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UNIVERSITY OF LONDON

[E1.11 2004]

B.ENG. AND M.ENG. EXAMINATIONS 2004

For Internal Students of Imperial College London

This paper is also taken for the relevant examination for the Associateship of the City & Guilds of London Institute

INFORMATION SYSTEMS ENGINEERING E1.11

MATHEMATICS

Date Wednesday 2nd June 2004 10.00 am - 1.00 pm

Answer SEVEN questions

Answers to Section A *questions must be written in a different answer book from answers to* Section B *questions.*

Corrected Copy

[Before starting, please make sure that the paper is complete. There should be SIX pages, with a total of NINE questions. Ask the invigilator for a replacement if this copy is faulty.]

SECTION A [E1.11 2004]

1. (i) Express each of the following complex numbers in the form $x + iy$ (with x and y real) :

$$\text{(a)}\ \ \frac{1+i}{(1-2i)^2}\ ; \qquad \text{(b)}\ \ e^{i2\pi/3}\ ;$$

$$\text{(c)}\ \ (1+i)^5\ ; \qquad \text{(d)}\ \ \sinh\left(1+\frac{i\pi}{2}\right)\ .$$

(ii) Find all the solutions of the equation $\sin z = 4$.

Give your answer in the form $z = x + iy$ (with x and y real).

2. (i) (a) Use Leibnitz's rule to find $\dfrac{d^5}{dx^5}\left(x^2 e^{-2x}\right)$.

(b) Differentiate $(\cosh x)^x$.

(c) If $y + y^3 + \ln y = 5x$, find $\dfrac{dy}{dx}$ as a function of y.

(ii) Find the limits :

(a)
$$\lim_{x\to 3} \frac{x^2 - x - 6}{x^2 - 4x + 3}\ ;$$

(b)
$$\lim_{x\to 5} \frac{\sin(x-5)}{x^2 - 6x + 5}\ ;$$

(c)
$$\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{e^x - 1}\right)\ .$$

In (b), you may use the result $\displaystyle\lim_{y\to 0} \frac{\sin y}{y} = 1$, without proof.

PLEASE TURN OVER

3. (i) (a) Evaluate (*correct to* 3 *decimal places*) $\sum_{n=1}^{10} (\ln 2)^n$.

(b) Evaluate (*correct to* 3 *decimal places*) $\sum_{n=1}^{\infty} (\ln 2)^n$.

(ii) Use standard tests to determine whether the following series converge or diverge:

(a) $\sum_{n=1}^{\infty} \frac{e^{n^2}}{n!}$;

(b) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n!}}$.

(iii) Using the Maclaurin expansion for the exponential function, find the Maclaurin expansion for $\cosh(x^2)$ up to the third non-zero term.

(iv) Find the interval of convergence of $\sum_{n=1}^{\infty} \frac{2^n(2x+1)^n}{n^2}$.

Investigate also the endpoints of the interval.

4. Evaluate the following indefinite integrals:

(i) $$\int \frac{2x+9}{x^2+9x+4}\, dx \ ;$$

(ii) $$\int x^3 \ln x \, dx \ ;$$

(iii) $$\int \frac{dx}{\sqrt{x^2-4x-5}} \ ;$$

(iv) $$\int \frac{(x+3)dx}{(x+2)(x-1)} \ .$$

PLEASE TURN OVER

5. Find the general solution of the following differential equations:

(i)
$$\frac{dy}{dx} = \frac{y^2 + 2xy}{x^2} \;;$$

(ii)
$$\frac{dy}{dx} + \frac{y}{1+x^2} = x \exp(-\tan^{-1} x) \;;$$

(iii)
$$y'' - 10y' + 25y = e^{3x} \,.$$

(iv)
$$y'' - 11y' + 30y = 0 \,.$$

For (iv) find also the solution subject to the conditions $y(0) = y'(0) = 1$.

PLEASE TURN OVER

SECTION B

6. Let $f(x, y) = x^2y - 9y + y^3$.

   (i) Find the four stationary points of f.

   (ii) Determine the nature (maximum, minimum or saddle point) of each of these stationary points.

   (iii) Sketch the contours of f which pass through the saddle points.

   (iv) Make a rough sketch of some further contours of f.

7. (i) Consider the three planes

$$\begin{aligned} \mathbf{r}.\,(1,\,1,\,1) &= 1\,, \\ \mathbf{r}.\,(1,\,2,\,a) &= 0\,, \\ \mathbf{r}.\,(3,\,2,\,a) &= b\,, \end{aligned}$$

   where $\mathbf{r} = (x,\, y,\, z)$.

   Giving your reasoning, determine for which values of a and b these three planes

   (a) meet in exactly one point,

   (b) meet in a line,

   (c) do not meet at all.

   (ii) Let

$$A = \begin{pmatrix} 4 & 3 \\ 3 & -4 \end{pmatrix}.$$

   Find the eigenvalues and eigenvectors of A.

   Find an invertible 2×2 matrix P such that $P^{-1}AP$ is diagonal.

8. Find the Fourier series of the function

$$f(x) = \pi^2 - x^2, \qquad -\pi \le x < \pi .$$

Use Parseval's formula to deduce from this that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} .$$

9. (i) Find the inverse Laplace transforms of the following functions:

$$\text{(a)} \quad \frac{s+1}{s^2+4}, \qquad\qquad \text{(b)} \quad \frac{e^{-2s}}{s^4} .$$

(ii) Use Laplace transforms to find functions x, y of t satisfying the following simultaneous differential equations:

$$\begin{aligned} 8\frac{dx}{dt} - 5\frac{dy}{dt} + 2x &= 0, \\ 2\frac{dx}{dt} - \frac{dy}{dt} &= -2\sin 2t, \end{aligned}$$

with $x(0) = 2$, $y(0) = 3$.

END OF PAPER

MATHEMATICS DEPARTMENT

MATHEMATICAL FORMULAE

1. VECTOR ALGEBRA

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1,\, a_2,\, a_3)$$

Scalar (dot) product: $\mathbf{a}\,.\,\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

Vector (cross) product:

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Scalar triple product:

$$[\mathbf{a},\,\mathbf{b},\,\mathbf{c}] = \mathbf{a}\,.\,\mathbf{b}\times\mathbf{c} = \mathbf{b}\,.\,\mathbf{c}\times\mathbf{a} = \mathbf{c}\,.\,\mathbf{a}\times\mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product: $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{c}\,.\,\mathbf{a})\mathbf{b} - (\mathbf{b}\,.\,\mathbf{a})\mathbf{c}$

2. SERIES

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \ldots \quad (\alpha \text{ arbitrary, } |x|<1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \cdots,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n\frac{x^{2n}}{(2n)!} + \cdots,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + \cdots,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^n\frac{x^{n+1}}{(n+1)} + \ldots (-1 < x \le 1)$$

3. TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin(a+b) = \sin a\,\cos b + \cos a\,\sin b\;;$$

$$\cos(a+b) = \cos a\,\cos b - \sin a\,\sin b\,.$$

$$\cos iz = \cosh z\,;\quad \cosh iz = \cos z\,;\quad \sin iz = i\sinh z\,;\quad \sinh iz = i\sin z\,.$$

4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f\,D^n g + \tbinom{n}{1}\,Df\,D^{n-1}g + \ldots + \tbinom{n}{r}\,D^r f\,D^{n-r}g + \ldots + D^n f\;g\,.$$

(b) Taylor's expansion of $f(x)$ about $x=a$:

$$f(a+h) = f(a) + hf'(a) + h^2f''(a)/2! + \ldots + h^nf^{(n)}(a)/n! + \epsilon_n(h)\,,$$

where $\epsilon_n(h) = h^{n+1}f^{(n+1)}(a+\theta h)/(n+1)!,\quad 0<\theta<1\,.$

(c) Taylor's expansion of $f(x,\,y)$ about $(a,\,b)$:

$$f(a+h,\,b+k) = f(a,\,b) + [hf_x + kf_y]_{a.b} + 1/2!\,\left[h^2f_{xx} + 2hkf_{xy} + k^2f_{yy}\right]_{a.b} + \cdots$$

(d) Partial differentiation of $f(x,\,y)$:

i. If $y=y(x)$, then $f=F(x)$, and $\dfrac{dF}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dx}$.

ii. If $x=x(t)$, $y=y(t)$, then $f=F(t)$, and $\dfrac{dF}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$.

iii. If $x=x(u,\,v)$, $y=y(u,\,v)$, then $f=F(u,\,v)$, and

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}\,,\qquad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}\,.$$

(e) Stationary points of $f(x,\,y)$ occur where $f_x=0$, $f_y=0$ simultaneously.
Let $(a,\,b)$ be a stationary point: examine $D = [f_{xx}f_{yy} - (f_{xy})^2]_{a.b}$.
If $D>0$ and $f_{xx}(a,\,b)<0$, then $(a,\,b)$ is a maximum;
If $D>0$ and $f_{xx}(a,\,b)>0$, then $(a,\,b)$ is a minimum;
If $D<0$ then $(a,\,b)$ is a saddle-point.

(f) Differential equations:

i. The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating factor $I(x) = \exp[\int P(x)(dx]$, so that $\dfrac{d}{dx}(Iy) = IQ$.

ii. $P(x,\,y)dx + Q(x,\,y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y$.

5. INTEGRAL CALCULUS

(a) An important substitution: $\tan(\theta/2) = t$:
$\sin\theta = 2t/(1+t^2)$, $\cos\theta = (1-t^2)/(1+t^2)$, $d\theta = 2\,dt/(1+t^2)$.

(b) Some indefinite integrals:

$$\int (a^2 - x^2)^{-1/2} dx = \sin^{-1}\left(\frac{x}{a}\right), \quad |x| < a .$$

$$\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1}\left(\frac{x}{a}\right) = \ln\left\{\frac{x}{a} + \left(1 + \frac{x^2}{a^2}\right)^{1/2}\right\} .$$

$$\int (x^2 - a^2)^{-1/2} dx = \cosh^{-1}\left(\frac{x}{a}\right) = \ln\left|\frac{x}{a} + \left(\frac{x^2}{a^2} - 1\right)^{1/2}\right| .$$

$$\int (a^2 + x^2)^{-1} dx = \left(\frac{1}{a}\right)\tan^{-1}\left(\frac{x}{a}\right) .$$

6. NUMERICAL METHODS

(a) Approximate solution of an algebraic equation:

If a root of $f(x) = 0$ occurs near $x = a$, take $x_0 = a$ and
$x_{n+1} = x_n - [f(x_n)/f'(x_n)]$, $n = 0, 1, 2 \ldots$

(Newton Raphson method).

(b) Formulae for numerical integration: Write $x_n = x_0 + nh$, $y_n = y(x_n)$.

i. Trapezium rule (1-strip): $\int_{x_0}^{x_1} y(x)dx \approx (h/2)\,[y_0 + y_1]$.

ii. Simpson's rule (2-strip): $\int_{x_0}^{x_2} y(x)dx \approx (h/3)\,[y_0 + 4y_1 + y_2]$.

(c) Richardson's extrapolation method: Let $I = \int_a^b f(x)dx$ and let I_1, I_2 be two estimates of I obtained by using Simpson's rule with intervals h and $h/2$.

Then, provided h is small enough,

$$I_2 + (I_2 - I_1)/15 ,$$

is a better estimate of I .

7. LAPLACE TRANSFORMS

Function	Transform	Function	Transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)dt$	$af(t) + bg(t)$	$aF(s) + bG(s)$
df/dt	$sF(s) - f(0)$	d^2f/dt^2	$s^2F(s) - sf(0) - f'(0)$
$e^{at}f(t)$	$F(s-a)$	$tf(t)$	$-dF(s)/ds$
$(\partial/\partial\alpha)f(t,\alpha)$	$(\partial/\partial\alpha)F(s,\alpha)$	$\int_0^t f(t)dt$	$F(s)/s$
$\int_0^t f(u)g(t-u)du$	$F(s)G(s)$		
1	$1/s$	$t^n (n = 1, 2 \ldots)$	$n!/s^{n+1}$, $(s > 0)$
e^{at}	$1/(s-a)$, $(s > a)$	$\sin\omega t$	$\omega/(s^2+\omega^2)$, $(s > 0)$
$\cos\omega t$	$s/(s^2+\omega^2)$, $(s > 0)$	$H(t-T) = \begin{cases} 0, & t < T \\ 1, & t > T \end{cases}$	e^{-sT}/s , $(s, T > 0)$

8. FOURIER SERIES

If $f(x)$ is periodic of period $2L$, then $f(x+2L) = f(x)$, and

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \text{ where}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad n = 0, 1, 2, \ldots, \text{ and}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1, 2, 3, \ldots$$

Parseval's theorem

$$\frac{1}{L}\int_{-L}^{L} [f(x)]^2\, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right) .$$

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MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

2003 – 2004

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PAPER ISE 16

QUESTION

SOLUTION 1

(i) (a) $\dfrac{1+i}{(1-2i)^2} = \dfrac{1+i}{1-4i+4i^2} = \dfrac{1+i}{-3-4i} = \dfrac{-(1+i)(3-4i)}{(3+4i)(3-4i)}$

$$= \frac{-[3+3i-4i-4i^2]}{[9-16i^2]} = \frac{-[7-i]}{25} = -\frac{7}{25} + \frac{i}{25}$$ 3

(b) $e^{i2\pi/3} = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ 2

(c) $(1+i)^5 = \left(\sqrt{2}e^{i\pi/4}\right)^5 = (\sqrt{2})^5 e^{i\frac{5\pi}{4}} = 4\sqrt{2}\left(-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)$

$$= -4(1+i)$$ 3

(d) $\sinh\left(1+\frac{i\pi}{2}\right) = \dfrac{e^{1+\frac{i\pi}{2}} - e^{-(1+\frac{i\pi}{2})}}{2} = \dfrac{e\,i - e^{-1}(-i)}{2}$

$$= \frac{i(e+1/e)}{2} = \frac{i(e^2+1)}{2e}$$ 3

(ii) $\sin z = 4 \Rightarrow \dfrac{e^{iz}-e^{-iz}}{2i} = 4 \Rightarrow e^{2iz} - 8ie^{iz} - 1 = 0$

Let $u = e^{iz} \Rightarrow u^2 - 8iu - 1 = 0 \Rightarrow u = \dfrac{8i \pm \sqrt{-64+4}}{2}$

$$= 4i \pm \sqrt{-15} = i(4 \pm \sqrt{15})$$ 9

So $e^{iz} = i(4\pm\sqrt{15}) \Rightarrow iz = \ln\left[e^{\frac{i\pi}{2} + i2n\pi}(4\pm\sqrt{15})\right]$ $n = 0, \pm1, \pm2, \ldots$

$$= i\left[\frac{\pi}{2} + 2n\pi\right] + \ln(4\pm\sqrt{15})$$

$$\Rightarrow z = \pi\left[\frac{1}{2}+2n\right] - i\ln(4\pm\sqrt{15})$$

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PAPER ISE 1.6
QUESTION
SOLUTION 2

(i)

(a) $$\frac{d^5}{dx^5}(x^2e^{-2x}) = x^2\frac{d^5}{dx^5}(e^{-2x}) + 5\frac{d}{dx}(x^2)\frac{d^4}{dx^4}(e^{-2x}) + 10\frac{d^2}{dx^2}(x^2)\frac{d^3}{dx^3}(e^{-2x})$$

$$= x^2(-2)^5e^{-2x} + 5\cdot 2x\,(-2)^4e^{-2x} + 10.\,2.(-2)^3e^{-2x}$$

$$= -32x^2e^{-2x} + 160xe^{-2x} - 160e^{-2x}$$ 3

(b) $y = (\cosh x)^x \Rightarrow \ln y = x\ln(\cosh x)$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \ln(\cosh x) + \frac{x\sinh x}{\cosh x}$$

$$\Rightarrow \frac{dy}{dx} = (\cosh x)^x\left[\ln(\cosh x) + x\tanh x\right]$$ 4

(c) $y + y^3 + \ln y = 5x \Rightarrow \frac{dy}{dx} + 3y^2\frac{dy}{dx} + \frac{1}{y}\frac{dy}{dx} = 5$

$$\Rightarrow \frac{dy}{dx} = \frac{5}{1+3y^2+\frac{1}{y}}$$ 3

(ii)

(a) $$\lim_{x\to 3}\frac{x^2-x-6}{x^2-4x+3} = \lim_{x\to 3}\frac{(x-3)(x+2)}{(x-3)(x-1)} = \lim_{x\to 3}\frac{x+2}{x-1} = \frac{5}{2}$$ 3

(b) $$\lim_{x\to 5}\frac{\sin(x-5)}{x^2-6x+5} = \lim_{x\to 5}\frac{\sin(x-5)}{(x-5)(x-1)} = \frac{1}{4}\lim_{x\to 5}\frac{\sin(x-5)}{(x-5)}$$ 3

Let $y = x-5$ $$= \frac{1}{4}\lim_{y\to 0}\frac{\sin y}{y} = \frac{1}{4}$$

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MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

2003 – 2004

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PAPER ISE 1.6

QUESTION

SOLUTION 2

(c) $$\lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{e^x - 1} \right) = \lim_{x \to 0} \left(\frac{e^x - 1 - x}{x(e^x - 1)} \right)$$

4

$$= \lim_{x \to 0} \left(\frac{1 + x + \frac{x^2}{2} + \dots - 1 - x}{x(1 + x + \dots - 1)} \right)$$

$$= \lim_{x \to 0} \left(\frac{x^2/2}{x^2} \right) = \frac{1}{2}.$$

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PAPER ISE 1·6

QUESTION

SOLUTION 3

(i) (a) $\sum_{n=1}^{10} (\ln 2)^n = \dfrac{(\ln 2)\left[(\ln 2)^{10} - 1\right]}{\ln 2 - 1} \approx 2.201$ 2

(b) $\sum_{n=1}^{\infty} (\ln 2)^n = \dfrac{\ln 2}{1-\ln 2} \approx 2.259$ 2

(ii) (a) For $\sum_{n=1}^{\infty} \dfrac{e^{n^2}}{n!}$ Let ρ_n be ratio between $(n+1)$th & nth terms of series 4

$$\rho = \lim_{n\to\infty} \rho_n = \lim_{n\to\infty} \left(\frac{e^{(n+1)^2}}{(n+1)!}\right) \Big/ \left(\frac{e^{n^2}}{n!}\right) = \lim_{n\to\infty} \frac{e^{n^2+2n+1}}{e^{n^2}} \frac{n!}{(n+1)n!}$$

$$= \lim_{n\to\infty} \frac{e^{2n+1}}{(n+1)}$$ which diverges $\Rightarrow$ series is divergent

(b) For $\sum_{n=1}^{\infty} \dfrac{1}{\sqrt[3]{n!}}$ 3

$$\rho = \lim_{n\to\infty} \rho_n = \lim_{n\to\infty} \frac{1}{\sqrt[3]{(n+1)!}} \Big/ \frac{1}{\sqrt[3]{n!}} = \lim_{n\to\infty} \sqrt[3]{\frac{n!}{(n+1)!}} = \lim_{n\to\infty} \sqrt[3]{\frac{1}{n+1}} = 0$$

$\Rightarrow$ series converges

(iii) $$\cosh x^2 = \frac{e^{x^2} + e^{-x^2}}{2} = \frac{1}{2}\left[1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \dots\right]$$ 3

$$+ \frac{1}{2}\left[1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} + \dots\right] = 1 + \frac{x^4}{2!} + \frac{x^8}{4!} + \dots$$

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PAPER ISE 1.6

QUESTION

SOLUTION 3

(ii) $\rho = \lim_{n\to\infty} \rho_n = \lim_{n\to\infty} \dfrac{2^{n+1}(2x+1)^{n+1}}{(n+1)^2} \Big/ \dfrac{2^n(2x+1)^n}{n^2}$

$$= \lim_{n\to\infty} \frac{2(2x+1)n^2}{(n+1)^2} = \lim_{n\to\infty} \frac{2(2x+1)}{(1+\frac{1}{n})^2} = 2(2x+1)$$

Converges if $|2(2x+1)| < 1$

$\Rightarrow |2x+1| < \frac{1}{2} \Rightarrow -\frac{3}{4} < x < -\frac{1}{4}$

6

At endpoints $x = -\frac{1}{4}$, series is $\sum_{n=1}^{\infty} \frac{2^n(\frac{1}{2})^n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ which converges

$x = -\frac{3}{4}$, series is $\sum_{n=1}^{\infty} \frac{2^n(-\frac{1}{2})^n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ which also converges

$\Rightarrow$ ~~series~~ convergent for $-\frac{3}{4} \leq x \leq -\frac{1}{4}$

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(i) $\int \frac{2x+9}{x^2+9x+4}\,dx$ Let $u = x^2+9x+4$, $\frac{du}{dx} = 2x+9$ 3

$= \int \frac{du}{u} = \ln u + c = \ln(x^2+9x+4)+c$

(ii) $\int x^3 \ln x\, dx$ $\quad v = \ln x \quad \frac{du}{dx} = x^3$ (Integrate by parts)

$\Rightarrow \frac{dv}{dx} = \frac{1}{x} \quad u = \frac{1}{4}x^4$ 4

$= \frac{1}{4}x^4 \ln x - \int \frac{1}{4}x^4 \cdot \frac{1}{x}\,dx = \frac{1}{4}x^4 \ln x - \frac{1}{4}\int x^3\,dx$

$= \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + c$

(iii) $\int \frac{dx}{\sqrt{x^2-4x-5}} = \int \frac{dx}{\sqrt{(x-2)^2-9}}$ Let $x-2 = 3\cosh\Theta$, $\Rightarrow \frac{dx}{d\Theta} = 3\sinh\Theta$ 6

$= \int \frac{3\sinh\Theta\, d\Theta}{\sqrt{9\cosh^2\Theta-9}} = \int \frac{3\sinh\Theta\, d\Theta}{3\sinh\Theta} = \Theta + c = \cosh^{-1}\left(\frac{x-2}{3}\right) + c$

(iv) $\int \frac{(x+3)\,dx}{(x+2)(x-1)}$ $\quad \frac{1}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$ $\Rightarrow (A+B)x - A + 2B = 1$ $\Rightarrow A = -B$ & $3B = 1$ $\Rightarrow B = \frac{1}{3}, A = -\frac{1}{3}$ 7

$= -\frac{1}{3}\int \left(\frac{x+3}{x+2} - \frac{x+3}{x-1}\right) dx$

$= -\frac{1}{3}\int \left[1+\frac{1}{x+2} - \left(1+\frac{4}{x-1}\right)\right]dx = -\frac{1}{3}\int \left[\frac{1}{x+2} - \frac{4}{x-1}\right]dx = \frac{1}{3}\ln\frac{(x-1)^4}{x+2} + c$

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(i) $\frac{dy}{dx} = \frac{y^2+2xy}{x^2}$ Use $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\Rightarrow v + x\frac{dv}{dx} = v^2 + 2v \Rightarrow x\frac{dv}{dx} = v(v+1) \Rightarrow \int \frac{dv}{v(v+1)} = \int \frac{dx}{x}$

$\Rightarrow \ln x = \int \left[\frac{1}{v} - \frac{1}{v+1}\right] dv = \ln \frac{v}{v+1} + c$ 6

$\Rightarrow x = \frac{Av}{v+1} = \frac{A\,y/x}{\frac{y}{x}+1} = \frac{Ay}{x+y} \Rightarrow x = \frac{Ay}{x+y}$

(ii) $\frac{dy}{dx} + \frac{y}{1+x^2} = xe^{-\tan^{-1}x}$

I.F. $e^{\int \frac{dx}{1+x^2}} = e^{\tan^{-1}x} \Rightarrow e^{\tan^{-1}x}\frac{dy}{dx} + \frac{e^{\tan^{-1}x}}{1+x^2}y = x$ 5

$\Rightarrow \frac{d}{dx}\left(ye^{\tan^{-1}x}\right) = x \Rightarrow ye^{\tan^{-1}x} = \frac{1}{2}x^2 + c$

$\Rightarrow y = e^{-\tan^{-1}x}\left[\frac{1}{2}x^2 + c\right]$

(iii) $y'' - 10y' + 25y = e^{3x}$

Auxiliary equation: $\lambda^2 - 10\lambda + 25 = 0 \Rightarrow (\lambda-5)^2 = 0 \Rightarrow \lambda = 5$ (repeated)

C.F. $(Ax+B)e^{5x}$ 4

P.I. Try $ae^{3x} \Rightarrow ae^{3x}[9-30+25] = e^{3x} \Rightarrow a = \frac{1}{4}$

Solution $y = (Ax+B)e^{5x} + \frac{1}{4}e^{3x}$

(iv) $y'' - 11y' + 30y = 0$

Auxiliary equation: $\lambda^2 - 11\lambda + 30 = 0 \Rightarrow (\lambda-6)(\lambda-5) = 0 \Rightarrow \lambda = 6$ or $\lambda = 5$ 3

$\Rightarrow$ General solution: $y = Ae^{6x} + Be^{5x}$

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PAPER ISE 1.6

QUESTION

SOLUTION 5

$y(0) = y'(0) = 1$

$\Rightarrow A + B = 1 \quad \Rightarrow B = 1 - A$

$\& \quad 6A + 5B = 1$

$\Rightarrow 6A + 5 - 5A = 1 \quad \Rightarrow A = -4$

$\Rightarrow B = 5$

Specific solution : $y = -4e^{6x} + 5e^{5x}$

2

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$f_x = 2xy$, $f_y = x^2 - 9 + 3y^2$.

At stationary pts $f_x = f_y = 0$, so $xy = 0$.

[illegible] $x = 0$ [illegible] $-9 + 3y^2 = 0$ [illegible] $y = \pm\sqrt{3}$ [illegible]

[illegible]

[illegible] $y = 0$ [illegible] $x = \pm 3$ [illegible]

[illegible] the stationary pts are

$(3, 0)$, $(-3, 0)$, $(0, \sqrt{3})$, $(0, -\sqrt{3})$. 7

[illegible] $f_{xx} = 2y$, $f_{xy} = 2x$ & $f_{yy} = 6y$

[illegible] $f_{xy}^2 - f_{xx} f_{yy}$ [illegible]

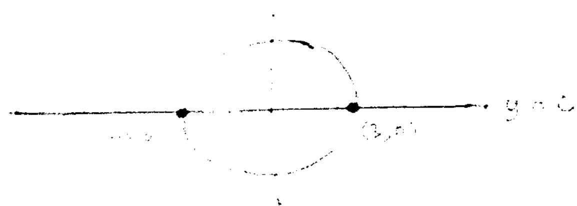
pt.	f_{xx}	Δ	nature
$(3,0)$	0	>0	saddle
$(-3,0)$	0	>0	saddle
$(0,\sqrt{3})$	>0	<0	minimum
$(0,-\sqrt{3})$	<0	<0	maximum

7

[illegible] Saddles are $(\pm 3, 0)$, at which $f = 0$.

[illegible] $f(x,y) = 0$ [illegible] $y(x^2 + y^2 - 9) = 0$

[illegible] together with circle $x^2 + y^2 = 9$: 3



Setter : [illegible]   Setter's signature : [illegible]

Checker :   Checker's signature : [illegible]

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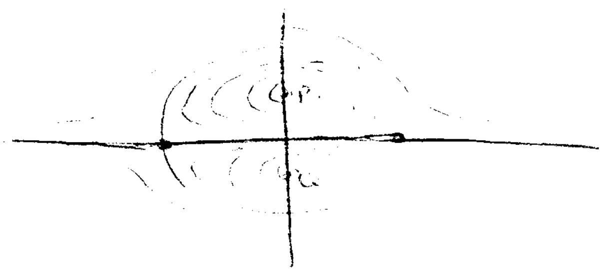
QUESTION

SOLUTION

(d)



3

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(a) Need to solve simultaneously the system

$$x + y + z = 1$$
$$x + 2y + az = 0$$
$$3x + 2y + az = b.$$

Reduce to echelon form:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & a & 0 \\ 3 & 2 & a & b \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & a-1 & -1 \\ 0 & -1 & a-3 & b-3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & a-1 & -1 \\ 0 & 0 & 2a-4 & b-4 \end{pmatrix}.$$ 6

Last eqn. is now $(2a-4)z = b-4$.

So

(i) unique solution if $a \neq 2$ (i.e. planes meet in 1 point) 2

(ii) solutions form a line if $a = 2,\ b = 4$ 2

(iii) no solutions if $a = 2,\ b \neq 4$. 2

(b) Char. poly. is $\begin{vmatrix} 4-\lambda & 3 \\ 3 & -4-\lambda \end{vmatrix} = \lambda^2 - 25$

So eigenvalues are $\underline{5,\ -5}$ 2

$\underline{\lambda = 5}$ Evectors are solns of $\begin{pmatrix} -1 & 3 \\ 3 & -9 \end{pmatrix} x = 0$, evecs. $a\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ $(a \neq 0)$ 2

$\underline{\lambda = -5}$ $\begin{pmatrix} 9 & 3 \\ 3 & 1 \end{pmatrix} x = 0$, evecs. $a\begin{pmatrix} -1 \\ 3 \end{pmatrix}$ $(a \neq 0)$. 2

Take $P = \begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix}$. Then $P^{-1}AP = \begin{pmatrix} 5 & 0 \\ 0 & -5 \end{pmatrix}$. 2

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2003 – 2004

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QUESTION

SOLUTION 8

This is an even function, so Fourier series is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$, $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$. 4

So

$$a_0 = \frac{2}{\pi}\left[\pi^2 x - \frac{x^3}{3}\right]_0^{\pi} = \frac{2}{\pi}\cdot\frac{2\pi^3}{3} = \frac{4\pi^2}{3}.$$ 2

Also

$$a_n = \frac{2}{\pi}\int_0^{\pi}(\pi^2 - x^2)\cos nx\,dx$$

Now $\int_0^{\pi}\pi^2\cos nx\,dx = \left[\frac{\pi^2}{n}\sin nx\right]_0^{\pi} = 0$

And

$$\int_0^{\pi} x^2 \cos nx\,dx = \left[x^2\cdot\frac{1}{n}\sin nx\right]_0^{\pi} - \int_0^{\pi} 2x\cdot\frac{1}{n}\sin nx\,dx$$

$$= -\frac{2}{n}\left(\left[x\cdot -\frac{1}{n}\cos nx\right]_0^{\pi} - \int_0^{\pi} -\frac{1}{n}\cos nx\,dx\right)$$

$$= \frac{2}{n^2}(\pi\cos n\pi) - \frac{2}{n^2}\left[\frac{1}{n}\sin nx\right]_0^{\pi}$$ 7

$$= \frac{2\pi}{n^2}(-1)^n$$

So $a_n = \frac{2}{\pi}\cdot\frac{2\pi}{n^2}(-1)^{n+1} = \frac{4\cdot(-1)^{n+1}}{n^2}$.

Fourier series is therefore

$$\frac{2\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}\cos nx.$$

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QUESTION

SOLUTION: 8 cld.

Parseval's formula says

$$\frac{2}{\pi}\int_0^{\pi} f(x)^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$$ 2

LHS is $\frac{2}{\pi}\int_0^{\pi}(\pi^2-x^2)^2\,dx = \frac{2}{\pi}\int_0^{\pi}(\pi^4+x^4-2\pi^2x^2)\,dx$

$$= \frac{2}{\pi}\left[\pi^4 x + \frac{x^5}{5} - \frac{2\pi^2x^3}{3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\cdot\pi^5\cdot\frac{8}{15} = \frac{16\pi^4}{15}.$$ 5

RHS is $\frac{8\pi^4}{9} + 16\sum_{n=1}^{\infty}\frac{1}{n^4}$

Hence

$$16\sum\frac{1}{n^4} = \pi^4\left(\frac{16}{15}-\frac{8}{9}\right)$$

so

$$\sum\frac{1}{n^4} = \pi^4\left(\frac{1}{15}-\frac{1}{18}\right) = \frac{\pi^4}{90}.$$

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PAPER ISE 1.6

QUESTION

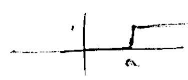
SOLUTION 9.

a)(i) $L^{-1}\left(\frac{s+1}{s^2+4}\right) = L^{-1}\left(\frac{s}{s^2+4}\right) + L^{-1}\left(\frac{1}{s^2+4}\right) = \underline{\cos 2t + \tfrac{1}{2}\sin 2t}$ → 3

(ii) Use Shift Rule $L\left(H_a(t) f(t-a)\right) = e^{-as} L(f(t))$.

Now $L\left(\frac{t^3}{6}\right) = \frac{1}{s^4}$, so

$$L^{-1}\left(\frac{e^{-2s}}{s^4}\right) = \underline{H_2(t)\,\frac{(t-2)^3}{6}}$$ 4

(where $H_a(t)$ is Heaviside step function )

b) Take Laplace transforms of both eqns, using

$L\left(\frac{dx}{dt}\right) = -x(0) + s\,L(x)$:

① $8\left(-2 + s\,L(x)\right) - 5\left(-3 + s\,L(y)\right) + 2L(x) = 0$

② $2\left(-2 + s\,L(x)\right) - \left(-3 + s\,L(y)\right) = -\frac{4}{s^2+4}$

i.e. ① $(8s+2)\,L(x) - 5s\,L(y) = 1$ 6

② $2s\,L(x) - s\,L(y) = \frac{-4}{s^2+4} + 1 = \frac{s^2}{s^2+4}$.

Then $(5\times②) - ①$ gives

$$(2s-2)\,L(x) = \frac{5s^2}{s^2+4} - 1 = \frac{4s^2-4}{s^2+4} = \frac{4(s^2-1)}{s^2+4}.$$

Thus

$$L(x) = \frac{2(s+1)}{s^2+4}$$

So from (a)(i) we get $\underline{x = 2\cos 2t + \sin 2t}$

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2003 – 2004

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QUESTION

SOLUTION 9 ctd.

Then from original 2nd eqn,

$$\frac{dy}{dt} = 2\frac{dx}{dt} + 2\sin 2t$$

$$= -6\sin 2t + 4\cos 2t$$

So $y = 3\cos 2t + 2\sin 2t + C$.

Since $y(0) = 3$, $C = 0$ so soln is

$$x = 2\cos 2t + \sin 2t$$
$$y = 3\cos 2t + 2\sin 2t$$

7

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