DIGITAL SIGNAL PROCESSING

1. a) Figure 1.1 shows a digital signal processing system containing 4 subsystems, A, B, C and D with impulse responses as shown in Table 1.

Subsystem	Impulse response
A	$h_a(n)$
B	$h_b(n)$
C	$h_c(n)$
D	$h_d(n)$

Table 1

Write an expression for $h(n)$, where $h(n)$ is the impulse response of the complete LTI system with input $x(n)$ and output $y(n)$. [2]

Determine $h(n)$ for the following definitions of $h_a(n)$, $h_b(n)$, $h_c(n)$, $h_d(n)$. [3]

$$h_a(n) = \{0.5, 0.3, 0.5\}$$
$$h_b(n) = (n+1)u(n)$$
$$h_c(n) = h_b(n)$$
$$h_d(n) = \delta(n-1).$$

Determine the output of the LTI system, $y(n)$, when the input is

$$x(n) = \delta(n+2) + 3\delta(n-1) - 4\delta(n-3).$$

[4]

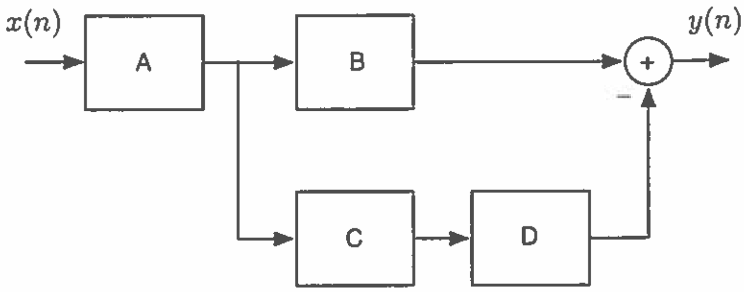


Figure 1.1

Solution: Temporarily omitting the dependence on n for brevity,

$$\begin{aligned}
h &= h_a * [h_b - h_c * h_d] \\
h_c * h_d &= (n+1)u(n) * \delta(n-1) \\
&= nu(n-1) \\
h_b - h_c * h_d &= (n+1)u(n) - nu(n-1) = u(n) \\
h &= h_a * u(n) \\
&= [0.5\delta(n) + 0.3\delta(n-1) + 0.5\delta(n-2)] * u(n) \\
&= 0.5u(n) + 0.3u(n-1) + 0.5u(n-2).
\end{aligned}$$

n	x	h	y
-3	0	0	0
-2	1	0	0.5
-1	0	0	0.8
0	0	0.5	1.3
1	3	0.8	$0.5 \times 3 + 1.3 = 2.8$
2	0	1.3	$0.8 \times 3 + 1.3 = 3.7$
3	-4	1.3	$-4 \times 0.5 + 3 \times 1.3 + 1.3 = 3.2$
4	0	1.3	2
5	0	1.3	0
6	0	1.3	0
⋮	⋮	⋮	⋮

b) An LTI system, with input signal $x(n)$ and output signal $y(n)$ is characterized by the difference equation

$$y(n) = 0.7y(n-1) - 0.1y(n-2) + 2x(n) - x(n-2).$$

Determine the impulse response and the unit step response of this system.
[6]

Solution: Writing

$$y(n) - 0.7y(n-1) + 0.1y(n-2) = 2x(n) - x(n-2)$$

we obtain the characteristic equation

$$\lambda^2 - 0.7\lambda + 0.1 = 0$$

for which $\lambda = 0.5, 0.2$. so that the general solutions is

$$y(n) = a_1(0.5)^n + a_2(0.2)^n.$$

For the impulse response $x(n) = \delta(n)$ so that

$$\begin{aligned} y(0) &= 2 \\ y(1) - 0.7y(0) &= 0, \Rightarrow y(1) = 1.4 \\ a_1 + a_2 &= 2 \\ 0.5a_1 + 0.2 &= 7/5 \\ \Rightarrow a_1 + 2/5a_2 &= 14/5. \end{aligned}$$

So that finally

$$\begin{aligned} a_1 &= 10/3 \\ a_2 &= -4/3 \end{aligned}$$

and

$$h(n) = [10/3 \times (0.5)^n - 4/3 \times (0.2)^n]u(n).$$

For the step response $s(n)$, we can consider this as the sum of impulse responses with increasing delays so that

$$\begin{aligned} s(n) &= \sum_{k=0}^{n} h(n-k) \\ &= (10/3) \sum_{k=0}^{n} (0.5)^{n-k} - (4/3) \sum_{k=0}^{n} (0.2)^{n-k} \\ &= (10/3)(0.5)^n \sum_{k=0}^{n} 2^k - (4/3)(0.2)^n \sum_{k=0}^{n} 5^k \\ &= (10/3)(0.5)^n(2^{n+1} - 1)u(n) - (4/3)(0.2)^n(5^{n+1} - 1)u(n). \end{aligned}$$

c) Consider two classes of discrete-time systems: recursive and nonrecursive. State the important differences between these two classes of systems. [1]

Derive the difference equation of a recursive system for which the output $y(n)$ is the cumulative average of a signal $x(n)$. Draw the signal flow diagram of this system. [4]

Solution: Recursive systems use feedback from output to input such that the difference equation includes past outputs. This has implications for storage of past output values in the system.

$$
\begin{aligned}
y(n) &= \frac{1}{n+1}\sum_{k=0}^{n} x(k) \\
(n+1)y(n) &= \sum_{k=0}^{n-1} x(k) + x(n) \\
y(n) &= \frac{1}{n+1}\left(ny(n-1) + x(n)\right).
\end{aligned}
$$

2. a) i) Given an M-fold decimator with input $x(n)$ and output $y_D(n)$, write down the expression for $y_D(n)$ in terms of $x(n)$. [2]

ii) Given an L-fold expander with input $x(n)$ and output $y_E(n)$, write down the expression for $y_E(n)$ in terms of $x(n)$. [2]

Solution:

$$y_D(n) = x(Mn)$$

$$y_E(n) = \begin{cases} x(n/L) & \text{if n is an integer multiple of L} \\ 0 & \text{otherwise.} \end{cases}$$

b) Next consider the multirate signal processing system shown in Fig. 2.1.

i) Write down $Y(z)$ in terms of $X(z)$ for this system. [4]

ii) In Fig. 2.2, the magnitude spectrum of $X(z)$ is shown for the frequency range $-\pi \leq \omega \leq \pi$. Now consider the case when the LTI system $H(z)$ is an ideal lowpass filter with normalized cut-off frequency $\frac{\pi}{2}$.

Draw a labelled sketch of the magnitude spectrum of $Y(z)$ covering the range of frequencies $-2\pi \leq \omega \leq 4\pi$. Explain the main features of this magnitude spectrum sketch.

Does aliasing occur? Explain your answer. [4]

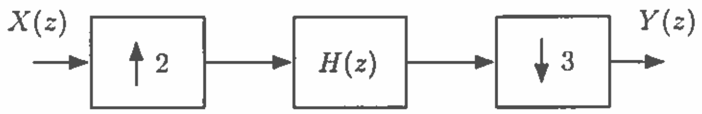


Figure 2.1

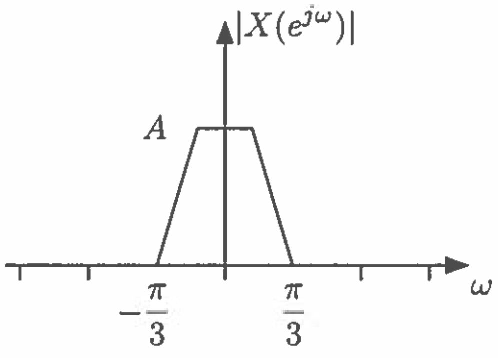
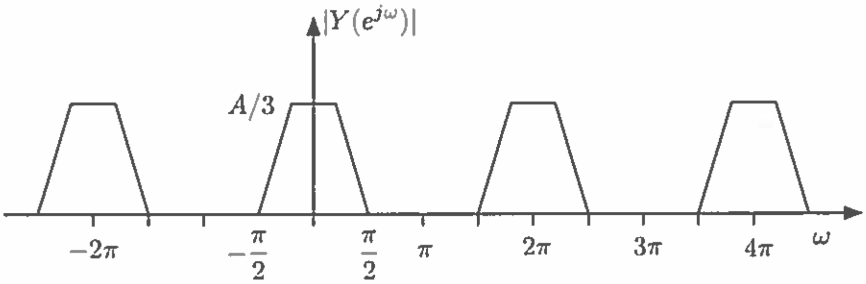


Figure 2.2

Solution:

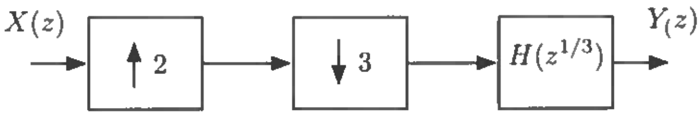
$$Y(z) = \frac{1}{3}\sum_{k=0}^{2} X(z^{2/3}W^{2k})H(z^{1/3}W^{k}) \quad W = e^{-j2\pi/3}$$

Main features of the sketch are the stretching of the original spectrum and the inclusion of the additional spectral images. Aliasing does not occur because of the correct ordering of expansion and decimation so that the Nyquist sampling criterion is always satisfied. The filter $H(z)$ ensures this.



c) State the Noble identities relating to multirate signal processing. Hence, redraw the system in Fig. 2.1 such that the computation of $H(z)$ is performed at the lowest possible sampling frequency. [4]

Solution:



d) i) Write down an expression for the 3-phase Type 1 polyphase decomposition of a filter $H(z)$.

ii) Redraw Fig. 2.1 using the 3-phase Type 1 polyphase decomposition.

iii) Redraw your solution to part c) using 3-phase Type 1 polyphase filtering.

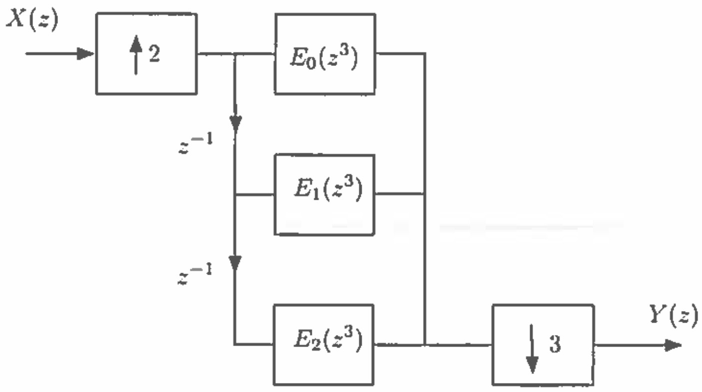
iv) State the relative merits of the structures resulting from (ii) and (iii).

[4]

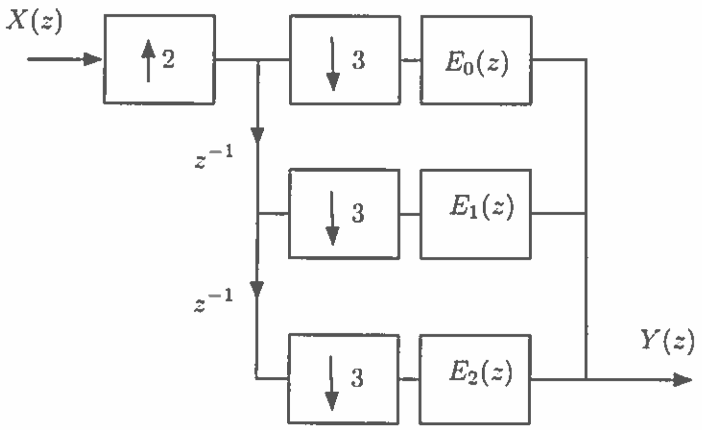
Solution:
Type 1 polyphase decomposition:

$$H(z) = \sum_{l=0}^{M-1} z^{-l} E_l(z^M) \quad E_l(z) = \sum_{n=-\infty}^{\infty} e_l(n) z^{-n} \quad e_l(n) = h(Mn+l) \quad M = 3.$$

Block diagram of direct implementation using Type 1 polyphase.



Block diagram of efficient implementation using Type 1 polyphase.



The polyphase solution in (iii) is more computationally efficient. Credit will be given if the computational reduction is quantified.

3. a) Using a few sentences and one or more relevant diagrams, describe the similarities and differences between the continuous-time Fourier transform, the DTFT and the DFT? [6]

> *Solution:* Marks are award for (i) clarity on continuous-time vs discrete-time and the corresponding representation in the s and z planes, (ii) clarity on continuous-frequency vs discrete-frequency for DTFT and DFT. Example sketches of the s-plane, z-plane and an illustration of the effect in the DFT of sampling the DTFT at uniformly spaced frequencies around the unit circle.

b) i) With discrete-time index n, consider a signal $x(n)$ for $n = 0, 1, \ldots, 5$. State and explain a sufficient condition on $x(n)$ such that its Fourier transform $X(e^{j\omega})$ is real valued. [2]

ii) Find $x(n)$ for $X(e^{j\omega}) = \{6, -5, 3, 4, 3, -5\}$. [5]

> *Solution:*
> A discussion of symmetry in the time domain is expected. Additional clarity should be introduced either using a general formulation or a simple numerical example.
> $x = [1, -1, 2, 3, 2, -1]$

c) A continuous-time analogue signal containing an audio recording of bird song has a bandwidth of 4 kHz. It is required to compute the spectrum of this signal using an N-point DFT with $N = 2^m$, for integer m, and a frequency resolution $\Delta f \leq 50$ Hz.

i) What is the minimum sampling frequency required?

ii) What is the minimum number of samples required?

iii) What is the minimum duration of the analogue signal required?

[3]

> *Solution:*
> $F_s = \frac{1}{T_s} = 8000$ Hz.
> $\frac{1}{N_s T_s} \leq 50 \Rightarrow N_s \geq \frac{8000}{50} = 160$ samples.
> Total time of sampling $= 160 * \frac{1}{8000} = 0.02$ s.

d) By exploiting the DFT, prove that

$$\sum_{l=-\infty}^{\infty} \delta(n + lN) = \frac{1}{N} \sum_{k=0}^{N-1} e^{j2\pi kn/N}.$$

[4]

Solution:
Note that the LHS $= 1$ for $n =$ (any integer multiple of N) and 0 otherwise. This is a periodic time-domain signal. Denoting the LHS to be $x(n)$ and taking the DFT of $x(n)$ over the range $n = 0 \ldots N-1$ gives

$$\begin{aligned} X(k) &= \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \\ &= 1, \quad \forall k \text{ in } k = 0, \ldots, N-1. \end{aligned}$$

Next writing the inverse DFT

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi kn/N}$$

and substituting $X(k) = 1$ leads to

$$x(n) = \sum_{l=-\infty}^{\infty} \delta(n + lN) = \frac{1}{N} \sum_{k=0}^{N-1} e^{j2\pi kn/N}.$$

4. a) Consider a general discrete-time system with system function

$$H(z) = \frac{Y(z)}{X(z)}.$$

With reference to the polynomials $X(z)$ and $Y(z)$, explain the meaning of, and define criteria for, the underlined terms in the following statements.

i) The system $H(z)$ is an <u>unstable</u> system.

ii) The system $H(z)$ is a <u>non-causal</u> system.

iii) The system $H(z)$ is a <u>maximum phase</u> system.

[3]

Solution:
Bookwork

b) Consider a linear time-invariant system

$$H(z) = \frac{(z-\frac{1}{3})(z+2)(z^2+\frac{1}{9})}{(z^2+2z+5)(z^2-4z+13)}.$$

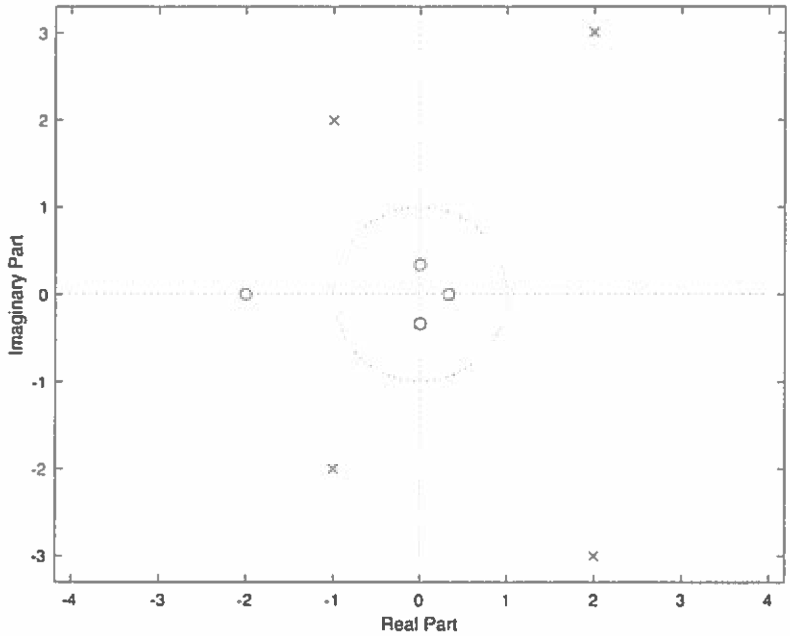
i) Find the roots of $H(z)$ and draw a labelled sketch showing the roots on the z-plane. [6]

ii) Determine any regions of convergence of $H(z)$ and state whether the inverse z-transform associated with each region of convergence is left-sided, right-sided or two-sided. Comment on the stability of $H(z)$. [5]

Solution:

$$H(z) = \frac{(z-1/3)(z+2)(z^2+1/9)}{(z^2+2z+5)(z^2-4z+13)}.$$

has zeros at $z = 1/3, \quad -2, \quad \pm j/3$
and poles at $z = -1 \pm j2, \quad 2 \pm j3$



The moduli of the pole pairs are 2.24 and 3.61. Hence the possible ROCs are

$$\begin{array}{ll} \text{(i)} & |z| < 2.24 \\ \text{(ii)} & 2.24 < |z| < 3.61 \\ \text{(iii)} & |z| > 3.61 \end{array}$$

For (i), the ROC is interior to all poles, hence inverse z-transform is left-sided. For (iii), the ROC is exterior to all poles, hence inverse z-transform is right-sided. For (ii), the ROC is bounded by the two complex pole-pairs, hence inverse z-transform is two-sided.
For stability, the ROC must included the unit circle, hence only 1 is stable.

c) Given the signal representation in the z-domain

$$X(z) = z^3 + 2z^2 - 2 + \frac{2}{z^2+5z+4}, \quad |z| > 4$$

find the corresponding signal representation in the time domain using the inverse z-transform. [6]

Solution:

$$
\begin{aligned}
X(z) &= z^3 + 2z^2 - 2 + \frac{2}{z^2 + 5z + 4} \\
&= z^3 + 2z^2 - 2 + \frac{2/3}{z+1} - \frac{2/3}{z+4} \\
x(n) &= \delta(n+3) + 2\delta(n+2) - 2\delta(n) + \frac{2}{3}(-1)^{n-1}u(n-1) - \frac{2}{3}(-4)^{n-1}u(n-1).
\end{aligned}
$$