Imperial College London

BSc/MSci EXAMINATION May 2012

This paper is also taken for the relevant Examination for the Associateship

PHYSICS COMPREHENSIVE I

For Third and Fourth Year Physics Students

3 May 2012: 14:00 to 17:10

You may attempt as many questions as you wish. Only the answers to the best EIGHT questions over the two papers will contribute to your mark.

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in all answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

1. This question is about hydrostatic pressure and floating objects

(i) Show that if atmospheric pressure can be ignored, then in a static fluid of uniform density ρ_F, the pressure P, at a depth h, is given by $P = \rho_F g h$, where g is the acceleration due to gravity. [3 marks]

(ii) A thin pencil of cross sectional area A and length ℓ is fully submerged in a static fluid with the long edge aligned parallel to the gravitational field. Verify that Archimedes' Principle (the buoyancy force equals the weight of the fluid displaced) holds for the pencil. [2 marks]

(iii) The pencil is of uniform density ρ_P. If this is less than the fluid density it will rise to the surface. If the fluid exerts a resistive drag force proportional to the pencil's velocity v (i.e. $drag = -kv$) sketch a free body force diagram for the pencil when fully submerged. Apply Newton's Second Law to derive an expression for the pencil's velocity as a function of time t. Determine the terminal velocity of the pencil and sketch a graph of v against t. Assume the pencil starts from rest.

[7 marks]

(iv) When the pencil reaches the surface it will float. Assuming the pencil remains vertical, show that if it is pushed downwards a little from its equilibrium position and frictional forces are negligible, it will oscillate with simple harmonic motion with period

$$T = 2\pi\sqrt{\frac{\rho_P}{\rho_F}\frac{\ell}{g}}$$

[4 marks]

(v) The tectonic plates that comprise the Earth's crust can be thought of as solid objects floating in a liquid mantle. If a section of crust gets pushed downwards (e.g. by a mass of frozen water during an ice age) and then gets "suddenly" released (when the ice melts - a process that can take up to 100 years) then the crust returns to equilibrium with *over-damped harmonic motion*. If the initial depression is w_0 then the depression as a function of time can be shown to be $w = w_0 e^{-t/\tau}$. The time constant τ is given by

$$\tau = \frac{4\pi\mu}{\rho_C g \lambda}$$

where μ is the mantle's viscosity in Pascal.seconds, ρ_C is the crustal density and λ is the linear dimension of the section of crust.

(a) Find the value of the time constant for a section of crust 1 km in length if the viscosity of the surrounding mantle is 10^{18} Pa.s and the crustal density is 2800 kgm^{-3}. Comment on the assertion that the depressed crust is "suddenly" released. [2 marks]

(b) If an isolated section of the Earth's crust measuring 1 km $\times$ 1 km is depressed by 100 m due to the weight of an ice sheet, determine an upper limit to the speed of the "postglacial rebound" in cm.year^{-1} when the ice age ends. [2 marks]

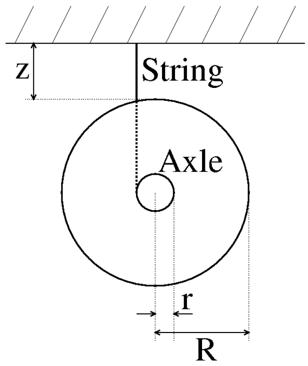
[Total 20 marks]

2. (i) For a body free to rotate with a moment of inertia I and an angular velocity ω, write down a relationship between these quantities and the applied torque T.

[2 marks]

(ii) The rotating body has mass m and its centre of mass has a linear velocity of magnitude v. What is the total kinetic energy of the body? [3 marks]

(iii) A yoyo is constructed from two heavy disks of radius R connected by a light axle of radius r, as shown in the figure below. The total mass of the yoyo is m and its moment of inertia for rotations around the axle can be taken to be $I = mR^2/2$. A thin, light, inextensible string is fixed to the axle and then wound around it. The yoyo is dropped from rest at $t = 0$.



(a) Find an expression for the fraction of the kinetic energy which is due to the rotational motion. [2 marks]

(b) Consider the yoyo after a length z of the string has unwound. Assuming energy loss to friction is negligible, use an energy conservation argument to derive an expression giving the linear velocity of the yoyo in terms of z. Integrate this expression to show that z changes as a function of time according to

$$z = \frac{r^2 g t^2}{2r^2 + R^2}.$$

[4 marks]

(c) Hence deduce that the acceleration of the yoyo is constant. How does the value of the acceleration compare with that due to gravity, g? [2 marks]

(d) By considering the forces on the yoyo, find an expression for the tension in the string. [2 marks]

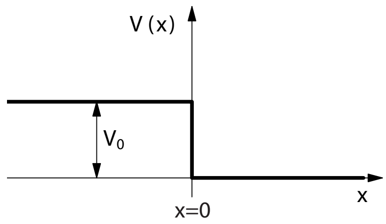
(iv) The yoyo eventually reaches the end of the string.

(a) Assuming no loss due to friction and neglecting effects due to the non-zero size of the axle radius r, qualitatively state how the vertical component of the linear momentum of the yoyo changes around this time. Sketch this linear momentum as a function of time. Also sketch the string tension as a function of time. [3 marks]

(b) Under the same assumptions, qualitatively state how the angular momentum changes around this time. Sketch this dependence. [2 marks]

[Total 20 marks]

3. In a number of cartoon films, hilarity ensues when a character is chased towards the edge of a cliff and, rather than plunging over the edge, the character is invariably able to turn around in mid-air and return to the safety of the cliff. Whilst this is impossible classically, this question explores how much the film makers knew about quantum mechanics. Consider the potential shown in the figure.



(i) Consider first the classical behaviour of a particle in this potential. Explain what happens if a particle with energy $E > V_0$ travels towards the potential step from the left or with energy $E < V_0$ from the right. [2 marks]

(ii) Write down the time-independent Schrödinger equation for a particle in the two regions $x < 0$ and $x > 0$. [1 mark]

(iii) Show that the solutions to the Schrödinger equation just given can be constructed from the set

$$\psi(x) = \begin{cases} e^{\pm iqx}, & x < 0 \\ e^{\pm ikx}, & x > 0 \end{cases}$$

where $k = \sqrt{2mE}/\hbar$ and $q = \sqrt{2m(E - V_0)}/\hbar$. [2 marks]

(iv) Now assume that a particle approaches the potential step from the left with $E > V_0$. Its wavefunction can then be written as

$$\psi(x) = \begin{cases} e^{iqx} + re^{-iqx}, & x < 0 \\ te^{ikx}, & x > 0 \end{cases} .$$

What boundary conditions must a wavefunction obey at $x = 0$? Write down the conditions for the given wavefunction $\psi(x)$. [4 marks]

(v) Show that the reflection and transmission coefficients $R = |r|^2$ and $T = |t|^2 k/q$ are given by

$$R = \left(\frac{q-k}{q+k}\right)^2 , \quad T = \frac{4kq}{(q+k)^2} .$$

[4 marks]

(vi) Show that the relation $R + T = 1$ holds. What is the physical meaning of this relation? [2 marks]

(vii) If the energy, E, of the incoming particle is just above the potential, V_0, show that, to first order in $E - V_0$, the reflection coefficient is

$$R \approx 1 - 4\sqrt{\frac{E - V_0}{V_0}}$$

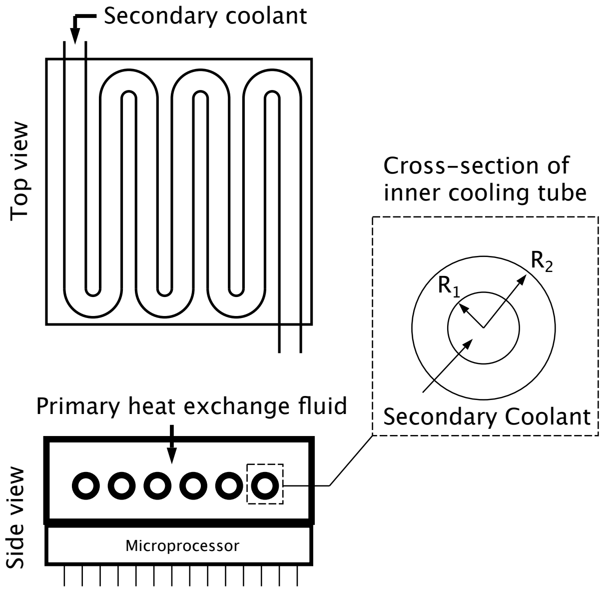
[3 marks]

[This question continues on the next page . . .]

(viii) For what value of E is the reflection probability maximal? Hence comment on how fast a body must travel to optimise the chance of reflection. [2 marks]

[Total 20 marks]

4. In the cooling system of the microprocessor shown schematically in the figure, the primary heat exchange fluid is housed in an enclosure in thermal contact with the processor together with an inner cooling tube carrying the secondary coolant. The temperature of the primary heat exchange fluid can be considered to be the same as that of the processor. The secondary coolant is pumped through the inner cooling tube which has a circular cross section (inner radius R_1, outer radius R_2) as shown in the insert.



The inner cooling tube is constructed from a material with thermal conductivity κ, mass density ρ and specific heat capacity c. The radial temperature variation $\theta(r, t)$ in the walls of the tube is governed by the following equation:

$$\kappa\left[\frac{\partial^2\theta(r,t)}{\partial r^2} + \frac{1}{r}\frac{\partial\theta(r,t)}{\partial r}\right] = c\rho\frac{\partial\theta(r,t)}{\partial t}\ . \tag{1}$$

You may assume that the temperature distribution has cylindrical symmetry and is independent of position along the length of the tube.

(i) Solve Eq. (1) in the steady state to show that the temperature gradient in the wall of the tube satisfies

$$\frac{\mathrm{d}\theta}{\mathrm{d}r} = Cr^{-1} \tag{2}$$

where C is a constant.

Sketch the steady state temperature as a function of r, identifying regions of strong and weak temperature gradients. [5 marks]

Under an intensive but steady computational task, the processor consumes 60W of power and runs at a temperature of 80°C. The secondary coolant enters the inner cooling tube at a temperature of 20°C and exits with a temperature of 25°C. The inner cooling tube has an outside radius R_2 = 0.50mm and inner radius R_1 = 0.25mm.

[This question continues on the next page . . .]

(ii) Assuming that the processor is running at 3GHz and each change of state of a logic element (i.e. a transistor) consumes $100k_BT$ worth of energy, estimate the average number of logic elements participating in the calculation during each clock cycle. [3 marks]

(iii) Make an estimate of the maximum temperature gradient. [4 marks]

(iv) If the secondary coolant is water with $c = 4180$ J kg^{-1}K^{-1} and $\rho = 1000$ kg m^{-3} and there is no turbulent flow, estimate the linear speed of water in the tube. [3 marks]

(v) If the length of the inner cooling tube is 0.4m, make an estimate of the minimum thermal conductivity, κ, of the material used in the tube. [3 marks]

(vi) Suggest possible ways to reduce the linear speed of the coolant. [2 marks]

[Total 20 marks]

5. (i) It is sometimes said that according to special relativity 'moving clocks go slow'. Make this statement precise by identifying events in two co-oriented frames S and S' in relative motion (where S' moves with speed u along the $+x$ axis of S), and deriving the time dilation formula from the inverse Lorentz transformation $t = \gamma\left(t' + ux'/c^2\right)$, where $\gamma = \left(1 - u^2/c^2\right)^{-1/2}$ and c is the speed of light in vacuum. [2 marks]

(ii) A light source emits into vacuum a wave of wavelength λ_0 and period τ_0 in the source's rest frame. The source is moving towards an observer with speed u.

(a) How is λ_0 related to τ_0? How much time τ elapses in the observer's frame when a clock attached to the source ticks τ_0 seconds? According to the observer, how far will the source move in τ seconds? [3 marks]

(b) Hence show that the wavelength λ according to the observer is given by

$$\lambda = \left(\frac{c-u}{c+u}\right)^{1/2} \lambda_0 ,$$

and that this formula is consistent with the energy-momentum (E, p) Lorentz transformation, $E' = \gamma(E - up)$. [4 marks]

Imagine that the Universe is spatially periodic such that in the frame of the Earth $x + L = x, y + L = y, z + L = z$. This means that travelling a distance L in the x, y or z directions brings you back to the same place. Consider motion just along the x-axis. A traveller leaves the Earth with velocity $u = 0.4c$ emitting a monochromatic wave in both the forward ($+x$) and backward ($-x$) directions. Ignore gravitational effects, and any effects due to the orbital motion of the Earth.

(iii) According to an observer who stays on the Earth, how long will it take the traveller to make his first return to Earth? At this time, how many images of the traveller will the Earth observer see when he looks (a) in the $+x$ direction and (b) in the $-x$ direction. Give reasons for your answers and describe any differences in color between the images. [4 marks]

(iv) What will the time be according to the traveller when he returns to Earth? Check consistency with part (iii) by using the spacetime coordinates of this event in the traveller's frame to calculate the return time of the traveller according to the Earth observer. [2 marks]

(v) What is the extent of the Universe according to the traveller? [1 mark]

(vi) Comment briefly on these results in respect of comparing with the usual twin paradox where one twin turns around before returning home. In what way might this model of the Universe be said to violate a principle of Special Relativity? [4 marks]

[Total 20 marks]

6. A new soft drink, FizzyInnit, is tested against a rival product, OldCola. Members of the public at a shopping centre are asked to compare the products in a blind test. The trial involved 900 people. The outcome of the trial is that 472 preferred FizzyInnit and the remaining 428 preferred OldCola.

(i) Suppose these consumers have no preference for either product. In other words, they chose randomly and independently with equal probabilities for the two products. Under this hypothesis, what are the expected mean and standard deviation for the number of people (out of the sample of 900) who chose FizzyInnit? You may find information given below useful. [4 marks]

(ii) Show that the outcome of the trial is 1.5 standard deviations away from the number expected from the hypothesis that there is no underlying preference. [3 marks]

(iii) Under the hypothesis of no preference, what is the probability that 472 *or more* people chose FizzyInnit? Can one conclude that there is a preference for FizzyInnit among the people who took part in the test? (See information given below.) [4 marks]

FizzyInnit is developed to target a young market while OldCola is known to have broad appeal. The FizzyInnit development team claims that adults have no preference but 70% of the young market express a preference for FizzyInnit. The test team did not record the responses individually but they know that, out of the 900 testers, 576 were adults and 324 were non-adults.

(iv) Assuming that adults have no preference, estimate from the trial results the fraction of the young market that prefers FizzyInnit to OldCola. [3 marks]

(v) Show that, according to the claim of the development team, the number of testers preferring FizzyInnit should have a mean of 515 and a standard deviation of 14.6. [3 marks]

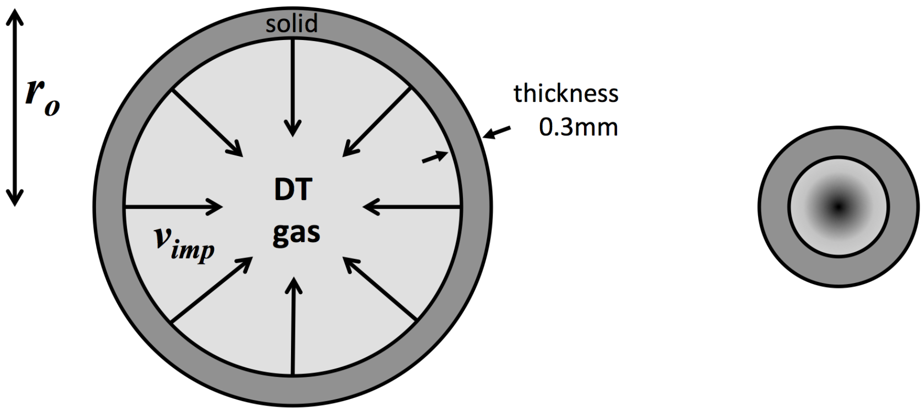
(vi) Hence, discuss quantitatively whether the outcome of the trial (472 choosing FizzyInnit) is consistent with the claim of the development team. [3 marks]

For N independent trials of a random process with success probability p, the probability P_n for n successes is given by the binomial distribution. The number of successes has a mean of Np and a variance of $Np(1-p)$.
For $N \gg 1$, the binomial distribution is well approximated by a Normal distribution with the same mean and variance. Let $p(x)$ be the probability density function for a random variable x which follows a Normal distribution with zero mean and unit variance.

z	0	0.5	1.0	1.5	2.0	2.5	3.0
$\int_z^\infty p(x)\,dx$	0.5	0.31	0.16	0.067	0.023	0.0062	0.0013

If x and y are two independent random variables following Normal distributions, the sum $x + y$ also follows a Normal distribution. [Total 20 marks]

7. In inertial confinement fusion (ICF), a thin, hollow spherical shell of solid material surrounding a volume of deuterium-tritium (D-T) gas, is imploded by intense laser light (see figure). This sphere quickly accelerates to high velocity then essentially acts as a piston to adiabatically compress the D-T gas to the very high densities and temperatures required for thermonuclear fusion of the nuclei ($k_B T > 10\,\text{keV}$).



(i) Explain what the term 'adiabatic' means. [2 marks]

(ii) Starting from the 1st law of thermodynamics (relating changes in internal energy to changes in volume and entropy), or otherwise, show that $PV^{5/3} = const.$ for an adiabatic process involving an ideal monatomic gas of pressure P and volume V. [3 marks]

The ICF shell has density $\rho_{sh} = 2 \times 10^3\ \text{kg m}^{-3}$, and initially a radius of $r_o = 5$ mm, thickness 0.3 mm and kinetic energy of $K_{sh} = 100$ kJ. The deuterium gas has an initial temperature of $T_o = 10^4$ K and an initial atomic number density of $n_o = 2 \times 10^{26}\ \text{m}^{-3}$.

(iii) Calculate the initial implosion velocity of the shell. [3 marks]

(iv) At maximum compression, the shell comes temporarily to a halt. The gas temperature increases substantially and becomes so hot that the D atoms ionize forming a gas containing free electrons and D ions (i.e. a plasma). The plasma behaves like an ideal gas formed of free electrons and deuterons at equal temperatures. Show that the temperature T_f of the gas at maximum compression is

$$T_f = \frac{K_{sh}}{3N_D k_B}$$

where N_D is the number of D atoms in the fuel. Estimate the value of T_f.

[3 marks]

(v) Show that the factor by which the radius is compressed is approximately 50. [3 marks]

To try and achieve even higher compression, the initial gas temperature is dropped to $T_c = 100$ K.

[This question continues on the next page . . .]

(vi) The importance of quantum degeneracy during compression can be deduced by comparing the Fermi energy with the particle kinetic energy that would occur if the gas were to behave like a classical adiabatic gas. Show that ratio of the Fermi energy (see below) to the "classical" energy at any instant during the compression is given by

$$\alpha = \frac{c_F n_o^{\frac{2}{3}}}{k_B T_o}$$

where c_F is a factor formed from fundamental constants.

[3 marks]

(vii) Deduce whether or not the D-ions, and also the electrons, are degenerate.

[3 marks]

Useful formula:

The Fermi energy of a system of particles of number density n and mass m is given by $E_F = \frac{\hbar^2}{2m}(3\pi^2 n)^{2/3}$.

[Total 20 marks]

8. (i) State Stefan's law, which gives the radiation intensity (radiation power per unit area) emitted by a certain type of hot body. Define all physical quantities entering this law and state the nature of the body. [3 marks]

The temperature distribution in the Solar System can be found to a fair approximation assuming that the Sun, the planets and moons absorb and emit radiation as **black bodies** and that the Sun is the only source of heat. Use this assumption to calculate the quantities listed below. For simplicity, you can also assume that the astronomical bodies are spherical and their orbits are circular. Give your answers with an accuracy of **3 significant figures** and in **SI units**. Take care to keep track of the units in intermediate results.

(ii) Calculate the radiation intensity at the surface of the Sun. [2 marks]

(iii) Show that the rate of solar mass loss due to radiation emission is the following function of the solar radius and surface temperature:

$$\frac{dM_\odot}{dt} = AR_\odot^2 T_\odot^4 ,$$

and find the constant A in terms of fundamental physical constants.

[3 marks]

(iv) Calculate the solar constant, i.e. the sunlight intensity falling on the Earth ignoring atmospheric absorption. [3 marks]

(v) Draw a diagram of the radiation fluxes emitted by the Sun, absorbed by the Earth and emitted by the Earth. Use the drawing to help deduce an expression for the Earth's temperature and calculate its value. [5 marks]

(vi) Considering a more realistic model of the Earth than a black body, take into account the Earth's albedo (the fraction of solar radiation reflected back into space) and its emissivity in infrared (the ratio of energy radiated by Earth to energy radiated by a black body of the same temperature) and:

(a) Derive the formula for the Earth's temperature as a function of the Earth's albedo, its emissivity in infrared and the Sun's surface temperature.

[2 marks]

(b) Deduce from the formula what would happen to the Earth's temperature if the albedo was kept constant but the emissivity tended to zero. Explain why such a conclusion is incorrect. [2 marks]

[Total 20 marks]

In your calculations you may use any of the following data as well as data on your formula sheet:
Sun radius, $R_\odot = 696 \times 10^3$ km
Sun mass, $M_\odot = 1.99 \times 10^{30}$ kg
Sun surface temperature, $T_\odot = 5510$ °C
Sun–Earth mean distance $r = 150 \times 10^6$ km
Parsec, 1 pc $= 30.9 \times 10^{15}$ m
Earth radius, $R_E = 6.37 \times 10^3$ km

[This question continues on the next page . . .]

Earth mass, $M_E = 5.97 \times 10^{24}$ kg
Earth albedo, $\alpha = 0.306$
Earth emissivity in infrared, $\epsilon = 0.612$
Absolute zero, $T_0 = -273.15$ °C

9. The static form of the wave-equation describing the propagation in free space of spinless particles of mass m is given by

$$\nabla^2\phi - \frac{m^2c^2}{\hbar^2}\phi = 0 \ . \tag{1}$$

The form of the potential for a force produced by the exchange of a quantum of mass, m, between two objects at a distance r is

$$\phi_Y = g\frac{e^{-r/r_0}}{r} \ . \tag{2}$$

(i) Verify that the potential in Eqn. (2) satisfies the wave equation in (1) and show that $r_0 = \hbar/mc$. You may find the following expression useful:

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) \ .$$

[4 marks]

(ii) Briefly describe what is represented by g and r_0 in Eq. (2). [2 marks]

(iii) If the range of the nuclear force is $\approx$ 1 fm ($= 10^{-15}$m), show that the mass of the exchange quantum responsible for the force is 200 MeVc^{-2} (you may find it convenient to use $\hbar c = 200$ MeV.fm). [2 marks]

(iv) The mass of the W or Z boson is $\sim$ 100GeVc^{-2}. Show that the range of the weak force mediated by these bosons is $\sim 2 \times 10^{-3}$ fm. [2 marks]

(v) Write down the reaction responsible for the first stage of burning in stars comprising hydrogen. Briefly explain the two processes which hinder this reaction. [3 marks]

(vi) Consider a bound system comprising a proton and an anti-proton. Write down the expressions for the Coulomb and gravitational potential energies for this system. Show that the ratio of the strengths of the Coulomb to the gravitational energy is $\sim 10^{36}$. You may find it convenient to compute using natural units:
Fine structure constant $e^2/4\pi\epsilon_0\hbar c = 1/137$,
Mass of proton $m_p \approx 1$GeVc^{-2},
$G/\hbar c = 6.7 \times 10^{-39}(\text{GeVc}^{-2})^{-2}$. [3 marks]

After the discovery of the nucleus by E. Rutherford one hundred years ago, it was conjectured that the nucleus (A, Z) contained A protons and $(A - Z)$ electrons. Note that this conjecture was rejected after the discovery of the neutron.

(vii) Using Heisenberg's uncertainty principle, calculate the momentum that the electrons would have if confined inside a strontium nucleus. Comment on the Rutherford model in the light of your result. (Assume that the radius of a nucleus is given by $R = a_0A^{1/3}$ where $a_0 = 1.2$ fm. The (A, Z) of strontium is $(90, 38)$. In the β-decay of (90,38) strontium the maximum energy that an electron can take is measured to be 2.28 MeV.)

[4 marks]

[Total 20 marks]

10. Write an essay about ONE of the following subjects:

(i) 100 years of superconductivity: the applications.

(ii) Design a school syllabus for physics that promotes appreciation and understanding of science and the scientific method.

(iii) Space elevator: orbital mechanics and material challenges.

(iv) Scientists have a duty to explain their work to the public. Discuss, with specific examples from physics.

(v) The role of particle accelerators in science, technology and medicine.

(vi) Large terrestrial optical telescopes: challenge and science.

[Total 20 marks]