EE1-10A Mathematics I- Solutions 2017

The below comments apply to common errors. Where there is no comment, the question was done well by most students.

1. (a) Express in the form $x + iy$:

$$(i) \quad \frac{1-2i}{i-2}, \qquad (ii) \quad \left(\frac{1-\sqrt{3}i}{2}\right)^{2017}.$$

SOLUTION

$$\text{(i) } z = \frac{-1+2i}{2-i} \cdot \frac{2+i}{2+i} = \frac{-2+3i+2i^2}{4+1} = -\frac{4}{5} + \frac{3}{5}i$$

$$\left(\frac{1-\sqrt{3}i}{2}\right)^{2017} = \left(e^{-i\pi/3}\right)^{2017} = e^{-i2017\pi/3} = e^{-i\pi/3} = \frac{1-\sqrt{3}i}{2}.$$

Many left the answer in (ii) in the form $e^{-i2017\pi/3}$.

(b) Sketch the locus of the complex number z satisfying

$$z - \bar{z} = \frac{1}{\bar{z}} - \frac{1}{z}.$$

SOLUTION Rewrite as

$$x + iy - (x - iy) = \frac{1}{x-iy} - \frac{1}{x+iy} \Rightarrow 2iy = \frac{x+iy}{x^2+y^2} - \frac{x-iy}{x^2+y^2} = \frac{2iy}{x^2+y^2}$$

so either $y = 0$ or $x^2 + y^2 = 1$. Sketch: union of unit circle and $x-$axis.

Very few people saw that $y = 0$ *is a solution, simply cancelling the* $2yi$ *terms. One even wrote: "as* z *is complex", clearly rejecting the real solution. But a real number is a complex number with zero imaginary part!*

(c) Obtain all complex solutions z, when

$$(i) \quad \sinh z = -i, \qquad (ii) \quad \sin^2(iz) = 1.$$

SOLUTION

(i) Rewrite as

$$e^z - e^{-z} = -2i \Rightarrow e^{2z} + 2ie^z - 1 = 0 \Rightarrow e^z = \frac{-2i \pm \sqrt{(2i)^2 - 4}}{2} = -i\,,$$

so that

$$e^{x+iy} = e^x(\cos y + i\sin y) = -i \Rightarrow e^x \cos y = 0 \text{ and } e^x \sin y = -1\,,$$

equating real and imaginary parts. As $e^x \neq 0$ first equation gives

$$\cos y = 0 \Rightarrow y = (2n+1)\frac{\pi}{2}$$

for integers n. The second equation gives $\sin y = \pm 1$ for the given values of y, so we must have $e^x = 1 \to x = 0$ and hence $\sin y = -1 \Rightarrow y = -\pi/2 + 2n\pi$, where n is any integer.

Alternative approach, but longer:

$$e^z - e^{-z} = -2i \Rightarrow e^x e^{iy} - e^{-x}e^{-iy} = -2i$$

and using Euler's formula, get

$$e^x(\cos y + i\sin y) - e^{-x}(\cos y - i\sin y) = -2i$$

and equate real and imaginary parts to obtain

$$\cos y(e^x - e^{-x}) = 0\,, \quad \sin y(e^x + e^{-x}) = -2 \qquad (*)$$

and solve the first equation, then substitute into the second to get the same result.

Lots of people got to $e^z = -i$ *and then left it as* $z = \ln(-i)$. *Others gave single solution* $y = -\pi/2$. *Many obtained the two equations in* $(*)$, *but solved only one equation. Many rewrote in terms of* $\sin(iz)$ *and took the inverse sine. This approach ignores that* z *is complex and will not give the correct result in almost all cases, for example* $\sin(iz) = 2$.

(ii)

$$\sin^2(iz) = 1 \Rightarrow \sin(iz) = i\sinh z = \pm 1 \Rightarrow \sinh z = \pm i$$

Of the two equations, we have solved $\sinh z = -i$ in (i). As sinh is an odd function, we can write

$$\sinh z = i \Rightarrow \sinh(-z) = -i$$

so using (i) again we have the same solutions for $-z$, giving $x = 0$ and $y = \pi/2 + 2n\pi$. Combining the two solutions we get $x = 0$ and $y = \pi/2 + n\pi$.

Few saw the shortcut linking this to (i). Many rewrote in terms of sinh *and then squared* $(e^{-z} - e^z)/2i$, *complicating the process. Many took the inverse sine, as in (i).*

(d) Obtain the limits

$$(i)\ \lim_{x\to 0} x\cos(\cot x)\,, \quad (ii)\ \lim_{x\to 0} \frac{x^2}{\ln(\cos x)}\,, \quad (iii)\ \lim_{x\to\pi/6} \frac{1-\sin(3x)}{\cot x - \sqrt{3}}\,.$$

SOLUTION

(i) As $|\cos(\cot x)| \leq 1$ we can write $-x \leq x\cos(\cot x) \leq x$ and the sandwich theorem gives that $0 = \lim_{x\to 0}(-x) \leq \lim_{x\to 0} x\cos(\cot x) \leq \lim_{x\to 0} x = 0$ so the limit is zero.

Lots of people tried l'Hopital's rule. Many decided the limit didn't exist as $\cot x$ *is not defined at* 0. *Many argued that the result is zero times* $\lim_{x\to 0}\cos(\cot x)$ *which is, again, incorrect, as that limit does not exist.*

(ii) Use l'Hopital's rule, given "0/0":

$$\lim_{x\to 0}\frac{x^2}{\ln(\cos x)} = \lim_{x\to 0}\frac{2x}{-\tan x}$$

which is still "0/0", so apply l'Hopital again:

$$\lim_{x\to 0}\frac{2x}{-\tan x} = \lim_{x\to 0}\frac{2}{-\sec^2 x} = -2$$

(iii) Need to see that it's "0/0" given $\sin(3\pi/6) = 1$ and $\cot(\pi/6) = \sqrt{3}$, then apply

$$\lim_{x\to\pi/6}\frac{1-\sin(3x)}{\cot x - \sqrt{3}} = \lim_{x\to\pi/6}\frac{-3\cos(3x)}{-\csc^2 x} = 0$$

as the denominator is non-zero.

2. (a) Obtain the value of q for which the folowing limit exists and is non-zero, and state the value of the limit:

$$\lim_{x\to\infty} x^q\left[(x+1)^{2/3} - (x-1)^{2/3}\right].$$

SOLUTION

Rewrite as $x^q\left[x^{2/3}\left(1+\frac{1}{x}\right)^{2/3} - x^{2/3}\left(1-\frac{1}{x}\right)^{2/3}\right]$

and as $x\to\infty, 1/x\to 0$, so we can use the Binomial expansion:

$$= x^{q+2/3}\left[\left(1+\frac{2}{3}\frac{1}{x}+\frac{\frac{2}{3}(\frac{2}{3}+1)}{2}\frac{1}{x^2}+\dots\right)-\left(1-\frac{2}{3}\frac{1}{x}+\frac{\frac{2}{3}(\frac{2}{3}+1)}{2}\frac{1}{x^2}-\dots\right)\right]$$

$$= x^{q+2/3}\left(\frac{4}{3x}+k\frac{1}{x^3}+\dots\right), \quad \text{(some } k\text{)}$$

and choosing $q = 1/3$ we ensure existence of the non-zero finite limit $4/3$, as all other terms vanish

Many tried the approach $A^3 - B^3 = (A-B)(A^2+AB+B^2)$ *with* $A = (x+1)^{2/3}$*, etc. More complicated, but works fine. Others ignored the limit to infinity and expanded incorrectly:*
$(1+x)^{2/3} = 1 + \frac{2}{3}x + [\frac{2}{3}(\frac{2}{3}-1)/2]x^2 + \ldots$

(b) Differentiate to obtain $\dfrac{dy}{dx}$:

$$(i) \quad y = (\sin x)^{\cos x}\,, \qquad (ii) \quad \cos(x) = \sin(y)\,, \qquad (iii) \quad y^2 = \cos(xy)\,.$$

Solution

(i) Logarithmic differentiation:
$\ln y = \cos x \ln(\sin x) \Rightarrow \dfrac{1}{y}\dfrac{dy}{dx} = -\sin x \ln(\sin x) + \cos x \dfrac{1}{\sin x}\cos x$
so that $\dfrac{dy}{dx} = (\sin x)^{\cos x - 1}\cos^2 x - (\sin x)^{\cos x + 1}\ln(\sin x)\,.$

(ii) and (iii): Differentiate implicitly:

$$(ii) \qquad -\sin x = \cos y \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = -\frac{\sin x}{\cos y}\,.$$

Many noticed that $\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-\cos^2 x} = \pm\sin x$*, and correctly wrote* $\frac{dy}{dx} = \pm 1$*. Others ignored the negative root.*

$$(iii) \qquad 2y\frac{dy}{dx} = -\sin(xy)\left(y + x\frac{dy}{dx}\right) \Rightarrow \frac{dy}{dx} = -\frac{y\sin(xy)}{2y + x\sin(xy)}\,.$$

(c) Given the function

$$f(x) = \frac{2x^2 - 5x + 1}{x+1}\,,$$

find all stationary points and their nature, obtain any asymptotes and give a sketch showing these and any other relevant features.

Solution To find stationary points, differentiate:

$$f'(x) = \frac{(4x-5)(x+1) - (2x^2 - 5x + 1)}{(x+1)^2} = \frac{2x^2 + 4x - 6}{(x+1)^2} = 0$$

giving stationary points at $x = -3, 1$.

There is a vertical asymptote at $x = -1$ and given that at -1, the numerator $2x^2 - 5x + 1 = 8$ we have the asymptotic behaviour on either side:

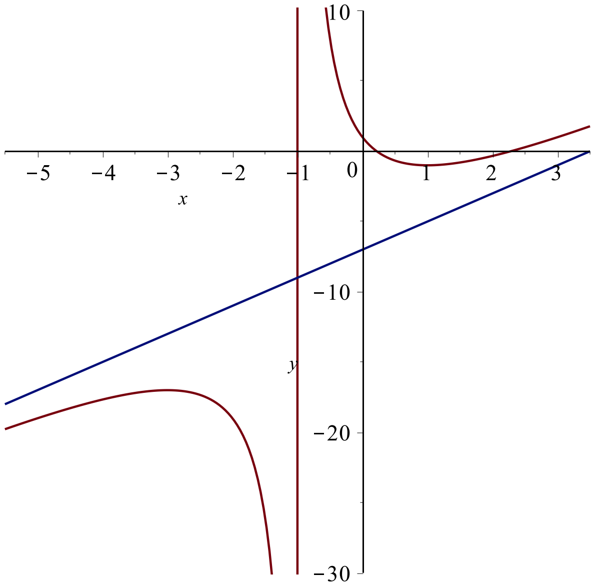
as $x \to -1^+$ we have $f(x) \to \infty$, and as $x \to -1^-$ we have $f(x) \to -\infty$.

Using polynomial division we have

$$f(x) = 2x - 7 + \frac{8}{x+1}$$

giving a diagonal asymptote $g(x) = 2x-7$. As $x \to \infty$, $1/(x+1) \to 0^+$ so the function is approaching the asymptote from above, and vice-versa in the other direction. For $x > -1$ we have $x \to \infty$ on both sides of the stationary point at $x = 1$: it must be a minimum. For $x < -1$, we have $x \to -\infty$ on both sides of the stationary point at $x = -3$: it must be a maximum. The alternative to this is to calculate the second derivative and evaluate it at $x = -3, 1$, but the argument with asymptotics is sufficient.

Intercepts are at $(0, 1)$ and $\left(\frac{5\pm\sqrt{17}}{4}, 0\right) \approx (1/4, 0), (9/4, 0)$. The stationary points are at $(-3, -17)$ and $(1, -1)$ and we can sketch the function:



Many people ignored some of the detail such as intercepts. More than half of students ignored the diagonal asymptote. Only one student (!) checked the detail that "as $x \to \infty$, $1/(x+1) \to 0^+$ so the function is approaching the asymptote from above, and vice-versa in the other direction."

(d) Obtain the n^{th} derivative $\dfrac{d^n y}{dx^n}$ for

$$y = x^2 e^{-x}\,.$$

Solution

Using Leibnitz' Theorem we get

$$y^{(n)} = x^2 D^n e^{-x} + n(2x)D^{n-1}e^{-x} + \frac{n(n-1)}{2}(2)D^{n-2}e^{-x}$$

and reasoning that

$$D^n e^{-x} = (-1)^n e^{-x}$$

we conclude that

$$y^{(n)} = (-1)^n e^{-x}\left[x^2 - 2nx + n(n-1)\right]\,.$$

Many differentiated three or four times and tried to deduce the general form. Some of these found the $2nx$ and x^2 terms, but none got the $n(n-1)$ term.

3. (a) Evaluate the indefinite integrals:

$$(i) \int \frac{4x-6}{x^2-3x+4}\, dx \quad (ii) \int \frac{1}{x \ln x}\, dx\,, \quad (iii) \int \frac{1}{4\sin x - 3\cos x - 5}\, dx\,.$$

SOLUTION

(i) Observing that $(x^2-3x+4)' = 2x-3$ we get

$$\int \frac{2(2x-3)}{x^2-3x+4}\, dx = 2\ln(x^2-3x+4) + C\,.$$

(ii) Given that $(\ln x)' = 1/x$ we substiture $u = \ln x$ to get

$$\int \frac{1}{x \ln x}\, dx = \int \frac{1}{u}\, du = \ln(\ln x) + C\,.$$

A number tried integration by parts, unsuccessfully.

(iii) Using the substitution $t = \tan(x/2)$ (formula sheet) we have $\sin x = 2t/(1+t^2)\,, \cos x = (1-t^2)(1+t^2)$ and $dx = 2dt/(1+t^2)$ and the integral becomes

$$\int \frac{1}{\left(4\frac{2t}{1+t^2} - 3\frac{1-t^2}{1+t^2} - 5\right)} \frac{2dt}{(1+t^2)} = \int \frac{2\ dt}{4(2t) - 3(1-t^2) - 5(1+t^2)}$$

$$= -\int \frac{2}{2t^2-8t+8}\, dt = -\frac{1}{(t-2)^2}\, dt = \frac{1}{t-2} + C = \frac{1}{\tan(x/2)-2} + C\,.$$

Lots left the answer in terms of t.

(b) Use a substitution to integrate $\dfrac{1}{\sqrt{x^2-1}}$ and hence show that

$$\cosh^{-1} x = \ln(x + \sqrt{x^2-1})\,.$$

SOLUTION The required substitution is $x = \cosh u \Rightarrow dx = \sinh u\, du$ and $x^2-1 = \cosh^2 u - 1 = \sinh^2 u$ so that

$$\int \frac{1}{\sqrt{x^2-1}}\, dx = \int \frac{1}{\sqrt{\sinh^2 u}} \sinh u\, du = \int 1\ du = u + C = \cosh^{-1} x + C\,.$$

If two functions are equal, they have the same derivative. From the above, the Fundamental Theorem of Calculus gives that

$$\frac{d}{dx}\left(\cosh^{-1}\right) = \frac{1}{\sqrt{x^2-1}}\,.$$

If the last exprssion is also equal to the derivative of $\ln(x+\sqrt{x^2-1})$ then the two functions are equal, up to a constant:

$$\frac{d}{dx}\left(\ln(x+\sqrt{x^2-1})\right) = \frac{1}{x+\sqrt{x^2-1}}\left(1+\frac{2x}{2\sqrt{x^2-1}}\right) = \frac{1}{x+\sqrt{x^2-1}}\left(\frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}}\right),$$

and the last cancellation gives the desired result. The alternative is to let $y=\cosh^{-1}x \Rightarrow x=\cosh y$ and solve this for $y=\ln(x+\sqrt{x^2-1})$, but this loses marks, as the instruction is to use the result of the integration.

Many did not differentiate $\ln(x+\sqrt{x^2-1})$. *Lots tried the substitution* $x=\sinh u$. *Others tried* $x=\cos u$ *or* $\sin u$*, which led to complex roots and then attempted to establish a (non-existing) relationship between* $\cos^{-1}(ix)$ *and* $\cosh^{-1}x$

(c) Obtain the Maclaurin series of $\dfrac{1}{e^{-x}+1}$ to first order with remainder term. Explain how the error estimate from the remainder term can be improved without any more terms in the series. Obtain the improved error estimate.

SOLUTION

To obtain the Maclaurin series to order one we need to differentiate twice:

$$f'(x) = \frac{e^{-x}}{(1+e^{-x})^2} \Rightarrow f''(x) = \frac{-e^{-x}(1+e^{-x})^2-(e^{-x})2(1+e^{-x})(-e^{-x})}{(1+e^{-x})^4}$$

which simplifies to

$$f''(x) = \frac{e^{-x}(e^{-x}+1)}{(1+e^{-x})^3}$$

so that

$$f(0)=\frac{1}{2}\,, f'(0)=\frac{1}{4} \Rightarrow f(x)=\frac{1}{2}+\frac{1}{4}x+R_1$$

where the Lagrange remainder is

$$R_1 = \frac{e^{-c}(e^{-c}+1)}{2(1+e^{-c})^3}x^2\,, \quad \text{with} \quad 0<|c|<|x|\,.$$

We can improve the error estimate without adding further terms to the series by observing that $f''(0)=0$, so that

$$f(x) = \frac{1}{2}+\frac{1}{4}x+0x^2+R_2$$

where careful differentiation gives

$$f'''(x) = \frac{e^{-x}(e^{-2x}-4e^{-x}+1)}{(1+e^{-x})^4}$$

and the remainder term is

$$R_2 = \frac{e^{-c}(e^{-2c} - 4e^{-c} + 1)}{6(1+e^{-c})^4} x^3 \,,$$

where near zero, the higher power of x makes $|R_2|$ smaller than $|R_1|$ and hence an improved error estimate.

Most people got the series correct, sometimes with minor arithmetic errors in the derivatives. The idea of improving the error estimate without more terms in the series was interpreted by nearly everyone to mean looking at c to narrow down the interval of uncertainty, but this only works if we have a specific x and are able to determine the behaviour of the second derivative on the interval between zero and the given x: too complicated in this case!

(d) Use the integral test to find constants A,B such that

$$A < \sum_{n=1}^{\infty} \frac{1}{n^3} < B \,.$$

SOLUTION

The integral test gives

$$\sum_{n=2}^{\infty} \frac{1}{n^3} < \int_1^{\infty} \frac{1}{x^3}\, dx < \sum_{n=1}^{\infty} \frac{1}{n^3} \Longrightarrow \int_1^{\infty} \frac{1}{x^3}\, dx = \left[-\frac{1}{2x^2} \right]_1^{\infty} = \frac{1}{2} \,,$$

so immediately $A = 1/2$ can be chosen. Then

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = 1 + \sum_{n=2}^{\infty} \frac{1}{n^3} < 1 + \int_1^{\infty} \frac{1}{x^3}\, dx = 1 + \frac{1}{2} = \frac{3}{2} = B$$

An alternative is to pick any partial sum, or even zero, for the lower bound, or compare with the known $\sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6$, for the upper bound, but use of the integral test at least once is required for full marks.

Most people did the integral, finding $1/2$, and thus concluded that A or B is $1/2$, correct for A, but not for B.

4. (a) Find the radius and interval of convergence of the infinite series

$$\sum_{n=2}^{\infty} \frac{(3x)^n}{n(n-1)} \,,$$

SOLUTION

Begin with the ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(3x)^{n+1}}{(n+1)n}}{\frac{(3x)^n}{n(n-1)}}\right| = 3\frac{n-1}{n+1}|x|$$

so that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 3|x| \lim_{n\to\infty}\frac{1-\frac{1}{n}}{1+\frac{1}{n}} = 3|x|$$

and we require $3|x| < 1$ for convergence, so the radius of convergence is $1/3$. The interval of convergence given by the ratio test is $-1/3 < x < 1/3$, and we know the series diverges for $|x| > 1/3$, but the ratio test gives no information regarding the convergence for $x = \pm 1/3$, which need to be tested separately. Letting $x = 1/3$ we have

$$\sum_{n=2}^{\infty}\frac{1}{n(n-1)}\,,$$

a telescoping sum, converging to a known value, as shown in lectures. Letting $x = -1/3$ we have

$$\sum_{n=2}^{\infty}\frac{(-1)^n}{n(n-1)}\,,$$

which converges absolutely, comparing to the case $x = 1/3$, or by the alternating series test. Hence the interval of converges is extended to $-1/3 \le x \le 1/3$.

Many people did not consider the endpoints. Of those that did, many thought the series diverges in the case $x = 1/3$. A very few of them found divergence for $x = -1/3$.

(b) Without obtaining the Fourier Series of the function

$$f(x) = \begin{cases} x+2\,, & 0 \le x < 1.5 \\ 4-x\,, & 1.5 \le x < 3 \end{cases} \quad \text{and} \quad f(x+3) = f(x)\,, \forall x\,,$$

find the values of the Fourier Series at $x = 0$ and $x = 1.5$.

Solution At discontinuities x_0, the FS converges to the average of the limiting values:

$\frac{1}{2}\left(\lim_{x\to x_0^+} f(x) + \lim_{x\to x_0^-} f(x)\right)$. It is often best to sketch the function.

At $x = 0$: $\frac{1}{2}\left([0+2]+[4-3]\right) = \frac{3}{2}$, as $f(0)^- = f(3)^-$.

At $x = 1.5$: $\frac{1}{2}\left([1.5+2]+[4-1.5]\right) = 3$.

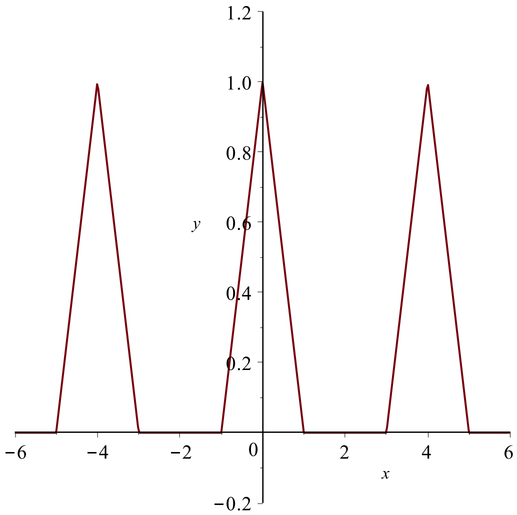
Most people incorrectly calculated $f(0) = 0+2$ and $f(1.5) = 4-1.5$. Some claimed that at discontinuities, the FS is not defined or equal to infinity. Many drew the correct picture, but still did not find the averages.

(c) A function is defined as

$$f(x) = \begin{cases} 1-x & 0 \le x < 1 \\ 0 & 1 \le x < 2 \end{cases}$$

(i) Obtain $g(x)$, the even extension of $f(x)$, with period $T = 4$ and sketch $g(x)$ for $-6 \le x \le 6$.

Solution



Many strange pictures, many non-periodic ones. Many had the correct shape but the other "tents" were on $x \in [2,4]$ and $x \in [-4,-2]$.

(ii) Obtain the Fourier cosine series of $g(x)$.
[You may assume that $\cos(n\pi/2) = (-1)^{n/2}$ for even n.]

Solution

It's an even function, so all $b_n = 0$ and the series is a Fourier cosine series. For period $T = 2L$, the half-range formula is

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx$$

so with $T = 4 = 2L$ we have

$$a_0 = \int_0^2 f(x)\,dx = \int_0^1 1 - x\,dx + \int_1^2 0\,dx = \left[x - \frac{x^2}{2}\right]_0^1 = \frac{1}{2}$$

and

$$\begin{aligned} a_n &= \int_0^2 f(x)\cos\left(\frac{n\pi x}{2}\right)\,dx = \int_0^1 (1-x)\cos\left(\frac{n\pi x}{2}\right)\,dx \\ &= \frac{2}{n\pi}\left[2(1-x)\sin\left(\frac{n\pi x}{2}\right)\right]_0^1 + \frac{2}{n\pi}\int_0^1 \sin\left(\frac{n\pi x}{2}\right)\,dx \end{aligned}$$

$$= \quad 0 - 0 - \frac{4}{n^2\pi^2}\left[\cos\left(\frac{n\pi x}{2}\right)\right]_0^1 = \frac{4}{n^2\pi^2}\left[1 - \cos\left(\frac{n\pi}{2}\right)\right]$$

When n is odd, $\cos(n\pi/2) = 0$ so $a_n = 4/(n^2\pi^2)$. For even n we use the hint: $\cos(n\pi/2) = (-1)^{n/2}$ and so

$$a_n = \frac{4}{n^2\pi^2}\left[1 - (-1)^{n/2}\right]$$

and so it is more convenient to begin with two series, one for even n and one for odd n:

$$f(x) = \frac{1}{4} + \frac{4}{\pi^2}\left[\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} \frac{1}{n^2}\cos\left(\frac{n\pi x}{2}\right) + \sum_{\substack{n=2\\ n \text{ even}}}^{\infty} \frac{1-(-1)^{n/2}}{n^2}\cos\left(\frac{n\pi x}{2}\right)\right]$$

Full marks for the above or equivalent. Simplify by letting $n = 2m$ in the second sum:

$$= \frac{1}{4} + \frac{4}{\pi^2}\left[\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} \frac{1}{n^2}\cos\left(\frac{n\pi x}{2}\right) + \sum_{m=1}^{\infty} \frac{1-(-1)^{m}}{(2m)^2}\cos(m\pi x)\right]$$

finally, revert to n, and use $1 - (-1)^n = 2$ for odd n, and zero for even n:

$$\begin{aligned}
&= \frac{1}{4} + \frac{4}{\pi^2}\left[\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} \frac{1}{n^2}\cos\left(\frac{n\pi x}{2}\right) + \frac{1}{2}\sum_{\substack{n=1\\ n \text{ nodd}}}^{\infty} \frac{1}{n^2}\cos(n\pi x)\right] \\
&= \frac{1}{4} + \frac{4}{\pi^2}\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} \frac{1}{n^2}\left[\cos\left(\frac{n\pi x}{2}\right) + \frac{1}{2}\cos(n\pi x)\right] \\
&= \frac{1}{4} + \frac{4}{\pi^2}\left(\cos\left(\frac{\pi x}{2}\right) + \frac{1}{2}\cos(\pi x) + \frac{1}{9}\cos\left(\frac{3\pi x}{2}\right)\right. \\
&\quad \left. + \frac{1}{18}\cos(3\pi x) + \frac{1}{25}\cos\left(\frac{5\pi x}{2}\right) + \frac{1}{50}\cos(5\pi x) + \dots\right)
\end{aligned}$$

The most challenging question. Few fully correct answers. Some integrated $(1-x)\cos(n\pi x)$, many integrated $(1-x)\cos(n\pi x/2)$, but over $x = 0..2$. Some integrated $(1+x)\cos(n\pi x/2)$ over $x = -1..0$ and the corresponding integral on $x = 0..1$ but the additional complication led to arithmetic errors. Many integrated $\cos(n\pi x/2)$ and $x\cos(n\pi x/2)$ separately, again complicating calculations and leading to sine terms not cancelling. Of those that got the integration right, most took the hint $\cos(n\pi/2) = (-1)^{n/2}$ for even n and incorrectly concluded that $a_n = 0$ for odd n, thus losing the main point: the more complicated form of a_n, resulting in the two series.

(iii) By careful choice of a value of x, using the results of (ii) or otherwise, calculate the infinite series

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$$

Solution The needed value is $x = 0$, where $f(x) = 1$ and all cosine terms are equal to 1, so that

$$1 = \frac{1}{4} + \frac{4}{\pi^2} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n^2} \left(1 + \frac{1}{2}\right) \Rightarrow \frac{3}{4} = \frac{6}{\pi^2} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n^2}$$ and multiplying gives the result $\pi^2/8$.

Many attempts to solve this using results obtained in (i) which had series over even n converting to series over odd n, with no explanation given. Some did the required work to obtain a series over odd n, but had incorrect constants. A few realized that this wouldn't work and correctly argued

$$\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{\substack{n=2 \\ n \text{ even}}}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} - \sum_{m=1}^{\infty} \frac{1}{(2m)^2} = \frac{\pi^2}{6} - \frac{1}{4}\frac{\pi^2}{6} .$$