Paper Number(s): **E3.10**
ISE3.7

IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE
UNIVERSITY OF LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2001

EEE/ISE PART III/IV: M.Eng., B.Eng. and ACGI

MATHEMATICS FOR SIGNALS AND SYSTEMS

Wednesday, 9 May 10:00 am

There are FIVE questions on this paper.

Answer THREE questions.

Time allowed: 3:00 hours

Corrected Copy

Examiners: Weiss,G. and Allwright,J.C.

Special instructions to invigilators: None

Information for candidates: None

1. Consider the space $Z = \mathbb{C}^{3\times 2}$ of matrices with three rows and two columns. We define an inner product on Z by $< A, B > \; = \text{trace}\, AB^*$, where B^* is the complex conjugate of the transpose of B, and we define $\|A\|_Z^2 \; = \; < A, A >$.

(a) What is the dimension of Z?

(b) If

$$A = \alpha \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

find a number α and two matrices B and C in Z such that $\{A, B, C\}$ is an orthonormal set.

(c) Give an example of a subspace $V \subset Z$ with $V \neq Z$ which contains A, B and C.

(d) Give an example of a subspace $W \subset Z$ with $W \neq \{0\}$ which does not contain any of A, B and C.

(e) If

$$D = \begin{bmatrix} 3 & 0 \\ 0 & 4 \\ 0 & 0 \end{bmatrix},$$

compute $\|D\|_Z$. Compute also the norm $\|D\|$ when D is regarded as an operator from $\mathbb{C}^2$ to $\mathbb{C}^3$ (be careful, this is not the same as $\|D\|_Z$).

(f) We define on $\mathbb{C}^2$ the functions p, q and r by

$$p(x) = \|Ax\|, \qquad q(x) = \|Dx\|, \qquad r(x) = \|Dx\|^2,$$

where A and D are those specified above and $\|\cdot\|$ is the usual Euclidean norm on $\mathbb{C}^3$. State, without proof, which of the functions p, q and r is a norm on $\mathbb{C}^2$? For each function (if any) which is not a norm, explain briefly why it is not a norm.

(g) Which of the norms that you found in part (f) is derived from an inner product? If there is such a norm, indicate the corresponding inner product.

2. Recall that $\mathbf{c}_0$ is the space of sequences convergent to zero, and $\mathbf{c}$ is the space of convergent sequences (indices run from 1 to ∞).

(a) Give an example of a sequence $q \in l^1$.

(b) Give an example of a sequence $u \in l^2$ such that $u \notin l^1$.

(c) Give an example of a sequence $v \in \mathbf{c}_0$ such that $v \notin l^2$.

(d) Give an example of a sequence $w \in \mathbf{c}$ such that $w \notin \mathbf{c}_0$.

(e) Give an example of a sequence $y \in l^\infty$ such that $y \notin \mathbf{c}$.

(f) The $\mathcal{Z}$ transform of a sequence $(d_1, d_2, d_3 ...)$ is defined by

$$(\mathcal{Z}d)(z) = \sum_{k=1}^{\infty} d_k z^{-k}$$

Compute the $\mathcal{Z}$ transform of **one** of the sequences q, u, v, w, y above. Indicate a domain (the largest domain that you can determine) where the series defining your $\mathcal{Z}$ transform is convergent.

(g) For the sequences w and y above, compute their norm in l^∞.

(h) If $a \in l^1$ and $b \in l^2$, is it true that the series $\sum_{k=1}^{\infty} a_k b_k$ converges to a finite sum? If yes, then say why, if no, then give a counterexample.

3. In this question, $\mathbf{S}_\tau$ denotes the right shift operator by τ on $L^2[0,\infty)$ and $*$ denotes the convolution product.

(a) Explain, in your words, what is the space $L^2[0,\infty)$. Do not forget to define the inner product and the norm. It is advisable to start by introducing the space $L^2[a,b]$ for a finite interval $[a,b]$. Do not write more than one page of explanation!

(b) In the sequel, consider $f(t) = te^{-t}$ and $g(t) = e^{-3t}$, $t \geq 0$. Compute the Laplace transforms $F = \mathcal{L}f$ and $G = \mathcal{L}g$.

(c) Compute $\|f\|_2$, $< f, g >$ and $\|g\|_2$ and check that the Cauchy-Schwarz inequality holds for them.

(d) Define $h = \mathbf{S}_5 g$, i.e., h is obtained by delaying g by 5 time units. Compute

$$H = \mathcal{L}h, \qquad \|h\|_2 \qquad \text{and} \quad P = \mathcal{L}(h * g).$$

(e) Compute

$$\|G\|_2, \qquad \|H\|_2 \quad \text{and} \quad < F, G > ,$$

where the norms and the scalar products correspond to the Hardy space $H^2(\mathbb{C}_+)$ and F, G, H are as defined above.

4. Consider the system described by

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u,$$

$$y = [1 \;\; K]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

where u is the input signal, x is the state (with two components), y is the output signal and K is a real constant.

(a) For which values of K is the system stable?

(b) Compute the transfer function $\mathbf{G}$ of this system.

(c) Determine a value of K for which $\mathbf{G}$ becomes a first order transfer function.

(d) For $K = 0$, compute $\|\mathbf{G}\|_\infty$.

(e) If $u(t) = e^{-2t} - e^{-3t}$ and $x(0) = 0$, compute the Laplace transform of the output signal, $\mathcal{L}y$.

(f) Consider the cascade connection of the system with a delay line of 0.5 time units. Thus, if z is the output signal of the delay line, then $z(t) = y(t - 0.5)$. Compute the transfer function $\mathbf{H}$ from u to z.

(g) For $K = 0.5$, compute $\|\mathbf{H}\|_\infty$.

(h) Suppose now that K is a function of t: $K(t) = \cos t$. Does the system with input u and output y have a transfer function? Explain very briefly your answer.

5. In parts (a) and (c) of this question you are asked to explain a concept and in (b) and (d) you are asked to state a theorem and to comment on it. You may state the two theorems in your own words. Try to add comments about the significance and the applications of the two theorems, but do not exceed one page per theorem (including the comments).

   (a) Explain briefly what is meant by a time-invariant operator on $L^2[0, \infty)$.

   (b) State the Fourés-Segal theorem (continuous-time version) and, if possible, make some comments about its connections with systems theory.

   (c) Define the space $BL(\omega_b)$ of band-limited functions with angular frequencies not higher than ω_b.

   (d) State the sampling theorem and, if possible, make some comments about its significance for the transmission and storage of signals.

[END]

E3.10
ISE 3.7

Mathematics for Signals & Systems

Exam of ~~June~~ MAY 2001

SOLUTIONS

Question 1 (a) ② 6. (b) ⑤ $\alpha = \frac{1}{\sqrt{2}}$

$$B = \alpha \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} \qquad C = \alpha \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

(there are many other solutions, of course).

(c) ② $V = \text{span}\{A, B, C\}$ (d) ③ $W = V^{\perp}$

(e) ⑤

$$\|D\|_Z = \sqrt{3^2 + 4^2} = 5.$$

$$\|D\|^2 = \max \sigma(D^*D) = \max \sigma \begin{bmatrix} 9 & 0 \\ 0 & 16 \end{bmatrix}$$

$$\Longrightarrow \|D\| = 4.$$

(f) ④ q is a norm. p is not a norm because $p\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = 0$, r is not a norm because $r(\lambda x) \neq \lambda r(x)$.

(g) ④ $q(x) = \sqrt{\langle\langle x, x \rangle\rangle}$, where $\langle\langle \cdot, \cdot \rangle\rangle$ is an inner product, $\langle\langle x, y \rangle\rangle = \langle Dx, Dy \rangle$.

Question 2 (a)(2) $q_k = \frac{1}{2^k}$

(b)(3) $u_k = \frac{1}{k}$ (c)(3) $v_k = \frac{1}{\sqrt{k}}$

(d)(2) $w_k = 1 + \frac{1}{k}$ (e)(3) $y_k = (-1)^k$

(f)(5) $(\mathcal{Z} q)(z) = \frac{1}{2z-1}$, $(\mathcal{Z} u)(z) = \ln \frac{z}{z-1}$

$(\mathcal{Z} w)(z) = \frac{1}{z-1} + \ln \frac{z}{z-1}$

$(\mathcal{Z} y)(z) = \frac{-1}{z+1}$

(Only one of these is required.)

For all of these, the series converges for $|z| > 1$ (in the case of q, it actually converges for $|z| > \frac{1}{2}$).

(g)(2) $\|w\|_\infty = 2$, $\|y\|_\infty = 1$.

(h)(5) $\sum_{k=1}^{\infty} a_k b_k$ converges to a finite sum because the series with the absolute values of the terms converges. Indeed, since $\ell^1 \subset \ell^2$, we have by the Cauchy-Schwarz inequality, for any $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} |a_k b_k| \leq \sqrt{\sum_{k=1}^{n} |a_k|^2} \cdot \sqrt{\sum_{k=1}^{n} |b_k|^2} \leq \|a\|_2 \cdot \|b\|_2 .$$

Question 3 (a) ⑥ For a finite interval $[a,b]$, $L^2[a,b]$ is defined as the completion of the continuous functions $C[a,b]$ with respect to the norm

$$\|f\|_2 = \sqrt{\int_a^b |f(t)|^2\,dt} \qquad (f \in C[a,b]).$$

The elements of $L^2[a,b]$ may also be regarded as functions defined on $[a,b]$, but with the convention that we do not distinguish between two functions g_1 and g_2 in $L^2[a,b]$ if $\int_a^b |g_1(t) - g_2(t)|\,dt = 0$

(this property of g_1 and g_2 is called "$g_1(t) = g_2(t)$ for almost every $t \in [a,b]$"). $L^2[a,b]$ is a Hilbert space with the inner product

$$\langle g,h \rangle = \int_a^b g(t)\,\overline{h(t)}\,dt$$

and the corresponding norm $\|g\|_2 = \sqrt{\langle g,g \rangle}$.

A function u defined on $[0,\infty)$ is in $L^2[0,\infty)$ if for every $n \in \mathbb{N}$, its restriction to $[0,n]$, $u|_{[0,n]}$ is in $L^2[0,n]$ and

$$\|u\|_2 = \lim_{n\to\infty} \| u|_{[0,n]} \|_2 < \infty.$$

We do not distinguish between two functions u_1, u_2 in $L^2[0,\infty)$ if $\int_0^\infty |u_1(t) - u_2(t)|\,dt = 0$. (This is equivalent to $\int_0^n |u_1(t) - u_2(t)|\,dt = 0$ for every $n \in \mathbb{N}$.) $L^2[0,\infty)$ is a Hilbert space with the inner product

$$\langle g,h \rangle = \int_0^\infty g(t)\,\overline{h(t)}\,dt$$

and the norm is derived from here in the obvious w[ay] which is the same as $\|u\|_2$ defined 7 lines above. —

(b)④

$$F(s)=\frac{1}{(s+1)^2}\ , \quad G(s)=\frac{1}{s+3}\ .$$

(c)⑥ $$\|f\|_2^2=\int_0^\infty e^{-2t}\,t^2\,dt=(\mathcal{L}\,t^2)(2)\,.$$

Since $(\mathcal{L}\,t^2)(s)=\frac{2}{s^3}$, we get

$$\|f\|_2^2=\frac{2}{2^3}=\frac{1}{4}\ , \text{ so } \|f\|_2=\frac{1}{2}\,.$$

$$\langle f,g\rangle=\int_0^\infty e^{-4t}\,t\,dt=\mathcal{L}(t)(4)=\frac{1}{16}\,.$$

$$\|g\|_2^2=\int_0^\infty e^{-6t}\,dt=\frac{1}{6}\ , \text{ so } \|g\|_2=\frac{1}{\sqrt{6}}\,.$$

Cauchy-Schwarz: $\frac{1}{16}<\frac{1}{2}\cdot\frac{1}{\sqrt{6}}$.

(d)⑤ $$H(s)=e^{-5s}\cdot\frac{1}{s+3}\ , \quad \|h\|_2=\|g\|_2=\frac{1}{\sqrt{6}}\,,$$

$$P(s)=H(s)\cdot G(s)=\frac{e^{-5s}}{(s+3)^2}\,.$$

(e)④ $$\|G\|_2=\|g\|_2=\frac{1}{\sqrt{6}}\ , \quad \|H\|_2=\|h\|_2=\frac{1}{\sqrt{6}}\,,$$

$$\langle F,G\rangle=\langle f,g\rangle=\frac{1}{16}\,.$$

Question 4 (a) ③ $\dot{x} = Ax + Bu$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$,

$A = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$, $\det(sI - A) = s^2 + 5s + 6$ (characteristic polynomial)

$= (s+2)(s+3)$, so that the eigenvalues of A are -2 and -3. The system is stable regardless of K.

(b) ④ $G(s) = [1 \ K](sI - A)^{-1} B = \dfrac{Ks + 1}{s^2 + 5s + 6}$.

(c) ③ For $K = \frac{1}{2}$, $G(s) = \dfrac{0.5(s+2)}{(s+2)(s+3)}$

$= \dfrac{0.5}{s+3}$. Another possibility: $K = \frac{1}{3}$.

(d) ④ $\left\| \dfrac{1}{s^2 + 5s + 6} \right\|_\infty = \dfrac{1}{6}$, because the maximum is attained at $s = 0$.

(e) ③ $(\mathcal{L} u)(s) = \dfrac{1}{s+2} - \dfrac{1}{s+3} = \dfrac{1}{s^2 + 5s + 6}$,

$(\mathcal{L} y)(s) = G(s)(\mathcal{L} u)(s) = \dfrac{Ks + 1}{(s^2 + 5s + 6)^2}$.

(f) ② $H(s) = e^{-0.5s} \dfrac{Ks + 1}{s^2 + 5s + 6}$.

(g) ③ $\|H\|_\infty = \|G\|_\infty = \left\| \dfrac{0.5}{s+3} \right\|_\infty = \dfrac{1}{6}$ (maximum at $s = 0$).

(h) ③ If $K(t) = \cos t$, then the system (its input-output operator) is not time-invariant, hence it has no transfer function.

Question 5 (a) (4) We denote by S_τ the operator of right shift by τ on $L^2[0,\infty)$ (delay by τ time units). A bounded operator T from $L^2[0,\infty)$ to $L^2[0,\infty)$ is called <u>time-invariant</u> if $TS_\tau = S_\tau T$ for all $\tau > 0$.

(b) (8) Theorem (Fourés-Segal). Let T be a bounded linear operator from $L^2[0,\infty)$ to $L^2[0,\infty)$. T is time-invariant iff there exists $G \in H^\infty(\mathbb{C}_+)$ such that
$$T = \mathcal{L}^{-1} G \mathcal{L} \quad (\mathcal{L} = \text{Laplace tr.}).$$
If this is the case, then $\|T\| = \|G\|_\infty$.

Consider a linear system described by
$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases} \quad \left| \begin{array}{l} u = \text{input signal} \\ x = \text{state} \\ y = \text{output signal} \end{array} \right.$$
where A, B, C, D are constant matrices and A is stable, i.e., its eigenvalues are in $\mathbb{C}_-$. If $x(0) = 0$, then $y = Tu$, where T is the <u>input-output operator</u> of the system. This is a bounded and time-invariant operator from $L^2[0,\infty)$ to $L^2[0,\infty)$. It is of the form indicated in the theorem, with $G(s) = C(sI-A)^{-1}B + D$. Such a system is called a finite-dimensional LTI system, and for this situation, $T = \mathcal{L}^{-1} G \mathcal{L}$ is easy to prove directly from the equations, without using the Fourés-Segal theorem. For more complicated systems (such as those involving propagating waves), a direct proof becomes more difficult. Even for finite-dimensional systems, it is not easy to prove directly that $\|T\| = \|G\|_\infty$.

(c)[4] $BL(\omega_b)$ is the subspace of $L^2(-\infty,\infty)$ consisting of those functions whose Fourier transform is in $L^2[-i\omega_b, i\omega_b]$ (in other words, $u \in BL(\omega_b)$ if $(\mathcal{F}u)(i\omega) = 0$ for $|\omega| > \omega_b$). Such functions u are infinitely differentiable (actually, analytic).

(d)[9]

Theorem (Whittaker–Kotelnikov–Shannon).

If $u \in BL(\omega_b)$ and $\tau \in \left(0, \frac{\pi}{\omega_b}\right]$, then for all $t \in \mathbb{R}$

$$u(t) = \sum_{k \in \mathbb{Z}} u(k\tau) \frac{\sin \omega_b (t - k\tau)}{\omega_b (t - k\tau)} .$$

This shows that if we sample the signal at the time instants $k\tau$, $k \in \mathbb{Z}$, where τ is the sampling period, then u can be completely reconstructed from these samples. It is easier to store and/or transmit samples of a signal than the whole signal.

In practice, signals are not exactly bandlimited, just "almost" bandlimited. This means that $u = v + e$, where $v \in BL(\omega_b)$ and e is a small error (deviation). Also, the samples $u(k\tau)$ cannot be taken for all $k \in \mathbb{Z}$, only for a finite (but possibly very large) set of integers. Then, the formula will hold approximately, for values of t which are not close to the end of the time interval in which samples were taken. The condition $\tau \leq \frac{\pi}{\omega_b}$ means that the sampling frequency $\frac{1}{\tau} \geq 2$ times the highest frequency components of u.

END