

IMPERIAL COLLEGE LONDON

E4.07  
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DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING  
EXAMINATIONS 2009

MSc and EEE/ISE PART IV: MEng and ACGI

Corrected Copy

**CODING THEORY**

Wednesday, 6 May 10:00 am

Time allowed: 3:00 hours

**There are SIX questions on this paper.**

**Answer FOUR questions.**

*All questions carry equal marks*

**Any special instructions for invigilators and information for candidates are on page 1.**

Examiners responsible	First Marker(s) :	A.A. Ivanov
	Second Marker(s) :	C. Ling

A table of the field of order 16

log	0	1	12	2	9	13	7	3	4	10	5	14	11	8	6
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	3	4	6	8	10	12	14	9	11	13	15	1	3	5	7
3	2	1	5	12	15	10	9	1	2	7	4	13	14	11	8
4	5	6	7	9	13	1	5	11	15	3	7	2	6	10	14
5	4	7	6	1	8	7	2	3	6	9	12	14	11	4	1
6	7	4	5	2	3	13	11	2	4	14	8	3	5	15	9
7	6	5	4	3	2	1	12	10	13	4	3	15	8	1	6
8	9	10	11	12	13	14	15	15	7	6	14	4	12	13	5
9	8	11	10	13	12	15	14	1	14	12	5	8	1	3	10
10	11	8	9	14	15	12	13	2	3	11	1	5	15	8	2
11	10	9	8	15	14	13	12	3	2	1	10	9	2	6	13
12	13	14	15	8	9	10	11	4	5	6	7	6	10	7	11
13	12	15	14	9	8	11	10	5	4	7	6	1	7	9	4
14	15	12	13	10	11	8	9	6	7	4	5	2	3	2	12
15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	3

Below diagonal  $a + b$ , on or above  $a \times b$ ,  
 $0 + a = a$ ,  $a + a = 0$ ,  $0 \times a = 0$

1. Let  $D$  be the binary  $(7, 4)$ -code given by the check matrix

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

and let  $C$  be the *shortened* code obtained from  $D$  by removing the final zero from all the code words that end in zero. Thus  $u \in C$  if and only if  $(u, 0) \in D$ .

Construct (with justification) generator and check matrices for  $C$ . You should state but need not prove the relation between standard form generator and check matrices of a binary linear code.

[10]

Using your check matrix construct (with justification) a syndrome/coset leader decoding table for  $C$ .

[10]

Where there is a choice for the coset leader, indicate all possible values.

[5]

2. Let  $C$  be a (not necessarily binary) linear  $(n, m)$ -code and let  $C'$  be the *punctured* code obtained by deleting a particular position from all the code words of  $C$ . Show that if  $C$  has minimal distance  $d > 1$ , then  $C'$  is a  $(n - 1, m)$ -code with minimal distance  $d' \geq d - 1$ .

[7]

Hence or otherwise show that the minimal distance of any linear  $(n, m)$ -code  $C$  is at most  $n - m + 1$ . This is the Singleton bound. If the minimal distance is exactly equal to  $n - m + 1$  we shall say the code meets the Singleton bound.

[7]

Define an  $r$ -perfect code.

[1]

Show that the binary Hamming code  $\text{Ham}(4)$  is 1-perfect, but does not meet the Singleton bound.

[5]

Show that the triple error correcting Reed–Solomon code  $\text{RS}(4, 3)$  considered as a code over  $\text{GF}(16)$  meets the Singleton bound but it is not perfect. You may assume that  $\text{RS}(4, 3)$  corrects 3 errors and that there are exactly  $1559476 = 4 \times 389869$  words at distance  $\leq 3$  from a code word of  $\text{RS}(4, 3)$ .

[5]

3. Explain how the field table of the field of order 16 on page 1 is constructed, emphasizing the properties of the polynomial  $x^4 + x^3 + 1$  that ensure that the element 2 is a *primitive element*, and explaining how the 'logarithms' at the head of the table can be used to multiply and find inverses in the field.

[10]

Prove that every finite field contains a primitive element.

[15]

4. Describe Euclid's algorithm in its four column version.

[10]

Prove that Euclid's algorithm determines the highest common factor of its input.

[5]

Let  $t(x)$  be an irreducible binary polynomial and let  $s(x)$  be a non-zero binary polynomial such that  $\deg(s(x)) < \deg(t(x))$ . Stating clearly any properties of Euclid's algorithm you require, show how that algorithm can be used to determine an inverse polynomial  $r(x)$  of  $s(x) \bmod t(x)$  that is a polynomial  $r(x)$  such that  $r(x)s(x) \equiv 1 \bmod t(x)$ .

[5]

Illustrate your method by finding an inverse of  $x^6 + x^4 + x^3 + x + 1 \bmod x^7 + x + 1$  (you may assume that  $x^7 + x + 1$  is irreducible).

[5]

5. Explain how the binary,  $t$ -error correcting BCH code  $\text{BCH}(k, t)$  is constructed by extending the check matrix of the Hamming code  $\text{Ham}(k)$ , describing the two check matrices  $H_{k,t}$  and  $V_{k,t}$ .

[5]

Show that the matrices  $H_{k,t}$  and  $V_{k,t}$  define the same code.

[8]

Use the matrices  $H_{k,t}$  and  $V_{k,t}$ , and the properties of check matrices in general, and of Vandermonde type matrices in particular (which you should state, but need not prove) to give estimates of the parameters of the codes.

[4]

Suppose now that the check matrix of the Hamming code  $H_4$  is written in the form

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

Construct the check matrix  $H_{4,2}$  of the code  $\text{BCH}(4, 2)$ .

[8]

6. The 4-error correcting code  $RS(4,4)$  based on the primitive element 2 is used to transmit a message. One received word is

$$u = (1, 10, 3, 4, 4, 2, 2, 13, 15, 3, 6, 7, 1, 1, 0).$$

The first 7 syndromes of  $S_1, \dots, S_7$  of  $u$  are 10, 8, 12, 4, 7, 0, 1.

Calculate the remaining syndrome  $S_8$ .

[5]

Using the syndromes calculate the error locator and the error evaluator for the word.

[10]

Verify that the error locator has roots 6, 11, 12, 15 and calculate a code word at distance  $\leq 4$  from  $u$ .

[10]

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## Coding Theory 2009

## SOLUTION 1

We use the fact that the standard form matrix  $\begin{pmatrix} I \\ A \end{pmatrix}$  is a generator for a binary code if and only if the standard form matrix  $(A, J)$  is a check matrix for that code.

Thus a generator matrix for  $D$  is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}^T$$

A code word of  $D$  is the sum of a subset of the columns of this matrix and it will end in 0 iff that set does not include the last column. Hence a generator matrix  $G$  for  $C$ , and the corresponding check matrix  $H$  are given by

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}^T, \quad H = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

The syndromes of single errors are the columns of  $H$  and ambiguities occur only if the same column occurs several times. Thus the first part of the syndrome/coset leader table has the following form (the syndromes and error patterns are written as rows for convenience):

syndr.	error
000	000000
001	010000 or 000001
010	100000 or 000010
100	000100
011	001000

That exhausts all the columns of  $H$  but there are still three syndromes left. These we represent as sums of two columns of  $H$ . It is helpful to note, that the only column of  $H$  with a 1 in the first row is the 4th. Thus an error producing syndrome starting with 1 must include have a 1 in its 4th position. An error word  $abcdef$  will produce the syndrome  $lyz$  iff the error word  $abc0ef$  produces the syndrome  $0yz$ . Since all the outstanding syndromes do start with 1 this makes it straightforward to determine all the corresponding error patterns of weight 2.

syndr.	error
101	010100 or 000101
110	100100 or 000110
111	001100

Unseen 10

Unseen 10

Unseen 5



## SOLUTION 2

Since the minimum distance of  $C$  is  $> 1$  two distinct code words of  $C$  cannot yield the same word when a single symbol is deleted. Thus  $C'$  has exactly the same number of code words as  $C$ . It is also obviously still linear. Since the number of code words of a linear code on  $q$  symbols of rank  $k$  is  $q^k$ , it follows that  $C'$  has the same rank as  $C$ .

Let  $u$  and  $v$  be two code words of  $C$  and  $u'$  and  $v'$  the corresponding code words of  $C'$ . Then if the deleted symbols of  $u$  and  $v$  are the same we have  $d(u', v') = d(u, v)$  and otherwise  $d(u', v') = d(u, v) - 1$ . Hence the minimum distance of  $C'$  is  $d - 1$  if there are two code words  $u, v$  of  $C$  with  $d(u, v) = d$  and different symbols in the punctured location. Otherwise the minimum distance of  $C'$  is  $d$ .

Exerc. 5

Now we puncture the code repeatedly until the minimum distance is 1 that happens at the latest when the block length equals the rank  $m$  of  $C$  which would occur after  $n - m$  steps. At each step the minimum distance has dropped by at most 1 so the original minimum distance is at most  $1 + n - m$  as claimed.

A code of block length  $n$  is  $r$ -perfect if to every word  $v$  of length  $n$  there is exactly one code word  $u$  with  $d(u, v) \leq r$ .

seen 10

The parameters of the code  $\text{Ham}(4)$  are 15, 11, 3 since its check matrix  $H_4$  has as its columns all non-zero binary words of length 4. It is perfect single-error correcting because for any word  $v$  of length 15, we have either  $H_4 v = 0$  in which case  $v \in \text{Ham}(4)$  or  $H_4 v$  is a unique column of  $H_4$ . Correcting the corresponding bit of  $v$  will then produce a code word and changing any other or none will not do so. This also proves that the minimum distance of  $\text{Ham}(4)$  is 3 rather than 5 which would correspond to the Singleton bound.

unseen 5

The code  $\text{RS}(4)$  has as its generator polynomial  $\prod_{k=1}^6 (x - \alpha^k)$ , where  $\alpha$  is a primitive element of  $\text{GF}(16)$ . Since its code words are represented by polynomials of degree 14 it has block length 15 and rank  $14 - 6 + 1 = 9$ . Since it can correct three errors it has minimum distance at least 7. As  $7 = 15 - 9 + 1$  the minimum distance cannot be greater and the code meets the Singleton bound. If it is perfect then  $|\text{RS}(4, 3)| \times 1559476 = 16^{15}$ . However  $|\text{RS}(4, 3)| = 16^9$  which would imply that 1559476 is a power of 16 which is not the case. so the code is not perfect.

unseen 5

### SOLUTION 3

The number  $n$  represents its binary 4-tuple  $a, b, c, d$ , which in turn represents the polynomial  $ax^3 + bx^2 + cx + d$ . Thus  $13 \sim 1, 1, 0, 1 \sim x^3 + x^2 + 1$ . Addition (the lower half of the table) is ordinary addition over  $\mathbb{B}$  (i.e. XOR). In particular  $\alpha + \alpha = 0$  for any  $\alpha$ , so this does not appear in the table. Multiplication (the upper half of the table) is multiplication in  $\mathbb{B}[x]$ , followed if necessary, by taking the remainder after division by  $x^4 + x^3 + 1$ .

The fact that 2 which corresponds to  $x$  is a primitive root is equivalent to the statement that  $x^4 + x^3 + 1$  divides  $x^{15} - 1$  but not  $x^k - 1$  for any smaller power  $k$ . Then the powers  $2^0, \dots, 2^{14}$  must all be distinct and so they cover all the non-zero elements of the field. These powers are the logarithms at the head of the table. To multiply field elements  $\alpha$  and  $\beta$  we represent them as powers of 2, say  $2^k$  and  $2^\ell$ . Then  $\alpha\beta = 2^{k+\ell}$ . Since  $2^{15} = 1$  we can reduce  $k + \ell \pmod{15}$ . Then the product is the element with that value as its logarithm. In particular,  $\alpha^{-1} = 2^{15-k}$

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**THEOREM.** Every finite field has a primitive element.

We first prove a lemma:

**LEMMA.** If  $\beta, \gamma \in F$  and  $\text{ord}(\beta) = m$  and  $\text{ord}(\gamma) = n$ , and their highest common factor  $(m, n) = 1$ , then  $\text{ord}(\beta\gamma) = mn$ .

*Proof.* Certainly  $(\beta\gamma)^{mn} = 1$ . Suppose  $(\beta\gamma)^k = 1$ . Then  $\beta^{kn} = \beta^{kn}\gamma^{kn} = (\beta\gamma)^{kn} = 1$ . Hence  $m$  divides  $kn$ . So  $n$  divides  $km$ . Now  $m$  and  $n$  have no prime factors in common, so  $m$  divides  $kn$  only if it divides  $k$ . Similarly  $\gamma^{km} = 1$  and  $n$  divides  $km$  only if it divides  $k$ . So both  $m$  and  $n$  divide  $k$ . Thus their least common multiple,  $mn$  divides  $k$ .  $\square$

*Proof.* Let  $p_1, \dots, p_k$  be the prime factors of  $q - 1$  and let  $s_i$  be the highest power of  $p_i$  that divides the order of some element  $\gamma_i$  of  $F$ . By taking  $\gamma_i$  to a suitable power we may assume that  $\text{ord}(\gamma_i) = s_i$ . Then  $\alpha = \gamma_1 \cdots \gamma_k$  has order  $s_1 \cdots s_k = u$  by the lemma. Now by the construction of  $u$  any  $\beta \in F$  has order dividing  $u$ . So the non-zero elements of  $F$  are all roots of  $x^u - 1$ . Hence  $q - 1 \leq u$ . But each  $s_i$  is a factor of  $q - 1$  (by Lemma 1a), and they are powers of distinct primes. Therefore  $u$  divides  $q - 1$ . Thus they are equal.  $\square$

back to

# SOLUTION 4

ALGORITHM Euclid's Algorithm.

Step 1. Set up a table with 4 columns (5 if you count the row number) headed Q, R, U, V. Fill in the first two rows (numbers -1 and 0) as follows:

ROW	Q	R	U	V
-1	-	$a$	1	0
0	-	$b$	0	1

Step 2. Calculate the Q and R entries for row 1 by dividing  $a$  by  $b$ :  $a = q_1 b + r_1$ . The U entry is 1 and the V entry is  $-q_1$ .

1	$q_1$	$r_1$	$u_1 = 1$	$v_1 = -q_1$
---	-------	-------	-----------	--------------

Note that  $r_1 = 1a + (-q_1)b$ .

Step 3. Suppose we have calculated up to row  $k$  and the last two rows are as follows

$k-1$	$q_{k-1}$	$r_{k-1}$	$u_{k-1}$	$v_{k-1}$
$k$	$q_k$	$r_k$	$u_k$	$v_k$

If  $r_k = 0$  Stop.

Otherwise divide  $r_{k-1}$  by  $r_k$ :  $r_{k-1} = q_{k+1}r_k + r_{k+1}$ .

That gives the Q and R entries of row  $k+1$ .

Using the Q entry just calculated, put  $u_{k+1} = u_{k-1} - q_{k+1}u_k$  and  $v_{k+1} = v_{k-1} - q_{k+1}v_k$ .

back to

PROPOSITION. The last non-zero element of the R-column is a highest common factor of  $a$  and  $b$ .

Proof. Let the last non-zero element be  $r_n$ . Then  $r_{n-1} = q_{n+1}r_n + 0$ . So  $r_n \mid r_{n-1}$ . Next,  $r_{n-2} = q_n r_{n-1} + r_n$ . Since  $r_n$  divides both summands on the right hand side it divides  $r_{n-2}$ . Now suppose we have shown  $r_n$  divides  $r_{k+1}$  and  $r_k$ . As  $r_{k-1} = q_{k+1}r_k + r_{k+1}$ , it follows in the same way that  $r_n$  divides  $r_{k-1}$ . Finally,  $r_n$  divides  $r_0 = b$  and  $r_{-1} = a$ . So  $r_n$  is a common factor of  $a$  and  $b$ .

back to

Conversely, it will follow from (b) that if  $c \mid a$  and  $c \mid b$ , then  $c \mid r_n = u_n a + v_n b$ . So  $r_n$  is indeed  $(a, b)$ .  $\square$

seen 5

Since  $t(x)$  is irreducible and cannot divide  $s(x)$ , the highest common factor of the two polynomials must be a non-zero constant, and the only binary non-zero constant is 1. So the last non-zero entry in the R column must be 1. Now for each row of the table the entries satisfy  $r_k = au_k + bv_k$ . Therefore in the last row we have  $1 = t(x)u_k(x) + s(x)v_k(x)$ . That implies that  $v_k(x)s(x) \equiv 1 \pmod{t(x)}$ . So  $v_k$  is the required polynomial  $r(x)$ .

We execute Euclid's algorithm with the two given polynomials below

Q	R	U	V
	1 0 0 0 0 0 1 1	1	0
	1 0 1 1 0 1 1	0	1
1 0	1 1 0 1 0 1	1	1 0
1 1	1 0 0	1 1	1 1 1
1 1 0 1		1 1 0 1 1 0	1 0 0 0 0 1

seen 5

The final row shows that  $(x^5 + 1)(x^6 + x^4 + x^3 + x + 1) \equiv 1 \pmod{x^7 + x + 1}$ . So  $x^5 + 1$  is the required polynomial.

## SOLUTION 5

We first consider the columns of the check matrix of  $\text{Ham}(k)$  as the binary representation of the integers  $1, \dots, 2^k - 1$ , which in turn we take to represent the non-zero elements of  $\text{GF}(2^k)$ . Using this interpretation, the check matrix  $V_{k,t}$  extends the column  $\alpha$  to

$$\begin{pmatrix} \alpha \\ \alpha^2 \\ \vdots \\ \alpha^{2^t} \end{pmatrix}.$$

the check matrix  $H_{k,t}$  is obtained by deleting the rows corresponding to even powers of  $\alpha$ .

PROPOSITION.  $V_{k,t}$  is a check matrix for the same code as  $H_{k,t}$ .

Proof. We need only show that a binary word  $u$  of length  $2^k - 1$  satisfying  $H_{k,t}u = \underline{0}$  also satisfies  $V_{k,t}u = \underline{0}$ . Write the Hamming check matrix as  $\alpha_1, \dots, \alpha_{2^k-1}$ . Then the condition  $H_{k,t}u = \underline{0}$  is equivalent to

$$\sum_{r=1}^{2^k-1} \alpha_r^s u_r = 0 \quad \text{for all odd } s \leq 2t.$$

Since the field we are calculating in has characteristic 2 we have

$$\sum_{r=1}^{2^k-1} \alpha_r^{2s} u_r^2 = \left( \sum_{r=1}^{2^k-1} \alpha_r^s u_r \right)^2 = 0.$$

But since  $u_r = 0, 1$  we have  $u_r^2 = u_r$  so

$$\sum_{r=1}^{2^k-1} \alpha_r^{2s} u_r = 0,$$

proving that  $V_{k,t}u = \underline{0}$ .

The block length of  $\text{BCH}(k, t)$  is the number of columns in either of its check matrices,  $2^k - 1$ . Since each binary row of a check matrix can only reduce the rank of a code by at most one using  $H_{k,t}$  we see that the rank of  $\text{BCH}(k, t)$  is at least  $2^k - 1 - kt$ .

Noting that each  $(2t \times 2t)$ -submatrix of  $V_{k,t}$  considered over  $GF(2^k)$  is a Vandermonde matrix with non-zero determinant, we see that no  $2t$  columns of  $V_{k,t}$  are linearly dependent, even when they are considered as binary vectors. It follows that the code has minimum distance at least  $2t + 1$ .

Writing the matrix over  $GF(16)$  it takes the form

$$(15 \ 14 \ 13 \ 12 \ 11 \ 10 \ 9 \ 8 \ 7 \ 6 \ 5 \ 4 \ 3 \ 2 \ 1)$$

Adding the cubes beneath it gives the matrix

$$\begin{pmatrix} 15 & 14 & 13 & 12 & 11 & 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 8 & 5 & 8 & 3 & 1 & 1 & 3 & 5 & 15 & 5 & 3 & 15 & 15 & 8 & 1 \end{pmatrix}.$$

In binary this is

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

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## SOLUTION 6

Received Word:

1 10 3 4 4 2 2 13 15 3 6 7 1 1 0

Syndrome Calculation using Horner's Scheme (*the candidate need only calculate the last row*):

	1	10	3	4	4	2	2	13	15	3	6	7	1	1	0
2:	1	8	10	9	15	5	8	4	7	13	5	13	2	5	10
4:	1	14	9	11	3	14	8	6	14	9	9	8	10	2	8
8:	1	2	10	2	13	14	15	8	0	3	7	13	13	13	12
9:	1	3	1	13	5	4	13	12	7	14	5	1	8	6	4
11:	1	1	8	10	5	14	4	10	14	5	10	6	9	4	7
15:	1	5	2	3	12	9	8	8	10	1	9	13	5	0	0
7:	1	13	11	7	8	8	8	7	3	10	2	9	12	14	1
14:	1	4	9	7	5	6	13	4	5	7	7	6	14	3	11

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The syndromes are the entries in the final column.

Syndrome Polynomial: 11 1 0 7 4 12 8 10

Euclid's Algorithm:

0 0 0 0 0 0	1 0 0 0 0 0 0 0 0 0	0 0 0 0 1	0 0 0 0 0
0 0 0 0 0 0	0 11 1 0 7 4 12 8 10	0 0 0 0 0	0 0 0 0 1
0 0 0 10 0	0 10 0 4 3 5 6 11 0	0 0 0 0 1	0 0 0 10 0
0 0 0 0 11	0 0 11 4 0 2 15 5 1	0 0 0 0 1	0 0 0 10 11
0 0 0 1 0	0 0 5 0 5 11 9 9 10	0 0 0 1 0	0 0 10 11 1
0 0 0 0 9	0 0 0 15 5 0 3 15 3	0 0 0 1 9	0 0 10 7 4
0 0 0 12 0	0 0 0 10 0 15 4 8 1	0 0 12 8 1	0 5 15 8 11
0 0 0 0 9	0 0 0 0 6 15 6 2 3	0 0 12 1 15	0 5 3 5 4
0 0 0 14 0	0 0 0 0 9 15 6 4 3	0 7 14 13 9	4 11 14 13 4
0 0 0 0 15	0 0 0 0 0 12 15 3 11	0 7 5 2 10	4 10 6 12 10

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The third column is optional. The last entry in the second column is the error evaluator (it is the first time the entry in this column drops below degree 4). The last entry in the final column is the error locator.

Verification of the zeros of the error locator and calculation of derivatives at the zeros:

	4	10	6	12	10
6:	4	11	14	3	0
	4	10	0	3	
11:	4	13	4	11	0
	4	10	5	7	
12:	4	8	2	13	0
	4	10	7	2	
15:	4	4	8	9	0
	4	10	10	11	

The entries in the last column are the values of the error locator and as expected they are all zero. The final entries of the shorter rows are the values of derivative of the error locator (which could also be found by differentiating and calculating directly). Roots: 6 11 12 15. Corresponding error locations: 2 10 1 9

Error Evaluator:

	12	15	3	11
6:	12	12	0	11
11:	12	6	11	1
12:	12	9	11	2
15:	12	4	13	15

Error Values (error evaluator/derivative of error locator): 14 14 1 2

Corrected Word:

1 10 3 4 10 0 2 13 15 3 6 7 15 0 0.

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