E4.04
SC6
ISE4.9

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2007

MSc and EEE/ISE PART IV: MEng and ACGI

Corrected Copy

ADVANCED DATA COMMUNICATIONS

Monday, 23 April 10:00 am

Time allowed: 3:00 hours

There are FOUR questions on this paper.

Answer THREE questions.

All questions carry equal marks. The maximum mark for each subquestion is shown in brackets.

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s) :	M.K. Gurcan
	Second Marker(s) :	C. Ling

Instructions to the candidates:

Assume that the target argument of the Q function is 13.8 dB and 15 dB for average error rates of 10^{-6} and 1.8×10^{-8} respectively when using M-ary pulse amplitude modulation (PAM) and quadrature amplitude modulation (QAM) systems.

The Questions

1. Answer the following subquestions.

   (a) Prove that [5]

$$\phi_1(t) = \sqrt{\frac{2}{T}} \cos\left(2\pi\left(f_c + \frac{k}{T}\right)t\right) \quad \text{for } 0 \le t \le T \text{ and}$$
$$\phi_2(t) = \sqrt{\frac{2}{T}} \sin\left(2\pi\left(f_c + \frac{k}{T}\right)t\right) \quad \text{for } 0 \le t \le T$$

   are orthonormal basis functions for any integer $k = 1, \cdots, N$.

   (b) Consider the orthonormal basis functions

$$\phi_1(t) = \begin{cases} \sqrt{\frac{2}{T}} & \text{for } 0 \le t \le \frac{T}{2} \\ 0 & \text{otherwise} \end{cases}, \text{ and}$$
$$\phi_2(t) = \begin{cases} \sqrt{\frac{2}{T}} & \text{for } \frac{T}{2} \le t \le T \\ 0 & \text{otherwise} \end{cases},$$

   and also the vectors

$$\mathbf{x}_1 = \left(A\sqrt{\frac{T}{2}}, A\sqrt{\frac{T}{2}}\right)$$
$$\mathbf{x}_2 = \left(A\sqrt{\frac{T}{2}}, -A\sqrt{\frac{T}{2}}\right).$$

   Plot the time waveforms $x_1(t)$ and $x_2(t)$ which are constructed using the above vectors and orthonormal basis functions. Calculate the signal energies ε_1 and ε_2 and also the Euclidean distance between $x_1(t)$ and $x_2(t)$. [5]

   (c) Consider an AWGN system with a SNR $\frac{\bar{\varepsilon}_x}{\sigma^2}$ of 22 dB, a target probability of error P_e=10^{-6}, and a symbol rate $\frac{1}{T} = 8$ KHz.

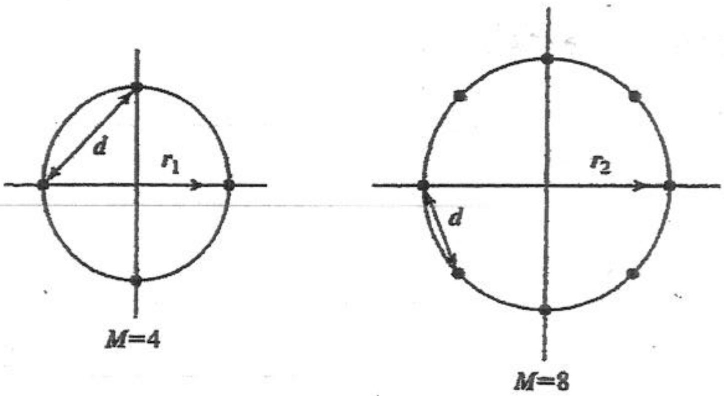
      i. Find the maximum data rate $R = \frac{b}{T}$ kHz that can be transmitted when using
         A. Pulse Amplitude Modulation (PAM) . [1]
         B. Quadrature Amplitude Modulation (QAM). [1]
      ii. What is the Nearest Neighbour Union Bound (NNUB) normalized probability of error $\bar{P}_e$ for the systems used in part 1.c. above ? [3]

   (d) A speech signal is sampled at a rate of 8 kHz, using 8 bits/sample. The Pulse Code Modulation (PCM) encoded data is transmitted through an AWGN baseband channel via an M-level PAM encoder. Determine the symbol data rate required for transmission when [5]

      i. M= 4,
      ii. $M = 8$, and
      iii. M=16.

2. Answer the following subquestions.

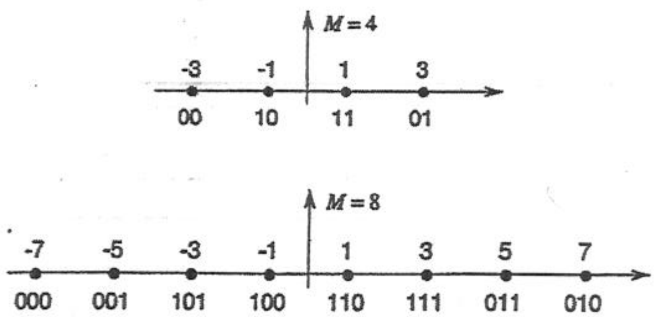
(a) Consider the four-level and eight-level phase modulation system constellations shown in the following figure



i. Determine r_1 and r_2 if the minimum distance between two adjacent points in the two constellations is to be d. [3]

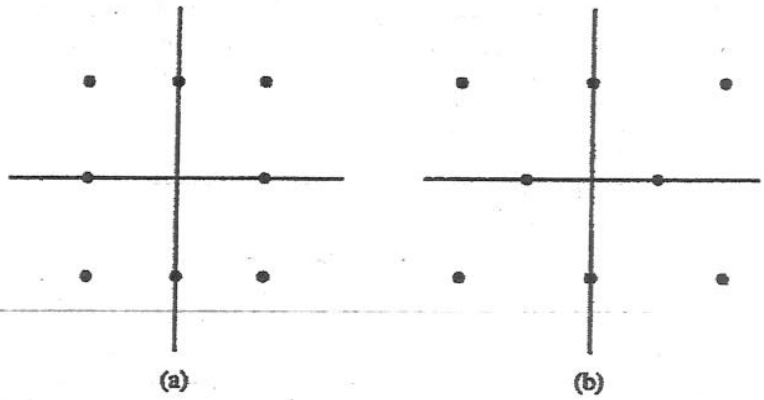
ii. From this result, determine the **additional** transmitted energy required when using the 8-PSK system if the same error probability is to be achieved when using a 4-PSK system. You may assume that only errors associated with adjacent points are significant. [4]

(b) Consider the four-level and eight-level PAM signal constellations shown in the following figure



Plot the baseband signals, $x(t)$, associated with the use of these constellations for $M = 4$ and 8 when transmitting the signal sequence is $(1, 0, 1, 0, 0, 0, 1, 1, 0, 0, 1, 1, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1)$. [6]

(c) Consider the two 8-point QAM signal constellations shown in the following figure



The minimum distance between adjacent points is $2A$. Determine the average transmitted energy for each constellation assuming that the signal points are equally probable. Which constellation is more power efficient? [7]

3. Answer the following subquestions.

(a) A voice-band telephone transmission is limited to the frequency range $300 \leq f \leq 3000$ Hz. Arrive at a symbol rate and the size of power efficient constellation to achieve a data rate of 9600 bits/sec explaining carefully the reasons for your choice. [2]
If a square-root raised cosine pulse, $g_T(t)$, is used for the transmitter pulse select the roll-off factor. [2]

(b) In a binary PAM system, the input to the detector is

$$y_m = x_m + n_m + i_m$$

where $x_m \in \{\pm 1\}$ is the desired signal, n_m is a zero-mean Gaussian random variable of variance σ_n^2 and i_m represents the Inter-Symbol-Interference (ISI) due to channel distortion. The ISI term is a random variable which takes the values $\frac{1}{2}$, 0,-$\frac{1}{2}$ with probabilities $\frac{1}{4}$, $\frac{1}{2}$, $\frac{1}{4}$ respectively. Determine the average probability of error as a function of σ_n^2. [3]

(c) Consider a communication channel with the pulse energy, p, and sampled auto-correlation function $Q(D)$ given as follows

$$|p|^2 = 1 + aa^* \qquad 0 \leq |a| < 1$$
$$Q(D) = \frac{a^* D^{-1} + |p|^2 + aD}{|p|^2}.$$

i. Assume that the matched filter bound is SNR_{MFB}. Find the coefficients for the zero forcing equaliser, $W_{ZFE}(D)$, and minimum mean square error linear equalizer, $W_{MMSE-LE}(D)$. Use the variable $b = |p|^2\left(1 + \frac{1}{SNR_{MFB}}\right)$ in your expression for $W_{MMSE-LE}(D)$. [3]

ii. Find the roots r_1, r_2 of the polynomial [2]

$$aD^2 + bD + a^*.$$

Show that $(b^2 - 4aa^*)$ is always a positive real number provided $|a| \neq 1$. Let r_2 be the root for which $|r_2| \leq |r_1|$. Show that $r_1 r_2^* = 1$.

iii. Use the previous results to show that for the MMSE-LE [2]

$$W_{MMSE-LE}(D) = \frac{|p|}{a(r_1 - r_2)}\left(\frac{r_1}{D - r_1} - \frac{r_2}{D - r_2}\right).$$

iv. For the channel considered in part c.i above, show that the canonical factorization is [2]

$$Q(D) + \frac{1}{SNR_{MFB}} = \gamma_0 \left(1 - r_2 D^{-1}\right)\left(1 - r_2^* D\right).$$

What is γ_0 in terms of a and b? [2]

v. Find the feedback $B(D)$ and feedforward $W(D)$ filter coefficients for the MMSE DFE. [2]

4. Answer the following subquestions.

(a) The data rate, $R = b/T$, of a multi-tone system with a set of 8 sub channels is to be maximized. In the system, $1/T$ is the symbol rate, and the term b

$$b = \frac{1}{2}\sum_{n=1}^{8} b_n = \frac{1}{2}\sum_{n=1}^{8} \log_2\left(1 + \varepsilon_n g_n\right)$$

is the largest number of bits that can be transmitted over the parallel set of 8 channels. In the equation for b, the term $g_n = \left|H_n\right|^2/\sigma_n^2$ represents the subchannel signal-to-noise ratio when the transmitter applies unit energy to that subchannel. The terms ε_n, $\left|H_n^2\right|$ and σ_n^2 correspond to the energy, the channel gain and noise variance in the n^{th} subchannel respectively. Using Lagrange multiplier method show that the following set of linear equations [6]

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & -1 \\ 0 & 1 & 0 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & -1 \\ 1 & 1 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_8 \\ K \end{bmatrix} = \begin{bmatrix} -1/g_1 \\ -1/g_2 \\ \vdots \\ -1/g_8 \\ 8\bar{\varepsilon}_x \end{bmatrix}$$

provide solutions for energy distributions in each subchannel, where K is a constant and $8\bar{\varepsilon}_x = \sum_{n=1}^{8} \varepsilon_n$.

(b) While the SNR gap may be fixed for often-used constellations in multi-tone modulation systems, this gap is a function of two other parameters.

i. What are these two parameters? [1]

ii. Based on your answer to part b.i above, how can the gap be reduced? [1]

iii. A 16 ($b = 4$,) QAM channel having $SNR = 25$dB is to have an average error probability, $\bar{P}_e = 10^{-6}$. What is the margin for this transmission system? [2]

(c) An N =8 dimensional multi-tone modulation signal is transmitted over a channel with the gain

$$H(f) = 1 + 0.5e^{j2\pi f}.$$

The signal SNR is $\bar{\varepsilon}_x \left|h\right|^2/\sigma^2 = 10$ dB and the average energy $\bar{\varepsilon}_x = 1$. Assuming that target argument of Q-function is 9 dB, calculate the average number of bits $\bar{b}$ per dimension if the total energy is distributed equally among each dimension. [5]

(d) For the channel in problem (4.c),

i. calculate the multi-channel signal-to-noise-ratio, $SNR_{m,u}$, for a set of parallel channels using the exact formula and the geometric mean SNR_{geo} for a gap value of $\Gamma = 8.8$dB. [2]

ii. Compare the difference between he SNR_{MFB} and the exact $SNR_{m,u}$ for the channels with the transfer functions $1 + 0.5D^{-1}$ and $1 + 0.9D^{-1}$. [3]

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MODEL ANSWER and MARKING SCHEME

First Examiner M.K. GURCAN	Paper Code E4.04 / ISE4.9 / SC6
Second Examiner T. STATHAKI	Question 1.a Page 1 out of 17

Question labels in left margin | **Marks allocations in right margin**

1.a

$$\phi_1(t) = \sqrt{\frac{2}{T}} \cos\left(2\pi\left(f_c + \frac{K}{T}\right)t\right)$$

$$\phi_2(t) = \sqrt{\frac{2}{T}} \sin\left(2\pi\left(f_c + \frac{K}{T}\right)t\right)$$

CONSIDER

$$\int_0^T \phi_1^2(t)\,dt = \int_0^T \frac{2}{T} \cos^2\left(2\pi\left(f_c + \frac{K}{T}\right)t\right) dt$$

$$= \int_0^T \frac{1}{T} \left[\cos\left(4\pi\left(f_c + \frac{K}{T}\right)t\right) + \cos(0)\right] dt$$

$$= \int_0^T \frac{1}{T} \cos\left(4\pi\left(f_c + \frac{K}{T}\right)t + \int_0^T \frac{dt}{T}\right.$$

$$= 0 + \frac{t}{T}\Big|_0^T = 1$$

ALSO CONSIDER

$$\int_0^T \phi_2^2(t)\,dt = \int_0^T \frac{2}{T} \sin^2\left(2\pi\left(f_c + \frac{K}{T}\right)t\right) dt$$

$$= \int_0^T \frac{2}{2T} \cos(0)\,dt - \int_0^T \frac{1}{T} \cos\left(4\pi\left(f_c + \frac{K}{T}\right)t\right) dt$$

$$= \frac{t}{T}\Big|_0^T - 0 = 1$$

CONSIDER

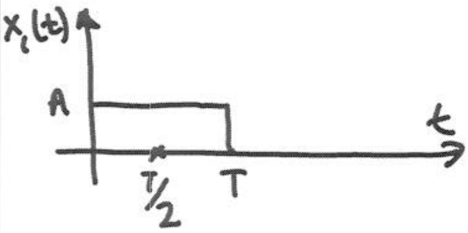
$$\int_0^T \phi_1(t)\,\phi_2(t)\,dt = \int_0^T \frac{2}{T} \cos\left(2\pi\left(f_c + \frac{K}{T}\right)t\right) \sin\left(2\pi\left(f_c + \frac{K}{T}\right)t\right) dt$$

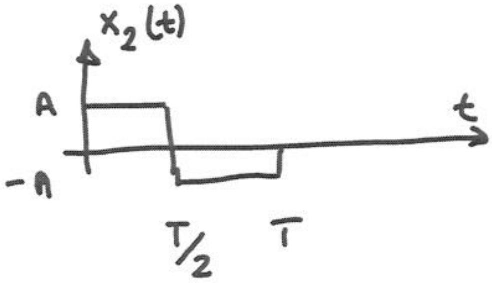
$$= \int_0^T \frac{1}{T} \sin\left(4\pi\left(f_c + \frac{K}{T}\right)t\right) dt - \int_0^T \frac{1}{T} \sin 0 \, dt$$

$$= 0$$

The wave forms $\phi_1(t)$ and $\phi_2(t)$ are orthogonal.

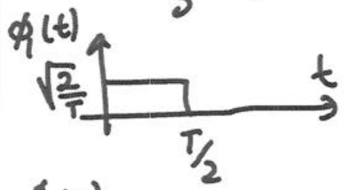
1.b

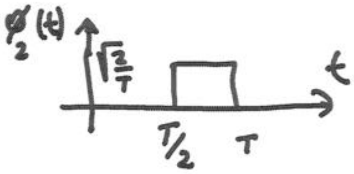


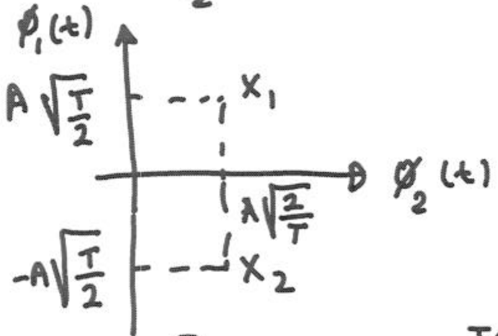


$$\mathcal{E}_1 = \int_0^T x_1^2(t)\,dt^2 = A^2 T$$

$$\mathcal{E}_2 = \int_0^T x_2^2(t)\,dt^2 = A^2 T$$







$$d = 2A\sqrt{\frac{T}{2}} = A\sqrt{2T}$$

$$\boxed{d = A\sqrt{2T}}$$

$$x_{1,1} = \int_0^T x_1(t)\,\phi_1(t)\,dt = \int_0^{T/2} A\sqrt{\frac{2}{T}}\,dt = A\sqrt{\frac{T}{2}}$$

$$x_{1,2} = \int_0^T x_1(t)\,\phi_2(t)\,dt = \int_{T/2}^{T} A\sqrt{\frac{2}{T}}\,dt = A\sqrt{\frac{T}{2}}$$

$$x_{2,1} = \int_0^T x_2(t)\,\phi_1(t)\,dt = \int_0^{T/2} A\sqrt{\frac{2}{T}}\,dt = A\sqrt{\frac{T}{2}}$$

$$x_{2,2} = \int_0^T x_2(t)\,\phi_2(t)\,dt = -A\int_{T/2}^{T} \sqrt{\frac{2}{T}}\,dt = -A\sqrt{\frac{T}{2}}$$

1c THE $\bar{P}_e$ FORMULA FOR PAM, AND QAM IS IDENTICAL

$$\bar{P}_e \leq 2.\,Q\left(\sqrt{\frac{3\,SNR}{2^{2\bar{b}}-1}}\right) < 10^{-6}$$

$$\therefore \quad \frac{3.SNR}{2^{2\bar{b}}-1} = 13.8\ dB$$

$$2^{2\bar{b}}-1 = 4.77+22-13.8 = 12.97\ dB$$

$$\boxed{\bar{b} = 2.19}$$

$\bar{b}$ MUST BE AN INTEGER, SO THE HIGHEST DATA RATE CORRESPONDS TO $\bar{b}=2$ IN ALL CASES. THUS

$$\text{PAM} \quad R = \frac{\bar{b}.N}{T} = 2\times1\times8k = 16\ kbps$$

$$\text{QAM} \quad R = 2\times2\times8k = 32\ kbps.$$

—///—

IN BOTH CASES

$$\frac{3.SNR}{2^{2\bar{b}}-1} = 4.77+22-11.76 = 15.01\ dB$$

$$\bar{P}_e \leq 2\ Q(15.01dB) = 1.8\times10^{-8}$$

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MODEL ANSWER and MARKING SCHEME

First Examiner M.K. GURCAN Paper Code 4.04

Second Examiner T. STATHAKI Question 1.d Page 4 out of 17

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1.d THE BANDWIDTH REQUIRED FOR TRANSMISSION OF AN M-ARY PAM SIGNAL IS

$$W = \frac{R_b}{2 \log_2 M} \text{ Hz}$$

SINCE

$$R_b = 8 \times 10^3 \frac{\text{SAMPLES}}{\text{SEC}} \times 8 \frac{\text{bits}}{\text{SAMPLE}}$$

$$= 64 \times 10^3 \text{ bits/sec}$$

WE OBTAIN

$$W = \begin{cases} 16 \text{ KHZ} & \text{for } M = 4 \\ 10.66 \text{ KHZ} & M = 8 \\ 8 \text{ KHZ} & M = 16 \end{cases}$$

2.a USING THE PHYTHAGOREAN THEOREM FOR THE FOUR PHASE CONSTELLATION, WE FIND

$$r_1^2 + r_1^2 = d^2 \Rightarrow r_1 = \frac{d}{\sqrt{2}}$$

THE RADIUS OF THE 8-PSK CONSTELLATION IS FOUND USING THE COSINE RULE. THUS

$$d^2 = r_2^2 + r_2^2 - 2r_2^2 \cos(45^\circ) \Rightarrow r_2 = \frac{d}{\sqrt{2-\sqrt{2}}}$$

THE AVERAGE TRANSMITTED POWER OF THE 4-PSK AND THE 8-PSK CONSTELLATION IS GIVEN BY

$$P_{4,AV} = \frac{d^2}{2}, \quad P_{8,AV} = \frac{d^2}{2-\sqrt{2}}$$

THUS THE ADDITIONAL TRANSMITTED POWER NEEDED BY THE 8-PSK SIGNAL IS

$$P = 10 \log_{10} \frac{2d^2}{(2-\sqrt{2})d^2} = 5.3329 \text{ dB}$$

WE OBTAIN THE SAME RESULTS IF WE USE THE PROBABILITY OF ERROR GIVEN BY

$$P_M = 2Q\left[\sqrt{2\rho_s} \sin\frac{\pi}{M}\right]$$

WHERE ρ_s IS THE SNR PER SYMBOL. IN THIS CASE EQUAL ERROR PROBABILITY FOR THE TWO SIGNALLING SCHEMES IMPLIES THAT.

$$\rho_{4,s} \sin^2\frac{\pi}{4} = \rho_{8,s} \sin^2\frac{\pi}{8} \Rightarrow 10 \log_{10} \frac{\rho_{8,s}}{\rho_{4,s}}$$

$$= 20 \log_{10} \frac{\sin \pi/4}{\sin \pi/8} = 5.3329 \text{ dB}.$$

2.b

$$\frac{10}{-1} \quad \frac{10}{-1} \quad \frac{00}{-3} \quad \frac{11}{1} \quad \frac{00}{-3} \quad \frac{11}{1} \quad \frac{01}{3} \quad \frac{00}{-3} \quad \frac{01}{3} \quad \frac{00}{-3} \quad \frac{00}{-3} \quad \frac{01}{3} \quad \frac{01}{3}$$

$$\frac{101}{-3} \quad \frac{000}{-7} \quad \frac{110}{1} \quad \frac{011}{5} \quad \frac{010}{7} \quad \frac{001}{-5} \quad \frac{000}{-7} \quad \frac{001}{-5}$$

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MODEL ANSWER and MARKING SCHEME

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2.C THE CONSTELLATION OF FIGURE 3.a HAS FOUR POINTS AT A DISTANCE 2A FROM THE ORIGIN AND FOUR POINTS AT A DISTANCE $2\sqrt{2}A$. THUS, THE AVERAGE TRANSMITTED POWER OF THE CONSTELLATION IS

$$P_{AV} = \frac{1}{4}\left[4\times(2A)^2 + 4(2\sqrt{2}\,A)^2\right] = 6A^2$$

THE SECOND CONSTELLATION HAS FOUR POINTS AT A DISTANCE $\sqrt{7}\,A$ FROM THE ORIGIN, TWO POINTS AT A DISTANCE $\sqrt{3}\,A$ AND TWO POINTS AT A DISTANCE A. THUS THE AVERAGE TRANSMITTED POWER OF THE SECOND CONSTELLATION IS

$$P_b = \frac{1}{8}\left[4\times(\sqrt{7}\,A)^2 + 2\times(\sqrt{3}\,A)^2 + 2A^2\right] = \frac{9}{2}A^2$$

SINCE $P_b < P_a$ THE SECOND CONSTELLATION IS MORE POWER EFFICIENT.

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MODEL ANSWER and MARKING SCHEME

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3.a THE BANDWIDTH OF THE CHANNEL IS

$$W = 3000 - 300 = 2700 \text{ Hz}$$

SINCE THE MINIMUM TRANSMISSION BANDWIDTH REQUIRED FOR BANDPASS SIGNALLING IS R, WHERE R IS THE RATE OF TRANSMISSION, WE CONCLUDE THAT THE MAXIMUM VALUE OF THE SYMBOL RATE FOR THE GIVEN CHANNEL IS $R_{MAX} = 2700$. IF AN M-ARY PAM MODULATION IS USED FOR TRANSMISSION, THEN IN ORDER TO ACHIEVE A BIT RATE OF 9600 bps WITH MAXIMUM RATE OF R_{MAX}, THE MINIMUM SIZE OF THE CONSTELLATION IS $M = 2^k = 16$. IN THIS CASE THE SYMBOL RATE IS

$$R = \frac{9600}{k} = 2400 \text{ SYMBOLS/SEC}$$

AND THE SYMBOL INTERVAL $T = \frac{1}{R} = \frac{1}{2400}$ SEC.

THE ROLL OF FACTOR α OF THE RAISED COSINE PULSE USED FOR TRANSMISSION IS DETERMINED BY NOTHING THAT $1200(1+\alpha) = 1350$, AND HENCE, $\alpha = 0.125$. THEREFORE, THE SQUARED ROOT RAISED COSINE PULSE CAN HAVE A ROLL-OFF $\alpha = 0.125$.

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3.6 WE HAVE

$$y = \begin{cases} a+n-\frac{1}{2} & \text{WITH PROB. } 1/4 \\ a+n+\frac{1}{2} & \text{WITH PROB } 1/4 \\ a+n & \text{WITH PROB } 1/2 \end{cases}$$

BY SYMMETRY

$$P_e = P(e|a=1) = P(e\,|\,a=-1), \text{ HENCE}$$

$$P_e = P(e|a=-1) = \frac{1}{2}P(n-1>0) + \frac{1}{4}P\left(n-\frac{3}{2}>0\right) + \frac{1}{4}P\left(n-\frac{1}{2}>0\right)$$

$$= \frac{1}{2}Q\left(\frac{1}{\sigma_n}\right) + \frac{1}{4}Q\left(\frac{3}{2\sigma_n}\right) + \frac{1}{4}Q\left(\frac{1}{2\sigma_n}\right)$$

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MODEL ANSWER and MARKING SCHEME

First Examiner M.K. GURCAN Paper Code 4.04

Second Examiner T. STATHAKI Question 3.C Page 10 out of 17

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3.C i) ZFE

$$W_{ZFE} = \frac{1}{|P|\,Q(D)} = \frac{|P|}{a^* D^{-1} + |P|^2 + aD} = \frac{\sqrt{1+aa^*}}{a^* D^{-1} + (1+aa^*) + aD}$$

MMSE-LE

$$W_{MMSE-LE} = \frac{1}{|P|\left(Q(D) + \frac{1}{SNR_{MFB}}\right)}$$

$$= \frac{|P|}{a^* D^{-1} + |P|^2\left(1 + \frac{1}{SNR_{MFB}}\right) + aD}$$

$$= \frac{|P|}{a^* D^{-1} + b + aD}$$

WHERE $b = |P|^2\left(1 + \frac{1}{SNR_{MFB}}\right)$

—————////—

3.C.ii THE ROOTS r_1, r_2 OF THE EQUATION $aD^2 + bD + a^* = 0$ ARE GIVEN BY

$$r_1 = \frac{-b - \sqrt{b^2 - 4aa^*}}{2a}, \quad r_2 = \frac{-b + \sqrt{b^2 - 4aa^*}}{2a}$$

SINCE b and aa^* REAL, $b^2 - 4aa^*$ IS REAL. TO PROVE THAT IT IS ALWAYS POSITIVE

$$b^2 - 4aa^* = \left(|P|^2\left(1 + \frac{1}{SNR_{MFB}}\right)\right)^2 - 4aa^*$$

$$\geq |P|^2 - 4aa^* = |1 + aa^*|^2 - 4aa^*$$

$$= |1 - aa^*|^2 > 0$$

3.c ii

FINALLY $r_1 r_2^*$ IS COMPUTED

$$r_1 r_2^* = \frac{1}{4aa^*}\left(-b-\sqrt{b^2-4aa^*}\right)\left(-b+\sqrt{b^2-4aa^*}\right)$$

$$= \frac{1}{4aa^*}\left(b^2-\left(\sqrt{b^2-4aa^*}\right)^2\right) = 1$$

iii)

FOR MMSE-LE

$$W_{MMSE-LE} = \frac{|P|\ D}{a^* + bD + aD^2} = \frac{|P|}{a}\ \frac{D}{(D-r_1)(D-r_2)}$$

BY EXPANDING $W_{MMSE-LE}$ INTO PARTIAL FRACTION

$$W_{MMSE-LE}(D) = \frac{|P|}{a}\left[\frac{A}{D-r_1} + \frac{B}{D-r_2}\right]$$

WE FIND THAT

$$A = \frac{r_1}{r_1 - r_2} \quad \text{AND} \quad B = \frac{r_2}{r_2 - r_1} \quad \text{SO}$$

$$W_{MMSE-LE}(D) = \frac{|P|}{a(r_1-r_2)}\left[\frac{r_1}{D-r_1} - \frac{r_2}{D-r_2}\right]$$

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MODEL ANSWER and MARKING SCHEME

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3.C IV

$$W_{MMSE-LE}(D) = \frac{1}{|P|\left[Q(D) + \frac{1}{SNR_{MFB}}\right]}$$

$$\Leftrightarrow \quad Q(D) + \frac{1}{SNR_{MFB}} = \frac{1}{|P|}\frac{1}{W_{MMSE-LE}(D)}$$

BUT

$$W_{MMSE-LE}(\gamma) = \frac{|P|}{a}\frac{D}{(D-r_1)(D-r_2)}$$

$$Q(D) + \frac{1}{SNR_{MFB}} = \frac{a}{|P|^2}(D-r_1)(1-D^{-1}r_2)$$

$$= -\frac{ar_1}{|P|^2}(1-Dr_2^*)(1-D^{-1}r_2)$$

WHERE $r_1 r_2^* = 1$, HENCE

$$\gamma_0 = -\frac{ar_1}{|P|^2} = \frac{b+\sqrt{b^2-4aa^*}}{2(1+aa^*)}$$

———///———

V.

$$B(D) = 1 - Dr_2^* = 1 - \frac{-b+\sqrt{b^2-4aa^*}}{2a^*}D$$

$$w(D) = \frac{1}{|P|\gamma_0 G^*(D^{-*})} = -\frac{|P|}{ar_1}\frac{1}{1-r_2D^{-1}}$$

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MODEL ANSWER and MARKING SCHEME

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4.a $R = \frac{b}{T}$, WITH SYMBOL RATE $1/T$.

WE NEED TO MAXIMISE $b = \sum_n b_n$ OVER b_n AND $\mathcal{E}_n$.

$$b = \frac{1}{2} \sum \log_2 \left(1 + \frac{\mathcal{E}_n g_n}{\Gamma}\right)$$

WHERE $g_n = \frac{|H_n|^2}{\sigma_n^2}$.

$$\sum_{n=1}^{N} \mathcal{E}_n = N \bar{\mathcal{E}}_x$$

USING LA GRANGE MULTIPLIERS, THE COST FUNCTION TO MAXIMIZE

$$b = \frac{1}{2} \sum_{n=1}^{N} \log_2 \left(1 + \frac{\mathcal{E}_n g_n}{\Gamma}\right)$$

SUBJECT TO THE CONSTRAINT IN

$$\frac{1}{2\ln(2)} \sum_n \ln\left(1 + \frac{\mathcal{E}_n g_n}{\Gamma}\right) + \lambda \left(\sum_n \mathcal{E}_n - N \bar{\mathcal{E}}_x\right)$$

DIFFERENTIATING WRT $\mathcal{E}_n$ PRODUCES

$$\frac{1}{2\ln(2)} \frac{1}{\mathcal{E}_n + \frac{\Gamma}{g_n}} = -\frac{\lambda \Gamma}{g_n}$$

THUS

$$b = \frac{1}{2} \sum_{n=1}^{N} \log_2 \left(1 + \frac{\mathcal{E}_n g_n}{\Gamma}\right) \text{ IS}$$

MAXIMIZED WHEN

$$\mathcal{E}_n + \frac{\Gamma}{g_n} = \text{CONSTANT}$$

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4.a WHEN $\Gamma = 1$ (0 dB) THE MAXIMUM DATA RATE

$$\mathcal{E}_1 + \frac{\Gamma}{g_1} = K$$

$$\mathcal{E}_2 + \frac{\Gamma}{g_2} = K$$

$$\vdots$$

$$\mathcal{E}_N + \frac{\Gamma}{g_n} = K$$

$$\mathcal{E}_1 + \mathcal{E}_2 + \cdots + \mathcal{E}_N = N\bar{\mathcal{E}}_x$$

IN MATRIX FORM

$$\begin{bmatrix} 1 & & 0 & 0 & 0 \\ 0 & 1 & & & -1 \\ 0 & & 1 & & -1 \\ 0 & & & 1 & -1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \mathcal{E}_1 \\ \mathcal{E}_2 \\ \vdots \\ \mathcal{E}_N \\ K \end{bmatrix} = \begin{bmatrix} -\frac{\Gamma}{g_1} \\ \vdots \\ -\frac{\Gamma}{g_N} \\ N\bar{\mathcal{E}}_x \end{bmatrix}$$

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4.b i) THE GAP DEPENDS ON P_e AND THE CODING EMPLOYED.

ii). IMPROVE CODING SCHEME WHEN P_e IS FIXED OR ACCEPT A HIGHER P_e.

iii.) ASSUMING QAM TRANSMISSION WE OBTAIN. TARGET ARGUMENT OF Q FUNCTION IS 13.8dB for $P_e = 10^{-6}$.

THE GAP VALUE $\Gamma = \frac{10^{1.38}}{3} = 7.99 \Rightarrow \Gamma = 8.8\text{dB}$.

$$\gamma_m = \frac{SNR/\Gamma}{2^b - 1} = 25 - 8.8 - 11.7\text{dB} = 4.5\text{dB}$$

4.c

$$H(D) = 1 + 0.5D \Rightarrow \quad h = [1 \quad 0.5]^T$$

$$|h|^2 = \sum_{i=1}^{2} h_i^2 = (1)^2 + (0.5)^2 = 1.25 \quad , \quad \bar{\mathcal{E}}_x = 1$$

$$SNR_{MFB} = 10 \text{ dB} \qquad SNR_{MFB} = 10 = \frac{\bar{\mathcal{E}}_x |h|^2}{\sigma^2}$$

$$\sigma^2 = \frac{1 \times 1.25}{10} = 0.125$$

$$SNR_n = \frac{\mathcal{E}_n |H_n|^2}{\sigma^2}$$

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FOR $N = 8$

n	0	1	2	3
ε_n	8/7	16/7	16/7	16/7
$\lvert H_n \rvert$	1.5	1.4	1.118	0.737
SNR_n	20.6	17.9	11.43	4.97
$\bar{b}_n$	1.566	1.479	1.205	0.761

WHERE

$$\bar{b}_n = \frac{1}{2}\log_2\left(1 + \frac{SNR_n}{\Gamma}\right) = \frac{1}{2}\log_2\left(1 + SNR_n \times 3/A_{rg}\right)$$

$$A_{rg} = 10^{0.9} = 7.94$$

$$\bar{b} = \frac{1 \times 1.566 + 2 \times 1.479 + 2 \times 1.205 + 2 \times 0.761}{8}$$

$$\boxed{\bar{b} = 1.057.}$$

4.d

$$SNR_{m,u} = \left[\left(\prod_{n=1}^{N} (1 + SNR_n/\Gamma)\right)^{1/N} - 1\right]\Gamma$$

$$SNR_{geo} = \left[\prod_{n=1}^{N} (SNR_n)\right]^{1/N}$$

FOR $1 + 0.5D^{-1}$ CHANNEL

$SNR_{m,u} = 9.68 = 9.86$ dB.

$SNR_{geo} = 8.24 = 9.16$ dB

FOR $1 + 0.9D^{-1}$ CHANNEL

$SNR_{m,u} = 9.53 = 9.8$ dB

$SNR_{geo} = 7.75 = 8.89$ dB

SNR_{geo} IS CLOSER TO $SNR_{m,u}$ for $1 + 0.5D^{-1}$ CASE. THAT IS BECAUSE IT SHOWS A FLATTER RESPONSE AND THE ROLL-OFF IS NOT SEVERE. THIS IMPLIES THAT THE WORST SNR_n/Γ IS BETTER FOR $1 + 0.5D^{-1}$ CHANNEL, WHICH RESULTS IN A BETTER APPROXIMATION.