UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 2003

MEng Honours Degree in Information Systems Engineering Part IV
MSci Honours Degree in Mathematics and Computer Science Part IV
MEng Honours Degrees in Computing Part IV
MSc in Advanced Computing
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the Associateship of the City and Guilds of London Institute
This paper is also taken for the relevant examinations for the Associateship of the Royal College of Science

PAPER C477=I4.20

COMPUTING FOR OPTIMAL DECISIONS

Friday 16 May 2003, 14:30
Duration: 120 minutes

Answer THREE questions

Paper contains 4 questions
Calculators not required

1 a Solve the equality constrained quadratic programming problem

$$\underset{x}{\text{minimise}} \left\{ \sum_{i=1}^{n} i\, x_i^2 \;\middle|\; \sum_{i=1}^{n} x_i = K \right\},$$

for $K > 0$, by formulating the appropriate Lagrangian and optimality conditions.

b Does this solution minimise the objective function subject to the constraints

$$x_i \geq 0, i = 1, \ldots, n\,; \; \sum_{i=1}^{n} x_i \geq K\ ?$$

(All parts carry equal marks)

2 a Consider the problem

$$\min \left\{ a^T x + \tfrac{1}{2} x^T \mathcal{Q}\, x \;\middle|\; \mathcal{H}^T x = h \right\}$$

where $a \in \Re^n$; $\mathcal{H} \in \Re^{n \times m}$, $\mathcal{Q} \in \Re^{n \times n}$, $h \in \Re^m$ are given values and $x \in \Re^n$ is to be determined. The matrix $\mathcal{Q}$ is not assured to be strictly positive definite. Write the optimality conditon for this problem, derive the optimal solution x and the Lagrange multiplier vector associated with $\mathcal{H}^T x = h$.

b Consider the inequality constrained problem

$$\min \left\{ a^T x + \tfrac{1}{2} x^T \mathcal{Q}\, x \;\middle|\; \mathcal{A}^T x \leq h_{\mathcal{A}} \right\}$$

where $\mathcal{A}$, $h_{\mathcal{A}}$ are given and the equality constraints $\mathcal{H}^T x = h$ in part (a) are an active subset of $\mathcal{A}^T x \leq h_{\mathcal{A}}$. Suppose the optimum in part (a), in the intersection of the active set $\mathcal{H}^T x = h$, has been attained and is feasible with respect to all the constraints in $\mathcal{A}^T x \leq h_{\mathcal{A}}$. Establish a test for the existence of a descent direction for the inequality constrained problem at the optimum obtained in part (a).

(All parts carry equal marks)

3 a Two stocks are being considered for inclusion in an investment portfolio. The estimated mean and variance of the return on each share is given by

Stock	mean(%)	variance
1	9	8
2	15	17

while the covariance of the returns is 0.1. The price per share of stocks 1 and 2 are £250 and £700 respectively. The amount budgeted for the portfolio is £90,000. Formulate the quadratic programming model for the problem of maximising return while minimising portfolio risk with the added constraint that only portfolios yielding 11% or over should be considered. [Hint: Let x_i be the number of stock i purchased and allow the purchase of fractions of a stock.]

b Suppose you only have access to linear programming software. Discuss an algorithm for solving the quadratic programming problem in part (a) using this software. [Do not attempt to compute the solution of the problem in part (a).]

(All parts carry equal marks)

4 a Consider the problem of finding a feasible solution for the system of linear inequalities

$$Ax \leq b.$$

where $x \in \Re^n$, $b \in \Re^m$, $n \geq m$ and A is an $n \times m$ matrix. Use slack variables $x_s \in \Re^m$ to write the equivalent sytem of equalities and inequalities

$$Ax + x_s = 0,\ x_s \geq 0.$$

Formulate a quadratic programming problem for determining $x \in \Re^n$ that satisfies $Ax \leq b$. [Hint: for all $v \in \Re^m$, we have $v^T v \geq 0$; $v^T v = 0 \Leftrightarrow v = 0$; the least value of $v^T v$ is $\min_v \left\{v^T v\right\} = 0$.]

b Consider the direction of search $d = -\nabla \mathcal{W}(y_j)^T \mathcal{W}(y_j)$ for the system of nonlinear equations $\mathcal{W}(y)$, where $\mathcal{W} \in \Re^m$, $y \in \Re^n$. Establish that at y_j, d is a descent direction for the merit function $\frac{1}{2} \| \mathcal{W}(y) \|_2^2$.

c Describe the Newton algorithm for computing y such that $\mathcal{W}(y) = 0$ with $\frac{1}{2} \| \mathcal{W}(y) \|_2^2$ used as the merit function for determining progress.

(All parts carry equal marks)