E4.25
C4.1
ISE4.23

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2005

MSc and EEE/ISE PART IV: MEng and ACGI

Corrected Copy

DESIGN OF LINEAR MULTIVARIABLE CONTROL SYSTEMS

Tuesday, 17 May 10:00 am

Time allowed: 3:00 hours

There are SIX questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s) :	I.M. Jaimoukha
	Second Marker(s) :	D.J.N. Limebeer

Special Information for Invigilators :	None
Information for Candidates :	None

1. (a) Let

$$G(s) = \frac{1}{s-1} \begin{bmatrix} 1 & s+2 \\ s+3 & 12 \end{bmatrix}$$

(i) Find the McMillan form of $G(s)$. **[4]**

(ii) Determine the pole and zero polynomials of $G(s)$. **[2]**

(iii) Find the poles and zeros of $G(s)$, specifying the multiplicity of each. **[2]**

(b) Consider a state–variable model described by the dynamics

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t). \end{aligned}$$

(i) Suppose that there exists $P = P' > 0$ such that

$$AP + PA' + BB' < 0.$$

Prove that A is stable. **[6]**

(ii) Suppose that there exists $Q = Q' > 0$ such that

$$A'Q + QA + C'C < 0.$$

Prove that the pair (A, C) is observable. **[6]**

2. (a) Define internal stability for the feedback loop shown in Figure 2 below, and derive necessary and sufficient conditions for which this feedback loop is internally stable.

[4]

(b) Suppose that the transfer matrix $G(s)$ in the feedback loop in Figure 2 is stable. Derive a parameterization of all internally stabilizing controllers for the feedback loop.

[4]

(c) Suppose that

$$G(s) = \frac{s-1}{s+2}.$$

Let $C(s)$ denote the transfer matrix from the reference signal $r(s)$ to the output signal $y(s)$ in Figure 2.

(i) Show that there does not exist an internally stabilizing controller $K(s)$ such that $C(s)$ is minimum–phase.

[6]

(ii) Design an internally stabilising controller $K(s)$ such that $C(s)$ is allpass (that is, $|C(j\omega)| = 1$ for all real ω).

[6]

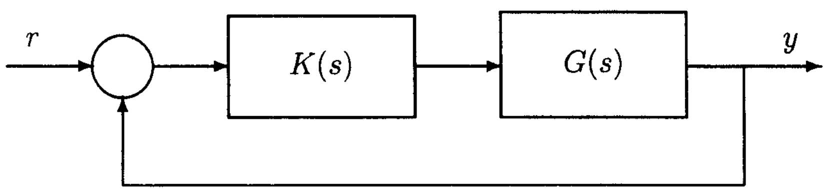


Figure 2

3. Figure 3 illustrates the implementation of the control law $u = -Kx$ which minimises

$$J(x_0, u) = \int_0^\infty \|Cx(t)\|^2 + \|u(t)\|^2 \, dt$$

subject to the nominal dynamics $\dot{x} = Ax(t) + Bu(t)$, $x(0) = x_0$. Here $K = B'P$ and $P = P'$ is the unique positive definite solution of $A'P + PA - PBB'P + C'C = 0$. Assume that the triple (A, B, C) is minimal. Let $G(s) = (sI - A)^{-1}B$.

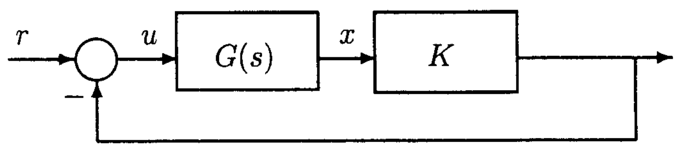


Figure 3

(a) Let $L(s) = I + KG(s)$. Show that

$$L(j\omega)'L(j\omega) = I + G(j\omega)'C'CG(j\omega).$$

[6]

(b) Suppose that the nominal model $G(s)$ is stable and that in the actual implementation of the loop, we use $K_a(s) = K + \Delta_1(s)$ where $\Delta_1(s)$ is a stable perturbation. Derive the maximal stability radius (using the $\mathcal{L}_\infty$-norm as a measure) for the feedback loop when K is replaced by $K_a(s)$. The stability radius should be given in terms of $\|G\|_\infty$. **[7]**

(c) Suppose that the nominal model $G(s)$ is stable and that the actual system is given by $G_a(s) = G(s)(I + \Delta_2(s))$ where $\Delta_2(s)$ is a stable perturbation. Derive the maximal stability radius (using the $\mathcal{L}_\infty$-norm as a measure) for the feedback loop when $G(s)$ is replaced by $G_a(s)$. **[7]**

4. (a) State the small gain theorem concerning the internal stability of a feedback loop having a forward transfer matrix Δ and a feedback transfer matrix S. [4]

(b) Consider the feedback loop shown in Figure 4.1 where $G(s)$ represents a plant model and $K(s)$ represents an internally stabilizing compensator. Suppose that

$$K(s) \stackrel{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{ccc|cc} -1 & -1 & 0 & 1 & 1 \\ -1 & -1.25 & 0.4 & 0.6 & 0.8 \\ 0 & 0.4 & -10 & 1 & -1 \\ \hline 1 & 0.6 & 1 & 0 & 0 \\ 1 & 0.8 & -1 & 0 & 0 \end{array}\right] \in \mathcal{RH}_\infty.$$

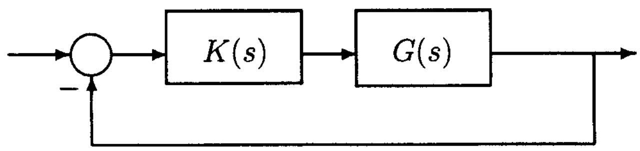


Figure 4.1

(i) Show that the given realization for $K(s)$ is balanced and evaluate the Hankel singular values of $K(s)$. [6]

(ii) The graph in Figure 4.2 shows the singular value plot of $(I + GK)^{-1}G$. Obtain the lowest order balanced truncation of $K(s)$ such that the loop in Figure 4.1 remains stable when $K(s)$ is replaced by its balanced truncation. [10]

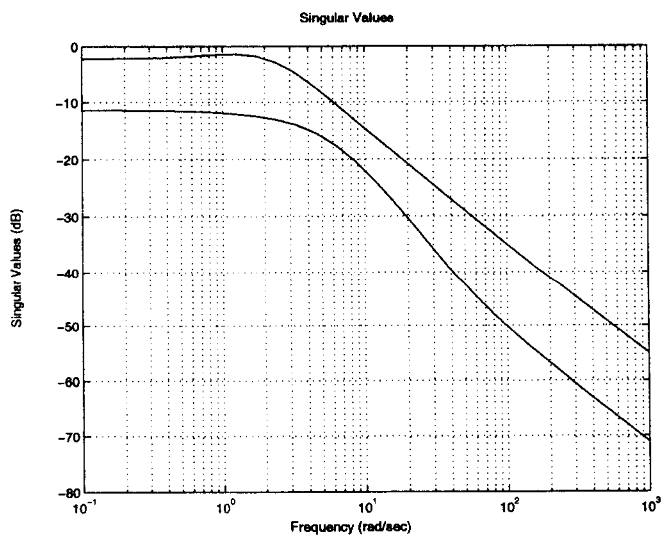


Figure 4.2

5. Consider the feedback configuration shown in Figure 5. Here, $G(s)$ represents a nominal plant model and $K(s)$ represents a compensator. The actual plant is given by $G_a(s) = (I + \Delta_2(s))(G(s) + \Delta_1(s))$ where $\Delta_1(s)$ and $\Delta_2(s)$ are stable transfer matrices that represent uncertainties. The design specification are to synthesize a compensator $K(s)$ such that the feedback loop is internally stable when:

(i) $\Delta_1 = 0$ and $||\Delta_2(j\omega)|| \leq |w_2(j\omega)|, \forall\omega$, and,

(ii) $\Delta_2 = 0$ and $||\Delta_1(j\omega)|| \leq |w_1(j\omega)|, \forall\omega$,

where

$$w_1(s) = 2\frac{(s+1)^2}{(s+5)^2}, \qquad w_2(s) = 10\frac{(s+10)^2}{(s+50)^2}.$$

(a) Derive conditions, in terms of $G(s), K(s), w_1(s)$ and $w_2(s)$ that are sufficient to achieve the design specifications.

[7]

(b) Derive a generalized regulator formulation of the design problem that captures the sufficient conditions in part (a).

[7]

(c) Assume that a compensator $K(s)$ achieves the design specifications. Comment on the performance properties (tracking, disturbance rejection, noise attenuation and control effort) for the resulting feedback loop.

[6]

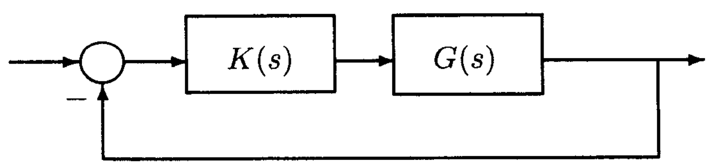


Figure 5

6. Consider the regulator shown in Figure 6 for which it is assumed that the triple (A, B, C) is minimal and $x(0) = 0$.

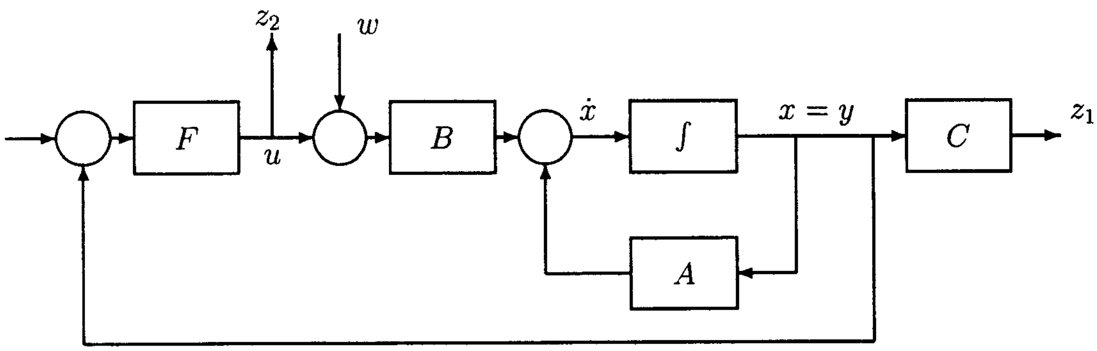


Figure 6

Let $z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ and let H denote the transfer matrix from w to z. A stabilizing state-feedback gain matrix F is to be designed such that, for given $\gamma > 0$, $\|H\|_\infty < \gamma$.

(a) Write down the generalized regulator system for this design problem. **[6]**

(b) By using the Lyapunov function $V(t) = x(t)^T X x(t)$, where X is to be determined, derive sufficient conditions for the solution of the design problem. Your conditions should be in the form of the existence of a certain solution to an algebraic Riccati inequality. It should also include an expression for F and an expression for the worst-case disturbance w.
Use the identity

$$(\alpha R - \alpha^{-1} S)^T (\alpha R - \alpha^{-1} S) = \alpha^2 R^T R + \alpha^{-2} S^T S - R^T S - S^T R,$$

for scalar $\alpha \neq 0$ and matrices R and S to complete the squares. **[9]**

(c) Suggest an algorithm for solving the algebraic Riccati inequality derived in part (b) using linear matrix inequality techniques. Ignore any issues associated with stability. **[5]**

[END]

E4.25
ISE 4.27
C4.1

Design of Linear Multivariable Control Systems

Solutions 2004/2005

1. (a) (i) By performing the following elementary operations:

(A) $r_2 \leftrightarrow r_2 - (s+3)r_1$

(B) $c_2 \leftrightarrow c_2 - (s+2)\, c_1$

(C) $c_2 \leftrightarrow -c_2$

the McMillan form of $G(s)$ is given by

$$\boxed{G(s) = \begin{bmatrix} 1 & 0 \\ s+3 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{s-1} & 0 \\ 0 & s+6 \end{bmatrix} \begin{bmatrix} 1 & s+2 \\ 0 & -1 \end{bmatrix} = L(s)M(s)R(s)}$$

where $L(s)$ and $R(s)$ are unimodular.

(ii) The pole and zero polynomials are given by

$$\boxed{p(s) = s - 1,} \qquad \boxed{z(s) = s + 6}$$

respectively.

(iii) It follows that the system has a $\boxed{\text{simple pole at } 1}$ and a $\boxed{\text{simple zeros at } -6}$.

(b) (i) Let $z' \neq 0$ be a left eigenvector of A and let λ be the corresponding eigenvalue. Then multiplying the controllability Lyapunov inequality from the left by z' and from the right by z gives $(\lambda + \bar{\lambda})z'Pz + z'BB'z < 0$. Since $P > 0$ it follows that $z'Pz > 0$ and since $z'BB'z \geq 0$ it follows that $\lambda + \bar{\lambda} < 0$ so that A is stable.

(ii) Let $z' \neq 0$ be a left eigenvector of A and let λ be the corresponding eigenvalue. Then multiplying the observability Lyapunov inequality from the left by z' and from the right by z gives $(\lambda + \bar{\lambda})z'Qz + z'C'Cz < 0$. Since $Q > 0$, a dual proof to that given above shows that A is stable, so that $\lambda + \bar{\lambda} < 0$. Since $Q > 0$ and $z \neq 0$, $z'Qz > 0$. Thus $z'C'Cz > 0$ and so $Cz \neq 0$. It follows that the pair (A, C) is controllable by the PBH test.

2. (a) Inject a signal d in between $G(s)$ and $K(s)$ and call the input to $G(s)$, u and the input to $K(s)$, e. The loop is internally stable if and only if the transfer matrix from $\begin{bmatrix} d \\ r \end{bmatrix}$ to $\begin{bmatrix} u \\ e \end{bmatrix}$ is stable (no poles in the closed right half plane). Since

$$\begin{bmatrix} d \\ r \end{bmatrix} = \begin{bmatrix} I & -K \\ -G & I \end{bmatrix} \begin{bmatrix} u \\ e \end{bmatrix} =: T(s) \begin{bmatrix} u \\ e \end{bmatrix}$$

$\boxed{\text{the loop is internally stable if and only if } T^{-1}(s) \text{ is stable.}}$

(b) Since $G(s)$ is stable, we proceed as follows. Note that

$$\begin{bmatrix} I & -K \\ -G & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ -G & I \end{bmatrix} \begin{bmatrix} I & -K \\ 0 & I - GK \end{bmatrix}.$$

Hence

$$\begin{bmatrix} I & -K \\ -G & I \end{bmatrix}^{-1} = \begin{bmatrix} I & -K \\ 0 & I-GK \end{bmatrix}^{-1} \begin{bmatrix} I & 0 \\ -G & I \end{bmatrix}^{-1} = \begin{bmatrix} I & K(I-GK)^{-1} \\ 0 & (I-GK)^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ G & I \end{bmatrix}.$$

Finally, since $(I - GK)^{-1} = I + GK(I - GK)^{-1}$, it follows that if G is stable, then the loop is internally stable if and only if $Q := K(I - GK)^{-1}$ is stable. Rearranging terms shows that K is internally stabilizing if and only if $\boxed{K = Q(I + GQ)^{-1} \text{ for some stable } Q.}$

(c) Since in both cases $K(s)$ is required to be internally stabilizing,

$$K = Q(I + GQ)^{-1}$$

for some stable Q. A simple calculation now shows that

$$C(s) = GK(I - GK)^{-1} = GQ.$$

(i) Since Q is required to be stable, it follows that: $\boxed{\text{the nonminimum–phase zero of } G \text{ cannot be cancelled.}}$

(ii) Since $C(s)$ is required to be allpass, we set

$$\boxed{Q(s) = \tfrac{s+2}{s+1}.}$$

3. (a) By direct evaluation, $L(j\omega)'L(j\omega) =$

$$I + K(j\omega I - A)^{-1}B + B'(-j\omega I - A')^{-1}K' + B'(-j\omega I - A')^{-1}K'K(j\omega I - A)^{-1}B$$

But

$$K'K = A'P + PA + C'C = -(-j\omega I - A')P - P(j\omega I - A) + C'C$$

from the Riccati equation. So, $L(j\omega)'L(j\omega)$

$$\begin{aligned}
&= I + K(j\omega I - A)^{-1}B + B'(-j\omega I - A')^{-1}K' \\
&\quad + B'(-j\omega I - A')^{-1}[-(-j\omega I - A')P - P(j\omega I - A) + C'C](j\omega I - A)^{-1}B \\
&= I + [K - B'P](j\omega I - A)^{-1}B + B'(-j\omega I - A')^{-1}[K' - PB] \\
&\quad + B'(-j\omega I - A')^{-1}C'C(j\omega I - A)^{-1}B \\
&= I + G(j\omega)'C'CG(j\omega)
\end{aligned}$$

(b) Let ϵ be the input to Δ_1 and δ be the output of Δ_1. Then

$$\epsilon = -G(\delta + K\epsilon) = -(I + GK)^{-1}G\delta = -G(I + KG)^{-1}\delta.$$

Using the small gain theorem (since $G(s)$ and the regulator are stable and the perturbation is assumed stable), the loop is stable if $\|\Delta_1 G(I + KG)^{-1}\|_\infty < 1$. But part (a) implies that $\underline{\sigma}[I + KG(j\omega)] \geq 1$ which implies $\|(I + KG)^{-1}\|_\infty \leq 1$. This shows that the loop will tolerate perturbations of size $\boxed{\|\Delta_1\|_\infty < \|G\|_\infty^{-1}}$ without losing internal stability since

$$\|\Delta_1 G(I + KG)^{-1}\|_\infty < 1$$

(c) Let ϵ be the input to Δ_2 and δ be the output of Δ_2. Then

$$\epsilon = -KG(\delta + \epsilon) = -(I + KG)^{-1}KG\delta = L^{-1}(I - L)\delta = (L^{-1} - I)\delta$$

Using the small gain theorem (since the regulator is stable and the perturbation is assumed stable), the loop is stable if $\|\Delta_2(L^{-1} - I)\|_\infty < 1$. But part (a) implies that

$$\bar{\sigma}[L(j\omega)^{-1} - I] \leq 1 + \bar{\sigma}[L(j\omega)^{-1}] \leq 1 + \frac{1}{\underline{\sigma}[L(j\omega)]} \leq 2$$

This shows that the loop will tolerate perturbations Δ_2 of size $\boxed{\|\Delta_2\|_\infty < 0.5}$ without losing internal stability.

4. (a) $\boxed{\text{Suppose that both } \Delta(s) \text{ and } S(s) \text{ are stable.}}$ Then the feedback loop with forward transfer matrix $\Delta(s)$ and feedback transfer matrix $S(s)$ is stable if

$$\boxed{\|\Delta(s)S(s)\|_\infty < 1.}$$

(b) (i) The realisation is balanced if

$$A\Sigma + \Sigma A' + BB' = A'\Sigma + \Sigma A + C'C = 0$$

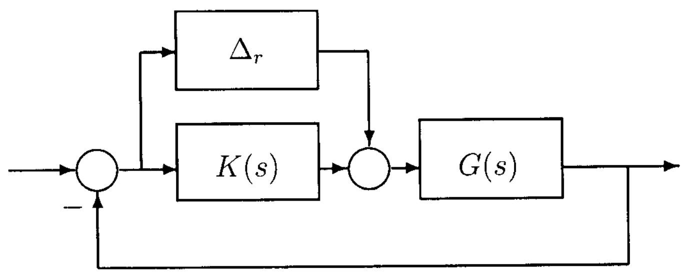
for $\Sigma = \text{diag}(\sigma_1, \sigma_2, \sigma_3) > 0$ and where the $\sigma_i's$ are the Hankel singular values of $K(s)$. A simple calculation gives

$$\boxed{\Sigma = \text{diag}(1, 0.4, 0.1).}$$

(ii) Let $K_r(s)$ denote an rth order balanced truncation of $K(s)$. Then $K_r(s) = K(s) + \Delta_r(s)$ where

$$\|\Delta_r\|_\infty \le 2 \sum_{i=r+1}^{3} \sigma_i. \tag{1}$$

Then replacing $K(s)$ be $K_r(s)$ in the loop is equivalent to:



Let ϵ be the input to Δ_r and δ be the output of Δ_r. Then

$$\epsilon = -(I + GK)^{-1}G\delta$$

and so the loop is stable if $\|\Delta_r\|_\infty \|(I + GK)^{-1}G\|_\infty < 1$. However,

$$\|(I + GK)^{-1}G\|_\infty < 1$$

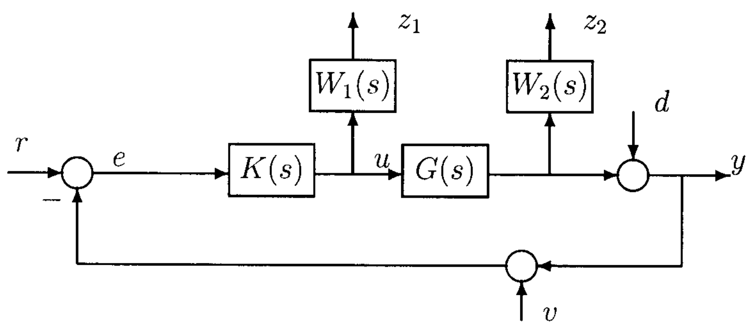
from the graph. It follows from Equation (1) above that $r = 1$ will guarantee that $\|\Delta_r\|_\infty \le 2(0.4 + 0.1) = 1$ and the loop will be stable. So

$$\boxed{K_r(s) \stackrel{s}{=} \left[\begin{array}{c|cc} -1 & 1 & 1 \\ \hline 1 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right]}$$

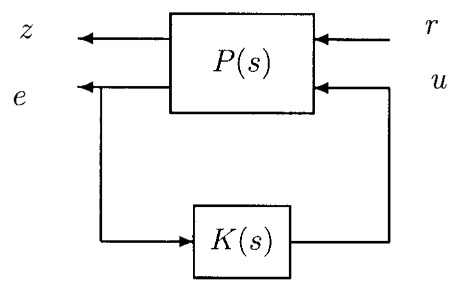
is a first order internally stabilising controller for $G(s)$.

5. (a) It is clear that we require K to internally stabilize the nominal model. Suppose that $\Delta_1 = 0$ and let the input to Δ_2 be ϵ while the output from Δ_2 be δ. Then a calculation shows that $\epsilon = -C\delta$ where $C = (I + GK)^{-1}GK$ is the complementary sensitivity which is stable. Using the small gain theorem, to satisfy the first design requirement, it is sufficient that $\|\Delta_2(j\omega)C(j\omega)\| < 1, \forall\omega$. This can be satisfied if $\boxed{\|W_2C\|_\infty < 1}$, where $W_2 = w_2 I$. An analogous procedure shows that to satisfy the second design requirement, it is sufficient that $\|\Delta_1(j\omega)K(j\omega)S(j\omega)\| < 1, \forall\omega$ where $S = (I + GK)^{-1}$ is the sensitivity. This can be satisfied if $\boxed{\|W_1KS\|_\infty < 1}$, where $W_1 = w_1 I$. To satisfy both design requirements, it is sufficient that $\boxed{\left\| \begin{bmatrix} W_1KS \\ W_2C \end{bmatrix} \right\|_\infty < 1.}$

(b) The design specifications reduce to the requirement that the transfer matrix from r to $z = [z_1^T \ \ z_2^T]^T$ in the following diagram has $\mathcal{H}_\infty$-norm less than 1.



The corresponding generalized regulator formulation is to find an internally stabilizing K such that $\|\mathcal{F}_l(P, K)\| < 1$:



$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \left[\begin{array}{c|c} 0 & W_1G \\ 0 & W_2G \\ \hline I & -G \end{array}\right].$$

(c) Now, $\|K(j\omega)S(j\omega)\| < |w_1^{-1}(j\omega)|$ and $\|C(j\omega)\| < w_2^{-1}(j\omega)|, \forall\omega$. Since w_1^{-1} and w_2^{-1} are low pass, we expect $\boxed{\text{a limited bandwidth of } u}$ (since $u(j\omega) = -K(j\omega)S(j\omega)$), which implies low control effort (up to 1 radians/second) and $\boxed{\text{good noise attenuation}}$ beyond 10 radians/second (since $\|y(j\omega)\| \le \|C(j\omega)\|\|v(j\omega)\|$ with $r=0, d=0$). $\boxed{\text{Nothing can be said about the tracking and disturbance rejection}}$ properties of the loop which may therefore be unacceptable.

6. (a) The generalized regulator formulation is given by

$$\boxed{\begin{bmatrix} z(s) \\ y(s) \end{bmatrix} = P(s)\begin{bmatrix} w(s) \\ u(s) \end{bmatrix},\ u(s) = Fy(s),\ P(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{c||c|c} A & B & B \\ \hline C & 0 & 0 \\ 0 & 0 & I \\ \hline I & 0 & 0 \end{array}\right].}$$

(b) The requirement $\|H\|_\infty < \gamma$ is equivalent to $J := \|z\|_2^2 - \gamma^2\|w\|_2^2 < 0$, with $\|v\|_2^2 := \int_0^\infty \|v(t)\|^2 dt$. Let $V = x^TXx$ and set $u = Fx$. Provided that $X = X^T > 0$ and $\dot{V} < 0$ along closed loop trajectory, we can assume $\lim_{t\to\infty} x(t) = 0$. Then

$\dot{V} = \dot{x}^TXx + x^TX\dot{x} = x^T\left(A^TX + XA + F^TB^TX + XBF\right)x + x^TXBw + w^TB^TXx.$

Integrating from 0 to ∞ and using $x(0) = x(\infty) = 0$,

$$0 = \int_0^\infty [x^T\left(A^TX + XA + F^TB^TX + XBF\right)x + x^TXBw + w^TB^TXx]dt.$$

Using the definition of J and adding the last equation, $J =$

$$\int_0^\infty\{x^T[A^TX + XA + C^TC + F^TF + F^TB^TX + XBF]x - [\gamma^2w^Tw - x^TXBw - w^TB^TXx]\}dt.$$

Completing the squares by using

$$(F + B^TX)^T(F + B^TX) = F^TF + F^TB^TX + XBF + XBB^TX$$
$$\|(\gamma w - \gamma^{-1}B^TXx)\|^2 = \gamma^2w^Tw - w^TB^TXx - x^TXBw + \gamma^{-2}x^TXBB^TXx,$$

$$J = \int_0^\infty\{x^T[A^TX + XA + C^TC - (1-\gamma^{-2})XBB^TX]x + \left\|(F + B^TX)x\right\|^2 - \left\|\gamma w - \gamma^{-1}B^TXx\right\|^2\}dt.$$

Thus two sufficient conditions for $J < 0$ are the existence of X such that

$$\boxed{A^TX + XA + C^TC - (1-\gamma^{-2})XBB^TX < 0,} \qquad \boxed{X = X^T > 0.}$$

The state feedback gain is $\boxed{F = -B^TX}$ and the worst case disturbance is $w^* = \gamma^{-2}B^TXx$. The closed-loop with these feedback laws is $\dot{x} = [A-(1-\gamma^{-2})BB^TX]x$ and a third condition is therefore $\boxed{Re\ \lambda_i[A - (1-\gamma^{-2})BB^TX] < 0,\ \forall\ i.}$

It remains to prove $\dot{V} < 0$ along state-trajectory with $u = Fx$ and $w = 0$. But

$$\boxed{\dot{V} = x^T\left(A^TX + XA + F^TB^TX + XBF\right)x < -x^T(C^TC + (1 + \gamma^{-2})XBB^TX)x < 0}$$

for all $x \neq 0$ (since (A, B, C) is assumed minimal) proving closed-loop stability.

(c) Multiply the Riccati inequality from the left and right by X^{-1} to get $AX^{-1} + X^{-1}A^T - BB^T + X^{-1}C^TCX^{-1} + \gamma^{-2}BB^T < 0$. Using a Schur complement argument this can be linearized as $\boxed{\begin{bmatrix} AX^{-1} + X^{-1}A^T + (\gamma^{-2} - 1)BB^T & X^{-1}C^T \\ CX^{-1} & -I \end{bmatrix} < 0.}$