IMPERIAL COLLEGE LONDON

B.Eng and M.Eng Examinations 2015 - 2016
Part 1

BE1-HMATH1 Mathematics I

Wednesday, 03 June 2016 14:30-16:00
(duration: 90 minutes)

All three questions are compulsory.
Please answer each question in separate answer book.
A list of formulae is provided separately.
Each question is worth 100 marks.
Marks for questions and parts of questions are shown next to the question. The marks for questions (and parts thereof) are indicative, and they may be slightly moderated at the discretion of the Examiner.

Question 1 This question has two parts.

a) Consider the function

$$f(x) = \int_0^x \frac{\ln(1+w)}{w} dw \ .$$

i) Find the power series expansion of the function $f(x)$.

40 marks

ii) Determine the interval in which the series expansion absolutely converges.

10 marks

b) **i)** Find all the solutions to the complex equation

$$\left| \frac{z-i}{z+i} \right| = 1 \ .$$

20 marks

ii) Find all the solutions to the complex equation

$$z^5 + 32 = 0 \ .$$

20 marks

iii) Draw the solutions from i) and ii) on an Argand diagram.

10 marks

The two parts carry equal marks.

Question 2 This question has two parts.

a) Find the general solution, $y(x)$, of the following differential equation.

$$y'' + 4y - 10e^x \sin x = 0$$

40 marks

b) **i)** Solve the following differential equation in the domain $D \equiv [1/2, \infty)$:

$$x\frac{dy}{dx} + 2y = x^2 + 1$$

with the condition $y(1) = 1$.

30 marks

ii) Sketch the solution $y(x)$. Identify clearly within the domain D, if exist, the intercepts, local maxima/minima, intervals in which $y(x)$ is increasing and decreasing, intervals in which $y(x)$ is concave up and down, and asymptotes.

30 marks

The two parts carry, respectively, 40%, and 60% of the marks.

Question 3 Consider the two points $A(2, 1, 0)$ and $B(1, 3, 0)$.

a) Find a point C, such that the set of the three vectors $\{\underline{\text{OA}}, \underline{\text{OB}}, \underline{\text{OC}}\}$ is linear independent.

10 marks

b) Find the area of the parallelogram which is spanned by the vectors $\underline{\text{OA}}$ and $\underline{\text{OB}}$.

20 marks

c) Calculate $\det \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$.

10 marks

d) Find an equation of the plane Π containing the points A, B and C.

10 marks

e) Find the minimal distance from the origin to the plane Π in Question 3d).

20 marks

f) State whether the two functions, $f(x) = x + x^{-1}$ and $g(x) = x - x^{-1}$ are even, odd, or nether, and explain why.

10 marks

g) Evaluate $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ and $\lim_{x\to 0} \frac{f(x)}{g(x)}$.

20 marks

The seven parts carry, respectively, 10%, 20%, 10%, 10%, 20%, 10%, and 20% of the marks.

Question 1 This question has two parts.

a) Consider the function

$$f(x) = \int_0^x \frac{\ln(1+w)}{w} dw \ .$$

i) Find the power series expansion of the function $f(x)$.

40 marks

Marks: $\overline{40}$

ii) Determine the interval in which the series expansion absolutely converges.

10 marks

Marks: $\overline{10}$

b) **i)** Find all the solutions to the complex equation

$$\left| \frac{z-i}{z+i} \right| = 1 \ .$$

20 marks

Let $z = a + bi$, then the equation leads to $a^2 + (b-1)^2 = a^2 + (b+1)^2$. The solutions are therefore $z \in \mathbf{R}$.

Marks: $\overline{20}$

ii) Find all the solutions to the complex equation

$$z^5 + 32 = 0 \ .$$

20 marks

Let $z = re^{i\theta}$ to find that $r = 2$ and $\theta = \pi/5 + 2\pi k/5$ where $k = 0, 1, 2, 3, 4$.

Marks: $\overline{20}$

iii) Draw the solutions from i) and ii) on an Argand diagram.

10 marks

i) The $Re[z]$-axis on the Argand diagram. ii) The vertices of the a pentagon centred at the origin with one of one vertex at $(-2, 0)$.

Marks:

$\overline{\mathbf{10}}$

The two parts carry equal marks.

Question 2 This question has two parts.

a) Find the general solution, $y(x)$, of the following differential equation.

$$y'' + 4y - 10e^x \sin x = 0$$

40 marks

Marks: $\overline{40}$

b) **i)** Solve the following differential equation in the domain $D \equiv [1/2, \infty)$:

$$x\frac{dy}{dx} + 2y = x^2 + 1$$

with the condition $y(1) = 1$.

30 marks

The equation is linear, so we first find the integrating factor, which is x^2. The equation thus becomes $(x^2y)' = x^3 + x$ and the solution is therefore $y = x^2/4 + 1/2 + C/x^2$, where $C = 1/4$ from to the initial condition.

Marks: $\overline{30}$

ii) Sketch the solution $y(x)$. Identify clearly within the domain D, if exist, the intercepts, local maxima/minima, intervals in which $y(x)$ is increasing and decreasing, intervals in which $y(x)$ is concave up and down, and asymptotes.

30 marks

No y-intercept. Local minimum at $x = 1$. y is decreasing in $[1/2, 1)$ and increasing in $(1, \infty)$, and is always concave up. There are no asymptotes.

Marks: $\overline{30}$

The two parts carry, respectively, 40%, and 60% of the marks.

Question 3 Consider the two points $A(2, 1, 0)$ and $B(1, 3, 0)$.

a) Find a point C, such that the set of the three vectors $\{\underline{OA}, \underline{OB}, \underline{OC}\}$ is linear independent.

10 marks

For example, $C(0, 0, 1)$.

Marks: $\mathbf{10}$

b) Find the area of the parallelogram which is spanned by the vectors <u>OA</u> and <u>OB</u>.

20 marks

$$\text{The area} = |\underline{OA} \times \underline{OB}| = |\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix}|.$$

Marks: $\mathbf{20}$

c) Calculate $\det \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$.

10 marks

5

Marks: $\mathbf{10}$

d) Find an equation of the plane Π containing the points A, B and C.

10 marks

Either

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}.$$

or $2x + y + 5z = 5$.

Marks: $\mathbf{10}$

e) Find the minimal distance from the origin to the plane Π in Question 3d).

20 marks

$D = \frac{|-5|}{\sqrt{4+1+25}} = \frac{5}{\sqrt{30}} = \frac{1}{6}\sqrt{30}$.

Marks:

$\overline{20}$

f) State whether the two functions, $f(x) = x + x^{-1}$ and $g(x) = x - x^{-1}$ are even, odd, or nether, and explain why.

10 marks

$f(-x) = -f(x)$ *and* $g(-x) = -g(x)$. *Therefore they are both odd.*

Marks:

$\overline{10}$

g) Evaluate $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)}$ and $\lim\limits_{x\to 0} \frac{f(x)}{g(x)}$.

20 marks

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1 \text{ and } \lim_{x\to 0} \frac{f(x)}{g(x)} = -1.$$

Marks:

$\overline{20}$

The seven parts carry, respectively, 10%, 20%, 10%, 10%, 20%, 10%, and 20% of the marks.