

B.ENG. and M.ENG. EXAMINATIONS 2009

MATHEMATICS (INFORMATION SYSTEMS ENGINEERING E1.11)

Date Wednesday 3rd June 2009 10.00 am - 1.00 pm

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

Answer ANY SEVEN questions.

Answers to questions from Section A and Section B should be written in different answer books.

CALCULATORS MAY NOT BE USED.

A mathematical formulae sheet is provided.

[Before starting, please make sure that the paper is complete; there should be 7 pages, with a total of 9 questions. Ask the invigilator for a replacement if your copy is faulty.]

SECTION A

1. (i) Find the real and imaginary parts of

$$\frac{2 + 3i}{3 + 2i} .$$

What are the modulus and argument of this number?

- (ii) If x and y are real, find the real and imaginary parts of

$$\sinh (x + i y) .$$

Hence show that

$$| \sinh (x + i y) |^2 = \sinh^2(x) + \sin^2(y) .$$

PLEASE TURN OVER

2. (i) Evaluate the partial sum

$$\sum_{n=1}^N \ln \left(\frac{n+2}{n+1} \right) .$$

Evaluate the limit

$$\lim_{n \rightarrow \infty} \ln \left(\frac{n+2}{n+1} \right) .$$

State whether the infinite series

$$\sum_{n=1}^{\infty} \ln \left(\frac{n+2}{n+1} \right)$$

is convergent or not.

Is the series

$$\sum_{n=1}^{\infty} (-1)^n \ln \left(\frac{n+2}{n+1} \right)$$

convergent?

- (ii) Explain what is meant by the *radius of convergence* of a power series

$$\sum_{n=0}^{\infty} a_n z^n .$$

Calculate the radii of convergence of the following two power series:

(a)
$$\sum_{n=0}^{\infty} \frac{2n+1}{\sqrt{n^2+1}} z^n ,$$

(b)
$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} z^n .$$

Do not attempt to sum either series.

PLEASE TURN OVER

3. (i) Evaluate the limits

(a) $\lim_{n \rightarrow 1} \frac{n^2 - 1}{n^2 - 3n + 2} ,$

(b) $\lim_{x \rightarrow \pi/2} (\sec x - \tan x) ,$

(c) $\lim_{n \rightarrow \infty} \left[n \left((n^3 + 3n)^{1/3} - (n^4 + n^2)^{1/4} \right) \right].$

(ii) Using L'Hôpital's rule, evaluate :

$$\lim_{x \rightarrow 0} \frac{\exp(x^2) - 1}{\sin^2(4x)} .$$

4. Evaluate the definite integrals

(i)

$$\int_0^{\frac{\pi}{2}} x^2 \sin(x) \, dx,$$

(ii)

$$\int_0^{\infty} \exp(-3x) \cos(4x) \, dx$$

and

$$\int_0^{\infty} \exp(-3x) \sin(4x) \, dx ,$$

(iii)

$$\int_0^{\infty} \frac{dx}{(x+1)(x^2+4)} .$$

PLEASE TURN OVER

5. Solve the ordinary differential equations

(i)

$$\frac{dy}{dx} = -\frac{4x+3y}{x+y};$$

(ii)

$$\frac{dy}{dx} - \tan(x)y = 1, \quad \text{with } y(0) = 1;$$

(iii)

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = \exp(-2x), \quad \text{with } y(0) = 0, \quad \text{and } y'(0) = 0.$$

In each case, find the most general solution possible.

PLEASE TURN OVER

SECTION B

6. (i) Let $u = u(x, y)$, where $x = r \cos \theta$, $y = r \sin \theta$. Show that

$$\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 = \left(\frac{\partial u}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial u}{\partial \theta} \right)^2 .$$

- (ii) Find the stationary points of the function

$$f(x, y) = (x^2 + 2y) e^{x+y}$$

and determine their nature.

7. (i) Find the inverse Laplace transform of the function

$$\frac{e^{-s}}{(s-2)^2} .$$

- (ii) Find functions $y(t)$ and $z(t)$ satisfying the simultaneous differential equations

$$\frac{d^2 y}{dt^2} - 2y + z = 0 ,$$

$$\frac{d^2 z}{dt^2} - 2z + y = 0 ,$$

such that $y(0) = z(0) = 0$, $y'(0) = z'(0) = 1$.

For (i) you may assume the shift rule

$$L (H_a(t) f(t-a)) = e^{-as} L (f(t)) ,$$

where $H_a(t)$ is the Heaviside function

$$H_a(t) = \begin{cases} 1, & t > a, \\ 0, & t \leq a. \end{cases}$$

For (ii) you may assume that

$$L \left(\frac{d^2 f}{dt^2} \right) = -f'(0) - sf(0) + s^2 L(f(t)) .$$

PLEASE TURN OVER

8. The function $f(x)$ is defined in the range $-\pi \leq x < \pi$ by

$$f(x) = \begin{cases} 1 + \frac{x}{\pi}, & -\pi \leq x \leq 0, \\ 1 - \frac{x}{\pi}, & 0 \leq x < \pi. \end{cases}$$

Sketch the graph of $f(x)$.

Find the Fourier series of $f(x)$ in the range $-\pi \leq x < \pi$.

By substituting a suitable value of x , deduce that

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

9. (i) Find, in terms of K , the determinant of the matrix

$$\begin{pmatrix} 1 & 5 & 3 \\ 5 & 1 & -K \\ 1 & 2 & K \end{pmatrix}.$$

- (ii) Let A be the matrix of part (i) with $K = 1$. Find all solutions $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ of the system of linear equations

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

- (iii) Let A be as in part (ii). Find all vectors $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ such that the system

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

has no solutions.

END OF PAPER

MATHEMATICAL FORMULAE

1. VECTOR ALGEBRA

$$a = a_1i + a_2j + a_3k = (a_1, a_2, a_3)$$

Scalar (dot) product: $a \cdot b = a_1b_1 + a_2b_2 + a_3b_3$

Vector (cross) product:

$$a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix};$$

Scalar triple product:

$$[a, b, c] = a \cdot b \times c = b \cdot c \times a = c \cdot a \times b = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product: $a \times (b \times c) = (c \cdot a)b - (b \cdot a)c$

2. SERIES

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \dots \quad (\alpha \text{ arbitrary, } |x| < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^n \frac{x^{2n}}{(2n)!} + \dots,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \dots,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^n \frac{x^{n+1}}{(n+1)} + \dots \quad (-1 < x \leq 1)$$

3. TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin(a+b) = \sin a \cos b + \cos a \sin b;$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b.$$

$$\cos iz = \cosh z; \quad \cosh iz = \cos z; \quad \sin iz = i \sinh z; \quad \sinh iz = i \sin z.$$

4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f D^n g + \binom{n}{1} Df D^{n-1} g + \dots + \binom{n}{n} D^n f D^{n-n} g + \dots + D^n f g.$$

(b) Taylor's expansion of $f(x)$ about $x = a$:

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \dots + \frac{h^n}{n!}f^{(n)}(a) + \epsilon_n(h),$$

where $\epsilon_n(h) = h^{n+1}f^{(n+1)}(u + \theta h)/(n+1)!$, $0 < \theta < 1$.

(c) Taylor's expansion of $f(x, y)$ about (a, b) :

$$f(a+h, b+k) = f(a, b) + [hf_x + kf_y]_{a,b} + \frac{1}{2!} [h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}]_{a,b} + \dots$$

(d) Partial differentiation of $f(x, y)$:

i. If $y = y(x)$, then $f = F(x)$, and $\frac{dF}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx}$.

ii. If $x = x(t)$, $y = y(t)$, then $f = F(t)$, and $\frac{dF}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$.

iii. If $x = x(u, v)$, $y = y(u, v)$, then $f = F(u, v)$, and

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u}, \quad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial v}.$$

(e) Stationary points of $f(x, y)$ occur where $f_x = 0$, $f_y = 0$ simultaneously.

Let (a, b) be a stationary point: examine $D = [f_{xx}f_{yy} - (f_{xy})^2]_{a,b}$.

If $D > 0$ and $f_{xx}(a, b) < 0$, then (a, b) is a maximum;

If $D > 0$ and $f_{xx}(a, b) > 0$, then (a, b) is a minimum;

If $D < 0$ then (a, b) is a saddle-point.

(f) Differential equations:

i. The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating factor $I(x) = \exp[\int P(x)(dx)]$, so that $\frac{d}{dx}(Iy) = IQ$.

ii. $P(x, y)dx + Q(x, y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y$.

5. INTEGRAL CALCULUS

(a) An important substitution: $\tan(\theta/2) = t$;
 $\sin \theta = 2t/(1+t^2)$, $\cos \theta = (1-t^2)/(1+t^2)$, $d\theta = 2dt/(1+t^2)$.

(b) Some indefinite integrals:

$$\int (a^2 - x^2)^{-1/2} dx = \sin^{-1} \left(\frac{x}{a} \right), \quad |x| < a.$$

$$\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1} \left(\frac{x}{a} \right) = \ln \left\{ \frac{x}{a} + \left(1 + \frac{x^2}{a^2} \right)^{1/2} \right\}.$$

$$\int (x^2 - a^2)^{-1/2} dx = \cosh^{-1} \left(\frac{x}{a} \right) = \ln \left| \frac{x}{a} + \left(\frac{x^2}{a^2} - 1 \right)^{1/2} \right|.$$

$$\int (a^2 + x^2)^{-1} dx = \left(\frac{1}{a} \right) \tan^{-1} \left(\frac{x}{a} \right).$$

6. NUMERICAL METHODS

(a) Approximate solution of an algebraic equation:

If a root of $f(x) = 0$ occurs near $x = a$, take $x_0 = a$ and
 $x_{n+1} = x_n - \{f(x_n)/f'(x_n)\}$, $n = 0, 1, 2, \dots$

(Newton Raphson method).

(b) Formulae for numerical integration: Write $x_n = x_0 + nh$, $y_n = y(x_n)$.

i. Trapezium rule (1-strip): $\int_{x_0}^{x_1} y(x) dx \approx (h/2) [y_0 + y_1]$.

ii. Simpson's rule (2-strip): $\int_{x_0}^{x_2} y(x) dx \approx (h/3) [y_0 + 4y_1 + y_2]$.

(c) Richardson's extrapolation method: Let $I = \int_a^b f(x) dx$ and let I_1, I_2 be two

estimates of I obtained by using Simpson's rule with intervals h and $h/2$.

Then, provided h is small enough,

$$I_2 + (I_2 - I_1)/15,$$

is a better estimate of I .

7. LAPLACE TRANSFORMS

Function	Transform	Function	Transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t) dt$	$af(t) + bg(t)$	$aF(s) + bG(s)$
df/dt	$sF(s) - f(0)$	$d^2 f/dt^2$	$s^2 F(s) - sf(0) - f'(0)$
$e^{at} f(t)$	$F(s-a)$	$tf(t)$	$-dF(s)/ds$
$(\partial/\partial\alpha)f(t, \alpha)$	$(\partial/\partial\alpha)F(s, \alpha)$	$\int_0^t f(t) dt$	$F(s)/s$
$\int_0^t f(u)g(t-u)du$	$F(s)G(s)$		
1	$1/s$	t^n ($n = 1, 2, \dots$)	$n!/s^{n+1}$, ($s > 0$)
e^{at}	$1/(s-a)$, ($s > a$)	$\sin \omega t$	$\omega/(s^2 + \omega^2)$, ($s > 0$)
$\cos \omega t$	$s/(s^2 + \omega^2)$, ($s > 0$)	$II(t-T) = \begin{cases} 0, & t < T \\ 1, & t > T \end{cases}$	e^{-sT}/s , ($s, T > 0$)

8. FOURIER SERIES

If $f(x)$ is periodic of period $2L$, then $f(x+2L) = f(x)$, and

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \text{ where}$$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx, \quad n = 0, 1, 2, \dots, \text{ and}$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx, \quad n = 1, 2, 3, \dots$$

Parseval's theorem

$$\frac{1}{L} \int_{-L}^L |f(x)|^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

A1. (i) To find the real and imaginary parts of

$$\frac{2+3i}{3+2i},$$

multiply numerator and denominator by the conjugate of the denominator:

$$\begin{aligned}\frac{2+3i}{3+2i} &= \frac{(2+3i)(3-2i)}{(3+2i)(3-2i)} \\ &= \frac{12+5i}{13}.\end{aligned}$$

So

$$\Re\left(\frac{2+3i}{3+2i}\right) = 12/13, \quad \Im\left(\frac{2+3i}{3+2i}\right) = 5/13.$$

The modulus and argument of this number are respectively

$$\left|\frac{2+3i}{3+2i}\right| = 1, \quad \arg\left(\frac{2+3i}{3+2i}\right) = \tan^{-1}(5/12).$$

(ii) Using the addition formula for \sinh , we have

$$\sinh(x+iy) = \sinh(x)\cos(y) + i\cosh(x)\sin(y).$$

Hence the modulus we need is, for real (x, y) ,

$$|\sinh(x+iy)|^2 = \sinh^2(x)\cos^2(y) + \cosh^2(x)\sin^2(y).$$

Substitute $\cos^2(y) = 1 - \sin^2(y)$, $\cosh^2(x) = 1 + \sinh^2(x)$, getting

$$\begin{aligned}|\sinh(x+iy)|^2 &= \sinh^2(x)(1 - \sin^2(y)) + (1 + \sinh^2(x))\sin^2(y) = \\ &= \sinh^2(x) + \sin^2(y).\end{aligned}$$

(Total
20)

ISE 1

②

(b) For

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} z^n,$$

we have

$$\begin{aligned} R &= \lim_{n \rightarrow \infty} \left| \frac{(n!)^2 (2(n+1))!}{(2n)! ((n+1)!)^2} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(2n+2)(2n+1)}{(n+1)^2} \right| = 4. \end{aligned}$$

4

(Total
20)

A2. (i) The partial sum

$$\sum_{n=1}^N \ln \left(\frac{n+2}{n+1} \right).$$

is

$$\begin{aligned} \sum_{n=1}^N \ln(n+2) - \ln(n+1) \\ = \ln(N+2) - \ln(2). \end{aligned}$$

The limit

$$\begin{aligned} \lim_{n \rightarrow \infty} \ln \left(\frac{n+2}{n+1} \right) \\ = \lim_{n \rightarrow \infty} \ln \left(1 + \frac{1}{n+1} \right) \\ = \ln(1) = 0. \end{aligned}$$

This is a necessary, but not sufficient, condition for the series to converge.

The infinite series

$$\sum_{n=1}^{\infty} \ln \left(\frac{n+2}{n+1} \right)$$

is convergent if and only if the sequence of partial sums, here

$$S_N = \ln(N+2) - \ln(2),$$

converges - but $\ln(N+2)$ is unbounded - so the series diverges. However, the series

$$\sum_{n=1}^{\infty} (-1)^n \ln \left(\frac{n+2}{n+1} \right)$$

does converge, by the alternating series test.

(ii) By the *radius of convergence* of a power series

$$\sum_{n=0}^{\infty} a_n z^n,$$

we mean that R such that for $|z| < R$, the series converges, and for $|z| > R$, it diverges. If the limit

$$\lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right|$$

exists, it is equal to the radius of convergence R .

(a) For

$$\sum_{n=0}^{\infty} \frac{2n+1}{\sqrt{n^2+1}} z^n,$$

the radius of convergence is

$$\begin{aligned} R &= \lim_{n \rightarrow \infty} \left| \frac{2n+1}{\sqrt{n^2+1}} \frac{\sqrt{(n+1)^2+1}}{2(n+1)+1} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{2+1/n}{\sqrt{1+1/n^2}} \frac{\sqrt{1+1/(n+1)^2}}{2+1/(n+1)} \right| = 1. \end{aligned}$$

(3)

A3. (a) (i)

$$\begin{aligned} & \lim_{n \rightarrow 1} \frac{n^2 - 1}{n^2 - 3n + 2} \\ &= \lim_{n \rightarrow 1} \frac{(n+1)(n-1)}{(n-2)(n-1)} \\ &= \lim_{n \rightarrow 1} \frac{(n+1)}{(n-2)} = -2. \end{aligned}$$

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(ii)

$$\begin{aligned} & \lim_{x \rightarrow \pi/2} (\sec x - \tan x) \\ &= \lim_{x \rightarrow \pi/2} \frac{(\sec^2 x - \tan^2 x)}{(\sec x + \tan x)} \\ &= \lim_{x \rightarrow \pi/2} \frac{1}{(\sec x + \tan x)} = 0. \end{aligned}$$

5

(iii)

$$\begin{aligned} & \lim_{n \rightarrow \infty} [n((n^3 + 3n)^{1/3} - (n^4 + n^2)^{1/4})] \\ &= \lim_{n \rightarrow \infty} [n^2((1 + 3/n^2)^{1/3} - (1 + 1/n^2)^{1/4})] \end{aligned}$$

5

Using the binomial series, expanding for large n , this becomes

$$\lim_{n \rightarrow \infty} [n^2 ((1 + 1/n^2 + O(1/n^4)) - (1 + 1/(4n^2) + O(1/n^4)))] = \frac{3}{4}.$$

(b) Using L'Hopital's rule,

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{\exp(x^2) - 1}{\sin^2(4x)} \\ &= \lim_{x \rightarrow 0} \frac{2x \exp(x^2)}{8 \sin(4x) \cos(4x)} \\ &= \lim_{x \rightarrow 0} \frac{(4x^2 + 2) \exp(x^2)}{32 \cos^2(4x) - 32 \sin^2(4x)} = \frac{1}{16}. \end{aligned}$$

5

(Total 20)

A4. (i) To evaluate

$$\int_0^{\frac{\pi}{2}} x^2 \sin(x) dx,$$

integrate by parts:

$$\begin{aligned} & \int_0^{\frac{\pi}{2}} x^2 \sin(x) dx \\ &= \int_0^{\frac{\pi}{2}} 2x \cos(x) dx - [x^2 \cos(x)]_0^{\frac{\pi}{2}} \\ &= \int_0^{\frac{\pi}{2}} 2x \cos(x) dx \end{aligned}$$

integrate by parts again,

$$\begin{aligned} &= - \int_0^{\frac{\pi}{2}} 2 \sin(x) dx + [2x \sin(x)]_0^{\frac{\pi}{2}} \\ &= [2 \cos(x)]_0^{\frac{\pi}{2}} + [2x \sin(x)]_0^{\frac{\pi}{2}} = -2 + \pi. \end{aligned}$$

(ii) To evaluate

$$u = \int_0^{\infty} \exp(-3x) \cos(4x) dx,$$

and

$$v = \int_0^{\infty} \exp(-3x) \sin(4x) dx,$$

consider

$$\begin{aligned} u + iv &= \int_0^{\infty} \exp(-3x) \exp(4ix) dx \\ &= \int_0^{\infty} \exp(-(3 - 4i)x) dx \\ &= \left[-\frac{\exp(-(3 - 4i)x)}{3 - 4i} \right]_0^{\infty} = \frac{1}{3 - 4i} = \frac{3 + 4i}{25}. \end{aligned}$$

Hence

$$\begin{aligned} \int_0^{\infty} \exp(-3x) \cos(4x) dx &= \frac{3}{25}, \\ \int_0^{\infty} \exp(-3x) \sin(4x) dx &= \frac{4}{25}. \end{aligned}$$

(iii) To evaluate

$$\int_0^{\infty} \frac{1}{(x+1)(x^2+4)} dx,$$

expand in partial fractions:

$$\frac{1}{(x+1)(x^2+4)} = \frac{a}{x+1} + \frac{bx+c}{x^2+4},$$

provided

$$1 = a(x^2+4) + (bx+c)(x+1),$$

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(4)

giving $a + b = 0$, $b + c = 0$, and $4a + c = 1$. Thus $a = 1/5$, $b = -1/5$, $c = 1/5$, and

$$\begin{aligned}\int_0^\infty \frac{1}{(x+1)(x^2+4)} &= \frac{1}{5} \int_0^\infty \frac{1}{x+1} + \frac{1-x}{x^2+4} dx \\ &= \frac{1}{5} \left[\ln(x+1) - \ln(\sqrt{x^2+4}) + \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) \right]_0^\infty \\ &= \frac{1}{5} \left(0 + \ln(2) + \frac{\pi}{4} \right) = \frac{\ln(2)}{5} + \frac{\pi}{20}.\end{aligned}$$

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(Total
20)

A5. (i) To solve the ordinary differential equation

$$\frac{dy}{dx} = -\frac{4x+3y}{x+y},$$

note that it is homogeneous- set $y = xu(x)$. Then

$$u(x) + x \frac{du}{dx} = -\frac{4+3u}{1+u},$$

or equivalently

$$x \frac{du}{dx} = -\frac{4+4u+u^2}{1+u},$$

which is separable:

$$\begin{aligned} \int^{u(x)} \frac{(1+u)du}{4+4u+u^2} &= -\int^x \frac{dx}{x}. \\ \int^{u(x)} \frac{((2+u)-1)du}{(2+u)^2} &= -\int^x \frac{dx}{x}. \\ \int^{u(x)} \frac{1du}{(2+u)} - \int^{u(x)} \frac{du}{(2+u)^2} &= -\int^x \frac{dx}{x}. \end{aligned}$$

Thus

$$\ln\left(2 + \frac{y(x)}{x}\right) + \frac{x}{2x+y(x)} = -\ln x + c.$$

(ii)

$$\frac{dy}{dx} - \tan(x)y = 1, \quad \text{with } y(0) = 1;$$

Here the integrating factor is seen to be $\cos(x)$:

$$\frac{d(y \cos(x))}{dx} = \cos(x), \quad \text{with } y(0) = 1.$$

Integrating from 0 to x

$$y(x) \cos(x) - 1 = \sin(x),$$

so that

$$y(x) = \sec(x) + \tan(x).$$

(iii) The homogeneous equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$$

is linear, with constant coefficients - the auxiliary equation has the root $\lambda = -2$, repeated, so the CF is seen to be

$$y_{CF} = \exp(-2x)(A + Bx).$$

The forcing term $\exp(-2x)$ on the RHS of the given ode

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = \exp(-2x), \quad \text{with } y(0) = 0, \text{ and } y'(0) = 0.$$

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has the same exponential dependence as the CF, so we look for a PI of the form,

$$y_{PI} = \alpha x^2 \exp(-2x).$$

Substituting in, we find $2\alpha = 1$ so the general solution of the given equation is

$$y(x) = \exp(-2x)\left(A + Bx + \frac{x^2}{2}\right).$$


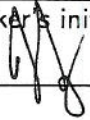
The initial condition $y(0) = 0$ gives $A = 0$; then the other condition $y'(0) = 0$ gives $B = 0$. Hence the most general solution satisfying these conditions is



$$y(x) = \exp(-2x)\frac{x^2}{2}.$$

7

(Total
20)

	EXAMINATION QUESTIONS/SOLUTIONS 2008-09	Course <u>ISE 1</u>
Question <u>Solution 7.</u>		Marks & seen/unseen
Parts	<p>(i) Using tables provided: $(LT)^{-1}\left(\frac{1}{s^2}\right) = t$ Shift rule $\Rightarrow (LT)^{-1}\frac{1}{(s-2)^2} = e^{2t} \cdot t$ Using 2nd shift rule $(LT)^{-1}\left(\frac{e^{-s}}{(s-2)^2}\right) = \underline{H(t-1)(t-1)e^{2(t-1)}}$ given in question</p> <p>(ii) Take LT's of both ODEs $-y'(0) - sy(0) + s^2L(y) - 2L(y) + L(z) = 0$ & $-z'(0) - sz(0) + s^2L(z) - 2L(z) + L(y) = 0$ but $y(0) = z(0) = 0$ & $y'(0) = z'(0) = 1$, $\therefore (s^2-2)L(y) + L(z) = 1$ (1) & $L(y) + (s^2-2)L(z) = 1$ (2)</p> <p>Eliminating $L(z)$: $((s^2-2)^2-1)L(y) = s^2-3$ i.e. $(s^2-1)(s^2-3)L(y) = s^2-3$ $\Rightarrow L(y) = \frac{1}{s^2-1} \equiv \frac{1}{2}\left(\frac{1}{s-1} - \frac{1}{s+1}\right)$</p> <p>Inverting: (using tables) $y = \underline{\frac{1}{2}(e^x - e^{-x})}$ ($= \sinh x$)</p> <p>& then: $z = 2y - y'' = e^x - e^{-x} - \frac{1}{2}(e^x - e^{-x})$ $= \underline{\frac{1}{2}(e^x - e^{-x})}$</p>	<p>6</p> <p>4</p> <p>4</p> <p>4</p> <p>2</p> <p>(Total 20)</p>
	Setter's initials <u>MWL</u>	Checker's initials <u>JG</u>
		Page number

	EXAMINATION QUESTIONS/SOLUTIONS 2008-09	Course ISE 1
Question Soln 8, ch	Page 2 of 2	Marks & seen/unseen
Parts	<p>So</p> $a_n = \begin{cases} 0, & n \text{ even} \\ \frac{4}{n^2 n^2}, & n \text{ odd} \end{cases}$ <p>So Fourier series is</p> $\frac{1}{2} + \frac{4}{\pi^2} \left(\cos n + \frac{\cos 3n}{3^2} + \frac{\cos 5n}{5^2} + \dots \right)$ <p>Putting $n = 0$ (at which f is continuous),</p> $1 = f(0) = \frac{1}{2} + \frac{4}{\pi^2} \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right)$ <p>Hence</p> $1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$	<p>9</p> <p>2</p> <p>4</p> <p>20</p>
	Setter's initials 	Checker's initials 
		Page number

	EXAMINATION QUESTIONS/SOLUTIONS 2008-09	Course ISE 1
Question Soln 9	Page 1 of 2	Marks & seen/unseen
Parts	<p>(a) $\begin{vmatrix} 1 & 5 & 3 \\ 5 & 1 & -k \\ 1 & 2 & k \end{vmatrix} = \begin{vmatrix} 1 & 5 & 3 \\ 0 & -24 & -k-15 \\ 0 & -3 & k-3 \end{vmatrix}$</p> $= -24(k-3) - 3(k+15)$ $= -27k + 27 = \underline{27(1-k)}.$ <p>(b) System is $\begin{pmatrix} 1 & 5 & 3 \\ 5 & 1 & -1 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$</p> <p>Solve: $\left(\begin{array}{ccc c} 1 & 5 & 3 & 0 \\ 5 & 1 & -1 & 0 \\ 1 & 2 & 1 & 0 \end{array} \right) \rightarrow \left(\begin{array}{ccc c} 1 & 5 & 3 & 0 \\ 0 & -24 & -16 & 0 \\ 0 & -3 & -2 & 0 \end{array} \right)$</p> <p>So system reduces to</p> $x + 5y + 3z = 0 \quad (1)$ $3y + 2z = 0 \quad (2)$ <p>General soln:</p> $\underline{z = 3t, \quad y = -2t, \quad x = t \quad (\text{any } t)}$	<p>4</p> <p>8</p>
	Setter's initials 	Checker's initials 
		Page number

