EE3-09

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2015

EEE/EIE PART III/IV: MEng, BEng and ACGI

Corrected copy

CONTROL ENGINEERING

Friday, 18 December 9:00 am

Time allowed: 3:00 hours

Correction at the start of exam 9:00am

Q4 c) ii)

There are FOUR questions on this paper.

Answer ALL questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : A. Astolfi

Second Marker(s) : D. Angeli

CONTROL ENGINEERING

1. Consider a linear, single-input, single-output , discrete-time, system of dimension $n = 3$, that is $x = [x_1, x_2, x_3]'$, with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}, \quad D = 1,$$

and let x_0 be the initial state of the system, that is $x_0 = x(0)$.

a) Show that the system is non-reachable, controllable and observable. [8 marks]

b) Assume $u(0) = 0$. Determine all initial conditions $x(0)$ such that $x(1) = 0$. [2 marks]

c) Assume $u(0) = u(1) = 0$. Determine all initial conditions $x(0)$ such that $x(2) = 0$. [2 marks]

d) Let $x(0) = 0$. Assume that $y(0) = 0$, $y(1) = 0$ and $y(2) = 1$. Determine input values $u(0)$, $u(1)$ and $u(2)$ which generate this output sequence. Suppose now that the input sequence determined above is *completed* with the values $u(t) = 0$, for all $t \geq 3$. Determine the output sequence $y(t)$ for all $t \geq 3$. Explain why the output sequence is constant for all $t \geq 3$. [8 marks]

2. Consider a linear, continuous-time, system described by the equations

$$\dot{x}_1 = x_1 + \alpha x_2 + u,$$

$$\dot{x}_2 = x_1 + x_2 - 2\alpha x_2,$$

$$y = x_1,$$

with $\alpha \in \mathbb{R}$ and constant, $x(t) = [x_1(t), x_2(t)]' \in \mathbb{R}^2$ and $u(t) \in \mathbb{R}$.

a) Let $u = 0$. Compute the equilibrium points of the system as a function of α. [4 marks]

b) Assume now $u(t) = \bar{u}$, for all $t \geq 0$, with $\bar{u} \neq 0$. Compute the equilibrium points of the system as a function of α. [4 marks]

c) Discuss similarities and differences between the results in part a) and part b). [2 marks]

d) Study the stability properties of the system as a function of α. [6 marks]

e) Let $u = -ky$, with k constant. Write the equations of the closed-loop system and determine conditions on α and k such that the closed-loop system is asymptotically stable. Argue that if $\alpha > 1/2$ there always exists a k which renders the closed-loop system asymptotically stable. [4 marks]

3. Consider a linear, continuous-time, system with state $x(t) \in I\!R^n$, input $u(t) \in I\!R$ and output $y(t) \in I\!R$ described by the equations

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx. \end{aligned}$$

The system can be approximated by a discrete-time system which describes the evolution of the variables $x(t)$, $u(t)$ and $y(t)$ at the sampling times $t = kT$, with $k = 0, 1, 2, \ldots.$ and $T > 0$ the constant sampling period, using the equations

$$\begin{aligned} x(k+1) &= \left(I + TA + \frac{T^2}{2}A^2\right)x(k) + TBu(k) = A_d x(k) + B_d u(k), \\ y(k) &= Cx(k) = C_d x(k). \end{aligned}$$

The discrete-time approximation may however not preserve some of the structural properties of the original continuous-time system. To see how this happens, consider the case in which

$$A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

with λ constant.

a) Show that the continuous-time system with the given matrices A, B and C is controllable and observable for all values of λ. [4 marks]

b) Determine the matrices A_d, B_d and C_d of the discrete-time approximating system as a function of T. [4 marks]

c) Show that if $\lambda < 0$ there exists a value of T such that the discrete-time system is not reachable and not observable. [6 marks]

d) For the value of T determined in part c) write the equations of the approximating discrete-time system and argue that the input-output behaviour of the system is given by the relation $y(k) = 0$, for all $k \geq 0$.
(Hint: assume that $x(0) = 0$ and compute $y(k)$ for all $k \geq 0$.) [6 marks]

4. Consider a linear, continuous-time, system described by the equations

$$\begin{aligned}
\dot{x}_1 &= \lambda_1 x_1 + b_1 u, \\
\dot{x}_2 &= \lambda_2 x_2 + b_2 u, \\
&\vdots \\
\dot{x}_n &= \lambda_n x_n + b_n u,
\end{aligned}$$

where the variables $x_i(t) \in \mathbb{R}$, for all $i = 1, \cdots, n$, describe the state of the system and $u(t) \in \mathbb{R}$ is the input. The coefficients b_i, for all $i = 1, \cdots, n$, are constant and the coefficients λ_i, for all $i = 1, \cdots, n$, are also constant.

a) Write the matrices A and B of a state space description of the system.
[2 marks]

b) Determine conditions on the coefficients λ_i and b_i, for all $i = 1, \cdots, n$, such that the system is controllable.
Hint: use the PBH test. [10 marks]

c) Assume $n = 2$, $\lambda_1 = -1$, $\lambda_2 = 1$, $b_1 = 2$ and $b_2 = 1$.

i) Compute a state feedback control law $u = Kx$, with $x = [x_1, x_2]'$, such that the closed-loop system has all eigenvalues equal to -1.
[6 marks]

ii) Suppose that the first equation of the system is perturbed as

$$\dot{x}_1 = \lambda_1 x_1 + b_1 u + \delta x_2 = -x_1 + 2u + \delta x_2,$$ ⇐ correction

with δ constant. Show that the state feedback determined in part c.i) asymptotically stabilizes the perturbed closed-loop system for any δ.
[2 marks]