Imperial College London

MSci EXAMINATION May 2012

This paper is also taken for the relevant Examination for the Associateship

QUANTUM OPTICS

For 4th-Year Physics Students

Tuesday, 22nd May 2012: 10:00 to 12:00

Answer a total of THREE out of Six questions.
At least ONE question must be from SECTION A and at least ONE question must be from SECTION B.
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the THREE answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

SECTION A

1. The frequency dependent electromagnetic scattering cross-section from a classical oscillator is of the following form:

$$\sigma(\omega) = \frac{8\pi r_0^2}{3} \frac{\omega^4}{\left(\omega_0^2 - \omega^2\right)^2 + \Gamma_{cl}^2 \omega^6/\omega_0^4}$$

where r_0 is the classical radius of the electron, ω_0 the resonance frequency of the oscillator and Γ_{cl} is the classical damping.

(i) (a) Derive the form of the scattering valid for the high frequency limit $\omega >> \omega_0$ (You may assume that $\Gamma_{cl}\omega/\omega_0^2 << 1$),

[1 mark]

(b) Derive the form of the scattering valid for the low frequency limit $\omega << \omega_0$,

[1 mark]

(c) Show that the form of the scattering cross-section close to the resonance wavelength λ_0 (assuming the resonance frequency is much larger than the classical damping Γ_c) is:

$$\sigma(\omega) = \frac{3\lambda_0^2}{2\pi} \frac{\Gamma_{cl}^2/4}{(\omega_0 - \omega)^2 + \Gamma_{cl}^2/4}$$

[3 marks]

(d) Sketch the form of the scattering cross-section over a range of frequencies from far below to far above resonance. Indicate on your sketch the vaue of the scattering cross-section at resonance and at high frequencies.

[3 marks]

(ii) Estimate the scattering cross-section from a hydrogen atom

(a) at the 1s - 2p resonance,

[5 marks]

(b) for 1keV photon energy x-rays.

[3 marks]

(iii) How is this picture of light scattering modified when we go to a fully quantum treatment of the light-atom interaction?

[4 marks]

$\Gamma_{cl} = 2r_0\omega_0^2/3c = 4\pi\omega_0 r_0/3\lambda_0$
$r_0 = 2.818x10^{-15}m$

[Total 20 marks]

2. For a two level atom, comprising a ground state $|a>$ at energy $E_a = 0$ and an excited state $|b>$ at energy E_b, that is initially (t=0) in the ground state, but evolves under a near resonant field for a time t, the probability amplitudes have the form:

$$c_a(t) = \cos\left(\frac{\Omega t}{2}\right) - i\frac{\delta}{\Omega}\sin\left(\frac{\Omega t}{2}\right)$$

$$c_b(t) = i\frac{\Omega_0}{\Omega}e^{-i\phi}sin\left(\frac{\Omega t}{2}\right)e^{iE_b t/\hbar}$$

Where δ is the angular frequency detuning from resonance, Ω_0 the resonant Rabi frequency and $\Omega = \sqrt{\Omega_0^2 + \delta^2}$.

(i) Sketch the probability of occupation of the excited state $P_b(t)$ as a function of time in the resonant case and for finite detuning.

[4 marks]

(ii) (a) Evaluate the magnitude of the Rabi frequency in a particular two-level atom where the relevant component of the dipole moment is $d = e < r >= 10^{-29}Cm$ and which is driven by a linearly polarized resonant field of intensity $10^9 Wm^{-2}$.

[2 marks]

(b) For the case just introduced what is the probability of occupation of state $|b>$ (assuming the population was all initially in the groundstate) at t= 5 ps?

[2 marks]

(c) Sketch how the occupation probability of state $|b>$ varies over the period t=0 to t=50 ps in the case of interaction with this resonant field.

[3 marks]

(iii) The expressions for the real and imaginary components of the susceptibility of a two-level system in the weak field limit is given by:

$$\chi' = \frac{N}{V}\frac{d^2}{\epsilon_0\hbar}\frac{\delta}{\Gamma^2 + \delta^2}$$

$$\chi'' = \frac{N}{V}\frac{d^2}{\epsilon_0\hbar}\frac{\Gamma}{\Gamma^2 + \delta^2}$$

which lead to the dispersion k' and absorption k'' of the medium:

$$k' = \left(1 + \frac{\chi'}{2}\right)\frac{\omega}{c}$$

$$k'' = \frac{\chi''}{2}\frac{\omega}{c}$$

where δ is the detuning from resonance, N/V the number density of atoms and Γ the damping of the transition.

[This question continues on the next page . . .]

(a) From these expressions evaluate the maximum value of the absorption and dispersion in a gas of these two-level atoms where the number density is $N/V = 10^{20} m^{-3}$, the resonant frequency $\omega_0 = 3.77x10^{15}$ rad s^{-1} and the damping $\Gamma = 5x10^{8}$ rad s^{-1}.

[3 marks]

(b) At what angular frequency will the maximum values of absorption and dispersion occur?

[2 marks]

(c) How are the absorption and dispersion modified in a 3-level system in which electromagnetically induced transparency (EIT) has been created?

[2 marks]

(d) Identify two potential applications of the modified optical properties that arise from EIT.

[2 marks]

Electric field $E_0 = 27.4 I^{1/2}$ in SI units.

[Total 20 marks]

3. (i) Sketch and describe the motion of the Bloch vector on the Bloch sphere in the following cases (assume that any decay is much less rapid than the imposed coherent evolution of the system)

(a) A resonant coupling is applied of pulse area π.

[2 marks]

(b) A resonant coupling is applied of pulse area $\pi/2$

[1 mark]

(c) A strong non-resonant coupling is applied.

[2 marks]

(ii) (a) Draw a sequence of annotated sketches using the Bloch vector to illustrate the principle of the Ramsey interferometer method.

[5 marks]

(b) Draw a scaled sketch of the resulting fluoresence signal from the final excited state as a function of detuning in the case where the time between Ramsey pulses is 1 ms.

[3 marks]

(iii) (a) In a Ramsey experiment in which thermal sodium atoms (T 400K) are investigated estimate the limits to resolution if the laser beam diameter used is 2mm.

[5 marks]

(b) How might this result be improved upon?

[2 marks]

Mass of a sodium atom $m_{Na} = 23m_p$

[Total 20 marks]

SECTION B

4. The creation and annihilation operators for photons are defined as

$$\hat{a} = \frac{1}{\sqrt{2\hbar}}\left[\hat{Q} + i\hat{P}\right] \quad \hat{a}^{\dagger} = \frac{1}{\sqrt{2\hbar}}\left[\hat{Q} - i\hat{P}\right]$$

where $\hat{Q}$ and $\hat{P}$ obey the usual coordinate-momentum commutation relationship. The quadrature operators of the electromagnetic field are defined as

$$\hat{E}_Q = \mathcal{E}^{(1)}[\hat{a} + \hat{a}^{\dagger}] \quad \hat{E}_P = -i\mathcal{E}^{(1)}[\hat{a} - \hat{a}^{\dagger}]$$

where $\mathcal{E}^{(1)}$ is the field-strength per one photon, $\mathcal{E}^{(1)} = \sqrt{\hbar\omega/(2\epsilon_0 V)}$, with V the quantization volume, ω the radiation frequency, and ϵ_0 the vacuum permitivity.

(i) Show that

$$[\hat{a}, \hat{a}^{\dagger}] = 1$$

[4 marks]

(ii) Let $|n\rangle$ be the Fock state with n photons in it.

(a) Show that $\langle n|\hat{E}_Q|n\rangle = 0$

[2 marks]

(b) Give an example of a quantum state $|\psi\rangle$ which (1) is a superposition of two Fock states with equal amplitudes, and (2) has $\langle\psi|\hat{E}_Q|\psi\rangle = 0$ at all times.

[2 marks]

(iii) Let $|\alpha\rangle$ be coherent state defined as $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$. Its evolution as a function of time is denoted as $|\alpha(t)\rangle$.

(a) Show that

$$\langle\alpha(t)|\hat{E}_Q|\alpha(t)\rangle = \mathcal{E}^{(1)}(\alpha e^{-i\omega t} + \alpha^* e^{i\omega t})$$

where $\hbar\omega$ is the photon energy. [6 marks]

(b) Show that for arbitrary superposition of Fock states $|\psi\rangle$, such that $\langle\psi|\hat{E}_Q|\psi\rangle \neq 0$, the following is true:

$$\langle\psi(t)|\hat{E}_Q|\psi(t)\rangle = E_0 \cos(\omega t + \phi)$$

where $\hbar\omega$ is the photon energy. Give expressions for E_0 and ϕ. [6 marks]

[Total 20 marks]

5. Let the laboratory axes x and y be associated with two possible orthogonal polarizations of radiation. The polarization sensitive beam-splitter outputs x-polarized radiation in port (+1) and y-polarized radiation in port (-1), when its main axis is aligned with x direction. Its action is described by the operator

$$\hat{A}(0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with eigenstates, representing x-polarized and y-polarized radiation,

$$|x\rangle \equiv |+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \textit{and} \qquad |y\rangle \equiv |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

and eigenvalues +1 and −1 correspondingly.

(i) The polarizer is rotated by an angle θ_a relative to the x axis. The new eigenstate $|+_a\rangle$ corresponding to the (+1) output of the rotated polarizer is

$$|+_a\rangle = \cos\theta_a|x\rangle + \sin\theta_a|y\rangle$$

Write down the second eigenstate $|-_a\rangle$.

[2 marks]

(ii) Consider polarization-entangled state of two photons, 1 and 2:

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left[|x_1, x_2\rangle + |y_1, y_2\rangle\right]$$

The first photon is sent to the polarizer A, rotated by angle θ_a relative to the laboratory x axis. The second photon is sent to the polarizer B, rotated by angle θ_b relative to the laboratory x axis.

(a) Show that the joint probability $P_{+,+}(a, b)$ of detecting the two photons in the output ports +1 and +1 of the two polarizers is

$$P_{+,+}(a, b) = \frac{1}{2}\cos^2(\theta_a - \theta_b)$$

[4 marks]

(b) Show that the joint probability $P_{+,-}(a, b)$ of detecting the photons in the output ports +1 and −1 of the two polarizers is

$$P_{+,-}(a, b) = \frac{1}{2}\sin^2(\theta_a - \theta_b)$$

[4 marks]

(c) Taking into account that $P_{+,-}(a, b) = P_{-,+}(a, b) = \frac{1}{2}\sin^2(\theta_a - \theta_b)$ and $P_{+,+}(a, b) = P_{-,-}(a, b) = \frac{1}{2}\cos^2(\theta_a - \theta_b)$, consider the correlation between the measurement outcome $A(a)$ obtained by Alice with polarizer A and the measurement outcome $B(b)$ obtained by Bob with polarizer B. The correlation is defined as

$$E(a, b) = \overline{A(a)B(b)}$$

[This question continues on the next page . . .]

where the overline means averaging over different outcomes. Show that

$$E(a, b) = \cos 2(\theta_a - \theta_b)$$

[4 marks]

(d) Alice and Bob perform 4 sets of measurements with polarizers A and B, changing the angles θ_a and θ_b. In the first set of measurements they set $\theta_a = 0$, $\theta_b = \theta$, and they measure the correlation $E(0, \theta) = E_1$. In the second set $\theta_a = \theta$, $\theta_b = \theta$, and the measured correlation is $E(\theta, \theta) = E_2$. In the third set $\theta_a = \theta$, $\theta_b = 2\theta$, and the measured correlation is $E(\theta, 2\theta) = E_3$. In the fourth and final set, $\theta_a = 0$, $\theta_b = 2\theta$, and the measured correlation is $E(0, 2\theta) = E_4$.

At the end of their experiments Alice and Bob compute

$$S = E_1 + E_2 + E_3 - E_4.$$

What angle θ should they choose for their experiments to maximize their chance of observing the violation of Bell's inequalities?

[6 marks]

[Total 20 marks]

6. A two-level atom interacts with the electromagnetic field in a resonant, or near-resonant, high-finesse cavity which supports a single radiation mode. The mode has frequency ω and is linearly polarized. The energy splitting between the states $|a\rangle$ and $|b\rangle$ of the two-level atom is $\hbar\omega_0$, $|\omega - \omega_0| \ll \omega$. The energy of the lower atomic state $|a\rangle$ is E_a.

The Hamiltonian of the system is

$$\hat{H} = H_{Atom} + \hbar\omega\left(\hat{a}^{\dagger}\hat{a} + \frac{1}{2}\right) + \hat{H}_I$$
$$\hat{H}_I = -\frac{q}{m}(\hat{\mathbf{p}} \cdot \mathbf{e})\frac{\mathcal{E}^{(1)}}{\omega}[\hat{a} + \hat{a}^{\dagger}]$$

where $\mathcal{E}^{(1)}$ is the field strength per one photon for this mode, $\mathcal{E}^{(1)} = \sqrt{\hbar\omega/(2\epsilon_0 V)}$, with V the quantization volume, ω the radiation frequency, $\mathbf{e}$ the polarization vector, $\hat{\mathbf{p}}$ the momentum operator, and ϵ_0 the vacuum permitivity.

(i) Show that the interaction operator H_I can be re-written as

$$\hat{H}_I = \frac{\hbar\Omega_1}{2}\left[|a\rangle\langle b| + |b\rangle\langle a|\right]\left[\hat{a} + \hat{a}^{\dagger}\right]$$

within the limited 2-level sub-space of our atom. Find an explicit expression for $\hbar\Omega_1$.

[3 marks]

(ii) Let us write the wave-function of the total system (radiation + atom) as

$$|\Psi(t)\rangle = e^{-i\frac{E_a}{\hbar}t - i\frac{\omega}{2}t}\left[\sum_{n'} a_{n'}(t)|a, n'\rangle + \sum_{n'} b_{n'}(t)|b, n' - 1\rangle\right]$$

where n' correspond to different number of photons in the cavity. Using the time-dependent Schroedinger equation, one can derive the following equations of motion for the amplitudes $a_n(t)$ and $b_n(t)$

$$i\frac{d}{dt}b_n = \left[(n-1)\omega + \omega_0\right] b_n + \frac{\Omega_1}{2}\left[\sqrt{n}a_n + \sqrt{n-1}a_{n-2}\right]$$
$$i\frac{d}{dt}a_n = n\omega a_n + \frac{\Omega_1}{2}\left[\sqrt{n}b_n + \sqrt{n+1}b_{n+2}\right]$$

In the approximation often referred to as 'Rotating Wave Approximation' or 'Resonance Approximation', the terms $\propto \sqrt{n-1}a_{n-2}$ in the equation for $(d/dt)b_n$ and the term $\propto \sqrt{n+1}b_{n+2}$ in the equation for $(d/dt)a_n$ are dropped. Identify the physical process expressed by the term proportional to $\sqrt{n-1}a_{n-2}$ in the equation for db_n/dt. Sketch a diagram describing this process, identify the energy detuning associated with it, explain why this process can be neglected and write down the condition for which the approximation is justified.

[6 marks]

[This question continues on the next page . . .]

(iii) At $t = 0$ an atom with equally populated states $|a\rangle$ and $|b\rangle$ is introduced into the cavity, which has exactly one photon.

(a) Using the 'Rotating Wave Approximation' and assuming exact resonance between the photon energy and the transition energy in the atom, write down equations of motion for the time-dependent probability amplitudes a_1, b_1, a_2, b_2.

[2 marks]

(b) Solve these equations of motion and find the time-dependent amplitudes $a_1(t)$, $b_1(t)$, $a_2(t)$, $b_2(t)$, subject to the initial conditions at $t = 0$ stated above.

[5 marks]

(c) Using these equations of motion, calculate the probability $P_1(t)$ of finding one photon in the cavity at later times. What is the value of this probability when $\Omega_1 t = 5\pi$? Is there a moment of time when $P_1(t) = 0$?

[4 marks]

[Total 20 marks]