Imperial College London

M3P6

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May-June 2016

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Probability

Date: **Tuesday 24th May 2016**

Time: **14.00 – 16.00**

Time Allowed: **2 Hours**

This paper has Four Questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the relevant main book(s) are full.

Statistical tables will not be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers to each question as per the table below.

Raw Mark	Up to 12	13	14	15	16	17	18	19	20
Extra Credit	0	½	1	1 ½	2	2 ½	3	3 ½	4

- Each question carries equal weight.
- Calculators may not be used.

Notation
$\mathbb{N}$ set of natural numbers;
$\mathbb{R}$ set of real numbers;
$\lambda(dx)$ the Lebesgue measure in $\mathbb{R}^d$;
2^A family of all subsets of a set A;
$\circ$ denotes composition of functions;
χ_A characteristic function of a set A;
$A^{\mathbb{N}}$ Cartesian infinite product $\equiv\{(\omega_i \in A)_{i\in\mathbb{N}}\}$.

Q1.

(1.i) Give the definition of a probability space explaining carefully the notions involved. Which of the following are probability spaces ? Justify your answer.

(1.i.a) $(\mathbb{N}, 2^{\mathbb{N}}, f)$, where $f: 2^{\mathbb{N}} \to [0,1]$ is given by $f(A) = \sum_{k \in A} q(k)$ with $q(k) = 2^{-k}$, for $k \in \mathbb{N}$;

(1.i.b) $(\Omega \equiv \{1,2,3,4\}, \Sigma, \nu)$ where $\Sigma = \{\Omega, \{i\}, i \in \Omega, \{i,j\} \text{ for } i \neq j, i,j \in \Omega\}$ and $\nu(\{i\}) = \frac{1}{4}$, $\nu(\{i,j\}) = 0$.

(1.ii) Give the definition of a (finite) family of mutually independent random variables explaining carefully the notions involved.

(1.iii)

(1.iii.a) Let μ be a probability measure on Σ and, for a given $B \in \Sigma$, suppose $\mu(B) = 0$.
Prove or disprove that, for any $A \in \Sigma$, the characteristic functions χ_A and χ_B are mutually independent random variables;

(1.iii.b) Let ν be a probability measure on $2^{\{0,1\}}$ such that $\nu(\{0\})$ is neither zero nor one. Consider the corresponding product probability space $(\{0,1\}^{\mathbb{N}}, \mathcal{B}, \nu^{\otimes \mathbb{N}})$, where $\mathcal{B}$ is the Borel σ-algebra in $\{0,1\}^{\mathbb{N}}$.
For $i \in \mathbb{N}$, let $\pi_i: \{0,1\}^{\mathbb{N}} \to \{0,1\}$ denote the projection map defined by

$$\omega \equiv (\omega_j)_{j \in \mathbb{N}} \quad \mapsto \quad \pi_i(\omega) := \omega_i \ .$$

Let ξ and η be real functions on $\{0,1\}$. Prove or disprove that for $i \neq j$, $\xi \circ \pi_i$ and $\eta \circ \pi_j$ are mutually independent random variables on $(\{0,1\}^{\mathbb{N}}, \mathcal{B}, \nu^{\otimes \mathbb{N}})$.

Q2.

(2.i) Let $d\gamma \equiv e^{-\frac{1}{2}x^2}\lambda(dx)/\sqrt{2\pi}$ be the Gaussian measure on the real line. Consider the corresponding infinite product space $(\mathbb{R}^{\mathbb{N}}, \mathcal{B}, \gamma^{\otimes \mathbb{N}})$, with $\mathcal{B}$ denoting the Borel σ-algebra in $\mathbb{R}^{\mathbb{N}}$. For $i \in \mathbb{N}$, let π_i be the projection map defined on the product space $\mathbb{R}^{\mathbb{N}}$ by

$$\omega \equiv (\omega_j \in \mathbb{R})_{j \in \mathbb{N}} \quad \mapsto \quad \pi_i(\omega) := \omega_i \ .$$

Let H_i denote the ith Hermite polynomial normalised by $\gamma(H_i^2) = 1$.

Verify the Weak Law of Large Numbers for the following random variables :

$$X_i \equiv (H_i \circ \pi_i) \cdot (H_{i+1} \circ \pi_{i+1}), \ i \in \mathbb{N}.$$

(2.ii) State and prove the Kolmogorov inequality for a sequence of mutually independent random variables with mean zero and finite 2nd moment.

Q3.

(3.i) State the Borel - Cantelli lemmas.

(3.ii) Prove that the random variables $H_j \circ \pi_j$, introduced in (2.i), satisfy the conclusion of the Strong Law of Large Numbers.

(3.iii) Let $(\Omega, \Sigma, \mu) \equiv (\mathcal{A}, 2^{\mathcal{A}}, \nu)^{\otimes \mathbb{N}}$, where $\mathcal{A}$ denotes the set of letters (lower case and capitals) in the English alphabet and punctuation (including spaces) signs, and ν is the uniform probability measure. Prove that in an infinite sequence of elements of $\mathcal{A}$ chosen independently at random one can find the pattern

Improbable Probability

infinitely many times with probability one .

Q4.

(4.i) Give the definition of the characteristic function of a random variable.

(4.ii) Prove or disprove that the following function is a characteristic function of a random variable.

$$\phi(t) \equiv \frac{1}{\cosh(\pi t)}$$

Hint: You can use the following representation

$\cosh(x) = \prod_{k=1}^{\infty} \left(1 + \frac{4x^2}{\pi^2(2k-1)^2}\right)$

You can also use a formula for the characteristic function associated to the following distribution function

$F(s) \equiv \int_{-\infty}^{s} \frac{1}{2} e^{-|x|} \lambda(dx)$,

where $\lambda(dx)$ *denotes the Lebesgue measure on real line.*

(4.iii) Verify the conclusion of the central limit theorem in the special case of independent identically distributed random variables X_i, $i \in \mathbb{N}$, for which the characteristic function $\phi(t)$ is given in (4.ii).

Imperial College
London

Course: M34P6/M5P6	
Setter:	
Checker:	
Editor:	
External:	
Date:	February 23, 2016

BSc and MSci EXAMINATIONS (MATHEMATICS)
May-June 2016

M34P6/M5P6

PROBABILITY THEORY

SOLUTIONS

Setter's signature ....

Checker's signature ....

Editor's signature ....

Imperial College

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2016

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

M34P6/M5P6

PROBABILITY THEORY

SOLUTIONS

Date: examdate Time: examtime

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.
Calculators may not be used.

Notation

$\mathbb{N}$ set of natural numbers;

$\mathbb{R}$ set of real numbers;

$\lambda(dx)$ the Lebesgue measure in $\mathbb{R}^d$;

2^A family of all subsets of a set A

$\circ$ denotes composition of functions

χ_A characteristic function of a set A

$A^{\mathbb{N}}$ Cartesian infinite product $\equiv\{(\omega_i \in A)_{i\in\mathbb{N}}\}$

Solution Q1.

(1.i)
seen 4pts

A triple (Ω, Σ, μ) consisting of a nonempty set Ω, a σ-algebra Σ of subsets of Ω and a probability measure μ on Σ is called a probability space.
By definition the σ-algebra Σ satisfies:
(a) $\Omega \in \Sigma$;
(b) $\forall A \in \Sigma, \quad \Omega \setminus A \in \Sigma$;
(c) $\forall A_n \in \Sigma, (n \in \mathbb{N}), \quad \bigcup_{n\in\mathbb{N}} A_n \in \Sigma$.
A function $\mu: \Sigma \to [0,1]$ is called a probability measure iff
(a') $\forall A_n \in \Sigma, (n \in \mathbb{N}), A_i \cap A_j = \emptyset$ for $i \neq j$

$$\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \mu(A_n)$$

and

(b') $\mu(\Omega) = 1$.

(1.i.a) ;
unseen 3pts

Yes. $2^{\mathbb{N}}$ by definition contains all the subsets of $\mathbb{N}$ and so satisfies all requirements of the σ-algebra. Since $\sum_{k\in\mathbb{N}} 2^{-k} = 1$ we have $f(\mathbb{N}) = 1$. Moreover for any family $A_n \in \Sigma, (n \in \mathbb{N}), A_i \cap A_j = \emptyset$ for $i \neq j$, we have

$$f(\bigcup_{n\in\mathbb{N}} A_n) = \sum_{k\in\bigcup_{n\in\mathbb{N}} A_n} q(k) = \sum_{n\in\mathbb{N}} \sum_{k\in A_n} q(k) = \sum_{n\in\mathbb{N}} f(A_n).$$

(1.i.b)

unseen 3pts

No. Σ is not closed with respect to the complement operation and $1 = \nu(\Omega) = \nu(\{1,2\} \cup \{3,4\}) \neq \nu(\{1,2\}) + \nu(\{3,4\}) = 0$

(1.ii)

seen 4pts

Random variable X_j , $j \in I$, (with some nonempty finite index set I), on a probability space (Ω, Σ, μ) into $\mathbb{R}$ are called mutually independent iff the σ-algebras $\Sigma_j \equiv$ smallest σ-algebra in Σ containing all sets $\{X_j \leq a\}$, $a \in \mathbb{R}$, are mutually independent. The σ-algebras $\Sigma_j \subset \Sigma$, $j \in I$, are mutually independent iff

$\forall A_j \in \Sigma_j, j \in I, \quad \mu(\cap_j A_j) = \prod_j \mu(A_j)$.

(1.iii)

(1.iii.a)

unseen 3pts

We have

$$\{\chi_A \leq a\} = \begin{cases} \Omega & \text{for} \quad a \geq 1 \\ \Omega \setminus A & \text{for} \quad a < 1 \end{cases}$$

and so the associated smallest σ-algebra is $\Sigma_A \equiv (\Omega, \Omega \setminus A, A, \emptyset)$, and similarly for the set B. One can check that for any $C \in \Sigma_A$ we have

$0 \leq \mu(C \cap B) \leq \mu(B) = 0 = \mu(C) \cdot \mu(B)$

and so $\mu(C \cap B) = \mu(C) \cdot \mu(B)$. On the other hand

$\mu(C \cap (\Omega \setminus B)) = \mu(C \setminus (C \cap B)) = \mu(C) - \mu(C \cap B) = \mu(C) = \mu(C) \cdot \mu(\Omega \setminus B)$.

Similar properties hold replacing the set B by Ω and $\emptyset$.

Thus χ_A and χ_B are mutually independent.

(1.iii.b)

unseen 3pts

By definition of the product measure, if $i \neq j$, we have

$$\nu^{\otimes\mathbb{N}}(\{\xi \circ \pi_i \leq a\} \cap \{\eta \circ \pi_j \leq b\}) = \nu^{\otimes\mathbb{N}}(\{\xi(\omega_i) \leq a\} \cap \{\eta(\omega_j) \leq b\})$$

$$= \nu(\{x \in \{0,1\} : \xi(x) \leq a\}) \cdot \nu(\{x \in \{0,1\} : \eta(x) \leq b\})$$

$$= \nu^{\otimes\mathbb{N}}(\{\xi(\omega_i) \leq a\}) \cdot \nu^{\otimes\mathbb{N}}(\{\eta(\omega_j) \leq b\})$$

which implies independence of the random variables under consideration.

Solution Q2.

(2.i)

unseen 11 pts

We note that, using that $H_j \circ \pi_j$ and $H_{j+1} \circ \pi_{j+1}$ are mutually independent, we have

$\gamma^{\otimes \mathbb{N}}(X_j) = \gamma^{\otimes \mathbb{N}}(H_j \circ \pi_j)\gamma^{\otimes \mathbb{N}}(H_{j+1} \circ \pi_{j+1}) = \gamma(H_j) \cdot \gamma(H_{j+1}) = 0$

(since at least one of the expectations on the right hand side is equal to zero), and
$\gamma^{\otimes \mathbb{N}}(X_j^m) = \gamma^{\otimes \mathbb{N}}((H_j \circ \pi_j)^m)\gamma^{\otimes \mathbb{N}}((H_{j+1} \circ \pi_{j+1})^m) = \gamma((H_j)^m) \cdot \gamma((H_{j+1})^m)$
which for $m = 2$ is equal to one by the assumption of normalisation of the polynomials H_j.
The random variables X_j , $j \in \mathbb{N}$, are not mutually independent, but we have
$\gamma^{\otimes \mathbb{N}}\big((\frac{1}{2n}\sum_{j=1}^{2n} X_j)^2\big) \leq 2\gamma^{\otimes \mathbb{N}}\big((\frac{1}{2n}\sum_{j=1}^{n} X_{2j})^2\big) + 2\gamma^{\otimes \mathbb{N}}\big((\frac{1}{2n}\sum_{j=1}^{n} X_{2j-1})^2\big)$
and since X_{2j}, $j \in \mathbb{N}$, are mutually independent as well as X_{2j-1} $j \in \mathbb{N}$, are mutually independent, we have

$\gamma^{\otimes \mathbb{N}}\big((\frac{1}{n}\sum_{j=1}^{n} X_{2j})^2\big) = \frac{1}{n^2}\sum_{i,j=1}^{n} \gamma^{\otimes \mathbb{N}}(X_{2i}X_{2j}) = \frac{1}{n^2}\sum_{j=1}^{n} \gamma^{\otimes \mathbb{N}}((X_{2j})^2) = \frac{1}{n}$
and similarly
$\gamma^{\otimes \mathbb{N}}\big((\frac{1}{n}\sum_{j=1}^{n} X_{2j-1})^2\big) = \frac{1}{n}$.

Hence
$\gamma^{\otimes \mathbb{N}}\big((\frac{1}{2n}\sum_{j=1}^{2n} X_j)^2\big) \leq 4\frac{1}{n} \to 0$ as $n \to \infty$.

By Chebyshev inequality this implies that $\frac{1}{2n}\sum_{j=1}^{2n} X_j \to 0$ in probability.

(2.ii)

seen 9 pts

Theorem: Let $X_k, k \in \mathbb{N}$, be mutually independent random variables on a probability space (Ω, Σ, μ) with mean zero and finite 2nd moment.
Then for any $n \in \mathbb{N}$, we have

$\mu(\{\max_{1\leq k\leq n} |\sum_{l=1}^{k} X_l| \geq t\}) \leq \frac{1}{t^2}\sum_{k=1}^{n} \mathrm{Var}(X_k)$
for any $t \in (0, \infty)$.

Proof : Note that

$\{\max_{1\le k\le n}|\sum_{l=1}^{k}X_l|\ge t\}=\bigcup_{j=1}^{n}A_j\equiv A$
where

$$A_j=\bigcap_{l<j}\{|\sum_{k=1}^{l}X_k|<t\}\cap\{|\sum_{k=1}^{j}X_k|\ge t\}.$$

Then A_j's are pairwise disjoint and we have

$$\mu(\sum_{l=1}^{n}X_l)^2\ge\mu((\sum_{l=1}^{n}X_l)^2\chi_A)=\sum_{j=1}^{n}\mu((\sum_{l=1}^{k}X_l)^2\chi_{A_j})$$

and
$\mu((\sum_{l=1}^{n}X_l)^2\chi_{A_j}) = \mu((\sum_{l=1}^{j}X_l)^2\chi_{A_j}) + 2\mu((\sum_{l=j+1}^{n}X_l)\cdot(\sum_{l=1}^{j}X_l)\chi_{A_j}) + \mu((\sum_{l=j+1}^{n}X_l)^2\chi_{A_j})$.
Since $(\sum_{l=1}^{j}X_l)\chi_{A_j}$ and $(\sum_{l=j+1}^{n}X_l)$ are independent and $\mu(\sum_{l=j+1}^{n}X_l)=0$, so

$$\mu((\sum_{l=j+1}^{n}X_l)\cdot(\sum_{l=1}^{j}X_l)\chi_{A_j})=0.$$

Using this together with the definition of the sets A_j we get
$\mu(\sum_{l=1}^{n}X_l)^2\ge\sum_{j=1}^{n}\mu((\sum_{l=1}^{j}X_l)^2\chi_{A_j})\ge t^2\sum_{j=1}^{n}\mu(\chi_{A_j})=t^2\,\mu(\chi_A)$

which concludes the proof.

Solution Q3.

(3.i)

seen 6 pts

Theorem :
(1) Let $C_n \in \Sigma, k \in \mathbb{N}$. If $\sum_{n\in\mathbb{N}} \mu(C_n) < \infty$, then

$\mu(\bigcap_{k\in\mathbb{N}} \bigcup_{n\geq k} C_n) = 0$.

(2)
Let $C_n \in \Sigma, n \in \mathbb{N}$, be mutually independent. If $\sum_{n\in\mathbb{N}} \mu(C_n) = \infty$, then

$\mu(\bigcap_{k\in\mathbb{N}} \bigcup_{n\geq k} C_n) = 1$.

(3.ii)
unseen 9 pts

By our assumption the random variables $Z_j \equiv H_j \circ \pi_j$, $j \in \mathbb{N}$, are mutually independent (although not identically distributed).
Moreover $\gamma^{\otimes\mathbb{N}}(Z_j) = \gamma(H_j) = 0$ and $\gamma^{\otimes\mathbb{N}}(Z_j^2) = \gamma(H_j^2) \leq 1$ and so

$\sum_j \frac{1}{j^2} \operatorname{Var}(Z_j) < \infty$.
Therefore using Kolmogorov inequality:

$\gamma^{\otimes\mathbb{N}}(\{\max_{1\leq k\leq n} |\sum_{k=1}^n Z_k| \geq t\}) \leq \frac{1}{t^2} \sum_{k=1}^n \operatorname{Var}(Z_k)$,

together with

$\sum_{m=1}^\infty \gamma^{\otimes\mathbb{N}}\big(\big\{\max_{2^{m-1}\leq n<2^m} \big|\frac{1}{n}\sum_{k=1}^n Z_k\big| \geq \varepsilon\big\}\big) \leq \frac{16}{\varepsilon^2} \sum_{j=1}^\infty \frac{1}{j^2} \operatorname{Var}(Z_j) < \infty$

Thus by the Borel-Cantelli lemma one concludes that for any $\varepsilon > 0$,
for almost all $\omega \in \mathbb{R}^\mathbb{N}$ exists $m_0 \equiv m_0(\omega) \in \mathbb{N}$, such that
$\max_{2^{m-1}\leq n<2^m} \big|\frac{1}{n}\sum_{k=1}^n Z_k\big| < \varepsilon$
for all $m > m_0$. Therefore for any $\varepsilon > 0$

$\mathcal{D}(\varepsilon) \equiv \bigcup_{m=1}^\infty \bigcap_{n\geq m} \big\{\omega : \big|\frac{1}{n}\sum_{k=1}^n Z_k\big| < \varepsilon\big\}$

has probability one. Hence also

$\gamma^{\otimes\mathbb{N}}\big(\bigcap_k \mathcal{D}(\frac{1}{k})\big) = 1$, which means that
$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^n Z_k = 0$
almost everywhere.

(3.iii)
unseen 5 pts
For $n \in \mathbb{N}$, consider an event that the indicated pattern starts from n-th place,
$T_{100n} \equiv \{\omega \in \Omega: \omega_l = a_{l-100n}, l = 100n, ..., 100n + L\}$
where L denotes number of symbols in the pattern and a_i denotes the $i-$th symbol in the pattern.
It follows from the definition of the product measure that these events are independent in the given probability space and moreover we have
$\mu(T_{100n}) = \prod_{l=100n,...,100n+L} \nu(\{a_l\}) = \text{const} > 0,$
since by our assumption $\nu(\{a\}) > 0$.
Hence
$\sum_{n\in\mathbb{N}} \mu(T_{100n}) = \infty$
and by the 2nd Borel-Cantelli lemma this implies
$\mu(\bigcap_{k\in\mathbb{N}} \bigcup_{n\geq k} T_{100n}) = 1$
that is the desired pattern shows up infinitely often with probability one.

Solution Q4.

(4.i)
seen 5 pts

The characterisitc function of a random variable X on a probability space (Ω, Σ, μ) is defined as follows

$$\phi_X(t) := \mu(e^{itX}) \equiv \int e^{itX} d\mu.$$

(4.ii)

unseen 8 pts

Note that by direct computation we have

$\int e^{i\alpha tx} \frac{1}{2} e^{-|x|} \lambda(dx) = \int_0^\infty e^{i\alpha tx} \frac{1}{2} e^{-x} \mathrm{dx} + \int_0^\infty e^{-i\alpha tx} \frac{1}{2} e^{-x} dx$
$= \frac{1}{2}\left(\frac{1}{1-i\alpha t} + \frac{1}{1+i\alpha t}\right) = \frac{1}{1+\alpha^2 t^2}$

Hence in particular the function

$\varphi_\alpha(t) \equiv \frac{1}{1+\alpha^2 t^2}$

is positive definite for any $\alpha \in (0, \infty)$. Since any product of positive definite functions is positive definite, so in particular $\phi_n(t) \equiv \prod_{k=1}^n \varphi_{\alpha_k}(t)$ is positive definite. For $\alpha_k \equiv \frac{4}{\pi^2(2k-1)^2}$, a sequence $\phi_n(t)$ converges pointwise (uniformly of compact intervals) as $n \to \infty$ and so its limit $\phi(t)$ is also positive definite. It follows from the definition of $\phi(t)$ that it is equal to one at $t = 0$. We notice also that

$|\phi(t+\delta) - \phi(t)| = \left|\int_0^1 \mathrm{ds} \frac{\sinh(\pi(t+s\delta))}{(\cosh(\pi(t+s\delta)))^2} \pi\delta\right| \leq \pi\delta$

and so $\phi(t)$ is uniformly continuous and in particular continuous at 0. Using Lévy's continuity theorem (the part for characteristic functions) concludes the proof.

(4.iii)

unseen 7 pts

By independence and identical distribution assumption, we have
$\int e^{\mathrm{it}\left(\frac{1}{\sqrt{n}}\sum_{j=1}^n X_j\right)} d\mu = \prod_{j=1}^n \int e^{it\left(\frac{1}{\sqrt{n}} X_j\right)} d\mu \equiv \phi\left(\frac{t}{\sqrt{n}}\right)^n$

Next we have
$\phi\left(\frac{t}{\sqrt{n}}\right)^n = \exp\left(-n\log\left(\cosh\left(\pi\frac{t}{\sqrt{n}}\right)\right)\right)$
and since for $\left|\frac{t}{\sqrt{n}}\right| \leq 1$
$1+\frac{\pi^2t^2}{2n} \leq \cosh\left(\pi\frac{t}{\sqrt{n}}\right) \leq 1+\frac{\pi^2t^2}{2n} + \left(\frac{t}{\sqrt{n}}\right)^3\frac{1}{3!}e^{\pi}$
we get

$\lim_{n\to\infty} n\log\left(\cosh\left(\pi\frac{t}{\sqrt{n}}\right)\right) = \frac{\pi^2t^2}{2}.$

Hence, using Lévy's continuity theorem, we conclude that $\frac{1}{\sqrt{n}}\sum_{j=1}^{n} X_j$ converges in the sense of distribution to a Gaussian random variable with mean zero and variance $\frac{\pi^2}{2}$.

Solution Q5.

(5.i)
unseen 8 pts

First we note that

$\|\sum_{j=1}^{n} X_j \int_0^t f_j(s)\mathrm{ds}\,\|_2^2 = \sum_{j=1}^{n} |\int_0^t f_j(s)\mathrm{ds}|^2 = \sum_{j=1}^{n} |\langle f_j, \chi([0,t])\rangle|^2$
Since $\chi([0,t])$ is in $L_2([0,1],\lambda)$, the last series converges.

(By Hölder inequality, this implies convergence in $\mathbb{L}_p(\mu)$, for any $p \in [1,2)$.)

Using the mutual independence of the Gaussian variables X_j's , we have
$\mu(\exp\{\varepsilon \sum_{j=1}^{n} X_j \int_0^t f_j(s)\mathrm{ds}\}) \leq \exp\left\{\frac{\varepsilon^2}{2} \sum_{j\in\mathbb{N}} |\langle f_j, \chi([0,t])\rangle|^2\right\}$.

Hence

$\mu(\,|\exp\{\varepsilon \sum_{j=1}^{m} X_j \int_0^t f_j(s)\mathrm{ds}\} - \exp\{\varepsilon \sum_{j=1}^{n} X_j \int_0^t f_j(s)\mathrm{ds}\}|)$

$=\varepsilon\mu\{(\,|\int_0^1 d\tau \exp\{\varepsilon\tau \sum_{j=1}^{m} X_j \int_0^t f_j(s)\mathrm{ds} + \varepsilon\,(1-\tau)\sum_{j=1}^{n} X_j \int_0^t f_j(s)\mathrm{ds}\}|$

$\cdot(|\sum_{j=n+1}^{m} X_j \int_0^t f_j(s)\mathrm{ds}|\,\}$

$\leq \varepsilon \exp\left\{\frac{\varepsilon^2}{2} \sum_{j\in\mathbb{N}} |\langle f_j, \chi([0,t])\rangle|^2\right\} \sum_{j=n+1}^{m} |\langle f_j, \chi([0,t])\rangle|^2$

which implies that in fact $\exp\{\varepsilon \sum_{j=1}^{n} X_j \int_0^t f_j(s)\mathrm{ds}\}, n \in \mathbb{N}$, converges in $\mathbb{L}_1(\mu)$.
This implies convergence in $\mathbb{L}_p(\mu)$, for any $p \in [1,\infty)$.

(5.ii)

unseen 6 pts

We note that for any $\alpha, \beta \in \mathbb{C}$, we have
$\mu\exp\{\alpha B_{t_1} + \beta(B_{t_2} - B_{t_1})\} = \mu\exp\{\sum_{j\in\mathbb{N}} X_j\,(\langle f_j, \alpha\chi([0,t_1])\rangle + \beta\chi([t_1,t_2])\rangle)\}$

$=\exp\{\frac{1}{2}\sum_{j\in\mathbb{N}} |\langle f_j, \alpha\chi([0,t_1]) + \beta\chi([t_1,t_2])\rangle|^2\}$

$=\exp\{\frac{1}{2}\sum_{j\in\mathbb{N}} (|\langle f_j, \alpha\chi([0,t_1])\rangle|^2 + |\langle f_j, \beta\chi([t_1,t_2])\rangle|^2)\}$

$\mu\exp\{\alpha B_{t_1}\} \cdot \mu\exp\{\beta(B_{t_2} - B_{t_1})\}$.

This implies independence for the Gaussian random variables B_{t_1} and $B_{t_2} - B_{t_1}$.

We have

$\mu(B_{t_2}-B_{t_1})^2=\mu(\sum_{j\in\mathbb{N}}X_j\,\langle f_j,\chi([t_1,t_2])\rangle)^2=\sum_{j\in\mathbb{N}}|\langle f_j,\chi([t_1,t_2])\rangle|^2$

$=\int\chi([t_1,t_2])^2\mathrm{dt}=t_2-t_1$

(5.iii)

seen 6 pts

(M1)
Let $A\in\mathbb{R}^n$ be an open set.
Then $\forall t,s>0$

$$\mathbb{P}_x(W_{t+s}\in A)=\mathbb{E}_x(\mathbb{P}_{W_t}(W_s\in A))$$

where $\mathbb{P}_x$ and $\mathbb{P}_{W_t(\omega)}$ denotes the Wiener measure for the process starting at x and $W_t(\omega)$ (where ω denotes the integration variable), respectively,
and $\mathbb{E}_x$ denotes the expectation associated to $\mathbb{P}_x$,
$(W_s\in A)\equiv\{\omega\colon W_s(\omega)\in A\}$.