EE4-45
EE9-CS7-21
EE9-SO22

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2015

MSc and EEE/EIE PART IV: MEng and ACGI

WAVELETS AND APPLICATIONS

Corrected Copy

Thursday, 30 April 10:00 am

Time allowed: 3:00 hours

There are FOUR questions on this paper.

Answer ALL questions.

All questions carry equal marks.

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : P.L. Dragotti
Second Marker(s) : A. Manikas

Special Information for the Invigilators: NONE

Information for Candidates:

Sub-sampling by an integer N

$$x_{\downarrow N}[n] \longleftrightarrow \frac{1}{N}\sum_{k=0}^{N-1} X(e^{j(\omega-2\pi k)/N}) = \frac{1}{N}\sum_{k=0}^{N-1} X(W_N^k z^{1/N}),$$

where

$$W_N^k = e^{-j2\pi k/N}.$$

Dual Basis:
Given a basis $\{\varphi_i(t)\}_{i\in\mathbf{Z}}$, the dual basis is given by the set of elements $\{\tilde{\varphi}_i(t)\}_{i\in\mathbf{Z}}$ satisfying:

$$\langle\varphi_i(t), \tilde{\varphi}_j(t)\rangle = \delta_{i,j}.$$

The Questions

1. *Multirate Signal Processing*

   (a) Design a filter $H(z)$ that annihilates the sequence $x[n] = \cos(\omega_0 n)$, where ω_0 is an arbitrary constant. [7]

   (b) You are asked to approximate a sequence $x[n]$ with a sequence $y[n]$ having the following form:

$$y[n] = \sum_{k \in \mathbf{Z}} \beta[k]\phi[n-2k]$$

   where $\phi[n]$ is a length-4 sequence with z-transform given by

$$\Phi(z) = (1 + 3z^{-1} + 3z^{-2} - z^{-3})/(2\sqrt{5}).$$

   i. Draw a block-diagram of a system having $\beta[n]$ as input and $y[n]$ as output. [6]

   ii. We compute $\beta[n] = (x[2n] + 3x[2n+1] + 3x[2n+2] - x[2n+3])/2\sqrt{5})$. Draw a block diagram of a system having $x[n]$ as input and $y[n]$ as output. Explain what that system does. [6]

   iii. You wish to find a sequence $z[n]$ which, when added to $y[n]$ exactly recovers $x[n]$. That is, $y[n] + z[n] = x[n]$. The sequence $z[n]$ can be written as follows:

$$z[n] = \sum_{k \in \mathbf{Z}} \alpha[k]\psi[n-2k].$$

   Find the sequence $\psi[n]$ and give an expression for the coefficients $\alpha[k]$ in terms of $x[n]$. [6]

2. Consider the two-channel filter bank of Figure 2. The analysis high-pass branch has a delay since we want the down-sampler to retain the odd-indexed terms of the sequence.

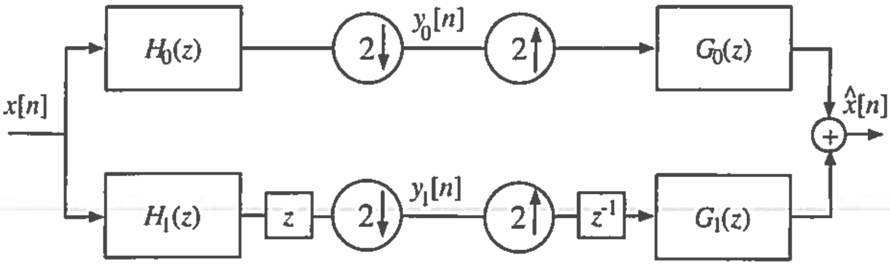

Figure 2: Two-channel filter bank with a delay.

(a) Express $\hat{X}(z)$ as a function of $X(z)$ and the filters. Then derive the two perfect reconstruction (PR) conditions the filters have to satisfy. [7]

(b) Derive the conditions the filters have to satisfy in order to have an orthogonal perfect reconstruction filter bank. [Hint: Use the condition $\langle g_0[n], g_1[n-2k-1]\rangle = 0$]. [6]

(c) Using the conditions of part (b), design the shortest orthogonal filter bank with one vanishing moment. [6]

(d) Assuming that the filters $H_0(z)$ and $G_0(z)$ of Figure 2 satisfy

$$H_0(z)G_0(z) = \frac{1}{16}(1+z)^2(1+z^{-1})^2(-z+4-z^{-1}),$$

design a biorthogonal perfect-reconstruction filter bank with symmetric $h_0[n]$ and $g_0[n]$. [6]

3. Consider the interval $t \in [0,3]$ and let

$$\varphi_1(t) = \begin{cases} 1, & \text{for } t \in [0, 3/2) \\ 0, & \text{for } t \in [3/2, 3]. \end{cases}$$

Denote with $V = span(\{\varphi_1(t), \varphi_2(t), \varphi_3(t)\})$ the sub-space generated by $\varphi_1(t)$ and its circular shifts by 1 over the interval $t \in [0,3]$. The three basis functions are shown in Fig. 3.

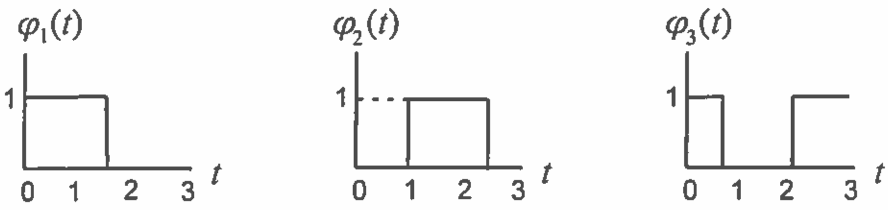

Figure 3: Three functions $\varphi_1, \varphi_2, \varphi_3$ defined for $t \in [0,3]$ and related by circular shifts.

Given a signal $x(t)$ defined for $t \in [0,3]$, the aim is to compute the orthogonal projection of $x(t)$ onto V. Recall that this is given by:

$$x_v(t) = \sum_{i=1}^{3} \langle x(t), \tilde{\varphi}_i(t) \rangle \varphi_i(t)$$

where $\{\tilde{\varphi}_i(t)\}_{i=1}^3$ are the three dual-basis functions.

(a) Since $\tilde{\varphi}_i(t) \in V$ we can write $\tilde{\varphi}_i(t) = \sum_{k=1}^3 \alpha_{i,k}\varphi_k(t)$. Using this fact

i. Determine the three dual-basis functions $\tilde{\varphi}_i(t)$, $i = 1,2,3$. That is, find the coefficients $\alpha_{i,k}$, $i = 1,2,3$; $k = 1,2,3$. [5]

ii. Sketch and dimension $\tilde{\varphi}_i(t)$, $i = 1,2,3$. [5]

(b) Given the dual basis and the signal

$$x(t) = \begin{cases} 1, & \text{for } t \in [0,1] \\ 0 & \text{for } t \in (1,3]. \end{cases}$$

i. Compute the inner products $\langle x(t), \tilde{\varphi}_i(t) \rangle$, $i = 1,2,3$. [5]

ii. Sketch and dimension $x_v(t) = \sum_{i=1}^{3} \langle x(t), \tilde{\varphi}_i(t) \rangle \varphi_i(t)$. [5]

iii. Verify that the error $e(t) = x(t) - x_v(t)$ is orthogonal to V. [5]

4. Consider a filter bank specified by the following signal equations:

$$\begin{aligned}
y_0 &= D_2 G D_2 G x \\
y_1 &= D_2 G D_2 H D_2 G x \\
y_2 &= D_2 H D_2 H D_2 G x \\
y_3 &= D_2 G D_2 H x \\
y_4 &= D_2 G D_2 H D_2 H x \\
y_5 &= D_2 H D_2 H D_2 H x,
\end{aligned}$$

where G and H are the infinite matrix representations for filtering with a lowpass filter g_n and a highpass filter h_n, respectively, and D_2 is the matrix representation of downsampling by 2.

(a) Draw a block diagram of the system using two-channel filter banks. [8]

(b) Draw the equivalent single-level six-channel filter bank clearly specifying the downsampling factors and transfer functions of the filters in each branch. [7]

(c) Consider now a filter bank specified by the following signal equations:

$$\begin{aligned}
y_0 &= D_2 G D_2 G x \\
y_1 &= D_2 H D_2 G x \\
y_2 &= D_2 H x.
\end{aligned}$$

i. Draw the equivalent single-level three-channel filter bank clearly specifying the downsampling factors and transfer functions of the filters in each branch. [5]

ii. Assume now that

$$G(z) = (1+z)(1+z^{-1})/(2\sqrt{2})$$

and that

$$H(z) = \frac{\sqrt{2}}{8}(z^2 + 2z - 6 + 2z^{-1} + z^{-2}).$$

Moreover, assume that $x[n] = n$ and ignore any boundary effect. Which of the signals $y_0[n], y_1[n], y_2[n]$ is nonzero? (Justify your answer). [5]