IMPERIAL COLLEGE LONDON

E3.07
AU1
ISE3.11

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2006

EEE/ISE PART III/IV: MEng, BEng and ACGI

Corrected Copy

None

DIGITAL SIGNAL PROCESSING

Wednesday, 10 May 10:00 am

Time allowed: 3:00 hours

There are SIX questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : P.A. Naylor
Second Marker(s) : D.P. Mandic

Special Instructions for Invigilators: None

Information for Candidates:

Sequence	z-transform
$\delta(n)$	1
$a^n u(n)$	$\dfrac{1}{1-az^{-1}}$
$\left(r^n \cos \omega_0 n\right)u(n)$	$\dfrac{1-(r\cos\omega_0)z^{-1}}{1-(2r\cos\omega_0)z^{-1}+r^2z^{-2}}$

Table 1 : z-transform pairs

$\delta(n)$ is defined to be the unit impulse function.
$u(n)$ is defined to be the unit step function.

Numbers in square brackets against the right margin of the following pages are a guide to the marking scheme.

1. (a) Explain what is meant by the following terms in the context of discrete-time systems: [4]
   (i) causal
   (ii) impulse response
   (iii) step response
   (iv) linear time invariant

   (b) What are the main differences between Finite Impulse Response (FIR) and Infinite Impulse Response (IIR) systems? [4]

   (c) A causal linear time-invariant discrete-time system $H(z)$ has impulse response $h(n)$ and step response $g(n)$.

   (i) Derive a relationship between $h(n)$ and $g(n)$. [4]

   (ii) For $H(z) = b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_N z^{-N}$, find expressions for the values of the samples $h(2N+4)$ and $g(N-1)$. [4]

   (iii) Write an expression for $\left|H(e^{j\omega})\right|$ for $\omega = \pi$. [4]

2.

   (a) Describe the overlap-save method for convolution of a time series $x(n)$ with a system impulse response $h(n)$. Include an illustrative diagram. [8]

   (b) Use the overlap-save block filtering method to calculate the output of a filter with impulse response [8]

$$h(n) = [1, 1]$$

   and input signal

$$x(n) = [1, 2, 3, -3, -2, -1, 0, -1, -2, -3, -2, -1, 0, 3, 2, 1, \ldots].$$

   Use only the first two data blocks to calculate the output. Explain your method clearly and include your working and any relevant diagrams.

   (c) Verify your answer to (b) using time-domain convolution. [4]

3.

(a) Consider the Hamming and Hanning window functions.

(i) Give their definitions and draw labelled sketches of each. [4]

(ii) Draw an illustrative sketch of the magnitude spectrum of each window. Hence describe the differences that would be expected in the performance of FIR filters designed using the window method employing the Hamming and Hanning window functions. [4]

(b) Consider a Bartlett window for *N* odd,

$$w_B(n) = \begin{cases} 1 - \dfrac{|2n-N+1|}{N+1} & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

and a rectangular window function

$$w_r(n) = \begin{cases} 1, & 0 \le n < (N+1)/2 \\ 0, & \text{otherwise} \end{cases}.$$

Let $N = 21$.

(i) Show that the Bartlett window can be obtained from [5]

$$\frac{2}{N+1}\left(w_r * w_r\right)$$

where * indicates convolution.

(ii) Draw a labelled sketch of the Bartlett window. [4]

(iii) Show that the Fourier Transform of $w_B(n)$ can be written [3]

$$W_B(k) = \frac{2\sin^2\left(0.25\omega(N+1)\right)}{(N+1)\sin^2\left(0.5\omega\right)} e^{-j\omega(N-1)/2}.$$

You may use the result that the sum of the first *M* terms of a geometric series $a + ar + ar^2 + \ldots + ar^{M-1}$ can be written as $\dfrac{a(1-r^M)}{1-r}$.

4.

Consider a continuous-time signal, $x_a(t)$, and the discrete-time signal, $x(n) = x_a(nT)$ for sampling period T and integer *n*, $-\infty < n < \infty$.

(a) By considering the Fourier Transform of $x_a(nT)$ and using the Discrete-time Fourier Transform, show that the Fourier Transform of $x(n)$ is periodic with period 2π. Include an illustrative sketch. [5]

(b) Show how $x_a(t)$ can be reconstructed from $x(n)$ stating any necessary conditions. [5]

(c) The input to a linear time invariant system is $x(n) = \begin{cases} 1, & n = 0 \\ 0, & \text{otherwise} \end{cases}$. [6]

The output from the system is $h(n) = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$.

Determine the frequency response of the system using the DTFT and sketch the magnitude response for $N = 3$.

You may use the identity $1 + a + a^2 + \ldots + a^{N-1} = \dfrac{1 - a^N}{1 - a}$.

(d) Consider the signal $y(n)$ which is formed from $x(n)$ with every m^{th} sample of $x(n)$ removed. For example, with $m = 4$ [4]

$$y(n) = [x(0), x(1), x(2), x(4), x(5), x(6), x(8), x(9), \ldots].$$

What is the maximum bandwidth of $y(n)$ that can be recovered without error?

5.

(a) Given a discrete-time signal $x(n)$, write down the expression for the Discrete Fourier Transform (DFT) $X(k)$. [2]

(b) State the symmetry properties of $X(k)$ if $x(n)$ is real. [3]

(c) Let $W_N = e^{-j2\pi/N}$. Show that [5]

$$\sum_{n=0}^{N-1} W_N^{-(k-l)n} = \begin{cases} N, & \text{for } k-l=rN, \ \ r \text{ is an integer} \\ 0, & \text{otherwise} \end{cases}$$

(d) By writing $x(n)$ in terms of complex exponentials, find the N-point DFT of the sequence [10]

$$x(n) = \cos(2\pi\, r\, n / N) \qquad\qquad n = 0,1,\ldots,N-1$$

where r is any integer in the range $0 \le r \le N-1$.

Determine $X(k)$ for the specific cases

(i) $k = r$

(ii) $k = N - r$.

6.

(a) Consider the M-fold decimator shown in Figure 1.

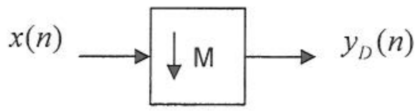

Figure 1

(i) Write an expression for $y_D(n)$ in terms of $x(n)$. [2]

(ii) Derive an expression for $Y_D(e^{j\omega})$ in terms of $X(e^{j\omega})$ and explain the effect of decimation in the frequency domain using appropriate diagrams for an example of M = 3. [7]

(iii) State the conditions on $x(n)$ and M that must be satisfied in order that $y_D(n)$ contains no aliasing errors. [2]

(b) Consider an input signal $x(n)$ with magnitude spectrum as shown in Figure 2. Draw the block diagram of a signal processing system containing multirate processing elements and a single filter $H(z)$ to perform decimation of $x(n)$ by a factor of M = 1.5. Sketch the magnitude spectra of all intermediate signals and of the output signal. [4]

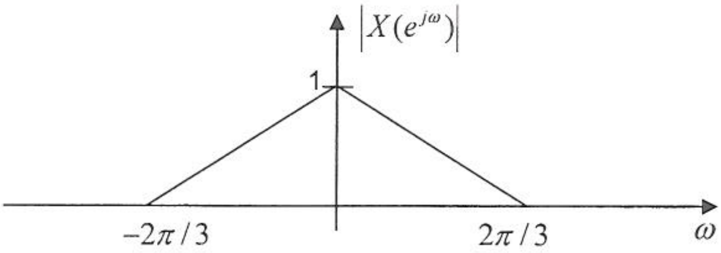

Figure 2

(c) Bellanger's formula for estimating the order of linear phase equiripple lowpass filters is given by [5]

$$N \approx \frac{2\log_{10}\left(\frac{1}{10\delta_1\delta_2}\right)}{3\,\Delta f}$$

where δ_1 and δ_2 are the passband and stopband ripple respectively, and Δf is the filter transition bandwidth normalized to the sampling frequency.

Use this formula to estimate the minimum order required for the filter $H(z)$ in part (b). You may assume that -40 dB of passband and stopband ripple is adequate.

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DIGITAL SIGNAL PROCESSING

DSP Solutions

2005/2006

1.

(a)
Causal systems have zero impulse response before the application of the input impulse.

Impulse response is the output of a system when the input is zero for all n except n=0.

Step response is the output of a system when the input is 1 for n>=0, 0 elsewhere.

Linear time invariant systems have a linear transfer function that is independent of time.

(b)
FIR has finite length impulse response; no poles other than at z=0; inherently stable; linear phase with appropriately symmetric coefficients

IIR has infinite length impulse response; poles anywhere on the z plane so can be unstable; non-linear phase response

(c)
Z-transform of input impulse is $X_H(z) = 1$

Z-transform of step input is $X_G(z) = \dfrac{1}{1 - z^{-1}}$

$$G(z) = H(z)X_G(z)$$

$$G(z) = H(z)\frac{1}{1 - z^{-1}}$$

$$G(z)(1 - z^{-1}) = H(z)$$

$$g(n) - g(n-1) = h(n)$$

(d)

$$h(n) = 0 \text{ for } n > N$$

We can write $g(n) = h(n) + g(n-1)$ so that $g(n) = \sum_{r=0}^{N} h(r)$ and therefore

$$g(N-1) = \sum_{r=0}^{N-1} h(r) = b_0 + b_1 + \ldots + b_{N-1}$$

(e)
Given $H(z) = b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_N z^{-N}$ and evaluating $H(z)$ at $z = -1$, corresponding to $\omega = \pi$ leads to

$$\left|H(e^{j\pi})\right| = \left|b_0 - b_1 + b_2 - \ldots b_N\right| = \left|\sum_{r=0}^{N} (-1)^r b_r\right|$$

2.
a)

1. Determine the FFT size, N, the data block size, L, and the impulse response block size, M such that N is an integer power of 2 and greater than $L+M-1$.
2. Form each data block using the last $M-1$ points from previous block followed by the L samples of the current block.
3. Zero-pad the impulse response block with $L-1$ zeros.
4. Compute and store the FFT of the zero-padded impulse response.
5. For each data block, calculate the IFFT of the product of the FFT of the zero-padded impulse response with the FFT of the zero-padded data block.
6. Determine the final output from the IFFTs discarding the first $M-1$ samples of each block and then concatenating.

b)
Let $x(n)$ be divided up into data blocks of length L=3 so that N = L + M -1 = 4.

The first two data blocks are therefore
x1 = [0, 1, 2, 3]
x2 = [3, -3, -2, -1]

Now we have the zero padded impulse response as
h = [1, 1, 0, 0]
and by FFT
H = [2, 1-j, 0, 1+j]

X1=[6, -2+j2, -2, -2-j2]
X2=[-3, 5+j2, 5, 5-j2]]

Next for the products we obtain

Y1 = H .* X1 = [12, j4, 0, -j4]
Y2 = H * X2 = [6, 7-j3, 0, 7+j3]

Next for the IFFTs we obtain

y1 = [3, 1, 3, 5]
y2 = [2, 0, -5, -3]

These outputs are then overlap-saved as follows:
1, 3, 5
            0, -5, -3
y = [1, 3, 5, 0, -5, -3]

3.

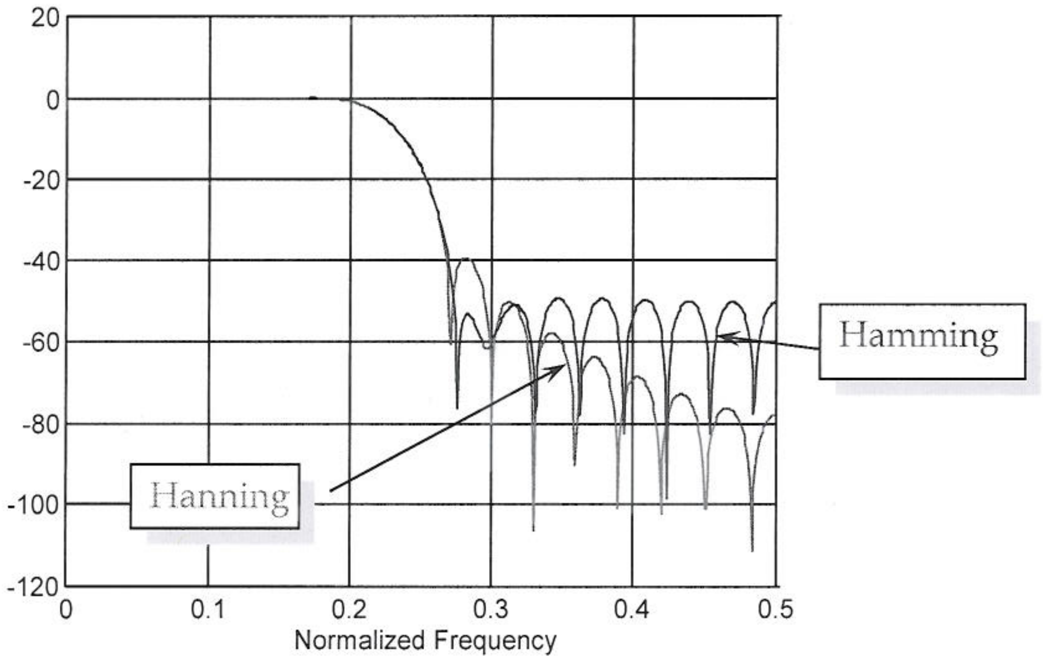
(a)

Hamming:

$$w(n) = \begin{cases} 0.54 + 0.46\cos\left(\dfrac{n\pi}{I}\right), & -I \le n \le I \\ 0, & \text{otherwise} \end{cases}$$

Hanning:

$$w(n) = \begin{cases} 0.5 + 0.5\cos\left(\dfrac{n\pi}{I}\right), & -I \le n \le I \\ 0, & \text{otherwise} \end{cases}$$



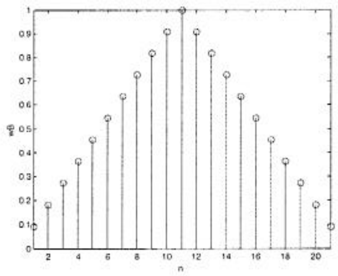
Hamming gives constant stopband attenuation.
Hanning gives increasing stopband attenuation with frequency but is worse close to the transition band.

b)

From the convolution sum

$$y(n) = w_r * w_r = \sum_m w_r(m) w_r(n-m)$$

or the graphical method we obtain two straight line segments of unit gradient. The maximum amplitude is equal to the number of points in w_r which can be adjust to 1 by the term $2/(N+1)$



To derive the FT, first consider the rectangular window of length *M* which has FT given by

$$W_r(k) = \sum_{n=0}^{M-1} e^{-jn\omega} = \frac{1-e^{-j\omega M}}{1-e^{-j\omega}} = \frac{2e^{-j\omega M/2}\sin(\omega M/2)}{2e^{-j\omega/2}\sin(\omega/2)} = \frac{\sin(\omega M/2)}{\sin(\omega/2)} e^{-j\frac{\omega}{2}(M-1)}$$

The convolution in the time domain gives rise to multiplication in the frequency domain so that

$$W_B(k) = \frac{2}{N+1}\left(\frac{\sin(\omega M/2)}{\sin(\omega/2)} e^{-j\frac{\omega}{2}(M-1)}\right)^2$$

and for our case we require $M = \dfrac{N+1}{2}$ hence

$$W_B(k) = \frac{2}{N+1}\left(\frac{\sin(\omega(N+1)/4)}{\sin(\omega/2)} e^{-j\frac{\omega}{4}(N-1)}\right)^2$$

$$= \frac{2}{N+1}\left(\frac{\sin^2(\omega(N+1)/4)}{\sin^2(\omega/2)} e^{-j\frac{\omega}{2}(N-1)}\right)$$

4.

(a)

From the FT of the continuous time signal $x_a(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X_a(j\Omega)e^{j\Omega t}d\Omega$

From the DTFT of the discrete-time signal $x(n)=\frac{1}{2\pi}\int_{n=-\pi}^{\pi}X(e^{j\omega})e^{j\omega n}d\omega$

Write the expression for $x_a(nT)$ as the sum of integrals over ranges of length $2\pi/T$

$$x(n)=\frac{1}{2\pi}\sum_{r=-\infty}^{\infty}\int_{(2r-1)\pi/T}^{(2r+1)\pi/T}X_a(j\Omega)e^{j\Omega nT}d\Omega$$

Change variable so as to replace Ω with $\Omega'+2\pi r/T$

$$x(n)=\frac{1}{2\pi}\sum_{r=-\infty}^{\infty}\int_{-\pi/T}^{\pi/T}X_a\left(j\left(\Omega'+\frac{2\pi r}{T}\right)\right)e^{j\Omega' nT}e^{2\pi nr}d\Omega'$$

Use $e^{j2\pi rn}=1 \quad \forall(r,n)$ integer, reverse order of sum and integration and use $\Omega'=\omega/T$

$$x(n)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{1}{T}\sum_{r=-\infty}^{\infty}X_a\left(j\left(\frac{\omega}{T}+\frac{2\pi r}{T}\right)\right)\right]e^{j\omega n}d\omega$$

Note that this is in the form of an IDFT so that

$$X(e^{j\omega})=\frac{1}{T}\sum_{r=-\infty}^{\infty}X_a\left(j\left(\frac{\omega}{T}+\frac{2\pi r}{T}\right)\right)$$

Which is periodic with period 2π

(b)

If the signal is bandlimited to Ω_0 such that $\Omega_0/2<\pi/T$ then

$X(e^{j\Omega T})=\frac{1}{T}X_a(j\Omega)$ in the range $-\pi/T\le\Omega\le\pi/T$.

Considering only this range and using the expression for the Fourier transform,

$$x_a(t)=\frac{1}{2\pi}\int_{-\pi/T}^{\pi/T}TX(e^{j\Omega T})e^{j\Omega t}d\Omega$$

Using the DTFT

$$X(e^{j\Omega T})=\sum_{n=-\infty}^{\infty}x_a(nT)e^{-j\Omega nT}$$

we obtain

$$x_a(t)=\frac{T}{2\pi}\int_{-\pi/T}^{\pi/T}\left[\sum_{n=-\infty}^{\infty}x_a(nT)e^{-j\Omega nT}\right]e^{j\Omega t}d\Omega$$

$$x_a(t) = \sum_{n=-\infty}^{\infty} x_a(nT)\left[\frac{T}{2\pi}\int_{-\pi/T}^{\pi/T} e^{j\Omega(t-nT)}d\Omega\right]$$

$$= \sum_{n=-\infty}^{\infty} x_a(nT)\frac{\sin\left(\frac{\pi}{T}(t-nT)\right)}{\frac{\pi}{T}(t-nT)}$$

Which represents a lowpass filtering operation through an ideal bandpass filter with cutoff frequency π/T

(c)

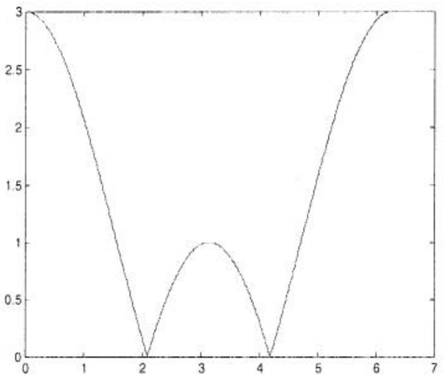
$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h(n)e^{-j\omega n} = \sum_{n=0}^{N-1} e^{-j\omega n}$$

$$= \frac{1-e^{-j\omega N}}{1-e^{j\omega}} = \frac{1-\cos\omega N + j\sin\omega N}{1-\cos\omega + j\sin\omega}$$

For the magnitude response

$$\left|H(e^{j\omega})\right| = \frac{\sqrt{(1-\cos\omega N)^2 + (\sin\omega N)^2}}{\sqrt{(1-\cos\omega)^2 + (\sin\omega)^2}} = \frac{\sin\omega^{N}\!/\!_2}{\sin{}^{\omega}\!/\!_2}$$

This response has nulls at $\omega = \pm\frac{2\pi}{3}, \pm\frac{4\pi}{3}$ etc



(d) We are here considering non-uniform sampling and we note that the requirement is for the Nyquist sampling criterion to be satisfied on average.

In this case we are left with 3 samples every 4T so that the effective sampling period is on average 4T/3. Therefore the effective sampling frequency is 3/4T and the bandwidth that satisfies the Nyquist sampling criterion on average is 3/8T.

In general, we can write the Nyquist bandwidth as (m-1)/(2mT).

5.

(a)

$$X(k)=\sum_{n=0}^{N-1}x(n)e^{-j\omega nk/N}$$

(b)

When the input signal is real $X(k)=X^*(-k)$

(c)

$$\sum_{n=0}^{N-1}W_N^{-(k-l)n} \quad = \sum_{n=0}^{N-1}e^{j2\pi(k-l)n/N}$$

$$=\sum_{n=0}^{N-1}\left(e^{j2\pi kn/N}.e^{-j2\pi ln/N}\right)$$

$$=\sum_{n=0}^{N-1}\left(\cos(2\pi kn/N)+j\sin(2\pi kn/N)\right)\left(\cos(2\pi ln/N)-j\sin(2\pi ln/N)\right)$$

$$=\sum_{n=0}^{N-1}\left(\cos(2\pi(k-l)n/N)+j\sin(2\pi(k-l)n/N)\right)$$

Using symmetry of the cos and sin, the sum gives zero except when k-l i=rN in which case the cos terms are all 1 and the sin terms all zero so that we obtain

$$\sum_{n=0}^{N-1}1=N, \quad \text{for } k-l=rN, \;\; r \text{ is an integer}$$

$$\sum_{n=0}^{N-1}\left(e^{j2\pi kn/N}.e^{-j2\pi ln/N}\right)=0, \quad \text{otherwise}$$

(d)

$$x(n)=\cos(2\pi rn/N)$$

$$=\frac{1}{2}\left(e^{j2\pi rn/N}+e^{-j2\pi rn/N}\right)=\frac{1}{2}\left(W_N^{-rn}+W_N^{rn}\right)$$

This leads to the DFT

$$X(k)=\frac{1}{2}\left(\sum_{n=0}^{N-1}W_N^{-(r-k)n}+\sum_{n=0}^{N-1}W_N^{(r+k)n}\right)$$

And simplifies to

$$X(k)=\begin{cases}N/2 & \text{for } k=r\\ N/2 & \text{for } k=N-r\\ 0 & \text{otherwise}\end{cases}$$

6.

For the decimator, $y_D(n) = x(Mn)$.

To analyze in the frequency domain, first write

$$Y_D(z) = \sum_{n=-\infty}^{\infty} x(Mn) z^{-n}$$

Then introduce the intermediate signal

$$x_1(n) = \begin{cases} x(n), & \text{when n is a multiple of M} \\ 0, & \text{otherwise} \end{cases}$$

Then

$$Y_D(z) = \sum_{n=-\infty}^{\infty} x_1(Mn) z^{-n} = \sum_{k=-\infty}^{\infty} x_1(k) z^{-k/M}$$

So that

$$Y_D(z) = X_1(z^{1/M})$$

And

$$X_1(z) = \frac{1}{M} \sum_{k=0}^{M-1} \sum_{n=-\infty}^{\infty} x(n) W^{-kn} z^{-n} = \frac{1}{M} \sum_{k=0}^{M-1} \sum_{n=-\infty}^{\infty} x(n)(zW^k)^{-n}$$

Thus

$$Y_D(z) = \frac{1}{M} \sum_{k=0}^{M-1} X(z^{1/M} W^k)$$

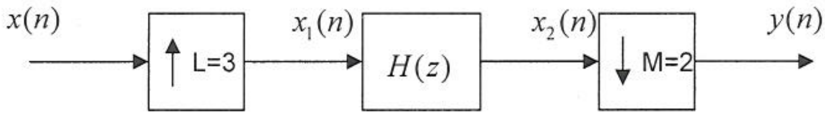
Which leads to

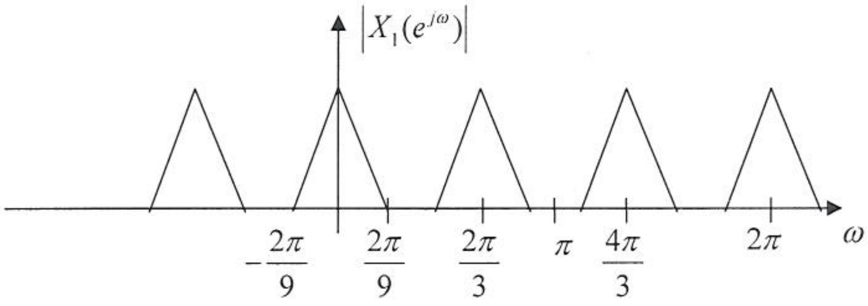
$$Y_D(e^{jw}) = \frac{1}{M} \sum_{k=0}^{M-1} X(e^{j(\omega - 2\pi k)/M})$$

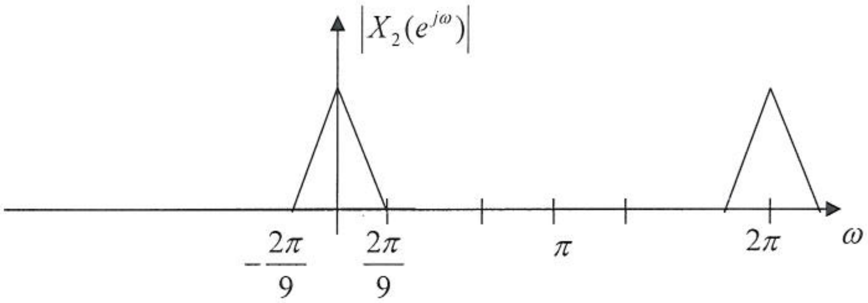
This is equivalent to superimposing M-1 copies of the spectrum after stretching in the frequency axis the original spectrum by an factor M. An amplitude factor of 1/M is also applied.

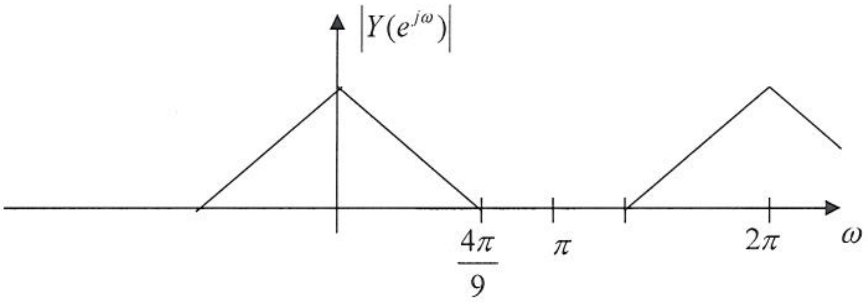
For no aliasing in $y_D(n)$ we require that $x(n)$ is bandlimited to $\pi/3$.

(b) Here we require expansion by 3 and decimation by 2.









(c) Transition bandwidth required is pi/3 corresponding to $\Delta f = 1/9$. For -40 dB ripple we need $\delta_1 = \delta_2 = 0.01$. This leads to $N \approx 18$.