UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 2003

MEng Honours Degree in Information Systems Engineering Part IV
MSci Honours Degree in Mathematics and Computer Science Part IV
MEng Honours Degrees in Computing Part IV
MSc in Advanced Computing
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the Associateship of the City and Guilds of London Institute
This paper is also taken for the relevant examinations for the Associateship of the Royal College of Science

PAPER C478=I4.37

ADVANCED OPERATIONS RESEARCH

Thursday 1 May 2003, 15:30
Duration: 120 minutes

Answer THREE questions

Paper contains 4 questions
Calculators required

1a An oil refinery can produce four types of petrol, P-1, ..., P-4 using three different ingredients, I-1, I-2 and I-3, which are available in limited quantities. As part of the company's production planning system the following problem has to be solved.

$$\begin{array}{lllllllllll}
\max & \text{Revenue } z = & 7x_1 & + & 10x_2 & + & 11x_3 & + & 5x_4 & & \\
\text{s.t.} & \text{I-1} & 2x_1 & + & 3x_2 & + & 3x_3 & + & 1x_4 & \leq & 180 \\
 & \text{I-2} & 4x_1 & + & 2x_2 & + & 3x_3 & + & 3x_4 & \leq & 200 \\
 & \text{I-3} & 4x_1 & + & 4x_2 & + & 6x_3 & + & 2x_4 & \leq & 250 \\
 & & & & & x_1,\, x_2,\, x_3,\, x_4 \geq 0. & & & & &
\end{array}$$

An optimal solution is achieved with $x_2 = 55$, $x_4 = 15$ and $y_2 = 45$ giving an objective value of $z = 625$, where y_2 is the logical variable of constraint 2. It can also be referred to as x_6.

The optimal basis is $B = \{4, 6, 2\}$. The basis matrix is

$$\boldsymbol{B} = \begin{bmatrix} 1 & 0 & 3 \\ 3 & 1 & 2 \\ 2 & 0 & 4 \end{bmatrix} \qquad \text{its inverse} \qquad \boldsymbol{B}^{-1} = \begin{bmatrix} -2 & 0 & \frac{3}{2} \\ 4 & 1 & -\frac{7}{2} \\ 1 & 0 & -\frac{1}{2} \end{bmatrix}$$

(i) Determine the simplex multiplier associated with $\boldsymbol{B}$ and the reduced costs of nonbasic variables.

(ii) Determine the ranges of the original right-hand-side coefficients within which the basis remains optimal.

b An LP problem is being solved by the simplex method: $\min\{\boldsymbol{c}^T\boldsymbol{x} \mid \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}\}$ where the variables are subject to type specifications. At an intermediate stage a basic feasible solution (BFS) with $z = 1$ and a type-2 incoming variable with reduced cost $d_q = 5$ are given. Also, $\boldsymbol{\alpha}_q = \boldsymbol{B}^{-1}\boldsymbol{a}_q$ is available. The table below shows the relevant part of the problem:

i	x_{Bi}	type(x_{Bi})	u_{Bi}	α_{iq}	t_i
1	0	3	$+\infty$	2	
2	2	2	$+\infty$	-1	
3	2	1	4	1	
4	6	2	$+\infty$	2	
5	1	1	5	-2	

Is the BFS degenerate? Determine the ratios, the value of the incoming variable, the variable leaving the basis (if any), the new BFS and the new value of the objective function. Is the new BFS degenerate?

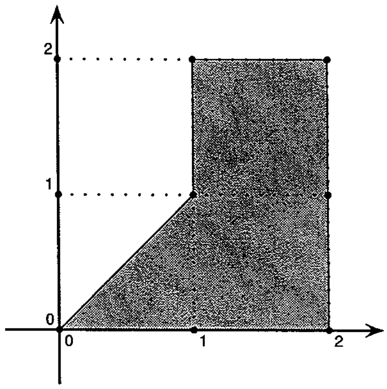
c All LP constraints can be converted into equalities. How? You are given the following set of linear programming (LP) constraints. Simplify and convert them into standard LP equalities. Indicate the type of the associated logical variable for each constraint. Try to combine constraints if possible.

$$\begin{aligned}
2x_1 - x_2 + 3x_3 - 4x_5 &\geq -5 - x_1 + x_4 && (1)\\
x_1 + x_2 + x_3 + x_4 &<> 0 && (2)\\
-x_1 + x_2 &\geq 2x_3 - x_4 && (3)\\
3x_1 - x_2 - 4 &\leq -2x_3 + x_4 && (4)\\
6 \geq 3x_1 - 2x_2 + x_3 - 4x_4 &\geq -3 && (5)\\
2x_1 - 3x_4 &= x_1 + x_2 - x_3 && (6)\\
x_2 - 2x_3 &\leq 3 + 3x_1 - x_4 && (7)
\end{aligned}$$

(The three parts carry, respectively, 40%, 30% and 30% of the marks).

2a Show that the reduced cost of every basic variable is zero for any feasible basis $\boldsymbol{B}$ of a general linear programming problem.

b Nonconvex sets can often be described by techniques of mixed integer programming. The figure below shows a nonconvex region R. Write the appropriate linear inequalities and logic expressions to describe the points in the shaded area. Introduce indicator variable(s) if needed and give a MIP formulation of the region. Verify your solution by showing that point $P_1(1.5,\ 0.5)$ satisfies your constraints and point $P_2(0.5,\ 1.5)$ does not.



c Determine the type of each variable in the following two problems and answer the following questions: Are the given solutions feasible? Do they satisfy the optimality conditions? Which, if any, of the solutions is degenerate. What is the size of the problems in terms of m and n? How do you know that? (Note, B/N refers to Basic/Nonbasic.)

(i) Problem: min $\boldsymbol{c}^T\boldsymbol{x},\ \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$,

	x_1	x_2	x_3	x_4	x_5	x_6
ℓ_j	0	$-\infty$	0	0	0	$-\infty$
u_j	$+\infty$	$+\infty$	5	5	0	$+\infty$
type(x_j)						
In the solution:						
B/N	N	B	N	B	N	N
Value	0	0	5	5	0	0
d_j	-2	0	1	0	-2	-1
Opt. cond. Y/N						

(ii) Problem: max $\boldsymbol{c}^T\boldsymbol{x},\ \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$,

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
ℓ_j	0	0	0	$-\infty$	0	0	0
u_j	5	$+\infty$	$+\infty$	$+\infty$	10	$+\infty$	0
type(x_j)							
In the solution:							
B/N	N	B	N	B	N	N	N
Value	5	1	0	-7	10	0	0
d_j	-1	0	11	0	-1	10	1
Opt. cond. Y/N							

(The three parts carry, respectively, 20%, 50% and 30% of the marks).

3a (i) In an optimization model you have to represent a discrete variable that can take only the following 6 values: $\{-1.2, -0.6, 0, 1.2, 2.4, 5.3\}$. How can you include this requirement in the model? Write the necessary equation(s). Explain your work.

(ii) There is an integer variable in a model that is limited to the $[-5, 10]$ range. Express this condition using as few 0/1 variables as possible.

b The Father & Son haulage company is planning an extension of its fleet. Three types of trucks are included in the plan with the following characteristics:

Type	Load capacity (tons)	Cost (£1000)
Light	5	18
Medium	10	34
Large	20	55

Market analysis shows it would be desirable to add 10 light, 12 medium and 8 large models. The total capacity expansion should be around 300 tons and the total cost is limited to £1,000,000.

Write a goal programming model for the above problem if

- the financial constraint cannot be exceeded,
- it is equally undesirable to underachieve the number of light and medium models and overachieve the number of large models,
- it is undesirable to overachieve or underachieve the 300 to goal of capacity expansion, underachievement being twice as bad as overachievement,

Explain your work.

c Investigate the following LP problem. Make conversions or transformations on it if necessary to bring it to a form suitable for the dual simplex method. Solve the problem with the dual algorithm.

$$
\begin{array}{lrcrcrcrcr}
\max & 2x_1 & - & x_2 & - & 4x_3 & - & x_4 & & \\
\text{s. t.} & -x_1 & - & 2x_2 & + & x_3 & + & \frac{1}{2}x_4 & \leq & 3 \\
 & x_1 & + & 2x_2 & - & x_3 & & & \leq & -1 \\
 & -x_1 & & & - & 4x_3 & + & x_4 & \geq & 2 \\
\end{array}
$$
$$x_1 \leq 0,\ x_j \geq 0,\ j = 2, 3, 4$$

Discuss the conversions and the steps of the solution in detail. Express the solution in terms of the original variables and constraints. At the end, determine the dual solution from its defining equation. Show work.

(The three parts carry, respectively, 20%, 30% and 50% of the marks).

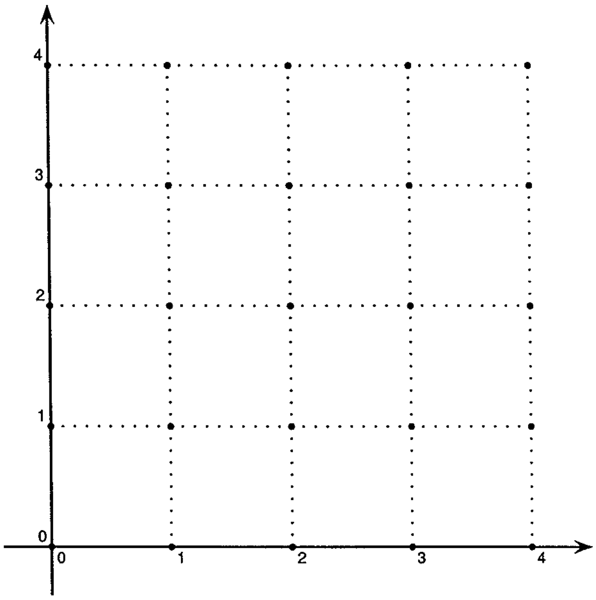
4a (i) Discuss the notion of convex hull in terms of integer programming (IP). Explain its importance for the efficient solution of IP problems.

(ii) Discuss and interpret the possible outcomes of the simplex method.

b Graphical solution of mixed integer linear programming problems (MILP) is possible if there are not more than two variables.

Solve the following MILP problem using the graph printed here or your own drawing. It need not be very accurate. If in doubt, rely on the numerical data given below.

The objective is to maximize $z = -x_1 + 2x_2$, where x_1 is a general nonnegative integer, x_2 is nonnegative. The feasible region of the LP relaxation of the problem is determined by the polygon with vertices: $P_1(0,0)$, $P_2(0,1)$, $P_3(1,3)$, $P_4(3,4)$, $P_5(4,3)$ and $P_6(2,0)$. Where are the feasible solutions of the problem located? Determine an optimal solution. Is it unique? If not, can you find them all? How many are there? Compare the situation with continuous LP.



c Formulate the dual of the following two problems. Explain your work.

(i)

$$
\begin{array}{rrrrrcr}
\min\ z = & 2x_1 - & 3x_2 + & & - \quad x_4 & & \\
\text{s.t.} & x_1 + & x_2 + & x_3 + & x_4 & \geq & 4 \\
 & -2x_1 + & x_2 + & 2x_3 & & = & 1 \\
 & 3x_1 & - & x_3 + & 2x_4 & \leq & 6 \\
\end{array}
$$
$$x_1 \leq 0,\ x_2 \text{ free},\ x_3, x_4 \geq 0.$$

(ii)

$$
\begin{array}{rrrrrcr}
\max\ z = & 2x_1 - & 3x_2 + & 4x_3 - & 5x_4 & & \\
\text{s.t.} & 2x_1 + & 5x_2 - & 4x_3 + & 9x_4 & \leq & 9 \\
 & x_1 + & 4x_2 + & 2x_3 - & 6x_4 & \geq & -9 \\
 & 4x_1 - & 3x_2 - & 6x_3 + & 4x_4 & = & 1 \\
\end{array}
$$
$$x_1, \ldots, x_4 \geq 0.$$

(The three parts carry, respectively, 30%, 40% and 30% of the marks).