

[E2.8 (Maths 3) 2012]

B.ENG. AND M.ENG. EXAMINATIONS 2012

**PART II Paper 3 : MATHEMATICS (ELECTRICAL AND INFORMATION
SYSTEMS ENGINEERING)**

Date Thursday 7th June 2012 2.00 - 4.00 pm

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.

Answer FOUR questions.

A mathematical formulae sheet is provided.

[Before starting, please make sure that the paper is complete; there should be 6 pages, with a total of SIX questions. Ask the invigilator for a replacement if your copy is faulty.]

1. (i) Consider the function

$$f(x, y) = x^3 - 6x^2 + 7x - y^2 + 2yx - 3.$$

Find the gradient vector $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$.

Use this to identify the direction of the constant contour at $(0, 0)$. Identify the locations of the stationary points of $f(x, y)$. Classify each stationary point as a maximum, minimum or saddle point.

- (ii) Sketch the function $g(x, y) = x^6 + y^6$.

Write down the location of its stationary point.

Explain, in words, why classifying this stationary point is difficult and explain what would be required in order to provide a classification.

2. (i) Consider the function u below:

$$u = x^3/6 + 4x^2 - xy^2/2 - 4y^2.$$

Show that it satisfies Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Use the Cauchy-Riemann equations to construct a conjugate function v .

Construct from u and v an analytic function $f(z)$.

Show that $f(z)$ can be put in the form

$$f(z) = az^3 + bz^2 + c$$

and find a and b (c is an arbitrary constant).

- (ii) Consider the map $w = z^2$ (which is conformal everywhere except at $z = 0$). The straight line $u = 1$ (for all v) in the w -plane meets the straight line $v = 0$ (for all u) at right angles.

Explain what the angle of intersection of these curves is when they are mapped to the z -plane.

PLEASE TURN OVER

3. (i) Find the residue of

$$F(z) = \frac{1}{(1+z^2)} \quad \text{at } z = i .$$

- (ii) Consider the function

$$\frac{1}{(z^3 - 8)(z - 1)^2} .$$

State the location of all of its poles in the z -plane and state if they are single or multiple poles.

Evaluate the contour integral

$$\oint \frac{1}{(z^3 - 8)(z - 1)^2} dz ,$$

when the contour is a counter-clockwise circle of radius 1 with centre at $z = \frac{3}{2}$.

Sketch the contour and the location of the poles.

Evaluate the integral when the contour is a counter-clockwise circle of radius 1 with centre $z = 100$.

Hint: The residue of a complex function $f(z)$ at a pole $z = a$ of multiplicity m is given by the expression

$$\lim_{z \rightarrow a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \{(z-a)^m f(z)\} .$$

4. (i) Defining the convolution of $f(t)$ with $g(t)$ as $\int_{-\infty}^{\infty} f(t') g(t-t') dt'$, find the convolution of $f(t) = \sum_{n=-\infty}^{\infty} \delta(t-t_n)$ (the Shannon sampling function) with a function $g(t)$.

Show that the Fourier transform of this convolution is $\sum_{n=-\infty}^{\infty} e^{-i\omega t_n} \bar{g}(\omega)$.

- (ii) It is the case that :

$$\int_{-\infty}^{\infty} \frac{e^{ip\omega}}{\omega} d\omega = \begin{cases} +i\pi & p > 0 ; \\ -i\pi & p < 0 . \end{cases}$$

Consider a counter-clockwise contour integral in the z -plane $\oint_C \frac{e^{ipz}}{z} dz$ where

$p > 0$ and where C is a semi-circular contour in the upper half-plane of radius R , with a suitable semi-circular indentation into the upper half-plane of radius r around $z = 0$. Use Jordan's Lemma to show that in the limits $R \rightarrow \infty$ and $r \rightarrow 0$

$$\int_{-\infty}^{\infty} \frac{e^{ip\omega}}{\omega} d\omega = +i\pi \quad \text{where } w, p \text{ are real and } p > 0.$$

Explain briefly why a different contour must be used for $p < 0$.

Use the above integral to show that the inverse Fourier transform of $F(\omega) = \frac{\cos \omega}{\omega}$ is :

$$F(t) = \begin{cases} +i/2 & t > 1 ; \\ 0 & -1 \leq t \leq 1 ; \\ -i/2 & t < -1 . \end{cases}$$

It might be useful to consider the three cases when $t > 1$, $-1 \leq t \leq 1$ and $t < -1$ separately.

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5. (i) Show that the Fourier transform of $f(t+a)$ is $e^{ia\omega} \bar{f}(\omega)$ and that the Fourier transform of $e^{\alpha t} f(t)$ is $\bar{f}(\omega + i\alpha)$ (where α can be real, complex or imaginary).

- (ii) Using Plancherel's integral relation between $f(t)$ and $g(t)$:

$$\int_{-\infty}^{\infty} f(t)g^*(t)dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{f}(\omega)\bar{g}(\omega)d\omega$$

and using Fourier transforms, show that

$$\int_{-\infty}^{\infty} e^{-|t|} \cos(\omega_0 t + \phi) dt = \frac{e^{i\phi} + e^{-i\phi}}{1 + \omega_0^2}, \quad (1)$$

where ω_0 and ϕ are real constants. You may need to use the definition

$$\delta(\Omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\pm i\Omega\tau} d\tau.$$

Find the right hand side of equation (1) when $\phi = -\pi/2$.

Why, when $\phi = -\pi/2$, is this result obvious from inspection of the integrand of the left hand side of equation (1)?

6. Show that the Laplace transform of $\frac{d^3x}{dt^3}$ is

$$s^3 \bar{x}(s) - s^2 x(0) - s \frac{dx(0)}{dt} - \frac{d^2x(0)}{dt^2} ,$$

where you may use the result that the Laplace transform of $\frac{d^2x}{dt^2}$ is

$$s^2 \bar{x}(s) - sx(0) - \frac{dx(0)}{dt} .$$

Recall the notation that $\bar{x}(s)$ is the Laplace transform of $x(t)$ and

$$x(0) = x(t=0), \quad \frac{dx(0)}{dt} = \frac{dx(t=0)}{dt}, \quad \frac{d^2x(0)}{dt^2} = \frac{d^2x(t=0)}{dt^2} .$$

Prove that the Laplace transform of t^2 is $2/s^3$ (for $\text{Re}(s) > 0$).

Find the Laplace transform of $e^{-2t} t^2$.

Consider the third order differential equation

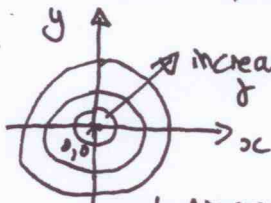

$$\frac{d^3x}{dt^3} + 6 \frac{d^2x}{dt^2} + 12 \frac{dx}{dt} + 8x = f(t) ,$$

with $x = \frac{dx}{dt} = \frac{d^2x}{dt^2} = 0$ at $t = 0$.

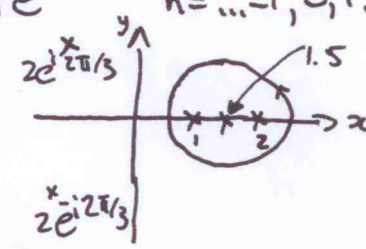

Show that

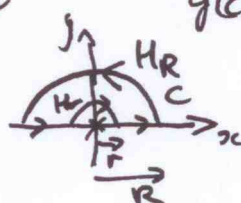
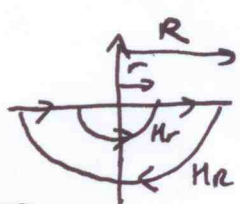
$$x(t) = \frac{1}{2} \int_0^t e^{-2v} v^2 f(t-v) dv .$$

END OF PAPER

	EXAMINATION QUESTIONS/SOLUTIONS 2011-2012	Course <u>EE II</u> (3)
Question 1.	TOPIC Real Valued Functions	Marks & seen/unseen
Parts	<p>a) $-\frac{\partial f}{\partial x} = \underline{3x^2 - 12x + 7 + 2y}$; $\frac{\partial f}{\partial y} = \underline{-2y + 2x}$.</p> <p>- At $(0,0)$ $\nabla f = (7,0)$. ∇f is perpendicular to the constant contours.</p> <p>- ∇f is parallel to x-axis so constant contour is parallel to the y-axis at $(0,0)$.</p> <p>- Stationary points have $\nabla f = 0$ it follows that $x=y$ and so $3x^2 - 10x + 7 = (3x-7)(x-1) = 0$.</p> <p>- Therefore $(7/3, 7/3)$ and $(1,1)$ are stationary pts.</p> <p>- $f_{xx} = 6x - 12$; $f_{xy} = 2$; $f_{yy} = -2$</p> <p>- at $(1,1)$ $f_{xx} = -6 < 0$ and $f_{xy}^2 - f_{xx}f_{yy} = 4 - 12 = -8 < 0$ $\Rightarrow (1,1)$ is a Maximum.</p> <p>- at $(7/3, 7/3)$ $f_{xy}^2 - f_{xx}f_{yy} = 4 - 2 \cdot 2 = 8 > 0$ \Rightarrow saddle point.</p> <p>b)  Good Sketch + statement of minimum</p> <p>- This is hard to classify as all of its second differentials are zero at the origin, so the classification tests for stationarity are unclear. In order to classify this one needs to also consider higher order terms in a Taylor expansion about $(0,0)$.</p>	<p>2</p> <p>2 } 4</p> <p>2 } 4</p> <p>2 } 4</p> <p>2 } 4</p> <p>2</p>
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	EXAMINATION QUESTIONS/SOLUTIONS 2011-2012	Course EE II (3)
Question 2	TOPIC Complex Variables	Marks & seen/unseen
Parts	<p>a) $-u_x = \frac{3x^2}{6} + 8x - \frac{y^2}{2}; u_y = -\frac{2yx}{2} - 8y$ $u_{xx} = x+8; u_{yy} = -x-8$ $u_{xx} + u_{yy} = 0.$</p> <p>$-v_y = u_x = \frac{x^2}{2} - \frac{y^2}{2} + 8x; v_x = -u_y = yx + 8y$ ② <u>$v = \frac{x^2 y}{2} - \frac{y^3}{6} + 8xy + A(x)$ integrating ① w.r.t. y</u> <u>$v = \frac{y x^2}{2} + 8yx + B(y)$ integrating ② w.r.t. x.</u> $\Rightarrow A(x) = C; B(y) = -y^3/6 + C.$ $\Rightarrow v = x^2 y/2 - y^3/6 + 8xy + C.$</p> <p>$-f(z) = \frac{z^3}{6} - \frac{y^3}{6} + 4x^2 - 4y^2 + i(\frac{x^2 y}{2} - \frac{y^3}{6} + 8xy + C)$ + intermediate step</p> <p>$-f(z) = \frac{1}{6} z^3 + 4z^2 + C; a=1/6; b=4.$</p> <p>b) - Since the map is conformal at the intersection of $u=1$ and $v=0$ and conformal maps are angle preserving then since the curves meet at right-angles in the w-plane <u>the same is true</u> in the z-plane. or $-u=1$ and $v=0$ are "curves of constant u and v" and it is true that such curves meet at <u>right angles</u> in z (in regions of analyticity). or $-x^2 - y^2 + 2ixy = u + iv; v=0 \Rightarrow x=0$ and $y=0$ curves in z. $u=1 \Rightarrow x^2 - 1 = y^2$. These curves intersect at $z=(1,0)$. Accept any reasonable argument that they meet at <u>right angles</u>. (including a sketch). (if candidates find a circle/ellipse $\Rightarrow 2/4$)</p>	<p>4</p> <p>7</p> <p>2 } 5 3 } 2 } 4 2 }</p>
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	EXAMINATION QUESTIONS/SOLUTIONS 2011-2012	Course EE II (3)
Question 3	TOPIC Contour Integrals	Marks & seen/unseen
Parts	<p>a) $F(z) = \frac{1}{1+z^2} = \frac{1}{(z-i)(z+i)}$ so $\lim_{z \rightarrow i} \frac{z-i}{(z-i)(z+i)} = \frac{1}{2i}$ (simple pole at $z=i$)</p> <p>or $F(z) = \frac{h(z)}{g(z)} = \frac{1}{1+z^2}$ so $\frac{h(z)}{g'(z)} = \frac{1}{2z} = \frac{1}{2i}$ at $z=i$ simple pole.</p> <p>b) - Pole of multiplicity 2 at $z=1$ - Pole of multiplicity 1 at $z^3=8$ must get all three poles correct to score $z^3 = 8e^{i2\pi n}$ $n = \dots, -1, 0, 1, \dots$ $z = 2, 2e^{\pm i2\pi/3}$</p>  <p>- if the calculated poles, even wrong ones, are displayed then awarded - but need either contour sense or axes labels.</p> <p>- Residue at 1: $\frac{1}{1!} \frac{d}{dz} \left[\frac{(z-1)^2}{(z-1)^2 (z^3-8)} \right] = \frac{-3z^2}{(z^3-8)^2} = -\frac{3}{49}$</p> <p>- Residue at 2: write $(z^3-8) = (z-2)(z^2+2z+4)$ \Rightarrow Residue $\lim_{z \rightarrow 2} \frac{(z-2)}{(z-2)(z^2+2z+4)(z-1)^2} = \frac{1}{(4+4+4) \cdot 1^2} = \frac{1}{12}$</p> <p>- Residue theorem states that the value of the integral is $2\pi i \times$ (sum of residues enclosed) <small>if not stated clearly but used the 1/2</small></p> <p>- $2\pi i \times \left(\frac{1}{12} - \frac{3}{49} \right) = i\pi \frac{13}{294}$ <small>mark correct use 1 mark correct answer</small></p> <p>- No poles are enclosed by this contour. Cauchy's theorem tells us this integral is zero.</p>	<p>2</p> <p>2</p> <p>1</p> <p>3</p> <p>3</p> <p>2</p> <p>2</p> <p>2</p>
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	EXAMINATION QUESTIONS/SOLUTIONS 2011-2012	Course EE II (3)
Question 4	TOPIC Fourier Transforms	Marks & seen/unseen
Parts	<p>a) $\int_{-\infty}^{\infty} f(t') g(t-t') dt' = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(t'-t_n) g(t-t') dt' = \sum_{n=-\infty}^{\infty} g(t-t_n)$</p> <p>$\int_{-\infty}^{\infty} e^{-i\omega t} \sum_{n=-\infty}^{\infty} g(t-t_n) dt = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i\omega(t_n+t_n)} g(t_n) dt_n = \sum_{n=-\infty}^{\infty} e^{-i\omega t_n} \bar{g}(\omega)$</p> <p>b) $p > 0$ consider contour </p> <p>$\oint_C \frac{e^{ipz}}{z} dz = 0$ (Cauchy's theorem)</p> <p>$\lim_{R \rightarrow \infty, r \rightarrow 0} \left[\int_{H_R} \frac{e^{ipz}}{z} dz + \int_{H_r} \frac{e^{ipz}}{z} dz + \int_{-\infty}^{\infty} \frac{e^{ipw}}{w} dw \right] = 0$</p> <p>By Jordan's lemma (since $p > 0$; $\frac{1}{z} \rightarrow 0$ as $R \rightarrow \infty$; no singularities) $\int_{H_R} \rightarrow 0$.</p> <p>On contour \int_{H_r} substitute $z = re^{i\theta}$ as $r \rightarrow 0$ $e^{ipz} = e^{ipre^{i\theta}} \rightarrow [e^0]$ so $\int_{H_r} \frac{e^{ipz}}{z} dz = \int_{\pi}^0 \frac{1}{re^{i\theta}} i r e^{i\theta} d\theta = -i\pi$.</p> <p>From $\textcircled{*}$ $\int_{-\infty}^{\infty} \frac{e^{ipw}}{w} dw = i\pi$ $p > 0$.</p> <p>For $p < 0$ consider contour </p> <p>Some arguments show $\int_{H_R} \rightarrow 0$ and $\int_{-\pi}^0 \frac{1}{re^{i\theta}} i r e^{i\theta} d\theta = \int_{\pi}^0 = i\pi$</p> <p>For $p < 0$ cannot use Jordan's lemma with a contour in the upper half plane. Need $e^{ipz} \rightarrow 0$ so contour in lower half plane.</p>	<p>2</p> <p>2</p> <p>2</p> <p>2</p> <p>2</p> <p>2</p> <p>2</p> <p>2</p>
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	EXAMINATION QUESTIONS/SOLUTIONS 2011-2012	Course EE II (3)
Question 4 cont	TOPIC Fourier Transforms.	Marks & seen/unseen
Parts	<p>b) $F(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} \cos \omega \, d\omega$</p> $= \frac{1}{2\pi} \left[\underbrace{\int_{-\infty}^{\infty} \frac{e^{i(t+1)\omega}}{2\omega} d\omega}_{I_1} + \underbrace{\int_{-\infty}^{\infty} \frac{e^{i(t-1)\omega}}{2\omega} d\omega}_{I_2} \right]$ <p>From above integral. when $t > 1$ I_1 $t+1 > 0$ and $t-1 > 0$ so $I_1 = I_2 = i\frac{\pi}{2} F(t)$ $\Rightarrow F(t) = \frac{1}{2\pi} \times 2i\frac{\pi}{2} = i/2$.</p> <p>When $-1 \leq t \leq 1$ $I_1 = i\frac{\pi}{2}$; $I_2 = -i\pi/2 \Rightarrow F(t) = 0$ When $t < -1$ $I_1 = I_2 = -i\pi/2 \Rightarrow F(t) = -i/2$.</p>	<p>2</p> <p>2 } 3 1 }</p>
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	EXAMINATION QUESTIONS/SOLUTIONS 2011-2012	Course EE II (3)
Question 5	TOPIC Fourier Transforms	Marks & seen/unseen
Parts	<p>a) $\int_{-\infty}^{\infty} e^{-i\omega t} f(t+a) dt = \int_{-\infty}^{\infty} e^{-i\omega(\tau-a)} f(\tau) d\tau = \bar{f}(\omega) e^{i\omega a}$ change of variables $t+a=\tau$.</p> <p>$\int_{-\infty}^{\infty} e^{-i\omega t} e^{i\omega' t} f(t) dt = \int_{-\infty}^{\infty} e^{-i\omega' t} f(t) dt = \bar{f}(\omega') = \bar{f}(\omega+i\omega')$ writing $+i(\omega+i\omega') = +i\omega' = i\omega - \omega'$ $\omega' = \omega + i\omega'$ $-i\omega' = -i\omega + \omega'$</p> <p>b) $\text{FT}[e^{- t }] = \int_{-\infty}^{\infty} e^{- t } e^{-i\omega t} dt = \int_{-\infty}^0 e^{- t } e^{-i\omega t} dt + \int_0^{\infty} e^{- t } e^{-i\omega t} dt$ $= I_1 + I_2$ defining $t' = -t$; $I_2 = -\int_{\infty}^0 e^{-t'} e^{i\omega t'} dt' = \int_0^{\infty} e^{-t'} e^{i\omega t'} dt'$ $= \left[\frac{1-e^{-t'}}{1-i\omega} \right]_0^{\infty} = \frac{1}{1-i\omega}$ $I_1 = \frac{1}{i\omega+1}$; $I_1 + I_2 = \frac{2}{1+\omega^2}$</p> <p>$\text{FT}[\cos(\omega_0 t + \phi)] = \int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{2} e^{i(\omega_0 t + \phi)} dt + \int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{2} e^{-i(\omega_0 t + \phi)} dt$ $= \pi [e^{i\phi} \delta(\omega - \omega_0) + e^{-i\phi} \delta(\omega + \omega_0)]$ (using defn of δ-function).</p> <p>$\int_{-\infty}^{\infty} e^{- t } \cos(\omega_0 t + \phi) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2\pi}{1+\omega^2} [e^{i\phi} \delta(\omega - \omega_0) + e^{-i\phi} \delta(\omega + \omega_0)] d\omega$ $= \frac{e^{i\phi} + e^{-i\phi}}{1+\omega_0^2}$</p> <p>If $\phi = -\pi/2$ RHS is zero. In this case the integrand is an odd function and so must integrate to zero [$e^{- t }$ even and $\sin \omega_0 t$ odd]</p>	<p>3</p> <p>3</p> <p>4</p> <p>4</p> <p>3</p>
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	EXAMINATION QUESTIONS/SOLUTIONS 2011-2012	Course EE II (3)
Question 6	TOPIC Laplace Transforms	Marks & seen/unseen
Parts	<p> $-LT[\frac{d^3 x(t)}{dt^3}] = \int_0^{\infty} e^{-st} \ddot{x} dt = [\ddot{x} e^{-st}]_0^{\infty} - \int_0^{\infty} -s e^{-st} \ddot{x} dt$ $= -\ddot{x}(t=0) + s \times LT[\ddot{x}]$ $= s^3 \bar{x}(s) - s^2 x(0) - s \dot{x}(0) - \ddot{x}(0)$. </p> <p> $-LT[t^2] = \int_0^{\infty} e^{-st} t^2 dt = [-\frac{e^{-st}}{s} t^2]_0^{\infty} + \frac{1}{s} \int_0^{\infty} 2t e^{-st} dt$ $= \frac{2}{s} [\frac{e^{-st}}{s} t]_0^{\infty} + \frac{1}{s} \int_0^{\infty} e^{-st} dt = 2/s^3$ </p> <p> $-LT[e^{2t} t^2]$ well $LT[e^{at} f(t)] = F(s-a)$ so $LT[e^{-2t} f(t)] = \bar{f}(s+2)$ so $LT[e^{-2t} t^2] = 2/(s+2)^3$. </p> <p> $- \frac{2}{s^3} s^3 \bar{x}(s) + 6 s^2 \bar{x}(s) + 12 s \bar{x}(s) + 8 \bar{x}(s) = \bar{f}(s)$ $\bar{x}(s) = \frac{\bar{f}(s)}{(s+2)^3}$. </p> <p> Treat $\bar{x}(s) = \bar{g}(s) \bar{f}(s) \Rightarrow \bar{g}(s) = 1/(s+2)^3$. From the Laplace convolution theorem and noting $g(t) = e^{-2t} t^2 / 2$ $x(t) = \int_0^t e^{-2v} \frac{v^2}{2} f(t-v) dv$. </p> <p> $\int_0^t t^2 e^{-(s+2)t} dt = \frac{2}{(s+2)^3} \sim 2/4$ one more line of argument required some where for full marks. </p>	<p>4</p> <p>5</p> <p>4</p> <p>7</p>
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