Answers Communications II

1. a) i) The mean $E[X(t)]$ of a wide-sense stationary random process doesn't depend on t, and the autocorrelation function $R_X(t_1, t_2)$ depends only on $\tau = t_1 - t_2$. [2, bookwork]

The Wiener-Khinchine theorem says that the power spectral density is the Fourier transform of $R_X(\tau)$. [1, bookwork]

ii) Give two statistically independent Gaussian random variables X and Y, [2, bookwork]

Rayleigh random variable $Z_1 = \sqrt{X^2 + Y^2}$ [2, bookwork]

Ricean random variable $Z_2 = \sqrt{(A+X)^2 + Y^2}$, A a constant

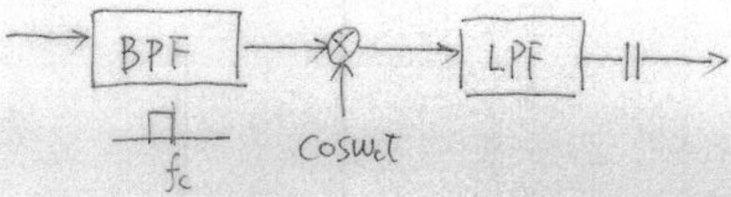
iii) Ergodicity: A wide-sense stationary random process is ergodic if:

- its time average = ensemble average [2, bookwork]
- time autocorrelation function = ensemble autocorrelation function.

Yes, it is ergodic. [1, bookwork]

b) i)
- Synchronous detection: needs a local carrier that is synchronized with the incoming carrier. [1, bookwork]
- envelope detection: tracks the envelope of the signal; no local carrier needed. [1, bookwork]
- Coherent detection: the same as synchronous detection, a term used more often in digital communications. [1, bookwork]
- Noncoheret detection: does not require phase synchronization at the receiver. [1, bookwork]

ii)



[3, bookwork]

iii) NO. [3, bookwork]

In DPSK, the information symbols are differentially encoded, thus permitting differential detection.

c) i) Information of a symbol s: $I(s) = \log_2\left(\frac{1}{p}\right)$

Entropy of an information source $S = \{s_1, s_2, \cdots, s_K\}$

$$H(S) = -\sum_{k=1}^{K} p_k \log_2(p_k) \quad \text{bits/symbol}$$

[5, bookwork]

ii) Mutual information

$$I(X;Y) = H(X) - H(X|Y)$$

Channel capacity

$$C = \max_{p(x)} I(X;Y)$$

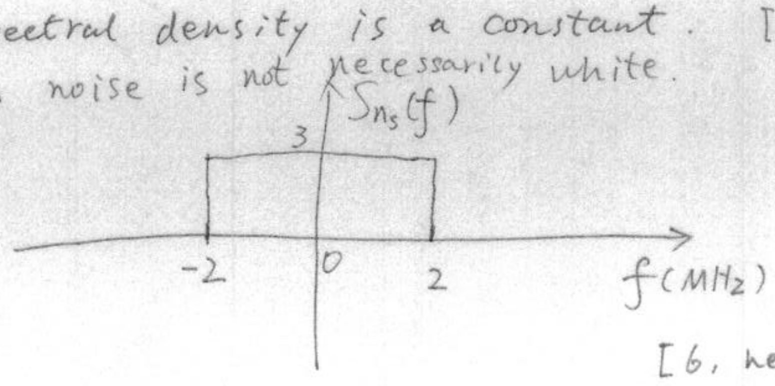
[5, bookwork]

Capacity formula $C = B\log_2(1 + \frac{S}{N})$

d) i) "Additive white Gaussian noise" means noise is added to the signal, it has Gaussian distribution, and its power spectral density is a constant. [4, bookwork]

Gaussian noise is not necessarily white.

ii)

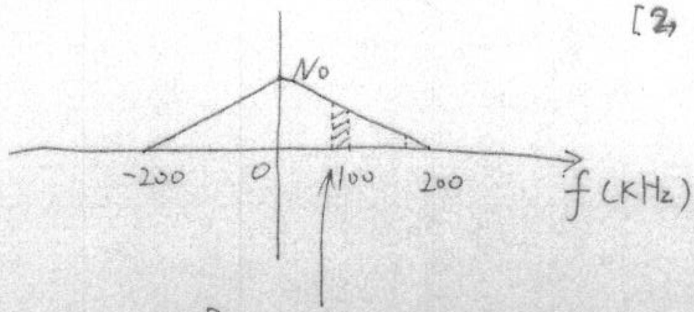


[6, new example]

2\. a) i) Transmitted power $P_T = \frac{A^2 P}{4} = 2.5\ W$

Received power $P_R = \frac{2.5}{1000} = 2.5 \times 10^{-3}\ W = 2.5\ mW$

[2, book work]



Noise power P_N = the area $\times 2$

$$= 2 \times \left(\frac{N_0}{2} \times 10\,kHz + \frac{0.05 N_0}{2} \times 10\,kHz \right)$$

$$= 10.5\ \mu W$$ [3, new example]

$$SNR = \frac{P_R}{P_N} = \frac{2.5 \times 10^{-3}}{10.5 \times 10^{-6}} = 238 \quad (23.8\,dB)$$

[3, new example]

ii) Noise power $P_N = 2 \times \frac{0.05 N_0}{2} \times 10\ kHz$

$$= 0.5 \times 10^{-6}\ W$$ [3, new example]

$$= 0.5\ \mu W$$

$$SNR = \frac{2.5 \times 10^{-3}}{0.5 \times 10^{-6}} = 5000 \quad (37\,dB)$$

[3, new example]

b) i) 3dB bandwidth

$$|H_{de}(f_{3dB})| = \frac{1}{\sqrt{1 + (2\pi f_{3dB} RC)^2}} = \frac{1}{\sqrt{2}}$$

$$f_{3dB} = \frac{1}{2\pi RC}$$

equivalent bandwidth

$$B_{eq} = \frac{\int_0^\infty |H_{de}(f)|^2 df}{|H_{de}(0)|^2} = \int_0^\infty \frac{1}{1 + (2\pi f RC)^2} df$$

$$= \frac{1}{2\pi RC} \tan^{-1}(x) \Big|_0^\infty = \frac{\frac{\pi}{2}}{2\pi RC} = \frac{1}{4RC}$$

[4, new example]

ii) After de-emphasis, noise PSD becomes

$$S_D(f) = \frac{f^2}{A^2} N_0 \frac{1}{1+(f/f_{3dB})^2}$$

$$P_N = \int_{-W}^{W} S_D(f)\,df = \frac{N_0}{A^2}\int_{-W}^{W} \frac{f^2}{1+(f/f_{3dB})^2}\,df$$

$$= \frac{N_0}{A^2} f_{3dB}^2 \int_{-W}^{W}\left[1 - \frac{1}{1+(f/f_{3dB})^2}\right] df$$

$$= \frac{N_0}{A^2} f_{3dB}^2 \left[2W - 2f_{3dB}\tan^{-1}\left(\frac{W}{f_{3dB}}\right)\right]$$

$$= 2\frac{N_0}{A^2} f_{3dB}^3 \left[\frac{W}{f_{3dB}} - \tan^{-1}\left(\frac{W}{f_{3dB}}\right)\right]$$

[6, new theory]

iii) Without de-emphasis

$$P_N = \frac{2N_0W^3}{3A^2}$$

[3, bookwork]

iv) Improvement

$$I = \frac{\frac{2N_0W^3}{3A^2}}{2\frac{N_0}{A^2} f_{3dB}^2\left[\frac{W}{f_{3dB}} - \tan^{-1}\left(\frac{W}{f_{3dB}}\right)\right]}$$

$$= \frac{W^3}{3f_{3dB}^3\left[\frac{W}{f_{3dB}} - \tan^{-1}\left(\frac{W}{f_{3dB}}\right)\right]}$$

$$RC = 6\times10^{-5} \Rightarrow f_{3dB} = \frac{1}{2\pi RC} = \frac{1}{2\pi\times6\times10^{-5}} = 2.65\ \text{kHz}$$

$$I = \frac{15^3}{3\times2.65^3\times\left[\frac{15}{2.65} - 1.4\right]} = \frac{3375}{238} = 14.2$$

(11.5 dB)

[3, new application]

3. a) i) $m(t) = A_m(\cos w_m t + \sin w_m t)$

$$= \sqrt{2} A_m \cos(w_m t - \tfrac{\pi}{4})$$

$$P_S = \tfrac{1}{2}(\sqrt{2}A_m)^2 = A_m^2 \qquad \Delta = \frac{2\sqrt{2}A_m}{2^n}$$

$$P_N = \frac{\Delta^2}{12} = \frac{8A_m^2}{12 \times 2^{2n}} = \frac{2A_m^2}{3 \times 2^{2n}}$$

$$SNR = \frac{P_S}{P_N} = \frac{3}{2} \times 2^{2n} \Rightarrow 6.02n + 1.76 \text{ dB}$$

[6, new example]

ii) $6.02n + 1.76 = 62$

$$n = 10$$

[4, book work]

b) i) $A/\sigma = 1/0.2 = 5$

$$P_{e,ASK} = Q\left(\frac{A}{2\sigma}\right) = Q(2.5) \approx \frac{e^{-2.5^2/2}}{\sqrt{2\pi} \times 2.5} = 7 \times 10^{-3}$$

$$P_{e,FSK} = Q\left(\frac{A}{\sqrt{2}\sigma}\right) = Q\left(\frac{5}{\sqrt{2}}\right) = 2.1 \times 10^{-4}$$

$$P_{e,PSK} = Q\left(\frac{A}{\sigma}\right) = Q(5) = 3 \times 10^{-7}$$

[5, bookwork]

ii) Noncoherent detection

$$P_{e,ASK} = \frac{1}{2} e^{-\frac{A^2}{8\sigma^2}} = 2.2 \times 10^{-2}$$

$$P_{e,FSK} = \frac{1}{2} e^{-\frac{A^2}{4\sigma^2}} = 9.7 \times 10^{-4}$$

$$P_{e,PSK} = \frac{1}{2} e^{-\frac{A^2}{2\sigma^2}} = 1.9 \times 10^{-6}$$

[5, bookwork]

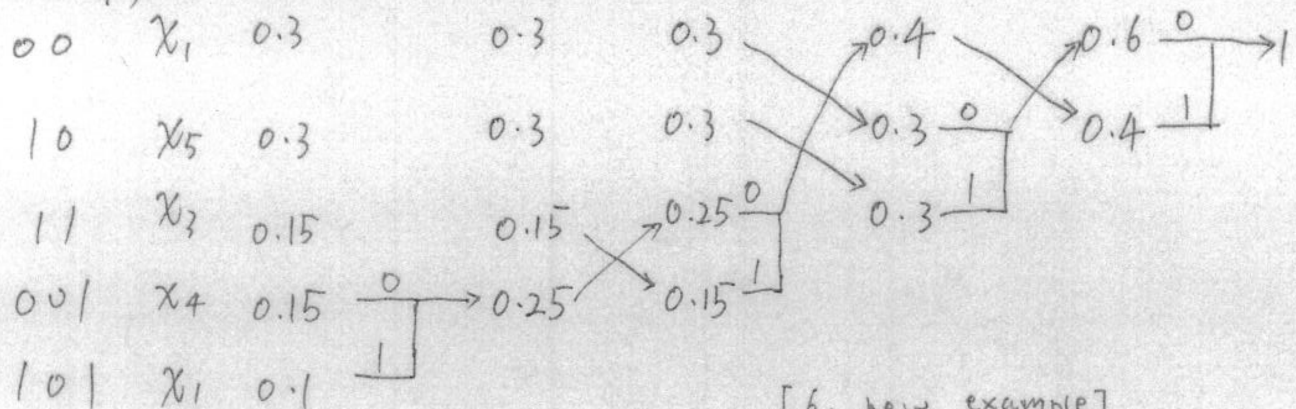
c) i)
$$Q(x) = \frac{1}{\pi}\int_0^{\pi/2} e^{-\frac{x^2}{2\sin^2\theta}}\, d\theta$$
$$\le \frac{1}{\pi}\int_0^{\pi/2} e^{-\frac{x^2}{2}}\, d\theta \quad \text{since } \sin^2\theta \le 1$$
$$= \frac{\pi/2}{\pi} e^{-\frac{x^2}{2}}$$
$$= \frac{1}{2} e^{-\frac{x^2}{2}}$$

[4, new theory]

ii)
$$Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\, dt$$
$$= -\int_x^\infty \frac{1}{\sqrt{2\pi}\, t}\, d e^{-t^2/2}$$
$$= -\frac{e^{-t^2/2}}{\sqrt{2\pi}\, t}\Bigg|_x^\infty - \int_x^\infty \frac{e^{-t^2/2}}{\sqrt{2\pi}\, t^2}\, dt$$
$$= \frac{e^{-x^2/2}}{\sqrt{2\pi}\, x} - \int_x^\infty \frac{e^{-t^2/2}}{\sqrt{2\pi}\, t^2}\, dt$$
$$\le \frac{e^{-x^2/2}}{\sqrt{2\pi}\, x} \quad \text{since the second term is non negative.}$$

[6, new theory]

4. a) i)



[6, new example]

ii) Average code word length

$$\bar{L} = 2\times 0.3 + 2\times 0.3 + 2\times 0.15 + 3\times 0.15 + 3\times 0.1$$

$$= 2.25$$

Entropy

$$H(S) = -\sum p_i \log_2(p_i)$$

$$= 2\times(-0.3\times(\log_2(0.3)) + 2\times(-0.15\times\log_2(0.15)) + (-0.1\times\log_2(0.1))$$

$$= 2.19$$

$$\eta = \frac{2.19}{2.25} = 97.3\%$$ [6, new application]

b) i)

$$G = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{\text{row 4 + row 3}} \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[\text{row3}]{\text{row2}+} \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[\text{row4}]{\text{row1}+\ \text{row2}+} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

[6, new application]

Systematic form

ii) $g(z)\,h(z) = z^7 + 1$

$$\begin{array}{r l} & z^4 + z^3 + z^2 + 1 \\ z^3+z^2+1 \,\big) & z^7 + 1 \\ & z^7 + z^6 + z^4 \\ \hline & z^6 + z^4 + 1 \\ & z^6 + z^5 + z^3 \\ \hline & z^5 + z^4 + z^3 + 1 \\ & z^5 + z^4 + z^2 \\ \hline & z^3 + z^2 + 1 \end{array}$$

$h(z) = z^4 + z^3 + z^2 + 1$

[4, new example]

iii) $$H = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$d_{min} = 3$ because the smallest number of dependent rows is 3.

error detection $t = d_{min} - 1 = 2$

error correction $t = \frac{d_{min}-1}{2} = 1$

[4, bookwork]

iv) Hamming Bound $r = n - k \geq \log_2 \zeta(n,t)$

$$\zeta(n,t) = \sum_{i=0}^{t} \binom{n}{i} \underset{n=7}{\overset{t=1}{=\!=}} \binom{7}{0} + \binom{7}{1} = 8$$

$\log_2 \zeta(n,t) = 3$

$r = n - k = 3$

$\therefore$ Yes, it's a 'perfect' code in this sense.

[4, new application]