Imperial College London

BSc/MSci EXAMINATION June 2012

This paper is also taken for the relevant Examination for the Associateship

ELECTROMAGNETISM AND OPTICS

For Second Year Physics Students

Monday, 11th June 2012: 10:00 to 12:00

Answer ALL parts of Section A, TWO questions from Section B and ONE question from Section C.
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the 5 answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in 5 answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

SECTION A

1. (i) The equation:

$$\mathbf{E} = E_0 \cos(kz - \omega t)\,\hat{\mathbf{x}}$$

describes the electric field component of a plane electromagnetic wave travelling in vacuum.

For this wave write down expressions for the

(a) wavelength

(b) frequency

(c) phase velocity

(d) direction of travel

(e) direction of polarisation

[2 marks]

(ii) Show that the vector potential of this wave can be written as:

$$\mathbf{A} = A_0 \sin(kz - \omega t)\,\hat{\mathbf{x}}$$

and hence give an expression for A_0 in terms of E_0.

[2 marks]

(iii) Hence calculate the magnetic component of this wave.

[2 marks]

(iv) Obtain an expression for the energy flux carried by the wave in terms of A_0.

[2 marks]

(v) Give an expression for the kinetic energy of an electron oscillating in this field also in terms of A_0, (you may assume $v \ll c$).

[2 marks]

[Total 10 marks]

2. The closest point to the eye at which the eye can focus on an object is known as the *nearpoint* and is taken to be 250 mm. An object of height y is placed at the near point and viewed by the naked eye. Sketch a diagram to show this, labeling the angle subtended by the object at the eye α.

A simple positive lens of focal length f (in mm) is used as a magnifying glass. Sketch a diagram to show the ray paths for the situation when the object is placed in the focal plane of the lens, labeling the angle subtended by the object as viewed through the lens α'.

By comparing the situations with and without the lens show that the angular magnification, M, when using the magnifying glass is given by

$$M = \frac{250}{f}$$

[5 marks]

[Total 5 marks]

SECTION B

3. Consider cylindrically symmetric fields $\mathbf{B} = B_r(r)\hat{\mathbf{r}} + B_\theta(r)\hat{\mathbf{e}}_\theta + B_z(r)\hat{\mathbf{z}}$ where $\hat{\mathbf{r}}$, $\hat{\mathbf{e}}_\theta$ and $\hat{\mathbf{z}}$ are unit vectors in the r, θ and z directions. The field is caused by a current in a plasma, with current density:

$$\mathbf{J} = J_\theta\,\hat{\mathbf{e}}_\theta = \frac{B_0}{\mu_0}\frac{4r}{a^2}\left(1 - \frac{r^2}{a^2}\right)\hat{\mathbf{e}}_\theta \qquad \text{for } r < a$$

and $\mathbf{J} = \mathbf{B} = 0$, for $r > a$.

Note For these symmetric fields $\nabla \cdot \mathbf{B} = \frac{1}{r}\frac{\mathrm{d}(rB_r)}{\mathrm{d}r}$ and $\nabla \times \mathbf{B} = -\frac{\mathrm{d}B_z}{\mathrm{d}r}\hat{\mathbf{e}}_\theta + \frac{1}{r}\frac{\mathrm{d}(rB_\theta)}{\mathrm{d}r}\hat{\mathbf{z}}$

(i) Show that for $r < a$, $B_r = 0$ and:

$$\mathbf{B} = B_z\,\hat{\mathbf{z}} = B_0\left(1 - \frac{r^2}{a^2}\right)^2\hat{\mathbf{z}} \tag{1}$$

[4 marks]

(ii) Obtain an expression for the total magnetic flux along the cylinder in terms of B_0 and a.

[3 marks]

(iii) Obtain an expression for the magnetic energy for a cylinder that is one metre long (in z), also in terms of B_0 and a.

The cylinder of plasma and field is surrounded by a perfect conductor in which $\mathbf{E} = 0$. The conductor, plasma and field are compressed rapidly by cylindrically arranged explosives, so that the radius a shrinks and B_0 changes but the form of the field Eq. (1) and the total magnetic flux stay constant.

[2 marks]

(iv) Suppose that $a = a_0$ and $B_0 = B_{00}$ before compression. Show that $B_0\,a^2$ stays constant during compression and hence show that when the radius is a:

$$\mathbf{B} = \frac{B_{00}\,a_0^2}{a^2}\left(1 - \frac{r^2}{a^2}\right)^2\hat{\mathbf{z}}. \tag{2}$$

[3 marks]

(v) Suppose the initial field $B_{00} = 10$ Tesla and $a_0 = 1$ m. The radius is shrunk by a factor of ten so that the final radius $a_f = a_0/10 = 0.1$ m. What is the field at $r = 0$ after compression. Calculate the energy for a one metre length cylinder before and after compression. Where does the extra magnetic energy come from?
(Note: $\mu_0 = 4\pi \times 10^{-7}$).

[3 marks]

[Total 15 marks]

4. (i) Write down Maxwell's equations in differential form for a (collisionless) plasma.

[2 marks]

(ii) Derive the three-dimensional wave equation for transverse waves that is a solution to these equations.

(You may need the identity: $\nabla \times (\nabla \times \mathbf{a}) = \nabla(\nabla \cdot \mathbf{a}) - \nabla^2 \mathbf{a}$)

[4 marks]

(iii) Show that the dispersion relation for transverse electromagnetic waves in a plasma is given by;

$$\omega^2 = c^2 k^2 + {\omega_p}^2$$

Give an expression for ω_p.

[3 marks]

(iv) Give the condition that the solutions found describe a travelling wave. Hence find an expression for the (*critical*) density, n_{cr}, below which an electromagnetic wave of angular frequency ω can propagate freely in a plasma.

[2 marks]

(v) Describe what would happen to an electromagnetic wave that propagates (normally) up a linear density ramp whose maximum plasma density is greater than the critical density.

[2 marks]

(iv) A spaceship communicates with mission control at a frequency of 2.6 GHz. On return to earth, the spaceship experiences a communication black-out due to the plasma generated around the ship by the intense heat of re-entry. Calculate the minimum plasma density that was generated around the ship.

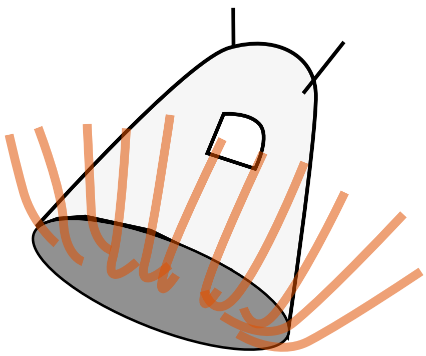


Figure 1: Spaceship on re-entry.

[2 marks]

[Total 15 marks]

5. (i) Give the boundary conditions for both perpendicular and parallel components of both the electric $\mathbf{E}$ and magnetic field $\mathbf{B}$ at the boundary of a *simple* dielectric.

[3 marks]

(ii) An electromagnetic wave is incident in the xz plane from vacuum on to a dielectric of refractive index n at $z = 0$, at an angle θ_i to the surface normal, (see figure).

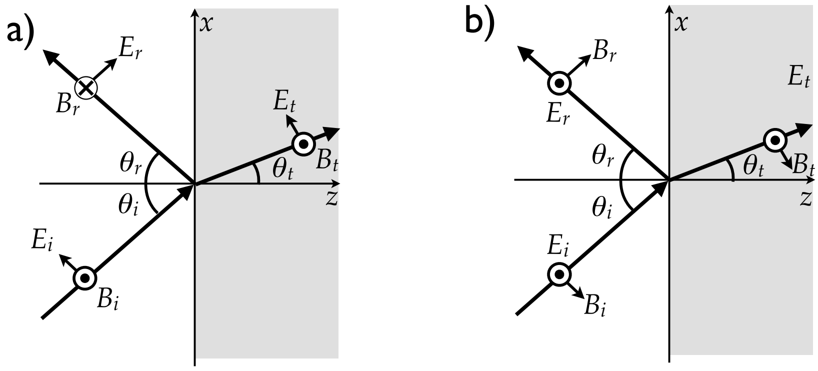


Figure 1: Electromagnetic wave incident on dielectric with a) p and b) s-polarisation.

The electric field inside and outside the dielectric for incidence with p-polarisation (fig. 1a) are:

$$\mathbf{E} = E_i\,(\cos\theta_i\,\hat{\mathbf{x}} - \sin\theta_i\,\hat{\mathbf{z}})e^{i(\mathbf{k_i}\cdot\mathbf{r}-\omega t)} + E_r\,(\cos\theta_r\,\hat{\mathbf{x}} + \sin\theta_r\,\hat{\mathbf{z}})e^{i(\mathbf{k_r}\cdot\mathbf{r}-\omega t)} \quad \text{for } z < 0$$

$$\mathbf{E} = E_t\,(\cos\theta_t\,\hat{\mathbf{x}} - \sin\theta_t\,\hat{\mathbf{z}})e^{i(\mathbf{k_t}\cdot\mathbf{r}-\omega t)} \qquad \text{for } z > 0$$

By considering phase matching at the boundary (or otherwise), show that $\theta_r = \theta_i$ and find the corresponding expression for θ_t.

You may use without proof that for a material with refractive index n, $k = n\,\omega/c$.

[3 marks]

(iii) For incidence with p-polarisation, as in fig. 1a, show that the reflection and transmission coefficients, $r = E_r/E_i$ and $t = E_t/E_i$, are given by;

$$t = \frac{2\cos\theta_i}{\cos\theta_t + n\cos\theta_i} \quad \text{and} \quad r = \frac{\cos\theta_t - n\cos\theta_i}{\cos\theta_t + n\cos\theta_i}$$

You may use without proof that $E = cB/n$. [5 marks]

(iv) The corresponding reflection and transmission coefficients for s-polarisation, (polarisation parallel to the surface as in fig. 1b), are given by:

$$t = \frac{2\cos\theta_i}{\cos\theta_i + n\cos\theta_t} \quad \text{and} \quad r = \frac{\cos\theta_i - n\cos\theta_t}{\cos\theta_i + n\cos\theta_t}$$

A (clockwise) circularly polarised beam is incident on a surface with $n = 1.5$, at $\theta_i = 60^\circ$. Calculate the r and t for *both* polarisations of this beam. Hence sketch the temporal variation of the polarisation vector of the reflected and transmitted beams, indicating both amplitudes and direction of rotation.

[4 marks]

[Total 15 marks]

6. (i) The electric field produced in the *far-field* by an accelerating point charge is given (for $\dot{z} \ll c$) by;

$$\mathbf{E_{rad}} = \frac{q}{4\pi\varepsilon_0} \frac{[\ddot{z}]}{rc^2} \sin\theta\,\hat{\theta}$$

where θ is the angle between the acceleration (which is along $\hat{\mathbf{z}}$) and $\mathbf{r}$.

Explain the significance of the square brackets around $\ddot{z}$ in the above equation, and hence rewrite the expression explicitly as a function of the variable(s) upon which it depends.

[2 marks]

(ii) Give an expression for the electric field of an electron oscillating harmonically with displacement $z = z_0 \sin(\omega t)$ for $z_0 \ll 2\pi c/\omega$ and $z_0 \ll r$.

[2 marks]

(iii) Calculate *explicitly* the corresponding magnetic field, (i.e. do not write down the result directly).

You may require the identity:

$$\nabla \times \mathbf{a} = \frac{1}{r\sin\theta}\left(\frac{\partial(a_\phi \sin\theta)}{\partial\theta} - \frac{\partial a_\theta}{\partial\phi}\right)\hat{\mathbf{r}} + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial a_r}{\partial\phi} - \frac{\partial(r a_\phi)}{\partial r}\right)\hat{\theta} + \frac{1}{r}\left(\frac{\partial(r a_\theta)}{\partial r} - \frac{\partial a_r}{\partial\theta}\right)\hat{\phi}$$

[4 marks]

(iv) By integrating the Poynting flux over all angles show that the total mean radiated power is;

$$P = \frac{q^2 {z_0}^2 \omega^4}{12\pi\varepsilon_0 c^3}$$

(You will need $\langle \sin^2(t - r/c)\rangle = \frac{1}{2}$ and $\oint \sin^2\theta\, \mathrm{d}\Omega = \frac{8}{3}\pi$, where the latter integral is over all solid angle.)

[4 marks]

(v) In the Bohr model of the hydrogen atom, the electron is treated as performing circular motion around the nucleus. By treating the circular motion as two $\pi/2$ out-of-phase simple harmonic oscillations, calculate the total power radiated. Hence calculate the time in which the electron would radiate an amount of energy equivalent to its initial kinetic energy T.

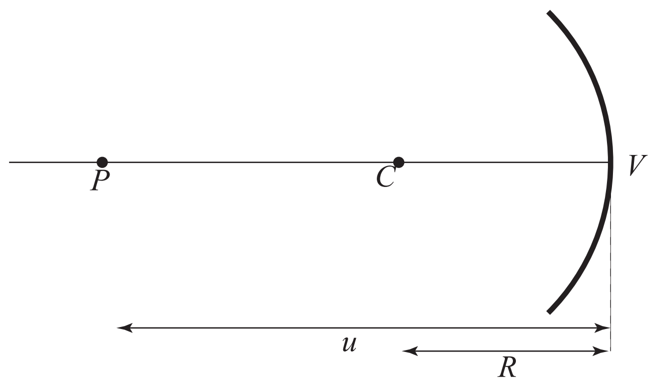
For a “classical” hydrogen atom circular orbit, $T = 13.6$ eV, and $z_0 = a = 5.29 \times 10^{-11}$ m (the Bohr radius).

[3 marks]

[Total 15 marks]

SECTION C

7. A point P is a distance u from a concave reflecting surface of radius R, with the centre of curvature of the surface, C, on the line joining P and the vertex, V, of the reflecting surface.



(i) By considering reflection at the surface of the curved surface, show that the relationship between the object distance u and image distance v is given by:

$$\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$$

where f is the focal length of the mirror and is given by $R/2$.

[5 marks]

(ii) Write down four ray tracing rules that may be used to locate the position and size of the image of an extended object when it is reflected in a curved mirror.

Sketch a diagram and apply these rules to find the image (y') of a small extended object (y) (of size much less than the diameter of the mirror) placed a distance 3R/2 from the concave mirror. Using the geometry of the diagram find a general expression for the lateral magnification M=y'/y in terms of the object and image position.

An object of height 10 mm is placed on axis a distance 100 mm from a concave mirror of radius of curvature 300 mm. Find the position, size and form of the image.

[6 marks]

(iii) An application of curved mirrors is in astronomical telescopes. Sketch a possible arrangement of a large concave mirror and a smaller convex mirror to form an astronomical telescope. Comment on the relative merits of a mirror based telescope in comparison to a lens based telescope.

[4 marks]

[Total 15 marks]

8. A transmission diffraction grating of N slits, each of width a and separated by a distance d, is to be used to resolve two closely spaced wavelengths.

(i) By considering the optical path difference between adjacent slits as a function of angle of diffraction, θ, show that the condition for a maximum of intensity is given by the grating equation

$$dsin(\theta) = m\lambda$$

where λ is the wavelength of the light and m is an integer designating the order of the diffraction.

[3 marks]

(ii) The intensity distribution as a function of angle is described by the following equation (DO NOT DERIVE THIS EQUATION)

$$I = I_0 \left[\frac{sin(\beta)}{\beta}\right]^2 \left[\frac{sin(\frac{N\delta}{2})}{sin(\frac{\delta}{2})}\right]^2$$

where

$$\beta = \frac{\pi}{\lambda} asin(\theta), \quad \delta = \frac{2\pi}{\lambda} dsin(\theta)$$

Briefly explain the significance of the two main terms in this equation and, without calculation, sketch this intensity distribution labeling the key features.

[3 marks]

(iii) To determine how well different wavelengths can be resolved, the change in angle, from the centre of the m^{th} order diffraction peak to the first zero of that diffraction order is required. Using the equation given or otherwise, and justifying any assumptions or approximations made, show that the change in angle is given by:

$$\Delta\theta = \frac{\lambda}{Ndcos(\theta_m)}$$

where θ_m is the angular position of the m^{th} diffraction order.
State the Rayleigh criterion for resolving two closely spaced peaks. Hence, by differentiating the grating equation and applying this criterion, show that the resolving power $R = \lambda/\Delta\lambda$, is given by

$$R = mN$$

[6 marks]

(iv) The two wavelengths to be resolved are the mercury lines at 577 nm and 579 nm. If the grating has 600 lines/mm, calculate the size of the grating required to resolve these wavelengths in the second order.

[3 marks]

[Total 15 marks]