THE ANSWERS

Notations:

(a) B - Bookwork

(b) E - New example

(c) A - New application

1. This question checks your understanding of basic concepts of probability. It is very similar to many problems we have seen in the lectures/classes. Most students answered the question correctly.

a) i) $P(X \leq Y) = 0.05+0.05+0.15+0.05+0.25+0.05 = 1-0.05-0.15-0.20 = 0.60$

[1 - E]

$P(X < Y) = 0.05+0.15+0.25 = 0.45 = P(X \leq Y) = P(X = Y)$

[1 - E]

ii)

x	0	1	2
$P(X=x)$	0.25	0.35	0.40

[1 - E]

y	0	1	2
$P(Y=y)$	0.25	0.30	0.45

[1 - E]

iii) $E(X) = 0 \times 0.25 + 1 \times 0.35 + 2 \times 0.40 = 1.15$ [1 - E]

$E(Y) = 1.20$ [1 - E]

iv) $\text{Var}(X) = E(X^2) - E(X)^2 = 1 \times 0.35 + 4 \times 0.40 - (1.15)^2 = 0.6275,$

[1 - E]

$\text{Var}(Y) = 0.66,$ [1 - E]

$\text{Cov}(X,Y) = E(XY) - E(X)E(Y) = 1 \times 0.05 + 2 \times 0.20 + 2 \times 0.25 + 4 \times 0.05 - 1.15 \times 1.20 = -0.23$ [1 - E]

$\text{Corr}(X,Y) = \frac{-0.23}{\sqrt{0.6275 \times 0.66}} = -0.3574.$ [1 - E]

v) X and Y are correlated since $\text{Corr}(X,Y) \neq 0$. [1 - E]

Since they are correlated, they are also dependent. Dependency can also be seen from $P(X=1, Y=1) - 0.05 \neq P(X=1)P(Y=1) = 0.35 \times 0.30$ [1 - E]

vi) Compute the conditional probability mass function of X given that $Y = 0, 1, 2$.

x	0	1	2	
$P(X=x\|Y=0)$	$\frac{0.05}{0.25}=0.20$	$\frac{0.05}{0.25}=0.20$	$\frac{0.15}{0.25}=0.60$	[1 - E]

x	0	1	2	
$P(X=x\|Y=1)$	$\frac{0.05}{0.30}=1/6$	$\frac{0.05}{0.30}=1/6$	$\frac{0.20}{0.30}=2/3$	[1 - E]

x	0	1	2	
$P(X=x\|Y=2)$	$\frac{0.15}{0.45}=1/3$	$\frac{0.25}{0.45}=0.5555$	$\frac{0.05}{0.45}=0.1111$	[1 - E]

vii) Compute the conditional expectation of X given that $Y=0,1,2$.

$E(X|Y=0)=0\times 0.20+1\times 0.20+2\times 0.60=1.4$ [1 - E]

$E(X|Y=1)=0\times \frac{0.05}{0.30}+1\times \frac{0.05}{0.30}+2\times \frac{0.20}{0.30}=1.5$ [1 - E]

$E(X|Y=2)=0\times \frac{0.15}{0.45}+1\times \frac{0.25}{0.45}+2\times \frac{0.05}{0.45}=7/9$ [1 - E]

viii) $E(X)=E(E(X|Y))=1.4\times 0.25+1.5\times 0.30+7/9\times 0.45=1.15$

[2 - E]

b) We can re-express the argument as the pdf of a Normal distribution

$$\int_{-\infty}^{2.35}\sqrt{\frac{2}{\pi}}e^{-2(u-2)^2}du=\int_{-\infty}^{2.35}\frac{1}{\sqrt{2\pi\frac{1}{4}}}e^{-\frac{1}{2}(\frac{u-2}{\frac{1}{2}})^2}du.$$

Hence this is the CDF of a normal distribution with mean $\mu=2$ and $\sigma^2=\frac{1}{4}$.

[2 - A]

By standardizing the normal distribution, we can write

$$\int_{-\infty}^{2.35}\frac{1}{\sqrt{2\pi\frac{1}{4}}}e^{-\frac{1}{2}(\frac{u-2}{\frac{1}{2}})^2}du=\int_{-\infty}^{\frac{2.35-2}{1/2}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}dz.$$

[2 - A]

Last integral is obtained from the table

$$\int_{-\infty}^{0.7}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}dz=0.758.$$

[1 - E]

2. a) This question is not complicated and makes use of concepts seen in the lectures/classes but can be puzzling for some students as it deals with application of probablity to communication. You however do not need to know anything about communication to answer the question. The main problem where students have had problems is on part i). The problem of min and max has been seen in several exercices in classes.

i) $F_P(S) = P(P \leq S) = P(P_1 \leq S \cap P_2 \leq S)$. [1 - A]
From independence, we write $P(P_1 \leq S \cap P_2 \leq S) = P(P_1 \leq S)P(P_2 \leq S)$ [1 - A]
From the exponential distribution, we get $F_P(S) = \begin{cases} (1-e^{-\lambda S})^2 & S > 0 \\ 0 & \text{otherwise} \end{cases}$
[2 - A]

ii) $f_P(p) = \frac{dF_P(p)}{dp}$ [2 - A]
$f_P(p) = \begin{cases} 2\lambda(1-e^{-\lambda p})e^{-\lambda p} & p > 0 \\ 0 & \text{otherwise} \end{cases}$
[2 - A]

iii) The error probability approximates as $m_P(-d) = E(e^{-dP})$. [1 - A]
Hence $m_P(-d) = \int_0^\infty e^{-dP} 2\lambda(1-e^{-\lambda p})e^{-\lambda p} dp = \frac{2\lambda^2}{(d+\lambda)(d+2\lambda)}$. [3 - A]

iv) $E(P) = m_P'(0)$. [2 - A]
$E(P) = m_P'(0) = \frac{3}{2\lambda}$. [2 - A]

b) Part i) is straightforward and comes from the notes. Part ii) has not been correctly answered by most students despite the fact that it is clearly explained in the notes/lectures. Make sure that the reasoning is rigorous. Most students provided a proof for the mean and variance but failed to explain why the resultant pdf is Normal distributed.

i) The MGF of a Normal random variable $X \sim N(\mu, \sigma^2)$ is given as

$$m_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f_X(x)\, dx$$
$$= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

[1 - B]

Hence

$$m_X(t) = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\frac{x^2-2(\mu+t\sigma^2)x+\mu^2}{\sigma^2}} dx$$
$$= e^{t\mu+t^2\sigma^2/2} \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\frac{x^2-2(\mu+t\sigma^2)x+\mu^2+2t\mu\sigma^2+t^2\sigma^4}{\sigma^2}} dx.$$

[2 - B]

Hence

$$m_X(t) = e^{t\mu+t^2\sigma^2/2} \int_{-\infty}^{\infty} \underbrace{\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-(\mu+t\sigma^2)}{\sigma}\right)^2}}_{N(\mu+t\sigma^2,\sigma^2)} dx = e^{t\mu+t^2\sigma^2/2}.$$

[1 - B]

ii) No, it is not correct. If $X_1 \sim N(\mu_1, \sigma_1^2)$ and $X_2 \sim N(\mu_2, \sigma_2^2)$ and X_1, X_2 are independent random variables, we have $2X_1 - X_2 \sim N(2\mu_1 - \mu_2, 4\sigma_1^2 + \sigma_2^2)$.

[1 - A]

Now, we can show (using independence)

$$m_{2X_1-X_2} = E(e^{t(2X_1-X_2)}) = E(e^{t2X_1})E(e^{-tX_2})$$

[2 - A]

Hence,

$$m_{2X_1-X_2} = e^{2t\mu_1+4t^2\sigma_1^2/2}e^{-t\mu_2+t^2\sigma_2^2/2} = e^{t(2\mu_1-\mu_2)+t^2(4\sigma_1^2/2+\sigma_2^2/2)}.$$

This is the MGF of a Normal distribution with mean $2\mu_1 - \mu_2$ and variance $4\sigma_1^2 + \sigma_2^2$. Since the pdf uniquely identifies the pdf, $2X_1 - X_2 \sim N(2\mu_1 - \mu_2, 4\sigma_1^2 + \sigma_2^2)$. [2 - A]