IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2018

MSc Control Systems

Corrected Copy.

GAME THEORY

Q1 A (b) 10:45am.

Wednesday, 2 May 10:00 am

Time allowed: 3:00 hours

There are FIVE questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : R.B. Vinter
Second Marker(s) : D. Angeli

Information for candidates:

No special instructions for candidates.

correction written on white board at start of exam but most students unaware. Professor made announcement at 10-45.

1(A): Consider non-zero sum game with two players (Player A and Player B), for which the pay-off matrices are:

$$S^A = \begin{array}{c|cc} A\backslash B & & \\ \hline & 1 & -1 \\ & -1 & +2 \end{array} \qquad \text{and} \qquad S^B = \begin{array}{c|cc} A\backslash B & & \\ \hline & 0 & 1 \\ & 1 & -2 \end{array} \; .$$

Player A chooses a row (or mixture of rows), and player B chooses a column (or mixture of columns) to minimize the pay-offs given by S^A and S^B respectively.

(a): Using graphical means, find a pair of mixed strategies $((1-\bar{\lambda}, \bar{\lambda}), (1-\bar{\mu}, \bar{\mu}))$ such that $(1-\bar{\lambda}, \bar{\lambda})$ is a mixed safety strategy for player A, and $(1-\bar{\mu}, \bar{\mu})$ is a mixed safety strategy for player B. Evaluate the pay-offs for this pair of strategies. [6]

Show that$((1-\bar{\lambda}, \bar{\lambda}), (1-\bar{\mu}, \bar{\mu}))$ is not a Nash Equilibrium. no bar over each letter [2]

(b): The above game has a Nash equilibrium $((1-\lambda', \lambda'), (1-\mu', \mu'))$ in mixed strategies, which is strictly mixed, in the sense

Professor announced at 10-45.

$$0 < \lambda' < 1 \text{ and } 0 < \mu' < 1 \, .$$

Determine $((1-\lambda', \lambda'), (1-\mu', \mu'))$ and evaluate the pay-offs. [6]

Comment on the values of the pay-offs, for the mixed safety strategies and the mixed Nash equilibrium strategies. [1]

1(B): Consider a zero sum game, involving two players, with strategy spaces X and Y, respectively, and pay-off

$$L(x, y) \, .$$

(Players A and B seek to minimize and maximize the pay-off respectively.)

Assume that there exists a Nash equilibrium $(\bar{x}, \bar{y})$.

Show that

$$\operatorname*{Max}_{y \in Y} \left(\operatorname*{Min}_{x \in X} L(x, y) \right) \le \operatorname*{Min}_{x \in X} \left(\operatorname*{Max}_{y \in Y} L(x, y) \right) \qquad (\text{'Weak Duality'}) \, .$$

[5]

2. Consider the multistage zero sum game shown in Fig. 1, involving players A and B, and also a third player N ('Nature'), who plays a random strategy known to players A and B.

A chooses L or R. B observes A's action and chooses L or R. N chooses L or R, with a probability that depends on A's action.

The probability rule governing Nature's action is

$$\textit{Scenario 1:} \quad N \text{ chooses} \begin{cases} L & \text{if } A \text{ chooses L} \\ R & \text{if } A \text{ chooses R} \end{cases} \quad \text{with probability } 1/2$$

and

$$\textit{Scenario 2:} \quad N \text{ chooses} \begin{cases} R & \text{if } A \text{ chooses L} \\ L & \text{if } A \text{ chooses R} \end{cases} \quad \text{with probability } 1/2$$

The pay-offs for players A and B, which each seek to minimize, are shown below the tree.

Construct two trees, involving only the players A and B, which take account of scenario 1 and scenario 2, respectively. [6]

Calculate the average pay-offs. [2]

Reduce the game to normal form for players A and B. [8]

Identify any Nash equilibria, and state whether they are admissible. [4]

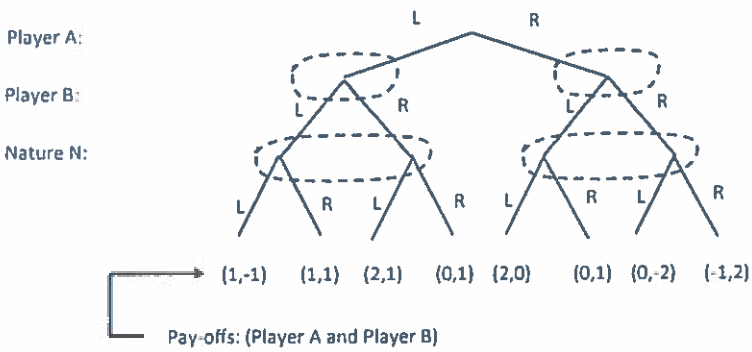

Figure 1

3: Mr. Big, the head of a criminal organization, is in a trial, the outcome of which is that he will either go to prison or not go to prison. Each of N jurors will vote 'guilty' or 'not guilty'. Mr. Big will go to prison if only 2, or fewer, jurors vote 'not guilty'.

Satisfaction Level	Scenario
1	Mr. Big goes to prison and the i'th juror votes 'not guilty'
1-c	Mr. Big goes to prison and the i'th juror votes 'guilty'
0	Mr. Big does not go to prison

(Here, c is a given number $0 < c < 1$.)

Interpretation: each juror believes that Mr. Big is guilty, and want him to go to prison as he is a danger to society. But, each juror also fears that Mr. Big's organization will find out how they voted and punish their family if the vote was 'guilty'. c is the reduction in satisfaction, if Mr. Big goes to prison, due to fear.

Consider an N-person game, in which each juror is regarded as a player. For each i, the i'th juror chooses a probability to vote 'not guilty'. The pay-off for each juror is the expected value of that juror's satisfaction level, when all jurors vote randomly and independently, according to their chosen probabilities. All jurors wish to maximize their pay-offs.

Suppose that each juror votes 'not guilty' with the same probability $\bar{\alpha}$, $0 < \bar{\alpha} < 1$. Show that this collection of strategies is a Nash equilibrium if

$$\big(p_{\bar{\alpha}}(0; N-1) + p_{\bar{\alpha}}(1; N-1) + p_{\bar{\alpha}}(2; N-1)\big) \times c \;=\; p_{\bar{\alpha}}(2; N-1)$$

Here [16]

$$p_\alpha(k; M) := \text{Probability } \{ \text{ exactly } k \text{ jurors will vote 'not guilty', when each of } M \text{ jurors votes 'not guilty', independently, w.p. } \alpha \} \;.$$

Using the information,

$$p_\alpha(0; M) = (1-\alpha)^M, p_\alpha(1; M) = M\alpha(1-\alpha)^{M-1} \text{ and } p_\alpha(2; M) = \frac{M(M-1)}{2}\alpha^2(1-\alpha)^{M-2},$$

show that the probability that Mr. Big goes to prison is 1, in the limit as $c \to 0$. [4]

4: Consider 2 companies A and B that sell identical products. The production levels of Companies A and B is a, $(a > 0)$, and b, $(b > 0)$, respectively. The price of the product $p(a, b)$ is the same for both companies, and is given by the demand curve

$$p(a, b) = 1 - (a + b)$$

(The price decreases with increased total production.) The pay-offs for company A and B, which they both seek to maximize, are

$$L^A(a, b) = a \times p(a, b) - a^2 \text{ and } L^B(a, b) = b \times p(a, b) - b^2$$

('profits from sales minus production costs'), respectively.

(a): Determine the Nash equilibrium $(\bar{a}, \bar{b})$ for the production levels of the two companies, and the pay-off. [5]

(b): Sketch the response curves for the two companies, and show that they are consistent with the Nash equilibrium calculated in part (a). [5]

(c): Take Company A as the leader. Determine the Steckelberg optimizing production level for Company A and the pay-off, and the corresponding production level for Company B. [5]

(d): Now assume that the production level of Company B is restricted by the constraint

$$0 < b \leq 1/8\,.$$

No constraint is placed on the production level of Company A.

Show how the response curves are modified, and determine the new Nash equilibrium and pay-off. [5]

5(A): Consider the optimal control problem:

$$\begin{cases} \text{Minimize } \sum_{t=0}^{N-1} u_t^2 + \lambda^T x_N \\ \text{over sequences } \{u_t\}_0^{N-1} \text{ and } \{x_t\}_{t=0}^{N} \text{ such that} \\ x_{t+1} = Ax_t + bu_t \quad \text{for } t = 0, \ldots, N-1 \\ x_0 \text{ a given } \cancel{\text{number}}, \text{vector} \end{cases}$$

in which A is a given $n \times n$ matrix, b is a given n-vector and λ is a given n-vector.

What is the Bellman equation for the value function $V_t(x)$, $t = 0, \ldots, N$? [3]

Assuming that the value function is of the form

$$V_t = x_t^T P_t x + 2r_t^T x \quad \text{for } t = 0, \ldots, N,$$

derive recursive equations for the symmetric matrices P_t and the n-vectors $r_t, t = 0, \ldots N$ in these relations. [8]

Hence show that the solution $\{x_t^\lambda, u_t^\lambda\}$ to the optimal control problem is expressible, in feedback form, by

$$\begin{cases} x_{t+1}^\lambda = Ax_t^\lambda + k^{\lambda T} u_t^\lambda \quad \text{for } t = 0, \ldots, N-1 \\ u_t^\lambda = -k_t^{\lambda\, T} x_t^\lambda + d_t^\lambda \\ x_0 \text{ a given number.} \end{cases}$$

and provide formulae for k_t^λ and d_t^λ, $t = 0, 1 \ldots, N-1$. [4]

5(B): Let $\bar{\lambda}$ be an n-vector such that the solution $\{x_t^{\bar{\lambda}}, u_t^{\bar{\lambda}}, \}\}$ to the optimal control satisfies, for given initial state x_0,

$$x_N^{\bar{\lambda}} = 0\,.$$

Show that $\{u_t^{\bar{\lambda}}\}$ is a control that transfers the given initial state from x_0 at time 0 to the final state 0 at time N, for which the control energy

$$\sum_{t=0}^{N-1} u_t^{2}$$

is minimized. [5]