Paper Number(s): **E4.12**
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IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE
UNIVERSITY OF LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2002

MSc and EEE/ISE PART IV: M.Eng. and ACGI

DIGITAL SIGNAL PROCESSING AND DIGITAL FILTERS

Tuesday, 23 April 10:00 am

There are FIVE questions on this paper.

Answer THREE questions.

Corrected Copy

Time allowed: 3:00 hours

Examiners responsible:

First Marker(s): Constantinides,A.G.

Second Marker(s): Stathaki,T.

Special instructions for invigilators: none

Information for candidates: none

1. The transfer function of an ideal real coefficient lowpass filter $H_{LP}(z)$ has cutoff frequency θ_p and impulse response $h_{LP}(n)$. Show that $H_{LP}(-z)$ is a highpass filter and of cutoff frequency $\pi - \theta_p$. Find an expression for the impulse response $h_{HP}(n)$ of this highpass filter in terms of the impulse response $h_{LP}(n)$ of the original lowpass filter. [7]

Let $\theta_P < \frac{\pi}{2}$ and form $G(z) = H_{LP}(ze^{j\theta_0}) + H_{LP}(ze^{-j\theta_0})$. Show that $G(z)$ is a real coefficient bandpass filter with a passband centred at θ_0. Determine the bandwidth and the impulse response $g(n)$ of the bandpass filter in terms of the impulse response $h(n)$ of the lowpass filter and the centre frequency θ_0. [9]

Explain why such a transformation for realisable systems may not produce the bandpass response. [4]

2. A polyphase digital signal processing system is shown in Figure 2.1. The individual transfer functions may expressed in 2-phase form

$$F_0(z) = R_0(z^2) + z^{-1}R_1(z^2)$$
$$H_0(z) = E_0(z^2) + z^{-1}E_1(z^2)$$

Determine the condition that the individual transfer functions must fulfil so that the output $y(n)$ is a scaled and delayed replica of the input $x(n)$. [12]

When $H_o(z) = 1 + z^{-1}$ evaluate the minimum order Finite Impulse Response transfer function $F_o(z)$ such that $y(n) = x(n-1)$. [8]

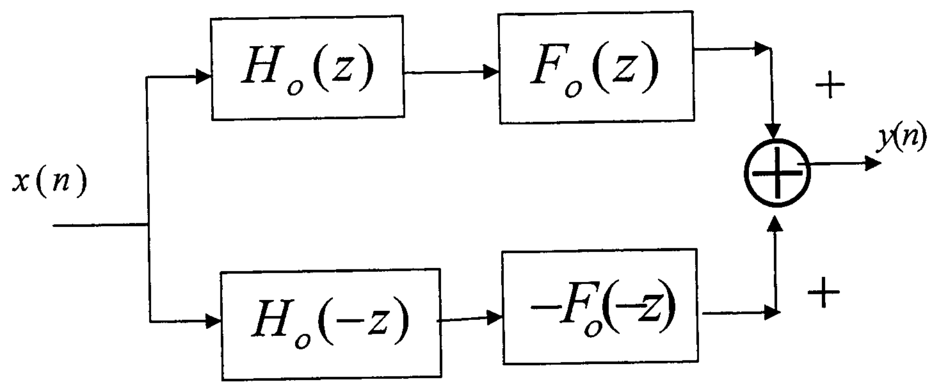

Figure 2.1

3. An infinite impulse response digital filter transfer function $H(z)$ is to be realised by the structure shown in Figure 3.1 as $H(z) = Y_1 / X_1$. The constraining transfer function is $z^{-1}C(z)$ and the relationship between $H(z)$ and $z^{-1}C(z)$ is given by

$$H(z) = \frac{a + z^{-1}C(z)}{1 + az^{-1}C(z)}$$

where $-1 < \alpha < 1$.

Determine a set of [a b c d] parameters for the interconnecting system. Comment on the choice available and finally suggest a good selection. Justify your answer. [10]

Show that when $|C(z)| \leq 1$ on $|z| = 1$ then $|H(z)| \leq 1$. [10]

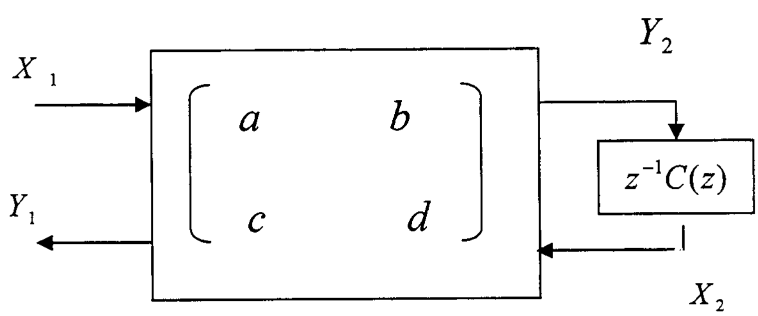

Figure 3.1

4. A stable second order transfer function is given by

$$A_2(z) = \frac{a_2 + a_1 z^{-1} + z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

(a) Show that $A_2(z)$ is allpass. [5]

(b) It is proposed that realise $A_2(z)$ is an embedded form as

$$A_2(z) = \frac{A_1(z) + \beta}{1 + \beta A_1(z)}$$

$$A_1(z) = z^{-1}\frac{z^{-1} + \alpha}{1 + \alpha z^{-1}}$$

Determine expressions for $a_i \; i = 1, 2$ in terms of α, β and hence determine an expression for the non-trivial frequency θ_0 in terms of α, β at which $A_2(z)$ is completely real. [5]

(c) A notch filter transfer $T(z)$ is to be produced using $A_2(z)$ as

$$T(z) = a + bA_2(z)$$

The attenuation at the notch frequency θ_0 is infinite and at $\theta = 0$ and π , is zero dB. [4]

Determine the parameters a and b .

(d) Show that the 3 dB points of the notch filter $T(z)$ occur at the solution of [6]

$$\operatorname{Re} A_2(z) \,\big|_C = 0$$

5. The signal flow graph of an oversampling A/D converter is shown in Figure 5.1. The connecting block S is two-input, single-output linear system described by $V = \alpha X + \beta U$ where α and β are appropriate transfer functions. The block labelled $Q[.]$ is a bipolar one-bit quantiser, which introduced quantisation noise Q as indicated.

By making appropriate assumptions derive an expression for the output Y in terms of X, α and β and the quantisation noise Q. Comment on the validity of your assumptions in practice. [5]

In a specific realisation it is required that a) the output has real unity gain with respect to the input and b) the noise shaping transfer function is $F(z)$.

Show that under these conditions $\alpha = \dfrac{1}{F(z)}$ and $\beta = \dfrac{F(z)-1}{F(z)}$.

Give an account of the factors that influence the choice for $F(z)$. [10]

Draw the signal flow graph of the interconnecting block S when $F(z) = (1 - z^{-1})$ and reduce it to a form that contains only one accumulator. [5]

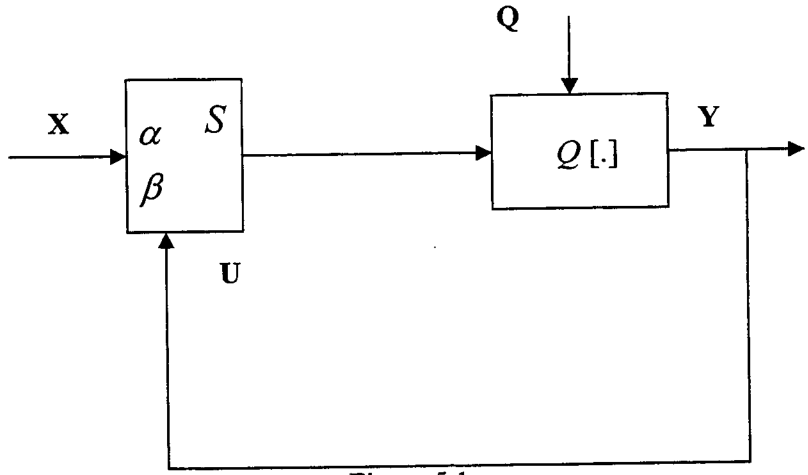

Figure 5.1

Solutions: DSP + DIGITAL FILTERS 2002

Q.1

Replacing z to -z is a z-plane rotation by 180°.

Hence the passband $-\theta_p \le \theta \le \theta_p$ will be rotated to be around the point -1 2

i.e $\pi-\theta_p \le \theta \le \pi+\theta_p$

This is a highpass filter response 2

h_{HP} = inverse z-transform of $H_{LP}(-z)$

Let $H_{LP}(z) = \sum_{n=-\infty}^{+\infty} h_{LP}(n)\, z^{-n}$

Thus $H_{HP}(z) = H_{LP}(-z) = \sum_{n=-\infty}^{+\infty} h_{LP}(n)\, (-1.z)^{-n}$

or $h_{HP} = (-1)^n . h_{LP}(n)$ 3

$$G(z) = \sum_{r=-\infty}^{+\infty} h_{LP}(n)(e^{-j\theta_0}z)^{-n} + \sum_{n=-\infty}^{+\infty} h_{LP}(n)\,(e^{+j\theta_0}z)^{-n}$$

$$= \left(\sum_{r=-\infty}^{+\infty} h_{LP}(n)\cdot(e^{-jn\theta_0} + e^{jn\theta_0} \right) . z^{-n}$$ 5

ie. $g(n) = 2\, h_{LP}(n) . \cos n\theta_0$

The bandwidth is $= (\theta_0+\theta_p)-(\theta_0-\theta_p) = 2\theta_p$ 4

In realisable systems the amplitude response does not fall off to zero outside the passband. Hence the addition of shifted versions of the LP response will produce interactions between the passband of one & stop band of the other which can be destructively combined due to phase. 4

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AGC

TS

Solutions:

Q2: 2/5

Consider the two branches separately

$$H_0F_0 = [E_0 + z^{-1}E_1][R_0 + z^{-1}R_1]$$
$$= E_0R_0 + z^{-1}[E_0R_1 + E_1R_0] + z^{-2}E_1R_1$$

and
$$H_0^* \cdot F_0^* = E_0R_0 - z^{-1}[E_0R_1 + E_1R_0] + z^{-2}E_1R_1$$

where $H_0^* = H_0(-z)$, $F_0^* = F_0(-z)$

Hence the transfer function is

$$G(z) = H_0F_0 - H_0^*F_0^* = 2z^{-1}[E_0(z^2)R_1(z^2) + E_1(z^2)R_0(z^2)]$$

The condition requires

$G(z) = Az^{-m}$ A = scaling m = +ve integer

i.e.
$$2z^{-1}[E_0(z^2)R_1(z^2) + E_1(z^2)R_0(z^2)] = Az^{-m}$$

for $A = 1$, $m = 1$ and $H_0(z) = 1 + z^{-1}$ we have
$$2z^{-1}[1.R_1(z^2) + 1.R_0(z^2)] = z^{-1}$$
or $R_1(z^2) + R_0(z^2) = \frac{1}{2}$ 12

Hence for minimum order solution

$$\left.\begin{aligned} R_0(z^2) &= a_0 \\ R_1(z^2) &= a_1 \end{aligned}\right\} \quad a_0 + a_1 = \frac{1}{2}$$

or $F_0(z) = (a_0 + a_1z^{-1})$ with $a_0 + a_1 = \frac{1}{2}$ 8

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ACC

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Solutions

7/5

Q 3

Write $\left.\begin{aligned} Y_1 &= aX_1 + bX_2 \\ Y_2 &= cX_1 + dX_2 \end{aligned}\right\}$ and $X_2 = z^{-1}C$

Hence $Y_2 = cX_1 + d.z^{-1}C.Y_2$

or $Y_2 = \dfrac{c}{1 - dz^{-1}C} . X_1(z)$

Thus $Y_1(z) = aX_1(z) + bz^{-1}C(z) . \dfrac{c}{1 - dz^{-1}C(z)} . X_1(z)$

or $\dfrac{Y_1(z)}{X_1(z)} = \dfrac{a + (bc - ad).z^{-1}C(z)}{1 - dz^{-1}C(z)}$

Compare expressions $\quad d = -\alpha$
$\quad a = \alpha$
$\quad bc - ad = 1$

$bc - ad \Rightarrow \quad bc = 1 + da = 1 - \alpha^2$ 10

3 equations with four parameters. For minimal multiplier solution either $b = 1$ & $c = 1 - \alpha^2$ or v.v.

Consider $1 - |H(z)|^2 = 1 - \left| \dfrac{\alpha + z^{-1}C(z)}{1 + \alpha z^{-1}C(z)} \right|^2$

$$= 1 - \frac{[\alpha + z^{-1}C(z)][\alpha + zC^*(z)]}{[1 + \alpha z^{-1}C(z)][1 + \alpha zC^*(z)]}$$

$$= 1 - \frac{(\alpha^2 + \alpha(z^{-1}C(z) + zC^*(z)) + |C(z)|^2)}{(1 + \alpha(z^{-1}C(z) + zC^*(z)) + \alpha^2|C(z)|^2)}$$

$$= \frac{1 + \alpha^2|C(z)|^2 - \alpha^2 - |C(z)|^2}{\text{Squared Real Quantity}} = \frac{(1-\alpha^2)[1 - |C(z)|^2]}{\text{Positive}}$$ 10

i.e $1 - |H(z)|^2 \geq 0$ when $|C(z)|^2 \leq 1$ on $|z| = 1$ Ace 20

TS

Solutions

4/5

Q 4

(a) Write

$$A_2(z) = \frac{a_2 + a_1 z^{-1} + z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}} \cdot \frac{z}{z}$$

$$= \frac{az + a_1 + z^{-1}}{z + a_1 + a_2 z^{-1}} = \frac{\overset{*}{P}(z)}{P(z)} \quad \text{for } |z| = 1$$

Hence $|A_2(z)| = 1$ allpass 5

(b)

$$A_2(z) = \frac{z^{-1} \cdot \frac{z^{-1}+\alpha}{1+\alpha z^{-1}} + \beta}{1 + \beta z^{-1} \cdot \frac{z^{-1}+\alpha}{1+\alpha z^{-1}}} = \frac{z^{-2} + (\alpha + \alpha\beta)z^{-1} + \beta}{\beta z^{-2} + (\alpha + \alpha\beta)z^{-1} + 1}$$

Thus on comparing we have

$$a_2 = \beta \qquad a_1 = \alpha(1+\beta)$$

Since $|A_2(z)| = 1$ there are two real values $+1$ & -1

For $A_2 = +1$ we have $\beta z^{-2} + \alpha(1+\beta)z^{-1} + 1 = z^{-2} + \alpha(\beta+1)z^{-1} + \beta$

or $(\beta-1)z^2 = \beta - 1$ or $z = \pm 1$ $\Rightarrow$ Trivial solutions

For $A_2 = -1$ we have $-\beta z^{-2} - \alpha(1+\beta)z^{-1} - 1 = z^{-2} + \alpha(1+\beta)z^{-1} + \beta$

or $(1+\beta) + 2\alpha(1+\beta)z^{-1} + (1+\beta) = 0$

For $\beta \neq 1$ $\quad z^{-2} + 2\alpha z^{-1} + 1 = 0$. 5

since $|\alpha| < 1$ let $\alpha = -\cos\mu$

i.e. $z^{-2} - 2\cos\mu \cdot z^{-1} + 1 = 0 = (z^{-1} - e^{-j\mu})(z^{-1} - e^{+j\mu})$

or $\theta_0 = \pm\mu = \cos^{-1}\alpha$.

(c) $T(z) = a + b A_2(z)$ with notch at θ_0 $T(e^{j\theta_0}) = 0$ 4

i.e. $a = -b = 1$ produces solution

(d) at 3dB $\left(\frac{1}{2}\right)^2 \left|[1 - A_2]\right|^2 = \frac{1}{2}$ or $|1 - A_2|^2 = 2 = [1 - A_2][1 - A_2^*]$ 6

or $2 = |2 - (A_2 + A_2^*)|$ & since $A_2 = e^{j\phi}$ we have $A_2 + A_2^* = 0$ 20

or $\text{Real}\,[A_2(z)]_c = 0$

TS

Solutions

Q5 $\frac{5}{5}$

Assumptions

1) High sampling rate for X to be represented by z-transforms
2) Linear Quantisation model
3) Computable loops (ie at least one sample delay)
4) Stable Loops
5) Negigible latencies (ie mults & adds & $Q[\cdot]$ take no time)

1) realisable, 2) oversimplification, 3) achievable, 4) as 3 5) Can lead to problems. 5

From figure $\alpha X + \beta Y + Q = Y$ or $Y = \frac{\alpha}{1-\beta} X + \frac{1}{1-\beta} Q$

Requirement (a) needs $\alpha = (1-\beta)$ [if phase $180°$ is acceptable then $\alpha = -(1-\beta)$ is possible]

Requirement (b) yields $\frac{1}{1-\beta} = F(z) = \frac{1}{\alpha}$

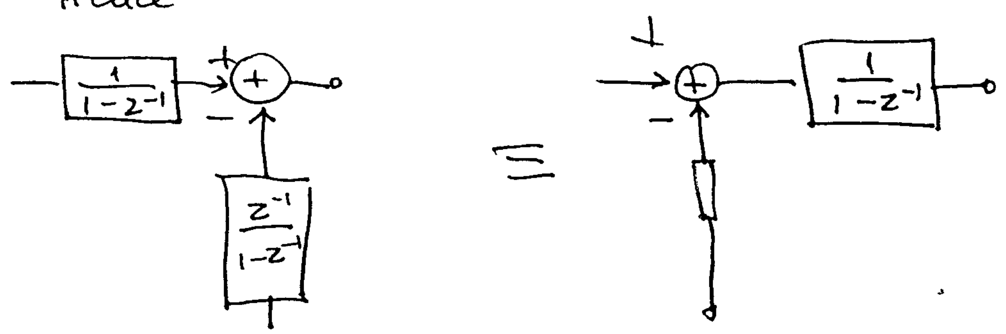
ie. $\beta = 1 - \frac{1}{F(z)} = (F(z) - 1) / F(z)$

Noise shaping filter $F(z)$ chosen to place noise energy at high end of spectrum to be removed by post-filtering. 10

If X is lowpass then $F(z)$ is highpass

for $F(z) = 1 - z^{-1}$ we have $\alpha = \frac{1}{1-z^{-1}}$ $\beta = -\frac{z^{-1}}{1-z^{-1}}$

Hence



(Two accumulators) (Single accumulator) 5

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