IMPERIAL COLLEGE LONDON Exam Copy

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2004

EEE/ISE PART II: MEng, BEng and ACGI

SIGNALS AND LINEAR SYSTEMS

Tuesday, 1 June 2:00 pm

Time allowed: 2:00 hours

There are FIVE questions on this paper.

Answer THREE questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : T. Stathaki
Second Marker(s) : A.G. Constantinides

1.

Consider the cascade interconnection of three linear time invariant (LTI) systems, illustrated in the following Figure 1. The impulse response $h_2[n]$ is

$$h_2[n] = u[n] - u[n-1],$$

where $u[n]$ is the discrete unit step function defined as

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

The overall impulse response is as shown in Figure 2.

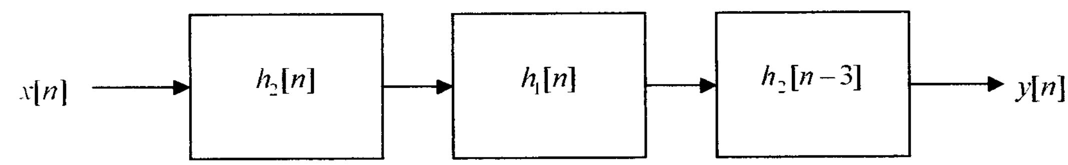


Figure 1

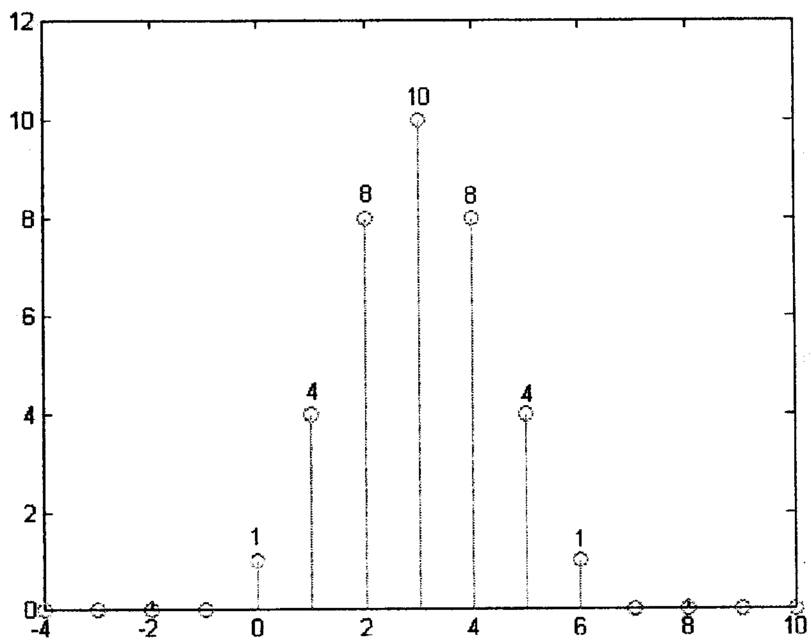


Figure 2

(a) Find the impulse response $h_1[n]$.

[10]

(b) Find the convolution

$$u[n - c_1] * u[n - c_2]$$

where $u[n]$ is the discrete unit step function defined above and c_1, c_2 are constant parameters.

[10]

2.

(a) Consider the continuous signal $x(t)$ that is periodic with period T and fundamental frequency $\omega_0 = \frac{2\pi}{T}$. Suppose that the Fourier series coefficients of $x(t)$ are c_k.

(i) Find the Fourier series coefficients of the signal $x^*(t)$.

[2]

(ii) Find the Fourier series coefficients of the signal $x(-t)$.

[2]

(b) Let $x(t)$ be a periodic signal whose Fourier series coefficients are

$$c_k = \begin{cases} 1, & k = 0 \\ -j\left(\frac{1}{3}\right)^{|k|}, & \text{otherwise} \end{cases}$$

(i) Is $x^*(t)$ real?

[3]

(ii) Is $x^*(t)$ odd?

[3]

(iii) Is $x(-t)$ real?

[3]

Justify your answers.

(c) The Parseval's relation for a discrete time periodic signal $x[n]$ is given by the following expression

$$\frac{1}{N}\sum_{n=\langle N\rangle} |x[n]|^2 = \sum_{k=\langle N\rangle} |c_k|^2$$

where N is the period of the discrete signal $x[n]$, c_k are the Fourier series coefficients of $x[n]$ and $n = \langle N\rangle$ indicates that n varies over a range of N successive integers. Suppose that we are given the following information about $x[n]$:

1. $x[n]$ is a real and even signal. In that case the Fourier series coefficients c_k of $x[n]$ are also real an even.
2. $x[n]$ has period 10 and Fourier coefficients c_k.
3. $c_{11} = 5$.
4. $\frac{1}{10}\sum_{n=\langle 10\rangle} |x[n]|^2 = 50$.

Using the Parseval's theorem with $-1 \le n \le 8$ find the Fourier series coefficients c_k of $x[n]$.

[7]

3.

(a) State the advantages that the Bode plots offer in terms of characterizing a frequency response.

[5]

(b) The output $y(t)$ of a continuous, causal LTI system is related to the input $x(t)$ by the differential equation

$$\frac{d^2y(t)}{dt^2}+6\frac{dy(t)}{dt}+9y(t)=\frac{dx(t)}{dt}+2x(t)$$

Determine the frequency response of the system, then find and sketch its Bode plots. Justify your answers.

[15]

4\.

(a) Consider the signal $w(t)$ with Laplace transform $W(s)$. Find the analytical expression of the Laplace transform of the signal $w(t-t_0)$, where t_0 is a constant parameter.

[1]

(b) Consider the signal $z(t)$ with Laplace transform $Z(s)$. Find the analytical expression of the Laplace transform of the signal $z(at)$, where a is a constant parameter.

[4]

(c) Consider a signal $y(t)$ which is related to two signals $x_1(t)$ and $x_2(t)$ by

$$y(t) = x_1(t-2) * x_2(-t+3)$$

where $x_1(t) = e^{-2t}u(t)$ and $x_2(t) = e^{-3t}u(t)$. Determine the analytical expression of the Laplace transform $Y(s)$ of $y(t)$.

[10]

(d) The system function of a causal Linear Time Invariant system is

$$H(s) = \frac{s+1}{s^2+2s+2}$$

Determine the analytical expression of the Laplace transform of the output when the input is $x(t) = e^{-t}u(t) + e^{t}u(-t)$.

The function $u(t)$ is the continuous unit step function defined as

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

[5]

5.

(a) Consider a discrete time Linear Time Invariant system. Find the response of the system to the input z_0^n, as a function of the z transform of the impulse response of the system. z_0 is a constant, generally complex number.

[7]

(b) Suppose that we are given the following information about a Linear Time Invariant system:

1. If the input to the system is $x_1[n] = \left(\frac{1}{6}\right)^n u[n]$, then the output is

$$y_1[n] = \left[a\left(\frac{1}{2}\right)^n + 10\left(\frac{1}{3}\right)^n \right] u[n]$$

   where a is a real number.

2. If the input to the system is $x_2[n] = (-1)^n$, then the output is

$$y_2[n] = \frac{7}{4}(-1)^n$$

The function $u[n]$ is the discrete unit step function defined as

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

Determine the analytical expression of the z transform of the impulse response of the system $H(z)$, consistent with the information above. There should be no unknown constant in your answer; that is, the constant a should not appear in your answer.

[13]

E 2.5 SIGNALS + LINEAR SYSTEMS

SOLUTIONS 2004

1.

(a) The impulse response $h_2[n]$ is $h_2[n] = u[n] - u[n-1] = \delta[n]$. This means that $h_2[n-3] = \delta[n-3]$. Moreover, $h_2[n] * h_2[n-3] = \delta[n] * \delta[n-3] = \delta[n-3]$

We call the overall impulse response with $h[n]$, and this is equal to

$h[n] = \delta[n] + 4\delta[n-1] + 8\delta[n-2] + 10\delta[n-3] + 8\delta[n-4] + 4\delta[n-5] + \delta[n-6]$. From this and $h_2[n-3] = \delta[n-3]$ we obtain that

$h_1[n] = h[n+3] = \delta[n+3] + 4\delta[n+2] + 8\delta[n+1] + 10\delta[n] + 8\delta[n-1] + 4\delta[n-2] + \delta[n-3]$

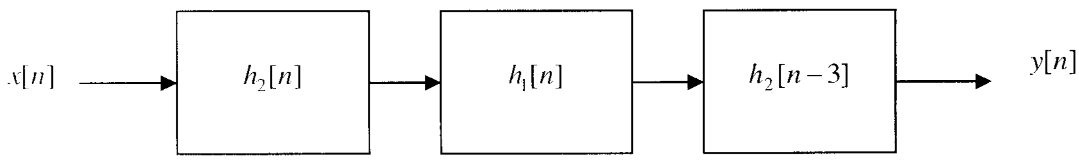


Figure 1

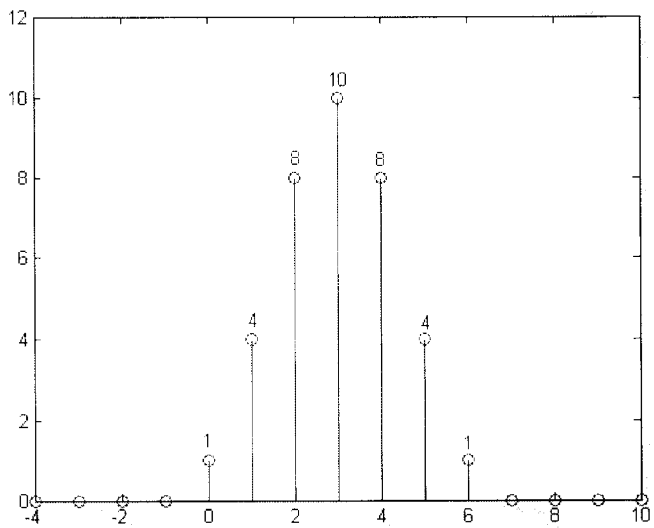


Figure 2

[10]

(b) Find the output of the overall system to the input $u[n-c_1] * u[n-c_2]$.

$$u[n-c_1] * u[n-c_2] = (n - c_1 - c_2 + 1)u[n - c_1 - c_2]$$

[10]

2.

(a) Consider the continuous signal $x(t)$ that is periodic with period T and fundamental frequency $\omega_0 = \frac{2\pi}{T}$. Suppose that the Fourier series coefficients of $x(t)$ are c_k.

(i) Find the Fourier series coefficients of the signal $x^*(t)$.

$x^*(t)=\sum_{k=-\infty}^{+\infty} c_k^* e^{-jk\omega_0 t}$. In that case the Fourier series of $x^*(t)$ are c_{-k}^* .

[2]

(ii) Find the Fourier series coefficients of the signal $x(-t)$.

$x(-t)=\sum_{k=-\infty}^{+\infty} c_k e^{-jk\omega_0 t}$. In that case the Fourier series of $x(-t)$ are c_{-k} .

[2]

(b) Let $x(t)$ be a periodic signal whose Fourier series coefficients are

$$c_k=\begin{cases}1, & k=0\\ -j\left(\frac{1}{3}\right)^{|k|}, & \text{otherwise}\end{cases}$$

(i) Is $x^*(t)$ real? The Fourier series of $x^*(t)$ are

$$c_k=\begin{cases}1, & k=0\\ j\left(\frac{1}{3}\right)^{|k|}, & \text{otherwise}\end{cases}$$

In that case $x^*(t)=1+\sum_{k=-\infty}^{+\infty} j\left(\frac{1}{3}\right)^{|k|} e^{jk\omega_0 t}=1+\sum_{k=-\infty}^{-1} j\left(\frac{1}{3}\right)^{|k|} e^{jk\omega_0 t}+\sum_{1}^{+\infty} j\left(\frac{1}{3}\right)^{|k|} e^{jk\omega_0 t}=$

$=1+\sum_{1}^{+\infty} j\left(\frac{1}{3}\right)^{|k|} e^{-jk\omega_0 t}+\sum_{1}^{+\infty} j\left(\frac{1}{3}\right)^{|k|} e^{jk\omega_0 t}=$

$=1+\sum_{1}^{+\infty} 2j\left(\frac{1}{3}\right)^{|k|} \cos(jk\omega_0 t)$. It is NOT real.

[3]

(ii) Is $x^*(t)$ odd? No, according to the above it is EVEN.

[3]

(iii) Is $x(-t)$ real? The Fourier series of $x(-t)$ are

$$c_k=\begin{cases}1, & k=0\\ -j\left(\frac{1}{3}\right)^{|k|}, & \text{otherwise}\end{cases}$$

In that case $x(-t)=1+\sum_{k=-\infty}^{+\infty} -j\left(\frac{1}{3}\right)^{|k|} e^{jk\omega_0 t}=1+\sum_{k=-\infty}^{-1} -j\left(\frac{1}{3}\right)^{|k|} e^{jk\omega_0 t}+\sum_{1}^{+\infty} -j\left(\frac{1}{3}\right)^{|k|} e^{jk\omega_0 t}=$

$=1+\sum_{1}^{+\infty} -j\left(\frac{1}{3}\right)^{|k|} e^{-jk\omega_0 t}+\sum_{1}^{+\infty} -j\left(\frac{1}{3}\right)^{|k|} e^{jk\omega_0 t}=$

$=1+\sum_{1}^{+\infty} -2j\left(\frac{1}{3}\right)^{|k|} \cos(k\omega_0 t)$. It is NOT real.

[3]

(c) The Parseval's relation for a discrete time periodic signal $x[n]$ is given by the following expression

$$\frac{1}{N}\sum_{n=\langle N\rangle} |x[n]|^2=\sum_{k=\langle N\rangle} |c_k|^2$$

where N is the period of the discrete signal $x[n]$, c_k are the Fourier series coefficients of $x[n]$ and $n=\langle N\rangle$ indicates that n varies over a range of N successive integers. Suppose that we are given the following information about $x[n]$:

1. $x[n]$ is a real and even signal. In that case the Fourier series coefficients c_k of $x[n]$ are also real an even.
2. $x[n]$ has period 10 and Fourier coefficients c_k.
3. $c_{11} = 5$. From this we get $c_1 = c_{-1} = 5$.
4. $\frac{1}{10}\sum_{n=\langle 10\rangle} |x[n]|^2 = 50$.

Using the Parseval's theorem with $-1 \le n \le 8$ find the Fourier series coefficients c_k of $x[n]$.

$$\sum_{1}^{8} c_k^2 = 50 \Rightarrow c_{-1}^2 + c_1^2 + c_0^2 + \sum_{2}^{8} c_k^2 = 50 \Rightarrow c_0^2 + \sum_{2}^{8} c_k^2 = 0 \Rightarrow c_0 = 0 \text{ and } c_i = 0, i = 2, \ldots, 8$$

[7]

3.

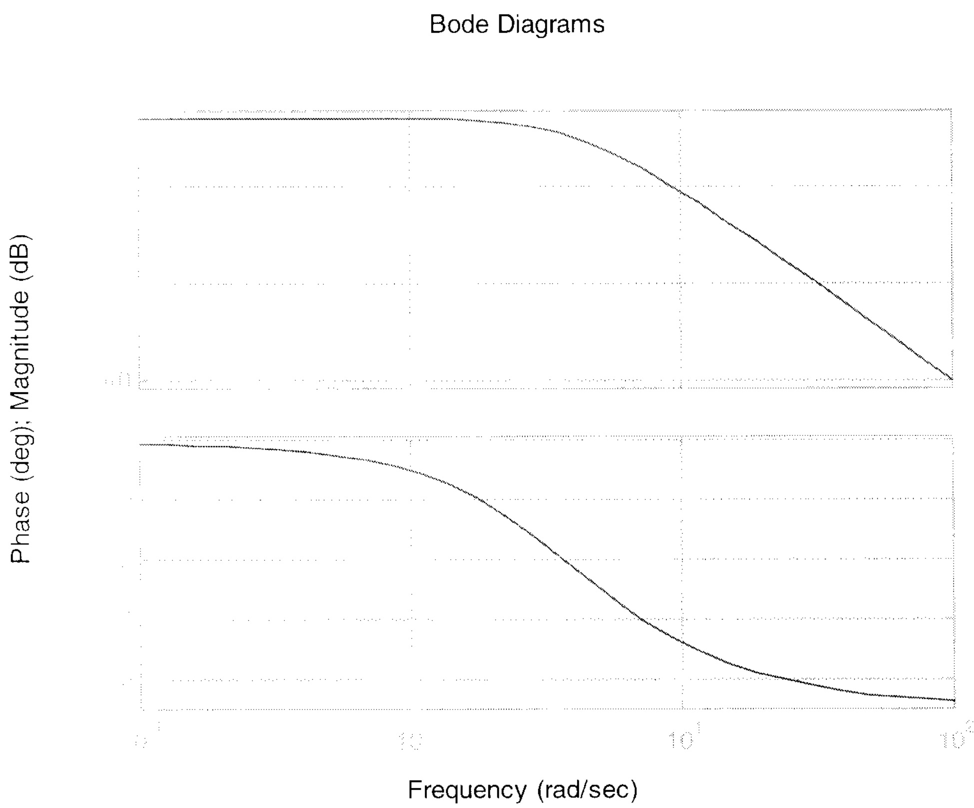
(a) State the advantages that the Bode plots offer in terms of characterizing a frequency response.
Multiplication becomes addition for both amplitude and phase.
Division becomes subtraction for both amplitude and phase.

[5]

(b) The output $y(t)$ of a causal LTI system is related to the input $x(t)$ by the differential equation

$$\frac{d^2 y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 9y(t) = \frac{dx(t)}{dt} + 2x(t) \Rightarrow H(j\omega) = \frac{j\omega + 2}{(j\omega + 3)^2}$$

Determine the frequency response of the system, then find and sketch its Bode plots. **[15]**



4.

(a) Consider the signal $w(t)$ with Laplace transform $W(s)$. Find the analytical expression of the Laplace transform of the signal $w(t-t_0)$, where t_0 is a constant.

This is $e^{-st_0}W(s)$ The Laplace transform of the function $x(t-t_0)$ is given by

$$\int_{-\infty}^{\infty} w(t-t_0)e^{-st}dt = e^{-j\omega\tau}\int_{-\infty}^{\infty} w(t-t_0)e^{-s(t-t_0)}d(t-t_0) = e^{-st_0}W(s)$$

with $W(s)$ the Fourier transform of the signal $w(t)$.

[1]

(b) Consider the signal $z(t)$ with Laplace transform $Z(s)$. Find the analytical expression of the Laplace transform of the signal $z(at)$, where a is a constant.

The Laplace transform of the function $x(at)$ is given by $\int_{-\infty}^{\infty} x(at)e^{-st}dt = \int_{-\infty}^{\infty} \frac{1}{a}x(at)e^{-j\frac{s}{a}(at)}d(at)$. If we use the transformation $at = u$ then we get the following.

If $a \geq 0$ then $t \to \infty \Rightarrow u \to \infty$ and $t \to -\infty \Rightarrow u \to -\infty$.

In that case $\int_{-\infty}^{\infty} \frac{1}{a}x(at)e^{-j\frac{s}{a}(at)}d(at) = \int_{-\infty}^{\infty} \frac{1}{a}x(u)e^{-j\frac{s}{a}u}du$ and since a is positive $\int_{-\infty}^{\infty} \frac{1}{a}x(u)e^{-j\frac{s}{a}u}du = \int_{-\infty}^{\infty} \frac{1}{|a|}x(u)e^{-j\frac{s}{a}u}du$.

If $a \leq 0$ then $t \to \infty \Rightarrow u \to -\infty$ and $t \to -\infty \Rightarrow u \to \infty$.

In that case $\int_{-\infty}^{\infty} \frac{1}{a}x(at)e^{-j\frac{s}{a}(at)}d(at) = \int_{\infty}^{-\infty} \frac{1}{a}x(u)e^{-j\frac{s}{a}u}du = -\int_{-\infty}^{\infty} \frac{1}{a}x(u)e^{-j\frac{s}{a}u}du$ and since a is negative $-\int_{-\infty}^{\infty} \frac{1}{a}x(u)e^{-j\frac{s}{a}u}du = \int_{-\infty}^{\infty} \frac{1}{-a}x(u)e^{-j\frac{s}{a}u}du = \int_{-\infty}^{\infty} \frac{1}{|a|}x(u)e^{-j\frac{s}{a}u}du$.

Hence, the Laplace transform of the signal $x(at)$ is $\frac{1}{|a|}X(\frac{s}{a})$ with $X(s)$ the Laplace transform of the signal $x(t)$.

[4]

(c) Consider a signal $y(t)$ which is related to two signals $x_1(t)$ and $x_2(t)$ by

$$y(t) = x_1(t-2) * x_2(-t+3)$$

where $x_1(t) = e^{-2t}u(t)$ and $x_2(t) = e^{-3t}u(t)$. Determine the Laplace transform $Y(s)$ of $y(t)$.

The Laplace transform of $x_1(t) = e^{-2t}u(t)$ is $\frac{1}{s+1}$.

The Laplace transform of $x_1(t-2)$ is $e^{-2s}\frac{1}{s+1}$.

The Laplace transform of $x_2(t)$ is $\frac{1}{s+3}$.

The Laplace transform of $x_2(-t)$ is $\frac{1}{-s+3}$.

The Laplace transform of $x_2(-t+3) = x_2[-(t-3)]$ is $e^{-3s}\frac{1}{-s+3}$.

$$Y(s) = e^{2s}\frac{1}{s-2} + e^{-3s}\frac{1}{-s+3}$$

[10]

(d) The system function of a causal Linear Time Invariant system is

$$H(s) = \frac{s+1}{s^2+2s+2}$$

Determine the Laplace transform of the output when the input is $x(t) = e^{-t}u(t) + e^{t}u(-t)$.

$X(s) = \frac{1}{s+1} - \frac{1}{s-1}$. The Laplace transform of the output is $H(s)X(s)$

[5]

5.

(a) Consider a discrete time Linear Time Invariant system. Find the response of the system to the input z_0^n, as a function of the z transform of the impulse response of the system. z_0 is a constant, generally complex number.

If we call the output $y[n]$ then $y[n] = \sum_{k=-\infty}^{k=+\infty} x[n-k]h[k] = z_0^n \sum_{k=-\infty}^{k=+\infty} h[k]z_0^{-k} = z_0^n H(z_0)$

[7]

(b) Suppose that we are given the following information about a Linear Time Invariant system:

1. If the input to the system is $x_1[n] = \left(\frac{1}{6}\right)^n u[n]$, then the output is

$$y_1[n] = \left[a\left(\frac{1}{2}\right)^n + 10\left(\frac{1}{3}\right)^n\right]u[n]$$

where a is a real number.

$$X_1(z) = \frac{1}{1-\frac{1}{6}z^{-1}}, |z| > \frac{1}{6}$$

$$Y_1(z) = \frac{a}{1-\frac{1}{2}z^{-1}} + \frac{10}{1-\frac{1}{3}z^{-1}} = \frac{(a+10)-(5+\frac{a}{3})z^{-1}}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{3}z^{-1})}, |z| > \frac{1}{2}$$

Furthermore, $H(z) = \frac{Y_1(z)}{X_1(z)} = \frac{[(a+10)-(5+\frac{a}{3})z^{-1}](1-\frac{1}{6}z^{-1})}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{3}z^{-1})}$

2. If the input to the system is $x_2[n] = (-1)^n$, then the output is

$$y_2[n] = \frac{7}{4}(-1)^n$$

From 2. we know that $H(-1) = \frac{7}{4} \Rightarrow a = -9$, so that

$$H(z) = \frac{[1-2z^{-1}](1-\frac{1}{6}z^{-1})}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{3}z^{-1})}$$

[13]