Master version, 3. 6. 03

Paper Number(s): **ISE2.9, E2.6**

IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY & MEDICINE
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DEPARTMENT OF ELECTRICAL & ELECTRONIC ENGINEERING
EXAMINATIONS 2003

EEE and ISE PART II: M.Eng., B.Eng. and ACGI

CONTROL ENGINEERING

Friday, 30 May 2003, 14:00

There are FIVE questions on this paper.
Answer THREE questions.

Time allowed: 2:00 hours

Examiners: Jaimoukha, I.M. and Clark, J.M.C.
2nd marker - Allwright, J.C.

1. The figure shows a mass-spring system. The coefficients K, D, and M represent, respectively, the spring constant, the damping coefficient and the mass. The signal $u(t)$ represents an externally applied force and $y(t)$ the displacement of the mass from its rest position.

   (a) By considering the balance of forces on the mass, derive the differential equation relating $u(t)$ to $y(t)$. **[2]**

   (b) Derive a state-variable model

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned}$$

[4]

   (c) Determine the transfer function relating u to y. **[4]**

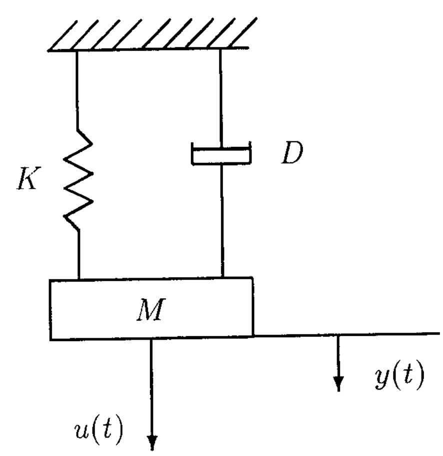
   (d) Set $M = 1$ and suppose that $u(t)$ is a unit step input applied at $t = 0$. Derive the values of K and D so that the following design specifications are satisfied:

      i. The displacement of the mass $y(t)$ settles to its steady state value in the least time without oscillation.

      ii. The settling time is 4 seconds.

   For these values of K and D, evaluate the steady state value of $y(t)$. **[10]**

   (Hint: You may take the settling time to be four times the time constant associated with the rate of decay of responses.)



2. Consider the feedback system below for the speed control of a DC motor. The shaft drives a load with inertia J and is connected to a tacho generator. Here, $v_r(t)$ is the reference voltage, $i_a(t)$ and R_m are the armature current and resistance, respectively, $v_t(t)$ is the tacho voltage, $w(t)$ is the shaft speed and $E(t)$ is the generated EMF. Assume that the current through R_1 is much smaller than that through R_m and:
   - The field flux is constant so that that $E(t)$ is proportional to $w(t)$ and the developed torque, $T(t)$, is proportional to $i_a(t)$. Take the constant of proportionality to be the same and equal to k_e.
   - The power op-amp has negligible output impedance and dynamics and large input impedance and gain, so we can make the 'virtual earth' assumption.
   - Torque disturbances and friction are negligible.
   - The tacho voltage is proportional to speed with proportionality constant k_t.

   In parts (a), (b) and (c) below, all references are to Laplace transforms of signals.

   (a) Derive the transfer function $G(s) = w(s)/i_a(s)$. [3]

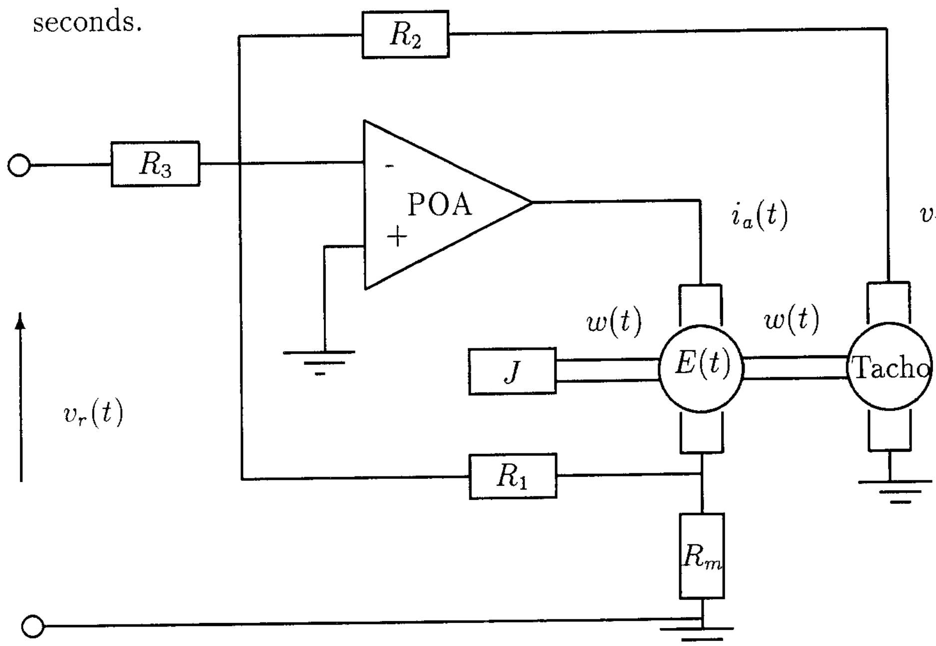
   (b) Derive an expression for $i_a(s)$ in terms of $v_r(s)$ and $w(s)$. [3]

   (c) Hence, derive and clearly draw a block diagram representation of the feedback loop. Take the reference signal to be $-v_r(s)$ and the output signal to be $w(s)$. Indicate clearly the signals $v_t(s)$ and $i_a(s)$ on the block diagram. [6]

   (d) Set $R_2 = R_3 = R_m = J = k_e = k_t = 1$. Suppose that $v_r(t) = -V, t \geq 0$ where V is constant. Derive the values of V and R_1 so that

   i. The steady-state value of the shaft speed is equal to 1.

   ii. The shaft speed settles to within ±2% of its steady-state value in 4 seconds. [8]



3. Consider the feedback loop in the figure below. Here

$$G(s) = \frac{1}{s^2 + s - 1}$$

and

$$K(s) = K_P + \frac{K_I}{s}$$

is a PI compensator where K_I and K_P are design parameters.

(a) Derive the range of values of K_I and K_P for which the closed loop is stable. **[4]**

(b) For this part, take $K_I = 0$.

i. Derive the value of K_P so that the closed loop is marginally stable.

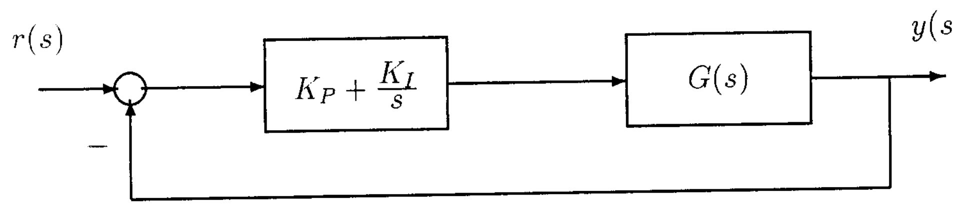
ii. For this value of K_P, derive the steady-state value of the output $y(t)$ for a unit impulse reference signal $r(t)$.

[6]

(c) For this part, take $K_I \neq 0$. Derive the values of K_I and K_P so that the following design specifications are satisfied:

i. The closed loop is marginally stable.

ii. The response $y(t)$ to a unit impulse reference input $r(t)$ is oscillatory with the frequency of oscillation being 1 radians per second. **[10]**



4. Consider the feedback loop in the figure below. Here

$$G(s) = \frac{1}{(s+4)(s+5)}$$

and $K(s)$ is a compensator.

(a) Take $K(s) = k$ where $k > 0$ is a constant gain. Draw the root-locus accurately as k varies in the range $0 < k < \infty$. **[4]**

(b) Suppose that

$$K(s) = k\frac{s-z}{s-p}$$

is a phase-lag compensator where the compensator zero z and pole p are design parameters. Design a stabilizing controller $K(s)$ as follows:

i. Choose the pole p so that $\lim_{t\to\infty} r(t) - y(t) = 0$, when $r(t)$ is a step reference signal.

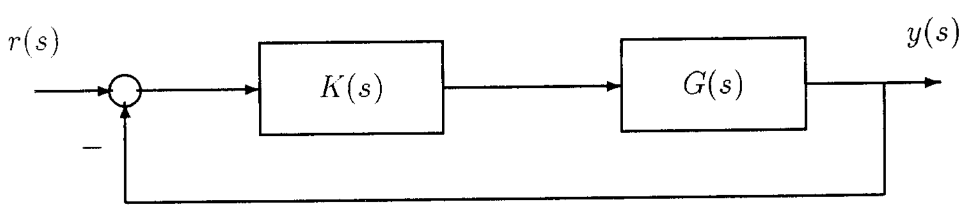
ii. Choose the zero z so that for all $k > 0$ the closed loop system has at least one pole whose real part is to the right of -1 and at least one pole whose real part is to the left of -4. **[8]**

(c) For the values of z and p designed in Part (b), draw the root-locus of the compensated system

$$\frac{s-z}{(s-p)(s+4)(s+5)}$$

as k varies in the range $0 < k < \infty$. Your answer should include the centre and angles of the asymptotes and the real-axis intercepts.

(*Hint: for the real-axis intercepts, you might find it useful to use the fact that the polynomial* $P(s) = s^3 + 6s^2 + 9s + 10$ *has a real root at* -4.492.) **[8]**



5. Consider the feedback control system in the figure below. Here,

$$G(s) = \frac{1}{(s+1)^3}$$

and $K(s)$ is the transfer function of a feedforward compensator.

(a) Sketch the Nyquist diagram of $G(s)$, clearly indicating the low and high frequency portions, as well as the real-axis intercepts. **[4]**

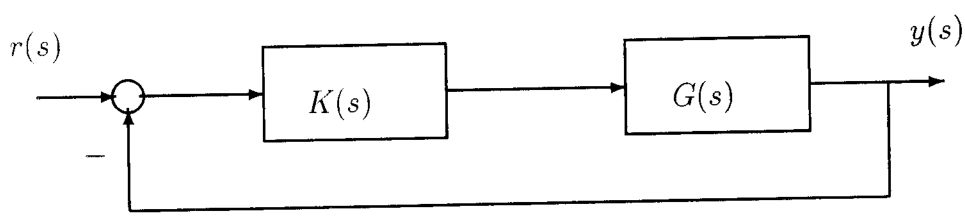
(b) Set $K(s) = K$, a constant compensator. Give the number of unstable closed loop poles for all (positive and negative) K. **[4]**

(c) Take $K = 1$. Determine the gain margin. **[4]**

(d) Without doing any actual design, briefly describe how a PI compensator,

$$K(s) = K_P + \frac{K_I}{s}$$

would improve the steady-state tracking properties without deteriorating the stability margins. **[8]**



SOLUTIONS (E2.6/ISE2.9, Control Engineering, 2003)

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1. (a) Applying Newton's laws on the mass,

$$\boxed{u(t) = M\ddot{y}(t) + D\dot{y}(t) + Ky(t).}$$

(b) Take $x_1(t) = y(t)$ and $x_2(t) = \dot{y}(t)$. Then,

$$\boxed{\begin{aligned} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -\frac{K}{M} & -\frac{D}{M} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}. \end{aligned}}$$

(c) Taking the Laplace transform of the differential equation relating $u(t)$ to $y(t)$,

$$(s^2M + sD + K)y(s) = u(s) \quad \Rightarrow \quad \boxed{G(s) = \frac{y(s)}{u(s)} = \frac{1}{s^2M + sD + K}.}$$

(d) Setting $M = 1$ and comparing the transfer function $G(s)$ with the standard second order form

$$G(s) = \frac{1}{K}\frac{K}{s^2 + sD + K} = \frac{1}{K}\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

It follows that $\omega_n = \sqrt{K}$ and $\zeta = \frac{D}{2\omega_n}$. The first specification demands $\zeta = 1$ for critical damping while the second demands $\zeta\omega_n = 1$. It follows that $\omega_n = 1$ and so $\boxed{K = 1 \text{ and } D = 2.}$ The steady state output is simply $G(0)$ which is $\frac{1}{K}$ and so $\boxed{y_{ss} = 1.}$

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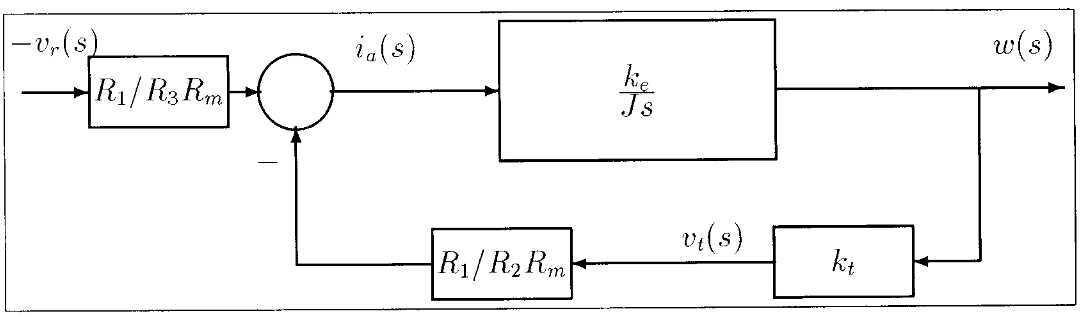
2. (a) The developed torque is $T(t) = k_e i_a(t)$ and the generated EMF is $E(t) = k_e w(t)$. Since friction is negligible and all the developed torque is supplied to the load, we have that $T(t) = J\dot{w}(t)$ or $k_e i_a(t) = J\dot{w}(t)$. Taking Laplace transforms (assuming zero initial conditions),

$$\boxed{G(s) = \frac{k_e}{Js}.}$$

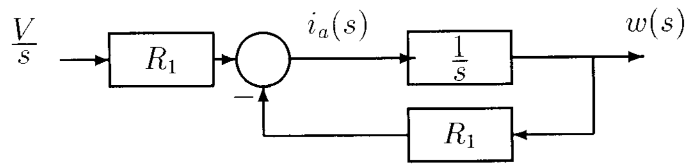
(b) Making the virtual earth assumption: $\frac{R_m i_a(t)}{R_1} + \frac{k_t w(t)}{R_2} + \frac{v_r(t)}{R_3} = 0$, since $v_t(t) = k_t w(t)$ and since it is assumed that the current through R_1 is much smaller than that through R_m. Taking Laplace transforms and rearranging,

$$\boxed{i_a(s) = \frac{R_1}{R_3 R_m}(-v_r(s)) - \frac{R_1}{R_2 R_m} k_t w(s).}$$

(c) Using the last equation and the expression for $G(s)$, the block diagram becomes,



(d) Putting in the numbers, the block diagram simplifies to



It follows that the closed loop transfer function is given by

$$H(s) = \frac{1}{1 + s/R_1}$$

and the steady state value is V. So $\boxed{V = 1.}$ The settling time is four time constants and is therefore $4R_1$. So $\boxed{R_1 = 1.}$

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3. (a) Taking

$$K(s) = K_P + \frac{K_I}{s} = \frac{K_P s + K_I}{s}$$

gives the closed loop characteristic equation as

$$1 + \frac{K_P s + K_I}{s(s^2+s-1)} = 0 \Rightarrow s^3 + s^2 + (K_P - 1)s + K_I = 0.$$

The Routh array is then

$$\begin{array}{c|cc} s^3 & 1 & K_P - 1 \\ s^2 & 1 & K_I \\ s & K_P - K_I - 1 & \\ 1 & K_I & \end{array}$$

For stability, we require no sign changes in the first column. Thus the closed loop will be stable for $\boxed{K_I > 0 \text{ and } K_P > K_I + 1.}$

(b) i. The closed loop system is marginally stable when all the elements of a row of the Routh array are equal to zero and all the other elements in the first column have the same signs. When $K_I = 0$, $K(s) = K_P$ and the Routh array becomes

$$\begin{array}{c|cc} s^2 & 1 & K_P - 1 \\ s & 1 & 0 \\ 1 & K_P - 1 & \end{array}$$

For marginal stability we require $\boxed{K_P = 1}$.

ii. For this value of K_P, the closed loop transfer function is

$$H(s) = \frac{1}{s(s+1)}$$

The steady-state response to a unit impulse is

$$\boxed{\lim_{s\to 0} sH(s) = 1.}$$

(c) If $K_I \neq 0$, marginal stability occurs when $K_I > 0$ and $K_P = K_I + 1$. The auxiliary polynomial is given by $s^2 + K_I$ and it follows that the frequency of oscillations is given by $\sqrt{K_I}$ and so $\boxed{K_I = 1.}$ Thus $\boxed{K_P = 2.}$

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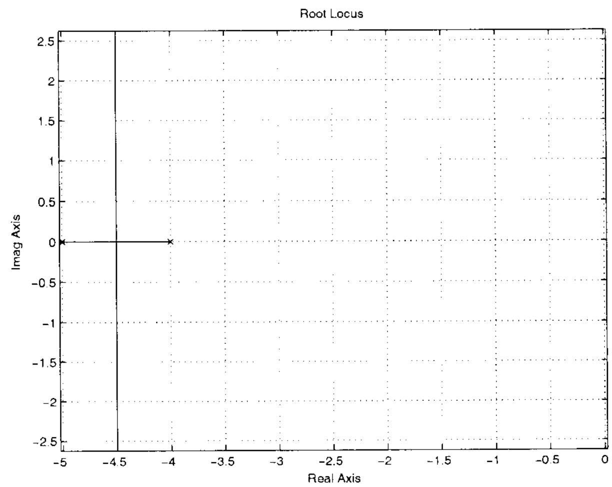
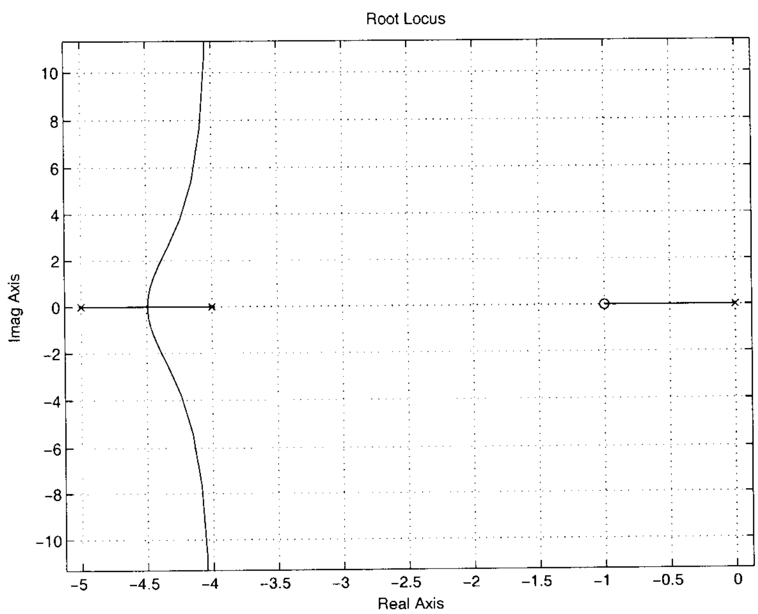
4. (a) The root-locus plot is shown below.



(b) i. For zero steady-state error against a step reference signal, we need an integrator in the loop so $\boxed{p = 0.}$

ii. The requirements are satisfied by $\boxed{z = -1}$ since one branch of the root locus will be to the right of -1 and another to the left of -4.

(c) The root-locus of the compensated system is shown below. The centre and



angles of the asymptotes are $\boxed{\sigma = -4 \ \& \ \psi = \pm 90^o.}$ For the real axis intercepts, we search for real roots of $\frac{d}{ds}K(s)G(s) = 0$ or $s^3 + 6s^2 + 9s + 10 = 0$, which has a real root at $\boxed{-4.492}$ according to the hint in the question.

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5. (a) The Nyquist plot, together with the unit circle centred on the origin is shown below. The real-axis intercepts can be found by setting the imaginary part of $G(j\omega)$ to zero. This gives intercepts at $\boxed{\omega_i = 0, \pm\sqrt{3}, \infty}$ and it follows that $\boxed{G(j\omega_i) = 1, -0.125, -0.125, 0.}$

(b) The number of unstable closed loop poles associated with gain K can be determined by the number of encirclements by $G(s)$ of the point $-\frac{1}{K}$. Thus

$$\boxed{\begin{array}{rcl} 0 < k < 8 & \Rightarrow & \text{no unstable poles} \\ k > 8 & \Rightarrow & \text{2 unstable poles} \\ -1 < k < 0 & \Rightarrow & \text{no unstable poles} \\ k < -1 & \Rightarrow & \text{1 unstable pole.} \end{array}}$$

(c) Since the intercept with the negative real axis is at $-1/8$ the $\boxed{\text{gain margin is 8.}}$

(d) The PI compensator can be written as

$$K(s) = K_P + \frac{K_I}{s} = K_I \frac{1 + \dfrac{s}{K_I/K_P}}{s}$$

and is a special form of phase-lag compensation. It has high gain at frequencies below $\omega_0 = K_I/K_P$ and gain close to K_P beyond ω_0. The phase is negative and large below ω_0 but insignificant above. It follows that by varying K_I and K_P we can use PI compensation to increase low frequency gain (hence improving tracking properties) without introducing phase-lag at high frequency (which would reduce the phase margin) by placing w_0 in the 'middle' frequency range.