The Solutions to Exam 2017

B—bookwork, E—new example, T—new theory

1.

a) We have the following outcomes, each with probability ¼:

$X_1 X_2$	00	01	10	11
$Y = \max(X_1, X_2)$	0	1	1	1

i) Thus $P(Y=0) = ¼$, $P(Y=1) = ¾$,

$H(Y) = -¼\log(1/4) - ¾\log(¾) = ½ + 0.31 = 0.81$ [3E]

ii) We have the joint distribution

$X_1 \ Y$	0	1
0	¼	¼
1	0	½

$H(Y|X_1) = ½\, H(½) + ½\, H(1) = ½$ [3E]

$I(X_1 ; Y) = H(Y) - H(Y| X_1) = 0.81 - ½ = 0.31$

iii) $I(X_{1:2} ; Y) = H(Y) - H(Y| X_{1:2})$

$= 0.81 - 0$ $\quad$ Y is a function of X_1 and X_2

$= 0.81$ [3E]

b) Recall

$$D(\mathbf{p}\,||\,\mathbf{q}) = \sum_i p_i \log_2\left(\frac{p_i}{q_i}\right) \geq 0$$

$D(\mathbf{p}||\mathbf{q}) = \frac{1}{2}\log\frac{3}{2} + \frac{1}{4}\log\frac{3}{4} + \frac{1}{4}\log\frac{3}{4} = \log 3 - 1.5 = 0.085.$ [3E]

$D(\mathbf{q}||\mathbf{p}) = \frac{1}{3}\log\frac{2}{3} + \frac{1}{3}\log\frac{4}{3} + \frac{1}{3}\log\frac{4}{3} = \frac{5}{3} - \log 3 = 0.082$ [3E]

c) Recall Fano's inequality $H(X|Y) \leq P(X \neq Y) \log M + H(P(X \neq Y))$. [1B]

$I(X ; Y) = H(X) - H(X \mid Y)$ $\quad$ definition [1B]

$\geq \log M - [P(X \neq Y) \log M - H(P(X \neq Y))]$ $\quad$ Fano [2E]

$\geq \log M - [(1 - P(X = Y)) \log M + H(P(X \neq Y))]$ $\quad$ algebra [2E]

$\geq P(X = Y) \log M - H(P(X \neq Y))$ $\quad$ algebra [2E]

$= P(X = Y) \log M - H(P(X = Y))$ $\quad$ because $H(P(X \neq Y)) = H(P(X = Y))$ [2E]

2.

a) [1B each]

(1) total probability of the jointly typical set

(2) $p(\mathbf{x},\mathbf{y}) \le \max_{\mathbf{x},\mathbf{y}\in J_\varepsilon^{(n)}} p(\mathbf{x},\mathbf{y})$

(3) $\max_{\mathbf{x},\mathbf{y}\in J_\varepsilon^{(n)}} p(\mathbf{x},\mathbf{y}) \le 2^{-n(H(X,Y)-\varepsilon)}$, from definition of the jointly typical set

(4) algebra

(5) total probability of the jointly typical set <= 1

(6) $p(\mathbf{x},\mathbf{y}) \ge \min_{\mathbf{x},\mathbf{y}\in J_\varepsilon^{(n)}} p(\mathbf{x},\mathbf{y})$

(7) $\min_{\mathbf{x},\mathbf{y}\in J_\varepsilon^{(n)}} p(\mathbf{x},\mathbf{y}) = 2^{-n(H(X,Y)+\varepsilon)}$, from definition of the jointly typical set

(8) algebra

b)

i) Note that the marginal distribution of X is

$$P(X=0)=4/7, \qquad P(X=1)=3/7. \qquad [2E]$$

Because $\varepsilon = 0$, $\mathbf{x} \in T_\mathbf{x}$ if and only if exactly 4 of the X_i's are equal to 0, and the other 3 of the X_i's are equal to 1. Hence

$$P(\mathbf{x} \in T_\mathbf{x}) = \binom{7}{4}\left(\frac{4}{7}\right)^4\left(\frac{3}{7}\right)^3 = 0.294. \qquad [3E]$$

ii) If, in addition, $(\mathbf{x},\mathbf{y}) \in J_0^{(7)}$, we require that $Y_i = 0$ for 3 out of 4 index i's for which $X_i = 0$, and $Y_i = 0$ for 1 out of 3 index i's for which $X_i = 1$.
Thus

$$P(\mathbf{x},\mathbf{y} \in J_0^{(7)} \mid \mathbf{x} \in T_\mathbf{x}) = \binom{4}{3}\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^1 \times \binom{3}{1}\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^2 = 0.188. \qquad [3E]$$

iii) Hence we determine the value

$$\begin{aligned}
P(\mathbf{x},\mathbf{y} \in J_0^{(7)}) &= P(\mathbf{x},\mathbf{y} \in J_0^{(7)} \mid \mathbf{x} \in T_\mathbf{x})P(\mathbf{x} \in T_\mathbf{x}) + P(\mathbf{x},\mathbf{y} \in J_0^{(7)} \mid \mathbf{x} \notin T_\mathbf{x})P(\mathbf{x} \notin T_\mathbf{x}) \\
&= P(\mathbf{x},\mathbf{y} \in J_0^{(7)} \mid \mathbf{x} \in T_\mathbf{x})P(\mathbf{x} \in T_\mathbf{x}) + 0 \\
&= 0.188 \times 0.294 \\
&= 0.055
\end{aligned}$$

[3E]

iv) We also require that $Z_i = 0$ for 3 out of 4 index i's for which $X_i = 0$, and $Z_i = 0$ for 1 out of 3 index i's for which $X_i = 1$. But, since $\mathbf{z}$ is independent of $\mathbf{x}$,

$$P(\mathbf{x},\mathbf{z} \in J_0^{(7)} \mid \mathbf{x} \in T_\mathbf{x}) = \binom{4}{3}\left(\frac{4}{7}\right)^3\left(\frac{3}{7}\right)^1 \times \binom{3}{1}\left(\frac{4}{7}\right)^1\left(\frac{3}{7}\right)^2 = 0.101 \qquad [3E]$$

Thus

$$
\begin{aligned}
P(\mathbf{x},\mathbf{z} \in J_0^{(7)}) &= P(\mathbf{x},\mathbf{y} \in J_0^{(7)} \mid \mathbf{x} \in T_{\mathbf{x}}) P(\mathbf{x} \in T_{\mathbf{x}}) \\
&= 0.101 \times 0.294 \\
&= 0.030
\end{aligned}
$$

[3E]

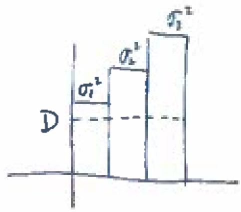
3.

a)

(1) definition of mutual info	[1B]
(2) chain rule	[1B]
(3) x_i is a function of y_1^{i-1} and w	[1B]
(4) channel is memoryless	[1B]
(5) indep. bound, or chain rule + conditioning reduces entropy	[1B]
(6) definition of mutual info	[1B]
(7) mutual info. <= capacity	[1B]
(8) Fano's inequality	[1B]
(9) algebra	[1B]
(10) taking limit and $P_e^{(n)} \to 0$	[1B]

b)

i) In this case, all 3 sources are encoded. [5E]

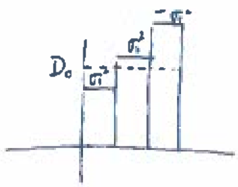


$0 < D < \sigma_1^2$

$R(D) = \sum \frac{1}{2} \log \frac{\sigma_i^2}{D_i}$

$$R(D) = \frac{1}{2}\log\frac{1}{0.5} + \frac{1}{2}\log\frac{2}{0.5} + \frac{1}{2}\log\frac{4}{0.5} = 3$$

ii) In this case, 2 sources are encoded. [5E]



$\sigma_1^2 < D_0 < \sigma_2^2$

$\sigma_1^2 + 2D_0 = 3D \Rightarrow D_0 = \frac{1}{2}(3D - \sigma_1^2)$

$\Rightarrow \sigma_1^2 < D < \frac{\sigma_1^2 + 2\sigma_2^2}{3}$

$R(D) = \frac{1}{2} \log \frac{\sigma_2^2}{D} + \frac{1}{2} \log \frac{\sigma_3^2}{D}$

$$R(D) = \frac{1}{2}\log\frac{2}{1} + \frac{1}{2}\log\frac{4}{1} = \frac{3}{2}$$

iii) In this case, only 1 source is encoded. [5E]

$$\sigma_2^2 < D_0 < \sigma_3^2$$

$$\sigma_1^2 + \sigma_2^2 + D_0 = 3D \Rightarrow D_0 = 3D - \sigma_1^2 - \sigma_2^2$$

$$\frac{\sigma_1^2 + 2\sigma_2^2}{3} < D < \frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2}{3}$$

$$R(D) = \frac{1}{2}\log\frac{\sigma_3^2}{D}$$

$$R(D) = \frac{1}{2}\log\frac{4}{2} = 1$$

4.

a)

i) Denote by N_1 the variance of Z_1, N'_2 the variance of Z'_2. We are supposed to verify

$$f(y_1, y_2|x) = f(y_1|x)f(y_2|y_1)$$

Definition of Markov chain [1E]

This is so because

$$f(y_1, y_2|x) = f(y_1, y_1 + z_2'|x)$$

From channel model [1E]

$$= f(y_1|x)f(y_1 + z_2'|x, y_1)$$

Chain rule [1E]

$$= f(y_1|x)f(y_1 + z_2'|y_1)$$

Z'_2 is independent of x [1E]

$$= f(y_1|x)f(y_2|y_1)$$

Obvious [1E]

ii)

Encoding: The sender uses the first codebook with power αP at rate R_1, and the second codebook with power $(1-\alpha)P$ at rate R_2, sends the sum of two codewords. Both codebooks are i.i.d. Gaussian. [3B]

Decoding: Bad receiver Y_2 treats Y_1 as noise, yielding a rate [2B]

$$R_2 \le C\left(\frac{(1-\alpha)P}{\alpha P + N_2}\right)$$

Good receiver Y_1 first decodes the second message X_2. It is able to do so because its channel is better: [2B]

$$R_2 \le C\left(\frac{(1-\alpha)P}{\alpha P + N_2}\right) \le C\left(\frac{(1-\alpha)P}{\alpha P + N_1}\right)$$

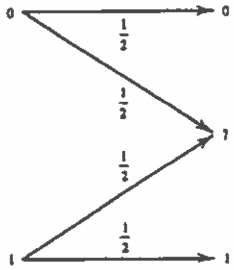
Then it subtracts out X_2, and decodes his own message. Since the channel is clean now, the rate is given by [3B]

$$R_1 \le C\left(\frac{\alpha P}{N_1}\right)$$

b)

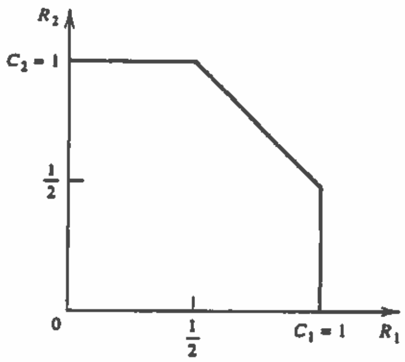
Setting $X_2 = 0$, we can send at a rate of 1 bit per transmission from sender 1. Similarly, setting $X_1 = 0$, we can send at a rate $R_2 = 1$. This gives us two extreme points of the capacity region. [2T]

Can we do better? Let us assume that $R_1 = 1$, so that the codewords of X_1 must include all possible binary sequences; X_1 would look like a Bernoulli(1/2) process. This acts like noise for the transmission from X_2. For X_2, the channel looks like a binary erasure channel in the following figure, whose capacity is 1/2 bit per transmission. [3T]



Hence when sending at maximum rate 1 for sender 1, we can send an additional 1/2 bit from sender 2, and vice versa. We can verify that these rates are the best that can be achieved. [3T]

The capacity region for this multi-access channel is illustrated as follows.



[2T]