UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 2002

MSci Honours Degree in Mathematics and Computer Science Part IV
MEng Honours Degrees in Computing Part IV
MSc in Advanced Computing
for Internal Students of the Imperial College of Science, Technology and Medicine

*This paper is also taken for the relevant examinations for the Associateship of the City and Guilds of London Institute*
*This paper is also taken for the relevant examinations for the Associateship of the Royal College of Science*

# PAPER C477=I4.20

# COMPUTING FOR OPTIMAL DECISIONS

Tuesday 23 April 2002, 14:00
Duration: 120 minutes

*Answer THREE questions*

Paper contains 4 questions
Calculators not required

1 a Let f(x) and g(x) be two convex functions of $x \in \Re^n$ and let $\mathcal{R}$ be a convex feasible set for x. Consider the problem

$$\min \left\{ f(x) + g(x) \;\middle|\; x \in \mathcal{R} \right\}.$$

Establish the number of possible optima for this problem.

b Suggest a modification to the steepest descent algorithm for solving the constrained problem

$$\min_x \left\{ f(x) \;\middle|\; \mathcal{L} \le x \le \mathcal{U} \,;\; x \in \Re^n \right\},$$

where $\mathcal{L}, \mathcal{U} \in \Re^n$ are upper and lower bounds on x. Establish the descent property of the direction generated at each iteration of the algorithm for the constrained problem.

*(All parts carry equal marks)*

2 Consider the quadratic programming problem

$$\min \left\{ a^T x + \tfrac{1}{2} x^T Q x \;\middle|\; \mathcal{H}^T x \le h \right\}$$

where $x \in \Re^n$, a and h are respectively n- and m-dimensional fixed vectors, Q is a fixed symmetric positive definite matrix and $\mathcal{H}$ is an $n \times m$ dimensional matrix.

a Suppose we are at $x_1$, with $\mathcal{H}^T x_1 \le h$ and let $x_1$ be the solution of the equality constrained quadratic programming problem

$$\min \left\{ a^T x + \tfrac{1}{2} x^T Q x \;\middle|\; \mathcal{H}_1^T x = h_1 \right\}$$

where $\mathcal{H}_1^T x = h_1$ is a subset of the original constraints (i.e. $\mathcal{H}^T x \le h$) treated as equalities. Write the optimality condition of this equality constrained problem.

b Suppose some of the Lagrange multipliers associated with $\mathcal{H}_1^T x = h_1$ are negative and we derive the system $\mathcal{H}_2^T x = h_2$ by dropping one of the constraints in $\mathcal{H}_1^T x = h_1$ associated with a negative multiplier. Let $x_2$ denote the solution of

$$\min \left\{ a^T x + \tfrac{1}{2} x^T Q x \;\middle|\; \mathcal{H}_2^T x = h_2 \right\}.$$

Show that $x_2 - x_1$ is a descent direction at $x_1$ and that it is a feasible direction for $\mathcal{H}_1^T x \le h_1$.

c Write the first order optimality conditions of the inequality constrained problem.

*(All parts carry equal marks)*

 

3 a Consider the Newton algorithm for solving the following system of nonlinear equations

$$\mathcal{W}(\,y) = \begin{bmatrix} y^{1} + y^{2} - 3 \\ (y^{2})^{2} + (y^{2})^{2} - 9 \end{bmatrix} = 0; \quad y = \begin{bmatrix} y^{1} \\ y^{2} \end{bmatrix}.$$

Starting at $y_0 = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$, formulate the system of linear equations that determine the direction used in the Newton algorithm. [Do not solve the system of linear equations.]

b Consider the following merit function for the Newton algorithm $\frac{1}{2} \| \mathcal{W}(y) \|_2^2$. Establish that the Newton direction in part (a) is a descent direction for this merit function.

c Let $\nabla \mathcal{W}(y_j)$ be the (nonsingular) matrix of gradients of the elements of $\mathcal{W}(y_j)$ at $y_j$. Show that there exists a $\tau \in (0, 1]$, which satisfies the inequality

$$\tfrac{1}{2} \| \mathcal{W}(y_j + \tau\, d_j) \|_2^2 - \tfrac{1}{2} \| \mathcal{W}(y_j) \|_2^2 \;\leq\; \tau_j\, \mu < [\nabla \mathcal{W}(y_j)]^T \, \mathcal{W}(y_j),\, d_j >, \;\; \mu \in (0, \tfrac{1}{2}].$$

*(All parts carry equal marks)*

4 Two production lines of a computer company produce the same computer at different costs. The first production line produces $q_1$ computers at a cost of

$$£\ (\alpha\, q_1 + \beta).$$

The second production line produces $q_2$ computers at a cost of

$$£\ (\gamma\, q_2^2 + \mu)$$

where $\alpha$, $\beta$, $\gamma$, $\mu$ are given constant values. If a total of $q_1+q_2$ computers are produced by both production lines together, consumers are expected to pay

$$£\ (\mathcal{A} - \eta\, (q_1+q_2) + \epsilon\ )$$

for each computer. The parameters $\mathcal{A}$ and $\eta$ are given constant values whereas $\epsilon$ is a random variable, $\epsilon \sim \mathcal{N}\ (0, \mathcal{V})$. $\mathcal{V}$ is the variance of $\epsilon$. Demand for the computer is estimated at 30000 units.

a If the objective is to choose $q_1,q_2$ to maximise the expected value of the profit, formulate the corresponding problem. [Formulate the profit function and subsequently evaluate its mean.]

b If the objective is to choose $q_1,q_2$ to minimise the variance of the profit at a fixed value of the profit, say $\pi$, given the limited demand, formulate the corresponding problem. [Evaluate the variance of the profit function and use the mean obtained in (a).]

*(All parts carry equal marks)*