EE4-23

SOLUTIONS: STABILITY AND CONTROL OF NONLINEAR SYSTEMS MASTER IN CONTROL

1. Exercise

a) We are dealing with a second order equation, hence we may choose our state vector as: $[y(t),\dot{y}(t)]' = [x_1(t),x_2(t)]'$. Correspondingly, our equations read:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1+(1-x_1^2-x_2^2)x_2 \end{bmatrix}.$$

b) The function f is polynomial, hence it is locally Lipschitz continuous and this guarantees existence and uniqueness of solutions.

c) Equilibria are real solutions of the following set of algebraic equations:

$$\begin{cases} x_2 &= 0 \\ -x_1+(1-x_1^2+x_2^2)x_2 &= 0 \end{cases}.$$

From the first equation we see $x_2 = 0$; hence, substituting into the second yields $x_1 = 0$. The origin is the only equilibrium of the system.

d) Computing the Jacobian of f yields:

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -1-2x_1x_2 & 1-x_1^2-3x_2^2 \end{bmatrix}.$$

At the equilibrium point we have:

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}.$$

The eigenvalues of A are in $(1 \pm \sqrt{3}j)/2$ and are therefore complex conjugate. The local phase portrait is an unstable focus.

e) Let $g(x) = x_1^2+x_2^2-R^2$. Taking derivatives of g along solutions of the system gives:

$$\begin{aligned}\frac{\partial g}{\partial x}f(x) = [2x_1,2x_2]\cdot \begin{bmatrix} x_2 \\ -x_1+(1-x_1^2-x_2^2)x_2 \end{bmatrix} &= \\ = 2(1-x_1^2-x_2^2)x_2^2.\end{aligned}$$

Hence for $g(x) = 0$ we have:

$$\dot{g}(x) = 2(1-R^2)x_2^2 \leq 0$$

provided $R \geq 1$. This shows that the set is forward invariant for all $R \geq 1$.

f) A similar argument shows that for all $R \leq 1$ the set $\{x_1^2+x_2^2 \leq R^2\}$ is backwards invariant. As a consenquence the annular region $\{(x_1,x_2) : r^2 \geq x_1^2+x_2^2 \leq R^2\}$ is forward invariant for all $r < 1$ and all $R > 1$. This region does not contain equilibria, and therefore, by Poincarè-Bendixson theory every solution initiated in it converges to a periodic solution. The global phase-portrait is sketched in Figure 1.1.

g) Notice that, for $R = 1$ and $g(x) = 0$ we see that $\dot{g}(x) = 0$. Hence, $g(x(t)) = g(x(0)) = 0$ for all t. This shows that the unit circle is an invariant set.

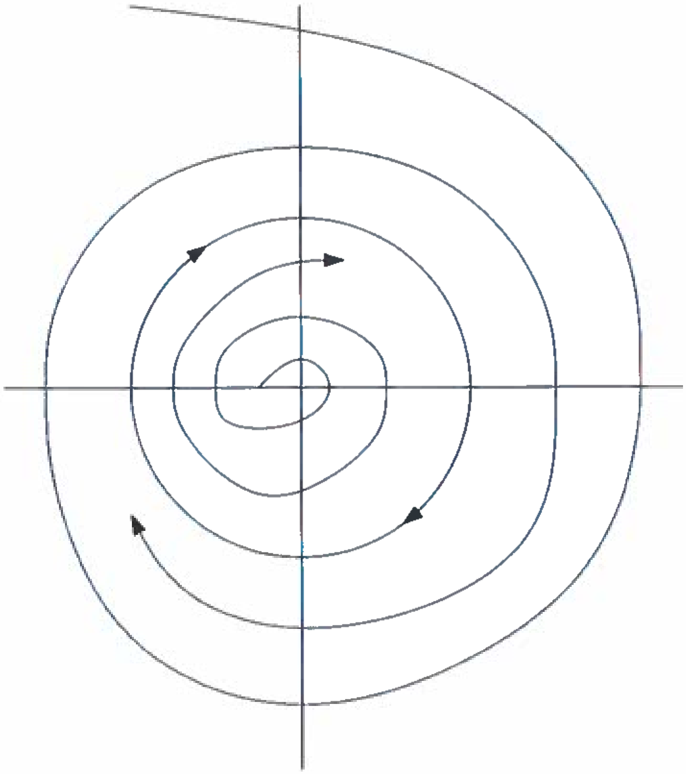

Figure 1.1 Global phase portrait

2. Exercise

a) We choose $x(t) = [y(t), \dot{y}(t)]' = [x_1(t), x_2(t)]'$. Accordingly we see that:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -x_2(t) - \text{atan}(x_1(t)) + d(t) \end{bmatrix}.$$

b) For $d = 0$ the equilibrium are computed by solving the following system of nonlinear equations:

$$\begin{cases} x_2 & = & 0 \\ -x_2 - \text{atan}(x_1) & = & 0 \end{cases}$$

Frome the first equation we see $x_2 = 0$ and substituing into the second yields $\text{atan}(x_1) = 0$. As the latter only admits the solution $x_1 = 0$, the system has a unique equilibrium $x_e = [0, 0]'$. We take as a candidate Lyapunov function the following:

$$V(x) = \frac{x_2^2}{2} + \int_0^{x_1} \text{atan}(\xi)\, d\xi.$$

Notice that $V(x)$ is smooth and positive definite; in fact:

$$x \neq 0 \Rightarrow (x_1 \neq 0) \text{ or } (x_2 \neq 0).$$

In the first case we have:

$$V(x) \geq \int_0^{x_1} \text{atan}(\xi)\, d\xi > 0.$$

In the second case, similarly:

$$V(x) \geq x^2/2 > 0.$$

Moreover, it is possible to verify that $V(x)$ is radially unbounded. In fact:

$$V(x) \leq M \Rightarrow \frac{x_2^2}{2} \leq M \text{ and } \int_0^{x_1} \text{atan}(\xi)d\xi \leq M$$

From the first inequality we see that:

$$|x_2| \leq \sqrt{2M},$$

that is the x_2 component of the state is indeed bounded. From the second inequality consider first the case of $x_1 \geq 1$:

$$M \geq \int_0^{x_1} \text{atan}(\xi)d\xi \geq \int_1^{x_1} \text{atan}(\xi)d\xi \geq \text{atan}(1)(x_1 - 1).$$

Hence, $x_1 \leq 1 + M/\text{atan}(1)$. A similar argument holds for the case of negative x_1. We consider next the derivative:

$$\dot{V}(x) = x_2\dot{x}_2 + \text{atan}(x_1)\dot{x}_1 = -x_2^2 - x_2\text{atan}(x_1) + x_2\text{atan}(x_1) = -x_2^2 \leq 0.$$

Hence $\dot{V}(x)$ is negative semi-definite. The kernel of $\dot{V}(x)$ can be expressed as $K_0 = \{x : x_2 = 0\}$. Any invariant set contained in K_0 is also contained in $K_1 := \{x : x_2 = 0, -x_2 - \text{atan}(x_1) = 0\} = \{0\}$. Hence, the origin is the largest invariant set contained in the kernel of $\dot{V}(x)$. It follows that the equilibrium is globally asymptotically stable by virtue of the Lasalle's invariance principle.

c) i) For the considered selection of $d(t)$ we see that:

$$\ddot{y}(t) = \frac{1}{1+t^2}.$$

ii) These can be seen as the equations of a double integrator (a linear system) forced by the input $1/(1+t^2)$. Direct integration leads to the following expressions:

$$\dot{y}(t) = \dot{y}(0) + \int_0^t \frac{1}{1+t^2}dt = \dot{y}(0) + \text{atan}(t).$$

Integrating once more yields:

$$y(t) = y(0) + \dot{y}(0)t + \int_0^t \text{atan}(t)dt.$$

iii) Notice that

$$|\dot{y}(t)| \leq |\dot{y}(0)| + |\text{atan}(t)| \leq |\dot{y}(0)| + \frac{\pi}{2}$$

and it is therefore a bounded signal.

iv) Similarly $|\text{atan}(y(t))| \leq \pi/2$ so that for $d(t)$ it holds:

$$d(t) = \dot{y}(t) + \text{atan}(y(t)) + \frac{1}{1+t^2} \leq |\dot{y}(0)| + \pi + 1.$$

The signal $d(t)$ is therefore bounded.

v) Notice that $y(t)$ fulfills:

$$y(t) \geq y(0) + \int_0^t \text{atan}(t)dt \geq y(0) + \int_1^t \text{atan}(t)dt \geq y(0) + (t-1)\text{atan}(1),$$

where the second inequality follows considering that atan is non-negative for positive values of its argument, and the last inequality is a consequence of increasingness of atan. Hence, $y(t)$ diverges to $+\infty$. This shows that our original system is not Input-to-State stable as we could find a bounded input $d(t)$ giving raise to an unbounded solution.

3. Exercise

a) Consider the quadratic function:

$$V(x) = \frac{x_1^2 + x_2^2}{2}.$$

Differentiating along solutions of our system yields:

$$\dot{V} = x_1\dot{x}_1 + x_2\dot{x}_2 = \sin(x_1)x_2x_1 - \sin(x_1)x_1x_2 + x_2u = yu.$$

Hence the system is passive from u to y and lossless.

b) Notice that the storage function V is positive-definite, and radially unbounded. Moreover, for $u = -\mathrm{atan}(y)$ we have:

$$\dot{V}(x) = yu = -x_2\mathrm{atan}(x_2) \leq 0.$$

Hence, $\dot{V}$ is negative semidefinite. Moreover, the kernel of $\dot{V}(x)$ equals $\{x : x_2 = 0\}$. Any invariant sent contained in $K_0 = \{x : x_2 = 0\}$ is also contained in $K_1 = \{x : x_2 = 0, -\sin(x_1)x_1 - \mathrm{atan}(x_2) = 0\}$. This set in turns equal $K_1 = \{k\pi, k \in \mathbb{Z}\}$. Notice that these are in fact all equilibria of the closed-loop system. Since the origin is the largest invariant set contained in a sufficiently small neighborhood of 0, this implies that it is locally asymptotically stable by the Lasalle's invariance principle.

c) Consider the quadratic candidate Lyapunov function:

$$W(x) = V(x) + \varepsilon x_1 x_2.$$

The function $W(x)$ is a quadratic form and it is radially unbounded and positive definite for all sufficiently small ε, in particular $|\varepsilon| < 1$. Taking derivatives along solutions of the system yields for $u = -x_1 - x_2 + d$:

$$\dot{W} = x_1\dot{x}_1 + x_2\dot{x}_2 + \varepsilon\dot{x}_1x_2 + \varepsilon x_1\dot{x}_2 = [\varepsilon\sin(x_1) - 1]x_2^2 - \varepsilon x_1^2 - \varepsilon x_1x_2 + (x_2 + \varepsilon x_1)d.$$

Notice that $|\sin(x_1)| \leq 1$, therefore:

$$\dot{W}(x) \leq -x'Px + (x_2 + \varepsilon x_1)d,$$

where P is the symmetric matrix defined below:

$$P = \begin{bmatrix} \varepsilon & \varepsilon/2 \\ \varepsilon/2 & (1-\varepsilon) \end{bmatrix}.$$

Notice that P is positive definite provided $0 < \varepsilon < 4/5$. In particular then, by completion of squares, we can show that $W(x)$ is for all such values of ε an ISS Lyapunov function.

d) For $u(t) = 0$ the system fulfills:

$$\dot{V}(x) = 0.$$

Hence $V(x(t)) = V(x(0))$ for all t. Lyapunov stability follows, but asymptotic stability is not fulfilled since

$$\forall x(0) \neq 0. \qquad \lim_{t \to +\infty} V(x(t)) = V(x(0)) \neq 0.$$

4. Exercise

a) Consider the output $y = x_1$. Differentiating once along solution of the control system yields:

$$\dot{y} = \sin(x_2) - x_1.$$

Notice that u does not appear in this expression. In particular then:

$$\ddot{y} = \cos(x_2)[\sin(x_3) - x_2] - \sin(x_2) + x_1.$$

Taking an extra round of derivatives yields:

$$y^{(3)} = -\sin(x_2)[\sin(x_3) - x_2]^2 + \cos(x_2)[\cos(x_3)(u + \sin(x_1)) - \sin(x_3) - x_2)].$$

Hence, the relative degree (local) is equal to 3.

b) We may let:

$$\cos(x_2)\cos(x_3)u - \sin(x_2)[\sin(x_3) - x_2]^2 + \cos(x_2)[\cos(x_3)\sin(x_1) - \sin(x_3) - x_2)] = v.$$

This (locally) results in the equation $y^{(3)} = v$.

c) A stabilizing feedback as requested is given, for instance, by letting $v = -\ddot{3}y^{(2)} - 3\dot{y} - y$.

d) With the new choice of output we see that

$$\dot{y} = \dot{x}_3 = u + \sin(x_1).$$

Hence the relative degree is global and equal to 1.

e) The internal dynamics are given as:

$$\begin{aligned} \dot{x}_1 &= \sin(x_2) - x_1 \\ \dot{x}_2 &= \sin(x_3) - x_2 \end{aligned}$$

where the variable x_3 has now to be regarded as an exogenous input. The zero-dynamics are obtained from the internal dynamics by letting $x_3 = 0$.

f) It is possible to prove Input-to-State Stability of the Internal Dynamics by using the candidate ISS Lyapunov function

$$V(x) = x_1^2 + x_2^2$$

and completion of squares to bound the derivative $\dot{V}$. In fact:

$$\dot{V} = 2x_1[-x_1 + \sin(x_2)] + 2x_2[-x_2 + \sin(x_3)]$$

$$= -2x_1^2 - 2x_2^2 + 2\sin(x_2)x_1 + 2x_2\sin(x_3) \leq -2x_1^2 - 2x_2^2 + 2|x_1||x_2| + 2|x_2||x_3|$$

$$\leq -x_1^2 - x_2^2 + 2|x_2||x_3| \leq -\frac{3}{4}(x_1^2 + x_2^2) + 4x_3^2$$