

Paper Number(s): **E3.09**
ISE3.9

IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE
UNIVERSITY OF LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2001

EEE/ISE PART III/IV: M.Eng., B.Eng. and ACGI

CONTROL ENGINEERING

Tuesday, 8 May 10:00 am

There are SIX questions on this paper.

Answer FOUR questions.

Time allowed: 3:00 hours

Examiners: Vinter,R.B. and Astolfi,A.

Corrected Copy

Special instructions for invigilators:

None

Information for candidates:

None

1. State Nyquist's Theorem relating the Nyquist diagram of the forward path transfer function of a unity feedback control system to the number of 'unstable' open loop and closed loop poles of the system.

Consider the closed loop control system of *Figure 1(a)*, in which

$$G(s) = \frac{1}{(s-1)^2}.$$

Here, k is an adjustable parameter (the 'velocity feedback gain'). Investigate the effects on system stability of increasing k , using Nyquist's Theorem.

You should use the following method.

- (i) Show that the closed loop poles of the system of *Figure 1(a)* coincide with the closed loop poles of the unity feedback system of *Figure 1(b)*, in which

$$\tilde{G} = \frac{sG(s)}{1+G(s)}.$$

- (ii) Sketch the Nyquist Diagram of $\tilde{G}(s)$. (You are required to calculate the intercepts with the real axis.)
- (iii) By interpreting the Nyquist diagram of $\tilde{G}(s)$, describe how the closed loop stability properties of the system of *Figure 1(a)* are affected, as k increases over the range $0 \leq k < \infty$.

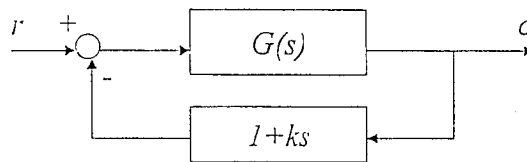


Figure 1(a)

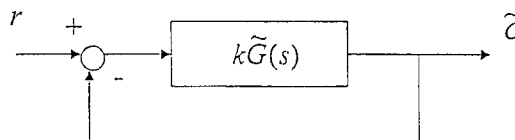


Figure 1(b)

2. Figure 2 illustrates a cart of mass M , attached to a rigid support by a spring (spring constant K). The cart carries a mechanical accelerometer, comprising a mass m , spring (spring constant k) and a damper (damper constant K_d). The absolute displacement of the cart is z . The displacement of the accelerometer relative to the cart is y .

Show that z and y are governed by the equations

$$\begin{aligned} d^2z/dt^2 &= -\left(\frac{K}{M}\right)z + \left(\frac{k}{M}\right)y + \frac{K_d}{M}dy/dt \\ d^2y/dt^2 &= -k\left(\frac{1}{m} + \frac{1}{M}\right)y - K_d\left(\frac{1}{m} + \frac{1}{M}\right)dy/dt + \left(\frac{K}{M}\right)z. \end{aligned}$$

Derive (control free) state space equations,

$$dx/dt = Ax \quad \text{and} \quad y = c^T x, \quad (1)$$

for the system, with state vector $x = (z, dz/dt, y, dy/dt)$ and scalar output y . Show that (1) is *not* observable, if $K = 0$. Why?

(10%)

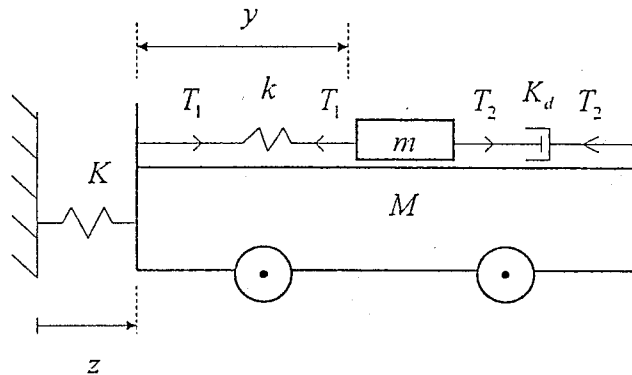


Figure 2

3. Sketch the amplitude and phase frequency response of a phase lag compensator $D_{lag}(s)$ and of a phase advance compensator $D_{adv}(s)$:

$$D_{lag}(s) = \frac{1 + s/\omega_1}{1 + s/\omega_0}, \quad \omega_0 < \omega_1, \quad \text{and} \quad D_{adv}(s) = \frac{1 + s/\omega_2}{1 + s/\omega_3}, \quad \omega_2 < \omega_3.$$

Explain why there is a practical design limitation on the size of ω_3/ω_2 .

Consider the control system of *Figure 3*, in which

$$G(s) = \frac{2}{s(s+1)^2}.$$

Design a lag lead compensator

$$D(s) = D_{lag}(s)D_{adv}(s)$$

(with $D_{lag}(s), D_{adv}(s)$ as above), to achieve the following specifications for the compensated system.

- (a) $\omega_c = 0.9 \text{ rads}^{-1}$,
- (b) $\phi = 60^\circ$,
- (c) $\omega_3/\omega_2 \leq 10$

10 ²⁷

where ω_c is the gain crossover frequency, i.e. the frequency ω_c such that

$$|D(j\omega_c)G(j\omega_c)| = 1,$$

and ϕ is the phase margin.

To carry out your design, you should use the following steps.

Step 1. Design phase advance compensation $D_{adv}(s)$ such that $\omega_c = \omega_{max}$ and $\angle D_{adv}(j\omega_c)G(j\omega_c)$ gives a phase margin of 60° . Check (c).

Step 2. Choose the phase lag compensation $D_{lag}(s)$ such that

$$|D_{lag}(j\omega_c)D_{adv}(j\omega_c)G(j\omega_c)| = 1.$$

You can quote the facts that the maximum phase advance of $D_{adv}(j\omega)$ is $90^\circ - 2 \times \tan^{-1} \sqrt{\omega_2/\omega_3}$, and occurs at $\omega = \sqrt{\omega_2\omega_3}$.

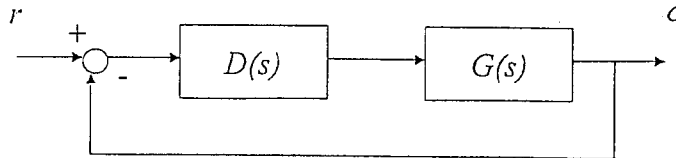


Figure 3

4(a). Consider the unity feedback system of *Figure 4*, in which

$$G(s) = \frac{k}{s+a}.$$

Here, $a > 0$ is a modeling constant and $k > 0$ is a variable gain.

Determine the open loop gain cross-over frequency ω_c of $G(j\omega)$, i.e. the frequency ω_c such that $|G(j\omega_c)| = 1$.

Determine also the rise time t_r of the closed loop system, defined by

$$y(t_r) = 0.99 \times y(t = \infty),$$

where $y(t)$ is the unit step response of the closed loop system, initially at rest.

Show that, for all k , ω_c and t_r are related according to

$$\omega_c^2 = \frac{\log_e(100)}{t_r} \cdot \left[\frac{\log_e(100)}{t_r} - 2a \right].$$

Deduce that, for t_r small,

$$\omega_c t_r = \text{constant}.$$

(‘gain cross-over frequency is inversely proportional to rise time’) What is the value of the constant?

4(b). Consider again the unity feedback system of *Figure 4*, for general $G(s)$. ~~Assume $k=1$.~~

(9⁵⁰)

Fix a number $N \geq 0$. Suppose $\bar{\omega}$ is a frequency for which the closed loop phase frequency response satisfies

$$\angle \frac{G(j\bar{\omega})}{1 + G(j\bar{\omega})} = \tan^{-1}(N).$$

Show that $G(j\bar{\omega})$ lies on an ‘ N ’ circle in the complex plane, namely the set of points with coordinates (X, Y) which satisfy the equation

$$\left(X + \frac{1}{2}\right)^2 + \left(Y - \left(\frac{1}{2N}\right)\right)^2 = \frac{1}{4} + \left(\frac{1}{2N}\right)^2.$$

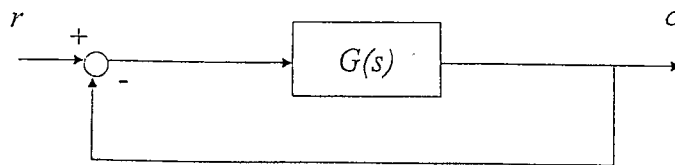


Figure 4

- 5(a). Consider the dynamic system of *Figure 5(a)*, relating the input u to the output y , in which the transfer function is

$$G(s) = \frac{1}{(s+3)(s-1)s^2}$$

Derive a state space model with states $x_1 = y$, $x_2 = dy/dt$, $x_3 = d^2y/dt^2$, $x_4 = d^3y/dt^3$.

Choose the parameters k_1 , k_2 and k_3 in the proportional + velocity + acceleration controller

$$u = -k_1x_1 - k_2x_2 - k_3x_3$$

to arrange that the closed loop characteristic polynomial is of the form

$$(s + \alpha(1 + j))^2(s + \alpha(1 - j))^2$$

for some $\alpha \geq 0$, i.e. all closed loop eigenvalues have damping factor $1/\sqrt{2}$ and are equidistant from the origin.

- 5(b). Consider now the control system of *Figure 5(b)* to stabilize the orientation θ of a rocket in the plane. The control system provides proportional + velocity + acceleration control, except that the accelerometer hardware includes a first order lag.

Use the results of part (a) to choose k , k_v and k_a , such that all the closed loop eigenvalues have damping factor $1/\sqrt{2}$ and are equidistant from the origin.

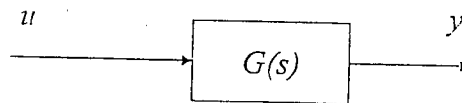


Figure 5(a)

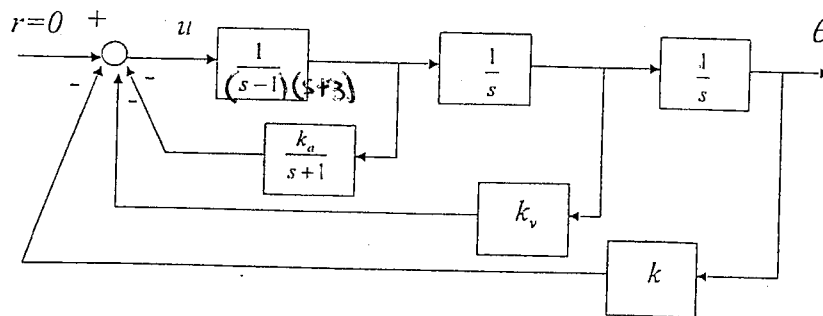


Figure 5(b)

(1130)

6. Derive the describing function $N(A)$ of the amplifier with gain K and off-set at the origin a , whose characteristic is shown in Figure 6(a).

Hint: decompose the nonlinearity ~~as~~ the sum of a pure gain and an ideal relay.

(9.50)

Such a device is present in the forward path of the control system of Figure 6(b), in which

$$G(s) = \frac{48}{(s+2)^3}.$$

Estimate the frequency of limit cycle oscillations, predicted by describing function analysis.

It is known that $K = 1$. It is observed that the amplitude of limit cycle oscillations of the output $y(t)$ is 0.01 units. Determine the magnitude of the amplifier offset a .

Assess whether the limit cycle is stable.

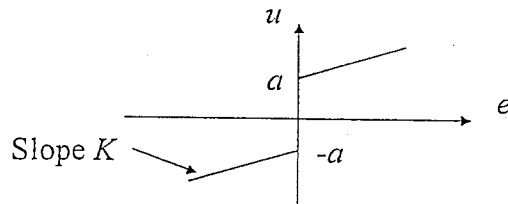


Figure 6(a)

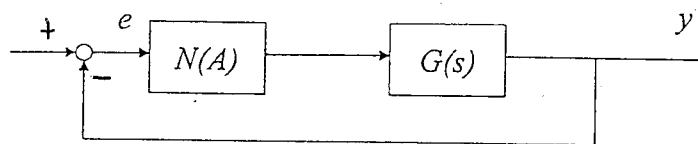


Figure 6(b)

1. 2 $N = C - O$ where $N = \#$ clockwise encirclements, $C = \#$ poles, $O = \#$ zeros.
The closed loop poles of $1(s)$ are the zeros of $1 + (1+ks)G(s)$.
These coincide with the zeros of $1+G(s) + ksG(s)$ and therefore

$$\frac{1 + ksG(s)}{1+G(s)} = 1 + k\tilde{G}(s).$$

But the zeros of $1 + k\tilde{G}(s)$ are the closed loop poles of:

4 $\frac{k\tilde{G}(s)}{1 + k\tilde{G}(s)}$ ($\tilde{G} = sG/(1+G)$)

$$\tilde{G}(s) = \frac{s/(s-1)^2}{1 + 1/(s-1)^2} = \frac{s}{s^2 - 2s + 2}.$$

$$\tilde{G}(j\omega) = \frac{j\omega}{-\omega^2 - 2j\omega + 2} \text{ is real when } \omega = \pm\sqrt{2}.$$

Then $\tilde{G}(j(\omega=\sqrt{2})) = -1/2$

For $j\omega$ on path segment (a), we have

$$\angle \tilde{G}(j\omega) \quad |\tilde{G}(j\omega)|$$

$\omega=0 \quad +90^\circ - (360^\circ)$

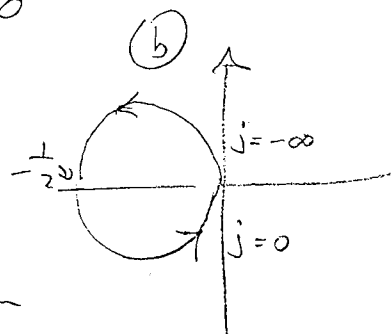
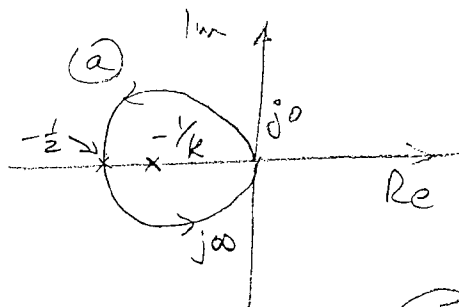
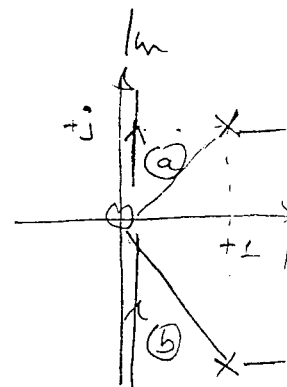
0

$\omega=\sqrt{2} \quad +90^\circ - 270^\circ$

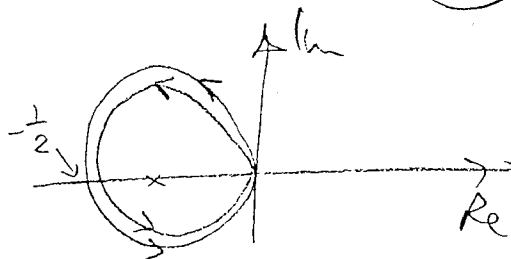
$-1/2$

$\omega=+\infty \quad +90^\circ - 180^\circ$

0



Overall:



goes round twice

For $k > 2$

(encirclements) $N = -2$, # open loop poles $O = 2$

So # closed loop poles = $C = N + O = 0$

Stable for $k > 2$

For $k < 2$

$$N = 0, O = 2, C = O = 2$$

4 Unstable (2 unstable poles) for $k < 2$.

2. Motion of cart: $M\ddot{z} = -Kz + T_1 - T_2$

Motion of load: $m(\ddot{z} + \ddot{y}) = T_2 - T_1$

Spring and damper: $T_1 = Ky$ and $T_2 = -K_d \dot{y}$.

Hence

$$\begin{cases} \ddot{z} = -\frac{K}{M}z + \frac{K}{M}y + \frac{K_d}{M}\dot{y} \\ m\ddot{y} = -m\left[-\frac{K}{M}z + \frac{K}{M}y + \frac{K_d}{M}\dot{y}\right] - K_d\dot{y} - Ky \end{cases}$$

\Rightarrow

12 $\ddot{z} = -\left(\frac{K}{M}\right)z + \left(\frac{K}{M}\right)y + \left(\frac{K_d}{M}\right)\dot{y}$ and $\ddot{y} = -k\left(\frac{1}{M} + \frac{1}{m}\right)y - K_d\left(\frac{1}{M} + \frac{1}{m}\right)\dot{y} + \left(\frac{K}{m}\right)z$

Set $x_1 = z, x_2 = \dot{z}, x_3 = y, x_4 = \dot{y}$. Then

$$\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 \\ -\left(\frac{K}{M}\right) & 0 & \left(\frac{K}{M}\right) & \left(\frac{K_d}{M}\right) \\ 0 & 0 & 0 & 1 \\ +\frac{K}{M} & 0 & -k\left(\frac{1}{M} + \frac{1}{m}\right) & -K_d\left(\frac{1}{M} + \frac{1}{m}\right) \end{pmatrix}}_A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

and

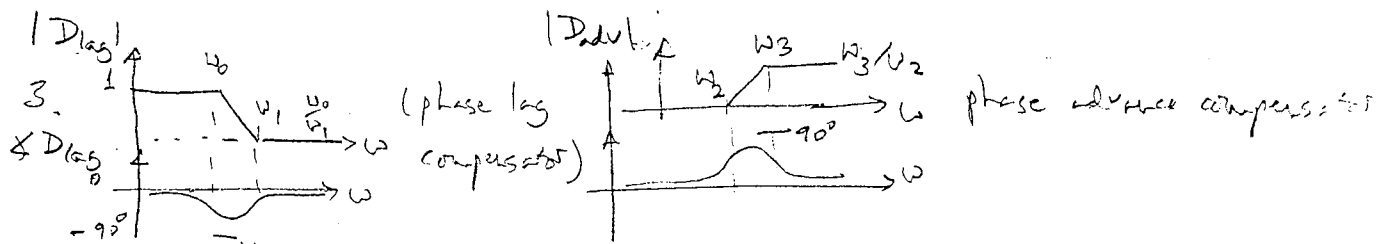
4 $y = \underbrace{[0 \quad 0 \quad 0 \quad 1]}_{C^T} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$

The observability matrix is

$$\begin{bmatrix} C^T \rightarrow \\ C^T A \rightarrow \\ C^T A^2 \rightarrow \\ C^T A^3 \rightarrow \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ +\left(\frac{K}{M}\right) & 0 & -k\left(\frac{1}{M} + \frac{1}{m}\right) & -K_d\left(\frac{1}{M} + \frac{1}{m}\right) \\ -\frac{K_d K}{M}\left(\frac{1}{M} + \frac{1}{m}\right) & \frac{K}{M} & +kK_d\left(\frac{1}{M} + \frac{1}{m}\right)^2 & \left(K_d^2\left(\frac{1}{M} + \frac{1}{m}\right)^2 - k\left(\frac{1}{M} + \frac{1}{m}\right)\right) \\ -\frac{kK}{M}\left(\frac{1}{M} + \frac{1}{m}\right) & \sim & \sim & \sim \\ +\frac{K K_d^2}{M}\left(\frac{1}{M} + \frac{1}{m}\right) & \sim & \sim & \sim \end{bmatrix}$$

Notice that the first column is identically zero if $K=0$. In this case $\begin{bmatrix} C^T \\ C^T A \\ C^T A^2 \\ C^T A^3 \end{bmatrix}$ is singular and the system is not observable.

4 $K=0$ corresponds to "no spring attaching cart to support". In this case, replacing $z(t)$ by " $z(t) + \text{constant}$ " has no effect on the future transient behaviour of $y(t)$ (provided $\dot{z}(t)$ remains the same). So clearly we cannot determine $z(0)$ from the data record $y(t), 0 \leq t \leq T$, i.e. system is not observable.



5 Large $\frac{\omega_3}{\omega_2}$ gives very large control signals when there is a step change in the reference signal - this is often not acceptable.

$$G(s) = \frac{2}{s(s+1)^2}. \text{ We require } \angle G(j\omega_c) D_{adv}(j\omega_c) = -90^\circ - 2 \tan^{-1} \omega_c + \angle D_{adv}(j\omega_c) = -180^\circ + 60^\circ$$

$$\text{Hence } \angle D_{adv}(j\omega_c) = 2 \tan^{-1} 0.9 - 30^\circ = 53.974^\circ \text{ (when } \omega_c = 0.9 \text{)}.$$

$$\text{Since } \omega_c = \omega_{max}, \angle D_{adv}(j\omega_c) = 90^\circ - 2 \tan^{-1} (\omega_2/\omega_3). \text{ It follows that } \omega_2/\omega_3 = 0.3251664. \text{ Also } \omega_2 \omega_3 = 0.9^2.$$

$$\Rightarrow \omega_2^2 = 0.3251664 \times 0.81. \text{ Hence } \omega_2 = 0.51321. \text{ Also } \omega_3 = 1.5783$$

$$\text{Also, } |G(j\omega_c) D_{adv}(j\omega_c)| = \frac{2}{0.9 \times (1+0.81)} \cdot \frac{(1 + (\omega_3/\omega_2)^2)^{1/2}}{(1 + (\omega_2/\omega_3)^2)^{1/2}} = \frac{2}{0.9 \times 1.81} \cdot \frac{(1 + 3.07534 \times 10^5)}{(1 + 0.125166^2)} = 3.7249715$$

$G(s) D_{adv}(s)$ has the correct phase at $s = j\omega_c$. But its gain is 3.7249715. We reduce this gain to unity by means of phase lag compensation.

$$\text{We require } \omega_0 < \omega_1 < \omega_c \text{ (to ensure } D_{lag}(j\omega_c) \approx 0^\circ \text{) and } \omega_1/\omega_0 = 3.7249715.$$

Choose

$$5 \quad \omega_1 = \frac{0.9}{20} \text{ (rad/s)} = 4.5 \times 10^{-2} \text{ and } \omega_0 = \frac{\omega_1}{3.7249715} = 1.208$$

Final compensator design

$$D(s) = \underbrace{\frac{1 + s/(4.5 \times 10^{-2})}{1 + s/(1.208 \times 10^{-2})}}_{D_{lag}} \times \underbrace{\frac{1 + s/0.51321}{1 + s/1.5783}}_{D_{adv}}$$

4(a) The gain cross-over frequency is given by

$$\frac{k^2}{|j\omega_c + a|^2} = 1$$

i.e. $\omega_c = \sqrt{k^2 - a^2}$ — (A)

The unit step closed loop step response is

$$L.T.^{-1} \left\{ \frac{k}{s+a+k} + \frac{1}{s} \right\} = \frac{k}{a+k} [1 - e^{-(a+k)t}]$$

Hence t_r is given by

$$1 - e^{-(a+k)t_r} = 0.99 \Rightarrow e^{-(a+k)t_r} = 0.01$$

$$\Rightarrow t_r = \log_e(100) / (a+k) \quad \text{--- (B)}$$

Eliminate k from (A) and (B)

$$\omega_c^2 = k^2 - a^2, \quad k = \log_e(100) / t_r - a$$

$$\Rightarrow \omega_c^2 = \left(\frac{\log_e(100)}{t_r} - a \right)^2 - a^2$$

For t_r small, $\log_e(100) / t_r \gg a$ and $\frac{\log_e(100)}{t_r} - 2a \approx \frac{\log_e(100)}{t_r}$

Hence $\omega_c^2 \approx \left(\frac{\log_e(100)}{t_r} \right)^2$, i.e. $\omega_c t_r = \text{const.}$, with $\text{const.} = \log_e(10)$

8

(b) Write $G(j\omega) = X + jY$. We require

$$\tan(X + jY) / (1 + X + jY) = \tan \phi$$

i.e. $\tan \left[\tan^{-1} \left(\frac{Y}{X} \right) - \tan^{-1} \left(\frac{Y}{1+X} \right) \right] = \tan \phi$

Since $\tan(A-B) = (\tan A - \tan B) / (1 + \tan A \tan B)$, we have

$$\frac{\frac{Y}{X} - \frac{Y}{1+X}}{1 + \frac{Y^2}{X(1+X)}} = \frac{Y + XY - XY}{X^2 + Y^2 + X} = \tan \phi = N$$

i.e.

$$X^2 + Y^2 + X - \frac{1}{N} Y = 0$$

This equation can be expressed

$$\left(X + \frac{1}{2} \right)^2 + \left(Y - \frac{1}{2N} \right)^2 = \frac{1}{4} + \left(\frac{1}{2N} \right)^2$$

12 (Circle, centre $(-\frac{1}{2}, +\frac{1}{2N})$, radius $\frac{1}{2} \sqrt{1 + \frac{1}{N^2}}$)

can be expressed $\frac{1}{\sin \phi}$

5(a) $G(s) = (s^4 + 2s^3 - 3s^2)^{-1}$. If $x_1 = y, x_2 = \dot{y}, \dots, x_4 = y^{(3)}$ then $\dot{x}_1 = x_2, \dot{x}_2 = x_3, \dot{x}_3 = x_4, \dot{x}_4 = -2x_3 + 3x_2 + u$. In state space form:

$$\frac{d}{dt} \underline{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -3 & 2 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u \quad \equiv \quad \dot{\underline{x}} = A \underline{x} + b u$$

Closed loop dynamics:

$$\dot{\underline{x}} = (A - b[k_1, k_2, k_3, 0]) \underline{x}$$

We require $\det \left[sI - \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k_1 & -k_2 & 3-k_3 & -2 \end{pmatrix} \right] = \frac{(s+\alpha(1+j))^2}{(s+\alpha(1-j))^2}$

for some $\alpha \geq 1$, i.e.

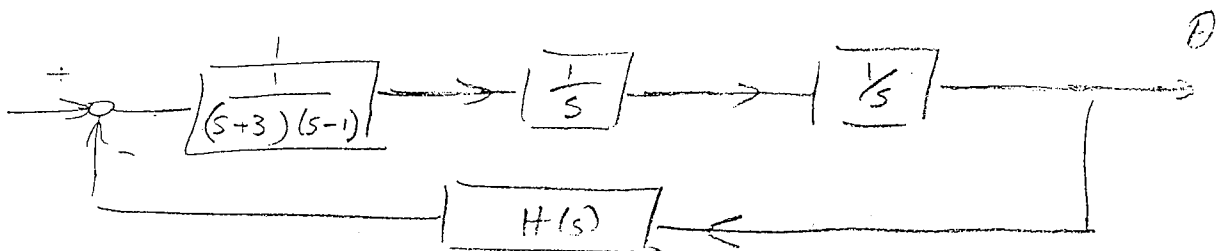
$$s^4 + 2s^3 + (k_3 - 3)s^2 + k_2s + k_1 =$$

$$s^4 + 4\alpha s^3 + 8\alpha^2 s^2 + 8\alpha^3 s + 4\alpha^4.$$

Matching gives

$$\alpha = \frac{1}{2}, \quad k_1 = \frac{1}{4}, \quad k_2 = 1, \quad k_3 = 5$$

(b) The block diagram can be re-arranged as



in which $H(s) = k(s+3) + k_v(s+3)s + k_a s^2$

Reorganising terms gives

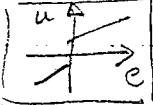
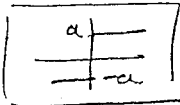
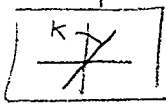
$$H(s) = 3k + 3(k+k_v)s + (k_v+k_a)s^2$$

To locate Φ poles as required, set

$$3k = \frac{1}{4}, \quad k+k_v = \frac{1}{3} \quad \text{and} \quad k_v+k_a = 5$$

This gives

$$k = \frac{1}{12}, \quad k_v = \frac{1}{4}, \quad k_a = 4\frac{3}{4}$$

6. . This characteristic can be expressed as  + 

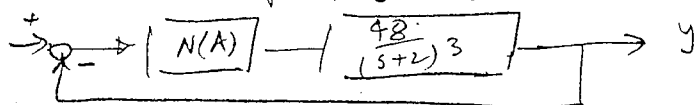
(a pure relay, with output amplitude a and a pure gain of K)

If $N_1(A)$ is the describing function of the relay

$$N_1(A) \cdot A = \frac{2}{T} \left(\int_0^{T/2} a \sin \omega t dt - \int_{T/2}^T a \sin \omega t dt \right) \quad (T = \frac{2\pi}{\omega})$$

$$= -\frac{4\omega}{2\pi} \times \frac{1}{\omega} a \cos \omega t \Big|_0^{T/2} = \frac{4a}{\pi} \quad \text{So } N_1(A) = \frac{4a}{\pi A}$$

6 / Adding the effect of the pure gain gives $N(A) = \frac{4a}{\pi A} + K$



If $\bar{\omega}$ and A are the frequency and amplitude of limit cycle oscillations respectively, limit cycle analysis predicts

$$N(A) \times \frac{48}{(j\bar{\omega}+2)^3} = -1 + j0.$$

Since $N(A)$ is real,

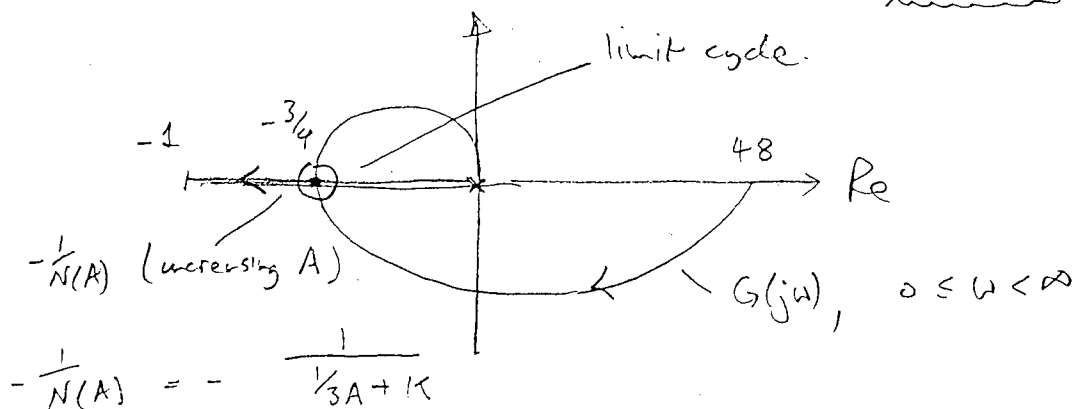
4 we deduce $-j\bar{\omega}^3 - 6\bar{\omega}^2 + 12j\bar{\omega} + 8 = 0$. Hence $\bar{\omega} = \sqrt{12}$. Also

$$-\left(\frac{4a}{\pi A} + K \right) \times \frac{48}{64} = -1$$

$$\text{i.e. } \frac{4a}{\pi A} = \frac{4}{3} - K = \frac{1}{3}$$

It follows

$$4 \quad a = \frac{\pi A}{1.2} = \pi / 1200 = \underline{2.62 \times 10^{-3}}$$



4 We see that, as A increases, locus of $-\frac{1}{N(A)}$ passes from "unstable" to stable region. It follows that the limit cycle is stable.