EE4-47

Modelling and control of multibody mechanical systems

Model answers

Question 1

a) i) Three single-axis-rotation transformation matrices are needed.

$$D_\psi = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which is the rotation matrix by angle ψ about a z axis.

$$C_\theta = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix},$$

which is the rotation matrix by angle θ about a y axis.

$$D_\phi = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which is the rotation matrix by angle ϕ also about a z axis.
The complete transformation from earth-fixed coordinates to body-fixed coordinates is $A = D_\phi C_\theta D_\psi$ and it amounts to

$$A = \begin{bmatrix} \cos\phi\cos\theta\cos\psi - \sin\phi\sin\psi & \cos\phi\cos\theta\sin\psi + \sin\phi\cos\psi & -\cos\phi\sin\theta \\ -\sin\phi\cos\theta\cos\psi - \cos\phi\sin\psi & -\sin\phi\cos\theta\sin\psi + \cos\phi\cos\psi & \sin\phi\sin\theta \\ \sin\theta\cos\psi & \sin\theta\sin\psi & \cos\theta \end{bmatrix}$$

[4 marks]

ii)

$$\Omega = D_\phi C_\theta \begin{bmatrix} 0 \\ 0 \\ \dot\psi \end{bmatrix} + D_\phi \begin{bmatrix} 0 \\ \dot\theta \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot\phi \end{bmatrix} = \begin{bmatrix} -\dot\psi\cos\phi\sin\theta + \dot\theta\sin\phi \\ \dot\psi\sin\phi\sin\theta + \dot\theta\cos\phi \\ \dot\psi\cos\theta + \dot\phi \end{bmatrix}. \quad (1)$$

[4 marks]

b) The unit vector of the earth-fixed z-axis in body-fixed coordinates can be found by using the transformation matrix A, as follows:

$$\hat{n}_z = A \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\cos\phi\sin\theta \\ \sin\phi\sin\theta \\ \cos\theta \end{bmatrix}.$$

The total external moment acting on the rigid body due to gravity is

$$N = fk' \times mgk,$$

where $\boldsymbol{k}'$ and $\boldsymbol{k}$ are the unit vectors in the body-fixed and earth-fixed z-axes respectively. Therefore

$$\boldsymbol{N} = f\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times mg\begin{bmatrix} -\cos\phi\sin\theta \\ \sin\phi\sin\theta \\ \cos\theta \end{bmatrix} = mgf\begin{bmatrix} -\sin\phi\sin\theta \\ -\cos\phi\sin\theta \\ 0 \end{bmatrix} = -mgf\sin\theta\begin{bmatrix} \sin\phi \\ \cos\phi \\ 0 \end{bmatrix} \tag{2}$$

in body-fixed coordinates. [5 marks]

c) The Euler equations of motion are given by

$$\begin{aligned} I_{xx}\dot{\Omega}_x - \Omega_y\Omega_z(I_{xx} - I_{zz}) &= N_x, \\ I_{xx}\dot{\Omega}_y - \Omega_z\Omega_x(I_{zz} - I_{xx}) &= N_y, \\ I_{zz}\dot{\Omega}_z - \Omega_x\Omega_y(I_{xx} - I_{xx}) &= N_z, \end{aligned}$$

and by substitution of the Ω and N components from equations (1) and (2), they are given by

$$\begin{aligned} I_{xx}\frac{\mathrm{d}}{\mathrm{d}t}\left(-\dot{\psi}\cos\phi\sin\theta + \dot{\theta}\sin\phi\right) - (\dot{\psi}\sin\phi\sin\theta + \dot{\theta}\cos\phi)(\dot{\psi}\cos\theta + \dot{\phi})(I_{xx} - I_{zz}) \\ = -mgf\sin\phi\sin\theta, \\ I_{xx}\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{\psi}\sin\phi\sin\theta + \dot{\theta}\cos\phi\right) - (\dot{\psi}\cos\theta + \dot{\phi})(-\dot{\psi}\cos\phi\sin\theta + \dot{\theta}\sin\phi)(I_{zz} - I_{xx}) \\ = -mgf\cos\phi\sin\theta, \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{\psi}\cos\theta + \dot{\phi}\right) = 0. \end{aligned}$$

[7 marks]

Question 2

a) The position vector of the large wheel is $\boldsymbol{r_1} = x\boldsymbol{i}$ and the position vector of the small wheel is $\boldsymbol{r_2} = x\boldsymbol{i} + (r_1 + r_2)\boldsymbol{e_r}$. [2 marks]

b) The velocity vectors are found by differentiating the corresponding position vectors. The velocity vector of the large wheel is $\dot{\boldsymbol{r}}_1 = \dot{x}\boldsymbol{i}$ and the velocity vector of the small wheel is

$$\dot{\boldsymbol{r}}_2 = \dot{x}\boldsymbol{i} + (r_1 + r_2)\dot{\theta}\boldsymbol{e_\theta} = \dot{x}\cos\theta\boldsymbol{e_r} + (-\dot{x}\sin\theta + (r_1 + r_2)\dot{\theta})\boldsymbol{e_\theta}.$$

[3 marks]

c) The total kinetic energy of the system is the sum of the kinetic energies of the two wheels, as follows:

$$\begin{aligned}
T &= \frac{1}{2}m_1\dot{\boldsymbol{r}}_1 \cdot \dot{\boldsymbol{r}}_1 + \frac{1}{2}I_1\dot{\phi}_1^2 + \frac{1}{2}m_2\dot{\boldsymbol{r}}_2 \cdot \dot{\boldsymbol{r}}_2 + \frac{1}{2}I_2\dot{\phi}_2^2 \\
&= \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}I_1\dot{\phi}_1^2 + \frac{1}{2}m_2\left(\dot{x}^2 - 2\dot{x}\dot{\theta}(r_1 + r_2)\sin\theta + (r_1 + r_2)^2\dot{\theta}^2\right) + \frac{1}{2}I_2\dot{\phi}_2^2 \\
&= \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2}I_1\dot{\phi}_1^2 - m_2\dot{x}\dot{\theta}(r_1 + r_2)\sin\theta + \frac{1}{2}m_2(r_1 + r_2)^2\dot{\theta}^2 + \frac{1}{2}I_2\dot{\phi}_2^2.
\end{aligned}$$

Only the potential energy of the small wheel is changing, therefore the potential energy of the system is $V = mg(r_1 + r_2)\sin\theta$ with the centre of the large wheel the zero potential energy level.
Hence the Lagrangian is $L = T - V$, which gives

$$L = \frac{1}{2}(m_1+m_2)\dot{x}^2+\frac{1}{2}I_1\dot{\phi}_1^2-m_2\dot{x}\dot{\theta}(r_1+r_2)\sin\theta+\frac{1}{2}m_2(r_1+r_2)^2\dot{\theta}^2+\frac{1}{2}I_2\dot{\phi}_2^2-mg(r_1+r_2)\sin\theta.$$

[5 marks]

d) $\dot{x} + r_1\dot{\phi}_1 = 0$ or $x + r_1\phi_1 = 0$. [2 marks]

e) The velocity of the centre of mass of the wheel is $\dot{\boldsymbol{r}}_2 = \dot{x}\boldsymbol{i}+(r_1+r_2)\dot{\theta}\boldsymbol{e_\theta}$ and the velocity of the contact point caused by the rotation of the small wheel is $-r_2\dot{\phi}_2\boldsymbol{e_\theta}$. Therefore, the velocity of the material contact point on the small wheel is $\dot{x}\boldsymbol{i}+(r_1+r_2)\dot{\theta}\boldsymbol{e_\theta}-r_2\dot{\phi}_2\boldsymbol{e_\theta}$. The velocity of the material contact point on the large wheel is $\dot{x}\boldsymbol{i}+r_1\dot{\phi}_1\boldsymbol{e_\theta}$. The relative velocity between the two material contact points on the two wheels is zero, therefore

$$\dot{x}\boldsymbol{i} + (r_1 + r_2)\dot{\theta}\boldsymbol{e_\theta} - r_2\dot{\phi}_2\boldsymbol{e_\theta} - \left(\dot{x}\boldsymbol{i} + r_1\dot{\phi}_1\boldsymbol{e_\theta}\right) = 0,$$

which yields the constraint equation

$$(r_1 + r_2)\dot{\theta} - r_2\dot{\phi}_2 - r_1\dot{\phi}_1 = 0.$$

Therefore, $A_1 = r_1 + r_2$, $A_2 = -r_1$, and $A_3 = -r_2$. [3 marks]

f) The two rolling constraint equations represent holonomic constraints and therefore it is possible to use these equations to eliminate the x and ϕ_2 coordinates in the Lagrangian function. After elimination the Lagrangian becomes

$$L = \frac{1}{2}\left((m_1 + m_2)r_1^2 + I_1 + \frac{r_1^2}{r_2^2}I_2\right)\dot{\phi}_1^2 + r_1(r_1 + r_2)\left(m_2\sin\theta - \frac{I_2}{r_2^2}\right)\dot{\theta}\dot{\phi}_1+$$

$$+\frac{(r_1+r_2)^2}{2}\left(m_2+\frac{I_2}{r_2^2}\right)\dot{\theta}^2 - m_2 g(r_1+r_2)\sin\theta.$$

The Lagrangian equation with respect to the generalised coordinate ϕ_1 is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\phi}_1}\right) - \frac{\partial L}{\partial \phi_1} = T_c,$$

yielding

$$\left((m_1+m_2)r_1^2 + I_1 + \frac{r_1^2}{r_2^2}I_2\right)\ddot{\phi}_1 + m_2 r_1(r_1+r_2)\cos\theta\dot{\theta}^2 + r_1(r_1+r_2)\left(m_2\sin\theta - \frac{I_2}{r_2^2}\right)\ddot{\theta} = T_c.$$

The Lagrangian equation with respect to the generalised coordinate θ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0,$$

which gives

$$r_1\left(m_2\sin\theta - \frac{I_2}{r_2^2}\right)\ddot{\phi}_1 + (r_1+r_2)\left(m_2+\frac{I_2}{r_2^2}\right)\ddot{\theta} + m_2 g\cos\theta = 0.$$

[5 marks]

Question 3

a) The generalised coordinates are ψ, θ_l, θ_r. [2 marks]

b) Consider the case of the left wheel. Three single-axis-rotation transformation matrices are needed.

$$D_\psi = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which is the rotation matrix by an angle ψ about a z axis.

$$B_\phi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which is the rotation matrix by an angle ϕ about an x axis, but $\phi = 0$ since the axle has only one degree of freedom.

$$C_\theta = \begin{bmatrix} \cos\theta_l & 0 & -\sin\theta_l \\ 0 & 1 & 0 \\ \sin\theta_l & 0 & \cos\theta_l \end{bmatrix}.$$

which is the rotation matrix by an angle θ_l about a y axis.

The angular velocity vector of the left wheel in body-fixed coordinates is

$$\boldsymbol{\Omega}_{wl} = C_\theta B_\phi D_\psi \begin{bmatrix} 0 \\ 0 \\ \dot\psi \end{bmatrix} + C_\theta \begin{bmatrix} 0 \\ \dot\theta_l \\ 0 \end{bmatrix} = C_\theta B_\phi \begin{bmatrix} 0 \\ 0 \\ \dot\psi \end{bmatrix} + \begin{bmatrix} 0 \\ \dot\theta_l \\ 0 \end{bmatrix} = \begin{bmatrix} -\dot\psi\sin\theta_l \\ \dot\theta_l \\ \dot\psi\cos\theta_l \end{bmatrix}.$$

Similarly, in the case of the right wheel:

$$\boldsymbol{\Omega}_{wr} = \begin{bmatrix} -\dot\psi\sin\theta_r \\ \dot\theta_r \\ \dot\psi\cos\theta_r \end{bmatrix}.$$

[4 marks]

c) The kinetic energy of the left wheel is the sum of the kinetic energies due to the translation of the centre of mass and the rotation about the centre of mass of the left wheel, as follows:

$$T_{lw} = \frac{1}{2}m\left(\frac{l\dot\psi}{2}\right)^2 + \frac{1}{2}I_{xx}\dot\psi^2\sin^2\theta_l + \frac{1}{2}I_{yy}\dot\theta_l^{\,2} + \frac{1}{2}I_{xx}\dot\psi^2\cos^2\theta_l,$$

which yields

$$T_{lw} = \frac{1}{2}\left(\frac{ml^2}{4} + I_{xx}\right)\dot\psi^2 + \frac{1}{2}I_{yy}\dot\theta_l^{\,2}.$$

[4 marks]

d) The kinetic energy of the right wheel is similarly given as:

$$T_{rw} = \frac{1}{2}\left(\frac{ml^2}{4} + I_{xx}\right)\dot{\psi}^2 + \frac{1}{2}I_{yy}\dot{\theta_r}^2.$$

The total kinetic energy of the system is the sum of the kinetic energies of the two wheels and the axle, as follows:

$$T = T_{lw} + T_{axle} + T_{rw},$$

which is evaluated as

$$T = \frac{1}{2}\left(\frac{ml^2}{4} + I_{xx}\right)\dot{\psi}^2 + \frac{1}{2}I_{yy}\dot{\theta_l}^2 + \frac{1}{2}\left(\frac{ml^2}{4} + I_{xx}\right)\dot{\psi}^2 + \frac{1}{2}I_{yy}\dot{\theta_r}^2 + \frac{1}{2}I_{axle}\dot{\psi}^2,$$

and which yields

$$T = \left(\frac{ml^2}{4} + I_{xx} + \frac{1}{2}I_{axle}\right)\dot{\psi}^2 + \frac{1}{2}I_{yy}\dot{\theta_l}^2 + \frac{1}{2}I_{yy}\dot{\theta_r}^2.$$

[3 marks]

e) i) The velocity of the road contact point of the left wheel is

$$\frac{l}{2}\dot{\psi} + R\dot{\theta}_l = 0, \tag{3}$$

which represents the first constraint equation. The velocity of the road contact point of the right wheel is

$$\frac{l}{2}\dot{\psi} - R\dot{\theta}_r = 0, \tag{4}$$

which represents the second constraint equation. [2 marks]

ii) The Lagrangian function of the system is $L = T - V = T$, as the potential energy of the system does not change. Also, since the two constraint equations (3) and (4) are integrable (and therefore holonomic) they can be used to substitute $\dot{\theta}_l$ and $\dot{\theta}_r$, and simplify the Lagrangian function, as follows:

$$L = \left(\frac{ml^2}{4} + I_{xx} + \frac{1}{2}I_{axle}\right)\dot{\psi}^2 + \frac{1}{2}I_{yy}\left(\frac{l\dot{\psi}}{2R}\right)^2 + \frac{1}{2}I_{yy}\left(\frac{l\dot{\psi}}{2R}\right)^2,$$

which results in

$$L = \left(\frac{ml^2}{4} + I_{xx} + \frac{1}{2}I_{axle} + I_{yy}\frac{l^2}{4R^2}\right)\dot{\psi}^2.$$

The only equation of motion is therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\psi}}\right) - \frac{\partial L}{\partial \psi} = M,$$

which yields

$$\left(2\left(\frac{ml^2}{4} + I_{xx}\right) + I_{axle} + I_{yy}\frac{l^2}{2R^2}\right)\ddot{\psi} = M.$$

[5 marks]

Question 4

a) The generalised coordinates are x, ϕ_1, ϕ_2 and θ. [2 marks]

b) The position vector of the large wheel is $\boldsymbol{r_1} = x\boldsymbol{i}$ and the position vector of the small wheel is $\boldsymbol{r_2} = x\boldsymbol{i} + (r_1 + r_2)\boldsymbol{e_r}$. By differentiating respectively twice each of the position vectors, the acceleration vector of the large wheel is given by $\ddot{\boldsymbol{r}}_1 = \ddot{x}\boldsymbol{i}$ and of the small wheel is given by

$$\ddot{\boldsymbol{r}}_2 = \ddot{x}\boldsymbol{i} - (r_1 + r_2)\dot{\theta}^2\boldsymbol{e_r} + (r_1 + r_2)\ddot{\theta}\boldsymbol{e_\theta}.$$

[3 marks]

c) The total force on the large wheel in vector form is

$$\boldsymbol{F_{total1}} = -F_{roll1}\boldsymbol{i} - R_1\boldsymbol{k} - R_2\boldsymbol{e_r} - F_{roll2}\boldsymbol{e_\theta} + m_1 g\boldsymbol{k}.$$

The total force on the small wheel in vector form is

$$\boldsymbol{F_{total2}} = R_2\boldsymbol{e_r} + F_{roll2}\boldsymbol{e_\theta} + m_2 g\boldsymbol{k}.$$

[3 marks]

d) By considering the translational motion of the centre of mass of the small wheel

$$R_2\boldsymbol{e_r} + F_{roll2}\boldsymbol{e_\theta} + m_2 g\boldsymbol{k} = m_2\left(\ddot{x}\boldsymbol{i} - (r_1 + r_2)\dot{\theta}^2\boldsymbol{e_r} + (r_1 + r_2)\ddot{\theta}\boldsymbol{e_\theta}\right),$$

and by decomposing all the force and acceleration vector components along $\boldsymbol{e_r}$ and $\boldsymbol{e_\theta}$

$$R_2 = m_2\left(\ddot{x}\cos\theta - (r_1 + r_2)\dot{\theta}^2 + g\sin\theta\right), \tag{5}$$

$$F_{roll2} = m_2\left(-\ddot{x}\sin\theta + (r_1 + r_2)\ddot{\theta} + g\cos\theta\right). \tag{6}$$

By considering the translational motion of the centre of mass of the large wheel

$$-F_{roll1}\boldsymbol{i} - R_1\boldsymbol{k} - R_2\boldsymbol{e_r} - F_{roll2}\boldsymbol{e_\theta} + m_1 g\boldsymbol{k} = m_1\ddot{x}\boldsymbol{i},$$

and by decomposing all the force and acceleration vector components along $\boldsymbol{i}$ and $\boldsymbol{k}$

$$-F_{roll1} + F_{roll2}\sin\theta - R_2\cos\theta = m_1\ddot{x},$$

$$-R_1 + F_{roll2}\cos\theta + R_2\sin\theta + m_1 g = 0.$$

By substitution using equations (5) and (6)

$$F_{roll1} = -(m_1 + m_2)\ddot{x} + m_2(r_1 + r_2)\left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right), \tag{7}$$

$$R_1 = (m_1 + m_2)g + m_2(r_1 + r_2)\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right).$$

[6 marks]

e) By considering the motion about the small wheel centre of mass according to $\frac{dH}{dt} = \boldsymbol{N}$,

$$I_2\ddot{\phi}_2 = -F_{roll2}r_2, \tag{8}$$

and by substitution using equation (6)

$$I_2\ddot{\phi}_2 + m_2 r_2\left(-\ddot{x}\sin\theta + (r_1+r_2)\ddot{\theta} + g\cos\theta\right) = 0.$$

By considering the motion about the large wheel centre of mass according to $\frac{d\boldsymbol{H}}{dt} = \boldsymbol{N}$,

$$I_1\ddot{\phi}_1 = -F_{roll1}r_1 - F_{roll2}r_1 + T_c,$$

and by substitution using equations (7) and (8)

$$I_1\ddot{\phi}_1 - \frac{r_1}{r_2}I_2\ddot{\phi}_2 - (m_1+m_2)r_1\ddot{x} + m_2 r_1(r_1+r_2)\left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right) = T_c.$$

[6 marks]