IMPERIAL COLLEGE LONDON

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DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2018

MSc and EEE/EIE PART IV: MEng and ACGI

Corrected copy

DIGITAL SIGNAL PROCESSING AND DIGITAL FILTERS

Correction 10.am 1(d) ii
Correction 11am
Q3 b

Tuesday, 1 May 10:00 am

Time allowed: 3:00 hours

There are FOUR questions on this paper.

Answer Question 1 and any TWO other questions

Question 1 is worth 40% of the marks and other questions are worth 30%

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : D.M. Brookes
Second Marker(s) : P.T. Stathaki

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DIGITAL SIGNAL PROCESSING AND DIGITAL FILTERS

Information for Candidates:

Where a question requires a numerical answer, it must be given as a fully evaluated decimal number and not as an unevaluated arithmetic expression.

Notation

- All signals and filter coefficients are real-valued unless explicitly noted otherwise.
- Unless otherwise specified, upper and lower case letters are used for sequences and their z-transforms respectively. The signal at a block diagram node V is $v[n]$ and its z-transform is $V(z)$.
- $x[n] = [a, b, c, d, e, f]$ means that $x[0] = a, \ldots x[5] = f$ and that $x[n] = 0$ outside this range.
- $\Re(z)$, $\Im(z)$, z^*, $|z|$ and $\angle z$ denote respectively the real part, imaginary part, complex conjugate, magnitude and argument of a complex number z.
- The expected value of x is denoted $E\{x\}$.
- In block diagrams: solid arrows denote the direction of signal flow; an open triangle denotes a gain element with the gain indicated adjacently; a "+" in a circle denotes an adder/subtractor whose inputs may be labelled "+" or "−" according to their sign; the sample rate, f, of a signal in Hz may be indicated in the form "@f".

Abbreviations

BIBO	Bounded Input, Bounded Output	IIR	Infinite Impulse Response
CTFT	Continuous-Time Fourier Transform	LTI	Linear Time-Invariant
DCT	Discrete Cosine Transform	MDCT	Modified Discrete Cosine Transform
DFT	Discrete Fourier Transform	PSD	Power Spectral Density
DTFT	Discrete-Time Fourier Transform	SNR	Signal-to-Noise Ratio
FIR	Finite Impulse Response		

A datasheet is included at the end of the examination paper.

1. a) The signal $x[n]$ is zero outside the range $n \in [0, N-1]$. The impulse response $h[n]$ is zero outside the range $n \in [0, M]$.

i) The convolution of $x[n]$ and $h[n]$ is given by $y[n] = \sum_{r=0}^{M} h[r]x[n-r]$. Giving your reasons fully, determine the range of n for which $y[n]$ may be non-zero. [3]

ii) The signal $w[n] = \sum_{r=0}^{M} h[r]x[(n-r)_{\text{mod}\,K}]$ is the length-K circular convolution of $x[n]$ and $h[n]$. Giving your reasons fully, determine the smallest value of K to ensure that $w[n] = y[n]$ for $n \in [0, K-1]$ where $y[n]$ is defined in part i). Note that $m_{\text{mod}\,K}$ is the remainder when m is divided by K and satisfies $m_{\text{mod}\,K} \in [0, K-1]$. [3]

b) i) Determine the z-transform of $x[n] = a^n u[n-1]$ and its region of convergence when a is a positive real number and $u[n]$ is the unit step function. [3]

ii) Show that $y[n] = -a^n u[-n]$ has the same z-transform but a different region of convergence. [2]

c) A first-order IIR filter is given by $H(z) = \frac{1-z^{-1}}{1-az^{-1}}$ where a is real with $|a| < 1$.

i) If $a = 0.8$, sketch a diagram of the complex plane showing the unit circle, $|z| = 1$, and indicating any poles and zeros of $H(z)$ by crosses and circles respectively. [2]

ii) Calculate $|H(e^{j\omega})|$ for $\omega = \{0, \frac{\pi}{2}, \pi\}$ and hence sketch a dimensioned graph of $|H(e^{j\omega})|$ for $\omega \in [0, \pi]$. [3]

iii) Determine the value of a such that $|H(e^{j0.1})|^2 = \frac{1}{2}|H(e^{j\pi})|^2$. [4]

d) i) The frequency response of an ideal lowpass filter is given by

$$H(e^{j\omega}) = \begin{cases} 1 & |\omega| \leq \omega_0 \\ 0 & |\omega| > \omega_0 \end{cases}.$$

By taking the inverse DTFT of $H(e^{j\omega})$, show that the corresponding impluse response is $h[n] = \dfrac{\sin \omega_0 n}{\pi n}$. [3]

ii) By multiplying an ideal filter impulse response by a Hamming window, determine an expression for the real-valued coefficients of a causal FIR high-pass filter of even order, M, whose passband is $\cancel{\omega \geq 1}$ $1 \leq |\omega| \leq \pi$. [announced at start]

For even M, a non-causal symmetric Hamming window is given by $w[n] = 0.54 + 0.46 \cos \frac{2\pi n}{M+1}$ for $-0.5M \leq n \leq 0.5M$. [3]

iii) The lowest value of $\omega > 0$ for which $W(e^{j\omega}) = 0$ is $\omega \approx \frac{4\pi}{M+1}$ where $W(e^{j\omega})$ is the DTFT of $w[n]$. Estimate the filter order, M, that results in a transition width, $\Delta\omega$, of approximately 0.2. [2]

e) Figure 1.1 shows the power spectral density (PSD) of a real-valued signal $x[n]$. The horizontal portions of the PSD have values 2 and 1 respectively.

The signal $y[n]$ is obtained by downsampling $x[n]$ by a factor of 2. Draw a dimensioned sketch showing the PSD of $y[n]$ for $0 \le \omega \le \pi$. You should assume that components of $x[n]$ at different frequencies are uncorrelated and may assume without proof that $Y(z) = \frac{1}{2}\left\{X(z^{\frac{1}{2}}) + X(-z^{\frac{1}{2}})\right\}$.

Determine the value of each horizontal portion of the PSD of $y[n]$ and each of the angular frequencies at which its value changes. [3]

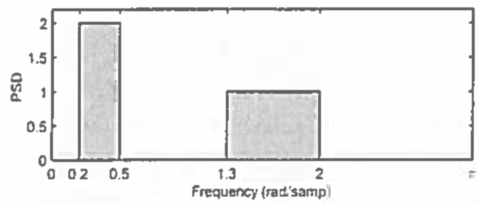

Figure 1.1

f) Figure 1.2 shows the block diagram of a two-band analysis and synthesis processor. You may assume without proof that, for $m = 0$ or 1, $U_m(z) = \frac{1}{2}\left\{V_m(z^{\frac{1}{2}}) + V_m(-z^{\frac{1}{2}})\right\}$ and $W_m(z) = U_m(z^2)$.

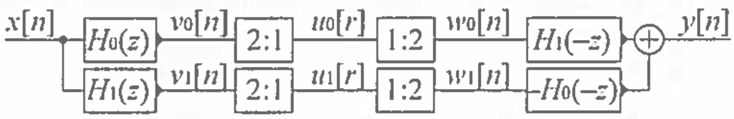

Figure 1.2

i) Derive a simplified expression for $Y(z)$ in terms of $X(z)$. [3]

ii) Show that the transfer function, $\frac{Y(z)}{X(z)}$, may be written in the form $\frac{1}{2}\{G(z) - G(-z)\}$ and describe how the coefficients of $\frac{Y(z)}{X(z)}$ are related to those of $G(z)$. [2]

iii) If $H_0(z) = 4\left(1 + z^{-1}\right)^2$ and $H_1(z) = -\left(1 - z^{-1}\right)^2\left(1 + 4z^{-1} + z^{-2}\right)$, determine the transfer function $\frac{Y(z)}{X(z)}$. [4]

2. A causal symmetric high-pass FIR filter, $H(z)$, of even order, M, is to be designed such that its magnitude response lies within the unshaded region of the graph in Figure 2.1. The impulse response of $H(z)$ satisfies $h[M-n] = h[n]$. We define $\overline{H}(\omega) \triangleq H(e^{j\omega})e^{0.5jM\omega}$.

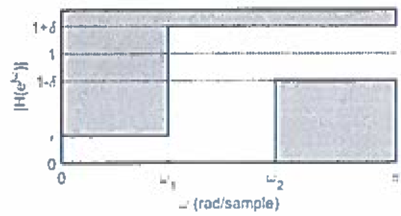

Figure 2.1

a) i) Show that $\overline{H}(\omega)$ (defined above) is given by the real-valued expression $\overline{H}(\omega) = h[0.5M] + 2\sum_{n=1}^{0.5M} h[n+0.5M]\cos(n\omega)$. [4]

ii) Consider the hypothesis that $\cos(n\omega) = T_n(\cos\omega)$ where $T_n(\cdot)$ is a polynomial of order n. From the trigonometrical identity,

$$\cos((n+1)\omega) + \cos((n-1)\omega) = 2\cos(\omega)\cos(n\omega),$$

show that if the hypothesis is true for both n and $n-1$ then it is also true for $n+1$. Hence prove by induction that the hypothesis is true for all $n \geq 0$. [3]

iii) Show that $\overline{H}(\omega)$ has at most $0.5M+1$ stationary values within the range $\omega \in [0, \pi]$. [3]

b) For a target response of $d(\omega) = \begin{cases} 0 & \text{for } \omega < \omega_2 \\ 1 & \text{for } \omega \geq \omega_2 \end{cases}$, the weighted error of $\overline{H}(\omega)$ is defined as $e(\omega) = s(\omega)\left(\overline{H}(\omega) - d(\omega)\right)$. Determine the positive real-valued weighting function, $s(\omega)$, in each of the three frequency bands bounded by $\omega_i = \{0, \omega_1, \omega_2, \pi\}$ so that $|e(\omega)| \leq 1\ \forall\omega$ if and only if the filter satisfies its specification. [3]

c) If $\delta = \varepsilon = 0.25$ in Figure 2.1, determine the values in dB of the the maximum stopband gain and of the maximum passband ripple. [2]

d) A polynomial fit, $y = f(x)$, to a set of data pairs, (x_i, y_i) is "minimax-optimal" if the worst-case absolute error is as small as possible for a given polynomial order, n. If the error is given by $e_i = f(x_i) - y_i$, the Alternation Theorem states that $f(x)$ is a minimax-optimal fit if and only if the maximum value of $|e_i|$ occurs for $n+2$ values of i with the errors having alternating signs.

By constructing equations of the form $y_i = mx_i + c \pm e$, determine the constants m and c such that $y = mx + c$ is the minimax-optimal linear fit to the data pairs $(1, 3)$, $(4, 5)$, $(8, 17)$. [5]

e) Explain why, if the frequency range $0 \leq \omega \leq \pi$ is partitioned into a finite number of bands and $d(\omega)$ and $s(\omega)$ are both constant within each band, then all maxima of $|e(\omega)|$ must occur either at the band edges or at stationary values of $\overline{H}(\omega)$. [2]

f) If $M = 2$, determine a minimax-optimal solution, $\overline{H}(\cos^{-1} x) = mx + c$, corresponding to the filter design problem specified in Figure 2.1 using the values $\omega_1 = 1$, $\omega_2 = 2$, $\delta = \varepsilon = 0.25$ and $s(\omega) = 4\ \forall\omega$. You may assume without proof that, in the minimax-optimal solution, the maximal values of $|e(\omega)|$ occur at $\omega = \{\omega_1, \omega_2, \pi\}$. [6]

g) Give the coefficients, $h[0]$, $h[1]$ and $h[2]$ of the filter found in part f). [2]

3. a) The transfer function $G(z)$ is given by $G(z) = 1 + g_1 z^{-1} + g_2 z^{-2}$ where g_1 and g_2 are real-valued.

i) By considering $G(z)G(z^{-1})$, or otherwise, show that [4]

$$|G(e^{j\omega})|^2 = 4g_2 \cos^2 \omega + 2g_1 (1 + g_2) \cos \omega + g_1^2 + (1 - g_2)^2.$$

ii) Hence determine a condition for $|G(e^{j\omega})|$ to have a stationary value in the range $0 \le \omega \le \pi$ and show that, if it exists, its value is given by $|G(e^{j\omega})| = \sqrt{g_1^2 + (1 - g_2)^2 - 0.25 g_1^2 g_2^{-1} (1 + g_2)^2}$. [4]

b) The block diagram in Figure 3.1 has two input signals, $x[n]$ and $u[n]$ and one output, $y[n]$. The five multipliers have gains $k_0, \cdots, k_4$ as shown. Determine the transfer function of the block diagram in the form $Y(z) = \frac{B(z)}{A(z)} X(z) + \frac{D(z)}{C(z)} U(z)$ where $A(z), \cdots, D(z)$ are polynomials in z^{-1}. Give the coefficients of the polynomials in terms of the k_i. [4]

11.am

z^{-1}

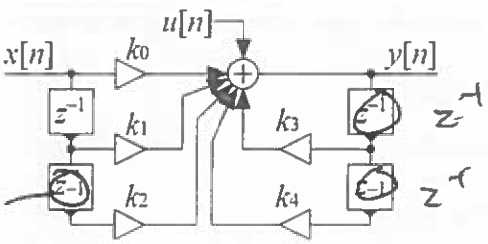

Figure 3.1

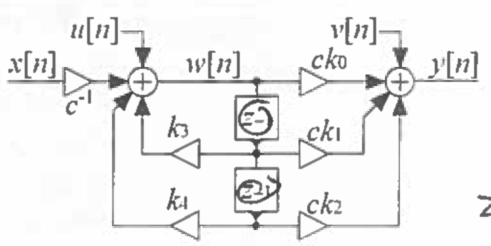

Figure 3.2

z^{-1}

c) The transfer function of a second-order lowpass elliptic filter with cutoff frequency $\omega_0 = 1$ is $H(z) = \frac{B(z)}{A(z)}$ where $B(z) = 0.1696 + 0.082z^{-1} + 0.1696z^{-2}$, and $A(z) = 1 - 0.9887z^{-1} + 0.5837z^{-2}$.
You are given that $\frac{1}{2\pi}\int_0^{2\pi} |H(e^{j\omega})|^2 d\omega = 0.2508$ and $\frac{1}{2\pi}\int_0^{2\pi} |A(e^{j\omega})|^{-2} d\omega = 2.4854$.

i) Determine the poles and zeros of $H(z)$ and express them in polar form to 3 decimal places. Explain the relationship between the positions of the poles and zeros on the complex plane and the shapes of the magnitude responses $|B(e^{j\omega})|$, $|A(e^{j\omega})|^{-1}$ and $|H(e^{j\omega})|$. [4]

ii) When $H(z)$ is implemented as in Figure 3.1 on a fixed point processor, the effect of truncation error is to add zero-mean white noise of variance σ^2 at the input $u[n]$. Explain why the variance of the resultant noise at $y[n]$ will equal $2.4854\sigma^2$. [4]

d) If the filter, $H(z)$, is instead implemented as in Figure 3.2, the truncation error introduces uncorrelated zero-mean white noise signals of variance σ^2 at both $u[n]$ and $v[n]$.

i) Determine the value of the smallest constant c in Figure 3.2 that will ensure that the magnitude gain, $\left|\frac{W(e^{j\omega})}{X(e^{j\omega})}\right|$, is ≤ 1 at all frequencies. [4]

ii) Using the value of c calculated in the previous part, determine the variance of the total noise at $y[n]$ due to $u[n]$ and $v[n]$. [3]

e) Using the appropriate z-plane transformation from the datasheet, determine the transformation needed to convert $H(z)$ into a highpass filter, $\widehat{H}(\hat{z})$, with a cutoff frequency of $\omega = 1.5$. Give the value of the transformation parameter λ to 3 decimal places and give an expression for $\widehat{H}(\hat{z})$. It is not necessary to calculate the numerical values of the coefficients of $\widehat{H}(\hat{z})$. [3]

4. Figure 4.1 shows the block diagram of a sample rate converter. The input signal, $x[n]$, has a bandwidth of 4 kHz with a sample rate of 21 kHz while the output signal, $y[m]$, has a sample rate of 12 kHz. The causal lowpass FIR filter, $H(z)$, has coefficients $h[0], \cdots, h[M]$ where $M = 59$.

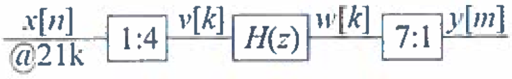

Figure 4.1

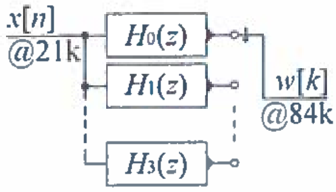

Figure 4.2

a) i) Explain the purpose of the lowpass filter, $H(z)$. Giving reasons for your choices, determine appropriate values in radians per sample for its passband edge and transition width. [4]

ii) Assume that the order of $H(z)$ is given by $M = \frac{a}{3.5\Delta\omega}$ where a is the stopband attenuation in dB and $\Delta\omega$ is the transition width in radians/sample. Given that $M = 59$, determine the stopband attenuation, a. [2]

iii) Suppose that the signal to noise ratio of the input signal, $x[n]$, is 60 dB and that the noise spectrum is white. Stating any assumptions that you make, estimate the signal to noise ratio of the output signal, $y[n]$. [5]

iv) Estimate the number of mutiplications per second required for a direct implementation of Figure 4.1. [2]

b) i) Prove that, if k is written as $k = 4r + p$ with $0 \leq p \leq 3$, then $w[k] = \sum_{s=0}^{(M-3)/4} h[4s+p]x[r-s]$. [5]

ii) If the first two blocks in Figure 4.1 are replaced by the polyphase implementation shown in Figure 4.2, determine the number of coefficients in each of the subfilters and give an expression for the coefficients, $h_3[n]$, of $H_3(z)$ in terms of the coefficients, $h[k]$, of $H(z)$. [3]

c) i) Explain how, by reordering the subfilters and changing the commutator frequency in Figure 4.2 it is possible to eliminate the downsampler and to generate the samples, $y[m]$, directly. Explaining your answer fully, give the revised order of the subfilters. [4]

ii) Estimate the number of multiplications per second required for the resultant system. [2]

iii) Determine which of the input samples, $x[n]$, are used to calculate the output sample $y[90]$ and identify which subfilter, $H_p(z)$, is used. [3]

Datasheet:

Standard Sequences

- $\delta[n] = 1$ for $n = 0$ and 0 otherwise.
- $\delta_{\text{condition}}[n] = 1$ whenever "condition" is true and 0 otherwise.
- $u[n] = 1$ for $n \geq 0$ and 0 otherwise.

Geometric Progression

- $\sum_{n=0}^{r} \alpha^n z^{-n} = \frac{1-\alpha^{r+1}z^{-r-1}}{1-\alpha z^{-1}}$ provided that $\alpha z^{-1} \neq 1$.
- $\sum_{n=0}^{\infty} \alpha^n z^{-n} = \frac{1}{1-\alpha z^{-1}}$ provided that $\left|\alpha z^{-1}\right| < 1$.

Forward and Inverse Transforms

z:	$X(z) = \sum_{-\infty}^{\infty} x[n]z^{-n}$	$x[n] = \frac{1}{2\pi j}\oint X(z)z^{n-1}dz$
CTFT:	$X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt$	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)e^{j\Omega t}d\Omega$
DTFT:	$X(e^{j\omega}) = \sum_{-\infty}^{\infty} x[n]e^{-j\omega n}$	$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$
DFT:	$X[k] = \sum_{0}^{N-1} x[n]e^{-j2\pi\frac{kn}{N}}$	$x[n] = \frac{1}{N}\sum_{0}^{N-1} X[k]e^{j2\pi\frac{kn}{N}}$
DCT:	$X[k] = \sum_{n=0}^{N-1} x[n]\cos\frac{2\pi(2n+1)k}{4N}$	$x[n] = \frac{X[0]}{N} + \frac{2}{N}\sum_{n=1}^{N-1} X[k]\cos\frac{2\pi(2n+1)k}{4N}$
MDCT:	$X[k] = \sum_{n=0}^{2N-1} x[n]\cos\frac{2\pi(2n+1+N)(2k+1)}{8N}$	$y[n] = \frac{1}{N}\sum_{0}^{N-1} X[k]\cos\frac{2\pi(2n+1+N)(2k+1)}{8N}$

Convolution

DTFT: $v[n] = x[n] * y[n] \triangleq \sum_{r=-\infty}^{\infty} x[r]y[n-r] \Leftrightarrow V\left(e^{j\omega}\right) = X\left(e^{j\omega}\right)Y\left(e^{j\omega}\right)$

$v[n] = x[n]y[n] \Leftrightarrow V\left(e^{j\omega}\right) = \frac{1}{2\pi}X\left(e^{j\omega}\right) \circledast Y\left(e^{j\omega}\right) \triangleq \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j\theta}\right)Y\left(e^{j(\omega-\theta)}\right)d\theta$

DFT: $v[n] = x[n] \circledast_N y[n] \triangleq \sum_{r=0}^{N-1} x[r]y[(n-r)_{\bmod N}] \Leftrightarrow V[k] = X[k]Y[k]$

$v[n] = x[n]y[n] \Leftrightarrow V[k] = \frac{1}{N}X[k] \circledast_N Y[k] \triangleq \frac{1}{N}\sum_{r=0}^{N-1} X[r]Y[(k-r)_{\bmod N}]$

Group Delay

The group delay of a filter, $H(z)$, is $\tau_H(e^{j\omega}) = -\frac{d\angle H(e^{j\omega})}{d\omega} = \Re\left(\frac{-z}{H(z)}\frac{dH(z)}{dz}\right)\Big|_{z=e^{j\omega}} = \Re\left(\frac{\mathscr{F}(nh[n])}{\mathscr{F}(h[n])}\right)$ where $\mathscr{F}()$ denotes the DTFT.

Order Estimation for FIR Filters

Three increasingly sophisticated formulae for estimating the minimum order of an FIR filter with unity gain passbands:

1. $M \approx \frac{a}{3.5\Delta\omega}$
2. $M \approx \frac{a-8}{2.2\Delta\omega}$
3. $M \approx \frac{a-1.2-20\log_{10} b}{4.6\Delta\omega}$

where a =stop band attenuation in dB, b = peak-to-peak passband ripple in dB and $\Delta\omega$ = width of smallest transition band in radians per sample.

z-plane Transformations

A lowpass filter, $H(z)$, with cutoff frequency ω_0 may be transformed into the filter $H(\hat{z})$ as follows:

Target $H(\hat{z})$	Substitute	Parameters
Lowpass $\hat{\omega} < \hat{\omega}_1$	$z^{-1} = \frac{\hat{z}^{-1}-\lambda}{1-\lambda\hat{z}^{-1}}$	$\lambda = \frac{\sin\left(\frac{\omega_0-\hat{\omega}_1}{2}\right)}{\sin\left(\frac{\omega_0+\hat{\omega}_1}{2}\right)}$
Highpass $\hat{\omega} > \hat{\omega}_1$	$z^{-1} = -\frac{\hat{z}^{-1}+\lambda}{1+\lambda\hat{z}^{-1}}$	$\lambda = \frac{\cos\left(\frac{\omega_0+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\omega_0-\hat{\omega}_1}{2}\right)}$
Bandpass $\hat{\omega}_1 < \hat{\omega} < \hat{\omega}_2$	$z^{-1} = -\frac{(\rho-1)-2\lambda\rho\hat{z}^{-1}+(\rho+1)\hat{z}^{-2}}{(\rho+1)-2\lambda\rho\hat{z}^{-1}+(\rho-1)\hat{z}^{-2}}$	$\lambda = \frac{\cos\left(\frac{\hat{\omega}_2+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)}$, $\rho = \cot\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)\tan\left(\frac{\omega_0}{2}\right)$
Bandstop $\hat{\omega}_1 \not< \hat{\omega} \not< \hat{\omega}_2$	$z^{-1} = \frac{(1-\rho)-2\lambda\hat{z}^{-1}+(\rho+1)\hat{z}^{-2}}{(\rho+1)-2\lambda\hat{z}^{-1}+(1-\rho)\hat{z}^{-2}}$	$\lambda = \frac{\cos\left(\frac{\hat{\omega}_2+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)}$, $\rho = \tan\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)\tan\left(\frac{\omega_0}{2}\right)$

Noble Identities

$$-\boxed{Q{:}1}-\boxed{H(z)}- \quad = \quad -\boxed{H(z^Q)}-\boxed{Q{:}1}-$$

$$-\boxed{H(z)}-\boxed{1{:}Q}- \quad = \quad -\boxed{1{:}Q}-\boxed{H(z^Q)}-$$

Multirate Spectra

Upsample: $\xrightarrow{v[n]}\boxed{1{:}Q}\xrightarrow{x[r]}$ $\Rightarrow$ $x[r] = \begin{cases} v\left[\frac{r}{Q}\right] & \text{if } Q \mid r \\ 0 & \text{if } Q \nmid r \end{cases}$ $\Rightarrow$ $X(z) = V(z^Q)$

Downsample: $\xrightarrow{v[n]}\boxed{Q{:}1}\xrightarrow{y[m]}$ $\Rightarrow$ $y[m] = v[Qm]$ $\Rightarrow$ $Y(z) = \frac{1}{Q}\sum_{k=0}^{Q-1} V\left(e^{\frac{-j2\pi k}{Q}} z^{\frac{1}{Q}}\right)$

Multirate Commutators