1 (a) Let $x(t) =: \begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix} \Rightarrow y(t) = \alpha(t) \Rightarrow \alpha(t) = q(t)$ [1]

$$\dot{x}(t) = \begin{bmatrix} \beta(t) \\ -u(t)\beta(t) \end{bmatrix} = \begin{bmatrix} \dot\alpha(t) \\ \dot\beta(t) \end{bmatrix} \Rightarrow \dot\alpha(t) = \dot q(t) \Rightarrow \beta(t) = \dot q(t) \quad [1]$$

Check: $\dot\beta(t) = \ddot q(t) = -u(t)\dot q(t)$, same as ODE.

(b) $\exp(M) := I + M + M^2/2! + \frac{M^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{M^n}{n!}$ [1]

(c) $M := \begin{bmatrix} 0 & h \\ 0 & 0 \end{bmatrix} \Rightarrow M^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow M^n = 0 \ \forall n \geq 2$ [1]

$$\Rightarrow \exp\begin{bmatrix} 0 & h \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & h \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} \quad [1]$$

(d) Let $X := \begin{bmatrix} 0 & h \\ 0 & -\alpha h \end{bmatrix} \Rightarrow X^2 = \begin{bmatrix} 0 & -h^2\alpha \\ 0 & h^2\alpha^2 \end{bmatrix} \Rightarrow X^3 = \begin{bmatrix} 0 & h^3\alpha^2 \\ 0 & -h^3\alpha^3 \end{bmatrix}$ ....etc.

$$\Rightarrow X^n = \begin{bmatrix} 0 & (-1)^{n-1}h^n\alpha^{n-1} \\ 0 & (-1)^n h^n \alpha^n \end{bmatrix} \ \forall n \geq 1 \quad [2]$$

Now $e^{-\alpha h} = \sum_{n=0}^{\infty} \frac{1}{n!}(-\alpha h)^n = 1 + \sum_{n=1}^{\infty} \frac{1}{n!}(-1)^n \alpha^n h^n$ [1]

$$\Rightarrow \frac{1-e^{-\alpha h}}{\alpha} = \sum_{n=1}^{\infty} \frac{1}{n!}(-1)^{n-1}\alpha^{n-1}h^n \quad [1]$$

$$\exp(X) = I + \sum_{n=1}^{\infty} \frac{1}{n!} \begin{bmatrix} 0 & (-1)^{n-1}\alpha^{n-1}h^n \\ 0 & (-1)^n \alpha^n h^n \end{bmatrix}$$

$$\Rightarrow F(\alpha) = \exp(X) = \begin{bmatrix} 1 & \sum_{n=1}^{\infty} \frac{1}{n!}(-1)^{n-1}\alpha^{n-1}h^n \\ 0 & 1 + \sum_{n=1}^{\infty}\frac{1}{n!}(-1)^n\alpha^n h^n \end{bmatrix} = \begin{bmatrix} 1 & (1-e^{-\alpha h})/\alpha \\ 0 & e^{-\alpha h} \end{bmatrix} \quad [1]$$

(e) $\forall t \in [kh, (k+1)h)$ we have

$$\dot x(t) = \begin{bmatrix} 0 & 1 \\ 0 & -u_k \end{bmatrix} x(t) \Rightarrow x((k+1)h) = \exp\left(\begin{bmatrix} 0 & 1 \\ 0 & -u_k \end{bmatrix} h\right) x(k) \quad [2]$$

$$\Rightarrow x_{k+1} = \exp\Bigg(\overbrace{\underbrace{\begin{bmatrix} 0 & h \\ 0 & -u_k h \end{bmatrix}}_{= X \text{ if } u_k = \alpha \neq 0}}^{= M \text{ if } u_k = 0}\Bigg) x(k) \quad [2]$$

(f) $y_k = [1 \ 0]\, x_k = \dots \Rightarrow y_0 = [1 \ 0]\overbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}^{x_0} = 1$ [1]

$x_1 = A(u_0)x_0 = A(1)x_0 = \begin{pmatrix} 2-e^{-1} \\ e^{-1} \end{pmatrix} \Rightarrow y_1 = 2 - e^{-1} = 1.6321$ [2]

$\Rightarrow x_2 = A(u_1)x_1 = A(0)x_1 = \begin{pmatrix} 2 \\ e^{-1} \end{pmatrix} \Rightarrow y_2 = 2$ [2]

where $A(0) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$\Rightarrow x_3 = A(u_2)x_1 = A(1)x_2 = \begin{pmatrix} 2-e^{-2}+e^{-1} \\ e^{-2} \end{pmatrix} \Rightarrow y_3 = 2 - e^{-2} + e^{-1} = 2.325$ [1]

{in above, $A(1) = \begin{bmatrix} 1 & 1-e^{-1} \\ 0 & e^{-1} \end{bmatrix}$}

2(a) $\theta := (u_0' \;\; u_1' \; \cdots \; u_{N-1}' \;\; s_0' \;\; s_1' \; \cdots \; s_{N-1}')$ [2]

where each $u_k \in \mathbb{R}^m$ and slack variable $s_k \in \mathbb{R}^p$ [1]

(b) H is square, so has as many rows as columns [1]

H has as many columns as θ and $\theta \in \mathbb{R}^{mN+Np}$ [1]

$\Rightarrow H$ as $(p+m)N$ rows

(c) Cost $\sum_{k=0}^{N-1} \|d(Fx_{k+1}-g)\|_1 = \min_{\substack{s_k \\ k=0,\dots,N-1}} \sum_{k=0}^{N-1} s_k$ s.t. $\begin{cases} Fx_{k+1}-g \leq s_k \\ s_k \geq 0 \end{cases} \forall k \in \{0,\dots,N-1\}$

$\Rightarrow$ there are $2pN$ constraints to do with 1-norm stage cost [1]

and $2mN$ constraints due to input constraints [1]

$\Rightarrow M$ has $2(p+m)N$ rows [1]

(d) Many ways to answer. The following is probably the most likely way that students will attempt.

$$\bar{x} := \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} = \Phi \hat{x} + \Gamma \underbrace{\begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{pmatrix}}_{\bar{u}}, \text{ where } \underbrace{\Phi := \begin{pmatrix} A \\ A^2 \\ \vdots \\ A^N \end{pmatrix}, \Gamma := \begin{pmatrix} B & 0 & \cdots & 0 \\ AB & \ddots & & \vdots \\ \vdots & & & 0 \\ A^{N-1}B & A^{N-2}B & \cdots\ AB & B \end{pmatrix}}_{\text{no need to derive} \Rightarrow \text{can just quote as recalled from lectures}}$$

Input constraints can be written as $\mathbb{1}_N \otimes u_\ell \leq \bar{u} \leq \mathbb{1}_N \otimes u_h$, where $\mathbb{1}_N$ is a column vector of ones and $\otimes$ denotes Kronecker product

$$\Leftrightarrow \begin{bmatrix} I_{Nm} \\ -I_{Nm} \end{bmatrix} \bar{u} \leq \begin{bmatrix} \mathbb{1}_N \otimes u_h \\ -\mathbb{1}_N \otimes u_\ell \end{bmatrix}$$

From part (c), the constraints due to the slack variables can be written as:

$$(I_N \otimes F)\bar{x} - I_{Np}\bar{s} \leq \mathbb{1}_N \otimes g \quad (*)$$

$$-I_{Np}\bar{s} \leq 0$$

$$\theta = \begin{pmatrix} \bar{u} \\ \bar{s} \end{pmatrix}$$

where $\bar{s} := (s_0' \;\; s_1' \;\; s_2' \; \cdots \; s_{N-1}')'$

$$(*) \Leftrightarrow (I_N \otimes F)(\Phi\hat{x} + \Gamma\bar{u}) - I_{Np}\bar{s} \leq \mathbb{1}_N \otimes g$$

$$\Rightarrow M = \begin{pmatrix} I_{Nm} & 0_{Nm \times Np} \\ -I_{Nm} & 0_{Nm \times Np} \\ (I_N \otimes F)\Gamma & -I_{Np} \\ 0_{Np \times Nm} & -I_{Np} \end{pmatrix} \begin{matrix} [1] \\ [1] \\ [1] \\ [1] \end{matrix}$$

e) $$f = \begin{pmatrix} \mathbb{1}_N \otimes u_h \\ -\mathbb{1}_N \otimes u_l \\ \mathbb{1}_N \otimes g - (I_N \otimes F)\Phi \hat{x} \\ 0_{Np \times 1} \end{pmatrix} \quad \begin{matrix} \}\,[1] \\ \\ \}\,[1] \\ \end{matrix}$$

(f) $$\text{Cost} = \|\underbrace{(I_N \otimes Q)}_{\bar{Q}} \bar{x}\|_2^2 + \|\underbrace{(I_N \otimes R)}_{\bar{R}} \bar{u}\|_2^2 + \rho \sum_{k=0}^{N-1} s_k$$

$$= \left\| \begin{pmatrix} \bar{Q}\Phi\hat{x} + \bar{Q}\Gamma\bar{u} \\ \bar{R}\bar{u} \end{pmatrix} \right\|_2^2 + \rho \mathbb{1}_{Np}^T \bar{s}$$

$$= \left\| \underbrace{\begin{pmatrix} \bar{Q}\Gamma \\ \bar{R} \end{pmatrix}}_{W} \bar{u} + \underbrace{\begin{pmatrix} \bar{Q}\Phi\hat{x} \\ 0 \end{pmatrix}}_{g} \right\|_2^2 + \rho \mathbb{1}_{Np}^T \bar{s}$$

$$= \bar{u}^T W^T W \bar{u} + 2g^T W \bar{u} + g^T g + \rho \mathbb{1}_{Np}^T \bar{s}$$

$$= \underbrace{\begin{pmatrix} \bar{u} \\ \bar{s} \end{pmatrix}^T}_{\theta} \begin{pmatrix} W^T W & 0 \\ 0 & 0 \end{pmatrix} \underbrace{\begin{pmatrix} \bar{u} \\ \bar{s} \end{pmatrix}}_{\theta} + \begin{pmatrix} 2g^T W & \rho \mathbb{1}_{Np}^T \end{pmatrix} \underbrace{\begin{pmatrix} \bar{u} \\ \bar{s} \end{pmatrix}}_{\theta} + \underbrace{g^T g}_{\text{not function of } \theta, \text{ so can ignore.}}$$

$$\Rightarrow H = \begin{bmatrix} 2W^T W & 0_{mN \times pN} \\ 0_{pN \times mN} & 0_{pN \times pN} \end{bmatrix} \quad \begin{matrix} [1] \\ [1] \end{matrix}$$

(Check defn of W by student is correct)

(g) $$\Rightarrow c = \begin{pmatrix} 2W^T g \\ \rho \mathbb{1}_{Np} \end{pmatrix} \quad \begin{matrix} [1] \\ [2] \end{matrix}$$

(Check defn of g by student is correct. Size of $\mathbb{1}$ also has to be correct)

3 (a) The system is open-loop asymptotically stable $\Rightarrow$ all e/values of A inside unit disk [1]

$\Rightarrow I-A$ has no e/values at zero $\Rightarrow (I-A)^{-1}$ exists [1]

(b) At equilibrium $x_e = Ax_e + BFr$

$$\Leftrightarrow x_e = (I-A)^{-1}BFr$$

Want $Hx_e = r \Leftrightarrow H(I-A)^{-1}BFr = r \quad \forall r$

$$\Leftrightarrow H(I-A)^{-1}BF = 1$$

$$\Leftrightarrow F = 1/\left(H(I-A)^{-1}B\right) \text{ and } H(I-A)^{-1}B \neq 0$$

(c) We can eliminate s_j, w_j from constraints, so only inequality constraints remain. There are 2 inequality constraints from

$$0 \le \rho \le 1 \Leftrightarrow \left\{\begin{matrix} \rho \le 1 \\ -\rho \le 0 \end{matrix}\right\}.$$

Since $C \in \mathbb{R}^{q\times n}$, it follows there are qN remaining inequality constraints from $Cs_j \le d$, $j = 1,\dots,N$

$\Rightarrow M = qN + 2$ [2]

(d) Define the augmented state $\tilde{x}_j := \begin{pmatrix} s_j \\ w_{j-1} \end{pmatrix} \Rightarrow \tilde{x}_0 = \begin{pmatrix} x_k \\ v_{k-1} \end{pmatrix}$

$$\Rightarrow \tilde{x}_{j+1} = \begin{pmatrix} s_{j+1} \\ w_j \end{pmatrix} = \underbrace{\begin{pmatrix} A & BF \\ 0 & 1 \end{pmatrix}}_{\tilde{A}} \begin{pmatrix} s_j \\ w_{j-1} \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ r - v_{k-1} \end{pmatrix}}_{\tilde{B}} \rho$$

Note $s_j = [I_n \ 0]\tilde{x}_j \Rightarrow Cs_j = C[I_n \ 0]\tilde{x}_j$

$$\tilde{x}_1 = \tilde{A}\tilde{x}_0 + \tilde{B}\rho$$

$$\tilde{x}_2 = \tilde{A}\tilde{x}_1 + \tilde{B}\rho = \tilde{A}^2\tilde{x}_0 + \tilde{A}\tilde{B}\rho + \tilde{B}\rho$$

$$\Rightarrow \tilde{x}_j = \tilde{A}^j\tilde{x}_0 + \sum_{i=0}^{j-1}\tilde{A}^i\tilde{B}\rho$$

$$\begin{pmatrix} Cs_1 \\ Cs_2 \\ \vdots \\ Cs_N \end{pmatrix} = \begin{pmatrix} C[I_n \ 0]\tilde{x}_1 \\ C[I_n \ 0]\tilde{x}_2 \\ \vdots \\ C[I_n \ 0]\tilde{x}_N \end{pmatrix} = I_N \otimes \left[C[I_n \ 0]\right] \left[\begin{pmatrix} \tilde{A} \\ \tilde{A}^2 \\ \vdots \\ \tilde{A}^N \end{pmatrix}\tilde{x}_0 + \begin{pmatrix} \cancel{A}\tilde{B} \\ \tilde{A}\tilde{B}+\tilde{B} \\ \vdots \\ \sum_{i=0}^{N-1}\tilde{A}^i\tilde{B} \end{pmatrix}\rho\right]$$

3 d) continued..

therefore the inequality constraints $Cs_j \le d$, $j = 1, \dots, N$ becomes

$$\underbrace{I_N \otimes \left[C[I_n\ 0]\right]}_{\bar{C}} \left[\underbrace{\begin{pmatrix} \tilde{A} \\ \tilde{A}^2 \\ \vdots \\ \tilde{A}^N \end{pmatrix}}_{\Phi} \tilde{x}_0 + \underbrace{\begin{pmatrix} \tilde{B} \\ \tilde{A}\tilde{B} + \tilde{B} \\ \vdots \\ \sum_{i=0}^{N-1} \tilde{A}^i \tilde{B} \end{pmatrix}}_{\Gamma} \rho \right] \le \mathbb{1}_N \otimes d$$

Recall also that we have the constraints $0 \le \rho \le 1$

$$\Rightarrow G = \begin{bmatrix} \overbrace{I_N \otimes \left[C[I_n\ 0]\right]}^{\bar{C}} \overbrace{\begin{bmatrix} \tilde{B} \\ \tilde{A}\tilde{B} + \tilde{B} \\ \vdots \\ \sum_{i=0}^{N-1} \tilde{A}^i \tilde{B} \end{bmatrix}}^{\Gamma} \\ +1 \\ -1 \end{bmatrix}$$

[2] for Γ

[2] for $\bar{C}$

[1]

and

$$h = \begin{bmatrix} \mathbb{1}_N \otimes d - \bar{C}\,\Phi \begin{pmatrix} x_k \\ v_{k-1} \end{pmatrix} \\ +1 \\ 0 \end{bmatrix}$$

[1]

[1]

G is a function of r, since $\tilde{B}$ is a function of r, but h is not, since it is not a function of $\tilde{B}$. [1]

There could be many variations on above derivation.
Important point to note is that G is a column vector, not a matrix

4)

(a) At each sample instant, a new measurement is taken to produce a new state estimate. This is used to set up a new optimization problem, and only the first part of the solution is implemented as a new control input. At the next sample, a new measurement is used to recompute the control. [3]

(b) To compensate for uncertainty. [1]

(c) Feedback can destabilize an ~~already~~ open-loop stable system. [2]
Feedback can inject measurement noise into a system where there previously was no noise.

(d) In receding horizon control, an optimal control problem with a fixed horizon is solved at each sample instant. In variable horizon control, the horizon is also a decision/optimization variable. [2]

(e) In a satellite docking / aircraft landing example, the time taken may need to be optimized and there is a chance in a receding horizon formulation that the target may never be reached in finite time. (Any other sensible example & motivation acceptable). [2]

(f) When minimizing fuel in a satellite control example. Fuel used is a linear and not quadratic function of thrust. (Any other is OK) [2]

g) An aeroplane coming into land. If you exceed the constraint, that the aircraft stay above the ground, you have a crash! [2]
(Any other ok).

h) Temperature in a room. There are comfort ranges, but exceeding these for a short while is often acceptable. (Any other is OK) [2]

(i) The value function is used as a Lyapunov function. The terminal weight of the cost function is a Control Lyapunov Function and the terminal constraint has to define a control invariant set associated with the same control law as the terminal weight. Using a shifting argument and appending the terminal control law, one can prove that ~~the~~ the control sequence is feasible and that the value function decreases. (Has to have at least 4 different keywords / ideas as highlighted) [4]