IMPERIAL COLLEGE LONDON

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AO2
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ISE4.7

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2005

MSc and EEE/ISE PART IV: MEng and ACGI

DIGITAL SIGNAL PROCESSING AND DIGITAL FILTERS

Tuesday, 3 May 10:00 am

Time allowed: 3:00 hours

There are FIVE questions on this paper.

Answer THREE questions.

All questions carry equal marks

Corrected Copy

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s) :	P.T. Stathaki
	Second Marker(s) :	A.G. Constantinides

1. (a) An M^{th}-order, causal, real transfer function is given as

$$A_M(z) = \frac{a_0 + a_1 z^{-1} + \ldots + a_M z^{-M}}{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}, \; M > 0$$

(i) Derive the conditions on the numerator and denominator coefficients to yield $\left|A_M(e^{j\omega})\right|^2 = 1$, for all ω.

[3]

(ii) Comment on the location of the zeros of a causal, real, stable, allpass transfer function within the z-plane with respect to the unit circle and the poles. Justify your answer.

[3]

(b) Consider the first-order, causal, real, stable, allpass transfer function $A_1(z)$.

(i) Determine the expression for $1 - |A_1(z)|^2$.

[5]

(ii) Show that

$$|A_1(z)| \begin{cases} <1 & \text{for} \quad |z| > 1 \\ =1 & \text{for} \quad |z| = 1 \\ >1 & \text{for} \quad |z| < 1 \end{cases}$$

[5]

(c) Discuss a possible application of allpass filters.

[4]

2\. (a) Show that the transfer function

$$H_1(z) = \frac{1-\alpha}{2}\frac{1+z^{-1}}{1-\alpha z^{-1}}, |\alpha| < 1 \quad (1)$$

has a lowpass magnitude response. Determine the 3-dB cut-off frequency ω_c at which the gain response is 3 dB below the maximum value of 0 dB at $\omega = 0$.

[5]

(b) Show that the transfer function

$$H_2(z) = \frac{1+\alpha}{2}\frac{1-z^{-1}}{1-\alpha z^{-1}}, |\alpha| < 1 \quad (2)$$

has a highpass magnitude response. Determine the 3-dB cut-off frequency ω_c at which the gain response is 3 dB below the maximum value of 0 dB at $\omega = \pi$.

[5]

(c) The lowpass transfer function $H_1(z)$ of equation (1) and the highpass transfer function $H_2(z)$ of equation (2) can be expressed in the form

$$H_1(z) = \frac{1}{2}[A_1(z) - A_2(z)],\ H_2(z) = \frac{1}{2}[A_1(z) + A_2(z)]$$

where $A_1(z)$ and $A_2(z)$ are stable, allpass transfer functions. Determine $A_1(z)$ and $A_2(z)$.

[10]

(a) Show that a Type 1 linear-phase FIR transfer function $H(z)$ of length $2M+1=7$ can be expressed as

$$H(z)=z^{-M}\left[h[M]+\sum_{n=1}^{M}h[M-n](z^{n}+z^{-n})\right]$$

[4]

(b) By using the relation

$$z^{r}+z^{-r}=2T_r\left(\frac{z+z^{-1}}{2}\right)$$

where $T_r(x)$ is the $r-$th order Chebychev polynomial in x, express $H(z)$ of Part (a) in the form

$$H(z)=z^{-M}\sum_{n=0}^{M}a[n]\left(\frac{z+z^{-1}}{2}\right)^{n} \tag{1}$$

Determine the relation between $a[n]$ and $h[n]$.

[8]

(c) Develop a realization of $H(z)$ based on equation (1) in the form of Figure 1, where $F_1(z^{-1})$ and $F_2(z^{-1})$ are causal structures. Determine the form of $F_1(z^{-1})$ and $F_2(z^{-1})$.

[8]

[The Chebychev polynomials satisfy the following recursive relationship:

$T_r(x)=2xT_{r-1}(x)-T_{r-2}(x), r\geq 2$

$T_0(x)=1$, $T_1(x)=x$]

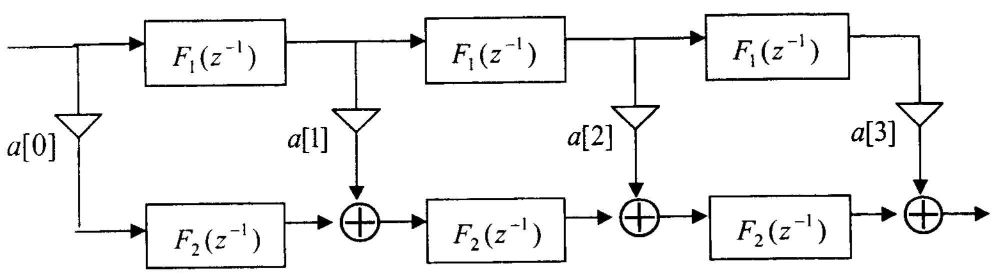

Figure 1

4. (a) The bilinear transformation from the $s-$plane to the $z-$plane is given by

$$s = \frac{1 - z^{-1}}{1 + z^{-1}}$$

(i) Explain using mathematical relations, where each point $s = \sigma + j\Omega$ on the $s-$plane is mapped on the $z-$plane through the bilinear transformation.
[6]

(ii) Prove that the relation between the continuous frequency Ω and the discrete frequency ω is non-linear.
[4]

(b) A given real lowpass digital IIR filter has a rational transfer function $H_L(z)$ and a digital cutoff frequency ω_c. The transfer function is transformed by replacing z by $F(\hat{z})$, to another real lowpass rational form $H_D(\hat{z}) = H_L(F(\hat{z}))$ of the same order as $H_L(z)$ and cutoff frequency $\hat{\omega}_c$.

(i) Give a full account of the properties of $F(\hat{z})$ in order to effect the transformation.
[3]

(ii) Starting with a general expression for $F(\hat{z})$ find a specific expression for $F(\hat{z})$ which has all its parameters derivable from the requirements of the problem.
[7]

5 (a) (i) Explain the function of a down sampler with a down sampling factor M where M , is a positive integer. Give the relation between the output sequence $y[n]$ of a down sampler with a down sampling factor M and its input sequence $x[n]$.

[4]

(ii) Derive the relation between the spectrums of the input and the output of a down sampler.

[8]

(b) Determine the condition under which a cascade of a factor of M down sampler with a factor of L up sampler is interchangeable.

[8]

1. (a) (i) An M^{th}-order causal real-coefficient allpass transfer function is of the form

$$A_M(z) = \pm \frac{d_M + d_{M-1}z^{-1} + \ldots + d_1 z^{-M+1} + z^{-M}}{1 + d_1 z^{-1} + \ldots + d_{M-1}z^{-M+1} + d_M z^{-M}}$$

If we denote the denominator polynomial of the allpass function $A_M(z)$ as $D_M(z)$ with

$$D_M(z) = 1 + d_1 z^{-1} + \ldots + d_{M-1}z^{-M+1} + d_M z^{-M}$$

then it follows that $A_M(z)$ can be written as

$$A_M(z) = \pm \frac{z^{-M} D_M(z^{-1})}{D_M(z)}$$

From the above we see that

$$A_M(z^{-1}) = \pm \frac{z^{M} D_M(z)}{D_M(z^{-1})}$$

$$A_M(z)A_M(z^{-1}) = \frac{z^{-M} D_M(z^{-1})}{D_M(z)} \frac{z^{M} D_M(z)}{D_M(z^{-1})} = 1 \Rightarrow$$

$$\left|A_M(e^{j\omega})\right|^2 = A_M(z)A_M(z^{-1}) = 1$$

(ii) The poles of a causal stable transfer function must lie inside the unit circle. As a result, all zeros of a causal stable allpass transfer function lie outside the unit circle in a mirror-image symmetry with its poles situated inside the unit circle.

(b) Consider the first-order causal and stable allpass transfer function $A_1(z)$ with its pole located at d_1.

(i) Determine the expression for $1 - |A_1(z)|^2$.

$$A_1(z) = \frac{d_1 + z^{-1}}{1 + d_1 z^{-1}} = \frac{zd_1 + 1}{z + d_1} \text{ with } d_1 \text{ real and } |d_1| < 1.$$

$$A_1^*(z) = \frac{z^* d_1 + 1}{z^* + d_1}$$

$$A_1(z)A_1^*(z) = \frac{zd_1 + 1}{z + d_1} \cdot \frac{z^* d_1 + 1}{z^* + d_1} = \frac{|z|^2 d_1^2 + d_1(z + z^*) + 1}{|z|^2 + d_1(z + z^*) + d_1^2}$$

$$1 - A_1(z)A_1^*(z) = 1 - \frac{|z|^2 d_1^2 + d_1(z + z^*) + 1}{|z|^2 + d_1(z + z^*) + d_1^2} = \frac{(|z|^2 - 1)(1 - d_1^2)}{|z + d_1|^2}$$

(ii) If $|z| > 1$ then $|z|^2 - 1 > 0$. Since $\frac{(1 - d_1^2)}{|z + d_1|^2} > 0$ we see that

$$1 - A_1(z)A_1^*(z) = \frac{(|z|^2 - 1)(1 - d_1^2)}{|z + d_1|^2} > 0 \Rightarrow |A_1(z)|^2 < 1.$$

The rest are proved in a similar fashion.

(c) A simple but often used application of an allpass filter is as a ***delay equalizer***. Let $G(z)$ be the transfer function of a digital filter that has been designed to meet a prescribed magnitude response. The nonlinear phase response of this filter can be corrected by cascading it with an allpass filter section $A(z)$ so that the overall cascade with transfer function $G(z)A(z)$ as a constant group delay over the frequency domain of interest. Since the allpass filter has a unity magnitude response, the magnitude response of the cascade is still equal to $|G(e^{j\omega})|$, while the overall delay is given by the sum of the group delays of $G(z)$ and $A(z)$. The allpass is designed so that the overall group delay is approximately a constant in the frequency region of interest.

2 (a) For the transfer function

$$H_1(z) = \frac{1-\alpha}{2}\frac{1+z^{-1}}{1-\alpha z^{-1}}, |\alpha| < 1 \quad (1)$$

we have $H_1(j\omega) = \frac{1-\alpha}{2}\frac{1+e^{-j\omega}}{1-\alpha e^{-j\omega}}, |\alpha| < 1$,

$$|H_1(j\omega)|^2 = \left(\frac{1-\alpha}{2}\right)^2 \frac{(1+\cos\omega)^2 + \sin^2\omega}{(1-\alpha\cos\omega)^2 + \alpha^2\sin^2\omega} = \frac{(1-\alpha)^2}{2}\frac{1+\cos\omega}{1+\alpha^2 - 2\alpha\cos\omega}.$$

The above has a lowpass magnitude response with a monotonically decreasing magnitude since

$$|H_1(j0)|^2 = \left(\frac{1-\alpha}{2}\right)^2 \frac{(1+\cos 0)^2 + \sin^2 0}{(1-\alpha\cos 0)^2 + \alpha^2\sin^2 0} = \left(\frac{1-\alpha}{2}\right)^2 \frac{4}{(1-\alpha)^2} = 1,$$

$$|H_1(j\pi)|^2 = \left(\frac{1-\alpha}{2}\right)^2 \frac{(1+\cos\pi)^2 + \sin^2\pi}{(1-\alpha\cos\pi)^2 + \alpha^2\sin^2\pi} = \left(\frac{1-\alpha}{2}\right)^2 \frac{(1-1)^2 + 0}{(1+\alpha)^2 + 0} = 0$$

$$\frac{d(|H_1(j\omega)|^2)}{d\omega} = \frac{(1-\alpha)^2}{2}\frac{(-\sin\omega)(1+\alpha^2 - 2\alpha\cos\omega) - (1+\cos\omega)2\alpha\sin\omega}{(1+\alpha^2 - 2\alpha\cos\omega)^2}$$

$$= \frac{(1-\alpha)^2}{2}\frac{\sin\omega(-1-\alpha^2 + 2\alpha\cos\omega - 2\alpha - 2\alpha\cos\omega)}{(1+\alpha^2 - 2\alpha\cos\omega)^2} = -\frac{(1-\alpha)^2}{2}\frac{(1+\alpha)^2\sin\omega}{(1+\alpha^2 - 2\alpha\cos\omega)^2}$$

which is monotonically decreasing within the range $\omega = [0, \pi]$.

The 3-dB cut-off frequency ω_c at which the gain response is 3 dB below the maximum value of 0 dB at $\omega = 0$ is found below

$$|H_1(j\omega_c)|^2 = \frac{(1-\alpha)^2}{2}\frac{1+\cos\omega_c}{1+\alpha^2 - 2\alpha\cos\omega_c} = \frac{1}{2} \Rightarrow \cos\omega_c = \frac{2\alpha}{1+\alpha^2}$$

(b) For the transfer function

$$H_2(z) = \frac{1+\alpha}{2}\frac{1-z^{-1}}{1-\alpha z^{-1}}, |\alpha| < 1 \quad (2)$$

we have

$$|H_2(j\omega)|^2 = \left(\frac{1+\alpha}{2}\right)^2 \frac{(1-\cos\omega)^2 + \sin^2\omega}{(1-\alpha\cos\omega)^2 + \alpha^2\sin^2\omega} = \frac{(1+\alpha)^2}{2}\frac{1-\cos\omega}{1+\alpha^2 - 2\alpha\cos\omega}.$$

The above has a highpass magnitude response with a monotonically increasing magnitude since

$$|H_2(j0)|^2 = \left(\frac{1+\alpha}{2}\right)^2 \frac{(1-\cos 0)^2 + \sin^2 0}{(1-\alpha\cos 0)^2 + \alpha^2 \sin^2 0} = 0,$$

$$|H_2(j\pi)|^2 = \left(\frac{1+\alpha}{2}\right)^2 \frac{(1-\cos \pi)^2 + \sin^2 \pi}{(1-\alpha\cos \pi)^2 + \alpha^2 \sin^2 \pi} = \left(\frac{1+\alpha}{2}\right)^2 \frac{(1+1)^2}{(1+\alpha)^2} = 1$$

$$|H_2(j\omega)|^2 = \left(\frac{1+\alpha}{2}\right)^2 \frac{(1-\cos \omega)^2 + \sin^2 \omega}{(1-\alpha\cos \omega)^2 + \alpha^2 \sin^2 \omega} = \frac{(1+\alpha)^2}{2} \frac{1-\cos\omega}{1+\alpha^2-2\alpha\cos\omega}$$

$$\frac{d(|H_2(j\omega)|^2)}{d\omega} = \frac{(1+\alpha)^2}{2} \frac{\sin\omega(1+\alpha^2-2\alpha\cos\omega) - (1-\cos\omega)2\alpha\sin\omega}{(1+\alpha^2-2\alpha\cos\omega)^2}$$

$$= \frac{(1+\alpha)^2}{2} \frac{\sin\omega(1+\alpha^2-2\alpha\cos\omega - 2\alpha + 2\alpha\cos\omega)}{(1+\alpha^2-2\alpha\cos\omega)^2} = \frac{(1+\alpha)^2}{2} \frac{(1-\alpha)^2\sin\omega}{(1+\alpha^2-2\alpha\cos\omega)^2}$$

which is monotonically increasing within the range $\omega = [0, \pi]$.

The 3-dB cut-off frequency ω_c at which the gain response is 3 dB below the maximum value of 0 dB at $\omega = 0$ is found below

$$|H_1(j\omega_c)|^2 = \frac{(1+\alpha)^2}{2} \frac{1-\cos\omega_c}{1+\alpha^2-2\alpha\cos\omega_c} = \frac{1}{2} \Rightarrow \cos\omega_c = \frac{2\alpha}{1+\alpha^2}$$

(c) $A_1(z) = 1$ $A_2(z) = \dfrac{\alpha - z^{-1}}{1-\alpha z^{-1}}$ $|A_1(j\omega)| = 1$ $|A_2(j\omega)| = 1$

3. (a)

$$H(z) = h[0] + h[1]z^{-1} + h[2]z^{-2} + h[3]z^{-3} + h[4]z^{-4} + h[5]z^{-5} + h[6]z^{-6}$$

$$h[6] = h[0], h[5] = h[1], h[4] = h[2]$$

$$H(z) = h[0](1+z^{-6}) + h[1](z^{-1}+z^{-5}) + h[2](z^{-2}+z^{-4}) + h[3]z^{-3} \Rightarrow$$

$$H(z) = z^{-3}\{h[3] + h[0](z^3+z^{-3}) + h[1](z^2+z^{-2}) + h[2](z+z^{-1})\}$$

Thus

$$H(z) = z^{-3}\left[h[3] + \sum_{n=1}^{3} h[3-n](z^n + z^{-n})\right]$$

(b) The Chebychev polynomials satisfy the following recursive relationship:

$$z^r + z^{-r} = 2T_r\left(\frac{z+z^{-1}}{2}\right)$$

with $T_r(x) = 2xT_{r-1}(x) - T_{r-2}(x), r \ge 2$ $T_0(x) = 1$ $T_1(x) = x$.

$$T_2(x) = 2x^2 - 1$$

$$T_3(x) = 2x(2x^2-1) - x = 4x^3 - 3x$$

Thus,

$$z^2 + z^{-2} = 2T_2\left(\frac{z+z^{-1}}{2}\right) = 4\left(\frac{z+z^{-1}}{2}\right)^2 - 2$$

$$z^{3} + z^{-3} = 2T_3\left(\frac{z+z^{-1}}{2}\right) = 8\left(\frac{z+z^{-1}}{2}\right)^3 - 6\left(\frac{z+z^{-1}}{2}\right)$$

For reasons of simplicity I call $\frac{z+z^{-1}}{2} = x$

$$H(z(x)) = z^{-3}(h[3]x^0 + h[0](8x^3 - 6x) + h[1](4x^2 - 2) + 2h[2]x)$$
$$= z^{-3}[(h[3] - 2h[1])x^0 + (2h[2] - 6h[0])x + 4h[1]x^2 + 8h[0]x^3]$$
$$H(z) = z^{-M}\sum_{n=0}^{M} a[n]\left(\frac{z+z^{-1}}{2}\right)^n$$
$$a[0] = h[3] - 2h[1], a[1] = 2h[2] - 6h[0], a[2] = 4h[1], a[3] = 8h[0]$$

(c) Develop a realization of $H(z)$ based on equation (1) in the form of Figure 1, where $F_1(z^{-1})$ and $F_2(z^{-1})$ are causal structures. Determine the form of $F_1(z^{-1})$ and $F_2(z^{-1})$.

$$F_1(z^{-1}) = \frac{z+z^{-1}}{2},\; F_2(z^{-1}) = z^{-1}$$

4. (a) (i) $z = \frac{1+s}{1-s}$

(ii) For $s = j\Omega_0$ we have that $z = \frac{1+j\Omega_0}{1-j\Omega_0}$ which has a unity magnitude. This implies that a point on the imaginary axis in the $s-$ plane is mapped onto a point on the unit circle in the $z-$ plane where $|z| = 1$. In the general case, for $s = \sigma_0 + j\Omega_0$,

$$z = \frac{1 + (\sigma_0 + j\Omega_0)}{1 - (\sigma_0 + j\Omega_0)} \Rightarrow |z|^2 = \frac{(1+\sigma_0)^2 + \Omega_0^2}{(1-\sigma_0)^2 + \Omega_0^2}$$

A point in the left-half $s-$ plane with $\sigma_0 < 0$ is mapped onto a point inside the unit circle in the $z-$ plane as $|z| < 1$. Likewise, a point in the right-half $s-$ plane with $\sigma_0 > 0$ is mapped onto a point outside the unit circle in the $z-$ plane as $|z| > 1$.

(iii) Prove that the relation between the continuous frequency Ω and the discrete frequency ω is non-linear.

$$j\Omega = \frac{1 - e^{-j\omega}}{1 + e^{-j\omega}} = j\tan(\frac{\omega}{2})$$
$$\Omega = \tan(\frac{\omega}{2})$$

(b) (i) $F(\hat{z})$ must be real, rational function $\hat{z}$, of order 1 and stable.

(ii) Starting with a general expression for $F(\hat{z})$ as follows

$$F(\hat{z}) = \frac{a\hat{z} + b}{c\hat{z} + d},\; a, b, c, d \text{ constant parameters}$$

For $\hat{\omega}=0$, $\hat{z}=e^{j0}=1$ we require $H_D(\hat{z})=H_L(F(\hat{z}))=1$. Thus

$$F(1)=\frac{a+b}{c+d}=1 \Rightarrow a+b=c+d$$

For $\hat{\omega}=\pi$, $\hat{z}=e^{j\pi}=-1$ we require $H_D(\hat{z})=H_L(F(\hat{z}))=0$. Thus

$$F(-1)=\frac{-a+b}{-c+d}=-1 \Rightarrow -a+b=c-d$$

From the above conditions we see that $b=c$ and $a=d$.

Therefore, we have $F(\hat{z})=\dfrac{a\hat{z}+b}{b\hat{z}+a}=\dfrac{\alpha\hat{z}+1}{\hat{z}+\alpha}$, with $\alpha=\dfrac{a}{b}$.

Therefore, $F(\hat{z})$ is an allpass filter.

5. (a) (i) The down sampler with a down sampling factor M, where M is a positive integer, develops an output sequence $y[n]$ with a sampling rate that is $(1/M)^{\text{th}}$ of that of the input sequence $x[n]$. The down sampling operation is implemented by keeping every M^{th} sample of the input sequence and removing $M-1$ in-between samples, to generate the output sequence according to the relation

$$y[n]=x[nM]$$

(ii)

$$Y(z)=\sum_{n=-\infty}^{+\infty} x[Mn]z^{-n}$$

$$x_{\text{int}}[n]=\begin{cases} x[n], & n=0,\pm M,\pm 2M,\ldots, \\ 0, & \text{otherwise} \end{cases}$$

$$x_{\text{int}}[n] \quad x[n]$$

$$Y(z)=\sum_{n=-\infty}^{+\infty} x[Mn]z^{-n}=\sum_{n=-\infty}^{+\infty} x_{\text{int}}[Mn]z^{-n}=\sum_{k=-\infty}^{+\infty} x_{\text{int}}[k]z^{-k/M}=X_{\text{int}}(z^{1/M})$$

$$x_{\text{int}}[n]=c[n]x[n]$$

$$c[n]$$

$$c[n]=\begin{cases} 1, & n=0,\pm M,\pm 2M,\ldots, \\ 0, & \text{otherwise} \end{cases}$$

$$W_M=e^{-\frac{j2\pi}{M}}$$

$$X_{\text{int}}(z)=\sum_{n=-\infty}^{+\infty} c[n]x[n]z^{-n}=\frac{1}{M}\sum_{n=-\infty}^{+\infty}\left(\sum_{k=0}^{M-1} W_M^{kn}\right)x[n]z^{-n}$$

$$=\frac{1}{M}\sum_{k=0}^{M-1}\left(\sum_{n=-\infty}^{+\infty} x[n]W_M^{kn}z^{-n}\right)=\frac{1}{M}\sum_{k=0}^{M-1} X(zW_M^{-k})$$

$$Y(z)=\frac{1}{M}\sum_{k=0}^{M-1} X(z^{1/M}W_M^{-k})$$

(b) Determine the condition under which a cascade of a factor of M down sampler with a factor of L up sampler is interchangeable.

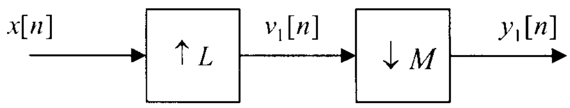

Figure 1

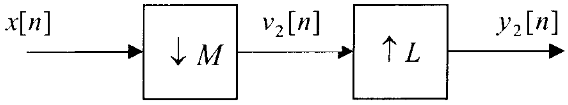

Figure 2

For Figure 1 we have $V_1(z) = X(z^L)$ and $Y_1(z) = \frac{1}{M}\sum_{k=0}^{M-1} X(z^{L/M} W_M^{-kL})$

For Figure 2 we have

$V_2(z) = \frac{1}{M}\sum_{k=0}^{M-1} X(z^{1/M} W_M^{-k})$ and $Y_2(z) = \frac{1}{M}\sum_{k=0}^{M-1} X(z^{L/M} W_M^{-k})$

The functions W_M^{-k} and W_M^{-kL} are the same only if the numbers L and M are relatively prime.