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| | <p>EXAMINATION QUESTIONS/SOLUTIONS 2007-08</p> <p>EEZ - MATHS PAPER 3 -</p> <p>SOLUTIONS 2008</p> | <p>Course</p> <p>①</p> <p>CORE</p> |
| <p>Question</p> <p>8</p> | | <p>Marks & seen/unseen</p> |
| <p>Parts</p> | <p>a) Take the principal branch $-\pi < \arg z \leq \pi$. <i>which is well-defined everywhere except at $z=0$.</i></p> <p>b) Yes, since $\frac{dw}{dz} = 1/z$ <i>mapping is conformal except at $z=0$.</i></p> <p>c) If $z = re^{i\theta}$ then $w = u + iv = \log r + i\theta \Rightarrow \begin{matrix} u = \log r \\ v = \theta \end{matrix} \Rightarrow$</p> <p>if If $\theta = 0$, the positive real axis is mapped into the u-axis.</p> <p>d) A straight line $\theta = \alpha$, $-\pi < \theta \leq \pi$ in the z-plane is mapped into the straight line $v = \alpha$ in the w-plane.</p> <p>e) A circle ^{circle} with radius $r = a > 0$ & center at the origin is mapped onto the line segment $\log a + i\theta$ $-\pi < \theta \leq \pi$ in the w-plane.</p> | <p>②</p> <p>④</p> <p>①</p> <p>②</p> <p>②</p> <p>④</p> |
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Checker's initials

RLJ

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| | EXAMINATION QUESTIONS/SOLUTIONS 2007-08 | Course EE2 (2) Pap 3 |
| Question 2 | | Marks & seen/unseen |
| Parts | $f(z) = \frac{z}{(z-1)^2(z-i)^2}$ <p>i) Res at the double pole at $z=1$ is</p> $\lim_{z \rightarrow 1} \frac{d}{dz} \{ (z-1)^2 f(z) \} = \lim_{z \rightarrow 1} \frac{d}{dz} \left[\frac{z}{(z-i)^2} \right]$ $= \lim_{z \rightarrow 1} \left\{ \frac{(z-i)^2 - 2z(z-i)}{(z-i)^4} \right\} = - \lim_{z \rightarrow 1} \frac{(z+i)}{(z-i)^3}$ $= - \frac{(1+i)}{(1-i)^3} = -(1+i)(1-3i-3+i)^{-1} = \frac{1}{2}$ <p>ii) Res at the double pole at $z=i$ is</p> $\lim_{z \rightarrow i} \frac{d}{dz} \{ (z-i)^2 f(z) \} = \lim_{z \rightarrow i} \frac{d}{dz} \left[\frac{z}{(z-1)^2} \right]$ $= \lim_{z \rightarrow i} \frac{(z-1)^2 - 2z(z-1)}{(z-1)^4} = - \lim_{z \rightarrow i} \frac{z+1}{(z-1)^3}$ $= - \frac{i+1}{(i-1)^3} = \frac{i+1}{(1-i)^3} = -1/2 \text{ from (i)}$ <p>iii)</p> $\oint_C f(z) dz = 2\pi i \left\{ \frac{1}{2} - \frac{1}{2} \right\} \text{ by R.T.}$ $= 0$ <p>as both poles at $z=1$ & $z=i$ lie within C.</p> | <p>7</p> <p>7</p> <p>6</p> |
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EXAMINATION QUESTIONS/SOLUTIONS 2007-08

Course

3

EE2

pag 3

Question

3

Marks &

seen/unseen

Parts

i)

$$F(z) = \frac{e^{iz}}{z(z^2+4)}$$

Pole at $z=0$

Poles at $z = \pm 2i$

3

ii)

$$\text{Res at } z=0 \text{ is } \lim_{z \rightarrow 0} \left\{ \frac{ze^{iz}}{z(z^2+4)} \right\} = \frac{1}{4}$$

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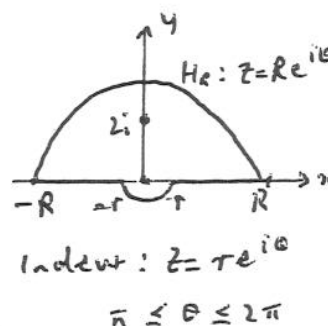
$$\text{" " } z=2i \text{ is } \lim_{z \rightarrow 2i} \left\{ \frac{(z-2i)e^{iz}}{z(z^2+4)} \right\} = -\frac{e^{-2}}{8}$$

3

iii)

$$\oint_C \frac{e^{iz}}{z(z^2+4)} = 2\pi i \left\{ \frac{1}{4} - \frac{e^{-2}}{8} \right\}$$

Now split up the contour into



2

$$\oint_C = \left(\int_{-R}^{-r} + \int_r^R \right) \frac{e^{ix} dx}{x(x^2+4)}$$

$$+ \int_{H_R} \frac{e^{iz} dz}{z(z^2+4)} + \int_{h_r} \frac{e^{iz} dz}{z(z^2+4)}$$

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By Jordan's Lemma $\lim_{R \rightarrow \infty} \int_{H_R} f(z) e^{iz} dz = 0$

because a) only singularities are poles

b) $n=1 > 0$

c) $|f(z)| \rightarrow 0$ fast enough as $R \rightarrow \infty$.

2

Thus we take the 2 limits $R \rightarrow \infty, r \rightarrow 0$

$$\therefore 2\pi i \left\{ \frac{1}{4} - \frac{e^{-2}}{8} \right\} = \int_{-\infty}^{\infty} \frac{e^{ix} dx}{x(x^2+4)} + \lim_{r \rightarrow 0} \int_{h_r} \frac{e^{iz} dz}{z(z^2+4)}$$

$$\text{and } \lim_{r \rightarrow 0} \int_{h_r} \frac{e^{iz} dz}{z(z^2+4)} = \frac{i}{4} \int_{\pi}^{2\pi} d\theta = \frac{\pi i}{4}$$

2

$$\therefore \int_{-\infty}^{\infty} \frac{e^{ix} dx}{x(x^2+4)} = \pi i \left\{ \frac{1}{4} - \frac{e^{-2}}{4} \right\}$$

Now $\int_{-\infty}^{\infty} f(x) e^{ix} dx = 0$ as $f(x)$ is odd, thus

$$\int_{-\infty}^{\infty} \frac{\sin x dx}{x(x^2+4)} = \frac{\pi}{4} \{1 - e^{-2}\}$$

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Setter's initials

JDG

Checker's initials

Agw

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| | EXAMINATION QUESTIONS/SOLUTIONS 2007-08 | Course 5 EE2 Pap 3 |
| Question 5 | | Marks & seen/unseen |
| Parts | <p>i) $\int_{-\infty}^{\infty} f(t) g^*(t) dt$</p> $= \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \bar{f}(\omega) e^{+i\omega t} d\omega \right) \left(\int_{-\infty}^{\infty} \bar{g}(\omega') e^{i\omega' t} d\omega' \right)^* dt$ $= \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \bar{f}(\omega) \bar{g}^*(\omega') e^{i(\omega-\omega')t} dt d\omega' d\omega$ <p>Given that $\int_{-\infty}^{\infty} e^{i(\omega-\omega')t} dt = 2\pi \delta(\omega-\omega')$ we have</p> $\int_{-\infty}^{\infty} f(t) g^*(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \bar{f}(\omega) \bar{g}^*(\omega') \delta(\omega-\omega') d\omega' d\omega$ $= \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{f}(\omega) \bar{g}^*(\omega) d\omega \quad \square$ <p>ii) With $f(t) = e^{- t }$ $t = \begin{cases} t & t > 0 \\ -t & t < 0 \end{cases}$</p> $\bar{f}(\omega) = \int_{-\infty}^0 e^{+(1-i\omega)t} dt + \int_0^{\infty} e^{-(1+i\omega)t} dt$ $= \frac{1}{1-i\omega} + \frac{1}{1+i\omega} = \frac{2}{1+\omega^2}$ $\bar{g}(\omega) = \frac{1}{t} \int_{-\infty}^{\infty} e^{-i\omega t} (e^{i\omega_0 t} + e^{-i\omega_0 t}) dt$ $= \pi \{ \delta(\omega - \omega_0) + \delta(\omega + \omega_0) \}$ $\therefore \int_{-\infty}^{\infty} e^{- t } \cos \omega_0 t dt = \frac{\pi}{2\pi} \int_{-\infty}^{\infty} \frac{2[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]}{1+\omega^2} d\omega$ $= \frac{2}{1+\omega_0^2}$ | <p>Seen.</p> <p>4</p> <p>4</p> <p>Unseen.</p> <p>4</p> <p>4</p> <p>4</p> |
| | <p>Setter's initials JDE</p> <p>Checker's initials Agy</p> | Page number 1 |

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| | EXAMINATION QUESTIONS/SOLUTIONS 2007-08 | Course 6 EE2 Pap3 |
| Question 6 | | Marks & seen/unseen |
| Parts | <p> $\mathcal{L}(x) = s\bar{x}(s) - x(0)$ Take $\mathcal{L}(\ddot{x}) = s^2\bar{x}(s) - sx(0) - \dot{x}(0)$ $\mathcal{L}(\ddot{x}) = \int_0^\infty e^{-st}\ddot{x} dt = \int_0^\infty e^{-st}d(\dot{x})$ $= [\dot{x}e^{-st}]_0^\infty + s\mathcal{L}(\dot{x})$ $= s^3\bar{x}(s) - s^2x(0) - s\dot{x}(0) - \ddot{x}(0)$ </p> <p>Now $x(0) = \dot{x}(0) = \ddot{x}(0) = 0$; thus LT. the ODE</p> $(s^3 + 3s^2 + 3s + 1)\bar{x}(s) = \bar{f}(s)$ $\therefore \bar{x}(s) = \frac{\bar{f}(s)}{(s+1)^3}$ <p>Now we know i) $\mathcal{L}(t^n) = n!/s^{n+1}$ (take) ii) Shift Thm $\mathcal{L}(e^{at}f(t)) = \bar{f}(s-a)$ (take)</p> <p>$\therefore \mathcal{L}[\frac{1}{2}e^{-t}t^2] = \frac{1}{(s+1)^3}$ choosing $n=2$ in (i) and $a=-1$ in (ii)</p> <p>If $\bar{x}(s) = \bar{f}(s)\bar{g}(s)$ where $f(t)$ given the Lap. conv. thm says $g(t) = \frac{1}{2}e^{-t}t^2$</p> $x(t) = \int_0^t f(t-u)g(u)du = f * g$ <p>Thus $x(t) = \frac{1}{2}\int_0^t f(t-u)e^{-u}u^2du.$</p> | <p>5</p> <p>5</p> <p>2</p> <p>2</p> <p>2</p> <p>4</p> |
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EXAMINATION QUESTIONS/SOLUTIONS 2007-08

Course **7**

EE2

Pap3

Question

7

Marks & seen/unseen

Parts

Choose $\underline{v} = iP + jQ \Rightarrow \text{curl } \underline{v} = \underline{k}(Q_x - P_y)$

$\therefore \underline{v} \cdot d\underline{r} = Pdx + Qdy$
 $\underline{k} \cdot \text{curl } \underline{v} = Q_x - P_y \} \therefore \text{G.T.} \rightarrow 2D, \text{ Stokes' Thm.}$

When $\underline{v} = \frac{1}{2}(y^2 \underline{i} + x^2 \underline{j}) : \text{curl } \underline{v} = \underline{k}(x - y)$

$\therefore \iint_R (x - y) dx dy = \frac{1}{2} \oint_C (y^2 dx + x^2 dy)$

$C_1: y=0 \quad 0 \leq x \leq 2$

$\int_{C_1} = 0$

$C_2: x=2 \quad \int_{C_2} = 2 \int_0^{1/2} dy = 1$

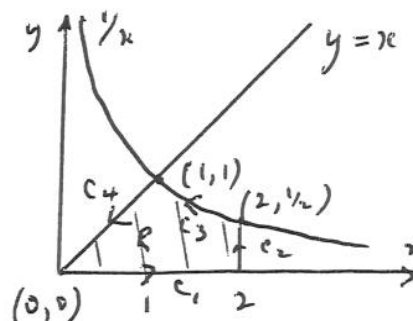
$C_3: y=1/x \quad dy = -dx/x^2$

$\int_{C_3} = \frac{1}{2} \int_2^1 \left(\frac{dx}{x^2} - dx \right) = -\frac{1}{2} \left[x + \frac{1}{x} \right]_2^1 = \frac{1}{2} \left[x + \frac{1}{x} \right]_1^2$
 $= \frac{1}{2} \left(2 + \frac{1}{2} \right) - \frac{1}{2} (1 + 1) = 1/4$

$C_4: y=x : dy = dx \quad \int_{C_4} = \frac{1}{2} \int_1^0 x^2 dx + x^2 dx$

$\therefore \int_{C_4} = -\int_0^1 x^2 dx = -1/3$

Total = $0 + 1 + 1/4 - 1/3 = 5/4 - 1/3 = 11/12$.



6

2

2 (pic)

4x2 (for each \int_{C_i})

2

Via the double integral is also acceptable:

$$\begin{aligned} \iint_R (x-y) dx dy &= \int_0^1 \left\{ \int_0^x (x-y) dy \right\} dx + \int_1^2 \left\{ \int_0^{1/x} (x-y) dy \right\} dx \\ &= \int_0^1 \left\{ x^2 - \frac{1}{2} x^2 \right\} dx + \int_1^2 \left\{ 1 - \frac{1}{2} \frac{1}{x} \right\} dx \\ &= \frac{1}{6} + 1 + \frac{1}{2} \left(\frac{1}{2} - 1 \right) = \frac{1}{6} + \frac{3}{4} = \frac{22}{24} = \frac{11}{12} \end{aligned}$$

Setter's initials

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Course

EGE (3)

8

2nd Year.

Solution

Question

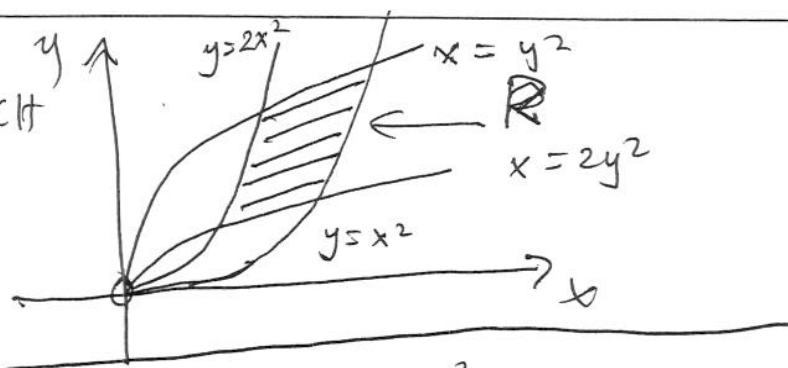
C3

Marks &
seen/unseen

Parts

(1)

SKETCH



When $y = x^2$ $u = \frac{x^2}{y} = 1$

when $y = 2x^2$ $u = \frac{x^2}{y} = \frac{1}{2}$

when $x = y^2$ $v = \frac{y^2}{x} = 1$

when $x = 2y^2$ $v = \frac{y^2}{x} = \frac{1}{2}$

$$x^3 = \left(\frac{x^2}{y}\right)^2 \left(\frac{y^2}{x}\right) = u^2 v \Rightarrow x = (u^2 v)^{1/3}$$

$$y^3 = \left(\frac{y^2}{x}\right)^2 \left(\frac{x^2}{y}\right) = v^2 u \Rightarrow y = (u v^2)^{1/3}$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{2}{3} u^{-1/3} v^{1/3} & \frac{1}{3} u^{2/3} v^{-2/3} \\ \frac{1}{3} u^{-2/3} v^{2/3} & \frac{2}{3} u^{1/3} v^{-1/3} \end{vmatrix}$$

$$= \frac{4}{9} - \frac{1}{9} = \frac{1}{3}$$

Or else use $J = \frac{1}{J'}$

where $J' = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = 3$

R

Setter's initials

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Checker's initials

JB

Page number

EXAMINATION QUESTIONS/SOLUTIONS 2007-08

 Course
 EE 1(3)

8

Solution

Question

C3

 Marks &
 seen/unseen

Parts

Further $x, y = uv$

So Int = $\int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 v e^{uv} J du dv$

$$= \frac{1}{3} \int_{\frac{1}{2}}^1 v \left[\frac{1}{v} e^{uv} \right]_{u=\frac{1}{2}}^1 dv$$

$$= \frac{1}{3} \int_{\frac{1}{2}}^1 (e^v - e^{\frac{1}{2}v}) dv$$

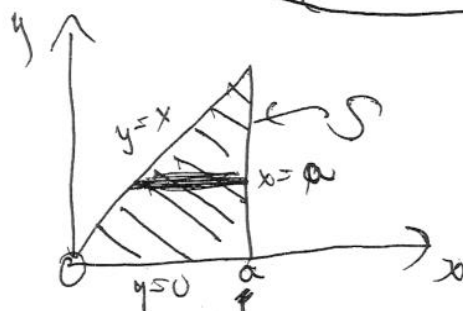
$$= \frac{1}{3} \left[e^v - 2e^{\frac{1}{2}v} \right]_{\frac{1}{2}}^1 = \frac{(e - 2e^{\frac{1}{2}} - e^{\frac{1}{2}} + 2e^{\frac{1}{4}})}{3}$$

$$= (e - 3e^{\frac{1}{2}} + 2e^{\frac{1}{4}})/3.$$

4

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(ii)



$$I_{02} = \int_{y=0}^{y=a} \int_{x=y}^{x=a} (x^2 + y^2) dx dy$$

$$= \int_{y=0}^a \left[\frac{x^3}{3} + xy^2 \right]_{x=y}^{x=a} dy$$

$$= \int_0^a \left(\frac{a^3}{3} + ay^2 - \frac{y^3}{3} - y^3 \right) dy$$

$$= a^3 \left(\frac{1}{3} + \frac{1}{3} - \frac{1}{12} - \frac{1}{4} \right) = \frac{1}{3} a^4$$

omit

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Setter's initials

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| | EXAMINATION QUESTIONS/SOLUTIONS 2007-08 | Course EEII(3) 9 |
| Question C4 | <u>Solution</u> | Marks & seen/unseen |
| Parts | $\text{grad } \varphi = \left(\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}, \frac{\partial \varphi}{\partial z} \right)$ $\text{curl grad } \varphi = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} & \frac{\partial \varphi}{\partial z} \end{vmatrix}$ $= \underline{i} (\partial_y \partial_z \varphi - \partial_z \partial_y \varphi) - \underline{j} (\partial_x \partial_z \varphi - \partial_z \partial_x \varphi) + \underline{k} (\partial_x \partial_y \varphi - \partial_y \partial_x \varphi) = 0.$ <hr/> $\text{curl } \underline{E} = \underline{i} (\partial_y E_3 - \partial_z E_2) - \underline{j} (\partial_x E_3 - \partial_z E_1) + \underline{k} (\partial_x E_2 - \partial_y E_1)$ $\text{div curl } \underline{E} = \partial_x (\partial_y E_3 - \partial_z E_2) + \partial_y (\partial_x E_3 - \partial_z E_1) + \partial_z (\partial_x E_2 - \partial_y E_1)$ <p>cancellation of terms in pairs gives result.</p> <hr/> <p>From $\underline{E} = \underline{A} + \text{grad } \varphi$ it follows that $\text{curl } \underline{E} = \text{curl } \underline{A} + \text{curl grad } \varphi$ Last term is zero so result follows follows.</p> <hr/> $\text{curl } \underline{E} = \underline{i}(0) - \underline{j}(0) + \underline{k}(a e^{ax} \sin y + e^{ax} \sin y)$ $\text{curl } \underline{A} = \underline{i}(1-1) - \underline{j}(0) + \underline{k}(+2 e^{ax} \sin y)$ <p>These are equal if $a = 1$.</p> | <p>4</p> <p>4</p> <p>3</p> <p>4</p> |
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| Question C4 | | Marks & seen/unseen |
| Parts | <p>If $a = 1$ $\text{grad } \phi = \underline{\tilde{E}} - \underline{\tilde{A}}$ so</p> $\frac{\partial \phi}{\partial x} = -e^x \cos y - x$ $\frac{\partial \phi}{\partial y} = e^x \sin y - z$ $\frac{\partial \phi}{\partial z} = z^2 - y$ <p>Thus $\phi(x, y, z) = -e^x \cos y - \frac{x^2}{2} + f(y, z)$</p> $\phi(x, y, z) = -e^x \cos y - zy + g(x, z)$ $\phi(x, y, z) = \frac{z^3}{3} - yz + h(x, y)$ <p>By comparison (or otherwise)</p> $\phi(x, y, z) = -e^x \cos y + \frac{z^3}{3} - yz - \frac{x^2}{2} + C.$ | <p>5</p> <p>20</p> |
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Course

EEII(3)

Sol

10

Question

C5

Marks &
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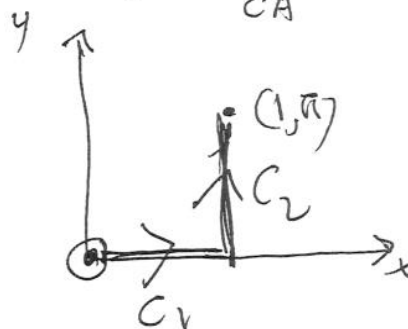
Parts

$$\text{Int} = e(1.0 - 1.0 - (-1)) - 1(0.1 - 0.0 - 1)$$

$$= 1$$

$$\text{Second integral} = \text{First integral} + \int_{CA}^B (x dx + \sin \frac{y}{2} dy)$$

CA path shown



$$\therefore \text{Second int} = 1 + \int_{C_1} + \int_{C_2}$$

On C_1 $y=0$, $dy=0$ x ranges from 0 to 1

On C_2 $x=1$, $dx=0$ y ranges from 0 to π

$$\text{Thus Second int} = 1 + \int_0^1 x dx + \int_0^\pi \sin \frac{y}{2} dy$$

$$= 1 + \frac{1}{2} + \left[-2 \cos \frac{y}{2} \right]_0^\pi$$

$$= 1 + \frac{1}{2} + 2 = 3\frac{1}{2}$$

7

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Setter's initials

RL

Checker's initials

PB

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| | EXAMINATION SOLUTIONS 2007-08 | Course EE2(3) 11 |
| Question 11 | | Marks & seen/unseen |
| Parts | <p>(i) $1 = \int_0^\pi k \sin(x) dx = -k \cos(x) _{x=0}^\pi = 2k$. Hence, $k = \frac{1}{2}$.</p> <p>(ii) $P(1 \leq X \leq 4) = \int_1^\pi \frac{\sin(x)}{2} dx = -\frac{1}{2} \cos(x) _{x=1}^\pi = \frac{1}{2}(\cos(1) + 1) \approx 0.770$</p> <p>(iii) $P(X > 2 X > 1) = \frac{P(X > 2)}{P(X > 1)} = \frac{\int_2^\pi \frac{\sin(x)}{2} dx}{\frac{1}{2}(\cos(1) + 1)} = \frac{\frac{1}{2}(\cos(2) + 1)}{\frac{1}{2}(\cos(1) + 1)} \approx 0.380$</p> <p>(iv) $E(X) = \int_0^\pi x \sin(x)/2 dx = -x \cos(x)/2 _{x=0}^\pi - \int_0^\pi -\cos(x)/2 dx$ $= \frac{\pi}{2} + \sin(x)/2 _{x=0}^\pi = \frac{\pi}{2} \approx 1.57$ $E(X^2) = \int_0^\pi x^2 \sin(x)/2 dx = \frac{1}{2}[(2 - x^2) \cos(x) + 2x \sin(x)] _{x=0}^\pi$ $= \frac{1}{2}[(2 - \pi^2)(-1) - 2] = \pi^2/2 - 2 \approx 2.93$ $\text{Var}(X) = E(X^2) - (E(X))^2 = \pi^2/2 - 2 - \pi^2/4 = \frac{1}{4}\pi^2 - 2 \approx 0.467$</p> | <p>3</p> <p>4</p> <p>4</p> <p>4</p> <p>3</p> <p>2</p> <p style="text-align: right;">Seen similar</p> |
| | <p>Setter's initials AG</p> <p>Checker's initials MJC</p> | Page number |

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| Question 12 | | Marks & seen/unseen |
| Parts | <p>(i) A time series $\{e_t\}$ is called white noise if $E(e_t) = 0$ for all t, $\text{cov}(e_t, e_s) = 0$ for all $t \neq s$ $\text{Var}(e_t)$ does not depend on t.</p> <p>(ii) $\gamma(t, t) = \text{Var}(y_t) = 0.3^2 \text{Var}(e_t) + 0.5^2 \text{Var}(e_{t-1}) + 0.2^2 \text{Var}(e_{t-2})$ $= 0.3^2 + 0.5^2 + 0.2^2 = 0.38$ $\gamma(t, t+1) = \text{cov}(y_t, y_{t+1}) = 0.3 \cdot 0.5 \text{cov}(e_t, e_t) + 0.5 \cdot 0.2 \text{cov}(e_{t-1}, e_{t-1})$ $= 0.3 \cdot 0.5 + 0.5 \cdot 0.2 = 0.25$ $\gamma(t, t+2) = 0.3 \cdot 0.2 = 0.06$ $\gamma(t, t+k) = 0$ for $k = 3, 4, \dots$</p> <p>(iii) The covariance $\gamma(t, t+s)$ is independent of t by (ii). $\mu_t = E(y_t) = 0.3 E(e_t) + 0.5 E(e_{t-1}) + 0.2 E(e_{t-2}) = 0$ Since both μ_t and $\gamma(t, t+s)$ does not depend on t, the time series is stationary.</p> <p>(iv) $\gamma_1 = 0.25/0.38 \approx 0.658$ $\gamma_2 = 0.06/0.38 \approx 0.157$ $\gamma_k = 0$ for $k = 3, 4, \dots$</p> <p>(v) The spectrum is given by</p> $f(\omega) = \gamma_0 + 2 \sum_{k=1}^{\infty} \gamma_k \cos(k\omega) = 0.38 + 0.5 \cos(\omega) + 0.12 \cos(2\omega)$ | <p>3</p> <p>8</p> <p>3</p> <p>3</p> <p>3</p> <p>Seen Similar</p> |
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| | <p>EXAMINATION QUESTIONS/SOLUTIONS 2007-08</p> <p>EE2 - MATHS PAPER 4 - SOLUTIONS 2008</p> | <p>Course ① EE2 Pap 4</p> |
| <p>Question 1</p> | | <p>Marks & seen/unseen</p> |
| <p>Parts</p> | <p>i) $A \underline{e}_i = \lambda_i \underline{e}_i$ ^① Transpose & c.c. $\underline{e}_i^{*T} A^T = \lambda_i^* \underline{e}_i^{*T}$ RH multiply by \underline{e}_i $\therefore \underline{e}_i^{*T} A^T \underline{e}_i = \lambda_i^* \underline{e}_i^{*T} \underline{e}_i$ — ② Now take ① & LH multiply by \underline{e}_i^{*T} $\underline{e}_i^{*T} A \underline{e}_i = \lambda_i \underline{e}_i^{*T} \underline{e}_i$ — ③ Compare ② & ③ using $A^T = A : \Rightarrow \lambda_i = \lambda_i^*$ Hence λ_i are real.</p> <p>ii) Take ① & LH multiply by \underline{e}_j^T $\underline{e}_j^T A \underline{e}_i = \lambda_i \underline{e}_j^T \underline{e}_i$ — ④ & consider $A \underline{e}_j = \lambda_j \underline{e}_j \Rightarrow \underline{e}_j^T A = \lambda_j \underline{e}_j^T$ RH multiply this by \underline{e}_i $\underline{e}_j^T A \underline{e}_i = \lambda_j \underline{e}_j^T \underline{e}_i$ — ⑤ Subtract ⑤ from ④ : $\underline{e}_j^T \underline{e}_i (\lambda_j - \lambda_i) = 0$ We know $\lambda_i \neq \lambda_j \Rightarrow \underline{e}_j^T \underline{e}_i = 0$ — ⑥</p> <p>iii) $P = \{ \underline{e}_1 \ \underline{e}_2 \ \dots \ \underline{e}_n \}$ a row of col-vecs $P^T = \begin{pmatrix} \underline{e}_1^T \\ \underline{e}_2^T \\ \vdots \\ \underline{e}_n^T \end{pmatrix}$ col of row-vecs. $\therefore P^T P = \{ \underline{e}_i^T \underline{e}_j \} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} = I$ from ⑥</p> | <p>4 4 4</p> |
| | <p>Setter's initials JDC</p> | <p>Checker's initials Agw</p> |
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Parts

$$A = \begin{pmatrix} 1 & \sqrt{2} & 0 \\ \sqrt{2} & 1 & \sqrt{2} \\ 0 & \sqrt{2} & 1 \end{pmatrix} \Rightarrow \begin{vmatrix} 1-\lambda & \sqrt{2} & 0 \\ \sqrt{2} & 1-\lambda & \sqrt{2} \\ 0 & \sqrt{2} & 1-\lambda \end{vmatrix} = 0$$

$$\therefore (1-\lambda) [(1-\lambda)^2 - 2] = \sqrt{2} [\sqrt{2}(1-\lambda)]$$

$$\therefore \lambda = 1 \text{ and } (1-\lambda)^2 = 4 \quad \therefore \lambda^2 - 2\lambda - 3 = 0 \quad \lambda = -1$$

$$\lambda = 3$$

$$\lambda_1 = 3 \quad \lambda_2 = 1 \quad \lambda_3 = -1$$

$$\lambda_1 = 3 \quad \begin{pmatrix} -2 & \sqrt{2} & 0 \\ \sqrt{2} & -2 & \sqrt{2} \\ 0 & \sqrt{2} & -2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \quad \begin{matrix} \sqrt{2}a = b \\ c + a = \sqrt{2}b \\ b = \sqrt{2}c \end{matrix} \quad \underline{e}_1 = \frac{1}{2} \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$$

$$\lambda_2 = 1 \quad \begin{pmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \quad \begin{matrix} b = 0 \\ c = -a \\ b = 0 \end{matrix} \quad \underline{e}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$\lambda_3 = -1 \quad \begin{pmatrix} 2 & \sqrt{2} & 0 \\ \sqrt{2} & 2 & \sqrt{2} \\ 0 & \sqrt{2} & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \quad \begin{matrix} b = -\sqrt{2}a \\ a + c = -\sqrt{2}b \\ b = -\sqrt{2}c \end{matrix} \quad \underline{e}_3 = \frac{1}{2} \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

$$Q = \underline{x}^T A \underline{x} \text{ with } A \text{ as above; write } \underline{x} = P \underline{y}$$

$$\underline{x}^T = \underline{y}^T P^T \Rightarrow Q = \underline{y}^T (P^T A P) \underline{y}$$

$$\left. \begin{matrix} P^T A P = \Lambda \\ \text{if } P = \{ \underline{e}_1, \underline{e}_2, \underline{e}_3 \} \end{matrix} \right\} \begin{matrix} \text{reason } AP = P\Lambda \\ \Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \\ P^T A P = \Lambda \end{matrix}$$

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix} : Q = \underline{y}^T \Lambda \underline{y} = 3y_1^2 + y_2^2 - y_3^2$$

$$\underline{y} = P^T \underline{x} = P^T \underline{x} \quad y_1 = \frac{1}{2} x_1 + \frac{1}{\sqrt{2}} x_2 + \frac{1}{2} x_3$$

$$y_2 = \frac{1}{\sqrt{2}} (x_1 - x_3)$$

$$y_3 = \frac{1}{2} x_1 - \frac{1}{\sqrt{2}} x_2 + \frac{1}{2} x_3$$

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| | EXAMINATION SOLUTIONS 2007-08 | Course EE2(4) 3 |
| Question 3 | | Marks & seen/unseen |
| Parts (i) | <p>Empirical cdf $F_n(x)$</p> <p>(ii) $\bar{x} = \frac{1}{7}(-0.2 + 0.3 + \dots + 0.5) = 1.1/7 \approx 0.157$ Median: 0.2</p> <p>(iii) Let x_1, \dots, x_7 denote the observed values.</p> $s^2 = \frac{1}{7-1} \left(\sum_{k=1}^7 x_k^2 - \frac{1}{7} \left(\sum_{j=1}^7 x_j \right)^2 \right) = \frac{1}{6} \left(0.69 - \frac{1}{7}(1.1)^2 \right)$ $\approx \frac{1}{6}(0.69 - 0.1729) = \frac{0.5171}{6} \approx 0.0862$ <p>so $s = \sqrt{s^2} \approx 0.2936$.</p> <p>(iv) From the Student t table we find $t_0 = t_{7-1,0.1} = 1.94$. The 90% confidence interval for μ is thus</p> $\left(\bar{x} - t_0 \frac{s}{\sqrt{7}}, \bar{x} + t_0 \frac{s}{\sqrt{7}} \right) = \left(0.157 - 1.94 \frac{0.2936}{\sqrt{7}}, 0.157 + 1.94 \frac{0.2936}{\sqrt{7}} \right)$ $\approx (-0.058, 0.372)$ <p>(v) The test statistic is $t = \frac{\bar{x}-0}{s/\sqrt{7}} = \frac{1.1/7}{0.2936/\sqrt{7}} \approx 1.42$ Since we have one-sided hypotheses one should use a one-sided test. Taking into account that the formula sheet only gives two-sided values, the critical value is given by $t_0 = t_{6,0.1} = 1.94$. Since $t < 1.94$ the hypothesis H_0 is not rejected.</p> | <p>3</p> <p>2</p> <p>1</p> <p>3</p> <p>1</p> <p>2</p> <p>2</p> <p>2</p> <p>2</p> <p>2</p> |
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| | EXAMINATION SOLUTIONS 2007-08 | Course EE2(4) 4 |
| Question 4 | | Marks & seen/unseen |
| Parts | | |
| (i) | <p>We know that $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dy dx = 1$. Hence,</p> $ \begin{aligned} 1 &= k \int_0^1 \int_0^{1-y} xy^2 dx dy = k \int_0^1 \frac{1}{2} y^2 (1-y)^2 dy \\ &= k \frac{1}{2} \int_0^1 (y^4 - 2y^3 + y^2) dy = k \frac{1}{2} \left[\frac{1}{5} y^5 - \frac{1}{2} y^4 + \frac{1}{3} y^3 \right]_{y=0}^1 \\ &= k \frac{1}{2} \left(\frac{1}{5} - \frac{2}{4} + \frac{1}{5} \right) = k \frac{1}{60} \end{aligned} $ <p>Hence, $k=60$.</p> | <div style="text-align: right; transform: rotate(90deg);"> Seen similar </div> |
| (ii) | <p>$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy$. Hence, $f_X(x) = 0$ for $x < 0$ or $x > 1$. For $0 \leq x \leq 1$,</p> $f_X(x) = \int_0^{1-x} kxy^2 dy = kx[y^3/3]_{y=0}^{1-x} = 20x(1-x)^3.$ | 4 |
| (iii) | $ \begin{aligned} E[X] &= \int_{-\infty}^{\infty} xf_X(x) dx = 20 \int_0^1 x^2(1-x)^3 dx \\ &= 20 \int_0^1 (-x^5 + 3x^4 - 3x^3 + x^2) dx \\ &= 20[-x^6/6 + 3x^5/5 - 3x^4/4 + x^3/3]_{x=0}^1 \\ &= 20(-1/6 + 3/5 - 3/4 + 1/3) = 1/3 \end{aligned} $ | 3 |
| (iv) | $ \begin{aligned} E(XY) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xyf(x, y) dx dy = 60 \int_0^1 \int_0^{1-y} x^2 y^3 dx dy \\ &= 60 \int_0^1 y^3 (1-y)^3 / 3 dy = 20 \int_0^1 (-y^6 + 3y^5 - 3y^4 + y^3) dy \\ &= 20(-1/7 + 3/6 - 3/5 + 1/4) = 1/7 (\approx 0.143) \end{aligned} $ <p>Hence,</p> $\text{cov}(X, Y) = E(XY) - E(X)E(Y) = \frac{1}{7} - \frac{1}{3} \frac{1}{2} = -\frac{1}{42} \approx -0.024$ | 4 |
| (v) | No, since $\text{cov}(X, Y) \neq 0$. | 2 |
| (vi) | No, since they are correlated. | 2 |
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| | EXAMINATION SOLUTIONS 2007-08 | Course EE2(4) 5 |
| Question 5 | | Marks & seen/unseen |
| Parts | | |
| (i) | <p>For the ith student let Y_i denote the result of the spinner, let $Z_i = 1$ if the student has cheated and $Z_i = 0$ otherwise and let X_i denote the number the student reports.</p> <p>One can assume that Y_i and Z_i are independent. Hence, $P(X_i = 1) = P(Y_i = 1, Z_i = 1) + P(Y_i = 0, Z_i = 0) = \theta p + (1 - \theta)(1 - p) = \theta(2p - 1) + (1 - p).$</p> | <p>→ Unseen</p> <p>3</p> |
| (ii) | $P(X_i = 0) = 1 - P(X_i = 1) = 1 - (\theta(2p - 1) + (1 - p)) = \theta(1 - 2p) + p$ | 1 |
| (iii) | <p>The likelihood function is given by</p> $L(\theta) = P(X_i = 1)^{\sum_{i=1}^n X_i} P(X_i = 0)^{n - \sum_{i=1}^n X_i}$ $= (\theta(2p - 1) + (1 - p))^{\sum_{i=1}^n X_i} (\theta(1 - 2p) + p)^{n - \sum_{i=1}^n X_i}$ <p>Differentiating the loglikelihood leads to the equation</p> $\sum_{i=1}^n X_i \frac{2p - 1}{\hat{\theta}(2p - 1) + (1 - p)} + (n - \sum_{i=1}^n X_i) \frac{1 - 2p}{\hat{\theta}(1 - 2p) + p} = 0$ <p>for the maximum likelihood estimator $\hat{\theta}$. Since $p \neq 1/2$ this is equivalent to</p> $-\sum_{i=1}^n X_i (\hat{\theta}(1 - 2p) + p) + (n - \sum_{i=1}^n X_i) (\hat{\theta}(2p - 1) + (1 - p)) = 0$ <p>Hence,</p> $\hat{\theta} = \frac{p \sum_{i=1}^n X_i - (n - \sum_{i=1}^n X_i)(1 - p)}{-\sum_{i=1}^n X_i(1 - 2p) + (n - \sum_{i=1}^n X_i)(2p - 1)}$ $= \frac{\sum_{i=1}^n X_i - n(1 - p)}{n(2p - 1)} = \frac{\frac{1}{n} \sum_{i=1}^n X_i - (1 - p)}{2p - 1}$ | <p>4</p> <p>4</p> |
| (iv) | $E_{\theta}(\hat{\theta}) = \frac{1}{2p-1} (\frac{1}{n} \sum_{i=1}^n E(X_i) + p - 1) = \frac{1}{2p-1} (\theta(2p - 1) + 1 - p + p - 1)$ $= \theta$ <p>Hence, $\hat{\theta}$ is an unbiased estimator for θ.</p> | <p>4</p> <p>4</p> |
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| | EXAMINATION SOLUTIONS 2007-08 | Course EE2(4) 6 |
| Question 6 | | Marks & seen/unseen |
| Parts | <p>(i) For $0 < x < 1$:</p> $F_{X^5}(x) = P(X^5 \leq x) = P(X \leq x^{1/5}) = F_X(x^{1/5}) = x^{1/5}$ <p>Hence, $f_{X^5}(x) = \frac{d}{dx}x^{1/5} = x^{-4/5}/5$.</p> <p>For $x \leq 0$ or $x \geq 1$ we have $f_{X^5}(x) = 0$.</p> <p>(ii) Since X and Y are independent, $\text{cov}(X, Y) = 0$.</p> <p>Hence, from the formula sheet: $X + Y$ is $N(0 + 5, 1 + 3)$, i.e. $N(5, 4)$.</p> <p>(iii) For $t > 0$,</p> $\begin{aligned} f_{X+Y}(t) &= (f_X * f_Y)(t) \\ &= \int_0^{\min(t,1)} \lambda e^{-\lambda(t-y)} dy \\ &= e^{-\lambda t} \int_0^{\min(t,1)} \lambda e^{\lambda y} dy = e^{-\lambda t} (e^{\lambda \min(t,1)} - 1) \\ &= \begin{cases} e^{-\lambda t} (e^\lambda - 1) & \text{if } 1 \leq t \\ 1 - e^{-\lambda t} & \text{if } 0 < t < 1 \end{cases} \\ f_{X+Y}(t) &= 0 \text{ for } t \leq 0. \end{aligned}$ | <p>4 3 1</p> <p>4</p> <p>8</p> <p>Seen similar</p> |
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