EE1-10A

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2015

EEE PART I: MEng, BEng and ACGI

before exam started.
(correction 4(b).

MATHEMATICS 1A (E-STREAM AND I-STREAM)

Corrected Copy

Thursday, 28 May 10:00 am

Time allowed: 2:00 hours

There are FOUR questions on this paper.

Answer ALL questions. Question 1 is worth 40%. Questions 2-4 are each worth 20%.

NO CALCULATORS ALLOWED

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) :

Second Marker(s) :

EE1-10A MATHEMATICS I

Information for Candidates:

Calculators are not permitted in this exam.

1. a) Express the following complex numbers in the form $x+iy$: [4]

$$\text{(i)}\,(1-i)^3, \quad \text{(ii)}\,\frac{1-i}{1+i}, \qquad \text{(iii)}\,\left(\frac{1+\sqrt{3}i}{2}\right)^{10}$$

b) Find all values of $(2+2i)^{1/3}$. [4]

c) Obtain the limit: [4]

$$\lim_{x\to\pi/6}\frac{\cos(3x)}{\tan(2x)-\sqrt{3}}.$$

d) Obtain the limit: [4]

$$\lim_{x\to 0} x^x.$$

e) Differentiate: $y = x^{\ln x}$. [4]

f) Integrate: $\displaystyle\int_0^1 \frac{2x}{(1-3x^2)^{1/3}}\,dx$. [4]

g) Use a trigonometric substitution to integrate $\displaystyle\int_0^1 \sqrt{1-x^2}\,dx$. [4]

h) The convolution of two sets of complex Fourier coefficients, X_n and Y_n, is defined as $X_n * Y_n = \sum_{m=-\infty}^{\infty} X_{n-m}Y_m$.

Prove that $X_n * Y_n = Y_n * X_n$. [4]

i) The waveform $x(t)$ is defined as $x(t) = \begin{cases} e^{-2t} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$.

Calculate $y(t) = x(t) \otimes x(t) = \int_{-\infty}^{\infty} x^*(s-t)x(s)ds$. [4]

j) Suppose that the Fourier transform of a waveform $x(t)$ is given by

$$X(f) = 2i\,(\delta(f+20) - \delta(f-20))$$

where $\delta(\ )$ is the Dirac delta function.

Determine an expression for $x(t)$. [4]

2. Given the function

$$f(x) = \frac{x}{1+x^4} - \frac{x^3}{1+x^4}$$

a) Show that the area under the graph of the function $f(x)$, for $0 \leq x \leq b$ is given by

$$A(b) = \frac{1}{2}\arctan(b^2) - \frac{1}{4}\log(1+b^4).$$

[6]

b) Find the stationary points of $A(b)$ with $b > 0$ and determine whether they are maxima or minima. [6]

c) Assume that b is a function of time given by

$$b(t) = e^{-t}.$$

i) Use the chain rule to determine $\frac{dA}{dt}$ as a function of t. [4]

ii) Find the limit of $A(b(t))$ as t tends to $+\infty$. [4]

3. a) i) Show that $\sinh(x+iy) = \sinh x \cos y + i \cosh x \sin y$. [4]

ii) Hence, or otherwise, show that [4]

$$|\sinh(x+iy)|^2 = \frac{1}{2}(\cosh 2x - \cos 2y).$$

b) Obtain the limit:

$$\lim_{x \to -3} \frac{3 - \sqrt{-3x}}{x+3}.$$

[Do NOT use l'Hopital's rule.] [4]

c) Show that $\int \frac{dx}{\sqrt{x^2-1}} = \cosh^{-1} x + C$. [4]

d) Integrate $\int \frac{1}{1 - \sin x - \cos x}\, dx$. [4]

4. a) The real-valued waveform $x(t)$ is periodic with period $T = \frac{1}{F}$ and its complex Fourier series coefficients are denoted X_n.

Prove that $\frac{1}{T}\int_0^T |x(t)|^2 = \sum_{n=-\infty}^{\infty} |X_n|^2$. [5]

b) The waveform $x(t)$ has period $T = 2$ and is defined by

$$x(t) = \begin{cases} t & \text{for } 0 \le t < 1 \\ 2-t & \text{for } 1 \le t < 2 \end{cases}.$$

By evaluating the formula $X_n = \langle x(t)e^{-i2\pi nFt}\rangle$, show that for $n \neq 0$ [6]

$$X_n = \frac{(-1)^n - 1}{\pi^2 n^2}.$$

c) Determine the value of X_n in simplified form for each of the coefficient indices $n = -4, -3, \cdots, 3, 4$. [4]

d) For each of the following properties that are satisfied by $x(t)$, state the corresponding properties of X_n that can be deduced:

i) $x(t) = x(-t)$, [2]

ii) $x(t) + x(t+1) = 1$. [3]

