E4.25
C4.1
ISE4.23

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2004

MSc and EEE/ISE PART IV: MEng and ACGI

DESIGN OF LINEAR MULTIVARIABLE CONTROL SYSTEMS

Wednesday, 12 May 10:00 am

Time allowed: 3:00 hours

There are SIX questions on this paper.

Answer FOUR questions.

Corrected Copy

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s) :	I.M. Jaimoukha
	Second Marker(s) :	D.J.N. Limebeer

Special Information for Invigilators :	None
Information for Candidates :	None

1. (a) Let

$$G(s) = \begin{bmatrix} \frac{(s+1)}{(s+2)(s+4)} & \frac{(s+1)}{(s+4)} \\ \frac{(s+3)}{(s+2)(s+4)} & \frac{1}{(s+4)} \end{bmatrix}$$

(i) Find the McMillan form of $G(s)$. **[6]**

(ii) Determine the pole and zero polynomials of $G(s)$. **[2]**

(iii) Find the poles and zeros of $G(s)$ specifying the multiplicity of each. **[2]**

(b) Consider a state–variable model described by the dynamics

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

(i) Suppose that the pair (A, C) is observable and that there exists $Q = Q' > 0$ such that

$$A'Q + QA + C'C = 0$$

Prove that A is stable. **[5]**

(ii) Suppose that A is stable and that there exists $P = P' > 0$ such that

$$AP + PA' + BB' = 0$$

Prove that the pair (A, B) is controllable. **[5]**

2. (a) Define internal stability for the feedback loop shown in Figure 2, and derive necessary and sufficient conditions (in terms of $G(s)$ and $K(s)$) for which this loop is internally stable.

[4]

(b) Suppose that $G(s)$ is stable. Derive a parametrisation of all internally stabilising controllers for $G(s)$.

[6]

(c) Suppose that $G(s)$ and $G^{-1}(s)$ are stable transfer matrices. Using the answer to part (b), or otherwise, design an internally stabilising controller $K(s)$ such that

$$y(s) = \frac{1}{s+1} r(s).$$

The controller $K(s)$ should be given in terms of $G(s)$.

[10]

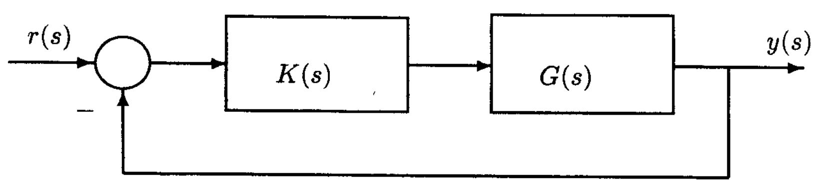


Figure 2

3. Figure 3.1 illustrates the implementation of the control law $u(t) = -Kx(t)$ which minimises

$$J(x_0, u) = \int_0^{\infty} \|Cx(t)\|^2 + \|u(t)\|^2 \, dt$$

subject to $\dot{x}(t) = Ax(t) + Bu(t)$, $x(0) = x_0$. Here $K = B'P$ and $P = P'$ is the unique positive definite solution of $A'P + PA - PBB'P + C'C = 0$. Assume that the triple (A, B, C) is minimal.

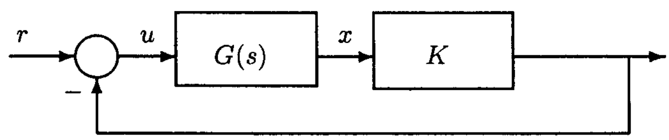


Figure 3.1

(a) Write the closed–loop dynamics as $\dot{x}(t) = A_c x(t) + Br(t)$. Find A_c and prove that it is stable. **[6]**

(b) Let $G(s) = (sI - A)^{-1}B$ and $L(s) = I + KG(s)$. Show that

$$L(j\omega)'L(j\omega) = I + G(j\omega)'C'CG(j\omega).$$ **[6]**

(c) Suppose that stable perturbations Δ_1 and Δ_2 are introduced as shown in Figure 3.2. Derive the maximal stability radius (using the $\mathcal{L}_\infty$-norm as a measure):

(i) for Δ_1 when $\Delta_2 = 0$,

(ii) for Δ_2 when $\Delta_1 = 0$. **[8]**

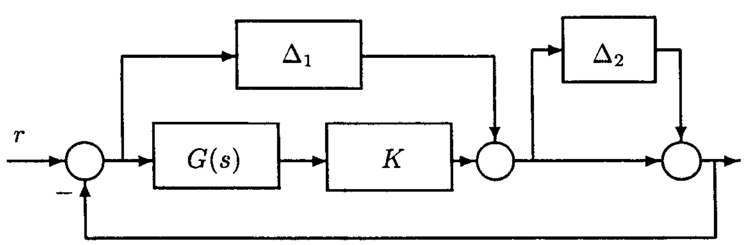


Figure 3.2

4. Consider the feedback configuration shown in Figure 4. Here, $G(s)$ represents a nominal plant model and $K(s)$ represents a compensator. $\Delta_1(s)$ and $\Delta_2(s)$ are stable transfer matrices that represent uncertainties. The design specification are to synthesise a compensator $K(s)$ such that the feedback loop is internally stable when:

(i) $\Delta_1 = 0$ and $\|\Delta_2(j\omega)\| \leq |w_2(j\omega)|, \forall\omega$, and,

(ii) $\Delta_2 = 0$ and $\|\Delta_1(j\omega)\| \leq |w_1(j\omega)|, \forall\omega$,

where

$$w_1(s) = 0.5\frac{(s+5)^2}{(s+1)^2}, \qquad w_2(s) = 10\frac{(s+10)^2}{(s+50)^2}.$$

(a) Derive conditions, in terms of $G(s), K(s), w_1(s)$ and $w_2(s)$ that are sufficient to achieve the design specifications. [5]

(b) Derive a generalised regulator formulation of the design problem that captures the sufficient conditions in Part (a). [10]

(c) Assume that a compensator $K(s)$ achieves the design specifications. Comment on the performance properties (tracking, disturbance rejection, noise attenuation and control effort) for the resulting feedback loop. [5]

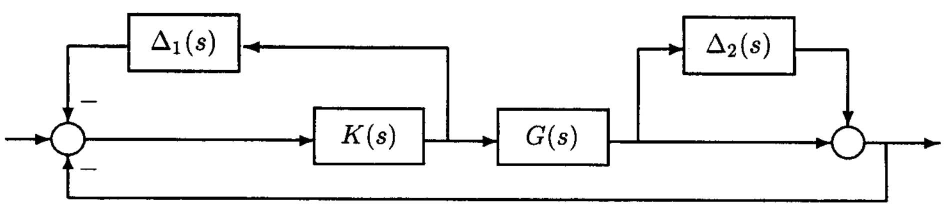


Figure 4

5. (a) State the small gain theorem concerning the internal stability of a loop with forward transfer matrix Δ and feedback transfer matrix S. [4]

(b) Consider the feedback loop shown in Figure 5 where $G(s)$ represents a plant model and $K(s)$ represents an internally stabilising compensator. Suppose that

$$K(s) \overset{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{ccc|cc} -1 & -1 & 0 & 1 & 1 \\ -1 & -1.25 & 0 & 0.6 & 0.8 \\ 0 & 0 & -10 & 0 & 0 \\ \hline 1 & 0.6 & 0 & 0 & 0 \\ 1 & 0.8 & 0 & 0 & 0 \end{array}\right] \in \mathcal{RH}_\infty.$$

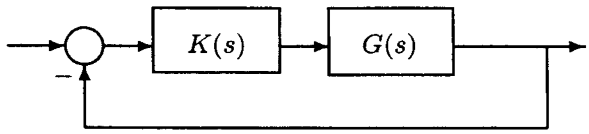
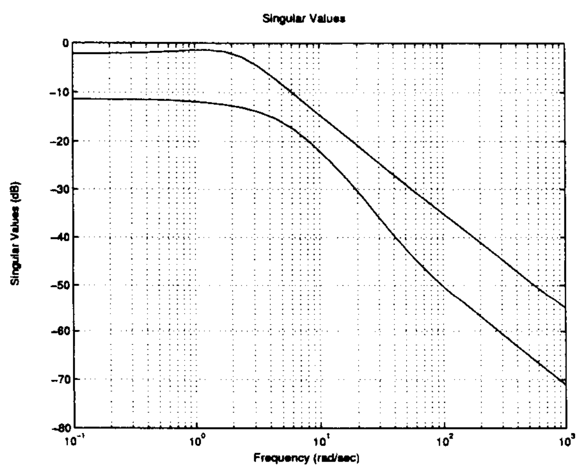


Figure 5

(i) Show that the given realisation for $K(s)$ is balanced and evaluate the Hankel singular values of $K(s)$. [5]

(ii) Find a 2nd order compensator that achieves the same design specifications as $K(s)$. [5]

(iii) The graph below shows the singular value plot of $(I+GK)^{-1}G$. Find a first order compensator $K_r(s)$, such that the loop is stable when $K(s)$ is replaced by $K_r(s)$. Justify your answer. [6]



6. (a) Consider the regulator shown in Figure 6 for which it is assumed that the triple (A, B, C) is minimal and $x(0)=0$.

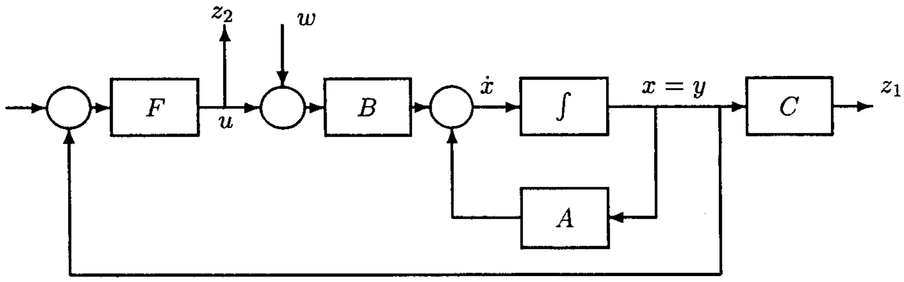


Figure 6

Let $z = \begin{bmatrix} z_1^T & z_2^T \end{bmatrix}^T$ and let H denote the transfer matrix from w to z. A stabilizing state-feedback gain matrix F is to be designed such that, for given $\gamma > 0$, $\|H\|_\infty < \gamma$.

(i) Derive the generalized regulator system for this problem. **[6]**

(ii) By using the Lyapunov function $V(t) = x(t)^T X x(t)$, where X is to be determined, derive sufficient conditions for the solution of the design problem. Your conditions should be in the form of the existence of a certain solution to an algebraic Riccati equation. It should also include an expression for F and an expression for the worst-case disturbance w. Use the identity
$(\alpha R - \alpha^{-1} S)^T (\alpha R - \alpha^{-1} S) = \alpha^2 R^T R + \alpha^{-2} S^T S - R^T S - S^T R,$
for scalar $\alpha \neq 0$ and matrices R and S to complete the squares. **[8]**

(b) Consider the dynamics

$$\dot{x} = Ax + B(w_1 + u), \qquad y = Cx + w_2$$

where variables have the standard interpretation and the estimator

$$\dot{\hat{x}} = A\hat{x} + Bu - u_e, \qquad \hat{y} = C\hat{x}$$

Define $x_e = x - \hat{x}$, $y_e = y - \hat{y}$, $z_e = Cx_e$ and $u_e = Ky_e$ where K is a constant matrix to be designed. Using the principle of duality and the answer to part (a), or otherwise, find an internally stabilising K such that the $\mathcal{H}_\infty$-norm of the transfer matrix from $w_e = \begin{bmatrix} w_1^T & w_2^T \end{bmatrix}^T$ to z_e is smaller than γ. **[6]**

[END]

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IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

UNIVERSITY OF LONDON

DEPARTMENT OF ELECTRICAL & ELECTRONIC ENGINEERING

MEng and ACGI EXAMINATIONS 2004

PART IV

DESIGN OF LINEAR MULTIVARIABLE CONTROL SYSTEMS

SOLUTIONS

Day, Date: 10:00-13:00

There are SIX questions. Answer *FOUR*.

Examiners responsible: I.M. Jaimoukha and D.J.N. Limebeer.

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1. (a) (i) By performing the operations: $r_2 := r_2 - r_1$, $r_1 \leftrightarrow r_2$, $r_2 := r_2 - 0.5(s+1)r_1$, $c_2 := c_2 + 0.5s(s+2)c_1$, $c_1 := 0.5c_1$, $c_2 := 2c_2$, we get the McMillan form $G(s) = L(s)M(s)R(s)$ where

$$\boxed{L(s) = \begin{bmatrix} 0.5(s+1) & 1 \\ 0.5(s+3) & 1 \end{bmatrix}}$$

$$\boxed{M(s) = \begin{bmatrix} \frac{1}{(s+2)(s+4)} & 0 \\ 0 & \frac{(s+1)(s+2)}{(s+4)} \end{bmatrix}}$$

$$\boxed{R(s) = \begin{bmatrix} 2 & -s(s+2) \\ 0 & 0.5 \end{bmatrix}}$$

(ii) The pole polynomial is given by $\boxed{p(s) = (s+2)(s+4)^2}$ and the zero polynomial is given by $\boxed{z(s) = (s+1)(s+2)}$.

(iii) $\boxed{\text{The poles are at } -2, -4, -4}$ and $\boxed{\text{the zeros are at } -1, -2}$. All poles and zeros have $\boxed{\text{multiplicity } 1}$.

(b) (i) Let $z \neq 0$ be an eigenvector of A and let λ be the corresponding eigenvalue. Multiplying the observability equation by z' from the left and z from the right gives $(\lambda + \bar{\lambda})z'Qz + z'C'Cz = 0$. Since $Q > 0$ it follows that $z'Qz > 0$ and since the pair (A, C) are observable it follows that $Cz \neq 0$ by the PBH test. This proves that $\lambda + \bar{\lambda} < 0$ and so A is stable.

(ii) Let $z \neq 0$ be an eigenvector of A and let λ be the corresponding eigenvalue. Multiplying the controllability equation by z' from the left and z from the right gives $(\lambda + \bar{\lambda})z'Pz + z'BB'z = 0$. Since A is stable $(\lambda + \bar{\lambda}) < 0$ and since $P > 0$ and $z \neq 0$, $z'Pz > 0$. It follows that $z'BB'z > 0$ and so $z'B \neq 0$ and so the pair (A, B) are observable by the PBH test.

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2. (a) Inject a signal d in between $G(s)$ and $K(s)$ and call the input to $G(s)$ u. The loop is internally stable if and only if the transfer matrix from $\begin{bmatrix} d \\ r \end{bmatrix}$ to $\begin{bmatrix} u \\ e \end{bmatrix}$ is stable (no poles in the closed right half plane). Since

$$\begin{bmatrix} d \\ r \end{bmatrix} = \begin{bmatrix} I & -K \\ G & I \end{bmatrix} \begin{bmatrix} u \\ e \end{bmatrix} =: S \begin{bmatrix} u \\ e \end{bmatrix}$$

the $\boxed{\text{loop is internally stable if and only if } S^{-1} \text{ is stable.}}$

(b) Since $G(s)$ is stable, we proceed as follows. Note that

$$\begin{bmatrix} I & -K \\ G & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ G & I \end{bmatrix} \begin{bmatrix} I & -K \\ 0 & I+GK \end{bmatrix}$$

Hence

$$\begin{aligned} \begin{bmatrix} I & -K \\ G & I \end{bmatrix}^{-1} &= \begin{bmatrix} I & -K \\ 0 & I+GK \end{bmatrix}^{-1} \begin{bmatrix} I & 0 \\ G & I \end{bmatrix}^{-1} \\ &= \begin{bmatrix} I & K(I+GK)^{-1} \\ 0 & (I+GK)^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -G & I \end{bmatrix} \end{aligned}$$

Since $(I+GK)^{-1} = I - GK(I+GK)^{-1}$ and G is stable, the loop is $\boxed{\text{internally stable if and only if } Q := K(I+GK)^{-1} \text{ is stable.}}$ Rearranging terms shows that K is internally stabilising if and only if $\boxed{K = Q(I-GQ)^{-1} \text{ for some stable } Q.}$

(c) Since K is required to be internally stabilising, $K = Q(I-GQ)^{-1}$ for some stable Q from part (b). We search for a stable Q to satisfy the design requirements. Now $y = GK(I+GK)^{-1}r = GQr$, and since $G^{-1}(s)$ is stable, we can take

$$Q(s) = \frac{1}{s+1} G^{-1}(s)$$

which is stable to give

$$y(s) = \frac{1}{s+1} r(s)$$

which satisfies the design requirement. Finally,

$$\boxed{K(s) = Q(s)[I - G(s)Q(s)]^{-1} = \tfrac{1}{s} G^{-1}(s)}.$$

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3. (a) A little calculation shows that $\boxed{A_c = A - BB'P}$. Let $A_c z = \lambda z$ with $z \neq 0$. We prove $\lambda + \bar{\lambda} < 0$. Rearrange the Riccati equation as

$$A_c'P + PA_c + PBB'P + C'C = 0$$

Multiply from the left by z' and from the right by z to get

$$(\lambda + \bar{\lambda})z'Pz + z'PBB'Pz + z'C'Cz = 0$$

Then either $\boxed{\lambda + \bar{\lambda} < 0}$, in which case we are done, or else

$$\lambda + \bar{\lambda} = 0, \qquad B'Pz = 0, \qquad Cz = 0$$

So $\boxed{\lambda+\bar{\lambda}=0 \Rightarrow Az=\lambda z \ \&\ Cz=0 \text{ which contradicts observability}}$ of (A, C) by the PBH test and proves the result.

(b) By direct evaluation, $L(j\omega)'L(j\omega) = I + K(j\omega I - A)^{-1}B$

$$+B'(-j\omega + B'(-j\omega I - A')^{-1}K'K(j\omega I - A)^{-1}B$$

But $K'K = -(-j\omega I - A')P - P(j\omega I - A) + C'C$ from the Riccati equation. So, $L(j\omega)'L(j\omega)$

$$\begin{aligned} &= I + K(j\omega I - A)^{-1}B + B'(-j\omega I - A')^{-1}K' + \\ &\quad B'(-j\omega I - A')^{-1}[(j\omega I + A')P - P(j\omega I - A) + C'C](j\omega I - A)^{-1}B \\ &= I + [K - B'P](j\omega I - A)^{-1}B + B'(-j\omega I - A')^{-1}[K' - PB] \\ &\quad +B'(-j\omega I - A')^{-1}C'C(j\omega I - A)^{-1}B \boxed{= I + G(j\omega)'C'CG(j\omega)} \end{aligned}$$

(c) (i) Set $\Delta_2 = 0$. Let ϵ be the input to and δ the output of, Δ_1. Then

$$\epsilon = -(\delta + KG\epsilon) = -(I + KG)^{-1}\delta$$

Using the small gain theorem (since the regulator and the perturbation are stable), the loop is stable if $\left\|\Delta_1(I + KG)^{-1}\right\|_\infty < 1$. But part (b) implies that $\underline{\sigma}[I + KG(j\omega)] \geq 1$ which implies $\left\|(I + KG)^{-1}\right\|_\infty \leq 1$. This shows that the loop will tolerate perturbations of size $\boxed{\|\Delta_1\|_\infty < 1}$ without losing internal stability.

(ii) Set $\Delta_1 = 0$. Let ϵ be the input to and δ the output of, Δ_2. Then

$$\epsilon = -KG(\delta + \epsilon) = -(I + KG)^{-1}KG\delta = L^{-1}(I - L)\delta = (L^{-1} - I)\delta$$

Using the small gain theorem (since the regulator and the perturbation are stable), the loop is stable if $\left\|\Delta_2(L^{-1} - I)\right\|_\infty < 1$. But part (b) implies that

$$\bar{\sigma}[L(j\omega)^{-1} - I] \leq 1 + \bar{\sigma}[L(j\omega)^{-1}] \leq 1 + \frac{1}{\underline{\sigma}[L(j\omega)]} \leq 2$$

This shows that the loop will tolerate perturbations Δ_2 of size $\boxed{\|\Delta_2\|_\infty < 0.5}$ without losing internal stability.

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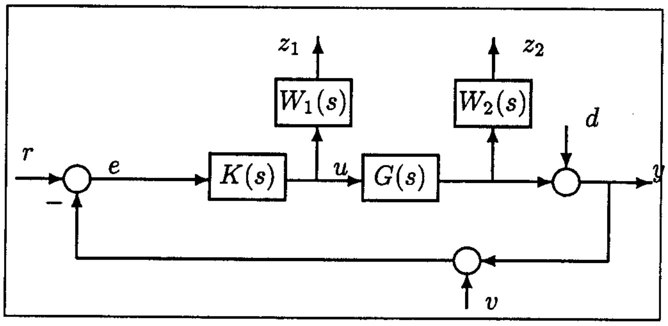
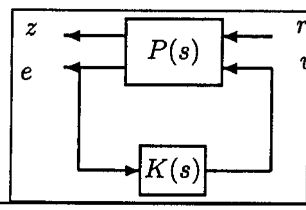
4. (a) We require K to internally stabilise the nominal model. Suppose that $\Delta_1 = 0$ and let the input to Δ_2 be ϵ while the output be δ. Then $\epsilon = -C\delta$ where $C = (I+GK)^{-1}GK$ is the complementary sensitivity which is stable. Using the small gain theorem, to satisfy the first requirement, it is sufficient that $\|\Delta_2(j\omega)C(j\omega)\| < 1, \forall\omega$. This is satisfied if $\boxed{\|W_2C\|_\infty < 1, \text{ where } W_2 = w_2 I.}$ An analogous procedure shows that to satisfy the second requirement, it is sufficient that $\|\Delta_1(j\omega)K(j\omega)S(j\omega)\| < 1, \forall\omega$ where $S = (I+GK)^{-1}$. This can be satisfied if $\boxed{\|W_1KS\|_\infty < 1, \text{ where } W_1 = w_1 I}$. To satisfy both requirements, it is sufficient (but not necessary) that

$$\boxed{\left\| \begin{bmatrix} W_1KS \\ W_2C \end{bmatrix} \right\|_\infty < 1.}$$

(b) The specifications can be met if the transfer matrix from r to $z = [z_1^T \ z_2^T]^T$ in the diagram below has $\mathcal{H}_\infty$-norm less than 1.



The corresponding generalised regulator formulation is to find an internally stabilising K such that $\|\mathcal{F}_l(P,K)\| < 1$:



$$\boxed{P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \left[\begin{array}{c|c} 0 & W_1 \\ 0 & W_2G \\ \hline I & -G \end{array}\right].}$$

(c) Since w_1 and w_2^{-1} are low pass filters, we expect $\boxed{\text{limited controller bandwidth}}$ (since $\|u(j\omega)\| \leq \|K(j\omega)S(j\omega)\|\|r(j\omega)\|$, and $\boxed{\text{good noise attenuation}}$ beyond 10 radians/second (since $\|y(j\omega)\| \leq \|C(j\omega)\|\|v(j\omega)\|$.
$\boxed{\text{Nothing can be said about the tracking and disturbance rejection}}$ properties of the loop which therefore may be unacceptable.

5. (a) Suppose that both $\Delta(s)$ and $S(s)$ are stable. Then the feedback loop with forward transfer matrix $\Delta(s)$ and feedback transfer matrix $S(s)$ is stable if $\boxed{\|\Delta(s)S(s)\|_\infty < 1.}$

(b) (i) The realisation is balanced if

$$A\Sigma + \Sigma A' + BB' = A'\Sigma + \Sigma A + C'C = 0$$

for $\Sigma = \text{diag}(\sigma_1, \sigma_2, \sigma_3) \geq 0$ and where the $\sigma_i' s$ are the Hankel singular values of $K(s)$. A calculation gives $\boxed{\Sigma = \text{diag}(1, 0.4, 0).}$

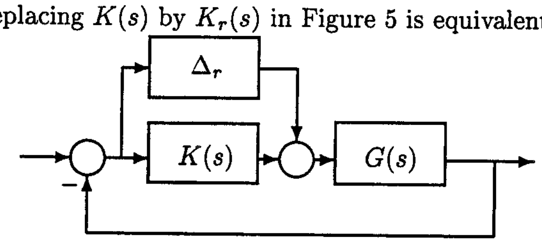
(ii) Sine one of the Hankel singular values is zero, the realisation for K is nonminimal and one state can be removed without changing the loop performance. Hence

$$\boxed{K_2(s) \stackrel{s}{=} \left[\begin{array}{cc|cc} -1 & -1 & 1 & 1 \\ -1 & -1.25 & 0.6 & 0.8 \\ \hline 1 & 0.6 & 0 & 0 \\ 1 & 0.8 & 0 & 0 \end{array}\right]}$$

(iii) Let $K_r(s)$ denote an rth order balanced truncation of $K(s)$. Then $K_r(s) = K(s) + \Delta_r(s)$ where

$$\|\Delta_r\|_\infty \leq 2 \sum_{i=r+1}^{3} \sigma_i. \qquad (1)$$

Then replacing $K(s)$ by $K_r(s)$ in Figure 5 is equivalent to:



Let ϵ be the input to Δ_r and δ be the output of Δ_r. Then

$$\epsilon = -(I + GK)^{-1}G\delta$$

and so the loop is stable if $\|\Delta_r\|_\infty \|(I+GK)^{-1}G\|_\infty < 1$. But,

$$\|(I + GK)^{-1}G\|_\infty < 1$$

from the graph. It follows from (1) that $r = 1$ will guarantee that $\|\Delta_r\|_\infty \leq 2(0.4 + 0) = .8$ and the loop will be stable. So

$$\boxed{K_r(s) \stackrel{s}{=} \left[\begin{array}{c|cc} -1 & 1 & 1 \\ \hline 1 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right]}$$

is a first order internally stabilising controller for $G(s)$.

6. (a) (i) The generalized regulator formulation is given by

$$\boxed{\begin{bmatrix} z \\ y \end{bmatrix} = P \begin{bmatrix} w \\ u \end{bmatrix}, \; u = Fy, \; P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{c||c|c} A & B & B \\ \hline\hline C & 0 & 0 \\ 0 & 0 & I \\ \hline I & 0 & 0 \end{array}\right].}$$

(ii) The requirement $\|H\|_\infty < \gamma$ is equivalent to $J := \|z\|_2^2 - \gamma^2 \|w\|_2^2 < 0$, with $\|v\|_2^2 := \int_0^\infty \|v(t)\|^2 dt$. Let $V = x^T X x$ and set $u = Fx$. Provided that $X = X^T > 0$ and $\dot{V} < 0$ along closed loop trajectory, we can assume $\lim_{t\to\infty} x(t) = 0$. Then $\dot{V} = \dot{x}^T X x + x^T X \dot{x}$

$$= x^T \left(A^T X + XA + F^T B^T X + XBF\right) x + x^T X B w + w^T B^T X x.$$

Integrating from 0 to ∞ and using $x(0) = x(\infty) = 0$,

$$0 = \int_0^\infty [x^T \left(A^T X + XA + F^T B^T X + XBF\right) x + x^T X B w + w^T B^T X x] dt.$$

Using the definition of J and adding the last equation,

$$\begin{aligned} J &= \int_0^\infty \{x^T [A^T X + XA + C^T C + F^T F + F^T B^T X + XBF] x \\ &\quad - [\gamma^2 w^T w - x^T X B w - w^T B^T X x]\} dt. \end{aligned}$$

Completing the squares by using

$$\begin{aligned} (F + B^T X)^T (F + B^T X) &= F^T F + F^T B^T X + XBF + XBB^T X \\ \|(\gamma w - \gamma^{-1} B^T X x)\|^2 &= \gamma^2 w^T w - w^T B^T X x - x^T X B w + \gamma^{-2} x^T X B B^T X x, \end{aligned}$$

$$\begin{aligned} J &= \int_0^\infty \{x^T [A^T X + XA + C^T C - (1 - \gamma^{-2}) X B B^T X] x \\ &\quad + \left\|(F + B^T X) x\right\|^2 - \left\|\gamma w - \gamma^{-1} B^T X x\right\|^2\} dt. \end{aligned}$$

So 2 sufficient conditions for $J < 0$ are the existence of X s.t.

$$\boxed{A^T X + XA + C^T C - (1 - \gamma^{-2}) X B B^T X = 0,} \qquad \boxed{X = X^T > 0.}$$

The state feedback gain is $\boxed{F = -B^T X}$ and the worst case disturbance is $\boxed{w^* = \gamma^{-2} B^T X x.}$ The closed-loop with these feedback laws is $\dot{x} = [A - (1 - \gamma^{-2}) B B^T X] x$ and a third condition is therefore $\boxed{Re\ \lambda_i[A-(1-\gamma^{-2})BB^TX]<0,\ \forall\, i.}$ It remains to prove $\dot{V}<0$ along state-trajectory with $u=Fx$ and $w=0$. But

$$\begin{aligned}\dot{V} &= x^T\left(A^TX+XA+F^TB^TX+XBF\right)x\\ &= \boxed{-x^T(C^TC+(1+\gamma^{-2})XBB^TX)x<0}\end{aligned}$$

for all $x\neq 0$ (since (A,B,C) is minimal) proving closed-loop stability.

(b) The dynamics of the state estimation error system are given by

$$\dot{x}_e = Ax_e + Bw_1 + u_e, \quad z_e = Cx_e, \quad y_e = Cx_e + w_2$$

which has the generalised regulator formulation

$$Q \stackrel{s}{=} \left[\begin{array}{c||cc|c} A & B & 0 & I \\ \hline\hline C & 0 & 0 & 0 \\ \hline C & 0 & I & 0 \end{array}\right] \Longrightarrow Q^T \stackrel{s}{=} \left[\begin{array}{c||c|c} A^T & C^T & C^T \\ \hline\hline B^T & 0 & 0 \\ 0 & 0 & I \\ \hline I & 0 & 0 \end{array}\right].$$

Noting that $\boxed{Q^T \text{ has the same structure as the generlised regulator } P}$ of part (a), we can obtain the solution for the $\mathcal{H}_\infty$ estimator from that of the solution of part (a) using the duality principle by substituting $\boxed{A:=A^T,\ B:=C^T,\ C:=B^T \text{ and substituting } K=F^T.}$