Imperial College
London

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2013

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Measure & Integration

Date: Friday, 31 May 2013. Time: 2.00pm. Time allowed: 2 hours.

This paper has FOUR questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the main book is full.

Statistical tables will not be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Answer all the questions. Each question carries equal weight.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Calculators may not be used.

1. i. Consider sets of the plane in $E = \{0 \leq x \leq 1, 0 \leq y \leq 1\}$. Give the definition of the outer measure μ^*, and formulate its property of subadditivity.
ii. Let $f : X \to \mathbb{R}$, be a function on a complete measure space.

a) If f is measurable, does it follow that the absolute value $|f|$ is measurable?

b) If $|f|$ is measurable, does it follow that f is measurable?

Justify your answer.

2. i. State and prove the Chebyshev inequality.
ii. Explaining your reasoning, calculate the limit of the Lebesgue integrals

$$\lim_{n\to\infty} \int_0^1 \frac{\log(x+n)}{n\sqrt{x}} dx.$$

3. i. Give the definition of a finite signed measure and of a negative set with respect to it.
ii. Prove that if f is of bounded variation on $[a, b]$, and $v(x) = V_a^x(f)$ (the variation of f on $[a, x]$) is absolutely continuous on $[a, b]$, then f is absolutely continuous on $[a, b]$.

4. Formulate and prove Lebesgue theorem (dominated convergence theorem) in a space of finite measure.

EXAMINATION SOLUTIONS 2012-13

Course P19

Question 1

Marks & seen/unseen

Parts

i

The outer measure $\mu^*(A)$ of a set $A \subset E$ is

$$\mu^*(A) = \inf_{A \subset \cup P_k} \sum_k m(P_k)$$

over all coverings of A by finite or countable number of rectangles P_k; $m(P_k)$ - measure of the rectangle P_k.

seen 5

μ^* is subadditive, which means that if $A \subset \bigcup_{n\in I} A_n$, the set I - finite or countable, then $\mu^*(A) \leq \sum_{n\in I} \mu^*(A_n)$

Setter's initials

Checker's initials

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	EXAMINATION SOLUTIONS 2012-13	Course
Question 1		Marks & seen/unseen
Parts		
ii a	Since f is measurable and $\lvert\cdot\rvert$ is continuous, $\lvert f\rvert$ is measurable as a continuous function of a measurable function.	unseen 15
ii b	Let $A \subset X$ be nonmeasurable (such sets exist as shown in lectures). Let $f(x) = \begin{cases} -1, & x \in A \\ 1, & x \notin A \end{cases}$ Then f is not measurable as the preimage of the Borel set $\{-1\}$ is not measurable. However, $\lvert f\rvert \equiv 1$ is clearly measurable.	
	Setter's initials	Checker's initials

	EXAMINATION SOLUTIONS 2012-13	Course
Question 2		Marks & seen/unseen
Parts		
i.	Let $f(x)$ be integrable on A and nonnegative, let $C>0$. Then $\mu\{x\in A : f(x)\geq C\} \leq \frac{1}{C}\int_A f\,d\mu$	seen 5
	<u>Proof</u>	
	Let $A'=\{x\in A : f(x)\geq C\}$	
	Then	
	$\int_A f\,d\mu = \int_{A'} f\,d\mu + \int_{A\setminus A'} f\,d\mu \geq$	
	$\geq \int_{A'} f\,d\mu \geq C\,\mu(A')$.	

	EXAMINATION SOLUTIONS 2012-13	Course
Question 2		Marks & seen/unseen
Parts ii	$$\lim_{n\to\infty}\int_0^1 \frac{\log(x+n)}{n\sqrt{x}}\,dx = 0$$	unseen 15

Proof. For any $\varepsilon > 0$

$$\left|\frac{\log(x+n)}{n}\right| \leq \frac{\log(n+1)}{n} < \varepsilon$$

for all $n > N_0(\varepsilon)$. Hence

$$\left|\frac{\log(x+n)}{n\sqrt{x}}\right| \leq \frac{\varepsilon}{\sqrt{x}}, \quad n > N_0(\varepsilon)$$

Since the Riemann integrals

$$\int_0^1 \frac{\log(x+n)}{n\sqrt{x}}\,dx, \quad \int_0^1 \frac{dx}{\sqrt{x}} \quad \text{converge}$$

absolutely, $\frac{\log(x+n)}{n\sqrt{x}}$ and $\frac{1}{\sqrt{x}}$ are integrable (in Lebesque sense) and by Lebesque theorem

$$\lim_{n\to\infty}\int_0^1 \frac{\log(x+n)}{n\sqrt{x}}\,dx = \int_0^1 \lim_{n\to\infty}\frac{\log(x+n)}{n\sqrt{x}}\,dx = 0.$$

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	EXAMINATION SOLUTIONS 2012-13	Course
Question 3		Marks & seen/unseen
Parts i	A finite signed measure is a finite real-valued σ-additive function of a set $\Phi(A)$ defined on a σ-algebra $\mathcal{S}$ of subsets of a space X. A set $E \in \mathcal{S}$ is called negative with respect to Φ if $\Phi(E \cap F) \leq 0 \ \forall F \in \mathcal{S}$.	seen 5
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	EXAMINATION SOLUTIONS 2012-13	Course
Question 3		Marks & seen/unseen
Parts ii	Let $f \in$ B.V. on $[a,b]$, $V(x) = V_a^x(f)$ be a.c. on $[a,b]$, i.e. $\forall \varepsilon > 0\ \exists \delta > 0$ such that $\sum_{j=1}^{n} \lvert V(b_j) - V(a_j) \rvert < \varepsilon$ for any family of disjoint subintervals (a_j, b_j) of $[a,b]$ satisfying $\sum_{j=1}^{n} \lvert b_j - a_j \rvert < \delta$. However, $V(b_j) - V(a_j) = V_{a_j}^{b_j}(f) = \sup\limits_{\substack{\text{finite} \\ \text{subdiv. of } [a_j, b_j]}} \sum_{k=1}^{n} \lvert f(x_k) - f(x_{k-1}) \rvert \geq \lvert f(b_j) - f(a_j) \rvert$, so $\sum_{j=1}^{n} \lvert f(b_j) - f(a_j) \rvert < \varepsilon \Rightarrow f$ is a.c.	unseen 15
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	EXAMINATION SOLUTIONS 2012-13	Course
Question 4		Marks & seen/unseen
Parts	Lebesgue thm. $\mu(A) < \infty$ Let f_n, $n \in \mathbb{N}$, be integrable on A, $f_n \to f$ a.e. on A, and $\lvert f_n(x) \rvert \leq \varphi(x)$ a.e. on A, where $\varphi(x)$ is integrable. Then f is integrable and $\int_A f_n \, d\mu \to \int_A f \, d\mu$	5 seen
	<u>Proof</u>. We have $\lvert f(x) \rvert \leq \varphi(x)$, and, moreover, f is measurable as an a.e. limit of measurable f_n's. Therefore, f is integrable Fix $\varepsilon > 0$. By absolute continuity of the integral $\exists \, \delta > 0$ s.t. $\int_B \varphi \, d\mu < \varepsilon/4$	15 seen
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	EXAMINATION SOLUTIONS 2012-13	Course
Question 4		Marks & seen/unseen
Parts		

for any set B satisfying $\mu(B)<\delta$.

By Egorov's theorem, set B here can be chosen s.t. $f_n \to f$ uniformly on $C = A \setminus B$, so $\exists N$ s.t. $\forall n \geq N$, $\forall x \in C$:

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2\mu(C)}.$$

Then $\int_A f\,d\mu - \int_A f_n\,d\mu =$

$$= \int_C (f-f_n)\,d\mu + \int_B f\,d\mu - \int_B f_n\,d\mu$$

$\Longrightarrow$

$$\left|\int_A f\,d\mu - \int_A f_n\,d\mu\right| \leq \frac{\varepsilon}{2\mu(C)}\mu(C) +$$

$$+ \underbrace{\int_B |f|\,d\mu}_{\leq \int_B \varphi\,d\mu} + \underbrace{\int |f_n|\,d\mu}_{\leq \int_B \varphi\,d\mu} \leq \varepsilon, \quad n>N.$$

$\varepsilon>0$ is arbitrary

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