Imperial College
London

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2013

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Time Series

Date: Thursday, 30 May 2013. Time: 10.00am. Time allowed: 2 hours.

This paper has FOUR questions.

Candidates should use TWO main answer books (A & B) for their solutions as follows:
book A - solutions to questions 1 & 2; book B - solutions to questions 3 & 4.

Supplementary books may only be used after the relevant main book(s) are full.

Statistical tables will not be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Answer all the questions. Each question carries equal weight.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Calculators may not be used.

Note: Throughout this paper $\{\epsilon_t\}$ is a sequence of uncorrelated random variables (white noise) having zero mean and variance σ_ϵ^2, unless stated otherwise. The unqualified term "stationary" will always be taken to mean second-order stationary. All processes are real-valued unless stated otherwise. The sample interval is unity unless stated otherwise.

1. (a) (i) What is meant by saying that a stochastic process is stationary?

   (ii) Let $\{Y_t\}$ be a stationary process with mean zero, and define

$$X_t = \alpha + \beta t + \nu_t + Y_t$$

where α and β are non-zero constants and ν_t is a deterministic seasonal component with period 2. Define $W_t = (1-B^2)(1-B)X_t$ where B is the backward shift operator. Express the autocovariance sequence $\{s_{W,\tau}\}$ of $\{W_t\}$ in terms of the autocovariance sequence $\{s_{Y,\tau}\}$ of $\{Y_t\}$.

(b) A continuous-time process $\{X(t)\}$, with t in seconds, has spectral density function

$$S_{X(t)}(f) = \begin{cases} 1 - \frac{2}{3}|f|, & |f| \le 3/2, \\ 0, & \text{otherwise,} \end{cases}$$

with f in cycles/second. It is sampled with a sample interval $\Delta t = 1/2$ second to produce the discrete-time process $\{X_t\}$.
What is the spectral density function $S_{X_t}(f)$ of $\{X_t\}$ for $|f| < f_{\mathcal{N}}$, where $f_{\mathcal{N}}$ is the Nyquist frequency?

(c) Given a sequence $X_1, \ldots, X_N$ from a real-valued stationary time series, the periodogram incorporating a taper $\{h_t\}$ takes the form

$$\widehat{S}^{(t)}(f) = \left| \sum_{t=1}^{N} h_t X_t \mathrm{e}^{-\mathrm{i}2\pi f t} \right|^2.$$

By writing $\widehat{S}^{(t)}(f)$ in the form $\widehat{S}^{(t)}(f) = \sum_{\tau=-(N-1)}^{N-1} \widehat{s}_\tau^{(t)} \mathrm{e}^{-\mathrm{i}2\pi f \tau}$, identify the form of $\widehat{s}_\tau^{(t)}$.

(d) Suppose the stationary process $\{X_t\}$ can be written as a one-sided linear process, $X_t = \sum_{k=0}^{\infty} \psi_k \epsilon_{t-k}$. We wish to construct the l-step ahead forecast

$$X_t(l) = \sum_{k=0}^{\infty} \delta_k \varepsilon_{t-k}.$$

Show that the l-step prediction variance $\sigma^2(l) = E\{(X_{t+l} - X_t(l))^2\}$ is minimized by setting $\delta_k = \psi_{k+l}$, $k \ge 0$.

2. (a) (i) Suppose $\{X_t\}$ is an MA(q) process with zero mean, i.e., X_t can be expressed in the form

$$X_t = -\theta_{0,q}\epsilon_t - \theta_{1,q}\epsilon_{t-1} - \ldots - \theta_{q,q}\epsilon_{t-q},$$

where the $\theta_{j,q}$'s are constants ($\theta_{0,q} \equiv -1, \theta_{q,q} \neq 0$). Show that its autocovariance sequence is given by

$$s_\tau = \begin{cases} \sigma_\epsilon^2 \sum_{j=0}^{q-|\tau|} \theta_{j,q}\theta_{j+|\tau|,q}, & \text{if } |\tau| \leq q, \\ 0, & \text{if } |\tau| > q. \end{cases}$$

(ii) Let $\{X_t\}$ be the moving average process of order 2 given by

$$X_t = \epsilon_t - \theta_{2,2}\epsilon_{t-2}$$

where $\theta_{1,2} = 0$. Use the autocovariance sequence of $\{X_t\}$ to find the variance of the sample mean $(X_1 + X_2 + X_3 + X_4)/4$ when $\theta_{2,2} = 0.8$.

(b) Consider the process

$$Y_t = \epsilon_t \cos(ct) + \epsilon_{t-1}\sin(ct),$$

where c is a non-zero constant. By finding $\operatorname{cov}\{Y_t, Y_{t+\tau}\}$ determine the values of c for which the process is stationary.

(c) Let $\{X_t\}$ be a Gaussian (normal) stationary process with a mean of zero. Define $Y_t = X_t X_{t-1}$.

(i) Find the autocovariance sequence $s_{Y,\tau}$ of $\{Y_t\}$ in terms of the autocovariance sequence $s_{X,\tau}$ of $\{X_t\}$.
You will need to use the following version of the Isserlis Theorem: If X_j, X_k, X_l, X_m are any four real-valued Gaussian random variables with zero mean then

$$E\{X_jX_kX_lX_m\} = E\{X_jX_k\}E\{X_lX_m\} + E\{X_jX_l\}E\{X_kX_m\} + E\{X_jX_m\}E\{X_kX_l\}.$$

(ii) If $\{X_t\}$ is an MA(1) process, give the form of $s_{Y,\tau}$ in terms of $\theta_{1,1}$ and σ_ϵ^2.

3. (a) (i) State the three defining properties of a linear time-invariant digital filter.

(ii) Consider the simple moving-average filter $L\{W_t\} = \sum_{j=-q}^{q} g_j W_{t-j}$ with weights $g_j = 1/(2q+1)$, $-q \leq j \leq q$. If $W_t = \alpha + \beta t$ where α and β are non-zero constants, show that

$$L\{W_t\} = W_t.$$

(iii) Now let $L\{\epsilon_t\} = X_t$, (with L the moving average filter defined in (ii)). By finding the mean and variance of X_t describe the behaviour of X_t when q is large.

(b) Consider the ARMA(p, q) process

$$\Phi(B)X_t = \Theta(B)\epsilon_t$$

where $\Phi(B) = 1 - \phi_{1,p}B - \ldots - \phi_{p,p}B^p$ and $\Theta(B) = 1 - \theta_{1,q}B - \ldots - \theta_{q,q}B^q$.

(i) Derive the form of the spectral density function $S(f)$ of $\{X_t\}$.

(ii) Hence, or otherwise, identify the *stationary* and *invertible process* $\{X_t\}$ having the spectral density

$$S(f) = \frac{17 - 8\cos 2\pi f}{13 - 12\cos 2\pi f},$$

obtaining values for any autoregressive parameters $\{\phi_{j,p}\}$, any moving average parameters $\{\theta_{j,q}\}$, and σ_ϵ^2.

[Hint: Start by considering the moving average part (numerator).]

4. (a) (i) Let $\{X_t\}$ be a Gaussian (normal) zero-mean AR(p) process, and $X_1, X_2, \ldots, X_N$, be a realization of the process. Formulate the equations

$$\boldsymbol{X}_B = \boldsymbol{B}\boldsymbol{\phi} + \boldsymbol{\epsilon}_B$$

from which the backward least squares estimator of the parameter vector $\boldsymbol{\phi} = [\phi_{1,p}, \ldots, \phi_{p,p}]^T$ is derived, giving the exact forms of the vectors $\boldsymbol{X}_B$ and $\boldsymbol{\epsilon}_B$ and the matrix $\boldsymbol{B}$.

(ii) Suppose $p = 1$ and $X_1 = 2, X_2 = 2, X_3 = 0, X_4 = 1$. Find the backward least squares estimate of $\phi_{1,1} = \phi$ and the corresponding estimate of σ_ϵ^2.

(iii) For a stationary Gaussian (normal) AR(p) process, why is it not surprising that forward/backward least squares performs better than either forward least squares or backward least squares by themselves?

(b) Suppose we are given two real-valued zero mean jointly stationary processes $\{Y_t\}$ and $\{Z_t\}$ defined by

$$Y_t = aX_{t-d} + \epsilon_t \qquad \text{and} \qquad Z_t = X_{t-\ell},$$

where the stationary process $\{X_t\}$ and the white noise process $\{\epsilon_t\}$ are uncorrelated with each other, and a is a real-valued non-zero constant and $d, \ell \, (d \neq \ell)$ are integer constants.

(i) What is meant by saying two real-valued discrete time stochastic processes $\{Y_t\}$ and $\{Z_t\}$ are jointly stationary stochastic processes?

(ii) Find the cross-covariance sequence $s_{YZ,\tau}$ between $\{Y_t\}$ and $\{Z_t\}$, and determine the lag τ at which it is maximized.

(iii) Derive the cross-spectrum $S_{YZ}(f)$, and hence find the phase spectrum $\theta(f)$.

(iv) The quantity $-\frac{1}{2\pi}\frac{\mathrm{d}\theta(f)}{\mathrm{d}f}$ is called the group delay. When it is a constant, the group delay is said to measure where $s_{YZ,\tau}$ is concentrated in terms of the lag τ. Compute the group delay and comment on its form.

1. (a) (i) $\{X_t\}$ is second-order stationary if $E\{X_t\}$ is a finite constant for all t, var$\{X_t\}$ is a finite constant for all t, and cov$\{X_t, X_{t+\tau}\}$, is a finite quantity depending only on τ and not on t. $\boxed{\text{seen} \Downarrow}$ $\boxed{4}$

(ii) Write $(1-B^2)(1-B) = 1 - B - B^2 + B^3$ then $\boxed{\text{sim. seen} \Downarrow}$

$$\begin{aligned} W_t = (1 - B - B^2 + B^3)X_t &= X_t - X_{t-1} - X_{t-2} + X_{t-3} \\ &= \alpha + \beta t + \nu_t + Y_t - \alpha - \beta(t-1) - \nu_{t-1} - Y_{t-1} \\ &- \alpha - \beta(t-2) - \nu_{t-2} - Y_{t-2} \\ &+ \alpha + \beta(t-3) + \nu_{t-3} + Y_{t-3} \\ &= Y_t - Y_{t-1} - Y_{t-2} + Y_{t-3}. \end{aligned}$$

$\boxed{2}$

So $E\{W_t\} = 0$. Then

$$\begin{aligned} E\{W_t W_{t+\tau}\} &= E\{(Y_t - Y_{t-1} - Y_{t-2} + Y_{t-3})(Y_{t+\tau} - Y_{t+\tau-1} - Y_{t+\tau-2} + Y_{t+\tau-3})\} \\ &= s_\tau - s_{\tau-1} - s_{\tau-2} + s_{\tau-3} - (s_{\tau+1} - s_\tau - s_{\tau-1} + s_{\tau-2}) \\ &- (s_{\tau+2} - s_{\tau+1} - s_\tau + s_{\tau-1}) + (s_{\tau+3} - s_{\tau+2} - s_{\tau+1} + s_\tau) \\ &= 4s_{Y,\tau} - (s_{Y,\tau-1} + s_{Y,\tau+1}) - 2(s_{Y,\tau-2} + s_{Y,\tau+2}) + (s_{Y,\tau-3} + s_{Y,\tau+3}) \\ &= s_{W,\tau}. \end{aligned}$$

$\boxed{2}$

(b) The Nyquist frequency for the discrete process is $f_\mathcal{N} = 1/(2\Delta t) = 1$ cycle/second. This is the folding frequency; the aliased spectrum is got by folding $S_{X(t)}(f)$ about $\pm f_\mathcal{N} = \pm 1$. Only $1/2 \le |f| < 1$ is affected by the folding (because elsewhere there is nothing to fold in). Since the spectrum is triangular and takes the value $1/3$ at ± 1, the folding addition produces : $\boxed{\text{unseen} \Downarrow}$

$$S_{X_t}(f) = \begin{cases} 1 - \frac{2}{3}|f|, & |f| \le 1/2, \\ \frac{2}{3}, & 1/2 \le |f| < 1. \end{cases}$$

$\boxed{4}$

(c) We use the method of replacing row sums of a matrix by diagonal sums: $\boxed{\text{sim. seen} \Downarrow}$

$$\begin{aligned} \widehat{S}^{(t)}(f) &= \left| \sum_{t=1}^{N} h_t X_t e^{-i2\pi f t} \right|^2 = \sum_{j=1}^{N} \sum_{k=1}^{N} h_j X_j h_k X_k e^{-i2\pi f(k-j)} \\ &= \sum_{\tau=-(N-1)}^{N-1} \left[\sum_{t=1}^{N-|\tau|} h_t X_t h_{t+|\tau|} X_{t+|\tau|} \right] e^{-i2\pi f \tau}, \end{aligned}$$

so

$$\widehat{s}_\tau^{(t)} = \sum_{t=1}^{N-|\tau|} h_t X_t h_{t+|\tau|} X_{t+|\tau|}.$$

$\boxed{4}$

(d) We want to minimize, $\boxed{\text{seen} \Downarrow}$

$$
\begin{aligned}
E\{(X_{t+l} - X_t(l))^2\} &= E\left\{\left(\sum_{k=0}^{\infty}\psi_k\epsilon_{t+l-k} - \sum_{k=0}^{\infty}\delta_k\epsilon_{t-k}\right)^2\right\} \\
&= \mathsf{E}\left\{\left(\sum_{k=0}^{l-1}\psi_k\epsilon_{t+l-k} + \sum_{k=0}^{\infty}[\psi_{k+l} - \delta_k]\epsilon_{t-k}\right)^2\right\} \\
&= \sigma_\epsilon^2\left\{\left(\sum_{k=0}^{l-1}\psi_k^2\right) + \sum_{k=0}^{\infty}(\psi_{k+l} - \delta_k)^2\right\}.
\end{aligned}
$$

The first term is independent of the choice of the $\{\delta_k\}$ and the second term is clearly minimized by choosing $\delta_k = \psi_{k+l}, k = 0, 1, 2, \ldots$. $\boxed{4}$

2. (a) (i) Since $E\{\epsilon_t \epsilon_{t+\tau}\} = 0 \ \forall \tau \neq 0$ we have for $\tau \geq 0$. seen ⇓

$$s_\tau = \text{cov}\{X_t, X_{t+\tau}\} \ = \ \sum_{j=0}^{q}\sum_{k=0}^{q} \theta_{j,q}\theta_{k,q} E\{\epsilon_{t-j}\epsilon_{t+\tau-k}\}.$$

This is always identically zero if $\tau > q$. For $q \geq \tau \geq 0$, the double sum is only non-zero along the diagonal specified by $k = j + \tau$ so $s_\tau = \sigma_\epsilon^2 \sum_{j=0}^{q-\tau} \theta_{j,q}\theta_{j+\tau,q}$.

Now, $s_\tau = s_{-\tau}$, and so the autocovariance sequence is given by

$$s_\tau = \begin{cases} \sigma_\epsilon^2 \sum_{j=0}^{q-|\tau|} \theta_{j,q}\theta_{j+|\tau|,q}, & \text{if } |\tau| \leq q, \\ 0, & \text{if } |\tau| > q. \end{cases}$$

4

(ii) From (i) sim. seen ⇓

$$s_\tau = \begin{cases} \sigma_\epsilon^2(1+\theta_{2,2}^2) & \tau = 0, \\ -\sigma_\epsilon^2 \theta_{2,2}, & |\tau| = 2, \\ 0, & \text{otherwise.} \end{cases}$$

So,

$$\begin{aligned} \text{var}\{\bar{X}\} &= \text{var}\{\frac{1}{4}(X_1 + X_2 + X_3 + X_4)\} \\ &= \frac{1}{16}[\text{var}\{X_1\} + \text{var}\{X_2\} + \text{var}\{X_3\} + \text{var}\{X_4\} \\ &+ \ 2\,\text{cov}\{X_1, X_3\} + 2\,\text{cov}\{X_2, X_4\}] = \frac{1}{4}[s_0 + s_2]. \end{aligned}$$

(The other covariances have lags of 1 or 3 and so are zero.) Now $s_0 = \text{var}\{X_t\} = \sigma_\epsilon^2(1+\theta_{2,2}^2) = 1.64\sigma_\epsilon^2$ and $s_2 = -0.8\sigma_\epsilon^2$ so that $\text{var}\{\bar{X}\} = \sigma_\epsilon^2[1.64 - 0.8]/4 = 0.21\sigma_\epsilon^2$.

4

(b) sim. seen ⇓

$$E\{Y_t\} = E\{\epsilon_t\}\cos(ct) + E\{\epsilon_{t-1}\}\sin(ct) = 0.$$

For the covariance (which for $\tau = 0$ gives the variance),

$$\begin{aligned} \text{cov}\{Y_t, Y_{t+\tau}\} &= E\{[\epsilon_t \cos(ct) + \epsilon_{t-1}\sin(ct)][\epsilon_{t+\tau}\cos(c[t+\tau]) + \epsilon_{t+\tau-1}\sin(c[t+\tau])\} \\ &= E\{\epsilon_t\epsilon_{t+\tau}\}\cos(ct)\cos(c[t+\tau]) + E\{\epsilon_t\epsilon_{t+\tau-1}\}\cos(ct)\sin(c[t+\tau])] \\ &+ \ E\{\epsilon_{t-1}\epsilon_{t+\tau}\}\sin(ct)\cos(c[t+\tau])] + E\{\epsilon_{t-1}\epsilon_{t+\tau-1}\}\sin(ct)\sin(c[t+\tau])] \\ &= \sigma_\epsilon^2\cos^2(ct)\delta_{0,\tau} + \sigma_\epsilon^2\cos(ct)\sin(c[t+1])]\delta_{1,\tau} \\ &+ \ \sigma_\epsilon^2\sin(ct)\cos(c[t-1])]\delta_{-1,\tau} + \sigma_\epsilon^2\sin^2(ct)\delta_{0,\tau} \\ &= \begin{cases} \sigma_\epsilon^2, & \tau = 0, \\ \sigma_\epsilon^2\sin(ct)\cos(c[t-1])], & \tau = -1, \\ \sigma_\epsilon^2\cos(ct)\sin(c[t+1])], & \tau = 1, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

4

When $c = k\pi, k \in \mathbb{Z}\setminus\{0\}$, ($c$ is non-zero) $\{Y_t\}$ is stationary and $s_{Y,\tau} = \sigma_\epsilon^2\delta_{0,\tau}$. For other values of c the process $\{Y_t\}$ depends on t and is hence non-stationary.

2

(c) (i) $\boxed{\text{unseen} \Downarrow}$

$$
\begin{aligned}
s_{Y,\tau} &= E\{Y_t Y_{t+\tau}\} - E\{Y_t\}E\{Y_{t+\tau}\} \\
&= E\{X_t X_{t-1} X_{t+\tau} X_{t+\tau-1}\} - E\{X_t X_{t-1}\}E\{X_{t+\tau} X_{t+\tau-1}\} \\
&= E\{X_t X_{t-1}\}E\{X_{t+\tau} X_{t+\tau-1}\} + E\{X_t X_{t+\tau}\}E\{X_{t-1} X_{t+\tau-1}\} \\
&+ E\{X_t X_{t+\tau-1}\}E\{X_{t-1} X_{t+\tau}\} - E\{X_t X_{t-1}\}E\{X_{t+\tau} X_{t+\tau-1}\} \\
&= \operatorname{cov}\{X_t, X_{t+\tau}\} \operatorname{cov}\{X_{t-1}, X_{t+\tau-1}\} + \operatorname{cov}\{X_t, X_{t+\tau-1}\} \operatorname{cov}\{X_{t-1}, X_{t+\tau}\} \\
&= s_{X,\tau}^2 + s_{X,\tau-1} s_{X,\tau+1}.
\end{aligned}
$$

$\boxed{4}$

(ii) For the MA(1) we have from (a)(i) that $\boxed{\text{unseen} \Downarrow}$

$$
s_{X,\tau} = \begin{cases} \sigma_\epsilon^2(1+\theta_{1,1}^2) & \tau = 0, \\ -\sigma_\epsilon^2 \theta_{1,1}, & |\tau| = 1, \\ 0, & \text{otherwise.} \end{cases}
$$

So

$$
s_{Y,\tau} = \begin{cases} \sigma_\epsilon^4(1+3\theta_{1,1}^2+\theta_{1,1}^4) & \tau = 0, \\ \sigma_\epsilon^4 \theta_{1,1}^2, & |\tau| = 1, \\ 0, & \text{otherwise.} \end{cases}
$$

$\boxed{2}$

3. (a) (i) Let $\{x_t\}, \{y_t\}, \{x_{1,t}\}$ and $\{x_{2,t}\}$ be discrete-time sequences. seen ⇓

[1] Scale-preservation: Given a non-zero constant α,

$$L\{\alpha x_t\} = \alpha L\{x_t\}.$$

[2] Superposition:

$$L\{x_{1,t} + x_{2,t}\} = L\{x_{1,t}\} + L\{x_{2,t}\}.$$

[3] Time invariance: If $y_t = L\{x_t\}$ then

$$L\{x_{t+\tau}\} = y_{t+\tau}.$$

3

(ii) unseen ⇓

$$\begin{aligned}
L\{W_t\} &= \sum_{j=-q}^{q} g_j W_{t-j} = \frac{1}{2q+1} \sum_{j=-q}^{q} [\alpha + \beta(t-j)] \\
&= \frac{1}{2q+1} \left[\alpha(2q+1) + \beta \sum_{j=-q}^{q} (t-j) \right] \\
&= \alpha + \frac{\beta}{2q+1} \left[t(2q+1) - \sum_{j=-q}^{q} j \right] \\
&= \alpha + \beta t - \frac{\beta}{2q+1} \left[\sum_{j=1}^{q} j + \sum_{j=1}^{q} -j \right] = W_t.
\end{aligned}$$

4

(iii) $E\{X_t\} = E\{\sum_{j=-q}^{q} g_j \epsilon_{t-j}\} = \sum_{j=-q}^{q} g_j E\{\epsilon_{t-j}\} = 0$. Since $\{\epsilon_t\}$ is white noise,

$$\text{var}\{X_t\} = \sum_{j=-q}^{q} g_j^2 \,\text{var}\{\epsilon_{t-j}\} = \frac{1}{(2q+1)^2} \sum_{j=-q}^{q} \sigma_\epsilon^2 = \frac{\sigma_\epsilon^2}{2q+1}.$$

For large q the variance is small and X_t is close to its mean of zero. 3

(b) (i) seen ⇓

$$X_t - \phi_{1,p} X_{t-1} - \ldots - \phi_{p,p} X_{t-p} = \epsilon_t - \theta_{1,q}\epsilon_{t-1} - \ldots - \theta_{q,q}\epsilon_{t-q}$$

If we write this as

$$X_t - \phi_{1,p} X_{t-1} - \ldots - \phi_{p,p} X_{t-p} = Y_t;$$

$$Y_t = \epsilon_t - \theta_{1,q}\epsilon_{t-1} - \ldots - \theta_{q,q}\epsilon_{t-q},$$

then we have

$$|G_\phi(f)|^2 S_X(f) = S_Y(f),$$

where $G_\phi(f) = 1 - \phi_{1,p} e^{-i2\pi f} - \ldots - \phi_{p,p} e^{-i2\pi f p}$, and

$$S_Y(f) = |G_\theta(f)|^2 S_\epsilon(f),$$

where $G_\theta(f) = 1 - \theta_{1,q}e^{-i2\pi f} - \ldots - \theta_{q,q}e^{-i2\pi fq}$, so that

$$\begin{aligned} S_X(f) &= S_\epsilon(f)\frac{|G_\theta(f)|^2}{|G_\phi(f)|^2} \\ &= \sigma_\epsilon^2 \frac{|1 - \theta_{1,q}e^{-i2\pi f} - \ldots - \theta_{q,q}e^{-i2\pi fq}|^2}{|1 - \phi_{1,p}e^{-i2\pi f} - \ldots - \phi_{p,p}e^{-i2\pi fp}|^2}. \end{aligned}$$

4

(ii) The form of the spectral density is that of an ARMA(1,1) process and thus is of the form unseen $\Downarrow$

$$S(f) = \frac{17 - 8\cos 2\pi f}{13 - 12\cos 2\pi f} = \sigma_\epsilon^2 \frac{(1 + \theta_{1,1}^2 - 2\theta_{1,1}\cos(2\pi f))}{(1 + \phi_{1,1}^2 - 2\phi_{1,1}\cos(2\pi f))}.$$

Consider the moving average part. Since $1 + \theta_{1,1}^2 = 17$ and $2\theta_{1,1} = 8$ we see that a solution is $\theta_{1,1} = 4$. The characteristic polynomial is $1 - \theta_{1,1}z$ which has a root of $z = 1/\theta_{1,1}$. So this solution does not produce an invertible process since the root is inside the unit circle. However, an inversion of the root does provide an invertible process, so take $\theta_{1,1} = 1/4$ [idea seen in class]. Then 2

$$S(f) = 16\Big[\frac{(17/16) - (8/16)\cos 2\pi f}{13 - 12\cos 2\pi f}\Big].$$

In a similar way we can take out a scaling term from the denominator. But which? We require $1 + \phi_{1,1}^2 = 13a$ and $2\phi_{1,1} = 12a$ where a is the scale factor. Hence $\phi_{1,1} = 6a$, and therefore $1 + 36a^2 = 13a$ so that we need solutions of $36a^2 - 13a + 1 = 0$. The two solutions are $a = 1/4$ and $a = 1/9$. The latter is what we seek since $|\phi_{1,1}| = 6a = 2/3 < 1$. So we get

$$S(f) = (16/9)\Big[\frac{(17/16) - (8/16)\cos 2\pi f}{(13/9) - (12/9)\cos 2\pi f}\Big].$$

The model is $X_t - (2/3)X_{t-1} = \epsilon_t - (1/4)\epsilon_{t-1}$, with $\sigma_\epsilon^2 = (16/9)$.

4

4. (a) (i) Write down the defining equation

sim. seen $\Downarrow$

$$X_t = \phi_{1,p} X_{t+1} + \phi_{2,p} X_{t+2} + \ldots \phi_{p,p} X_{t+p} + \epsilon_t$$

for $t = 1, \ldots, N-p$. This gives $\boldsymbol{X}_B = \boldsymbol{B}\boldsymbol{\phi} + \boldsymbol{\epsilon}_B$, where,

$$\boldsymbol{X}_B = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_{N-p} \end{bmatrix}; \quad \boldsymbol{B} = \begin{bmatrix} X_2 & X_3 & \ldots & X_{p+1} \\ X_3 & X_4 & \ldots & X_{p+2} \\ \vdots & & & \vdots \\ X_{N-p+1} & X_{N-p+2} & \ldots & X_N \end{bmatrix}; \quad \boldsymbol{\epsilon}_B = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_{N-p} \end{bmatrix}.$$

$\boxed{4}$

(ii) Here $N = 4$ and the order $p = 1$ so $N - p = 3$. Write

$$\begin{aligned} X_1 &= \phi X_2 + \epsilon_1 \\ X_2 &= \phi X_3 + \epsilon_2 \\ X_3 &= \phi X_4 + \epsilon_3 \end{aligned}$$

So $\boldsymbol{X}_B = [X_1, X_2, X_3]^T$ and $\boldsymbol{B} = [X_2, X_3, X_4]^T$ and $\boldsymbol{\epsilon}_B = [\epsilon_1, \epsilon_2, \epsilon_3]^T$. Then $\boldsymbol{X}_B = \phi \boldsymbol{B} + \boldsymbol{\epsilon}_B$ and so the backward least-squares estimate is

$$\widehat{\phi} = (\boldsymbol{B}^T\boldsymbol{B})^{-1}\boldsymbol{B}^T\boldsymbol{X}_B = \frac{\sum_{t=2}^4 X_t X_{t-1}}{\sum_{t=2}^4 X_t^2} = \frac{X_1X_2 + X_2X_3 + X_3X_4}{X_2^2 + X_3^2 + X_4^2} = 0.8.$$

The same estimate is obtained by minimizing $\sum_{t=1}^3 (X_t - \phi X_{t+1})^2$.

Since $N - 2p = 4 - 2 = 2$, the estimate of the innovations variance is

$$\widehat{\sigma}_\epsilon^2 = \frac{1}{2}\sum_{t=1}^3 (X_t - \widehat{\phi} X_{t+1})^2 = \frac{1}{2}[(0.4)^2 + 2^2 + (0.8)^2] = 2.4.$$

$\boxed{4}$

(iii) A stationary Gaussian AR(p) process has a time-reversed formulation which has equal validity to the standard (forward) formulation. It is not surprising therefore that forward/backward least squares, which takes both formulations equally into account, is better than either forward least squares or backward least squares by themselves.

$\boxed{2}$

seen ⇓

(b) (i) Two real-valued discrete time stochastic processes $\{Y_t\}$ and $\{Z_t\}$ are said to be jointly stationary stochastic processes if $\{Y_t\}$ and $\{Z_t\}$ are each, separately, second-order stationary processes, and cov$\{Y_t, Z_{t+\tau}\}$ is a function of τ only. 2

(ii) Since $\{Y_t\}$ and $\{Z_t\}$ are zero mean jointly stationary we have unseen ⇓

$$s_{YZ,\tau} = E\{Y_t Z_{t+\tau}\} = aE\{X_{t-d}X_{t+\tau-\ell}\}$$

because $\{X_t\}$ and $\{\epsilon_t\}$ are uncorrelated processes. So

$$s_{YZ,\tau} = as_{X,\tau-(\ell-d)}.$$

Now $s_{X,\tau}$ has a maximum at $\tau = 0$ so $s_{YZ,\tau}$ has a maximum at $\tau = \ell - d$. 2

(iii)

$$\begin{aligned} S_{YZ}(f) &= \sum_{\tau=-\infty}^{\infty} s_{YZ,\tau}\mathrm{e}^{-\mathrm{i}2\pi f\tau} = a\sum_{\tau=-\infty}^{\infty} s_{X,\tau-(\ell-d)}\mathrm{e}^{-\mathrm{i}2\pi f\tau} \\ &= a\mathrm{e}^{-\mathrm{i}2\pi f(\ell-d)}\sum_{\tau=-\infty}^{\infty} s_{X,\tau-(\ell-d)}\mathrm{e}^{-\mathrm{i}2\pi f[\tau-(\ell-d)]} \\ &= a\mathrm{e}^{-\mathrm{i}2\pi f(\ell-d)}S_X(f). \end{aligned}$$

2

If we write $S_{YZ}(f) = |S_{YZ}(f)|\mathrm{e}^{\mathrm{i}(f)}$ then $\theta(f)$ is the phase spectrum. So we see that

$$\theta(f) = -2\pi f(\ell - d) = 2\pi f(d - \ell).$$

[It is not necessary to state that this is modulo 2π.] 2

(iv) The group delay is thus

$$-\frac{1}{2\pi}\frac{\mathrm{d}\theta(f)}{\mathrm{d}f} = \ell - d.$$

1

This agrees with the lag at which $s_{YZ,\tau}$ is a maximum and so does indeed provide a measure of where $s_{YZ,\tau}$ is concentrated in terms of τ, as expected. 1