Final

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ISE3.11

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2010

EEE/ISE PART III/IV: MEng, BEng and ACGI

DIGITAL SIGNAL PROCESSING

Friday, 30 April 10:00 am

Time allowed: 3:00 hours

There are SIX questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : P.A. Naylor
Second Marker(s) : D.P. Mandic

DIGITAL SIGNAL PROCESSING

1. a) i) Write down the formula for the discrete Fourier transform (DFT) of a discrete time sequence $x(n)$ of length N. [2]

   ii) Write down the formula for the corresponding inverse discrete Fourier transform (IDFT). [2]

   iii) Prove that your formula in part (ii) is the inverse for your formula in part (i). [4]

b) Let $z(n)$ represent the circular convolution of two sequences $x(n)$ and $y(n)$, given by

$$z(n) = x(n) \circledast y(n).$$

Using appropriate properties of the DFT, prove that

$$Z(k) = X(k)Y(k)$$

where $Z(k)$, $X(k)$ and $Y(k)$ are the DFTs of $z(n)$, $x(n)$ and $y(n)$ respectively. [5]

c) Compute the circular convolution of the following discrete-time sequences: $x(n) = \{2,3,1,5\}$, $y(n) = \{1,4,-3,-5\}$. [5]

d) Briefly state how circular convolution and linear convolution differ. [2]

2. Consider the multirate signal processing operations of decimation, denoted $\downarrow M$, and expansion, denoted $\uparrow L$, where M and L are the decimation and expansion factors respectively. When the real discrete-time signal $x(n)$, with z transform $X(z)$, is decimated by a factor M it is denoted $x_D(n)$ and when expanded by a factor L it is denoted $x_E(n)$.

a) Show that the z-transform of $x_D(n)$ is given by [6]

$$X_D(z) = \frac{1}{M}\sum_{k=0}^{M-1} X\left(z^{1/M}W_M^k\right)$$

and define W_M^k.

b) Let the samples of the signal $x(n)$ be denoted $\{x(0),\ x(1),\ x(2),\ \ldots\ \}$. The signal $x(n)$ is input to the system shown in Fig. 2.1.

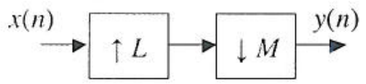

Figure 2.1

Write down the first 10 samples of $y(n)$ for [6]

i) $L = 2,\ M = 3$

ii) $L = 4,\ M = 5$

and explain whether $y(n)$ would be altered by changing the order of the decimation and expansion.

c) Let $x(n) = \alpha^n$, $n \geq 0$. [8]

Write down expressions for

i) $X(z)$, using the z-transform of $x(n)$,

ii) $x_D(n)$,

iii) $X_D(z)$, using the z-transform of $x_D(n)$.

Show that $X_D(z)$ can also be written in the form

$$\frac{1}{M}\sum_{m=0}^{M-1} \frac{z^{1/M}W_M^{-m}}{z^{1/M}W_M^{-m} - \alpha}.$$

3. Let $x(n)$ be the input and $y(n)$ be the output of a finite impulse response (FIR) digital filter where n is the discrete time index and

$$y(n) = 0.5x(n) - 0.45x(n-2).$$

a) Find all the poles and zeros associated with the transfer function of this filter and sketch a plot of them on the z-plane. [4]

b) Find an expression for the magnitude of the frequency response of this filter. [3]

c) Draw a sketch of the magnitude of the frequency response. [3]

d) Determine the gain of the filter at frequencies of 0 and $\pi/4$. [4]

e) Define the group delay of a digital filter and state the units in which group delay is measured. [3]

f) If the digital filter above operates with a sampling frequency of 44.1 kHz, what is the group delay of the filter measured at a frequency of 10 kHz? [3]

4. Consider the discrete time sequence $x(n)$ and its z-transform $X(z)$.

a) Explain what is meant by the *Region of Convergence* of the z-transform. [2]

Describe any significant similarities and differences between the z-transform and the Laplace transform. Illustrate your answer using sketches in the z and s domains. Comment on the way in which the spectrum of discrete time signals is represented in the z domain. [6]

b) The bilinear transformation between the z domain and s domain employs the relation

$$s = \frac{z-1}{z+1}.$$

Employing the notation $s = \sigma + j\Omega$, show how the following regions in the s domain are mapped to the z domain and state the significance of these regions. [5]

i) $\sigma < 0$,

ii) $\sigma = 0$,

iii) $\sigma > 0$.

Applying the bilinear transformation directly as stated above, and assuming any consideration of frequency scaling can be ignored, transform

$$H(s) = \frac{1}{s^2 + 0.6s + 0.18}$$

to the z domain to obtain an expression for $H(z)$. (Answers correct to within an arbitrary scale factor will receive full credit.) [5]

Further using the bilinear transformation directly as stated above, write down the relationship between the frequency, ω, in the frequency domain representation of a discrete-time signal and the corresponding value of Ω as defined in part b). [2]

5. a) Write down definitions of the following terms when used to describe discrete-time linear systems: [3]

i) impulse response,

ii) step response,

iii) causal.

b) Write down an example of a transfer function $H(z)$ for a discrete-time linear time-invariant system exhibiting two zeros, two poles and with gain at d.c. of 25. Make reasonable choices for the zeros and poles. [4]

c) A causal linear time-invariant discrete-time system has an impulse response $h(n)$ and a step response $g(n)$. Derive a relationship between the sample values of $h(n)$ and those of $g(n)$. [7]

d) Consider a system defined by

$$H(z) = a_0 + a_1 z^{-1} + a_2 z^{-2} + \ \ldots \ + a_N z^{-N}.$$

If N is finite, what are the sample values for

i) $h(2N+2)$

ii) $g(2N+2)$?

[6]

6. a) Consider a time series $x(n)$ applied as input to a linear time invariant system with impulse response $h(n)$.

Describe and compare the *overlap-save* method and the *overlap-add* method for computing the output of the system. Give illustrative diagrams. [8]

b) Use the overlap-save block filtering method to calculate the output of a filter for the case when the input signal is

$$x(n) = [1,\ 2,\ 3,\ 3,\ 2,\ 1,\ 0\ ,-1,\ -2,\ -3,\ -3,\ -2]$$

and the impulse response of the filter is

$$h(n) = [1,\ -1].$$

Use only the first 2 data blocks in your calculation. [9]

c) Verify your solution to b) using time domain convolution. [3]

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DIGITAL SIGNAL PROCESSING
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1. a) i) $X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N}$

ii) $x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi nk/N}$

iii)

$$\begin{aligned}
\frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{j2\pi nk/N} &= \frac{1}{N}\sum_{k=0}^{N-1}\left[\sum_{m=0}^{N-1} x(m) e^{-j2\pi mk/N}\right] e^{j2\pi nk/N} \\
&= \frac{1}{N}\sum_{m=0}^{N-1} x(m)\left[\sum_{k=0}^{N-1} e^{j2\pi(n-m)k/N}\right] \\
&= \frac{1}{N}\sum_{m=0}^{N-1} x(m) N\delta[(n-m)_{\bmod N}] \\
&= x(n).
\end{aligned}$$

b) Circular convolution can be written $z(n) = \sum_{m=0}^{N-1} x(m) y(n-m)_{\bmod N}$. We can then exploit linearity and the shift property to give

$$\begin{aligned}
Z(k) &= \sum_{m=0}^{N-1} x(m) W_N^{-mk} Y(k) \\
&= Y(k) \sum_{m=0}^{N-1} x(m) W_N^{-mk} = X(k)Y(k)
\end{aligned}$$

c)

$$\begin{pmatrix} 1 & -5 & -3 & 4 \\ 4 & 1 & -5 & -3 \\ -3 & 4 & 1 & -5 \\ -5 & -3 & 4 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 4 \\ -9 \\ -18 \\ -10 \end{pmatrix}$$

d) Circular convolution can be used to perform linear convolution by first zero padding the two sequences. If $x(n)$ is of length L and $y(n)$ is of length M, pad respectively with $M-1$ and $L-1$ zeros. Circular convolution of the two zero padded sequences is equal to linear convolution of the two original sequences.

Patrick Naylor

2. a) From the definition of the z-transform

$$X_D(z) = \sum_{n=-\infty}^{\infty} x_D(n)z^{-n} = \sum_{n=-\infty}^{\infty} x(Mn)z^{-n}.$$

Let

$$x_1(n) = \begin{cases} x(n) & \text{if } n \text{ is an integer multiple of } M \\ 0 & \text{otherwise} \end{cases}$$

then

$$X_D(z) = \sum_{n=-\infty}^{\infty} x_1(Mn)z^{-n} = \sum_{k=-\infty}^{\infty} x_1(k)z^{-k/M}$$

since $x_1(k) = 0$ unless k is a multiple of M.

Therefore

$$X_D(z) = X_1(z^{1/M}) = \frac{1}{M}\sum_{k=0}^{M-1} X\left(z^{1/M}W_M^k\right)$$

as will be shown using $W_M^k = e^{-j2\pi k/M}$

We arrive at the previous expression for $X_D(z)$ by considering a new sequence

$$c_M(n) = \begin{cases} 1 & \text{if } n \text{ is an integer multiple of } M \\ 0 & \text{otherwise} \end{cases}$$

and then writing

$$x_1(n) = c_M(n)x(n)$$

Further consideration of $c_M(n)$ tells us that $c_M(n)$ is the inverse Fourier transform of unity and can be written

$$c_M(n) = \frac{1}{M}\sum_{k=0}^{M-1} W_M^{-kn}$$

Then

$$\begin{aligned} X_1(z) &= \frac{1}{M}\sum_{k=0}^{M-1}\sum_{n=-\infty}^{\infty} x(n)W_M^{-kn}z^{-n} \\ &= \frac{1}{M}\sum_{k=0}^{M-1}\sum_{n=-\infty}^{\infty} x(n)\left(W_M^k z\right)^{-n} \\ &= \frac{1}{M}\sum_{k=0}^{M-1} X\left(zW_M^k\right) \end{aligned}$$

from the definition of the z-transform. So finally

$$X_D(z) = \frac{1}{M}\sum_{k=0}^{M-1} X\left(z^{1/M}W_M^k\right)$$

b) i) $\{x(0),\ 0,\ x(3),\ 0,\ x(6),\ 0,\ x(9),\ 0,\ x(12),\ 0,\ \ldots\}$

ii) $\{x(0),\ 0,\ 0,\ 0,\ x(5),\ 0,\ 0,\ 0,\ x(10),\ 0,\ \ldots\}$

The values of the samples of $y(n)$ would be unaltered in case (i) but altered in case (ii). Only when L and M are co-prime is the order of the decimation and expansion arbitrary.

c) i) $X(z) = \frac{z}{z-\alpha}$

ii) $x_D(n) = \alpha^{Mn},\ n \geq 0$

iii) $X_D(z) = \frac{z}{z-\alpha^M}$

The final result is obtained by substituting the solution of (i) into the expression of (a) and replacing z with $z^{1/M}W_M^{-m}$

3. a) Zeros at $z = \pm 0.95$. Poles (x2) at $z = 0$.

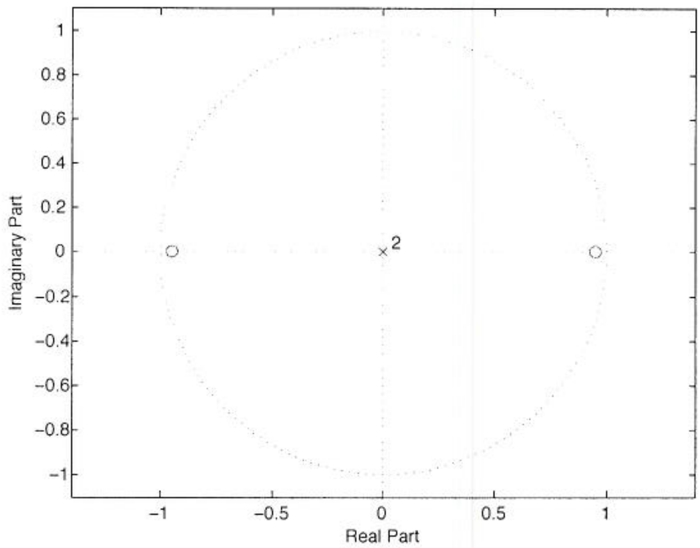

Figure 3.1

$$y(n) = 0.5x(n) - 0.45x(n-2)$$
$$H(z) = \frac{Y(z)}{X(z)} = 0.5 - 0.45z^{-2} = 0.5 \times \frac{(z-0.95)(z+0.95)}{z^2}$$

b) The frequency response is found by setting $z = e^{j\omega}$ giving

$$\begin{aligned} H(e^{j\omega}) &= 0.5 - 0.45e^{-j2\omega} \\ &= 0.5 - 0.45\cos(2\omega) + 0.45j\sin(2\omega) \\ |H(e^{j\omega})| &= \sqrt{(0.5 - 0.45\cos(2\omega))^2 + (0.45\sin(2\omega))^2} \\ &= \sqrt{0.25 + 0.203\cos^2(2\omega) - 0.45\cos(2\omega) + 0.203\sin^2(2\omega)} \\ &= \sqrt{0.453 - 0.45\cos(2\omega)} \end{aligned}$$

c) See Fig. 3.2.

d) At $\omega = 0$, $|H(e^{j\omega})| = \sqrt{0.003} = 0.0548 \approx -25$ dB.
At $\omega = \pi/4$, $|H(e^{j\omega})| = \sqrt{0.453 - 0.45\cos(\pi/2)} = 0.673 \approx -3.4$ dB.

e) Group delay is the negative derivative of phase with frequency: $-\frac{d\phi}{d\omega}$, measured in units of time (s).

f) Here we are concerned with $\omega = 2\pi * 10/44.1 = \pi/2.205$. The phase response is found from

$$\angle H(e^{j\omega}) = \arctan\frac{0.45\sin(2\omega)}{0.5 - 0.45\cos(2\omega)}$$

Near $\pi/2.205$ we are a long way, in terms of distance around the unit circle, from either of the zeros. In this case, a first order difference will adequately

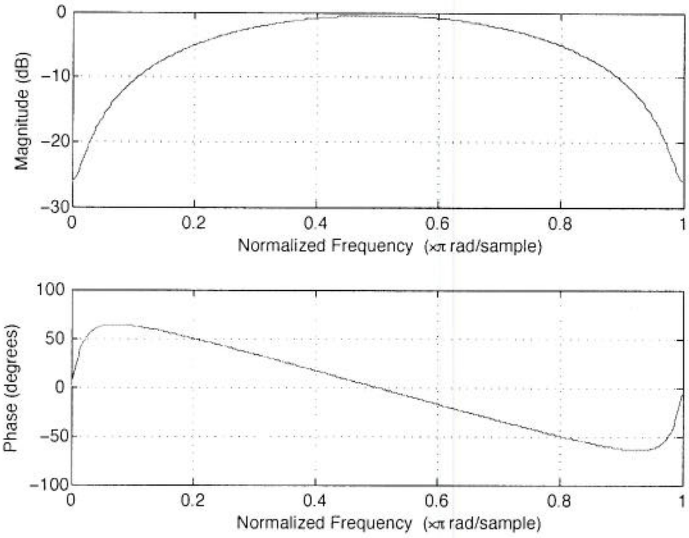

Figure 3.2

estimate the derivative using frequencies of, for example, 1.32 and 1.52, giving approx

$$\frac{\pi/5 \text{ (radians)}}{0.2\pi \text{ (radians/sample)} * fs \text{ (samples/second)}}$$

giving approximate one sample period or 22 μs.

4. a) ROC encloses the range of z for which the z-transform expression converges. Outside the ROC the z-transform does not exist in any meaningful way.

The z-transform in the discrete-time case corresponds to the Laplace transform in the continuous time case. Sketches should show the significance of the unit circle vs. the $j\omega$ axis. The frequency response of discrete-time signals is periodic. These are compactly represented by multiple rotations around the unit circle in z.

b) $\sigma < 0$ is the left-hand half-plane of s which maps to the interior of the unit circle in z.
$\sigma = 0$ maps to the unit circle in z.
$\sigma > 0$ is the right-hand half-plane of s which maps to the exterior of the unit circle in z.

The significance comes from consideration of stability of causal systems which is satisfied for poles inside the unit circle in z, marginally satisfied for poles on the unit circle itself and not satisfied for poles outside the unit circle in z.

$$H(z) = \frac{z^2 + 2z + 1}{1.78z^2 - 1.64z + 0.58}$$

$$\Omega = \tan\left(\frac{\omega}{2}\right)$$

5. a) i) Impulse response is the output of a system when the input is 1 at $n = 0$ and zero for all other n.

ii) Step response is the output of a system when the input is 1 for $n \geq 0$ and zero for all other n.

iii) Causal systems have zero impulse response before the application of the input impulse.

b) Any bilinear transfer functions are acceptable subject to -1 mark if the d.c. gain is not 25 and -1 mark if the system is unstable.

c) $h(n) = g(n) - g(n-1)$

d) Given

$$H(z) = a_0 + a_1 z^{-1} + a_2 z^{-2} + \ \ldots \ + a_N z^{-N},$$

then $h(n) = a_n$ for $n = 0,\ 1,\ \ldots,\ N$ and $h(n) = 0$ for $n > N$. There it is easy to see that $h(2N+2) = 0$.

We can rewrite the expression for $h(n)$ as $g(n) = h(n) + g(n-1)$

so that

$$\begin{aligned} g(2N+2) &= \sum_{n=0}^{2N+2} h(n) = \sum_{n=0}^{N} h(n) \\ &= a_0 + a_1 z^{-1} + a_2 z^{-2} + \ \ldots \ + a_N z^{-N}. \end{aligned}$$

6. a) The description is bookwork but, as an example for the overlap save method, the level of detail required is seen in the following. Full marks will be awarded for this level if accompanied by an appropriate and clear block diagram.

1. Determine the FFT size, N, the data block size, L, and the impulse response block size, M such that N is an integer power of 2 and greater than $L+M-1$.

2. Form each data block using the last $M-1$ points from previous block followed by the L samples of the current block.

3. Zero-pad the impulse response block with $L-1$ zeros.

4. Compute and store the FFT of the zero-padded impulse response.

5. For each data block, calculate the IFFT of the product of the FFT of the zero-padded impulse response with the FFT of the zero-padded data block.

6. Determine the final output from the IFFTs discarding the first $M-1$ samples of each block and then concatenating.

b) Let $x(n)$ be divided up into data blocks of length $L=3$ so that $N=L+M-1=4$.

The first two data blocks are therefore

$x1=[0,\ 1,\ 2,\ 3]$ with FFT $X1=[6,\ -2+j2,\ -2,\ -2-j2]$
$x2=[3,\ 3,\ 2,\ 1]$ with FFT $X2=[9,\ 1-j2,\ 1,\ 1+j2]$

Now we have the zero padded impulse response as

$h(n)=[1,-1,0,0]$ with FFT $H(k)=[0,1+j,2,1-j]$

Next for the products we obtain

$Y1=H.*X1=[0,\ -4,\ -4,\ -4]$
$Y2=H.*X2=[0,\ 3-j,\ 2,\ 3+j]$

Next for the IFFTs we obtain

$y1=[-3,\ 1,\ 1,\ 1]$
$y2=[2,\ 0,\ -1,\ -1]$

These outputs are then overlap-saved as follows:

1, 1, 1
. . . 0, -1, -1
y = [1, 1, 1, 0, -1, -1]