EE3-09

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2014-15

EEE/EIE PART III/IV: MEng, BEng and ACGI

Corrected Copy

**CONTROL ENGINEERING**

9:35 AM
CORRECTION ON
QUESTION 2 a.ii)
SEE INSIDE

Friday, 19 December 9:00 am

Time allowed: 3:00 hours

**There are FOUR questions on this paper.**

**Answer ALL questions.**

*All questions carry equal marks*

**Any special instructions for invigilators and information for candidates are on page 1.**

Examiners responsible First Marker(s) : A. Astolfi
Second Marker(s) : D. Angeli

## CONTROL ENGINEERING

1. Consider a linear, single-input, discrete-time, system of dimension $n = 3$, that is $x = [x_1, x_2, x_3]'$, with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

a) Study the reachability and controllability properties of the system. [ 4 marks ]

b) Compute the set of states $\mathscr{R}_2$ that can be reached from $x(0) = 0$ in two steps. [ 4 marks ]

c) Determine the set of states $\mathscr{C}_2$ that can be controlled to zero in two steps. Explain why the set $\mathscr{C}_2$ is larger than the set $\mathscr{R}_2$. [ 6 marks ]

d) Consider a state feedback control law $u = Kx$ and determine the set of all matrices $K$ such that the matrix $A + BK$ has all eigenvalues equal to 0. Explain why the matrix assigning the closed-loop eigenvalues exists, but it is not unique, and determine the matrix $K$ which is such that $KK^T$ is minimal. [ 6 marks ]

2. Consider the problem of regulating the temperature of a shower. The system can be modeled as a linear discrete-time system described by the equation

$$T(t+h) = u(t),$$

in which $h$ is a positive integer, $T$ denotes the temperature of the water, and $u$ is the user selected control signal. The constant $h$ models how fast the boiler is to react to the user command. Most people update $u$ depending on the difference between the current temperature and their favourite temperature $T_0$, thus implementing the equation

$$u(t) = u(t-1) - \alpha(T(t) - T_0),$$

where $\alpha \in (1,2)$.

a) Assume $h = 1$, that is the boiler is *fast*.

i) By eliminating the variable $u$ write the equation of the closed-loop system as a difference equation in the variable $T(t)$ with input $T_0$. [ 2 marks ]

ii) Study the stability properties of the system determined in part a.i) as a function of the parameter $\alpha \in (1,2)$. [ 2 marks ]

b) Assume $h = 2$, that is the boiler is *slow*.

i) By eliminating the variable $u$ write the equation of the closed-loop system as a difference equation in the variable $T(t)$ with input $T_0$. [ 4 marks ]

ii) Let $T(t)$ be the output of the system. Determine a state space representation for the system, that is write the equation of the system in the form

$$x^+ = Ax + Bu, \qquad y = Cx,$$

where $x$ is the state vector with two components, $y(t) = T(t)$ and $u(t) = T_0$. Write explicitly the matrices $A$, $B$ and $C$. [ 6 marks ]

iii) Study the stability properties of the system determined in part c.ii) as a function of the parameter $\alpha \in (1,2)$. [ 4 marks ]

c) Explain why a fast boiler gives a better shower. [ 2 marks ]

3. A linear, continuous-time, descriptor system is a system described by the equations

$$E\dot{x} = Ax + Bu, \qquad y = Cx,$$

in which $x(t) \in I\!R^n$ denotes the state of the system, $u(t) \in I\!R^m$ the input signal and $y(t) \in I\!R^p$ the output signal. The matrix $E \in I\!R^{n\times n}$ is singular, whereas the matrices $A$, $B$ and $C$ are defined as for standard linear systems.

Descriptor systems often arise as the result of a modeling process whenever too many states have been introduced and can be transformed, under certain conditions, into standard linear systems using the simple procedure described in what follows for a specific example.

Let $x(t) = \begin{bmatrix} x_1(t) & x_2(t) & x_3(t) \end{bmatrix}'$

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}.$$

a) Write explicitly, that is one by one, the three equations

$$E\dot{x} = Ax + Bu.$$

[ 2 marks ]

b) Observe that one of the equations in part a) does not contain any time derivative. Solve this equation for the variable $x_3$ and use the solution to eliminate the variable $x_3$ from the differential equations and from the equation that defines the output signal. [ 2 marks ]

c) Write the differential equations and the output equation computed in part b) in the form

$$\dot{\chi} = A_r\chi + B_r u, \qquad y = C_r\chi + D_r u,$$

with $\chi(t) \in I\!R^2$. Compute explicitly the matrices $A_r$, $B_r$ $C_r$ and $D_r$. [ 4 marks ]

d) The eigenvalues of the descriptor system are the solution of the equation $\det(\lambda E - A) = 0$. Compute the eigenvalues of the considered descriptor system and show that these coincide with the eigenvalues of the matrix $A_r$. [ 4 marks ]

e) Show that the system determined in part c) is observable. The descriptor system is observable if

$$\text{rank}\begin{bmatrix} \lambda E - A \\ C \end{bmatrix} = 3,$$

for all $\lambda$ which are eigenvalues of the descriptor system. Show that the descriptor system is observable. [ 4 marks ]

f) Show that the system determined in part c) is controllable. Show using the definition of controllability and the simplest possible argument that the descriptor system is also controllable. [ 4 marks ]

4. Consider a nonlinear, continuous-time, system described by the equations

$$\dot{x} = f(x) = \begin{bmatrix} x_2^2 \\ x_3^2 \\ 0 \end{bmatrix},$$

with $x(t) = [x_1(t), x_2(t), x_3(t)]' \in \mathbb{R}^3$.

a) Compute all equilibrium points of the system. [ 2 marks ]

b) Compute the linearization of the system around the equilibrium points determined in part a) and write explicitly the matrix $A$ of the linearized system. (Hint: the matrix $A$ is the same at all equilibrium points.)

[ 4 marks ]

c) Study the stability properties of the linearized system. [ 2 marks ]

d) Solve the differential equations of the nonlinear system to determine $x(t)$ as a function of $x(0)$, hence argue that all equilibrium points of the system are unstable. [ 6 marks ]

e) Consider the output equation $y = x_1$. Show that

$$\frac{d^4y}{dt^4} = 0,$$

hence argue that the nonlinear system with output can be rewritten as a linear system with equations

$$\dot{x}_e = A_e x_e, \qquad y = C_e x_e,$$

with $x_e(t) \in \mathbb{R}^4$. Write explicitly the matrices $A_e$ and $C_e$. [ 6 marks ]