UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 2000

MEng Honours Degrees in Computing Part IV
MSc in Advanced Computing
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the Associateship of the City and Guilds of London Institute

PAPER C477=I4.20

COMPUTING FOR OPTIMAL DECISIONS

Friday 5 May 2000, 10:00
Duration: 120 minutes

Answer THREE questions

Paper contains 4 questions

1 a Consider the problem

$$\underset{x \in \Re^n}{\text{minimise}} \left\{ \sum_{i=1}^{n} x_i f_i(x_i) \;\middle|\; \sum_{i=1}^{n} x_i = 1 \right\}$$

where

$$f_i(x_i) = \begin{cases} x_i^2 + 3x_i & \text{if } x_i \le 0 \\ i\, x_i & \text{if } x_i > 0 \end{cases}.$$

Formulate this as a nonlinear programming problem with continuous variables, differentiable objective function and linear constraints. Do not solve the nonlinear programming problem. [Hint: Let x_i be expressed as the difference of two numbers $x_i = x_i^+ - x_i^-$; $x_i^+, x_i^- \ge 0$ and $x_i^+ \times x_i^- = 0$.]

b Consider the optimisation problem

$$\underset{x \in \Re^n}{\text{minimise}} \left\{ f(x) \;\middle|\; A\,x \le b \right\},$$

where f is a differentiable function of x, A is a coefficient matrix, b a constant vector of appropriate dimension. At a given feasible (but nonoptimal) point x_k, the direction of search is defined by the solution of the linear programming (LP) subproblem

$$\underset{x \in \Re^n}{\text{minimise}} \left\{ \nabla f(x_k)^T (x - x_k) \;\middle|\; A\,x \le b \right\}.$$

Let $\hat{x}$ denote the solution of the LP. Establish and discuss the property of the direction $(\hat{x} - x_k)$ which may be used in the search for the solution of the original optimisation problem.

(All parts carry equal marks)

2 a Consider the general nonlinear programming problem

$$\underset{x \in \Re^n}{\text{minimise}} \left\{ f(x) \;\middle|\; g(x) \le b \right\},$$

where f, g are differentiable convex functions of x, g is an m-dimensional vector and b is a given constant m-dimensional vector. Write the first-order necessary conditions for optimality for this problem.

b Given that f and g in (a) above are convex functions, establish the number of optima for the problem.

c Let the right hand side vector b in (a) above be replaced by $(b + \Delta b)$, for some small Δb. Evaluate the corresponding change in the optimal value of the objective function, in terms of the optimal solution of the original problem and Δb.

(All parts carry equal marks)

3 Consider the problem

$$\underset{x}{\text{maximise}} \left\{ -2x_1 - (x_2 - 3)^2 \;\middle|\; x_1 \ge 3 \;;\; x_2 \ge 3 \right\}.$$

a Formulate the barrier function used by the SUMT algorithm for this problem and derive analytically the maximising solution (x_1, x_2) as a function of the barrier parameter. [Hint: Do not use the logarithmic barrier function and adopt positive roots of polynomials as maximiser.]

b Describe the application of two iterations of the algorithm to the above problem. You need not perform the arithmetic computation.

c Consider imposing further the constraint $x_1 + x_2 = 10$ to the above problem. Describe how you would incorporate this equality constraint within the SUMT framework.

d Suggest any other algorithm for solving the above problem (with linear inequality and equality constraints).

(All parts carry equal marks)

4 a Consider the investment problem with three stocks. Let r_i, $i = 1, 2, 3$, denote the random variable return plus capital for stock i. We are given two scenarios for the expected value of r_i, $E(r_i)$: $\hat{r}_i^1$ and $\hat{r}_i^2$. The values are

$$\begin{bmatrix} \hat{r}_1^1 \\ \hat{r}_2^1 \\ \hat{r}_3^1 \end{bmatrix} = \begin{bmatrix} 1.15 \\ 1.21 \\ 1.09 \end{bmatrix}; \quad \begin{bmatrix} \hat{r}_1^2 \\ \hat{r}_2^2 \\ \hat{r}_3^3 \end{bmatrix} = \begin{bmatrix} 1.22 \\ 1.05 \\ 1.11 \end{bmatrix}.$$

Also, $\text{var}(r_1) = .08$, $\text{var}(r_2) = .05$, $\text{var}(r_3) = .07$, $\text{cov}(r_1 r_2) = .05$, $\text{cov}(r_1 r_3) = -.03$, $\text{cov}(r_2 r_3) = .01$. We have £1000 to invest and wish to have an expected return of at least 17%. Formulate the portfolio of minimum variance that will attain the expected return desired given the two scenarios. Do not solve the optimisation problem.

b A computer manufacturer produces two types of computers: A and B. If the unit price charged is p_1 for A and p_2 for B, the number of computers that can be sold are q_1 of A and q_2 of B, where

$$q_1 = .9\,p_2 - 10\,p_1 + 5000 + \epsilon_1\,;$$

$$q_2 = .7\,p_1 - 8\,p_2 + 3000 + \epsilon_2$$

where ϵ_1, ϵ_2 are zero mean, normally distributed random uncertainties with $\text{var}(\epsilon_1) = .06$, $\text{var}(\epsilon_2) = .07$, $\text{cov}(\epsilon_1 \epsilon_2) = .01$. Furthermore, the prices of the two computers are related such that

$$p_1 = 1500 - \frac{2}{3}\,p_2.$$

Formulate the mean-variance optimisation problem that will maximise the revenue for the manufacturer and minimise risk due to the uncertainties. Do not solve the optimisation problem.

(All parts carry equal marks)