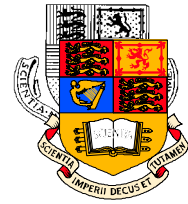


IMPERIAL COLLEGE OF SCIENCE TECHNOLOGY AND MEDICINE
UNIVERSITY OF LONDON

[E303/ISE3.3]



DEPARTMENT of ELECTRICAL and ELECTRONIC ENGINEERING
EXAMINATIONS 2002

EEE/ISE PART III/IV: M.Eng., B.Eng. and ACGI

SOLUTIONS 2002

COMMUNICATION SYSTEMS

There are FOUR questions (Q1 to Q4)

Answer question ONE (in separate booklet) and TWO other questions.

Question 1 has 20 multiple choice questions numbered 1 to 20, all carrying equal marks.
There is only one correct answer per question.

Distribution of marks

Question-1: 40 marks

Question-2: 30 marks

Question-3: 30 marks

Question-4: 30 marks

The following are provided:

- *A table of Fourier Transforms*
- *A "Gaussian Tail Function" graph*

Examiner: Dr A. Manikas

ANSWER to Q1

- | | | | | | |
|-----|---|---|---|---|---|
| 1) | A | B | C | D | E |
| 2) | A | B | C | D | E |
| 3) | A | B | C | D | E |
| 4) | A | B | C | D | E |
| 5) | A | B | C | D | E |
| 6) | A | B | C | D | E |
| 7) | A | B | C | D | E |
| 8) | A | B | C | D | E |
| 9) | A | B | C | D | E |
| 10) | A | B | C | D | E |
| 11) | A | B | C | D | E |
| 12) | A | B | C | D | E |
| 13) | A | B | C | D | E |
| 14) | A | B | C | D | E |
| 15) | A | B | C | D | E |
| 16) | A | B | C | D | E |
| 17) | A | B | C | D | E |
| 18) | A | B | C | D | E |
| 19) | A | B | C | D | E |
| 20) | A | B | C | D | E |

ANSWER to Q2

a)

$$P_e = \mathbb{P}\left\{\sqrt{(1-p)EUE}\right\}$$

$$E_b = \frac{1}{2} \int_0^{T_{cs}} (s_0^2(t) + s_1^2(t)) dt = \frac{1}{2} \int_0^{8h} ((1m)^2 + (1m)^2) dt = 8 \times 10^{-12}$$

$$p = \frac{1}{E_b} \int_0^{8h} s_0(t)s_1(t) dt =$$

$$= \frac{1}{8 \times 10^{-12}} \left(\int_0^{2h} 10^{-6} dt + \int_{2h}^{4h} (-10^{-6}) dt + \int_{4h}^{6h} (-10^{-6}) dt + \int_{6h}^{8h} 10^{-6} dt \right)$$

$$= 0$$

$$\text{i.e. } \left\{ \begin{array}{l} E_b = 8 \times 10^{-12} \\ p = 0 \\ N_0 = 2 \times 10^{-12} \end{array} \right\} \Rightarrow EUE = \frac{8 \times 10^{-12}}{2 \times 10^{-12}} = 4$$

$$\Rightarrow P_e = \mathbb{P}\{\sqrt{4}\} = \mathbb{P}\{2\} = 2.2 \times 10^{-2}$$

$$\text{Hamming}(7,4) \Rightarrow d_{\min} = 3 \Rightarrow \left\{ \begin{array}{l} 1 \text{ bit in error can be} \\ \text{corrected} \dots \end{array} \right.$$

$$\downarrow \left(\frac{d_{\min}-1}{2} = 1 \right) \downarrow$$

$$\dots \text{ by moving to the nearest codeword}$$

$$\therefore \Pr(\text{correctly decoded}) = \Pr\left(\begin{array}{l} 0 \text{ bits in error} \\ \text{in a 7 bit sequ.} \end{array}\right) + \Pr\left(\begin{array}{l} 1 \text{ bit in error} \\ \text{in a 7 bit sequ.} \end{array}\right)$$

$$= \binom{7}{0} P_e^0 (1-P_e)^{7-0} + \binom{7}{1} P_e^1 (1-P_e)^{7-1}$$

$$= 0.868 + 0.124$$

$$= 0.992$$

b)

$$P_e = \underbrace{\Pr(r_2|m_1) \cdot \Pr(m_1)}_{\Pr(r_2, m_1)} + \underbrace{\Pr(r_1|m_2) \cdot \Pr(m_2)}_{\Pr(r_1, m_2)}$$

$$= \underbrace{0.1 \times 0.25}_{0.025} + \underbrace{0.2 \times 0.75}_{0.15}$$

$$= 0.175$$

$$\underline{q} = \begin{bmatrix} \Pr(r_1) \\ \Pr(r_2) \end{bmatrix} = \underline{\mathbb{F}} \cdot \underline{p} = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix} \begin{bmatrix} 0.25 \\ 0.75 \end{bmatrix} = \begin{bmatrix} 0.375 \\ 0.625 \end{bmatrix}$$

$$\underline{\mathbb{J}} = \underline{\mathbb{F}}, \text{diag}(\underline{p}) = \begin{bmatrix} 0.9 \times 0.25 & 0.2 \times 0.75 \\ 0.1 \times 0.25 & 0.8 \times 0.75 \end{bmatrix} = \begin{bmatrix} 0.225 & 0.15 \\ 0.025 & 0.6 \end{bmatrix}$$

$$H_{M41} = H_R - H_{R|M}$$

$$\text{where } H_R = \underline{q}^T \cdot \log_2(\underline{q}) = 0.9544$$

$$H_{R|M} = -\| \underline{\mathbb{J}} \odot \log_2 \underline{\mathbb{F}} \|_{1*} = 0.658695$$

$$\therefore H_{M41} = 0.9544 - 0.658695 = 0.295705$$

ANSWER to Q3

a)

$$SNR_q \geq 42 \text{ dB with } SNR_q = 4.77 + 6\gamma - a \text{ dB}$$

PCM system with uniform quantizer:

$a = \text{CREST FACTOR in dBs} = 13$

$$\therefore 4.77 + 6\gamma - \underset{13}{a} \geq 42 \Rightarrow 6\gamma \geq 42 + 13 - 4.77$$

$$\Rightarrow \gamma \geq 8.37 \Rightarrow \boxed{\gamma = 9 \text{ bits}}$$

μ -law ($\mu = 255$)

$$a = 20 \log_{10} (\underset{255}{\mu} (1 + \mu)) = 14.878$$

$$\Rightarrow 6\gamma = 42 + 14.878 - 4.77 \Rightarrow \gamma \geq 8.68 \text{ bits} \Rightarrow \boxed{\gamma = 9 \text{ bits}}$$

A-law ($A = 87.6$)

$$a = 20 \log_{10} (1 + \ln A) = 14.764$$

$$\Rightarrow 6\gamma = 42 + 14.764 - 4.77 \Rightarrow \gamma \geq 8.665 \text{ bits} \Rightarrow \boxed{\gamma = 9 \text{ bits}}$$

diff. quant.

$$-10.23 \text{ dB} < a < 7.77 \text{ dB}$$

$$\therefore 4.77 + 6\gamma + 10.23 \geq 42 \Rightarrow \gamma \geq 4.5 \Rightarrow \gamma = 5 \text{ bits}$$

$$\text{or } 4.77 + 6\gamma - 7.77 \geq 42 \Rightarrow \gamma \geq 7.5 \Rightarrow \gamma = 8 \text{ bits}$$

$$\therefore \boxed{5 \leq \gamma \leq 8 \text{ bits}}$$

ie. μ -law, A-law $\Rightarrow Q = 2^8 = 256$ levels

diff. quant $\Rightarrow 32 \leq Q \leq 256$ levels.

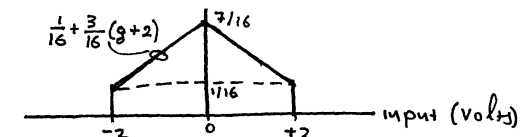
symbol rate: $r_s = \gamma \cdot F_s = 9 \times 2 \times 15 \text{ K} = 270 \text{ K}$ for $\begin{cases} \mu\text{-law} \\ \text{A-law} \\ \mu\text{-law} \end{cases}$

and $\frac{5 \times 30 \text{ K}}{150 \text{ K}} \leq r_s \leq \frac{8 \times 30 \text{ K}}{240 \text{ K}}$ for diff. quant.

$$\therefore B_{\text{PCM}} = \frac{r_s}{2} \Rightarrow \left\{ \begin{array}{l} B_{\text{PCM}}^{\mu\text{-law/A-law}} = 135 \text{ KHz.} \\ 75 \text{ KHz} \leq B_{\text{diff}} \leq 120 \text{ KHz} \end{array} \right.$$

b) i)

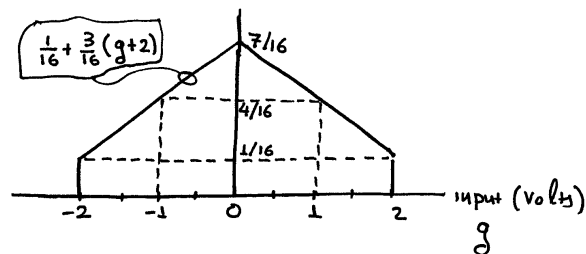
input pdf:



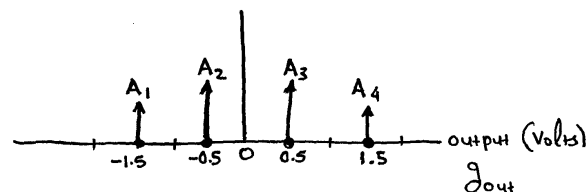
$$\begin{aligned} \therefore \text{Power of } g(t) &= P_g = \int_{-\infty}^{+\infty} g^2 \cdot \text{pdf}_g(g) dg = \\ &= 2 \int_{-2}^0 g^2 \cdot \left[\frac{1}{16} + \frac{3}{16}(g+2) \right] dg \\ &= 2 \int_{-2}^0 \left(\frac{g^2}{16} + \frac{3}{16}g^3 + \frac{6}{16}g^2 \right) dg \\ &= 2 \int_{-2}^0 \left(\frac{7}{16}g^2 + \frac{3}{16}g^3 \right) dg \\ &= \frac{2 \times 7}{16} \left[\frac{g^3}{3} \right]_{-2}^0 + \frac{2 \times 3}{16} \left[\frac{g^4}{4} \right]_{-2}^0 \\ &= \frac{5}{6} = 0.8333 \end{aligned}$$

ii)

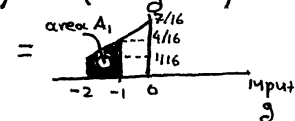
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output pdf:



where $A_1 = \Pr(g_{out} = 1.5V) = \Pr(-2 < g < -1)$

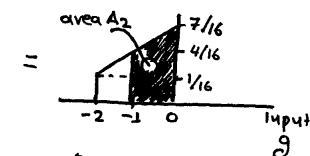


$$= \int_{-2}^{-1} \left(\frac{1}{16} + \frac{3}{16}(g+2) \right) dg =$$

$$= \left[\frac{1}{16}g + \frac{3}{16} \frac{(g+2)^2}{2} \right]_{-2}^{-1}$$

$$= -\frac{1}{16} + \frac{3}{32} + \frac{2}{16} - 0 = \frac{5}{32} = 0.1563$$

$$A_2 = \Pr(g_{out} = -0.5V) = \Pr(-1 < g < 0)$$



$$= \int_{-1}^0 \left(\frac{1}{16} + \frac{3}{16}(g+2) \right) dg$$

$$= \left[\frac{1}{16}g + \frac{3}{16} \frac{(g+2)^2}{2} \right]_{-1}^0$$

$$= \frac{6}{16} + \frac{1}{16} - \frac{3}{32} = \frac{11}{32} = 0.3438$$

also

$$A_3 = A_2 = \frac{11}{32}$$

$$A_4 = A_1 = \frac{5}{32}$$

iii)

$$SNR_{out} = \frac{P_{g_{out}}}{P_{n_g}} = \frac{2 \times ((-1.5)^2 \times 0.1563 + (-0.5)^2 \times 0.3438)}{\Delta^2/12}$$

$$= \frac{0.8752}{1/12} = 10.5030$$

iv)

$$r_s = 2 \times 2 \times 4K = 16 \frac{\text{K symbols}}{\text{sec}} = 16K \frac{\text{levels}}{\text{sec}}$$

v)

$$w_1 = -1.5V; w_2 = -0.5V; w_3 = 0.5V; w_4 = 1.5V$$

$$(M \times M, g) = \begin{bmatrix} (w_1 w_1, \frac{25}{1024}) & (w_2 w_1, \frac{55}{1024}) & (w_3 w_1, \frac{55}{1024}) & (w_4 w_1, \frac{25}{1024}) \\ (w_1 w_2, \frac{55}{1024}) & (w_2 w_2, \frac{121}{1024}) & (w_3 w_2, \frac{121}{1024}) & (w_4 w_2, \frac{55}{1024}) \\ (w_1 w_3, \frac{55}{1024}) & (w_2 w_3, \frac{121}{1024}) & (w_3 w_3, \frac{121}{1024}) & (w_4 w_3, \frac{55}{1024}) \\ (w_1 w_4, \frac{25}{1024}) & (w_2 w_4, \frac{55}{1024}) & (w_3 w_4, \frac{55}{1024}) & (w_4 w_4, \frac{25}{1024}) \end{bmatrix}$$

ANSWER to Q4

a)

$$\begin{aligned} F_g &= 4 \times 10^3 \text{ Hz} \\ F_s &= 2 \times F_g = 8 \times 10^3 \text{ Hz} \\ Q &= 2 \end{aligned}$$

$$\begin{aligned} \Pr(-2V) &= 3/4 \\ \Pr(+2V) &= 1/4 \end{aligned}$$

$$N_0 = 2 \times 10^{-3}$$

symbols	probabilities	Huffman	l_i (bits)
$x_1 x_1 x_1$	27/64	1	1
$x_1 x_1 x_2$	9/64	001	3
$x_1 x_2 x_1$	9/64	010	3
$x_2 x_1 x_1$	9/64	011	3
$x_1 x_2 x_2$	3/64	00000	5
$x_2 x_1 x_2$	3/64	00001	5
$x_2 x_2 x_1$	3/64	00010	5
$x_2 x_2 x_2$	1/64	00011	5

$$\begin{aligned} \bar{l} &= 1 \times 27/64 + 3 \times 9/64 + 3 \times 9/64 + 3 \times 9/64 + 5 \times 3/64 + 5 \times 3/64 \\ &\quad + 5 \times 3/64 + 5 \times 1/64 = 2.46875 \text{ bits/} \underbrace{\text{triple-level}}_{\text{(or 3-samples)}} \end{aligned}$$

$$\text{Alphabet: } \underline{X} = \left\{ \begin{matrix} x_1 = 1 \\ x_2 = 0 \end{matrix} \right\} \text{ (since } \Pr(x_1) > \Pr(x_2) \text{)}$$

$$\text{with probabilities: } \underline{p} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} \Pr(x_1) \\ \Pr(x_2) \end{bmatrix} = \begin{bmatrix} 0.6344 \\ 0.3656 \end{bmatrix}$$

$$\text{Note: } \Pr(x_2) = \frac{2}{3} \times \frac{9}{64} + \frac{2}{3} \times \frac{9}{64} + \frac{1}{3} \times \frac{9}{64} + \frac{5}{5} \times \frac{3}{64} + \frac{4}{5} \times \frac{3}{64} + \frac{4}{5} \times \frac{3}{64} + \frac{3}{5} \times \frac{1}{64} = 0.3656$$

$$\begin{aligned} p_e &= \Pr(y_2, x_1) + \Pr(y_1, x_2) \\ &= \Pr(y_2|x_1)\Pr(x_1) + \Pr(y_1|x_2)\Pr(x_2) \\ &= 0.05 \times 0.6344 + 0.2 \times 0.3656 \\ &= 0.1048 \end{aligned}$$

$$\text{b) } H_x = - \sum_{m=1}^2 p_m \cdot \log_2 p_m = - \underline{p}^T \cdot \log_2 \underline{p} = 0.9473 \frac{\text{bits}}{\text{symbol}}$$

data rate:

$$r_{\text{data}} = r_b = F_s \frac{1}{3} \bar{l} = 6583.3 \text{ bits/sec}$$

information rate:

$$r_{\text{inf}} = r_b \times H_x = r_b \times 0.9473 = 6236.4 \text{ bits/sec}$$

c)

$M = 2$ i.e. binary CS

$$\text{Therefore: } T_{cs} = \frac{1}{r_{cs}} = 1.5190 \times 10^{-4} \text{ sec}$$

$$E_b = \frac{0.5^2}{2} T_{cs} \times \Pr(x_1) = 1.2046 \times 10^{-5}$$

$$\text{EUE} = \frac{E_b}{N_0} = 6.0228 \times 10^{-3} \text{ (data EUE)}$$

$$\text{BUE} = \frac{B}{r_{cs}} = \frac{B}{2B \times \log_2(M)} = \frac{1}{2} \text{ (data BUE with } B \text{ denoting the baseband bandwidth)}$$

$$\text{data point} = (\text{EUE}, \text{BUE}) = (6.0228 \times 10^{-3}, \frac{1}{2})$$

d)

$$\text{CS} = \text{inf.point} = (\text{EUE}_{\text{inf}}, \text{BUE}_{\text{inf}}) = (\text{data point}) \times \frac{\log_2(M)}{H_{\text{mut}}}$$

Therefore we have to estimate the mutual information H_{mut}

$$H_{\text{mut}} = H_Y - H_{Y|X} \text{ or } (H_{\text{mut}} = H_X - H_{X|Y})$$

i.e.

$$\underline{p} = \begin{bmatrix} 0.6344 \\ 0.3656 \end{bmatrix} \quad \mathbb{F} = \begin{bmatrix} 0.95, & 0.2 \\ 0.05, & 0.8 \end{bmatrix} \quad \underline{q} = \mathbb{F} \cdot \underline{p} = \begin{bmatrix} 0.6758 \\ 0.3242 \end{bmatrix}$$

$$\mathbb{B} = \text{diag}(\underline{q})^{-1} \cdot \mathbb{F} \cdot \text{diag}(\underline{p}) = \begin{bmatrix} 0.8918, & 0.1082 \\ 0.0978, & 0.9022 \end{bmatrix}$$

$$\mathbb{J} = \mathbb{F} \cdot \text{diag}(\underline{p}) = \text{diag}(\underline{q}) \cdot \mathbb{B} = \begin{bmatrix} 0.6027, & 0.0731 \\ 0.0317 & 0.2925 \end{bmatrix}$$

$$H_X = - \sum_{m=1}^2 p_m \cdot \log_2(p_m) = - \underline{p}^T \log_2(\underline{p}) = 0.9473 \frac{\text{bits}}{\text{symbol}}$$

$$H_Y = - \sum_{k=1}^2 p_k \cdot \log_2(p_k) = - \underline{q}^T \log_2(\underline{q}) = 0.9089 \frac{\text{bits}}{\text{symbol}}$$

$$H_{X \times Y} = - \sum_{m=1}^2 \sum_{k=1}^2 J_{km} \cdot \log_2(J_{km}) = - \left\| \mathbb{J} \odot \log_2(\mathbb{J}) \right\|_{1*} = 1.3929 \frac{\text{bits}}{\text{symbol}}$$

$$H_{Y|X} = H_{Y|X}(\mathbb{J}) \equiv - \sum_{m=1}^2 \sum_{k=1}^2 J_{km} \cdot \log_2\left(\frac{J_{km}}{p_m}\right)$$

$$= - \left\| \mathbb{J} \odot \log_2\left(\underbrace{\mathbb{J} \cdot \text{diag}(\underline{p})^{-1}}_{\mathbb{F}}\right) \right\|_{1*} = 0.4456 \frac{\text{bits}}{\text{symbol}}$$

$$\Rightarrow \mathbf{H}_{\text{mut}} = \mathbf{H}_Y - \mathbf{H}_{Y|X} = 0.4633$$

Therefore $\text{CS} = \text{inf.point} = (0.013, 1.0792)$

e)

$$\Rightarrow \text{CS is not realizable} \quad (\text{since } \text{EUE}_{inf} = 0.013 < 0.693)$$

f)

$$\text{SNR}_{\text{in}} = \frac{\text{EUE}}{\text{BUE}} = 0.012 \Rightarrow \text{SNR}_{\text{in}} = -19.2082 \text{ dB}$$