Imperial College
London

M4/5P6

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May-June 2017

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Probability Theory

Date: Tuesday 23 May 2017

Time: 10:00 - 12:30

Time Allowed: 2.5 Hours

This paper has 5 Questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the relevant main book(s) are full.

All required additional material will be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers to each question as per the table below.

Raw Mark	Up to 12	13	14	15	16	17	18	19	20
Extra Credit	0	½	1	1 ½	2	2 ½	3	3 ½	4

- Each question carries equal weight.
- Calculators may not be used.

1. (i) Give the definition of a random variable, its distribution and distribution function.

   (ii) Let $F(x)$ be the distribution function of a random variable. Let $a \in \mathbb{R}$. Is it possible that the limit $\lim_{x\to a} F(x)$ exists but is not equal to $F(a)$? Give a reason for your answer.

   (iii) Formulate the central limit theorem for independent identically distributed random variables.

2. (i) Give any two definitions of what it means that a sequence of random variables ζ_n converges to a random variable ζ in distribution.

   (ii) Are $f_1(t) = \cos(t/2)$, $f_2(t) = \cos^6(t)$, $f_3(t) = \exp\{-t^6\}$ characteristic functions of some random variables? In each case, if yes: (a) describe this random variable, for example, by providing its distribution; (b) find its expectation and variance.

3. (i) Let $\mathcal{P}$ be a family of probability measures on $\mathbb{R}$ which is relatively compact with respect to weak convergence. Prove that $\mathcal{P}$ is tight.

   (ii) Let ζ_n, $n = 1, 2, \ldots$, be independent Bernoulli random variables with parameters $1/n^2$, namely $P(\zeta_n = 1) = 1/n^2$, $P(\zeta_n = 0) = 1 - 1/n^2$. Prove or disprove that $\zeta_n \to 0$ a.s.

4. Let X_k, $k = 1, 2, \ldots$, be positive independent identically distributed random variables, $Y_k = \frac{X_{2k-1}}{X_{2k-1}+X_{2k}}$, and $S_n = Y_1 + \cdots + Y_n$. Is it true that $\frac{S_n}{n} \to a$ a.s. for some constant a? If yes, what is the value of a? Justify your answers.

5. (i) Let ζ_n, $n = 1, 2, \ldots$, be independent identically distributed random variables such that $E\zeta_1 = 0$, $0 < E\zeta_1^2 < \infty$. Let $S_n = \zeta_1 + \cdots + \zeta_n$. Show that $\lim_{n\to\infty} E\frac{|S_n|}{n} = 0$.

(ii) Let ζ, η be independent identically distributed random variables such that $\zeta + \eta$ has the same distribution as $\zeta + 1$. What can be said about the distribution of ζ? Justify your answer.

	EXAMINATION SOLUTIONS 2016-17	Course Prob.
Question 1		Marks & seen/unseen
Parts		
i	Let $(\Omega, \mathcal{F}, P)$ be a probability space. A function $\xi : \Omega \to R$ is called random variable if $\xi^{-1}(B) \in \mathcal{F}$ for any $B \in \mathcal{B}(R)$ – Borel sets.	2 seen
	The distribution P_ξ of ξ is a probability measure on $(R, \mathcal{B}(R))$ given by $P_\xi(B) = P(\xi^{-1}(B))$ $\forall B \in \mathcal{B}(R)$	2 seen
	The distribution function $F_\xi(x) = P_\xi(-\infty, x]$ $\forall x \in R$	2 seen
ii	No, since distribution function is nondecreasing	6 unseen
	Setter's initials	Checker's initials	Page number 1

iii

Let $\xi_1, \xi_2, \dots$ be i.i.d. r.v. with $0 < E\xi_1^2 < \infty$; $V\xi_1 \neq 0$; $S_n = \xi_1 + \xi_2 + \dots + \xi_n$. — 2 seen

Then

$$P\left(\frac{S_n - ES_n}{\sqrt{VS_n}} \le x\right) \xrightarrow[n\to\infty]{} \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2}\, du, \quad \forall x \in R$$

— 6 seen

i.e.

$$\frac{S_n - ES_n}{\sqrt{VS_n}} \xrightarrow{d} \mathcal{N}(0,1).$$

Question		Marks & seen/unseen
2		

Parts		
i	$\xi_n \xrightarrow{d} \xi$ if $E f(\xi_n) \to E f(\xi)$ for any bounded continuous function $f(x)$.	2 seen
	Equivalently, if the distribution functions $F_{\xi_n}(x) \to F_\xi(x)$ at any point of continuity of $F_\xi(x)$.	2 seen
ii	$f_1(t) = \cos t/2$ is the characteristic function of Bernoulli r.v. ξ such that $P(\xi = 1/2) = P(\xi = -1/2) = 1/2$	6 unseen

Setter's initials | Checker's initials

	EXAMINATION SOLUTIONS 2016-17	Course Prob
Question 2		Marks & seen/unseen
Parts	$E\xi = 0$, $V\xi = E\xi^2 = 1/4$. $f_2(t) = \cos^6 t$ is the characteristic function of the sum of 6 i.i.d. Bernoulli r.v. such that $P(\xi = 1) = P(\xi = -1) = 1/2$; $E(\sum_{j=1}^{6} \xi_j) = 0$, $V(\sum_{j=1}^{6} \xi_j) = \sum_{j=1}^{6} V\xi_j =$ $= 6 \cdot 1 = 6$ (by independence) $f_3(t) = e^{-t^6}$ is not a characteristic function by Marcinkiewicz theorem.	6 unseen 4 unseen
	Setter's initials Checker's initials	Page number 4

	EXAMINATION SOLUTIONS 2016-17	Course Prob
Question 3		Marks & seen/unseen
Parts		
i	Suppose $\mathcal{P}$ is not tight, i.e. $\exists\, \varepsilon > 0$ s.t. for any compact $K \subset \mathbb{R}$, $\sup_{\alpha} P_\alpha(\mathbb{R}\setminus K) > \varepsilon$, where $\mathcal{P} = \{P_\alpha\}$. Take $K = [-n, n]$. Therefore, for any n $\exists\, P_{\alpha_n} \in \mathcal{P}$ s.t. $P_{\alpha_n}(\mathbb{R}\setminus(-n,n)) > \varepsilon$. (1) By relative compactness, there exists a subsequence $P_{\alpha_{n_k}}$ converging weakly to a probability measure Q.	10 seen
	Setter's initials	Checker's initials
		Page number 5

	EXAMINATION SOLUTIONS 2016-17	Course Prob.
Question 3		Marks & seen/unseen
Parts	By one of the definitions of weak convergence, $\limsup P_{\alpha_{n_k}}(R \setminus (-n,n)) \leq$ $\leq Q(R \setminus (-n,n))$ But $Q(R \setminus (-n,n)) \to 0$ as $n \to \infty$, which contradicts inequality (1). Therefore, $\mathcal{P}$ is tight.	
	Setter's initials Checker's initials	Page number 6

	EXAMINATION SOLUTIONS 2016-17	Course Prob
Question 3		Marks & seen/unseen
Parts ii	We have that $\sum_{n=1}^{\infty} P(\zeta_n = 1) = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$. Therefore, by Borel-Cantelli lemma, $P(\zeta_n = 1 \ i.o.) = 0$ The probability of the complement event $P(\zeta_n = 0 \ ev.) = 1$ i.e. $\zeta_n \to 0$ a.s.	10 unseen
	Setter's initials Checker's initials	Page number 7

	EXAMINATION SOLUTIONS 2016-17	Course Prob.
Question 4		Marks & seen/unseen

Parts

1) y_k's are identically distributed, indeed

$$P(y_k \in B) = \int \chi_{\{\frac{t}{t+s} \in B\}} \, dP_{(X_{2k-1}, X_{2k})}(t,s)$$

$$= \int \chi_{\{\frac{t}{t+s} \in B\}} \, dP(t) \, dP(s)$$

since X_k's are independent, identically distributed.

2) y_k's are independent, since, similarly,

$$P(y_k \in B_k, y_j \in B_j) =$$

$$= \int \chi_{\{\frac{t}{t+s} \in B_k\}} \chi_{\{\frac{u}{u+v} \in B_j\}} \, dP_{(X_{2k-1}, X_{2k}, X_{2j-1}, X_{2j})}$$

$$= P(y_k \in B_k) \, P(y_j \in B_j),$$

$k \neq j$.

Setter's initials	Checker's initials	Page number 8

	EXAMINATION SOLUTIONS 2016-17	Course Prob
Question 4		Marks & seen/unseen

Parts

3) $E|Y_k| = E\left|\frac{X_{2k-1}}{X_{2k-1}+X_{2k}}\right| \le$ unseen

$\le E(1) = 1 < \infty$

Therefore, by Kolmogorov's strong LLN,

$S_n/n \longrightarrow EY_1$ a.s. 12

$$2EY_1 = 2\int \frac{t}{t+s}\,dP(t)\,dP(s)$$

$$= \int \frac{t}{t+s}\,dP(t)\,dP(s) + \int \frac{s}{t+s}\,dP(t)\,dP(s) = 1$$

$\Rightarrow EY_1 = 1/2$. 8

Setter's initials Checker's initials

EXAMINATION SOLUTIONS 2016-17

Course: Prob, Masters' q.

Question 5

Marks & seen/unseen

Parts

i

$\zeta_1, \zeta_2, \dots$ i.i.d.

$E\zeta_1 = 0, \quad 0 < E\zeta_1^2 < \infty.$

$S_n = \zeta_1 + \dots + \zeta_n$

By Lyapunov inequality,

$$\left(E\frac{|S_n|}{n}\right)^2 \leq E\frac{S_n^2}{n^2} =$$

$$= \frac{1}{n^2} E\left(\sum_{j=1}^{n} \zeta_j^2 + \sum_{j \neq k} \zeta_j \zeta_k\right)$$

$$= \frac{1}{n^2}\left(n E\zeta_1^2 + \sum_{j\neq k} E\zeta_j \cdot E\zeta_k\right)$$

(by independence)

$$= \frac{1}{n} E\zeta_1^2, \quad (\text{since } E\zeta_j = 0)$$

$$\to 0, \text{ as } n \to \infty$$

Thus, $E\frac{|S_n|}{n} \to 0, \; n \to \infty.$ 12

unseen

Setter's initials Checker's initials

Parts

ii

We have for the characteristic functions:

$$\varphi_\xi^2(t) = \varphi_{\xi+\eta}(t) = \varphi_{\xi+1}(t) = e^{it}\varphi_\xi(t).$$

8

unseen

Therefore for any t, either $\varphi_\xi(t) = e^{it}$ or $\varphi_\xi(t) = 0$

But since $\varphi_\xi(t)$ is a characteristic function, $\varphi_\xi(0) = 1$, and since $|e^{it}| = 1$, continuity of characteristic function implies that $\varphi_\xi(t) = e^{it}$ $\forall t$.

Hence, by uniqueness of characteristic functions, $\xi = 1$ a.s.