Imperial College London

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2015

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Algebra & Analysis

Date: Tuesday, 19 May 2014. Time: 10.00am – 12.00noon. Time allowed: 2 hours.

This paper has FOUR questions.

Candidates should use TWO main answer books (A & B) for their solutions as follows: book A - solution to question 1; book B - solutions to questions 2, 3 & 4.

Supplementary books may only be used after the relevant main book(s) are full.

Statistical tables will not be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers to each question as per the table below.

Raw mark	up to 12	13	14	15	16	17	18	19	20
Extra credit	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4

- Each question carries equal weight.
- Calculators may not be used.

1. (i). Let G be a finite group with group operation $*$. Say what it means for a subset H of G to be a subgroup.

(ii). Let H be a subgroup of G and $g \in G$. Define the left coset gH.

(iii). Prove that if $x, y \in G$ and $xH \cap yH \neq \emptyset$ then $xH = yH$.

(iv). Use this to prove Lagrange's theorem: that if n is the number of distinct cosets of H in G then $|G| = n|H|$.

(v). You are told that

$$A_4 = \{e, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3), (1,2,3), (1,3,2), (1,2,4), (1,4,2),$$
$$(1,3,4), (1,4,3), (2,3,4), (2,4,3)\}$$

is a subgroup of the symmetric group S_4. For each $1 \leq r \leq 12$ either write down a subgroup of A_4 with order r or prove that none exists.

2. (i). Let V be a vector space. Define what it means for elements $\mathbf{v}_1, \ldots, \mathbf{v}_n$ of V to be

(a) linearly independent

(b) a spanning set

(c) a basis of V.

(ii). Prove that if the elements $\mathbf{v}_1, \ldots, \mathbf{v}_n$ of V are linearly independent and $\mathbf{u}$ is an element of V not in the span of $\mathbf{v}_1, \ldots, \mathbf{v}_n$ then $\mathbf{u}, \mathbf{v}_1, \ldots, \mathbf{v}_n$ is linearly independent.

(iii). Let $\mathbb{R}[x]_{\leq 2}$ denote the vector space of all polynomials of degree at most two in the variable x, which has a basis $\mathcal{B} = \{1, x, x^2\}$. Find a basis of $\mathbb{R}[x]_{\leq 2}$ containing the set $\{x^2+2, x^2+1\}$, justifying your answer.

(iv). (a) Show that the map $T : \mathbb{R}[x]_{\leq 2} \to \mathbb{R}[x]_{\leq 2}$ defined by $T(f(x)) = f(x+1)$ is a linear map.

(b) Find the matrix of T with respect to initial basis $\mathcal{B}$ and final basis $\mathcal{B}$.

(c) Is T invertible?

3. (i). Prove that if r is a real number with $|r| < 1$ then $\sum_{n=0}^{\infty} r^n$ converges. You may use any results from the algebra of limits for sequences *so long as you state them clearly.*

(ii). State the comparison test for summability of series.

(iii). Prove that $\sum_{n=1}^{\infty}(e/n)^n$ is convergent.

(iv). Define the radius of convergence of a power series $\sum_{n=1}^{\infty} a_n z^n$.

(v). Find the radius of convergence for the following power series, justifying your answer.

(a) $\sum_{n=1}^{\infty}(n^2+n)z^n$

(b) $\sum_{n=1}^{\infty} n! z^n$.

You may use any series convergence tests you like *so long as they are stated clearly.*

(vi). The series $\sum_{n=1}^{\infty} z^n/n$ converges if $z = -1$ and diverges if $z = 1$. What does this tell you about its radius of convergence?

4. (i). Give the $\epsilon - \delta$ definition of continuity of a function $f : \mathbb{R} \to \mathbb{R}$ at a point $a \in \mathbb{R}$.

(ii). Prove that if f is continuous at $a \in \mathbb{R}$ and $\lim\limits_{n\to\infty} a_n = a$ then $\lim\limits_{n\to\infty} f(a_n) = f(a)$.

(iii). By considering the sequence $a_n = \frac{1}{(2n+1/2)\pi}$ or otherwise, prove that the function

$$f(x) = \begin{cases} \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is not continuous at 0.

(iv). You are told that the sequence $a_n = \log(n)/n$ tends to a limit L. By considering the subsequence (a_{2n}) or otherwise, show that $L = 0$. You may use any results from the algebra of limits *if they are stated clearly.*

(v). Deduce that $\lim_{n\to\infty} n^{1/n} = 1$.