UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 2000

MEng Honours Degree in Mathematics and Computer Science Part IV
MEng Honours Degrees in Computing Part IV
MSc in Advanced Computing
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the Associateship of the City and Guilds of London Institute
This paper is also taken for the relevant examinations for the Associateship of the Royal College of Science

PAPER C499

MODAL AND TEMPORAL LOGIC

Thursday 18 May 2000, 10:00
Duration: 120 minutes

Answer THREE questions

Paper contains 4 questions

1a i) Briefly explain the meaning of the notation $\vdash A$, for a modal formula A. State, without proof, a soundness and completeness theorem for $\vdash$.

ii) What is the *finite model property?* Briefly outline how the finite model property for the modal logic K follows from the proof of soundness and completeness for $\vdash$.

b Let B be the temporal formula

$$B = p \wedge PG(p \rightarrow Fq) \rightarrow Gq.$$

Using a temporal tableau, either show that $\vdash B$, or find a model of $\neg B$.

c Let C be the modal formula

$$C = \Diamond p \rightarrow \Box\Diamond p.$$

i) Use Sahlqvist's algorithm to find a first-order frame condition that is true in a Kripke frame F if and only if C is valid in F.

ii) What does your frame condition say in English?

2a i) Let A be a modal formula, and F a Kripke frame. Define the meaning of the statement that *A is valid in F.*

ii) Explain what a modal *p-morphism* is, carefully distinguishing between frame p-morphisms and model p-morphisms.

iii) State and prove a theorem showing that frame p-morphisms preserve frame validities. You may assume without proof a result establishing that model p-morphisms preserve truth of formulas, if you state it clearly.

b Let N, C_3, and K_3 be the following Kripke frames:

$N = (\mathbb{N}, <) = (\{0,1,2,3,\ldots\}, <)$,
$K = (\{0, 1, 2\}, R)$, where $R(x,y)$ holds for all worlds $x,y \in \{0,1,2\}$,
$C = (\{0, 1, 2\}, S)$, where S holds on all pairs of worlds except (1,0), (2,1), (0,2).

i) Show that every modal formula that is valid in N is also valid in K.

By considering frame properties, or otherwise,

ii) find a modal formula that is valid in K but not in N;

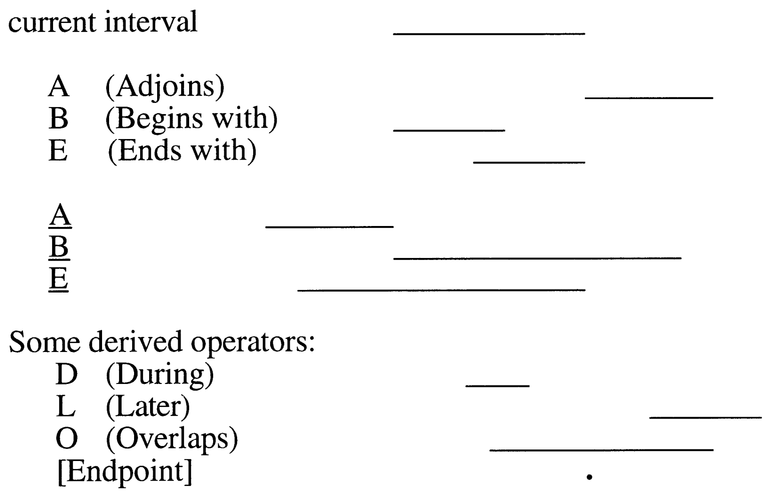
iii) find a modal formula that is valid in K but not in C;

iv) find a modal formula that is valid in N but not in C.

c Briefly compare and contrast the notions of *p-morphism* and *bisimulation.*

The three parts carry, respectively, 50%, 30%, 20% of the marks.

3 Halpern and Shoham's interval temporal logic is a multi-modal logic in which each of six basic relations A, B, E, $\underline{A}$, $\underline{B}$, $\underline{E}$, is used to define a normal modality over a set of intervals on a partially ordered set of time points. For time points which are real numbers a typical instance of each relation is illustrated in the figure below, relative to a current interval.



As usual, for each such relation R, the modal formula $\langle R\rangle p$ is defined to hold on the current interval i, if and only if the formula p holds on some interval j which is accessible to i under relation R, *i.e.* such that $i\mathrm{R}j$. Again, as usual, for each R, $[R]p \leftrightarrow \neg\langle R\rangle\neg p$, for any formula p. In general the temporal scale need not be continuous and may branch. An interval can be a point in the degenerate case, but the end point (or beginning point) of the current interval is not accessible to the current interval under the relation A (or $\underline{A}$).

a Provide definitions for the conditions $\langle D\rangle p$, $\langle L\rangle p$, and $\langle O\rangle p$ in terms of the basic operators.

b Determine whether each of the following is an axiom of the system, giving brief explanation for your answer:

i) $[A]\,p \rightarrow [A][A]\,p$

ii) $[B]\,p \rightarrow [B][B]\,p$

c Explain why in the illustrated case above, the following are also axioms:

i) $[B][B]\,p \rightarrow [B]\,p$

ii) $\langle A\rangle[A]\,p \rightarrow [A]\langle A\rangle p$

d Let a be a term for an action of finite duration and let p, q be a precondition and a consequent for successful execution of a. Suppose *exec*(a) to be true on exactly the interval of execution of a. Provide a formula which relates the execution to the pre and post conditions.

The four parts of this question carry, respectively, 40%, 20%, 20%, and 20% of the marks.

4 a State and justify the three axioms that are accepted for a normal modal logic of:
i) knowledge, and ii) belief.

b Show for the case of knowledge these axioms make iterated modalities redundant, i.e. so that $\mathrm{KK}p \leftrightarrow \mathrm{K}p$, $\neg\mathrm{K}\neg\mathrm{K}p \leftrightarrow \mathrm{K}p$ *etc.*

c Let $\mathrm{K}_i\, p$ express the fact that agent i knows that p holds and let $send_{ij}\, p$ be a message from i to j which conveys the fact that p holds. Suppose that *everyone knows p* be defined by $\mathrm{E}p \leftrightarrow \mathrm{K}_i\, p \wedge \mathrm{K}_j\, p$. Explain the auxiliary presumptions and communications needed in order to strengthen $\mathrm{E}p$ into a form of common knowledge.

Parts a, b and c carry, respectively, 30%, 40% and 30% of the marks.