Imperial College London

[E1.14 (Maths 2) 2013]

B.ENG. AND M.ENG. EXAMINATIONS 2013

PART I : MATHEMATICS 2 (ELECTRICAL ENGINEERING)

Date Friday 31st May 2013 10.00 - 12.00

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.

Answer Question 1 and THREE of the remaining five questions.

Answer Section A and Section B in different answerbooks.

Question 1 carries twice the marks of each of the other questions.

CALCULATORS MAY **NOT** BE USED.

A mathematical formulae sheet is provided.

[*Before starting, please make sure that the paper is complete; there should be 4 pages, with a total of SIX questions. Ask the invigilator for a replacement if your copy is faulty.*]

1. (i) Find x such that the vectors $\boldsymbol{a} = (1,1,1)$ and $\boldsymbol{b} = (x,2,3)$ are perpendicular.

(ii) Let
$$
\begin{aligned}
A &= \{x \in \mathbb{R} | -2 < x \leq 2\} \\
B &= \left\{q \in \mathbb{Z} | \ \ q^2 < 10\right\}
\end{aligned}
$$
Determine $A \cap B$ and $A \cup B$.

(iii) Determine for which fixed $b \in \mathbb{R}$ the following propositions are true:

(a) $\exists\, x \in \mathbb{R} \quad \wedge \quad x^2 + x + b = 0$
(b) $\exists\, x \in \mathbb{C} \quad \wedge \quad x^2 + x + b = 0$

(iv) Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Determine the matrix B such that $AB = I$

(v) Construct the truth table for
$$(P \oplus Q) \wedge (P \vee Q)$$

(vi) Show that $f(x,y) = xe^{xy}$ satisfies the equation
$$\frac{\partial f}{\partial x} + \frac{\partial^2 f}{\partial y^2} - (x^2 + y)f = e^{xy}$$

(vii) Suppose function $f(x)$ is even and periodic with period 2π. Show that its Fourier series contains cos-terms only.

(viii) Determine t such that the point $P = (1,2,t)$ belongs to the plane through the points $A = (1,0,0)$, $B = (0,1,0)$ and $C = (0,0,1)$.

(ix) Determine the Taylor expansion to second order (without remainder term) about the point $(x_0, y_0) = (1,0)$ of the function
$$f(x,y) = \ln(x^2 + y)$$

(x) Calculate the angle between the vector $\boldsymbol{c} = (1,1,0)$ and the vector normal to the two vectors
$$\boldsymbol{a} = (1,-1,0)$$
$$\boldsymbol{b} = (0,-1,1).$$

PLEASE TURN OVER

2. (i) Let $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5\}$.
   Determine the sets $A \cap B$, $B - A$, $A \cup B$ and $P(B)$.

   (ii) Write down the truth tables for $P \Rightarrow Q$ and $P \wedge Q$.

   (iii) Determine the truth value of the following propositions:
   (a) $(\forall x > 0 \quad \wedge \quad \forall y < 0) \Rightarrow xy < 0$
   (b) $(\exists f : A \mapsto B \mid f \text{ is injective }) \quad \wedge \quad (|A| > |B|)$.

   (iv) Use proof by induction to show that

$$\sum_{k=0}^{n} 3^k = \frac{3^{n+1} - 1}{2}, \quad \forall n \in \mathbb{N}.$$

3. Consider the function $f : \mathbb{R}^2 \mapsto \mathbb{R}$ given by $f(x, y) = x^2 y - y - \frac{1}{2}y^2$.

   (i) Find all stationary points.
   (ii) Classify all the stationary points.
   (iii) Sketch a contour plot of the function. (Neatness is essential)

4. (i) Let f and g be differentiable functions on $\mathbb{R}$. Show that

$$u(x, t) = f(x - vt) + g(x + vt),$$

   where $v \in \mathbb{R}$ is a constant, solves the wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 u}{\partial t^2}$$

   (ii) Show that the function

$$f(x, y, t) = \frac{1}{t} e^{-\frac{x^2 + y^2}{4t}}$$

   solves the heat equation

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

   (iii) Show that

$$(\sin x \sin y - x^2) \frac{dy}{dx} - (2xy + \cos x \cos y) = 0$$

   is exact and derive an implicit expression for y as a function of x.

5. (i) A periodic function of period 4 is defined by

$$f(x) = 4 - x^2 \quad \text{for } -2 \leq x \leq 2$$
$$\text{and} \quad f(x+4) = f(x)$$

Sketch the function over three periods.

(ii) Find the Fourier expansion of $f(x)$.

(iii) Use the result in (ii) to derive the formula

$$\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

6. Let

$$M = \begin{pmatrix} -2 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 6 \end{pmatrix}.$$

(i) Write down the characteristic equation for M and show that the eigenvalues are equal to $\lambda = 1, -3, 7$.

(ii) Find the eigenvectors corresponding to $\lambda = 1, -3, 7$.

(iii) Express the vector $\boldsymbol{a} = \begin{pmatrix} 0 \\ 5 \\ -2 \end{pmatrix}$ in terms of the eigenvectors found in (ii) and use this to compute $M\boldsymbol{a}$.

END OF PAPER

	EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2012-13	Course EE 1 Part 2
Question 1	TOPIC General	Marks & seen/unseen

Parts

(i) $\underline{a} \perp \underline{b} \Leftrightarrow \underline{a} \cdot \underline{b} = 0 \Rightarrow x + 2 + 3 = 0$
$\Rightarrow x = -5$ 4

(ii) Note $B = \{-3, -2, -1, 0, 1, 2, 3\}$, hence
$A \cap B = \{-1, 0, 1, 2\}$ 4
$A \cup B = \{-3, 3, -2 \le x \le 2\}$

(iii) (a) To make the proposition true we need a real x that satisfy $x^2 + x + b = 0$, which can only be done if the roots are real, i.e. $D = +1 - 4b \geq 0$
$\Rightarrow \underline{b \le \frac{1}{4}}$ 2

(b) Since $\mathbb{C} \not\subset \mathbb{R}$ and since the polynomial always have either two complex or real roots this proposition is true for all values of $b \in \mathbb{R}$. 2

(iv) $\underline{\underline{B}} = \underline{\underline{A}}^{-1}$, $\det \underline{\underline{A}} = 1$, $\underline{\underline{A}}^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$ 4

Seen similar

/5

Question 1 TOPIC General

Marks & seen/unseen

Parts

(v)

P	Q	$P \oplus Q$	$P \vee Q$	$(P \oplus Q) \wedge (P \vee Q)$
T	T	F	T	F
T	F	T	T	T
F	T	T	T	T
F	F	F	F	F

4

(vi) $\partial_x f = e^{xy} + xye^{xy} = e^{xy} + yf$

$\partial_y^2 f = x^2 e^{xy} = x^2 f$

$\Downarrow$

$\partial_x f + \partial_y^2 f = x^2 f + yf + e^{xy}$. OK

4

(vii) General form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx \, dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{2\pi}\int_{-2\pi}^{2\pi} \underbrace{f(x)}_{\text{even}} \underbrace{\sin nx \, dx}_{\text{odd}}$$

(Since periodic: integrate over two periods and divide by 2

$= 0$

4

Seen similar

EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2012-13

Course EE1 Part 2

Question 1

TOPIC General

Marks & seen/unseen

Parts

$P = (1, 2, t)$

(viii) $A = (1,0,0)$, $B = (0,1,0)$, $C = (0,0,1)$

want $\underline{AP}$ to be perpendicular to a normal $\underline{n}$ to the plane.

Chose $\underline{n} = \underline{AB} \times \underline{AC}$ determine t such that $\underline{n} \cdot \underline{AP} = 0$: — 2

$\underline{AB} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, $\underline{AC} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, $\underline{AP} = \begin{pmatrix} 0 \\ 2 \\ t \end{pmatrix}$

$\underline{n} = \underline{AB} \times \underline{AC} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = (1, 1, 1)$

$0 = \underline{n} \cdot \underline{AP} = 2 + t \Rightarrow \underline{t = -2}$ — 2

seen similar

Setter's initials H.J.J.

Checker's initials

Page number S3

EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2012-13

Course EE1 Part2

Question 1

TOPIC General

Marks & seen/unseen

Parts (ix)

$f(x,y) = \ln(x^2+y) \quad , (x_0, y_0) = (1,0)$

$\partial_x f = \frac{2x}{x^2+y} \rightarrow \partial_x f|_{(1,0)} = 2$

$\partial_y f = \frac{1}{x^2+y} \rightarrow \partial_y f|_{(1,0)} = 1$

$\partial_x^2 f = \frac{2(x^2+y) - 2x \cdot 2x}{(x^2+y)^2} \rightarrow \partial_x^2 f|_{(1,0)} = -2$

$\partial_y^2 f = -\frac{1}{(x^2+y)^2} \rightarrow \partial_y^2 f|_{(1,0)} = -1$

$\partial_{xy}^2 f = \frac{-2x}{(x^2+y)^2} \rightarrow \partial_{xy}^2 f|_{(1,0)} = -2$ — 2

Taylor expan:

$f(x,y) = f(x_0,y_0) + \partial_x f\,(x-x_0) + \partial_y f\,(y-y_0)$

$+ \frac{1}{2}\left[\partial_x^2 f\,(x-x_0)^2 + 2\partial_{xy}^2 f\,(x-x_0)(y-y_0)\right.$

$\left. + \partial_y^2 f\,(y-y_0)^2\right]$

$= 0 + 2(x-1) + y + \frac{1}{2}\left[-2(x-1)^2\right.$

$\left. + 2(-2)(x-1)y - y^2\right]$

$= (x-1)[3 - 4y - x] + y(1-\cancel{y})$ $\wedge \frac{1}{2}$ — 2

$= (x-1)[3-4y-x] + y(1-\frac{1}{2}y)$

Seen similar

Setter's initials H.J. Checker's initials h.L.

	EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2012-13	Course EE 1 Part 2
Question 1	TOPIC General	Marks & seen/unseen

Parts

(x) The normal to the plane given by the vectors $\underline{a}$ and $\underline{b}$ is given by

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \end{vmatrix}$$

$= (-1, -1, -1)$ or use normal $\underline{n} = (1,1,1)$

$$\text{Angle} = \arccos\left(\frac{\underline{c} \cdot \underline{n}}{|\underline{c}||\underline{n}|}\right)$$

$$= \arccos\left(\frac{2}{\sqrt{2}\sqrt{3}}\right) = \arccos\left(\frac{2}{\sqrt{6}}\right)$$

$\approx$ ~~[illegible]~~ $\simeq$ ~~[illegible]~~

≈ 0.61548 rad $\approx 35.26°$

unseen 4

40

Setter's initials [illegible] Checker's initials [illegible]

Page number S5

EXAMINATION QUESTIONS/SOLUTIONS 2012-13

Course: EE1 Part 2

Question: 2

TOPIC: Discrete Math

Marks & seen/unseen

Parts

(i)

$A \cap B = \{3,4\}$ (1)

$B - A = \{5\}$ (1)

$A \cup B = \{1,2,3,4,5\}$ (1)

$P(B) = \{\emptyset, \{3\}, \{4\}, \{5\}, \{3,4\}, \{3,5\}, \{4,5\}, \{3,4,5\}\}$ (2)

5

5

(ii) (iii) → ~~6~~ 4

P	Q	$P \Rightarrow Q$
T	T	T
T	F	~~T~~ F
F	T	~~F~~ T
F	F	~~F~~ T

(2)

So (a) is true since $\forall x>0 \wedge \forall y<0$ is true when x is positive and y is negative in which case xy is negative and therefore $xy<0$ true. (3)

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

(2)

(b) is false since if f is injective from A to B $|A| \leq |B|$. Hence we cannot have $\exists f: A \to B \mid f$ is injective and $|A| > |B|$ true at the same time (3)

4

Similar seen

Setter's initials

Checker's initials

Page number S6

Should really start with $n=1$ (but allow $n=0$) since $n \in \mathbb{N}$ in Q.

Question 1

TOPIC Discrete Maths

Marks & seen/unseen

Parts

(iii) For $n = 0$ we have Left hand side $= 1$ and Right hand side $= \frac{3-1}{2} = 1$

Induction step: per assumption

$$\sum_{k=0}^{n+1} 3^k = \sum_{k=0}^{n} 3^k + 3^{n+1} = \frac{3^{n+1}-1}{2} + 3^{n+1}$$

$$= \frac{3^{n+1} - 1 + 2\, 3^{n+1}}{2} = \frac{3^{n+1}(1+2) - 1}{2}$$

$$= \frac{3^{n+2}-1}{2}. \quad \text{OK}$$

unseen 5

5

20

EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2012-13

Course EE1 Part 2

Question 3

TOPIC Partial Derivatives

Marks & seen/unseen

Parts

(i) Stationary points

$\frac{\partial f}{\partial x} = 2xy = 0 \quad (a)$

$\frac{\partial f}{\partial y} = x^2 - 1 - y = 0 \quad (b)$

from (a) $x = 0$ or $y = 0$

If $x = 0$ then (b) $\Rightarrow y = -1$

If $y = 0$ then (b) $\Rightarrow x = \pm 1$

I.e. three stationary points

$(0,-1)$, $(1,0)$ and $(-1,0)$ — 5

seen similar 5

(ii) Classification

$\partial_x^2 f = 2y$, $\partial_y^2 f = -1$, $\partial_{xy}^2 f = 2x$

$\Delta = \partial_x^2 f \, \partial_y^2 f - [\partial_{xy}^2 f]^2 = -2y - 4x^2$ — 5

5

Point	$\partial_x^2 f$	$\partial_y^2 f$	Δ	classification
$(0,-1)$	$-2<0$	-1	$\cancel{0}2$	Max $f = \frac{1}{2}$
$(1,0)$	0	-1	-4 $\cancel{2}$	saddle $f = 0$
$(-1,0)$	0	-1	-4 $\cancel{+2}$	saddle $f = 0$

5

5

EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2012-13

Course: EE1 Part 2

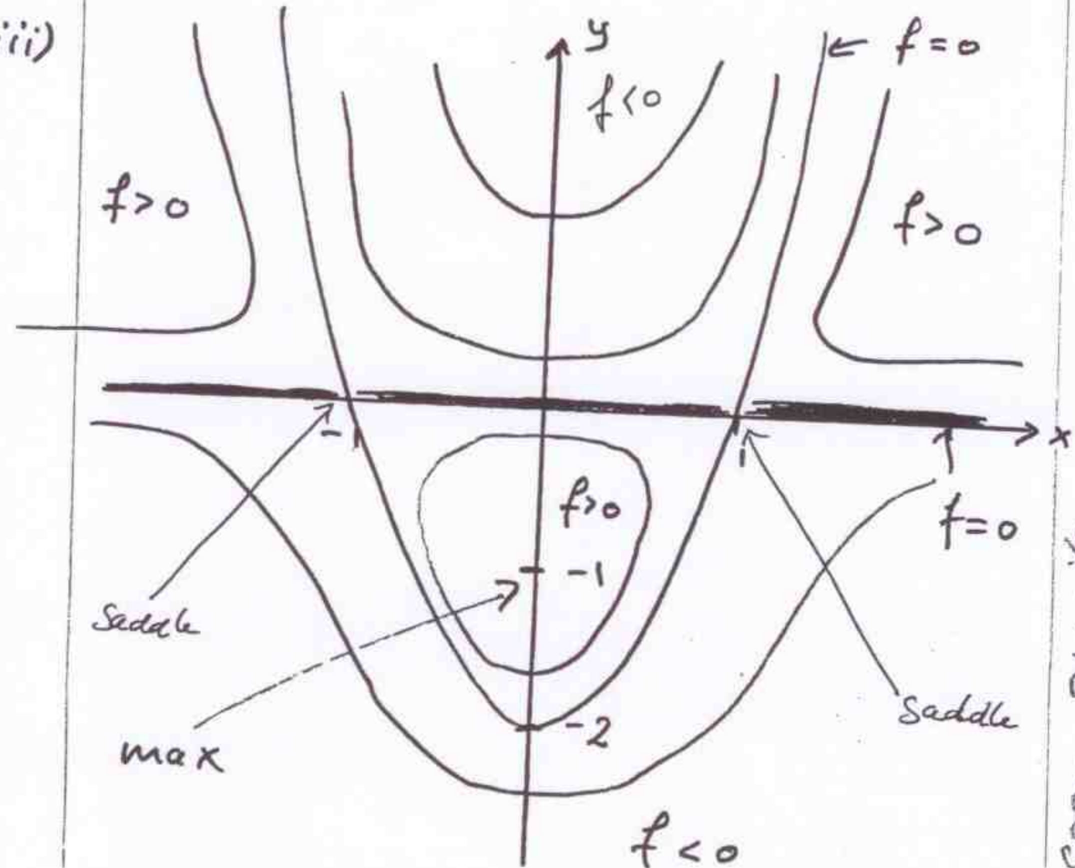
Question: 3

TOPIC: Partial Derivatives

Marks & seen/unseen

Parts

(iii)



3

$$f(x,y) = x^2 y - y - y^2/2 = 0$$

$\Rightarrow$ $y = 0$ (1) or if $y \neq 0$

$$x^2 - 1 - \frac{y}{2} = 0$$

$$\Rightarrow y = 2x^2 - 2 \quad (1)$$

5

5

seen similar

20

Setter's initials: H.J.

Checker's initials: L.L.

Page number: S9

EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2012-13

Course: EE1 Part 2

Question: 4

TOPIC: Diff. Eq.

Marks & seen/unseen

Parts

(i) $u(x,t) = f(x-vt) + g(x+vt)$

$$\partial_x u = f'(x-vt) + g'(x+vt)$$

$$\partial_x^2 u = f''(x-vt) + g''(x+vt)$$

$$\partial_t u = -v f'(x-vt) + v g'(x+vt)$$

$$\partial_t^2 u = v f''(x-vt) + v^2 g''(x+vt)$$

hence

$$\frac{1}{v^2} \partial_t^2 u = f''(x-vt) + g''(x+vt)$$

$$= \partial_x^2 f$$

5 5

seen similar 5

(ii) $f(x,y,t) = \frac{1}{t} e^{-\frac{x^2+y^2}{4t}}$

$$\partial_t f = \left[-\frac{1}{t^2} + \frac{1}{t}\left(\frac{x^2+y^2}{4t^2}\right)\right] e^{-\frac{x^2+y^2}{4t}}$$

$$= \left[-\frac{1}{t} + \frac{x^2+y^2}{4t^2}\right] f$$

$$\partial_x f = \frac{1}{t}\left(-\frac{2x}{4t}\right) e^{-\frac{x^2+y^2}{4t}} = -\frac{x}{2t} f$$

$$\partial_x^2 f = -\frac{1}{2t} f + \left(\frac{x}{2t}\right)^2 f$$

$$= \left(-\frac{1}{2t} + \frac{x^2}{4t^2}\right) f$$

Setter's initials: H.J.

Checker's initials

EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2012-13

Course EE1

Part 2

Question 4

TOPIC Differential Equation

Marks & seen/unseen

Parts

(ii) cont.

By symmetry

$$\partial_y^2 f = \left[-\frac{1}{2t} + \frac{y^2}{4t^2}\right] f$$

Therefore

$$(\partial_x^2 + \partial_y^2) f = \left(-\frac{1}{t} + \frac{x^2+y^2}{4t^2}\right) f$$

$$= \partial_t f$$

5

seen similar

5

Setter's initials HJ.

Checker's initials

EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2012-13

Course: EE1 Part 2

Question 4

TOPIC: ODE

Marks & seen/unseen

Parts

(i) $(x+1)\frac{dy}{dx} - 3y = (x+1)^5$

$\Downarrow$

$\frac{dy}{dx} - \frac{3}{x+1}y = (x+1)^4$

I.e. $P(x) = -\frac{3}{x+1}$ and $Q(x) = (1+x)^4$

Hence

$I(x) = \exp\{-\int \frac{3}{x+1}dx\} = \exp\{-3\ln|x+1|\}$

$= \pm(x+1)^{-3}$

We multiply through and get

$(x+1)^{-3}\frac{dy}{dx} - \frac{3}{(x+1)^4}y = (x+1)$

$\Downarrow$

$\frac{d}{dx}\left[\frac{y}{(x+1)^3}\right] = x+1 \Rightarrow \frac{y}{(x+1)^3} = \frac{1}{2}(x+1)^2 + C$

$\Downarrow$

$y = \frac{1}{2}(x+1)^5 + C(x+1)$

(ii) $(\sin x \sin y - x^2)\frac{dy}{dx} - (2xy + \cos x \cos y) = 0$

$P = -(2xy + \cos x \cos y)$

$Q = \sin x \sin y - x^2$

Check:

$\frac{\partial P}{\partial y} = -2x + \cos x \sin y$

$\frac{\partial Q}{\partial x} = \cos x \sin y - 2x$ i.e. $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

hence exact

4

Setter's initials: H.H.

Checker's initials: L.L.

Page number: 5/2

Parts

Marks & seen/unseen

(iii cont) Determine $u(x,y)$:

$$\frac{\partial u}{\partial x} = P = -2xy - \cos x \cos y$$

$$\Rightarrow u = -x^2 y - \sin x \cos y + g(y)$$

$$\frac{\partial u}{\partial y} = Q = \sin x \sin y - x^2$$

$$\Rightarrow u = -\sin x \cos y - x^2 y + f(x)$$

by comparison we conclude

$f(x) = g(y)$ for all x and y and therefore

$$f(x) = g(y) = \text{constant} \equiv C$$

$$\Downarrow$$

$$u(x,y) = -x^2 y - \sin x \cos y + C.$$

The general solution for y as function of x is given implicitly by $u(x,y) =$ constant or

$$x^2 y + \sin x \cos y = \text{constant}$$

seen similar

8

3

20

EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2012-13

Course EE1 Part 2

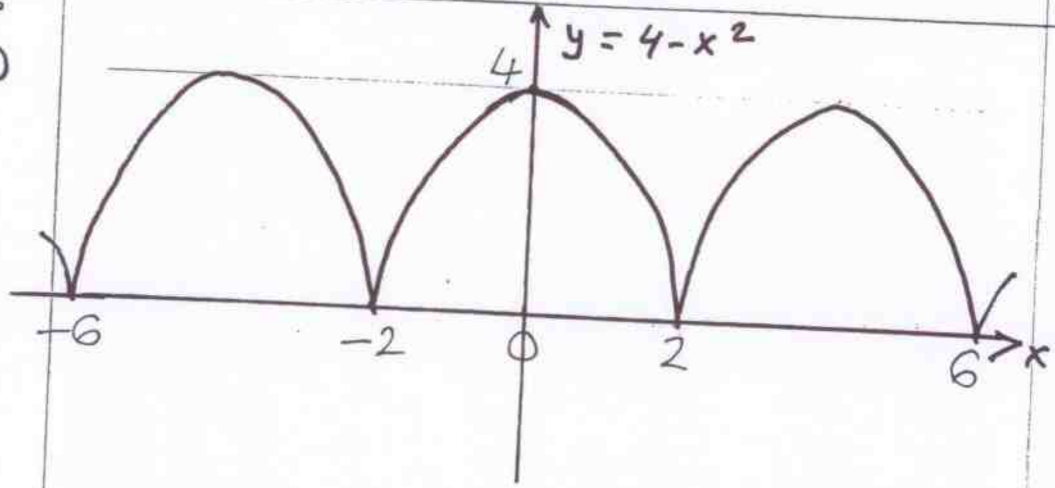
Question 5

TOPIC Fourier Series

Marks & seen/unseen

Parts

(i)



3 for graph

(ii) $f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$

$a_n = \frac{1}{L}\int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}$ Here

$b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}$ $L = 2$

All $b_n = 0$ since $f(x)$ is even ← 2

$a_n = \frac{1}{2}\int_{-2}^{2} (4-x^2) \cos \frac{n\pi x}{2}\, dx$ ← 2

$= 2\int_{-2}^{2} \cos \frac{n\pi x}{2}\, dx - \frac{1}{2}\int_{-2}^{2} x^2 \cos \frac{n\pi x}{2}\, dx$

Integration by parts

$v = x^2$, $v' = 2x$

$u' = \cos \frac{n\pi x}{2}$, $u = \frac{2}{n\pi} \sin \frac{n\pi x}{2}$

Seen similar

Setter's initials

Checker's initials

Page number S14

Marks & seen/unseen

Parts

(iii cont)

$$a_0 = 4\int_0^2 dx - \int_0^2 x^2 dx = 8 - \left[\frac{x^3}{3}\right]_0^2 = \frac{16}{3}$$ (2)

For $n \neq 0$ we have

$$\int_{-2}^{2} \cos\frac{n\pi x}{2}\, dx = 2\int_0^2 \cos\frac{n\pi x}{2}\, dx =$$

$$= 2\,\frac{2}{n\pi}\left[\sin\frac{n\pi x}{2}\right]_0^2 = \frac{4}{n\pi}\sin(n\pi) = 0$$

and

$$\int_{-2}^{2} x^2 \cos\frac{n\pi x}{2}\, dx = 2\int_0^2 x^2\cos\frac{n\pi x}{2}\, dx$$

$$= 2\left\{\left[x^2\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right]_0^2 - \int_0^2 2x\,\frac{2}{n\pi}\sin\frac{n\pi x}{2}\,dx\right\}$$

$$= -\frac{8}{n\pi}\int_0^2 x\sin\frac{n\pi x}{2}\,dx$$

$$= -\frac{8}{n\pi}\left\{\left[-x\,\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right]_0^2 + \underbrace{\int_0^2 \frac{2}{n\pi}\cos\frac{n\pi x}{2}\,dx}_{=0 \text{ from above}}\right\}$$

Seen similar 4

$$= \frac{16}{n^2\pi^2}\left[x\cos\frac{n\pi x}{2}\right]_0^2$$

$$= \frac{32}{n^2\pi^2}\cos(n\pi) = \frac{32}{n^2\pi^2}(-1)^n$$

and then

$$a_n = -\frac{1}{2}\,\frac{32}{n^2\pi^2}(-1)^n = \frac{16}{n^2\pi^2}(-1)^{n+1}$$ (4)

So

$$f(x) = \frac{8}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos\left(\frac{n\pi}{2}x\right)$$ (2)

EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2012-13

Course EE1 Part 2

Question 5

TOPIC

Marks & seen/unseen

Parts

(iii) From $f(0) = 4$ we get

$$4 = \frac{8}{3} + \frac{16}{\pi^2}\left(1 - \frac{1}{2^2} + \cdots\right.$$

$$\Downarrow$$

$$1 - \frac{1}{2^2} + \cdots = \frac{4 \cdot 3 - 8}{3} \cdot \frac{\pi^2}{16} = \frac{4}{3}\,\frac{\pi^2}{16}$$

$$= \frac{\pi^2}{12}.$$

unseen 5

5

20

Setter's initials

Checker's initials

Page number 5/6

EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2012-13

Course: EE1 Part 2

Question 6

TOPIC Linear Algebra

Parts | Marks & seen/unseen

(i) Characteristic eqn: $|\underline{\underline{M}} - \lambda \underline{\underline{I}}| = 0$

$$\begin{vmatrix} -2-\lambda & 2 & 1 \\ 2 & 1-\lambda & 2 \\ 1 & 2 & 6-\lambda \end{vmatrix} = 0$$

$$\Downarrow -(2+\lambda)(1-\lambda)(6-\lambda) + 4 + 4 - (1-\lambda) + 4(2+\lambda) - 4(6-\lambda) =$$

$$-(2+\lambda)(1-\lambda)(6-\lambda) - 9(1-\lambda) = 0$$

I.e. $\lambda = 1$ or $(2+\lambda)(6-\lambda) + 9 = 0$

$$\Downarrow \quad \lambda^2 - 4\lambda - 21 = 0$$

$$\Downarrow \quad \lambda = \tfrac{1}{2}[4 \pm \sqrt{16+84}]$$

factorizes $(\lambda-7)(\lambda+3)$ $\quad = \begin{cases} 7 \\ -3 \end{cases}$

5

(ii) Eigenvectors

$\underline{\lambda = 1}$ ① $-2x + 2y + z = x$

② $2x + y + 2z = y \longrightarrow x = -z$

③ $x + 2y + 6z = z$

hence from ① $-2x + 2y - x = x \Rightarrow y = 2x$

So an eigenvector is $\underline{v}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

$\lambda = -3$

① $-2x + 2y + z = -3x$ (1a) $x + 2y + z = 0$

② $2x + y + 2z = -3y$ $\Rightarrow$ (2a) $2x + 4y + 2z = 0$

③ $x + 2y + 6z = -3z$ (3a) $x + 2y + 9z = 0$

Steer Similar

Setter's initials: [signature]   Checker's initials: [signature]

Page number: 518

EXAMINATION QUESTIONS/SOLUTIONS 2012-13

Course: EE1 Part 2

Question: 6

TOPIC: Linear Algebra

Parts | | Marks & seen/unseen

(ii cont) (1a) and (2a) are equivalent.

(2a) -2 (3a) $\rightarrow -16z = 0 \Rightarrow z = 0$.

Then from (1a) $x = -2y$

An eigenvector $\underline{v}_{-3} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$ — 3

$\lambda = 7$

(1) $-2x + 2y + z = 7x$ | (1a) $-9x + 2y + z = 0$

(2) $2x + y + 2z = 7y$ $\Rightarrow$ (2a) $2x - 6y + 2z = 0$

(3) $x + 2y + 6z = 7z$ | (3a) $x + 2y - z = 0$

(2a) $+ 2$ (3a) $\Rightarrow 4x - 2y = 0 \Rightarrow x = \frac{1}{2}y$

Sub into (1a): $-\frac{9}{2}y + 2y + z = 0$

$\Rightarrow y = \frac{2}{5}z$

An eigenvector

$$\underline{v}_7 = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$$ — 3

Seen similar — 9 for eigenvectors

(iii) $\underline{\underline{M}}$ symmetric $\rightarrow$ Eigenvectors orthogonal. Normalise eigenvectors:

$\underline{e}_1 = \frac{\underline{v}_1}{|v_1|}$, $\underline{e}_2 = \frac{\underline{v}_{-3}}{|\underline{v}_{-3}|}$, $\underline{e}_3 = \frac{\underline{v}_7}{|v_7|}$

Setter's initials | Checker's initials | Page number 5/9

EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2012-13

Course EE1

Question 6

TOPIC Linear Algebra

Part 2

Parts

Marks & seen/unseen

(iii) cont.

$|\underline{v}_1| = \sqrt{1+4+1} = \sqrt{6} \rightarrow \underline{e}_1 = \frac{1}{\sqrt{6}}\begin{pmatrix}1\\2\\-1\end{pmatrix}$

$|\underline{v}_{-3}| = \sqrt{4+1} = \sqrt{5} \rightarrow \underline{e}_2 = \frac{1}{\sqrt{5}}\begin{pmatrix}2\\-1\\0\end{pmatrix}$

$|\underline{v}_7| = \sqrt{1+4+25} = \sqrt{30} \rightarrow \underline{e}_3 = \frac{1}{\sqrt{30}}\begin{pmatrix}1\\2\\5\end{pmatrix}$

<u>Expand</u>

$\underline{a} = x\,\underline{e}_1 + y\,\underline{e}_2 + z\,\underline{e}_3$

$x = \underline{a}\cdot\underline{e}_1 = \frac{1}{\sqrt{6}}(10+2) = \frac{12}{\sqrt{6}}$

$y = \underline{a}\cdot\underline{e}_2 = \frac{1}{\sqrt{5}}(-5) = -\frac{5}{\sqrt{5}}$

$z = \underline{a}\cdot\underline{e}_3 = \frac{1}{\sqrt{30}}\cdot 0 = 0$ i.e. $\begin{pmatrix}0\\5\\-2\end{pmatrix} = 2\begin{pmatrix}1\\2\\-1\end{pmatrix} - \begin{pmatrix}2\\-1\\0\end{pmatrix}$

unseen

4

$\underline{\underline{M}}\,\underline{a} = \frac{12}{\sqrt{6}}\underline{\underline{M}}\,\underline{e}_1 - \frac{5}{\sqrt{5}}\underline{\underline{M}}\,\underline{e}_2$

$= \frac{12}{\sqrt{6}}\underline{e}_1 - \frac{5}{\sqrt{5}}(-3)\,\underline{e}_2$

$= \frac{12}{\sqrt{6}}\frac{1}{\sqrt{6}}\begin{pmatrix}1\\2\\-1\end{pmatrix} + \frac{15}{\sqrt{5}}\frac{1}{\sqrt{5}}\begin{pmatrix}2\\-1\\0\end{pmatrix}$

$= \begin{pmatrix}2\\4\\-2\end{pmatrix} + \begin{pmatrix}6\\-3\\0\end{pmatrix} = \begin{pmatrix}8\\1\\-2\end{pmatrix}$

2

2

20

Setter's initials

Checker's initials