EE1-10A Mathematics I- Solutions 2017

1. (a) Express in the form $x+iy$:

$$(i)\quad \frac{1-2i}{i-2}\,, \qquad (ii)\quad \left(\frac{1-\sqrt{3}i}{2}\right)^{2017}.$$

SOLUTION

$$\text{(i) } z = \frac{-1+2i}{2-i}\cdot\frac{2+i}{2+i} = \frac{-2+3i+2i^2}{4+1} = -\frac{4}{5}+\frac{3}{5}i$$

$$\left(\frac{1-\sqrt{3}i}{2}\right)^{2017} = \left(e^{-i\pi/3}\right)^{2017} = e^{-i2017\pi/3} = e^{-i\pi/3} = \frac{1-\sqrt{3}i}{2}.$$

(b) Sketch the locus of the complex number z satisfying

$$z-\bar{z} = \frac{1}{\bar{z}} - \frac{1}{z}.$$

SOLUTION Rewrite as

$$x+iy-(x-iy) = \frac{1}{x-iy} - \frac{1}{x+iy} \Rightarrow 2iy = \frac{x+iy}{x^2+y^2} - \frac{x-iy}{x^2+y^2} = \frac{2iy}{x^2+y^2}$$

so either $y=0$ or $x^2+y^2=1$. Sketch: union of unit circle and $x-$axis.

(c) Obtain all complex solutions z, when

$$(i)\quad \sinh z = -i\,, \qquad (ii)\quad \sin^2(iz) = 1\,.$$

SOLUTION

(i) Rewrite as

$$e^z - e^{-z} = -2i \Rightarrow e^{2z} + 2ie^z - 1 = 0 \Rightarrow e^z = \frac{-2i \pm \sqrt{(2i)^2-4}}{2} = -i\,,$$

so that

$$e^{x+iy} = e^x(\cos y + i\sin y) = -i \Rightarrow e^x\cos y = 0 \text{ and } e^x \sin y = -1\,,$$

equating real and imaginary parts. As $e^x \neq 0$ first equation gives

$$\cos y = 0 \Rightarrow y = (2n+1)\frac{\pi}{2}$$

for integers n. The second equation gives $\sin y = \pm 1$ for the given values of y, so we must have $e^x = 1 \rightarrow x = 0$ and hence $\sin y = -1 \Rightarrow y = -\pi/2 + 2n\pi$, where n is any integer.

Alternative approach, but longer:

$$e^z - e^{-z} = -2i \Rightarrow e^x e^{iy} - e^{-x} e^{-iy} = -2i$$

and using Euler's formula, get

$$e^x(\cos y + i \sin y) - e^{-x}(\cos y - i \sin y) = -2i$$

and equate real and imaginary parts to obtain

$$\cos y (e^x - e^{-x}) = 0\,, \quad \sin y (e^x + e^{-x}) = -2$$

and solve the first equation, then substitute into the second to get the same result.

(ii)

$$\sin^2(iz) = 1 \Rightarrow \sin(iz) = i \sinh z = \pm 1 \Rightarrow \sinh z = \pm i$$

Of the two equations, we have solved $\sinh z = -i$ in (i). As sinh is an odd function, we can write

$$\sinh z = i \Rightarrow \sinh(-z) = -i$$

so using (i) again we have the same solutions for $-z$, giving $x = 0$ and $y = \pi/2 + 2n\pi$. Combining the two solutions we get $x = 0$ and $y = \pi/2 + n\pi$.

(d) Obtain the limits

$$(i)\ \lim_{x\to 0} x\cos(\cot x)\,, \quad (ii)\ \lim_{x\to 0} \frac{x^2}{\ln(\cos x)}\,, \quad (iii)\ \lim_{x\to\pi/6} \frac{1-\sin(3x)}{\cot x - \sqrt{3}}\,.$$

Solution

(i) As $|\cos(\cot x)| \leq 1$ we can write $-x \leq x\cos(\cot x) \leq x$ and the sandwich theorem gives that $0 = \lim_{x\to 0}(-x) \leq \lim_{x\to 0} x\cos(\cot x) \leq \lim_{x\to 0} x = 0$ so the limit is zero.

(ii) Use l'Hopital's rule, given "0/0":

$$\lim_{x\to 0} \frac{x^2}{\ln(\cos x)} = \lim_{x\to 0} \frac{2x}{-\tan x}$$

which is still "0/0", so apply l'Hopital again:

$$\lim_{x\to 0} \frac{2x}{-\tan x} = \lim_{x\to 0} \frac{2}{-\sec^2 x} = -2$$

(iii) Need to see that it's "0/0" given $\sin(3\pi/6) = 1$ and $\cot(\pi/6) = \sqrt{3}$, then apply

$$\lim_{x\to\pi/6} \frac{1-\sin(3x)}{\cot x - \sqrt{3}} = \lim_{x\to\pi/6} \frac{-3\cos(3x)}{-\csc^2 x} = 0$$

as the denominator is non-zero.

2. (a) Obtain the value of q for which the folowing limit exists and is non-zero, and state the value of the limit:

$$\lim_{x\to\infty} x^q\left[(x+1)^{2/3}-(x-1)^{2/3}\right].$$

SOLUTION

Rewrite as $x^q\left[x^{2/3}\left(1+\frac{1}{x}\right)^{2/3}-x^{2/3}\left(1-\frac{1}{x}\right)^{2/3}\right]$

and as $x\to\infty, 1/x\to 0$, so we can use the Binomial expansion:

$$=x^{q+2/3}\left[\left(1+\frac{2}{3}\frac{1}{x}+\frac{\frac{2}{3}(\frac{2}{3}+1)}{2}\frac{1}{x^2}+\ldots\right)-\left(1-\frac{2}{3}\frac{1}{x}+\frac{\frac{2}{3}(\frac{2}{3}+1)}{2}\frac{1}{x^2}-\ldots\right)\right]$$

$$=x^{q+2/3}\left(\frac{4}{3x}+k\frac{1}{x^3}+\ldots\right), \quad \text{(some } k\text{)}$$

and choosing $q=1/3$ we ensure existence of the non-zero finite limit $4/3$, as all other terms vanish

(b) Differentiate to obtain $\dfrac{dy}{dx}$:

$$(i)\quad y=(\sin x)^{\cos x}, \qquad (ii)\quad \cos(x)=\sin(y), \qquad (iii)\quad y^2=\cos(xy).$$

SOLUTION

(i) Logarithmic differentiation:

$\ln y=\cos x\ln(\sin x)\Rightarrow\dfrac{1}{y}\dfrac{dy}{dx}=-\sin x\ln(\sin x)+\cos x\dfrac{1}{\sin x}\cos x$

so that $\dfrac{dy}{dx}=(\sin x)^{\cos x-1}\cos^2 x-(\sin x)^{\cos x+1}\ln(\sin x)$.

(ii) and (iii): Differentiate implicitly:

$$(ii)\qquad -\sin x=\cos y\frac{dy}{dx}\Rightarrow\frac{dy}{dx}=-\frac{\sin x}{\cos y}.$$

$$(iii)\qquad 2y\frac{dy}{dx}=-\sin(xy)\left(y+x\frac{dy}{dx}\right)\Rightarrow\frac{dy}{dx}=-\frac{y\sin(xy)}{2y+x\sin(xy)}.$$

(c) Given the function

$$f(x) = \frac{2x^2 - 5x + 1}{x + 1},$$

find all stationary points and their nature, obtain any asymptotes and give a sketch showing these and any other relevant features.

SOLUTION To find stationary points, differentiate:

$$f'(x) = \frac{(4x-5)(x+1) - (2x^2 - 5x + 1)}{(x+1)^2} = \frac{2x^2 + 4x - 6}{(x+1)^2} = 0$$

giving stationary points at $x = -3, 1$.

There is a vertical asymptote at $x = -1$ and given that at -1, the numerator $2x^2 - 5x + 1 = 8$ we have the asymptotic behaviour on either side:

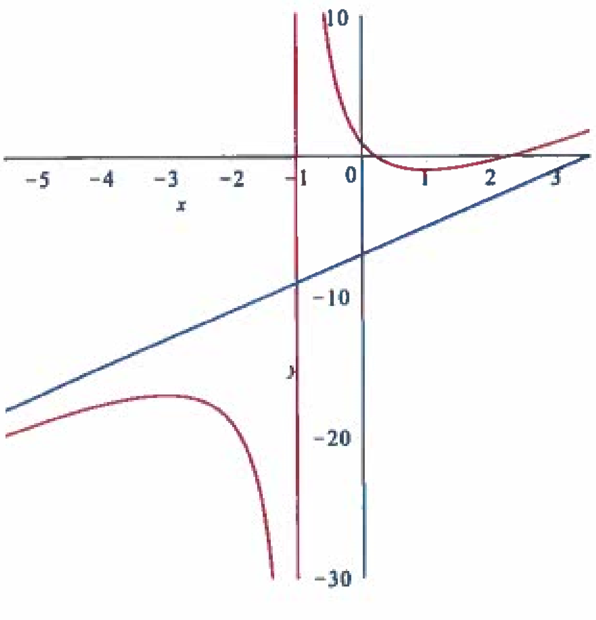
as $x \to -1^+$ we have $f(x) \to \infty$, and as $x \to -1^-$ we have $f(x) \to -\infty$.

Using polynomial division we have

$$f(x) = 2x - 7 + \frac{8}{x+1}$$

giving a diagonal asymptote $g(x) = 2x - 7$. As $x \to \infty$, $1/(x+1) \to 0^+$ so the function is approaching the asymptote from above, and vice-versa in the other direction. For $x > -1$ we have $x \to \infty$ on both sides of the stationary point at $x = 1$: it must be a minimum. For $x < -1$, we have $x \to -\infty$ on both sides of the stationary point at $x = -3$: it must be a maximum. The alternative to this is to calculate the second derivative and evaluate it at $x = -3, 1$, but the argument with asymptotics is sufficient.

Intercepts are at $(0, 1)$ and $\left(\frac{5 \pm \sqrt{17}}{4}, 0\right) \approx (1/4, 0), (9/4, 0)$. The stationary points are at $(-3, -17)$ and $(1, -1)$ and we can sketch the function:



(d) Obtain the n^{th} derivative $\dfrac{d^n y}{dx^n}$ for

$$y = x^2 e^{-x}\,.$$

SOLUTION

Using Leibnitz' Theorem we get

$$y^{(n)} = x^2 D^n e^{-x} + n(2x) D^{n-1} e^{-x} + \frac{n(n+1)}{2}(2) D^{n-2} e^{-x}$$

and reasoning that

$$D^n e^{-x} = (-1)^n e^{-x}$$

we conclude that

$$y^{(n)} = (-1)^n e^{-x}\left[x^2 - 2nx + n(n+1)\right]\,.$$

3. (a) Evaluate the indefinite integrals:

$$(i)\ \int \frac{4x-6}{x^2-3x+4}\,dx \quad (ii)\ \int \frac{1}{x\ln x}\,dx\,, \quad (iii)\ \int \frac{1}{4\sin x - 3\cos x - 5}\,dx\,.$$

SOLUTION

(i) Observing that $(x^2-3x+4)' = 2x-3$ we get

$$\int \frac{2(2x-3)}{x^2-3x+4}\,dx = 2\ln(x^2-3x+4) + C\,.$$

(ii) Given that $(\ln x)' = 1/x$ we substiture $u = \ln x$ to get

$$\int \frac{1}{x\ln x}\,dx = \int \frac{1}{u}\,du = \ln(\ln x) + C\,.$$

(iii) Using the substitution $t = \tan(x/2)$ (formula sheet) we have $\sin x = 2t/(1+t^2)\,, \cos x = (1-t^2)(1+t^2)$ and $dx = 2dt/(1+t^2)$ and the integral becomes

$$\int \frac{1}{\left(4\frac{2t}{1+t^2} - 3\frac{1-t^2}{1+t^2} - 5\right)}\frac{2dt}{(1+t^2)} = \int \frac{2\,dt}{4(2t) - 3(1-t^2) - 5(1+t^2)}$$

$$= -\int \frac{2}{2t^2-8t+8}\,dt = -\frac{1}{(t-2)^2}\,dt = \frac{1}{t-2} + C = \frac{1}{\tan(x/2)-2} + C\,.$$

(b) Use a substitution to integrate $\dfrac{1}{\sqrt{x^2-1}}$ and hence show that

$$\cosh^{-1} x = \ln(x + \sqrt{x^2-1})\,.$$

SOLUTION The required substitution is $x = \cosh u \Rightarrow dx = \sinh u\, du$ and $x^2-1 = \cosh^2 u - 1 = \sinh^2 u$

so that

$$\int \frac{1}{\sqrt{x^2-1}}\, dx = \int \frac{1}{\sqrt{\sinh^2 u}} \sinh u\, du = \int 1\, du = u + C = \cosh^{-1} x + C\,.$$

If two functions are equal, they have the same derivative. From the above, the Fundamental Theorem of Calculus gives that

$$\frac{d}{dx}\left(\cosh^{-1}\right) = \frac{1}{\sqrt{x^2-1}}\,.$$

If the last exprssion is also equal to the derivative of $\ln(x + \sqrt{x^2-1})$ then the two functions are equal, up to a constant:

$$\frac{d}{dx}\left(\ln(x+\sqrt{x^2-1})\right) = \frac{1}{x+\sqrt{x^2-1}}\left(1 + \frac{2x}{2\sqrt{x^2-1}}\right) = \frac{1}{x+\sqrt{x^2-1}}\left(\frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}}\right)\,,$$

and the last cancellation gives the desired result. The alternative is to let $y = \cosh^{-1} x \Rightarrow x = \cosh y$ and solve this for $y = \ln(x + \sqrt{x^2-1})$, but this loses marks, as the instruction is to use the result of the integration.

(c) Obtain the Maclaurin series of $\dfrac{1}{e^{-x}+1}$ to first order with remainder term. Explain how the error estimate from the remainder term can be improved without any more terms in the series. Obtain the improved error estimate.

SOLUTION

To obtain the Maclaurin series to order one we need to differentiate twice:

$$f'(x) = \frac{e^{-x}}{(1+e^{-x})^2} \Rightarrow f''(x) = \frac{-e^{-x}(1+e^{-x})^2 - (e^{-x})2(1+e^{-x})(-e^{-x})}{(1+e^{-x})^4}$$

which simplifies to

$$f''(x) = \frac{e^{-x}(e^{-x}+1)}{(1+e^{-x})^3}$$

so that

$$f(0) = \frac{1}{2}\,, f'(0) = \frac{1}{4} \Rightarrow f(x) = \frac{1}{2} + \frac{1}{4}x + R_1$$

where the Lagrange remainder is

$$R_1 = \frac{e^{-c}(e^{-c}+1)}{2(1+e^{-c})^3}x^2\,, \quad \text{with} \quad 0 < |c| < |x|\,.$$

We can improve the error estimate without adding further terms to the series by observing that $f''(0) = 0$, so that

$$f(x) = \frac{1}{2} + \frac{1}{4}x + 0x^2 + R_2$$

where careful differentiation gives

$$f'''(x) = \frac{e^{-x}(e^{-2x} - 4e^{-x} + 1)}{(1 + e^{-x})^4}$$

and the remainder term is

$$R_2 = \frac{f'''(c)}{6}x^2 ,$$

where near zero, the higher power of x makes $|R_2|$ smaller than $|R_1|$ and hence an improved error estimate.

(d) Use the integral test to find constants A,B such that

$$A < \sum_{n=1}^{\infty} \frac{1}{n^3} < B .$$

SOLUTION

To find the lower bound, it's sufficient to note that the infinite sum is greater than any partial sum, as the terms are all positive, so that (for example)

$$A = \frac{1}{1^3} + \frac{1}{2^3} = \frac{9}{8} < \sum_{n=1}^{\infty} \frac{1}{n^3} ,$$

but other values of A are clearly available. The integral test gives

$$\sum_{n=2}^{\infty} \frac{1}{n^3} < \int_1^{\infty} \frac{1}{x^3}\, dx < \sum_{n=1}^{\infty} \frac{1}{n^3}$$

so that

$$\sum_{n=1}^{\infty} \frac{1}{n^3} < 1 + \int_1^{\infty} \frac{1}{x^3}\, dx = 1 + \left[-\frac{1}{2x^2}\right]_1^{\infty} = \frac{3}{2} = B$$

An alternative is to compare with the known $\sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6$, but use of the integral test is required for full marks.

4. (a) Find the radius and interval of convergence of the infinite series

$$\sum_{n=2}^{\infty} \frac{(3x)^n}{n(n-1)},$$

SOLUTION

Begin with the ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(3x)^{n+1}}{(n+1)n}}{\frac{(3x)^n}{n(n-1)}}\right| = 3\frac{n-1}{n+1}|x|$$

so that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 3|x| \lim_{n\to\infty} \frac{1-\frac{1}{n}}{1+\frac{1}{n}} = 3|x|$$

and we require $3|x| < 1$ for convergence, so the radius of convergence is $1/3$. The interval of convergence given by the ratio test is $-1/3 < x < 1/3$, and we know the series diverges for $|x| > 1/3$, but the ratio test gives no information regarding the convergence for $x = \pm 1/3$, which need to be tested separately. Letting $x = 1/3$ we have

$$\sum_{n=2}^{\infty} \frac{1}{n(n-1)},$$

a telescoping sum, converging to a known value, as shown in lectures. Letting $x = -1/3$ we have

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n(n-1)},$$

which converges absolutely, comparing to the case $x = 1/3$, or by the alternating series test. Hence the interval of converges is extended to $-1/3 \le x \le 1/3$.

(b) Without obtaining the Fourier Series of the function

$$f(x) = \begin{cases} x+2\,, & 0 \le x < 1.5 \\ 4-x\,, & 1.5 \le x < 3 \end{cases} \quad \text{and} \quad f(x+3) = f(x)\,, \forall x\,,$$

find the values of the Fourier Series at $x = 0$ and $x = 1.5$.

SOLUTION At discontinuities x_0, the FS converges to the average of the limiting values: $\frac{1}{2}\left(\lim_{x\to x_0^+} f(x) + \lim_{x\to x_0^-}\right.$

At $x = 0$: $\frac{1}{2}([0+2] + [4-3]) = \frac{3}{2}$, as $f(0) = f(3)$.

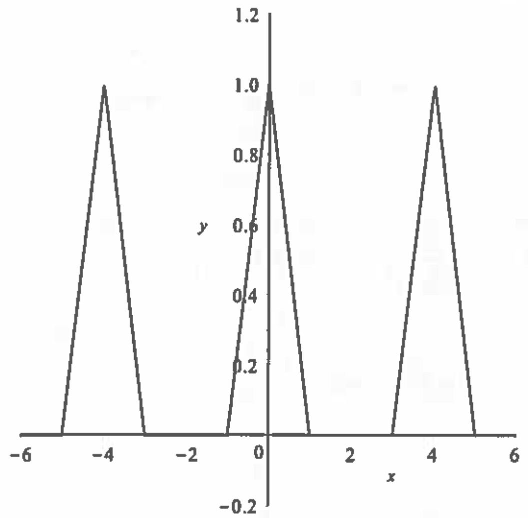
At $x = 1.5$: $\frac{1}{2}([1.5+2] + [4-1.5]) = 3\,.$

(c) A function is defined as

$$f(x) = \begin{cases} 1-x & 0 \le x < 1 \\ 0 & 1 \le x < 2 \end{cases}$$

(i) Obtain $g(x)$, the even extension of $f(x)$, with period $T = 4$ and sketch $g(x)$ for $-6 \le x \le 6$.

Solution



(ii) Obtain the Fourier cosine series of $g(x)$.
[You may assume that $\cos(n\pi/2) = (-1)^{n/2}$ for even n.]

Solution

It's an even function, so all $b_n = 0$ and the series is a Fourier cosine series. For period $T = 2L$, the half-range formula is

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx$$

so with $T = 4 = 2L$ we have

$$a_0 = \int_0^2 f(x)\,dx = \int_0^1 1-x\,dx + \int_1^2 0\,dx = \left[x - \frac{x^2}{2}\right]_0^1 = \frac{1}{2}$$

and

$$\begin{aligned} a_n &= \int_0^2 f(x)\cos\left(\frac{n\pi x}{2}\right)\,dx = \int_0^1 (1-x)\cos\left(\frac{n\pi x}{2}\right)\,dx \\ &= \frac{2}{n\pi}\left[2(1-x)\sin\left(\frac{n\pi x}{2}\right)\right]_0^1 + \frac{2}{n\pi}\int_0^1 \sin\left(\frac{n\pi x}{2}\right)\,dx \\ &= 0 - 0 - \frac{4}{n^2\pi^2}\left[\cos\left(\frac{n\pi x}{2}\right)\right]_0^1 = \frac{4}{n^2\pi^2}\left[1 - \cos\left(\frac{n\pi}{2}\right)\right] \end{aligned}$$

When n is odd, $\cos(n\pi/2) = 0$ so $a_n = 4/(n^2\pi^2)$. For even n we use the hint: $\cos(n\pi/2) = (-1)^{n/2}$ and so

$$a_n = \frac{4}{n^2\pi^2}\left[1 - (-1)^{n/2}\right]$$

and so it is more convenient to begin with two series, one for even n and one for odd n:

$$f(x) = \frac{1}{4} + \frac{4}{\pi^2}\left[\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} \frac{1}{n^2}\cos\left(\frac{n\pi x}{2}\right) + \sum_{\substack{n=2\\ n \text{ even}}}^{\infty} \frac{1-(-1)^{n/2}}{n^2}\cos\left(\frac{n\pi x}{2}\right)\right]$$

Full marks for the above or equivalent. Simplify by letting $n = 2m$ in the second sum:

$$= \frac{1}{4} + \frac{4}{\pi^2}\left[\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} \frac{1}{n^2}\cos\left(\frac{n\pi x}{2}\right) + \sum_{m=1}^{\infty} \frac{1-(-1)^{m}}{(2m)^2}\cos(m\pi x)\right]$$

finally, revert to n, and use $1 - (-1)^n = 2$ for odd n, and zero for even n:

$$\begin{aligned}
&= \frac{1}{4} + \frac{4}{\pi^2}\left[\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} \frac{1}{n^2}\cos\left(\frac{n\pi x}{2}\right) + \frac{1}{2}\sum_{\substack{n=1\\ n \text{ nodd}}}^{\infty} \frac{1}{n^2}\cos(n\pi x)\right]\\
&= \frac{1}{4} + \frac{4}{\pi^2}\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} \frac{1}{n^2}\left[\cos\left(\frac{n\pi x}{2}\right) + \frac{1}{2}\cos(n\pi x)\right]\\
&= \frac{1}{4} + \frac{4}{\pi^2}\left(\cos\left(\frac{\pi x}{2}\right) + \frac{1}{2}\cos(\pi x) + \frac{1}{9}\cos\left(\frac{3\pi x}{2}\right)\right.\\
&\qquad\left. + \frac{1}{18}\cos(3\pi x) + \frac{1}{25}\cos\left(\frac{5\pi x}{2}\right) + \frac{1}{50}\cos(5\pi x) + \ldots\right)
\end{aligned}$$

(iii) By careful choice of a value of x, using the results of (ii) or otherwise, calculate the infinite series

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$$

SOLUTION The needed value is $x = 0$, where $f(x) = 1$ and all cosine terms are equal to 1, so that

$$1 = \frac{1}{4} + \frac{4}{\pi^2}\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} \frac{1}{n^2}\left(1 + \frac{1}{2}\right) \Rightarrow \frac{3}{4} = \frac{6}{\pi^2}\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} \frac{1}{n^2}$$

and multiplying gives the result $\pi^2/8$.