BSc, MSc and MSci EXAMINATIONS (MATHEMATICS)

May/June 2018

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Probability and Statistics

Date: Friday, 25 May 2018
Time: 14:00 – 16:00
Time Allowed: 2 Hours
This paper has *4 Questions*.
Candidates should use ONE main answer book.
Supplementary books may only be used after the relevant main book(s) are full.
Statistical tables are provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted.
- Each question carries equal weight.
- Calculators may not be used.

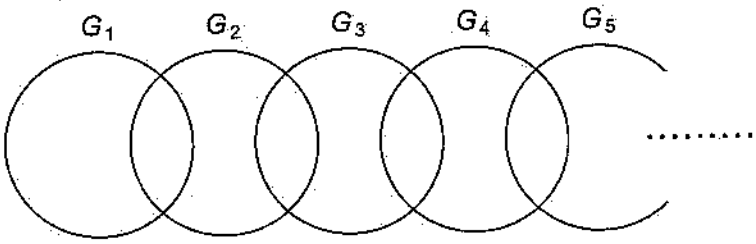
1. (a) (i) State the three axioms of probability for events defined on a sample space Ω.

(ii) For events E and F, prove from the axioms that if E and F^C are disjoint then $P(F) \geq P(E)$.

(iii) Consider the sequence of events $G_1, G_2, G_3, \ldots$ with the following properties:

$$G_1 \cap G_j = \phi \; \forall \; j > 2;$$
$$G_i \cap G_j = \phi \; \forall \; i > 1; \; j > i+1 \text{ and } j < i-1,$$

i.e. the events intersect as follows:



You are given that, for $i \geq 1$:

$$\mathsf{P}(G_i \cap G_{i+1}) = \left(\frac{1}{5}\right)^i; \quad \mathsf{P}(G_{i-1}^C \cap G_i \cap G_{i+1}^C) = \left(\frac{3}{7}\right)^i,$$

where $G_0 = \phi$.

Determine $\mathsf{P}(G_i \mid G_j)$, for all values of i and j with $i > j$, and show that $\bigcup_{i=1}^{\infty} G_i$ is exhaustive for Ω.

(b) At the end of each week during the autumn term a particular student either has status "up-to-date" with the problem sheets for M1S or has status "behind" with the problem sheets for M1S. Their status at the end of a week depends on their status at the end of the previous week, but does not depend on their status prior to the previous week. If they are up-to-date at the end of a given week, the probability that they will be up-to-date at the end of the next week is 0.9 otherwise they will be behind. If they are behind at the end of a given week, the probability that they will be up-to-date at the end of the next week is 0.4 otherwise they will be behind. They are up-to-date at the end of the week before the start of term.

(i) What is the probability that the student is up-to-date at the end of week two?

(ii) Given the student is up-to-date at the end of week two, what is the probability they were up-to-date at the end of week one?

(iii) What is the probability that the student is up-to-date for the whole of a ten week term?

2. The number of mistakes, X_i, on pages i, $i = 1, \ldots, n$, of a particular textbook with n pages, are independently distributed with the following probability mass functions (pmf):

$$f_{X_i}(x) = \begin{cases} \dfrac{e^{-\lambda_i}\lambda_i^x}{x!} & x = 0, 1, 2, \ldots; \\ 0 & \text{otherwise,} \end{cases}$$

with $\lambda_i > 0$, $i = 1, \ldots, n$.

(a) (i) Verify that each $f_{X_i}(x)$ is a valid pmf.

(ii) Determine, p, the probability that there is at least one mistake on the first page.

(iii) What is the maximum value of λ_1 which ensures that the value of p calculated in part (a)(ii) does not exceed 0.5.

(b) (i) Define the probability generating function (pgf), $G_X(t)$, of a discrete random variable X with range $\{x_1, x_2, x_3, \ldots\}$.

(ii) Show that $G_X(1) = 1$.

(iii) Prove that the pgf of X_i is,

$$G_{X_i}(t) = e^{\lambda_i(t-1)}.$$

(iv) Determine the pgf of $T = \sum_{i=1}^{n} X_i$, the total number of mistakes. Hence name the distribution of T and identify its parameter(s).

(v) Determine the pmf of $Z = T/n$, the average number of mistakes per page.

(vi) If $\lambda_i = \beta$, $\forall\ i$ where $\beta > 0$, find an expression for the probability that the average number of mistakes per page is not more than one.

3. The continuous random variables X and Y are independent with probability density functions (pdfs) given by,

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0; \\ 0, & \text{otherwise}, \end{cases} \qquad f_Y(y) = \begin{cases} \lambda^2 y e^{-\lambda y}, & y > 0; \\ 0, & \text{otherwise}, \end{cases}$$

with $\lambda > 0$. i.e. $X \sim Exponential(\lambda)$ and $Y \sim Gamma(2, \lambda)$.

(a) Derive $\mathsf{E}_{f_X}(X)$ and $\mathsf{E}_{f_Y}(Y)$ directly from the definition of expectation. Hence show that $\mathsf{E}_{f_Y}(Y) = 2\mathsf{E}_{f_X}(X)$.

(b) The reliability, $R_{f_Z}(z)$, of a random variable Z is defined as $R_{f_Z}(z) = \mathsf{P}(Z > z)$. Determine $R_{f_X}(x)$ and $R_{f_Y}(y)$.

(c) Determine $\mathsf{P}(X > x_1 + x_2 \mid X > x_1)$ and $\mathsf{P}(Y > y_1 + y_2 \mid Y > y_1)$ with $x_1, x_2, y_1, y_2 > 0$.

(d) If the continuous random variable Z is "memoryless" i.e. has the property that $\mathsf{P}(Z > z_1 + z_2 \mid Z > z_1) = \mathsf{P}(Z > z_2)$, for all z_1 and z_2, then it can be shown that $R_{f_Z}(t) = [R_{f_Z}(1)]^t$, $t \geq 0$. Use this result to show that Z must have the same named distribution as X, and identify its parameter.

(e) Let $V = \exp(-X)$, determine $f_V(v)$, the pdf of V.

(f) Let $Z = \min(X, Y)$, by considering the reliability of Z, determine $f_Z(z)$, the pdf of Z.

4. (a) I have 12 alternative routes that I can take for my cycle ride into work, the routes are classified into three different type: 3 are dangerous (but short), 4 are safe (but longer) and 5 are scenic. To determine my daily routes, I choose 2 different routes at random (unordered) from the 12 possibilities. Out of the 2 routes chosen, let X denote the number of dangerous routes and Y denote the number of safe routes.

(i) Complete a two-way table with entries given by $f_{X,Y}(x, y)$, the joint pmf of X and Y.

(ii) Are X and Y independent?

(iii) Find $E_{f_X}(X)$.

(b) Prove that, for continuous random variables X and Y,

$$E_{f_X}(X) = E_{f_Y}\left[\mathsf{E}_{f_{X|Y}}(X \mid Y = y)\right].$$

[You may assume that $E_{f_X}(X)$ is finite].

(c) The continuous random variables X and Y have joint pdf given by

$$f_{X,Y}(x, y) = \begin{cases} 2y(3-y)(1+x)^{-(y+1)}, & 2 < y < 3,\ \ x > 0; \\ 0, & \text{otherwise.} \end{cases}$$

(i) Prove that, $f_Y(y)$, the marginal distribution of Y is,

$$f_Y(y) = 2(3-y), \quad 2 < y < 3.$$

(ii) Find $\mathsf{E}_{f_Y}(Y)$.

(iii) Find $f_{X|Y}(x|y)$ the conditional pdf of X given Y.

(iv) Determine the conditional mean, $\mathsf{E}_{f_{X|Y}}(X|Y = y)$, of X given $Y = y$, and hence determine the marginal mean, $\mathsf{E}_{f_X}(X)$ of X.

DISCRETE DISTRIBUTIONS

	RANGE	PARAMETERS	MASS FUNCTION f_X	CDF F_X	$\mathsf{E}_{f_X}[X]$	$\mathsf{Var}_{f_X}[X]$	MGF M_X
$Bernoulli(\theta)$	$\{0,1\}$	$\theta \in (0,1)$	$\theta^x(1-\theta)^{1-x}$		θ	$\theta(1-\theta)$	$1-\theta+\theta e^t$
$Binomial(n,\theta)$	$\{0,1,...,n\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{n}{x}\theta^x(1-\theta)^{n-x}$		$n\theta$	$n\theta(1-\theta)$	$(1-\theta+\theta e^t)^n$
$Poisson(\lambda)$	$\{0,1,2,...\}$	$\lambda \in \mathbb{R}^+$	$\dfrac{e^{-\lambda}\lambda^x}{x!}$		λ	λ	$\exp\{\lambda(e^t-1)\}$
$Geometric(\theta)$	$\{1,2,...\}$	$\theta \in (0,1)$	$(1-\theta)^{x-1}\theta$	$1-(1-\theta)^x$	$\dfrac{1}{\theta}$	$\dfrac{(1-\theta)}{\theta^2}$	$\dfrac{\theta e^t}{1-e^t(1-\theta)}$
$NegBinomial(n,\theta)$	$\{n,n+1,...\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{x-1}{n-1}\theta^n(1-\theta)^{x-n}$		$\dfrac{n}{\theta}$	$\dfrac{n(1-\theta)}{\theta^2}$	$\left(\dfrac{\theta e^t}{1-e^t(1-\theta)}\right)^n$
or	$\{0,1,2,...\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{n+x-1}{x}\theta^n(1-\theta)^x$		$\dfrac{n(1-\theta)}{\theta}$	$\dfrac{n(1-\theta)}{\theta^2}$	$\left(\dfrac{\theta}{1-e^t(1-\theta)}\right)^n$

For **CONTINUOUS** distributions (see over), define the **GAMMA FUNCTION**

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx \qquad \Gamma(\alpha+1) = \alpha\Gamma(\alpha) \qquad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

and the LOCATION/SCALE transformation $Y = \mu + \sigma X$ gives

$$f_Y(y) = f_X\left(\frac{y-\mu}{\sigma}\right)\frac{1}{\sigma} \qquad F_Y(y) = F_X\left(\frac{y-\mu}{\sigma}\right) \qquad M_Y(t) = e^{\mu t}M_X(\sigma t) \qquad \mathsf{E}_{f_Y}[Y] = \mu+\sigma\mathsf{E}_{f_X}[X] \qquad \mathsf{Var}_{f_Y}[Y] = \sigma^2\mathsf{Var}_{f_X}[X]$$

CONTINUOUS DISTRIBUTIONS							
	$\mathbb{X}$	PARAMS.	PDF f_X	CDF F_X	$\mathsf{E}_{f_X}[X]$	$\mathsf{Var}_{f_X}[X]$	MGF M_X
$Uniform(\alpha,\beta)$ (stand. model $\alpha=0,\beta=1$)	(α,β)	$\alpha<\beta\in\mathbb{R}$	$\dfrac{1}{\beta-\alpha}$	$\dfrac{x-\alpha}{\beta-\alpha}$	$\dfrac{(\alpha+\beta)}{2}$	$\dfrac{(\beta-\alpha)^2}{12}$	$\dfrac{e^{\beta t}-e^{\alpha t}}{t(\beta-\alpha)}$
$Exponential(\lambda)$ (stand. model $\lambda=1$)	$\mathbb{R}^+$	$\lambda\in\mathbb{R}^+$	$\lambda e^{-\lambda x}$	$1-e^{-\lambda x}$	$\dfrac{1}{\lambda}$	$\dfrac{1}{\lambda^2}$	$\left(\dfrac{\lambda}{\lambda-t}\right)$
$Gamma(\alpha,\beta)$ (stand. model $\beta=1$)	$\mathbb{R}^+$	$\alpha,\beta\in\mathbb{R}^+$	$\dfrac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}$		$\dfrac{\alpha}{\beta}$	$\dfrac{\alpha}{\beta^2}$	$\left(\dfrac{\beta}{\beta-t}\right)^\alpha$
$Weibull(\alpha,\beta)$ (stand. model $\beta=1$)	$\mathbb{R}^+$	$\alpha,\beta\in\mathbb{R}^+$	$\alpha\beta x^{\alpha-1}e^{-\beta x^\alpha}$	$1-e^{-\beta x^\alpha}$	$\dfrac{\Gamma(1+1/\alpha)}{\beta^{1/\alpha}}$	$\dfrac{\Gamma\left(1+\frac{2}{\alpha}\right)-\Gamma\left(1+\frac{1}{\alpha}\right)^2}{\beta^{2/\alpha}}$	
$Normal(\mu,\sigma^2)$ (stand. model $\mu=0,\sigma=1$)	$\mathbb{R}$	$\mu\in\mathbb{R}$ $\sigma\in\mathbb{R}^+$	$\dfrac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\dfrac{(x-\mu)^2}{2\sigma^2}\right\}$		μ	σ^2	$e^{\{\mu t+\sigma^2t^2/2\}}$
$Student(\nu)$	$\mathbb{R}$	$\nu\in\mathbb{R}^+$	$\dfrac{(\pi\nu)^{-\frac{1}{2}}\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\left\{1+\frac{x^2}{\nu}\right\}^{(\nu+1)/2}}$		0 (if $\nu>1$)	$\dfrac{\nu}{\nu-2}$ (if $\nu>2$)	
$Pareto(\theta,\alpha)$	$\mathbb{R}^+$	$\theta,\alpha\in\mathbb{R}^+$	$\dfrac{\alpha\theta^\alpha}{(\theta+x)^{\alpha+1}}$	$1-\left(\dfrac{\theta}{\theta+x}\right)^\alpha$	$\dfrac{\theta}{\alpha-1}$ (if $\alpha>1$)	$\dfrac{\alpha\theta^2}{(\alpha-1)(\alpha-2)}$ (if $\alpha>2$)	
$Beta(\alpha,\beta)$	$(0,1)$	$\alpha,\beta\in\mathbb{R}^+$	$\dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$		$\dfrac{\alpha}{\alpha+\beta}$	$\dfrac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	

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$\boxed{\text{seen} \Downarrow}$

1. (a) (i) **Axioms of Probability**

Given a $\sigma-$field, $\mathcal{F}$ (a set of subsets of the sample space Ω.) For events $E, E_1, E_2, \ldots \in \mathcal{F}$, then the probability function, $\mathsf{P}(\cdot)$, must satisfy:

(I) $\mathsf{P}(E) \geq 0$.

(II) $\mathsf{P}(\Omega) = 1$.

(III) If $E_1, E_2, \ldots$, are pairwise disjoint then $\mathsf{P}(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mathsf{P}(E_i)$ (Countable additivity).

(Do not need to specify $\sigma-$field, could instead say: for events $E, E_1, \ldots \subseteq \Omega$. Lose 1 mark if finite rather than countable additivity specified, but they do need to specify the meaning of finite/countable additivity). $\boxed{3}$

$\boxed{\text{sim seen} \Downarrow}$

(ii) We are given that $E \cap F^C = \phi \Rightarrow \mathsf{P}(E \cap F^C) = 0$. From Axiom III we have,

$$\begin{aligned} &\mathsf{P}(F) = \mathsf{P}(F \cap E^C) + \mathsf{P}(F \cap E) \text{ as } (F \cap E^C) \text{ and } (F \cap E) \text{ are disjoint} \\ &\mathsf{P}(E) = \mathsf{P}(E \cap F^C) + \mathsf{P}(E \cap F) \text{ as } (E \cap F^C) \text{ and } (E \cap F) \text{ are disjoint} \\ \Rightarrow\ &\mathsf{P}(E) = \mathsf{P}(E \cap F) \Rightarrow \mathsf{P}(F) = \mathsf{P}(F \cap E^C) + \mathsf{P}(E) \\ \Rightarrow\ &\mathsf{P}(F) \geq \mathsf{P}(E) \text{ as } \mathsf{P}(F \cap E^C) \geq 0 \text{ from Axiom I.} \end{aligned}$$

$\boxed{3}$

$\boxed{\text{unseen} \Downarrow}$

(iii) We have,

$$\mathsf{P}(G_{i+1} \mid G_i) = \frac{\mathsf{P}(G_i \cap G_{i+1})}{\mathsf{P}(G_i)} = \frac{(1/5)^i}{\mathsf{P}(G_i)}.$$

Now, for $i > 1$,

$$\begin{aligned} \mathsf{P}(G_i) &= \mathsf{P}(G_{i-1} \cap G_i \cap G_{i+1}^C) + \mathsf{P}(G_{i-1}^C \cap G_i \cap G_{i+1}) + \mathsf{P}(G_{i-1}^C \cap G_i \cap G_{i+1}^C) \\ &= \mathsf{P}(G_{i-1} \cap G_i) + \mathsf{P}(G_i \cap G_{i+1}) + \mathsf{P}(G_{i-1}^C \cap G_i \cap G_{i+1}^C). \\ &= \left(\frac{1}{5}\right)^{i-1} + \left(\frac{1}{5}\right)^i + \left(\frac{3}{7}\right)^i \\ &= 6\left(\frac{1}{5}\right)^i + \left(\frac{3}{7}\right)^i, \end{aligned}$$

and for $i = 1$,

$$\mathsf{P}(G_1) = \mathsf{P}(G_1 \cap G_2) + \mathsf{P}(G_1 \cap G_2^C) = \frac{1}{5} + \frac{3}{7} = \frac{22}{35}.$$

Giving,

$$\mathsf{P}(G_i \mid G_j) = \frac{\mathsf{P}(G_i \cap G_j)}{\mathsf{P}(G_j)} = \begin{cases} 0, & i > j+1; \\ \frac{7}{22}, & i = 2, j = 1; \\ \frac{(\frac{1}{5})^j}{6(\frac{1}{5})^j + (\frac{3}{7})^j}, & j > 1, i = j+1. \end{cases}$$

$\boxed{5}$

For $\cup_{i=1}^{\infty} G_i$ to be exhaustive for Ω, we need to show $\mathsf{P}(\cup G_i) = 1$. Rewrite as the disjoint union:

$$\begin{aligned}
\mathsf{P}\left(\bigcup_{i=1}^{\infty} G_i\right) &= \sum_{i=1}^{\infty} \mathsf{P}(G_i \cap G_{i+1}^C) \\
&= \sum_{i=1}^{\infty} \mathsf{P}(G_i) - \mathsf{P}(G_i \cap G_{i+1}) \\
&= \left(\frac{3}{7}\right) + \sum_{i=2}^{\infty} \left[6\left(\frac{1}{5}\right)^i + \left(\frac{3}{7}\right)^i - \left(\frac{1}{5}\right)^i\right] \\
&= \left(\frac{3}{7}\right) + \sum_{i=2}^{\infty} \left[\left(\frac{1}{5}\right)^{i-1} + \left(\frac{3}{7}\right)^i\right] \\
&= \sum_{i=1}^{\infty} \left(\frac{1}{5}\right)^i + \sum_{i=1}^{\infty} \left(\frac{3}{7}\right)^i \\
&= \left(\frac{1}{1-\frac{1}{5}} - 1\right) + \left(\frac{1}{1-\frac{3}{7}} - 1\right) = \left(\frac{5}{4} - 1\right) + \left(\frac{7}{4} - 1\right) = 1,
\end{aligned}$$

$\boxed{3}$

as required.

sim seen $\Downarrow$

(b) Let U_i = event up to date at the end of week i and B_i = event behind at the end of week i.

(i) We have, for $i > 1$,

$$\begin{aligned}
\mathsf{P}(U_1) = 0.9,\ \mathsf{P}(B_1) = 0.1 \\
\mathsf{P}(U_i \mid U_{i-1}) = 0.9,\ \mathsf{P}(B_i \mid U_{i-1}) = 0.1 \\
\mathsf{P}(U_i \mid B_{i-1}) = 0.4,\ \mathsf{P}(B_i \mid B_{i-1}) = 0.6.
\end{aligned}$$

So, from the theorem of total probability,

$$\begin{aligned}
U_2 &= (U_2 \cap B_1) \cup (U_2 \cap U_1) \\
\Rightarrow \mathsf{P}(U_2) &= \mathsf{P}(U_2 \mid B_1)\mathsf{P}(B_1) + \mathsf{P}(U_2 \mid U_1)\mathsf{P}(U_1) \\
&= 0.4 \cdot 0.1 + 0.9 \cdot 0.9 = 0.85.
\end{aligned}$$

$\boxed{3}$

(ii)

$$\mathsf{P}(U_1 \mid U_2) = \frac{\mathsf{P}(U_2 \mid U_1)\mathsf{P}(U_1)}{\mathsf{P}(U_2)} = \frac{0.9^2}{0.85} = \frac{81}{85}.$$

$\boxed{2}$

(iii)

$$\mathsf{P}\left(\bigcap_{i=1}^{10} U_i\right) = \mathsf{P}(U_1) \prod_{i=2}^{10} \mathsf{P}(U_i \mid U_{i-1}) = 0.9^{10}.$$

$\boxed{1}$

2. (a) (i) For a valid pmf we must have

seen ⇓

$$f_{X_i}(x) \geq 0 \text{ and } \sum_x f_{X_i}(x)\,\mathrm{d}x = 1$$

$$f_{X_i}(x) \geq 0 \text{ as } \lambda_i > 0, x \geq 0 \text{ and } \mathrm{e}^{-\lambda_i} > 0$$

$$\sum_x f_{X_i}(x) = \sum_{x=0}^{\infty} \frac{\mathrm{e}^{-\lambda_i}\lambda_i{}^x}{x!} = \mathrm{e}^{-\lambda_i}\sum_{x=0}^{\infty}\frac{\lambda_i{}^x}{x!} = \mathrm{e}^{-\lambda_i}\mathrm{e}^{\lambda_i} = 1,$$

as required. 2

(ii)

unseen ⇓

$$p = \mathrm{P}(X_1 \geq 1) = 1 - \mathrm{P}(X_1 = 0) = 1 - \mathrm{e}^{-\lambda_1}.$$

2

(iii)

seen ⇓

$$p \leq 0.5 \Rightarrow 1 - \mathrm{e}^{-\lambda_1} \leq 0.5 \Rightarrow \mathrm{e}^{-\lambda_1} \geq 0.5 \Rightarrow \lambda_1 \leq \log(2).$$

2

(b) (i) The pgf is defined as

$$G_X(t) = \mathrm{E}(t^X) = \sum_{i=1}^{\infty} t^{x_i} f_X(x_i).$$

2

(ii)

$$G_X(1) = \sum_{i=1}^{\infty} 1^{x_i} f_X(x_i) = \sum_{i=1}^{\infty} f_X(x_i) = 1,$$

as f_X is a pmf. 1

(iii)

$$G_{X_i}(t) = \sum_{x=0}^{\infty} t^x \frac{\mathrm{e}^{-\lambda_i}\lambda_i{}^x}{x!}$$

$$= \mathrm{e}^{-\lambda_i}\sum_{x=0}^{\infty}\frac{(t\lambda_i)^x}{x!} = \mathrm{e}^{-\lambda_i}\mathrm{e}^{t\lambda_i}$$

$$\Rightarrow G_{X_i}(t) = \mathrm{e}^{\lambda_i(t-1)},$$

as required. 2

(iv) As the X_i are independent, we have

meth seen ⇓

$$G_T(t) = \prod_{i=1}^{n} G_{X_i}(t) = \prod_{i=1}^{n} \mathrm{e}^{\lambda_i(t-1)} = \exp\left(\sum_{i=1}^{n}\lambda_i(t-1)\right).$$

2

We recognise this as the pgf of a Poisson distribution with parameter $\sum_{i=1}^{n}\lambda_i$, i.e. $T \sim Poisson(\sum_{i=1}^{n}\lambda_i)$ and

$$f_T(x) = \frac{\mathrm{e}^{-\lambda}\lambda^x}{x!}, x = 0, 1, 2, \ldots$$

where $\lambda = \sum_{i=1}^{n}\lambda_i$. 2

unseen $\Downarrow$

(v) The range of T is $\{0,1,2,\ldots\}$, hence the range of Z is $\{0,\frac{1}{n},\frac{2}{n},\ldots\}$.

$$f_Z(z) = \mathsf{P}(Z=z) = \mathsf{P}\left(\frac{T}{n}=z\right) = \mathsf{P}(T=zn) = f_T(zn)$$
$$= \frac{e^{-\lambda}\lambda^{zn}}{(zn)!}, z \in \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots\right\},$$

where $\lambda = \sum_{i=1}^n \lambda_i$. 3

(vi) For the probability that the average number of mistakes per page is not more than 1, we need to calculate $\mathsf{P}(Z \leq 1) = \mathsf{P}(T \leq n)$. Where $\lambda = \sum_{i=1}^n \lambda_i = n\beta$.

$$\mathsf{P}(T \leq n) = \sum_{i=0}^{n} f_T(i) = \sum_{i=0}^{n} \frac{e^{-\lambda}\lambda^i}{i!} = \sum_{i=0}^{n} \frac{e^{-n\beta}(n\beta)^i}{i!}.$$

2

seen $\Downarrow$

3. (a)

$$\mathsf{E}_{f_X}(X) = \int_0^\infty x f_X(x)\,dx = \int_0^\infty \lambda x e^{-\lambda x}\,dx$$
$$= \left[-xe^{-\lambda x}\right]_0^\infty + \int_0^\infty e^{-\lambda x}\,dx = \left[-\frac{e^{-\lambda x}}{\lambda}\right]_0^\infty$$
$$\Rightarrow \mathsf{E}_{f_X}(X) = \frac{1}{\lambda}.$$

2

meth seen $\Downarrow$

$$\mathsf{E}_{f_Y}(Y) = \int_0^\infty y f_Y(y)\,dy = \int_0^\infty \lambda^2 y^2 e^{-\lambda y}\,dy = \left[-\lambda y^2 e^{-\lambda y}\right]_0^\infty + \int_0^\infty 2\lambda y e^{-\lambda y}\,dy$$
$$= \left[-2ye^{-\lambda y}\right]_0^\infty + \int_0^\infty 2e^{-\lambda y}\,dy = \left[-\frac{2e^{-\lambda y}}{\lambda}\right]_0^\infty$$
$$\Rightarrow \mathsf{E}_{f_Y}(Y) = \frac{2}{\lambda},$$

3

and hence $\mathsf{E}_{f_Y}(Y) = 2\mathsf{E}_{f_X}(X)$ as required.

(b) $R_{f_X}(x) = 1,\ x \leq 0$ and $R_{f_Y}(y) = 1,\ y \leq 0$.

$$R_{f_X}(x) = \mathsf{P}(X > x) = \int_x^\infty f_X(y)\,dy = \int_x^\infty \lambda e^{-\lambda y}\,dy = \left[-e^{-\lambda y}\right]_x^\infty$$
$$\Rightarrow R_{f_X}(x) = e^{-\lambda x}, x > 0.$$

2

$$R_{f_Y}(y) = \mathsf{P}(Y > y) = \int_y^\infty f_Y(x)\,dx = \int_y^\infty \lambda^2 x e^{-\lambda x}\,dx$$
$$= \left[-\lambda x e^{-\lambda x}\right]_y^\infty + \int_y^\infty \lambda e^{-\lambda x}\,dx = \lambda y e^{-\lambda y} + \left[-e^{-\lambda x}\right]_y^\infty$$
$$\Rightarrow R_{f_Y}(y) = e^{-\lambda y}(1+\lambda y),\ \ y > 0.$$

2

seen ⇓

(c)

$$\mathsf{P}(X > x_1 + x_2 \mid X > x_1) = \frac{\mathsf{P}((X > x_1 + x_2) \cap (X > x_1))}{\mathsf{P}(X > x_1)} = \frac{\mathsf{P}(X > x_1 + x_2)}{\mathsf{P}(X > x_1)}$$
$$= \frac{\mathrm{e}^{-\lambda(x_1+x_2)}}{\mathrm{e}^{-\lambda x_1}} = \mathrm{e}^{-\lambda x_2}.$$

2

meth seen ⇓

$$\mathsf{P}(Y > y_1 + y_2 \mid Y > y_1) = \frac{\mathsf{P}((Y > y_1 + y_2) \cap (Y > y_1))}{\mathsf{P}(Y > y_1)} = \frac{\mathsf{P}(Y > y_1 + y_2)}{\mathsf{P}(Y > y_1)}$$
$$= \frac{\mathrm{e}^{-\lambda(y_1+y_2)}(1 + \lambda(y_1 + y_2))}{\mathrm{e}^{-\lambda y_1}(1 + \lambda y_1)} = \frac{\mathrm{e}^{-\lambda y_2}(1 + \lambda(y_1 + y_2))}{(1 + \lambda y_1)}.$$

2

meth seen ⇓

(d) Given $R_{f_Z}(t) = R_{f_Z}(1)^t$, $t \geq 0$, and $0 \leq R_{f_Z}(t) \leq 1$ is a non-increasing function,

$$\log(R_{f_Z}(t)) = t \log(R_{f_Z}(1)) \Rightarrow R_{f_Z}(t) = \exp(t \log(R_{f_Z}(1))).$$

Recognise this as the reliability of an exponential distribution with parameter $-\log(R_{f_Z}(1))$.

1

(e) Given $V = \exp(-X)$, the range of $V = (0, 1)$.

$$F_V(v) = \mathsf{P}(V \leq v) = \mathsf{P}(\exp(-X) \leq v) = \mathsf{P}(X \geq -\log(v))$$
$$= R_{f_X}(-\log(v)) = \exp(\lambda \log(v)) = v^{\lambda}.$$
$$\Rightarrow f_V(v) = \lambda v^{\lambda - 1}, \quad 0 < v < 1.$$

3

(f) If $Z = \min(X, Y)$,

$$\mathsf{P}(Z > z) = \mathsf{P}(\min(X, Y) > z) = \mathsf{P}((X > z) \cap (Y > z)) = \mathsf{P}(X > z)\mathsf{P}(Y > z)$$
$$= \mathrm{e}^{-\lambda z}\mathrm{e}^{-\lambda z}(1 + \lambda z)$$
$$\Rightarrow F_Z(z) = 1 - \mathrm{e}^{-2\lambda z}(1 + \lambda z)$$
$$\Rightarrow f_Z(z) = -\lambda \mathrm{e}^{-2\lambda z} + 2\lambda \mathrm{e}^{-2\lambda z}(1 + \lambda z)$$
$$\Rightarrow f_Z(z) = \lambda \mathrm{e}^{-2\lambda z}(1 + 2\lambda z), \quad z > 0.$$

3

sim seen ⇓

4. (a) (i) Let event E be the event that I choose 2 routes with x dangerous, y safe and $2-(x+y)$ scenic. Then $f_{X,Y}(x,y) = \frac{n_E}{n_\Omega}$, where n_E is the number of routes which satisfy E (for $x+y \leq 2$), and n_Ω is the number of ways I can choose 2 routes with no restrictions.

$$f_{X,Y}(x,y) = \mathsf{P}(X=x, Y=x) = \frac{\binom{3}{x}\binom{4}{y}\binom{5}{2-(x+y)}}{\binom{12}{2}}, \quad 0 \leq x+y \leq 2.$$

	$X=0$	$X=1$	$X=2$
$Y=0$	$\frac{10}{66}$	$\frac{15}{66}$	$\frac{3}{66}$
$Y=1$	$\frac{20}{66}$	$\frac{12}{66}$	0
$Y=2$	$\frac{6}{66}$	0	0

4

(ii) The marginal distributions are given by,

	$X=0$	$X=1$	$X=2$	
$Y=0$	$\frac{10}{66}$	$\frac{15}{66}$	$\frac{3}{66}$	$\frac{28}{66}$
$Y=1$	$\frac{20}{66}$	$\frac{12}{66}$	0	$\frac{32}{66}$
$Y=2$	$\frac{6}{66}$	0	0	$\frac{6}{66}$
	$\frac{36}{66}$	$\frac{27}{66}$	$\frac{3}{66}$	1

For independence, need $f_{X,Y}(x,y) = f_X(x)f_Y(y)$.
Consider $f_{X,Y}(2,2) = 0 \neq f_X(x)f_Y(y) = \frac{3}{66} \times \frac{6}{66}$, so X and Y are not independent (as expected given the context!). 1

(iii)

$$\mathsf{E}_{f_X}(X) = \sum_{x=0}^{2} x f_X(x) = 1 \cdot \frac{27}{66} + 2 \cdot \frac{3}{66} = \frac{33}{66} = \frac{1}{2}.$$

2

seen ⇓

(b) Consider

$$\begin{aligned}
\mathsf{E}_{f_Y}\left[\mathsf{E}_{f_{X|Y}}(X|Y=y)\right] &= \int_y \mathsf{E}_{f_{X|Y}}(X|Y=y)\ f_Y(y)\,\mathrm{d}y \\
&\int_y \int_x x f_{X|Y}(x|y) f_Y(y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_x x \int_y f_{X,Y}(x,y)\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_x x f_X(x)\,\mathrm{d}x \\
&= \mathsf{E}_{f_X}(X),
\end{aligned}$$

2

as required.

(c) (i) $\boxed{\text{meth seen} \Downarrow}$

$$f_Y(y) = \int_0^\infty 2y(3-y)(1+x)^{-(y+1)}\,\mathrm{d}x = 2(3-y)\left[-(1+x)^{-y}\right]_0^\infty$$
$$= 2(3-y), \quad y \in (2,3),$$

as required. $\boxed{2}$

(ii)

$$\mathsf{E}_{f_Y}(Y) = \int_2^3 y f_Y(y)\,\mathrm{d}y = \int_2^3 2y(3-y)\,\mathrm{d}y = \left[3y^2 - \frac{2y^3}{3}\right]_2^3$$
$$= 27 - 18 - \left(12 - \frac{16}{3}\right) = \frac{7}{3}.$$

$\boxed{2}$

(iii)

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{2y(3-y)(1+x)^{-(y+1)}}{2(3-y)} = y(1+x)^{-(y+1)}, x > 0.$$

$\boxed{2}$

(iv)

$$\mathsf{E}_{f_{X|Y}}(X|Y) = \int_0^\infty xy(1+x)^{-(y+1)}\,\mathrm{d}x = \left[-x(1+x)^{-y}\right]_0^\infty + \int_0^\infty (1+x)^{-y}\,\mathrm{d}y$$
$$= \left[\frac{(1+x)^{-y+1}}{-y+1}\right]_0^\infty = \frac{1}{y-1}.$$

$\boxed{2}$

Hence,

$$E_{f_X}(X) = \mathsf{E}_{f_Y}\left[\mathsf{E}_{f_{X|Y}}(X|Y=y)\right]$$
$$\Rightarrow \mathsf{E}_{f_X}(X) = \int_2^3 \mathsf{E}_{f_{X|Y}}(X|Y=y) f_Y(y)\,\mathrm{d}y = \int_2^3 \frac{1}{y-1} 2(3-y)\,\mathrm{d}y$$
$$= 2\int_2^3 \frac{-(y-3)}{y-1}\,\mathrm{d}y = 2\int_2^3 = 2\int_2^3 \frac{-(y-1)+2}{y-1}\,\mathrm{d}y$$
$$= 2\int_2^3 -1 + \frac{2}{y-1}\,\mathrm{d}y = 2\left[-y + 2\log(|y-1|)\right]_2^3$$
$$= 2(-3 + 2\log(2) - (-2 + 2\log(1))) = 4\log(2) - 2.$$

$\boxed{3}$