Paper Number(s): **E2.6**

IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE
UNIVERSITY OF LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2000

EEE PART II: M.Eng., B.Eng. and ACGI

CONTROL ENGINEERING

Wednesday, 21 June 2000, 2:00 pm

There are FIVE questions on this paper.

Answer THREE questions.

All questions carry equal marks.

Time allowed: 2:00 hours

Corrected Copy

Examiners: Prof D.J.N. Limebeer, Dr A. Astolfi

(1) Consider the system of interconnected tanks given in figure 1. The water levels in the three tanks are $h_1(t), h_2(t)$ and $h_3(t)$ metres respectively. Water from an external source flows into the first tank at a rate $q_{in}(t)\ m^3 s^{-1}$. The object is to maintain the water level $h_3(t)$ in the third tank close to a prescribed level h^* by controlling the flow rate $q_{in}(t)$.

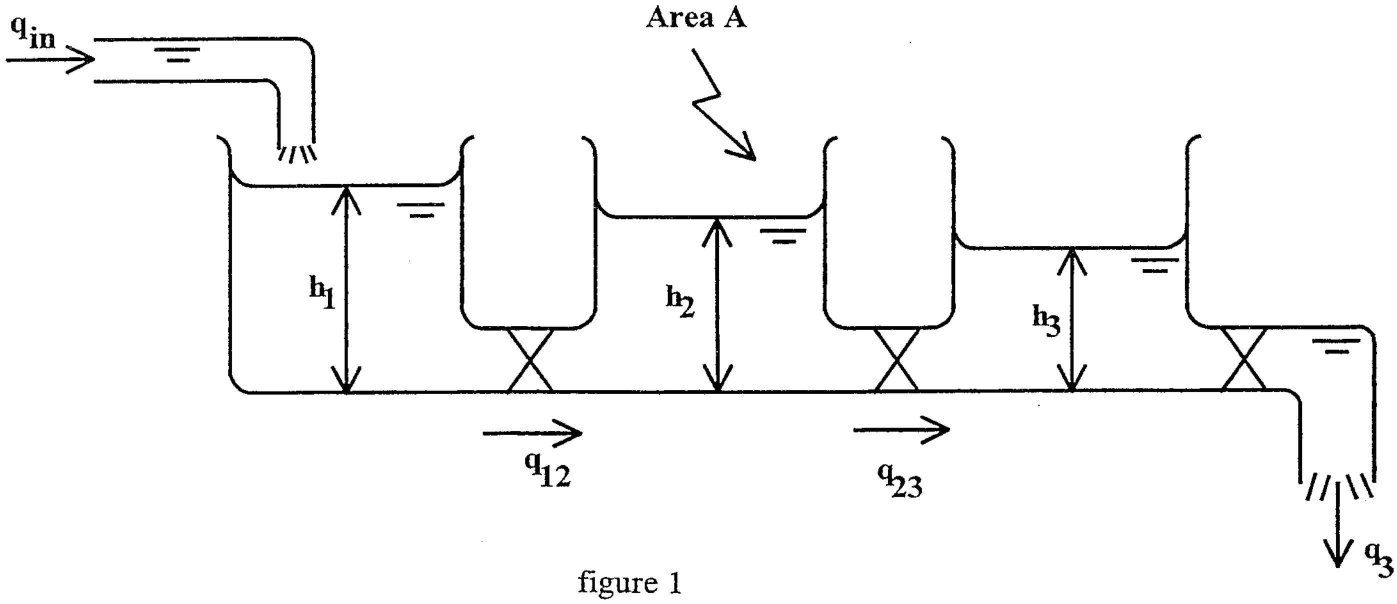

figure 1

The flow rates $q_{12}(t)$ and $q_{23}(t)$ between the tanks, and the flow rate $q_3(t)$ from the third tank are given by:

$$
\begin{aligned}
q_{12}(t) &= k(h_1(t) - h_2(t)) \\
q_{23}(t) &= k(h_2(t) - h_3(t)) \\
q_3(t) &= kh_3(t)
\end{aligned}
$$

in which k is a constant. The cross-sectional area of each tank is $A\ m^2$.

(a) By using the three heights as state-variables, derive a state-space model for the system which relates the input $q_{in}(t)$ to the output $h_3(t)$. [7]

(b) What is the constant input flow rate q^* that results in the constant output height $h_3(t) = h^*$? [3]

(c) The input flow rate $q_{in}(t)$ is now controlled according to the feedback law:

$$q_{in}(t) = q^* + F(h^* - h_3(t)).$$

What is the range of values of the gain parameter F for which the closed-loop is stable? [10]

(2) Large space structures have to be stabilised against vibration. One way to achieve this is to use a vibration absorber such as the one illustrated in Figure 2.

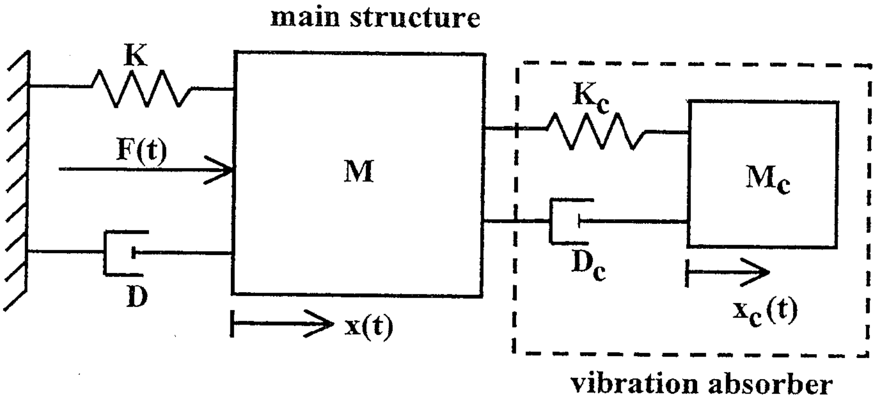

figure 2

a) Derive a state-space model which links the input force $F(t)$ to the structure displacement $x(t)$. [10]

b) Show that the structure may be represented by the feedback loop shown in figure 3 below:

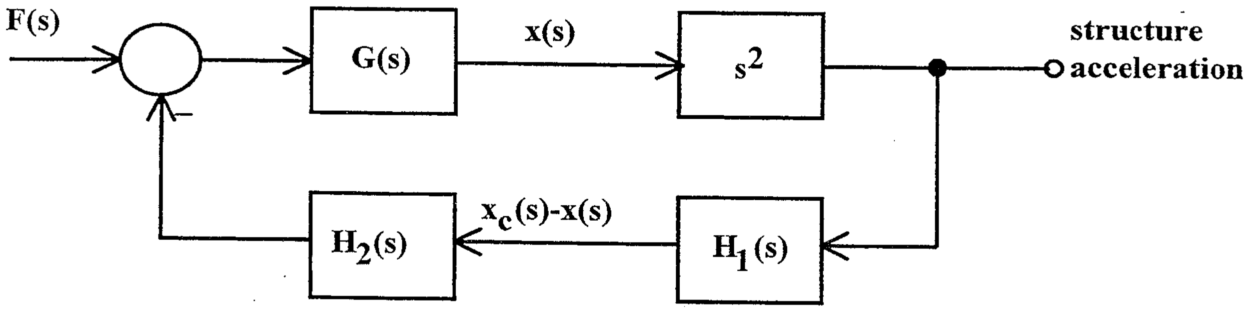

figure 3

and find $G(s)$, $H_1(s)$ and $H_2(s)$. [8]

c) Briefly comment on the stability of this system – argue by analogy with electric circuits. (*Hint:* What do you know about the stability of passive circuits – circuits comprising inductors, capacitors and resistors alone?) [2]

(3) Consider an ecology that is comprised of rabbits and foxes. The number of rabbits is denoted x_1, and if left alone the rabbit population would grow indefinitely according to:

$$\dot{x}_1 = kx_1 \qquad\qquad k > 0\,.$$

However, with foxes present this equation becomes

$$\dot{x}_1 = kx_1 - ax_2\,,$$

where x_2 is the number of foxes – the foxes eat rabbits! Now, since foxes must have rabbits to feed on, we have

$$\dot{x}_2 = -hx_2 + bx_1\,.$$

a) What are the requirements on a, b, h and k for a stable system? [4]

b) What will happen if $k > h$? [4]

c) Suppose $a=b=2$, $k=1$ and $h=4$. What is the equilibrium composition of rabbits and foxes? [4]

d) Suppose we take the rabbit food supply into account so that

$$\begin{aligned}\dot{x}_1 &= kx_1 - ax_2 + ax_3\\ \dot{x}_2 &= bx_1 - hx_2\\ \dot{x}_3 &= -\gamma x_1 + \beta x_3\end{aligned}$$

where x_3 represents the supply of rabbit food. If $k=1$, $h=3$, $\beta=-0.5$ and $a=b=2$, what values of γ will cause the demise of the rabbits? [8]

(4) Consider the automatic gain control for the compact disc radial position loop illustrated in Figure 4:

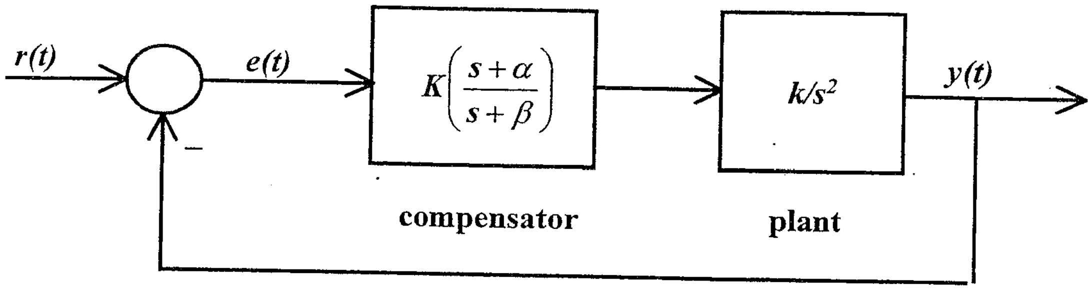

figure 4

The plant gain k is unknown, variable, but always positive, while the compensator gain K can be adjusted. The input to the loop is a sinusoidal "wobble" signal of the form $r(t) = \sin \omega_b t$.

(a) Show that the transfer function between $r(t)$ and $e(t)$ is

$$\frac{e(s)}{r(s)} = \frac{s^2(s+\beta)}{s^2(s+\beta)+kK(s+\alpha)}.$$ [6]

(b) To ensure stability, show that the compensator must be a lead network and that the loop gain must be positive. [8]

(c) The aim of the automatic gain control is to ensure the $e(t)$ lags r(t) *by* 90^o when $r(t) = \sin \omega_b t$. Find the value of the gain K that will achieve this. [6]

(5) Figure 5 below shows a plant equipped with a temperature controller, $K(s)=1/s\tau$, and which is subjected to disturbances D(s).

a) What is the value of τ that will give the closed-loop system a gain margin of 3? [8]

b) Find the transfer function that relates $\theta_o(s)$ to $D(s)$. [6]

b) Suppose the disturbance D(s) is sinusoidal with frequency 0.3r/s.
How much will the feedback loop amplify the disturbances at this frequency. [6]

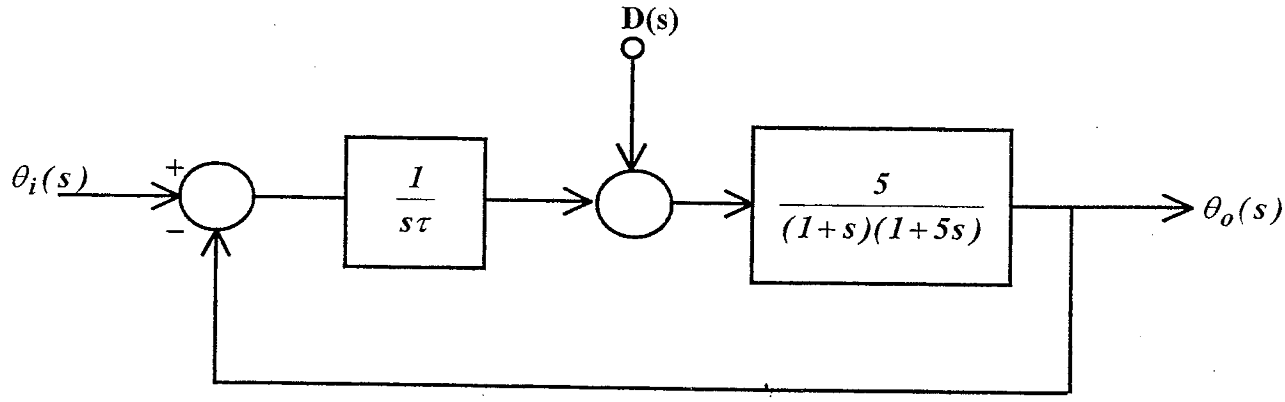

figure 5

Question 1

(a) Using the fact that $A\dot{h}(t)$ is the net flow-rate into a tank we have

$$A\dot{h}_1 = q_{in} - k(h_1 - h_2)$$

$$A\dot{h}_2 = k(h_1 - h_2) - k(h_2 - h_3)$$

$$A\dot{h}_3 = k(h_2 - h_3) - kh_3$$

Setting $x_1 = h_1$; $x_2 = h_2$; $x_3 = h_3$

we get:

$$\dot{x}_1 = -\frac{k}{A}(x_1 - x_2) + q_{in}/A$$

$$\dot{x}_2 = \frac{k}{A}(x_1 - 2x_2 + x_3)$$

$$\dot{x}_3 = \frac{k}{A}(x_2 - x_3) - \frac{k}{A}x_3$$

or in matrix form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -k/A & k/A & 0 \\ k/A & -2k/A & k/A \\ 0 & k/A & -2k/A \end{bmatrix} x + \begin{bmatrix} 1/A \\ 0 \\ 0 \end{bmatrix} [q_{in}]$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

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(b) Under steady-state conditions all the flow rates must be the same. Also

$$q_3^* = kh^*$$

Hence

$$\underline{q_{in}^* = kh^*}$$

3/20

(c) If

$$q_{in} = q^* + K(h^* - h_3)$$

we get

$$q_{in} = q^* + K(h^* - x_3)$$
$$= (k+K)h^* - Kx_3$$

and so

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -k/A & k/A & -K/A \\ k/A & -2k/A & k/A \\ 0 & k/A & -2k/A \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} \frac{k+K}{A} \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} h^* \end{bmatrix}$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The closed-loop poles are given by:

$$\begin{vmatrix} \lambda + k/A & -k/A & K/A \\ -k/A & \lambda + 2k/A & -k/A \\ 0 & -k/A & \lambda + 2k/A \end{vmatrix}$$

$$= (\lambda + k/A)\left[(\lambda + 2k/A)(\lambda + 2k/A) - k^2/A^2\right]$$

$$+ k/A\left[-\frac{k}{A}(\lambda + 2k/A) + kK/A^2\right]$$

$$= \lambda^3 + \lambda^2\left(\frac{4k}{A} + \frac{k}{A}\right) + \lambda\left(\frac{4k^2}{A^2} - \frac{k^2}{A^2} + \frac{4k^2}{A^2} - \frac{k^2}{A^2}\right) + 3\frac{k^3}{A^3} - \frac{2k^3}{A^3} + \frac{k^2K}{A^3}$$

$$= \lambda^3 + \frac{5k}{A}\lambda^2 + 6\frac{k^2}{A^2}\lambda + \frac{k^3}{A^3} + \frac{k^2K}{A^3}$$

From the Routh criterion:

$$\begin{array}{ll} 1 & \frac{6k^2}{A^2} \\ \frac{5k}{A} & \frac{k^3}{A^3} + \frac{k^2K}{A^3} \\ \left(30k^3/A^3 - \frac{k^3}{A^3} - \frac{k^2K}{A^3}\right)\frac{A}{5k} & = \left(\frac{29k^3}{A^3} - \frac{k^2K}{A_3}\right)\left(\frac{A}{5k}\right) = (29k - K)\left(\frac{k}{A^2 5}\right) \\ \frac{k^2}{A^3}(k + K) & \end{array}$$

$$\Rightarrow \quad 29k > +K > -k$$

10/20

Question 2

(a) Balancing forces on M and M_c gives:

$$F - Kx - D\dot{x} + K_c(x_c - x) + D_c(\dot{x}_c - \dot{x}) = M\ddot{x} \quad (1)$$

$$(x - x_c)K_c + (\dot{x} - \dot{x}_c)D_c = \ddot{x}_c M_c \quad (2)$$

$x_1 = x; \quad x_2 = \dot{x}; \quad x_3 = x_c; \quad x_4 = \dot{x}_c$

$\dot{x}_1 = x_2$

$\dot{x}_3 = x_4$

$$\dot{x}_2 = F/M - K/M\, x_1 - D/M\, x_2 + (x_3 - x_1)\frac{K_c}{M} + (x_4 - x_2)D_c/M$$

$$\dot{x}_4 = K_c/M_c\,(x_1 - x_3) + D_c/M_c\,(x_2 - x_4)$$

from which there results

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -(K+K_c)/M & -(D+D_c)/M & K_c/M & D_c/M \\ 0 & 0 & 0 & 1 \\ K_c/M & D_c/M & -K_c/M & -D_c/M \end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} + \begin{bmatrix}0\\ 1/M\\ 0\\ 0\end{bmatrix}F$$

$$y = \begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}$$

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(b) From (1) we see that

$$F + (K_c + sD_c)(x_c - x) = x(s^2M + sD + K)$$

$$\therefore \quad H_2(s) = -(K_c + sD_c)$$

$$G(s) = 1/(s^2 M + sD + K)$$

From (2)

$$x(K_c + sD_c) = (s^2 M_c + sD_c + K_c)\, x_c$$

$$\therefore \quad x_c = \frac{(K_c + sD_c)\,x}{s^2 M_c + sD_c + K_c}$$

$$\therefore \quad x - x_c = \frac{(s^2 M_c + sD_c + K_c - K_c - sD_c)\,x}{s^2 M_c + sD_c + K_c}$$

$$= \frac{s^2 M_c\, x}{s^2 M_c + sD_c + K_c}$$

$$\therefore \quad H_1(s) = \frac{-M_c}{s^2 M_c + sD_c + K_c}$$

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(c) System stability follows from passivity

2/20

Question 3

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(a) $$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}k & -a\\ b & -h\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

$$\begin{vmatrix}\lambda - k & a\\ -b & \lambda + h\end{vmatrix} = (\lambda-k)(\lambda+h) + ab$$
$$= \lambda^2 + \lambda(h-k) + ab - kh.$$

For stability: $h > k$

$ab > kh$

4/20

(b) if $k > h$, the system is unstable and the rabbit population will explode. 4/20

(c) $$\begin{bmatrix}0\\ 0\end{bmatrix} = \begin{bmatrix}1 & -2\\ 2 & -4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

$\Rightarrow \quad x_1 = 2x_2$

4/20

There will be two rabbits per fox.

(d) $$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}k & -a & a\\ b & -h & 0\\ -\gamma & 0 & \beta\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

$$= \begin{bmatrix}1 & -2 & 2\\ 2 & -3 & 0\\ -\gamma & 0 & -0.5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

$$\begin{vmatrix} \lambda - 1 & 2 & -2 \\ -2 & \lambda + 3 & 0 \\ \gamma & 0 & \lambda + \frac{1}{2} \end{vmatrix}$$

$$= (\lambda - 1)(\lambda + 3)(\lambda + \tfrac{1}{2}) + 4(\lambda + \tfrac{1}{2}) + 2\gamma(\lambda + 3)$$

$$= \lambda^3 + 2.5\lambda^2 + \lambda(2 + 2\gamma) - 0.5 + 6\gamma$$

Routh table:

1	$2+2\gamma$
2.5	$6\gamma - \frac{1}{2}$
$(4\frac{1}{2} - \gamma)/2.5$	
$6\gamma - \frac{1}{2}$	

For stability

$$-\tfrac{1}{12} < \gamma < 4\tfrac{1}{2}$$

8/20

Question 4

(a)

$$\frac{e(s)}{r(s)} = \frac{1}{1 + C(s)\,G(s)}$$

$$= \frac{1}{1 + K\left(\frac{s+\alpha}{s+\beta}\right) k/s^2}$$

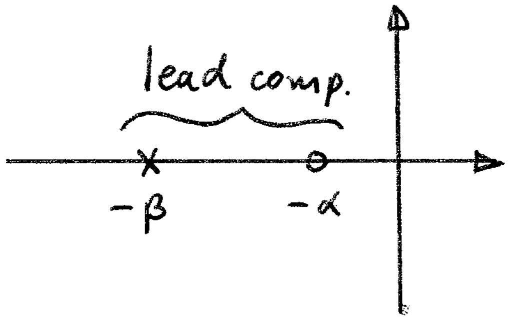
$$= \frac{s^2(s+\beta)}{s^2(s+\beta) + kK(s+\alpha)}$$

6/20

(b) $\text{clcp} = s^3 + \beta s^2 + kKs + kK\alpha$

$$\begin{array}{|ll} 1 & kK \\ \beta & kK\alpha \\ kK(\beta-\alpha)/\beta & \\ kK\alpha & \end{array}$$

For stability $\beta > 0$; $Kk\alpha > 0$; $\beta > \alpha$



8/20

(c)

$$\frac{e}{r}(j\omega_b) = \frac{-\omega_b^2\,(\beta + j\omega_b)}{-\omega_b^2\,(j\omega_b + \beta) + kK\,(\alpha + j\omega_b)}$$

$$= \frac{-\omega_b^2\,(\beta + j\omega_b)}{j\omega_b\,(kK - \omega_b^2) + kK\alpha - \omega_b^2\beta}$$

We need 90° of phase lead from

$$(\beta + j\omega_b)\left[(kK\alpha - \omega_b^2\beta) - j\omega_b\,(kK - \omega_b^2)\right]$$

and so the real part must be zero. That is

$$0 = \beta(kK\alpha - \omega_b^2\beta) + \omega_b^2\,(kK - \omega_b^2)$$

$$0 = K\,(\beta k\alpha + \omega_b^2 k) - \omega_b^2\beta^2 - \omega_b^4$$

$$K = \frac{\omega_b^2\,(\beta^2 + \omega_b^2)}{k\,(\omega_b^2 + \beta\alpha)}$$

6/20

Question 5

$\frac{10}{11}$

(a) $$\frac{\theta_o(s)}{\theta_i(s)} = \frac{5}{s\tau(1+s)(1+5s) + 5}$$

$$= \frac{5}{5\tau s^3 + 6\tau s^2 + s\tau + 5}$$

If τ is to have a gain margin of 3, it must cause

$$5\tau s^3 + 6\tau s^2 + s\tau + 5\times 3$$

to be marginally stable.

5τ	τ
6τ	15
$(6\tau^2 - 75\tau)/6\tau$	
15	

$$6\tau = 75$$

$$\underline{\tau = 12.5}$$

$\frac{8}{20}$

(b) $$\theta_o(s) = \frac{5}{(1+s)(1+5s)}\left(D - \theta_o/\tau s\right)$$

$$\theta_o(s)\left[(1+s)(1+5s)\tau s + 5\right] = 5\tau s D$$

and so

$$\Theta_o(s)/D(s) = \frac{5\tau s}{(1+s)(1+5s)\tau s + 5}$$

$$= \frac{5\tau s}{5\tau s^3 + 6\tau s^2 + \tau s + 5}$$

6/20

(c)

$$\left|\frac{\Theta_o(j\omega)}{D(j\omega)}\right|_{\omega=0.3} = \left|\frac{5 \times 12.5 \times j0.3}{5 \times 12.5 \times (j.3)^3 + 6 \times 12.5 \times (j.3)^2 + 12.5 \times 0.3j + 5}\right|$$

$$= 6.9319$$

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