THE ANSWERS

Notations:

(a) B - Bookwork

(b) E - New example

(c) A - New application

1. a) $P(X+Y \leq 0.5) = 3\int_{x=0}^{0.5}\int_{y=0}^{0.5-x}(x+y)dydx$ [2 - E]

$P(X+Y \leq 0.5) = \frac{3}{2}\int_{x=0}^{0.5}\frac{1}{4} - x^2 dx = \frac{1}{8}$ [2 - E]

b) $f_X(x) = \begin{cases} \int_0^{1-x} 3(x+y)dy, & 0 < x < 1, \\ 0, & otherwise. \end{cases}$ [2 - E]

$f_X(x) = \begin{cases} \frac{3}{2}(1-x^2), & 0 < x < 1, \\ 0, & otherwise. \end{cases}$ [2 - E]

c) $E(X) = \int_0^1 x\frac{3}{2}(1-x^2)dx = \frac{3}{8}$ [2 - E]

$E(X^2) = \int_0^1 x^2\frac{3}{2}(1-x^2)dx = \frac{1}{5}$ [1 - E]

$\mathrm{Var}(X) = E(X^2) - E(X)^2 = \frac{19}{320}$ [1 - E]

d) In view of the joint pdf and the domain, we can easily reuse b) to write

$f_Y(y) = \begin{cases} \frac{3}{2}(1-y^2), & 0 < y < 1, \\ 0, & otherwise. \end{cases}$ [2 - E]

e) We find the same values as in c).

$E(Y) = \frac{3}{8}$ [1 - E]

$\mathrm{Var}(Y) = \frac{19}{320}$ [1 - E]

f) $E(XY) = \int_{x=0}^{1}\int_{y=0}^{1-x} xy3(x+y)dydx = \int_{x=0}^{1}\frac{x}{2}(2-3x+x^3)dx = \frac{1}{10}$

$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y) = \frac{1}{10} - \frac{3}{8}\frac{3}{8} = \frac{-26}{640} = -0.0406$ [1 - E]

$\mathrm{Corr}(X,Y) = \frac{-26/640}{19/320} = -\frac{13}{19} = -0.684$ [1 - E]

g) X and Y are correlated since $\mathrm{Corr}(X,Y) \neq 0$. [1 - E]

Since they are correlated, they are also dependent. [1 - E]

h) $f_{Y|X}(y|x) = \begin{cases} \frac{3(x+y)}{\frac{3}{2}(1-x^2)}, & 0 < x < 1,\ 0 < y < 1,\ 0 < x+y < 1, \\ 0, & otherwise. \end{cases}$ [2 - E]

i) $E[Y|X=x] = \begin{cases} \int_{y=0}^{1-x} y\frac{2(x+y)}{(1-x^2)}dy & 0 < x < 1, \\ 0, & otherwise. \end{cases}$ [2 - E]

$E[Y|X=x] = \begin{cases} \frac{(1-x)(2+x)}{3(1+x)} & 0 < x < 1, \\ 0, & otherwise. \end{cases}$ [1 - E]

2. a) i) Let us take $Y_i = \log X_i$ and write the probability

$$P\left(\prod_{i=1}^{n} X_i \le e^a\right) = P\left(\sum_{i=1}^{n} Y_i \le a\right)$$

with $a = -\frac{n}{2} + 0.5\sqrt{n}$. [1 - A]

We can then use the CLT to approximate the probability as

$$P\left(\sum_{i=1}^{n} Y_i \le a\right) \approx P\left(Z \le \frac{a - nE(Y)}{\sqrt{n\text{Var}(Y)}}\right)$$

with $Z \sim N(0,1)$. [2 - A]

We can compute $E(Y)$ and $\text{Var}(Y)$ as follows.
$E(Y) = \int_0^1 2x\log x dx = \left[x^2 \log x\right]_0^1 - \int_0^1 x dx = -\frac{1}{2}$.
(Use integration by part with $u = \log x$ and $dv = 2xdx$. Use L'Hospital rule to compute $\lim_{x\to 0} x^2 \log x$).
[1 - A]
$E(Y^2) = \int_0^1 2x(\log x)^2 dx = \left[x^2(\log x)^2\right]_0^1 - \int_0^1 2x\log x dx = \frac{1}{2}$.
(Use integration by part with $u = (\log x)^2$ and $dv = 2xdx$. Use L'Hospital rule to compute $\lim_{x\to 0} x^2(\log x)^2$).
$\text{Var}(Y) = E(Y^2) - E(Y)^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$. [1 - A]

We finally get $P\left(Z \le \frac{a-nE(Y)}{\sqrt{n\text{Var}(Y)}}\right) = P\left(Z \le \frac{-n/2+0.5\sqrt{n}+n/2}{\sqrt{n}/2}\right) = P(Z \le 1) =$ 0.841 from the tables. [2 - A]

ii) We can make the change of variables $U = X_1X_2$ and $V = X_1$. [1 - A]

We can compute the Jacobian matrix

$$J = \begin{bmatrix} \frac{\partial X_1}{\partial U} & \frac{\partial X_1}{\partial V} \\ \frac{\partial X_2}{\partial U} & \frac{\partial X_2}{\partial V} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{1}{V} & \frac{-U}{V^2} \end{bmatrix}$$

and make use of the change of variables relationship

$$f_{U,V}(u,v) = |\det J|\, f_{X_1,X_2}(x_1,x_2).$$

[2 - A]

We first get the joint pdf of X_1,X_2. Since X_1 and X_2 are independent, we have

$$f_{X_1,X_2}(x_1,x_2) = \begin{cases} (2x_1)(2x_2), & 0 < x_1 < 1,\ 0 < x_2 < 1, \\ 0, & otherwise. \end{cases}$$

Since $X_1 = V$ and $X_2 = \frac{U}{V}$, we get from the change of variables relationship

$$f_{U,V}(u,v) = \begin{cases} \left(\frac{1}{v}\right)(2v)\left(2\frac{u}{v}\right) = 4\frac{u}{v}, & 0 < v < 1,\ 0 < u < v, \\ 0, & otherwise. \end{cases}$$

Note the change of variables implies that the new domain is characterized by $0 < v < 1$ and $0 < u < v$. [2 - A]

The probability density function of U is finally obtained as

$$f_U(u) = \begin{cases} \int_u^1 4\frac{u}{v}dv = -4u\log u, & 0 < u < 1, \\ 0, & otherwise. \end{cases}$$

[1 - A]

iii) $E(U) = \int_0^1 u(-4u\log u)du = \left[-4(u^3/3\log u - u^3/9)\right] = \frac{4}{9}$. We could have obtained the same result by noting that since X_1 and X_2 are independent, $E(U) = E(X_1)E(X_2)$. [2 - A]

b) i) X and Y are clearly not independent since $Y = X^4$, i.e. if we know X, we also know Y. [2 - E]
To see whether X and Y are uncorrelated, we compute $\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y) = E(X^5) - E(X)E(X^4) = E(X^5)$ since $E(X) = 0$. Since the distribution of X is symmetric around 0, $E(X^5) = 0$ and $\mathrm{Cov}(X,Y) = \mathrm{Corr}(X,Y) = 0$. X and Y are not independent but are uncorrelated. [3 - E]

ii) No, it is incorrect. The correct statement is: If X is a continuous random variable with first moment m_1 and second moment m_2, then we have $m_1^2 \le m_2$. [2 - B]
Recall that $\mathrm{Var}(X) = E\left[(X - m_1)^2\right] \ge 0$. Moreover $\mathrm{Var}(X) = m_2 - m_1^2$. Hence $m_2 - m_1^2 \ge 0$ and $m_1^2 \le m_2$. [3 - B]