EE4-10
EE9-CS5-1
EE9-SO26

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2014

MSc and EEE PART IV: MEng and ACGI

Corrected Copy

PROBABILITY AND STOCHASTIC PROCESSES

Tuesday, 20 May 10:00 am

Time allowed: 3:00 hours

There are FOUR questions on this paper.

Answer ALL questions. All questions carry equal marks.

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : C. Ling
Second Marker(s) : D. Angeli

Information for students

Each of the four questions has 25 marks.

The Questions

1. Random variables.

a) A rare disease affects one person in 10^4. A test for this disease shows positive with probability 9/10 when applied to an ill person, and with probability 1/10 when applied to a healthy person. What is the probability that you have the disease given that the test shows positive?

[5]

b) Suppose the random variable X has a Cauchy density

$$f_X(x) = \frac{\alpha/\pi}{\alpha^2 + x^2}$$

and $Y = \tan^{-1} X$, derive the probability density function of Y, and determine the value of α such that Y is uniformly distributed.

[5]

c) X and Y are independent, identically distributed (i.i.d.) random variables with common probability density function

$$f_X(x) = e^{-x}, \qquad x > 0$$
$$f_Y(y) = e^{-y}, \qquad y > 0$$

Find the probability density function of the following random variables:

i) $Z = X + Y$. [5]

ii) $Z = X - Y$. [5]

iii) $Z = XY$. [5]

2. Estimation.

a) The random variable X has the density $f(x) \sim c^4 x^3 e^{-cx}, \ \ x > 0$. We observe the i.i.d. samples $x_i = 6.1, 5.7, 6.3, 5.7, 6.2$. Find the maximum-likelihood estimate of parameter c.

[8]

b) Consider the auto-regressive process

$$Y(n) = \alpha Y(n-1) + Z(n)$$

where is α a real number satisfying $|\alpha| < 1$, and $Z(n)$ is an i.i.d. sequence with zero mean and unit variance.

i) Show that the autocorrelation function of $Y(n)$ is given by

$$R_Y(n) = \frac{\alpha^{|n|}}{1-\alpha^2}$$

[7]

ii) Suppose we wish to predict $Y(n+1)$ from $Y(n)$, $Y(n-1)$, …, $Y(1)$. The coefficients of the linear MMSE estimator

$$Y(n+1) = \sum_{i=1}^{n} c_i Y(i)$$

are given by the Wiener-Hopf equation

$$\boldsymbol{Rc} = \boldsymbol{r}$$

where $\boldsymbol{c} = [c_1, c_2, \dots, c_n]^T$, $\boldsymbol{r} = [R_Y(n), R_Y(n-1), \dots, R_Y(1)]^T$, and $\boldsymbol{R}$ is a n-by-n matrix whose (i,j)th entry is $R_Y(i-j)$. Find the best coefficients and the associated mean-square error.

[10]

3. Random processes.

a) Consider the random process

$$X(t) = A_t \cos(\omega t + \theta)$$

where t is continuous time and A_t are i.i.d. random variables with $E[A_t] = 0, Var[A_t] = \sigma^2$.

i) Let θ be a constant. Calculate the mean, variance of $X(t)$ and determine whether it is stationary or not.

[5]

ii) Now let θ be uniformly distributed on [-π, π], and also independent of A_t. Calculate the mean, autocorrelation function of $X(t)$ and determine whether it is wide-sense stationary or not.

[5]

b) The random process $X(t)$ has autocorrelation $R(\tau)$.

i) If $X(t)$ is real-valued, show that

$$P\{|X(t+\tau) - X(t)| \geq a\} \leq 2[R(0) - R(\tau)]/a^2.$$

[5]

ii) From the fact that $R(\tau)$ is the inverse Fourier transform of the power spectral density $S(\omega)$, show that $R(\tau)$

$$\sum_{i,k} a_i a_k^* R(\tau_i - \tau_k) \geq 0$$

for all a_i.

[5]

iii) If $X(t)$ is a normal (i.e., Gaussian) process with zero mean and $Y(t) = Ie^{aX(t)}$, show that

$$E[Y(t)] = I\exp\left\{\frac{a^2}{2} R(0)\right\}$$

$$R_Y(\tau) = I^2 \exp\{a^2[R(0) + R(\tau)]\}$$

Hint: Use the characteristic function of two jointly Gaussian random variables $N(0,0,\sigma_1^2,\sigma_2^2,\rho)$, which is given by

$$\Phi(\omega_1,\omega_2) = \exp\left\{-\frac{\sigma_1^2\omega_1^2 + 2\rho\sigma_1\ \sigma_2\ \omega_1\ \omega_2\ + \sigma_2^2\omega_2^2}{2}\right\}$$

[5]

4. Martingale and Markov chains.

a) Show that the sums $S_n = X_1 + X_2 + \cdots + X_n$ of independent zero mean random variables form a martingale.

[5]

b) Consider a Markov chain with states $e_1, e_2, \ldots, e_m$ and the following transition matrix

$$P = \begin{pmatrix} q & p & 0 & \cdots & 0 \\ 0 & q & p & \cdots & 0 \\ \vdots & \vdots & q & \cdots & 0 \\ 0 & 0 & \cdots & q & p \\ p & 0 & \cdots & 0 & q \end{pmatrix}$$

Find the limiting distribution.

[5]

c) Consider a stationary Markov chain $\ldots, X_{n-1}, X_n, X_{n+1}, \ldots$ with transition probabilities $\{p_{ij}\}$.

i) Assuming the chain has reached the steady state with limiting distribution $\{q_i\}$, show that the reversed sequence is also a stationary Markov chain with transition probabilities

$$P(X_n = j | X_{n+1} = i) \triangleq p_{ij}^* = \frac{q_j p_{ji}}{q_i}$$

[5]

ii) A Markov chain is said to be reversible if $p_{ij}^* = p_{ij}$ for all i, j. Show that a necessary condition for reversibility is

$$p_{ij} p_{jk} p_{ki} = p_{ik} p_{kj} p_{ji}, \quad \text{for all } i, j, k.$$

[5]

iii) In general, a Markov chain may or may not have a steady state distribution. Yet, show that if it is reversible for some distribution $\{q_i\}$, then $\{q_i\}$ is just the steady state distribution.

[5]