

Please write on this side only, legibly and neatly, between the margins

$$f(x, y) = y^4 + 4xy^2 - 2y^2 + 2x^2 - 1$$

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0 \Rightarrow$$

$$4x(2y^2 + 1) = 0$$

$$4y(y^2 + 2x^2 - 1) = 0$$

From 1st equation $x = 0$. Sub into

2nd eqn $\Rightarrow y = 0, y = \pm 1$. Hence

stationary pts are $(0, 0), (0, 1), (0, -1)$

To determine nature, require

$$f_{xx} = 4(2y^2 + 1)$$

$$f_{xy} = 16xy$$

$$f_{yy} = 12y^2 + 2x^2 - 4$$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} \text{ etc}$$

$$(0, 0): f_{xx}(0, 0) = 4, f_{xy}(0, 0) = 0, f_{yy}(0, 0) = -4$$

$$\Delta = f_{xy}^2(0, 0) - f_{xx}(0, 0)f_{yy}(0, 0) = 16 > 0$$

\therefore saddle.

$$(0, \pm 1): f_{xx}(0, \pm 1) = 12, f_{xy}(0, \pm 1) = 0$$

$$f_{yy}(0, \pm 1) = 8$$

For either pt

$$\Delta = f_{xy}^2(0, \pm 1) - f_{xx}(0, \pm 1)f_{yy}(0, \pm 1) = -96 < 0$$

Since f_{xx} and $f_{yy} > 0$ for both pts, each

is a minimum

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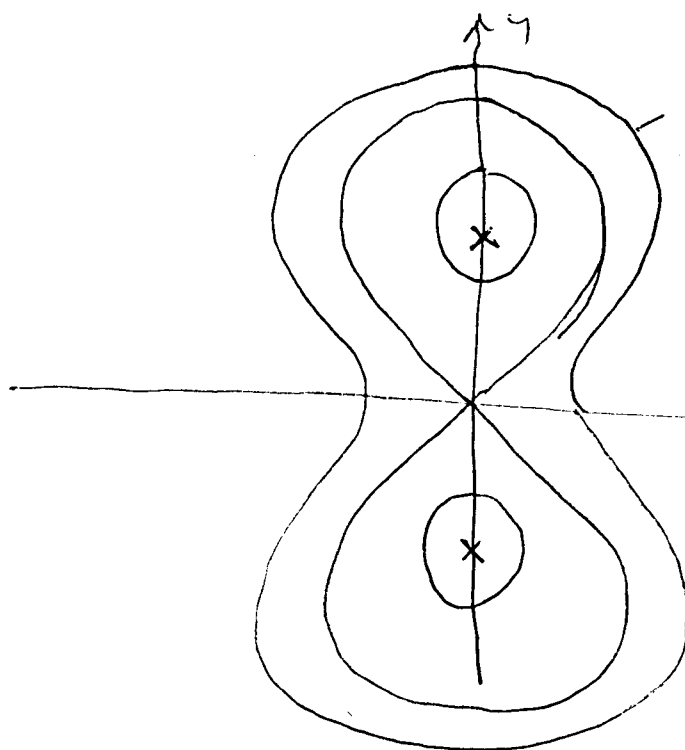
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$$f(0,0) = -1, \quad f(0, \pm 1) = -2$$

$$\text{On } y = 0, \quad f(x,0) = 2x^2 - 1$$

$$\begin{aligned} \text{On } x = 0, \quad f(0,y) &= y^4 - 2y^2 - 1 \\ &= (y^2 - 1)^2 - 2 \end{aligned}$$

Hence, contour plot looks like



$f(x,y) = \text{const}$
increasing away
from $(0, \pm 1)$

$$f(x,y) = f(\pm x, \pm y)$$

any combination

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$$x = r \cosh \theta, \quad y = r \sinh \theta$$

$$\Rightarrow \sqrt{x^2 - y^2} = r, \quad \tanh^{-1}(y/x) = \theta$$

$$\text{Then, } \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = -\frac{y}{r}$$

$$\frac{\partial \theta}{\partial x} = -\frac{y}{r^2}, \quad \frac{\partial \theta}{\partial y} = \frac{x}{r^2}$$

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial r}{\partial x} \frac{\partial f}{\partial r} + \frac{\partial \theta}{\partial x} \frac{\partial f}{\partial \theta} \\ &= \frac{x}{r} \frac{\partial f}{\partial r} - \frac{y}{r^2} \frac{\partial f}{\partial \theta} \\ &= \cosh \theta \frac{\partial f}{\partial r} - \frac{\sinh \theta}{r} \frac{\partial f}{\partial \theta} \end{aligned}$$

Similarly,

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial r}{\partial y} \frac{\partial f}{\partial r} + \frac{\partial \theta}{\partial y} \frac{\partial f}{\partial \theta} \\ &= -\frac{y}{r} \frac{\partial f}{\partial r} + \frac{x}{r^2} \frac{\partial f}{\partial \theta} \\ &= -\sinh \theta \frac{\partial f}{\partial r} + \frac{\cosh \theta}{r} \frac{\partial f}{\partial \theta} \end{aligned}$$

$$\text{ii) With } f = \sqrt{x^2 - y^2} \tanh^{-1}(y/x) = r\theta$$

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{x}{\sqrt{x^2 - y^2}} \tanh^{-1} y/x + \sqrt{x^2 - y^2} \left(-\frac{y}{x^2 + y^2} \right) \\ &= \frac{x}{\sqrt{x^2 - y^2}} \tanh^{-1} y/x - \frac{y}{\sqrt{x^2 - y^2}} \end{aligned}$$

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$$= 0 \cosh \theta - \sinh \theta$$

$$\text{Also, } \frac{\partial f}{\partial \theta} = r, \quad \frac{\partial f}{\partial r} = 0$$

$$\therefore \cosh \theta \frac{\partial f}{\partial \theta} - \frac{\sinh \theta}{r} \frac{\partial f}{\partial \theta} = 0 \cosh \theta - \sinh \theta$$

$$\therefore \frac{\partial f}{\partial x} = \cosh \theta \frac{\partial f}{\partial \theta} - \frac{\sinh \theta}{r} \frac{\partial f}{\partial \theta}$$

$$\text{Similarly, } \frac{\partial f}{\partial y} = \frac{-y}{\sqrt{x^2 - y^2}} \text{Tanh}^{-1} \frac{y}{x} + \frac{\sqrt{x^2 - y^2}}{x^2 - y^2} x$$

$$= \frac{-y}{\sqrt{x^2 - y^2}} \text{Tanh}^{-1} \left(\frac{y}{x} \right) + \frac{x}{\sqrt{x^2 - y^2}}$$

$$= -0 \sinh \theta + \cosh \theta$$

$$\text{Also, } -\sinh \theta \frac{\partial f}{\partial r} + \frac{\cosh \theta}{r} \frac{\partial f}{\partial \theta}$$

$$= -0 \sinh \theta + \frac{\cosh \theta}{r} \cdot r$$

$$= \cosh \theta - 0 \sinh \theta$$

$$\text{ie, } \frac{\partial f}{\partial y} = -\sinh \theta \frac{\partial f}{\partial r} + \frac{\cosh \theta}{r} \frac{\partial f}{\partial \theta}$$

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(15)

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$$I_2 = \frac{\pi/4 - 0}{2} \left[0 + \pi/4 \sin \pi/4 \right]$$

$$= \pi/8 \left[0 + \pi/4 \sin \pi/4 \right] = 0.2180$$

(3)

$$I_3 = \frac{\pi/8 - 0}{2} \left[0 + \pi/8 \sin \pi/8 \right] + \frac{\pi/8}{2} \left[\frac{\pi}{8} \sin \pi/8 + \frac{\pi}{4} \sin \pi/4 \right]$$

$$= \frac{\pi}{16} \left[2 \times 0.1503 + 0.5554 \right] = 0.1680$$

(3)

If we were to use Simpson's Rule immediately

$$I = \frac{\pi}{24} \left[f(0) + 4f(\pi/8) + f(\pi/4) \right]$$

$$= \frac{\pi}{24} \left[0 + 4 \times (0.1503) + 0.5554 \right] = 0.1513$$

(3)

If we extrapolate the Trapezium Rule we have

$$I_n = (4I_3 - I_2)/3 = 0.1513$$

(3)

This agrees with the Simpson Rule approximation

To find the true solution we integrate by parts

$$I_{\text{true}} = - \left[\theta \cos \theta \right]_0^{\pi/4} + \int_0^{\pi/4} \cos \theta \, d\theta$$

$$= - \left[\theta \cos \theta \right]_0^{\pi/4} + \left[\sin \theta \right]_0^{\pi/4}$$

$$= (1 - \pi/4) \cdot 1/2 = 0.1517$$

(3)

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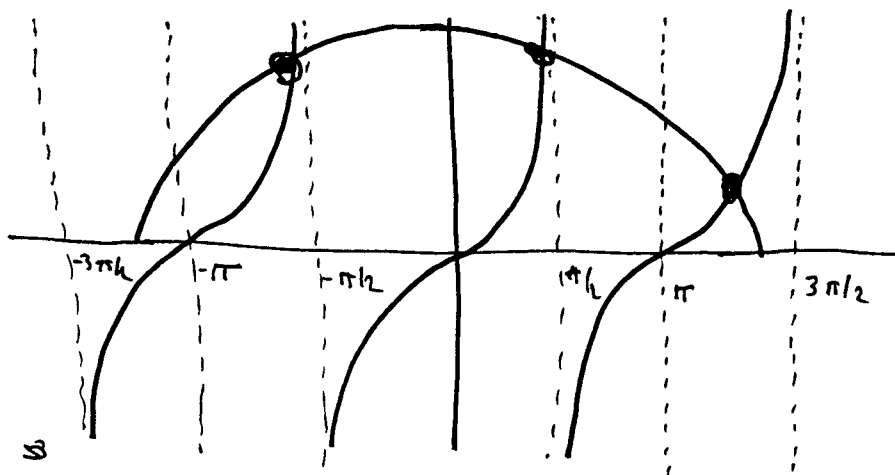
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QUESTION NO.

SOLUTION NO.

E 4

MARKSCHEME



Rewrite equation as

$$0 = f(x) = \tan x - \sqrt{a^2 - x^2}$$

Then $f'(x) = \sec^2 x + \frac{x}{\sqrt{a^2 - x^2}}$.

So Newton-Raphson is

$$x_{n+1} = x_n - \frac{(\tan x_n - \sqrt{16 - x_n^2})}{\sec^2 x_n + \frac{x_n}{\sqrt{16 - x_n^2}}}$$

Iterates are $x_0 = 1.3$, $x_1 = 1.31262$, $x_2 = 1.312086$, $x_3 = 1.31208 = x_4$

$x_0 = 3.8$, $x_1 = 3.80245$, $x_2 = 3.89079$, $x_3 = 3.89049 = x_4$

(4)

(3)

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Std equation of plane is

$$\vec{n} \cdot \vec{r} = n_1x + n_2y + n_3z = p$$

where \vec{n} is unit vector \perp to plane,
 p is perpendicular distance from origin.

i) Plane $x + y - 2z = 3$ can be
 written

$$\frac{x}{(1^2+1^2+(-2)^2)} + \frac{y}{1^2+1^2+(-2)^2} + \frac{-2z}{1^2+1^2+(-2)^2} = \frac{3}{1^2+1^2+(-2)^2}$$

$$\therefore \vec{n}_1 \cdot \vec{r} = \frac{3}{\sqrt{6}} = \sqrt{\frac{3}{2}} = p_1$$

2

Plane $2x - 2y + z = 1$ can
 be written

$$\vec{n}_2 \cdot \vec{r} = \frac{1}{\sqrt{9}} = \frac{1}{3} = p_2$$

2

$$\text{Hence } p_1 = \sqrt{\frac{3}{2}}, \quad p_2 = \frac{1}{3}.$$

ii) let direction cosine of req'd st. line
 be (l, m, n) . Since this lies in both
 planes, it must be \perp to \vec{n}_1 & $\vec{n}_2 \Rightarrow$

$$\begin{aligned} l + m - 2n &= 0 \\ 2l - 2m + n &= 0 \end{aligned}$$

$$\text{leading to } \frac{l}{3} = \frac{m}{5} = \frac{n}{4}$$

The direction ratios are therefore $(3, 5, 4)$

6

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and hence

$$\underline{t} = \frac{1}{\sqrt{50}} [3, 5, 4]$$

is a unit vector along req'd line.

Pt. on line : This will also lie on both planes. Put $x=0$ in original eqns for planes \Rightarrow

$$\begin{aligned} y - 2z &= 3 \\ -2y + z &= 1 \end{aligned} \Rightarrow \begin{aligned} y &= -5/3 \\ z &= -7/3 \end{aligned}$$

Req'd vector to this pt on line is

$$\underline{a} = [0, -5/3, -7/3]$$

Hence, req'd vector equation for line is

$$\underline{r} = \underline{a} + \lambda \underline{t}, \text{ parameter } \lambda$$

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$$(i) A^2 = \begin{pmatrix} a^2 + bc & b(a+d) \\ c(a+d) & d^2 + bc \end{pmatrix}$$

$$\text{So } A^2 = I \text{ is equivalent to } \left. \begin{aligned} a^2 + bc &= d^2 + bc = 1 \\ b(a+d) &= c(a+d) = 0 \end{aligned} \right\} (*)$$

(a) If $b = c = 0$, (*) reduces to $a^2 = d^2 = 1$, giving the four solutions

$$A = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$$

(b) If neither b nor c is 0, (*) becomes $d = -a$, $1 - a^2 = bc \neq 0$, giving the solutions

$$A = \begin{pmatrix} a & b \\ (1-a^2)/b & -a \end{pmatrix},$$

$a \neq \pm 1$, b arbitrary $\neq 0$.

(c) If $b \neq 0$, $c = 0$, (*) becomes

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$d = -a$, $a^2 = 1$, giving solutions

$$A = \begin{pmatrix} 1 & b \\ 0 & -1 \end{pmatrix}, A = \begin{pmatrix} -1 & b \\ 0 & 1 \end{pmatrix}, b \text{ arbitrary.}$$

Similarly if $b = 0$, $c \neq 0$ we get

$$A = \begin{pmatrix} 1 & 0 \\ c & -1 \end{pmatrix}, A = \begin{pmatrix} -1 & 0 \\ c & 1 \end{pmatrix}, c \text{ arbitrary}$$

(ii) (a) Not valid, because $(A+B)(A-B)$

$$= A^2 + BA - AB - B^2, \text{ and in general}$$

$$BA \neq AB.$$

(b) Valid, because $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$

$$= AIA^{-1} = AA^{-1} = I, \text{ and similarly}$$

$$(B^{-1}A^{-1})(AB) = I.$$

(c) Not valid even for scalars: $A = B = I$

is a counterexample.

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Denoting the missing elements as l_{ij} and U_{ij} we have.

$$U_{11} = 1, U_{12} = 1, U_{22} = 1$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & \cdot & \cdot \\ 0 & 0 & \cdot \end{pmatrix}$$

$$\text{So } l_{21} = 1, 1 + U_{22} = a \Rightarrow U_{22} = a - 1, 1 + U_{23} = 2 \Rightarrow U_{23} = 1$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & a-1 & 1 \\ 0 & 0 & \cdot \end{pmatrix}$$

$$\text{Finally } l_{31} = 1, l_{31} + (a-1)l_{32} = a \Rightarrow l_{32} = 1 \quad 2 + U_{33} = 2a \Rightarrow U_{33} = 2a - 2$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & a-1 & 1 \\ 0 & 0 & 2a-2 \end{pmatrix}$$

$$\text{So we need to solve } LU \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ b \end{pmatrix}$$

$$\text{Put } Ux = y \quad \text{So } \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ b \end{pmatrix}; y = \begin{pmatrix} 0 \\ 0 \\ b \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & a-1 & 1 \\ 0 & 0 & 2a-2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ b \end{pmatrix}$$

$$x_3 = b / (2a-2)$$

$$(a-1)x_2 + b / (2(a-1)) = 0$$

$$\therefore x_2 = -b / (2(a-1)^2)$$

$$x_1 - b / (2(a-1)^2) + b / (2(a-1)) = 0$$

$$x_1 = \frac{b}{2(a-1)} \left[\frac{1}{a-1} - 1 \right] = \frac{b(2-a)}{2(a-1)^2}$$

If $a=1, b \neq 0$ No solution

$$\text{If } a=1, b=0 \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore x_3 = 0$ infinitely many solutions.
 $x_1 = -x_2$

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$$(i) \quad (y^2 - x^2) \frac{dy}{dx} = x^2 + 2xy \rightarrow \frac{dy}{dx} = \frac{x^2 + 2xy}{(y^2 - x^2)} = \frac{1 + 2\frac{y}{x}}{\left(\frac{y}{x}\right)^2 - 1}$$

$$\therefore \text{ Put } y(x) = xv(x) \rightarrow x \frac{dv}{dx} + v = \frac{1 + 2v}{v^2 - 1}$$

$$\therefore x \frac{dv}{dx} = \frac{1 + 2v}{v^2 - 1} - v = \frac{1 + 2v - v^3 + v}{v^2 - 1} = \frac{1 + 3v - v^3}{v^2 - 1}$$

$$\therefore \int \frac{dx}{x} = \int \frac{v^2 - 1}{1 + 3v - v^3} dv \quad \therefore \ln x = -\frac{1}{3} \ln(1 + 3v - v^3) + c_1$$

$$\rightarrow \ln(1 + 3v - v^3)x^3 = c_2. \quad \therefore x^3 + 3x^2y - y^3 = c_3.$$

$$\text{Given that } y(1) = 2, \quad c_3 = 1 + 6 - 8 = -1.$$

$$\therefore \text{ Required solution is : } y^3 - x^3 - 3x^2y = 1$$

$$(ii) \quad (x+1) \frac{dy}{dx} - 3y = (x+1)^5 \rightarrow \frac{dy}{dx} - \frac{3}{(x+1)}y = (x+1)^4.$$

$$\text{Integrating factor : } \exp\left(-\int \frac{3dx}{(x+1)}\right) = \exp(-3 \ln(x+1)) = (x+1)^{-3}.$$

$$\therefore \text{ ODE } \rightarrow \frac{d((x+1)^{-3}y)}{dx} = x+1. \quad \therefore (x+1)^{-3}y = \frac{1}{2}(x+1)^2 + c$$

$$\therefore \text{ The general solution is } y(x) = \frac{1}{2}(x+1)^5 + c(x+1)^3 \\ = (x+1)^3\left(\frac{1}{2}x^2 + x + c_1\right)$$

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(i) The auxiliary equation is $m^2 + m - 2 = 0$.

Hence $m_1 = 1$, $m_2 = -2$. The general solution of the homogeneous equation is $ae^x + be^{-2x}$.

3 mark

To find a solution of the inhomogeneous equation try $y = \alpha x + \beta$. Then we get

$$\alpha - 2\alpha x - 2\beta = x. \text{ Whence } \alpha = -\frac{1}{2}, \beta = -\frac{1}{4}.$$

3 marks

The answer is $ae^x + be^{-2x} - \frac{1}{2}x - \frac{1}{4}$.

(ii) The auxiliary equation is $m^2 + 2m = 0$.

The general solution of the homogeneous equation is thus $a + be^{-2x}$. Now try

3 marks

$y(x) = \alpha \cos x + \beta \sin x$. We obtain

$$-\alpha \cos x - \beta \sin x + 2\alpha \sin x + 2\beta \cos x = \cos x.$$

Hence $-\alpha + 2\beta = 1$, $\beta + 2\alpha = 0$. We get

$$\alpha = -\frac{1}{5}, \beta = \frac{2}{5}. \text{ The general solution is}$$

3 marks

$$a + be^{-2x} - \frac{1}{5} \cos x + \frac{2}{5} \sin x.$$

The initial condition says that

$$a + b - \frac{1}{5} = 1, \quad -2b + \frac{2}{5} = 0, \text{ whence } b = \frac{1}{5},$$

$\alpha = 1$. The answer is

3 marks

$$1 + \frac{1}{5}e^{-2x} - \frac{1}{5} \cos x + \frac{2}{5} \sin x.$$

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a) Identify as Fourier Cosine series $\therefore b_n = 0$

$$\begin{aligned} \therefore a_n &= \frac{2}{\pi} \int_0^{\pi} x \cos nx \, dx = \frac{2}{\pi} \left(\frac{x \sin nx}{n} \Big|_0^{\pi} - \frac{1}{n} \int_0^{\pi} \sin nx \, dx \right) \\ &= \frac{2}{\pi} \left[\frac{-1 + (-1)^n}{n^2} \right] = \begin{cases} \frac{4}{\pi n^2} & , n \text{ odd} \\ 0 & , n \text{ even.} \end{cases} \end{aligned}$$

$$a_0 = \frac{2}{\pi} \frac{\pi^2}{2} = \pi$$

$$\therefore x = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n, \text{ odd}} \frac{\cos nx}{n^2}, \quad 0 \leq x < \pi$$

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b) Fourier Sine series $\therefore a_n = 0$

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^{\pi} x \sin nx \, dx = \frac{2}{\pi} \left(-\frac{x \cos nx}{n} \Big|_0^{\pi} + \frac{1}{n} \int_0^{\pi} \cos nx \, dx \right) \\ &= -\frac{2}{\pi} \frac{\pi (-1)^n}{n} \end{aligned}$$

$$\therefore \text{for } x = -2 \sum_{n=1}^{\infty} \frac{(-1)^n \sin nx}{n}, \quad 0 \leq x < \pi$$

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In a), even f_n is continuous at $x=0$

Put $x=0$

$$c) \Rightarrow 0 = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n, \text{ odd}} \frac{1}{n^2}$$

$$\therefore \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$$

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MATHEMATICS DEPARTMENT

MATHEMATICAL FORMULAE

1. VECTOR ALGEBRA

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1, a_2, a_3)$$

Scalar (dot) product:

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

Vector (cross) product:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Scalar triple product:

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = \mathbf{b} \cdot \mathbf{c} \times \mathbf{a} = \mathbf{c} \cdot \mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c} \cdot \mathbf{a})\mathbf{b} - (\mathbf{b} \cdot \mathbf{a})\mathbf{c}$$

2. SERIES

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \dots \quad (\alpha \text{ arbitrary, } |x| < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^n \frac{x^{2n}}{(2n)!} + \dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \dots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^n \frac{x^{n+1}}{(n+1)} + \dots \quad (-1 < x \leq 1)$$

3. TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin(a+b) = \sin a \cos b + \cos a \sin b;$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b.$$

$$\cos iz = \cosh z; \quad \cosh iz = \cos z; \quad \sin iz = i \sinh z; \quad \sinh iz = i \sin z.$$

4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f D^n g + \binom{n}{1} Df D^{n-1} g + \dots + \binom{n}{n-1} D^{n-1} f Dg + \dots + D^n f g.$$

(b) Taylor's expansion of $f(x)$ about $x = a$:

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \dots + \frac{h^n}{n!}f^{(n)}(a) + \epsilon_n(h),$$

where $\epsilon_n(h) = h^{n+1}f^{(n+1)}(a + \theta h)/(n+1)!$, $0 < \theta < 1$.

(c) Taylor's expansion of $f(x, y)$ about (a, b) :

$$f(a+h, b+k) = f(a, b) + [hf_x + kf_y]_{a,b} + 1/2! [h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}]_{a,b} + \dots$$

(d) Partial differentiation of $f(x, y)$:

$$\text{i. If } y = y(x), \text{ then } f = F(x), \text{ and } \frac{dF}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx}.$$

$$\text{ii. If } x = x(t), y = y(t), \text{ then } f = F(t), \text{ and } \frac{dF}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}.$$

$$\text{iii. If } x = x(u, v), y = y(u, v), \text{ then } f = F(u, v), \text{ and}$$

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u}, \quad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial v}.$$

(e) Stationary points of $f(x, y)$ occur where $f_x = 0, f_y = 0$ simultaneously.

Let (a, b) be a stationary point: examine $D = [f_{xx}f_{yy} - (f_{xy})^2]_{a,b}$.

If $D > 0$ and $f_{xx}(a, b) < 0$, then (a, b) is a maximum;

If $D > 0$ and $f_{xx}(a, b) > 0$, then (a, b) is a minimum;

If $D < 0$ then (a, b) is a saddle-point.

(f) Differential equations:

i. The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating factor $I(x) = \exp[\int P(x)(dx)]$, so that $\frac{d}{dx}(Iy) = IQ$.

ii. $P(x, y)dx + Q(x, y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y$.