SOLUTIONS: ESTIMATION AND FAULT DETECTION

1. Solution

a) By defining $x_1 = z_s$, $x_2 = \dot{z}_s$, $x_3 = z_u$, $x_4 = \dot{z}_u$, $u = F$, we obtain

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{k_s}{m_s}x_1 + \frac{k_s}{m_s}x_3 + \frac{1}{m_s}u \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = \frac{k_s}{m_u}x_1 - \frac{k_s+k_t}{m_u}x_3 - \frac{1}{m_u}u + \frac{k_t}{m_u}z_r \\ y = x_1 \end{cases}$$

Then, the state equations can be written in matrix form as follows

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_s}{m_s} & 0 & \frac{k_s}{m_s} & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_s}{m_u} & 0 & -\frac{k_s+k_t}{m_u} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m_s} \\ 0 \\ -\frac{1}{m_u} \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{k_t}{m_u} \end{bmatrix} z_r \\ y = [1\,0\,0\,0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \end{cases}$$

[4 marks]

b) By substituting the parameters with the given values, we can rewrite the matrices of the state equations as

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -200 & 0 & 200 & 0 \\ 0 & 0 & 0 & 1 \\ 1250 & 0 & -1450 & 0 \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ 0.4 \\ 0 \\ -2.5 \end{bmatrix}; \quad C = [1\,0\,0\,0].$$

The observability matrix is given by

$$Q = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -200 & 0 & 200 & 0 \\ 0 & -200 & 0 & 200 \end{bmatrix}$$

It is easy to observe that Q is full rank. Therefore the input-free system is completely observable.

Furthermore, we know from the theory that an algebraic output feedback law does not change the observabilities properties of the system. Therefore, the system remains completely observable for any value of the scalar h.

[6 marks]

c) The observer canonical form has to be determined. We have

$$\det(sI - A) = s^4 + 1650s^2 + 40000$$

using the expansion of cofactors. Then, the matrices A_o and C_o of the observer canonical form are

$$A_o = \begin{bmatrix} 0 & 0 & 0 & -40000 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1650 \\ 0 & 0 & 1 & 0 \end{bmatrix}; \quad C_o = [0\,0\,0\,1]$$

The observability matrix Q_o computed on the basis of the pair (A_o, C_o) is

$$Q_o = \begin{bmatrix} C_o \\ C_oA_o \\ C_oA_o^2 \\ C_oA_o^3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1650 \\ 1 & 0 & -1650 & 0 \end{bmatrix}$$

Using Q determined in the answer to Question 1-b), the matrix T_o transforming the original state equations into the observer canonical form

$$\begin{cases} \dot{z} = A_oz + B_ou \\ y = C_oz \end{cases} \quad \text{with} \quad A_O = T_o^{-1}AT_o; B_o = T_o^{-1}B; C_o = CT_o$$

is given by

$$T_o = Q^{-1}Q_o = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0.005 & 0 & -7.25 \\ 0.005 & 0 & -7.25 & 0 \end{bmatrix}$$

Now considering

$$L_o = \begin{bmatrix} l_o^{(1)} \\ l_o^{(2)} \\ l_o^{(3)} \\ l_o^{(4)} \end{bmatrix}$$

we have

$$\det[sI - (A_o - L_oC_o)] = s^4 + l_o^{(4)}s^3 + (l_o^{(3)} + 1650)s^2 + l_o^{(2)}s + l_o^{(1)} + 40000$$

and by equating this polynomial with the polynomial having the desired observer eigenvalues as roots, that is

$$\alpha_d(s) = (s+10)^2(s+5)^2 = s^4 + 30s^3 + 325s^2 + 1500s + 2500$$

we immediately obtain

$$l_o^{(1)} = -37500; \quad l_o^{(2)} = 1500; \quad l_o^{(3)} = -1325; \quad l_o^{(4)} = 30$$

Finally, the observer gain L such that $F = A - LC$ has the desired eigenvalues $\lambda_1 = -5, \lambda_2 = -5, \lambda_3 = -10, \lambda_4 = -10$ is given by

$$L = T_oL_o = \begin{bmatrix} 30 \\ -1325 \\ -210 \\ 9418.8 \end{bmatrix}$$

[7 marks]

d) Using the state-space description obtained in the answer to Question 1-a), we note that matrix C has rank equal to $p = 1$. Therefore, the order of the reduced-order state observer is $n - p = 3$. [3 marks]

2. Solution

a) By observing Fig. 1.1 of the text of the exam paper, we obtain

$$\dot{h}_1 = \frac{u_1 - v}{A_1},$$

$$\dot{h}_2 = \frac{u_2 + v - y}{A_2},$$

$$y = \alpha h_2,$$

$$v = \beta(h_1 - h_2).$$

We can then write the state equations

$$\begin{cases} \begin{bmatrix} \dot{h}_1 \\ \dot{h}_2 \end{bmatrix} = \begin{bmatrix} -\frac{\beta}{A_1} & \frac{\beta}{A_1} \\ \frac{\beta}{A_2} & -\frac{\beta+\alpha}{A_2} \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{A_1} & 0 \\ 0 & \frac{1}{A_2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ y = [0\ \alpha] \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \end{cases},$$

where

$$A = \begin{bmatrix} -\frac{\beta}{A_1} & \frac{\beta}{A_1} \\ \frac{\beta}{A_2} & -\frac{\beta+\alpha}{A_2} \end{bmatrix}; \quad B = \begin{bmatrix} \frac{1}{A_1} & 0 \\ 0 & \frac{1}{A_2} \end{bmatrix}; \quad C = [0\ \alpha].$$

We can then compute the vector transfer function from the inputs u_1, u_2 to the output y, as

$$G(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B = [0\ \alpha] \begin{bmatrix} s + \frac{\beta}{A_1} & -\frac{\beta}{A_1} \\ -\frac{\beta}{A_2} & s + \frac{\beta+\alpha}{A_2} \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{A_1} & 0 \\ 0 & \frac{1}{A_2} \end{bmatrix} =$$

$$= \frac{1}{s^2 + (\frac{\beta}{A_1} + \frac{\beta+\alpha}{A_2})s + \frac{\beta\alpha}{A_1 A_2}} \begin{bmatrix} \frac{\beta\alpha}{A_1 A_2} & \frac{\alpha}{A_2}(s + \frac{\beta}{A_1}) \end{bmatrix}.$$

[3 marks]

b) The observability matrix is given by

$$Q = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & \alpha \\ \frac{\alpha\beta}{A_2} & -\frac{\alpha(\beta+\alpha)}{A_2} \end{bmatrix}$$

In order for the system to be completely observable, matrix Q has to be full-rank. As

$$\det Q = -\frac{\alpha^2\beta}{A_2}$$

we conclude that the system is completely observable if $\alpha^2\beta \neq 0$, that is, if $\alpha \neq 0$ and $\beta \neq 0$.

[3 marks]

c) Setting $A_1 = A_2 = 1$, $\alpha = 1/2$, $\beta = 2$, we obtain

$$\begin{cases} \dot{x} = \begin{bmatrix} -2 & 2 \\ 2 & -5/2 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u \\ \\ y = [0 \;\; 1/2] x \end{cases},$$

The Luenberger observer takes on the form:

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}) \\ \hat{y} = C\hat{x} \end{cases}$$

where L denotes the observer gain matrix to be designed. The pair (A, C) is completely observable according to the answer to Question 2-b). We let

$$F = A - LC = \begin{bmatrix} -2 & 2 - 1/2 l_1 \\ 2 & -5/2 - 1/2 l_2 \end{bmatrix} \quad \text{where} \quad L = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}$$

After some algebra, we obtain that by selecting

$$L = \begin{bmatrix} 7 \\ 7 \end{bmatrix}$$

we have $\det(sI - F) = s^2 + 8s + 15$ and hence $\lambda_1 = -3, \lambda_2 = -5$.

Finally

$$e(t) = \mathcal{L}^{-1}[(sI - F)^{-1}]\tilde{e} = \mathcal{L}^{-1} \begin{bmatrix} \dfrac{s+6}{s^2+8s+15} & \dfrac{-3/2}{s^2+8s+15} \\ \dfrac{2}{s^2+8s+15} & \dfrac{s+2}{s^2+8s+15} \end{bmatrix} \begin{bmatrix} \tilde{e}_1 \\ \tilde{e}_2 \end{bmatrix}$$

and hence, after some algebra and computing the inverse Laplace transforms, we get

$$\begin{aligned} \varepsilon(t) &= Ce(t) = \tfrac{1}{2}\mathcal{L}^{-1}\left[\tfrac{2}{s^2+8s+15}\right]\tilde{e}_1 + \tfrac{1}{2}\mathcal{L}^{-1}\left[\tfrac{s+2}{s^2+8s+15}\right]\tilde{e}_2 \\ &= \tfrac{1}{2}e^{-3t}(\tilde{e}_1 - \tfrac{1}{2}\tilde{e}_2) + \tfrac{1}{2}e^{-5t}(-\tilde{e}_1 + \tfrac{3}{2}\tilde{e}_2). \end{aligned}$$

[6 marks]

d) Consider an abrupt actuator fault f occurring at some finite time T_0 in Tank 1. Then, the state equations after fault time T_0 are

$$\begin{cases} \dot{x} = \begin{bmatrix} -2 & 2 \\ 2 & -5/2 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u + f \\ \\ y = [0 \;\; 1/2] x \end{cases},$$

with

$$f = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} u.$$

Then, the dynamics of the state estimation error after the occurrence of the fault (that is, for $t \geq T_0$) can be written as

$$\dot{e}(t) = \dot{x}(t) - \dot{\hat{x}}(t) = Fe(t) + \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} u(t)$$

and thus, exploiting the answer to Question 2-c), we have

$$\varepsilon(t) = \frac{1}{2}e^{-3t}(\tilde{e}_1 - \frac{1}{2}\tilde{e}_2) + \frac{1}{2}e^{-5t}(-\tilde{e}_1 + \frac{3}{2}\tilde{e}_2) + \mathscr{L}^{-1}\left[\frac{-1}{s^2+8s+15}U(s)\right]$$

where $U(s)$ denotes the Laplace transform of the input function $u(t)$. From this relationship, it follows that the residual $\varepsilon(t)$ does not vanish asymptotically under the action of fault f.

Now, using the answer to Question 2-c), and as $|x(0)| \leq 4$, for $t < T_0$ we can write

$$\begin{aligned} |\varepsilon(t)| = \; & \left|\tfrac{1}{2}e^{-3t}(\tilde{e}_1 - \tfrac{1}{2}\tilde{e}_2) + \tfrac{1}{2}e^{-5t}(-\tilde{e}_1 + \tfrac{3}{2}\tilde{e}_2)\right| \\ & \leq \tfrac{1}{2}\left|e^{-3t} - e^{-5t}\right| |\tilde{e}_1| + \tfrac{1}{2}\left|3e^{-5t} - e^{-3t}\right| |\tilde{e}_2| \end{aligned}$$

By noting that

$$e^{-3t} - e^{-5t} > 0, \;\; \forall t > 0,$$

$$3e^{-5t} - e^{-3t} \geq 0, \;\; 0 < t \leq \frac{1}{2}ln(1/3),$$

$$3e^{-5t} - e^{-3t} < 0, \;\; t > \frac{1}{2}ln(1/3),$$

we can define the fault detection threshold as

$$\bar{\varepsilon}(t) = \begin{cases} e^{-3t} + 2e^{-5t} & 0 < t \leq 0.5493 \\ 3e^{-3t} - 4e^{-5t} & t > 0.5493 \end{cases}$$

and observing that $\bar{\varepsilon}(t) \to 0$ for $t \to \infty$, it is immediate to conclude that for $t \geq T_0$

$$\exists T_d > T_0 : \varepsilon(T_d) > \bar{\varepsilon}(T_d)$$

[8 marks]

3. Solution

a) By analysing the block diagram in Question 3, we obtain the following discrete-time dynamic system affected by state and output disturbances:

$$\begin{cases} x_1(t+1) = 2x_1(t) + 3x_2(t) + u_1(t) + \xi_1(t) \\ x_2(t+1) = -4x_2(t) + u_2(t) + \xi_2(t) \\ y_1(t) = 3x_1(t) + \eta_1(t) \\ y_2(t) = 4x_2(t) + \eta_2(t) \end{cases}$$

which can be rewritten as

$$\begin{cases} x(t+1) = \begin{bmatrix} 2 & 3 \\ 0 & -4 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u(t) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xi(t) \\ y(t) = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \eta(t) \end{cases}$$

[3 marks]

b) By considering the input signals in the Question 3-b), the state equations become

$$\begin{cases} x_1(t+1) = -\frac{1}{2}x_1(t) + \xi_1(t) \\ x_2(t+1) = \frac{1}{3}x_2(t) + \xi_2(t) \end{cases}$$

It is easy to see that the two states are decoupled. It is then possible to separately compute the optimal steady-state Kalman predictor for each state component. Let us first consider variable x_1:

$$\begin{cases} x_1(t+1) = -\frac{1}{2}x_1(t) + \xi_1(t) \\ y_1(t) = 3x_1(t) + \eta_1(t) \end{cases}$$

where $\xi_1(\cdot) \sim WGN(0,4)$ and $\eta_1(\cdot) \sim WGN(0,1)$. The general algebraic Riccati equation is

$$P = F\left[P - PH^\top \left(V_2 + HPH^\top\right)^{-1} HP\right] F^\top + V_1$$

Letting $F_1 = -1/2$, $H_1 = 3$, $V_{1,1} = 4$, $V_{1,2} = 1$, we have

$$P_1 = \frac{1}{4}\left(P_1 - \frac{9P_1^2}{1+9P_1}\right) + 4 \Longrightarrow 9P_1^2 - \frac{141}{4}P_1 - 4 = 0$$

thus obtaining the two solutions

$$\tilde{P}_{1,1} \simeq -0.11 \quad \text{and} \quad \tilde{P}_{1,2} \simeq 4.03$$

Clearly, the only admissible solution is the positive one. Thus $\tilde{P}_1 = \tilde{P}_{1,2} \simeq 4.03$. Accordingly:

$$\tilde{K}_1 = F_1\tilde{P}_1 H_1^\top \left(V_{1,2} + H_1\tilde{P}_1 H_1^\top\right)^{-1} \simeq -0.16$$

Let us know repeat the same procedure for the second state variable:

$$\begin{cases} x_2(t+1) = \frac{1}{3}x_2(t) + \xi_2(t) \\ y_2(t) = 4x_2(t) + \eta_2(t) \end{cases}$$

where $\xi_2(\cdot) \sim WGN(0,1)$ and $\eta_2(\cdot) \sim WGN(0,9)$. Letting $F_2 = 1/3$, $H_2 = 4$, $V_{2,1} = 1$, $V_{2,2} = 9$, the algebraic Riccati equation is

$$P_2 = \frac{1}{9}\left(P_2 - \frac{16P_2^2}{9+16P_2}\right) + 1 \Longrightarrow 16P_2^2 - 8P_2 - 9 = 0$$

thus obtaining the two solutions

$$\bar{P}_{2,1} \simeq -0.54 \quad \text{and} \quad \bar{P}_{2,2} \simeq 1.04$$

Again, the only admissible solution is the positive one. Thus $\bar{P}_2 = \bar{P}_{2,2} \simeq 1.04$. Accordingly:

$$\bar{K}_2 = F_2\bar{P}_2H_2^\top \left(V_{2,2} + H_2\bar{P}_2H_2^\top\right)^{-1} \simeq 0.054$$

The steady state Kalman predictor obeys to the following equations:

$$\begin{cases} \hat{x}(t+1|t) = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \hat{x}(t|t-1) + \bar{K}e(t) \\ \hat{y}(t+1|t) = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} \hat{x}(t+1|t) \\ e(t) = y(t) - \hat{y}(t|t-1) \end{cases}$$

where $\bar{K} \simeq \begin{bmatrix} -0.16 & 0 \\ 0 & 0.054 \end{bmatrix}$.

[6 marks]

c) We have

$$\mathrm{Cov}[x(t) - \hat{x}(t|t-1)] = \bar{P} = \begin{bmatrix} \bar{P}_1 & 0 \\ 0 & \bar{P}_2 \end{bmatrix} \simeq \begin{bmatrix} 4.03 & 0 \\ 0 & 1.04 \end{bmatrix}$$

Let us now compute $\mathrm{Cov}[x(t)]$. The stochastic process $x(\cdot)$ generated by the system

$$\begin{cases} x_1(t+1) = -\frac{1}{2}x_1(t) + \xi_1(t) \\ x_2(t+1) = \frac{1}{3}x_2(t) + \xi_2(t) \end{cases}$$

is stationary because $\xi_1(\cdot) \sim WGN(0,4)$ and $\xi_2(\cdot) \sim WGN(0,1)$ are independent, and because the system is asymptotically stable.

Because of the stationarity of $x(\cdot)$, $\mathrm{var}[x_i(t)] = \mathrm{var}[x_i(t-1)]$, $i = 1,2$, and since $\mathbb{E}[x_1(t)] = \mathbb{E}[x_2(t)] = 0$, hence

$$\mathrm{var}[x_1(t)] = \frac{1}{4}\mathrm{var}[x_1(t)] + 4 \Longrightarrow \mathrm{var}[x_1(t)] = \frac{16}{3}$$

$$\mathrm{var}[x_2(t)] = \frac{1}{9}\mathrm{var}[x_1(t)] + 1 \Longrightarrow \mathrm{var}[x_2(t)] = \frac{9}{8}$$

Therefore $\text{Cov}[x(t)] = \begin{bmatrix} \frac{16}{3} & 0 \\ 0 & \frac{9}{8} \end{bmatrix}$. Now:

$$\text{var}[x_1(t) - \hat{x}_1(t|t-1)] \simeq 4.03 < \frac{16}{3} = \text{var}[x_1(t)]$$

$$\text{var}[x_2(t) - \hat{x}_2(t|t-1)] \simeq 1.04 < \frac{9}{8} = \text{var}[x_2(t)]$$

As can be seen, the Kalman estimator allows to reduce the variance with respect to the a-priori one thanks to the use of the measurements $y(t)$.

[5 marks]

d) The steady-state Kalman filter obeys to the following equation:

$$\hat{x}(t|t) = \hat{x}(t|t-1) + K_0 e(t)$$

The following relationship exists between the Kalman predictor gain $K(t)$ and the Kalman filter gain $K_0(t)$:

$$K(t) = FK_0(t).$$

Therefore, we compute the constant gain vector $\bar{K}_0$ of the optimal steady-state Kalman filter as

$$\bar{K}_0 = F^{-1}\bar{K} \simeq \begin{bmatrix} 0.32 & 0 \\ 0 & 0.162 \end{bmatrix}$$

Since $\hat{x}(t+1|t) = F\hat{x}(t|t)$, then we can write

$$\hat{x}(t|t) = F^{-1}\hat{x}(t+1|t).$$

We can then compute the covariance matrix of the filtering error $\text{Cov}[x(t) - \hat{x}(t|t)]$. We can write the following

$$v(t) = x(t) - \hat{x}(t|t) = x(t) - \hat{x}(t|t-1) - K_0(t)e(t) =$$
$$= (I - K_0 H)(x(t) - \hat{x}(t|t-1)) - K_0(t)\eta(t)$$

Since both $x(t)$ and $\hat{x}(t|t-1)$ are not correlated with the measurement noise $\eta(t)$, then we compute

$$\text{Cov}[x(t) - \hat{x}(t|t)] = (I - K_0 H)\text{Cov}[x(t) - \hat{x}(t|t-1)](I - K_0 H)^\top + K_0 V_2 K_0^\top$$

$$\simeq \begin{bmatrix} 0.0016 & 0 \\ 0 & 0.124 \end{bmatrix} \bar{P} + \begin{bmatrix} 0.1024 & 0 \\ 0 & 0.0262 \end{bmatrix} V_2 \simeq \begin{bmatrix} 0.11 & 0 \\ 0 & 0.365 \end{bmatrix}$$

Now:

$$\text{var}[x_1(t) - \hat{x}_1(t|t)] \simeq 0.11 < \text{var}[x_1(t) - \hat{x}_1(t|t-1)] \simeq 4.03 < \frac{16}{3} = \text{var}[x_1(t)]$$

$$\text{var}[x_2(t) - \hat{x}_2(t|t)] \simeq 0.365 < \text{var}[x_2(t) - \hat{x}_2(t|t-1)] \simeq 1.04 < \frac{9}{8} = \text{var}[x_2(t)]$$

As can be seen, the Kalman filter allows a further reduction of the variance of the estimation error with respect to the variance obtained by the Kalman one-step-ahead predictor in the answer to Question 3-c), thanks to the use of an additional output measurement.

[6 marks]

4. Solution

a) Referring to the block-diagram shown in Fig. 4.1 of the text of the exam paper, one assigns to the sub-system "Int. n. 1" a state variable denoted by x_1, to the sub-system "Int. n. 2" a state variable denoted by x_2, to the sub-system "Int. n. 3" a state variable denoted by x_3, and to the sub-system "Int. n. 4" a state variable denoted by x_4.

After inspection of the block-diagram shown in Fig. 4.1 of the text of the exam paper, one immediately gets:

$$\begin{cases} \dot{x}_1 = -4x_2 \\ \dot{x}_2 = 4x_1 \\ \dot{x}_3 = 15x_2 - x_4 \\ \dot{x}_4 = -2x_2 + x_3 - 2x_4 \\ y = x_2 + x_4 \end{cases}$$

Then, the matrix form of the state-space description of the whole interconnected system is given by:

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = A \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 & -4 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 0 & 15 & 0 & -1 \\ 0 & -2 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \\ y = C \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = [0\ 1\ 0\ 1] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \end{cases}$$

[3 marks]

b) i) After some algebra, the observability matrix is

$$Q = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 4 & -2 & 1 & -2 \\ -8 & 3 & -2 & 3 \\ 12 & -4 & 3 & -4 \end{bmatrix}$$

Clearly

$$\text{rank}(Q) = 2 .$$

Hence, we conclude that the system is not completely observable.

[3 marks]

ii) To answer Question 4-b)-ii) of the exam paper, the observability canonical form has to be determined starting from the state-space description determined in the answer to Question 4-a). In particular, recall that the observability matrix computed in the answer to Question 4-b)-i) is:

$$Q = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 4 & -2 & 1 & -2 \\ -8 & 3 & -2 & 3 \\ 12 & -4 & 3 & -4 \end{bmatrix}$$

Since $\text{rank}(Q) = 2$, a basis $\{\alpha, \beta\}$ for $\ker(Q)$ can be determined computing a basis for the vector subspace of solutions of:

$$Qv = 0 \Longrightarrow \begin{bmatrix} 0 & 1 & 0 & 1 \\ 4 & -2 & 1 & -2 \\ -8 & 3 & -2 & 3 \\ 12 & -4 & 3 & -4 \end{bmatrix} v = 0$$

A possible choice is:

$$\alpha = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \quad \beta = \begin{bmatrix} -1 \\ 0 \\ 4 \\ 0 \end{bmatrix}$$

A basis $\{\delta, \gamma\}$ for the orthogonal complement to $\ker(Q)$ has to be determined. After a simple algebra, we obtain, for example:

$$\delta = \begin{bmatrix} 4 \\ -17/2 \\ 1 \\ -17/2 \end{bmatrix}; \; \gamma = \begin{bmatrix} 4 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Now, select the matrix

$$T = [\delta | \gamma | \alpha | \beta] = \begin{bmatrix} 4 & 4 & 0 & -1 \\ -17/2 & 1 & -1 & 0 \\ 1 & 1 & 0 & 4 \\ -17/2 & 1 & 1 & 0 \end{bmatrix}$$

and compute the inverse

$$T^{-1} = \begin{bmatrix} 8/323 & -1/19 & 2/323 & -1/19 \\ 4/19 & 1/19 & 1/19 & 1/19 \\ 0 & -1/2 & 0 & 1/2 \\ -1/17 & 0 & 4/17 & 0 \end{bmatrix}$$

By setting $x = Tz$, the following equivalent observability canonical form is obtained:

$$\begin{cases} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = T^{-1}ATz = \begin{bmatrix} -49/19 & -225/323 & 0 & 0 \\ 68/19 & 11/19 & 0 & 0 \\ 19/2 & -19/2 & 0 & 4 \\ -30 & 60/17 & -4 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} \\ y = CTz = [-17\;2\;0\;0] \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} \end{cases}$$

Upon the Inspection of the observability canonical form, it is immediate to conclude that the sub-system described by the state variables z_1 and z_2 is observable whereas the sub-system described by the state variables z_3 and z_4 is not observable.

[5 marks]

iii) Since the system is not observable according to the answer to Question 4-b)-i), the non-observable subspace X_{no} is not trivial, that is $X_{no} \neq \{0\}$.

A basis for the non-observable subspace X_{no} is a basis for $\ker(Q)$, that is (see the answer to Question 4-b)-ii) a possible basis is given by the two following linearly-independent vectors:

$$\alpha = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \quad \beta = \begin{bmatrix} -1 \\ 0 \\ 4 \\ 0 \end{bmatrix}$$

[3 marks]

c) According to the answer to Question 4-b)-ii), the observable sub-system is described by:

$$\begin{cases} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \widetilde{A}_{11} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} -49/19 & -225/323 \\ 68/19 & 11/19 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \\ y = \widetilde{C}_1 \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = [-17\ 2] \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \end{cases}$$

which is fully observable by construction.

Then:

$$Y(s) = \mathscr{L}[y(t)] = \widetilde{C}_1 \left(sI - \widetilde{A}_{11} \right)^{-1} T^{-1} x(0^-)$$

and, after some algebra, one gets:

$$Y(s) = \left[\frac{-17(s-1)}{s^2+2s+1}, \frac{2s+17}{s^2+2s+1} \right] \cdot T^{-1} x(0^-)$$

The degree of the denominators is 2 whereas the dimension of the state vector is 4. This is consistent with the dimension of the non-observable subspace which is 2 which, in turns implies that the non-observable eigenvalues do not appear as roots of the polynomials at the denominator of the expression of $Y(s)$. Indeed, $\operatorname{rank}(Q) = 2$ coincides with the degree of the above denominators which is consistent with the observability theory.

[3 marks]

d) After inspection of the block-diagram shown in Fig. 4.1 of the text of the exam paper, one immediately gets:

$$\begin{cases} \dot{x}_1 = -4x_2 \\ \dot{x}_2 = 4x_1 \\ w = x_2 \end{cases}$$

The eigenvalues of this subsystem are the roots of the characteristic polynomial $s^2 + 4$ which are located on the imaginary axis of the complex plane thus giving rise to a sinusoidal state mode of response.

This sinusoidal mode of response does not show up in the output y because the eigenvalues of the above sub-system are clearly non-observable as can be immediately ascertained by inspecting the canonical form obtained in the answer to Question 4-b)-ii).

[3 marks]