UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 1998

BEng Honours Degree in Computing Part I
MEng Honours Degrees in Computing Part I
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the Associateship of the City and Guilds of London Institute

PAPER 1.3

DISCRETE MATHEMATICS
Thursday, May 7th 1998, 2.00 - 3.30

Answer THREE questions

For admin. only: paper contains 4 questions

1 *Notation.* For any set A, $|A|$ is the cardinality of A. Also, if $f: A \rightarrow B$ is a function, then $f[A]$ is the image of A under f, that is,

$$f[A] = \{f(a) \mid a \in A\}$$

a Let $f: A \rightarrow B$ be a function.

i) Explain what it means for f to be *onto*; for f to be *one-one*; and for f to be a *bijection.*

ii) Assume that A is finite. What, in general, is the relation between $|A|$ and $|f[A]|$? Prove that, if f is not one-one, then $|f[A]| < |A|$.

b Further notation : Σ^* is the set of strings (or words) in a given finite alphabet Σ; $\Sigma^{(n)}$ is the set of strings of length $\leq n$. Tail $:\Sigma^* \rightarrow \Sigma^*$ is the function which deletes the first character of a non-null string. More precisely :

Tail(ax) = x, where $a \in \Sigma$; Tail(ε) = ε, where ε is the null string.

i) Show that Tail is onto, but that it is not a bijection.

ii) Is it possible to define a function $h:\Sigma^{(n)} \rightarrow \Sigma^{(n)}$ that is onto but not a bijection? Justify your answer.

c Let V be a set of voters, and P be a set of proposals to be voted on. Each voter must vote for or against each proposal.

i) State the Pigeonhole Principle.

ii) If there are three proposals and nine voters, show that at least two voters must be voting exactly the same way.

iii) In general, if there are n proposals, what is the least value of $|V|$ which ensures that at least two voters vote the same way?

2a i) Define the terms *transitive, symmetric,* and *anti-symmetric* as applied to a relation R.

ii) Let R be the relation $\{(1,2),(3,1),(4,5),(5,6),(5,1)\}$ on the set $\{1,2,...,6\}$. Show the relation R, and its transitive closure R^+, as a directed graph.

iii) Is the relation R^+ (same as in ii)) anti-symmetric? In general, how would you tell, from the directed graph of a relation S, whether S is anti-symmetric?

iv) Let S,T be relations on a set A. Decide whether each of the following statements is true or false, giving a proof or a counterexample as appropriate:

if S,T are transitive, then $S \cup T$ is transitive

if S,T are symmetric, then $S \cap T$ is symmetric.

b A *pre-order* is a reflexive, transitive relation.

i) What needs to be added to this definition, so as to yield definitions of *partial order* and *total order* ?

Let $\leq$ be the usual order defined on the natural numbers $\mathbb{N}$. It is desired to define an "order of magnitude" for *sets* of natural numbers. (The sets may be assumed finite and non-empty.) The relations R_1, R_2, as follows, represent two attempts to define such an order:

$$A\, R_1\, B \quad \text{iff} \quad \forall x \in A\ \forall y \in B.\ x \leq y$$

$$A\, R_2\, B \quad \text{iff} \quad \forall y \in B\ \exists x \in A.\ x \leq y.$$

(For example, $\{2,9\}\ R_2\ \{2,3,7\}$.)

ii) Show that the relation R_1 is not a pre-order.

iii) Show that R_2 is a pre-order. Determine whether it is also a partial order.

The two parts carry, respectively, 60% and 40% of the marks.

turn over

3a Let G be an undirected graph.

i) What is a *path* of G? What is a *cycle* of G?

ii) Let k be some positive integer. What does it mean for G to be *k-colourable*?

iii) State the Four Colour Theorem for map colouring.
What is the equivalent statement for 4-colouring graphs?

iv) Give an example of a map which requires 4 colours (i.e. cannot be 3-coloured). Give also the corresponding graph.

v) Show how to construct for each positive integer k a graph which cannot be k-coloured.

b Let G be an undirected graph.

i) Suppose that G is 2-colourable. Show that G has no odd length cycles.

ii) Suppose instead that G has no odd length cycles. Show that G is 2-colourable.

4a i) Describe the Binary Search algorithm for searching an ordered list L.

ii) Give a decision tree for Binary Search on a list of length 10 where the element x being searched for is known to be in the list.

iii) Calculate the average number of comparisons for your algorithm of (ii), assuming x is equally likely to be in any position in the list.

b You are given 4 coins, each of which is either genuine or counterfeit. You are not told how many are genuine—it could be any number between 0 and 4. The task is to split the coins into two groups according to whether they are genuine or counterfeit. You are not asked to say which group has the genuine coins—in fact you will not be able to determine this.

The coins can be told apart because the counterfeit coins have a different weight from the genuine ones (all the counterfeit coins weigh the same). You are given a pair of scales of the balance type, which will tell you whether two coins weigh the same or differently (so that a weighing has two possible outcomes).

i) Give a decision tree for an algorithm using weighings to split the coins into two groups. Your algorithm should be optimal in the sense of using as few weighings as possible.

ii) What is the worst case number of weighings?

iii) Show by consideration of decision trees that your algorithm is indeed optimal.

End of paper

