Imperial College London

M4/5S8

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May-June 2018

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Time Series

Date: Thursday, 10 May 2018

Time: 10:00 AM - 12:30 PM

Time Allowed: 2.5 hours

This paper has 5 questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the relevant main book(s) are full.

All required additional material will be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Each question carries equal weight.
- Calculators may not be used.

Note: Throughout this paper $\{\epsilon_t\}$ is a sequence of uncorrelated random variables (white noise) having zero mean and variance σ_ϵ^2, unless stated otherwise. The unqualified term "stationary" will always be taken to mean second-order stationary. All processes are real-valued unless stated otherwise. The sample interval is unity unless stated otherwise. B denotes the backward shift operator. The autocovariance sequence for a stationary process is denoted by $\{s_\tau\}$.

1. (a) What is meant by saying that a stochastic process is stationary?

   (b) Are the following statements true or false?:

   (i) a strictly stationary time series $\{X_t\}$ which is not Gaussian/normal is second-order stationary;

   (ii) a time series which is the sum of of a seasonal component of period $s = 12$, a linear trend, and a stationary process, can be made stationary by applying the operator $(1 - B - B^{12} + B^{13})$;

   (iii) a time series $\{X_t\}$ with general linear process form $X_t = G(B)\epsilon_t$, will be invertible if the z-polynomial $G(z)$ is analytic for $|z| \leq 1$;

   (iv) a process with a purely discrete spectrum (or line spectrum) has an autocovariance sequence $\{s_\tau\}$ such that $s_\tau \to 0$ as $|\tau| \to \infty$;

   (v) as more tapering is performed with direct spectral estimators, sidelobe leakage decreases.

   (c) The zero mean, stationary, AR(1) process $Y_t = \phi Y_{t-1} + \epsilon_t$ may be written in the form $Y_t = \sum_{j=0}^{\infty} \phi^j \epsilon_{t-j}$. By applying the same steps used in the derivation of the Yule-Walker equations, and utilising this result, show that

$$s_{Y,\tau} = \frac{\phi^{|\tau|}}{1-\phi^2}\sigma_\epsilon^2,$$

   where $s_{Y,\tau} = \text{cov}\{Y_t, Y_{t+\tau}\}$.

   (d) Show that the zero mean and stationary ARMA(p,q) process

$$X_t = \phi_{1,p}X_{t-1} + \cdots + \phi_{p,p}X_{t-p} + \epsilon_t - \theta_{1,q}\epsilon_{t-1} - \cdots - \theta_{q,q}\epsilon_{t-q}$$

   can be decomposed as

$$X_t = -\sum_{j=0}^{q} \theta_{j,q} Y_{t-j} \quad \text{where} \quad Y_t = \sum_{k=1}^{p} \phi_{k,p} Y_{t-k} + \epsilon_t \quad \text{and} \quad \theta_{0,q} = -1.$$

   Hence express $s_{X,\tau} = \text{cov}\{X_t, X_{t+\tau}\}$ in terms of $\{s_{Y,\tau}\}$.

   (e) Using the results in parts (c) and (d), show that, for a zero mean and stationary ARMA(1,1) process, $s_{X,0} = \text{var}\{X_t\}$ is given by

$$s_{X,0} = [1 + c]\sigma_\epsilon^2,$$

   where the form of c is to be found.

2. Consider the zero mean and stationary MA(2) process $\{X_t\}$ given by

$$X_t = \epsilon_t - \theta_{1,2}\epsilon_{t-1} - \theta_{2,2}\epsilon_{t-2}.$$

(a) (i) Use the filtering approach to derive the spectral density function of $\{X_t\}$ and show it may be written as

$$S(f) = \sigma_\epsilon^2[1 + \theta_{1,2}^2 + \theta_{2,2}^2 - 2\theta_{1,2}(1 - \theta_{2,2})\cos(2\pi f) - 2\theta_{2,2}\cos(4\pi f)]$$

(ii) The spectral density function of $\{X_t\}$ is the Fourier transform of its autocovariance sequence $\{s_\tau\}$. Using just this relationship show that

$$s_0 = (1 + \theta_{1,2}^2 + \theta_{2,2}^2)\sigma_\epsilon^2; \quad s_1 = -\theta_{1,2}(1 - \theta_{2,2})\sigma_\epsilon^2; \quad s_2 = -\theta_{2,2}\sigma_\epsilon^2.$$

(b) Let $\theta_{1,2} = 1, \theta_{2,2} = -1/2$.

(i) Determine whether $\{X_t\}$ is invertible, and give the meaning of invertibility.

(ii) Find the values of the autocorrelation sequence elements ρ_1 and ρ_2.

(c) Now consider a zero mean and stationary MA(2) process $\{Y_t\}$ for which its characteristic polynomial has roots $\frac{1}{2} \pm \frac{1}{2}\mathrm{i}$.

(i) Find the parameters $\theta_{1,2}$ and $\theta_{2,2}$.

(ii) Find the autocorrelation sequence elements ρ_1 and ρ_2.

(d) Carefully explain the relationship you observe between the autocorrelation sequence of $\{X_t\}$ defined in part (b) and the autocorrelation sequence $\{Y_t\}$ defined in part (c). Is $\{Y_t\}$ invertible? What do you conclude?

3. (a) Given a sample, $X_1, \ldots, X_N$, from a zero mean stationary time series $\{X_t\}$, the periodogram spectral estimator, $\widehat{S}^{(p)}(f)$, of the spectral density function, $S(f)$, is given by

$$\widehat{S}^{(p)}(f) = \left| \sum_{t=1}^{N} \frac{1}{\sqrt{N}} X_t \mathrm{e}^{-\mathrm{i}2\pi f t} \right|^2 .$$

(i) Use the spectral representation theorem to show that the mean of the periodogram, $\widehat{S}^{(p)}(f)$, is given by

$$E\{\widehat{S}^{(p)}(f)\} = \int_{-1/2}^{1/2} \mathcal{F}(f - f') S(f') \mathrm{d}f' ,$$

where $\mathcal{F}(f)$ denotes Fejér's kernel given by $\mathcal{F}(f) = \frac{1}{N} \left| \sum_{t=1}^{N} \mathrm{e}^{-\mathrm{i}2\pi f t} \right|^2$.

Is the periodogram biased or unbiased if $\{X_t\}$ is white noise? Justify your answer.

(ii) Consider the case when $X_t = \epsilon_t$, where $\{\epsilon_t\}$ is Gaussian/normal distributed, i.e., the process is Gaussian/normal white noise. By writing the periodogram, $\widehat{S}^{(p)}(f_j)$, *at the Fourier frequencies* $f_j = j/N, 1 \le j < N/2$, in the form $\widehat{S}^{(p)}(f_j) = |Y_1(f_j) + \mathrm{i}Y_2(f_j)|^2$, where $Y_1(f_j), Y_2(f_j)$ are real-valued random variables, show that

$$\widehat{S}^{(p)}(f_j) \stackrel{\mathrm{d}}{=} \frac{\sigma_\epsilon^2}{2} \chi_2^2,$$

i.e., $\widehat{S}^{(p)}(f_j)$ is distributed as a scaled version of a chi-square random variable with 2 degrees of freedom.

You will need the following results:

$$\sum_{t=1}^{N} \cos^2(2\pi f_j t) = \sum_{t=1}^{N} \sin^2(2\pi f_j t) = \frac{N}{2}; \qquad \sum_{t=1}^{N} \cos(2\pi f_j t) \sin(2\pi f_j t) = 0.$$

(b) The autocovariance sequence $\{s_{Z,\tau}\}$ for a *complex-valued* stationary time series $\{Z_t\}$ with mean zero is defined as $s_{Z,\tau} = \mathrm{cov}\{Z_t, Z_{t+\tau}\} = E\{Z_t^* Z_{t+\tau}\}$, where * denotes complex conjugation. A second quantity, called the complementary covariance, denoted $\{r_{Z,\tau}\}$, is defined as $r_{Z,\tau} = \mathrm{cov}\{Z_t^*, Z_{t+\tau}\} = E\{Z_t Z_{t+\tau}\}$, and is the covariance sequence between $\{Z_t\}$ and its complex-conjugate. It is an important quantity in areas such as communications. If $\{r_{Z,\tau}\}$ is zero for all $\tau \in \mathbb{Z}$ then $\{Z_t\}$ is called *proper*.

Consider the time series $Z_t = X_t \mathrm{e}^{\mathrm{i}Y_t}$. Here $\{X_t\}$ is a real-valued, zero mean, unit variance, stationary process with autocovariance $\{s_{X,\tau}\}$. $\{Y_t\}$ is a sequence of independent random variables drawn from the uniform distribution on $[-\pi, \pi]$. The sequences $\{X_t\}$ and $\{Y_t\}$ are assumed independent of each other, (i.e., the random variables $X_{t_1}, \ldots, X_{t_n}$ and $Y_{t'_1}, \ldots, Y_{t'_n}$ are mutually independent for any $n \ge 1$).

(i) Find the form of the sequence $\{s_{Z,\tau}\}$. [Express the values in integers, to be found.]

(ii) Determine if $\{Z_t\}$ is proper.

4. (a) Consider the bivariate white noise process

$$\boldsymbol{X}_t = \begin{bmatrix} X_{1,t} \\ X_{2,t} \end{bmatrix} = \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix} = \boldsymbol{\epsilon}_t$$

where $E\{\boldsymbol{\epsilon}_t\} = 0$ and $E\{\boldsymbol{\epsilon}_s \boldsymbol{\epsilon}_t^T\} = \Sigma$ if $s = t$ and zero otherwise. (Here ^{T} denotes transpose.)

(i) Show, for any nonzero real numbers a_1, a_2, that $\sum_{j=1}^{2} \sum_{k=1}^{2} \sigma_{jk} a_j a_k \geq 0$, where σ_{jk} is the (j,k)th element of Σ, i.e., that Σ is positive semidefinite.

(ii) Show that $\{X_{1,t}\}$ and $\{X_{2,t}\}$ are jointly stationary stochastic processes.

Now assume $\Sigma = \Sigma_1 \stackrel{\text{def}}{=} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

(iii) Verify that Σ_1 is positive semidefinite.

(iv) Find the coherence $\gamma^2_{X_1,X_2}(f)$ for $|f| \leq 1/2$.

(v) Are $\{X_{1,t}\}$ and $\{X_{2,t}\}$ in fact identical? Justify your answer.

(b) (i) Suppose the stationary process $\{X_t\}$ can be written as a one-sided linear process, $X_t = \sum_{k=0}^{\infty} \psi_k \epsilon_{t-k}$, with $\psi_0 = 1$. We wish to construct the l-step ahead forecast $X_t(l) = \sum_{k=0}^{\infty} \delta_k \varepsilon_{t-k}$. Show that the linear least squares predictor, which minimizes $E\{(X_{t+l} - X_t(l))^2\}$, corresponds to setting $\delta_k = \psi_{k+l}$, $k \geq 0$.

Now assume a zero mean stationary AR(1) process, $X_t = \phi X_{t-1} + \epsilon_t$, and linear least squares prediction for which the l-step prediction variance is $\sigma^2(l) = \sigma_\epsilon^2 \sum_{k=0}^{l-1} \psi_k^2$.

(ii) Find the resulting 2-step prediction variance $\sigma^2(2)$ in terms of σ_ϵ^2 and ϕ.

(iii) From the course notes on forecasting, we know that for linear least squares prediction, the l-step ahead forecast $X_t(l)$ of X_{t+l} may be obtained by setting future innovations to zero. Calculate the 2-step prediction variance again, this time by evaluating $E\{(X_{t+2} - X_t(2))^2\}$.
Hint: Recall the required result in Q1(c).

5.

(a) Consider a *continuous* parameter real-valued stationary process $\{X(t)\}$ with a Lorenzian spectral density function (SDF) given by

$$S(f) = \frac{2L\sigma^2}{1+(2\pi f L)^2}, \qquad f \in \mathbb{R},$$

where $\sigma^2 > 0$ is the process variance, and $L > 0$ is a real-valued parameter. Show, with full justification, that the autocovariance function for $\{X(t)\}$ is given by

$$s(\tau) = \sigma^2 \mathrm{e}^{-|\tau|/L}, \qquad \tau \in \mathbb{R}.$$

Hint: for $m \in \mathbb{R}$,

$$\int_0^\infty \frac{\cos(mx)}{1+x^2}\mathrm{d}x = \frac{\pi}{2}\mathrm{e}^{-|m|}.$$

(b) Consider a *continuous* parameter real-valued stationary process $\{X(t)\}$ with a spectral density function (SDF) given by

$$S(f) = \begin{cases} C, & |f| \le 2; \\ 0, & |f| > 2, \end{cases}$$

where $C > 0$ is a real-valued constant (a process with the above SDF is known as band-limited white noise). For a given sampling interval $\Delta t > 0$, define the associated discrete time process by $X_t = X(t\,\Delta t)$, $t \in \mathbb{Z}$.

(i) Starting from the standard aliasing formula, (which you are not required to prove), explain why the SDF $S_{X_t}(f;\Delta t)$ of $\{X_t\}$, for $f \in [0, f_N]$, can be written in terms of $S(f)$ given above, without error, as

$$S_{X_t}(f;\Delta t) = S(f) + \sum_{k=1}^{m} S(f+\tfrac{k}{\Delta t}) + \sum_{k=1}^{\ell} S(f-\tfrac{k}{\Delta t}) = \sum_{k=-\ell}^{m} S\left(f+\tfrac{k}{\Delta t}\right), \qquad \text{for} \quad f \in [0, f_N],$$

where ℓ and m are suitable integers and $f_N = 1/(2\Delta t)$ is the Nyquist frequency.
Show that $m = \lfloor 2\Delta t \rfloor$, where $\lfloor x \rfloor$ is the largest integer less than or equal to x.
Express ℓ in the form $\ell = \lfloor y \rfloor$, where y is to be found.

(ii) Determine the SDFs for $\{X_t\}$ for $f \in [0, f_N]$, when $\Delta t = \frac{2}{3}$, $\Delta t = \frac{1}{3}$ and when $\Delta t = \frac{1}{5}$.

(iii) What is $S_{X_t}(f;\Delta t)$ for $f \in [-f_N, 0)$? How can $S_{X_t}(f;\Delta t)$ be found for $f \notin [-f_N, f_N]$?

(iv) Verify that each of the integrals over $[-f_N, f_N]$ for the SDFs in (b)(ii) is the same as the integral of $S(\cdot)$ over $f \in \mathbb{R}$.

(v) The terms 'red noise' and 'blue noise' are used to describe spectra with certain dominant frequencies. Classify the SDFs in (b)(ii) as 'red noise' or 'blue noise'.

Course: M3S8/M4S8/M5S8
Setter:
Checker:
Editor:
External:
Date: March 16, 2018

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2018

M3S8/M4S8/M5S8

Time Series [FINAL SOLUTIONS]

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Note: In the mark scheme the following categorization has been adopted
Routine (A), Sound (B), Borderline (C), Challenging (D). seen ⇓

1. (a) $\{X_t\}$ is second-order stationary if $E\{X_t\}$ is a finite constant for all t, var$\{X_t\}$ is a finite constant for all t, and cov$\{X_t, X_{t+\tau}\}$, is a finite quantity depending only on τ and not on t. 4 A

(b) (i) TRUE, a strictly stationary process always has first and second joint moments which are time invariant, whatever its distributional structure. sim. seen ⇓

(ii) TRUE, we need to apply the operators $(1-B)$ and $(1-B^{12})$ one after the other, which is equivalent to the stated operation.

(iii) FALSE, for invertibility (rather than stationarity) we need that $G^{-1}(B)X_t = \epsilon_t$, is well-defined, i.e., that $G^{-1}(z)$ is analytic for $|z| \leq 1$;

(iv) FALSE, the stated property holds for purely continuous spectra; for a purely discrete spectrum the ACVS does not damp down;

(v) TRUE, the main purpose of tapering is the reduction of sidelobe leakage, and the more tapering, the bigger the decrease. 5 B

(c) Firstly multiply the defining equation through by $Y_{t-\tau}$ for $\tau > 0$, sim. seen ⇓

$$\begin{aligned} Y_t Y_{t-\tau} &= \phi Y_{t-1} Y_{t-\tau} + \epsilon_t Y_{t-\tau} \\ \Rightarrow Y_t Y_{t-\tau} &= \phi Y_{t-1} Y_{t-\tau} + \epsilon_t \sum_{j=0}^{\infty} \phi^j \epsilon_{t-\tau-j} \\ \Rightarrow s_{Y,\tau} &= \phi s_{Y,\tau-1} \Rightarrow s_{Y,\tau} = \phi^\tau s_{Y,0}, \end{aligned}$$

since the expectation of the rightmost term is zero. Next multiply the defining equation by Y_t

$$\begin{aligned} Y_t^2 &= \phi Y_{t-1} Y_t + \epsilon_t \sum_{j=0}^{\infty} \phi^j \epsilon_{t-j} \\ \Rightarrow s_{Y,0} &= \phi s_{Y,1} + \sigma_\epsilon^2 = \phi^2 s_{Y,0} + \sigma_\epsilon^2 \Rightarrow s_{Y,0} = \frac{\sigma_\epsilon^2}{1-\phi^2}, \end{aligned}$$

which, combined with the fact that $s_{Y,\tau}$ is symmetric, gives the required result. 4 A

(d) We have

$$X_t = -\sum_{j=0}^{q} \theta_{j,q} Y_{t-j} \quad \text{with} \quad Y_{t-j} = \sum_{k=1}^{p} \phi_{k,p} Y_{t-j-k} + \epsilon_{t-j}.$$

So

$$\begin{aligned} X_t &= -\sum_{k=1}^{p} \sum_{j=0}^{q} \phi_{k,p} \theta_{j,q} Y_{t-j-k} - \sum_{j=0}^{q} \theta_{j,q} \epsilon_{t-j} = \sum_{k=1}^{p} \phi_{k,p} \left(-\sum_{j=0}^{q} \theta_{j,q} Y_{t-j-k} \right) - \sum_{j=0}^{q} \theta_{j,q} \epsilon_{t-j} \\ &= \sum_{k=1}^{p} \phi_{k,p} X_{t-k} - \sum_{j=0}^{q} \theta_{j,q} \epsilon_{t-j}, \quad \text{which is of the required form.} \end{aligned}$$

2 C

Next,

$$\begin{aligned} s_{X,\tau} &= \operatorname{cov}\{X_t, X_{t+\tau}\} = \operatorname{cov}\{-\sum_{j=0}^{q} \theta_{j,q} Y_{t-j}, -\sum_{k=0}^{q} \theta_{k,q} Y_{t+\tau-k}\} \\ &= \sum_{j=0}^{q}\sum_{k=0}^{q} \theta_{j,q}\theta_{k,q} \operatorname{cov}\{Y_{t-j}, Y_{t+\tau-k}\} \\ &= \sum_{j=0}^{q}\sum_{k=0}^{q} \theta_{j,q}\theta_{k,q}\, s_{Y,\tau-k+j}. \end{aligned}$$

$\boxed{2\text{ C}}$

(e) From the last part of (d) with $q = 1$, and using the expression derived in part (c), with $s_{Y,1} = s_{Y,-1}$, we get $\boxed{\text{unseen} \Downarrow}$

$$\begin{aligned} s_{X,0} = \operatorname{var}\{X_t\} &= (-1)^2 s_{Y,0} - \theta s_{Y,-1} - \theta s_{Y,1} + \theta^2 s_{Y,0} = (1+\theta^2) s_{Y,0} - 2\theta s_{Y,1} \\ &= \frac{(1+\theta^2)}{1-\phi^2}\sigma_\epsilon^2 - \frac{2\theta\phi}{1-\phi^2}\sigma_\epsilon^2 = \left[1 + \frac{(\theta-\phi)^2}{1-\phi^2}\right]\sigma_\epsilon^2, \end{aligned}$$

so $c = (\theta-\phi)^2/(1-\phi^2)$. $\boxed{3\text{ D}}$

2. (a) (i) From linear filtering, input $e^{i2\pi ft}$ to the filter $L(\epsilon_t) = \epsilon_t - \theta_{1,2}\epsilon_{t-1} - \theta_{2,2}\epsilon_{t-2}$ to obtain the frequency response function $G(f)$: $\boxed{\text{seen} \Downarrow}$

$$\begin{aligned} L\{e^{i2\pi ft}\} &= e^{i2\pi ft}(1 - \theta_{1,2}e^{-i2\pi f} - \theta_{2,2}e^{-i4\pi f}) \\ \Rightarrow G(f) &= (1 - \theta_{1,2}e^{-i2\pi f} - \theta_{2,2}e^{-i4\pi f}) \\ \Rightarrow |G(f)|^2 &= |1 - \theta_{1,2}e^{-i2\pi f} - \theta_{2,2}e^{-i4\pi f}|^2. \end{aligned}$$

The output spectrum is the input spectrum times $|G(f)|^2$:

$$S(f) = |G(f)|^2 S_\epsilon(f) = \sigma_\epsilon^2 |1 - \theta_{1,2}e^{-i2\pi f} - \theta_{2,2}e^{-i4\pi f}|^2.$$

Then

$$\begin{aligned} S(f) &= \sigma_\epsilon^2[1 + \theta_{1,2}^2 + \theta_{2,2}^2 - \theta_{1,2}\left(e^{i2\pi f} + e^{-i2\pi f}\right) - \theta_{2,2}\left(e^{i4\pi f} + e^{-i4\pi f}\right) \\ &\quad + \theta_{1,2}\theta_{2,2}\left(e^{i2\pi f} + e^{-i2\pi f}\right) \\ &= \sigma_\epsilon^2[1 + \theta_{1,2}^2 + \theta_{2,2}^2 - 2\theta_{1,2}(1 - \theta_{2,2})\cos(2\pi f) - 2\theta_{2,2}\cos(4\pi f)] \qquad (*) \end{aligned}$$

$\boxed{4\ \mathrm{A}}$

(ii) Since the process is an MA(2) its autocovariance cuts-off at $|\tau| = 2$, so

$$\begin{aligned} S(f) = \sum_{\tau=-\infty}^{\infty} s_\tau e^{-i2\pi f} &= s_0 + s_1\left(e^{i2\pi f} + e^{-i2\pi f}\right) + s_2\left(e^{i4\pi f} + e^{-i4\pi f}\right) \\ &= s_0 + 2s_1\cos(2\pi f) + 2s_2\cos(4\pi f) \qquad (**) \end{aligned}$$

A comparison of (*) and (**) gives the required expressions. $\boxed{4\ \mathrm{A}}$

$\boxed{\text{sim. seen} \Downarrow}$

(b) (i) The characteristic polynomial is $\Theta(z) = 1 - z + \frac{1}{2}z^2$ which has roots $z_1, z_2 = 1 \pm i$, both of which have modulus greater than one, so $\{X_t\}$ is invertible. This means that it can be rewritten as a well-defined autoregressive process. $\boxed{2\ \mathrm{A}}$

(ii) Now

$$\rho_1 = -\frac{\theta_{1,2}(1 - \theta_{2,2})}{1 + \theta_{1,2}^2 + \theta_{2,2}^2} \text{ and } \rho_2 = -\frac{\theta_{2,2}}{1 + \theta_{1,2}^2 + \theta_{2,2}^2}$$

Putting $\theta_{1,2} = 1, \theta_{2,2} = -1/2$ gives $\rho_1 = -2/3$ and $\rho_2 = 2/9$. $\boxed{2\ \mathrm{A}}$

$\boxed{\text{unseen} \Downarrow}$

(c) (i) Write the characteristic polynomial in root form, (roots are $1/a, 1/b$):

$$1 - \theta_{1,2}z - \theta_{2,2}z^2 = (1 - az)(1 - bz) = 1 - (a + b)z + abz^2$$

so $\theta_{1,2} = (a + b)$ and $\theta_{2,2} = -ab$. From the stated roots we have

$$a = \frac{2}{1 + i}, \qquad b = \frac{2}{1 - i}$$

and therefore

$$\theta_{1,2} = a + b = \frac{2(1 - i) + 2(1 + i)}{(1 + i)(1 - i)} = \frac{4}{2} = 2; \quad \theta_{2,2} = -ab = -\frac{4}{2} = -2.$$

$\boxed{4\ \mathrm{D}}$

(ii) Putting these values into ρ_1 and ρ_2 as given in (b)(ii) above, we again get $\rho_1 = -2/3$ and $\rho_2 = 2/9$, as for $\{X_t\}$. $\boxed{1\ \mathrm{C}}$

(d) We have met the idea that inverting the roots of the characteristic polynomial of a MA does not change its autocorrelation sequence. If we invert the roots specified in (c) for $\{Y_t\}$ we get

$$z_1 = \frac{2}{1+\mathrm{i}} = 1 - \mathrm{i}; \quad z_1 = \frac{2}{1-\mathrm{i}} = 1 + \mathrm{i},$$

which are the roots for $\{X_t\}$. So indeed the two MA processes here have roots which are inverses, and the equal autocorrelations follow. The process $\{Y_t\}$ has roots inside the unit circle and so, unlike $\{X_t\}$, it is not invertible. The two processes have the same autocorrelation sequence, but different invertibility properties. $\boxed{3\ \mathrm{C}}$

3. (a) (i) Let $J(f) \equiv (1/\sqrt{N})\sum_{t=1}^{N} X_t e^{-i2\pi ft}$. By the spectral representation theorem $X_t = \int_{-1/2}^{1/2} e^{i2\pi f't}\, dZ(f')$, where $\{Z(\cdot)\}$ is a process with orthogonal increments, and $E\{dZ(f)\} = 0$. Thus seen ⇓

$$\begin{aligned} J(f) &= (1/\sqrt{N})\sum_{t=1}^{N}\left(\int_{-1/2}^{1/2} e^{i2\pi f't}\, dZ(f')\right) e^{-i2\pi ft} \\ &= (1/\sqrt{N})\int_{-1/2}^{1/2}\sum_{t=1}^{N} e^{-i2\pi(f-f')t}\, dZ(f') \\ &= \int_{-1/2}^{1/2} F(f-f')\, dZ(f'), \end{aligned}$$

where $F(f) = (1/\sqrt{N})\sum_{t=1}^{N} e^{-i2\pi ft}$. Now $\widehat{S}^{(p)}(f) = |J(f)|^2$, and since $\{Z(\cdot)\}$ has orthogonal increments, and $E\{|dZ(f')|^2\} = S(f')df'$,

$$E\{\widehat{S}^{(p)}(f)\} = \int_{-1/2}^{1/2} \mathcal{F}(f-f')S(f')\, df',$$

where $\mathcal{F}(f) \equiv |F(f)|^2 = (1/N)\left|\sum_{t=1}^{N} e^{-i2\pi ft}\right|^2$. 4 A

For white noise $S(f) = \sigma_\epsilon^2$, so $E\{\widehat{S}^{(p)}(f)\} = \sigma_\epsilon^2 \int_{-1/2}^{1/2} \mathcal{F}(f-f')\, df' = \sigma_\epsilon^2$, since $\mathcal{F}$ has a period of unity and integrates to 1. So **unbiased**. 2 A

(ii) From the question we set $\widehat{S}^{(p)}(f_j) = |Y_1(f_j) + iY_2(f_j)|^2 = Y_1(f_j)^2 + Y_2^2(f_j)$ and unseen ⇓

$$Y_1(f_j) = \frac{1}{\sqrt{N}}\sum_{t=1}^{N} \epsilon_t \cos(2\pi f_j t); \qquad Y_2(f_j) = -\frac{1}{\sqrt{N}}\sum_{t=1}^{N} \epsilon_t \sin(2\pi f_j t).$$

Using the shorthand $Y_1 = Y_1(f_j)$ and $Y_2 = Y_2(f_j)$ we have $E\{Y_1\} = E\{Y_2\} = 0$. Since the white noise terms are uncorrelated,

$$\mathrm{var}\{Y_1\} = \frac{\sigma_\epsilon^2}{N}\sum_{t=1}^{N}\cos^2(2\pi f_j t) = \frac{\sigma_\epsilon^2}{2} = \frac{\sigma_\epsilon^2}{N}\sum_{t=1}^{N}\sin^2(2\pi f_j t) = \mathrm{var}\{Y_2\}.$$

Also,

$$\begin{aligned} \mathrm{cov}\{Y_1, Y_2\} = E\{Y_1 Y_2\} &= -\frac{1}{N}\sum_{t=1}^{N}\sum_{t'=1}^{N} E\{\epsilon_t\epsilon_{t'}\}\cos(2\pi f_j t)\sin(2\pi f_j t') \\ &= -\frac{\sigma_\epsilon^2}{N}\sum_{t=1}^{N}\cos(2\pi f_j t)\sin(2\pi f_j t) = 0. \end{aligned}$$

Since the process is normal, Y_1, Y_2 are jointly normal and uncorrelated and hence independent. So, as required,

$$\frac{2}{\sigma_\epsilon^2}[Y_1^2(f_j) + Y_2^2(f_j)] \stackrel{d}{=} \chi_2^2 \Rightarrow \widehat{S}^{(p)}(f_j) \stackrel{d}{=} \frac{\sigma_\epsilon^2}{2}\chi_2^2.$$

6 B

(b) (i) Now $Z_t = X_t e^{iY_t}$ and $Y_t \stackrel{d}{=} U[-\pi, \pi]$. $\{X_t\}, \{Y_t\}$ are independent of each other. Since $\{X_t\}$ has a zero mean, $E\{Z_t\} = E\{X_t\}E\{e^{iY_t}\} = 0$. unseen ⇓ 1 D

Then, by independence of $\{X_t\}, \{Y_t\}$,

$$s_{Z,\tau} = E\{Z_t^* Z_{t+\tau}\} = E\{X_t e^{-iY_t} \cdot X_{t+\tau} e^{iY_{t+\tau}}\} = s_{X,\tau} \cdot E\{e^{-iY_t} e^{iY_{t+\tau}}\}.$$

When $\tau = 0$, $s_{Z,0} = s_{X,0} \cdot E\{e^{-iY_t} e^{iY_t}\} = \sigma_X^2 \cdot E\{1\} = \sigma_X^2 = 1$.
When $\tau \neq 0$, $s_{Z,\tau} = s_{X,\tau} \cdot E\{e^{-iY_t}\} \cdot E\{e^{iY_{t+\tau}}\}$, by independence. But

$$E\{e^{-iY_t}\} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(y) dy - i \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin(y) dy = 0$$

and likewise for $E\{e^{iY_{t+\tau}}\}$, (identically distributed). So,

$$s_{Z,\tau} = \begin{cases} 1, & \tau = 0; \\ 0, & \tau \neq 0. \end{cases}$$

4 D

(ii) Next,

$$r_{Z,\tau} = E\{Z_t Z_{t+\tau}\} = E\{X_t e^{iY_t} \cdot X_{t+\tau} e^{iY_{t+\tau}}\} = s_{X,\tau} \cdot E\{e^{i(Y_t + Y_{t+\tau})}\}.$$

When $\tau = 0$, $r_{Z,0} = s_{X,0} \cdot E\{e^{i2Y_t}\} = 0$, since $\int_{-\pi}^{\pi} \cos(2y) dy = 0$, and similarly for sin.
When $\tau \neq 0$, $r_{Z,\tau} = s_{X,\tau} \cdot E\{e^{i(Y_t + Y_{t+\tau})}\} = s_{X,\tau} \cdot E\{e^{iY_t}\} E\{e^{iY_{t+\tau}}\} = s_{X,\tau} \cdot 0 = 0$.
So $r_{Z,\tau} = 0$ for all τ, and therefore $\{Z_t\}$ is **proper**. 3 D

4. (a) (i) Let $\boldsymbol{a}^T = [a_1, a_2]$. Then $\boxed{\text{sim. seen} \Downarrow}$

$$\text{var}\{\boldsymbol{a}^T \boldsymbol{\epsilon}_t\} = \boldsymbol{a}^T \Sigma \boldsymbol{a} = \sum_{j=1}^{2} \sum_{k=1}^{2} \sigma_{jk} a_j a_k \geq 0.$$

$\boxed{1\text{ B}}$

(ii) Two real-valued discrete time stochastic processes $\{X_t\}$ and $\{Y_t\}$ are said to be jointly stationary stochastic processes if $\{X_t\}$ and $\{Y_t\}$ are each, separately, second-order stationary processes, and cov$\{X_t, Y_{t+\tau}\}$ is a function of τ only. $\boxed{\text{seen} \Downarrow}$ $\boxed{2\text{ A}}$

For the example, for $j, k \in \{1, 2\}$, $\boxed{\text{unseen} \Downarrow}$

$$\text{cov}\{X_{j,t}, X_{k,t+\tau}\} = E\{\epsilon_{j,t}\epsilon_{k,t+\tau}\} = \begin{cases} \sigma_{jk} & \tau = 0; \\ 0 & \tau \neq 0. \end{cases}$$

so they are jointly stationary. $\boxed{2\text{ B}}$

(iii) For $\Sigma = \Sigma_1$, the eigenvalues are the solutions of

$$\det\left\{\begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix}\right\} = 0.$$

These are $\lambda_1, \lambda_2 = 0, 2$. So the matrix is positive semidefinite. Alternatively, observe that the principal minors/subdeterminants are all nonnegative. $\boxed{1\text{ B}}$

(iv) The coherence is defined as

$$\gamma^2_{X_1,X_2}(f) = \frac{|S_{X_1X_2}(f)|^2}{S_{X_1}(f)S_{X_2}(f)}.$$

From part (ii) and the form of Σ_1,we know

$$s_{X_1,\tau} = s_{X_2,\tau} = s_{X_1X_2,\tau} = \begin{cases} 1 & \tau = 0; \\ 0 & \tau \neq 0. \end{cases}$$

So $S_{X_1}(f) = \sum_{\tau=-\infty}^{\infty} s_{X_1,\tau} e^{-i2\pi f\tau} = 1$, $|f| \leq 1/2$, and likewise for $S_{X_2}(f)$ and $S_{X_1X_2}(f)$. Hence, $\gamma^2_{X_1,X_2}(f) = 1$, $|f| \leq 1/2$. $\boxed{2\text{ B}}$

(v) From (iv) we know that $\{X_{1,t}\}$ and $\{X_{2,t}\}$ are perfectly correlated at all frequencies so they are related through a linear filtering, i.e., $X_{2,t} = \sum_u g_u X_{1,t-u}$. Multiplying through by $X_{1,t}$ and taking expectations gives $1 = g_0 \cdot 1$ so $g_0 = 1$. Multiplying through by $X_{1,t+\tau}$ for $|\tau| \neq 0$ and taking expectations gives $0 = g_{-\tau} \cdot 1$, so only g_0 is non-zero. Thus $X_{2,t} = X_{1,t}$, they are identical. [Other valid justifications are fine.] $\boxed{2\text{ D}}$

(b) (i) We want to minimize, $\boxed{\text{seen} \Downarrow}$

$$
\begin{aligned}
E\{(X_{t+l}-X_t(l))^2\} &= E\left\{\left(\sum_{k=0}^{\infty}\psi_k\epsilon_{t+l-k}-\sum_{k=0}^{\infty}\delta_k\epsilon_{t-k}\right)^2\right\}\\
&= E\left\{\left(\sum_{k=0}^{l-1}\psi_k\epsilon_{t+l-k}+\sum_{k=0}^{\infty}[\psi_{k+l}-\delta_k]\epsilon_{t-k}\right)^2\right\}\\
&= \sigma_\epsilon^2\left\{\left(\sum_{k=0}^{l-1}\psi_k^2\right)+\sum_{k=0}^{\infty}(\psi_{k+l}-\delta_k)^2\right\}.
\end{aligned}
$$

The first term is independent of the choice of the $\{\delta_k\}$ and the second term is clearly minimized by choosing $\delta_k=\psi_{k+l}, k=0,1,2,\ldots$. $\boxed{4\text{ A}}$

(ii) $X_t=(1-\phi B)^{-1}\epsilon_t$. So $\boxed{\text{unseen} \Downarrow}$

$$\Psi(z)=1+\phi z+\phi^2z^2+\cdots=\psi_0+\psi_1 z+\psi_2 z^2+\cdots$$

so $\psi_k=\phi^k$. Then $\sigma^2(l)=\sigma_\epsilon^2\sum_{k=0}^{l-1}\psi_k^2 \Rightarrow \sigma^2(2)=\sigma_\epsilon^2(\psi_0^2+\psi_1^2)=\sigma_\epsilon^2(1+\phi^2)$. $\boxed{2\text{ B}}$

(iii) We set future innovations to zero: $X_t(1)=\phi X_t$ and $X_t(2)=\phi X_t(1)=\phi^2 X_t$. So

$$
\begin{aligned}
E\{(X_{t+2}-X_t(2))^2\} &= E\{X_{t+2}^2\}+E\{\phi^4X_t^2\}-2E\{X_{t+2}\cdot\phi^2X_t\}\\
&= \frac{\sigma_\epsilon^2}{1-\phi^2}+\phi^4\frac{\sigma_\epsilon^2}{1-\phi^2}-2\phi^2 s_2\\
&= \frac{\sigma_\epsilon^2}{1-\phi^2}+\phi^4\frac{\sigma_\epsilon^2}{1-\phi^2}-2\phi^2\frac{\sigma_\epsilon^2\phi^2}{1-\phi^2}\\
&= \frac{\sigma_\epsilon^2(1-\phi^2)(1+\phi^2)}{1-\phi^2}=\sigma_\epsilon^2(1+\phi^2),
\end{aligned}
$$

as before. $\boxed{4\text{ C}}$

5. (a) unseen $\Downarrow$

$$
\begin{aligned}
s(\tau) &= \int_{-\infty}^{\infty} S(f) \mathrm{e}^{\mathrm{i}2\pi f\tau}\,\mathrm{d}f = \int_{-\infty}^{\infty} \frac{2L\sigma^2}{1+(2\pi f L)^2} \mathrm{e}^{\mathrm{i}2\pi f\tau}\,\mathrm{d}f \\
&= 4L\sigma^2 \int_0^{\infty} \frac{\cos(2\pi f\tau)}{1+(2\pi f L)^2}\,\mathrm{d}f,
\end{aligned}
$$

since the imaginary part of the integral is zero by symmetry because its integrand is the product of the even function $S(\cdot)$ and the odd function $\sin(\cdot)$. For the real part, the integrand is an even function, and thus we can rewrite as twice an integral ranging from 0 to ∞. Next, make the change of variable $x = 2\pi f L$ and set $m = \tau/L$ in the definite integral in the hint, so

$$
2\pi L \int_0^{\infty} \frac{\cos(2\pi f\tau)}{1+(2\pi f L)^2}\,\mathrm{d}f = \int_0^{\infty} \frac{\cos(mx)}{1+x^2}\,\mathrm{d}x = \frac{\pi}{2} \mathrm{e}^{-|\tau|/L},
$$

i.e.,

$$
\int_0^{\infty} \frac{\cos(2\pi f\tau)}{1+(2\pi f L)^2}\,\mathrm{d}f = \frac{1}{4L} \mathrm{e}^{-|\tau|/L},
$$

from which we obtain $s(\tau) = \sigma^2 \mathrm{e}^{-|\tau|/L}$, as required. 4

(b) (i) Let $S_{X_t}(f;\Delta t)$ denote the spectral density function (SDF) of $\{X_t\}$ for a given sampling interval Δt. So initially assume $f \in [0, f_N]$ in what follows. Then from the given reading material, seen $\Downarrow$

$$
S_{X_t}(f;\Delta t) = \sum_{k=-\infty}^{\infty} S(f+\tfrac{k}{\Delta t}) = S(f) + \sum_{k=1}^{\infty} S(f+\tfrac{k}{\Delta t}) + \sum_{k=1}^{\infty} S(f-\tfrac{k}{\Delta t}).
$$

For a given Δt, and the form of $S(f)$ given, the two sums on the right will only have a finite number of terms, (none if no aliasing), since $S(f)$ has finite support. 1 unseen $\Downarrow$

In the first summation on the right-hand side, the SDF is non-zero only when $f + \frac{k}{\Delta t} \le 2$, i.e., when $k \le \lfloor (2-f)\,\Delta t \rfloor$, where $\lfloor x \rfloor$ is the largest integer less than or equal to x. Since $f \in [0, f_N]$, we can replace the upper limit of the summation by $m = \lfloor 2\,\Delta t \rfloor$. 1

In the second summation, the SDF is non-zero only when $f - \frac{k}{\Delta t} \ge -2$, i.e., when $k \le \lfloor (2+f)\,\Delta t \rfloor$. Since $f \in [0, f_N]$, we can replace the upper limit of the summation by $\ell = \lfloor 2\,\Delta t + \frac{1}{2} \rfloor$, (so $y = 2\,\Delta t + \frac{1}{2}$). 2

[Hence we have

$$
S_{\Delta t}(f) = \sum_{k=-\ell}^{m} S(f+\tfrac{k}{\Delta t}).
$$

with $\ell = \lfloor 2\,\Delta t + \frac{1}{2} \rfloor$ and $m = \lfloor 2\,\Delta t \rfloor$.]

(ii) Now specialize to the cases of interest. When $\Delta t = \frac{2}{3}$, then $f_{\mathcal{N}} = \frac{3}{4}$, and $\ell = 1, m = 1$, and

$$S_{X_t}(f; \tfrac{2}{3}) = S(f) + S(f + \tfrac{3}{2}) + S(f - \tfrac{3}{2}) = \begin{cases} 3C, & f \in [0, \frac{1}{2}]; \\ 2C, & f \in (\frac{1}{2}, \frac{3}{4}]. \end{cases}$$

When $\Delta t = \frac{1}{3}$ we have $f_{\mathcal{N}} = \frac{3}{2}$ and $\ell = 1, m = 0$. So

$$S_{X_t}(f; \tfrac{1}{3}) = S(f - 3) + S(f) = \begin{cases} C, & f \in [0, 1); \\ 2C, & f \in [1, \frac{3}{2}]; \end{cases}$$

When $\Delta t = \frac{1}{5}$, then $f_{\mathcal{N}} = \frac{5}{2}$, and $\ell = 0, m = 0$, and there is no aliasing:

$$S_{X_t}(f; \tfrac{1}{5}) = S(f) = \begin{cases} C, & f \in [0, 2]; \\ 0, & f \in (2, \frac{5}{2}]. \end{cases}$$

$\boxed{6}$

(iii) Firstly, $S_{X_t}(f; \Delta t) = S_{X_t}(-f; \Delta t)$ for $f \in [-f_{\mathcal{N}}, 0)$. seen ⇓
Secondly, $S_{X_t}(f; \Delta t)$ for f outside of $[-f_{\mathcal{N}}, f_{\mathcal{N}}]$ is defined by periodic extension, (period of $2f_{\mathcal{N}}$). $\boxed{3}$

(iv) The integral of $S(\cdot)$ over $f \in \mathbb{R}$ is $4C$, and the integrals over $[-f_{\mathcal{N}}, f_{\mathcal{N}}]$ of $S_{X_t}(f; \frac{2}{3}), S_{X_t}(f; \frac{1}{3})$ and $S_{X_t}(f; \frac{1}{5})$ are also $4C$. unseen ⇓ $\boxed{1}$

(v) $S_{X_t}(f; \frac{2}{3})$ is dominated by low frequencies, so red noise; $S_{X_t}(f; \frac{1}{3})$ is dominated by high frequencies, so blue noise; $S_{X_t}(f; \frac{1}{5})$, is dominated by low frequencies, so red noise. $\boxed{2}$