Imperial College
London

[E2.8 (Maths 3) 2011]

B.ENG. AND M.ENG. EXAMINATIONS 2011

PART II Paper 3 : MATHEMATICS (ELECTRICAL ENGINEERING)

Date Wednesday 8th June 2011 2.00 - 5.00 pm

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.

Answer EIGHT questions.

Please answer questions from Section A and Section B in separate answer-books.

A mathematical formulae sheet is provided.

Statistical data sheets are provided.

[Before starting, please make sure that the paper is complete; there should be NINE pages, with a total of TWELVE questions. Ask the invigilator for a replacement if your copy is faulty.]

1. (i) Consider the mapping

$$w = \frac{1}{(z-1)^2}$$

from the z-plane to the w-plane where $w = u + iv$. Show that the circle

$$(x-1)^2 + y^2 = R^2$$

in the z-plane, maps to the circle

$$u^2 + v^2 = R^{-4}$$

in the w-plane.

(ii) Now consider

$$w = \frac{1}{z}.$$

For the class of circles

$$(x-1)^2 + (y-1)^2 = r^2,$$

show that when $r = \sqrt{2}$ the circle maps to the straight line $v = u - \frac{1}{2}$ in the $u - v$ plane whereas when $r = 1$ it maps to the circle $(u-1)^2 + (v+1)^2 = 1$.

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2. (i) Use the Residue Theorem to show that

$$\oint_C \frac{z\,dz}{(z-1)^2(z-i)} = 0\,,$$

where the contour C is the circle of radius 2 centred at the origin. What is the answer when C is changed to be the rectangle with vertices at $\pm\frac{1}{2}+2i$ and $\pm\frac{1}{2}-2i$?

(ii) The closed contour C is a circle of arbitrary finite radius r centred at the origin, Show that the value of the complex integral

$$\oint_C \frac{dz}{z} = 2\pi i$$

in three different ways:

(a) By the Residue Theorem;

(b) By evaluating the integral as a line integral around the circle $z = r\exp(i\theta)$;

(c) By direct integration.

The residue of a complex function $f(z)$ at a pole $z = a$ of multiplicity m is given by the expression

$$\lim_{z\to a}\frac{1}{(m-1)!}\left[\frac{d^{m-1}}{dz^{m-1}}\left\{(z-a)^m f(z)\right\}\right].$$

3. Consider the contour integral

$$\oint_C \frac{e^{iz}}{(z^2+1)^2}\,dz\,,$$

where the closed contour C consists of a semi-circle in the upper half of the complex plane and the real axis. Use the Residue Theorem to show that

$$\int_{-\infty}^{\infty}\frac{\cos x}{(x^2+1)^2}\,dx = \pi/e\,.$$

The residue of a complex function $f(z)$ at a pole $z = a$ of multiplicity m is given by

$$\lim_{z\to a}\frac{1}{(m-1)!}\left[\frac{d^{m-1}}{dz^{m-1}}\left\{(z-a)^m f(z)\right\}\right].$$

4. The sawtooth function $\Pi(t)$, the tent function $\Lambda(t)$, and the sinc-function $\text{sinc}\,(t)$ are defined respectively by

$$\Pi(t) = \begin{cases} 1, & -1/2 \le t \le 1/2, \\ 0, & \text{otherwise}, \end{cases}$$

$$\Lambda(t) = \begin{cases} 1+t, & -1 \le t \le 0, \\ 1-t, & 0 \le t \le 1, \end{cases}$$

and

$$\text{sinc}\,(t) = \frac{\sin(t/2)}{(t/2)}.$$

Show that the Fourier transforms of the above functions are

$$(i) \qquad \overline{\Pi}(\omega) = \text{sinc}\,(\omega)\,,$$

$$(ii) \qquad \overline{\Lambda}(\omega) = \text{sinc}^2\,(\omega)\,.$$

Use Parseval's equality to show that

$$(iii) \qquad \int_{-\infty}^{\infty} \text{sinc}^2\,(\omega)\,d\omega = 2\pi\,,$$

$$(iv) \qquad \int_{-\infty}^{\infty} \text{sinc}^4\,(\omega)\,d\omega = 4\pi/3\,.$$

Parseval's equality is given by

$$\int_{-\infty}^{\infty} |f(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\overline{f}(\omega)|^2\,d\omega\,.$$

5. Given that $\overline{f}(s) = \mathcal{L}\{f(t)\}$ is the Laplace transform of $f(t)$, prove that when a is a real constant

$$\mathcal{L}\{e^{at}f(t)\} \;=\; \overline{f}(s-a)\,, \qquad\qquad Re(s) > a\,.$$

A 2nd order ordinary differential equation, with initial values, takes the form

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 8x = \delta(t-1)\,, \qquad x = \frac{dx}{dt} \;=\; 0 \text{ when } t = 0\,,$$

where δ represents the Dirac delta function. Use the Laplace convolution theorem to show that the solution for $x(t)$ is

$$x(t) = \begin{cases} \frac{1}{2}e^{-2(t-1)}\sin\left[2(t-1)\right] & t > 1\,, \\ 0 & 0 \le t \le 1\,. \end{cases}$$

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6. A function $f(t)$ has Laplace transform $\overline{f}(s) = \mathcal{L}\{f(t)\}$. Prove that

$$\mathcal{L}\left\{\int_0^t f(u)\,du\right\} = \frac{\overline{f}(s)}{s}\,.$$

Show that if

$$\overline{f}(s) = \frac{1}{(1+s^2)^2}$$

then

$$f(t) = \frac{1}{2}\left(\sin t - t\cos t\right)\,.$$

Hence show that

$$\mathcal{L}^{-1}\left\{\frac{1}{s(1+s^2)^2}\right\} = 1 - \cos t - \frac{1}{2}t\sin t.$$

7. P and Q are continuous functions of x and y with continuous first partial derivatives in a simply connected region R with a piecewise smooth boundary C. Green's Theorem in a plane states that

$$\oint_C (Pdx + Qdy) = \int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dxdy\,.$$

Find a two-dimensional vector $\boldsymbol{u}$, defined in terms of P and Q, to show that Green's theorem can be re-expressed as the two-dimensional version of the Divergence Theorem

$$\oint_C \boldsymbol{u}\cdot\boldsymbol{n}\,ds = \int\int_R (\mathrm{div}\,\boldsymbol{u})\,dxdy\,,$$

where $\boldsymbol{n}$ is the unit normal to the curve C.

If $\boldsymbol{u}$ is given by $\boldsymbol{u} = \boldsymbol{i}x^2 + \boldsymbol{j}y^2$ and R is the first quadrant of the circle of unit radius, evaluate the right hand side of the Divergence Theorem to show that

$$\int\int_R (\mathrm{div}\,\boldsymbol{u})\,dxdy = 4/3\,.$$

8. Consider a metal plate occupying a region $\mathcal{R}$ in the first quadrant of the xy-plane bounded by the curves

$$xy = 1,\ x^2 - y^2 = 1,\ y = 0 \ \text{ and } \ y = x\ .$$

If the mass/unit area of the plate is $f(x, y) = \exp(x^2 - y^2)$ then the moment of inertia of the plate about the z-axis is

$$I_{0z} \ = \ \int\!\!\int_{\mathcal{R}} (x^2 + y^2)\, \exp(x^2 - y^2)\, dx\, dy\ .$$

Using new variables u and v such that $u = xy$ and $v = x^2 - y^2$, show that the Jacobian of the transformation from (u, v) to (x, y) satisfies

$$|J| \ = \ 2(x^2 + y^2)\ ,$$

and hence that the Jacobian for the transformation from (x, y) to (u, v) is

$$|J'| \ = \ \frac{1}{2(x^2 + y^2)}\ .$$

Then show that

$$I_{0z} \ = \ \int\!\!\int_{\mathcal{R}*} \frac{1}{2}\, e^v\, du\, dv\ ,$$

where the region $\mathcal{R}^*$ in the uv-plane is to be determined.

Sketch the region $\mathcal{R}$ in the xy-plane and the new region $\mathcal{R}^*$ in the uv-plane and then evaluate I_{0z}.

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9. (i) The scalar field $\phi(\boldsymbol{r})$ is given by

$$\phi(\boldsymbol{r}) = \frac{\phi_0}{8\pi R}\left(3 - \frac{r^2}{R^2}\right) \quad \text{for} \quad r < R$$

and

$$\phi(\boldsymbol{r}) = \frac{\phi_0}{4\pi r} \quad \text{for} \quad r > R$$

where ϕ_0 is a constant, R is a fixed radius, $\boldsymbol{r} = x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}$ and $r = (x^2 + y^2 + z^2)^{1/2}$.

Calculate grad ϕ when $r < R$ and also when $r > R$.

Show that div grad $\phi = -\dfrac{3\phi_0}{4\pi R^3}$ for $r < R$ and div grad $\phi = 0$ for $r > R$.

(ii) The vector field $\boldsymbol{A}(\boldsymbol{r})$ is given by

$$\boldsymbol{A}(\boldsymbol{r}) = -\frac{I}{4\pi\, a^2}\,(x^2 + y^2)\boldsymbol{k} \quad \text{for} \quad x^2 + y^2 < a^2$$

and

$$\boldsymbol{A}(\boldsymbol{r}) = -\frac{I}{4\pi}\left[1 + \ln\left(\frac{x^2 + y^2}{a^2}\right)\right]\boldsymbol{k} \quad \text{for} \quad x^2 + y^2 > a^2,$$

where I is a constant and a is a fixed radius.

Calculate curl $\boldsymbol{A}$ when $x^2 + y^2 < a^2$ and also when $x^2 + y^2 > a^2$.

N.B. $\nabla \cdot \boldsymbol{A} = \text{div}\boldsymbol{A}$, $\nabla \times \boldsymbol{A} = \text{curl}\, A$, $\nabla\phi = \text{grad}\,\phi$.

10. Write down the condition for the line integral

$$\int_{\mathcal{P}} \left[\, f(x,\, y)\, dx + g(x,\, y)\, dy \,\right]$$

to be independent of the path P joining the initial point A to the final point B.

Show that

$$\int_{\mathcal{P}} \left[\, (x\, e^{y} + b\, \sin x)\, dx \;+\; (ax^2\, e^{y} + c\cos y)\, dy \,\right]$$

is independent of the path $\mathcal{P}$ for any values of b and c if $a = 1/2$.

Find the potential function $V(x,\, y)$ such that

$$\frac{\partial V}{\partial x} \;=\; xe^{y} \;+\; b\, \sin x$$

and

$$\frac{\partial V}{\partial y} \;=\; a\, x^2\, e^{y} \;+\; c\, \cos y$$

for $a = 1/2$ and arbitrary b and c.

Hence evaluate the integral when A is the point $(0,\, \pi/2)$ and B is the point $(\pi/2,\, 0)$.

Evaluate the integral also by using a particular path $\mathcal{P}$ of your own choice joining A to B and verify that the two methods give the same answer.

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SECTION B [E 2.8 (Maths 3) 2011]

11. A factory produces engine piston and cylinder units. The diameter of the pistons is normally distributed with mean 88mm and standard deviation 0.7mm. Independently, the diameter of the cylinders is normally distributed with mean 91mm and standard deviation 1mm. A piston and cylinder unit is acceptable if the piston fits inside the cylinder, and its diameter is within 5mm of that of the cylinder.

   (i) We have a cylinder with diameter 89.7mm and we select a piston at random. What is the probability they form an acceptable unit?

   (ii) We select a piston and a cylinder at random. Show that the probability that they form an acceptable unit is 0.942.

   (iii) For a sample of 10 such piston/cylinder pairs (as in part (ii)), what is the distribution of the number of acceptable units? Compute the probability there are more than two unacceptable units in our sample.

12. Consider the autoregressive process

$$y_t = \frac{5}{6}\, y_{t-1} - \frac{1}{6}\, y_{t-2} + \frac{1}{6}\, e_t \ ,$$

where $\{e_t\}$ is white noise with $E(e_t) = 0$ and $\mathrm{Var}(e_t) = 1$.

   (i) What is the order of the autoregression?

   (ii) Define the backshift operator B and show that

$$(2 - B)(3 - B) y_t = e_t .$$

   What is $E(y_t)$?

   (iii) Prove that

$$[(2 - B)(3 - B)]^{-1} = \frac{1}{2 - B} - \frac{1}{3 - B}$$

   and hence show that $\mathrm{Var}(y_t) = 7/120$. You may use the result

$$(k - B)^{-1} = \frac{1}{k} \sum_{j=0}^{\infty} \left(\frac{B}{k}\right)^j .$$

END OF PAPER

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MATHEMATICAL FORMULAE

1. VECTOR ALGEBRA

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1,\, a_2,\, a_3)$$

Scalar (dot) product: $\mathbf{a}\,.\,\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

Vector (cross) product:

$$\mathbf{a}\times\mathbf{b} \;=\; \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Scalar triple product:

$$[\mathbf{a},\,\mathbf{b},\,\mathbf{c}] = \mathbf{a}\,.\,\mathbf{b}\times\mathbf{c} = \mathbf{b}\,.\,\mathbf{c}\times\mathbf{a} = \mathbf{c}\,.\,\mathbf{a}\times\mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product: $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{c}\,.\,\mathbf{a})\mathbf{b} - (\mathbf{b}\,.\,\mathbf{a})\mathbf{c}$

2. SERIES

$$(1+x)^\alpha = 1 + \alpha\, x + \frac{\alpha(\alpha-1)}{2!}\,x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}\,x^3 + \ldots \quad (\alpha \text{ arbitrary},\ |x|<1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots\ ,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n\frac{x^{2n}}{(2n)!} + \ldots\ ,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + \ldots\ ,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^n\frac{x^{n+1}}{(n+1)} + \ldots (-1 < x \le 1)$$

3. TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin(a+b) = \sin a \,\cos b + \cos a \,\sin b \;;$$

$$\cos(a+b) = \cos a \,\cos b - \sin a \,\sin b \,.$$

$$\cos iz = \cosh z \,; \quad \cosh iz = \cos z \,; \quad \sin iz = i \sinh z \,; \quad \sinh iz = i \sin z \,.$$

4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f\, D^n g + \tbinom{n}{1}\, Df\, D^{n-1}\, g + \ldots + \tbinom{n}{r}\, D^r f\, D^{n-r} g + \ldots + D^n f\; g \,.$$

(b) Taylor's expansion of $f(x)$ about $x = a$:

$$f(a+h) = f(a) + hf'(a) + h^2 f''(a)/2! + \ldots + h^n f^{(n)}(a)/n! + \epsilon_n(h)\,,$$

where $\quad \epsilon_n(h) = h^{n+1} f^{(n+1)}(a+\theta h)/(n+1)!, \quad 0 < \theta < 1\,.$

(c) Taylor's expansion of $f(x, y)$ about (a, b):

$$f(a+h,\, b+k) = f(a,\, b) + \left[hf_x + kf_y\right]_{a.b} + 1/2!\, \left[h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}\right]_{a.b} + \cdots$$

(d) Partial differentiation of $f(x, y)$:

i. If $y = y(x)$, then $f = F(x)$, and $\dfrac{dF}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dx}\,.$

ii. If $x = x(t)$, $y = y(t)$, then $f = F(t)$, and $\dfrac{dF}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}\,.$

iii. If $x = x(u,\, v)$, $y = y(u,\, v)$, then $f = F(u,\, v)$, and

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}\,, \quad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}\,.$$

(e) Stationary points of $f(x, y)$ occur where $f_x = 0$, $f_y = 0$ simultaneously.
Let (a, b) be a stationary point: examine $D = [f_{xx} f_{yy} - (f_{xy})^2]_{a.b}$.
If $D > 0$ and $f_{xx}(a, b) < 0$, then (a, b) is a maximum;
If $D > 0$ and $f_{xx}(a, b) > 0$, then (a, b) is a minimum;
If $D < 0$ then (a, b) is a saddle-point.

(f) Differential equations:

i. The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating factor $\quad I(x) = \exp[\int P(x)(dx]$, so that $\dfrac{d}{dx}(Iy) = IQ$.

ii. $P(x, y)dx + Q(x, y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y\,.$

5. INTEGRAL CALCULUS

(a) An important substitution: $\tan(\theta/2) = t$:
$\sin\theta = 2\,t/(1+t^2)$, $\cos\theta = (1-t^2)/(1+t^2)$, $d\theta = 2\,dt/(1+t^2)$.

(b) Some indefinite integrals:

$$\int (a^2 - x^2)^{-1/2} dx = \sin^{-1}\left(\frac{x}{a}\right), \quad |x| < a\,.$$

$$\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1}\left(\frac{x}{a}\right) = \ln\left\{\frac{x}{a} + \left(1 + \frac{x^2}{a^2}\right)^{1/2}\right\}.$$

$$\int (x^2 - a^2)^{-1/2} dx = \cosh^{-1}\left(\frac{x}{a}\right) = \ln\left|\frac{x}{a} + \left(\frac{x^2}{a^2} - 1\right)^{1/2}\right|.$$

$$\int (a^2 + x^2)^{-1} dx = \left(\frac{1}{a}\right)\tan^{-1}\left(\frac{x}{a}\right).$$

6. NUMERICAL METHODS

(a) Approximate solution of an algebraic equation:

If a root of $f(x) = 0$ occurs near $x = a$, take $x_0 = a$ and
$x_{n+1} = x_n - [f(x_n)/f'(x_n)]$, $n = 0, 1, 2 \ldots$

(Newton Raphson method).

(b) Formulae for numerical integration: Write $x_n = x_0 + nh$, $y_n = y(x_n)$.

i. Trapezium rule (1-strip): $\int_{x_0}^{x_1} y(x)dx \approx (h/2)\,[y_0 + y_1]$.

ii. Simpson's rule (2-strip): $\int_{x_0}^{x_2} y(x)dx \approx (h/3)\,[y_0 + 4y_1 + y_2]$.

(c) Richardson's extrapolation method: Let $I = \int_a^b f(x)dx$ and let I_1, I_2 be two estimates of I obtained by using Simpson's rule with intervals h and $h/2$. Then, provided h is small enough,

$$I_2 + (I_2 - I_1)/15\,,$$

is a better estimate of I.

7. LAPLACE TRANSFORMS

Function	Transform	Function	Transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)dt$	$af(t) + bg(t)$	$aF(s) + bG(s)$
df/dt	$sF(s) - f(0)$	d^2f/dt^2	$s^2F(s) - sf(0) - f'(0)$
$e^{at}f(t)$	$F(s-a)$	$tf(t)$	$-dF(s)/ds$
$(\partial/\partial\alpha)f(t,\alpha)$	$(\partial/\partial\alpha)F(s,\alpha)$	$\int_0^t f(t)dt$	$F(s)/s$
$\int_0^t f(u)g(t-u)du$	$F(s)G(s)$		
1	$1/s$	$t^n (n = 1, 2\ldots)$	$n!/s^{n+1}$, $(s > 0)$
e^{at}	$1/(s-a)$, $(s > a)$	$\sin\omega t$	$\omega/(s^2+\omega^2)$, $(s > 0)$
$\cos\omega t$	$s/(s^2+\omega^2)$, $(s > 0)$	$H(t-T) = \begin{cases} 0, & t < T \\ 1, & t > T \end{cases}$	e^{-sT}/s, $(s, T > 0)$

8. FOURIER SERIES

If $f(x)$ is periodic of period $2L$, then $f(x+2L) = f(x)$, and

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \text{ where}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad n = 0, 1, 2, \ldots, \text{ and}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1, 2, 3, \ldots$$

Parseval's theorem

$$\frac{1}{L}\int_{-L}^{L} [f(x)]^2\, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right).$$

1. $\boxed{\text{Probabilities for events}}$

For events A, B, and C $\quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$

More generally $\quad P(\bigcup A_i) = \sum P(A_i) - \sum P(A_i \cap A_j) + \sum P(A_i \cap A_j \cap A_k) - \cdots$

The odds in favour of A $\quad P(A) \,/\, P(\overline{A})$

Conditional probability $\quad P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$ provided that $P(B) > 0$

Chain rule $\quad P(A \cap B \cap C) = P(A)\,P(B \mid A)\,P(C \mid A \cap B)$

Bayes' rule $\quad P(A \mid B) = \dfrac{P(A)\,P(B \mid A)}{P(A)\,P(B \mid A) + P(\overline{A})\,P(B \mid \overline{A})}$

A and B are independent if $\quad P(B \mid A) = P(B)$

A, B, and C are independent if $\quad P(A \cap B \cap C) = P(A)P(B)P(C)$, and

$$P(A \cap B) = P(A)P(B), \quad P(B \cap C) = P(B)P(C), \quad P(C \cap A) = P(C)P(A)$$

2. $\boxed{\text{Probability distribution, expectation and variance}}$

The probability distribution for a discrete random variable X is called the probability mass function (pmf) and is the complete set of probabilities $\{p_x\} = \{P(X = x)\}$

Expectation $\quad E(X) = \mu = \sum_x x p_x$

For function $g(x)$ of x, $\quad E\{g(X)\} = \sum_x g(x) p_x$, so $E(X^2) = \sum_x x^2 p_x$

Sample mean $\quad \overline{x} = \dfrac{1}{n} \sum_k x_k$ estimates μ from random sample $x_1, x_2, \ldots, x_n$

Variance $\quad \operatorname{var}(X) = \sigma^2 = E\{(X - \mu)^2\} = E(X^2) - \mu^2$

Sample variance $\quad s^2 = \dfrac{1}{n-1}\left\{ \sum_k x_k^2 - \dfrac{1}{n}\Big(\sum_j x_j\Big)^2 \right\}$ estimates σ^2

Standard deviation $\quad \operatorname{sd}(X) = \sigma$

If value y is observed with frequency n_y

$$n = \sum_y n_y, \quad \sum_k x_k = \sum_y y n_y, \quad \sum_k x_k^2 = \sum_y y^2 n_y$$

Skewness $\quad \beta_1 = E\left(\dfrac{X - \mu}{\sigma}\right)^3$ is estimated by $\dfrac{1}{n-1} \sum \left(\dfrac{x_i - \overline{x}}{s}\right)^3$

Kurtosis $\quad \beta_2 = E\left(\dfrac{X - \mu}{\sigma}\right)^4 - 3$ is estimated by $\dfrac{1}{n-1} \sum \left(\dfrac{x_i - \overline{x}}{s}\right)^4 - 3$

Sample median $\quad \tilde{x}$ or x_{med}. Half the sample values are smaller and half larger

If the sample values $x_1, \ldots, x_n$ are ordered as $x_{(1)} \le x_{(2)} \le \cdots \le x_{(n)}$,
then $\tilde{x} = x_{(\frac{n+1}{2})}$ if n is odd, and $\tilde{x} = \frac{1}{2}\left(x_{(\frac{n}{2})} + x_{(\frac{n+2}{2})}\right)$ if n is even

α-quantile $Q(\alpha)$ is such that $P(X \leq Q(\alpha)) = \alpha$

Sample α-quantile $\widehat{Q}(\alpha)$ Proportion α of the data values are smaller

Lower quartile Q1 $= \widehat{Q}(0.25)$ one quarter are smaller

Upper quartile Q3 $= \widehat{Q}(0.75)$ three quarters are smaller

Sample median $\widetilde{x} = \widehat{Q}(0.5)$ estimates the population median $Q(0.5)$

3. Probability distribution for a continuous random variable

The cumulative distribution function (cdf) $F(x) = P(X \leq x) = \int_{x_0=-\infty}^{x} f(x_0)\,\mathrm{d}x_0$

The probability density function (pdf) $f(x) = \dfrac{\mathrm{d}F(x)}{\mathrm{d}x}$

$E(X) = \mu = \int_{-\infty}^{\infty} x\, f(x)\mathrm{d}x$, $\mathrm{var}\,(X) = \sigma^2 = E(X^2) - \mu^2$, where $E(X^2) = \int_{-\infty}^{\infty} x^2\, f(x)\mathrm{d}x$

4. Discrete probability distributions

Discrete Uniform *Uniform* (n)

$$p_x = \frac{1}{n} \quad (x = 1, 2, \ldots, n) \qquad \mu = (n+1)/2,\ \ \sigma^2 = (n^2-1)/12$$

Binomial distribution *Binomial* (n, θ)

$$p_x = \binom{n}{x} \theta^x (1-\theta)^{n-x} \quad (x = 0, 1, 2, \ldots, n) \qquad \mu = n\theta\,,\ \ \sigma^2 = n\theta(1-\theta)$$

Poisson distribution *Poisson* (λ)

$$p_x = \frac{\lambda^x e^{-\lambda}}{x!} \quad (x = 0, 1, 2, \ldots) \quad (\text{with } \lambda > 0) \qquad \mu = \lambda\,,\ \ \sigma^2 = \lambda$$

Geometric distribution *Geometric* (θ)

$$p_x = (1-\theta)^{x-1}\theta \quad (x = 1, 2, 3, \ldots) \qquad \mu = \frac{1}{\theta}\,,\ \ \sigma^2 = \frac{1-\theta}{\theta^2}$$

5. Continuous probability distributions

Uniform distribution *Uniform* (α, β)

$$f(x) = \begin{cases} \dfrac{1}{\beta - \alpha} & (\alpha < x < \beta), \\ 0 & (\text{otherwise}). \end{cases} \qquad \mu = (\alpha+\beta)/2\,,\ \ \sigma^2 = (\beta-\alpha)^2/12$$

Exponential distribution *Exponential* (λ)

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & (0 < x < \infty), \\ 0 & (-\infty < x \leq 0). \end{cases} \qquad \mu = 1/\lambda\,,\ \ \sigma^2 = 1/\lambda^2$$

Normal distribution $N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2 \right\} \quad (-\infty < x < \infty), \quad E(X) = \mu, \quad \mathrm{var}(X) = \sigma^2$$

Standard normal distribution $N(0,1)$

If X is $N(\mu, \sigma^2)$, then $Y = \dfrac{X - \mu}{\sigma}$ is $N(0,1)$

6. Reliability

For a device in continuous operation with failure time random variable T having pdf $f(t)$ $(t > 0)$

The reliability function at time t	$R(t)$	$= P(T > t)$	
The failure rate or hazard function	$h(t)$	$= f(t)/R(t)$	
The cumulative hazard function	$H(t)$	$= \int_0^t h(t_0)\, \mathrm{d}t_0$	$= -\ln\{R(t)\}$
The Weibull(α, β) distribution has	$H(t)$	$= \beta t^\alpha$	

7. System reliability

For a system of k devices, which operate independently, let

$R_i = P(D_i) = P($"device i operates"$)$

The system reliability, R, is the probability of a path of operating devices

A system of devices in series operates only if every device operates

$$R = P(D_1 \cap D_2 \cap \cdots \cap D_k) = R_1 R_2 \cdots R_k$$

A system of devices in parallel operates if any device operates

$$R = P(D_1 \cup D_2 \cup \cdots \cup D_k) = 1 - (1 - R_1)(1 - R_2) \cdots (1 - R_k)$$

8. Covariance and correlation

The covariance of X and Y $\quad \mathrm{cov}(X,Y) = E(XY) - \{E(X)\}\{E(Y)\}$

From pairs of observations $(x_1, y_1), \ldots, (x_n, y_n)$ $\quad S_{xy} = \sum_k x_k y_k - \frac{1}{n}(\sum_i x_i)(\sum_j y_j)$

$$S_{xx} = \sum_k x_k^2 - \frac{1}{n}(\sum_i x_i)^2, \qquad S_{yy} = \sum_k y_k^2 - \frac{1}{n}(\sum_j y_j)^2$$

Sample covariance	s_{xy}	$= \dfrac{1}{n-1} S_{xy}$	estimates $\mathrm{cov}(X,Y)$
Correlation coefficient	ρ	$= \mathrm{corr}(X,Y)$	$= \dfrac{\mathrm{cov}(X,Y)}{\mathrm{sd}(X) \cdot \mathrm{sd}(Y)}$
Sample correlation coefficient	r	$= \dfrac{S_{xy}}{\sqrt{S_{xx} S_{yy}}}$	estimates ρ

9. Sums of random variables

$$E(X+Y) = E(X)+E(Y)$$

$$\text{var}\,(X+Y) = \text{var}\,(X)+\text{var}\,(Y)+2\,\text{cov}\,(X,Y)$$

$$\text{cov}\,(aX+bY,\ cX+dY) = (ac)\,\text{var}\,(X)+(bd)\,\text{var}\,(Y)+(ad+bc)\,\text{cov}\,(X,Y)$$

If X is $N(\mu_1,\sigma_1^2)$, Y is $N(\mu_2,\sigma_2^2)$, and $\text{cov}\,(X,Y)=c$, then $X+Y$ is $N(\mu_1+\mu_2,\ \sigma_1^2+\sigma_2^2+2c)$

10. Bias, standard error, mean square error

If t estimates θ (with random variable T giving t)

Bias of t	$\text{bias}\,(t)$	$= E(T)-\theta$
Standard error of t	$\text{se}\,(t)$	$= \text{sd}\,(T)$
Mean square error of t	$\text{MSE}(t)$	$= E\{(T-\theta)^2\} = \{\text{se}\,(t)\}^2+\{\text{bias}\,(t)\}^2$

If $\overline{x}$ estimates μ, then $\text{bias}\,(\overline{x})=0$, $\text{se}\,(\overline{x})=\sigma/\sqrt{n}$, $\text{MSE}(\overline{x})=\sigma^2/n$, $\widehat{\text{se}}\,(\overline{x})=s/\sqrt{n}$

Central limit property If n is fairly large, $\overline{x}$ is from $N(\mu,\ \sigma^2/n)$ approximately

11. Likelihood

The likelihood is the joint probability as a function of the unknown parameter θ.

For a random sample $x_1,x_2,\ldots,x_n$

$$\ell(\theta;x_1,x_2,\ldots,x_n) = P(X_1=x_1\,|\,\theta)\cdots P(X_n=x_n\,|\,\theta) \qquad \text{(discrete distribution)}$$

$$\ell(\theta;x_1,x_2,\ldots,x_n) = f(x_1\,|\,\theta)\,f(x_2\,|\,\theta)\cdots f(x_n\,|\,\theta) \qquad \text{(continuous distribution)}$$

The maximum likelihood estimator (MLE) is $\widehat{\theta}$ for which the likelihood is a maximum

12. Confidence intervals

If $x_1,x_2,\ldots,x_n$ are a random sample from $N(\mu,\sigma^2)$ and σ^2 is known, then the 95% confidence interval for μ is $(\overline{x}-1.96\frac{\sigma}{\sqrt{n}},\ \overline{x}+1.96\frac{\sigma}{\sqrt{n}})$

If σ^2 is estimated, then from the Student t table for t_{n-1} we find $t_0=t_{n-1,0.05}$

The 95% confidence interval for μ is $(\overline{x}-t_0\frac{s}{\sqrt{n}},\ \overline{x}+t_0\frac{s}{\sqrt{n}})$

13. Standard normal table Values of pdf $\phi(y) = f(y)$ and cdf $\Phi(y) = F(y)$

y	$\phi(y)$	$\Phi(y)$	y	$\phi(y)$	$\Phi(y)$	y	$\phi(y)$	$\Phi(y)$	y	$\Phi(y)$
0	.399	.5	.9	.266	.816	1.8	.079	.964	2.8	.997
.1	.397	.540	1.0	.242	.841	1.9	.066	.971	3.0	.999
.2	.391	.579	1.1	.218	.864	2.0	.054	.977	0.841	.8
.3	.381	.618	1.2	.194	.885	2.1	.044	.982	1.282	.9
.4	.368	.655	1.3	.171	.903	2.2	.035	.986	1.645	.95
.5	.352	.691	1.4	.150	.919	2.3	.028	.989	1.96	.975
.6	.333	.726	1.5	.130	.933	2.4	.022	.992	2.326	.99
.7	.312	.758	1.6	.111	.945	2.5	.018	.994	2.576	.995
.8	.290	.788	1.7	.094	.955	2.6	.014	.995	3.09	.999

14. Student t table Values $t_{m,p}$ of x for which $P(|X| > x) = p$, when X is t_m

m	$p=$ 0.10	0.05	0.02	0.01	m	$p=$ 0.10	0.05	0.02	0.01
1	6.31	12.71	31.82	63.66	9	1.83	2.26	2.82	3.25
2	2.92	4.30	6.96	9.92	10	1.81	2.23	2.76	3.17
3	2.35	3.18	4.54	5.84	12	1.78	2.18	2.68	3.05
4	2.13	2.78	3.75	4.60	15	1.75	2.13	2.60	2.95
5	2.02	2.57	3.36	4.03	20	1.72	2.09	2.53	2.85
6	1.94	2.45	3.14	3.71	25	1.71	2.06	2.48	2.78
7	1.89	2.36	3.00	3.50	40	1.68	2.02	2.42	2.70
8	1.86	2.31	2.90	3.36	∞	1.645	1.96	2.326	2.576

15. Chi-squared table Values $\chi^2_{k,p}$ of x for which $P(X > x) = p$, when X is χ^2_k and $p = .995, .975,$ *etc*

k	.995	.975	.05	.025	.01	.005	k	.995	.975	.05	.025	.01	.005
1	.000	.001	3.84	5.02	6.63	7.88	18	6.26	8.23	28.87	31.53	34.81	37.16
2	.010	.051	5.99	7.38	9.21	10.60	20	7.43	9.59	31.42	34.17	37.57	40.00
3	.072	.216	7.81	9.35	11.34	12.84	22	8.64	10.98	33.92	36.78	40.29	42.80
4	.207	.484	9.49	11.14	13.28	14.86	24	9.89	12.40	36.42	39.36	42.98	45.56
5	.412	.831	11.07	12.83	15.09	16.75	26	11.16	13.84	38.89	41.92	45.64	48.29
6	.676	1.24	12.59	14.45	16.81	18.55	28	12.46	15.31	41.34	44.46	48.28	50.99
7	.990	1.69	14.07	16.01	18.48	20.28	30	13.79	16.79	43.77	46.98	50.89	53.67
8	1.34	2.18	15.51	17.53	20.09	21.95	40	20.71	24.43	55.76	59.34	63.69	66.77
9	1.73	2.70	16.92	19.02	21.67	23.59	50	27.99	32.36	67.50	71.41	76.15	79.49
10	2.16	3.25	13.31	20.48	23.21	25.19	60	35.53	40.48	79.08	83.30	88.38	91.95
12	3.07	4.40	21.03	23.34	26.22	28.30	70	43.28	48.76	90.53	95.02	100.4	104.2
14	4.07	5.63	23.68	26.12	29.14	31.32	80	51.17	57.15	101.9	106.6	112.3	116.3
16	5.14	6.91	26.30	28.85	32.00	34.27	100	67.33	74.22	124.3	129.6	135.8	140.2

16. The chi-squared goodness-of-fit test

The frequencies n_y are grouped so that the fitted frequency $\widehat{n}_y$ for every group exceeds about 5.

$X^2 = \sum_y \frac{(n_y - \widehat{n}_y)^2}{\widehat{n}_y}$ is referred to the table of χ^2_k with significance point p,

where k is the number of terms summed, less one for each constraint, *eg* matching total frequency, and matching $\overline{x}$ with μ

17. Joint probability distributions

Discrete distribution $\{p_{xy}\}$, where $p_{xy} = P(\{X = x\} \cap \{Y = y\})$.

Let $p_{x\bullet} = P(X = x)$, and $p_{\bullet y} = P(Y = y)$, then

$$p_{x\bullet} = \sum_y p_{xy} \quad \text{and} \quad P(X = x \,|\, Y = y) = \frac{p_{xy}}{p_{\bullet y}}$$

Continuous distribution

Joint cdf $\quad F(x,y) = P(\{X \le x\} \cap \{Y \le y\}) = \int_{x_0=-\infty}^{x} \int_{y_0=-\infty}^{y} f(x_0\,, y_0)\,\mathrm{d}x_0\,\mathrm{d}y_0$

Joint pdf $\quad f(x,y) = \frac{\mathrm{d}^2 F(x,y)}{\mathrm{d}x\,\mathrm{d}y}$

Marginal pdf of X $\quad f_X(x) = \int_{-\infty}^{\infty} f(x, y_0)\,\mathrm{d}y_0$

Conditional pdf of X given $Y = y$ $\quad f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$ (provided $f_Y(y) > 0$)

18. Linear regression

To fit the linear regression model $y = \alpha + \beta x$ by $\widehat{y}_x = \widehat{\alpha} + \widehat{\beta} x$ from observations $(x_1, y_1), \ldots, (x_n, y_n)$, the least squares fit is $\quad \widehat{\alpha} = \overline{y} - \overline{x}\widehat{\beta}\,, \quad \widehat{\beta} = \frac{S_{xy}}{S_{xx}}$

The residual sum of squares $\quad \mathrm{RSS} = S_{yy} - \frac{S_{xy}^2}{S_{xx}}$

$\widehat{\sigma^2} = \frac{\mathrm{RSS}}{n-2} \qquad \frac{n-2}{\sigma^2}\,\widehat{\sigma^2}$ is from χ^2_{n-2}

$E(\widehat{\alpha}) = \alpha\,, \quad E(\widehat{\beta}) = \beta\,,$

$$\mathrm{var}\,(\widehat{\alpha}) = \frac{\sum x_i^2}{n\,S_{xx}}\sigma^2\,, \quad \mathrm{var}\,(\widehat{\beta}) = \frac{\sigma^2}{S_{xx}}\,, \quad \mathrm{cov}\,(\widehat{\alpha}, \widehat{\beta}) = -\frac{\overline{x}}{S_{xx}}\,\sigma^2$$

$$\widehat{y}_x = \widehat{\alpha} + \widehat{\beta} x\,, \qquad E(\widehat{y}_x) = \alpha + \beta x\,, \qquad \mathrm{var}\,(\widehat{y}_x) = \left\{\frac{1}{n} + \frac{(x - \overline{x})^2}{S_{xx}}\right\}\sigma^2$$

$\frac{\widehat{\alpha} - \alpha}{\widehat{\mathrm{se}}\,(\widehat{\alpha})}\,, \quad \frac{\widehat{\beta} - \beta}{\widehat{\mathrm{se}}\,(\widehat{\beta})}\,, \quad \frac{\widehat{y}_x - \alpha - \beta x}{\widehat{\mathrm{se}}\,(\widehat{y}_x)}$ are each from t_{n-2}

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Solution

(i) $w = u + iv = [(x-1) + iy]^{-2}$ gives

$$u = \frac{(x-1)^2 - y^2}{[(x-1)^2 + y^2]^2} \qquad v = -\frac{2(x-1)y}{[(x-1)^2 + y^2]^2} [\mathbf{3}]$$

$$u^2 + v^2 = \frac{[(x-1)^2 + y^2]^2}{[(x-1)^2 + y^2]^4} = \frac{1}{[(x-1)^2 + y^2]^2} \,.[\mathbf{3}]$$

Thus on $(x-1)^2 + y^2 = R^{-2}$ we have $u^2 + v^2 = R^{-4}$. [2]

(ii) For $w = z^{-1}$ we have $u = \frac{x}{x^2+y^2}$ & $v = -\frac{y}{x^2+y^2}$ and $u^2 + v^2 = (x^2+y^2)^{-1}$ [2], which makes

$$(x-1)^2 + (y-1)^2 = r^2 \qquad *$$

into

$$\frac{1 - 2(u-v)}{u^2+v^2} = r^2 - 2\,.[\mathbf{4}]$$

Thus when $r^2 = 2$, we have $v = u - 1/2$. [3]

When $r^2 = 1$, we have $u^2 + v^2 + 1 - 2(u - v) = 0$ from which we obtain

$$(u-1)^2 + (v+1)^2 = 1\,.[\mathbf{3}]$$

It is possible to do this the other way round (but longer) by writing $z = w^{-1}$ where $x = \frac{u}{u^2+v^2}$ and $y = -\frac{v}{u^2+v^2}$ and then substitute this into * and re-arrange.

EXAMINATION QUESTIONS/SOLUTIONS 2010-11

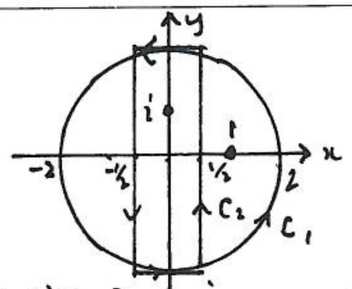
Course: EE paper 3

Question: EE2

Marks & seen/unseen: Unseen

Parts

a) $F(z) = \dfrac{z}{(z-1)^2(z-i)}$



Picture — 3

$z=1$ is a double pole

$z=i$ is a single pole

Residue at $z=i$ is $\lim\limits_{z\to i}\left\{\dfrac{(z-i)z}{(z-1)^2(z-i)}\right\} = \dfrac{i}{(i-1)^2} = -\frac{1}{2}$ — 2

" " $z=1$ is $\lim\limits_{z\to 1}\dfrac{d}{dz}\left\{\dfrac{(z-1)^2 z}{(z-1)^2(z-i)}\right\} = \dfrac{-i}{(i-1)^2} = \frac{1}{2}$ — 3

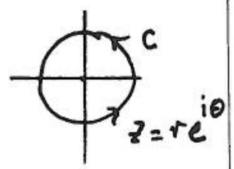
From the Residue Theorem $\oint_{C_1} = 2\pi i\left(-\frac{1}{2}+\frac{1}{2}\right) = 0$ — 1

The rectangle C_2 contains only $z=i$

so $\oint_{C_2} = 2\pi i\left(-\frac{1}{2}\right) = -\pi i$ — 2

b) i) $\oint_C \dfrac{dz}{z} = 2\pi i \times \{\text{Res of } z^{-1} \text{ at } z=0\}$

$= 2\pi i$ — 3



or

ii) $\oint_C \dfrac{dz}{z} = i\oint \dfrac{z\,d\theta}{z} = i\oint d\theta \qquad z=re^{i\theta},\ dz = iz\,d\theta$

$= i\int_0^{2\pi} d\theta = 2\pi i$ — 3

or

iii) $\oint_C \dfrac{dz}{z} = \left[\ln z\right]_C = \left[\ln r + i\theta\right]_C$

$= i\left[\theta\right]_0^{2\pi} = 2\pi i$ — 3

Setter's initials: JDG

Checker's initials: ADG

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EXAMINATION QUESTIONS/SOLUTIONS 2010-11

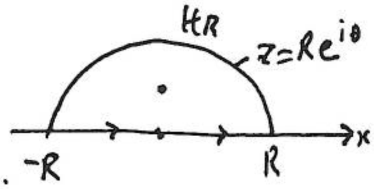
Course: EE2, EE paper 3

Question: EE 3

Marks & seen/unseen: unseen

Parts

$\oint_c e^{iz}F(z)\,dz$ with $F(z) = \dfrac{1}{(z^2+1)^2}$



4 with picture

$F(z)$ has a double pole at $z=i$ in the upper ½-plane. Thus the Residue of $e^{iz}F(z)$ at $z=i$ is

$$\lim_{z\to i}\left\{\frac{d}{dz}\frac{e^{iz}(z-i)^2}{(z-i)^2(z+i)^2}\right\} = \left\{e^{iz}\left[\frac{i(z+i)-2}{(z+i)^3}\right]\right\}_{z=i} = -\tfrac{1}{2}ie^{-1}$$ 4

Using the R.T. we have

$$2\pi i \times\left\{-\tfrac{1}{2}ie^{-1}\right\} = \oint_C e^{iz}F(z)\,dz = \int_{-R}^{R} e^{ix}F(x)\,dx + \oint_{H_R}$$ 4

Now $\lim_{R\to\infty}\oint_{H_R} e^{iz}F(z)\,dz = 0$ Jordan's Lemma

i) $m = 1 > 0$
ii) Only sings are poles
iii) $|F(z)| \to 0$ as $R\to\infty$.

4

Taking $R\to\infty$

$$\therefore \int_{-\infty}^{\infty} e^{ix}F(x)\,dx = \pi/e$$

Moreover $\int_{-\infty}^{\infty}\sin x\,F(x)\,dx = 0$ because the integrand is odd resulting in cancellation. 4

$$\therefore \int_{-\infty}^{\infty}\frac{\cos x}{(x^2+1)^2}\,dx = \pi/e$$

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	EXAMINATION QUESTIONS/SOLUTIONS 2010-11	Course EE2 Paper 3
Question EE4		Marks & seen/unseen

Parts

i) $\bar{\Pi}(\omega) = \int_{-1/2}^{1/2} e^{-i\omega t} 1.dt = -\frac{1}{i\omega}\left(e^{-\frac{1}{2}i\omega} - e^{\frac{1}{2}i\omega}\right)$ — 2

$= \frac{2}{\omega}\sin\frac{1}{2}\omega = \text{sinc}\,\omega$

ii) $\bar{\Lambda}(\omega) = \int_{-1}^{0}(1+t)e^{-i\omega t}dt + \int_0^1 (1-t)e^{-i\omega t}dt$

$= \int_{-1}^{1} e^{-i\omega t}dt - 2\int_0^1 t\cos\omega t\,dt$

$= \frac{1}{i\omega}\left(e^{i\omega} - e^{-i\omega}\right) - \frac{2}{\omega}\int_0^1 t\,d(\sin\omega t)$

$= \frac{2}{\omega}\sin\omega - \frac{2}{\omega}\left[[t\sin\omega t]_0^1 - \int_0^1 \sin\omega t\,dt\right]$

$= \frac{2}{\omega}\left\{\sin\omega - \sin\omega + \frac{1}{\omega}(1-\cos\omega)\right\}$ — 10

$= 4/\omega^2 \sin^2\frac{1}{2}\omega = \text{sinc}^2\omega$

iii) Parseval gives $\int_{-\infty}^{\infty}\text{sinc}^2\omega\,d\omega = 2\pi\int_{-\infty}^{\infty}\Pi^2 dt$ — 2

$= 2\pi\int_{-1/2}^{1/2} 1.dt = 2\pi$

iv) Also $\int_{-\infty}^{\infty}\text{sinc}^4\omega\,d\omega = 2\pi\int_{-\infty}^{\infty}|\Lambda(\omega)|^2 d\omega$

$= 2\pi\int_{-1}^{0}(1+t)^2dt + 2\pi\int_0^1(1-t)^2dt$

$= 4\pi\int_0^1(1-t)^2dt$ — 6

$= 4\pi\left[t - t^2 + \frac{1}{3}t^3\right]_0^1$

$= 4\pi/3$

Setter's initials	Checker's initials	Page number
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	EXAMINATION QUESTIONS/SOLUTIONS 2010-11	Course EE2 paper 3
Question EE5		Marks & seen/unseen
Parts		Unseen

$$\mathcal{L}\left(e^{at}f(t)\right) = \int_0^\infty e^{-(s-a)} f(t)\,dt$$

$$= \bar{f}(s-a) \qquad Re(s) > a \qquad 2$$

Now L.T. the ODE with ICs $x(0) = \dot{x}(0) = 0$

$$\mathcal{L}(\ddot{x} + 4\dot{x} + 8x) = \mathcal{L}\left[\delta(t-1)\right] = \int_0^\infty e^{-st}\delta(t-1)\,dt \qquad 3$$

$$\therefore (s^2 + 4s + 8)\bar{x}(s) = e^{-s}$$

$$\therefore \bar{x}(s) = \frac{e^{-s}}{(s+2)^2+4} = \underbrace{\left(\tfrac{1}{2}e^{-s}\right)}_{\bar{f}(s)} \underbrace{\frac{2}{(s+2)^2+2^2}}_{\bar{g}(s)}$$ Completing square 3

$$\therefore \bar{f}(s) = \tfrac{1}{2}e^{-s} \rightarrow f(t) = \tfrac{1}{2}\delta(t-1) \qquad 1$$

$$\bar{g}(s) = \frac{2}{(s+2)^2+2^2} \rightarrow g(t) = e^{-2t}\sin 2t$$ using Shift Thm 3

$\therefore$ If $\bar{x}(s) = \bar{f}(s)\bar{g}(s)$ then by the Convol Thm

$$x(t) = \mathcal{L}^{-1}\{\bar{f}(s)\bar{g}(s)\} = \int_0^t f(t')g(t-t')\,dt' \qquad 4$$

$$= \int_0^t \tfrac{1}{2}\delta(t'-1)\, e^{-2(t-t')}\sin 2(t-t')\,dt'$$

$$= \begin{cases} \tfrac{1}{2} e^{-2(t-1)}\sin 2(t-1) & t > 1 \\ 0 & 0 \le t \le 1 \end{cases} \qquad 4$$

Setter's initials	Checker's initials	Page number
JDG	ADG	

EXAMINATION QUESTIONS/SOLUTIONS 2010-11

Course: EE2 Paper 3

Question: EE6

Marks & seen/unseen

Parts

$$\mathcal{L}\left(\int_0^t f(u)\,du\right) = \mathcal{L}\left(\int_0^t 1.f(u)\,du\right) = \mathcal{L}(1 * f) \quad \text{Conv. Thm}$$ Seen 5

$$= \mathcal{L}(1)\mathcal{L}[f(t)] = \frac{1}{s}\bar{f}(s)$$

If $\bar{f}(s) = [\bar{g}(s)]^2$ with $\bar{g}(s) = \frac{1}{s^2+1} \to g(t) = \sin t$ Unseen

$$\therefore f(t) = \mathcal{L}^{-1}[\bar{g}(s)\bar{g}(s)] = g(t) * g(t)$$

$$= \int_0^t \sin u \sin(t-u)\,du$$ 7

$$= \frac{1}{2}\int_0^t \{\cos(2u-t) - \cos t\}\,du$$

$$= \frac{1}{2}\left\{\left[\frac{1}{2}\sin(2u-t) - u\cos t\right]_0^t\right\}$$

$$f(t) = \frac{1}{2}(\sin t - t\cos t) \qquad \text{—(1)}$$

Thus Unseen

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\bar{f}(s)\right\} = \int_0^t f(u)\,du$$

with $\bar{f}(s) = \frac{1}{(1+s^2)^2}$ & $f(t)$ as in (1)

$$\therefore \mathcal{L}^{-1}\left\{\frac{\bar{f}(s)}{s}\right\} = \int_0^t \frac{1}{2}(\sin u - u\cos u)\,du$$ 8

$$= -\frac{1}{2}[\cos u]_0^t - \frac{1}{2}\int_0^t u\cos u\,du$$

$$= \frac{1}{2}(1-\cos t) - \frac{1}{2}[u\sin u + \cos u]_0^t$$

$$= 1 - \cos t - \frac{1}{2}t\sin t$$

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Checker's initials: AOG

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EXAMINATION QUESTIONS/SOLUTIONS 2010-11

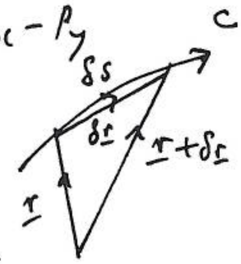
Course: EE2 Paper 3

Question: EE7

Marks & seen/unseen

Parts

Choose $\underline{u} = \hat{\underline{i}}Q - \hat{\underline{j}}P$ so $\operatorname{div}\underline{u} = Q_x - P_y$ — seen



Tangent unit vector is $\hat{\underline{t}} = \dfrac{d\underline{r}}{ds}$ — 2 (pic)

$\therefore \hat{\underline{t}} = \hat{\underline{i}}\dfrac{dx}{ds} + \hat{\underline{j}}\dfrac{dy}{ds}$ & use $\hat{\underline{t}}\cdot\hat{\underline{n}}$ — 2

$\hat{\underline{n}} = \pm\left(\hat{\underline{i}}\dfrac{dy}{ds} - \hat{\underline{j}}\dfrac{dx}{ds}\right)$ to construct — 2

$\therefore ds(\underline{u}\cdot\hat{\underline{n}}) = P\,dx + Q\,dy$

Thus G.T. becomes $\displaystyle\oint_C \underline{u}\cdot\hat{\underline{n}}\,ds = \iint_R (Q_x - P_y)\,dx\,dy$ — 2

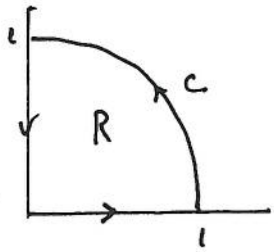
$\displaystyle = \iint_R \operatorname{div}\underline{u}\,dx\,dy.$

---

unseen

With $\underline{u} = \hat{\underline{i}}x^2 + \hat{\underline{j}}y^2$ and

$\operatorname{div}\underline{u} = 2(x+y)$; thus



$\displaystyle\iint_R (\operatorname{div}\underline{u})\,dx\,dy = 2\iint_R (x+y)\,dx\,dy$ — 2+2 (pic)

Use polars where $dx\,dy = r\,dr\,d\theta$. — 2

$\displaystyle\therefore \iint_R \operatorname{div}\underline{u}\,dx\,dy = 2\iint_R r(\cos\theta+\sin\theta)\,r\,dr\,d\theta.$

$\displaystyle = 2\int_0^1 r^2\,dr \int_0^{\pi/2}(\cos\theta+\sin\theta)\,d\theta$ — 4

$\displaystyle = \frac{2}{3}\left[\sin\theta - \cos\theta\right]_0^{\pi/2}$

$\displaystyle = \frac{2}{3}\left[(1-0)-(0-1)\right]$

$= 4/3$ — 2

(The line integral $\oint_C -y^2dx + x^2dy = 4/3$ if evaluated over all 3 segments.)

Setter's initials: JDG

Checker's initials: ADG

Page number

	EXAMINATION QUESTIONS/SOLUTIONS 2010-2011	Course
Solution ~~Question~~ C3		Marks & seen/unseen

Parts

$$J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} y & x \\ 2x & -2y \end{vmatrix}$$

ALL UNSEEN

$$= -2y^2 - 2x^2$$

$$\therefore |J| = 2(x^2+y^2)$$

Now $du\,dv \Rightarrow |J|\,dx\,dy$

$dx\,dy \rightarrow |J'|\,du\,dv$

Thus $|J'| = 1/|J| = \frac{1}{2(x^2+y^2)}$ 2

$$I_{OZ} = \iint_{R^*} (x^2+y^2)\exp(x^2-y^2)|J'|\,du\,dv$$

where x, y must be written in terms of u and v and R^* must be determined. 3

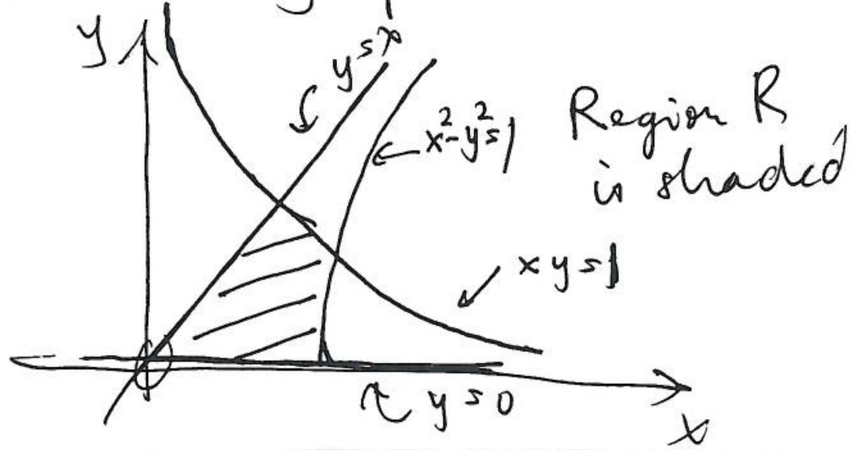
Thus $I_{OZ} = \iint_{R^*} \cancel{(x^2+y^2)} \frac{1}{2\cancel{(x^2+y^2)}} e^{v}\,du\,dv$

Setter's initials	Checker's initials	Page number
RW	AOG	1/3

EE 8

	EXAMINATION QUESTIONS/SOLUTIONS 2010-2011	Course
Question		Marks & seen/unseen
Parts		

Sketch in xy-plane



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Sketch in uv-plane is obtained by noting that

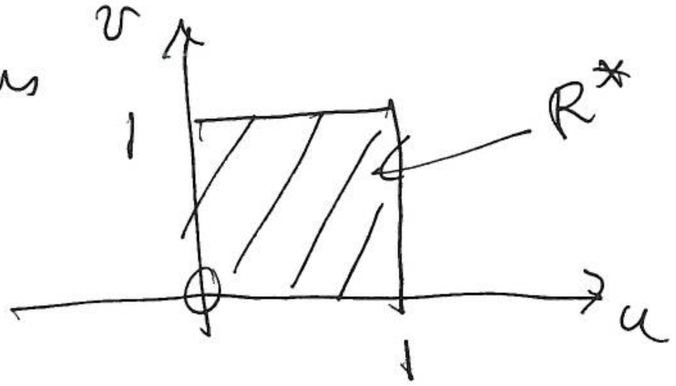
when (1) $y=0 \rightarrow u=0$

(2) $y=x \rightarrow v=0$

(3) $xy=1 \rightarrow u=1$

(4) $x^2-y^2=1 \rightarrow v=1$

Thus



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Setter's initials RWJ

Checker's initials ADG

EE 8 ~~Core 3~~

	EXAMINATION QUESTIONS/SOLUTIONS 2010-2011	Course
Solution ~~Question~~ C3		Marks & seen/unseen
Parts	Finally $$P_{0Z} = \int_0^1 \int_0^1 \frac{1}{2} e^{v} \, du \, dv$$ $$= \frac{1}{2} \cdot 1 \left[e^{v}\right]_0^1 = \frac{1}{2}(e-1).$$	3
		Total 20
	Setter's initials RLJ — Checker's initials ADG	Page number 3/3

EE II (3) 9

	EXAMINATION QUESTIONS/SOLUTIONS 2010-2011	Course Core 4
Solution C4		Marks & seen/unseen

Parts

(a) For $r<R$ $\quad \varphi(\underset{\sim}{r}) = \frac{Q}{8\pi R}\left(3 - \frac{x^2+y^2+z^2}{R^2}\right)$ (ALL UNSEEN)

$\therefore \text{grad}\ \varphi(\underset{\sim}{r}) = -\frac{Q}{8\pi R}\ \frac{2(x\underset{\sim}{i}+y\underset{\sim}{j}+z\underset{\sim}{k})}{R^2}$ — (1) $\quad$ ~~2~~ 3

For $r>R$ $\quad \varphi(\underset{\sim}{r}) = \frac{Q}{4\pi}\ \frac{1}{(x^2+y^2+z^2)^{1/2}}$

$\therefore \text{grad}\ \varphi = \frac{Q}{4\pi}\left(-\frac{1}{2}\right)\cdot\frac{2(x\underset{\sim}{i}+y\underset{\sim}{j}+z\underset{\sim}{k})}{(x^2+y^2+z^2)^{3/2}}$

$= -\frac{Q}{4\pi}\ \frac{\underset{\sim}{r}}{r^3}$ — (2) $\quad$ ~~2~~ 3

For $r<R$ use (1) to get

$\text{div grad}\ \varphi = -\frac{Q}{8\pi R^3}\ 2.3 = -Q/(4\pi R^3/3)$ $\quad$ 3 ~~2~~

For $r>R$ use (2) to get

$\text{div grad}\ \varphi = -\frac{Q}{4\pi}\left\{\frac{\partial}{\partial x}\left[\frac{x}{(x^2+y^2+z^2)^{3/2}}\right] + \frac{\partial}{\partial y}\left[\frac{y}{(x^2+y^2+z^2)^{3/2}}\right] + \frac{\partial}{\partial z}\left[\frac{z}{(x^2+y^2+z^2)^{3/2}}\right]\right\}$

⇒ Next page

EXAMINATION QUESTIONS/SOLUTIONS 2010-2011

Course: Core 4

~~Question~~ Solution: C4

Parts | Marks & seen/unseen

$$\operatorname{div}\operatorname{grad}\varphi = -\frac{Q}{4\pi}\left\{\frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3}{2}\frac{x\,2x}{(x^2+y^2+z^2)^{5/2}} + \frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3}{2}\frac{y\,2y}{(x^2+y^2+z^2)^{5/2}} + \frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3}{2}\frac{z\,2z}{(x^2+y^2+z^2)^{5/2}}\right\}$$

$$= -\frac{Q}{4\pi}\left\{\frac{3}{r^3} - 3\frac{(x^2+y^2+z^2)}{r^5}\right\} = 0$$

~~4~~ 5

(b) For $x^2+y^2<a^2$ $\operatorname{curl}\underset{\sim}{A} = \begin{vmatrix} \underset{-}{i} & \underset{-}{j} & \underset{-}{k} \\ \partial_x & \partial_y & \partial_z \\ 0 & 0 & -\frac{I}{4\pi a^2}(x^2+y^2) \end{vmatrix}$

$$= -\frac{I}{4\pi a^2}\left(\underset{\sim}{i}\,2y - \underset{\sim}{j}\,2x\right) \longrightarrow 3$$

~~2~~ 3

For $x^2+y^2>a^2$

$\operatorname{curl}\underset{\sim}{A} = \begin{vmatrix} \underset{-}{i} & \underset{-}{j} & \underset{-}{k} \\ \partial_x & \partial_y & \partial_z \\ 0 & 0 & -\frac{I}{4\pi}\left[1+\ln\frac{x^2+y^2}{a^2}\right] \end{vmatrix}$

$$= -\frac{I}{4\pi}\,\frac{2y\,\underset{\sim}{i} - 2x\,\underset{\sim}{j}}{x^2+y^2} \longrightarrow 4$$

~~2~~ 3

Total 20

Setter's initials: RW | Checker's initials: AOG | Page number 2/2

EXAMINATION QUESTIONS/SOLUTIONS 2010-2011	Course: Core 5

Solution
~~Question~~ Q1

Marks & seen/unseen

Parts

The condition is that $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$ everywhere inside a simply-connected region R containing A, B and the path P. (SEEN) 2

In the particular case $f = xe^y + b\sin x$ (UNSEEN)

$g = ax^2e^y + c\cos y$

so $\frac{\partial f}{\partial y} = xe^y$, $\frac{\partial g}{\partial x} = 2axe^y$ 5

Thus for condition to be satisfied $a = \frac{1}{2}$ but b and c can be anything.

Potential $V(x,y)$ satisfies (UNSEEN)

$\frac{\partial V}{\partial x} = xe^y + b\sin x$ — ①, $\frac{\partial V}{\partial y} = \frac{1}{2}x^2e^y + c\cos y$ — ②

Thus $V = \frac{1}{2}x^2e^y - b\cos x + h(y)$ from ① 5

Using ② $\cancel{\frac{1}{2}x^2e^y} + \frac{dh}{dy} = c\cos y + \cancel{\frac{1}{2}x^2e^y}$

$\therefore h = c\sin y + d \leftarrow$ arbitrary const.

$\therefore V(x,y) = \frac{1}{2}x^2e^y - b\cos x + c\sin y + d$ (UNSEEN)

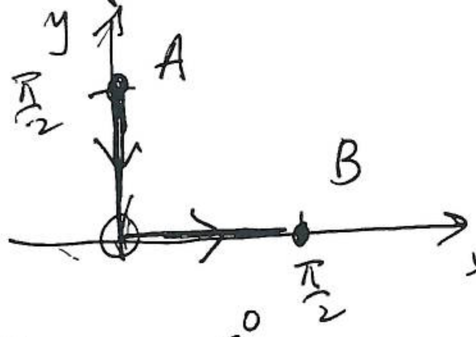
$\text{Int} = V(\pi/2, 0) - V(0, \pi/2) = \frac{1}{2}\left(\frac{\pi}{2}\right)^2 + b - c$ (✱) 2

EEII(3) (10)

	EXAMINATION QUESTIONS/SOLUTIONS 2010-2011	Course
solution		Core 5
~~Question~~ CJ		Marks & seen/unseen

Parts

Simplest path



$$\text{Int} = \int_{\pi/2}^{0} c\cos y\,dy + \int_0^{\pi/2} (x + b\sin x)\,dx$$

$$= \left[c\sin y\right]_{\pi/2}^{0} + \left[\frac{x^2}{2} - b\cos x\right]_0^{\pi/2}$$

$$= -c + \frac{1}{2}\left(\frac{\pi}{2}\right)^2 + b$$

Same as previous (*)

(UNSEEN)

6

TOTAL 20

Setter's initials RLJ

Checker's initials AOb

Page number 2/2

10) i) $g_x = f_y$ everywhere inside a S-conn region R containing A, B & the path P. 2

ii) $f = xe^y + b\sin x$; $g = ax^2e^y + c\cos y$ 5

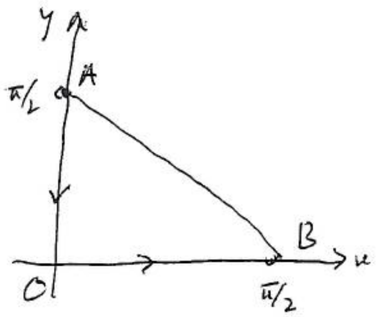
$2axe^y = xe^y \Rightarrow a = \frac{1}{2}$, b & c anything.

iii) $V_x = xe^y + b\sin x \rightarrow V = \frac{1}{2}x^2e^y - b\cos x + h(y)$

$V_y = ax^2e^y + c\cos y \rightarrow V = ax^2e^y + c\sin y + i(x)$

$\therefore\ a = \frac{1}{2}$ & $h(y) = \underbrace{(b\cos x + c\sin y + i(x))}_{d}$ 5

$\therefore\ V = \frac{1}{2}x^2e^y - b\cos x + c\sin y + d$



a) $A = (0, \pi/2)$ & $B = (\pi/2, 0)$

$f = V_x$, $g = V_y$

$\therefore\ I = \int_P V_x dx + V_y dy = \int_P dV = [V]_A^B$

$= V[\pi/2, 0] - V[0, \pi/2] = \left[\frac{1}{2}\left(\frac{\pi}{2}\right)^2 + d\right] - [-b + c + d]$

$= \frac{\pi^2}{8} + b - c.$ 2

b) Path $\vec{AO}$ then $\vec{OB}$: $\int_{AO} = V[0,0] - V[0, \pi/2]$ ⟩ as before. 6

Actual evaluation $\int_{OB} = V[\pi/2, 0] - V[0,0]$

$\int_{AO} = \int\left\{(xe^y + b\sin x)dx + \left(\frac{1}{2}x^2e^y + c\cos y\right)dy\right\} = c\int^0 \cos y\, dy = -c$ 6

$\int_{OB} = \int\left\{(\quad)dx + (\quad)dy\right\} = \int_0^{\pi/2}(x + b\sin x)dx = \pi^2/8 + b$ ⟩ as before

E 2.8 (Maths 3) 2011 — Solutions

11. (a) Pistons: $X \sim N(88, 0.7^2)$.

$$\begin{aligned}
P(\text{'acceptable'}) &= P(89.7 - 5 < X < 89.7) \\
&= P(\frac{84.7 - 88}{0.7} < X < \frac{89.7 - 88}{0.7}) \\
&= P(-4.71 < Z < 2.43) \\
&= \Phi(2.43) - \Phi(-4.71) = 0.992 - 0 = 0.992
\end{aligned}$$

Seen similar — 5 MARKS

(b) Cylinders: $Y \sim N(91, 1)$, so

$$Y - X \sim N(91 - 88, 1 + 0.7^2) \Rightarrow Y - X \sim N(3, 1.49)\,.$$

$$\begin{aligned}
P(\text{'acceptable'}) = P(0 < Y - X < 5) &= P(\frac{0-3}{\sqrt{1.49}} < Y - X < \frac{5-3}{\sqrt{1.49}}) \\
&= P(-2.46 < Z < 1.64) = 0.949 - 0.007 = 0.942
\end{aligned}$$

Unseen — 6 MARKS

(c) If A is the number of acceptable units, then

$$A \sim \text{Bin}(10, 0.942)$$

and the probability of there being more than 2 unacceptable units is

$$\begin{aligned}
P(A < 8) &= 1 - P(A \geq 8) \\
&= 1 - P(A = 8) - P(A = 9) - P(A = 10) \\
&= 1 - \binom{10}{8} 0.942^8 (1 - 0.942)^2 + \binom{10}{9} 0.942^9 (1 - 0.942)^1 + 0.942^{10} \\
&= 1 - 0.094 - 0.339 - 0.550 \\
&= 0.017
\end{aligned}$$

Seen similar — 9 MARKS

PTO

12. (a) This is a second-order autoregression (or AR(2) process), because of the presence of the term y_{t-2}.

Seen similar — 1 MARKS

(b) For any process $\{x_t\}$, the backshift operator B is defined as $Bx_t = x_{t-1}$.
From the LHS:

$$
\begin{aligned}
(2-B)(3-B)y_t &= (6-5B+B^2)y_t = 6y_t - 5y_{t-1} + y_{t-2} \\
&= 6(y_t - \frac{5}{6}\,y_{t-1} + \frac{1}{6}\,y_{t-2}) = 6\frac{1}{6}e_t = e_t
\end{aligned}
$$

$E(y_t) = 0$, because the errors $\{e_t\}$ all have mean 0.

Seen similar — 5 MARKS

(c) From the LHS:

$$
\frac{1}{2-B} - \frac{1}{3-B} = \frac{3-B-(2-B)}{(2-B)(3-B)} = \frac{1}{(2-B)(3-B)}.
$$

Hence:

$$
\begin{aligned}
y_t &= (2-B)^{-1}e_t - (3-B)^{-1}e_t \\
&= \frac{1}{2}\sum_{j=0}^{\infty}\left(\frac{B}{2}\right)^j e_t - \frac{1}{3}\sum_{j=0}^{\infty}\left(\frac{B}{3}\right)^j e_t \\
&= \sum_{j=0}^{\infty}\left(\frac{1}{2}\right)^{j+1} B^j e_t - \sum_{j=0}^{\infty}\left(\frac{1}{3}\right)^{j+1} B^j e_t \\
&= \sum_{j=0}^{\infty}\left(\frac{1}{2}\right)^{j+1} e_{t-j} - \sum_{j=0}^{\infty}\left(\frac{1}{3}\right)^{j+1} e_{t-j} \\
&= \sum_{j=0}^{\infty}\left[\left(\frac{1}{2}\right)^{j+1} - \left(\frac{1}{3}\right)^{j+1}\right] e_{t-j}\,.
\end{aligned}
$$

Seen similar — 7 MARKS

The errors are uncorrelated, so the variance of their sum is equal to the sum of their variances. The variance of y_t is thus:

$$
\begin{aligned}
\mathrm{Var}(y_t) &= \sum_{j=0}^{\infty}\left[\left(\frac{1}{2}\right)^{j+1} - \left(\frac{1}{3}\right)^{j+1}\right]^2 \mathrm{Var}(e_{t-j}) \\
&= \sum_{j=0}^{\infty}\left[\left(\frac{1}{4}\right)^{j+1} - 2\left(\frac{1}{6}\right)^{j+1} + \left(\frac{1}{9}\right)^{j+1}\right] \\
&= \frac{1/4}{1-1/4} - 2\frac{1/6}{1-1/6} + \frac{1/9}{1-1/9} = \frac{7}{120}
\end{aligned}
$$

Seen similar — 7 MARKS