IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2013

EEE/EIE PART III/IV: MEng, Beng and ACGI

Corrected Copy

DIGITAL SIGNAL PROCESSING

Wednesday, 9 January 10:00 am

Time allowed: 3:00 hours

There are FOUR questions on this paper.

Answer ALL questions.

All questions carry equal marks.

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : P.A. Naylor
Second Marker(s) : W. Dai

DIGITAL SIGNAL PROCESSING

1. a) i) Draw a signal flow diagram representing a two-band maximally decimated analysis filter bank. Denoting the input signal to the filter bank $x(n)$ with z-transform $X(z)$, write an expression for the signals in each subband $Y_0(z)$ and $Y_1(z)$ in terms of $X(z)$. [6]

ii) Consider a lowpass prototype filter $H(z)$ with cut-off frequency $\frac{\pi}{2}$. Write down suitable expressions for the filters in your filter bank in terms of $H(z)$. Give reasons for your choices. [4]

iii) Draw a signal flow diagram representing a two-band synthesis filter bank corresponding to the analysis filter bank in part (a)(i). Denoting the output signal of the synthesis filter bank $\hat{x}(n)$ with z-transform $\hat{X}(z)$, write an expression for $\hat{X}(z)$ in terms of $X(z)$ when the analysis and synthesis filter banks are connected in series. [5]

b) The magnitude spectrum of a signal $s(n)$ is shown in Fig. 1.1. The signal $s(n)$ is applied to the input of the system shown in Fig. 1.2. Determine and draw a labelled sketch of the magnitude spectrum of the signal $q(n)$, which has z-transform $Q(z)$, as indicated in Fig. 1.2, over the range of frequencies $-2\pi \leq \omega \leq 2\pi$.

[5]

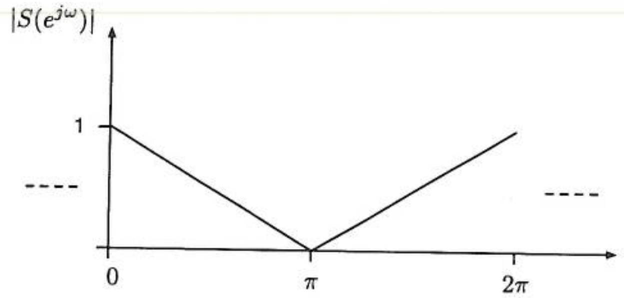


Figure 1.1 Signal magnitude spectrum

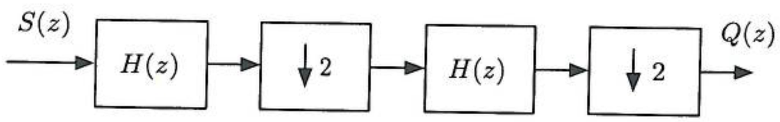


Figure 1.2 System block diagram

2. Consider a complex discrete-time signal $x(n)$ with N samples with discrete Fourier transform (DFT) $X(k)$.

a) State the expression for $X(k)$ in terms of $x(n)$ and the number of real multiply operations to compute $X(k)$. [3]

b) State what is meant by the term basis function in the content of transforms such as the Fourier transform. Write out the expressions for the basis functions employed in a 4-point DFT and find each of their values. [5]

c) Now consider the vectors

$\mathbf{x} = [x(0)x(1)\ldots x(N-1)]^T$ and

$\mathbf{X} = [X(0)X(1)\ldots X(N-1)]^T$,

where $[\ ]^T$ indicates the matrix transpose operation. Deduce and write out the elements of a matrix $\mathbf{D}_N$, known as the DFT matrix, such that

$$\mathbf{X} = \mathbf{D}_N\mathbf{x}$$

showing clearly the elements of the matrix. [5]

d) Derive the Decimation-in-Frequency radix-2 FFT algorithm and give the relevant expressions for $X(2k)$ and $X(2k+1)$. Draw a signal flow graph of this algorithm for the case $N = 4$. Illustrate the operation of the algorithm by using it to calculate $X(k)$ for the sampled data sequence $x(n) = [\ -2 \quad 1 \quad 0 \quad -1\ ]$. [7]

3. Consider $X(z)$ to be the z-transform of the discrete-time signal $x(n)$.

a) State an expression for the z-transform of $x(n)$, denoted $X(z) = Z\{x(n)\}$. [2]

b) Prove that the z-transform of $nx(n)$ is given by $Z\{nx(n)\} = -z\frac{dX(z)}{dz}$.

[3]

c) Prove that the z-transform of $b^{-n}x(n)$ is given by $Z\{b^{-n}x(n)\} = X(bz)$.

[3]

d) For the case of

$$X(z) = a_0 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3} + \dots$$
$$Y(z) = a_0 z^2 - a_1 b^1 z^3 - 2a_2 b^2 z^4 - 3a_3 b^3 z^5 - \dots$$

use the properties of part b) and c) to find a compact expression for $Y(z)$ in terms of $\frac{dX(z)}{dz}$. [12]

4. a) i) Define and give brief descriptions of the operations of linear convolution and circular convolution in discrete time. Clearly explain any similarities and differences between these operations. State in which circumstances each would be used. [4]

ii) Compute the linear and circular convolution of the two sampled data sequences $x(n) = [0\ -2\ 1\ 3]$ and $y(n) = [5\ -1\ -2\ 0]$. [2]

iii) Show a technique by which circular convolution can be used to compute linear convolution. Illustrate the use of the technique using the sequences $x(n)$ and $y(n)$ from part a) (ii). [2]

b) Consider a continuous-time signal $x_a(t)$ with Fourier transform $X_a(j\Omega)$.

i) Show how sampling of $x_a(t)$ to give the discrete-time signal $x(n)$ can be represented mathematically. Assume a sampling frequency Ω_s. Sketch a simple block diagram of the sampling process corresponding to your mathematical representation. [3]

ii) Prove that the Fourier transform of $x(n)$ can be written

$$X\left(e^{j\omega}\right) = \frac{1}{T}\sum_{r=-\infty}^{\infty} X_a\left(j\left(\frac{\omega+2\pi r}{T}\right)\right).$$

[3]

iii) For the case of $|X_a(j\Omega\}|$ as illustrated in Fig. 4.1, draw a labelled sketch of $\left|X(e^{j\omega})\right|$ clearly showing the range $-2 \le r \le 2$. Comment on any significant features of the sampled signal's spectrum.

[3]

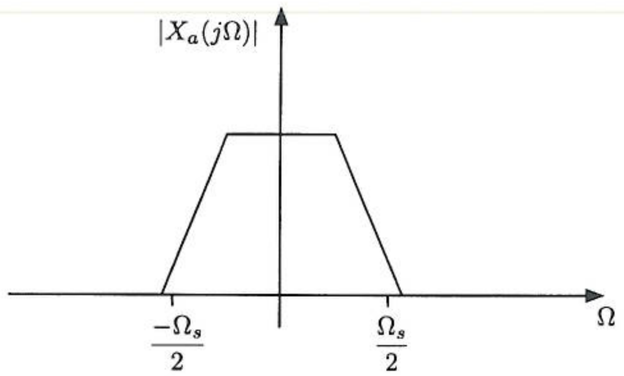


Figure 4.1 Signal magnitude spectrum.

iv) Describe and give an accompanying proof of the method for reconstructing a continuous-time signal from its samples. [3]

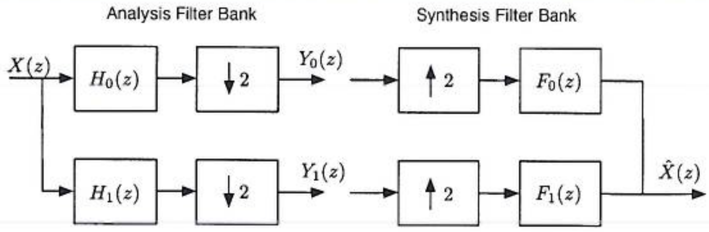
EE3-07

DIGITAL SIGNAL PROCESSING   Solutions 2013

1. a) i) Draw a signal flow diagram representing a two-band maximally decimated analysis filter bank. Denoting the input signal to the filter bank $x(n)$ with z-transform $X(z)$, write an expression for the signals in each subband $Y_0(z)$ and $Y_1(z)$ in terms of $X(z)$. [6]

Solution:

The signal flow graphs for the analysis and synthesis filter banks are shown in Fig. 1.1.



[3]

Figure 1.1 Maximally decimated two-band filter bank

$$Y_k(z) = \frac{1}{2}\left(X_k(z^{1/2}) + X_k(-z^{1/2})\right) \quad k = 0, 1$$

[3]

ii) Consider a lowpass prototype filter $H(z)$ with cut-off frequency $\frac{\pi}{2}$. Write down suitable expressions for the filters in your filter bank in terms of $H(z)$. Give reasons for your choices. [4]

Solution:

Quadrature Mirror Filters (QMF) are straightforward to design and offer perfect reconstruction if the synthesis filters are chosen correctly.

Lowpass: $H_0(z) = H(z)$ [2]

Highpass: $H_1(z) = H_0(-z)$ [2]

iii) Draw a signal flow diagram representing a two-band synthesis filter bank corresponding to the analysis filter bank in part (a)(i). Denoting the output signal of the synthesis filter bank $\hat{x}(n)$ with z-transform $\hat{X}(z)$, write an expression for $\hat{X}(z)$ in terms of $X(z)$ when the analysis and synthesis filter banks are connected in series. [5]

Solution:

The signal flow graphs for the analysis and synthesis filter banks are shown in Fig. 1.1. [2]

$$\hat{X}(z) = \frac{1}{2}\left(H_0(z)F_0(z) + H_1(z)F_1(z)\right)X(z) + \frac{1}{2}\left(H_0(-z)F_0(z) + H_1(-z)F_1(z)\right)X(-z)$$

[3]

b) The magnitude spectrum of a signal $s(n)$ is shown in Fig. 1.2. The signal $s(n)$ is applied to the input of the system shown in Fig. 1.3. Determine and draw a labelled sketch of the magnitude spectrum of the signal $q(n)$, which has z-transform $Q(z)$, as indicated in Fig. 1.3, over the range of frequencies $-2\pi \leq \omega \leq 2\pi$.

[5]

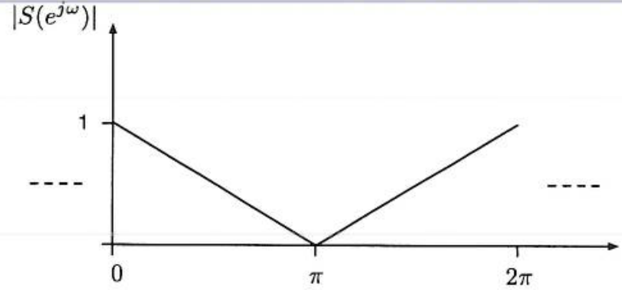


Figure 1.2 Signal magnitude spectrum

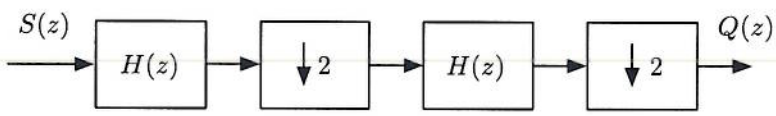


Figure 1.3 System block diagram

Solution:

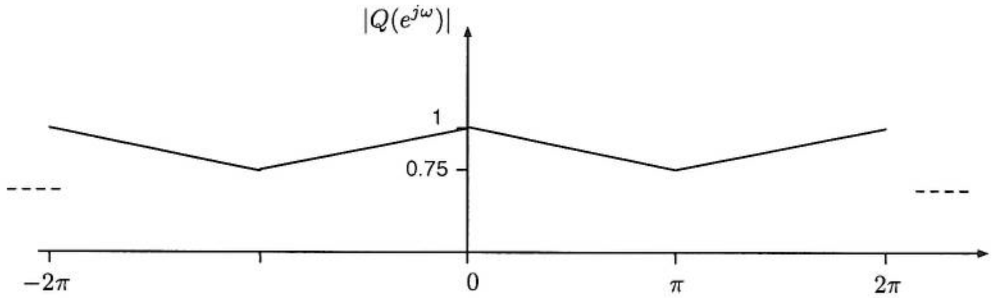


Figure 1.4 Signal magnitude spectrum for $Q(z)$

Diagram [3]

Labels [2]

2. Consider a complex discrete-time signal $x(n)$ with N samples with discrete Fourier transform (DFT) $X(k)$.

a) State the expression for $X(k)$ in terms of $x(n)$ and the number of real multiply operations to compute $X(k)$. [3]

Solution:

$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} \quad k = 0, 1, ..., N-1$

This requires N^2 complex multiplications corresponding to $4N^2$ real multiplies.

b) State what is meant by the term basis function in the content of transforms such as the Fourier transform. Write out the expressions for the basis functions employed in a 4-point DFT and find each of their values. [5]

Solution:

The basis functions are an orthogonal set of functions from which the signal can be synthesized using a linear combination. For the Fourier transform, the basis function are $W_N = e^{-j2\pi kn/N}$. For the case of $N = 4$, this gives

$$
\begin{array}{l}
k=0:[\ 1 \quad 1 \quad 1 \quad 1\ ] \\
k=1:[\ 1 \quad -j \quad -1 \quad j\ ] \\
k=2:[\ 1 \quad -1 \quad 1 \quad -1\ ] \\
k=3:[\ 1 \quad j \quad -1 \quad -j\ ]
\end{array}
$$

[-1 for each error]

c) Now consider the vectors

$\mathbf{x} = [x(0)x(1)\ldots x(N-1)]^T$ and

$\mathbf{X} = [X(0)X(1)\ldots X(N-1)]^T$,

where $[\ ]^T$ indicates the matrix transpose operation. Deduce and write out the elements of a matrix $\mathbf{D}_N$, known as the DFT matrix, such that

$$\mathbf{X} = \mathbf{D}_N\mathbf{x}$$

showing clearly the elements of the matrix. [5]

Solution:

The DFT matrix is given by

$$
\begin{bmatrix}
1 & 1 & 1 & \cdots & 1 \\
1 & W_N & W_N^2 & & W_N^{N-1} \\
1 & W_N^2 & & & \\
\vdots & \vdots & & & \vdots \\
1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)}
\end{bmatrix}
$$

for $W_N = e^{-j2\pi/N}$.

General structure [2]

then [-1 for each error]

d) Derive the Decimation-in-Frequency radix-2 FFT algorithm and give the relevant expressions for $X(2k)$ and $X(2k+1)$. Draw a signal flow graph of this algorithm for the case $N = 4$. Illustrate the operation of the algorithm by using it to calculate $X(k)$ for the sampled data sequence $x(n) = [\ -2 \quad 1 \quad 0 \quad -1\ ]$. [7]

Solution:

The DIF algorithm writes

$$X(2k) = \sum_{n=0}^{N/2-1} g_1(n) W_{N/2}^{kn}$$
$$X(2k+1) = \sum_{n=0}^{N/2-1} g_2(n) W_{N/2}^{kn}$$

for

$$g_1(n) = x(n) + x(n+N/2)$$
$$g_2(n) = (x(n) - x(n+N/2))\, W_N^n$$

[2]

The signal flow graph follows in Fig. 2.1 as

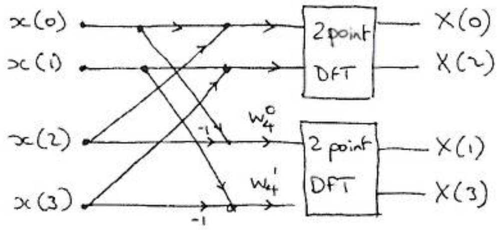


Figure 2.1 4-point Decimation in Frequency Radix-2 FFT

[2]

The FFT of the given sequence is found for $N = 4$ using

$$g_1(n) = [\ (-2+0) \quad (1-1)\ ] = [\ -2 \quad 0\ ]$$
$$g_2(n) = [\ (-2-0) \quad (1+1).-j\ ]$$

for which

$$
\begin{aligned}
X(0) &= g_1(0) + g_1(1) = -2 \\
X(2) &= g_1(0) - g_1(1) = -2 \\
X(1) &= g_2(0) + g_2(1) = -2 - 2j \\
X(3) &= g_2(0) - g_2(1) = -2 + 2j
\end{aligned}
$$

[3]

3. Consider $X(z)$ to be the z-transform of the discrete-time signal $x(n)$.

a) State an expression for the z-transform of $x(n)$, denoted $X(z) = Z\{x(n)\}$. [2]

Solution: $X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$

b) Prove that the z-transform of $nx(n)$ is given by $Z\{nx(n)\} = -z\frac{dX(z)}{dz}$.

[3]

Solution:

$$\sum_{n=-\infty}^{\infty} nx(n)z^{-n} = z\sum_{n=-\infty}^{\infty} nx(n)z^{-n-1} = -z\sum_{n=-\infty}^{\infty} x(n)\left(-nz^{-n-1}\right)$$

$$= -z\sum_{n=-\infty}^{\infty} x(n)\frac{d\{z^{-n}\}}{dz} = -z\frac{dX(z)}{dz}$$

c) Prove that the z-transform of $b^{-n}x(n)$ is given by $Z\{b^{-n}x(n)\} = X(bz)$.

[3]

Solution:

$$\sum_{n=-\infty}^{\infty} b^{-n}x(n)z^{-n} = \sum_{n=-\infty}^{\infty} x(n)(bz)^{-n} = X(bz).$$

d) For the case of

$$X(z) = a_0 + a_1z^{-1} + a_2z^{-2} + a_3z^{-3} + \dots$$
$$Y(z) = a_0z^2 - a_1b^1z^3 - 2a_2b^2z^4 - 3a_3b^3z^5 - \dots$$

use the properties of part b) and c) to find a compact expression for $Y(z)$ in terms of $\frac{dX(z)}{dz}$. [12]

The first step is to apply the property a) to obtain

$$X_1(z) = -z\frac{dX(z)}{dz}$$
$$= a_1z^{-1} + 2a_2z^{-2} + 3a_3z^{-3} + \dots$$

The second step is to use the property b) to obtain

$$X_2(z) = X_1\left(\frac{z}{b}\right) = a_1bz^{-1} + 2a_2b^2z^{-2} + 3a_3b^3z^{-3} + \dots$$

Next multiply by z^{-2} to obtain

$$X_3(z) = a_1bz^{-3} + 2a_2b^2z^{-4} + 3a_3b^3z^{-5} + \dots$$

Next use $z \leftarrow z^{-1}$ to obtain

$$X_4(z) = a_1 b z^3 + 2a_2 b^2 z^4 + 3a_3 b^3 z^5 + \dots$$

noting that all of the above steps comprise the relationship

$$X_4(z) = \frac{z}{b} \frac{dX(z)}{dz}\bigg|_{z=(zb)^{-1}}$$

from which is can be seen that the required transform $Y(z)$ can be written

$$\begin{aligned} Y(z) &= a_0 z^2 - a_1 b z^3 - 2a_2 b^2 z^4 - 3a_3 b^3 z^5 + \dots \\ &= a_0 z^2 - X_4(z) \end{aligned}$$

so that

$$Y(z) = a_0 z^2 + \frac{z}{b} \frac{dX(z)}{dz}\bigg|_{z=(zb)^{-1}}.$$

Each of 5 steps [2]

Final deduction [2]

4. a) i) Define and give brief descriptions of the operations of linear convolution and circular convolution in discrete time. Clearly explain any similarities and differences between these operations. State in which circumstances each would be used. [4]

Solution:

For linear convolution we have $y(n) = \sum_{m=-\infty}^{\infty} x(m)h(n-m)$. *This represents the operation of an LTI system with impulse response* $h(n)$ *acting on input* $x(m)$.

For circular convolution we have $y(n) = \sum_{m=0}^{N-1} x(m)h(m-n)_N$ *for* $n = 0, 1, 2 \ldots N-1$. *This does not provide the output of an LTI system but can be made equivalent to linear convolution by the process of zero extending.*

Description [2]

Similarities/difference and applications [2]

ii) Compute the linear and circular convolution of the two sampled data sequences $x(n) = [0\ \ -2\ 1\ 3]$ and $y(n) = [5\ \ -1\ \ -2\ 0]$. [2]

Solution:

The linear and circular convolution result in

$[0\ \ -10\ 7\ 18\ \ -5\ \ -6\ 0]$ *and*

$[18\ \ -5\ \ -16\ 7]$ *respectively.*

iii) Show a technique by which circular convolution can be used to compute linear convolution. Illustrate the use of the technique using the sequences $x(n)$ and $y(n)$ from part a) (ii). [2]

Solution:

The technique is to zero-extend each sequence. Sequence x *is extended with* $N-1$ *zeros and sequence* y *with* $M-1$ *zeros, where* N *is the length of* y *and* M *is the length of* x. *In this case the result of circular convolution is identical to linear convolution of the original unextended sequences.*

b) Consider a continuous-time signal $x_a(t)$ with Fourier transform $X_a(j\Omega)$.

i) Show how sampling of $x_a(t)$ to give the discrete-time signal $x(n)$ can be represented mathematically. Assume a sampling frequency Ω_s. Sketch a simple block diagram of the sampling process corresponding to your mathematical representation. [3]

Solution:

Using $T = \frac{2\pi}{\Omega_s}$

$$x(n) = x_a(nT) = x_a(t) \times \sum_{m=-\infty}^{\infty} \delta(t - mT)$$

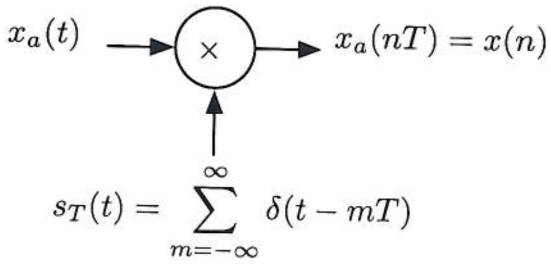


Figure 4.1 Sampling

ii) Prove that the Fourier transform of $x(n)$ can be written

$$X\left(e^{j\omega}\right) = \frac{1}{T} \sum_{r=-\infty}^{\infty} X_a\left(j\left(\frac{\omega + 2\pi r}{T}\right)\right).$$

[3]

Solution:

Bookwork.

iii) For the case of $|X_a(j\Omega\}|$ as illustrated in Fig. 4.2, draw a labelled sketch of $|X(e^{j\omega})|$ clearly showing the range $-2 \le r \le 2$. Comment on any significant features of the sampled signal's spectrum.

[3]

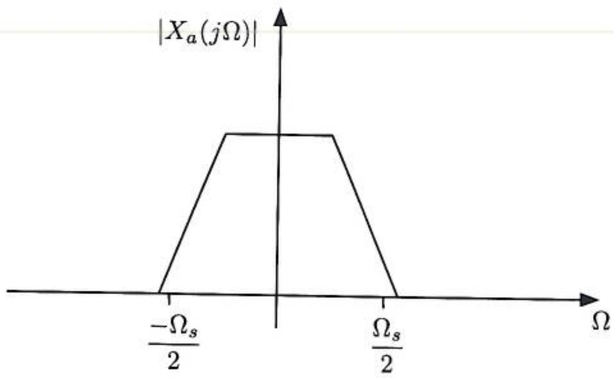


Figure 4.2 Signal magnitude spectrum.

Solution:

The spectrum is periodic and in this case aliasing occurs because the signal is not sufficiently bandlimited.

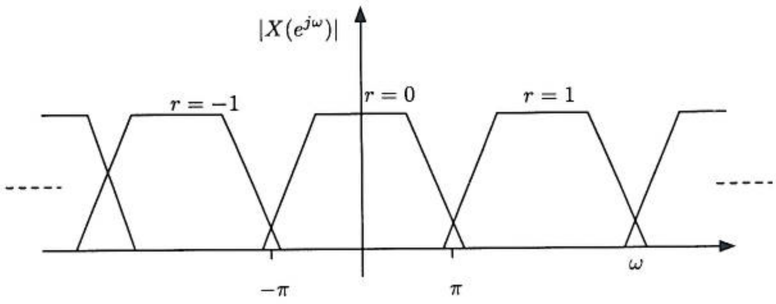


Figure 4.3 Discrete-time signal magnitude spectrum.

iv) Describe and give an accompanying proof of the method for reconstructing a continuous-time signal from its samples. [3]

Solution:

This is achieved in a DAC with an output reconstruction filter specified with impulse response $\frac{\sin(\pi t/T}{\pi t/T}$. *The proof is bookwork.*