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MSc Degree in Advanced Computing
for Internal Students of the Imperial College of Science, Technology and Medicine

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PAPER A4.12

LOGIC - MODEL THEORY AND PROOF THEORY

Thursday, May 8th 1997, 10.00 - 12.00

Answer THREE questions

For admin. only: paper contains 4 questions

1 Recursion and Structural Induction.

a Define a recursive function SF which computes the set of constant and function symbols in a term. Examples:

$SF(a) = \{a\}$ if a is a constant symbol
$SF(x) = \emptyset$ if x is a variable symbol
$SF(f(g(x, a, a))) = \{f, g, a\}$.

b Let NS be the function that counts the number of symbols in a term (as defined in the lecture). State the definition of NS and prove by structural induction: For all terms t: $NS(t) \geq |SF(t)|$ where $|\ldots|$ denotes the set cardinality function.

c Outline the proof technique *structural induction*

i) for terms
ii) for formulae.

d Define an algorithm NDS that computes the number of *different* symbols (variables, constants and functions) in a term *without* first collecting the symbols in a list list or a set and then computing the length of the list.

Example: $NDS(f(f(a, a), a)) = 2$) (the f and the a are counted only once.)

Hint: choose an appropriate representation (data structure) for symbols and terms. (This question appeals to your intuition as computer scientist, and not as a logician.)

All parts carry 25% of the marks

2 **Semantics of First-Order Predicate Logic (PL1).**

Consider the following interpretation for PL1 (with binary truth value semantics):

- The domain consists of the natural numbers $0, 1, 2, \ldots$
- The constant symbol a is mapped to 0.
- The constant symbol b is mapped to 25.
- The function symbol s is mapped to the successor function $(\ldots + 1)$.
- The function symbol f is mapped to the $+$ function (addition).
- The function symbol g is mapped to the $*$ function (multiplication).
- The predicate symbol P is mapped to the $<$ relation.

a For each of the following formulae check whether they are true or false in this interpretation.

i) $P(a, a)$
ii) $P(a, s(a))$
iii) $\exists x\ P(a, s(x))$
iv) $\exists x\ P(s(x), x)$
v) $\forall y \exists x\ P(x, f(x, y))$
vi) $\forall x\ (x = b \Rightarrow \exists y\ g(y, y) = x)$ ('=' is equality)
vii) $\forall x\ (P(x, b) \Rightarrow \exists y\ g(y, y) = x)$.

b Find for the formulae i-vii interpretations (maybe a different one for each formula) where the formulae have just the opposite truth value as in the interpretation above. Hint: use basically the above interpretation, but change the meaning of some symbol.

c Give a semantic definition of the notion *soundness of an inference rule.*

d Suppose $\mathcal{L}$ is a first-order predicate logic language with a *fixed* number of constant and function symbols. Is the following inference rule:

$$\text{From } \exists x \psi[x] \text{ infer } \psi[x/t]$$

sound or not in the language $\mathcal{L}$? ($\psi[x]$ is an arbitrary formula containing the variable x somewhere. $\psi[x/t]$ means replacing x with t where t is an *existing* term of the language $\mathcal{L}$.) If the inference rule is sound, prove it, if not, give a counter example.

a, b and d each carry 30% of the marks and c carries 10%.

Turn over ...

3 **Hilbert systems.**

A Hilbert system for the implicational fragment of classical propositional logic is:

Axioms:

A1 $\vdash F \Rightarrow F$

A2 $\vdash F \Rightarrow (G \Rightarrow F)$

A3 $\vdash (F \Rightarrow G) \Rightarrow ((G \Rightarrow H) \Rightarrow (F \Rightarrow H))$

A4 $\vdash (F \Rightarrow (G \Rightarrow H)) \Rightarrow ((F \Rightarrow G) \Rightarrow (F \Rightarrow H))$

Modus Ponens Inference Rule: From $\vdash F$ and $\vdash F \Rightarrow G$ infer $\vdash G$.

a Is a Hilbert system a decision procedure for figuring out whether a formula is a theorem in that logic? If not, why not?

b Prove the deduction theorem

$$H \vdash (F \Rightarrow G) \text{ iff } H \cup \{F\} \vdash G.$$

(which says: in order to prove $F \Rightarrow G$, assume F and derive G) for the above Hilbert system.

c Prove $\vdash F \Rightarrow ((F \Rightarrow G) \Rightarrow G)$ using the Deduction Theorem.

d Prove $\vdash F \Rightarrow ((F \Rightarrow G) \Rightarrow G)$ with A1-A4 and the Modus Ponens rule, not using the Deduction Theorem.
Hint: Use A2 and A3 first, then A1 and A4, and then combine the results. You can rename and instantiate the predicate symbols as you like.

a carries 10%, b 40%, c 20% and d carries 30% of the marks.

4 **Tableaux systems and many-valued logics.**

The following truth tables for a logic with the three connectives $\neg_4$, $\wedge_4$ and $\vee_4$ define a logic with four truth values (T, B, N, F):

$\neg_4$	
T	F
B	B
N	N
F	T

$\wedge_4$	T	B	N	F
T	T	B	N	F
B	B	B	N	F
N	N	N	N	N
F	F	F	N	F

$\vee_4$	T	B	N	F
T	T	T	T	T
B	T	B	B	B
N	T	B	N	F
F	T	B	F	F

An intuitive interpretation of this logic is as follows: Suppose somebody sends out a number of questionnaires asking people to answer 'yes' or 'no' to a given question Q. The result of this action is T if all questionnaires are returned with answer 'yes'. The result is F if all are returned with answer 'no'. If some answers are 'yes' and some others are 'no', or some, but not all questionnaires are returned, then the result is labelled B (for both). If the questionnaires are not returned at all (because the people can't decide on it) then the result is N. The negation $\neg_4$ is to be interpreted as the result of asking $\neg Q$. (E.g. if people can't decide on Q, they can't decide on $\neg Q$ as well. Therefore the result for $\neg_4 Q$ is the same as for Q.) Conjunction and disjunction are to be interpreted as the result of asking questions $Q \wedge R$ and $Q \vee R$ respectively. (For example, if the result for Q is B (both answers) and for R is N (people can't decide on R) then the result for $Q \wedge_4 R$ is N as well (the people who can't decide on R can't decide on $Q \wedge R$ either.)

a Give a short (one line) argument to show that the connectives $\wedge_4$ and $\vee_4$ are commutative in this four-valued logic.

b Define a tableaux calculus for this logic.

c Check with this tableaux calculus whether $(p \vee_4 q) \wedge_4 r$ entails $p \wedge_4 r$ or not, by analyzing the tableaux for $T : (p \vee_4 q) \wedge_4 r$ and *not* $T : p \wedge_4 r$. If the entailment holds, give a tableaux proof, if not, give a counter-model (an open tableaux branch representing an assignment which yields T for $(p \vee_4 q) \wedge_4 r$ and some other truth value for $p \wedge_4 r$.)
Hint: start the tableaux with $B : p \wedge_4 r$ (i.e. $B : p \wedge_4 r$ is one of the choices for *not* $T : p \wedge_4 r$.)

a carries 20% of the marks, b 50% and c carries 30%.

End of Paper

