EE4-66
EE9-SO28

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2017

MSc and EEE PART IV: MEng and ACGI

Corrected copy

TOPICS IN LARGE DIMENSIONAL DATA PROCESSING

Tuesday, 9 May 10:00 am

Time allowed: 3:00 hours

There are FOUR questions on this paper.

Answer ALL questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

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EE4-66 Topics in Large Dimensional Data Processing

Instructions for Candidates

Answer all questions. Each question carries 20 marks.

1. (Linear Algebra and Restricted Isometry Property)

(a) Let $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ be a matrix with rank $r < \min(m, n)$. Let $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T$ be its singular value decomposition (SVD) and $\boldsymbol{A} = \boldsymbol{U}_r\boldsymbol{\Sigma}_r\boldsymbol{V}_r^T$ be its ***compact*** SVD, where $\boldsymbol{\Sigma}_r$ is a diagonal matrix containing the nonzero singular values.

i Specify the dimension of the matrices $\boldsymbol{U}_r$, $\boldsymbol{\Sigma}_r$, and $\boldsymbol{V}_r$. [2]

ii Give the singular value decomposition of $\boldsymbol{A}^T\boldsymbol{A}$ and $\boldsymbol{A}\boldsymbol{A}^T$ using $\boldsymbol{U}_r$, $\boldsymbol{\Sigma}_r$, and $\boldsymbol{V}_r$. [2]

iii Let $\boldsymbol{q} \in \mathbb{R}^n$ be an arbitrary vector with $\|\boldsymbol{q}\|_2 = 1$. Prove that $\left\|\boldsymbol{V}^T\boldsymbol{q}\right\|_2 = 1$ and $\left\|\boldsymbol{V}_r^T\boldsymbol{q}\right\|_2 \leq 1$. [2]

iv Let $\boldsymbol{q} \in \mathbb{R}^n$ be an arbitrary vector with $\|\boldsymbol{q}\|_2 = 1$. Find upper and lower bounds on $\|\boldsymbol{A}\boldsymbol{q}\|_2$. Prove your results and explain the choices of $\boldsymbol{q}$ to achieve the upper and lower bounds respectively. [4]

v For any given vector $\boldsymbol{x} \in \mathbb{R}^m$, the projection of $\boldsymbol{x}$ on the subspace $\mathrm{span}(\boldsymbol{A})$ is given by $\mathrm{proj}(\boldsymbol{x}, \boldsymbol{A}) = \boldsymbol{A}\boldsymbol{A}^\dagger\boldsymbol{x}$ where $\boldsymbol{A}^\dagger$ is the pseudoinverse of $\boldsymbol{A}$. Decompose the matrices $\boldsymbol{A}^\dagger$ and $\boldsymbol{A}\boldsymbol{A}^\dagger$ using the (compact) SVD of $\boldsymbol{A}$. [2]

vi Prove that $\boldsymbol{x}_{\mathrm{res}} = \mathrm{resid}(\boldsymbol{x}, \boldsymbol{A}) := \boldsymbol{x} - \mathrm{proj}(\boldsymbol{x}, \boldsymbol{A})$ is orthogonal to $\boldsymbol{A}$. [2]

(b) Let $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ be a flat matrix, i.e., $m < n$. Let S be a positive integer such that $2S < m$. Suppose that $\boldsymbol{A}$ satisfies Restricted Isometry Property (RIP) with Restricted Isometry Constant (RIC) δ_{2S}.

i Give the definition of RIP (with RIC δ_{2S}) using singular values of submatrices. [2]

ii Suppose that $\mathcal{I}, \mathcal{J} \subset [n]$, $|\mathcal{I}| = |\mathcal{J}| = S$, and $\mathcal{I} \bigcap \mathcal{J} = \phi$. Let $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{R}^S$ be arbitrary vectors with $\|\boldsymbol{p}\|_2 = \|\boldsymbol{q}\|_2 = 1$. Find lower and upper bounds on $\|\boldsymbol{A}_{\mathcal{I}}\boldsymbol{p} + \boldsymbol{A}_{\mathcal{J}}\boldsymbol{q}\|_2$ and $\|\boldsymbol{A}_{\mathcal{I}}\boldsymbol{p} - \boldsymbol{A}_{\mathcal{J}}\boldsymbol{q}\|_2$. [2]

iii Based on the above bounds, show that $\sigma_{\max}\left(\boldsymbol{A}_{\mathcal{I}}^T\boldsymbol{A}_{\mathcal{J}}\right) \leq \delta_{2S}$ where $\sigma_{\max}$ denotes the largest singular value. [2]

In other words, RIP implies that 'disjoint' submatrices are near orthogonal.

2. (SVM)

(a) A hyperplane in $\mathbb{R}^n$ can be defined as

$$\mathcal{P} = \left\{ \boldsymbol{x} : \boldsymbol{\beta}^T \boldsymbol{x} = b \right\} \subset \mathbb{R}^n,$$

for some $\boldsymbol{\beta} \in \mathbb{R}^n$ with $\boldsymbol{\beta} \neq \mathbf{0}$. Show how the distance between an $\boldsymbol{x} \in \mathbb{R}^n$ and $\mathcal{P}$ can be computed? Proof is not required. [2]

(b) Let $\{\boldsymbol{x}_i, y_i\}$, $i = 1, \cdots, m$, be the training dataset where $y_i \in \{-1, 1\}$. Suppose that this training dataset is linearly separable. Give the mathematical definition of linearly separable. [2]

(c) Define the primal optimization problem for Support Vector Machine (SVM). [2]

(d) Find the corresponding Lagrangian $L(\boldsymbol{\beta}, b, \boldsymbol{\lambda})$ where $\boldsymbol{\lambda} = [\cdots, \lambda_i, \cdots]^T$ is the vector of Lagrange multipliers. Specify the constraints of λ_is. [2]

(e) Show that the Lagrange dual function $L_D(\boldsymbol{\lambda}) := \min_{\boldsymbol{\beta}, b} L(\boldsymbol{\beta}, b, \boldsymbol{\lambda}) = -\frac{1}{2}\boldsymbol{\lambda}^T \boldsymbol{K} \boldsymbol{\lambda} + \mathbf{1}^T \boldsymbol{\lambda}$ where $K_{i,j} = y_i \boldsymbol{x}_i^T \boldsymbol{x}_j y_j$. [4]

(f) Give the dual optimization problem. [2]

(g) Give the KKT conditions for the optimal solutions $\boldsymbol{\beta}$, b, and $\boldsymbol{\lambda}$. [4]

(h) Discuss how to find the supporting vectors for the resulting SVM using KKT conditions. [2]

3. (ℓ_1-minimization)

(a) (Iterative Shrinkage-Thresholding Algorithm)

i Let $f(x) = |x|$. Find the subdifferential of $f(x)$. [2]

ii Define $f: \mathbb{R}^n \to \mathbb{R}$ by $f(\boldsymbol{x}) = \frac{1}{2}\|\boldsymbol{x} - \boldsymbol{z}\|_2^2 + \lambda\|\boldsymbol{x}\|_1$ where $\boldsymbol{z} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}^+$ are given. Find the optimal $\boldsymbol{x}^*$ to minimize $f(\boldsymbol{x})$. Express your answer using the famous soft thresholding function η. [2]

iii Let $g(\boldsymbol{x}) = \frac{1}{2}\|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}\|_2^2$. Find its gradient $\nabla g(\boldsymbol{x})$. (No proof is needed.) [2]

iv Let $g(\boldsymbol{x}) = \frac{1}{2}\|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}\|_2^2$. For any given $\boldsymbol{x}^k$, define $\tilde{g}(\boldsymbol{x}) = g(\boldsymbol{x}^k) + \langle \boldsymbol{x} - \boldsymbol{x}^k, \nabla g(\boldsymbol{x}^k)\rangle + \frac{1}{2t_k}\|\boldsymbol{x} - \boldsymbol{x}^k\|_2^2$ for some given constant $t_k \in \mathbb{R}^+$. Simplify $\tilde{g}(\boldsymbol{x})$ into the form $\tilde{g}(\boldsymbol{x}) = \frac{1}{c_1}\|\boldsymbol{x} - \boldsymbol{z}\|_2^2 + c_2$. Find c_1 and $\boldsymbol{z}$. [3]

v Note that in above $\tilde{g}$ is a local approximation of g at $\boldsymbol{x}^k$. Based on $\tilde{g}$, derive the so called Iterative Shrinkage-Thresholding Algorithm to find the global minimum of $f(\boldsymbol{x}) = \frac{1}{2}\|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}\|_2^2 + \lambda\|\boldsymbol{x}\|_1$. [3]

(b) Let $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}_0 + \boldsymbol{w}$ where $\boldsymbol{x}_0$ is an S-sparse signal supported at $\mathcal{T}_0 \subset [n]$ and the noise satisfies $\|\boldsymbol{w}\|_2 \le \epsilon$. Consider the constrained optimization problem

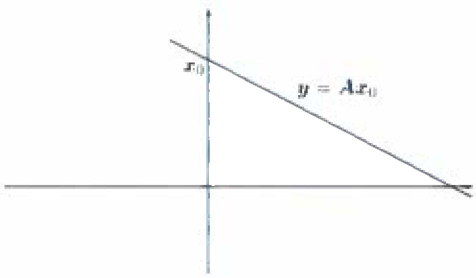
$$\min_{\boldsymbol{x}} \|\boldsymbol{x}\|_1 \quad \text{subject to } \|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}\|_2 \le \epsilon.$$

Let $\boldsymbol{x}^{\#}$ be the output of this optimization program. Define the estimate error $\boldsymbol{h} := \boldsymbol{x}^{\#} - \boldsymbol{x}_0$.

i Prove the tube constraint $\|\boldsymbol{A}\boldsymbol{h}\|_2 \le 2\epsilon$. [2]

ii Prove the cone constraint $\|\boldsymbol{h}_{\mathcal{T}_0^c}\|_1 \le \|\boldsymbol{h}_{\mathcal{T}_0}\|_1$. [3]

iii Based on the 2-dimensional illustration below, illustrate the regions corresponding to $\|\boldsymbol{A}\boldsymbol{h}\|_2 \le 2\epsilon$ and $\|\boldsymbol{h}_{\mathcal{T}_0^c}\|_1 \le \|\boldsymbol{h}_{\mathcal{T}_0}\|_1$ and use your illustration to explain the intuition why $\boldsymbol{h}$ should be small (under some mathematical assumptions for example RIP). [3]



4. (a) Consider a matrix $\boldsymbol{X}$ with rank 2. The partial observations of $\boldsymbol{X}$ are given as follows:

$$\begin{bmatrix} 1 & 1 & -1 & -1 & 1 & 1 & ? & ? \\ ? & ? & -1 & -1 & 1 & 1 & -1 & -1 \\ 2 & 2 & 0 & 0 & ? & ? & 0 & 0 \\ 1 & 1 & 1 & ? & 1 & 1 & 1 & ? \end{bmatrix}$$

Find the complete $\boldsymbol{X}$ and explain your steps briefly. [5]

(b) (Gaussian Random Vectors) The Gaussian Conditioning Lemma is as follows. Suppose that $\boldsymbol{X} = \left[\boldsymbol{X}_{\mathcal{A}}^T, \boldsymbol{X}_{\mathcal{B}}^T\right]^T \sim \mathcal{N}(\mathbf{0}, \Sigma)$ and

$$\boldsymbol{K} := \Sigma^{-1} = \begin{bmatrix} \boldsymbol{K}_{\mathcal{A}\mathcal{A}} & \boldsymbol{K}_{\mathcal{A}\mathcal{B}} \\ \boldsymbol{K}_{\mathcal{B}\mathcal{A}} & \boldsymbol{K}_{\mathcal{B}\mathcal{B}} \end{bmatrix}.$$

Then $\boldsymbol{X}_{\mathcal{A}}|\boldsymbol{X}_{\mathcal{B}} \sim \mathcal{N}\left(-\boldsymbol{K}_{\mathcal{A}\mathcal{A}}^{-1}\boldsymbol{K}_{\mathcal{A}\mathcal{B}}\boldsymbol{X}_{\mathcal{B}},\ \boldsymbol{K}_{\mathcal{A}\mathcal{A}}^{-1}\right)$.

i Consider a linear system $\boldsymbol{Y} = \boldsymbol{H}\boldsymbol{X} + \boldsymbol{W}$ where $\boldsymbol{X} \sim \mathcal{N}(\mathbf{0}, \Sigma_x)$ and $\boldsymbol{W} \sim \mathcal{N}(\mathbf{0}, \Sigma_w)$. What is the distribution of $\boldsymbol{Y}$? What is the joint distribution of $\left[\boldsymbol{X}^T, \boldsymbol{Y}^T\right]^T$? In many signal processing applications, one is asked to estimate $\boldsymbol{X}$ from the given observation $\boldsymbol{Y} = \boldsymbol{y}$. Describe how to use Gaussian conditioning lemma to estimate $\boldsymbol{X}$ (no detailed computations are required). [5]

ii In the Gaussian Graphical model, under what condition will nodes a and b have a connection? Explain your answer using Gaussian conditioning lemma. [5]

iii Now given data points $\boldsymbol{x}_{(1)}, \cdots, \boldsymbol{x}_{(m)}$, we would like to use these data points to estimate the underlying sparse graph. In particular, we assume that $\mathrm{E}\left[X_a|\boldsymbol{X}_{\sim\{a\}}\right] = \sum_{b\neq a}\theta_{ba}X_b$ and want to find θs to minimize $\mathrm{E}\left[\sum_a\left(X_a - \sum_{b\neq a}\theta_{ba}X_b\right)^2\right]$. Let $\boldsymbol{X} = \begin{bmatrix} \boldsymbol{x}_{(1)}^T \\ \vdots \\ \boldsymbol{x}_{(m)}^T \end{bmatrix}$ be the data matrix. Give the optimization problem for the sparse graph estimation. [5]