EE4-45
EE9CS7-21
EE9SO22

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2012

MSc and EEE/ISE PART IV: MEng and ACGI

**WAVELETS AND APPLICATIONS**

Tuesday, 15 May 10:00 am

Time allowed: 3:00 hours

**There are FOUR questions on this paper.**

**Answer ALL questions.**

*All questions carry equal marks.*

**Any special instructions for invigilators and information for candidates are on page 1.**

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|---|---|---|
| Examiners responsible | First Marker(s) : | P.L. Dragotti |
| | Second Marker(s) : | K.D. Harris |

**Special Information for the Invigilators: NONE**

**Information for Candidates:**

*Poisson summation formula:*

$$\sum_{n=-\infty}^{\infty} f(t-n) = \sum_{k=-\infty}^{\infty} \hat{f}(2\pi k) e^{j2\pi kt}.$$

# The Questions

1. Consider a set of linearly independent functions $\{\varphi_i\}_{i=1,2,..,N}$ that covers the sub-space $V$, that is, $V = span\{\varphi_i\}_{i=1,2,...N}$. Consider a function $f(t) \in L_2(\mathbb{R})$, the orthogonal projection of $f(t)$ onto $V$ is:

$$\hat{f}(t) = \sum_{i=1}^{N} c_i \varphi_i(t),$$

with $c_i = \langle f(t), \tilde{\varphi}_i(t) \rangle$, $i = 1, 2, ...N$. Here $\{\tilde{\varphi}_i\}_{i=1,2,..,N}$ is the dual basis of $\{\varphi_i\}_{i=1,2,..,N}$. Based on the above formula and assuming

$$f(t) = \begin{cases} t & \text{for} \quad 0 \leq t < 1 \\ 0 & \qquad \text{otherwise} \end{cases}$$

(a) compute the coefficients $c_i$ of the orthogonal projection of $f(t)$ onto the space spanned by $\varphi(t), \psi(t), \sqrt{2}\psi(2t), \sqrt{2}\psi(2t-1)$ with

$$\varphi(t) = \begin{cases} 1 & \text{for} \quad 0 \leq t < 1 \\ 0 & \qquad \text{otherwise} \end{cases}$$

and

$$\psi(t) = \begin{cases} 1 & \text{for} \quad 0 \leq t < 1/2 \\ -1 & \text{for} \quad 1/2 \leq t < 1 \\ 0 & \qquad \text{otherwise,} \end{cases}$$

[7]

(b) sketch and dimension the function $\hat{f}(t)$,

[6]

(c) compute the energy of the error function $\epsilon(t) = f(t) - \hat{f}(t)$,

[6]

(d) verify that $\|f(t)\|^2 = \|\hat{f}(t)\|^2 + \|\epsilon(t)\|^2$, where $\|x(t)\|^2 = \int_{-\infty}^{\infty} x^2(t)dt$.

[6]

2. Consider a filter bank specified by the following signal equations:

$$\begin{aligned}
y_0 &= D_2 G D_2 G x \\
y_1 &= D_2 G D_2 H D_2 G x \\
y_2 &= D_2 H D_2 H D_2 G x \\
y_3 &= D_2 G D_2 G D_2 H x \\
y_4 &= D_2 H D_2 G D_2 H x \\
y_5 &= D_2 H D_2 H x,
\end{aligned}$$

where $G$ and $H$ are the infinite matrix representations for filtering with a lowpass filter $g_n$ and a highpass filter $h_n$, respectively, and $D_2$ is the matrix representation of down-sampling by 2.

(a) Draw a block diagram of the system using two-channel filter banks.

[8]

(b) Draw the equivalent single-level six-channel filter bank clearly specifying the down-sampling factors and transfer functions of the filters in each branch.

[8]

(c) Consider now a filter bank specified by the following signal equations:

$$\begin{aligned}
y_0 &= D_2 G D_2 G x \\
y_1 &= D_2 H D_2 G x \\
y_2 &= D_2 H x.
\end{aligned}$$

Draw the equivalent single-level three-channel filter bank and derive the exact transfer functions of the equivalent filters assuming that $g_n$ and $h_n$ are the low-pass and highpass Haar filters respectively. Specifically, the $z$-transform of $g_n$ is $G(z) = (1+z)/\sqrt{2}$ and $H(z) = (1-z)/\sqrt{2}$.

[9]

3. Consider the tree-structured filter bank shown in Fig. 2.

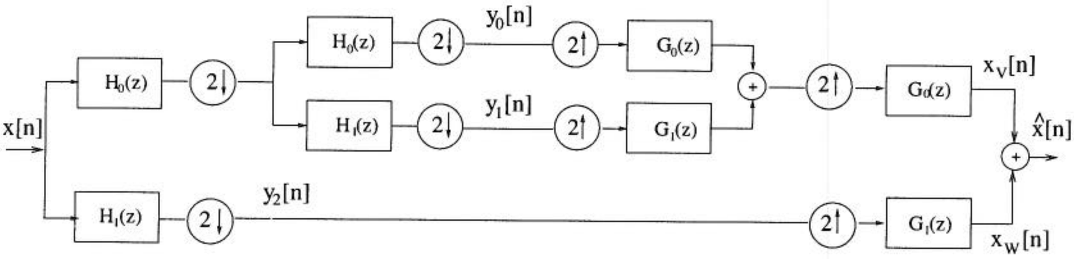



Figure 2: Tree-structured filter bank.

(a) Let $G_1(z) = \frac{3\sqrt{2}}{5}\left(\frac{1}{2} + \frac{1}{6}z^{-1} + \frac{1}{3}z^{-2} - z^{-3}\right)$. Design $G_0(z), H_0(z), H_1(z)$ in order to obtain an orthogonal perfect reconstruction filter bank.

[8]

(b) Find the zeros of $G_1(z)$ [Hint: if you correctly guess one of the zeros, you will be left with an easy factorization].

[7]

(c) Assume $x[n] = 1$ and ignore any boundary effect. Which of the signals $y_0[n], y_1[n], y_2[n]$ is nonzero? (Justify your answer).

[5]

(d) Assume now that $x[n] = n$ and again ignore any boundary effect. Which of the signals $y_0[n], y_1[n], y_2[n]$ is nonzero? (Justify your answer).

[5]

4. Suppose you are given a two-channel FIR filter bank with real coefficients and synthesis lowpass filter

$$g_0[n] = \frac{1}{2\sqrt{2}}(\delta_n + 2\delta_{n-1} + \delta_{n-2}).$$

(a) Is it possible that you were given an orthogonal filter bank? Justify your answer. [6]

(b) Consider the equivalent filter

$$G_0^{(i)}(z) = \prod_{k=0}^{i-1} G_0(z^{2^k})$$

obtained by iterating the filter bank decomposition $i$ times. Consider the function

$$\varphi^{(i)}(t) = 2^{i/2} g_0^{(i)}[n], \qquad n/2^i \leq t < (n+1)/2^i.$$

Can you say anything about the convergence of $\lim_{i\to\infty} \varphi^{(i)}(t)$? [6]

(c) Assume that $\varphi(t) = \lim_{i\to\infty} \varphi^{(i)}(t)$ exists. We know that, in the case of convergence, $\varphi(t)$ is a valid scaling function. Therefore, by operating the frequency domain, show that $\varphi(t)$ satisfies partition of unity:

$$\sum_{n=-\infty}^{\infty} \varphi(t-n) = 1.$$

[6]

(d) Show that $\varphi(t)$ satisfy the two scale equation

$$\varphi(t) = \sqrt{2} \sum_{n=-\infty}^{\infty} g_0[n]\varphi(2t-n).$$

[6]

1.

(a) SINCE THE FUNCTIONS $\varphi(t)$, $\psi(t)$, $\sqrt{2}\psi(2t)$ $\sqrt{2}\psi(2t-1)$ ARE ORTHONORMAL, WE DO NOT NEED TO FIND THE DUAL BASIS: $\varphi_i(t) = \tilde{\varphi}_i(t)$.

THEREFORE

$$c_1 = \langle f(t), \varphi(t) \rangle = \int_0^1 t\,dt = \frac{t^2}{2}\Big|_0^1 = \frac{1}{2}$$

$$c_2 = \langle f(t), \psi(t) \rangle = \int_0^1 t\,\psi(t)\,dt = \int_0^{1/2} t\,dt - \int_{1/2}^1 t\,dt$$

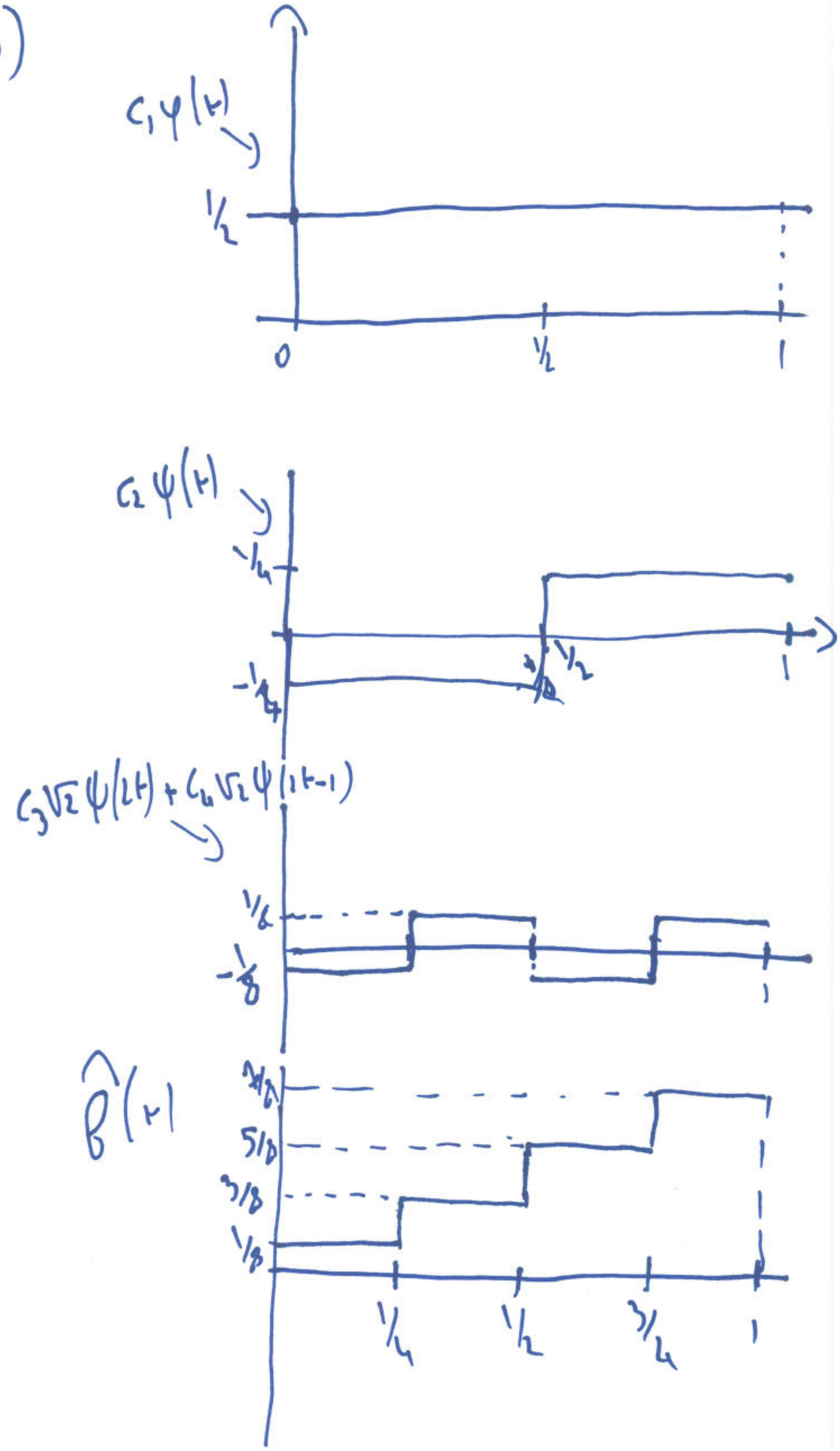
$$= \frac{1}{2}\left(\frac{1}{4} - 1 + \frac{1}{4}\right) = -\frac{1}{4}$$

$$c_3 = \langle f(t), \sqrt{2}\,\psi(2t) \rangle = \sqrt{2}\int_0^{1/4} t\,dt - \sqrt{2}\int_{1/4}^{1/2} t\,dt$$

$$= -\frac{\sqrt{2}}{16}$$

$$c_4 = c_3 = -\frac{\sqrt{2}}{16}$$

(b)


$c_1\psi(t)$
1/2
0
1/2
1
$c_2\psi(t)$
1/4
-1/4
1/2
1
$c_3\sqrt{2}\psi(2t)+c_4\sqrt{2}\psi(2t-1)$
1/8
-1/8
1
$\hat{f}(t)$
7/8
5/8
3/8
1/8
1/4
1/2
3/4
1

(c)

$$\|\varepsilon(t)\|^2 = 4\int_0^{1/4}\left(t-\frac{1}{8}\right)^2 dt$$

$$= \int_0^{1/4}\left(t^2+\frac{1}{16}-\frac{t}{4}\right)dt$$

$$= \frac{t^3}{3}\Big|_0^{1/4} + \frac{t}{64}\Big|_0^{1/4} - \frac{1}{4}\frac{t^2}{2}\Big|_0^{1/4}$$

$$= \frac{1}{3}\cdot\frac{1}{64}$$

(d)

USING PARSEVAL, WE HAVE THAT:

$$\|\hat{f}\|^2 = c_1^2+c_2^2+c_3^2+c_4^2 = \frac{1}{4}+\frac{1}{16}+\frac{4}{(16)^2} = \frac{1}{4}+\frac{1}{16}+\frac{1}{64} = \frac{21}{64}$$

MOREOVER

$$\|f\|^2 = \int_0^1 t^2\,dt = \frac{1}{3}$$

THUS

$$\|\hat{f}\|^2+\|\varepsilon\|^2 = \frac{1}{3}\cdot\frac{1}{64}+\frac{21}{64} = \frac{1}{3} = \|f\|^2$$

$\square$

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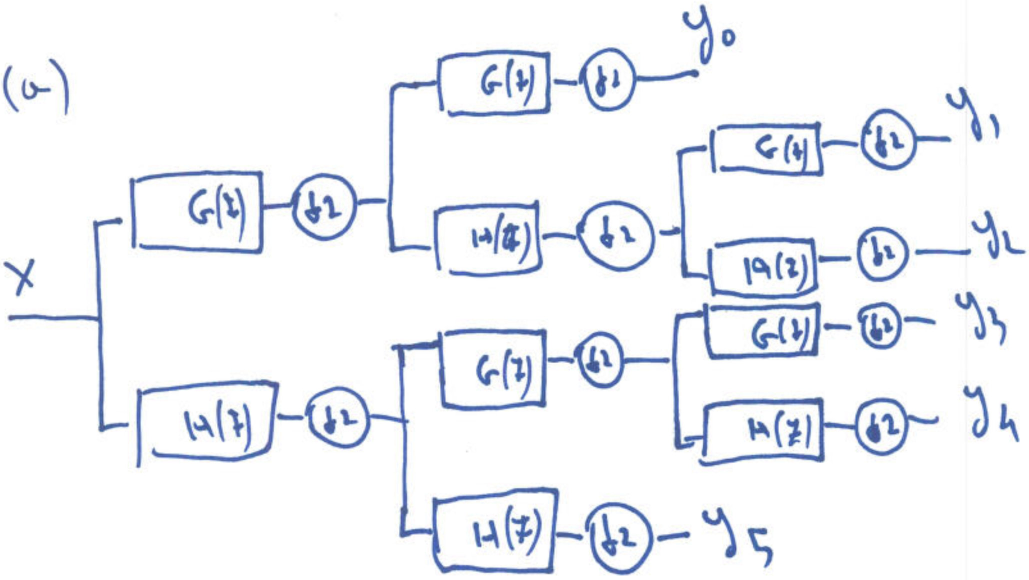
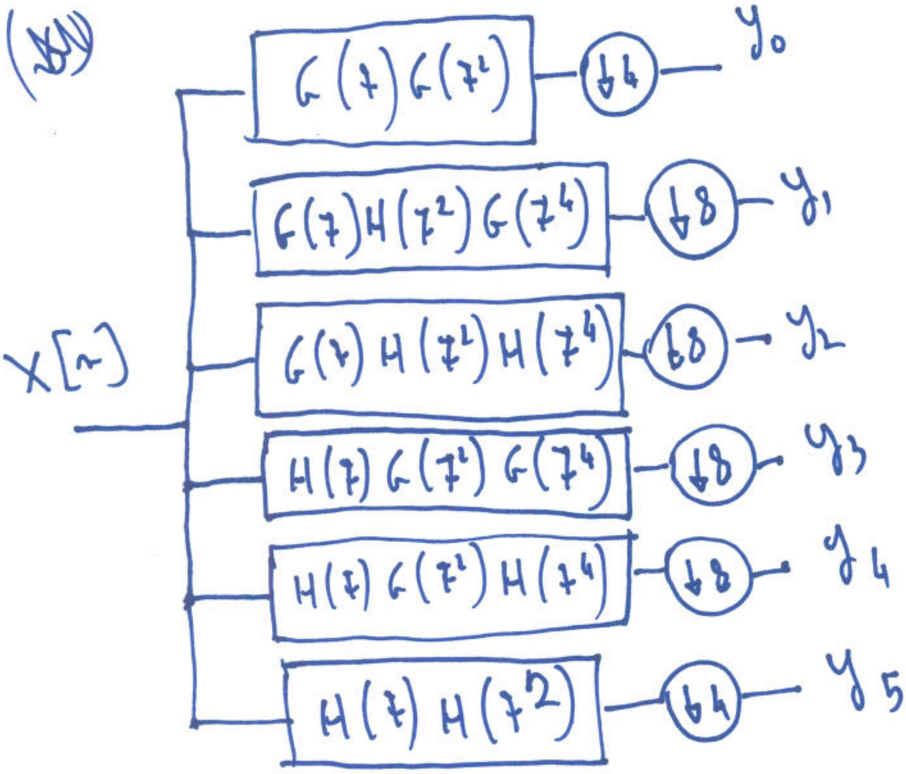
2.

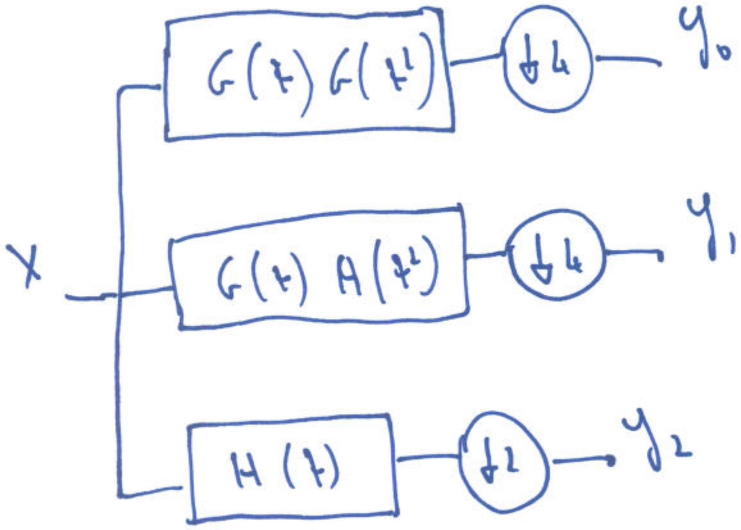
(a)




(b) USING NOBLE IDENTITY WE GET:



(c)




$$G(z)\,G(z^2) = \left(\frac{1+z}{\sqrt{2}}\right)\left(\frac{1+z^2}{\sqrt{2}}\right) = \frac{1}{2}\left(1+z+z^2+z^3\right)$$

$$G(z)\,H(z^2) = \left(\frac{1+z}{\sqrt{2}}\right)\left(\frac{1-z^2}{\sqrt{2}}\right) = \frac{1}{2}\left(1+z-z^2-z^3\right)$$

$$H(z) = \frac{1-z}{\sqrt{2}}$$

3

6

USING SHIFT AND MODULATION

(a) WE OBTAIN

$$G_0(z) = -z^{-1} G_1(-z^{-1})$$

$$= \left(\frac{1}{2} z^{-1} - \frac{1}{6} + \frac{1}{3} z + z^2\right) \cdot \frac{3\sqrt{2}}{5}$$

THE OTHER TWO FILTERS ARE

$$H_0(z) = G_0(z^{-1}) = \frac{3\sqrt{2}}{5}\left(\frac{1}{2} z - \frac{1}{6} + \frac{1}{3} z^{-1} + z^{-2}\right)$$

$$H_1(z) = G_1(z^{-1}) = \frac{3\sqrt{2}}{5}\left(\frac{1}{2} + \frac{1}{6} z + \frac{1}{3} z^2 - z^3\right)$$

(b)

WE TRY $z = 1$

$$G_1(1) = \left(\frac{1}{2} + \frac{1}{6} + \frac{1}{3} - 1\right) \cdot \frac{3\sqrt{2}}{5} = 0 \quad \checkmark$$

THUS $G_1(z)$ HAS A FACTOR $(1 - z^{-1})$ SO

$$G_1(z) = (-z^{-1} + 1)\left(\frac{1}{2} + \frac{2}{3} z^{-1} + z^{-2}\right)$$

THE OTHER TWO ROOTS ARE

$$z_{1,2} = \left(-\frac{1}{3} \pm j\frac{\sqrt{14}}{6}\right)^{-1} = -\frac{2}{3} \pm j\frac{\sqrt{14}}{3}$$

(c) SINCE $G_1(z)$ HAS ONLY ONE ZERO AT $z=1$, ALSO $H_1(z)$ HAS ONLY ONE ZERO AT $z=1$

THUS THE HIGHPASS FILTER ANNIHILATES CONSTANTS BUT NOT HIGHER DEGREE POLYNOMIALS.
THUS $y_1[n]$ AND $y_2[n]$ ARE ZERO, BUT $y_0[n] \neq 0$

(d) AS DESCRIBED ABOVE $H(z)$ DOES NOT ANNIHILATES POLYNOMIALS WITH DEGREE GREATER THAN ZERO. THEREFORE

$y_0[n] \neq 0$, $y_1[n] \neq 0$, $y_2[n] \neq 0$.

4. 

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(a) THE ANSWER IS NO, BECAUSE THE GIVEN $g_0[n]$ IS SYMMETRIC AND WE KNOW THAT WITH THE ONLY EXCEPTION OF THE HAAR FILTER, IT IS NOT POSSIBLE TO DESIGN PERFECT-RECONSTRUCTION REAL-VALUED LINEAR-PHASE ORTHOGONAL FILTER BANKS

(b)

$\cancel{G_0(z)}$

$$g_0[n] = \frac{1}{2\sqrt{2}}\left(\delta_n + 2\delta_{n-1} + \delta_{n-2}\right)$$

THUS

$$G_0(z) = \frac{1}{2\sqrt{2}}\left(1 + 2z^{-1} + z^{-2}\right)$$

$$= \frac{1}{2\sqrt{2}}\left(1 + z^{-1}\right)^2$$

$$G_0(e^{j\omega})\Big|_{\omega=0} = \frac{1}{\sqrt{2}} \quad \text{AND} \quad G_0(e^{j\omega})\Big|_{\omega=\pi} = 0$$

THE TWO NECESSARY CONDITIONS FOR THE LIMIT TO EXIST ARE SATISFIED.

MOREOVER,

IF WE DENOTE WITH $M_0(w) = \frac{G_0(e^{jw})}{\sqrt{2}}$

WE HAVE,

THAT $M_0(w) = \left(\frac{1+e^{-jw}}{2}\right)^N \cdot R(w)$

WITH $R(w)=1$ AND $N=2$.

WE DENOTE WITH $B = \sup_w R(w) = 1$

AND SINCE

$$B < 2^{N-1} = 2^{2-1} = 2$$

WE KNOW THAT ~~THE~~ A SUFFICIENT CONDITION FOR $\varphi(t)$ TO CONVERGE TO A CONTINUOUS FUNCTION IS SATISFIED.

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(c)

[redacted]y using Poisson summation formula, we can show that $\varphi(t)$ satisfies partition of unity. The Poisson summation formula says that:

$$\sum_{n=-\infty}^{\infty} \varphi(t-nT) = \frac{1}{T}\sum_{k=-\infty}^{\infty} \hat{\varphi}\left(\frac{2\pi k}{T}\right) e^{j2\pi kt/T}$$

and we want to verify that:

$$\sum_{n=-\infty}^{\infty} \varphi(t-n) = 1.$$

Thus by combining the two equations and for $T=1$, we obtain the following:

$$\sum_{n=-\infty}^{\infty} \varphi(t-n) = \sum_{k=-\infty}^{\infty} \hat{\varphi}(2\pi k) e^{j2\pi kt} = 1.$$

The condition $\sum_{k=-\infty}^{\infty} \hat{\varphi}(2\pi k) e^{j2\pi kt} = 1$ is then clearly satisfied. Indeed, by using the infinite product formula [redacted] and since $G(e^{j\omega}) = \sqrt{2}$ for $\omega = 0$ and $G(e^{j\omega}) = 0$ for $\omega = \pi$, we have that $\hat{\varphi}(2\pi k) = 1$ for $k=0$ and $\hat{\varphi}(2\pi k) = 0$ otherwise. $\square$

(d) IN FREQUENCY DOMAIN THE TWO-SCALE EQUATION CAN BE WRITTEN AS FOLLOWS:

$$\varphi(t) = \sqrt{2}\sum_m g_0[m]\varphi(2t-m) \Longleftrightarrow \frac{1}{\sqrt{2}} G_0\left(e^{j\omega/2}\right)\hat{\varphi}\left(\frac{\omega}{2}\right)$$

MOREOVER WE KNOW THAT

$$\hat{\varphi}(\omega) = \lim_{i\to\infty} \hat{\varphi}^{(i)}(\omega) = \prod_{k=1}^{\infty} M_0\left(\frac{\omega}{2^k}\right)$$

WITH $M_0(\omega) \triangleq \frac{G_0\left(e^{j\omega}\right)}{\sqrt{2}}$

THEREFORE

$$\hat{\varphi}(\omega) = \prod_{k=1}^{\infty} M_0\left(\frac{\omega}{2^k}\right) = M_0\left(\frac{\omega}{2}\right)\prod_{k=2}^{\infty} M_0\left(\frac{\omega}{2^k}\right)$$

$$= M_0\left(\frac{\omega}{2}\right)\hat{\varphi}\left(\frac{\omega}{2}\right) = \frac{1}{\sqrt{2}} G_0\left(e^{j\omega/2}\right)\hat{\varphi}\left(\frac{\omega}{2}\right)$$

$\square$