Paper Number(s): **ISE2.9**

IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE
UNIVERSITY OF LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2000

ISE PART I: M.Eng. and B.Eng. and ACGI

CONTROL SYSTEMS

Friday, May 12 2000, 2:00 pm

There are FIVE questions on this paper.

Answer THREE questions.

All questions carry equal marks.

Corrected Copy

Time allowed: 2:00 hours

None HCJ

Examiners: Dr I.M. Jaimoukha, Dr J.M.C. Clark

1. Consider the mass-spring-damper system shown in Figure 1 below, in which $y(t)$ denotes the displacement of the mass M from its rest position. A force $u(t)$ is applied to the mass M as shown.

   (a) By considering the balance of forces on the mass, derive the differential equations relating $u(t)$ to $y(t)$. [5]

   (b) Derive a state-variable model in the standard form:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t). \end{aligned}$$

[5]

   (c) Derive the transfer function between $u(s)$ and $y(s)$. [5]

   (d) Take $M = 1Kg, K = 1N/m$ and $D_1 = D_2 = 1Ns/m$. Suppose that $u(t) = \sin \omega t$. Find the steady-state response $y_{ss}(t)$. [5]

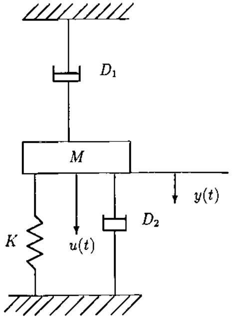


Figure 1

2. Figure 2 below depicts a feedback control system with

$$G(s) = \frac{k}{(s+1)^2}$$

where k is a design parameter. Design a stabilising compensator $K(s)$ as follows:

(a) Choose $K(s)$ so that when $r(t)$ is a unit step,

$$r(t) = 1, \quad t \geq 0,$$

applied at $t = 0$, the steady-state error must satisfy

$$\lim_{t\to\infty} e(t) = 0.$$

[6]

(b) Find the range of values of k such that the closed-loop is stable. [7]

(c) Find the minimum value of k such that when $r(t)$ is a unit ramp,

$$r(t) = t, \quad t \geq 0,$$

applied at $t = 0$, the steady-state error must satisfy

$$\lim_{t\to\infty} e(t) \leq 1.$$

[7]

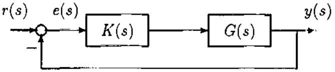


Figure 2

3. Consider the feedback control system shown in Figure 3 below. Here,

$$G(s) = \frac{1}{(s+1)(s+2)}$$

and $K(s)$ is the transfer function of the compensator.

(a) For $K(s) = k$, a constant compensator, draw the root locus accurately as k varies in the range $0 \leq k \leq \infty$. [4]

(b) Find the constant compensator $K(s) = k$ which gives a critically damped response to a unit step reference $r(t)$. [6]

(c) Design a first order compensator $K(s) = \frac{k}{s-p}$ as follows:

i. Choose the compensator pole p so that the root locus of the compensated system passes through the point $-1 + j$.

ii. Draw a rough sketch of the root locus of the compensated system.

iii. Choose the constant gain k so that the closed-loop transfer function has a pole at $-1 + j$. [10]

Figure 3

4. Consider the feedback control system in Figure 4 below. Here,

$$G(s) = \frac{4(s+1)}{(s-1)^2}$$

and $K(s)$ is the transfer function of a compensator.

(a) Sketch the Nyquist diagram of $G(s)$, clearly indicating the low and high frequency portions, as well as the real-axis intercepts. [7]

(b) Suppose that $K(s) = 1$. Use the Nyquist diagram to show that the closed-loop system is stable and determine the phase margin. [7]

(c) Without doing any actual design, briefly describe how a phase-lead compensator,

$$K(s) = \frac{1 + s/\omega_0}{1 + s/\omega_p}, \qquad 0 < \omega_0 < \omega_p,$$

would affect the phase margin. Indicate in which frequency range should ω_0 and ω_p be chosen. [6]

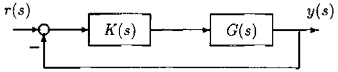


Figure 4

5. Consider the feedback control system shown in Figure 5 below. Here,

$$G(s) = \frac{1}{(s+1)^2}$$

and $K(s)$ is the transfer function of the compensator.

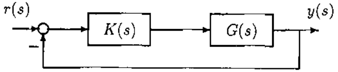


Figure 5

(a) For $K(s) = k$, a constant compensator, draw the root locus accurately as k varies in the range $0 \leq k \leq \infty$. [5]

(b) Design a proportional-plus-derivative compensator $K(s) = k(s - z)$ as follows:

i. Choose the compensator zero z so that the root locus of the compensated system $(s - z)G(s)$ passes through the point $-2 + j2$.
ii. Draw a rough sketch of the root locus of the compensated system.
iii. Choose the constant gain k so that the closed-loop transfer function has a pole at $-2 + j2$.

[10]

(c) Suppose now that the input to the compensator $K(s)$ designed in Part (b) is corrupted by a noise signal

$$v(t) = V_0 \sin \omega t,$$

as shown in Figure 6 below, where V_0 and ω are constants. Comment on the likely impact of this noise on the performance of the control system in Part (b).

[5]

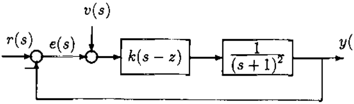


Figure 6

SOLUTIONS (ISE2.9, 2000)

1. (a) Applying Newton's laws on the mass,

$$u(t) = M\ddot{y}(t) + (D_1 + D_2)\dot{y}(t) + Ky(t).$$

(b) Take $x_1(t) = y(t)$, $x_2(t) = \dot{y}(t)$. Then,

$$\begin{aligned} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -\frac{K}{M} & -\frac{(D_1 + D_2)}{M} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}. \end{aligned}$$

(c) Taking the Laplace transform of the differential equation in part (a):

$$[Ms^2 + (D_1 + D_2)s + K]y(s) = u(s).$$

The transfer function is then given by

$$g(s) = \frac{1}{Ms^2 + (D_1 + D_2)s + K}.$$

(d) Putting in the numbers, we get,

$$g(s) = \frac{1}{s^2 + 2s + 1} = \frac{1}{(s+1)^2}.$$

Since $g(s)$ is stable, the steady-state response to a sinusoid of frequency ω is also a sinusoid of the same frequency, with an amplitude $|g(j\omega)|$ and phase $\angle g(j\omega)$. Since $\omega = 1$, we have that, in the steady-state,

$$\begin{aligned} y_{ss}(t) &= |g(j)| \sin(t + \angle g(j)) \\ &= 0.5 \sin(t - \frac{\pi}{2}) \\ &= -0.5 \cos t \end{aligned}$$

2. (a) After some block diagram manipulations,

$$\frac{e(s)}{r(s)} = \frac{1}{1 + K(s)\frac{k}{(s+1)^2}}$$

Using the final value theorem of the Laplace transform,

$$\begin{aligned} e_{ss} &= \lim_{t\to\infty} e(t) = \lim_{s\to 0} se(s) = \lim_{s\to 0} \frac{sr(s)}{1 + K(s)\frac{k}{(s+1)^2}} \\ &= \lim_{s\to 0} \frac{1}{1 + K(s)\frac{k}{(s+1)^2}} \end{aligned}$$

since $r(s) = 1/s$. For zero steady-state error, we need $K(s) = 1/s$, that is, a type 1 system, provided k is chosen so that the closed-loop system is stable.

(b) Taking $K(s) = 1/s$, gives the characteristic equation as

$$1 + \frac{k}{s(s+1)^2} = 0$$

or

$$s^3 + 2s^2 + s + k = 0$$

The Routh array is then

$$\begin{array}{c|cc} s^3 & 1 & 1 \\ s^2 & 2 & k \\ s & 1 - 0.5k & \\ 1 & k & \end{array}$$

For stability, we require no sign changes in the first column. Thus the closed-loop will be stable for

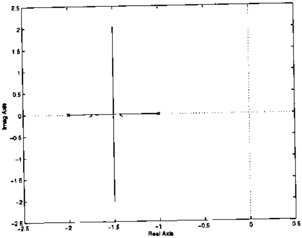
$$0 < k < 2$$

(c) When $r(s) = 1/s^2$ (unit ramp), we have

$$\begin{aligned} e_{ss} &= \lim_{s\to 0} \frac{1}{s + \frac{k}{(s+1)^2}} \\ &= k. \end{aligned}$$

Therefore, the minimum value of k so that $e_{ss} \leq 1$ is $k = 1$.

3. (a) The plot is shown below.



(b) For a critically damped response, the closed-loop poles must be equal and real. The characteristic equation is given by

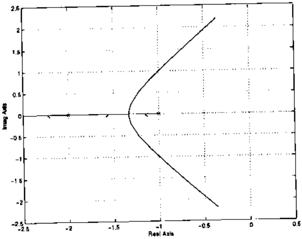
$$1+\frac{k}{s^2+3s+2}=0 \Rightarrow s^2+3s+2+k=0 \Rightarrow (s+1.5)^2+k-0.25=0 \Rightarrow k=0.25.$$

(c) i. Let the angle between $(-1+j)$ and p be θ. Applying the angle criterion:

$$0-(90^\circ+45^\circ+\theta)=\pm 180^\circ$$

or $\theta = 45^\circ$. Thus $p=-2$.

ii. The root locus is shown below.



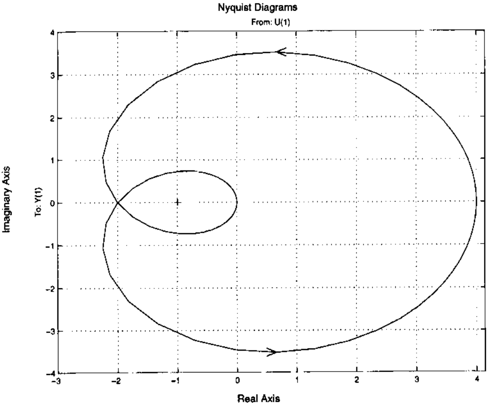
iii. Since $(-1+j)$ lies on the root locus, we use the gain criterion to find k:

$$k=-\frac{1}{(s+1)(s+2)^2}|_{s=-1+j}=2$$

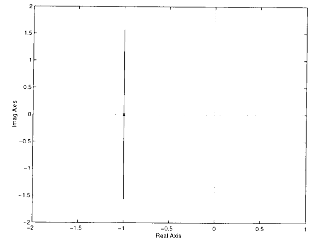
4. (a) The Nyquist plot is shown below. The real-axis intercepts can be found by setting the imaginary part of $G(j\omega)$ to zero. This gives intercepts at $\omega_i = 0, \pm\sqrt{3}, \infty$ and so $G(j\omega_i) = 4, -2, 0$, respectively.

   (b) Since the intercept with the negative real axis is at -2, the number of anticlockwise encirclements of the $-1+j0$ point is 2. Since the open-loop system has two unstable poles, it follows from the Nyquist stability criterion that the closed-loop system is stable. For the phase margin, we need the intercept with the unit circle centred on the origin. We solve $|G(j\omega)| = 1$, this gives $\omega_1 = \pm\sqrt{15}$ and arg $[G(j\omega_1)] = -133.4°$. The phase margin is then $46.6°$.

   (c) The phase-lead compensator has positive and large phase between ω_0 and ω_p which tends to improve the phase margin. We should therefore place w_p and w_0 in the crossover frequency range (when $|G(j\omega)| \approx 1$).



5. (a) The plot is shown below.

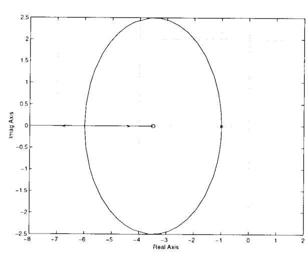


(b) i. Let the angle between $(-2+j2)$ and z be θ. Applying the angle criterion:

$$\theta - 2(116.565°) = \pm 180°$$

or $\theta = 53.13°$. Thus $z = -3.5$.

ii. The root locus is shown below.



iii. Since $(-2+j2)$ lies on the root locus, we use the gain criterion to find k:

$$k = -\frac{(s+1)^2}{(s+3.5)}\Big|_{s=-2+j2} = 2.$$

(c) The signal at the input of the compensator is given by $e(t) + V_0 \sin \omega t$. Thus the signal at the input of the plant is $k\left(\dot{e}(t) - ze(t)\right) + k\left(\omega V_0 \cos \omega t - zV_0 \sin \omega t\right)$. If the frequency ω is too large, then the noise term at the input of the plant may be too large and is likely to deteriorate the performance of the control system.