UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 2003

MEng Honours Degrees in Computing Part IV
MSc in Advanced Computing
PhD
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the Associateship of the City and Guilds of London Institute

PAPER C499

MODAL AND TEMPORAL LOGIC

Thursday 15 May 2003, 10:00
Duration: 120 minutes

Answer THREE questions

Paper contains 4 questions
Calculators not required

1a Let $\mathcal{F} = (W, R)$ be a Kripke frame. Prove that $\Box p \to \Diamond p$ is valid in $\mathcal{F}$ if, and only if, the possible-world relation R of $\mathcal{F}$ is *serial*: for all $w \in W$, there is some $w' \in W$ with $(w, w') \in R$.

b Consider the formula

$$\phi \stackrel{\text{def}}{=} \Diamond\Box\Box p \to \Diamond\Diamond p$$

of modal logic. First show that ϕ is equivalent to a Sahlqvist formula ψ and justify why ψ is a Sahlqvist formula. Second, apply Sahlqvist's algorithm to extract the frame condition $\forall t \neg\alpha(t)$ of ϕ in a form where quantifiers are moved to the front whenever possible and negation is applied to R only.

c Consider the Kripke structure M with $W = \{0, 1, 2, \ldots\}$, $R = \{(n, n+1) \mid n \geq 0\}$, $h(p) = \{0, 2, 4, \ldots\}$, $h(q) = \{1, 3, 5, \ldots\}$, and $h(r) = \{\}$ for every other atom $r \notin \{p, q\}$. Prove that there is a Kripke structure M' with a finite set of worlds W' and a world $0' \in W'$ such that

i) $M, 0$ is bisimilar to $M', 0'$ and
ii) the size of W' is minimal with respect to the existence of such a $0'$: in every Kripke structure with smaller set of worlds, no world is bisimilar to $M, 0$.

d For a Kripke structure $M = (W, R, h)$ consider $w \cong w'$ if, and only if, there is a bisimulation $B \subseteq W \times W$ between M, w and M, w'. Prove that $\cong$ is reflexive and transitive on W.

The four parts carry, respectively, 25%, 25%, 25%, and 25% of the marks.

2a Which of the formulas

$$\eta \stackrel{\text{def}}{=} \mathsf{E}\,[(p_1 \mathsf{U}\, q_1) \wedge (p_2 \mathsf{U}\, q_2)] \tag{1}$$
$$\gamma \stackrel{\text{def}}{=} \mathsf{E}[(p_1 \wedge p_2)\mathsf{U}(q_1 \wedge \mathsf{E}[p_2 \mathsf{U}\, q_2])] \vee \mathsf{E}[(p_1 \wedge p_2)\mathsf{U}(q_2 \wedge \mathsf{E}[p_1 \mathsf{U}\, q_1])], \tag{2}$$

if either, are formulas of CTL (computation-tree logic)? Justify your answer. Explain — using an appropriate mix of formal notation and plain English — when $M, s \models \eta$ holds for the η in (1).

b Determine whether for all Kripke structures M and all states s we have $M, s \models \eta$ if, and only if, $M, s \models \gamma$.

c Specification patterns are often expressible in CTL. For example,

> "There is an execution trace along which p holds invariably."

may be written as $\mathsf{EG}\,p$. Translate the following CTL formulas into specification patterns in plain English; if the last formula appears to be too complex, you may state a natural property that is *entailed* by this CTL formula.

i) $\mathsf{AG}\,(p \to \mathsf{AF}\,q)$
ii) $\mathsf{AG}\,(p \to \mathsf{EF}\,q)$
iii) $\mathsf{EF}(p \wedge \mathsf{E}[p\mathsf{U}\,(\neg p \wedge \mathsf{E}[\neg q \mathsf{U}\, p])])$.

d Consider the following SMV program

```
MODULE main
VAR
        a : boolean;
        b : boolean;
ASSIGN
        init(a) = 0;
        init(b) = 0;
        next(a) = !b;
        next(b) =
                  case
                        a | b : !b;
                        !a    : b;
                        1     : {0,1};
                  esac;
SPEC AG (AF b)
```

Draw the Kripke structure determined by this SMV program, indicate its initial states, and determine, justifying your answer, whether the specification of the program holds.

The four parts carry, respectively, 25%, 25%, 25%, and 25% of the marks.

3a Consider a card game for n players named $1, 2, \ldots, n$ ($n \geq 2$). Each player receives one of m cards ($m \geq n$). No two cards are the same. Each player can see its own card but not the card of any other player. Let $(c_1, c_2, \ldots, c_n)$ represent the state of the game in which player 1 holds card c_1, player 2 holds card c_2, ..., player n holds card c_n.

Use this example to explain how the formalism of *interpreted systems* can be used to model an information-theoretic notion of knowledge for analysing epistemic properties of distributed and multi-agent systems. Your answer should explain

i) how to define truth conditions for expressions of the form $\mathrm{K_i}\,A$ (agent i 'knows that' A) in the example;

ii) why each $\mathrm{K_i}$ is of type $S5$ ($= KT45$);

iii) how to formulate that when player i holds card c, i knows that player j ($j \neq i$) does not hold card c;

iv) how to show that this property holds in the example.

It is not necessary to model any temporal aspects of the game.

b Consider a language with modal operators O and $\Box$. Models are Kripke structures $\langle W, R_{\mathrm{O}}, R_{\Box}, h \rangle$ where W is a set of worlds, h is a valuation function for the atoms, and R_{O} and $R_{\Box}$ are the accessibility relations for the operators O and $\Box$ respectively.

Suppose further that O is a normal modality of type KD (R_{O} is serial) and $\Box$ is a normal modality of type KT ($R_{\Box}$ is reflexive).

Suppose now that the language is extended with another modal operator Oblig defined as follows:

$$\mathrm{Oblig}\,A \stackrel{\mathrm{def}}{=} \mathrm{O A} \land \neg\Box\mathrm{A}$$

Show that Oblig has the following properties

noN. $\neg\,\mathrm{Oblig}\,\top$

D. $\mathrm{Oblig}\,A \rightarrow \neg\,\mathrm{Oblig}\,\neg A$, i.e. $\neg(\mathrm{Oblig}\,A \land \mathrm{Oblig}\,\neg A)$

C. $(\mathrm{Oblig}\,A \land \mathrm{Oblig}\,B) \rightarrow \mathrm{Oblig}(A \land B)$

You may use any standard properties of normal logics, such as $\Box(A \land B) \rightarrow \Box A$, as long as you identify them clearly.

c i) Define what is meant by a *normal* system of modal logic and a *classical* system of modal logic.

ii) Demonstrate that the Oblig operator of part (b) is not normal.

iii) What is the structure of models and the associated truth conditions in *neigbourhood semantics* for classical systems?

iv) Identify a suitable model condition for a neighbourhood model that will validate the schema C.

The three parts carry, respectively, 35%, 35%, 30% of the marks.

4a Let Σ be a system (not necessarily normal) of modal logic.

i) Define Σ-consistency.

ii) Define what is meant by a maximal Σ-consistent set (a maxi-consistent set of Σ).

iii) State (without proof) Lindenbaum's Lemma.

b Explain what is meant by a *canonical model* for a *normal* system of modal logic Σ and how such a model can be used to prove completeness of Σ with respect to a given class of Kripke models.

Your answer should identify clearly (without proof) the key properties of canonical models required for such completeness proofs.

c The system $S4$ is a normal modal logic of type $KT4$, i.e., the smallest normal system containing the following schemas:

T. $\Box A \to A$

4. $\Box A \to \Box\Box A$

Prove completeness of $S4 = KT4$ with respect to the class of reflexive, transitive Kripke models by using the canonical model of $KT4$ whose accessibility relation R_c^{KT4} is defined as follows:

$$w \, R_c^{KT4} \, w' \quad \text{iff} \quad \{A \mid \Box A \in w\} \subseteq w'$$

(Assume, without proof, that this model is a canonical model for the system $KT4$.)

d The system $S4.2$ is a normal modal logic of type $KT4G$, i.e., the smallest normal system containing the schemas T and 4 and the following schema:

G. $\Diamond\Box A \to \Box\Diamond A$

A relation R is said to be *strongly convergent* when, for all w, w' there exists a v such that $w \, R \, v$ and $w' \, R \, v$.

Following the method of part (c), show that $S4.2 = KT4G$ is complete with respect to the class of reflexive, transitive, strongly convergent Kripke models.

You may assume without proof that

$$\{A \mid \Box A \in \Gamma\} \cup \{A \mid \Box A \in \Gamma'\}$$

is $KT4G$-consistent for any maximal $KT4G$-consistent sets Γ and Γ'.

The four parts carry, respectively, 25%, 25%, 25%, 25% of the marks.