

Paper Number(s): **E4.22**  
**C1.2**

IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE  
UNIVERSITY OF LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING  
EXAMINATIONS 2002

MSc and EEE PART IV: M.Eng. and ACGI

**LINEAR OPTIMAL CONTROL**

Monday, 22 April 10:00 am

There are SIX questions on this paper.

Answer FOUR questions.

Time allowed: 3:00 hours

**Corrected Copy**

**Examiners responsible:**

First Marker(s): Astolfi, A.

Second Marker(s): Weiss, G.

Special instructions for invigilators:

None

Information for candidates:

System:  $\dot{x}(t) = Ax(t) + Bu(t), x(0) = x_0.$

Quadratic cost function:  $J(x_0, u) = \int_0^\infty [x(t)'Qx(t) + u(t)'Ru(t)] dt,$   
 $Q = Q' \geq 0, R = R' > 0.$

Riccati equation:  $A'P + PA + Q - PBR^{-1}B'P = 0.$

Optimal control law:  $u(t) = -R^{-1}B'Px(t) = -Kx(t).$

Minimum cost:  $x_0'Px_0.$

Return difference inequality for scalar  $u$ :  $|1 + K(j\omega I - A)^{-1}B| \geq 1,$

Minimum principle:  $\dot{x} = f(x, u), u \in \mathcal{U}$

$$J(x_0, u) = \int_0^{t_f} L(x(t), u(t))dt,$$

$$H(x, u, \lambda_0, \lambda) = \lambda_0 L(x, u) + \lambda^T f(x, u),$$

$$\dot{\lambda}^* = -\frac{\partial H}{\partial x} \Big|_{(x^*, u^*, \lambda_0^*, \lambda^*)}^T,$$

$$H(x^*, \omega, \lambda_0^*, \lambda^*) \geq H(x^*, u^*, \lambda_0^*, \lambda^*), \quad \forall \omega \in \mathcal{U},$$

$$H(x^*, u^*, \lambda_0^*, \lambda^*) = k.$$

1. Consider the linear electric network in Figure 1, with  $R_1 > 0$ ,  $R_2 > 0$ ,  $C > 0$  and  $L > 0$ . Denote by  $u$  be the driving voltage, by  $x_1$  the current through the inductor  $L$ , by  $x_2$  the voltage across the capacitor  $C$ , and by  $y$  the current through the voltage source.

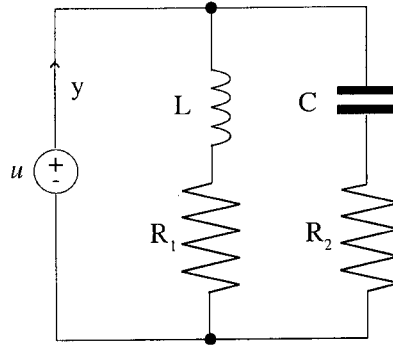


Figure 1.

- (a) Using Kirchhoff's laws, or otherwise, express the dynamics of the circuit in the standard state-space form, regarding  $u$  as the input and  $y$  as the output. [6]
- (b) Study the controllability and the observability of the dynamical system determined in part (a). [6]
- (c) Compute the transfer function from the input  $u$  to the output  $y$ . [4]
- (d) Compute values of  $R_1$ ,  $R_2$ ,  $C$  and  $L$  such that in the transfer function computed in part (c) there is a pole-zero cancellation. Interpret your answer in the light of your answer to part (b). [4]

2. A linear system is described by the differential equations

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= \alpha x_3 + u_1 \\ \dot{x}_3 &= \beta x_1 + \gamma x_2 + u_2\end{aligned}$$

where  $u_1 \in \mathbb{R}$  and  $u_2 \in \mathbb{R}$  are control inputs and  $\alpha$ ,  $\beta$  and  $\gamma$  are constant parameters.

- (a) Suppose that only  $u_1$  is used for control (i.e.  $u_2 = 0$ ). Study the controllability and stabilizability properties of the system as a function of  $\alpha$ ,  $\beta$  and  $\gamma$ . [4]
- (b) Suppose that both  $u_1$  and  $u_2$  can be used for control. Show that the system is controllable for any  $\alpha$ ,  $\beta$  and  $\gamma$ . [7]
- (c) Set  $\alpha = 1$ ,  $\beta = 0$  and  $\gamma = -1$ . Let  $u_1 = k_1 x_1 + k_3 x_3$  and  $u_2 = f_1 x_1 + f_3 x_3$ . Find values of  $k_1$ ,  $k_3$ ,  $f_1$  and  $f_3$  such that the closed-loop system has all three eigenvalues at  $-\lambda^*$ , with  $\lambda^* > 0$ . Show that such values of  $k_1$ ,  $k_3$ ,  $f_1$  and  $f_3$  are not unique. Compute the values of  $k_1$ ,  $k_3$  and  $f_3$ , which together with  $f_1 = 0$ , assign all the eigenvalues of the closed-loop system at  $-\lambda^*$ . [9]

3. Consider the system

$$\dot{x} = \begin{bmatrix} 1 & -2 \\ 3 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u$$

with initial state  $x_0$ , with the quadratic cost to be minimised

$$J(x_0, u) = \int_0^\infty (q_{11}x_1^2 + q_{22}x_2^2 + ru^2) dt,$$

with  $x = [x_1, x_2]'$ ,  $q_{11} > 0$ ,  $q_{22} > 0$  and  $r > 0$ .

- (a) Verify that the conditions for the existence and uniqueness of an optimal feedback control law are met. [4]
- (b) Write the ARE associated with the above optimal control problem. Find  $q_{11}$ ,  $q_{22}$  and  $p_{22}$  such that the ARE is satisfied by a matrix of the form

$$P = \begin{bmatrix} 2 & 0 \\ 0 & p_{22} \end{bmatrix}.$$

Verify that the resulting  $Q$  and  $P$  are positive definite for all  $r \in (0, 1)$ . [8]

- (c) For  $q_{11}$  and  $q_{22}$  as determined in part (b) compute the optimal feedback law. Show that the eigenvalues of the optimal closed-loop system go to  $-3$  and to  $-\infty$  as  $r \rightarrow 0$ . (Hint: Re-write the optimal closed-loop system in the new state variable  $z = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} x$ .) [8]

4. Consider the system

$$\dot{x} = x^3 + u,$$

the initial state  $x(0) = x_0 \neq 0$ , the final state  $x(T) = 0$  and the cost (to be minimized)

$$J(x_0, u) = \int_0^T \frac{1}{2} u^2 dt.$$

- (a) Write the necessary conditions of optimality in the case of normal extremals. (Note that there is no constraint on  $u$ .) [6]
- (b) Compute the optimal control as a function of the costate  $\lambda$ . Then, using the condition that the Hamiltonian is zero along any extremal, compute the optimal control as a function of  $x$ . [6]
- (c) Integrate the state equation and find the optimal control law as a function of  $t$ . Finally, compute the time  $T$  at which the condition  $x(T)$  is met, and compute the optimal cost for any  $x_0$ . [8]

5. Consider the system

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx\end{aligned}$$

with  $A + A^T = 0$  and  $C = B^T$ , and the quadratic cost (to be minimised)

$$J(x_0, u) = \int_0^\infty \left( \alpha y^T(t)y(t) + u^T(t)u(t) \right) dt,$$

with  $\alpha > 0$ .

(a) Show that the system is controllable if and only if it is observable. [4]

(b) Write the ARE associated with the above optimal control problem. [2]

(c) Find the positive definite solution  $P$  of the ARE derived in part (b) and compute the optimal state feedback control law and the optimal closed-loop system.  
(Hint: consider a diagonal  $P$ .) [6]

(d) Let

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad B = C^T = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Using the results in part (c), compute the optimal closed-loop system as a function of  $\alpha$ . Study the behaviour of the eigenvalues of the optimal closed-loop system when  $\alpha$  goes to infinity. Compute the transfer function  $G(s)$  of the system  $(A, B, C)$  and verify that, as  $\alpha \rightarrow \infty$ , one eigenvalue of the closed-loop system approaches the zero of  $G(s)$ . [8]

6. Consider the system

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= u\end{aligned}$$

and the problem of finding a bounded control law  $|u(t)| \leq 1$  that drives the state of the system from  $(x_1(0), x_2(0)) = (-1/2, 1)$  to  $(x_1(t_f), x_2(t_f)) = (0, 0)$  in minimum time.

- (a) Write the necessary conditions of optimality for normal extremals. [4]
- (b) Write the optimal control as a function of the optimal costate  $\lambda^*(t)$ . [2]
- (c) Assume that the optimal control has constant sign in the interval  $t \in [0, t_f]$ , integrate the state equations and compute the optimal control  $u^*(t)$  as a function of  $t$ . Thus evaluate  $t_f$ . [6]
- (d) Integrate the costate equations and verify that there exists an initial condition  $\lambda(0)$  for the costate such that the Hamiltonian is equal to zero along the optimal solution computed in part (c). [4]
- (e) Show that the control that drives the state of the system from  $(x_1(0), x_2(0)) = (-1/2, 1)$  to  $(x_1(t_f), x_2(t_f)) = (0, 0)$  in minimum time is  $-u^*(t)$ , where  $u^*(t)$  is the control computed in part (c). [4]  
(Hint: Define a new state variable  $z = -x_1$ .)



# Linear Optimal Control - Model answers 2002

## Question 1

- (a) Let  $i_1$  and  $i_2$  be the currents through  $R_1$  and  $R_2$ , respectively. Then  $y = i = i_1 + i_2$ .

Moreover,

$$i_1 = x_1 = \frac{u - L\dot{x}_1}{R_1} \quad i_2 = C\dot{x}_2 = \frac{u - x_2}{R_2}.$$

Hence,

$$A = \begin{bmatrix} -\frac{R_1}{L} & 0 \\ 0 & -\frac{1}{R_2 C} \end{bmatrix} \quad B = \begin{bmatrix} \frac{1}{L} \\ \frac{1}{R_2 C} \end{bmatrix} \quad C = \begin{bmatrix} 1 & -\frac{1}{R_2} \end{bmatrix} \quad D = \frac{1}{R_2}.$$

- (b) The controllability matrix is

$$\mathcal{C} = \begin{bmatrix} \frac{1}{L} & -\frac{R_1}{L^2} \\ \frac{1}{R_2 C} & -\frac{1}{R_2^2 C^2} \end{bmatrix}$$

and it is full rank if  $R_1 R_2 C \neq L$ . The observability matrix is

$$\mathcal{O} = \begin{bmatrix} 1 & -\frac{1}{R_2} \\ -\frac{R_1}{L} & \frac{1}{R_2^2 C} \end{bmatrix}$$

and it is full rank if  $R_1 R_2 C \neq L$ .

- (c) The transfer function is

$$W(s) = C(sI - A)^{-1}B + D = \frac{s^2 LC + sC(R_1 + R_2) + 1}{(Ls + R_1)(CR_2 s + 1)}.$$

- (d) There is a pole-zero cancellation if

$$(s^2 LC + sC(R_1 + R_2) + 1)_{s=-R_1/L} = 0$$

or

$$(s^2 LC + sC(R_1 + R_2) + 1)_{s=-1/(CR_2)} = 0$$

and this is the case if  $R_1 R_2 C = L$ . This is expected, in fact, from part (b) we know that if  $R_1 R_2 C = L$  then the system is neither controllable nor observable.

## Question 2

- (a) The controllability matrix is

$$\mathcal{C} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & \alpha\gamma \\ 0 & \gamma & \beta \end{bmatrix}$$

and it is full rank if  $\beta \neq 0$ . Hence the system is controllable if and only if  $\beta \neq 0$ . To check stabilizability, consider the matrix

$$T = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & \gamma & 1 \end{bmatrix}.$$

Then, with  $\beta = 0$ ,

$$\tilde{A} = T^{-1}AT = \begin{bmatrix} 0 & \alpha\gamma & \alpha \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \tilde{B} = T^{-1}B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

showing that the system is not stabilizable (the un-controllable mode has eigenvalue equal to 0).

- (b) The controllability matrix is now composed of six columns ( $\star$  denotes an element which does not need to be computed)

$$\mathcal{C} = \begin{bmatrix} 0 & 0 & 1 & 0 & \star & \star \\ 1 & 0 & 0 & \alpha & \star & \star \\ 0 & 1 & \gamma & 0 & \star & \star \end{bmatrix}.$$

This matrix has always rank three, as the first three columns are linearly independent, for any  $\alpha$ ,  $\beta$  and  $\gamma$ . The system is controllable for any  $\alpha$ ,  $\beta$  and  $\gamma$ .

- (c) The closed-loop system is

$$\dot{x} = A_{cl}x = \begin{bmatrix} 0 & 1 & 0 \\ k_1 & 0 & k_3 + 1 \\ f_1 & -1 & f_3 \end{bmatrix} x$$

and the characteristic polynomial of  $A_{cl}$  is

$$\lambda^3 - f_3\lambda^2 + (k_3 + 1 - k_1)\lambda + (k_1f_3 - f_1 - f_1k_3).$$

This has to be equal to  $(\lambda + \lambda^*)^3$ . Hence, equating coefficients with equal power yields

$$f_3 = -3\lambda^* \quad k_3 + 1 - k_1 = 3(\lambda^*)^2 \quad k_1f_3 - f_1 - f_1k_3 = (\lambda^*)^3$$

and this system of equations admits infinitely many solutions  $k_1$ ,  $k_3$ ,  $f_1$  and  $f_3$ . However, if  $f_1 = 0$  then the only solution is

$$k_1 = -\frac{(\lambda^*)^2}{3} \quad k_3 = 3(\lambda^*)^2 - 1 - \frac{(\lambda^*)^2}{3} \quad f_3 = -3\lambda^*.$$

### Question 3

- (a) The pair  $(A, B)$  is controllable, the pair  $(A, Q^{1/2})$ , with  $Q = \text{diag}(q_{11}, q_{22})$ , is observable,  $Q > 0$  and  $R = r > 0$ .

(b)

$$A'P + PA - PBR^{-1}B'P + Q = \begin{bmatrix} 4 - \frac{4}{r} + q_{11} & 3p_{22} - 4 + 2\frac{p_{22}}{r} \\ 3p_{22} - 4 + 2\frac{p_{22}}{r} & q_{22} - 2p_{22} - \frac{p_{22}^2}{r} \end{bmatrix} = 0.$$

The above equation has the solution

$$q_{11} = 4\frac{1-r}{r} \quad q_{22} = \frac{8r(3r+4)}{9r^2+12r+4} \quad p_{22} = \frac{4r}{3r+2}.$$

Note that  $P > 0$  for any  $r > 0$  and  $Q > 0$  for any  $r \in (0, 1)$ .

- (c) The optimal feedback is  $u = -kx$  with

$$k = \frac{1}{r}[1 \quad -1]\text{diag}(2, \frac{4r}{3r+2})$$

and the optimal closed-loop system is

$$\dot{x} = A_{cl}x = \begin{bmatrix} 1 - \frac{2}{r} & -2 + \frac{4}{3r+2} \\ 3 + \frac{2}{r} & -1 - \frac{4}{3r+2} \end{bmatrix} x.$$

Note that in the variable  $z$  one has

$$A_{cl} = \begin{bmatrix} \frac{9r^2 - 4r - 4}{r(3r+2)} & -\frac{6r}{3r+2} \\ 7 & -3 \end{bmatrix}$$

and for  $r \rightarrow 0$

$$A_{cl} \approx \begin{bmatrix} -2/r & 0 \\ 7 & -3 \end{bmatrix}.$$

This shows that one eigenvalue goes to  $-3$  and the other to  $-\infty$ .

## Question 4

(a) Let

$$H = \frac{1}{2}u^2 + \lambda(x^3 + u).$$

The necessary conditions of optimality, for normal extremals, are

$$\dot{x} = x^3 + u \quad \dot{\lambda} = -3\lambda x^2 \quad 0 = \frac{\partial H}{\partial u} = u + \lambda \quad 0 = H.$$

(b) The optimal control as a function of  $\lambda$  is  $u = -\lambda$ . Replacing this into  $H = 0$  yields

$$\frac{1}{2}\lambda^2 + \lambda x^3 - \lambda^2 = 0,$$

hence, either

$$\lambda = 0 \quad u = 0$$

or

$$\lambda = 2x^3 \quad u = -2x^3.$$

Note that the solution  $u = 0$  is not admissible, in fact, if  $u = 0$  then  $\dot{x} = x^3$ , and if  $x(0) = x_0 \neq 0$  the condition  $x(T) = 0$  cannot be met for any  $T$ . The only admissible solution is therefore  $u = -2x^3$ .

(c) The optimal closed-loop system is

$$\dot{x} = -x^3,$$

and integrating this differential equation with  $x(0) = 0$  yields

$$x(t) = \frac{x_0}{\sqrt{2x_0^2 t + 1}}.$$

We conclude that the condition  $x(T) = 0$  is met for  $T = +\infty$ . The optimal control, as a function of  $t$  and  $x_0$ , is

$$u(t) = -\frac{2x_0^3}{(\sqrt{2x_0^2 t + 1})^3},$$

and

$$J = \int_0^\infty \frac{1}{2}u^2(t)dt = \frac{1}{2}x_0^4.$$

## Question 5

- (a) Controllability of  $(A, B)$  is equivalent to

$$\text{rank} \begin{bmatrix} sI - A & B \end{bmatrix} = n$$

for all  $s \in \mathcal{C}$ . However,

$$\text{rank} \begin{bmatrix} sI - A & B \end{bmatrix} = \text{rank} \begin{bmatrix} sI - A' \\ B' \end{bmatrix} = \text{rank} \begin{bmatrix} sI + A \\ C \end{bmatrix},$$

hence  $(A, C)$  is observable if and only if  $(A, B)$  is controllable.

- (b)

$$0 = A'P + PA - PBB'P + \alpha C'C = -AP + PA - PBB'P + \alpha BB'.$$

- (c) Let  $P = \lambda I$ , with  $\lambda > 0$ . Then the ARE becomes

$$0 = -\lambda A + \lambda A - \lambda^2 BB' + \alpha BB',$$

hence  $P = \sqrt{\alpha}I$  is a solution of the ARE and it is positive definite. The optimal state feedback control law is

$$u = -Kx = -R^{-1}B'Px = -\sqrt{\alpha}B'x$$

and the optimal closed-loop system is

$$\dot{x} = A_{cl}x = (A - \sqrt{\alpha}BB')x.$$

- (d) For the specified  $A$  and  $B$ , the optimal closed-loop system is

$$\dot{x} = A_{cl}x = \begin{bmatrix} -\sqrt{\alpha} & -1 - \sqrt{\alpha} \\ 1 - \sqrt{\alpha} & -\sqrt{\alpha} \end{bmatrix} x$$

and the characteristic polynomial of the matrix  $A_{cl}$  is

$$s^2 + 2s\sqrt{\alpha} + 1.$$

The roots of the characteristic polynomial are

$$-\sqrt{\alpha} + \sqrt{\alpha - 1} \quad -\sqrt{\alpha} - \sqrt{\alpha - 1}.$$

As  $\alpha \rightarrow +\infty$  the first tends to 0 and the second to  $-\infty$ . Finally, the transfer function of the system is

$$W(s) = C(sI - A)^{-1}B = \frac{2s}{s^2 + 1}.$$

This has a zero for  $s = 0$ , and this is where one of the eigenvalues of the optimal closed-loop system tends as  $\alpha \rightarrow +\infty$ .

## Question 6

(a) Let

$$H = 1 + \lambda_1 x_2 + \lambda_2 u.$$

The necessary conditions of optimality, for normal extremals, are

$$\dot{x}_1 = x_2 \quad \dot{x}_2 = u \quad \dot{\lambda}_1 = 0 \quad \dot{\lambda}_2 = -\lambda_1 \quad 0 = 1 + \lambda_1 x_2 + \lambda_2 u$$

$$1 + \lambda_1 x_2 + \lambda_2 u \leq 1 + \lambda_1 x_2 + \lambda_2 \omega, \quad \forall \omega \in [-1, 1].$$

(b) From the last condition the optimal control is

$$u = -\text{sign}(\lambda_2),$$

hence  $u = \pm 1$ .

(c) If  $u = c$ , for some constant  $c$ , then

$$x_1(t) = x_1(0) + x_2(0)t + ct^2/2 \quad x_2(t) = x_2(0) + ct.$$

If  $u = +1$ ,  $x_1(0) = -1/2$  and  $x_2(0) = 1$ , then

$$x_1(t) = -1/2 + t + t^2/2 \quad x_2(t) = 1 + t$$

and the condition  $x_2(t_f) = 0$  cannot be met for any  $t_f$ . If  $u = -1$ ,  $x_1(0) = -1/2$  and  $x_2(0) = 1$

$$x_1(t) = -1/2 + t - t^2/2 \quad x_2(t) = 1 - t$$

and the condition  $x_2(t_f) = x_1(t_f) = 0$  holds with  $t_f = 1$ .

(d) Integration of the costate equations yields

$$\lambda_1(t) = \lambda_1(0) \quad \lambda_2(t) = \lambda_2(0) - \lambda_1(0)t.$$

Substituting into the Hamiltonian yields

$$H = 1 + \lambda_1(0)(1 - t) - \lambda_2(0) + \lambda_1(0)t = 1 + \lambda_1(0) - \lambda_2(0).$$

Hence,  $H = 0$  for all  $t$  (along the optimal solutions) if

$$1 + \lambda_1(0) - \lambda_2(0) = 0.$$

(e) Let  $z = -x$  and note that

$$\dot{z}_1 = z_2 \quad \dot{z}_2 = -u$$

and  $z(0) = -x(0)$ . So the optimization problem in the  $z$  variables is solved by  $-u = -1$  for  $t \in [0, 1]$ .