CONTROL ENGINEERING

1. a) The system is already in reachability canonical form, hence it is reachable regardless of the value of α. Alternatively, the rechability matrix is

$$\mathscr{R} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

which does not depend upon α and has rank two. [2 marks]

b) The observability matrix is

$$\mathscr{O} = \begin{bmatrix} -\delta & 1 \\ -\alpha & -\delta \end{bmatrix}.$$

Note that det $\mathscr{O} = \delta^2 + \alpha$, hence the system is observable for all α and δ such that $\delta^2 + \alpha \neq 0$. [4 marks]

c)

i) A state space realization of the interconnected system is described by the matrices

$$A_i = \begin{bmatrix} 0 & 1 & 0 \\ -\alpha & 0 & 0 \\ -\delta & 1 & 0 \end{bmatrix}, \qquad B_i = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad C_i = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}.$$

[2 marks]

ii) The reachability matrix of the interconnected system is

$$\mathscr{R}_i = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -\alpha \\ 0 & 1 & -\delta \end{bmatrix}$$

and this has rank three provided $\delta \neq 0$.
For $\delta = 0$ a direct calculation shows that

$$\text{Im}\, A_i^3 \subset \text{Im}\, \mathscr{R}_i,$$

hence the system is controllable for any δ (and α). [8 marks]

iii) The observability matrix of the interconnected system is

$$\mathscr{O}_i = \begin{bmatrix} 0 & 0 & 1 \\ -\delta & 1 & 0 \\ -\alpha & -\delta & 0 \end{bmatrix}$$

and this has rank three provided $\delta^2 + \alpha \neq 0$, that is provided the system with state x is observable. [2 marks]

iv) Note that

$$\det \begin{bmatrix} zI - A & B \\ C & -D \end{bmatrix} = \delta - z,$$

hence the system with state x has a zero at $z = \delta$. The eigenvalue of the system with state ξ is at $z = 0$. When $\delta = 0$ the zero and the eigenvalue coincide (that is they cancel each other). As noted in part c.ii) for $\delta = 0$ the interconnected system is not reachable. It remains controllable since the *cancellation* is at $z = 0$. [2 marks]

2. a) The observability matrix is

$$\mathcal{O} = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix},$$

which is full rank (the determinant is equal to 6), hence the system is observable. [2 marks]

b) Note that

$$\dot{z} - \dot{x}_2 = fz + gu + h(x_1 - x_2) - (-2x_1 - 3x_2 + u).$$

Selecting $g = 1$, $h = -2$ and $f = -5$ yields

$$\dot{z} - \dot{x}_2 = -5\,(z - x_2),$$

that is the desired equation with $k = -5$. The state z is such that

$$z(t) - x_2(t) = e^{-5t}(z(0) - x_2(0)),$$

hence

$$z(t) = x_2(t) + e^{-5t}(z(0) - x_2(0)),$$

that is $z(t)$ converges exponentially to $x_2(t)$. The variable z can then be used to estimate x_2. [6 marks]

c) To build an asymptotic estimate of the state x_1 note that the output equation is $y = x_1 - x_2$, hence an asymptotic estimate of x_1 is given by

$$y(t) + z(t).$$

[2 marks]

d) The system with state $[x_1, x_2, z]'$ can be rewritten in the coordinate $[x_1, x_2, e]'$, with $e = z - x_2$ as

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -2x_1 - 3x_2 + u \qquad \dot{e} = -5e,$$

which clearly shows that the system is not controllable and the uncontrollable mode is at $s = -5$. This is a consequence of the fact that the $\dot{z}$ equation has been designed such that $\dot{e} = -5e$, that is the observer has to have converging properties which do not depend upon x_1, x_2 and u. [4 marks]

e) The A matrix of the closed-loop system is

$$A_{cl} = \begin{bmatrix} 0 & 1 & 0 \\ -2-p & -3+p & -q \\ -2-p & 2+p & -q-5 \end{bmatrix},$$

the characteristic polynomial of which is

$$(s+5)(s^2 + (3+q-p)s + p + 2).$$

Selecting $p = 2$ and $q = 3$ yields the desired eigenvalues. Note that, consistently with the design in part b) and the analysis in part d), one of the eigenvalues of the closed-loop system is *fixed* at -5. [6 marks]

3. a) The matrices A and B are given by

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

[2 marks]

b) The system is in controllability canonical form, hence it is controllable. The control objective is to *steer* the initial state $[\bar{x}, 0]$ to the origin, and this can be always achieved, for any $T > 0$, by the very definition of controllability.

[2 marks]

c)

i) The differential equation of the system for $u = 1$ are

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = 1,$$

yielding (recall the considered initial conditions)

$$x_1(t) = x_1(0) + x_2(0)t + \frac{1}{2}t^2 = \bar{x} + \frac{1}{2}t^2, \qquad x_2(t) = x_2(0) + t = t,$$

as indicated in the exam paper. Note that $t = x_2$, hence replacing this in the x_1 equation yields

$$x_1 = \bar{x} + \frac{1}{2}x_2^2,$$

that is the family of parabolas, parameterized by $\bar{x}$, in red-dashed lines in the figure. The arrow of time is pointing upward since $\dot{x}_2 = 1 > 0$, that is the state x_2 is monotonically increasing with time. Clearly, the parabola with equation $x_1 = \frac{1}{2}x_2^2$ is the only one that goes through the origin (the parabola is drawn in bold in the figure). [4 marks]

ii) Similarly to the previous point, the differential equation of the system for $u = -1$ are

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -1,$$

yielding (recall the considered initial conditions)

$$x_1(t) = x_1(0) + x_2(0)t - \frac{1}{2}t^2 = \bar{x} - \frac{1}{2}t^2, \qquad x_2(t) = x_2(0) - t = -t.$$

Note that $t = -x_2$, hence replacing this in the x_1 equation yields

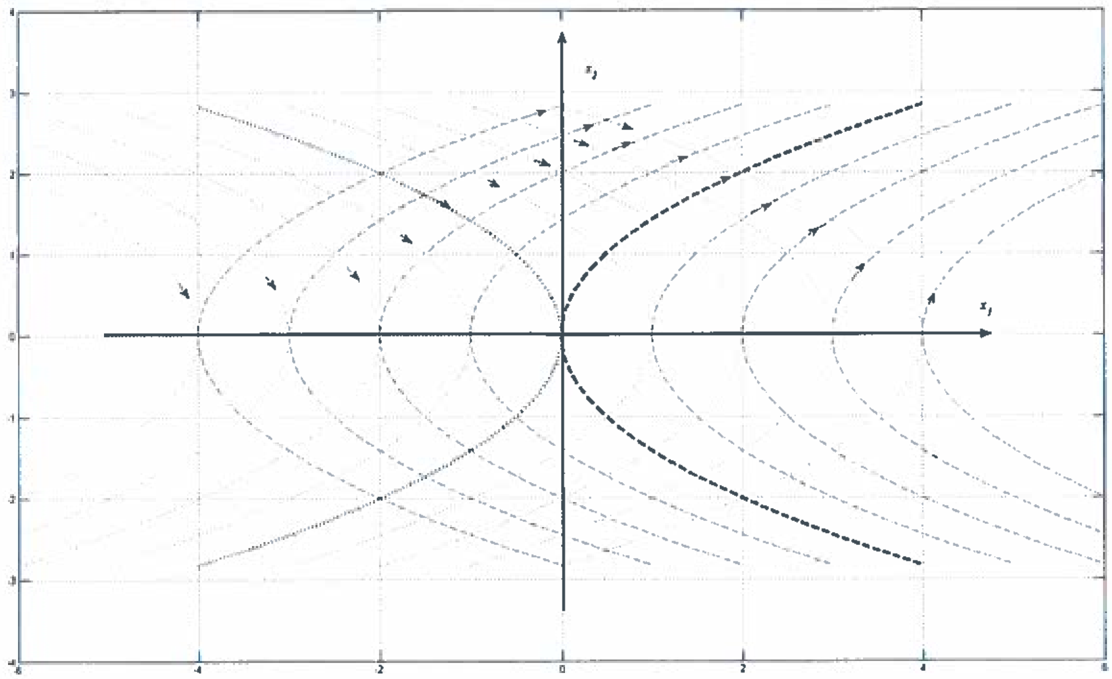
$$x_1 = \bar{x} - \frac{1}{2}x_2^2,$$

that is the family of parabolas, parameterized by $\bar{x}$, in blue-dotted lines in the figure. The arrow of time is pointing downward since $\dot{x}_2 = -1 < 0$, that is the state x_2 is monotonically decreasing with time. As above the parabola with equation $x_1 = -\frac{1}{2}x_2^2$ is the only one that goes through the origin (the parabola is drawn in bold in the figure). [4 marks]

iii) If $\bar{x} < 0$ one could follow the red-dashed trajectory starting from $(\bar{x}, 0)$ till the trajectory meets the blue-dotted trajectory described by $x_1 = -\frac{1}{2}x_2^2$. At that point, the sign of the input signal is switched and the state follows the blue-dottet trajectory till the origin. Similarly, for $\bar{x} > 0$. [6 marks]

iv) Consider a trajectory with $\bar{x} < 0$. Note that for the first part of the trajectory one has $x_2(\bar{t}) - x_2(0) = \bar{t}$ and then $x_2(\bar{t}) - x_2(T) = -(t-T)$, which shows that the time T to reach the origin is twice the maximum value achieved by $x_2(t)$ along the considered trajectory, hence it is finite. Similar considerations apply for trajectories starting with $\bar{x} > 0$.

The bound on the acceleration is trivially satisfied since $\ddot{x} = u$, hence $|\ddot{x}| = 1$, for all $t \in [0, T)$. At $t = T$ one sets $u(t) = 0$, for $t \geq T$ which, since the origin is an equilibrium for the system, yields a trajectory which remains at the origin for all $t \geq 0$. [2 marks]



4. a) The matrices A and B are given by

$$A = \begin{bmatrix} 0 & \varepsilon \\ 1 & 2, \end{bmatrix} \qquad B = \begin{bmatrix} -\varepsilon \\ 1 \end{bmatrix}.$$

[2 marks]

b) The reachability matrix is

$$\mathscr{R} = \begin{bmatrix} -\varepsilon & \varepsilon \\ 1 & 2-\varepsilon \end{bmatrix}.$$

Note that $\det \mathscr{R} = \varepsilon^2 - 3\varepsilon$, hence the system is reachable for all $\varepsilon \neq 0$ and $\varepsilon \neq 3$. The unreachable modes can be computed using the Hautus test: for $\varepsilon = 0$ it is at $z = 0$, whereas for $\varepsilon = 3$ it is at $z = 3$. [6 marks]

c) The system is controllable, by reachability, for all $\varepsilon \neq 0$ and $\varepsilon \neq 3$. For $\varepsilon = 0$, since the unreachable mode is at $z = 0$ it is controllable, whereas it is not controllable for $\varepsilon = 3$. (One could check controllability using alternative conditions, which however would require longer computations.) [4 marks]

d) Note that

$$A_{cl} = A + BK = \begin{bmatrix} -\varepsilon k_1 & \varepsilon - \varepsilon k_2 \\ 1+k_1 & 2+k_2 \end{bmatrix},$$

which has the characteristic polynomial

$$\det\,(zI - A_{cl}) = z^2 + z(\varepsilon k_1 - k_2 - 2) + (\varepsilon k_2 - 3\varepsilon k_1 - \varepsilon).$$

This should be equal to z^2, yielding

$$k_1 = \frac{3}{\varepsilon - 3}, \qquad k_2 = \frac{\varepsilon + 6}{\varepsilon - 3}.$$

Note that

$$\lim_{\varepsilon = 0} k_1 = -1 \qquad \lim_{\varepsilon = 0} k_2 = -2,$$

whereas k_1 and k_2 are not defined for $\varepsilon = 3$. This is consistent with the fact that for $\varepsilon = 0$ the unreachable mode is at $z = 0$, that is it coincides with one of the desired closed-loop eigenvalues, whereas for $\varepsilon = 3$ the unreachable mode does not coincide with one of the desired closed-loop eigenvalues. [6 marks]

e) The closed-loop matrix with $K = K_0$ is

$$A + BK_0 = \begin{bmatrix} \varepsilon & 3\varepsilon \\ 0 & 0 \end{bmatrix},$$

which has eigenvalues equal to 0 and ε. Hence the gain K_0 is stabilizing for all ε such that $|\varepsilon| < 1$. [2 marks]