EE4-45
EE9-SO22

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2014

MSc and EEE/EIE PART IV: MEng and ACGI

Corrected Copy

WAVELETS AND APPLICATIONS

Thursday, 15 May 10:00 am

Time allowed: 3:00 hours

There are FOUR questions on this paper.

Answer ALL questions.

All questions carry equal marks.

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : P.L. Dragotti
Second Marker(s) : A. Manikas

Special Information for the Invigilators: NONE

Information for Candidates:

Causal spline $\beta_n^+(t)$

The causal spline $\beta_n^+(t)$ of order n is obtained from the $(n+1)$-fold convolution of the causal box function $\beta_0^+(t)$. Specifically,

$$\beta_n^+(t) = \underbrace{\beta_0^+(t) * \beta_0^+(t) \ldots * \beta_0^+(t)}_{n+1 \text{ times}}$$

where $*$ denotes convolution and with

$$\beta_0^+(t) = \begin{cases} 1 & t \in [0,1) \\ 0 & \text{otherwise.} \end{cases}$$

Dual Basis:
Given a basis $\{\varphi_i(t)\}_{i\in\mathbf{Z}}$, the dual basis is given by the set of elements $\{\tilde{\varphi}_i(t)\}_{i\in\mathbf{Z}}$ satisfying:

$$\langle \varphi_i(t), \tilde{\varphi}_j(t) \rangle = \delta_{i,j}.$$

The Questions

1. Consider the celebrated Laplacian pyramid (LP) of Burt and Adelson shown in Figure 1.

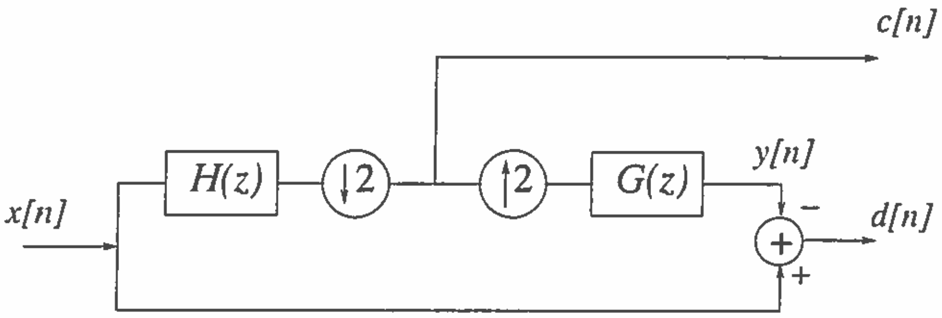

Figure 1: Decomposition of $x[n]$ using the Laplacian Pyramid.

(a) State the conditions $G(z)$ and $H(z)$ have to satisfy in order for the operator P that converts $x[n[$ into $y[n]$ to be idempotent. That is, $P^2 = P$.

[9[

(b) Assume $H(z) = G(z^{-1})$. Design a 4-tap filter $G(z)$ with two zeros at $z = -1$ such that the idempotent constraint is met.

[8[

(c) Construct now a complementary wavelet branch.

[8[

2. Consider the two-channel filter bank of Figure 2.

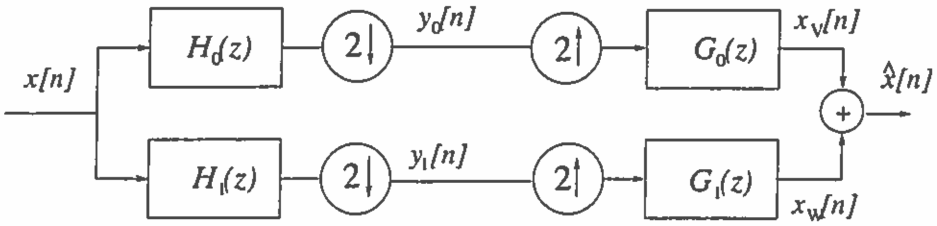

Figure 2: Two-channel filter bank.

(a) Assume that $G_0(z) = \frac{1}{2\sqrt{2}}(1+z^{-1})(1+z)$ and assume that $H_0(z) = (1+z)(1+z^{-1})B(z)$. Determine the shortest symmetric polynomial $B(z)$ such that $P(z) + P(-z) = 2$, where $P(z) = H_0(z)G_0(z)$.

[7]

(b) Given the filters $G_0(z)$ and $H_0(z)$ of part (a), design the filters $H_1(z)$ and $G_1(z)$ in order to have a perfect reconstruction biorthogonal filter bank.

[6]

(c) Based on the polynomial $P(z)$ of part (a), construct an orthogonal filter bank.

[6]

(d) Based of the polynomial $P(z)$ of part (a), construct a biorthogonal filter bank where $H_1(z)$ is able to annihilate polynomials of maximum degree $d = 2$.

[6[

3. Consider the interval $t \in [0,3]$ and let

$$\varphi_1(t) = \begin{cases} t, & \text{for } t \in [0,1] \\ 2-t & \text{for } t \in (1,2[\\ 0, & \text{for } t \in (2,3[. \end{cases}$$

Denote with $V = span(\{\varphi_1(t), \varphi_2(t), \varphi_3(t)\})$ the sub-space generated by $\varphi_1(t)$ and its circular shifts by 1 over the interval $t \in [0,3[$. The three basis functions are shown in Fig. 3.

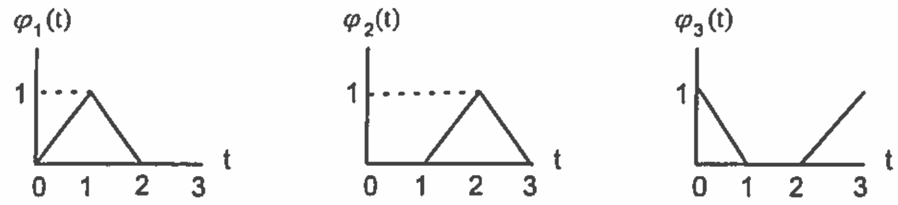

Figure 3: Three functions $\varphi_1, \varphi_2, \varphi_3$ defined for $t \in [0,3[$ and related by circular shifts.

Given a signal $x(t)$ defined for $t \in [0,3]$, the aim is to compute the orthogonal projection of $x(t)$ onto V. Recall that this is given by:

$$x_v(t) = \sum_{i=1}^{3} \langle x(t), \tilde{\varphi}_i(t) \rangle \varphi_i(t)$$

where $\{\tilde{\varphi}_i(t)\}_{i=1}^3$ are the three dual-basis functions.

(a) Since $\tilde{\varphi}_i(t) \in V$ we can write $\tilde{\varphi}_i(t) = \sum_{k=1}^3 \alpha_{i,k}\varphi_k(t)$. Using this fact

i. Determine the three dual-basis functions $\tilde{\varphi}_i(t)$, $i = 1,2,3$. That is, find the coefficients $\alpha_{i,k}$, $i = 1,2,3$; $k = 1,2,3$.

[5[

ii. Sketch and dimension $\tilde{\varphi}_i(t)$ $i = 1,2,3$.

[5]

(b) Given the dual basis and the signal

$$x(t) = \begin{cases} 1, & \text{for } t \in [0,1] \\ 0 & \text{for } t \in (1,3[. \end{cases}$$

i. Compute the inner products $\langle x(t), \tilde{\varphi}_i(t) \rangle$, $i = 1,2,3$.

[5]

ii. Sketch and dimension $x_v(t) = \sum_{i=1}^3 \langle x(t), \tilde{\varphi}_i(t) \rangle \varphi_i(t)$.

[5]

iii. Verify that the error $e(t) = x(t) - x_v(t)$ is orthogonal to V.

[5[

4. *Continuous-time Wavelets and Scaling functions*

(a) Suppose you are given a two-channel FIR filter bank with real coefficients and synthesis lowpass filter

$$g_0[n[= \frac{1}{4\sqrt{2}}(\delta_n + 3\delta_{n-1} + 3\delta_{n-2} + \delta_{n-3}).$$

Consider the equivalent filter

$$G_0^{(i)}(z) = \prod_{k=0}^{i-1} G_0(z^{2^k})$$

obtained by iterating the filter bank decomposition i times. Consider the function

$$\varphi^{(i)}(t) = 2^{i/2} g_0^{(i)}[n], \qquad n/2^i \leq t < (n+1)/2^i.$$

i. Can you say anything about the convergence of $\lim_{i \to \infty} \varphi^{(i)}(t)$?

[5[

ii. Assume that $\varphi(t) = \lim_{i \to \infty} \varphi^{(i)}(t)$ exists. We know that, in the case of convergence, $\varphi(t)$ is a valid scaling function. Can you say anything about continuity of this function?

[5]

iii. State the number of vanishing moments of the analysis wavelet function obtained from $\varphi(t)$.

]5[

(b) Suppose that your are given a signal $f(t)$ which is uniformly Lipschitz $\alpha \geq 0$ over $[a, b]$. This means that given $t_0 \in (a, b)$, there exists a constant $K > 0$ and a polynomial $p_{t_0}(t)$ of degree $d = \lfloor \alpha \rfloor$ such that

$$|f(t) - p_{t_0}(t)| \leq K|t - t_0|^\alpha.$$

Show that the wavelet coefficients $|\langle f, \psi_{m,n} \rangle|$ in the cone of influence of t_0 with a wavelet with $d+1$ vanishing moments decay like $2^{m(\alpha+1/2)}$. That is show that

$$|\langle f, \psi_{m,n} \rangle| \leq C 2^{m(\alpha+1/2)}$$

for some constant C.

[5[

(c) Assume now that $f(t) = \delta(t - t_0)$. Analyse the behaviour of the wavelet coefficients in the cone of influence of t_0.

[5[