IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2014-15

MSc and EEE/EIE PART III/IV: MEng, BEng and ACGI

Corrected Copy

MATHEMATICS FOR SIGNALS AND SYSTEMS

Wednesday, 10 December 9:00 am

Time allowed: 3:00 hours

There are THREE questions on this paper.

Answer ALL questions.   All questions carry equal marks.

NO Calculators allowed

Any special instructions for invigilators and information for
candidates are on page 1.

Examiners responsible   First Marker(s) :   P.T. Stathaki

Second Marker(s) :   D. Angeli

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1. a) Suppose A is a matrix of size $m \times n$ of rank r. Let $R = \text{rref}(A)$ be the row reduced echelon form of A.

   (i) If we assume that the first r columns of R are the pivot columns and R can be written as $R = \begin{bmatrix} B & C \\ D & F \end{bmatrix}$ where B, C, D, F are sub-matrices, give the form of all matrices B, C, D, F and their dimensions. Justify your answer. **[2]**

   (ii) Find a general form for the null space matrix N. This is defined as a matrix whose columns are the special solutions of the system $Ax = 0$. In that representation 0 is a column vector with zero elements. **[2]**

   (iii) Suppose you know that the 3×4 matrix A has the vector $\begin{bmatrix} 4 & 2 & 0 & 2 \end{bmatrix}^T$ as a basis for its null space. What is the row reduced echelon form R of A? **[2]**

   (iv) Show that a basis for the left null space of a rectangular matrix A of size $m \times n$ and rank r consists of the last $m - r$ rows of the corresponding elimination matrix E. **[2]**

   b) Suppose that the matrix A is the product
   $$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

   (i) Give a basis for the null space of A and a basis for the row space of A. **[2]**

   (ii) Express row 3 of A as a combination of your basis vectors in your answer to (i). **[2]**

   (iii) What is the dimension of the null space of A^T? Justify your answer. **[2]**

   c) Assume that the matrix A is invertible and permutations are not required in elimination.

   (i) Show that the product of pivots of A^{-1} is related to the product of pivots of A. **[2]**

   (ii) Apply block elimination to the $2n \times 2n$ block-matrix
   $$M = \begin{bmatrix} A & I \\ -I & 0 \end{bmatrix}$$

   The elements $A, I, 0$ are sub-matrices of size $n \times n$. The sub-matrix I is the identity matrix and the sub-matrix 0 is a matrix with zero elements. **[2]**

   (iii) Find the determinant of M by exchanging the first n columns of M with the last n columns and using properties of determinants. **[2]**

2. a) Consider the system of equations $Ax = b$, where A is a matrix of size $m \times n, m > n$ and the vector b does not belong to the column space of the matrix A. In an attempt to find an approximate solution to the above system we seek for $\hat{x}$ that satisfies the system $A\hat{x} = p$ where p is the projection of b onto the column space of A.

   (i) Give the matrix P that projects b onto the column space of A for the general case where A is a matrix of size $m \times n, m > n$ and $A^T A$ is invertible. Give the matrix P for the specific case where A is a square invertible matrix. [2]

   (ii) Show that P is symmetric and furthermore, that $P^2 = P$. [2]

   (iii) Show that the projection of b onto to the column space of A is zero if b is perpendicular to the column space of A. [1]

   (iv) Give the error of the approximation and show that the error belongs to the null space of A^T. [1]

   b) Consider the following matrix:
   $$P = \frac{1}{14} \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -3 & -6 & 9 \end{bmatrix}$$

   (i) Is P a projection matrix? Justify your answer. [2]

   (ii) What subspace does P project onto? [1]

   (iii) What is the distance from that subspace to $b = \begin{bmatrix} -1 & 1 & 1 \end{bmatrix}^T$? [1]

   (iv) What are the eigenvalues of P ? [2]

   (v) Is P diagonalizable? By observing the structure of P give the type of decomposition that P is amenable to. [2]

   c) We are seeking to fit the 5 two-dimensional points $(-2,0), (-1,0), (0,1), (1,1), (2,1)$ onto a straight line.

   (i) Give the system of equations that we must solve in order to achieve the above requirement. Explain why the system doesn't have a solution. [2]

   (ii) Find an approximate solution of the system using the least squares approach and give the equation for the required straight line. [2]

   (iii) Calculate the magnitude of the error of the approximation. [2]

3. a) Consider the matrix A:

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

(i)   Find the null space of A.   [2]

(ii) Consider the matrix

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Calculate its eigenvalues.   [2]

(iii) Find the relationship between the matrix $A^T A$ and B and calculate the eigenvalues of the matrix $A^T A$.   [2]

(iv) For the Singular Value Decomposition of matrix A, $A = U\Sigma V^T$, find the nonzero entries in the diagonal matrix Σ and one column of the matrix V.   [2]

b) Consider a square matrix Z that has orthonormal columns.

(i)   Show that Z satisfies $Z^T Z = I$.   [2]

(ii) Show that $\|Zx\|^2 = \|x\|^2$ where x is any vector.   [2]

c) Find the QR decomposition of the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

[8]