Final

Imperial College London

[E1.10 (Maths 1) 2010]

B.ENG. and M.ENG. EXAMINATIONS 2010

PART I : MATHEMATICS 1 (ELECTRICAL ENGINEERING)

Date **Wednesday 2nd June 2010** **10.00 am - 1.00 pm**

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.

Answer EIGHT questions.

CALCULATORS MAY **NOT** BE USED.

A mathematical formulae sheet is provided.

[*Before starting, please make sure that the paper is complete; there should be 6 pages, with a total of 10 questions. Ask the invigilator for a replacement if your copy is faulty.*]

1. (i) Explain why the function

$$f(x) = \begin{cases} \frac{1}{2} - x^2 & \text{if } x \le 0 , \\ x^2 + C & \text{if } x > 0 , \end{cases}$$

is differentiable at $x = 0$ only for the choice $C = 1/2$. For this value of C, prove that f is differentiable at $x = 0$ by computing the limits

$$\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} \quad \text{and} \quad \lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} .$$

(ii) Sketch the graph of $\cos^{-1}(x)$ and find the derivative of $\cos^{-1}(x)$.

(iii) Differentiate

$$(\cos(x))^{\cos^{-1}(x)} .$$

2. (i) Calculate

$$\lim_{x \to 0} \frac{\ln(1+x)}{x} .$$

(ii) By performing the substitution

$$y = \frac{1}{x} ,$$

or otherwise, calculate

$$\lim_{y \to \infty} \left(1 + \frac{1}{y}\right)^y .$$

(iii) The power series

$$\sum_{n=1}^{\infty} u_n \, x^n$$

has coefficients given by

$$u_n = \frac{n^n}{n!} .$$

Show that the $(n+1)$th term divided by the nth term is

$$\left(\frac{n+1}{n}\right)^n x ,$$

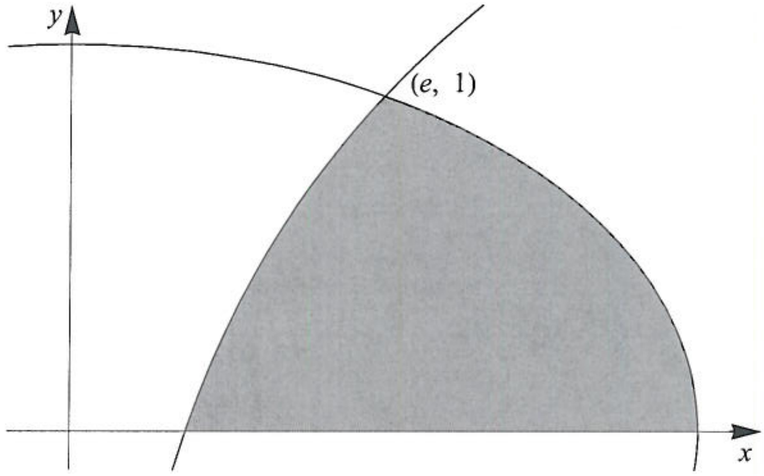
and hence calculate the radius of convergence of this power series.

PLEASE TURN OVER

3. The graph of the logarithm $y = \ln(x)$ meets the ellipse

$$\frac{x^2}{4e^2} + \frac{3y^2}{4} = 1$$

at $(e,\ 1)$. Find the area of the finite region enclosed in the first quadrant by these curves and the x-axis, as in the figure below



4. (i) If

$$\begin{aligned} z_1 &= -1 + 2i\ , \\ z_2 &= 3 + 3i\ , \end{aligned}$$

find $z_1\, z_2$ and $\dfrac{z_1}{z_2}$ in the form $x + iy$ for some real x and y and verify that:

$$|\, z_1\, z_2\,| = |\, z_1\,|\,|\, z_2\,|\ .$$

(ii) Find all complex solutions of $z^3 = 1$. Show that the sum of all these solutions is 0. Indicate the position of each solution in the complex plane.

(iii) De Moivre's theorem states

$$(\cos\theta\ +\ i\ \sin\theta)^n\ =\ \cos(n\theta)\ +\ i\ \sin(n\theta)\ .$$

Using this theorem, or otherwise, find an expression for $\sin(3\theta)$ in terms of powers of $\sin\theta$.

5. Consider the function

$$f(x) \;=\; \frac{e^{-x^2}}{2-x^2}\;.$$

(i) State, giving reasons, whether $f(x)$ is even, odd or neither.

(ii) Find all vertical and horizontal asymptotes.

(iii) Use (ii) to determine the sign of $f(x)$, for all x.

(iv) Find all the points where $f'(x) = 0$.

(v) Determine any local minima and maxima of $f(x)$.

(vi) Sketch the graph of $f(x)$.

6. (i) Find a vector equation for the line of intersection of the planes $x+y+z \;=\; 1$ and $2x-y+3z \;=\; 0$.

Find the equation of a plane through the origin which is perpendicular to the line of intersection found above.

(ii) Given two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in three dimensional space, show that the vector

$$\boldsymbol{a}_p \;=\; \frac{(\boldsymbol{a}\cdot\boldsymbol{b})}{|\,\boldsymbol{b}\,|^2}\;\boldsymbol{b}$$

is the projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$. Verify that $\boldsymbol{a}-\boldsymbol{a}_p$ is a vector perpendicular to $\boldsymbol{b}$.

Decompose the vector (1, 2, 3) into a sum of two vectors, one of which is parallel to (1, 1, 1) and the other perpendicular to (1, 1, 1).

PLEASE TURN OVER

7. Let

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ -1 & 2 & 3 \end{pmatrix} .$$

(i) Find A^2 and A^3. Show that

$$A^3 - 5A^2 + 7A + I = 0 .$$

(ii) Hence, or otherwise, show that $|A| \neq 0$, and hence that

$$A^{-1} = -A^2 + 5A - 7I .$$

Using this result, find A^{-1}.

(iii) Check your answer in (ii) by finding A^{-1}, using row operations on A.

8. (i) Find, in the form $y = f(x)$, the solution of the differential equation

$$\frac{dy}{dx} = \frac{4x - y}{x + 2y} ,$$

for which $y(0) = 1$.

(ii) Find the general solution of the linear differential equation

$$\frac{dy}{dx} \cos x + y \sin x = 1 ,$$

using an integrating factor, and hence prove that all solution curves pass through the point $(\pi/2, 1)$.

9. A forced linear oscillator with adjustable damping has equation of motion

$$\frac{d^2x}{dt^2} + k\,\frac{dx}{dt} + x = f(t)\,.$$

Given that $x(0) = 0.24$ and $\dot{x}(0) = 0$, and that $f(t)$ is identically zero, find the solution of this equation in the following cases:-

$$\begin{array}{rrcl} \text{(i)} & k & = & 2.5\,; \\ \text{(ii)} & k & = & 2.0\,; \\ \text{(iii)} & k & = & 1.2\,. \end{array}$$

For the case $k = 2.0$, find also the solution, with the same initial conditions, given that

$$f(t) = e^{-t}\,.$$

10. A function $y(x)$ satisfies the equation

$$y'' + (4x^2 + 1)\,y = 0$$

and the initial conditions $y(0) = 0,\ y'(0) = 1$.

Differentiate the equation m times using the Leibnitz formula to show that for $m \geq 2$

$$y^{(m+2)}(0) = -\left[y^{(m)}(0) + 4m(m-1)\,y^{(m-2)}(0)\right]\,.$$

Deduce that

$$y(x) = \sum_{m=0}^{\infty} c_m\, x^{2m+1}\,,$$

where c_m are constants.

Find the first three non-zero terms of this series.

END OF PAPER

MATHEMATICS DEPARTMENT

MATHEMATICAL FORMULAE

1. VECTOR ALGEBRA

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1, a_2, a_3)$$

Scalar (dot) product: $\mathbf{a}\,.\,\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

Vector (cross) product:

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix};$$

Scalar triple product:

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a}\,.\,\mathbf{b}\times\mathbf{c} = \mathbf{b}\,.\,\mathbf{c}\times\mathbf{a} = \mathbf{c}\,.\,\mathbf{a}\times\mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product: $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{c}\,.\,\mathbf{a})\mathbf{b} - (\mathbf{b}\,.\,\mathbf{a})\mathbf{c}$

2. SERIES

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \ldots \quad (\alpha \text{ arbitrary}, |x| < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n\frac{x^{2n}}{(2n)!} + \ldots,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + \ldots,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^n\frac{x^{n+1}}{(n+1)} + \ldots (-1 < x \le 1)$$

3. TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin(a+b) = \sin a\cos b + \cos a\sin b\,;$$

$$\cos(a+b) = \cos a\cos b - \sin a\sin b\,.$$

$$\cos iz = \cosh z\,;\quad \cosh iz = \cos z\,;\quad \sin iz = i\sinh z\,;\quad \sinh iz = i\sin z\,.$$

4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f\,D^n g + \binom{n}{1} Df\,D^{n-1}g + \ldots + \binom{n}{r} D^r f\,D^{n-r}g + \ldots + D^n f\,g\,.$$

(b) Taylor's expansion of $f(x)$ about $x = a$:

$$f(a+h) = f(a) + hf'(a) + h^2 f''(a)/2! + \ldots + h^n f^{(n)}(a)/n! + \epsilon_n(h)\,,$$

where $\epsilon_n(h) = h^{n+1}f^{(n+1)}(a+\theta h)/(n+1)!, \quad 0 < \theta < 1\,.$

(c) Taylor's expansion of $f(x, y)$ about (a, b):

$$f(a+h, b+k) = f(a,b) + [hf_x + kf_y]_{a,b} + 1/2!\left[h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}\right]_{a,b} + \ldots$$

(d) Partial differentiation of $f(x, y)$:

i. If $y = y(x)$, then $f = F(x)$, and $\dfrac{dF}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dx}$.

ii. If $x = x(t)$, $y = y(t)$, then $f = F(t)$, and $\dfrac{dF}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$.

iii. If $x = x(u, v)$, $y = y(u, v)$, then $f = F(u, v)$, and

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}, \quad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$

(e) Stationary points of $f(x, y)$ occur where $f_x = 0$, $f_y = 0$ simultaneously.
Let (a, b) be a stationary point: examine $D = [f_{xx}f_{yy} - (f_{xy})^2]_{a,b}$.
If $D > 0$ and $f_{xx}(a, b) < 0$, then (a, b) is a maximum;
If $D > 0$ and $f_{xx}(a, b) > 0$, then (a, b) is a minimum;
If $D < 0$ then (a, b) is a saddle-point.

(f) Differential equations:

i. The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating factor $I(x) = \exp[\int P(x)(dx]$, so that $\dfrac{d}{dx}(Iy) = IQ$.

ii. $P(x, y)dx + Q(x, y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y$.

5. INTEGRAL CALCULUS

(a) An important substitution: $\tan(\theta/2) = t$:
$\sin\theta = 2t/(1+t^2)$, $\cos\theta = (1-t^2)/(1+t^2)$, $d\theta = 2\,dt/(1+t^2)$.

(b) Some indefinite integrals:

$$\int (a^2 - x^2)^{-1/2} dx = \sin^{-1}\left(\frac{x}{a}\right), \quad |x| < a .$$

$$\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1}\left(\frac{x}{a}\right) = \ln\left\{\frac{x}{a} + \left(1 + \frac{x^2}{a^2}\right)^{1/2}\right\} .$$

$$\int (x^2 - a^2)^{-1/2} dx = \cosh^{-1}\left(\frac{x}{a}\right) = \ln\left|\frac{x}{a} + \left(\frac{x^2}{a^2} - 1\right)^{1/2}\right| .$$

$$\int (a^2 + x^2)^{-1} dx = \left(\frac{1}{a}\right) \tan^{-1}\left(\frac{x}{a}\right) .$$

6. NUMERICAL METHODS

(a) Approximate solution of an algebraic equation:

If a root of $f(x) = 0$ occurs near $x = a$, take $x_0 = a$ and
$x_{n+1} = x_n - [f(x_n)/f'(x_n)]$, $n = 0, 1, 2 \ldots$

(Newton Raphson method).

(b) Formulae for numerical integration: Write $x_n = x_0 + nh$, $y_n = y(x_n)$.

i. Trapezium rule (1-strip): $\int_{x_0}^{x_1} y(x)dx \approx (h/2)\,[y_0 + y_1]$.

ii. Simpson's rule (2-strip): $\int_{x_0}^{x_2} y(x)dx \approx (h/3)\,[y_0 + 4y_1 + y_2]$.

(c) Richardson's extrapolation method: Let $I = \int_a^b f(x)dx$ and let I_1, I_2 be two estimates of I obtained by using Simpson's rule with intervals h and $h/2$.

Then, provided h is small enough,

$$I_2 + (I_2 - I_1)/15 ,$$

is a better estimate of I.

7. LAPLACE TRANSFORMS

Function	Transform	Function	Transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)dt$	$af(t) + bg(t)$	$aF(s) + bG(s)$
df/dt	$sF(s) - f(0)$	d^2f/dt^2	$s^2F(s) - sf(0) - f'(0)$
$e^{at}f(t)$	$F(s-a)$	$tf(t)$	$-dF(s)/ds$
$(\partial/\partial\alpha)f(t,\alpha)$	$(\partial/\partial\alpha)F(s,\alpha)$	$\int_0^t f(t)dt$	$F(s)/s$
$\int_0^t f(u)g(t-u)du$	$F(s)G(s)$		
1	$1/s$	$t^n (n = 1, 2 \ldots)$	$n!/s^{n+1}$, $(s > 0)$
e^{at}	$1/(s-a)$, $(s > a)$	$\sin\omega t$	$\omega/(s^2+\omega^2)$, $(s > 0)$
$\cos\omega t$	$s/(s^2+\omega^2)$, $(s > 0)$	$H(t-T) = \begin{cases} 0, & t < T \\ 1, & t > T \end{cases}$	e^{-sT}/s, $(s, T > 0)$

8. FOURIER SERIES

If $f(x)$ is periodic of period $2L$, then $f(x + 2L) = f(x)$, and

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \text{ where}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad n = 0, 1, 2, \ldots, \text{ and}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1, 2, 3, \ldots$$

Parseval's theorem

$$\frac{1}{L}\int_{-L}^{L} [f(x)]^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right) .$$

EE1 - Maths Paper 1 (E1.10) EE 1

	EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2009-2010	Course EE I(1)
Question C1		Marks & seen/unseen

Parts

(a) The function is not continuous at $x = 0$ if $C \neq 1/2$, so f can only be differentiable if $C = 1/2$. 2

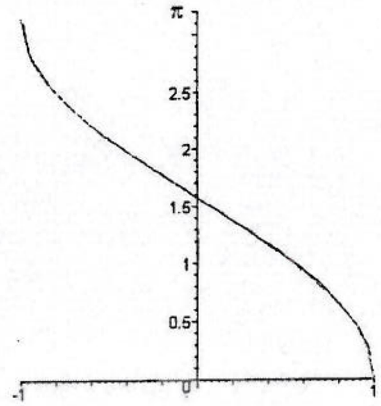
For $C = 1/2$,

$$\lim_{h\to 0+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0+} \frac{h^2+\frac{1}{2}-\frac{1}{2}}{h} = \lim_{h\to 0+} h = 0,$$ 2

$$\lim_{h\to 0-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0-} \frac{\frac{1}{2}-h^2-\frac{1}{2}}{h} = \lim_{h\to 0-} -h = 0.$$ 2

Since the lateral limits $\lim_{h\to 0+} \frac{f(0+h)-f(0)}{h}$ and $\lim_{h\to 0-} \frac{f(0+h)-f(0)}{h}$ coincide, f is differentiable $x = 0$ and $f'(0) = 0$. 1

(b) The graph of $\cos^{-1}(x)$ is



3

Using the inverse function rule

$$(\cos^{-1})'(x) = \frac{1}{\cos'(\cos^{-1}(x))} = \frac{1}{-\sin(\cos^{-1}(x))} = -\frac{1}{\sqrt{1-\cos^2(\cos^{-1}(x))}} = -\frac{1}{\sqrt{1-x^2}}.$$ 4

[Other methods also acceptable].

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EE 1

	EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2009-2010	Course EE I(1)
Question C1		Marks & seen/unseen

(c) In order to differentiate $f(x) = \cos(x)^{\cos^{-1}(x)}$ we need first to take logarithms

$$\ln(f(x)) = \ln\left(\cos(x)^{\cos^{-1}(x)}\right) = \cos^{-1}(x)\ln(\cos(x))$$ 2

so that we can then differentiate

$$\begin{aligned}\tfrac{1}{f(x)}f'(x) &= (\cos^{-1})'(x)\ln(\cos(x)) + \cos^{-1}(x)(\ln\circ\cos)'(x)\\ &= -\tfrac{1}{\sqrt{1-x^2}}\ln(\cos(x)) - \cos^{-1}(x)\tfrac{\sin(x)}{\cos(x)}.\end{aligned}$$ 2

Therefore

$$\begin{aligned}f'(x) &= -f(x)\left(\tfrac{1}{\sqrt{1-x^2}}\ln(\cos(x)) + \cos^{-1}(x)\tan(x)\right)\\ &= -\cos(x)^{\cos^{-1}(x)}\left(\tfrac{1}{\sqrt{1-x^2}}\ln(\cos(x)) + \cos^{-1}(x)\tan(x)\right).\end{aligned}$$ 2

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Parts

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EXAMINATION QUESTIONS/SOLUTIONS 2009-2010

Course ~~EIA~~ FIRST YR ENG CORE EE1 (1)

Question C2

Marks & seen/unseen

Parts

(i)

$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = \lim_{x\to 0} \frac{x - x^2/2 + x^3/3 - \cdots}{x}$$

$$= \lim_{x\to 0} 1 - x/2 + x^2/3 - \cdots$$

$$= \underline{\underline{1.}}$$

OR l'Hôpital's rule

$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = \lim_{x\to 0} \frac{1/(1+x)}{1}$$

$$= \underline{\underline{1.}}$$

5

(ii)

$$\lim_{y\to\infty} \left(1+\frac{1}{y}\right)^y = \lim_{x\to 0^+} (1+x)^{1/x}$$

$$= \lim_{x\to 0^+} \left[e^{1/x \ln(1+x)}\right]$$

$$= e^1 \text{ from (i) } = \underline{\underline{e.}}$$

5

(iii)

$$\frac{a_{n+1}}{a_n} = \frac{u_{n+1} x^{n+1}}{u_n x^n}$$

$$= \frac{(n+1)^{n+1}}{(n+1)!} \times \frac{n!}{n^n} \times x$$

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EE 2

	EXAMINATION QUESTIONS/SOLUTIONS 2009-2010	Course FIRST YR ENG CORE EE I (1)
Question C2		Marks & seen/unseen

Parts

$$= \frac{(n+1)^{n+1}}{(n+1)\, n^n} \times x$$

$$= \frac{(n+1)^n}{n^n} x = \left(\frac{n+1}{n}\right)^n x .$$ 5

Now,

$$\lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n x = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n x$$

$$= e x .$$

For convergence, we require

$$|e x| < 1$$

that is, $|x| < \frac{1}{e}$.

Radius of convergence is $\frac{1}{e}$. 5

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EE 3

	EXAMINATION QUESTIONS/SOLUTIONS 2009-2010	Course EE I (1)
Question C3		Marks & seen/unseen

The area of the figure that we want to calculate is delimited by the horizontal x-axis, the graph of $y = \ln(x)$, and the graph of

$$y = \tfrac{2}{\sqrt{3}}\sqrt{1 - \tfrac{x^2}{4e^2}}.$$ 1

Therefore, the area is given by

$$\int_1^e \ln(x)\,dx + \tfrac{2}{\sqrt{3}}\int_e^{2e}\sqrt{1 - \tfrac{x^2}{4e^2}}dx$$ 2

(Correct identification of integration limits) 2

Parts

On the one hand, to integrate $\int_1^e \ln(x)\,dx$ we apply integration by parts

$$\begin{matrix} v' = 1 \\ u = \ln(x) \end{matrix} \implies \begin{matrix} v = x \\ u' = 1/x \end{matrix}$$ 3

so that

$$\int_1^e \ln(x)\,dx = [x\ln(x)]_1^e - \int_1^e x\tfrac{1}{x}dx = \mathrm{e} - [x]_1^e = 1.$$ 3

On the other hand, to evaluate $\frac{2}{\sqrt{3}}\int_e^{2e}\sqrt{1 - \frac{x^2}{4e^2}}dx$ we apply the following change of variables

$$\tfrac{x}{2e} = \sin(u) \implies dx = 2e\cos(u)\,du$$ 2

which implies that the limits of integration change as follows:

$$\begin{matrix} x = e \implies \tfrac{1}{2} = \sin(u) \implies u = \tfrac{\pi}{6} \\ x = 2e \implies 1 = \sin(u) \implies u = \tfrac{\pi}{2} \end{matrix}$$ 2

Therefore

$$\tfrac{2}{\sqrt{3}}\int_e^{2e}\sqrt{1 - \tfrac{x^2}{4e^2}}dx = \tfrac{4e}{\sqrt{3}}\int_{\pi/6}^{\pi/2}\sqrt{1 - \sin^2(u)}\cos(u)du = \tfrac{4e}{\sqrt{3}}\int_{\pi/6}^{\pi/2}\cos^2(u)\,du$$

$$= \tfrac{4e}{\sqrt{3}}\int_{\pi/6}^{\pi/2}\tfrac{1+\cos(2u)}{2}du = \tfrac{2e}{\sqrt{3}}\left[u + \tfrac{\sin(2u)}{2}\right]_{\pi/6}^{\pi/2}$$ 3

$$= \tfrac{2e}{\sqrt{3}}\left[\tfrac{\pi}{3} - \tfrac{\sqrt{3}}{4}\right]$$

The final result is

$$\text{Area} = \tfrac{2e}{\sqrt{3}}\left[\tfrac{\pi}{3} - \tfrac{\sqrt{3}}{4}\right] + 1 \qquad \text{(correct value)}$$ 2

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EE 4

	EXAMINATION QUESTIONS/SOLUTIONS 2009-2010	Course EE I(1)
Question C4	Solutions (1/2)	Marks & seen/unseen

Parts

(i)

$$z_1 z_2 = (-1+2i)(3+3i)$$

$$= \cancel{\ } -3+6i^2+6i-3i = -9+3i.$$ 2

$$\frac{z_1}{z_2} = \frac{(-1+2i)}{(3+3i)} = \frac{(-1+2i)(3-3i)}{(3+3i)(3-3i)}$$

$$= \frac{-3-6i^2+3i+6i}{18} = \frac{-3+9i}{18}$$

$$= \frac{1}{6} + \frac{1}{2}i.$$ 2

$$|z_1| = \sqrt{(-1+2i)(-1-2i)} = \sqrt{5}$$ 1

$$|z_2| = \sqrt{(3+3i)(3-3i)} = 3\sqrt{2}$$ 1

$$|z_1 z_2| = \sqrt{(-9+3i)(-9-3i)} = \sqrt{90} = 3\sqrt{2}.\sqrt{5} = |z_1||z_2|$$ 1

(ii)

$$z = re^{i\theta} \Rightarrow z^3 = r^3 e^{i3\theta} = 1e^{2k\pi i} \Rightarrow$$

$$r = 1, \quad \theta = 2k\pi \quad k = 0, 1, 2$$

Solutions : $z_1 = 1$, $z_2 = e^{\frac{2\pi}{3}i}$, $z_3 = e^{\frac{4\pi}{3}i}$ 3

$$z_1 + z_2 + z_3 = 1 + e^{\frac{2\pi}{3}i} + e^{\frac{4\pi i}{3}} = \left(1 + \cos\frac{2\pi}{3} + \cos\frac{4\pi}{3}\right)$$

$$+ i\left(\sin\frac{2\pi}{3} + \sin\frac{4\pi}{3}\right) = (1 - \cos\pi/3 - \cos\pi/3) +$$

$$i(\cancel{\sin\pi/3} - \cancel{\sin\pi/3}) = \left(1 - \frac{1}{2} - \frac{1}{2}\right) = 0$$ 2

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EE 4

EXAMINATION QUESTIONS/SOLUTIONS 2009-2010

Course: EE I(1)

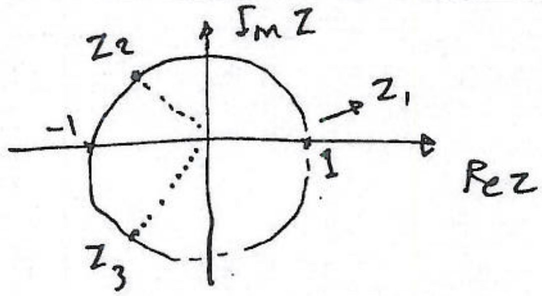
Question: C4

Solutions (2/2)

Marks & seen/unseen

Parts

(ii) continued



2

(iii) For $-\theta$ De Moivre's theorem states

$$(\cos\theta - i\sin\theta)^n = \cos(n\theta) - i\sin(n\theta) \quad (*)$$

For $n=3$ by subtracting $(*)$ from the original De Moivre's theorem we get,

$$\sin 3\theta = \frac{1}{2i}\left[(\cos\theta + i\sin\theta)^3 - (\cos\theta - i\sin\theta)^3\right] =$$

3

$$\frac{1}{2i}\left[(\cos^3\theta + 3i\sin\theta\cos^2\theta + 3i^2\sin^2\theta\cos\theta + i^3\sin^3\theta)\right.$$

$$\left.-(\cos^3\theta - 3i\sin\theta\cos^2\theta + 3i^2\sin^2\theta\cos\theta - i^3\sin^3\theta)\right] =$$

$$\frac{1}{2i}\left[6i\sin\theta\cos^2\theta - 2i\sin^3\theta\right] =$$

$$3\sin\theta(1-\sin^2\theta) - \sin^3\theta =$$

$$3\sin\theta - 4\sin^3\theta.$$

3

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Setter's initials: V.S.

Checker's initials: RJR.

Page number: 2

EXAMINATION QUESTIONS/SOLUTIONS 2009-2010

Course: EE I (1)

Question: C5

Solutions

Marks & seen/unseen

Parts

(i) $f(-x) = f(x)$ even. — 2

(ii) $\lim_{x \to \infty} f(x) = 0 \Rightarrow y = 0$ horizontal asymtote. — 1

$\lim_{x \to \sqrt{2}} f(x) = \infty \Rightarrow x = \sqrt{2}$ vertical asymtote — 1

$\lim_{x \to -\sqrt{2}} f(x) = \infty \Rightarrow x = -\sqrt{2}$ vertical asymtote — 1

(iii) Since $f(x) = 0$ has no solutions and also e^{-x^2} is always positive we have: — 1

$2-x^2$: $- \; - \; - \; - \quad + + + + + + \quad - - - - - -$ (axis x, marked at -2 and $+2$)

then

$f(x)$: $- - - - - - \quad + + + + + + + + \quad - - - - -$ (axis x, marked at -2 and -2) — 2

(iv)

$$f'(x) = \frac{-2x e^{-x^2}(2-x^2) - (-2x)e^{-x^2}}{(2-x^2)^2} =$$

$$\frac{2x e^{-x^2}(x^2-1)}{(2-x^2)^2}$$

— 2

Setter's initials: V.S.

Checker's initials: DTP

EE 5

	EXAMINATION QUESTIONS/SOLUTIONS 2009-2010	Course EE I(1)
Question C5	Solutions	Marks & seen/unseen

Parts

(iv) continued

$f'(x) = 0 \Rightarrow x(x^2-1) = 0 \Rightarrow$ — 2

$x = 0, \pm 1$

(v) We should find the sign of $f''(x)$ at $x = 0, \pm 1$:

$$f''(x) = \frac{2(2-x^2)^2 e^{-x^2}\left(-2x(x^3-x) + 3x^2 - 1\right) + 8x(2-x^2)(x^3-x)e^{-x^2}}{(2-x^2)^4}$$

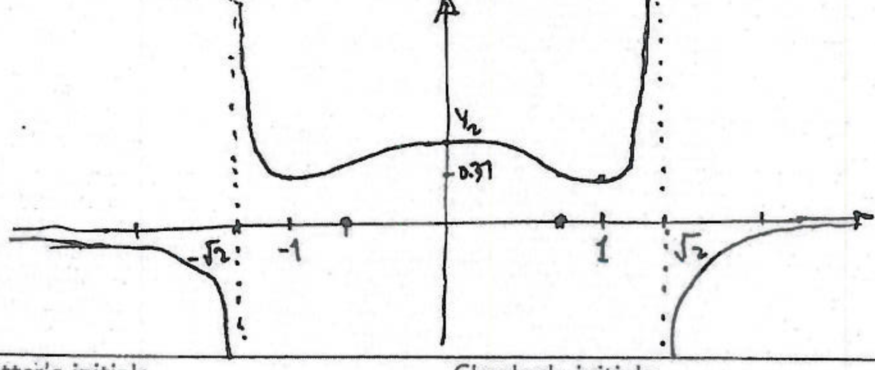
$$= \frac{2e^{-x^2}\left(2x^6 - 5x^4 + 7x^2 - 2\right)}{(2-x^2)^3}$$ — 2

$f''(0) = \frac{2 \times -2}{8} = -\frac{1}{2} < 0 \Rightarrow x = 0$ maximum — 1

$f''(1) = f''(-1) = \frac{2e^{-1}(2-5+7-2)}{1} = 4e^{-1} > 0 \Rightarrow$ — 1

$x = \pm 1$ minima

(vi) $f(0) = \frac{1}{2}$ $\qquad f(1) = f(-1) = e^{-1} \simeq 0.37$ — 1



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EE 6

EXAMINATION QUESTIONS/SOLUTIONS 2009-2010

Course: CORE EE I (1)

Question: C6

ANSWER

Marks & seen/unseen

Parts

Note Any reasonable method accepted.

(a) Normal to $x+y+z=1$ is $\underline{n}_1 = (1,1,1)$ — 1

" " $2x-y+3z=0$ is $\underline{n}_2 = (2,-1,3)$ — 1

Direction of line is $\underline{n}_1 \times \underline{n}_2 = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 1 \\ 2 & -1 & 3 \end{vmatrix}$

$= (4,-1,-3)$ — 2

Equation is $\underline{x} = \underline{x}_0 + t(4,-1,-3)$ — 2

where $\underline{x}_0$ is any point on the line

Find $\underline{x}_0$: Put $x=0$ in equs of the planes

$$\left.\begin{array}{l} y+z=1 \\ -y+3z=0 \end{array}\right\} \Rightarrow \underline{x}_0 = \left(0, \tfrac{3}{4}, \tfrac{1}{4}\right)$$ — 2

Plane perpendicular to the line has normal $\underline{n} = (4,-1,-3)$

$\Rightarrow$ equation is $4x-y-3z = \text{const.}$

Since plane passes through the origin the equation is $4x-y-3z=0$ — 4

Setter's initials: DTP

Checker's initials: JWB

EZE 6

EXAMINATION QUESTIONS/SOLUTIONS 2009-2010

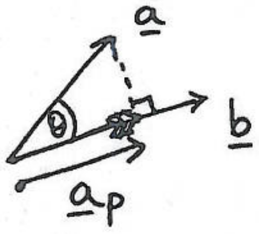
Course: CORE EE1 (1)

Question: C6 — ANSWER

Marks & seen/unseen

Parts

(b)



$$\cos\theta = \frac{(\underline{a}\cdot\underline{b})}{|\underline{a}||\underline{b}|}$$

$$\underline{a}_p = |\underline{a}|\cos\theta\,\frac{\underline{b}}{|\underline{b}|} = \frac{(\underline{a}\cdot\underline{b})}{|\underline{b}|^2}\,\underline{b}$$ 4

To show $\underline{a}-\underline{a}_p \perp \underline{b}$, compute

$$(\underline{a}-\underline{a}_p)\cdot\underline{b} = \underline{a}\cdot\underline{b} - \frac{(\underline{a}\cdot\underline{b})}{|\underline{b}|^2}(\underline{b}\cdot\underline{b}) = 0$$ 2

Here $\underline{a} = (1,2,3)$, $\underline{b} = (1,1,1)$

$$\underline{a}_p = \frac{6}{3}(1,1,1) = (2,2,2)$$ 1

other vector is $\underline{a}-\underline{a}_p = (-1,0,1)$ 1

(Check $(-1,0,1)\cdot(1,1,1) = 0$)

$\Rightarrow$ required decomposition is

$$(1,2,3) = (2,2,2) + (-1,0,1)$$

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Checker's initials: JWD

EE 7

EXAMINATION QUESTIONS/SOLUTIONS 2009-2010

Course: EE I(1)

~~Question~~ Answer

7

Marks & seen/unseen

Parts

(i) $A^2 = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ -1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ -1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} -3 & 12 & 20 \\ -2 & 5 & 8 \\ -4 & 6 & 9 \end{pmatrix}$ 2

$A^3 = \begin{pmatrix} -3 & 12 & 20 \\ -2 & 5 & 8 \\ -4 & 6 & 9 \end{pmatrix}\begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ -1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} -23 & 46 & 72 \\ -10 & 17 & 26 \\ -13 & 16 & 23 \end{pmatrix}$ 2

$A^3 - 5A^2 + 7A + I = \begin{pmatrix} -23+15+7+1 & 46-60+14 & 72-100+28 \\ -10+10+0 & 17-25+7+1 & 26-40+14 \\ -13+20-7 & 16-30+14 & 23-45+21+1 \end{pmatrix}$ 2

$= 0.$

(ii) $|A| = 1\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} - \begin{vmatrix} 2 & 4 \\ 1 & 2 \end{vmatrix} = -1 \neq 0 \Rightarrow A^{-1}$ exists 1

$A^{-1}(A^3 - 5A^2 + 7A + I) = 0 \Rightarrow A^{-1} = -A^2 + 5A - 7I$ 2

$A^{-1} = \begin{pmatrix} 3+5-7 & -12+10 & -20+20 \\ 2+0 & -5+5-7 & -8+10 \\ 4-5 & -6+10 & -9+15-7 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 0 \\ 2 & -7 & 2 \\ -1 & 4 & -1 \end{pmatrix}.$ 2

(iii) $\left(\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ -1 & 2 & 3 & 0 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 4 & 7 & 1 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & -4 & 1 \end{array}\right)$ 4

$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 4 & -1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 5 & -16 & 4 \\ 0 & 1 & 0 & 2 & -7 & 2 \\ 0 & 0 & 1 & -1 & 4 & -1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 0 \\ 0 & 1 & 0 & 2 & -7 & 2 \\ 0 & 0 & 1 & -1 & 4 & -1 \end{array}\right)$ 5

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Setter's initials: JWB

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Page number

	EXAMINATION QUESTIONS/SOLUTIONS 2009-2010	Course 1ST YR ENG CORE EE I(1)
Question C9		Marks & seen/unseen

Parts

(i) Homogeneous. Let $y(x) = xu(x)$; then

$$y' = u + xu' \text{ by product rule.}$$

This gives $u + xu' = \frac{4-u}{1+2u}$

and thus
$$x\frac{du}{dx} = \frac{(4-u)-(u+2u^2)}{1+2u}$$
$$= \frac{-2(u^2+u-2)}{2u+1}$$ 4

Separating the variables,

$$\int \frac{2u+1}{u^2+u-2}\,du = -\int \frac{2}{x}\,dx$$

$$\therefore \ln(u^2+u-2) = \ln(x^{-2}) + c$$

hence $u^2+u-2 = \frac{A}{x^2}$

and $y^2+xy-2x^2 = A$. 4

From quadratic formula (or otherwise)

$$y = \frac{-x \pm \sqrt{x^2+8x^2+4A}}{2}$$
$$= \frac{-x \pm \sqrt{9x^2+4A}}{2}$$

$y(0)=1$ implies +ve branch, $A=1$ and we have

$$y = \frac{-x+\sqrt{9x^2+4}}{2}$$ 4

[Accept any valid method. Penalise selecting wrong branch or both branches: 2 marks]

[continues...

	EXAMINATION QUESTIONS/SOLUTIONS 2009-2010	Course 1ST YR ENG CORE
Question C9 (cont)		EE I(1) Marks & seen/unseen
Parts (ii)		

Linear. Rewrite as

$$\frac{dy}{dx} + y\tan x = \sec x.$$

$\int \tan x\,dx = \ln|\sec x| + c$ ∴ suitable integrating factor is $\sec x$

Rewrite as

$$\sec x\frac{dy}{dx} + y\sec x\tan x = \sec^2 x$$ 3

Exact LHS:

$$\frac{d}{dx}\left[y\sec x\right] = \sec^2 x$$

$\therefore\ y\sec x = \tan x + C$, C constant.

giving $\underline{\underline{y = \sin x + C\cos x}}$ 3

$x = \pi/2 \Rightarrow y = \sin\pi/2 + C\cos\pi/2$

$= 1 + 0C = 1.$ ✓

(for all values of C) 2

[Accept any valid method.]

/20

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Checker's initials: VS

EXAMINATION QUESTIONS/SOLUTIONS 2009-2010

Course: 1ST YR ENG CORE

Question: C10

Marks & seen/unseen: EE I(1)

Parts

Aux eqn $\lambda^2 + k\lambda + 1 = 0$, roots $\lambda = \frac{-k \pm \sqrt{k^2-4}}{2}$ — 2

(i) $k = 2.5$: $\lambda = \frac{-2.5 \pm \sqrt{1.25}}{2} = -0.5, -2.0$

genl soln: $x = Ae^{-0.5t} + Be^{-2.0t}$

$\therefore \dot{x} = -0.5Ae^{-0.5t} - 2.0Be^{-2.0t}$

$x(0) = 0.24 \Rightarrow A + B = 0.24$

$\dot{x}(0) = 0 \Rightarrow A = -4B$

Hence $-3B = 0.24$, giving $B = -0.08$, $A = 0.32$

and $\underline{\underline{x = 0.32e^{-0.5t} - 0.08e^{-2.0t}}}$ — 4

(ii) $k = 2.0$: $\lambda = -1.0$, repeated

genl soln: $x = e^{-t}(At + B)$

$\therefore \dot{x} = e^{-t}(-At - B + A)$

$x(0) = 0.24 \Rightarrow B = 0.24$

$\dot{x}(0) = 0 \Rightarrow A = B$

and $\underline{\underline{x = 0.24e^{-t}(t+1)}}$ — 4

(iii) $k = 1.2$: $\lambda = \frac{-1.2 \pm \sqrt{-2.56}}{2} = -0.6 \pm 0.8i$

genl soln: $x = e^{-0.6t}(A\cos 0.8t + B\sin 0.8t)$

$\therefore \dot{x} = e^{-0.6t}(-0.6A\cos 0.8t - 0.6B\sin 0.8t$
$- 0.8A\sin 0.8t + 0.8B\cos 0.8t)$

$x(0) = 0.24 \Rightarrow A = 0.24$

$\dot{x}(0) = 0 \Rightarrow B = 3/4 A = 0.18$

and $\underline{\underline{x = e^{-0.6t}(0.24\cos 0.8t + 0.18\sin 0.8t)}}$ — 4

Last bit: comp. fn. is $x = e^{-t}(At + B)$

Seek partic. sl in the form $x = Mt^2e^{-t}$

$\dot{x} = e^{-t}(-Mt^2 + 2Mt)$

$\ddot{x} = e^{-t}(Mt^2 - 4Mt + 2M)$

[continues...

Setter's initials: PJR

Checker's initials: DJB

EE 9

	EXAMINATION QUESTIONS/SOLUTIONS 2009-2010	Course Core I EE I(1)
Question C10		Marks & seen/unseen

Parts

$\ddot{x} + 2\dot{x} + x = e^{-t}(\cancel{Mt^2} - \cancel{4Mt} + 2M - \cancel{2Mt^2} + \cancel{4Mt} + \cancel{Mt^2})$

$= 2Me^{-t}$ and therefore $M = \frac{1}{2}$. — 3

gen. soln $x = e^{-t}(\frac{1}{2}t^2 + At + B)$

$\dot{x} = e^{-t}(-\frac{1}{2}t^2 - At - B + t + A)$

$x(0) = 0.24 \Rightarrow B = 0.24$

$\dot{x}(0) = 0.24 \Rightarrow A = B$

and $\underline{\underline{x = e^{-t}(\frac{1}{2}t^2 + 0.24t + 0.24)}}$ — 3

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Setter's initials | Checker's initials | Page number 2 of 2

EE 10

	EXAMINATION QUESTIONS/SOLUTIONS 2009-2010	Course CORE EE I (1)
Question 11	Answer	Marks & seen/unseen
Parts		Seen Similar

$$y''(x) = -(4x^2+1)\,y(x)$$

Leibnitz for $m \geq 2 \Rightarrow$

$$y^{(m+2)}(x) = -\left[(4x^2+1)y(x)\right]^{(m)} \qquad 2$$

$$= -\left[(4x^2+1)y^{(m)}(x) + m\,8x\,y^{(m-1)}(x) + \frac{m(m-1)}{2}\,8\,y^{(m-2)}(x)\right] \qquad 3$$

$\Rightarrow$

$$y^{(m+2)}(0) = -\left[y^{(m)}(0) + 4m(m-1)\,y^{(m-2)}(0)\right] \qquad 2$$

for $m \geq 2$

Leibnitz for $m=1 \Rightarrow$

$$y'''(x) = -\left[(4x^2+1)y'(x) + 8x\,y(x)\right] \qquad 2$$

$$\Rightarrow y'''(0) = -y'(0) \qquad 1$$

$y(0)=0,\ y'(0)=1 \Rightarrow y''(0) = -y(0) = 0$ 1

and $y'''(0) = -1.$ 1

$y(0)=y''(0) \Rightarrow y^{(2m)}(0) = 0 \quad m \geq 2$ 2

$$\Rightarrow y(x) = \sum_{m=0}^{\infty} c_m x^{2m+1} \text{ with } c_m = \frac{y^{(2m+1)}(0)}{(2m+1)!} \qquad 2$$

$$c_0 = y'(0) = 1, \quad c_1 = \frac{y^{(3)}(0)}{3!} = -\frac{1}{6}$$

$$c_2 = \frac{y^{(5)}(0)}{5!} = -\frac{1}{120}\left[y^{(3)}(0) + 4.3.2\,y'(0)\right] = \frac{-23}{120} \qquad 3$$

$$y(x) = x - \frac{1}{6}x^3 - \frac{23}{120}x^5 \ldots \qquad 1$$

/20

Setter's initials JWB	Checker's initials DTP	Page number