Imperial College London

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2012

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

M3/4/5 S4

Applied Probability

Date: Thursday, 10th May 2012 Time: 10 am – 12 noon

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. (a) Suppose $\{N_t\}_{t\geq 0}$ is a counting process. Give a formal definition, if it is to be a homogeneous Poisson process of rate $\lambda > 0$.

(b) Let $\{N_t\}_{t\geq 0}$ denote a homogeneous Poisson process of rate $\lambda > 0$. Let X_1 denote the corresponding first inter–arrival time. Show that X_1 is exponentially distributed with parameter λ.

(c) Let $\{N_t\}_{t\geq 0}$ be a homogeneous Poisson process of rate $\lambda > 0$. Let $0 < t_1 < t_2 < t_3$ and $x_1 \leq x_2 \leq x_3$ for $x_1, x_2, x_3 \in \{0, 1, 2, \dots\}$. Show that

$$\mathbb{P}(N_{t_1} = x_1, N_{t_2} = x_2, N_{t_3} = x_3) = \exp(-\lambda t_3)\lambda^{x_3} \frac{t_1^{x_1}(t_2 - t_1)^{x_2 - x_1}(t_3 - t_2)^{x_3 - x_2}}{x_1!(x_2 - x_1)!(x_3 - x_2)!}.$$

(d) You go on a summer holiday. Your airline has two check–in desks at the airport. The service times are independent and exponentially distributed with parameter $\lambda > 0$. When the check–in starts at 10am, you get to the check–in area with two other tourists. You are generous and let the two other tourists proceed to be served. You will then be served by the next available check–in desk. What is the probability that, of the three tourists, you will be the last to leave the check–in area?
Hint: There is a clever (and quick!) solution to the problem. Alternatively, express the event in question in terms of three independent random variables which are exponentially distributed with parameter λ.

2. (a) Suppose $\{N_t\}_{t\geq 0}$ is a counting process. Give a formal definition, if it is to be a non–homogeneous Poisson process with intensity function $\lambda(t)$, $t \geq 0$.

For (b)-(d) assume that $N = \{N_t\}_{t\geq 0}$ is a non–homogeneous Poisson process with intensity function $\lambda(t)$, $t \geq 0$.

(b) Let $p_n(t) = \mathbb{P}(N_t = n)$ for $n \in \{0, 1, 2, \dots\}$. Show that the forward equations are given by

$$\begin{aligned}
\frac{dp_0(t)}{dt} &= -\lambda(t)p_0(t), \\
\frac{dp_n(t)}{dt} &= -\lambda(t)p_n(t) + \lambda(t)p_{n-1}(t), \qquad \text{for } n \geq 1.
\end{aligned}$$

(c) Compute $p_0(t)$ and $p_1(t)$.
Hint: Recall that a one–dimensional ordinary differential equation

$$\frac{df(t)}{dt} + \alpha(t)f(t) = g(t), \qquad t \geq 0$$

with continuous functions α, g and initial condition $f(0) = C$ has solution

$$f(t) = \frac{\int_0^t g(u)M(u)du + C}{M(t)},$$

where M is the integrating factor

$$M(t) = \exp\left\{\int_0^t \alpha(u)du\right\}.$$

(d) Let X_1, X_2 denote the first two inter–arrival times of N, i.e. X_1 is the time from 0 to the first event, and X_2 is the time from the first event to the second event. Define $m(t) := \int_0^t \lambda(s)ds$.

(i) Derive the probability density function of X_1;

(ii) Derive the conditional probability density function of $X_2|X_1$;

(iii) Hence, find the marginal probability density function of X_2, leaving your answer as an integral.

3. (a) Let $\{X_n\}_{n\in\{0,1,2,\dots\}}$ denote a discrete–time stochastic process taking values in a state space $E \subseteq \mathbb{Z}$. Under which condition is $\{X_n\}_{n\in\{0,1,2,\dots\}}$ a Markov chain on E?

(b) Let X_n be the minimum observation obtained in the first of n rolls of a fair die (for $n \in \{1, 2, 3, \dots\}$). Show that $\{X_n\}_{n\in\{1,2,3,\dots\}}$ is a Markov chain, and give the transition probabilities.

(c) Consider a Markov chain with state space $E = \{1, 2, 3, 4, 5\}$ and transition matrix

$$\mathbf{P} = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1/3 & 1/3 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1/2 & 1/2 \end{pmatrix}.$$

(i) Specify the communicating classes and determine whether they are transient, null recurrent or positive recurrent.

(ii) Is the Markov chain irreducible?

(iii) Let $\boldsymbol{\pi}$ denote a 5–dimensional row vector. Which conditions have to be satisfied in order for $\boldsymbol{\pi}$ to be a stationary distribution of the Markov chain?

(iv) Find a stationary distribution for the Markov chain.

(v) Decide whether or not the stationary distribution is unique and justify your answer.

4. (a) Let $\{X_t\}_{t\geq 0}$ denote a continuous–time stochastic process taking values in a state space $E \subseteq \mathbb{Z}$. Under which condition is $\{X_t\}_{t\geq 0}$ a Markov chain on E?

(b) Let $X = \{X_t\}_{t\geq 0}$ denote a continuous–time Markov chain on the state space $E = \{0, 1, \ldots\}$. Suppose X is a birth–death process with birth rates $\lambda_0, \lambda_1, \ldots$ and death rates $\mu_0, \mu_1, \ldots$ satisfying

$$\lambda_i, \geq 0 \quad \mu_i \geq 0, \text{ for all } i \in E, \quad \mu_0 = 0.$$

State the infinitesimal transition probabilities

$$\mathbb{P}(X_{t+\delta} = n + m | X_t = n)$$

for $n, m \in \{0, 1, 2, \ldots\}$ and $t \geq 0, \delta > 0$.

(c) What is the generator $\mathbf{G}$ of the birth–death process defined in (b)?

(d) Let $N = \{N_t\}_{t\geq 0}$ denote a pure birth process with $N_0 = 0$ and

$$\mathbb{P}(\text{one event happens in } (t, t+\delta] | N_t \text{ is odd}) = \alpha\delta + o(\delta),$$
$$\mathbb{P}(\text{one event happens in } (t, t+\delta] | N_t \text{ is even}) = \beta\delta + o(\delta),$$

for $t \geq 0$, $\delta, \alpha, \beta > 0$.

(i) Derive the forward equations for $p_n(t) := \mathbb{P}(N_t = n)$ for $n \in \{0, 1, 2, \ldots\}$.

(ii) Find the following probabilities:

$$P_e(t) := \mathbb{P}(N_t \text{ is even}), \qquad P_o(t) := \mathbb{P}(N_t \text{ is odd}).$$

Hint 1: Derive the following differential equations

$$P_e'(t) = \alpha P_o(t) - \beta P_e(t), \qquad P_o'(t) = -\alpha P_o(t) + \beta P_e(t),$$

and solve them by using the identity $P_e(t) + P_o(t) = 1$.

Hint 2: Recall that a one–dimensional ordinary differential equation

$$\frac{df(t)}{dt} + \alpha(t) f(t) = g(t), \qquad t \geq 0$$

with continuous functions α, g *and initial condition* $f(0) = C$ *has solution*

$$f(t) = \frac{\int_0^t g(u) M(u) du + C}{M(t)},$$

where M *is the integrating factor*

$$M(t) = \exp\left\{ \int_0^t \alpha(u) du \right\}.$$

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BSc and MSci EXAMINATIONS (MATHEMATICS)

May UPDATE! 2012

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M3/4/5 S4

Applied Probability (Solutions)

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1. (a) Either of the following definitions, are appropriate for full marks. A counting Process, $\{N_t\}_{t\geq 0}$, is a Poisson Process of rate $\lambda > 0$ if $\boxed{\text{seen} \Downarrow}$

   1. $N_0 = 0$.
   2. The increments are independent.
   3. The increments are stationary: for any $0 < s < t$, $k \in \mathbb{Z}_+$

$$\mathbb{P}(\{N_t - N_s = k\}) = \mathbb{P}(\{N_{t-s} = k\})$$

   4. There is a 'single arrival', i.e. for $t \geq 0, \delta > 0$

$$\begin{aligned}\mathbb{P}(\{N_{t+\delta} - N_t = 0\}) &= 1 - \lambda\delta + o(\delta)\\ \mathbb{P}(\{N_{t+\delta} - N_t = 1\}) &= \lambda\delta + o(\delta)\\ \mathbb{P}(\{N_{t+\delta} - N_t \geq 2\}) &= o(\delta)\end{aligned}$$

(Note that you could argue that 4. implies 3. Also, in 4. it is sufficient to give the probabilities for 1 and more than 2 events or for 0 and more than 2 events to obtain full marks.)

Or: A counting Process, $\{N_t\}_{t\geq 0}$, is a Poisson Process of rate $\lambda > 0$ if

   1. $N_0 = 0$
   2. The increments are independent
   3. For any $0 \leq s < t$, $k \in \mathbb{Z}_+$ we have

$$\mathbb{P}(N_t - N_s = k) = \frac{(\lambda(t-s))^k e^{-\lambda(t-s)}}{k!}.$$

That is, the number of events in $[s, t]$ is a Poisson random variable, with mean $\lambda(t - s)$. $\boxed{5}$

(b) For $t > 0$ we have $\boxed{\text{seen} \Downarrow}$

$$\mathbb{P}(X_1 > t) = \mathbb{P}(\text{no events in}[0, t]) = \mathbb{P}(N_t = 0) = e^{-\lambda t}.$$

Either you argue that $e^{-\lambda t}$ is the survival function of an exponential random variable with parameter λ, or you compute the density function, which is given by

$$f_{X_1}(t) = \lambda e^{-\lambda t},$$

which is an exponential density function of rate λ. $\boxed{5}$

(c) Using the notation given in the problem, we have $\boxed{\text{unseen} \Downarrow}$

$$\begin{aligned}p &:= \mathbb{P}(N_{t_1} = x_1, N_{t_2} = x_2, \ldots, N_{t_n} = x_n)\\ &= \mathbb{P}(N_{t_1} = x_1, N_{t_2} - N_{t_1} = x_2 - x_1, N_{t_3} - N_{t_2} = x_3 - x_2 \ldots, N_{t_n} - N_{t_{n-1}} = x_n - x_{n-1})\\ &= \mathbb{P}(N_{t_1} = x_1)\mathbb{P}(N_{t_2} - N_{t_1} = x_2 - x_1)\mathbb{P}(N_{t_3} - N_{t_2} = x_3 - x_2)\cdots\mathbb{P}(N_{t_n} - N_{t_{n-1}} = x_n - x_{n-1}),\end{aligned}$$

where we used the independent increment property. $\boxed{3}$

Next, we use the fact that the increments of a Poisson process are stationary and have Poisson distribution. Hence:

$$\begin{aligned}p &= \frac{e^{-\lambda t_1}(\lambda t_1)^{x_1}}{x_1!}\frac{e^{-\lambda(t_2-t_1)}(\lambda(t_2 - t_1))^{x_2-x_1}}{(x_2 - x_1)!}\cdots\frac{e^{-\lambda(t_n-t_{n-1})}(\lambda(t_n - t_{n-1}))^{x_n-x_{n-1}}}{(x_n - x_{n-1})!}\\ &= \exp(-\lambda t_n)\lambda^{x_n}\frac{t_1^{x_1}(t_2 - t_1)^{x_2-x_1}\cdots(t_n - t_{n-1})^{x_n-x_{n-1}}}{x_1!(x_2 - x_1)!\cdots(x_n - x_{n-1})!}.\end{aligned}$$

(Note that this is the *finite–dimensional distribution* of a Poisson process.) $\boxed{2}$

(d) Either of the following solutions is appropriate for full marks: $\boxed{\text{unseen} \Downarrow}$

You can use the property that the exponential distribution is memoryless. Hence, as soon as the first tourist leaves, your service time and the one by the other tourists have the same distribution. By symmetry, you have probability 0.5 of being the last to leave the check–in area.

Alternatively, you can argue as follows: Let X, Y denote the service times for the other two tourists, and let Z denote your service time. Then X, Y, Z are independent and exponentially distributed with parameter $\lambda > 0$. Hence the joint density function is given by

$$f_{X,Y,Z}(x,y,z) = \lambda^3 \exp(-\lambda x)\exp(-\lambda y)\exp(-\lambda z), \qquad x, y, z \geq 0.$$

You need to compute the probability of the event

$$B := \{X < Y < X + Z\} \cup \{Y < X < Y + Z\},$$

which is the union of two disjoint events, which have the same probability (by symmetry). Then

$$\begin{aligned}
\mathbb{P}(X < Y < X + Z) &= \int_0^\infty \int_x^\infty \int_{y-x}^\infty \lambda^3 \exp(-\lambda x)\exp(-\lambda y)\exp(-\lambda z) dz dy dx \\
&= \int_0^\infty \lambda \exp(-\lambda x) \int_x^\infty \lambda \exp(-\lambda y) \int_{y-x}^\infty \lambda \exp(-\lambda z) dz dy dx \\
&= \int_0^\infty \lambda \exp(-\lambda x) \int_x^\infty \lambda \exp(-\lambda y)\exp(-\lambda(y-x)) dy dx \\
&= \int_0^\infty \lambda \exp(-\lambda x)\exp(\lambda x) \int_x^\infty \lambda \exp(-\lambda y)\exp(-\lambda y) dy dx \\
&= \int_0^\infty \lambda \int_x^\infty \lambda \exp(-2\lambda y) dy dx.
\end{aligned}$$

Note that

$$\int_x^\infty \lambda \exp(-2\lambda y) dy = \frac{-1}{2}\exp(-2\lambda y)\Big|_x^\infty = \frac{1}{2}\exp(-2\lambda x),$$

and

$$\int_0^\infty \lambda \frac{1}{2}\exp(-2\lambda x) dx = \frac{-1}{4}\exp(-2\lambda x)\Big|_0^\infty = \frac{1}{4},$$

Hence $\mathbb{P}(X < Y < X + Z) = \frac{1}{4} = \mathbb{P}(Y < X < Y + Z)$ and $\mathbb{P}(B) = 0.5$. $\boxed{5}$

2. (a) A non–homogeneous Poisson process with intensity function $(\lambda(t))$ is a stochastic process $N = \{N_t\}_{t\geq 0}$, which satisfies the following properties: $\boxed{\text{seen} \Downarrow}$

1. $N_0 = 0$.
2. N has independent increments. $\boxed{2}$
3. 'Single arrival' property: For $t \geq 0$, $\delta > 0$:

$$\begin{aligned}
\mathbb{P}(N_{t+\delta} - N_t = 0) &= 1 - \lambda(t)\delta + o(\delta), \\
\mathbb{P}(N_{t+\delta} - N_t = 1) &= \lambda(t)\delta + o(\delta), \\
\mathbb{P}(N_{t+\delta} - N_t \geq 2) &= o(\delta),
\end{aligned}$$

$\boxed{2}$

(As in Question 1, it is sufficient to state the probability for 0 and more than 2 events or for 1 and more than 2 events. Also you could work with the alternative definition that $N_0 = 0$, N has independent increments and that $N_t - N_s \sim$ Poi$(m(t) - m(s))$ for $0 \leq s < t$, where $m(t) := \int_0^t \lambda(u)du$.) $\boxed{\text{seen} \Downarrow}$

(b) Let $n = 0$. Then

$$\begin{aligned}
p_0(t+\delta) &= \mathbb{P}(N_{t+\delta} = 0) = \mathbb{P}(\text{no event in}[0, t+\delta]) \\
&= \mathbb{P}(\{\text{no event in } [0,t]\} \cap \{\text{no event in } (t, t+\delta]\}) \\
&= \mathbb{P}\Big(\text{no event in } [0,t]\Big)\mathbb{P}\Big(\text{no event in } (t, t+\delta]\Big),
\end{aligned}$$

where we applied the independent increments property, then

$$p_0(t+\delta) = p_0(t)[1 - \lambda(t)\delta + o(\delta)].$$

Hence we have
$$\frac{p_0(t+\delta) - p_0(t)}{\delta} = -\lambda(t)p_0(t) + \frac{o(\delta)}{\delta}$$
letting $\delta \downarrow 0$ we get
$$\frac{dp_0(t)}{dt} = -\lambda(t)p_0(t).$$

$\boxed{2}$

For $n \geq 1$ we have

$$\begin{aligned}
p_n(t+\delta) &= \mathbb{P}(N_{t+\delta} = n) \\
&= \sum_{k=0}^{\infty} \mathbb{P}(N_{t+\delta} = n | N_t = k)\mathbb{P}(N_t = k) \\
&= \sum_{k=0}^{\infty} \mathbb{P}((n-k) \text{ events in } (t, t+\delta])\mathbb{P}(N_t = k) \quad \text{(by independent increments property)} \\
&= \mathbb{P}(1 \text{ event in } (t, t+\delta])\mathbb{P}(N_t = n-1) \\
&\quad + \mathbb{P}(0 \text{ events in } (t, t+\delta])\mathbb{P}(N_t = n) + o(\delta) \\
&= p_{n-1}(t)\lambda(t)\delta + p_n(t)(1 - \lambda(t)\delta) + o(\delta) \\
&= p_n(t)(1 - \lambda(t)\delta) + p_{n-1}(t)\lambda(t)\delta + o(\delta).
\end{aligned}$$

Re-arranging and letting $\delta \downarrow 0$ we have $\boxed{3}$

$$\frac{dp_n(t)}{dt} = -\lambda(t)p_n(t) + \lambda(t)p_{n-1}(t).$$

$\boxed{1}$

(c) We solve

$$\frac{dp_0(t)}{dt} = -\lambda(t)p_0(t)$$

with $p_0(0) = \mathbb{P}(N_0 = 0) = 1$. Hence, we obtain

$$p_0(t) = \exp\left(-\int_0^t \lambda(s)ds\right).$$

seen ⇓

1

Let $n = 1$, then we have the ordinary differential equation (ODE)

$$\frac{dp_1(t)}{dt} + \lambda(t)p_1(t) = \lambda(t)\exp\left\{-\int_0^t \lambda(s)ds\right\}.$$

In our case, the integrating factor is

$$M(t) = \exp\left\{\int_0^t \lambda(u)du\right\}.$$

Since $p_1(0) = \mathbb{P}(N_0 = 1) = 0$, the solution is

$$p_1(t) = \left[\int_0^t \lambda(s)ds\right]\exp\left\{-\int_0^t \lambda(s)ds\right\}.$$

3

(d) Let $m(t) := \int_0^t \lambda(s)ds$ for $t \geq 0$.

meth seen ⇓

(i) Let $t > 0$. Then

$$\mathbb{P}(X_1 > t) = \mathbb{P}(\text{no events in } [0,t]) = \mathbb{P}(N_t = 0) = \exp(-m(t)).$$

Hence the cumulative distribution function is given by $F_{X_1}(t) = 1 - \exp(-m(t))$. Differentiating yields

$$f_{X_1}(t) = -\exp(-m(t))(-1)\lambda(t) = \exp(-m(t))\lambda(t).$$

2

(ii) Let $t, t_1 > 0$. Then

$$\begin{aligned}\mathbb{P}(X_2 > t|X_1 = t_1) &= \mathbb{P}(\text{no events in } (t_1, t_1+t]|\text{one event in } [0,t_1]) \\ &= \mathbb{P}(N_{t_1+t} - N_{t_1} = 0|N_{t_1} = 1) = \mathbb{P}(N_{t_1+t} - N_{t_1} = 0) \\ &= \exp\left(-\int_{t_1}^{t_1+t} \lambda(u)du\right)\end{aligned}$$

where we used the independent increment property and the fact that the increments are Poisson distributed. Hence differentiating the conditional cumulative distribution function yields

$$\begin{aligned}f_{X_2|X_1}(t|t_1) &= \exp\left(-\int_{t_1}^{t_1+t} \lambda(u)du\right)\lambda(t_t + t) \\ &= \exp\left(m(t_1) - m(t_1+t)\right)\lambda(t_1+t).\end{aligned}$$

2

(iii) Let $t > 0$. Then

unseen ⇓

$$\begin{aligned}f_{X_2}(t) &= \int_0^\infty f_{X_2|X_1}(t|t_1)f_{X_1}(t_1)dt_1 \\ &= \int_0^\infty \exp\left(m(t_1) - m(t_1+t)\right)\lambda(t_1+t)\exp(-m(t_1))\lambda(t_1)dt_1 \\ &= \int_0^\infty \exp\left(-m(t_1+t)\right)\lambda(t_1+t)\lambda(t_1)dt_1.\end{aligned}$$

2

3. (a) A discrete-time stochastic process $\{X_n\}_{n\in\{0,1,2,\dots\}}$ is a Markov chain on E if it satisfies the Markov condition: seen $\Downarrow$

$$\mathbb{P}(X_n = s | X_{n-1} = x_{n-1}, \dots, X_0 = x_0) = \mathbb{P}(X_n = s | X_{n-1} = x_{n-1})$$

for all integers $n \geq 1$ and for all $s, x_0, \dots, x_{n-1} \in E$. 4

(b) Let D_n be the score of the die at time n. Then D_n is a uniform random variable on $E = \{1, \dots, 6\}$. Then sim. seen $\Downarrow$

$$X_n = \min\{D_1, \dots, D_n\} = \min\{X_{n-1}, D_n\}.$$

Clearly X_n depends on $(X_1, \dots, X_{n-1})$ only through X_{n-1} so it is a Markov chain, i.e. $\mathbb{P}(X_n = x_n | X_{n-1} = x_{n-1}, \dots, X_1 = x_1) = \mathbb{P}(X_n = x_n | X_{n-1} = x_{n-1})$ for all integers $n \geq 2$ and for all $x_1, \dots, x_n \in E$. The transition matrix is given by

$$\mathbf{P} = \begin{pmatrix} 6/6 & 0 & 0 & 0 & 0 & 0 \\ 1/6 & 5/6 & 0 & 0 & 0 & 0 \\ 1/6 & 1/6 & 4/6 & 0 & 0 & 0 \\ 1/6 & 1/6 & 1/6 & 3/6 & 0 & 0 \\ 1/6 & 1/6 & 1/6 & 1/6 & 2/6 & 0 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \end{pmatrix}.$$

6

(c) (i) There are three communicating classes: $C_1 = \{1, 2\}$ and $C_2 = \{4, 5\}$ are closed and hence positive recurrent. The class $T = \{3\}$ is not closed and hence transient. sim. seen $\Downarrow$ 2

(ii) The Markov chain is not irreducible since it has more than one communicating class. 1

(iii) A vector $\boldsymbol{\pi}$ is a stationary distribution of the Markov chain on E if:

(1) for each $j \in E$, $\pi_j \geq 0$ and $\sum_{j\in E} \pi_j = 1$.

(2) $\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P}$, that is, for each $j \in E$, $\pi_j = \sum_{i\in E} \pi_i p_{ij}$. 2

(iv) We consider the transition matrices restricted to the essential communicating classes:

$$\mathbf{P}(C_1) := \begin{pmatrix} 1/2 & 1/2 \\ 1 & 0 \end{pmatrix}, \qquad \mathbf{P}(C_2) := \begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix}$$

Let $\boldsymbol{\pi}(C_1), \boldsymbol{\pi}(C_2)$ denote 2–dimensional row vectors. Solve

$$\boldsymbol{\pi}(C_1)\mathbf{P}(C_1) = \boldsymbol{\pi}(C_1), \qquad \boldsymbol{\pi}(C_2)\mathbf{P}(C_2) = \boldsymbol{\pi}(C_2).$$

Then $\boldsymbol{\pi}(C_1) = (a, 0.5a)$ and $\boldsymbol{\pi}(C_2) = (0.5b, b)$ for constants $a, b \in \mathbb{R}$. Now we define $\boldsymbol{\pi} := (a, 0.5a, 0, 0.5b, b)$ for constants $a, b \geq 0$ such that $1.5a + 1.5b = 1$. Then $\boldsymbol{\pi}$ is a stationary distribution, since $\pi_i \geq 0$ for $i = 1, \dots, 5$ and $\sum_{i=1}^{5} \pi_i = 1$. Also $\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P}$. 4

(v) The stationary distribution is not unique since we have two essential communicating classes. 1

4. (a) The process $\{X_t\}_{t\geq 0}$ has to satisfy the Markov property: seen ⇓

$$\mathbb{P}(X_{t_n} = j | X_{t_1} = i_1, \dots, X_{t_{n-1}} = i_{n-1}) = \mathbb{P}(X_{t_n} = j | X_{t_{n-1}} = i_{n-1})$$

for all $j, i_1, \dots, i_{n-1} \in E$ and for any sequence $0 \leq t_1 < \cdots < t_n < \infty$ of times (with $n > 1$). 4

(b) seen ⇓

$$\mathbb{P}(X_{t+\delta} = n + m | X_t = n) = \begin{cases} 1 - (\lambda_n + \mu_n)\delta + o(\delta), & \textit{if } m = 0, \\ \lambda_n \delta + o(\delta) & \textit{if } m = 1 \\ \mu_n \delta + o(\delta) & \textit{if } m = -1 \\ o(\delta) & \textit{if} |m| > 1 \end{cases}.$$

4

(c) The generator is seen ⇓

$$\mathbf{G} = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & 0 & \cdots \\ \mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 & 0 & 0 & \cdots \\ 0 & \mu_2 & -(\lambda_2 + \mu_2) & \lambda_2 & 0 & \cdots \\ 0 & 0 & \mu_3 & -(\lambda_3 + \mu_3) & \lambda_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

3

(d) (i) Let $p_n(t) := \mathbb{P}(N_t = n)$ for $n \in \{0, 1, 2, \dots\}$. We derive the forward equations: unseen ⇓

$$p_0(t + \delta) = \mathbb{P}(N_{t+\delta} = 0 | N_t = 0)\mathbb{P}(N_t = 0) = (1 - \beta\delta)p_0(t) + o(\delta).$$

Subtract $p_0(t)$ on both sides, divide by δ and consider $\delta \to 0$. Then we get

$$p_0'(t) = -\beta p_0(t).$$

Now, let $n \geq 1$. Then

$$\begin{aligned} p_{2n}(t + \delta) &= \sum_{l=0}^{\infty} \mathbb{P}(N_{t+\delta} = 2n | N_t = l) p_l(t) \\ &= (\alpha\delta + o(\delta))p_{2n-1}(t) + (1 - \beta\delta + o(\delta))p_{2n}(t) + o(\delta). \end{aligned}$$

Subtract $p_{2n}(t)$ on both sides, divide by δ and consider $\delta \to 0$. Then we get

$$p_{2n}'(t) = \alpha p_{2n-1}(t) - \beta p_{2n}(t).$$

Similarly, we get for $n \geq 0$:

$$p_{2n+1}'(t) = -\alpha p_{2n+1}(t) + \beta p_{2n}(t).$$

That is, altogether we have

$$\begin{aligned} p_0'(t) &= -\beta p_0(t) \\ p_n'(t) &= -\alpha p_n(t) + \beta p_{n-1}(t), \qquad n = 1, 3, 5, \dots \\ p_n'(t) &= \alpha p_{n-1}(t) - \beta p_n(t), \qquad n = 2, 4, 6, \dots \end{aligned}$$

5

(ii) Then $P_e(t) = \sum_{n=0}^{\infty} p_{2n}(t)$ and

$$\begin{aligned}
P_e'(t) &= \sum_{n=0}^{\infty} p_{2n}'(t) = -\beta p_0'(t) + \sum_{n=1}^{\infty} (\alpha p_{2n-1}(t) - \beta p_{2n}(t)) \\
&= \alpha \sum_{n=1}^{\infty} p_{2n-1}(t) - \beta \sum_{n=0}^{\infty} p_{2n}(t) = \alpha P_o(t) - \beta P_e(t).
\end{aligned}$$

Using $1 = P_e(t) + P_o(t)$, we also get $P_o'(t) = -\alpha P_o(t) + \beta P_e(t)$.
Now we solve

$$P_e'(t) = \alpha P_o(t) - \beta P_e(t) = \alpha(1 - P_e(t)) - \beta P_e(t) = \alpha - (\alpha+\beta) P_e(t),$$

using the integrating factor approach: Then $M(t) = \exp((\alpha+\beta)t)$. Note that $P_e(0) = P(N_0 \text{ is even}) = 1$. Altogether we have

$$\begin{aligned}
P_e(t) &= \left(\int_0^t \alpha \exp((\alpha+\beta)u) du + 1 \right) \exp(-(\alpha+\beta)t) \\
&= \left(\frac{\alpha}{\alpha+\beta} \exp((\alpha+\beta)u) \Big|_0^t + 1 \right) \exp(-(\alpha+\beta)t) \\
&= \frac{\alpha}{\alpha+\beta} + \frac{\beta}{\alpha+\beta} \exp(-(\alpha+\beta)t).
\end{aligned}$$

Then

$$P_o(t) = 1 - P_e(t) = \frac{\beta}{\alpha+\beta} \left(1 - \exp(-(\alpha+\beta)t)\right).$$

$\boxed{4}$