

OPTIMISATION

1. The company XYZ has invested £20000 to develop a new product. The product can be manufactured for £2 per unit. The company then performs a marketing research. The conclusion of the research is that if the company spends £ a on advertising then it can sell the product at price £ p per unit and it will sell

$$2000 + 4\sqrt{a} - 20p$$

units.

- a) Compute the revenue for sales as a function of a and p . [2 marks]
- b) Compute the overall costs associated to the production and commercialization of the product, that is the development cost plus the production cost and the advertising cost, as a function of a and p . [2 marks]
- c) Compute the company's profit as a function of a and p . [2 marks]
- d) The company wishes to select a and p to maximize the profit. Pose this problem as an unconstrained optimization problem (disregard the non-negativity conditions on a and p). [2 marks]
- e) Compute the unique stationary point of the profit. Using second order sufficient conditions of optimality show that the stationary point is a local maximizer. [4 marks]
- f) Assume that the company is forced to fix the sale price of the product to $p = \bar{p}$, with $\bar{p} > 2$.
 - i) Determine the optimal advertising cost as a function of \bar{p} . [4 marks]
 - ii) Determine the optimal profit as a function of \bar{p} . [2 marks]
 - iii) Plot the optimal profit as a function of the fixed price \bar{p} and show that, as \bar{p} increases the profit becomes negative. [2 marks]

2. A man launches his boat from point A (see Figure 2) on a bank of a straight river, 3 km wide, and wants to reach the point B which is on the opposite bank and it is a km, with $a > 0$, distant from the point C (which is, clearly, 3 km from point A), as quickly as possible. He can row 6 km/h and run 8 km/h.

He has three options:

- row the boat across the river to point C and then run to point B ;
- row directly to point B ;
- row to some point D between C and B and then run to point B .

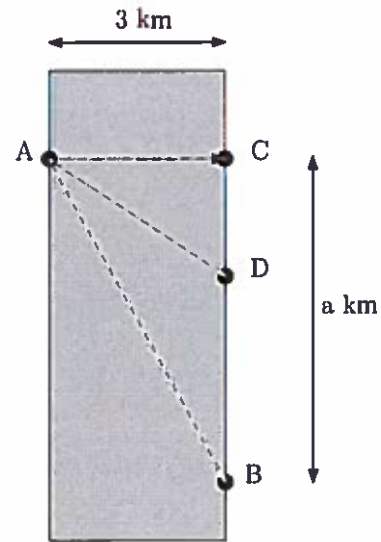


Figure 2

Determine, following the steps described below, the minimum time to reach point B .

- a) Let x be the distance from C to D . Note that $x \in [0, a]$. Express the rowing distance D_{row} , that is the distance A to D , and the running distance D_{run} , that is the distance D to B , as a function of x and a . [2 marks]
- b) Recall that, at constant speed, $\text{time} = \frac{\text{distance}}{\text{speed}}$. Determine the time T from A to B as a function of x and a , that is express the time from A to B as a function $T(x, a)$. [2 marks]
- c) Assume $a = 8$ and solve the optimization problem

$$\begin{aligned} \min T(x, a)|_{a=8}, \\ 0 \leq x \leq a = 8. \end{aligned}$$

[4 marks]

- d) Consider again the problem

$$\begin{aligned} \min T(x, a), \\ 0 \leq x \leq a, \end{aligned}$$

with $a \geq 0$. Determine the optimal solution of the problem, as a function of a , and plot the optimal value of x as a function of a . [6 marks]

- e) Consider now the problem of maximizing the distance travelled, by rowing and running, in one hour, that is consider the optimization problem

$$\begin{aligned} \max D_{run}(x, a) + D_{row}(x, a), \\ T(x, a) = 1. \end{aligned}$$

Solve the problem using the constraint elimination method. Comment on the obtained result. [6 marks]

3. Two sets of two positive numbers $\{p_1, p_2\}$ and $\{q_1, q_2\}$ are given. The numbers describe probability distributions, that is

$$p_1 + p_2 = 1 \quad q_1 + q_2 = 1.$$

In addition

$$0 < p_1 < \frac{1}{2} < p_2 < 1.$$

Consider the optimization problem

$$\min_{x_1, x_2} 1 - (x_1 q_1 + x_2 q_2),$$

$$x_1 p_1 + x_2 p_2 = \frac{1}{2},$$

$$0 \leq x_1 \leq 1,$$

$$0 \leq x_2 \leq 1.$$

- a) Sketch on the (x_1, x_2) -plane the admissible set. [2 marks]
 b) Using the sketch in part a), show that there are three types of admissible points:

$$Type_1 = (0, \star), \quad Type_2 = (1, \star), \quad Type_3 = (\star, \star),$$

where \star denotes some numbers such that $0 < \star < 1$. [4 marks]

- c) Write first order necessary conditions of optimality for the problem. [4 marks]
 d) Exploiting the results in part b) and the conditions in part c) show that the following holds.
- i) Assume $q_1 p_2 - q_2 p_1 < 0$. Show that $Type_1$ points are optimal and compute explicitly the optimal solution. [4 marks]
 - ii) Assume $q_1 p_2 - q_2 p_1 > 0$. Show that $Type_2$ points are optimal and compute explicitly the optimal solution. [2 marks]
 - iii) Assume $q_1 p_2 - q_2 p_1 = 0$. Show that all admissible points are optimal and explain why. [4 marks]

4. Consider the optimization problem

$$\begin{aligned} \min_{x,y} \quad & x^2 + y^2, \\ & 6 - x - 2y \leq 0. \end{aligned}$$

- a) Sketch in the (x,y) -plane the admissible set and the level lines of the objective function. Hence, using only graphical considerations determine the optimal solutions of the considered problem. [4 marks]
- b) Consider the so-called log-barrier function of the problem defined as

$$B_\varepsilon(x,y) = x^2 + y^2 - \varepsilon \ln \left(- (6 - x - 2y) \right),$$

with $\varepsilon > 0$ and small.

- i) Compute the stationary points of the function $B_\varepsilon(x,y)$. Show that $B_\varepsilon(x,y)$ has two stationary points: one admissible for all $\varepsilon \geq 0$ and one not admissible for all $\varepsilon \geq 0$. [8 marks]
- ii) Show that the admissible stationary point determined in part b.i) is a local minimizer of the function $B_\varepsilon(x,y)$. [6 marks]
- iii) Show that the admissible stationary point determined in part b.i) converges, as ε converges, to zero to the graphical solution determined in part a). [2 marks]