UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 2001

BEng Honours Degree in Computing Part II
MEng Honours Degrees in Computing Part II
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the Associateship of the City and Guilds of London Institute

PAPER C233

COMPUTATIONAL TECHNIQUES

Thursday 3 May 2001, 14:00
Duration: 90 minutes
(Reading time 5 minutes)

Answer THREE questions

Paper contains 4 questions
Calculators required

1a For each of the following statements, determine whether it is true or false. Justify your answer.

(i) Any set of m vectors containing the null vector is linearly dependent.

(ii) The dot product of two, linearly dependent, nonzero vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^m$ is always equal to zero.

(iii) If $\det(\mathbf{A}) < 0$ then at least one eigenvalue of $\mathbf{A}$ is negative.

b Solve $\mathbf{Ax} = \mathbf{b}$ if

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & -1 & 0 \\ 5 & 2 & -1 & 1 \\ 9 & 3 & -2 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 5 \\ 10 \end{bmatrix}$$

(i) Which method do you use?

(ii) Characterize all solutions.

(iii) Identify a particular solution for which all components of $\mathbf{x}$ are nonnegative.

c (i) Let $\mathbf{A}$ be an $m \times m$ positive definite matrix. Prove that all diagonal elements (a_{ii}, for $i = 1, \ldots, m$) are strictly positive. Explain your work.

(ii) Show that for any $m \times n$ matrix $\mathbf{A}$, both $\mathbf{A}^T\mathbf{A}$ and $\mathbf{A}\mathbf{A}^T$ are symmetric. Give the dimensions of these matrices.

(The three parts carry, respectively, 30%, 40% and 30% of the marks).

2a Which of the following two sets of vectors is linearly independent (if any)?

$$\begin{array}{lcl} \mathbf{a}_1 = [3, -1, 2]^T & & \mathbf{b}_1 = [2, -3/2, 1]^T \\ \mathbf{a}_2 = [0, 0, 0]^T & \text{or} & \mathbf{b}_2 = [-1, 4, -3/2]^T \\ & & \mathbf{b}_3 = [3, 1, 1/2]^T \end{array}$$

Explain your work.

b Given matrix $\mathbf{A}$, determine its ℓ_1, ℓ_2 and ℓ_∞ norm:

$$\mathbf{A} = \begin{bmatrix} -3 & 1 \\ 3 & -1 \\ 0 & 4 \end{bmatrix}.$$

c Given three large positive numbers x, y, z that are just within the range of the floating point mantissa of a given computer. The sum of any two of them results in floating point overflow. You are expected to determine the average of x, y, z. Will the average be within the range? Can you compute it? If yes, how, if not, why not?

(The three parts carry, respectively, 30%, 50% and 20% of the marks).

3a Given matrix $\mathbf{A}$:

$$\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 0 & -1 \\ 1 & 2 \end{bmatrix}$$

(i) Determine the range space of $\mathbf{A}$.

(ii) Determine the null space of $\mathbf{A}^T$.

(iii) Verify that any vector in range($\mathbf{A}$) is orthogonal to any vector in null($\mathbf{A}^T$).

(iv) Determine the rank of range($\mathbf{A}$) and null($\mathbf{A}$).

b Matrix $\mathbf{A}$ is called *skew symmetric* if $\mathbf{A}^T = -\mathbf{A}$. What is the shape of $\mathbf{A}$? What are the diagonal elements of $\mathbf{A}$? Show that if $\mathbf{A}$ is skew symmetric then $\mathbf{A}^T\mathbf{A} = \mathbf{A}\mathbf{A}^T$.

c Let $\mathbf{B}$ an $m \times m$ nonsingular matrix with inverse $\mathbf{B}^{-1}$, $\mathbf{c}$ an m-vector and $\mathbf{0}$ the m dimensional null vector. $\mathbf{A}$ is given in the following partitioned form:

$$\mathbf{A} = \begin{bmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{c}^T & 1 \end{bmatrix}$$

Determine $\mathbf{A}^{-1}$ symbolically in a partitioned form. Verify the correctness of your answer. What is the dimension of $\mathbf{A}$? What are the dimensions of the submatrices in $\mathbf{A}^{-1}$?

(The three parts carry, respectively, 50%, 20% and 30% of the marks).

4a Given the following matrix $\mathbf{A}$:

$$\mathbf{A} = \begin{bmatrix} 9 & 0 & -3 \\ 0 & 1 & 1 \\ -3 & 1 & 6 \end{bmatrix}$$

(i) Verify it is positive definite by applying the Cholesky factorization algorithm. Explain your answer.

(ii) Using the results of (i) and solve $\mathbf{AX} = \mathbf{B}$, where $\mathbf{A}$ is as in (i) and

$$\mathbf{B} = \begin{bmatrix} 6 & -3 \\ 3 & 6 \\ 5 & -1 \end{bmatrix}$$

(*Hint: view it as two systems with identical matrix but with different right hand sides.*)

b Comparing vectors (not their norms). Below, there are four pairs of vectors. For each pair, determine whether the two vectors are equal, one of them is greater than the other, no relationship can be determined, or cannot be compared for other reasons. Explain your answer.

(i) $\mathbf{u} = [-2, -1, 2, -2]^T$, $\mathbf{v} = [-1, 1, 3, 0]^T$.

(ii) $\mathbf{u} = [4, 3, 0, 0]^T$, $\mathbf{v} = [4, 3, 0]^T$.

(iii) $\mathbf{u} = [3, 2, -1, 4]^T$, $\mathbf{v} = [2, 1, 1, 0]^T$.

(iv) $\mathbf{u} = [-1, 3, 1.5, -2]^T$, $\mathbf{v} = [-1, 3, 3/2, -4/2]^T$.

c Find a local minimum or maximum for

$$f(x, y) = x^2 - 2xy + \frac{1}{2}y^4.$$

Explain your work.

(The three parts carry, respectively, 40%, 20% and 40% of the marks).