# Digital Signal Processing and Digital Filters

********* **Solutions** *********

**Information for Candidates:**

## Notation

- All signals and filter coefficients are real-valued unless explicitly noted otherwise.
- Unless otherwise specified, upper and lower case letters are used for sequences and their $z$-transforms respectively. The signal at a block diagram node $V$ is $v[n]$ and its $z$-transform is $V(z)$.
- $x[n] = [a, b, c, d, e, f]$ means that $x[0] = a, \ldots x[5] = f$ and that $x[n] = 0$ outside this range.
- $\Re(z)$, $\Im(z)$, $z^*$, $|z|$ and $\angle z$ denote respectively the real part, imaginary part, complex conjugate, magnitude and argument of a complex number $z$.

## Abbreviations

| | |
|---|---|
| BIBO | Bounded Input, Bounded Output |
| CTFT | Continuous-Time Fourier Transform |
| DCT | Discrete Cosine Transform |
| DFT | Discrete Fourier Transform |
| DTFT | Discrete-Time Fourier Transform |
| LTI | Linear Time-Invariant |
| MDCT | Modified Discrete Cosine Transform |
| SNR | Signal-to-Noise Ratio |

## Standard Sequences

- $\delta[n] = 1$ for $n = 0$ and 0 otherwise.
- $\delta_{\text{condition}}[n] = 1$ whenever "condition" is true and 0 otherwise.
- $u[n] = 1$ for $n \geq 0$ and 0 otherwise.

## Geometric Progression

- $\sum_{n=0}^{r} \alpha^n z^{-n} = \frac{1-\alpha^{r+1}z^{-r-1}}{1-\alpha z^{-1}}$ or, more generally, $\sum_{n=q}^{r} \alpha^n z^{-n} = \frac{\alpha^q z^{-q} - \alpha^{r+1}z^{-r-1}}{1-\alpha z^{-1}}$

## Forward and Inverse Transforms

| | | |
|---|---|---|
| z: | $X(z) = \sum_{-\infty}^{\infty} x[n]z^{-n}$ | $x[n] = \frac{1}{2\pi j}\oint X(z)z^{n-1}dz$ |
| CTFT: | $X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt$ | $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)e^{j\Omega t}d\Omega$ |
| DTFT: | $X(e^{j\omega}) = \sum_{-\infty}^{\infty} x[n]e^{-j\omega n}$ | $x[n[ = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$ |
| DFT: | $X[k] = \sum_{0}^{N-1} x[n]e^{-j2\pi\frac{kn}{N}}$ | $x[n] = \frac{1}{N}\sum_{0}^{N-1} X[k]e^{j2\pi\frac{kn}{N}}$ |
| DCT: | $X[k] = \sum_{n=0}^{N-1} x[n]\cos\frac{2\pi(2n+1)k}{4N}$ | $x[n] = \frac{X[0]}{N} + \frac{2}{N}\sum_{n=1}^{N-1} X[k]\cos\frac{2\pi(2n+1)k}{4N}$ |
| MDCT: | $X[k] = \sum_{n=0}^{2N-1} x[n]\cos\frac{2\pi(2n+1+N)(2k+1)}{8N}$ | $y[n] = \frac{1}{N}\sum_{0}^{N-1} X[k]\cos\frac{2\pi(2n+1+N)(2k+1)}{8N}$ |

## Convolution

DTFT: $v[n] = x[n] * y[n] \triangleq \sum_{r=-\infty}^{\infty} x[r]y[n-r] \quad \Leftrightarrow \quad V\left(e^{j\omega}\right) = X\left(e^{j\omega}\right)Y\left(e^{j\omega}\right)$

$v[n[ = x[n]y[n] \quad \Leftrightarrow \quad V\left(e^{j\omega}\right) = \frac{1}{2\pi}X\left(e^{j\omega}\right) \circledast Y\left(e^{j\omega}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j\theta}\right)Y\left(e^{j(\omega-\theta)}\right)d\theta$

DFT: $v[n] = x[n] \circledast_N y[n[ \triangleq \sum_{r=0}^{N-1} x[r]y[(n-r)_{\bmod N}] \quad \Leftrightarrow \quad V[k[ = X[k]Y[k]$

$v[n] = x[n]y[n[ \quad \Leftrightarrow \quad V[k[ = \frac{1}{N}X[k] \circledast_N Y[k[ \triangleq \frac{1}{N}\sum_{r=0}^{N-1} X[r]Y[(k-r)_{\bmod N}]$

## Group Delay

The group delay of a filter, $H(z)$, is $\tau_H(e^{j\omega}) = -\frac{d\angle H(e^{j\omega})}{d\omega} = \Re\left(\frac{-z}{H(z)}\frac{dH(z)}{dz}\right)\Big|_{z=e^{j\omega}} = \Re\left(\frac{\mathscr{F}(nh[n])}{\mathscr{F}(h[n])}\right)$ where $\mathscr{F}()$ denotes the DTFT.

## Order Estimation for FIR Filters

Three increasingly sophisticated formulae for estimating the minimum order of an FIR filter with unity gain passbands:

1. $M \approx \frac{a}{3.5\Delta\omega}$
2. $M \approx \frac{a-8}{2.2\Delta\omega}$
3. $M \approx \frac{a-1.2-20\log_{10}b}{4.6\Delta\omega}$

where $a =$stop band attenuation in dB, $b =$ peak-to-peak passband ripple in dB and $\Delta\omega =$ width of smallest transition band in normalized rad/s.

**z-plane Transformations**

A lowpass filter, $H(z)$, with cutoff frequency $\omega_0$ may be transformed into the filter $H(\hat{z})$ as follows:

| Target $H(\hat{z})$ | Substitute | Parameters |
|---|---|---|
| Lowpass<br>$\hat{\omega} < \hat{\omega}_1$ | $z^{-1} = \frac{\hat{z}^{-1}-\lambda}{1-\lambda\hat{z}^{-1}}$ | $\lambda = \frac{\sin\left(\frac{\omega_0-\hat{\omega}_1}{2}\right)}{\sin\left(\frac{\omega_0+\hat{\omega}_1}{2}\right)}$ |
| Highpass<br>$\hat{\omega} > \hat{\omega}_1$ | $z^{-1} = -\frac{\hat{z}^{-1}+\lambda}{1+\lambda\hat{z}^{-1}}$ | $\lambda = \frac{\cos\left(\frac{\omega_0+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\omega_0-\hat{\omega}_1}{2}\right)}$ |
| Bandpass<br>$\hat{\omega}_1 < \hat{\omega} < \hat{\omega}_2$ | $z^{-1} = -\frac{(\rho-1)-2\lambda\rho\hat{z}^{-1}+(\rho+1)\hat{z}^{-2}}{(\rho+1)-2\lambda\rho\hat{z}^{-1}+(\rho-1)\hat{z}^{-2}}$ | $\lambda = \frac{\cos\left(\frac{\hat{\omega}_2+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)}, \rho = \cot\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)\tan\left(\frac{\omega_0}{2}\right)$ |
| Bandstop<br>$\hat{\omega}_1 \nless \hat{\omega} \nless \hat{\omega}_2$ | $z^{-1} = \frac{(1-\rho)-2\lambda\hat{z}^{-1}+(\rho+1)\hat{z}^{-2}}{(\rho+1)-2\lambda\hat{z}^{-1}+(1-\rho)\hat{z}^{-2}}$ | $\lambda = \frac{\cos\left(\frac{\hat{\omega}_2+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)}, \rho = \tan\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)\tan\left(\frac{\omega_0}{2}\right)$ |

********* Questions and Solutions *********

1. a) i) Explain what is meant by saying that a linear time-invariant system is "BIBO stable". [ 1 ]

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*An LTI system is BIBO stable if any bounded input sequence $x[n]$ always results in an output sequence $y[n]$ that is also bounded. A sequence $x[n]$ is bounded iff $\exists B < \infty$ such that $|x[n]| < B\,\forall n$.*

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ii) The impulse response, $h[n]$, of a linear time-invariant system satisfies $\sum_{n=-\infty}^{\infty} |h[n]| = S$ where $S < \infty$. Prove that the system is BIBO stable and also that $H(z)$ converges for $|z| = 1$. [ 2 ]

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*Suppose that $x[n]$ is any bounded sequence with $|x[n]| < B < \infty\,\forall n$. We need to show that the output, $y[n] = \sum_{r=-\infty}^{\infty} h[r]x[n-r]$ is also bounded. We have*

$$\begin{aligned} |y[n]| &= \left|\sum_{r=-\infty}^{\infty} h[r]x[n-r]\right| \\ &\leq \sum_{r=-\infty}^{\infty} |h[r]x[n-r]| \\ &= \sum_{r=-\infty}^{\infty} |h[r]|\,|x[n-r]| \\ &< B\sum_{r=-\infty}^{\infty} |h[r]| = BS \end{aligned}$$

*We have $H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$. If $|z| = 1$ we can write*

$$\begin{aligned} |H(z)| &= \left|\sum_{n=-\infty}^{\infty} h[n]z^{-n}\right| \\ &\leq \sum_{n=-\infty}^{\infty} |h[n]|\,|z^{-n}| \\ &= \sum_{n=-\infty}^{\infty} |h[n]| = S \end{aligned}$$

*Hence $H(z)$ is convergent for $|z| = 1$.*

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b) A filter is defined by the difference equation

$$y[n] = \alpha y[n-1] + (1-\alpha)x[n]$$

where $0 < \alpha < 1$ is a real constant.

i) Determine the system function of the filter, $H(z)$, and the impulse response, $h[n]$, for $n = -1, 0, 1, 2$. [ 2 ]

*The system function is* $H(z) = \frac{1-\alpha}{1-\alpha z^{-1}}$.
$h[-1] = 0$, $h[0] = 1-\alpha$, $h[1] = (1-\alpha)\alpha$, $h[1[ = (1-\alpha)\alpha^2$.

ii) State the values of $z$ at which $H(z)$ has a pole or zero. [ 2 ]

*The system function is* $H(z) = \frac{1-\alpha}{1-\alpha z^{-1}} = \frac{(1-\alpha)z}{z-\alpha}$. *This has a pole at* $z = \alpha$ *and a zero at* $z = 0$.

iii) Determine the frequency at which the filter has a gain of $-3\,\mathrm{dB}$. [ 3 ]

*We want* $\left|H(e^{j\omega})\right|^2 = H(e^{j\omega}) \times H^*(e^{j\omega}) = 0.5$ . *Thus*

$$\begin{aligned} 0.5 &= \frac{1-\alpha}{1-\alpha e^{-j\omega}} \times \frac{1-\alpha}{1-\alpha e^{j\omega}} \\ &= \frac{(1-\alpha)^2}{1-2\alpha\cos\omega+\alpha^2} \end{aligned}$$

*From this,* $\cos\omega = \frac{1+\alpha^2-2(1-\alpha)^2}{2\alpha} = 1 - \frac{(1-\alpha)^2}{2\alpha}$ *and so* $\omega_{3\mathrm{dB}} = \cos^{-1}\left(1 - \frac{(1-\alpha)^2}{2\alpha}\right)$.

iv) If the sample frequency is $f_s$, show that, for $n \geq 0$, the impulse response, $h[n]$, is equal to a sampled version of $h(t) = Ae^{-\frac{t}{\tau}}$ and determine the values of the constants $A$ and $\tau$. [ 2 ]

*For* $n \geq 0$, *we have from part i) that* $h[n[ = (1-\alpha)\alpha^n$. *This can be proved by induction from the given recurrence relation assuming that* $x[n]$ *is an impulse at* $n = 0$. *If we now substitute* $t = \frac{n}{f_s}$ *into* $h(t) = Ae^{-\frac{t}{\tau}}$ *we obtain* $h[n] = h\left(\frac{t}{f_s}\right) = Ae^{-\frac{n}{\tau f_s}}$ *from which* $A = 1-\alpha$ *and* $\alpha = e^{-\frac{1}{\tau f_s}}$. *Rearranging the later equation gives* $\tau = \frac{-1}{f_s \ln\alpha}$.

c) Figure 1.1 shows the block diagram of a filter implementation comprising two delays, five multipliers with real-valued coefficients $c_1, \cdots, c_5$ and four adder elements.

i) Show that transfer function $\frac{Y(z)}{X(z)} = \frac{c_3+c_4z^{-1}+c_5z^{-2}}{1-c_1z^{-1}-c_2z^{-2}}$. [ 3 ]

*The two delay elements form a shift register whose input is* $u[n]$. *Hence the inputs to the* $c_1$ *and* $c_2$ *multipliers are* $u[n-1[$ *and* $u[n-2]$ *respectively. It follows directly from the diagram that* $u[n] = x[n] + c_1u[n-1] + c_2u[n-2]$. *Taking z-transforms gives* $U(z) = X(z) + c_1z^{-1}U(z) + c_2z^{-1}U(z)$ *from which* $\frac{U(z)}{X(z)} = \frac{1}{1-c_1z^{-1}-c_2z^{-2}}$. *From the diagram* $y[n] = c_3u[n] + c_4u[n-1] + c_5u[n-2]$ *from which* $\frac{Y(z)}{U(z)} = c_3 + c_4z^{-1} + c_5z^{-2}$.

*Combining this with the previous result gives* $\frac{Y(z)}{X(z)} = \frac{Y(z)}{U(z)} \times \frac{U(z)}{X(z)} = \frac{c_3+c_4z^{-1}+c_5z^{-2}}{1-c_1z^{-1}-c_2z^{-2}}$.

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ii) Suppose that each multiplier introduces independent additive white noise at its output with power spectral density $S(\omega) = S_0$ and that the noise signals are uncorrelated with $x[n]$. Show that the combined effect of the five noise sources is equivalent to two additive white noise signals at $x[n]$ and $y[n]$ respectively. Hence determine the overall power spectral density, $N(\omega)$, of the noise at $y[n[$. [ 3 ]

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*The noise components added by multipliers $c_1$ and $c_2$ merely add onto the input and so add noise with power spectral density $2S_0$onto the input. Similarly, the remaining noise sources add noise with power spectral density $3S_0$onto the output. Hence the overall noise power spectral density at $y[n]$ is*

$$\begin{aligned}
N(\omega) &= 3S_0 + 2S_0 \times \left|H(e^{j\omega})\right|^2 \\
&= S_0\left(3+2\frac{\left(c_3+c_4z^{-1}+c_5z^{-2}\right)\left(c_3+c_4z^{1}+c_5z^{2}\right)}{\left(1-c_1z^{-1}-c_2z^{-2}\right)\left(1-c_1z^{1}-c_2z^{2}\right)}\right) \\
&= S_0\left(3+2\frac{c_3^2+c_4^2+c_5^2+(c_3+c_5)c_4\cos\omega+c_3c_5\cos 2\omega}{1+c_1^2+c_2^2+(c_2-1)c_1\cos\omega-c_2\cos 2\omega}\right).
\end{aligned}$$

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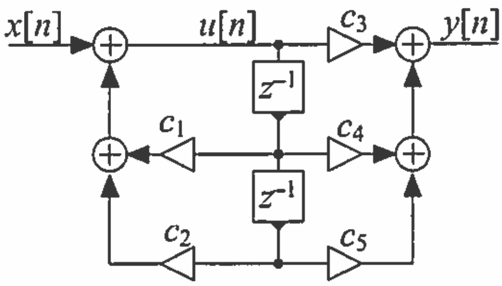


Figure 1.1

d) The impulse response of an antisymmetric FIR filter, $H(z)$, of order $M$ satisfies the relation $h[n[ = -h[M-n\}$.

i) Show that the magnitude response $|H(e^{j\omega})|$ can be expressed as the absolute value of the sum of $N$ sine waves where $N = \frac{M}{2}$ if $M$ is even and $N = \frac{M+1}{2}$ if $M$ is odd. [ 3 ]

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*If M is odd, there is an even number of coefficients so we can write*

$$\begin{aligned}
H(z) &= \sum_{n=0}^{\frac{M-1}{2}} h[n]\left(z^{-n} - z^{n-M}\right) \\
&= z^{-0.5M} \sum_{n=0}^{\frac{M-1}{2}} h[n]\left(z^{-n+0.5M} - z^{n-0.5M}\right) \\
H(e^{j\omega}) &= -je^{-j0.5M\omega} \sum_{n=0}^{\frac{M-1}{2}} 2h[n]\sin\left((n-0.5M)\,\omega\right) \\
\left|H(e^{j\omega})\right| &= \left|\sum_{n=0}^{\frac{M-1}{2}} 2h[n]\sin\left((n-0.5M)\,\omega\right)\right|
\end{aligned}$$

*The right side is the absolute value of the sum of* $\frac{M+1}{2}$ *sine waves as required.*

*If M is even, there is an odd number of coefficients but the central coefficient must be zero. Hence*

$$\begin{aligned}
H(z) &= \sum_{n=0}^{\frac{M}{2}} h[n]\left(z^{-n} - z^{n-M}\right) \\
\Rightarrow \left|H(e^{j\omega})\right| &= \left|\sum_{n=0}^{\frac{M}{2}} 2h[n]\sin\left((n-0.5M)\,\omega\right)\right|
\end{aligned}$$

*The derivation is identical to that for odd M except for the upper summation limit.*

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ii) Show that $H(e^{j\omega})$ is necessarily zero at $\omega = 0$ but may be non-zero at $\omega = \pi$ if $M$ is odd. Give an example of a filter for which this is the case. [ 2 ]

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*When* $\omega = 0$ *then* $\sin\alpha\omega = 0$ *for any* $\alpha$. *Hence all the summation terms are zero for both odd and even M.*

*When* $\omega = \pi$ *then* $\sin\alpha\omega = 0$ *if* $\alpha$ *is an integer. So, if M is even,* $\sin\left((n-0.5M)\,\omega\right)$ *will always be* $0$ *at* $\omega = \pi$. *However, if M is odd then this is not necessarily true.*

*An example, for* $M = 1$, *is* $H(z) = 1 - z^{-1}$. *For this case,* $H(e^{j\omega}) = 1 - e^{-j\omega} = 2je^{-0.5j\omega}\sin 0.5\omega$. *When* $\omega = pi$, $H(e^{j\omega}) = 2$ *which is non-zero.*

iii) Derive an expression for the phase response, $\angle H(e^{j\omega})$ and determine the group delay, $\tau_H\left(e^{j\omega}\right) = -\frac{d\angle H(e^{j\omega})}{d\omega}$. [2]

*From part d), the phase response is*

$$\begin{aligned}\angle H(e^{j\omega}) &= \angle\left(-je^{-j0.5M\omega}\right) + \pi\,\text{sgn}\left(\sum_{n=0}^{\frac{M-1}{2}} 2h[n[\sin\left(\left(n-0.5M\right)\omega\right)\right) \\ &= -0.5\left(M\omega+\pi\right) + \pi\,\text{sgn}\left(\sum_{n=0}^{\frac{M-1}{2}} 2h[n]\sin\left(\left(n-0.5M\right)\omega\right)\right)\end{aligned}$$

*Differentiating this gives* $\tau_H\left(e^{j\omega}\right) = -\frac{d\angle H(e^{j\omega})}{d\omega} = 0.5M$. *Note that the* sgn() *function is piecewise constant and so its derivative is zero.*

e) Figure 1.2 shows the analysis and synthesis sections of a subband processing system. The input and output signals are $x[n]$ and $y[n]$ respectively and the intermediate signals are $v_m[n]$, $u_m[r]$ and$w_m[n]$ where $m = 0$ or 1 according to the subband. The corresponding z-transforms are $X(z)$, $Y(z)$ etc.

i) Show that it is possible to express the overall transfer function in the form $Y(z) = \begin{bmatrix} T(z) & A(z) \end{bmatrix}\begin{bmatrix} X(z) \\ X(-z) \end{bmatrix}$ and determine expressions for $T(z)$ and $A(z)$.

You may assume without proof that for $m = 0$ or 1, [3]

$$\begin{aligned}U_m(z) &= \frac{1}{2}\left\{V_m\left(z^{\frac{1}{2}}\right) + V_m\left(-z^{\frac{1}{2}}\right)\right\} \\ W_m(z) &= U_m\left(z^2\right).\end{aligned}$$

*Combining the two given equations we get*

$$\begin{aligned}W_m(z) = U_m\left(z^2\right) &= \frac{1}{2}\left\{V_m\left(z\right) + V_m\left(-z\right)\right\} \\ &= = \frac{1}{2}\left\{H_m(z)X\left(z\right) + H_m(-z)X\left(-z\right)\right\}.\end{aligned}$$

*Hence*

$$\begin{aligned}Y(z) &= G_0(z)W_0(z) + G_1(z)W_1(z) \\ &= \frac{1}{2}\left\{\left(G_0(z)H_0(z) + G_1(z)H_1(z)\right)X\left(z\right) + \left(G_0(z)H_0(-z) + G_1(z)H_1(-z)\right)X\left(-z\right)\right\} \\ &= \begin{bmatrix} T(z) & A(z) \end{bmatrix}\begin{bmatrix} X(z) \\ X(-z) \end{bmatrix}\end{aligned}$$

*where* $T(z) = \frac{1}{2}\left(G_0(z)H_0(z) + G_1(z)H_1(z)\right)$ *and* $A(z) = \frac{1}{2}\left(G_0(z)H_0(-z) + G_1(z)H_1(-z)\right)$.

ii) Explain why it is normally desirable to have $A(z) \equiv 0$. [ 2 ]

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*The factor $A(z)$ multiplies $X(-z)$. The spectrum of $X(-z)$ is $X(-e^{j\omega}) = X(e^{j(\omega+\pi)})$ is an aliased version of the spectrum $X(e^{j\omega})$ in which the spectrum is shifted by $\pi$ (or equivalently conjugated and reflected around $\frac{\pi}{2}$ for the case of a conjugate symmetric spectrum). These aliased spectral images are normally unwanted and so we would like $A(e^{j\omega}) \equiv 0 \Leftrightarrow A(z) \equiv 0$ (assuming $A(z)$ is analytic).*

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iii) Suppose that $H_0(z) = H_1(-z) = G_0(z) = -G_1(-z)$. Show that in this case $A(z) = 0$ and demonstrate how the frequency responses $H_1(e^{j\omega})$, $G_0(e^{j\omega})$ and $G_1(e^{j\omega})$ are related to $H_0(e^{j\omega})$ assuming that $H_0(z)$ is an FIR or IIR filter with real coefficients. [ 2 ]

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*We the relations given, we can express all the blocks in terms of $H_0(z)$ to obtain $G_0(z)H_0(-z)+G_1(z)H_1(-z) = H_0(z)H_0(-z) - H_0(-z)H_0(z) = 0$. Our no-alias condition is therefore met.*

*Clearly $G_0(e^{j\omega})$ is identical to $H_0(e^{j\omega})$. $H_1(e^{j\omega}) = H_0(-e^{j\omega}) = H_0(e^{j(\omega-\pi)}) = H_0^*(e^{j(\pi-\omega)})$. This is therefore the frequency response of $H_0(e^{j\omega})$ but conjugated and reflected around $\omega = \frac{\pi}{2}$. $G_1(e^{j\omega})$ is the same as $H_1(e^{j\omega})$ but negated or, equivalently, with $\pi$ added onto the phase response.*

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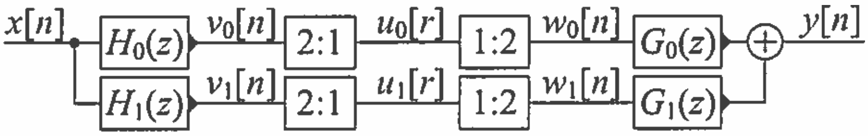


Figure 1.2

f) Figure 1.3 shows an upsampler with real-valued input $x[n[$ and output

$$y[r] = \begin{cases} x\left[\frac{r}{K}\right] & \text{if } K \mid r \\ 0 & \text{otherwise} \end{cases}$$

where $K \mid r$ means $K$ is a factor of $r$.

i) Show that $Y(z) = X\left(z^K\right)$. [ 1 ]

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$Y(z) = \sum_{r=-\infty}^{\infty} y[r] z^{-r} = \sum_{\{r:K \mid r\}} y[r[z^{-r} = \sum_{n=-\infty}^{\infty} y[nK] z^{-nK} = \sum_{n=-\infty}^{\infty} x[n] \left(z^K\right)^{-n} = X\left(z^K\right)$.

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ii) The energy and average power of $x[n]$ are defined respectively as

$$\begin{aligned} E_x &= \sum_{n=-\infty}^{\infty} |x[n]|^2 \\ P_x &= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 . \end{aligned}$$

Give expressions for the energy and average power of $y[r[$ in terms of $E_x$ and $P_x$. [ 2 ]

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*The non-zero samples of $y[n]$ are identical to the samples of $x[n[$ and so the energy of the two signals is the same: $E_Y = E_X$. However $y[n]$ has K times as many samples, so its power is $P_Y = \frac{1}{K} P_X$.*

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iii) Figure 1.4 shows the power spectral density of $x[n]$ which comprises white noise of unit magnitude together with a bandpass signal component occupying the range $0.5 < \omega < 1$. Sketch the power spectral density of $y[r[$ when $K = 3$ and give the magnitudes of its white noise component and the magnitude and frequency range of all bandpass components. [ 3 ]

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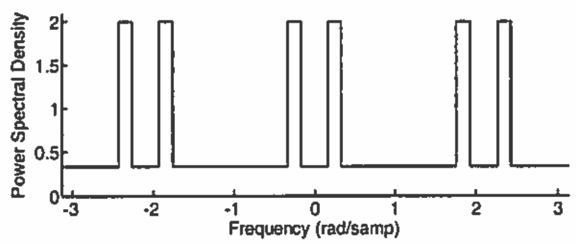
*The upsampling horizontally compresses the 2-sided spectrum by a factor of K and replicates it K times at increments of $\Delta\omega = \frac{2\pi}{K}$. For the DTFT and the energy spectrum, each image is the same magnitude as the original. However, for the power spectrum, the magnitudes will be reduced by a factor of K.*




*Thus the magnitude of the white component will be 0.33 and that of the bandpass components will be 2.*

*The frequency ranges of the bandpass components will be* $\{-\frac{2\pi}{3}, 0, \frac{2\pi}{3}\} \pm \frac{1}{4} \pm \frac{1}{12}$. *That is, each will be of width* $\frac{1}{6}$ *and centered* $\pm\frac{1}{4}$ *either side for the image centre.*

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iv) The diagram of Fig. 1.3 is followed by a lowpass filter to remove spectral images. If $K = 3$ and $x[n[$ is as specified in part iii) above, determine the transition bandwidth and transition band centre frequency of a suitable lowpass filter and explain the reasons for your choices. [ 2 ]

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*The required filter order is inversely proportional to the transition width,* $\Delta\omega$ *which we therefore wish to make as wide as possible. We therefore make the transition width from the edge of the wanted signal component to the start of the first image component:* $\frac{1}{3} = 0.333$ *to* $\frac{2\pi-1}{3} = 1.761$. *The centre of the transition band is the average of these two values and corresponds to the old Nyquist frequency, i.e.* $\frac{\pi}{3} = 1.047$.

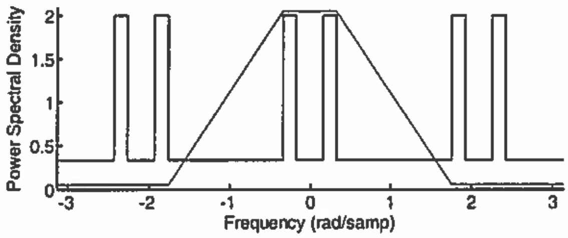


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Figure 1.3

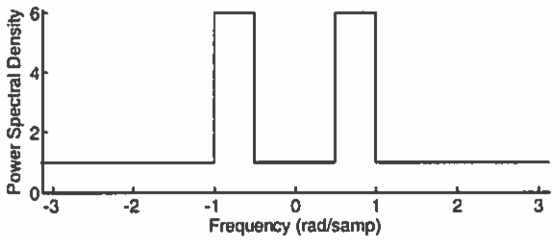



Figure 1.4

2. a) Suppose that $G_1(z) = 1 - pz^{-1}$ and $G_2(z) = 1 - qz^{-1}$ where the constants $p$ and $q$ may be complex. If $q = \frac{1}{p^*}$ show that $|G_1(e^{j\omega})| = \alpha |G_2(e^{j\omega})|$ for all $\omega$ and determine an expression for the constant $\alpha$. [ 4 ]

---

*For $z = e^{j\omega}$, we have $z^* = z^{-1}$ and so we can write $\frac{|G_1(z)|^2}{|G_2(z)|^2} = \frac{(1-pz^{-1})(1-p^*z)}{(1-qz^{-1})(1-q^*z)} = \frac{pp^*(1-pz^{-1})(1-p^*z)}{(1-pz^{-1})(1-p^*z)} = pp^*$ (independent of $\omega$). Hence $\alpha = \sqrt{pp^*} = |p|$.*

---

b) Suppose that $H_1(z) = 4 + 14z^{-1} - 8z^{-2}$. Determine the coefficients of $H_2(z)$ such that $|H_1(e^{j\omega})| = |H_2(e^{j\omega})|$ for all $\omega$ and that all the zeros of $H_2(z)$ lie inside the unit circle. [ 4 ]

---

*The roots of $H_1(z)$ are 0.5 and $-4$ and so the roots of $H_2(z)$ must be 0.5 and $-0.25$ which implies that $H_2(z) = \alpha\left(1 - 0.5z^{-1}\right)\left(1 + 0.25z^{-1}\right)$. To determine alpha, we substitute a suitable value of $z$ that lies on the unit circle, e.g. $z = 1$, to obtain $|H_2(1)| = |\alpha(1-0.5)(1+0.25)| = 0.625|\alpha| = |H_1(1)| = |4 + 14 - 8| = 10$. From this we get $\alpha = \pm\frac{10}{0.625} = \pm 16$ and so $\pm H_2(z) = 16\left(1 - 0.25z^{-1} - 0.125z^{-2}\right) = 16 - 4z^{-1} - 2z^{-2}$.*

---

c) When designing an IIR filter $H(e^{j\omega}) = \frac{B(e^{j\omega})}{A(e^{j\omega})}$ to approximate a complex target response $D(\omega)$ two error measures that may be used are the weighted solution error, $E_S(\omega)$, and the weighted equation error, $E_E(\omega)$, defined respectively by

$$\begin{aligned} E_S(\omega) &= W_S(\omega)\left(\frac{B(e^{j\omega})}{A(e^{j\omega})} - D(\omega)\right) \\ E_E(\omega) &= W_E(\omega)\left(B(e^{j\omega}) - D(\omega)A(e^{j\omega})\right). \end{aligned}$$

Explain the relative advantages of the two error measures and explain the purpose of the real-valued non-negative weighting functions $W_S(\omega)$ and $W_E(\omega)$. [ 2 ]

---

*The solution error is a direct measure of the error in the frequency response but has the disadvantage that it gives rise to a set of non-linear simultaneous equations which do not have a closed form solution. Although the equation error gives rise to a set of linear equations which are straightforward to solve, it multiplies the true frequency response errors by $A(e^{j\omega})$ and so, unless $W_E(\omega)$ is adjusted accordingly, gives a higher weight to spectral regions in which $|A(e^{j\omega})|$ is large.*

*The weighting functions, $W_S(\omega)$ and $W_E(\omega)$ are used to control the relative importance of errors in different parts of the spectrum. A high weight will result in a lower error.*

---

d) Suppose that $0 \le \omega_1 < \omega_2 < \ldots < \omega_K \le \pi$ is a set of $K$ frequencies and that $A(z) = 1 + [z^{-1}\ z^{-2}\ \cdots\ z^{-N}]\mathbf{a}$ and $B(z) = [1\ z^{-1}\ z^{-2}\ \cdots\ z^{-M}]\mathbf{b}$ where $\mathbf{a}$ and $\mathbf{b}$ are real-valued coefficient vectors.

i) Show that it is possible to express the equations $E_E(\omega_k) = 0$ for $1 \le k \le K$ as a set of $K$ simultaneous linear equations in the form

$(\mathbf{P}\ \mathbf{Q})\begin{pmatrix}\mathbf{a}\\ \mathbf{b}\end{pmatrix}=\mathbf{d}.$

State the dimensions of the matrices **P** and **Q** and of the vector **d** and derive expressions for the elements of **P**, **Q** and **d**. [ 4 ]

---

*If $E_E(\omega_k)=0$ then $W_E(\omega_k)\left(B(e^{j\omega_k})-D(\omega_k)A(e^{j\omega_k})\right)=0$. Substituting in the expressions for $A(z)$ and $B(z)$ gives*

$$\begin{aligned}W_E(\omega_k)\,([1\ e^{-j\omega_k}\ e^{-j2\omega_k}\ \ldots\ e^{-jM\omega_k}]\mathbf{b}&\\ -D(\omega_k)-D(\omega_k)[e^{-j\omega_k}\ e^{-j2\omega_k}\ \ldots\ e^{-jN\omega_k}]\mathbf{a})&=0.\end{aligned}$$

*Rearranging this equation gives* $(\mathbf{p}_k^T\ \mathbf{q}_k^T)\begin{pmatrix}\mathbf{a}\\ \mathbf{b}\end{pmatrix}=W_E(\omega_k)D(\omega_k)$ *where*
$\mathbf{p}_k^T=-W_E(\omega_k)D(\omega_k)[e^{-j\omega_k}\ e^{-j2\omega_k}\ \ldots\ e^{-jN\omega_k}]$
*and* $\mathbf{q}_k^T=W_E(\omega_k)[1\ e^{-j\omega_k}\ e^{-j2\omega_k}\ \ldots\ e^{-jM\omega_k}]$.

*Thus the dimensions of* **P**, **Q** *and* **d** *are $K\times N$, $K\times(M+1)$ and $K\times 1$. and their $k^{\text{th}}$ rows are $\mathbf{p}_k^T$, $\mathbf{q}_k^T$ and $W_E(\omega_k)D(\omega_k)$ respectively.*

---

ii) Explain how, by separating the real and imaginary parts of **P**, **Q** and **d**, it is possible to obtain a set of simultaneous linear equations for $\begin{pmatrix}\mathbf{a}\\ \mathbf{b}\end{pmatrix}$ in which all coefficients are real-valued. Explain the circumstances under which some of the resultant equations will necessarily have all-zero coefficients. [ 4 ]

---

*By treating the real and imaginary parts of each row as two separate equations, we can write* $\begin{pmatrix}\Re(\mathbf{P}) & \Re(\mathbf{Q})\\ \Im(\mathbf{P}) & \Im(\mathbf{Q})\end{pmatrix}\begin{pmatrix}\mathbf{a}\\ \mathbf{b}\end{pmatrix}=\begin{pmatrix}\Re(W_E(\omega_k)D(\omega_k))\\ \Im(W_E(\omega_k)D(\omega_k))\end{pmatrix}$ *in which all coefficients are real.*

*If $W_E(\omega_k)=0$ for any particular $k$, then the corresponding pair of equations will have all-zero coefficients. In addition, if $\omega_k=0$ or $\pi$ then $\mathbf{p}_k^T$ and $\mathbf{q}_k^T$ will both be real-valued, so the equation formed from their imaginary parts will have all-zero coefficients.*

---

iii) Explain why it may be desirable to apply the transformation of part b) after obtaining the solution to the equations of part d)ii). [ 2 ]

---

*To obtain a stable filter, the zeros of $A(z)$ must lie inside the unit circle. There is no guarantee that the solutions of the equations will meet this requirement so the transformation of part b) may be applied to $A(z)$ to enforce stability. This will change the phase response but not the magnitude response of the resultant filter.*

---

iv) Assuming that $\omega_1=0$ and $\omega_K=\pi$, determine the minimum value of

$K$ to ensure that the equations of part d) ii) are not underdetermined.

[ 4 ]

---

*Since $\mathbf{p}_k^T$ and $\mathbf{q}_k^T$ are real for $k=1$ and $k=K$, we have a total of $2K-2$ equations with $M+N+1$ unknowns.*

*Therefore we require $2K-2 \geq M+N+1$ which implies that $K \geq \frac{M+N+3}{2}$.*

---

e) Suppose now that $H(z) = \frac{b}{1+az^{-1}}$, that $K=3$, that $\omega_k = 0.5\,(k-1)\,\pi$, that

$$\begin{aligned} D(\omega) &= \begin{cases} 2 & \text{for } \omega \leq 0.25\pi \\ 1 & \text{for } \omega > 0.25\pi \end{cases}. \\ W_E(\omega) &\equiv 1 \end{aligned}$$

Determine the numerical values of the elements of $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{d}$ and hence determine the numerical values of $a$ and $b$ that minimize $\sum_k |E_E(\omega_k)|^2$. [ 6 ]

You may assume without proof that the least squares solution to an overdetermined set of real-valued linear equations, $\mathbf{Rx}=\mathbf{q}$, is given by $\mathbf{x} = \left(\mathbf{R}^T\mathbf{R}\right)^{-1}\mathbf{R}^T\mathbf{q}$ assuming that $\mathbf{R}$ has full column rank.

---

*Since $K=3$ the values of $\omega_k$ are $\{0\; 0.5\pi\; \pi\}$ and hence $e^{-j\omega_k} = \{1\; -j\; -1\}$. At these frequencies $D(\omega_k) = \{2\,1\,1\}$. The complex equations from part d)i) are therefore given by* $\begin{pmatrix} -2 & 1 \\ j & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$. *From this, we get a set of four real equations* $\begin{pmatrix} -2 & 1 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}$. *Hence, using the formula given in the question, we can write*

$$\begin{aligned} \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} &= \left( \begin{pmatrix} -2 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix} \right)^{-1} \begin{pmatrix} -2 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 6 & -1 \\ -1 & 3 \end{pmatrix}^{-1} \begin{pmatrix} -3 \\ 4 \end{pmatrix} \\ &= \frac{1}{17}\begin{pmatrix} 3 & 1 \\ 1 & 6 \end{pmatrix}\begin{pmatrix} -3 \\ 4 \end{pmatrix} = \frac{1}{17}\begin{pmatrix} -5 \\ 21 \end{pmatrix} \end{aligned}$$

*From which $a = -0.294$ and $b = 1.235$.*

---

3. a) Figure 3.1 shows the block diagram of a system that multiplies the input sample rate by $\frac{P}{Q}$ where $P$ and $Q$ are coprime with $P < Q$.

i) Explain why the cutoff frequency of the lowpass filter $H(z)$ should be placed at the Nyquist rate of the output signal, $y[m]$ and give the normalized cutoff frequency, $\omega_0$, in rad/sample in terms of $P$ and/or $Q$.

Using the approximation formula $M \approx \frac{a}{3.5\Delta\omega}$, determine the required filter order $M$ in terms of $P$ and/or $Q$ if the stopband attenuation in dB is $a = 60$ and the normalized transition bandwidth is $\Delta\omega = 0.1\omega_0$. [ 4 ]

---

*The lowpass filter must eliminate the images introduced by the upsampler and the alias components introduced by the down sampler and must therefore eliminate all frequencies above the lower of the input and output Nyquist frequencies. Since $Q > P$ this is the output Nyquist frequency. The normalized cutoff frequency is therefore $\omega_0 = \frac{\pi}{Q}$.*

*We have* $M = \frac{60}{3.5\times 0.1\omega_0} = \frac{60Q}{3.5\times 0.1\pi} = 54.6Q$.

---

ii) Using the value of $M$ from part a)i), estimate the average number of multiplications per input sample, $x[n]$, needed to implement the system. [ 2 ]

---

*The filter requires $M+1$ multiplications per sample which equals $(M+1)P$ per input sample. Substituting $M = 54.6Q$ gives $(54.6Q+1)P \approx 54.6PQ$ multiplications per input sample.*

---

iii) The filter $H(z)$ has a symmetrical impulse response $h[r] = g[r]w[r]$ for $0 \le r \le M$ where $g[r]$ is the impulse response of an ideal lowpass filter with cutoff frequency $\omega_0$ and $w[r]$ is a symmetrical window function. Derive an expression for the ideal response, $g[r]$, in terms of $\omega_0$, $M$ and $r$. [ 4 ]

---

*The ideal response (centered on $r = 0$) is $H(e^{j\omega}) = 1$ for $|\omega| < \omega_0$ and zero otherwise. To this ideal response, we need to add a delay of $\frac{M}{2}$ samples which corresponds to a phase shift of $e^{-j0.5M\omega}$. Using the inverse DTFT (available in the formula sheet)*

$$
\begin{aligned}
g[r] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{-j0.5M\omega}e^{j\omega r}d\omega \\
&= \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} e^{j\omega(r-0.5M)}d\omega = \frac{1}{j2(r-0.5M)\pi}\left[e^{j\omega(r-0.5M)}\right]_{-\omega_0}^{\omega_0} \\
&= \frac{1}{j2(r-0.5M)\pi}\times 2j\sin((r-0.5M)\omega_0) = \frac{\sin((r-0.5M)\omega_0)}{\pi(r-0.5M)}
\end{aligned}
$$

---

b) The filter $H(z)$ is now implemented as a polyphase filter as shown in Fig. 3.2.

The filter implementation uses a single set of delays and multipliers with commutated coefficients.

i) State the length of the filter impulse response $h_0[n[$ in terms of $M$, $P$ and/or $Q$ and give an expression for the coefficients $h_0[n]$ in terms of $h[r]$. [ 2 ]

---

*The length of the filter $h_0[n[$ is $\frac{M+1}{P}$ rounded up to the nearest integer. The coefficeints are given by $h_0[n[ = h[nP]$.*

---

ii) If $x[n] = 0$ for $n < 0$, give expressions for $v[0]$, $v[1]$, $v[2P+1]$ in terms of the input $x[n[$ and the coefficients $h_p[n]$. [ 2 ]

---

*Since $x[n[$ is causal, $v[0] = h_0[0]x[0]$, $v[1] = h_1[0[x[0[$ and $v[2P+1] = h_1[0]x[2[+h_1[1]x[1]+h_1[2[x[0]$.*

---

iii) Explain how it is possible to eliminate the output decimator by changing both the order and rate at which the coefficient sets, $h_p[n]$ are accessed.

Determine the new coefficient set order for the case $P = 5$ and $Q = 7$. [ 3 ]

---

*The output decimator selects every $Q^{\text{th}}$ sample of $v[r]$ and discards the others. Therefore if we access the coefficient sets in the order $p = mQ \bmod P$ for $m = 0, 1, \cdots$ and reduce the rate by a factor of $Q$ will will generate only the wanted output samples.*

*For the specific values $P = 5$ and $Q = 7$, $Q \bmod P = 2$ so the coefficient set order becomes $h_0[n[$, $h_2[n]$, $h_4[n]$, $h_1[n]$, $h_3[n]$.*

---

iv) Determine the number of multiplications per input sample for the system of part b) iii) and the number of distinct coefficients that must be stored. You may assume that $M+1$ is a multiple of $P$. [ 2 ]

---

*We require $\frac{M+1}{P}$ multiplications per output sample and therefore $\frac{M+1}{P} \times \frac{P}{Q} = \frac{M+1}{Q}$ multiplications per input sample. Because of symmetry, the number of distinct coefficients is only $\frac{M+1}{2}$. [Athough not explicitly requested, it is interesting to note that since $M \approx 54.6Q$, the number of multiplications per input sample is approximately 54.6 independently of P or Q.]*

---

c) Suppose now that the sample rate of the input, $x[n]$, is 18 kHz and that the system is implemented as in part b) iii) with the values of $a$ and $\Delta\omega$ as given in part a) i). Determine the values of $P$, $Q$ and $M$ when the sample rate of the output, $y[m]$, is (i) 10 kHz and (ii) 10.1 kHz [note that 101 is a prime number].

For each of these cases estimate the number of multiplications per input sample and the number of distinct coefficients that must be stored. [ 5 ]

---

*(i) For an output sample rate of* $10\text{kHz} = \frac{5}{9} \times 18\text{kHz}$, $P = 5$ *and* $Q = 9$. $M = 54.6Q = 491$. *The number of multiplications per input sample is therefore* $\frac{M+1}{Q} = 54.7$. *The total number of distinct coefficients is* $\frac{M+1}{2} = 246$.

*(ii) For an output sample rate of* $10.1\text{kHz} = \frac{101}{180} \times 18\text{kHz}$, $P = 101$ *and* $Q = 180$. $M = 54.6Q = 9822$. *The number of multiplications per input sample is therefore* $\frac{M+1}{Q} = 54.6$ *(virtually unchanged). The total number of distinct coefficients is* $\frac{M+1}{2} = 4912$ *(much increased).*

---

d) In a Farrow filter, the coefficients, $h_p[n]$, are approximated by a low-order polynomial $f_n(t)$ where $t = \frac{p}{P}$ for $0 \le p \le P-1$.

i) Assuming that a rectangular window, $w[r] \equiv 1$, is used in the design of $H(z)$ and that $\omega_0 = \frac{\pi}{P}$, give an expression for the target value of $f_0(t)$ in terms of $t$, $M$ and $P$. [ 3 ]

---

*From the answer to part i)iii)* $h[r] = \frac{\sin(r-0.5M)\omega_0}{\pi(r-0.5M)}$ *so* $h_p[0] = h[p] = \frac{\sin(p-0.5M)\omega_0}{\pi(p-0.5M)}$ *and* $f_n(t) = \frac{\sin(Pt-0.5M)\frac{\pi}{P}}{\pi(Pt-0.5M)} = \frac{\sin\left(\pi t-0.5\pi\frac{M}{P}\right)}{\pi(Pt-0.5M)}$.

---

ii) If the polynomials, $f_n(v)$, are of order $K = 5$, determine the number of coefficients that must be stored for each of the cases defined in part c). [ 3 ]

---

*The* $\frac{M+1}{P}$ *polynomials* $f_n(t)$ *each require* $K = 6$ *coefficients, so we require a total of* $\frac{6(M+1)}{P}$ *coefficients. For the two cases, this gives (i)* $\frac{6\times 492}{5} = 590$ *(somewhat larger than before) and (ii)* $\frac{6\times 9823}{101} = 584$ *(much less than before).*

---

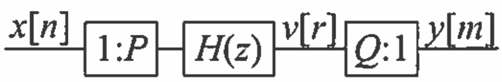



Figure 3.1

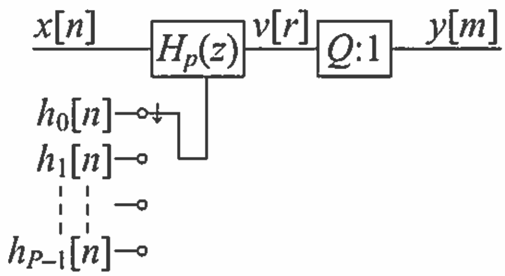



Figure 3.2

4. A complex-valued frequency-modulated signal, $x(t) = a(t)e^{j\phi(t)}$, has a $0\,Hz$ carrier frequency and a peak frequency deviation of $d = 75\text{kHz}$. The amplitude, $a(t)$, is approximately constant with $a(t) \approx 1$ and the phase is $\phi(t) = k\int_0^t m(\tau)d\tau$ where $k$ is a constant and $m(t)$ is a baseband audio signal with bandwidth $b = 15\text{kHz}$. The signal $x(t)$ is sampled at 400kHz to obtain the discrete-time signal $x[n]$.

a) Carson's rule for the bandwidth of a double-sideband FM signal is $B = 2(d+b)$. Use this to determine the single-sided bandwidth, $\omega_0$, of $x[n]$ in radians/sample. [2]

---

*From Carson's rule,* $B = 180\text{kHz}$. *This bandwidth includes both sidebands, so* $\omega_0 = 2\pi \times \frac{90\text{kHz}}{400\text{kHz}} = 0.45\pi = 1.41\,\text{rad/samp}$.

---

b) Show that $m(t) = k^{-1}a^{-2}(t)\Im\left(x^*(t)\frac{dx(t)}{dt}\right)$ where $\Im\,()$ denotes the imaginary part. [4]

---

*From the definition of* $\phi(t)$, $m(t) = k^{-1}\frac{d\phi}{dt}$. *Differentiating* $x(t)$ *gives* $\frac{dx}{dt} = \frac{da}{dt}e^{j\phi(t)} + ja(t)\frac{d\phi}{dt}e^{j\phi(t)}$ *from which (since* $a(t)$ *is real-valued),* $x^*(t)\frac{dx(t)}{dt} = a(t)\frac{da}{dt} + ja^2(t)\frac{d\phi}{dt}$ *and hence* $\Im\left(x^*(t)\frac{dx(t)}{dt}\right) = a^2(t)\frac{d\phi}{dt}$. *It follows that* $k^{-1}a^{-2}(t)\Im\left(x^*(t)\frac{dx(t)}{dt}\right) = m(t)$ *as required.*

---

c) Figure 4.1 shows a block diagram that implements the equation of part b) in discrete time. Complex-valued signals are shown as bold lines and are represented using their real and imaginary parts. The block labelled "Conj" takes the complex conjugate of its input. The differentiation block, $D(z)$, is designed as an FIR filter using the window method with a target response

$$\bar{D}(e^{j\omega}) = \begin{cases} jc\omega & \text{for } |\omega| \le \omega_1 \\ 0 & \text{otherwise} \end{cases}$$

where $c$ is a scaling constant.

i) Determine the impulse response $\bar{d}[n[$ of $\bar{D}(z)$ in simplified form. [4]

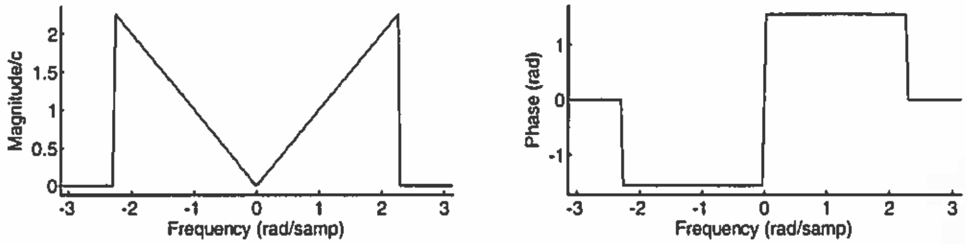
---

*From the inverse DTFT (included in the formula sheet) we use integration by parts to obtain*

$$\begin{aligned}
\bar{d}[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\bar{D}\left(e^{j\omega}\right)e^{j\omega n}d\omega \\
&= \frac{jc}{2\pi}\int_{-\omega_1}^{\omega_1}\omega e^{j\omega n}d\omega \\
&= \frac{jc}{2\pi}\left[\frac{\omega}{jn}e^{j\omega n} - \frac{1}{(jn)^2}e^{j\omega n}\right]_{-\omega_1}^{\omega_1} \\
&= \frac{jc}{2\pi}\left(\frac{\omega_1}{jn}2\cos n\omega_1 - \frac{1}{(jn)^2}2j\sin n\omega_1\right) \\
&= \frac{c}{\pi n^2}\left(n\omega_1\cos n\omega_1 - \sin n\omega_1\right).
\end{aligned}$$

ii) Assuming that $\omega_1 = \frac{\omega_0+\pi}{2}$, draw dimensioned sketches showing the magnitude and phase responses of $\overline{D}(e^{j\omega})$ over the range $-\pi \le \omega \le \pi$. [3]

*For $\omega > \omega_1$, $\overline{D}(e^{j\omega}) = 0$ and so the phase is indeterminate (shown here as zero).*




iii) Assume that the DTFT of the window function used when designing $D(z)$ has a main lobe width of $\omega = \pm\frac{18}{M+1}$ for a window of length $M+1$. If $\omega_1$ is chosen as $\omega_1 = \frac{\omega_0+\pi}{2}$, determine the smallest value of $M$ that will ensure that the transition in the response of $D(e^{j\omega})$ near $\omega = \omega_1$ lies completely within the range $(\omega_0, \pi)$. [3]

*The transition in the response of $D(e^{j\omega})$ near the discontinuity in $\overline{D}(e^{j\omega})$ near $\omega = \omega_1$ will extend for $\frac{18}{M+1}$ either side of the discontinuity. So we need $\frac{18}{M+1} \le \pi - \omega_1 = \frac{\pi-\omega_0}{2} = 0.275\pi$ from which $M \ge \frac{18}{0.275\pi} - 1 = \frac{18}{0.864} - 1 = 22.92 - 1 = 21.92 \approx 22$.*

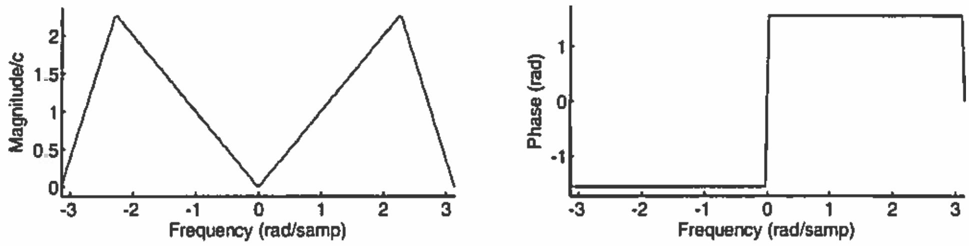
iv) Stating any assumptions, determine the maximum value of $c$ that will ensure $|w[n]| \le 1$. [4]

*We assume that $a(t) \equiv 1$ and that $D(e^{j\omega}) = \overline{D}(e^{j\omega})$. Then, at the maximum frequency deviation of 75 kHz, $x[n] = e^{j\omega_f n}$ where $\omega_f = 2\pi \times \frac{75}{400}$. To ensure $|w[n]| \le 1$, we require $\left|D(e^{j\omega_f})\right| = c\omega_f \le 1$. Hence $c \le \frac{400}{2\pi\times 75} = 0.849$.*

d) An alternative choice for the target response is

$$\widetilde{D}(e^{j\omega}) = \begin{cases} \frac{-jc\omega_1(\pi+\omega)}{\pi-\omega_1} & \text{for } -\pi < \omega \le -\omega_1 \\ jc\omega & \text{for } |\omega| \le \omega_1 \\ \frac{jc\omega_1(\pi-\omega)}{\pi-\omega_1} & \text{for } \omega_1 < \omega \le \pi \end{cases}$$

i) Assuming that $\omega_1 = \frac{\omega_0+\pi}{2}$, draw dimensioned sketches showing the magnitude and phase responses of $\widetilde{D}(e^{j\omega})$ over the range $-\pi \le \omega \le \pi$. [4]

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ii) Outline the relative advantages and disadvantages of using $\widetilde{D}(e^{j\omega})$ rather than $\overline{D}(e^{j\omega})$ as the target response when designing $D(e^{j\omega})$. [ 2 ]

---

*The advantage is that since the $\widetilde{D}(e^{j\omega})$ is a continuous function of $\omega$ it will not be affected by Gibbs phenomenon and the coefficients will decay $\propto n^{-2}$ instead of $\propto n^{-1}$. For any given filter length, $M+1$, the errors will be much smaller for $\widetilde{D}(e^{j\omega})$ than for $\overline{D}(e^{j\omega})$.*

*The disadvantage is that the gain is no longer approximately zero for frequencies above $\omega_1$ and so it may be necessary to include additional filtering in the channel selection process in order to remove frequency components between $\omega_0$ and $\pi$.*

---

e) An alternative structure that avoids any divisions is shown in Fig. 4.2 where the polynomial $f(v)$ is the truncated Taylor series for $v^{-1}$ expanded around $v=1$. Determine $f(v)$ for the cases when it is (i) a linear expression and (ii) a quadratic expression. In each case determine the gain error (expressed in dB) resulting from the approximation when $a(t)=1.1$. [ 4 ]

---

*(i) Linear case: $f(v)=1-(v-1)=2-v$. Thus we multiply by $2-a^2$ instead of $a^{-2}$. The error is therefore $\frac{2-a^2}{a^{-2}}=0.959=-0.392\,\text{dB}$.*

*(ii) Quadratic case: $f(v)=1-(v-1)+(v-1)^2=3-3v+v^2$. The error is therefore $\frac{3-3a^2+a^4}{a^{-2}}=1.009=+0.0801\,\text{dB}$.*

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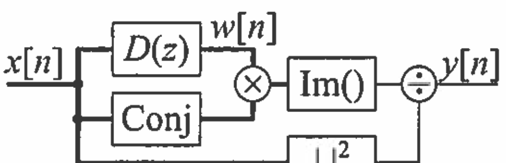


Figure 4.1

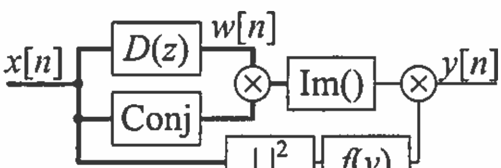


Figure 4.2