IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2016

EEE PART II: MEng, BEng and ACGI

Corrected Copy

MATHEMATICS 2A (E-STREAM AND I-STREAM)

Monday, 23 May 2:00 pm

Time allowed: 1:30 hours

There are TWO questions on this paper.

Answer TWO questions.

Answer both questions

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : D. Nucinkis
Second Marker(s) : B. Clerckx

Table of Laplace transforms

$f(t)$	$\mathcal{L}\{f(t)\} \equiv F(s)$
A	$\dfrac{A}{s}\,, \qquad \mathrm{Re}(s) > 0$
e^{at}	$\dfrac{1}{s-a}\,, \qquad \mathrm{Re}(s) > a$
$t^n, \qquad n = 1, 2 \ldots$	$\dfrac{n!}{s^{n+1}}\,, \qquad \mathrm{Re}(s) > 0$
$\sin \omega t$	$\dfrac{\omega}{s^2+\omega^2}\,, \qquad \mathrm{Re}(s) > 0$
$\cos \omega t$	$\dfrac{s}{s^2+\omega^2}\,, \qquad \mathrm{Re}(s) > 0$
$e^{at} f(t)$	$F(s-a)$
$t^n f(t)$	$(-1)^n \dfrac{d^n F}{ds^n}$
$\dfrac{df}{dt}$	$sF(s) - f(0)$
$\dfrac{d^2 f}{dt^2}$	$s^2 F(s) - sf(0) - \dfrac{df}{dt}(0)$
$H(t-a)$	$\dfrac{e^{-as}}{s}$
$\delta(t-a)$	$e^{-as}\,, \qquad a > 0$
$f(t-a)H(t-a)$	$e^{-as}F(s)$

EE2-08A MATHEMATICS

1. a) i) Show that the function

$$u(x,y) = \sinh x \cos y + 2\cosh x \sin y$$

satisfies Laplace's equation [4]

ii) Integrate the Cauchy-Riemann equations to find the conjugate function $v(x,y)$. [4]

iii) Show that $w = u + iv$ can be expressed as

$$w = C_1 \sin(C_2 z) + C_3$$

and determine the complex constants C_1, C_2, C_3 [5]

b) The complex function $f(z) = \dfrac{e^{iz}}{z(z^2+9)}$ has simple poles at $z = \pm 3i$ and $z = 0$.

(i) Show that the residue at $z = 0$ is $1/9$; find the residue at $z = 3i$. [4]

Consider the contour integral $I = \oint_{\Gamma} \dfrac{e^{iz}}{z(z^2+9)}\, dz$,

where Γ is taken to be the union of a semi-circle of radius R, lying in the upper half-plane, with a small semi-circle of radius r indented into the lower half-plane, both centred at $z = 0$, and the real intervals $[-R, -r]$ and $[r, R]$.

(ii) Show that the contribution to I from the indented semi-circle of radius r, in the limit $r \to 0$, is $i\pi/9$. [5]

(iii) Show that the contribution to I from the arc of the larger semi-circle, in the limit $R \to \infty$, is zero. [3]

(iv) Hence use the Residue Theorem to show that

$$\int_{-\infty}^{\infty} \frac{\sin x}{x(x^2+9)}\, dx = \frac{\pi}{9}\left(1 - e^{-3}\right).$$

[5]

Recall that the residue of a complex function $F(z)$ at a pole $z = a$ of multiplicity m is given by the expression

$$\lim_{z \to a} \left\{ \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m F(z) \right] \right\}.$$

2. a) Derive the formula for the Laplace transform of the Heaviside function:

$$\mathscr{L}\left[H(t-a)\right] = \frac{e^{-as}}{s}. \qquad [4]$$

b) Hence, or otherwise, obtain the Laplace transform of the rectangular pulse fuction:

$$f(t) = \begin{cases} 3 & \text{for} \quad 0 \le t \le 2 \\ 0 & \text{for} \quad t > 2 \end{cases} \qquad [3]$$

c) Use partial fractions to obtain the inverse Laplace transform:

$$\mathscr{L}^{-1}\left[\frac{1}{s\,(s^2+4s+3)}\right]. \qquad [5]$$

d) Use (b) and (c) and Laplace transforms to show that the differential equation

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 3x = \begin{cases} 3 & \text{for} \quad 0 \le t \le 2 \\ 0 & \text{for} \quad t > 2 \end{cases},$$

which satifies the initial conditions $x = 1$ and $\dfrac{dx}{dt} = 0$ when $t = 0$, has solution

$$x(t) = \begin{cases} 1 & \text{for} \quad 0 \le t \le 2 \\ -\frac{1}{2}e^{-3(t-2)} + \frac{3}{2}e^{-(t-2)} & \text{for} \quad t > 2 \end{cases} \qquad [8]$$