UNIVERSITY OF LONDON

IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 1996

BEng Honours Degree in Computing Part III
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the Associateship of the City and Guilds of London Institute

PAPER 3.14

NUMERICAL ANALYSIS

Monday, April 29th 1996, 10.00 - 12.00

Answer THREE questions

For admin. only: paper contains
4 questions
4 pages (excluding cover page)

1. Let $F : \Re \mapsto \Re$ be a twice continuously differentiable function, $x^\star$ a point such that

$$F(x^\star) = F'(x^\star) = 0 \qquad F''(x^\star) \neq 0,$$

and I an interval centred on $x^\star$ for which

$$(\dagger) \quad \mu \equiv \frac{\max_{x\in I} |F''(x)|}{\min_{x\in I} |F''(x)|} < 2.$$

a Prove that Newton's method can be written in the form

$$x^{(k+1)} - x^\star = \frac{[x^{(k)} - x^\star]F'(x^{(k)}) - F(x^{(k)}) + F(x^\star)}{F'(x^{(k)}) - F'(x^\star)}.$$

b Use a and $(\dagger)$ to deduce that $x^{(k)} \in I$ implies that $x^{(k+1)} \in I$.

c Hence prove that Newton's method converges to $x^\star$ for any $x^{(0)} \in I$.

d Obtain the convergence rate result

$$(\ddagger) \quad \lim_{k\to\infty} \left| \frac{x^{(k+1)} - x^\star}{x^{(k)} - x^\star} \right| = \frac{1}{2}.$$

Finally, assume that $x^\star$ is a root of higher multiplicity, i.e. F is p times continuously differentiable with

$$F(x^\star) = F'(x^\star) = \ldots = F^{(p-1)}(x^\star) = 0 \qquad F^{(p)}(x^\star) \neq 0.$$

e Explain what condition should replace $(\dagger)$ and what is the convergence rate analogue to $(\ddagger)$ in this case.

[You may use without proof the following error term in Taylor's series for a function G with p continuous derivatives:-

$$G(x) = \sum_{m=0}^{p-1} \frac{1}{m!}[x - y]^m G^{(m)}(y) + \frac{1}{p!}[x - y]^p G^{(p)}(\xi)$$

for some ξ between x and y.]

The five parts carry, respectively, 10%,30%,20%,15%,25% of the total marks.

2. Let $F : \Re \mapsto \Re$ be a twice differentiable function with

$$(\dagger) \quad |F''(x)| \leq \gamma \qquad \forall x \in \Re$$

and $x^{(0)} \in \Re$ a point for which $F'(x^{(0)}) \neq 0$. If $x^{(1)}$ is the Newton iterate, $x^{(1)} = x^{(0)} - F(x^{(0)})/F'(x^{(0)})$, and

$$(\ddagger) \quad |F'(x^{(0)})| \geq 2|\ell_0|\gamma,$$

where $\ell_0 \equiv x^{(1)} - x^{(0)}$, you are asked to obtain the following results.

a Use $(\dagger)$ to prove that $|F'(x^{(1)}) - F'(x^{(0)})| \leq |\ell_0|\gamma$ and then $(\ddagger)$ to show that $|F'(x^{(1)}) - F'(x^{(0)})| \leq \frac{1}{2}|F'(x^{(0)})|$. Hence deduce that $F'(x^{(1)}) \neq 0$ and

$$|F'(x^{(1)})| \geq \tfrac{1}{2}|F'(x^{(0)})|.$$

b If $x^{(2)}$ is the next Newton iterate, $x^{(2)} = x^{(1)} - F(x^{(1)})/F'(x^{(1)})$, prove that

$$x^{(2)} - x^{(1)} = \frac{F(x^{(0)}) - F(x^{(1)}) + F'(x^{(0)})[x^{(1)} - x^{(0)}]}{F'(x^{(1)})}$$

and hence use a.$(\dagger)$ and $(\ddagger)$ to deduce that

$$|\ell_1| \leq \tfrac{1}{2}|\ell_0|,$$

where $\ell_1 \equiv x^{(2)} - x^{(1)}$.

c Use a,b and $(\ddagger)$ to prove that

$$|F'(x^{(1)})| \geq 2|\ell_1|\gamma.$$

d Explain carefully how the above results may be used in an inductive proof to show that, under conditions $(\dagger)$ and $(\ddagger)$, Newton's method $x^{(k+1)} = x^{(k)} - F(x^{(k)})/F'(x^{(k)})$ will never breakdown and the sequence $\{x^{(k)}\}$ must converge to a root of F.

[You may use without proof the result that $(\dagger)$ implies

$$|F(x) - F(y) - F'(y)[x-y]| \leq \frac{\gamma}{2}|x-y|^2 \qquad \forall x, y \in \Re.]$$

The four parts carry, respectively, 25%,30%,15%,30%, of the total marks.

Turn over ...

3. Let A be a real, symmetric $n \times n$ matrix.

   a Prove that every eigenvalue of A is real.

   b Prove that eigenvectors corresponding to distinct eigenvalues are orthogonal.

   c Define the Rayleigh quotient $\rho(\mathbf{u})$ for a non-zero $\mathbf{u} \in \Re^n$.

Assume, in addition, that the eigenvalues of A are ordered

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n,$$

with their corresponding eigenvectors $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ forming an orthonormal set.

   d Prove that, for any $1 \leq p \leq q \leq n$,

$$\mathbf{v} = \sum_{i=p}^{q} \alpha_i \mathbf{u}_i \neq \mathbf{0} \qquad \Longrightarrow \qquad \lambda_p \geq \rho(\mathbf{v}) \geq \lambda_q.$$

   e If $|\lambda_1| \geq |\lambda_n|$ and $\mathbf{v} = \sum_{i=1}^{n} \alpha_i \mathbf{u}_i \neq \mathbf{0}$, prove that

$$0 \leq \lambda_1 - \rho(\mathbf{v}) \leq 2\|A\|_2 \frac{\sum_{i=2}^{n} \alpha_i^2}{\sum_{i=1}^{n} \alpha_i^2}.$$

The five parts carry, respectively, 20%,20%,10%,20%,30% of the total marks.

4. Let A be a real $n \times n$ non-singular matrix, with eigenvalues

$$\lambda_1, \lambda_2, \ldots, \lambda_n$$

(possibly complex) and a linearly independent set of eigenvectors

$$\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\},$$

and let $t \in \Re$ be fixed and non-zero.

a Prove that the iteration

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + t\left(\mathbf{b} - A\mathbf{x}^{(k)}\right)$$

will converge to the unique solution $\mathbf{x}^\star$ of

$$A\mathbf{x} = \mathbf{b}$$

from any starting value $\mathbf{x}^{(0)}$ *if and only if*

$$(\dagger) \qquad \max_{1 \leq i \leq n} \{|1 - t\lambda_i|\} < 1.$$

b Explain what values of t will satisfy ($\dagger$) if :-

i) $\lambda_1, \lambda_2, \ldots, \lambda_n$ are real and positive,
ii) $\lambda_1, \lambda_2, \ldots, \lambda_n$ are real and negative,
iii) $\lambda_1, \lambda_2, \ldots, \lambda_n$ are real and of both signs,
iv) the real parts of $\lambda_1, \lambda_2, \ldots, \lambda_n$ are strictly positive,
v) the real parts of $\lambda_1, \lambda_2, \ldots, \lambda_n$ are strictly negative,
vi) the real parts of $\lambda_1, \lambda_2, \ldots, \lambda_n$ are non-zero but of both signs.

c For cases i) and ii) in b, deduce what values of t will make

$$\max_{1 \leq i \leq n} \{|1 - t\lambda_i|\}$$

smallest?

The eight parts carry, respectively, 30%,10%,5%,5%,15%,5%,10%,20% of the total marks.

End of Paper