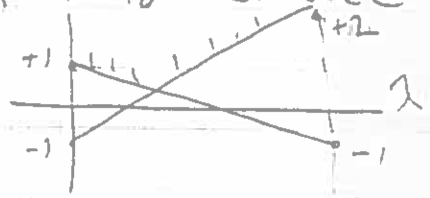
Game Theory Exam 2018
(Model Answers)

1(A): (a) If A chooses a mixed strategy $((1-\lambda), \lambda)$ A's pay-off is: $(1-\lambda)-\lambda$ if B chooses col. 1, and $-(1-\lambda)+2\lambda$ if B chooses col 1. So A's safety strategy is to minimize

$$\max\{1-2\lambda, -1+3\lambda\}$$

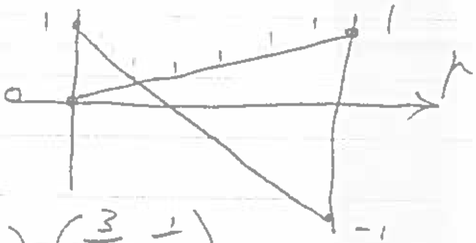


This is minimized at "cross-over" when

$$1-2\bar{\lambda} = -1+3\bar{\lambda} \implies \bar{\lambda} = 2/5$$

Likewise, a mixed safety strategy $((1-\bar{\mu}), \bar{\mu})$ minimizes

$$\max\{\mu, (1-\mu)-2\mu\}.$$



The cross-over condition gives

$$\bar{\mu} = 1-3\bar{\mu} \implies \bar{\mu} = \frac{1}{4}$$

Pay-off $\left(\frac{2}{5}, \frac{1}{4}\right)$

The pair of mixed safety strategies is

$$(1-\bar{\lambda}, \bar{\lambda}), (1-\bar{\mu}, \bar{\mu}) = \left(\left(\frac{3}{5}, \frac{2}{5}\right), \left(\frac{3}{4}, \frac{1}{4}\right)\right)$$

We see $[(1-\lambda), \lambda]\, S^A \begin{bmatrix} 1-\bar{\mu} \\ \bar{\mu} \end{bmatrix} = \frac{1}{2}(1-\lambda) - \frac{1}{4}\lambda = \frac{1}{2} - \frac{3}{4}\lambda$

This is not minimized at $\bar{\lambda} = \frac{1}{2}$, so this not a Nash equilibrium.

(b) Since $((1-\lambda', \lambda'), (1-\mu', \mu'))$ is a Nash equilibrium in mixed strategies, we must have

$$\begin{cases} [(1-\lambda), \lambda]\, S^A \begin{bmatrix} 1-\mu' \\ \mu' \end{bmatrix} \geq [(1-\lambda'), \lambda']\, S^A \begin{bmatrix} 1-\mu' \\ \mu' \end{bmatrix} & \text{for all } \lambda \quad (1) \\ [(1-\lambda'), \lambda']\, S^B \begin{bmatrix} 1-\mu \\ \mu \end{bmatrix} \geq [(1-\lambda'), \lambda']\, S^B \begin{bmatrix} 1-\mu' \\ \mu' \end{bmatrix} & \text{for all } \mu \quad (2) \end{cases}$$

(1) $\Rightarrow [(1-\lambda), \lambda] \begin{bmatrix} (1-\mu')-\mu' \\ -(1-\mu')+2\mu' \end{bmatrix} = (1-\lambda)(1-2\mu') - \lambda(-1+3\mu')$ is minimized at $\lambda = \lambda'$.

Because λ' is an interior minimizer slope must be zero $\Rightarrow \mu' = \frac{2}{5}$

(2) $\Rightarrow [\lambda', (1-\lambda')-2\lambda'] \begin{bmatrix} 1-\mu \\ \mu \end{bmatrix} = \lambda'(1-\mu) + (1-3\lambda')\mu$ is minimized at $\mu = \mu'$

Since the minimizer is interior, slope is zero $\Rightarrow \lambda' = 1/4$

Pay-off $\left(\frac{1}{5}, \frac{1}{4}\right)$

The Nash equilibrium pay-offs are not worse than the safety pay-offs as you would expect since Nash equilibria take account of the other players rational behaviour.

(B): Take any pair of strategies (x, y). Then

$$\operatorname{Min}_{x'} L(x', y) \leq L(x, y) \leq \operatorname{Max}_{y'} L(x, y')$$

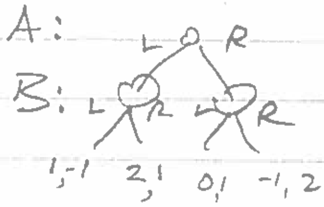
Since the right side does not depend on y

$$\operatorname{Max}_{y'}(\operatorname{Min}_{x'} L(x', y')) \leq \operatorname{Max}_{y'} L(x, y')$$

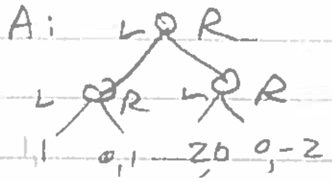
Since the left side does not depend on x,

$$\operatorname{Max}_{y'}(\operatorname{Min}_{x'} L(x', y')) \leq \operatorname{Min}_{x'}(\operatorname{Max}_{y'} L(x', y'))$$

2. The trees associated with the 2 chance scenarios are



and



The strategies for A are (L, R)

B's strategies are $\begin{pmatrix} L \\ R \end{pmatrix}^{B_1}, \begin{pmatrix} R \\ L \end{pmatrix}^{B_2}, \begin{pmatrix} R \\ R \end{pmatrix}^{B_3}, \begin{pmatrix} L \\ L \end{pmatrix}^{B_4}$ when $A = \begin{pmatrix} L \\ R \end{pmatrix}$

The pay-off matrices are obtained from the paths

A\B	B_1	B_2	B_3	B_4
L	LL	LR	LR	LL
R	RR	RL	RR	RL

They are

scenario ①

A\B	B_1	B_2	B_3	B_4
L	(1,-1)	(2,1)	(2,1)	(1,-1)
R	(-1,2)	(0,1)	(-1,2)	(0,1)

and scenario ②

A\B	B_1	B_2	B_3	B_4
L	(1,1)	(0,1)	(0,1)	(1,1)
R	(0,-2)	(2,0)	(0,-2)	(2,0)

Mixing them $\frac{1}{2} \times \{\text{scenario } 1\} + \frac{1}{2} \times \{\text{scenario } 2\}$ gives

A\B	B_1	B_2	B_3	B_4
L	(1,0)	(1,1)	(1,1)	(1,0)
R	$(-\frac{1}{2}, 0)$	$(1, \frac{1}{2})$	$(-\frac{1}{2}, 0)$	$(1, \frac{1}{2})$

The Nash equilibria are (L, B_4), $(R B_1)$, $(R B_3)$

These give $(L L)$ (pay-off $(1,0)$), $(R R)$ (pay-off $(-1,2)$)

(Notice $(R B_3)$ duplicates (R, B_1) and gives (R, R))

(L, B_4) is weakly dominated, so not admissible

(R, B_1) ($\equiv (R B_3)$) is not weakly dominated, so is admissib

3. Assume the i^{th} juror votes "not guilty" w.p. α and all other $(N-1)$ jurors vote "not guilty" w.p. $\bar{\alpha}$. $(\alpha, \bar{\alpha} \in (0,1))$

We must calculate

$A = $ Prob{ juror i votes 'not guilty' and 0 or 1 other jurors vote 'not guilty' }

$B = $ Prob{ juror i votes 'guilty' and 0, 1 or 2 other jurors vote 'not guilty' }

But

$$A = \text{Prob}\{\text{juror } i \text{ votes 'not guilty'}\} \times \big(\text{Prob}\{0 \text{ other jurors vote 'not guilty'}\} + \text{Prob}\{\text{exactly 1 other juror votes 'not guilty'}\}\big)$$

$$= \alpha\left(P_{\bar{\alpha}}(0; N-1) + P_{\bar{\alpha}}(1; N-1)\right)$$

$$B = \text{Prob}\{\text{juror } i \text{ votes guilty}\} \times \big(\text{Prob}\{0 \text{ jurors vote 'not guilty'}\} + \text{Prob}\{1 \ldots\} + \text{Prob}\{2 \ldots\}\big)$$

$$= (1-\alpha)\left(P_{\bar{\alpha}}(0; N-1) + P_{\bar{\alpha}}(1; N-1) + P_{\bar{\alpha}}(2; N-1)\right)$$

The pay-off is $A + (1-c)B$

$$J(\alpha; \bar{\alpha}) := \alpha\left(P_{\bar{\alpha}}(0; N-1) + P_{\bar{\alpha}}(1; N-1)\right) + (1-c)(1-\alpha)\left(P_{\bar{\alpha}}(0; N-1) + P_{\bar{\alpha}}(1; N-1) + P_{\bar{\alpha}}(2; N-1)\right)$$

If $(\bar{\alpha}, \ldots, \bar{\alpha})$ is a mixed Nash equilibrium, we require then

$$J(\alpha; \bar{\alpha}) \leq J(\bar{\alpha}; \bar{\alpha})$$

$\alpha \to J(\alpha; \bar{\alpha})$ is linear. So this is only possible if 'slope $= 0$', i.e.

$$P_{\bar{\alpha}}(0; N-1) + P_{\bar{\alpha}}(1; N-1) - (1-c)\left(P_{\bar{\alpha}}(0; N-1) + P_{\bar{\alpha}}(1; N-1) + P_{\bar{\alpha}}(2; N-1)\right) = 0$$

This gives

$$\left(P_{\bar{\alpha}}(0; N-1) + P_{\bar{\alpha}}(1; N-1) + P_{\bar{\alpha}}(2; N-1)\right) \times c = P_{\bar{\alpha}}(2; N-1)$$

Using data

$$c = \frac{\frac{(N-1)(N-2)}{2} \bar{\alpha}^2 (1-\bar{\alpha})^{N-3}}{(1-\bar{\alpha})^{N-1} + (N-1)\bar{\alpha}(1-\bar{\alpha})^{N-2} + \frac{1}{2}(N-1)(N-2)\bar{\alpha}^2(1-\bar{\alpha})^{N-3}}$$

We see the right is 0 when $\bar{\alpha} = 0$. The 'by continuity',

$$\bar{\alpha} \to 0 \text{ as } c \to 0,$$

4 (a) The pay-offs are $L^A(a,b) = a(1-[a+b]) - a^2$ and $L^B(a,b) = b(1-[a+b]) - b^2$

If $(\bar{a}, \bar{b})$ is a Nash equilibrium for which $\bar{a}, \bar{b} > 0$, then, $\partial/\partial a\, L^A(\bar{a},\bar{b}) = 0$ and $\partial/\partial b\, L^B(\bar{a},\bar{b}) = 0$.

This gives

$$(1-[\bar{a}+\bar{b}]) - \bar{a} - 2\bar{a} = 0 \Rightarrow \bar{a} = \tfrac{1}{4}(1-\bar{b})$$

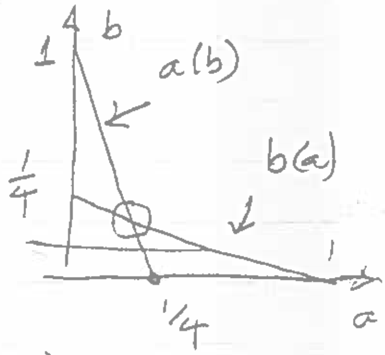
By symmetry $\bar{b} = \frac{1}{4}(1-\bar{a})$.

Hence $\bar{a} = \frac{1}{4}(1 - \frac{1}{4}(1-\bar{a})) \Rightarrow \underline{\bar{a} = \frac{1}{5} \text{ and } \bar{b} = \frac{1}{5}}$ (by symmetry)

(b) The response curves for company A and B resp are given by $\partial/\partial a\, L^A(a = a(b), b) = 0 \Rightarrow a(b) = \frac{1}{4}(1-b)$ and also $b(a) = \frac{1}{4}(1-a)$

The intersection is at the solution of $a = \frac{1}{4}(1-b)$ and $b = \frac{1}{4}(1-a) \Rightarrow (\frac{1}{5}, \frac{1}{5})$.

This is <u>consistent</u> with the calculated Nash equilibrium $(\bar{a}, \bar{b}) = (\frac{1}{5}, \frac{1}{5})$



(c) The Stackelberg optimizer maximizes

$$a \to L^A(a, b(a)) = a - 2a^2 - \tfrac{1}{4}(1-a)$$

$$\partial/\partial a(\ldots) = 0 \Rightarrow 5/4 - 4\tilde{a} = 0 \Rightarrow \tilde{a} = \tfrac{5}{16}$$

B's price is then $\tilde{b} = \frac{1}{4}(1-\tilde{a}) \Rightarrow \tilde{b} = 11/72$

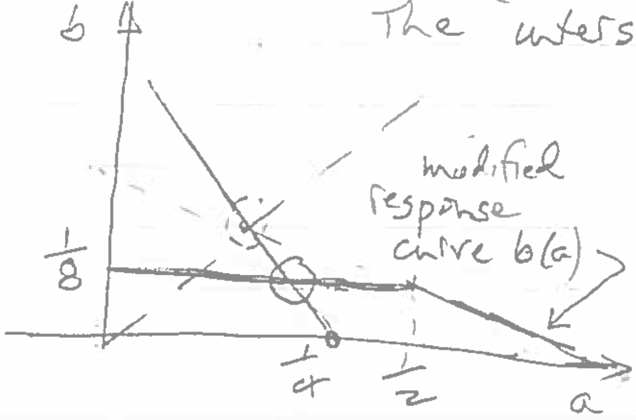
(d) Assume B production is constrained by $0 < b \le \frac{1}{8}$.

Then B's response curve becomes

$$b_{\text{modified}}(a) = \max\{\tfrac{1}{4}(1-a), \tfrac{1}{8})$$

Noting the 'cut-off' occurs when $\frac{1}{4}(1-a') = \frac{1}{8} \Rightarrow a' = \frac{1}{2}$

We can now draw the modified response curve $b_{\text{modified}}(a)$. ($a(b)$ remains the same.)



The intersection of response curves is now $a' = \frac{1}{4}(1-\frac{1}{8})$, $b' = \frac{1}{8}$

Modified Nash equilibrium is $(\frac{7}{32}, \frac{1}{8})$

5A The Bellman equation is $V_t(x) = \min\{ V_{t+1}(Ax+bu) + u^2 \mid u \in$

with boundary condition $V_N(x) = \lambda^T x$ — (*)

Assume $V_t(x) = x^T P_t x + 2 r_t^T x + s_t$

Then $x^T P_t x + 2 r_t^T x + s_t = \min_u \{(Ax+bu)^T P_{t+1} (Ax+bu) + 2 r_{t+1}^T (Ax+b$

$= \min_u \{ x^T A^T P_{t+1} A x + 2(x^T A^T P_{t+1} b + r_{t+1}^T b) u + 2 r_{t+1}^T A x + (b^T P_{t+1} b + 1) u^2$

The minimizing $u = -(b^T P_{t+1} b + 1)^{-1} (b^T P_{t+1} A x + b^T r_{t+1})$

Plug this into Bellman equation

$x^T P_t x + 2 r_t^T x =$

$x^T (A^T P_{t+1} A + 2 r_{t+1} A x - (x^T A^T P_{t+1} b + r_{t+1}^T b)^2 (b^T P_{t+1} b + 1)^{-1}$

Equating 1st and 2nd powers of x on both sides gives:

$P_t = A^T P_{t+1} A - (b^T P_{t+1} b + 1)^{-1} A^T P_{t+1} b b^T P_{t+1} A$ — (1)

$r_t = A^T r_{t+1} - (b^T P_{t+1} b + 1)^{-1} A^T P_{t+1} b b^T r_{t+1}$ — (2)

$s_t = (b^T P_{t+1} b + 1)^{-1} r_{t+1}^T b b^T r_{t+1}$ — (3)

The boundary conditions are

$P_N = 0$, $r_N = \lambda$ and $s_N = 0$

The optimal control $\{u_t^\lambda\}$ (and state $\{x_t^\lambda\}$ are given by the feedback relation

$u_t = -k_t^T x_t - d_t^T$

$k_t = -(b^T P_{t+1} b + 1)^{-1} b^T P_{t+1} A$, $d_t = -(b^T P_{t+1} b + 1)^{-1} b^T r_{t+1}$

5B Suppose $\bar\lambda$ is such that, for the given initial state x_0, $x_N^{\bar\lambda} =$

Take any control/state pair $\{x_t, u_t\}$ such that $x_N = 0$. Then, because $\{x_t^{\bar\lambda}, u_t^{\bar\lambda}\}$ is optimal for $\lambda = \bar\lambda$

$$\sum_{t=0}^{N-1} u_t^2 + \bar\lambda^T x_N \geq \sum_{t=0}^{N-1} |u_t^{\bar\lambda}|^2 + \bar\lambda^T x_N^{\bar\lambda}.$$

Since $x_N = 0$ and $x_N^{\bar\lambda} = 0$, it follows

$$\sum_{t=0}^{N-1} u_t^2 \geq \sum_{t=0}^{N-1} |u_t^{\bar\lambda}|^2$$

This inequality tells us that $\{x_t^\lambda, u_t^\lambda\}$ minimizes the control energy for transferring the state from x_0 to 0