EE4-54

EE9CS4-1

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2012

MSc and EEE/ISE PART IV: MEng and ACGI

PREDICTIVE CONTROL

Friday, 18 May 10:00 am

Time allowed: 3:00 hours

There are SIX questions on this paper.

Answer FOUR questions.

All questions carry equal marks.

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s) :	E.C. Kerrigan
	Second Marker(s) :	S. Evangelou

1. a) Formulate the following problem as a linear program:

$$\min_{\theta} \|C\theta - d\|_\infty$$

where $C \in \mathbb{R}^{m\times n}$, $\theta \in \mathbb{R}^n$, $d \in \mathbb{R}^m$. [8]

b) Consider the following finite-horizon discrete-time optimal control problem:

$$\min_{u_0,u_1,\ldots,u_{N-1}} \sum_{k=0}^{N-1} (\|Qx_k\|_\infty + \|Ru_k\|_\infty)$$

where the system dynamics are given by

$$x_{k+1} = Ax_k + Bu_k, \quad k = 0,1,\ldots,N-1,$$

where the states $x_k \in \mathbb{R}^n$, inputs $u_k \in \mathbb{R}^m$ and weighting matrices $Q \in \mathbb{R}^{p\times n}$ and $R \in \mathbb{R}^{q\times m}$. N is the horizon length and an estimate of the current state $x_0 = \hat{x}$ is given.

Formulate the above optimal control problem as an equivalent linear program with inequality constraints only. Pay particular attention to defining the sizes of the various matrices and vectors that define the optimisation problem. [12]

2. Consider the discrete-time system given by

$$x_{k+1} = Ax_k + Bu_k, \quad k = 0, 1, \ldots, N-1,$$

where the states $x_k \in \mathbb{R}^n$, inputs $u_k \in \mathbb{R}^m$ and an estimate of the current state $x_0 = \hat{x}$ is given. N is the horizon length.

We also define the following vectors:

$$\overline{x} := \begin{bmatrix} x_0 \\ \vdots \\ x_N \end{bmatrix}, \quad \overline{u} := \begin{bmatrix} u_0 \\ \vdots \\ u_{N-1} \end{bmatrix}.$$

a) Show that

$$\left(I_{(N+1)n} - \begin{bmatrix} 0 & 0 \\ I_N \otimes A & 0 \end{bmatrix} \right) \overline{x} = \begin{bmatrix} I_n \\ 0 \end{bmatrix} \hat{x} + \begin{bmatrix} 0 \\ I_N \otimes B \end{bmatrix} \overline{u},$$

where 0 represents a zero matrix of compatible size. [10]

b) Show that

$$\left(I_{(N+1)n} - \begin{bmatrix} 0 & 0 \\ I_N \otimes A & 0 \end{bmatrix} \right)^{-1} = \begin{bmatrix} I_n & 0 & \cdots & \cdots & 0 \\ A & I_n & \ddots & \ddots & \vdots \\ A^2 & A & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ A^N & A^{N-1} & \cdots & A & I_n \end{bmatrix}.$$

[6]

c) Using the results in parts a) and b), show that

$$\overline{x} = \begin{bmatrix} I_n \\ A \\ A^2 \\ \vdots \\ A^N \end{bmatrix} \hat{x} + \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ B & \ddots & \ddots & \vdots \\ AB & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ A^{N-1}B & \cdots & AB & B \end{bmatrix} \overline{u}.$$

[4]

3. We are interested in solving the following optimal control problem :

$$\min_{u_0,\dots,u_{N-1}} \sum_{k=0}^{N-1} \left(\|Qx_{k+1}\|_2^2 + \|Ru_k\|_2^2\right)$$

subject to the constraints

$$\begin{aligned} x_0 &= \hat{x}, \\ x_{k+1} &= Ax_k + Bu_k, \quad k = 0,1,\dots,N-1 \\ \delta_\ell \le u_k - u_{k-1} &\le \delta_h, \quad k = 0,1,\dots,N-1 \end{aligned}$$

where the states $x_k \in \mathbb{R}^n$, inputs $u_k \in \mathbb{R}^m$and weights $Q \in \mathbb{R}^{p\times n}$ and $R \in \mathbb{R}^{m\times m}$. N is the horizon length. The previous value of the input u_{-1} is known and an estimate of the current state $\hat{x}$ is given.

We also define the following vectors:

$$\overline{x} := \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}, \quad \overline{u} := \begin{bmatrix} u_0 \\ \vdots \\ u_{N-1} \end{bmatrix}.$$

a) Give a condition on R such that the optimal control problem has a unique solution. [2]

b) Show that

$$\sum_{k=0}^{N-1} \left(\|Qx_{k+1}\|_2^2 + \|Ru_k\|_2^2\right) = \|(I_N \otimes Q)\overline{x}\|_2^2 + \|(I_N \otimes R)\overline{u}\|_2^2.$$

[4]

c) Give an expression for the matrix C and vector d such that

$$\sum_{k=0}^{N-1} \left(\|Qx_{k+1}\|_2^2 + \|Ru_k\|_2^2\right) = \|C\overline{u} - d\|_2^2.$$

[6]

d) Give an expression for the matrix E and vector f such that one can solve the above optimal control problem by solving the constrained least squares problem

$$\min_{\overline{u}} \|C\overline{u} - d\|_2^2$$

subject to

$$E\overline{u} \le f.$$

[8]

4. a) Suppose we are given a set of inequality constraints $c(\theta) \leq 0$ and equality constraints $d(\theta) = 0$, where $\theta \in \mathbb{R}^n$ and the functions $c : \mathbb{R}^n \to \mathbb{R}^m$ and $d : \mathbb{R}^n \to \mathbb{R}^p$ are differentiable.

Assume that a feasible point exists such that the equality constraints can be satisfied. Show that one can determine whether a point θ exists that also satisfies the inequality constraints by solving a suitably-defined optimisation problem of the form

$$\min_x f(x)$$

subject to

$$g(x) \leq 0, \quad h(x) = 0,$$

where f, g and h are all differentiable functions. [10]

b) Suppose that the inequality constraints are in the form

$$c(\theta) := \begin{bmatrix} c_H(\theta) \\ c_S(\theta) \end{bmatrix} \leq 0,$$

were $c_H(\theta) \leq 0$ represents the hard constraints and $c_S(\theta) \leq 0$ represents q soft constraints.

Referring to your answer in part a), what would you do if a feasible point does not exist and you want to find a point that minimises the worst case violation of the soft constraints, but still satisfies the hard constraints and equality constraints? [4]

c) Explain how hard and soft constraints arise in predictive control. Show how you would define θ, $c(\cdot)$ and $d(\cdot)$ so that you can use your answers in parts a) and b) to compute an input sequence that minimises the worst case violation of the soft constraints over the whole prediction horizon, if the system and constraints are linear? [6]

5. a) Give a sufficient condition on the matrix M for which one can guarantee that the system of linear equations $M\theta = d$ has a solution for any vector d. Justify your condition using mathematical arguments. [4]

b) Discuss, with the aid of a block diagram and equations, how you would implement a control scheme so that a given linear combination of the states can track constant, non-zero setpoints. Assume the dynamical system is linear and discrete-time, the setpoints are such that they can be tracked, measurements of all states are available and a stabilising linear state feedback gain matrix K is given. [6]

c) Recall that a system is defined to be reachable if there exists an input sequence that can take the system from any state to any other state in finite time. Give an example of a reachable linear discrete-time system for which it is not possible for all the states to track arbitrary constant, non-zero setpoints. Use detailed mathematical arguments to justify your example. [10]

6. a) State the Popov-Bellevitch-Hautus (PBH) detectability test for the discrete-time linear system

$$x_{k+1} = Ax_k + Bu_k, \quad y_k = Cx_k,$$

where the states $x_k \in \mathbb{R}^n$, inputs $u_k \in \mathbb{R}^m$ and measured outputs $y_k \in \mathbb{R}^p$. [4]

b) Suppose you now have constant, unknown disturbances $d \in \mathbb{R}^\ell$ acting on the system such that

$$x_{k+1} = Ax_k + Bu_k + B_d d, \quad y_k = Cx_k.$$

Derive a necessary and sufficient condition such that one can construct a stable observer to estimate the disturbances. [8]

c) Referring to your answer in part b), explain why a necessary condition for a stabilising observer to exist is that there should be at least as many measured variables as disturbances. [2]

d) Assume the system in part b) is stabilizable. Derive a necessary and sufficient condition on the system and disturbances in part b) such that one can find an input sequence that asymptotically drives the state of the system to zero. [6]

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Predictive Control - Solutions 2012

EE4-54
EE9CS4-1

Question 1

(a) Bookwork.

$$\min_{\theta,t} \left\{ t \;\middle|\; -1_m t \le C\theta - d \le 1_m t \right\}$$

$$= \min_{\binom{\theta}{t}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}^T \begin{pmatrix} \theta \\ t \end{pmatrix} \quad \text{s.t.} \begin{cases} C\theta - d \le 1_m t \\ -C\theta + d \le +1_m t \end{cases}$$

$$= \min_{\binom{\theta}{t}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}^T \begin{pmatrix} \theta \\ t \end{pmatrix} \quad \text{s.t.} \begin{pmatrix} C & -1_m \\ -C & -1_m \end{pmatrix} \begin{pmatrix} \theta \\ t \end{pmatrix} \le \begin{pmatrix} d \\ -d \end{pmatrix}$$

1_m is a column vector of m ones.

(b) New problem.

~~Cost function becomes~~ Ignoring the equality constraints for now, ~~the~~ we get

$$\min_{\substack{u_0,\dots,u_{N-1} \\ t_0,\dots,t_{N-1} \\ s_0,\dots,s_{N-1}}} \sum_{k=0}^{N-1} (t_k + s_k) \quad \text{s.t.} \quad \begin{aligned} -1_p t_k &\le Qx_k \le 1_p t_k \\ -1_q s_k &\le Ru_k \le 1_q s_k \\ k &= 0,1,\dots,N-1 \end{aligned}$$

$$= \min_{\bar{u},\bar{s},\bar{t}} 1_N^T \bar{t} + 1_N^T \bar{s} \quad \text{s.t.} \quad \begin{aligned} -(I_N \otimes 1_p)\bar{t} &\le (I_N \otimes Q)\bar{x} \\ (I_N \otimes Q)\bar{x} &\le (I_N \otimes 1_p)\bar{t} \\ -(I_N \otimes 1_q)\bar{s} &\le (I_N \otimes R)\bar{u} \\ (I_N \otimes R)\bar{u} &\le (I_N \otimes 1_q)\bar{s} \end{aligned}$$

where $\bar{x} = \begin{pmatrix} x_0 \\ \vdots \\ x_{N-1} \end{pmatrix}$, $\bar{u} = \begin{pmatrix} u_0 \\ \vdots \\ u_{N-1} \end{pmatrix}$, $\bar{t} = \begin{pmatrix} t_0 \\ \vdots \\ t_{N-1} \end{pmatrix}$, $\bar{s} = \begin{pmatrix} s_0 \\ \vdots \\ s_{N-1} \end{pmatrix}$

Equality constraints are $\bar{x} = \Phi \hat{x} + \Gamma \bar{u}$, where

$$\Phi = \begin{pmatrix} I \\ A \\ \vdots \\ A^{N-1} \end{pmatrix}, \quad \Gamma = \begin{pmatrix} 0 & 0 & & 0 \\ B & 0 & & \vdots \\ AB & B & & \\ \vdots & \vdots & \ddots & \\ A^{N-2}B & \cdots & B & 0 \end{pmatrix}.$$

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1b) continued... $\frac{2}{12}$

Inequality constraints now become:

$$
\begin{aligned}
-(I_N \otimes Q)(\Phi \hat{x} + \Gamma \bar{u}) - (I_N \otimes 1_p)\,\bar{t} &\le 0\\
+(I_N \otimes Q)(\Phi \hat{x} + \Gamma \bar{u}) - (I_N \otimes 1_p)\,\bar{t} &\le 0\\
-(I_N \otimes R)\,\bar{u} - (I_N \otimes 1_q)\,\bar{s} &\le 0\\
+(I_N \otimes R)\,\bar{u} - (I_N \otimes 1_q)\,\bar{s} &\le 0
\end{aligned}
$$

$$
\Longleftrightarrow \quad \underbrace{\begin{pmatrix} -(I_N \otimes Q)\Gamma & -I_N \otimes 1_p & 0 \\ +(I_N \otimes Q)\Gamma & -I_N \otimes 1_p & 0 \\ -(I_N \otimes R) & 0 & -I_N \otimes 1_q \\ +(I_N \otimes R) & 0 & -I_N \otimes 1_q \end{pmatrix} \begin{pmatrix} \bar{u} \\ \bar{s} \\ \bar{t} \end{pmatrix} \le \begin{pmatrix} (I_N \otimes Q)\Phi \hat{x} \\ -(I_N \otimes Q)\Phi \hat{x} \\ 0 \\ 0 \end{pmatrix}}
$$

~~Cost function is~~ LP is $\min\limits_{\begin{pmatrix}\bar{u}\\ \bar{s}\\ \bar{t}\end{pmatrix}} \begin{pmatrix} 0_{Nm} \\ 1_N \\ 1_N \end{pmatrix}^T \begin{pmatrix} \bar{u} \\ \bar{s} \\ \bar{t} \end{pmatrix}$ s.t.

$\longrightarrow$.

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$\frac{7}{12}$

Question 2

(a) ~~Equality~~ New problem.

Equality constraints become:

$$x_0 = \hat{x}$$
$$x_1 = Ax_0 + Bu_0$$
$$x_2 = Ax_1 + Bu_1$$
$$\vdots$$
$$x_N = Ax_{N-1} + Bu_{N-1}$$

$$\Leftrightarrow \begin{array}{llll} x_0 & & & = \hat{x} \\ -Ax_0 + x_1 & & & = Bu_0 \\ & -Ax_1 + x_2 & & = Bu_1 \\ & \ddots & & \ddots \\ & & -Ax_{N-1} + x_N & = Bu_{N-1} \end{array}$$

$$\Rightarrow \begin{pmatrix} I_n & & \\ & I_n & \\ & & \ddots & \\ & & & I_n \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_N \end{pmatrix} + \begin{pmatrix} 0 & 0 & \cdots & 0 \\ -A & & & \vdots \\ & -A & & \\ 0 & & \ddots \; -A & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_N \end{pmatrix}$$

$$= \begin{pmatrix} I_n \\ 0 \\ \vdots \\ 0 \end{pmatrix} \hat{x} + \begin{pmatrix} 0 & \cdots & 0 \\ B & & \vdots \\ & \ddots & 0 \\ 0 & & B \end{pmatrix} \begin{pmatrix} u_0 \\ \vdots \\ u_{N-1} \end{pmatrix}$$

$$\Leftrightarrow I_{(N+1)n}\bar{x} - \begin{pmatrix} 0 & 0 \\ I_N \otimes A & 0 \end{pmatrix} \bar{x} = \begin{pmatrix} I_n \\ 0 \end{pmatrix} \hat{x} + \begin{pmatrix} 0 \\ I_N \otimes B \end{pmatrix} \bar{u}$$

$\longrightarrow$

(b) ~~Show that~~

$$\left[I - \begin{pmatrix} 0 & 0 \\ I_N \otimes A & 0 \end{pmatrix} \right] = \begin{pmatrix} I & 0 & & \\ -A & I & & \\ 0 & -A & I & \\ & & \ddots & 0 \\ & & 0 & -A \; I \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} I & 0 & & 0 \\ -A & I & & \vdots \\ 0 & -A & I & 0 \\ 0 & 0 & -A & I \end{pmatrix} \begin{pmatrix} I & 0 & & 0 \\ A & I & & \vdots \\ \vdots & \vdots & \ddots & 0 \\ A^N & A^{N-1} & & I \end{pmatrix} = \underbrace{\begin{pmatrix} I & 0 & 0 & \cdots & 0 \\ -A+A & I & 0 & & \\ -A^2+A^2 & -A+A & & \ddots & \vdots \\ \vdots & \vdots & & & \\ -A^{N-1}+A^{N-1} & -A^{N-2}+A^{N-2} & & & I \end{pmatrix}}_{= I}$$

$\longrightarrow$

(c) Using (a),

$$\bar{x} = \begin{bmatrix} I_{(N+1)n} - \begin{matrix} 0 & 0 \\ I_N \otimes A & 0 \end{matrix} \end{bmatrix}^{-1} \left(\begin{bmatrix} I_n \\ 0 \end{bmatrix} \hat{x} + \begin{bmatrix} 0 \\ I_N \otimes B \end{bmatrix} \bar{u} \right)$$

$$= \begin{bmatrix} I & 0 & \cdots & & 0 \\ A & I & & & \vdots \\ A^2 & A & \ddots & & \\ \vdots & & & \ddots & 0 \\ A^N & A^{N-1} & \cdots & A & I \end{bmatrix} \left(\begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix} \hat{x} + \begin{pmatrix} 0 & \cdots & & 0 \\ B & 0 & \cdots & \\ 0 & B & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & & B \end{pmatrix} \bar{u} \right)$$

$$= \begin{pmatrix} I \\ A \\ A^2 \\ \vdots \\ A^N \end{pmatrix} \hat{x} + \begin{bmatrix} 0 & \cdots & & & 0 \\ B & & & & \vdots \\ AB & & & & \vdots \\ \vdots & & & & \\ & & & & 0 \\ A^{N-1}B & & \cdots & AB & B \end{bmatrix} \bar{u}$$

by just multiplying blocks.

2012

$\frac{5}{12}$

Question 3

(a) Book-work. R full column rank $\Leftrightarrow$ R positive definite.

(b) New problem, slightly different than one done in class.

$$\sum_{k=0}^{N-1} \|Q x_{k+1}\|_2^2 = \left\| \begin{pmatrix} Qx_1 \\ Qx_2 \\ \vdots \\ Qx_N \end{pmatrix} \right\|_2^2 = \left\| \begin{pmatrix} Q & & 0 \\ & Q & \\ 0 & & \ddots \; Q \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} \right\|_2^2$$

$$= \|(I_N \otimes Q)\bar{x}\|_2^2$$

$$\sum_{k=0}^{N-1} \|R u_k\|_2^2 = \left\| \begin{pmatrix} Ru_0 \\ \vdots \\ Ru_{N-1} \end{pmatrix} \right\|_2^2 = \left\| \begin{pmatrix} R & & 0 \\ & R \ddots & \\ 0 & & R \end{pmatrix} \begin{pmatrix} u_0 \\ \vdots \\ u_{N-1} \end{pmatrix} \right\|_2^2$$

$$= \|(I_N \otimes R)\bar{u}\|_2^2$$

~~Combi~~ Adding the above, we get the result.

(c) Can write. $\bar{x} = \Phi \hat{x} + \Gamma \bar{u}$, $\Phi = \begin{pmatrix} A \\ \vdots \\ A^N \end{pmatrix}$

$$\Gamma = \begin{pmatrix} B & 0 & \cdots & 0 \\ AB & B & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ A^{N-1}B & \cdots & AB & B \end{pmatrix}$$

$$\Rightarrow \|(I_N \otimes Q)\bar{x}\|_2^2 + \|(I_N \otimes R)\bar{u}\|_2^2 = \left\| \begin{pmatrix} (I_N \otimes Q)\bar{x} \\ (I_N \otimes R)\bar{u} \end{pmatrix} \right\|_2^2$$

$$= \left\| \begin{pmatrix} (I_N \otimes Q)(\Phi\hat{x} + \Gamma\bar{u}) \\ (I_N \otimes R)(\bar{u}) \end{pmatrix} \right\|_2^2 = \left[\begin{pmatrix} (I_N \otimes Q)\Gamma \\ I_N \otimes R \end{pmatrix} \bar{u} - \begin{pmatrix} -I_N \otimes Q\Phi \\ 0 \end{pmatrix} \right.,$$

note

$$\Rightarrow C = \begin{bmatrix} (I_N \otimes Q)\Gamma \\ I_N \otimes R \end{bmatrix}, \quad d = \begin{pmatrix} -(I_N \otimes Q)\Phi\hat{x} \\ 0 \end{pmatrix}$$

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3 d) New problem.

Inequality constraints become: Note: u_{-1} given/known.

~~$\delta l \le u_0$~~

$$\begin{aligned} \delta l + u_{-1} &\le u_0 \le \delta h - u_{-1} \\ \delta l &\le u_1 - u_0 \le \delta h \\ &\vdots \\ \delta l &\le u_{N-1} - u_{N-2} \le \delta h. \end{aligned}$$

$\Longrightarrow$

$$\begin{pmatrix} \delta l \\ \delta l \\ \vdots \\ \delta l \end{pmatrix} + \begin{pmatrix} u_{-1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \le \begin{pmatrix} I & & & \\ -I & I & & \\ & -I & I & \\ & & \ddots & \ddots \\ & & -I & I \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-2} \\ u_{N-1} \end{pmatrix} \le \begin{pmatrix} \delta h \\ \delta h \\ \delta h \\ \vdots \\ \delta h \end{pmatrix} - \begin{pmatrix} u_{-1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$\Longrightarrow$

$$1_N \otimes \delta_l + \begin{pmatrix} u_{-1} \\ 0 \end{pmatrix} \le \left(I - \begin{bmatrix} 0 & 0 \\ I_{(N-1)m} & 0 \end{bmatrix} \right) \bar{u} \le 1_N \otimes \delta_h - \begin{pmatrix} u_{-1} \\ 0 \end{pmatrix}$$

$$\Longleftrightarrow \underbrace{\begin{bmatrix} I - \begin{bmatrix} 0 & 0 \\ I_{(N-1)m} & 0 \end{bmatrix} \\ -I + \begin{bmatrix} 0 & 0 \\ I_{(N-1)m} & 0 \end{bmatrix} \end{bmatrix}}_{E} \bar{u} \le \underbrace{\begin{pmatrix} 1_N \otimes \delta_h - \begin{pmatrix} u_{-1} \\ 0 \end{pmatrix} \\ -1_N \otimes \delta_l - \begin{pmatrix} u_{-1} \\ 0 \end{pmatrix} \end{pmatrix}}_{f}$$

$\longrightarrow$

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Question 4 $\frac{7}{12}$

Mostly application of bookwork.

(a)

~~[illegible]~~ We would introduce a slack variable t and solve the problem.

$$t^* := \min_{\theta, t} \; t \quad \text{s.t.} \quad c(\theta) \leq 1_m t \;, \quad d(\theta) = 0$$
$$t \geq 0$$

~~any [illegible]~~

$$\iff \min_{\binom{\theta}{t}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}^T \begin{pmatrix} \theta \\ t \end{pmatrix} \quad \text{s.t.} \quad c(\theta) - 1_m t \leq 0$$
$$-t \leq 0$$
$$d(\theta) = 0$$

i.e. if we let $x := \begin{pmatrix} \theta \\ t \end{pmatrix}$,

$$f(x) := \begin{pmatrix} 0 \\ 1 \end{pmatrix}^T \begin{pmatrix} \theta \\ t \end{pmatrix}$$
$$g(x) := \begin{pmatrix} c(\theta) - 1_m t \\ t \end{pmatrix}$$
$$h(x) := d(\theta)$$

A feasible point exists $\iff t^* = 0$

(b) Solve the following problem instead:

$$t^* := \min_{(\theta, t)} \; t \quad \text{s.t.} \quad C_H(\theta) \leq 0$$
$$C_S(\theta) \leq 1_q t$$
$$+t \geq 0$$
$$d(\theta) = 0 \,.$$

(c) Hard constraints often arise due to physical limits, such as actuator constraints of the form:

$$u_l \leq u_k \leq u_h \;, \quad k = 0, 1, \ldots, N-1$$

4 c) continued...

$\frac{8}{12}$

Soft constraints are usually on the outputs and represent desirable limits that we would ~~be~~ like to satisfy, but which might be violated, if necessary, e.g.

$$y_l \leq y_k \leq y_h, \quad k = 1, \ldots, N$$

If our system dynamics are of the form:

$$x_{k+1} = A x_k + B u_k, \quad k = 0, \ldots, N-1 \quad (I)$$
$$y_k = C x_k, \quad k = 1, \ldots, N, \quad (II)$$

then we can define $\Theta = \begin{pmatrix} x_1 \\ \vdots \\ x_N \\ u_0 \\ \vdots \\ u_{N-1} \end{pmatrix}$, x_0 given

and define a suitable function $d(\Theta)$ to implement (I).

One can convert the input constraints above into the form
$$u_k - u_h \leq 0, \quad k = 0, \ldots, N-1$$
$$-u_k + u_l \leq 0$$
and use the left hand side to define $G_H(\Theta)$.

Similarly with the output constraints:
$$C x_k - y_h \leq 0$$
$$-C x_k + y_l \leq 0$$
and use the left hand side to define $G_S(\Theta)$.

Any sensible answer similar to above will be acceptable.

$\longrightarrow$

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Question 5

(a) Bookwork. ~~R M should be full row rank~~.
If M is full row rank $\Rightarrow$ a solution exists for any d. This is because then $\operatorname{rank} M = \operatorname{rank}(M \mid d)$ ~~for any~~ which is a necessary and sufficient condition for $M\Theta = d$ to have a solution.

(b) ~~For~~ Consider the system: $x_{k+1} = Ax_k + Bu_k$.
$y_k = Cx_k$. output

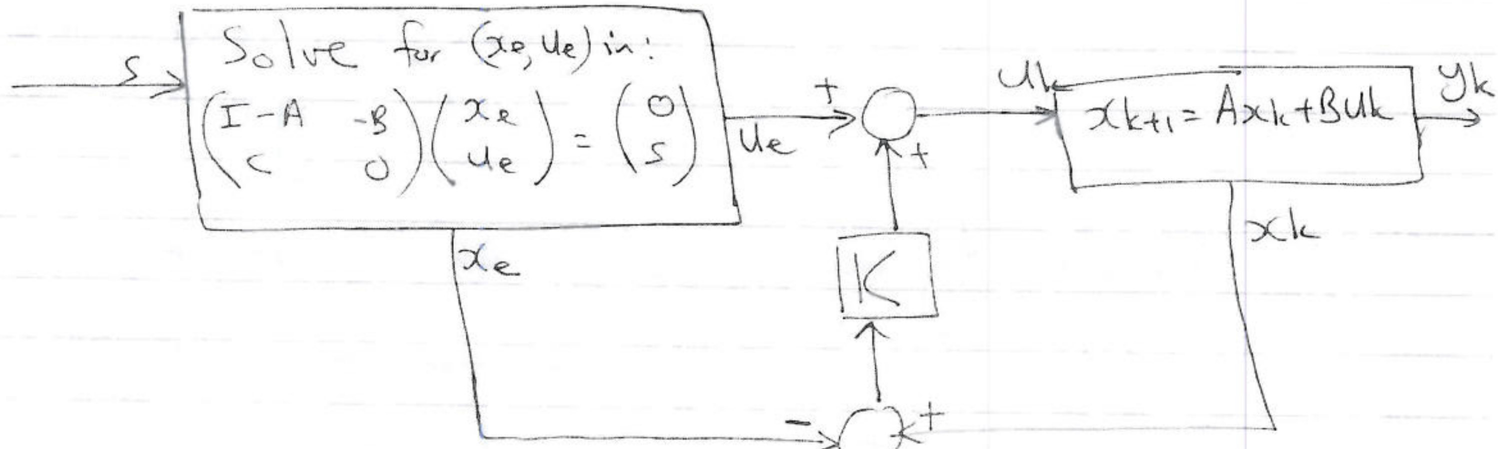
and we have a setpoint s, and we want $y_k = s$ in steady-state.

First, compute a target state-equilibrium pair by solving the linear system:

$$\begin{aligned} x_e &= Ax_e + Bu_e \\ Cx_e &= s \end{aligned} \iff \begin{pmatrix} I-A & -B \\ C & 0 \end{pmatrix}\begin{pmatrix} x_e \\ u_e \end{pmatrix} = \begin{pmatrix} 0 \\ s \end{pmatrix}$$

Then, implement the control law

$$u_k = K(x_k - x_e) + u_e.$$



If $\rho(A+BK) < 1 \Rightarrow \lim_{k\to\infty} y_k = s$ $\longrightarrow$

2012

10/12

(c) Application of theory. To construct such a system, we need to find an example ~~Consider the clos syst~~ of a reachable system and such that

$$\begin{pmatrix} I-A & -B \\ C & 0 \end{pmatrix}\begin{pmatrix} x_e \\ u_e \end{pmatrix} = \begin{pmatrix} 0 \\ s \end{pmatrix}$$

does not have a solution: this will be the case if

$$\text{rank}\begin{pmatrix} I-A & -B \\ C & 0 \end{pmatrix} < \text{rank}\begin{pmatrix} I-A & -B & 0 \\ C & 0 & s \end{pmatrix}$$

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} I-A & -B \\ C & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$, which has rank 3 by inspection.

$$\Rightarrow \begin{pmatrix} I-A & -B & 0 \\ C & 0 & s \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$, which has ~~non zero~~ det = -1 hence rank 4.

$s = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Check: Reachability matrix is $(B \;\; AB)$ ~~...~~

$= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, ~~so system~~ which has full rank, so system is reachable.

Any ~~sensible~~ other example will be acceptable.

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Question 6

11/12

a) Bookwork. The system is reachable iff.

$$\text{rank} \begin{pmatrix} C \\ \lambda I - A \end{pmatrix} = n \quad \text{for all e/values } \lambda \text{ of } A$$

on or outside the unit disk.

(b) We can construct the augmented system

$$x_{k+1} = A x_k + B_d d_k + B u_k.$$

$$d_{k+1} = d_k$$

$$y_k = C x_k + 0 \, d_k$$

$$\Leftrightarrow \begin{pmatrix} x_{k+1} \\ d_{k+1} \end{pmatrix} = \underbrace{\begin{pmatrix} A & B_d \\ 0 & I \end{pmatrix}}_{\tilde{A}} \begin{pmatrix} x_k \\ d_k \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u_k$$

$$y_k = \begin{pmatrix} C & 0 \end{pmatrix} \begin{pmatrix} x_k \\ d_k \end{pmatrix}$$

This system is detectable $\Leftrightarrow$

$$\text{rank} \begin{pmatrix} C & 0 \\ \lambda I - A & -B_d \\ 0 & \lambda I - I \end{pmatrix} = n \quad \text{for all e/values of } \tilde{A} \text{ on / outside unit disk.}$$

e/values of $\tilde{A}$ = e/values of A and $\{1, \dots, 1\}$.

~~Note that~~ Consider all e/values $\lambda \neq 1$ on or outside the unit disk $\Rightarrow \text{rank} \begin{pmatrix} C \\ \lambda I - A \\ 0 \end{pmatrix} = n$ for all $\lambda \neq 1$, $|\lambda| \geq 1$

$\Leftrightarrow (C, A)$ detectable.

$\text{rank}(\lambda I - I) = m \ \forall \lambda \neq 1.$

(b) continued   20/12

$\frac{12}{12}$

Considering all $\lambda = 1$

$$\Rightarrow \quad \operatorname{rank}\begin{pmatrix} C & 0 \\ I - A & -B_d \\ 0 & 0 \end{pmatrix} = \operatorname{rank}\begin{pmatrix} C & 0 \\ I-A & -B_d \end{pmatrix}$$

So we require that this matrix be full column rank as well

$\Rightarrow$ One can construct a stable observer

$\Leftrightarrow$ (C, A) detectable and

$$\operatorname{rank}\begin{pmatrix} C & 0 \\ I-A & -B_d \end{pmatrix} = n + l$$

$\longrightarrow$

(c) The matrix $\begin{pmatrix} C & 0 \\ I-A & -B_d \end{pmatrix}$ has $n+p$ rows and $n+l$ columns.

The matrix will be full colum rank only if it is skinny, ie. $n+p \geq n+l \Leftrightarrow p \geq l$.

$\longrightarrow$

d) We need to guarantee that a state-input equilibrium pair exists such that (x_e, u_e)

$$x_e = A x_e + B u_e + B_d d \quad \begin{pmatrix}\text{equilibrium} \\ \text{target/} \\ \text{setpoint}\end{pmatrix}$$

and $x_e = 0$

$$\Leftrightarrow \left.\begin{aligned} B u_e &= -B_d d \\ x_e &= 0 \end{aligned}\right\} \text{Note decoupled.}$$

~~In other words, we require rank B = ra~~

A solution will exist if and only if

$$\operatorname{rank}(B) = \operatorname{rank}(-B_d d)$$

$\longrightarrow$