UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 1999

BEng Honours Degree in Computing Part III
BSc Honours Degree in Mathematics and Computer Science Part III
MSci Honours Degree in Mathematics and Computer Science Part III
MSc Degree in Advanced Computing
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the
Diploma of Membership of Imperial College
Associateship of the Royal College of Science
Associateship of the City and Guilds of London Institute

PAPER 3.78

MATHEMATICAL STRUCTURES IN COMPUTER SCIENCE
Friday, May 14th 1999, 10.00 – 12.00

Answer THREE questions

For admin. only:
paper contains 4 questions

Section A *(Use a separate answer book for this Section)*

1a i) Given a poset (partially ordered set) P and a subset A of P, define the notions *lower bound, greatest lower bound*, of A. Prove that, if $x,y \in P$, then $x \leq y$ if and only if x is the greatest lower bound of $\{x,y\}$.

ii) State what it means for two posets to be isomorphic. Carefully define the (usual) partial order on the partial functions from a set A to a set B.

Let Q be the poset of partial functions from $\mathbb{N}$ to the one-element poset $\{1\}$. Show that $(\wp(\mathbb{N}), \subseteq)$ is isomorphic to Q but not isomorphic to Tapes(0,1) (that is, binary strings with the prefix order).

b i) Define the terms *lattice, sublattice* (notions to do with partial order need not be defined). State a criterion, in terms of "forbidden sublattices", for a lattice to be distributive.

ii) Let S be the usual three-dimensional space. Consider the poset P whose elements are the points, straight lines and planes contained in S, together with S itself and the empty set, the ordering being subset inclusion. Show that P is a lattice (just say what the meet and join are). Also, determine whether this lattice is distributive. (Hint: consider a suitable arrangement of three lines, all passing through a given point.)

2a i) Define the terms *signature, Σ-algebra* (for a signature Σ), *homomorphism* of Σ-algebras.

Let Σ be the signature {0,succ}, where 0, succ have arities 0,1 respectively. $\mathbb{N}$ is to be considered as a Σ-algebra in the usual way.

ii) Determine how many distinct (non-isomorphic) Σ-algebras there are which have exactly two elements.

iii) Carefully determine for which of these two-element Σ-algebras B there is a homomorphism from $\mathbb{N}$ *onto* B.

b A *right zero* of a monoid (M,e,*) is an element z such that, for all $x \in M$, $x*z = z$. *Left zero* is defined similarly. If z is both a right and a left zero, it is called a *zero.* Let R,F be the monoids of relations and of functions, under composition, on a given (non-empty) set X. (Take function composition as ; , so as to agree with that of relations.)

i) What are the units of the monoids R,F?

ii) Show that a monoid cannot have more than one zero. If a monoid has a left zero 0, and a right zero 0', can anything be concluded about whether 0 = 0' ?

iii) Show that R has a zero. Show also that, in F, every constant function is a right zero.

iv) Show that every right zero of F is a constant function.

Section B (Use a separate answer book for this Section)

3a Let X be a set, let (X*, [], ++) be the monoid of lists over X, and let η: X $\rightarrow$ X* be defined by letting η(x) be the singleton list [x].

i) What is meant by saying that X* (equipped with η) is the *free* monoid over X?

ii) Explain how the length function len: X* $\rightarrow$ **nat** can be defined using the free monoid property. Prove from your definition that

len(x:xs) = 1+len(xs)

b Let digit be the set {0,1,2,3,4,5,6,7,8,9}. The function val: digit* $\rightarrow$ **nat** is to be the usual conversion function – for instance, val([5,7]) = 57, val([0,0,7]) = 7; also val([]) = 0. It can be characterized using a recursion equation

val(ds++[d]) = 10*val(ds) + d (*)

i) Write down an expression to show how val(ds1++ds2) can be calculated from the results of applying val and len to ds1 and ds2.

ii) Let α: digit* $\rightarrow$ **nat**$\times$**nat** be the paired function $\langle$val, len$\rangle$. Define a binary operation $\cdot$ on **nat**$\times$**nat** for which α(ds1++ds2) = α(ds1)$\cdot\alpha$(ds2), and prove that this equation does indeed hold. Show that, with an appropriate unit element, the operation makes **nat**$\times$**nat** a monoid.

iii) Define α using the free monoid property of digit*.

iv) Starting from your definition of α in (iii), let $\alpha = \langle\alpha_1,\alpha_2\rangle$ (i.e. α_1 = α;fst and α_2 = α;snd). Show from your definitions that α_1 satisfies the recursion equation (*) for val, and that α_2 = len.

The two parts carry, respectively, 30%, 70% of the marks.

Turn over ...

4a Let $\mathcal{C}$ be a category.

i) If X and Y are objects of $\mathcal{C}$, what is meant by a *product* $X \times Y$ in $\mathcal{C}$?

ii) What is a *terminal object* (or *final object* or *nullary product*) in $\mathcal{C}$?

iii) Describe products and terminal objects when $\mathcal{C}$ is the category **Sets** of sets (with functions as morphisms) and when $\mathcal{C}$ is the opposite category $\mathbf{Sets}^{op}$.

b Let $\mathcal{C}$ be a category in which every pair X, Y of objects has a product $X \times Y$.

i) If U, X and Y are objects in $\mathcal{C}$, show that every morphism from U to $X \times Y$ can be written uniquely in the form $\langle x, y \rangle$. How does $Id_{X \times Y}$ appear in this form, in the case when $U = X \times Y$?

ii) Show how to define, for every X, Y, a morphism $\sigma_{XY}: X \times Y \rightarrow Y \times X$ such that for every U, x and y, $\sigma_{XY} \circ \langle x,y \rangle = \langle y,x \rangle$. Prove that your σ_{XY} has the required property. Deduce that σ_{XY} is an isomorphism with $\sigma_{XY}^{-1} = \sigma_{YX}$. (*Hint:* calculate $\sigma_{XY} \circ Id_{X \times Y}$.)

c Give an example of a single category $\mathcal{C}$ that has all of the following four properties and explain why the properties hold:

- $\mathcal{C}$ has a product $X \times Y$ for every pair of objects X, Y.
- For every object X, the diagonal morphism $\Delta: X \rightarrow X \times X$ is an isomorphism.
- $\mathcal{C}$ has at least one object.
- $\mathcal{C}$ has no terminal object.

The three parts carry, respectively, 35%, 40%, 25% of the marks.

End of paper