Modelling and control of multibody mechanical systems

Model answers

Question 1

a) $\boldsymbol{r_M} = x\boldsymbol{i}$ and $\boldsymbol{r_m} = x\boldsymbol{i} + l\boldsymbol{e_r} = (x + l\sin\theta)\boldsymbol{i} + l\cos\theta\boldsymbol{k}$.

b) By differentiating the position vector $\dot{\boldsymbol{r}}_{\boldsymbol{M}} = \dot{x}\boldsymbol{i}$ and

$$\dot{\boldsymbol{r}}_{\boldsymbol{m}} = \dot{x}\boldsymbol{i} + l\dot{\theta}\boldsymbol{e_\theta} = \dot{x}\sin\theta\boldsymbol{e_r} + (\dot{x}\cos\theta + l\dot{\theta})\boldsymbol{e_\theta}.$$

c)

$$T = \frac{1}{2}M\dot{\boldsymbol{r}}_{\boldsymbol{M}}\cdot\dot{\boldsymbol{r}}_{\boldsymbol{M}} + \frac{1}{2}m\dot{\boldsymbol{r}}_{\boldsymbol{m}}\cdot\dot{\boldsymbol{r}}_{\boldsymbol{m}} = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x}^2 + 2\dot{x}\dot{\theta}l\cos\theta + l^2\dot{\theta}^2\right).$$

d) The horizontal level at O is taken as the zero potential energy level, therefore

$$V = -m\boldsymbol{r_m}\cdot\boldsymbol{g} + \frac{1}{2}kx^2 = -mgl\cos\theta + \frac{1}{2}kx^2.$$

e)

$$L = T - V = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x}^2 + 2\dot{x}\dot{\theta}l\cos\theta + l^2\dot{\theta}^2\right) + mgl\cos\theta - \frac{1}{2}kx^2.$$

f) The Lagrangian equation with respect to the generalised coordinate x is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{x}}\right) - \frac{\partial L}{\partial x} = -c\dot{x},$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(M\dot{x} + m\dot{x} + m\dot{\theta}l\cos\theta\right) + kx = -c\dot{x},$$

or

$$(M+m)\ddot{x} + ml\cos\theta\ddot{\theta} - ml\dot{\theta}^2\sin\theta + c\dot{x} + kx = 0.$$

The Lagrangian equation with respect to the generalised coordinate θ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0,$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m\dot{x}l\cos\theta + ml^2\dot{\theta}\right) + m\dot{x}\dot{\theta}l\sin\theta + mgl\sin\theta = 0,$$

or

$$\cos\theta\ddot{x} + l\ddot{\theta} + g\sin\theta = 0.$$

Question 2

a) $\dot{\boldsymbol{r}} = \dot{r}\boldsymbol{e_r} + r\dot{\theta}\boldsymbol{e_\theta}$.

b) The kinetic energy is $T = \frac{1}{2}m\dot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$.
The potential energy is $V = -m\boldsymbol{r} \cdot \boldsymbol{g} = -mr\boldsymbol{e_r} \cdot g\boldsymbol{k} = -mgr\cos\theta$, with the level of point O corresponding to zero gravitational potential energy.
The Lagrangian is $L = T - V = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + mgr\cos\theta$.

c) The constraint equation is

$$r = \alpha + (r_0 - \alpha)\cos\theta, \tag{1}$$

by differentiating

$$\dot{r} + (r_0 - \alpha)\sin\theta\dot{\theta} = 0, \tag{2}$$

and by differentiating once again

$$\ddot{r} = -(r_0 - \alpha)\cos\theta\dot{\theta}^2 - (r_0 - \alpha)\sin\theta\ddot{\theta}. \tag{3}$$

The Lagrangian equation with respect to the generalised coordinate r is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} + \lambda = 0,$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{r}) - mr\dot{\theta}^2 - mg\cos\theta + \lambda = 0,$$

or

$$m\ddot{r} - mr\dot{\theta}^2 - mg\cos\theta + \lambda = 0,$$

or by using Equations (1) and (3)

$$\lambda = m\left((\alpha + 2(r_0 - \alpha)\cos\theta)\dot{\theta}^2 + (r_0 - \alpha)\sin\theta\ddot{\theta} + g\cos\theta\right). \tag{4}$$

The Lagrangian equation with respect to the generalised coordinate θ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} + \lambda(r_0 - \alpha)\sin\theta = 0,$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(mr^2\dot{\theta}\right) + mgr\sin\theta + \lambda(r_0 - \alpha)\sin\theta = 0,$$

or

$$mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} + mgr\sin\theta + \lambda(r_0 - \alpha)\sin\theta = 0,$$

or by using Equations (1), (2) and (4)

$$\left(\alpha^2 + 2\alpha(r_0 - \alpha)\cos\theta + (r_0 - \alpha)^2\right)\ddot{\theta} - \alpha(r_0 - \alpha)\sin\theta\dot{\theta}^2 + (\alpha + 2(r_0 - \alpha)\cos\theta)\,g\sin\theta = 0. \tag{5}$$

d) The force in the wire, F_{wire}, is given by $-\lambda$, therefore

$$F_{wire} = -m\left((\alpha + 2(r_0 - \alpha)\cos\theta)\dot{\theta}^2 + (r_0 - \alpha)\sin\theta\ddot{\theta} + g\cos\theta\right).$$

e) For small θ the equation of motion is

$$r_0{}^2\ddot{\theta} + (r_0 + (r_0 - \alpha))g\theta = 0,$$

and therefore the mass executes simple harmonic motion with angular frequency

$$\sqrt{\frac{r_0 + (r_0 - \alpha)}{r_0}\frac{g}{r_0}}.$$

For fixed wire length, $r = r_0$, $r_0 = \alpha$ and therefore the frequency of oscillations is $\omega_0 = \sqrt{g/r_0}$. For $r_0 > \alpha$, $\omega > \omega_0$ and for $r_0 < \alpha$, $\omega < \omega_0$.

Question 3

a) The angular velocity of the system about the vertical axis is $\dot{\psi}$ and in the $\boldsymbol{i}$ direction it is $\dot{\theta}$. All together it is

$$\boldsymbol{\Omega} = \dot{\theta}\boldsymbol{i} + \dot{\psi}\sin\theta\boldsymbol{j} + \dot{\psi}\cos\theta\boldsymbol{k}.$$

The position vector of the lower mass (in the position shown in the diagram) is

$$\boldsymbol{r_1} = \frac{l}{2}\boldsymbol{j}.$$

The velocity vector is

$$\boldsymbol{v_1} = \dot{\boldsymbol{r_1}} = \boldsymbol{\Omega} \times \boldsymbol{r_1}$$

which gives

$$\boldsymbol{v_1} = -\frac{l}{2}\dot{\psi}\cos\theta\boldsymbol{i} + \frac{l}{2}\dot{\theta}\boldsymbol{k}.$$

The velocity vector of the other mass is given by

$$\boldsymbol{v_2} = -\boldsymbol{v_1}.$$

b) The acceleration vector of the lower mass is

$$\boldsymbol{a_1} = \dot{\boldsymbol{v_1}} = \frac{l}{2}\ddot{\theta}\boldsymbol{k} - \left(\frac{l}{2}\ddot{\psi}\cos\theta - \frac{l}{2}\dot{\psi}\dot{\theta}\sin\theta\right)\boldsymbol{i} + \boldsymbol{\Omega} \times \boldsymbol{v_1},$$

or

$$\boldsymbol{a_1} = \left(\frac{l}{2}\ddot{\psi}\cos\theta + l\dot{\psi}\dot{\theta}\sin\theta\right)\boldsymbol{i} - \left(\frac{l}{2}\dot{\psi}^2\cos^2\theta + \frac{l}{2}\dot{\theta}^2\right)\boldsymbol{j} + \left(\frac{l}{2}\ddot{\theta} + \frac{l}{2}\dot{\psi}^2\sin\theta\cos\theta\right)\boldsymbol{k}.$$

The acceleration vector of the other mass is

$$\boldsymbol{a_2} = -\boldsymbol{a_1}.$$

c) The force vector acting on the lower mass is

$$\boldsymbol{F_1} = -F_N\boldsymbol{i} - F_r\boldsymbol{j},$$

where F_N is the magnitude of the force on each mass due to the moment N acting on the rod. This is given by

$$F_N = \frac{N}{l},$$

therefore

$$\boldsymbol{F_1} = -\frac{N}{l}\boldsymbol{i} - F_r\boldsymbol{j}.$$

The force vector on the other mass is

$$\boldsymbol{F_2} = -\boldsymbol{F_1}.$$

d) The motion of the system can be found by considering the motion of one of the masses. For the lower mass

$$\boldsymbol{F_1} = m\boldsymbol{a_1}$$

or by substituting the force and acceleration expressions from the equations above and collecting the terms with respect to $\boldsymbol{i}$ and $\boldsymbol{k}$

$$\ddot{\theta} + \dot{\psi}^2\sin\theta\cos\theta = 0,$$

and

$$\frac{1}{2}ml^2\left(\ddot{\psi}\cos\theta - 2\dot{\psi}\dot{\theta}\sin\theta\right) = N.$$

e) By using again the equation $\boldsymbol{F_1} = m\boldsymbol{a_1}$ and collecting the terms with respect to $\boldsymbol{j}$ we obtain

$$F_r = m\left(\frac{l}{2}\dot{\psi}^2\cos^2\theta + \frac{l}{2}\dot{\theta}^2\right).$$

Question 4

a) The velocity vector of mass M is $\dot{\boldsymbol{r}}_M = \dot{x}\boldsymbol{i}$. By differentiating the velocity expression we obtain the acceleration vector,

$$\ddot{\boldsymbol{r}}_M = \ddot{x}\boldsymbol{i}.$$

b) The velocity vector of mass m is $\dot{\boldsymbol{r}}_m = \dot{x}\boldsymbol{i} + l\dot{\theta}\boldsymbol{e_\theta} = \dot{x}\sin\theta\boldsymbol{e_r} + (\dot{x}\cos\theta + l\dot{\theta})\boldsymbol{e_\theta}$. By differentiating the velocity expression we obtain the acceleration vector,

$$\ddot{\boldsymbol{r}}_m = \left(\ddot{x}\sin\theta - l\dot{\theta}^2\right)\boldsymbol{e_r} + \left(\ddot{x}\cos\theta + l\ddot{\theta}\right)\boldsymbol{e_\theta}.$$

c) The equation of motion of mass m in vector form is

$$\boldsymbol{F_m} = m\ddot{\boldsymbol{r}}_m,$$

or

$$-F_r\boldsymbol{e_r} + mg\cos\theta\boldsymbol{e_r} - mg\sin\theta\boldsymbol{e_\theta} = m\left(\ddot{x}\sin\theta - l\dot{\theta}^2\right)\boldsymbol{e_r} + m\left(\ddot{x}\cos\theta + l\ddot{\theta}\right)\boldsymbol{e_\theta}.$$

i) The first equation of motion is found by collecting the $\boldsymbol{e_\theta}$ terms

$$\ddot{x}\cos\theta + l\ddot{\theta} + g\sin\theta = 0.$$

ii) The force in the rod, F_r, is found by collecting the $\boldsymbol{e_r}$ terms and it is given by

$$F_r = m\left(-\ddot{x}\sin\theta + l\dot{\theta}^2 + g\cos\theta\right).$$

d) The equation of motion of mass M in vector form is

$$\boldsymbol{F_M} = M\ddot{\boldsymbol{r}}_M,$$

or

$$(-kx - c\dot{x} + F_r\sin\theta)\boldsymbol{i} + (Mg + F_r\cos\theta - R)\boldsymbol{k} = M\ddot{x}\boldsymbol{i},$$

where R is the normal reaction from the surface on the cart. By collecting the $\boldsymbol{i}$ terms we obtain the second equation of motion

$$M\ddot{x} - F_r\sin\theta + c\dot{x} + kx = 0,$$

or

$$\left(M + m\sin^2\theta\right)\ddot{x} - ml\dot{\theta}^2\sin\theta - mg\sin\theta\cos\theta + c\dot{x} + kx = 0.$$

e) For $k_s = 0$, $c = 0$ and small x and θ the two equations of motion become

$$M\ddot{x} - mg\theta = 0,$$

and

$$\ddot{x} + l\ddot{\theta} + g\theta = 0.$$

By a simple manipulation these two equations give

$$Ml\ddot{\theta} + (M + m)g\theta = 0,$$

or

$$\ddot{\theta} + \frac{M+m}{M}\frac{g}{l}\theta = 0,$$

which is simple harmonic motion in θ for the pendulum with frequency of oscillation $\sqrt{\frac{M+m}{M}\frac{g}{l}}$. By another simple manipulation the equations of motion give

$$(M+m)\ddot{x} + ml\ddot{\theta} = 0,$$

or

$$\ddot{x} = -\frac{ml}{M+m}\ddot{\theta},$$

which can be integrated to give

$$x = -\frac{ml}{M+m}\theta + x_0,$$

for initial $\dot{x} = 0$ and $\dot{\theta} = 0$. Therefore the cart also executes simple harmonic motion about some position x_0 with the same frequency as for the pendulum, and with its amplitude scaled by $-\frac{ml}{M+m}$ as compared to the amplitude of the motion of the pendulum.