# UNIVERSITY OF LONDON
# IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

## EXAMINATIONS 1998

BEng Honours Degree in Computing Part III
for Internal Students of the Imperial College of Science, Technology and Medicine

*This paper is also taken for the relevant examinations for the Associateship of the City and Guilds of London Institute*

## PAPER 3.14

## NUMERICAL ANALYSIS
Friday, May 8th 1998, 10.00 - 12.00

*Answer THREE questions*

For admin. only: paper contains 4 questions

1 Give a clear description of the modern Gauss elimination strategy for solving a general $n \times n$ non-singular system of linear equations $A\mathbf{x} = \mathbf{b}$. No proofs are expected, but you should include the following topics.

a The role played by triangular systems of equations, together with algorithms for forward and backward substitution (overwriting the right-hand side with the solution).

b Assuming no breakdown occurs, display the algorithm for overwriting $A$ by the elements of triangular matrices $L$ and $U$, where $A = LU$. In addition, explain how the solution of $A\mathbf{x} = \mathbf{b}$ may then be obtained, including algorithms for computing $\mathbf{x}$ by overwriting $\mathbf{b}$. State the approximate number of operations used by each algorithm.

c Explain briefly why the process in b may breakdown and how this can be avoided. (No algorithms required.)

d Explain briefly the strategy of partial pivoting and its purpose when inexact arithmetic is used. (No algorithms required.)

[No discussion of row/column ordering or pre-/post-multiplication is required, but you should make a consistent choice throughout.]

*The four parts carry, respectively, 20%,40%,20%,20% of the total marks.*

2a Define the 1-norm, 2-norm and $\infty$-norm for a general vector $\mathbf{x} \in \Re^n$.

b Draw the line $\begin{pmatrix} 1-2t \\ t \end{pmatrix}$ $t \in [0,1]$ on a two-dimensional graph; together with the three sets of points

$$\|\mathbf{x}\|_1 = 1, \quad \|\mathbf{x}\|_2 = 1, \quad \|\mathbf{x}\|_\infty = 1,$$

for $\mathbf{x} \in \Re^2$. For each of the three norms, calculate which value of $t$ gives the closest point on the line to the origin.

c Define the $\infty$-norm of a general matrix $A \in \Re^{n \times n}$ in terms of the vector $\infty$-norm on $\Re^n$ and state (proof <u>not</u> required) its characterisation in terms of the elements of $A$.

d Consider a matrix $B \in \Re^{n \times n}$ with $\|B\|_\infty < 1$.

i) Use the equation

$$(I - B)\mathbf{y} = \mathbf{0}$$

to deduce that $I - B$ is non-singular.

ii) Verify that

$$(I-B)^{-1} = I + B(I-B)^{-1}$$

and hence prove that

$$\|(I-B)^{-1}\|_\infty \le \frac{1}{1 - \|B\|_\infty}.$$

e Use d above to show that the matrix

$$\begin{pmatrix} 1/2 & 0 & 1/4 & 0 \\ 1/3 & 1 & 0 & -1/3 \\ 0 & 1/3 & 2/3 & -1/4 \\ 1/2 & 0 & 0 & 5/4 \end{pmatrix}$$

is non-singular, and calculate a bound on the $\infty$-norm of its inverse.

*The six parts carry, respectively, 10%,20%,10%,15%,25%,20% of the total marks.*

*Turn over ...*

3 You are reminded of the following two results for Gauss elimination without pivoting applied to an $n \times n$ matrix $A$.

- Gauss elimination will break down if and only if a leading principal $k \times k$ minor of $A$, $1 \le k \le n-1$, is singular.
- If $A$ is non-singular and Gauss elimination does not break down, $A$ can be factored uniquely into the form
$$(\dagger) \quad A = L_1 D U_1,$$
where $L_1$ is an $n \times n$ unit lower triangular matrix, $U_1$ is an $n \times n$ unit upper triangular matrix and $D$ is an $n \times n$ diagonal matrix.

a If $A$ is positive-definite, i.e.
$$\mathbf{y} \neq \mathbf{0} \Rightarrow \mathbf{y}^T A \mathbf{y} > 0,$$
prove that:-

i) $A$ is non-singular,

ii) Gauss elimination applied to $A$ will not break down.

b If $A$ is both symmetric and positive-definite:-

i) deduce that $L_1 = U_1^T$ in $(\dagger)$;

ii) establish that the diagonal elements of $D$ in $(\dagger)$ are strictly greater than zero;

iii) hence show that
$$(\ddagger) \quad A = U^T U,$$
where $U$ is an upper triangular matrix with strictly positive diagonal elements;

iv) write down the component form of $(\ddagger)$ for the elements in the upper triangular part of $A$;

v) explain carefully how these equations may be solved, column-by-column, for the corresponding elements of $U$.

c Calculate the Cholesky factorisation of
$$A \equiv \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 8 & 11 \\ 3 & 8 & 14 & 20 \\ 4 & 11 & 20 & 30 \end{pmatrix}.$$

*The eight parts carry, respectively, 10%,15%,10%,15%,10%,10%,15%,15% of the total marks.*

4a Give the definition for an $m \times m$ matrix $Q$ to be orthogonal and deduce that, when $Q$ is orthogonal,

$$\|Q\mathbf{x}\|_2 = \|\mathbf{x}\|_2$$

for every $\mathbf{x} \in \Re^m$.

b Verify that the $m \times m$ matrix

$$H(\mathbf{w}) \equiv I - 2\frac{\mathbf{w}\mathbf{w}^T}{\mathbf{w}^T\mathbf{w}}$$

is orthogonal for every non-zero $\mathbf{w} \in \Re^m$ and that, if the first $k-1$ components of $\mathbf{w}$ are zero, then:-

i) $H(\mathbf{w})\mathbf{x}$ leaves the first $k-1$ components of $\mathbf{x}$ unchanged,

ii) $H(\mathbf{w})\mathbf{x} = \mathbf{x}$ if the last $m-k+1$ components of $\mathbf{x}$ are zero.

c You are reminded that:-

- if $\mathbf{y} \in \Re^m$ with $\sum_{i=2}^m y_i^2 \neq 0$ then

$$\mathbf{w} = \|\mathbf{y}\|_2\mathbf{e}_1 + \mathbf{y} \Rightarrow H(\mathbf{w})\mathbf{y} = -\|\mathbf{y}\|_2\mathbf{e}_1,$$

  where $\mathbf{e}_1 \in \Re^m$ is the first unit vector;
- if $\mathbf{y} \in \Re^m$ with $\sum_{i=k+1}^m y_i^2 \neq 0$ then $\mathbf{w} = \|\hat{\mathbf{y}}\|_2\mathbf{e}_k + \hat{\mathbf{y}}$, where

$$\hat{\mathbf{y}} \equiv (0, \ldots, 0, y_k, y_{k+1}, \ldots, y_m)^T,$$

  means that

$$H(\mathbf{w})\mathbf{y} = (y_1, \ldots, y_{k-1}, -\|\hat{\mathbf{y}}\|_2, 0, \ldots, 0)^T.$$

Use these results to explain how, if $A$ is an $m \times n$ matrix with $m \geq n$, orthogonal matrices $Q_1, Q_2, \ldots, Q_n$ may be generated so that

$$Q_n \ldots Q_2 Q_1 A = U,$$

with the $m \times n$ matrix $U$ satisfying $u_{ij} = 0$ if $i > j$. Deduce that, if $A^TA$ is non-singular, then so is $U^TU$ and hence that the $n \times n$ matrix $\hat{U}$, where $\hat{U}_{ij} = U_{ij} \quad i,j = 1, \ldots, n$, is also non-singular.

d Explain briefly how the above theory can be used to solve the linear least squares problem

$$\min_{\mathbf{x} \in \Re^n} \|\mathbf{b} - A\mathbf{x}\|_2,$$

where $\mathbf{b}$ is a given vector in $\Re^m$ and $A$ is a given $m \times n$ matrix with $m \geq n$ and $A^TA$ non-singular.

*The four parts carry, respectively, 10%,20%,40%,30% of the total marks.*

*End of Paper*