Imperial College
London

[E1.14 (Maths 2) 2009]

B.ENG. AND M.ENG. EXAMINATIONS 2009

PART I : MATHEMATICS 2 (ELECTRICAL ENGINEERING)

Date Thursday 4th June 2009 10.00 am - 1.00 pm

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

Answer EIGHT questions.

A mathematical formulae sheet is provided.

[*Before starting, please make sure that the paper is complete; there should be 7 pages, with a total of 10 questions. Ask the invigilator for a replacement if your copy is faulty.*]

1. (i) State whether each of the following functions is invertible or not. If not, change the domain in order that the resulting function has an inverse and give the range of your function:

   (a) $f(x) = x^2$ with domain $(-2, 1)$,

   (b) $f(x) = \cos(x)$ with domain $(-1, 1)$,

   (c) $f(x) = \sin(x)$ with domain $(-1, 1)$,

   (d) $f(x) = x^{1/3}$ with domain $(-10, 10)$,

   (e) $f(x) = x^3$ with domain $(-10, 10)$.

   (ii) The function $p(x)$ is defined by a power series

$$p(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} .$$

   If $i^2 + 1 = 0$ and x is real, is

$$\mathrm{Re}(p(ix))$$

   an even or an odd function of x? Find $\mathrm{Im}(p(ix))$, and determine whether it is even or odd as a function of x.

2. Sketch the graph of the function

$$f(x) = -x\,(\ln(x) - 1),$$

   with domain $(0, \infty)$, specifying

   (i) $\lim_{x \to 0} f(x)$,

   (ii) $\lim_{x \to \infty} f(x)$,

   (iii) all stationary points and their nature,

   (iv) all zeros of $f(x)$.

PLEASE TURN OVER

3. (i) Differentiate the following functions with respect to x:

(a) x^x,

(b) $\sin^{-1}(\sin(x)) + \sin(\sin^{-1}(x)) \quad \text{for} \quad -1 < x < 1$,

(c) $\int_0^x \sqrt{1+t^2}\, dt$,

(d) $\sum_{n=1}^{100} \frac{1}{n} \cos(nx)$.

(ii) If $f(x)$ is a function defined on a domain (a, b) and

$$\frac{df}{dx}(x) = (f(x))^2 ,$$

prove by induction, or otherwise, that

$$\frac{d^n f}{dx^n}(x) = n!\ (f(x))^{n+1}$$

for all x between a and b and all integers $n \geq 1$.

4. (i) Evaluate the integral

$$\int_{-1}^{1} \sqrt{1-x^2}\, dx$$

with a trigonometric substitution.

(ii) The hyperbolic cosine and sine are functions defined respectively by

$$\cosh(x) = \frac{1}{2}(e^x + e^{-x}) \quad \text{and} \quad \sinh(x) = \frac{1}{2}(e^x - e^{-x}) ,$$

whose domains are given by the set of all real numbers. Find

$$\frac{d}{dx}\cosh(x) \quad \text{and} \quad \frac{d}{dx}\sinh(x)$$

in terms of $\cosh(x)$ and $\sinh(x)$, and determine the constant A satisfying $\cosh^2(x) - \sinh^2(x) = A$, for all real x.

Hence write the indefinite integral

$$\int \sqrt{1+x^2}\, dx$$

in terms of $\sinh^{-1}(x)$ and x using a substitution based on the hyperbolic sine.

You may use the identity $\sinh(2x) = 2\sinh(x)\cosh(x)$.

5. (i) Prove that

$$\sum_{n=0}^{N} x^n = \frac{1 - x^{N+1}}{1 - x}$$

for all $x \neq 1$ where $N \geq 0$ is a positive integer. Hence deduce that

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1 - x}$$

converges for all x in some interval. Determine the upper and lower limits of this interval.

(ii) Evaluate the integral

$$\frac{1}{2} \int \left[\frac{1}{1+x} + \frac{1}{1-x} \right] dx.$$

Using this and the result of part (i), or otherwise, deduce a Maclaurin series for the function

$$\ln \left| \frac{1+x}{1-x} \right|^{1/2}.$$

If you are told that

$$(3/5)^3 \simeq 0.216 \quad \text{and} \quad (3/5)^5 \simeq 0.0778,$$

using only the first few terms of your series, find an approximation to $\ln(2)$.

6. (i) Write $\cos(x)$ and $\sin(x)$ in terms of e^{ix} and e^{-ix}.

(ii) For any real number x, write $e^{\cos(x)+i\sin(x)}$ in the form $a+ib$ where a and b are real. Hence use a substitution of the form $z = e^{ix}$ in the power series

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

to find Fourier series representations for both the functions

$$e^{\cos(x)}\cos(\sin(x)) \quad \text{and} \quad e^{\cos(x)}\sin(\sin(x)).$$

(iii) You are given that the geometric series

$$\frac{1}{1-z} = 1 + z + z^2 + \dots = \sum_{n=0}^{\infty} z^n$$

converges for all complex z with $|z| < 1$. Use a substitution of the form $z = re^{i\theta}$ for real θ and $r > 0$ to deduce that there are real numbers A and B (to be found) such that

$$\sum_{n=0}^{\infty} r^n \cos(n\theta) = \frac{A - r\cos(\theta)}{B - 2r\cos(\theta) + r^2}, \qquad (0 < r < 1) \;;$$

and obtain an analogous result for the series $\sum_{n=0}^{\infty} r^n \sin(n\theta)$.

7. (i) Cylindrical polar coordinates (r, θ, z) are defined in terms of Cartesian coordinates (x, y, z) by

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z\,.$$

Suppose

$$f(x, y, z) = x^2 + 3xy + y^2 + z^2 = g(r, \theta, z)\,.$$

Use the chain rule to evaluate $\dfrac{\partial g}{\partial r}$ and $\dfrac{\partial g}{\partial \theta}$, in terms of r, θ and z.

Verify your results by substituting for x, y, z in terms of r, θ, z into f and differentiating directly.

(ii) The pressure P of a volume of gas V at temperature T satisfies the equation $PV = RT$, where R is a constant. Suppose that $T = 300 \pm 10$, $V = 2 \pm 0.3$ and $R = 8$.

Show that the estimated range for P is between 980 and 1420.

8. (i) Find $10^{1/3}$ correct to 5 decimal places using Newton's method with initial guess $5/2$.

(ii) Show that Newton's method for approximating roots of

$$x^4 + 4x^3 + 4x^2 - x - 1 = 0$$

leads to the recurrence relation

$$a_n = \frac{3a_{n-1}^4 + 8a_{n-1}^3 + 4a_{n-1}^2 + 1}{4a_{n-1}^3 + 12a_{n-1}^2 + 8a_{n-1} - 1}\,, \qquad n \geq 2\,.$$

What happens if $a_1 = 0$ is used as the initial guess?

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9. (i) Show the equation

$$(2x^2y + 4)\,\frac{dy}{dx} + 2xy^2 - 3 = 0$$

is exact. Hence find the solution of the equation that satisfies $y(1) = 1$.

(ii) Given the differential equation

$$x\,\frac{dy}{dx} + y = x^2\,y^2\,,$$

make the substitution $u = y^{1-b}$ to obtain a differential equation for u of the form

$$\frac{du}{dx} + (1-b)\,\frac{u}{x} = x\beta u^{\alpha}\,,$$

where α and β are constants to be determined.

Assuming $b = 2$, solve this equation for u. Hence determine y satisfying $y = 1/2$ when $x = 1/2$.

10. Using integration by parts, or otherwise, evaluate the integral $\int xe^{kx}dx$, where k is any constant, possibly complex, and x is real.

If $f(x)$ is the 2π-periodic function that coincides with x on $(-\pi, \pi)$, determine the Fourier series of f and hence evaluate

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

in terms of π.

Hint: *Use either the real or the complex Fourier series representation of the real, 2π-periodic function $f(x)$. The latter is given by*

$$f(x) = c_0 + 2\sum_{n=1}^{\infty} Re(c_n e^{inx}) \quad \text{with} \quad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx,$$

with Parseval's theorem stated as follows:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} [f(x)]^2\,dx = c_0^2 + 2\sum_{n=1}^{\infty} |c_n|^2\,.$$

END OF PAPER

MATHEMATICS DEPARTMENT

MATHEMATICAL FORMULAE

1. VECTOR ALGEBRA

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1, a_2, a_3)$$

Scalar (dot) product: $\mathbf{a}\,.\,\mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

Vector (cross) product:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Scalar triple product:

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a}\,.\,\mathbf{b} \times \mathbf{c} = \mathbf{b}\,.\,\mathbf{c} \times \mathbf{a} = \mathbf{c}\,.\,\mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product: $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c}\,.\,\mathbf{a})\mathbf{b} - (\mathbf{b}\,.\,\mathbf{a})\mathbf{c}$

2. SERIES

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \ldots \quad (\alpha \text{ arbitrary}, |x| < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \ldots,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^n \frac{x^{n+1}}{(n+1)} + \ldots (-1 < x \le 1)$$

3. TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin(a+b) = \sin a \cos b + \cos a \sin b\,;$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b\,.$$

$$\cos iz = \cosh z\,; \quad \cosh iz = \cos z\,; \quad \sin iz = i \sinh z\,; \quad \sinh iz = i \sin z\,.$$

4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f\,D^n g + \binom{n}{1} Df\,D^{n-1}g + \ldots + \binom{n}{r} D^r f\,D^{n-r}g + \ldots + D^n f\,g\,.$$

(b) Taylor's expansion of $f(x)$ about $x = a$:

$$f(a+h) = f(a) + hf'(a) + h^2 f''(a)/2! + \ldots + h^n f^{(n)}(a)/n! + \epsilon_n(h)\,,$$

where $\epsilon_n(h) = h^{n+1} f^{(n+1)}(a+\theta h)/(n+1)!, \quad 0 < \theta < 1\,.$

(c) Taylor's expansion of $f(x, y)$ about (a, b):

$$f(a+h, b+k) = f(a,b) + [hf_x + kf_y]_{a,b} + 1/2!\,\left[h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}\right]_{a,b} + \ldots$$

(d) Partial differentiation of $f(x, y)$:

i. If $y = y(x)$, then $f = F(x)$, and $\dfrac{dF}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dx}$.

ii. If $x = x(t)$, $y = y(t)$, then $f = F(t)$, and $\dfrac{dF}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$.

iii. If $x = x(u, v)$, $y = y(u, v)$, then $f = F(u, v)$, and

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}, \quad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$

(e) Stationary points of $f(x, y)$ occur where $f_x = 0$, $f_y = 0$ simultaneously.
Let (a, b) be a stationary point: examine $D = [f_{xx}f_{yy} - (f_{xy})^2]_{a,b}$.
If $D > 0$ and $f_{xx}(a, b) < 0$, then (a, b) is a maximum;
If $D > 0$ and $f_{xx}(a, b) > 0$, then (a, b) is a minimum;
If $D < 0$ then (a, b) is a saddle-point.

(f) Differential equations:

i. The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating factor $I(x) = \exp[\int P(x)(dx)]$, so that $\dfrac{d}{dx}(Iy) = IQ$.

ii. $P(x, y)dx + Q(x, y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y$.

5. INTEGRAL CALCULUS

(a) An important substitution: $\tan(\theta/2) = t$:
$\sin\theta = 2t/(1+t^2)$, $\cos\theta = (1-t^2)/(1+t^2)$, $d\theta = 2\,dt/(1+t^2)$.

(b) Some indefinite integrals:

$$\int (a^2 - x^2)^{-1/2} dx = \sin^{-1}\left(\frac{x}{a}\right), \quad |x| < a.$$

$$\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1}\left(\frac{x}{a}\right) = \ln\left\{\frac{x}{a} + \left(1 + \frac{x^2}{a^2}\right)^{1/2}\right\}.$$

$$\int (x^2 - a^2)^{-1/2} dx = \cosh^{-1}\left(\frac{x}{a}\right) = \ln\left|\frac{x}{a} + \left(\frac{x^2}{a^2} - 1\right)^{1/2}\right|.$$

$$\int (a^2 + x^2)^{-1} dx = \left(\frac{1}{a}\right)\tan^{-1}\left(\frac{x}{a}\right).$$

6. NUMERICAL METHODS

(a) Approximate solution of an algebraic equation:

If a root of $f(x) = 0$ occurs near $x = a$, take $x_0 = a$ and
$x_{n+1} = x_n - [f(x_n)/f'(x_n)]$, $n = 0, 1, 2 \ldots$

(Newton Raphson method).

(b) Formulae for numerical integration: Write $x_n = x_0 + nh$, $y_n = y(x_n)$.

i. Trapezium rule (1-strip): $\int_{x_0}^{x_1} y(x)dx \approx (h/2)[y_0 + y_1]$.

ii. Simpson's rule (2-strip): $\int_{x_0}^{x_2} y(x)dx \approx (h/3)[y_0 + 4y_1 + y_2]$.

(c) Richardson's extrapolation method: Let $I = \int_a^b f(x)dx$ and let I_1, I_2 be two estimates of I obtained by using Simpson's rule with intervals h and $h/2$. Then, provided h is small enough,

$$I_2 + (I_2 - I_1)/15,$$

is a better estimate of I.

7. LAPLACE TRANSFORMS

Function	Transform	Function	Transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)dt$	$af(t) + bg(t)$	$aF(s) + bG(s)$
df/dt	$sF(s) - f(0)$	d^2f/dt^2	$s^2F(s) - sf(0) - f'(0)$
$e^{at}f(t)$	$F(s-a)$	$tf(t)$	$-dF(s)/ds$
$(\partial/\partial\alpha)f(t,\alpha)$	$(\partial/\partial\alpha)F(s,\alpha)$	$\int_0^t f(t)dt$	$F(s)/s$
$\int_0^t f(u)g(t-u)du$	$F(s)G(s)$		
1	$1/s$	$t^n (n = 1, 2 \ldots)$	$n!/s^{n+1}$, $(s > 0)$
e^{at}	$1/(s-a)$, $(s > a)$	$\sin\omega t$	$\omega/(s^2+\omega^2)$, $(s > 0)$
$\cos\omega t$	$s/(s^2+\omega^2)$, $(s > 0)$	$H(t-T) = \begin{cases} 0, & t < T \\ 1, & t > T \end{cases}$	e^{-sT}/s, $(s, T > 0)$

8. FOURIER SERIES

If $f(x)$ is periodic of period $2L$, then $f(x + 2L) = f(x)$, and

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \text{ where}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad n = 0, 1, 2, \ldots, \text{ and}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1, 2, 3, \ldots$$

Parseval's theorem

$$\frac{1}{L}\int_{-L}^{L} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right).$$

EE1.14 - Maths 2 - Solutions 2009
(EE 1st yr)

	EXAMINATION QUESTIONS/SOLUTIONS 2008-09	Course EE1(2)
Question 1	Page 1 of 2	Marks & seen/unseen

Parts

i) a) $f(x) = x^2$ is <u>NOT</u> invertible on $(-2,1)$ but it is on $(0,1)$, with range $(0,1)$. — 3

b) $f(x) = \cos x$ is <u>NOT</u> invertible on $(-1,1)$ but it is on $(0,1)$ with range $(1, \cos(1))$. — 3

c) $f(x) = \sin x$ on $(-1,1)$ <u>is</u> invertible — 3

d) $f(x) = x^{1/3}$ on $(-10,10)$ <u>is</u> invertible. — 3

e) $f(x) = x^3$ on $(-10,10)$ <u>is</u> invertible — 3

ii) Given that

$$p(x) = \sum_{n=0}^{\infty} x^{2n}/(2n)! \quad \text{then}$$

$$p(ix) = \sum_{n=0}^{\infty} (ix)^{2n}/(2n)!$$

$$= \sum_{n=0}^{\infty} i^{2n} x^{2n}/(2n)!$$

$$= \sum_{n=0}^{\infty} (i^2)^n \frac{x^{2n}}{2n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

on but. sheet

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	EXAMINATION QUESTIONS/SOLUTIONS 2008-09	Course EEI (2)
Question 1	Page 2 of 2.	Marks & seen/unseen
Parts	and so $Re(p(ix)) = p(ix)$ is an <u>EVEN</u> function. Hence $Im(p(ix)) = 0$ for all x, which is ~~also~~ EVEN. AND ODD award full marks for either	5 Total 20
	Setter's initials CB / Checker's initials	Page number

	EXAMINATION QUESTIONS/SOLUTIONS 2008-09	Course EE1(2)
Question 2	Page 1 of 2.	Marks & seen/unseen

Parts

standard

Given $f(x) = -x(\ln x - 1)$

note that $\frac{df}{dx} = -\ln x + 1 - x \cdot \frac{1}{x}$
$= -\ln x$.

Moreover $\frac{d^2f}{dx^2} = -1/x < 0$ if $x > 0$.

Hence $\frac{df}{dx} = 0$ iff $x = 1$

and so $x = 1$ is a local MAX. — 6

i) to find $\lim_{x\to 0} -x(\ln x - 1)$

tutorial sheet

note $\lim_{x\to 0} -x(\ln x - 1) = -\lim_{x\to 0} x \ln x$

$= \lim_{x\to 0} \frac{\ln x}{1/x} = +\lim_{x\to 0} \frac{1/x}{1/x^2}$ (l'Hôpital)

$= \lim_{x\to 0} \frac{x^2}{x} = \lim_{x\to 0} x = 0$. — 4

ii) $\lim_{x\to\infty} -x(\ln x - 1) = -\infty$ and — 2

iii) $x = 1$ is the only stationary (MAX) point for $x > 0$.
(Marks given above for this)

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Parts

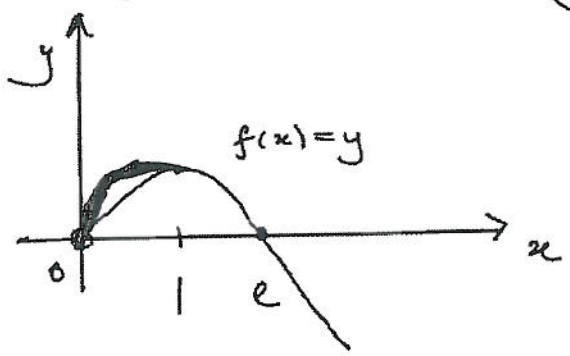
"e"

iv) $f(x) = 0$ if $x = e$. ← 2 for part (iv)

Now $\frac{df}{dx}(x) = -\ln(x)$ and

so $\lim_{x \to 0} \frac{df}{dx} = -\infty$

however f has the following graph:



6 for graph

~~and so $x=0$ is a local MIN even though $f'(0)$ is not defined.~~

~~2~~

unseen

(Total 20)

	EXAMINATION QUESTIONS/SOLUTIONS 2008-09	Course EE1(2)
Question 3	Page 1 of 2	Marks & seen/unseen
Parts	i) a) To find $\frac{d}{dx}(x^x)$ let $y = x^x$	basic unseen
	and then $\ln y = x \ln x \implies$	
	$\frac{1}{y}\frac{dy}{dx} = \ln x + \frac{x}{x} \implies$	
	$\frac{dy}{dx} = (\ln x + 1)x^x$.	3
	b) Note that	
	$\frac{d}{dx}\left(\sin(\sin^{-1}(x)) + \sin^{-1}(\sin x)\right)$	
	$= \frac{d}{dx}(x + x) = 2$	3
	c) $\frac{d}{dx}\int_0^x \sqrt{1+t^2}\, dt = \sqrt{1+x^2}$	
	by the fundamental theorem of calculus.	3
	d) $\frac{d}{dx}\sum_{n=1}^{100} \frac{1}{n}\cos nx = \sum_{n=1}^{100} -\sin(nx)$.	3
	Setter's initials KB	Checker's initials	Page number

	EXAMINATION QUESTIONS/SOLUTIONS 2008-09	Course EE1(2)
Question 3	Page 2 of 2	Marks & seen/unseen
Parts	ii) Let $P(n)$ be the statement $P(n): \frac{d^n f}{dx^n}(x) = n!\, f(x)^{n+1}$ (for all x and $n \geq 1$). Then $P(1)$ is true by the definition of f. Assume $P(k)$ is true, then $\frac{d^k f}{dx^k}(x) = k!\, f(x)^{k+1}$ and so $\frac{d^{k+1} f}{dx^{k+1}} = \frac{d}{dx}\frac{d^k f}{dx^k}$ $= \frac{d}{dx} k!\, f(x)^{k+1} = (k+1)\, k!\, f(x)^k \frac{df}{dx}$ $= (k+1)!\, f(x)^k f(x)^2 = (k+1)!\, f(x)^{k+2}$ and so $P(k+1)$ is true. Hence $P(n)$ is true for all $n \geq 1$ by the principle of induction.	standard 2 6 (Total 20)
	Setter's initials KB Checker's initials	Page number

	EXAMINATION QUESTIONS/SOLUTIONS 2008-09	Course EE I(2)
Question 4	Page 1 of 2	Marks & seen/unseen
Parts		Seen tutorial sheet

i) In order to integrate $\int_{-1}^{1} \sqrt{1-x^2}\,dx$

let $x = \sin\theta$, so $dx = \cos\theta\, d\theta$

and therefore

$$\int_{-1}^{1}\sqrt{1-x^2}\,dx = 2\int_0^1 \sqrt{1-x^2}\,dx$$

$$= 2\int_0^{\pi/2} \sqrt{1-\sin^2\theta}\,\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \cos\theta\,|\cos\theta|\,d\theta = 2\int_0^{\pi/2} \cos^2\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \frac{1}{2}(1+\cos 2\theta)\,d\theta$$

$$= \int_0^{\pi/2} (1+\cos 2\theta)\,d\theta = \frac{\pi}{2}$$ 6

ii) Now $\frac{d}{dx}\cosh x = \sinh x$, $\frac{d}{dx}\sinh x = \cosh x$ 2

and $\cosh^2 x - \sinh^2 x = 1$. 2

seen

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	EXAMINATION QUESTIONS/SOLUTIONS 2008-09	Course EE I (2)
Question 4	Page 2 of 2.	Marks & seen/unseen

Parts

unseen similar to tut sheet

(ii) To find $\int \sqrt{1+x^2}\,dx$ set $x = \sinh\theta$ and then $dx = \cosh\theta \cdot d\theta$ so

$$\int \sqrt{1+x^2}\,dx = \int \sqrt{1+(\sinh\theta)^2}\,\cosh\theta\,d\theta$$

$$= \int (\cosh\theta)^2\,d\theta = \int \left(\frac{e^{\theta}+e^{-\theta}}{2}\right)^2 d\theta \qquad 4$$

(as $\cosh\theta > 0$)

$$= \int \frac{1}{4}\left(e^{2\theta} + e^{-2\theta} + 2\right) d\theta$$

$$= \frac{1}{2}\int 1 + \cosh(2\theta)\,d\theta$$

$$= \frac{1}{2}\theta + \frac{\sinh 2\theta}{4} + C \qquad 4$$

$$= \frac{1}{2}\sinh^{-1}x + \frac{1}{2}\sinh\theta\cosh\theta + C$$

$$= \frac{1}{2}\sinh^{-1}x + \frac{1}{2}x\sqrt{1+x^2} + C \qquad 2$$

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Question 5

Marks & seen/unseen

Parts

seen in lecture

i) Define the function $\sigma_N(x) = 1 + x + x^2 + \ldots + x^N$, then

$$x\sigma_N(x) - \sigma_N(x)$$

$$= x + x^2 + x^3 + \ldots + x^N + x^{N+1}$$

$$-1 - x - x^2 - x^3 - \ldots - x^N$$

$$= x^{N+1} - 1$$

$$\Rightarrow \sigma_N(x) \cdot (x-1) = x^{N+1} - 1$$

$$\Rightarrow \sigma_N(x) = \frac{1 - x^{N+1}}{1 - x}, \quad x \neq 1.$$ 8

(Induction is also possible)

If $-1 < x < 1$ 2

we can take the limit as $N \to \infty$ to deduce

$$\lim_{N\to\infty} \sigma_N(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

Hence $$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots$$

$$= \sum_{n=0}^{\infty} (-1)^n x^n$$

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Question 5		Marks & seen/unseen

Parts

ii)

So $\frac{1}{2}\int \frac{1}{1+x} + \frac{1}{1-x}\, dx = c + \frac{1}{2}\left(\ln|1+x| - \ln|1-x|\right)$

$= c + \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|$

$= \int (1 + x + x^2 + x^3 + \dots + 1 - x + x^2 - x^3 + \dots)\, \frac{1}{2}\, dx$

$= \int 1 + x^2 + x^4 + x^6 + \dots\, dx$

$= x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \dots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}$ — 6

from where $c = 0$. — unseen

To find $\ln 2$ note that

$\left(\frac{1+x}{1-x}\right)^{1/2} = 2 \iff \frac{1+x}{1-x} = 4$

$\iff 1+x = 4-4x \Rightarrow 5x = 3$

$\Rightarrow x = 3/5$.

Hence $\ln 2 = \ln\left(\frac{1+x}{1-x}\right)^{1/2}$ if $x = \frac{3}{5}$

$\Rightarrow \ln 2 = \frac{1}{2}\ln\left(\frac{1+3/5}{1-3/5}\right)$

$\simeq \frac{3}{5} + \frac{1}{3}\left(\frac{3}{5}\right)^3 + \frac{1}{5}\left(\frac{3}{5}\right)^5 = 0.6 + \frac{0.216}{3} + \frac{0.0778}{5} \simeq 0.69$

Students have a calculator for this exam, so OK — 4

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Parts

i) $\cos x = \dfrac{e^{ix}+e^{-ix}}{2}$, $\sin x = \dfrac{e^{ix}-e^{-ix}}{2i}$. seen 2

ii) Now
$$e^{e^{ix}} = e^{\cos x + i\sin x}$$
$$= e^{\cos x}\cdot e^{i\sin x}$$
$$= e^{\cos x}\left(\cos(\sin x) + i\sin(\sin x)\right)$$
$$= e^{\cos x}\cdot\cos(\sin x) + i\,e^{\cos x}\sin(\sin x),$$
as required. 6

Using $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ as given,

on setting $z = e^{i\theta}$ we obtain
$$e^{e^{i\theta}} = e^{\cos\theta + i\sin\theta}$$
$$= \sum_{n=0}^{\infty} \frac{(e^{i\theta})^n}{n!} = \sum_{n=0}^{\infty} \frac{e^{in\theta}}{n!}$$
$$= \sum_{n=0}^{\infty} \frac{\cos n\theta}{n!} + i\sum_{n=0}^{\infty} \frac{\sin n\theta}{n!}.$$
seen

Marks & seen/unseen

Parts

Hence comparing real and imaginary parts,

$$\sum_{n=0}^{\infty} \frac{\cos n\theta}{n!} = e^{\cos\theta} \cdot \cos(\sin\theta)$$

$$\sum_{n=0}^{\infty} \frac{\sin(n\theta)}{n!} = e^{\cos\theta} \cdot \sin(\sin\theta).$$ 6

(iii) Given that $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ unseen

an setting $z = re^{i\theta}$, for $|z|<1$
we obtain

$$\sum_{n=0}^{\infty} r^n e^{in\theta} = \frac{1}{1-re^{i\theta}} = \frac{1}{1-re^{i\theta}} \cdot \frac{1-re^{-i\theta}}{1-re^{-i\theta}}$$

$$= \frac{1-re^{-i\theta}}{1-r(e^{i\theta}+e^{-i\theta})+r^2}$$ 3

[i.e. $A=B=\underline{1}$]

$$= \frac{1-re^{-i\theta}}{1-2r\cos\theta+r^2}.$$

Hence $\sum_{n=0}^{\infty} r^n \cos n\theta = \frac{1-r\cos\theta}{1-2r\cos\theta+r^2}$

and $\sum_{n=0}^{\infty} r^n \sin n\theta = \frac{+r\sin\theta}{1-2r\cos\theta+r^2}$. 3

Marks & seen/unseen

Parts

(i)

$$\frac{\partial g}{\partial r} = f_x x_r + f_y y_r + f_z z_r$$

$$\frac{\partial g}{\partial \theta} = f_x x_\theta + f_y y_\theta + f_z z_\theta$$

$$\frac{\partial g}{\partial z} = f_x x_z + f_y y_z + f_z z_z$$

2

$$f_x = 2x + 3y = r(2\cos\theta + 3\sin\theta)$$

$$f_y = 3x + 2y = r(3\cos\theta + 2\sin\theta)$$

$$f_z = 2z$$

2

$x_r = (\cos\theta)$, $x_\theta = -r\sin\theta$, $y_r = \sin\theta$, $y_\theta = r\cos\theta$

$z_r = z_\theta = 0$, $z_z = 1$, $x_z = y_z = 0$,

$$\therefore g_r = 2r(1 + 3\sin\theta\cos\theta)$$

$$g_\theta = 3r^2(\cos 2\theta)$$

5

OR Directly $g = r^2\cos^2\theta + 3r^2\sin\theta\cos\theta + r^2\sin^2\theta + z^2$

$$= r^2 + 3r^2\sin\theta\cos\theta + z^2$$

$$\therefore g_r = 2r + 6r\sin\theta\cos\theta$$

$$g_\theta = 3r^2(\cos 2\theta)$$

3

as required.

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Parts

(ii)

$$\frac{\partial P}{\partial T} = \frac{R}{V}, \quad \frac{\partial P}{\partial V} = -\frac{RT}{V^2}$$ 2

$$\therefore |\delta P| \approx \left|\frac{R}{V}\delta T\right| + \left|\frac{RT}{V^2}\delta V\right|$$ 3

$$\therefore P = \frac{RT}{V} \pm \left(\frac{R}{V}|\delta T| + \frac{RT}{V^2}|\delta V|\right)$$

$$\therefore P = \frac{8 \times 300}{2} \pm (40 + 180)$$ 3

$$= 1200 \pm 220.$$

(Total 20)

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Course: EE I paper 2

Question 8

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Parts

(i) $f = x^3 - 10$, $f' = 3x^2$ — 2

$\therefore$ Newtons method gives sequence of approximations

$$a_n = a_{n-1} - \left(\frac{a_{n-1}^3 - 10}{3a_{n-1}^2}\right) = \frac{2a_{n-1}^3 + 10}{3a_{n-1}^2}$$ — 5

Now take $a_1 = 5/2$

then $a_2 = 2.2$

$a_3 \simeq 2.1553719$

$a_4 \simeq 2.1544351$

$a_5 \simeq 2.1544347$ — 4

$\therefore$ $10^{1/3} \approx 2.15443$ correct to 4 decimal places

(ii) If $f = x^4 + 4x^3 + 4x^2 - x - 1$, $f' = 4x^3 + 12x^2 + 8x - 1$ — 2

$$\therefore a_n = a_{n-1} - \frac{a_{n-1}^4 + 4a_{n-1}^3 + 4a_{n-1}^2 - a_{n-1} - 1}{4a_{n-1}^3 + 12a_{n-1}^2 + 8a_{n-1} - 1}$$

$$= \frac{3a_{n-1}^4 + 8a_{n-1}^3 + 4a_{n-1}^2 + 1}{4a_{n-1}^3 + 12a_{n-1}^2 + 8a_{n-1} - 1}$$ — 4

If $a_1 = 0$

$a_2 = -1$

$a_3 = 0$

$a_4 = -1$, ... so no convergence — 3

(Total 20)

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Parts

(i) $P\frac{dy}{dx} + Q = 0$ is exact if $P_x = Q_y$. — 2

and $P_x = 4xy$, $Q_y = 4xy$ so equation is exact. — 2

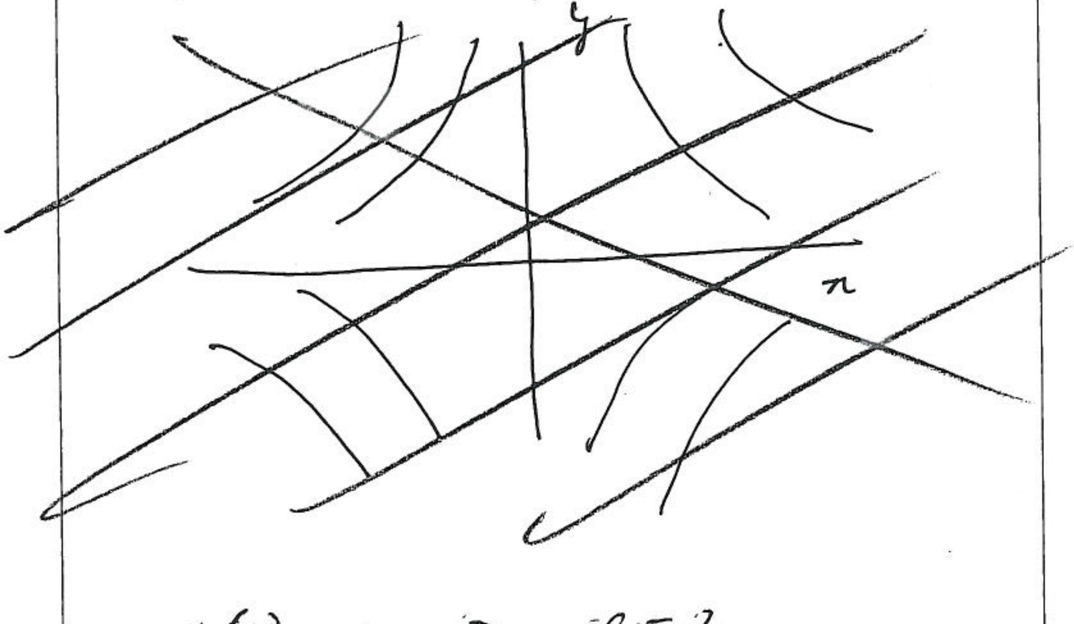
$\therefore f(x,y)$ exists such that

$$\frac{\partial f}{\partial x} = 2y^2x - 3 \Rightarrow f = x^2y^2 - 3x + A(y)$$

$$\frac{\partial f}{\partial y} = 2yx^2 + 4 \Rightarrow f = x^2y^2 + 4y + B(x)$$

$$\Rightarrow f = y^2x^2 - 3x + 4y + C$$

$\therefore$ Solution is $y^2x^2 - 3x + 4y = \text{constant } c$ — 4



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$y(1) = 1 \Rightarrow \underline{c = 2}$

$\Rightarrow y^2x^2 - 3x + 4y = 2.$ — 2

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Parts 2(ii)

$$\frac{dy}{dx} + \frac{y}{x} = xy^2$$

Let $u = y^{1-b}$ $\therefore \frac{du}{dx} = (1-b)y^{-b}\frac{dy}{dx}$ 2

$$\therefore \underbrace{\frac{du}{dx}\cdot\frac{1}{1-b}\cdot y^{b}}_{dy/dx} + \frac{y}{x} = xy^2$$ 2

$$\therefore \frac{du}{dx}\frac{1}{1-b} + \frac{y^{1-b}}{x} = xy^{2-b}$$ 1

$$\therefore \frac{du}{dx} + (1-b)\frac{u}{x} = xu^{(2-b)/(1-b)}(1-b)$$

So, req'd ODE is $\frac{du}{dx} + (1-b)\frac{u}{x} = xu^{(2-b)/(1-b)}(1-b)$ (and so $\alpha = \frac{2-b}{1-b}$ & $\beta = (1-b)$)

Taking $b=2$ we have

$$-\frac{du}{dx} + \frac{u}{x} = x$$

$$\therefore \frac{d}{dx}\left(\frac{u}{x}\right) = -1$$ 2

$$\frac{u}{x} = -x + A$$ 2

Since $u = y^{-1}$ we have $\therefore y = \frac{1}{x(A-x)}$

$$\therefore \tfrac{1}{2} = \frac{1}{\tfrac{1}{2}(A-\tfrac{1}{2})} \Rightarrow A = 9/2$$

$$\therefore y = \frac{1}{x(9/2 - x)}$$ 1

Total 20

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Parts

Using integration by parts

$$\int \underset{v}{x}\,\underset{u'}{e^{kx}}\,dx = x\frac{e^{kx}}{k} - \int \frac{e^{kx}}{k}\cdot 1\,dx$$

$$= x\frac{e^{kx}}{k} - \frac{e^{kx}}{k^2} + \text{const.} \qquad \begin{pmatrix}3\\+\\1\end{pmatrix}$$

$\left[(k=0) \quad \int x\,dx = \frac{1}{2}x^2 + C \quad \text{holds when } k=0\right]$

Seen

4

Hence,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} x e^{-inx}\,dx = \left[\frac{xe^{-inx}}{-in} - \frac{e^{-inx}}{(-in)^2}\right]_{-\pi}^{\pi}\left(\frac{1}{2\pi}\right)$$

Seen

$$= \left(\frac{\pi e^{-in\pi}}{-in} - \frac{\pi e^{in\pi}}{in}\right)\frac{1}{2\pi}$$

$$= \frac{1}{2\pi}\cdot\frac{+i}{n}\pi\left(e^{-in\pi} + e^{in\pi}\right) = \frac{i}{2n}\left((-1)^n + (-1)^n\right)$$

$$= \frac{i}{n}(-1)^n = c_n \quad \text{and}$$

6

$$f(x) = c_0 + 2\,\mathrm{Re}\sum_{n=1}^{\infty} c_n e^{inx}$$

$$= c_0 + 2\,\mathrm{Re}\sum_{n=1}^{\infty} \frac{i}{n}(-1)^n e^{inx}$$

$$= c_0 + 2\,\mathrm{Re}\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\, i\,(\cos nx + i\sin nx)$$

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Parts

$$= \quad 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot \sin nx$$ 4

because $c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f \, dx = 0.$

Parseval now states that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 dx = c_0^2 + 2 \sum_{n=1}^{\infty} |c_n|^2$$

$$\Rightarrow \frac{1}{\pi} \int_0^{\pi} x^2 dx = \frac{\pi^3}{3} \cdot \frac{1}{\pi}$$

$$= 2 \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$ 6 seen