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UNIVERSITY OF LONDON

[I(1) 2004]

B.ENG. AND M.ENG. EXAMINATIONS 2004

For Internal Students of Imperial College London

This paper is also taken for the relevant examination for the Associateship of the City & Guilds of London Institute

PART I : MATHEMATICS 1

Wednesday 2nd June 2004 10.00 am - 1.00 pm

Answer EIGHT questions.

Corrected Copy

[Before starting, please make sure that the paper is complete; there should be 7 pages, with a total of 10 questions. Ask the invigilator for a replacement if your copy is faulty.]

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1. (i) Define what it is meant by :

- (a) an even function ,
- (b) an odd function .

Let $f(x)$ be an odd function.

What can you say about $f(x)^2$?

- (ii) Classify the following functions as odd, even or neither :

- (a) e^x ; (b) e^{x^2} ; (c) e^{x^3} ;
- (d) $\sin(x)$; (e) $\sin(x^2)$; (f) $\sin(x^3)$.

- (iii) Find the inverse function of

$$f(x) = \frac{x+1}{x+2} .$$

- (iv) Write $\frac{1}{x+2}$ as the sum of an even function and an odd function.

2. Let

$$f(x) = \frac{1}{x^2 - 1} .$$

- (i) Find the stationary point of $f(x)$.

- (ii) Find the second derivative of $f(x)$.

Determine whether the stationary point is a maximum or a minimum.

- (iii) Sketch the graph of $f(x)$.

Indicate clearly the vertical and horizontal asymptotes and the location of the stationary point.

PLEASE TURN OVER

3. Find $\frac{dy}{dx}$ in terms of x in the following three cases, simplifying your answer where necessary :

(i)
$$y = \frac{\sin x}{1 + \cos x} ;$$

(ii)
$$y = e^{x+x^2} \cosh x ;$$

(iii)
$$y = x^{\ln x} ;$$

Show that if $y = \tan^{-1} x$ then $\frac{dy}{dx} = \frac{1}{1+x^2}$.

4. (i) Suppose that $x(t) = \cos^2 t$ and $y(t) = \tan t$. Find $\frac{dy}{dx}$ first in terms of t and then in terms of x and y .

Check your answer by finding a relation between x and y and then differentiating.

- (ii) Suppose that $y - 2x + \sin(xy) = 0$.

Prove that
$$\frac{dy}{dx} = \frac{2 - y \cos(xy)}{1 + x \cos(xy)} .$$

Given also that $y = 0$ when $x = 0$, use the formula

$$\frac{dy}{dx} = \lim_{\delta x \rightarrow 0} \frac{y(x + \delta x) - y(x)}{\delta x}$$

to find the approximate value of y for $x = 0.01$.

5. (i) You are given that

$$\lim_{\theta \rightarrow 0} \frac{\sin(\theta)}{\theta} = 1 .$$

Evaluate the following limits:

(a)
$$\lim_{\theta \rightarrow 0} \frac{\sin(2\theta)}{\theta} ,$$

(b)
$$\lim_{\theta \rightarrow 0} \frac{(\sin(\theta))^2}{\theta} ,$$

(c)
$$\lim_{\theta \rightarrow 0} \frac{\sin(\theta^2)}{\theta \sin(\theta)} .$$

- (ii) State whether or not the following limits exist; if so, give the limiting value :

(a)
$$\lim_{x \rightarrow 0} x \sin(1/x) ,$$

(b)
$$\lim_{x \rightarrow \infty} \left(\sqrt{1+x} - \sqrt{x} \right) ,$$

(c)
$$\lim_{x \rightarrow \infty} \sin(1/x) ,$$

(d)
$$\lim_{x \rightarrow 1} \frac{x-1}{x^n-1} \text{ where } n \geq 2 \text{ is an integer ,}$$

(e)
$$\lim_{n \rightarrow \infty} \frac{2^n}{n!} \text{ where } n \text{ is an integer .}$$

PLEASE TURN OVER

6. (i) Using a trigonometric substitution, evaluate the indefinite integral

$$\int \frac{x}{\sqrt{1-x^2}} dx .$$

- (ii) Given the definite integral

$$I_n = \int_0^{\pi/2} (\cos x)^n dx ,$$

where $n \geq 0$ is an integer, evaluate I_0 and I_1 and show that

$$I_n = \frac{n-1}{n} I_{n-2} , \quad (n \geq 2) .$$

Hence evaluate

$$\int_0^{\pi/2} (\cos x)^3 (\sin x)^2 dx .$$

7. Evaluate the following indefinite integrals :

(i)
$$\int \frac{4x-8}{x^2-4x+5} dx ;$$

(ii)
$$\int \frac{\sec^2 x}{\tan x} dx ;$$

(iii)
$$\int x^2 \ln x dx ;$$

(iv)
$$\int \frac{x+1}{x^2-3x+2} dx .$$

8. (i) Decide whether each of the following series is convergent or divergent :

$$(a) \quad \sum_1^{\infty} \frac{2n+5}{100n} ; \quad (b) \quad \sum_1^{\infty} \frac{n^{100}}{2^n+5} .$$

- (ii) Find the radius of convergence of the following power series :

$$(a) \quad \sum_1^{\infty} \frac{|x-2|^n}{5^n} ; \quad (b) \quad \sum_1^{\infty} \frac{(n+2)!}{(2n)!} x^n .$$

- (iii) Find the first three non-zero terms of the Maclaurin series of $\ln(1+x)$.

9. (i) Write

$$(a) \quad 5 + i12 \quad \text{and} \quad (b) \quad 1/(5 + i12)$$

in polar form and indicate the position of each point in the complex plane, stating the tangent of the argument in each case.

- (ii) De Moivre's theorem states

$$(\cos \theta + i \sin \theta)^n = \cos(n\theta) + i \sin(n\theta)$$

for any value of the real parameter n .

Use this to prove the trigonometric identities

$$\cos(n\theta) \cos \theta - \sin(n\theta) \sin \theta = \cos(n+1)\theta ,$$

$$\cos(n\theta) \sin \theta + \sin n\theta \cos \theta = \sin(n+1)\theta .$$

PLEASE TURN OVER

10. (i) The hyperbolic functions $\cosh x$ and $\sinh x$ are defined by

$$\cosh x = \frac{e^x + e^{-x}}{2}, \quad \sinh x = \frac{e^x - e^{-x}}{2}.$$

Use these to derive the hyperbolic form of de Moivre's theorem

$$(\cosh x + \sinh x)^n = \cosh nx + \sinh nx.$$

- (ii) Prove that

$$\cosh(x + iy) = \cosh x \cos y + i \sinh x \sin y.$$

Hence or otherwise, find all the roots of the equation

$$\cosh^2 z = -1,$$

where $z = x + iy$.

END OF PAPER

MATHEMATICAL FORMULAE

1. VECTOR ALGEBRA

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1, a_2, a_3)$$

Scalar (dot) product:

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

Vector (cross) product:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Scalar triple product:

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = \mathbf{b} \cdot \mathbf{c} \times \mathbf{a} = \mathbf{c} \cdot \mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product: $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c} \cdot \mathbf{a})\mathbf{b} - (\mathbf{b} \cdot \mathbf{a})\mathbf{c}$

2. SERIES

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \dots \quad (\alpha \text{ arbitrary, } |x| < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^n \frac{x^{2n}}{(2n)!} + \dots,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \dots,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^n \frac{x^{n+1}}{(n+1)} + \dots \quad (-1 < x \leq 1)$$

3. TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin(a+b) = \sin a \cos b + \cos a \sin b;$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b.$$

$$\cos iz = \cosh z; \quad \cosh iz = \cos z; \quad \sin iz = i \sinh z; \quad \sinh iz = i \sin z.$$

4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f D^n g + \binom{n}{1} Df D^{n-1} g + \dots + \binom{n}{n} D^n f D^0 g + \dots + D^n f g.$$

(b) Taylor's expansion of $f(x)$ about $x = a$:

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \dots + \frac{h^n}{n!}f^{(n)}(a) + \epsilon_n(h),$$

where $\epsilon_n(h) = h^{n+1}f^{(n+1)}(a + \theta h)/(n+1)!$, $0 < \theta < 1$.

(c) Taylor's expansion of $f(x, y)$ about (a, b) :

$$f(a+h, b+k) = f(a, b) + [hf_x + kf_y]_{a,b} + 1/2! [h^2f_{xx} + 2hkf_{xy} + k^2f_{yy}]_{a,b} + \dots$$

(d) Partial differentiation of $f(x, y)$:

i. If $y = y(x)$, then $f = F(x)$, and $\frac{dF}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx}$.

ii. If $x = x(t)$, $y = y(t)$, then $f = F(t)$, and $\frac{dF}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$.

iii. If $x = x(u, v)$, $y = y(u, v)$, then $f = F(u, v)$, and

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u}, \quad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial v}.$$

(e) Stationary points of $f(x, y)$ occur where $f_x = 0$, $f_y = 0$ simultaneously.

Let (a, b) be a stationary point: examine $D = [f_{xx}f_{yy} - (f_{xy})^2]_{a,b}$.

If $D > 0$ and $f_{xx}(a, b) < 0$, then (a, b) is a maximum;

If $D > 0$ and $f_{xx}(a, b) > 0$, then (a, b) is a minimum;

If $D < 0$ then (a, b) is a saddle-point.

(f) Differential equations:

i. The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating

factor $I(x) = \exp[\int P(x)(dx)]$, so that $\frac{d}{dx}(Iy) = IQ$.

ii. $P(x, y)dx + Q(x, y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y$.

5. INTEGRAL CALCULUS

(a) An important substitution: $\tan(\theta/2) = t$:
 $\sin \theta = 2t/(1+t^2)$, $\cos \theta = (1-t^2)/(1+t^2)$, $d\theta = 2dt/(1+t^2)$.

(b) Some indefinite integrals:

$$\int (a^2 - x^2)^{-1/2} dx = \sin^{-1} \left(\frac{x}{a} \right), \quad |x| < a.$$

$$\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1} \left(\frac{x}{a} \right) = \ln \left\{ \frac{x}{a} + \left(1 + \frac{x^2}{a^2} \right)^{1/2} \right\}.$$

$$\int (x^2 - a^2)^{-1/2} dx = \cosh^{-1} \left(\frac{x}{a} \right) = \ln \left| \frac{x}{a} + \left(\frac{x^2}{a^2} - 1 \right)^{1/2} \right|.$$

$$\int (a^2 + x^2)^{-1} dx = \left(\frac{1}{a} \right) \tan^{-1} \left(\frac{x}{a} \right).$$

6. NUMERICAL METHODS

(a) Approximate solution of an algebraic equation:

If a root of $f(x) = 0$ occurs near $x = a$, take $x_0 = a$ and
 $x_{n+1} = x_n - [f(x_n)/f'(x_n)]$, $n = 0, 1, 2 \dots$

(Newton Raphson method).

(b) Formulae for numerical integration: Write $x_n = x_0 + nh$, $y_n = y(x_n)$.

- i. Trapezium rule (1-strip): $\int_{x_0}^{x_1} y(x) dx \approx (h/2) [y_0 + y_1]$.
- ii. Simpson's rule (2-strip): $\int_{x_0}^{x_2} y(x) dx \approx (h/3) [y_0 + 4y_1 + y_2]$.

(c) Richardson's extrapolation method: Let $I = \int_a^b f(x) dx$ and let I_1 , I_2 be two

estimates of I obtained by using Simpson's rule with intervals h and $h/2$.

Then, provided h is small enough,

$$I_2 + (I_2 - I_1)/15,$$

is a better estimate of I .

7. LAPLACE TRANSFORMS

Function	Transform	Function	Transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t) dt$	$af(t) + bg(t)$	$aF(s) + bG(s)$
df/dt	$sF(s) - f(0)$	$d^2 f/dt^2$	$s^2 F(s) - sf(0) - f'(0)$
$e^{at} f(t)$	$F(s-a)$	$tf(t)$	$-dF(s)/ds$
$(\partial/\partial \alpha) f(t, \alpha)$	$(\partial/\partial \alpha) F(s, \alpha)$	$\int_0^t f(t) dt$	$F(s)/s$
$\int_0^t f(u) g(t-u) du$	$F(s)G(s)$		
1	$1/s$	$t^n (n = 1, 2 \dots)$	$n!/s^{n+1}, (s > 0)$
e^{at}	$1/(s-a), (s > a)$	$\sin \omega t$	$\omega/(s^2 + \omega^2), (s > 0)$
$\cos \omega t$	$s/(s^2 + \omega^2), (s > 0)$	$H(t-T) = \begin{cases} 0, & t < T \\ 1, & t > T \end{cases}$	$e^{-sT}/s, (s, T > 0)$

8. FOURIER SERIES

If $f(x)$ is periodic of period $2L$, then $f(x+2L) = f(x)$, and

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \text{ where}$$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx, \quad n = 0, 1, 2, \dots, \text{ and}$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx, \quad n = 1, 2, 3, \dots$$

Parseval's theorem

$$\frac{1}{L} \int_{-L}^L [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

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(i) A function $f(x)$ is even if $f(x) = f(-x)$.
 A function $f(x)$ is odd if $f(x) = -f(-x)$.
 In that case $f(-x)^2 = (-f(x))^2 = f(x)^2$
 hence $f(x)^2$ is an even function.

(ii) (a) neither, (b) even, (c) neither,
 (d) odd, (e) even, (f) odd.

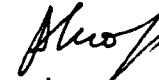

(iii) Write $y = \frac{x+1}{x+2}$. Then $(y-1)x = 1-2y$
 hence the inverse function is $\frac{1-2y}{y-1}$.

$$(iv) \frac{1}{x+2} = \frac{1}{2} \left(\frac{1}{x+2} + \frac{1}{-x+2} \right) + \frac{1}{2} \left(\frac{1}{x+2} - \frac{1}{-x+2} \right) =$$

$$= -\frac{2}{x^2-4} + \frac{x}{x^2-4} \quad \text{Here}$$

$-\frac{2}{x^2-4}$ is even, and $\frac{x}{x^2-4}$ is odd.

Setter : A. Skorobogatov
 Checker : S. LUTTATTO

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 Checker's signature : 

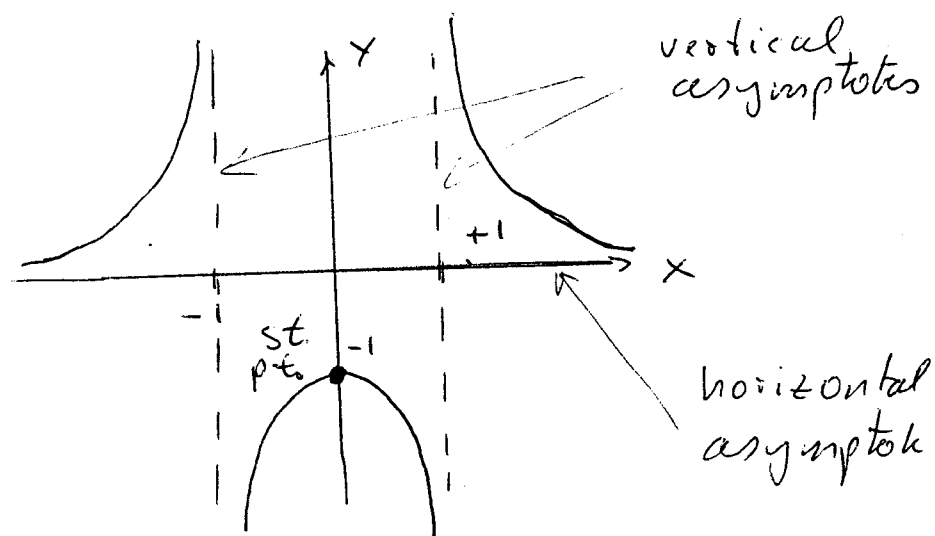
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a) $f'(x) = \frac{-2x}{(x^2-1)^2}$, for stationary point: $f'(x) = 0$, hence $x = 0$

b) $f''(x) = \frac{-2}{(x^2-1)^2} + \frac{8x^2}{(x^2-1)^3}$

for $x=0$, $f''(0) = -2 < 0$, stationary point is maximum.

c) vertical asymptotes at $x = \pm 1$
horizontal asymptote is $y = 0$
stationary point at $(0, -1)$



Setter : S. REICH

Setter's signature :

Checker : S. Salamon

Checker's signature :

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$$\begin{aligned}
 (i) \quad \frac{dy}{dx} &= \frac{\cos x (1 + \cos x) - \sin x (-\sin x)}{(1 + \cos x)^2} \\
 &= \frac{\cos^2 x + \sin^2 x + \cos x}{(1 + \cos x)^2} \\
 &= \frac{1 + \cos x}{(1 + \cos x)^2} = \frac{1}{1 + \cos x}.
 \end{aligned}$$

4

$$\begin{aligned}
 (ii) \quad \frac{dy}{dx} &= (1+2x) e^{x+x^2} \cosh x + e^{x+x^2} \sinh x \\
 &= e^{x+x^2} \{ (1+2x) \cosh x + \sinh x \}.
 \end{aligned}$$

3

$$(iii) \quad \ln y = (\ln x)^2.$$

$$\frac{dy}{dx} \frac{1}{y} = \frac{2}{x} \ln x. \quad \frac{dy}{dx} = \frac{2y}{x} \ln x = 2 x^{-1+\ln x} \ln x.$$

4

$$\text{If } x = \tan y, \quad \frac{dx}{dy} = \sec^2 y = 1 + \tan^2 y = 1 + x^2.$$

$$\therefore \frac{dy}{dx} = \frac{1}{1+x^2}.$$

4

Setter : C J RIDLER - Rowe

Setter's signature : C J Ridler

Checker : J. R. CASH

Checker's signature : J R Cash

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$$(i) \quad \frac{dy}{dx} = \left(\frac{dy}{dt}\right) / \left(\frac{dx}{dt}\right) = \frac{\sec^2 t}{-2 \cos t \sin t} = -\frac{1}{2 \cos^3 t \sin t}$$

2

$$\frac{dy}{dx} = -\frac{1}{2 \cos t \tan t} = -\frac{1}{2 x^2 y}$$

2

$$\tan^2 t = 1 + \sec^2 t, \quad \therefore y^2 = 1 + \frac{1}{x^2},$$

2

$$2y \frac{dy}{dx} = -\frac{1}{x^2}, \quad \therefore \frac{dy}{dx} = -\frac{1}{2x^2 y}$$

2

$$(ii) \quad \frac{dy}{dx} - \left[1 + y \cos(xy) + x \frac{dy}{dx} \cos(xy)\right] = 0$$

giving $\frac{dy}{dx}$ as required.

3

$$y(x + \delta x) \approx y(x) + \left(\frac{dy}{dx}\right) \delta x \quad \therefore \text{Putting } x=0, \delta x=0.01,$$

$$y(0.01) \approx 0 + 2 \times 0.01 = 0.02.$$

4

Setter : C J RIDLER-ROWE

Setter's signature :

Checker : J R. CASH

Checker's signature :

P. Ridler-Rowe
JR Cash

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A5 a) i) $\lim_{\theta \rightarrow 0} \frac{\sin(2\theta)}{\theta} = 2 \lim_{\theta \rightarrow 0} \frac{\sin(2\theta)}{2\theta} = 2.$

ii) $\lim_{\theta \rightarrow 0} \frac{(\sin(\theta))^2}{\theta} = \lim_{\theta \rightarrow 0} \frac{\sin(\theta)}{\theta} \lim_{\theta \rightarrow 0} (\sin \theta) = 1 \cdot 0 = 0.$

iii) $\lim_{\theta \rightarrow 0} \frac{\sin(\theta^2)}{\theta \sin \theta} = \lim_{\theta \rightarrow 0} \frac{\sin(\theta^2)}{\theta^2} \lim_{\theta \rightarrow 0} \frac{\theta}{\sin \theta} = 1.$

b) i) $\lim_{x \rightarrow 0} \sin(1/x)$ does not exist as the function in this limit takes every value between -1 and 1 infinitely often in every interval of $x=0$.

ii) Using $|x \sin(1/x)| \leq |x|$ we obtain $\lim_{x \rightarrow 0} x \sin(1/x) = 0.$

iii) Using

$$\sqrt{1+x} - \sqrt{x} = (\sqrt{1+x} - \sqrt{x}) \frac{\sqrt{1+x} + \sqrt{x}}{\sqrt{1+x} + \sqrt{x}} = \frac{1}{\sqrt{1+x} + \sqrt{x}}$$

we obtain $\lim_{x \rightarrow \infty} (\sqrt{1+x} - \sqrt{x}) = 0.$

iv) Using l'Hopital's rule we find

$$\lim_{x \rightarrow 1} \frac{x-1}{x^n-1} = \lim_{x \rightarrow 1} \frac{1}{nx^{n-1}} = \frac{1}{n}.$$

v) $\lim_{x \rightarrow \infty} \sin(1/x) = \lim_{y \rightarrow 0} \sin(y) = 0.$

vi) Using

$$\frac{2^n}{n!} = \frac{2 \cdot 2 \dots 2}{n(n-1) \dots 1} = \frac{2}{n} \cdot \frac{2}{n-1} \dots \frac{2}{2} \frac{2}{1} \leq \frac{4}{n},$$

we have $\lim_{n \rightarrow \infty} \frac{2^n}{n!} = 0.$

Setter : L. BOARDMAN

Checker: J ELGIN

Setter's signature :

Checker's signature :

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A6 a) To evaluate

$$I \equiv \int \frac{x}{\sqrt{1-x^2}} dx,$$

let $x = \sin \theta$ and then

$$I = \int \frac{\sin \theta \cos \theta}{\sqrt{1-\sin^2 \theta}} d\theta = \int \frac{\sin \theta \cos \theta}{|\cos \theta|} d\theta = \int \sin \theta d\theta = -\cos \theta,$$

provided $\cos \theta > 0$. Hence $I = -\sqrt{1-x^2} + c$.

b) If $I_n = \int_0^{\pi/2} (\cos x)^n dx$ then integrating by parts we find

$$I_n = \int_0^{\pi/2} (\cos x)^{n-1} \cos x dx = \sin x (\cos x)^{n-1} \Big|_0^{\pi/2} - \int_0^{\pi/2} \sin x (n-1) (\cos x)^{n-2} \cdot (-\sin x) dx.$$

so that

$$I_n = (n-1) \int_0^{\pi/2} (\sin x)^2 (\cos x)^{n-2} dx = (n-1) \int_0^{\pi/2} (1-\cos^2 x) (\cos x)^{n-2} dx$$

and therefore

$$I_n = (n-1)(I_{n-2} - I_n) \implies I_n(1+n-1) = (n-1)I_{n-2} \implies I_n = \frac{n-1}{n} I_{n-2}.$$

Now $I_0 = \int_0^{\pi/2} 1 dx = \pi/2$ and $I_1 = \int_0^{\pi/2} \cos x dx = \sin(\pi/2) = 1$, and

$$\int_0^{\pi/2} (\cos x)^3 (\sin x)^2 dx = \int_0^{\pi/2} (\cos x)^3 (1-\cos^2 x) dx = I_3 - I_5.$$

But $I_3 = \frac{2}{3} I_1 = \frac{2}{3}$ and $I_5 = \frac{4}{5} I_3 = \frac{8}{15}$, so that

$$I_3 - I_5 = \frac{2}{3} - \frac{8}{15} = \frac{6}{45} = \frac{2}{15}.$$

Setter : R. BONDARE

Checker : J. ELGIN

Setter's signature :

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$$\begin{aligned} \text{i)} \quad \int \frac{4x-8}{x^2-4x+5} dx &= 2 \int \frac{2x-4}{x^2-4x+5} dx \\ &= 2 \ln(x^2-4x+5) + C \end{aligned}$$

2

$$\text{ii)} \quad \int \frac{\sec^2 x}{\tan x} dx = \ln |\tan x| + C$$

2

$$\begin{aligned} \text{iii)} \quad \text{Let } u &= \ln x, \quad \frac{du}{dx} = 1/x \\ v &= x^3/3, \quad \frac{dv}{dx} = x^2 \end{aligned}$$

2

$$\int x^2 \ln x dx = \int u dv = uv - \int v du$$

2

$$= \ln x \left(\frac{x^3}{3} \right) - \frac{1}{3} \int x^3 \frac{1}{x} dx$$

$$= \frac{x^3}{3} \ln x - \frac{1}{3} \int x^2 dx$$

$$= \frac{x^3}{3} \ln x - \frac{1}{3} \frac{x^3}{3} + C$$

$$= \frac{x^3}{3} \left(\ln x - \frac{1}{3} \right) + C$$

2

Setter : *W. L. L. L. L.*Setter's signature : *W. L. L. L. L.*Checker : *A. Shorobogator*Checker's signature : *Shorobogator*

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(v)

$$\frac{x+1}{x^2-3x+2} = \frac{x+1}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$$

$$x+1 = A(x-2) + B(x-1)$$

$$\therefore x=1 \Rightarrow A = -2$$

$$x=2 \Rightarrow B = 3$$

$$\int \frac{x+1}{x^2-3x+2} dx = \int \frac{3}{x-2} dx - \int \frac{2}{x-1} dx$$

$$= 3 \ln(x-2) - 2 \ln(x-1) + C$$

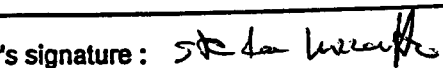
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2

1

Setter : LUTZANSE

Checker: Skoro bogator

Setter's signature : 

Checker's signature :



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i)

(a) Terms do not tend to zero, so series diverges.

(b) Ratio test

$$\frac{|U_{n+1}|}{|U_n|} = \frac{(n+1)^{100}}{2^{n+1}+5} \cdot \frac{2^n+5}{n^{100}} = \left(\frac{n+1}{n}\right)^{100} \cdot \frac{2^n+5}{2^{n+1}+5}$$

$$= \left(\frac{n+1}{n}\right)^{100} \frac{1+5/2^n}{2+5/2^n}$$

$\rightarrow 1/2 < 1$ converges

$$(ii) \frac{|U_{n+1}|}{|U_n|} = \frac{(x-2)^{n+1}}{5^{n+1}} \cdot \frac{5^n}{(x-2)^n} = \frac{|x-2|}{5}$$

$$< 1 \Leftrightarrow |x-2| < 5 \quad R=5$$

$$(b) \frac{|U_{n+1}|}{|U_n|} = \frac{(n+3)!}{(2n+2)!} \cdot \frac{2n!}{(n+2)!} \cdot \frac{x^{n+1}}{x^n}$$

$$= \frac{n+3}{(2n+1)(2n+2)} \cdot x = \frac{1}{2(2n+1)} \cdot x \rightarrow 0$$

$$\therefore R = \infty$$

Setter : LUZZATTO

Checker : Skovborg

Setter's signature : *Stefano Luzzatto*

Checker's signature : *Plus*

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(iii)

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots$$

$$f(x) = \ln(1+x)$$

$$f(0) = 0$$

$$f'(x) = \frac{1}{1+x} \quad f'(0) = 1$$

$$f''(x) = \frac{-1}{(1+x)^2} \quad f''(0) = -1$$

$$f'''(x) = \frac{2}{(1+x)^3} \quad f'''(0) = 2$$

$$\therefore f(x) = x - \frac{x^2}{2} + \frac{x^3}{3} \dots$$

Setter : LUZATTO

Setter's signature : *Stefano Mazzetta*Checker : *Stefano Mazzetta*Checker's signature : *Stefano*

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a)

$$i) (5+i12) = \sqrt{5^2+12^2} (\cos\theta + i\sin\theta), \quad \theta = \tan^{-1}\left(\frac{12}{5}\right)$$

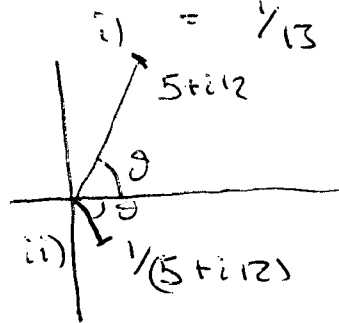
$$= 13(\cos\theta + i\sin\theta) = 13e^{i\theta}, \quad \tan\theta = \frac{12}{5}$$

$$ii) \frac{1}{5+i12} = \frac{5-i12}{(5+i12)(5-i12)} = \frac{5-i12}{13^2}$$

$$= \frac{1}{13} (\cos\theta - i\sin\theta)$$

$$e^{-i\theta}$$

$$\tan\theta = \frac{12}{5}$$



b) Have

$$(1) \cos n\theta \cos\theta - \sin n\theta \sin\theta = x, \text{ say}$$

$$(2) \cos n\theta \sin\theta + \sin n\theta \cos\theta = y, \text{ say}$$

$$\text{Then } (1) + i(2) \Rightarrow$$

$$(\cos n\theta \cos\theta - \sin n\theta \sin\theta) + i(\cos n\theta \sin\theta + \sin n\theta \cos\theta) = x + iy$$

LHS can be written

$$(\cos n\theta + i\sin n\theta)(\cos\theta + i\sin\theta)$$

$$= (\cos\theta + i\sin\theta)^n (\cos\theta + i\sin\theta) \text{ de Moivre's thm}$$

$$= (\cos\theta + i\sin\theta)^{n+1}$$

$$= \cos(n+1)\theta + i\sin(n+1)\theta \text{ de Moivre's thm}$$

$$\Rightarrow x = \cos(n+1)\theta, \quad y = \sin(n+1)\theta \quad \text{Q.E.D.}$$

Setter : J ELGIN

Setter's signature :

Checker : R BEARDMORE

Checker's signature :

Q10
Mark scheme

$$(i) \quad \cosh x = \frac{e^x + e^{-x}}{2}, \quad \sinh x = \frac{e^x - e^{-x}}{2}$$

$$\therefore \cosh x + \sinh x = e^x$$

$$\therefore (\cosh x + \sinh x)^n = e^{nx}$$

$$= \cosh nx + \sinh nx \quad 4$$

$$(ii) \quad \begin{aligned} \cosh(x+iy) &= \frac{1}{2} \left[e^{x+iy} + e^{-(x+iy)} \right] \\ &= \frac{1}{2} \left[e^x (\cos y + i \sin y) + e^{-x} (\cos y - i \sin y) \right] \\ &= \frac{1}{2} \cosh x \cos y + i \sinh x \sin y \quad (*) \end{aligned}$$

$$\cosh^2 z = -1 \Rightarrow \cosh z = \pm i$$

Then Matches (*) if ① $\sinh x = 1, \sin y = -1, \cos y = 0$
or ② $\sinh x = -1, \sin y = +1, \cos y = 0$

$$① \Rightarrow y = (2n+1)\pi/2 \quad \text{with } n = \pm 1, \pm 3, \dots \quad \left. \begin{array}{l} x = \sinh^{-1} 1 \\ x = -\sinh^{-1} 1 \end{array} \right\}$$

$$② \Rightarrow y = (2n+1)\pi/2 \quad \text{with } n = 0, \pm 2, \dots$$