E2.5

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2006

EEE/ISE PART II: MEng, BEng and ACGI

SIGNALS AND LINEAR SYSTEMS

Corrected Copy

None

Wednesday, 7 June 2:00 pm

Time allowed: 2:00 hours

There are FOUR questions on this paper.

Q1 is compulsory.
Answer Q1 and any two of questions 2-4.
Q1 carries 40% of the marks. Questions 2 to 4 carry equal marks (30% each).

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : P.T. Stathaki,
Second Marker(s) : A.G. Constantinides,

1.

Consider the discrete-time system with the following input-output relationship

$$y[n] = \frac{x[n] - x[n-1]}{2} \tag{1}$$

with $x[n]$ the input of the system and $y[n]$ the output of the system.

(i) Is this system linear and time-invariant? Justify your answer.

[5]

(ii) Find the impulse response $h[n]$ of the system and express it compactly in a mathematical form. Sketch the impulse response.

[5]

(iii) Find the step response $s[n]$ of the system and express it compactly in a mathematical form. Sketch the step response.

[5]

(iv) By performing the discrete time convolution $y[n] = x[n] * h[n]$ find the output $y[n]$ of the system when the input is given by

$$x[n] = \begin{cases} n, & n = 0,1,2,3,4 \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

Verify that the output is indeed the output expected from the filter defined in Equation (1) above, when the input is the signal $x[n]$ defined in Equation (2) above.

[5]

(v) Consider a discrete signal $x[n]$ with Discrete Time Fourier Transform $X(e^{j\omega})$. Find the Discrete Time Fourier Transform of the signal $x[n-n_0]$ with n_0 any integer.

[5]

(vi) Find the Discrete Time Fourier Transform of the signal $x[n]$ defined in (iv).

[5]

(vii) Consider a discrete signal $x[n]$ with z-transform $X(z)$. Find the z-transform of the signal $x[n-n_0]$ with n_0 any integer.

[5]

(viii) Find the z-transform of the output $y[n]$ of the system defined in Equation (1) above, when the input is the function $x[n]$ defined in (iv).

[5]

2.

(a) Consider a discrete, real and even signal $x_1[n]$ that is periodic with period $N = 7$ and fundamental frequency $\omega_0 = \dfrac{2\pi}{N}$.

(i) Prove that the Fourier series coefficients c_k of $x_1[n]$ given by

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x_1[n] e^{-jk(2\pi/N)n}$$

are real and even.

[6]

(ii) Show that the coefficients c_k are periodic with respect to k with period $N = 7$.

[6]

(iii) Given that

$$c_{15} = 1,\ c_{16} = 2,\ c_{17} = 3,$$

determine the values of c_{-1}, c_{-2}, c_{-3}.

[6]

(**Definition:** A discrete signal $x[n]$ is even if $x[-n] = x[n]$).

(b) Find the Discrete Time Fourier Transform of the discrete signal $x_2[n] = a^n u[n], |a| < 1$, where $u[n]$ is the discrete unit step function defined as

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

You may wish to use the relationship $\sum_{n=0}^{+\infty} x^n = \dfrac{1}{1-x}$, if $|x| < 1$.

[6]

(c) The input $x[n]$ and output $y[n]$ of a stable and causal linear, time-invariant system are related by the difference equation

$$y[n] - \frac{5}{6}y[n-1] + \frac{1}{6}y[n-2] = -x[n].$$

Find the impulse response of this system.

[6]

3.

(a) Consider a continuous-time signal $x(t)$ which is sampled uniformly with sampling period T_s to obtain the signal $x_s(t) = x(t) \sum_{k=-\infty}^{+\infty} \delta(t - kT_s)$, where $\delta(t)$ is the continuous-time impulse function.

(i) Prove that $x_s(t) = \frac{1}{T_s} x(t) \sum_{k=-\infty}^{+\infty} e^{jk\frac{2\pi}{T_s}t}$. **[7]**

(ii) Prove that the Fourier transform of the sampled signal $x_s(t)$ is $X_s(\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{k=+\infty} X(\omega + k\omega_s), \omega_s = \frac{2\pi}{T_s}$ where $X(\omega)$ is the Fourier transform of the original signal $x(t)$. **[8]**

(b) Consider a continuous-time signal $x(t)$ with Fourier transform $X(\omega) = (1 - \frac{|\omega|}{2\pi \times 10^3})\Pi(\frac{\omega}{4\pi \times 10^3})$ where ω is the angular frequency and $\Pi(\omega)$ is defined as:

$$\Pi(\omega) = \begin{cases} 1 & |\omega| < 0.5 \\ 0.5 & |\omega| = 0.5 \\ 0 & \text{otherwise.} \end{cases}$$

We sample $x(t)$ uniformly with sampling period T_s to obtain the signal $x_s(t) = x(t) \sum_{k=-\infty}^{+\infty} \delta(t - kT_s)$.

(i) How large can T_s be and yet allow perfect reconstruction of the continuous-time signal from its samples? **[7]**

(ii) Sketch the Fourier transform of $x_s(t)$, $X_s(\omega)$, assuming $T_s = 0.1\text{ms}$. **[8]**

4.

(a) (i) Find the analytical expression and the region of convergence (ROC) of the z-transform of the discrete causal signal $x[n] = a^n u[n-1]$, with a real and $u[n]$ the discrete unit step function defined as

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

[6]

(ii) Find the analytical expression and the region of convergence (ROC) of the z-transform of the discrete anti-causal signal $x[n] = -a^n u[-n]$, with a real and $u[n]$ the discrete unit step function.

[6]

For parts (a) (i), (a) (ii) you may wish to use the relationship $\sum_{n=0}^{+\infty} x^n = \frac{1}{1-x}$, if $|x| < 1$.

(b) Consider a linear, time-invariant system with input $x[n]$ and output $y[n]$ related by the difference equation

$$y[n] - \frac{9}{2} y[n-1] + 2y[n-2] = -7x[n].$$

Determine the impulse response and its z-transform in the following three cases:

(i) The system is causal.

(ii) The system is stable.

(iii) The system is neither causal nor stable.

[18]

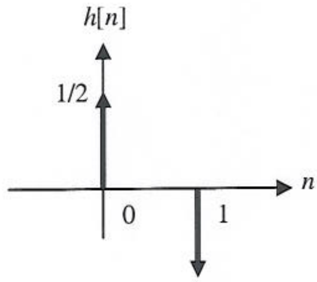
E2.5 Signals + Systems Solutions. 2006

1/4

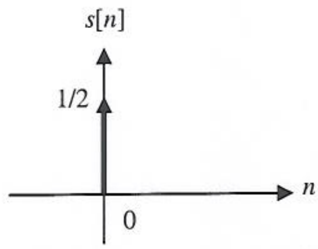
Answer

(i) Yes, since if the inputs $x_1[n]$ and $x_2[n]$ produce the outputs $y_1[n]$ and $y_2[n]$ respectively, the input $a_1x_1[n]+a_2x_2[n]$ will produce the output $a_1y_1[n]+a_2y_2[n]$.

(ii) The impulse response of the system $h[n]$ is defined as the output of the system when the input is the impulse function $\delta[n]$. Therefore, $h[n]=\dfrac{\delta[n]-\delta[n-1]}{2}$. This function is shown below:



(iii) The step response of the system $s[n]$ is defined as the output of the system when the input is the unit step function $u[n]$. Therefore, $s[n]=\dfrac{u[n]-u[n-1]}{2}=\dfrac{\delta[n]}{2}$. This function is shown below:



(iv) This is defined as $y[n]=\sum_{k=-\infty}^{+\infty}x[k]h[n-k]$. In that case $x[k]$ is non-zero if $0\le k\le 4$ and $h[n-k]$ is non-zero if $0\le n-k\le 1\Rightarrow -1\le -n+k\le 0\Rightarrow n-1\le k\le n$. We may find three separate cases for which the two intervals overlap, and therefore the convolution is non-zero.

1. The lower bound of the function $x[k]$ lies within the bounds of the function $h[n-k]$, i.e., $n-1\le 0\le n\Rightarrow 0\le n\le 1$.

   In that case $y[n]=\sum_{k=-\infty}^{+\infty}x[k]h[n-k]=\sum_{k=0}^{n}\frac{1}{2}(\delta[n-k]-\delta[n-k-1])k=\frac{1}{2}n$. Therefore, $y[0]=0$ and $y[1]=\frac{1}{2}$.

2. The bounds of the function $h[n-k]$ are included within the bounds of the function $x[k]$, $n-1>0\Rightarrow n>1$ and $n\le 4$, i.e., $1<n\le 4$

   In that case $y[n]=\sum_{k=n-1}^{n}\frac{1}{2}(\delta[n-k]-\delta[n-k-1])k=-\frac{(n-1)}{2}+\frac{n}{2}=\frac{1}{2}$.

3. The upper bound of the function $x[k]$ lies within the bounds of the function $h[n-k]$, i.e., $n-1\le 4<n\Rightarrow 4<n\le 5\Rightarrow n=5$.

   In that case $y[5]=\sum_{k=-\infty}^{+\infty}x[k]h[5-k]=x[5]h[0]+x[4]h[1]=5\frac{1}{2}-4\frac{1}{2}=\frac{1}{2}$.

   Thus,

2/4

$$y[n]=\begin{cases}0 & n=0\\ \frac{1}{2} & n=1,2,3,4,5\ .\\ 0 & \text{otherwise}\end{cases}$$

From the relationship $y[n]=\dfrac{x[n]+x[n-1]}{2}$ we see that for the given function $x[n]$ we can get $y[n]$ as above, since:

$$y[0]=\frac{x[0]-x[-1]}{2}=0,\ y[1]=\frac{x[1]-x[0]}{2}=\frac{1}{2},\ \text{etc.}$$

(v) $\sum_{n=-\infty}^{\infty}x[n-n_0]e^{-j\omega n}=\sum_{n=-\infty}^{\infty}x[n-n_0]e^{-j\omega(n-n_0)}e^{-j\omega n_0}=e^{-j\omega n_0}X(e^{j\omega})$

(vi) $\sum_{n=1}^{4}ne^{-j\omega n}=e^{-j\omega}+2e^{-2j\omega}+3e^{-3j\omega}+4e^{-4j\omega}$

(vii) $\sum_{n=-\infty}^{\infty}x[n-n_0]z^{-n}=\sum_{n=-\infty}^{\infty}x[n-n_0]z^{-(n-n_0)}z^{-n_0}=z^{-n_0}X(z)$

(viii) $Y(z)=X(z)\dfrac{1-z^{-1}}{2}=(z^{-1}+2z^{-2}+3z^{-3}+4z^{-4})\dfrac{1-z^{-1}}{2}$

2.

(a) Consider the discrete, real and even signal $x[n]$ that is periodic with period $N=7$ and fundamental frequency $\omega_0=\dfrac{2\pi}{N}$. Suppose that the Fourier series coefficients of $x(t)$ are c_k.

(i) $c_k=\dfrac{1}{N}\sum_{n=0}^{N-1}x_1[n]e^{-jk(2\pi/N)n}$ and also

$$c_k=\frac{1}{N}\sum_{n=-(N-1)}^{0}x_1[n]e^{-jk(2\pi/N)n}=\frac{1}{N}\sum_{n=0}^{N-1}x_1[-n]e^{jk(2\pi/N)n}=\frac{1}{N}\sum_{n=0}^{N-1}x_1[n]e^{jk(2\pi/N)n}=c_k^*$$

$$c_{-k}=\frac{1}{N}\sum_{n=0}^{N-1}x_1[n]e^{-jk(2\pi/N)(-n)}=\frac{1}{N}\sum_{n=0}^{-(N-1)}x_1[-n]e^{-jk(2\pi/N)n}=\frac{1}{N}\sum_{n=0}^{-(N-1)}x_1[n]e^{-jk(2\pi/N)n}=c_k$$

(ii)

$$c_{k+N}=\frac{1}{N}\sum_{n-0}^{N-1}x_1[n]e^{-j(k+N)(2\pi/N)n}=\frac{1}{N}\sum_{n-0}^{N-1}x_1[n]e^{-jk(2\pi/N)n}e^{-j2\pi n}=c_k$$

(iii) Since the Fourier series coefficients c_k will be real and even. Given that

$$c_{15}=1,\ c_{16}=2,\ c_{17}=3$$

determine the values of c_{-1}, c_{-2}, c_{-3}.

$c_{-1}=c_1=c_{15}=1$

$c_{-2}=c_2=c_{16}=2$

$c_{-3}=c_3=c_{17}=3$

(b) Find the Fourier transform of the discrete signal $x[n]=a^nu[n], |a|<1$.

In this case $X(e^{j\omega})=\sum_{n=-\infty}^{+\infty}a^nu[n]e^{-j\omega n}=\sum_{n=0}^{+\infty}(ae^{-j\omega})^n=\dfrac{1}{1-ae^{-j\omega}},\ |a|<1$

(c) The input and output of a stable and causal LTI system are related by the differential equation

$$y[n]-\frac{5}{6}y[n-1]+\frac{1}{6}y[n-2]=-x[n]$$

Find the impulse response of this system.

We take the Fourier transform in both sides:

3/4

$$Y(e^{j\omega}) - \frac{5}{6}e^{-j\omega}Y(e^{j\omega}) + \frac{1}{6}e^{-2j\omega}Y(e^{j\omega}) = -X(e^{j\omega}) \Rightarrow$$

$$\frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{-1}{1-\frac{5}{6}e^{-j\omega}+\frac{1}{6}e^{-2j\omega}} = \frac{-1}{(1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{3}e^{-j\omega})} =$$

$$\frac{2}{(1-\frac{1}{3}e^{-j\omega})} - \frac{3}{(1-\frac{1}{2}e^{-j\omega})} \Rightarrow$$

$$h[n] = [2(\frac{1}{3})^n - 3(\frac{1}{2})^n]u[n]$$

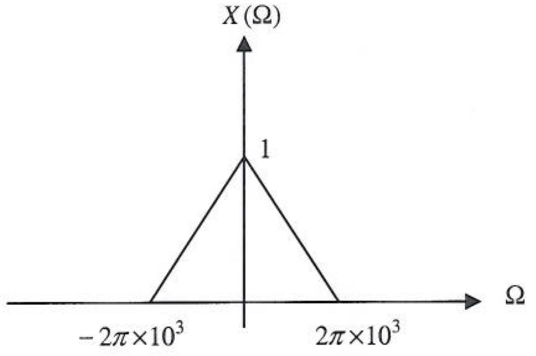
3.

(a) (i) The continuous-time impulse train $\sum_{k=-\infty}^{\infty}\delta(t-kT)$ is a periodic function and therefore it can be written using Fourier series. The Fourier series coefficients are $c_k = \frac{1}{T_s}\int_{-\frac{T_s}{2}}^{\frac{T_s}{2}}\delta(t)e^{-jk\omega_s t}dt = \frac{1}{T_s}$.

Therefore, $x_s(t) = x(t)\sum_{k=-\infty}^{+\infty}\delta(t-kT_s) = \frac{1}{T_s}x(t)\sum_{k=-\infty}^{+\infty}e^{jk\frac{2\pi}{T_s}t}$

(ii) The Fourier transform of the signal $e^{j\omega_s t}x(t)$ is $X(j(\omega-\omega_s))$ and therefore the Fourier transform of the sampled signal $x_s(t)$ is $X_s(e^{j\omega}) = \frac{1}{T_s}\sum_{k=-\infty}^{k=+\infty}X(\omega+k\omega_s), \omega_s = \frac{2\pi}{T_s}$.

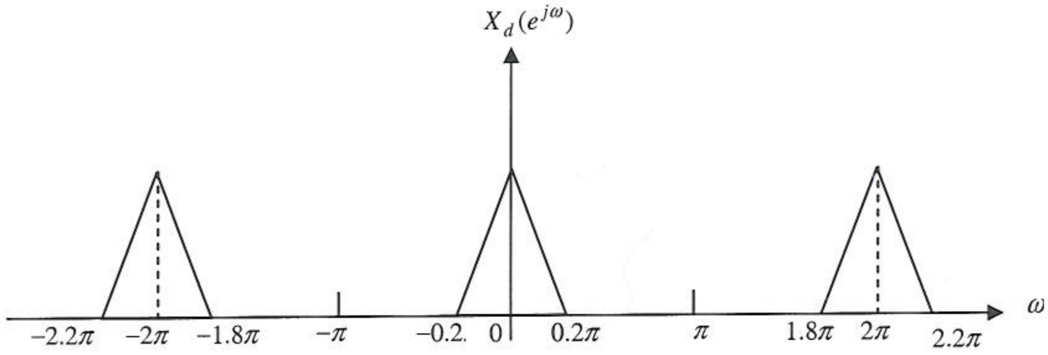
(b) (i) The function $\Pi(\frac{\Omega}{4\pi\times10^3})$ is equal to 1 if $-\frac{1}{2} < \frac{\Omega}{4\pi\times10^3} < \frac{1}{2} \Rightarrow -2\pi\times10^3 < \Omega < 2\pi\times10^3$, equal to $\frac{1}{2}$ if $\Omega = \pm2\pi\times10^3$ and 0 otherwise. Therefore, $X(\Omega)$ has the form shown below.



Based on the sampling theorem (called also Nyquist criterion) the sampling frequency must be at twice the maximum frequency of the signal. In that case:

$$\Omega_s = 2\pi f_s \geq 2(2\pi\times10^3) \Rightarrow f_s \geq 2\times10^3 \Rightarrow T_s = \frac{1}{f_s} \leq 0.5\times10^{-3}$$

(ii) Based on the analysis given in the first section of this set of notes, the DTFT of $x[n]$ has the form shown below. The horizontal axis ω is the axis Ω shown above, multiplied by $T_s = 10^{-4}$ sec



4.

(a) (i) The z transform expression is

$$X(z)=\sum_{n=1}^{+\infty}a^n z^{-n}=\sum_{n=0}^{+\infty}(az^{-1})^n-1$$

If $\left|az^{-1}\right|<1$, or equivalently, $|z|>|a|$, the above sum converges and

$$X(z)=\frac{1}{1-az^{-1}}-1=\frac{z}{z-a}-1=\frac{a}{z-a},|z|>|a|$$

$$x(n)=\begin{cases}-a^n & n<0\\ 0 & n\geq 0\end{cases}$$

(ii) The z transform expression is

$$X(z)=-\sum_{n=-\infty}^{0}a^n z^{-n}=-\sum_{n=0}^{+\infty}a^{-n}z^n$$

If $\left|a^{-1}z\right|<1$, or equivalently, $|z|<|a|$, the above sum converges and

$$X(z)=-\frac{1}{1-a^{-1}z}=\frac{a}{z-a},|z|<|a|$$

(b) $y[n]-\frac{9}{2}y[n-1]+2y[n-2]=-7x[n]\Rightarrow Y(z)-\frac{9}{2}z^{-1}Y(z)+2z^{-2}Y(z)=-7X(z)$

$$\Rightarrow H(z)=\frac{-7}{1-\frac{9}{2}z^{-1}+2z^{-2}}=\frac{1}{1-\frac{1}{2}z^{-1}}-\frac{8}{1-4z^{-1}}\Rightarrow$$

Causal $\quad h(n)=(\frac{1}{2})^n u[n]+4^n u[n]$

Stable $\quad h(n)=(\frac{1}{2})^n u[n]-4^n u[-n-1]$

Nether causal nor stable $\quad h(n)=-(\frac{1}{2})^n u[-n-1]+4^n u[n]$ or

$$h(n)=-(\frac{1}{2})^n u[-n-1]-4^n u[-n-1]$$