DTS AND COMPUTER CONTROL

1. *This is a standard exercise which requires to compute several inverse Z transforms. Unfortunately, many students were not able to do so. Many computational mistakes have been done. It is somewhat worrying that so many students failed to compute inverse Z transforms since we have done many exercises during the lectures and Z transforms are at the basis of the module. Some alternative solutions (based on the assumption of the information sheet rather than the instructions in this exercise) have been considered correct in points b.i and b.ii.*

a) Applying the forward difference property of the z-transform we obtain

$$z^2X(z) - z^2x(0) - zx(1) + \alpha(zX(z) - zx(0)) + \beta X(z) = U(z).$$

Solving this equation with respect to $X(z)$ yields

$$X(z) = \frac{z^2+\alpha z}{z^2+\alpha z+\beta}x(0) + \frac{z}{z^2+\alpha z+\beta}x(1) + \frac{1}{z^2+\alpha z+\beta}U(z).$$

[3 marks]

b) For $\alpha = -2$, $\beta = 1$, $x(1) = 0$ and the input

$$U(z) = \frac{z(z-1)}{(z-\frac{1}{2})^2},$$

the function $X(z)$ becomes

$$X(z) = \frac{z^2-2z}{z^2-2z+1}x(0) + \frac{1}{z^2-2z+1}\frac{z(z-1)}{(z-\frac{1}{2})^2}.$$

i) Note that $X(z)$ can be rewritten as

$$X(z) = \frac{z^2-2z}{(z-1)^2}x(0) + \frac{z}{(z-1)(z-\frac{1}{2})^2}.$$

Computing the partial fraction expansion of $\dfrac{X(z)}{z}$ we obtain

$$X(z) = \frac{z}{z-1}x(0) - \frac{z}{(z-1)^2}x(0) + 4\frac{z}{z-1} - 4\frac{z}{z-\frac{1}{2}} - 2\frac{z}{(z-\frac{1}{2})^2}.$$

The inverse z-transform of this last expression is

$$x(k) = (1-k)x(0) + 4 - 4(1+k)\left(\frac{1}{2}\right)^k.$$

[3 marks]

ii) Note that

$$\lim_{k\to\infty}(1+k)\left(\frac{1}{2}\right)^k \to 0,$$

Hence, $x(k)$ behaves as $(1-k)x(0)+4$. If $x(0) = 0$ the solution $x(k)$ converges asymptotically to 4. If $x(0) \neq 0$, the solution $x(k)$ diverges to ∞.

[3 marks]

c) For $\alpha = -\frac{3}{2}$, $\beta = \frac{1}{2}$, $x(0) = 0$, $x(1) = 0$ and the input

$$u(k) = \sin\left(\frac{\pi}{2}k\right),$$

the function $X(z)$ becomes

$$X(z) = \frac{1}{z^2 - \frac{3}{2}z + \frac{1}{2}} \frac{z}{z^2+1} = \frac{z}{(z-1)(z-\frac{1}{2})(z^2+1)}.$$

i) Using the partial fraction expansion we obtain

$$X(z) = \frac{1}{5}\frac{3z^2 - z}{z^2+1} + \frac{z}{z-1} - \frac{8}{5}\frac{z}{z-\frac{1}{2}}.$$

The inverse z-transform of this last expression gives

$$x(k) = -\frac{1}{5}\sin\left(\frac{\pi}{2}k\right) + \frac{3}{5}\cos\left(\frac{\pi}{2}k\right) + 1 - \frac{8}{5}\left(\frac{1}{2}\right)^k.$$

Hence, the transient response is

$$x_{tr}(k) = -\frac{8}{5}\left(\frac{1}{2}\right)^k,$$

whereas the steady-state response is

$$x_{ss}(k) = 1 - \frac{1}{5}\sin\left(\frac{\pi}{2}k\right) + \frac{3}{5}\cos\left(\frac{\pi}{2}k\right).$$

[3 marks]

ii) In this case the function $X(z)$ is

$$X(z) = \frac{1}{(z-1)(z-\frac{1}{2})}\left(\frac{z}{z-1} + \frac{z}{z^2+1}\right).$$

To compute the response to the new input we just need to compute the contribution of 1 because using linearity the contribution due to $\sin\left(\frac{\pi}{2}k\right)$ remains unchanged. Note that

$$\frac{z}{(z-1)^2(z-\frac{1}{2})} = -4\frac{z}{z-1} + 2\frac{z}{(z-1)^2} + 4\frac{z}{z-\frac{1}{2}}.$$

So the new contribution is

$$-4 + 2k + 4\left(\frac{1}{2}\right)^k.$$

Adding this contribution to the previous result yields

$$x(k) = -\frac{1}{5}\sin\left(\frac{\pi}{2}k\right) + \frac{3}{5}\cos\left(\frac{\pi}{2}k\right) - 3 + \frac{12}{5}\left(\frac{1}{2}\right)^k + 2k.$$

The steady-state response is not properly defined as $x(k) - x_{tr}(k) \to \infty$. The reason of this change is that the input 1 "resonates" with the pole in 1 already in the system. The interconnected system has two poles in 1. This produces terms in k and, similarly to part b), the solution diverges to ∞.

[3 marks]

d) i) For $a = b$,

$$X(z) = \frac{z}{(z-a)^2}.$$

From the formula sheet it follows that

$$x(k) = ka^{k-1}.$$

If $a = b = 1$, then $x(k) = k$ and the solution diverges.

[2 marks]

ii) For $a \neq b$

$$X(z) = \frac{z}{(z-a)(z-b)}.$$

It follows that

$$x(k) = \frac{1}{a-b}a^k + \frac{1}{b-a}b^k.$$

When $a = 1$, this last expression becomes

$$x(k) = \frac{1}{1-b} + \frac{1}{b-1}b^k.$$

If $|b| > 1$, then $x(k)$ diverges. If $|b| < 1$, then $x(k)$ converges to $\frac{1}{1-b}$. If $b = -1$, then

$$x(k) = \begin{cases} 0, & k \text{ even}, \\ \dfrac{2}{1-b} = 1, & k \text{ odd}. \end{cases}$$

[3 marks]

2. *Also this is a standard exercise which we have solved in a different form during the lecture. Most students were able to complete part a) and part b), although some students were not able to solve part b.iii). Most students were not able to solve part c) which was identical to part b) but with more complex computations.*

a) The closed-loop characteristic polynomial is $s+k+2$, hence the closed-loop system is asymptotically stable for all $k>0$.

[1 marks]

b) i) The equivalent discrete-time model is

$$\begin{aligned} H_0P(z) &= \frac{z-1}{z}Z\left(\frac{P(s)}{s}\right) = k\frac{z-1}{z}Z\left(\frac{1}{2}\frac{1}{s}-\frac{1}{2}\frac{1}{s+2}\right) \\ &= k\frac{z-1}{z}\left(\frac{1}{2}\frac{1}{1-z^{-1}}-\frac{1}{2}\frac{1}{1-e^{-2T}z^{-1}}\right) \\ &= \frac{k}{2}\frac{1-e^{-2T}}{z-e^{-2T}}. \end{aligned}$$

[2 marks]

ii) The characteristic polynomial of the closed-loop system is

$$z+\frac{k}{2}(1-e^{-2T})-e^{-2T}.$$

The roots of this polynomial are all inside the unity circle if

$$k<2\frac{1+e^{-2T}}{1-e^{-2T}}=\overline{K}.$$

[2 marks]

iii) Note that $\lim_{T\to 0}\overline{K}=\infty$ and $\lim_{T\to\infty}\overline{K}=2$.

[2 marks]

c) i) The equivalent discrete-time model is

$$\begin{aligned} H_0P(z) &= \frac{(z-1)^2}{z^2}Z\left(\frac{1+Ts}{T}\frac{P(s)}{s^2}\right) \\ &= \frac{k}{T}\frac{(z-1)^2}{z^2}Z\left(\frac{1}{2}\frac{1}{s^2}-\frac{1}{4}\frac{2T-1}{s+2}+\frac{1}{4}\frac{2T-1}{s}\right) \\ &= \frac{k}{T}\frac{(z-1)^2}{z^2}\left(\frac{1}{2}\frac{Tz}{(z-1)^2}+\frac{1}{4}\frac{(2T-1)z}{z-1}-\frac{1}{4}\frac{(2T-1)z}{z-e^{-2T}}\right) \\ &= \frac{k}{4T}\frac{(-1+4T-2e^{-2T}T+e^{-2T})z+1-2T-e^{-2T}}{z(z-e^{-2T})}. \end{aligned}$$

[4 marks]

ii) The characteristic polynomial of the closed-loop system is

$$z^2+\left(\frac{k}{4T}(-1+4T-2e^{-2T}T+e^{-2T})-e^{-2T}\right)z+\frac{k}{4T}(1-2T-e^{-2T}).$$

To determine the location of the roots of this polynomial we can use the bilinear transformation $z=\frac{w+1}{w-1}$. A quicker and safer route is to recall that the polynomial

$$z^2+\alpha z+\beta$$

has all roots in the unit circle if

$$\begin{aligned} \beta &> -1 + \alpha, \\ \beta &< 1, \\ \beta &> -1 - \alpha. \end{aligned} \tag{2.1}$$

These conditions corresponds, respectively, to

$$\begin{aligned} &k < \frac{2(1 + e^{-2T})}{(3 - e^{-2T} - \frac{1}{T}(1 - e^{-2T}))}, \\ &k > \frac{4}{(\frac{1}{T} - 2 - \frac{1}{T}e^{-2T})}, \\ &k > -2. \end{aligned}$$

Note that the last two conditions are verified for $k > 0$. Hence, the roots of the characteristic polynomial are all inside the unity circle if

$$k < \frac{2(1 + e^{-2T})}{(3 - e^{-2T} - \frac{1}{T}(1 - e^{-2T}))} = \overline{\overline{K}}.$$

[5 marks]

iii) For $T \to 0$

$$\overline{\overline{K}} \approx \frac{2(1 + 1 - 2T)}{(3 - 1 + 2T - \frac{1}{T}(1 - 1 + 2T)} \approx \frac{2}{T}.$$

Hence, $\lim\limits_{T\to 0} \overline{\overline{K}} = \infty$ and $\lim\limits_{T\to\infty} \overline{\overline{K}} = \dfrac{2}{3}$.

[3 marks]

d) For sufficiently small values of T, the behavior of the system interconnected to a ZOH or to a FOH approaches the behavior of the continuous-time system. As the sampling time increases, the discretized systems becomes increasingly unstable for larger values of k. The maximum k achievable using the FOH is $\dfrac{1}{3}$ of the maximum value achievable using the ZOH.

[1 marks]

3. *This exercise requires to show the equivalence between state-space and Z-domain. Some students failed to compute the matrices A and B despite the hint. Some students failed to compute the transfer function. Few students attempted the control questions, but some succeeded. At least one student obtained the top mark.*

a) The relation between the matrices (A,B,C) and (F,G,H) are the following

$$A = e^{FT}, \qquad B = \int_0^T e^{F\lambda} G d\lambda, \qquad C = H.$$

To compute these matrices we first compute $(sI-F)^{-1}$, namely

$$(sI-F)^{-1} = \begin{bmatrix} \frac{1}{s} & \frac{1}{s(s+1)} \\ 0 & \frac{1}{s+1} \end{bmatrix}.$$

It follows that

$$A = \begin{bmatrix} 1 & 1-e^{-1} \\ 0 & e^{-1} \end{bmatrix}, \qquad B = \begin{bmatrix} e^{-1} \\ 1-e^{-1} \end{bmatrix}.$$

[4 marks]

b) To compute the transfer function we first determine the term

$$(zI-A)^{-1} = \begin{bmatrix} \frac{1}{z-1} & \frac{1-e^{-1}}{(z-1)(z-e^{-1})} \\ 0 & \frac{1}{z-e^{-1}} \end{bmatrix}.$$

The input-output transfer function is given by

$$\frac{Y(z)}{U(z)} = C(zI-A)^{-1}B = \frac{e^{-1}z+1-2e^{-1}}{(z-1)(z-e^{-1})}.$$

[3 marks]

c) Exploiting the result of part a), the computation of the transfer function is straightforward

$$\frac{Y(s)}{U(s)} = P(s) = H(sI-F)^{-1}G = \frac{1}{s(s+1)}.$$

[2 marks]

d) The equivalent discrete-time model is

$$\begin{aligned} HP(z) &= \frac{z-1}{z} Z\left(\frac{P(s)}{s}\right) = \frac{z-1}{z} Z\left(\frac{1}{s+1} - \frac{1}{s} + \frac{1}{s^2}\right) \\ &= \frac{z-1}{z}\left(\frac{1}{1-e^{-1}z^{-1}} - \frac{1}{1-z^{-1}} + \frac{z^{-1}}{(1-z^{-1})^2}\right) \\ &= \frac{e^{-1}z+1-2e^{-1}}{(z-1)(z-e^{-1})}. \end{aligned}$$

as in b).

[3 marks]

e) Selecting $u(k) = -Kx(k)$ we obtain the closed-loop state equation

$$x(k+1) = (A - BK)x(k).$$

To achieve a deadbeat response we place the eigenvalues of $(A - BK)$ in zero. Let $K = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$. Then,

$$\det(\lambda I - (A - BK)) = \lambda^2 - (1 - K_2 + e^{-1})\lambda - ((e^{-1} - 1)K_1 + K_2 - e^{-1}).$$

To place the eigenvalues in zero we solve the equations

$$\begin{aligned} &1 - K_2 + e^{-1} = 0, \\ &(e^{-1} - 1)K_1 + K_2 - e^{-1} = 0, \end{aligned}$$

which gives

$$K_1 = \frac{1}{1 - e^{-1}}, \qquad K_2 = 1 + e^{-1}.$$

[4 marks]

f) The closed-loop transfer function of the block diagram is given by

$$\frac{Y(z)}{R(z)} = \frac{C(z)HP(z)}{1 + C(z)HP(z)}.$$

Thus, the characteristic polynomial is

$$\text{num}(1 + C(z) + HP(z)).$$

The problem has a unique solution if we select the control $C(z)$ as

$$C(z) = \frac{s_0 z + s_1}{z + v_1},$$

because $m = \deg(\text{den}(HP(z))) - 1$. Thus,

$$\begin{aligned} &\text{num}(1 + C(z)HP(z)) = z^3 + (v_1 - (1 + e^{-1}) + s_0 e^{-1})z^2 + \\ &+(-v_1(1 + e^{-1}) + e^{-1} + e^{-1}s_1 + (1 - 2e^{-1})s_0)z + (e^{-1}v_1 + (1 - 2e^{-1})s_1). \end{aligned}$$

To achieve a deadbeat response, we require that $\text{num}(1 + C(z)HP(z)) = z^3$, which yields

$$\begin{bmatrix} e^{-1} & 0 & 1 \\ 1 - 2e^{-1} & e^{-1} & -1 - e^{-1} \\ 0 & 1 - 2e^{-1} & e^{-1} \end{bmatrix} \begin{bmatrix} s_0 \\ s_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} 1 + e^{-1} \\ -e^{-1} \\ 0 \end{bmatrix}.$$

(For the sake of completeness the solution of this linear system is $s_0 = 2.3055$, $s_1 = -0.7236$, $v_1 = 0.5197$. However, it was not required to determine these values.)

[4 marks]

4. *This exercise is a special addition for the year 2017/2018. This exercise covers basic state-space control and topics that the students are expected to know as a prerequisite since this material is covered in the third year. This material was covered again in this module. However, many students were not able to complete this exercise. As a result, change of structure in the MSc has been suggested and implemented, to make these topics compulsory starting in year 2018/2019.*
Typical mistakes include missing some conditions in point b) and some computational mistakes in part c) and d). At least one student obtained the top mark.

a) i) The reachability matrix is

$$P = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} d & ad \\ 0 & bd \end{bmatrix}.$$

We easily see that full rank is achieved if and only if $b \neq 0$ and $d \neq 0$. If either b or d is zero, the system is not reachable.

[2 marks]

ii) The observability matrix is

$$Q = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & e \\ be & ce \end{bmatrix}.$$

We easily see that full rank is achieved if and only if $b \neq 0$ and $e \neq 0$. If either b or e is zero, the system is not observable.

[2 marks]

b) The system is controllable if and only if

$$\operatorname{rank}\left(\begin{bmatrix} P & A^2 \end{bmatrix}\right) = \operatorname{rank}(P),$$

which yields

$$\operatorname{rank}\left(\begin{bmatrix} d & ad & a^2 & 0 \\ 0 & bd & ab+cb & c^2 \end{bmatrix}\right) = \operatorname{rank}\left(\begin{bmatrix} d & ad \\ 0 & bd \end{bmatrix}\right).$$

We distinguish three cases.

- If $b \neq 0$ and $d \neq 0$, the system is reachable. This implies that it is also controllable.
- If $b = 0$ and $d \neq 0$, $\operatorname{rank}(P) = 1$. If $c = 0$, then $\operatorname{rank}\left(\begin{bmatrix} P & A^2 \end{bmatrix}\right) = 1$, which implies that the system is controllable. If $c \neq 0$, then $\operatorname{rank}\left(\begin{bmatrix} P & A^2 \end{bmatrix}\right) = 2$, which implies that the system is not controllable.
- If $d = 0$, $\operatorname{rank}(P) = 0$. If $a = c = 0$, then $\operatorname{rank}\left(\begin{bmatrix} P & A^2 \end{bmatrix}\right) = 0$, which implies that the system is controllable. If $a \neq 0$ or $c \neq 0$, then $\operatorname{rank}\left(\begin{bmatrix} P & A^2 \end{bmatrix}\right)$ is either 1 or 2, which implies that the system is not controllable.

[8 marks]

c) i) Since $b = d = 1$, the system is reachable. Let $x(0) = \begin{bmatrix} x_1(0) & x_2(0) \end{bmatrix}^\top$. Then

$$\begin{bmatrix} x_1(1) \\ x_2(1) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(0) = \begin{bmatrix} u(0) \\ x_1(0) + x_2(0) \end{bmatrix}$$

and

$$\begin{bmatrix} x_1(2) \\ x_2(2) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(1) \\ x_2(1) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(1) = \begin{bmatrix} u(1) \\ u(0) + x_1(0) + x_2(0) \end{bmatrix}.$$

Hence, the problem is solved selecting $u(0) = 1 - x_1(0) - x_2(0)$ and $u(1) = 1$.

[2 marks]

ii) The equations

$$y(0) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}, \qquad y(1) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(1) \\ x_2(1) \end{bmatrix}$$

imply $x_2(0) = 1$ and $x_2(1) = 2$. Substituting these values in the equation

$$\begin{bmatrix} x_1(1) \\ x_2(1) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(0)$$

yields $x_1(1) = 2$ and $x_1(0) = 1$. Finally, $x(2)$ can be computed directly from the state equation, yielding $x_1(2) = 1$ and $x_2(2) = 4$.

[2 marks]

d) Recall from part b) that since $b = 0$, the controllability of the system depends on the value of c.

i) If $c = 0$, then the system is controllable. In fact,

$$\begin{bmatrix} x_1(1) \\ x_2(1) \end{bmatrix} = \begin{bmatrix} x_{10} + u(0) \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} x_1(2) \\ x_2(2) \end{bmatrix} = \begin{bmatrix} x_{10} + u(0) + u(1) \\ 0 \end{bmatrix}.$$

The system can be controlled to zero in one step selecting $u(0) = -x_{10}$ and $u(k) = 0$ for all $k > 0$.

[2 marks]

ii) If $c \neq 0$, then the system is not controllable. However, note that there is still a set of initial states that can be controlled to zero. In fact,

$$\begin{bmatrix} x_1(1) \\ x_2(1) \end{bmatrix} = \begin{bmatrix} x_{10} + u(0) \\ x_{20} \end{bmatrix} \qquad \begin{bmatrix} x_1(2) \\ x_2(2) \end{bmatrix} = \begin{bmatrix} x_{10} + u(0) + u(1) \\ x_{20} \end{bmatrix}$$

implies that any initial state such that $x_{20} = 0$ can be controlled to zero in one step selecting $u(0) = -x_{10}$ and $u(k) = 0$ for all $k > 0$.

[2 marks]