EE4-12
EE9-AO2
EE9-SC1

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2015

MSc and EEE/EIE PART IV: MEng and ACGI

DIGITAL SIGNAL PROCESSING AND DIGITAL FILTERS

Friday, 1 May 10:00 am

Time allowed: 3:00 hours

Corrected Copy

There are FOUR questions on this paper.

Answer Question 1 and any TWO other questions

Question 1 is worth 40% of the marks and other questions are worth 30%

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : D.M. Brookes

Second Marker(s) : P.T. Stathaki

DIGITAL SIGNAL PROCESSING AND DIGITAL FILTERS

Information for Candidates:

Notation

- All signals and filter coefficients are real-valued unless explicitly noted otherwise.
- Unless otherwise specified, upper and lower case letters are used for sequences and their z-transforms respectively. The signal at a block diagram node V is $v[n]$ and its z-transform is $V(z)$.
- $x[n] = [a, b, c, d, e, f]$ means that $x[0] = a, \ldots x[5] = f$ and that $x[n] = 0$ outside this range.
- $\Re(z)$, $\Im(z)$, z^*, $|z|$ and $\angle z$ denote respectively the real part, imaginary part, complex conjugate, magnitude and argument of a complex number z.

Abbreviations

BIBO	Bounded Input, Bounded Output
CTFT	Continuous-Time Fourier Transform
DCT	Discrete Cosine Transform
DFT	Discrete Fourier Transform
DTFT	Discrete-Time Fourier Transform
FIR	Finite Impulse Response

IIR	Infinite Impulse Response
LTI	Linear Time-Invariant
MDCT	Modified Discrete Cosine Transform
PSD	Power Spectral Density
SNR	Signal-to-Noise Ratio

Standard Sequences

- $\delta[n] = 1$ for $n = 0$ and 0 otherwise.
- $\delta_{\text{condition}}[n] = 1$ whenever "condition" is true and 0 otherwise.
- $u[n] = 1$ for $n \geq 0$ and 0 otherwise.

Geometric Progression

- $\sum_{n=0}^{r} \alpha^n z^{-n} = \frac{1-\alpha^{r+1}z^{-r-1}}{1-\alpha z^{-1}}$ provided that $\alpha z^{-1} \neq 1$.
- $\sum_{n=0}^{\infty} \alpha^n z^{-n} = \frac{1}{1-\alpha z^{-1}}$ provided that $\left|\alpha z^{-1}\right| < 1$.

Forward and Inverse Transforms

z:	$X(z) = \sum_{-\infty}^{\infty} x[n] z^{-n}$	$x[n] = \frac{1}{2\pi j}\oint X(z) z^{n-1} dz$
CTFT:	$X(j\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt$	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega) e^{j\Omega t} d\Omega$
DTFT:	$X(e^{j\omega}) = \sum_{-\infty}^{\infty} x[n] e^{-j\omega n}$	$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$
DFT:	$X[k] = \sum_{0}^{N-1} x[n] e^{-j2\pi\frac{kn}{N}}$	$x[n] = \frac{1}{N}\sum_{0}^{N-1} X[k] e^{j2\pi\frac{kn}{N}}$
DCT:	$X[k] = \sum_{n=0}^{N-1} x[n] \cos\frac{2\pi(2n+1)k}{4N}$	$x[n] = \frac{X[0]}{N} + \frac{2}{N}\sum_{n=1}^{N-1} X[k] \cos\frac{2\pi(2n+1)k}{4N}$
MDCT:	$X[k] = \sum_{n=0}^{2N-1} x[n] \cos\frac{2\pi(2n+1+N)(2k+1)}{8N}$	$y[n] = \frac{1}{N}\sum_{0}^{N-1} X[k] \cos\frac{2\pi(2n+1+N)(2k+1)}{8N}$

Convolution

DTFT: $v[n] = x[n] * y[n] \triangleq \sum_{r=-\infty}^{\infty} x[r] y[n-r] \quad \Leftrightarrow \quad V\left(e^{j\omega}\right) = X\left(e^{j\omega}\right) Y\left(e^{j\omega}\right)$

$v[n] = x[n] y[n] \quad \Leftrightarrow \quad V\left(e^{j\omega}\right) = \frac{1}{2\pi} X\left(e^{j\omega}\right) \circledast Y\left(e^{j\omega}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j\theta}\right) Y\left(e^{j(\omega-\theta)}\right) d\theta$

DFT: $v[n] = x[n] \circledast_N y[n] \triangleq \sum_{r=0}^{N-1} x[r] y[(n-r)_{\bmod N}] \quad \Leftrightarrow \quad V[k] = X[k] Y[k]$

$v[n] = x[n] y[n] \quad \Leftrightarrow \quad V[k] = \frac{1}{N} X[k] \circledast_N Y[k] \triangleq \frac{1}{N}\sum_{r=0}^{N-1} X[r] Y[(k-r)_{\bmod N}]$

Group Delay

The group delay of a filter, $H(z)$, is $\tau_H(e^{j\omega}) = -\frac{d\angle H(e^{j\omega})}{d\omega} = \Re\left(\frac{-z}{H(z)}\frac{dH(z)}{dz}\right)\Big|_{z=e^{j\omega}} = \Re\left(\frac{\mathscr{F}(nh[n])}{\mathscr{F}(h[n])}\right)$ where $\mathscr{F}()$ denotes the DTFT.

Order Estimation for FIR Filters

Three increasingly sophisticated formulae for estimating the minimum order of an FIR filter with unity gain passbands:

1. $M \approx \frac{a}{3.5\Delta\omega}$
2. $M \approx \frac{a-8}{2.2\Delta\omega}$
3. $M \approx \frac{a-1.2-20\log_{10}b}{4.6\Delta\omega}$

where a =stop band attenuation in dB, b = peak-to-peak passband ripple in dB and $\Delta\omega$ = width of smallest transition band in normalized rad/s.

z-plane Transformations

A lowpass filter, $H(z)$, with cutoff frequency ω_0 may be transformed into the filter $H(\hat{z})$ as follows:

Target $H(\hat{z})$	Substitute	Parameters
Lowpass $\hat{\omega} < \hat{\omega}_1$	$z^{-1} = \frac{\hat{z}^{-1}-\lambda}{1-\lambda\hat{z}^{-1}}$	$\lambda = \frac{\sin\left(\frac{\omega_0-\hat{\omega}_1}{2}\right)}{\sin\left(\frac{\omega_0+\hat{\omega}_1}{2}\right)}$
Highpass $\hat{\omega} > \hat{\omega}_1$	$z^{-1} = -\frac{\hat{z}^{-1}+\lambda}{1+\lambda\hat{z}^{-1}}$	$\lambda = \frac{\cos\left(\frac{\omega_0+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\omega_0-\hat{\omega}_1}{2}\right)}$
Bandpass $\hat{\omega}_1 < \hat{\omega} < \hat{\omega}_2$	$z^{-1} = -\frac{(\rho-1)-2\lambda\rho\hat{z}^{-1}+(\rho+1)\hat{z}^{-2}}{(\rho+1)-2\lambda\rho\hat{z}^{-1}+(\rho-1)\hat{z}^{-2}}$	$\lambda = \frac{\cos\left(\frac{\hat{\omega}_2+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)}, \rho = \cot\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)\tan\left(\frac{\omega_0}{2}\right)$
Bandstop $\hat{\omega}_1 \not< \hat{\omega} \not< \hat{\omega}_2$	$z^{-1} = \frac{(1-\rho)-2\lambda\hat{z}^{-1}+(\rho+1)\hat{z}^{-2}}{(\rho+1)-2\lambda\hat{z}^{-1}+(1-\rho)\hat{z}^{-2}}$	$\lambda = \frac{\cos\left(\frac{\hat{\omega}_2+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)}, \rho = \tan\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)\tan\left(\frac{\omega_0}{2}\right)$

Noble Identities

$$-\boxed{Q:1}-\boxed{H(z)}- \;=\; -\boxed{H(z^Q)}-\boxed{Q:1}-$$

$$-\boxed{H(z)}-\boxed{1:Q}- \;=\; -\boxed{1:Q}-\boxed{H(z^Q)}-$$

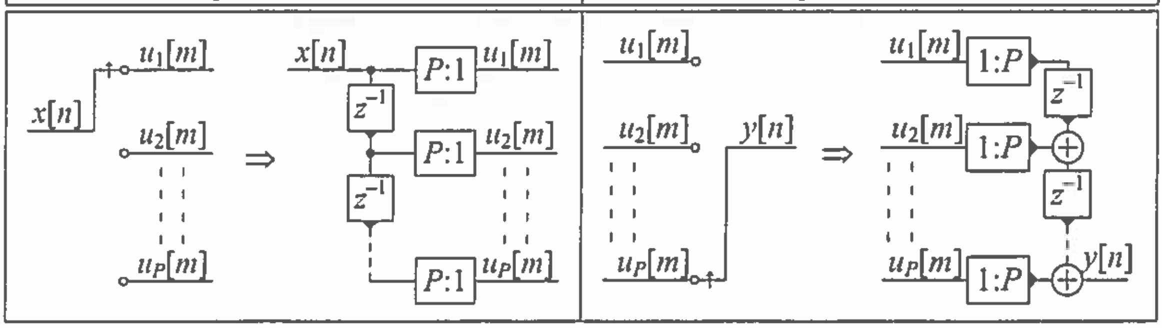
Multirate Spectra

Upsample $v[n]$ by Q: $\quad x[r] = \begin{cases} v\left[\frac{r}{Q}\right] & \text{if } Q \mid r \\ 0 & \text{if } Q \nmid r \end{cases} \quad \Rightarrow \quad X(z) = V(z^Q)$

Downsample $v[n]$ by Q: $\quad y[m] = v[Qm] \quad \Rightarrow \quad Y(z) = \frac{1}{Q}\sum_{k=0}^{Q-1} V\left(e^{\frac{-j2\pi k}{Q}} z^{\frac{1}{Q}}\right)$

Multirate Commutators

Input Commutator	Output Commutator



1. a) The finite length signals $u[0], \cdots, u[M-1]$ and $v[0], \cdots, v[N-1]$ are of length M and N respectively where $M < N$.

The signals $x[n] = u[n] * v[n]$ and $y[n] = u[n] \circledast_N v[n]$ are respectively the convolution and circular convolution of $u[n]$ and $v[n]$ as defined in the data sheet.

i) Prove that $y[n] = x[n]$ for $M-1 \leq n \leq N-1$. [3]

ii) Determine an expression for $y[n]$ in terms of the $\{x[n]\}$ that is valid for $0 \leq n \leq M-2$. [2]

iii) If $M = 3$ and $N = 4$ with $u[n] = [1, 2, -1]$ and $v[n] = [1, 1, -1, -1]$ determine both $x[n]$ and $y[n]$ for $0 \leq n \leq 7$. [3]

b) i) Show that, if $u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$ and a is a complex-valued constant, then $x[n] = a^n u[n]$ and $y[n] = -a^n u[-n-1]$ have the same z-transform but with different regions of convergence. You may use without proof the geometric progression formulae given in the datasheet. [3]

ii) The z-transform $H(z)$ is given by

$$H(z) = \frac{2 + 17z^{-1}}{(2 - z^{-1})(1 + 4z^{-1})}.$$

By expressing $H(z)$ in partial fraction form, determine the sequence, $h[n]$, whose z-transform is $H(z)$ and whose region of convergence includes $|z| = 1$. [4]

c) i) The frequency response of an ideal lowpass filter is given by

$$H(e^{j\omega}) = \begin{cases} 1 & |\omega| \leq \omega_0 \\ 0 & |\omega| > \omega_0 \end{cases}.$$

By taking the inverse DTFT of $H(e^{j\omega})$, show that the corresponding impluse response is $h[n] = \frac{\sin \omega_0 n}{\pi n}$ [3]

ii) By multiplying an ideal filter response by a Hamming window, determine an expression for the coefficients of an FIR causal bandpass filter of even order M whose passband is $1 \leq \omega \leq 2$.

For even M, a symmetric Hamming window is given by $w[n] = 0.54 + 0.46 \cos \frac{2\pi n}{M+1}$ for $-0.5M \leq n \leq 0.5M$. [3]

d) i) Show that if

$$H(z) = \frac{B(z)}{A(z)} = \frac{\sum_{r=0}^{M} a[M-r] z^{-r}}{\sum_{r=0}^{M} a[r] z^{-r}}$$

then $|H(e^{j\omega})| \equiv 1$ and $\angle H(e^{j\omega}) = -M\omega - 2\angle A(e^{j\omega})$. [3]

ii) If $H(z) = \frac{2 - 4z^{-1}}{2 - z^{-1}}$, sketch graphs of the magnitude and phase of $H(e^{j\omega})$ for $-\pi \leq \omega \leq \pi$. [3]

e) Figure 1.1 shows the power spectral density (PSD) of a real-valued signal $x[n]$. The horizontal portions of the PSD have values 3, 2 and 1 respectively. The signal $y[n]$ is then obtained by downsampling $x[n]$ by a factor of 3.

Draw a dimensioned sketch showing the PSD of $y[n]$ for $0 \le \omega \le \pi$. You should assume that components of $x[n]$ at different frequencies are uncorrelated and may assume without proof that $Y(z) = \frac{1}{3}\sum_{k=0}^{2} X\left(e^{\frac{-j2\pi k}{3}} z^{\frac{1}{3}}\right)$.

Determine the value of each horizontal portion of the PSD and each of the angular frequencies at which its value changes. [5]

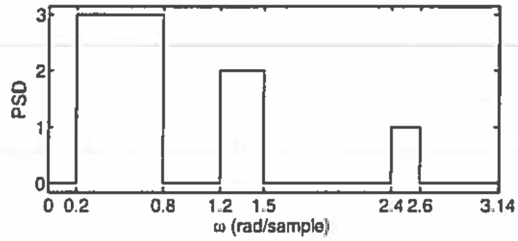


Figure 1.1

f) Figure 1.2 shows the block diagram of a two-band analysis and synthesis processor. You may assume without proof that, for $m = 0$ or 1, $W_m(z) = U_m(z^2)$ and $U_m(z) = \frac{1}{2}\left\{V_m(z^{\frac{1}{2}}) + V_m(-z^{\frac{1}{2}})\right\}$.

i) Derive an expression for $Y(z)$ in terms of $X(z)$. [4]

ii) Explain the relationship between the magnitude responses of the filters $H(z)$ and $H(-z)$. [2]

iii) Explain what is meant by saying that the analysis-synthesis processor shown in Figure 1.2 is "alias-free". [2]

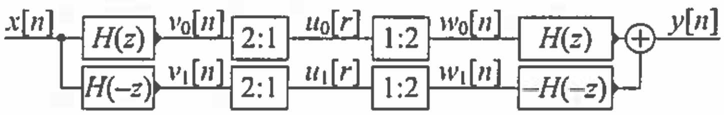


Figure 1.2

2. In this question, filters should be expressed in the standard form $g \times \frac{1+b_1 z^{-1}+\cdots}{1+a_1 z^{-1}+\cdots}$ with numerical values given for all coefficients.

a) A bilinear transformation of the z-plane is given by $z = \frac{\hat{z}-\lambda}{1-\lambda\hat{z}}$ where the real-valued constant λ satisfies $|\lambda| < 1$.

i) Show that $|z|^2 = 1 + \frac{\left(|\hat{z}|^2-1\right)\left(1-\lambda^2\right)}{|1-\lambda\hat{z}|^2}$.

Hence show that $|z| < 1$ if and only if $|\hat{z}| < 1$. [4]

ii) Explain why the property shown in part i) is important when using the transformation for filter design. [2]

b) A first-order lowpass filter has the transfer function $G(z) = 1 + z^{-1}$.

i) Determine the gain of the filter at $\omega = 0$ and show that the magnitude of the gain has decreased by a factor of $\sqrt{2}$ at the cutoff frequency, $\omega_G = \frac{\pi}{2}$. [2]

ii) By considering the value of $z^{\frac{1}{2}}G(z)$, determine a trigonometrical expression for $\left|G(e^{j\omega})\right|$ and draw a dimensioned sketch of its value over the range $0 \leq \omega \leq \pi$. [4]

iii) Using the appropriate z-plane transformation from the datasheet, transform $G(z)$ to a lowpass filter, $H(z)$, with a cutoff frequency of $\omega_H = 0.2$. Calculate the numerical values of the filter coefficients when expressed in the standard form given in the first line of the question. [5]

iv) Draw a dimensioned sketch of $\left|H(e^{j\omega})\right|$ over the range $0 \leq \omega \leq \pi$. [2]

c) A quadratic transformation of the z-plane is given by $z = -\tilde{z}^2$.

i) Show that $|z| < 1$ if and only if $|\tilde{z}| < 1$. [2]

ii) If $z = e^{j\omega}$ and $\tilde{z} = e^{j\tilde{\omega}}$ sketch a graph of ω versus $\tilde{\omega}$ over the range $-\pi \leq \tilde{\omega} \leq \pi$. The value of ω should be restricted to $-\pi < \omega \leq \pi$. [2]

iii) A new filter is defined by $P(\tilde{z}) = H(z)$. Determine the numerical values of the coefficients of $P(\tilde{z})$ when expressed in the standard form given in the first line of the question. [3]

iv) Draw a dimensioned sketch of $\left|P(e^{j\omega})\right|$ over the range $0 \leq \omega \leq \pi$ and determine the values of ω within this range for which $\left|P(e^{j\omega})\right| = \sqrt{2}$.

Explain the relationship between the bandwidth of the filter $P(e^{j\omega})$ and the cutoff frequency of the filter $H(e^{j\omega})$. [4]

3. a) The filter $H(z) = \dfrac{1}{1+a_1z^{-1}+a_2z^{-2}}$ where $a_1 = -1.56$ and $a_2 = 0.64$.

i) By multiplying $H(z)$ by its complex conjugate and using the identity $\cos 2\omega = 2\cos^2\omega - 1$, express $\left|H(e^{j\omega})\right|^{-2}$ as a polynomial in $\cos\omega$ giving the coefficients to 5 significant figures. [4]

ii) The filter $H_1(z)$ is the same as $H(z)$ but with coefficient a_1 increased in magnitude by 1% (i.e. multiplied by 1.01). Similarly, the filter $H_2(z)$ is the same as $H(z)$ but with coefficient a_2 increased in magnitude by 1% .

For $\omega_0 = 0.2$, determine the ratios $\left|\dfrac{H_1(e^{j\omega_0})}{H(e^{j\omega_0})}\right|$ and $\left|\dfrac{H_2(e^{j\omega_0})}{H(e^{j\omega_0})}\right|$ in dB. [6]

b) In the block diagram of Figure 3.1 the outputs of all adders are on the right and solid arrows indicate the direction of information flow. Multiplier gains are written adjacent to each multiplier symbol. The parameter p is strictly positive.

i) Show that $G(z) = \dfrac{Y(z)}{X(z)} = \dfrac{1}{1+(p^2-pq-2)z^{-1}+(pq+1)z^{-2}}$. [6]

ii) Determine the conditions on p and q for the filter $G(z)$ to be BIBO stable.

You may assume without proof that the filter $\dfrac{1}{1+b_1z^{-1}+b_2z^{-2}}$ is BIBO stable if and only if $|b_1| - 1 < b_2 < 1$. [6]

iii) If

$$G(z) = \frac{1}{1+b_1z^{-1}+b_2z^{-2}},$$

determine expressions for p and q as functions of b_1 and b_2. Calculate the numerical values of p and q if $b_1 = -1.56$ and $b_2 = 0.64$. [3]

iv) The filter $G_p(z)$ is the same as $G(z)$ but with coefficient p increased by 1% (i.e. multiplied by 1.01) from the value determined in part iii). Similarly, the filter $G_q(z)$ is the same as $G(z)$ but with coefficient q increased by 1% from the value determined in part iii).

For $\omega_0 = 0.2$, determine the ratios $\left|\dfrac{G_p(e^{j\omega_0})}{G(e^{j\omega_0})}\right|$ and $\left|\dfrac{G_q(e^{j\omega_0})}{G(e^{j\omega_0})}\right|$ in dB. [5]

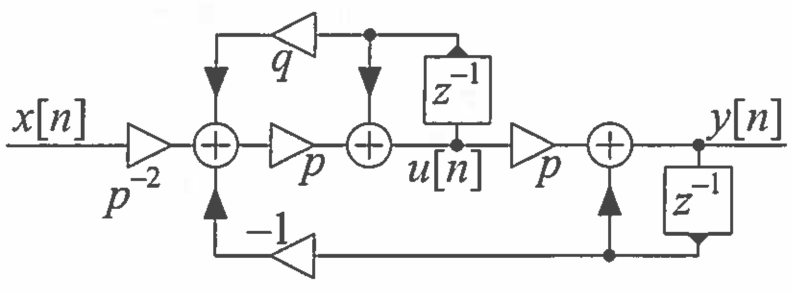


Figure 3.1

4. The FM radio band extends from 87.5 to 108 MHz. Within this band, an FM channel occupies ± 100 kHz around a centre frequency of $c \times 100$ kHz where the channel index, c, is an integer in the range $876 \leq c \leq 1079$. Figure 4.1 shows the block diagram of an FM radio front-end in which bold lines denote complex-valued signals. The diagram includes a bandpass filter (BPF) whose passband is 87.5 to 108 MHz and an analogue-to-digital converter (ADC) with a sample rate of 78 MHz.

a) Assume the bandpass filter is ideal and the power spectral density of the received signal is constant within the FM band. Sketch the power spectrum of $u[n]$ over the unnormalized frequency range -39 to $+39$ MHz. Determine the maximum width of both the lower transition region and the upper transition region of the BPF block in order to ensure that the FM band image is uncorrupted by aliasing. [3]

b) In Figure 4.1, $u[n]$ is multiplied by the complex-valued $v[n] = \exp(-j\omega_c n)$ where ω_c is the normalized centre frequency of the wanted channel.

i) Give a formula for ω_c in terms of c and state how many multiplications are required per second to multiply $u[n]$ and $v[n]$ (where one multiplication calculates the product of two real numbers). [2]

ii) Assume now that only the FM channels with centre frequencies 99.5, 100 and 100.4 MHz are present. Using an unnormalized frequency axis in kHz, draw a dimensioned sketch of the power spectrum of $w[n]$ when $c = 1000$ covering the range -700 to $+700$ kHz. On your sketch, label the centre frequency of each of the occupied spectral regions. [3]

c) i) Explain the purpose of the lowpass FIR filter, $H(z)$ in Figure 4.1. [2]

ii) Assuming that the centre frequencies of active channels are always at least 400 kHz apart, determine the cutoff frequency and maximum transition width of the filter $H(z)$ in radians/sample. Hence use the formula $M = \frac{a}{3.5\Delta\omega}$ from the datasheet to determine the order of the filter to give a stopband attenuation of 50 dB. [3]

iii) Suppose that $H(z)$ is implemented as a polyphase filter as shown in Figure 4.3. Determine the order of the sub-filters assuming they all have the same order. Give an expression for $h_p[r]$, the impulse response of the sub-filter $H_p(z)$, in terms of $h[n]$, the impulse response of $H(z)$. [2]

iv) Calculate the number of multiplications per second needed to implement Figure 4.3 assuming that all sub-filters have the same order. [3]

d) i) Determine the impulse response of $G_c(z)$ such that Figures 4.1 and 4.2 are functionally identical. [3]

ii) If $G_c(z)$ is implemented as a conventional polyphase filter, give an expression for the impulse response, $g_{c,p}[r]$, of the sub-filter $G_{c,p}(z)$. Show that if $\alpha_c = \exp\left(\frac{j2\pi c}{780}\right)$, then each coefficient, $\alpha_c^{-p} g_{c,p}[r]$, of $\alpha_c^{-p} G_{c,p}(z)$ is either purely real or purely imaginary. [3]

iii) In Figure 4.4, the subfilter $G_{c,p}(z)$ is implemented as $\alpha_c^{-p} G_{c,p}(z)$ followed by a multiplication by α_c^p. Determine a simplified expression for $s[r]$ so that Figure 4.4 is functionally equivalent to Figure 4.3. [3]

iv) Giving your reasons fully, determine the number of multiplications per second required to implement Figure 4.4. You may exclude negation operations from the multiplication count. [3]

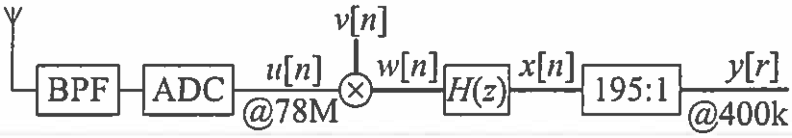


Figure 4.1

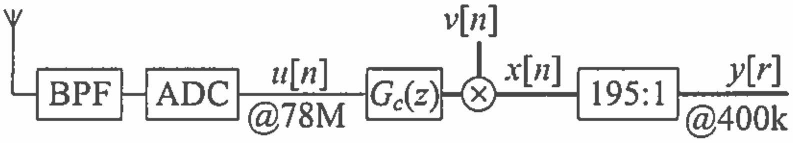


Figure 4.2

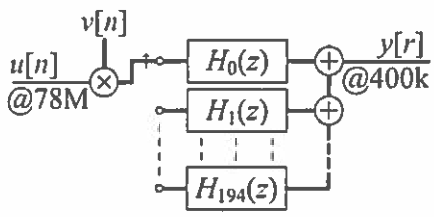


Figure 4.3

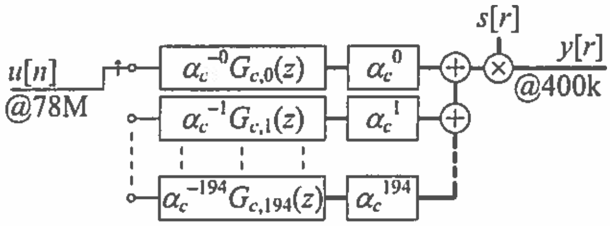


Figure 4.4