EE1-10A MATHEMATICS I

1. a) i) $(1-i)^3 = \left(\sqrt(2)e^{-\pi i/4}\right)^3 = 2^{3/2}e^{-3\pi i/4} = -2-2i$ [1]

ii) $\dfrac{1-i}{1+i}\dfrac{(1-i)}{(1-i)} = \dfrac{-2i}{2} = -i$ [1]

iii) $\left(\dfrac{1+\sqrt{3}i}{2}\right)^{10} = \left(e^{\pi i/3}\right)^{10} = e^{10\pi i/3} = e^{4\pi i/3} = -\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i$ [2]

b) $(2+2i)^{1/3} = \left(\sqrt{8}e^{i(\pi/4+2n\pi)}\right)^{1/3} = \sqrt{2}e^{i(\pi/12+2n\pi/3)} \quad n = 0,1,2$ [2]

$\sqrt{2}e^{\pi i/12}, \sqrt{2}e^{3\pi i/4}, \sqrt{2}e^{17\pi i/12}$ [2]

c) $\lim\limits_{x\to\pi/6} \dfrac{\cos(3x)}{\tan(2x)-\sqrt{3}} = \text{"}\dfrac{0}{0}\text{"} \Rightarrow l'Hopital$

$= \lim\limits_{x\to\pi/6} \dfrac{-3\sin(3x)}{2\sec^2(2x)} = -\dfrac{3}{2}\sin(\pi/2)\cos^2(\pi/3) = -\dfrac{3}{2}\left(\dfrac{1}{2}\right)^2 = -\dfrac{3}{8}$ [4]

d) Let $y = x^x$, so $\ln y = x\ln x \to 0$, as $x \to 0$; hence $y \to 1$ and $\lim\limits_{x\to 0} x^x = 1$ [4]

e) $\ln y = \ln\left(x^{\ln x}\right) = (\ln x)^2$. Differentiate:

$\dfrac{1}{y}\dfrac{dy}{dx} = 2\ln x\left(\dfrac{1}{x}\right) \Rightarrow \dfrac{dy}{dx} = 2(\ln x)x^{\ln x - 1}$. [4]

f) Let $u = 1-3x^2 \Rightarrow du = -6xdx$ so

$\displaystyle\int \frac{2x}{(1-3x^2)^{1/3}}\,dx = -\frac{1}{3}\int u^{-1/3}du = -\frac{1}{3}\frac{u^{2/3}}{2/3} = -\frac{1}{2}u^{2/3}.$

With limits, $-\dfrac{1}{2}\left[(1-3x^2)^{2/3}\right]_0^1 = \dfrac{1}{2}\left(2^{2/3}+1\right)$. [4]

g) Let $x = \sin u \Rightarrow dx = \cos u\; du$ and integral becomes

$\displaystyle\int_0^{\pi/2} \sqrt{1-\sin^2 u}\cos u\; du = \int_0^{\pi/2} \cos^2 u\; du$

$= \frac{1}{2}\int_0^{\pi/2} \cos(2u)+1\; du = \frac{1}{2}\left[\frac{1}{2}\sin(2u)+u\right]_0^{\pi/2} = \frac{\pi}{4}$ [4]

h) From the definition,

$$Y_n * X_n = \sum_{m=-\infty}^{\infty} Y_{n-m}X_m$$

We now make a substitution: $m = n-r$. For any fixed n this is a 1-1 mapping between m and r with the range $-\infty \le m \le \infty$ mapping to $\infty \le r \le -\infty$.

Therefore

$$
\begin{aligned}
Y_n * X_n &= \sum_{m=-\infty}^{\infty} Y_{n-m} X_m \\
&= \sum_{r=\infty}^{-\infty} Y_r X_{n-r} \\
&= \sum_{r=-\infty}^{\infty} Y_r X_{n-r} \\
&= \sum_{r=-\infty}^{\infty} X_{n-r} Y_r = X_n * Y_n
\end{aligned}
$$

[4]

i) Since $x(t)$ is real, $x^*(t) = x(t)$ so we can write $y(t) = x(t) \otimes x(t) = \int_{-\infty}^{\infty} x(s-t)x(s)ds$. We know that $x(s)$ is zero for $s < 0$ and hence $x(s-t)$ is zero for $s < t$. It follows that the integrand will be zero for $s < \max(0, t)$.

Thus we get two different lower integration limits depending on whether or not $t < 0$. When $t < 0$

$$
\begin{aligned}
y(t) &= \int_0^{\infty} e^{-2(s-t)} e^{-2s} ds \\
&= e^{2t} \int_0^{\infty} e^{-4s} ds \\
&= -\frac{1}{4} e^{2t} \left[e^{-4s} \right]_0^{\infty} = \frac{1}{4} e^{2t}
\end{aligned}
$$

When $t \geq 0$ the integrand is the same but the lower limit is now t , so

$$
\begin{aligned}
y(t) &= -\frac{1}{4} e^{2t} \left[e^{-4s} \right]_t^{\infty} \\
&= \frac{1}{4} e^{2t} e^{-4t} = \frac{1}{4} e^{-2t}
\end{aligned}
$$

We can neatly combine these into $y(t) = \frac{1}{4} e^{-2|t|}$. [4]

j) From the formula sheet,

$$
\begin{aligned}
x(t) &= \int_{-\infty}^{\infty} X(f) e^{i2\pi ft} df \\
&= 2i \left(\int_{-\infty}^{\infty} \delta(f+20) e^{i2\pi ft} df - \int_{-\infty}^{\infty} \delta(f-20) e^{i2\pi ft} df \right) \\
&= 2i \left(e^{-i40\pi t} - e^{i40\pi t} \right) \\
&= 2i \times -2i \sin 40\pi t \\
&= 4 \sin 40\pi t
\end{aligned}
$$

[4]

[*(a-g, i,j: Similar to examples seen in class, h: bookwork*]

2. Given the function

$$f(x) = \frac{x}{1+x^4} - \frac{x^3}{1+x^4} \tag{2.1}$$

a) Show that the area under the graph of the function $f(x)$, for $0 \leq x \leq b$ is given by

$$A(b) = \frac{1}{2}\arctan(b^2) - \frac{1}{4}\log(1+b^4). \tag{2.2}$$

The integral is made of two parts, the first requires a trigonometric substitution, the second is an almost exact differential.
For the first integral:

$$A_1(b) = \int_0^b \frac{x}{1+x^4}\, dx$$
$$\text{substitute } x^2 = u \rightarrow 2x\,dx = du$$
$$A_1(b) = \frac{1}{2}\int \frac{1}{1+u^2}\, du =$$
$$= \frac{1}{2}\arctan(u)$$
$$= \frac{1}{2}\left[\arctan(x^2)\right]_0^b =$$
$$= \frac{1}{2}\arctan(b^2)$$

[*Similar to examples seen in class.*] [3]

For the second integral:

$$A_2(b) = -\int_0^b \frac{x^3}{1+x^4}\, dx =$$
$$-\frac{1}{4}\int_0^b \frac{4x^3}{1+x^4}\, dx =$$
$$-\frac{1}{4}\left[\log(1+x^4)\right]_0^b = -\frac{1}{4}\log(1+b^4).$$

[*Similar to examples seen in class.*] [3]

The result follows.

b) Find the stationary points of $A(b)$ with $b > 0$ and determine whether they are maxima or minima.

Must now calculate the first derivative of $A(b)$ and its roots, but the result (the first derivative) is given by the initial integrand and the calculation is superfluous.

$$\frac{dA}{db} = \frac{b-b^3}{1+b^4}.$$

Critical points are given by

$$b - b^3 = 0 \rightarrow b_c = 0, \pm 1 \rightarrow b_c = +1.$$

(as $b > 0$, only the positive root is accepted). [3]

To check the nature of the stationary points, calculate the second derivative

$$\frac{d^2A}{db^2} = \frac{(1-3b^2)(1+b^4)-4b^3(b-b^3)}{(1+b^4)^2},$$

which, at the critical point $b_c = +1$ gives

$$\frac{d^2A}{db^2} = -1 < 0$$

therefore b_c is a maximum. [*Similar to examples seen in class.*] [3]

c) Assume that b is a function of time given by

$$b(t) = e^{-t}. \tag{2.3}$$

i) Use the chain rule to determine $\frac{dA}{dt}$ as a function of t.

Using the chain rule and the results of the previous part of the question:

$$\frac{dA}{dt} = \frac{dA}{db}\frac{db}{dt} = \frac{b-b^3}{1+b^4}\left(-e^{-t}\right) = e^{-2t}\frac{e^{-2t}-1}{e^{-4t}+1}.$$

[*Similar to examples seen in class.*] [4]

ii) Find the limit of $A(b(t))$ as t tends to $+\infty$.

$$\lim_{t\to+\infty} b(t) = \lim_{t\to+\infty} e^{-t} = 0, \text{ thus}$$

$$\lim_{t\to+\infty} \frac{1}{2}\arctan(b^2) - \frac{1}{4}\log(1+b^4) =$$

$$\lim_{b\to 0} \frac{1}{2}\arctan(b^2) - \frac{1}{4}\log(1+b^4) = 0.$$

[*Unseen.*] [4]

3. a) i) Show that $\sinh(x+iy) = \sinh x\cos y + i\cosh x\sin y$.

$$\begin{aligned}
\sinh(x+iy) &= \frac{1}{2}\left(e^{x+iy} - e^{-x-iy}\right) \\
&= \frac{1}{2}\left[e^x(\cos y + i\sin y) - e^{-x}(\cos y - i\sin y)\right] \\
&= \cos y\left(\frac{e^x - e^{-x}}{2}\right) + i\sin y\left(\frac{e^x+e^{-x}}{2}\right) \\
&= \cos y\sinh x + i\sin y\cosh x
\end{aligned}$$

as required. [*Similar to examples seen in class.*] [4]

ii) Hence, or otherwise, show that $|\sinh(x+iy)|^2 = \frac{1}{2}(\cosh 2x - \cos 2y)$.

From (i) we have

$$\begin{aligned}
|\sinh(x+iy)|^2 &= \cos^2 y\sinh^2 x + \sin^2 y\cosh^2 x \\
&= (1-\sin^2 y)\sinh^2 x + \sin^2 y(1+\sinh^2 x) \\
&= \sinh^2 x + \sin^2 y \\
&= \frac{1}{2}(\cosh 2x - 1) + \frac{1}{2}(1-\cos 2y)
\end{aligned}$$

and the result follows. [*Unseen.*] [4]

b) Obtain the limit: $\lim_{x \to -3} \frac{3-\sqrt{-3x}}{x+3}$. [Do NOT use l'Hopital's rule.]

$$\begin{aligned}
\lim_{x\to-3}\frac{3-\sqrt{-3x}}{x+3} &= \lim_{x\to-3}\frac{(3-\sqrt{-3x})}{(x+3)}\frac{(3+\sqrt{-3x})}{(3+\sqrt{-3x})}\\
&= \lim_{x\to-3}\frac{9+3x}{(x+3)(3+\sqrt{-3x})}\\
&= \lim_{x\to-3}\frac{3}{3+\sqrt{-3x}}\\
&= \frac{1}{2}
\end{aligned}$$

[*Similar to examples seen in class.*] [4]

c) Show that $\int \frac{dx}{\sqrt{x^2-1}} = \cosh^{-1}x + C$.

Let $x = \cosh u$, then $dx = \sinh u \, du$ and we get

$$\begin{aligned}
\int \frac{dx}{\sqrt{1-x^2}} &= \int \frac{\sinh u \, du}{\sqrt{\cosh^2 u - 1}}\\
&= \int \frac{\sinh u \, du}{\sqrt{\sinh^2 u}}\\
&= \int 1 \, du\\
&= u + C\\
&= \cosh^{-1} x + C
\end{aligned}$$

[*Similar to examples seen in class.*] [4]

d) Integrate $\int \frac{1}{(1-\sin x - \cos x} dx$.

Use the standard substitution from formula sheet with $t = \tan(x/2)$ then

$$\begin{aligned}
\int \frac{1}{(1-\sin x-\cos x} dx &= \int \frac{1}{1-\frac{1-t^2}{1+t^2}-\frac{2t}{1+t^2}}\left(\frac{2\,dt}{1+t^2}\right)\\
&= \int \frac{2}{1+t^2-(1-t^2)-2t} dt\\
&= \int \frac{1}{t^2-t} dt\\
&= \int \frac{1}{t-1} - \frac{1}{t} dt\\
&= \ln|t-1| - \ln|t| + C\\
&= \ln\left|\tan\left(\frac{x}{2}\right)-1\right| - \ln\left|\tan\left(\frac{x}{2}\right)\right| + C
\end{aligned}$$

(May complete square instead of partial fractions, obtain equivalent answer in terms of arctanh.) [*Similar to examples seen in class.*] [4]

4. a) We know that $x(t) = \sum_{n=-\infty}^{\infty} X_n e^{-i2\pi nFt}$. Hence

$$\begin{aligned}
\frac{1}{T}\int_0^T |x(t)|^2 &= \langle x(t)x^*(t)\rangle \\
&= \left\langle \sum_{n=-\infty}^{\infty} X_n e^{-2\pi nFt} \sum_{m=-\infty}^{\infty} X_m^* e^{i2\pi mFt} \right\rangle \\
&= \sum_{n=-\infty}^{\infty} X_n \sum_{m=-\infty}^{\infty} X_m^* \left\langle e^{i2\pi(m-n)Ft} \right\rangle
\end{aligned}$$

But $\left\langle e^{i2\pi kFt} \right\rangle = \langle \cos 2\pi kFt\rangle + i\langle \sin 2\pi kFt\rangle = \begin{cases} 1 & \text{for } k=0 \\ 0 & \text{for } k \neq 0 \end{cases}$

So all terms of $\sum_{m=-\infty}^{\infty} X_m^* \left\langle e^{i2\pi(m-n)Ft} \right\rangle$ are zero except for the one with $m = n$ which means that the entire sum is equal to X_n^*.

Hence

$$\frac{1}{T}\int_0^T |x(t)|^2 = \sum_{n=-\infty}^{\infty} X_n X_n^* = \sum_{n=-\infty}^{\infty} |X_n|^2$$

[*Bookwork.*] [5]

b) For convenience, we will write $a = -i2\pi nF = -i\pi n$. Then

$$\begin{aligned}
X_n &= \frac{1}{2}\int_0^2 x(t)e^{at}\,dt \\
&= \frac{1}{2}\int_0^1 te^{at}\,dt + \frac{1}{2}\int_1^2 (2-t)e^{at}\,dt \\
&= \frac{1}{2}\left[\left(\frac{t}{a} - \frac{1}{a^2}\right)e^{at}\right]_0^1 + \frac{1}{2}\left[\left(\frac{2-t}{a} + \frac{1}{a^2}\right)e^{at}\right]_1^2 \\
&= \frac{1}{2}\left(\left(\frac{1}{a} - \frac{1}{a^2}\right)e^{a} + \frac{1}{a^2}\right) + \frac{1}{2}\left(\frac{1}{a^2}e^{2a} - \left(\frac{1}{a} + \frac{1}{a^2}\right)e^{a}\right) \\
&= \frac{1}{2}\left(\frac{1}{a^2}e^{2a} - \frac{2}{a^2}e^{a} + \frac{1}{a^2}\right) = \frac{1}{2a^2}\left(e^{2a} - 2e^{a} + 1\right)
\end{aligned}$$

Now we substitute for $a = -i\pi n$ to get

$$\begin{aligned}
X_n &= \frac{-1}{2\pi^2 n^2}\left(e^{-i2\pi n} - 2e^{-i\pi n} + 1\right) \\
&= \frac{-1}{2\pi^2 n^2}\left(1 - 2(-1)^n + 1\right) \\
&= \frac{(-1)^n - 1}{\pi^2 n^2}
\end{aligned}$$

[*Similar to examples seen in class.*] [6]

c) Note that the expression for X_n is even, so $X_{-n} = X_n$. Also the expression gives $X_0 = \frac{0}{0}$ so we need to calculate X_0 directly from the definition.

$X_0 = \left\langle x(t)e^{0t}\right\rangle = \langle x(t)\rangle = 0.5$ where this value is deduced from the average value of the waveform. Alternatively, an algebraic calculation gives

$$
\begin{aligned}
X_0 = \langle x(t) \rangle &= \frac{1}{2} \int_0^2 x(t)dt \\
&= \frac{1}{2} \left(\int_0^1 t dt + \int_1^2 (2-t) dt \right) \\
&= \frac{1}{2} \left(\left[\frac{t^2}{2} \right]_0^1 + \left[2t - \frac{t^2}{2} \right]_1^2 \right) \\
&= \frac{1}{2} \left(\left(\frac{1}{2} - 0 \right) + \left(2 - \frac{3}{2} \right) \right) \\
&= \frac{1}{2} = 0.5
\end{aligned}
$$

For the other values of n, we can use the formula: $X_{\pm 1} = \frac{-2}{\pi^2}$, $X_{\pm 2} = 0$, $X_{\pm 3} = \frac{-2}{9\pi^2}$, $X_{\pm 4} = 0$. [*Unseen.*] [4]

d) i) If $x(t)$ is even, then the X_n are also even, i.e. $X_n = X_{-n}$. If $x(t)$ is both even and real-valued, then the X_n are also even and real-valued.

[*Seen in class.*] [2]

ii) We can deduce two things.

Firstly

$$
\begin{aligned}
& x(t) + x(t+1) = 1 \\
\Rightarrow \quad & \langle x(t) \rangle + \langle x(t+1) \rangle = 1 \\
\Rightarrow \quad & \langle x(t) \rangle = 0.5 \\
\Rightarrow \quad & X_0 = 0.5
\end{aligned}
$$

where the third line follows because $\langle x(t) \rangle = \langle x(t+a) \rangle$ for any a.

Secondly we can write

$$
\begin{aligned}
& (x(t) - 0.5) + (x(t+1) - 0.5) = 0 \\
\Rightarrow \quad & (x(t) - 0.5) = -\left(x(t + \frac{T}{2}) - 0.5 \right) \\
\Rightarrow \quad & (x(t) - X_0) = -\left(x(t + \frac{T}{2}) - X_0 \right)
\end{aligned}
$$

which means that $(x(t) - X_0)$ is anti-periodic and so all its even numbered coefficients are zero. It follows that all the even numbered coefficients of $x(t)$ except for X_0 are zero.

[*Unseen.*] [3]