1. a) i) Let $y(t)$ be the position of the mass M. The force equations are

$$f(t) = K_1 z(t) + D\dot{z}(t) + K_2(z(t) - y(t)), \quad M\ddot{y}(t) + K_2(y(t) - z(t)) = 0.$$

Taking Laplace transforms, substituting and eliminating $y(t)$ gives

$$G(s) = \frac{s^2+1}{s^3 + (1+K_1)s^2 + s + K_1},$$

so, $n(s) = s^2 + 1$. [5]

ii) The Routh array is:

$$\begin{array}{c|cc} s^3 & 1 & 1 \\ s^2 & 1+K_1 & K_1 \\ s & \frac{1}{1+K_1} & \\ 1 & K_1 & \end{array}$$

So $K_1 > 0$ for stability. [5]

iii) When $K_1 = 0$ the closed–loop is marginally stable. Substituting $K_1 = 0$ in $G(s)$ gives the poles as the roots of $s(s^2+s+1)$ which are $0, \frac{-1 \pm j\sqrt{3}}{2}$. [5]

iv) Using the final value theorem and the fact that $f(s) = 1/s$,

$$z_{ss} := \lim_{t\to\infty} z(t) = \lim_{s\to 0} sz(s) = \lim_{s\to 0} sG(s)f(s) = \lim_{s\to 0} \frac{sG(s)}{s} = G(0) = \frac{1}{K_1}.$$

So for $y_{ss} = 2$, we need $K_1 = 0.5$. [5]

b) i) The characteristic equation (CE) is $1 + K(s)G(s) = 0$.

I. When $K(s) = K_P$, a proportional controller, the CE becomes $s^2 + 2K_P - 1 = 0$. The closed–loop cannot be stabilised since the coefficient of s is zero. [5]

II. When $K(s) = K_P + K_I s^{-1}$, a PI controller, the CE becomes $s^3 + (2K_P - 1)s + sK_I = 0$. The closed–loop cannot be stabilised since the coefficient of s^2 is zero. [5]

III. When $K(s) = K_P + K_D s$, a PD controller, the CE becomes $s^2 + 2K_D s + (2K_P - 1) = 0$. The Routh array:

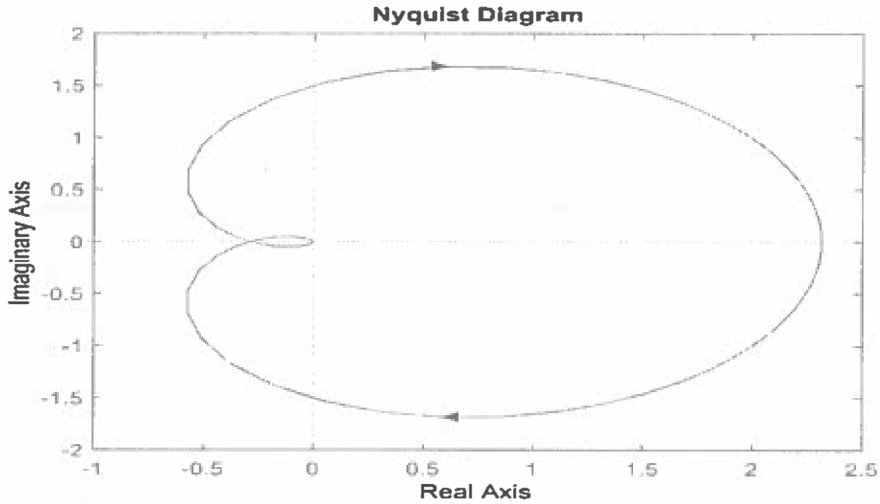
$$\begin{array}{c|cc} s^2 & 1 & 2K_P - 1 \\ s & 2K_D & \\ 1 & 2K_P - 1 & \end{array}$$

So, the closed–loop can be stabilised by any $K_D > 0$, $K_P > 0.5$. [5]

ii) A PD compensator has the form $K(s) = K_P + K_D s$. For critical damping with a pole at $s = -1$, the closed–loop poles must be placed at -1,-1, and so the CE must be $s^2 + 2s + 1$. So we need $s^2 + 2K_D s + (2K_P - 1) = s^2 + 2s + 1$, and so $K_D = K_P = 1$. [5]

2. The transfer function used in fact was $G(s) = \frac{|j+a|^3}{(s+a)^3}$, where $a = 2/\sqrt{3}$, although this is not required.

a) The real axis intercepts can be obtained from the frequency response (when the phase is 0, -180° and -270° and are approximately given by 2.3, -0.3 and 0. The Nyquist plot is given below. [5]



b) From the intercepts above, the gain margin is ~ 3.5. The phase margin can be obtained from the frequency response (by inspecting the phase when the gain is 1) and is approximately 57°. Thus, the stability margins are adequate. [5]

c) i) Let $K(s)=k$. The Nyquist criterion states that $N=Z-P$, where N is the number of clockwise encirclements by the Nyquist diagram of $-k^{-1}$, P is the number of unstable open–loop poles and Z is the number of unstable closed–loop poles. [5]

ii) Since $G(s)$ is stable, $P=0$. An inspection of the Nyquist diagram shows that

I. When $k=1$, $N=0$ so $Z=0$. [5]

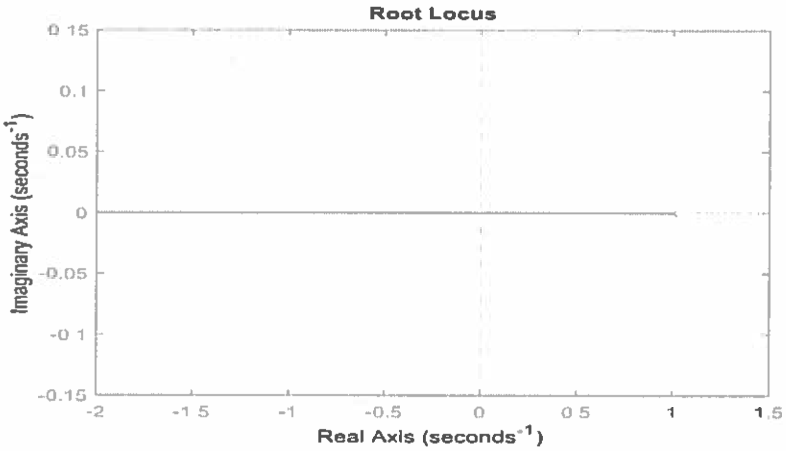
II. When $k=10$, $N=2$ so $Z=2$. [5]

d) An inspection of the frequency response reveals this is a proportional–plus–integral (PI) compensator. This can be written as

$$K(s) = K_P + \frac{K_I}{s} = \frac{K_I}{s}(1 + \frac{s}{K_I/K_P})$$

It has high gain at frequencies below $\omega_0 = K_I/K_P$ and gain close to K_P beyond ω_0. The phase is negative and large below ω_0 but insignificant above. It follows that by varying K_I and K_P we can use PI compensation to increase low frequency gain (hence improving tracking properties) without introducing phase-lag at high frequency (which would reduce the phase margin) by placing w_0 in the 'middle' frequency range. Since the cross–over frequency for $G(s)$ is approximately 1 and ω_0 for $K(s)$ is approximately 0.2, this condition is satisfied. [5]

3. a) The root–locus is shown below. [6]



b) i) The root–locus of $\hat{G}(s)$ is shown below for $z = 4/5$ (which happens to be the correct answer). [6]

ii) The root–locus shows that there are two possible values of K_P for which the closed–loop has double poles. These are the breakaway and break–in points. [6]

iii) To solve for z, we set $\frac{d\hat{G}(s)}{ds} = 0$ at $s = -2$ to get

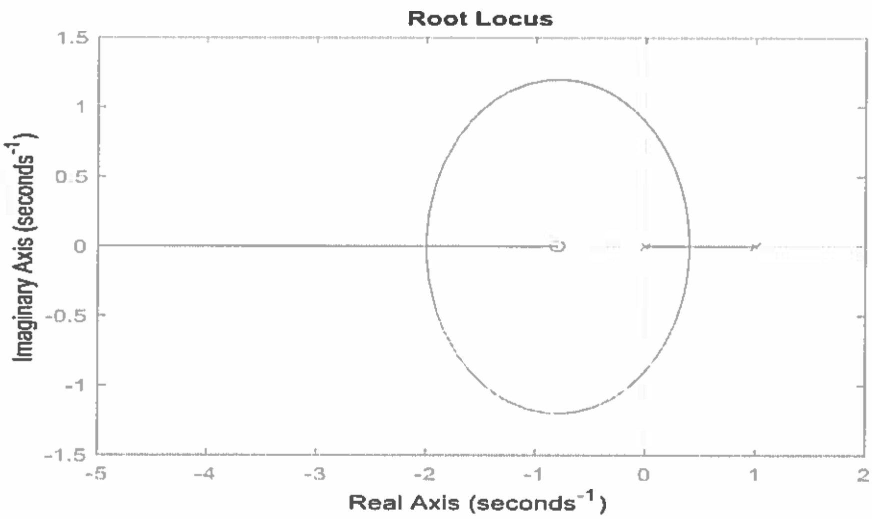
$$\frac{s^2 - s - (s+z)(2s-1)}{(s^2 - s)^2} = 0 \Rightarrow s^2 + 2sz - z = 0 \Rightarrow 4 - 4z - z = 0 \Rightarrow z = 4/5.$$

[6]

iv) We use the gain criterion to find K_P since

$$K_P = -1/\hat{G}(-2) = -(-2)(-2-1)/(-2+z) = 5.$$

It follows that $K_I = zK_P = 4$. [6]


Root Locus
1.5
1
0.5
0
-0.5
-1
-1.5
Imaginary Axis (seconds^-1)
-5
-4
-3
-2
-1
0
1
2
Real Axis (seconds^-1)