UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 2000

BEng Honours Degree in Computing Part I
MEng Honours Degrees in Computing Part I
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the Associateship of the City and Guilds of London Institute

PAPER C142

DISCRETE MATHEMATICS

Friday 19 May 2000, 16:00
Duration: 90 minutes
(Reading time 5 minutes)

Answer THREE questions

Paper contains 4 questions

1a A hospital administration decides to analyse some of the information in its data base using set algebra. Let P be the set of patients treated in a particular ward W during 1999, and D the set of doctors treating patients in W. Doctors a,b, and c are three elements of D. P_a is the set of patients (members of P) treated by a; similarly for P_b,P_c. P_l is the set of patients who stay in the ward for less than one week. $T \subseteq P \times D$ is the relation "treated by", so pTd means "p is treated by d".

In parts (ii),(iii),(v),(vi) you are asked to give the answer as an expression built up by set (or relation) operations from the above data. For example, the set of patients who stay in the ward for less than one week and are not treated by b could be given as $P_l \cap (P - P_b)$.

i) What is meant by the statement that $\{P_a, P_b, P_c\}$ is a partition of P?

ii) Give: the set of patients who are treated by exactly one of b,c.

iii) Give: the set of patients who are treated by exactly one of a,b,c and stay in the ward for at least one week. How may the expression be simplified, if it is known that $\{P_a, P_b, P_c\}$ is a partition of P?

iv) Let $R \subseteq A \times B$ and $S \subseteq B \times C$ be relations. Define: the *inverse* R^{-1} and the *composition* $R \circ S$.

v) Give: the relation R on P, where xRy means "x and y are treated by the same doctor".

vi) Give: the relation S on D, where xSy means "x and y do not treat the same patient". Can your expression be simplified, if it is known that $\{P_a,P_b,P_c\}$ is a partition of P? Explain.

b i) Define: *partial order* and *equivalence relation*. (Basic properties of relations such as transitivity can be used without being defined.)

ii) Give an example of a set S and a relation R on S that is a pre-order (that is, a reflexive transitive relation) but not an equivalence relation or a partial order. (Hint: it can be done with a set S having three elements.)

iii) Determine whether the relation R on the set of integers, defined by

$$xRy \qquad \text{iff } x + 2y \text{ is divisible by } 3$$

is a pre-order, partial order or equivalence relation (or none of these).

The two parts carry, respectively, 65%, 35%, of the marks

2a A function $f:A \to A$ will be called *symmetric* if $f(x) = y$ implies that $f(y) = x$, for all $x,y \in A$ (so that the function can be thought of as "swapping" the two values). Examples are the identity function, and negation on the integers (given by $f(x) = -x$).

i) List all the symmetric functions, in the case that A is the set $\{a,b,c\}$.

ii) Define the terms *onto* and *one-one*, as applied to a function $h:B \to C$.

iii) Prove: every symmetric function is onto and one-one.

iv) Let S_k be the number of distinct symmetric functions on a set $A_k = \{a_1,...,a_k\}$ with k elements. If $k > 0$, prove that

$$S_{k+2} = S_{k+1} + (k+1)\, S_k .$$

(Hint. Assuming the symmetric functions on A_{k+1}, as well as on k-element subsets, to be known, divide the "new" functions we get by adding a_{k+2} into two cases: those in which a_{k+2} is mapped to itself, and those in which it is mapped to one of $a_1,...,a_{k+1}$.)

v) Find S_4.

b Let $g:A \to B$ be a function. For any subset X of A, the *image* $g[X]$ is $\{g(a) | a \in X\}$.

i) Show that for all subsets X and Y of A

$$g[X \cap Y] \subseteq g[X] \cap g[Y]$$

ii) Show that, if g is one-one, then for all subsets X and Y of A

$$g[X] \cap g[Y] \subseteq g[X \cap Y]$$

iii) Give an example (i.e. specify A,B,g,X,Y: a diagram is sufficient) in which

$$g[X \cap Y] \neq g[X] \cap g[Y].$$

The two parts carry, respectively, 60%, 40%, of the marks

3a Define the following:

i) Simple graph

ii) Euler circuit

iii) Give a necessary and sufficient condition for a connected graph to hve an Euler circuit.

iv) Explain why your condition in (iii) is necessary.

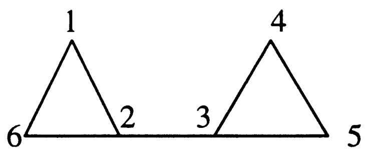
b A *Hamiltonian circuit* (HC) is a path through a graph which visits every node exactly once and returns to the start node.

Give two simple graphs G_1 and G_2 such that:

i) G_1 has an EC but no HC

ii) G_2 has a HC but no EC

c An *automorphism* is an isomorphism from a graph to itself. How many automorphisms does the following graph with 6 nodes have? Justify your answer.



d Construct a graph with exactly three automorphisms (including the identity).

The four parts carry, respectively, 30%, 25%, 20%, 25% of the marks.

4a Draw the decision tree for binary search applied to a sorted list of 13 elements.

b It is desired to merge two sorted lists L_1 and L_2 of distinct natural numbers to produce a single sorted list L. L_1 and L_2 have lengths m and n, respectively.

i) Describe an algorithm to perform the merge.

ii) What is the worst-case number of comparisons for merging the lists? Justify your answer briefly.

iii) Give two different examples for the case where m=4 and n=2, to show that the worst case as stated in (ii) may or may not occur. State the number of comparisons in each case.

iv) Suggest a criterion on L_1 and L_2 which is necessary and sufficient for the worst case for your merge algorithm to arise.

c i) Obtain (with brief explanation) a recurrence relation for the worst-case number of comparisons W(n) taken by the usual Mergesort algorithm when applied to a sorted list of length n. For convenience, assume that n is a power of 2.

ii) Solve the recurrence relation from (i) above.

The three parts carry, respectively, 20%, 50%, 30% of the marks.