Imperial College London

MEng (Engineering) Examination 2017

Year 1

AE1-107 Mathematics Term 1

Monday 16th January 2017: 10.00 to 12.00
[2 hours]

There are ***FOUR*** questions.
All questions carry equal weight.
Full marks may be obtained for
complete answers to ***ALL FOUR*** questions.

A data sheet is provided.

The use of lecture notes is NOT allowed.

Question 1

(a) Consider the function $f(x)$ defined as $f(x) = \frac{1}{x^2 - 1}$.

i. Using the definition of derivative as a limit, find the first derivative $f'(x)$ of the function $f(x)$. [15%]

ii. Sketch the curve $f(x)$ and identify, if any, extrema (maximum and/or minimum), inflection points and asymptotes. [25%]

iii. Recover the first derivative $f'(x)$ of the function $f(x)$ using the quotient rule. [10%]

(b) Determine $\frac{dy}{dx}$ in each of the following cases:

i. $y = \frac{\sin(x)}{1 + \cos(x)}$. [15%]

ii. $y^2 = \sin(xy)$. [15%]

(c) The equation $x^3 - 3x - 4 = 0$ is of the form $f(x) = 0$ where $f(1) < 0$ and $f(3) > 0$. It means that there is a solution to the equation between 1 and 3. Using the Newton-Raphson method (see Data Sheet for formula), determine the root close to 2 correct to three decimal places. [20%]

Question 2

(a) Evaluate the following limits:

i. $\lim_{x \to +\infty} \left(\sqrt{1+x} - \sqrt{x}\right)$. [12.5%]

ii. $\lim_{x \to +\infty} x \left(\sqrt{x^2+4} - x\right)$. [12.5%]

iii. $\lim_{x \to 1} \frac{x^2 - 1}{x^2 + x - 2}$. [12.5%]

(b) Determine the following integrals:

i. $\int \frac{\sec^2(x)}{\tan(x)} dx$. [12.5%]

ii. $\int \frac{2x^2 - x + 2}{x^3 - x} dx$. [12.5%]

iii. $\int \sin^2(5x) dx$. [12.5%]

(c) Using the recursive iteration method, evaluate the following integral:

$$\int_0^{+\infty} x^5 \mathrm{e}^{-x^2} dx.$$

[25%]

Question 3

(a) Determine if the following series converge

i. $\sum_{n=1}^{+\infty} \frac{(n+3)x^n}{3^n}$. [15%]

ii. $\sum_{n=2}^{+\infty} \frac{1}{n\ln(n)}$. [15%]

(b) The power series

$$\sum_{n=1}^{\infty} u_n x^n$$

has coefficients given by

$$u_n = \frac{n^n}{n!} \ .$$

i. Show that the $(n+1)^{\text{th}}$ term divided by the n^{th} term is

$$\left(\frac{n+1}{n}\right)^n x \ .$$

[10%]

ii. Calculate the radius of convergence of this power series (Hint: you can use $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \mathrm{e}$). [10%]

(c) If $u = x + y$, $v = xy$, and f is a function of x and y, express $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ in terms of $\frac{\partial f}{\partial u}$, $\frac{\partial f}{\partial v}$ and prove that [30%]

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial u^2} + u\frac{\partial^2 f}{\partial u \partial v} + v\frac{\partial^2 f}{\partial v^2} + \frac{\partial f}{\partial v}.$$

[20%]

Question 4

(a) De Moivre's theorem states that $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$. Using this result, or otherwise, express the complex number

$$(1+\sqrt{3}i)^5 + (1-\sqrt{3}i)^5$$

in the form $a + ib$. [30%]

(b) Give the definition of:

i. A periodic function, which has period $L > 0$;

ii. An odd function; and an even function. Given one example of each. [10%]

(c) Consider the function

$$f(x) = \begin{cases} -\cos(x) & \text{for} \quad -\pi \le x \le 0 \\ \cos(x) & \text{for} \quad 0 \le x \le \pi, \end{cases}$$

where $f(x)$ is defined on $-\pi < x \le \pi$ with period, 2π. Find its Fourier series. [40%]

(d) Hence show that, for $x = \frac{\pi}{4}$, the series may be written as

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2n+1)}{(4n-1)(4n-3)} = \frac{\pi}{8\sqrt{2}}.$$

[20%]

1

Question 1

(a) i. From the definition of derivatives as a limit, we have

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
f'(x) &= \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{(x+h)^2-1} - \frac{1}{x^2-1}\right] \\
f'(x) &= \lim_{h\to 0} \frac{1}{h}\left[\frac{(x^2-1)-(x+h)^2+1}{(x^2-1)[(x+h)^2-1]}\right] \\
f'(x) &= \lim_{h\to 0} \frac{1}{h}\left[\frac{x^2-1-x^2-2xh-h^2+1}{(x^2-1)[(x+h)^2-1]}\right] \\
f'(x) &= \lim_{h\to 0} \frac{1}{h}\left[\frac{-2xh-h^2}{(x^2-1)[(x+h)^2-1]}\right] \\
f'(x) &= \lim_{h\to 0} \frac{h}{h}\left[\frac{(-2x-h)}{(x^2-1)[(x+h)^2-1]}\right] \\
f'(x) &= \lim_{h\to 0} \frac{-2x-h}{(x^2-1)[(x+h)^2-1]}
\end{aligned}$$

$$f'(x) = \frac{-2x}{(x^2-1)^2}$$

[15%]

ii. Extreme and inflexion points can be obtained from $f'(x)=0$ hence $x=0$. Note that there is no non-stationnary points. The second derivative can clarify the nature of the extreme points (maximum or minimum):

$$f''(x) = \frac{8x^2}{(x^2-1)^3} - \frac{2}{(x^2-1)^2} = \frac{2(3x^2+1)}{(x^2-1)^3}$$

When $x=0$, $f''(0) = -2 < 0$, so that the stationary point is a maximum.
Since $f(x)$ is defined as a quotient, we have two vertical asympotes when the denominator is equal to zero, for $x = \pm 1$.
Because

$$\lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} \frac{\frac{1}{x^2-1}}{x} = \lim_{x\to\pm\infty} \frac{1}{x(x^2-1)} = 0$$

we also have an horizontal asymptote $y=n$. The value of n is given by

$$n = \lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{1}{x^2-1} = 0$$

Therefore we have an horizontal asymptote $y=0$. See figure 1 for the sketch of the curve. [25%]

iii. Using the quotient rule with $h(x)=1$ and $g(x)=x^2-1$ we have

$$\begin{aligned}
f'(x) =& \frac{h'(x)g(x)-g'(x)h(x)}{g^2(x)} \\
f'(x) =& \frac{-2x}{(x^2-1)^2}
\end{aligned}$$

[10%]

2

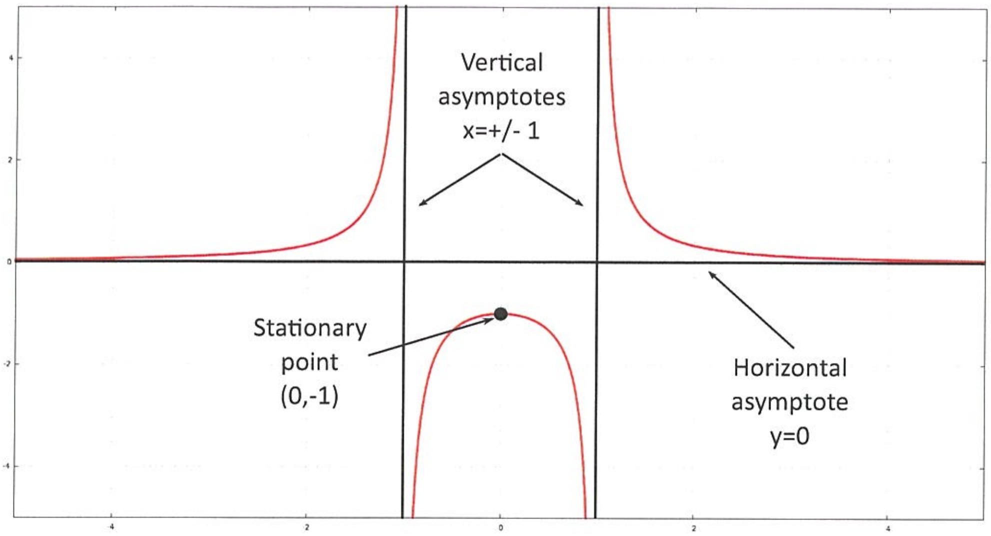


Figure 1: Sketch of the function $f(x) = \dfrac{1}{x^2 - 1}$.

(b) i.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{\cos(x)[1+\cos(x)] - \sin(x)[-\sin(x)]}{(1+\cos(x))^2} \\
&= \frac{\cos^2(x) + \sin^2(x) + \cos(x)}{(1+\cos(x))^2} \\
&= \frac{1+\cos(x)}{(1+\cos(x))^2} \\
&= \frac{1}{1+\cos(x)}
\end{aligned}$$

[15%]

ii.

$$\begin{aligned}
\frac{d}{dx}y^2 &= \frac{d}{dx}\sin(xy) \\
\frac{d}{dy}y^2\frac{dy}{dx} &= \frac{d}{dx}\sin(xy) \\
2y\frac{dy}{dx} &= \cos(xy)[y + x\frac{dy}{dx}] \\
\frac{dy}{dx} &= \frac{y\cos(xy)}{2y - x\cos(xy)}
\end{aligned}$$

[15%]

(c) We have $f(x) = x^3 - 3x - 4 = 0$ and $f'(x) = 3x^2 - 3$. If the first approximation is $x_0 = 2$, then

$$f(x_0) = f(2) = -2 \qquad \text{and} \qquad f'(x_0) = f'(2) = 9$$

Using the Newton-Raphson formula, we get

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{-2}{9} = 20/9 \approx 2.222$$

If we now start from $x_1 = 20/9$, we can get a better approximation by repeating the process to get x_2:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 2.196$$

Using $x_2 = 2.196$ as a starter value, we can continue the process until the value is correct to three decimal places. Because we have $x_3 \approx 2.196$, then we can say that the root close to 2 correct to three decimal places is 2.196. [20%]

4

Question 2

(a) i. Using

$$\left(\sqrt{1+x}-\sqrt{x}\right) = \left(\sqrt{1+x}-\sqrt{x}\right)\frac{\sqrt{1+x}+\sqrt{x}}{\sqrt{1+x}+\sqrt{x}} = \frac{1+x-x}{\sqrt{1+x}+\sqrt{x}} = \frac{1}{\sqrt{1+x}+\sqrt{x}}$$

we obtain

$$\lim_{x\to+\infty}\left(\sqrt{1+x}-\sqrt{x}\right) = 0$$

[12.5%]

ii.

$$\begin{aligned} x\left(\sqrt{x^2+4}-x\right) =& \frac{x(x^2+4-x^2)}{\sqrt{x^2+4}+x} \\ =& \frac{4x}{\sqrt{x^2+4}+x} \\ =& \frac{4}{\sqrt{1+\frac{4}{x^2}}+1} \end{aligned}$$

We obtain

$$\lim_{x\to+\infty} x\left(\sqrt{x^2+4}-x\right) = 2$$

(It is also possible to develop the $\sqrt{}$ as a series.) [12.5%]

iii.

$$\lim_{x\to1}\frac{x^2-1}{x^2+x-2} = \lim_{x\to1}\frac{(x+1)(x-1)}{(x-1)(x+2)} = \frac{x+1}{x+2} = \frac{2}{3}$$

(It is also possible to use l'Hôpital's rule.) [12.5%]

(b) i. Using the change of variables $u=\tan(x)$ and $du=\sec^2(x)dx$, we get

$$\int\frac{\sec^2(x)}{\tan(x)}dx = \int\frac{du}{u} = \ln|\tan(x)|+C$$

[12.5%]

ii. We can write

$$\frac{2x^2-x+2}{x^3-x} = \frac{2x^2-x+2}{x(x-1)(x+1)} = \frac{A}{x}+\frac{B}{x-1}+\frac{C}{x+1}$$

In order to find A, B and C we use

$$2x^2-x+2 = A(x-1)(x+1)+Bx(x+1)+Cx(x-1)$$

which leads to

$$A=-2 \qquad B=\frac{3}{2} \qquad C=\frac{5}{2}$$

We finally have

$$\begin{aligned} \int\frac{2x^2-x+2}{x^3-x}dx =& -2\int\frac{dx}{x}+\frac{3}{2}\int\frac{dx}{x-1}+\frac{5}{2}\int\frac{dx}{x+1} \\ =& -2\ln|x|+\frac{3}{2}\ln|x-1|+\frac{5}{2}\ln|x+1|+C \end{aligned}$$

[12.5%]

iii. Using trigonometric identity $2\sin^2(a) = 1 - \cos(2a)$, we get

$$\int \sin^2(5x)dx = \int \frac{1-\cos(10x)}{2}dx$$

which can be easily integrated

$$\int \sin^2(5x)dx = \frac{1}{2}\left(x - \frac{\sin(10x)}{10}\right) + C$$

[12.5%]

(c) We can define I_5 as

$$I_5 = \int_0^{+\infty} x^5 \mathrm{e}^{-x^2} dx$$

Using an integration by part with $u = x^4$ and $v' = x\mathrm{e}^{-x^2}$, we get

$$\begin{aligned} I_5 =& \left[\frac{-x^4}{2}\mathrm{e}^{-x^2}\right]_0^{+\infty} + \frac{4}{2}\int_0^{+\infty} x^3\mathrm{e}^{-x^2}dx \\ I_5 =& 2I_3 \end{aligned}$$

By repeating the same process for I_3 with $u = x^2$ and $v' = x\mathrm{e}^{-x^2}$, we get

$$\begin{aligned} I_3 =& \left[\frac{-x^2}{2}\mathrm{e}^{-x^2}\right]_0^{+\infty} + \int_0^{+\infty} x\mathrm{e}^{-x^2}dx \\ I_3 =& I_1 \end{aligned}$$

Hence we have $I_5 = 2I_1 = 2\int_0^{+\infty} x\mathrm{e}^{-x^2}dx = 1$, Using the change of variables $u = -x^2$ and $du = -2xdx$, we get

$$I_1 = -\frac{1}{2}\int_0^{+\infty} \mathrm{e}^u du = -\frac{1}{2}[\mathrm{e}^u]_0^{-\infty} = -\frac{1}{2}[\mathrm{e}^{-x^2}]_0^{+\infty} = \frac{1}{2}$$

Hence we have $I_5 = 2I_1 = 1$. [25%]

6

Question 3

(a) i. Using the ratio term we obtain

$$\lim_{n\to+\infty}\frac{|x_{n+1}|}{|x_n|} = \lim_{n\to+\infty}\frac{|3^n(n+4)x^{n+1}|}{|3^{n+1}(n+3)x^n|}$$
$$=\frac{|x|}{3}\lim_{n\to+\infty}\frac{n+4}{n+3}$$
$$=\frac{|x|}{3}\lim_{n\to+\infty}\frac{1+\frac{4}{n}}{1+\frac{3}{n}}$$
$$=\frac{|x|}{3}$$

Therefore the series will converge if

$$\frac{|x|}{3} < 1$$

If $x = 3$ or $x = -3$ the test will fail and we obtain

$$\sum_{n=1}^{\infty}(n+3)1^n$$

and

$$\sum_{n=1}^{\infty}(n+3)(-1)^n$$

respectively. Both case clearly diverge, therefore the series converges only for

$$-3 < x < 3$$

[15%]

ii. Using the positive and decreasing function $f(x) = \dfrac{1}{x\ln(x)}$ and the integral test, the convergence of the series can be determine with the following integral

$$\int_2^{+\infty}\frac{1}{x\ln(x)}dx = \lim_{t\to+\infty}\int_2^t\frac{1}{x\ln(x)}dx.$$

Using the change of variables $u = \ln(x)$ and $du = \dfrac{dx}{x}$ we have

$$\lim_{t\to+\infty}\int_2^t\frac{1}{x\ln(x)}dx = \lim_{t\to+\infty}\left[\ln(\ln(x))\right]_2^{+\infty} = +\infty$$

The integral is divergent and so the series is also divergent by the integral test. [15%]

(b) i.

$$\frac{u_{n+1}x^{n+1}}{u_nx^n} = \frac{(n+1)^{n+1}n!}{n^n(n+1)!}\frac{x^{n+1}}{x^n}$$
$$=\frac{(n+1)^{n+1}n!}{n^n(n+1)n!}x$$
$$=\frac{(n+1)^n}{n^n}x = \left(\frac{n+1}{n}\right)^n x$$

[10%]

7

ii.

$$\lim_{n\to+\infty}\left(\frac{n+1}{n}\right)^n x = \lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n x = \mathrm{e}\,x$$

Using the ratio test we require

$$|\mathrm{e}\,x| < 1$$

for the series to converge, that is

$$|x| < \frac{1}{\mathrm{e}}$$

The radius of convergence is $\frac{1}{\mathrm{e}}$.
Note that the radius of convergence can also be obtain by evaluating

$$\lim_{x\to+\infty}\frac{u_{n+1}}{u_n}$$

[10%]

8

Course Code and Title (Required): AE1-107 Mathematics

Setter (Required): JFM

Qu 3 (1)

Write on this side only (in ink) between the margins, not more than one solution per sheet please. Solutions must be signed and dated by both exam setter and referee.

	Marks

(a) $u = x + y \qquad v = xy$

$$\frac{\partial u}{\partial x} = 1 \qquad \frac{\partial v}{\partial x} = y$$

$$\frac{\partial u}{\partial y} = 1 \qquad \frac{\partial v}{\partial y} = x$$

extend $f(u,v) = f(x+y, xy) = f(x,y)$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}$$

$$= \frac{\partial f}{\partial u} + y\frac{\partial f}{\partial v}$$ 15

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y} = \frac{\partial f}{\partial u} + x\frac{\partial f}{\partial v}$$ 15

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial u}\right) + \frac{\partial f}{\partial v} + x\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial v}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial u}\right) = \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial u}\right) + y\frac{\partial}{\partial v}\left(\frac{\partial f}{\partial u}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial v}\right) = \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial v}\right) + x\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial v}\right)$$

$$\therefore \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial u^2} + y\frac{\partial^2 f}{\partial v \partial u} + \frac{\partial f}{\partial v} + x\frac{\partial^2 f}{\partial u \partial v} + xy\frac{\partial^2 f}{\partial v^2}$$

$$= \frac{\partial^2 f}{\partial u^2} + u\frac{\partial^2 f}{\partial u \partial v} + \frac{\partial f}{\partial v} + v\frac{\partial^2 f}{\partial v^2}$$ 20/50

JFM 10/10/16

Course Code and Title (Required): AE1-107 Mathematics

Setter (Required): JFM

Qu 4(1)

Write on this side only (in ink) between the margins, not more than one solution per sheet please. Solutions must be signed and dated by both exam setter and referee.

	Marks

(a) Note $w = 1 - \sqrt{3}\,i$ has

$$\text{modulus } |w| = \sqrt{1+3} = 2$$

$$\text{argument}(w) = \arctan(-\sqrt{3}) = -\pi/3.$$

De Moivre:

$$(1+\sqrt{3}\,i)^5 = 2^5\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^5 = 2^5\cos\frac{5\pi}{3} + i\,2^5\sin\frac{5\pi}{3}$$

$$(1-\sqrt{3}\,i)^5 = 2^5\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)^5 = 2^5\cos\frac{5\pi}{3} - i\,2^5\sin\frac{5\pi}{3}$$

Adding $(1+\sqrt{3}\,i)^5 + (1-\sqrt{3}\,i)^5$

$$= 2^6\cos\frac{5\pi}{3} = 2^5 = \underline{\underline{32}}$$

30

10

Course Code and Title (Required): AE1-107 Mathematics

Qu 4 (2)

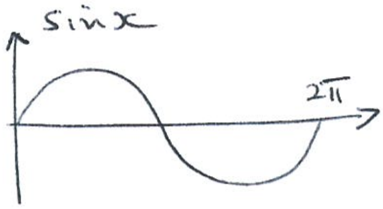
Setter (Required): JFM

Write on this side only (in ink) between the margins, not more than one solution per sheet please. Solutions must be signed and dated by both exam setter and referee.

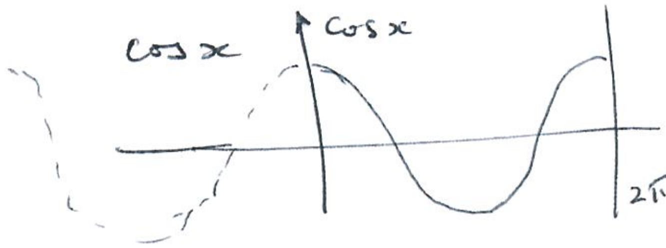
Marks

(b) (i) even for period L if

$$f(x) = f(x+L)$$



(ii) odd for period L if $f(x) = -f(-x)$



(iii) above.

10

(c)

$$f(x) = \begin{cases} -\cos x & -\pi \le x \le 0 \\ \cos x & 0 \le x \le \pi \end{cases}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} -\cos x\,dx + \frac{1}{\pi}\int_{0}^{\pi}\cos x\,dx$$

$$= \frac{1}{\pi}\left\{ \left[-\sin x\right]_{-\pi}^{0} + \left[\sin x\right]_{0}^{\pi} \right\}$$

$$= \frac{1}{\pi}0 = \underline{0}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} -\cos x \cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi}\cos x\cos nx\,dx = 0$$

JFM 10/10/16

Course Code and Title (Required): AE1-107 Mathematics

Setter (Required): JFM

Qu 4 (3)

Write on this side only (in ink) between the margins, not more than one solution per sheet please. Solutions must be signed and dated by both exam setter and referee.

Marks

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0} -\cos x \sin nx \, dx + \frac{1}{\pi}\int_{0}^{\pi} \cos x \sin nx \, dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{0} -\sin(n-1)x - \sin(n+1)x \, dx + \frac{1}{2\pi}\int_{0}^{\pi} \sin(n-1)x + \sin(n+1)x \, dx$$

$$= \frac{1}{2\pi}\left[\frac{1}{n-1}\cos(n-1)x + \frac{1}{n+1}\cos(n+1)x\right]_{-\pi}^{0} - \left[\frac{1}{n-1}\cos(n-1)x + \frac{1}{n+1}\cos(n+1)x\right]_{0}^{\pi}$$

n even:
$$= \frac{1}{2\pi}\left[+\frac{1}{n-1} + \frac{1}{n+1} + \frac{1}{n-1} + \frac{1}{n+1}\right] - \frac{1}{2\pi}\left[-\frac{1}{n-1} - \frac{1}{n+1} - \frac{1}{n-1} - \frac{1}{n+1}\right]$$

$$= \frac{1}{2\pi}\left[\frac{4}{n-1} + \frac{4}{n+1}\right] = \frac{1}{\pi}\left[\frac{4n}{n^2-1}\right] = \frac{4}{\pi}\left[\frac{n}{n^2-1}\right]$$

n odd:
$$= \frac{1}{2\pi}\left[\frac{1}{n-1} + \frac{1}{n+1} - \frac{1}{n-1} - \frac{1}{n+1}\right] - \frac{1}{2\pi}\left[\frac{1}{n-1} + \frac{1}{n+1} - \frac{1}{n-1} - \frac{1}{n+1}\right] = 0$$

$$f(x) = \frac{4}{\pi}\sum_{n \text{ even}}^{\infty} \frac{n}{n^2-1} \sin nx.$$

40

(d)

$$= \frac{4}{\pi}\left\{\frac{2}{3}\sin 2x + \frac{4}{15}\sin 4x + \frac{6}{35}\sin 6x \ldots\right\}$$

$$x = \pi/4 \quad f(x) = \frac{1}{\sqrt{2}}$$

JFM 10/10/16.

LAST PAGE (12)

Course Code and Title (Required): AE1-107 Mathematics

Setter (Required): JFM

Qu 4 (4)

Write on this side only (in ink) between the margins, not more than one solution per sheet please. Solutions must be signed and dated by both exam setter and referee.

Marks

$$\therefore \quad \frac{1}{\sqrt{2}} = \frac{4}{\pi}\left\{ \frac{2}{3} - \frac{6}{35} + \frac{10}{99} \cdots \right\}$$

$$\frac{\pi}{8\sqrt{2}} = \left\{ \frac{1}{3} - \frac{3}{35} + \frac{5}{99} \cdots \right\}$$

$$= \left\{ \frac{1}{3} - \frac{3}{5 \times 7} + \frac{5}{9 \times 11} - \frac{7}{13 \times 15} \right.$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2n-1)}{(4n-1)(4n-3)}$$

20

100

JFM 10/10/16.

First Year Mathematics (AE1-107)

Department of Aeronautics
Imperial College London

TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin^2(a) + \cos^2(a) = 1 \qquad 1 + \tan^2(a) = \sec^2(a) = \frac{1}{\cos^2(a)}$$

$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$$

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

$$\sin(2a) = 2\sin(a)\cos(a); \qquad \cos(2a) = \cos^2(a) - \sin^2(a);$$

$$\sin^2(a) = \frac{1-\cos(2a)}{2}; \qquad \cos^2(a) = \frac{1+\cos(2a)}{2}$$

$$\cos(a)\cos(b) = \frac{\cos(a-b)+\cos(a+b)}{2}; \qquad \sin(a)\sin(b) = \frac{\cos(a-b)-\cos(a+b)}{2}$$

$$\sin(a)\cos(b) = \frac{\sin(a+b)+\sin(a-b)}{2}; \qquad \sin(a) \pm \sin(b) = 2\sin\left(\frac{a \pm b}{2}\right)\cos\left(\frac{a \mp b}{2}\right)$$

$$\cos(a) - \cos(b) = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right); \qquad \cos(a) + \cos(b) = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$

$$\sin[\arccos(x)] = \sqrt{1-x^2}; \qquad \tan[\arcsin(x)] = \frac{x}{\sqrt{1-x^2}}$$

$$\sin[\arctan(x)] = \frac{x}{\sqrt{1+x^2}}; \qquad \tan[\arccos(x)] = \frac{\sqrt{1-x^2}}{x}$$

$$\cos[\arctan(x)] = \frac{1}{\sqrt{1+x^2}}; \qquad \cos[\arcsin(x)] = \sqrt{1-x^2}$$

$$\cosh^2(x) - \sinh^2(x) = 1; \qquad \sinh(x) = \frac{e^x - e^{-x}}{2}; \qquad \cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$\cos iz = \cosh z; \qquad \cosh iz = \cos z; \qquad \sin iz = i\sinh z; \qquad \sinh iz = i\sin z$$

DIFFERENTIAL CALCULUS

1. Inverse function differentiation:
$$\frac{dx}{dy} = \frac{1}{dy/dx}.$$

2. Parametric differentiation:
$$\frac{dy}{dx} = \frac{dy}{ds}|_{s=s(x)} \left(\frac{dx}{ds}|_{s=s(x)}\right)^{-1} = \frac{dy/ds}{dx/ds}|_{s=s(x)}$$

3. Estimating small changes:
$$\delta f \approx f'(x)\delta x$$

4. Leibniz's formula:
$$(fg)^{(n)} = {}^nC_0 f^{(n)} g + {}^nC_1 f^{(n-1)} g' + \ldots + {}^nC_r f^{(n-r)} g^{(r)} + \ldots + {}^nC_n f g^{(n)} = \sum_{r=0}^{n} {}^nC_r f^{(n-r)} g^{(r)},$$
where ${}^nC_r$ is defined as $\frac{n!}{r!(n-r)!}$.

5. Taylor's expansion of $f(x)$ about $x = a$:
$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x).$$
where
$$R_n = \frac{f^{(n+1)}(x_0)}{(n+1)!}(x-a)^{n+1}$$

6. Taylor's expansion of $f(x,y)$ about (a,b):
$$\begin{aligned} f(x,y) &= f(a,b) + (x-a)f_x(a,b) + (y-b)f_y(a,b) \\ &+ \frac{1}{2}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right] + \cdots \end{aligned}$$

7. Partial differentiation:
   i. If $u = f(x,y)$ and $y = y(x)$, then $\frac{du}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$.
   ii. If $x = x(t),\ y = y(t)$ and $u(t) = f(x(t), y(t))$, then $\frac{du}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$.
   iii. If $x = x(s,t),\ y = y(s,t)$ and $u(s,t) = f(x(s,t),\ y(s,t))$, then
$$\frac{\partial u}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}, \qquad \frac{\partial u}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}.$$

8. Stationary points of $f(x,y)$ occur where $f_x = 0,\ f_y = 0$ simultaneously.
   Let (a,b) be a stationary point and examine $\Delta(a,b) = (f_{xy}(a,b))^2 - f_{xx}(a,b)f_{yy}(a,b)$. Then:
   i. If $\Delta(a,b) < 0$ and either $f_{xx}(a,b) < 0$ or $f_{yy}(a,b) < 0$, then (a,b) is a maximum;
   ii. If $\Delta(a,b) < 0$ and either $f_{xx}(a,b) > 0$ or $f_{yy}(a,b) > 0$, then (a,b) is a minimum;

iii. If $\Delta(a,b) > 0$ then (a,b) is a saddle-point.

9. Approximate solution of an algebraic equation:
If a root of $f(x)=0$ occurs near $x=a$, take $x_0 = a$ and

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad n = 0, 1, 2 \ldots$$

(Newton Raphson method).

INTEGRAL CALCULUS

1. Integration by parts:

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx.$$

2. Integration by partial fractions:

$$\begin{aligned}
\text{1 term} \rightarrow (ax-b) &\rightarrow \frac{A}{ax-b} \\
r \text{ terms} \rightarrow (ax-b)^r &\rightarrow \frac{A_1}{ax-b} + \frac{A_2}{(ax-b)^2} + \cdots + \frac{A_r}{(ax-b)^r} \\
\text{No real roots, 1 term} \rightarrow (ax^2+bx+c) &\rightarrow \frac{Ax+B}{(ax^2+bx+c)} \\
\text{No real roots, } r \text{ terms} \rightarrow (ax^2+bx+c)^r &\rightarrow \frac{A_1x+B_1}{(ax^2+bx+c)} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} \\
&+ \cdots + \frac{A_rx+B_r}{(ax^2+bx+c)^r}
\end{aligned}$$

3. An important substitution: $\tan(\theta/2) = t$:

$$\sin(\theta) = \frac{2t}{(1+t^2)}; \qquad \cos(\theta) = \frac{(1-t^2)}{(1+t^2)}; \qquad d\theta = \frac{2dt}{(1+t^2)}.$$

4. Some indefinite integrals:

$$\int (a^2-x^2)^{-1/2}dx = \sin^{-1}\left(\frac{x}{a}\right), \ |x| < a.$$

$$\int \left(a^2+x^2\right)^{-1/2} dx = \sinh^{-1}\left(\frac{x}{a}\right) = \ln\left|\frac{x}{a} + \left(1+\frac{x^2}{a^2}\right)^{1/2}\right|$$

$$\int \left(x^2-a^2\right)^{-1/2} dx = \cosh^{-1}\left(\frac{x}{a}\right) = \ln\left|\frac{x}{a} + \left(\frac{x^2}{a^2}-1\right)^{1/2}\right|$$

$$\int \left(a^2+x^2\right)^{-1} dx = \left(\frac{1}{a}\right)\tan^{-1}\left(\frac{x}{a}\right)$$

5. Binomial Theorem:

$$(a+b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b + \ldots + {}^nC_ra^{n-r}b^r + \ldots + {}^nC_nb^n = \sum_{r=0}^{n} {}^nC_ra^{n-r}b^r.$$

where ${}^nC_r$ is defined as $\frac{n!}{r!(n-r)!}$.

SERIES

Common Maclaurin expansions:

$$(1+x)^{\alpha} = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \ldots \qquad (\alpha \text{ arbitrary, } |x| < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \ldots,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^n \frac{x^{n+1}}{(n+1)} + \ldots (-1 < x \leq 1)$$

FOURIER SERIES

1. If $f(x)$ is periodic of period $2L$, then $f(x+2L) = f(x)$ and

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right),$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \qquad n = 0, 1, 2, \ldots$$

and

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \qquad n = 1, 2, 3, \ldots$$

2. Parseval's Theorem:

$$\frac{1}{L}\int_{-L}^{L} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$