IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2005

EEE/ISE PART II: MEng, BEng and ACGI

Corrected Copy

**CONTROL ENGINEERING**

Friday, 27 May 2:00 pm

Time allowed: 2:00 hours

**There are FOUR questions on this paper.**

**Q1 is compulsory.**
**Answer Q1 and any two of questions 2-4.**
**Q1 carries 40% of the marks. Questions 2 to 4 carry equal marks (30% each).**

**Any special instructions for invigilators and information for candidates are on page 1.**

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1. Consider the mass-spring-damper system shown in Figure 1 below, in which $y(t)$ denotes the displacement of the mass $M$ from its rest position. A force $f(t)$ is applied to the mass $M$ as shown. Take $M = 1Kg$ and $D_1 = D_2 = 1Ns/m$.

   (a) By considering the balance of forces on the mass, derive the differential equation relating $f(t)$ to $y(t)$. [6]

   (b) Derive the transfer function $G(s)$ between $f(s)$ and $y(s)$. [6]

   (c) Sketch the locus of the poles of $G(s)$ for $0 \leq K < \infty$. [8]

   (d) Find the value of $K$ for which the response is (i) critically damped, (ii) marginally stable. [6]

   (e) Take $K = 1N/m$. Suppose that $f(t) = \cos t$. Find the steady-state response $y_{ss}(t)$. [7]

   (f) Take $K = 1N/m$. Sketch the Nyquist diagram of $G(s)$. [7]

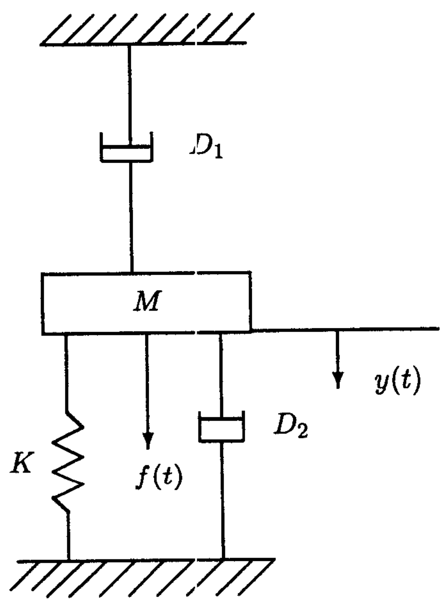



*Figure 1*

2. Consider the feedback system shown in Figure 2 below for the regulation of a voltage supply. Here, $v_r(t)$ is the reference voltage, $v_o(t)$ is the supplied output voltage and $i(t)$ is the load current. $R_o$ is the output resistances of the op–amp. The op–amp voltage is modelled as

$$E(s) = -\frac{A}{(\tau s + 1)^3} v_e(s)$$

where $A > 0$ is the op–amp dc–gain, $\tau > 0$ is a time constant and $v_e(s)$ is the the Laplace transform of the voltage at the op–amp negative terminal.

(a) Derive an expression for $v_e(s)$ in terms of $v_r(s)$ and $v_o(s)$. **[6]**

(b) Derive an expression for $v_o(s)$ in terms of $v_e(s)$ and $i(s)$. **[6]**

(c) Hence, derive and draw a block diagram representation of the feedback loop. Take the reference to be $-v_r(s)$ and the output to be $v_o(s)$. Indicate the signals $v_e(s)$ and $i(s)$ on the block diagram. **[6]**

(d) Find the maximum value of the op–amp gain $A$ for which the voltage regulator is stable. **[12]**

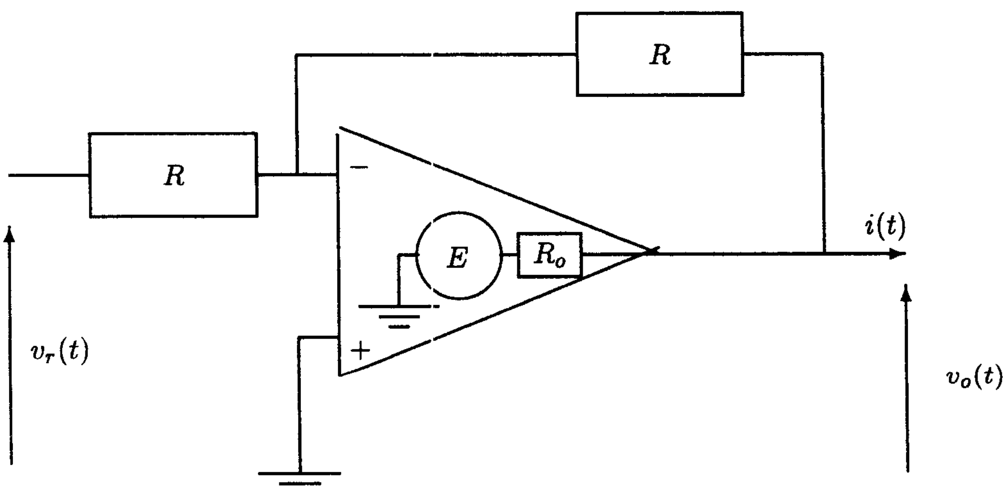



*Figure 2*

3. Consider the feedback loop shown in Figure 3 below. Here

$$G(s) = \frac{1}{s(s-1)}$$

and $K(s)$ is a compensator.

(a) Take $K(s) = k$ where $k$ is a constant gain. Draw the root-locus accurately as $k$ varies in the range $0 \leq k < \infty$. Your answer should include the centre and angles of the asymptotes, the breakaway points and the range of values of $k$ for closed-loop stability. **[10]**

(b) Design a proportional-plus-derivative ($PD$) compensator $K(s)$ for the feedback loop shown in Figure 3 such that

i. the closed-loop is marginally stable, and

ii. the closed-loop response is oscillatory with a frequency of oscillation of 1 $rad/s$.

Sketch of the root-locus for the compensated system. **[10]**

(c) Design a $PD$ compensator $K(s)$ for the feedback loop shown in Figure 3 such that

i. the closed-loop is stable, and

ii. the closed-loop has a double pole at $-2$.

Sketch of the root-locus for the compensated system. **[10]**

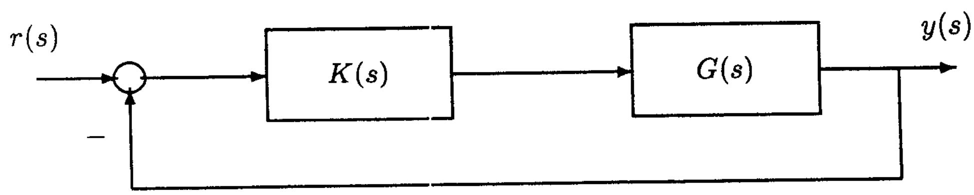



*Figure 3*

4. Consider the feedback control system in Figure 4 below. Here,

$$G(s) = \frac{4}{(s+1)^3}$$

and $K(s)$ is the transfer function of a compensator.

(a) Sketch the Nyquist diagram of $G(s)$, indicating the low and high frequency portions. Also, calculate the real–axis intercepts. **[7]**

(b) Take $K = 1$. Show that the closed–loop is stable and determine the gain and phase margins. **[7]**

(c) Without doing any actual design, briefly describe how a phase–lag compensator,

$$K(s) = \frac{1 + s/\omega_0}{1 + s/\omega_p}, \qquad 0 < \omega_p < \omega_0$$

would effect the stability margins and the steady–state tracking properties of the loop. **[8]**

(d) Without doing any actual design, briefly describe how a phase–lead compensator,

$$K(s) = \frac{1 + s/\omega_0}{1 + s/\omega_p}, \qquad 0 < \omega_0 < \omega_p,$$

would affect the gain and phase margins. Your answer should emphasize the difficulties involved in the design. **[8]**

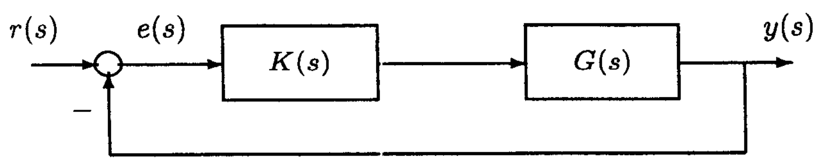


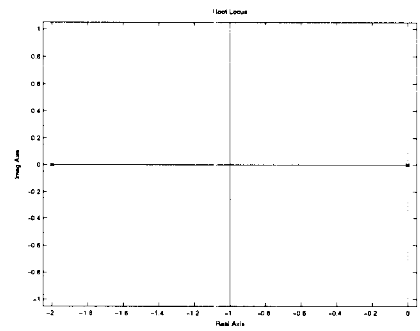

*Figure 4*

**SOLUTIONS (E2.6/ISE2.9, Control Engineering, 2005)**

1. (a) $\boxed{f(t) = M\ddot{y}(t) + (D_1 + D_2)\dot{y}(t) + Ky(t) = \ddot{y}(t) + 2\dot{y}(t) + Ky(t).}$

   (b) Taking Laplace transform: $(Ms^2+(D_1+D_2)s+K)y(s)=f(s)$. So $\boxed{G(s) = \frac{1}{Ms^2+(D_1+D_2)s+K} = \frac{1}{s^2+2s+K}.}$

   (c) The poles are the roots of $s^2 + 2s + K = 0$, or equivalently, of $1 + K\hat{G}(s) = 0$ where $\hat{G}(s) = 1/s(s + 2)$. The locus of the poles of $G(s)$ is then the root locus of $\hat{G}(s)$ which is shown below.

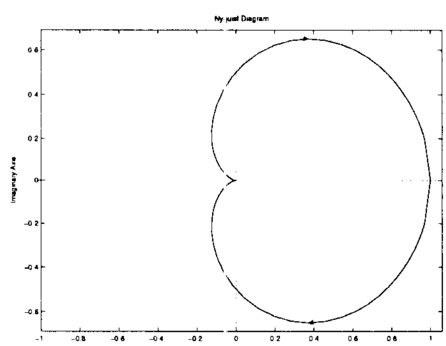


   (d) (i) $\boxed{K = 1.}$ (ii) $\boxed{K = 0.}$

   (e) Since $G(s) = 1/(s + 1)^2$ is stable, the steady-state response to a sinusoid of frequency $\omega$ is also a sinusoid of the same frequency, with an amplitude $|g(j\omega)|$ and phase $\angle g(j\omega)$. Since $\omega = 1$,

$$\boxed{y_{ss}(t) = |g(j)| \cos(t + \angle g(j)) = 0.5 \cos(t - \tfrac{\pi}{2}) = 0.5 \sin t.}$$

   (f) The Nyquist diagram is shown below:



2/4

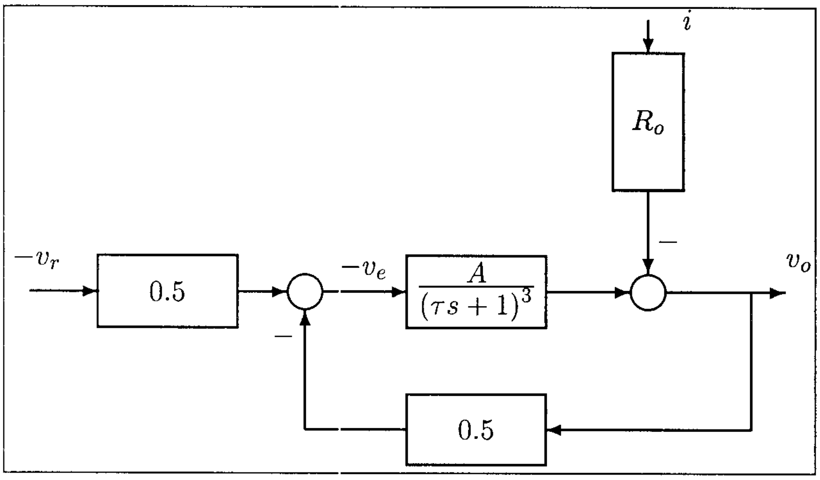
2. (a) Using a potential divider rule at the op–amp input gives

$$\frac{v_e(s) - v_r(s)}{v_o(s) - v_r(s)} = 0.5 \Rightarrow \boxed{-v_e(s) = -0.5v_r(s) - 0.5v_o(s).}$$

(b) At the op–amp output we have

$$E(s) - v_o(s) = R_o i(s) \Rightarrow \boxed{v_o(s) = -\frac{A}{(\tau s + 1)^3} v_e(s) - R_o i(s).}$$

(c) Using parts (a) and (b), the block diagram becomes,




(d) The closed–loop characteristic equation is

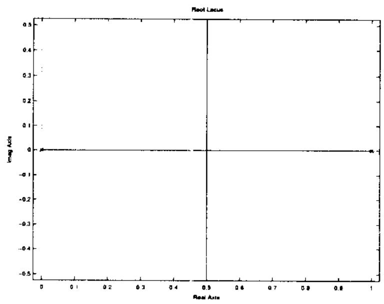
$$1 + \frac{0.5A}{(\tau s + 1)^3} = 0 \Rightarrow \tau^3 s^3 + 3\tau^2 s^2 + 3\tau s + 1 + 0.5A = 0.$$

The Routh array is then

$$\begin{array}{c|cc} s^3 & \tau^3 & 3\tau \\ s^2 & 3\tau^2 & 1 + 0.5A \\ s & \frac{8 - 0.5A}{3}\tau & \\ 1 & 1 + 0.5A & \end{array}$$

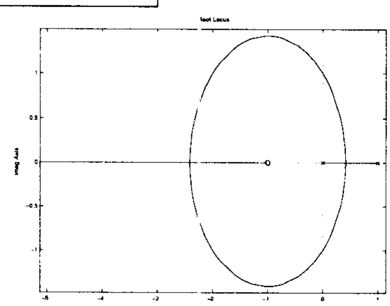
For stability, we require no sign changes in the first column. Since $\tau > 0$ we require (1): $1 + 0.5A > 0$ and (2): $8 - 0.5A > 0$. Since $A$ is positive, this reduces to $\boxed{A < 16.}$

3. (a) The root-locus plot is shown below. The centre and angles of the

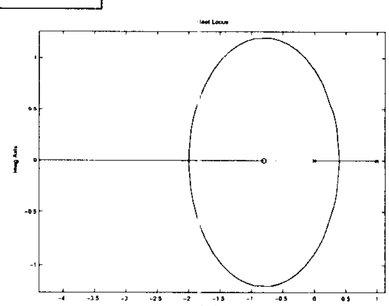


asymptotes are $\boxed{\sigma = 0.5 \ \& \ \psi = \pm 90^o}$ and the breakaway point is $\boxed{\sigma_b = 0.5.}$ The closed-loop is $\boxed{\text{unstable for all } k \geq 0.}$

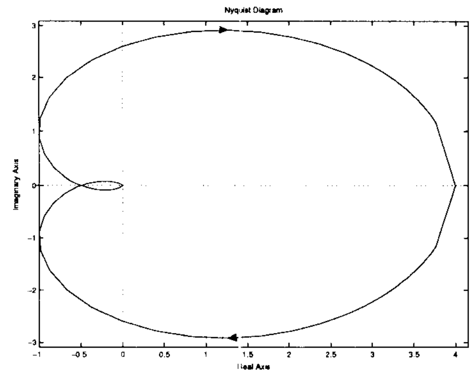
(b) A $PD$ compensator has the form $K(s) = k(s+z)$ where $k > 0$ and $z > 0$. The required locations of the closed-loop poles are at $\pm j$. The angle criterion gives $z = 1$ and the gain criterion gives $k = 1$ so $\boxed{K(s) = s+1}$. The root-locus is shown below.



(c) The required locations of the closed-loop poles are at $-2$, which is a break-in point. Setting $K(s) = k(s+z)$ and $dG(s)K(s)/ds = 0$ for $s = -2$ gives $z = 0.8$. The gain criterion gives $k = 5$. So $\boxed{K(s) = 5(s+0.8)}$. The root-locus is shown below.



4. (a) The Nyquist plot is shown below. The real-axis intercepts can be found by setting the imaginary part of $G(j\omega)$ to zero. This gives intercepts at $\boxed{\omega_i = 0, \pm\sqrt{3}, \infty}$ and so $\boxed{G(j\omega_i) = 4, -0.5, -0.5, 0.}$




(b) The number of unstable closed-loop poles is determined by the number of encirclements by $G(s)$ of the point $-1$, which is zero. $\boxed{\text{Thus the closed-loop is stable}}$ since $G(s)$ has no unstable poles. Since the real-axis intercept is at $-0.5$, $\boxed{\text{the gain margin is } 2}$. For the phase margin, we need the intercept with the unit circle centred on the origin. We solve $|G(j\omega)| = 1$, this gives $\omega_1 = \sqrt{4^{\frac{2}{3}} - 1}$ and $\arg[G(j\omega_1)] = -153°$. $\boxed{\text{The phase margin is then } 27°.}$

(c) The phase-lag compensator has gain close to one for frequencies below $\omega_p$ and close to $\frac{\omega_p}{\omega_0} < 1$ for frequencies beyond $\omega_0$. The phase is negative and large between these two frequencies but insignificant elsewhere. Thus phase-lag compensation can reduce high frequency gain (and so improve stability margins) without reducing low frequency gain (and hence degrading steady-state tracking since $|e(j\omega) = |\frac{G(j\omega)K(j\omega)}{1+G(j\omega)K(j\omega)}||r(j\omega)|)$ or introducing phase lag at high frequency (which destabilizes the loop). We should place $w_p$ and $w_0$ in the 'middle' frequency range.

(d) The phase-lead has gain close to 1 for $\omega < \omega_0$ and close to $\frac{\omega_p}{\omega_0} > 1$ for $\omega > \omega_p$. The phase is positive and large between $\omega_0$ and $\omega_p$ but small elsewhere. Thus the gain increase for $\omega > \omega_p$ degrades stability margins while the phase-lead increases the phase margin. It is important to balance the destabilizing increase in gain and the stabilizing increase in phase. We should place $w_p$ and $w_0$ in the crossover frequency range (when $|G(j\omega)| \approx 1$).