CONTROL ENGINEERING

1. a) The system is already in reachability canonical form, hence it is reachable regardless of the value of α. Alternatively, the rechability matrix is

$$\mathscr{R} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

which does not depend upon α and has rank two.

Typical mistakes: incorrect computation of the reachability matrix, or of its rank.

[2 marks]

b) The observability matrix is

$$\mathscr{O} = \begin{bmatrix} -\delta & 1 \\ -\alpha & -\delta \end{bmatrix}.$$

Note that det $\mathscr{O} = \delta^2 + \alpha$, hence the system is observable for all α and δ such that $\delta^2 + \alpha \neq 0$.

Typical mistakes: incorrect computation of the observability matrix, or of its rank. The condition for a full rank observability matrix has to be assessed on the "reals", not on the set of complex numbers!

[4 marks]

c)

i) A state space realization of the interconnected system is described by the matrices

$$A_i = \begin{bmatrix} 0 & 1 & 0 \\ -\alpha & 0 & 0 \\ -\delta & 1 & 0 \end{bmatrix}, \qquad B_i = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad C_i = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}.$$

Some students have not identified the correct state space representation, that is have not been able to exploit the interconnection equation in the construction of the matrices A, B, and C.

[2 marks]

ii) The reachability matrix of the interconnected system is

$$\mathscr{R}_i = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -\alpha \\ 0 & 1 & -\delta \end{bmatrix}$$

and this has rank three provided $\delta \neq 0$.
For $\delta = 0$ a direct calculation shows that

$$\text{Im } A_i^3 \subset \text{Im } \mathscr{R}_i,$$

hence the system is controllable for any δ (and α). [8 marks]

Typical mistakes: incorrect computation of the reachability matrix, or of its rank, and incorrect study of the controllability property.

iii) The observability matrix of the interconnected system is

$$\mathscr{O}_i = \begin{bmatrix} 0 & 0 & 1 \\ -\delta & 1 & 0 \\ -\alpha & -\delta & 0 \end{bmatrix}$$

and this has rank three provided $\delta^2 + \alpha \neq 0$, that is provided the system with state x is observable.

Typical mistakes: incorrect computation of the observability matrix, or of its rank. Some students have failed to understand the connection between the observability of the subsystem with state x and the observability of the interconnected system.

[2 marks]

iv) Note that

$$\det \begin{bmatrix} zI - A & B \\ C & -D \end{bmatrix} = \delta - z,$$

hence the system with state x has a zero at $z = \delta$. The eigenvalue of the system with state ξ is at $z = 0$. When $\delta = 0$ the zero and the eigenvalue coincide (that is they cancel each other). As noted in part c.ii) for $\delta = 0$ the interconnected system is not reachable. It remains controllable since the *cancellation* is at $z = 0$.

Typical mistakes: incorrect computation of the indicated determinant and of the eigenvalues of the system. some students have also not understood the role of δ in the pole-zero cancellation process.

[2 marks]

2. a) The observability matrix is

$$\mathscr{O} = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix},$$

which is full rank (the determinant is equal to 6), hence the system is observable.

Typical mistakes: incorrect computation of the observability matrix, or of its rank (which can be performed computing the determinant of the matrix).

[2 marks]

b) Note that

$$\dot{z} - \dot{x}_2 = fz + gu + h(x_1 - x_2) - (-2x_1 - 3x_2 + u).$$

Selecting $g = 1$, $h = -2$ and $f = -5$ yields

$$\dot{z} - \dot{x}_2 = -5\ (z - x_2),$$

that is the desired equation with $k = -5$. The state z is such that

$$z(t) - x_2(t) = e^{-5t}(z(0) - x_2(0)),$$

hence

$$z(t) = x_2(t) + e^{-5t}(z(0) - x_2(0)),$$

that is $z(t)$ converges exponentially to $x_2(t)$. The variable z can then be used to estimate x_2.

Typical mistakes: incorrect expression of the time derivative of the signal $z - x_2$ and of the parameters f, g and h. Some students have also struggled to understand how to use z to estimate x_2.

[6 marks]

c) To build an asymptotic estimate of the state x_1 note that the output equation is $y = x_1 - x_2$, hence an asymptotic estimate of x_1 is given by

$$y(t) + z(t).$$

This requires a bit of imagination: the students should have not used the lecture notes verbatim, but should have observed that to estimate two states of a two dimensional system it is sufficient to have two known signals.

[2 marks]

d) The system with state $[x_1, x_2, z]'$ can be rewritten in the coordinate $[x_1, x_2, e]'$, with $e = z - x_2$ as

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -2x_1 - 3x_2 + u \qquad \dot{e} = -5e,$$

which clearly shows that the system is not controllable and the uncontrollable mode is at $s = -5$. This is a consequence of the fact that the $\dot{z}$ equation has been designed such that $\dot{e} = -5e$, that is the observer has to have converging properties which do not depend upon x_1, x_2 and u.

This is a separation principle with a reduced order observer. The principle is exactly the same as what discussed in the lectures, but the application of the principle is not standard.

[4 marks]

e) The A matrix of the closed-loop system is

$$A_{cl} = \begin{bmatrix} 0 & 1 & 0 \\ -2-p & -3+p & -q \\ -2-p & 2+p & -q-5 \end{bmatrix},$$

the characteristic polynomial of which is

$$(s+5)(s^2+(3+q-p)s+p+2).$$

Selecting $p=2$ and $q=3$ yields the desired eigenvalues. Note that, consistently with the design in part b) and the analysis in part d), one of the eigenvalues of the closed-loop system is *fixed* at -5.

While the computation of the characteristic polynomial of the matrix A_{cl} seems hard, the students should have recognized from part d) that one of the eigenvalues is fixed, that is does not depend upon p and q.

[6 marks]

3. a) The matrices A and B are given by

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Some students have failed to recall the relations between position, velocity and acceleration in a linear motion!

[2 marks]

b) The system is in controllability canonical form, hence it is controllable. The control objective is to *steer* the initial state $[\bar{x}, 0]$ to the origin, and this can be always achieved, for any $T > 0$, by the very definition of controllability.

Hard to make mistakes here!

[2 marks]

c)

i) The differential equation of the system for $u = 1$ are

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = 1,$$

yielding (recall the considered initial conditions)

$$x_1(t) = x_1(0) + x_2(0)t + \frac{1}{2}t^2 = \bar{x} + \frac{1}{2}t^2, \qquad x_2(t) = x_2(0) + t = t,$$

as indicated in the exam paper. Note that $t = x_2$, hence replacing this in the x_1 equation yields

$$x_1 = \bar{x} + \frac{1}{2}x_2^2,$$

that is the family of parabolas, parameterized by $\bar{x}$, in red-dashed lines in the figure. The arrow of time is pointing upward since $\dot{x}_2 = 1 > 0$, that is the state x_2 is monotonically increasing with time. Clearly, the parabola with equation $x_1 = \frac{1}{2}x_2^2$ is the only one that goes through the origin (the parabola is drawn in bold in the figure).

Typical mistakes: incorrect integration of the ode's, incorrect elimination of time and computation of the "parabola", and wrong graphs!

[4 marks]

ii) Similarly to the previous point, the differential equation of the system for $u = -1$ are

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -1,$$

yielding (recall the considered initial conditions)

$$x_1(t) = x_1(0) + x_2(0)t - \frac{1}{2}t^2 = \bar{x} - \frac{1}{2}t^2, \qquad x_2(t) = x_2(0) - t = -t.$$

Note that $t = -x_2$, hence replacing this in the x_1 equation yields

$$x_1 = \bar{x} - \frac{1}{2}x_2^2,$$

that is the family of parabolas, parameterized by $\bar{x}$, in blue-dotted lines in the figure. The arrow of time is pointing downward since $\dot{x}_2 = -1 < 0$, that is the state x_2 is monotonically decreasing with time. As above the parabola with equation $x_1 = -\frac{1}{2}x_2^2$ is the only one that goes through the origin (the parabola is drawn in bold in the figure).

This is essentially the same as the previous point with a sign difference. some students have failed to see this!

[4 marks]

iii) If $\bar{x} < 0$ one could follow the red-dashed trajectory starting from $(\bar{x}, 0)$ till the trajectory meets the blue-dotted trajectory described by $x_1 = -\frac{1}{2}x_2^2$. At that point, the sign of the input signal is switched and the state follows the blue-dottet trajectory till the origin. Similarly, for $\bar{x} > 0$.

This part requires some geometric vision and an understanding of the previous points: both missed by many students.

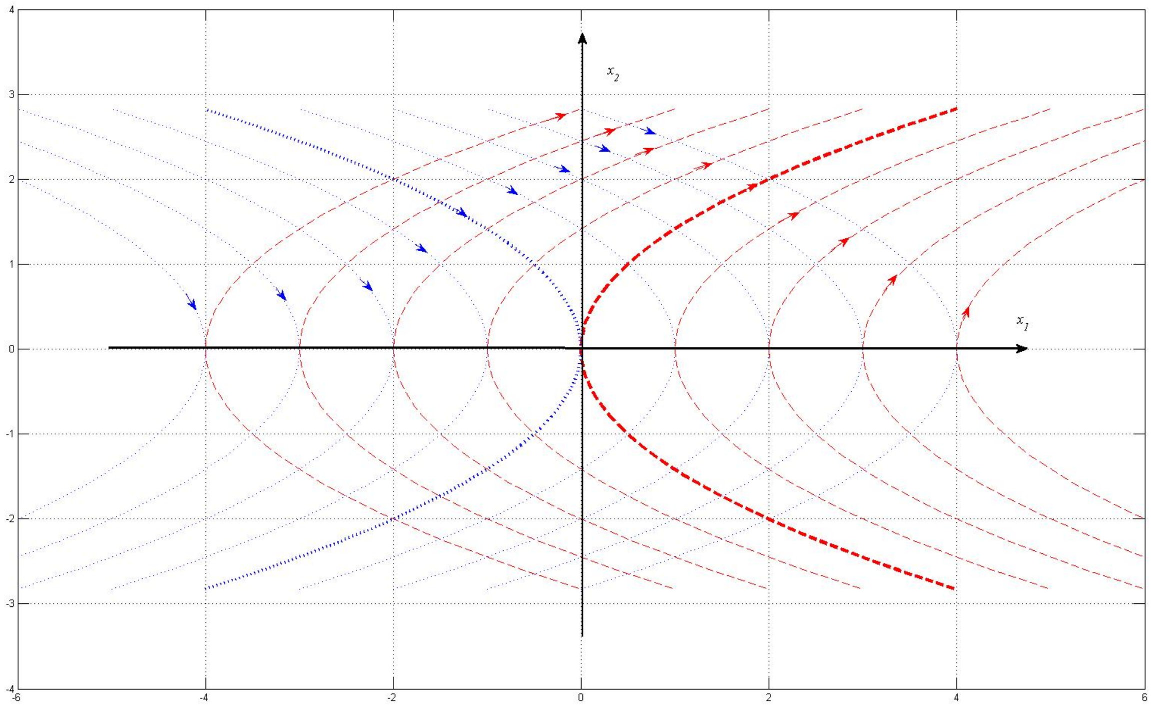
[6 marks]

iv) Consider a trajectory with $\bar{x} < 0$. Note that for the first part of the trajectory one has $x_2(\bar{t}) - x_2(0) = \bar{t}$ and then $x_2(\bar{t}) - x_2(T) = -(t - T)$, which shows that the time T to reach the origin is twice the maximum value achieved by $x_2(t)$ along the considered trajectory, hence it is finite. Similar considerations apply for trajectories starting with $\bar{x} > 0$.

The bound on the acceleration is trivially satisfied since $\ddot{x} = u$, hence $|\ddot{x}| = 1$, for all $t \in [0, T)$. At $t = T$ one sets $u(t) = 0$, for $t \geq T$ which, since the origin is an equilibrium for the system, yields a trajectory which remains at the origin for all $t \geq 0$.

Again, this is trivial once all previous parts have been solved correctly. This is clearly a non-standard question and the student have struggled to use the tools taught in the lectures in a non-standard exercise.

[2 marks]



x2
x1
4
3
2
1
0
-1
-2
-3
-4
-6
-4
-2
0
2
4
6

4. a) The matrices A and B are given by

$$A = \begin{bmatrix} 0 & \varepsilon \\ 1 & 2, \end{bmatrix} \qquad B = \begin{bmatrix} -\varepsilon \\ 1 \end{bmatrix}.$$

Typical mistakes: incorrect expressions for the matrices A and B.

[2 marks]

b) The reachability matrix is

$$\mathscr{R} = \begin{bmatrix} -\varepsilon & \varepsilon \\ 1 & 2-\varepsilon \end{bmatrix}.$$

Note that $\det \mathscr{R} = \varepsilon^2 - 3\varepsilon$, hence the system is reachable for all $\varepsilon \neq 0$ and $\varepsilon \neq 3$. The unreachable modes can be computed using the Hautus test: for $\varepsilon = 0$ it is at $z = 0$, whereas for $\varepsilon = 3$ it is at $z = 3$.

Typical mistake: incorrect computation of the reachability matrix and of the unreachable modes.

[6 marks]

c) The system is controllable, by reachability, for all $\varepsilon \neq 0$ and $\varepsilon \neq 3$. For $\varepsilon = 0$, since the unreachable mode is at $z = 0$ it is controllable, whereas it is not controllable for $\varepsilon = 3$. (One could check controllability using alternative conditions, which however would require longer computations.)

Typical mistake: incorrect understanding of the relation between reachability and controllability.

[4 marks]

d) Note that

$$A_{cl} = A + BK = \begin{bmatrix} -\varepsilon k_1 & \varepsilon - \varepsilon k_2 \\ 1 + k_1 & 2 + k_2 \end{bmatrix},$$

which has the characteristic polynomial

$$\det\,(zI - A_{cl}) = z^2 + z(\varepsilon k_1 - k_2 - 2) + (\varepsilon k_2 - 3\varepsilon k_1 - \varepsilon).$$

This should be equal to z^2, yielding

$$k_1 = \frac{3}{\varepsilon - 3}, \qquad k_2 = \frac{\varepsilon + 6}{\varepsilon - 3}.$$

Note that

$$\lim_{\varepsilon = 0} k_1 = -1 \qquad \lim_{\varepsilon = 0} k_2 = -2,$$

whereas k_1 and k_2 are not defined for $\varepsilon = 3$. This is consistent with the fact that for $\varepsilon = 0$ the unreachable mode is at $z = 0$, that is it coincides with one of the desired closed-loop eigenvalues, whereas for $\varepsilon = 3$ the unreachable mode does not coincide with one of the desired closed-loop eigenvalues.

Typical mistakes: incorrect computation of the characteriztic polynomial, of the gains k_1 and k_2 and of the limiting values of the gains. Some students have *also failed to see the connection between the finite limits and the controllability of the system.*

[6 marks]

e) The closed-loop matrix with $K = K_0$ is

$$A + BK_0 = \begin{bmatrix} \varepsilon & 3\varepsilon \\ 0 & 0 \end{bmatrix},$$

which has eigenvalues equal to 0 and ε. Hence the gain K_0 is stabilizing for all ε such that $|\varepsilon| < 1$.

Typical mistakes: use of the incorrect gain matrix, wrong characteristic polynomial and incorrect use of the stability conditions for discrete-time systems. On the subtle side, some students have provided the strict inequality $|\varepsilon| \leq 1$ *and, on the less subtle side, some students have identified only half of the admissible* ε*'s, that is* $0 \leq \varepsilon < 1$.

[2 marks]