UNIVERSITY OF LONDON [I(2)E 2001]

B.ENG. AND M.ENG. EXAMINATIONS 2001

For Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examination for the Associateship.

PART I : MATHEMATICS 2 (ELECTRICAL ENGINEERING)

Thursday 7th June 2001 10.00 am - 1.00 pm

Answer EIGHT questions.

[*Before starting, please make sure that the paper is complete; there should be 6 pages, with a total of 10 questions. Ask the invigilator for a replacement if your copy is faulty.*]

1. Sketch the contour $f(x,\, y) \;=\; 0$ for the function

$$f(x,\, y) \;=\; x(\, y^2 - [x^2 - 1]^2\,)\ ,$$

showing the coordinates of the points where the contour's components intersect. By considering the partial derivative of f with respect to y, show that at each of the stationary points of f, either $x = 0$ or $y = 0$.

Find the x- and y- coordinates of each of the six stationary points of f.

With the aid of your sketch, or otherwise, classify each stationary point as a saddle point, maximum or minimum.

2. (i) The polar coordinates r and θ are related to the Cartesian coordinates x and y via the relations

$$x \;=\; r\cos\theta, \qquad y \;=\; r\sin\theta.$$

For a function $f(x,\, y)$, find $\dfrac{\partial f}{\partial r}$ and $\dfrac{\partial f}{\partial \theta}$ in terms of $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ and hence show that

$$\frac{\partial f}{\partial x} \;=\; \cos\theta\frac{\partial f}{\partial r} \;-\; \frac{\sin\theta}{r}\,\frac{\partial f}{\partial \theta}$$

and

$$\frac{\partial f}{\partial y} \;=\; \sin\theta\,\frac{\partial f}{\partial r} \;+\; \frac{\cos\theta}{r}\,\frac{\partial f}{\partial \theta}\ .$$

(ii) Find the value of the constant α such that the function $V(t,\, x) \;=\; t^{\alpha}e^{-x^2/4t}$ satisfies the equation

$$\frac{\partial V}{\partial t} \;=\; \frac{\partial^2 V}{\partial x^2}\ .$$

PLEASE TURN OVER

3. (i) Find the two points at which the curves $y = x^3$ and $y^2 = 32x$ intersect.

   (ii) Use Simpson's rule to estimate the area enclosed by the two curves between their points of intersection, using function values at 3 values of x (including the points of intersection). Give your estimate correct to 4 significant figures.

   (iii) Repeat the application of Simpson's rule to obtain an improved estimate for the area (to the same precision), using function values at 5 values of x.

   (iv) Use Richardson's extrapolation to find a further improved estimate for the area.

   (v) Find the exact value of the area, and hence find the percentage error in your estimate from part (iv).

4. (i) Given the position vectors

$$\mathbf{a} = (1,\, 1,\, 2), \qquad \mathbf{b} = (0,\, 2,\, 1)\,,$$

   find the lengths of $\mathbf{a}$ and $\mathbf{b}$, and the angle between them. Evaluate $\mathbf{a} \times \mathbf{b}$ and hence find the area of the triangle with sides $\mathbf{a}$ and $\mathbf{b}$.

   (ii) The vector $\mathbf{r}$ satisfies the equation

$$\mathbf{r} + \mathbf{c} \times \mathbf{r} = \mathbf{d}\,,$$

   where $\mathbf{c}$ and $\mathbf{d}$ are given vectors. By taking the scalar product and the vector product of this equation with $\mathbf{c}$, or otherwise, solve for $\mathbf{r}$.

   Verify your result for the special case where $\mathbf{c} = (2,\, 0,\, 1)$, $\mathbf{d} = (1,\, 2,\, 0)$.

5. The straight lines L_1 and L_2 are given by

$$L_1 \; : \qquad (x,\, y,\, z) \;=\; (-1,\, -2,\, 5) \;+\; s(1,\, 1,\, -2)\,,$$

$$L_2 \; : \qquad (x,\, y,\, z) \;=\; (5,\, 4,\, 0) \;+\; t(3,\, 1,\, -3)\,.$$

Let $\mathbf{x}_1(s)$ and $\mathbf{x}_2(t)$ denote points on L_1 and L_2 respectively. Find values of s and t so that $\mathbf{x}_1(s) - \mathbf{x}_2(t)$ is perpendicular to both L_1 and L_2.

Hence, or otherwise, find the line L that passes through L_1 and L_2 and is perpendicular to each of them. Find the shortest distance p between the lines L_1 and L_2.

Find the distance d of the line L from the origin.

6. A matrix A is given in terms of a constant a as

$$A \;=\; \begin{bmatrix} 3 & 3 & 4 \\ 6 & 7 & 9 \\ 3 & 4 & a \end{bmatrix}\,.$$

(i) Express A in the form $A = LU$ where L is a lower triangular matrix with 1's on the diagonal and U is upper triangular.

(ii) Hence find the conditions on a for which

$$A\mathbf{x} \;=\; \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

has

(a) exactly one solution;

(b) infinitely many solutions.

(iii) Find the solution to the simultaneous equations

$$\begin{array}{rcrcrcl} 3x & + & 3y & + & 4z & = & 1 \\ 6x & + & 7y & + & 9z & = & 1 \\ 3x & + & 4y & + & 4z & = & 1. \end{array}$$

PLEASE TURN OVER

7. (i) Find the solution of the differential equation

$$2xy\,\frac{dy}{dx} = x^2 + y^2$$

which satisfies the condition $y = 1$ at $x = 6$.

(ii) Show that the equation

$$\left[x - \frac{1}{1+x^2}\right] dx + 2y^2\,dy = 0$$

is exact and hence, or otherwise, find its general solution.

8. (i) Find the general solution of the homogeneous equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0\,.$$

(ii) Find the solution of the equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x$$

which satisfies the boundary conditions

$$y = 1 \text{ and } \frac{dy}{dx} = -1 \text{ at } x = 0\,.$$

9. The function $y(x)$ satisfies:

$$y \;=\; 1 \;\text{ and }\; \frac{dy}{dx} \;=\; 0 \;\text{ at }\; x = 0\,,$$

$$(1-x^2)\,\frac{d^2y}{dx^2} \;-\; 2x\,\frac{dy}{dx} \;+\; 2y \;=\; 0\,.$$

Differentiate the above differential equation n times using the Leibniz formula. Hence show that

$$y^{(n+2)}\,(0) \;=\; (n^2+n-2)\,y^{(n)}\,(0)\,.$$

Use this formula to write down the Maclaurin series expansion for y up to and including the term of order x^{10}. Write down the general term of the series and find the radius of convergence of the series using the ratio test.

10. Find the Fourier series for each of the following functions:

(i) $$f(x) \;=\; |x|, \qquad -\pi < x < \pi\,,$$

(ii) $$f(x) \;=\; x, \qquad -\pi < x < \pi\,.$$

From the series obtained for (i), deduce that

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \;=\; \frac{\pi^2}{8}\,.$$

By application of Parseval's Identity to the series obtained for (ii), deduce that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \;=\; \frac{\pi^2}{6}\,.$$

END OF PAPER

MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

SESSION : 2000 - 2001

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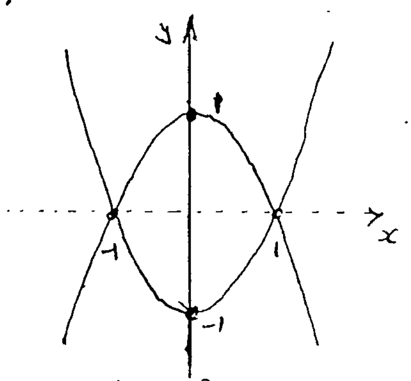
PAPER 2

QUESTION —

SOLUTION 1

$f(x,y) = x(y-[x^2-1])(y+[x^2-1])$

$\therefore f(x,y)=0 \iff x=0$ or $y=x^2-1$ or $y=1-x^2$ 2



2

$f_y = 2xy \quad \therefore \quad f_y = 0 \Rightarrow x=0$ or $y=0$ 1

Now, $f(x,y) = xy^2 - (x^5-2x^3+x)$

and therefore $f_x = y^2 - (5x^4-6x^2+1)$ 1

$\underline{x=0,\ f_x=0} \Rightarrow y^2-1=0$: sp's at $\underline{(0,\pm1)}$ 2

$\underline{y=0,\ f_x=0} \Rightarrow 5x^4-6x^2+1=0 \iff (5x^2-1)(x^2-1)=0$

$\iff x=\pm\frac{1}{\sqrt5},\ x=\pm1$

sp's at $\underline{(\pm\frac{1}{\sqrt5},0)}$, $\underline{(\pm1,0)}$ 4

From sketch, it is clear that $(0,\pm1)$ and $(\pm1,0)$ are <u>saddles</u>, and that $(\pm\frac{1}{\sqrt5},0)$ are <u>extrema</u>. 1

Now, if $y=0$ and $0<x<1$ then $f(x,y)=-x(1-x^2)^2<0$, and similarly if $y=0$ and $-1<x<0$ then $f(x,y)>0$. Since f is continuous, it follows that $(\frac{1}{\sqrt5},0)$ is a <u>min</u> and that $(-\frac{1}{\sqrt5},0)$ is a <u>max</u>. 2

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MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

SESSION : 2000 - 2001

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PAPER I(2)

QUESTION

SOLUTION 2

(i) $\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = \cos\theta\frac{\partial f}{\partial x} + \sin\theta\frac{\partial f}{\partial y}$

$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = -r\sin\theta\frac{\partial f}{\partial x} + r\cos\theta\frac{\partial f}{\partial y}$ 4

$\therefore \cos\theta\frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}(\cos^2\theta + \sin^2\theta) + \frac{\partial f}{\partial y}(\sin\theta\cos\theta - \cos\theta\sin\theta)$

$= \frac{\partial f}{\partial x}$; 4

$\sin\theta\frac{\partial f}{\partial r} + \frac{\cos\theta}{r}\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}(\sin\theta\cos\theta - \cos\theta\sin\theta) + \frac{\partial f}{\partial y}(\cos^2\theta + \sin^2\theta)$

$= \frac{\partial f}{\partial y}$.

(ii) $V = t^\alpha e^{-(x^2/4t)}$

$\therefore V_t = t^\alpha\left(\frac{x^2}{4t^2}\right)e^{-(x^2/4t)} + \alpha t^{\alpha-1}e^{-(x^2/4t)}$ 2

$V_x = t^\alpha\left(-\frac{x}{2t}\right)e^{-(x^2/4t)}$ 2

$\therefore V_{xx} = -\frac{1}{2}t^{\alpha-1}e^{-(x^2/4t)} + t^\alpha\left(-\frac{x}{2t}\right)^2 e^{-(x^2/4t)}$ 2

$\therefore V_t = V_{xx}$ if $\alpha = -\frac{1}{2}$. 1

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EXAMINATION QUESTION / SOLUTION

SESSION : 2000 - 2001

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PAPER I (ii)

~~QUESTION~~

SOLUTION 3

(i) $\left.\begin{aligned} y &= x^3 \\ y^2 &= 32x \end{aligned}\right\}$ intersect at $x^6 = 32x$

$x(x^5 - 32) = 0$

$x = 0$ or 2

(the only real solutions) 2

(ii) $A_1 = \frac{1}{3}\left[0 + 4(\sqrt{32} - 1) + 0\right] = 6.209$ 3

(iii) $A_2 = \frac{1}{2\times3}\left[0 + 4\left(\sqrt{16} - \frac{1}{8}\right) + 2(\sqrt{32} - 1) + 4\left(\sqrt{48} - \frac{27}{8}\right) + 0\right]$

$= 6.504$ 3

(iv) $A_3 = 6.504 + \dfrac{6.504 - 6.209}{15} = 6.524$ 2

(v) $A_4 = \displaystyle\int_0^2 \left(\sqrt{32}\,x^{\frac{1}{2}} - x^3\right) dx$

$= \left[\frac{2}{3}\sqrt{32}\,x^{\frac{3}{2}} - \frac{x^4}{4}\right]_0^2$

$= \frac{2}{3}\sqrt{32}\,2^{\frac{3}{2}} - 4 = 6.667$ 4

Error in $A_3 = \dfrac{(6.667 - 6.524)}{6.667} \times 100 = 2.1\%$ 1

Setter : R.T. FENNER Setter's signature : R.T. Fenner

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EXAMINATION QUESTION / SOLUTION

SESSION : 2000 - 2001

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PAPER 2

QUESTION

SOLUTION 5

(a) $a^2 = 1+1+4 = 6$, $b^2 = 0+4+1 = 5$ 2

$\underline{a}\cdot\underline{b} = 0+2+2 = 4 = ab\cos\theta \Rightarrow \cos\theta = \frac{4}{\sqrt{30}}$

$\theta = \cos^{-1} 4/\sqrt{30}$ 2

$\underline{a}\times\underline{b} = (-3,-1,2) \Rightarrow |\underline{a}\times\underline{b}| = \sqrt{14}$ 2

Now $|\underline{a}\times\underline{b}| = ab\sin\theta$ & Area $= \frac{1}{2}ab\sin\theta$ $\therefore$ $\underline{A = \frac{1}{2}\sqrt{14}}$

(b) $\underline{c}\cdot\underline{r} = \underline{c}\cdot\underline{d}$ (noting $\underline{c}\cdot(\underline{c}\times\underline{r}) = 0$) 1

Also $\underline{c}\times\underline{r} + \underline{c}\times(\underline{c}\times\underline{r}) = \underline{c}\times\underline{d}$

$\Leftrightarrow \underline{c}\times\underline{r} + (\underline{c}\cdot\underline{r})\underline{c} - c^2\underline{r} = \underline{c}\times\underline{d}$ 1

Subst. for $\underline{c}\times\underline{r}$ (in terms of $\underline{r}$) & $\underline{c}\cdot\underline{r} = \underline{c}\cdot\underline{d} \Rightarrow$

$-\underline{r} + \underline{d} + (\underline{c}\cdot\underline{d})\underline{c} - c^2\underline{r} = \underline{c}\times\underline{d} \Rightarrow$

$\underline{r} = \frac{1}{1+c^2}\left\{\underline{d} + (\underline{c}\cdot\underline{d})\underline{c} - \underline{c}\times\underline{d}\right\}$ 3

Special case: $\underline{c} = (2,0,1)$, $\underline{d} = (1,2,0)$

$\Rightarrow \underline{c}\times\underline{d} = (-2,1,4)$, $\underline{c}\cdot\underline{d} = 2$, $c^2 = 5$

So solution is $\underline{r} = \frac{1}{6}(7,1,-2)$ 2

Check it satisfies original equation

$\underline{c}\times\underline{r} = \frac{1}{6}(-1,11,2)$ so

$\underline{r} + (\underline{c}\times\underline{r}) = \frac{1}{6}(6,12,0) = (1,2,0) = \underline{d}$ ✓ 2

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MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

SESSION : 2000 - 2001

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PAPER	QUESTION	SOLUTION
2		6

$\underline{x}_2 - \underline{x}_1 = (6+3t-s,\ 6+t-s,\ -5-3t+2s)$ 1

$\underline{x}_2 - \underline{x}_1 \perp$ to $L_1 \Leftrightarrow (\underline{x}_2-\underline{x}_1)\cdot(1,1,-2) = 0 \Rightarrow$

$$22 + 10t - 6s = 0.$$ 2

$\underline{x}_2 - \underline{x}_1 \perp$ to $L_2 \Leftrightarrow (\underline{x}_2-\underline{x}_1)\cdot(3,1,-3) = 0 \Rightarrow$ 1

$$39 + 19t - 10s = 0$$

Solution: $\underline{t=-1}$, $\underline{s=2}$ 2

Feet of perps. at $\underline{x}_1\,(s=2) = (1,0,1)$
and $\underline{x}_2\,(t=-1) = (2,3,3)$

Shortest dist. $p = |\underline{x}_2 - \underline{x}_1| = (1+9+4)^{\frac{1}{2}} = \underline{\underline{\sqrt{14}}}$ 2

To find L: Direction is $\underline{x}_2 - \underline{x}_1 = (1,3,2)$
& passes through $\underline{x}_1 = (1,0,1)$ so
$L: \quad \underline{x} = (1,0,1) + \lambda(1,3,2)$ 2

If $q(\lambda)$ denotes dist of point $\underline{x}(\lambda)$ from origin, then

$$q^2 = (\lambda+1)^2 + 9\lambda^2 + (2\lambda+1)^2$$

Minimise w.r. to $\lambda \Rightarrow 2(\lambda+1) + 18\lambda + 4(2\lambda+1) = 0$

(or Fix λ so $\underline{x}(\lambda)\cdot(1,3,2) = 0 \Rightarrow (\lambda+1) + 9\lambda + 2(2\lambda+1) = 0$

so $\underline{\lambda = -3/14}$ 5

$d = q(\lambda = -3/14) = \underline{\underline{\dfrac{\sqrt{266}}{14}}}$

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MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

SESSION : 2000 - 2001

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PAPER	QUESTION	SOLUTION
I(2)		8

(i)

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \qquad U = \begin{bmatrix} 3 & 3 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & a-5 \end{bmatrix}$$ 6

(ii) Hence (a) if $a \neq 5$ and (b) otherwise. 4

(iii) The equations have system matrix A with $a = 4$. Hence

$$L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}, \quad L^{-1}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

and 5

$$U^{-1} = \begin{bmatrix} 1/3 & -1 & 1/3 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}, \quad \mathbf{x} = \begin{pmatrix} 5/3 \\ 0 \\ -1 \end{pmatrix}$$

Setter : Reich

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Checker : CROWDY

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MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

SESSION : 2000 - 2001

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PAPER 1 (2)

QUESTION

SOLUTION 9

$$2xy\frac{dy}{dx} = x^2+y^2$$

So $\frac{dy}{dx} = \frac{x^2+y^2}{2xy}$. ②

Note this is homogeneous so put $y = vx$ ②

$$\frac{dy}{dx} = x\frac{dv}{dx} + v = \frac{v}{2} + \frac{1}{2v}$$

So $x\frac{dv}{dx} = \frac{1}{2v} - \frac{v}{2}$

$= \frac{1}{2}\left(\frac{1-v^2}{v}\right)$ i.e $\frac{dx}{x} = \frac{2v\,dv}{1-v^2}$ ②

$\ln|x| = -\ln|1-v^2| + C$ ①

$x(1 - y^2/x^2) = A \Rightarrow x^2 - y^2 = Ax$

When $x = 6$, $y = 1 \Rightarrow 36 - 1 = 6A \Rightarrow A = 35/6$ ①

So $x^2 - y^2 = 35x/6$. ①

Test for exactness is $\frac{\partial}{\partial y}\left[x - \frac{1}{1+x^2}\right] = \frac{\partial}{\partial x} 2y^2$ ✓ ②

$\therefore \frac{\partial U}{\partial y} = 2y^2 \quad U = \frac{2}{3}y^3 + f(x)$ ①

$\frac{\partial U}{\partial x} = f'(x) = x - \frac{1}{1+x^2}$ ②

$\therefore f(x) = x^2/2 - \tan^{-1}x$.

$\therefore$ General solution is $\frac{1}{2}x^2 - \tan^{-1}x + \frac{2}{3}y^3 = C$ ①

Setter : J. R. CASH Setter's signature : JR Cash

Checker : C. J. RIDLER-ROWE Checker's signature : CJ Ridler Rowe

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MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

SESSION : 2000 - 2001

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PAPER 2

QUESTION

SOLUTION 11

i) $$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$$

Aux equ: $y \sim e^{mx} \Rightarrow m^2 - 2m + 1 = 0 \quad \therefore m = 1$, twice 2

Double root, $\therefore$ soln has form

$$y = (Ax + B)e^{x}, \quad A, B \text{ arb. consts.}$$ 2

ii) $$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^{x}$$ 1

C.F. $\quad y_c = (Ax + B)e^{x}$, as in i)

PI : Try $\quad y_p(x) = p(x)e^{x}$ 2

$$\Rightarrow \quad y_p' = (p' + p)e^{x}$$

$$y_p'' = (p'' + 2p' + p)e^{x}$$

Hence, ode becomes

$$(p'' + 2p' + p)e^{x} - 2(p' + p)e^{x} + pe^{x} = xe^{x}$$

which simplifies to 3

$$p'' = x$$

$$\therefore \quad p' = \frac{x^2}{2}, \quad p = \frac{x^3}{6}$$ 2

$\therefore$ Part. Sol. $\quad y_p(x) = \frac{x^3}{6}e^{x}$.

Gen. Solⁿ is $\quad y = y_c + y_p$

$$= \left(\frac{x^3}{6} + Ax + B\right)e^{x}$$ 1

$$y(0) = 1 \quad \Rightarrow \quad B = 1$$

Also, $\quad y'(x) = y(x) + \left(\frac{x^2}{2} + A\right)e^{x} \quad \therefore\, y'(0) = -1$

$$\Rightarrow \quad y(0) + A = -1 \quad \text{or} \quad A = -2$$

$\therefore$ Req'd Sol is $\quad y(x) = \left(\frac{x^3}{6} - 2x + 1\right)e^{x}$. 2

Setter : J ELGIN Setter's signature : J Elgin

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MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

SESSION : 2000 - 2001

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PAPER I (ii)

QUESTION

SOLUTION 12

Differentiate n times to give

$$(1-x^2)y^{(n+2)}+n(-2x)y^{(n+1)}+\frac{n(n-1)(-2)}{2}y^{(n)}$$

$$-2xy^{(n+1)}-2ny^{(n)}+2y^{(n)}=0.$$ 4

Now set $x=0$ to give

$$y^{(n+2)}(0)-ny^{(n)}(0).(n-1)-2ny^{(n)}(0)+2y^{(n)}(0)=0,$$ 4

which we simplify to give

$$y^{(n+2)}(0)=(n^2+n-2)y^{(n)}(0)=(n+2)(n-1)y^{(n)}(0).$$

Now given that $y(0)=1$, $y'(0)=0$, $y^{n}(0)=0$ for n odd, 3

and $y^{(2)}(0)=2$, $y^{4}(0)=-8$, $y^{6}(0)=-144$, $y^{8}(0)=-5760$,

$y^{(10)}(0)=-(70)(5760)$, ..

$$\therefore y=1-x^2-x^3/3-x^6/5-x^8/7-x^{10}/9+\dots$$ 2

$\therefore$ general term is $\dfrac{-x^{2n-1}}{2n-1}$. Ratio of $\left|\dfrac{n+1^{\text{th}}\text{ term}}{n^{\text{th}}\text{ term}}\right|\to|x|^2$ 2

when $n\to\infty$ so converges $|x|<1$, diverges $|x|>1$.

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MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

SESSION : 2000 - 2001

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PAPER 2

QUESTION

SOLUTION 15

i) $f(x) = |x|, \quad -\pi < x < \pi$

Even fn; $\therefore b_n = 0$, 1

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \, dx = \frac{2}{\pi}\int_0^{\pi} x\cos nx \, dx$

$= \frac{2}{\pi}\left(x\frac{\sin nx}{n}\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin nx \, dx\right)$

$= -\frac{2}{\pi}\cdot\frac{1}{n}\frac{(-\cos nx)}{n}\Big|_0^{\pi} = \frac{-2}{\pi n^2}(1-(-1)^n)$.

$\therefore a_n = \begin{cases} -\frac{4}{\pi n^2}, & n \text{ odd} \\ 0, & n \text{ even.}\end{cases}$ 2

$a_0 = \frac{2}{\pi}\int_0^{\pi} x \, dx = \pi$ 1

$\therefore f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\cos((2n+1)x)}{(2n+1)^2}$ 1

$0 \le x < \pi$: $\quad x = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\cos((2n+1)x)}{(2n+1)^2}$

Put $x = 0$, $\Rightarrow \frac{\pi^2}{8} = \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$. 2

ii) $f(x) = x, \quad -\pi < x < \pi$. 1

Odd fn $\therefore a_n = 0, \quad a_0 = 0$

$b_n = \frac{2}{\pi}\int_0^{\pi} x\sin nx \, dx = \frac{2}{\pi}\left(-\frac{x\cos nx}{n}\Big|_0^{\pi} + \int_0^{\pi}\frac{\cos nx}{n}dx\right)$

$= -\frac{2(-1)^n}{n} + 0 \quad \therefore b_n = \frac{2(-1)^{n+1}}{n}$ 3

$\therefore f(x) = \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\sin nx$ 1

Parseval: $\frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\,dx = \sum_{n=1}^{\infty} b_n^2 = 4\sum_{n=1}^{\infty}\frac{1}{n^2}$ 1

LHS $= \frac{2}{\pi}\int_0^{\pi} x^2 \, dx = \frac{2\pi^2}{3}$

$\therefore \frac{2\pi^2}{3} = 4\sum_{n=1}^{\infty}\frac{1}{n^2} \Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$ 2

Setter : J ELGIN Setter's signature : J Elgin

Checker : F LEPPINGTON Checker's signature : F.U. Leppington