Imperial College
London

M3P19

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May-June 2019

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Measure and Integration

Date: Wednesday 22 May 2019

Time: 10.00 - 12.00

Time Allowed: 2 Hours

This paper has 4 Questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the relevant main book(s) are full.

All required additional material will be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Calculators may not be used.

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M4/5P19

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Measure and Integration

Date: Wednesday 22 May 2019

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This paper has 5 Questions.

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1. In this question, let μ be the Lebesgue measure on $[0,1]$.

(a) Give a definition of the corresponding outer measure μ^* of a subset of $[0,1]$, a measurable set and its Lebesgue measure μ.

(b) Let $A \subset [0,1]$ be an arbitrary (not necessarily Lebesgue measurable) set. Show that there exists a Lebesgue measurable set $B \subset [0,1]$ such that $A \subset B$ and $\mu(B) = \mu^*(A)$.

(c) Show that any increasing sequence $A_1 \subset A_2 \subset \cdots$ of arbitrary (not necessarily Lebesgue measurable) subsets of $[0,1]$ satisfies

$$\mu^*(\cup_{k=1}^{\infty} A_k) = \lim_{n\to\infty} \mu^*(A_n)$$

Hint: Find an appropriate increasing sequence of measurable sets using part (b) to show inequality in one of the directions ($\leq$).

2. (a) Define what it means for a sequence of measurable functions f_n to converge in measure to a function f.

(b) If a sequence of measurable functions f_n converges in measure to a function f, does it follow that this sequence converges to f a.e.? Give either a proof, or a counterexample without proof.

3. (a) Let g be a simple function on a set of finite measure. Define what it means for it to be integrable, and define the integral.

   (b) Let f be a measurable function on a set of finite measure. Define what it means for it to be integrable, and define the integral.

   (c) Let $(X, \mathcal{M}, \mu)$ be a measure space and $f : X \to \mathbb{R}$ be integrable. Show that

$$\lim_{n\to\infty} \int_{A_n} f d\mu = 0$$

   for any sequence of measurable sets such that $A_1 \supset A_2 \supset \cdots$ and $\cap_{n=1}^{\infty} A_n = \emptyset$.

4. In this question let $f : [a, b] \to \mathbb{R}$, where $[a, b]$ is a nonempty bounded interval.

   (a) Define what it means for f to be of bounded variation.

   (b) Define what it means for f to be absolutely continuous.

   (c) (i) Is any function of bounded variation absolutely continuous?

   (ii) Is any absolutely continuous function of bounded variation?

   (iii) Is any function of bounded variation continuous?

   (iv) Is any nondecreasing function on $[a, b]$ of bounded variation?

   Answer 'yes' or 'no' in each case without proof.

   (d) Prove that any absolutely continuous function can be represented as an indefinite integral.

5. (a) Let $(X, \mathcal{M}, \mu)$ be a measure space such that $\mu(X) = 1$, $T : X \to X$. Define what it means for T to be measure-preserving.

(b) Let $(X, \mathcal{M}, \mu)$ be a measure space such that $\mu(X) = 1$, $T : X \to X$. Define what it means for T to be ergodic.

(c) Let $\alpha \in (0, 1)$. Consider the Lebesgue measure on the unit circle extended from the semiring of arcs (to each arc we associate its length divided by 2π), and the transformation $Tz = ze^{2\pi i\alpha}$, $|z| = 1$.

(i) If α is rational, give an explicit example of a set invariant under T which demonstrates that T is not ergodic.

(ii) Let α be irrational. Let $f(z) = 1$ if $\arg z \subset (a, b)$, $0 \le a < b < 2\pi$, and $f(z) = 0$ otherwise. Determine

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j z)$$

for a.e. z on the circle. Justify your answer. You can assume without proof that T is ergodic (it is so).

	EXAMINATION SOLUTIONS 2018-19	Course M34P19
Question 1		Marks & seen/unseen
Parts		
a	The outer measure of a set A $$\mu^*(A) = \inf_{A \subset \bigcup_n I_n} \sum_n m(I_n)$$ over finite or countable unions of intervals I_n, where $m(I_n)$ is the length of I_n. A set A is measurable if $\forall \varepsilon > 0 \ \exists B \in \mathcal{R}$ s.t. $\mu^*(A \triangle B) < \varepsilon$, where $\mathcal{R}$ is the minimal ring generated by intervals. μ^* restricted to measurable sets is called the Lebesgue measure.	4/seen
	Setter's initials	Checker's initials
		Page number 1

	EXAMINATION SOLUTIONS 2018-19	Course M34P19
Question 1		Marks & seen/unseen
Parts b	Let $A \subset [0,1]$. Fix $n \geq 1$. By definition of the outer measure there is at most countable family of intervals whose union cover A $\{I_j^{(n)}\}_{j=1}^{\infty}$, $I_j^{(n)} \subset [0,1]$, s.t. $\sum_j m(I_j^{(n)}) \leq \mu^*(A) + 1/n$. By subadditivity, $\mu^*(\bigcup_j I_j^{(n)}) \leq \mu^*(A) + 1/n$. Let $B = \bigcap_{n=1}^{\infty} \bigcup_j I_j^{(n)}$. Then B is measurable as $I_j^{(n)}$ are measurable and measurable sets form a σ-algebra and, by subadditivity $\mu(B) \leq \mu^*(A) + 1/n \quad \forall n$ $\Rightarrow \mu(B) \leq \mu^*(A)$.	8/ seen/unseen
	Setter's initials	Checker's initials	Page number 2

Parts

On the other hand,

$A \subset B \Rightarrow \mu^*(A) \leq \mu(B)$.

Thus, $A \subset B$, $\mu^*(A) = \mu(B)$.

c. First, by subadditivity, 8 unseen

$\mu^*(A_n) \leq \mu^*(\bigcup_k A_k)$ $\forall n$, so

$\lim_{n\to\infty} \mu^*(A_n) \leq \mu^*(\bigcup_k A_k)$.

To show the opposite, choose <u>measurable</u> $B_n \supset A_n$, $\mu(B_n) = \mu^*(A_n)$, by Part b.

Let $C_n = \bigcap_{m=n}^{\infty} B_m$. Then $C_n \supset A_n$ by definition of A_n's, and moreover, $C_1 \subset C_2 \subset \cdots$

Note

i) $\mu^*(A_n) \leq \mu^*(C_n) \leq \mu(B_n) = \mu^*(A_n)$

$\Rightarrow \mu^*(A_n) = \mu(C_n)$

	EXAMINATION SOLUTIONS 2018-19	Course M34P19
Question 1		Marks & seen/unseen
Parts		

2) Using continuity of measure, we obtain

$$\mu^*\Big(\bigcup_k A_k\Big) \le \mu^*\Big(\bigcup_k C_k\Big) = \mu\Big(\bigcup_k C_k\Big)$$

$$= \lim_{n\to\infty} \mu(C_n) = \lim_{n\to\infty} \mu^*(A_n).$$

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Checker's initials

	EXAMINATION SOLUTIONS 2018-19	Course M34P19
Question 2		Marks & seen/unseen
Parts		
a.	A sequence of measurable functions $f_n(x)$ is said to converge in measure to $f(x)$ if $\forall \delta > 0 \ \lim_{n\to\infty} \mu\{x : \lvert f - f_n \rvert \geq \delta\} = 0$	5/seen
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EXAMINATION SOLUTIONS 2018-19	Course M34P19
Question 2	Marks & seen/unseen

Parts b

15/seen

No.

Counterexample:

The sequence

$$f_1^{(1)}, f_1^{(2)}, f_2^{(2)}, \dots,$$

$$\dots, f_1^{(k)}, f_2^{(k)}, \dots, f_k^{(k)}, \dots$$

where

$$f_j^{(k)}(x) = \begin{cases} 1, & \frac{j-1}{k} < x \le j/k \\ 0, & \text{otherwise}, \end{cases}$$

$1 \le j \le k$, $x \in (0,1]$.

This sequence converges to 0 in measure but does not converge to any point of $(0,1)$

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	EXAMINATION SOLUTIONS 2018-19	Course M34P19
Question 3		Marks & seen/unseen
Parts		
a	Let g take values y_j on A_j, $j=1,2,\dots$, $y_j \neq y_k$ if $j \neq k$. For a measurable set A let $\int_A g\,d\mu = \sum_n y_n \mu(A_n)$, $A_n = \{x \in A : f(x) = y_n\}$. g is called integrable if the series converge absolutely. Its value is called then the integral of g over A.	5/seen
b	f is called integrable on A if there is a sequence of simple integrable on A f_n's which converges uniformly to f. The integral is then $\int_A f\,d\mu = \lim_{n\to\infty} \int_A f_n\,d\mu$	5/seen
	Setter's initials	Checker's initials
		Page number 7

EXAMINATION SOLUTIONS 2018-19	Course M34P19

Question 3

Marks & seen/unseen

Parts

c

Let $f_n(x) = \begin{cases} f(x), & x \in A_n \\ 0, & \text{otherwise} \end{cases}$ 10/unseen

Since $\bigcap_n A_n = \emptyset$, $A_1 \supset A_2 \supset \cdots$,

$\lim_{n\to\infty} f_n(x) = 0 \quad \forall x$.

Moreover, $|f_n(x)| \leq |f(x)|$

and $|f(x)|$ is integrable as f is integrable.

Therefore, by dominated convergence thm (Lebesgue),

$$\int_{A_n} f \, d\mu = \int_A f_n \, d\mu \longrightarrow$$

$$\longrightarrow \int_A \lim f_n \, d\mu = \int_A 0 \, d\mu = 0.$$

Setter's initials

Checker's initials

	EXAMINATION SOLUTIONS 2018-19	Course M34P19
Question 4		Marks & seen/unseen

Parts

a Let $V_a^b(f) = \sup \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|$ over all finite subdivisions of $[a,b]$. If $V_a^b(f) < \infty$, f is called the function of bounded variation over $[a,b]$ — 2/seen

b f is called absolutely continuous on $[a,b]$ if $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $\sum_{j=1}^{n} |f(b_j) - f(a_j)| < \varepsilon$ for any finite family of disjoint subintervals of $[a,b]$ $\{(a_j, b_j)\}$ satisfying $\sum_{j=1}^{n} |b_j - a_j| < \delta$. — 3/seen

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EXAMINATION SOLUTIONS 2018-19	Course M34P19

Question 4

Marks & seen/unseen

Parts

seen/unseen

c

i no 2

ii yes 2

iii no 2

iv yes 2

d Let Φ be a.c. Then $\Phi \in$ B.V. 7/seen

So Φ' exists a.e. and integrable.

Let $f(x) = \Phi(x) - \int_a^x \Phi'(t)\,dt$.

This function is a.c. and

$f' = \Phi' - \Phi' = 0$ a.e.

Therefore $f = \text{const}$. Thus

$\Phi(x) = \int_a^x \Phi'(t)\,dt + \text{const}$,

$\text{Const} = \Phi(a)$.

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EXAMINATION SOLUTIONS 2018-19

Course: M34P19

Question: 5

Marks & seen/unseen

Parts

a $T: X \to X$ is called measure-preserving if for any measurable $A \subset X$,

$$\mu(T^{-1}(A)) = \mu(A).$$

3

b $T: X \to X$ is called ergodic if it is measure-preserving and if for any measurable A satisfying $T^{-1}(A) = A$, it follows that $\mu(A) = 0$ or $\mu(A) = 1$.

3

c i Let $A_0 = \{e^{i\theta} : 0 < \theta < \frac{2\pi}{2q}\}$;

$$A = \bigcup_{j=0}^{q-1} T^j(A), \quad \text{if}$$

$\alpha = p/q$, p, q - coprime.

4

Setter's initials

Checker's initials

	EXAMINATION SOLUTIONS 2018-19	Course M34P19
Question		Marks & seen/unseen
Parts		

Then $T(A) = A$,

and $\mu(A) = \sum_{j=0}^{q-1} \mu(T^j A) =$

$= q \cdot \mu(A_0) = 1/2$ since all

$T^j(A)$ are disjoint and of

measure $\mu(A_0)$.

Since A is invariant and

$\mu(A) \neq 0, 1$, T is not ergodic.

C ii. By Birkhoff's theorem, and since T is ergodic, 4

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j z) \equiv f^*(z)$$

is constant, f^*, a.e. and

$$\int_0^{2\pi} f^* \frac{d\theta}{2\pi} = f^* = \int_0^{2\pi} f \frac{d\theta}{2\pi} = b-a$$

So $f^*(z) = b-a$ a.e.

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