ANALYSIS OF CIRCUITS

**** Solutions 2018 ****

Information for Candidates:

- Numerical answers must be given as fully evaluated decimal values and not as unevaluated arithmetic expressions.

Notation

The following notation is used in this paper:

1. The voltage waveform at node X in a circuit is denoted by $x(t)$, the phasor voltage by X and the root-mean-square (or RMS) phasor voltage by $\widetilde{X} = \frac{X}{\sqrt{2}}$. The complex conjugate of X is X^*.
2. Component and source values in a circuit are normally given in Ohms, Farads, Henrys, Volts or Amps with the unit symbol omitted. Where an imaginary number is specified, it represents the complex impedance or phasor value.
3. Times are given in seconds unless otherwise stated.
4. Unless otherwise indicated, frequency response graphs should use a linear axis for phase and logarithmic axes for frequency and magnitude.
5. The real and imaginary parts of a complex number, X, are written $\Re(X)$ and $\Im(X)$ respectively.

Key: B=bookwork, U=unseen example

1. a) Using nodal analysis, calculate the voltages at nodes X and Y of Figure 1.1. [5]

[U] KCL at node X gives

$$\begin{aligned} \frac{X}{5} + \frac{X-11}{3} + \frac{X-Y}{2} &= 0 \\ \Rightarrow 6X + 10X - 110 + 15X - 15Y &= 0 \\ \Rightarrow 31X - 15Y &= 110 \end{aligned}$$

KCL at node Y gives

$$\begin{aligned} \frac{Y-X}{2} + \frac{Y}{1} + \frac{Y-11}{4} &= 0 \\ \Rightarrow -2X + 7Y &= 11 \end{aligned}$$

Solving these simultaneous equations gives

$$X = 5, \qquad Y = 3.$$

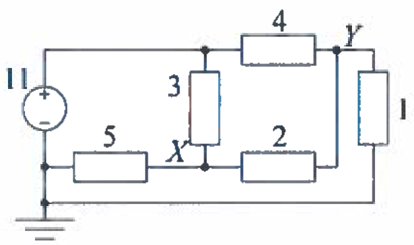

Figure 1.1

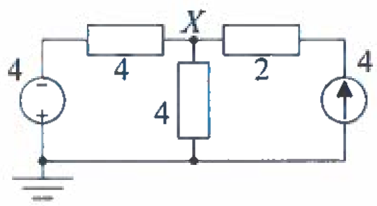

Figure 1.2

b) Use the principle of superposition to find the voltage X in Figure 1.2. [5]

[U] If we open-circuit the current source, the two 4 Ω resistors form a potential divider, so $X_1 = -2\,\text{V}$.

If we now short-circuit the voltage source, the two 4 Ω resistors are in parallel and are equivalent to a 2 Ω resistor. The current flowing through it is 4 A and so $X_2 = +8\,\text{V}$.

By superposition, the total voltage is therefore $X = X_1 + X_2 = -2 + 8 = 6\,\text{V}$.

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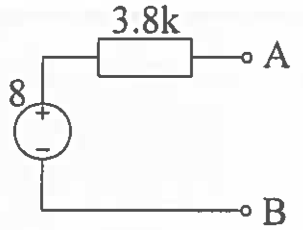
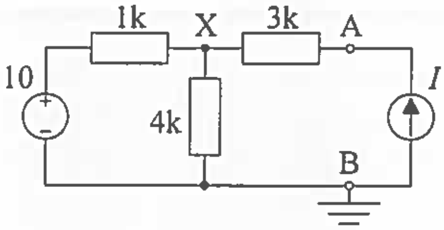
c) Draw the Thévenin equivalent circuit of the two-terminal network in Figure 1.3 and find the values of its components. [5]

[U] We can find the Thévenin resistance by short-circuiting the voltage source. This leaves two resistors in parallel with an equivalent resistance of $R_P = \frac{1\times 4}{1+4} = 0.8\,\mathrm{k}\Omega$ in series with a $3\,\mathrm{k}\Omega$ resistor. The total resistoance is therefore $R_{Thev} = 3.8\,\mathrm{k}\Omega$.

To find the open circuit voltage, we note that there is no current through the $3\,\mathrm{k}\Omega$ resistor and hence no voltage across it. The other two resistors form a potential divider and the voltage across the $4\,\mathrm{k}\Omega$ resistor is therefore $V_{Thev} = 8\,V$. Thus we get the diagram on the left below.

Alternatively we can ground node B and append a current source, I, as shown in the rightmost diagram below. Now doing KCL at node A gives $\frac{A-X}{3} - I = 0$ from which $X = A - 3I$ and KCL at node X gives $\frac{X-10}{1} + \frac{X}{4} + \frac{X-A}{3} = 0$ from which $19X = 4A - 120$. Eliminating X between these equations gives $19A - 57I = 4A - 120$ from which $A = 8 + \frac{57}{15}I = 8 + 3.8I$ which gives V_{Thev} and R_{Thev} directly.





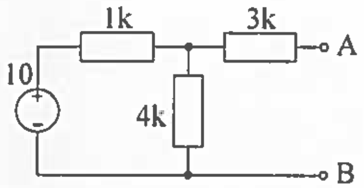


Figure 1.3

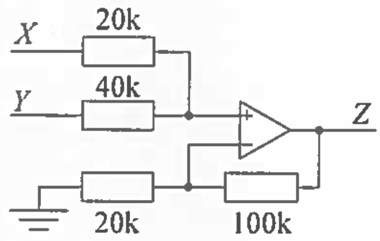


Figure 1.4

d) Assuming the opamp in the circuit of Figure 1.4 is ideal, give an expression for Z in terms of X and Y. [5]

[U] The upper two resistors form a weighted average circuit, so $V_+ = \frac{2X+Y}{3}$. The remainder of the circuit is a non-inverting amplifier with gain of $1 + \frac{100}{20} = 6$. Thus $Z = \frac{2X+Y}{3} \times 6 = 4X + 2Y$.

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e) Determine R_1 and R_2 in Figure 1.5 so that $Y = 0.25X$ and the parallel combination of R_1 and R_2 has an impedance of $75\,\Omega$. [5]

[U] The gain of the potential divider is $0.25 = \frac{R_2}{R_1+R_2}$ which implies that $0.25R_1 = (1-0.25)R_2 = 0.75R_2$ from which $R_1 = 3R_2$.

Substituting this relationship into the parallel impedance formula gives $75 =$

$\frac{R_1 R_2}{R_1+R_2} = \frac{3R_2^2}{4R_2} = 0.75R_2$ *from which* $R_2 = 100\,\Omega$ *and* $R_1 = 3R_2 = 300\,\Omega$.

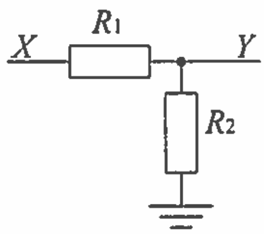


Figure 1.5

f) The circuit of Figure 1.6 shows a 50 Hz voltage source, with RMS voltage phasor $\widetilde{V} = 230$, driving a load of impedance $Z_L = 20 + 10j\,\Omega$ through a line of impedance $Z_T = 0.2 + 0.8j\,\Omega$. Calculate the complex power, $\widetilde{V} \times \widetilde{I}^*$, absorbed by (i) Z_T and (ii) Z_L. [5]

[U] The current phasor is $\widetilde{I}_L = \frac{230}{Z_L+Z_T} = \frac{230}{20.2+10.8j} = 8.855 - 4.734j$. *From this* $\left|\widetilde{I}_L\right| = 10.04$ *and* $\left|\widetilde{I}_L\right|^2 = 100.8$.

The complex power absorbed by Z_L *is*

$$\left|\widetilde{I}_L\right|^2 Z_L = 100.8\,(20 + 10j) = 2016 + 1008j\,\text{VA} = 2255\angle 26.6^\circ$$

.

The complex power absorbed by Z_T *is*

$$\left|\widetilde{I}_L\right|^2 Z_T = 100.8\,(0.2 + 0.8j) = 20.16 + 80.66j\,\text{VA} = 83.1\angle 76.0^\circ.$$

Alternatively, if you want to use the $\widetilde{V} \times \widetilde{I}^*$ *formula directly, you must calculate*

$$\widetilde{V}_L = \widetilde{I}_L Z_L = (8.855 - 4.734j)\,(20 + 10j) = 224.4 - 6.1j$$

and

$$\widetilde{V}_T = \widetilde{I}_L Z_T = (8.855 - 4.734j)\,(0.2 + 0.8j) = 230 - \widetilde{V}_L = 5.6 + 6.1j.$$

From this we get the absorbed powers as

$$\widetilde{V}_L \widetilde{I}_L^* = (224.4 - 6.1j)\,(8.855 - 4.734j) = 2016 + 1008j\,\text{VA}$$

and

$$\widetilde{V}_T \widetilde{I}_L^* = (5.6 + 6.1j)\,(8.855 - 4.734j) = 43.3 + 216.6j\,\text{VA}.$$

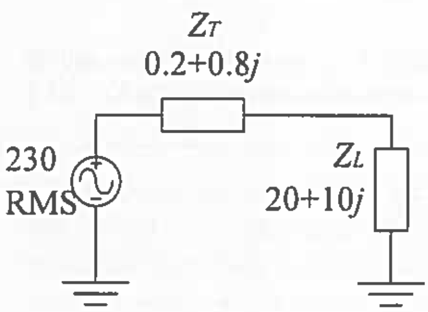


Figure 1.6

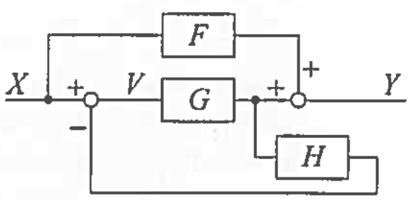


Figure 1.7

g) Determine the gain, $\frac{Y}{X}$, for the block diagram shown in Figure 1.7. The rectangular blocks are drawn with inputs at the left and outputs at the right and have gains of F, G and H respectively. The open circles represents adder/subtractors whose inputs have the signs indicated on the diagram and whose outputs are V and Y respectively. [5]

[U] We can write down the following equations from the block diagram:

$$\begin{aligned} V &= X - GHV \\ Y &= FX + GV \end{aligned}$$

We need to eliminate V from these equations:

$$\begin{aligned} V(1+GH) &= X \\ \Rightarrow V &= \frac{1}{1+GH} X \\ Y &= FX + GV \\ &= \left(F + \frac{G}{1+GH}\right) X \\ \Rightarrow \frac{Y}{X} &= F + \frac{G}{1+GH} \\ &= \frac{F+G+FGH}{1+GH} \end{aligned}$$

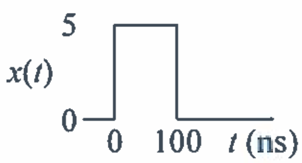


Figure 1.8

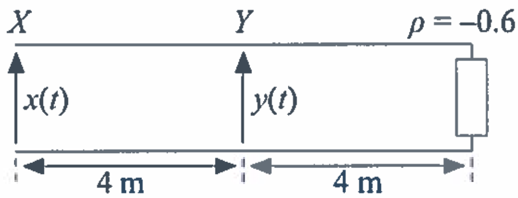
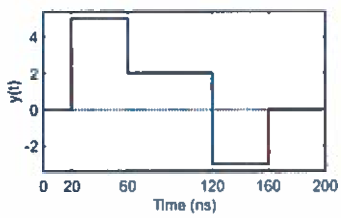


Figure 1.9

h) Figure 1.9 shows a transmission line of length 8 m that is terminated in a resistive load with reflection coefficient $\rho = -0.6$. The line has a propagation velocity of $u = 2 \times 10^8$ m/s. At time $t = 0$, a forward-travelling (i.e. left-to-right) pulse arrives at X with amplitude 5 V and duration 100 ns, as shown in Figure 1.8.

Draw a dimensioned sketch of the waveform at Y, a point 4 m from the end of the line, for $0 \le t \le 200$ ns. Assume that no reflections occur at point X. [5]

[U] The time taken to travel 4 m is $T = \frac{400}{20} = 20$ ns. So the forward wave (of amplitude 5 V) arrives at Y at $t = T = 20$ ns and ends at $t = 120$ ns. The reflected wave (of amplitude $5\rho = -3$ V) has to travel an additional 8 m and so arrives at $t = 3T = 60$ ns and ends at $t = 160$ ns. Adding the two waves together gives the following graph:



2. The frequency response of a highpass filter circuit is given by

$$H(j\omega) = \frac{k(j\omega)^2}{(j\omega)^2 + 2\zeta\omega_0 j\omega + \omega_0^2}$$

where k, ζ and ω_0 are positive real numbers.

a) i) Give a simplified expression for the value of $H(j\omega)$ at the frequency $\omega = \omega_0$. [2]

[U] At $\omega = \omega_0$, $(j\omega)^2 + \omega_0^2 = -\omega^2 + \omega_0^2 = 0$ so these two terms cancel out in the denominator. So $H(j\omega) = \frac{-k\omega_0^2}{2\zeta\omega_0 j\omega_0} = \frac{-k}{2j\zeta} = 0.5jk\zeta^{-1}$.

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ii) Determine the low and high frequency asymptotes of $H(j\omega)$. [2]

[U] LF asymptote is $H(j\omega) \to k\omega_0^{-2}(j\omega)^2$.

HF asymptote is $H(j\omega) \to k$.

The asymptotes cross at $\omega = \omega_0$ [not requested].

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iii) By finding the squared magnitudes of the numerator and denominator expressions in $H(j\omega)$, show that [5]

$$|H(j\omega)|^2 = \frac{k^2}{\left(\frac{\omega_0^2}{\omega^2} - 1\right)^2 + 4\zeta^2\frac{\omega_0^2}{\omega^2}}.$$

[U] If $z = a + jb$, then $|z|^2 = (a+jb)(a-jb) = a^2 + b^2$. We can write

$$\begin{aligned}
|H(j\omega)|^2 &= \frac{\left|k(j\omega)^2\right|^2}{\left|(j\omega)^2 + 2\zeta\omega_0 j\omega + \omega_0^2\right|^2} \\
&= \frac{k^2\omega^4}{\left|(\omega_0^2 - \omega^2) + 2\zeta\omega_0\omega j\right|^2} \\
&= \frac{k^2\omega^4}{(\omega_0^2 - \omega^2)^2 + (2\zeta\omega_0\omega)^2} \\
&= \frac{k^2\omega^4}{(\omega_0^2 - \omega^2)^2 + 4\zeta^2\omega_0^2\omega^2} \\
&= \frac{k^2}{\left(\frac{\omega_0^2}{\omega^2} - 1\right)^2 + 4\zeta^2\frac{\omega_0^2}{\omega^2}}
\end{aligned}$$

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iv) By writing the denominator of the previous expression in terms of $x = \frac{\omega_0^2}{\omega^2}$, show that the denominator has a minimum when $x = 1 - 2\zeta^2$.

Hence determine the value of ω at which $|H(j\omega)|$ is maximum and the value of $|H(j\omega)|$ at this frequency. [5]

[U] Making the suggested substitution, the denominator of $|H(j\omega)|^2$ becomes $(x-1)^2+4\zeta^2 x$. The coefficient of x^2 in this quadratic expression is positive and so the expression has a minimum. Setting its derivative to zero gives $2(x-1)+4\zeta^2=0$ from which $x=1-2\zeta^2$. Note that, since we must have $x\geq 0$, this only has a solution in ω if $\zeta<\sqrt{0.5}=0.707$ [not requested].

Thus the maximum of $|H(j\omega)|^2$ and hence of $|H(j\omega)|$ occurs at when $x_p=\frac{\omega_p^2}{\omega^2}=1-2\zeta^2 \Rightarrow \omega_p=\frac{\omega_0}{\sqrt{1-2\zeta^2}}$.

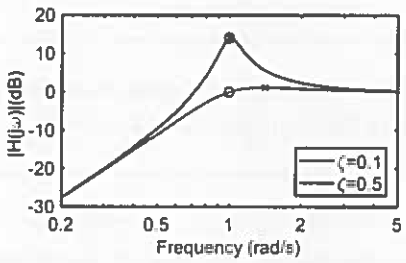
When $x=1-2\zeta^2$, the denominator of $|H(j\omega_p)|^2$ equals $(x-1)^2+4\zeta^2 x=\left(2\zeta^2\right)^2+4\zeta^2\left(1-2\zeta^2\right)=4\zeta^2\left(1-\zeta^2\right)$. So $|H(j\omega_p)|^2=\frac{k^2}{4\zeta^2(1-\zeta^2)}$ and $|H(j\omega_p)|=\frac{k}{2\zeta\sqrt{1-\zeta^2}}$.

b) Assuming $\omega_0=k=1$, draw a dimensioned plot showing the magnitude response, $|H(j\omega)|$, in dB for the two cases: (A) $\zeta=0.1$ and (B) $\zeta=0.5$. Show both lines on the same set of axes. For each case, calculate the maximum value of $|H(j\omega)|$ in dB and the frequency, ω_p, at which it occurs. [6]

[U] Using the formulae from the answers to parts i) and iv) above, we can construct the following table (note that numerical values for $|H(j\omega_0)|$ were not requested):

ζ	$\lvert H(j\omega_0)\rvert$	ω_p	$\lvert H(j\omega_p)\rvert$
0.1	$5=14\,\text{dB}$	1.01	$5.03=14\,\text{dB}$
0.5	$1=0\,\text{dB}$	1.41	$1.15=1.2\,\text{dB}$

This results in the following graph where o and x denote $|H(j\omega_0)|$ and $|H(j\omega_p)|$:



c) In the highpass filter circuit of Figure 2.1, the opamp is ideal, the capacitors have value C and the resistors have values P, Q, R and $(k-1)R$ respectively.

i) Explain why $Y=kV$. [1]

[U] The resistors $(k-1)R$ and R form a potential divider and, assuming the opamp draws no input current, the voltage at its negative input is therefore $\frac{R}{R+(k-1)R}Y=k^{-1}Y$. Since the circuit has negative feedback, the opamp inputs will have the same voltage and so $V=k^{-1}Y$. Alternatively, just recognise that it is a standard non-inverting opamp configuration whose gain is $1+\frac{(k-1)R}{R}=k$.

ii) By applying Kirchoff's current law at nodes U and V show that the transfer function $H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}$ is given by [6]

$$H(j\omega) = \frac{kPQC^2(j\omega)^2}{PQC^2(j\omega)^2 + (2P + (1-k)Q)Cj\omega + 1}.$$

[U] Applying KCL and U and $V = k^{-1}Y$ gives

$$(U-X)\,j\omega C + \left(U - k^{-1}Y\right) j\omega C + \frac{U-Y}{P} = 0$$
$$\left(k^{-1}Y - U\right) j\omega C + \frac{k^{-1}Y}{Q} = 0$$

from which

$$(2j\omega PC + 1)U = j\omega PCX + \left(j\omega k^{-1}PC + 1\right)Y$$
$$j\omega QCU = (j\omega QC + 1)k^{-1}Y$$

Crossmultiplying to eliminate U gives

$$(2j\omega PC + 1)(j\omega QC + 1)Y = j\omega kQC\left(j\omega PCX + \left(j\omega k^{-1}PC + 1\right)Y\right)$$
$$\Rightarrow \left(2PQC^2(j\omega)^2 + (2PC + QC)\,j\omega + 1\right)Y$$
$$= kPQC^2(j\omega)^2X + \left(PQC^2(j\omega)^2 + kQCj\omega\right)Y$$
$$\Rightarrow \left(PQC^2(j\omega)^2 + (2PC + (1-k)QC)\,j\omega + 1\right)Y = kPQC^2(j\omega)^2X$$
$$\Rightarrow \frac{Y}{X} = \frac{kPQC^2(j\omega)^2}{PQC^2(j\omega)^2 + (2P + (1-k)Q)Cj\omega + 1}.$$

iii) Determine simplified expressions for ζ and ω_0 when $H(j\omega)$ is written in the form given at the start of the question. [3]

[U] In order to make the coefficient of $(j\omega)^2$ equal to k (to match the equation for $H(j\omega)$), we divide numerator and denominator by PQC^2 to obtain

$$\frac{Y(j\omega)}{X(j\omega)} = \frac{k(j\omega)^2}{(j\omega)^2 + (2Q^{-1} + (1-k)P^{-1})C^{-1}j\omega + P^{-1}Q^{-1}C^{-2}}.$$

Matching coefficients gives

$$\omega_0 = \sqrt{P^{-1}Q^{-1}C^{-2}} = C^{-1}\sqrt{P^{-1}Q^{-1}}$$
$$2\zeta\omega_0 = \left(2Q^{-1} + (1-k)P^{-1}\right)C^{-1}$$
$$\Rightarrow \zeta = \sqrt{\frac{P}{Q}} + 0.5(1-k)\sqrt{\frac{Q}{P}}$$

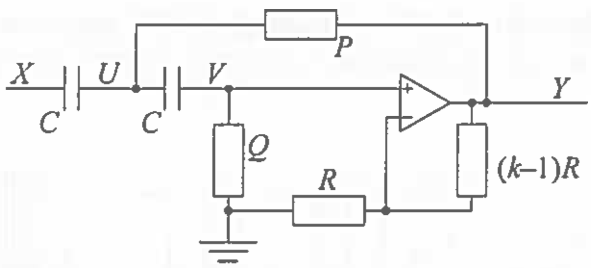


Figure 2.1

3. The diode in Figure 3.1 has a forward voltage of 0.7V when it is conducting. The voltage waveforms at nodes X and Y are $x(t)$ and $y(t)$ respectively and the diode current is $i(t)$ as shown.

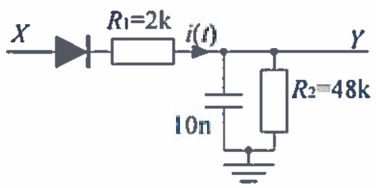


Figure 3.1

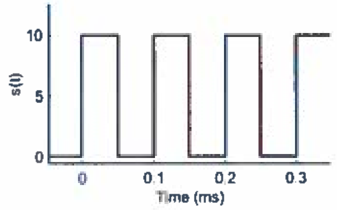


Figure 3.2

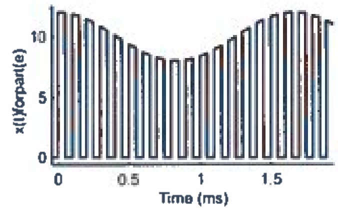


Figure 3.3

a) Assuming that node X is connected to a voltage source, calculate the time constant of the circuit when (a) the diode is conducting and (b) the diode is non-conducting. [4]

[U] (A) When the diode is conducting, the Thevenin resistance of the circuit driving the capacitance is $R_p = 2||48 = 1.92\,\mathrm{k}\Omega$. The time constant is therefore $\tau_{on} = R_pC = 19.2\,\mu\mathrm{s}$.

(B)When the diode is off, the Thevenin resistance of the circuit driving the capacitance is $R_2 = 48\,\mathrm{k}\Omega$. The time constant is therefore $\tau_{off} = R_2C = 480\,\mu\mathrm{s}$.

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b) If $x(t)$ has a constant voltage of 10V, determine the steady-state values of $i(t)$ and $y(t)$. [3]

[U] If the input voltage is constant, there is no current through the capacitor and so the circuit just consists of a diode and two resistors in series. The diode voltage is 0.7V and so the diode current is $i = \frac{9.3}{R_1+R_2} = 186\,\mu\mathrm{A}$. Hence $y = i_D R_2 = 8.928\,\mathrm{V}$.

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c) Suppose $x(t) = \begin{cases} 0 & t<0 \\ 10 & t \geq 0 \end{cases}$. Determine an expression for $y(t)$ for $t \geq 0$. [4]

[U] For $t < 0$, $y(t) = 0$ and hence, since the capacitor voltage cannot change instantly, $y(0+) = 0$.

For $t \geq 0$, the diode will be on and the steady state value is $y_{SS}(t) = 8.928$ (from part b)) and so the full expression for $y(t)$ is $y(t) = 8.928\left(1 - e^{-\frac{t}{\tau_{on}}}\right)$.

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d) Suppose now that $x(t) = s(t)$ as shown in Figure 3.2 where $s(t)$ is a positive-valued squarewave of period $T = 100\,\mu\mathrm{s}$ and amplitude 10V.

i) Determine an expression for $y(t)$ for $0 \leq t < 0.5T$ assuming that the diode is conducting throughout this interval and that the value of y at the start of the interval is $y(0) = A$. Hence, derive and simplify an equation relating A and B where $B = y(0.5T)$ is the value of y at the end of the interval. [4]

[U] This is the same as part c) except that $y(0+) = A$. *The formula is therefore* $y(t) = 8.928 + (A - 8.928)\,e^{-\frac{t}{\tau_{on}}}$. *Substituting* $t = 0.5T = 50\,\mu s$ *gives*

$$\begin{aligned} y(0.5T) = B &= 8.928 + (A - 8.928)\,e^{-\frac{0.5T}{\tau_{on}}} \\ &= 8.928 + (A - 8.928)\,0.074 \\ &\Rightarrow -0.074A + B - 8.2676 = 0 \end{aligned}$$

.

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ii) Determine an expression for $y(t)$ for $0.5T \le t < T$ assuming that the diode is non-conducting throughout this interval and that the value of y at the start of the interval is $y(0.5T) = B$. Hence derive and simplify a second equation relating A and B assuming that the value of y at the end of the interval is $y(T) = A$. [3]

[U] For the interval $0.5T \le t < T$, *the diode is off and so the steady state is* $y_{SS}(t) = 0$. *It follows that* $y(t) = Be^{-\frac{t-0.5T}{\tau_{off}}}$. *Substituting* $t = T = 100\,\mu s$ *gives* $y(T) = A = Be^{-\frac{0.5T}{\tau_{off}}} = 0.9011B \Rightarrow A = 0.9011B$.

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iii) By combining the equations determined in parts i) and ii), determine the numerical values of both A and B. [2]

[U] Substituting $A = 0.9011B$ *into* $-0.074A + B - 8.2676 = 0$ *gives* $-0.0666B + B = 8.2676$ *from which* $B = \frac{8.2676}{0.9334} = 8.858$. *Hence* $A = 0.9011B = 7.9817$.

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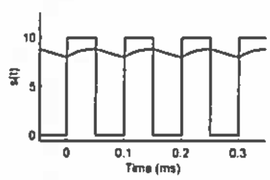
iv) Sketch a dimensioned graph of $y(t)$ for $t \in [0, 200\,\mu s]$. [3]

[U] The waveform of $y(t)$ *oscillates between* $A = 7.9817$ *and* $B = 8.858$ *with each segment a negative exponential having the appropriate time constant. Because of the difference in time constants, the rising portion of the waveform is much more curved than the falling part. The waveform is shown below:*



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e) Suppose now that $R_2 = 500\,\text{k}\Omega$ and that $x(t) = (1 + 0.2\cos(2\pi ft))\,s(t)$ is a modulated squarewave as illustrated in Figure 3.3 for $f = 600\,\text{Hz}$.

i) Assuming that $y(t) \leq 11.3$, determine an upper bound on the current through R_2. [1]

[U] Since $y(t) \leq 11.3$, $i_{R2}(t) = \frac{y(t)}{R_2} \leq \frac{11.3}{500\text{k}} = 22.3\,\mu\text{A}$.

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ii) Explain why the average value of $i(t)$ must equal the average current through R_2. Hence find an upper bound on the average voltage across R_1 during the times that the diode is conducting. [3]

[U] The average current through the capacitor is zero (when averaged over one cycle) and so, by KCL, the average value of $i(t)$ *must equal the average current through* R_2. *Since* $i(t) = 0$ *when the diode is off (i.e. half the time), the average value of* $i(t)$ *when the diode is conducting must be* $\leq 2 \times 22.3 = 44.6\,\mu\text{A}$. *Hence the average voltage across* R_1 *when the diode is conducting must be* $\leq 2k \times 44.6\mu = 89.2\,\text{mV}$.

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iii) Sketch the waveform $y(t)$ for a modulating frequency of $f = 20\,\text{Hz}$. It is not necessary to calculate the value of $y(t)$ precisely. [3]

[U] When the diode is on, the capacitor will charge to slightly less than $x(t) - 0.7$ *(since we know that the average value of* $(x(t) - 0.7) - y(t) \leq 89.2\,\text{mV}$*). When the diode turns off, the voltage will drop by* $\Delta V = \frac{\Delta Q}{C} \leq \frac{22.3\,\mu\text{A} \times 50\,\mu\text{s}}{10\,\text{nF}} = 0.11\,\text{V}$. *So the waveform is approximately* $y(t) = 9.3 + 2\cos(2\pi f t) + r(t)$ *where* $r(t)$ *is a* $10\,kHz$ *negative-valued ripple of amplitude* $\leq 0.11\,\text{V}$.
