Digital Signal Processing

1. Consider an FIR filter of order $M-1$.

a) Write down the z-domain system function for this filter and the expression for the frequency response of this filter. [2]

Solution:
The number of coefficients is M so that

$$H(z) = \sum_{k=0}^{M-1} h_k z^{-k}.$$

For frequency response, write $z = e^{j\omega}$ so that

$$H(e^{j\omega}) = \sum_{k=0}^{M-1} h_k e^{-j\omega k}.$$

b) Explain the key properties of a linear phase FIR filter and describe the corresponding characteristics of the filter coefficients and the roots of the system function. [4]

Solution:
In a linear phase filter, the phase response varies linearly with frequency. As a consequence
- all frequency components are subject to the same time shift,
- the filter coefficients satisfy $h(n) = h(N-1-n)$,
- the zeros in the z-domain, being the roots of the transfer function, occur in mirror image pairs such that if a zero exists at $z0$ then another zero also exists at $1/z0$.

c) Show that the frequency response of a linear phase FIR filter with M being odd can be written

$$H(e^{j\omega}) = e^{j\lambda}\left\{h\left(\frac{M-1}{2}\right) + \sum_{n=0}^{\alpha} 2h(n)\cos\left(\omega(n-\beta)\right)\right\}$$

and give expressions for λ, α and β. [6]

Solution:
From the system function $H(z) = \sum_{n=0}^{M-1} h(n)z^{-n}$, for M odd and exploiting symmetry we can write

$$H(z) = \sum_{n=0}^{((M-1)/2)-1} h(n)\left[z^{-n} + z^{-(M-1-n)}\right] + h\left(\frac{M-1}{2}\right) z^{-(M-1)/2}.$$

To obtain the frequency response, we replace z with $e^{j\omega}$ to give

$$H(e^{j\omega}) = e^{-j\omega(M-1)/2}\left\{h\left(\frac{M-1}{2}\right) + \sum_{n=0}^{(M-3)/2} 2h(n)\cos\left(\omega\left(n - \frac{M-1}{2}\right)\right)\right\}.$$

Thus $\lambda = -\omega(M-1)/2$, $\alpha = (M-3)/2$ and $\beta = (M-1)/2$.

d) Explain the meaning of group delay and state an expression for the group delay of a linear phase FIR filter. [3]

Solution:
The group delay describes the time delay introduced into signals passing through a system. Group delay is the negative derivative of phase with frequency: $-\frac{d\phi}{d\omega}$, measured in units of time (s). From the phase term in the frequency response expression above, the group delay is given by $(M-1)/2$,

e) Find the coefficients of a linear phase FIR filter with $M = 5$ satisfying the specification

$$|H(e^{j0})| = 0 \quad \text{dB}$$
$$|H(e^{j\frac{\pi}{3}})| = -6 \quad \text{dB}$$
$$|H(e^{j\pi}) = -\infty \quad \text{dB}.$$

[5]

Solution:
From the expression for the frequency response and ignoring the phase term $e^{-j\omega(M-1)/2}$ we can write:

$$\omega = 0 : h_2 + 2h_0 + 2h_1 = 1$$
$$\omega = \frac{\pi}{3} : h_2 - h_0 + h_1 = 0.5$$
$$\omega = \pi : h_2 + 2h_0 - 2h_1 = 0.$$

Solving simultaneously gives:

$$h_1 = \frac{1}{4}$$
$$h_0 = \frac{1}{4} - \frac{0.5}{3} = \frac{1}{12}$$
$$h_2 = \frac{1}{3}.$$

Lastly the set of coefficients is constructed:

$$b = [h_0, h_1, h_2, h_1, h_0].$$

2. a) Give the definition of the z-transform $X(z)$ of a discrete-time signal $x(n)$. State the meaning of the *region of convergence* of the z-transform. [3]

> The z-transform is expressed as an infinite series. The ROI defines the region of the z-plane for which this infinite series converges.
>
> $$X(z) = \sum_{-\infty}^{\infty} x(n) z^{-n}.$$

b) Write the two-sided signal $x(n) = a^n$ as the sum of two one-side functions. By considering the z-transform of each one-sided function, show that the z-transform of $x(n) = a^n$ has no region of convergence. [4]

> The first term converges for $|z| < |a|$ and the second term converges for $|z| > |a|$. Hence it can be seen that the two ROCs have no common regions.

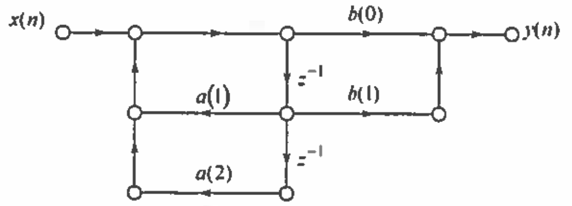
c) Consider the linear system

$$H(z) = \frac{1 - 1.2z^{-1}}{1 + 0.4z^{-1} - 0.12z^{-2}}.$$

Calculate the first 5 non-zero sample amplitudes of the impulse response of the system by finding the inverse z-transform of $H(z)$ using long division. [7]

> Using long division we obtain the first 5 samples of the impulse response as:
> [1.0, -1.6, 0.76, -0.496, 0.2896].

d) Construct the signal flow graph for $H(z)$ using the minimum number of delay elements. [6]

> The signal flow graph can be draw as
>
> 
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>
> with $a(1) = 0.4$, $a(2) = -0.12$, $b(0) = 1$, $b(1) = -1.2$.

3. a) Let $x(n)$ be a discrete time signal and let $X(k)$ be the DFT of $x(n)$. Consider $\hat{x}(n)$ to be the inverse DFT of $X(k)$. By considering the formulae for the forward and inverse DFT, prove that

$$\hat{x}(n) = x(n).$$

[4]

$$\begin{aligned}
X(k) &= \sum_{n=0}^{N-1} x(n)e^{-j2\pi nk/N} \\
\hat{x}(n) &= \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j2\pi nk/N} \\
&= \frac{1}{N}\sum_{k=0}^{N-1}\left[\sum_{m=0}^{N-1} x(m)e^{-j2\pi mk/N}\right]e^{j2\pi nk/N} \\
&= \frac{1}{N}\sum_{m=0}^{N-1} x(m)\left[\sum_{k=0}^{N-1} e^{j2\pi(n-m)k/N}\right] \\
&= \frac{1}{N}\sum_{m=0}^{N-1} x(m)N\delta\left[(n-m)_{\text{mod } N}\right] \\
&= x(n).
\end{aligned}$$

Equivalently

$$\begin{aligned}
\hat{x}(n) &= \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j2\pi nk/N} \\
&= \frac{1}{N}\sum_{k=0}^{N-1}\sum_{p=0}^{N-1} x(p)e^{-j2\pi pk/N}e^{j2\pi nk/N} \\
&= \frac{1}{N}\sum_{k=0}^{N-1}\sum_{p=0}^{N-1} x(p)e^{j2\pi(n-p)k/N} \\
&= x(n)
\end{aligned}$$

since

$$\sum_{k=0}^{N-1}\sum_{p=0}^{N-1} x(p)e^{j2\pi(n-p)k/N} = \begin{cases} 0, & n \neq p \\ Nx(p), & n = p \end{cases}.$$

b) Consider two finite duration discrete time signals, $p(n)$ and $q(n)$, given by

$$\begin{aligned}
p(n) &= [2, -1, 1, 0] \\
q(n) &= [-3, -1, 0, -2].
\end{aligned}$$

i) Give the formula for circular convolution of $p(n) \circledast q(n)$ and hence compute the sample values of

$$r(n) = p(n) \circledast q(n)$$

where $\circledast$ represents circular convolution. Show and explain your working. [6]

Circular convolution is formulated using

$$r(n) = \sum_{m=0}^{N-1} p(m)q(n-m)_{\text{mod}N} \quad n = 0, 1, \ldots, N-1.$$

The computation can be written

$$\begin{pmatrix} 2 & 0 & 1 & -1 \\ -1 & 2 & 0 & 1 \\ 1 & -1 & 2 & 0 \\ 0 & 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} -3 \\ -1 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} -4 \\ -1 \\ -2 \\ -5 \end{pmatrix}$$

ii) Given that $R(k)$, $P(k)$ and $Q(k)$ are the DFTs of $r(n)$, $p(n)$ and $q(n)$ respectively, prove that $R(k) = P(k)Q(k)$. [4]

Using

$$r(n) = \sum_{m=0}^{N-1} p(m)q(n-m)_{\text{mod } N}$$

taking DFTs and using the shift property we obtain

$$\begin{aligned} R(k) &= \sum_{m=0}^{N-1} p(m)Q(k)W_N^{-mk} \\ &= Q(k)\sum_{m=0}^{N-1} p(m)W_N^{-mk} \\ &= Q(k)P(k) \end{aligned}$$

where $W_N = e^{j2\pi/N}$

c) Briefly explain how linear convolution can be implemented using circular convolution. [4]

Linear convolution can be implemented by performing the circular convolution on zero-extended versions of the two signals.
Given signals $x_1(n)$ of length N_1 and $x_2(n)$ of length N_2, the signal $x_1(n)$ is zero extended using $N_2 - 1$ zeros to form $x_1'(n)$ and $x_2(n)$ is zero extended using $N_1 - 1$ zeros to form $x_2'(n)$. Linear convolution can then be obtained from the circular convolution of

$$x_1'(n) \circledast x_2'(n).$$

d) Briefly explain how circular cross-correlation can be implemented using circular convolution. [2]

Circular cross-correlation can be implemented using circular convolution by first time-reversing one of the two signals.

$$r_{pq}(l) = p(l) \circledast q(-l)$$

for circular index l and for the example of the signals p and q above.
A brief explanation should include the time reversal of one of the input sequences followed by circulation convolution. Matlab code is also very compact:

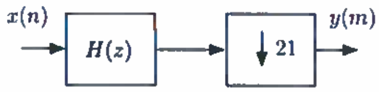
```
cconv(p,conj(fliplr(q)),7);
```

4. All filters in this question can be assumed ideal.

Consider a signal with sampling frequency $fs = 336$ kHz applied at the input to a downsampling process employing decimation and ideal filtering. The downsampling factor is 21.

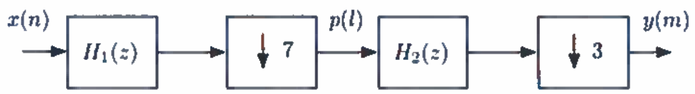
a)

i) Draw a labelled block diagram of a DSP system that performs the downsampling process given an input signal $x(n)$ and an output signal $y(m)$. Write an expression for $y(m)$ in terms of $x(n)$. [2]



$$y(m) = x(21n)$$

ii) Draw a labelled block diagram of a DSP system that performs the same operation as in (i) but employing two downsampling processes in cascade. [2]



iii) For the DSP system in (ii), give the sampling frequency of all signals in the block diagram. Also state the passband edge frequencies of filters employed. [3]

Whereas in the input signal sampling frequency is 336 kHz, after downsampling by 7 the sample frequency of the signal $p(l)$ is 48 kHz and after a further downsampling by a factor of 3 the sampling frequency of $y(m)$ is 16 kHz.
The required filters are lowpass and, in the case that they are ideal filters as stated, have band edges at 24 kHz and 8 kHz respectively.

b) Consider the system of Fig. 4.1. The input and output signals are denoted $x(n)$ and $y(m)$ respectively and two intermediate signals are shown as $p(l)$ and $q(l)$.

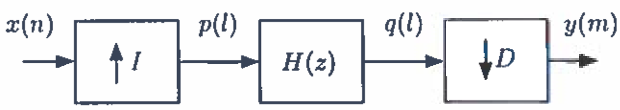


Figure 4.1

i) Write expressions for $p(l)$ and $q(l)$ and $y(m)$. [4]

$$
\begin{aligned}
p(l) &= \begin{cases} x(l/I), & l = 0, \pm I, \pm 2I, \dots \\ 0, & \text{otherwise.} \end{cases} \\
q(l) &= \sum_{k=-\infty}^{\infty} h(l-k)p(k) \\
&= \sum_{k=-\infty}^{\infty} h(l-kI)x(k) \\
y(m) &= q(mD) \\
&= \sum_{k=-\infty}^{\infty} h(mD-kI)x(k)
\end{aligned}
$$

ii) Find an expression in the frequency domain for the output signal in terms of the input signal. [4]

$$
Y(e^{j\omega}) = \begin{cases} \frac{I}{D}X(e^{j\omega/D}) & 0 \le |\omega| \le \min\left(\pi, \frac{\pi D}{I}\right) \\ 0, & \text{otherwise.} \end{cases}
$$

c) Given an appropriate lowpass filter $H(z)$ with impulse response

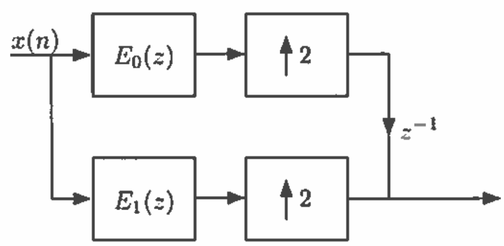
$$h(n) \quad n = 0, 1, \dots, N-1,$$

design an interpolator with interpolation factor $L = 2$ that exploits the Type 2 polyphase decomposition

$$H(z) = E_1(z^2) + z^{-1}E_0(z^2)$$

to achieve computational efficiency.

Show the signal flow diagram for your design and specify fully all processing functions and filter impulse responses. [5]



For the example of N even, $E_0(z)$ has impulse response $\{h(1), h(3), h(5) \dots, h(N-1)\}$ and $E_1(z)$ has impulse response $\{h(0), h(2), h(4) \dots, h(N-2)\}$.