EE4-27

SOLUTIONS: SYSTEMS IDENTIFICATION

1. Solution

a) Inspecting the block-scheme shown in Fig. 1.1 in the text of the exam paper, one immediately gets:

$$\begin{cases} x_1(t+1) = -\frac{1}{2}x_1(t) + \frac{7}{6}x_2(t) + u(t) + e(t) \\ x_2(t+1) = \frac{1}{3}x_1(t) + \frac{1}{3}x_2(t) + 3e(t) \\ y(t) = x_2(t) \end{cases}$$

With the usual interpretation of z as a one-step forward shift operator, one has:

$$\begin{cases} (z+1/2)x_1(t) = \frac{7}{6}x_2(t) + u(t) + e(t) \\ (z-1/3)x_2(t) = \frac{1}{3}x_1(t) + 3e(t) \\ y(t) = x_2(t) \end{cases}$$

Eliminating the variables x_1 and x_2 from equations above, after some algebra, one gets:

$$y(t) = \frac{1/3}{(z+5/6)(z-2/3)}u(t) + \frac{3(z+11/18)}{(z+5/6)(z-2/3)}e(t)$$

Equivalently, one can write

$$A(z)y(t) = B(z)u(t) + C(z)e(t)$$

where

$$A(z) = 1 + \frac{1}{6}z^{-1} - \frac{5}{9}z^{-2};\ B(z) = \frac{1}{3}z^{-2};\ C(z) = 3z^{-1} + \frac{33}{18}z^{-2}$$

By defining

$$\xi(t) := e(t-1)$$

one gets

$$A(z)y(t) = B(z)u(t) + \tilde{C}(z)\xi(t)$$

where

$$\tilde{C}(z) = 3 + \frac{33}{18}z^{-1}$$

Finally, by introducing

$$\overline{C}(z) = 1 + \frac{11}{18}z^{-1};\quad \eta(\cdot) \sim WN(0,9)$$

the following ARMAX model in canonical form can be obtained:

$$A(z)y(t) = B(z)u(t) + \overline{C}(z)\eta(t)$$

[6 Marks]

b) For the ARMAX model in canonical form determined in the answer to Question 1a), the optimal one-step ahead prediction has the form

$$\widehat{y}(t+1|t) = \frac{\overline{C}(z) - A(z)}{\overline{C}(z)} y(t+1) + \frac{B(z)}{\overline{C}(z)} u(t+1)$$

By replacing the numerical values obtained in the answer to Question 1a), after some calculations, one obtains:

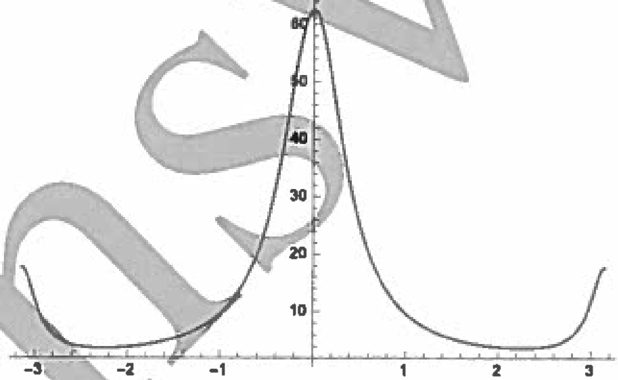
$$\widehat{y}(t+1|t) = -\frac{11}{18}\widehat{y}(t|t-1) + \frac{4}{9}y(t) + \frac{5}{9}y(t-1) + \frac{1}{3}u(t-1)$$

[3 Marks]

c) i) To sketch the behaviour of the spectrum $\Gamma_y(\omega)$ of the process $y(\cdot)$ described by the model obtained in the answer to Question 1a), one computes a few values of $\Gamma_y(\omega)$ by geometric considerations:

$$\Gamma_y: \begin{cases} \Gamma_y(0) = 9 \cdot \dfrac{\left(1+\frac{11}{18}\right)^2}{\frac{1}{9}\cdot\left(1+\frac{5}{6}\right)^2} = \dfrac{7569}{121} \simeq 62.55 \\ \Gamma_y(\pi/2) = 9 \cdot \dfrac{1+\left(\frac{11}{18}\right)^2}{\left(1+\frac{4}{9}\right)\left(1+\frac{25}{36}\right)} = \dfrac{4005}{793} \simeq 5.05 \\ \Gamma_y(\pi) = 9 \cdot \dfrac{\left(1-\frac{11}{18}\right)^2}{\frac{1}{36}\cdot\left(1+\frac{2}{3}\right)^2} = \dfrac{441}{25} \simeq 17.64 \end{cases}$$

The behaviour of the spectrum $\Gamma_y(\omega)$ is plotted in the figure below.



Note for the student. For the sake of completeness, the analytical expression of the spectrum $\Gamma_y(\omega)$ is given below. **This is not part of the answer to be provided for the exam, that is, the student is NOT asked to provide this expression.**

$$\Gamma_y(\omega) = 9 \cdot \frac{\left|e^{j\omega}\left(e^{j\omega}+11/18\right)\right|^2}{\left|e^{j\omega}+5/6\right|^2 \cdot \left|e^{j\omega}-2/3\right|^2} = \frac{\frac{445}{324}+\frac{11}{9}\cos(\omega)}{\left(\frac{61}{36}+\frac{5}{3}\cos(\omega)\right)\left(\frac{13}{9}-\frac{4}{3}\cos(\omega)\right)}$$

[4 Marks]

ii) Consider the model of the process $y(\cdot)$ in canonical form determined in the answer to Question 1a), that is:

$$A(z)y(t) = \overline{C}(z)\eta(t), \quad \text{with } \eta(\cdot) \sim WN(0,9)$$

where the term $B(z)u(t)$ has been dropped since $u(t) = 0, \forall t \geq 0$ in Question 1c), and where

$$A(z) = 1 + \frac{1}{6}z^{-1} - \frac{5}{9}z^{-2}; \quad \overline{C}(z) = 1 + \frac{11}{18}z^{-1}.$$

By carrying out two iterations of polynomial division of $\overline{C}(z)$ by $A(z)$ one gets:

$$\begin{array}{llllllll}
1 & \frac{11}{18}z^{-1} & & & \quad & 1 & \frac{1}{6}z^{-1} & -\frac{5}{9}z^{-2} \\
-1 & -\frac{1}{6}z^{-1} & \frac{5}{9}z^{-2} & & & 1+\frac{4}{9}\mathbf{z}^{-1} & & \\
// & \frac{4}{9}z^{-1} & \frac{5}{9}z^{-2} & & & & & \\
// & -\frac{4}{9}z^{-1} & -\frac{2}{27}z^{-2} & \frac{20}{81}z^{-3} & & & & \\
// & // & \frac{13}{27}\mathbf{z}^{-2} & \frac{20}{81}\mathbf{z}^{-3} & & & &
\end{array}$$

and thus one obtains

$$\widehat{W}(z) = \frac{\overline{C}(z)}{A(z)} = 1 + \frac{4}{9}z^{-1} + z^{-2}\frac{\frac{13}{27} + \frac{20}{81}z^{-1}}{1 + \frac{1}{6}z^{-1} - \frac{5}{9}z^{-2}}$$

Therefore, the transfer function of the two-steps ahead predictor of $y(t+2)$ from the white noise process $\eta(t)$ is given by

$$\widehat{W}_2(z) = \frac{\frac{13}{27} + \frac{20}{81}z^{-1}}{1 + \frac{1}{6}z^{-1} - \frac{5}{9}z^{-2}}$$

and the transfer function of the two-steps ahead predictor of $y(t+2)$ from the past data $y(t)$ is

$$W_2(z) = \frac{\frac{13}{27} + \frac{20}{81}z^{-1}}{1 + \frac{11}{18}z^{-1}}.$$

Finally, the difference equation implementing the two-step ahead predictor of $y(t+2)$ from the data $y(t)$ is

$$\hat{y}(t+2|t) = -\frac{11}{18}\hat{y}(t+1|t-1) + \frac{13}{27}y(t) + \frac{20}{81}y(t-1).$$

[4 Marks]

iii) The prediction error $\varepsilon_1(t)$ associated with the optimal one-step ahead predictor determined in the answer to Question 1b) is

$$\varepsilon_1(t) = y(t+1) - \hat{y}(t+1|t) = \eta(t+1)$$

Hence

$$\text{var}[\varepsilon_1(t)] = 9.$$

Concerning the prediction error $\varepsilon_2(t)$ associated with the optimal two-step sahead predictor determined in the answer to Question 1c)ii), one has:

$$\varepsilon_2(t) = y(t+2) - \hat{y}(t+2|t) = \eta(t) + \frac{4}{9}\eta(t-1)$$

and thus

$$\begin{aligned}\mathrm{var}[\varepsilon_2(t)] &= \mathrm{var}[y(t+2) - \hat{y}(t+2|t)] = \\ &= 1 \cdot \mathrm{var}[\eta(t+2)] + \left(\frac{4}{9}\right)^2 \cdot \mathrm{var}[\eta(t+1)] = \frac{97}{9} \simeq 10.77\,.\end{aligned}$$

The comparison between $\mathrm{var}[\varepsilon_1(t)]$ and $\mathrm{var}[\varepsilon_2(t)]$ gives

$$\mathrm{var}[\varepsilon_2(t)] = \frac{97}{9} > 9 = \mathrm{var}[\varepsilon_1(t)]$$

This confirms that the variance of the prediction error $\mathrm{var}[\varepsilon_r(t)]$ increases with the number r of steps-ahead of the prediction that is computed.

[3 Marks]

2. Solution

a) One refers to **Case 1** of the text of the exam paper with data generated by the model

$$v(t) = e(t) + \frac{1}{2}e(t-1), \quad e(\cdot) \sim WN(0,1)$$

i) The model in prediction form concerning the family of models $\mathcal{M}_1(\theta_1)$ is

$$\widehat{\mathcal{M}_1}(\theta_1): \quad \hat{v}(t|t-1) = av(t-1)$$

The estimate $\widehat{\theta}_1(N) = \widehat{a}(N)$ converges almost surely to the minima of

$$\begin{aligned}
\overline{J}(a) &= \mathbb{E}\left\{[v(t) - \hat{v}(t|t-1)]^2\right\} = \mathbb{E}\left\{[v(t) - av(t-1)]^2\right\} \\
&= \mathbb{E}\left\{\left[e(t) + \frac{1}{2}e(t-1) - a\left(e(t-1) + \frac{1}{2}e(t-2)\right)\right]^2\right\} \\
&= \left(\frac{5}{4}a^2 - a + \frac{5}{4}\right)\text{var}(e)
\end{aligned}$$

Therefore, $\overline{J}(a)$ has a single minimum attained for $\overline{a} = \frac{2}{5}$ and hence $\overline{\theta}_1^{(1)} = \frac{2}{5}$.

[3 Marks]

ii) The model in prediction form concerning the family of models $\mathcal{M}_2(\theta_2)$ is

$$\widehat{\mathcal{M}_2}(\theta_2): \quad \hat{v}(t|t-1) = a_1v(t-1) + a_2v(t-2)$$

The estimate $\widehat{\theta}_2(N) = [\widehat{a}_1(N), \widehat{a}_2(N)]^\top$ converges almost surely to the minima of

$$\begin{aligned}
\overline{J}(\theta_2) &= \mathbb{E}\left\{[v(t) - \hat{v}(t|t-1)]^2\right\} = \mathbb{E}\left\{[v(t) - a_1v(t-1) - a_2v(t-2)]^2\right\} \\
&= \mathbb{E}\left\{\left[e(t) + \frac{1}{2}e(t-1) - a_1\left(e(t-1) + \frac{1}{2}e(t-2)\right)\right.\right. \\
&\qquad \left.\left. - a_2\left(e(t-2) + \frac{1}{2}e(t-3)\right)\right]^2\right\} \\
&= \left(\frac{5}{4}a_1^2 + \frac{5}{4}a_2^2 - a_1 + a_1a_2 + \frac{5}{4}\right)\text{var}(e)
\end{aligned}$$

From the stationarity conditions

$$\frac{\partial}{\partial a_1}\overline{J}(\theta_2) = 0; \quad \frac{\partial}{\partial a_2}\overline{J}(\theta_2) = 0$$

it follows that

$$\begin{cases} \frac{5}{2}a_1 + a_2 - 1 = 0 \\ \frac{5}{2}a_2 + a_1 = 0 \end{cases} \Longrightarrow \begin{cases} \overline{a}_1 = \frac{10}{21} \simeq 0.48 \\ \overline{a}_2 = -\frac{4}{21} \simeq -0.19 \end{cases}$$

and hence $\overline{\theta}_2^{(1)} = \left[\frac{10}{21}, -\frac{4}{21}\right]^\top$.

[3 Marks]

iii) Concerning the model $\mathcal{M}_1(\overline{\theta}_1^{(1)})$, the variance of the prediction error is given by $\overline{J}(\overline{\theta}_1^{(1)})$, that is

$$\overline{J}(\overline{\theta}_1^{(1)}) = \left(\frac{5}{4}\left(\frac{2}{5}\right)^2 - \frac{2}{5} + \frac{5}{4}\right)\text{var}(e) = \frac{21}{20} = 1.05$$

Concerning the model $\mathcal{M}_2(\overline{\theta}_2^{(1)})$, the variance of the prediction error is given by $\overline{J}(\overline{\theta}_2^{(1)})$, that is

$$\overline{J}(\overline{\theta}_2^{(1)}) = \left(\frac{5}{4}\left(\frac{10}{21}\right)^2 + \frac{5}{4}\left(\frac{4}{21}\right)^2 - \frac{10}{21} - \frac{10}{21}\frac{4}{21} + \frac{5}{4}\right)\text{var}(e) = \frac{85}{84} \simeq 1.01$$

The variance of the prediction error associated with the model $\mathcal{M}_2(\overline{\theta}_2^{(1)})$ is slightly smaller than the one associated with the model $\mathcal{M}_1(\overline{\theta}_1^{(1)})$ because of the higher order of model $\mathcal{M}_2(\overline{\theta}_2^{(1)})$ and thus its improved capability to capture the dynamic characteristics of the stochastic process $v(\cdot)$.

[3 Marks]

iv) The prediction error associated $\varepsilon(\cdot)$ when the model $\mathcal{M}_1(\overline{\theta}_1^{(1)})$ is used is given by

$$\varepsilon(t) = v(t) - \hat{v}(t|t-1) = e(t) + \frac{1}{10}e(t-1) - \frac{1}{5}e(t-2)$$

Accordingly, the correlation function $\gamma_\varepsilon(\tau)$ is given by:

$$\gamma_\varepsilon(0) = 1 + \left(\frac{1}{10}\right)^2 + \left(\frac{1}{5}\right)^2 = \frac{21}{20};\ \gamma_\varepsilon(1) = \frac{1}{10} - \frac{1}{50} = \frac{2}{25};\ \gamma_\varepsilon(2) = -\frac{1}{5}$$

and $\gamma_\varepsilon(\tau) = 0, \forall \tau \geq 3$.

The analytical expression of the spectrum $\Gamma_\varepsilon(\omega)$, $\omega \in [-\pi, \pi]$ can be immediately obtained from the above expression of $\gamma_\varepsilon(\tau)$, that is:

$$\begin{aligned}\Gamma_\varepsilon(\omega) &= \gamma_\varepsilon(0) + \gamma_\varepsilon(1)\left(e^{j\omega} + e^{-j\omega}\right) + +\gamma_\varepsilon(2)\left(e^{j2\omega} + e^{-j2\omega}\right) \\ &= \frac{21}{20} + \frac{4}{25}\cos(\omega) - \frac{2}{5}\cos(2\omega)\end{aligned}$$

[3 Marks]

b) Analogously to the answer to Question 2a), one refers to Case 2 of the text of the exam paper with data generated by the model

$$v(t) = e(t) + \frac{1}{2}e(t-1) + \frac{1}{4}e(t-2), \quad e(\cdot) \sim WN(0,1)$$

i) The model in prediction form concerning the family of models $\mathcal{M}_1(\theta_1)$ is

$$\widehat{\mathcal{M}_1}(\theta_1): \quad \hat{v}(t|t-1) = av(t-1)$$

The estimate $\widehat{\theta}_1(N) = \widehat{a}(N)$ converges almost surely to the minima of

$$\bar{J}(a) = \mathbb{E}\left\{[v(t) - \hat{v}(t|t-1)]^2\right\} = \mathbb{E}\left\{[v(t) - av(t-1)]^2\right\}$$
$$= \mathbb{E}\left\{\left[e(t) + \frac{1}{2}e(t-1) + \frac{1}{4}e(t-2) - a\left(e(t-1) + \frac{1}{2}e(t-2) + \frac{1}{4}e(t-3)\right)\right]^2\right\}$$
$$= \left(\frac{21}{16}a^2 - \frac{5}{4}a + \frac{21}{16}\right)\mathrm{var}(e)$$

Therefore, $\bar{J}(a)$ has a single minimum attained for $\bar{a} = \dfrac{10}{21}$ and hence $\bar{\theta}_1^{(2)} = \dfrac{10}{21}$.

[4 Marks]

ii) The model in prediction form concerning the family of models $\mathcal{M}_2(\theta_2)$ is

$$\widehat{\mathcal{M}_2}(\theta_2): \quad \hat{v}(t|t-1) = a_1 v(t-1) + a_2 v(t-2)$$

The estimate $\widehat{\theta}_2(N) = [\widehat{a}_1(N), \widehat{a}_2(N)]^\top$ converges almost surely to the minima of

$$\bar{J}(\theta_2) = \mathbb{E}\left\{[v(t) - \hat{v}(t|t-1)]^2\right\} = \mathbb{E}\left\{[v(t) - a_1v(t-1) - a_2v(t-2)]^2\right\}$$
$$= \mathbb{E}\left\{\left[e(t) + \frac{1}{2}e(t-1) + \frac{1}{4}e(t-2) - a_1\left(e(t-1) + \frac{1}{2}e(t-2) + \frac{1}{4}e(t-3)\right)\right.\right.$$
$$\left.\left. - a_2\left(e(t-2) + \frac{1}{2}e(t-3) + \frac{1}{4}e(t-4)\right)\right]^2\right\}$$
$$= \left(\frac{21}{16}a_1^2 + \frac{21}{16}a_2^2 - \frac{5}{4}a_1 + \frac{5}{4}a_1a_2 - \frac{1}{2}a_2 + \frac{21}{16}\right)\mathrm{var}(e)$$

From the stationarity conditions

$$\frac{\partial}{\partial a_1}\bar{J}(\theta_2) = 0; \quad \frac{\partial}{\partial a_2}\bar{J}(\theta_2) = 0$$

it follows that

$$\begin{cases} \dfrac{21}{8}a_1 + \dfrac{5}{4}a_2 - \dfrac{5}{4} = 0 \\[2ex] \dfrac{21}{8}a_2 + \dfrac{5}{4}a_1 - \dfrac{1}{2} = 0 \end{cases} \Longrightarrow \begin{cases} \bar{a}_1 = \dfrac{170}{341} \simeq 0.5 \\[2ex] \bar{a}_2 = -\dfrac{16}{341} \simeq -0.05 \end{cases}$$

and hence $\bar{\theta}_2^{(1)} = \left[\dfrac{170}{341}, -\dfrac{16}{341}\right]^\top$.

[4 Marks]

3. Solution

a) Inspecting the block-scheme shown in Fig. 3.1 in the text of the exam paper, one immediately gets:

$$\begin{cases} v(t+1) = -\frac{1}{2}v(t) + \eta(t) \\ w(t+1) = -\frac{1}{4}w(t) + 4v(t) + \xi(t) \\ y(t) = v(t) + w(t) \end{cases}$$

Consider the first equation above:

$$v(t+1) = -\frac{1}{2}v(t) + \eta(t)$$

Clearly, with the usual interpretation of z as a one-step forward shift operator, one has:

$$v(t) = \frac{1}{z+\frac{1}{2}}\eta(t)$$

Since $\eta(\cdot)$ is a stationary process and $\frac{1}{z+\frac{1}{4}}$ is an asympotically stable discrete-time transfer function, it follows that $v(\cdot)$ is a stationary stochastic process.

Now, one considers the second equation:

$$w(t+1) = -\frac{1}{4}w(t) + 4v(t) + \xi(t)$$

Again, with the usual interpretation of z as a one-step forward shift operator, one obtains:

$$w(t) = \frac{1}{z+\frac{1}{4}}\left[4v(t) + \xi(t)\right]$$

Introducing a new input stochastic process $\rho(t) = 4v(t) + \xi(t)$, it follows immediately that $\rho(\cdot)$ is a stationary process. Since $\frac{1}{z+\frac{1}{4}}$ is an asympotically stable discrete-time transfer function, it follows that $w(\cdot)$ is a stationary stochastic process.

Finally, as

$$y(t) = v(t) + w(t)$$

the stationarity of $v(\cdot)$ and $w(\cdot)$ implies the stationarity of $y(\cdot)$.

[3 Marks]

b) One considers again the dynamic equation for v, that is:

$$v(t+1) = -\frac{1}{2}v(t) + \eta(t)$$

and applies the expected value operator $\mathbb{E}[\cdot]$ on both sides, thus getting

$$\mathbb{E}[v(t+1)] = -\frac{1}{2}\mathbb{E}[v(t)] + \mathbb{E}[\eta(t)]$$

Due to the stationarity of $v(\cdot)$ established in the answer to Question 3a), it follows that

$$m_v = -\frac{1}{2}m_v + 1 \Longrightarrow m_v = \frac{2}{3}$$

[3 Marks]

c) From the answer to Question 3a) one gets:

$$w(t+1) = -\frac{1}{4}w(t) + 4v(t) + \xi(t)$$

The expected value operator $\mathbb{E}[\cdot]$ applied on both sides gives

$$\mathbb{E}[w(t+1)] = -\frac{1}{4}\mathbb{E}[w(t)] + 4m_v + \mathbb{E}[\xi(t)]$$

Again, due to the stationarity of $w(\cdot)$ established in the answer to Question 3a), one gets

$$m_w = -\frac{1}{4}m_w + 4\cdot\frac{2}{3} + 3 \Longrightarrow m_w = \frac{68}{15} \simeq 4.53$$

[3 Marks]

d) For the sake of convenience, one introduces the following zero-mean stochastic processes:

$$\begin{aligned}\widetilde{v}(t) &= v(t) - m_v\\ \widetilde{w}(t) &= w(t) - m_w\\ \widetilde{\eta}(t) &= \eta(t) - 1\\ \widetilde{\xi}(t) &= \xi(t) - 3\end{aligned}$$

Then, the original equations determined in the answer to Question 3a) can be equivalently rewritten as

$$\begin{cases}\widetilde{v}(t+1) + m_v = -\frac{1}{2}(\widetilde{v}(t) + m_v) + \widetilde{\eta}(t) + 1\\[2ex] \widetilde{w}(t+1) + m_w = -\frac{1}{4}(\widetilde{w}(t) + m_w) + 4(\widetilde{v}(t) + m_v) + \widetilde{\xi}(t) + 3\end{cases}$$

and hence, using the values of m_v and m_w computed in the answers to Question 3b) and 3c), one gets respectively:

$$\begin{cases}\widetilde{v}(t+1) = -\frac{1}{2}\widetilde{v}(t) + \widetilde{\eta}(t)\\[2ex] \widetilde{w}(t+1) = -\frac{1}{4}\widetilde{w}(t) + 4\widetilde{v}(t) + \widetilde{\xi}(t)\end{cases}$$

Now, one considers the above equation for $\widetilde{v}(t)$, that is

$$\widetilde{v}(t+1) = -\frac{1}{2}\widetilde{v}(t) + \widetilde{\eta}(t),$$

and applies the variance operator var$[\cdot]$ on both sides. Since the process $\widetilde{v}(\cdot)$ is stationary according to the answer to Question 3a), and owing to the fact that $\widetilde{v}(t)$ is uncorrelated with $\widetilde{\eta}(t)$, it follows that

$$\mathrm{var}[\widetilde{v}(t)] = \left(-\frac{1}{2}\right)^2 \mathrm{var}[\widetilde{v}(t)] + 1$$

and hence

$$\lambda_{vv} = \mathbb{E}[(v(t) - m_v)^2] = \mathrm{var}[\widetilde{v}(t)] = \frac{4}{3} \simeq 1.33$$

Now, one has to compute

$$\lambda_{vw} = \mathrm{cov}[v(t), w(t)] = \mathbb{E}[(v(t) - m_v)(w(t) - m_w)] = \mathbb{E}[\widetilde{v}(t)\widetilde{w}(t)] = \mathrm{cov}[\widetilde{v}(t), \widetilde{w}(t)]$$

As a first step, it is convenient to multiply the equations giving $\widetilde{v}(t+1)$ and $\widetilde{w}(t+1)$:

$$\widetilde{v}(t+1)\widetilde{w}(t+1) = \left[-\frac{1}{2}\widetilde{v}(t) + \widetilde{\eta}(t)\right] \cdot \left[-\frac{1}{4}\widetilde{w}(t) + 4\widetilde{v}(t) + \widetilde{\xi}(t)\right]$$

Recalling the stationarity of $\widetilde{v}(\cdot)$ and $\widetilde{w}(\cdot)$ and the mutual un-correlation between $\widetilde{v}(t)$ and $\widetilde{\eta}(t)$ and $\widetilde{\xi}(t)$ and between $\widetilde{w}(t)$ and $\widetilde{\eta}(t)$ and $\widetilde{\xi}(t)$, applying the expected value operator $\mathbb{E}[\cdot]$ on both sides gives

$$\mathbb{E}[\widetilde{v}(t+1)\widetilde{w}(t+1)] = \frac{1}{8}\mathbb{E}[\widetilde{v}(t)\widetilde{w}(t)] - 2\mathbb{E}[\widetilde{v}(t)^2]$$

Due to stationarity, it follows that

$$\mathbb{E}[\widetilde{v}(t+1)\widetilde{w}(t+1)] = \mathbb{E}[\widetilde{v}(t)\widetilde{w}(t)]$$

and hence

$$\lambda_{vw} = \frac{1}{8}\lambda_{vw} - \frac{8}{3} \Longrightarrow \lambda_{vw} = -\frac{64}{21} \simeq -3.05$$

Finally, to compute λ_{ww}, one applies the variance operator var[$\cdot$] to both sides of the equation

$$\widetilde{w}(t+1) = -\frac{1}{4}\widetilde{w}(t) + 4\widetilde{v}(t) + \widetilde{\xi}(t)$$

thus getting

$$\begin{aligned}\mathrm{var}[\widetilde{w}(t+1)] &= \mathbb{E}\left\{\left[-\frac{1}{4}\widetilde{w}(t) + 4\widetilde{v}(t) + \widetilde{\xi}(t)\right]^2\right\} \\ &= \frac{1}{16}\mathrm{var}[\widetilde{w}(t)] + 16\mathrm{var}[\widetilde{v}(t)] + 1 - 2\mathbb{E}[\widetilde{w}(t)\cdot\widetilde{v}(t)]\end{aligned}$$

and thus

$$\lambda_{ww} = \frac{1}{16}\lambda_{ww} + \frac{32}{21} + 1 + \frac{128}{21} \Longrightarrow \lambda_{ww} = \frac{3184}{105} \simeq 30.32$$

[7 Marks]

e) The transfer function $G_{\eta y}(z)$ is obtained by setting $\xi(t) = 0, \forall t \geq 0$. Inspecting the block-scheme reported in Fig. 3.1 of the text of the exam paper, one immediately gets:

$$y(t) = \frac{4}{(z+1/4)(z+1/2)}\eta(t) + \frac{1}{z+1/2}\eta(t) = \frac{z+17/4}{(z+1/4)(z+1/2)}\eta(t)$$

Thus

$$G_{\eta y}(z) = \frac{z+17/4}{(z+1/4)(z+1/2)}$$

The transfer function $G_{\xi y}(z)$ is obtained by setting $\eta(t) = 0, \forall t \geq 0$. Inspecting again the block-scheme reported in Fig. 3.1 of the text of the exam paper, one trivially gets:

$$G_{\xi y}(z) = \frac{1}{z+1/4}$$

[4 Marks]

4. Solution

a) The empirical mean estimator defined in the text of the exam paper (Question 4a)) is given by:

$$\tilde{x} = \frac{1}{3}(x_1 + x_2 + x_3)$$

One applies the expected value operator:

$$\mathbb{E}[\tilde{x}] = \frac{1}{3}\mathbb{E}(x_1) + \frac{1}{3}\mathbb{E}(x_2) + \frac{1}{3}\mathbb{E}(x_3) = \frac{1}{3}\bar{x} + \frac{1}{3}(0.1 + \bar{x}) + \frac{1}{3}\bar{x} = \bar{x} + \frac{1}{30}.$$

It can be noticed that the empirical mean estimator is biased.

Now, one addresses the computation of $\text{var}(\tilde{x})$. Since the data x_1, x_2, and x_3 are supposed to be mutually independent (see the text of Question 4), one can write:

$$\mathbb{E}[(x_1 - \bar{x})(x_2 - 0.1 - \bar{x})] = 0; \; \mathbb{E}[(x_1 - \bar{x})(x_3 - \bar{x})] = 0; \; \mathbb{E}[(x_2 - 0.1 - \bar{x})(x_3 - \bar{x})] = 0.$$

Therefore:

$$\text{var}(\tilde{x}) = \mathbb{E}\left\{\left[\frac{1}{3}(x_1 - \bar{x}) + \frac{1}{3}(x_2 - 0.1 - \bar{x}) + \frac{1}{3}(x_3 - \bar{x})\right]^2\right\}$$
$$= \frac{1}{9}\mathbb{E}\left\{[(x_1 - \bar{x}) + (x_2 - 0.1 - \bar{x}) + (x_3 - \bar{x})]^2\right\} = \frac{1}{9}[\text{var}(x_1) + \text{var}(x_2) + \text{var}(x_3)]$$

Since

$$x_1 \sim \mathcal{G}(\bar{x}, 1/3), \quad x_2 \sim \mathcal{G}(0.1 + \bar{x}, 2), \quad x_3 \sim \mathcal{G}(\bar{x}, 1),$$

one gets $\text{var}(\tilde{x}) = \dfrac{10}{27} \simeq 0.37$.

b) The estimator considered in the text of Question 4b) is:

$$\hat{x}(a, b, c, d) = a x_1 + b x_2 + c x_3 + d.$$

The application of the expected value operator to both sides gives:

$$\mathbb{E}[\hat{x}(a, b, c, d)] = a\mathbb{E}(x_1) + b\mathbb{E}(x_2) + c\mathbb{E}(x_3) + d = (a + b + c)\bar{x} + 0.1b + d.$$

Then, the estimator $\hat{x}(a, b, c, d)$ is unbiased if $a + b + c = 1$ and $0.1b + d = 0$.

[4 Marks]

c) The estimator considered in the text of Question 4c) is:

$$\check{x}(\alpha, \beta, \gamma) = \alpha x_1 + \beta x_2 + \gamma x_3.$$

The application of the expected value operator to both sides gives:

$$\mathbb{E}[\check{x}(\alpha, \beta, \gamma)] = \alpha\mathbb{E}(x_1) + \beta\mathbb{E}(x_2) + \gamma\mathbb{E}(x_3) = (\alpha + \beta + \gamma)\bar{x} + 0.1\beta.$$

Then, the estimator $\check{x}(\alpha, \beta, \gamma)$ is unbiased if $\alpha + \gamma = 1$ and $\beta = 0$.

[4 Marks]

d) With reference to the estimator $\widehat{x}(a,b,c,d)$, one has:

$$\begin{aligned}\operatorname{var}[\widehat{x}(a,b,c,d)] &= \mathbb{E}\left\{[a(x_1-\bar{x})+b(x_2-0.1-\bar{x})+c(x_3-\bar{x})]^2\right\}\\ &= a^2\operatorname{var}(x_1)+b^2\operatorname{var}(x_2)+c^2\operatorname{var}(x_3)+2ab\mathbb{E}[(x_1-\bar{x})(x_2-0.1-\bar{x})]\\ &\quad +2ac\mathbb{E}[(x_1-\bar{x})(x_3-\bar{x})]+2bc\mathbb{E}[(x_2-0.1-\bar{x})(x_3-\bar{x})]\end{aligned}$$

Since the samples x_1, x_2, and x_3 are supposed to be mutually independent, it follows that:

$$\mathbb{E}[(x_1-\bar{x})(x_2-0.1-\bar{x})]=0;\ \mathbb{E}[(x_1-\bar{x})(x_3-\bar{x})]=0;\ \mathbb{E}[(x_2-0.1-\bar{x})(x_3-\bar{x})]=0.$$

Then:

$$\operatorname{var}[\widehat{x}(a,b,c,d)] = a^2\operatorname{var}(x_1)+b^2\operatorname{var}(x_2)+c^2\operatorname{var}(x_3)$$

Owing to

$$x_1\sim\mathscr{G}(\bar{x},1/3),\quad x_2\sim\mathscr{G}(0.1+\bar{x},2),\quad x_3\sim\mathscr{G}(\bar{x},1),$$

one gets

$$\operatorname{var}[\widehat{x}(a,b,c,d)] = \frac{1}{3}a^2+2b^2+c^2.$$

To ensure unbiasedness according to the answer to Question 4b) the constraints $a+b+c=1$ and $d=-0.1b$ have to be satisfied thus leading to

$$\operatorname{var}(\widehat{x}) = \frac{1}{3}a^2+2b^2+(1-a-b)^2 = \frac{4}{3}a^2+3b^2+1-2a-2b+2ab.$$

To minimise $\operatorname{var}(\widehat{x})$ with respect to a,b, one first computes:

$$\frac{\partial}{\partial a}\operatorname{var}(\widehat{x}) = \frac{8}{3}a+2b-2;\quad \frac{\partial}{\partial b}\operatorname{var}(\widehat{x}) = 2a+6b-2$$

From the stationarity conditions

$$\frac{\partial}{\partial a}\operatorname{var}(\widehat{x}) = 0;\quad \frac{\partial}{\partial b}\operatorname{var}(\widehat{x}) = 0$$

it follows that

$$\begin{cases}\frac{8}{3}a+2b-2=0\\ 2a+6b-2=0\end{cases} \Longrightarrow \begin{cases}a^\circ=\frac{2}{3}\simeq 0.67\\ b^\circ=\frac{1}{9}\simeq 0.11\end{cases}$$

and $c^\circ=1-a^\circ-b^\circ=\frac{2}{9}\simeq 0.22$, $d^\circ=-0.1b^\circ=-1/90\simeq -0.01$.

Concerning the estimator $\check{x}(\alpha,\beta,\gamma)$ analogous calculations can be carried out. Specifically, one gets

$$\operatorname{var}[\check{x}(\alpha,\beta,\gamma)] = \frac{1}{3}\alpha^2+2\beta^2+\gamma^2.$$

In this case, the unbiasedness conditions (see the answer to Question 4c)) give $\alpha+\gamma=1$ and $\beta=0$ and hence one gets

$$\operatorname{var}(\check{x}) = \frac{1}{3}\alpha^2+(1-\alpha)^2 = \frac{4}{3}\alpha^2-2\alpha+1.$$

The minimisation of $\operatorname{var}(\check{x})$ with respect to α immediately gives $\alpha^\circ=3/4=0.75$ and $\gamma^\circ=1-\alpha^\circ=1/4=0.25$.

[6 Marks]

e) From the answer to Question 4d) one has:

$$\text{var}[\hat{x}(a,b,c,d)] = \frac{1}{3}a^2 + 2b^2 + c^2$$

and

$$\text{var}[\check{x}(\alpha,\beta,\gamma)] = \frac{1}{3}\alpha^2 + 2\beta^2 + \gamma^2$$

Replacing into the above formulas the optimal values $a^\circ = \frac{2}{3}, b^\circ = \frac{1}{9}, c^\circ = \frac{2}{9}, d^\circ = -\frac{1}{90}$ and $\alpha^\circ = \frac{3}{4}, \beta^\circ = 0, \gamma^\circ = \frac{3}{4}$, respectively, it follows that

$$\text{var}[\hat{x}(a^\circ,b^\circ,c^\circ,d^\circ)] = \frac{2}{9} \simeq 0.22; \quad \text{var}[\check{x}(\alpha^\circ,\beta^\circ,\gamma^\circ)] = \frac{1}{4} = 0.25$$

Moreover, from the answer to Question 4a), one has

$$\text{var}(\tilde{x}) = \frac{10}{27} \simeq 0.37$$

The comparison between the above three variances leads to the conclusion that the best estimator is $\hat{x}(a^\circ,b^\circ,c^\circ,d^\circ)$. This is not surprising as this unbiased estimator is characterised by more parameters compared to the unbiased estimator $\check{x}(\alpha^\circ,\beta^\circ,\gamma^\circ)$. The fact that the empirical mean estimator $\tilde{x}$ is worse is also not surprising since no free parameters to be optimised are available (and this estimator is also biased, as already noted in the answer to Question 4a).

[3 Marks]