Imperial College London

BSc/MSci EXAMINATION May 2012

This paper is also taken for the relevant Examination for the Associateship

INTRODUCTION TO QUANTUM INFORMATION

For Third-Year, Fourth-Year and MSc Physics Students

Wednesday, 16th May 2012: 10.00 to 12.00

Answer ALL of Section A, and TWO questions from Section B

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Write your CANDIDATE NUMBER clearly on each of the THREE answer books provided.

If an electronic calculator is used, write its serial number in the box at the top right hand corner of the front cover of each answer book.

USE ONE ANSWER BOOK FOR PART A

USE TWO ANSWER BOOKS FOR PART B – ONE BOOK PER QUESTION

Enter the number of each question attempted in the horizontal box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

You are reminded that the Examiners attach great importance to legibility, accuracy and clarity of expression.

Possibly useful information:

All states and matrices are given in the computational basis unless otherwise specified.

$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$|\pm\rangle \equiv (|0\rangle \pm |1\rangle)/\sqrt{2}$, $\left|\phi^{\pm}\right\rangle = (|00\rangle \pm |11\rangle)/\sqrt{2}$, $\left|\psi^{\pm}\right\rangle = (|01\rangle \pm |10\rangle)/\sqrt{2}$,

$\mathbb{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$\sigma_x \equiv X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\sigma_y \equiv Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$, $\sigma_z \equiv Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$,

Controlled-NOT=CNOT= $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

Controlled-Z=C-Z= $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

Toffoli= $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$.

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PART A

Answer all of the following

Question 1

(i) Write short notes (about 200 words) discussing *teleportation.* Points you could consider addressing include: "state swapping" Star-Trek style versus genuine quantum teleporation; the quantum and classical resources required; experimental challenges; teleporting more than just one qubit. [5]

(ii) Write short notes (about 200 words) discussing *the no-cloning theorem.* Points you could consider addressing include: the implications if cloning were possible; a simple proof of the theorem; do measurements help?; probabilistic cloning on limited sets of input states; no-cloning of classical probability distributions. [5]

(iii) Write short notes (about 200 words) discussing *entanglement distillation.* Points you could consider addressing include: why distill entanglement?; different distillation methods and how they differ in terms of numbers of copies of the states required, amounts of classical communication and probabilities of success. [5]

(iv) Write short notes (about 200 words) discussing *quantum error correction.* Points you could consider addressing include: why is it necessary?; imprecision versus noise, classical error correction, simple codes and their associated encoding and decoding circuits. [5]

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PART B

Question 2

Consider joint conditional probability distributions $\Pr(ab|xy)$ where Alice gets a measurement output labelled $a \in \{0,1\}$ when she performs measurement labelled $x \in \{0,1\}$ and similarly for Bob given by:

$$\text{Distribution 1: } \begin{bmatrix} \Pr(ab|xy) & a=0,b=0 & a=1,b=0 & a=0,b=1 & a=1,b=1 \\ x=0,y=0 & 0 & 3/4 & 1/4 & 0 \\ x=0,y=1 & 2/3 & 0 & 0 & 1/3 \\ x=1,y=0 & 1/3 & 0 & 0 & 2/3 \\ x=1,y=1 & 0 & 1/4 & 3/4 & 0 \end{bmatrix}$$

$$\text{Distribution 2: } \begin{bmatrix} \Pr(ab|xy) & a=0,b=0 & a=1,b=0 & a=0,b=1 & a=1,b=1 \\ x=0,y=0 & 0 & 1/2 & 1/2 & 0 \\ x=0,y=1 & 0 & 1/2 & 1/2 & 0 \\ x=1,y=0 & 0 & 1/2 & 1/2 & 0 \\ x=1,y=1 & 1/2 & 0 & 0 & 1/2 \end{bmatrix}$$

(i) Show Distribution 1 allows for superluminal signalling. [3]

(ii) Show that Distribution 2 violates a Bell inequality. That is it would allow two parties ('psychics') playing the game discussed in class where they must give opposite answers when local coin flips come up heads and the same answer when one or other come up tails, to win with probability better than any local classical strategy. [4]

(iii) Alice holds the first qubit of the 2 qubits prepared in state

$$|\Phi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle|+\rangle - |1\rangle|-\rangle\right)$$

and Bob holds the second qubit. Show explicitly that Alice and Bob cannot superluminally signal using a method where to send bit 0 Alice measures in the Z eigenbasis and to send bit 1 she performs no measurement at all. [4]

(iv) Imagine you are given a magical box M that implements the following transformation on a single qubit:

$$|1\rangle \xrightarrow{M} |0\rangle$$
$$|-\rangle \xrightarrow{M} |+\rangle$$

but leaves every other state unchanged, i.e.

$$|\psi\rangle \xrightarrow{M} |\psi\rangle \text{ for } |\psi\rangle \neq |1\rangle, |-\rangle$$

Show that you could devise a method to use this box for superluminal signalling. [4]

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Question 3

Alice sends Bob a qubit prepared with equal probability in either the state $|\psi_0\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}|1\rangle$ or $|\psi_1\rangle = \cos\frac{\theta}{2}|0\rangle - \sin\frac{\theta}{2}|1\rangle$ with $0 < \theta < \pi/2$. Bob measures the qubit and announces which of the two states $|\psi_0\rangle$ or $|\psi_1\rangle$ he thinks the qubit was prepared in. If he is right he wins W pounds. If he is wrong he pays Alice L pounds.

(i) Show that if

$$\frac{L}{W} < \frac{1+\sin\theta}{1-\sin\theta}$$

Bob can, on average, make money playing this game. [3]

(ii) The game is now modified so that Bob can pay D pounds and decline to announce one of the two states. Assuming he performs unambiguous discrimination, what should the ratio W/D be in terms of θ for Bob to, on average, make money? [3]

(iii) Show that in this modified game of (ii), if $L = 4W$ and θ is small, then choosing

$$\frac{D}{W} \approx \frac{3}{2} - \frac{5}{2}\theta$$

results in a game for which Bob's expected winnings are the same regardless of which (maximum-likelihood or unambiguous) discrimination strategy he uses. [4]

(iv) Show that unambiguous discrimination on two copies of a state, i.e. $|\psi_0\rangle|\psi_0\rangle$ or $|\psi_1\rangle|\psi_1\rangle$ has no greater probability of success if a single joint measurement is made on the systems, than if measurements are made on each particle separately. [2]

(v) Consider a situation wherein Bob receives a qubit prepared with equal probability in either the state $|\psi_0\rangle$ or $|\psi_1\rangle$ and he performs the standard optimal maximum likelihood estimation measurement on it. He then receives a second qubit, that he is told is prepared in the same state as the first qubit was. Explain qualitatively how and why Bob's optimal measurement for maximum-likelihood estimation on this second qubit will differ from that of the first. You do not need to derive the new optimal measurement! [3]

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Question 4

Alice and Bob share a pair of entangled qubits. Instead of being given a system prepared in some unknown state, as for teleportation, Alice is told (classically) an angle τ that is unknown to Bob. The goal is to implement a protocol such that Bob's qubit ends up in the state $|\tau\rangle \equiv \cos\frac{\tau}{2}|0\rangle + \sin\frac{\tau}{2}|1\rangle$ without Alice having to transmit the (possibly infinite) number of classical bits of τ. This task is called remote state preparation.

(i) Consider the entangled pair of qubits Alice and Bob share are in the state

$$\left|\phi^+\right\rangle = (|0\rangle|0\rangle + |1\rangle|1\rangle)/\sqrt{2}.$$

Show that if Alice performs the measurement on her qubit described by projectors $\{|\tau\rangle\langle\tau|, |\bar{\tau}\rangle\langle\bar{\tau}|\}$, with $\langle\bar{\tau}|\tau\rangle = 0$, then they can achieve remote state preparation of $|\tau\rangle$ using only 1 bit of classical communication. What is the unitary operation Bob must apply if Alice obtains the $|\bar{\tau}\rangle\langle\bar{\tau}|$ outcome? [4]

(ii) Consider instead that the entangled pair of qubits Alice and Bob share are in the state the non-maximally-entangled state

$$|\chi\rangle = \frac{\sqrt{3}}{2}|0\rangle|0\rangle + \frac{1}{2}|1\rangle|1\rangle.$$

If Alice and Bob do procrustean distillation and then follow the procedure in (i), what is the probability they successfully achieve remote state preparation of $|\tau\rangle$? How many classical bits of communication are required? [2]

For questions (iii) and (iv) consider Alice and Bob share $|\chi\rangle$, but they are only allowed one bit of classical communication. When she is told τ Alice performs a measurement $\{|\theta\rangle\langle\theta|, |\bar{\theta}\rangle\langle\bar{\theta}|\}$ with $|\theta\rangle \equiv \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}|1\rangle$, $\langle\bar{\theta}|\theta\rangle = 0$, choosing θ carefully such that when she obtains the $|\theta\rangle\langle\theta|$ outcome Bob's qubit is collapsed to $|\tau\rangle$ as desired.

(iii) For the particular values of $\tau = 0, \pi/3, \pi/2$ determine the corresponding values of θ Alice should use, and the probability of success for each case. [6]

(iv) Compare your answers from (iii) with the procedure in (ii), and discuss when doing procrustean distillation first is not optimal. Why can Bob not apply a unitary to correct his qubit in those cases that Alice obtains the $|\bar{\theta}\rangle\langle\bar{\theta}|$ outcome, as could be done in the procedure of (i)? [3]

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Question 5

(i) Draw a quantum circuit constructed from standard gates such as CNOT, Toffoli, Pauli or Hadamard gates, that implements the "phase kickback" quantum evaluation of a function:

$$|x\rangle \longrightarrow (-1)^{g(x)}\,|x\rangle\,,$$

for the two-bit function $g : \{0,1\}^2 \to \{0,1\}$ defined by [3]

$$g(00) = 1,\; g(01) = 0,\; g(10) = 0,\; g(11) = 0.$$

Consider a simple case of Grover's algorithm where the "black box" function $f : \{0,1\}^n \to \{0,1\}$ is only used once, and Grover's inversion about average operation is only applied once.

(ii) Show that if the number of target inputs (the number of inputs for which $f(x) = 1$) is $2^n/2$ then there is no advantage over simply choosing a random input and evaluating f. [2]

(iii) Show that if the number of target inputs (the number of inputs for which $f(x) = 1$) is $2^n/4$ then a target state will be found with certainty. [3]

You are given a "black box" that takes in two qubits and you are promised it will act one of the following 4 unitary operations on them:

$$U_0 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, U_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, U_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, U_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

(iv) Sketch a 2 qubit circuit that *with only one use* of the black box and one use of Grover's inversion about average operator G will allow for determination of which of the 4 unitary gates has been applied. [3]

(v) Show that feeding a two qubit cluster state into the black box can also be used to determine which of the 4 gates is being applied. [4]

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END OF EXAMINATION