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EE9AO11

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2012

MSc and EEE PART IV: MEng and ACGI

MEMS AND NANOTECHNOLOGY

Thursday, 3 May 10:00 am

Time allowed: 3:00 hours

There are FIVE questions on this paper.

Answer Question 1.
Answer Question 2 OR Question 3.
Answer Question 4 OR Question 5.

Question 1 carries 40% of the marks. Remaining questions carry 30% each.

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s) :	Z. Durrani, A.S. Holmes, Z. Durrani
	Second Marker(s) :	A.S. Holmes, Z. Durrani, A.S. Holmes

Information for Candidates

The steady state, 1-dimensional heat flow equation for a current-carrying beam is:

$$\kappa A^2 \frac{d^2T}{dx^2} + I^2\rho = 0$$

where $T(x)$ is the temperature variation along the beam, A is the beam cross-sectional area, κ and ρ are the thermal conductivity and electrical resistivity of the beam material respectively, and I is the current.

The fractional resistance change for a <110>-aligned piezoresistor on a <100>-oriented silicon wafer, subject to longitudinal and transverse axial stresses σ_L, σ_T, can be expressed as:

$$\frac{\Delta R}{R} = \pi_L \sigma_L + \pi_T \sigma_T$$

where π_L and π_T are the reduced piezoelectric coefficients.

This question is compulsory

1. a) Consider an enhancement mode, n-channel, Si MOSFET, with gate oxide thickness t_{ox}, and energy separation $e\psi_B$ between the Fermi level E_F and the intrinsic level E_i. Here, e is the electron charge.

   (i) Sketch for 'flat-band' conditions the energy band diagram (E_c, E_v, E_i, and E_F) along a line perpendicular to the gate oxide - bulk Si plane. Your diagram should show the energy bands in the bulk Si, oxide, and gate regions. [3]

   (ii) Sketch the energy band diagram (E_c, E_v, E_i, and E_F) along the line perpendicular to the gate oxide – bulk Si plane, for the following condition: There is a trapped charge $+Q_f$ at the gate oxide/Si interface, and the work function in the gate metal $\phi_m < \phi_s$, the work function in the Si. [2]

b) A Si (100) surface with an oxide layer of thickness X_i = 20 nm is to be further oxidised at 1000°C. At this temperature, the oxidation 'rate' constants in the Deal-Grove equation for oxide growth:

$$X^2 + AX = B(t + \tau)$$

are $A = 0.33 \times 10^4$ Å and $B = 2.86 \times 10^4$ Å^2/min, with X the thickness of the oxide layer.

Calculate the total thickness of the oxide grown under these conditions after 60 minutes. [5]

Question 1 continues on the next page.

Question 1 continued.

c) Figure 1.1 shows schematically the cross-section through a back-gated Si nanowire FET. The Si nanowire length $L = 1$ μm and Si core radius $R = 20$ nm. An oxide shell of thickness $t_1 = 10$ nm surrounds the nanowire. The back-gate is covered by a planar oxide layer of thickness $t_2 = 10$ nm. The gate capacitance for the device can be approximated by:

$$C_g = \frac{2\pi\varepsilon_r\varepsilon_0 L}{\ln\left(\frac{2H}{R}\right)}$$

where H is the minimum separation between the nanowire Si core and the back gate.

Calculate the carrier concentration n, per cubic metre, in the nanowire at threshold voltage $V_{th} = 2$ V.

You may use $\varepsilon_0 = 8.854 \times 10^{-12}$ F/m and $e = 1.6 \times 10^{-19}$ C. The dielectric constants for Si and SiO_2 are $\varepsilon_{si} = 11.9$ and $\varepsilon_{ox} = 3.9$ respectively. [5]

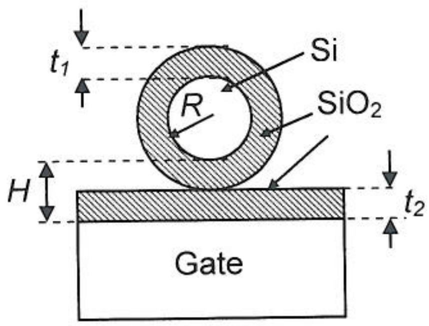

Figure 1.1

d) Using suitable diagrams, explain how the unit cell in a Si crystal may be constructed from two face centred cubic cells. [5]

Question 1 continues on the next page.

e) By considering the image of a periodic line/space pattern, show that the minimum resolvable half-pitch R of a projection lithography system with on-axis, plane wave illumination is expected to be:

$$R = 0.5\frac{\lambda}{NA}$$

where λ is the illumination wavelength and NA is the numerical aperture. Why do off-axis components in the illumination allow R to be reduced below this limit? [6]

f) Derive an expression for the electrostatic force developed by a parallel plate electrostatic actuator. Hence obtain a scaling law for the electrostatic force assuming constant maximum electric field. Also comment on the validity of the constant field assumption at different length scales. [5]

g) You are required to design a mask for a silicon alignment groove for an optical fibre, as shown in Figure 1.2 below. The fibre has a diameter of 125 μm, and its axis is to be positioned 15 μm above the wafer surface. Calculate the width of the mask aperture required if the groove is to be etched to a depth of 65 μm using an EDP etch with an anisotropy of 40. [4]

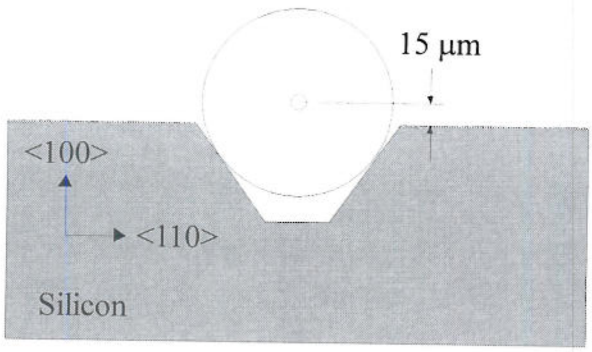

Figure 1.2

h) Two common types of elastic suspension are the hammock flexure and the folded flexure. Sketch both types and explain the most important difference between their elastic behaviours. [5]

2. This question relates to a three-dimensional potential well where the potential energy $V(x,y,z) = V_x + V_y + V_z$ is given by:

$$V_x(x) = 0 \quad \text{for } 0 \le x \le L_x \quad , \quad V(x) = \infty \quad \text{elsewhere}$$
$$V_y(y) = 0 \quad \text{for } 0 \le y \le L_y \quad , \quad V(y) = \infty \quad \text{elsewhere}$$
$$V_z(z) = 0 \quad \text{for } 0 \le z \le L_z \quad , \quad V(z) = \infty \quad \text{elsewhere}$$

a) Demonstrate that the time-independent Schrödinger Equation in three-dimensions

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi$$

can be separated into three one-dimensional equations. [5]

b) Hence, solve these equations for $V(x,y,z)$ and derive expressions for the particle energy levels E_{ngl} in the well, where n, g, and l are integers greater than zero. [10]

c) Derive an expression for the normalised wave function ψ_{ngl}. [10]

d) For $L_x = 2L_y = 4L_z$, write down expressions, in terms of L_x, for the three lowest energy levels in the well. [5]

3. a) Figure 3.1 shows the energy band diagram (in one-dimension) for a resonant tunnelling diode at drain-source bias $V_{ds} = 0$ V. E_F and E_c are the Fermi energy, and the energy of the bottom of the conduction band, respectively. The potential barrier and potential well regions have equal widths L, and the first electron level lies at energy E_1 above E_c. You may assume that the first potential barrier, the potential well, and the second potential barrier regions have equal resistances R. Figure 3.2 shows schematically the I_{ds}-V_{ds} current-voltage characteristics of the device.

(i) For the voltage points V_{ds1} and V_{ds2} marked in Figure 3.2, draw energy band diagrams for the device. [8]

(ii) Hence, or otherwise, show that $V_{ds1} = 3(E_1 - E_F)/e$ and $V_{ds2} = 3(E_1 - E_c)/e$, where e is the electron charge. [8]

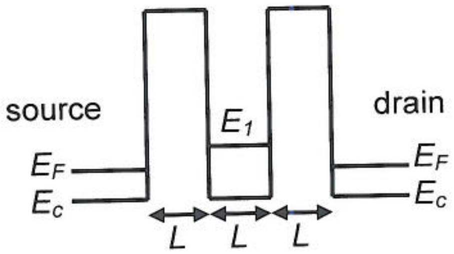

Figure 3.1

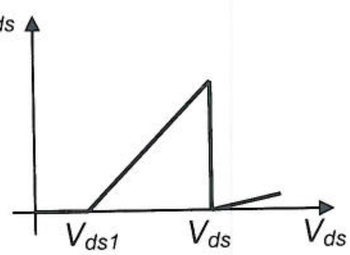

Figure 3.2

b) Consider in one-dimension an ideal conductor of length L, lying between metal source and drain regions with Fermi energies E_{F1} and E_{F2} respectively. A voltage V is applied across the conductor. If the density of states (no. of states in k-space, per unit volume) in 1-D is given by $n(k) = 1/\pi$, show that the conductance of a single electron channel is given by $G = 2e^2/h$, where h is Plank's constant. [14]

4. a) Sketch a diagram of a flexure subject to a transverse end load, showing all the external forces and couples acting on the beam. By solving the bending equation, show that the transverse stiffness is given by:

$$k_T = \frac{w^3 hE}{L^3}$$

where w is the width of the beam in the direction of deflection, h and L are its depth and length, and E is Young's modulus. Also give an expression for the axial stiffness, k_A of the beam. [10]

b) Figure 4.1 shows a so-called v-beam electrothermal actuator which consists of two flexures connected in series in a chevron configuration i.e. with an angle between them. When current is passed through the structure, frustrated thermal expansion causes the beams to deflect as shown in Figure 4.2.

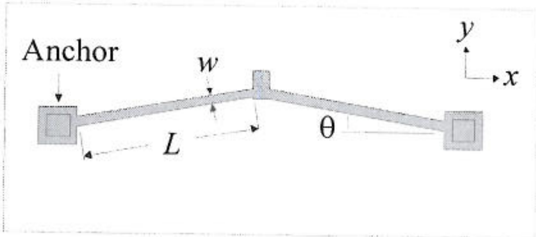

Figure 4.1

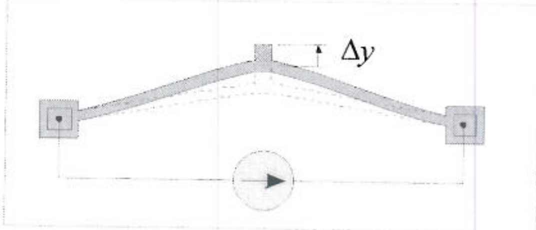

Figure 4.2

Considering the deflected structure, explain why the end forces P acting on the flexures must lie along the x-direction. Hence show that the following relations apply:

$$v_A = L\varepsilon_{th} - \frac{P\cos\theta}{k_A} \quad ; \quad k_T v_T = P\sin\theta$$

where v_A and v_T are the axial and transverse end displacements of either flexure, and ε_{th} is the thermally induced strain. You should ignore the effect of axial loading on the transverse stiffness. Using these equations, and the geometrical relationships between v_A and v_T and Δy, show that the actuator deflection may be written as:

$$\Delta y = \frac{L\varepsilon_{th}}{\sin\theta[1+(w/L)^2\cot^2\theta]}$$

Hence determine the average temperature rise required to produce a 2 μm deflection in a nickel actuator where L = 200 μm, w = 4 μm and θ = 2°. The thermal expansion coefficient for nickel is 13.3 x 10^{-6} K^{-1}. [12]

c) Assuming heat flow occurs only by conduction in the beams, and that the substrate is a perfect heat sink, sketch the temperature profile along the structure when current is applied. Also, by solving the 1D heat flow equation, determine the current required to deflect the actuator by 2 μm if the nickel layer is 10 μm thick. Assume values of 90.9 $Wm^{-1}K^{-1}$ and 6.9 μΩcm for the thermal conductivity and electrical resistivity of nickel. [8]

5. Figure 5.1 below shows a simple silicon micromachined accelerometer comprising a proof mass supported by a cantilever. A p-type piezoresistive bridge is used to measure the strain at the cantilever root. The length, width and depth of the cantilever are $l = 500$ μm, $w = 1000$ μm and $h = 20$ μm respectively, and the corresponding parameters for the proof mass are $L = 2$ mm, $W = 2$ mm, and $H = 500$ μm.

a) Write down an expression for the bending moment as a function of position along the cantilever when the accelerometer is subject to an upward acceleration a normal to the wafer. You should assume the cantilever is mass-less and neglect the gravitational force on the proof mass. [4]

b) Show that the strain at the location of the piezoresistive bridge may be written as:

$$\varepsilon = \frac{6WHL(l + L/2)}{Ewh^2}\rho a$$

Hence calculate the sensitivity (output voltage per unit acceleration) of the accelerometer, assuming the piezoresistive bridge is connected to a 5 V supply. The density of silicon is $\rho = 2330$ kg/m^3, and you should assume a value of $E = 160$ GPa for Young's modulus. Use the following values for the reduced piezoresistive coefficients for p-type silicon: $\pi_L = 72 \times 10^{-11}$ Pa^{-1}; $\pi_T = -66 \times 10^{-11}$ Pa^{-1}. [10]

c) Solve the bending equation for the cantilever to obtain its deflection profile. Hence determine the deflection *at the far end of the proof mass* when $a = 100$ ms^{-2}. You should assume the proof mass is rigid. [10]

d) Show that the sensitivities of the accelerometer in Figure 5.1 and a piezoresistive device based on a simple cantilever with length $(l + L/2)$, width w and depth h are in the approximate ratio $2m_p/m$, where m_p is the proof mass and m is the mass of the simple cantilever. [6]

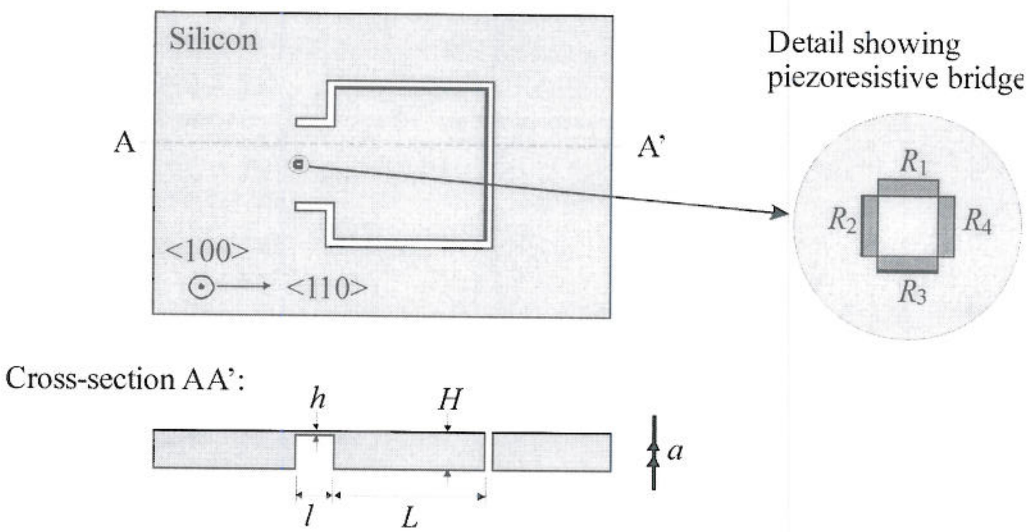

Figure 5.1

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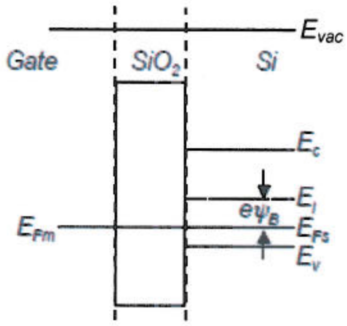
MEMS/NANO Examination, 2012 - Answers

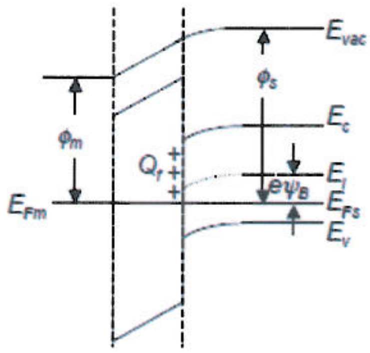
Question 1

a)(i) Band diagram for 'flat–band' conditions:



[2]

(ii) Band diagram with trapped charge $+Q_f$ at the gate oxide/Si interface, and work function in the gate metal $\phi_m < \phi_s$, the work function in the Si.



[3]

(b) In the Deal-Grove equation, τ corresponds to the 'time' corresponding to the initial oxide thickness $X_i = 20$ nm. Working in Angstrom, we first calculate τ:

$$\tau = (X_i^2 + AX_i)/B = (200^2 + 0.33 \times 10^4 \times 200) / 2.86 \times 10^4 = 24.5 \text{ min.}$$

[2]

Therefore, at $t = 60$ min., we have:

$$X^2 + AX = B(t + \tau)$$

$\Rightarrow$ $X^2 + (0.33 \times 10^4)X = (2.86 \times 10^4)(60 + 24.5) = 2.417 \times 10^6$

$\Rightarrow$ $X = (-3300 \pm \sqrt{(3300^2 - 4(-2.417 \times 10^6)})/2$

$\Rightarrow$ $X = 61$ nm

Here we have used the positive root only. [3]

(c) From Figure 1.1, $H = t_1 + t_2 = 10$ nm + 10 nm = 20 nm. The gate capacitance is then given by:

$$C_g = \frac{2\pi\varepsilon_r\varepsilon_0 L}{\ln\left(\frac{2H}{R}\right)} = \frac{2\pi \times 3.9 \times 8.854 \times 10^{-12} \times 10^{-6}}{\ln\left(\frac{2 \times 20}{20}\right)} = 3.128 \times 10^{-16} F$$

[2]

At threshold voltage $V_{th} = 2$ V, the charge induced in the nanowire is given by:

$$Q = C_g V_{th} = 3.128 \times 10^{-16} \times 2 = 6.256 \times 10^{-16} \text{ C}$$

[1]

We can then find the total number of charge carriers in the nanowire:

$$N = Q/e = 6.256 \times 10^{-16} / 1.6 \times 10^{-19} = 4.14 \times 10^3 \text{ carriers.}$$

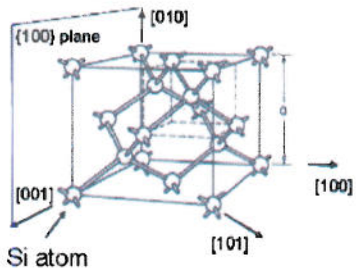
[1]

Therefore the carrier concentration is:

$$n = N/\text{volume} = N/(\pi R^2 \times L) = 4.14 \times 10^3/(\pi \times 20 \times 20 \times 10^{-18} \times 10^{-6})$$
$$= 3.29 \times 10^{24} \text{ m}^{-3}$$

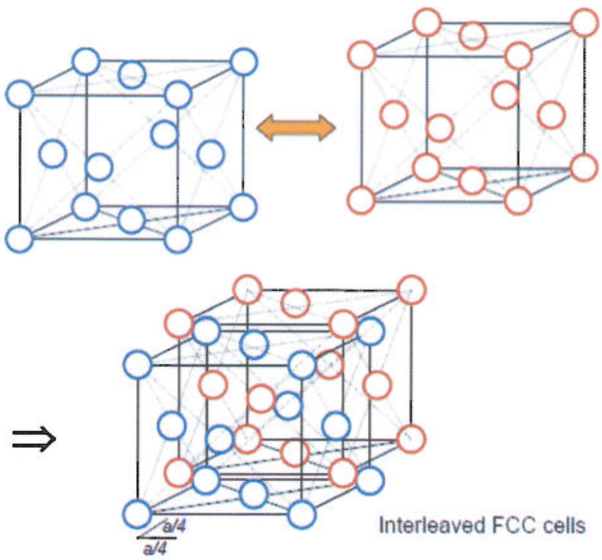
[1]

(d) The unit cell in a Si lattice, with side length a, is shown below:



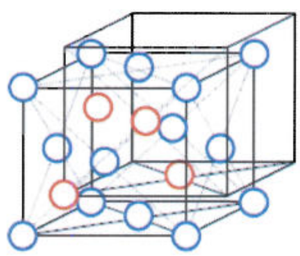
[2]

This may be constructed by interleaving two FCC cells, displaced by $a/4$ along the x, y and z axis, as follows:



[2]

Dropping atoms outside the unit cell gives us the Si unit cell:



[1]

e) A grating mask of period d will generate diffracted beams at angles θ_n given by the Bragg condition:

$$\sin\theta_n = n\frac{\lambda}{d} \qquad n = 0, \pm 1, \pm 2 \ldots$$

The finest grating that will be resolved is the one for which the ±1 diffraction orders just pass through the projection lens. In this case we have $\theta_1 = \theta_m$ where θ_m is the maximum acceptance angle of the lens. But $\sin\theta_m = NA_o$ where NA_o is the numerical aperture on the object side, so for the finest resolvable grating we have:

$$d_{\min} = \frac{\lambda}{NA_o}$$

and the minimum resolvable half-pitch on the object side is:

$$R_o = \frac{d_{\min}}{2} = \frac{\lambda}{2NA_o}$$

The corresponding half-pitch on the image side is $R = R_o/m$, where m is the magnification, while the image side numerical aperture is $NA = mNA_o$, so we can recast the above formula for the image side as:

$$R = \frac{\lambda}{2NA}$$ [5]

With-off axis illumination, the angle between the 0 and +1 (or -1) diffraction orders can be increased beyond θ_m, allowing information about finer patterns to be transmitted through the project lens to the image. [1]

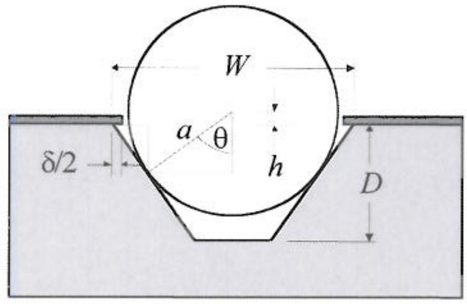
f) Using virtual work, the electrostatic force is $F_e = \partial U / \partial g$, where U is the electrostatic energy stored in the capacitor, and g is the gap between the plates. U is given by: $U = CV^2/2 = \varepsilon_0 AV^2/(2g)$, where A is the plate area, so $F_e = -\varepsilon_0 AV^2/(2g^2)$. At constant maximum electric field, V/g remains constant, so F_e scales as $A \propto L^2$. [3]

At cm scale and larger, the maximum field is limited by avalanche breakdown, and it is reasonable to assume that the threshold field for this is constant. At smaller scales, the density of gas molecules is insufficient to support avalanche breakdown, and the maximum field increases with decreasing size. At the scale of a few microns, the maximum field becomes constant again, but at a much higher level (set by field emission). [2]

g) From the diagram below, the width of the final V-groove at the wafer surface needs to satisfy:

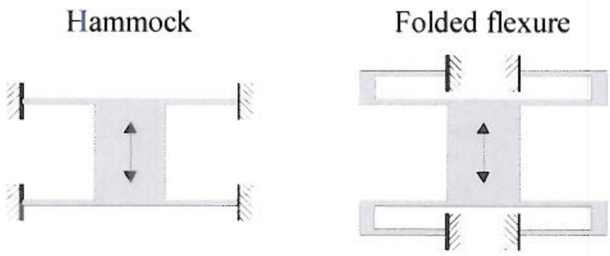
$$W/2 = a\sin\theta + (a\cos\theta - h)\cot\theta = (\sqrt{3}a - h)/\sqrt{2}$$

where we have used the fact that the angle between the V-groove sidewalls and the wafer surface is $\theta = \tan^{-1}(\sqrt{2})$. With $a = 62.5$ μm and $h = 15$ μm, this gives $W = 131.9$ μm. The undercut is given by $\delta = \sqrt{6}D / S$ where S is the anisotropy, and the mask aperture needs to be narrower than the groove by this amount. With $D = 65$ μm and $S = 40$, we get $\delta = 4.0$ μm, so the required mask width is $W_m = 127.9$ μm.



[4]

h)



The hammock suspension has fixed span, and consequently when the mass moves the flexures experience axial stress (because their arc lengths are forced to increase). As a result the suspension is non-linear, becoming stiffer with increasing deflection. The folded flexure does not suffer from this effect because the links at the end of the folded flexures are free to move inwards when the mass deflects. [5]

Question 2

Q2 (a)

Schrödinger's Equation in 3-D is :-

$$\frac{-\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi$$

If ψ is seperable, we have $\psi = u(x)v(y)w(z)$

$$E = E_x + E_y + E_z$$

$$V = V_x + V_y + V_z$$ [2]

Substituting $\Rightarrow$

$$\frac{-\hbar^2}{2m}\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\right)uvw + (V_x+V_y+V_z)uvw = (E_x+E_y+E_z)uvw$$

$$\Rightarrow \frac{-\hbar^2}{2m}\left(vw\frac{d^2u}{dx^2} + uw\frac{d^2v}{dy^2} + uv\frac{d^2w}{dz^2}\right) + (V_x+V_y+V_z)uvw = (E_x+E_y+E_z)uvw$$

$$\Rightarrow \frac{-\hbar^2}{2m}\left(\frac{1}{u}\frac{d^2u}{dx^2} + \frac{1}{v}\frac{d^2v}{dy^2} + \frac{1}{w}\frac{d^2w}{dz^2}\right) + (V_x+V_y+V_z) = E_x+E_y+E_z$$

$$\Rightarrow \frac{-\hbar^2}{2m}\frac{d^2u}{dx^2} + (V_x - E_x)u = 0$$

$$\frac{-\hbar^2}{2m}\frac{d^2v}{dy^2} + (V_y - E_y)v = 0$$

These are three 1-D Schrödinger Equations

$$\frac{-\hbar^2}{2m}\frac{d^2w}{dz^2} + (V_z - E_z)w = 0$$ [3]

(b) For all three direction x, y, z, V is zero within the well. The problem then reduces to three 1-D solutions, e.g. for x-direction:

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dx^2} + (V_x - E_x)u = 0$$

Within the well, $0 < x < L_x$, $V_x = 0$

$$\Rightarrow \frac{-\hbar^2}{2m}\frac{d^2u}{dx^2} - E_x u = 0$$ [2]

This has a general solution of the form:

$$u = A\sin k_x x + B\cos k_x x \quad \text{where } k_x = \sqrt{\frac{2mE_x}{\hbar^2}}$$ [2]

Substituting boundary conditions $u=0$ at $x=0$

$$\Rightarrow B = 0 \Rightarrow u = A\sin k_x x$$

Substituting $u=0$ at $x = L_x$

$$\Rightarrow 0 = A\sin k_x L_x \Rightarrow k_x L_x = n\pi, \quad n = 1, 2, \ldots$$

$$\Rightarrow E_n = \frac{\hbar^2 k_x^2}{2m} = \frac{n^2\pi^2\hbar^2}{2mL_x^2}$$ [2]

Similarly, $E_g = \frac{g^2\pi^2\hbar^2}{2mL_y^2}$ for y-direction, $g = 1, 2, \ldots$

$E_l = \frac{l^2\pi^2\hbar^2}{2mL_z^2}$ for z-direction, $l = 1, 2, \ldots$

$$\therefore E_{ngl} = E_n + E_g + E_l = \frac{\pi^2\hbar^2}{2m}\left(\frac{n^2}{L_x^2} + \frac{g^2}{L_y^2} + \frac{l^2}{L_z^2}\right)$$ [4]

(c) Full wavefunction $\psi = uvw = A_x A_y A_z \sin k_x x.\sin k_y y.\sin k_z z$
$= C\sin k_x x.\sin k_y y.\sin k_z z.$

where $k_x = \frac{n\pi}{L_x}$, $k_y = \frac{g\pi}{L_y}$, $k_z = \frac{l\pi}{L_z}$

Normalizing: $\int_0^{L_x}\int_0^{L_y}\int_0^{L_z} C^2 \sin^2 k_x x \sin^2 k_y y \sin^2 k_z z \, dx\,dy\,dz = 1$ [4]

$$\Rightarrow C^2 \int_0^{L_x} \sin^2 k_x x\,dx \int_0^{L_y} \sin^2 k_y y\,dy \int_0^{L_z} \sin^2 k_z z\,dz = 1$$

$$\Rightarrow C^2 \int_0^{L_x}\left(\frac{1-\cos 2k_x x}{2}\right)dx \int_0^{L_y}\left(\frac{1-\cos 2k_y y}{2}\right)dy \int_0^{L_z}\left(\frac{1-\cos 2k_z z}{2}\right)dz = 1$$

$$\Rightarrow C^2\left(\frac{L_x}{2}\right)\left(\frac{L_y}{2}\right)\left(\frac{L_z}{2}\right) = 1 \Rightarrow C = \sqrt{\frac{8}{L_x L_y L_z}}$$

$$\therefore \psi_{ngl} = \sqrt{\frac{8}{L_x L_y L_z}} \sin\frac{n\pi}{L_x} \sin\frac{g\pi}{L_y} \sin\frac{l\pi}{L_z}$$

[6]

(d) For $L_x = 2L_y = 4L_z$

$$E_{ngl} = \frac{\pi^2\hbar^2}{2m}\left(\frac{n^2}{L_x^2} + \frac{4g^2}{L_x^2} + \frac{16\,l^2}{L_x^2}\right)$$

$$= \frac{\pi^2\hbar^2}{2mL_x^2}\left(n^2 + 4g^2 + 16\,l^2\right)$$ [2]

Lowest three energy levels :- (i) $n = g = l = 1 \Rightarrow E_{111} = \frac{\pi^2\hbar^2}{2mL_x^2}(1 + 4 + 16)$

$$= (10.5)\frac{\pi^2\hbar^2}{mL_x^2}$$ [1]

(ii) $n=2,\ g=1,\ l=1$

$$\Rightarrow E_{211} = \frac{\pi^2\hbar^2}{2mL_x^2}(4 + 4 + 16)$$

$$= 12\,\frac{\pi^2\hbar^2}{mL_x^2}$$ [1]

(iii) $n=3,\ g=1,\ l=1$

$$\Rightarrow E_{311} = \frac{\pi^2\hbar^2}{2mL_x^2}(9 + 4 + 16)$$

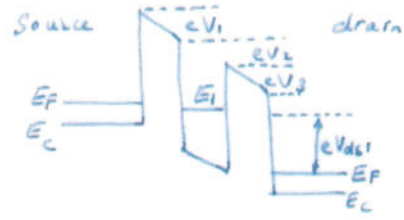
$$= (14.5)\frac{\pi^2\hbar^2}{mL_x^2}$$

Note that $E_{121} > E_{311}$. [1]

Question 3

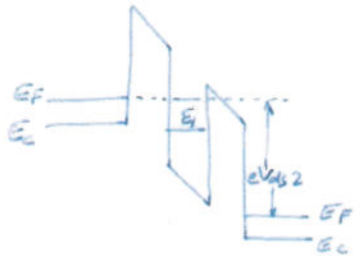
Q3(a)(i)

As the resistances of the potential barrier and potential well regions are equal, band bending is linear across the device. Voltage drops V_1, V_2 and V_3 across the three regions are equal.



Band diagram at V_{ds1}
E_1 aligned with E_F in source

[4]



Band diagram at V_{ds2}
E_1 aligned with E_C in source

[4]

Q3(a)(ii)

Equal resistances R $\Rightarrow$ equal voltage drops across the potential barrier and potential well regions

$$\Rightarrow \quad V_1 = V_2 = V_3 = V_x$$
$$\text{and} \quad V_1 + V_2 + V_3 = V_{ds}$$
$$\Rightarrow \quad 3V_x = V_{ds}$$
$$\Rightarrow \quad V_x = \frac{V_{ds}}{3}$$

[4]

For $V_{ds} = V_{ds1}$, $E_1 - E_F = eV_1 = eV_x = \frac{eV_{ds1}}{3}$ (see Fig 3.1 and above)

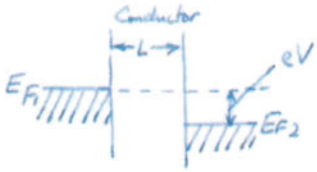
$$\Rightarrow V_{ds1} = \frac{3(E_1 - E_F)}{e}$$

and at $V_{ds} = V_{ds2}$, $E_1 - E_C = eV_1 = eV_x = \frac{eV_{ds2}}{3}$

$$\Rightarrow V_{ds2} = \frac{3(E_1 - E_C)}{e}$$

[4]

Q3(b)



Current I, across the conductor at applied voltage V is given by:

$$I = I_{right} - I_{left}$$

$$= e\int_0^{\infty} v(k)n(k)f(k)\,dk - e\int_0^{\infty} v(k')n(k')f(k')\,dk'$$ [4]

where $v(k)$ = electron velocity

$n(k) = 1/\pi$

$f(k)$ = Fermi-Dirac function

Using $v = \frac{p}{m} = \frac{\hbar k}{m}$,

$$I = e\int_0^{\infty} \frac{\hbar k}{m}\cdot\frac{1}{\pi}\cdot f(E)\,\frac{dk}{dE}dE - e\int_0^{\infty} \frac{\hbar k'}{m}\cdot\frac{1}{\pi}\cdot f(E')\frac{dk'}{dE'}dE'$$

$$= e\int_0^{E_{F1}} \frac{\hbar k}{m}\frac{1}{\pi}\frac{dk}{dE}dE - e\int_0^{E_{F2}} \frac{\hbar k'}{m}\cdot\frac{1}{\pi}\cdot\frac{dk'}{dE'}dE' \quad \text{(Assume } T = 0\,K\text{)}$$ [4]

Furthermore, $E = \frac{p^2}{2m} = \frac{(\hbar k)^2}{2m} \Rightarrow \frac{dE}{dk} = \frac{\hbar^2 k}{m}$

$$\Rightarrow I = e\int_0^{E_{F1}} \frac{\hbar k}{m}\cdot\frac{1}{\pi}\cdot\frac{m}{\hbar^2 k}dE - e\int_0^{E_{F2}} \frac{\hbar k'}{m}\cdot\frac{1}{\pi}\cdot\frac{m}{\hbar^2 k'}dE'$$

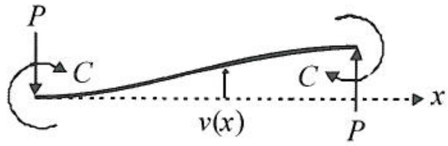
$$= e\int_{E_{F2}}^{E_{F1}} \frac{1}{\hbar\pi}dE = \frac{e}{\hbar\pi}(E_{F1} - E_{F2})$$

$$\Rightarrow I = \frac{e}{\frac{h}{2\pi}\cdot\pi}(eV) = \frac{2e^2}{h}\cdot V$$

$$\therefore G = \frac{I}{V} = \frac{2e^2}{h} \;//$$ [6]

Question 4

a) Sketch showing a flexure subject to a transverse load P:



[4]

The bending moment a distance x along the beam is $M = P(L - x) - C$, and we know that the couple C must satisfy $2C = PL$ for equilibrium, so the bending equation is:

$$M = P(L/2 - x) = EIv''$$

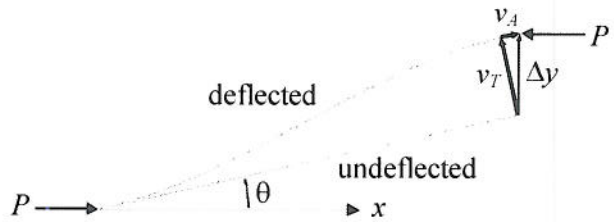
Integrating twice, and applying the boundary conditions $v(0) = v'(0) = 0$, the deflection profile is obtained as:

$$v(x) = P(Lx^2/4 - x^3/6)$$

The end deflection is therefore $v(L) = PL^3/12EI$, and the transverse stiffness is $k_T = \dfrac{P}{v(L)} = \dfrac{12EI}{L^3} = \dfrac{w^3hE}{L^3}$, where we have used $I = \dfrac{w^3h}{12}$. [4]

From Hooke's law, an axial stress of F/wh will lead to an axial strain $\delta L/L = F/(whE)$. The axial stiffness is therefore $k_A = \dfrac{F}{\delta L} = \dfrac{whE}{L}$. [2]

b) Considering first the reaction forces at the anchors, if these had y-components then these components would need to be equal and opposite, resulting in an overall moment on the structure that could not be balanced by couples at the anchors (the latter must also be equal and opposite from symmetry). We know, therefore, that the anchor reactions lie along x. Now considering the equilibrium of an individual flexure, since the force at one end lies along the x-direction, the same must apply to the force at the opposite end.



Since the forces P lie along x, the flexure is subject to an axial load of $P\cos\theta$ and a transverse load of $P\sin\theta$. If the flexure were unconstrained then it would have expanded thermally by an amount $L\varepsilon_{th}$. However, it will also be compressed axially by an amount $P\cos\theta/k_A$ due to the axial load. Taking both of these effects into account, the net axial end displacement will be: $v_A = L\varepsilon_{th} - P\cos\theta/k_A$. The transverse deflection arises only from the transverse load, and so is given simply by: $v_T = P\sin\theta/k_T$. [4]

From the diagram, the relationships between the axial and transverse displacements and Δy are: $v_A = \Delta y \sin\theta$, $v_T = \Delta y \cos\theta$. Combining these with the given equations we have:

$$\Delta y \sin\theta = L\varepsilon_{th} - P\cos\theta / k_A \quad ; \quad \Delta y \cos\theta = P\sin\theta / k_T$$

Eliminating P from these equations gives:

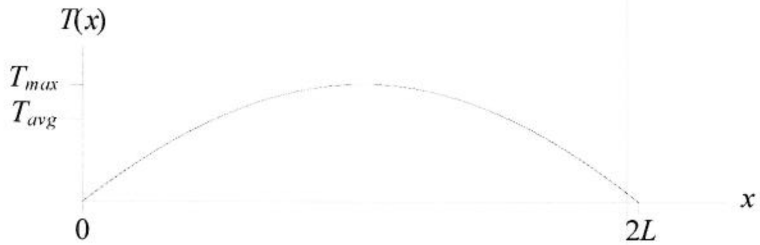
$$\Delta y \sin\theta = L\varepsilon_{th} - \Delta y k_T \cot\theta \cdot \cos\theta / k_A .$$

Rearranging, and using the fact that $k_T / k_A = (w/L)^2$, we obtain the required result:

$$\Delta y = \frac{L\varepsilon_{th}}{\sin\theta\,[1 + (w/L)^2 \cot^2\theta]} \qquad [4]$$

With $\theta = 2° = 0.035$ rads, $w = 4$ μm, $L = 200$ μm, for a deflection of 2 μm we require a thermal strain of $\varepsilon_{th} = 4.64 \times 10^{-4}$. Since $\varepsilon = \alpha T_{avg}$, where T_{avg} is the average temperature rise, with $\alpha = 13.3 \times 10^{-6}$ we have $T_{avg} = 34.9$ °C. [4]

c) Under the given assumptions, the temperature profile will be parabolic, going to zero at the anchors:



[4]

In the case of uniform beams, the heat flow equation (given on the *information for candidates* page) can be simply integrated twice to give:

$$T(x) = -cx^2/2 + Ax + B$$

where $c = I^2\rho / \kappa A^2$ and A and B are constants of integration. (Strictly x should now be distance measured along the beams.)

With $T = 0$ at $x = 0$ and $x = 2L$, this becomes $T(x) = cx(2L - x)/2$. The peak temperature rise is $T_{\max} = cL^2/2$ and the average rise is $T_{avg} = 2T_{\max}/3 = cL^2/3 = \dfrac{I^2\rho L^2}{3\kappa A^2}$. The current is therefore given by: $I = \dfrac{A}{L}\sqrt{\dfrac{3\kappa T_{avg}}{\rho}}$

With $T_{avg} = 34.9$ °C, $\kappa = 90.9$ Wm^{-1}K^{-1}, $\rho = 6.9$ μΩcm (= 6.9×10^{-8} Ωm), $A = 40$ μm^2 (= 4×10^{-11} m^2), $L = 200$ μm, we obtain $I = 74$ mA. [4]

Question 5

a) The accelerating proof mass m_p can be treated as a point load of $P = -m_p a$ at a distance of $(l + L/2)$ from the root of the cantilever. The bending moment at an arbitrary position x is therefore:

$$M = -m_p(l + L/2 - x)a$$ [4]

b) The strain in a bent beam is $\varepsilon = -y/R$ where y is the distance from neutral axis and R is the radius of curvature. Combining this with the relation $M = EI/R$, and noting the fact that the piezoresistive bridge is at the top surface of the cantilever where $y = h/2$, we have:

$$\varepsilon = -\frac{h}{2}\frac{M}{EI} = \frac{h}{2}\frac{m_p(l + L/2)}{EI}a$$

where we have evaluated M at the root of the cantilever, i.e. at $x = 0$. Using $m_p = WHL\rho$ and $I = wh^3/12$ this becomes:

$$\varepsilon = \frac{h}{2}\frac{WHL\rho(l + L/2)}{E(wh^3/12)}a = \frac{6WHL(l + L/2)}{Ewh^2}\rho a$$ [5]

Considering the orientations of the piezoresistors, we see that R_1 and R_3 will have $\sigma_L = E\varepsilon, \sigma_T = 0$ while R_2 and R_4 will have $\sigma_T = E\varepsilon, \sigma_L = 0$. Assuming the resistors have the same nominal value R, their values will change to:

$$R_1 = R_3 = R(1 + \pi_L E\varepsilon) \ ; \ R_2 = R_4 = R(1 + \pi_T E\varepsilon)$$

The bridge output will then be:

$$V_{out} = \frac{R_1R_3 - R_2R_4}{(R_1 + R_2)(R_3 + R_4)}V_s = \frac{(1 + \pi_L E\varepsilon)^2 - (1 + \pi_T E\varepsilon)^2}{(2 + \pi_L E\varepsilon + \pi_T E\varepsilon)^2}V_s$$

Using the formula given in part b), the stress $E\varepsilon$ at unit acceleration is 104.85 kPa, so we have $\pi_L E\varepsilon = 7.55 \times 10^{-5}$ and $\pi_T E\varepsilon = -6.92 \times 10^{-5}$. With $V_s = 5$ V, the above formula gives a sensitivity of 362 μV/(ms^{-2}). [5]

c) The bending equation is $M = EIv'' = -m_p(l + L/2 - x)a$ where $v(x)$ is the deflection profile. Integrating once and applying the boundary condition $v'(0) = 0$, the slope is obtained as:

$$v'(x) = -\frac{m_p a}{EI}[(l + L/2)x - x^2/2]$$

A second integration, with the boundary condition $v(0) = 0$, gives the deflection profile:

$$v(x) = -\frac{m_p a}{EI}[(l + L/2)x^2/2 - x^3/6]$$ [6]

The deflection at the end of the proof mass is approximately $v_{end} = v(l) + v'(l)L$. The proof mass is $m_p = 2 \times 2 \times 0.5 \times 10^{-9} \times 2330 = 4.66 \times 10^{-6}$ kg, and $I = 1 \times 10^{-3} \times (20 \times 10^{-6})^3 = 8 \times 10^{-18}$ m^4. With $E = 160$ GPa, $a = 100$ ms^{-2}, $x = l = 500$ μm and $L = 2$ mm, the above formulae give $v(L) = -0.0607$ μm and $v'(L) = -2.28 \times 10^{-4}$, so that $v_{end} = -0.517$ μm. [4]

d) For the simple cantilever, the bending moment at the root is $M = -m(l + L/2)/2$, so the strain at the location of the piezoresistive bridge is:

$$\varepsilon_{cant} = \frac{h}{2}\frac{m(l + L/2)}{2EI}a$$

The ratio of sensitivities is equal to the ratio of the strains, and using the first form of the strain in the answer to part a) above, we have:

$$\frac{\varepsilon}{\varepsilon_{cant}} = \frac{h}{2}\frac{m_p(l + L/2)}{EI}a \cdot \left(\frac{h}{2}\frac{m(l + L/2)}{2EI}a\right)^{-1} = \frac{2m_p}{m}$$

The result is approximate because the strain calculation for the device with a proof mass neglects the mass of the cantilever. [6]