UNIVERSITY OF LONDON [E2.11 2005]

B.ENG. AND M.ENG. EXAMINATIONS 2005

For Internal Students of Imperial College London

This paper is also taken for the relevant examination for the Associateship of the City & Guilds of London Institute

INFORMATION SYSTEMS ENGINEERING E2.11

MATHEMATICS

Date **Thursday 2nd June 2005 2.00 - 4.00 pm**

Answer FOUR questions, to include at least one from Section B

Answers to Section A *questions must be written in a different answer book from answers to* Section B *questions.*

A statistics formula sheet is provided

[Before starting, please make sure that the paper is complete. There should be SIX pages, with a total of SIX questions. Ask the invigilator for a replacement if this copy is faulty.]

[E2.11 2005]

Section A

1. (i) If the Fourier transform of $f(t)$, $-\infty < t < \infty$, is given by

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt ,$$

show that the Fourier transform of $f(at)$ $(a > 0)$ is $\dfrac{1}{a}\, \hat{f}\left(\dfrac{\omega}{a}\right)$ and that of $t f(t)$ is $i \dfrac{d\hat{f}}{d\omega}(\omega)$.

(ii) If the convolution of two functions $f(t)$ and $g(t)$ is defined by

$$(f * g)(t) = \int_{-\infty}^{\infty} f(t-u)\, g(u)\, du ,$$

show that $(\widehat{f * g})(\omega) = \hat{f}(\omega)\, \hat{g}(\omega)$.

(iii) Find the Fourier transform of te^{-at^2}, $(a > 0)$ either by using the fact that

$$\int_{-\infty}^{\infty} e^{-u^2} du = \sqrt{\pi} ,$$

or otherwise.

PLEASE TURN OVER

2. (i) The Laplace transform $\overline{y}(p)$ of a function $y(t)$ is

$$\overline{y}(p) = \int_0^\infty e^{-pt}\, y(t)\, dt\ .$$

Show that, assuming $y(t)$ behaves suitably at infinity, the Laplace transform of $\quad y'(t) \equiv \dfrac{dy}{dt} \quad$ is

$$\overline{y'}(p) = p\overline{y}(p) - y(0)\,.$$

Hence or otherwise, solve the coupled differential equations

$$\begin{aligned} x' + y &= e^{-t}\,, \\ -x + y' &= e^{t}\,, \end{aligned}$$

for the functions $x(t)$ and $y(t)$, with initial conditions $x(0) = y(0) = 0$.

(ii) What is the condition on the functions $P(x,\, y)$ and $Q(x,\, y)$ in the path integral

$$I = \int_C (P dx + Q\, dy)$$

for them to admit a potential? Explain why this leads to I being independent of the path C.

Do the functions

$$P(x,\, y) = \frac{-y^2}{2} \sin x\,, \qquad Q(x,\, y) = y \cos x + 1$$

admit a potential?

If so, what is it?

3. (i) Make a sketch of the region of the $x-y$ plane over which the integral

$$I = \int_0^1 dx \int_{x^2}^1 x\, e^{-y^2} dy$$

is taken. Reverse the order of integration, using your sketch as needed, and hence evaluate the integral.

(ii) Let

$$I = \int_0^\infty e^{-x^2}\, dx .$$

We could also write

$$I = \int_0^\infty e^{-y^2} dy .$$

Evaluate I^2 (and hence I) by considering

$$I^2 = \int_0^\infty dx \int_0^\infty e^{-(x^2+y^2)}\, dy$$

and changing to polar coordinates.

4. (i) Find all the poles, and the residue at each pole, of the function

$$f(z) = \frac{2z-1}{z(z^2+1)} .$$

(ii) Evaluate

$$\int_{C_1} f(z)\, dz ,$$

where C_1 is a counterclockwise-oriented circle of radius 2, centred on the origin.

(iii) Same as for (ii), but for a radius of $1/2$.

PLEASE TURN OVER

5. (i) In a binary symmetric channel, where X denotes the digit transmitted and Y denotes the digit received, the following transmissions probabilities hold, with all transmissions independent.

$$\begin{aligned} P(Y=1 \mid X=1) &= 0.9 & \qquad P(Y=0 \mid X=0) &= 0.9 \\ P(Y=1 \mid X=0) &= 0.1 & \qquad P(Y=0 \mid X=1) &= 0.1 \end{aligned}$$

The probability of a 1 being transmitted is 0.7.

(a) Find the probability that a 0 is received.

(b) If a 0 is received, find the probability that a 0 was transmitted.

(c) If a 5 bit string of all zeros is transmitted, what is the probability that the received string will contain at most one error?

(ii) In a study to design an email SPAM filter, the following events are defined

S: email is SPAM
A_1: email contains the string "cheapest"
A_2: email contains the string "meds"
A_3: email contains the string "credit"

It is found that,

$$\begin{aligned} P(A_1 \mid S) &= 0.2 & P(A_2 \mid S) &= 0.4 & P(A_3 \mid S) &= 0.2 \\ P(A_1 \mid \overline{S}) &= 0.05 & P(A_2 \mid \overline{S}) &= 0.1 & P(A_3 \mid \overline{S}) &= 0.01 \end{aligned}$$

Assume that A_1, A_2 and A_3 are independent, both conditional on S and $\overline{S}$.

If $P(S) = 0.2$, find the probability that the email is SPAM if

(a) A_1 occurs ($= p_1$, say).

(b) both A_1 and A_2 occur ($= p_2$, say).

(c) A_1, A_2 and A_3 occur ($= p_3$, say).

(d) Explain why $p_3 > p_2 > p_1$.

6. (i) The lifetime, T, of a particular component is normally distributed with mean 6 years and variance 0.25 years^2.

What is the reliability of the component at 7 years?

(ii) The lifetimes, T_A and T_B of components of type A and B in hours, have probability density function

$$f(t) = \lambda e^{-\lambda t} \qquad t > 0,$$

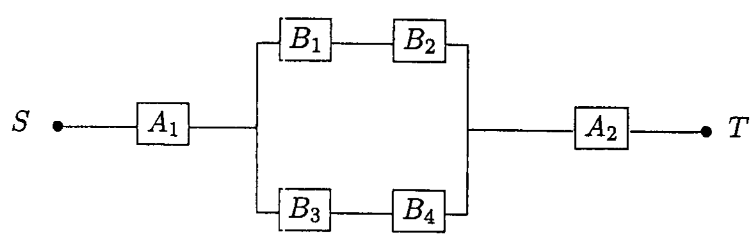
with $\lambda = 0.1$ and $\lambda = 0.5$ for components A and B respectively.

(a) Show that $f(t)$ is a valid probability density function.

(b) Determine the reliability fınctions and hazard rates associated with T_A and T_B.

(c) Determine the reliability of each type of component at 90 minutes.

(d) A system is made up of six components, A_1 and A_2, of type A and B_1, B_2, B_3 and B_4, of type B. All components operate independently and each have lifetimes as described above. The system functions as long as there a path of functioning components between S and T.



Determine the reliability of the system at 90 minutes.

END OF PAPER

I H - E2.11 2005

MATHEMATICS DEPARTMENT

MATHEMATICAL FORMULAE

1. VECTOR ALGEBRA

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1,\, a_2,\, a_3)$$

Scalar (dot) product: $\mathbf{a}\,.\,\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

Vector (cross) product:

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Scalar triple product:

$$[\mathbf{a},\,\mathbf{b},\,\mathbf{c}] = \mathbf{a}\,.\,\mathbf{b}\times\mathbf{c} = \mathbf{b}\,.\,\mathbf{c}\times\mathbf{a} = \mathbf{c}\,.\,\mathbf{a}\times\mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product: $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{c}\,.\,\mathbf{a})\mathbf{b} - (\mathbf{b}\,.\,\mathbf{a})\mathbf{c}$

2. SERIES

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \ldots \quad (\alpha \text{ arbitrary, } |x| < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots\,,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n\frac{x^{2n}}{(2n)!} + \ldots\,,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + \ldots\,,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^n\frac{x^{n+1}}{(n+1)} + \ldots (-1 < x \le 1)$$

3. TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin(a+b) = \sin a\,\cos b + \cos a\,\sin b\,;$$

$$\cos(a+b) = \cos a\,\cos b - \sin a\,\sin b\,.$$

$$\cos iz = \cosh z\,; \quad \cosh iz = \cos z\,; \quad \sin iz = i\sinh z\,; \quad \sinh iz = i\sin z\,.$$

4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f\,D^n g + \tbinom{n}{1} Df\,D^{n-1}g + \ldots + \tbinom{n}{r} D^r f\,D^{n-r}g + \ldots + D^n f\,g\,.$$

(b) Taylor's expansion of $f(x)$ about $x = a$:

$$f(a+h) = f(a) + hf'(a) + h^2f''(a)/2! + \ldots + h^nf^{(n)}(a)/n! + \epsilon_n(h)\,,$$

where $\epsilon_n(h) = h^{n+1}f^{(n+1)}(a+\theta h)/(n+1)!, \quad 0 < \theta < 1\,.$

(c) Taylor's expansion of $f(x,\,y)$ about $(a,\,b)$:

$$f(a+h,\,b+k) = f(a,\,b) + [hf_x + kf_y]_{a.b} + 1/2!\,\left[h^2f_{xx} + 2hkf_{xy} + k^2f_{yy}\right]_{a.b} + \ldots$$

(d) Partial differentiation of $f(x,\,y)$:

i. If $y = y(x)$, then $f = F(x)$, and $\dfrac{dF}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dx}$.

ii. If $x = x(t)$, $y = y(t)$, then $f = F(t)$, and $\dfrac{dF}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$.

iii. If $x = x(u,\,v)$, $y = y(u,\,v)$, then $f = F(u,\,v)$, and

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}\,, \quad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}\,.$$

(e) Stationary points of $f(x,\,y)$ occur where $f_x = 0$, $f_y = 0$ simultaneously.
Let $(a,\,b)$ be a stationary point: examine $D = [f_{xx}f_{yy} - (f_{xy})^2]_{a.b}$.
If $D > 0$ and $f_{xx}(a,\,b) < 0$, then $(a,\,b)$ is a maximum;
If $D > 0$ and $f_{xx}(a,\,b) > 0$, then $(a,\,b)$ is a minimum;
If $D < 0$ then $(a,\,b)$ is a saddle-point.

(f) Differential equations:

i. The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating factor $I(x) = \exp[\int P(x)(dx)]$, so that $\dfrac{d}{dx}(Iy) = IQ$.

ii. $P(x,\,y)dx + Q(x,\,y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y$.

5. INTEGRAL CALCULUS

(a) An important substitution: $\tan(\theta/2) = t$.
$\sin\theta = 2t/(1+t^2)$, $\cos\theta = (1-t^2)/(1+t^2)$, $d\theta = 2\,dt/(1+t^2)$.

(b) Some indefinite integrals:

$$\int (a^2 - x^2)^{-1/2} dx = \sin^{-1}\left(\frac{x}{a}\right), \quad |x| < a .$$

$$\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1}\left(\frac{x}{a}\right) = \ln\left\{\frac{x}{a} + \left(1 + \frac{x^2}{a^2}\right)^{1/2}\right\} .$$

$$\int (x^2 - a^2)^{-1/2} dx = \cosh^{-1}\left(\frac{x}{a}\right) = \ln\left|\frac{x}{a} + \left(\frac{x^2}{a^2} - 1\right)^{1/2}\right| .$$

$$\int (a^2 + x^2)^{-1} dx = \left(\frac{1}{a}\right)\tan^{-1}\left(\frac{x}{a}\right) .$$

6. NUMERICAL METHODS

(a) Approximate solution of an algebraic equation:

If a root of $f(x) = 0$ occurs near $x = a$, take $x_0 = a$ and
$x_{n+1} = x_n - [f(x_n)/f'(x_n)]$, $n = 0, 1, 2 \ldots$

(Newton Raphson method).

(b) Formulae for numerical integration: Write $x_n = x_0 + nh$, $y_n = y(x_n)$.

i. Trapezium rule (1-strip): $\int_{x_0}^{x_1} y(x)dx \approx (h/2)[y_0 + y_1]$.

ii. Simpson's rule (2-strip): $\int_{x_0}^{x_2} y(x)dx \approx (h/3)[y_0 + 4y_1 + y_2]$.

(c) Richardson's extrapolation method: Let $I = \int_a^b f(x)dx$ and let I_1, I_2 be two estimates of I obtained by using Simpson's rule with intervals h and $h/2$.

Then, provided h is small enough,

$$I_2 + (I_2 - I_1)/15 ,$$

is a better estimate of I .

7. LAPLACE TRANSFORMS

Function	Transform	Function	Transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)dt$	$af(t) + bg(t)$	$aF(s) + bG(s)$
df/dt	$sF(s) - f(0)$	d^2f/dt^2	$s^2F(s) - sf(0) - f'(0)$
$e^{at}f(t)$	$F(s-a)$	$tf(t)$	$-dF(s)/ds$
$(\partial/\partial\alpha)f(t,\alpha)$	$(\partial/\partial\alpha)F(s,\alpha)$	$\int_0^t f(t)dt$	$F(s)/s$
$\int_0^t f(u)g(t-u)du$	$F(s)G(s)$		
1	$1/s$	$t^n (n = 1, 2 \ldots)$	$n!/s^{n+1}$, $(s > 0)$
e^{at}	$1/(s-a)$, $(s > a)$	$\sin\omega t$	$\omega/(s^2+\omega^2)$, $(s > 0)$
$\cos\omega t$	$s/(s^2+\omega^2)$, $(s > 0)$	$H(t-T) = \begin{cases} 0, & t < T \\ 1, & t > T \end{cases}$	e^{-sT}/s , $(s, T > 0)$

8. FOURIER SERIES

If $f(x)$ is periodic of period $2L$, then $f(x+2L) = f(x)$, and

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} , \text{ where}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad n = 0, 1, 2, \ldots , \text{ and}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1, 2, 3, \ldots$$

Parseval's theorem

$$\frac{1}{L}\int_{-L}^{L} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right) .$$

1. Probabilities for events

For events A, B, and C $\quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$

More generally $\quad P(\bigcup A_i) = \sum P(A_i) - \sum P(A_i \cap A_j) + \sum P(A_i \cap A_j \cap A_k) - \cdots$

The odds in favour of A $\quad P(A) / P(\overline{A})$

Conditional probability $\quad P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$ provided that $P(B) > 0$

Chain rule $\quad P(A \cap B \cap C) = P(A)\, P(B \mid A)\, P(C \mid A \cap B)$

Bayes' rule $\quad P(A \mid B) = \dfrac{P(A)\, P(B \mid A)}{P(A)\, P(B \mid A) + P(\overline{A})\, P(B \mid \overline{A})}$

A and B are independent if $\quad P(B \mid A) = P(B)$

A, B, and C are independent if $\quad P(A \cap B \cap C) = P(A)P(B)P(C)$, and

$$P(A \cap B) = P(A)P(B), \quad P(B \cap C) = P(B)P(C), \quad P(C \cap A) = P(C)P(A)$$

2. Probability distribution, expectation and variance

The probability distribution for a discrete random variable X is the complete set of probabilities $\{p_x\} = \{P(X = x)\}$

Expectation $\quad E(X) = \mu = \sum_x x p_x$

Sample mean $\quad \overline{x} = \dfrac{1}{n} \sum_k x_k$ estimates μ from random sample $x_1, x_2, \ldots, x_n$

Variance $\quad \mathrm{var}\,(X) = \sigma^2 = E\{(X - \mu)^2\} = E(X^2) - \mu^2$, where $E(X^2) = \sum_x x^2 p_x$

Sample variance $\quad s^2 = \dfrac{1}{n-1} \left\{ \sum_k x_k^2 - \dfrac{1}{n} \left(\sum_j x_j \right)^2 \right\}$ estimates σ^2

Standard deviation $\quad \mathrm{sd}\,(X) = \sigma$

If value y is observed with frequency n_y

$$n = \sum_y n_y, \quad \sum_k x_k = \sum_y y n_y, \quad \sum_k x_k^2 = \sum_y y^2 n_y$$

For function $g(x)$ of x, $\quad E\{g(X)\} = \sum_x g(x) p_x$

Skewness $\quad \beta_1 = E\left(\dfrac{X - \mu}{\sigma}\right)^3$ is estimated by $\dfrac{1}{n-1} \sum \left(\dfrac{x_i - \overline{x}}{s}\right)^3$

Kurtosis $\quad \beta_2 = E\left(\dfrac{X - \mu}{\sigma}\right)^4 - 3$ is estimated by $\dfrac{1}{n-1} \sum \left(\dfrac{x_i - \overline{x}}{s}\right)^4 - 3$

Sample median $\tilde{x}$. If the sample values $x_1, \ldots, x_n$ are ordered $x_{(1)} \le x_{(2)} \le \cdots \le x_{(n)}$

$$\tilde{x} = x_{(\frac{n+1}{2})} \text{ if } n \text{ is odd, and } \tilde{x} = \tfrac{1}{2}\left(x_{(\frac{n}{2})} + x_{(\frac{n+2}{2})}\right) \text{ if } n \text{ is even.}$$

α-quantile $Q(\alpha)$ is such that $P(X \le Q(\alpha)) = \alpha$

Sample α-quantile $\widehat{Q}(\alpha)$ is the sample value for which the proportion of values $\le \widehat{Q}(\alpha)$ is α (using linear interpolation between values on either side)

The sample median $\tilde{x}$ estimates the population median $Q(0.5)$.

3. Probability distribution for a continuous random variable

The cumulative distribution function (cdf) $\quad F(x) = P(X \le x) = \int_{x_0=-\infty}^{x} f(x_0)\mathrm{d}x_0$

The probability density function (pdf) $\quad f(x) = \dfrac{\mathrm{d}F(x)}{\mathrm{d}x}$

$$E(X) = \mu = \int_{-\infty}^{\infty} x\, f(x)\mathrm{d}x, \quad \mathrm{var}\,(X) = \sigma^2 = E(X^2) - \mu^2,$$

$$\text{where } E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\mathrm{d}x$$

4. Discrete probability distributions

Discrete Uniform *Uniform* (n)

$$p_x = \frac{1}{n} \quad (x = 1, 2, \ldots, n) \qquad \mu = \tfrac{1}{2}(n+1), \ \sigma^2 = \tfrac{1}{12}(n^2 - 1)$$

Binomial distribution *Binomial* (n, θ)

$$p_x = \binom{n}{x} \theta^x (1-\theta)^{n-x} \quad (x = 0, 1, 2, \ldots, n) \qquad \mu = n\theta, \ \sigma^2 = n\theta(1-\theta)$$

Poisson distribution *Poisson* (λ)

$$p_x = \frac{\lambda^x e^{-\lambda}}{x!} \quad (x = 0, 1, 2, \ldots) \ \text{(with } \lambda > 0) \qquad \mu = \lambda, \ \sigma^2 = \lambda$$

Geometric distribution *Geometric* (θ)

$$p_x = (1-\theta)^{x-1}\theta \quad (x = 1, 2, 3, \ldots) \qquad \mu = \frac{1}{\theta}, \ \sigma^2 = \frac{1-\theta}{\theta^2}$$

5. Continuous probability distributions

Uniform distribution *Uniform* (α, β)

$$f(x) = \begin{cases} \dfrac{1}{\beta - \alpha} & (\alpha < x < \beta), \\ 0 & (\text{otherwise}). \end{cases} \qquad \begin{aligned} \mu &= (\alpha+\beta)/2, \\ \sigma^2 &= (\beta-\alpha)^2/12. \end{aligned}$$

Exponential distribution $\textit{Exponential}\,(\lambda)$

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & (0 < x < \infty), \\ 0 & (-\infty < x \le 0). \end{cases} \qquad \begin{aligned} &\mu = 1/\lambda\,, \\ &\sigma^2 = 1/\lambda^2\,. \end{aligned}$$

Normal distribution $N\,(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2 \right\} \quad (-\infty < x < \infty)$$

$$E(X) = \mu\,, \quad \mathrm{var}\,(X) = \sigma^2$$

Standard normal distribution $N\,(0, 1)$

If X is $N(\mu, \sigma^2)$, then $Y = \dfrac{X - \mu}{\sigma}$ is $N(0, 1)$

6. Reliability

For a device in continuous operation with failure time random variable T having pdf $f(t)$ $(t > 0)$

The reliability function at time t	$R(t)$	$=$	$P(T > t)$		
The failure rate or hazard function	$h(t)$	$=$	$f(t)/R(t)$		
The cumulative hazard	$H(t)$	$=$	$\int_0^t h(t_0)\,\mathrm{d}t_0$	$=$	$-\ln\{R(t)\}$
The Weibull(α, β) distribution has	$H(t)$	$=$	βt^α		

7. System reliability

For a system of k devices, which operate independently, let

$R_i = P(D_i) = P($"device i operates"$)$

The system reliability, R, is the probability of a path of operating devices

A system of devices in series operates only if every device operates

$$R = P(D_1 \cap D_2 \cap \cdots \cap D_k) = R_1 R_2 \cdots R_k$$

A system of devices in parallel operates if any device operates

$$R = P(D_1 \cup D_2 \cup \cdots \cup D_k) = 1 - (1 - R_1)(1 - R_2) \cdots (1 - R_k)$$

8. Covariance and correlation

The covariance of X and Y $\quad \mathrm{cov}\,(X, Y) = E(XY) - \{E(X)\}\{E(Y)\}$

From pairs of observations $(x_1, y_1), \ldots, (x_n, y_n)$ $\quad S_{xy} \;=\; \sum_k x_k y_k - \frac{1}{n}(\sum_i x_i)(\sum_j y_j)$

$$S_{xx} \;=\; \sum_k x_k^2 - \frac{1}{n}(\sum_i x_i)^2, \qquad S_{yy} \;=\; \sum_k y_k^2 - \frac{1}{n}(\sum_j y_j)^2$$

Sample covariance $\quad s_{xy} \;=\; \frac{1}{n-1} S_{xy}$ estimates $\operatorname{cov}(X, Y)$

Correlation coefficient $\quad \rho \;=\; \operatorname{corr}(X, Y) \;=\; \frac{\operatorname{cov}(X, Y)}{\operatorname{sd}(X) \cdot \operatorname{sd}(Y)}$

Sample correlation coefficient $\quad r \;=\; \frac{S_{xy}}{\sqrt{S_{xx} S_{yy}}}$ estimates ρ

9 Sums of random variables

$$E(X+Y) \;=\; E(X) + E(Y)$$
$$\operatorname{var}(X+Y) \;=\; \operatorname{var}(X) + \operatorname{var}(Y) + 2\operatorname{cov}(X, Y)$$
$$\operatorname{cov}(aX + bY,\; cX + dY) \;=\; (ac)\operatorname{var}(X) + (bd)\operatorname{var}(Y) + (ad + bc)\operatorname{cov}(X, Y)$$

If X is $N(\mu_1, \sigma_1^2)$, Y is $N(\mu_2, \sigma_2^2)$, and $\operatorname{cov}(X, Y) = c$,
then $X + Y$ is $N(\mu_1 + \mu_2,\; \sigma_1^2 + \sigma_2^2 + 2c)$

10. Bias, standard error, mean square error

If t estimates θ (with random variable T giving t)

Bias of t $\quad \operatorname{bias}(t) \;=\; E(T) - \theta$

Standard error of t $\quad \operatorname{se}(t) \;=\; \operatorname{sd}(T)$

Mean square error of t $\quad \mathsf{MSE}(t) \;=\; E\{(T-\theta)^2\} \;=\; \{\operatorname{se}(t)\}^2 + \{\operatorname{bias}(t)\}^2$

If $\bar{x}$ estimates μ, then $\operatorname{bias}(\bar{x}) = 0$, $\operatorname{se}(\bar{x}) = \sigma/\sqrt{n}$, $\mathsf{MSE}(\bar{x}) = \sigma^2/n$, $\widehat{\operatorname{se}}(\bar{x}) = s/\sqrt{n}$

Central limit property $\quad$ if n is fairly large, $\bar{x}$ is from $N(\mu,\; \sigma^2/n)$ approximately

11 Likelihood

The likelihood is the joint probability as a function of the unknown parameter θ.
For a random sample $x_1, x_2, \ldots, x_n$

$$\ell(\theta;\, x_1, x_2, \ldots, x_n) \;=\; P(X_1 = x_1 \mid \theta) \cdots P(X_n = x_n \mid \theta) \quad \text{(discrete distribution)}$$

$$\ell(\theta;\, x_1, x_2, \ldots, x_n) \;=\; f(x_1 \mid \theta) f(x_2 \mid \theta) \cdots f(x_n \mid \theta) \quad \text{(continuous distribution)}$$

The maximum likelihood estimator (MLE) is $\widehat{\theta}$ for which the likelihood is a maximum.

12. Confidence intervals

If $x_1, x_2, \ldots, x_n$ are a random sample from $N(\mu, \sigma^2)$ and σ^2 is known, then the 95% confidence interval for μ is $(\bar{x} - 1.96\frac{\sigma}{\sqrt{n}},\ \bar{x} + 1.96\frac{\sigma}{\sqrt{n}})$

If σ^2 is estimated, then from the Student t table for t_{n-1} we find $t_0 = t_{n-1,0.05}$

The 95% confidence interval for μ is $(\bar{x} - t_0\frac{s}{\sqrt{n}},\ \bar{x} + t_0\frac{s}{\sqrt{n}})$

13. Standard normal table

Values of pdf $\phi(y) = f(y)$ and cdf $\Phi(y) = F(y)$

y	$\phi(y)$	$\Phi(y)$	y	$\phi(y)$	$\Phi(y)$	y	$\phi(y)$	$\Phi(y)$	y	$\Phi(y)$
0	.399	.5	.9	.266	.816	1.8	.079	.964	2.8	.997
.1	.397	.540	1.0	.242	.841	1.9	.066	.971	3.0	.998
.2	.391	.579	1.1	.218	.864	2.0	.054	.977	0.841	.8
.3	.381	.618	1.2	.194	.885	2.1	.044	.982	1.282	.9
.4	.368	.655	1.3	.171	.903	2.2	.035	.986	1.645	.95
.5	.352	.691	1.4	.150	.919	2.3	.028	.989	1.96	.975
.6	.333	.726	1.5	.130	.933	2.4	.022	.992	2.326	.99
.7	.312	.758	1.6	.111	.945	2.5	.018	.994	2.576	.995
.8	.290	.788	1.7	.094	.955	2.6	.014	.995	3.09	.999

14. Student t table

Values $t_{m,p}$ of x for which $P(|X| > x) = p$, when X is t_m

	p	.10	.05	.02	0.01		p	.10	.05	.02	0.01
m	1	6.31	12.71	31.82	63.66	m	9	1.83	2.26	2.82	3.25
	2	2.92	4.30	6.96	9.92		10	1.81	2.23	2.76	3.17
	3	2.35	3.18	4.54	5.84		12	1.78	2.18	2.68	3.05
	4	2.13	2.78	3.75	4.60		15	1.75	2.13	2.60	2.95
	5	2.02	2.57	3.36	4.03		20	1.72	2.09	2.53	2.85
	6	1.94	2.45	3.14	3.71		25	1.71	2.06	2.48	2.78
	7	1.89	2.36	3.00	3.50		40	1.68	2.02	2.42	2.70
	8	1.86	2.31	2.90	3.36		∞	1.645	1.96	2.326	2.576

15. Chi-squared table

Values $\chi^2_{k,p}$ of x for which $P(X > x) = p$, when X is χ^2_k and $p = .995,\ .975,\ etc$

k	.995	.975	.05	.025	.01	.005	k	.995	.975	.05	.025	.01	.005
1	.000	.001	3.84	5.02	6.63	7.88	18	6.26	8.23	28.87	31.53	34.81	37.16
2	.010	.051	5.99	7.38	9.21	10.60	20	7.43	9.59	31.42	34.17	37.57	40.00
3	.072	.216	7.81	9.35	11.34	12.84	22	8.64	10.98	33.92	36.78	40.29	42.80
4	.207	.484	9.49	11.14	13.28	14.86	24	9.89	12.40	36.42	39.36	42.98	45.56
5	.412	.831	11.07	12.83	15.09	16.75	26	11.16	13.84	38.89	41.92	45.64	48.29
6	.676	1.24	12.59	14.45	16.81	18.55	28	12.46	15.31	41.34	44.46	48.28	50.99
7	.990	1.69	14.07	16.01	18.48	20.28	30	13.79	16.79	43.77	46.98	50.89	53.67
8	1.34	2.18	15.51	17.53	20.09	21.95	40	20.71	24.43	55.76	59.34	63.69	66.77
9	1.73	2.70	16.92	19.02	21.67	23.59	50	27.99	32.36	67.50	71.41	76.15	79.49
10	2.16	3.25	13.31	20.48	23.21	25.19	60	35.53	40.48	79.08	83.30	88.38	91.95
12	3.07	4.40	21.03	23.34	26.22	28.30	70	43.28	48.76	90.53	95.02	100.4	104.2
14	4.07	5.63	23.68	26.12	29.14	31.32	80	51.17	57.15	101.9	106.6	112.3	116.3
16	5.14	6.91	26.30	28.85	32.00	34.27	100	67.33	74.22	124.3	129.6	135.8	140.2

16. The chi-squared goodness-of-fit test

The frequencies n_y are grouped so that the fitted frequency $\widehat{n}_y$ for every group exceeds about 5.

$X^2 = \sum_y \dfrac{(n_y - \widehat{n}_y)^2}{\widehat{n}_y}$ is referred to the table of χ^2_k with significance point p,

where k is the number of terms summed, less one for each constraint, *eg* matching total frequency, and matching $\overline{x}$ with μ.

17. Joint probability distributions

Discrete distribution $\{p_{xy}\}$, where $p_{xy} = P(\{X = x\} \cap \{Y = y\})$.

Let $p_{x\bullet} = P(X = x)$, and $p_{\bullet y} = P(Y = y)$, then

$p_{x\bullet} = \sum_y p_{xy}$, and $P(X = x \,|\, Y = y) = \dfrac{p_{xy}}{p_{\bullet y}}$.

Continuous distribution

Joint cdf $\quad F(x,y) = P(\{X \le x\} \cap \{Y \le y\}) = \int_{x_0=-\infty}^{x} \int_{y_0=-\infty}^{y} f(x_0, y_0)\, dx_0\, dy_0$

Joint pdf $\quad f(x,y) = \dfrac{d^2 F(x,y)}{dx\, dy}$

Marginal pdf of X $\quad f_X(x) = \int_{-\infty}^{\infty} f(x, y_0)\, dy_0$

Conditional pdf of X given $Y = y$ $\quad f_{X|Y}(x|y) = \dfrac{f(x,y)}{f_Y(y)}$ (provided $f_Y(y) > 0$)

18. Linear regression

To fit the linear regression model $y = \alpha + \beta x$ by $\widehat{y}_x = \widehat{\alpha} + \widehat{\beta} x$ from observations $(x_1, y_1), \ldots, (x_n, y_n)$, the least squares fit is

$$\widehat{\alpha} = \overline{y} - \overline{x}\widehat{\beta}, \quad \widehat{\beta} = S_{xy}/S_{xx}$$

The residual sum of squares RSS $= S_{yy} - \dfrac{S_{xy}^2}{S_{xx}}$

$$\widehat{\sigma^2} = \frac{\text{RSS}}{n-2}, \qquad \frac{n-2}{\sigma^2}\widehat{\sigma^2} \text{ is from } \chi^2_{n-2}$$

$$E(\widehat{\alpha}) = \alpha, \quad E(\widehat{\beta}) = \beta,$$

$$\text{var}(\widehat{\alpha}) = \frac{\sum x_i^2}{n S_{xx}}\sigma^2, \quad \text{var}(\widehat{\beta}) = \frac{\sigma^2}{S_{xx}}, \quad \text{cov}(\widehat{\alpha}, \widehat{\beta}) = -\frac{\overline{x}}{S_{xx}}\sigma^2$$

$$\widehat{y}_x = \widehat{\alpha} + \widehat{\beta} x, \qquad E(\widehat{y}_x) = \alpha + \beta x, \qquad \text{var}(\widehat{y}_x) = \left\{\frac{1}{n} + \frac{(x-\overline{x})^2}{S_{xx}}\right\}\sigma^2$$

$$\frac{\widehat{\alpha} - \alpha}{\widehat{\text{se}}(\widehat{\alpha})}, \quad \frac{\widehat{\beta} - \beta}{\widehat{\text{se}}(\widehat{\beta})}, \quad \frac{\widehat{y}_x - \alpha - \beta x}{\widehat{\text{se}}(\widehat{y}_x)} \quad \text{are each from } t_{n-2}$$

19. Design matrix for factorial experiments

With 3 factors each at 2 levels

$$\mathbf{X} = \begin{pmatrix} 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

ISE 2
MATHS
(E2.11)
2005

MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

2004 – 2005

Please write on this side only, legibly and neatly, between the margins

PAPER	QUESTION	SOLUTION
ISE2.6		1

(i) F.T. of $f(at) = \int_{-\infty}^{\infty} f(at)e^{-i\omega t}dt$ Let $s=at$

$$= \int_{-\infty}^{\infty} f(s)e^{-i(\omega/a)s}\frac{ds}{a} = \frac{1}{a}\hat{f}(\omega/a)$$ 3

F.T. of $t f(t) = \int_{-\infty}^{\infty} t f(t)e^{-i\omega t}dt = i\frac{d}{d\omega}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$

$$= i\frac{d\hat{f}(\omega)}{d\omega}$$ 3

(ii) $\widehat{(f*g)}(\omega) = \int_{-\infty}^{\infty}(f*g)(t)e^{-i\omega t}dt$

$$= \int_{-\infty}^{\infty}dt\int_{-\infty}^{\infty} f(t-u)g(u)\, e^{-i\omega t}du$$

Let $s=t-u$: (instead of t)

$$= \int_{-\infty}^{\infty}ds\int_{-\infty}^{\infty} f(s)g(u)e^{-i\omega(s+u)}du$$

$$= \left(\int_{-\infty}^{\infty} f(s)e^{-i\omega s}ds\right)\left(\int_{-\infty}^{\infty} g(u)e^{-i\omega u}du\right) = \hat{f}(\omega)\hat{g}(\omega)$$ 6

(iii) F.T. of $e^{-at^2} = \int_{-\infty}^{\infty} e^{-at^2}e^{-i\omega t}dt = e^{\frac{-\omega^2}{4a}}\int_{-\infty}^{\infty} e^{-\left(\sqrt{a}t+\frac{i\omega}{2\sqrt{a}}\right)^2}dt$

$$= \sqrt{\frac{\pi}{a}}\, e^{\frac{-\omega^2}{4a}}$$ after letting $u=\sqrt{a}t+\frac{i\omega}{2\sqrt{a}}$

From (i), $\widehat{te^{-at^2}} = i\frac{d}{d\omega}\left[\sqrt{\frac{\pi}{a}}\, e^{\frac{-\omega^2}{4a}}\right] = \frac{-2i}{4a}\sqrt{\frac{\pi}{a}}\,\omega e^{-\omega^2/4a}$

$$= -\frac{i}{2a}\sqrt{\frac{\pi}{a}}\,\omega e^{-\omega^2/4a}$$ 8

Setter : J-Luc

Setter's signature :

Checker : RLJ

Checker's signature :

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(i) $\overline{y'}(p) = \int_0^\infty y'(t)\, e^{-pt}\, dt$ Integrate by parts

$$= \left[y(t)\, e^{-pt}\right]_0^\infty + p\int_0^\infty y(t)\, e^{-pt}\, dt$$

$$= -y(0) + p\,\bar{y}(p),$$ 3

assuming $\lim_{t\to\infty} y(t)\, e^{-pt} \to 0$.

Take Laplace transform of equations:

$$p\bar{x} + \bar{y} = \overline{e^{-t}} = \frac{1}{p+1}$$

$$-\bar{x} + p\bar{y} = \overline{e^{t}} = \frac{1}{p-1}$$

Solve for $\bar{x} = \frac{1}{p^2+1}\left(\frac{p}{p+1} - \frac{1}{p-1}\right) = \frac{p^2-2p-1}{(p^2+1)(p+1)(p-1)}$

$$= \frac{p+1}{p^2+1} - \frac{1}{2}\frac{1}{(p+1)} + \frac{1}{2}\frac{1}{(p-1)}$$ (partial fractions) 3

Can now easily invert.

$$x(t) = \cos t + \sin t - \frac{1}{2}e^{-t} + \frac{1}{2}e^{t}$$

Then $y(t) = e^{-t} - x' = e^{-t} + \sin t - \cos t - \frac{1}{2}e^{-t} + \frac{1}{2}e^{t}$

$$= \sin t - \cos t + \frac{1}{2}e^{-t} + \frac{1}{2}e^{t}$$ 4

(ii) In order for Φ to exist such that

$$P = \frac{\partial \Phi}{\partial x}, \quad Q = \frac{\partial \Phi}{\partial y}$$

need $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, for then $\frac{\partial^2 \Phi}{\partial y \partial x} = \frac{\partial^2 \Phi}{\partial x \partial y}$. 3

Then $I = \int_C \frac{\partial \Phi}{\partial x} dx + \frac{\partial \Phi}{\partial y} dy = \int_C d\Phi = \Phi(\text{end}) - \Phi(\text{start})$ 3

$\Rightarrow$ path-independent.

The functions $P = -\frac{y^2}{2} \sin x$ and $Q = y \cos x + 1$

admit a potential, since $\frac{\partial P}{\partial y} = -y \sin x = \frac{\partial Q}{\partial x}$.

The potential is $\Phi = \frac{y^2}{2} \cos x + y + \underbrace{\text{const.}}_{\text{const } \to 0}$. 4

Setter : J-Luc Setter's signature :

Checker : R.L.J Checker's signature :

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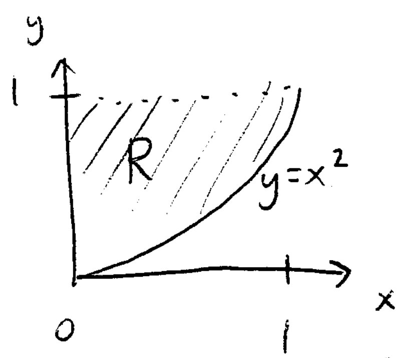
(i)



$$I = \int_0^1 dx \int_{x^2}^1 x\, e^{-y^2}\, dy$$ 2 (diagram)

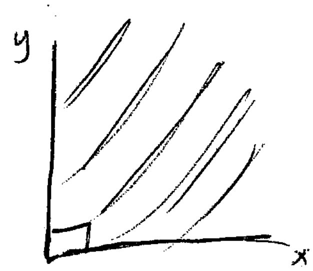
$$= \int_0^1 dy \int_0^{\sqrt{y}} x\, e^{-y^2}\, dx$$ 3

$$I = \int_0^1 dy\; e^{-y^2}\left[\frac{x^2}{2}\right]_0^{\sqrt{y}} = \frac{1}{2}\int_0^1 y\, e^{-y^2}\, dy$$

Put $u = -y^2$, $du = -2y\,dy$

$$= \frac{1}{2}\int_0^{-1} e^u \frac{du}{(-2)} = -\frac{1}{4}\left[e^u\right]_0^{-1} = \frac{1}{4}\left(1 - e^{-1}\right)$$ 5

(ii) $$I^2 = \int_0^\infty dx \int_0^\infty e^{-(x^2+y^2)}\, dy$$



Let $r^2 = x^2 + y^2$ 3

$$I^2 = \int_0^\infty dr \int_0^{\frac{\pi}{2}} e^{-r^2}\, r\, dr\, d\theta$$ 2

$$= \frac{\pi}{2}\;\frac{1}{2}\left[-e^{-r^2}\right]_0^\infty = \frac{\pi}{4}$$ 5

Hence, $I = \dfrac{\sqrt{\pi}}{2}$.

(i) The poles are at $z = 0, \pm i$. 3

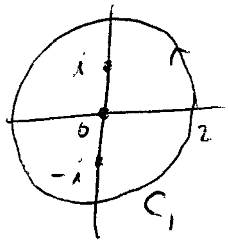
The residues are

$\underline{z=0}$: $\dfrac{2(0)-1}{0^2+1} = -1$ 2

$\underline{z=+i}$: $\dfrac{2i-1}{i(i+i)} = \dfrac{2i-1}{-2} = \dfrac{1}{2} - i$ 2

$\underline{z=-i}$: $\dfrac{2(-i)-1}{(-i)(-i-i)} = \dfrac{-2i-1}{-2} = \dfrac{1}{2} + i$ 2

(ii)



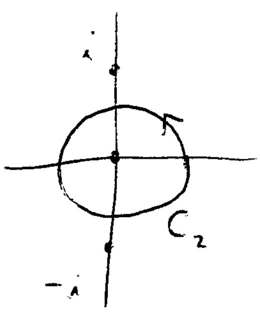
The circle encloses all 3 poles. 3

Hence

$$I = \int_{C_1} f(z)\,dz = 2\pi i\left(-1 + \frac{1}{2} - i + \frac{1}{2} + i\right)$$ 3

$$= 0.$$

(iii)



The circle encloses only one pole. 3

Hence,

$$I = \int_{C_2} f(z)\,dz = 2\pi i(-1) = -2\pi i.$$ 2

Setter : J-Luc

Setter's signature :

Checker : R.L.J

Checker's signature :

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5. (i) (a)

$$\begin{aligned} P(Y=0) &= P(Y=0 \mid X=0)P(X=0) + P(Y=0 \mid X=1)P(X=1) \\ &= 0.9 \times 0.3 + 0.1 \times 0.7 = 0.34 \end{aligned}$$

2

(b)

$$P(X=0 \mid Y=0) = \frac{P(Y=0 \mid X=0)P(X=0)}{P(Y=0)} = \frac{0.9 \times 0.3}{0.34} = 0.7941$$

3

(c) Let N = number of digits in error, then $N \sim \mathit{Binomial}(5, 0.1)$ and $P(N=n) = \binom{5}{n}(0.1)^n(0.9)^{5-n}$.

2

$$P(N \leq 1) = P(N=0) + P(N=1) = (0.9)^5 + 5(0.1)(0.9)^4 = 0.9185$$

2

(ii) (a)

$$\begin{aligned} P(S \mid A_1) &= \frac{P(A_1 \mid S)P(S)}{P(A_1)} = \frac{P(A_1 \mid S)P(S)}{P(A_1 \mid S)P(S) + P(A_1 \mid \overline{S})P(\overline{S})} \\ &= \frac{0.2 \times 0.2}{0.2 \times 0.2 + 0.05 \times 0.8} = 0.5 \end{aligned}$$

2

(b)

$$\begin{aligned} P(S \mid A_1 \cap A_2) &= \frac{P(A_1 \cap A_2 \mid S)P(S)}{P(A_1 \cap A_2)} \\ &= \frac{P(A_1 \mid S)P(A_2 \mid S)P(S)}{P(A_1 \cap A_2 \mid S)P(S) + P(A_1 \cap A_2 \mid \overline{S})P(\overline{S})} \\ &= \frac{P(A_1 \mid S)P(A_2 \mid S)P(S)}{P(A_1 \mid S)P(A_2 \mid S)P(S) + P(A_1 \mid \overline{S})P(A_2 \mid \overline{S})P(\overline{S})} \\ &= \frac{0.2 \times 0.4 \times 0.2}{0.2 \times 0.4 \times 0.2 + 0.05 \times 0.1 \times 0.8} = 0.8 \end{aligned}$$

3

Setter : E. McCoy Setter's signature : EM

Checker : M. Crowder Checker's signature : M J Crowder

5. (ii) (c)

$$\begin{aligned}
P(S \mid A_1 \cap A_2 \cap A_3) &= \frac{P(A_1 \cap A_2 \cap A_3 \mid S)P(S)}{P(A_1 \cap A_2 \cap A_3)} \\
&= \frac{P(A_1 \mid S)P(A_2 \mid S)P(A_3 \mid S)P(S)}{P(A_1 \cap A_2 \cap A_3 \mid S)P(S) + P(A_1 \cap A_2 \cap A_3 \mid \overline{S})P(\overline{S})} \\
&= \frac{P(A_1 \mid S)P(A_2 \mid S)P(A_3 \mid S)P(S)}{P(A_1 \mid S)P(A_2 \mid S)P(A_3 \mid S)P(S) + P(A_1 \mid \overline{S})P(A_2 \mid \overline{S})P(A_3 \mid \overline{S})P(\overline{S})} \\
&= \frac{0.2 \times 0.4 \times 0.2 \times 0.2}{0.2 \times 0.4 \times 0.2 \times 0.2 + 0.05 \times 0.1 \times 0.01 \times 0.8} = 0.9877
\end{aligned}$$

$\boxed{4}$

(d) The probabilities increase as we have $P(A_i \mid S) > P(A_i \mid \overline{S})$, so as we include more terms, we introduce more information about whether the email is SPAM or not.

$\boxed{2}$

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Checker : M. Crowder

Setter's signature : *EM*

Checker's signature : *MJ Crowder*

6 (i)

$$\begin{aligned}
T &\sim N(6, 0.25)\\
\frac{T-6}{\sqrt{0.25}} &\sim N(0,1)\\
R(t) = \mathrm{P}(T > t) &= \mathrm{P}\left(\frac{T-6}{0.5} > \frac{t-6}{0.5}\right)\\
&= 1 - \Phi\left(\frac{t-6}{0.5}\right)\\
R(7) &= 1 - \Phi(2) = 1 - 0.977 = 0.023.
\end{aligned}$$

4

(ii) (a)

$$\int_{-\infty}^{\infty} f(t)\,dt = \int_0^{\infty} \lambda e^{-\lambda t}\,dt = \left[-e^{-\lambda t}\right]_0^{\infty} = 1.$$

Also, $f(t) \geq 0\ \forall t$, so $f(t)$ is a valid pdf.

3

(b)

$$\begin{aligned}
R(t) &= \mathrm{P}(T > t) = \int_t^{\infty} f(u)\,du\\
&= \int_t^{\infty} \lambda e^{-\lambda u}\,du = \left[-e^{-\lambda u}\right]_t^{\infty}\\
&= e^{-\lambda t}
\end{aligned}$$

For component A: $R_A(t) = e^{-0.1t}$
For component B: $R_B(t) = e^{-0.5t}$

3

$$h(t) = \frac{f(t)}{R(t)} = \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}} = \lambda$$

For component A: $h_A(t) = 0.1$
For component B: $h_B(t) = 0.5$

3

(c) when $t = 1.5$ (90 minutes):
For component A: $R_A(1.5) = e^{-0.1\times 1.5} = e^{-0.15} = 0.8607$
For component B: $R_B(1.5) = e^{-0.5\times 1.5} = e^{-0.75} = 0.4724$

2

(d) Let T = lifetime of system.
Let A_i = event that component A_i, $i = 1, 2$ is functioning at 90 minutes.
B_i = event that component B_i, $i = 1, 2, 3, 4$ is functioning at 90 minutes.

$$\begin{aligned}
R(1.5) &= \mathrm{P}(T > 1.5) = \mathrm{P}(A_1 \cap A_2 \cap ((B_1 \cap B_2) \cup (B_3 \cap B_4))\\
&= \mathrm{P}(A_1)\mathrm{P}(A_2)[\mathrm{P}(B_1 \cap B_2) + \mathrm{P}(B_3 \cap B_4) - \mathrm{P}(B_1 \cap B_2 \cap B_3 \cap B_4)]\\
&= \mathrm{P}(A_1)\mathrm{P}(A_2)[\mathrm{P}(B_1)\mathrm{P}(B_2) + \mathrm{P}(B_3)\mathrm{P}(B_4) - \mathrm{P}(B_1)\mathrm{P}(B_2)\mathrm{P}(B_3)\mathrm{P}(B_4)]\\
&= e^{-0.15}e^{-0.15}[e^{-0.75}e^{-0.75} + e^{-0.75}e^{-0.75} - e^{-0.75}e^{-0.75}e^{-0.75}e^{-0.75}]\\
&= e^{-0.3}(2e^{-1.5} - e^{-3}) = 0.2937.
\end{aligned}$$

5

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Checker : M. Crowder — Checker's signature : MJ Crowder