Imperial College
London

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2013

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Probability Theory

Date: Tuesday, 21 May 2013. Time: 2.00pm. Time allowed: 2 hours.
This paper has FOUR questions.
Candidates should use ONE main answer book.
Supplementary books may only be used after the main book is full.
Statistical tables will not be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Answer all the questions. Each question carries equal weight.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Calculators may not be used.

Q1. (1.a) Define a random variable on a probability space describing carefully all notions involved.

(1.b) Define independent events and independent random variables. Explain giving reasons which of the following are independent random variables and which are not.

- (1.b.i) Rademacher functions H_n, $n \in \mathbb{N}$, on $([0,1], \Sigma_{Leb} \cap [0,1], \lambda)$, where Σ_{Leb} denotes the σ-algebra of Lebesgue measurable sets on real line ;
- (1.b.ii) Functions $\sin(2^n x)$, $n \in \mathbb{N}$, on $([0,2\pi), \Sigma_{Leb} \cap [0,2\pi), \frac{1}{2\pi}\lambda)$.

(1.c) Prove or disprove the Strong Law of Large Numbers in each of the cases in (1.b).

Q2. (2.a) State and prove Borel-Cantelli Lemmas.

(2.b) Prove or disprove that the following events happen infinitely many times.

- (2.b.i) In $((0,2\pi], \Sigma_{Leb} \cap (0,2\pi], \frac{1}{2\pi}\lambda)^{\mathbb{N}}$, at two consecutive sites the angle is between $\pi/4$ and $\pi/2$;
- (2.b.ii) In $(\Omega \equiv \{0,1\}, 2^{\Omega}, \nu_0)^{\mathbb{N}}$, with uniform probability measure ν_0, the intervals with a size proportional to the logarithm of the distance from the origin have all the points occupied by zeros;

Q3. (3.a) Define convergence in probability, convergence almost everywhere and in p-th power.

(3.b) Prove or disprove the following implication: For a sequence of random variables $(X_n)_{n\in\mathbb{N}}$ and a random variable X on a probability space (Ω, Σ, μ)

$$\forall \varepsilon > 0 \qquad \sum_{n\in\mathbb{N}} \mu\{|X_n - X| > \varepsilon\} < \infty \Longrightarrow X_n \longrightarrow_{n\in\mathbb{N}} X, \qquad \mu - a.e.$$

Q4. (4.a) Define a characteristic function of a random variable on a given probability space. State and prove the fundamental properties of the characteristic function.

(4.b)

- (4.b.i) Compute the characteristic function ϕ_n of sum of n independent Gaussian random variables with distribution $N(m, \sigma^2)$ normalised by $\frac{1}{n}$. Find the limit $\lim_{n\to\infty} \phi_n$.
- (4.b.ii) Compute the characteristic function φ_n of sum of n independent Gaussian random variables with distribution $N(0, \sigma^2)$ normalised by $\frac{1}{\sqrt{n}}$. Find the limit $\lim_{n\to\infty} \varphi_n$;

SOLUTIONS. P6

Sln1

Sln 1.

4pts

(1.a) Let (Ω, Σ, μ) be a probability space, where Σ is a family of subsets of a nonempty set Ω, containing this set, and closed with respect of taking complements in Ω and countable unions, and $\mu : \Sigma \to [0,1]$ is a function such that $\mu(\Omega) = 1$ which is σ-additive, i.e. for any countable family of disjoint sets $(A_n \in \Sigma)_{n\in\mathbb{N}}$ we have

$$\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \mu(A_n).$$

A function $X : \Omega \to \mathbb{R}$ is called a random variable iff

$$\forall x \in \mathbb{R} \qquad f^{-1}((-\infty, x)) \in \Sigma.$$

seen

4pts

(1.b) Two events $A, B \in \Sigma$ are called independent iff

$$\mu(A \cap B) = \mu(A) \cdot \mu(B).$$

Random variables f and g on the probability space (Ω, Σ, μ) are called independent iff

$$\forall x, y\mathbb{R} \qquad f^{-1}((-\infty, x)) \text{ } and \text{ } g^{-1}((-\infty, y))$$

seen

are independent.

4pts

- (1.b.i) By definition an n-th Rademacher function H_n is associated with a partition of the unit interval into 2^n equal intervals with value $(-1)^k$ on k-th interval, $k = 0, .., 2n-1$. Thus for any finite collection H_{j_l}, $l = 1, .., N$, $N \in \mathbb{N}$, with $j_l < j_{l+1}$, we have

$$\int \prod_{l=1,..,N} f_l(H_{j_l}) d\lambda = \prod_{l=1,..,N} \int f_l(H_{j_l}) d\lambda$$

which can be seen by representing the integral as a sum over the intervals where H_{j_1} is constant (and therefore factorizes from the integral), and continuing by induction.

seen

4pts

- (1.b.ii) The random variables $\sin(2^n x)$, $n \in \mathbb{N}$, on $([0, 2\pi), \Sigma_{Leb} \cap [0, 2\pi), \frac{1}{2\pi}\lambda)$ are not independent. For example one has

$$\int_{[0,2\pi)} sin^4(2x) sin^4(4x) d\lambda \neq \int_{[0,2\pi)} sin^4(2x) d\lambda \cdot \int_{[0,2\pi)} sin^4(4x) d\lambda$$

unseen

1.c
4pts

(1.c) In both cases we note that the corresponding random variables have 4-th moment. One notices that

$$E(S_n)^2 \equiv E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)^4 = \frac{1}{n^4}\sum_{i=1}^{n} EX_i^4 + 6\frac{1}{n^4}\sum_{1\le i\ne j\le n} E\left(X_i^2 X_j^2\right) + \frac{1}{n^4} EX_3X_2X_1^2 \le Const\frac{1}{n^2}$$

This is because for distinct i, j, k and l, we have

$$E(X_iX_j^3) = E(X_iX_jX_kX_l) = 0$$

and

$$E(X_iX_jX_k^2) = 0, \qquad for\ i, j, k \neq 1, 2, 3,\ in\ (1.b.ii)$$

(In the first case (1.b.i) using independence and in the second case (1.b.ii) using trigonometric identities.) Hence

$$\sum_{n\in\mathbb{N}} (S_n)^2 < \infty,\ a.e.$$

which implies

unseen

$$\lim_{n\to\infty} S_n = 0,\ a.e.$$

$\boxed{\texttt{Sln2}}$

Sln 2. (2.a)

$\boxed{\texttt{4 pts}}$ B-C.1: Let $A_n \in \Sigma$ be a sequence of events. Then

$$\sum_{n\in\mathbb{N}} \mu(A_n) < \infty \Longrightarrow \mu\left(\bigcap_{n\in\mathbb{N}}\bigcup_{k\geq n} A_k\right) = 0$$

$\boxed{\texttt{seen}}$

$\boxed{\texttt{4 pts}}$ B-C.2: Let $A_n \in \Sigma$ be a sequence of independent events. Then

$$\sum_{n\in\mathbb{N}} \mu(A_n) = \infty \Longrightarrow \mu\left(\bigcap_{n\in\mathbb{N}}\bigcup_{k\geq n} A_k\right) = 1$$

$\boxed{\texttt{seen}}$

Proof of (B-C.1): We have

$$\mu\left(\bigcap_{n\in\mathbb{N}}\bigcup_{k\geq n} A_k\right) \leq \inf_{n\in\mathbb{N}} \mu\left(\bigcup_{k\geq n} A_k\right) \leq \inf_{n\in\mathbb{N}} \sum_{k\geq n} \mu(A_n) = 0$$

Proof of (B-C.2):

$$1-\mu\left(\bigcap_{n\in\mathbb{N}}\bigcup_{k\geq n} A_k\right) = \mu\left(\{\bigcap_{n\in\mathbb{N}}\bigcup_{k\geq n} A_k\}^c\right) = \mu\left(\bigcup_{n\in\mathbb{N}}\bigcap_{k\geq n} A_k^c\right) \leq \sum_{n\in\mathbb{N}} \mu\left(\bigcap_{k\geq n} A_k^c\right)$$

By monotonicity of the measure and independence of the events, we have

$$\mu\left(\bigcap_{k\geq n} A_k^c\right) \leq \mu\left(\bigcap_{N\geq k\geq n} A_k^c\right) = \prod_{N\geq k\geq n} \mu(A_k^c)$$

Thus, using inequality $\log(1-x) \leq -x$ and our assumption, we get for any $n \in \mathbb{N}$

$$\mu\left(\bigcap_{k\geq n} A_k^c\right) \leq \exp\{\sum_{N\geq k\geq n} \log\left(1-\mu(A_k)\right)\} \leq \exp\{-\sum_{N\geq k\geq n} \mu(A_k)\} \longrightarrow_{N\to\infty} 0$$

Hence the (B-C.2) follows.

(2.b)

5 pts

– (2.b.i) Consider the following independent events $A_n \equiv \{\omega_i \in (\pi/4, \pi/2) : i = 4n, 4n+1\}$. Then for each n the probability of A_n to appear is given by $P(A_n) = \frac{1}{8}$ and hence $\sum_n P(A_n) = \infty$. Hence by Borel-Cantelli lemma (B-C.2), we have

$$P\left(\bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} A_k\right) = 1;$$

i.e. for two consecutive sites the angle is in $(\pi/4, \pi/2)$ happens infinitely many times.

seen

7 pts

– (2.b.ii)
Consider the following events.

$$A_n \equiv \{\omega_{2n+l} = 0, for l = 1, .., [\varepsilon \log(2n)]\}$$

where $[x]$ means integer part of x and $\varepsilon \in (0, \infty)$, for $n \in \mathbb{N}$ with $n > [\varepsilon \log(2n)]$. We note that this events are independent, as condition in the definitions concern disjoint sets of l's for different n's. Next we have that

$$P(A_n) = 2^{-[\varepsilon \log(2n)]}$$

and

$$\frac{1}{2} \cdot 2^{-\varepsilon \log(2n)} < P(A_n)$$

Hence for $\varepsilon \log 2 \leq 1$

$$\sum_n P(A_n) = \infty$$

and by (B-C.2) we get that there is infinitely many intervals of logarithmic size with respect to the distance to the origin which are occupied by zeros.

unseen

Sln3

Sln 3. (3.a) A sequence or random variables X_n, $n \in \mathbb{N}$, on a probability space (Ω, Σ, μ) converges to a random variable X :

2pts

- in probability iff

$$\forall \varepsilon > 0 \quad \mu\{|X_n - X| > \varepsilon\} \longrightarrow_{n\to\infty} 0$$

seen

2pts

- almost everywhere iff

$$\mu\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\} = 1$$

seen

2pts

- in p-th moment iff

$$\lim_{n\to\infty} \int |X_n - X|^p d\mu = 0$$

seen

14pts

(3.b)

By Borel-Cantelli lemma (BC.1) for any given $\varepsilon > 0$

$$\sum_{n\in\mathbb{N}} \mu\{|X_n - X| > \varepsilon\} < \infty$$

implies that with probability one the inequality

$$|X_n - X| > \varepsilon$$

appears only for finitely many times, i.e. a set Ω_ε where for given $\varepsilon > 0$ and any $\omega \in \Omega$ there exists $N \equiv N(\omega)$ such that for all $n \geq N$ we have

$$|X_n(\omega) - X(\omega)| < \varepsilon.$$

Next we notice that

$$\mu\left(\bigcap_{k\in\mathbb{N}} \Omega_{1/k}\right) = 1$$

unseen

and hence the definition of convergence is satisfied with probability one.

Sln4

Sln 4.

5pts

(4.a) For a random variable X on a probability space (Ω, Σ, μ), a function

$$\mathbb{R} \ni t \longmapsto \phi_X(t) \equiv \mu e^{itX}$$

seen

is called a characteristic function of X.

10pts

The characteristic function has the following properties

$$\begin{array}{ll} (a) & \phi_X(t=0) = 1 \\ (b) & \mathbb{R} \ni t \longmapsto \phi_X(t) \; is \; uniformly \; continuous \\ (c) & \forall n \in \mathbb{N}, \forall z_j \in \mathbb{C}, t_j \in \mathbb{R}, \; j = 1,..,n \qquad \sum_{j,k=1,..,n} \phi_X(t_j - t_k)\bar{z}_j \, z_k \geq 0 \end{array}$$

The first follows from the fact that μ is a probability measure. To prove the second one we note that

$$|\phi_X(t+\delta) - \phi_X(t)| = \left|\mu\left(e^{itX}(e^{i\delta X} - 1)\right)\right| \leq \mu\left|e^{i\delta X} - 1\right|$$

The integrand on the right hand side converges to zero $\mu - a.e.$ and is bounded, so by the Lebesgue dominated convergence theorem the right hand side converges to zero.
To prove the last one we note

$$\sum_{j,k=1,..,n} \phi_X(t_j - t_k)\bar{z}_j \, z_k = \mu \left| \sum_{j=1,..,n} e^{it_j X} z_j \right|^2 \geq 0$$

seen

5pts

(4.b)

- (4.b.i) Let X_n, $n \in \mathbb{N}$, be independent Gaussian random variables with distribution $N(m, \sigma^2)$. We have, by independence of the random variables

$$\phi_n(t) \equiv \mu(e^{it\frac{1}{n}\sum_{j=1,..,n} X_j}) = \prod_{j=1,..,n} \mu(e^{it\frac{1}{n}X_j}) = \left(e^{itm/n}e^{-\frac{1}{2}\sigma^2 t^2/n^2}\right)^n = e^{itm}e^{-\frac{1}{2}\sigma^2 t^2/n}$$

  where μ denotes the corresponding Gaussian product measure. Hence

$$\lim_{n\to\infty} \phi_n(t) = e^{itm}$$

not seen

– (4.b.ii) We have, by independence of the random variables

$$\mu(e^{it\frac{1}{\sqrt{n}}\sum_{j=1,..,n} X_j}) = \prod_{j=1,..,n} \mu(e^{it\frac{1}{\sqrt{n}}X_j}) = \left(e^{-\frac{1}{2}\sigma^2 t^2/n}\right)^n = e^{-\frac{1}{2}\sigma^2 t^2}$$

not seen

Thus the characteristic function in question is independent of n.