IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING

EXAMINATIONS 2017

WAVELETS AND APPLICATIONS SOLUTIONS

SOLUTIONS

1. (a) Since the sequence $p[n]$ doesn't change $x[n]$, we have for the upper branch:

$$Y_0(z) = \frac{1}{2}\left[H_0(z^{1/2})X(z^{1/2}) + H_0(-z^{1/2})X(-z^{1/2})\right].$$

Since the z-transform of $q[n]x[n] = (-1)^n x[n]$ is $X(-z)$, we have for the lower branch.

$$Y_1(z) = \frac{1}{2}\left[H_1(z^{1/2})X(-z^{1/2}) + H_1(-z^{1/2})X(z^{1/2})\right].$$

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Consequently,

$$\hat{X}(z) = \frac{G_0(z)}{2}\left[H_0(z)X(z) + H_0(-z)X(-z)\right] + \frac{G_1(z)}{2}\left[H_1(z)X(-z) + H_1(-z)X(z)\right]$$

and the two PR conditions are:

$$G_0(z)H_0(z) + G_1(z)H_1(-z) = 2$$

and

$$G_0(z)H_0(-z) + G_1(z)H_1(z) = 0.$$

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(b) If you replace $H_1(z)$ with $\hat{H}_1(-z)$, the PR conditions found in part (a) converge to the traditional ones:

$$G_0(z)H_0(z) + G_1(z)\hat{H}_1(z) = 2$$

and

$$G_0(z)H_0(-z) + G_1(z)\hat{H}_1(-z) = 0.$$

Therefore, if $G_0(z) = (1+z^{-1})/\sqrt{2}$ and $H_0(z) = (1+z)/\sqrt{2}$, the perfect reconstruction is achieved with $\hat{H}_1(z) = (1-z)/\sqrt{2}$ and $G_1(z) = (1-z^{-1})/\sqrt{2}$. These are also the shortest possible filters. Consequently $H_1(z) = \hat{H}_1(-z) = (1+z)/\sqrt{2}$. So the end effect of the modulation is that the two analysis filters are now the same. Please note that this is not always the case.

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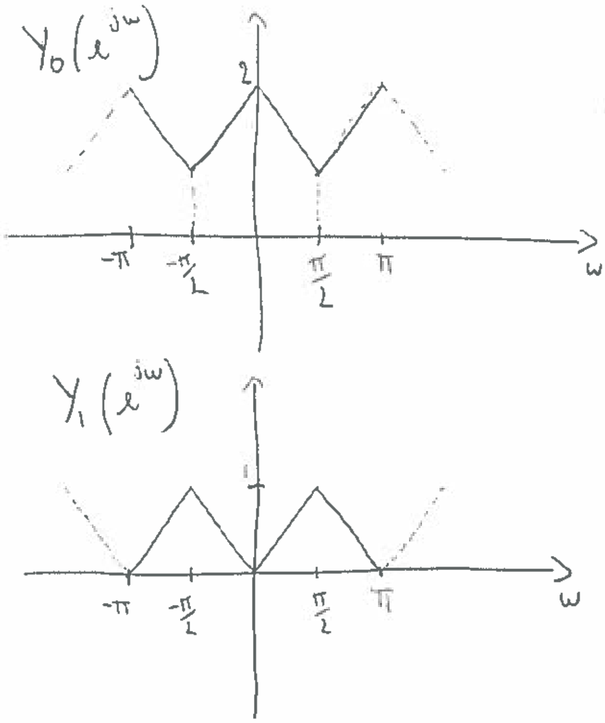
(c) Similarly to the case of part (b), PR conditions are satisfied when $H_1(z) = H_0(z)$. That means that $H_1(z)$ is a half-band ideal low-pass filter with gain 2..

(d) The sketch of $Y_0(e^{j\omega})$ and $Y_1(e^{j\omega})$ is as follows:

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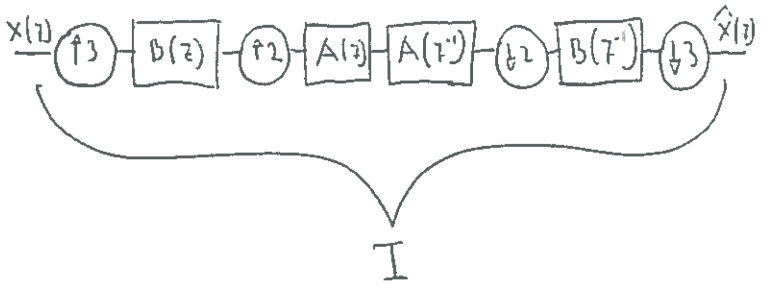
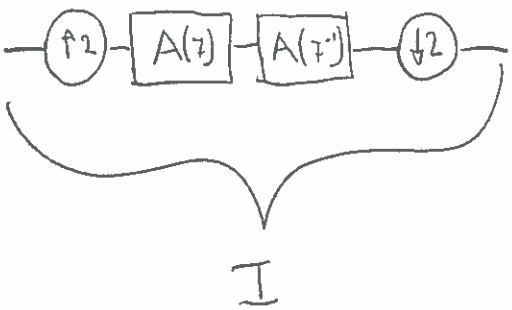
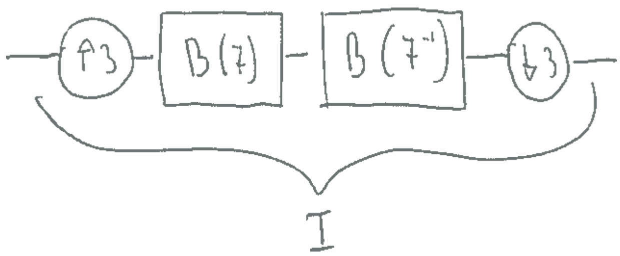
2. (a) This requires merely application of the Noble Identities. We obtain $M = N = 6$, $C(z) = A(z)B(z^2)$, and $D(z) = C(z^{-1})$.

(b) The whole system is idempotent if and only if $\langle a[n], a[n-2k]\rangle = \delta_k$. This can be shown by noticing that the system is idempotent if and only if the subsystem shown below is the identity.




This is satisfied when the two following subsystems are themselves identity.

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The requirement that the first of these two systems be the identity implies:

$$\frac{X(z)}{2}\left[A(z^{1/2})A(z^{-1/2}) + A(-z^{1/2})A(-z^{-1/2})\right] = X(z)$$

which yields

$$A(z)A(z^{-1}) + A(-z)A(-z^{-1}) = 2$$

and this last condition is satisfied when $a[n]$ is such that $\langle a[n], a[n-2k]\rangle = \delta_k$.

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The second sub-system is by construction idempotent.
First note that

$$\langle b[n], b[n-3k]\rangle = \delta_k \Leftrightarrow \frac{1}{3}\sum_{n=0}^{2} B(W_3^n z^{1/3}) = 1 \qquad (1)$$

with $W_N^k = e^{-j2\pi k/N}$. Then the identity condition on the system implies

$$\frac{X(z)}{3}\left[\sum_{n=0}^{2} B(W_3^n z^{1/3})\right] = X(z)$$

and this condition is satisfied when (1) is satisfied.

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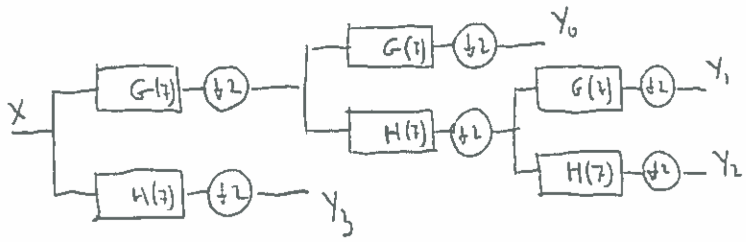
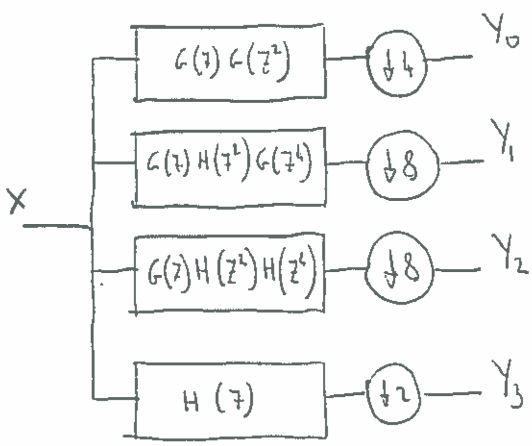
(c) The block diagram of the system using two channel filter banks and the equivalent single-level four-channel filter bank are given below:




(c) i.




(c) ii.

Solutions

3. (a) By the projection theorem, we know that if $p_s(t)$ is the orthogonal projection then $\epsilon_s(t) = x(t) - p_s(t)$ must be orthogonal to $V = \text{span}(\{\varphi_1(t), \varphi_2(t)\})$. In our case $p_s(t) = x(0) + x'(0)t = 1 + t$ and it is immediately evident that this is not the case since, for example, $\langle e_s(t), \varphi_1(t)\rangle \neq 0$.
Specifically:

$$\langle e_s(t), \varphi_1(t)\rangle = \int_0^1 (e^t - 1 - t)dt = e - 2.5 \neq 0.$$

(b) i. We have:

$$\langle \varphi_1, \varphi_2\rangle = \langle \varphi_2, \varphi_1\rangle = \int_0^1 t dt = \frac{1}{2}.$$

Moreover, $\|\varphi_1\|^2 = 1$ and $\|\varphi_2\|^2 = \frac{1}{3}$. Consequently,

$$1 = \langle \varphi_1(t), \tilde{\varphi}_1(t)\rangle = a_{1,1}\langle \varphi_1, \varphi_1\rangle + a_{1,2}\langle \varphi_1, \varphi_2\rangle = a_{1,1} + \frac{a_{1,2}}{2}$$

$$0 = \langle \varphi_2(t), \tilde{\varphi}_1(t)\rangle = a_{1,1}\langle \varphi_2, \varphi_1\rangle + a_{1,2}\langle \varphi_2, \varphi_2\rangle = \frac{a_{1,1}}{2} + \frac{a_{1,2}}{3}$$

and $a_{1,1} = 4$, $a_{1,2} = -6$.
Similarly, we have: $a_{2,1} = -6$, $a_{2,2} = 12$.
The two dual-basis functions can then be written as:

$$\tilde{\varphi}_1(t) = 4 - 6t$$

and

$$\tilde{\varphi}_2(t) = 12t - 6$$

ii. A.

$$\langle x(t), \tilde{\varphi}_1(t)\rangle = 4\int_0^1 e^t dt - 6\int_0^1 te^t dt = 4e - 10$$

$$\langle x(t), \tilde{\varphi}_2(t)\rangle = 12\int_0^1 te^t dt - 6\int_0^1 e^t dt = 18 - 6e$$

B.

$$x_v(t) = \sum_{i=1}^{3} \langle x(t), \tilde{\varphi}_i(t)\rangle \varphi_i(t) = 4e - 10 + (18 - 6e)t.$$

C. We need to verify that $\langle \epsilon_v(t), \varphi_1(t)\rangle = 0$ and that $\langle \epsilon_v(t), \varphi_2(t)\rangle = 0$.
We have

$$\langle \epsilon_v(t), \varphi_1(t)\rangle = \int_0^1 e^t dt - (4e-10)\int_0^1 dt - (18-6e)\int_0^1 t dt = e - 1 - 4e + 10 - 9 + 3e = 0$$

and

$$\langle \epsilon_v(t), \varphi_2(t)\rangle = \int_0^1 te^t dt - (4e-10)\int_0^1 t dt - (18-6e)\int_0^1 t^2 dt = 0$$

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4. (a) Since $f(t)$ is non-zero in an interval of size 1/8, then the right scale is $k = -3$ and

$$f(t) = c_{-3,2}\varphi_{-3,2}(t)$$

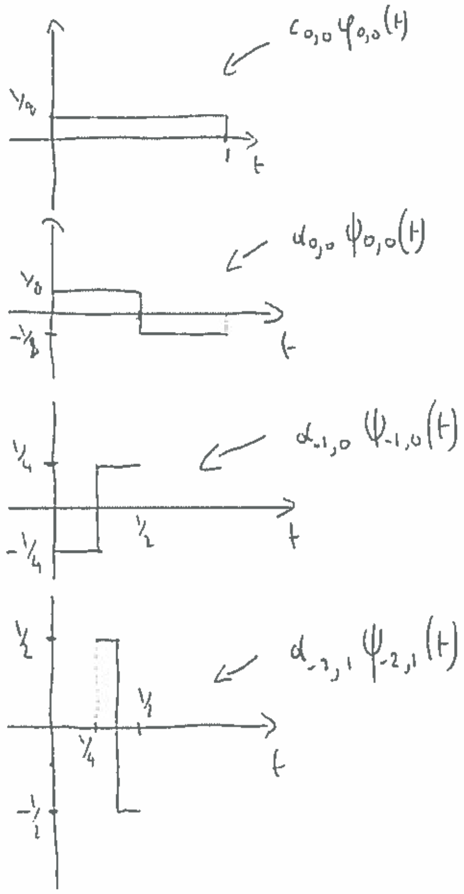
with $c_{-3,2} = \sqrt{2}/4$ (recall that $\varphi_{-3,2}(t) = 2^{3/2}\varphi(8t-2)$).

(b) Since $k = -3$, we need to consider only the subspaces $W_{-2}$, $W_{-1}$, $W_0$, and $V_0$. Therefore

$$f(t) = \sum_{n\in\mathbb{Z}} c_{0,n}\varphi_{0,n}(t) + \sum_{j=-2}^{0}\sum_{n\in\mathbb{Z}} d_{j,n}\psi_{j,n}(t).$$

By inspection, we see that only $c_{0,0}, d_{0,0}$, $d_{-1,0}$ and $d_{-2,1}$ are non-zero. Moreover, $c_{0,0} = \langle f(t), \varphi_{0,0}(t)\rangle = 1/8$, $d_{0,0} = \langle f(t), \psi_{0,0}(t)\rangle = 1/8$ $d_{-1,0} = \langle f(t), \psi_{-1,0}(t)\rangle = -\sqrt{2}/8$. and $d_{-2,1} = \langle f(t), \psi_{-1,0}(t)\rangle = 1/4$.

(c) The plots are shown below:




(d) We have that $|c_{0,0}|^2 + |d_{0,0}|^2 + |d_{-1,0}|^2 + |d_{-2,1}|^2 = 1/64 + 1/64 + 2/64 + 1/16 = 1/8$ which clearly corresponds to $\|f(t)\|^2 = 1/8$.