

EEE/EIE PART III/IV: MEng, Beng and ACGI

Time allowed: 3:00 hours

**Answer ALL questions.**

**Any special instructions for invigilators and information for candidates are on page 1.**

Examiners responsible      First Marker(s) :      A. Astolfi  
Second Marker(s) :      D. Angeli



## CONTROL ENGINEERING

1. The mathematical model of a simple mobile robot is described by the equations

$$\dot{x} = \cos \theta \, v, \quad \dot{y} = \sin \theta \, v, \quad \dot{\theta} = \omega,$$

in which  $x$  and  $y$  denote the position of the robot with respect to a fixed reference frame,  $\theta$  denotes its orientation with respect to the  $x$ -axis of the reference frame,  $v$  denotes its forward velocity, and  $\omega$  denotes its angular velocity.

The robot is therefore a system with state  $(x, y, \theta)$  and input  $(v, \omega)$ .

- a) Assume  $v(t) = v_0$ , with  $v_0$  constant, and  $\omega(t) = 0$ . Compute the solution of the differential equations describing the robot with initial condition  $x(0) = 0$ ,  $y(0) = 0$ , and  $\theta(0) = 0$ . Argue that the solution describes a motion of the robot along a rectilinear path. [ 6 marks ]
- b) Compute the linearization of the equations of the mobile robot along the motion determined in part a). (Hint: the linearized system is time-invariant!) [ 4 marks ]
- c) Compute the reachability matrix of the linearized system in part b). [ 2 marks ]
- d) Show that the reachability matrix determined in part c) has rank equal to three for all  $v_0 \neq 0$ . Hence conclude that the linearized system is controllable. [ 4 marks ]
- e) Show that the reachability matrix determined in part c) has rank equal to two for  $v_0 = 0$ . Hence conclude that the linearized system is not controllable and compute the unreachable mode. [ 4 marks ]

2. Consider the so-called Collatz iteration, which can be described by the discrete-time system

$$x_{k+1} = \begin{cases} 3x_k + 1 & \text{if } x_k \text{ is odd,} \\ \frac{x_k}{2} & \text{if } x_k \text{ is even,} \end{cases}$$

with state  $x$  which is assumed to be an integer.

- a) Show that if  $x_0$  is an integer then  $x_k$  is an integer for all  $k \geq 0$ . [ 2 marks ]
- b) Show that the system does not have any equilibrium (not even if  $x$  is a real number). [ 4 marks ]
- c) Show that selecting  $x_0 = 1$  yields a period sequence  $x_k$ . [ 2 marks ]
- d) What happens if  $x_0 = 3$  or  $x_0 = 7$ ?  
(Hint: compute no more than 10 elements of the sequence  $x_k$ .) [ 2 marks ]
- e) Consider the modified Collatz systems, with  $x_k \in \mathbb{R}$ , (note that 0 is even)

$$C_1 : x_{k+1} = \begin{cases} 3x_k + 1 & \text{if } k \text{ is odd,} \\ \frac{x_k}{2} & \text{if } k \text{ is even,} \end{cases} \quad C_2 : x_{k+1} = \begin{cases} 3x_k + 1 & \text{if } k \text{ is even,} \\ \frac{x_k}{2} & \text{if } k \text{ is odd.} \end{cases}$$

- i) Show that the system  $C_1$  can be described by the equation

$$x_{k+2} = \frac{3}{2}x_k + 1.$$

Determine the equilibria of the system and study their stability properties. [ 8 marks ]

- ii) Show that the system  $C_2$  can be described by the equation

$$x_{k+2} = \frac{3}{2}x_k + \frac{1}{2}.$$

Determine the equilibria of the system and study their stability properties. [ 2 marks ]

(As a side comment, Collatz conjecture states that every sequence generated by the Collatz iteration *converges* to the periodic sequence 1, 4, 2, 1, ..... but this is only a conjecture and “Mathematics is not yet ready for such problems”!)

3. Consider a linear, continuous-time, system described by the equations

$$\dot{x} = Ax + Bu \quad y = Cx$$

with

$$A = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

- a) Show that the system is not controllable and compute the uncontrollable modes. Show that the system is observable. [ 4 marks ]
- b) Design a state feedback control law  $u = Kx$  such that the matrix  $A + BK$  has eigenvalues equal to  $-2$  and  $-3$ . Explain why this problem is solvable despite the fact that the system is not controllable. [ 4 marks ]
- c) Design an observer such that the matrix  $A + LC$  has eigenvalues equal to  $-2$  and  $-1$ . [ 4 marks ]
- d) Using the separation principle, and the results in parts b) and c) write the equations of a dynamic, output feedback, control law which stabilizes the closed-loop system. Determine the eigenvalues of the resulting closed-loop system. [ 4 marks ]
- e) Suppose the response of the closed-loop system in part d) is *too slow*, hence it is necessary to modify the design to achieve a faster response. Suppose, in addition that the designer can either redesign the state feedback or the observer (he/she cannot redesign both). Discuss which design has to be modified, and determine a new design achieving the fastest possible response. [ 4 marks ]

4. Consider a nonlinear, continuous-time, system described by the equations

$$\dot{x}_1 = -x_1 + x_1 x_2 \quad \dot{x}_2 = -x_2 + x_1 x_2.$$

- a) Compute the equilibrium points of the system. [ 4 marks ]
- b) Compute the linearizations of the system around the equilibrium points determined in part a). [ 4 marks ]
- c) Study the stability properties of the linearized systems determined in part b), hence establish (if possible) stability properties for the equilibrium points computed in part a). [ 4 marks ]
- d) Consider the change of coordinates

$$x_1 = \rho \cos \theta, \quad x_2 = \rho \sin \theta,$$

with  $\rho \geq 0$  and  $\theta \in (-\pi, \pi]$ .

- i) Write a differential equation for the variable  $\rho^2$ . [ 4 marks ]

- ii) Show, exploiting the facts that  $|\sin \theta \cos \theta| \leq 1/2$ , and  $|\sin \theta + \cos \theta| \leq 3/2$ , that

$$\frac{d}{dt} \rho^2 \leq -2\rho^2 + \frac{3}{2}\rho^3.$$

[ 2 marks ]

- iii) Using the inequality in part d.ii) show that all trajectories of the system starting from initial conditions  $(x_1(0), x_2(0))$  such that

$$x_1^2(0) + x_2^2(0) \leq 1$$

converge to zero.

[ 2 marks ]

## Control engineering exam paper - Model answers

### Question 1

- The controllability matrix is

$$\mathcal{R} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

which has rank three. Hence the system is controllable.

[2 marks]

- As indicated in the question, to evaluate the transmission zeros we build the matrix

$$\Sigma(s) = \begin{bmatrix} s+1 & -1 & 0 & 0 \\ 0 & s & -1 & 0 \\ -1 & 0 & s & 1 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

The determinant of  $\Sigma(s)$  is  $s+1$ , which shows that the system has  $n-2 = 1$  transmission zero equal to  $s = -1$ : the transmission zero has negative real part.

[6 marks]

- Since  $y = Cx = x_2$ , then  $\dot{y} = CAx + CBu = x_3$ . Note that  $CB = 0$ .

[2 marks]

- The feedback is given by

$$u = -k^2 x_2 - k x_3.$$

[2 marks]

- The closed-loop system is described by the equation

$$\dot{x} = A_{cl}x = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -k^2 & -k \end{bmatrix} x.$$

The characteristic polynomial of  $A_{cl}$  is

$$\det(\lambda I - A_{cl}) = \lambda^3 + (k+1)\lambda^2 + k(k+1)\lambda + (k^2 - 1).$$

The Routh test shows that the roots of the polynomial have all negative real part for all  $k > 1 = k_*$ .

[8 marks]



## Question 2

- The controls are described by the equations

$$u_1 = k_1(x_2 - x_1), \quad u_2 = k_2(x_1 - x_2) + k_3(x_3 - x_2), \quad u_3 = k_4(x_2 - x_3).$$

[2 marks]

- The equations are

$$\dot{x}_1 = k_1(x_2 - x_1), \quad \dot{x}_2 = k_2(x_1 - x_2) + k_3(x_3 - x_2), \quad \dot{x}_3 = k_4(x_2 - x_3).$$

Hence

$$A = \begin{bmatrix} -k_1 & k_1 & 0 \\ k_2 & -k_2 - k_3 & k_3 \\ 0 & k_4 & -k_4 \end{bmatrix}.$$

[4 marks]

- Note that

$$\det A = 0,$$

which shows that  $A$  has a zero eigenvalue.

[2 marks]

- The characteristic polynomial of  $A$  is (recall that it has a zero eigenvalue)

$$\det(\lambda I - A) = \lambda(\lambda^2 + (k_1 + k_2 + k_3 + k_4)\lambda + (k_1k_4 + k_4k_2 + k_1k_3)).$$

Selecting, for example,

$$k_1 = 1, \quad k_2 = 1, \quad k_3 = 1, \quad k_4 = 1,$$

yields

$$\det(\lambda I - A) = \lambda(\lambda^2 + 4\lambda + 3) = \lambda(\lambda + 3)(\lambda + 1).$$

[4 marks]

- The differential equations are

$$\dot{z}_{12} = 3z_{12} - 2z_{11} - z_{23}, \quad \dot{z}_{23} = z_{11} - 3z_{12} + 2z_{23}.$$

[2 marks]

These can be rewritten as

$$\dot{z}_{12} = -2z_{12} + z_{23}, \quad \dot{z}_{23} = z_{12} - 2z_{23}.$$

As a result,

$$F = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}.$$

The characteristic polynomial of  $F$  is

$$\det(\lambda I - F) = (\lambda + 3)(\lambda + 1),$$

which shows that the matrix  $F$  has eigenvalues equal to  $-3$  and  $-1$ .

[4 marks]



The above implies that

$$\lim_{t \rightarrow \infty} x_1(t) - x_2(t) = \lim_{t \rightarrow \infty} x_2(t) - x_3(t) = 0,$$

which is the same as condition  $(\star)$ .

[2 marks]

### Question 3

- The state equations are

$$x_1(k+1) = a_1x_1(k) + x_2(k) + a_1b_2u(k), \quad x_2(k+1) = a_0x_1(k) + a_0b_2u(k),$$

$$y(k) = x_1(k) + b_2u(k).$$

As a result

$$y(k+1) = x_1(k+1) + b_2u(k+1) = a_1x_1(k) + x_2(k) + a_1b_2u(k) + b_2u(k+1),$$

and

$$\begin{aligned} y(k+2) &= a_1x_1(k+1) + x_2(k+1) + a_1b_2u(k+1) + b_2u(k+2) \\ &= a_1^2x_1(k) + a_1x_2(k) + a_1^2b_2u(k) + a_0x_1(k) + a_0b_2u(k) + a_1b_2u(k+1) + b_2u(k+2). \end{aligned}$$

The same expression is obtained replacing the expression of  $y(k)$  and  $y(k+1)$ , as a functions of  $x_1(k)$ ,  $x_2(k)$ ,  $u(k)$ ,  $u(k+1)$  and  $u(k+2)$ , in the equation

$$y(k+2) = a_1y(k+1) + a_0y(k) + b_2u(k+2),$$

which proves that the state-space description is equivalent to the input-output description.

[8 marks]

- The reachability matrix is

$$\mathcal{R} = \begin{bmatrix} a_1b_2 & a_1^2b_2 + a_0b_2 \\ a_0b_2 & a_0a_1b_2 \end{bmatrix},$$

and

$$\det \mathcal{R} = -a_0^2b_2^2.$$

As a result the system is reachable if  $a_0$  and  $b_2$  are both non-zero.

If  $b_2 = 0$ , the system is non-reachable. It is controllable if  $a_1 = a_0 = 0$ , and uncontrollable otherwise.

If  $a_0 = 0$  and  $b_2 \neq 0$  the system is controllable.

[6 marks]

- The observability matrix is

$$\mathcal{O} = \begin{bmatrix} 1 & 0 \\ a_1 & 1 \end{bmatrix},$$

hence the system is observable for any value of the constants  $a_0$  and  $a_1$ .

[2 marks]

- Selecting  $a_0 = 0$  yields

$$y(k+2) + a_1y(k+1) = b_2u(k+2).$$

Hence, selecting  $a_1 = \alpha$  and  $b_2 = 1 - \alpha$ , and replacing  $k+1$  with  $k$  yields the first order smoother.

[4 marks]

## Question 4

- The equilibria of the system satisfy the equations

$$x_1 = k \sin x_2, \quad x_2 = \sin x_1.$$

Eliminating  $x_2$  yields

$$x_1 = k \sin(\sin x_1).$$

This equation, for  $k \in [-1, 1]$  has the unique solution  $x_1 = 0$ , hence  $(0, 0)$  is the only equilibrium of the system.

[4 marks]

The linearization of the system around the zero equilibrium is given by

$$\delta_x^+ = A\delta_x + B\delta_u,$$

with

$$A = \begin{bmatrix} 0 & k \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

[4 marks]

The characteristic polynomial of the matrix  $A$  is

$$\det(\lambda I - A) = \lambda^2 - k,$$

which has roots inside the unity disk for all  $|k| < 1$ . For  $k = 1$ , the roots are  $\pm 1$ , whereas for  $k = -1$  the roots are  $\pm j$ . As a result, the linearized system is asymptotically stable for all  $|k| < 1$ , and stable for  $|k| = 1$ .

[4 marks]

Let  $K = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$  and note that

$$A + BK = \begin{bmatrix} K_1 & k + K_2 \\ 1 & 0 \end{bmatrix}.$$

Hence, selecting  $K_1 = 0$  and  $K_2 = -k$ , yields two eigenvalues at zero.

[4 marks]

- For  $k = 1$ , and  $u$  constant, the equilibria are solutions of the equations

$$x_1 = \sin x_2 + u, \quad x_2 = \sin x_1.$$

Eliminating  $x_2$  yields

$$x_1 = \sin(\sin x_1) + u,$$

which is the same as

$$x_1 - \sin(\sin x_1) = u.$$

Note that

$$\sin(\sin x_1) = \alpha(x_1)x_1,$$

with  $|\alpha(x_1)| < 1$  for all  $x_1 \neq 0$ . Hence

$$x_1 - \sin(\sin x_1) = (1 - \alpha(x_1))x_1,$$

which shows that

$$\lim_{x_1 \rightarrow \pm\infty} (x_1 - \sin(\sin x_1)) = \pm\infty.$$

As a result, the equation  $x_1 - \sin(\sin x_1) = u$  has at least one solution for any  $u$ .

[4 marks]



## Control engineering exam paper - Model answers 2013

## Question 1

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- b) As indicated in the question, to evaluate the transmission zeros we build the matrix

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The determinant of  $\Sigma(s)$  is  $s+1$ , which shows that the system has  $n-2 = 1$  transmission zero equal to  $s = -1$ : the transmission zero has negative real part.

- c) Since  $y = Cx = x_2$ , then  $\dot{y} = CAx + CBu = x_3$ . Note that  $CB = 0$ .

- d) The feedback is given by

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- e) The closed-loop system is described by the equation

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## Question 2

a) The controls are described by the equations

$$u_1 = k_1(x_2 - x_1), \quad u_2 = k_2(x_1 - x_2) + k_3(x_3 - x_2), \quad u_3 = k_4(x_2 - x_3).$$

b) The equations are

$$\dot{x}_1 = k_1(x_2 - x_1), \quad \dot{x}_2 = k_2(x_1 - x_2) + k_3(x_3 - x_2), \quad \dot{x}_3 = k_4(x_2 - x_3).$$

Hence

$$A = \begin{bmatrix} -k_1 & k_1 & 0 \\ k_2 & -k_2 - k_3 & k_3 \\ 0 & k_4 & -k_4 \end{bmatrix}.$$

c) Note that

$$\det A = 0,$$

which shows that  $A$  has a zero eigenvalue.

d) The characteristic polynomial of  $A$  is (recall that it has a zero eigenvalue)

$$\det(\lambda I - A) = \lambda(\lambda^2 + (k_1 + k_2 + k_3 + k_4)\lambda + (k_1k_4 + k_4k_2 + k_1k_3)).$$

Selecting, for example,

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e) The differential equations are

$$\dot{z}_{12} = 3x_2 - 2x_1 - x_3, \quad \dot{z}_{23} = x_1 - 3x_2 + 2x_3,$$

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which ~~is~~ <sup>implies</sup> the same as condition  $(*)$ .

### Question 3

a) The state equations are

$$\begin{aligned}x_1(k+1) &= a_1x_1(k) + x_2(k) + a_1b_2u(k), & x_2(k+1) &= a_0x_1(k) + a_0b_2u(k), \\y(k) &= x_1(k) + b_2u(k).\end{aligned}$$

As a result

$$y(k+1) = x_1(k+1) + b_2u(k+1) = a_1x_1(k) + x_2(k) + a_1b_2u(k) + b_2u(k+1),$$

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The same expression is obtained replacing the expression of  $y(k)$  and  $y(k+1)$ , as a functions of  $x_1(k)$ ,  $x_2(k)$ ,  $u(k)$ ,  $u(k+1)$  and  $u(k+2)$ , in the equation

$$y(k+2) = a_1y(k+1) + a_0y(k) + b_2u(k+2),$$

which proves that the state-space description is equivalent to the input-output description.

b) The reachability matrix is

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hence the system is observable for any value of the constants  $a_0$  and  $a_1$ .

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Hence, selecting  $a_1 = \alpha$  and  $b_2 = 1 - \alpha$ , and replacing  $k+1$  with  $k$  yields the first order smoother.



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Note that

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As a result, the equation  $x_1 - \sin(\sin x_1) = u$  has at least one solution for any  $u$ .