IMPERIAL COLLEGE LONDON

EE4-40
EE9-CS7-26
EE9-SO20

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2016

MSc and EEE/EIE PART IV: MEng and ACGI

Corrected copy

INFORMATION THEORY

Thursday, 5 May 10:00 am

Time allowed: 3:00 hours

There are FOUR questions on this paper.

Answer ALL questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : C. Ling

Second Marker(s) : D. Gunduz

Information for students

Notation:

(a) Random variables are shown in Tahoma font. x, $\mathbf{\mathsf{x}}$, $\mathbf{\mathsf{X}}$ denote a random scalar, vector and matrix respectively.

(b) The size of a set A is denoted by $|A|$.

(c) By default, the logarithm is to the base 2.

(d) $\oplus$ denotes the exclusive-or operation, or modulo-2 addition.

(e) "i.i.d." means "independent identically distributed".

(f) $H(\cdot)$ is the entropy function.

(g) $C(x) = \frac{1}{2}\log_2(1+x)$ is the capacity function for the Gaussian channel in bits/channel use.

The Questions

1. Basics of information theory.

a) Let X be the outcome of a throw of a fair dice which has six faces marked with 1, 2, 3, 4, 5, 6, respectively, and let Y be Even, if X is even, and Odd, otherwise. Calculate

i) $H(X)$, $H(Y)$

ii) $H(X, Y)$, $H(X|Y)$, $H(Y|X)$

iii) $I(X ; Y)$

[10]

b) Let $p(x,y)$ be the joint probability distribution of random variables X and Y. Show that the mutual information $I(X ; Y)$ is always nonnegative. State the condition when $I(X ; Y) = 0$. You may assume without proof that the relative entropy is always non-negative.

[5]

c) Consider a sequence of n binary random variables $X_1, X_2, \ldots, X_n$. Each n-sequence with an even number of 1's has probability $2^{-(n-1)}$ and each n-sequence with an odd number of 1's has probability 0. Find the mutual information $I(X_1 ; X_2)$, $I(X_2 ; X_3 | X_1), \ldots, I(X_{n-1} ; X_n | X_1, \ldots, X_{n-2})$.

[10]

2. Markov chains.

a) Consider the probability distribution of a random variable X:

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ 0.49 & 0.26 & 0.12 & 0.04 & 0.04 & 0.03 & 0.02 \end{pmatrix}$$

i) Find a binary Huffman code for X. [5]

ii) Find the expected code length for this code. [3]

b) Lempel-Ziv coding. Give the parsing and encoding of the following sequence:

101001100101001000101011

[Note: For this question, you will see less than 15 phrases; so ALWAYS use four bits to represent the location of a phrase. Do not worry about how to save such bits.]

[7]

c) Consider a source with memory modelled by a three-state Markov chain with output symbols $\{A, B, C\}$ and with the state-transition diagram shown in Figure 2.1.

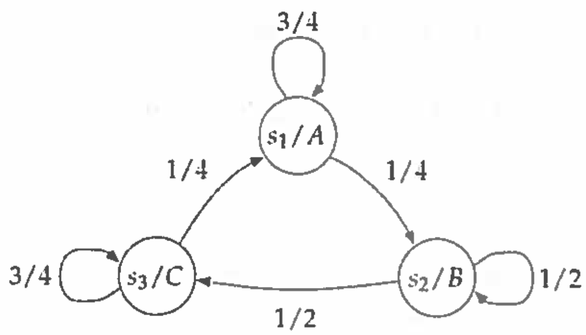

Fig. 2.1. State-transition diagram.

i) Calculate the entropy rate of the source. [7]

ii) Consider a memoryless source with the same probability distribution as the stationary distribution. Calculate its entropy and comment on your finding. [3]

3. Gaussian sources and channels.

a) Justify each step in the following proof of the converse of the coding theorem for the additive Gaussian noise channel where

$$y_i = x_i + z_i$$

Here z_i is independent Gaussian noise.

Fig. 3.1. Coding for the Gaussian channel.

Assume error probability $P_e^{(n)} \to 0$ and $n^{-1}\mathbf{x}^T\mathbf{x} < P$ for each $\mathbf{x}(w)$. We derive

$$nR = H(W) \overset{(1)}{=} I(W; y_{1:n}) + H(W \mid y_{1:n})$$

$$\overset{(2)}{\le} I(x_{1:n}; y_{1:n}) + H(W \mid y_{1:n})$$

$$\overset{(3)}{=} h(y_{1:n}) - h(y_{1:n} \mid x_{1:n}) + H(W \mid y_{1:n})$$

$$\overset{(4)}{\le} \sum_{i=1}^{n} h(y_i) - h(z_{1:n}) + H(W \mid y_{1:n})$$

$$\overset{(5)}{\le} \sum_{i=1}^{n} I(x_i; y_i) + 1 + nRP_e^{(n)}$$

$$\overset{(6)}{\le} \sum_{i=1}^{n} ½\log\left(1 + PN^{-1}\right) + 1 + nRP_e^{(n)}$$

$$R \overset{(7)}{\le} ½\log\left(1 + PN^{-1}\right) + n^{-1} + RP_e^{(n)} \overset{(8)}{\to} ½\log\left(1 + PN^{-1}\right)$$

[8]

b) Evaluate the differential entropy $h(X) = -\int f \ln f$ for the following:

i) The exponential density $f(x) = \lambda e^{-\lambda x}, x \ge 0$. [4]

ii) The sum of X_1 and X_2; where X_1 and X_2 are independent normal random variables with means μ_i and variances σ_i^2 for $i = 1, 2$. [3]

c) Prove that for any continuous source with differential entropy h, the mean square distortion D must satisfy the inequality

$$D \ge \frac{2^{2h}}{2\pi e} 2^{-2R}$$

where R is the compression rate.

[10]

4. Network information theory.

a) Consider the inference channel in Fig. 4.1. There are two senders with equal power P, two receivers, with crosstalk coefficient a. The noise is Gaussian with zero mean and variance N. Show that the capacity under very strong interference (i.e., $a^2 \geq 1 + P/N$) is equal to the capacity under no interference at all.

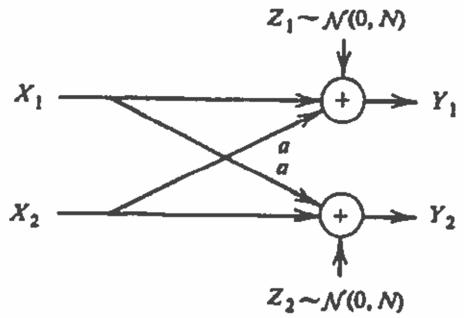

Fig. 4.1. Interference channel.

[10]

b) Slepian-Wolf coding. Consider a stereo system where the sum and the difference of the right and left ear signals are to be individually compressed for a common receiver. Let Z_1 be Bernoulli (p_1) and Z_2 be Bernoulli (p_2) and suppose Z_1 and Z_2 are independent. We have $P(Z_i = 0) = p_i$ and $P(Z_i = 1) = 1 - p_i$. Let $X = Z_1 + Z_2$, and $Y = Z_1 - Z_2$.

i) What is the Slepian-Wolf rate region of achievable (R_X, R_Y)? [10]

ii) Is this larger or smaller than the rate region of (R_{Z_1}, R_{Z_2})? Why? [5]