Imperial College London

BSc/MSci EXAMINATION May 2012

This paper is also taken for the relevant Examination for the Associateship

GROUP THEORY

For 3rd and 4th Year Physics Students

Tuesday, 22nd May 2012: 10:00 to 12:00

Answer ALL parts of Section A and TWO questions from section B.
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the Three answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in Three answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

SECTION A

1. (i) Define the representation of a finite group in terms of linear maps $GL(V)$ on vector space V over $\mathbb{F}$. What is the dimension of the representation and what is a faithful representation? [5 marks]

(ii) Explain what is meant by G-invariant sub-space of a representation and hence define what is meant by an irreducible representation of the group. [4 marks]

(iii) Explain how a matrix representation may be formed on vector space V and what is meant by equivalent matrix representations. [4 marks]

(iv) If G is the symmetry group of the Hamiltonian H of some system, by making use of Schurs's first and second lemma, show that the matrix representation H with respect to the energy eigenstates is diagonal. State clearly any theorem you use. [5 marks]

(v) The electronic states of valence electrons in carbon have quantum number $n = 2$. An isolated carbon atom has spherical symmetry. If we assume the states are hydrogen like, calculate the number of states the valence electron can occupy. You may assume that the dimension of representation of $SO(3)$ group D_l is $2l+1$. [2 marks]

[Total 20 marks]

SECTION B

2. (i) (a) Define a equivalence relation on a set of objects. What is an equivalent class? [3 marks]

(b) Show that an equivalence relation partitions a set into classes such that no members belong to more than one classes. [3 marks]

(ii) (a) Define a group. [3 marks]

(b) Define the conjugacy relation between two members of a group G and show that it is an equivalence relation. [3 marks]

(iii) (a) Define what is a sub group and what is a self conjugate (invariant) sub group. [4 marks]

(b) Show that invariant subgroup contains entire conjugacy classes. [4 marks]

[Total 20 marks]

3. (i) A group can be defined in terms of three elements a, b and c where

$$a^4 = b^2 = c^2 = (ab)^2 = (ac)^2 = e.$$

These elements also obey the relations $a^3b = ba$, $a^3c = ca$ and $bc = a$. There are 8 elements in the group which can be written as $\{e, a, a^2, a^3, b, a^2b, c, a^2c\}$. Show that this group has five equivalence classes and find what they are.

[8 marks]

(ii) Complete the following character table for this group. You must clearly state any general results or theorems you use but you need not prove them.

	$\{e\}$	$\{a, a^3\}$	$\{a^2\}$	$\{?, ?\}$	$\{?, ?\}$
Γ_1			1		
Γ_2			1		
Γ_3			1		
Γ_4			1		
Γ_5			-2		

[12 marks]

[Total 20 marks]

4. The direct product of two matrices A and B is defined by:

$$C = A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots \\ a_{21}B & a_{22}B & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

$$C_{ik;jl} = A_{ij}B_{kl}$$

(i) If $D^{\mu}(g)$ and $D^{\nu}(g)$ are matrix representations of group G, show that the set of matrices $D^{\gamma}(g) = D^{\mu}(g) \otimes D^{\nu}(g)$ also from a representation of the group.

[6 marks]

(ii) Show that the character of the product representation $D^{\gamma}(g)$ is the product of characters of representations $D^{\mu}(g)$ and $D^{\nu}(g)$. [4 marks]

Conductivity tensor of a semiconducting material is a symmetric second rank cartesian tensor and transforms according the relevant symmetric product representations of the symmetry group. The character of symmetric and anti-symmetric product representations may be written as

$$\chi^{\pm}(g) = \frac{1}{2}\left[\chi^{\mu}(g)^2 \pm \chi^{\mu}(g^2)\right]$$

where μ refers to the vector representation. A two dimensional semiconductor structure has the symmetry group of C_{6v} whose character table is given below:

	E	C_2	$2C_3$	$2C_6$	$3\sigma_d$	$3\sigma_v$
Γ_1	1	1	1	1	1	1
Γ_2	1	1	1	1	−1	−1
Γ_3	1	−1	1	−1	1	−1
Γ_4	1	−1	1	−1	−1	1
Γ_5	2	−2	−1	1	0	0
Γ_6	2	2	−1	−1	0	0

where the label of the class has their usual meaning. Γ_5 is the vector representation for transformation of $\{x, y\}$ coordinates. The order of group $|G| = 12$.

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where μ refers to the vector representation.

(iii) (a) Find the decomposition of symmetric and antisymmetric direct product representation from $\Gamma_5 \otimes \Gamma_5$ into irreducible representations. State clearly all the theorems or formulae you use but you don't need to prove them.

[5 marks]

[This question continues on the next page . . .]

(b) Determine the number of independent constants that uniquely specifies the conductivity tensor in this material and write down its general form.

[5 marks]

[Total 20 marks]

5. (i) (a) Define Lie groups and their dimensions. [4 marks]

(b) Define a Lie Algebra. [3 marks]

(c) State, without proof, the matrix representation $D(\epsilon)$ of an element of Lie Group specified by parameter $\boldsymbol{\epsilon}$ in terms of the matrix representation of the element of Lie Algebra $\Gamma(\boldsymbol{\epsilon})$. [2 marks]

(d) State, without proof, the generator of Lie Algebra in terms of the representation matrix of a Lie group. [2 marks]

The exponential of an matrix $\boldsymbol{\Gamma}$ can be defined as

$$\exp\{\boldsymbol{\Gamma}\} = \sum_{n=0}^{\infty} \frac{\boldsymbol{\Gamma}^n}{n!}$$

The Campbell-Baker-Hausdroff (CBH) formula relates the products of two matrix exponentials to a third one

$$\begin{aligned}\exp\{\boldsymbol{A}\}\exp\{\boldsymbol{B}\} &= \exp\{\boldsymbol{C}\}\\ \boldsymbol{C} &= \boldsymbol{A} + \boldsymbol{B} + \frac{1}{2}[\boldsymbol{A}, \boldsymbol{B}] + \frac{1}{12}\{[\boldsymbol{A}, [\boldsymbol{A}, \boldsymbol{B}]] + [\boldsymbol{B}, [\boldsymbol{B}, \boldsymbol{A}]]\} + \cdots\end{aligned}$$

where higher order terms involve higher orders of commutators of $\boldsymbol{A}$ and $\boldsymbol{B}$.

(ii) (a) By making use of CBH, show that if the representation of a Lie group is unitary, then the representation of the Lie Algebra is anti-hermittian. [5 marks]

(b) Using result of part (a), deduce the dimensions of $SO(3)$ group and $SO(4)$ group. [4 marks]

[Total 20 marks]