MODELLING AND CONTROL OF MULTIBODY MECHANICAL SYSTEMS

1. A compound 'ball-jointed' pendulum of mass m consists of a symmetric rigid body fixed at one of its end points along its axis of symmetry (point O in Figure 1.1), and which is free to rotate about the fixed point in any direction under the influence of gravity. The centre of mass of the pendulum is on the axis of symmetry at distance f from the fixed point. A body-fixed Cartesian coordinate system has its origin at the fixed point and its z-axis (Z') along the axis of symmetry of the body. The body-fixed x-, y-, and z-axes ($X'Y'Z'$) correspond to principal axes with principal moments of inertia I_{xx}, I_{xx}, and I_{zz} respectively. In the nominal configuration the body-fixed x-, y-, and z-axes are respectively aligned with the x-, y-, and z-axes (XYZ) of an earth-fixed Cartesian coordinate system, in which the x-axis is horizontal and the z-axis points downwards.

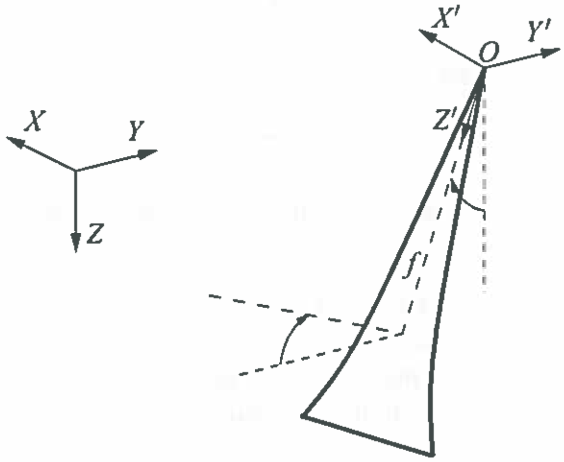

Figure 1.1 Compound 'ball-jointed' pendulum.

The angular configuration of the body is given by three Euler angles in the 3-2-3 convention. In this convention the body is first rotated from its nominal configuration by an angle ψ about the z-axis, then by an angle θ about the intermediate y-axis of the body and finally by an angle ϕ about the new z-axis of the body.

a) By making use of the standard single-axis-rotation transformation matrices, calculate

i) the complete transformation matrix A from earth-fixed to body-fixed coordinates, and [4]

Many students did this correct

ii) the body angular velocity vector in the body-fixed coordinate system, [4]

Many did this correctly but some got the sequence of matrix multiplication of the different velocity components wrong.

in terms of the generalised coordinates.

b) By expressing the unit vector of the earth-fixed z-axis in body-fixed coordinates, compute the total external moment N acting on the rigid body due to gravity in the body-fixed coordinate system. [5]

A small number of students got this right but most did not realise that it was simply $\underline{r} \times \underline{F}$. They either used projection of some force or they tried to get N from $\frac{dH}{dt} + \Omega \times H$.

c) Make use of the Euler equations of motion for a rigid body with one point fixed to derive the equations of motion of the compound pendulum in terms of the generalised coordinates. [7]

Many students got this right even though their N and Ω expressions were wrong and got most of the marks

A number of students rederived the Euler eq. from $\frac{dH}{dt} + \Omega \times H = N$ but that was unnecessary as they could just take them from the notes.

2. A 2-wheeled self-balancing robot comprises a small uniform wheel of radius r_2 that rolls by an angle ϕ_2 without slipping on a large uniform wheel of radius r_1 under the influence of gravity, as shown in Figure 2.1; the small wheel is constrained to always touch the large wheel by a massless link (not shown) joining the centres of the two wheels. The large wheel rolls by an angle ϕ_1 without slipping on a horizontal surface, and the large wheel centre moves horizontally by a distance x. The mass and spin inertia of the large wheel are m_1 and I_1 respectively, and the corresponding quantities for the small wheel are m_2 and I_2 respectively. A control torque T_c is applied on the large wheel and reacts on earth.

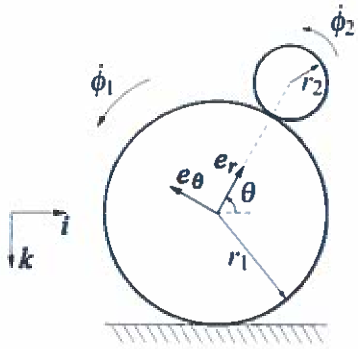

Figure 2.1 Self-balancing robot with two wheels.

A fixed Cartesian coordinate system with unit vectors $\boldsymbol{i}$ and $\boldsymbol{k}$, and a moving Cartesian coordinate system with unit vectors $\boldsymbol{e_r}$ and $\boldsymbol{e_\theta}$ (see Figure 2.1) are used to analyse the motion of the two wheels. The moving coordinate system has its origin at the centre of the large wheel and it rotates with the line joining the centres of the two wheels by an angle θ.

a) Write the position vectors of the centre of mass of the large wheel and of the centre of mass of the small wheel. [2]

most did it correctly

b) Determine the velocity vectors of the centre of mass of the large wheel and of the centre of mass of the small wheel. [3]

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c) Compute the total kinetic energy and potential energy of the system, and hence determine the Lagrangian function. [5]

generally well but some common mistakes are: taking the square of the velocity by squaring two components that are not orthogonal

d) Derive the equation of the rolling constraint between the large wheel and the horizontal surface. [2]

most did it correctly

e) The equation of the rolling constraint between the two wheels is given by

$$A_1\dot{\theta} + A_2\dot{\phi}_1 + A_3\dot{\phi}_2 = 0.$$

By considering the velocity of the instantaneous contact point between the two wheels, determine A_1, A_2, and A_3. [3]

mixed results for this one

f) Use the two rolling constraint equations to eliminate the x and ϕ_2 generalised coordinates in the Lagrangian function. Use the Lagrangian approach to derive the equations of motion of the wheel in terms of the generalised coordinates ϕ_1 and θ. [5]

Some did not follow the instruction of eliminating first x and ϕ_2.
Many got the RHS of the Lagrangian eq. wrong, or they added Lagrange multipliers unnecessarily

3. Two identical uniform wheels of mass m and radius R each are held at their centre of mass respectively at the left and right ends of a uniform axle of length l, as shown in Figure 3.1. The axle has only one rotational degree of freedom about a vertical axis through its middle point, A, represented by an angle ψ. The axle moment of inertia about its axis of rotation is I_{axle}. The left and right wheels spin relative to the axle by angles θ_l and θ_r respectively. Each of the wheels is axisymmetric with spin moment of inertia I_{yy} and radial moment of inertia, passing through the wheel centre of mass, I_{xx}.

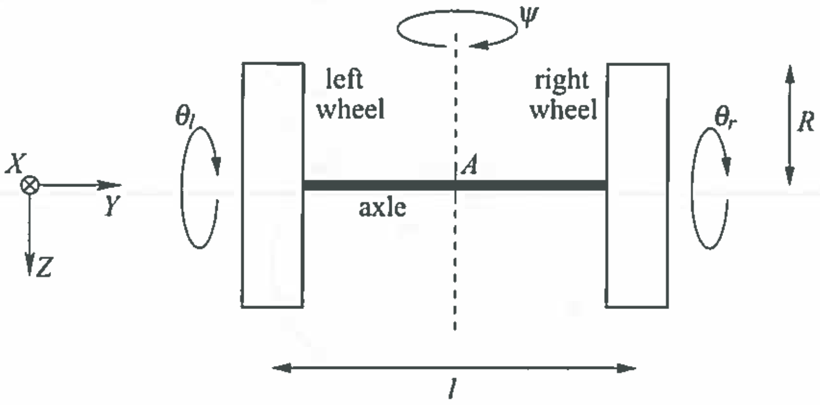

Figure 3.1 Two wheels on an axle.

Three body-fixed Cartesian coordinate systems, fixed in the left wheel, the axle and the right wheel respectively, are used to analyse the motion of the system. These coordinate systems have their origin at the left wheel centre of mass, point A, and the right wheel centre of mass respectively, and their unit vectors are initially respectively aligned with the earth-fixed axes X, Y and Z, in which X is into the page, as shown in Figure 3.1.

a) Write the generalised coordinates of the system. [2]

Most students got this right but surprisingly some didn't

b) The rotation of each of the left and right wheels is represented respectively by three Euler angles in the yaw-roll-pitch convention. By making use of the standard single-axis-rotation transformation matrices, calculate the body angular velocity vector of each wheel in its respective body-fixed coordinate system, in terms of the generalised coordinates. [4]

Many got this right but many also got the sequence of matrix multiplic wrong, and also many did not real that $\varphi=0$

c) Compute the kinetic energy of the left wheel. [4]

The most common mistake here is to only consider the K.E. due to the rotation and not take int account the K.E. due to translation of the wheel C.O.M

d) Calculate the total kinetic energy of the system. [3]

Mostly done correctly at the level of $T = T_{left} + T_{right} + T_{axle}$ some forgot to include T_{axle} and some used expressions with θ for both left and right wheels instead of distinguishing θ_l and θ_r

e) Assume that the wheels rest on a rough horizontal road such that their contact points with the road are instantaneously at rest, as the axle rotates.

i) Write the constraint equation for each wheel due to their road contact point being at rest. [2]

Mostly done correctly with only some cases of sign mistakes.

ii) Use the <u>Lagrangian</u> approach to derive the equations of motion of the system when a moment M is acting on the axle about the vertical axis. [5]

a relatively small number of students did this correctly. Many students confused how the Lagrange multipliers should be used and in which Lagrangian equation M enters.

4. A 2-wheeled self-balancing robot comprises a small uniform wheel of radius r_2 that rolls by an angle ϕ_2 without slipping on a large uniform wheel of radius r_1 under the influence of gravity, as shown in Figure 4.1; the small wheel is constrained to always touch the large wheel by a massless link (not shown) joining the centres of the two wheels. The large wheel rolls by an angle ϕ_1 without slipping on a horizontal surface, and the large wheel centre moves horizontally by a distance x. The mass and spin inertia of the large wheel are m_1 and I_1 respectively, and the corresponding quantities for the small wheel are m_2 and I_2 respectively. A control torque T_c is applied on the large wheel and reacts on earth.

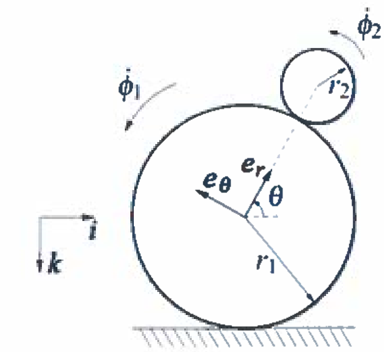

Figure 4.1 Self-balancing robot with two wheels.

A fixed Cartesian coordinate system with unit vectors $\boldsymbol{i}$ and $\boldsymbol{k}$, and a moving Cartesian coordinate system with unit vectors $\boldsymbol{e_r}$ and $\boldsymbol{e_\theta}$ (see Figure 4.1) are used to analyse the motion of the two wheels. The moving coordinate system has its origin at the centre of the large wheel and it rotates with the line joining the centres of the two wheels by an angle θ.

a) Write the generalised coordinates of the system. ← surprisingly many students got this wrong by showing only two coord. or by including φ_1, φ_2 [2]

This was mostly done correctly

b) Determine the acceleration vectors of the centre of mass of the large wheel and of the centre of mass of the small wheel. [3]

c) F_{roll1} and R_1 are the unknown constraint forces acting on the large wheel and reacting on earth that respectively maintain the rolling constraint between the large wheel and the horizontal surface, and prevent the large wheel from sinking into the horizontal surface. F_{roll2} and R_2 are the unknown constraint forces acting on the small wheel and reacting on the large wheel that respectively maintain the rolling constraint between the two wheels and the radial contact between the two wheels. Write the total force that acts on each of the wheels in vector form in terms of the unknown constraint forces. [3]

Some students got this one right. Common mistakes were to include F_{roll2} only in the total force expression for the small wheel without any reaction on the large wheel, or not to include R_2 and F_{roll2} at all in the large wheel force expression. Some also forgot to include the gravity forces.

d) By considering the translational motion of the centre of mass of each of the wheels, use the vectorial approach to derive F_{roll2}, R_2, F_{roll1}, and R_1 in terms of the generalised coordinates. [6]

This was generally well attempted but often the solutions had careless mistakes, and often they did not decompose in the most convenient bas vectors and the resulting expression were too difficult to handle.

e) By considering the rotational motion about the centre of mass of each of the wheels, use the vectorial approach to derive the equations of motion of the system with respect to the generalised coordinates. [6]

many students did this correctly (assuming the force expressions substituted from the solution in part (d) were correct, but some got it wrong by not including T_c in the equation for the large wheel or by missing out either F_{roll1} or F_{roll2} in the expression for the small wheel.