

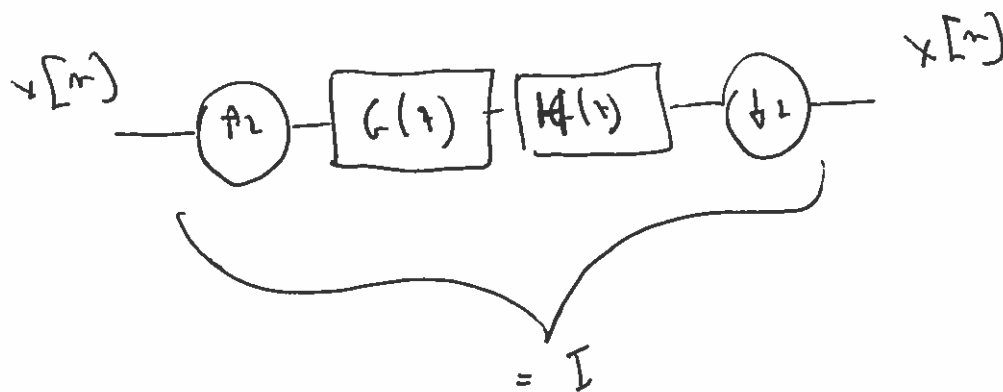
ANSWER 2

EE4-45

Wavelets & Applications

QUESTION 1

(a) THE SYSTEM IS IDENTICAL IF AND ONLY IF



THIS IMPLIES:

$$\frac{1}{2} x(z) \left[G(z^{1/2}) H(z^{1/2}) + G(-z^{1/2}) H(-z^{1/2}) \right] = x(z)$$

$$\Rightarrow H(z) G(z) + G(-z) H(-z) = 2 \quad (*)$$

(b)

SINCE WE ARE ASSUMING $H(z) = G(z^{-1})$

CONDITION (*) BECOMES

$$G(z) G(z^{-1}) + G(-z) G(-z^{-1}) = 2$$

$G(z)$ HAS ~~4~~ 4-TAPS AND TWO ZEROS AT $z = -1$

THIS IMPLIES

$$L(z) = a(1+z)^2 \left(\frac{a}{z} + b \right)$$

DEFINE

$$P(z) = L(z)L(z^{-1}) = (1+z)^2 (1+z^{-1})^2 (a z^{-1} + b + a z)$$

WE FIND a AND b USING

$$P(z) + P(-z) = 2$$

$$P(z) = (a z^{-3} + b z^{-2} + a z^{-1} + 4a z^{-2} + 4b z^{-1} + 4a + 6a z^{-1} + 6b + 6a z + 4a + 4b z + 4a z^2 + a z + b z^2 + a z^3)$$

AND USING $P(z) + P(-z) = 2$

WE HAVE

$$\begin{cases} b + 4a = 0 \\ 8a + 6b = 1 \end{cases} \Rightarrow \begin{aligned} a &= -\frac{1}{16} \\ b &= \frac{1}{4} \end{aligned}$$

$$\text{SO } P(z) = \frac{1}{16} (1+z)^2 (1+z^{-1})^2 (-z^{-1} + 4 - z)$$

~~PROVE~~ $-z^2 + 4z - 1 = 0 \Rightarrow$
 $(-z^{-1} + 4 - z) = (z - 4)(z^{-1} - 4)$

WITH $a = 2 \pm \sqrt{3}$

THUS

$$G(z) = \frac{1}{4} (1+z^{-1})^2 (z^{-1} - a)$$

$$H(z) = \frac{1}{4} (1+z)^2 (z - a)$$

WITH $a = 2 \pm \sqrt{3}$

(6) ~~THE~~ BECAUSE OF ORTHOGONALITY
THE WAVELET BRANCH
IS

$$G_1(z) = -z^{-1} G_a(z^{-1})$$

$$H_1(z) = G_1(z^{-1})$$

QUESTION 2

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$$(a) \quad P(z) = \frac{1}{2\sqrt{2}} (1+z)^2 (1+z^{-1})^2 B(z)$$

$$\text{IF } B(z) = a \rightarrow P(z) + P(-z) \neq 2$$

So LET'S TRY

$$B(z) = (a z^{-1} + b + a z)$$

$$\begin{aligned} P(z) &= \frac{1}{2\sqrt{2}} (1+2z+z^2) (1+2z^{-1}+z^{-2}) (a z^{-1} + b + a z) \\ &= \left[a z^{-3} + (b+4a) z^{-2} + (7a+4b) z^{-1} + (8a+6b) \right. \\ &\quad \left. + (7a+4b) z + (b+4a) z^2 + a z^3 \right] / 2\sqrt{2} \end{aligned}$$

THE HALF-BAND CONDITION IMPLIES THAT:

$$\left. \begin{aligned} b+4a &= 0 \\ \frac{1}{2\sqrt{2}} (8a+6b) &= 1 \end{aligned} \right\} \Rightarrow \begin{aligned} a &= -\frac{2\sqrt{2}}{16} \\ b &= \frac{2\sqrt{2}}{4} \end{aligned}$$

$$P(z) = \frac{1}{16} (1+z)^2 (1+z^{-1})^2 (-z^{-1} + 4 - z)$$

And

$$H_0(z) = \frac{1}{4\sqrt{2}} (1+z) (1+z^{-1}) (-z + 4 - z^{-1})$$

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$$(b) \quad H_1(z) = z G_0(-z) = \frac{1}{2\sqrt{2}} z (1-z)(1-z^{-1})$$

$$G_1(z) = z^{-1} H_0(-z) = \frac{z^{-1}}{4\sqrt{2}} (1-z)(1-z^{-1})(z+4+z^{-1})$$

(c) roots of $z^2 + 4z + 1$ ARE

$$z_0 = -2 - \sqrt{3} \quad \text{AND} \quad z_1 = -2 + \sqrt{3}$$

NOTE THAT $z_0 = 1/z_1$

THUS

$$p(z) = \frac{1}{16a} (1+z)^2 (1+z^{-1})^2 (1-z_0 z^{-1})(1-z_0 z)$$

WITH $a = (-2 + \sqrt{3})$

THUS

$$G_0(z) = \frac{1}{4\sqrt{a}} (1+z)^2 (1-z_0 z^{-1})$$

$$H_0(z) = G_0(z^{-1})$$

$$G_1(z) = -z^{-1} G_0(-z^{-1})$$

$$H_1(z) = G_1(z^{-1})$$

(d) WE NEED $H_1(z)$ TO HAVE 3 ZEROS
 AT $\omega=0$ $\Rightarrow G_0(z)$ MUST HAVE
 3 ZEROS AT $\omega=\pi$.

SOLUTION

$$G_0(z) = \frac{1}{4\sqrt{2}} (1+z^{-1})^2 (1+z)$$

$$H_0(z) = \frac{\sqrt{2}}{4} (1+z) (-z+4-z^{-1})$$

$$H_1(z) = z G_0(-z) = \frac{1}{4\sqrt{2}} z (1-z^{-1}) (1-z) \quad \left[\begin{array}{l} 3 \text{ ZEROS} \\ \text{AT } \omega=0 \end{array} \right]$$

$$G_1(z) = z^{-1} H_0(-z)$$

QUESTION 3

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(a) LET'S START WITH $\tilde{\psi}_1(t)$

i. WE WANT

$$\langle \tilde{\psi}_1(t), \psi_j(t) \rangle = \delta_{1,j} \quad j=1,2,3.$$

SINCE

$$\tilde{\psi}_1(t) = \sum_{k=1}^3 a_{1,k} \psi_k(t)$$

WE HAVE

$$\sum_{k=1}^3 a_{1,k} \langle \psi_k(t), \psi_j(t) \rangle = \delta_{1,j} \quad j=1,2,3.$$

WE NOTE THAT

$$\langle \psi_1(t), \psi_1(t) \rangle = \frac{2}{3}$$

$$\langle \psi_2(t), \psi_1(t) \rangle = \frac{1}{6}$$

$$\langle \psi_3(t), \psi_1(t) \rangle = \frac{1}{6}$$

BECAUSE OF CIRCULAR SYMMETRY WE CONCLUDE THAT

$$\langle \psi_i(t), \psi_j(t) \rangle = \begin{cases} \frac{2}{3} & i=j \\ \frac{1}{3} & i \neq j \end{cases}$$

CONSEQUENTLY THE SYSTEM OF EQUATIONS:

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$$\sum_{k=1}^3 \alpha_{1,k} \langle \varphi_k(t), \varphi_j(t) \rangle = \delta_{1,j} \quad j=1,2,3$$

REDUCES TO:

$$\begin{cases} \frac{2}{3} \alpha_1 + \frac{\alpha_2}{6} + \frac{\alpha_3}{6} = 1 \\ \frac{\alpha_1}{6} + \frac{2}{3} \alpha_2 + \frac{\alpha_3}{6} = 0 \\ \frac{\alpha_1}{6} + \frac{\alpha_2}{6} + \frac{2}{3} \alpha_3 = 0 \end{cases}$$

WITH SOLUTION:

$$\alpha_1 = \frac{5}{3} \quad \alpha_2 = -\frac{1}{3} \quad , \quad \alpha_3 = -\frac{1}{3} .$$

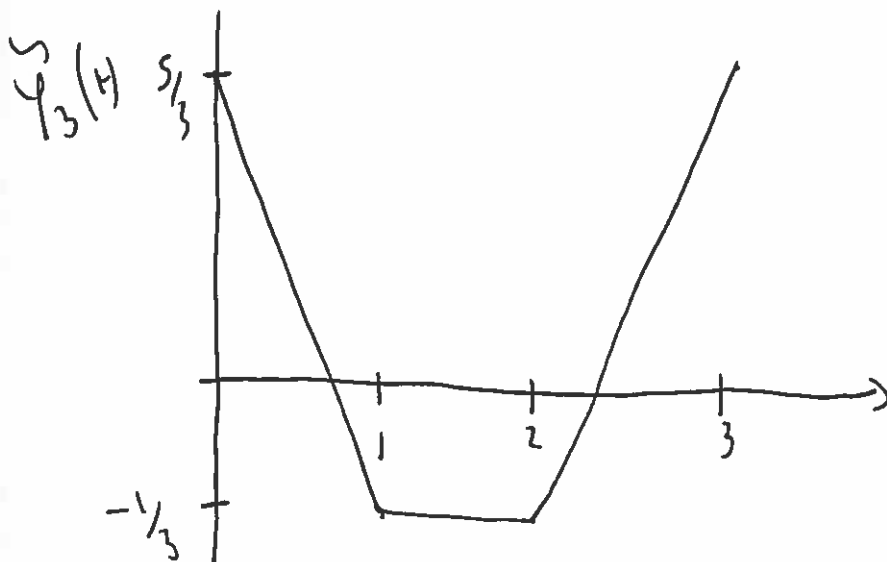
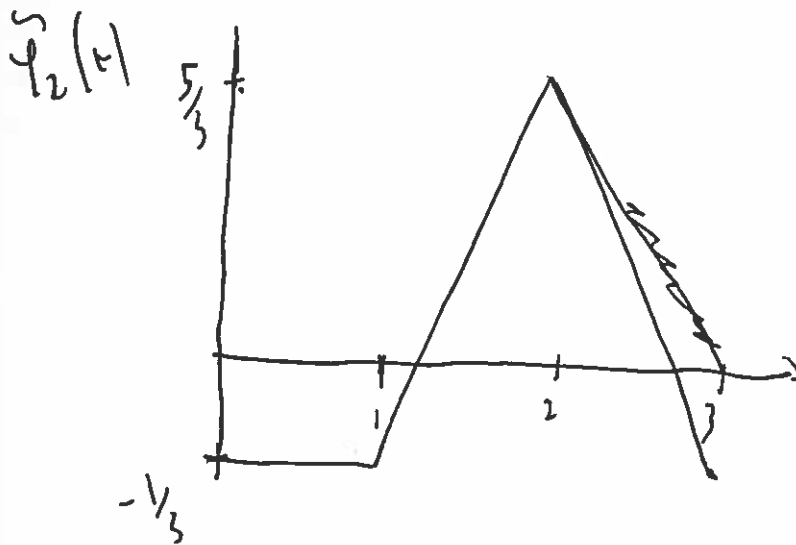
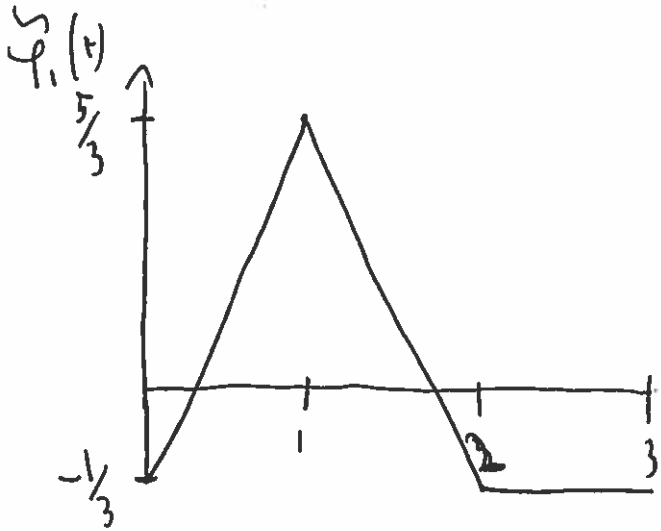
THIS IMPLIES

$$\tilde{\varphi}_2(t) = \frac{5}{3} \varphi_1(t) - \frac{1}{3} \varphi_2(t) - \frac{1}{3} \varphi_3(t) .$$

$\tilde{\varphi}_2(t)$ AND $\tilde{\varphi}_3(t)$ ARE OBTAINED BY
CIRCULAR SHIFT BY ONE OF $\tilde{\varphi}_1(t)$

ii.

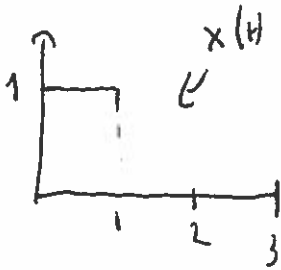
THE SKETCH OF THE THREE FUNCTIONS IS
AS FOLLOWS



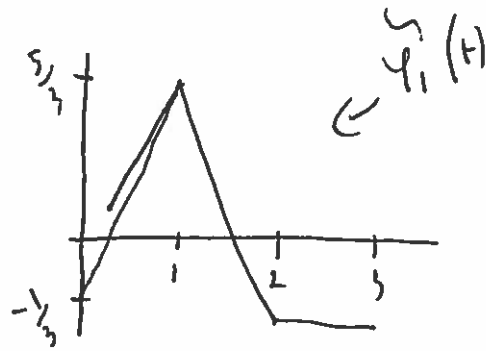
(b)

$$i. \langle x(t), \tilde{\varphi}_1(t) \rangle$$

SINCE



AND



WE HAVE THAT

$$\langle x(t), \tilde{\varphi}_1(t) \rangle = \int_0^1 (2t - \frac{1}{3}) dt = 1 - \frac{1}{3} = \frac{2}{3}$$

SIMILARLY

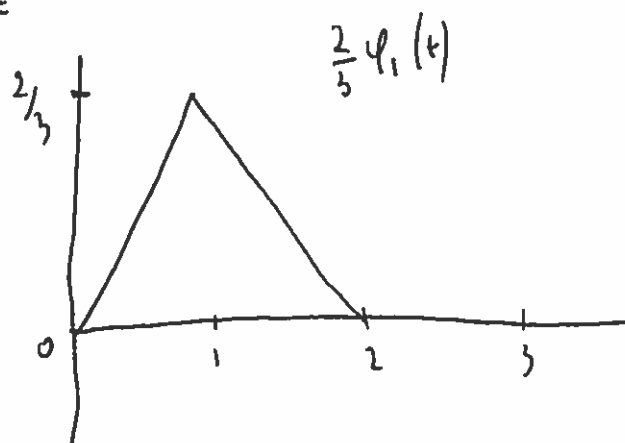
$$\langle x(t), \tilde{\varphi}_2(t) \rangle = -\frac{1}{3} \int_0^1 dt = -\frac{1}{3}$$

AND

$$\langle x(t), \tilde{\varphi}_3(t) \rangle = \int_0^1 (-2t + \frac{5}{3}) dt = -1 + \frac{5}{3} = \frac{2}{3}$$

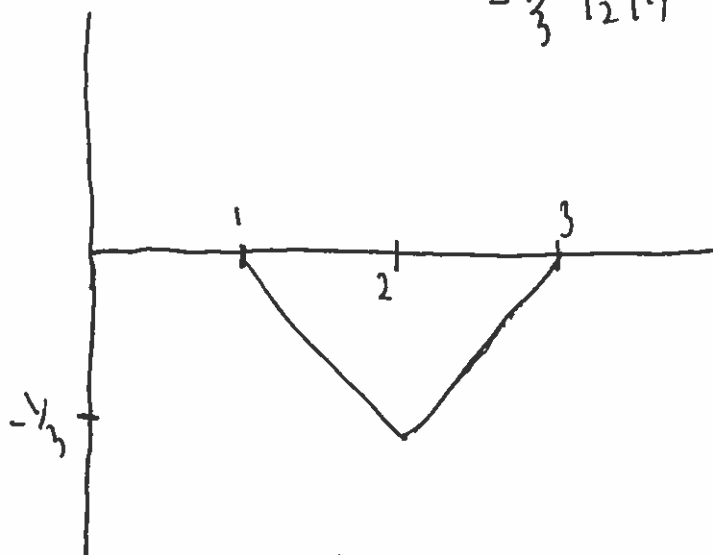
ii. GRAPHICALLY

$$x_v(t) =$$

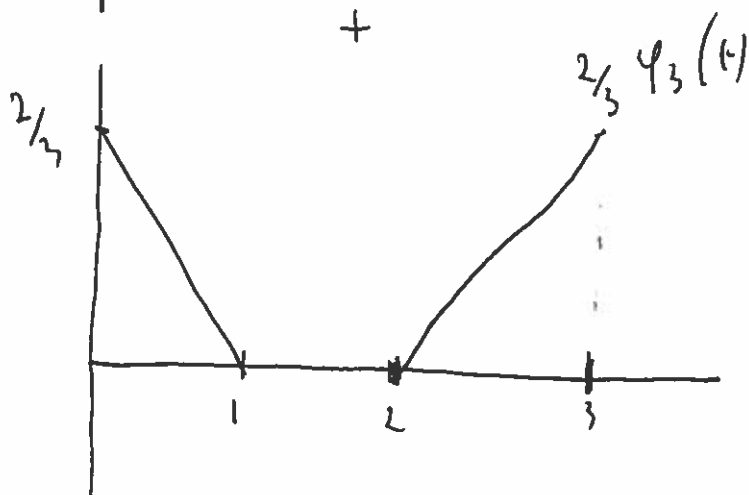


+

$$-\frac{1}{3} \varphi_2(t)$$

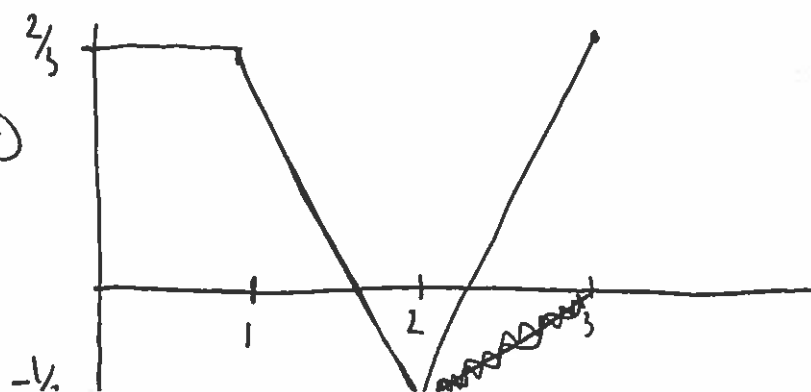


+



=

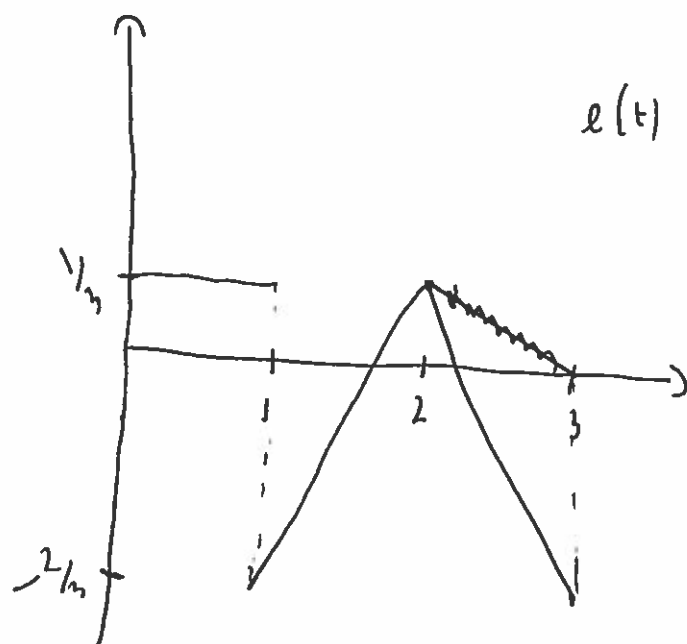
$$x_v(t)$$



(iii)

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$$e(t) = x(t) - x_V(t)$$



$$e(t) \perp V \quad \text{iff} \quad \langle e(t), \varphi_i(t) \rangle = 0 \quad i=1, 2, 3.$$

$$\begin{aligned} \langle e(t), \varphi_1(t) \rangle &= \frac{1}{3} \int_0^1 t \, dt + \int_1^2 \left(t - \frac{5}{3}\right)(2-t) \, dt \\ &= \frac{1}{6} - \frac{1}{6} = 0 \end{aligned}$$

for symmetry $\langle e(t), \varphi_3(t) \rangle = 0$

by

$$\langle e(t), \varphi_2(t) \rangle = 2 \int_1^2 \left(t-1\right)\left(t-\frac{2}{3}-1\right) dt = \int_0^1 t \left(t-\frac{2}{3}\right) dt = 0$$

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QUESTION 4

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i)

$$g_0[n] = \frac{1}{4\sqrt{2}} (\delta_n + 3\delta_{n-1} + 3\delta_{n-2} + \delta_{n-3})$$

THUS

$$G_0(z) = \frac{1}{4\sqrt{2}} (1 + z^{-1})^3$$

AND

$$G_0(e^{j\omega}) \Big|_{\omega=0} = \frac{1}{\sqrt{2}} \quad \text{AND} \quad G_0(e^{j\omega}) \Big|_{\omega=\pi} = 0$$

THE TWO NECESSARY CONDITIONS FOR THE LIMIT TO EXIST ARE SATISFIED

ii) MOREOVER,

IF WE DENOTE WITH $\Pi_0(\omega) = \frac{G_0(e^{j\omega})}{\sqrt{2}}$

WE HAVE THAT

$$\Pi_0(\omega) = \left(1 + \frac{e^{-j\omega}}{2}\right)^N R(\omega)$$

WITH $B = \sup R(\omega) = 1$

SINCE $B < 2^{N-2} = 2$

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WE KNOW THAT A SUFFICIENT CONDITION
FOR $\psi(t)$ TO CONVERGE TO A ~~CONTINUOUS~~
FUNCTION ~~IS~~ ~~SATISFIED~~ CONTINUOUS WITH
ITS FIRST ORDER DERIVATIVE IS SATISFIED.

iii)

THE ANALYSIS WAVELET TRANSFORM

IS

$$\tilde{\psi}(t) = \sum_m h_0^*[m] \psi(2t-m)$$

WITH

$$h_0[n] = (-1)^n g_0[1-n] \Leftrightarrow H_0(t) = -t \underbrace{G_0(-t)}$$

NOTE THAT
BOTH

$h_0[n]$ AND $g_0[n]$
ARE SYMMETRIC

$H_0(t)$ HAS THREE ZEROS AT $\omega=0 \Rightarrow \tilde{\psi}(t)$

HAS THREE VANISHING MOMENTS.

(b)

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$$\begin{aligned}
 |\langle f, \psi_{m,n} \rangle| &= \left| \underbrace{\langle p_{t_0}(t), \psi_{m,n}(t) \rangle}_{=0} + \langle \epsilon(t), \psi_{m,n}(t) \rangle \right| \\
 &\stackrel{(a)}{\leq} K 2^{-m/2} \int_{-\infty}^{\infty} |t - t_0|^\alpha \psi(2^{-m}t - n) dt \\
 &= K 2^{m/2} \int_{-\infty}^{\infty} |x 2^m + n 2^m - t_0|^\alpha \psi(x) dx \\
 &\stackrel{(b)}{\leq} K C 2^{m(\alpha+1/2)} \underbrace{\int_{-\infty}^{\infty} (|x| + |C|)^\alpha \psi(x) dx}_{=A} \\
 &= C_1 2^{m(\alpha+1/2)}
 \end{aligned}$$

THE VANISHING
PROPERTY
OF $\psi(t)$

where (a) follows from ~~the~~ and (b) from the fact that we are in the cone of influence of t_0 and therefore $|n 2^m - t_0| \leq C 2^m$.

$$(c) \quad |\langle \phi(t), \psi_{m,n}(t) \rangle| = \left| \int_{-\infty}^{\infty} \frac{1}{\sqrt{2^m}} \psi(2^{-m}t - n) \delta(t - t_0) dt \right|$$

$$= \frac{1}{\sqrt{2^m}} |\psi(2^{-m}t_0 - n)| \quad \text{WHEN}$$

$\psi_{m,n}(t)$ IS IN THE CONE OF INFLUENCE OF t_0 .

DEFINING WITH $B = \max |\psi(t)|$

WE HAVE THAT

$$|\langle \phi, \psi_{m,n} \rangle| \leq B 2^{-\frac{m}{2}}.$$

SO THE COEFFICIENTS INCREASE FOR LARGE
NEGATIVE m RATHER THEN DECREASING AS
IN (b).