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EE3-07

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2011

EEE/ISE PART III/IV: MEng, BEng and ACGI

DIGITAL SIGNAL PROCESSING

Monday, 9 May 10:00 am

Time allowed: 3:00 hours

There are FOUR questions on this paper.

Answer ALL questions.

All questions carry equal marks.

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s) :	P.A. Naylor
	Second Marker(s) :	D.P. Mandic

DIGITAL SIGNAL PROCESSING

1. a) Given a complex signal $x(n)$ with DFT $X(k)$, write an expression for $X(k)$ in terms of $x(n)$ and define all terms. State and justify an expression for the computational complexity of direct computation of the DFT. [3]

   b) Consider a complex signal $x(n)$ of length N samples with real part $x_r(n)$ and imaginary part $x_i(n)$ and denote the DFT of $x(n)$ as $X(k)$. Write down expressions for the DFT of $x_r(n)$ and the DFT of $x_i(n)$ in terms of $X(k)$. [5]

   c) Consider a real signal $y(n)$ of length $2N$ samples and denote its DFT as $Y(k)$. Show how $Y(k)$ can be computed using a single N-point DFT. Include a clear mathematical description of all the steps involved. [9]

   d) Give an example of a practical application in which the approach of c) would be beneficial and indicate with a numeric example the amount of benefit that might be obtained. [3]

2. a) What can be deduced about a linear time-invariant (LTI) system from its impulse response? Explain and consider left-sided, right-sided and two-sided impulse responses. [4]

b) A loudspeaker manufacturer has developed an LTI model for one of its products in the form

$$H(z) = \frac{z(z+2.1)}{(z-0.2)(z+0.6)}$$

which is the transfer function between the input voltage and the output sound level.

i) Draw the poles and zeros of this model $H(z)$ on a plot in the z-domain. [4]

ii) Comment on any general characteristics of the system that can be deduced from your plot. [2]

iii) Calculate the gain in dB and phase in radians of $H(z)$ at frequencies $\omega = 0$ and $\omega = \pi$. [4]

iv) Using partial fraction expansion, find a closed-form expression for the impulse response of the loudspeaker modelled by $H(z)$. [6]

3. a) i) State and explain the Noble Identities in the context of multirate signal processing. [5]

ii) A cascaded integrator comb (CIC) filter has the transfer function

$$H(z) = \frac{1 - z^{-M}}{1 - z^{-1}}.$$

Consider the case when a CIC filter with $M = D$ is followed by decimation by a factor D. Show how the Noble Identities can be used to implement efficiently this combination. [5]

b) Consider the multirate signal processing structure shown in Fig. 3.1 in which the blocks from left to right represent an expander, a filter and a decimator respectively.

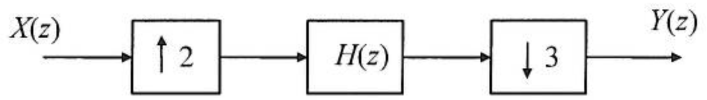

Figure 3.1

i) Derive an expression for $Y(z)$ in terms of $X(z)$. [5]

ii) Figure 3.2 shows part of the magnitude spectrum corresponding to $X(z)$. Draw a labelled sketch of the magnitude spectrum corresponding to $Y(z)$ over the range of frequency ω spanning $-4\pi < \omega < 4\pi$. [5]

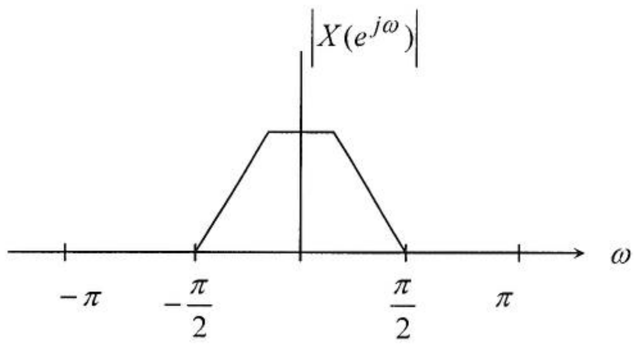

Figure 3.2

4. The system function of an FIR filter is given by

$$H(z) = \sum_{k=0}^{N-1} b_k z^{-k}.$$

a) What is the order of this filter? [1]

b) Explain the key properties of a linear phase FIR filter in terms of the filter coefficients and in terms of the z-domain. [3]

c) Explain the meaning of group delay and state an expression for the group delay of a linear phase FIR filter. [3]

d) Show that the frequency response of a linear phase FIR filter with $N = 2M$ can be written

$$H(e^{j\omega}) = e^{j\lambda} \sum_{n=0}^{M-1} 2h(n)\sin(\gamma)$$

and give expressions for λ and γ. [5]

e) Describe how a linear phase FIR filter can be efficiently implemented and state quantitatively the reduction in computational complexity achieved by the efficient implementation compared to Direct Form 1. [3]

f) Draw and label the signal flow diagram of the efficient implementation of a linear phase FIR filter for $N = 8$. [5]

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DIGITAL SIGNAL PROCESSING
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1. a)

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N} \quad 0 \leq k < N$$

k is the frequency index.
n is the discrete time index.
N is the number of points in the $x(n)$.
j is the square root of -1.

This computation executes 2 loops, one for n and one for k. Each loop executes N MAC operations.

b)

$$X_r(k) = \frac{1}{2}(X(k) + X^*(-k))$$
$$X_i(k) = \frac{1}{2j}(X(k) - X^*(-k))$$

with all indexing modulo N.

c) First consider two real signals obtained from $y(n)$:

$$g(n) = y(2n) \quad h(n) = y(2n+1) \quad 0 \leq n < N.$$

with DFTs $G(k)$ and $H(k)$ respectively.

Next, write the complex signal

$$x(n) = g(n) + jh(n)$$

such that $x(n) = \{y(0) + jy(1),\ y(2) + jy(3),\ \ldots\}$

Find $G(k)$ and $H(k)$ by first using a single N point DFT to find $X(k)$ and then exploiting the symmetry relations to obtain

$$G(k) = \frac{1}{2}(X(k) + X^*(-k))$$
$$H(k) = \frac{1}{2j}(X(k) - X^*(-k))$$

with all indexing modulo N.

To obtain $Y(k)$ from $G(k)$ and $H(k)$ we write

$$\begin{aligned} Y(k) &= \sum_{n=0}^{2N-1} y(n) W_{2N}^{nk} \\ &= \sum_{n=0}^{N-1} y(2n) W_{2N}^{2nk} + \sum_{n=0}^{N-1} y(2n+1) W_{2N}^{(2n+1)k} \\ &= \sum_{n=0}^{N-1} g(n) W_N^{nk} + W_{2N}^{k} \sum_{n=0}^{N-1} h(n) W_N^{nk} \end{aligned}$$

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given that $W_{2N}^{2nk} = W_N^{nk}$.

Finally it can be seen that

$$Y(k) = G(k_{modN}) + W_{2N}^{k} H(k_{modN}) \quad 0 \leq k \leq 2N-1.$$

d) Any example in which the input signal is intrinsically real such as audio signals will obtain credit. The benefit of reduction from $2N$ to N point DFTs scales as N^2. If the FFT is employed, scaling is proportional to $\log_2 N$. Full marks require a numeric example of the student's own devising such as for an example of $N = 512$ or any other reasonable value. The computation associated with the recombination in the final equation should also be mentioned but is not required to be included in the quantitative example for full marks.

2. a) The impulse response of an LTI system completely characterizes the system in the steady state. This includes stability and causality. The impulse response also contains complete information about the poles and zeros of the system which can be obtained by the inverse z-transform.

Right and left sided impulse responses indicate causal and anti-causal systems respectively. Two-sided impulse responses indicate non-causal systems.

b) i) See Fig. 2.1.

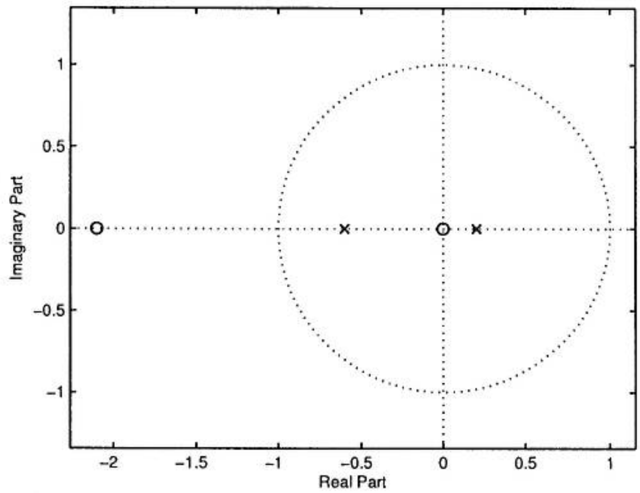

Figure 2.1

ii) The system is stable and mixed phase.

iii)

$$20\log_{10}|H(e^{j\omega})|_{\omega=0} = 20*\log_{10}(3.1/1.28) = 7.7 \text{ dB}$$
$$20\log_{10}|H(e^{j\omega})|_{\omega=\pi} = 20*\log_{10}(1.1/0.48) = 7.2 \text{ dB}$$
$$\angle H(e^{j\omega})_{\omega=0} = 0 \text{ radians}$$
$$\angle H(e^{j\omega})_{\omega=\pi} = \pi \text{ radians.}$$

iv)

$$H(z) = \frac{z(z+2.1)}{(z-0.2)(z+0.6)} = \frac{1+2.1z^{-1}}{(1-0.2z^{-1})(1+0.6z^{-1})}.$$

We are then looking for the PFE in the form

$$H(z) = \frac{A}{1-0.2z^{-1}} + \frac{B}{1+0.6z^{-1}}$$

which is solved from $A(1+0.6z^{-1}) + B(1-0.2z^{-1}) = 1+2.1z^{-1}$.

At $z = 0.2$ we obtain the solution $A = 2.875$.

At $z = -0.6$ we obtain the solution $B = -1.875$.

Thus the PFE is

$$H(z) = \frac{2.875}{1-0.2z^{-1}} - \frac{1.875}{1+0.6z^{-1}}.$$

By inspection and using the standard z-transform pair:

$$a^n u(n) \longleftrightarrow \frac{1}{1 - az^{-1}}$$

we finally obtain $h(n) = 2.875(0.2)^n u(n) - 1.875(-0.6)^n u(n)$.

3. a) i) The Noble Identities can be expressed in the following way as shown in Fig. 3.1. The identities show the manner in which the order of filtering and decimation/expansion can be switched.

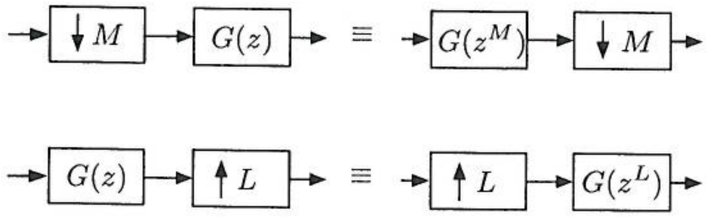

Figure 3.1

ii) The CIC transfer function is separable and can be drawn in the block diagram shown in upper part of Fig. 3.2. The Noble Identities can be used to change the order of the comb filter and the decimator in order that the comb filtering is done at the lower sampling rate. Interestingly, this doesn't save computations but does help in terms of memory.

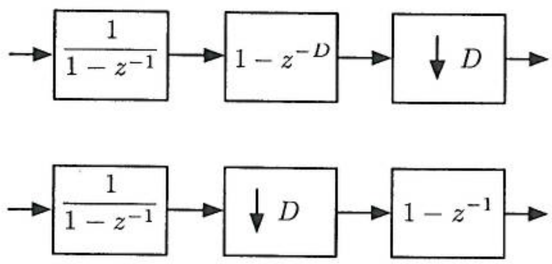

Figure 3.2

b)

$$Y(z) = \frac{1}{3}\sum_{k=0}^{2} X(z^{2/3}W^{2k})H(z^{1/3}W_3^k)$$

with $W_3 = e^{-j2\pi/3}$

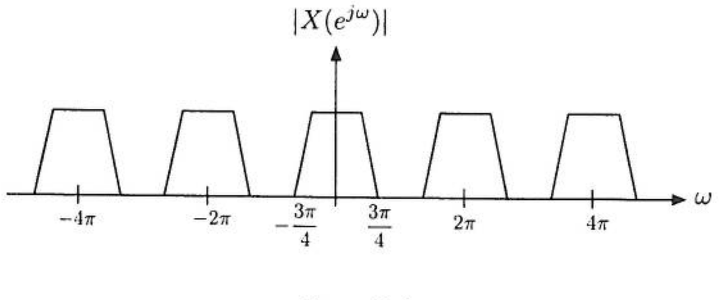

Figure 3.3

4. a) Order is $N-1$.

b) The filter coefficients satisfy $h(n) = h(N-1-n)$. The zeros in the z-domain occur in mirror image pairs such that if a zero exists at z_0 then another zero also exists at $\frac{1}{z_0^*}$.

c) Group delay, measured in seconds, is a measure of the delay of signals propagating through the filter. It is defined as the negative derivative of the phase with respect to frequency:

$$-\frac{\partial\phi(\omega)}{\partial\omega}.$$

In the case in question of linear phase filters, the delay is constant for any frequency since the phase response is linearly proportional to frequency.

d) For $N = 2M$ we have

$$\begin{aligned}
H(e^{j\omega}) &= \sum_{n=0}^{M-1} h(n)e^{-jn\omega} + \sum_{n=M}^{2M-1} h(n)e^{-jn\omega} \\
&= \sum_{n=0}^{M-1} h(n)e^{-jn\omega} + \sum_{n=0}^{M-1} h(N-n)e^{-j(N-n)\omega} \\
&= e^{-jM\omega}\left(\sum_{n=0}^{M-1} h(n)e^{+j\omega(M-n)} + \sum_{n=0}^{M-1} h(N-n)e^{-j\omega(M-n)}\right) \\
&= e^{-jM\omega}\sum_{n=0}^{M-1} 2jh(n)\sin(\omega(M-n)) \\
&= e^{+j(\pi/2-M\omega)}\sum_{n=0}^{M-1} 2h(n)\sin(\omega(M-n))
\end{aligned}$$

which is of the required form with

$$\begin{aligned}
\lambda &= (\pi/2 - M\omega) \\
\gamma &= \omega(M-n).
\end{aligned}$$

e) The efficiency is achieved by performing the multiplication only once for each symmetric pair of coefficients. This can be written mathematically as:

$$H(z) = h(0)(1+z^{-7}) + h(1)(z^{-1}+z^{-6}) + h(2)(z^{-2}+z^{-5})h(3)(z^{-3}+z^{-4})$$

f) See Fig. 4.1

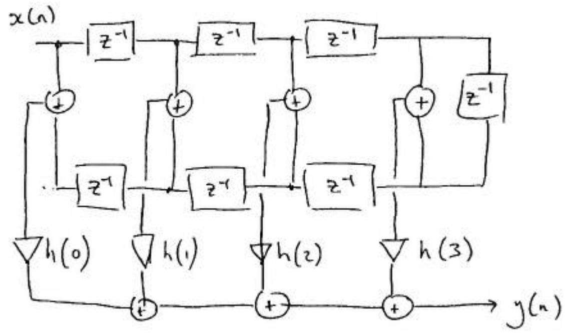

Figure 4.1