EE3-09

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2016

EEE/EIE PART III/IV: MEng, BEng and ACGI

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**CONTROL ENGINEERING**

Friday, 16 December 9:00 am

Time allowed: 3:00 hours

**There are FOUR questions on this paper.**

**Answer ALL questions.**

*All questions carry equal marks*

**Any special instructions for invigilators and information for candidates are on page 1.**

Examiners responsible First Marker(s) : A. Astolfi
Second Marker(s) : I.M. Jaimoukha

## CONTROL ENGINEERING

1. Consider a linear, single-input, single-output, discrete-time, with state $x = [x_1, x_2]'$, input $u$, output $y$,

$$A = \begin{bmatrix} 0 & 1 \\ -\alpha & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} -\delta & 1 \end{bmatrix}, \quad D = 0,$$

and $\alpha$ and $\delta$ constant.

a) Show that the system is reachable, hence controllable for all values of $\alpha$. [ 2 marks ]

b) Determine a condition on $\alpha$ and $\delta$ such that the system is observable. [ 4 marks ]

c) Consider a second discrete-time system, with state $\xi$, input $v$ and output $w$, described by the equations

$$\xi^+ = v, \qquad w = \xi.$$

Note that this system is reachable, controllable and observable.

i) Consider the interconnection of the two discrete-time systems resulting from the interconnection equation

$$v = y,$$

yielding a new system with input $u$ and output $w$. Write a state space representation for the interconnected system. [ 2 marks ]

ii) Study the reachability and controllability properties of the interconnected system as a function of $\alpha$ and $\delta$. [ 8 marks ]

iii) Show that the interconnected system is observable provided the system with state $x$ is observable. [ 2 marks ]

iv) The zeros of the system with state $x$ are the complex numbers $z$ such that

$$\det \begin{bmatrix} zI - A & B \\ C & -D \end{bmatrix} = 0.$$

Show that the system with state $x$ has one zero and that when this coincides with the eigenvalue of the system with state $\xi$ the interconnected system is not reachable. Why does the interconnected system remain controllable? [ 2 marks ]

2. Consider a linear, continuous-time, system described by the equations

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = -2x_1 - 3x_2 + u,$$
$$y = x_1 - x_2,$$

with $x(t) = [x_1(t), x_2(t)]' \in \mathbb{R}^2$, $u(t) \in \mathbb{R}$ and $y(t) \in \mathbb{R}$.

a) Show that the system is observable. [ 2 marks ]

b) One could design an observer for the state $x_2$ alone considering the dynamical system

$$\dot{z} = fz + gu + hy$$

and selecting the constant $f$, $g$ and $h$ such that

$$\frac{d(z - x_2)}{dt} = -k(z - x_2),$$

for some $k > 0$. Determine the values of $f$, $g$ and $h$ that achieve this objective and hence the value of $k$. Show that indeed the state $z$ can be used as an asymptotic estimate for $x_2$. [ 6 marks ]

c) Using the results of part b) show how to build an asymptotic estimate of the state $x_1$ using $y$ and $z$. [ 2 marks ]

d) Show that the system with state $[x_1(t), x_2(t), z(t)]'$ and input $u$ is not controllable. Determine the uncontrollable mode and show that the system is stabilizable. Comment on the connection between the results of this part and the results of part b). [ 4 marks ]

e) Let $u = -py - qz$. Determine $p$ and $q$ such that the closed-loop system has two eigenvalues at $-2$. Where is the third eigenvalue of the closed-loop system? [ 6 marks ]

3. Consider the problem of firing a rocket so that it arrives to its final destination with zero velocity in a minimum amount of time, and with its acceleration bounded in absolute value by one. Assume that the rocket has unity mass, and its motion is described by the differential equation $\ddot{x} = u$, where $x(t)$ is the position of the rocket, $\dot{x}(t)$ is its velocity and $u(t)$ the control action. Assume that the initial condition is $x(0) = \bar{x}$, $\dot{x}(0) = 0$ and the final condition, at some time $T > 0$ to be determined, is $x(T) = \dot{x}(T) = 0$.

a) Write a state space representation for the system with state $x_1(t) = x(t)$ and $x_2(t) = \dot{x}(t)$, that is determine the matrices $A$ and $B$ describing the system. [ 2 marks ]

b) Show that the system is controllable hence argue that there exists an input signal $u$ which drives the state of the system from the initial condition to the final condition in any given time $T > 0$. [ 2 marks ]

c) The input signal determined in part b) may violate the requirement that the acceleration is always bounded in absolute value by one, or may not achieve the state transfer in minimum time. To satisfy the acceleration constrain and to minimize the transfer time one could follow the following procedure.

i) Set $u(t) = 1$ and show that (recall the considered initial condition) $x_1(t) = \bar{x} + \frac{1}{2}t^2$ and $x_2(t) = t$. Eliminate the variable $t$ in the above equations to determine a relation between $x_1$ and $x_2$ and use this relation to show that the trajectories of the system are left facing parabolas, parameterized by $\bar{x}$, which are *travelled* upward. Plot this family of parabolas on the state space using $\bar{x}$ as a parameter and $\rightarrow$'s to indicate the positive direction of $t$. Show that there is only one parabola that goes through the origin of the state space. [ 4 marks ]

ii) Set $u(t) = -1$ and (recall the considered initial condition) compute $x_1(t)$ and $x_2(t)$. Eliminate the variable $t$ in the equations of $x_1(t)$ and $x_2(t)$ to determine a relation between $x_1$ and $x_2$. Use this relation to show that the trajectories of the system are right facing parabolas, parameterized by $\bar{x}$, which are *travelled* downward. Plot this family of parabolas on the state space using $\bar{x}$ as a parameter and $\rightarrow$'s to indicate the positive direction of $t$. Show that there is only one parabola that goes through the origin of the state space. [ 4 marks ]

iii) Using the plots of trajectories determined in parts c.i) and c.ii) argue that from any initial condition $x_1(0) = \bar{x}$, $x_2(0) = 0$ the state can be controlled to the origin using the input sequence (for some $T > \bar{t} > 0$)

$$u(t) = \begin{cases} 1 & t \in [0, \bar{t}) \\ -1 & t \in [\bar{t}, T) \end{cases}$$

for $\bar{x} < 0$, and the input sequence

$$u(t) = \begin{cases} -1 & t \in [0, \bar{t}) \\ 1 & t \in [\bar{t}, T) \end{cases}$$

for $\bar{x} > 0$. [ 6 marks ]

iv) While it is difficult to show that the above control strategy drives the state of the system from the considered initial conditions to zero in minimum time, show that the state is indeed driven to the origin in finite time and that the constraint on the acceleration is satisfied. Finally, explain why the selection $u(t) = 0$, for $t \geq T$ guarantees that the state of the system remains at the origin for all $t \geq T$. [ 2 marks ]

4. Consider a linear, discrete-time, system described by the equations

$$\begin{aligned} x_1^+ &= \varepsilon x_2 - \varepsilon u, \\ x_2^+ &= x_1 + 2x_2 + u, \end{aligned}$$

with state $x = [x_1, x_2]'$ and input $u$. The parameter $\varepsilon$ is constant.

a) Write the matrices $A$ and $B$ of a state space description of the system. [ 2 marks ]

b) Determine conditions on the coefficient $\varepsilon$ such that the system is reachable. Identify the unreachable mode, whenever one exists. [ 6 marks ]

c) Using the results in part b) determine conditions on $\varepsilon$ such that the system is controllable. [ 4 marks ]

d) Let $u = Kx$. Compute $K$ such that all eigenvalues of the closed-loop system are at 0. Note that $K$ is a function of $\varepsilon$. Show that $\lim_{\varepsilon \to 0} K$ is well-defined, whereas $\lim_{\varepsilon \to 3} K$ is not. Explain why. [ 6 marks ]

e) Let $K_0 = \lim_{\varepsilon \to 0} K$. Suppose that one uses this feedback gain, that is $u = K_0 x$, to control the system for all values of $\varepsilon$. Write the equations of the closed-loop system and determine for which values of $\varepsilon$ the closed-loop system is asymptotically stable. [ 2 marks ]