UNIVERSITY OF LONDON [II(3)E 2000]

B.ENG. AND M.ENG. EXAMINATIONS 2000

For Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examination for the Associateship.

PART II : MATHEMATICS 3 (ELECTRICAL ENGINEERING)

Wednesday 7th June 2000 2.00 - 5.00 pm

Answer EIGHT questions.

[*Before starting, please make sure that the paper is complete; there should be 8 pages, with a total of 12 questions. Ask the invigilator for a replacement if your copy is faulty.*]

1. Consider the mapping

$$w = \frac{1}{z-2}$$

from the z-plane $(z = x + iy)$ to the w-plane $(w = u + iv)$.

Find u and v in terms of x and y.

(i) Show that the circle in the z-plane

$$(x-2)^2 + y^2 = a^2$$

maps to a circle centred at $(0, 0)$ and of radius a^{-1} in the w-plane.

(ii) Show that the straight line $y = x - 2$ maps to the straight line $v = -u$ in the w-plane.

(iii) To what does the straight line $x = 0$ map in the w-plane?

(iv) To what does the straight line $x = 2$ map in the w-plane?

(v) Where are the fixed points of this mapping?

2. Consider the contour integral

$$\oint_C \frac{e^{imz}}{(z^2+1)^2}\, dz$$

where the closed contour C consists of a semi-circle in the upper half of the complex plane and $m > 0$.

Use the Residue Theorem to show that

$$\int_{-\infty}^{\infty} \frac{\cos mx}{(x^2+1)^2}\, dx = \frac{\pi}{2}(m+1)\, e^{-m} .$$

The residue of a complex function $f(z)$ *at a pole* $z = a$ *of multiplicity* n *is given by*

$$\lim_{z \to a} \frac{1}{(n-1)!} \left[\frac{d^{n-1}}{dz^{n-1}} \left\{ (z-a)^n f(z) \right\} \right] .$$

PLEASE TURN OVER

3. Consider the real integral

$$I = \int_0^{2\pi} \frac{d\theta}{(5 - 4\cos\theta)^2} \,.$$

Taking the contour C as the unit circle $z = e^{i\theta}$, show that

$$I = -i \oint_C \frac{z dz}{(2z-1)^2(z-2)^2} \,.$$

Hence show that

$$I = \frac{10\pi}{27} \,.$$

The residue of a complex function $f(z)$ at a pole $z = a$ of multiplicity n is given by

$$\lim_{z \to a} \frac{1}{(n-1)!} \left[\frac{d^{n-1}}{dz^{n-1}} \{(z-a)^n f(z)\} \right] \,.$$

4. Two functions $f(t)$ and $g(t)$ have Laplace transforms $\overline{f}(s) = \mathcal{L}\{f(t)\}$ and $\overline{g}(s) = \mathcal{L}\{g(t)\}$ respectively. If the convolution of $f(t)$ with $g(t)$ is defined as

$$f * g = \int_0^t f(u)g(t-u)du \,,$$

prove that

$$\mathcal{L}\{f * g\} = \overline{f}(s)\overline{g}(s) \,.$$

Show also that if

$$\overline{g}(s) = \frac{1}{(1+s^2)^2}$$

then

$$g(t) = \frac{1}{2}(\sin t - t \cos t).$$

Hence show that for $s > 0$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(1+s^2)^2}\right\} = 1 - \cos t - \frac{1}{2} t \sin t.$$

5. If $\overline{f}(\omega)$ is the Fourier transform of $f(t)$, prove Parseval's equality

$$\int_{-\infty}^{\infty} |f(t)|^2 \, dt \; = \; \frac{1}{2\pi} \int_{-\infty}^{\infty} |\overline{f}(\omega)|^2 \, d\omega \, .$$

The squarewave function $\Pi\,(t)$, the tent function $\Lambda\,(t)$, and the sinc-function $\text{sinc}(t)$ are defined respectively by

$$\Pi\,(t) = \begin{cases} 1, & -1/2 \le t \le 1/2, \\ 0 & \text{otherwise}, \end{cases}$$

$$\Lambda\,(t) = \begin{cases} 1+t, & -1 \le t \le 0, \\ 1-t, & 0 \le t \le 1, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$\text{sinc}(t) = \frac{\sin(t/2)}{t/2}\,, \qquad -\infty < t < \infty\,.$$

Show that $\overline{\Pi}(\omega) = \text{sinc}(\omega)$ and $\overline{\Lambda}(\omega) = \text{sinc}^2(\omega)$.

Also show that

$$\int_{-\infty}^{\infty} \text{sinc}^2(\omega) d\omega = 2\pi \qquad \text{and} \qquad \int_{-\infty}^{\infty} \text{sinc}^4(\omega) d\omega = 4\pi/3.$$

[*The identity*

$$\delta\,(\omega - \omega') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\pm i(\omega-\omega')t} \, dt$$

may be assumed, where δ *represents the Dirac delta function.*]

PLEASE TURN OVER

6. Given that $\overline{f}\,(s) = \mathcal{L}\,\{f\,(t)\}$ is the Laplace transform of $f\,(t)$, prove that when a is a constant
$$\mathcal{L}\left\{e^{at}\,f\,(t)\right\} = \overline{f}\,(s-a) \qquad Re(s) > a.$$

A 2nd order ordinary differential equation, with initial values, takes the form

$$\frac{d^2x}{dt^2} + 4\,\frac{dx}{dt} + 8x = \delta\,(t-1)\,, \qquad x = \frac{dx}{dt} = 0 \;\text{ when }\; t = 0\,,$$

where δ represents the Dirac delta function. Use the Laplace convolution theorem to show that
$$x\,(t) = \begin{cases} \frac{1}{2}e^{-2(t-1)}\,\sin 2(t-1) & t > 1 \\ \\ 0 & 0 \le t \le 1 \end{cases}$$

satisfies the differential equation and its initial conditions.

7. The double integral I_n is given by

$$I_n = \iint_{R_n} x\,y\,\mathbf{e}^{-(x^2/a^2+y^2/b^2)}\,dxdy\,, \qquad a,\,b > 0\,,$$

for $n = 1$ and 2, where the finite regions of integration R_n are given as follows:

R_1 is the region bounded by the lines $x = 0,\; x = a,\; y = 0$ and $y = b$;

R_2 is the region in the positive quadrant enclosed by the lines $x = 0,\; y = 0$ and the curve $x^2/a^2 + y^2/b^2 = 1$.

(i) Sketch the regions of integration R_1 and R_2.

(ii) Show that
$$I_1 = \frac{1}{4}\,a^2\,b^2\,\left(1 - \frac{1}{\mathbf{e}}\right)^2\,.$$

(iii) Calculate I_2 by making the transformation

$$x = ar\,\cos\theta,\;\; y = br\,\sin\theta\,,$$

and demonstrate that
$$I_1 - I_2 = \left(\frac{ab}{2\mathbf{e}}\right)^2\,.$$

8. If $\phi = xyz^2$, $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $f(r)$ is an arbitrary function of $r = |\mathbf{r}|$, evaluate

   (i) grad ϕ,

   (ii) div $\mathbf{r}$,

   (iii) div $(\phi\,\mathbf{r})$,

   (iv) curl $(f(r)\,\mathbf{r})$.

9. The curve C is given in parametric form by

$$x = 2 + \cos\theta\,, \quad y = 1 + \sin\theta\,, \quad |\,\theta\,| \le \pi/2\,,$$

and the vector function $\mathbf{F}$ is defined by

$$\mathbf{F} = \frac{x\mathbf{i} + y\mathbf{j}}{x^2 + y^2}\,, \qquad x^2 + y^2 \neq 0\,.$$

   (i) Sketch the curve C.

   (ii) Show that along C:

$$x\,dx + y\,dy = (\cos\theta - 2\sin\theta)\,d\theta\,.$$

   (iii) Prove that curl $\mathbf{F} = \mathbf{0}$, and find a potential function Φ such that $\mathbf{F} = \nabla\Phi$.

   (iv) Calculate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where C is tranversed anti-clockwise, by each of the following methods:

      (a) use of the potential function found in (iii),

      (b) direct evaluation, making use of the result obtained in (ii).

PLEASE TURN OVER

10. $P\,(x,\,y)$ and $Q\,(x,\,y)$ are continuous functions of x and y with continuous first partial derivatives in a simply connected region R with a piecewise smooth boundary C. Green's Theorem in a plane states that

$$\oint_C (Pdx + Qdy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy\,.$$

If the vector $\mathbf{u}\,(x,\,y)$ is defined in terms of P and Q by

$$\mathbf{u}\,(x,\,y) = \mathbf{i}P\,(x,\,y) + \mathbf{j}Q\,(x,\,y)\,,$$

show that Green's Theorem can be re-expressed as the two-dimensional version of Stokes' Theorem

$$\oint_C \mathbf{u}\cdot d\mathbf{r} = \iint_R (\mathbf{k}\cdot \mathrm{curl}\,\mathbf{u})\, dxdy.$$

If $Q = \frac{1}{2}x^2$, $P = \frac{1}{2}y^2$ and R is defined as being the area lying between the parabola $y = x^2$ and the straight line $y = x$, evaluate both sides of Stokes' Theorem showing that they each take the value $1/60$.

11. Let $A_1, \ldots, A_k$ form a partition of a sample space and B be some event. Use the definition of conditional probability and the theorem of total probabilities to derive Bayes' formula for $P\,(A_i\,|\,B)$.

It is estimated that 0.5% of computer hard disks produced by a manufacturer are faulty. A method has been designed to test the disks to try to ascertain whether they are faulty or not. This test has a probability of 0.95 of giving a diagnosis of 'faulty' when applied to a faulty disk, and a probability of 0.10 of giving the same diagnosis when applied to a perfect disk.

A disk is chosen at random and tested.

(i) What is the probability that the test gives a diagnosis of 'faulty'?

(ii) Given a diagnosis of 'faulty', what is the probability the disk is in fact faulty?

(iii) Given a diagnosis of 'not faulty', what is the probability the disk is in fact perfect?

(iv) What is the probability the disk will be misclassified?

12. The annual profit, Y, (in millions of pounds) of a computer manufacturer is a function $g(X)$ of the availability, X, of microchips during the year. The availability X in a given year has an exponential distribution with probability density function

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0\,; \\ 0\,, & \text{otherwise}\,, \end{cases}$$

with $\lambda > 0$. The profit Y is given by $Y = g(X) = 2(1 - e^{-2X})$.

(i) Write down the cumulative distribution function of X, $F_X(x) = P(X \leq x)$.

(ii) Show that the cumulative distribution function of Y, $F_Y(y) = P(Y \leq y)$ is given by $F_Y(y) = F_X(-\frac{1}{2}\ln[1 - \frac{y}{2}])$, $0 \leq y < 2$.

(iii) Using the results in (i) and (ii), show that $F_Y(y)$ can thus be written as $F_Y(y) = 1 - [1 - \frac{y}{2}]^{\lambda/2}$, $0 \leq y < 2$.

(iv) Hence find the probability density function, $f_Y(y)$, of Y.

(v) Use the fact that $E\{Y\} = E\{g(X)\}$ to find the mean annual profit.

END OF PAPER

MATHS 3 - 2000

MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

SESSION : 1999/2000

E1

PAPER 3

QUESTION

SOLUTION 14

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(14) $w = \frac{1}{z-2} = \frac{1}{x-2+iy} = \frac{x-2-iy}{(x-2)^2+y^2}$

$\therefore \quad u = \frac{x-2}{(x-2)^2+y^2} \qquad v = \frac{-y}{(x-2)^2+y^2}$ 2

(i) $u^2+v^2 = \frac{1}{(x-2)^2+y^2} = 1/a^2$ on $(x-2)^2+y^2 = a^2$ 3

$u^2+v^2 = a^{-2}$ is a circle centred at $(0,0)$ radius a^{-1}.

(ii) $y = x-2$ means

$u = \frac{x-2}{2(x-2)^2} = \frac{1}{2(x-2)}$

$v = \frac{-(x-2)}{2(x-2)^2} = -\frac{1}{2(x-2)}$ 3

Hence $v = -u$.

(iii) $x=0$ means $u = \frac{-2}{y^2+4}$

$v = \frac{-y}{y^2+4}$

$\therefore \quad u^2+v^2 = \frac{1}{y^2+4} = -u/2$ 3

$\therefore \; (u+1/4)^2+v^2 = (1/4)^2$

A Circle, centred at $(-1/4, 0)$, radius $1/4$.

(iv) $x=2 \qquad u=0 \qquad v = -1/y$

A Line (vertical) which is the v-axis. 2

(v) Fixed points of the map $w = \frac{1}{z-2}$ lie at solutions of

$z = \frac{1}{z-2}$ 2

$\therefore \; z^2-2z=1$ or $(z-1)^2 = 2$

ie $z = 1 \pm \sqrt{2}$ Two points on the real axis.

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Checker : C ATKINSON Checker's signature : C. Atkinson

MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

SESSION : 1999/2000

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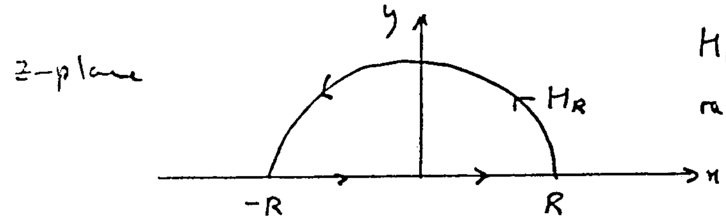
E 2

PAPER 3

QUESTION

SOLUTION 15

(15)



H_R is the semi-circle radius R: $z = Re^{i\theta}$, $0 \le \theta \le \pi$

Contour **C** is complete semi-circle

$$\oint_C \frac{e^{imz}}{(z^2+1)^2}\,dz = \int_{-R}^{R} \frac{e^{imx}}{(1+x^2)^2}\,dx + \int_{H_R} \frac{e^{imz}}{(1+z^2)^2}\,dz$$ 2

Using Jordan's Lemma:

$$\lim_{R\to\infty} \int_{H_R} \frac{e^{imz}}{(1+z^2)^2}\,dz = 0$$ because i) $m \ge 0$ ii) $f(z) \to 0$ as $R \to \infty$ iii) Sings are poles. 3

$$\therefore \int_{-\infty}^{\infty} \frac{e^{imx}}{(1+x^2)^2}\,dx = \int_{-\infty}^{\infty} \frac{\cos mx\,dx}{(1+x^2)^2}.$$ (Sine-part vanishes as an odd fn)

$$= \lim_{R\to\infty} \oint_C \frac{e^{imz}}{(1+z^2)^2}\,dz$$

Using the Residue Thm. $= 2\pi i \times \{$Sum of Residues of poles in the <u>upper</u> ½-plane$\}$ 5

There is one pole (double) at $z = +i$ in the upper ½-plane.

Residue at $z = i$ is $\displaystyle \lim_{z\to i} \left[\frac{d}{dz}\left\{(z-i)^2 \frac{e^{imz}}{(z^2+1)^2}\right\}\right]$ 3

$$= \lim_{z\to i} \frac{d}{dz}\left\{\frac{e^{imz}}{(z+i)^2}\right\}$$

$$= \lim_{z\to i} \left\{e^{imz}\left[\frac{im}{(z+i)^2} - \frac{2}{(z+i)^3}\right]\right\}$$

$$= e^{-m}\left\{\frac{im}{(2i)^2} - \frac{2}{(2i)^3}\right\}$$

$$= -\frac{i}{4} e^{-m}\{m+1\}$$

$$\therefore \int_{-\infty}^{\infty} \frac{\cos mx}{(1+x^2)^2}\,dx = \frac{\pi}{2} e^{-m}(m+1)$$ 2

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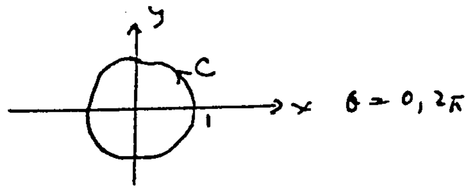
(16) $I = \int_0^{2\pi} \frac{d\theta}{(5-4\cos\theta)^2}$ $\qquad z = e^{i\theta} \qquad \cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)$

$dz = iz\,d\theta$

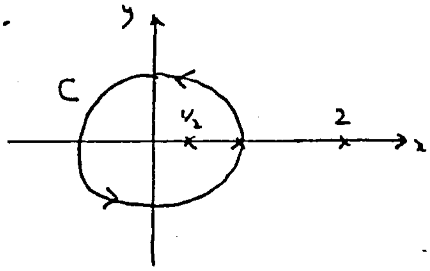
$= \oint_C \frac{dz}{iz} \cdot \frac{1}{\left[5-2\left(z+\frac{1}{z}\right)\right]^2}$



$= -i \oint_C \frac{z\,dz}{(5z-2z^2-2)^2}$

$\therefore I = -i \oint_C \frac{z\,dz}{(2z-1)^2(z-2)^2}$ $\qquad$ C is the unit circle 5

Now note that the integrand has 2 double poles, one at $z = 1/2$ & the other at $z = 2$. The latter lies outside C so it doesn't count. 3



Residue of $f(z)$ at the double pole at $z = 1/2$ is 2

$\lim_{z\to 1/2} \frac{d}{dz}\left\{ (z-1/2)^2 \cdot \frac{z}{(2z-1)^2(z-2)^2} \right\}$

$= \frac{1}{4} \lim_{z\to 1/2} \frac{d}{dz}\left\{\frac{z}{(z-2)^2}\right\} = -\frac{1}{4} \lim_{z\to 1/2} \left(\frac{z+2}{(z-2)^3}\right)$

$= -\frac{1}{4} \cdot \frac{5/2}{(-3/2)^3} = \frac{5}{27}$ 5

$\therefore I = 2\pi i \times -i \times \frac{5}{27} = \frac{10\pi}{27}.$

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MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

SESSION : 1999/2000

E4

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PAPER 3

QUESTION

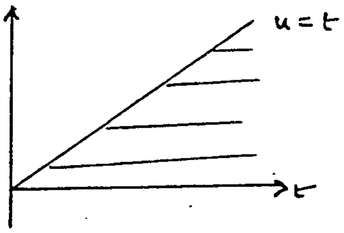
SOLUTION 18

18 (a)

$$\mathcal{L}(f*g) = \int_0^\infty e^{-st}\left\{\int_0^t f(u)\,g(t-u)\,du\right\}dt$$

Exchange integration order in the double integral

$$\therefore \mathcal{L}(f*g) = \int_0^\infty f(u)\left(\int_u^\infty e^{-st} g(t-u)\,dt\right)du$$



Put $t-u=\theta$, then

$$\mathcal{L}(f*g) = \int_0^\infty f(u)\left(\int_0^\infty e^{-s(\theta+u)} g(\theta)\,d\theta\right)du$$

$$= \int_0^\infty f(u)\,e^{-su}du \int_0^\infty e^{-s\theta} g(\theta)\,d\theta = \bar{f}(s)\,\bar{g}(s)$$ 5

(b) Given that

$$\mathcal{L}(\sin\omega t) = \frac{\omega}{s^2+\omega^2} \qquad \text{Diffn. w.r.t. } \omega$$

$$\therefore \mathcal{L}(t\cos\omega t) = \frac{1}{s^2+\omega^2} - \frac{2\omega^2}{(s^2+\omega^2)^2}$$

Now put $\omega=1$ to get

$$\mathcal{L}(t\cos t) - \mathcal{L}(\sin t) = -2\cdot\frac{1}{(s^2+1)^2}$$

$$\therefore \mathcal{L}^{-1}\left(\frac{1}{(s^2+1)^2}\right) = g(t) = \tfrac{1}{2}(\sin t - t\cos t)$$

(b) Alternatively,
Let $\bar{f}(s)=\bar{g}(s)=\frac{1}{1+s^2}$.
Then, by convolution
$$\mathcal{L}^{-1}\left(\frac{1}{(1+s^2)^2}\right) = \int_0^t \sin u\,\sin(t-u)\,du$$
$$= \tfrac{1}{2}(\sin t - t\cos t)$$ 5

(c) Using the convolution Thm & choosing

$$\bar{f}(s) = 1/s \longrightarrow f(t)=1$$

$$\bar{g}(s) = \frac{1}{(1+s^2)^2} \longrightarrow g(t) = \tfrac{1}{2}(\sin t - t\cos t) \quad \text{as above}$$

$$\therefore \mathcal{L}^{-1}\left(\frac{1}{s(1+s^2)^2}\right) = f*g = \tfrac{1}{2}\int_0^t (\sin u - u\cos u).1.du$$

$$= \tfrac{1}{2}\left[[-\cos u]_0^t - \int_0^t u\,d(\sin u)\right]$$

$$= \tfrac{1}{2}\left\{1-\cos t - [u\sin u]_0^t - [\cos u]_0^t\right\}$$

$$= 1-\cos t - \tfrac{1}{2}t\sin t$$ 5

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MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

SESSION : 1999/2000

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E5

PAPER 3

QUESTION

SOLUTION

17 (i) $\bar{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} f(t)\,dt \qquad f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\bar{f}(\omega)\,d\omega$

$$\int_{-\infty}^{\infty} f(t) f^*(t)\,dt = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{i\omega t}\bar{f}(\omega)\,d\omega\right)\left(\int_{-\infty}^{\infty} e^{-i\omega' t}\bar{f}^*(\omega')\,d\omega'\right)dt$$

$$= \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} \bar{f}(\omega)\left\{\int_{-\infty}^{\infty}\bar{f}^*(\omega')\underbrace{\left(\int_{-\infty}^{\infty} e^{i(\omega-\omega')t}\,dt\right)}_{2\pi\,\delta(\omega-\omega')}d\omega'\right\}d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\bar{f}(\omega)\bar{f}^*(\omega)\,d\omega$$ 5

because $\int_{-\infty}^{\infty}\bar{f}^*(\omega')\,\delta(\omega-\omega')\,d\omega' = \bar{f}^*(\omega)$

(ii) $\bar{\Pi}(\omega) = \int_{-1/2}^{1/2} e^{-i\omega t}.1.\,d\omega$ (zero on rest of the t-axis) 2

$$= -\frac{1}{i\omega}\left(e^{-i\omega/2} - e^{i\omega/2}\right) = \text{sinc}(\omega)$$

(iii) $\bar{\Lambda}(\omega) = \int_{-1}^{0}(1+t)e^{-i\omega t}dt + \int_0^1 (1-t)e^{-i\omega t}dt$

$$= \int_{-1}^{1} e^{-i\omega t}dt - 2\int_0^1 t\cos\omega t\,dt$$

$$= \frac{2}{\omega}\sin\omega - \frac{2}{\omega}\int_0^1 t\,d(\sin\omega t)$$ 4

$$= \frac{2}{\omega}\sin\omega - \frac{2}{\omega}\left[\left[t\sin\omega t\right]_0^1 + \left[\frac{\cos\omega t}{\omega}\right]_0^1\right]$$

$$= \frac{2}{\omega^2}(1-\cos\omega) = \text{sinc}^2\omega \quad \text{by double angle form}$$

(iii) Use Parseval above $\frac{1}{2\pi}\int_{-\infty}^{\infty}\text{sinc}^2\omega\,d\omega = \int_{-\infty}^{\infty}\Pi^2(t)\,dt = 1.$ 2

(iv) $\frac{1}{2\pi}\int_{-\infty}^{\infty}\text{sinc}^4\omega\,d\omega = \int_{-\infty}^{\infty}\Lambda^2(t)\,dt = \int_{-1}^{0}(1+t)^2dt + \int_0^1(1-t)^2dt$ 2

$$= \int_{-1}^{1}(1+t^2)\,dt + 2\int_{-1}^{0} t\,dt - 2\int_0^1 t\,dt$$

$$= 8/3 + \left[t^2\right]_{-1}^{0} - \left[t^2\right]_0^1 = 8/3 - 2 = 2/3.$$

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MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

SESSION : 1999/2000

E 6

PAPER 3

QUESTION

SOLUTION 19

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(19) $\mathcal{L}\left(e^{at}f(t)\right) = \int_0^\infty e^{-st}e^{at}f(t)\,dt$

$= \int_0^\infty e^{-(s-a)t}f(t)\,dt$

$= \bar{f}(s-a) \qquad s > a.$ 3

$\mathcal{L}(\ddot{x} + 4\dot{x} + 8x) = \mathcal{L}(\delta(t-1)) \qquad x(0) = \dot{x}(0) = 0.$

From Tables $\mathcal{L}\ddot{x} = s^2\bar{x}(s) - s\,x(0) - \dot{x}(0) = s^2\bar{x}(s)$

$\mathcal{L}\dot{x} = s\bar{x}(s) - x(0) \qquad = s\bar{x}(s)$

$\therefore (s^2 + 4s + 8)\bar{x}(s) = \int_0^\infty e^{-st}\delta(t-1)\,dt = e^{-s}$

$\therefore \quad \bar{x}(s) = \dfrac{e^{-s}}{(s+2)^2+4}$

$= \frac{1}{2}e^{-s}\cdot\left(\dfrac{2}{(s+2)^2+2^2}\cdot\right) \qquad (*)$ 4

Now $\mathcal{L}^{-1}\left(\dfrac{2}{s^2+2^2}\right) = \sin 2t$

& so $\mathcal{L}^{-1}\left(\dfrac{2}{(s+2)^2+2^2}\right) = \sin(2t)e^{-2t}$ where $a=2$ 4

If $f(t) = \frac{1}{2}e^{-2t}\sin 2t \qquad g(t) = \delta(t-1)$

where $\bar{g}(s) = e^{-s}$

then

$x(t) = \int_0^t g(u)f(t-u)\,du$ (Solving (*) by Conv. Thm.)

$= \frac{1}{2}\int_0^t \delta(u-1)\,e^{-2(t-u)}\sin 2(t-u)\,du$ 4

$$x(t) = \begin{cases} \frac{1}{2}e^{-2(t-1)}\sin 2(t-1) & t>1 \\ 0 & 0\le t\le 1 \end{cases}$$

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PAPER III
QUESTION
SOLUTION 20

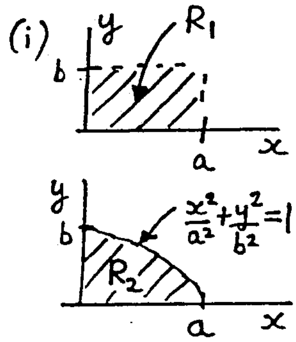
(ii) $$I_1 = \int_{x=0}^{a}\int_{y=0}^{b} xy e^{-\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)}\,dx\,dy$$

$$= \left(\int_0^a x e^{-x^2/a^2}dx\right)\left(\int_0^b y e^{-y^2/b^2}dy\right)$$

$$= \left[-\tfrac{1}{2}a^2 e^{-x^2/a^2}\right]_0^a \left[-\tfrac{1}{2}b^2 e^{-y^2/b^2}\right]_0^b$$

$$= \underline{\underline{\tfrac{1}{4}a^2b^2\left(\tfrac{1}{e}-1\right)^2}}$$

(i)



2 (sketch)

3 (part (ii))

(iii) $x = ar\cos\theta$; $y = br\sin\theta$. $r^2 = \frac{x^2}{a^2}+\frac{y^2}{b^2}$ require $0 \leq \theta \leq \frac{\pi}{2}$ 2

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} a\cos\theta & -ar\sin\theta \\ b\sin\theta & br\cos\theta \end{vmatrix} = abr(\cos^2\theta+\sin^2\theta) = \underline{abr}$$ 2

so, $dx\,dy = abr\,dr\,d\theta$

and $$I_2 = \int_{\theta=0}^{\pi/2}\int_{r=0}^{1} (ar\cos\theta)(br\sin\theta)\, e^{-r^2}\, abr\,dr\,d\theta$$

$$= a^2b^2\left(\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta\right)\left(\int_0^1 \overbrace{r^3e^{-r^2}}^{r^2\cdot re^{-r^2}}dr\right)$$

$$= a^2b^2\left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2}\left\{\left[-\tfrac{1}{2}r^2e^{-r^2}\right]_0^1 + \int_0^1 2r\cdot\tfrac{1}{2}e^{-r^2}dr\right\}$$

$$= \frac{a^2b^2}{2}\left\{-\tfrac{1}{2}e^{-1}+\left[-\tfrac{1}{2}e^{-r^2}\right]_0^1\right\}$$

$$= \underline{\underline{\frac{a^2b^2}{4}(1-2e^{-1})}}$$ 4

Then $$I_1 - I_2 = \frac{a^2b^2}{4}\left(\frac{1}{e^2}+1-\cancel{\frac{2}{e}}+\cancel{\frac{2}{e}}-1\right)$$

$$= \underline{\underline{\left(\frac{ab}{2e}\right)^2}}$$ 2

Total 15

Setter : A.G. WALTON Setter's signature : Andrew Walton

Checker : R-L. Jacobs Checker's signature : R. JACOBS

MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

SESSION : 1999/2000

E8

PAPER BEng/Pt II 3

QUESTION

SOLUTION 22

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3 (i) $\nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right) = (yz^2, xz^2, 2xyz)$

since $\phi = xyz^2$. 3

3 (ii) $\operatorname{div} \underline{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$ 3

4 (iii) $\operatorname{div}(\phi \underline{r}) = \frac{\partial}{\partial x}(\phi x) + \frac{\partial}{\partial y}(\phi y) + \frac{\partial}{\partial z}(\phi z)$

$= 2xyz^2 + 2yxz^2 + 3xyz^2 = 7xyz^2$ 4

(iv) $\operatorname{Curl}(f(r)\,\underline{r}) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f(r)x & f(r)y & f(r)z \end{vmatrix}$

$= \underline{i}\left(z\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial z}\right) - \underline{j}\left(z\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial z}\right)$

$+ \underline{k}\left(y\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial y}\right)$ 5

5 but $\frac{\partial f}{\partial y} = f'(r)\cdot\frac{\partial r}{\partial y} = \frac{yf'}{r}$ etc. since $\frac{\partial r}{\partial y} = \frac{y}{r}$

so $\operatorname{Curl}(f(r)\,\underline{r}) = \underline{i}\left(\frac{zy}{r}f' - \frac{yz}{r}f'\right) - \underline{j}\left(\frac{zx}{r}f' - \frac{xz}{r}f'\right)$

$+ \underline{k}\left(\frac{yx}{r}f' - \frac{xy}{r}f'\right)$

$= \underline{0}$

Setter : C. ATKINSON Setter's signature : C. Atkinson

Checker : J D GIBSON Checker's signature : J.D. Gibson

E9

MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

SESSION : 1999/2000

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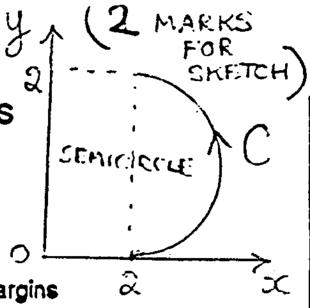
PAPER III

QUESTION

SOLUTION 23

(i) (2 MARKS FOR SKETCH)



(ii) $$x\,dx + y\,dy = (2+\cos\theta)(-\sin\theta\,d\theta) + (1+\sin\theta)\cos\theta\,d\theta$$
$$= \underline{\underline{(\cos\theta - 2\sin\theta)\,d\theta}}$$ 2

(iii) $$\text{Curl } \underline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{x}{x^2+y^2} & \frac{y}{x^2+y^2} & 0 \end{vmatrix} = \hat{k}\left(\frac{-2xy}{(x^2+y^2)^2} - \frac{(-2xy)}{(x^2+y^2)^2}\right)$$
$$= \underline{\underline{0}}$$ 2

Write $\underline{F} = \nabla \underline{\Phi}$. Then $\dfrac{\partial \Phi}{\partial x} = \dfrac{x}{x^2+y^2}$

$$\Rightarrow \Phi = \tfrac{1}{2}\ln(x^2+y^2) + g(y)$$

& $$\frac{\partial \Phi}{\partial y} = \frac{y}{x^2+y^2}$$

$$\Rightarrow \Phi = \tfrac{1}{2}\ln(x^2+y^2) + h(x)$$

Thus $\underline{\underline{\Phi = \tfrac{1}{2}\ln(x^2+y^2) + C}}$ 2

(iv) (b) $$\int_C \underline{F}\cdot d\underline{r} = \int_C \frac{x\,dx + y\,dy}{x^2+y^2} = \int_{-\pi/2}^{\pi/2} \frac{(\cos\theta - 2\sin\theta)\,d\theta}{(2+\cos\theta)^2 + (1+\sin\theta)^2}$$ 2

($d\underline{r} = dx\,\hat{i} + dy\,\hat{j}$; using (i))

$$= \int_{-\pi/2}^{\pi/2} \frac{(\cos\theta - 2\sin\theta)\,d\theta}{4+\cos^2\theta + 4\cos\theta + 1 + 2\sin\theta + \sin^2\theta}$$

[or use $t = \tan\frac{1}{2}\theta$ substn : longer]

$$= \int_{-\pi/2}^{\pi/2} \frac{\cos\theta - 2\sin\theta}{6+4\cos\theta+2\sin\theta}\,d\theta = \left[\tfrac{1}{2}\ln\left(4\cos\theta + 2\sin\theta + 6\right)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$
$$= \tfrac{1}{2}\ln\left(\tfrac{8}{4}\right) = \underline{\underline{\tfrac{1}{2}\ln 2}}$$ 3

(a) $$\int_C \underline{F}\cdot d\underline{r} = \int_C \nabla\Phi\cdot d\underline{r} = [\Phi]_C = \Phi(2,2) - \Phi(2,0)$$ (using (ii))

$$\left.\begin{array}{l}\text{when } \theta = -\frac{\pi}{2}: \ x=2, y=0 \\ \theta = \frac{\pi}{2}: \ x=2, y=2\end{array}\right\} = \tfrac{1}{2}\ln 8 - \tfrac{1}{2}\ln 4$$
$$= \underline{\underline{\tfrac{1}{2}\ln 2}}$$ 2

(Parts (a) & (b) interchanged).

Total 15

Setter : A.G. WALTON Setter's signature : Graham Walton

Checker : R.L. Jacobs Checker's signature : R.JACOBS

MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

SESSION : 1999/2000

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E 10

PAPER II (3)

QUESTION

SOLUTION 29

G.T. $\oint (P\,dx + Q\,dy) = \iint_R (Q_x - P_y)\,dx\,dy$

$\underline{u} = \hat{\underline{i}} P + \hat{\underline{j}} Q \quad \therefore \quad \text{curl}\,\underline{u} = \hat{\underline{k}}\,(Q_x - P_y)$

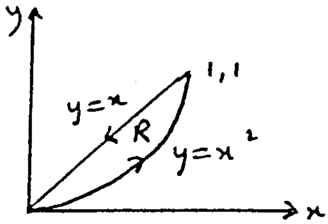
$\therefore \quad \text{RHS} = \iint_R (\hat{\underline{k}} \cdot \text{curl}\,\underline{u})\,dx\,dy$

$\underline{r} = \hat{\underline{i}} x + \hat{\underline{j}} y \Rightarrow d\underline{r} = \hat{\underline{i}}\,dx + \hat{\underline{j}}\,dy$ 7

$\therefore \quad \text{LHS} = \oint_c \underline{u} \cdot d\underline{r}$

Now $\quad Q = \tfrac{1}{2} x^2, \quad P = \tfrac{1}{2} y^2$

$\therefore \quad \hat{\underline{k}} \cdot \text{curl}\,\underline{u} = x - y$



$\therefore \iint_R \hat{\underline{k}} \cdot \text{curl}\,\underline{u}\;dx\,dy = \iint_R (x-y)\,dx\,dy$

$= \int_0^1 \left\{ \int_{x^2}^{x} (x-y)\,dy \right\} dx$ 4

$= \int_0^1 \left[xy - \tfrac{1}{2} y^2 \right]_{x^2}^{x} dx$

$= \int_0^1 \left(\tfrac{1}{2} x^2 - x^3 + \tfrac{1}{2} x^4 \right) dx = \tfrac{1}{60}$

$\oint_c \underline{u} \cdot d\underline{r} = \tfrac{1}{2} \oint_c (y^2\,dx + x^2\,dy)$

$= \tfrac{1}{2} \int_0^1 (x^4\,dx + 2x^3\,dx) + \tfrac{1}{2} \int_1^0 (x^2\,dx + x^2\,dx)$

$= \tfrac{1}{2} \left(\tfrac{1}{5} + \tfrac{1}{2} \right) - \tfrac{1}{3}$ 4

$= \tfrac{7}{20} - \tfrac{1}{3} = \dfrac{21-20}{60}$

$= \tfrac{1}{60}$

Setter : J. D. GIBBON Setter's signature : J. D. Gibbon

Checker : A. G. WALTON Checker's signature : Andrew Walton

$P(A_i|B) = P(A_i \cap B)/P(B) = P(B|A_i)P(A_i)/P(B)$ and

by law of total probabilities $P(B) = \sum_{j=1}^{k} P(B|A_j)P(A_j)$.

Hence $P(A_i|B) = \dfrac{P(B|A_i)P(A_i)}{\sum_{j=1}^{k} P(B|A_j)P(A_j)}$. 4

Let T = test is positive ; F = disk is faulty. 2

$P(T|F) = 0.95 \qquad P(F) = 0.005$

$P(T|\bar{F}) = 0.10$

(i) $P(T) = P(T|F)P(F) + P(T|\bar{F})P(\bar{F})$

$= (0.95 \times 0.005) + (0.10 \times 0.995) = \underline{\underline{0.10425}}$ ✓ 2

(ii) $P(F|T) = \dfrac{P(T|F)P(F)}{P(T|F)P(F) + P(T|\bar{F})P(\bar{F})} = \dfrac{0.95 \times 0.005}{0.10425} = \underline{\underline{0.04556}}$ ✓ 2

(iii) $P(\bar{F}|\bar{T}) = \dfrac{P(\bar{T}|\bar{F})P(\bar{F})}{P(\bar{T})} = \dfrac{0.9 \times 0.995}{1 - 0.10425} = \underline{\underline{0.99972}}$ ✓ 2

(iv) $P(\text{misclassified}) = P(T \cap \bar{F}) + P(\bar{T} \cap F)$ 3

$= P(T|\bar{F})P(\bar{F}) + P(\bar{T}|F)P(F)$

$= (0.1 \times 0.995) + (0.05 \times 0.005)$

$= \underline{\underline{0.09975}}$ ✓

15

Setter : A T Walden Setter's signature : A T Walden

Checker : S G Walker Checker's signature : [signature]

(i) $F_X(x) = \int_0^x \lambda e^{-\lambda y}\,dy = -e^{-\lambda y}\Big|_0^x = 1-e^{-\lambda x}$ $\quad 0 \le x < \infty$ 3

(ii) $F_Y(y) = P(Y \le y) = P(1-e^{-2X} \le y/2)$

$= P(e^{-2X} \geq 1 - y/2)$

$= P(X \le -\tfrac{1}{2}\ln(1-y/2))$

$= F_X(-\tfrac{1}{2}\ln[1-y/2])$. 3

Note $0 \le x < \infty \Rightarrow 0 < e^{-2x} \le 1 \Rightarrow 0 \le 1-e^{-2x} < 1 \Rightarrow 0 \le y < 2$.

(iii) $F_Y(y) = F_X(-\tfrac{1}{2}\ln[1-y/2])$

$= 1 - \exp\left(\tfrac{\lambda}{2}\ln\left(1-\tfrac{y}{2}\right)\right) = 1 - \exp\left(\ln\left(1-\tfrac{y}{2}\right)^{\lambda/2}\right)$

$= 1 - \left[1-\tfrac{y}{2}\right]^{\lambda/2}, \quad 0 \le y < 2.$ 3

(iv) $f_Y(y) = F_Y'(y) = \frac{d}{dy}\left\{1 - \left[1-\tfrac{y}{2}\right]^{\lambda/2}\right\}$

$= \frac{\lambda}{4}\left[1-\tfrac{y}{2}\right]^{(\lambda/2)-1} \quad 0 \le y < 2.$ 2

(v) $E\{Y\} = E\{g(X)\} = E\{2(1-e^{-2X})\}$

$= \int_0^\infty 2(1-e^{-2x})\lambda e^{-\lambda x}\,dx$

$= 2\lambda \int_0^\infty e^{-\lambda x} - e^{-(2+\lambda)x}\,dx$ 4

$= 2\lambda\left[\frac{1}{\lambda} - \frac{1}{2+\lambda}\right] = \frac{4}{2+\lambda}.$

15

Setter : AWalden
Checker : SGW

Setter's signature : AWalden
Checker's signature : SGWaller