Final copy - June 08

E4.29
C1.1
ISE4.55

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2008

MSc and EEE PART IV: MEng and ACGI

OPTIMIZATION

Thursday, 8 May 2:30 pm

Time allowed: 3:00 hours

There are SIX questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : A. Astolfi

Second Marker(s) : M.M. Draief

OPTIMISATION

1. Consider the problem of minimizing the function

$$f(x_1,x_2,\cdots,x_n,y)=\frac{1}{4}x_1^4+\frac{1}{4}x_2^4+\cdots+\frac{1}{4}x_n^4-(x_1+x_2+\cdots+x_n)y+\frac{n}{2}y^2,$$

where n is a positive integer.

a) Compute all stationary points of the function. [4 marks]

b) Using second order sufficient conditions *classify* the stationary points determined in part a), *i.e.* say which is a local minimum, or a local maximum, or a saddle point. [8 marks]

c) Show that the function f is radially unbounded and hence compute the global minimum of f. Is the global minimizer unique? [4 marks]

d) Consider the points $P_p=(1,1,\cdots,1,1)$ and $P_m=(-1,-1,\cdots,-1,-1)$ and the direction d from P_p to P_m. Show that this is an ascent direction for f at P_p. [4 marks]

2. The problem of minimizing a differentiable function $f:\mathbb{R}^n\to\mathbb{R}$ can be solved with the so-called heavy ball algorithm, which is a modification of the gradient algorithm, and it is described (in its simplest form) by the iteration

$$x_{k+1}=x_k-\alpha\nabla f(x_k)+\beta(x_k-x_{k-1}),$$

where $\alpha>0$ is a constant and $\beta\in[0,1)$ is the *heavy ball* parameter.

a) Assume $x_{-1}=x_0$. Show that the iteration of the heavy ball algorithm can be written as

$$x_{k+1}=x_k-\alpha\left(\nabla f(x_k)+\beta\nabla f(x_{k-1})+\beta^2\nabla f(x_{k-2})+\cdots+\beta^k\nabla f(x_0)\right),$$

for $k\geq 1$. [4 marks]

b) Consider the function

$$f(x_1,x_2)=2x_1^2+\frac{1}{2}x_2^2,$$

which has a unique (global) minimizer at $x_1=x_2=0$.

i) Consider the heavy ball algorithm with $\beta=0$, *i.e.* the gradient algorithm with a constant line search parameter α. Show that the sequence $\{x_k\}=\{x_{1,k},x_{2,k}\}$ generated by this algorithm converges to the minimizer of f if and only if

$$0<\alpha<\frac{1}{2}.$$

Select $\alpha=1/4$. Show that $x_{1,k}=0$, for all $k\geq 1$, and determine the speed of convergence of the sequence $\{x_k\}$. [8 marks]

ii) Consider the heavy ball algorithm described above with $x_{-1}=x_0$, $\alpha=1/4$ and $\beta=3/4$. Show that the sequence $\{x_k\}=\{x_{1,k},x_{2,k}\}$ generated by this algorithm is such that $x_{1,k}=0$ for all $k\geq 1$. Evaluate $x_{2,k}$ for $k=1,\cdots,4$. Estimate the speed of convergence of the sequence $\{x_k\}$. [8 marks]

3. (Please use the enclosed Figure 3.1 to answer part b) of this question and attach it to your answer book.)

Consider the problem of minimizing the function

$$f(x_1,x_2) = x_2^2 - \delta x_2(x_1^2+x_2^2) + (x_1^2+x_2^2)^2.$$

a) Compute all stationary points of the function as a function of δ. [8 marks]

b) Assume $\delta = \sqrt{32}/3$. Figure 3.1 shows the level lines of the function f for this value of δ.

i) Determine the stationary points of the function f, indicate them on Figure 3.1, and *classify* the stationary points *i.e.* say which is a local minimum, or a local maximum, or a saddle point, without computing the Hessian matrix of f. [4 marks]

ii) Determine, from inspection of Figure 3.1, a set of points such that the gradient algorithm with exact line search initialized at such points yields a sequence which converges to the global minimum in one step. Sketch the obtained set on Figure 3.1. [2 marks]

iii) Determine, analytically, all points such that the gradient algorithm with exact line search initialized at such points yields a sequence which converges to the global minimum in one step. Sketch the obtained set on Figure 3.1. [6 marks]

4. Consider the optimization problems

$$P_{min} \begin{cases} \min\limits_{x_1,\, x_2} |x_1| + |x_2|, \\ x_1^2 + x_2^2 = 1, \end{cases}$$

and

$$P_{max} \begin{cases} \max\limits_{x_1,\, x_2} |x_1| + |x_2|, \\ x_1^2 + x_2^2 = 1. \end{cases}$$

a) Sketch in the (x_1,x_2)-plane the admissible set and the level sets of the function $|x_1| + |x_2|$. [6 marks]

b) Using only graphical considerations determine the solutions of the considered problems. [4 marks]

c) State first order necessary conditions of optimality for these constrained optimization problems. Show that the optimal solutions determined in part b) satisfy the necessary conditions of optimality.
(Hint: use the fact that $\text{sign}(0) = 0$.) [4 marks]

d) Write a penalty function F_ε for problem P_{max}. Show that, for $\varepsilon > 0$ and sufficiently small, the stationary points of F_ε approach the optimal solutions determined in part b). (Do not compute explicitly the stationary points of F_ε.)
(Hint: for ε sufficiently small, the stationary points of F_ε are such that $x_1 \neq 0$ and $x_2 \neq 0$.) [6 marks]

5. Consider the optimization problem

$$\begin{cases} \min_{x_1,\, x_2} x_1^3 - x_1^2 x_2 + 2x_2^2, \\ x_1 \geq 0, \\ x_2 \geq 0. \end{cases}$$

a) State first order necessary conditions of optimality for this constrained optimization problem. [2 marks]

b) Using the conditions derived in part a) compute candidate optimal solutions. Show that there is one candidate solution on the boundary of the admissible set and one in the interior of the admissible set. [6 marks]

c) Using second order sufficient conditions of optimality show that the candidate solution inside the admissible set is not a local minimizer. [4 marks]

d) Show that the candidate optimal solution on the boundary of the admissible set is a local minimizer.
(Hint: show that the function to be minimized is zero at the candidate optimal solution, and it is strictly positive in all admissible points in a neighborhood of the candidate optimal solution). [6 marks]

e) Show that the function to be minimized is not bounded from below in the admissible set. Hence, argue that the problem does not have a global solution.
(Hint: consider the function to be minimized along the line $x_2 = 2x_1$, and study its behaviour for $x_1 > 0$ and large.) [2 marks]

6. Consider the optimization problem

$$\begin{cases} \max_{x_1,\, x_2,\, x_3} (x_1x_2 + x_2x_3 + x_1x_3), \\ x_1 + x_2 + x_3 = 3. \end{cases}$$

a) State first order necessary conditions of optimality for this constrained optimization problem and show that there exists only one candidate optimal solution. [4 marks]

b) Using second order sufficient conditions of optimality show that the candidate solution is a local maximizer. [6 marks]

c) Consider the use of an exact penalty function for the solution of the problem.

   i) Write an exact penalty function $G(x_1, x_2, x_3)$ for the problem. [2 marks]

   ii) Show that the function is well-defined for every (x_1, x_2, x_3). [2 marks]

   iii) Show that the exact penalty function has only one stationary point and this coincides with the optimal solution of the problem determined in part b). [6 marks]

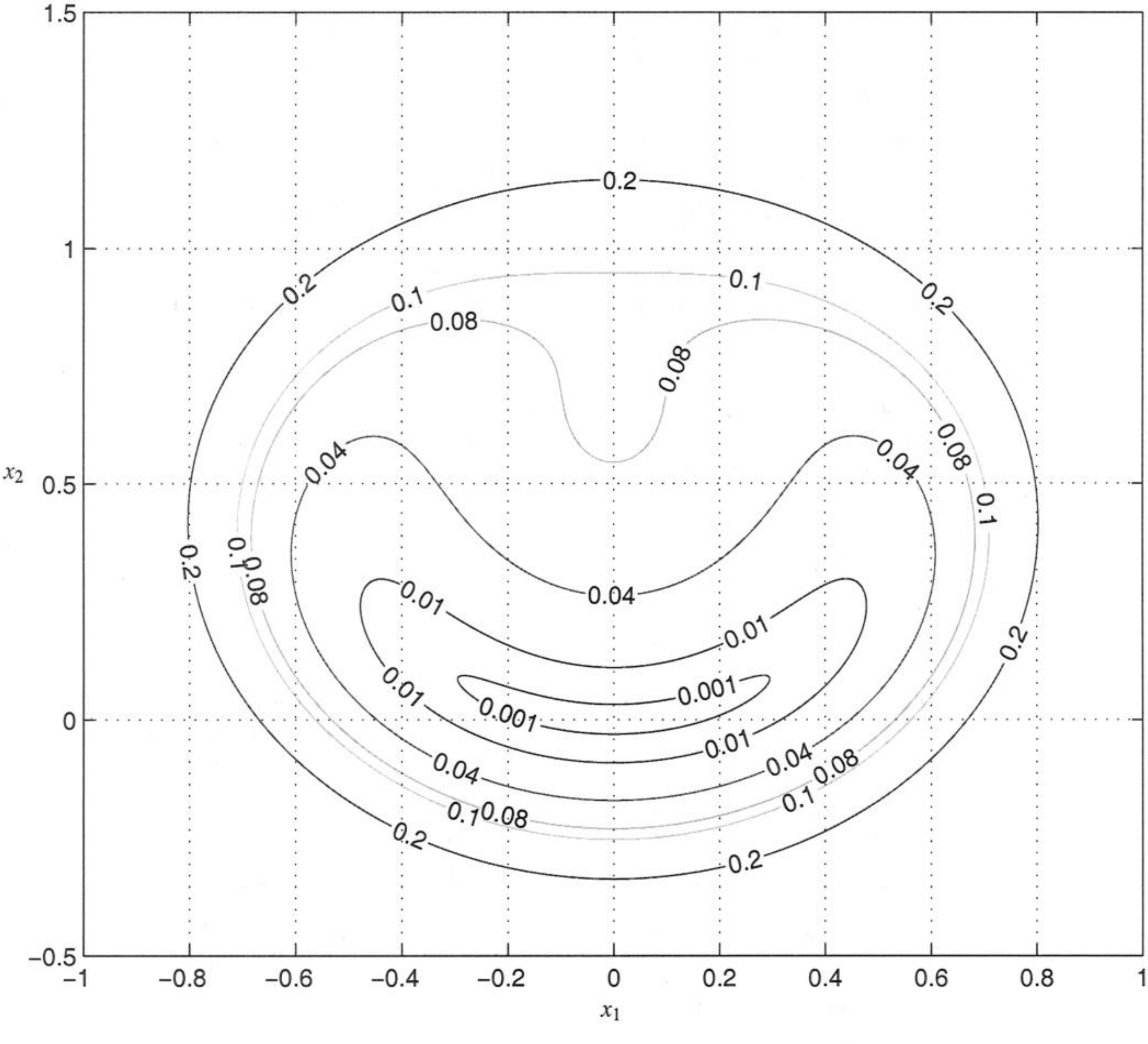

Figure 3.1: The level lines of the function $f(x_1, x_2)$.