EE1-10A MATHEMATICS I - SOLUTIONS

1. a) Express in the form $x+iy$:

$$(i)\quad \frac{3i+2}{2i-3}, \qquad (ii)\quad \left(\frac{\sqrt{3}-i}{2}\right)^{2018}.$$

SOLUTION

(i) Multiply by conjugate of denominator:

$$\frac{2+3i}{-3+2i}\left(\frac{-3-2i}{-3-2i}\right) = \frac{6-6-4i-9i}{4+9} = -i.$$

[2]

(ii) Express in complex exponential form and proceed:

$$\left(e^{-i\pi/6}\right)^{2018} = e^{-i\pi\ (2018/6)} = e^{-i\pi(336+\frac{1}{3})} = e^{-i\pi/3} = \frac{1}{2} - \frac{\sqrt{3}}{2}i.$$

[2]

b) Obtain all complex z such that $\sin(z^2)$ is purely imaginary. [5]

SOLUTION

Let $z^2 = (x+iy)^2 = x^2 - y^2 + 2xyi$, then

$\sin(z^2) = \sin(x^2-y^2+2xyi) = \sin(x^2-y^2)\cos(2xyi) + \cos(x^2-y^2)\sin(2xyi)$

If this is to be purely imaginary, we require the real part to be equal to zero, hence

$$\sin(x^2-y^2) = 0 \quad \text{or} \quad \cos(2xyi) = \cosh(2xy) = 0.$$

The hyperbolic cosine is always positive, so the second equation has no solutions, leaving only the first which gives

$$x^2 - y^2 = n\pi$$

where n can be any integer, and the solution can be written as

$$z = x \pm i\sqrt{x^2 - n\pi}$$

where x is any real number.

c) (i) Show that if complex z satisfies $\cot z = k$, where k is real, then

$$e^{2iz} = \frac{k^2-1+2ki}{k^2+1}. \qquad [4]$$

SOLUTION

Begin by writing in terms of complex exponentials:

$$\cot z = \frac{\cos z}{\sin z} = \frac{2i(e^{iz}+e^{-iz})}{2(e^{iz}-e^{-iz})} = k \Rightarrow \frac{e^{iz}+e^{-iz}}{e^{iz}-e^{-iz}} = -ki$$

now multiply top/bottom of lhs by e^{iz}:

$$\frac{e^{2iz}+1}{e^{2iz}-1} = -ki \Rightarrow e^{2iz}+1 = -ki(e^{2iz}-1) \Rightarrow e^{2iz} = \frac{-1+ki}{1+ki}\left(\frac{1-ki}{1-ki}\right) = \frac{k^2-1+2ki}{k^2+1},$$

as required.

(ii) Hence, or otherwise, find all solutions of $\cot z = -1$. [3]

SOLUTION

With $k=-1$, the result above gives $e^{2iz} = -i$ and we write the rhs as a complex exponential:

$$e^{2iz} = e^{i(-\pi/2+2n\pi)} \Rightarrow 2z = -\frac{\pi}{2}+2n\pi \Rightarrow z = -\frac{\pi}{4}+n\pi,$$

where n is any integer.

d) Obtain the limits

$$(i)\ \lim_{x\to\pi/2}\left(x-\frac{\pi}{2}\right)\sin(\sec x), \quad (ii)\ \lim_{x\to\pi/4}\frac{1-\tan x}{2\sin x-\sqrt{2}}.$$

SOLUTION

(i) Given that $-1 \le \sin(\sec x) \le 1$ wherever $\sec x$ is defined, we can write

$$-\left(x-\frac{\pi}{2}\right) \le \left(x-\frac{\pi}{2}\right)\sin(\sec x) \le \left(x-\frac{\pi}{2}\right)$$

and so

$$0 = \lim_{x\to\pi/2} -\left(x-\frac{\pi}{2}\right) \le \lim_{x\to\pi/2}\left(x-\frac{\pi}{2}\right)\sin(\sec x) \le \lim_{x\to\pi/2}\left(x-\frac{\pi}{2}\right) = 0$$

so by the sandwich theorem, the required limit is zero. [2]

(ii) Check that this is a case of "0/0", so we can apply l'Hopital's rule:

$$\lim_{x\to\pi/4}\frac{1-\tan x}{2\sin x-\sqrt{2}} = \lim_{x\to\pi/4}\frac{-\sec^2 x}{2\cos x} = \lim_{x\to\pi/4} -\frac{1}{2\cos^3 x} = -\frac{1}{2\cos^2(\pi/4)} = -\sqrt{2}.$$

[3]

e) A function is defined as

$$f(x) = \begin{cases} \frac{\sqrt{2x+5}-\sqrt{x+7}}{x-2} & \text{for} \quad x \neq 2\,; \\ k & \text{for} \quad x = 2\,, \end{cases}$$

for $x \geq -2.5$. What value of k makes $f(x)$ a continuous function? [4]

SOLUTION

We need the limit of $f(x)$ as $x \to 2$, but the form is "0/0". Begin by multiplying appropriately and cancelling:

$$f(x) = \frac{\sqrt{2x+5}-\sqrt{x+7}}{x-2}\left(\frac{\sqrt{2x+5}+\sqrt{x+7}}{\sqrt{2x+5}+\sqrt{x+7}}\right) = \frac{1}{\sqrt{2x+5}+\sqrt{x+7}}$$

Hence

$$\lim_{x\to 2} f(x) = \frac{1}{\sqrt{9}+\sqrt{9}} = \frac{1}{6}$$

and letting $k = 1/6$ will make the function continuous at $x = 2$. As this is the only discontinuity, the function is continuous for all $x \geq -2.5$.

2. a) Differentiate to obtain $\dfrac{dy}{dx}$: [3] each

$$(i) \quad y = x^{\ln x}, \qquad (ii) \quad y^2\sqrt{x} - \ln(x+y) = 1\,.$$

SOLUTION

(i) Take logarithms

$$y = x^{\ln x} \Rightarrow \ln y = \ln\left(x^{\ln x}\right) = (\ln x)^2$$

and differentiate implicitly:

$$\frac{1}{y}\frac{dy}{dx} = \frac{2\ln x}{x} \Rightarrow \frac{dy}{dx} = 2\ln x\left(x^{\ln x - 1}\right).$$

(ii) Differentiate implicitly:

$$\frac{d}{dx}\left(y^2\sqrt{x}\right) - \frac{d}{dx}\left(\ln(x+y)\right) = \frac{d}{dx}(1) \Rightarrow 2y\frac{dy}{dx}x^{1/2} + \frac{y^2}{2x^{1/2}} - \frac{1}{x+y}\left(1+\frac{dy}{dx}\right) = 0$$

collecting terms and simplifying we have

$$\frac{dy}{dx} = \frac{2x^{1/2} - xy^2 - y^3}{2yx^{3/2} + 2y^2x^{1/2} - 1}\,.$$

b) Differentiate from first principles to show that the derivative of $\sin x$ is $\cos x$. [You may quote the result for $\lim_{x\to 0}\frac{\sin x}{x}$.] [5]

SOLUTION

We require

$$\begin{aligned}\frac{df}{dx} &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}\\ &= \lim_{h\to 0}\frac{\sin x\cos h+\sin h\cos x-\sin(x)}{h}\\ &= \lim_{h\to 0}\sin x\frac{\cos h-1}{h}+\cos x\frac{\sin h}{h}\end{aligned}$$

For the second term we quote $\lim_{h\to 0}\frac{\sin h}{h}=1$. The first term is zero:

$$\lim_{h\to 0}\frac{\cos h-1}{h}\left(\frac{\cos h+1}{\cos h+1}\right)=\lim_{h\to 0}\frac{-\sin^2 h}{h}=-\lim_{h\to 0}(\sin h)\lim_{h\to 0}\frac{\sin h}{h}=0,$$

Hence we have

$$\frac{df}{dx}=0\sin x+1\cos x=\cos x.$$

(Using l'Hopital's rule for the first term is acceptable as well, even though it uses the result.)

c) (i) Show that if $y=\left(\cos^{-1}x\right)^2$, then $\sqrt{1-x^2}\frac{dy}{dx}=-2\cos^{-1}x$. [3]

[Recall $\frac{d}{dx}\left(\cos^{-1}x\right)=-\frac{1}{\sqrt{1-x^2}}$.]

SOLUTION

Given the hint,

$$\frac{d}{dx}\left[\left(\cos^{-1}x\right)^2\right]=-2\cos^{-1}x\frac{1}{\sqrt{1-x^2}}$$

and multiplying by $\sqrt{1-x^2}$ gives the desired result.

(ii) Hence, or otherwise, deduce that $(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}-2=0$. [3]

SOLUTION

Now differentiate both sides:

$$\frac{d}{dx}\left(\sqrt{1-x^2}\frac{dy}{dx}\right)=\frac{d}{dx}\left(-2\cos^{-1}x\right)\Rightarrow\frac{-2x}{2\sqrt{1-x^2}}\frac{dy}{dx}+\sqrt{1-x^2}\frac{d^2y}{dx^2}=\frac{2}{\sqrt{1-x^2}}$$

and multiplying by $\sqrt{1-x^2}$ gives the result.

d) For the function $f(x) = (x^2 - 3)e^x$, determine all stationary points and classify these using the second derivative test. Obtain all asymptotes. Sketch the graph of the function. You do not need to find points of inflection, but should indicate on your graph where other information allows you to deduce them. [8]

SOLUTION

Begin by differentiating $f(x) = (x^2 - 3)e^x$:

$$f'(x) = (x^2 - 3)e^x + 2xe^x = (x^2 + 2x - 3)e^x = (x+3)(x-1)e^x$$

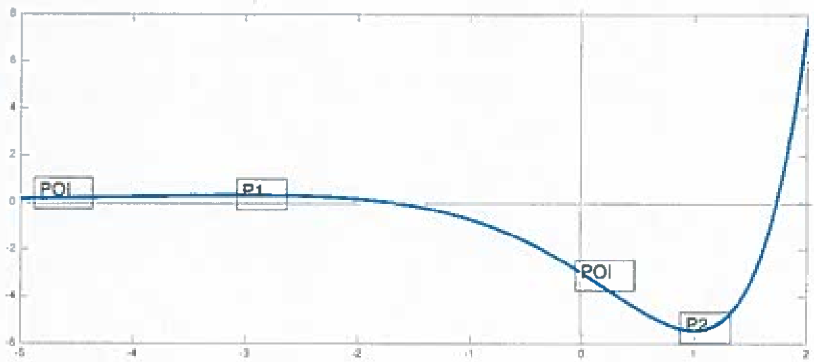
and as $e^x \neq 0$ we deduce that stationary points, given by $f'(x) = 0$ are at $x = 1$ and $x = -3$. Evaluating, we have stationary points $P_1(-3, 6e^{-3})$ and $P_2(1, -2e)$.

Differentiating again, we obtain $f''(x) = (x^2 + 4x - 1)e^x$ and evaluating at the stationary points we get

$f''(1) = 4e > 0 \Rightarrow P_1$ is a local minimimum;

$f''(-3) = -4e^{-3} < 0 \Rightarrow P_2$ is a local maximum.

There is one asymptote: as $x \to -\infty$, the exponential goes to zero and so does $f(x)$. Intercepts are $(\pm\sqrt{3}, 0)$ and $(0, -3)$. This is sufficient to plot the function:



Stationary points and intercepts are shown as expected. Between max and min curvature changes, and we deduce a point of inflection; to the left of P_1, curvature must change for the asymptote and we infer another point of inflection, approximate position indicated on plot.

3. a) Evaluate the following integrals.

i) $$\int \frac{18x + 12}{\sqrt{3x^2 + 4x - 7}}\, dx,$$ [3]

SOLUTION

Let $u = 3x^2 + 4x - 7$, then $du = 6x + 4\, dx$ and we have

$$\int \frac{18x+12}{\sqrt{3x^2+4x-7}}\, dx = \int \frac{3}{\sqrt{u}}\, du = 6\sqrt{u} + C = 6\sqrt{x^2+4x-7} + C.$$

ii) $$\int_3^4 \frac{2x+3}{x^2-x-2}\,dx,$$ [3]

SOLUTION

Partial fractions gives

$$\frac{2x+3}{x^2-x-2} = \frac{7}{3(x-2)} - \frac{1}{x+1}$$

and so

$$\int_3^4 \frac{7}{3(x-2)} - \frac{1}{x+1}\,dx = \left[\frac{7}{3}\ln(x-2) - \ln(x+1)\right]_3^4 = \frac{13}{3}\ln 2 - \ln 5\,.$$

iii) $$\int \frac{\cosh^{-1}x}{(x^2-1)^{1/2}}\,dx,$$ [3]

SOLUTION

Note that the derivative of $\cosh^{-1}x$ is present, so let

$$u = \cosh^{-1}x \Rightarrow du = \frac{1}{(x^2-1)^{1/2}}dx$$

and the integral becomes

$$\int u\,du = \frac{u^2}{2} + C = \frac{\left(\cosh^{-1}x\right)^2}{2} + C\,.$$

iv) $$\int_{-2}^{2} \sqrt{4-x^2}\,dx,$$ using a substitution. [4]

SOLUTION

We observe that the integrand is the upper semi-circle of a circle of radius two, centred at the origin, so the area should be $\frac{1}{2}(\pi 2^2) = 2\pi$, but we need to use a substitution. The usual argument suggest a trigonometric one: let

$$x = 2\sin u \Rightarrow 4 - x^2 = 4 - 4\sin^2 u = 4\cos^2 u$$

with limits $x = \pm 2 \Rightarrow \sin u = \pm 1 \Rightarrow u = \pm\pi/2$ and $dx = 2\cos u\,du$ so the integral becomes

$$\int_{-\pi/2}^{\pi/2} (\sqrt{4\cos^2 u})\,2\cos u\,du = 4\int_{-\pi/2}^{\pi/2} \cos^2 u\,du = 4\int_{-\pi/2}^{\pi/2} \frac{\cos 2u + 1}{2}\,du = 2\left[\frac{\sin 2u}{2} + u\right]_{-\pi/2}^{\pi/2} = 2\pi\,.$$

b) Let

$$I_n = \int_0^\infty x^{2n} e^{-x}\, dx,$$

where n is a positive integer. Find I_0 and show that $I_n = (2n)!I_0$. Hence, or otherwise, obtain I_6 in terms of a factorial. [7]

SOLUTION

Finding I_0 is trivial:

$$I_0 = \lim_{k\to\infty} \int_0^k e^{-x}\, dx = \lim_{k\to\infty} \left[-e^{-x}\right]_0^k = \lim_{k\to\infty} (1-e^{-k}) = 1\,.$$

For I_n we integrate by parts:

$$I_n = \left[x^{2n}e^{-x}\right]_0^\infty + 2n\int_0^\infty x^{2n-1}e^{-x}\, dx$$

The first two terms vanish with the zero and infinite limits. As we need an even power of x, we integrate by parts again:

$$I_n = 2n\left\{\left[-x^{2n-1}e^{-x}\right]_0^\infty + (2n-1)\int_0^\infty x^{2n-2}e^{-x}\, dx\right\}$$

The first two terms vanish as before, so

$$I_n = 2n(2n-1)\int_0^\infty x^{2(n-1)}e^{-x}\, dx = 2n(2n-1)I_{n-1}$$

The recursion formula gives

$$I_n = 2n(2n-1)\left[(2n-2)(2n-3)I_{n-2}\right]$$

and so on, with the last terms

$$I_2 = 4\cdot 3I_1 = 4\cdot 3\cdot 2\cdot 1I_0$$

so

$$I_n = (2n)!I_0$$

as required. Hence $I_6 = (12!)I_0 = 12!$ and we leave the solution as factorial.

c) Show that the harmonic series $\sum_{n=1}^\infty \frac{1}{n}$ diverges.

Hence, or otherwise, show that $\sum_{n=1}^\infty \frac{1}{\sqrt{n}}$ also diverges. [5]

SOLUTION

For the harmonic series we apply the integral test. Given

$$\int_1^\infty \frac{1}{x}\, dx < \sum_{n=1}^\infty \frac{1}{n}$$

we have

$$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{k\to\infty}[\ln x]_1^k = \lim_{k\to\infty} \ln k$$

which diverges, and so does the series which is greater than the integral.

We observe

$$n > \sqrt{n} \Rightarrow \frac{1}{n} < \frac{1}{\sqrt{n}}$$

for all $n > 1$. Hence

$$\sum_{n=1}^{\infty} \frac{1}{n} < \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

and the second series diverges by the comparison test.

4. a) Find the radius and interval of convergence of the infinite series [5]

$$\sum_{n=0}^{\infty} (-1)^n (n+1) x^n,$$

SOLUTION

We apply the ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1}(n+2)x^{n+1}}{(-1)^n(n+1)x^n}\right| = \frac{n+2}{n+1}|x|$$

and taking the limit, we get

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{n+2}{n+1}|x| = \lim_{n\to\infty}\frac{1+2/n}{1+1/n}|x| = \frac{1+0}{1+0}|x| < 1$$

for convergence, so the radius of convergence is 1 and the interval of convergence is $-1 < x < 1$. Testing at the endpoint $x = 1$, we see

$$\sum_{n=0}^{\infty} (-1)^n (n+1) = 1 - 2 + 3 - 4 + 5 - \cdots$$

which diverges. Similarly, letting $x = -1$

$$\sum_{n=0}^{\infty} (-1)^n (n+1)(-1)^n = \sum_{n=0}^{\infty} (-1)^{2n}(n+1) = 1 + 2 + 3 + 4 + \cdots$$

which also diverges, so the interval of convergence cannot be extended.

b) Obtain the n^{th} derivative of $f(x) = (1+x)^{-2}$ and hence show that the Maclaurin series of $f(x)$ is given by the series in (a). [5]

SOLUTION

Differentiating a few times:

$$f(x) = \frac{1}{(1+x)^2} \Rightarrow f'(x) = -2\frac{1}{(1+x)^3} \Rightarrow f''(x) = 6\frac{1}{(1+x)^2} \Rightarrow f'''(x) = -24\frac{1}{(1+x)^2}$$

until we can spot the pattern:

$$f^{(n)}(x) = (n+1)!(-1)^n \frac{1}{(1+x)^n} \Rightarrow f^{(n)}(0) = (n+1)!(-1)^n$$

and the Maclaurin series is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{(n+1)!(-1)^n}{n!} x^n$$

and the factorials cancel, leaving only $n+1$ in the numerator, so that

$$f(x) = \sum_{n=0}^{\infty} (n+1)(-1)^n x^n \,,$$

matching the series from (a).

c) i) Find the real Fourier Series for the functions with period 2, defined on $[-1,1]$ as $f(x) = |x|$ and $g(x) = x$. [7]

SOLUTION

For $f(x)$ this is an even function and $b_n = 0$ so that with $T = 2$

$$\begin{aligned}
a_n &= \frac{2}{T}\int_{-T/2}^{T/2} f(x)\cos(2n\pi x/T)\,dx = \int_{-1}^{1} |x|\cos(n\pi x)\,dx \\
&= 2\int_0^1 x\cos(n\pi x)\,dx = 2\left[\frac{x\sin(n\pi x)}{n\pi}\right]_0^1 - \frac{2}{n\pi}\int_0^1 \sin(n\pi x)\,dx \\
&= [0-0] + \frac{2}{n\pi}\left[\frac{\cos(n\pi x)}{n\pi}\right]_0^1 = \frac{2}{(n\pi)^2}[\cos(n\pi) - 1] \\
&= \frac{2}{(n\pi)^2}[(-1)^n - 1]
\end{aligned}$$

so that $a_n = -4/(n\pi)^2$ for odd n and zero for even n. Calculate a_0 separately: $a_0 = 2\int_0^1 x\,dx = 1$ and the series is

$$f(x) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos(n\pi x)\,.$$

For $g(x)$, an odd function, we have $b_n = 0$ and having done the set-up above, we proceed directly to

$$\begin{aligned}
b_n &= 2\int_0^1 x\sin(n\pi x)\,dx \\
&= 2\left[-x\frac{\cos(n\pi x)}{n\pi}\right]_0^1 + \frac{2}{n\pi}\int_0^1 \cos(n\pi x)\,dx \\
&= \frac{2}{n\pi}[-\cos(n\pi) - 0 + 0 - 0] = \frac{2}{n\pi}(-1)^{n+1}
\end{aligned}$$

where we omitted the detail on the integral vanishing, as we have done this for the first series already. Hence the Fourier series is

$$g(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(n\pi x)\,.$$

ii) Choosing an appropriate value for x in the Fourier series for $g(x)$ above, obtain the value of the infinite sum [4]

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\cdots$$

SOLUTION

To find the correct value of x, we reason that the Fourier series does have the alternating pattern, but involves all fractions

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \cdots$$

whereas we only want the odd denominators. We need a value of x so that $\sin(n\pi x) = 0$ for even n: this must be $x = 1/2$. Trying it out we have

$$\frac{1}{2}=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(n\pi/2)=\frac{2}{\pi}\left(1-0-\frac{1}{3}+0+\frac{1}{5}-0-\frac{1}{7}\cdots\right)$$

which fits, so the required series has sum $\pi/4$.

iii) Using Parseval's theorem on the Fourier series for $f(x)$ above, obtain the value of the infinite sum [4]

$$1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\frac{1}{6^4}+\cdots$$

SOLUTION Using Parseval's theorem for period 2 we have

$$\int_{-1}^{1}[f(x)]^2\,dx=\frac{1}{2}a_0^2+\sum_{n=1}^{\infty}a_n^2$$

so

$$\int_{-1}^{1}x^2\,dx=\left[\frac{x^3}{3}\right]_{-1}^{1}=\frac{2}{3}=\frac{1}{2}1^2+\sum_{n=1}^{\infty}\left(\frac{4}{n^2\pi^2}\right)^2\Rightarrow\frac{1}{6}=\frac{16}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4}$$

so the series has the value $\pi^4/96$.