E1.14 MATHS 2
(EE stream - 1st year)

UNIVERSITY OF LONDON

[I(2)E 2005]

B.ENG. AND M.ENG. EXAMINATIONS 2005

For Internal Students of Imperial College London

This paper is also taken for the relevant examination for the Associateship of the City & Guilds of London Institute

PART I : MATHEMATICS 2 (ELECTRICAL ENGINEERING)

Thursday 2nd June 2005 10.00 am - 1.00 pm

Answer EIGHT questions.

Formulae sheet provided.

[Before starting, please make sure that the paper is complete; there should be 6 pages, with a total of 10 questions. Ask the invigilator for a replacement if your copy is faulty.]

1. (i) Find A and B in the following:

$$\frac{4x-5}{x(x-1)} = \frac{A}{x} - \frac{B}{x-1} .$$

(ii) Find the two stationary points of

$$f(x) = \frac{4x-5}{x(x-1)}$$

and identify them as local maxima or minima.

(iii) Sketch the graph of $f(x)$, indicating the point of intersection with the x-axis and including any asymptotes.

2. Cylindrical polar coordinates (r, θ, t) in three dimensions are related to Cartesian coordinates by the following expressions:

$$\begin{aligned} x &= r\cos\theta , \\ y &= r\sin\theta , \\ z &= t . \end{aligned}$$

For a function $f(x, y, z)$:

(i) find $\dfrac{\partial f}{\partial r}$, $\dfrac{\partial f}{\partial \theta}$, $\dfrac{\partial f}{\partial t}$ in terms of $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$, $\dfrac{\partial f}{\partial z}$.

Hence show that

(ii) $$\frac{\partial f}{\partial x} = \cos\theta \frac{\partial f}{\partial r} - \frac{\sin\theta}{r} \frac{\partial f}{\partial \theta} .$$

(iii) $$\frac{\partial f}{\partial y} = \sin\theta \frac{\partial f}{\partial r} + \frac{\cos\theta}{r} \frac{\partial f}{\partial \theta} .$$

(iv) $$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial z} .$$

PLEASE TURN OVER

3. Show that the equation

$$x^3 + 4x^2 - 10 = 0$$

has a solution satisfying $1 < x < 2$.

It is required to compute this solution using a fixed point scheme

$$x_{n+1} = g(x_n) .$$

Show that possible choices for g are

(i)
$$g(x) = x - x^3 - 4x^2 + 10 ,$$

(ii)
$$g(x) = \left(\frac{10}{4+x}\right)^{1/2} .$$

Show that for choice (i) of g, we have $|g'(x)| > 1$ for $x \in [1, 2]$, while for choice (ii), $|g'(x)| < 1$ for $x \in [1, 2]$.

What does this tell you about the convergence of these iteration schemes?

Write down the Newton-Raphson scheme for computing this solution.

Starting with $x_0 = 1.5$, compute x_1 for all three schemes.

4. (i) Given $\boldsymbol{a} = (0, -1, \alpha)$, $\boldsymbol{b} = (1, 2, \beta)$ and $\boldsymbol{c} = (2, 1, \gamma)$, find the relation which must be satisfied by the scalars α, β, γ so that $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = 0$.

When this relation is satisfied, determine the scalars λ and μ such that $\boldsymbol{c} = \lambda \boldsymbol{a} + \mu \boldsymbol{b}$.

(ii) (a) Show that for any two given vectors $\boldsymbol{a}$ and $\boldsymbol{b}$

$$|\boldsymbol{a} \times \boldsymbol{b}|^2 = |\boldsymbol{a}|^2 |\boldsymbol{b}|^2 - (\boldsymbol{a} \cdot \boldsymbol{b})^2 .$$

(b) Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be two vectors, and let $\boldsymbol{c} = \boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{a})$.

Show that $\boldsymbol{c}$ lies in the plane containing $\boldsymbol{a}$ and $\boldsymbol{b}$ and show that

$$\boldsymbol{b} \cdot \boldsymbol{c} = |\boldsymbol{a}|^2 |\boldsymbol{b}|^2 - (\boldsymbol{a} \cdot \boldsymbol{b})^2 .$$

5. Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 4 & 0 \\ 5 & 4 & 7 \end{pmatrix} .$$

(i) Compute A^2 and A^3.

(ii) Using row operations, or otherwise, compute the inverse matrix A^{-1}.

6. (i) Show that the substitution $u = x + y + xy$ reduces the differential equation

$$\left(\frac{dy}{dx} + \frac{1+y}{1+x}\right) e^{x} = \frac{e^{-y(1+x)}}{1+x} \quad \text{for} \quad x > -1$$

to the form $\dfrac{du}{dx} = e^{-u}$.

Hence find the solution $y = y(x)$ for which $y(1) = 1$.

(ii) Find the general solution of the differential equation

$$xy\frac{dy}{dx} + (x^2 + y^2) = 0 .$$

7. (i) Solve the differential equation

$$x\frac{dy}{dx} - y = 1, \quad y(1) = 1,$$

using an integrating factor.

(ii) Use the fact that

$$\int \frac{1}{1+z^2}\,dz = \tan^{-1} z + c$$

to find the general solution of the differential equation

$$\frac{dy}{dx} = \frac{x^2+y^2}{x^2}.$$

8. Find the general solution of the differential equations

(i)

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 16\sin 2x,$$

subject to the boundary conditions

$$y(0) = 0, \quad y\left(\frac{\pi}{4}\right) = 2e^{-\frac{\pi}{2}};$$

(ii)

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = (2+x)\,e^{-2x},$$

subject to the initial conditions

$$y(0) = 1, \quad \frac{dy}{dx}(0) = 0.$$

9. The function $f(x)$ is defined as

$$f(x) \;=\; (1-x^2)^{\frac{1}{3}} \; .$$

Calculate the derivative $f'(x)$ and show that $f'(0) = 0$.

Calculate the second derivative $f''(x)$ and show that f satisfies the differential equation

$$(1-x^2)\, f'' \;-\; \frac{4}{3}\, x\, f' \;+\; \frac{2}{3} f \;=\; 0 \; .$$

Use the Leibnitz formula to differentiate this equation n times, and show that

$$f^{(n+2)}(0) \;=\; \left(n^2 + \frac{1}{3}\, n - \frac{2}{3}\right) f^{(n)}(0) \quad \text{for} \quad n \geq 0.$$

Here $f^{(n)}$ denotes the nth derivative of f and $f^{(0)}(0) \equiv f(0)$.

Hence find the first three non-zero terms in the Maclaurin expansion for $f(x)$.

Use the binomial expansion to check your result.

10. Find the Fourier expansion of the function $f(x)$ given by

$$f(x) \;=\; \begin{cases} x^2 \, , & 0 \leq x < \pi \, , \\ -x^2 \, , & -\pi < x \leq 0 \, , \end{cases}$$

and $f(x)$ periodic with period 2π.

Show that

$$x^2 \;=\; \frac{2}{\pi} \left\{ (\pi^2 - 4) \sin x \;-\; \frac{\pi^2}{2} \sin 2x \;+\; \left(\frac{\pi^2}{3} - \frac{4}{3^3}\right) \sin 3x \;-\; \frac{\pi^2}{4} \sin 4x \;+ \ldots \right\}$$

in $0 \leq x < \pi$.

Sketch $f(x)$ over $-2\pi < x \leq 2\pi$.

END OF PAPER

MATHEMATICS DEPARTMENT

MATHEMATICAL FORMULAE

1. VECTOR ALGEBRA

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1, a_2, a_3)$$

Scalar (dot) product: $\mathbf{a}\,.\,\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

Vector (cross) product:

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Scalar triple product:

$$[\mathbf{a},\, \mathbf{b},\, \mathbf{c}] = \mathbf{a}\,.\,\mathbf{b}\times\mathbf{c} = \mathbf{b}\,.\,\mathbf{c}\times\mathbf{a} = \mathbf{c}\,.\,\mathbf{a}\times\mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product: $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{c}\,.\,\mathbf{a})\mathbf{b} - (\mathbf{b}\,.\,\mathbf{a})\mathbf{c}$

2. SERIES

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \ldots \quad (\alpha \text{ arbitrary, } |x| < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \cdots,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n\frac{x^{2n}}{(2n)!} + \cdots,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + \cdots,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^n\frac{x^{n+1}}{(n+1)} + \ldots (-1 < x \le 1)$$

3 TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin(a+b) = \sin a\,\cos b + \cos a\,\sin b\,;$$

$$\cos(a+b) = \cos a\,\cos b - \sin a\,\sin b$$

$$\cos iz = \cosh z\,;\quad \cosh iz = \cos z\,;\quad \sin iz = i\sinh z\,;\quad \sinh iz = i\sin z\,.$$

4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f\,D^n g + \binom{n}{1} Df\,D^{n-1}g + \ldots + \binom{n}{r} D^r f\,D^{n-r}g + \ldots + D^n f\,g\,.$$

(b) Taylor's expansion of $f(x)$ about $x = a$:

$$f(a+h) = f(a) + hf'(a) + h^2 f''(a)/2! + \ldots + h^n f^{(n)}(a)/n! + \epsilon_n(h)\,,$$

where $\epsilon_n(h) = h^{n+1}f^{(n+1)}(a+\theta h)/(n+1)!$, $0 < \theta < 1$.

(c) Taylor's expansion of $f(x, y)$ about (a, b):

$$f(a+h,\, b+k) = f(a,\, b) + [hf_x + kf_y]_{a.b} + 1/2!\left[h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}\right]_{a.b} + \cdots$$

(d) Partial differentiation of $f(x, y)$:

i. If $y = y(x)$, then $f = F(x)$, and $\dfrac{dF}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dx}$.

ii. If $x = x(t)$, $y = y(t)$, then $f = F(t)$, and $\dfrac{dF}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$.

iii. If $x = x(u, v)$, $y = y(u, v)$, then $f = F(u, v)$, and

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}, \quad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$

(e) Stationary points of $f(x, y)$ occur where $f_x = 0$, $f_y = 0$ simultaneously.
Let (a, b) be a stationary point: examine $D = [f_{xx}f_{yy} - (f_{xy})^2]_{a.b}$.
If $D > 0$ and $f_{xx}(a, b) < 0$, then (a, b) is a maximum;
If $D > 0$ and $f_{xx}(a, b) > 0$, then (a, b) is a minimum;
If $D < 0$ then (a, b) is a saddle-point.

(f) Differential equations:

i. The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating factor $I(x) = \exp[\int P(x)(dx]$, so that $\dfrac{d}{dx}(Iy) = IQ$.

ii. $P(x, y)dx + Q(x, y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y$.

5 INTEGRAL CALCULUS

(a) An important substitution: $\tan(\theta/2) = t$:
$\sin\theta = 2t/(1+t^2)$, $\cos\theta = (1-t^2)/(1+t^2)$, $d\theta = 2\,dt/(1+t^2)$.

(b) Some indefinite integrals:

$$\int (a^2 - x^2)^{-1/2} dx = \sin^{-1}\left(\frac{x}{a}\right), \quad |x| < a.$$

$$\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1}\left(\frac{x}{a}\right) = \ln\left\{\frac{x}{a} + \left(1 + \frac{x^2}{a^2}\right)^{1/2}\right\}.$$

$$\int (x^2 - a^2)^{-1/2} dx = \cosh^{-1}\left(\frac{x}{a}\right) = \ln\left|\frac{x}{a} + \left(\frac{x^2}{a^2} - 1\right)^{1/2}\right|.$$

$$\int (a^2 + x^2)^{-1} dx = \left(\frac{1}{a}\right)\tan^{-1}\left(\frac{x}{a}\right).$$

6. NUMERICAL METHODS

(a) Approximate solution of an algebraic equation:

If a root of $f(x) = 0$ occurs near $x = a$, take $x_0 = a$ and
$x_{n+1} = x_n - [f(x_n)/f'(x_n)]$, $n = 0, 1, 2 \ldots$

(Newton Raphson method).

(b) Formulae for numerical integration: Write $x_n = x_0 + nh$, $y_n = y(x_n)$.

i. Trapezium rule (1-strip): $\int_{x_0}^{x_1} y(x)dx \approx (h/2)[y_0 + y_1]$.

ii. Simpson's rule (2-strip): $\int_{x_0}^{x_2} y(x)dx \approx (h/3)[y_0 + 4y_1 + y_2]$.

(c) Richardson's extrapolation method: Let $I = \int_a^b f(x)dx$ and let I_1, I_2 be two estimates of I obtained by using Simpson's rule with intervals h and $h/2$.

Then, provided h is small enough,

$$I_2 + (I_2 - I_1)/15,$$

is a better estimate of I.

7. LAPLACE TRANSFORMS

Function	Transform	Function	Transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)dt$	$af(t) + bg(t)$	$aF(s) + bG(s)$
df/dt	$sF(s) - f(0)$	d^2f/dt^2	$s^2F(s) - sf(0) - f'(0)$
$e^{at}f(t)$	$F(s-a)$	$tf(t)$	$-dF(s)/ds$
$(\partial/\partial\alpha)f(t,\alpha)$	$(\partial/\partial\alpha)F(s,\alpha)$	$\int_0^t f(t)dt$	$F(s)/s$
$\int_0^t f(u)g(t-u)du$	$F(s)G(s)$		
1	$1/s$	$t^n (n = 1, 2 \ldots)$	$n!/s^{n+1}$, $(s > 0)$
e^{at}	$1/(s-a)$, $(s > a)$	$\sin\omega t$	$\omega/(s^2+\omega^2)$, $(s > 0)$
$\cos\omega t$	$s/(s^2+\omega^2)$, $(s > 0)$	$H(t-T) = \begin{cases} 0, & t < T \\ 1, & t > T \end{cases}$	e^{-sT}/s, $(s, T > 0)$

8. FOURIER SERIES

If $f(x)$ is periodic of period $2L$, then $f(x+2L) = f(x)$, and

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \text{ where}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad n = 0, 1, 2, \ldots, \text{ and}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1, 2, 3, \ldots$$

Parseval's theorem

$$\frac{1}{L}\int_{-L}^{L} [f(x)]^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right).$$

i)

$$f(x) = \frac{5}{x} - \frac{1}{x-1}$$ 3

ii) $$f'(x) = -\frac{5}{x^2} + \frac{1}{(x-1)^2}$$

Setting equal to 0 gives stationary points at

$$\frac{5}{4} \pm \frac{\sqrt{5}}{4}$$ 3

Checking the sign of the derivative on either side of stationary pts gives 4

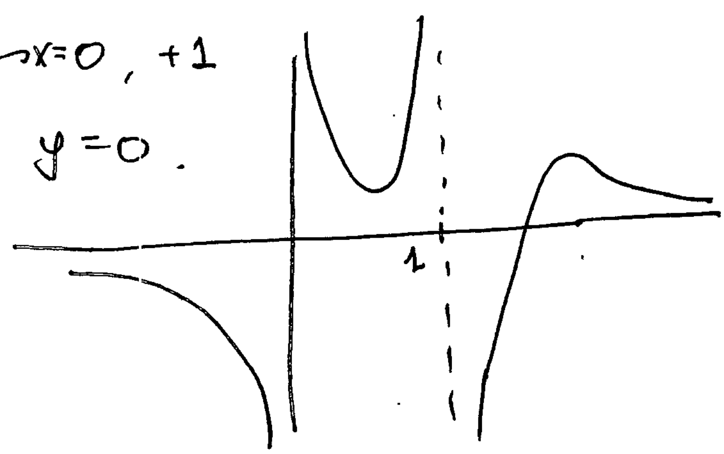
$\frac{5}{4} - \frac{\sqrt{5}}{4}$: local minima

$\frac{5}{4} + \frac{\sqrt{5}}{4}$: local maxima

iii) $f(x) = 0$ at $x = 5/4$

Vertical Asymptote $x = 0, +1$

Horizontal " $y = 0$.



5

i) $\frac{\partial x}{\partial r} = \cos\theta$, $\frac{\partial x}{\partial \theta} = -r\sin\theta$ 3

$\frac{\partial y}{\partial r} = \sin\theta$, $\frac{\partial y}{\partial \theta} = r\cos\theta$

so

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta$$ 5

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}(-r\sin\theta) + \frac{\partial f}{\partial y}(r\cos\theta)$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial z}$$

ii) $\cos\theta\frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\cos^2\theta +$ 3

$+ \frac{\partial f}{\partial y}\sin\theta\cos\theta + \frac{\partial f}{\partial x}\sin^2\theta - \frac{\partial f}{\partial y}\sin\theta\cos\theta$

$= \frac{\partial f}{\partial x}(\sin^2\theta + \cos^2\theta) = \frac{\partial f}{\partial x}$

iii) $\sin\theta\frac{\partial f}{\partial r} + \frac{\cos\theta}{r}\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial y}(\sin^2\theta + \cos^2\theta) = \frac{\partial f}{\partial y}$ 3

iv) $\frac{\partial f}{\partial t} = \frac{\partial f}{\partial z}$ as above. 1

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MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

2004 – 2005

E3

PAPER 2

QUESTION

SOLUTION 4

Please write on this side only, legibly and neatly, between the margins

$f(x) = x^3 + 4x^2 - 10 \qquad f(1) = -5, \; f(2) = 14$

Hence f has root in the interval $(1,2)$. 1

(i) $x = g(x) = x - x^3 - 4x^2 + 10 \Rightarrow f(x) = 0$ 1

(ii) $x = g(x) = \left(\frac{10}{4+x}\right)^{1/2} \Rightarrow x^2(4+x) = 10$

$\Rightarrow f(x) = 0$ 1

(i) $g'(x) = 1 - 3x^2 - 8x$

$-10 = g'(1) \geq g'(x) \geq g'(2) = -17 \Rightarrow |g'(x)| > 1$ for $x \in [1,2]$ 2

(ii) $g'(x) = \frac{1}{2}\left(\frac{10}{4+x}\right)^{-1/2} \frac{-10}{(4+x)^2}$

$= -\frac{1}{2}\frac{\sqrt{10}}{(4+x)^{3/2}}$

$\Rightarrow |g'(x)| \leq \frac{1}{2}\frac{\sqrt{10}}{5^{3/2}} = \frac{1}{5\sqrt{2}} < 1$ for $x \in [1,2]$ 2

(i) diverges, (ii) converges if $x_0 \in (1,2)$ 1

NR

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{(x_n^3 + 4x_n^2 - 10)}{3x_n^2 + 8x_n}$

$= \frac{2x_n^3 + 4x_n^2 + 10}{3x_n^2 + 8x_n}$ 2

(i) $x_1 = 1.5 - (1.5)^3 - 4(1.5)^2 + 10 = \frac{12 - 27 - 72 + 80}{8}$

$= -\frac{7}{8}$ 2

(ii) $x_1 = \left(\frac{20}{11}\right)^{1/2} \approx 1.3484$ 1

(iii) $x_1 = \frac{2(1.5)^3 + 4(1.5)^2 + 10}{3(1.5)^2 + 12} = \frac{27 + 36 + 40}{27 + 48} = \frac{103}{75}$ 2

$\simeq 1.3733$

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MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

2004 – 2005

E4

PAPER 2

QUESTION

SOLUTION 5-1-

Please write on this side only, legibly and neatly, between the margins

(i) $\vec{a}\cdot(\vec{b}\times\vec{c}) = 0 \iff$

$$\begin{vmatrix} 0 & -1 & \alpha \\ 1 & 2 & \beta \\ 2 & 1 & \gamma \end{vmatrix} = 0$$ 3

$\Rightarrow 3\alpha + 2\beta - \gamma = 0$ 2

$\vec{c} = \lambda\vec{a} + \mu\vec{b} \iff$

$(2 \quad 1 \quad \gamma) = \lambda(0 \quad -1 \quad \alpha) + \mu(1 \quad 2 \quad \beta)$

$(2 \quad 1 \quad 3\alpha+2\beta) = (0 \quad -\lambda \quad \lambda\alpha) + (\mu \quad 2\mu \quad \mu\beta)$

or

$(2 \quad 1 \quad 3\alpha+2\beta) = (\mu \quad -\lambda+2\mu \quad \lambda\alpha+\mu\beta)$ 2

$\Rightarrow \lambda = 3$ and $\mu = 2$

(ii)

a) $\|\vec{a}\times\vec{b}\|^2 = \|\vec{a}\|^2\cdot\|\vec{b}\|^2 - (\vec{a}\cdot\vec{b})^2$

$\iff \|\vec{a}\|^2\|\vec{b}\|^2\sin^2\theta = \|\vec{a}\|^2\cdot\|\vec{b}\|^2 - \|a\|^2\cdot\|b\|^2\cos^2\theta$ 1

where $\theta = \measuredangle(\vec{a},\vec{b})$

$\iff \|\vec{a}\|^2\cdot\|\vec{b}\|^2(\sin^2\theta + \cos^2\theta) = \|\vec{a}\|^2\cdot\|\vec{b}\|^2$

$\iff \|\vec{a}\|^2\cdot\|\vec{b}\|^2 = \|\vec{a}\|^2\|\vec{b}\|^2$ 1

b) $\vec{c}$ lies in the plane of $\vec{a}$ and $\vec{b}$

$\Rightarrow \vec{c}\cdot(\vec{a}\times\vec{b}) = 0 \iff$ 1

(since $\vec{c} = \vec{a}\times(\vec{b}\times\vec{a})$)

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$\Leftrightarrow \left(\vec{a} \times (\vec{b} \times \vec{a})\right) \cdot (\vec{a} \times \vec{b}) = 0$

$\Leftrightarrow \underbrace{\left((\vec{b} \times \vec{a}) \times (\vec{a} \times \vec{b})\right)}_{=0} \cdot \vec{a} = 0 \Leftrightarrow 0 = 0$ ✓ 2

$\vec{b} \cdot \vec{c} = \|\vec{a}\|^2 \|\vec{b}\|^2 - (\vec{a} \cdot \vec{b})^2 \Leftrightarrow$

$\vec{b} \cdot \left(\vec{a} \times (\vec{b} \times \vec{a})\right) = \|\vec{a}\|^2 \|\vec{b}\|^2 - (\vec{a} \cdot \vec{b})^2$

$\Leftrightarrow (\vec{b} \times \vec{a}) \cdot (\vec{b} \times \vec{a}) = \|\vec{a}\|^2 \cdot \|\vec{b}\|^2 - (\vec{a} \cdot \vec{b})^2$

$\Leftrightarrow \|\vec{b} \times \vec{a}\|^2 = \|\vec{a}\|^2 \cdot \|\vec{b}\|^2 - (\vec{a} \cdot \vec{b})^2$

$\Leftrightarrow \|\vec{a} \times \vec{b}\|^2 = \|\vec{a}\|^2 \cdot \|\vec{b}\|^2 - (\vec{a} \cdot \vec{b})^2$ ✓ 2

TRUE by point a).

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Please write on this side only, legibly and neatly, between the margins

(1) $$A^2 = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 4 & 0 \\ 5 & 4 & 7 \end{pmatrix}\begin{pmatrix} 1 & 2 & 1 \\ -1 & 4 & 0 \\ 5 & 4 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 14 & 8 \\ -5 & 14 & -1 \\ 36 & 54 & 54 \end{pmatrix}$$ 3

$$A^3 = \begin{pmatrix} 4 & 14 & 8 \\ -5 & 14 & -1 \\ 36 & 54 & 54 \end{pmatrix}\begin{pmatrix} 1 & 2 & 1 \\ -1 & 4 & 0 \\ 5 & 4 & 7 \end{pmatrix}$$ 3

$$= \begin{pmatrix} 30 & 96 & 60 \\ -24 & 42 & -12 \\ 252 & 504 & 414 \end{pmatrix}$$

(2) $$(A\,|\,I) = \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ -1 & 4 & 0 & 0 & 1 & 0 \\ 5 & 4 & 7 & 0 & 0 & 1 \end{array}\right)$$

row3 − 5·row1
$\sim$
row2 + row1 $$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 6 & 1 & 1 & 1 & 0 \\ 0 & -6 & 2 & -5 & 0 & 1 \end{array}\right)$$

row3 + row2
$\sim$ $$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 6 & 1 & 1 & 1 & 0 \\ 0 & 0 & 3 & -4 & 1 & 1 \end{array}\right)$$ row1 − $\frac{1}{3}$·row3
$\sim$
row2 − $\frac{1}{3}$·row3 $$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 7/3 & -1/3 & -1/3 \\ 0 & 6 & 0 & 7/3 & 2/3 & -1/3 \\ 0 & 0 & 3 & -4 & 1 & 1 \end{array}\right)$$ 9

row1 − $\frac{1}{3}$·row2
$\sim$ $$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{14}{9} & -\frac{5}{9} & -\frac{2}{9} \\ 0 & 6 & 0 & 7/3 & 2/3 & -1/3 \\ 0 & 0 & 3 & -4 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{14}{9} & -\frac{5}{9} & -\frac{2}{9} \\ 0 & 1 & 0 & 7/18 & 1/9 & -1/18 \\ 0 & 0 & 1 & -4/3 & 1/3 & 1/3 \end{array}\right)$$

$$= (I\,|\,A^{-1}) \Rightarrow A^{-1} = \begin{pmatrix} 14/9 & -5/9 & -2/9 \\ 7/18 & 1/9 & -1/18 \\ -4/3 & 1/3 & 1/3 \end{pmatrix}$$

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Checker : J. R. CASH Checker's signature : [signature]

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(i) $u = x + y + xy \Rightarrow$

$$\Rightarrow \frac{du}{dx} = 1 + \frac{dy}{dx} + y + x\frac{dy}{dx} \Rightarrow$$

$$\Rightarrow \frac{du}{dx} = (1+y) + (1+x)\frac{dy}{dx} \Rightarrow$$

$$\Rightarrow \frac{dy}{dx} + \frac{1+y}{1+x} = \frac{1}{1+x}\frac{du}{dx}$$ 2

Substitute into $\left(\frac{dy}{dx} + \frac{1+y}{1+x}\right)e^x = \frac{e^{-y(1+x)}}{1+x}$

$$\Rightarrow \frac{1}{1+x}\frac{du}{dx}e^x = \frac{e^{-y(1+x)}}{1+x}$$

or $$\frac{1}{1+x}\frac{du}{dx} = \frac{e^{-(x+y+xy)}}{1+x}$$

or $$\frac{du}{dx} = e^{-u}$$ 2

$$\Rightarrow u(x) = \ln|x+c| \qquad x > -c$$

but $u = x + y + xy \Rightarrow y = \dfrac{u-x}{1+x} \Rightarrow$

$$\Rightarrow y(x) = \frac{\ln|x+c| - x}{1+x}$$

$$y(1) = 1 \Rightarrow 1 = \frac{\ln|1+c| - 1}{1+1} \Rightarrow c = e^3 - 1$$

$$\Rightarrow y(x) = \frac{\ln|x + (e^3 - 1)| - x}{1+x}$$ 3

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Setter's signature : Stoica

Checker : BARRETT

Checker's signature :

(ii) $xy\frac{dy}{dx} + (x^2+y^2) = 0$

homog of deg. 2

Substitute $y = vx \Rightarrow$ 2

$\Rightarrow \frac{dy}{dx} = \frac{dv}{dx}x + v$ 1

The equation becomes

$$x \cdot vx\left(x\frac{dv}{dx} + v\right) + x^2 + v^2x^2 = 0$$

or $x^2\left(vx\frac{dv}{dx} + v^2 + 1 + v^2\right) = 0$

or $x^2\left(vx\frac{dv}{dx} + 2v^2+1\right) = 0$ 2

$\Rightarrow$ 1) $x \equiv 0$ 1

or

2) $vx\frac{dv}{dx} + 2v^2 + 1 = 0 \Rightarrow$

$\Rightarrow \frac{v}{2v^2+1}dv + x\,dx = 0 \Rightarrow$ 1

$\Rightarrow \frac{1}{4}\ln(2v^2+1) + \ln|x| = C \Rightarrow \ln(2v^2+1) = 4(C - \ln|x|)$

$\Rightarrow 2v^2+1 = e^{4C}\cdot e^{-4\ln|x|} \Rightarrow v^2 = \frac{1}{2}(Kx^{-4} - 1)$ for $0<x^4<K$

$K>0$ constant

$\Rightarrow \frac{y^2}{x^2} = \frac{1}{2}(Kx^{-4} - 1) \Rightarrow y(x) = \pm\left[\frac{x^2}{2}\left(\frac{K}{x^4} - 1\right)\right]^{1/2}$ 1

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Checker : BVARACG

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PAPER 2

QUESTION

SOLUTION 10a

(NB) Invert the 2 parts of this question

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(ii) To solve $\dfrac{dy}{dx} = \dfrac{x^2+y^2}{x^2}$ put $y = xv$,

then $y' = xv' + v$ so that

$$x\frac{dv}{dx} = 1 + v^2 - v, \text{ whence}$$

$$\int \frac{1}{v^2-v+1}\,dv = \int \frac{1}{x}\,dx .$$

Now $v^2 - v + 1 = (v - 1/2)^2 + 3/4$

and therefore

$$\int \frac{dv}{v^2-v+1} = \frac{4}{3}\int \frac{dv}{\frac{4}{3}(v-\frac{1}{2})^2+1} = \frac{4}{3}\int \frac{dv}{\left(\frac{2v-1}{\sqrt{3}}\right)^2+1}$$

and so putting $z = (2v-1)/\sqrt{3}$ yields the integral

$$\frac{4}{3}\int \frac{\sqrt{3}/2 \cdot dz}{z^2+1} = \frac{2}{\sqrt{3}}\tan^{-1} z + c'.$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2v-1}{\sqrt{3}}\right) + c'.$$

[6]

Hence, $\ln x + c = \dfrac{2}{\sqrt{3}}\tan^{-1}\left(\dfrac{2v-1}{\sqrt{3}}\right)$

and re-arranging this leads to

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Checker : R. CRASTER Checker's signature : RVC

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$$\frac{y}{x} = v$$

$$= \frac{\sqrt{3}}{2} \tan\left(\frac{\sqrt{3}}{2}(\ln x + c)\right) + \frac{1}{2},$$

so

$$y(x) = \frac{x}{2} + \frac{x\sqrt{3}}{2} \tan\left(\frac{\sqrt{3}}{2}(\ln x + c)\right),$$

where $x > 0$ and c is any real constant. [3]

(i) To solve $xy' - y = 1$, note that (using the integrating factor $1/x$).

$$x \frac{d}{dx}\left(\frac{1}{x}y\right) = y' - \frac{1}{x}y = 1/x$$

and so

$$\frac{1}{x}y = \int x^{-2}\,dx + c,$$

$$= -x^{-1} + c.$$

Thus $y(x) = -1 + cx$ and $y(1) = 1 \Rightarrow c = 2$ [6]

giving $y(x) = 2x - 1$.

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(i) $y_c = e^{\lambda x}$ $\quad \lambda^2 + 4\lambda + 4 = 0 \Rightarrow (\lambda+2)^2 = 0$

$\Rightarrow \quad y_c(x) = (A + Bx)e^{-2x}$ 2

$y_p(x) = C\sin 2x + D\cos 2x$ 1

$y_p'(x) = 2C\cos 2x - 2D\sin 2x$

$y_p''(x) = -4C\sin 2x - 4D\cos 2x$

$\Rightarrow$

$(-4C - 8D + 4C)\sin 2x + (-4D + 8C + 4D)\cos 2x$

$= 16\sin 2x \Rightarrow \begin{matrix} D = -2 \\ C = 0 \end{matrix}$ 2

$\Rightarrow \quad y(x) = y_c(x) + y_p(x) = (A+Bx)e^{-2x} - 2\cos 2x$

$y(0) = A - 2 = 0 \quad y(\frac{\pi}{4}) = (A + \frac{B\pi}{4})e^{-\frac{\pi}{2}} = 2e^{-\frac{\pi}{2}}$

$\Rightarrow A = 2, \; B = 0 \Rightarrow y(x) = 2(e^{-2x} - \cos 2x).$ 2

(ii) $y_p(x) = x^2(C + Dx)e^{-2x} = (Cx^2 + Dx^3)e^{-2x}$ 2

$y_p'(x) = (2Cx + (3D - 2C)x^2 - 2Dx^3)e^{-2x}$

$y_p''(x) = (2C + (6D - 8C)x + (4C - 12D)x^2 + 4Dx^3)e^{-2x}$ 2

$\Rightarrow \quad y_p'' + 4y_p' + 4y_p = (2C + 6Dx)e^{-2x} = (2+x)e^{-2x}$

$\Rightarrow \quad C = 1, \; D = \frac{1}{6}$ 2

$\Rightarrow \quad y(x) = y_c(x) + y_p(x) = (A + Bx + x^2 + \frac{1}{6}x^3)e^{-2x}$

$y'(x) = ((B - 2A) + (2 - 2B)x - \frac{3}{2}x^2 - \frac{1}{3}x^3)e^{-2x}$

$y(0) = A = 1 \;, \quad y'(0) = B - 2A = 0 \Rightarrow B = 2$

$\Rightarrow \quad y(x) = (1 + 2x + x^2 + \frac{1}{6}x^3)e^{-2x}$ 2

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Checker : STOICA $\qquad$ Checker's signature :

$f'(x) = -\frac{2}{3}x(1-x^2)^{-\frac{2}{3}}$, hence $f'(0)=0$. 1

$f''(x) = -\frac{2}{9}(x^2+3)(1-x^2)^{-\frac{5}{3}}$, hence 2

$(1-x^2)f'' - \frac{4}{3}xf' + \frac{2}{3}f = -\frac{2}{9}(x^2+3)(1-x^2)^{-\frac{2}{3}} +$

$+ \frac{8}{9}x^2(1-x^2)^{-\frac{2}{3}} + \frac{2}{3}(1-x^2)(1-x^2)^{-\frac{2}{3}} = 0.$ 2

Note that $(1-x^2)' = -2x$ vanishes at $x=0$,

thus $\left((1-x^2)f''\right)^{(n)}(0) = f^{(n+2)}(0) - 2\binom{n}{2}f^{(n)}(0)$.

We also have $\left(\frac{4}{3}xf'\right)^{(n)}(0) = \frac{4}{3}\cdot n f^{(n)}(0)$,

so finally $f^{(n+2)}(0) = n(n-1)f^{(n)}(0) +$

$+\frac{4}{3}nf^{(n)}(0) - \frac{2}{3}f^{(n)}(0) = \left(n^2+\frac{1}{3}n-\frac{2}{3}\right)f^{(n)}(0)$. 6

It follows that $f^{(m)}(0)=0$ if m is odd.

We obtain $f(x) = 1 + \frac{1}{2}f''(0)x^2 +$

$+ \frac{1}{24}f^{(4)}(0)x^2 + \ldots = 1 - \frac{1}{3}x^2 - \frac{1}{9}x^4 + \ldots$ 2

The binomial formula confirms this:

$(1-x^2)^{\frac{1}{3}} = 1 + \frac{1}{3}(-x^2) + \frac{1}{2}\cdot\frac{1}{3}\left(\frac{1}{3}-1\right)\cdot(-x^2)^2 + \ldots =$ 2

$= 1 - \frac{1}{3}x^2 - \frac{1}{9}x^4 + \ldots$

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Checker : LUZZATTO Checker's signature : Stefano Luzzatto

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(14)

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This is an odd function of x so:

Fourier sine series

$$f(x) = \sum_{r=1}^{\infty} b_r \sin rx$$ 2

$$b_r = \frac{2}{\pi}\int_0^{\pi} x^2 \sin rx \, dx$$ 2

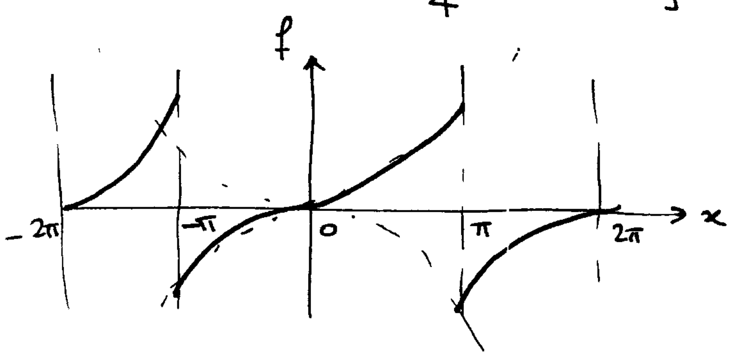
$$\int_0^{\pi} x^2 \sin rx = -\frac{x^2}{r}\cos rx\Big|_0^{\pi} + \int_0^{\pi}\frac{2x}{r}\cos rx\,dx$$

$$= \left(\frac{2}{r^3} - \frac{x^2}{r}\cos rx\right)\Big|_0^{\pi} + \frac{2}{r^2}x\sin rx\Big|_0^{\pi}$$ 3

$$= \left(\frac{2}{r^3} - \frac{\pi^2}{r}\right)(-1)^r - \frac{2}{r^3}$$

$$\therefore\ b_r = \frac{2}{\pi}\left[\frac{2}{r^3}\left[(-1)^r - 1\right] - \frac{\pi^2}{r}(-1)^r\right]$$ 3

$$\therefore\ x^2 = \frac{2}{\pi}\left\{(\pi^2-4)\sin x - \frac{\pi^2}{2}\sin 2x + \left(\frac{\pi^3}{3} - \frac{4}{3^3}\right)\sin 3x - \frac{\pi^2}{4}\sin 4x + \dots\right\}$$ 2



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(15)

Setter : R CRASTER Setter's signature : RC

Checker : R. BEARDMORE Checker's signature : R.B.