EE4-27
EE9-CS5-2

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2015

MSc and EEE/EIE PART IV: MEng and ACGI

SYSTEMS IDENTIFICATION

Corrected Copy

Monday, 11 May 10:00 am

Time allowed: 3:00 hours

There are FOUR questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s) :	T. Parisini
	Second Marker(s) :	S.A. Evangelou

1. Given a stationary stochastic process $v(\cdot)$ with $\mathbb{E}(v) = 0$, consider its correlation function $\gamma(\tau)$ of which in Fig. 1.1 the first six values $\gamma(\tau)$, $\tau = 0, 1, \dots, 5$ are shown.

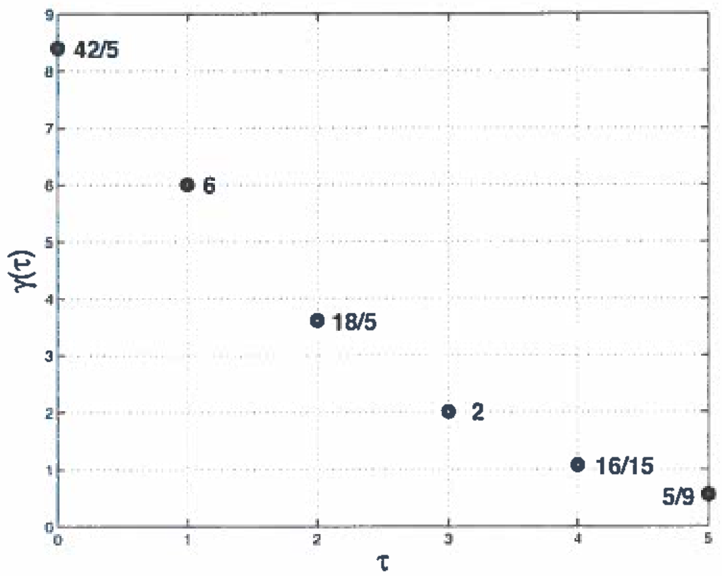

Figure 1.1 Plot of the first six values $\gamma(\tau)$, $\tau = 0, 1, \dots, 5$ of the correlation function of the stochastic process $v(\cdot)$. The numerical values are indicated close to each point.

a) Establish whether the family of moving-average stochastic models $MA(n)$, $n = 1, 2$ has a correlation function structure which is formally consistent with $\gamma(\tau)$ plotted in Fig. 1.1. Justify your answer.

[4 Marks]

b) Consider the following auto-regressive stochastic models $AR(n)$, $n = 1, 2$:

$$AR(1): \quad v(t) = av(t-1) + \eta(t), \quad \eta(\cdot) \sim WN(0, \lambda_1^2)$$

$$AR(2): \quad v(t) = a_1 v(t-1) + a_2 v(t-2) + \eta(t), \quad \eta(\cdot) \sim WN(0, \lambda_2^2)$$

Determine which of the AR models above has the correlation function plotted in Fig. 1.1 and compute its parameters (that is, a and λ_1^2 for $AR(1)$ or a_1, a_2 and λ_2^2 for $AR(2)$).

[8 Marks]

c) Determine the spectrum $\Gamma(\omega)$ of the process $v(\cdot)$ obtained in your answer to Question 1b).

[4 Marks]

d) Sketch the behaviour of the spectrum $\Gamma(\omega)$ mentioned in Question 1c) in the interval $\omega \in [-\pi, \pi]$.

[4 Marks]

2. Consider an *unknown* function $y = f(x) : [0,2] \mapsto \Re$ and assume that N noisy samples of y (denoted as $\hat{y}(i)$, $i = 0,\dots,N-1$) are available according to the following relationship:

$$\hat{y}(i) = f[x(i)] + \varepsilon(i), \quad i = 0,\dots,N-1,$$

for N given values of $x(i)$, $i = 0,\dots,N-1$, where $\varepsilon(\cdot) \sim WN(0,\lambda^2)$.

a) Consider the problem of determining a linear approximation $g(x,\theta)$ of $f(x)$:

$$g(x,\theta) = ax + b, \quad \text{with } \theta = [a,b]^\top .$$

The linear function $g(x,\theta)$ has to be determined using a *generic* set $\mathscr{M}$ of N noisy measurements:

$$\mathscr{M} = \{(x(i),\hat{y}(i)), i = 0,1,\dots,N-1\}.$$

Formulate a least-squares problem the solution of which provides the optimal (in the least-squares sense) linear approximation $g(x,\theta^\circ)$ of $f(x)$, where $\theta^\circ = [a^\circ,b^\circ]^\top$. [*Hint: Just formulate the problem, but do not attempt to determine the general solution*].

[6 Marks]

b) Provide the general least-squares solution of the problem formulated in your answer to Question 2a), that is, provide an analytical expression for the optimal vector of parameters $\theta^\circ = [a^\circ,b^\circ]^\top$ as a function of the above generic set $\mathscr{M}$ of noisy measurements.

[8 Marks]

c) Using the general solution obtained in your answer to Question 2b), compute the optimal linear approximation (that is, compute θ°) using the set $\mathscr{M}$ of 5 measurements whose numerical values are specified below (see also Fig. 2.1):

$$\mathscr{M} = \{(0,-0.34),(0.5,0.54),(1,0.81),(1.5,2.28),(2,3.66)\}$$

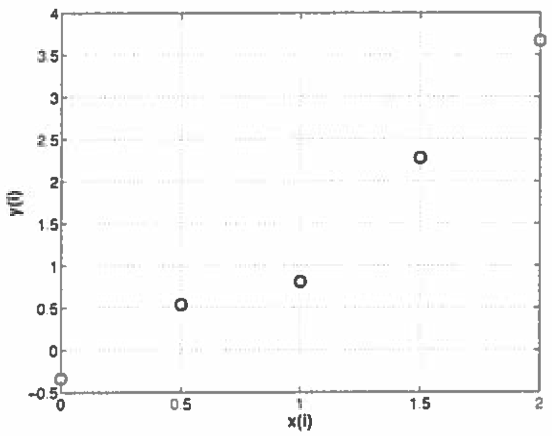

Figure 2.1 Plot of the set $\mathscr{M}$ of noisy samples of the unknown function $f(x)$.

Moreover, sketch the plot of the linear approximation that you have obtained adding on the same plot the noisy measurements of set $\mathscr{M}$. Comment on your findings.

[6 Marks]

3. Consider a sensor providing three samples T_i, $i = 1,2,3$ of the temperature of a given object. The samples are realisations of the following *Gaussian* and *independent* random variables, respectively:

$$T_1 \sim \mathscr{G}(\overline{T},2), \quad T_2 \sim \mathscr{G}(\overline{T},4), \quad T_3 \sim \mathscr{G}(\overline{T},9/4).$$

The objective is to determine an estimate $\widehat{T}$ of the mean temperature $\overline{T}$ through a suitably designed estimator.

a) Consider the family of linear estimators

$$\widehat{T}(\alpha,\beta,\gamma) = \alpha T_1 + \beta T_2 + \gamma T_3,$$

where $\alpha,\beta,\gamma \in \mathbb{R}$ are three parameters to be determined. Establish conditions on α,β,γ such that the estimator $\widehat{T}(\alpha,\beta,\gamma)$ is unbiased.

[4 Marks]

b) Among the values of α,β,γ determined in your answer to Question 3a), find

$$(\alpha^\circ,\beta^\circ,\gamma^\circ) = \arg\min_{\alpha,\beta,\gamma} \mathrm{var}[\widehat{T}(\alpha,\beta,\gamma)]$$

that is, find the values $(\alpha^\circ,\beta^\circ,\gamma^\circ)$ of (α,β,γ) such that $\mathrm{var}(\widehat{T})$ is the smallest possible.

[7 Marks]

c) Compute $\mathrm{var}[\widehat{T}(\alpha^\circ,\beta^\circ,\gamma^\circ)]$.

[3 Marks]

d) Consider the empirical mean estimator $\widetilde{T}$ of $\overline{T}$ given by:

$$\widetilde{T} = \frac{1}{3}(T_1 + T_2 + T_3)$$

Show that the estimator $\widetilde{T}$ is unbiased. Moreover, compute $\mathrm{var}(\widetilde{T})$ and compare it with $\mathrm{var}[\widehat{T}(\alpha^\circ,\beta^\circ,\gamma^\circ)]$ that has been computed in your answer to Question 3c). Discuss your findings.

[6 Marks]

4. Consider the stochastic process $v(\cdot)$ generated by the ARMA(1,1) model:

$$v(t) = \frac{1}{5}v(t-1) + \frac{1}{3}\eta(t) + \frac{5}{3}\eta(t-1), \tag{4.1}$$

where $\eta(\cdot) \sim WN(0,9)$.

a) Show that the stochastic process $v(\cdot)$ is stationary and determine its canonical form.

[5 Marks]

b) Determine the difference equation yielding the optimal one-step ahead prediction $\hat{v}(t+1|t)$ of $v(t+1)$ on the basis of past values $v(t), v(t-1), v(t-2), \ldots$ of the process $v(\cdot)$ and compute the variance of the one-step ahead prediction error $\text{var}[\varepsilon_1(t)] = \text{var}[v(t+1) - \hat{v}(t+1|t)]$.

[4 Marks]

c) Determine the difference equation yielding the optimal two-steps ahead prediction $\hat{v}(t+2|t)$ of $v(t+2)$ on the basis of past values $v(t), v(t-1), v(t-2), \ldots$ of the process $v(\cdot)$ and compute the variance of the two-steps ahead prediction error $\text{var}[\varepsilon_2(t)] = \text{var}[v(t+2) - \hat{v}(t+2|t)]$. Moreover, compare $\text{var}[\varepsilon_1(t)]$ computed in your answer to Question 4b) with $\text{var}[\varepsilon_2(t)]$ computed above. Comment on your findings.

[6 Marks]

d) Suppose now that the stochastic process $v(\cdot)$ is generated by the following *different* ARMA(1,1) model:

$$v(t) = \frac{1}{5}v(t-1) + \frac{1}{3}\xi(t) - \frac{5}{3}\xi(t-1), \tag{4.2}$$

where $\xi(\cdot) \sim WN(0,9)$. Discuss how the answers to Questions 4a), 4b), and 4c) have to be modified in the case of the stochastic process generated by the model (4.2) instead of (4.1).

[5 Marks]