E4.05
C5.22
SO7

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2007

MSc and EEE PART IV: MEng and ACGI

TRAFFIC THEORY & QUEUEING SYSTEMS

Thursday, 10 May 10:00 am

Time allowed: 3:00 hours

There are SIX questions on this paper.

Corrected Copy

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : J.A. Barria
Second Marker(s) : M.M. Draief

Special instructions for students

1. Erlang Loss formula recursive evaluation:

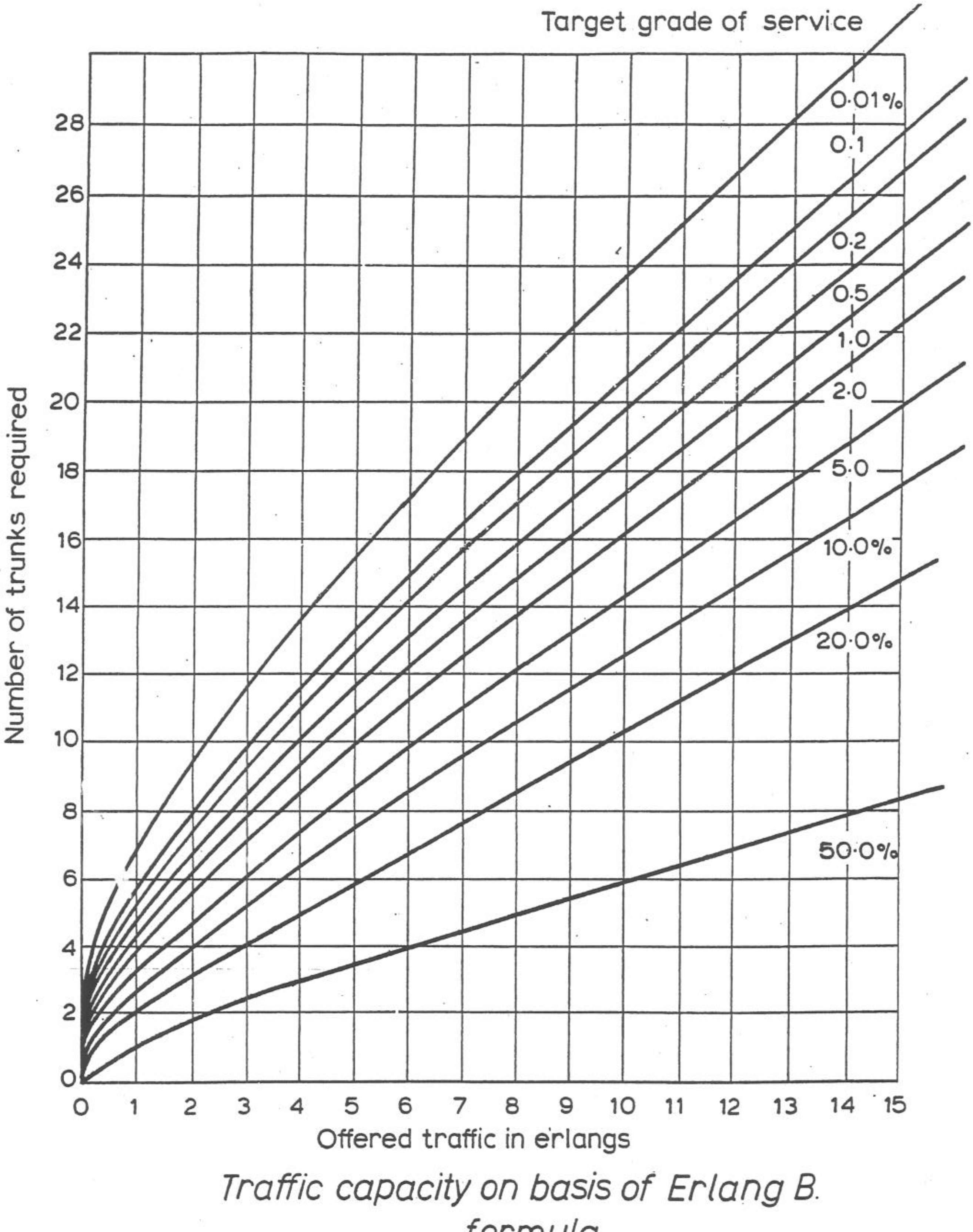
$$E_N(\rho) = \frac{\rho E_{N-1}(\rho)}{N + \rho E_{N-1}(\rho)}$$
$$E_0(\rho) = 1.$$

2. Engset Loss formula recursive evaluation (for a fixed M and $p = \alpha/1 + \alpha$):

$$e_N = \frac{(M - N + 1)\alpha e_{N-1}}{N + (M - N + 1)\alpha e_{N-1}}$$
$$e_0 = 1.$$
$$\alpha = \lambda/\mu.$$

3. Traffic capacity on basis of Erlang B formula (next page).

Note: for large ρ, N is approximately linear: $N \approx 1.33\rho + 5$



Traffic capacity on basis of Erlang B. formula.

1.

a) For an Engset traffic model with M sources and N channels

i) Describe and discuss the underlying assumption of an Engset model. [4]

ii) Derive the birth and death coefficients of the system described in i). [4]

iii) Assuming $M > N$, derive the steady state distribution for the system described in i). [5]

b) The total offered traffic of 45 Erlangs is offered to a link with a loss probability of 0.005.

i) Estimate the total income call rate if the average call duration is 150 seconds. [2]

ii) Estimate the number of trunks of the link. [3]

iii) Derive the total carried traffic. [2]

2.

In an M/M/K system let the variable Q_t represent the number of items in the buffer.

i) Derive the unconditional queue length distribution $P[Q_t = i]$.

[10]

ii) Derive $E[Q_t]$.

[10]

3.

a) In a K-channel message transmission link the arriving message stream consists of 2 separate arrival streams:

- Arrivals from stream 1 are Poisson with rate λ_1 and are allowed to wait if all channels are busy.

- Arrivals from stream 2 are Poisson with rate λ_2 and are discarded if K channels are busy.

- All messages have exponentially distributed length with mean length h.

Find the probability that an arrival message will not be transmitted immediately

[10]

b) Assume that the offered traffic to the system under analysis is pure chance traffic with parameters (λ, μ).

i) Define the state space for a 2-D Birth/Death model for overflow traffic if you know that

- the firth choice link has a maximum of M channels

- the overflow link has a maximum of N channels

[5]

ii) Draw the state transition diagram for the system.

[5]

4.

a) Consider a discrete-state, continuous-time Markov chain $\{N_t\}$ with state space $E = \{0,1,2,...,N\}$. The process can be further characterised by the following transition probabilities.

$$P[N_{t+\Delta t} = j \mid N_t = i] = \begin{cases} \lambda_i \Delta t & j = i+1 \\ \mu_i \Delta t & j = i-1 \\ 0 & |j-i| > 1 \end{cases}$$

And $\lambda_N = 0, \quad \mu_0 = 0$.

i) Draw the transition diagram.

[2]

ii) Obtain the equilibrium balance equations.

[3]

iii) Obtain the steady state distribution of N_t.

[3]

iv) Is the process reversible? Discuss your answer.

[2]

Question 4 b) Continues next page

4.

b) A 3-processor / 2-buffer stage system is shown in Figure 4.2.
The system can only be repaired if it is in a Faulty state.
The system is in a Faulty state if any one buffer or any one processor is not operational.

For the Failure and Repair processes:

- Failure rate of a buffer stage: h_b
- Failure rate of one processor: h_p
- Repair time of j faulty components: j/R

i) Define the state space of the system. [5]

ii) Derive the state space transition diagram of the system. [5]

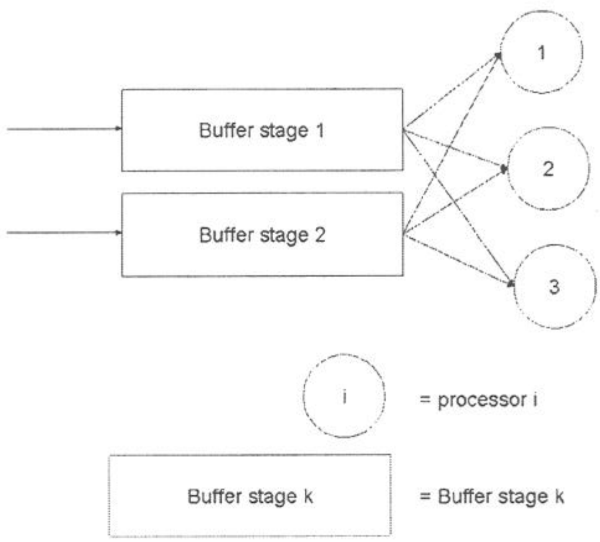


Figure 4.2: Three-processor / two-buffer stage system.

5.

a) The generic rate algorithm (GRA) proposed by ATM Forum has a number of equivalent representations one of which is the Leaky Bucket algorithm.

i) Using the close queueing system of Figure 5.1. Explain the main underlying characteristics of the Leaky Bucket algorithm.

[3]

ii) Discuss the meaning of the Queue 1 service rate λ.

[2]

iii) Discuss the meaning of the Queue 2 service rate r.

[2]

iv) Discuss how this model represents a finite token buffer of maximum capacity M cells.

[3]

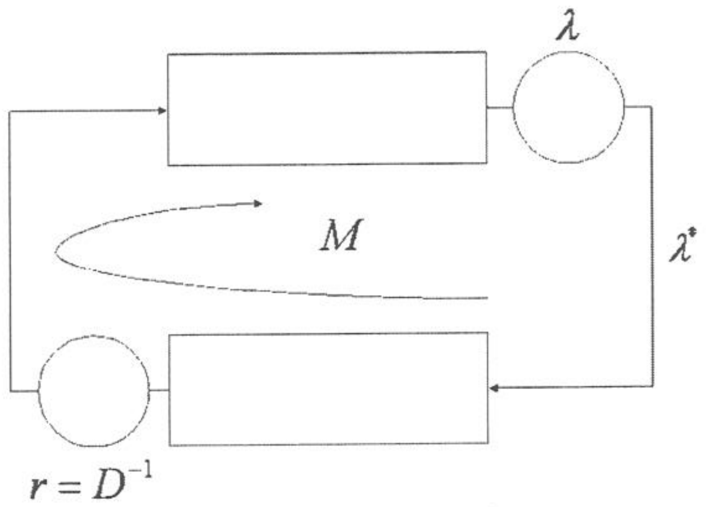


Figure 5.1: Closed queueing network model. Leaky Bucket.

Question 5 b) Continues next page

5.

b) Figure 5.2 represents an MMPP model of N multiplexed voice sources.

i) Define and discuss the meaning of the parameters λ, α and β.

[3]

ii) Assume that the voice source defined in i) is offered to a voice multiplexer with service rate ν cells/second.

- define a state space which can account for the number of voice sources and the state of the multiplexer.

[3]

- draw the state transition diagram of the multiplexer and clearly identify the transition rates.

[4]

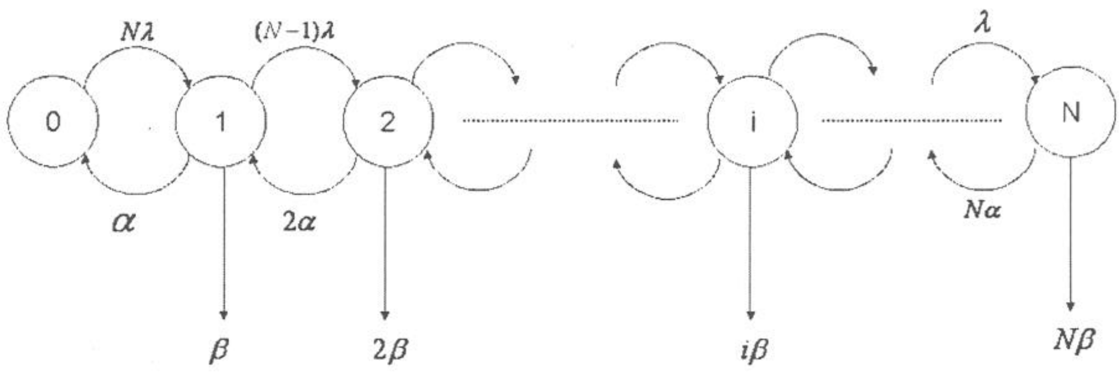


Figure 5.2: MMPP model N multiplexed voice sources.

6. A simple equivalent capacity expression is given by

$$C_L = (m + K\sigma)R_p$$

Where m and σ^2 can be obtained from an ON-OFF source model and K is dependent on a specified QoS.

a) Using an ON-OFF source model as the underlying traffic model; derive a simple expression for m and σ.

[6]

a) K is dependent on a specified QoS. Here QoS can be regarded as a measure of cell loss probability P_L or the probability of being in an overload state ε.

i) The probability of being in an overload state can be estimated by:

$$\varepsilon = \sum_{i=J_0}^{N} \pi_i$$

Derive an approximation to ε assuming that a large number of sources are multiplexed. That is N >> 1 and p << 1.

[7]

ii) The cell loss probability can be conservatively estimated by:

$$P_L = \sum_{i=J_0}^{N} \frac{(i - C)\pi_i}{m}$$

Where π_i is the probability that the system is in state i and J_0 is called the overload state.

Derive an approximation to P_L assuming that a large number of sources are multiplexed. That is N >> 1 and p << 1.

[7]

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Model Answers and Mark Schemes — First Examiner: TRAFFIC THEORY 2007 — 1/11

Paper Code: E4.05 — Second Examiner:

Q1

Question Number etc. in left margin — Mark allocation in right margin

a i) Engset assumptions

- Each idle source is a Poisson source:

$P[\text{new demand in } (t, t+\Delta t) \mid \text{source is idle}] = \lambda \Delta t$

- channel holding times are exponential with mean $1/\mu$
- Full availability access

Since there is no buffering j channels busy $\Leftrightarrow$ j sources busy — 3

the total arrival rate to the N channel link will fall as N_t increases.

ii) - Birth coefficients

The number of idle sources in state i is $(M-i)$

$$\lambda_i = (M-i)\lambda = \left(1 - \frac{i}{M}\right) M\lambda = \left(1 - i/M\right)\lambda_0$$ — 3

- Death coefficients

As in Erlang model $\mu_i = i\mu \quad i \geq 0$

iii) $M > N$

$$\pi_i = \left[\frac{\binom{M}{i} p^i (1-p)^{M-i}}{\sum_{j=0}^{N} \binom{M}{j} p^j (1-p)^{M-j}} \right] \quad i = 0, 1, \ldots, N$$

clearly state all steps of this derivation — 4

(bookwork extension)

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Model Answers and Mark Schemes

First Examiner:

2

Paper Code:

Second Examiner:

Q1

Question Number etc. in left margin

Mark allocation in right margin

b i) 45 Erlangs $\longrightarrow$ $B_C = 0.005$

offered traffic = Total calling rate * mean call duration

45 = Total calling rate * 150 s

Total calling rate = 18 calls / min 3

ii) carried traffic = offered traffic $(1 - B_C)$

$= 45\,(1 - B_C) = 44.8$ Erlangs 4

iii) From Erlang chart

$N \simeq 1.33\rho + 5$

$N \approx 65$ channels for $\rho = 45$ Erlangs. 3

(calculation of new example)

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Model Answers and Mark Schemes

First Examiner:

3

Paper Code:

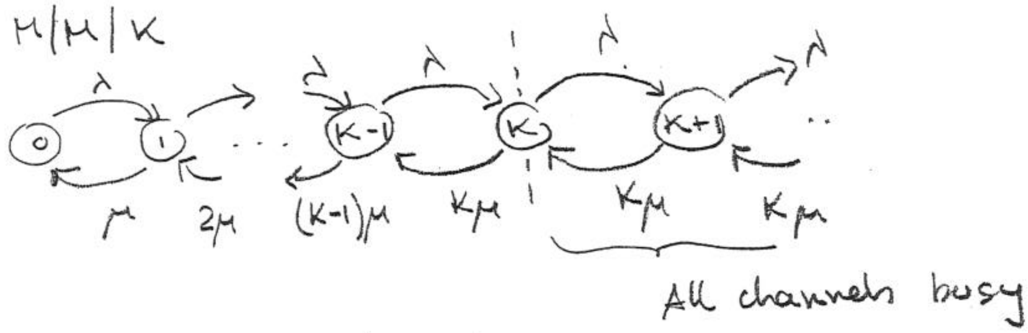
Second Examiner:

Q2

Question Number etc. in left margin

Mark allocation in right margin

i) M|M|K



5

- queue length distribution for delayed arrivals

$P[Q_t = i \mid \text{all server busy}]$

$$P[Q_t = i \mid N_t \geq K] = \left\{ \frac{P[N_t = K+i]}{\sum_{j=0}^{\infty} P[N_t = K+j]} \right\}$$

since $\Pi_{K+i} = \Pi_K \rho^i \Rightarrow \quad = \left\{ \frac{\Pi_K \rho^i}{\sum_{j=0}^{\infty} \Pi_K \rho^j} \right\}$

so $P[Q_t = i \mid \text{Delay}] = (1-\rho)\rho^i \qquad i = 0, 1, 2, \dots$ 5

(geometric distribution)

- unconditioned queue length Distribution

$$P[Q_t = i] = P[\text{delay}]\, P[Q_t = i \mid \text{delay}] + P[\text{No delay}]\, \underbrace{P[Q_t = i \mid \text{No delay}]}_{\substack{=1,\ i=0 \\ =0,\ i>0}}$$

$P[\text{Delay}] = D_K(A)$ so:

$$P[Q_t = i] = \begin{cases} [D_K(A)]\,(1-\rho)\rho^i & i > 0 \\ [1 - \rho D_K(A)] & i = 0 \end{cases}$$ 5

(bookwork extension) + derivate

$(\ D_K(A)\,(1-\rho)\rho^0 + (1 - D_K(A)) = 1 - \rho D_K(A)$

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Model Answers and Mark Schemes | First Examiner:

Paper Code: | Second Examiner: | 4

Q2

Question Number etc. in left margin | Mark allocation in right margin

iii)

$$E(Q_t) = \sum i \, P[Q_t = i] = \sum_{i=1}^{\infty} i(1-\rho)\rho^i D_K(A)$$

$$= D_K(A) \underbrace{\sum_{i=1}^{\infty} i(1-\rho)\rho^i}_{\frac{\rho}{1-\rho}}$$

$$= D_K(A) \frac{\rho}{1-\rho}$$

(Bookwork extension) + derivation 5

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Paper Code: | Second Examiner:

Q3

Question Number etc. in left margin | Mark allocation in right margin

a) This is a 1-D B/D process (both arrival streams have the same service time distribution)

- Birth coefficients $\lambda_i = (\lambda_1 + \lambda_2) \quad i \le K$

$= \lambda_1 \quad i > K$

- Death coefficients $\mu_i = i\mu \quad i \le K$

$= K\mu \quad i > K$

$(h = 1/\mu)$ 2

Equilibrium

$$\pi_i = \left[\frac{(\rho_1 + \rho_2)}{i}\right] \pi_{i-1} \quad i \le K$$

$$= \left[\frac{\rho_1}{K}\right] \pi_{i-1} \quad i > K$$ 3

$\rho_1 = \lambda_1 h$

$\rho_2 = \lambda_2 h$ and $\rho_1 / K < 1$

Recursive solution + normalisation gives

$$P[\text{congestion}] = \sum_K^{\infty} \pi_i = \left[\frac{E_K(\rho_1+\rho_2)}{1 - \frac{\rho_1}{K}\{1 - E_K(\rho_1+\rho_2)\}}\right]$$ 5

(bookwork extension + calculation new example)

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Paper Code: | Second Examiner: | 6

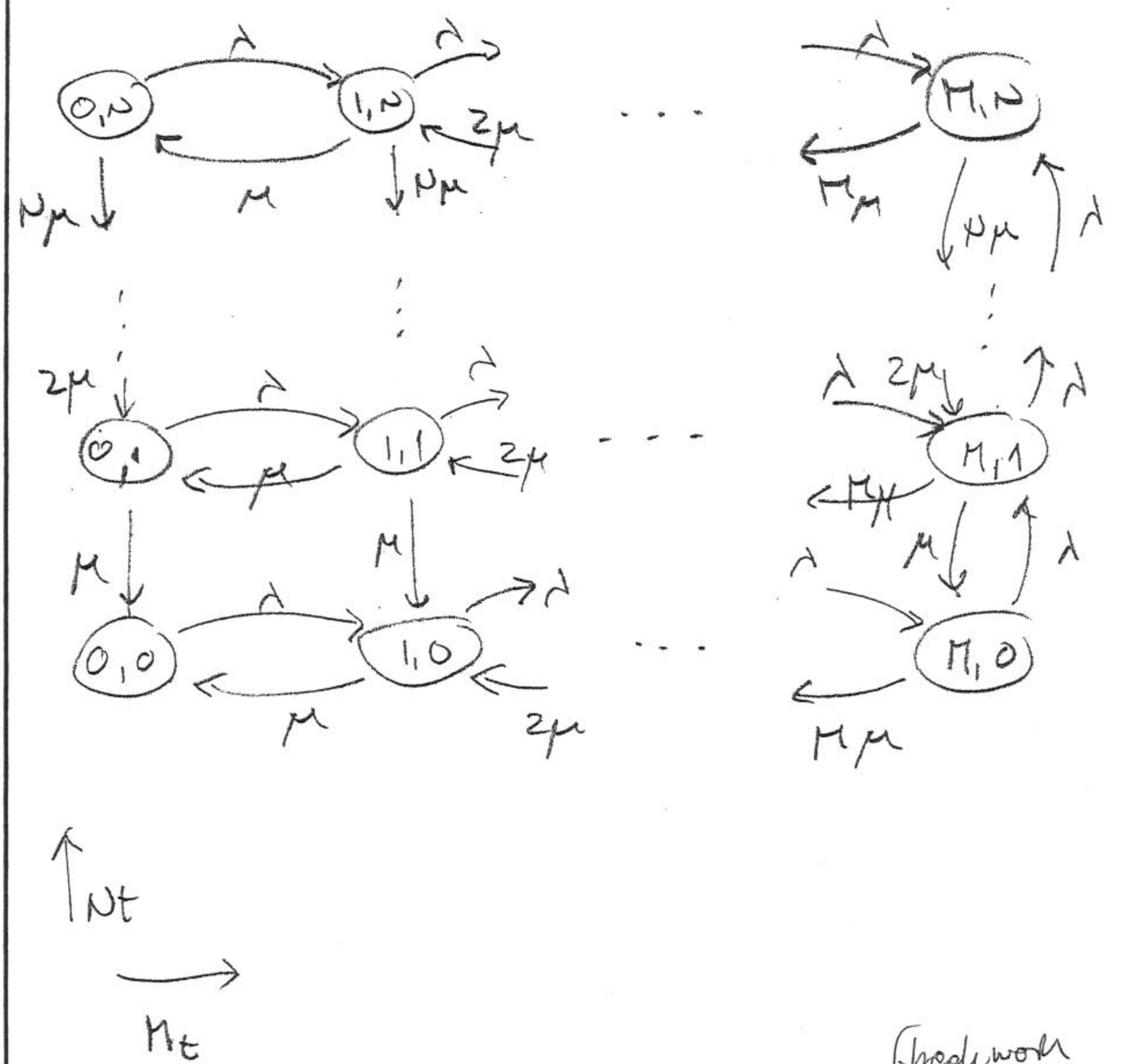
Q3

Question Number etc. in left margin | Mark allocation in right margin

b) - assume that the offered traffic is pure chance traffic with parameters (λ, μ)

- M_t = nr of busy channels on link ①
- N_t = nr of busy channels on link ②

Then $\{(M_t, N_t)\}$ is a 2-D birth/death process with state space $E = \{(i,j) : 0 \le i \le M,\ 0 \le j \le N\}$ 5



5

(bookwork extension)

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Model Answers and Mark Schemes — First Examiner:

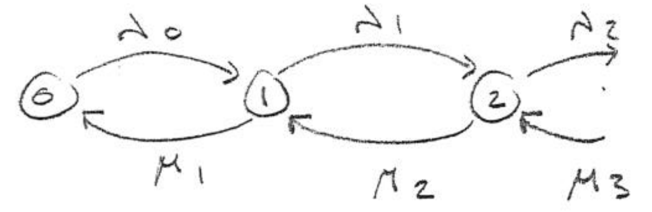
Paper Code: — Second Examiner:

7

Q4

Question Number etc. in left margin — Mark allocation in right margin

a) i)



2

ii)

$$\lambda_0\pi_0 = \mu_1\pi_1$$

$$\lambda_0\pi_0 + \mu_2\pi_2 = (\lambda_1+\mu_1)\pi_1$$

$$\lambda_1\pi_1 + \mu_3\pi_3 = (\lambda_2+\mu_2)\pi_2$$

$$\vdots$$

3

can be simplified $\mu_i\pi_i = \lambda_{i-1}\pi_{i-1} \quad i=1,2\ldots$

iii) Recursive solution from $i=0$ gives

$$\pi_i = \left(\frac{\lambda_{i-1}\lambda_{i-2}\cdots\lambda_0}{\mu_i\mu_{i-1}\cdots\mu_1}\right)\pi_0$$

and $\sum_i \pi_i = 1$

3

$$\pi_0 = \left[\frac{1}{1+\sum_{i\geq 1}\left(\frac{\lambda_{i-1}\cdots\lambda_0}{\mu_i\cdots\mu_1}\right)}\right] = \frac{1}{S}$$

iv) yes

local balance equation holds for $\{X_t\}$

2

$$\mu_i\pi_i = \lambda_{i-1}\pi_{i-1}$$

(bookwork extension)

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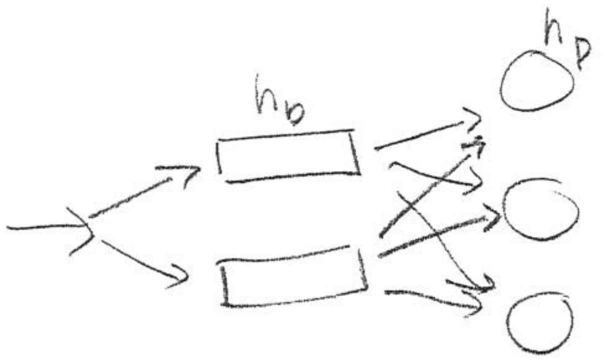
Model Answers and Mark Schemes First Examiner:

Paper Code: Second Examiner: 8

Q4

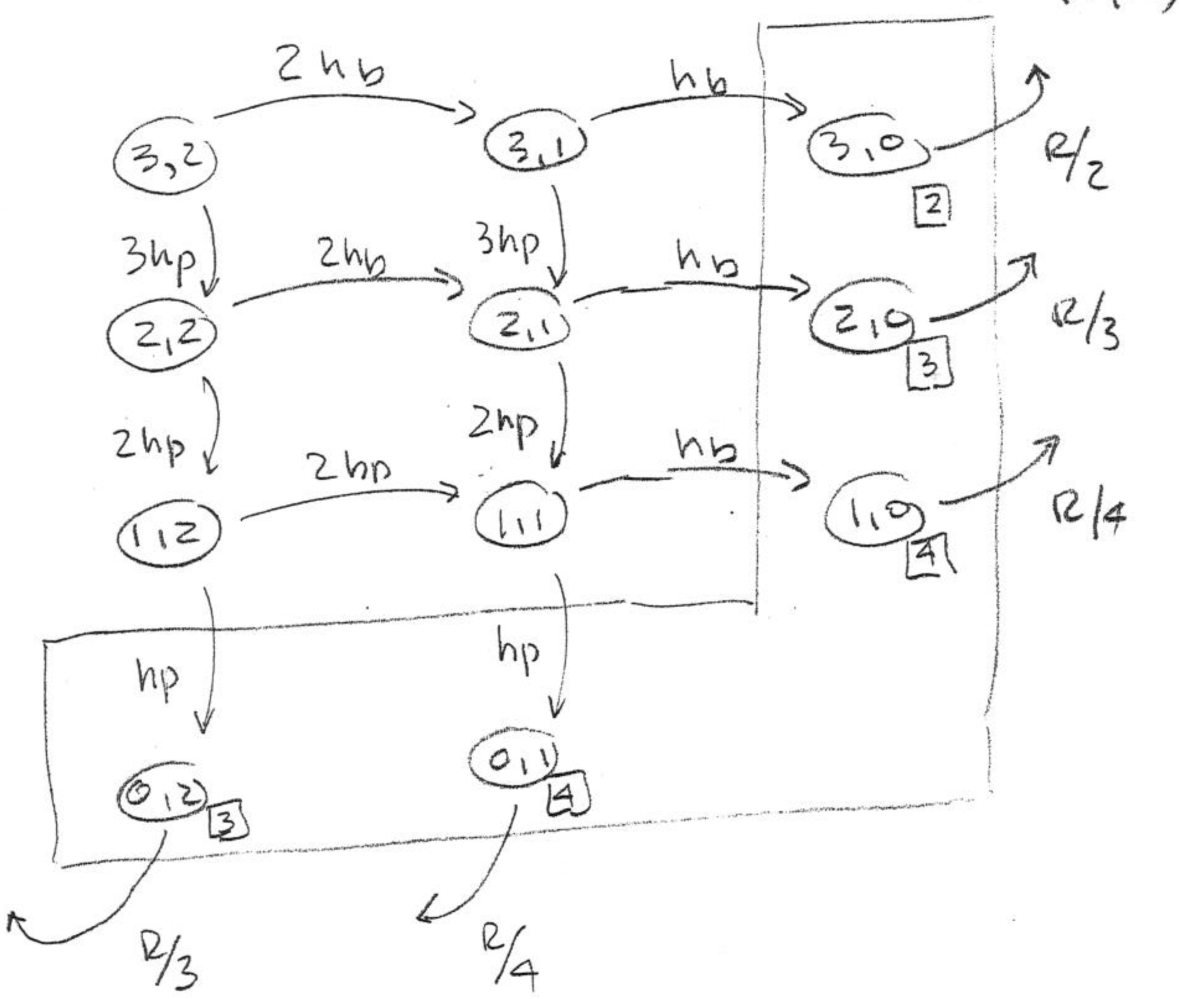
Question Number etc. in left margin — Mark allocation in right margin

4b



i) (no processors, no buffer) failure (x,0) or (0,x)

ii) 5



5

(calculate of new example)

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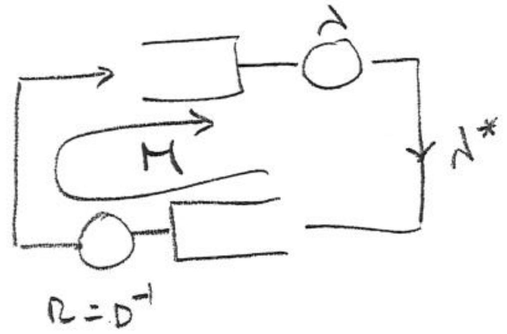
Paper Code: | Second Examiner:

9

Q5

Question Number etc. in left margin | Mark allocation in right margin

a) i)



An interpretation of a GRA involve the use of a "token pool" buffer. A cell must have a token waiting to be transmitted. Tokens are generated once per D seconds, and wait in the buffer until buffer fills. At this time no further token is generated. In this case the average throughput λ^* differs from the load λ because of possible cell loss 3

ii) $c\lambda$ is the system load 2

iii) $R = D^{-1}$ is the rate at which tokens are generated 2

iv) If the closed network has M tokens circulating at most M cells can thus be served in succession which represent the size of the "token pool" buffer 3

(bookwork extension)

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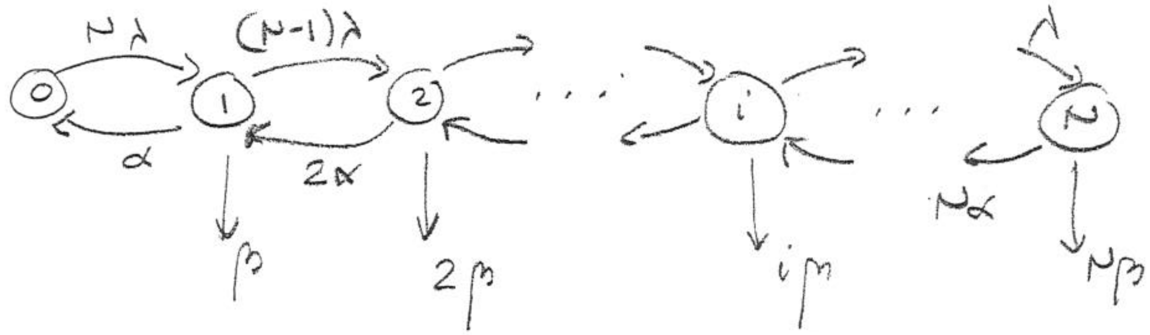
Paper Code: | Second Examiner:

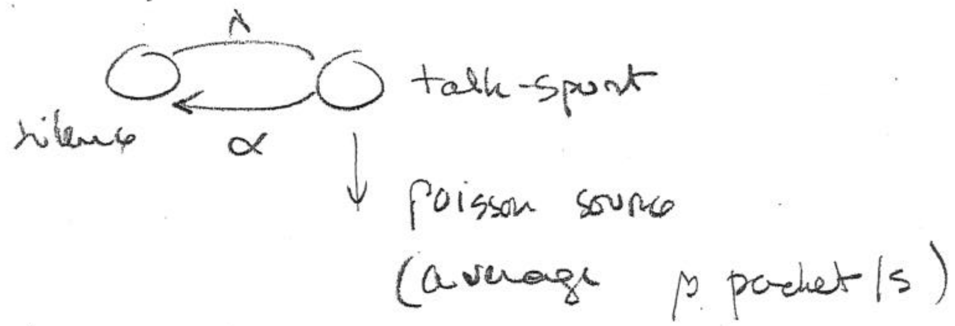
Q 5

Question Number etc. in left margin | Mark allocation in right margin

b) i)

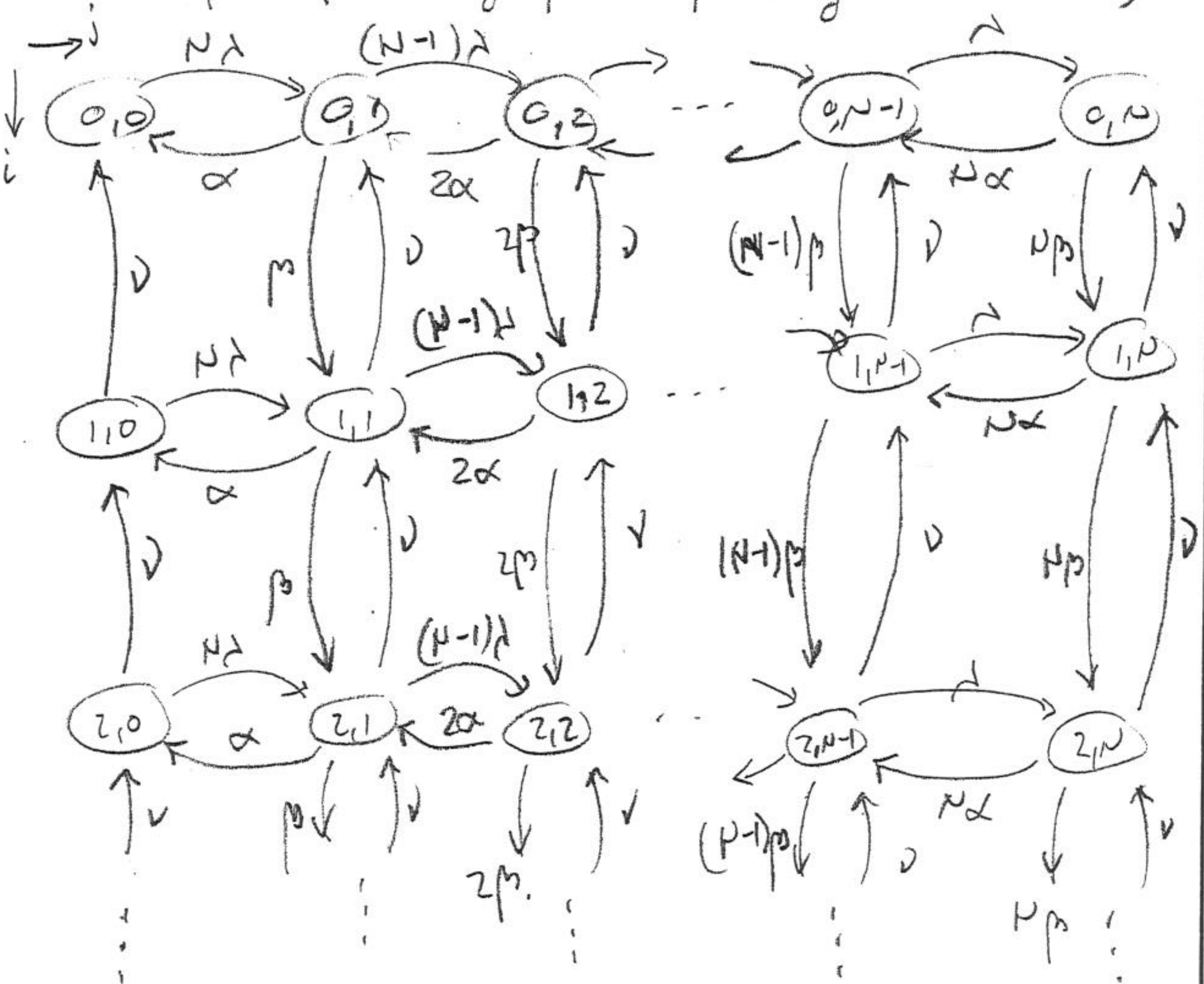


IPP single voice



3

ii) state space (state of queue, nr of sources on) 3



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Model Answers and Mark Schemes — First Examiner:

Paper Code: — Second Examiner:

11/11

Q6

Question Number etc. in left margin — Mark allocation in right margin

a) $N \gg 1$, $p \ll 1$ $\quad \pi_i = \binom{N}{i} p^i (1-p)^{N-i}$ can be approximated quite closely by the Normal distribution ($m = Np$, $\sigma^2 = Np(1-p)$) 6

b i)

$$\mathcal{E} = \int_{J_0}^{\infty} \frac{e^{-(x-m)^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}\, dx$$

if $(C-m) > 3\sqrt{2}\,\sigma$

$$\mathcal{E} = \frac{\sigma\, e^{-(C-m)^2/2\sigma^2}}{\sqrt{2\pi}\,(C-m)}$$

b. ii)

$$P_L = \frac{1}{m} \int_{J_0}^{\infty} \frac{e^{-(x-m)^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}} (x-C)\, dx$$ 7

if $(C-m) > 3\sqrt{2}\,\sigma$

$$P_L = \frac{1-p}{C-m}\, \mathcal{E}$$

(Derivates + book work explanation) 7