EE4-29
EE9-CS3-2
EE9-FPN1-04

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2016

MSc and EEE/EIE PART IV: MEng and ACGI

OPTIMIZATION

Thursday, 28 April 10:00 am

Time allowed: 3:00 hours

There are FOUR questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : A. Astolfi
Second Marker(s) : P.L. Dragotti

OPTIMISATION

1. Consider the problem of computing the average of four numbers, a_1, a_2, a_3 and a_4. This problem can be posed as an unconstrained optimization problem as follows

$$\min_x f(x),$$

with

$$f(x) = (x-a_1)^2 + (x-a_2)^2 + (x-a_3)^2 + (x-a_4)^2.$$

a) Compute the unique stationary point $x_\star$ of the function f and show that $x_\star$ is indeed the average of a_1, a_2, a_3 and a_4. [2 marks]

b) Using second order sufficient conditions of optimality show that the stationary point determined in part a) is a local minimizer. Hence, show that f is radially unbounded and that the stationary point determined in part a) is the global minimizer of f. [4 marks]

c) Assume $a_1 = 1$, $a_2 = 2$, $a_3 = 3$ and $a_4 = -6$.

   i) Write the gradient method for the minimization of the function f and determine the exact line search parameter $\alpha^\star$. [4 marks]

   ii) Consider the gradient method with line search parameter $\gamma\,\alpha^\star$, with $\gamma \in [0,3]$. Determine for which values of γ the iteration yields a converging sequence and, for these values of γ determine the speed of convergence of the sequence. [10 marks]

2. Consider the optimization problem

$$\min_{x_1,x_2} (x_1-6)^2+(x_2-7)^2,$$
$$x_1+x_2-7 \le 0.$$

a) State first order necessary conditions of optimality for this constrained optimisation problem. [2 marks]

b) Using the conditions derived in part a) compute a candidate optimal solution and argue, using geometrical terms, that this is indeed the solution of the optimization problem. Sketch the admissible set, the level lines of the cost function and identify the optimal solution on the plot. [8 marks]

c) Consider the penalty function

$$P_c(x_1,x_2) = (x_1-6)^2+(x_2-7)^2+c(\max\{0,x_1+x_2-7\})^2,$$

with $c > 0$.

i) Determine the stationary points of the function P_c. Let $x_c^\star$ be the stationary point which depends on c. [6 marks]

ii) Show that the point $x_c^\star$ is a global minimizer of P_c when the argument of the "max" function returns the constraint. [2 marks]

iii) Show that, as $c \to \infty$, $x_c^\star$ converges to the optimal solution of the considered problem. [2 marks]

3. Consider the optimization problem

$$\min_{x,y} \; -xy^2,$$

$$y - x + \varepsilon = 0,$$

with ε a non-negative constant.

a) State first order necessary conditions of optimality for such a constrained optimization problem. [2 marks]

b) Using the conditions derived in part a) determine the candidate optimal local solutions. Compute also the corresponding optimal multipliers.
(Hint: note that the candidate optimal solutions $(x^\star, y^\star)$ and the optimal multipliers $\lambda^\star$ may be functions of ε.) [6 marks]

c) Using second order sufficient conditions of optimality show that one of the candidate local optimal solutions is indeed a solution of the considered problem. [6 marks]

d) Evaluate the optimal cost as a function of ε, that is the function

$$J^\star(\varepsilon) = -x^\star(\varepsilon)(y^\star(\varepsilon))^2$$

and show that

$$\lambda^\star(\varepsilon) = \frac{\partial J^\star}{\partial \varepsilon}.$$

Hence argue that the Lagrange multiplier provides a measure of the sensitivity of the optimal cost to *constraint violation*, with ε regarded as measure of the violation. [6 marks]

4. Consider the optimization problem

$$\min_{x_1,x_2} 2x_1^2 + 9x_2,$$

$$x_1 + x_2 - 4 = 0,$$

$$-x_1 \leq 0.$$

a) Show that all points are regular points for the constraints. [2 marks]

b) State first order necessary conditions of optimality for this constrained optimisation problem. [2 marks]

c) Using the conditions derived in part b) compute candidate optimal solutions. [4 marks]

d) The problem can be solved using the mixed barrier-penalty function

$$B_r(x_1,x_2) = 2x_1^2 + 9x_2 + r\log x_1 + \frac{1}{r}(x_1 + x_2 - 4)^2,$$

with r sufficiently small and $r > 0$.

i) Determine the stationary points of the function B_r. [4 marks]

ii) Show that one of the stationary points determined in part d.i) converges, as $r \to 0$, to the optimal solution of the problem. [4 marks]

iii) Show that one of the stationary points determined in part d.i) converges, as $r \to 0$, to $x_1 = 0$ and $x_2 = 4$. Explain the *nature* of this point. [4 marks]

