Imperial College London

M4/5P6

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May-June 2018

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Probability Theory

Date: Wednesday, 23 May 2018

Time: 2:00 PM - 4:30 PM

Time Allowed: 2.5 hours

This paper has 5 questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the relevant main book(s) are full.

All required additional material will be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Each question carries equal weight.
- Calculators may not be used.

1. (1.a) Give the definition of a probability space explaining carefully all notions involved.

(1.b) Explain giving reasons which of the following is a probability space and which is not.

(1.b.i) $((0,1),\mathcal{O},\lambda_0)$,
where
$\mathcal{O} \equiv$ set containing all open intervals $(a,b) \subset (0,1)$, with $a<b$, and all countable unions of such intervals
and
$\lambda_0(A) \equiv \inf_{\{(a_i,b_i)\subseteq(0,1)\}_{i\in\mathbb{N}}}\{\sum_i |b_i-a_i| : A \subset \bigcup_{i\in\mathbb{N}}(a_i,b_i)\}$.

(1.b.ii) $(\mathbb{N},2^{\mathbb{N}},\kappa)$,
where $\mathbb{N}$ are natural numbers, $2^{\mathbb{N}}$ denotes the family of all subsets in $\mathbb{N}$
and, for $p_i \in (0,1]$ such that $\sum_{i\in\mathbb{N}} p_i = 1$, one defines
$\kappa : 2^{\mathbb{N}} \to \mathbb{R}^+$ by

$$\kappa(A) \equiv \begin{cases} \sum_{i\in A} p_i & \text{if } A \text{ is finite} \\ 1 \text{ if } A = \mathbb{N} \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

(1.c)

Let (Ω,Σ,μ) be a probability space. Prove the following statements.

(1.c.i) If $A_n \in \Sigma$, $n\in\mathbb{N}$, such that $A_n \subset A_{n+1}$, then

$$\mu\left(\bigcup_n A_n\right) = \lim_{k\to\infty} \mu\left(A_k\right).$$

(1.c.ii) If $A_n \in \Sigma$, $n\in\mathbb{N}$, such that $A_{n+1} \subset A_n$, then

$$\mu\left(\bigcap_n A_n\right) = \lim_{k\to\infty} \mu\left(A_k\right).$$

2.

(2.a) Give the definition of mutually independent random variables explaining carefully all notions involved.

(2.b) Prove or disprove that Hermite polynomials in the space $(\mathbb{R}, \Sigma_{Leb}, \mu)$, where $d\mu \equiv \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} d\lambda$, are mutually independent.

(2.c) State and prove the basic Weak Law of Large Numbers.

(2.d) Suppose X_j, $j \in \mathbb{N}$, are random variables on a probability space (Ω, Σ, μ), for which the expectations of fourth power are uniformly bounded. Suppose for $|j - i| \geq 2$ they are mutually independent.

Prove that the following sequence of random variables

$$s_n \equiv \frac{1}{n} \sum_{j=1,\dots,n} (X_j - E_\mu X_j)$$

converges almost everywhere to 0.

3.

(3.a) State and prove the Borel-Cantelli Lemmas.

(3.b) Let Y_j, $n \in \mathbb{N}$, be i.i.d. random variables on a probability space (Ω, Σ, μ). For $L_n \in \mathbb{N}$, define

$$A_n \equiv \bigcap_{j=2n}^{2n+L_n} \{Y_j = 1\}.$$

Assume $\mu(\{Y_1 = 1\}) = q \in (0,1)$. Prove or disprove the following claims.

(3.b.i) If $\forall n \in \mathbb{N}, \quad L_n = 2n$, then

$$\mu\left(\bigcap_{n=1} \bigcup_{k \geq n} A_k\right) = 0.$$

(3.b.ii) If $\forall n \in \mathbb{N}, \quad L_n \leq \frac{\log(n+1)}{\log \frac{1}{q}}$, then

$$\mu\left(\bigcap_{n=1} \bigcup_{k \geq n} A_k\right) = 1.$$

4.

(4.a) State Lévy's continuity theorem explaining carefully all notions involved.

(4.b) Let Z_j, $j \in \mathbb{N}$, be random variables on a probability space (Ω, Σ, μ), with joint Gaussian distribution of mean zero and covariance

$$C_{jk} \equiv E_\mu(Z_j Z_k).$$

Using the following integration by parts formula for Gaussian random variables

$$\int \sum_i C_{ji} \partial_i F d\mu = \int Z_j F d\mu,$$

or otherwise, prove that the characteristic function $\varphi(t)$ of

$$V_n \equiv \sum_{j=1}^{n} \alpha_j Z_j.$$

is equal to

$$\varphi(t) = \exp\left\{ -\frac{t^2}{2} \sum_{j,k=1}^{n} \alpha_j C_{jk} \alpha_k \right\}.$$

(4.c) Suppose $\sum_{k=1}^{\infty} C_{jk}$ is convergent, uniformly with respect to j, to a number $C \in \mathbb{R}$ independent of j. For each $\beta \in (0, \infty)$, prove or disprove that for $\alpha_j \equiv \frac{1}{n^\beta}$, the corresponding sequence of characteristic functions converges to a characteristic function.

5.

(5.a) State the Poincaré and Log-Sobolev inequalities for a probability measure in $\mathbb{R}^n$.

(5.b) Prove that the Poincaré inequality satisfy the product property.

(5.c) Assuming that the Log-Sobolev inequality holds, prove that the distribution of any Lipschitz random variable has Gaussian tails.

(5.d) Let (Ω, Σ, μ) be a probability space, where $\Omega \equiv \mathbb{R}^{\mathbb{Z}^d}$ and Σ is a σ-algebra including the Borel σ-algebra in Ω.

Assume that Poincaré inequality holds. Let φ be a Lipschitz function of one real variable. Let π_j denote a projection $\Omega \ni \omega = (\omega_i \in \mathbb{R})_{i \in \mathbb{Z}^d} \longmapsto \pi_j(\omega) \equiv \omega_j$. Define a random variable

$$X_j \equiv \varphi \circ \pi_j.$$

Define

$$s_n \equiv \frac{1}{(2n+1)^d} \sum_{|j| \leq n} (X_j - E_\mu X_j).$$

Prove that

$$s_n \longrightarrow_{n \to \infty} 0$$

almost everywhere.

Solutions

S.1.

(S.1.a) Suppose $\Omega \neq \emptyset$. Let Σ be a σ-algebra in Ω, that is a family of subsets of Ω such that : 4pts seen

(a) $\Omega \in \Sigma$;

(b) $A \in \Sigma \Longrightarrow \Omega \setminus A \in \Sigma$;

(c) $\forall A_n \in \Sigma, n \in \mathbb{N}, \qquad \bigcup_{n\in\mathbb{N}} A_n \in \Sigma$.

A probability measure is a function $\mu : \Sigma \longrightarrow [0,1]$ satisfying

(i) $\mu(\Omega) = 1$;

(ii) $\forall A_n \in \Sigma, n \in \mathbb{N}, A_n \cap A_k = \emptyset$ if $n \neq k \Longrightarrow \mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \mu(A_n)$.

A triple (Ω, Σ, μ), which members are described above, is called a probability space.

(S.1.b)

(S.1.b.i) 4pts unseen

The family of open sets $\mathcal{O}$ does not contain closed sets which are by definition complements of open sets. Hence $\mathcal{O}$ is not a σ-algebra and so $((0,1), \mathcal{O}, \lambda_0)$ is not a probability space.

(S.1.b.ii) 5pts unseen

Since $p_i \in (0,1]$ is such that $\sum_{i\in\mathbb{N}} p_i = 1$, there exists an $N \in \mathbb{N}$ such that for $n > N$ we have

$$\sum_{i=1}^{n} p_i > \frac{1}{2}$$

Consider an infinite set $\mathbb{I} \subsetneq \mathbb{N}$ which contains $i = 1, .., n$. The $\mathbb{I}$ is countable union of pairwise disjoint one point sets

$$\bigcup_{k\in\mathbb{I}} \{k\} = \mathbb{I}.$$

Hence, according to the definition of the function $\kappa(\cdot)$, we have $\kappa(\mathbb{I}) = \frac{1}{2}$ which is not equal to $\sum_{k\in\mathbb{I}} \kappa(\{k\}) = \sum_{k\in\mathbb{I}} p_k > \frac{1}{2}$. Thus κ is not countably additive.

(S.1.c)

(S.1.c.i)

Suppose $A_n \in \Sigma$, $n \in \mathbb{N}$, are such that $A_n \subset A_{n+1}$. Define $B_1 \equiv A_1$ and for $n > 1$, define $B_n \equiv A_{n+1} \setminus A_n$. By this definition the sets B_n are pairwise disjoint and 3pts seen

$$\bigcup_n A_n = \bigcup_n B_n.$$

Hence using the σ-additivity of the probability measure, we have

$$\mu\left(\bigcup_n A_n\right) = \mu\left(\bigcup_n B_n\right)$$
$$= \sum_{n\in\mathbb{N}} \mu(B_n) = \lim_{n\to\infty} \sum_{k=1}^{n-1} \mu(B_k)$$
$$= \lim_{n\to\infty} \mu\left(\bigcup_{k=1}^{n-1} B_k\right) = \lim_{n\to\infty} \mu(A_n).$$

(S.1.c.ii)

Proof of (S.1.c.i)$\Longrightarrow$ (S.1.c.ii):

4Pts unseen

Let (Ω, Σ, μ) be a probability space. We note first that if $A_n \subset A_{n+1}$, then $\Omega \setminus A_{n+1} \subset \Omega \setminus A_n$. Next because of de'Morgan Law, we have

$$\Omega \setminus \bigcup_n A_n = \bigcap_n \Omega \setminus A_n.$$

Hence, using the fact that μ is a probability measure, we have

$$\mu\left(\Omega \setminus \bigcup_n A_n\right) = \mu(\Omega) - \mu\left(\bigcup_n A_n\right) = 1 - \mu\left(\bigcup_n A_n\right).$$

Thus, if the first statement(S.1.c.i) is true, we have

$$\mu\left(\bigcap_n \Omega \setminus A_n\right) = 1-\mu\left(\bigcup_n A_n\right) = 1- \lim_{n\to\infty} \mu(A_n) = \lim_{n\to\infty}(1-\mu(A_n)) = \lim_{n\to\infty} \mu(\Omega \setminus A_n).$$

This means (S.1.c.i) $\Longrightarrow$ (S.1.c.ii).

S.2.

(S.2.a)

4pts seen

Let (Ω, Σ, μ) be a probability space. Random variables $X_j : (\Omega, \Sigma) \longrightarrow (\mathbb{R}, \mathcal{B})$, $j = 1, .., n$, $n \in \mathbb{N}$, are called mutually independent iff the following σ-algebras

$$X_j^{-1}(\mathcal{B}), \quad j = 1, .., n$$

are mutually independent, that is for any $A_j \in X_j^{-1}(\mathcal{B}), \quad j = 1, .., n$, one has

$$\mu\left(\bigcap_{j=1,..,n} A_j\right) = \prod_{j=1,..,n} \mu(A_j).$$

(S.2.b)

5pts unseen

Mutual independence of random variables $X_j : (\mathbb{R}, \Sigma_{Leb}) \longrightarrow (\mathbb{R}, \mathcal{B})$, $j = 1, 2$, implies that

$$E_\mu (f(X_1)g(X_2)) = E_\mu (f(X_1)) E_\mu (g(X_2))$$

for all Borel measurable real functions f and g for which all the integrals are well defined. One can check by direct calculation with $H_1 = x$ and $H_2 = \alpha(x^2 - 1)$, where $\alpha \equiv \frac{1}{\sqrt{Var(x^2)}}$, that

$$0 < E_\mu \left(H_1^2 H_2\right) = \frac{1}{\sqrt{2\pi}} \int \alpha(x^4 - x^2) e^{-\frac{1}{2}x^2} dx = 2\alpha \neq E_\mu \left(H_1^2\right) E_\mu (H_2) = 0,$$

where on the left hand side one uses integration by parts formula

$$\int x^4 e^{-\frac{1}{2}x^2} dx = \int x^3 \left(-\frac{d}{dx} e^{-\frac{1}{2}x^2}\right) dx = \int 3x^2 e^{-\frac{1}{2}x^2} dx.$$

Thus Hermite polynomials are in general not mutually independent.
Although this is not required, one can show (by induction) a more general statement

$$0 < E_\mu (H_1^n H_n) \neq E_\mu (H_1^n) E_\mu (H_n) = 0$$

5pts seen

(S.2.c) Theorem (WLLN) : Let X_n, $n \in \mathbb{N}$ be real valued random variables on a probability space (Ω, Σ, μ). Assume $\sup_n (E_\mu(X_n^2)) < \infty$ and, for $j \neq k$, $E_\mu((X_j - E_\mu X_j)(X_k - E_\mu X_k)) = 0$. Then

$$\frac{1}{n} \sum_{k=1}^{n} (X_k - E_\mu X_k) \underset{n \to \infty}{\longrightarrow} 0$$

in probability.

Proof: We need to show that $\forall\varepsilon > 0$

$$\lim_{n\to\infty}\mu\left(\left\{\left|\frac{1}{n}\sum_{k=1}^{n}(X_k - E_\mu X_k)\right| > \varepsilon\right\}\right) = 0.$$

To this end, we use Chebyshev inequality

$$\mu\left(\left\{\left|\frac{1}{n}\sum_{k=1}^{n}(X_k - E_\mu X_k)\right| > \varepsilon\right\}\right) \leq \frac{1}{\varepsilon^2}E_\mu\left|\frac{1}{n}\sum_{k=1}^{n}(X_k - E_\mu X_k)\right|^2.$$

Using the condition $E_\mu((X_j - E_\mu X_j)(X_k - E_\mu X_k)) = 0$, for $k \neq j$ we get

$$\begin{aligned}E_\mu\left|\frac{1}{n}\sum_{k=1}^{n}(X_k - E_\mu X_k)\right|^2 &\leq \frac{1}{n^2}\sum_{k=1}^{n}E_\mu\left(X_k - E_\mu X_k\right)^2\\ &\leq \frac{1}{n}\sup_{k\in\mathbb{N}}E_\mu\left(X_k - E_\mu X_k\right)^2 \underset{n\to\infty}{\longrightarrow} 0.\end{aligned}$$

This together with the Chebyshev inequality, the above implies that

$$\frac{1}{n}\sum_{k=1}^{n}(X_k - E_\mu X_k) \underset{n\to\infty}{\longrightarrow} 0$$

in probability.

6pts unseen

(S.2.d) We have

$$\Sigma_{j=1,..,n}(X_j - E_\mu X_j) = s_1 + s_2$$

where s_k for $k = 0$ and $k = 1$ denote the sum over the odd and even indices, respectively. With this notation we have

$$E_\mu\left(|\Sigma_{j=1,..,n}(X_j - E_\mu X_j)|^4\right) \leq 8E_\mu\left(|s_1|^4\right) + 8E_\mu\left(|s_2|^4\right).$$

By our assumption each of the sums s_k consists of independent random variables. Using this, with χ_k denoting characteristic function of indices being odd if $k = 0$ and even if $k = 1$ indices, we have

$$\begin{aligned}E_\mu\left(|s_k|^4\right) \leq &\sum_{j=1,..,n}\chi_k(j)E_\mu\left(|X_j - E_\mu X_j|^4\right)\\ &+\sum_{\substack{j,i=1,..,n\\ j\neq i}}\chi_k(j)\chi_k(i)E_\mu\left(|X_j - E_\mu X_j|^2\right)E_\mu\left(|X_i - E_\mu X_i|^2\right)\end{aligned}$$

since for $j \neq i$, we have

$$E_\mu\left(|X_j - E_\mu X_j|^2\,|X_i - E_\mu X_i|^2\right) \leq E_\mu\left(|X_j - E_\mu X_j|^2\right)E_\mu\left(|X_i - E_\mu X_i|^2\right).$$

Hence

$$E_\mu\left(|s_k|^4\right) \leq n \sup_{j\in\mathbb{N}} E_\mu\left(|X_j - E_\mu X_j|^4\right)$$
$$+ n^2\left(\sup_{j\in\mathbb{N}} E_\mu\left(|X_j - E_\mu X_j|^2\right)\right)^2$$
$$\leq 2n^2 \sup_{j\in\mathbb{N}} E_\mu\left(|X_j - E_\mu X_j|^4\right)$$

(where in last step we used Cauchy-Schwartz inequality). From the above we conclude that

$$\sum_n E_\mu\left(\left|\frac{1}{n}\sum_{j=1,..,n}(X_j - E_\mu X_j)\right|^4\right) \leq \sum_n \frac{1}{n^4}\left(16n^2 \sup_{j\in\mathbb{N}} E_\mu\left(|X_j - E_\mu X_j|^4\right)\right)$$

converges. Hence by monotone convergence theorem, the series

$$\sum_n \left|\frac{1}{n}\Sigma_{j=1,..,n}(X_j - E_\mu X_j)\right|^4$$

converges almost everywhere. Hence, by the necessary condition of the convergence of a series, we have

$$\frac{1}{n}\sum_{j=1,..,n}(X_j - E_\mu X_j) \underset{n\to\infty}{\longrightarrow} 0$$

almost everywhere.

S.3.

7pts
seen

(S.3.a) Borel-Cantelli Lemma:
Let (Ω, Σ, μ) be a probability space. Suppose $A_n \in \Sigma$, $n \in \mathbb{N}$.

(S.3.a.i) Suppose

$$\sum_{n\in\mathbb{N}} \mu(A_n) < \infty.$$

Then

$$\mu\left(\bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} A_k\right) = 0.$$

(S.3.a.ii) Suppose the events $A_n \in \Sigma$, $n \in \mathbb{N}$, are mutually independent and

$$\sum_{n\in\mathbb{N}} \mu(A_n) = \infty$$

Then

$$\mu\left(\bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} A_k\right) = 1.$$

Proof of (S.3.a.i)
By monotonicity and subadditivity of the probability, we have

$$\begin{aligned}\mu\left(\bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} A_k\right) &\leq \mu\left(\bigcup_{k\geq n} A_k\right)\\ &\leq \sum_{k\geq n} \mu(A_k)\end{aligned}$$

Since by our assumption

$$\sum_{n\in\mathbb{N}} \mu(A_n) < \infty$$

this implies that

$$\sum_{k\geq n} \mu(A_k) \underset{n\to\infty}{\longrightarrow} 0$$

Proof of (S.3.a.ii)
It is sufficient to show that the complement

$$\Omega \setminus \bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} A_k = \bigcup_{n\in\mathbb{N}} \bigcap_{k\geq n} (\Omega \setminus A_k)$$

of the event of interest to us has probability zero. This will be achieved if we show that

$$\forall n \in \mathbb{N} \quad \mu\left(\bigcap_{k\geq n} \Omega \setminus A_k\right) = 0.$$

By mutual independence of the events A_k's, also $\Omega \setminus A_k$'s are mutually independent. Therefore, for any $m > n$, we have

$$\mu\left(\bigcap_{k=n}^{m} \Omega \setminus A_k\right) = \prod_{k=n}^{m} \mu(\Omega \setminus A_k) = \prod_{k=n}^{m} (1 - \mu(A_k))$$

Using inequality $1 - x \leq e^{-x}$, for $x \in [0,1]$, we get

$$\mu\left(\bigcap_{k=n}^{m} \Omega \setminus A_k\right) \leq \exp\{-\sum_{k=n}^{m} \mu(A_k)\}$$

Thus if

$$\sum_{k=n}^{\infty} \mu(A_k) = \infty$$

we get

$$\mu\left(\bigcap_{k\geq n} \Omega \setminus A_k\right) = \lim_{m\to\infty} \mu\left(\bigcap_{k=n}^{m} \Omega \setminus A_k\right) \leq \lim_{m\to\infty} \exp\{-\sum_{k=n}^{m} \mu(A_k)\} = 0$$

which ends the proof of the second part of Borel - Cantelli lemma.

(S.3.b)

5pts unseen

(S.3.b.i) Suppose $\forall n \in \mathbb{N} \quad L_n = 2n$. By mutual independence of the random variables Y_j, $n \in \mathbb{N}$, and the definition of A_n's, we have

$$\mu(A_n) = \prod_{j=2n}^{2n+L_n} \mu(\{Y_j = 1\}) = q^{L_n} = q^{2n}.$$

Hence for $q \in (0,1)$, one has

$$\sum_{n\in\mathbb{N}} \mu(A_n) = \frac{q^2}{1-q^2} < \infty$$

Hence by the first part of the Borel-Cantelli lemma

$$\mu\left(\bigcap_{n=1} \bigcup_{k\geq n} A_k\right) = 0$$

holds.

8pts unseen

(S.3.bii) Suppose $\forall n \in \mathbb{N} \quad L_n \leq \frac{\log(n+1)}{\log\frac{1}{q}}$. By mutual independence of the random variables Y_j, $n \in \mathbb{N}$, and the definition of A_n's, we have

$$\mu(A_n) = \prod_{j=2n}^{2n+L_n} \mu(\{Y_j = 1\}) = q^{L_n} = q^{\frac{\log(n+1)}{\log\frac{1}{q}}} = \frac{1}{n+1}.$$

Hence , one has

$$\sum_{n\in\mathbb{N}} \mu(A_n) = \infty.$$

Since the events A_n, $n \in \mathbb{N}$, are mutually independent by the second part of the Borel-Cantelli lemma we have

$$\mu\left(\bigcap_{n=1} \bigcup_{k\geq n} A_k\right) = 1.$$

S.4.

(S.4.a) **5pts** seen

Let X be a real valued random variable on a probability space (Ω, Σ, μ), i.e. $X : \Omega \to \mathbb{R}$ is a function with a property that $X^{-1}(\mathcal{B}) \subset \Sigma$, where $\mathcal{B}$ denotes Borel σ-algebra of sets in $\mathbb{R}$. A distribution function F_X of the random variable X is by definition given by

$$F_X(z) \equiv \mu\left(\{X \leq z\}\right).$$

A characteristic function φ_X of the random variable X is by definition given by

$$\varphi(t) \equiv E_\mu\left(e^{itX}\right).$$

Lévy's continuity theorem: Let F_n, $n \in \mathbb{N}$, and F be a distribution function with a characteristic function φ_n, $n \in \mathbb{N}$, and φ, respectively. If $F_n \to F$ as $n \to \infty$ at all points of continuity of F, then $\varphi_n \to \varphi$ uniformly on finite intervals.
Conversely, suppose φ_n is the characteristic function corresponding to a distribution function F_n, $n \in \mathbb{N}$. If $\varphi_n \to \varphi$ as $n \to \infty$ for all $t \in \mathbb{R}$, where φ is continuous at 0, then φ is a characteristic function of some distribution F and $F_n \to F$ as $n \to \infty$.

(S.4.b) **8pts** unseen

Let Z_j, be the Gaussian variables on a probability space (Ω, Σ, μ), with mean zero and strictly positive covariance

$$C_{jk} \equiv E_\mu(Z_j Z_k)$$

For $n \in \mathbb{N}$, let $d\gamma_n$ denote the Gaussian probability measure on $\mathbb{R}^n$ corresponding to the joint distribution of X_j, $j = 1, .., n$. That is, with a positive definite operator $A \equiv C^{-1}$, we have

$$d\gamma_n = \frac{1}{D} e^{-\frac{1}{2}\langle x, Ax\rangle} \lambda_n(dx)$$

where $\lambda_n(dx)$ denotes the n-dimensional Lebesgue measure and $D \in (0, \infty)$ is the normalisation factor. Using this one can derive the following formula for integration by parts for Gaussian measure in $\mathbb{R}^n$

$$\int \sum_i C_{ji} \partial_i F d\mu = \int Z_j F d\mu$$

for a differentiable function $F \equiv F(Z_1, .., Z_n)$ for which the integrals are well defined.
By definition the characteristic function of

$$V_n \equiv \sum_{j=1}^{n} \alpha_j Z_j$$

is given by

$$\varphi(t) \equiv \int e^{itV_n} d\mu.$$

For Gaussian random variables $\varphi(t)$ is differentiable and we have

$$
\begin{aligned}
-i\frac{d}{dt}\varphi(t) &= \int V_n e^{itV_n}\,d\mu \\
&= \sum_{j=1}^{n}\alpha_j \int Z_j e^{itV_n}\,d\mu.
\end{aligned}
$$

Using integration by parts formula, one gets

$$
\begin{aligned}
\sum_{j=1}^{n}\alpha_j \int Z_j e^{itV_n}\,d\mu &= i\sum_{j,k=1}^{n}\alpha_j C_{jk}\int \partial_k e^{itV_n}\,d\mu \\
&= -t\sum_{j,k=1}^{n}\alpha_j C_{jk}\alpha_k\varphi(t).
\end{aligned}
$$

Hence we get

$$
\frac{d}{dt}\varphi(t) = -t\left(\sum_{j=1,k}^{n}\alpha_j C_{jk}\alpha_k\right)\varphi(t).
$$

That is we have

$$
\frac{d}{dt}\left(\exp\left\{+\frac{t^2}{2}\sum_{j,k=1}^{n}\alpha_j C_{jk}\alpha_k\right\}\varphi(t)\right) = 0.
$$

Integrating this relation and taking into the account that for a characteristic function $\varphi(t=0)=1$, one arrives at

$$
\varphi(t) = \exp\left\{-\frac{t^2}{2}\sum_{j,k=1}^{n}\alpha_j C_{jk}\alpha_k\right\}.
$$

7pts

(S.4.c) Using the formula for characteristic function described above with $\alpha_j \equiv \frac{1}{n^\beta}$, we need to discuss behaviour of **unseen**

$$
\frac{1}{n^{2\beta}}\sum_{j,k=1}^{n} C_{jk}.
$$

To this end we note that

$$
\sum_{j,k=1}^{n} C_{jk} = nC - \left(\sum_{j=1}^{n}\sum_{k=n+1}^{\infty} C_{jk}\right).
$$

By our assumption the series $\sum_{k=1}^{\infty} C_{jk}$ converges uniformly with respect to j to a number $C \in \mathbb{R}$ independent of j. Hence for any $\varepsilon \in (0,1)$, exists $N \in \mathbb{N}$ such that for any $n > N$, we have

$$
\left|\sum_{k=n+1}^{\infty} C_{jk}\right| < \varepsilon.
$$

Therefore for sufficiently large n, we have

$$
n(C-\varepsilon) \le \sum_{j,k=1}^{n} C_{jk} \le n(C+\varepsilon).
$$

This implies that for $\beta = \frac{1}{2}$ the sequence of the characteristic functions in question converges to the characteristic functions of a Gaussian random variable given by

$$\Phi(t) = e^{-\frac{Ct^2}{2}}$$

For $\beta > \frac{1}{2}$ the corresponding sequence converges to 1, that the characteristic function of zero random variable.

For $0 < \beta < \frac{1}{2}$ the corresponding sequence converges to 0, which is not a characteristic function.

S.5.

(S.5.a) We say that a probability measure μ in $\mathbb{R}^n$ satisfies Poincaré inequality iff 3pts seen

$$\exists m \in (0,\infty) \qquad m \cdot Var_\mu(f) \leq E_\mu\left(|\nabla f|^2\right)$$

for any function f for which the right hand side is well defined, and that it satisfies Logarithmic Sobolev inequality iff

$$\exists c \in (0,\infty) \qquad Ent_\mu(f^2) \leq cE_\mu\left(|\nabla f|^2\right)$$

where

$$Ent_\mu(f^2) \equiv E_\mu\left(f^2 \log \frac{f^2}{E_\mu f^2}\right)$$

for any function f for which the right hand side is well defined.

(S.5.b) [Proof of product property of Poincaré inequality]

Suppose probability measure μ_i, satisfy Poincaré inequality with a constant $m_i \in (0,\infty)$, for $i=1,2$. Let $\mu \equiv \mu_1 \otimes \mu_2$. Then by a property of variance, for the product measure μ, we have 4pts seen

$$E_\mu(f - E_\mu f)^2 = E_{\mu_2}E_{\mu_1}(f - E_{\mu_1}f)^2 + E_{\mu_2}(E_{\mu_1}f - E_{\mu_2}E_{\mu_1}f)^2$$

for any square integrable function f. Suppose f is differentiable with square integrable gradient. Applying Poincaré inequality for the measures μ_i, $i=1,2$, to each term on the right side, we get

$$E_\mu(f - E_\mu f)^2 = E_{\mu_2}\frac{1}{m_1}E_{\mu_1}|\nabla_1 f|^2 + \frac{1}{m_2}E_{\mu_2}|\nabla_2 E_{\mu_1} f|^2$$

where ∇_i denote the gradient with respect to the integration variables of μ_i, $i=1,2$, respectively. Next using the following bound

$$|\nabla_2 E_{\mu_1} f|^2 = |E_{\mu_1}\nabla_2 f|^2 \leq E_{\mu_1}|\nabla_2 f|^2$$

which is a consequence of Cauchy-Schwartz inequality, we arrive at

$$\min(m_1, m_2)E_\mu(f - E_\mu f)^2 \leq E_\mu\left(|\nabla_1 f|^2 + |\nabla_2 f|^2\right)$$

(S.5.c)

We prove in part I below that, under the assumption of Log-Sobolev inequality, for Lipschitz random variables a Gaussian exponential bound holds. Then, in part II, we show that the desired estimate of probability tails follows from Chebyshev inequality. **6pts unseen**

Part I:
Let f be a Lipschitz random variable which is bounded. If Log-Sobolev inequality holds, then in particular for a function $e^{\frac{1}{2}tf}$, with $t \in \mathbb{R}^+$, we have

$$E_\mu\left(e^{tf}\log\frac{e^{tf}}{E_\mu e^{tf}}\right) \le \frac{1}{4}t^2 c E_\mu\left(|\nabla f|^2 e^{tf}\right).$$

and hence

$$E_\mu\left(e^{tf}\log\frac{e^{tf}}{E_\mu e^{tf}}\right) \le \frac{1}{4}t^2 c \cdot \||\nabla f|^2\|_\infty \cdot E_\mu\left(e^{tf}\right).$$

This can be transformed into the following relation

$$\frac{d}{dt}\left(\frac{1}{t}\log\left(E_\mu e^{tf}\right)\right) \le \frac{1}{4}c \cdot \||\nabla f|^2\|_\infty.$$

Integrating this inequality from $\varepsilon \in (0,1)$ to $t \in \mathbb{R}^+$, after simple transformations, one gets

$$\log\left(E_\mu e^{tf}\right) \le \frac{1}{4}t^2 c \cdot \||\nabla f|^2\|_\infty + t\log\left(\left(E_\mu e^{\varepsilon f}\right)^{\frac{1}{\varepsilon}}\right).$$

Using the following limiting behaviour for the last term on the right hand side

$$\left(E_\mu e^{\varepsilon f}\right)^{\frac{1}{\varepsilon}} = \left(1 + \varepsilon E_\mu f + \mathcal{O}(\varepsilon^2)\right)^{\frac{1}{\varepsilon}} \underset{\varepsilon \to 0}{\longrightarrow} e^{E_\mu f}.$$

we conclude with the following exponential bound

$$E_\mu e^{tf} \le e^{\frac{1}{4}t^2 c \cdot \||\nabla f|^2\|_\infty + tE_\mu f}.$$

For general Lipschits function f, we apply the above arguments first to a cutoff function $f\chi(|f| \le L)$, with some $L \in (0,\infty)$, and pass to the limit with $L \to \infty$ in the last bound.
Part II:
From this, applying Chebyshev inequality, we have the following estimate on the tails of the distribution

$$\mu(\{f > x\}) = \mu\left(\{e^{tf} > e^{tx}\}\right) \le e^{-tx}E_\mu e^{tf} \le e^{-tx}e^{\frac{1}{4}t^2 c \cdot \||\nabla f|^2\|_\infty + tE_\mu f}.$$

Optimising this with respect to t, we obtain

$$\mu(\{f > x\}) \le e^{-\frac{(x - E_\mu f)^2}{c^2 \cdot \||\nabla f|^2\|_\infty}}.$$

(S.5.d)

7pts unseen

We note that if μ satisfies Poincaré inequality, than in particular we have

$$E_\mu(G^4) = Var_\mu(G^2) + (E_\mu G^2)^2 \leq \frac{4}{m} E_\mu G^2 |\nabla G|^2 + (E_\mu G^2)^2$$

We will apply this relation to a function

$$G \equiv \sum_{|j|\leq n} (X_j - E_\mu X_j)$$

where

$$X_j \equiv \varphi \circ \pi_j,$$

with a Lipschitz function φ of one real variable. First we note that

$$\begin{aligned}|\nabla G|^2 &= \sum_{k\in\mathbb{Z}^d} \left| \nabla_k \sum_{|j|\leq n} (X_j - E_\mu X_j) \right|^2 \\ &= \sum_{k\in\mathbb{Z}^d} \left| \nabla_k \sum_{|j|\leq n} (\varphi \circ \pi_j) \right|^2 \\ &= \sum_{|j|\leq n} |(\nabla\varphi) \circ \pi_j|^2 \leq \| |\nabla\varphi|^2 \|_\infty \cdot \frac{4}{m}(2n+1)^d\end{aligned}$$

Hence we have

$$E_\mu \left(\sum_{|j|\leq n} (X_j - E_\mu X_j) \right)^4 \leq \frac{4}{m}(2n+1)^d E_\mu \left(\sum_{|j|\leq n} (X_j - E_\mu X_j) \right)^2 + \left(E_\mu \left(\sum_{|j|\leq n} (X_j - E_\mu X_j) \right)^2 \right).$$

Since by Poincaré inequality we have

$$E_\mu \left(\sum_{|j|\leq n} (X_j - E_\mu X_j) \right)^2 \leq \frac{1}{m} \sum_{k\in\mathbb{Z}^d} E_\mu \left| \nabla_k \sum_{|j|\leq n} (X_j - E_\mu X_j) \right|^2 \leq \frac{1}{m}(2n+1)^d \| |\nabla\varphi|^2 \|_\infty,$$

therefore we have

$$E_\mu \left(\sum_{|j|\leq n} (X_j - E_\mu X_j) \right)^4 \leq \frac{5}{m^2}(2n+1)^{2d} \| |\nabla\varphi|^2 \|_\infty^2.$$

For normalised sum

$$s_n \equiv \frac{1}{(2n+1)^d} \sum_{|j|\leq n} (X_j - E_\mu X_j)$$

this implies that

$$E_\mu \left(\frac{1}{(2n+1)^d} \sum_{|j|\leq n} (X_j - E_\mu X_j) \right)^4 \leq \frac{5}{m^2 (2n+1)^{2d}} \| |\nabla\varphi|^2 \|_\infty^2.$$

Hence

$$\sum_{n\in\mathbb{N}} E_\mu \left(\frac{1}{(2n+1)^d} \sum_{|j|\le n} (X_j - E_\mu X_j) \right)^4 < \infty.$$

Therefore the series

$$\sum_{n\in\mathbb{N}} \left(\frac{1}{(2n+1)^d} \sum_{|j|\le n} (X_j - E_\mu X_j) \right)^4$$

converges almost everywhere to a finite limit. This implies that

$$s_n \xrightarrow[n\to\infty]{} 0$$

almost everywhere.