Imperial College London

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2012

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

M1S

Probability and Statistics

Date: Monday, 21st May 2012 Time: 10 am – 12 noon

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

Statistical tables will not be available.

Formula sheets are provided on pages 4 & 5.

1. (a) (i) State the three axioms of probability for events defined on a sample space Ω.

(ii) Prove from the axioms that if $E \subset F$ then $\mathsf{P}(E) < \mathsf{P}(F)$, where $E, F \subseteq \Omega$.

(iii) Prove that

$$\mathsf{P}(A \cup B \cup C) = 1 - \mathsf{P}(A' \mid B' \cap C')\mathsf{P}(B' \mid C')\mathsf{P}(C').$$

(b) The discrete random variables X and Y are independently, identically distributed with range $\{-1, 1\}$ and $f_X(1) = 1/2$.

(i) Find the pmf of $Z = XY$.

(ii) Show that X, Y and Z are pairwise independent.

(iii) Are they mutually independent?

(c) Events E and F have $\mathsf{P}(E) = 5/6$ and $\mathsf{P}(F) = 1/4$.

(i) Show that

$$\frac{1}{12} \leq \mathsf{P}(E \cap F) \leq \frac{1}{4}.$$

(ii) Determine bounds for $\mathsf{P}(E \cup F)$.

2. (a) A biased coin with probability θ of obtaining a Head is flipped repeatedly. Let X be the discrete random variable representing the length of the initial run (i.e. the number of successive heads, including the first, if the first flip shows a Head. On the other hand, the number of successive Tails, including the first, if the first flip shows a Tail).

(i) Find and name the conditional distribution of X given the first flip is a Head (event H).

(ii) What is $\mathsf{E}_{f_{X|H}}(X \mid H)$?

(iii) Prove that the expected length of the initial run is given by

$$E_{f_X}(X) = \frac{1}{\theta(1-\theta)} - 2.$$

You may use the following result without proof:
$\mathsf{E}_{f_X}(X) = \mathsf{E}_{f_{X|H}}(X \mid H)\mathsf{P}(H) + \mathsf{E}_{f_{X|H'}}(X \mid H')\mathsf{P}(H')$.

(b) Two independent biased coins both with probability θ of obtaining a Head are flipped n times. Let X and Y be the respective number of Heads out of the n for the two coins.

(i) Name the distribution of X and Y, identifying its parameters.

(ii) Find the pmf of $Z = X + Y$.

(iii) Find the conditional pmf of X given that $Z = N,\ N = 0, 1, \ldots, 2n$. Explain why this takes the form of a hypergeometric distribution.

3. (a) The number of students, X, arriving in the 10 minutes before the start of a particular lecture follows a Poisson distribution with mean 150. The number, Y, arriving in the 10 minutes after the start of the lecture follows a Poisson distribution with mean 20.

(i) Prove that the pgf of X is given by

$$G_X(t) = \exp\{150(t-1)\}.$$

(ii) Assuming X and Y are independent, find the pmf of $Z = X + Y$.

(iii) What is the pmf of Z?

(iv) Find the conditional pmf of X given that $Z = n,\ n = 0, 1, 2, \ldots$.

(b) The continuous random variable X has a uniform distribution on $[\alpha, \beta]$, for $\alpha < \beta$.

(i) Prove that the mean and variance of X are given by

$$\mathsf{E}_{f_X}(X) = \frac{(\alpha+\beta)}{2}, \qquad \mathsf{var}_{f_X}(X) = \frac{(\beta-\alpha)^2}{12}.$$

(ii) Prove that the cdf of X is given by

$$F_X(x) = \begin{cases} 0 & x < \alpha \\ \frac{x-\alpha}{\beta-\alpha} & \alpha \le x \le \beta \\ 1 & x > \beta \end{cases}$$

(iii) By considering the cdf of Y, determine the form of the monotonically decreasing function g such that $Y = g(X)$ follows an exponential distribution with $\lambda = 1$.

4. The continuous random variables X and Y have joint pdf given by

$$f_{X,Y}(x,y) = \begin{cases} x\mathrm{e}^{-x(y+1)} & x, y \ge 0, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Determine (i) $f_X(x)$, the marginal pdf of X and (ii) $f_Y(y)$, the marginal pdf of Y.

(b) Are X and Y independent?

(c) Find $\mathsf{E}_{f_{X|Y}}(X \mid Y = y)$.

(d) Prove the general result for any continuous random variables X and Y that

$$\mathsf{E}_{f_Y}(\mathsf{E}_{f_{X|Y}}(X \mid Y = y)) = \mathsf{E}_{f_X}(X).$$

(e) Determine $\mathsf{E}_{f_X}(X)$ directly from $f_X(x)$ and show that this gives the same result as that determined using the result of part (d).

DISCRETE DISTRIBUTIONS

	RANGE $\mathbb{X}$	PARAMETERS	MASS FUNCTION f_X	CDF F_X	$\mathsf{E}_{f_X}[X]$	$\mathsf{Var}_{f_X}[X]$	MGF M_X
$Bernoulli(\theta)$	$\{0,1\}$	$\theta \in (0,1)$	$\theta^x(1-\theta)^{1-x}$		θ	$\theta(1-\theta)$	$1-\theta+\theta e^t$
$Binomial(n,\theta)$	$\{0,1,...,n\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{n}{x}\theta^x(1-\theta)^{n-x}$		$n\theta$	$n\theta(1-\theta)$	$\left(1-\theta+\theta e^t\right)^n$
$Poisson(\lambda)$	$\{0,1,2,...\}$	$\lambda \in \mathbb{R}^+$	$\dfrac{e^{-\lambda}\lambda^x}{x!}$		λ	λ	$\exp\left\{\lambda\left(e^t-1\right)\right\}$
$Geometric(\theta)$	$\{1,2,...\}$	$\theta \in (0,1)$	$(1-\theta)^{x-1}\theta$	$1-(1-\theta)^x$	$\dfrac{1}{\theta}$	$\dfrac{(1-\theta)}{\theta^2}$	$\dfrac{\theta e^t}{1-e^t(1-\theta)}$
$NegBinomial(n,\theta)$	$\{n,n+1,...\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{x-1}{n-1}\theta^n(1-\theta)^{x-n}$		$\dfrac{n}{\theta}$	$\dfrac{n(1-\theta)}{\theta^2}$	$\left(\dfrac{\theta e^t}{1-e^t(1-\theta)}\right)^n$
or	$\{0,1,2,...\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{n+x-1}{x}\theta^n(1-\theta)^x$		$\dfrac{n(1-\theta)}{\theta}$	$\dfrac{n(1-\theta)}{\theta^2}$	$\left(\dfrac{\theta}{1-e^t(1-\theta)}\right)^n$

For **CONTINUOUS** distributions (see over), define the **GAMMA FUNCTION**

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx$$

and the LOCATION/SCALE transformation $Y = \mu + \sigma X$ gives

$$f_Y(y) = f_X\left(\frac{y-\mu}{\sigma}\right)\frac{1}{\sigma} \qquad F_Y(y) = F_X\left(\frac{y-\mu}{\sigma}\right) \qquad M_Y(t) = e^{\mu t}M_X(\sigma t) \qquad \mathsf{E}_{f_Y}[Y] = \mu + \sigma\mathsf{E}_{f_X}[X] \qquad \mathsf{Var}_{f_Y}[Y] = \sigma^2\mathsf{Var}_{f_X}[X]$$

CONTINUOUS DISTRIBUTIONS

	$\mathbb{X}$	PARAMS.	PDF f_X	CDF F_X	$\mathsf{E}_{f_X}[X]$	$\mathsf{Var}_{f_X}[X]$	MGF M_X
$Uniform(\alpha,\beta)$ (stand. model $\alpha=0,\beta=1$)	(α,β)	$\alpha<\beta\in\mathbb{R}$	$\dfrac{1}{\beta-\alpha}$	$\dfrac{x-\alpha}{\beta-\alpha}$	$\dfrac{(\alpha+\beta)}{2}$	$\dfrac{(\beta-\alpha)^2}{12}$	$\dfrac{e^{\beta t}-e^{\alpha t}}{t(\beta-\alpha)}$
$Exponential(\lambda)$ (stand. model $\lambda=1$)	$\mathbb{R}^+$	$\lambda\in\mathbb{R}^+$	$\lambda e^{-\lambda x}$	$1-e^{-\lambda x}$	$\dfrac{1}{\lambda}$	$\dfrac{1}{\lambda^2}$	$\left(\dfrac{\lambda}{\lambda-t}\right)$
$Gamma(\alpha,\beta)$ (stand. model $\beta=1$)	$\mathbb{R}^+$	$\alpha,\beta\in\mathbb{R}^+$	$\dfrac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}$		$\dfrac{\alpha}{\beta}$	$\dfrac{\alpha}{\beta^2}$	$\left(\dfrac{\beta}{\beta-t}\right)^\alpha$
$Weibull(\alpha,\beta)$ (stand. model $\beta=1$)	$\mathbb{R}^+$	$\alpha,\beta\in\mathbb{R}^+$	$\alpha\beta x^{\alpha-1}e^{-\beta x^\alpha}$	$1-e^{-\beta x^\alpha}$	$\dfrac{\Gamma(1+1/\alpha)}{\beta^{1/\alpha}}$	$\dfrac{\Gamma\left(1+\frac{2}{\alpha}\right)-\Gamma\left(1+\frac{1}{\alpha}\right)^2}{\beta^{2/\alpha}}$	
$Normal(\mu,\sigma^2)$ (stand. model $\mu=0,\sigma=1$)	$\mathbb{R}$	$\mu\in\mathbb{R},\sigma\in\mathbb{R}^+$	$\dfrac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\dfrac{(x-\mu)^2}{2\sigma^2}\right\}$		μ	σ^2	$e^{\{\mu t+\sigma^2t^2/2\}}$
$Student(\nu)$	$\mathbb{R}$	$\nu\in\mathbb{R}^+$	$\dfrac{(\pi\nu)^{-\frac{1}{2}}\Gamma\left(\dfrac{\nu+1}{2}\right)}{\Gamma\left(\dfrac{\nu}{2}\right)\left\{1+\dfrac{x^2}{\nu}\right\}^{(\nu+1)/2}}$		0 (if $\nu>1$)	$\dfrac{\nu}{\nu-2}$ (if $\nu>2$)	
$Pareto(\theta,\alpha)$	$\mathbb{R}^+$	$\theta,\alpha\in\mathbb{R}^+$	$\dfrac{\alpha\theta^\alpha}{(\theta+x)^{\alpha+1}}$	$1-\left(\dfrac{\theta}{\theta+x}\right)^\alpha$	$\dfrac{\theta}{\alpha-1}$ (if $\alpha>1$)	$\dfrac{\alpha\theta^2}{(\alpha-1)(\alpha-2)}$ (if $\alpha>2$)	
$Beta(\alpha,\beta)$	$(0,1)$	$\alpha,\beta\in\mathbb{R}^+$	$\dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$		$\dfrac{\alpha}{\alpha+\beta}$	$\dfrac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	