UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 2003

BEng Honours Degree in Computing Part II
MEng Honours Degrees in Computing Part II
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the Associateship of the City and Guilds of London Institute

PAPER C233

COMPUTATIONAL TECHNIQUES

Tuesday 29 April 2003, 14:30
Duration: 120 minutes

Answer THREE questions

Paper contains 4 questions
Calculators required

1a Assume 6-digit decimal arithmetic. Below, there are two numbers, x and y, and their approximations.

Accurate value	Approximation
$x = 1.00000$	1.00199
$y = 9.00000$	8.99694

Determine the error of the approximations using all measures of error. Which approximation is better? Why?

b Given matrices $\boldsymbol{A}$ and $\boldsymbol{B}$:

$$\boldsymbol{A} = \begin{bmatrix} 1 & a \\ b & 1 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} c & 1 \\ 1 & d \end{bmatrix},$$

where a, b, c and d are scalars. Compute $\boldsymbol{AB} - \boldsymbol{BA}$. Give conditions for $\boldsymbol{AB} = \boldsymbol{BA}$.

c Let $\boldsymbol{S} \in \mathbb{R}^{m \times m}$ be a nonsingular matrix, $\boldsymbol{d} \in \mathbb{R}^m$ and $\boldsymbol{0}$ the m dimensional null vector. Matrix $\boldsymbol{A}$ is given in the following partitioned form:

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{S} \\ 1 & \boldsymbol{d}^T \end{bmatrix}$$

Determine the inverse of $\boldsymbol{A}$ symbolically in a partitioned form. What is the dimension of $\boldsymbol{A}$? What are the dimensions of the submatrices in $\boldsymbol{A}^{-1}$? Having determined $\boldsymbol{A}^{-1}$ symbolically, explain the necessary and sufficient condition for the existence of $\boldsymbol{A}^{-1}$.

(The three parts carry, respectively, 20%, 30% and 50% of the marks).

2a Assume that matrices $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ have appropriate dimensions for the operations below.

(i) Prove that $(\boldsymbol{ABC})^T = \boldsymbol{C}^T\boldsymbol{B}^T\boldsymbol{A}^T$.

(ii) Prove that $(\boldsymbol{A}+\boldsymbol{B})\boldsymbol{C} = \boldsymbol{AC}+\boldsymbol{BC}$.

b Which of the following two matrices is positive definite? How can you determine that? Explain your work.

$$\boldsymbol{A} = \begin{bmatrix} 16 & 0 & -8 \\ 0 & 1 & 3 \\ -8 & 3 & 17 \end{bmatrix} \qquad \boldsymbol{B} = \begin{bmatrix} 9 & -6 & 0 \\ -6 & 4 & 2 \\ 0 & 2 & 16 \end{bmatrix}$$

c Given the matrix $\boldsymbol{A}$ below, determine its ℓ_1, ℓ_2 and ℓ_∞ norm:

$$\boldsymbol{A} = \begin{bmatrix} -3 & 1 & 2 \\ -2 & 6 & 4 \end{bmatrix}.$$

(The three parts carry, respectively, 20%, 40% and 40% of the marks).

3a (i) Determine the limit of the following sequence, if it exists:

$$a_n = \frac{1}{8} - \frac{1}{4}\frac{1}{8} + \left(\frac{1}{4}\right)^2 \frac{1}{8} - \cdots + (-1)^n \left(\frac{1}{4}\right)^n \frac{1}{8}$$

(ii) Given the vectors

$$\boldsymbol{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{v} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

Determine their outer product with $\boldsymbol{u}$ as the first term then with $\boldsymbol{v}$ as the first term. What is the relationship between the two results?

- Using the ℓ_∞ norm, determine the condition number of the following matrix

$$\boldsymbol{A} = \begin{bmatrix} \dfrac{1}{n-1} & \dfrac{1}{n} \\ \dfrac{1}{n} & \dfrac{1}{n+1} \end{bmatrix},$$

  assuming $n > 1$.
  Using your result, determine $\kappa(\boldsymbol{A})$ for $n = 4$.
  Hint: The inverse of a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\dfrac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

- Let I be the half-open interval $(-1, 0]$ and $d(x, y) = |x - y|$ be the distance for any $x, y \in I$.
  (i) Show that d is a metric on I.
  (ii) Show that $x_n = \dfrac{1-n}{n}$, for $n = 1, 2, \ldots$ is a Cauchy sequence in the metric d.
  (iii) Is (I, d) complete? Justify your answer.

(The three parts carry, respectively 20%, 40% and 40% of the marks).

4a Find a vector $\boldsymbol{v}$ that is conjugate to $\boldsymbol{u} = [3, -2]^T$ with respect to

$$\boldsymbol{A} = \begin{bmatrix} 11 & 3 \\ 3 & 13 \end{bmatrix}$$

and has integer coordinates. Explain your work.

b Which of the following functions have local extreme points (minimum or maximum), and if so, where? Justify your answer. Show your work.

(i) $f(x, y) = 1 - xy$

(ii) $f(x, y) = x^2 - y^3$

(iii) $f(x, y) = x^2 + y^2$

c Consider two systems of linear equations, $\boldsymbol{A}_i\boldsymbol{x} = \boldsymbol{b}$, $i = 1, 2$ with the following matrices and an arbitrary $\boldsymbol{b}$:

$$\boldsymbol{A}_1 = \begin{bmatrix} 3 & 1 & 1 & -1 \\ -2 & 4 & 0 & 3 \\ 1 & 0 & 15 & 0 \\ 1 & -1 & 2 & 10 \end{bmatrix} \qquad \boldsymbol{A}_2 = \begin{bmatrix} 2 & 0 & -1 & 0 \\ 1 & 3 & 0 & -1 \\ 4 & -3 & 9 & 1 \\ -1 & 2 & 2 & 6 \end{bmatrix}$$

Can you guarantee that the Jacobi and the Gauss-Seidel methods will converge with these matrices? Justify your answer.

(The three parts carry, respectively 30%, 50% and 20% of the marks).