

## The Answers 2008

Q1 ANSWER: [all computed example]

$$(a) V = V_R \frac{SRR_L / (SR + R_L)}{(10 - S)R + SRR_L / (SR + R_L)} = \frac{V_R S \rho}{10S - S^2 + 10\rho} \quad [3]$$

$$(b) G = \frac{dV}{dS} = \frac{d}{dS} \frac{\rho V_R S}{10S - S^2 + 10\rho} = V_R \frac{S^2 \rho + 10\rho^2}{(10S - S^2 + 10\rho)^2}$$

$$\text{The mean gain is } \bar{G} = \frac{V(9)}{9} = \frac{\rho}{9 + 10\rho} V_R \quad [3]$$

(c) It is always monotonic since the gain is always positive. [2]

$$(d) V = V_R \frac{S\rho}{10S - S^2 + 10\rho} \approx V_R \frac{S}{10} \left( 1 - \frac{10S - S^2}{10\rho} \right) \quad [2]$$

$$(e) \text{ Absolute non-linearity: } \max \delta_A \approx \max \frac{10S - S^2}{10\rho} = \frac{2.5}{\rho} \quad [4]$$

$$(f) \text{ differential non-linearity: } \max \delta_A \approx \max \frac{\frac{d}{dS} \frac{10S^2 - S^3}{100\rho}}{\frac{1}{10}} = \max \frac{20S - 3S^2}{10\rho}$$

$$\text{the maximum of the last expression is at } S = 20/6 \approx 3, \delta_A \approx \frac{60 - 27}{10\rho} \approx 3\rho \quad [4]$$

$$(g) \text{ We require } \frac{10S - S^2}{10\rho} < .01 \quad \forall S \in [0, 9] \Rightarrow 2.5 / \rho < 0.01 \Rightarrow \rho > 250 \quad [2]$$

Q2 ANSWER [bookwork, examples and an extension]

a) Define, very briefly, each of the following quantities for a sensor:

i. Sensitivity: is the derivative of the output with respect to the input:  $S = \frac{\partial y}{\partial x}$  [1]

ii. Threshold is the minimum detectable input [1]

iii. Zero Offset is the input which results into a zero output [1]

iv. Absolute non-linearity:

Express the response as:  $y = Ax + g(x)$ . Then  $\delta_A = \max \frac{g(x)}{Ax}$  [1]

v. Differential Non-linearity

Express the response as:  $y = Ax + g(x)$ . Then  $\delta_D = \frac{1}{A} \max \frac{dg(x)}{dx}$  [1]

b) A sensor is monotonic if the gain does not change sign over the sensor's input range. A Phase detector is not monotonic, since its response is periodic in the input.

No phase detector is monotonic because of the periodicity of the input. A D/A converter IS monotonic.

[6]

c) A linear sensor's response function  $y = f(x)$  does not depend on past input. As a result the output is the convolution of the Fourier transform of the input and the fourier transform of the impulse response of the sensor.

A non-linear sensor's response function depends on its past input. As a result the output is a multiple integral over past inputs.

[2]

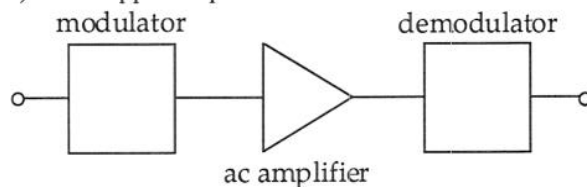
d) A sensor's response  $y$  is often dependant on other parameters  $z_i$  beside the intended stimulus

x. The partial derivatives  $\frac{\partial y}{\partial z_i}$  are called the cross sensitivities. By applying a signal on one of

the  $z_i$ 's the measurement can be AM modulated. Subsequent averaging (Low pass filtering) can increase the SNR of the measurement at the expense of a long observation time.

[3]

e) A chopper amplifier consists of a AM modulator-AM demodulator pair:



The modulator needs to be preceded by an anti-aliasing filter. This diagram is functionally identical to that of a communication link, suggesting that any type of modulation (FM, PM, QAM,...) can in principle be used for chopper amplifiers.

[4]

Q3. ANSWERS [computed example]

- a) Johnson Noise  $V_{j1}^2 = 4kTRB = 320 \times 10^{-15} V^2$ . Spectrally white, but we are told there is also pink noise (uncorrelated to this source) of magnitude  $V_{p1}^2 = 4kTR \frac{f_0}{f}$

The total contribution of the source  $V_{n1}$  is  $V_{n1}^2 = 4kTR \left( 1 + \frac{f_0}{f} \right)$  since these two sources are uncorrelated. Over the entire frequency band we need to integrate  $V_{n1}$  to get

$$V_{n1\_total}^2 = 4kTR \left( B + f_0 \ln(f_{up} / f_{low}) \right) \quad [5]$$

- b)  $V_{n2} = g_m R_L V_{n1} = g_m R_L \sqrt{4kTR \left( 1 + \frac{f_0}{f} \right)}$ .  $V_{n2\_total}^2 = 4kTR g_m^2 R_L^2 \left( B + \ln(f_{up} / f_{low}) \right)$  [3]

- c)  $V_{n3}^2 = 2eI_{DC} R^2$ , spectrally white, so that  $V_{n3\_total}^2 = 2eI_{DC} R^2 B$  [3]

- d)  $V_{n\_total}^2 = (V_{n1} + V_{n2})^2 + V_{n3}^2 = 4kTR \left( B + f_0 \ln(1000) \right) (1 + g_m R_L)^2 + 2eI_{DC} R^2 B$

The numbers given are  $4kTR = 1.6 \times 10^{-17}$ ,  $B + f_0 \ln 1000 = 20000 + 1000 \times 6.9 = 26900$

$$(1 + g_m R_L)^2 = 9$$

$$2eI_{DC} R^2 B = 6.4 \times 10^{-13} \text{ so that}$$

$$V_{n\_total}^2 = 3.87 \times 10^{-12} + 6.4 \times 10^{-13} = 4.51 \times 10^{-12} \Rightarrow V_{nRMS} = 2.12 \mu V \quad [5]$$

$$e) \quad F = 20 \log \frac{SNR_{in}}{SNR_{out}} = 20 \log \frac{\frac{S}{N}}{\frac{GS}{GN + N_A}} = 20 \log \frac{GN + N_A}{GN} = 20 \log \left( 1 + \frac{N_A}{GN} \right)$$

The source noise voltage is  $V_{NS}^2 = 4kTR_s B = 192 \times 10^{-15} V^2$

$$\text{And the noise figure is } F = 20 \log \left( 1 + \frac{4.51 \times 10^{-12}}{100 \times 192 \times 10^{-15}} \right) = 1.83 dB \quad [4]$$

Q4 ANSWER [Mostly bookwork, presented as computed example]

a) Gains:

Phase detector:  $K_d$ .

Filter  $F(s)$ .

VCO:  $K_o / s$ .

Multiplier:  $N$ .

$$B(s) = \frac{\varphi_{out}}{\varphi_{in}} = \frac{K_d K_o F(s)}{s + K_d K_o F(s) N}$$

The steady state response is :

$$B(0) = \frac{K_d K_o F(s)}{K_d K_o F(s) N} = \frac{1}{N}$$

This circuit divides the frequency of the input signal by a factor of  $N$ .

[4]

b) The loop gain for this circuit is  $G_L = K_d K_o F(s) N / s$ . If  $N \gg 1$  and  $F(s) = \frac{1}{1 + s\tau}$ ,

$$G_L = K_d K_o F(s) N / s = \frac{1}{1 + s\tau} K_d K_o N / s. \text{ If } N \rightarrow \infty, \text{ we can estimate } G_L \text{ at high frequencies as}$$

$G_L = K_d K_o N / s^2 \tau$  So that the phase margin is zero. Any delay in the loop will make this circuit unstable.

[4]

c) A device that emits one or two very narrow impulses in every period of an input signal. Consequently, its output spectrum contains power in high order harmonics ( $>100$ ) of the input signal. Used as high order frequency multiplier. If a comb generator is used in fig. 4 then the input can be a very high frequency and the phase detector can be a diode.

[4]

d) The circuit just described is called the transfer oscillator. It is a PLL with a multiplier in the feedback path. It functions as a frequency divider, and is used to generate sub-harmonics of extremely high frequencies. for precise measurements.

[4]

e) A transfer oscillator can be used to divide an input frequency by a high number (eg 100). A couple such sub-harmonic generation stages can bring the signal in a range where direct counters can measure it to any required precision.

[4]

#### Q5 ANSWER

a) [BOOKWORK] Linearity, offset, missing codes, monotonicity

[4]

b) [Computed example]

Input RMS power is  $P = 0.5 \text{ mW}$ . since  $SQNR = 10 \log(6 A^2 / q^2) = 6$ , for 1 bit system, the quantisation noise power is  $P_{NQ} = P_{sig} / SQNR = 0.5 \text{ mW} / 6 = 84 \mu\text{W}$  RMS. The power spectral density is  $\frac{dP_{QN}}{df} = \frac{P_{QN}}{2f_s} = \frac{84 \mu\text{W}}{10^7 \text{ s}^{-1}} = 8.33 \text{ pJ}$ . This is spectrally white.

c) [Computed example]

The thermal noise power over a bandwidth of  $B = 2f_s$  is simply  $P_J = kT = 4 \times 10^{-21} \text{ J}$  so the ratio is  $2 \times 10^9$ . This is a voltage ratio of  $44 \times 10^3$ , or 15.45 bits.

[4]

d) [Computed example]

By simple oversampling plus averaging we gain 0.5 bits/bit. An oversampling ratio of  $OSR = 2^{24} = 16.78 \times 10^6$  is required. This allows a maximum of 0.3 measurements / second.

[4]

e) [Computed example]

With a 1<sup>st</sup> order  $\Delta\Sigma$  modulator, we get 1.5 bits / bit of oversampling. So we need  $OSR = 2^8 = 256$  for a maximum sampling rate of  $f_s = 5 \times 10^6 / 256 = 19500$

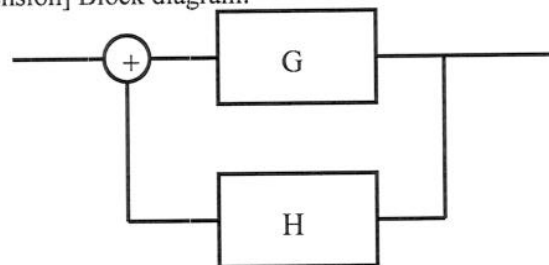
[4]

#### Q6 ANSWER

a) [bookwork] Frequency and phase can be defined for periodic signals. The period of a periodic signal is the smallest  $T$  such that  $f(t+T) = f(t)$ ,  $\forall t$ . The phase of a periodic signal is the fractional part of time normalised to the period. So we can write that for times between:  $nT \leq t \leq (n+1)T \Rightarrow t = (n + \phi / 2\pi)T$ . Phase is ambiguous and as such not a good measurement of time. The (fundamental) frequency is the inverse of the period.

[4]

b) [bookwork + extension] Block diagram:



Necessary condition: satisfy the Barkhausen criteria:  $G(s)H(s) = 1$ , or

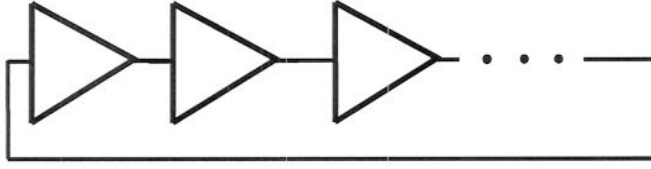
$$|G(s)H(s)| = 1, \arg(G(s)H(s)) = 1.$$

Sufficient condition:  $\exists f : |G(s)H(s)| > 1, \text{Re}(G(s)H(s)) > 0$ .

Example oscillator: Phase delay oscillator,  $G$  is an inverting amp,  $H$  is a number of 1<sup>st</sup> order low pass sections.

[4]

c) [new theory]



- i. Gain magnitude must exceed unity to start up.
- ii. With an odd number of inverters the condition of oscillation is:  $(2n+1)\tau = \frac{T}{2}$
- iii. With an even number of inverters the oscillation condition is satisfied at DC: no oscillation
- iv.  $(2n+1)\tau = \frac{T}{2} \Rightarrow (2n+1)\omega_0\tau = \pi$  . . The odd harmonics of  $f$  satisfy:  

$$\omega_k = (2k+1)\omega_0 \Rightarrow (2n+1)\omega_k\tau = (2n+1)(2k+1)\omega_0\tau = (2k+1)\pi$$

Since phase is understood mod  $\pi$ , the odd harmonics are supported (and as a result the oscillator supports square and triangle wave modes)

[6]

f) [Computed example]

The uncertainty in a counting experiment where an unknown signal at  $f_x$  is divided by N and used to gate a counter running at  $f_{ref}$ , is:

$$N = \text{int}\left(\frac{\Delta T}{T}\right) = \text{int}\left(\frac{Df_x}{f_{ref}}\right) \Rightarrow \frac{N}{D}f_{ref} < f_x < \frac{N+1}{D}f_{ref}$$

If we exchange the roles of  $f_{ref}$  and  $f_x$  we can similarly write

$$\frac{M}{D}f_x < f_{ref} < \frac{M+1}{D}f_x$$

$$\left. \begin{aligned} \frac{N}{D}f_{ref} < f_x < \frac{N+1}{D}f_{ref} \\ \frac{M}{D}f_x < f_{ref} < \frac{M+1}{D}f_x \Rightarrow \frac{D}{M+1}f_{ref} < f_x < \frac{D}{M}f_{ref} \end{aligned} \right\} \Rightarrow$$

$$\max\left(\frac{N}{D}, \frac{D}{M+1}\right) < \frac{f_x}{f_{ref}} < \min\left(\frac{N+1}{D}, \frac{D}{M}\right)$$

[6]