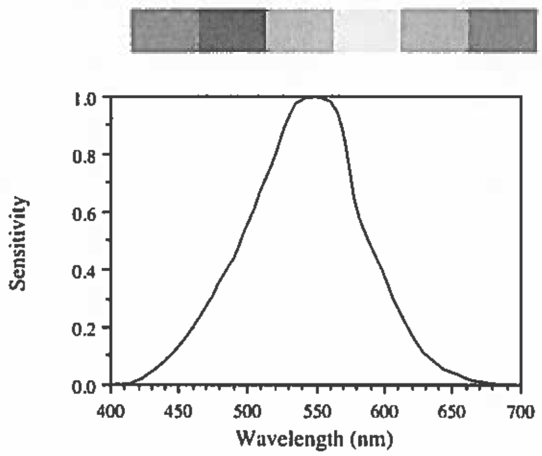
EE2-10B

Electromagnetic Fields 2016 – Solutions

1. a) The spectral range of sensitivity of the human eye is the so-called visible range, roughly 400 nm < λ < 700 nm. Wavelengths shorter than 400 nm fall into the ultraviolet regime, where there is large electronic absorption. Wavelengths longer than 700 nm fall into the near infrared regime. Although there is little absorption outside specific molecular absorption bands, the photons have too little energy to cause chemical changes that might be easily detectable.

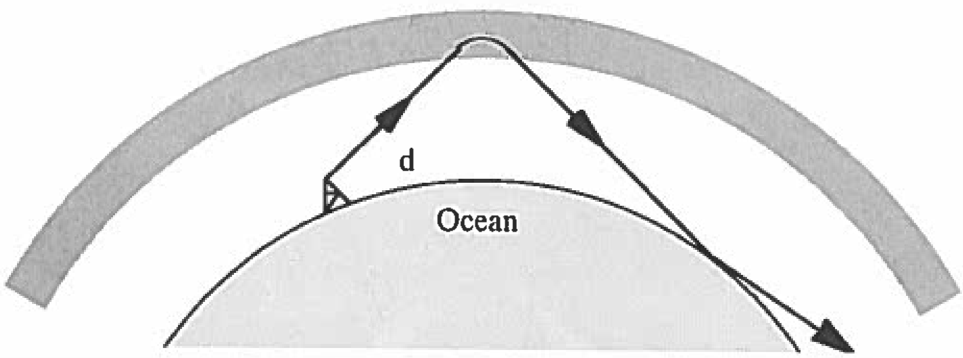
[4]



[4]

b) Marconi's achievement was to demonstrate transatlantic radio communication. Despite the lack of available components, he managed to modulated signals, and then de-modulate them using a simple envelope detector. He achieved over-the-horizon propagation of signals using the ionosphere and the ocean surface as weak reflectors. However, both are sensitive to weather conditions, rendering the communication method relatively unreliable.

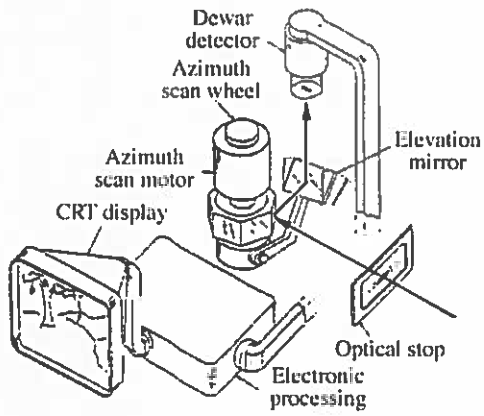
[4]



[4]

c) Night vision systems must detect electromagnetic waves emitted directly by an object, rather than waves from an external source that are scattered by the object. At normal temperatures, the emitted waves are infrared heat waves, which have relatively little energy. They can be detected using band-to-band transitions in a semiconductor with a small bandgap, such as cadmium mercury telluride. However, such transitions are easily generated by thermal energy in the detector so the system must be cryogenically cooled. It is difficult to make a pixellated array of CdHgTe detectors, and consequently the systems often operate by scanning the image over a single detector or a linear array.

[6]



[6]

d) The radar range equation gives the power received by a radar set with transmitter power P_T based on an antenna with efficiency η_T and effective area A_T from a target with cross section σ at a range r at wavelength λ. From the Friis transmission equation, the power intercepted by the target is:

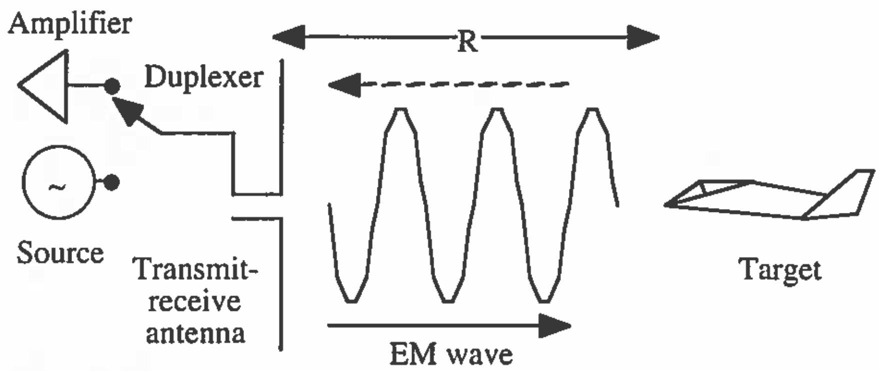
$P_{INT} = P_T(\eta_T A_T \sigma / R^2\lambda^2)$

Assuming the intercepted power is scattered as a spherical wave, the power received back at the transmit antenna is:

$P_R = P_{INT}\ (\eta_T A_T / 4\pi R^2) = P_T(\eta_T^2 A_T^2 \sigma / 4\pi R^4 \lambda^2)$

For a given detection limit P_R, the range then scales as $P_T^{1/4}$, so high transmitter powers are essential for long range.

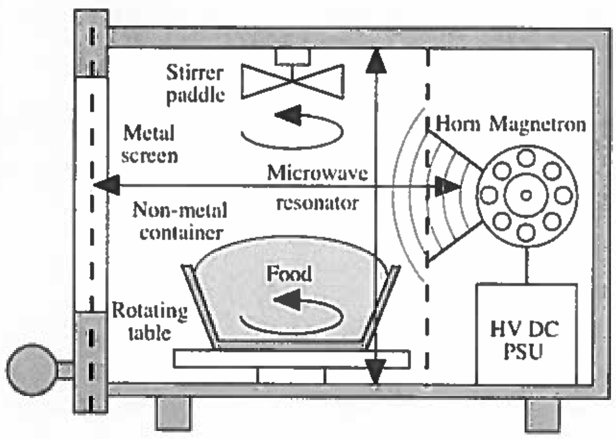
[6]



[2]

e) The microwave oven operates by absorption of microwave radiation in one of the molecular absorption bands of the water molecule. A cavity magnetron is used as a source of radiation at 2.45 GHz. Using a horn antenna, energy is coupled into a resonant metal enclosure, which acts as a cooking chamber. Wet food placed in the chamber will then heat up by absorption, but non-uniformly. Uniformity is improved by rotating the food on a table, and by 'stirring' the resonances using a rotating metal paddle.

[4]



[4]

2. a) The equations linking the voltages and currents V_n and I_n in the n^{th} section to the corresponding values in the $n+1^{th}$ section are:

$V_{n+1} = V_n - j\omega LI_n$

$I_{n+1} = I_n - j\omega CV_{n+1}$

[2]

Assuming wave solutions $V_n = V_0 \exp(-jnka)$ and $I_n = I_0 \exp(-jnka)$, where k is the propagation constant and a is the length of each section, we get:

$V_0 \exp\{-j(n+1)ka)\} = V_0 \exp(-jnka) - j\omega LI_0 \exp(-jnka)$

$I_0 \exp\{-j(n+1)ka)\} = I_0 \exp(-jnka) - j\omega CV_0 \exp\{-j(n+1)ka)\}$

Cancelling out terms exp(-jnka) on each side, we get:

$V_0 \exp(-jka) = V_0 - j\omega LI_0$

$I_0 \exp(-jka) = I_0 - j\omega CV_0 \exp(-jka)$

Re-arranging, we get:

$\{\exp(-jka) -1\}V_0 + j\omega LI_0 = 0$

$j\omega C \exp(-jka) V_0 + \{\exp(-jka) - 1\}I_0 = 0$

[4]

The result above is in the form $AV_0 + BI_0 = 0$, $CV_0 + DI_0 = 0$.

Consequently, there are only solutions with $V_0 \neq 0$ and $I_0 \neq 0$ if $AD - BC = 0$, i.e. if:

$\{\exp(-jka) -1\}\{\exp(-jka) - 1\} + \omega^2LC \exp(-jka) = 0$

$\exp(-jka) - 2 + \exp(+jka) + \omega^2LC = 0$

$\omega^2/\omega_0^2 = 2 - 2 \cos(ka) = 4 \sin^2(ka/2)$

Hence the dispersion relation is $\omega/\omega_0 = 2 \sin(ka/2)$

[4]

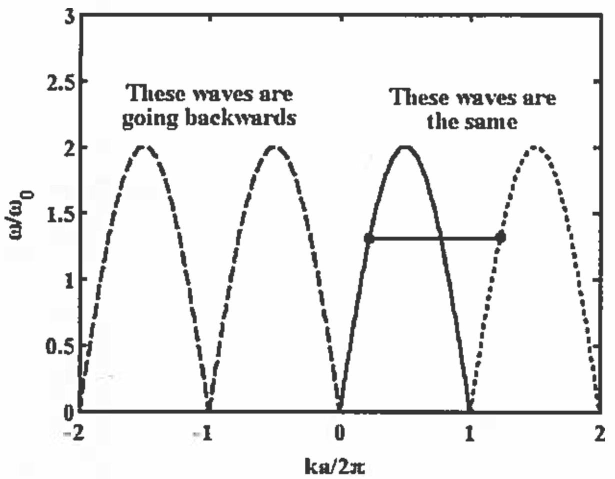
b) The dispersion relation is clearly sinusoidal. For positive ω, we can plot the dispersion characteristic as the modulus of the result above as shown below.

Waves with ka values in the range $0 \leq ka \leq +2\pi$ represent forward going waves.

Since $V_0 \exp\{-jn(ka + 2\pi)\} = V_0 \exp(-jnka) \exp(-jn2\pi) = V_0 \exp(-jnka)$, the waves in the dotted and full line sections cannot be distinguished from each other.

The waves in the dashed line sections represent backward-going waves.

[4]



[6]

c) The resonant frequency is $\omega_0 = 1/\sqrt{(LC)} = 1/\sqrt{(10 \times 10^{-9} \times 100 \times 10^{-12})} = 10^9$ rad/sec

The cutoff frequency is $\omega_c = 2\omega_0 = 2 \times 10^9$ rad/sec

The value above corresponds to a temporal frequency $f_c = \omega_c/2\pi = 318.3$ MHz

[2]

If $\omega/\omega_0 = 2 \sin(ka/2)$, ka can be found as $ka = 2 \sin^{-1}(\omega/2\omega_0)$

There is no real solution to $x = \sin^{-1}(y)$ if $y > 1$, the situation when $\omega > 2\omega_0$.

To resolve this, put $y = \sin(x)$ and substitute $x = \pi/2 - jz$.

Then $y = \sin(\pi/2) \cos(jz) + \cos(\pi/2) \sin(jz) = \cos(jz) = \cosh(z)$

Since $z = \cosh^{-1}(y)$ has a real solution for $y > 1$, $x = \pi/2 - j \cosh^{-1}(y)$

[4]

Consequently $ka = 2\{\pi/2 - j \cosh^{-1}(\omega/2\omega_0)\}$

Hence, k has a real and an imaginary part and can be written as $k = k' - jk''$.

Wave solutions then have the form $V_0 \exp(-jnka) = V_0 \exp(-jnk'a) \exp(-nk''a)$

Hence the wave decays as it propagates

[4]

3a) Ampere's law is:

$\iint_A \underline{\mathbf{D}} \cdot d\underline{a} = \iiint_V \rho \, dv$

Now, Gauss' theorem states that:

$\iint_A \underline{\mathbf{F}} \cdot d\underline{a} = \iiint_V \nabla \cdot \underline{\mathbf{F}} \, dv$

Applying this to Gauss' Law, we get:

$\iint_A \underline{\mathbf{D}} \cdot d\underline{a} = \iiint_V \nabla \cdot \mathbf{D} \, dv = \iiint_V \rho \, dv$

Since the volume is undefined, the last two integrands must be equal. Hence $\nabla \cdot \underline{\mathbf{D}} = \rho$

[3]

Similarly, Faraday's is:

$\int_L \underline{\mathbf{E}} \cdot d\underline{L} = - \iint_A \partial\underline{\mathbf{B}}/\partial t \cdot d\underline{a}$

Now, Stokes' theorem states that:

$\int_L \underline{\mathbf{F}} \cdot d\underline{L} = \iint_A (\nabla \times \underline{\mathbf{F}}) \cdot d\underline{a}$

Applying this to Faraday's Law, we get:

$\int_L \underline{\mathbf{E}} \cdot d\underline{L} = \iint_A (\nabla \times \mathbf{E}) \cdot da = - \iint_A \partial\underline{\mathbf{B}}/\partial t \cdot d\underline{a}$

Since the area is undefined, the last two integrands must be equal.

Hence $\nabla \times \mathbf{E} = - \partial\underline{\mathbf{B}}/\partial t$

[3]

b) i) The time-independent scalar wave equation is:

$\nabla^2\underline{E} = -\omega^2\mu_0\varepsilon_0\varepsilon_r \underline{E}$

Or:

$\partial^2\underline{E}/\partial x^2 + \partial^2\underline{E}/\partial y^2 + \partial^2\underline{E}/\partial z^2 = -\omega^2\mu_0\varepsilon_0\varepsilon_r \underline{E}$

Assuming the wave is polarized in the x-direction, $\underline{E} = E_x \underline{i}$ and

$\partial^2E_x/\partial x^2 + \partial^2E_x/\partial y^2 + \partial^2E_x/\partial z^2 = -\omega^2\mu_0\varepsilon_0\varepsilon_r E_x$

Assuming the field corresponds to an infinite plane wave propagating in the z-direction:

$\partial E_x/\partial x = \partial E_x/\partial y = 0$

E_x is then a function of z only, so

$d^2E_x/dz^2 + \omega^2\mu_0\varepsilon_0\varepsilon_r E_x = 0$

[3]

ii) If $E_x = E_0 \exp(-jk_0z)$

$dE_x/dz = -jk_0 E_x$

$d^2E_x/dz^2 = -k_0^2\,E_x$

Substituting into the wave equation, and assuming $\varepsilon_r = 1$ for free space,

$-k_0^2\,E_x + \omega^2\mu_0\varepsilon_0\,E_x = 0$

So the propagation constant is:

$k_0 = \omega\sqrt{(\mu_0\varepsilon_0)}$

[2]

The corresponding magnetic field can be found from the curl equation:

$\text{curl}(\underline{E}) = -j\omega\mu_0\underline{H}$

which gives

$\underline{H} = (-1/j\omega\mu_0)\,\text{curl}(\underline{E})$

In this case, we have:

$$\text{curl}(\underline{E}) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ E_x & 0 & 0 \end{vmatrix}$$

Or

$\text{curl}(\underline{E}) = \partial E_x/\partial z\,\underline{j} = -jk_0\,E_x\,\underline{j}$

The magnetic field is then:

$\underline{H} = (k_0/\omega\mu_0)\,E_x\,\underline{j} = (k_0/\omega\mu_0)\,E_0\,\exp(-jk_0z)\,\underline{j}$

[3]

If we write this as $\underline{H} = H_y\,\exp(-jk_0z)\,\underline{j}$ then

$H_y = (k_0/\omega\mu_0)\,E_x$

And the characteristic impedance of free space is:

$Z_0 = E_x/H_y = (\omega\mu_0/k_0) = \sqrt{(\mu_0/\varepsilon_0)}$

[2]

c) i) In the upper medium, there must be an incident and a reflected wave. In the lower medium, there must only be a transmitted wave. For TE incidence, both fields must be y-polarised. Hence:

$E_{y1} = E_I\,\exp[-jk_0n_1(z\,\sin(\theta_1) - x\,\cos(\theta_1))] + E_R\,\exp[-jk_0n_1(z\,\sin(\theta_1') + x\,\cos(\theta_1'))]$

$E_{y2} = E_T\,\exp[-jk_0n_2(z\,\sin(\theta_2) - x\,\cos(\theta_2))]$

[3]

ii) For the electric field, the boundary condition that must be satisfied is that $E_{y1} = E_{y2}$ on $x = 0$.

Hence:

$E_I \exp[-jk_0n_1z \sin(\theta_1)] + E_R \exp[-jk_0n_1z \sin(\theta_1')] = E_T \exp[-jk_0n_2z \sin(\theta_2)]$

[2]

iii) The only way this equation can be satisfied for all z is if:

$\exp[-jk_0n_1z \sin(\theta_1)] = \exp[-jk_0n_1z \sin(\theta_1')] = \exp[-jk_0n_2z \sin(\theta_2)]$

Hence

$\theta_1 = \theta_1'$ Alhazen's law

$n_1\sin(\theta_1) = n_2 \sin(\theta_2)$ Snell's law

[3]

d) Re-arranging Snell's law, we can obtain:

$\sin(\theta_2) = (n_1/n_2) \sin(\theta_1)$

Hence, $\cos(\theta_2) = \sqrt{\{1 - (n_1/n_2)^2 \sin^2(\theta_1)\}}$

When $(n_1/n_2)^2 \sin^2(\theta_1) > 1$ (i.e., beyond the critical angle) there is no real solution for θ_2

However, it is clear that $\cos(\theta_2)$ must be purely imaginary

In this case, we can write $\cos(\theta_2) = \pm j\alpha$

Taking the negative sign, the field in Medium 2 is

$E_{y2} = E_T \exp[-jk_0n_2(z \sin(\theta_2) - j\alpha x)]$ or

$E_{y2} = E_T \exp[-jk_0n_2z \sin(\theta_2)] \exp(k_0n_2\alpha x)]$

This field represents a wave travelling parallel to the boundary, whose amplitude decays exponentially as x becomes negative

[6]