UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 2001

MSci Honours Degree in Mathematics and Computer Science Part IV
MEng Honours Degrees in Computing Part IV
MSc in Advanced Computing
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the Associateship of the City and Guilds of London Institute
This paper is also taken for the relevant examinations for the Associateship of the Royal College of Science

PAPER C478=I4.37

ADVANCED OPERATIONS RESEARCH

Tuesday 1 May 2001, 10:00
Duration: 120 minutes

Answer THREE questions

Paper contains 4 questions
Calculators required

1a You are given an LP problem: $\min\{z = \boldsymbol{c}^T\boldsymbol{x} \mid \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}\}$ and variables are subject to type specifications. A basic feasible solution (BFS) with $z = -1$ and a type-2 incoming variable, $x_q = 0$ with $d_q = 5$, are given. Also, $\boldsymbol{\alpha}_q = \boldsymbol{B}^{-1}\boldsymbol{a}_q$ is available. Is the BFS degenerate?

Perform the ratio test. Determine the value of the incoming variable, the variable leaving the basis (if any), the new BFS and the new value of the objective function. Is the new BFS degenerate? Discuss the reasons in detail.

i	x_{Bi}	type(x_{Bi})	u_{Bi}	α_{iq}	t_i
1	2	2	$+\infty$	-1	
2	3	1	4	1	
3	8	2	$+\infty$	2	
4	2	1	6	-2	
5	4	2	$+\infty$	1	
6	0	3	$+\infty$	1	

b Solve the following linear programming problem with the dual simplex method. First, convert the problem into a form needed by the algorithm. Verify that the conditions of applying it are fulfilled. Discuss the solution in detail and explain your reasons.

Determine the dual solution from its defining equation.

$$\begin{array}{lrcrcrcrcr}
\min & x_1 & + & 4x_2 & + & 2x_3 & + & 2x_4 & & \\
\text{s. t.} & 2x_1 & - & x_2 & - & x_3 & & & \leq & -1 \\
 & -2x_1 & + & x_2 & + & x_3 & + & x_4 & \leq & 3 \\
 & & & 4x_2 & & x_3 & - & 2x_4 & \leq & -2 \\
\end{array}$$
$$x_j \geq 0,\ j = 1, \ldots, 4$$

c Explain the simplex multiplier, its derivation and its role in the revised simplex method.

(The three parts carry, respectively, 40%, 40% and 20% of the marks).

2a An exhibition centre is planning an extension. Three types of exhibition halls can be added.

Room	Size (m^2)	Cost (£1000)
Small	60	22
Medium	100	38
Large	150	50

Management thinks it would be desirable to add 5 small, 10 medium and 12 large halls. The expansion should be around $3000m^2$ and the total cost is limited to £1,000,000.

Write a goal programming model for the above problem if

- it is equally undesirable to underachieve the number of different halls (three desires!),
- it is undesirable to overachieve or underachieve the $3000m^2$ goal, underachievement being twice as bad as overachievement,
- the financial constraint cannot be exceeded.

Explain your work.

b Formulate the dual of the following two problems. Explain your work and findings.

(i)

$$\begin{array}{rrrrrl}
\max z = & 2x_1 - & 3x_2 + & 4x_3 - & 5x_4 & \\
\text{s.t.} & 2x_1 + & 5x_2 - & 4x_3 + & 9x_4 & \leq \quad 9 \\
& x_1 + & 4x_2 + & 2x_3 - & 6x_4 & \geq \ -9 \\
& 4x_1 - & 3x_2 - & 6x_3 + & 4x_4 & = \quad 1 \\
& & x_1, \ldots, x_4 \geq 0. & & &
\end{array}$$

(ii)

$$\begin{array}{rrrrl}
\min z = & -2x_1 + & x_2 + & 5x_3 & \\
\text{s.t.} & x_1 + & 3x_2 - & 2x_3 & \geq \ 0 \\
& -x_1 - & 2x_2 + & 5x_3 & = \ 0 \\
& 2x_1 + & 3x_2 + & 4x_3 & \leq \ 0 \\
& & x_1 \leq 0, \ x_2, x_3 \geq 0. & &
\end{array}$$

c Consider the assignment problem. It can briefly be described as follows. There are m jobs (or machines) and m assignees (workers, tasks, etc.) If worker i is assigned to task j it incurs a cost c_{ij} (training cost). Every worker has to be assigned to some task and every task has to be done. An allocation is sought that minimizes the training costs. Set up an appropriate integer programming model for the problem and show that if its LP relaxation is solved by the simplex method the solution will be all integer, thus an optimal solution to the problem.

What is the size of the problem in terms of constraints and variables? Show how this problem can be formulated as a minimal network flow problem.

(The three parts carry, respectively, 40%, 20% and 40% of the marks).

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3a Determine the type of each variable in the following two problems. Are the given solutions feasible? Do they satisfy the optimality conditions? Also, identify which solution is degenerate, if any, and say why..

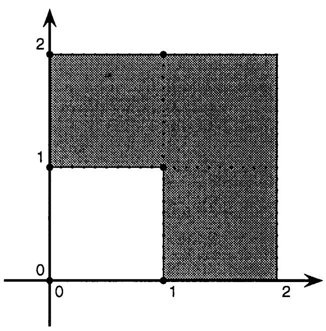
(i) Problem: max $\boldsymbol{c}^T\boldsymbol{x}$, $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$,

	x_1	x_2	x_3	x_4	x_5	x_6
ℓ_j	0	0	0	$-\infty$	0	0
u_j	0	$+\infty$	$+\infty$	$+\infty$	10	$+\infty$
type(x_j)						
In the solution:						
B/N	B	B	B	N	N	N
Value	0	1	11	0	10	0
d_j	0	0	0	0	10	10
Opt. cond. Y/N						

(ii) Problem: min $\boldsymbol{c}^T\boldsymbol{x}$, $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$,

	x_1	x_2	x_3	x_4	x_5	x_6
ℓ_j	0	0	$-\infty$	0	0	0
u_j	1	$+\infty$	$+\infty$	0	10	$+\infty$
type(x_j)						
In the solution:						
B/N	B	B	B	N	N	N
Value	1	1	-1	0	10	0
d_j	0	0	0	-10	10	0
Opt. cond. Y/N						

b Characterize the following, nonconvex, region shown in the figure. Write the appropriate linear inequalities and logic expressions to describe the points in the shaded area. Introduce indicator variable(s) if needed and give a MIP formulation of the region. Verify your solution by showing that point (1.5,1.5) satisfies your constraints and point (0.5,0.5) does not.



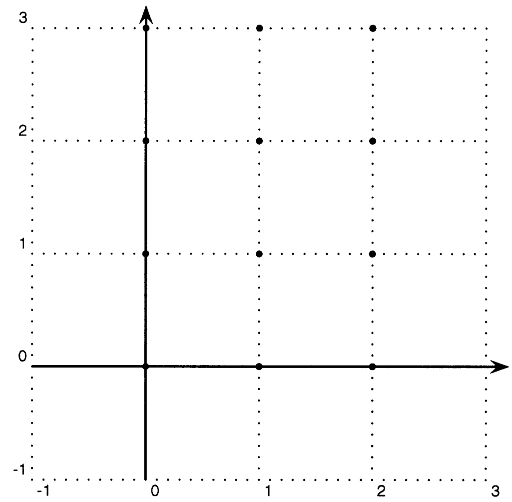
c Simplify and convert the following linear programming constraints into standard LP equalities. Indicate the type of the associated logical variable. Try to combine constraints if possible.

$$
\begin{aligned}
2x_1 - x_2 + 3x_3 - 4x_5 &\geq -5 - x_1 + x_4 && (1)\\
-x_1 + x_2 &\geq 2x_3 - x_4 && (2)\\
3x_1 - x_2 - 4 &\leq -2x_3 + x_4 && (3)\\
6 \geq 3x_1 - 2x_2 + x_3 - 4x_4 &\geq -6 && (4)\\
2x_1 - 3x_4 &= x_1 + x_2 - x_3 && (5)\\
x_2 - 2x_3 &\leq 3 + 3x_1 - x_4 && (6)
\end{aligned}
$$

(The three parts carry, respectively, 20%, 50% and 30% of the marks).

4a Solve the following mixed integer programming problem in two variables graphically. Use the graph printed here or your own drawing. It need not be very accurate. If in doubt, rely on the numerical data given below.

The objective is to maximize $z = x_1 + x_2$, where x_1 is a general nonnegative integer, x_2 is nonnegative. The feasible region of the LP relaxation of the problem is determined by the polygon with vertices: $O(0,0)$, $P_1(0,0.5)$, $P_2(1.95,3)$ and $P_3(1.05,0)$. Where are the feasible solutions located?



b A car manufacturing company can produce four different models. They want to maximize the profit by determining the quantity of each version (denoted by $x_1, \ldots, x_4$, in thousands) to be produced. Obviously, the models compete for the same resources. Therefore the production plan that achieves the goal has to be prepared with great care. Linear programming turned out to be an appropriate tool to assist the planners. Marketing research showed that any number of cars of each model can be sold. The high level model they set up is as follows.

$$\begin{array}{llrcrcrcrcr}
\text{max} & \text{Profit } z = & 4x_1 & + & 9x_2 & + & 6x_3 & + & 5x_4 & & \\
\text{s.t.} & \text{Body} & x_1 & + & 2x_2 & + & 1.5x_3 & + & x_4 & \leq & 160 \\
 & \text{Engine} & 5x_1 & + & 3x_2 & + & 2x_3 & + & 3x_4 & \leq & 240 \\
 & \text{Assembly} & x_1 & + & 2x_2 & + & 0.5x_3 & + & 2x_2 & \leq & 120 \\
 & & & & \multicolumn{5}{c}{x_1,\ x_2,\ x_3,\ x_4 \geq 0.} & &
\end{array}$$

The LP solver appended type-2 logical (slack) variables x_5, x_6, x_7 to constraints 1,2, and 3, respectively.

An optimal solution was achieved with $x_2 = 50$, $x_3 = 40$ and $x_6 = 10$ giving an objective value of $z = 690$.

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The optimal basis is $B = \{2, 6, 3\}$. The basis matrix is

$$\boldsymbol{B} = \begin{bmatrix} 2 & 0 & 3/2 \\ 3 & 1 & 2 \\ 2 & 0 & 1/2 \end{bmatrix} \qquad \text{its inverse} \qquad \boldsymbol{B}^{-1} = \begin{bmatrix} -\frac{1}{4} & 0 & \frac{3}{4} \\ -\frac{5}{4} & 1 & -\frac{1}{4} \\ 1 & 0 & -1 \end{bmatrix}$$

(i) Determine the simplex multiplier associated with $\boldsymbol{B}$ and the reduced costs of nonbasic variables.

(ii) Determine the ranges of all structural objective coefficients within which the solution remains optimal.

c (i) In some model you have to include a variable that can take only the following values: $(0, -1.1, 2.2, -3.3, 4.4)$. How can you formulate this requirement?

(ii) An integer variable x is limited to take a value in the interval $[-3, 11]$. How would you express this condition using as few $0/1$ variables as possible?

(The three parts carry, respectively, 30%, 50% and 20% of the marks).