EXAM QUESTIONS

Information for Students

Fourier Transform Pairs

Pair Number	$x(t)$	$X(f)$
1.	$\Pi\left(\frac{t}{\tau}\right)$	$\tau \operatorname{sinc} \tau f$
2.	$2W \operatorname{sinc} 2Wt$	$\Pi\left(\frac{f}{2W}\right)$
3.	$\Lambda\left(\frac{t}{\tau}\right)$	$\tau \operatorname{sinc}^2 \tau f$
4.	$\exp(-\alpha t)u(t),\ \alpha > 0$	$\frac{1}{\alpha + j2\pi f}$
5.	$t \exp(-\alpha t)u(t),\ \alpha > 0$	$\frac{1}{(\alpha + j2\pi f)^2}$
6.	$\exp(-\alpha\|t\|),\ \alpha > 0$	$\frac{2\alpha}{\alpha^2 + (2\pi f)^2}$
7.	$e^{-\pi(t/\tau)^2}$	$\tau e^{-\pi(f/\tau)^2}$
8.	$\delta(t)$	1
9.	1	$\delta(f)$
10.	$\delta(t - t_0)$	$\exp(-j2\pi f t_0)$
11.	$\exp(j2\pi f_0 t)$	$\delta(f - f_0)$
12.	$\cos 2\pi f_0 t$	$\frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)$
13.	$\sin 2\pi f_0 t$	$\frac{1}{2j}\delta(f - f_0) - \frac{1}{2j}\delta(f + f_0)$
14.	$u(t)$	$(j2\pi f)^{-1} + \frac{1}{2}\delta(f)$
15.	$\operatorname{sgn} t$	$(j\pi f)^{-1}$
16.	$\frac{1}{\pi t}$	$-j \operatorname{sgn}(f)$
17.	$\hat{x}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(\lambda)}{t - \lambda}\, d\lambda$	$-j \operatorname{sgn}(f) X(f)$
18.	$\sum_{m=-\infty}^{\infty} \delta(t - mT_s)$	$f_s \sum_{m=-\infty}^{\infty} \delta(f - mf_s)$, $f_s = T_s^{-1}$

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

$$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

$$\tan(\frac{\pi}{2} - x) = \cot x$$

$$\tan 2x = \frac{2\tan x}{1-\tan^2 x} = \frac{2\cot x}{\cot^2 x - 1} = \frac{2}{\cot x - \tan x}$$

$$\tan\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}} = \frac{\sin x}{1+\cos x} = \frac{1-\cos x}{\sin x}$$

$$\sin x \cos y = \frac{1}{2}[\sin(x+y) + \sin(x-y)]$$

$$\cos x \sin y = \frac{1}{2}[\sin(x+y) - \sin(x-y)]$$

$$\sin x \sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$$

$$\cos x \cos y = \frac{1}{2}[\cos(x+y) + \cos(x-y)]$$

$$\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$$

$$\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$$

$$\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$$

$$\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$$

Fourier Transform Theorems[a]

Name of Theorem		
1. Superposition (a_1 and a_2 arbitrary constants)	$a_1x_1(t) + a_2x_2(t)$	$a_1X_1(f) + a_2X_2(f)$
2. Time delay	$x(t - t_0)$	$X(f)e^{-j2\pi f t_0}$
3a. Scale change	$x(at)$	$\lvert a\rvert^{-1}X\left(\frac{f}{a}\right)$
b. Time reversal	$x(-t)$	$X(-f) = X*(f)$
4. Duality	$X(t)$	$x(-f)$
5a. Frequency translation	$x(t)e^{j\omega_0 t}$	$X(f - f_0)$
b. Modulation	$x(t)\cos\omega_0 t$	$\frac{1}{2}X(f - f_0) + \frac{1}{2}X(f + f_0)$
6. Differentiation	$\frac{d^n x(t)}{dt^n}$	$(j2\pi f)^n X(f)$
7. Integration	$\int_{-\infty}^{t} x(t')\,dt'$	$(j2\pi f)^{-1}X(f) + \frac{1}{2}X(0)\delta(f)$
8. Convolution	$\int_{-\infty}^{\infty} x_1(t - t')x_2(t')\,dt' = \int_{-\infty}^{\infty} x_1(t')x_2(t - t')\,dt'$	$X_1(f)X_2(f)$
9. Multiplication	$x_1(t)x_2(t)$	$\int_{-\infty}^{\infty} X_1(f - f')X_2(f')\,df' = \int_{-\infty}^{\infty} X_1(f')X_2(f - f')\,df'$

Differentiation Rule of Leibnitz

Let $F(z) = \int_{a(z)}^{b(z)} f(x,z)dx$. Then we have

$$\frac{dF(z)}{dz} = \frac{db(z)}{dz}f(b(z),z) - \frac{da(z)}{dz}f(a(z),z) + \int_{a(z)}^{b(z)} \frac{\partial f(x,z)}{\partial z}dx$$

Q-function

$$Q(x) = \frac{1}{\sqrt{2\pi}}\int_x^{\infty} e^{-t^2/2}dt$$

1. a) i) What is *modulation*? Describe the two types of modulation schemes known to you? Explain how the performance of each of these modulation schemes is measured. [3]

ii) Let $x(t)$ be a passband signal with the canonical representation as follows:

$$x(t) = x_I(t)\cos(2\pi f_c t) - x_Q(t)\sin(2\pi f_c t).$$

Write down the complex envelope of $x(t)$. How can you represent $x(t)$ in terms of its complex envelope? [3]

iii) Draw the diagram of a coherent receiver for a double sideband suppressed carrier (DSB-SC) modulation system, and explain the function of each component. [4]

iv) What is the advantage of single sideband (SSB) modulation compared to double sideband suppressed carrier (DSB-SC) modulation? [2]

b) i) Consider a wide-sense stationary (WSS) random process $X(t)$. Let $R_X(\tau)$ denote the autocorrelation function of $X(t)$, and $S_X(f)$ denote its power spectral density. State whether each of the following statements are true or false, and discuss your answer:

- $S_X(0) = \int_{-\infty}^{+\infty} R_X(\tau)d\tau$.
- $S_X(f) \geq 0, \ \forall f$.
- $S_X(f) + S_X(-f) = 0, \ \forall f$.
- $R_X(0) = \int_0^{+\infty} S_X(f)df$.
- If $Y(t)$ is the Hilbert transform of $X(t)$, then $E[Y(t)] = 0, \forall t$.

[6]

c) Consider an additive white Gaussian noise channel, characterised by

$$Y = X + N,$$

where X is the channel input, Y is the channel output, and N is the additive noise term, which is independent of the channel input, and is a zero-mean Gaussian random variable with variance 4, i.e., $N \sim \mathcal{N}(0,4)$.

i) If the channel input is $X = -4$, what is the probability of the received signal, Y, being greater than 3? Write this probability in terms of the Q function. [3]

ii) If the channel input is $X = 0$, what is the probability of the received signal, Y, being between 3 and 10? Write this probability in terms of the Q function. [4]

iii) Consider the AWGN channel given by

$$Y = X + 2 \cdot N_1 - N_2,$$

where X and Y are the channel input and output signals, respectively; and N_1 and N_2 are two noise terms, which are independent of the channel input, and each other. Assume that both noise terms are zero-mean Gaussian random variables with variance 5, i.e., $N_i \sim \mathcal{N}(0,5)$ for $i = 1,2$. If the channel input is $X = 1$, what is the probability of the received signal, Y, being less than or equal to 4? Write this probability in terms of the Q function. [5]

d) Consider the input signal $m(t) = A + B \cdot \cos(wt)$, where A, B, and w are constants. A uniform quantizer with 2^n levels is used to quantize this signal. Assume that the dynamic range of the quantizer is equal to that of the input signal.

i) What is the signal power? [2]

ii) Write down the probability density function of the quantization noise, and the quantization noise power? [2]

iii) Derive the signal-to-noise ratio (SNR) in dB at the output of the quantizer? [3]

iv) If $A = 2B$, what is the minimum value of n such that the output SNR is above 100 dB? [3]

2. a) Consider two correlated memoryless sources S_1 and S_2. Source S_1 produces symbols from the alphabet $\mathscr{A} = \{a, b, c, d\}$, while source S_2 produces symbols from the alphabet $\mathscr{X} = \{x, y, z, t\}$. The joint distribution of symbol pairs is given in the following table:

		S_2			
		x	y	z	t
S_1	a	1/32	0	0	0
	b	0	3/32	0	1/8
	c	1/16	0	7/32	0
	d	1/8	5/32	3/16	0

According to the table, we have, for example, $P(S_1 = a, S_2 = x) = 1/32$ and $P(S_1 = a, S_2 = t) = 0$.

i) What is the average code length per source symbol pair if we apply the Huffman coding procedure on the joint outcome of the sources? *(Hint: consider (S_1, S_2) as the outcome of a single source, which outputs symbol pairs.)* [4]

ii) Consider the following alternative code: We apply the Huffman procedure on each of the sources separately. In the combined code, for each source symbol pair, we assign the pair of codewords obtained from these two separate codes. What is the average code length per source symbol pair for the combined code? [5]

iii) Which of the two codes in i) and ii) has a smaller average code length? Why? What would be the answer if the two sources, S_1 and S_2, were independent? [4]

iv) Find the joint entropy of the two sources $H(S_1, S_2)$. How does it compare with the average code lengths of the codes in i) and ii) ? Can you reduce the average code length of these codes by coding across many source symbol pairs? Can you design a prefix code that has an average code length that is below the joint entropy $H(S_1, S_2)$? Discuss your answers. [5]

b) Assume that $X(t)$ is a real wide sense stationary (WSS) random process with autocorrelation function

$$R_X(\tau) = E[X(t+\tau)X(t)] = Ae^{-\alpha|\tau|} + \cos(\tau\frac{\pi}{2}) - 1,$$

for some $A > 0$ and $\alpha > 0$.

i) What is the second moment of the random variable $X(2) - X(-1)$? (*Hint: Second moment of a random variable Y is given by* $E[Y^2]$.) [4]

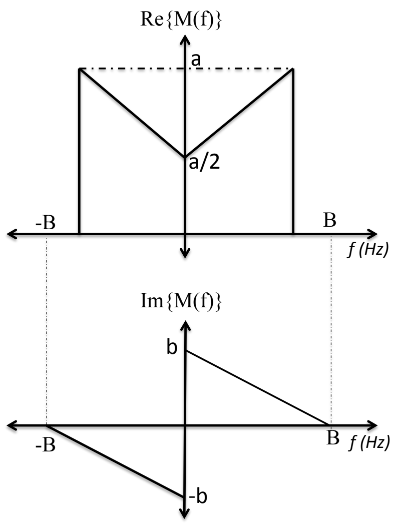
ii) What is the second moment of the random variable $X(97) - X(100)$? [3]

iii) What is the second moment of the following random variable?

$$X(-4) + X(-7) - X(-10)$$

[5]

3. a) Let $M(f)$ denote the spectrum of the finite energy real message signal $m(t)$. The real and imaginary components of $M(f)$ are shown below.



i) Let $\hat{m}(t)$ denote the Hilbert transform of $m(t)$, and let $M(f)$ and $\hat{M}(f)$ denote the Fourier transforms of $m(t)$ and $\hat{m}(t)$, respectively. Plot the real and imaginary components of $\hat{M}(f)$. [6]

ii) Consider the single-sideband (SSB) modulated signal given below:

$$s(t) = m(t)\cos(2\pi f_c t) + \hat{m}(t)\sin(2\pi f_c t),$$

Write down the spectrum of $s(t)$ in terms of $M(f)$ and $\hat{M}(f)$, and plot its real and imaginary components. [8]

iii) What is the bandwidth of the bandpass filter that should be used to demodulate this signal when it is transmitted over an additive white Gaussian noise channel? Plot the frequency response of the filter. [2]

iv) Assume that the signal $s(t)$ is demodulated with a coherent receiver which multiplies it with $2\cos(2\pi f_c t + \phi)$, where ϕ is the phase noise at the receiver's local oscillator. How can you recover the message signal if the phase noise is $\phi = 3\pi/2$? [4]

b) You are given a binary digital communication system. When a 0 is transmitted, it is decoded correctly with probability p_0, when a 1 is transmitted it is decoded correctly with probability p_1. It is given that $1 > p_0 > p_1 > 0.5$.

i) Assuming that bit 0 and bit 1 are transmitted with equal probability, if the decoder outputs 1, what is the probability of the input being 0? [2]

ii) You toss a biased coin which comes heads with probability 0.3 and tails with probability 0.7. You would like to transmit the outcome of the coin toss over the digital channel given above. If you can use the channel only once, what code would you use to minimise the error probability? What is the corresponding probability of error? [3]

iii) If you can use the channel three times to transmit the outcome of a single coin toss, what code would you use to minimise the error probability? What is the corresponding probability of error? [5]