

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2010

ISE PART II: MEng, BEng and ACGI

DISCRETE MATHEMATICS AND COMPUTATIONAL COMPLEXITY

Monday, 24 May 2:30 pm

Time allowed: 2:00 hours

There are FOUR questions on this paper.

Q1 is compulsory.

Answer Q1 and any two of questions 2-4.

Q1 carries 40% of the marks. Questions 2 to 4 carry equal marks (30% each).

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s) :	G.A. Constantinides, G.A. Constantinides
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NOTATION

The following notation may be used throughout this paper:

\mathbb{R} : The set of real numbers.

\mathbb{Z} : The set of integers.

\mathbb{N} : The set of natural numbers.

$\mathcal{P}(S)$: The power set of set S .

The Questions

1. [Compulsory]

a) For the sets $S_1 = \{\emptyset, a\}$, and $S_2 = \{2, 3\}$, list the elements of:

- i) $S_1 \cup S_2$,
- ii) $S_1 \cap S_2$,
- iii) $S_1 - S_2$,
- iv) $S_1 \times S_2$,
- v) $\mathcal{P}(S_1)$.

[9]

b) Provide one example of each of the following functions from \mathbb{N} to \mathbb{N} :

- i) An injection but not a surjection,
- ii) A surjection but not an injection,
- iii) A bijection,
- iv) Neither a surjection nor an injection.

[6]

c) Express each of these statements using predicate logic syntax. You may assume the existence of an addition operation “+” over integers, and an equality predicate “=”, both with the usual meaning, *e.g.* $4 + 1 = 3 + 2$. The universe of discourse should be the set of integers.

- i) “No matter which two integers a and b I choose, $a + b$ has the same value as $b + a$ ”.
- ii) “If I want to sum any three integers, it doesn’t matter whether I add the first two integers first, and then add the third, or whether I add the last two, and then add the first”.
- iii) “For every integer a , there is another integer b such that no matter which integer c I choose, if I add a to c and then add b to the result, I get back to c ”.

- iv) “There is an integer to which I can add any integer a , and I get the result a ”.

[6]

- d) Solve the following recurrence relations, in each case stating whether the resulting sequence a_n is $O(n)$.

- i) $a_n = a_{n-1}$ for $n > 1$ with $a_1 = 1$,
- ii) $a_n = 2a_{n-1} + 1$ for $n > 1$ with $a_1 = 1$,
- iii) $a_n = a_{n-1} + a_{n-2} + 1$ for $n > 1$ with $a_0 = 0, a_1 = 0$.
- iv) $a_n = \frac{3}{4}a_{n-1} - \frac{1}{8}a_{n-2}$ for $n > 1$ with $a_0 = 1, a_1 = 1$.

[9]

- e) Provide one example each of a relation on $A = \{1, 2, 3\}$ that has each of the following properties. The cardinality of the relation should be at least 1 in all cases.

- i) reflexive but not symmetric or transitive,
- ii) transitive but not reflexive or symmetric,
- iii) symmetric but not reflexive or transitive,
- iv) reflexive and transitive but not symmetric,
- v) reflexive and symmetric but not transitive,
- vi) transitive and symmetric but not reflexive,
- vii) reflexive, symmetric, and transitive.

[10]

2. Let A and B be finite sets.

- a) State a formula for the number of functions from A to B in terms of the cardinalities of A and B .

[3]

- b) Derive a formula for the number of injections from A to B in terms of the cardinalities of A and B .

[9]

- c) Derive a formula for the number of surjections from A to B in terms of the cardinalities of A and B .

[9]

- d) Derive a formula for the number of bijections from A to B in terms of the cardinalities of A and B .

[9]

3. a) Write a predicate logic expression for each of these statements, given that R is a relation on a set A . Use A as the universe of discourse.

i) “ R is a reflexive relation”.

ii) “ R is a transitive relation”.

[4]

R is said to be *antisymmetric* iff $\forall a \forall b (aRb \wedge bRa \rightarrow (a = b))$. A relation \preceq is a *partial order* iff it is reflexive, antisymmetric, and transitive. A relation \preceq is a *total order* if it is both a partial order and also $\forall a \forall b ((a \preceq b) \vee (b \preceq a))$.

Consider the relation $\preceq_1 = \{(a, b) | \exists k \in \mathbb{N} (b = ka)\}$ on the set \mathbb{N} and $\preceq_2 = \{(a, b) | \exists k \in \mathbb{N} (b = ka)\}$ on the set $B = \{1, 2, 3, 4, 6, 8, 12\}$.

- b) Show that \preceq_1 and \preceq_2 are both partial orders.

[12]

- c) Show that neither \preceq_1 nor \preceq_2 are total orders.

[6]

- d) Draw the digraph of $R_1 = \{(1, 2), (1, 3), (2, 4), (2, 6), (3, 6), (4, 8), (4, 12), (6, 12)\}$.

[4]

- e) Draw the digraph of $\{(b, b) | b \in B\} \cup R_1^*$, where R_1^* denotes the transitive closure of R_1 , and comment on its relationship to \preceq_2 .

[4]

4. a) State the Master Theorem.

[8]

- b) Write pseudo-code for four procedures, each operating on an array a of integers of length n , and respectively having execution time:
- i) that is a $\Omega(n)$ function,
 - ii) that is a $\Omega(2^n)$ function,
 - iii) that the Master Theorem shows to be a $O(n)$ function,
 - iv) that the Master Theorem shows to be a $O(n^2 \log n)$ function.

You may assume that no compiler optimizations would be performed on your code.

[14]

- c) Let Π denote the set of all problems. Let A denote the set of all algorithms. Let $Q(x, y, z)$ be the predicate “Algorithm x solves problem y in worst-case time $O(z)$ ”, where z is a function of the size, n , of the problem instance. For example, $Q(\text{myalg}, \text{myprob}, n^2)$ states that myalg solves myprob in worst-case quadratic time. Let P be the set of all tractable problems. Define P in terms of Q using predicate logic syntax.

[4]

- d) Give one example each of: a tractable problem, an unsolvable problem, and a solvable problem not known to be tractable.

[4]

Discrete Mathematics

and Computational Complexity

E 2.17

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Solutions 2010

1. a) (i) $S_1 \cup S_2 = \{\emptyset, a, 2, 3\}$

(ii) $S_1 \cap S_2 = \emptyset$

(iii) $S_1 - S_2 = \{\emptyset, a\}$

(iv) $S_1 \times S_2 = \{(\emptyset, 2), (\emptyset, 3), (a, 2), (a, 3)\}$

(v) $P(S_1) = \{\emptyset, \{\emptyset\}, \{a\}, \{\emptyset, a\}\}$

[9]

b) (i) $f(n) = n + 1$

(ii) $f(n) = \lfloor n/2 \rfloor$

(iii) $f(n) = n$

(iv) $f(n) = \lfloor n/2 \rfloor + 1$

[6]

c) (i) $\forall a \forall b (a + b = b + a)$

(ii) $\forall a \forall b \forall c ((a + b) + c = a + (b + c))$

(iii) $\forall a \exists b \forall c ((a + c) + b = c)$

(iv) $\exists b \forall a (b + a = a)$

[6]

d) (i) $a_n = 1 \quad (n \geq 1) \quad \text{is } O(n)$

(ii) $a_n = \alpha 2^n - 1 \quad (n \geq 1)$

$a_1 = 2\alpha - 1 = 1$

$\Rightarrow \alpha = 1 \quad \text{so} \quad a_n = 2^n - 1 \quad \text{Not } O(n)$

(iii) Form $r^2 - r - 1 = 0$

$1^2 - 4 \cdot 1 \cdot (-1) \neq 0$, so distinct roots.

Roots are $r_1 = \frac{1 - \sqrt{1+4}}{2} = \frac{1}{2}(1 - \sqrt{5})$

$r_2 = \frac{1}{2}(1 + \sqrt{5})$

$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n - \frac{1}{1 \mp 1 - 1}$

$= \alpha_1 r_1^n + \alpha_2 r_2^n - 1$

$\alpha_1 + \alpha_2 - 1 = 0 \quad \Rightarrow \quad \alpha_1 r_1 + \alpha_2 r_2 - r_1 = 0$

$\alpha_1 r_1 + \alpha_2 r_2 - 1 = 0$

$$\text{So } \alpha_2 = \frac{r_1 - 1}{r_1 - r_2}$$

$$\alpha_1 = \frac{1 - r_2}{r_1 - r_2}$$

a_n is not $O(n)$.

[9]

(iv) $(r - \frac{1}{2})(r - \frac{1}{4}) = 0 \rightarrow \text{distinct roots.}$

$$a_n = \alpha_1 \left(\frac{1}{2}\right)^n + \alpha_2 \left(\frac{1}{4}\right)^n$$

$$\left. \begin{array}{l} \alpha_1 + \alpha_2 = 1 \\ 2\alpha_1 + \alpha_2 = 4 \end{array} \right\} \Rightarrow \alpha_1 = 3, \alpha_2 = -2$$

$$a_n = 3\left(\frac{1}{2}\right)^n - 2\left(\frac{1}{4}\right)^n \quad n \geq 0 \quad \text{is } \underline{O(n)}.$$

e) (i) $R = \{(1,1), (2,2), (3,3), (1,2), (2,3)\}$

(ii) $R = \{(1,2), (2,3)\} \cup \{(1,3)\}$

(iii) $R = \{(1,2), (2,3), (2,1), (3,2)\}$

(iv) $R = \{(1,1), (2,2), (3,3), (1,2)\}$

(v) $R = \{(1,1), (2,2), (3,3), (1,2), (2,3), (2,1), (3,2)\}$

(vi) $R = \{ \emptyset \} \cup \{(1,1), (1,2), (2,1), (2,2)\}$

(vii) $R = \{1,2,3\} \times \{1,2,3\}$.

[10]

a) $|B|^{|A|}$ (book)

[3]

b) $\frac{|B|^{|A|}}{(|B|-|A|)!}$ $\left(|B| \times (|B|-1) \times \dots \times (|B|-|A|+1) \right)$ - each time we
 fewer co-domain element to select [9] ~~16~~

c) $\frac{|A|^{|B|}}{(|A|-|B|)!}$ $\left(|A| \times (|A|-1) \times \dots \times (|A|-|B|+1) \right)$ - each time we
 fewer domain element to select [9] ~~16~~

d) $|A|!$ (note that this is also $|B|!$, since $|A|=|B|$) [9]

3. a) (i) $\forall a (aRa)$
 (ii) $\forall a \forall b \forall c (aRb \wedge bRc \rightarrow aRc)$

b) (i) Reflexivity of \leq_1
 Consider $a \in \mathbb{Z}$. $(a, a) \in \leq_1$ because $\exists k$ ($k=1$)
 s.t. $a = k \cdot a$

(ii) Transitivity of \leq_1
 Consider $(a, b) \in \leq_1$ and $(b, c) \in \leq_1$

Then $\exists k_1, k_2$ s.t.

$$b = k_1 a \quad \text{and} \quad c = k_2 b$$

Since $c = k_2 k_1 a$, $\exists k$ s.t. ($k = k_1 k_2$)

$$\text{s.t. } c = k a$$

$(a, c) \in \leq_1 \Rightarrow \leq_1$ is transitive.

(iii) Antisymmetry.

$$a \leq b \quad \text{and} \quad b \leq a$$

$$\text{Then } b = k_1 a \quad \text{and} \quad a = k_2 b$$

$$\text{So } a = k_2 k_1 a \Rightarrow k_2 k_1 = 1$$

$$\text{Since } k_1, k_2 \in \mathbb{Z}, \quad k_1 = 1 \quad \text{and} \quad k_2 = 1$$

$$\text{Thus } a = b$$

$\therefore \leq_1$ is a partial order.

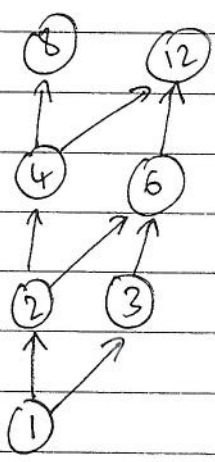
It follows that \leq_2 is a partial order, since
 $\{1, 2, 3, 4, 6, 8, 12\} \subseteq \mathbb{N}$

c) $(4, 6) \notin \leq_1$, as $\nexists k \in \mathbb{Z}$ s.t. $4 = 6k$ is not satisfiable
 for k integer.

Equally, $(6, 4) \notin \leq_1$, as $6 = 4k$ is not satisfiable for
 k integer.

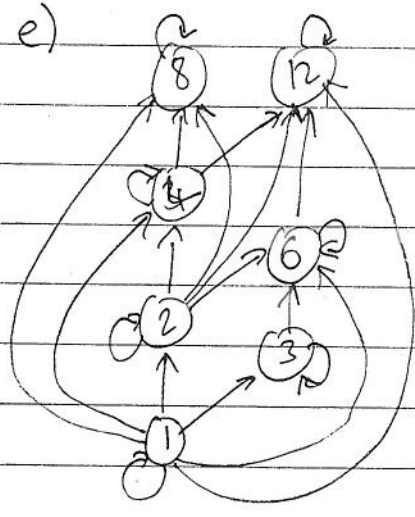
Since 4 & 6 are in the sets for both \leq_1 & \leq_2 ,
 the proof holds for both.

3. d)



[4]

e)



They are equal.

[4]

8. a) let $a \geq 1$ be a real number, $b > 1$ be an integer,
 $c > 0$ be a real number, and $d \geq 0$ be a real number.

let f be an increasing function satisfying $f(n) = a f(n/b) + cn^d$
 whenever $n = b^k$ for $k \in \mathbb{Z}^+$.

Then:

If $a < b^d$, $f(n)$ is $O(n^d)$

If $a = b^d$, $f(n)$ is $O(n^d \log n)$

If $a > b^d$, $f(n)$ is $O(n^{\log_b a})$

(bookwork)

-8]

b) (i) proc $p_1(a[n]: \text{integer})$

$t := 0$

for $i = 1$ to n

$t := t + a[i]$

end
 $\text{result} := t$

(ii) proc $p_2(a[n]: \text{integer})$

if $n > 1$
 $\text{result} := p_2(a[1 \text{ to } n-1]) + p_2(a[2 \text{ to } n])$

else
 $\text{result} := 1$

end

(iii) proc $p_3(a[n]: \text{integer})$

if $n > 1$

$\text{result} := p_3(a[1 \text{ to } \lfloor n/2 \rfloor])$

for $i = 1$ to n

$\text{result} := \text{result} + 1$

end

else

$\text{result} := 1$

end

(iv) proc $p_4(a[n]: \text{integer})$

if $n > 1$
 for $i = 1$ to n
 $\text{result} := \text{result} + p_4(a[1 \text{ to } \lfloor n/2 \rfloor])$

end
 $\text{result} := \text{result} + 1$

else
 $\text{result} := 1$

end

[4]

c) $P = \{y \mid \exists x \exists p \in Q(x, y, n)$

c) $P = \{y \mid \exists x \in A \exists b \in \mathbb{Z}^+ Q(x, y, n^b)\}$ [4]

d) tractable: matrix multiplication

unsolvable: halting problem

intertable (?): k -colouring of a graph. [4]