

Exam 2016 solutions

B—bookwork, E—new example, T—new theory

1.

Students did well.

a)

$$\begin{aligned} Z = X + Y > 0, \quad W = X - Y > 0 \\ x_1 = \frac{z + w}{2}, \quad y_1 = \frac{z - w}{2} \end{aligned}$$

is the only solution. Moreover

$$J = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2 \quad [2E]$$

so that

$$f_{ZW}(z, w) = \frac{1}{|J|} f_{XY}(x_1, y_1) = \frac{1}{2} e^{-(z+w)/2}, \quad 0 < w < z < \infty \quad [2E]$$

$$\begin{aligned} F_Z(z) &= \int_0^z f_{ZW}(z, w) dw = \frac{1}{2} e^{-z/2} \left. \frac{e^{-w/2}}{(-1/2)} \right|_0^z \\ &= \frac{1}{2} e^{-z/2} \left. \frac{e^{-w/2}}{(-1/2)} \right|_0^z = e^{-z/2} (1 - e^{-z/2}), \quad z > 0 \end{aligned} \quad [3E]$$

This is not an exponential distribution. Students were confused. In fact, question ii) should be:

ii) Derive the probability density function of Z.

To get exponential distribution, some students integrated from 0 to infinity and got $e^{-z/2}$.

In any case, students were given full marks.

b) We denote

$$I_k = \begin{cases} 1, & \text{the } k\text{-th post is rotten} \\ 0, & \text{otherwise} \end{cases}$$

$k+1, k+2, \dots, k+7$ are 7 consecutive posts (if $k+i > 17$, then $k+i \Leftarrow k+i-17$)

$$R_k = \sum_{i=1}^7 I_{k+i} \text{ is the number of rotten posts.} \quad [3B]$$

$$\text{So, } E(R_k) = E\left(\sum_{i=1}^7 I_{k+i}\right) = \sum_{i=1}^7 E(I_{k+i}) = 7E(I_k) = \frac{35}{17} > 2$$

$$\text{where } E(I_k) = \frac{5}{17} \text{ (5 of 17 posts are rotten)} \quad [3B]$$

Since R_k should be integer, it must be the case that $P(R_k \geq 3) > 0$, for some k , i.e. there necessarily exists a run of 7 consecutive posts at least 3 of which are rotten.

This is the probabilistic approach given in the tutorial, which most students followed.

Yet, a combinatorial approach is also possible (i.e., without using any probability).

Marks are given as long as your proof is logical.

- c) Part c is a bit tricky. Students did less well.
 Proof is not unique. Marks are given as long as your proof makes sense.

i)

$$\mathbb{E}(X) = \sum_{m=0}^{\infty} m \mathbb{P}(X = m) = \sum_{m=0}^{\infty} \sum_{n=0}^{m-1} \mathbb{P}(X = m) = \sum_{n=0}^{\infty} \sum_{m=n+1}^{\infty} \mathbb{P}(X = m) = \sum_{n=0}^{\infty} \mathbb{P}(X > n). \quad [6T]$$

- ii) Denote by r and b the numbers of red and blue balls, respectively. Let N be the number of balls drawn. From i), we have

$$\begin{aligned} \mathbb{E}(N) &= \sum_{n=0}^r \mathbb{P}(N > n) = \sum_{n=0}^r \mathbb{P}(\text{first } n \text{ balls are red}) \\ &= \sum_{n=0}^r \frac{r}{b+r} \frac{r-1}{b+r-1} \cdots \frac{r-n+1}{b+r-n+1} = \sum_{n=0}^r \frac{r!}{(b+r)!} \frac{(b+r-n)!}{(r-n)!} \end{aligned} \quad [2T]$$

$$= \frac{r!b!}{(b+r)!} \sum_{n=0}^r \binom{n+b}{b} = \frac{b+r+1}{b+1}, \quad [2T]$$

Substituting in $r=10$ and $b=10$, we obtain

$$E(N) = \frac{21}{11} \approx 2 \quad [2T]$$

2. A routine question. Students did well in general.

a) The joint distribution of the samples is given by

$$f_X(x_1, \dots, x_n; \lambda) = \prod_{i=1}^n e^{-\lambda} \frac{\lambda^{x_i}}{x_i!} = e^{-n\lambda} \frac{\lambda^{\sum x_i}}{\prod x_i!}$$

Then the log-likelihood function is

[3E]

$$\ln f_X(x_1, \dots, x_n; \lambda) = -n\lambda + \sum x_i \ln \lambda - \ln \prod x_i!$$

Now take the derivatives:

$$\frac{\partial}{\partial \lambda} \ln f_X(x_1, \dots, x_n; \lambda) = -n + \frac{\sum x_i}{\lambda}$$

[3E]

$$\frac{\partial^2}{\partial \lambda^2} \ln f_X(x_1, \dots, x_n; \lambda) = -\frac{\sum x_i}{\lambda^2}$$

Then the Fisher information is given by

$$I = -E \left[-\frac{\sum x_i}{\lambda^2} \right] = \frac{n\lambda}{\lambda^2} = \frac{n}{\lambda}$$

[2E]

So the Cramer-Rao bound is

$$\text{Var}[\hat{\lambda}] = \frac{1}{I(\lambda)} = \frac{\lambda}{n} = \frac{1}{10} = 0.1$$

[2E]

Similar to a question in coursework. Did well in general.

b) Recall the Wiener-Hopf equation

$$\mathbf{c} = \mathbf{R}^{-1} \mathbf{r}$$

$$\sigma^2 = r_0 - \mathbf{r}^T \mathbf{R}^{-1} \mathbf{r} \quad [2E]$$

i) If $n=1$, the Wiener-Hopf equation trivially reads

$$R_Y(0)c_1 = R_Y(1)$$

Therefore,

$$c_1 = \frac{R_Y(1)}{R_Y(0)} = \frac{J_0(2\pi f_d)}{J_0(0)} = J_0(2\pi f_d) \quad [2E]$$

because $J_0(0) = 1$. Therefore,

$$c_1 = J_0(2\pi \times 0.3) = 0.291 \quad [2E]$$

$$\sigma^2 = 1 - r_1^2 = 1 - 0.291^2 = 0.915$$

ii) When $n = 2$, we have

$$\mathbf{R} = \begin{bmatrix} 1 & 0.291 \\ 0.291 & 1 \end{bmatrix}$$

$$\mathbf{r} = \begin{bmatrix} -0.402 \\ 0.291 \end{bmatrix} \quad [3E]$$

Thus the coefficient vector

$$\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \mathbf{R}^{-1} \mathbf{r} = \begin{bmatrix} 1 & 0.291 \\ 0.291 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -0.402 \\ 0.291 \end{bmatrix} = \begin{bmatrix} -0.532 \\ 0.446 \end{bmatrix} \quad [3E]$$

Mean-square error

$$\sigma^2 = 1 - [-0.402 \quad 0.291] \begin{bmatrix} 1 & 0.291 \\ 0.291 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -0.402 \\ 0.291 \end{bmatrix} = 0.657 \quad [3E]$$

- Did well in general. Surprisingly, a non-negligible number of students stopped here (i.e., didn't simplify to $\cos(m\lambda)$), apparently couldn't use trigonometrical formula. Then, they concluded it is not stationary, which lost mark.

a)

i) It's clear that $E[X(n)] = 0$ and $\text{Var}[X(n)] = 1$. The autocorrelation function [2E]

$$R_X(m, m+n) = \cos(n\lambda) \cos((n+m)\lambda) + \sin(n\lambda) \sin((n+m)\lambda) = \cos(m\lambda) \quad [2E]$$

ii) Clearly it is wide-sense stationary. [2E]

iii) In general, this random process is not strict-sense stationary. To see this, let's consider an example where $\lambda = \frac{\pi}{2}$. Then

$$X(n) = A \cos(n \frac{\pi}{2}) + B \sin(n \frac{\pi}{2}) \quad [2E]$$

Obviously, the samples at different times

$$\begin{aligned} X(1) &= B \\ X(2) &= -A \end{aligned} \quad [2E]$$

have different distributions.

Again, approach is not unique for iii). Another approach, as demonstrated in lecture and used by many students, is to compute the third moment, i.e., $E[X(n_1)X(n_2)X(n_3)]$, and show it is not shift invariant.

b)

i) The autocorrelation function is

$$R_X(k) = \begin{cases} 1 + \alpha^2 & \text{if } k = 0 \\ \alpha & \text{if } |k| = 1 \\ 0 & \text{otherwise} \end{cases} \quad [3E]$$

Hence the power spectral density is

$$S_X(\omega) = R_X(0) + R_X(1)e^{-j\omega} + R_X(-1)e^{j\omega} = 1 + \alpha^2 + 2\alpha \cos(\omega), \quad |\omega| < \pi \quad [3E]$$

Did well in general

ii) In the general case, the autocorrelation function is

$$R_X(k) = \sum_{i=0}^r \alpha_i \alpha_{k+i} \quad [3E]$$

where the convention that $\alpha_s = 0$ if $s < 0$ or $s > r$. Thus,

[3E]

$$\begin{aligned}
S_X(\omega) &= \sum_{k=-\infty}^{\infty} R_X(k) e^{-jk\omega} = \sum_{k=-\infty}^{\infty} \sum_{i=0}^r \alpha_i \alpha_{k+i} e^{-jk\omega} \\
&= \sum_{k=-\infty}^{\infty} \alpha_{k+i} e^{-j(k+i)\omega} \sum_{i=0}^r \alpha_i e^{ji\omega} \\
&= \sum_{k'=0}^r \alpha_{k'} e^{-j(k')\omega} \sum_{i=0}^r \alpha_i e^{ji\omega} = |A(e^{j\omega})|^2, \quad |\omega| < \pi
\end{aligned} \tag{3E}$$

where $A(Z) = \sum_{i=0}^r \alpha_i Z^i$ (the last step holds because that $\alpha_s = 0$ if $s < 0$ or $s > r$).

4. **Standard question for Markov chains.**

a) From the equation $\pi = \pi P$, we obtain

$$\begin{aligned}\pi_1 &= \frac{3}{4}\pi_1 + \frac{1}{4}\pi_2 & \Rightarrow \pi_1 &= \pi_2 \\ \pi_2 &= \frac{1}{4}\pi_1 + \frac{1}{2}\pi_2 + \frac{1}{4}\pi_3 & \Rightarrow \pi_2 &= \pi_3 \\ &\dots \\ \pi_M &= \frac{1}{4}\pi_{M-1} + \frac{3}{4}\pi_M & \Rightarrow \pi_{M-1} &= \pi_M\end{aligned}\quad [3E]$$

So all the π_i 's are equal, and

$$\pi_i = \frac{1}{M} \text{ for all } i. \quad [2E]$$

b) **Part b is bookwork.**

i) $\{S_n\}$ is a martingale. This is because

$$E[S_{n+1}|X_1, \dots, X_n] = E\left[\sum_{i=1}^{n+1} X_i \mid X_1, \dots, X_n\right] = S_n + E[X_{n+1}] = S_n \quad [3B]$$

ii) $\{T_n\}$ is not a martingale. This is because

$$E[T_{n+1}|X_1, \dots, X_n] = E[S_n^2 + 2S_n X_{n+1} + X_{n+1}^2 | X_1, \dots, X_n] = T_n + E[X_{n+1}^2] \geq T_n \quad [4B]$$

Thus, $\{T_n\}$ is not a martingale.

iii) But it's a submartingale. [3B]

c)

i) Of the $2n$ steps, suppose the chain goes upward for i steps, leftward for j steps, and inward for k steps. It returns to the origin if and only if it also goes downward for i steps, rightward for j steps, and outward for k steps. Here we must have $i + j + k = n$.

[2E]

Therefore,

$$P\{X_{2n} = (0,0,0)\} = \left(\frac{1}{6}\right)^{2n} \sum_{i+j+k=n} \frac{(2n)!}{(i! j! k!)^2} \quad [2E]$$

ii) The above formula may be rewritten as

$$P\{X_{2n} = (0,0,0)\} = \left(\frac{1}{2}\right)^{2n} \binom{2n}{n} \sum_{i+j+k=n} \frac{(n)!}{3^n (i! j! k!)} \frac{(n)!}{3^n (i! j! k!)} \quad [2T]$$

$$\leq \left(\frac{1}{2}\right)^{2n} \binom{2n}{n} M \sum_{i+j+k=n} \frac{(n)!}{3^n (i! j! k!)} \quad (*)$$

Part c is hard. Very few students got the correct answer.

where

$$M = \max\left\{ \sum_{i+j+k=n} \frac{(n)!}{3^n (i! j! k!)} \right\} \approx \frac{(n)!}{3^n \left(\left(\frac{n}{3}\right)!\right)^3}$$

Further, the sum in (*) equals 1, since the summands form a probability distribution. It follows that

$$P\{X_{2n} = (0,0,0)\} \leq \left(\frac{1}{12}\right)^n \frac{(2n)!}{n! \left(\left(\frac{n}{3}\right)!\right)^3} \quad [2T]$$

Using Stirling's formula, we obtain

$$P\{X_{2n} = (0,0,0)\} \leq \left(\frac{1}{12}\right)^n \frac{\sqrt{2\pi(2n)} \left(\frac{2n}{e}\right)^{2n}}{\sqrt{2\pi(n)} \left(\frac{n}{e}\right)^n \left(\sqrt{2\pi \left(\frac{n}{3}\right)} \left(\frac{n}{3}\right)^{\frac{n}{3}} \left(\frac{n}{e}\right)^{\frac{n}{3}}\right)^3} = C n^{-\frac{3}{2}}$$

for some constant C . Finally, we find that

$$\sum_n P\{X_{2n} = (0,0,0)\} = C \sum_{n>1} n^{-\frac{3}{2}} < C \zeta\left(\frac{3}{2}\right) < \infty$$

because the sum converges. Here $\zeta(s)$ is the Riemann zeta function, which is well-known to converge for $s > 1$. We therefore conclude that the origin is a transient state.

Of course, this is not required; what matters is to recognize that the series converges.

PS. The famous Riemann hypothesis, considered one of the greatest unsolved problems in mathematics, asserts that any non-trivial zero s of $\zeta(s)$ on the complex plane has $\text{Re}(s) = 1/2$.