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C2.1
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IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE
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MSc and EEE PART IV: M.Eng. and ACGI

PROBABILITY AND STOCHASTIC PROCESSES

Friday, 3 May 10:00 am

There are SIX questions on this paper.

Answer FOUR questions.

Time allowed: 3:00 hours

Corrected Copy

Examiners responsible:

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Special Instructions for Invigilator: **None**

Information for Students: **None**

1. (a) The ring network of Figure 1.1, consisting of eight links, provides two possible paths between terminals A and B . Assume that the links fail independently, each with probability $1 - q$, $0 < q < 1$. What is the probability that a packet will be successfully transmitted from A to B ?

(Note that terminal A transmits the packet in both directions. B receives the packet if all links transmit in either path.) [10]

- (b) A signal $X(\omega)$ comes from one of two sources A or B . (See Figure 1.2.) Assume that:

if A is the source, the signal is normally distributed with mean $m_X = -1$ and variance $\sigma^2 = 1$.

if B is the source, the signal is normally distributed with mean $m_X = +1$ and variance $\sigma^2 = 1$.

A signal is received at R only if the switch that links it to its source is closed. One and only one switch is closed at transmission and

$$P(\text{'switch } a \text{ is closed'}) = 2 \times P(\text{'switch } b \text{ is closed'}) .$$

$$P[\text{'switch } A \text{ is closed'}] = 2 \times P[\text{'switch } B \text{ is closed'}]$$

- (i) Calculate the probability of the event $\{\omega : X(\omega) \geq -1\}$. [5]

- (ii) It is observed that $X(\omega) \geq -1$. What is the most likely source of the signal, A or B ? (The following table includes some relevant values of the distribution function $F(y) = P[Y \leq y]$, for a normally distributed random variable $Y(\omega)$ with zero mean and unit variance.) [5]

Normal Distribution $N(0, 1)$

x	-2	-1	0	+1	+2
$F(x)$	0.02276	0.15866	0.5	0.84134	0.97724

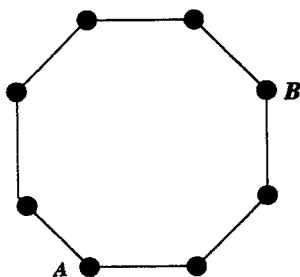


Figure 1.1

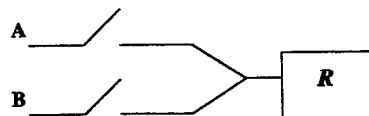


Figure 1.2

- 2 For a certain communication channel, the received signal $Y(\omega)$ is the transmitted signal $X(\omega)$ corrupted by additive noise

$$Y(\omega) = X(\omega) + N(\omega).$$

Assume that the noise is a zero mean normally distributed random variable with variance σ^2

$$f_N(n) = (2\pi\sigma^2)^{-1/2} \exp(-n^2/2\sigma^2).$$

10.35

Assume also that $X(\omega)$ and $N(\omega)$ are independent.

Determine the conditional probability density of Y given $X(\omega) = x$

$$f_{Y|X}(y|x).$$

[4]

Now suppose $X(\omega)$ is uniformly distributed on $[-\alpha, +\alpha]$ (for some $\alpha > 0$).

Derive a formula for the conditional probability density of X given $Y(\omega) = y$

$$f_{X|Y}(x|y).$$

[10]

Hence derive a formula for the (nonlinear) least squares estimate of X given $Y(\omega) = y$

$$E[X|Y(\omega) = y].$$

Show that, as $\alpha \rightarrow \infty$,

$$E[X|Y(\omega) = y] \rightarrow y.$$

[4]

Comment briefly on this last relationship.

[2]

3. (a) The generalized coordinates of a manoeuvring vehicle are represented by the n -vector random variable $X(\omega)$. Motion of the vehicle is affected by the manoeuvre 'mode' $R(\omega)$.

$R(\omega)$ is a discrete random variable, taking values $1, 2, \dots, n$. Let

$$w_j = P[R = j], \quad j = 1, 2, \dots, n.$$

For $j = 1, 2, \dots, n$ write

$$F_j(x) = P[X \leq x | R = j]$$

('the conditional probability distribution function of $X(\omega)$ given $R(\omega) = j$ '), and denote by m_j and P_j the mean and covariance matrix of $F_j(x)$, respectively.

Derive a formula for the probability distribution of $X(\omega)$, in terms of the $F_j(x)$'s and w_j 's. [2]

Show that the mean m and covariance matrix P of $X(\omega)$ are

$$m = \sum_j w_j m_j \quad \text{and} \quad P = \sum_j w_j (P_j + (x_j - m)(x_j - m)^T).$$

[10]

- (b) Henceforth assume that $X(\omega)$ is a scalar random variable and

$$n = 2, \quad m_1 = -1, \quad m_2 = +1, \quad P_1 = P_2 = 0, \quad w_1 = w_2 = 1/2.$$

Calculate the mean 'range' of the vehicle:

$$E[|X|]. \quad (3.1)$$

[4]

- (c) Sometimes, to simplify calculations, probability distributions are approximated by normal distributions having the same mean and covariance. Examine the effects of this approximation in calculating the mean range. Specifically:

determine the percentage error in the calculation of the mean range, when the normal probability density with (scalar) mean m and variance P ,

$$\tilde{f}(x) = (2\pi P)^{-\frac{1}{2}} \exp \left\{ -(x - m)^2 / 2P \right\},$$

is used in place of F_X to evaluate the expectation in (3.1). [4]

In part (c), you can use the fact that

$$\int_0^\infty (x/\sigma^2) \exp(-x^2/2\sigma^2) dx = 1 \quad \text{for } \sigma^2 > 0.$$

4. (a) A zero-mean scalar random variable $y(\omega)$ is correlated with a zero mean n -vector random variable $\mathbf{x}(\omega)$. Show that the random variable $\hat{\mathbf{x}}(\omega)$ given by

$$\hat{\mathbf{x}}(\omega) = y(\omega)\hat{\mathbf{a}}, \quad \hat{\mathbf{a}} = (E[y^2])^{-1}E[y\mathbf{x}],$$

is the linear least squares estimate of $\mathbf{x}(\omega)$ given $y(\omega)$, in the sense that $\hat{\mathbf{a}}$ minimizes

$$J(\mathbf{a}) := E[(\mathbf{x} - y\mathbf{a})^T(\mathbf{x} - y\mathbf{a})] .$$

[6]

Derive the following formula for the estimation error covariance matrix

$$\text{cov}\{\mathbf{x} - \hat{\mathbf{x}}\} = E[\mathbf{x}\mathbf{x}^T] - (E[y^2])^{-1}E[y\mathbf{x}]E[y\mathbf{x}^T] .$$

[4]

- (b) Consider now the one stage state space system, with scalar output:

$$\begin{aligned} \mathbf{x}_1(\omega) &= A\mathbf{x}_0(\omega) + \mathbf{e}(\omega) \\ y_1(\omega) &= \mathbf{c}^T\mathbf{x}_1(\omega) + v(\omega) . \end{aligned}$$

Here, A is a constant $n \times n$ matrix and \mathbf{c} is a constant n -vector. The n -vector random variables \mathbf{x}_0 , \mathbf{e} and the scalar random variable v are all uncorrelated.

Furthermore,

$$E[\mathbf{x}_0] = E[\mathbf{e}] = \mathbf{0}, \quad E[v] = 0, \quad E[\mathbf{x}_0\mathbf{x}_0^T] = P_0, \quad E[\mathbf{e}\mathbf{e}^T] = Q, \quad E[v^2] = w .$$

Using part (a), or otherwise, show that the linear least squares estimate of \mathbf{x}_1 given y_1 is

$$\hat{\mathbf{x}}_1 = y_1\mathbf{k}$$

where

$$\mathbf{k} = s^{-1}(AP_0A^T + Q)\mathbf{c} \quad \text{and} \quad s = (\mathbf{c}^T(AP_0A^T + Q)\mathbf{c} + w) .$$

[6]

Show, furthermore, that the covariance matrix of $\mathbf{x}(\omega) - \hat{\mathbf{x}}(\omega)$ is

$$P_1 = (AP_0A^T + Q) \left(I - s^{-1}\mathbf{c}\mathbf{c}^T(AP_0A^T + Q) \right) .$$

[4]

5. (a) Consider the scalar Auto-Regressive Moving Average (ARMA) process $\{y_k\}$, generated by the difference equation

$$y_k + gy_{k-2} = e_k + he_{k-1},$$

in which $\{e_k\}$ is a sequence of uncorrelated, zero mean random variables with variance σ^2 . $g, |g| < 1$, and h are constants.

Show that the covariance function $R_y(k)$, for $k = 0$, is

$$R_y(0) = \frac{1 + h^2}{1 - g^2} \sigma^2.$$

Determine also $R_y(1)$ and $R_y(2)$.

[12]

- (b) Now consider the controlled Auto-Regressive process $\{y_k\}$

$$y_k - ay_{k-1} = e_k + u_{k-2}, \quad (5.1)$$

in which e_k is as before and a is a constant, with $|a| < 1$. The control u_k , which depends on present and past values of $\{y_k\}$, is chosen to improve the statistical properties of the process $\{y_k\}$. Notice that there is a two sample period delay in control implementation.

For this system, a 'minimum variance' controller has the structure:

$$u_k + au_{k-1} = Ky_k, \quad (5.2)$$

in which K is a design parameter.

Derive the ARMA model for the process $\{y_k\}$ which results when the minimum variance controller (5.2) is inserted into (5.1). What conditions must K satisfy for this ARMA model to be stable?

10/10

Determine the value of K , satisfying the stability condition, which minimizes the output covariance:

$$E[y_k^2].$$

Show that, for this choice of K , $\{y_k\}$ is a Moving Average process.

[8]

6. Define the spectral density $\Phi(\omega)$ of a stationary, second order, zero mean, scalar stochastic process $\{y_k\}$. What conditions must $\Phi(\omega)$ satisfy if $\{y_k\}$ is to be the output of an Auto-Regressive Moving Average model

$$A(z^{-1})y_k = B(z^{-1})e_k \quad (6.1)$$

[4]

Here A and B are polynomials in the delay operator z^{-1} and $\{e_k\}$ is a sequence of zero mean, unit variance, uncorrelated scalar random variables. *FULL STOP 10.45*

Consider now the covariance function

$$R(k) = c_1 e^{-\lambda_1 |k|} + c_2 e^{-\lambda_2 |k|} \quad k = \dots, -1, 0, +1, \dots,$$

in which c_1 , c_2 , λ_1 and λ_2 are positive constants. Show that the corresponding spectral density function $\Phi(\omega)$ is

$$\Phi(\omega) = \sum_{i=1}^2 \frac{c_i (1 - e^{-2\lambda_i})}{(1 - e^{-\lambda_i} e^{-j\omega})(1 - e^{-\lambda_i} e^{+j\omega})}.$$

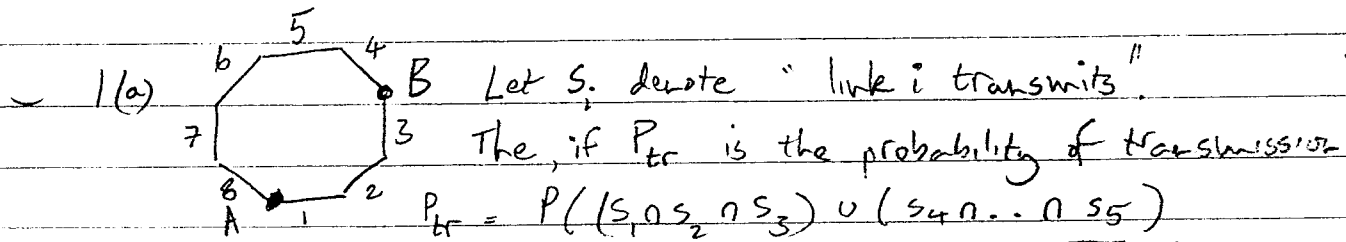
[8]

Now set

$$c_1 = 4/3, \lambda_1 = \log_e(2), c_2 = 9/8, \text{ and } \lambda_2 = \log_e(3),$$

For these values of the constants, determine an ARMA model (6.1) whose output $\{y_k\}$ has the covariance function $R(k)$. [8]

Note: in this question ω denotes a frequency, not a point in the sample space.



Hence $1 - P_{tr} = P(\overline{S_1 \cap S_2 \cap S_3} \cap \overline{S_4 \cap \dots \cap S_8})$

Independence of the S_i 's implies $\overline{S_1 \cap S_2 \cap S_3}$ and $\overline{S_4 \cap \dots \cap S_8}$ are independent, so

$$1 - P_{tr} = P(\overline{S_1 \cap S_2 \cap S_3}) \cdot P(\overline{S_4 \cap \dots \cap S_8}) = (1 - P(S_1 \cap S_2 \cap S_3)) (1 - P(S_4 \cap \dots \cap S_8))$$

$$= (1 - P(S_1) \times \dots \times P(S_3)) (1 - P(S_4) \times \dots \times P(S_8)) = (1 - q^3)(1 - q^5)$$

Hence,

probability of transmission = $1 - (1 - q^3)(1 - q^5)$

(b) Let A denote "A is source", etc. Then

$$P[X(w) \geq -1] = P[X(w) \geq -1 | A] \cdot P[A] + P[X(w) \geq -1 | B] \cdot P[B]$$

Since $P[A] + P[B] = 1$ and $P[A] = 2 \cdot P[B]$, $P[A] = \frac{2}{3}$, $P[B] = \frac{1}{3}$

Since $f_{X|A} \sim N(-1, 1)$, $P[X(w) \geq -1 | A] = 0.5$

Since $f_{X|B} \sim N(+1, 1)$

$$P[X(w) \geq -1 | B] = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-1} e^{-\frac{1}{2}(t-1)^2} dt = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-2} e^{-\frac{1}{2}t^2} dt$$

$$= 1 - 0.02276 = 0.97724$$

Hence $P[X(w) \geq -1] = 0.5 \times \frac{2}{3} + 0.97724 \times \frac{1}{3} = 0.65908$

By Bayes' Rule:

$$P[A | X(w) \geq -1] = \frac{P[X(w) \geq -1 | A] \cdot P[A]}{P[X(w) \geq -1]} = \frac{0.5 \cdot \frac{2}{3}}{0.65908} = 0.50576$$

and

$$P[B | X(w) \geq -1] = \frac{P[X(w) \geq -1 | B] \cdot P[B]}{P[X(w) \geq -1]} = \frac{0.97724 \cdot \frac{1}{3}}{0.65908} = 0.49424$$

We see that

$$P[A | X(w) \geq -1] > P[B | X(w) \geq -1].$$

It is therefore more likely that the source was "A"

$$2. \quad F_{Y|X}(y|x) = P[Y \leq y | X(\omega) = x] = P[X + N \leq y | X(\omega) = x] \\ = P[N \leq y - x | X(\omega) = x] = P[N \leq y - x] \quad (\text{by independence})$$

$$= F_N(y - x)$$

$$4. \quad \text{It follows } f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2} \frac{(y-x)^2}{\sigma^2}\right)$$

We know

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{\bar{f}_Y} = \frac{f_{Y|X}(y|x) \cdot \bar{f}_X(x)}{\bar{f}_Y(y)}$$

$$\text{But } f_X(x) = \begin{cases} \frac{1}{2\alpha} & -\alpha \leq x \leq +\alpha \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Also, for each } y \quad \int_{-\infty}^{+\infty} f_{X|Y}(x|y) dx = 1$$

This implies

$$\frac{1}{2\alpha} \int_{-\alpha}^{+\alpha} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2} \frac{(y-x')^2}{\sigma^2}\right) dx' = f_Y(y)$$

It follows

$$10. \quad f_{X|Y}(x|y) = \begin{cases} \frac{\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}|y-x|^2\right)}{\int_{-\alpha}^{+\alpha} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}|y-x'|^2\right) dx'} & \text{if } -\alpha \leq x \leq +\alpha \\ 0 & \text{otherwise} \end{cases}$$

The conditional expectation of $X(\omega)$ given $Y(\omega) = y$ is

$$4. \quad E[X(\omega) | Y(\omega) = y] = \frac{\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\alpha}^{+\alpha} x \exp\left(-\frac{1}{2\sigma^2}|y-x|^2\right) dx}{\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\alpha}^{+\alpha} \exp\left(-\frac{1}{2\sigma^2}|y-x'|^2\right) dx'} = \frac{(a)}{(b)}$$

$$2. \quad \text{Notice that } (a) \rightarrow \frac{1}{2\pi\sigma^2} \int_{-\infty}^{+\infty} x \exp\left(-\frac{1}{2\sigma^2}|y-x|^2\right) dx = y$$

$$(b) \rightarrow \frac{1}{2\pi\sigma^2} \int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2\sigma^2}|y-x|^2\right) dx = 1$$

as $\alpha \rightarrow \infty$ (by properties of the normal density). So $E[X|Y] \rightarrow y$

Note: ' $\alpha \rightarrow \infty$ ' implies we have no prior information about $X(\omega)$. It is natural, in these circumstances to estimate $X(\omega)$ as the value

3(a) We have

$$F_X(x) = P[X \leq x] = \sum_j P[X \leq x \text{ and } R=j]$$

$$= \sum_j P[X \leq x | R=j] P[R=j] = \sum_j \underbrace{F_j(x)}_{\text{cdf of } X \text{ given } R=j} \omega_j$$

2 So $m = \int x dF_X(x) = \sum_j \omega_j \int x dF_j(x) = \sum_j \omega_j \underbrace{m_j}_{\text{mean of } X \text{ given } R=j}$

Also,

$$P = \int (x-m)(x-m)^T dF_X(x)$$

$$= \sum_j \omega_j \int (x-m)(x-m)^T dF_j(x)$$

$$= \sum_j \omega_j \left[\int x x^T dF_j(x) - \int x dF_j(x) \cdot m^T - m \int x^T dF_j(x) + m m^T \int dF_j(x) \right]$$

$$= \sum_j \omega_j \left(\underbrace{P_j}_{\text{covariance of } X \text{ given } R=j} + \underbrace{m_j m_j^T}_{\text{outer product of mean}} - \underbrace{m m_j^T}_{\text{outer product of mean}} - \underbrace{m_j m^T}_{\text{outer product of mean}} + \underbrace{m m^T}_{\text{outer product of mean}} \right)$$

10

$$= \sum_j \omega_j (P_j + (m_j - m)(m_j - m)^T)$$

(b) When $m_1 = -1, m_2 = +1, P_1 = P_2 = 0, \omega_1 = \omega_2 = \frac{1}{2}$

$$m = \frac{1}{2}(-1) + \frac{1}{2}(+1) = 0 \text{ and}$$

$$P = \frac{1}{2}(1-0)^2 + \frac{1}{2}(1-0)^2 = 1$$

Also

$X|u$ is a discrete RV : $P(X=-1) = P(X=+1) = \frac{1}{2}$.

It follows

4

$$\text{mean range} = E|X| = 1-1 \cdot \frac{1}{2} + 1+1 \cdot \frac{1}{2} = \underline{1}$$

If we use the normal density to evaluate 'mean range'

$$\text{approx. mean range} = \int_{-\infty}^{+\infty} |x| \cdot \frac{1}{(2\pi)^{\frac{1}{2}}} \exp\left(-\frac{1}{2}x^2\right) dx$$

$$= \frac{2}{(2\pi)^{\frac{1}{2}}} \int_0^{+\infty} x \exp\left(-\frac{1}{2}x^2\right) dx = \sqrt{\frac{2}{\pi}}$$

4

$$\% \text{ error} = \frac{1 - \sqrt{\frac{2}{\pi}}}{1} \times 100 = \underline{20.2115 \%}$$

$$4 (a) J(a) = E[(x - y a)^T (x - y a)] = E[x^T x] - 2 \sum a_i E[y x_i] + (\sum a_i) E[y^2]$$

Since $J(\hat{a}_1, \dots, \hat{a}_n)$ is minimized at $a_i = \hat{a}_i$, for each i
 $0 = \partial_{a_i} J(\hat{a}) = 2 E[y x_i] + 2 \hat{a}_i E[y^2]$. Hence
 $\hat{a}_i = E[y^2]^{-1} E[y x_i]$, $i = 1, \dots, n$.

These relationships can be expressed:

$$\hat{a} = E[y^2]^{-1} E[y x]$$

Since x, \hat{x} and y have zero mean,

$$\begin{aligned} \text{cov}\{x - \hat{x}\} &= E[(x - \hat{x})(x - \hat{x})^T] \\ &= E[(x - y(E[y^2])^{-1} E[y x])(x - y(E[y^2])^{-1} E[y x])^T] \\ &= E[x x^T] - 2(E[y^2])^{-1} E[y x] E[y x^T] + (E[y^2])^{-1} E[y x] E[y x^T] \\ &= \text{cov}\{x\} - (E[y^2])^{-1} E[y x] E[y x^T] \end{aligned}$$

(b) System equations: $x_1 = A x_0 + e$ and $y_1 = c^T x_1 + v$

We must evaluate $E\{y_1^2\}$ and $E\{y_1 x_1^T\}$. Since x_0 and v are uncorrelated and x_0 and e are uncorrelated

$$E\{y_1^2\} = c^T E\{x_1 x_1^T\} c + 0 + E\{v^2\}.$$

$$\text{But } E\{x_1 x_1^T\} = A E\{x_0 x_0^T\} A^T + E\{e e^T\} = A P_0 A^T + Q$$

$$\text{Hence } E\{y_1^2\} = c^T (A P_0 A^T + Q) c + w.$$

Also

$$E\{y_1 x_1^T\} = E\{x_1 x_1^T\} c + 0 = (A P_0 A^T + Q) c$$

Furthermore,

$$\begin{aligned} \text{cov}\{x_1 - \hat{x}_1\} &= E\{x_1 x_1^T\} - (E\{y_1^2\})^{-1} E\{y_1 x_1^T\} E\{y_1 x_1^T\}^T \\ &= A P_0 A^T + Q - [c^T (A P_0 A^T + Q) c + w]^{-1} (A P_0 A^T + Q) c c^T (A P_0 A^T + Q). \end{aligned}$$

By part (a), $\hat{x}_1 = y_1 k$, where

$$k = (E\{y_1^2\})^{-1} E\{y_1 x_1^T\} = s^{-1} (A P_0 A^T + Q) c$$

$$\text{and } s = c^T (A P_0 A^T + Q) c + w$$

Also

$$\begin{aligned} \text{cov}(x_1 - \hat{x}_1) &= E\{x_1 x_1^T\} - (E\{y_1^2\})^{-1} E\{y_1 x_1^T\} E\{y_1 x_1^T\}^T \\ &= (A P_0 A^T + Q) [I - s^{-1} c c^T (A P_0 A^T + Q)] \end{aligned}$$

5(a) $y_k + g y_{k-2} = e_k + h e_{k-1}$

$E\{\dots \times y_k\} \Rightarrow R_y(0) + g R_y(2) = R_{ye}(0) + h R_{ye}(1)$

$E\{\dots \times y_{k-1}\} \Rightarrow R_y(1) + g R_y(1) = 0 + h R_{ye}(0)$

$E\{\dots \times y_{k-2}\} \Rightarrow R_y(2) + g R_y(0) = 0 + 0$

(We have used the facts that $R_y(1) = R_y(-1)$ and y_k is uncorrelated with $e_j, j > k$.)

$E\{\dots \times e_k\} \Rightarrow R_{ye}(0) = \sigma^2 + 0$

$E\{\dots \times e_{k-1}\} \Rightarrow R_{ye}(1) + 0 = h \sigma^2$

Also, $E\{\dots \times y_{k-j}\} (j > 2) \Rightarrow R_y(j) + g R_y(j-2) \dots (*)$

From these relationships

$(1+g) R(1) = h \sigma^2 \Rightarrow R(1) = \frac{h}{1+g} \sigma^2$

$R_y(0) + g R_y(2) = (1+h^2) \sigma^2 \Rightarrow R_y(0) (1-g^2) = (1+h^2) \sigma^2$

$R_y(2) + g R_y(0) = 0$
Hence $R_y(0) = \frac{1+h}{1-g^2} \sigma^2$ and $R_y(2) = \frac{-g(1+h^2)}{(1-g^2)} \sigma^2$

From (*) $R_y(k) = \begin{cases} -g^{\frac{|k|}{2}} \cdot \frac{1+h^2}{(1-g^2)} \sigma^2 & \text{for } k \text{ even} \\ -g^{\frac{|k|+1}{2}} \cdot \frac{h}{1+g} \sigma^2 & \text{for } k \text{ odd} \end{cases}$

(b) Inserting the control $u = \frac{-K}{(1+a z^{-1})} y$ into the system equations gives

$(1-a z^{-1}) y_k = e_k - \frac{K}{(1+a z^{-1})} y_k$

Rationalizing: $(1-a z^{-1})(1+a z^{-1}) y_k = (1+a z^{-1}) e_k - K y_k$

Hence $y_k - (a-K) y_{k-2} = e_k + a e_{k-1} \dots (**)$

This is stable if $|K-a| < 1$

By (a),

$R_y(0) = \frac{1+h^2}{1-g^2} \Big|_{h=a, g=K-a} = \frac{1+a^2}{1-(K-a)^2}$

Since $|K-a| < 1$, the minimizing K is $K=a$

For this choice of a , from (**),

$y_k = e_k + a e_{k-1}$

According to this relationship, $\{y_k\}$ is a moving average process.

6 If $R(l)$ is the covariance function of $\{y_k\}$, i.e. $R(l) = E\{y_k y_{k-l}\}$ then the spectral density is

$$\underline{\Phi}(\omega) = \sum_{l=-\infty}^{+\infty} R(l) e^{-j\omega l}$$

$\underline{\Phi}(\omega)$ is the spectral density of an ARMA process if and only if it can be factorized

$$\underline{\Phi}(\omega) = D(z) D(z^{-1}) \big|_{z=e^{-j\omega}}$$

in which $D(z)$ is a rational function of z .

If $R(k) = c_1 e^{-\lambda_1 |k|} + c_2 e^{-\lambda_2 |k|}$. Then

$$\underline{\Phi}(\omega) = c_1 \sum_{k=-\infty}^{+\infty} e^{-\lambda_1 |k|} \cdot e^{-j\omega k} + \dots \text{ (same but with } c_2, \lambda_2 \text{)}$$

$$= c_1 \left(\sum_{k=0}^{\infty} e^{-(\lambda_1 + j\omega)k} + \sum_{k=1}^{\infty} e^{-(\lambda_1 - j\omega)k} \right) + \dots$$

$$= c_1 \left(\frac{1}{1 - e^{-\lambda_1} e^{-j\omega}} + \frac{1}{1 - e^{-\lambda_1} e^{+j\omega}} - 1 \right) + \dots$$

$$= \frac{c_1 (1 - e^{-2\lambda_1})}{(1 - e^{-\lambda_1} e^{-j\omega})(1 - e^{-\lambda_1} e^{+j\omega})} + \frac{c_2 (1 - e^{-2\lambda_2})}{(1 - e^{-\lambda_2} e^{-j\omega})(1 - e^{-\lambda_2} e^{+j\omega})}$$

when $c_1 = \frac{4}{3}$, $\lambda_1 = \ln(2)$, $c_2 = \frac{9}{8}$, $\lambda_2 = \ln(3)$

$$\underline{\Phi}(\omega) = \frac{1}{(1 - \frac{1}{2} e^{-j\omega})(1 - \frac{1}{2} e^{+j\omega})} + \frac{1}{(1 - \frac{1}{3} e^{-j\omega})(1 - \frac{1}{3} e^{+j\omega})}$$

Hence $\underline{\Phi}(\omega) = \underline{\Psi}(z) \underline{\Psi}(z^{-1}) \big|_{z=e^{-j\omega}}$, with

$$\begin{aligned} \underline{\Psi}(z) &= \frac{(1 - \frac{1}{3} z)(1 - \frac{1}{3} z^{-1}) + (1 - \frac{1}{2} z)(1 - \frac{1}{2} z^{-1})}{(1 - \frac{1}{2} z^{-1})(1 - \frac{1}{3} z^{-1})(1 - \frac{1}{2} z)(1 - \frac{1}{3} z)} \\ &= \frac{85 - 30 z^{-1} - 30 z}{(2 - z^{-1})(3 - z^{-1})(2 - z)(3 - z)} \end{aligned}$$

The roots of $30z^2 - 85z + 30$ are $z = 2.419972$ and $\frac{1}{2.419972}$

$$\text{So } \underline{\Psi}(z) = \frac{(3.520915)^2 (2.419972 - z^{-1})}{(2 - z^{-1})(3 - z^{-1})} \cdot \frac{(2.419972 - z)}{(2 - z)(3 - z)}$$

It follows that the covariance function is realized by the ARMA model

$$(2 - z^{-1})(3 - z^{-1}) y_k = 3.520915 (2.419972 - z^{-1}) e_k$$