UNIVERSITY OF LONDON
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BEng Honours Degree in Computing Part III
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the Associateship of the City and Guilds of London Institute

PAPER 3.14

NUMERICAL ANALYSIS

Tuesday, April 22nd 1997, 10.00 - 12.00

Answer THREE questions

For admin. only: paper contains 4 questions

1a Let L be a non-singular $n \times n$ lower triangular matrix. Describe clearly two algorithms for solving

$$L\mathbf{x} = \mathbf{b};$$

one being preferable if it is more efficient to access the elements of L column-by-column, the other being preferable if it is more efficient to access the elements of L row-by-row. State a similar pair of algorithms for solving the system

$$U\mathbf{x} = \mathbf{b},$$

where U is a non-singular $n \times n$ upper triangular matrix.

b Let A be an $n \times n$ matrix with $a_{11} \neq 0$. Obtain the formula for the components of the vector $\boldsymbol{\ell}_1 \in \Re^n$ so that

$$J_1 A,$$

where J_1 is the matrix $I - \boldsymbol{\ell}_1 \mathbf{e}_1^T$ with $\mathbf{e}_1 \in \Re^n$ the first unit vector, has the same first row as A but otherwise zeroes in its first column. Similarly, obtain the formula for the components of the vector $\mathbf{u}_1 \in \Re^n$ so that

$$AK_1,$$

where K_1 is the matrix $I - \mathbf{e}_1 \mathbf{u}_1^T$, has the same first column as A but otherwise zeroes in its first row.

c Now consider the extension of the two algorithms in b.

i) Describe carefully the construction of the matrices $J_k \equiv I - \boldsymbol{\ell}_k \mathbf{e}_k^T$ so that

$$J_{n-1} \dots J_2 J_1 A$$

is an upper triangular matrix.

ii) Similarly, describe the construction of the matrices $K_k \equiv I - \mathbf{e}_k \mathbf{u}_k^T$ so that

$$AK_1 K_2 \dots K_{n-1}$$

is a lower triangular matrix.

iii) Note when the resulting algorithms will breakdown and state a condition on the original matrix A which characterises breakdown.

The five parts carry, respectively, 30%,10%,20%,20%,20% of the total marks.

2. Let $A \equiv \{a_{ij}\}$ be a given $m \times n$ matrix with $m > n$, $\mathbf{b} \equiv \{b_i\}$ a given vector in $\Re^m$, $\mathbf{x} \equiv \{x_i\}$ a general vector in $\Re^n$ and

$$\begin{aligned} g(\mathbf{x}) &\equiv \|\mathbf{b} - A\mathbf{x}\|_2^2 \\ &\equiv \sum_{i=1}^{m} \Big(b_i - \sum_{j=1}^{n} a_{ij}x_j \Big)^2 . \end{aligned}$$

a Deduce that

$$(\dagger) \qquad \frac{\partial}{\partial x_k} g(\mathbf{x}) = 0$$

if and only if

$$\mathbf{a}_k^T (\mathbf{b} - A\mathbf{x}) = 0,$$

where $\mathbf{a}_k$ is the k^{th} column of A, and hence establish that ($\dagger$) holds for $k = 1, \ldots, n$ if and only if

$$(\ddagger) \qquad A^T A\mathbf{x} = A^T\mathbf{b}.$$

b Assume that $A^T A$ is non-singular.

i) Deduce that $A\mathbf{z} = \mathbf{0}$ if and only if $\mathbf{z} = \mathbf{0}$.

ii) Use i) to prove that $\mathbf{z}^T A^T A\mathbf{z} > \mathbf{0}$ unless $\mathbf{z} = \mathbf{0}$.

iii) If $\mathbf{x}^\star$ denotes the solution of ($\ddagger$), use ii) to verify that

$$g(\mathbf{x}) > g(\mathbf{x}^\star) \quad \text{unless} \quad \mathbf{x} = \mathbf{x}^\star.$$

c For the particular case

$$A \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{pmatrix} \qquad \mathbf{b} \equiv \begin{pmatrix} 10 \\ 20 \\ 0 \end{pmatrix}.$$

i) Apply the normal equations approach to find the linear least squares solution of $A\mathbf{x} = \mathbf{b}$.

ii) Draw the three lines

$$a_{i1}x_1 + a_{i2}x_2 = b_i \qquad i = 1, 2, 3$$

on an $x_1 : x_2$ graph, together with your least squares solution.

The six parts carry, respectively, 20%,10%,10%,30%,20%,10% of the total marks.

Turn over ...

3. Let A be a symmetric $n \times n$ matrix with eigenvalues

$$\lambda_1 > \lambda_2 \geq \cdots \geq \lambda_{n-1} > \lambda_n$$

and a corresponding set of orthonormal eigenvectors

$$\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{n-1}, \mathbf{u}_n\}.$$

a Let $\{\mathbf{x}^{(k)}\}$ be generated, from $\mathbf{x}^{(0)} \equiv \sum_{i=1}^{n} \alpha_i \mathbf{u}_i$, by the iteration

$$\mathbf{x}^{(k+1)} = A\mathbf{x}^{(k)}.$$

i) Obtain a formula for the eigenvector expansion of $\mathbf{x}^{(k)}$ and calculate $\|\mathbf{x}^{(k)}\|_2$.

ii) If $\alpha_1 \neq 0$ and $\lambda_1 > |\lambda_n|$, use eigenvector expansions to explain why

$$\lim_{k\to\infty} \frac{\mathbf{x}^{(k)}}{\|\mathbf{x}^{(k)}\|_2} = \pm\mathbf{u}_1$$

and what factor determines the speed of convergence.

b Let $\{\mathbf{x}^{(k)}\}$ be generated, from $\mathbf{x}^{(0)} \equiv \sum_{i=1}^{n} \alpha_i \mathbf{u}_i$, by the iteration

$$\mathbf{x}^{(k+1)} = (A - \sigma I)\mathbf{x}^{(k)}.$$

i) Obtain a formula for the eigenvector expansion of $\mathbf{x}^{(k)}$ and calculate $\|\mathbf{x}^{(k)}\|_2$.

ii) If $\alpha_1 \neq 0$ and $\sigma < \frac{\lambda_1+\lambda_n}{2}$, use eigenvector expansions to explain why

$$\lim_{k\to\infty} \frac{\mathbf{x}^{(k)}}{\|\mathbf{x}^{(k)}\|_2} = \pm\mathbf{u}_1.$$

iii) If $\alpha_n \neq 0$ and $\sigma > \frac{\lambda_1+\lambda_n}{2}$, similarly explain why $\{\mathbf{x}^{(k)}\}$ will ultimately oscillate between $\mathbf{u}_n$ and $-\mathbf{u}_n$.

iv) State the optimal choice of σ, with respect to speed of convergence, for each of the cases ii) and iii).

c If $\mathbf{x} = \mathbf{u}_p + \sum_{\substack{i=1\\ i\neq p}}^{n} \beta_i \mathbf{u}_i$ with $\left(\sum_{\substack{i=1\\ i\neq p}}^{n} \beta_i^2\right)^{\frac{1}{2}} = \varepsilon$, use eigenvector expansions to show that the Rayleigh quotient $\rho(\mathbf{x}) \equiv \frac{\mathbf{x}^T A\mathbf{x}}{\mathbf{x}^T\mathbf{x}}$ satisfies

$$\rho(\mathbf{x}) = \frac{\lambda_p + \sum_{\substack{i=1\\ i\neq p}}^{n} \beta_i^2 \lambda_i}{1 + \sum_{\substack{i=1\\ i\neq p}}^{n} \beta_i^2}$$

and hence deduce that

$$|\rho(\mathbf{x}) - \lambda_p| \leq 2\varepsilon^2 \max_{i=1,\ldots,n} |\lambda_i|.$$

The seven parts carry, respectively, 10%,20%,10%,15%,15%,10%,20% of the total marks.

4a Give the definition for an $m \times m$ matrix Q to be orthogonal and deduce that, when Q is orthogonal,

$$\|Q\mathbf{x}\|_2 = \|\mathbf{x}\|_2$$

for every $\mathbf{x} \in \Re^m$.

b Verify that the $m \times m$ matrix

$$H(\mathbf{w}) \equiv I - 2\frac{\mathbf{w}\mathbf{w}^T}{\mathbf{w}^T\mathbf{w}}$$

is orthogonal for every non-zero $\mathbf{w} \in \Re^m$.

c If $\mathbf{y} \in \Re^m$ satisfies $\sum_{i=2}^m y_i^2 \neq 0$, verify that

$$H(\|\mathbf{y}\|_2\mathbf{e}_1 + \mathbf{y})\mathbf{y} = -\|\mathbf{y}\|_2\mathbf{e}_1,$$

where $\mathbf{e}_1 \in \Re^m$ is the first unit vector, by calculating $(\|\mathbf{y}\|_2\mathbf{e}_1 + \mathbf{y})^T\mathbf{y}$ and $(\|\mathbf{y}\|_2\mathbf{e}_1 + \mathbf{y})^T(\|\mathbf{y}\|_2\mathbf{e}_1 + \mathbf{y})$.

d If $\mathbf{y} \in \Re^m$ satisfies $\sum_{i=k+1}^m y_i^2 \neq 0$, then, with $\hat{\mathbf{y}} \equiv (0, \ldots, 0, y_k, y_{k+1}, \ldots, y_m)^T$,

i) explain why, for all $\mathbf{x} \in \Re^m$, $H(\|\hat{\mathbf{y}}\|_2\mathbf{e}_k + \hat{\mathbf{y}})\mathbf{x}$ does not alter the first $k-1$ components of $\mathbf{x}$,

ii) explain why

$$H(\|\hat{\mathbf{y}}\|_2\mathbf{e}_k + \hat{\mathbf{y}})\mathbf{x} = \mathbf{x}$$

when the last $n-k+1$ components of $\mathbf{x}$ are zero,

iii) verify that

$$H(\|\hat{\mathbf{y}}\|_2\mathbf{e}_k + \hat{\mathbf{y}})\mathbf{y} = (y_1, \ldots, y_{k-1}, -\|\hat{\mathbf{y}}\|_2, 0, \ldots, 0)^T.$$

e If A is an $m \times n$ matrix with $m \geq n$, explain briefly how c and d above may be used to generate orthogonal $m \times m$ matrices $Q_1, Q_2, \ldots, Q_n$ so that

$$Q_n, \ldots, Q_2, Q_1 A = U,$$

with the $m \times n$ matrix U satisfying $u_{ij} = 0$ if $i > j$.

The seven parts carry, respectively, 10%,10%,20%,10%,10%,20%,20% of the total marks.

End of Paper