IMPERIAL COLLEGE LONDON

EE4-25
EE9-CS1-2
EE9-FPN2-06

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2018

MSc and EEE/EIE PART IV: MEng and ACGI

Corrected copy

DESIGN OF LINEAR MULTIVARIABLE CONTROL SYSTEMS

Monday, 14 May 10:00 am

Time allowed: 3:00 hours

There are FOUR questions on this paper.

Answer ALL questions

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : I.M. Jaimoukha
Second Marker(s) : E.C. Kerrigan

1. a) Let the n-th order transfer matrix $G(s)$ have a state space realisation

$$G(s) \stackrel{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$$

and assume that A is stable. Let

$$AP + PA^T + BB^T = 0, \qquad A^T Q + QA + C^T C = 0,$$

for some

$$P = P^T = \begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad Q = Q^T = \begin{bmatrix} 0 & 0 \\ 0 & Q_2 \end{bmatrix},$$

where $\mathscr{R}^{n_1 \times n_1} \ni P_1 \succ 0$ and $\mathscr{R}^{n_2 \times n_2} \ni Q_2 \succ 0$ and where $n_1 + n_2 = n$. By partitioning the realisation for $G(s)$ compatibly with P and Q, prove that the realisation can be decomposed into two subsystems:

i) A subsystem with n_1 modes that are stable, controllable and unobservable. [4]

ii) A subsystem with n_2 modes that are stable, uncontrollable and observable. [4]

Draw a diagram illustrating the two subsystems of $G(s)$. [4]

b) Let

$$\mathscr{S} = \{(A_1, B_1), \ldots (A_N, B_N)\},$$

where $A_i \in \mathscr{R}^{n \times n}$ and $B_i \in \mathscr{R}^{n \times m}, i = 1, \ldots, N$ are given. The set $\mathscr{S}$ is called simultaneously stabilisable if there exists $K \in \mathscr{R}^{m \times n}$ such that $A_i + B_i K$ is stable for $i = 1, \ldots, N$.

i) Suggest a set of necessary and sufficient conditions, in the form of matrix inequalities, for the simultaneous stabilisability of $\mathscr{S}$. [4]

ii) Suggest a set of sufficient conditions, in the form of linear matrix inequalities, for the simultaneous stabilisability of $\mathscr{S}$. Explain how K may be obtained if these conditions are satisfied. [4]

2. a) Consider a state-variable model described by the dynamics

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t) + Du(t),$$

and let $H(s) = D + C(sI - A)^{-1}B$ denote the corresponding transfer matrix. Suppose there exists a $P = P^T \succ 0$ such that

$$\begin{bmatrix} A^TP + PA + C^TC & PB + C^TD \\ B^TP + D^TC & D^TD - \gamma^2 I \end{bmatrix} \prec 0.$$

i) Prove that A is stable. [4]

ii) By defining suitable Lyapunov and cost functions and completing a square, prove that

$$\|H\|_\infty < \gamma.$$

[4]

iii) By using the fact that $\|H\|_\infty = \|H^T\|_\infty$ derive a dual set of conditions for $H(s)$ stable and $\|H\|_\infty < \gamma$. [4]

b) Consider a state-variable model subject to disturbance $d(t)$ described by the dynamics

$$\dot{x}(t) = Ax(t) + Bu(t) + B_d d(t), \qquad z(t) = Cx(t) + Du(t) + D_d d(t).$$

A state-feedback controller of the form $u(t) = Kx(t)$ is to be designed such that, for a given $\gamma > 0$, $\|T_{zd}\|_\infty < \gamma$ where $T_{zd}(s)$ denotes the transfer matrix from d to z.

i) Derive a state space realization for $T_{zd}(s)$. [4]

ii) By using the answer to Part (aiii) above, or otherwise, derive sufficient conditions for the existence of a feasible K. Your conditions should be in the form of the existence of certain solutions to linear matrix inequalities. [4]

3. Consider the linear quadratic regulator (LQR) in Figure 3 for which $x(0) = x_0$ and where $d(t)$ denotes a disturbance. Let $z = \begin{bmatrix} z_1^T & z_2^T \end{bmatrix}^T$. A stabilizing state-feedback gain matrix F is to be designed such that, when $d(t) = 0, \forall t$, the cost function $J = \|z\|_2^2$ is minimized.

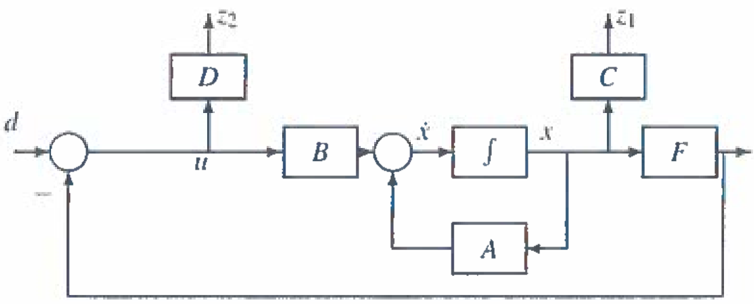

Figure 3

a) State the assumptions needed on the matrices A, B, C and D to obtain a solution and define a suitable Lyapunov function $V(t)$. [4]

b) Obtain an expression for $\int_0^\infty \dot{V}(t)dt$. State clearly any assumptions made. [4]

c) Find an expression for F that minimizes J. Give also the algebraic Riccati equation that needs to be satisfied. What is the minimum value of J? [4]

d) Prove now that the closed loop system in Figure 3 is stable. State the assumptions required to guarantee stability. [4]

e) Assume that $D = I$. By evaluating a state-space realisation for $T_{z_1 d}(s)$ in Figure 3 and using the Bounded Real Lemma (given in Question 2 above), prove that $\|T_{z_1 d}\|_\infty \leq 1$. Ignore issues relating to strict matrix inequalities. [4]

4. Consider the $\mathcal{H}_\infty$ regulator shown in Figure 4 for which it is assumed that the pair (A, B_2) is controllable, the triple (A, B_1, C_1) is minimal and $x(0) = x_0$. Let $H(s)$ denote the transfer matrix from w to $z = [\, z_1^T \;\; z_2^T \,]^T$ and let $\gamma > 0$ be given.

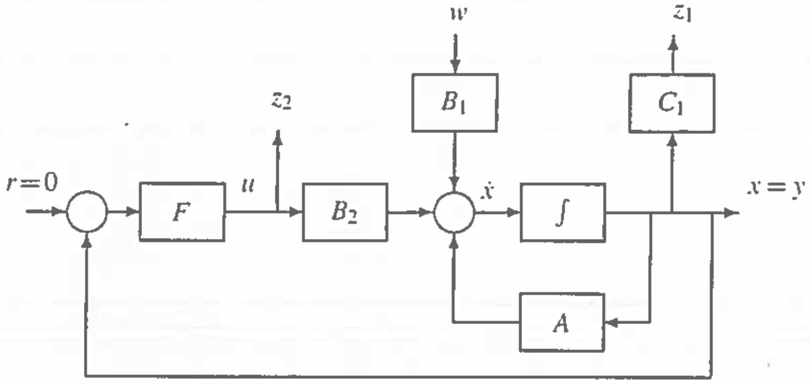

Figure 4

A stabilizing state-feedback gain matrix F is to be designed such that $\|H\|_\infty \leq \gamma$ for a given $\gamma > 0$.

a) Write down the generalized regulator formulation for this design problem. [5]

b) By defining a suitable Lyapunov function and a cost function J involving an auxiliary variable $X = X'$, and carrying out two completions of squares, derive an expression for J that can be used to solve the design problem. [5]

c) Use the expression for J to formulate the design problem and derive sufficient conditions for its solution. Your conditions should be in the form of the existence of a certain solution to an algebraic Riccati equation. It should also include an expression for F and an expression for the worst–case disturbance w^*. [5]

d) Let $\gamma \to \infty$. Write down the cost function and the Riccati equation and compare the solution of the $\mathcal{H}_\infty$ regulator to the solution of the LQR in Question 3 above. [5]