Formulae Sheet Provided

UNIVERSITY OF LONDON

[E1.11 2005]

B.ENG. AND M.ENG. EXAMINATIONS 2005

For Internal Students of Imperial College London

This paper is also taken for the relevant examination for the Associateship of the City & Guilds of London Institute

INFORMATION SYSTEMS ENGINEERING E1.11

MATHEMATICS

Date **Wednesday 1st June 2005** **10.00 am - 1.00 pm**

Answer SEVEN questions

Answers to Section A *questions must be written in a different answer book from answers to* Section B *questions.*

[Before starting, please make sure that the paper is complete. There should be SIX pages, with a total of NINE questions. Ask the invigilator for a replacement if this copy is faulty.]

1. (i) Express each of the following complex numbers in the form $x + iy$ (with x and y real) :

   (a) i^3 ;

   (b) $\dfrac{1}{1 - i + 2i^2}$;

   (c) $i^{1/3}$;

   (d) $\cos i$.

   (ii) Find all the solutions of the equation $\cosh z = 2i$.

   Give your answer in the form $z = x + iy$ (with x and y real).

2. (i) If $y = \sec^{-1} x$ (where $\sec^{-1} x$ is the inverse function of $\sec x$), show that

$$\frac{dy}{dx} = \cos y \cot y$$

   and hence that

$$\frac{dy}{dx} = \frac{1}{x\sqrt{x^2 - 1}} .$$

   (ii) Use Leibniz's Rule to find $\dfrac{d^7}{dx^7}\left(x^2 e^{x/2}\right)$.

   (iii) (a) Evaluate the limit $\displaystyle\lim_{x \to 0} \left[\frac{(\tan x)^2}{x} \right]$.

   You may assume that $\displaystyle\lim_{x \to 0} \frac{\sin x}{x} = 1$.

   (b) Evaluate the limit $\displaystyle\lim_{x \to 0} \left[\frac{1}{x^2} - \frac{1}{e^{x^2} - 1} \right]$.

PLEASE TURN OVER

3. (i) Use standard tests to determine whether the following series converge or diverge:

$$\text{(a)} \quad \sum_{n=1}^{\infty} \frac{n^3}{2^n} \; ; \qquad\qquad \text{(b)} \quad \sum_{n=1}^{\infty} \frac{n!}{3^{2n}} \; .$$

(ii) Find the intervals of convergence for the following series and investigate the endpoints:

$$\text{(a)} \quad \sum_{n=1}^{\infty} \frac{(-1)^n \, x^{2n}}{4^n} \; ; \qquad\qquad \text{(b)} \quad \sum_{n=1}^{\infty} \frac{e^{nx}}{2^n} \; .$$

(iii) Find the Maclaurin Series for $e^{-x} \cos x$ up to the third non-zero term.

You may use without proof the series for $\cos x$ *and* e^x .

4. (i) Using integration by parts, find $\int \ln x \, dx$.

(ii) Using integrating factors, find the general solution of the differential equation

$$\frac{dy}{dx} + y \ln x = x^{-x} \; .$$

(iii) Find the general solution of the differential equation

$$\frac{dy}{dx} = \frac{x^2 + y^2}{2x^2} \; .$$

5. (i) Find the solution of the differential equation

$$(x^2 + 6x + 9)\frac{dy}{dx} = \sqrt{16 - y^2}\,,$$

subject to the condition $x = 0$ at $y = 0$.

(ii) Find the general solution of the 2nd order differential equation

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{2x} + x^2 + 4\,.$$

PLEASE TURN OVER

SECTION B

6. (i) If $x = s^2 t$ and $y = s + e^{-t}$ and f is a function of x and y, then express $\dfrac{\partial f}{\partial s}$ and $\dfrac{\partial f}{\partial t}$ in terms of $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.

   Hence, or otherwise, find $\dfrac{\partial f}{\partial s}$ and $\dfrac{\partial f}{\partial t}$ when $f(x, y) = xy + ye^{-x}$

   and express your answer in terms of s and t.

   (ii) Find the stationary points of the function

$$f(x, y) = x^3 + y^3 - 6xy$$

   and determine their nature.

7. The Laplace transform of a function $f(t)$ is given by

$$\mathcal{L}(f(t)) \equiv F(s) \equiv \int_0^\infty e^{-st} f(t)\, dt\ .$$

   (i) Find $\mathcal{L}(\cos at)$.

   (ii) Use Laplace transforms to solve the simultaneous differential equations

$$\frac{d^2x}{dt^2} = y - 2x\ ,$$

$$\frac{d^2y}{dt^2} = x - 2y\ ,$$

   where x and y are functions of t satisfying the conditions

$$x(0) = 2, \qquad x'(0) = 0, \qquad y(0) = 4, \qquad y'(0) = 0\ .$$

   You may use the fact that $\mathcal{L}(f''(t)) = s^2\mathcal{L}(f(t)) - sf(0) - f'(0)$.

8. The function $f(x)$ has period 2π and satisfies

$$f(x) = x^2 \quad \text{for} \quad -\pi \le x < \pi .$$

Sketch the graph of $f(x)$ and calculate the Fourier series for $f(x)$.

Deduce that

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots = \frac{\pi^2}{12} .$$

By differentiating your Fourier series, deduce that

$$x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n} \sin nx \quad \text{for} \quad -\pi < x < \pi .$$

9. (i) A set of simultaneous equations takes the form $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{pmatrix} 1 & 5 & 2 \\ 2 & 6 & 9 \\ 3 & 8 & 15 \end{pmatrix} , \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} , \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} .$$

Use Gaussian elimination to find the solution for x_1, x_2, and x_3 in terms of b_1, b_2 and b_3.

(ii) Given the matrices

$$B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} , \qquad P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} ,$$

find the inverse, P^{-1}, of P and hence obtain the matrix

$$D = P^{-1} B P .$$

Show that for every positive integer n we have

$$D^n = P^{-1} B^n P .$$

Hence evaluate B^5 .

END OF PAPER

MATHEMATICS DEPARTMENT

MATHEMATICAL FORMULAE

1. VECTOR ALGEBRA

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1,\, a_2,\, a_3)$$

Scalar (dot) product: $\mathbf{a}\,.\,\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

Vector (cross) product:

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Scalar triple product:

$$[\mathbf{a},\,\mathbf{b},\,\mathbf{c}] = \mathbf{a}\,.\,\mathbf{b}\times\mathbf{c} = \mathbf{b}\,.\,\mathbf{c}\times\mathbf{a} = \mathbf{c}\,.\,\mathbf{a}\times\mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product: $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{c}\,.\,\mathbf{a})\mathbf{b} - (\mathbf{b}\,.\,\mathbf{a})\mathbf{c}$

2. SERIES

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \dots \quad (\alpha \text{ arbitrary, } |x| < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^n\frac{x^{2n}}{(2n)!} + \dots,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + \dots,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^n\frac{x^{n+1}}{(n+1)} + \dots (-1 < x \le 1)$$

3. TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin(a+b) = \sin a\,\cos b + \cos a\,\sin b\,;$$

$$\cos(a+b) = \cos a\,\cos b - \sin a\,\sin b\,.$$

$$\cos iz = \cosh z\,; \quad \cosh iz = \cos z\,; \quad \sin iz = i\sinh z\,; \quad \sinh iz = i\sin z\,.$$

4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f\,D^n g + \binom{n}{1} Df\,D^{n-1}g + \dots + \binom{n}{r} D^r f\,D^{n-r}g + \dots + D^n f\,g\,.$$

(b) Taylor's expansion of $f(x)$ about $x = a$:

$$f(a+h) = f(a) + hf'(a) + h^2f''(a)/2! + \dots + h^n f^{(n)}(a)/n! + \epsilon_n(h)\,,$$

where $\epsilon_n(h) = h^{n+1}f^{(n+1)}(a+\theta h)/(n+1)!, \quad 0 < \theta < 1\,.$

(c) Taylor's expansion of $f(x,\, y)$ about $(a,\, b)$:

$$f(a+h,\, b+k) = f(a,\, b) + [hf_x + kf_y]_{a.b} + 1/2!\,\left[h^2f_{xx} + 2hkf_{xy} + k^2f_{yy}\right]_{a.b} + \dots$$

(d) Partial differentiation of $f(x,\, y)$:

i. If $y = y(x)$, then $f = F(x)$, and $\dfrac{dF}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dx}\,.$

ii. If $x = x(t)$, $y = y(t)$, then $f = F(t)$, and $\dfrac{dF}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}\,.$

iii. If $x = x(u,\, v)$, $y = y(u,\, v)$, then $f = F(u,\, v)$, and

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}\,, \quad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}\,.$$

(e) Stationary points of $f(x,\, y)$ occur where $f_x = 0$, $f_y = 0$ simultaneously.
Let $(a,\, b)$ be a stationary point: examine $D = [f_{xx}f_{yy} - (f_{xy})^2]_{a.b}\,.$
If $D > 0$ and $f_{xx}(a,\, b) < 0$, then $(a,\, b)$ is a maximum;
If $D > 0$ and $f_{xx}(a,\, b) > 0$, then $(a,\, b)$ is a minimum;
If $D < 0$ then $(a,\, b)$ is a saddle-point.

(f) Differential equations:

i. The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating factor $I(x) = \exp[\int P(x)(dx]$, so that $\dfrac{d}{dx}(Iy) = IQ$.

ii. $P(x,\, y)dx + Q(x,\, y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y\,.$

5. INTEGRAL CALCULUS

(a) An important substitution: $\tan(\theta/2) = t$:
$\sin\theta = 2t/(1+t^2)$, $\cos\theta = (1-t^2)/(1+t^2)$, $d\theta = 2\,dt/(1+t^2)$.

(b) Some indefinite integrals:

$$\int (a^2 - x^2)^{-1/2} dx = \sin^{-1}\left(\frac{x}{a}\right), \quad |x| < a.$$

$$\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1}\left(\frac{x}{a}\right) = \ln\left\{\frac{x}{a} + \left(1 + \frac{x^2}{a^2}\right)^{1/2}\right\}.$$

$$\int (x^2 - a^2)^{-1/2} dx = \cosh^{-1}\left(\frac{x}{a}\right) = \ln\left|\frac{x}{a} + \left(\frac{x^2}{a^2} - 1\right)^{1/2}\right|.$$

$$\int (a^2 + x^2)^{-1} dx = \left(\frac{1}{a}\right)\tan^{-1}\left(\frac{x}{a}\right).$$

6. NUMERICAL METHODS

(a) Approximate solution of an algebraic equation:

If a root of $f(x) = 0$ occurs near $x = a$, take $x_0 = a$ and
$x_{n+1} = x_n - [f(x_n)/f'(x_n)]$, $n = 0, 1, 2 \ldots$

(Newton Raphson method).

(b) Formulae for numerical integration: Write $x_n = x_0 + nh$, $y_n = y(x_n)$.

i. Trapezium rule (1-strip): $\int_{x_0}^{x_1} y(x)dx \approx (h/2)[y_0 + y_1]$.

ii. Simpson's rule (2-strip): $\int_{x_0}^{x_2} y(x)dx \approx (h/3)[y_0 + 4y_1 + y_2]$.

(c) Richardson's extrapolation method: Let $I = \int_a^b f(x)dx$ and let I_1, I_2 be two estimates of I obtained by using Simpson's rule with intervals h and $h/2$.

Then, provided h is small enough,

$$I_2 + (I_2 - I_1)/15,$$

is a better estimate of I.

7. LAPLACE TRANSFORMS

Function	Transform	Function	Transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)dt$	$af(t) + bg(t)$	$aF(s) + bG(s)$
df/dt	$sF(s) - f(0)$	d^2f/dt^2	$s^2F(s) - sf(0) - f'(0)$
$e^{at}f(t)$	$F(s-a)$	$tf(t)$	$-dF(s)/ds$
$(\partial/\partial\alpha)f(t,\alpha)$	$(\partial/\partial\alpha)F(s,\alpha)$	$\int_0^t f(t)dt$	$F(s)/s$
$\int_0^t f(u)g(t-u)du$	$F(s)G(s)$		
1	$1/s$	$t^n (n = 1, 2 \ldots)$	$n!/s^{n+1}$, $(s > 0)$
e^{at}	$1/(s-a)$, $(s > a)$	$\sin\omega t$	$\omega/(s^2+\omega^2)$, $(s > 0)$
$\cos\omega t$	$s/(s^2+\omega^2)$, $(s > 0)$	$H(t-T) = \begin{cases} 0, & t < T \\ 1, & t > T \end{cases}$	e^{-sT}/s, $(s, T > 0)$

8. FOURIER SERIES

If $f(x)$ is periodic of period $2L$, then $f(x+2L) = f(x)$, and

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \text{ where}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad n = 0, 1, 2, \ldots, \text{ and}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1, 2, 3, \ldots$$

Parseval's theorem

$$\frac{1}{L}\int_{-L}^{L} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right).$$

MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

2004 – 2005

PAPER ISE 1·6

QUESTION

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SOLUTION 1

(i) (a) $i^3 = i \cdot i^2 = -i$ — 1

(b) $\frac{1}{1-i+2i^2} = \frac{1}{-1-i} = \frac{-(1-i)}{(1+i)(1-i)} = \frac{i-1}{1-i^2} = -\frac{1}{2} + \frac{i}{2}$ — 2

(c) $i^{1/3} = \left[e^{\frac{i\pi}{2} + i2n\pi}\right]^{1/3} = e^{\frac{i\pi}{6} + \frac{i2n\pi}{3}}$ — 4

$n=0 \qquad = e^{i\pi/6} = \frac{\sqrt{3}}{2} + \frac{i}{2}$

$n=1 \qquad = e^{i\frac{5\pi}{6}} = -\frac{\sqrt{3}}{2} + \frac{i}{2}$

$n=2 \qquad = e^{i\frac{3\pi}{2}} = -i$

(d) $\cos i = \frac{e^{i\cdot i} + e^{-i\cdot i}}{2} = \frac{e^{-1} + e}{2} = \frac{1+e^2}{2e} = \cosh 1$ — 3

(ii) $\cosh z = 2i \implies e^z + e^{-z} = 4i$

Let $u = e^z$ $\qquad u + \frac{1}{u} = 4i \implies u^2 - 4iu + 1 = 0$

$\implies u = \frac{4i \pm \sqrt{-16-4}}{2} = 2i \pm \frac{\sqrt{-20}}{2} = 2i \pm \sqrt{5}i$

$\implies e^z = (2 \pm \sqrt{5})i \qquad \implies z = \ln[(2\pm\sqrt{5})i]$ — 10

$z = \ln\left[(2+\sqrt{5})e^{\frac{i\pi}{2} + i2n\pi}\right]$ or $\ln\left[(\sqrt{5}-2)e^{-\frac{i\pi}{2} + i2n\pi}\right]$

$= \ln(2+\sqrt{5}) + \frac{i\pi}{2} + i2n\pi$ or $\ln(\sqrt{5}-2) - \frac{i\pi}{2} + i2n\pi$

for integer n.

Setter : Martin Howard — Setter's signature : [signature]

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(i) $y = \sec^{-1} x \Rightarrow \sec y = x \Rightarrow \frac{1}{\cos y} = x$

$$\Rightarrow \frac{\sin y}{\cos^2 y}\frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \cos y \frac{\cos y}{\sin y} = \cos y \cot y$$ 3

$$\Rightarrow \frac{dy}{dx} = \frac{1/x^2}{\sqrt{1 - 1/x^2}} = \frac{1}{x\sqrt{x^2-1}}$$ 2

(ii) $$\frac{d^7}{dx^7}\left(x^2 e^{\frac{1}{2}x}\right) = x^2 \frac{d^7}{dx^7}\left(e^{\frac{1}{2}x}\right) + 7\frac{d}{dx}(x^2)\frac{d^6}{dx^6}\left(e^{\frac{1}{2}x}\right) + \frac{7\times 6}{2}\frac{d^2}{dx^2}(x^2)\frac{d^5}{dx^5}\left(e^{\frac{1}{2}x}\right)$$ 5

$$= \frac{x^2 e^{\frac{1}{2}x}}{128} + \frac{14xe^{\frac{1}{2}x}}{64} + \frac{42e^{\frac{1}{2}x}}{32} = \frac{1}{128}x^2 e^{\frac{1}{2}x} + \frac{7}{32}xe^{\frac{1}{2}x} + \frac{21}{16}e^{\frac{1}{2}x}$$

(iii) (a) $$\lim_{x\to 0}\left[\frac{(\tan x)^2}{x}\right] = \lim_{x\to 0}\left[\frac{(\sin x)^2}{x(\cos x)^2}\right] = \lim_{x\to 0}\left[\frac{(\sin x)^2}{x^2}\frac{x}{(\cos x)^2}\right]$$ 4

$$= \lim_{x\to 0}\left[\left(\frac{\sin x}{x}\right)^2 x\right] = 0$$

(b) $$\lim_{x\to 0}\left[\frac{1}{x^2} - \frac{1}{e^{x^2}-1}\right] = \lim_{x\to 0}\left[\frac{1}{x^2} - \frac{1}{1+x^2+\frac{1}{2}x^4+\dots-1}\right]$$ 6

$$= \lim_{x\to 0}\left[\frac{1}{x^2} - \frac{1}{x^2+\frac{1}{2}x^4+\dots}\right] = \lim_{x\to 0}\left[\frac{1}{x^2}\left(1 - \left[1+\tfrac{1}{2}x^2+\dots\right]^{-1}\right)\right]$$

$$= \lim_{x\to 0}\left[\frac{1}{x^2}\left(1 - 1 + \tfrac{1}{2}x^2 - \dots\right)\right] = \frac{1}{2}$$

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MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

2004 -- 2005

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PAPER ISE 1.6

QUESTION

SOLUTION 3

(i)

(a) $\sum_{n=1}^{\infty} \frac{n^3}{2^n}$ Ratio test $\rho = \lim_{n\to\infty} \left|\frac{\rho_{n+1}}{\rho_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)^3}{2^{n+1}} \frac{2^n}{n^3}\right|$

where ρ_n is n-th term of series.

$= \lim_{n\to\infty} \left|\frac{(n+1)^3}{2n^3}\right| = \lim_{n\to\infty} \left(\frac{1}{2}\left(1+\frac{1}{n}\right)^3\right)$ 4

$= \frac{1}{2} \Rightarrow$ convergent

(b) $\sum_{n=1}^{\infty} \frac{n!}{3^{2n}}$ $\rho = \lim_{n\to\infty} \left|\frac{(n+1)!}{3^{2n+2}} \frac{3^{2n}}{n!}\right| = \lim_{n\to\infty} \left|\frac{n+1}{9}\right| = \infty$ 3

$\Rightarrow$ divergent.

(ii)

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{4^n}$ $\rho = \lim_{n\to\infty} \left|\frac{x^{2n+2}}{4^{n+1}} \frac{4^n}{x^{2n}}\right| = \lim_{n\to\infty} \left|\frac{x^2}{4}\right|$ 5

Convergent if $x^2 < 4 \Rightarrow -2 < x < 2$

If $x = \pm 2$, series becomes $\sum_{n=1}^{\infty} (-1)^n \frac{(\pm 2)^{2n}}{4^n} = \sum_{n=1}^{\infty} (-1)^n \frac{4^n}{4^n} = -1+1-1+1-1\cdots$

undefined.

(b) $\sum_{n=1}^{\infty} \frac{e^{nx}}{2^n}$ $\rho = \lim_{n\to\infty} \left|\frac{e^{(n+1)x}}{2^{n+1}} \frac{2^n}{e^{nx}}\right| = \lim_{n\to\infty} \left|\frac{e^x}{2}\right|$ 5

Convergent if $e^x < 2 \Rightarrow x < \ln 2$.

At $x = \ln 2$, series becomes $\sum_{n=1}^{\infty} \frac{e^{n\ln 2}}{2^n} = \sum_{n=1}^{\infty} 2^n/2^n = \sum_{n=1}^{\infty} 1 \Rightarrow$ divergent

(iii) $(\cos x)e^{-x} = \left[1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \cdots\right]\left[1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \cdots\right]$ 3

$= 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{2}x^2 + \frac{1}{2}x^3 + \cdots = 1 - x + \frac{1}{3}x^3 + \cdots$

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(i) $\int \ln x \, dx$ $\quad v = \ln x \quad \frac{du}{dx} = 1$

$\Rightarrow \frac{dv}{dx} = \frac{1}{x} \quad u = x$

$$= x \ln x - \int \frac{1}{x} \cdot x \, dx = x \ln x - x + c.$$

4

(ii) $\frac{dy}{dx} + y \ln x = x^{-x}$

I.F. $e^{\int \ln x} = e^{x \ln x - x}$

$$\Rightarrow e^{x\ln x - x} \frac{dy}{dx} + y e^{x \ln x - x} \ln x = e^{x \ln x - x} x^{-x}$$

$$= e^{x \ln x - x} e^{-x \ln x} = e^{-x}$$

$$\Rightarrow y e^{x \ln x - x} = \int e^{-x} dx$$

$$\Rightarrow y e^{-x} x^{x} = -e^{-x} + c \quad \Rightarrow y = e^{x} x^{-x}(c - e^{-x})$$

$$= x^{-x}(ce^{x} - 1)$$

8

(iii) $\frac{dy}{dx} = \frac{x^2 + y^2}{2x^2}$

Let $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{1}{2}(1 + v^2) \Rightarrow x\frac{dv}{dx} = \frac{1}{2}(v^2 - 2v + 1) = \frac{1}{2}(v-1)^2$$

$$\Rightarrow \int \frac{2dv}{(v-1)^2} = \int \frac{dx}{x} \Rightarrow \frac{-2}{v-1} = \ln x + c$$

$$\Rightarrow \frac{2}{1 - y/x} = \ln x + c \Rightarrow 1 - \frac{y}{x} = \frac{2}{\ln x + c} \Rightarrow y = x\left[1 - \frac{2}{\ln x + c}\right]$$

8

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(i) $(x^2+6x+9)\frac{dy}{dx} = \sqrt{16-y^2}$

$\Rightarrow \int \frac{dy}{\sqrt{16-y^2}} = \int \frac{dx}{(x+3)^2}$ 6

Let $y = 4\sin\theta$

$dy = 4\cos\theta\, d\theta$

$\Rightarrow \int \frac{4\cos\theta\, d\theta}{4\cos\theta} = \frac{-1}{x+3} + c$

$\Rightarrow \theta = \frac{-1}{x+3} + c \quad \Rightarrow \sin^{-1}\frac{y}{4} = c - \frac{1}{x+3}$

$\Rightarrow y = 4\sin\left[c - \frac{1}{x+3}\right]$

$x = y = 0 \Rightarrow 0 = 4\sin\left[c - \frac{1}{3}\right] \quad \Rightarrow c = \frac{1}{3}$ 2

$\Rightarrow y = 4\sin\left[\frac{1}{3} - \frac{1}{x+3}\right]$

(ii) $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{2x} + x^2 + 4$

Auxiliary equation : $\lambda^2 - 5\lambda + 6 = 0 \Rightarrow (\lambda-3)(\lambda-2) = 0 \Rightarrow \lambda = 3$ or 2.

C.F. $y = Ae^{3x} + Be^{2x}$

For exponential, try $y_{PI} = axe^{2x} \Rightarrow y'_{PI} = ae^{2x} + 2axe^{2x}$ 12

$y''_{PI} = 4axe^{2x} + 4ae^{2x}$

$\Rightarrow 4axe^{2x} + 4ae^{2x} - 5(ae^{2x} + 2axe^{2x}) + 6axe^{2x} = e^{2x}$

$\Rightarrow xe^{2x}[4a - 10a + 6a] + e^{2x}[4a - 5a] = e^{2x} \quad \Rightarrow a = -1$

For polynomial, try $y_{PI} = ax^2 + bx + c$

$\Rightarrow 2a - 5(2ax + b) + 6(ax^2 + bx + c) = x^2 + 4$

$\Rightarrow x^2(6a) + x(6b - 10a) + (6c - 5b + 2a) = x^2 + 4 \quad \Rightarrow a = 1/6,\ b = \frac{5}{18}$

$\Rightarrow c = \left[4 + \frac{25}{18} - \frac{2}{6}\right]\cdot\frac{1}{6} = \frac{91}{108}$

$\Rightarrow y = Ae^{3x} + Be^{2x} - xe^{2x} + \frac{1}{6}x^2 + \frac{5}{18}x + \frac{91}{108}$

Setter : Martin Howard

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PAPER ESE 1
QUESTION
SOLUTION 6

(i) We have $\frac{\partial x}{\partial s} = 2st \quad \frac{\partial x}{\partial t} = s^2 \quad \frac{\partial y}{\partial s} = 1 \quad \frac{\partial y}{\partial t} = -e^{-t}$ 2

Therefore,

$$\frac{\partial f}{\partial s} = 2st\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}$$ 3

$$\frac{\partial f}{\partial t} = s^2\frac{\partial f}{\partial x} - e^{-t}\frac{\partial f}{\partial y}$$

Let $f(x,y) = xy + ye^{-x}$. Then

$$\frac{\partial f}{\partial x} = y - ye^{-x} = (s+e^{-t})(1-e^{-s^2t})$$ 1

$$\frac{\partial f}{\partial y} = x + e^{-x} = s^2t + e^{-s^2t}$$ 1

Leave over for "Otherwise"

$$\therefore \frac{\partial f}{\partial s} = 2st(s+e^{-t})(1-e^{-s^2t}) + s^2t + e^{-s^2t}$$ 2

$$\frac{\partial f}{\partial t} = s^2(s+e^{-t})(1-e^{-s^2t}) - e^{-t}(s^2t + e^{-s^2t})$$

(ii) We have $\frac{\partial f}{\partial x} = 3x^2 - 6y \quad \frac{\partial f}{\partial y} = 3y^2 - 6x$ 2

At stationary points,

$$x^2 - 2y = 0 \quad y^2 - 2x = 0$$

Hence $\frac{x^4}{4} - 2x = 0$, so $x^4 - 8x = 0$, and $x = 0$ or 2

$\therefore$ stationary points are $(0,0)$ and $(2,2)$. 3

Now, $\frac{\partial^2 f}{\partial x^2} = 6x \quad \frac{\partial^2 f}{\partial y^2} = 6y \quad \frac{\partial^2 f}{\partial x \partial y} = -6$

$$\therefore \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 36xy - 36$$ 3

This is < 0 at $(0,0)$, so we have a saddle point.
At $(2,2)$ this is > 0 and $\frac{\partial^2 f}{\partial x^2} > 0$, so we have a minimum. 3

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ALTERNATIVE

"Otherwise"

$$f(x,y) = xy + ye^{-x} = s^3t + s^2te^{-t} + (s+e^{-t})e^{-s^2t}$$ 2

$$\frac{\partial f}{\partial s} = 3s^2t + 2ste^{-t} + e^{-s^2t} + (s+e^{-t})(-2st)e^{-s^2t}.$$

$$\left[= 3s^2t + 2ste^{-t} + e^{-s^2t} - 2s^2te^{-s^2t} - 2ste^{-t-s^2t}.\right]$$ 2

$$\frac{\partial f}{\partial t} = s^3 + s^2e^{-t} - s^2te^{-t} - e^{-t}e^{-s^2t} - s^2(s+e^{-t})e^{-s^2t}.$$ 2

$$\left[= s^3 + s^2e^{-t} - s^2te^{-t} - e^{-t-s^2t} - s^3e^{-s^2t} - s^2e^{-t-s^2t}\right]$$

MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

2004 – 2005

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PAPER	QUESTION	SOLUTION
ISE 1		7

(i) If $f(t) = \cos at$ then $F(s) = \int_0^\infty e^{-st}\cos at\,dt$

$$= \left[-\tfrac{1}{s}e^{-st}\cos at\right]_0^\infty - \int_0^\infty \tfrac{1}{s}e^{-st}a\sin at\,dt$$

$$= \frac{1}{s} - \frac{a}{s}\left[-\tfrac{1}{s}e^{-st}\sin at\right]_0^\infty + \frac{a}{s}\int_0^\infty -\tfrac{1}{s}e^{-st}a\cos at\,dt$$

$$= \frac{1}{s} - \frac{a^2}{s^2}F(s).$$

$\therefore\ F(s)\left(1+\frac{a^2}{s^2}\right) = \frac{1}{s}$ and $F(s) = \frac{s}{s^2+a^2}$. 6

(ii)
$$x'' = y - 2x$$
$$y'' = x - 2y$$

$\therefore\ -x'(0) - s\,x(0) + s^2\mathcal{L}(x) = \mathcal{L}(y) - 2\mathcal{L}(x)$ 2

$\quad -y'(0) - s\,y(0) + s^2\mathcal{L}(y) = \mathcal{L}(x) - 2\mathcal{L}(y)$

$\therefore\ (s^2+2)\mathcal{L}(x) - \mathcal{L}(y) = 2s$ 2

$\quad (s^2+2)\mathcal{L}(y) - \mathcal{L}(x) = 4s$

$\therefore\ (-1+(s^2+2)^2)\mathcal{L}(x) = 4s + 2s(s^2+2)$

$\mathcal{L}(x) = \dfrac{2s^3+8s}{(s^2+3)(s^2+1)} = \dfrac{As+B}{s^2+3} + \dfrac{Cs+D}{s^2+1} \Rightarrow \begin{array}{l} A+C=2 \\ B+D=0 \\ A+3C=8 \\ B+3D=0 \end{array} \Rightarrow \begin{array}{l} A=-1 \\ C=3 \end{array}$

$\mathcal{L}(x) = \dfrac{-s}{s^2+3} + \dfrac{3s}{s^2+1}$ 5

$\therefore\ x(t) = -\cos\sqrt{3}t + 3\cos t$

Also,

$(-1+(s^2+2)^2)\mathcal{L}(y) = 2s + 4s(s^2+2)$

$\mathcal{L}(y) = \dfrac{4s^3+10s}{(s^2+3)(s^2+1)}$

$\quad = \dfrac{s}{s^2+3} + \dfrac{3s}{s^2+1}$

$\therefore\ y(t) = \cos\sqrt{3}t + 3\cos t$ 5

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Checker : MARTIN HOWARD Checker's signature : M. [illegible]

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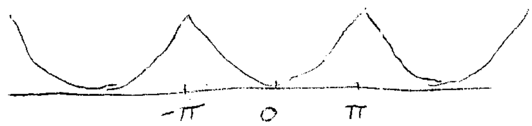

2

$f(x)$ is an even function, so

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

3

Here, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{2\pi^2}{3}$.

& for $n \geq 1$,

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx\,dx \\ &= \frac{1}{\pi}\left[x^2 \frac{\sin nx}{n}\right]_{-\pi}^{\pi} - \frac{1}{\pi}\int_{-\pi}^{\pi} 2x \frac{\sin nx}{n}\,dx \\ &= 0 - \frac{2}{n\pi}\left[-x\frac{\cos nx}{n}\right]_{-\pi}^{\pi} - \frac{2}{n\pi}\int_{-\pi}^{\pi} \frac{\cos nx}{n}\,dx \\ &= \frac{2}{n^2\pi}\left[\pi\cos n\pi + \pi\cos n\pi\right] - \frac{2}{n\pi}\left[\frac{\sin nx}{n^2}\right]_{-\pi}^{\pi} \\ &= \frac{4}{n^2}(-1)^n \end{aligned}$$

6

$$\therefore f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2} \cos nx$$

1

Put $x = 0$:

$$0 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$\therefore \frac{\pi^2}{12} = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$$

4

For $-\pi < x < \pi$,

$$f(x) = x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos nx$$

Differentiate:

$$2x = 4\sum_{n=1}^{\infty} -\frac{(-1)^n}{n} \sin nx$$

$$\therefore x = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{2}{n}\sin nx.$$

4

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(i) $A\underset{\sim}{x} = \underset{\sim}{b} \Rightarrow \begin{pmatrix} 1 & 5 & 2 \\ 0 & -4 & 5 \\ 0 & -7 & 9 \end{pmatrix} \underset{\sim}{x} = \begin{pmatrix} b_1 \\ b_2 - 2b_1 \\ b_3 - 3b_1 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 1 & 5 & 2 \\ 0 & -4 & 5 \\ 0 & 0 & \frac{1}{4} \end{pmatrix} \underset{\sim}{x} = \begin{pmatrix} b_1 \\ b_2 - 2b_1 \\ b_3 - \frac{7}{4}b_2 + \frac{1}{2}b_1 \end{pmatrix}$ 4

2

Hence $x_3 = 2b_1 - 7b_2 + 4b_3$

$\Rightarrow -4x_2 + 10b_1 - [illegible] + 2[illegible]b_3 = b_2 - 2b_1$

$\Rightarrow -4x_2$ [illegible] 2

$\Rightarrow x_2 = 3b_1 -$ [illegible] $+$ [illegible] b_3

$\Rightarrow$ [illegible] $+ 2.5b_3 + 2b_1 -$ [illegible] $+ 8b_3$ [illegible] 2

[illegible] $+ 59b_2 = 33b_3$

2

(ii) By some means or other, $P^{-1} = P$

$\therefore P^{-1}BP = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 3 & 3 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$. 2

$D^n = P^{-1}B^nP$ is true for $n = 1$.

Assume that $D^k = P^{-1}B^kP$. Then

$D^{k+1} = DD^k = P^{-1}BPP^{-1}B^kP = P^{-1}B^{k+1}P$. 2

Hence the required result, by induction.

Now, $D^5 = P^{-1}B^5P \Rightarrow B^5 = PD^5P^{-1}$, so

$B^5 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 3^5 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$= \frac{1}{2} \cdot \begin{pmatrix} 243 & 1 \\ 243 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$= \frac{1}{2}\begin{pmatrix} 244 & 242 \\ 242 & 244 \end{pmatrix}$

$= \begin{pmatrix} 122 & 121 \\ 121 & 122 \end{pmatrix}$ 4

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