IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2015

MSc and EEE PART IV: MEng and ACGI

Corrected Copy

TOPICS IN LARGE DIMENSIONAL DATA PROCESSING

Wednesday, 13 May 10:00 am

Time allowed: 3:00 hours

There are THREE questions on this paper.

Answer ALL questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : W. Dai
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EE4-66 Topics in Large Dimensional Data Processing

Instructions for Candidates

Answer all questions. Each question carries 20 marks.

1. (Convex Optimisation) For simplicity, it is assumed that domain $(f) = \mathbb{R}^n$ for any given function $f: \mathbb{R}^n \to \mathbb{R}$.

   (a) What is the definition that a function $f: \mathbb{R}^n \to \mathbb{R}$ is convex? [2]

   (b) Let $p: \mathbb{R}^n \to \mathbb{R}$ be a norm. Prove that p is convex. [3]

   (c) Let $f(x) = \frac{1}{2}(y-x)^2 + \lambda |x|$ where $y \in \mathbb{R}$ and $\lambda \in \mathbb{R}^+$ are given constants.

      i). What is the second condition for convexity? (Proof is not needed.) Use it to show that $\frac{1}{2}(y-x)^2$ is a convex function of x. [3]

      ii). Prove that a summation of convex functions is convex. Use this result to show that $f(x)$ is convex. [3]

      iii). Compute the subdifferential of $f(x)$, denoted by $\partial f(x)$, for all $x \in \mathbb{R}$. [3]

      iv). Find the $x^* \in \mathbb{R}$ that minimises $f(x)$. [3]

   (d) Let $f: \mathbb{R}^n \to \mathbb{R}$ be a convex function. Prove that if $x^* \in \mathbb{R}^n$ is a local minimiser of f, then x^* is also a global minimiser of f. [3]

2. (Restricted Isometry Property)

(a) Given a matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, what is the definition that this matrix satisfies the Restricted Isometry Property (RIP) for $K \leq m$ and $\delta \in (0, 1)$? What is the definition of Restricted Isometry Constant (RIC) δ_K? [4]

(b) Show that for all $K < K' \leq m$, it holds that $\delta_K \leq \delta_{K'}$. [4]

(c) Suppose that $\boldsymbol{A}$ satisfies RIP with RIC $\delta_{2K} < 1$. Let $\mathcal{I}, \mathcal{J} \subset \{1, \cdots, n\}$ be two disjoint sets, i.e., $\mathcal{I} \bigcap \mathcal{J} = \phi$ where ϕ is the empty set. Assume that $|\mathcal{I}| \leq K$ and $|\mathcal{J}| \leq K$. Let $\boldsymbol{A}_{\mathcal{I}}$ and $\boldsymbol{A}_{\mathcal{J}}$ be the sub-matrices of $\boldsymbol{A}$ composed of the columns indexed by $\mathcal{I}$ and $\mathcal{J}$ respectively.

i). Use the fact that $|\langle \boldsymbol{A}_{\mathcal{I}}\boldsymbol{a}, \boldsymbol{A}_{\mathcal{J}}\boldsymbol{b}\rangle| \leq \delta_{2K} \|\boldsymbol{a}\|_2 \|\boldsymbol{b}\|_2$, $\forall \boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^K$, to prove that $\left\|\boldsymbol{A}_{\mathcal{I}}^T \boldsymbol{A}_{\mathcal{J}} \boldsymbol{b}\right\|_2 \leq \delta_{2K} \|\boldsymbol{b}\|_2$ where the superscript T denotes the matrix transpose. [4]

ii). Let $\boldsymbol{y} = \boldsymbol{A}_{\mathcal{I}}\boldsymbol{x}$ where $\boldsymbol{x} \in \mathbb{R}^{|\mathcal{I}|}$. Define

$$\hat{\mathcal{I}} = \left\{K \text{ indices corresponding to the } K \text{ largest magnitudes of } \boldsymbol{A}^T\boldsymbol{y}\right\}.$$

This definitions suggests that $\left\|\boldsymbol{A}_{\hat{\mathcal{I}}}^T \boldsymbol{y}\right\|_2 \geq \left\|\boldsymbol{A}_{\mathcal{I}}^T \boldsymbol{y}\right\|_2$.
Prove that if $\delta_{2K} < \frac{1}{2}$ then $\hat{\mathcal{I}} \bigcap \mathcal{I} \neq \phi$. [4]

iii). Let $\boldsymbol{x}_{\hat{\mathcal{I}} \cap \mathcal{I}}$ be the sub-vector of $\boldsymbol{x}$ corresponding to $\boldsymbol{A}_{\hat{\mathcal{I}} \cap \mathcal{I}}$. Show that

$$\left\|\boldsymbol{A}_{\hat{\mathcal{I}}}^T \boldsymbol{y}\right\|_2 \leq \left\|\boldsymbol{A}_{\hat{\mathcal{I}} \cap \mathcal{I}}^T \boldsymbol{y}\right\|_2 + \delta_{2K} \|\boldsymbol{x}\|_2, \tag{2.1}$$

and

$$\left\|\boldsymbol{A}_{\hat{\mathcal{I}} \cap \mathcal{I}}^T \boldsymbol{y}\right\|_2 \leq (1 + \delta_{2K}) \left\|\boldsymbol{x}_{\hat{\mathcal{I}} \cap \mathcal{I}}\right\|_2 + \delta_{2K} \|\boldsymbol{x}\|_2. \tag{2.2}$$

Use the inequalities to show that

$$\left\|\boldsymbol{x}_{\hat{\mathcal{I}} \cap \mathcal{I}}\right\|_2 \geq \frac{1 - 3\delta_{2K}}{1 + \delta_{2K}} \|\boldsymbol{x}\|_2. \tag{2.3}$$

[4]

3. (Maximum Clique Problem) We consider the graph $G(n, 1/2)$ with n nodes where there is an edge between two distinct nodes with probability $1/2$ independently of other pair of nodes. For such graphs, we would like to study the *maximum clique problem*. We call a *clique* in a graph a subset S of nodes such that any pair of nodes $i, j \in S$ is connected. Let w_n be the size of the largest clique in $G(n, 1/2)$, i.e. the clique with the most number of nodes.

(a) We will first show that, for a given $\epsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left(w_n \geq (2+\epsilon)\log_2(n)\right) = 0. \tag{2.4}$$

i). Show that

$$\mathbb{P}\left(w_n \geq k\right) \leq 2^{k\log_2(n) - k(k-1)/2}.$$

Hint: Use the union bound and the fact that $\binom{n}{k} \leq n^k$. [3]

ii). For $k = (2+\epsilon)\log_2(n)$ show that (2.4) holds. [3]

(b) We now describe a greedy algorithm *greedy clique* that extracts a clique of size $(1-\epsilon)\log_2(n)$ which has half the size of the maximum clique.

Greedy clique: Pick a vertex v_1 in $G(n, 1/2)$, then pick a random neighbour of v_1 that you add to the set S'. Continue adding nodes to S' that are picked at random from the nodes that are neighbours to all nodes in S', i.e. have an edge to any node that has been so far included in S', as long as this can be done.

i). Explain that the nodes in S' form a a clique. [1]

ii). Let $q_k = \mathbb{P}(\text{Greedy clique terminates with a clique of size } k)$. Show that

$$q_k \leq \binom{n}{k}(1 - \frac{1}{2^k})^{n-k}.$$

Hint: Use the inequality $(1-x) \leq e^{-x}$. [5]

iii). Let $k_0 = (1-\epsilon)\log_2(n)$, show that $q_{k_0} = \exp(-Cn^{\epsilon}))$, for a given constant C (independent of n and ϵ) and large n. [4]

iv). Using the fact that the sequence q_k is increasing for $k \in \{1, \ldots, \lceil \frac{n-1}{2} \rceil\}$, show that

$$\lim_{n\to\infty} \mathbb{P}(\text{Greedy clique terminates with a clique of size less than } (1-\epsilon)\log_2(n)) = 0$$

[4]