CONTROL ENGINEERING

1. a) The state space description of the system, with the indicated state variables, is given by

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\omega_1^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\omega_2^2 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} u,$$

$$y = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x.$$

[4 marks]

b) The controllability matrix of the system is

$$\mathscr{C} = \begin{bmatrix} 0 & 1 & 0 & -\omega_1^2 \\ 1 & 0 & -\omega_1^2 & 0 \\ 0 & 1 & 0 & -\omega_2^2 \\ 1 & 0 & -\omega_2^2 & 0 \end{bmatrix}.$$

Note that $\det\mathscr{C} = (\omega_1 - \omega_2)^2(\omega_1 + \omega_2)^2$, hence the system is controllable if and only if $\omega_1 \neq \omega_2$ (recall that $\omega_1 > 0$ and $\omega_2 > 0$). [4 marks]

c) The observability matrix of the system is (there is no need to compute CA^2 and CA^3)

$$\mathscr{O} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\omega_1^2 & 0 & 0 & 0 \\ 0 & 0 & -\omega_2^2 & 0 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

Note that the first four rows of the observability matrix are linearly independent (recall that $\omega_1 > 0$ and $\omega_2 > 0$). [2 marks]

d) Note that $u = -k\begin{bmatrix} 1 & 1 \end{bmatrix} Cx$. Hence the closed-loop "A" matrix is

$$A_{cl} = A - kB\begin{bmatrix} 1 & 1 \end{bmatrix} C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\omega_1^2 & -k & 0 & -k \\ 0 & 0 & 0 & 1 \\ 0 & -k & -\omega_2^2 & -k \end{bmatrix}.$$

Its characteristic polynomial is

$$p(s) = s^4 + 2ks^3 + (\omega_1^2 + \omega_2^2)s^2 + k(\omega_1^2 + \omega_2^2)s + \omega_1^2\omega_2^2.$$

Applying Routh test yields that the closed-loop system is asymptotically stable for any $k > 0$ provided $\omega_1 \neq \omega_2$. [4 marks]

e) Differentiating E with respect to time yields

$$\dot{E} = \omega_1^2 x_1\dot{x}_1 + x_2\dot{x}_2 + \omega_2^2 x_3\dot{x}_3 + x_4\dot{x}_4,$$

hence

$$\dot{E} = (\dot{y}_1 + \dot{y}_2)u,$$

as indicated in the exam paper. Replacing the controller yields (recall that $k > 0$)

$$\dot{E} = -k(\dot{y}_1 + \dot{y}_2)^2 \le 0.$$

As a result, E is a non-increasing function of time, and i) it decreases either to its minimum, that is $E = 0$ or ii) it stops decreasing when $(\dot{y}_1 + \dot{y}_2) = 0$. In case i) $x_1 = x_2 = x_3 = x_4 = 0$, hence the equilibrium is attractive (and asymptotically stable, since the system is linear). In case ii) $\dot{y}_1 + \dot{y}_2 = 0$ implies $u = 0$, hence y_1 and y_2 are described by two independent linear oscillators. If the frequencies of the oscillators are different, then the only solutions such that $\dot{y}_1 + \dot{y}_2 = 0$ for all $t \ge 0$ is the trivial solution $y_1 = y_2 = 0$, hence again the equilibrium of the system is attractive (and asymptotically stable, since the system is linear).

[6 marks]

2. a) The A and B matrices of the system are

$$A = \begin{bmatrix} -b & b \\ c & -c \end{bmatrix}, \qquad B = \begin{bmatrix} a \\ 0 \end{bmatrix}.$$

The controllability matrix is

$$\mathscr{C} = \begin{bmatrix} a & -ba \\ 0 & ca \end{bmatrix},$$

hence the system is controllable provided $ca \neq 0$. Since, as stated in the exam paper, a and c are positive the system is always controllable. [4 marks]

b) Note that since $b + c > 0$ the variables z are independent, that is they qualify as a valid change of coordinates. The inverse transformation is

$$x_1 = \frac{z_1 + bz_2}{b+c} \qquad x_2 = \frac{z_1 - cz_2}{b+c}.$$

In the variables z one has

$$\dot{z}_1 = acu, \qquad \dot{z}_2 = -(b+c)z_2 + au.$$

[4 marks]

c) i) The "A" matrix of the closed-loop system is

$$A_{cl} = \begin{bmatrix} k_1 ac & k_2 ac \\ k_1 a & k_2 a - b - c \end{bmatrix}$$

and its characteristic polynomial is

$$s^2 + (b + c - k_2 a - k_1 ac)s - k_1 ac(b+c).$$

Hence the closed-loop system is asymptotically stable provided (recall that a, b and c are positive)

$$b + c - k_2 a - k_1 ac > 0 \qquad k_1 < 0.$$

[4 marks]

ii) The equilibrium of the closed-loop system is such that $\dot{z}_1 = \dot{z}_2 = 0$, yielding

$$z_{1,eq} = -\frac{u_\star}{k_1}, \qquad z_{2,eq} = 0.$$

Note that the equilibrium is well-defined provided $k_1 \neq 0$, condition which is implied by the asymptotic stability conditions in part c.ii).

[4 marks]

iii) In the x_1 and x_2 variables the feedback is

$$u = u_\star + k_1 z_1 + k_2 z_2 = u_\star + (ck_1 + k_2)x_1 + (bk_1 - k_2)x_2.$$

To have a function of x_1 only one has to select $k_2 = bk_1$. To have a stabilizing feebback one needs (recall again that a, b and c are positive)

$$1 - ak_1 > 0 \qquad k_1 < 0.$$

Hence, selecting $k_1 < 0$ and $k_2 = bk_1 < 0$ yields an asymptotically stable closed-loop system and the set of assignable closed-loop equilibrim is the set

$$z_{1,eq} = -\frac{u_\star}{k_1}, \qquad z_{2,eq} = 0,$$

that is the whole z_1 positive semi-axis. [4 marks]

3. a)

i) Reachability of the system implies that the reachability matrix

$$\mathscr{R} = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$$

is full rank. As a result, for any nonzero row vector v one has

$$v\mathscr{R} \neq 0,$$

that is the matrix $\mathscr{R}$ does not have a (left) kernel. Note now that when $p = 1$ the matrix C is a row vector: hence for any nonzero C

$$C\mathscr{R} = \begin{bmatrix} CB & CAB & \cdots & CA^{n-1}B \end{bmatrix} \neq 0,$$

which implies that the rank condition is satisfied, that is any reachable system is output controllable for any $C \neq 0$. [4 marks]

ii) If the system is output controllable for any $C \neq 0$ then $C\mathscr{R} \neq 0$ for any nonzero C, that is the matrix $\mathscr{R}$ does not have any (left) kernel, hence it is full rank. From this we conclude reachability of the system. [4 marks]

iii) Note that (recall that $D = 0$)

$$\begin{aligned} y(0) &= Cx(0), \\ y(1) &= Cx(1) = C(Ax(0) + Bu(0)) = CAx(0) + CBu(0). \end{aligned}$$

Hence, if $CB \neq 0$ the selection $u(0) = -\dfrac{CAx(0)}{CB}$ is such that $y(1) = 0$, for any $x(0)$, which proves the claim. [2 marks]

iv) One has to generalize the derivation in part a.iii). Note that if $CB = CAB = \cdots = CA^{i-2}B = 0$ and $CA^{i-1}B \neq 0$ then

$$\begin{aligned} y(0) &= Cx(0), \\ y(1) &= Cx(1) = CAx(0), \\ y(2) &= Cx(2) = CA^2x(0), \\ &\vdots \\ y(i-2) &= Cx(i-2) = CA^{i-2}x(0), \\ y(i-1) &= Cx(i-1) = CA^{i-1}x(0) + CA^{i-2}Bu(0). \end{aligned}$$

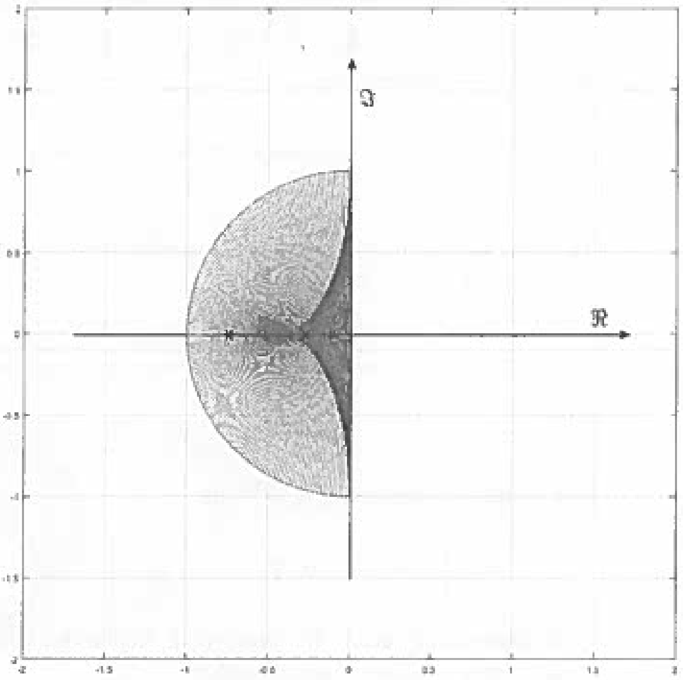
Hence, selecting $u(0)$ one can zero $y(i-1)$, that is the i-th component of the output sequence (recall that the sequence starts at $t = 0$). [8 marks]

b) If $D \neq 0$ then

$$\begin{aligned} y(0) &= Cx(0) & &+Du(0), \\ y(1) &= CAx(0) + CBu(0) & &+Du(1), \\ y(2) &= \cdots & &+Du(2), \\ &\vdots \\ y(k) &= \cdots & &+Du(k), \end{aligned}$$

hence $y(i)$ can be set to zero for all i by a selection of $u(i)$. [2 marks]

4. a) The set of stability is the intersection of the (open) left half of the complex plane with the interior of the unity disk, that is the half disk indicated in the figure below. Note that the red points do not belong to the set.



[4 marks]

b) The indicated points do belong to the admissible region in the figure above (they are indicated with x-marks). Note that

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}, \qquad A_{cl} = \begin{bmatrix} 0 & 1 \\ k_1 & k_2 \end{bmatrix}.$$

The characteristic polynomial of A_{cl} is

$$p(s) = s^2 - k_2 s - k_1$$

and this is equal to the desired closed-loop characteristic polynomical $(s+1/4)(s+3/4)$ provided $k_1 = -3/16$ and $k_2 = -1$. [4 marks]

c)

i) Since the system behaves like a continuous time system for $t \in [0,1)$ one has

$$x(t) = e^{A_{cl}t}x(0) \qquad t \in [0,1),$$

hence $x(1) = e^{A_{cl}}x(0)$. At $t = 1$ the system behaves like a discrete-time system, that is $x^+ = A_{cl}x$, hence, the state is updated to (note the abuse of notation for $x(1)$)

$$x(1) = A_{cl}x(1) = A_{cl}e^{A_{cl}}x(0).$$

The state then evolves as a continuous-time system for $t \in [1,2)$ yielding

$$x(2) = e^{A_{cl}}x(1) = e^{A_{cl}}A_{cl}e^{A_{cl}}x(0)$$

and, after one more step of the discrete-time iteration (note again the abuse of notation for $x(2)$)

$$x(2) = A_{cl}e^{A_{cl}}A_{cl}e^{A_{cl}}x(0) = (A_{cl}e^{A_{cl}})^2 x(0),$$

from which the general expression in the exam paper follows. The sequence $x(k)$ can be thought of as generated by a discrete-time system with "A" matrix given by

$$A_d = A_{cl}e^{A_{cl}}.$$

[4 marks]

ii) Let v be an eigenvector of A_{cl} and recall that A_{cl} has eigenvalues $\lambda_1 = -1/4$ and $\lambda_2 = -3/4$. As a result

$$e^{A_{cl}}v = e^{\lambda_i}v$$

and

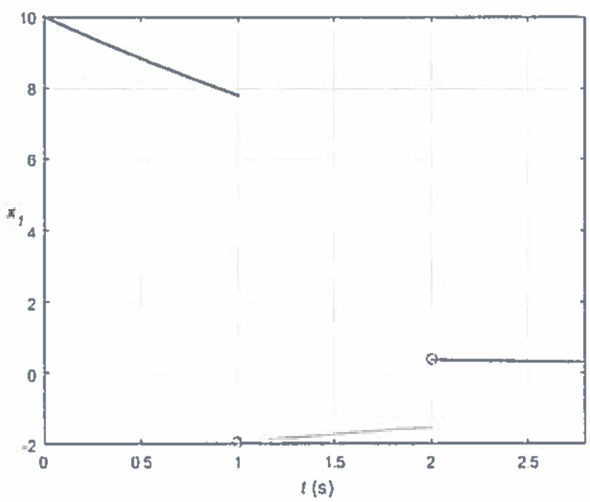
$$A_{cl}e^{A_{cl}}v = A_{cl}e^{\lambda_i}v = \lambda_i e^{\lambda_i}v,$$

that is the eigenvalues of $A_{cl}e^{A_{cl}}$ are

$$-\frac{1}{4}e^{-\frac{1}{4}} \approx= -0.19, \qquad -\frac{3}{4}e^{-\frac{3}{4}} \approx= -0.35.$$

Both eigenvalues are in the unity disk, hence the considered discrete-time system is asymptotically stable. [6 marks]

iii) A possible evolution of $x_1(t)$ is given in the figure below, in which the "o" sign indicate the *jumps* due to the discrete-time evolution. The change in sign at the *jumps* is due to the fact that both eigenvalues of $A_{cl}e^{A_{cl}}$ are negative.



[2 marks]