UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 2002

BEng Honours Degree in Computing Part I
MEng Honours Degrees in Computing Part I
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the Associateship of the City and Guilds of London Institute

PAPER C142

DISCRETE MATHEMATICS

Tuesday 7 May 2002, 16:00
Duration: 90 minutes
(Reading time 5 minutes)

Answer THREE questions

Paper contains 4 questions
Calculators not required

1a i) Define what it means for a function to be one-to-one.

ii) Define what it means for a function to be onto.

iii) Let $f : A \to B$ and $g : B \to C$ be functions. Define the composition function $g \circ f : A \to C$.

b Consider the following functions, where $\mathcal{N}$ denotes the set of natural numbers:

i) $\mathsf{max} : \mathcal{N} \times \mathcal{N} \to \mathcal{N}$ defined by

$$\begin{aligned} \mathsf{max}(n,m) &= n, \quad n \geq m \\ &= m, \quad m > n \end{aligned}$$

ii) $\mathsf{square} : \mathcal{N} \to \mathcal{N}$ defined by $\mathsf{square}(n) = n^2$;

iii) $\mathsf{pair} : \mathcal{N} \to \mathcal{N} \times \mathcal{N}$ defined by $\mathsf{pair}(n) = (n, n)$.

In each case, determine whether these functions are one-to-one or onto, and justify your answers.

c Let f and g be functions as in part 1aiii). Prove the following statements, or find and explain counter-examples using the functions given in part 1b:

i) f onto implies $g \circ f$ onto;

ii) $g \circ f$ one-to-one implies f one-to-one;

iii) $g \circ f$ one-to-one implies g one-to-one;

iv) f and g onto implies $g \circ f$ onto.

Also show that $g \circ \mathsf{max}$ cannot be one-to-one where $g : \mathcal{N} \to \mathcal{N}$.

The three parts carry, respectively, 15%, 30% and 55% of the marks.

2a Let R be a binary relation on a set A.

i) Using the composition operator for relations, define R^n for $n \geq 1$.

ii) Define the transitive closure of R, denoted R^+.

b i) Suppose $R = \{(a,b), (b,c), (c,d), (b,e), (e,f), (g,h), (h,e), (h,i), (i,f)\}$. State R^+, and illustrate both R and its transitive closure R^+ as directed graphs.

ii) Suppose that R is a binary relation on the natural numbers defined by $x\,R\,y$ if and only if $y = 2x$. State R^+.

iii) Let A be an arbitrary finite set with n elements. Give a *finite* description of R^+ in this case. Give an example to illustrate why it cannot be simplified further.

c Let R, S and T be arbitrary binary relations on a set A.

i) Show that the following distribution law holds:

$$(R \cup S) \circ T = (R \circ T) \cup (S \circ T)$$

ii) Let id be the identity relation on A. Using part 2ci) and the analogous distribution law

$$R \circ (S \cup T) = (R \circ S) \cup (R \circ T)$$

(which you need not prove), simplify the expression

$$(R \cup \mathsf{id}) \circ (R \cup \mathsf{id})$$

State clearly any other standard properties of composition that you use.

iii) Write down (without proof) an expression for $(R \cup \mathsf{id})^n$ for $n \geq 1$.

iv) Show that $(R \cup \mathsf{id})^+ = R^+ \cup \mathsf{id}$.

The three parts carry, respectively, 10%, 35% and 55% of the marks.

3a i) What does it mean for a graph to be *connected*?

ii) What is a *cycle* in a graph?

iii) What is the *degree* of a node in a graph?

b i) State a necessary and sufficient condition for a connected graph to have an Euler circuit (EC, for short). Explain why your condition is necessary.

ii) A *mail delivery circuit* (MDC, for short) is a path through a graph which uses every arc exactly twice and returns to the start node. Under what conditions does a graph have a MDC? Explain your answer.

c Let G be a connected graph. An *articulation point* is a node x of G such that if x is removed from G (together with all arcs incident on x) then the resulting graph is no longer connected. In the following, you may assume that a connected graph with n nodes has at least n-1 arcs.

i) Draw a connected graph with four nodes and exactly one articulation point. Mark the articulation point clearly.

ii) Show that for $n \geq 3$, a connected graph with n nodes has at least one node with degree ≥ 2.

iii) A graph G is *2-connected* if it is connected and it has no articulation point. Show that a 2-connected graph with n nodes has at least n arcs (all $n \geq 3$).

d A graph is *acyclic* if it has no cycles. A graph is a *tree* if it is connected and acyclic.

i) Show that a non-empty tree must have at least one node of degree one.

ii) Show by induction that for all $n \geq 1$, if a tree has n nodes then it has exactly n-1 arcs.

The four parts carry, respectively, 15%, 25%, 30%, 30% of the marks.

4a You are given four coins (numbered 1,2,3,4) of identical appearance. Two are genuine and two are counterfeit. The counterfeit coins have a different weight from the genuine coins. Both counterfeit coins weigh the same (as of course do the genuine coins). You are given scales of the balance type, where each weighing has two outcomes: the weights are the same or different.

i) Give a procedure in the form of a decision tree to place the coins into two groups of two, placing the genuine coins together and the counterfeit coins together (though you will not be able to determine which group is which). Your procedure should use as few weighings as possible.

ii) Explain why your procedure in (i) is optimal with respect to the number of weighings in worst case.

iii) Now consider the same problem, but where there are three genuine coins and three counterfeit coins. Obtain a lower bound on the worst case number of weighings. Explain your answer. There is no need to provide an actual weighing procedure.

b i) Derive a recurrence relation for the worst case number of comparisons for Quicksort on a list of length n. Explain your answer briefly.

ii) State the solution to your recurrence relation in (i).

iii) Derive a recurrence relation for the average case number of comparisons of Quicksort. Explain your answer briefly. Do not solve the recurrence relation.

iv) Suppose that the split element is chosen at the middle of the list (taking the left-hand middle element if the list is of even length). Give an example for length n=4 to show that the worst case can arise. Show the successive states of the list during the computation.

v) Suppose that as usual the split element is chosen at the start of the list. How many comparisons does Quicksort take when applied to the list [1,2,...,n,1,2,...,n] with 2n elements?

The two parts carry, respectively, 35%, 65% of the marks.