IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2006

EEE/ISE PART I: MEng, BEng and ACGI

COMMUNICATIONS 1

Corrected Copy

Friday, 26 May 10:00 am

Time allowed: 2:00 hours

There are FOUR questions on this paper.

Q1 is compulsory.
Answer Q1 and any two of questions 2-4.
Q1 carries 40% of the marks. Questions 2 to 4 carry equal marks (30% each).

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : P.L. Dragotti,
Second Marker(s) : M.K. Gurcan,

Special Information for the Invigilators: none

Information for Candidates

Some Fourier Transforms

$$\cos(\omega_0 t + \theta) \iff \pi[\delta(\omega - \omega_0)e^{-j\theta} + \delta(\omega + \omega_0)e^{j\theta}]$$

$$\text{rect}(\tfrac{t}{\tau}) \iff \tau\text{sinc}(\tfrac{\omega\tau}{2})$$

$$\tfrac{W}{\pi}\text{sinc}(Wt) \iff \text{rect}(\tfrac{\omega}{2W})$$

$$\tfrac{\alpha^2}{2\pi}\text{sinc}^2\left(\tfrac{\alpha t}{2}\right) \iff \Delta\left(\tfrac{\omega}{\alpha}\right)$$

where

$$\Delta(\omega) = \begin{cases} 1 - |\omega|, & |\omega| \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

Some useful trigonometric identities

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

$$\sin x \cos y = \tfrac{1}{2}\sin(x-y) + \tfrac{1}{2}\sin(x+y)$$

$$\cos x \cos y = \tfrac{1}{2}\cos(x-y) + \tfrac{1}{2}\cos(x+y).$$

Euler's formula

$$e^{jx} = \cos x + j \sin x.$$

Frequency modulation by a sinusoidal signal

$$\varphi_{FM}(t) = A \sum_{n=-\infty}^{\infty} J_n(\beta)\cos(\omega_c + n\omega_m)t.$$

where $\beta = \Delta f / B$.

Table of Bessel Function values (recall that $|J_n(\beta)| = |J_{-n}(\beta)|$):

n	$\beta = 1$	$\beta = 2$	$\beta = 5$	$\beta = 10$
0	0.765	0.224	-0.178	-0.246
1	0.440	0.577	-0.328	0.043
2	0.115	0.353	0.047	0.255

Roots of $J_0(x)$:

x	2.4048	5.5201	8.6537	11.7915

The Questions

1. This question is compulsory.

   (a) Consider the two signals $x_1(t)$ and $x_2(t)$ shown in Figure 1.1.

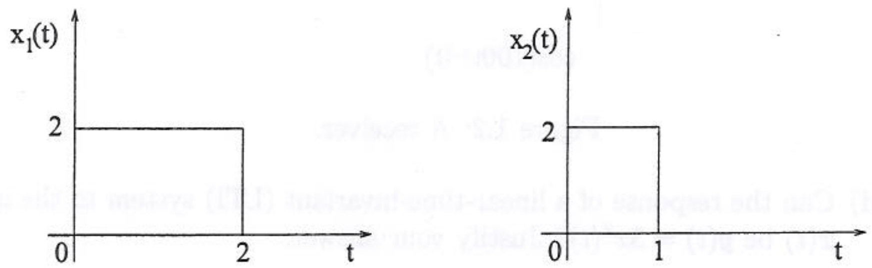


Figure 1.1: The two energy signals $x_1(t)$ and $x_2(t)$.

   i. Determine the correlation coefficient between $x_1(t)$ and $x_2(t)$.

[4]

   ii. Determine the energy of $y(t) = x_1(t) + x_2(t)$.

[4]

   (b) Compute the Fourier transform of $x(t) = e^{-2t}u(t)$, where $u(t)$ is the unit step function defined by

$$u(t) = \begin{cases} 1 & for \quad t \geq 0 \\ 0 & for \quad t < 0 \end{cases}$$

[4]

   (c) The received signal $s(t) = (\cos 10t)\cos 100t$ is multiplied by the local carrier $\cos(100t + \theta)$ and the result $x(t)$ is fed to a filter that has the frequency response

$$H(\omega) = \begin{cases} 1 & for \quad |\omega| \leq 30 \text{ rad/s} \\ 0 & \text{otherwise} \end{cases}$$

giving the output $y(t)$, as shown in Figure 1.2.

   i. For $\theta = 0$,

      A. Sketch and dimension the Fourier transform of $x(t)$.

[4]

      B. Sketch and dimension the Fourier transform of $y(t)$.

[4]

   ii. for $\theta = \pi/4$ and $\theta = \pi/2$, write the exact expression for the output $y(t)$.

[4]

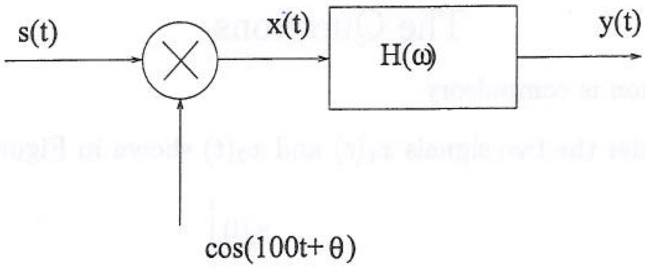


Figure 1.2: A receiver.

(d) Can the response of a linear-time-invariant (LTI) system to the input $x(t)$ be $y(t) = 3x^2(t)$? Justify your answer.

[4]

(e) Consider the FM signal

$$\varphi(t) = A\cos[2\pi f_c t + k_f \int_{-\infty}^{t} m(\alpha)d\alpha],$$

where $m(t) = 20\cos(200t)$, $k_f = 10\pi$ and $f_c = 1000$ Hz.

i. Compute the minimum and maximum instantaneous frequency of $\varphi(t)$.

[4]

ii. If the power of $\varphi(t)$ is 8, find the value A.

[4]

(f) A sinusoidal source $v(t) = 10\sin(500\pi t)$ Volts with internal resistance $R = 50\ \Omega$ is connected to a transmission line with characteristic impedance $Z_0 = 50\ \Omega$. The transmission line is connected to a load Z_L (see Figure 1.3).

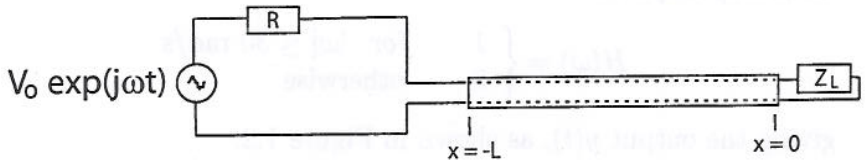


Figure 1.3: A transmission line connected to a sinusoidal source.

i. Choose Z_L so that there is no reflection in the line.

[2]

ii. For the value Z_L you found in part (i), find the exact expression for the current flowing in the circuit.

[2]

2. Consider the FM signal

$$\varphi(t) = 10\cos[2\pi f_c t + k_f \int_{-\infty}^{t} x(\alpha)d\alpha],$$

where $f_c = 2000$ Hz, $k_f = 16\pi$ and $x(t) = A\cos 16\pi t$.

(a) Using Carson's rule, the bandwidth of $\varphi(t)$ is $B_{FM} = 96$ Hz. Compute the amplitude A of $x(t)$.

[6]

(b) Sketch and dimension the Fourier transform of $x(t)$.

[6]

(c) Compute the power of the modulated signal $\varphi(t)$.

[6]

(d) The modulated signal is now passed through an ideal band-pass filter $H(\omega)$ centered at $w_c = 2\pi f_c$ rad/s with a bandwidth of 40π rad/s (see Figure 2.1). Determine the power of the output signal $y(t)$.

[6]

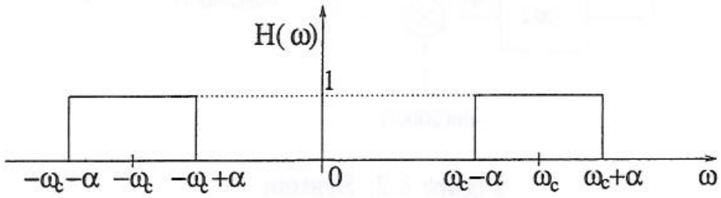


Figure 2.1: Ideal bandpass filter. In this case, $\omega_c = 2\pi f_c$ and $\alpha = 20\pi$ rad/s.

(e) Find the smallest value of k_f that guarantees no power is transmitted at the carrier frequency.

[6]

3. A lowpass signal $x(t)$ has the Fourier transform shown in Figure 3.1. This

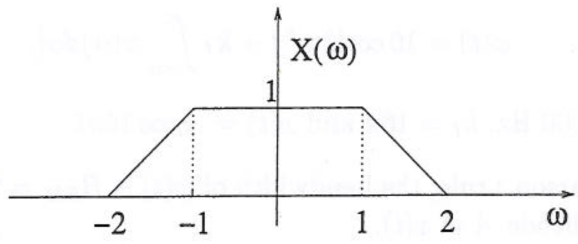


Figure 3.1: Fourier Transform of $x(t)$.

signal $x(t)$ is applied to the system shown in Figure 3.2. The block marked by $-\pi/2$ represents a block performing the Hilbert transform. The filter with transfer function $H(\omega)$ is an ideal lowpass filter with cut-off frequency $\omega_c = 1$ rad/s. That is, $H(\omega) = 1$ for $|\omega| \leq 1$ rad/s and is zero otherwise.

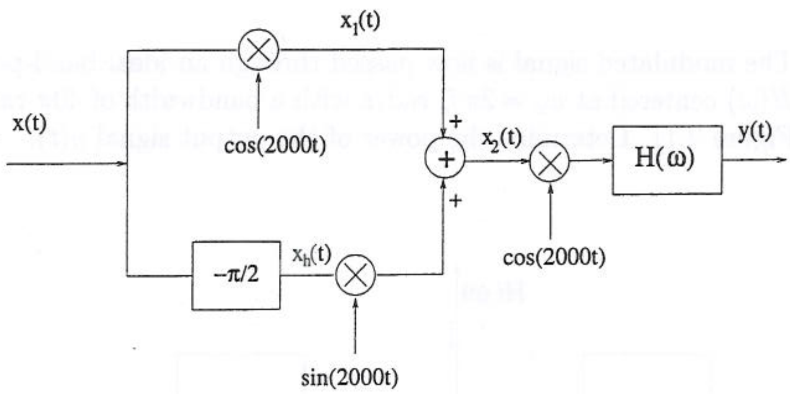


Figure 3.2: System

(a) Sketch and dimension the Fourier transform of $x_1(t)$.

[6]

(b) Sketch and dimension the Fourier transform of $x_2(t)$.

[6]

(c) Find an exact expression for $y(t)$.

[6]

(d) Find an exact expression for $x(t)$.

[12]

4. Consider a linear time-invariant system $h(t)$ where the input $x(t)$ and output $y(t)$ are related by the following linear differential equation:

$$\frac{d^2y(t)}{dt^2} + \frac{dy(t)}{dt} + \beta y(t) = \frac{dx(t)}{dt} + \alpha x(t).$$

(a) Find the transfer function of $h(t)$. Recall that the transfer function is defined as $Y(\omega) = H(\omega)X(\omega)$.

[6]

(b) Find the value of α such that $H(\omega) = 0$ for $\omega = 0$. Then choose β so that $|H(\omega)|^2$ has its maximum when $\omega = 2$ rad/s. (Assume β is real and $\beta > 0$).

[6]

(c) Assume that $x(t) = e^{-t}u(t)$. Compute the Energy Spectral Density (ESD) of $x(t)$.

[6]

(d) Compute the autocorrelation function $\psi(\tau) = \int_{-\infty}^{\infty} x(t)x(t+\tau)dt$. [Hint: Recall that in this case $\psi(\tau) = \psi(-\tau)$].

[6]

(e) Verify that the Fourier transform of $\psi(\tau)$ is equal to the ESD of $x(t)$. [Hint: Use the fact that $x(-t) \Leftrightarrow X(-\omega)$].

[6]

E1.6 COMMUNICATIONS 1

SOLUTIONS - 2006

QUESTION 1

(a)

i.) $E_{x_1} = 4 \cdot 2 = 8$

$E_{x_2} = 4$

$$c_{x_1 x_2} = \frac{1}{\sqrt{E_{x_1} E_{x_2}}} \int_{-\infty}^{\infty} x_1(t)\, x_2(t)\, dt =$$

$$= \frac{1}{4\sqrt{2}} \int_0^1 4\, dt = \frac{1}{\sqrt{2}}$$

Book

ii.) $y(t) = x_1(t) + x_2(t)$

$$E_y = E_{x_1} + E_{x_2} + 2\sqrt{E_{x_1} E_{x_2}} \cdot c_{x_1 x_2} =$$

$$= 12 + 8 = 20$$

Book

(b) NO. SUCH A SYSTEM IS NOT LINEAR.

FOR INSTANCE

$$x_1(t) \to y(t) = x_1^2(t)$$

$$x_2(t) \to y(t) = x_2^2(t)$$

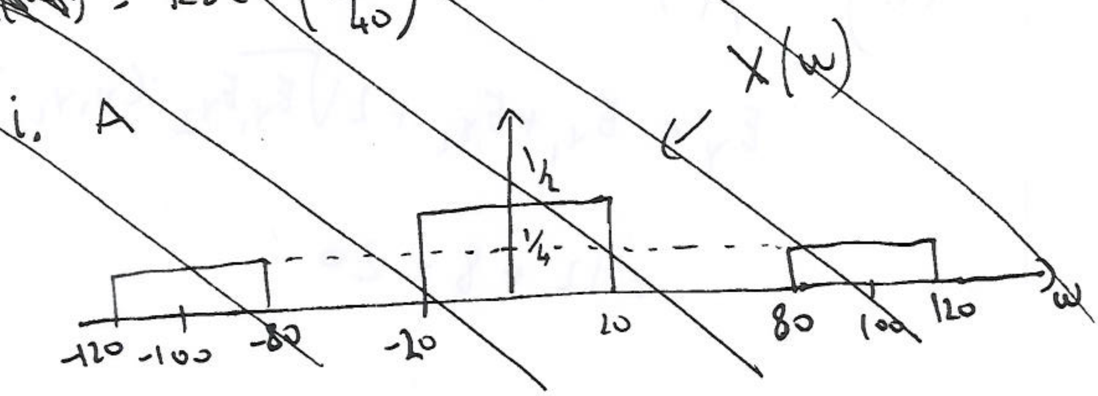
BUT

$$\alpha x_1(t) + \beta x_2(t) \to y(t) = \left(\alpha x_1(t) + \beta x_2(t)\right)^2 \neq \alpha y_1(t) + \beta y_2(t)$$

c) $$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \int_{0}^{\infty} e^{-2t} e^{-j\omega t}\, dt =$$

$$= \frac{1}{2+j\omega}$$

(ii) $H(\omega) = \mathrm{RECT}\left(\frac{\omega}{40}\right)$



3

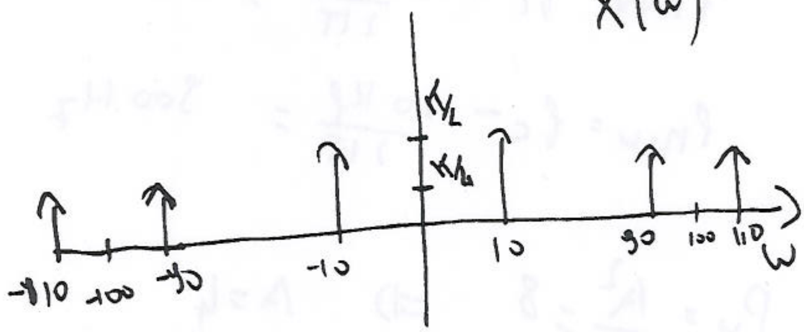
(d)

$$\cos 10t \Leftrightarrow \pi\left[\delta(w-10) + \delta(w+10)\right]$$

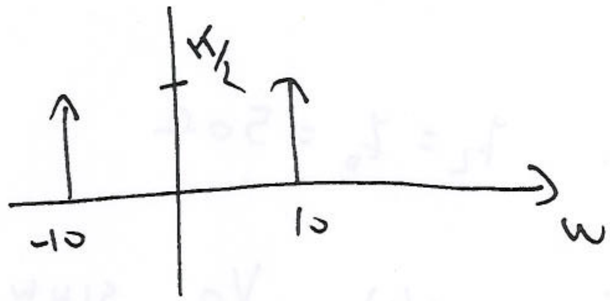
Book

i.

A.



B.



ii

NEW

$\theta \neq 0$

$$y(t) = \cos 10t \cdot \cos\theta$$

$\theta = \pi/2$

$y(t) = 0$

$\theta = \pi/4$

$y(t) = \frac{\sqrt{2}}{2}\cos 10t$

4

(a)

i. $\omega_i = 2\pi f_c + k_f m(t)$

$$f_i = f_c + \frac{k_f m(t)}{2\pi}$$

$$f_{max} = f_c + \frac{20 k_f}{2\pi} = 1000 + 100 = 1100\ Hz$$

$$f_{min} = f_c - \frac{20 k_f}{2\pi} = 900\ Hz$$

ii. $P_y = \frac{A^2}{2} = 8 \Rightarrow A = 4$

(b)

i. $Z_L = Z_0 = 50\Omega$

ii. $i(t) = \frac{V_0}{R + Z_0} \sin \omega_c t = 0.1 \sin(500\pi t)\ A$

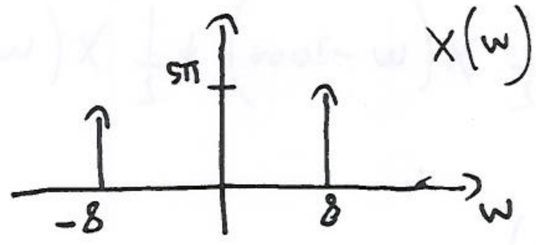
5

QUESTION 2

(a) $B_{FM} = 2(\Delta f + B) = 2\left(\frac{K_f m_p}{2\pi} + B\right) =$

$$= 2(8 \cdot m_p + 8) = 96 \Rightarrow 16 m_p = 80 \Rightarrow m_p = 5.$$

(b)



BOOK

(c) $P_\varphi = \frac{100}{2} = 50$

(d) $\beta = \frac{\Delta f}{B} = 5$. THE MODULATED SIGNAL CAN BE WRITTEN AS FOLLOWS:

NEW

$$\varphi_{FM}(t) = 10 \sum_{m=-\infty}^{\infty} J_m(\beta) \cos(w_c + m w_m) t$$

$w_m = 16\pi$ RAD/s & $w_c = 2\pi f_c$ WITH $f_c = 2000\,Hz$.

AFTER FILTERING

$$\varphi_{FM}(t) = 10 \sum_{m=-1}^{1} J_m(\beta) \cos(w_c + m w_m) t$$

THUS THE POWER IS

$$P_{\hat{\varphi}} = \frac{10^2}{2} J_0^2(\beta) + 2 \cdot \frac{10^2}{2} J_1^2(\beta) \cong$$

$$= 50(-0.178)^2 + 100 \cdot (-0.328)^2 = 12.34$$

(e) THIS HAPPENS WHEN $\beta \cong 2.4 \Rightarrow$

$$\beta = \frac{K_f \cdot m_p}{2\pi B} = 2.4 \Rightarrow K_f \cong 7.68\pi$$

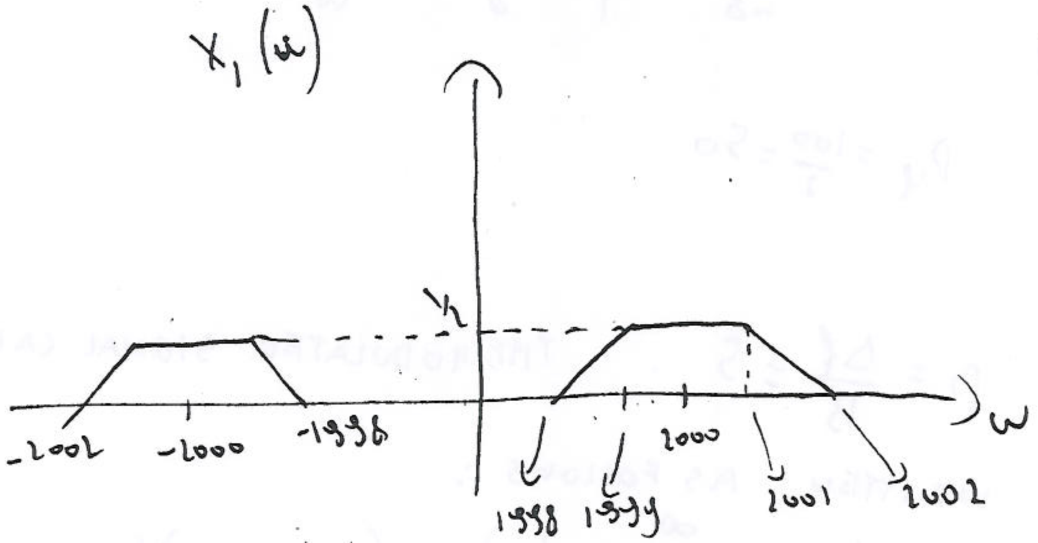
NEW

6

QUESTION 3

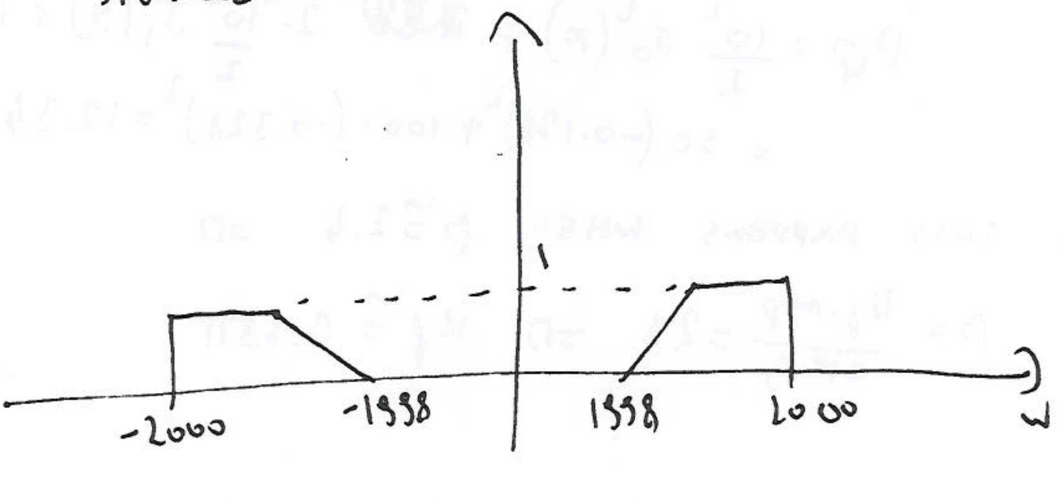
(a) $x_1(t) = x(t)\cos 2000t$

$$X_1(w) = \frac{1}{2}X(w-2000) + \frac{1}{2}X(w+2000)$$

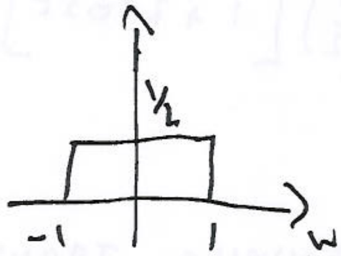


(b) $x_2(t) = x(t)\cos 2000t + x_H(t)\sin 2000t$

THUS, $x_2(t)$ IS AN SSB-LSB MODULATED SIGNAL.



(c) $$Y(w) = \frac{1}{2}\,\mathrm{RECT}\left(\frac{w}{2}\right)$$



$$y(t) = \frac{1}{2\pi}\,\mathrm{SINC}(t)$$

(d) $X(w)$ CAN BE WRITTEN AS FOLLOWS

$$\cancel{X(w) = \Delta\left(w+1\right) + \Delta\left(w\right) + \Delta\left(w-1\right)}$$

$$X(w) = \Delta(w+1) + \Delta(w) + \Delta(w-1)$$

WHERE

$$\Delta(w) = \begin{cases} 1-|w| & |w| \leq 1 \\ 0 & \text{OTHERWISE} \end{cases}$$

$$\frac{\alpha^2}{2\pi}\,\mathrm{SINC}^2\left(\frac{\alpha t}{2}\right) \Longleftrightarrow \Delta\left(\frac{w}{\alpha}\right)$$

THUS

$$x(t) = \frac{1}{2\pi}\,\mathrm{SINC}^2\left(\frac{t}{2}\right) + \frac{1}{2\pi}\,\mathrm{SINC}^2\left(\frac{t}{2}\right)\left[e^{jt} + e^{-jt}\right] =$$

$$= \frac{1}{2\pi} \operatorname{sinc}^2\left(\frac{t}{2}\right) + \frac{1}{\pi} \operatorname{sinc}^2\left(\frac{t}{2}\right) \cos t =$$

$$= \frac{1}{2\pi} \operatorname{sinc}^2\left(\frac{t}{2}\right)\left[1 + 2\cos t\right].$$

QUESTION 4

(a) TAKING THE FOURIER TRANSFORM ON BOTH SIDES

$$-w^2 Y(w) + jw Y(w) + \beta Y(w) = jw X(w) + \alpha X(w)$$

$$\Rightarrow \quad H(w) = \frac{Y(w)}{X(w)} = \frac{\alpha + jw}{-w^2 + jw + \beta}$$

(b)

$$H(0) = \frac{\alpha}{\beta} = 0 \quad \Rightarrow \quad \alpha = 0$$

$$|H(w)|^2 = \frac{w^2}{(\beta - w^2)^2 + w^2} = \frac{w^2}{\beta^2 + w^4 - 2\beta w^2 + w^2}$$

$$\frac{d(|H(w)|^2)}{dw} = \frac{2w\left(w^4 + (1-2\beta)w^2 + \beta^2\right) - w^2\left(4w^3 + 2(1-2\beta)w\right)}{w^4 + (1-2\beta)w^2 + \beta^2} =$$

$$= \frac{2w^5 + 2w\beta^2 - 4w^5}{(\quad)} = 0 \Rightarrow 2w(\beta^2 - w^4) = 0$$

$$\Rightarrow w_0 = \sqrt{\beta} = 2 \Rightarrow$$

$$\beta = 4$$

(c) $x(t) = e^{-t}u(t) \Leftrightarrow X(\omega) = \dfrac{1}{1+j\omega}$ BOOK

$$ESD = |X(\omega)|^2 = \frac{1}{1+\omega^2}$$

(d) $x(t) = e^{-t}u(t)$ AND $x(t+\tau) = e^{-(t+\tau)}u(t+\tau)$

THUS

$$\psi(\tau) = \int_{-\infty}^{\infty} x(t)\,x(t+\tau)\,dt = $$ NEW

$$= \int_0^{\infty} e^{-t} e^{-(t+\tau)}\,dt = e^{-\tau}\int_0^{\infty} e^{-2t}\,dt =$$

FOR $\tau > 0$

$$= \frac{1}{2}e^{-\tau}$$

SINCE $\psi(\tau) = \psi(-\tau) \Rightarrow \psi(\tau) = \frac{1}{2}e^{-|\tau|}$

(e) $\frac{1}{2}e^{-\tau}u(\tau) \Leftrightarrow \dfrac{1}{2(1+j\omega)}$ NEW

$$\frac{1}{2}e^{\tau}u(-\tau) \Leftrightarrow \frac{1}{2(1-j\omega)}$$

THUS $\frac{1}{2}e^{-|\tau|} = \frac{1}{2}e^{-\tau}u(\tau) + \frac{1}{2}e^{\tau}u(-\tau) \Leftrightarrow \dfrac{1}{(1+\omega^2)}$