Paper Number(s): **E4.29**
C1.1

IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE
UNIVERSITY OF LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2002

MSc and EEE PART IV: M.Eng. and ACGI

OPTIMIZATION

Wednesday, 24 April 10:00 am

There are SIX questions on this paper.

Answer FOUR questions.

Time allowed: 3:00 hours

Corrected Copy

Examiners responsible:

First Marker(s): Astolfi,A.

Second Marker(s): Clark,J.M.C.

Special instructions for invigilators: None

Information for candidates:

- All functions are sufficiently smooth.
- ∇f denotes the gradient of the function f. Note that ∇f is a column vector.
- $\nabla^2 f$ denotes the Hessian matrix of the function f. Note that $\nabla^2 f$ is a square matrix and that, under suitable regularity conditions, the Hessian matrix is symmetric.
- Let $f : I\!R^n \to I\!R$. A level set of f is any non-empty set described by

  $$\mathcal{L}(\alpha) = \{x \in I\!R^n \ : \ f(x) \leq \alpha\},$$

  with $\alpha \in I\!R$.

1. Consider the problem of minimizing a function $f : \mathbb{R}^n \to \mathbb{R}$ with the gradient method applied using at each step an exact line search. Let p_0 be the starting point of the algorithm and $\{p_k\}$ the sequence generated by the algorithm.

(a) Show that, for each $k \geq 0$, the search direction d_{k+1} is orthogonal to the search direction d_k. [4]

(b) Consider the function

$$f(x) = 10x^2 + y^2.$$

Show that the sequence $\{p_k\} = \{x_k, y_k\}$ (resulting from the application of the gradient method with exact line search) is such that

$$x_{k+1} = -9\frac{x_k y_k^2}{1000x_k^2 + y_k^2} \qquad y_{k+1} = 900\frac{y_k x_k^2}{1000x_k^2 + y_k^2}.$$

[8]

(c) Assume that the sequence $\{p_k\}$ converges to a minimum $p^\star$ of the function f. Using the result in part (b) compute such a minimum. [4]

(d) Compute the first element (*i.e.* p_1) of the sequence $\{p_k\}$ obtained from the starting point $p_0 = (1/10, 1)$. Let

$$C_1 = \frac{\|p_1 - p^\star\|}{\|p_0 - p^\star\|}$$

and show that C_1 is equal to the theoretical worst case value

$$\frac{\lambda_M - \lambda_m}{\lambda_M + \lambda_m},$$

where λ_M is the largest eigenvalue of $\nabla^2 f$ and λ_m is the smallest eigenvalue of $\nabla^2 f$. (Note that $\nabla^2 f$ is a constant matrix.) [4]

2. (a) Give necessary and sufficient *second order* conditions for a point $x^\star \in I\!R^n$ to be a strict local minimum for a C^2 function $f : I\!R^n \to I\!R$. [4]

(b) Consider the function

$$\phi(x_1, x_2) = e^{x_1^2 + x_1 x_2 + x_2^2 - 1}$$

and sketch its level sets. Hence perform two steps of the Newton algorithm for the local minimization of ϕ starting from the point $(x_1, x_2) = (1, 0)$. Let $(\bar{x}_1, \bar{x}_2)$ denote the point obtained by the application of the Newton algorithm. [8]

(c) Show that if $x^\star$ is a strict local minimum of the function

$$v(x) = e^{f(x)},$$

then $x^\star$ is also a strict local minimum of the function f. [4]

(d) Use the result in part (c) to compute a strict local minimum for the function ϕ in part (b). Compare the value of the exact local minimum with the value $(\bar{x}_1, \bar{x}_2)$ obtained in part (b). [4]

3. Consider the system of equations

$$r_1(x) = x_1^2 - x_2 - 1 = 0 \qquad r_2(x) = x_1 - x_2 = 0.$$

The solutions $(x_1^\star, x_2^\star)$ of such a system can be obtained minimizing the function $f : I\!R^2 \to I\!R$ defined as

$$f(x) = r_1^2(x) + r_2^2(x).$$

(a) Compute the stationary points of the function f. Show that two of these points are global minima, *i.e.* f is equal to zero at these points, and one is a saddle point. [8]

(b) Describe the method of coordinate directions for unconstrained minimization and discuss when it is convenient to use this method. [4]

(c) Starting from the point $x^0 = (1, -1)$ apply four iterations of the coordinate direction method with initial direction $d = [1, 0]'$ and selecting, at each step, the parameter α by inspection. Let x^1, x^2, x^3 and x^4 be the points obtained by the algorithm. Sketch on the (x_1, x_2) plane the position of such points and discuss if the sequence thus obtained approaches the global minimum of f. [8]

4. Consider an optimization problem of the form

$$\begin{cases} \min\limits_{x} f(x) \\ g(x) = 0 \end{cases}$$

(a) State first order necessary conditions and second order sufficient conditions of optimality for such a problem. [4]

(b) Let

$$f(x) = \frac{1}{2}((x_1 - 1)^2 + x_2^2)$$

and

$$g(x) = -x_1 + \beta x_2^2,$$

with β constant. Examine for what values of β it is possible to conclude that $x^\star = (0, 0)$ is a local minimum. [8]

(c) Solve the equation associated with the constraint and substitute the solution into the function f. Show that for $\beta \leq 1/2$ the point $x^\star = (0, 0)$ is a local minimum and for $\beta > 1/2$ it is a local maximum. [8]

5. Consider an optimization problem of the form

$$\begin{cases} \min\limits_{x} f(x) \\ g(x) = 0 \end{cases}$$

(a) Discuss the exact penalty function method for such an optimization problem. [4]

(b) Let

$$f(x) = x_1^2 + x_2^2 + 3x_1x_2$$

and

$$g(x) = x_1 + 3x_2 - 5.$$

Compute an exact penalty function for the minimization problem. [6]

(c) Compute the stationary points and the minima of the exact penalty function constructed in part (b). Hence construct a solution of the considered constrained optimization problem. [6]

(d) Let $x^\star$ be the constrained minimum computed in part (c). Using the first order necessary conditions of optimality construct the corresponding optimal multiplier $\lambda^\star$. [4]

6. Consider the function

$$f(x) = 10(1 - xe^{-x/3}\sin(x/2))$$

depicted in Figure 1 for $x \in [0, 20]$, and the problem of finding the global minimum of f for $x \in [0, 20]$.

(a) Show that the function is Lipshitz in the interval $[0, 20]$. (Hint: The Lipshitz constant is upper bounded by the maximum modulus of the derivative of f). [4]

(b) Assume that an estimate of the Lipshitz constant for the function f is $\tilde{L} = 5$. Starting from the point $x = 20$ apply (graphically) the algorithm for global minimization of Lipschitz functions and show that the algorithm converges to the global minimum. [16]

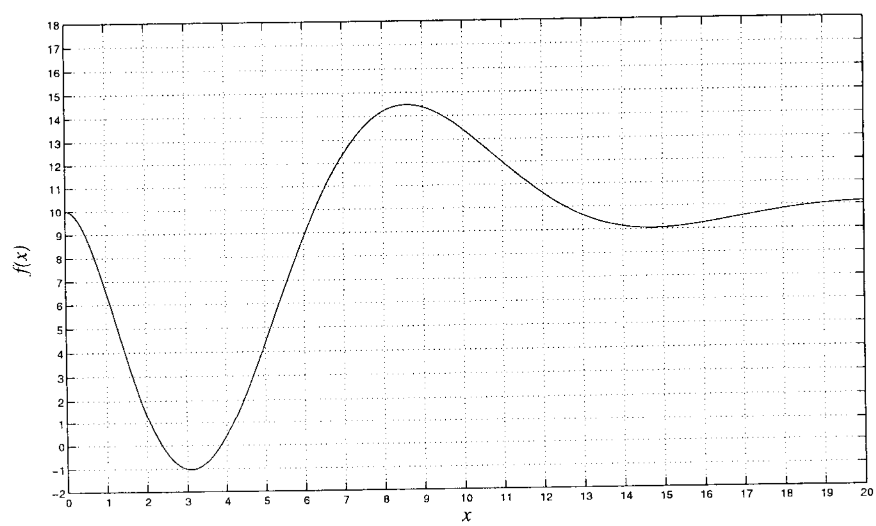

Figure 1: The function $f(x)$.

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(a) $$x_{k+1} = x_k + \alpha_k d_k$$

$$d_k = -\nabla f(x_k)$$

α_k is such that (exact line search)

$$\nabla_\alpha \left[f(x_k + \alpha d_k) \right] \Big|_{\alpha=\alpha_k} = 0$$

$$\Updownarrow$$

$$\nabla' f(x_k + \alpha_k d_k) \cdot d_k = 0$$

$$\Updownarrow$$

$$\nabla' f(x_k + \alpha_k d_k) \cdot \nabla f(x_k) = 0$$

$$\Updownarrow$$

$$\nabla' f(x_{k+1}) \, \nabla f(x_k) = 0 .$$

(b) $$f(x) = 10x^2 + y^2 \qquad \nabla f = \begin{bmatrix} 20x \\ 2y \end{bmatrix}$$

$$x_{k+1} = x_k - 20 x_k \alpha_k = x_k (1 - 20\alpha_k)$$

$$y_{k+1} = y_k - 2 y_k \alpha_k = y_k (1 - 2\alpha_k)$$

$$f(x_{k+1}, y_{k+1}) = 10 x_k^2 (1 - 20\alpha_k)^2 + y_k^2 (1 - 2\alpha_k)^2$$

$$= \alpha_k^2 \left[4000 x_k^2 + 4 y_k^2 \right] - \alpha_k \left[400 x_k^2 + 4 y_k^2 \right] + \ldots$$

optimal α_k: $\alpha_k^* = \dfrac{100 x_k^2 + y_k^2}{2(1000 x_k^2 + y_k^2)}$

$$x_{k+1} = x_k \left[1 - \frac{10\,(100 x_k^2 + y_k^2)}{1000 x_k^2 + y_k^2} \right] = -x_k \frac{9 y_k^2}{1000 x_k^2 + y_k^2}$$

$$y_{k+1} = y_k \left[1 - \frac{100 x_k^2 + y_k^2}{1000 x_k^2 + y_k^2} \right] = y_k \frac{900 x_k^2}{1000 x_k^2 + y_k^2}$$

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(c) If $\{p_k\} \to p^*$ then $p^*_{k+1} = p^*_k$

$$x^* = -\frac{9\, x^* {y^*}^2}{1000 (x^*)^2 + y} \quad , \; y^* = \cdots$$

$$\Downarrow$$

$$(x^*, y^*) = (0, 0).$$

(d) $p_0 \begin{bmatrix} 1/10 \\ 1 \end{bmatrix} \qquad p_1 = \begin{bmatrix} -0.0818 \\ 0.8182 \end{bmatrix}$

Use (b).

$$\frac{\| p_1 - p^* \|}{\| p_0 - p^* \|} = \frac{\| p_1 \|}{\| p_0 \|} = 0.8182 = \frac{10 - 1}{10 + 1} = \frac{\lambda_M - \lambda_m}{\lambda_M + \lambda_m}$$

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Question 2, Part (a)

Second order necessary condition

Let $f : \mathbb{R}^n \to \mathbb{R}$ and assume $\nabla^2 f$ exists and is continuous. The point $x^\star$ is a local minimum of f only if

$$\nabla f(x^\star) = 0$$

and

$$x' \nabla^2 f(x^\star) x \geq 0$$

for all $x \in \mathbb{R}^n$.

Second order sufficient condition

Let $f : \mathbb{R}^n \to \mathbb{R}$ and assume $\nabla^2 f$ exists and is continuous. The point $x^\star$ is a strict local minimum of f if

$$\nabla f(x^\star) = 0$$

and

$$x' \nabla^2 f(x^\star) x > 0$$

for all non-zero $x \in \mathbb{R}^n$.

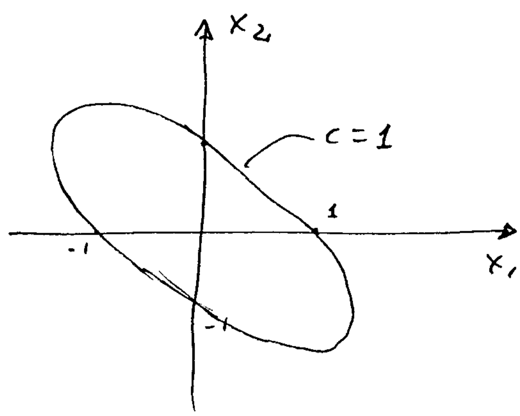
(b) Level sets of ϕ have the same form of level sets of $x_1^2 + x_1 x_2 + x_2^2 - 1$, but with different "levels".

$$f(x) = e^{x_1^2 + x_2^2 + x_1 x_2 - 1} = c > 0$$

$$\Updownarrow$$

$$x_1^2 + x_2^2 + x_1 x_2 - 1 = \lg c$$

Level sets are ellipses.



Newton's algorithm.

$$q_{k+1} = q_k - \left[\nabla^2 f(q_k)\right]^{-1} \nabla f(q_k)$$

$$q_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \nabla f(q_0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \nabla^2 f(q_0) = \begin{bmatrix} 6 & 3 \\ 3 & 3 \end{bmatrix}$$

Hence $q_1 = \begin{bmatrix} 2/3 \\ 0 \end{bmatrix}$

$$\nabla f(q_1) = \begin{bmatrix} 0.76 \\ 0.38 \end{bmatrix} \qquad \nabla^2 f(q_1) = \begin{bmatrix} 2.16 & 1.08 \\ 1.08 & 1.4 \end{bmatrix}$$

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Hence $q_2 = \begin{bmatrix} 0.315 \\ 0 \end{bmatrix}$

(c) If x^* is a strict local minimum of f

$$V(x) = e^{f(x)}$$

then $\nabla v(x^*) = 0 \implies \nabla v(x) = \nabla f(x) e^{f(x)} \implies \nabla f(x^*) = 0$

$$\nabla^2 v(x^*) \geq 0 \implies \nabla^2 v(x) = \nabla^2 f(x) e^{f(x)} + \nabla f(x) \cdot \nabla (e^{f(x)})$$

$$\implies \nabla^2 v(x^*) = \nabla^2 f(x^*) e^{f(x^*)} \implies \nabla^2 f(x^*) \geq 0.$$

Hence x^* is a strict local minimum for f.

(d) $f(x) = x_1^2 + x_1 x_2 + x_2^2 - 1$

$$\nabla f(x) = \begin{bmatrix} 2x_1 + x_2 \\ x_1 + 2x_2 \end{bmatrix} \qquad \nabla f(x) = 0 \implies \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\nabla^2 f(x) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} > 0 \implies \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} \text{ is a local minimum.}$$

The point q_2 in part (b) is not a local min.

The sequence is "slowly" converging to the min.

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(a) $\frac{\partial f}{\partial x_1} = 4\left(x_1^2 - x_2 - 1\right)x_1 + 2x_1 - 2x_2$

$\frac{\partial f}{\partial x_2} = -2x_1^2 + 4x_2 + 2 - 2x_1$

$x_2^* = \frac{1}{2}x_1^2 - \frac{1}{2} + \frac{1}{2}x_1$

$\left.\frac{\partial f}{\partial x_1}\right|_{x=x_1^*} = (2x_1 - 1)\left(x_1^2 - x_1 - 1\right)$

$x_1 = 1/2$	$x_1 = \frac{1+\sqrt{5}}{2}$	$x_1 = \frac{1-\sqrt{5}}{2}$
$x_2 = -1/8$	$x_2 = \frac{1+\sqrt{5}}{2}$	$x_2 = \frac{1-\sqrt{5}}{2}$
$f() > 0$	$f() = 0$	$f() = 0$
$H = \begin{bmatrix} 1.5 & -4 \\ -4 & 4 \end{bmatrix}$	$H = \begin{bmatrix} 22.9 & -8.4 \\ -8.4 & 4 \end{bmatrix} > 0$	$H = \begin{bmatrix} 5 & 0.4 \\ 0.4 & 4 \end{bmatrix};$
saddle point	Minimum	Minimum

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Question 3, Part (b)

The coordinate directions method can be described as follows.

Step 0. Given $x_0 \in I\!R^n$.

Step 1. Set $k = 0$.

Step 2. Set $j = 0$.

Step 3. Set $d_k = e_j$, where e_j is the j-th coordinate direction.

Step 4. Compute α_k performing a line search without derivatives along d_k.

Step 5. Set $x_{k+1} = x_k + \alpha_k d_k$, $k = k + 1$.

Step 6. If $j < n$ set $j = j + 1$ and go to **Step 3**. If $j = n$ go to **Step 2**.

It is easy to verify that the matrix

$$P_k = \left[\begin{array}{cccc} d_k & d_{k+1} & \cdots & d_{k+n-1} \end{array} \right]$$

is such that

$$|\text{det} P_k| = 1,$$

hence, if the line search is such that

$$\lim_{k\to\infty} \frac{\nabla f(x_k)' d_k}{\|d_k\|} = 0$$

and

$$\lim_{k\to\infty} \|x_{k+1} - x_k\| = 0,$$

convergence to stationary points is ensured.

(c)

$x_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\qquad$ $x_1 = \begin{bmatrix} 1+\alpha \\ -1 \end{bmatrix}$ $\qquad$ $\frac{8}{15}$

$f(x_0) = 5$ $\quad$ if $\alpha = -1$ $\quad$ $f(x_1) = 1 < f(x_0)$

Hence $\quad x_1 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$x_2 = \begin{bmatrix} 0 \\ \alpha - 1 \end{bmatrix}$ $\quad$ if $\alpha = 1/2$ $\quad$ $f(x_2) = 1/2 < f(x_1)$

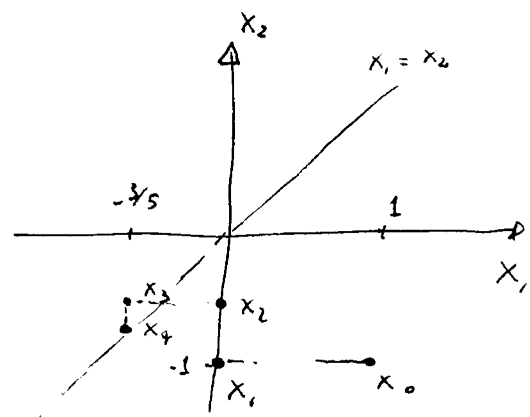
Hence $\quad x_2 = \begin{bmatrix} 0 \\ -1/2 \end{bmatrix}$

$x_3 = \begin{bmatrix} \alpha \\ -1/2 \end{bmatrix}$ $\quad$ if $\alpha = -3/5$ $\quad$ $f(x_3) = 0.03 < f(x_2)$

Hence $\quad x_3 = \begin{bmatrix} -3/5 \\ -1/2 \end{bmatrix}$

$x_4 = \begin{bmatrix} -3/5 \\ \alpha - 1/2 \end{bmatrix}$ $\quad$ if $\alpha = -3/25$ $\quad$ $f(x_4) = 0.008 < f(x_3)$

Hence $\quad x_4 = \begin{bmatrix} -3/5 \\ -0.62 \end{bmatrix}$



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Question 4, Part (a)

First order necessary condition

Consider the problem

$$P_1 \begin{cases} \min_x f(x) \\ g(x) = 0, \end{cases} \tag{1}$$

Suppose $x^\star$ is a local solution of the problem P_1, and $x^\star$ is a regular point for the constraints. Then there exist a (unique) multiplier $\lambda^\star$ such that

$$\begin{aligned} \nabla_x L(x^\star, \lambda^\star) &= 0 \\ g(x^\star) &= 0 \end{aligned} \tag{2}$$

with $L(x, \lambda, \rho) = f(x) + \lambda' g(x)$.

Second order sufficient condition

Consider the problem P_1. Assume that there exist $x^\star$ and $\lambda^\star$ satisfying conditions (2). Suppose that

$$x' \nabla^2_{xx} L(x^\star, \lambda^\star \quad) x > 0 \tag{3}$$

for all $x \neq 0$ such that

$$\left[\frac{\partial g(x^\star)}{\partial x} \right] x = 0.$$

Then $x^\star$ is a strict constrained local minimum of problem P_1.

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(b) $L(x,\lambda) = \frac{1}{2}\left[(x_1-1)^2 + x_2^2\right] + \lambda\left[\beta x_2^2 - x_1\right]$

$\nabla_{x_1} = x_1 - 1 - \lambda = 0$

$\nabla_{x_2} = x_2(1 + 2\lambda\beta) = 0$ Two solutions

$0 = \beta x_2^2 - x_1$

$x_1 = 0$	$x_1 = 1 - 1/2\beta$
$x_2 = 0$	$x_2 = \pm\sqrt{\frac{1}{\beta} - \frac{1}{2\beta^2}}$
$\lambda = -1$	$\lambda = -1/2\beta$

$\downarrow$

$\frac{\partial g}{\partial x} = [-1,\ 2x_2]$ $\qquad \nabla^2 L = \begin{bmatrix} 1 & 0 \\ 0 & 1+2\lambda\beta \end{bmatrix}$

If $\lambda = -1$

$\frac{\partial g}{\partial x}(x^*) = [-1,\ 0]$

$\frac{\partial g}{\partial x}(x^*)\, x = 0 \Rightarrow x = \begin{bmatrix} 0 \\ \alpha \end{bmatrix} \quad \alpha \neq 0$

$x'\, \nabla^2 L\, x = \alpha^2(1+2\lambda\beta) = \alpha^2(1-2\beta)$ ~~xo~~

This is > 0 for $\beta < 1/2$

Hence for $\beta < 1/2$ $(0,0)$ is a local minimum.

(c) $\quad f(x) = 0 \quad \Rightarrow \quad x_1 = \beta x_2^2$

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$$\varphi(x_2) = f(x)\Big|_{x_1 = \beta x_2^2} = \frac{1}{2}\beta^2 x_2^4 + \left(\frac{1}{2} - \beta\right) x_2^2 + \frac{1}{2}$$

$$\varphi'(x_2) = \nabla f(x)\Big|_{x_1 = \beta x_2^2} = 2\beta^2 x_2^3 + (1 - 2\beta) x_2$$

Hence $x_2 = 0$ is a stationary point.

$\varphi''(x_2)\Big|_{x_2 = 0} = 1 - 2\beta$ This is > 0 for $\beta < 1/2$

For $\beta > 1/2$ $\varphi''(x_2)\Big|_{x_2 = 0} < 0$ hence $x_2 = 0$ is a local max.

For $\beta = 1/2$ $\varphi(x_2) = \frac{1}{2}\beta^2 x_2^2 + \left(\frac{1}{2} - \beta\right) x_2^2 + 1/2 = \frac{1}{8} x_2^2 + \frac{1}{2}$

hence $x_2 = 0$ is a (global, strict) minimum.

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Question 5, Part (a)

Consider problem P_1, let $x^\star$ be a local solution and let $\lambda^\star$ be the corresponding multiplier. The basic idea of exact penalty functions methods is to determine the multiplier λ appearing in the augmented Lagrangian function as a function of x, *i.e.* $\lambda = \lambda(x)$, with $\lambda(x^\star) = \lambda^\star$. For, consider the augmented Lagrangian

$$L_a(x, \lambda(x)) = f(x) + \lambda(x)' g(x) + \frac{1}{\epsilon}\|g(x)\|^2.$$

The function $\lambda(x)$ is obtained considering the necessary condition of optimality

$$\nabla_x L_a(x^\star, \lambda^\star) = \nabla f(x^\star) + \frac{\partial g(x^\star)}{\partial x}\lambda^\star = 0 \tag{4}$$

and noting that, if $x^\star$ is a regular point for the constraints then equation (4) can be solved for $\lambda^\star$ yielding

$$\lambda^\star = -\left(\frac{\partial g(x^\star)}{\partial x}\frac{\partial g(x^\star)'}{\partial x}\right)^{-1}\frac{\partial g(x^\star)}{\partial x}\nabla f(x^\star).$$

This equality suggests to define the function $\lambda(x)$ as

$$\lambda(x) = -\left(\frac{\partial g(x)}{\partial x}\frac{\partial g(x)'}{\partial x}\right)^{-1}\frac{\partial g(x)}{\partial x}\nabla f(x),$$

and this is defined at all x where the indicated inverse exists, in particular at $x^\star$. It is possible to show that this selection of $\lambda(x)$ yields and exact penalty function for problem P_1. For, consider the function

$$G(x) = f(x) - g(x)'\left(\frac{\partial g(x)}{\partial x}\frac{\partial g(x)'}{\partial x}\right)^{-1}\frac{\partial g(x)}{\partial x}\nabla f(x) + \frac{1}{\epsilon}\|g(x)\|^2,$$

which is defined and differentiable in the set

$$\tilde{\mathcal{X}} = \{x \in I\!R^n \mid\mid \text{rank}\frac{\partial g(x)}{\partial x} = m\}. \tag{5}$$

For such a function the following fact holds.

Let $\bar{\mathcal{X}}$ be a compact subset of $\tilde{\mathcal{X}}$. Assume that $x^\star$ is the only global minimum of f in $\mathcal{X} \cap \bar{\mathcal{X}}$ and that $x^\star$ is in the interior of $\bar{\mathcal{X}}$. Then there exists $\bar{\epsilon} > 0$ such that, for any $\epsilon \in (0, \bar{\epsilon})$, $x^\star$ is the only global minimum of G in $\bar{\mathcal{X}}$.

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(b) The exact penalty function $G(x)$ is

$$G(x) = f(x) - g(x)\left[\nabla g(x)\,\nabla' g(x)\right]^{-1}\nabla g(x)\,\nabla' f + \frac{1}{\varepsilon}\,g(x)^2$$

$$= x_1^2 + x_2^2 + 3x_1x_2 -$$

$$(x_1 + 3x_2 - 5)\left[[1 \quad 3]\begin{bmatrix}1\\3\end{bmatrix}\right]^{-1}[1 \quad 3]\begin{bmatrix}2x_1 + 3x_2\\2x_2 + 3x_1\end{bmatrix} +$$

$$+\frac{1}{\varepsilon}(x_1 + 3x_2 - 5)^2$$

$$G(x) = x_1^2 + x_2^2 + 3x_1x_2 - \frac{1}{10}(x_1 + 3x_2 - 5)(11x_1 + 9x_2)$$

$$+\frac{(x_1 + 3x_2 - 5)^2}{\varepsilon}$$

(c)

$$\frac{\partial G}{\partial x_1} = -\frac{1}{5}x_1 - \frac{6}{5}x_2 + \frac{11}{2} + \frac{2(x_1 + 3x_2 - 5)}{\varepsilon}$$

$$\frac{\partial G}{\partial x_2} = -\frac{17}{5}x_2 - \frac{6}{5}x_1 + \frac{9}{2} + \frac{6(x_1 + 3x_2 - 5)}{2}$$

Stationary point of G is $x_1 = -\frac{35}{2}$

$x_2 = \frac{15}{2}$

$$\nabla^2 G = \begin{bmatrix} \frac{2}{\varepsilon} - 1/5 & \frac{6}{\varepsilon} - \frac{6}{5} \\ \frac{6}{\varepsilon} - \frac{6}{5} & \frac{18}{\varepsilon} - \frac{17}{5} \end{bmatrix}$$

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If $\varepsilon = 1/10$ $\nabla^2 G > 0$

The stationary point is a minimum.

(d) Necessary cond.

$$\mathcal{L} = f + \lambda g$$

$$\left.\frac{\partial \mathcal{L}}{\partial x_1}\right|^* = 0 \implies 2x_1^* + 3x_2^* + \lambda^* = 0$$

but x_1 and x_2 are as in (c). Hence $\lambda^* = 25/2$.

(Same conclusion from $\left.\frac{\partial \mathcal{L}}{\partial x_2}\right|^* = 0$).

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(a) $f(x) = 10\,(1 - x e^{-x/3} \sin \frac{x}{2})$

$\frac{15}{15}$

$f'(x) = \frac{5}{3} e^{-x/3} \left[2x \sin \frac{x}{2} - 6 \sin \frac{x}{2} - 3x \cos \frac{x}{2} \right]$

$f'(x)$ is bounded for $x \in [0, 20]$

Hence $f(x)$ is Lipschitz in $[0, 20]$

(b) Estimate of L: $\hat{L} = 5$.

See below.

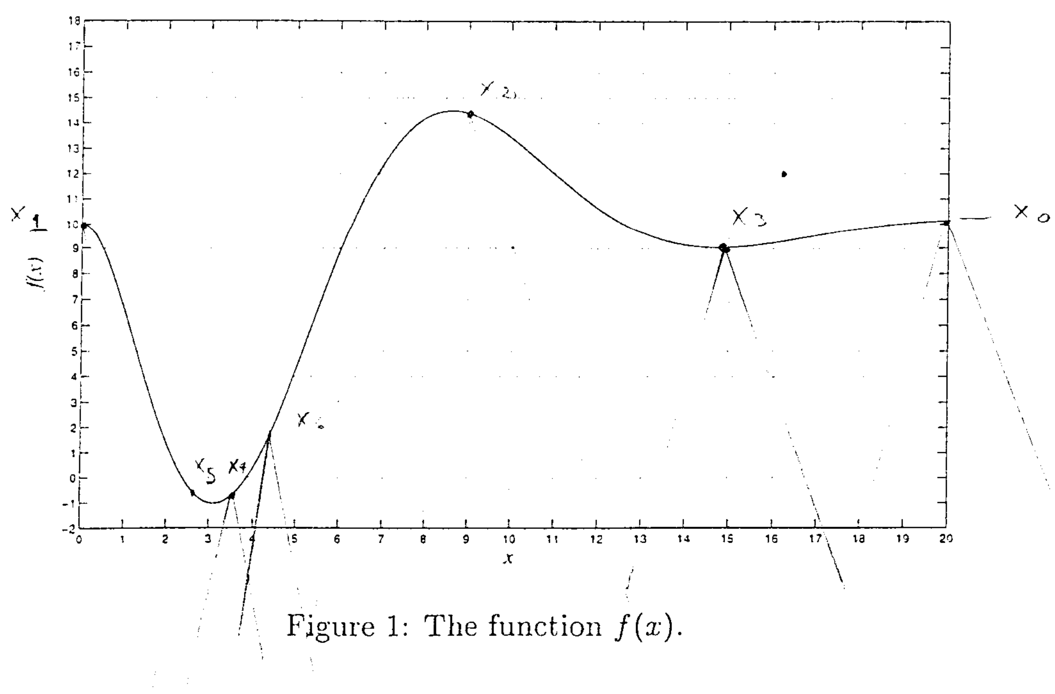

Figure 1: The function $f(x)$.