E3.10
CS3.1
ISE3.7

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2009

MSc and EEE/ISE PART III/IV: MEng, BEng and ACGI

Corrected Copy

**MATHEMATICS FOR SIGNALS AND SYSTEMS**

Wednesday, 6 May 10:00 am

Time allowed: 3:00 hours

**There are FIVE questions on this paper.**

**Answer THREE questions.**

Q2 –

*All questions carry equal marks*

**Any special instructions for invigilators and information for candidates are on page 1.**


Examiners responsible First Marker(s) : M.M. Draief
Second Marker(s) : D. Angeli

1. Consider the space $\mathbb{R}^{3\times 3}$ of three-by-three matrices. We define the inner product $\langle A, B\rangle = tr(B^T A)$, and we define the corresponding norm $N(A) = \sqrt{tr(A^T A)}$. Let

$$S = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

a) Find the range and the kernel of $S$. [ 1 ]

b) Compute $N(S)$ the norm of $S$ as defined before. [ 2 ]

c) Compute the matrix norm of $S$ given by $|||S||| = \sup_{||x||\leq 1} ||Sx||$ where $||x||^2 = x^T x$. [ 3 ]

d) A matrix $A$ is said to be $S$-invariant if $AS = SA$. Let $\mathcal{S}$ be the set of $S$-invariant matrices.

($i$) Show that $\mathcal{S}$ is a vector space. [ 2 ]

($ii$) Determine the dimension of $\mathcal{S}$. [ 4 ]

($iii$) Show that if $A$ and $B$ are two elements of $\mathcal{S}$ then $AB \in \mathcal{S}$ and $AB = BA$.

[ 3 ]

($iv$) Show that if $A \in \mathcal{S}$ then $A$ has a unique eigenvalue. [ 2 ]

($v$) Explicitly describe all the matrices $A \in \mathcal{S}$ for which $A^T \in \mathcal{S}$. [ 3 ]

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2. Consider $\mathcal{C}_0$ the space of real-valued ~~continuous~~ functions on the interval $[-1,1]$. For $f,g \in \mathcal{C}_0$, we define the inner product $\langle f,g\rangle = \int_{-1}^{1} f(x)g(x)dx$.

a) Check that the above inner product is indeed an inner product and find the expression of the corresponding norm. [ 3 ]

b) Let

$$\mathcal{E} = \{f \in \mathcal{C}_0, f(-x) = f(x)\}$$

be the set of even functions and

$$\mathcal{O} = \{f \in \mathcal{C}_0, f(-x) = -f(x)\}$$

be the set of odd functions.

(i) Show that $\mathcal{E}$ and $\mathcal{O}$ are two vector spaces. [ 3 ]

(ii) Show that $\mathcal{E}$ and $\mathcal{O}$ are orthogonal, i.e., for any $f \in \mathcal{E}$ and $g \in \mathcal{O}$, we have $\langle f,g\rangle = 0$. [ 4 ]

(iii) For $f \in \mathcal{C}_0$, and let $g(x) = f(x) + f(-x)$ and $h(x) = f(x) - f(-x)$. Show that $g \in \mathcal{E}$ and $h \in \mathcal{O}$. [ 3 ]

(iv) Show that any $f \in \mathcal{C}_0$ can be decomposed in a **unique** way as $f(x) = g(x) + h(x)$ where $g \in \mathcal{E}$ and $h \in \mathcal{O}$. [ 3 ]

(v) Determine the orthogonal projections on $\mathcal{E}$ and $\mathcal{O}$. [ 4 ]

3. Let $u_1, \ldots, u_n$ be a set of orthonormal vectors in $\mathbb{R}^n$, i.e., pairwise orthogonal

$$u_i^T u_j = 0\,,\ \text{for } i \neq j \quad \text{and} \quad u_i^T u_i = 1\,,\ \text{for all } i\,.$$

We denote by $||x||^2 = x^T x$.

a) Let $U = [u_1, \ldots, u_n]$ be a matrix in $\mathbb{R}^{n \times n}$.

$(i)$ Show that $U^T U = U U^T = I$, $I$ being the identity matrix. [ 2 ]

$(ii)$ Prove that $x = \sum_{i=1}^{n} (u_i^T x) u_i$ for $x$ in $\mathbb{R}^n$. [ 3 ]

$(iii)$ Show that $u_1, \ldots, u_n$ is an orthonormal basis of $\mathbb{R}^n$. [ 3 ]

$(iv)$ Show that $||Ux|| = ||x||$. [ 3 ]

b) Let $V_1 = \text{Span}(u_1, \ldots u_k)$ and $V_2 = \text{Span}(u_{k+1}, \ldots u_n)$.

$(i)$ Show that $\mathbb{R}^n = V_1 \oplus V_2$, i.e. $V_1$ and $V_2$ are complementary. [ 3 ]

$(ii)$ Let $p(x) = \sum_{i=1}^{k} (u_i^T x) u_i$; show that $p$ is a projection. [ 3 ]

$(iii)$ Let $s(x) = \sum_{i=1}^{k} (u_i^T x) u_i - \sum_{i=k+1}^{n} (u_i^T x) u_i$; show that $s$ is a reflexion.

[ 3 ]

4. Let $A$ be a matrix in $\mathbb{R}^{m\times n}$, $m \geq n$.

a) Show that if $A$ has a left inverse, i.e. there exists a $C$ such that $CA = I$ ($I$ the identity matrix in $\mathbb{R}^{n\times n}$), then $A$ has zero-null space. [ 3 ]

b) Assume that $A$ is zero-null space.

($i$) Show that $A^T A$ is a positive definite matrix. [ 3 ]

($ii$) Find a left inverse for $A$. [ 3 ]

($iii$) Let $y \in \mathbb{R}^n$. Find a condition on $y$ so that the equation $Ax = y$ admits a solution. [ 2 ]

c) For $y \in \mathbb{R}^n$, let $\hat{x} = \left(A^T A\right)^{-1} A^T y$. We consider the inner product $x^T y$ and the associated norm $||.||$

($i$) Show that, for any vector $x \in \mathbb{R}^n$, $A(x - \hat{x})$ is orthogonal to $A\hat{x} - y$. [ 3 ]

($ii$) Show that $||Ax - y|| \geq ||A\hat{x} - y||$. [ 3 ]

($iii$) Suppose that $y \notin \text{Range}(A)$. Relate the above to the linear least-square problem. [ 3 ]

5. a) Show that if $A$ is a positive definite matrix then if $\lambda$ is an eigenvalue of $A$ then $\lambda > 0$. [ 3 ]

b) Let $A$ be a symmetric matrix with $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ and $x_1, \ldots, x_n \in \mathbb{R}^n$ as its eigenvalues and eigenvectors respectively, i.e., $Ax_i = \lambda_i x_i$, $i = 1, \ldots, n$. Show that if for all $i = 1, \ldots, n$, $\lambda_i > 0$ then $A$ is positive definite. [ 4 ]

*Hint: Use the fact that if $A \in \mathbb{R}^{n\times n}$ is symmetric then $(x_1, \ldots, x_n)$ is an orthonormal basis, i.e., $x_i^T x_j = 0$ if $i \neq j$ and $x_i^T x_i = 1$.*

c) Let $A = \frac{1}{5}\begin{pmatrix} 3 & -6 & 26 \\ 4 & -8 & -7 \\ 0 & 4 & 4 \\ 0 & -3 & -3 \end{pmatrix}$.

Show that $A^T A = \begin{pmatrix} 1 & -2 & 2 \\ -2 & 5 & -3 \\ 2 & -3 & 30 \end{pmatrix}$. [ 2 ]

d) We now want to solve the linear least-square problem with $A$ above and $y = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

$(i)$ Show that is equivalent to solving the linear problem

$$\begin{pmatrix} 1 & -2 & 2 \\ -2 & 5 & -3 \\ 2 & -3 & 30 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7/5 \\ -13/5 \\ 4 \end{pmatrix}.$$

[ 2 ]

$(ii)$ Using the Cholesky decomposition, show that

$$\begin{pmatrix} 1 & -2 & 2 \\ -2 & 5 & -3 \\ 2 & -3 & 30 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & 1 & 5 \end{pmatrix}\begin{pmatrix} 1 & -2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 5 \end{pmatrix}.$$

[ 5 ]

$(iii)$ Show that the solution is $\hat{x} = \frac{1}{25}\begin{pmatrix} 41 \\ 4 \\ 1 \end{pmatrix}$. [ 4 ]

*Hint: Cholesky Decomposition: Let $A \in \mathbb{R}^{n\times n}$ such that*

$$A = \begin{pmatrix} a_{11} & A_{21}^T \\ A_{21} & A_{22} \end{pmatrix}$$

*where $a_{11}$ is a scalar, $A_{21} \in \mathbb{R}^{(n-1)\times 1}$, and $A_{22} \in \mathbb{R}^{(n-1)\times(n-1)}$ symmetric.*

- *Calculate the first column of $L$: $l_{11} = \sqrt{a_{11}}$ and $L_{21} = \frac{1}{l_{11}} A_{21}$,*
- *Compute the Cholesky factor $L_{22}$ of the matrix $A_{22} - \frac{1}{a_{11}} A_{21} A_{21}^T$.*
- *The Cholesky factor $L$ of a positive definite matrix $A$ is given by*

$$L = \begin{pmatrix} l_{11} & 0\ldots 0 \\ L_{21} & L_{22} \end{pmatrix}$$

# MATHEMATICS FOR SIGNAL & SYSTEMS



SOLUTIONS - 2009

## Q1.

a) Range $(S) =$ Span $\{$ vector columns of $S \}$.

$$\text{Span} \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

$x \in$ Kernel $(S)$.

$$S \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} 0 \\ x_1 \\ x_2 \end{pmatrix} = 0 \Rightarrow x_1 = x_2 = 0$$

$$\text{Kernel } (S) = \text{Span} \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

b)
$$S^T S = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$N(S) = \sqrt{tr(S^T S)} = \sqrt{2}.$$

c)
$$Sx = \begin{pmatrix} 0 \\ x_1 \\ x_2 \end{pmatrix} \qquad \|Sx\| = \sqrt{x_1^2 + x_2^2}$$

$$x \neq 0 \quad \frac{\|Sx\|}{\|x\|} \leq 1 \quad \text{and} \quad S \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

Hence $\|S\| = 1$.

Q1 

d) (i) We will show that $\mathcal{S}$ is a subspace of $\mathbb{R}^{3\times 3}$.

* $I \in \mathcal{S}$ ; $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ & $IS = SI = S$.

* $A, B \in \mathcal{S}$ ; $\lambda \in \mathbb{R}$

$$(\lambda A + B) S = S(\lambda A) + SB = S(\lambda A + B) \Rightarrow \lambda A + B \in \mathcal{S}$$

Hence $\mathcal{S}$ is a vector space.

(ii) $AS = \begin{pmatrix} a_{12} & a_{13} & 0 \\ a_{22} & a_{23} & 0 \\ a_{32} & a_{33} & 0 \end{pmatrix}$ ; $SA = \begin{bmatrix} 0 & 0 & 0 \\ a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$

$AS = SA \Rightarrow$ $A = \begin{pmatrix} \alpha & 0 & 0 \\ \beta & \alpha & 0 \\ \gamma & \beta & \alpha \end{pmatrix}$ ; $\alpha, \beta, \gamma \in$

Therefore $\dim(\mathcal{S}) = 3$.

(iii) If $A = \begin{pmatrix} \alpha_1 & 0 & 0 \\ \beta_1 & \alpha_1 & 0 \\ \gamma_1 & \beta_1 & \alpha_1 \end{pmatrix}$ & $B = \begin{pmatrix} \alpha_2 & 0 & 0 \\ \beta_2 & \alpha_2 & 0 \\ \gamma_2 & \beta_2 & \alpha_2 \end{pmatrix}$

then ; $\mathcal{S} \ni AB = BA = \begin{bmatrix} \alpha_1\alpha_2 & 0 & 0 \\ \beta_1\alpha_2 + \alpha_1\beta_2 & \alpha_1\alpha_2 & 0 \\ \gamma_1\alpha_2 + \beta_1\beta_2 + \alpha_1\gamma_2 & \beta_1\alpha_2 + \alpha_1\beta_2 & \alpha_1\alpha_2 \end{bmatrix}$

(iv) from description in d (ii) $A \in \mathcal{S}$

has a unique eigenvalue $\alpha$.

(v) $A \in \mathcal{S}$ ; $A^T = \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \alpha & \beta \\ 0 & 0 & \alpha \end{pmatrix} \in \mathcal{S} \Rightarrow$ $\beta = \gamma = 0$ $\Rightarrow A = \alpha I$.

Q2

a) 1) $\langle f, g\rangle = \langle g, f\rangle$

2) $\langle \lambda f + g, h\rangle = \lambda \langle f, h\rangle + \langle g, h\rangle$

3) $\langle f, f\rangle = \int_{-1}^{1} f(x) f(x)\, dx \geq 0$

4) $\langle f, f\rangle = 0 \Rightarrow f = 0$

$\|f\| = \sqrt{\int_{-1}^{1} f(x)^2\, dx}$

b)

(i) $\mathcal{E}, \mathcal{O}$ subspaces of $\mathcal{C}_0$.

$(x \mapsto 0) \in \mathcal{E}$ ; $(x \mapsto 0) \in \mathcal{O}$

$$\lambda f + g(-x) = \lambda f(-x) + g(-x) = \begin{cases} \lambda f(x) + g(x) & ; \ f, g \in \mathcal{E} \\ -\lambda f(x) - g(x) & ; \ f, g \in \mathcal{O}. \end{cases}$$

(ii) $f \in \mathcal{E}$, $g \in \mathcal{O}$ then $f(-x) g(-x) = -f(x) g(x)$;

$$\langle f, g\rangle = \int_{-1}^{1} f(x) g(x)\, dx = \int_{0}^{1} f(x) g(x)\, dx + \int_{-1}^{0} f(x) g(x)\, dx$$
$$= \int_{0}^{1} f(x) g(x)\, dx - \int_{0}^{1} f(x) g(x)\, dx$$
$$= 0.$$

(iii) $g(-x) = g(x)$ ; $h(-x) = -h(x)$.

Q2

b).

(iv) $$g(x) = \frac{f(x) + f(-x)}{2}$$

$$h(x) = \frac{f(x) - f(-x)}{2}$$

(if $f = g_1 + h_1$ then $g_1 = g$ & $h_1 = h$).

(v) orthogonal projection of $f$ on $\mathcal{E}$ is given by $x \mapsto \frac{f(x)+f(-x)}{2}$.

orthogonal projection of $f$ on $\mathcal{O}$ is given by $x \mapsto \frac{f(x)-f(-x)}{2}$.

Q3. Similar to problem solved during lectures.

a)

(i) $U^T U = \begin{pmatrix} u_1^T \\ \vdots \\ u_n^T \end{pmatrix} (u_1 \dots u_n) = \left( u_i^T u_j \right)_{i,j=1\dots n}$

$= I$

$U^{-1} = U^T \Rightarrow U^T U = U U^T = I.$

(ii) $x = I x = U U^T x$

$= U \begin{pmatrix} u_1^T \\ \vdots \\ u_n^T \end{pmatrix} x$

$= U \begin{pmatrix} u_1^T x \\ \vdots \\ u_n^T x \end{pmatrix} = \sum_{i=1}^{n} u_i^T x \, u_i$

(iii)

$u_1 \dots u_n$ linearly independent

since if $\lambda_1 u_1 + \dots + \lambda_n u_n$ then

$u_i^T (\lambda_1 u_1 + \dots + \lambda_n u_n) = \lambda_i \Rightarrow \lambda_i = 0$ for all $i$

& by (ii) $\text{Span}\{u_1 \dots u_n\} = \mathbb{R}^n$

$\Rightarrow (u_1 \dots u_n)$ is a basis of orthonormal vectors thus an orthonormal basis.

Q3

a)

(iv) $\|Ux\|_2^2 \quad (Ux)^T Ux = x^T \underbrace{U^T U}_{I} x$

$= x^T x = \|x\|^2$

b).

(i) $x \in V_1 \cap V_2 \qquad x = \sum_{i=1}^{k} \lambda_i v_i$

$= \sum_{j=k+1}^{n} \lambda_j v_j$

$x^T x = 0$ since $v_i$'s are othogonal

$\Rightarrow x = 0$.

and $V_1 + V_2 = \mathbb{R}^n$ by a) (ii)

$\Rightarrow V_1 \oplus V_2 = \mathbb{R}^n$.

(ii) $p^2(x) = p(x) \Rightarrow p$ projection

(iii) $s^2(x) = x \Rightarrow s$ reflexion

Q4

a) $\exists C \quad CA = I$

then $\quad Ax = 0 \quad \Rightarrow x = CAx = I0 = 0 \quad \Rightarrow x = 0.$

Kernel $\{A\} = \{0\}$.

b)

$$x^T (A^T A) x = (Ax)^T Ax \geq 0$$

if $\quad x^T (A^T A) x = 0 \quad \Rightarrow \quad Ax = 0 \quad \Rightarrow x = 0$

as $A$ has zero null-space

(ii) $$\left[ (A^T A)^{-1} A^T \right] A = I$$

(iii) $\quad Ax = y$ has a solution if $y \in \text{Range}(A)$

c) (i) $$\left[ A(x - \hat{x}) \right]^T (A\hat{x} - y) = (x - \hat{x})^T A^T (A\hat{x} - y).$$
$$= (x - \hat{x})^T (A^T A \hat{x} - A^T y)$$
$$= (x - \hat{x})^T (A^T y - A^T y) = 0$$

(ii) $$\| Ax - y \|^2 = \| (Ax - A\hat{x}) \|^2 + \| (A\hat{x} - y) \|^2$$
$$\geq \| A\hat{x} - y \|^2$$

Q4 8

(iii) $y \notin \text{Range}(A)$ then

$\hat{x}$ is the least square solution

(the one that minimises $\|Ax-y\|$).

for the linear problem $Ax=y$;

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Q5

a) $Ax = \lambda x \quad (x \neq 0) \qquad x^T A x = \lambda x^T x$

$\lambda = \dfrac{x^T A x}{x^T x} > 0 \qquad$ since $\quad x^T A x > 0$ & $x \neq 0$

& $x^T x$ "

b)

$x_1 \dots x_n$ is a basis.

for $x \in \mathbb{R}^n \qquad x = \alpha_1 x_1 + \dots + \alpha_n x_n$.

$A x = \alpha_1 \lambda_1 x_1 + \dots + \alpha_n \lambda_n x_n$.

$x^T A x = \alpha_1^2 \lambda_1 + \dots + \alpha_n^2 \lambda_n > 0$ if $x \neq 0$.

c)

~~First~~ $A^T A = \begin{pmatrix} 1 & -2 & 2 \\ -2 & 5 & -3 \\ 2 & -3 & 30 \end{pmatrix}$ by direct matrix multiplicat

d) (i) we want to find $\hat{x} = (A^T A)^{-1} A^T y$

that is to say solve

$$A^T A \hat{x} = A^T y$$

i.e. $\begin{pmatrix} 1 & -2 & 2 \\ -2 & 5 & -3 \\ 2 & -3 & 30 \end{pmatrix} x = A^T y = \begin{pmatrix} 7/5 \\ -13/5 \\ 4 \end{pmatrix}$

(ii) Immediate application of Cholesky.

(iii) First solve.

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & 1 & 5 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 7/5 \\ -13/5 \\ 4 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 7/5 \\ 1/5 \\ 1/5 \end{pmatrix}$$

and then solve.

$$\begin{pmatrix} 1 & -2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7/5 \\ 1/5 \\ 1/5 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 41/25 \\ 4/25 \\ 1/25 \end{pmatrix}$$