# UNIVERSITY OF LONDON

## IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

## EXAMINATIONS 1997

MEng Honours Degrees in Computing Part IV
MSc Degree in Advanced Computing
for Internal Students of the Imperial College of Science, Technology and Medicine

*This paper is also taken for the relevant examinations for the*
*Diploma of Membership of Imperial College*
*Associateship of the City and Guilds of London Institute*

PAPER 4.73

DOMAIN THEORY AND FRACTALS

Friday, May 9th 1997, 2.00 - 4.00

*Answer THREE questions*

For admin. only: paper contains 4 questions

1a Two chains $\langle x_i \rangle_{i \geq 0}$ and $\langle y_j \rangle_{j \geq 0}$ in a poset $(P, \sqsubseteq)$ are *cofinal* if

$$\forall i \geq 0. \exists j \geq 0.\, x_i \sqsubseteq y_j \quad \text{and} \quad \forall j \geq 0. \exists i \geq 0.\, y_j \sqsubseteq x_i.$$

i) Show that any two cofinal chains have the same set of upper bounds.

ii) Give an example of two chains in a poset which have the same set of upper bounds but are not cofinal.

b An iterated function system (IFS) consists of three maps

$$f_1, f_2, f_3 : \mathbb{R} \to \mathbb{R},$$

given by

$$f_1(x) = (x-3)/4, \qquad f_2(x) = x/4, \qquad f_3(x) = (x+3)/4.$$

i) Find the fixed points of the three maps $f_i$ $(i = 1, 2, 3)$ and the fixed point of $f_1 \circ f_2$.

ii) Determine a closed interval which is mapped into itself by each of these maps.

iii) Define a mapping $f : D \to D$ on a cpo $D$ of subsets of $\mathbb{R}$, ordered by reverse inclusion, whose fixed point gives the attractor of the IFS. Define a suitable metric $d$ on $D$ so that $f : (D, d) \to (D, d)$ is a contracting map. Deduce that $f$ has a unique fixed point.

iv) Obtain the first iterate $f(\perp)$.

v) Find the similarity dimension of the attractor.

vi) Show that, for a suitable set $\Sigma$, the points of the attractor are in one to one correspondence with the maximal elements of the cpo $\mathtt{Str}(\Sigma)$, the finite and infinite sequences of elements of $\Sigma$ with prefix ordering.

*The two parts carry, respectively, 25%, 75% of the marks.*

2a Let $2 = \{\bot, \top\}$ be the two element cpo with $\bot \sqsubseteq \top$. Find the least fixed point of the function:

$$\begin{array}{rccc} H: & (2 \to 2) & \to & (2 \to 2) \\ & g & \mapsto & \lambda x.x \sqcup g(x), \end{array}$$

i.e. $(H(g))(x) = x \sqcup g(x)$. How many other fixed points does $g$ have?

b The function $f : \mathrm{N} \to \mathrm{N}$ is defined by

$$f(x) := \text{if } x = 0 \text{ then } 1 \text{ else } 2 * f(x-1).$$

Obtain $f$ using the least fixed point of a continuous function

$$F : (\mathrm{N}_\bot \to_s \mathrm{N}_\bot) \to (\mathrm{N}_\bot \to_s \mathrm{N}_\bot),$$

where $\mathrm{N}_\bot \to_s \mathrm{N}_\bot$ is the strict function space of N. Compute $F^n(\bot_{\mathrm{N}_\bot \to_s \mathrm{N}_\bot})$ for $n \in \mathrm{N}$.

c Define a *fixed point operator* and a *uniform* fixed point operator. Show that the least fixed point operator is uniform.

*The three parts carry, respectively, 25%, 25%, 50% of the marks.*

Turn over...

3a i) Explain what is meant by an *isomorphism* in a category.

ii) Given two cpo's $D$ and $E$ define the function space $D \to E$.

iii) Given cpo's $A$, $B$, $C$ and $D$, assume $f : A \to B$ and $g : C \to D$ are isomorphic maps. Show that the cpo's $A \to C$ and $B \to D$ are isomorphic and construct an isomorphism $h : (A \to C) \to (B \to D)$ in terms of $f$ and $g$.

b Let $\Sigma = \{0, 1\}$. Consider the domain equation

$$X \cong (\Sigma_\perp \otimes X)_\perp.$$

i) Find the iterates $D_n = F^n(\{\perp\})$ for $n = 0, 1, 2, 3$ and the corresponding embedding-projection pairs

$$(e_n, p_n) : D_n \lhd D_{n+1}$$

for $n = 0, 1, 2$.

ii) Obtain the solution of the domain equation.

iii) Prove directly that your solution satisfies the domain equation by obtaining the corresponding isomorphism in the equation. Check that it is indeed an isomorphism.

*The two parts carry, respectively, 35%, 65% of the marks.*

4a i) Show that an embedding preserves finite elements.

ii) Let $P$ be a countable poset. Show that any principal ideal of $P$ is a finite element of the ideal completion of $P$.

b i) Consider the function space $\mathrm{N}_\perp \to \mathrm{N}_\perp$. Let $f : \mathrm{N}_\perp \to \mathrm{N}_\perp$ with $f(x) = \perp$ for all but a finite number of $x \in \mathrm{N}_\perp$. Show that $f$ is a finite element of $\mathrm{N}_\perp \to \mathrm{N}_\perp$.

ii) Construct an increasing chain of finite elements of $\mathrm{N}_\perp \to \mathrm{N}_\perp$ whose lub is the identity function on $\mathrm{N}_\perp$.

c Let $D = \{\perp, l, r, \top\}$ be the four element poset:

$$\begin{tikzcd} & \top & \\ l \arrow[ur] & & r \arrow[ul] \\ & \perp \arrow[ul] \arrow[ur] & \end{tikzcd}$$

i) Construct the upper power domain $\mathcal{U}(D)$ of $D$.

ii) Check directly that $\mathcal{U}(D)$ is a Scott domain.

*The three parts carry, respectively, 30%, 30%, 40% of the marks.*

*End of paper*