

ISE2  
Discrete Mathematics and  
Computational Complexity

Specimen Paper

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## 1. Compulsory Question

- (a) Which (if any) of these statements are propositions? Briefly justify your answers.

- (i)  $4x = 5$
- (ii)  $5x + 1 = 5$  if  $x = 1$
- (iii)  $x + y + z = y + 2z$  if  $x = z$

[3]

- (b) Show that each of these implications is a tautology using an appropriate truth table.

- (i)  $p \wedge q \rightarrow p$
- (ii)  $p \rightarrow p \vee q$
- (iii)  $\neg p \rightarrow (p \rightarrow q)$

[3]

- (c) Determine the truth of each of these propositions, where the universe of discourse is the set of integers. Briefly justify your answers.

- (i)  $\forall n(n^2 \geq n)$
- (ii)  $\exists n(n^2 = 2)$

[2]

- (d) Find the power set of each of the following sets.

- (i)  $\{a\}$
- (ii)  $\{a, b\}$
- (iii)  $\{\emptyset, \{\emptyset\}\}$

[3]

- (e) Let  $f(n)$  be the function from the set of integers to the set of integers such that  $f(n) = n^2 + 1$ . What are the domain, codomain, and range of this function?

[3]

- (f) Which (if any) of these functions are bijections from  $\mathbf{R}$  to  $\mathbf{R}$ ? Briefly justify your answers.

- (i)  $f(x) = 2x + 1$
- (ii)  $f(x) = x^2 + 1$
- (iii)  $f(x) = x^3$

[3]

- (g) Find an appropriate big-O expression for each of these recurrence relations.

- (i)  $f(n) = 2f(n-1)$ , with  $f(0) = 1$ .
- (ii)  $f(n) = 2f(n-1) + 1$ , with  $f(0) = 0$ .
- (iii)  $f(n) = 2f(n/3) + n^2$ , with  $f(0) = 0$ .

[3]

Total: 20 Marks

## 2. Logic

Let  $P(x)$  denote the statement “ $x$  owns a computer”.

Let  $Q(x)$  denote the statement “ $x$  can program a computer”.

Let  $R(x)$  denote the statement “ $x$  has studied computing”.

Let  $S(x,y)$  denote the statement “ $x$  knows  $y$ ”.

Let the universe of discourse be the set of all people.

(a) Write the following English statements using symbolic logic.

- (i) Everyone who owns a computer can program a computer.
- (ii) Someone can program a computer, but doesn't own one.
- (iii) Someone who has studied computing can't program a computer.
- (iv) Everyone who can program a computer has studied computing.
- (v) Steven has not studied computing.

[5]

(b) Use symbolic logic to construct a valid argument that results in the conclusion  $\neg P(\text{Steven})$ , given the premises derived in part (a). At each step of your argument, state the rule of inference used.

[7]

(c) Write the following statements using symbolic logic.

- (i) Someone knows everyone who can program a computer.
- (ii) Everyone knows someone who can program a computer.
- (iii) Everyone who has studied computing knows someone who owns a computer.

[3]

Total: 15 Marks

### 3. Algorithm Analysis

- a) Derive an expression for the number of multiplications performed by the code in Fig. 3.1, in terms of the input value  $n$ .

```
procedure p(n: integer)
begin
  total := 0
  for i := 1 to n
    for j := 1 to n
      total := total + i*j
  result := total
end
```

Figure 3.1

[2]

- b) Hence derive a recurrence relation for the number of multiplications performed by the code in Fig. 3.2, in terms of the input value  $n$ .

```
procedure q(n: integer)
begin
  if n = 0 then
    result := 1
  else
    result := q(n/2) + p(n)
  end
```

Figure 3.2

[3]

- c) State the Master Theorem.

[3]

- d) Derive a big-O expression for the number of multiplications performed by a call to **q**, in terms of  $n$ .

[1]

- e) Prove that if the recurrence relation  $f(n) = a f(n/b) + c$  is satisfied, where  $a > 1$ , then  $f(n)$  is  $O(n^{\log_b a})$ . You need only consider the case where  $n = b^k$  for some  $k \in \mathbb{Z}^+$ .

[6]

Total: 15 Marks

#### 4. Computability

- (a) Define what is meant if a problem is said to be *tractable*, and give an example of a tractable problem. [2]

- (b) Define what is meant if a problem is said to be *unsolvable*, and give an example of an unsolvable problem. [2]

- (c) Prove that the problem from part (b) is unsolvable. [6]

- (d) Let the set of ISE2 students be denoted  $S$ . Consider a symmetric relation  $D$  on  $S$ , such that  $s_1 D s_2$  iff student  $s_1$  dislikes student  $s_2$ .

The “student allocation” problem is defined as:

Can the set of students be partitioned into no more than  $n$  teams, such that no team contains any two students who dislike each other?

Prove that “student allocation” is at least as hard as  $k$ -colouring (defined below for convenience).

The  $k$ -colouring problem is defined as:

Given a set of nodes  $V$ , a set of edges  $E$ , and a positive integer  $k$ , does there exist a function  $p : V \rightarrow \{1, 2, \dots, k\}$  such that  $\forall v_1 \forall v_2 ( \{v_1, v_2\} \in E \rightarrow p(v_1) \neq p(v_2) )$ , where the universe of discourse is the set  $V$ ?

[5]

Total: 15 Marks.

# Model Answers

## Question 1.

(a)

- (i) is not a proposition. It is neither true nor false, as  $x$  is undefined.
- (ii) is a proposition, as it has a definite truth value (false).
- (iii) is a proposition, as it has a definite truth value (true).

(b) The truth tables are shown below. In each case, the final column is always true, thus the expression is a tautology.

(i)

$p$	$q$	$p \wedge q$	$p \wedge q \rightarrow p$
F	F	F	T
F	T	F	T
T	F	F	T
T	T	T	T

(ii)

$p$	$q$	$p \vee q$	$p \rightarrow p \vee q$
F	F	F	T
F	T	T	T
T	F	T	T
T	T	T	T

(iii)

$p$	$q$	$\neg p$	$p \rightarrow q$	$\neg p \rightarrow (p \rightarrow q)$
F	F	T	T	T
F	T	T	T	T
T	F	F	F	T
T	T	F	T	T

(c)

- (i) If  $n$  is negative, clearly  $n^2 \geq 0 \geq n$ . If  $n$  is zero, it is true. If  $n$  is positive, since  $n$  is an integer,  $n \geq 1$ , so multiplying both sides by  $n$ , we obtain  $n^2 \geq n$ . Thus this proposition is true.
- (ii) There is no integer  $n$  with  $n^2 = 2$ , so the proposition is false.

(d)

- (i)  $\{\emptyset, \{a\}\}$
- (ii)  $\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$
- (iii)  $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$

(e) Domain: the set of integers, Codomain: the set of integers,  
Range:  $\{n^2 + 1 \mid n \text{ is an integer}\}$ .

(f)

- (i) Yes. There is an inverse  $g(x) = (x-1)/2$ .
- (ii) No, as the range is  $\{x^2 + 1 \mid x \text{ is real}\}$ , which is not the set of reals, so the function is not a surjection. Alternatively, we can simply note that  $x = -1$  and  $x = +1$  have the same value of  $f(x)$ , so the function is not an injection.
- (iii) Yes. There is an inverse  $g(x) = x^{1/3}$ .

(g)

- (i)  $f(n) = 2^n$ , which is  $O(2^n)$ .
- (ii)  $f(n) = 2^n - 1$ , which is  $O(2^n)$ .
- (iii)  $f(n)$  is  $O(n^2)$  from the Master Theorem.

### Question 2.

(a)

- (i)  $\forall x( P(x) \rightarrow Q(x) )$
- (ii)  $\exists x( Q(x) \wedge \neg P(x) )$
- (iii)  $\exists x( R(x) \wedge \neg Q(x) )$
- (iv)  $\forall x( Q(x) \rightarrow R(x) )$
- (v)  $\neg R(\text{Steven})$

(b)

- 1.  $Q(\text{Steven}) \rightarrow R(\text{Steven})$  [universal instantiation, from premise (iv)]
- 2.  $\neg Q(\text{Steven})$  [modus tollens, when combined with premise (v)]
- 3.  $P(\text{Steven}) \rightarrow Q(\text{Steven})$  [universal instantiation, from premise (i)]
- 4.  $\neg P(\text{Steven})$  [modus tollens, when combined with (3) above]

(c)

- (i)  $\exists x \forall y( Q(y) \rightarrow S(x,y) )$
- (ii)  $\forall x \exists y( Q(y) \wedge S(x,y) )$
- (iii)  $\forall x \exists y( R(x) \rightarrow S(x,y) \wedge P(y) )$

### Question 3.

- (a) Each outer loop executes  $n$  times. Each inner loop executes  $n$  times per iteration of the outer loop. There is one multiplication per inner loop iteration. Thus the number of multiplications is  $n^2$ .
- (b)  $f(0) = 0$ : there are no multiplications in the base case.  $f(n) = f(n/2) + n^2$ .
- (c) [see notes]
- (d)  $O(n^2)$ . [A direct application of the Master Theorem]
- (e) [see notes]

Question 4.

- (a) [see notes]
- (b) [see notes]. The only unsolvable example studied in lectures is the halting problem.
- (c) [see notes]
- (d) There is a direct correspondence between the two problems. “Dislikes” corresponds to edges, and students correspond to nodes. The relation  $D$  is symmetric, so the digraph of the relation is equivalent to the graph to be coloured. The following reduction is appropriate:
  - (i) Set  $S = V$ .
  - (ii) Set  $D = \{(v_1, v_2) \mid \{v_1, v_2\} \in E\}$ .
  - (iii) Set  $n = k$ .