Master Copy.

Question 1 — Linear Optimal Control — E4.22 / C1.2 / ISE 4.5

Solutions - 2008

(a) Bookwork.

For step $k < N$

Dynamic programming iteration is:

$$J_k(x_k) = \min_{u_k} x_k' Q x_k + u_k' R u_k + J_{k+1}(Ax_k + Bu_k)$$

where the cost-to-go function is assumed to be quadratic, ie. $J_{k+1}(x_{k+1}) = x_{k+1}' P_{k+1} x_{k+1}$. Assume $P_{k+1} \geq 0$ (see below)

$$\Rightarrow J_k(x_k) = \min_{u_k} x_k' Q x_k + u_k' R u_k + (Ax_k + Bu_k)' P_{k+1} (Ax_k + Bu_k) \quad [3]$$

$$= x_k' Q x_k + \min_{u_k} u_k' R u_k + u_k' B' P_{k+1} B u_k + 2x_k' A' P_{k+1} B u_k + x_k' A' P_{k+1} A x_k.$$

Differentiating with respect to u_k and setting to zero: [2]

$$(R + B' P_{k+1} B) u_k + B' P_{k+1} A x_k = 0$$

Assume $P_{k+1} \geq 0$ and $R > 0 \Rightarrow R + B' P_{k+1} B > 0$ [2]

$$\Leftrightarrow u_k = -(R + B' P_{k+1} B)^{-1} B' P_{k+1} A x_k$$

(this guarantees that u_k is a minimum & unique) [2] & that if $P_{k+1} \geq 0 \Rightarrow P_k \geq 0$

Substitute the above into expression for $J_k(x_k)$, we get [3]

$$J_k(x_k) = x_k' \underbrace{\left[A' P_{k+1} A + Q - A' P_{k+1} B (B' P_{k+1} B + R)^{-1} B' P_{k+1} A\right]}_{P_k} x_k$$

$P_k \geq 0$ if, in addition, $Q \geq 0$.

$$\Rightarrow P_k = A'(P_{k+1} - P_{k+1} B (B' P_{k+1} B + R)^{-1} B' P_{k+1}) A + Q \quad [2]$$

step $k=N$: The same holds for the boundary condition with $P_N = Q$

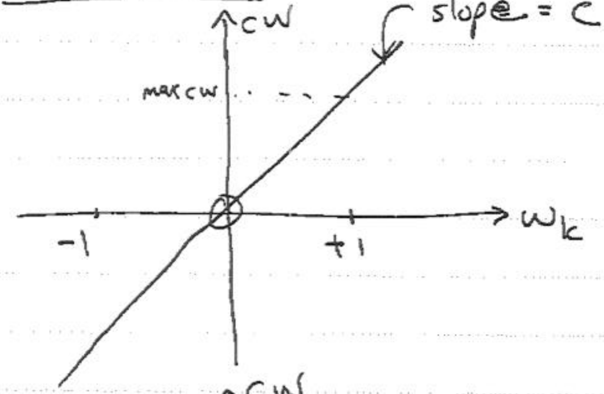
$$\Rightarrow P_{N-1} = A'(P_N - P_N B (B' P_N B + R)^{-1} B' P_N) A + Q \quad [2]$$

(1b) Replace Q with $\frac{1}{2}(Q+Q')$ and R with $\frac{1}{2}(R+R')$ [2]

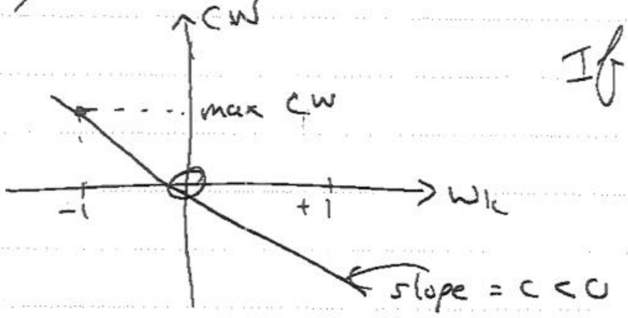
This is because $x'Mx = \frac{1}{2}x'(M+M')x$ for any matrix M, where $M+M'$ is symmetric. [2]

Question 2

a)



If $c \geq 0 \Rightarrow \max_{-1 \leq w_k \leq 1} c w_k = c$ [1]



If $c < 0 \Rightarrow \max_{-1 \leq w_k \leq 1} c w_k = -c$ [1]

If c is any scalar $\Rightarrow \max_{-1 \leq w_k \leq 1} c w_k = |c|$. [2]

(Partly Done ~~in class~~)

b) Partly done in class, but new problem

$\Rightarrow x_{k+1} = (a+L)x_k + w_k$.

$x_1 = (a+L)x_0 + w_0$

$x_2 = (a+L)x_1 + w_1 = (a+L)^2 x_0 + (a+L)w_0 + w_1$

$\vdots$

$x_k = (a+L)^k x_0 + \sum_{i=0}^{k-1} (a+L)^i w_{k-i-1}$

$\Rightarrow$ if $x_0 = 0 \Rightarrow x_k = \sum_{i=0}^{k-1} (a+L)^i w_{k-i-1}$ [2]

(from part a)

$\Rightarrow \max_{-1 \leq w_0, \dots w_{k-1} \leq 1} x_k = \sum_{i=0}^{k-1} |(a+L)^i|$ [2]

also, $\max_{-1 \leq w_0, \dots, w_{k-1} \leq 1} -x_k = \sum_{i=0}^{k-1} |-(a+L)^i| = \sum_{i=0}^{k-1} |(a+L)^i|$

$\Rightarrow \max_{w_0, \dots, w_{k-1}} x_k = \max_{w_0, \dots, w_{k-1}} -x_k \Rightarrow \max_{w_0, \dots, w_{k-1}} x_k = \max_{w_0, \dots, w_{k-1}} |x_k|$ [2]

Since $S_{k+1} - S_k = |(a+L)^k| \geq 0$, the sequence is non-decreasing. [1]

2b) continued...

The sequence $S_k = \sum_{i=0}^{k-1} |a+L|^i$ is convergent if and only if $|a+L|<1$. Since $\{S_k\}$ is an ~~in~~ increasing sequence, it is bounded iff $|a+L|<1$. [1]

~~2d)~~

2c) Since $\{S_k\}$ is non-decreasing, we have that

$$\max_{k=0,1,\dots} \{S_k\} = \lim_{k\to\infty} S_k = \lim_{k\to\infty} \sum_{i=0}^{k-1} |a+L|^i$$

$$= \sum_{i=0}^{\infty} |a+L|^i = \frac{1}{1-|a+L|} \iff |a+L|<1$$ [2]

We have two cases: [1]

I) $0 \le a+L < 1 \Rightarrow |a+L| = a+L$

$$\Rightarrow \arg\min_{-a\le L<1-a} \frac{1}{1-a-L} = L^*$$

$\frac{1}{1-a-L}$ is minimised if the denominator $1-a-L$ is maximised, ie.

we need to make L as small as possible $\Rightarrow L^* = -a$
(−L as large) [2]

II) $-1 < a+L \le 0 \Rightarrow |a+L| = -a-L$

$$\Rightarrow L^* = \arg\min_{-1-a\le L\le -a} \frac{1}{1+a+L}$$

$\frac{1}{1+a+L}$ is minimised if $1+a+L$ is maximised, ie. we need to make L as large as possible $\Rightarrow L^* = -a$ [2]

$$\Rightarrow L^* = -a$$ [1]

QED.

Question 3

a) Bookwork.

$(\bar{Q}^{1/2}, \bar{A})$ detectable
$(\bar{A}, \bar{B})$ stabilisable
$\bar{Q} \geq 0$
$\bar{R} > 0$

[4]

b) New problem.
With the given choice of $Q(t)$ and $R(t)$ we have

$$u^{*}(\cdot) = \arg\min_{u} \int_0^{\infty} e^{2\alpha t} x(t)' \bar{Q} x(t) + e^{2\alpha t} u(t)' \bar{R} u(t)\, dt$$

[2]

Comparing with the cost in part (a) and referring to the hint, these two problems are equivalent if we make the ~~subst~~ change of variables: [2]

$$z(t) = e^{\alpha t} x(t) \quad \text{and} \quad v(t) = e^{\alpha t} u(t). \quad [2]$$

$$\begin{aligned} \Rightarrow \dot{z} &= \alpha e^{\alpha t} x + e^{\alpha t} \dot{x} = \alpha z + e^{\alpha t}(Ax + Bu) \\ &= \alpha z + A e^{\alpha t} x + B e^{\alpha t} u \\ &= \alpha z + A z + B v \\ &= (A + \alpha I) z + B v \end{aligned}$$

[4]

$$\Rightarrow \bar{A} = (A + \alpha I) \quad \text{and} \quad \bar{B} = B$$

[2]

$$\begin{aligned} \Rightarrow u^{*}(t) &= e^{-\alpha t} v^{*}(t) = -e^{-\alpha t} \bar{R}^{-1} \bar{B}' \bar{P} z(t) \\ &= -\bar{R}^{-1} B' \bar{P} x(t) \quad (\text{because } x(t) = e^{-\alpha t} z(t)) \end{aligned}$$

[2]

where $\bar{P}$ is obtained from ARE in part a) with $\bar{A} = A + \alpha I$, $\bar{B} = B$.

$$\Rightarrow \cancel{\bar{B}}\,(A+\alpha I)' \bar{P} + \bar{P}(A + \alpha I) + \bar{Q} - \bar{P} B \bar{R}^{-1} B' \bar{P} = 0 \quad [2]$$

QED

Question #4

a) Bookwork.

Any control policy which is optimal over the interval $[i, N]$ is necessarily optimal over the interval $[i+1, N]$

[4]

b) New problem.

For step $k < N$

The Dynamic Programming iteration is:

$$J_k(x_k) = \min_{u_k} c' x_k + d(u_k) + \underset{\substack{w_k \\ A_k}}{E}\left[J_{k+1}\left(A_k x_k + \beta(u_k) + w_k \right) \right]$$

[2]

Assume that the ~~set~~ $J_{k+1}(\cdot)$ is linear + constant, ie.

$$J_{k+1}(x_{k+1}) = m_{k+1}' x_{k+1} + \eta_{k+1}$$

[1]

$$\Rightarrow J_k(x_k) = \min_{u_k} c' x_k + d(u_k) + \underset{\substack{w_k \\ A_k}}{E}\left[m_{k+1}' A_k x_k + m_{k+1}' \beta(u_k) + m_{k+1}' w_k + \eta_{k+1} \right]$$

$$= c' x_k + m_{k+1}' E[A_k] x_k + m_{k+1}' E[w_k] + \eta_{k+1}$$

$$+ \min_{u_k} \{ d(u_k) + m_{k+1}' \beta(u_k) \}$$

[3]

$$\Rightarrow m_k' = c' + (E[A_k])' m_{k+1}$$

and $\eta_k = m_{k+1}' E[w_k] + \eta_{k+1} + \min_{u_k} \{ d(u_k) + m_{k+1}' \beta(u_k) \}$

[2]

$$\Rightarrow J_k(x_k) = m_k' x_k + \eta_k .$$

~~For k = N~~

~~The same has to hold for $k = N$ where $m_N = c$ and $\eta_N = 0$~~

~~$\Rightarrow$ [illegible]~~

4b) continued...

For $k=N$

Check that the same holds with $m_N = c$ and $n_N = 0$

$$\Rightarrow J_{N-1}(x_{N-1}) = m_{N-1}' x_{N-1} + n_{N-1}$$

$$= \left(c' + c' E[A_{N-1}]\right) x_{N-1} + c E[w_{N-1}] + \min_{u_{N-1}} \{ d(u_{N-1}) + c'\beta(u_{N-1}) \}.$$

QED [2]

4c) "Certainty ~~to~~ equivalence" holds if the optimal control law for the non-deterministic problem is the same as the optimal control law for the deterministic problem, where the ~~the~~ uncertainty is replaced with the expected value of the uncertainty. [3]

In the above problem, we see that the minimisation with respect to u_k is

$$u_k^* = \arg\min_{u_k} d(u_k) + \left(\left(c' + m_{k+2}' E[A_k]\right)\beta(u_k)\right),$$

which is the same ~~solution~~ as in the deterministic case ~~where~~ if $E[A_k]$ is replaced with A_k.

$\Rightarrow$ Certainty equivalence does hold. [3].

Question 5

a) Application of theory $R > 0 \Rightarrow \min_u \ell(x,u)$ is obtained from. [1]

$$\frac{\partial \ell}{\partial u} = 2Ru + 2S'z = 0 \iff u = -R^{-1}S'z \quad [2]$$

$$\begin{aligned} \Rightarrow L(z) &= \\ &= z'Qz + z'SR^{-1}RR^{-1}S'z \\ &\quad - 2z'SR^{-1}S'z \\ &= z'(Q - SR^{-1}S')z \end{aligned} \quad [2]$$

$$L(z) \geq 0 \iff Q - SR^{-1}S' \geq 0 \quad [1].$$

QED

b) New problem.

$$z_{k+1} = \begin{pmatrix} x_{k+1} \\ u_k \end{pmatrix} = \begin{pmatrix} Ax_k + Bu_k \\ u_k \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_k \\ u_{k-1} \end{pmatrix} + \begin{pmatrix} B \\ I \end{pmatrix} u_k$$

$$= \bar{A} z_k + \bar{B} u_k, \text{ where } \bar{A} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, \quad \bar{B} = \begin{pmatrix} B \\ I \end{pmatrix}$$

[4]

$$\Rightarrow y_k = Cx_k = \begin{pmatrix} C & 0 \end{pmatrix} z_k.$$

The stage cost can be written as

$$z_k' \begin{pmatrix} C' \\ 0 \end{pmatrix} M \begin{pmatrix} C & 0 \end{pmatrix} z_k + u_k' V u_k + u_k' W u_k + u_{k-1}' W u_{k-1} - 2 u_k' W u_{k-1}$$

$$= z_k' \begin{pmatrix} C'MC & 0 \\ 0 & W \end{pmatrix} z_k + u_k'(V+W)u_k + 2u_k' \begin{pmatrix} 0 & -W \end{pmatrix} \begin{pmatrix} x_k \\ u_{k-1} \end{pmatrix}$$

$$\Rightarrow Q = \begin{pmatrix} C'MC & 0 \\ 0 & W \end{pmatrix}, \quad R = V + W, \quad S = \begin{pmatrix} 0 & -W \end{pmatrix}$$

[6]

c) $V \geq 0$, $W \geq 0$ and either $V > 0$ or $W > 0 \Rightarrow R > 0$

If, in addition $M \geq 0 \Rightarrow$ the stage cost is positive definite

$$\Rightarrow Q - S'R^{-1}S \geq 0 \quad [4].$$

Question 6

New computed example.

(a) 1) $R = \frac{1}{c} > 0$ [1]

2) $Q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \geq 0$ [1]

3) $A = \begin{pmatrix} 0 & 1 \\ \gamma & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow$ reachability matrix $= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\Rightarrow (A, B)$ reachable. [2]

4) $\begin{pmatrix} Q^{1/2} \\ Q^{1/2} A \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ is full rank $\Rightarrow (Q^{1/2}, A)$ observable. [2].

(b) $P = \begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix} \Rightarrow PA = \begin{pmatrix} p_2\gamma & p_1 \\ p_3\gamma & p_2 \end{pmatrix}$

$A'P = P^T A = \begin{pmatrix} p_2\gamma & p_3\gamma \\ p_1 & p_2 \end{pmatrix}$

$P. BR^{-1}BP = \begin{pmatrix} cp_2^2 & cp_2p_3 \\ cp_2p_3 & cp_3^2 \end{pmatrix}$

$\Rightarrow$ the individual terms of the ARE are:

$0 = 2p_2\gamma - cp_2^2 + 1$ ①
$0 = p_1 + p_3\gamma - cp_2p_3$ ②
$0 = 2p_2 - cp_3^2$ ③ [6]

① $\Leftrightarrow p_2 = \frac{\gamma \pm \sqrt{\gamma^2 + c}}{c}$

③ $\Leftrightarrow p_3 = \pm\frac{1}{\sqrt{c}}\sqrt{2p_2}$

$\Rightarrow p_2 = \frac{\gamma + \sqrt{\gamma^2 + c}}{c}$ because otherwise $p_2 < 0 \Leftrightarrow p_3$ imaginary (P is real.) [2]

6b) continued.

$$\Rightarrow P_3 = \pm \frac{\sqrt{2}}{c} \sqrt{\gamma + \sqrt{\gamma^2 + c}}$$ [2]

$$② \Leftrightarrow \rho_1 = P_3 \left(\sqrt{\gamma^2 + c} \right)$$

But for $P > 0$, we need $\rho_1 > 0 \Rightarrow \rho_3 > 0$ [2]

$$\Rightarrow \rho_3 = \frac{\sqrt{2}}{c} \sqrt{\gamma + \sqrt{\gamma^2 + c}}$$

$$u^*(t) = -\left(c P_2 \quad c P_3 \right) x(t)$$ [2]

$$\Rightarrow u^*(t) = -\left(\gamma + \sqrt{\gamma^2 + c} \quad \sqrt{2\left(\gamma + \sqrt{\gamma^2 + c}\right)} \right) x(t)$$

QED.