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Write on one side only, between the margins, double-spaced. Not more than one question or solution per sheet, please

SETTER  
Lupton/GW

QUESTION NO.

SOLUTION NO.

1

MARKSCHEME

2

(i)  $f$  is even if  $f(x) = f(-x)$  for all  $x$ ;

$f$  is odd if  $f(x) = -f(-x)$  for all  $x$ .

Examples:  $f(x) = x^2$  is even;  $f(x) = x$  is odd

(ii)  $e^{-x}$ : neither

$x \sin x$ : even

$x^2 \sin x$ : odd

$2x/(x^2-1)$ : odd.

(iii)  $f(g(x)) = e^{1/x^2}$ ,  $g(f(x)) = e^{-2x}$

$f^{-1}(x) = \ln x$ ,  $g^{-1}(x) = x^{-1/2}$

(iv) In general, we can write

$$f(x) = \underbrace{\frac{1}{2}(f(x) + f(-x))}_{\text{even}} + \underbrace{\frac{1}{2}(f(x) - f(-x))}_{\text{odd}}$$

When  $f(x) = \frac{2x}{x+1}$ , this gives

$$\frac{2x}{x+1} = \frac{-2x^2}{1-x^2} + \frac{2x}{1-x^2}$$

4

4

4

2

4

	EXAMINATION QUESTIONS/SOLUTIONS 2007-08	Course EEI(1) 2
Question Core 2 Solution		Marks & seen/unseen
Parts	<p>(i) <math>\lim_{x \rightarrow \infty} \frac{(2x-1)(x+3)}{(x+5)(3x-2)} = \lim_{x \rightarrow \infty} \frac{(2-\frac{1}{x})(1+\frac{3}{x})}{(1+\frac{5}{x})(3-\frac{2}{x})}</math></p> <p style="text-align: center;"><math>= \frac{2}{3}</math></p> <p>(ii) <math> x \sin(\omega t + x)  \leq  x </math></p> <p>So <math>\lim_{x \rightarrow 0} x \sin(\omega t + x) = 0</math></p> <p>iii) <math>\lim_{x \rightarrow 0} x^{-2} \ln(\cos x)</math></p> <p><math>= \lim_{x \rightarrow 0} \frac{\ln(1 - \frac{x^2}{2!} + \dots)}{x^2} = \lim_{x \rightarrow 0} \frac{-\frac{x^2}{2!} + \dots}{x^2}</math></p> <p style="text-align: center;"><math>= -\frac{1}{2}</math></p> <p>or use L'Hopital's rule</p> <p>iv <math>\lim_{x \rightarrow \infty} x^{-9} \{ (x+3)^{10} - (x+1)^{10} \}</math></p> <p><math>= \lim_{x \rightarrow \infty} x^{-9} \{ x^{10} (1+\frac{3}{x})^{10} - x^{10} (1+\frac{1}{x})^{10} \}</math></p> <p><math>= \lim_{x \rightarrow \infty} x^{-9} \{ x^{10} (1+\frac{30}{x} + \dots) - x^{10} (1+\frac{10}{x} + \dots) \}</math></p> <p><math>= \lim_{x \rightarrow \infty} x^{-9} \{ 30x^9 - 10x^9 + \dots \}</math></p> <p style="text-align: center;"><math>= 20</math></p>	<p>(2)</p> <p>(2)</p> <p>(4)</p> <p>(4)</p> <p>(2)</p> <p>(2)</p> <p>(2)</p> <p>(2)</p>
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	EXAMINATION QUESTIONS/SOLUTIONS 2007-08	Course EEI(1) 3
Question Solution C3		Marks & seen/unseen
Parts	<p>i) Use change of variable</p> $\int (3-2x)^{-5} dx = \frac{(3-2x)^{-4}}{4} \times \frac{1}{(-2)} + C$ $= \frac{(3-2x)^{-4}}{8} + C$ <p>ii) Partial fractions</p> $\frac{5x+2}{(3x+4)(x-1)} = \frac{A}{3x+4} + \frac{B}{x-1}$ $5x+2 = A(x-1) + B(3x+4) = (A+3B)x + (-A+4B)$ <p>So</p> $\begin{cases} A+3B = 5 \\ -A+4B = 2 \end{cases} \rightarrow \begin{cases} B=1 \\ A=2 \end{cases}$ $\int \frac{5x+2}{(3x+4)(x-1)} dx = \int \frac{2}{3x+4} dx + \int \frac{1}{x-1} dx$ $= \frac{2}{3} \ln 3x+4  + \ln x-1  + C$	<p>3</p> <p>3</p> <p>2</p>
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Question  
Solution  
C3 (cont)

Marks &  
seen/unseen

Parts

(iii) Integrate by parts

$$\int x \ln x dx = \ln x \left( \frac{x^2}{2} \right) - \frac{1}{2} \int x^2 \frac{1}{x} dx$$

$$= \frac{x^2 \ln x}{2} - \frac{1}{2} \int x dx$$

$$= \frac{x^2 \ln x}{2} - \frac{1}{2} \frac{x^2}{2} + C$$

$$= \frac{x^2}{2} \left[ \ln x - \frac{1}{2} \right] + C$$

(iv) Integrating by parts  $\because \int_1^2 x \ln x dx = 2 \ln 2 - \frac{3}{4}$

$$I_n = x^n e^x - n I_{n-1} \quad \leftarrow \text{extremely unlikely anyone will do it like this.}$$

$$I_0 = \int x^0 e^x dx = \int e^x dx = e^x + C$$

$$I_1 = x e^x - I_0 = x e^x - e^x + C$$

$$I_2 = x^2 e^x - 2[x e^x - e^x] + C$$

$$I_3 = x^3 e^x - 3[x^2 e^x - 2x e^x + 2e^x] + C$$

$$= e^x [x^3 - 3x^2 + 6x - 6] + C$$

$$\begin{aligned} \overline{I} &= \int x^3 e^x dx \stackrel{\text{parts}}{=} x^3 e^x - \int 3x^2 e^x dx \\ &= x^3 e^x - 3x^2 e^x + \int 6x e^x dx \\ &= x^3 e^x - 3x^2 e^x + 6x e^x - \int 6e^x dx \\ &= (x^3 - 3x^2 + 6x - 6) e^x + C \end{aligned}$$

Alternative  
(more  
likely)  
method.

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Question

Solution

C4

Parts

i) a)  $3+i5 = \sqrt{9+25} (\cos \theta + i \sin \theta)$

$$\theta = \tan^{-1}(5/3)$$

b)  $r = \sqrt{6^2+3^2} = \sqrt{45}$ ,  $\theta = \tan^{-1}(-3/6) + \pi$

c)  $r = \sqrt{4^2+5^2} = \sqrt{41}$ ,  $\theta = \tan^{-1}(5/4) + \pi$

ii) By De Moivre

$$\cos 3\theta + i \sin 3\theta = (\cos \theta + i \sin \theta)^3$$

$$= \cos^3 \theta + i 3 \cos^2 \theta \sin \theta - 3 \cos \theta \sin^2 \theta - i \sin^3 \theta$$

Equating real and imaginary parts

$$\cos 3\theta = \cos^3 \theta - 3 \cos \theta \sin^2 \theta$$

$$= \cos^3 \theta - 3 \cos \theta (1 - \cos^2 \theta)$$

$$= \cos^3 \theta - 3 \cos \theta + 3 \cos^3 \theta$$

$$= 4 \cos^3 \theta - 3 \cos \theta$$

iii)  $z+1 = (x+1) + iy$  so

$$\arg(z+1) = \tan^{-1} \left[ \frac{y}{x+1} \right] = \pi/3$$

implies

$$\frac{y}{x+1} = \tan \pi/3 = \sqrt{3}$$

and so

$$y = \sqrt{3}(x+1) \text{ for } x \geq -1$$

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Checker's initials

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3

4

4

3

other ways not  
many complex  
numbers are  
acceptable

e.g.  
 $\cos 3\theta$   
 $\equiv \cos(2\theta + \theta)$   
 $= \cos 2\theta \cos \theta$   
 $- \sin 2\theta \sin \theta$   
etc.

Question

Marks &amp;

seen/unseen

ts

(i)  $\frac{\partial f}{\partial x} = (y-2)^2 + 2x - 1 = 0$

$\frac{\partial f}{\partial y} = 2x(y-2) = 0.$

2nd  $\Rightarrow x=0$  or  $y=2$

1st  $\Rightarrow y-2 = \pm 1$  or  $x = \frac{1}{2}$

$\therefore$  S.P at  $(0, 1)$ ,  $(0, 3)$ ,  $(\frac{1}{2}, 2)$ .

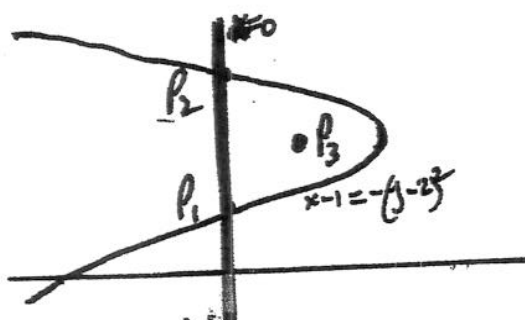
$\frac{\partial^2 f}{\partial x^2} = 2 \equiv A$     $\frac{\partial^2 f}{\partial x \partial y} = 2(y-2) \equiv B$     $\frac{\partial^2 f}{\partial y^2} = 2x \equiv C.$

$P_1(0, 1): A=2, B=-2, C=0: AC-B^2 < 0$  SADDLE Pt.

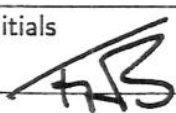

$P_2(0, 3): A=2, B=2, C=0: AC-B^2 < 0$  SADDLE Pt.

$P_3(\frac{1}{2}, 2): A=2, B=0, C=1: AC-B^2 > 0$  MINIMUM  
 $A > 0$

(ii)

ZERO  
CONTOUR

$x=0$   
AND  $(x-1) = -(y-2)^2$

	EXAMINATION QUESTIONS/SOLUTIONS 2007-08	Course BEI(1) 6
Question C6		Marks & seen/unseen
Parts	<p>(i) (a) eqns in std form are  <math display="block">(3, -5, -2) \cdot \underline{r} = \frac{2}{\sqrt{38}} \text{ and } (1, 1, 6) \cdot \underline{r} = -\frac{9}{\sqrt{38}}</math> <math display="block">\therefore \text{Distances are } \frac{2}{\sqrt{38}} \text{ and } \frac{9}{\sqrt{38}} \text{ respectively.}</math> </p> <p>(b) <math>N_1</math> given by <math>\left(\frac{2}{\sqrt{38}}\right) \frac{(3, -5, -2)}{\sqrt{38}} = \frac{(3, -5, -2)}{19}</math>  <math>N_2</math> given by <math>\left(-\frac{9}{\sqrt{38}}\right) \frac{(1, 1, 6)}{\sqrt{38}} = \frac{(-9, -9, -54)}{38}</math>  <math display="block">\vec{N}_1 \vec{N}_2 = (-9-6, -9+10, -54+4)/38</math> <math display="block">= (-15, 1, -50)/38</math> <math display="block">\therefore \underline{r} = \frac{(3, -5, -2)}{19} + \lambda \frac{(-15, 1, -50)}{38} \quad (-\infty &lt; \lambda &lt; +\infty)</math> </p> <p>(OR ALTERNATIVE)</p> <p>(ii) <math> \underline{u}  = 1 \Rightarrow x^2 + y^2 + z^2 = 1</math>  <math> \underline{u} - \underline{k}  = 1 \Rightarrow x^2 + y^2 + (z-1)^2 = 1</math>  <math>\Rightarrow z = \frac{1}{2}</math>  <math>\therefore \frac{1}{4} + x^2 + y^2 = 1</math>  <math>\Rightarrow \sqrt{x^2 + y^2} = \frac{\sqrt{3}}{2} \quad (x^2 + y^2 = 3/4)</math>          Circle in the plane <math>z = \frac{1}{2}</math>          Centre <math>(0, 0, \frac{1}{2})</math> and radius <math>\frac{\sqrt{3}}{2}</math>.       </p>	<p>2 2</p> <p>2</p> <p>2</p> <p>2</p> <p>2</p> <p>2</p> <p>1</p> <p>2</p> <p>1</p> <p>2</p> <p>20</p>
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	EXAMINATION QUESTIONS/SOLUTIONS 2007-08	Course EEI(1) 7
Question	C7 solution	Marks & seen/unseen
Parts	$L L^T = \begin{pmatrix} a & 0 & 0 \\ b & d & 0 \\ c & e & f \end{pmatrix} \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}$ $= \begin{pmatrix} a^2 & ab & ac \\ ab & b^2+d^2 & bc+de \\ ac & bc+de & c^2+e^2+f^2 \end{pmatrix} = A = \begin{pmatrix} 9 & 3 & -3 \\ 3 & 5 & 1 \\ -3 & 1 & 11 \end{pmatrix}$ $\Rightarrow a=3, b=1, c=-1, d=2, e=1$ <p>and <math>f=3</math> [Note that we could also have <math>a=-3, d=-2</math> - this will affect answer for <math>L^{-1}</math> below]</p> $\Rightarrow L = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{pmatrix}$ $A = L L^T \Rightarrow  A  =  L   L^T  =  L ^2 = (18)^2 = 324.$ $(L   I) = \left( \begin{array}{ccc ccc} 3 & 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ -1 & 1 & 3 & 0 & 0 & 1 \end{array} \right)$ $\Rightarrow \left( \begin{array}{ccc ccc} 1 & 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 2 & 0 & -\frac{1}{3} & 1 & 0 \\ 0 & 1 & 3 & \frac{1}{3} & 0 & 1 \end{array} \right)$ $\Rightarrow \left( \begin{array}{ccc ccc} 1 & 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{6} & \frac{1}{2} & 0 \\ 0 & 0 & 3 & \frac{1}{2} & -\frac{1}{2} & 1 \end{array} \right)$ $\therefore L^{-1} = \frac{1}{6} \begin{pmatrix} 2 & 0 & 0 \\ -1 & 3 & 0 \\ 1 & -1 & 2 \end{pmatrix}$ $A^{-1} = (L L^T)^{-1} = (L^T)^{-1} L^{-1} = (L^{-1})^T L^{-1}$ $A^{-1} = \frac{1}{36} \begin{pmatrix} 2 & -1 & 1 \\ 0 & 3 & -1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ -1 & 3 & 0 \\ 1 & -1 & 2 \end{pmatrix} = \frac{1}{36} \begin{pmatrix} 6 & -4 & 2 \\ -4 & 10 & -2 \\ 2 & -2 & 4 \end{pmatrix}$	<p>4</p> <p>2</p> <p>1</p> <p>3</p> <p>5</p> <p>2</p> <p>3</p>
Setter's initials	Checker's initials	Page number
JWB	JRC	20



	EXAMINATION QUESTIONS/SOLUTIONS 2007-08	Course ANCILLARY MATHEMATICS EE ICI)
Question C9		Marks & <b>8</b> seen/unseen
Parts		
(i)	<p>Substitute <math>y = xu</math>: <math>u + x \frac{du}{dx} = 2 \left( \frac{2+y/x}{2-y/x} \right) = \frac{4+2u}{2-u}</math> (1)</p> <p>(1) <math>\Leftrightarrow x \frac{du}{dx} = \frac{4+2u}{2-u} - u = \frac{4+u^2}{2-u}</math></p> <p><math>\Leftrightarrow \int \frac{4-2u}{4+u^2} du = \int \frac{2}{x} dx</math></p> <p><math>\Leftrightarrow \frac{4}{2} \tan^{-1}\left(\frac{u}{2}\right) - \ln(4+u^2) = \ln(x^2) + C</math></p> <p><math>\Leftrightarrow 2 \tan^{-1}\left(\frac{y}{2x}\right) = \ln\left(4 + \frac{y^2}{x^2}\right) + \ln(x^2) + C</math></p> <p><math>\quad \quad \quad = \ln(4x^2 + y^2) + C</math></p> <p><math>y\left(\frac{1}{2}\right) = 0 \Leftrightarrow 0 = 0 + C \therefore C = 0</math></p> <p>and we have <u><math>2 \tan^{-1}\left(\frac{y}{2x}\right) = \ln(4x^2 + y^2)</math></u></p>	<p>2</p> <p>2</p> <p>2</p> <p>2</p> <p>1</p> <p>1</p>
(ii)	<p><math>\frac{d}{dx}(\ln(\sec x + \tan x)) = \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}</math></p> <p><math>= \frac{\sec x (\tan x + \sec x)}{\sec x + \tan x} = \sec x</math></p> <p><math>\cos x \frac{dy}{dx} + y = 1 - \sinh x \Leftrightarrow \frac{dy}{dx} + y \sec x = \sec x - \tan x</math> (ii)</p> <p>Now <math>\int \sec x dx = \ln \sec x + \tan x  + \text{const}</math> from the above</p> <p>— integrating factor <u><math>\sec x + \tan x</math></u></p> <p>(ii) <math>\Leftrightarrow (\sec x + \tan x) \frac{dy}{dx} + y (\sec^2 x + \sec x \tan x) = \sec^2 x - \tan^2 x = 1</math></p> <p><math>\Leftrightarrow \frac{d}{dx}(y [\sec x + \tan x]) = 1</math></p> <p><math>\Leftrightarrow y (\sec x + \tan x) = x + C</math></p> <p><math>\Leftrightarrow y = \frac{x+C}{\sec x + \tan x}</math> or equivalent</p> <p>(In particular, accept <math>(x+C) \frac{(1-\sinh x)}{\cosh x}</math> or <math>\frac{(x+C) \cos x}{1+\sinh x}</math>)</p>	<p>2</p> <p>2</p> <p>1</p> <p>2</p> <p>1</p> <p>2</p>
	<p>Setter's initials PSR</p> <p>Checker's initials</p>	Page number 1 of 1

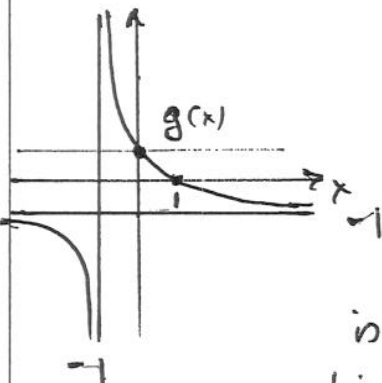
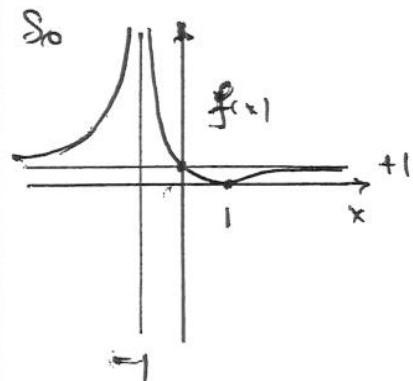
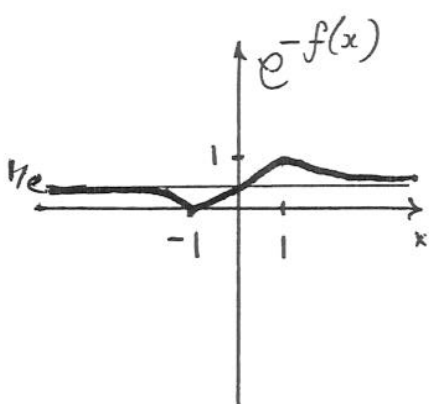
	EXAMINATION QUESTIONS/SOLUTIONS 2007-08	Course ANGLIAN MATHEMATICS <b>EX 1(1)</b>
Question C10		Marks & seen/unseen <b>9</b>
Parts (i)	<p>Aux. eqn <math>\mu^2 + 4\mu + 8 = 0 \Leftrightarrow \mu = -2 \pm i</math></p> <p>Comp. fn. <math>y = e^{-2x}(P \cos x + Q \sin x)</math></p> <p>Try PI <math>y = r e^{-2x} \Rightarrow y' = -2r e^{-2x} \Rightarrow y'' = 4r e^{-2x}</math></p> <p>substituting into the ODE:</p> <p><math>(4r - 8r + 5r) e^{-2x} = e^{-2x} \therefore r = 1</math></p> <p>general solution <math>y = e^{-2x}(1 + P \cos x + Q \sin x)</math></p> <p>whence <math>y' = e^{-2x}(-2 + [Q - 2P] \cos x - [P + 2Q] \sin x)</math></p> <p><math>y(0) = 0 \Leftrightarrow 1 + P = 0 \therefore P = -1</math></p> <p><math>y'(0) = 0 \Leftrightarrow -2 + Q - 2P = 0 \therefore Q = 0</math></p> <p>solution <u><math>y = e^{-2x}(1 - \cos x)</math></u></p>	<p>2</p> <p>1</p> <p>1</p> <p>1</p> <p>1</p> <p>1</p> <p>1</p>
(ii)	<p>Aux. eqn <math>\mu^2 + 4\mu + 4 = 0 \Leftrightarrow \mu = -2</math>, repeated</p> <p>Comp. fn. <math>y = e^{-2x}(Ax + B)</math></p> <p>Try PI <math>y = r x^2 e^{-2x} \Rightarrow y' = e^{-2x}(2rx - 2rx^2)</math></p> <p><math>\Rightarrow y'' = e^{-2x}(2r - 8rx + 4rx^2)</math></p> <p>substituting into the ODE:</p> <p><math>[(2r - 8rx + 4rx^2) + 4(2rx - 2rx^2) + 4rx^2] e^{-2x} = e^{-2x}</math></p> <p><math>\therefore r = 1/2</math></p> <p>general solution <math>y = e^{-2x}(\frac{1}{2}x^2 + Ax + B)</math></p> <p>whence <math>y' = e^{-2x}([A - 2B] + [1 - 2A]x - x^2)</math></p> <p><math>y(0) = 0 \Leftrightarrow B = 0</math></p> <p><math>y'(0) = 0 \Leftrightarrow A - 2B = 0 \therefore A = 0</math></p> <p>solution <u><math>y = \frac{1}{2}x^2 e^{-2x}</math></u></p>	<p>2</p> <p>2</p> <p>2</p> <p>2</p> <p>1</p> <p>1</p> <p>1</p> <p>1</p>
	<p>Setter's initials <b>PJR</b></p> <p>Checker's initials <b>JRC.</b></p>	Page number <b>191</b>

	EXAMINATION QUESTIONS/SOLUTIONS 2007-08	Course Solution C11
Question		Marks & seen/unseen
Parts	<p><math>y'' + xy = 0</math>; <math>y(0) = 1, y'(0) = 0</math>.</p> <p>Differentiate <math>n</math> times using Leibnitz:</p> $y^{(n+2)}(x) + xy^{(n)}(x) + ny^{(n-1)}(x) = 0.$ <p>Put <math>x=0</math></p> $y^{(n+2)}(0) + ny^{(n-1)}(0) = 0 \quad (*)$ <p>Put <math>n=1</math> in eqn (*) <math>y'''(0) = -y(0) = -1</math>  <math>n=2</math> " " " <math>y^{(4)}(0) = -2y'(0) = 0</math></p> <p>From the original eqn it follows <math>y''(0) = 0</math></p> <p><math>\therefore n=3</math> in eqn (*) <math>y^{(5)}(0) - 3y''(0) = 0 \therefore y^{(5)}(0) = 0</math>  <math>n=4</math> <math>y^{(6)}(0) + 4y'''(0) = 0 \quad y^{(6)}(0) = 4</math></p> <p>Clearly the only non-zero terms are <math>y(0), y'''(0), y^{(6)}(0)</math> etc</p> $\therefore y(x) = 1 - \frac{x^3}{3!} + \frac{4x^6}{6!} + \dots + \frac{y^{(3n)}(0) x^{3n}}{(3n)!}$ <p>By the ratio test <math>L = \lim_{n \rightarrow \infty} \left  \frac{y^{(3n)}(0) x^{3n}}{(3n)!} \cdot \frac{(3n-3)! x^3}{y^{(3n-3)}(0) x^{3n-3}} \right </math>  <math>= \lim_{n \rightarrow \infty} \left  \frac{(3n-2) x^3}{(3n)(3n-1)(3n-2)} \right </math> from (*)  <math>= 0 \therefore</math> Series converges <math>\forall x</math>.</p>	<p>(4)</p> <p>(2)</p> <p>(1)</p> <p>(1)</p> <p>(2)</p> <p>(2)</p> <p>(2)</p> <p>(2)</p> <p>(2)</p> <p>(2)</p>
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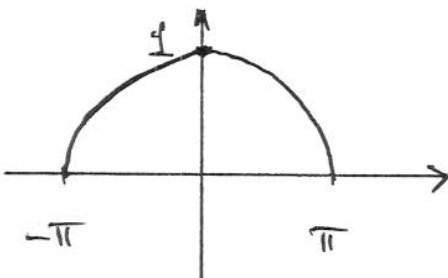
	EXAMINATION QUESTIONS/SOLUTIONS 2007-08 EE1 - Maths 2 - 2008	Course EE1(2) 1
Question 1		Marks & seen/unseen
Parts	<p>ii) Now <math>(\sinh(x))^2 = \left(\frac{e^x - e^{-x}}{2}\right)^2</math></p> $= \frac{e^{2x} + e^{-2x} - 2}{4} = \frac{\cosh 2x}{2} - \frac{1}{2}$ <p>&amp; <math>A = 1/2</math> and <math>B = -1/2</math>.</p> <p>To evaluate <math>\lim_{x \rightarrow 0} \frac{\sinh x}{x}</math> apply            l' Hopital's rule as <math>\sinh 0 = 0</math> :</p> $\lim_{x \rightarrow 0} \frac{\sinh x}{x} = \lim_{x \rightarrow 0} \cosh x = \cosh 0 = 1.$	<p>4</p> <p>4</p>
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Question 2		Marks & seen/unseen
Parts	<p>Given <math>f(x) = \left(1 - \frac{2x}{\pi}\right)^2</math>; first sketch</p> <p><math>g(x) := 1 - \frac{2x}{\pi} = \frac{x\pi - 2x}{x\pi} = \frac{1-x}{1+\pi}</math>;</p>  <p>When sketching <math>e^{-f}</math>, the only difficulty is at <math>x = -1</math> but</p> $\lim_{x \rightarrow -1} e^{-f(x)} = 0 \rightarrow$   <p><math>f</math> has a <u>min</u> at <math>x = 1</math>. <math>\Rightarrow</math></p> <p><math>e^{-f(x)}</math> has a <u>max</u> at <math>x = +1</math>.</p> <p><math>e^{-f(x)}</math> has a <u>min</u> at <math>x = -1</math></p>	<p>6 / 4</p> <p>} 2</p>
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	EXAMINATION QUESTIONS/SOLUTIONS 2007-08	Course EE 1(2) 2
Question 2		Marks & seen/unseen
Parts	<p>This uses <math>f(x) = g(x)^2</math> and so</p> $f'(x) = 2g \cdot g'$ $f'' = 2(g')^2 + 2g g''$ <p>As <math>g' = \frac{-1}{1+x} - \frac{1-x}{(1+x)^2}</math></p> $= \frac{-(1+x) - (1-x)}{(1+x)^2} = \frac{-2}{(1+x)^2}$ <p><math>f' = 0</math> iff <math>g = 0 \Rightarrow f</math> has extrema only when <math>g = 0</math> at <math>x = 1</math>.</p> <p>But then <math>g = 0 \Rightarrow f'' = 2(g')^2 &gt; 0</math> so <math>f</math> has a local min at <math>x = 1</math>.</p> <p>Then <math>\frac{d}{dx}(\bar{e}^f) = -f' \bar{e}^f \Rightarrow \bar{e}^f dx</math> has an extremum at <math>x = 1</math> as <math>f' = 0</math> there.</p> $\& \left(\frac{d}{dx}\right)^2(\bar{e}^f) = \frac{d}{dx}(-f' \bar{e}^f) = -(f'' - f'^2 \bar{e}^f)$ $= -f'' \cdot \bar{e}^f \text{ if } f' = 0 \& (f'') \bar{e}^f > 0$ <p>Hence <math>x = 1</math> is a local max for <math>\bar{e}^f</math> if <math>f'' &lt; 0</math></p>	<p>4</p> <p>4</p>
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	EXAMINATION QUESTIONS/SOLUTIONS 2007-08	Course EEI(2) 3
Question 3		Marks & seen/unseen
Parts	<p>a) <math>\frac{d}{dx} \text{sinc}(x) = \frac{\cos x}{x} - \frac{\sin x}{x^2} = \frac{x \cos x - \sin x}{x^2}</math></p> <p>and <math>\frac{d^2}{dx^2} \text{sinc}(x) = \frac{-x^2 \sin x - 2x \cos x + 2 \sin x}{x^3}</math></p> <p>b) Now <math>\left. \frac{d}{dx} \text{sinc}(x) \right _{x=0} = \lim_{x \rightarrow 0} \frac{x \cos x - \sin x}{x}</math></p> <p>this is in part (c) <math>\Rightarrow \lim_{x \rightarrow 0} \frac{\cos x - x \sin x - \cos x}{2x} = \lim_{x \rightarrow 0} \frac{-\sin x}{2} = 0.</math></p> <p>Also, <math>\lim_{x \rightarrow 0} \frac{\sin x}{x} = \lim_{x \rightarrow 0} \cos x = 1</math>, so that</p> <p><math>\left. \text{sinc}(x) \right _{x=0} = \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1.</math></p> <p>c) and <math>\left( \frac{d}{dx} \right)^2 \text{sinc}(x) \Big _{x=0} = \lim_{x \rightarrow 0} \left( \frac{d}{dx} \right)^2 \text{sinc}(x)</math></p> <p><math>= \lim_{x \rightarrow 0} \frac{-x^2 \sin x - 2x \cos x + 2 \sin x}{x^3}</math></p> <p><math>= \lim_{x \rightarrow 0} \frac{-2x \sin x - x^2 \cos x - 2 \cos x + 2 \sin x + 2 \cos x}{3x^2}</math></p> <p>? <math>\lim_{x \rightarrow 0} \frac{-4 \sin x + x \cos x}{3x} = -\frac{4}{3} + \frac{1}{3} = -1.</math></p>	<p>4</p> <p>4</p> <p>4</p>
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$\Rightarrow \lim_{x \rightarrow 0} \frac{-x^2 \cos x}{3x^2} = -\frac{1}{3} //$  (Corrected by AGW)

	EXAMINATION QUESTIONS/SOLUTIONS 2007-08	Course EE 1(2) 3
Question 3		Marks & seen/unseen
Parts	<p>d) From a) &amp; c) we find that <math>\text{sinc}</math> has a local max at <math>x=0</math></p>  <p>and <math>\text{sinc}(\pi) = \frac{\sin \pi}{\pi} = 0</math>, moreover <math>\text{sinc}</math> is an <u>even</u> function as the ratio of 2 odd functions.</p>	4
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	EXAMINATION QUESTIONS/SOLUTIONS 2007-08	Course EE1(2) 4
Question 4		Marks & seen/unseen
Parts	<p>a) The surface area of revolution of a function <math>y</math> is</p> $\int_a^b 2\pi y \sqrt{1+y'^2} dx$ <p>between limits <math>a</math> and <math>b</math> and using <math>y(x) = \sqrt{R^2 - x^2}</math> we get</p> $y' = -x(R^2 - x^2)^{-1/2}$ <p>and so the req'd surface area is</p> $2\pi \int_0^R y \sqrt{1+y'^2} dx = 2\pi \int_0^R \sqrt{R^2 - x^2} \left( \left( \frac{-x}{\sqrt{R^2 - x^2}} \right)^2 + 1 \right)^{1/2} dx$ $= 2\pi \int_0^R \sqrt{R^2 - x^2} \left( \frac{R^2}{R^2 - x^2} \right)^{1/2} dx = 4\pi R \int_0^R dx = 4\pi R^2$ <p>b) Now choose <math>y(x) = \frac{R}{h}(h-x)</math> and then</p> $2\pi \int_0^h y \sqrt{1+y'^2} dx = 2\pi \int_0^h \frac{R}{h}(h-x) \sqrt{1+(-R/h)^2} dx$ $= \frac{2\pi R}{h} \int_0^h (h-x) \sqrt{1+R^2/h^2} dx$ $= \frac{2\pi R}{h} \sqrt{1+R^2/h^2} \left[ hx - \frac{x^2}{2} \right]_0^h = \pi R \sqrt{h^2 + R^2}$	<p>4</p> <p>8</p> <p>2</p> <p>6</p>
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	EXAMINATION QUESTIONS/SOLUTIONS 2007-08	Course EE1(2) 5
Question 5		Marks & seen/unseen
Parts	<p>i) a) <math>\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}</math> <u>diverges</u> by the integral test as <math>\lim_{T \rightarrow \infty} \int_1^T x^{-1/2} dx</math> diverges</p> <p>b) <math>\sum_{n=1}^{\infty} (-1)^n / \sqrt{n+1}</math> <u>converges</u> by the alternating series test.</p> <p>c) <math>\sum_{n=1}^{\infty} \frac{n}{n^2+1}</math> <u>diverges</u> by comparison with <math>\sum_{n=m}^{\infty} \frac{1}{n}</math> (for large enough <math>m</math>)  <math>[n^2+1 \leq 2n^2 \text{ \&amp; so } \frac{n}{n^2+1} \geq \frac{n}{2n^2} = \frac{1}{2n} \text{ \&amp; } \sum \frac{1}{2n} \text{ diverges}]</math></p> <p>ii) The integral test states that  <math>-f(1) + \sum_{n=1}^{\infty} f(n) \leq \int_1^{\infty} f(x) dx \leq \sum_{n=1}^{\infty} f(n)</math>  and choosing <math>f(x) = \frac{1}{1+x^2}</math> we get  <math>-\frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{1+n^2} \leq \int_1^{\infty} \frac{dx}{1+x^2} \leq \sum_{n=1}^{\infty} \frac{1}{1+n^2}</math>  But <math>\int_1^T \frac{dx}{1+x^2} = \tan^{-1}(T) - \tan^{-1}(1)</math>  <math>\Rightarrow \sum_{n=1}^{\infty} \frac{1}{1+n^2} = \lim_{T \rightarrow \infty} \tan^{-1}(T) - \frac{\pi}{4} = \frac{\pi}{2} - \frac{\pi}{4} = \pi/4</math></p>	<p>3</p> <p>3</p> <p>3</p> <p>2</p> <p>8</p>
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$f(1) = \frac{1}{2}?$

	EXAMINATION QUESTIONS/SOLUTIONS 2007-08	Course EE1(2) 5
Question 5		Marks & seen/unseen
Parts $\infty$ u)	<p><del>and so</del></p> $\sum_{n=1}^{\infty} \frac{1}{n^2} \leq \frac{\pi}{4} + 1 < 1.79$ $\sum_{n=1}^{\infty} \frac{1}{n^2} \leq \frac{\pi}{4} + \frac{1}{2} < 1.29$	<p>1</p>
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	EXAMINATION QUESTIONS/SOLUTIONS 2007-08	Course EE1(2) 6
Question 6		Marks & seen/unseen
Parts	<p>The given function is <sup>(almost everywhere!)</sup> odd, so there are no cosine terms in the Fourier series. Now <math>b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx</math></p> $= \frac{1}{\pi} \int_{-\pi}^0 \sin nx dx + \frac{1}{\pi} \int_0^{\pi} (-1) \sin nx dx$ $= \frac{2}{\pi} \int_0^{\pi} \sin nx dx = \frac{2}{\pi} \left[ -\frac{\cos nx}{n} \right]_0^{\pi}$ $= \frac{-2}{n\pi} (\cos(n\pi) - 1) = \frac{-2}{n\pi} ((-1)^n - 1)$ $= \begin{cases} 0 & ; n \text{ even} \\ 4/n\pi & ; n \text{ odd} \end{cases}$ <p>hence <math>f(x) = \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} \frac{4}{n\pi} \sin(nx) (= f(x))</math> 8</p> <p>at those points where <math>f</math> is continuous. Otherwise,</p> $F(x_0) = \lim_{\epsilon \rightarrow 0} \frac{f(x_0 + \epsilon) + f(x_0 - \epsilon)}{2}$ <p>if <math>f</math> is discontinuous at <math>x_0</math>, provided these limits exist.</p>	2
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	EXAMINATION QUESTIONS/SOLUTIONS 2007-08	Course EE 1(2) 6
Question 6		Marks & seen/unseen
Parts	<p>Thus</p> $F(\pi) = \lim_{\epsilon \rightarrow 0} \frac{f(\pi+\epsilon) + f(\pi-\epsilon)}{2}$ $= \frac{1}{2}(-1+1) = 0$ <p>but <math>f(\pi) = 1 \neq F(\pi)</math> so  <math>x_0 = \pi</math> will do.</p> <p>Setting <math>x = \pi/2</math> into <math>f(x)</math> gives</p> $1 = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\sin((2n+1)\pi/2)}{2n+1}$ <p>but <math>\sin((2n+1)\pi/2) = (-1)^n \Rightarrow</math></p> $\frac{\pi}{4} = \sum_{n=0}^{\infty} (-1)^n / (2n+1).$	<p>4</p> <p>6</p>
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	EXAMINATION QUESTIONS/SOLUTIONS 2007-08	Course EE I (2) 7
Question		Marks & seen/unseen
Parts	<p style="text-align: center;"><b>SOLUTIONS</b></p> <p>Question 7</p> <p>(i) Let <math>V(r) = \ln(r)</math> and <math>r = \sqrt{x^2 + y^2}</math>. We have that</p> $V_x = \frac{1}{r} r_x, \quad V_{xx} = -\frac{1}{r^2} r_x^2 + \frac{1}{r} r_{xx}$ <p>and that</p> $r_x = \frac{x}{r}, \quad r_{xx} = \frac{1}{r} - \frac{x^2}{r^3}.$ <p>We combine these calculations to obtain</p> $\frac{\partial^2 V}{\partial x^2} = \frac{1}{r^2} - \frac{2x^2}{r^4}.$ <p>Similarly we have that</p> $\frac{\partial^2 V}{\partial y^2} = \frac{1}{r^2} - \frac{2y^2}{r^4}.$ <p>Consequently</p> $\begin{aligned} V_{xx} + V_{yy} &= \frac{1}{r^2} - \frac{2y^2}{r^4} + \frac{1}{r^2} - \frac{2y^2}{r^4} \\ &= \frac{2}{r^2} - 2 \frac{x^2 + y^2}{r^4} \\ &= \frac{2}{r^2} - \frac{2}{r^2} = 0. \end{aligned}$ <p>(ii) We have that</p> $\begin{aligned} \frac{dw}{dt} &= \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt} \\ &= y(-\sin(t)) + x \cos(t) + 1 \cdot 1 \\ &= -\sin^2(t) + \cos^2(t) + 1 \\ &= 1 + \cos(2t). \end{aligned}$	<p>7</p> <p>5</p> <p>8</p>
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	EXAMINATION QUESTIONS/SOLUTIONS 2007-08	Course EE1(2) 8
Question		Marks & seen/unseen
Parts		
Question 8	<p>(i) The trapezium rule with one interval gives:</p> $\int_0^1 (x \cos(x) + 1) dx \approx \frac{1}{2}(f(0) + f(1)) \approx 1.2715.$ <p>(ii) The trapezium rule gives:</p> $\int_0^1 (x \cos(x) + 1) dx \approx \frac{1}{4}(f(0) + 2f(1/2) + f(1)) \approx 1.3545.$ <p>(iii) Simpson's rule gives:</p> $\int_0^1 (x \cos(x) + 1) dx \approx \frac{1}{6}(f(0) + 4f(1/2) + f(1)) = 1.3826.$ <p>(iv) The correct value is</p> $\int_0^1 (\theta \cos(\theta) + 1) d\theta = (\theta \sin(\theta) + \cos(\theta) + \theta) \Big _0^1 = 1.3818. \quad (4dp) \quad 0.1116$ <p>The error when using the trapezium rule with 1 interval is <del>0.1103</del>. The error when using the trapezium rule with 2 intervals is 0.0273. The error when using Simpson's rule is 0.0008.</p>	<p>4</p> <p>4</p> <p>4</p> <p>4</p> <p>4</p>
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	EXAMINATION QUESTIONS/SOLUTIONS 2007-08	Course EE1(2) 9
Question		Marks & seen/unseen
Parts		
Question 9	<p>(i) The characteristic polynomial is</p> $p^2 - p - 2 = 0.$ <p>The roots are <math>p_{1,2} = -1, 2</math>. The solution of the homogeneous equation is</p> $y_h(x) = c_1 e^{-x} + c_2 e^{2x}.$ <p>We look for a particular integral of the form</p> $y_p(x) = A e^{3x}.$ <p>We substitute this into the equation to obtain</p> $4A e^{3x} = e^{3x} \Rightarrow A = \frac{1}{4}.$ <p>The general solution of the differential equation is</p> $y(x) = c_1 e^{-x} + c_2 e^{2x} + \frac{1}{4} e^{3x}.$ <p>(ii) The equation is of the form</p> $N(x, y) \frac{dy}{dx} + M(x, y) = 0$ <p>with <math>N(x, y) = y^2 - x^2</math> and <math>M(x, y) = 2xy</math>. We check that</p> $\frac{\partial M}{\partial y} = 2x \neq \frac{\partial N}{\partial x} = -2x,$ <p>and, hence, the equation is not exact. We multiply it by a function <math>I = I(y)</math>:</p> $\hat{N}(x, y) \frac{dy}{dx} + \hat{M}(x, y) = 0$ <p>where</p> $\hat{N}(x, y) = I(y)(y^2 - x^2)$ <p>and</p> $\hat{M}(x, y) = 2xyI(y).$ <p>We want to choose <math>I(y)</math> in such a way that the equation becomes exact. We have that <math>\hat{N}_x = -2xI</math> and <math>\hat{M}_y = 2xI + 2xyI'</math>. We choose <math>I</math> so that</p> $\hat{N}_x = \hat{M}_y \Rightarrow -2xI = 2xI + 2xyI'$ <p>and consequently <math>yI' = -2I</math>. The solution of this equation is</p> $I(y) = y^{-2}.$	<p>3</p> <p>3</p> <p>2</p> <p>3</p> <p>5</p>
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	EXAMINATION QUESTIONS/SOLUTIONS 2007-08	Course EE 1(2) 9
Question		Marks & seen/unseen
Parts	<p>To solve the equation in implicit form we look for <math>u(x, y)</math> so that</p> $u_x = \hat{M}, \quad u_y = \hat{N} \Rightarrow u = \frac{x^2}{y} + h(y) = y + \frac{x^2}{y} + H(x),$ <p>consequently</p> $u = \frac{x^2}{y} + y + C.$ <p>The solution of the differential equation, in implicit form, is</p> $\frac{x^2}{y} + y = C.$	4
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	EXAMINATION QUESTIONS/SOLUTIONS 2007-08	Course EE1 (2) 10
Question 10		Marks & seen/unseen
Parts	<p>Let <math>f(x) = e^{-x}</math> on <math>(-\pi, \pi)</math> and compute</p> $\int_{-\pi}^{\pi} e^{-x} e^{inx} dx$ $= \int_{-\pi}^{\pi} e^{(in-1)x} dx = \frac{e^{(in-1)x}}{in-1} \Big _{-\pi}^{\pi} \quad (*)$ $= \frac{e^{(in-1)\pi} - e^{-(in-1)\pi}}{in-1} \quad \text{but } e^{\pm in\pi} = (-1)^n$ <p>and <math>(*) = (-1)^n \left( \frac{e^{-\pi} - e^{\pi}}{in-1} \right) \cdot \frac{-in-1}{-in-1}</math></p> $= \frac{(-1)^n (e^{-\pi} - e^{\pi}) (-in-1)}{n^2+1} \Rightarrow$ $a_n = \int_{-\pi}^{\pi} e^{-x} \cos nx dx = 2(-1)^n \frac{\sinh \pi}{n^2+1} \quad \&$ $b_n = \int_{-\pi}^{\pi} e^{-x} \sin nx dx = (-1)^n \frac{2n \sinh \pi}{n^2+1}$ <p>and <math>c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-x} e^{-inx} dx</math></p> $= \frac{1}{2\pi} \frac{(-1)^n (e^{-\pi} - e^{\pi}) (in-1)}{(n^2+1)} = \frac{(-1)^n (1-in)}{n^2+1} \frac{\sinh \pi}{\pi}$	<p>4</p> <p>8</p>
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	EXAMINATION QUESTIONS/SOLUTIONS 2007-08	Course EE 1(2) 10
Question 10		Marks & seen/unseen
Parts	<p>So <math> c_n ^2 = \frac{(\sinh \pi)^2 (1+n^2)}{\pi^2 (1+n^2)^2}</math></p> <p><math>= \frac{(\sinh \pi)^2}{\pi^2 (1+n^2)}</math> and <math>c_0 = \frac{1}{2\pi} a_0</math></p> <p><math>= \frac{1}{2\pi} 2 \sinh \pi = \frac{\sinh \pi}{\pi}</math></p> <p><math>\therefore \frac{1}{2\pi} \int_{-\pi}^{\pi} f^2 dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-2x}</math></p> <p><math>= \frac{1}{2\pi} \frac{e^{2\pi} - e^{-2\pi}}{2} = \frac{\sinh 2\pi}{2\pi}</math></p> <p>(Parseval)</p> <p><math>= \underbrace{\frac{(\sinh \pi)^2}{\pi^2}}_{c_0^2} + 2 \sum_{n=1}^{\infty} \frac{(\sinh \pi)^2}{\pi^2 n^2}</math></p> <p>Hence</p> <p><math>\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi \sinh 2\pi}{4 (\sinh \pi)^2} - \frac{1}{2}</math></p> <p><math>(= \frac{1}{2} (\pi \coth \pi - 1))</math></p>	<p>8</p>
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