu}^{6\mu}(-10^{-6})\,dt + \int_{6\mu}^{8\mu}10^{6}\,dt\right)$$

$$= 0$$

$$\text{i.e.}\left\{\begin{array}{l} E_b = 8\times10^{-12} \\ \rho = 0 \\ N_0 = 2\times10^{-12}\end{array}\right\} \Rightarrow EUE = \frac{8\times10^{-12}}{2\times10^{-12}} = 4$$

$$\Rightarrow p_e = \mathbb{T}\{\sqrt{4}\} = \mathbb{T}\{2\} = 2.2\times10^{-2}$$

Hamming (7,4) $\Rightarrow d_{min} = 3 \Rightarrow$ $\boxed{\text{1 bit in error can be corrected...}}$

$\downarrow \left(\frac{d_{min}-1}{2} = 1\right) \downarrow$

$\boxed{\text{...by moving to the nearest codeword}}$

$$\therefore Pr(\text{correctly decoded}) = Pr\left(\begin{array}{c}\text{0 bits in error}\\ \text{in a 7bit sequ.}\end{array}\right) + Pr\left(\begin{array}{c}\text{1 bit in error}\\ \text{in a 7bit sequ.}\end{array}\right)$$

$$= \binom{7}{0} p_e^0 (1-p_e)^{7-0} + \binom{7}{1} p_e^1 (1-p_e)^{7-1}$$

$$= 0.868 \qquad + \qquad 0.124$$

$$= 0.992$$

b)

$$p_e = \underbrace{Pr(r_2|m_1)\cdot Pr(m_1)}_{Pr(r_2,m_1)} + \underbrace{Pr(r_1|m_2)\cdot Pr(m_2)}_{Pr(r_1,m_2)}$$

$$= \underbrace{0.1\times0.25}_{0.025} + \underbrace{0.2\times0.75}_{0.15}$$

$$= 0.175$$

$$\underline{q} = \begin{bmatrix} Pr(r_1) \\ Pr(r_2)\end{bmatrix} = \underline{\underline{F}}\cdot\underline{p} = \begin{bmatrix}0.9, & 0.2\\ 0.1, & 0.8\end{bmatrix}\begin{bmatrix}0.25\\0.75\end{bmatrix} = \begin{bmatrix}0.375\\0.625\end{bmatrix}$$

$$\underline{\underline{J}} = \underline{\underline{F}}\cdot \mathrm{diag}(\underline{p}) = \begin{bmatrix}0.9\times0.25, & 0.2\times0.75\\ 0.1\times0.25, & 0.8\times0.75\end{bmatrix} = \begin{bmatrix}0.225, & 0.15\\ 0.025, & 0.6\end{bmatrix}$$

$$\mathbb{H}_{MUT} = \mathbb{H}_R - \mathbb{H}_{R|M}$$

$$\text{where}\left\{\begin{array}{l}\mathbb{H}_R = \underline{q}^T.\log_2(\underline{q}) = 0.9544 \\ \mathbb{H}_{R|M} = -\left\|\underline{\underline{J}} \odot \log_2 \underline{\underline{F}}\right\|_{1*} = 0.658695\end{array}\right.$$

$$\therefore \mathbb{H}_{MUT} = 0.9544 - 0.658695 = 0.295705$$

ANSWER to Q3
a)

$\mathrm{SNR}_q \geq 42\,\mathrm{dB}$ with $\mathrm{SNR}_q = 4.77 + 6\gamma - \alpha$ dB

PCM system with uniform quantizer:

α = CREST FACTOR in dBs = 13

$$\therefore\ 4.77 + 6\gamma - \underset{13}{\underset{\uparrow}{\alpha}} \geq 42 \Rightarrow 6\gamma \geq 42 + 13 - 4.77$$

$$\Rightarrow \gamma \geq 8.37 \Rightarrow \boxed{\gamma = 9 \text{ bits}}$$

μ-law ($\mu = 255$)

$$\alpha = 20\log_{10}\big(\ln(1+\underset{255}{\underset{\uparrow}{\mu}})\big) = 14.878$$

$$\Rightarrow 6\gamma = 42 + 14.878 - 4.77 \Rightarrow \gamma \geq 8.68 \text{ bits}$$

$$\Rightarrow \boxed{\gamma = 9 \text{ bits}}$$

A-law ($A = 87.6$)

$$\alpha = 20\log_{10}(1 + \ln A) = 14.764$$

$$\Rightarrow 6\gamma = 42 + 14.764 - 4.77 \Rightarrow \gamma \geq 8.665 \text{ bits}$$

$$\Rightarrow \boxed{\gamma = 9 \text{ bits}}$$

diff. quant.

$-10.23\,\mathrm{dB} < \alpha < 7.77\,\mathrm{dB}$

$\therefore\ 4.77 + 6\gamma + 10.23 \geq 42 \Rightarrow \gamma \geq 4.5 \Rightarrow \gamma = 5$ bits

or $4.77 + 6\gamma - 7.77 \geq 42 \Rightarrow \gamma \geq 7.5 \Rightarrow \gamma = 8$ bits

$$\therefore\ \boxed{5 \leq \gamma \leq 8 \text{ bits}}$$

i.e. unif, μ-law, A-law $\Rightarrow Q = 2^{\gamma} = 2^9 = 512$ levels

diff. quant $\Rightarrow 32 \leq Q \leq 256$ levels.

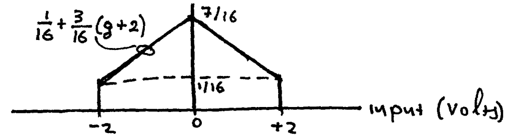
symbol rate: $r_s = \gamma \cdot F_s = 9 \times 2 \times 15\mathrm{K} = 270\mathrm{k}$ for $\begin{cases}\text{unif} \\ \text{A-law} \\ \mu\text{-law}\end{cases}$

and $\underbrace{5 \times 30\mathrm{k}}_{150\mathrm{k}} \leq r_s \leq \underbrace{8 \times 30\mathrm{k}}_{240\mathrm{k}}$ for diff. quant.

$$\therefore\ B_{PCM} = \frac{r_s}{2} \Rightarrow \begin{cases} B_{\substack{\text{unif} \\ \text{A-law} \\ \mu\text{-law}}} = 135\,\mathrm{kHz}. \\ \\ 75\,\mathrm{kHz} \leq B_{diff} \leq 120\,\mathrm{kHz} \end{cases}$$

b) *i)*

input pdf:



$$\therefore \text{ Power of } g(t) = P_g = \int_{-\infty}^{+\infty} g^2 \cdot \mathrm{pdf}_g(g)\, dg =$$

$$= 2\int_{-2}^{0} g^2 \cdot \left[\frac{1}{16} + \frac{3}{16}(g+2)\right] dg$$

$$= 2\int_{-2}^{0} \left(\frac{g^2}{16} + \frac{3}{16}g^3 + \frac{6}{16}g^2\right) dg$$

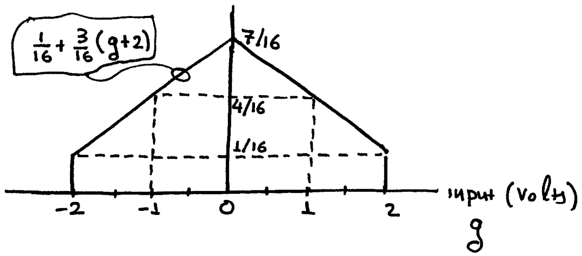
$$= 2\int_{-2}^{0} \left(\frac{7}{16}g^2 + \frac{3}{16}g^3\right) dg$$

$$= \frac{2\times 7}{16}\left.\frac{g^3}{3}\right]_{-2}^{0} + \frac{2\times 3}{16}\left.\frac{g^4}{4}\right]_{-2}^{0}$$

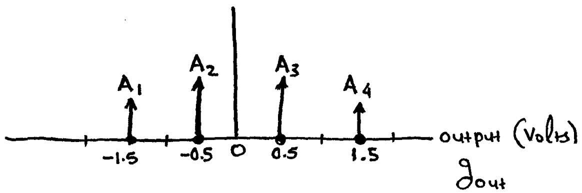
$$= \frac{5}{6} = 0.8333$$

ii)

input pdf:

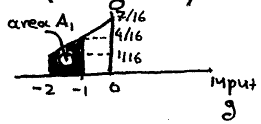


output pdf:



where $A_1 = \Pr(g_{out} = 1.5V) = \Pr(-2 < g < -1)$

$=$



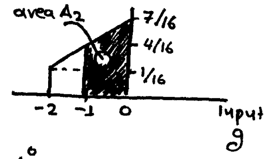
$$= \int_{-2}^{-1} \left(\frac{1}{16} + \frac{3}{16}(g+2) \right) dg =$$

$$= \left[\frac{1}{16} g + \frac{3}{16} \frac{(g+2)^2}{2} \right]_{-2}^{-1}$$

$$= -\frac{1}{16} + \frac{3}{32} + \frac{2}{16} - 0 = \frac{5}{32} = 0.1563$$

$A_2 = \Pr(g_{out} = -0.5V) = \Pr(-1 < g < 0)$

$=$



$$= \int_{-1}^{0} \left(\frac{1}{16} + \frac{3}{16}(g+2) \right) dg$$

$$= \left[\frac{1}{16} g + \frac{3}{16} \frac{(g+2)^2}{2} \right]_{-1}^{0}$$

$$= \frac{6}{16} + \frac{1}{16} - \frac{3}{32} = \frac{11}{32} = 0.3438$$

also

$$A_3 = A_2 = \frac{11}{32}$$

$$A_4 = A_1 = \frac{5}{32}$$

iii)

$$SNR_{out} = \frac{P_{g_{out}}}{P_{n_q}} = \frac{2 \times \left((-1.5)^2 \times 0.1563 + (-0.5)^2 \times 0.3438 \right)}{\Delta^2/12} =$$

$$= \frac{0.8752}{1/12} = 10.5030$$

iv)

$$r_s = 2 \times 2 \times 4K = 16 \frac{\text{ksymbols}}{\text{sec}} = 16k \frac{\text{levels}}{\text{sec}}$$

v)

$m_1 = -1.5V$; $m_2 = -0.5V$; $m_3 = 0.5V$; $m_4 = 1.5V$

$$(\underline{M} \times \underline{M}, \underline{g}) = \begin{bmatrix} (m_1 m_1, \frac{25}{1024}) & (m_2 m_1, \frac{55}{1024}) & (m_3 m_1, \frac{55}{1024}) & (m_4 m_1, \frac{25}{1024}) \\ (m_1 m_2, \frac{55}{1024}) & (m_2 m_2, \frac{121}{1024}) & (m_3 m_2, \frac{121}{1024}) & (m_4 m_2, \frac{55}{1024}) \\ (m_1 m_3, \frac{55}{1024}) & (m_2 m_3, \frac{121}{1024}) & (m_3 m_3, \frac{121}{1024}) & (m_4 m_3, \frac{55}{1024}) \\ (m_1 m_4, \frac{25}{1024}) & (m_2 m_4, \frac{55}{1024}) & (m_3 m_4, \frac{55}{1024}) & (m_4 m_4, \frac{25}{1024}) \end{bmatrix}$$

ANSWER to Q4

a)

$F_g = 4\,10^3$ Hz
$F_s = 2 \times F_g = 8 \times 10^3$ Hz
$Q = 2$

$\Pr(-2\mathrm{V}) = 3/4$
$\Pr(+2\mathrm{V}) = 1/4$

$N_0 = 2 \times 10^{-3}$

symbols	probabilities	Huffman	l_i (bits)
$x_1x_1x_1$	$27/64$	1	1
$x_1x_1x_2$	$9/64$	001	3
$x_1x_2x_1$	$9/64$	010	3
$x_2x_1x_1$	$9/64$	011	3
$x_1x_2x_2$	$3/64$	00000	5
$x_2x_1x_2$	$3/64$	00001	5
$x_2x_2x_1$	$3/64$	00010	5
$x_2x_2x_2$	$1/64$	00011	5

$$\bar{l} = 1 \times 27/64 + 3 \times 9/64 + 3 \times 9/64 + 3 \times 9/64 + 5 \times 3/64 + 5 \times 3/64 + 5 \times 3/64 + 5 \times 1/64 = 2.46875 \text{ bits/} \underbrace{\text{triple-level}}_{\text{(or 3-samples)}}$$

Alphabet: $\underline{X} = \left\{ \begin{matrix} x_1 = 1 \\ x_2 = 0 \end{matrix} \right\}$ (since $\Pr(x_1) > \Pr(x_2)$)

with probabilities: $\underline{p} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} \Pr(x_1) \\ \Pr(x_2) \end{bmatrix} = \begin{bmatrix} 0.6344 \\ 0.3656 \end{bmatrix}$

Note: $\Pr(x_2) = \frac{2}{3} \times \frac{9}{64} + \frac{2}{3} \times \frac{9}{64} + \frac{1}{3} \times \frac{9}{64} + \frac{5}{5} \times \frac{3}{64} + \frac{4}{5} \times \frac{3}{64} + \frac{4}{5} \times \frac{3}{64} + \frac{3}{5} \times \frac{1}{64}$
$= 0.3656$

$$\begin{aligned} p_e &= \Pr(y_2, x_1) + \Pr(y_1, x_2) \\ &= \Pr(y_2|x_1)\Pr(x_1) + \Pr(y_1|x_2)\Pr(x_2) \\ &= 0.05 \times 0.6344 + 0.2 \times 0.3656 \\ &= 0.1048 \end{aligned}$$

b)

$$\mathrm{H}_x = -\sum_{m=1}^{2} p_m.\log_2 p_m = -\underline{p}^T.\log_2 \underline{p} = 0.9473 \tfrac{\text{bits}}{\text{symbol}}$$

data rate:

$$r_{\text{data}} = r_b = F_s \tfrac{1}{3}\,\bar{l} = 6583.3 \text{ bits/sec}$$

information rate:

$$r_{\text{inf}} = r_b \times \mathrm{H}_x = r_b \times 0.9473 = 6236.4 \text{ bits/sec}$$

c)

$M = 2$ i.e. binary CS
Therefore: $T_{cs} = \frac{1}{r_{cs}} = 1.5190 \times 10^{-4}$ sec
$E_b = \frac{0.5^2}{2} T_{cs} \times \Pr(x_1) = 1.2046 \times 10^{-5}$

EUE $= \frac{E_b}{N_0} = 6.0228 \times 10^{-3}$ (data EUE)

BUE $= \frac{B}{r_{cs}} = \frac{B}{2B \times \log_2(M)} = \frac{1}{2}$ (data BUE with B denoting the baseband bandwidth)

data point $=$ (EUE,BUE) $= (6.0228 \times 10^{-3}, \frac{1}{2})$

d)

CS = inf.point = (EUE$_{inf}$, BUE$_{inf}$) = (data point) $\times \frac{\log_2(M)}{\mathbf{H}_{\text{mut}}}$

Therefore we have to estimate the mutual information $\mathbf{H}_{\text{mut}}$

$\mathbf{H}_{\text{mut}} = \mathbf{H}_Y - \mathbf{H}_{Y|X}$ or $(\mathbf{H}_{\text{mut}} = \mathbf{H}_X - \mathbf{H}_{X|Y})$

i.e.

$$\underline{p} = \begin{bmatrix} 0.6344 \\ 0.3656 \end{bmatrix} \qquad \mathbb{F} = \begin{bmatrix} 0.95, & 0.2 \\ 0.05, & 0.8 \end{bmatrix} \qquad \underline{q} = \mathbb{F}.\underline{p} = \begin{bmatrix} 0.6758 \\ 0.3242 \end{bmatrix}$$

$$\mathbb{B} = \text{diag}(\underline{q})^{-1}.\mathbb{F}.\text{diag}(\underline{p}) = \begin{bmatrix} 0.8918, & 0.1082 \\ 0.0978, & 0.9022 \end{bmatrix}$$

$$\mathbb{J} = \mathbb{F}.\text{diag}(\underline{p}) = \text{diag}(\underline{q}).\,\mathbb{B} = \begin{bmatrix} 0.6027, & 0.0731 \\ 0.0317 & 0.2925 \end{bmatrix}$$

$$\mathbf{H}_{\boldsymbol{X}} = -\sum_{m=1}^{2} p_m.\log_2\left(p_m\right) = -\underline{p}^T \log_2\left(\underline{p}\right) = 0.9473 \tfrac{\text{bits}}{\text{symbol}}$$

$$\mathbf{H}_{\boldsymbol{Y}} = -\sum_{k=1}^{2} p_k.\log_2\left(p_k\right) = -\underline{q}^T \log_2\left(\underline{q}\right) = 0.9089 \tfrac{\text{bits}}{\text{symbol}}$$

$$\mathbf{H}_{X \times \boldsymbol{Y}} = -\sum_{m=1}^{2}\sum_{k=1}^{2} J_{km}.\log_2\left(J_{km}\right) = -\left\|\mathbb{J} \odot \log_2\left(\mathbb{J}\right)\right\|_{1*} = 1.3929 \tfrac{\text{bits}}{\text{symbol}}$$

$$\mathbf{H}_{\boldsymbol{Y}|\boldsymbol{X}} = \mathbf{H}_{\boldsymbol{Y}|\boldsymbol{X}}(\mathbb{J}) \equiv -\sum_{m=1}^{2}\sum_{k=1}^{2} J_{km}.\log_2\left(\frac{J_{km}}{p_m}\right)$$

$$= -\left\|\mathbb{J} \odot \log_2\Big(\underbrace{\mathbb{J}.\text{diag}(\underline{p})^{-1}}_{\mathbb{F}}\Big)\right\|_{1*} = 0.4456 \tfrac{\text{bits}}{\text{symbol}}$$

$$\Rightarrow \boxed{\mathbf{H}_{\text{mut}} = \mathbf{H}_Y - \mathbf{H}_{Y|X} = 0.4633}$$

Therefore $\boxed{\text{CS} = \text{inf.point} = (0.013, 1.0792)}$

e)

$\Rightarrow$ $\boxed{\text{CS is not realizable}}$ (since $\text{EUE}_{inf} = 0.013 < 0.693$)

f)

$\text{SNR}_{\text{in}} = \frac{\text{EUE}}{\text{BUE}} = 0.012 \Rightarrow \text{SNR}_{\text{in}} = -19.2082\text{dB}$