CORRECTION
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M3/M4
S1

Imperial College
London

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2013

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Statistical Theory I

Date: Wednesday, 22 May 2013. Time: 2.00pm. Time allowed: 2 hours.
This paper has FOUR questions.
Candidates should use ONE main answer book.
Supplementary books may only be used after the main book is full.
Statistical tables will not be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Answer all the questions. Each question carries equal weight.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Calculators may not be used.

1. (a) What is meant by each of the following?

   (i) Parameter θ is *estimable.*

   (ii) Estimator T_n of θ based on a sample of size n is *consistent.*

   (iii) Statistic $t(\boldsymbol{x})$ is *sufficient* for a family of distributions parameterised by θ.

   (iv) Statistic $a(\boldsymbol{x})$ is *ancillary* for a family of distributions parameterised by θ.

   (b) Let $\boldsymbol{x} = \{x_1, x_2, \ldots, x_n\}$ be a random sample from *Poisson*(θ), with unknown $\theta > 0$.

   (i) Show that $z = \sum_1^n x_k$ is a minimal sufficient statistic for *Poisson*(θ).

   (ii) Find the maximum likelihood estimate of θ^2 and its bias.

   (iii) Obtain an unbiased estimator of θ^2 that has minimum variance.
   Give your reasoning.

2. (a) For a single random variable X from $N(\theta,\, c^2\theta)$, with $\theta > 0$ and known constant c, find

   (i) the efficient score $U(\theta)$,

   (ii) the Fisher information $I(\theta)$.

   (b) Let $\boldsymbol{x} = \{x_1, x_2, \ldots, x_n\}$ be a random sample from $N(\theta,\, c^2\theta)$, as in (a) above.

   (i) Show that the likelihood $\ell(\theta; \boldsymbol{x})$ is a function of sufficient statistics $(\overline{x},\, s^2)$.

   (ii) Show that the efficiency of the unbiased estimate $\overline{x}$ of θ is $\dfrac{2\theta}{2\theta + c^2}$.

   (iii) Find the efficiency of the unbiased estimate s^2/c^2 of θ.

   [*For a random sample of size* n *from* $\mathrm{N}(\mu,\, \sigma^2)$,
   $Z = (n-1)S^2/\sigma^2$ *is* χ^2_{n-1} *with* $E(Z) = n-1$ *and* $\operatorname{var}(Z) = 2(n-1)$.]

3. (a) Consider a size α test of composite hypotheses $H_0 : \theta \in \Theta_0$ against $H_1 : \theta \notin \Theta_0$.
   What is meant by each of the following?

   (i) The *size* of the test.

   (ii) The test is *similar.*

   (iii) The test is *unbiased.*

3. (b) Let $\boldsymbol{x} = \{x_1, x_2, \ldots, x_m\}$ and $\boldsymbol{y} = \{y_1, y_2, \ldots, y_n\}$ be independent random samples respectively from *Exponential*(ξ) and *Exponential*(η).

(i) To test $H_0 : \xi = \eta$ against $H_1 : \xi \neq \eta$, calculate the ratio of maximised likelihoods test statistic

$$\lambda(\boldsymbol{x}, \boldsymbol{y}) = \frac{\ell_{H_1}(\widehat{\xi_1}, \widehat{\eta_1}\,;\boldsymbol{x}, \boldsymbol{y})}{\ell_{H_0}(\widehat{\xi_0}\,;\boldsymbol{x}, \boldsymbol{y})},$$

where $\widehat{\xi_1}$ and $\widehat{\eta_1}$ are the maximum likelihood estimates of ξ and η respectively under H_1, and $\widehat{\xi_0}$ is the maximum likelihood estimate of the common value of ξ and η under H_0.
[*You may write down the maximum likelihood estimates without proof.*]

(ii) Show that $\lambda(\boldsymbol{x}, \boldsymbol{y})$ depends only on the ratio of the means, $z = \overline{x}/\overline{y}$.

(iii) Show that $\Lambda(\boldsymbol{x}, \boldsymbol{y}) = \ln \lambda(\boldsymbol{x}, \boldsymbol{y})$ has a unique minimum (at $z = 1$), which implies that the null hypothesis is rejected if z is too large or too small.

(iv) By observing that z has a sampling distribution proportional to an F-distributed random variable, show how you would find values c_1 and c_2 for the critical region $\{z : (z < c_1) \cup (z > c_2)\}$ of the test of size α.

4. (a) Let a single random variable X be from a distribution having probability density function

$$f(x\,|\,\theta) = \tfrac{1}{2}(1 + \theta x) \qquad (-1 < x < 1),$$

where $-1 < \theta < 1$.

(i) Obtain the efficient score $U(\theta)$.

(ii) Show that the Fisher information $I(\theta)$ is $\dfrac{1}{2\theta^3} \ln\left(\dfrac{1+\theta}{1-\theta}\right) - \dfrac{1}{\theta^2}$.

(b) Let $\boldsymbol{x} = \{x_1, x_2, \ldots, x_n\}$ be a random sample from the distribution in (a).

(i) Find the method of moments estimator $\widehat{\theta_0}$ of θ.

(ii) Find the efficiency of $\widehat{\theta_0}$ as an estimator of θ.

(iii) Write down the log likelihood function ℓ $\cancel{L}(\theta\,;\boldsymbol{x})$ and its derivative $L'(\theta\,;\boldsymbol{x}) = \dfrac{\partial}{\partial\theta} \ln L(\theta\,;\boldsymbol{x})$.

Observe that the minimal sufficient statistic is the entire set of order statistics, making calculation of the maximum likelihood estimate $\widehat{\theta}$ difficult.

Expand $L'(\widehat{\theta_0}\,|\,\boldsymbol{x})$ (*ie* $L'(\theta\,|\,\boldsymbol{x})$ evaluated at $\widehat{\theta_0}$) about the MLE $\widehat{\theta}$, and obtain a consistent estimate $\widehat{\theta_1}$ of θ that is more efficient than $\widehat{\theta_0}$.
[*You are not required to obtain the efficiency of* $\widehat{\theta_1}$.]

FINAL

	M3S1/M4S1 EXAMINATION SOLUTIONS 2012–13	Course M3S1 M4S1
Question 1		Marks & seen/unseen
Parts a) i)	There is at least one unbiased estimator of θ.	Bookwork 1
ii)	T_n converges to θ as $n \to \infty$.	1
iii)	When any statistic $z(\underset{\sim}{x})$ is such that $f_{Z\mid T,\theta}(z\mid t,\theta)$ does not depend on θ. I.e. $f_{\underset{\sim}{X}\mid T,\theta}(\underset{\sim}{x}\mid t,\theta)$ is the same for all $\theta \in \Theta$.	2
iv)	The conditional distribution of $a\mid\theta$ is the same for all θ.	2
b) i)	$f_{\underset{\sim}{X}\mid\theta}(\underset{\sim}{x}\mid\theta) = \prod_{k=1}^{n} \frac{e^{-\theta}\theta^{x_k}}{x_k!} = \frac{1}{\prod(x_k!)} e^{-n\theta}\theta^{\Sigma x_k} = h(\underset{\sim}{x})g(\Sigma x_k,\theta)$ so $z = \Sigma x_k$ is sufficient for θ by Neyman Factorisation, minimal since it has only one element - dimension - rank	Similar seen 3
ii)	• $Z = \Sigma X_k$ • $f_{Z\mid\theta}(z\mid\theta) = \frac{e^{-n\theta}(n\theta)^z}{z!}$ $\ln f_{Z\mid\theta}(z\mid\theta) = -n\theta + z\ln n + z\ln\theta - \ln(z!)$ $\frac{\partial \ln f_{Z\mid\theta}(z\mid\theta)}{\partial\theta} = -n + \frac{z}{\theta} = \frac{n}{\theta}\left(\frac{z}{n} - \theta\right)$ so $\hat\theta = \frac{z}{n}$ is MLE for θ • By invariance under transformation $\hat\theta^2 = \frac{z^2}{n^2}$ is MLE for θ^2 • $E(Z) = n\theta$, $var(Z) = n\theta$ so $E(Z^2) = var(Z) + \{E(Z)\}^2 = n\theta + n^2\theta^2$ so $E(\hat\theta^2) = \frac{\theta}{n} + \theta^2$ and $bias(\hat\theta^2) = \frac{\theta}{n}$	Unseen 3 3
iii)	$\frac{1}{n^2}E(Z^2 - Z) = \theta^2$ so $\frac{1}{n^2}(Z^2 - Z)$ is unbiased for θ^2 This unbiased estimator is a function only of minimal sufficient statistic z, so it is MVUE for θ^2 by Lehmann-Scheffé Theorem.	3 2
	Setter's initials	Checker's initials
		Page number 1

	M3S1/M4S1 EXAMINATION SOLUTIONS 2012–13	Course M3S1 M4S1
Question 2		Marks & seen/unseen

Parts

a) $$f(x|\theta) = \frac{1}{\sqrt{2\pi c^2\theta}} e^{-\frac{1}{2c^2\theta}(x-\theta)^2}$$ Unseen

$$\ln f(x|\theta) = \ln\left(\frac{1}{\sqrt{2\pi c^2}}\right) - \frac{1}{2}\ln\theta - \frac{1}{2c^2\theta}(x^2 - 2x\theta + \theta^2)$$

(i) $$U(\theta) = \frac{\partial}{\partial\theta}\ln f(X|\theta) = -\frac{1}{2\theta} + \left(\frac{1}{2c^2}\right)\frac{X^2}{\theta^2} - \frac{1}{2c^2}$$ 3

(ii) $$\frac{\partial U(\theta)}{\partial\theta} = \frac{1}{2\theta^2} - \frac{X^2}{c^2}\frac{1}{\theta^3}$$

$$I(\theta) = E\left(-\frac{\partial U(\theta)}{\partial\theta}\right) = -\frac{1}{2\theta^2} + \frac{1}{c^2\theta^3}\left(var(X) + \{E(X)\}^2\right)$$

$$= -\frac{1}{2\theta^2} + \frac{1}{c^2\theta^3}c^2\theta + \frac{1}{c^2\theta^3}\theta^2 = \frac{c^2+2\theta}{2c^2\theta^2}$$ 4

b) (i) $$f(\underset{\sim}{x}|\theta) = \left(\frac{1}{\sqrt{2\pi c^2}}\right)^n \theta^{-\frac{n}{2}} e^{-\frac{1}{2c^2\theta}\Sigma(x_i-\theta)^2}$$

$$\begin{aligned}\Sigma(x_i-\theta)^2 &= \Sigma\{(x_i-\bar{x})+(\bar{x}-\theta)\}^2 \\ &= \Sigma(x_i-\bar{x})^2 + 2(\bar{x}-\theta)\Sigma(x_i-\bar{x}) + n(\bar{x}-\theta)^2 \\ &= (n-1)s^2 + 0 + n(\bar{x}-\theta)^2\end{aligned}$$

so $$f(\underset{\sim}{x}|\theta) = \left(\frac{1}{\sqrt{2\pi c^2}}\right)^n \theta^{-\frac{n}{2}} \exp\left\{-\frac{1}{2c^2\theta}\left[(n-1)s^2 + n(\bar{x}-\theta)^2\right]\right\}$$

$= g(\bar{x}, s^2, \theta)$ so $(\bar{x}, s^2)$ are sufficient statistics by Neyman Factorisation Theorem 4

(ii) By (a) $\text{CRLB} = \frac{1}{nI(\theta)}$ & $var(\bar{X}) = \frac{c^2\theta}{n}$

$$\text{Efficiency}(\bar{x}) = \frac{1/c^2\theta}{(c^2+2\theta)/2c^2\theta^2} = \frac{2\theta}{2\theta+c^2}$$ 3

(iii) $$\text{Efficiency}\left(\frac{s^2}{c^2}\right) = \frac{1/var\left(\frac{S^2}{c^2}\right)}{nI(\theta)}$$

$$Z = \frac{(n-1)S^2}{\sigma^2} = \frac{(n-1)S^2}{c^2\theta} \qquad var(Z) = \frac{(n-1)^2}{\theta^2}var\left(\frac{S^2}{c^2}\right) = 2(n-1)$$

$$var\left(\frac{S^2}{c^2}\right) = \frac{2\theta^2}{n-1}$$

$$\text{Efficiency}\left(\frac{s^2}{c^2}\right) = \frac{n-1}{2\theta^2} \Big/ \frac{n(c^2+2\theta)}{2c^2\theta^2} = \frac{n-1}{n}\cdot\frac{c^2}{2\theta+c^2}$$ 6

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M3S1/M4S1 EXAMINATION SOLUTIONS 2012–13

Course: M3S1 M4S1

Question 3

Marks & seen/unseen

Parts

a) i) $\sup_{\theta \in \Theta_0} \alpha(\theta)$ where $\alpha(\theta) = P(\underset{\sim}{X} \in R \mid \theta \in \Theta_0)$ type I error probability — Bookwork 2

ii) $\alpha(\theta) = \alpha$ a constant for all $\theta \in \Theta$ — 1

iii) The power $\beta(\theta) = P(\underset{\sim}{X} \in R \mid \theta)$ satisfies

$$\beta(\theta) \begin{cases} \leq \alpha & \theta \in \Theta_0 \\ \geq \alpha & \theta \in \bar{\Theta}_0 \end{cases}$$

2

All unseen

b) i) $\hat{\xi}_1 = \frac{1}{\bar{x}}$ $\hat{\eta}_1 = \frac{1}{\bar{y}}$ $\hat{\xi}_0 = \frac{m+n}{m\bar{x}+n\bar{y}}$

$f_{H_1}(\underset{\sim}{x}, \underset{\sim}{y} \mid \xi_1, \eta) = \xi_1^m e^{-\xi_1(m\bar{x})} \cdot \eta_1^n e^{-\eta_1(n\bar{y})} \Rightarrow \text{MLE}\big|_{H_1}\ \hat{\xi}_1 = \frac{1}{\bar{x}}, \hat{\eta}_1 = \frac{1}{\bar{y}}$

$f_{H_0}(\underset{\sim}{x}, \underset{\sim}{y} \mid \xi_0) = \xi_0^{m+n} e^{-\xi_0(m\bar{x}+n\bar{y})} \Rightarrow \text{MLE}\big|_{H_0}\ \hat{\xi}_0 = \frac{m+n}{m\bar{x}+n\bar{y}}$ — 3

ii)
$$\lambda(\underset{\sim}{x}, \underset{\sim}{y}) = \frac{\left(\frac{1}{\bar{x}}\right)^m e^{-\frac{1}{\bar{x}}m\bar{x}} \cdot \left(\frac{1}{\bar{y}}\right)^n e^{-\frac{1}{\bar{y}}n\bar{y}}}{\left(\frac{m+n}{m\bar{x}+n\bar{y}}\right)^{m+n} e^{-\frac{m+n}{m\bar{x}+n\bar{y}}(m\bar{x}+n\bar{y})}}$$

$$= \frac{1}{\bar{x}^m} \cdot \frac{1}{\bar{y}^n} \cdot \frac{(m\bar{x}+n\bar{y})^{m+n}}{(m+n)^{m+n}} = \frac{m^{m+n}}{(m+n)^{m+n}} \left(\frac{\bar{x}}{\bar{y}} + \frac{n}{m}\right)^{m+n} \frac{1}{\left(\frac{\bar{x}}{\bar{y}}\right)^m}$$

$$= \frac{1}{\left(1+\frac{n}{m}\right)^{m+n}} \left(z + \frac{n}{m}\right)^{m+n} \frac{1}{z^m}$$

3

The test is reject H_0 if $\lambda(\underset{\sim}{x}, \underset{\sim}{y})$ is too large

iii) $\Lambda(\underset{\sim}{x}, \underset{\sim}{y}) = \ln \lambda(\underset{\sim}{x}, \underset{\sim}{y}) = -(m+n)\ln\left(1+\frac{n}{m}\right) + (m+n)\ln\left(z+\frac{n}{m}\right) - m\ln z$

$\frac{\partial \Lambda}{\partial z} = \frac{m+n}{z+\frac{n}{m}} - \frac{m}{z}$, $= 0$ when $zm + zn = zm + n$

i.e. when $z = 1$ if $\bar{x} = \bar{y}$

$\frac{\partial^2 \Lambda}{\partial z^2}\Big|_{z=1} = -\frac{m+n}{\left(z+\frac{n}{m}\right)^2} + \frac{m}{z^2}\Big|_{z=1} = m\left\{1 - \frac{1+\frac{n}{m}}{\left(1+\frac{n}{m}\right)^2}\right\} = m\left(1 - \frac{1}{1+\frac{n}{m}}\right) > 0$

so $\lambda(\underset{\sim}{x}, \underset{\sim}{y})$ has a unique minimum at $z = 1$

so reject if $z = \frac{\bar{x}}{\bar{y}}$ is too large or too small — 4

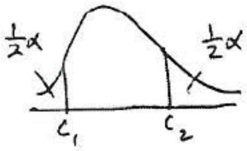
ctd.

Setter's initials — Checker's initials

b) iv) $2\xi(m\bar{X})$ is χ^2_{2m} independent of $2\eta(n\bar{Y})$ which is χ^2_{2n} (sums of exponential rvs)

so $$\frac{\frac{1}{2m}(2\xi m\bar{X})}{\frac{1}{2n}(2\eta n\bar{Y})} = \frac{\xi}{\eta}Z \quad \text{is } F_{2m,2n}$$

Under $H_0: \xi=\eta$, Z is $F_{2m,2n}$



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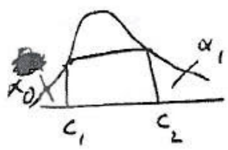

$\alpha_0+\alpha_1=1$

5

Question 4

Marks & seen/unseen

Parts

All unseen

a) $\ln f(x|\theta) = \ln\left(\tfrac{1}{2}\right) + \ln(1+\theta x)$

i) $U(\theta) = \dfrac{\partial \ln f(X|\theta)}{\partial\theta} = \dfrac{X}{1+\theta X}$ — 2

ii) $I(\theta) = E\{U^2(\theta)\} \quad \left(= E\left\{-\dfrac{\partial U(\theta)}{\partial\theta}\right\}\right)$

$$= \int_{-1}^{1} \frac{x^2}{(1+\theta x)^2}\cdot\frac{1}{2}(1+\theta x)\,dx = \frac{1}{2}\int_{-1}^{1}\frac{x^2}{1+\theta x}\,dx$$

$$= \frac{1}{2\theta^3}\int_{1-\theta}^{1+\theta}\frac{(z-1)^2}{z}\,dz \qquad (z = 1+\theta x)$$

$$= \frac{1}{2\theta^3}\int_{1-\theta}^{1+\theta}\left(z-2+\frac{1}{z}\right)dz = \frac{1}{2\theta^3}\Big[\underbrace{\tfrac{1}{2}z^2-2z}_{\frac{1}{2}(z-2)^2 - \cancel{4}} + \ln z\Big]_{1-\theta}^{1+\theta}$$

$\frac{1}{2}\{\cancel{}(\theta-1)^2-(-1-\theta)^2\}$

$$= \frac{1}{2\theta^3}\ln\left(\frac{1+\theta}{1-\theta}\right) - \frac{1}{\theta^2}$$ — 4

b) i) $E(X) = \int_{-1}^{1}\frac{1}{2}(\underset{\uparrow \text{odd}}{x}+\theta x^2)\,dx = \theta\int_0^1 x^2\,dx = \frac{\theta}{3}$ so $\theta = 3E(X)$

so $\hat\theta_0 = 3\bar{X}$

ii) $E(X^2) = \int_{-1}^{1}\frac{1}{2}(x^2+\underset{\uparrow \text{odd}}{\theta x^3})\,dx = \int_0^1 x^2\,dx = \frac{1}{3}$

$\mathrm{var}(X) = \frac{1}{3} - \frac{\theta^2}{9} = \frac{1}{9}(3-\theta^2)$ so $\mathrm{var}(3\bar{X}) = \frac{1}{n}(3-\theta^2)$

$$\text{Efficiency}(\hat\theta_0) = \frac{1/nI(\theta)}{\mathrm{var}(\hat\theta_0)} = \frac{1/I(\theta)}{3-\theta^2}$$ — 6

$\left[\text{Check: } \theta = \tfrac{1}{2} \Rightarrow \frac{2.53}{2.75} \sim 0.92\right]$

iii) $L(\theta;\underset{\sim}{x}) = \Sigma \ln f(x_i|\theta) = n\ln\left(\tfrac{1}{2}\right) + \sum_i \ln(1+\theta x_i)$

$\dfrac{\partial L(\theta;x)}{\partial\theta} = \sum_i \dfrac{x_i}{1+\theta x_i}$

For MLE $\hat\theta$ of θ requires solution of $\sum_i \frac{x_i}{1+\theta x_i} = 0$
— problematic

Expansion about MLE $\hat\theta$:

$L'(\hat\theta_0) = \left.\frac{\partial L}{\partial\theta}\right|_{\theta=\hat\theta_0=3\bar{x}} = \underset{0}{L'(\hat\theta)} + (\hat\theta_0-\hat\theta)L''(\hat\theta) + \cdots$

so $\hat\theta \approx \hat\theta_0 + \dfrac{L'(\hat\theta_0)}{-L''(\hat\theta)}$ $\qquad L''(\theta) = \sum_i \dfrac{x_i^2}{(1+\theta x_i)^2}$

$-L''(\hat\theta) \approx I_1(\hat\theta) \approx I_1(\hat\theta_0)$ by consistency of $\hat\theta_0$

ctd

Question 4 ctd

Marks & seen/unseen

Parts

b) iii) so $\hat{\theta}_1 \approx 3\bar{x} + \dfrac{\sum_i \frac{x_i}{1+(3\bar{x})x_i}}{n\,I(3\bar{x})}$ 8

Setter's initials Checker's initials