UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 1999

BEng Honours Degree in Computing Part III
BEng Honours Degree in Information Systems Engineering Part III
MEng Honours Degree in Information Systems Engineering Part III
BSc Honours Degree in Mathematics and Computer Science Part III
MSci Honours Degree in Mathematics and Computer Science Part III
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the
Associateship of the City and Guilds of London Institute
Associateship of the Royal College of Science

PAPER 3.43 / I 3.22

OPERATIONS RESEARCH

Tuesday, April 27th 1999, 10.00 – 12.00

Answer THREE questions

For admin. only:
paper contains 4 questions

1a You are given the following linear programming problem:

$$\begin{array}{llcccccc}
\max & x_1 & + & x_2 & & & & \\
\text{s.t.} & 4x_1 & + & x_2 & - & 7 & \leq & 0 \\
 & 2x_1 & - & x_2 & + & 1 & \geq & 0 \\
 & \multicolumn{7}{l}{x_1 \geq 0,\ x_2 \geq 0.}
\end{array}$$

Solve it using the simplex method. Graphical solution is not acceptable.

b Formulate the dual of the problem in part (a). Solve it using the simplex method.

c Compare the results you obtained in parts (a) and (b). What did you expect? What did you find? Explain.

(The three parts carry, respectively, 25%, 50% and 25% of the marks).

Turn over ...

2a A car manufacturing company can produce three types of cars: small, middle class and luxury. Selling one car of each results in a profit of £2200, £3500 and £6500, respectively. The company wants to determine its monthly production plan such that it maximises the profit while it takes into account the following constraints in the planning period.

- Production constraints:
  The total capacity of the assembly line is 300,000 manufacturing hours (there can be several cars on the line at the same time).
  The capital available to finance the production costs is £60 million.
- Marketing constraints:
  At least five times more middle class cars can be sold than luxury.
  Production of small cars cannot be more than three times that of the middle class.
- Labour constraint:
  There is an agreement with the trade unions that specialist workers must fully be employed and thus avoid redundancies. Their joint capacity is at least 40,000 hours.

The production of one car of each class requires the following quantities:

	Assembly hours	Production costs	Specialist hours
Small	16	3620	2
Middle class	30	6000	4
Luxury	45	11250	10

You are in charge of planning. Formulate a linear programming model that maximises the profit of the company for the planning period. Do not solve the problem.

b You are given the following linear programming problem.

$$\begin{array}{rrrrrrrl} \max\ x_0 = & -x_1 & - & 2x_2 & - & x_3 & & \\ \text{s.t.} & -x_1 & + & x_2 & + & x_3 & \geq & -1 \\ & x_1 & + & x_2 & + & 2x_3 & \leq & 4 \\ & x_1 \geq 0, & & x_2 \text{ free}, & & x_3 \geq 0, & & \end{array}$$

Use the simplex algorithm to solve the problem.

c Having solved the problem numerically, determine whether shadow prices can be easily obtained and, if yes, give them and explain where they belong to and what they mean.
(The parts carry, respectively, 35%, 45% and 20% of the marks).

3 a The search down a branch in the branch-and-bound algorithm of integer programming terminates under three basic conditions. One of these is when an optimal integer solution to the problem is achieved. Describe the other two conditions, explaining the reasoning when necessary.

b Consider the integer linear programming problem

$$\max \left\{ x_0 = 7\,x_1 + 9\,x_2 \;\middle|\; -x_1 + 3\,x_2 \le 6;\ 7\,x_1 + x_2 \le 35;\ x_1, x_2 \ge 0 \ \&\ \text{integer} \right\}.$$

Ignoring the integer requirement and applying the simplex algorithm to the resulting linear program (LP), we obtain the following optimal tableau:

	x_1	x_2	x_3	x_4	RHS
x_0	0	0	28/11	15/11	63
x_2	0	1	7/22	1/22	7/2
x_1	1	0	-1/22	3/22	9/2

Derive a Gomory cut based on the first constraint in the above tableau. Show that this cut excludes the optimal solution to the above LP. Formulate the next linear programming problem that needs to be solved. [Do not solve the integer programming problem.]

c Formulate the following model as a mixed integer linear programming problem

$$\text{maximise } \ 3x_1 + 2\,f_1(x_2) + 9\,x_3 + 7\,f_2(x_4)$$

subject to:

$x_j \ge 0$; $j = 1, \ldots, 4$; $x_1 = 1$, or 3, or 5

at least two of the following four inequalities hold:

$5\,x_1 + 4\,x_2 + 3\,x_3 + 2\,x_4 \le 50$; $3\,x_1 + 7\,x_2 - 2\,x_3 - 5\,x_4 \le 60$;

$2\,x_1 - 3\,x_2 - 8\,x_3 + 7\,x_4 \le 40$; $7\,x_1 + 6\,x_2 - 2\,x_3 - 4\,x_4 \le 70$;

$$f_1(x_2) = \begin{cases} -7 + 11\,x_2 & \text{if } x_2 > 0 \\ 0 & \text{if } x_2 = 0 \end{cases} \quad ; \quad f_2(x_4) = \begin{cases} -4 + 3\,x_4 & \text{if } x_4 > 0 \\ 0 & \text{if } x_4 = 0 \end{cases}$$

[Do not solve the resulting mixed integer programming problem. Hint: to account for f_1, f_2 the additional constraints $x_j \le M\,y_j$ are needed where y_j are binary variables, $j = 2, 4$ and M is a large positive number.]
(All parts carry equal marks)

4 a For a two-person zero-sum game with an $m \times n$ reward matrix, let

$$x^* = \begin{bmatrix} x_1^* \\ x_2^* \\ \vdots \\ x_m^* \end{bmatrix} \quad ; \qquad y^* = \begin{bmatrix} y_1^* \\ y_2^* \\ \vdots \\ y_n^* \end{bmatrix}$$

be the solutions of the row player's and column player's linear programs respectively. If the row player deviates from her optimal strategy, can she increase her expected reward against y^*? Briefly discuss and analyse. [Hint: let v^* be the value of the game for the row player and develop your analysis using this value.]

b Two players in a game simultaneously put out either one or two fingers. Each player must also announce the number of fingers that she believes her opponent has put out. If neither or both player correctly guess the number of fingers put out by each opponent, the game is a draw. Otherwise, the player who guesses correctly wins (from the other player) the sum (in £'s) of the fingers put out by the two players. Determine the reward matrix for the game; check if the game has a saddle point; and, if it has not, formulate the linear programming problems of the players with all variables restricted to be nonnegative. Do not solve the problem.
[Hint: let (i,j) represent the strategy of putting out i fingers and guessing that the opponent has put out j fingers. Thus, each player has four strategies.]
(All parts carry equal marks)