EE4-25
EE9CS1-2

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2011

MSc and EEE/ISE PART IV: MEng and ACGI

DESIGN OF LINEAR MULTIVARIABLE CONTROL SYSTEMS

Thursday, 12 May 10:00 am

Time allowed: 3:00 hours

There are SIX questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : I.M. Jaimoukha
Second Marker(s) : E.C. Kerrigan

1. Let the transfer matrix $G(s)$ have a state space realisation

$$G(s) \stackrel{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

and let

$$AP + PA^T + BB^T = 0$$

for some $P = P^T$.

Suppose that

$$P = \begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix}$$

where $P_1 = P_1^T \succ 0$.

a) Prove that the state space realisation for the transfer matrix $G(s)$ is uncontrollable. [4]

b) Derive a state space realisation for the controllable part. [4]

c) Prove that the controllable part is stable. [4]

d) State a condition on the uncontrollable part that guarantees that the realisation of $G(s)$ is stabilisable. [4]

e) Draw a diagram involving two subsystems of $G(s)$ illustrating the controllable and uncontrollable parts. [4]

Hint: You may want to partition the realisation of $G(s)$ compatibly with the partitioning of P.

2. Suppose that a state-space realisation of a transfer matrix $G(s)$ has the structure

$$G \stackrel{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cc|c} A_{11} & 0_2 & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & 0_2 & 0_2 \end{array}\right]$$

where

- A_{11} is stable and diagonal with diagonal elements a_1 and a_2.
- B_1 and C_1 are square and orthogonal.
- A_{22} is diagonal with diagonal elements a_3 and a_4.
- 0_2 denotes the 2×2 matrix of zeros.

a) Use the PBH test to show that the realisation of $G(s)$ is unobservable. What are the unobservable modes? [4]

b) Find the output decoupling zeros of $G(s)$. [3]

c) Give necessary and sufficient conditions for the unobservable modes to be detectable. [3]

d) By removing the unobservable modes, obtain a second order realisation of $G(s)$. [3]

e) For the second order realisation of $G(s)$ in Part (d):

i) Find the controllability and observability Grammians . [2]

ii) Show that the realization is balanced. [1]

iii) Evaluate the Hankel singular values. [1]

f) Write B_1 and C_1 as

$$B_1 = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \qquad C_1 = \begin{bmatrix} c_1 & c_2 \end{bmatrix}.$$

Suppose that $|a_1| > |a_2|$. Obtain a first order balanced truncation of $G(s)$. [3]

3. Consider the regulator in Figure 3.1 for which it is assumed that (A,B) is controllable and $x(0)=x_0$. A stabilizing state-feedback gain matrix F is to be designed such that the cost function $J := \int_0^\infty (u(t)'Ru(t) + z(t)'z(t))\,dt$ is minimized, where (A,C) is assumed to be observable.

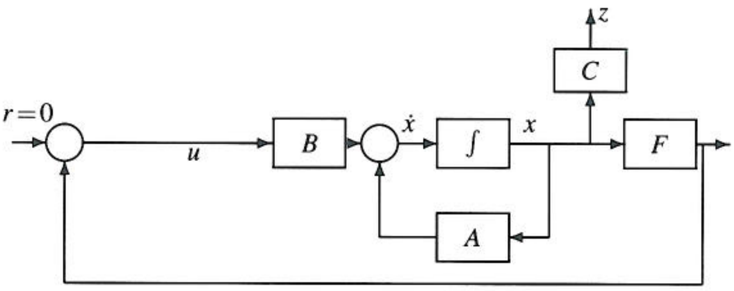

Figure 3.1

Let $V(t)=x(t)'Px(t)$ where $P=P'$ is the solution of an algebraic Riccati equation.

a) Assuming the closed loop is asymptotically stable, obtain an expression for $\int_0^\infty \dot{V}(t)dt$ in terms of x_0. [2]

b) Find an expression for F that minimizes J. Give also the minimum value of J and the algebraic Riccati equation satisfied by P. [4]

c) Prove that, for F chosen in part (b), the closed loop system in Figure 3.1 is stable. State clearly the assumption on P required to guarantee stability. [6]

d) Assume that $R = I$ and let $G(s) = (sI - A)^{-1}B$ and define $L(s) = I - FG(s)$. Using the algebraic Riccati equation show that

$$L(j\omega)'L(j\omega) = I + G(j\omega)'G(j\omega)$$

[4]

e) Let $G(s)$ be as defined in Part (d) and suppose that F chosen in Part (b) is given by $F = \begin{bmatrix} 1 & 1 \end{bmatrix}$. Using the answers to Parts (a)-(d) derive a robustness interpretation in terms of Figure 3.2. State clearly the assumptions needed on $\Delta(s)$. [4]

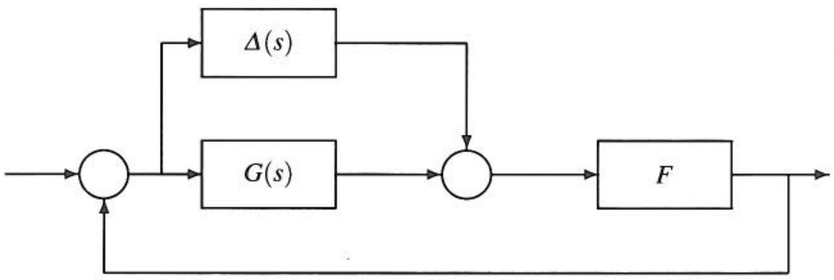

Figure 3.2

4. Consider the feedback configuration in Figure 4. Here, $G(s)$ is a nominal plant model and $K(s)$ is a compensator. The stable transfer matrices $\Delta_1(s)$ and $\Delta_2(s)$ represent uncertainties.

The design specification are to synthesize a compensator $K(s)$ such that the feedback loop is internally stable when:

(i) $\Delta_1 = 0$ and $\|\Delta_2\|_\infty < 0.5$, and,

(ii) $\Delta_2 = 0$ and $\|\Delta_1\|_\infty < 1$.

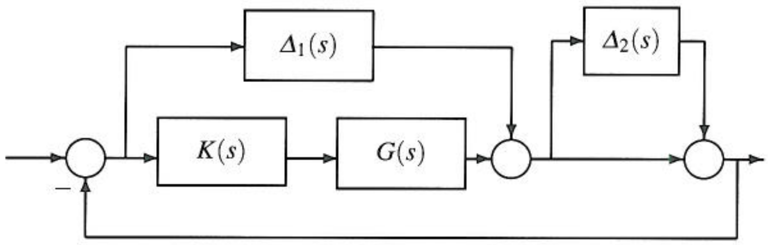

Figure 4

a) Derive $\mathscr{H}_\infty$–norm bounds, in terms of $G(s)$, $K(s)$, and two suitable weighting functions $W_1(s)$ and $W_2(s)$, that are sufficient to achieve the design specifications. [5]

b) Define suitable cost signals $z_1(s)$ and $z_2(s)$, external signal $w(s)$, measured signal $y(s)$ and control signal $u(s)$ and draw a block diagram, showing all these signals, as well as $W_1(s)$ and $W_2(s)$, that represents the design requirements in Part (a). [5]

c) Suppose that $G(s)$ is stable. Derive a parameterization of all internally stabilizing controllers for the loop in Figure 4 when $\Delta_1 = 0$ and $\Delta_2 = 0$. [5]

d) Let $G(s) = 1/(s+1)$. Use the answers to Parts (a) and (c) to find an internally stabilizing controller $K(s)$ that achieves the design specifications. [5]

5. a) Consider a state–variable model described by the dynamics

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t), \qquad x(0) = 0,$$

and let $H(s) = C(sI - A)^{-1}B$ denote the corresponding transfer matrix. Suppose there exists $P = P^T \succ 0$ such that

$$A^T P + PA + C^T C + \gamma^{-2} PBB^T P \prec 0. \tag{5.1}$$

i) Prove that A is stable. [4]

ii) By defining the Lyapunov function $V(t) = x(t)^T P x(t)$, the cost function

$$J := \int_0^\infty [y(t)^T y(t) - \gamma^2 u(t)^T u(t)] dt,$$

and using a property of the integral $\int_0^\infty \dot{V}(t)dt$, or otherwise, prove that $\|H\|_\infty < \gamma$. [6]

(*HINT:* Express J in the form $J = \int_0^\infty \begin{bmatrix} x(t)^T & u(t)^T \end{bmatrix} M \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt$, where the left side of (5.1) is the Schur complement of M.)

b) Consider the state feedback problem shown in Figure 5 for which $x(0) = 0$. Let $T_{yu}(s)$ denote the transfer matrix from u to y. An internally stabilizing state feedback gain matrix F is to be designed such that, for given $\gamma > 0$, $\|T_{yu}\|_\infty < \gamma$.

i) Derive a state space realization for $T_{yu}(s)$. [4]

ii) Using the answer to part (a), or otherwise, derive sufficient conditions for the existence of a feasible F. Your conditions should be in the form of the existence of solutions to linear matrix inequalities. [6]

(*HINT:* Consider a simple change of variables to linearize any nonlinear matrix inequalities resulting from the use of part (a).)

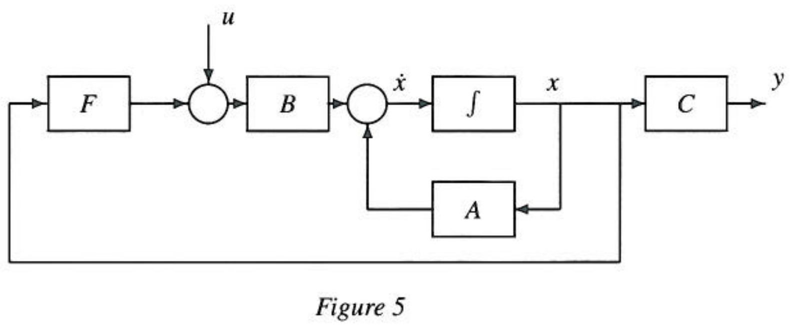

Figure 5

6. Consider the regulator shown in Figure 6 for which it is assumed that the pair (A, B) is controllable and $x(0) = 0$.

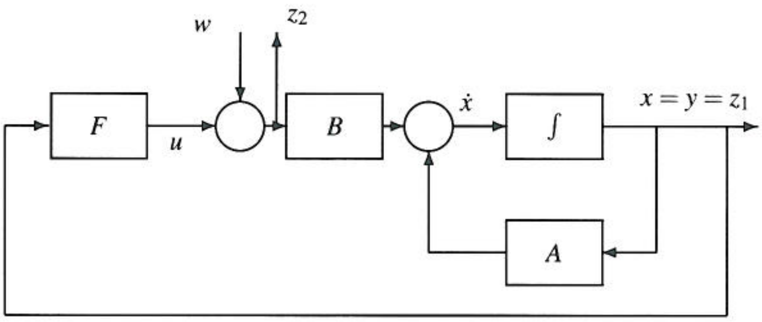

Figure 6

Let

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

and let $H(s)$ denote the transfer matrix from w to z. A stabilizing state-feedback gain matrix F is to be designed such that, for $\gamma > 0$, $\|H\|_\infty < \gamma$.

a) Write down the generalized regulator system for this design problem. [6]

b) Derive sufficient conditions for the solution of the design problem. Your conditions should be in the form of the existence of a certain solution to an algebraic Riccati equation. It should also include an expression for F and an expression for the worst-case disturbance w. [8]

c) Show that the state-feedback gain matrix F can be chosen to be independent of γ. [2]

d) What is the smallest γ for which your sufficient conditions guarantee the existence of F satisfying the design specifications. Justify your answer. [4]

Solutions - 2011

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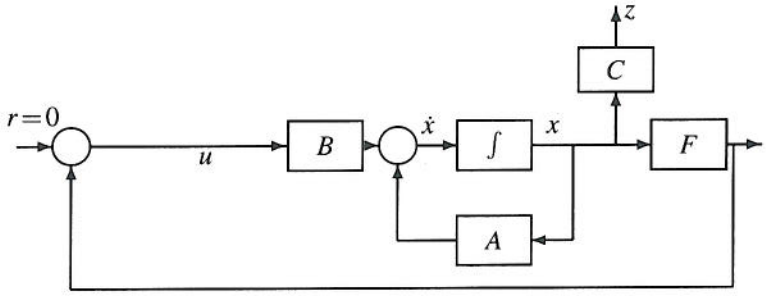

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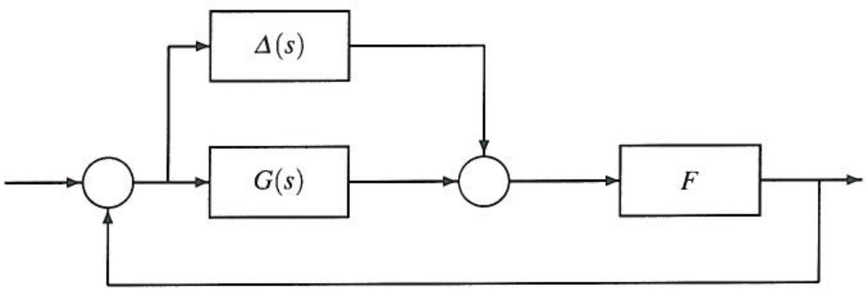

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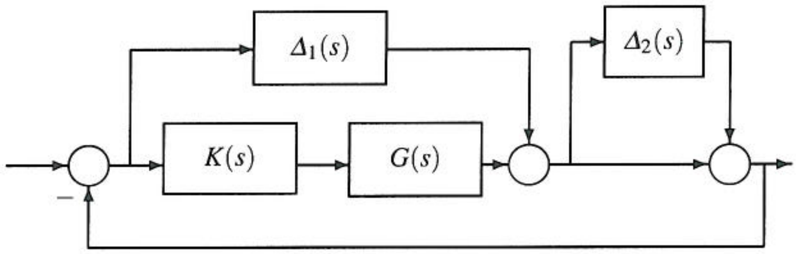

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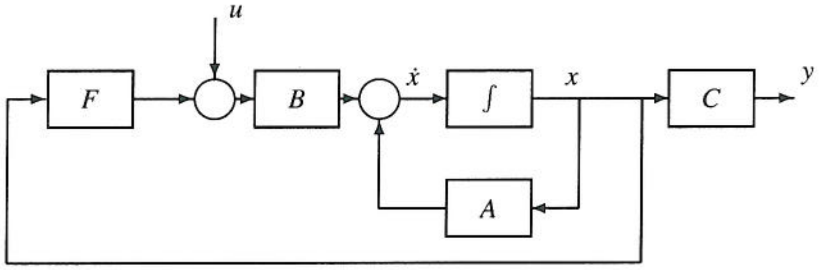

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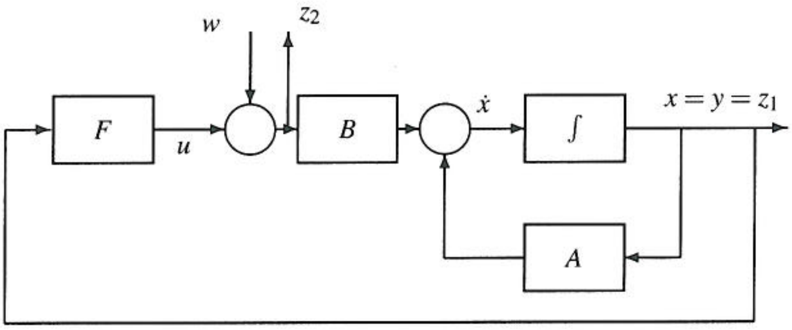

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1. Let the realisation of $G(s)$ be partitioned compatibly with the partitioning of P as

$$G(s) \stackrel{s}{=} \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right].$$

Then

$$AP + PA^T + BB^T = \begin{bmatrix} A_{11}P_1 + P_1A_{11}^T + B_1B_1^T & P_1A_{21} + B_1B_2^T \\ \star & B_2B_2^T \end{bmatrix} = 0 \tag{1.1}$$

It follows from the (2,2) entry that $B_2 = 0$. Also, it follows from the (1,2) entry and the assumption that $P_1 \succ 0$ that $A_{21} = 0$. So, the realisation for $G(s)$ has the form

$$G(s) \stackrel{s}{=} \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ 0 & A_{22} & 0 \\ \hline C_1 & C_2 & D \end{array}\right]. \tag{1.2}$$

a) Applying the PBH test, it is clear that

$$\left[\begin{array}{c|c} A - sI & B \end{array}\right] = \left[\begin{array}{cc|c} A_{11} - sI & A_{12} & B_1 \\ 0 & A_{22} - sI & 0 \end{array}\right]$$

loses rank when s is an eigenvalue of A_{22} so the realisation is uncontrollable.

b) By removing the uncontrollable part, a state space realisation of $G(s)$ is given as $G(s) \stackrel{s}{=} (A_{11}, B_1, C_1, D)$, which is controllable since Since $P_1 \succ 0$.

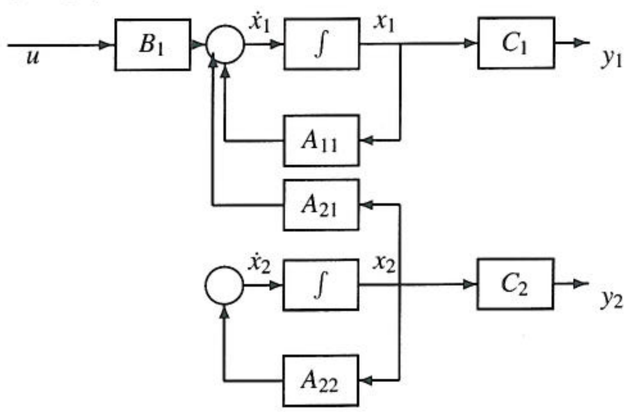
c) Suppose that λ is an eigenvalue of A_{11} and let $z \neq 0$ be the corresponding left eigenvector. Then $z'A_{11} = \lambda z'$. Pre- and post-multiplying the the (1,1) entry of (1.1) by z' and z, respectively, we get

$$(\lambda + \bar{\lambda})z'P_1z < 0.$$

Since $z \neq 0$ and $P_1 \succ 0$, $z'P_1z > 0$ so that $\lambda + \bar{\lambda} < 0$ and so A_{11} is stable.

d) The uncontrollable part must be stable in order to be stabilisable, and so a necessary condition is that the eigenvalues of A_{22} are in the open left half plane.

e) Setting $x = \begin{bmatrix} x_1^T & x_2^T \end{bmatrix}$ and $y = y_1 + y_2$, we get



2. a) Applying the PBH test, it is clear that

$$\left[\begin{array}{c} A - sI \\ \hline C \end{array}\right] = \left[\begin{array}{cc} A_{11} - sI & 0_2 \\ A_{21} & A_{22} - sI \\ \hline C_1 & 0_2 \end{array}\right]$$

loses rank when s is an eigenvalue of A_{22} so the realisation is unobservable. The unobservable modes are therefore the eigenvalues of A_{22} which are a_3 and a_4.

b) The output decoupling zeros are the unobservable modes and are therefore a_3 and a_4.

c) The realisation is detectable if and only if the unobservable modes are stable, equivalently, if and only if $a_3 < 0$ and $a_4 < 0$.

d) By removing the unobservable modes, a second order realisation of $G(s)$ is given as

$$G(s) \stackrel{s}{=} \left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & 0_2 \end{array}\right].$$

e) For the second order realisation of $G(s)$ in Part (d):

i) The controllability and observability Grammians are, respectively, the solutions P and Q of the Lyapunov equations

$$A_{11}P + PA_{11}^T + B_1B_1^T = A_{11}^TQ + QA_{11} + C_1^TC_1 = 0.$$

Since $A_{11} = \mathrm{diag}(a_1, a_2)$ and $B_1B_1^T = C_1^TC_1 = I$, it follows that $P = Q = \mathrm{diag}(-1/2a_1, -1/2a_2)$.

ii) Since $P = Q$ and is diagonal the realization is balanced.

iii) The Hankel singular values are the eigenvalues of PQ and are therefore $1/2|a_1|$ and $1/2|a_2|$.

f) Since $|a_1| > |a_2|$ we keep the realization corresponding to a_2 and so a first order balanced truncation of $G(s)$ is given by

$$G_b(s) \stackrel{s}{=} \left[\begin{array}{c|c} a_2 & b_2 \\ \hline c_2 & 0_2 \end{array}\right].$$

3. a) Let $V = x'Px$ and set $u = Fx$. Provided that $P = P' \succ 0$ and $\dot{V} < 0$ along closed-loop trajectories, we can assume $\lim_{t\to\infty} x(t) = 0$. Then

$$\dot{V} = \dot{x}'Px + x'P\dot{x} = x'\left(A'P + PA + F'B'P + PBF\right)x.$$

Integrating from 0 to ∞ and using $x(\infty) = 0$,

$$\int_0^\infty x'\left(A'P + PA + F'B'P + PBF\right)x dt = -x_0'Px_0.$$

b) Using the definition of J, adding the last equation and completing a square:

$$J = x_0'Px_0 + \int_0^\infty \{x'[A'P + PA + C'C - PBR^{-1}B'P]x + \left\|R^{\frac{1}{2}}(F + R^{-1}B'P)x\right\|^2\}dt.$$

Since the last term is always nonnegative, it follows that the minimizing value of F is given by $F = -R^{-1}B'P$. We can set the term in square brackets to zero provided P satisfies the Riccati equation,

$$A'P + PA + C'C - PBR^{-1}B'P = 0.$$

It follows that the minimum value of J is $x_0'Px_0$.

c) We need to prove that $A_c := A - BR^{-1}B'P$ is stable. The Riccati equation can be written as $A_c'P + PA_c + C'C + PBR^{-1}B'P = 0$. Let $\lambda \in \mathscr{C}$ be an eigenvalue of A_c and $z \neq 0$ be the corresponding eigenvector. Pre– and post–multiplying the Riccati equation be z' and z respectively gives $(\lambda + \bar{\lambda})z'Pz + z'C'Cz + z'PBR^{-1}B'Pz = 0$. Since $P > 0$ and $z \neq 0$, $z'Pz > 0$ and $z'PBR^{-1}B'Pz \geq 0$. Furthermore, if $B'Pz = 0$ and $Cz = 0$, then $A_cz = Az = \lambda z$ which, together with $Cz = 0$ contradicts the observability of (A, C). It follows that that $z'C'Cz + z'PBR^{-1}B'Pz > 0$ and so $\lambda + \bar{\lambda} < 0$ and the closed loop is therefore stable.

d) Setting $R = I$ and by direct evaluation, $L(j\omega)'L(j\omega) =$

$$I - F(j\omega I - A)^{-1}B - B'(-j\omega I - A')^{-1}F' + B'(-j\omega I - A')^{-1}F'F(j\omega I - A)^{-1}B$$

But $F'F = A'P + PA + I = -(-j\omega I - A')P - P(j\omega I - A) + I$ from the Riccati equation. So, $L(j\omega)'L(j\omega)$

$$\begin{aligned}
&= I - F(j\omega I - A)^{-1}B - B'(-j\omega I - A')^{-1}F' \\
&\quad + B'(-j\omega I - A')^{-1}[-(-j\omega I - A')P - P(j\omega I - A) + I](j\omega I - A)^{-1}B \\
&= I - [F + B'P](j\omega I - A)^{-1}B - B'(-j\omega I - A')^{-1}[F' + PB] \\
&\quad + B'(-j\omega I - A')^{-1}(j\omega I - A)^{-1}B = I + G(j\omega)'G(j\omega)
\end{aligned}$$

e) Assume that Δ is stable. Let ε be the input to Δ and δ the output. Then

$$\varepsilon(s) = F\left(\delta(s) + G(s)\varepsilon(s)\right) = (I - FG(s))^{-1}F\delta(s).$$

Using the small gain theorem (since the regulator is stable and the perturbation is assumed stable), the loop is stable if $\|\Delta(I - FG)^{-1}F\|_\infty < 1$. But Part (d) implies that $\|(I - FG)^{-1}\|_\infty \leq 1$. Furthermore, the largest singular value of F is equal to $\sqrt{2}$. Hence the loop will tolerate perturbations of size (measured in the $\mathscr{H}_\infty$–norm) at least $2^{-\frac{1}{2}}$ without losing internal stability, since $\|\Delta\|_\infty < 2^{-\frac{1}{2}}$ implies that $\|\Delta(I - FG)^{-1}F\|_\infty < 1$.

4. a) It is clear that we require $K(s)$ to be internally stabilising.

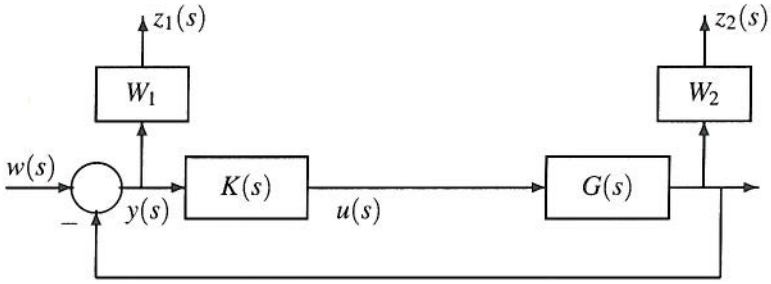
i) Suppose that $\Delta_1 = 0$ and let the input to Δ_2 be ε_2 and the output δ_2. A calculation shows that $\varepsilon_2 = -(I+GK)^{-1}GK\delta_2$. Using the small gain theorem, it is sufficient that $\|\Delta_2\,(I+GK)^{-1}\,GK\|_\infty < 1$. This can be satisfied if $\|W_2\,(I+GK)^{-1}\,GK\|_\infty < 1$, where $W_2 = 0.5I$.

ii) An analogous procedure shows that to satisfy the second design requirement, it is sufficient that $\|\Delta_1\,(I+GK)^{-1}\|_\infty < 1$. This can be satisfied if $\|W_1\,(I+GK)^{-1}\|_\infty < 1$, where $W_1 = I$.

Thus, to satisfy both design requirements, it is sufficient that

$$\left\| \begin{bmatrix} W_1\,(I+GK)^{-1} \\ W_2\,(I+GK)^{-1}\,GK \end{bmatrix} \right\|_\infty < 1.$$

b) All the requested signals are shown in the block diagram shown below.



c) Inject a signal r in between G and K. The loop is internally stable if the transfer matrix from $\begin{bmatrix} w^T & r^T \end{bmatrix}^T$ to $\begin{bmatrix} u^T & y^T \end{bmatrix}^T$ is stable. Since

$$\begin{bmatrix} w \\ r \end{bmatrix} = \begin{bmatrix} I & -K \\ G & I \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} =: T \begin{bmatrix} u \\ y \end{bmatrix}$$

the loop is internally stable if T^{-1} is stable. Since G is stable and

$$\begin{bmatrix} I & -K \\ G & I \end{bmatrix}^{-1} = \begin{bmatrix} I & K\,(I+GK)^{-1} \\ 0 & I - GK(I+GK)^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -G & I \end{bmatrix}$$

it follows that if G is stable, then the loop is internally stable if $Q := K\,(I+GK)^{-1}$ is stable. Rearranging shows that K is internally stabilizing if and only if $K = Q\,(I-GQ)^{-1}$ for some stable Q.

d) Since G is stable, using the parameterization in Part (c) gives $(I+GK)^{-1} = I - GQ$ and $GK(I+GK)^{-1} = GQ$. Thus Part (a) requires $\|GQ\|_\infty \le 2$ and $\|I-GQ\|_\infty \le 1$. Since $\|G\|_\infty = 1$, we can use $Q = 0.5$ which gives

$$K = (s+1)/(2s+1).$$

5. a) i) The inequality in (5.1) implies $A^TP+PA \prec 0$. Let $z \neq 0$ be a right eigenvector of A and let λ be the corresponding eigenvalue. Then multiplying the inequality from the left by z' and from the right by z gives $(\lambda+\bar{\lambda})z'Pz < 0$. Since $P \succ 0$ it follows that $z'Pz > 0$ and it follows that $\lambda+\bar{\lambda} < 0$ so that A is stable.

ii) Since A is stable, $\|H\|_\infty < \gamma$ if and only if, with $x(0)=0$, $J := \int_0^\infty [y^Ty - \gamma^2 u^Tu]dt < 0$, for all $u(t)$ such that $\|u\|_2 < \infty$. If $\|u\|_2$ is bounded, then $\lim_{t\to\infty} x(t)=0$. Now, $\int_0^\infty \frac{\mathrm{d}}{\mathrm{dt}}[x^TPx]dt = x(\infty)^TPx(\infty) - x(0)^TPx(0) = 0$. So,

$$0 = \int_0^\infty (\dot{x}^TPx + x^TP\dot{x})\,dt = \int_0^\infty [x^T(A^TP+PA)x + x^TPBu + u^TB^TPx]dt.$$

Use $y=Cx$ and add the last expression to J

$$\begin{aligned} J &= \int_0^\infty [x^T(A^TP+PA+C^TC)x + 2x^T(PB - u^T\gamma^2 I)u]dt \\ &= \int_0^\infty \begin{bmatrix} x^T & u^T \end{bmatrix} \overbrace{\begin{bmatrix} A^TP+PA+C^TC & PB \\ B^TP & -\gamma^2 I \end{bmatrix}}^{M} \begin{bmatrix} x \\ u \end{bmatrix} dt \end{aligned}$$

so that $J<0$ and $\|H\|_\infty < \gamma$, if $M \prec 0$. But this follows from (5.1).

b) i) Substituting $u := u + Fx$, into the state equation gives

$$\dot{x} = (A+BF)x + Bu, \qquad y = Cx.$$

It follows that $T_{yu}(s) = C(sI - (A+BF))^{-1}B$.

ii) Using the results of part (a), replacing A by by $A+BF$, there exists a feasible F if there exists $P = P^T \succ 0$ such that

$$\begin{bmatrix} (A+BF)^TP + P(A+BF) + C^TC & PB \\ B^TP & -\gamma^2 I \end{bmatrix} \prec 0.$$

Pre- and post-multiplying by $\mathrm{diag}(P^{-1}, I)$ gives

$$\begin{bmatrix} P^{-1}(A+BF)^T + (A+BF)P^{-1} + P^{-1}C^TCP^{-1} & B \\ B^T & -\gamma^2 I \end{bmatrix} \prec 0$$

and effecting a Schur complement

$$\begin{bmatrix} (A+BF)P^{-1} + P^{-1}(A+BF)^T & B & P^{-1}C^T \\ B^T & -\gamma^2 I & 0 \\ CP^{-1} & 0 & -I \end{bmatrix} \prec 0$$

Noting that the only nonlinearity is due to the product FP^{-1}, we define $Q = P^{-1}$ and $Z = FP^{-1}$ and so there exists a feasible F if there exists $Q = Q^T \succ 0$ and Z such that

$$\begin{bmatrix} AQ + QA^T + BZ + Z^TB^T & B & QC^T \\ B^T & -\gamma^2 I & 0 \\ CQ & 0 & -I \end{bmatrix} \prec 0.$$

6. a) The generalized regulator formulation is given by

$$\left[\begin{array}{c} z(s) \\ y(s) \end{array}\right]=P(s)\left[\begin{array}{c} w(s) \\ u(s) \end{array}\right], u(s)=Fy(s), P(s)=\left[\begin{array}{cc} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{array}\right] \stackrel{s}{=} \left[\begin{array}{c||c|c} A & B & B \\ \hline\hline I & 0 & 0 \\ 0 & I & I \\ \hline I & 0 & 0 \end{array}\right].$$

b) The requirement $\|H\|_\infty<\gamma$ is equivalent to $J:=\|z\|_2^2-\gamma^2\|w\|_2^2<0$. Let $V=x^TXx$ and set $u=Fx$. Provided that $X=X^T>0$ and $\dot{V}<0$ along the closed-loop trajectory, we can assume $\lim_{t\to\infty} x(t)=0$. Then

$$\dot{V}=\dot{x}^TXx+x^TX\dot{x}=x^T\left(A^TX+XA+F^TB^TX+XBF\right)x+x^TXBw+w^TB^TXx.$$

Integrating from 0 to ∞ and using $x(0)=x(\infty)=0$,

$$0=\int_0^\infty [x^T\left(A^TX+XA+F^TB^TX+XBF\right)x+x^TXBw+w^TB^TXx]dt.$$

Using the definition of J and adding the last equation, $J=$

$$\int_0^\infty\{x^T[A^TX+XA+I+F^TF+F^TB^TX+XBF]x-[\beta w^Tw-x^TZ^Tw-w^TZx]\}dt$$

where $Z=F+B^TX$ and $\beta=\gamma^2-1>0$ since $\gamma>1$ by assumption. Completing the squares by using

$$\begin{aligned} Z^TZ &= F^TF+F^TB^TX+XBF+XBB^TX \\ \|(\sqrt{\beta}w-\sqrt{\beta^{-1}}Zx)\|^2 &= \beta w^Tw-w^TZx-x^TZ^Tw+\beta^{-1}x^TZ^TZx, \end{aligned}$$

$$J=\int_0^\infty\{x^T[A^TX+XA+I-XBB^TX]x+(1+\beta^{-1})\|Zx\|^2-\left\|\sqrt{\beta}w-\sqrt{\beta^{-1}}Zx\right\|^2\}dt.$$

Thus two sufficient conditions for $J<0$ are the existence of X such that

$$A^TX+XA+I-XBB^TX=0, \qquad X=X^T>0.$$

The feedback gain is obtained by setting $Z=0$ so $F=-B^TX$. The worst case disturbance is $w^*=\beta^{-1}Zx=0$. The closed-loop with $u=Fx$ and $w=w^*$ is $\dot{x}=[A-BB^TX]x$ and a third condition is $Re\ \lambda_i[A-BB^TX]<0$, $\forall\ i$. It remains to prove $\dot{V}<0$ for $u=Fx$ and $w=0$. But

$$\dot{V}=x^T\left(A^TX+XA+F^TB^TX+XBF\right)x=-x^T(I+XBB^TX)x<0$$

for all $x\neq 0$ proving closed-loop stability.

c) Since X is independent of γ and $F=-B^TX$, then F is independent of γ.

d) It is clear that our procedure breaks down if $\gamma\le 1$ since in that case $\beta\le 0$. Thus the smallest value of γ is 1.

