Model Answers for E4.10 Probability & Stochastic Process

E4.10
C2.1
SC4

Q 1:

a) i) $P(\text{function after } t \text{ sec})$

$$= P\left(\text{function after } t \text{ sec} \,\middle|\, \text{bad device}\right) \cdot P(\text{bad device})$$

$$+ P\left(\text{function after } t \text{ sec} \,\middle|\, \text{good device}\right) P(\text{good device})$$

$$\Rightarrow P(\text{function after } t \text{ sec}) = \left[1 - F_b(t)\right] \cdot (1-p) + \left[1 - F_g(t)\right] \cdot p$$

ii) Let G be the event that a device is "good"
B be the event that a device is "bad"
F be the event that a device still functions

$$P(G|F) = 0.99$$

By Bayes' rule:

$$P(G|F) = \frac{P(GF)}{P(F)}$$

$$\Rightarrow P(G|F) = \frac{P(F|G)\,P(G)}{P(F|G)P(G) \cdot P(F|B)\,P(B)}$$

$$\Rightarrow P(G|F) = \frac{(1-F_g(t))\cdot p}{(1-F_g(t))\,p + (1-F_b(t))\cdot(1-p)}$$

$$\Rightarrow \frac{(1-F_g(t))\,p}{(1-F_g(t))\,p + (1-F_b(t))(1-p)} = 0.99$$

Q1

a) iii) For exponential distribution,

$$F_g(t) = 1 - e^{-\lambda t}$$

$$F_b(t) = 1 - e^{-1000\lambda t}$$

$$\Rightarrow \quad \frac{p e^{-\lambda t}}{p e^{-\lambda t} + (1-p) e^{-1000\lambda t}} = 0.99$$

$$\Rightarrow \quad 1 + \frac{1-p}{p} \cdot e^{-999\lambda t} = \frac{100}{99}$$

$$\Rightarrow \quad \frac{1-p}{p} e^{-999\lambda t} = \frac{1}{99}$$

$$\Rightarrow \quad e^{-999\lambda t} = \frac{p}{99(1-p)}$$

$$\Rightarrow \quad -999\lambda t = \ln\left(\frac{p}{99(1-p)}\right)$$

$$t = \frac{-1}{999\lambda} \ln\left(\frac{p}{99(1-p)}\right)$$

$$t = \frac{1}{999\lambda} \ln\left(\frac{99(1-p)}{p}\right)$$

Q 1

b) i) $Y = kX$

$$F_Y(y) = P(Y \le y) = P(kX \le y) = P(X \le \frac{y}{k})$$

$$\Rightarrow \quad F_Y(y) = F_X(y/k)$$

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{dF_X(y/k)}{dy}$$

$$\Rightarrow \quad f_Y(y) = \frac{dF_X(y')}{dy'} \cdot \frac{d(y/k)}{dy}$$

$$\Rightarrow \quad f_Y(y) = f_X(\frac{y}{k}) \cdot \frac{1}{k}$$

ii) $F_Y^*(s) = E[e^{-sY}] = E[e^{-skX}]$

$$\Rightarrow F_Y^*(s) = F_X^*(sk)$$

Q 2

a)

$$G^*(z) = \sum_{k=0}^{\infty} p_k z^k = \sum_{k=0}^{\infty} \frac{(\alpha z)^k}{k!} \cdot e^{-\alpha}$$

$$\Rightarrow \quad G^*(z) = e^{-\alpha} \cdot e^{\alpha z} = e^{-\alpha(1-z)}.$$

$$E(X) = \left.\frac{dG^*(z)}{dz}\right|_{z=1} = \left. e^{-\alpha} e^{\alpha z} \cdot \alpha \right|_{z=1} = \alpha$$

$$\left.\frac{d^2 G^*(z)}{dz^2}\right|_{z=1} = E\left(X^2 - X\right) = E(X^2) - E(X)$$

$$\left.\frac{d^2 G^*(z)}{dz^2}\right|_{z=1} = \left. e^{-\alpha} e^{\alpha z} \alpha^2 \right|_{z=1} = \alpha^2$$

$$\Rightarrow \quad \alpha^2 = E(X^2) - \alpha$$

$$\Rightarrow \quad E(X^2) = \alpha^2 + \alpha$$

$$VAR(X) = E(X^2) - \left(E(X)\right)^2 = \alpha^2 + \alpha - \alpha^2 = \alpha.$$

Q 2

b. Let N_k be the number of arrivals during time t from the k^{th} process and $N = N_1 + N_2 + \dots + N_k$

Consider $K = 2$. Let $N = N_1 + N_2$.

Then
$$P(N_1 + N_2 = n) = \sum_{j=0}^{n} P(N_1 = j) P(N_2 = n-j)$$

$$= \sum_{j=0}^{n} \frac{(\lambda_1 t)^j}{j!} e^{-\lambda_1 t} \cdot \frac{(\lambda_2 t)^{n-j}}{(n-j)!} e^{-\lambda_2 t}$$

Let $\lambda = \lambda_1 + \lambda_2$.

$$\Rightarrow P(N_1 + N_2 = n) = e^{-\lambda t} \cdot \frac{t^n}{n!} \sum_{j=0}^{n} \binom{n}{j} \lambda_1^j \lambda_2^{n-j}$$

$$= e^{-\lambda t} \cdot \frac{t^n}{n!} (\lambda_1 + \lambda_2)^n$$

$$\Rightarrow P(N_1 + N_2 = n) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}$$

The result for general K follows by induction.

Q2

c) Markov inequality

i)

$$P(X \geq a) \leq \frac{E(X)}{a}$$

$$\Rightarrow P(e^{sX} \geq \alpha) \leq \frac{E(e^{sX})}{\alpha} = \frac{\phi(s)}{\alpha}$$

ii) Set $\alpha = e^{sA}$ in part i.

We have $P(e^{sX} \geq e^{sA}) \leq \dfrac{\phi(s)}{e^{sA}}$

$$\Rightarrow P(e^{sX} \geq e^{sA}) \leq e^{-sA}\phi(s)$$

Since $s > 0$,

$$P(e^{sX} \geq e^{sA}) = P(X \geq A).$$

Therefore $P(X \geq A) \leq e^{-sA}\phi(s)$

Q3

a) i)

$$U = XY$$
$$V = X$$

The system $u = xy$ and $v = x$ has one single solution: $x = v$ and $y = u/v$.

The Jacobian is

$$|J(x,y)| = \begin{vmatrix} y & x \\ 1 & 0 \end{vmatrix} = -x = -v$$

Therefore, $f_{UV}(u,v) = \frac{1}{|v|} f_{XY}\left(v, \frac{u}{v}\right)$

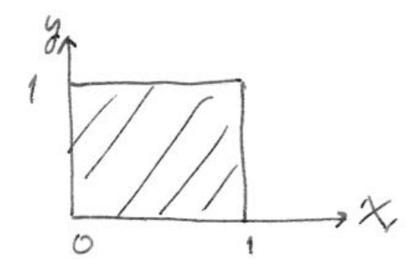
ii) The marginal pdf for U is

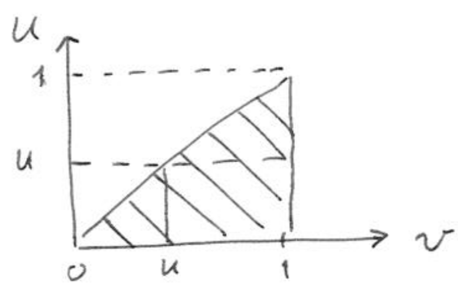
$$f_U(u) = \int_{-\infty}^{\infty} \frac{1}{|v|} \cdot f_{XY}\left(v, \frac{u}{v}\right) dv$$

iii) Given that X, Y are independent and each is uniformly distributed in $(0,1)$. Thus,

$$f_{XY}\left(v, \frac{u}{v}\right) = f_X(v) \cdot f_Y\left(\frac{u}{v}\right) = 1$$

for $u < v < 1$ and $0 < u < 1$, and 0 otherwise.





Q3

a) iii). Subs. $f_{XY}(v, \frac{u}{v})$ into the result in part ii, we obtain

$$f_U(u) = \int_u^1 \frac{1}{v}\, dv = \begin{cases} -\ln u & 0 < u < 1 \\ 0 & \text{otherwise} \end{cases}$$

b). i) Let the mean square error be f where

$$f = E\left[\left(X(n) - aX(n-2) - bX(n-1)\right)^2\right]$$

$$\frac{\partial f}{\partial a} = 2E\left[\left(X(n) - aX(n-2) - bX(n-1)\right) X(n-2)\right]$$

Set $\frac{\partial f}{\partial a} = 0$ for minimizing f. We obtain

$$E\left[X(n)\, X(n-2)\right] - aE\left[X(n-2)\, X(n-2)\right] - bE\left[X(n-1) \cdot X(n-2)\right] = 0$$

$$\Rightarrow \quad R(2) - aR(0) - bR(1) = 0$$

Given $R(\tau) = e^{-\tau^2}$, the above equation becomes

$$e^{-4} - a - e^{-1} b = 0$$

$$\Rightarrow \quad a + e^{-1} b = e^{-4} \quad \text{——} \quad (1)$$

Q3

b). i) Similarly, setting $\frac{\partial f}{\partial b} = 0$ leads to

$$E\left[\left(X(n) - aX(n-2) - bX(n-1)\right) \cdot X(n-1)\right] = 0$$

$$\Rightarrow E\left[X(n)X(n-1)\right] - aE\left[X(n-2) \cdot X(n-1)\right]$$

$$- bE\left[X(n-1) \cdot X(n-1)\right] = 0$$

$$\Rightarrow R(1) - aR(1) - bR(0) = 0$$

$$\Rightarrow aR(1) + bR(0) = R(1)$$

$$\Rightarrow e^{-1}a + b = e^{-1} \quad \text{——— (2)}$$

Solving (1) & (2) yields

$$a = -e^{-2} \quad \text{and} \quad b = e^{-1}(1 + e^{-2})$$

Q3

b.) ii.) The mean square error is

$$f = E\left[\left(X(n) - aX(n-2) - bX(n-1) \right)^2 \right]$$

$$= E\Big[X^2(n) + a^2 X^2(n-2) + b^2 X^2(n-1)$$

$$- 2a X(n)X(n-2) + 2ab X(n-2)X(n-1)$$

$$- 2b X(n)X(n-1) \Big]$$

$$= R(0) + a^2 R(0) + b^2 R(0) - 2a R(2)$$

$$+ 2ab R(1) - 2b R(1)$$

$$\Rightarrow \quad f = (1 + a^2 + b^2) R(0) - 2a R(2)$$

$$+ 2b R(1)(a-1)$$

$$= \left(1 + e^{-4} + e^{-2}(1+e^{-2})^2 \right) + 2e^{-2} e^{-4}$$

$$+ 2 \cdot e^{-1}(1+e^{-2}) \cdot e^{-1}(-e^{-2} - 1)$$

$$= 1 + e^{-4} + e^{-2}(1 + 2e^{-2} + e^{-4}) + 2e^{-6}$$

$$- 2e^{-2}(1 + 2e^{-2} + e^{-4})$$

$$= 1 + e^{-4} + 2e^{-6} - e^{-2}(1 + 2e^{-2} + e^{-4})$$

$$= 1 + e^{-4} + 2e^{-6} - e^{-2} - 2e^{-4} - e^{-6}$$

$$= 1 - e^{-2} - e^{-4} + e^{-6}$$

Q3
i) ii) $\Rightarrow$ Mean square error

$$f = (1-e^{-2})(1-e^{-4})$$

Q4

a) $\sigma_{X(t)}^2 = E\left(X^2(t)\right)$ since $E\left(X(t)\right) = 0$

$\Rightarrow \sigma_{X(t)}^2 = R(0) = 1.$

b) Since $X(t)$ has a normal distribution, we have

$$P\left(\frac{X(t) - E(X(t))}{\sigma_{X(t)}} \leq y\right) = F(y)$$

As $E(X(t)) = 0$ and $\sigma_{X(t)} = 1$, thus

$$P\left(X(t) \leq y\right) = F(y).$$

That is, $P\left(X(t) \leq 2\right) = F(2)$

c). $E\left[X(t+\tau) + X(t)\right] = E\left[X(t+\tau)\right] + E\left[X(t)\right]$

$= 0 \quad \because X(t)$ is WSS

d) $E\left[\left(X(t+\tau) + X(t)\right)^2\right]$

$= E\left[X^2(t+\tau) + X^2(t) + 2X(t+\tau)\,X(t)\right]$

$= R(0) + R(0) + 2R(\tau) \quad \because X(t)$ is WSS

$\Rightarrow E\left[\left(X(t+\tau) + X(t)\right)^2\right] = 2R(0) + 2R(\tau)$

$= 2\left(1 + e^{-2|\tau|}\right).$

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e.) Let $X = X(t)$, $Y = X(t+\tau)$

Define $S \equiv X + Y = X(t) + X(t+\tau)$

$\rho \equiv R(\tau)$

Since X and Y are jointly normal distributed, the pdf for S

$$f_S(s) = \int_{-\infty}^{\infty} f_{XY}(x, s-x)\, dx$$

Subs. the normal pdf for X,Y into the above, we get

$$f_S(s) = \int_{-\infty}^{\infty} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{ \frac{-1}{2(1-\rho^2)} \left[x^2 - 2\rho x(s-x) + (s-x)^2 \right] \right\} dx \quad (4.1)$$

Consider the expression inside the brackets

$$\begin{aligned} & x^2 - 2\rho x(s-x) + (s-x)^2 \\ &= x^2 + 2\rho x^2 - 2\rho s x + s^2 - 2xs + x^2 \\ &= 2(1+\rho)x^2 - 2(1+\rho)sx + s^2 \end{aligned}$$

Subs. this into (4.1)

$$f_S(s) = \int_{-\infty}^{\infty} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{ \frac{-1}{1-\rho} \left[x^2 - sx \right] - \frac{s^2}{2(1-\rho^2)} \right\} dx$$

$$\Rightarrow f_S(s) = \int_{-\infty}^{\infty} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{ \frac{-1}{1-\rho}\left(x - \frac{s}{2}\right)^2 + \frac{s^2}{4(1-\rho)} - \frac{s^2}{2(1-\rho^2)} \right\} dx$$

$$\Rightarrow f_S(s) = \int_{-\infty}^{\infty} \frac{1}{2\pi\sqrt{1-\rho^2}} \cdot \exp\left\{ \frac{-1}{2} \cdot \frac{(x - s/2)^2}{(1-\rho)/2} - \frac{s^2}{4(1+\rho)} \right\} dx$$

$$\Rightarrow f_S(s) = \frac{1}{\sqrt{2\pi}\sqrt{1+\rho}\sqrt{2}} \exp\left\{ \frac{-s^2}{4(1+\rho)} \right\} \cdot \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sqrt{1-\rho}/\sqrt{2}} \cdot \exp\left\{ \frac{-1}{2} \frac{(x - s/2)^2}{(1-\rho)/2} \right\} dx}_{=1}$$

$$\Rightarrow f_S(s) = \frac{1}{\sqrt{2\pi}\sqrt{1+\rho}\sqrt{2}} \cdot \exp\left\{ \frac{-s^2}{4(1+\rho)} \right\}$$

is a normal pdf

That is, $S = X(t+\tau) + X(t)$ is normally distributed.

Q4

f.) Let $S = X(t+\tau) + X(t)$

$$P(|S| \le 1) = P\left(\frac{|S - E(S)|}{\sigma_S} \le \frac{1}{\sigma_S} \right)$$

$\because$ $E(S) = 0$ from part c and $\sigma_S > 0$ which has been obtained in part d.

$$\Rightarrow P(|S| \le 1) = P\left(\frac{|S - E(S)|}{\sigma_S} \le \frac{1}{\sigma_S} \right) = F\left(\frac{1}{\sigma_S}\right) - F\left(\frac{-1}{\sigma_S}\right)$$

where $\sigma_S = \left[2(1 + e^{-2|\tau|}) \right]^{1/2}$

from part d.

Q5.

a.) i) $E(S_n) = n \cdot m$

$$VAR(S_n) = n \cdot m^2$$

ii) $P(950\,m < S_{1000} \leq 1050\,m)$

$$= P\left(\frac{950m - 1000m}{m\sqrt{1000}} \leq Z_{1000} \leq \frac{1050m - 1000m}{m\sqrt{1000}} \right)$$

by Central Limit Theorem

$\Rightarrow P(950\,m < S_{1000} \leq 1050\,m)$

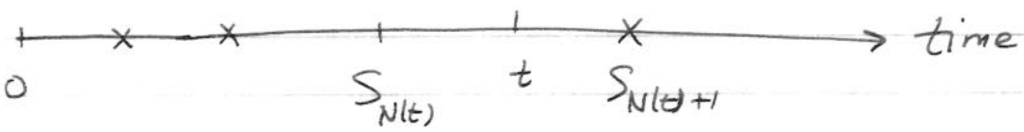
$$\doteq Q(1.58) - Q(-1.58)$$

$$= 1 - 2Q(1.58)$$

$$= 0.8866$$

Q5

b.) iii)



Since $S_{N(t)} \leq t < S_{N(t)+1}$, we have

$$\frac{S_{N(t)}}{N(t)} \leq \frac{t}{N(t)} < \frac{S_{N(t)+1}}{N(t)} \quad \text{—— } (5.1)$$

By the law of Large Number

$$\frac{S_{N(t)}}{N(t)} \to \mu \quad \text{as } t \to \infty \quad \text{— } (5.2)$$

Now, we write that

$$\frac{S_{N(t)+1}}{N(t)} = \left[\frac{S_{N(t)+1}}{N(t)+1}\right] \cdot \left[\frac{N(t)+1}{N(t)}\right]$$

By the same argument of the law of Large Number,

$$\frac{S_{N(t)+1}}{N(t)+1} \to \mu \quad \text{as } t \to \infty$$

Further, $\frac{N(t)+1}{N(t)} \to 1$ as $t \to \infty$

Therefore, $\frac{S_{N(t)+1}}{N(t)} \to \mu$ as $t \to \infty$ — (5.3)

Q5

b.) i) Note that $N(t) \geq n \iff S_n \leq t$

$$P(N(t)=n) = P(N(t) \geq n) - P(N(t) \geq n+1)$$
$$= P(S_n \leq t) - P(S_{n+1} \leq t)$$

Since X_i's are iid with a common PDF $F(x)$, the PDF for $S_n = \sum_{i=1}^{n} X_i$ is the n-fold convolution of $F(t)$ with itself.

Thus, $P(N(t)=n) = F_n(t) - F_{n+1}(t)$

where $F_n(t) \triangleq$ n-fold convolution of $F(t)$

ii) Define $N(t) = \sum_{n=1}^{\infty} I_n$

where $I_n = \begin{cases} 1 & \text{if the } n^{th} \text{ arrival occurs in } (0,t] \\ 0 & \text{otherwise} \end{cases}$

Therefore $E[N(t)] = E\left[\sum_{n=1}^{\infty} I_n\right]$
$$= \sum_{n=1}^{\infty} E[I_n]$$
$$= \sum_{n=1}^{\infty} P(I_n = 1)$$
$$= \sum_{n=1}^{\infty} P(S_n \leq t)$$
$$\Rightarrow E[N(t)] = \sum_{n=1}^{\infty} F_n(t)$$

Q5.

b) iii) Putting (5.2) & (5.3) into (r.1),

we must have

$$\frac{t}{N(t)} \to \mu \quad \text{as } t \to \infty .$$

Q.E.D.

Q6.

a.) For all $i = 0, 1, 2, \ldots$

$$\pi_i = \sum_{j=0}^{\infty} \pi_j \, p_{ji}$$

b) $$Q(z) = \sum_{i=0}^{\infty} \pi_i z^i = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \pi_j \, p_{ji} \, z^i$$

Subs. p_{ij} into the above, we have

$$Q(z) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \pi_j z^i e^{-\lambda} \sum_{n=0}^{i} \binom{j}{n} p^n q^{j-n} \frac{\lambda^{i-n}}{(i-n)!}$$

Since $\sum_{i=0}^{\infty} \sum_{n=0}^{i} \Leftrightarrow \sum_{n=0}^{\infty} \sum_{i=n}^{\infty}$, we have

$$Q(z) = e^{-\lambda} \sum_{j=0}^{\infty} \pi_j \sum_{n=0}^{\infty} \binom{j}{n} p^n q^{j-n} \sum_{i=n}^{\infty} z^i \frac{\lambda^{i-n}}{(i-n)!}$$

$$= e^{-\lambda} e^{\lambda z} \sum_{j=0}^{\infty} \pi_j \sum_{n=0}^{\infty} \binom{j}{n} (pz)^n q^{j-n}$$

Since $\binom{j}{n} = 0$ for $n > j$, then

$$Q(z) = e^{\lambda(z-1)} \cdot \sum_{j=0}^{\infty} \pi_j (pz + q)^j$$

$$\Rightarrow Q(z) = e^{\lambda(z-1)} Q(pz + q)$$

$$\Rightarrow Q(z) = e^{\lambda(z-1)} Q(1 + p(z-1)).$$

Q6.
d.)

$$Q(z) = e^{\lambda(z-1)(1+p^2+p^2+\dots+p^{n-1})} \cdot Q(1+p^n(z-1))$$

When $n=1$,

$$Q(z) = e^{\lambda(z-1)} Q(1+p(z-1))$$

which is valid as part b shows.

By induction, assume it is true that for some arbitrary integer n,

$$Q(z) = e^{\lambda(z-1)(1+p+p^2+\dots+p^{n-1})} \cdot Q(1+p^n(z-1)) \quad (6.1)$$

Now, substitute $1+p^n(z-1)$ for z in the result from part b; we then get

$$Q(1+p^n(z-1)) = e^{-\lambda p^n(z-1)} Q(1+p^{n+1}(z-1))$$

Subs. the above to the RHS of (6.1) gives

$$Q(z) = e^{\lambda(z-1)(1+p+p^2+\dots+p^{n-1}+p^n)} \cdot Q(1+p^{n+1}(z-1))$$

which completes the induction.

Q 6.

d.) $$Q(z) = e^{\lambda(z-1)(1+p+p^2+\cdots+p^n)} Q(1+p^{n+1}(z-1))$$

for all $n = 0, 1, 2, \ldots$ Consider $n \to \infty$. This gives

$$Q(z) = e^{\lambda(z-1)\sum_{n=0}^{\infty} p^n} Q(1)$$

as $p^{n+1} \to 0$ as $n \to \infty$ $\because p < 1$.

Since $Q(1) = 1$, we then have

$$Q(z) = e^{\lambda(z-1)\sum_{n=0}^{\infty} p^n}$$

$$\Rightarrow Q(z) = e^{\lambda(z-1)\cdot\frac{1}{1-p}}$$

or $$Q(z) = e^{\lambda(z-1)/q} = e^{\frac{\lambda}{q}(z-1)}$$

Expand $Q(z)$ in terms of z. We obtain

$$Q(z) = e^{-\frac{\lambda}{q}}\left(1 + \frac{\lambda}{q}z + \frac{1}{2!}\left(\frac{\lambda}{q}\right)^2 z^2 + \frac{1}{3!}\left(\frac{\lambda}{q}\right)^3 z^3 + \cdots\right)$$

By inspection, $$\pi_j = e^{-\frac{\lambda}{q}} \cdot \frac{(\lambda/q)^j}{j!}$$

for $\forall\, j = 0, 1, 2, \cdots$