EE2-06

SOLUTIONS: Control Engineering

1. a) i) The transfer function for the circuit in Figure 1.2 is given by

$$\frac{Z_f(s)}{Z_i(s)} = -\frac{C_i(s+1/R_iC_i)}{C_f(s+1/R_fC_f)} = -\frac{C_is+1/R_i}{C_fs+1/R_f}$$

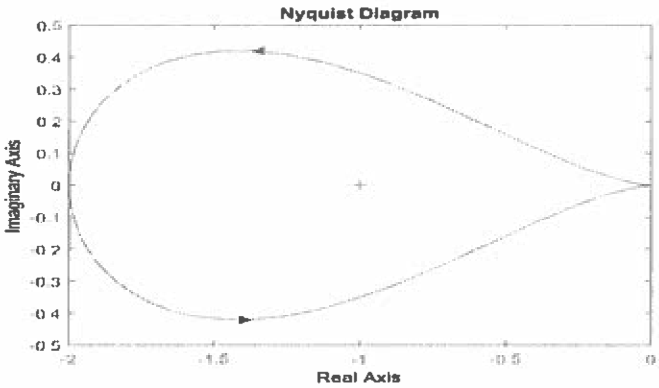
Putting in the values: $\boxed{G(s) = G_1(s)G_2(s) = \frac{1}{(s+1)(s+2)}}$.

ii) The DC gain of $G(s)$ is therefore $\boxed{G(0) = 0.5.}$

iii) $\boxed{y_{ss} := \lim_{t\to\infty} y(t) = \lim_{s\to 0} sy(s) = \lim_{s\to 0} sG(s)u(s) = G(0) = 0.5.}$

iv) $\lim_{t\to\infty} e(t) = \lim_{s\to 0} se(s) = \lim_{s\to 0} \frac{sr(s)}{1+K_pG(s)} = \frac{1}{1+K_pG(0)} \leq .01 \Rightarrow \boxed{K_p \geq 198.}$

b) i) The Nyquist diagram is shown below.



ii) The Nyquist criterion: $N = Z - P$ where N is the number of clockwise encirclements of $-1/K_p$, Z is the number of unstable closed loop poles and P is the number of unstable open loop poles ($= 1$). So for:

- $0 < K_p < 0.5$, $N = 0$ and so $Z = 1$,
- $0.5 < K_p < \infty$, $N = -1$ and so $Z = 0$
- $-\infty < K_p < 0$, $N = 0$ and so $Z = 1$.
- when $K_p = 0.5$ the closed loop is marginally stable.

iii) For $K_p = 1$, the closed loop is stable. The loop gain can be increased without bound and reduced by up to 50% without losing stability.

iv) The closed loop transfer function, DC gain and damping ratio are

$$\boxed{H(s) = \frac{4K_p}{s^2+s+2(2K_p-1)}}, \quad \boxed{H(0) = \frac{2K_p}{2K_p-1}}, \quad \boxed{\zeta = \frac{1}{2\sqrt{2(2K_p-1)}}}.$$

When $r(t)$ is a unit step, good steady–state response requires $H(0) \sim 1$, or equivalently, large K_p. However, large K_p will result in small ζ, which results in large overshoot and an oscillatory response.

2. a) The Nyquist diagram is shown in the next page. The real–axis intercepts can be found from the Routh arrray. The characteristic equation is

$$1+K_pG(s)=0 \Rightarrow s^2+(3-K_p)s+2+3K_p=0$$

The Routh array

$$\begin{array}{c|cc} s^2 & 1 & 2+3K_p \\ s^1 & 3-K_p & \\ 1 & 2+3K_p & \end{array}$$

The values $K_p=3$ and $K_p=-2/3$ result in a zero row, and so the real–axis intercepts are obtained as $-1/K_p$ and the corresponding frequencies are obtained from the auxiliary polynomials and are $\boxed{-1/3,\ \omega=\sqrt{11}}$ and $\boxed{3/2,\ \omega=0.}$

b) The $\boxed{\text{gain margin} = 3}$ from the Routh array. For the phase margin, we first find the real frequency ω such that $|G(j\omega)|=1$, or

$$9+\omega^2=(1+\omega^2)(4+\omega^2) \Leftrightarrow \omega^4+4\omega^2-5=0 \Leftrightarrow (\omega^2-1)(\omega^2+5)=0 \Rightarrow \omega=\pm1.$$

Therefore the cross–over frequency $\boxed{\omega_c=1}$ and the angle of $G(j\omega_c)=-j$ is $-90°$ and so the $\boxed{\text{phase margin} = 90°.}$ The stability margins are adequate so $\boxed{\text{the design is robust against uncertainties in the gain and phase of } G(s).}$ Furthermore, $\boxed{\text{we expect the transient response to be acceptable}}$ since the Nyquist plot is not too close to the point -1.

c) When $K(s)=1$, the closed loop transfer function is

$$H(s)=\frac{G(s)}{1+G(s)}=\frac{3-s}{s^2+2s+5} \Rightarrow H(0)=0.6.$$

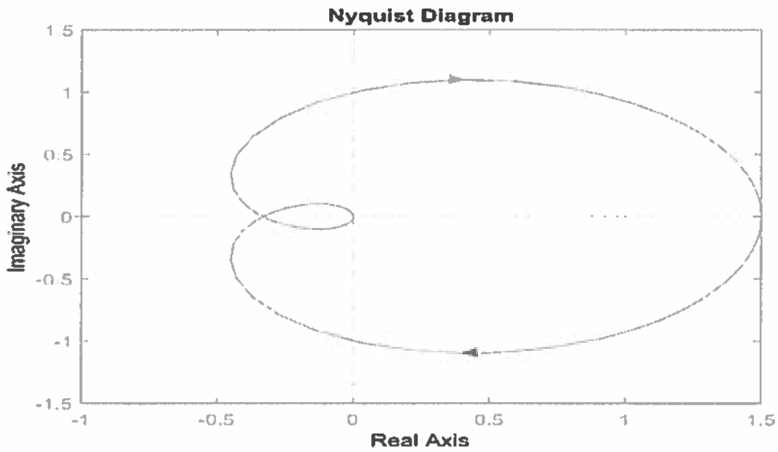
Therefore, $\boxed{\text{the steady–state output is 0.6.}}$ Since the reference signal is 1,

$\boxed{\text{the steady–state response of the system is not adequate.}}$

d) The Nyquist criterion: $N=Z-P$ where N is the number of clockwise encirclements of $-1/K_p$, Z is the number of unstable closed loop poles and P is the number of open loop poles (=0). So for

- $0<K_p<3$, $N=0$ and so $Z=0$,
- $3<K_p<\infty$, $N=2$ and so $Z=2$,
- $-\infty<K_p<-2/3$, $N=1$ and so $Z=1$,
- $-2/3<K_p<0$, $N=0$ and so $Z=0$,
- when $K_p=3$ and $K_p=-2/3$ the closed loop is marginally stable.

e) Note from Parts b and c that the transient response is adequate while the steady–state response is not. Hence $\boxed{\text{we need to improve the steady–state performance.}}$ Since phase–lag compensation increases low frequency gain, and so improves steady–state tracking it follows that $\boxed{\text{the system requires phase–lag compensation.}}$


Nyquist Diagram
Imaginary Axis
Real Axis
1.5
1
0.5
0
-0.5
-1
-1.5
-1
-0.5
0
0.5
1
1.5

3. a) The root–locus is shown below.

b) When $K(s) = K_p$ with $K_p > 0$, the closed loop poles lie on the root–locus.

i) For a critically damped response, the closed loop must have two real repeated poles. An inspection of the root–locus shows that this occurs at $s = -1$. Using the gain criterion we get that $\boxed{K_p = -1/G(-1) = 1.}$

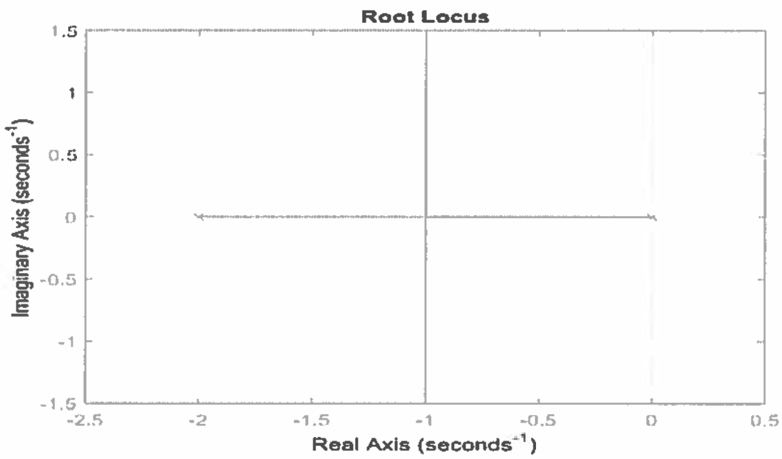
ii) Since the closed loop poles are at -1, the time constant is 1 s, and so the settling time T_s is 4 times that and is therefore $\boxed{T_s = 4\ s.}$

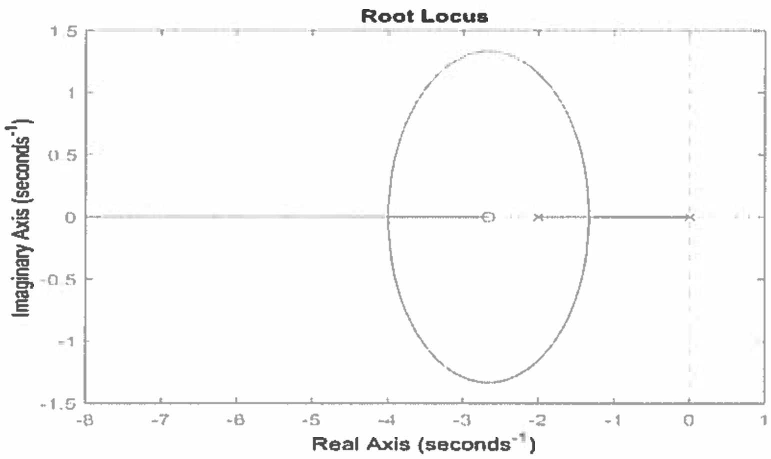
c) For a critically damped response with a settling time of approximately 1 s, the poles must be repeated and located at $\boxed{p_1 = p_2 = -4.}$ Such a compensator does not exist $\boxed{\text{since the root–locus does not pass through that point.}}$

d) The compensated system has the form $\hat{G}(s) = \frac{K(s+z)}{s(s+2)}$. Thus the point $s = -4$ must be a breakaway point. The breakaway points are solutions of $d\hat{G}(s)/ds = 0$. Carrying this out we get that $\boxed{z = 8/3}$. Using the gain criterion, we have $K = -1/\hat{G}(-4) = 6$. Thus $\boxed{K(s) = 6(s + 8/3).}$

e) The root–locus is shown on the next page.

f) Here $r(s) = 1/s^2$ and $e(s) = r(s)/\left(1 + 6\hat{G}(s)\right)$. Using the final value theorem of the Laplace transform: $\boxed{\lim_{t\to\infty} e(t) = \lim_{s\to 0} se(s) = 1/8.}$




Root Locus
1.5
1
0.5
0
-0.5
-1
-1.5
-8
-7
-6
-5
-4
-3
-2
-1
0
1
Imaginary Axis (seconds⁻¹)
Real Axis (seconds⁻¹)