UNIVERSITY OF LONDON [E2.11 2003]

B.ENG. AND M.ENG. EXAMINATIONS 2003

For Internal Students of Imperial College London

This paper is also taken for the relevant examination for the Associateship of the City & Guilds of London Institute

INFORMATION SYSTEMS ENGINEERING E2.11

MATHEMATICS

Date Thursday 5th June 2003 2.00 - 4.00 pm

Answer FOUR questions, to include at least one from Section B

Answers to Section A *questions must be written in a different answer book from answers to* Section B *questions.*

[*Before starting, please make sure that the paper is complete. There should be SIX pages, with a total of SIX questions. Ask the invigilator for a replacement if this copy is faulty.*]

Section A

1. We define the Fourier transform $\hat{f}(\omega)$ of a function $f(t)$ as

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} f(t) dt .$$

If f is smooth and $|f(t)| \to 0$ sufficiently fast as $t \to \pm\infty$, show that one can evaluate the Fourier transforms of $f'(t)$, $f''(t)$, $tf(t)$ and $tf'(t)$ in the following forms:

(i)
$$\widehat{f'(t)}(\omega) = a_1(\omega)\hat{f}(\omega) ;$$

(ii)
$$\widehat{f''(t)}(\omega) = a_2(\omega)\hat{f}(\omega) ;$$

(iii)
$$\widehat{tf(t)}(\omega) = i\frac{d\hat{f}(\omega)}{d\omega} ;$$

(iv)
$$\widehat{tf'(t)}(\omega) = -\hat{f}(\omega) + ia_1(\omega)\frac{d\hat{f}(\omega)}{d\omega} ;$$

and find $a_1(\omega)$ and $a_2(\omega)$.

Use Fourier transforms and the above results to solve the differential equation

$$\frac{d^2y}{dt^2} + 2t\frac{dy}{dt} + 2y = 0$$

for the solution $y(t)$ such that $y(t) \to 0$ as $t \to \pm\infty$ and $y(0) = 1$.

You may use the fact that $\int_{-\infty}^{\infty} e^{-u^2} du = \sqrt{\pi}$.

Check directly that your final solution $y(t)$ satisfies the differential equation.

PLEASE TURN OVER

2. Prove the convolution theorem for Laplace transforms

$$\overline{f * g} = \overline{f}(s)\overline{g}(s) \ ,$$

where the overbar indicates the Laplace transform, i.e.

$$\overline{f}(s) \equiv \int_0^\infty f(t)e^{-st}dt$$

and

$$f * g \equiv \int_0^t f(t-u)\, g(u)du \ .$$

Use the above result to find the function $y(t)$ which satisfies the equation

$$y(t) = e^{-3t} - \int_0^t y(t-u)e^{-u}\, du \quad \text{for} \quad t > 0.$$

Once you have found your solution, verify by direct substitution that it satisfies the equation.

3. (i) Make a sketch of the region R of the $x - y$ plane over which the integral

$$I \ = \ \int_0^1 dx \int_{x^3}^1 x^2 e^{y^2}\, dy$$

is taken. Change the order of integration, using your sketch as necessary, and hence evaluate the integral.

(ii) Use Green's theorem in R

$$\int_R\int \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy \ = \ \int_C (Pdx + Qdy)$$

to evaluate the integral in (i) by reducing it to a line integral.

Remember the choice of P and Q is not unique so it is up to you to make a convenient choice.

PLEASE TURN OVER

4. (i) Find the poles of

$$\frac{1}{(az^2 - (a^2+1)z + a)},$$

when a is a real positive constant and find the residue at each pole.

When $a > 0$ is a real constant, evaluate

$$I = \int_{-i\infty}^{i\infty} \frac{dz}{(az^2 - (a^2+1)z + a)},$$

the path of the integral being along the imaginary axis.

(ii) Show that

$$J = \int_0^{2\pi} \frac{d\theta}{2a\cos\theta - (a^2+1)}$$

can be expressed in terms of a complex path integral as

$$J = -i\int_C \frac{dz}{\alpha z^2 + \beta z + \gamma},$$

where C is the unit circle $|z| = 1$ described anticlockwise.

Identify the constants α, β and γ in terms of a and evaluate J for the case $a = 2$.

Section B **[E2.11 2003]**

1. The random variable X has a Normal distribution, with probability density function

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(\frac{-(x-\mu)^2}{2\,\sigma^2}\right), \qquad -\infty < x < \infty ,$$

where $-\infty < \mu < \infty$ and $\sigma > 0$.

(i) Describe the role of the parameters μ and σ.

(ii) For $X \sim N(20, 81)$, compute

(a) $P(17 < X < 24)$,

(b) $P(X > 20 \,|\, X > 19)$.

(iii) Suppose that random variables X_1 and X_2 are independent and distributed as

$$X_1 \sim N(\mu, \sigma_1^2) \qquad X_2 \sim N(\mu, \sigma_2^2) .$$

Write down the distribution of the random variable $Y = X_1 - X_2$.

(iv) In an experiment, a student moves the same large data file across the Internet at randomly selected times on each of 11 days. The times taken to move the file, recorded in seconds, were

22.95, 6.21, 21.04, 4.11, 12.28, 13.39, 43.64, 26.01, 20.42, 27.59, 15.48 .

(a) Compute the sample mean, sample median and sample standard deviation of these data.

(b) Assuming these data are a random sample from a Normal distribution, compute a 90% confidence interval for the mean communication time, using all the data.

(c) Suppose it is claimed (on the basis of physical characteristics of the network) that the mean communication time for a file of this size is 8 seconds. On the basis of your answer to (b), how would you respond to this claim?

PLEASE TURN OVER

2. The random variable T has an exponential distribution with probability density function

$$f(t) \;=\; \lambda e^{-\lambda t}\,, \qquad t > 0\,,$$

where $\lambda > 0$.

(i) Find the cumulative distribution function of T.

(ii) Show that $P(T > t + s \,|\, T > s) \;=\; e^{-\lambda t}$, where $s > 0$ and $t > 0$. What property does this demonstrate for the exponential distribution?

(iii) Show that the $E(T) \;=\; \dfrac{1}{\lambda}$.

(iv) Suppose we have a random sample of size n from an exponential distribution. Show (and verify) that the maximum likelihood estimator of λ is

$$\hat{\lambda} \;=\; \frac{n}{\sum_{i=1}^{n} x_i}\,.$$

END OF PAPER