IMPERIAL COLLEGE LONDON

E4.27
C2.2
ISE4.41

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2008

MSc and EEE/ISE PART IV: MEng and ACGI

SYSTEM IDENTIFICATION

Corrected Copy

Friday, 23 May 10:00 am

Time allowed: 3:00 hours

There are FIVE questions on this paper.

Answer THREE questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s) :	R.B. Vinter
	Second Marker(s) :	S. Evangelou

Special information for invigilators: none

Information for candidates:

$C(\tau) = E[(u(t)-\mu)(u(t+\tau)-\mu)]$

$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{x^2}{2\sigma^2}}$ $\qquad S_{yy} = |G|^2 S_{uu}$ $\qquad Z_L = sL \qquad Z_c = \frac{1}{Cs}$

$\Phi^{\#}=(\Phi^*\Phi)^{-1}\Phi^* \qquad P = \Phi\Phi^{\#} \qquad S = \frac{1}{N-\rho}\|y-\Phi\widehat{\theta}\|^2$

$A^d = e^{Ah} \qquad B^d = \left(e^{Ah}-I\right)A^{-1}B \qquad G^d(z) \approx G(\frac{2}{h}\,\frac{z-1}{z+1}) \qquad G(s) \approx G^d(\frac{1+sh/2}{1-sh/2})$

$C^{uu}_k g_0 + C^{uu}_{k-1} g_1 + C^{uu}_{k-2} g_2 + \ldots = C^{uy}_k$

$\mathrm{Cov}(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$

$E(X\cdot Y) = E(X)\cdot E(Y) + \mathrm{Cov}(X,Y)$

$\mathrm{Cov}(TX) = T\mathrm{Cov}(X)T^*$

$\widehat{v}(z) = \sum_{k=0}^{\infty} v_k z^{-k}$

$[(\Delta v)_k = v_{k+1}] \qquad \Rightarrow \Delta v\,(z) = z[\widehat{v}(z) - v_0]$

$[u_k = k v_k] \qquad \Rightarrow \widehat{u}(z) = -z\frac{d}{dz}\widehat{v}(z)$

$[v_k = \sin k\nu] \qquad \Rightarrow \widehat{v}(z) = \frac{z\sin\nu}{(z-e^{i\nu})(z-e^{-i\nu})}$

$\left[v_k = \rho^k\right] \qquad \Rightarrow \widehat{v}(z) = \frac{z}{z-\rho}$

$\left[v_k = \frac{1}{\rho}k\rho^k\right] \qquad \Rightarrow \widehat{v}(z) = \frac{z}{(z-\rho)^2}$

$P_n = \frac{1}{\lambda}\left[P_{n-1} - \frac{P_{n-1}\varphi_n^*\varphi_n P_{n-1}}{\lambda+\varphi_n P_{n-1}\varphi_n^*}\right]$

$\varepsilon_n = y_n - \varphi_n\hat{\theta}_{n-1}$

$\hat{\theta}_n = \hat{\theta}_{n-1} + P_n\varphi_n^*\varepsilon_n$

$$\begin{aligned} y_k + a_1 y_{k-1} \cdots + a_n y_{k-n} &= b_0 u_k + b_1 u_{k-1} \cdots + b_n u_{k-n} \\ &\quad + e_k + c_1 e_{k-1} \cdots + c_n e_{k-n} \end{aligned}$$

$$C(z) = 1 + c_1 z^{-1} \ldots + c_n z^{-n}$$

$$\hat{u}^F = C^{-1}\hat{u}, \qquad \hat{y}^F = C^{-1}\hat{y}$$

$$\begin{aligned} \overline{y_k} &= (c_1-a_1)y^F_{k-1} + (c_2-a_2)y^F_{k-2} \cdots + (c_n-a_n)y^F_{k-n} \\ &\quad + b_0 u^F_k + b_1 u^F_{k-1} \cdots + b_n u^F_{k-n} \end{aligned}$$

1. A random signal (u_k) has the structure

$$u_k = A\sin(0.01k+\varphi)+w_k\,, \qquad k\in\mathbb{Z},$$

where $A>0$ and $\varphi\in(-\pi,\pi]$ are unknown and (w_k) is a stationary ergodic Gaussian random signal. The measurements u_k are known for $k=1,2,3,\dots 10^6$. Our aim is to estimate $A,\varphi,E(w_k)$ and $\mathrm{Var}(w_k)$. In the first five parts below you are asked to prove certain statements. If you do not succeed to prove some of these statements, you can still use them to answer the other parts.

(a) Assume that (a_k) and (b_k) are independent normalized Gaussian white noise signals. Define the complex random sequence c_k by $c_k=\frac{1}{\sqrt{2}}(a_k+ib_k)$. Show that c_k is normalized Gaussian (we regard the complex plane as being equivalent to $\mathbb{R}^2$). [3]

(b) Let (ψ_k) be an arbitrary sequence of real numbers ($k\in\mathbb{Z}$) and let c_k be the random sequence from part (a). Show that the sequence $(e^{i\psi_k}c_k)$ is a normalized Gaussian white noise signal. [3]

(c) Show that for every $\nu\in\mathbb{R}$ we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}e^{i\nu k}c_k=0,$$

with probability 1. Here, (c_k) is the sequence from part (a). [3]

(d) Let (a_k) be the sequence introduced in part (a). Show that we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\cos(\nu k)a_k=0, \quad \lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\sin(\nu k)a_k=0,$$

with probability 1. Hint: justify and then use the fact that the result from part (c) is true also for the complex conjugate sequence $(\overline{c_k})$. [3]

(e) Let (a_k) be the sequence introduced in part (a). Assume for simplicity that (w_k) can be obtained from (a_k) by filtering it through a FIR filter with impulse response (g_k). Show that the two formulas from part (d) remain valid with w_k in place of a_k. [4]

(f) The statement from part (e) is true for $\nu\neq 0$ and any stationary ergodic random signal (w_k). Using this fact, propose a method to estimate A and φ. Hint: think of how we use a sinusoidal input to estimate the frequency response function at one point. [4]

2. The proposed mathematical model of a static system with two inputs, u and v, and with one output w is

$$w = \sqrt{\lambda + \left(\frac{u}{\alpha}\right)^2 + \left(\frac{v}{\beta}\right)^2}. \tag{1}$$

The variables u, v and w can be measured and α, β, λ are unknown positive parameters. We have 200 measurements available from experiments, $u_1, u_2, \ldots u_{200}$ and similarly for v and w. Because of measurement and modeling errors, the measurements do not fit any model of the form (1) exactly.

(a) By defining new variables if necessary, rewrite the model of the system in the form $y_k = \varphi_k \theta + e_k$, where y_k and φ_k are known, θ is the vector of unknown parameters and e_k are the equation errors. [3]

(b) State the condition under which a unique minimizing $\hat{\theta}$ exists for the cost $J(\theta) = e_1^2 + e_2^2 \ldots + e_{200}^2$. Assuming that this condition is satisfied, write the formula for the vector of estimated parameters $\hat{\theta}$ that minimizes $J(\theta)$. [3]

(c) Suppose our data are such that $u_k^2 - v_k^2 = 18$ for all $k \in \mathbb{N}$. Explain why in this case we cannot estimate α and β, but we can still estimate $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$. Explain how to estimate $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$. [4]

In the sequel, we assume that there is a unique minimizing $\hat{\theta}$ for the cost function $J(\theta)$ from part (*b*).

(d) Assume that e_k are independent and identically distributed random variables with $E(e_k) = 0$. Give a formula for an unbiased estimate of $Var(e_k)$ in terms of the values of φ_k and y_k from part (a). [3]

(e) Still assuming independent and identically distributed equation errors, give a formula for an unbiased estimate of $Cov(\hat{\theta})$, where $\hat{\theta}$ is the estimate from part (b). Note that $Var(e_k)$ is not known, but it can be estimated, as was required in part (d). [3]

(f) Suppose that each of the sequences u_k, v_k and e_k consists of independent and identically distributed random variables, and the three sequences are also independent of each other. Let $\hat{\theta}$ be the estimate from part (b). If, instead of 200 measurements, we have 800 measurements, approximately how many times do you expect $Cov(\hat{\theta})$ to decrease? Give, briefly, a reasoning for your answer. [4]

3. In this question, Σ is an unknown stable discrete-time LTI system with input signal u and output signal p. The measured output y is corrupted by the noise signal w, so that $y_k = p_k + w_k$. It is known that w is a stationary ergodic Gaussian signal with $E(w_k) = 0$ and the power spectral density of w, denoted by S^{ww}, satisfies $S^{ww}(e^{i\nu}) \geq 0.1$ for all $\nu \in (-\pi, \pi]$ (but otherwise S^{ww} is not known).

We have to identify Σ, based on the measurements of u_k and y_k. We would like to model Σ by a discrete-time transfer function of order 4:

$$p_k + a_1 p_{k-1} \dots + a_4 p_{k-4} = b_0 u_k + b_1 u_{k-1} \dots + b_4 u_{k-4} + v_k\,, \qquad (1)$$

where v_k is the equation error due to model mismatch. We assume that v is a stationary ergodic Gaussian random signal with $E(v_k) = 0$, and v is independent of w (i.e., v_k and w_j are independent for all integers k, j).

(a) Let α and β be two independent stationary random signals and $\gamma_k = \alpha_k + \beta_k$. How are the power spectral densities of α, β, γ related to each other? Give a short proof of your formula. [3]

(b) Consider the system with input u and output y. Describe this system by an ARMAX model with a white noise input denoted e. For this, introduce a new signal δ that accounts for the combined effect of w and v. Then represent δ as filtered white noise, where both the filter and its inverse are stable. Finally, approximate the filter by a FIR filter. Briefly, why is it possible to represent δ as described above, and why is it possible to approximate the filter by a FIR filter?

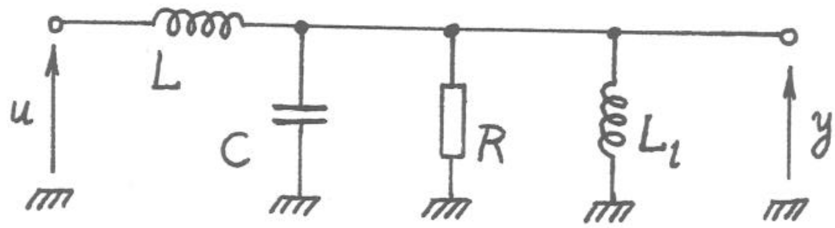
Hint: use the result from part (a). [5]

(c) Why is the ARMAX model of part (b) equivalent to an ARX model of very high order? Why do we need here that the inverse of the filter from (b) is stable? [4]

(d) Assuming that the measurements u_k and y_k are available for $k = 1, 2, \dots 20,000$, describe a least squares based method for estimating the unknown coefficients in the ARX model from part (c). [4]

(e) Use pseudolinear regression to explain how the unknown coefficients of the ARMAX model of part (b) can be estimated using the estimated coefficients of the ARX model of part (d). [4]

4. We want to model the output circuit of an inverter by the simplified circuit shown below, where the filter inductor $L > 0$ and the filter capacitor $C > 0$ are known, while the load resistor $R > 0$ and the load inductance $L_l > 0$ are unknown and should be estimated. We can choose the waveform of u, the output voltage of the inverter, and we can measure the load voltage y. We cannot expect a perfect match between the true circuit and this simplified circuit, but we would like to get a close match in a certain frequency range.



(a) Compute the transfer function $\mathbf{G}$ of the simplified circuit (from u to y), in terms of L, C, R and L_l. Is $\mathbf{G}$ stable? [4]

(b) Is the circuit shown in the figure a stable system? Hint: if we apply a constant input to a stable system, then the state variables converge to finite limit values as $t \to \infty$. Is this the case for our circuit? [3]

(c) Suppose that by measurements using sinusoidal u, we have obtained estimates for $\mathbf{G}$ at 25 angular frequencies $\omega_1, \ldots \omega_{25}$, in the frequency range of interest. By defining new variables, rewrite the model of the system in the form $y_k = \varphi_k \theta + e_k$, where y_k and φ_k are known (possibly complex), θ is the vector of unknown parameters and e_k are the equation errors (possibly complex). Hint: think carefully about what is known and what has to be estimated. [3]

(d) For the model constructed in part (c), explain how to find the *real* vector $\hat{\theta}$ which minimizes $J(\theta) = \sum_{k=1}^{25} |e_k|^2$. Explain how we can estimate R and L_l using $\hat{\theta}$. [4]

(e) Construct a minimal realization of $\mathbf{G}$ (from part (a)), of the form $\dot{x} = Ax + Bu$, $y = Cx + Du$, where A, B, C and D are matrices. [3]

(f) We connect a hold device (D/A converter) at the input of our system and we connect a sampler (A/D converter) at its output, both converters working with the sampling period T. How can we compute the transfer function $\mathbf{G}^d$ of the resulting discrete-time LTI system? Is $\mathbf{G}^d$ stable? There is no need to perform any computations to answer this part. [3]

5. We want to estimate the impulse response g of a stable discrete-time LTI plant but we do not have the possibility to apply input signals of our choice, we can only observe the existing signals. The input signal is denoted by $u = (u_k)$ and it can be measured. The output signal v is corrupted by a noise signal w, such that the measured output signal is given by $y_k = v_k + w_k$. Both u and w are assumed to be stationary and ergodic (but not necessarily independent of each other). The measurements of u_k and y_k are available for $k = 1, 2, 3, \ldots 6000$.

(a) Describe a method for estimating the auto-correlation function C_τ^{uu} and the cross-correlation function C_τ^{yu} for $0 \leq \tau \leq 30$. Explain very briefly how this problem is related to the concept of ergodicity. [3]

(b) Express C^{yu} in terms of C^{uu}, C^{wu} and g. [4]

(c) Assume now that u_k and w_j are independent of each other, for all $k, j \in \mathbb{Z}$. Describe a method for estimating the terms $g_0, g_1, g_2, \cdots g_{30}$ from the results of part (a). Show briefly how this method can be derived from your answer to part (b). [4]

(d) What is the meaning of a random signal being "persistent of order N"? What is the significance of this concept in the context of part (c) above? Explain the following: if u is persistent of order 30, then it is also persistent of order 20. [3]

(e) If $e = (e_k)$ is normalized white noise and $u_k = e_k - 0.3e_{k-1}$ (for all $k \in \mathbb{Z}$), show that u is persistent of any order. [3]

(f) After having estimated the first N terms of the impulse response, $g_0, g_1, \cdots g_{N-1}$, how can we build a FIR filter whose transfer function is a good approximation to the true transfer function? Write the corresponding difference equation. [3]

[END]