

B.ENG. AND M.ENG. EXAMINATIONS 2013

PART I : MATHEMATICS 2 (ELECTRICAL ENGINEERING)

Date Friday 31st May 2013 10.00 - 12.00

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.

Answer Question 1 and THREE of the remaining five questions.

Answer Section A and Section B in different answerbooks.

Question 1 carries twice the marks of each of the other questions.

CALCULATORS MAY NOT BE USED.

A mathematical formulae sheet is provided.

[Before starting, please make sure that the paper is complete; there should be 4 pages, with a total of SIX questions. Ask the invigilator for a replacement if your copy is faulty.]

1. (i) Find x such that the vectors $\mathbf{a} = (1, 1, 1)$ and $\mathbf{b} = (x, 2, 3)$ are perpendicular.

- (ii) Let

$$A = \{x \in \mathbb{R} \mid -2 < x \leq 2\}$$

$$B = \{q \in \mathbb{Z} \mid q^2 < 10\}$$

Determine $A \cap B$ and $A \cup B$.

- (iii) Determine for which fixed $b \in \mathbb{R}$ the following propositions are true:

(a) $\exists x \in \mathbb{R} \quad \wedge \quad x^2 + x + b = 0$

(b) $\exists x \in \mathbb{C} \quad \wedge \quad x^2 + x + b = 0$

(iv) Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Determine the matrix B such that $AB = I$

- (v) Construct the truth table for

$$(P \oplus Q) \wedge (P \vee Q)$$

- (vi) Show that $f(x, y) = xe^{xy}$ satisfies the equation

$$\frac{\partial f}{\partial x} + \frac{\partial^2 f}{\partial y^2} - (x^2 + y)f = e^{xy}$$

- (vii) Suppose function $f(x)$ is even and periodic with period 2π . Show that its Fourier series contains cos-terms only.

- (viii) Determine t such that the point $P = (1, 2, t)$ belongs to the plane through the points $A = (1, 0, 0)$, $B = (0, 1, 0)$ and $C = (0, 0, 1)$.

- (ix) Determine the Taylor expansion to second order (without remainder term) about the point $(x_0, y_0) = (1, 0)$ of the function

$$f(x, y) = \ln(x^2 + y)$$

- (x) Calculate the angle between the vector $\mathbf{c} = (1, 1, 0)$ and the vector normal to the two vectors

$$\mathbf{a} = (1, -1, 0)$$

$$\mathbf{b} = (0, -1, 1).$$

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2. (i) Let $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5\}$.

Determine the sets $A \cap B$, $B - A$, $A \cup B$ and $P(B)$.

- (ii) Write down the truth tables for $P \Rightarrow Q$ and $P \wedge Q$.

- (iii) Determine the truth value of the following propositions:

(a) $(\forall x > 0 \wedge \forall y < 0) \Rightarrow xy < 0$

(b) $(\exists f : A \mapsto B \mid f \text{ is injective}) \wedge (|A| > |B|)$.

- (iv) Use proof by induction to show that

$$\sum_{k=0}^n 3^k = \frac{3^{n+1} - 1}{2}, \quad \forall n \in \mathbb{N}.$$

3. Consider the function $f : \mathbb{R}^2 \mapsto \mathbb{R}$ given by $f(x, y) = x^2y - y - \frac{1}{2}y^2$.

- (i) Find all stationary points.
(ii) Classify all the stationary points.
(iii) Sketch a contour plot of the function. (Neatness is essential)

4. (i) Let f and g be differentiable functions on \mathbb{R} . Show that

$$u(x, t) = f(x - vt) + g(x + vt),$$

where $v \in \mathbb{R}$ is a constant, solves the wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 u}{\partial t^2}$$

- (ii) Show that the function

$$f(x, y, t) = \frac{1}{t} e^{-\frac{x^2 + y^2}{4t}}$$

solves the heat equation

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

- (iii) Show that

$$(\sin x \sin y - x^2) \frac{dy}{dx} - (2xy + \cos x \cos y) = 0$$

is exact and derive an implicit expression for y as a function of x .

5. (i) A periodic function of period 4 is defined by

$$f(x) = 4 - x^2 \quad \text{for } -2 \leq x \leq 2$$

and $f(x+4) = f(x)$

Sketch the function over three periods.

- (ii) Find the Fourier expansion of $f(x)$.

- (iii) Use the result in (ii) to derive the formula

$$\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

6. Let

$$M = \begin{pmatrix} -2 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 6 \end{pmatrix}.$$

- (i) Write down the characteristic equation for M and show that the eigenvalues are equal to $\lambda = 1, -3, 7$.
- (ii) Find the eigenvectors corresponding to $\lambda = 1, -3, 7$.
- (iii) Express the vector $\mathbf{a} = \begin{pmatrix} 0 \\ 5 \\ -2 \end{pmatrix}$ in terms of the eigenvectors found in (ii) and use this to compute $M\mathbf{a}$.

END OF PAPER

EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2012-13

Course

EE 1

Part 2

Question	TOPIC	Marks & seen/unseen
1	General	
Parts		
(i)	$\underline{a} \perp \underline{b} \Leftrightarrow \underline{a} \cdot \underline{b} = 0 \Rightarrow x + 2 + 3 = 0$ $\Rightarrow x = -5$	4
(ii)	<p>Note $B = \{-3, -2, -1, 0, 1, 2, 3\}$, hence</p> $A \cap B = \{-1, 0, 1, 2\}$ $A \cup B = \{-3, 3, -2 \leq x \leq 2\}$	4
(iii)	<p>(a) To make the proposition true we need a <u>real</u> x that satisfy $x^2 + x + b = 0$, which can only be done if the roots are real, i.e. $D = 1 - 4b \geq 0$</p> $\Rightarrow \underline{\underline{b \leq \frac{1}{4}}}$ <p>(b) Since $\mathbb{C} \not\subset \mathbb{R}$ and since the polynomial always have either two complex or real roots this proposition is true for all values of $b \in \mathbb{R}$.</p>	2
(iv)	$\underline{\underline{B}} = \underline{\underline{A}}^{-1}, \det \underline{\underline{A}} = 1, \underline{\underline{A}}^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$	4
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EXAMINATION QUESTIONS/SOLUTIONS 2012-13

Course

EE1

Part 2

Question

1

TOPIC

General

Marks &

seen/unseen

Parts

(v)

P	Q	$P \oplus Q$	$P \vee Q$	$(P \oplus Q) \wedge (P \vee Q)$
T	T	F	T	F
T	F	T	T	T
F	T	T	T	T
F	F	F	F	F

(vi) $\partial_x f = e^{xy} + xy e^{xy} = e^{xy} + y f$

$\partial_y^2 f = x^2 e^{xy} = x^2 f$

\Downarrow

$\partial_x f + \partial_y^2 f = x^2 f + y f + e^{xy} \quad \text{OK}$

(vii) General form

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

where

$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx$

$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \, dx$

$= \frac{1}{2\pi} \int_{-2\pi}^{2\pi} f(x) \sin nx \, dx$

even odd

Since periodic
integrate over
two periods
and divided by 2

$= 0$

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	EXAMINATION QUESTIONS /SOLUTIONS 2012-13	Course EF1 Part 2
Question 1	TOPIC General	Marks & seen/unseen
Parts	<p>$P = (1, 2, t)$</p> <p>(viii) $A = (1, 0, 0), B = (0, 1, 0), C = (0, 0, 1)$</p> <p>want \underline{AP} to be perpendicular to a normal \underline{n} to the plane.</p> <p>Choose $\underline{n} = \underline{AB} \times \underline{AC}$ determine t such that $\underline{n} \cdot \underline{AP} = 0$:</p> <p>$\underline{AB} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \underline{AC} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \underline{AP} = \begin{pmatrix} 0 \\ 2 \\ t \end{pmatrix}$</p> <p>$\underline{n} = \underline{AB} \times \underline{AC} = \begin{pmatrix} i & j & k \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} = (1, 1, 1)$</p> <p>$0 = \underline{n} \cdot \underline{AP} = 2 + t \Rightarrow \underline{t = -2}$</p>	<p>Seen similar 2</p> <p>2</p>
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	EXAMINATION QUESTIONS /SOLUTIONS 2012-13	Course EE1 Part 2
Question 1	TOPIC General	Marks & seen/unseen
Parts (ix)	$f(x,y) = \ln(x^2+y) \quad , (x_0, y_0) = (1, 0)$ $\partial_x f = \frac{2x}{x^2+y} \rightarrow \partial_x f _{(1,0)} = 2$ $\partial_y f = \frac{1}{x^2+y} \rightarrow \partial_y f _{(1,0)} = 1$ $\partial_x^2 f = \frac{2(x^2+y) - 2x \cdot 2x}{(x^2+y)^2} \rightarrow \partial_x^2 f _{(1,0)} = -2$ $\partial_y^2 f = -\frac{1}{(x^2+y)^2} \rightarrow \partial_y^2 f _{(1,0)} = -1$ $\partial_{xy}^2 f = \frac{-2x}{(x^2+y)^2} \rightarrow \partial_{xy}^2 f _{(1,0)} = -2$ <p>Taylor expansion:</p> $f(x,y) = f(x_0, y_0) + \partial_x f (x-x_0) + \partial_y f (y-y_0)$ $+ \frac{1}{2} [\partial_x^2 f (x-x_0)^2 + 2\partial_{xy}^2 f (x-x_0)(y-y_0) + \partial_y^2 f (y-y_0)^2]$ $= 0 + 2(x-1) + y + \frac{1}{2} [-2(x-1)^2 + 2(-2)(x-1)y - y^2]$ $= (x-1)[3-4y-x] + y(1-\frac{1}{2}y)$ $= (x-1)[3-4y-x] + y(1-\frac{1}{2}y)$	<div style="text-align: center;">2</div> <div style="text-align: center;">2</div>
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EE1

Part 2

Question

1

TOPIC

General

Marks &

seen/unseen

Parts

(X)

The normal to the plane given by the vectors \underline{a} and \underline{b} is given by

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \end{vmatrix}$$

$$= (-1, -1, -1) \text{ or } \text{use normal } \underline{n} = (1, 1, 1)$$

$$\text{Angle} = \arccos \left(\frac{\underline{c} \cdot \underline{n}}{|\underline{c}| |\underline{n}|} \right)$$

$$= \arccos \left(\frac{2}{\sqrt{2} \sqrt{3}} \right) = \arccos \left(\frac{2}{\sqrt{6}} \right)$$

$$\approx \text{unavailable} \approx 26.1579$$

$$\approx 0.61548 \text{ rad} \approx 35.26^\circ$$

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EXAMINATION QUESTIONS/SOLUTIONS 2012-13

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EE1

Part 2

Question

2

TOPIC

Discrete Math

Marks &

seen/unseen

Parts

(i)

$$A \cap B = \{3, 4\} \quad (1)$$

$$B - A = \{5\} \quad (1)$$

$$A \cup B = \{1, 2, 3, 4, 5\} \quad (1)$$

$$P(B) = \{\emptyset, \{3\}, \{4\}, \{5\}, \{3, 4\}, \{3, 5\}, \{4, 5\}, \{3, 4, 5\}\} \quad (2)$$

(ii)

2

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

(iii)

seen similar

So (a) is true since $\forall x > 0 \wedge \forall y < 0$ is true when x is positive and y is negative in which case xy is negative and therefore $xy < 0$ true. (3)

2

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

(b) is false since if f is injective from A to B $|A| \leq |B|$. Hence we cannot have $\exists f: A \rightarrow B \mid f$ is injective and $|A| > |B|$ true at the same time (3)

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Part 2

Should really start with $n=1$ (but allow $n=0$) since $n \in \mathbb{N}$ in \mathbb{Q} .

Question

1

TOPIC

Discrete Maths

Marks &

seen/unseen

Parts

(i)(ii)

For $n=0$ we have Left hand side = 1
and Right hand side = $\frac{3-1}{2} = 1$

Induction step:

per assumption

$$\sum_{k=0}^{n+1} 3^k = \sum_{k=0}^n 3^k + 3^{n+1} = \frac{3^{n+1} - 1}{2} + 3^{n+1}$$

$$= \frac{3^{n+1} - 1 + 2 \cdot 3^{n+1}}{2} = \frac{3^{n+1}(1+2) - 1}{2}$$

$$= \frac{3^{n+2} - 1}{2}$$

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EE1

Part 2

Question

3

TOPIC

Partial Derivatives

Marks &

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Parts

(i)

Stationary points

$$\frac{\partial f}{\partial x} = 2xy = 0 \quad (a)$$

$$\frac{\partial f}{\partial y} = x^2 - 1 - y = 0 \quad (b)$$

from (a) $x = 0$ or $y = 0$

If $x = 0$ then (b) $\Rightarrow y = -1$

If $y = 0$ then (b) $\Rightarrow x = \pm 1$

I.e. three stationary points

$(0, -1)$, $(1, 0)$ and $(-1, 0)$

(ii)

Classification

$$\partial_x^2 f = 2y, \partial_y^2 f = -1, \partial_{xy}^2 f = 2x$$

$$\Delta = \partial_x^2 f \partial_y^2 f - [\partial_{xy}^2 f]^2 = -2y - 4x^2$$

Point	$\partial_x^2 f$	$\partial_y^2 f$	Δ	classification
$(0, -1)$	$-2 < 0$	-1	2	Max $f = \frac{1}{2}$
$(1, 0)$	0	-1	-4	saddle $f = 0$
$(-1, 0)$	0	-1	-4	saddle $f = 0$

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TOPIC

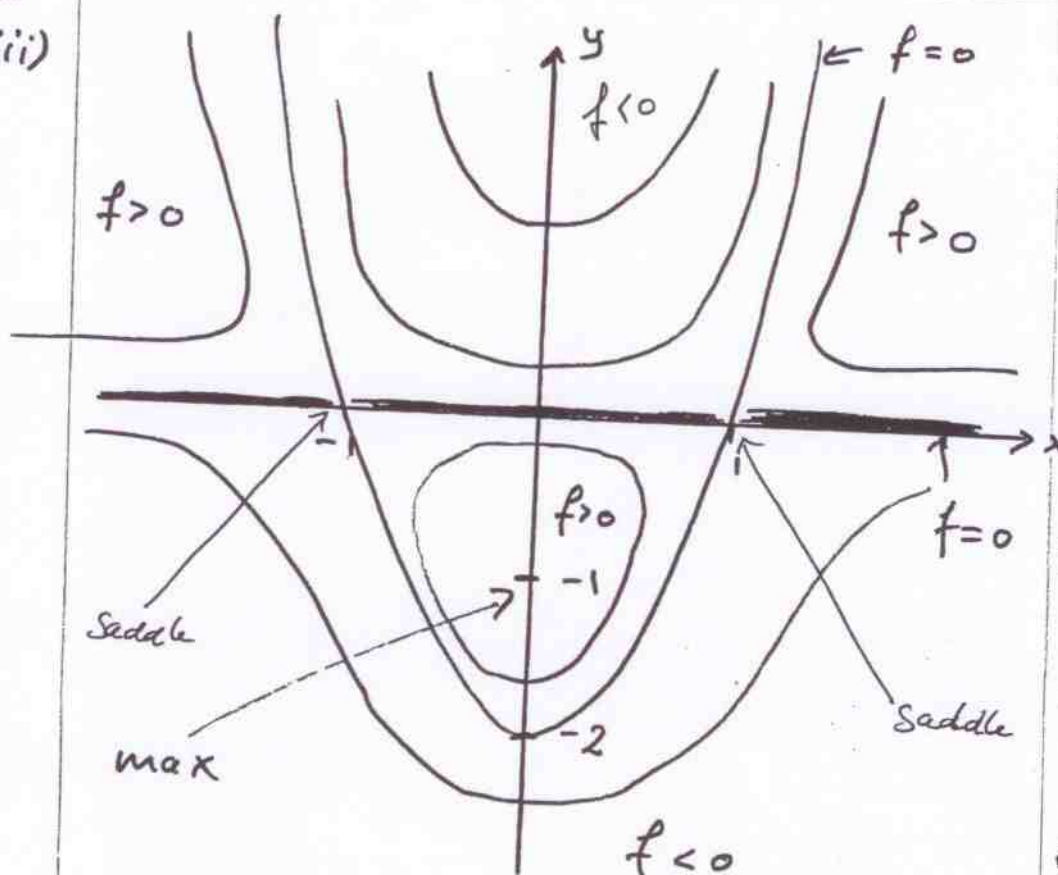
Partial Derivatives

Marks &

seen/unseen

Parts

(iii)



$$f(x, y) = x^2y - y - y^2/2 = 0$$

$$\Rightarrow y = 0 \quad \text{or if } y \neq 0$$

(1)

$$x^2 - 1 - \frac{y}{2} = 0$$

$$\Rightarrow y = 2x^2 - 2 \quad (1)$$

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4

TOPIC

Diff. Eq.

Marks &

seen/unseen

Parts

(i)

$$u(x,t) = f(x-vt) + g(x+vt)$$

$$\partial_x u = f'(x-vt) + g'(x+vt)$$

$$\partial_x^2 u = f''(x-vt) + g''(x+vt)$$

$$\partial_t u = -v f'(x-vt) + v g'(x+vt)$$

$$\partial_t^2 u = v f''(x-vt) + v^2 g''(x+vt)$$

hence

$$\frac{1}{v^2} \partial_t^2 u = f''(x-vt) + g''(x+vt) \\ = \partial_x^2 u$$

(ii) $f(x,y,t) = \frac{1}{t} e^{-\frac{x^2+y^2}{4t}}$

$$\partial_t f = \left[-\frac{1}{t^2} + \frac{1}{t} \left(\frac{x^2+y^2}{4t^2} \right) \right] e^{-\frac{x^2+y^2}{4t}} \\ = \left[-\frac{1}{t} + \frac{x^2+y^2}{4t^2} \right] f$$

$$\partial_x f = \frac{1}{t} \left(-\frac{2x}{4t} \right) e^{-\frac{x^2+y^2}{4t}} = -\frac{x}{2t} f$$

$$\partial_x^2 f = -\frac{1}{2t} f + \left(\frac{x}{2t} \right)^2 f \\ = \left(-\frac{1}{2t} + \frac{x^2}{4t^2} \right) f$$

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4

TOPIC

Differential Equation

Marks &

seen/unseen

Parts

(ii)
cont.

By symmetry

$$\partial_y^2 f = \left[-\frac{1}{2t} + \frac{y^2}{4t^2} \right] f$$

Therefore

$$\begin{aligned} (\partial_x^2 + \partial_y^2) f &= \left(-\frac{1}{t} + \frac{x^2 + y^2}{4t^2} \right) f \\ &= \partial_t f \end{aligned}$$

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EXAMINATION QUESTIONS/SOLUTIONS 2012-13

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EE1

Part 2

Question

4

TOPIC

ODE

Marks &

seen/unseen

Parts

(i)

$$(x+1) \frac{dy}{dx} - 3y = (x+1)^5$$

↓

$$\frac{dy}{dx} - \frac{3}{x+1} y = (x+1)^4$$

I.e. $P(x) = -\frac{3}{x+1}$ and $Q(x) = (x+1)^4$

hence

$$I(x) = \exp \left\{ - \int \frac{3}{x+1} dx \right\} = \exp \{ -3 \ln |x+1| \}$$

$$= (x+1)^{-3}$$

we multiply through and get

$$(x+1)^{-3} \frac{dy}{dx} = \frac{3}{(x+1)^4} y = (x+1)$$

$$\downarrow \frac{d}{dx} \left[\frac{y}{(x+1)^3} \right] = x+1 \Rightarrow \frac{y}{(x+1)^3} = \frac{1}{2} (x+1)^2 + C$$

$$\downarrow y = \frac{1}{2} (x+1)^5 + C(x+1)$$

(ii) $(\sin x \sin y - x^2) \frac{dy}{dx} - (2xy + \cos x \cos y) = 0$

$$P = -(2xy + \cos x \cos y)$$

$$Q = \sin x \sin y - x^2$$

Check:

$$\frac{\partial P}{\partial y} = -2x + \cos x \sin y$$

$$\frac{\partial Q}{\partial x} = \cos y \sin y - 2x$$

$$\text{i.e. } \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

hence exact

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(iii) (cont) Determine $u(x, y)$:

$$\frac{\partial u}{\partial x} = P = -2xy - \cos x \cos y$$

$$\Rightarrow u = -x^2y - \sin x \cos y + g(y)$$

$$\frac{\partial u}{\partial y} = Q = \sin x \sin y - x^2$$

$$\Rightarrow u = -\sin x \cos y - x^2y + f(x)$$

by comparison we conclude

$f(x) = g(y)$ for all x and y and

therefore

$$f(x) = g(y) = \text{constant} \equiv C$$

↓

$$u(x, y) = -x^2y - \sin x \cos y + C.$$

The general solution for y as function of x is given implicitly

by $u(x, y) = \text{constant}$ or

$$x^2y + \sin x \cos y = \text{constant}$$

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EXAMINATION QUESTIONS/SOLUTIONS 2012-13

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EE1

Part 2

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TOPIC

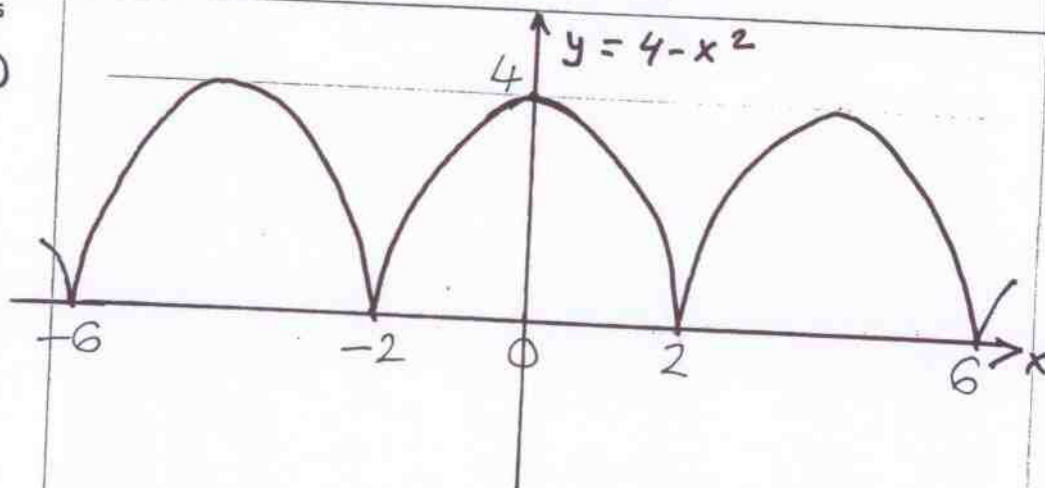
Fourier Series

Marks &

seen/unseen

Parts

(i)



③ for graph

$$(ii) f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx \quad \text{here}$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx \quad L = 2$$

All $b_n = 0$ since $f(x)$ is even

$$a_n = \frac{1}{2} \int_{-2}^2 (4 - x^2) \cos \frac{n\pi x}{2} dx$$

$$= 2 \int_{-2}^2 \cos \frac{n\pi x}{2} dx - \frac{1}{2} \int_{-2}^2 x^2 \cos \frac{n\pi x}{2} dx$$

Integration by parts

$$v = x^2, \quad v' = 2x$$

$$u' = \cos \frac{n\pi x}{2}, \quad u = \frac{2}{n\pi} \sin \frac{n\pi x}{2}$$

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EE1

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Question

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TOPIC

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Parts

(iii cont)

$$a_0 = 4 \int_0^2 dx - \int_0^2 x^2 dx = 8 - \left[\frac{x^3}{3} \right]_0^2 = \frac{16}{3}$$

For $n \neq 0$ we have

$$\begin{aligned} \int_{-2}^2 \cos \frac{n\pi x}{2} dx &= 2 \int_0^2 \cos \frac{n\pi x}{2} dx = \\ &= 2 \left[\frac{2}{n\pi} \sin \frac{n\pi x}{2} \right]_0^2 = \frac{4}{n\pi} \sin(n\pi) = 0 \end{aligned}$$

and

$$\begin{aligned} \int_{-2}^2 x^2 \cos \frac{n\pi x}{2} dx &= 2 \int_0^2 x^2 \cos \frac{n\pi x}{2} dx \\ &= 2 \left\{ \left[x^2 \frac{2}{n\pi} \sin \frac{n\pi x}{2} \right]_0^2 - \int_0^2 2x \frac{2}{n\pi} \sin \frac{n\pi x}{2} dx \right\} \\ &= -\frac{8}{n\pi} \int_0^2 x \sin \frac{n\pi x}{2} dx \\ &= -\frac{8}{n\pi} \left\{ \left[-x \frac{2}{n\pi} \cos \frac{n\pi x}{2} \right]_0^2 + \int_0^2 \frac{2}{n\pi} \cos \frac{n\pi x}{2} dx \right\} \\ &= \frac{16}{n^2 \pi^2} \left[x \cos \frac{n\pi x}{2} \right]_0^2 \quad \text{from above} \\ &= \frac{32}{n^2 \pi^2} \cos(n\pi) = \frac{32}{n^2 \pi^2} (-1)^n \end{aligned}$$

and then

$$a_n = -\frac{1}{2} \frac{32}{n^2 \pi^2} (-1)^n = \frac{16}{n^2 \pi^2} (-1)^{n+1}$$

So

$$f(x) = \frac{8}{3} + \frac{16}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos\left(\frac{n\pi x}{2}\right)$$

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Parts

(iii)

From $f(x) = 4$ we get

$$4 = \frac{8}{3} + \frac{16}{\pi^2} \left(1 - \frac{1}{2^2} + \dots\right)$$

\Downarrow

$$1 - \frac{1}{2^2} + \dots = \frac{4 \cdot 3 - 8}{3} \cdot \frac{\pi^2}{16} = \frac{4}{3} \cdot \frac{\pi^2}{16}$$

$$= \frac{\pi^2}{12}$$

Unseen

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TOPIC

Linear Algebra

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Parts

(i)

Characteristic eqn: $|M - \lambda I| = 0$

$$\begin{vmatrix} -2-\lambda & 2 & 1 \\ 2 & 1-\lambda & 2 \\ 1 & 2 & 6-\lambda \end{vmatrix} = 0$$

$$\Downarrow -(2+\lambda)(1-\lambda)(6-\lambda) + 4 + 4 - (1-\lambda) + 4(2+\lambda) - 4(6-\lambda) -$$

$$-(2+\lambda)(1-\lambda)(6-\lambda) - 9(1-\lambda) = 0$$

I.e. $\lambda = 1$ or $(2+\lambda)(6-\lambda) + 9 = 0$

$$\Downarrow \lambda^2 - 4\lambda + 21 = 0$$

$$\Downarrow \lambda = \frac{1}{2}[4 \pm \sqrt{16 + 84}]$$

factorizes

$$(\lambda - 7)(\lambda + 3) = \begin{cases} 7 \\ -3 \end{cases}$$

(ii)

Eigenvectors

$\lambda = 1$ ① $-2x + 2y + z = x$

② $2x + y + 2z = y \rightarrow x = -z$

③ $x + 2y + 6z = z$

hence from ① $-2x + 2y - x = x \Rightarrow y = 2x$

So an eigenvector is $\underline{v}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

$\lambda = -3$

① $-2x + 2y + z = -3x$

② $2x + y + 2z = -2y$

③ $x + 2y + 6z = -3z$

①a $x + 2y + z = 0$

②a $2x + 4y + 2z = 0$

③a $x + 2y + 9z = 0$

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(ii cont)

(1a) and (2a) are equivalent.

$$(2a) - 2(1a) \rightarrow -16z = 0 \Rightarrow z = 0.$$

Then from (1a) $x = -2y$

An eigenvector $\underline{u}_{-3} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$

$$\lambda = 7$$

$$(1) -2x + 2y + z = 7x$$

$$(1a) -9x + 2y + z = 0$$

$$(2) 2x + y + 2z = 7y$$

$$(2a) 2x - 6y + 2z = 0$$

$$(3) x + 2y + 6z = 7y$$

$$(3a) x + 2y - z = 0$$

$$(2a) + 2(3a) \Rightarrow 4x - 2y = 0 \Rightarrow x = \frac{1}{2}y$$

Sub into (1a) : $-\frac{9}{2}y + 2y + z = 0$

$$\Rightarrow y = \frac{2}{5}z$$

An eigenvector

$$\underline{u}_7 = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$$

(iii)

M symmetric \rightarrow Eigenvectors orthogonal. Normalise eigenvectors:

$$\underline{e}_1 = \frac{\underline{v}_1}{|\underline{v}_1|}, \underline{e}_2 = \frac{\underline{v}_3}{|\underline{v}_3|}, \underline{e}_3 = \frac{\underline{v}_2}{|\underline{v}_2|}$$

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9 for eigenvectors

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(i)(i)
cont.

$$|u_1| = \sqrt{1+4+1} = \sqrt{6} \rightarrow \underline{e}_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

$$|u_2| = \sqrt{4+1} = \sqrt{5} \rightarrow \underline{e}_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$$

$$|u_3| = \sqrt{1+4+25} = \sqrt{30} \rightarrow \underline{e}_3 = \frac{1}{\sqrt{30}} \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$$

Expand

$$\underline{a} = x \underline{e}_1 + y \underline{e}_2 + z \underline{e}_3$$

$$x = \underline{a} \cdot \underline{e}_1 = \frac{1}{\sqrt{6}} (10+2) = \frac{12}{\sqrt{6}}$$

$$y = \underline{a} \cdot \underline{e}_2 = \frac{1}{\sqrt{5}} (-5) = -\frac{5}{\sqrt{5}}$$

$$z = \underline{a} \cdot \underline{e}_3 = \frac{1}{\sqrt{30}} \cdot 0 = 0 \quad \text{i.e.} \quad \begin{pmatrix} 0 \\ 5 \\ -2 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$$

$$\underline{\underline{M}} \underline{\underline{a}} = \frac{12}{\sqrt{6}} \underline{\underline{M}} \underline{\underline{e}}_1 - \frac{5}{\sqrt{5}} \underline{\underline{M}} \underline{\underline{e}}_2$$

$$= \frac{12}{\sqrt{6}} \underline{e}_1 - \frac{5}{\sqrt{5}} (-3) \underline{e}_2$$

$$= \frac{12}{\sqrt{6}} \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \frac{15}{\sqrt{5}} \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix} + \begin{pmatrix} 6 \\ -3 \\ 0 \end{pmatrix} = \begin{pmatrix} 8 \\ 1 \\ -2 \end{pmatrix}$$

Setter's initials

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