EE4-26
EE9-CS5-3
EE9-FPN2-07

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2018

MSc and EEE PART IV: MEng and ACGI

Corrected copy

ESTIMATION AND FAULT DETECTION

Wednesday, 9 May 10:00 am

Time allowed: 3:00 hours

There are FOUR questions on this paper.

Answer ALL questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s) :	T. Parisini
	Second Marker(s) :	D. Angeli

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ESTIMATION AND FAULT DETECTION

Information for candidates:

- **One-step ahead Kalman predictor:**

$$\hat{x}(t+1|t) = F\hat{x}(t|t-1) + K(t)[y(t) - H\hat{x}(t|t-1)]$$

- **Kalman predictor gain**

$$K(t) = FP(t)H^\top \left(V_2 + HP(t)H^\top\right)^{-1}, \quad t = 1,2,\ldots$$

- **Riccati equation**

$$P(t+1) = F\left[P(t) - P(t)H^\top \left(V_2 + HP(t)H^\top\right)^{-1} HP(t)\right]F^\top + V_1, \quad t = 1,2,\ldots$$

- **Realization: observer canonical form**

Given

$$Y(s)/U(s) = \frac{b_m s^m + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots a_1 s + a_0} \quad \text{with} \quad m < n$$

then:

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & -a_{n-2} \\ 0 & \cdots & 0 & 1 & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_m \\ 0 \\ \vdots \\ 0 \end{bmatrix} u \\ y = [0 \cdots 0\,1] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} \end{cases}$$

- **Generic Structure of an Observer**

$$\begin{cases} \dot{z} = Fz + TBu + Ky \\ \hat{x} = z + Hy \end{cases}$$

- **Basic Conditions for UIOs**

$$(I - HC)N = 0\,, \quad N: \text{disturbance distribution matrix;}$$

$$F = A - HCA - K_1C\,;\; K = K_1 + FH\,;\; T = I - HC$$

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1. Consider the mechanical system drawn in Fig. 1.1.

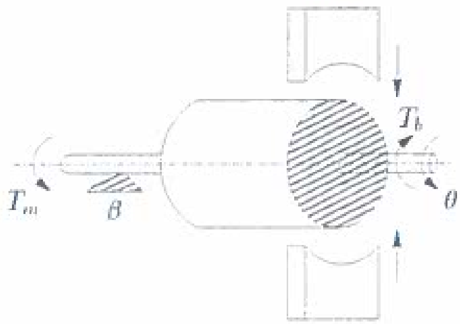

Figure 1.1 Scheme of the mechanical system of Question 1.

The mechanical shaft is connected to the cylinder with a known total inertia moment J and the rotation is imposed by the known motor torque T_m. The angular position is denoted by θ and is perfectly measurable by an ideal encoder. The linear viscous friction torque is characterised by a known friction parameter β. At time $t = t_0$, the braking system is activated generating a *constant and unknown* braking torque T_b.

a) Determine a state-space representation of the system valid for $t < t_0$ (that is, before the action of the braking system) supposing that the input is given by $u(t) = T_m(t)$ and the output is given by $y(t) = \theta(t)$.

[4 marks]

b) For all $t < t_0$, discuss the observability properties from the output $y(t)$.

[3 marks]

c) Consider the system for $t \geq t_0$ after the action of the braking system, that is consider the effect of the unknown and constant braking torque T_b. Establish whether it is possible to devise an observer scheme to estimate *simultaneously* the braking torque T_b and the state $x(t)$ associated with the state-space description derived in your answer to Question 1-a).

Hint: do not attempt to design such an observer. **[6 marks]**

d) In case the observer design of Question 1-c) is possible, denote by $\widehat{x}(t)$ the estimate of the state $x(t)$ associated with the state-space description derived in your answer to Question 1-a) and denote by $\widehat{T}_b(t)$ the estimate of the braking torque T_b described in Question 1-c). Let $e(t) := [(x(t) - \widehat{x}(t))^\top, T_b(t) - \widehat{T}_b(t)]^\top$ denote the total estimation error; moreover, suppose that its dynamics obeys

$$\dot{e}(t) = Fe(t).$$

Denote by $\lambda_i, i = 1, \ldots, n$ the eigenvalues of F, where n is the order of the observer scheme devised in your answer to Question 1-c). Suppose that the values of the mechanical parameters are given by: $J = 10$, $\beta = 1$. Design an n-order state observer such that

$$\lambda_i = -1, i = 1, \ldots, n.$$

[7 marks]

2. Consider the discrete-time dynamic system described in Fig. 2.1

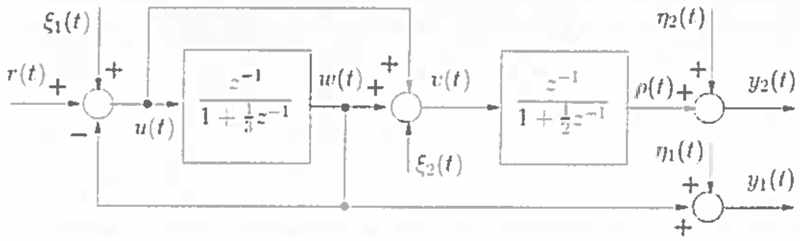

Figure 2.1 Block diagram for Question 2.

where $r(t)$ is a deterministic input variable, $\xi_1(\cdot) \sim WGN(0,1)$, $\xi_2(\cdot) \sim WGN(0,1)$, $\eta_1(\cdot) \sim WGN(0,9)$, $\eta_2(\cdot) \sim WGN(0.4)$ (Gaussian zero-mean stochastic processes) and the stochastic processes $\xi_1(\cdot)$, $\xi_2(\cdot)$, $\eta_1(\cdot)$, $\eta_2(\cdot)$ are supposed to be independent of each other.

a) Referring to the system sketched in Fig. 2.1, determine a stochastic state-space representation of the system supposing that the deterministic input is given by $r(t)$ and the output by $y(t) := [y_1(t), y_2(t)]^\top \in \Re^2$, respectively.

[4 Marks]

b) Consider the one-step ahead optimal steady-state Kalman predictor of the state x for the state-space description in Question 2-a). Using the asymptotic theory of steady-state Kalman estimation, show that the Algebraic Riccati Equation admits a feasible matrix solution $\bar{P}$ and compute it. Compute the corresponding gain vector $\bar{K}$ and write the difference equation yielding the steady-state Kalman prediction $\hat{x}(t|t-1)$.

[5 Marks]

c) Compute the covariance matrix of the prediction error, $\mathrm{Cov}[x(t)-\hat{x}(t|t-1)]$, and the covariance matrix of the process, $\mathrm{Cov}[x(t)]$. Compare $\mathrm{Cov}[x(t)-\hat{x}(t|t-1)]$ with $\mathrm{Cov}[x(t)]$ and comment on your findings.

[5 Marks]

d) Consider the optimal steady-state Kalman filter yielding $\hat{x}(t|t)$. From your answer to Question 2-b), compute the constant gain vector $\bar{K}_0$ of the optimal steady-state Kalman filter. Compute the covariance matrix of the filtering error, $\mathrm{Cov}[x(t)-\hat{x}(t|t)]$ and compare it with $\mathrm{Cov}[x(t)-\hat{x}(t|t-1)]$ and $\mathrm{Cov}[x(t)]$ computed in your answer to Question 2-c). Comment on your findings.

[6 Marks]

3. Consider the continuous-time dynamic system depicted in Fig. 3.1.

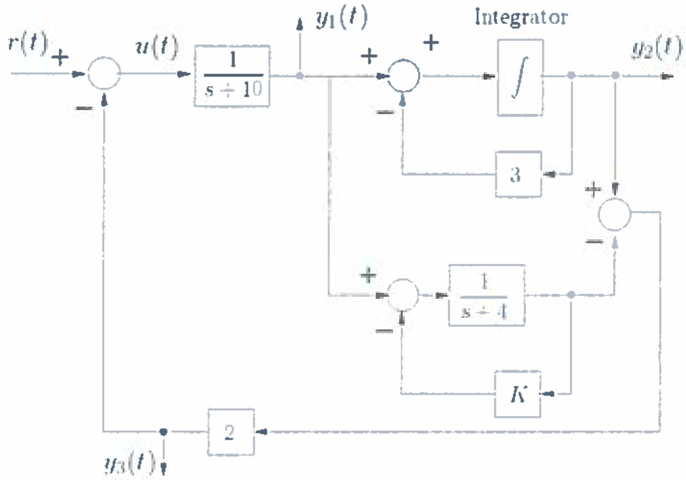

Figure 3.1 Block diagram for Question 3.

The system depicted in Fig. 3.1 has one input variable $r(t)$, three output variables $y_1(t)$, $y_2(t)$, and $y_3(t)$ and it also comprises transfer functions, an integrator, and several "gain" blocks (their output is given their input multiplied by the scalar shown inside the block). Moreover, $K \in \Re$ is a parameter.

a) Determine the transfer functions $G_1(s)$, $G_2(s)$, and $G_3(s)$ from the input $r(t)$ and the outputs $y_1(t)$, $y_2(t)$, and $y_3(t)$, respectively.

[5 marks]

b) Determine a state-space representation of the system in Fig. 3.1 supposing that the input variable is $r(t)$ and the output is given by $y(t) := [y_1(t), y_2(t), y_3(t)]^\top \in \Re^3$.

[3 marks]

c) Analyse the observability of the whole system in Fig. 3.1 from each output $y_i(t)$, $i = 1,2,3$ *taken separately* as a function of the parameter $K \in \Re$. Comment on your findings taking also into account your answer to Question 3-a).

[5 marks]

d) Consider the value of K determined in your answer to Question 3-c) making the system in Fig. 3.1 not observable from one of the outputs. Select this output variable and, *disregarding the other output variables*, for the above value of K determine a state-space description which is equivalent to that determined in your answer to Question 3-b) and in which the observable and the non-observable sub-systems are clearly identified.

[7 marks]

4. Consider the continuous-time dynamic system described by the following state equations:

$$\begin{cases} \dot{x}_1 = -3x_1 + x_2 + u + d \\ \dot{x}_2 = x_1 - 4x_2 + f \\ y_1 = x_1 + x_2 \\ y_2 = 2x_1 + x_2 \end{cases} \tag{4.1}$$

where $x = [x_1, x_2]^\top$ is the state vector, $y = [y_1, y_2]^\top$ is the output vector, u is a *known* input, d is an *unknown* disturbance, and f denotes a fault affecting the second state equation for $t \geq T_0$, where T_0 denotes the *unknown* time of fault occurrence. Suppose that $f(t) = 0, \forall t < T_0$ and assume that only one fault may occur during the whole time-horizon $t \in (0, \infty)$.

a) Consider the system (4.1) and suppose that no disturbance $d(t)$ nor fault $f(t)$ act on the system at any time, that is, suppose that $d(t) = 0, \forall t$ and $f(t) = 0, \forall t$. Denoting by $\hat{x}(t)$ an estimate of the state $x(t)$ of the system (4.1), let $e(t) = x(t) - \hat{x}(t)$ denote the state estimation error and suppose that its dynamics obeys

$$\dot{e}(t) = Fe(t),$$

where λ_1, λ_2 are the eigenvalues of F. Design a full-order state observer such that $\lambda_1 = -1, \lambda_2 = -2$. *Hint: the design of the observer matrix gain admits more than one solution. It is suggested to suitably restrict the number of free parameters of the observer gain.* **[5 marks]**

b) Suppose that no fault $f(t)$ acts on the system at any time, that is, suppose that $f(t) = 0, \forall t$. Using the state observer designed in your answer to Question 4-a), consider the time behaviour of the output residual

$$\varepsilon(t) = Ce(t), \ \forall t \geq 0$$

for a given value $\bar{e}_0$ of the initial estimation error $e(0)$ and assume that the disturbance is given by

$$d(t) = 0.1 \sin(t) \cdot 1(t),$$

where $1(t)$ denotes the unit step function. Determine the expression of the Laplace transform $\mathscr{L}[\varepsilon(t)]$. **[5 marks]**

c) Consider the system (4.1) for any $t \geq 0$ under the action of the disturbance considered in Question 4-b), that is, $d(t) = 0.1 \sin(t) \cdot 1(t)$ and also suppose that a fault f occurs at time $T_0 = 20$ sec taking on the following form:

$$f(t) = 2 \cdot 1(t - 20).$$

Design an *Unknown Input Observer* (UIO) such that the output residual $\tilde{\varepsilon}(t)$ is insensitive to the disturbance $d(t)$ but instead is sensitive to the fault f and where the eigenvalues characterising the dynamics of the estimation error are the same as in Question 4-a), that is $\lambda_1 = -1, \lambda_2 = -2$. *Hint: Similarly to Question 4-a), the design of the observer matrix gain admits more than one solution. It is suggested to suitably restrict the number of free parameters of the observer gain.* **[6 marks]**

d) Determine the expression of the Laplace transform $\mathscr{L}[\tilde{\varepsilon}(t)]$ when the UIO is used and compare it with the expression of the Laplace transform $\mathscr{L}[\varepsilon(t)]$ determined in your answer to Question 4-b). From this comparison, characterise the qualitative time-behaviour of $\varepsilon(t)$ and $\tilde{\varepsilon}(t)$ before the occurrence of the fault, that is, for $t < 20$, and comment on your findings in terms of the design of a fault detection scheme using the observers designed in your answers to Question 4-b) and Question 4-c), respectively. **[4 marks]**