

B.ENG. AND M.ENG. EXAMINATIONS 2008

PART I : MATHEMATICS 1 (ELECTRICAL ENGINEERING)

Date Wednesday 4th June 2008 10.00 am - 1.00 pm

Answer EIGHT questions.

A mathematical formulae sheet is provided.

[Before starting, please make sure that the paper is complete; there should be 6 pages, with a total of 10 questions. Ask the invigilator for a replacement if your copy is faulty.]

1. (i) Define what it means to say that a function f is odd or even, and give an example of each.
- (ii) Classify the following functions as odd, even or neither:
- (a) e^{-x} ;
 - (b) $x \sin x$;
 - (c) $x^2 \sin x$;
 - (d) $2x/(x^2 - 1)$.

(iii) Let $f(x) = e^x$ and $g(x) = 1/x^2$. Find $f(g(x))$ and $g(f(x))$. Find also the inverse functions $f^{-1}(x)$ and $g^{-1}(x)$.

(iv) Write

$$f(x) = \frac{2x}{x+1}$$

as the sum of an even function and an odd function.

2. Evaluate the following limits:

(i)
$$\lim_{x \rightarrow \infty} \frac{(2x-1)(x+3)}{(x+5)(3x-2)} ;$$

(ii)
$$\lim_{x \rightarrow 0} x \sin(\cot x) ;$$

(iii)
$$\lim_{x \rightarrow 0} x^{-2} \ln(\cos x) ;$$

(iv)
$$\lim_{x \rightarrow \infty} x^{-9} \left\{ (x+3)^{10} - (x+1)^{10} \right\}$$

PLEASE TURN OVER

3. Evaluate the following integrals ;

(i)
$$\int (3 - 2x)^{-5} dx ;$$

(ii)
$$\int \frac{5x + 2}{(3x + 4)(x - 1)} dx ;$$

(iii)
$$\int_1^2 x \ln x dx ;$$

(iv)
$$\int x^3 e^x dx .$$

4. (i) Express in polar form $re^{i\theta}$ with $0 \leq \theta < 2\pi$:

(a) $3 + 5i$; (b) $-6 + 3i$; (c) $-4 - 5i$.

(ii) Find an expression for $\cos 3\theta$ in terms of powers of $\cos \theta$.

(iii) Find the equation, in the form $y = f(x)$, for the locus of points which satisfy

$$\arg(z + 1) = \frac{\pi}{3}$$

where $z = x + iy$.

5. (i) Find and classify the stationary points of the function

$$f(x, y) = x(y - 2)^2 + x^2 - x.$$

- (ii) Sketch the locus $f(x, y) = 0$.

6. (i) Consider the planes $3x - 5y - 2z = 2$ and $x + y + 6z = -9$.

(a) Find the perpendicular distance from the origin to each plane.

(b) Find the vector equation of the straight line through the points N_1, N_2 which are the feet of the normals from the origin onto the two planes.

- (ii) Find all vectors $\mathbf{u} = (x, y, z)$ in 3-D space such that $|\mathbf{u}| = 1$ and $|\mathbf{u} - \mathbf{k}| = 1$, where $\mathbf{k} = (0, 0, 1)$.

Describe this set geometrically.

PLEASE TURN OVER

7. Let

$$A = \begin{pmatrix} 9 & 3 & -3 \\ 3 & 5 & 1 \\ -3 & 1 & 11 \end{pmatrix}.$$

Find the entries in

$$L = \begin{pmatrix} a & 0 & 0 \\ b & d & 0 \\ c & e & f \end{pmatrix}$$

so that

$$A = LL^T.$$

Show that $|A| = |L|^2 = 324$, where $|A|$ denotes the determinant of A .

Find L^{-1} , and hence A^{-1} .

8. (i) Find an implicit solution of the differential equation

$$\frac{dy}{dx} = 2 \frac{2x + y}{2x - y},$$

for which $y\left(\frac{1}{2}\right) = 0$.

(ii) Show that

$$\frac{d}{dx} (\ln[\sec x + \tan x]) = \sec x,$$

and hence find the general solution of the linear differential equation

$$\cos x \frac{dy}{dx} + y = 1 - \sin x$$

using an integrating factor.

9. For the differential equation

$$\frac{d^2y}{dx^2} + 4 \frac{dy}{dx} + ky = e^{-2x},$$

where k is a constant, find the solution that satisfies

$$\begin{aligned} y(0) &= 0, \\ y'(0) &= 0, \end{aligned}$$

in the case

(i) $k = 5$;

(ii) $k = 4$.

10. A function $y(x)$ satisfies the equation :

$$\frac{d^2y}{dx^2} + xy = 0$$

and the conditions $y(0) = 1$, $y'(0) = 0$.

Differentiate this equation n times to show that (for $n \geq 1$)

$$\frac{d^{n+2}y}{dx^{n+2}} + n \frac{d^{n-1}y}{dx^{n-1}} = 0 \text{ at } x = 0.$$

Hence show that the solution of the equation with the given initial conditions has the form

$$y(x) = \sum_{i=0}^{\infty} C_i x^{3i},$$

where the C_i are constants. Find the first three non-zero terms in this expansion, write down the general term and show by the ratio test that this series converges for all x .

END OF PAPER

MATHEMATICAL FORMULAE

I. VECTOR ALGEBRA

$$a = a_1i + a_2j + a_3k = (a_1, a_2, a_3)$$

Scalar (dot) product:

$$a \cdot b = a_1b_1 + a_2b_2 + a_3b_3$$

Vector (cross) product:

$$a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix};$$

Scalar triple product:

$$[a, b, c] = a \cdot b \times c = b \cdot c \times a = c \cdot a \times b = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product:

$$a \times (b \times c) = (c \cdot a)b - (b \cdot a)c$$

2. SERIES

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \dots \quad (\alpha \text{ arbitrary, } |x| < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^n \frac{x^{2n}}{(2n)!} + \dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \dots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^n \frac{x^{n+1}}{(n+1)} + \dots \quad (-1 < x \leq 1)$$

3. TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin(a+b) = \sin a \cos b + \cos a \sin b;$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b.$$

$$\cos iz = \cosh z; \quad \cosh iz = \cos z; \quad \sin iz = i \sinh z; \quad \sinh iz = i \sin z.$$

4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f D^n g + \binom{n}{1} Df D^{n-1} g + \dots + \binom{n}{n-1} D^{n-1} f Dg + \dots + D^n f g.$$

(b) Taylor's expansion of $f(x)$ about $x = a$:

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \dots + \frac{h^n}{n!}f^{(n)}(a) + \epsilon_n(h),$$

where $\epsilon_n(h) = h^{n+1}f^{(n+1)}(a + \theta h)/(n+1)!$, $0 < \theta < 1$.

(c) Taylor's expansion of $f(x, y)$ about (a, b) :

$$f(a+h, b+k) = f(a, b) + [hf_x + kf_y]_{a,b} + 1/2! [h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}]_{a,b} + \dots$$

(d) Partial differentiation of $f(x, y)$:

$$\text{i. If } y = y(x), \text{ then } f = F(x), \text{ and } \frac{dF}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx}.$$

$$\text{ii. If } x = x(t), y = y(t), \text{ then } f = F(t), \text{ and } \frac{dF}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}.$$

$$\text{iii. If } x = x(u, v), y = y(u, v), \text{ then } f = F(u, v), \text{ and}$$

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u}, \quad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial v}.$$

(e) Stationary points of $f(x, y)$ occur where $f_x = 0, f_y = 0$ simultaneously.

Let (a, b) be a stationary point: examine $D = [f_{xx}f_{yy} - (f_{xy})^2]_{a,b}$.

If $D > 0$ and $f_{xx}(a, b) < 0$, then (a, b) is a maximum;

If $D > 0$ and $f_{xx}(a, b) > 0$, then (a, b) is a minimum;

If $D < 0$ then (a, b) is a saddle-point.

(f) Differential equations:

i. The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating factor $I(x) = \exp[\int P(x)(dx)]$, so that $\frac{d}{dx}(Iy) = IQ$.

ii. $P(x, y)dx + Q(x, y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y$.

5. INTEGRAL CALCULUS

- (a) An important substitution: $\tan(\theta/2) = t$;
 $\sin \theta = 2t/(1+t^2)$, $\cos \theta = (1-t^2)/(1+t^2)$, $d\theta = 2dt/(1+t^2)$.
- (b) Some indefinite integrals:

$$\int (a^2 - x^2)^{-1/2} dx = \sin^{-1} \left(\frac{x}{a} \right), \quad |x| < a.$$

$$\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1} \left(\frac{x}{a} \right) = \ln \left\{ \frac{x}{a} + \left(1 + \frac{x^2}{a^2} \right)^{1/2} \right\}.$$

$$\int (x^2 - a^2)^{-1/2} dx = \cosh^{-1} \left(\frac{x}{a} \right) = \ln \left| \frac{x}{a} + \left(\frac{x^2}{a^2} - 1 \right)^{1/2} \right|.$$

$$\int (a^2 + x^2)^{-1} dx = \left(\frac{1}{a} \right) \tan^{-1} \left(\frac{x}{a} \right).$$

6. NUMERICAL METHODS

- (a) Approximate solution of an algebraic equation:

If a root of $f(x) = 0$ occurs near $x = a$, take $x_0 = a$ and
 $x_{n+1} = x_n - [f(x_n)/f'(x_n)]$, $n = 0, 1, 2, \dots$

(Newton Raphson method).

- (b) Formulae for numerical integration: Write $x_n = x_0 + nh$, $y_n = y(x_n)$.

i. Trapezium rule (1-strip): $\int_{x_0}^{x_1} y(x) dx \approx (h/2) [y_0 + y_1]$.

ii. Simpson's rule (2-strip): $\int_{x_0}^{x_2} y(x) dx \approx (h/3) [y_0 + 4y_1 + y_2]$.

- (c) Richardson's extrapolation method: Let $I = \int_a^b f(x) dx$ and let I_1, I_2 be two

estimates of I obtained by using Simpson's rule with intervals h and $h/2$.

Then, provided h is small enough,

$$I_2 + (I_2 - I_1)/15,$$

is a better estimate of I .

7. LAPLACE TRANSFORMS

Function	Transform	Function	Transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t) dt$	$a f(t) + b g(t)$	$a F(s) + b G(s)$
d/dt	$s F(s) - f(0)$	$d^2 f/dt^2$	$s^2 F(s) - s f(0) - f'(0)$
$e^{at} f(t)$	$F(s-a)$	$t f(t)$	$-dF(s)/ds$
$(\partial/\partial \alpha) f(t, \alpha)$	$(\partial/\partial \alpha) F(s, \alpha)$	$\int_0^t f(t) dt$	$F(s)/s$
$\int_0^t f(u) g(t-u) du$	$F(s) G(s)$		

1	1/s	$t^n (n = 1, 2, \dots)$	$n!/s^{n+1}, (s > 0)$
e^{at}	$1/(s-a), (s > a)$	$\sin \omega t$	$\omega/(s^2 + \omega^2), (s > 0)$
$\cos \omega t$	$s/(s^2 + \omega^2), (s > 0)$	$H(t-T) = \begin{cases} 0, & t < T \\ 1, & t > T \end{cases}$	$e^{-sT}/s, (s, T > 0)$

8. FOURIER SERIES

If $f(x)$ is periodic of period $2L$, then $f(x+2L) = f(x)$, and

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \text{ where}$$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx, \quad n = 0, 1, 2, \dots, \text{ and}$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx, \quad n = 1, 2, 3, \dots$$

Parseval's theorem

$$\frac{1}{L} \int_{-L}^L [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

B.ENG. AND M.ENG. EXAMINATIONS 2008

PART I : MATHEMATICS 2 (ELECTRICAL ENGINEERING)

Date Thursday 5th June 2008 10.00 am - 1.00 pm

Answer EIGHT questions.

A mathematical formulae sheet is provided.

[Before starting, please make sure that the paper is complete; there should be 6 pages, with a total of 10 questions. Ask the invigilator for a replacement if your copy is faulty.]

1. (i) The hyperbolic sine function is defined as follows:

$$f(x) = \sinh(x) = \frac{e^x - e^{-x}}{2}.$$

Sketch a graph of the following functions, stating whether each is even or odd:

(a) $f(x)$,

(b) $(f(x))^2$,

(c) $\frac{1}{2}(f(x) + f(-x))$.

By re-writing the equation

$$\sinh(x) = y$$

as a quadratic equation for e^x , find $\sinh^{-1}(y)$ in terms of a logarithm.

- (ii) The hyperbolic cosine is the function

$$\cosh(x) = \frac{1}{2}(e^x + e^{-x}).$$

Show that there are constants A and B , independent of x such that

$$(\sinh(x))^2 = A \cosh(2x) + B.$$

What are the values of A and B ?

Evaluate the limit

$$\lim_{x \rightarrow 0} \frac{\sinh(x)}{x}.$$

2. Consider the function

$$f(x) = \left(1 - \frac{2x}{x+1}\right)^2.$$

Find the stationary points of f and provide the details of the calculation that determines their nature. Hence draw a sketch of f on the entire real line, noting any horizontal and vertical asymptotes.

Use the information contained in your first graph to sketch a graph of the function $e^{-f(x)}$ also on the real line.

PLEASE TURN OVER

3. Let us assume that the sine function can be differentiated an arbitrary number of times at all points in its domain.

- (i) Find the first and second derivatives of the sinc function

$$\text{sinc}(x) = \frac{\sin(x)}{x}$$

for $x \neq 0$.

- (ii) Use l'Hôpital's rule to evaluate the limit

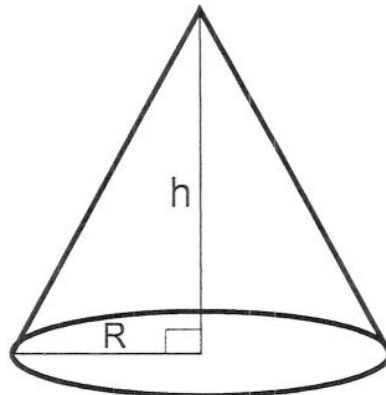
$$\lim_{x \rightarrow 0} \frac{\sin(x)}{x}.$$

Hence deduce the value of the sinc function at $x = 0$ if you are told that the sinc function is continuous at $x = 0$.

- (iii) Use l'Hôpital's rule to evaluate the value of the first and second derivatives of the sinc function at $x = 0$, both of which are continuous functions at $x = 0$.

- (iv) Draw a graph of the sinc function for $-\pi < x < \pi$.

4. (i) Find the surface area of revolution of the function $f(x) = \sqrt{R^2 - x^2}$ for $-R \leq x \leq R$ and hence deduce the surface area of a sphere of radius R .
- (ii) Use a similar method to determine the surface area of a circular cone of vertical height h and base radius R .



5. (i) Determine whether or not the following series converge:

(a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$,

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{2n+1}}$,

(c) $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$.

- (ii) Use the integral test to find a number $M < 1.29$ such that

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2} \leq M.$$

6. Obtain the Fourier series of the 2π -periodic, real function f defined for x in the interval $(-\pi, \pi]$ by

$$f(x) = \begin{cases} 1 & ; \quad 0 \leq x \leq \pi, \\ -1 & ; \quad -\pi < x < 0. \end{cases}$$

If the Fourier series of f is denoted by $F(x)$, explain why there exists at least one real number x_0 such that $F(x_0) \neq f(x_0)$ and provide a value of x_0 to illustrate this inequality between a function and its Fourier series.

Finally, find the value of

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}.$$

PLEASE TURN OVER

7. (i) If $V = \ln(r)$ where $r = \sqrt{x^2 + y^2}$ show that

$$\frac{\partial^2 V}{\partial x^2} = \frac{1}{r^2} - \frac{2x^2}{r^4} .$$

Verify that V satisfies the equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 .$$

- (ii) Let $w(x, y, z) = xy + z$ and let $x = \cos(t)$, $y(t) = \sin(t)$, $z = t$.

Show that

$$\frac{dw}{dt} = 1 + \cos(2t) .$$

8. Compute an approximation to the integral

$$I = \int_0^1 (\theta \cos(\theta) + 1) d\theta$$

using:

- (i) the trapezium rule with one interval;
- (ii) the trapezium rule with two intervals;
- (iii) Simpson's rule with two intervals.

Compare your results with the exact value of the integral.

You should work to 4 decimal places throughout.

9. (i) Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = e^{3x}.$$

- (ii) Find the solution of the differential equation

$$(y^2 - x^2) \frac{dy}{dx} + 2xy = 0$$

in implicit form.

10. The integral

$$\int e^{kx} dx = \frac{e^{kx}}{k} + \text{Const}$$

is known to hold for all complex k and real x . Use this result to deduce the values of

$$a_n = \int_{-\pi}^{\pi} e^{-x} \cos(nx) dx \quad \text{and} \quad b_n = \int_{-\pi}^{\pi} e^{-x} \sin(nx) dx$$

in terms of π where n is a fixed integer.

Using the complex Fourier series representation of the real, 2π -periodic function $f(x)$ that coincides with e^{-x} on $(-\pi, \pi)$,

$$f(x) = c_0 + 2 \sum_{n=1}^{\infty} \text{Re}(c_n e^{inx}) \quad \text{with} \quad c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$$

evaluate

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2}$$

in terms of e^{π} and $e^{-\pi}$ using the following version of Parseval's theorem

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} (f(x))^2 dx = c_0^2 + 2 \sum_{n=1}^{\infty} |c_n|^2.$$

END OF PAPER

MATHEMATICAL FORMULAE

1. VECTOR ALGEBRA

$$a = a_1i + a_2j + a_3k = (a_1, a_2, a_3)$$

Scalar (dot) product:

$$a \cdot b = a_1b_1 + a_2b_2 + a_3b_3$$

Vector (cross) product:

$$a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Scalar triple product:

$$[a, b, c] = a \cdot b \times c = b \cdot c \times a = c \cdot a \times b = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product:

$$a \times (b \times c) = (c \cdot a)b - (b \cdot a)c$$

2. SERIES

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \dots \quad (\alpha \text{ arbitrary, } |x| < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^n \frac{x^{2n}}{(2n)!} + \dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \dots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^n \frac{x^{n+1}}{(n+1)} + \dots \quad (-1 < x \leq 1)$$

3. TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin(a+b) = \sin a \cos b + \cos a \sin b;$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b.$$

$$\cos iz = \cosh z; \quad \cosh iz = \cos z; \quad \sin iz = i \sinh z; \quad \sinh iz = i \sin z.$$

4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f D^n g + \binom{n}{1} Df D^{n-1} g + \dots + \binom{n}{n} D^n f D^{n-n} g + \dots + D^n f g.$$

(b) Taylor's expansion of $f(x)$ about $x = a$:

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \dots + \frac{h^n}{n!}f^{(n)}(a) + \epsilon_n(h),$$

where $\epsilon_n(h) = h^{n+1}f^{(n+1)}(a + \theta h)/(n+1)!$, $0 < \theta < 1$.

(c) Taylor's expansion of $f(x, y)$ about (a, b) :

$$f(a+h, b+k) = f(a, b) + [hf_x + kf_y]_{a,b} + \frac{1}{2!} [h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}]_{a,b} + \dots$$

(d) Partial differentiation of $f(x, y)$:

i. If $y = y(x)$, then $f = F(x)$, and $\frac{dF}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx}$.

ii. If $x = x(t)$, $y = y(t)$, then $f = F(t)$, and $\frac{dF}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$.

iii. If $x = x(u, v)$, $y = y(u, v)$, then $f = F(u, v)$, and

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u}, \quad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial v}.$$

(e) Stationary points of $f(x, y)$ occur where $f_x = 0$, $f_y = 0$ simultaneously. Let (a, b) be a stationary point: examine $D = [f_{xx}f_{yy} - (f_{xy})^2]_{a,b}$.

If $D > 0$ and $f_{xx}(a, b) < 0$, then (a, b) is a maximum;

If $D > 0$ and $f_{xx}(a, b) > 0$, then (a, b) is a minimum;

If $D < 0$ then (a, b) is a saddle-point.

(f) Differential equations:

i. The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating factor $I(x) = \exp[\int P(x)(dx)]$, so that $\frac{d}{dx}(Iy) = IQ$.

ii. $P(x, y)dx + Q(x, y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y$.

5. INTEGRAL CALCULUS

(a) An important substitution: $\tan(\theta/2) = t$:

$$\sin \theta = 2t/(1+t^2), \quad \cos \theta = (1-t^2)/(1+t^2), \quad d\theta = 2dt/(1+t^2).$$

(b) Some indefinite integrals:

$$\int (a^2 - x^2)^{-1/2} dx = \sin^{-1} \left(\frac{x}{a} \right), \quad |x| < a.$$

$$\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1} \left(\frac{x}{a} \right) = \ln \left\{ \frac{x}{a} + \left(1 + \frac{x^2}{a^2} \right)^{1/2} \right\}.$$

$$\int (x^2 - a^2)^{-1/2} dx = \cosh^{-1} \left(\frac{x}{a} \right) = \ln \left| \frac{x}{a} + \left(\frac{x^2}{a^2} - 1 \right)^{1/2} \right|.$$

$$\int (a^2 + x^2)^{-1} dx = \left(\frac{1}{a} \right) \tan^{-1} \left(\frac{x}{a} \right).$$

6. NUMERICAL METHODS

(a) Approximate solution of an algebraic equation:

If a root of $f(x) = 0$ occurs near $x = a$, take $x_0 = a$ and $x_{n+1} = x_n - [f(x_n)/f'(x_n)]$, $n = 0, 1, 2, \dots$

(Newton Raphson method).

(b) Formulae for numerical integration: Write $x_n = x_0 + nh$, $y_n = y(x_n)$.

i. Trapezium rule (1-strip): $\int_{x_0}^{x_1} y(x) dx \approx (h/2) [y_0 + y_1]$.

ii. Simpson's rule (2-strip): $\int_{x_0}^{x_2} y(x) dx \approx (h/3) [y_0 + 4y_1 + y_2]$.

(c) Richardson's extrapolation method: Let $I = \int_a^b f(x) dx$ and let I_1, I_2 be two

estimates of I obtained by using Simpson's rule with intervals h and $h/2$.

Then, provided h is small enough,

$$I_2 + (I_2 - I_1)/15,$$

is a better estimate of I .

7. LAPLACE TRANSFORMS

Function	Transform	Function	Transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t) dt$	$a f(t) + b g(t)$	$a F(s) + b G(s)$
d/dt	$s F(s) - f(0)$	$d^2 f/dt^2$	$s^2 F(s) - s f(0) - f'(0)$
$e^{at} f(t)$	$F(s-a)$	$t f(t)$	$-dF(s)/ds$
$(\partial/\partial \alpha) f(t, \alpha)$	$(\partial/\partial \alpha) F(s, \alpha)$	$\int_0^t f(t) dt$	$F(s)/s$
$\int_0^t f(u) g(t-u) du$	$F(s) G(s)$		

1	1/s	$t^n (n = 1, 2, \dots)$	$n! / s^{n+1}, (s > 0)$
e^{at}	$1/(s-a), (s > a)$	$\sin \omega t$	$\omega / (s^2 + \omega^2), (s > 0)$
$\cos \omega t$	$s / (s^2 + \omega^2), (s > 0)$	$H(t-T)$	$e^{-sT} / s, (s, T > 0)$

8. FOURIER SERIES

If $f(x)$ is periodic of period $2L$, then $f(x+2L) = f(x)$, and

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \text{ where}$$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx, \quad n = 0, 1, 2, \dots, \text{ and}$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx, \quad n = 1, 2, 3, \dots$$

Parseval's theorem

$$\frac{1}{L} \int_{-L}^L [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$