EE1-10B MATHEMATICS II

1. a) Given the equations of three planes

$$\underline{r}\cdot(1,-1,2)=2, \qquad \underline{r}\cdot(0,1,-3)=\alpha, \qquad \underline{r}\cdot(2,1,-5)=1$$

show that when $\alpha=1$ the three planes do not intersect, but form the sides of a prism. Find the value of α so that the three planes intersect, and obtain the intersection.

Solution:

Can eliminate variables from equations, but more elegant to use Gaussian elimination. Each plane equation gives one row of the augmented coefficient matrix:

$$(A:\underline{b}) = \begin{pmatrix} 1 & -1 & 2 & 2 \\ 0 & 1 & -3 & \alpha \\ 2 & 1 & -5 & 1 \end{pmatrix} \overset{R_3-2R_1}{\sim} \begin{pmatrix} 1 & -1 & 2 & 2 \\ 0 & 1 & -3 & \alpha \\ 0 & 3 & -9 & -3 \end{pmatrix}$$

$$\overset{R_3-3R_2}{\sim} \begin{pmatrix} 1 & -1 & 2 & 2 \\ 0 & 1 & -3 & \alpha \\ 0 & 0 & 0 & -3-3\alpha \end{pmatrix}$$

Hence if $\alpha=1$, row 3 represents $0x+0y+0z=-6$, and the equations are inconsistent: no intersection exists. None of the planes are parallel, as none of the normal vectors are multiples of each other: the only possibility left is that the planes form the sides of a prism.

For an intersection to exist, require consistent equations: $\alpha=-1$. Row 2 then gives $y-3z=-1$.

Choose the free variable $z=\lambda\in\mathbb{R}\Rightarrow y=-1+3\lambda$ and then Row 1 gives $x=2+y-2z=2+(-1+3\lambda)-2\lambda=1-\lambda$. In vector form:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix},$$

the equation of a line in $\mathbb{R}^3$¿

[*Similar to examples seen in class.*]

[6]

b) Show that for any three vectors $\underline{\mathbf{u}}, \underline{\mathbf{v}}, \underline{\mathbf{w}}$: $[(\underline{\mathbf{u}}+\underline{\mathbf{v}}) \times (\underline{\mathbf{v}}-\underline{\mathbf{w}})] \cdot (\underline{\mathbf{u}}+\underline{\mathbf{w}}) = 0$.

Solution:

Begin with $(\underline{\mathbf{u}}+\underline{\mathbf{v}}) \times (\underline{\mathbf{v}}-\underline{\mathbf{w}}) = \underline{\mathbf{u}} \times \underline{\mathbf{v}} - \underline{\mathbf{u}} \times \underline{\mathbf{w}} - \underline{\mathbf{v}} \times \underline{\mathbf{w}}$, as $\underline{\mathbf{v}} \times \underline{\mathbf{v}} = \underline{0}$ Hence we have

$$(\underline{\mathbf{u}} \times \underline{\mathbf{v}} - \underline{\mathbf{u}} \times \underline{\mathbf{w}} - \underline{\mathbf{v}} \times \underline{\mathbf{w}}) \cdot \underline{\mathbf{u}} + (\underline{\mathbf{u}} \times \underline{\mathbf{v}} - \underline{\mathbf{u}} \times \underline{\mathbf{w}} - \underline{\mathbf{v}} \times \underline{\mathbf{w}}) \cdot \underline{\mathbf{w}}$$

$$= -(\underline{\mathbf{v}} \times \underline{\mathbf{w}}) \cdot \underline{\mathbf{u}} + (\underline{\mathbf{u}} \times \underline{\mathbf{v}}) \cdot \underline{\mathbf{w}},$$

as all the other terms are zero, e.g. $(\underline{\mathbf{v}} \times \underline{\mathbf{w}}) \cdot \underline{\mathbf{v}} = 0$, etc. Rearrange:

$$= -\underline{\mathbf{u}} \cdot (\underline{\mathbf{v}} \times \underline{\mathbf{w}}) + \underline{\mathbf{w}} \cdot (\underline{\mathbf{u}} \times \underline{\mathbf{v}})$$

Finally, the triple scalar product gives that $\underline{\mathbf{u}} \cdot (\underline{\mathbf{v}} \times \underline{\mathbf{w}}) = \underline{\mathbf{w}} \cdot (\underline{\mathbf{u}} \times \underline{\mathbf{v}})$ and the result follows. [*Similar to examples seen in class.*] [5]

c) Consider the matrix

$$A = \begin{pmatrix} 1 & -2 & 2 \\ 0 & -1 & 2 \\ -6 & 5 & -4 \end{pmatrix}.$$

i) Show that $\lambda = 1$ is an eigenvalue of A, and find the other eigenvalues.

Solution:

Set up

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -2 & 2 \\ 0 & -1-\lambda & 2 \\ -6 & 5 & -4-\lambda \end{vmatrix}$$

and expand by first column:

$$= (1-\lambda)[(-1-\lambda)(-4-\lambda) - 10] - 6[-4 - 2(-1-\lambda)]$$

Note $(1-\lambda)$ in first term, and don't multiply out:

$$= (1-\lambda)(\lambda^2 + 5\lambda - 6) + 12(1-\lambda) = (1-\lambda)(\lambda^2 + 5\lambda + 6) = 0$$

Hence $\lambda = 1$ is an eigenvalue and the quadratic term factorizes to give $\lambda = -2, -3$ as the other eigenvalues.

[*Similar to examples seen in class.*] [4]

ii) Find an eigenvector of A, corresponding to $\lambda = 1$.

Find $\underline{\mathbf{x}} = (x, y, z)^T$ such that $(A - 1I)\underline{\mathbf{x}} = \underline{0}$:

$$\begin{pmatrix} 0 & -2 & 2 \\ 0 & -2 & 2 \\ -6 & 5 & -5 \end{pmatrix} \underline{\mathbf{x}} = \underline{0}$$ and no further operations are required.

Row 1 and 2 both give $y = z$. Row 3 gives $-6x + 5y - 5z = 0 \Rightarrow x = 0$ and an eigenvector for $\lambda = 1$ is $(0, 1, 1)^T$.

[*Similar to examples seen in class.*] [3]

d) Let A be an invertible matrix, and λ an eigenvalue of A.

i) Show that $\lambda \neq 0$;

If $\lambda = 0$ were an eigenvalue of A, then $A\underline{\mathbf{x}} = 0\underline{\mathbf{x}}$ for a non-zero eigenvector $\underline{\mathbf{x}}$. Hence $A\underline{\mathbf{x}} = \underline{0}$ would have a nonzero solution. But if A is invertible, the only solution for $A\underline{\mathbf{x}} = \underline{0}$ is $\underline{\mathbf{x}} = \underline{0}$. Hence zero cannot be an eigenvalue, and $\lambda \neq 0$ [*Unseen.*] [3]

ii) Hence, or otherwise, show that $1/\lambda$ is an eigenvalue of A^{-1}.

Given that $\lambda \neq 0$ is an eigenvalue, we have non-zero $\underline{x}$ such that $A\underline{x} = \lambda\underline{x}$. As A^{-1} exists, we can multiply on left by A^{-1}:

$A^{-1}(A\underline{x}) = A^{-1}(\lambda\underline{x}) \Rightarrow (A^{-1}A)\underline{x} = \lambda A^{-1}\underline{x} \Rightarrow \underline{x} = \lambda A^{-1}\underline{x}$.

Now divide by $\lambda \neq 0$ to obtain $\frac{1}{\lambda}\underline{x} = A^{-1}\underline{x}$ and as $\underline{x}$ is non-zero, we see that $1/\lambda$ is an eigenvalue for A^{-1}.

[*Similar to examples seen in class.*] [4]

2. a) i) Evaluate the determinant of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 2 \\ 2 & -1 & 1 \\ \alpha & 1 & -5 \end{pmatrix}$$

and state the value of α for which $\mathbf{A}$ is singular.

Solution:

Expand by first column:

$\det(A) = 1(5-1) - 2(5-2) + \alpha(-1+2) = -2 + \alpha$.

So the matrix is singular, $\det(A) = 0$ when $\alpha = 2$.

[*Similar to examples seen in class.*] [3]

ii) Let $\alpha = 3$. Use Gauss-Jordan elimination (row operations) to find $\mathbf{A}^{-1}$.

Solution:

Set up the augmented matrix, and use row operations, one column at at time, left to right:

$$\begin{aligned}
(A:I) &= \left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 & 1 & 0 \\ 3 & 1 & -5 & 0 & 0 & 1 \end{array}\right) \\
&\underset{\sim}{\substack{R_2-2R_1 \\ R_3-3R_1}} \left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 4 & -11 & -3 & 0 & 1 \end{array}\right) \\
&\underset{\sim}{\substack{R_1+R_2 \\ R_3-4R_2}} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & -1 & 1 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -4 & 1 \end{array}\right) \\
&\underset{\sim}{\substack{R_1+R_3 \\ R_2+3R_3}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & -3 & 1 \\ 0 & 1 & 0 & 13 & -11 & 3 \\ 0 & 0 & 1 & 5 & -4 & 1 \end{array}\right)
\end{aligned}$$

So that $A^{-1} = \begin{pmatrix} 4 & -3 & 1 \\ 13 & -11 & 3 \\ 5 & -4 & 1 \end{pmatrix}$.

[*Similar to examples seen in class.*] [5]

iii) Use the inverse found in (ii) to solve the set of linear equations

$$\begin{array}{rcrcrcr} -x & + & 2y & + & z & = & 3 \\ -x & + & y & + & 2z & = & 0 \\ x & + & 3y & - & 5z & = & -8 \end{array}$$

Solution:

Spot the rearrangement of the equations:

$$\begin{array}{rcrcrcr} -x & + & y & + & 2z & = & 0 \\ -x & + & 2y & + & z & = & 3 \\ x & + & 3y & - & 5z & = & -8 \end{array}$$

and then

$$\begin{array}{rcrcrcr} y & - & x & + & 2z & = & 0 \\ 2y & - & x & + & z & = & 3 \\ 3y & + & x & - & 5z & = & -8 \end{array}$$

(can be done in one step!)

so the system is equivalent to $A\underline{\mathbf{x}} = (0,3,-8)^T$ where $\underline{\mathbf{x}} = (y,x,z)^T$.
[*Unseen.*]

Now use the inverse, $\underline{\mathbf{x}} = A^{-1}(0,3,-8)^T$:

$$\begin{pmatrix} y \\ x \\ z \end{pmatrix} = \begin{pmatrix} 4 & -3 & 1 \\ 13 & -11 & 3 \\ 5 & -4 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \\ -8 \end{pmatrix} = \begin{pmatrix} -17 \\ -57 \\ -20 \end{pmatrix}$$

so the solution is $x = -57$, $y = -17$ and $z = -20$.

[*Similar to examples seen in class.*] [4]

b) Given the function $f(x) = \dfrac{1}{\sqrt{1-x}}$,

i) Obtain the Maclaurin series for $f(x)$ up to the term in x^3 and state the remainder term

Solution:

$$\begin{aligned} f(x) &= (1-x)^{-1/2} &\Rightarrow\quad & f(0) = 1 \\ f'(x) &= -\frac{1}{2}(1-x)^{-3/2}(-1) &\Rightarrow\quad & f'(0) = \frac{1}{2} \\ f''(x) &= \frac{3}{4}(1-x)^{-5/2} &\Rightarrow\quad & f''(0) = \frac{3}{4} \\ f'''(x) &= \frac{15}{8}(1-x)^{-7/2} &\Rightarrow\quad & f'''(0) = \frac{15}{8} \\ f^{(4)}(x) &= \frac{105}{16}(1-x)^{-9/2} & & \end{aligned}$$

So the Maclaurin series is

$$\begin{aligned} (1-x)^{-1/2} &= 1 + \frac{1}{2}x + \frac{3}{4\cdot 2!}x^2 + \frac{15}{8\cdot 3!}x^3 + R_3 \\ &= 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + R_3 \end{aligned}$$

where the remainder term is

$$R_3(x) = \frac{35x^4}{128(1-c)^{9/2}}, \text{ with } 0 < |c| < |x|.$$

[*Similar to examples seen in class.*] [5]

ii) Find the maximum error incurred in using the series up to the term in x^4 to estimate $\frac{1}{\sqrt{0.9}}$. [You can leave the answer in terms of a fraction.]

Solution:

$\frac{1}{\sqrt{0.9}} = f(0.1)$, so the maximum error will be bound by $|R_3(0.1)|$.

Given that $\frac{1}{(1-c)^{9/2}}$ is maximal when the denominator is smallest, and $0 < c < 0.1$, then $c = 0.1$ gives the upper bound:

$$|R_3(0.1)| < \frac{35(0.1)^4}{128(0.9)^{9/2}}.$$

[*Similar to examples seen in class.*] [4]

(For information: The error is ≈ 0.00003 and the bound is ≈ 0.00004.)

c) Given the power series $\sum_{n=1}^{\infty} \frac{x^n}{2^n - 1}$, find all values of x for which the series converges. Use the ratio test:

$$\frac{a_{n+1}}{a_n} = \frac{\frac{x^{n+1}}{2^{n+1}-1}}{\frac{x^n}{2^n-1}} = \frac{2^n - 1}{2^{n+1} - 1} x = \frac{1 - 2^{-n}}{2 - 2^{-n}} x,$$

and, as $2^{-n} \to 0$, in the limit, we have

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{1 - 2^{-n}}{2 - 2^{-n}} |x| = \frac{1}{2}|x|.$$

So by the ratio test, the series converges for $\frac{1}{2}|x| < 1 \Rightarrow -2 < x < 2$ and diverges for $\frac{1}{2}|x| > 1 \Rightarrow x > 2$ or $x < -2$. When $\frac{1}{2}|x| = 1$, the ratio test is inconclusive. When $x = 2$ the series diverges as all terms a_n are positive and $a_n \to 1$. When $x = -2$, the series converges by the alternating series test. Thus the desired interval of convergence is $[-2, 2)$.

[*Similar to examples seen in class.*] [4]

3. a) The first two terms of y form the complementary function y_c. Therefore the roots of the auxiliary polynomial $a(\lambda)$ are 1 and 2 so $a(\lambda)=(\lambda-1)(\lambda-2)=\lambda^2-3\lambda+2$ and the LHS of the equation is $\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y$. The last term of y is the particular integral y_p and since it is not part of y_c the RHS of the differential equation has the form $f(x)=ce^{3x}$. Substituting $y_p(x)=e^{3x}$ into the LHS, we get $(3)^2-3(3)+2=2$ and so $f(x)=2e^{3x}$. Thus the differential equation is

$$\boxed{\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=2e^{3x}}$$

[6]

b) We recognise the equation as separable by writing the equation in the form

$$\frac{dy}{dx}=\frac{x}{y}e^{x-y}=\frac{x}{y}e^{x}e^{-y}=(xe^{x})\,(ye^{y})^{-1}$$

Cross–multiplying, integrating by parts and rearranging the solution is given as

$$ye^{y}dy=xe^{x}dx \quad\Rightarrow\quad \int ye^{y}dy=\int xe^{x}dx \Rightarrow ye^{y}-\int e^{y}dy=xe^{x}-\int e^{x}dx$$

$$\Rightarrow\quad \boxed{(y-1)e^{y}-(x-1)e^{x}+C=0}$$

[6]

c) i) Write the differential equation as $P(x,y)dx+Q(x,y)dy=0$ where $P(x,y)=2xy+e^x$ and $Q(x,y)=x^2+\cos y$. The equation is exact since

$$\boxed{\frac{\partial P}{\partial y}=2x=\frac{\partial Q}{\partial x}}$$

[3]

ii) Since the equation is exact, the solution has the form $f(x,y)=0$ where

$$\frac{\partial f}{\partial x}=P(x,y)=2xy+e^x, \qquad \frac{\partial f}{\partial y}=Q(x,y)=x^2+\cos y$$

Integrating in turn gives

$$f(x,y)=x^2y+e^x+g_1(y), \qquad f(x,y)=x^2y+\sin y+g_2(x)$$

It follows that $g_1(y)=\sin y$ and $g_2(x)=e^x$ so the general solution is

$$\boxed{x^2y+\sin y+e^x+C=0}$$

[3]

d) i) The transformation $y=\frac{dz}{dx}$ gives the linear first order ODE

$$\boxed{\frac{dy}{dx}-\frac{3}{x}y=-3x}$$

[3]

ii) Multiplying by the integrating factor $\mu(x)=e^{\int\frac{-3}{x}dx}=x^{-3}$ gives

$$x^{-3}\frac{dy}{dx}-3x^{-4}y=-3x^{-2}\Rightarrow\frac{d}{dx}\left(x^{-3}y\right)=-3x^{-2}$$

$$\Rightarrow\quad y=x^3\left(\int -3x^{-2}dx+C\right)=x^3\left(3x^{-1}+C\right)=3x^2+Cx^3$$

Finally $z=\int y\,dx=\int 3x^2+Cx^3dx$ and so $\boxed{z(x)=x^3+C_1x^4+C_2}$ [4]

4. a) i) $\boxed{\dfrac{\partial z}{\partial x} = e^{xy} + y(x+y)e^{xy}}$ [2]

ii) Since $dF = 0$ we have that $\dfrac{\partial F}{\partial x}dx + \dfrac{\partial F}{\partial y}dy + \dfrac{\partial F}{\partial z}dz = 0$. It follows that

$$\frac{\partial z}{\partial x} := \frac{dz}{dx}\Big|_{y=\text{constant}} = \frac{dz}{dx}\Big|_{dy=0} = -\frac{\partial F}{\partial x}\Big/\frac{\partial F}{\partial z} \Rightarrow \boxed{\frac{\partial z}{\partial x} = \frac{z\sin x}{e^{z}+\cos x}}$$

[3]

iii) Since the integration is with respect to x, $\boxed{\dfrac{\partial z}{\partial x} = \dfrac{xy+1}{x^2+y^3}}$ [3]

b) i) Evaluating the partial derivative for the chain rule

$$\begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{-y}{x^2+y^2} \\ \frac{y}{\sqrt{x^2+y^2}} & \frac{x}{x^2+y^2} \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial \rho} \\ \frac{\partial f}{\partial \phi} \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi/\rho \\ \sin\phi & \cos\phi/\rho \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial \rho} \\ \frac{\partial f}{\partial \phi} \end{bmatrix}$$

and so $\left(\dfrac{\partial f}{\partial x}\right)^2 + \left(\dfrac{\partial f}{\partial y}\right)^2$

$$= \begin{bmatrix} \frac{\partial f}{\partial \rho} & \frac{\partial f}{\partial \phi} \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi/\rho & \cos\phi/\rho \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi/\rho \\ \sin\phi & \cos\phi/\rho \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial \rho} \\ \frac{\partial f}{\partial \phi} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial f}{\partial \rho} & \frac{\partial f}{\partial \phi} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/\rho^2 \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial \rho} \\ \frac{\partial f}{\partial \phi} \end{bmatrix} = \boxed{\left(\frac{\partial f}{\partial \rho}\right)^2 + \frac{1}{\rho^2}\left(\frac{\partial f}{\partial \phi}\right)^2}$$

[4]

ii) Since $f_\rho = 0$ the equation transforms to

$$\left(\frac{\partial f}{\partial \phi}\right)^2 = \frac{x^2}{(x^2+y^2)} = \cos^2\phi \Rightarrow \frac{\partial f}{\partial \phi} = \pm\cos\phi \Rightarrow f(\phi) = \pm\sin\phi + C$$

and so $\boxed{f(x,y) = \pm\dfrac{y}{\sqrt{x^2+y^2}} + C}$ [4]

c) i) The gradient of f is given by

$$\boxed{\nabla f(x,y) = \begin{bmatrix} f_x \\ f_y \end{bmatrix} = \begin{bmatrix} 6x^2-4x-2y \\ 2(y-x) \end{bmatrix}}$$

The stationary points are obtained as the solutions of the set of equations $\nabla f(x,y) = 0$ and so $\boxed{(x_1,y_1) = (0,0),\ (x_2,y_2) = (1,1)}$ [4]

ii) The Hessian is given by $M(x,y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 12x-4 & -2 \\ -2 & 2 \end{bmatrix}$. It follows that $M(0,0) = \begin{bmatrix} -4 & -2 \\ -2 & 2 \end{bmatrix}$ and $M(1,1) = \begin{bmatrix} 8 & -2 \\ -2 & 2 \end{bmatrix}$. A calculation shows that the eigenvalues of $M(0,0)$ have opposite signs so $(0,0)$ is a saddle point and that the eigenvalues of $M(1,1)$ are both positive and so $(1,1)$ is a local minimiser. [5]