IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

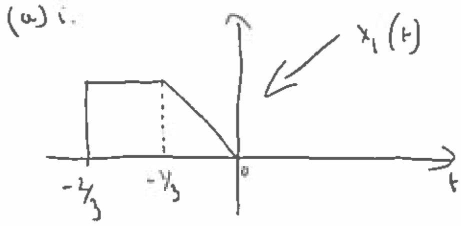
DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING

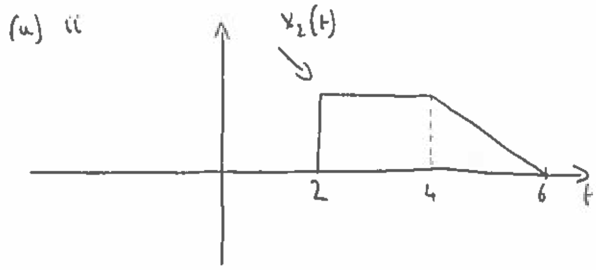
EXAMINATIONS 2017

SIGNALS AND LINEAR SYSTEMS SOLUTIONS

SOLUTIONS

1. (a) The two signals are as follows





(b) Any signal $x(t)$ can be expressed as a sum of even and odd components: $x(t) = x_o(t) + x_e(t)$ with

$$x_o(t) = \frac{1}{2}[x(t) - x(-t)] = \frac{1}{2}[e^{-t/2}u(t) - e^{t/2}u(-t)]\cos(t)$$

and

$$x_e(t) = \frac{1}{2}(x(t) + x(-t)) = \frac{1}{2}[e^{-t/2}u(t) + e^{t/2}u(-t)]\cos(t)$$

(c) A system is BIBO stable if and only if $\int_{-\infty}^{\infty} |h(t)|dt < \infty$ This condition is satisfied in our case since:

$$\int_{-\infty}^{\infty} |h(t)|dt = \int_{0}^{\infty} te^{-t}dt = 1$$

One can prove this also by looking at the root of $H(s)$.

(d) The two roots of $s^2 + 2s + 5$ are $s_{1,2} = -1 \pm 2j$. Consequently, the system is stable since both roots have negative real part.

(e) We look at the convolution $y(t) = h(t) * u(t)$ which is given by

$$y(t) = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau = \int_{-\infty}^{t} h(\tau)d\tau.$$

We also have that $\int_{-\infty}^{t} h(\tau)d\tau \geq 0$ since $h(t)$ is never negative. Consequently, for any $\epsilon \geq 0$,

$$y(t+\epsilon) = \int_{-\infty}^{t+\epsilon} h(\tau)d\tau = \int_{-\infty}^{t} h(\tau)d\tau + \int_{t}^{t+\epsilon} h(\tau)d\tau = y(t) + \int_{t}^{t+\epsilon} h(\tau)d\tau \geq y(t)$$

(f)

$$x_1(t) * x_2(t) = \begin{cases} 0 & t \leq 0 \\ \int_0^t e^{-\tau} e^{2(t-\tau)} d\tau = \frac{1}{3}(e^{2t} - e^{-t}) & 0 < t \leq 1 \\ \int_{t-1}^1 e^{-\tau} e^{2(t-\tau)} d\tau = \frac{e^3}{3}(e^{-t} - e^{-6+2t}) & 1 < t \leq 2 \\ 0 & t \geq 2 \end{cases}$$

(g) The key is to note that $x(t) = e^{-2}e^{-(t-2)}u(t-2)$. Consequently, by applying the time-shifting property, we obtain:

REMEMBER THAT YOU CANNOT APPLY THE TIME-SHIFTING PROPERTY DIRECTLY TO $e^{-t}u(t-2)$

$$X(s) = e^{-2}e^{-2s}\frac{1}{s+1}.$$

(h)

$$\begin{aligned} X(s) &= \frac{1}{(s^2+4s+3)(s^2+2s+1)} = \frac{1}{(s+1)^3(s+3)} \\ &= \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{(s+1)^3} + \frac{D}{s+3} \\ &= \frac{1}{8}\frac{1}{s+1} - \frac{1}{4}\frac{1}{(s+1)^2} + \frac{1}{2}\frac{1}{(s+1)^3} - \frac{1}{8}\frac{1}{s+3} \end{aligned}$$

Consequently

$$x(t) = \left(\frac{1}{8}e^{-t} - \frac{1}{4}te^{-t} + \frac{1}{4}t^2e^{-t} - \frac{1}{8}e^{-3t}\right)u(t)$$

(i) i. Characteristic polynomial: $s^2+7s+12$. Characteristic roots: $\lambda_1 = -3, \lambda_2 = -4$.

ii. $y(t) = c_1e^{-3t} + c_2e^{-4t}$. Therefore, since

$$y(0) = c_1 + c_2 = 1$$

and

$$\dot{y}(0) = -3c_1 - 4c_2 = 0,$$

we have that $c_1 = 4$ and $c_2 = -3$. So the zero-input response is

$$y(t) = [4e^{-3t} - 3e^{-4t}]u(t).$$

MOST STUDENT ANSWERED CORRECTLY TO THIS QUESTION

iii. In the Laplace domain

$$Y(s) = \frac{1}{(s+3)(s+4)(s+1)} = -\frac{1}{2}\frac{1}{(s+3)} + \frac{1}{3}\frac{1}{(s+4)} + \frac{1}{6}\frac{1}{(s+1)}.$$

Therefore,

$$y(t) = [-\frac{1}{2}e^{-3t} + \frac{1}{3}e^{-4t} + \frac{1}{6}e^{-t}]u(t)$$

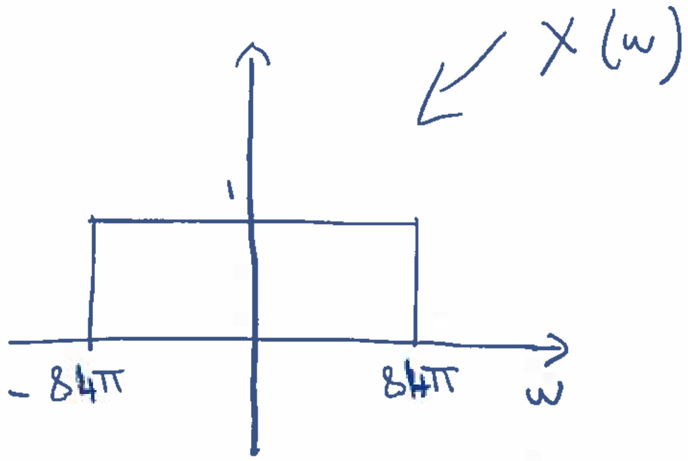
iv. The total response is the sum of the responses in part (ii) and (iii). Therefore,

$$\begin{aligned} y(t) &= [4e^{-3t} - 3e^{-4t} - \tfrac{1}{2}e^{-3t} + \tfrac{1}{3}e^{-4t} + \tfrac{1}{6}e^{-t}]u(t) \\ &= \left[\tfrac{7}{2}e^{-3t} - \tfrac{8}{3}e^{-4t} + \tfrac{1}{6}e^{-t}\right]u(t) \end{aligned}$$

REMEMBER THAT THE SAMPLING FREQUENCY IS AT LEAST TWICE THE BANDWIDTH OF THE SIGNAL

(j) i. Sampling frequency $f_s = 1/T = 100$ Hz, so $x(t)$ must be bandlimited to 50 Hz.

ii. $X(\omega)$ is as follows:



iii. The bandwidth of the signal is 42 Hz, so we need $f_s \geq 84$ Hz to avoid aliasing. This means that $T = 0.01$ is small enough since this leads to $f_s = 100$ Hz. In fact this already leads to oversampling.

2. (a) By applying the Laplace transform on both sides we obtain:

$$H_1(s) = \frac{1}{s^2 + a_1 s + a_2}$$

(b) Since the two systems are in parallel, we can write:

$$H_3(s) = H_1(s) + H_2(s) = \frac{(s + b_1) + (s^2 + a_1 s + a_2)}{(s^2 + a_1 s + a_2)(s + b_1)}.$$

(c) Using the final value theorem, we have that:

$$y_1(\infty) = \lim_{s \to 0} sH_1(s)X(s) = \lim_{s \to 0} sH_1(s)\frac{1}{s} = \lim_{s \to 0} \frac{1}{s^2 + a_1 s + a_2} = \frac{1}{2}.$$

Consequently, $a_2 = 2$. [3/8]

REMEMBER THAT ROC GIVES YOU SOME INFORMATION ABOUT THE LOCATION OF THE POLES

System S_1 is causal with ROC $Re\{s\} > -1$. This means that the real part of one of the two poles of $H_1(s)$ is -1 and that the other pole has a real part smaller than -1. The two poles are:

$$s_{1,2} = -\frac{a_1 \pm \sqrt{a_1^2 - 8}}{2}.$$

We can find a_1 by solving:

$$-\frac{a_1 \pm \sqrt{a_1^2 - 8}}{2} = -1.$$

Since we are told that both poles are real, $a_1^2 \geq 8$. Therefore, we need to solve

$$-a_1 + \sqrt{a_1^2 - 8} = -2.$$

We are considering only the sign $+$ since we want the greater of the two solutions to be equal to -1. This yields

$$a_1 = 3$$

and

$$H_1(s) = \frac{1}{s^2 + 3s + 2}.$$

[6/8]

We can apply the final value theorem again to obtain:

$$y(\infty) = \lim_{s \to 0} sH_3(s)X(s) = \lim_{s \to 0} \frac{(s + b_1) + (s^2 + a_1 s + a_2)}{(s^2 + a_1 s + a_2)(s + b_1)} = 1.$$

Given that $a_1 = 3$ and $a_2 = 2$, we obtain $b_1 = 2$. [8/8]

(d) We make the following preliminary observations: The frequency-shifting property states that

$$g(t)e^{s_0 t} \iff G(s - s_0)$$

where $G(s)$ is the Laplace transform of $g(t)$. Moreover, the amplitude Fourier transform of $g(t)$ is even and the phase is odd since $g(t)$ is real. We also introduce a new function $h(t) = g(t)e^{3t}$ and note that it is real since $g(t)$ is real.

THIS WAS A CHALLENGING QUESTION. REMEMBER THAT THE OUTPUT TO AN EVERLASTING EXPONENTIAL WITH FREQUENCY S_0 IS $H(S_0)e^{S_0 t}$ WHERE $H(S)$ IS THE TRANSFER FUNCTION OF THE LTI SYSTEM

[2/8]

We now compute the output $y(t)$ using the convolution formula for the input $x(t) = e^{-3t}\cos t = e^{-3t}(e^{jt} + e^{-jt})/2$:

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} g(\tau)x(t-\tau)d\tau = \frac{1}{2}\left[e^{(j-3)t}\int_{-\infty}^{\infty} g(\tau)e^{3\tau}e^{-j\tau}d\tau + e^{-(j+3)t}\int_{-\infty}^{\infty} g(\tau)e^{3\tau}e^{j\tau}d\tau\right] \\ &= \frac{e^{-3t}}{2}\left(\hat{H}(j)e^{jt} + \hat{H}(-j)e^{-jt}\right) \\ &= |\hat{H}(j)|e^{-3t}\cos(t+\varphi), \end{aligned}$$

where $\hat{H}(j)$ is the Fourier transform of $\hat{h}(t)$ at j and φ is the phase of $\hat{H}(j)$. Here we have also used the fact that the amplitude of the Fourier transform of $\hat{h}(t)$ is even and the phase is odd. We also note that since the question states that $y(t)$ is 'well defined', we are guaranteed that $\hat{H}(j)$ exists.

[6/8]

The knowledge of $\hat{H}(s)$ at $s = j$ is therefore sufficient to determine $y(t)$. Using the frequency shifting property, we realise that

$$\hat{H}(s) = G(s-3).$$

Consequently, the single value of s to pick is $s = j + 3$ since $G(s)$ at $s = j + 3$ corresponds to $\hat{H}(j)$.

[8/8]

(e) Using

$$\cos(t+\varphi) = \cos t\cos\varphi - \sin t\sin\varphi,$$

we have that

$$y(t) = |\hat{H}(j)|e^{-3t}\cos(t+\varphi) = e^{-3t}(A\cos t + B\sin t).$$

Consequently,

$$0 = y(0) = A$$

and

$$B = 1.$$

3. (a)

$$Y(s) = H_1(s)\,[X(s) - H_2(s)Y(s)]\,.$$

Therefore

$$H(s) = \frac{H_1(s)}{1 + H_1(s)H_2(s)} = \frac{(s+b)}{(s+a)(s+b)+1}$$

(b) Given that

$$H_3(s) = \frac{(s+2)}{(s^2+6s+9)},$$

we have that $b = 2$ and $a = 4$.

(c) The input $x(t)$ can be written as follows $x(t) = e^{-t}u(t) - e^{-5}e^{-(t-5)}u(t-5)$. Given that $y(t) = [-\frac{1}{4}e^{-3t} + \frac{1}{2}te^{-t} + \frac{1}{4}e^{-t}]u(t)$ when $x(t) = e^{-t}u(t)$, we can use the linearity of the Laplace transform together with the time-shifting property to arrive at the correct output given by:

$$y(t) = [-\frac{1}{4}e^{-3t} + \frac{1}{2}(t-5)e^{-t} + \frac{1}{4}e^{-t}]u(t) - e^{-5}[-\frac{1}{4}e^{-3(t-5)} + \frac{1}{2}te^{-(t-5)} + \frac{1}{4}e^{-(t-5)}]u(t-5)$$

(d) We note that:

$$x(t) = f(t) + f(t-T) \Longleftrightarrow X(s) = F(s)(1 + e^{-st}).$$

We also have:

$$Y(s) = \frac{H_1(s)}{1 + H_1(s)H_2(s)}X(s)$$

and we want $Y(s) = F(s)$ when the input is $X(s) = F(s)(1 + e^{-sT})$. Consequently, we need to design $H_1(s)$ and $H_2(s)$ such that the equality below is satisfied:

$$1 = \frac{H_1(s)}{1 + H_1(s)H_2(s)}(1 + e^{-sT})$$

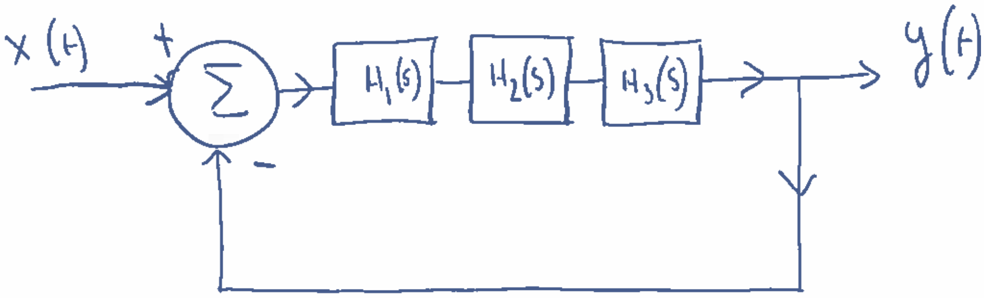
and this is achieved by imposing $H_1(s) = 1$ and $H_2(s) = e^{-sT}$. This means that $f(t-T)$ is removed by just imposing that $H_2(s)$ is a pure delay.

MOST STUDENTS COMPUTED THE OUTPUT FROM SCRATCH AND DIDN'T REALIZE THAT BY USING LINEARITY AND SHIFT INVARIANCE PROPERTIES THEY COULD GET THE ANSWER MUCH MORE QUICKLY.

(e) The desired transfer function is

$$H(s) = \frac{2}{s^2 + 5s + 6}.$$

Given that the numerator is 2, the best feedback configuration is shown below:



[5/8]

This yields

$$\hat{H}(s) = \frac{K}{(s+a)(s+b)+K}.$$

Consequently $K = 2$, moreover, by comparing the two denominators, we have that $a + b = 5$ and $ab + K = 6$. Because of the stability requirement, we have that $a > 0$ and $b > 0$. A possible, but not unique, solution is then $a = 1$, $b = 4$. [8/8]