

Modelling and control of multibody mechanical systems

Model answers

Question 1

a) $\mathbf{r}_M = x\mathbf{i}$ and $\mathbf{r}_m = x\mathbf{i} + l\mathbf{e}_r = (x + l \sin \theta)\mathbf{i} + l \cos \theta\mathbf{k}$.

b) By differentiating the position vector $\dot{\mathbf{r}}_M = \dot{x}\mathbf{i}$ and

$$\dot{\mathbf{r}}_m = \dot{x}\mathbf{i} + l\dot{\theta}\mathbf{e}_\theta = \dot{x} \sin \theta \mathbf{e}_r + (\dot{x} \cos \theta + l\dot{\theta})\mathbf{e}_\theta.$$

c)

$$T = \frac{1}{2}M\dot{\mathbf{r}}_M \cdot \dot{\mathbf{r}}_M + \frac{1}{2}m\dot{\mathbf{r}}_m \cdot \dot{\mathbf{r}}_m = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 + 2\dot{x}\dot{\theta}l \cos \theta + l^2\dot{\theta}^2).$$

d) The horizontal level at O is taken as the zero potential energy level, therefore

$$V = -m\mathbf{r}_m \cdot \mathbf{g} + \frac{1}{2}kx^2 = -mgl \cos \theta + \frac{1}{2}kx^2.$$

e)

$$L = T - V = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 + 2\dot{x}\dot{\theta}l \cos \theta + l^2\dot{\theta}^2) + mgl \cos \theta - \frac{1}{2}kx^2.$$

f) The Lagrangian equation with respect to the generalised coordinate x is

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = -c\dot{x},$$

or

$$\frac{d}{dt} (M\dot{x} + m\dot{x} + m\dot{\theta}l \cos \theta) + kx = -c\dot{x},$$

or

$$(M + m)\ddot{x} + ml \cos \theta \ddot{\theta} - ml\dot{\theta}^2 \sin \theta + c\dot{x} + kx = 0.$$

The Lagrangian equation with respect to the generalised coordinate θ is

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0,$$

or

$$\frac{d}{dt} (m\dot{x}l \cos \theta + ml^2\dot{\theta}) + m\dot{x}\dot{\theta}l \sin \theta + mgl \sin \theta = 0,$$

or

$$\cos \theta \ddot{x} + l\ddot{\theta} + g \sin \theta = 0.$$

Question 2

a) $\dot{\mathbf{r}} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta$.

b) The kinetic energy is $T = \frac{1}{2}m\dot{\mathbf{r}} \cdot \dot{\mathbf{r}} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$.

The potential energy is $V = -m\mathbf{r} \cdot \mathbf{g} = -mr\mathbf{e}_r \cdot g\mathbf{k} = -mgr \cos \theta$, with the level of point O corresponding to zero gravitational potential energy.

The Lagrangian is $L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + mgr \cos \theta$.

c) The constraint equation is

$$r = \alpha + (r_0 - \alpha) \cos \theta, \quad (1)$$

by differentiating

$$\dot{r} + (r_0 - \alpha) \sin \theta \dot{\theta} = 0, \quad (2)$$

and by differentiating once again

$$\ddot{r} = -(r_0 - \alpha) \cos \theta \dot{\theta}^2 - (r_0 - \alpha) \sin \theta \ddot{\theta}. \quad (3)$$

The Lagrangian equation with respect to the generalised coordinate r is

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{r}} \right) - \frac{\partial L}{\partial r} + \lambda = 0,$$

or

$$\frac{d}{dt} (m\dot{r}) - m\dot{\theta}^2 - mg \cos \theta + \lambda = 0,$$

or

$$m\ddot{r} - m\dot{\theta}^2 - mg \cos \theta + \lambda = 0,$$

or by using Equations (1) and (3)

$$\lambda = m \left((\alpha + 2(r_0 - \alpha) \cos \theta) \dot{\theta}^2 + (r_0 - \alpha) \sin \theta \ddot{\theta} + g \cos \theta \right). \quad (4)$$

The Lagrangian equation with respect to the generalised coordinate θ is

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} + \lambda(r_0 - \alpha) \sin \theta = 0,$$

or

$$\frac{d}{dt} (mr^2\dot{\theta}) + mgr \sin \theta + \lambda(r_0 - \alpha) \sin \theta = 0,$$

or

$$mr^2\ddot{\theta} + 2mrr\dot{\theta} + mgr \sin \theta + \lambda(r_0 - \alpha) \sin \theta = 0,$$

or by using Equations (1), (2) and (4)

$$\left(\alpha^2 + 2\alpha(r_0 - \alpha) \cos \theta + (r_0 - \alpha)^2 \right) \ddot{\theta} - \alpha(r_0 - \alpha) \sin \theta \dot{\theta}^2 + (\alpha + 2(r_0 - \alpha) \cos \theta) g \sin \theta = 0. \quad (5)$$

d) The force in the wire, F_{wire} , is given by $-\lambda$, therefore

$$F_{\text{wire}} = -m \left((\alpha + 2(r_0 - \alpha) \cos \theta) \dot{\theta}^2 + (r_0 - \alpha) \sin \theta \ddot{\theta} + g \cos \theta \right).$$

e) For small θ the equation of motion is

$$r_0^2 \ddot{\theta} + (r_0 + (r_0 - \alpha))g\theta = 0,$$

and therefore the mass executes simple harmonic motion with angular frequency

$$\sqrt{\frac{r_0 + (r_0 - \alpha)}{r_0} \frac{g}{r_0}}.$$

For fixed wire length, $r = r_0$, $r_0 = \alpha$ and therefore the frequency of oscillations is $\omega_0 = \sqrt{g/r_0}$. For $r_0 > \alpha$, $\omega > \omega_0$ and for $r_0 < \alpha$, $\omega < \omega_0$.

Question 3

- a) The angular velocity of the system about the vertical axis is $\dot{\psi}$ and in the \mathbf{i} direction it is $\dot{\theta}$. All together it is

$$\boldsymbol{\Omega} = \dot{\theta}\mathbf{i} + \dot{\psi}\sin\theta\mathbf{j} + \dot{\psi}\cos\theta\mathbf{k}.$$

The position vector of the lower mass (in the position shown in the diagram) is

$$\mathbf{r}_1 = \frac{l}{2}\mathbf{j}.$$

The velocity vector is

$$\mathbf{v}_1 = \dot{\mathbf{r}}_1 = \boldsymbol{\Omega} \times \mathbf{r}_1$$

which gives

$$\mathbf{v}_1 = -\frac{l}{2}\dot{\psi}\cos\theta\mathbf{i} + \frac{l}{2}\dot{\theta}\mathbf{k}.$$

The velocity vector of the other mass is given by

$$\mathbf{v}_2 = -\mathbf{v}_1.$$

- b) The acceleration vector of the lower mass is

$$\mathbf{a}_1 = \dot{\mathbf{v}}_1 = \frac{l}{2}\ddot{\theta}\mathbf{k} - \left(\frac{l}{2}\ddot{\psi}\cos\theta - \frac{l}{2}\dot{\psi}\dot{\theta}\sin\theta\right)\mathbf{i} + \boldsymbol{\Omega} \times \mathbf{v}_1,$$

or

$$\mathbf{a}_1 = \left(\frac{l}{2}\ddot{\psi}\cos\theta + l\dot{\psi}\dot{\theta}\sin\theta\right)\mathbf{i} - \left(\frac{l}{2}\ddot{\psi}^2\cos^2\theta + \frac{l}{2}\dot{\theta}^2\right)\mathbf{j} + \left(\frac{l}{2}\ddot{\theta} + \frac{l}{2}\dot{\psi}^2\sin\theta\cos\theta\right)\mathbf{k}.$$

The acceleration vector of the other mass is

$$\mathbf{a}_2 = -\mathbf{a}_1.$$

- c) The force vector acting on the lower mass is

$$\mathbf{F}_1 = -F_N\mathbf{i} - F_r\mathbf{j},$$

where F_N is the magnitude of the force on each mass due to the moment N acting on the rod. This is given by

$$F_N = \frac{N}{l},$$

therefore

$$\mathbf{F}_1 = -\frac{N}{l}\mathbf{i} - F_r\mathbf{j}.$$

The force vector on the other mass is

$$\mathbf{F}_2 = -\mathbf{F}_1.$$

- d) The motion of the system can be found by considering the motion of one of the masses. For the lower mass

$$\mathbf{F}_1 = m\mathbf{a}_1$$

or by substituting the force and acceleration expressions from the equations above and collecting the terms with respect to \mathbf{i} and \mathbf{k}

$$\ddot{\theta} + \dot{\psi}^2\sin\theta\cos\theta = 0,$$

and

$$\frac{1}{2}ml^2(\ddot{\psi}\cos\theta - 2\dot{\psi}\dot{\theta}\sin\theta) = N.$$

- e) By using again the equation $\mathbf{F}_1 = m\mathbf{a}_1$ and collecting the terms with respect to \mathbf{j} we obtain

$$F_r = m \left(\frac{l}{2} \dot{\psi}^2 \cos^2 \theta + \frac{l}{2} \dot{\theta}^2 \right).$$

Question 4

- a) The velocity vector of mass M is $\dot{\mathbf{r}}_M = \dot{x}\mathbf{i}$. By differentiating the velocity expression we obtain the acceleration vector,

$$\ddot{\mathbf{r}}_M = \ddot{x}\mathbf{i}.$$

- b) The velocity vector of mass m is $\dot{\mathbf{r}}_m = \dot{x}\mathbf{i} + l\dot{\theta}\mathbf{e}_\theta = \dot{x}\sin\theta\mathbf{e}_r + (\dot{x}\cos\theta + l\dot{\theta})\mathbf{e}_\theta$. By differentiating the velocity expression we obtain the acceleration vector,

$$\ddot{\mathbf{r}}_m = (\ddot{x}\sin\theta - l\dot{\theta}^2)\mathbf{e}_r + (\ddot{x}\cos\theta + l\ddot{\theta})\mathbf{e}_\theta.$$

- c) The equation of motion of mass m in vector form is

$$\mathbf{F}_m = m\ddot{\mathbf{r}}_m,$$

or

$$-F_r\mathbf{e}_r + mg\cos\theta\mathbf{e}_r - mg\sin\theta\mathbf{e}_\theta = m(\ddot{x}\sin\theta - l\dot{\theta}^2)\mathbf{e}_r + m(\ddot{x}\cos\theta + l\ddot{\theta})\mathbf{e}_\theta.$$

- i) The first equation of motion is found by collecting the \mathbf{e}_θ terms

$$\ddot{x}\cos\theta + l\ddot{\theta} + g\sin\theta = 0.$$

- ii) The force in the rod, F_r , is found by collecting the \mathbf{e}_r terms and it is given by

$$F_r = m(-\ddot{x}\sin\theta + l\dot{\theta}^2 + g\cos\theta).$$

- d) The equation of motion of mass M in vector form is

$$\mathbf{F}_M = M\ddot{\mathbf{r}}_M,$$

or

$$(-kx - c\dot{x} + F_r\sin\theta)\mathbf{i} + (Mg + F_r\cos\theta - R)\mathbf{k} = M\ddot{x}\mathbf{i},$$

where R is the normal reaction from the surface on the cart. By collecting the \mathbf{i} terms we obtain the second equation of motion

$$M\ddot{x} - F_r\sin\theta + c\dot{x} + kx = 0,$$

or

$$(M + m\sin^2\theta)\ddot{x} - ml\dot{\theta}^2\sin\theta - mg\sin\theta\cos\theta + c\dot{x} + kx = 0.$$

- e) For $k_s = 0$, $c = 0$ and small x and θ the two equations of motion become

$$M\ddot{x} - mg\theta = 0,$$

and

$$\ddot{x} + l\ddot{\theta} + g\theta = 0.$$

By a simple manipulation these two equations give

$$Ml\ddot{\theta} + (M + m)g\theta = 0,$$

or

$$\ddot{\theta} + \frac{M+m}{M} \frac{g}{l} \theta = 0,$$

which is simple harmonic motion in θ for the pendulum with frequency of oscillation $\sqrt{\frac{M+m}{M} \frac{g}{l}}$. By another simple manipulation the equations of motion give

$$(M+m)\ddot{x} + ml\ddot{\theta} = 0,$$

or

$$\ddot{x} = -\frac{ml}{M+m} \ddot{\theta},$$

which can be integrated to give

$$x = -\frac{ml}{M+m} \theta + x_0,$$

for initial $\dot{x} = 0$ and $\dot{\theta} = 0$. Therefore the cart also executes simple harmonic motion about some position x_0 with the same frequency as for the pendulum, and with its amplitude scaled by $-\frac{ml}{M+m}$ as compared to the amplitude of the motion of the pendulum.