UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 2004

BEng Honours Degree in Computing Part II
MEng Honours Degrees in Computing Part II
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the Associateship of the City and Guilds of London Institute

PAPER C233

COMPUTATIONAL TECHNIQUES

Thursday 29 April 2004, 10:00
Duration: 120 minutes

Answer THREE questions

Paper contains 4 questions
Calculators required

1a Prove that for any vector $u \in \mathbb{R}^n$ we have: $\|u\|_\infty \leq \|u\|_2 \leq \|u\|_1$.

b Determine the eigenvalues and eigenvectors of $A = \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix}$. Hence, obtain the eigenvalues and eigenvectors of A^n and A^{-n} for any positive integer n.

c Given the matrix $\mathbf{A}$ below, determine its ℓ_1, ℓ_2 and ℓ_∞ norm:

$$\mathbf{A} = \begin{bmatrix} 4 & 2 & 5 \\ 0 & 3 & 6 \end{bmatrix}.$$

(The three parts carry, respectively, 25%, 25% and 50% of the marks).

2a Show that for vectors $u \in \mathbb{R}^m, v \in \mathbb{R}^n$, the rank of the outer product uv^T is one if and only if $u \neq 0$ and $v \neq 0$

b Consider the following two sets of equations:

$$\begin{array}{rcrcrcr} 4x_1 & - & x_2 & + & 4x_3 & = & 6 \\ -x_1 & + & 9x_2 & - & 2x_3 & = & 15 \\ 4x_1 & - & x_2 & + & 4x_3 & = & 6 \end{array} \tag{1}$$

and

$$\begin{array}{rcrcrcr} 4x_1 & - & 2x_2 & + & 2x_3 & = & 6 \\ -2x_1 & + & 10x_2 & - & x_3 & = & 15 \\ 2x_1 & - & x_2 & + & 2x_3 & = & 6 \end{array} \tag{2}$$

One of them can be solved by Cholesky factorisation. Identify it, explain your choice and apply Cholesky factorisation to solve it.

c For

$$\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \\ 0 & -1 \end{bmatrix},$$

find

(a) the null space of $\mathbf{A}$,
(b) the null space of $\mathbf{A}^T$,
(c) the range of $\mathbf{A}$,
(d) the range of $\mathbf{A}^T$.
(e) Check that null($\mathbf{A}^T$) is orthogonal to range($\mathbf{A}$), and that null($\mathbf{A}$) is orthogonal to range($\mathbf{A}^T$).
(f) For $\mathbf{x} = [1, 1, 1]^T$, find the two vectors $\mathbf{x}_R \in$ range($\mathbf{A}$) and $\mathbf{x}_N \in$ null($\mathbf{A}^T$) which satisfy $\mathbf{x} = \mathbf{x}_R + \mathbf{x}_N$. Check that $\mathbf{x}_R$ and $\mathbf{x}_N$ are orthogonal!

(The three parts carry, respectively, 20%, 30% and 50% of the marks).

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3a Determine the limit of the following sequence:

$$a_n = \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\cdots\left(1 - \frac{1}{n^2}\right),$$

where $n = 2, 3, \ldots$.

Is this an increasing or decreasing sequence?

Hint: Try to investigate a general intermediate term in a_n in relationship with its neighbours.

b Determine the condition number of the following upper triangular matrix measured in the ℓ_∞ norm.

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 2 \\ & 2 & 0 \\ & & 1 \end{bmatrix}$$

c Let an iterative solution of a system of linear equations be defined by the following formula:

$$\mathbf{x}^{(k+1)} = \mathbf{G}\mathbf{x}^{(k)} + \mathbf{c}, \quad k = 1, \ldots \tag{3}$$

where

$$\mathbf{G} = \begin{bmatrix} 0.08 & 0 & 0.05 \\ 0.04 & 0.1 & 0.02 \\ 0 & 0 & 0.08 \end{bmatrix}$$

($\mathbf{c}$ is unimportant for this problem). Determine the rate of convergence for the iterative method defined by (3). Explain your work.

(The three parts carry, respectively 20%, 40% and 40% of the marks).

4a Find a vector $\mathbf{v}$ that is conjugate to $\mathbf{u} = [0, 2, 1]^T$ with respect to

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & -1 \\ 2 & -1 & 1 \\ 0 & 2 & -1 \end{bmatrix}.$$

Explain your work.

b Find a local minimum or maximum for

$$f(x, y) = -x^2 - 2xy - 3y + \frac{1}{3}y^3.$$

Explain your work.

c You are given two systems of linear equations, $\mathbf{Fx} = \mathbf{f}$, and $\mathbf{Gy} = \mathbf{g}$, with the following matrices:

$$\mathbf{F} = \begin{bmatrix} 7 & -1 & 0 & 0 \\ -1 & 5 & -2 & 3 \\ 1 & 1 & 9 & -1 \\ 1 & 2 & -2 & 5 \end{bmatrix} \qquad \mathbf{G} = \begin{bmatrix} 4 & 0 & 3 & 0 \\ 1 & 3 & 0 & -1 \\ -3 & 4 & 9 & 1 \\ 1 & -3 & -2 & 7 \end{bmatrix}$$

and $\mathbf{f}$ and $\mathbf{g}$ are some vectors in $\mathbb{R}^4$.

Assuming that an iterative solution algorithm is to be used, can you guarantee that the Jacobi and the Gauss-Seidel methods will converge with these matrices? What is the relevance of $\mathbf{f}$ and $\mathbf{g}$ in the proof of convergence? Justify your answer.

(The three parts carry, respectively 30%, 50% and 20% of the marks).

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