New problem EE4-54 Predictive Control
SOLUTIONS

(a) $x := \begin{pmatrix} q \\ v \end{pmatrix}$ ~~[illegible]~~

(b) $C := \begin{pmatrix} H & 0 \end{pmatrix}$

(c) $e^X := \sum_{k=0}^{\infty} \frac{X^k}{k!}$

(d) $\frac{d}{dt} x(t) = \begin{pmatrix} F & G \\ 0 & 0 \end{pmatrix} \begin{pmatrix} q(t) \\ v(t) \end{pmatrix} + \begin{pmatrix} 0 \\ I_m \end{pmatrix} u_k, \forall t \in [kh, kh+h)$

But $\frac{d}{dt} u_k = 0 \quad \forall t \in [kh, kh+h)$

$\Rightarrow$ augment state $z = \begin{pmatrix} q \\ v \\ u_k \end{pmatrix} = \begin{pmatrix} x \\ u_k \end{pmatrix}$

$$\Rightarrow \frac{d}{dt} z = \underbrace{\begin{pmatrix} F & G & 0 \\ 0 & 0 & I_m \\ 0 & 0 & 0 \end{pmatrix}}_{\tilde{A}} \begin{pmatrix} q \\ v \\ u_k \end{pmatrix} \quad \forall t \in [kh, kh+h)$$

$$\Rightarrow z(t) = e^{\tilde{A}t} z(kh) \quad \forall t \in [kh, kh+h)$$

$$\Rightarrow \lim_{t \to kh+h} z(t) = \begin{pmatrix} A & B \\ 0 & I_m \end{pmatrix} \begin{pmatrix} x(kh) \\ u_k \end{pmatrix}$$ as in lectures

partition exponential accordingly to

$$\Rightarrow \begin{pmatrix} A & B \\ 0 & I_m \end{pmatrix} = e^{\tilde{A}h}$$

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(e) Let $\bar{x} := \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} \Rightarrow \bar{x} = \underbrace{\begin{pmatrix} A \\ A^2 \\ \vdots \\ A^{N} \end{pmatrix}}_{\Phi} x(0) + \underbrace{\begin{pmatrix} B & 0 & \cdots & 0 \\ AB & B & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ A^{N-1}B & A^{N-2}B & \cdots & B \end{pmatrix}}_{\Gamma} \bar{u}$

(Can derive above or just state)

$q=0$ and $v=0 \Leftrightarrow x=0$

$\Rightarrow x_N = 0 \Leftrightarrow A^N x(0) + \left(A^{N-1}B \;\; A^{N-2}B \cdots B\right)\bar{u} = 0$

$\Rightarrow A^N =: (P \;\; Q)$ and $R := \left(A^{N-1}B \;\; A^{N-2}B \;\cdots\; B\right)$

$P \in \mathbb{R}^{(n+m)\times n}$, $Q \in \mathbb{R}^{(n+m)\times m}$

$R \in \mathbb{R}^{(n+m)\times Nm}$

(f) Inequality constraints are equivalently written as

① $-\mathbb{1}_m \le v(t) \le \mathbb{1}_m$

② $-\mathbb{1}_m \le \frac{dv(t)}{dt} \le \mathbb{1}_m$

$\Big\} \; \forall t \in [0, Nh)$

where $\mathbb{1}_m := \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{R}^m$

Consider ② $\Leftrightarrow -\mathbb{1}_m \le u_k \le \mathbb{1}_m \;\; \forall k \in \{0, \dots, N-1\}$

rewrite $\begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \\ -u_0 \\ -u_1 \\ \vdots \\ -u_{N-1} \end{pmatrix} \le \begin{pmatrix} \mathbb{1}_m \\ \mathbb{1}_m \\ \vdots \\ \mathbb{1}_m \\ -\mathbb{1}_m \\ -\mathbb{1}_m \\ \vdots \\ -\mathbb{1}_m \end{pmatrix} \Leftrightarrow \begin{pmatrix} I_{mN} \\ -I_{mN} \end{pmatrix} \bar{u} \le \begin{pmatrix} \mathbb{1}_{mN} \\ -\mathbb{1}_{mN} \end{pmatrix}$

1 f cont...)

Consider ①. An important point to note is that

$v(kh+h) = v(kh)$ think $\left(\frac{d}{dt} v(t) = u_k\right)$

$\Rightarrow v(\cdot)$ is piecewise linear and continuous (integral of piecewise constant signal)

$\Rightarrow \|v(t)\|_\infty \leq 1 \;\; \forall t \in [0, Nh)$

is equivalent to $\|v(t)\|_\infty \leq 1 \;\; \forall t \in \{0, h, 2h, \ldots, kh\}$

$\Rightarrow$ Can turn infinite set of constraints to finite set of constraints.

now $v(kh) = v(0) + h \sum_{i=0}^{k-1} u_k, \; k = 1, \ldots, N$

$$\Rightarrow v(kh) = v(0) + (I_m \; I_m \cdots I_m \; 0 \cdots 0)\,\bar{u}$$

$$\bar{v} := \begin{pmatrix} v(h) \\ v(2h) \\ v(3h) \\ \vdots \\ v(Nh) \end{pmatrix} = \underbrace{\begin{pmatrix} I_m \\ I_m \\ \vdots \\ I_m \end{pmatrix}}_{J} v(0) + h \underbrace{\begin{pmatrix} I_m & 0 & & 0 \\ I_m & I_m & \ddots & \\ I_m & I_m & & \\ \vdots & & \ddots & 0 \\ I_m & & & I_m \end{pmatrix}}_{K} \bar{u}$$

As with the constraints ②, one can rewrite ① as

$$\begin{pmatrix} I_{mN} \\ -I_{mN} \end{pmatrix} \bar{v} \leq \begin{pmatrix} \mathbb{1}_{mN} \\ -\mathbb{1}_{mN} \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} I_{mN} \\ -I_{mN} \end{pmatrix} (J v(0) + K\bar{u}) \leq \begin{pmatrix} \mathbb{1}_{mN} \\ -\mathbb{1}_{mN} \end{pmatrix}$$

$$\Rightarrow L = \begin{pmatrix} I_{mN} \\ -I_{mN} \\ K \\ -K \end{pmatrix} \in \mathbb{R}^{4mN \times mN}, \quad M = \begin{pmatrix} \mathbb{1}_{mN} \\ -\mathbb{1}_{mN} \\ \mathbb{1}_{mN} - J \\ -\mathbb{1}_{mN} - J \end{pmatrix} \in \mathbb{R}^{4mN \times m}$$

$\longrightarrow$

Bookwork + New problems

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(a) $f(x) = b^Tb + x^TA^TAx - 2b^TAx$

$\Rightarrow \left(\frac{\partial f}{\partial x}\right)^T = 2A^TAx^* - 2A^Tb = 0 \Leftrightarrow A^TAx^* = A^Tb$ normal eqns.

Solve for $x^* \Rightarrow f^* = f(x^*)$

(b) Equivalent to $\min_{x,s} (s^T\mathbb{1})(\mathbb{1}^Ts) = (\mathbb{1}^Ts)^2$ s.t. $-s \le Ax - b \le s$

$\Leftrightarrow \min_{x,s} \begin{pmatrix} x \\ s \end{pmatrix}^T \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{I}_m \end{pmatrix} \begin{pmatrix} x \\ s \end{pmatrix}$ s.t. $Ax - s \le b$, $-Ax - s \le -b$

$\Leftrightarrow (x^*, s^*) = \arg\min_{(x,s)} \left\{ \begin{pmatrix} x \\ s \end{pmatrix}^T \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{I}_m \end{pmatrix} \begin{pmatrix} x \\ s \end{pmatrix} \right.$

$\mathbb{I} := \mathbb{1}_m \mathbb{1}_m^T$ note !! (not identity)

s.t. $\begin{pmatrix} A & -I_m \\ -A & -I_m \end{pmatrix} \begin{pmatrix} x \\ s \end{pmatrix} \le \begin{pmatrix} b \\ -b \end{pmatrix}$

$\mathbb{1}_m = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \in \mathbb{R}^m$

if $A \in \mathbb{R}^{m \times n}$. $\Rightarrow f(x^*) = f^*$.

(c) Equivalent to $\min_{x,t} t^2$ s.t. $-\mathbb{1}t \le Ax - b \le \mathbb{1}t$

note

$(x^*, t^*) = \arg\min_{(x,t)} \begin{pmatrix} x \\ t \end{pmatrix}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}$

s.t. $\begin{pmatrix} A & -\mathbb{1}_m \\ -A & -\mathbb{1}_m \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} \le \begin{pmatrix} b \\ -b \end{pmatrix}$

& if $A \in \mathbb{R}^{m \times n} \Rightarrow f(x^*) = f^*$.

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(d) Equivalent to $\min_{x,s,t} \mathbb{1}_m^T s + t$

$$\text{s.t.} \quad -s \leq Ax - b \leq s$$
$$Cx - d \leq \mathbb{1}_p t$$
$$0 \leq t$$

$$\Leftrightarrow \quad (x^*, s^*, t^*) := \arg\min_{(x,s,t)} \begin{pmatrix} 0 \\ \mathbb{1}_m \\ 1 \end{pmatrix}^T \begin{pmatrix} x \\ s \\ t \end{pmatrix}$$

$$\text{s.t.} \quad \begin{pmatrix} A & -I_m & 0 \\ -A & -I_m & 0 \\ C & 0 & -\mathbb{1}_p \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ s \\ t \end{pmatrix} \leq \begin{pmatrix} b \\ -b \\ d \\ 0 \end{pmatrix}$$

(if $C \in \mathbb{R}^{p \times n}$)

Solve for $x^* \Rightarrow f^* = f(x^*)$.

(e) Equivalent to $\min_{x,s} b^T b - 2b^T A x + x^T A^T A x + s^T s$

($C \in \mathbb{R}^{p \times n}$)

$$\text{s.t.} \quad Cx - d \leq s$$
$$0 \leq s$$

$$(x^*, s^*) = \arg\min_{\binom{x}{s}} \begin{pmatrix} x \\ s \end{pmatrix}^T \begin{pmatrix} A^T A & 0 \\ 0 & I_p \end{pmatrix} \begin{pmatrix} x \\ s \end{pmatrix} + \begin{pmatrix} -2A^T b \\ 0 \end{pmatrix}^T \begin{pmatrix} x \\ s \end{pmatrix}$$

$$\text{s.t.} \quad \begin{pmatrix} C & -I_p \\ 0 & -I_p \end{pmatrix} \begin{pmatrix} x \\ s \end{pmatrix} \leq \begin{pmatrix} d \\ 0 \end{pmatrix}$$

Solve for x^*
$\Rightarrow f(x^*) = f^*$.

New problem

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(a) The cost function contains terms that are 1-norm and ∞-norm, hence is not purely quadratic.

(b) $\theta := \begin{pmatrix} u_0 \\ \vdots \\ u_{N-1} \\ s_0 \\ \vdots \\ s_{N-1} \\ t \end{pmatrix}$ where $u_k \in \mathbb{R}^m$

$s_k \in \mathbb{R}^m$

$t \in \mathbb{R}$

(c) ~~Cost func~~ Problem equivalent to

$$\min_{\theta} \quad t + \|\bar{Q}(\Phi\hat{x} + \Gamma\bar{u})\|_2^2 + \mathbb{1}^T \bar{s}$$

where $\bar{Q} = \begin{pmatrix} I_N \otimes Q & 0 \\ 0 & 0 \end{pmatrix}$

note

① s.t. $-s_k \leq Ru_k \leq s_k \quad k = 0, 1, \dots, N-1$

② $-\mathbb{1}_n t \leq P x_N \leq \mathbb{1}_n t$

where $\bar{u} := \begin{pmatrix} u_0 \\ \vdots \\ u_{N-1} \end{pmatrix}, \quad \bar{s} := \begin{pmatrix} s_0 \\ \vdots \\ s_{N-1} \end{pmatrix}$

note

$x_N = \begin{pmatrix} 0 & I_n \end{pmatrix}(\Phi\hat{x} + \Gamma\bar{u})$ where $\Phi := \begin{pmatrix} I \\ A \\ \vdots \\ A^N \end{pmatrix}$

$\Gamma := \begin{pmatrix} 0 & 0 & \cdots & 0 \\ B & 0 & & 0 \\ AB & B & \ddots & 0 \\ \vdots & & & \\ A^{N-1}B & A^{N-2}B & \cdots & B \end{pmatrix}$

① $\Leftrightarrow$ $\begin{pmatrix} Ru_0 \\ Ru_1 \\ \vdots \\ Ru_{N-1} \\ -Ru_0 \\ -Ru_1 \\ \vdots \\ -Ru_{N-1} \end{pmatrix} - \begin{pmatrix} s_0 \\ s_1 \\ \vdots \\ s_{N-1} \\ s_0 \\ s_1 \\ \vdots \\ s_{N-1} \end{pmatrix} \leq 0 \Leftrightarrow \begin{pmatrix} \bar{R} \\ -\bar{R} \end{pmatrix}\bar{u} - \begin{pmatrix} I \\ I \end{pmatrix}\bar{s} \leq 0$

$\Leftrightarrow \begin{pmatrix} \bar{R} & -I_{mN} & 0 \\ -\bar{R} & -I_{mN} & 0 \end{pmatrix}\theta \leq 0$

where $\bar{R} := I_N \otimes R$

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$$\textcircled{2} \iff \begin{aligned} &(0 \quad P)(\Phi \hat{x} + \Gamma \bar{u}) - \mathbb{1}_n t \le 0 \\ -&(0 \quad P)(\Phi \hat{x} + \Gamma \bar{u}) - \mathbb{1}_n t \le 0 \end{aligned}$$

$$\iff \begin{pmatrix} (0 \quad P)\Gamma \\ -(0 \quad P)\Gamma \end{pmatrix} \bar{u} - \begin{pmatrix} \mathbb{1}_n \\ \mathbb{1}_n \end{pmatrix} t \le \begin{pmatrix} -(0 \quad P)\Phi \hat{x} \\ +(0 \quad P)\Phi \hat{x} \end{pmatrix}$$

$$\iff \begin{pmatrix} (PA^{N-1}B \quad PA^{N-2}B \cdots PB) & 0 & -\mathbb{1}_n \\ -(PA^{N-1}B \quad -PA^{N-2}B \cdots PB) & 0 & -\mathbb{1}_n \end{pmatrix} \theta \le \begin{pmatrix} -PA^N \hat{x} \\ +PA^N \hat{x} \end{pmatrix}$$

$$\Rightarrow M = \begin{pmatrix} \bar{R} & -I_{mN} & 0 \\ -\bar{R} & -I_{mN} & 0 \\ (0 \quad P)\Gamma & 0 & -\mathbb{1}_n \\ -(0 \quad P)\Gamma & 0 & -\mathbb{1}_n \end{pmatrix}$$

(d) M has $2mN + 2n$ rows ~~and~~
" $2mN + 1$ colums

(e) $$f = \begin{pmatrix} 0 \\ -PA^N \hat{x} \\ +PA^N \hat{x} \end{pmatrix}$$

(f) $$\text{Cost} = \bar{u}^T \Gamma^T \bar{Q}^T \bar{Q} \Gamma \bar{u} + \hat{x}^T \Phi^T \bar{Q}^T \bar{Q} \Phi \hat{x} + 2 \hat{x}^T \Phi^T \bar{Q}^T \Gamma \bar{u} + \underline{\mathbb{1}^T \xi} + t$$

$$\Rightarrow H := \begin{pmatrix} 2\Gamma^T \bar{Q}^T \bar{Q} \Gamma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

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(g) $c = \begin{pmatrix} 2\Gamma^T \bar{Q} \Phi \hat{x} \\ \mathbb{1}_{N_m} \\ 1 \end{pmatrix}$

4 Bookwork & research papers that were part of reading list

(a) When the horizon is long enough and the closed-loop trajectory is equal to the open-loop input trajectory

(b) ~~(other~~ When there are constraints or the dynamics are nonlinear, then it is impossible or computationally impractical to compute the explicit ~~solution~~ control law, ~~hence~~

(c) An interior-point solver ~~uses~~ formulates a sequence of unconstrained optimization problems, ~~th~~ for which the optimal point is always in the interior of the ~~set~~ feasible set.
An active set solver uses a candidate set of inequality constraints, which are formulated as equality constraints, to solve a problem whose solution is on ~~one of a subset of the const~~ the boundary of the feasible set.

(d) In RHC, the horizon is ~~co~~ the same at each sampling ~~instand~~ instance. In DHC the horizon decreases by after each sample instead

(e) In spacecraft docking or aircraft landing ~~the~~ If RHC were used then there is a chance the craft may never reach the destination, cf. Zeno's ~~parad~~ paradox.

(f) I would formulate the road geometry as ~~constraints the~~ soft constraints, whereas the cost would be penalising the constraint violations.

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(g) The control law is nonlinear if there are ~~cor~~ inequality constraints or ~~a~~ a non-standard ~~cy~~ (non-quadratic) cost is used. Hence, the closed-loop is nonlinear.

(h) I would model the disturbance as an integrator and ~~ag~~ augment the state of the system with the state of the integrator. I would then design an observer to estimate the state and disturbance. I would include the current estimate of the disturbance in my prediction model used in my optimal control problem that is solved at ~~ech~~ each sample instance.