EE4-47
EE9-CS6-1

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2013

MSc and EEE PART IV: MEng and ACGI

MODELLING AND CONTROL OF MULTI-BODY MECHANICAL SYSTEMS

Friday, 3 May 10:00 am

Time allowed: 3:00 hours

There are FOUR questions on this paper.

Answer ALL questions

All questions carry equal marks.

This is an OPEN BOOK examination.

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s) :	S. Evangelou
	Second Marker(s) :	A. Astolfi

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MODELLING AND CONTROL OF MULTIBODY MECHANICAL SYSTEMS

1. The cart of mass M shown in Figure 1.1 moves on a smooth (frictionless) horizontal surface by a distance x, against a spring and damper force. The spring stiffness is k_s and the damping coefficient of the damper is c. A pendulum is attached on the cart at point O. The pendulum is a mass m suspended from the cart via a massless rod of length l, as shown in Figure 1.1. Assume that the pendulum is free to move in a vertical plane under the influence of gravity and the interaction with the cart, coming only from the joint O.

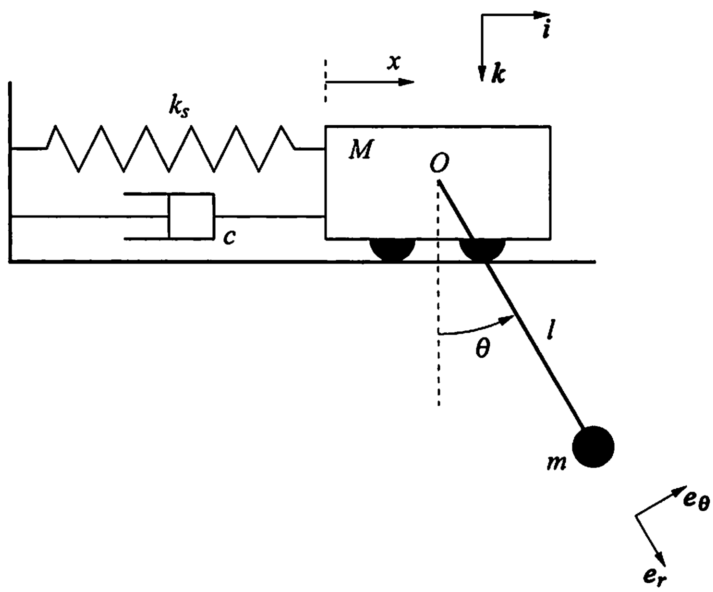

Figure 1.1 Cart and pendulum.

A fixed Cartesian coordinate system with unit vectors $\boldsymbol{i}$ and $\boldsymbol{k}$, and a moving Cartesian coordinate system with unit vectors $\boldsymbol{e_r}$ and $\boldsymbol{e_\theta}$ are used to analyse the motion of the two masses. The moving coordinate system has a cart-fixed origin O and it rotates by an angle θ.

a) Determine the position vector of each mass. [2]

b) Determine the velocity vector of each mass. [2]

c) Compute the kinetic energy of the system. [2]

d) Compute the potential energy of the system. [2]

e) Calculate the Lagrangian function. [2]

f) Use the Lagrangian approach to derive the equations of motion of the system. [10]

2. Consider a pendulum with mass m suspended from an inextensible massless wire of length r, as shown in Figure 2.1. Assume that m is free to move in a vertical plane under the influence of gravity, and that there is a small hole at the attachment point O through which the wire is pulled so that at any time $r = \alpha + (r_0 - \alpha)\cos\theta$, where r_0 is the length of the wire when the pendulum is in the vertical position, $\theta = 0$, and α is its length when the pendulum is in the horizontal position, $\theta = \pm\pi/2$.

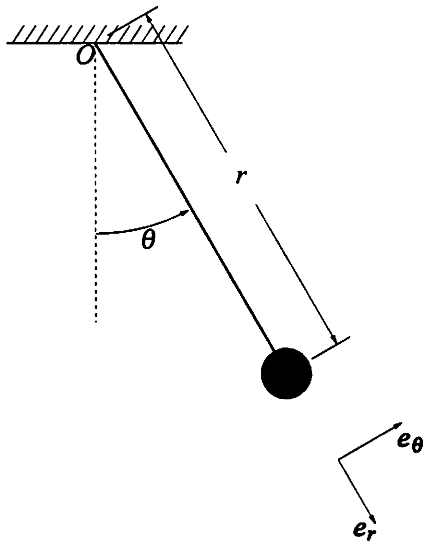

Figure 2.1 Pendulum.

A moving Cartesian coordinate system with unit vectors $\boldsymbol{e_r}$ and $\boldsymbol{e_\theta}$ is used to analyse the motion of the mass. This coordinate system has a fixed origin O but it rotates by an angle θ.

a) Determine the velocity vector of the mass. [2]

b) Compute the total kinetic energy and potential energy of the mass, and hence determine the Lagrangian function. [3]

c) Use the Lagrangian approach to derive the equation of motion of the mass. [6]

d) Calculate the force in the wire holding the mass. [3]

e) Determine the equation of motion of the system when θ is a small angle and hence write the angular frequency of the oscillations of the motion. What can you say about this frequency for the cases when $r_0 > \alpha$ and $r_0 < \alpha$, as compared to the frequency corresponding to a simple pendulum of the same mass and fixed wire length $r = r_0$? [6]

3. Two particles of mass m each are attached at the two ends of a rigid rod of length l and of negligible mass that is free to rotate by an angle ψ about the vertical axis and by an angle θ about a horizontal axis which is perpendicular to the rod, as shown in Figure 3.1. Both axes of rotation pass through the centre of the rod. A moving Cartesian coordinate system attached to the rod with fixed origin O and with unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ is used to analyse the motion of the system. The $\boldsymbol{i}$ vector has a direction into the page at the time instant shown (in Figure 3.1) and it is along the axis of the θ rotation. A moment N is applied onto the rod in the $\boldsymbol{k}$ direction. The effect of gravity is neglected.

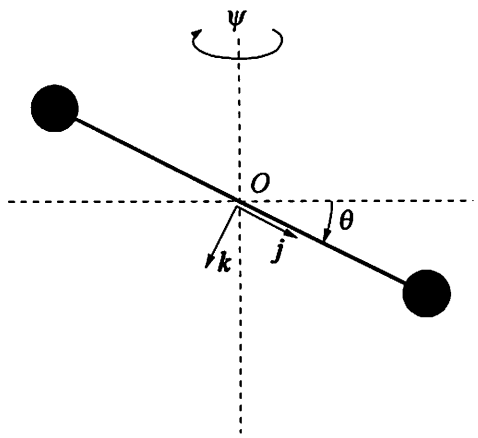

Figure 3.1 Two masses on a massless link.

a) Write an expression for the velocity vector of each mass in terms of $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$. [3]

b) Write an expression for the acceleration vector of each mass. [5]

c) Write an expression for the force vector acting on each mass, due to the interaction with the rod, in terms of N, the unknown radial force F_r (along the rod) and $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$. [3]

d) Use the vectorial approach to derive the equations of motion of the system. [6]

e) Compute the magnitude of the radial force, F_r. [3]

4. The cart of mass M shown in Figure 4.1 moves on a smooth (frictionless) horizontal surface by a distance x, against a spring and damper force. The spring stiffness is k_s and the damping coefficient of the damper is c. A pendulum is attached on the cart at point O. The pendulum is a mass m suspended from the cart via a massless rod of length l, as shown in Figure 4.1. Assume that the pendulum is free to move in a vertical plane under the influence of gravity and the interaction with the cart, coming only from the *joint O*.

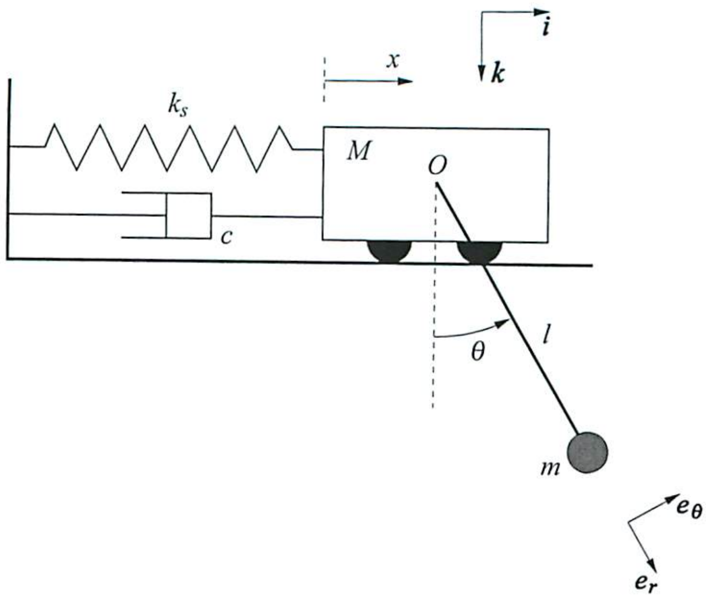

Figure 4.1 Cart and pendulum.

A fixed Cartesian coordinate system with unit vectors $\boldsymbol{i}$ and $\boldsymbol{k}$, and a moving Cartesian coordinate system with unit vectors $\boldsymbol{e_r}$ and $\boldsymbol{e_\theta}$ are used to analyse the motion of the two masses. The moving coordinate system has a cart-fixed origin O and it rotates by an angle θ.

a) Determine the acceleration vector of mass M in terms of $\boldsymbol{i}$ and $\boldsymbol{k}$. [1]

b) Calculate the acceleration vector of mass m in terms of $\boldsymbol{e_r}$ and $\boldsymbol{e_\theta}$. [4]

c) Write the equation of motion of mass m in vector form and hence:

   i) derive one of the equations of motion of the system. [3]

   ii) compute the force in the rod which holds the mass m. [3]

d) Write the equation of motion of the mass M in vector form and hence derive the second equation of motion of the system. [4]

e) Assume that $k_s = 0$, $c = 0$ and that x and θ are small. Determine the equations of motion of the system and hence specify the type of motion the cart and pendulum execute, when the masses are perturbed by a small amount and initially $\dot{x} = 0$ and $\dot{\theta} = 0$. [5]

Modelling and control of multibody mechanical systems

Model answers

Question 1

a) $\boldsymbol{r}_M = x\boldsymbol{i}$ and $\boldsymbol{r}_m = x\boldsymbol{i} + l\boldsymbol{e}_r = (x + l\sin\theta)\boldsymbol{i} + l\cos\theta\boldsymbol{k}$.

b) By differentiating the position vector $\dot{\boldsymbol{r}}_M = \dot{x}\boldsymbol{i}$ and

$$\dot{\boldsymbol{r}}_m = \dot{x}\boldsymbol{i} + l\dot{\theta}\boldsymbol{e}_\theta = \dot{x}\sin\theta\boldsymbol{e}_r + (\dot{x}\cos\theta + l\dot{\theta})\boldsymbol{e}_\theta.$$

c)

$$T = \frac{1}{2}M\dot{\boldsymbol{r}}_M\cdot\dot{\boldsymbol{r}}_M + \frac{1}{2}m\dot{\boldsymbol{r}}_m\cdot\dot{\boldsymbol{r}}_m = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x}^2 + 2\dot{x}\dot{\theta}l\cos\theta + l^2\dot{\theta}^2\right).$$

d) The horizontal level at O is taken as the zero potential energy level, therefore

$$V = -m\boldsymbol{r}_m\cdot\boldsymbol{g} + \frac{1}{2}kx^2 = -mgl\cos\theta + \frac{1}{2}kx^2.$$

e)

$$L = T - V = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x}^2 + 2\dot{x}\dot{\theta}l\cos\theta + l^2\dot{\theta}^2\right) + mgl\cos\theta - \frac{1}{2}kx^2.$$

f) The Lagrangian equation with respect to the generalised coordinate x is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{x}}\right) - \frac{\partial L}{\partial x} = -c\dot{x},$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(M\dot{x} + m\dot{x} + m\dot{\theta}l\cos\theta\right) + kx = -c\dot{x},$$

or

$$(M+m)\ddot{x} + ml\cos\theta\ddot{\theta} - ml\dot{\theta}^2\sin\theta + c\dot{x} + kx = 0.$$

The Lagrangian equation with respect to the generalised coordinate θ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0,$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m\dot{x}l\cos\theta + ml^2\dot{\theta}\right) + m\dot{x}\dot{\theta}l\sin\theta + mgl\sin\theta = 0,$$

or

$$\cos\theta\ddot{x} + l\ddot{\theta} + g\sin\theta = 0.$$

a) $\dot{\boldsymbol{r}} = \dot{r}\boldsymbol{e}_r + r\dot{\theta}\boldsymbol{e}_\theta$.

b) The kinetic energy is $T = \frac{1}{2}m\dot{\boldsymbol{r}}\cdot\dot{\boldsymbol{r}} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$.
The potential energy is $V = -m\boldsymbol{r}\cdot\boldsymbol{g} = -mr\boldsymbol{e}_r\cdot g\boldsymbol{k} = -mgr\cos\theta$, with the level of point O corresponding to zero gravitational potential energy.
The Lagrangian is $L = T - V = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + mgr\cos\theta$.

c) The constraint equation is

$$r = \alpha + (r_0 - \alpha)\cos\theta, \tag{1}$$

by differentiating

$$\dot{r} + (r_0 - \alpha)\sin\theta\dot{\theta} = 0, \tag{2}$$

and by differentiating once again

$$\ddot{r} = -(r_0 - \alpha)\cos\theta\dot{\theta}^2 - (r_0 - \alpha)\sin\theta\ddot{\theta}. \tag{3}$$

The Lagrangian equation with respect to the generalised coordinate r is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} + \lambda = 0,$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{r}) - mr\dot{\theta}^2 - mg\cos\theta + \lambda = 0,$$

or

$$m\ddot{r} - mr\dot{\theta}^2 - mg\cos\theta + \lambda = 0,$$

or by using Equations (1) and (3)

$$\lambda = m\left((\alpha + 2(r_0 - \alpha)\cos\theta)\dot{\theta}^2 + (r_0 - \alpha)\sin\theta\ddot{\theta} + g\cos\theta\right). \tag{4}$$

The Lagrangian equation with respect to the generalised coordinate θ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} + \lambda(r_0 - \alpha)\sin\theta = 0,$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(mr^2\dot{\theta}\right) + mgr\sin\theta + \lambda(r_0 - \alpha)\sin\theta = 0,$$

or

$$mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} + mgr\sin\theta + \lambda(r_0 - \alpha)\sin\theta = 0,$$

or by using Equations (1), (2) and (4)

$$\left(\alpha^2 + 2\alpha(r_0 - \alpha)\cos\theta + (r_0 - \alpha)^2\right)\ddot{\theta} - \alpha(r_0 - \alpha)\sin\theta\dot{\theta}^2 + (\alpha + 2(r_0 - \alpha)\cos\theta)\, g\sin\theta = 0. \tag{5}$$

d) The force in the wire, F_{wire}, is given by $-\lambda$, therefore

$$F_{wire} = -m\left((\alpha + 2(r_0 - \alpha)\cos\theta)\dot{\theta}^2 + (r_0 - \alpha)\sin\theta\ddot{\theta} + g\cos\theta\right).$$

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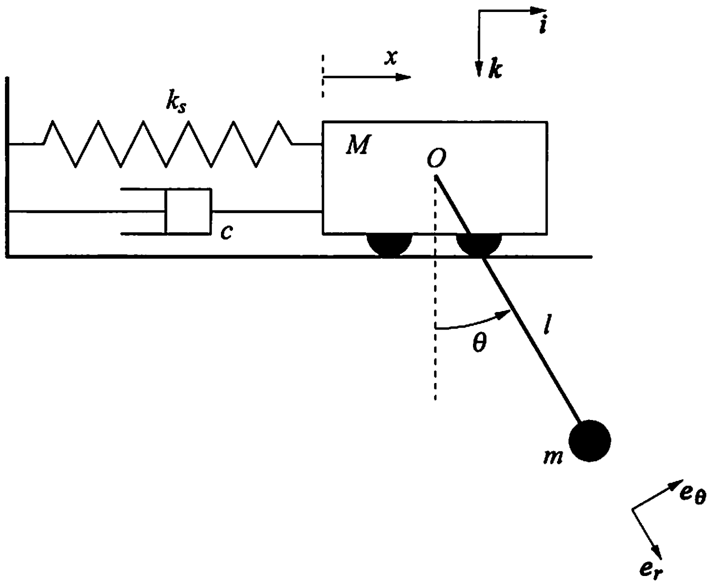

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A fixed Cartesian coordinate system with unit vectors $\boldsymbol{i}$ and $\boldsymbol{k}$, and a moving Cartesian coordinate system with unit vectors $\boldsymbol{e_r}$ and $\boldsymbol{e_\theta}$ are used to analyse the motion of the two masses. The moving coordinate system has a cart-fixed origin O and it rotates by an angle θ.

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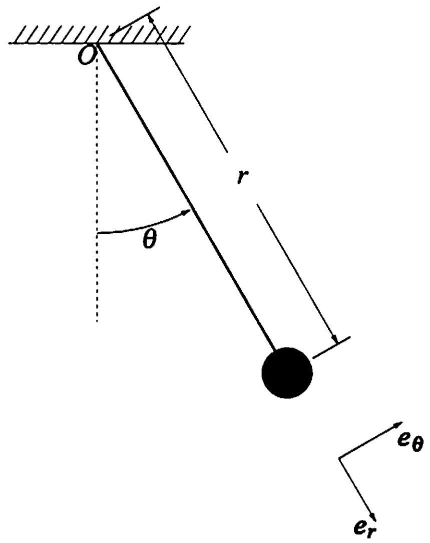

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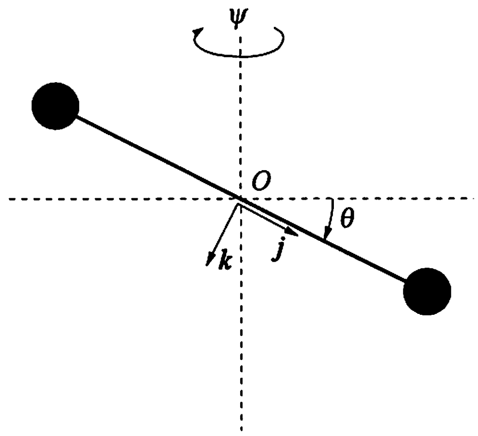

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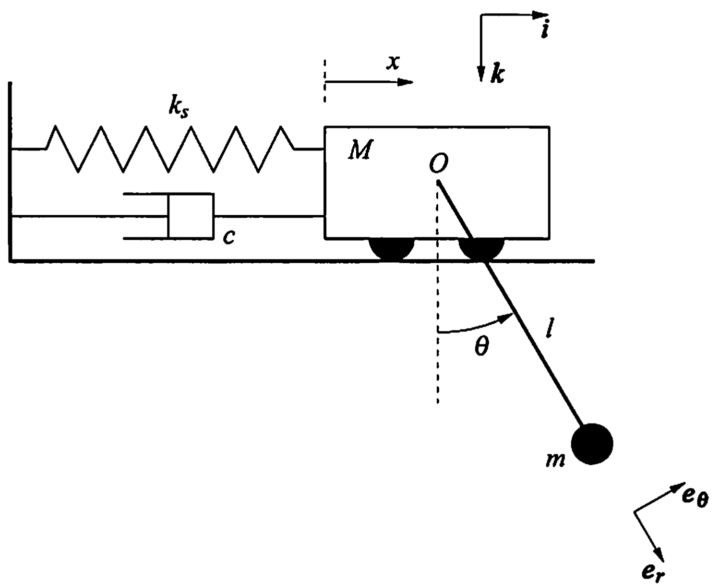

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Modelling and control of multibody mechanical systems

Model answers

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b) By differentiating the position vector $\dot{\boldsymbol{r}}_M = \dot{x}\boldsymbol{i}$ and

$$\dot{\boldsymbol{r}}_m = \dot{x}\boldsymbol{i} + l\dot{\theta}\boldsymbol{e}_\theta = \dot{x}\sin\theta\boldsymbol{e}_r + (\dot{x}\cos\theta + l\dot{\theta})\boldsymbol{e}_\theta.$$

c)

$$T = \frac{1}{2}M\dot{\boldsymbol{r}}_M\cdot\dot{\boldsymbol{r}}_M + \frac{1}{2}m\dot{\boldsymbol{r}}_m\cdot\dot{\boldsymbol{r}}_m = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x}^2 + 2\dot{x}\dot{\theta}l\cos\theta + l^2\dot{\theta}^2\right).$$

d) The horizontal level at O is taken as the zero potential energy level, therefore

$$V = -m\boldsymbol{r}_m\cdot\boldsymbol{g} + \frac{1}{2}kx^2 = -mgl\cos\theta + \frac{1}{2}kx^2.$$

e)

$$L = T - V = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x}^2 + 2\dot{x}\dot{\theta}l\cos\theta + l^2\dot{\theta}^2\right) + mgl\cos\theta - \frac{1}{2}kx^2.$$

f) The Lagrangian equation with respect to the generalised coordinate x is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{x}}\right) - \frac{\partial L}{\partial x} = -c\dot{x},$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(M\dot{x} + m\dot{x} + m\dot{\theta}l\cos\theta\right) + kx = -c\dot{x},$$

or

$$(M+m)\ddot{x} + ml\cos\theta\ddot{\theta} - ml\dot{\theta}^2\sin\theta + c\dot{x} + kx = 0.$$

The Lagrangian equation with respect to the generalised coordinate θ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0,$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m\dot{x}l\cos\theta + ml^2\dot{\theta}\right) + m\dot{x}\dot{\theta}l\sin\theta + mgl\sin\theta = 0,$$

or

$$\cos\theta\ddot{x} + l\ddot{\theta} + g\sin\theta = 0.$$

Question 2

a) $\dot{\boldsymbol{r}} = \dot{r}\boldsymbol{e}_r + r\dot{\theta}\boldsymbol{e}_\theta$.

b) The kinetic energy is $T = \frac{1}{2}m\dot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$.
The potential energy is $V = -m\boldsymbol{r} \cdot \boldsymbol{g} = -mr\boldsymbol{e}_r \cdot g\boldsymbol{k} = -mgr\cos\theta$, with the level of point O corresponding to zero gravitational potential energy.
The Lagrangian is $L = T - V = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + mgr\cos\theta$.

c) The constraint equation is

$$r = \alpha + (r_0 - \alpha)\cos\theta, \tag{1}$$

by differentiating

$$\dot{r} + (r_0 - \alpha)\sin\theta\dot{\theta} = 0, \tag{2}$$

and by differentiating once again

$$\ddot{r} = -(r_0 - \alpha)\cos\theta\dot{\theta}^2 - (r_0 - \alpha)\sin\theta\ddot{\theta}. \tag{3}$$

The Lagrangian equation with respect to the generalised coordinate r is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} + \lambda = 0,$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{r}) - mr\dot{\theta}^2 - mg\cos\theta + \lambda = 0,$$

or

$$m\ddot{r} - mr\dot{\theta}^2 - mg\cos\theta + \lambda = 0,$$

or by using Equations (1) and (3)

$$\lambda = m\left((\alpha + 2(r_0 - \alpha)\cos\theta)\dot{\theta}^2 + (r_0 - \alpha)\sin\theta\ddot{\theta} + g\cos\theta\right). \tag{4}$$

The Lagrangian equation with respect to the generalised coordinate θ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} + \lambda(r_0 - \alpha)\sin\theta = 0,$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(mr^2\dot{\theta}\right) + mgr\sin\theta + \lambda(r_0 - \alpha)\sin\theta = 0,$$

or

$$mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} + mgr\sin\theta + \lambda(r_0 - \alpha)\sin\theta = 0,$$

or by using Equations (1), (2) and (4)

$$\left(\alpha^2 + 2\alpha(r_0 - \alpha)\cos\theta + (r_0 - \alpha)^2\right)\ddot{\theta} - \alpha(r_0 - \alpha)\sin\theta\dot{\theta}^2 + (\alpha + 2(r_0 - \alpha)\cos\theta)\,g\sin\theta = 0. \tag{5}$$

d) The force in the wire, F_{wire}, is given by $-\lambda$, therefore

$$F_{wire} = -m\left((\alpha + 2(r_0 - \alpha)\cos\theta)\dot{\theta}^2 + (r_0 - \alpha)\sin\theta\ddot{\theta} + g\cos\theta\right).$$

e) For small θ the equation of motion is

$$r_0{}^2\ddot{\theta} + (r_0 + (r_0 - \alpha))g\theta = 0,$$

and therefore the mass executes simple harmonic motion with angular frequency

$$\sqrt{\frac{r_0 + (r_0 - \alpha)}{r_0}\frac{g}{r_0}}.$$

For fixed wire length, $r = r_0$, $r_0 = \alpha$ and therefore the frequency of oscillations is $\omega_0 = \sqrt{g/r_0}$. For $r_0 > \alpha$, $\omega > \omega_0$ and for $r_0 < \alpha$, $\omega < \omega_0$.

Question 3

a) The angular velocity of the system about the vertical axis is $\dot{\psi}$ and in the $\boldsymbol{i}$ direction it is $\dot{\theta}$. All together it is

$$\boldsymbol{\Omega} = \dot{\theta}\boldsymbol{i} + \dot{\psi}\sin\theta\boldsymbol{j} + \dot{\psi}\cos\theta\boldsymbol{k}.$$

The position vector of the lower mass (in the position shown in the diagram) is

$$\boldsymbol{r_1} = \frac{l}{2}\boldsymbol{j}.$$

The velocity vector is

$$\boldsymbol{v_1} = \dot{\boldsymbol{r_1}} = \boldsymbol{\Omega} \times \boldsymbol{r_1}$$

which gives

$$\boldsymbol{v_1} = -\frac{l}{2}\dot{\psi}\cos\theta\boldsymbol{i} + \frac{l}{2}\dot{\theta}\boldsymbol{k}.$$

The velocity vector of the other mass is given by

$$\boldsymbol{v_2} = -\boldsymbol{v_1}.$$

b) The acceleration vector of the lower mass is

$$\boldsymbol{a_1} = \dot{\boldsymbol{v_1}} = \frac{l}{2}\ddot{\theta}\boldsymbol{k} - \left(\frac{l}{2}\ddot{\psi}\cos\theta - \frac{l}{2}\dot{\psi}\dot{\theta}\sin\theta\right)\boldsymbol{i} + \boldsymbol{\Omega} \times \boldsymbol{v_1},$$

or

$$\boldsymbol{a_1} = \left(\frac{l}{2}\ddot{\psi}\cos\theta + l\dot{\psi}\dot{\theta}\sin\theta\right)\boldsymbol{i} - \left(\frac{l}{2}\dot{\psi}^2\cos^2\theta + \frac{l}{2}\dot{\theta}^2\right)\boldsymbol{j} + \left(\frac{l}{2}\ddot{\theta} + \frac{l}{2}\dot{\psi}^2\sin\theta\cos\theta\right)\boldsymbol{k}.$$

The acceleration vector of the other mass is

$$\boldsymbol{a_2} = -\boldsymbol{a_1}.$$

c) The force vector acting on the lower mass is

$$\boldsymbol{F_1} = -F_N\boldsymbol{i} - F_r\boldsymbol{j},$$

where F_N is the magnitude of the force on each mass due to the moment N acting on the rod. This is given by

$$F_N = \frac{N}{l},$$

therefore

$$\boldsymbol{F_1} = -\frac{N}{l}\boldsymbol{i} - F_r\boldsymbol{j}.$$

The force vector on the other mass is

$$\boldsymbol{F_2} = -\boldsymbol{F_1}.$$

d) The motion of the system can be found by considering the motion of one of the masses. For the lower mass

$$\boldsymbol{F_1} = m\boldsymbol{a_1}$$

or by substituting the force and acceleration expressions from the equations above and collecting the terms with respect to $\boldsymbol{i}$ and $\boldsymbol{k}$

$$\ddot{\theta} + \dot{\psi}^2\sin\theta\cos\theta = 0,$$

and

$$\frac{1}{2}ml^2\left(\ddot{\psi}\cos\theta - 2\dot{\psi}\dot{\theta}\sin\theta\right) = N.$$

e) By using again the equation $\boldsymbol{F}_1 = m\boldsymbol{a}_1$ and collecting the terms with respect to $\boldsymbol{j}$ we obtain

$$F_r = m\left(\frac{l}{2}\dot{\psi}^2\cos^2\theta + \frac{l}{2}\dot{\theta}^2\right).$$

Question 4

a) The velocity vector of mass M is $\dot{\boldsymbol{r}}_M = \dot{x}\boldsymbol{i}$. By differentiating the velocity expression we obtain the acceleration vector,

$$\ddot{\boldsymbol{r}}_M = \ddot{x}\boldsymbol{i}.$$

b) The velocity vector of mass m is $\dot{\boldsymbol{r}}_m = \dot{x}\boldsymbol{i} + l\dot{\theta}\boldsymbol{e}_\theta = \dot{x}\sin\theta\boldsymbol{e}_r + (\dot{x}\cos\theta + l\dot{\theta})\boldsymbol{e}_\theta$. By differentiating the velocity expression we obtain the acceleration vector,

$$\ddot{\boldsymbol{r}}_m = \left(\ddot{x}\sin\theta - l\dot{\theta}^2\right)\boldsymbol{e}_r + \left(\ddot{x}\cos\theta + l\ddot{\theta}\right)\boldsymbol{e}_\theta.$$

c) The equation of motion of mass m in vector form is

$$\boldsymbol{F}_m = m\ddot{\boldsymbol{r}}_m,$$

or

$$-F_r\boldsymbol{e}_r + mg\cos\theta\boldsymbol{e}_r - mg\sin\theta\boldsymbol{e}_\theta = m\left(\ddot{x}\sin\theta - l\dot{\theta}^2\right)\boldsymbol{e}_r + m\left(\ddot{x}\cos\theta + l\ddot{\theta}\right)\boldsymbol{e}_\theta.$$

i) The first equation of motion is found by collecting the $\boldsymbol{e}_\theta$ terms

$$\ddot{x}\cos\theta + l\ddot{\theta} + g\sin\theta = 0.$$

ii) The force in the rod, F_r, is found by collecting the $\boldsymbol{e}_r$ terms and it is given by

$$F_r = m\left(-\ddot{x}\sin\theta + l\dot{\theta}^2 + g\cos\theta\right).$$

d) The equation of motion of mass M in vector form is

$$\boldsymbol{F}_M = M\ddot{\boldsymbol{r}}_M,$$

or

$$(-kx - c\dot{x} + F_r\sin\theta)\boldsymbol{i} + (Mg + F_r\cos\theta - R)\boldsymbol{k} = M\ddot{x}\boldsymbol{i},$$

where R is the normal reaction from the surface on the cart. By collecting the $\boldsymbol{i}$ terms we obtain the second equation of motion

$$M\ddot{x} - F_r\sin\theta + c\dot{x} + kx = 0,$$

or

$$\left(M + m\sin^2\theta\right)\ddot{x} - ml\dot{\theta}^2\sin\theta - mg\sin\theta\cos\theta + c\dot{x} + kx = 0.$$

e) For $k_s = 0$, $c = 0$ and small x and θ the two equations of motion become

$$M\ddot{x} - mg\theta = 0,$$

and

$$\ddot{x} + l\ddot{\theta} + g\theta = 0.$$

By a simple manipulation these two equations give

$$Ml\ddot{\theta} + (M + m)g\theta = 0,$$

or

$$\ddot{\theta} + \frac{M+m}{M}\frac{g}{l}\theta = 0,$$

which is simple harmonic motion in θ for the pendulum with frequency of oscillation $\sqrt{\frac{M+m}{M}\frac{g}{l}}$. By another simple manipulation the equations of motion give

$$(M+m)\ddot{x} + ml\ddot{\theta} = 0,$$

or

$$\ddot{x} = -\frac{ml}{M+m}\ddot{\theta},$$

which can be integrated to give

$$x = -\frac{ml}{M+m}\theta + x_0,$$

for initial $\dot{x} = 0$ and $\dot{\theta} = 0$. Therefore the cart also executes simple harmonic motion about some position x_0 with the same frequency as for the pendulum, and with its amplitude scaled by $-\frac{ml}{M+m}$ as compared to the amplitude of the motion of the pendulum.