Imperial College London

[E1.10 (Maths 1) 2008]

B.ENG. AND M.ENG. EXAMINATIONS 2008

PART I : MATHEMATICS 1 (ELECTRICAL ENGINEERING)

Date Wednesday 4th June 2008 10.00 am - 1.00 pm

Answer EIGHT questions.

A mathematical formulae sheet is provided.

[Before starting, please make sure that the paper is complete; there should be 6 pages, with a total of 10 questions. Ask the invigilator for a replacement if your copy is faulty.]

1. (i) Define what it means to say that a function f is odd or even, and give an example of each.

   (ii) Classify the following functions as odd, even or neither:

   (a) e^{-x} ;

   (b) $x \sin x$;

   (c) $x^2 \sin x$;

   (d) $2x/(x^2-1)$.

   (iii) Let $f(x) = e^x$ and $g(x) = 1/x^2$. Find $f(g(x))$ and $g(f(x))$. Find also the inverse functions $f^{-1}(x)$ and $g^{-1}(x)$.

   (iv) Write

$$f(x) \;=\; \frac{2x}{x+1}$$

   as the sum of an even function and an odd function.

2. Evaluate the following limits:

(i)
$$\lim_{x\to\infty} \frac{(2x-1)\,(x+3)}{(x+5)\,(3x-2)} \;;$$

(ii)
$$\lim_{x\to 0} \; x\,\sin(\cot x) \;;$$

(iii)
$$\lim_{x\to 0} \; x^{-2}\,\ln(\cos x) \;;$$

(iv)
$$\lim_{x\to\infty} \; x^{-9}\,\left\{ (x+3)^{10} \;-\; (x+1)^{10} \right\}$$

PLEASE TURN OVER

3. Evaluate the following integrals ;

(i) $$\int (3-2x)^{-5}\,dx\ ;$$

(ii) $$\int \frac{5x+2}{(3x+4)\,(x-1)}\,dx\ ;$$

(iii) $$\int_1^2 x\ln x\,dx\ ;$$

(iv) $$\int x^3\,e^x\,dx\ .$$

4. (i) Express in polar form $re^{i\theta}$ with $0 \le \theta < 2\pi$:

(a) $3 + 5i$; (b) $-6 + 3i$; (c) $-4 - 5i$.

(ii) Find an expression for $\cos 3\theta$ in terms of powers of $\cos\theta$.

(iii) Find the equation, in the form $y = f(x)$, for the locus of points which satisfy

$$\arg(z+1) = \frac{\pi}{3}$$

where $z = x + iy$.

5. (i) Find and classify the stationary points of the function

$$f(x, y) = x(y-2)^2 + x^2 - x .$$

(ii) Sketch the locus $f(x, y) = 0$.

6. (i) Consider the planes $3x - 5y - 2z = 2$ and $x + y + 6z = -9$.

(a) Find the perpendicular distance from the origin to each plane.

(b) Find the vector equation of the straight line through the points N_1, N_2 which are the feet of the normals from the origin onto the two planes.

(ii) Find all vectors $\boldsymbol{u} = (x, y, z)$ in 3-D space such that $|\boldsymbol{u}| = 1$ and $|\boldsymbol{u}-\boldsymbol{k}| = 1$, where $\boldsymbol{k} = (0, 0, 1)$.

Describe this set geometrically.

PLEASE TURN OVER

7. Let

$$A = \begin{pmatrix} 9 & 3 & -3 \\ 3 & 5 & 1 \\ -3 & 1 & 11 \end{pmatrix}.$$

Find the entries in

$$L = \begin{pmatrix} a & 0 & 0 \\ b & d & 0 \\ c & e & f \end{pmatrix}$$

so that

$$A = L L^T .$$

Show that $|A| = |L|^2 = 324$, where $|A|$ denotes the determinant of A.

Find L^{-1}, and hence A^{-1}.

8. (i) Find an implicit solution of the differential equation

$$\frac{dy}{dx} = 2\,\frac{2x + y}{2x - y} ,$$

for which $y\left(\frac{1}{2}\right) = 0$.

(ii) Show that

$$\frac{d}{dx}\left(\ln\left[\sec x + \tan x\right]\right) = \sec x ,$$

and hence find the general solution of the linear differential equation

$$\cos x \, \frac{dy}{dx} + y = 1 - \sin x$$

using an integrating factor.

9. For the differential equation

$$\frac{d^2y}{dx^2} + 4\,\frac{dy}{dx} + ky = e^{-2x}\,,$$

where k is a constant, find the solution that satisfies

$$\begin{aligned} y\,(0) &= 0\,, \\ y'\,(0) &= 0\,, \end{aligned}$$

in the case

(i) $k = 5$;

(ii) $k = 4$.

10. A function $y(x)$ satisfies the equation :

$$\frac{d^2y}{dx^2} + xy = 0$$

and the conditions $y(0) = 1,\ \ y'(0) = 0$.

Differentiate this equation n times to show that (for $n \geq 1$)

$$\frac{d^{n+2}y}{dx^{n+2}} + n\,\frac{d^{n-1}y}{dx^{n-1}} = 0 \ \text{ at } \ x = 0\,.$$

Hence show that the solution of the equation with the given initial conditions has the form

$$y(x) = \sum_{i=0}^{\infty} C_i\, x^{3i}\,,$$

where the C_i are constants. Find the first three non-zero terms in this expansion, write down the general term and show by the ratio test that this series converges for all x.

END OF PAPER

MATHEMATICS DEPARTMENT

MATHEMATICAL FORMULAE

1. VECTOR ALGEBRA

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1, a_2, a_3)$$

Scalar (dot) product: $\mathbf{a}\,.\,\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

Vector (cross) product:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Scalar triple product:

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a}\,.\,\mathbf{b} \times \mathbf{c} = \mathbf{b}\,.\,\mathbf{c} \times \mathbf{a} = \mathbf{c}\,.\,\mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product: $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c}\,.\,\mathbf{a})\mathbf{b} - (\mathbf{b}\,.\,\mathbf{a})\mathbf{c}$

2. SERIES

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \ldots \quad (\alpha \text{ arbitrary, } |x| < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \ldots,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^n \frac{x^{n+1}}{(n+1)} + \ldots (-1 < x \le 1)$$

3. TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin(a+b) = \sin a \cos b + \cos a \sin b\,;$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b\,.$$

$$\cos iz = \cosh z\,; \quad \cosh iz = \cos z\,; \quad \sin iz = i \sinh z\,; \quad \sinh iz = i \sin z\,.$$

4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f\,D^n g + \binom{n}{1} Df\,D^{n-1} g + \ldots + \binom{n}{r} D^r f\,D^{n-r} g + \ldots + D^n f\,g\,.$$

(b) Taylor's expansion of $f(x)$ about $x = a$:

$$f(a+h) = f(a) + hf'(a) + h^2 f''(a)/2! + \ldots + h^n f^{(n)}(a)/n! + \epsilon_n(h)\,,$$

where $\epsilon_n(h) = h^{n+1} f^{(n+1)}(a + \theta h)/(n+1)!, \quad 0 < \theta < 1\,.$

(c) Taylor's expansion of $f(x, y)$ about (a, b):

$$f(a+h, b+k) = f(a, b) + [hf_x + kf_y]_{a,b} + 1/2!\left[h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}\right]_{a,b} + \ldots$$

(d) Partial differentiation of $f(x, y)$:

i. If $y = y(x)$, then $f = F(x)$, and $\dfrac{dF}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dx}$.

ii. If $x = x(t)$, $y = y(t)$, then $f = F(t)$, and $\dfrac{dF}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$.

iii. If $x = x(u, v)$, $y = y(u, v)$, then $f = F(u, v)$, and

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}, \quad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$

(e) Stationary points of $f(x, y)$ occur where $f_x = 0$, $f_y = 0$ simultaneously.
Let (a, b) be a stationary point: examine $D = [f_{xx}f_{yy} - (f_{xy})^2]_{a,b}$.
If $D > 0$ and $f_{xx}(a, b) < 0$, then (a, b) is a maximum;
If $D > 0$ and $f_{xx}(a, b) > 0$, then (a, b) is a minimum;
If $D < 0$ then (a, b) is a saddle-point.

(f) Differential equations:

i. The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating factor $I(x) = \exp[\int P(x)(dx)]$, so that $\dfrac{d}{dx}(Iy) = IQ$.

ii. $P(x, y)dx + Q(x, y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y$.

5. INTEGRAL CALCULUS

(a) An important substitution: $\tan(\theta/2) = t$:
$\sin\theta = 2t/(1+t^2)$, $\cos\theta = (1-t^2)/(1+t^2)$, $d\theta = 2\,dt/(1+t^2)$.

(b) Some indefinite integrals:

$$\int (a^2 - x^2)^{-1/2} dx = \sin^{-1}\left(\frac{x}{a}\right), \quad |x| < a.$$

$$\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1}\left(\frac{x}{a}\right) = \ln\left\{\frac{x}{a} + \left(1 + \frac{x^2}{a^2}\right)^{1/2}\right\}.$$

$$\int (x^2 - a^2)^{-1/2} dx = \cosh^{-1}\left(\frac{x}{a}\right) = \ln\left|\frac{x}{a} + \left(\frac{x^2}{a^2} - 1\right)^{1/2}\right|.$$

$$\int (a^2 + x^2)^{-1} dx = \left(\frac{1}{a}\right)\tan^{-1}\left(\frac{x}{a}\right).$$

6. NUMERICAL METHODS

(a) Approximate solution of an algebraic equation:

If a root of $f(x) = 0$ occurs near $x = a$, take $x_0 = a$ and
$x_{n+1} = x_n - [f(x_n)/f'(x_n)]$, $n = 0, 1, 2 \ldots$

(Newton Raphson method).

(b) Formulae for numerical integration: Write $x_n = x_0 + nh$, $y_n = y(x_n)$.

i. Trapezium rule (1-strip): $\int_{x_0}^{x_1} y(x)dx \approx (h/2)[y_0 + y_1]$.

ii. Simpson's rule (2-strip): $\int_{x_0}^{x_2} y(x)dx \approx (h/3)[y_0 + 4y_1 + y_2]$. $\leftarrow [y_0 + 4y_1 + y_2]$

(c) Richardson's extrapolation method: Let $I = \int_a^b f(x)dx$ and let I_1, I_2 be two estimates of I obtained by using Simpson's rule with intervals h and $h/2$. Then, provided h is small enough,

$$I_2 + (I_2 - I_1)/15,$$

is a better estimate of I.

7. LAPLACE TRANSFORMS

Function	Transform	Function	Transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)dt$	$af(t) + bg(t)$	$aF(s) + bG(s)$
df/dt	$sF(s) - f(0)$	d^2f/dt^2	$s^2F(s) - sf(0) - f'(0)$
$e^{at}f(t)$	$F(s-a)$	$tf(t)$	$-dF(s)/ds$
$(\partial/\partial\alpha)f(t,\alpha)$	$(\partial/\partial\alpha)F(s,\alpha)$	$\int_0^t f(t)dt$	$F(s)/s$
$\int_0^t f(u)g(t-u)du$	$F(s)G(s)$		
1	$1/s$	$t^n (n = 1, 2 \ldots)$	$n!/s^{n+1}$, $(s > 0)$
e^{at}	$1/(s-a)$, $(s > a)$	$\sin\omega t$	$\omega/(s^2+\omega^2)$, $(s > 0)$
$\cos\omega t$	$s/(s^2+\omega^2)$, $(s > 0)$	$H(t-T) = \begin{cases} 0, & t < T \\ 1, & t > T \end{cases}$	e^{-sT}/s, $(s, T > 0)$

8. FOURIER SERIES

If $f(x)$ is periodic of period $2L$, then $f(x+2L) = f(x)$, and

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \text{ where}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad n = 0, 1, 2, \ldots, \text{ and}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1, 2, 3, \ldots$$

Parseval's theorem

$$\frac{1}{L}\int_{-L}^{L} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right).$$

September 2000

Imperial College
London

[E1.14 (Maths 2) 2008]

B.ENG. AND M.ENG. EXAMINATIONS 2008

PART I : MATHEMATICS 2 (ELECTRICAL ENGINEERING)

Date Thursday 5th June 2008 10.00 am - 1.00 pm

Answer EIGHT questions.

A mathematical formulae sheet is provided.

[Before starting, please make sure that the paper is complete; there should be 6 pages, with a total of 10 questions. Ask the invigilator for a replacement if your copy is faulty.]

1. (i) The hyperbolic sine function is defined as follows:

$$f(x) \;=\; \sinh(x) \;=\; \frac{e^{x} - e^{-x}}{2} \;.$$

Sketch a graph of the following functions, stating whether each is even or odd:

(a) $f(x)$,

(b) $(f(x))^2$,

(c) $\frac{1}{2}\,(f(x) \;+\; f(-x))$.

By re-writing the equation

$$\sinh(x) \;=\; y$$

as a quadratic equation for e^{x}, find $\sinh^{-1}(y)$ in terms of a logarithm.

(ii) The hyperbolic cosine is the function

$$\cosh(x) \;=\; \frac{1}{2}\,(e^{x} \;+\; e^{-x}) \;.$$

Show that there are constants A and B, independent of x such that

$$(\sinh(x))^2 \;=\; A\,\cosh(2x) \;+\; B \;.$$

What are the values of A and B?

Evaluate the limit

$$\lim_{x\to 0} \frac{\sinh(x)}{x} \;.$$

2. Consider the function

$$f(x) = \left(1 \;-\; \frac{2x}{x+1}\right)^{2} \;.$$

Find the stationary points of f and provide the details of the calculation that determines their nature. Hence draw a sketch of f on the entire real line, noting any horizontal and vertical asymptotes.

Use the information contained in your first graph to sketch a graph of the function $e^{-f(x)}$ also on the real line.

3. Let us assume that the sine function can be differentiated an arbitrary number of times at all points in its domain.

   (i) Find the first and second derivatives of the sinc function

   $$\operatorname{sinc}(x) \;=\; \frac{\sin(x)}{x}$$

   for $x \neq 0$.

   (ii) Use l'Hôpital's rule to evaluate the limit

   $$\lim_{x\to 0} \frac{\sin(x)}{x} \, .$$

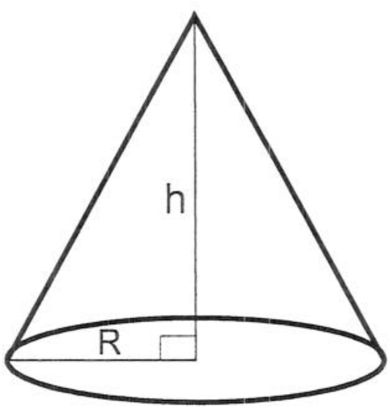
   Hence deduce the value of the sinc function at $x = 0$ if you are told that the sinc function is continuous at $x = 0$.

   (iii) Use l'Hôpital's rule to evaluate the value of the first and second derivatives of the sinc function at $x = 0$, both of which are continuous functions at $x = 0$.

   (iv) Draw a graph of the sinc function for $-\pi < x < \pi$.

4. (i) Find the surface area of revolution of the function $f(x) = \sqrt{R^2 - x^2}$ for $-R \leq x \leq R$ and hence deduce the surface area of a sphere of radius R.

   (ii) Use a similar method to determine the surface area of a circular cone of vertical height h and base radius R .



5. (i) Determine whether or not the following series converge:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$,

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{2n+1}}$,

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$.

(ii) Use the integral test to find a number $M < 1.29$ such that

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2} \leq M \, .$$

6. Obtain the Fourier series of the 2π-periodic, real function f defined for x in the interval $(-\pi, \pi]$ by

$$f(x) = \begin{cases} 1 \; ; & 0 \leq x \leq \pi \, , \\ -1 \; ; & -\pi < x < 0 \, . \end{cases}$$

If the Fourier series of f is denoted by $F(x)$, explain why there exists at least one real number x_0 such that $F(x_0) \neq f(x_0)$ and provide a value of x_0 to illustrate this inequality between a function and its Fourier series.

Finally, find the value of

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \, .$$

PLEASE TURN OVER

7. (i) If $V = \ln(r)$ where $r = \sqrt{x^2 + y^2}$ show that

$$\frac{\partial^2 V}{\partial x^2} = \frac{1}{r^2} - \frac{2x^2}{r^4} .$$

Verify that V satisfies the equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 .$$

(ii) Let $w\,(x,\, y,\, z) = xy + z$ and let $x = \cos(t)$, $y(t) = \sin(t)$, $z = t$.

Show that

$$\frac{dw}{dt} = 1 + \cos(2t) .$$

8. Compute an approximation to the integral

$$I = \int_0^1 (\theta \cos(\theta) + 1)\, d\theta$$

using:

(i) the trapezium rule with one interval;

(ii) the trapezium rule with two intervals;

(iii) Simpson's rule with two intervals.

Compare your results with the exact value of the integral.

You should work to 4 decimal places throughout.

9. (i) Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = e^{3x} .$$

(ii) Find the solution of the differential equation

$$(y^2 - x^2)\,\frac{dy}{dx} + 2xy = 0$$

in implicit form.

10. The integral

$$\int e^{kx}\,dx = \frac{e^{kx}}{k} + \text{Const}$$

is known to hold for all complex k and real x. Use this result to deduce the values of

$$a_n = \int_{-\pi}^{\pi} e^{-x} \cos(nx)\,dx \quad \text{and} \quad b_n = \int_{-\pi}^{\pi} e^{-x} \sin(nx)\,dx$$

in terms of π where n is a fixed *integer.*

Using the complex Fourier series representation of the real, 2π-periodic function $f(x)$ that coincides with e^{-x} on $(-\pi, \pi)$,

$$f(x) = c_0 + 2\sum_{n=1}^{\infty} Re\,(c_n\,e^{inx}) \quad \text{with} \quad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,e^{-inx}\,dx$$

evaluate

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2}$$

in terms of e^{π} and $e^{-\pi}$ using the following version of Parseval's theorem

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} (f(x))^2\,dx = c_0^2 + 2\sum_{n=1}^{\infty} |\,c_n\,|^2 .$$

END OF PAPER

MATHEMATICS DEPARTMENT

MATHEMATICAL FORMULAE

1. VECTOR ALGEBRA

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1, a_2, a_3)$$

Scalar (dot) product: $\mathbf{a}\,.\,\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

Vector (cross) product:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix};$$

Scalar triple product:

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a}\,.\,\mathbf{b} \times \mathbf{c} = \mathbf{b}\,.\,\mathbf{c} \times \mathbf{a} = \mathbf{c}\,.\,\mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product: $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c}\,.\,\mathbf{a})\mathbf{b} - (\mathbf{b}\,.\,\mathbf{a})\mathbf{c}$

2. SERIES

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \ldots \quad (\alpha \text{ arbitrary, } |x| < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \ldots,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^n \frac{x^{n+1}}{(n+1)} + \ldots (-1 < x \le 1)$$

3. TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin(a+b) = \sin a \cos b + \cos a \sin b\,;$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b\,.$$

$$\cos iz = \cosh z\,; \quad \cosh iz = \cos z\,; \quad \sin iz = i\sinh z\,; \quad \sinh iz = i \sin z\,.$$

4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f\,D^n g + \binom{n}{1} Df\,D^{n-1}g + \ldots + \binom{n}{r} D^r f\,D^{n-r}g + \ldots + D^n f\,g\,.$$

(b) Taylor's expansion of $f(x)$ about $x = a$:

$$f(a+h) = f(a) + hf'(a) + h^2 f''(a)/2! + \ldots + h^n f^{(n)}(a)/n! + \epsilon_n(h)\,,$$

where $\epsilon_n(h) = h^{n+1} f^{(n+1)}(a + \theta h)/(n+1)!, \quad 0 < \theta < 1\,.$

(c) Taylor's expansion of $f(x, y)$ about (a, b):

$$f(a+h, b+k) = f(a, b) + [hf_x + kf_y]_{a,b} + 1/2!\left[h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}\right]_{a,b} + \ldots$$

(d) Partial differentiation of $f(x, y)$:

i. If $y = y(x)$, then $f = F(x)$, and $\dfrac{dF}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dx}$.

ii. If $x = x(t)$, $y = y(t)$, then $f = F(t)$, and $\dfrac{dF}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$.

iii. If $x = x(u, v)$, $y = y(u, v)$, then $f = F(u, v)$, and

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}, \quad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$

(e) Stationary points of $f(x, y)$ occur where $f_x = 0$, $f_y = 0$ simultaneously.
Let (a, b) be a stationary point: examine $D = [f_{xx}f_{yy} - (f_{xy})^2]_{a,b}$.
If $D > 0$ and $f_{xx}(a, b) < 0$, then (a, b) is a maximum;
If $D > 0$ and $f_{xx}(a, b) > 0$, then (a, b) is a minimum;
If $D < 0$ then (a, b) is a saddle-point.

(f) Differential equations:

i. The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating factor $I(x) = \exp[\int P(x)(dx)]$, so that $\dfrac{d}{dx}(Iy) = IQ$.

ii. $P(x, y)dx + Q(x, y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y$.

5. INTEGRAL CALCULUS

(a) An important substitution: $\tan(\theta/2) = t$:
$\sin\theta = 2t/(1+t^2)$, $\cos\theta = (1-t^2)/(1+t^2)$, $d\theta = 2\,dt/(1+t^2)$.

(b) Some indefinite integrals:

$$\int (a^2 - x^2)^{-1/2} dx = \sin^{-1}\left(\frac{x}{a}\right), \quad |x| < a.$$

$$\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1}\left(\frac{x}{a}\right) = \ln\left\{\frac{x}{a} + \left(1 + \frac{x^2}{a^2}\right)^{1/2}\right\}.$$

$$\int (x^2 - a^2)^{-1/2} dx = \cosh^{-1}\left(\frac{x}{a}\right) = \ln\left|\frac{x}{a} + \left(\frac{x^2}{a^2} - 1\right)^{1/2}\right|.$$

$$\int (a^2 + x^2)^{-1} dx = \left(\frac{1}{a}\right)\tan^{-1}\left(\frac{x}{a}\right).$$

6. NUMERICAL METHODS

(a) Approximate solution of an algebraic equation:

If a root of $f(x) = 0$ occurs near $x = a$, take $x_0 = a$ and
$x_{n+1} = x_n - [f(x_n)/f'(x_n)]$, $n = 0, 1, 2 \ldots$

(Newton Raphson method).

(b) Formulae for numerical integration: Write $x_n = x_0 + nh$, $y_n = y(x_n)$.

i. Trapezium rule (1-strip): $\int_{x_0}^{x_1} y(x)dx \approx (h/2)[y_0 + y_1]$.

ii. Simpson's rule (2-strip): $\int_{x_0}^{x_2} y(x)dx \approx (h/3)[y_0 + 4y_1 + y_2]$. $\leftarrow [y_0 + \underline{\underline{4}} y_1 + y_2]$

(c) Richardson's extrapolation method: Let $I = \int_a^b f(x)dx$ and let I_1, I_2 be two estimates of I obtained by using Simpson's rule with intervals h and $h/2$.

Then, provided h is small enough,

$$I_2 + (I_2 - I_1)/15,$$

is a better estimate of I.

7. LAPLACE TRANSFORMS

Function	Transform	Function	Transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)dt$	$af(t) + bg(t)$	$aF(s) + bG(s)$
df/dt	$sF(s) - f(0)$	d^2f/dt^2	$s^2F(s) - sf(0) - f'(0)$
$e^{at}f(t)$	$F(s-a)$	$tf(t)$	$-dF(s)/ds$
$(\partial/\partial\alpha)f(t,\alpha)$	$(\partial/\partial\alpha)F(s,\alpha)$	$\int_0^t f(t)dt$	$F(s)/s$
$\int_0^t f(u)g(t-u)du$	$F(s)G(s)$		
1	$1/s$	$t^n (n = 1, 2 \ldots)$	$n!/s^{n+1}$, $(s > 0)$
e^{at}	$1/(s-a)$, $(s > a)$	$\sin\omega t$	$\omega/(s^2+\omega^2)$, $(s > 0)$
$\cos\omega t$	$s/(s^2+\omega^2)$, $(s > 0)$	$H(t-T) = \begin{cases} 0, & t < T \\ 1, & t > T \end{cases}$	e^{-sT}/s, $(s, T > 0)$

8. FOURIER SERIES

If $f(x)$ is periodic of period $2L$, then $f(x+2L) = f(x)$, and

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \text{ where}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad n = 0, 1, 2, \ldots, \text{ and}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1, 2, 3, \ldots$$

Parseval's theorem

$$\frac{1}{L}\int_{-L}^{L} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right).$$

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