IMPERIAL COLLEGE LONDON

E4.25
CS1.2
ISE4.23

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2009

MSc and EEE/ISE PART IV: MEng and ACGI

Corrected Copy

DESIGN OF LINEAR MULTIVARIABLE CONTROL SYSTEMS

Wednesday, 20 May 10:00 am

Time allowed: 3:00 hours

There are SIX questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s) :	I.M. Jaimoukha
	Second Marker(s) :	E.C. Kerrigan

DESIGN OF LINEAR MULTIVARIABLE CONTROL SYSTEMS

1. a) Let the transfer matrix $G(s)$ have a state space realisation

$$G(s) \stackrel{s}{=} \left[\begin{array}{ccc|cc} 1 & 2 & 0 & 1 & 2 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 4 & 3 & 4 \\ \hline 2 & 3 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 & 0 \end{array}\right].$$

i) Find the uncontrollable and/or unobservable modes and determine whether the realisation is detectable and stabilisable. [4]

ii) Obtain a minimum realisation of $G(s)$. [4]

b) Consider a state–variable model described by the dynamics

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t). \end{aligned}$$

i) Suppose there exists $Q = Q^T \succ 0$ such that

$$A^T Q + QA \prec 0.$$

Prove that A is stable. [6]

ii) Suppose there exist $Q = Q^T \succ 0$ and Y such that

$$A^T Q + QA + YC + C^T Y^T \prec 0.$$

Prove that the pair (A, C) is detectable. [6]

2. a) Define internal stability for the feedback loop shown in Figure 2.1 below and derive necessary and sufficient conditions for which this feedback loop is internally stable. [5]

b) Suppose that the transfer matrix $G(s)$ in the feedback loop in Figure 2.1 is stable. Derive a parameterization of all internally stabilizing controllers $K(s)$ for the feedback loop. [6]

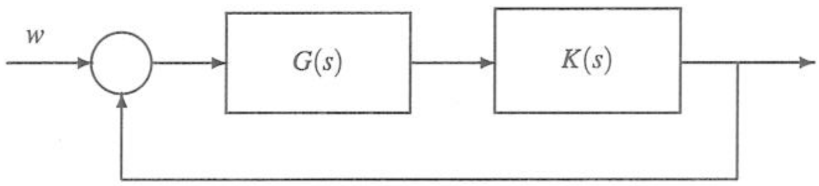


Figure 2.1

c) Consider the feedback loop in Figure 2.2. Suppose that $G(s) := D + C(sI - A)^{-1}B$ is square, stable and minimum–phase and that D is nonsingular. Let $\Delta(s)$ represent a stable uncertainty. Design an internally stabilising compensator $K(s)$ such that

i) The order of $K(s)$ is the same as that of $G(s)$. [3]

ii) The feedback loop in Figure 2.2 is internally stable for all $\Delta(s)$ satisfying

$$\|\Delta\|_\infty < 1.$$

[3]

iii) The DC loop gain satisfies $\bar{\sigma}(K(0)G(0)) = 2$, where $\bar{\sigma}(\cdot)$ denotes the largest singular value. [3]

The compensator $K(s)$ should be given in terms of $G(s)$.

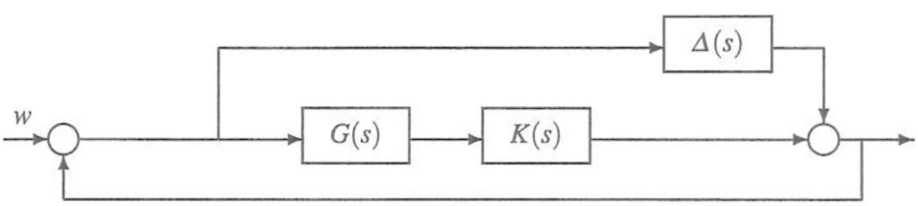


Figure 2.2

3. Figure 3.1 illustrates the implementation of the control law $u(t) = -Kx(t) + r(t)$ which (when $r(t) = 0$) minimises

$$J(x_0, u) = \int_0^\infty \left(x(t)^T C^T C x(t) + u(t)^T u(t) \right) \, dt$$

subject to $\dot{x}(t) = Ax(t) + Bu(t)$, $x(0) = x_0$ where $K = B^T P$ and $P = P^T$ is the unique stabilising solution of the Riccati equation $A^T P + PA - PBB^T P + C^T C = 0$. Assume that the triple (A, B, C) is minimal. Let $F(s) = (sI - A)^{-1} B$, $G(s) = C(sI - A)^{-1} B$ and $L(s) = I + KF(s)$.

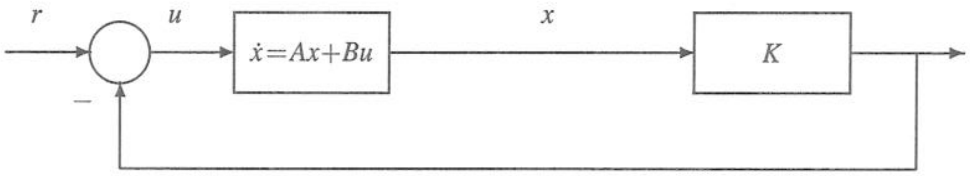


Figure 3.1

a) Let $S(s)$ denote the transfer matrix from r to u in Figure 3.1. By evaluating a return difference equality, or otherwise, prove that $\|S\|_\infty \leq 1$. [6]

b) Suppose that

$$G(s) = \begin{bmatrix} \frac{4}{s+3} & 0 \\ 0 & \frac{3}{s+4} \end{bmatrix}.$$

Derive a minimal state-space realisation $G(s) = C(sI - A)^{-1} B$ and evaluate K for this realisation. [6]

c) Let $G(s)$ and K be as in Part (b). Suppose a stable uncertainty $\Delta(s)$ is introduced as shown in Figure 3.2. Derive the maximal stability radius (using the $\mathscr{H}_\infty$–norm as a measure) for $\Delta(s)$ that can be deduced from Part (a) and the small gain theorem. [8]

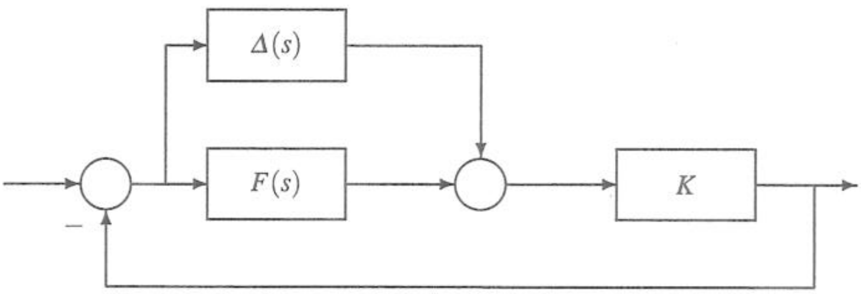


Figure 3.2

4. Consider the feedback configuration in Figure 4. Here, $G(s)$ is a plant model and $K(s)$ is a compensator. The signals $d_1(s)$ and $d_2(s)$ represent disturbance signals. Let

$$d(s) = \begin{bmatrix} d_1(s) \\ d_2(s) \end{bmatrix}.$$

The design specifications are to synthesize a compensator $K(s)$ such that the feedback loop is internally stable and, for all real ω,

- $\|y(j\omega)\| < |w_1(j\omega)^{-1}|\ \|d(j\omega)\|$,
- $\|u(j\omega)\| < |w_2(j\omega)^{-1}|\ \|d(j\omega)\|$,

where $w_1(s)$ and $w_2(s)$ are given filters.

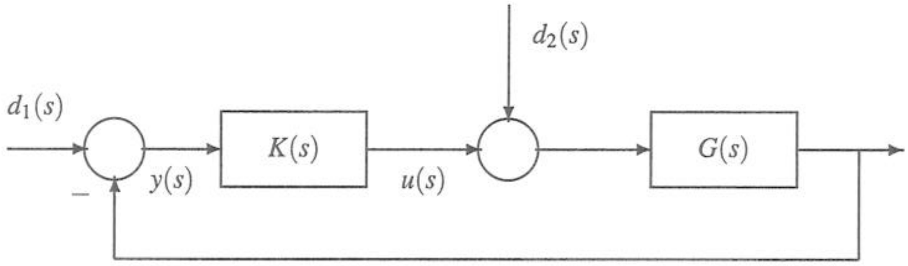


Figure 4

a) Derive $\mathscr{H}_\infty$–norm bounds, in terms of $G(s)$, $K(s)$, $w_1(s)$ and $w_2(s)$ that are sufficient to achieve the design specifications. [6]

b) Define suitable cost signals $z_1(s)$ and $z_2(s)$ and draw a block diagram, of the same form as Figure 4, showing $z_1(s)$ and $z_2(s)$ as well as suitable weighting functions. [6]

c) Hence derive a generalised regulator formulation of the design problem that captures the sufficient conditions. [8]

5. a) Let $G(s) = D + C(sI - A)^{-1}B$ and let $\gamma > 0$ be given.

i) Suppose there exists $P = P^T \succ 0$ such that

$$\begin{bmatrix} A^TP + PA + C^TC & C^TD + PB \\ D^TC + B^TP & D^TD - \gamma^2 I \end{bmatrix} \prec 0. \tag{5.1}$$

Show that A is stable and $\|G\|_\infty < \gamma$. [4]

ii) Using a Schur type argument show that (5.1) is satisfied if and only if

$$\begin{bmatrix} A^TP + PA & PB & C^T \\ B^TP & -\gamma^2 I & D^T \\ C & D & -I \end{bmatrix} \prec 0. \tag{5.2}$$

[4]

iii) By pre– and post–multiplying (5.2) by appropriate matrices show that A is stable and $\|G\|_\infty < \gamma$ if there exists $Q = Q^T \succ 0$ such that

$$\begin{bmatrix} AQ + QA^T & B & QC^T \\ B^T & -\gamma^2 I & D^T \\ CQ & D & -I \end{bmatrix} \prec 0. \tag{5.3}$$

[4]

b) Consider the regulator shown in Figure 5 for which it is assumed that the triple (A, B, C) is minimal and $x(0) = 0$. Let $z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ and let $H(s)$ denote the transfer matrix from w to z. A stabilizing state–feedback gain matrix F is to be designed such that, for $\gamma > 0$, $\|H\|_\infty < \gamma$.

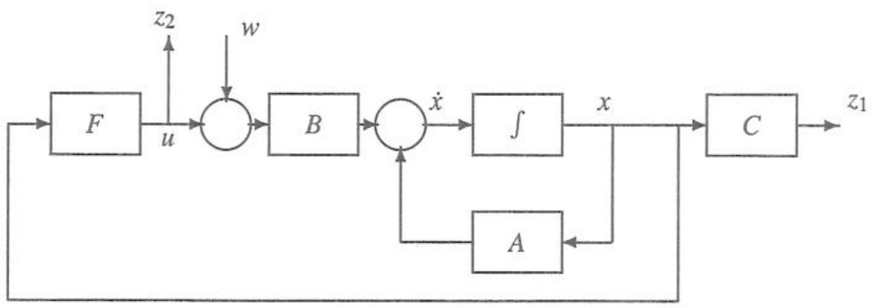


Figure 5

i) Derive a state–space realisation for the closed–loop system $H(s)$ in terms of A, B, C and F. [4]

ii) By using Part (a) above, or otherwise, derive sufficient conditions for the existence of a feasible F in the form of linear matrix inequality conditions. [4]

6. Consider the regulator shown in Figure 6 for which it is assumed that the triple (A, B, C) is minimal and $x(0) = 0$.

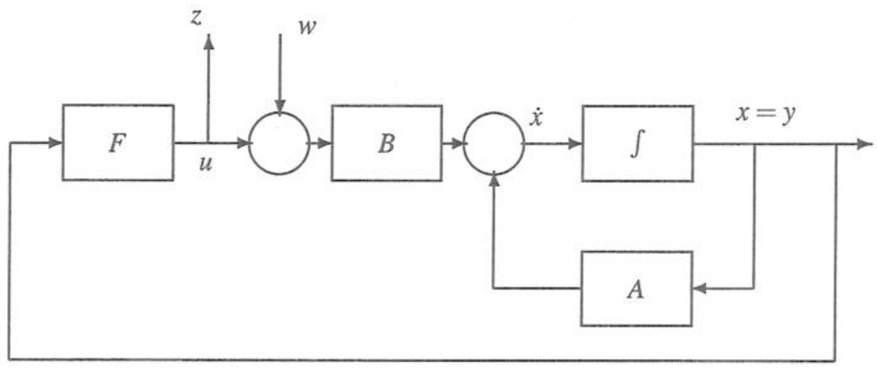


Figure 6

Let $H(s)$ denote the transfer matrix from w to z. A stabilizing state–feedback gain matrix F is to be designed such that, for $\gamma > 0$, $\|H\|_\infty < \gamma$.

a) Write down the generalized regulator system for this design problem. [4]

b) By using the Lyapunov function $V(t) = x(t)^T X x(t)$, where X is to be determined, derive sufficient conditions for the solution of the design problem. Your conditions should be in the form of the existence of a certain solution to an algebraic Riccati equation. It should also include an expression for F and an expression for the worst–case disturbance w. [10]

c) Suppose that A is stable. Show that the optimal value of γ is equal to 0. *(Hint: Look carefully at Figure 6.)* [3]

d) Suppose that A is unstable. Show that the optimal value of γ is greater than 1. *(Hint: Set $\gamma = 1$ and show that the closed–loop A–matrix is unstable.)* [3]

Master - April 09

E4.25
CS1.2
ISE 4.23

SOLUTIONS: DESIGN OF LINEAR MULTIVARIABLE CONTROL SYSTEMS 2009

1. a) i) Since $\begin{bmatrix} A-sI & B \end{bmatrix}$ loses rank for $s=3$, 3 is an uncontrollable mode, and since $\begin{bmatrix} A^T-sI & C^T \end{bmatrix}$ loses rank for $s=4$, 4 is an unobservable mode. Since the uncontrollable mode is unstable, the realisation is not stabilisable and since the unobservable mode is unstable, the realisation is not detectable.

ii) By removing the uncontrollable and unobservable parts we get the minimal realisation

$$G(s) \stackrel{s}{=} \left[\begin{array}{c|cc} 1 & 1 & 2 \\ \hline 2 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right].$$

b) i) Suppose that λ is an eigenvalue of A and let $z \neq 0$ be the corresponding eigenvector. Then $Az = \lambda z$. Pre– and post–multiplying the matrix inequality by z' and z, respectively, we get

$$(\lambda + \bar{\lambda})z'Qz < 0.$$

Since $z \neq 0$ and $Q \succ 0$, this implies that $z'Qz > 0$ so that $\lambda + \bar{\lambda} < 0$ and so A is stable.

ii) The pair (A,C) is detectable if and only if there exists L such that $A+LC$ is stable. That is, the pair (A,C) is detectable if and only if there exist L and $Q = Q^T \succ 0$ such that

$$(A+LC)^T Q + Q(A+LC) \prec 0.$$

Comparing this with the inequality in the question, it follows that the pair (A,C) is detectable by identifying Y with QL.

2. a) Inject a signal $r(s)$ in between $G(s)$ and $K(s)$ and let $u(s)$ be the input to $G(s)$ and $y(s)$ be the input to $K(s)$. The loop is internally stable if and only if the transfer matrix from $\begin{bmatrix} w(s) \\ r(s) \end{bmatrix}$ to $\begin{bmatrix} u(s) \\ y(s) \end{bmatrix}$ is stable (no poles in the closed right half plane). Since

$$\begin{bmatrix} w(s) \\ r(s) \end{bmatrix} = \begin{bmatrix} I & -K(s) \\ -G(s) & I \end{bmatrix} \begin{bmatrix} u(s) \\ y(s) \end{bmatrix} =: T(s) \begin{bmatrix} u(s) \\ y(s) \end{bmatrix}$$

the loop is internally stable if and only if $T(s)^{-1}$ is stable.

b) Since $G(s)$ is stable, we proceed as follows. Note that

$$\begin{bmatrix} I & -K(s) \\ -G(s) & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ -G(s) & I \end{bmatrix} \begin{bmatrix} I & -K(s) \\ 0 & I - G(s)K(s) \end{bmatrix}.$$

Hence

$$\begin{aligned} \begin{bmatrix} I & -K(s) \\ -G(s) & I \end{bmatrix}^{-1} &= \begin{bmatrix} I & -K(s) \\ 0 & I-G(s)K(s) \end{bmatrix}^{-1} \begin{bmatrix} I & 0 \\ -G(s) & I \end{bmatrix}^{-1} \\ &= \begin{bmatrix} I & K(s)\,(I-G(s)K(s))^{-1} \\ 0 & (I-G(s)K(s))^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ G(s) & I \end{bmatrix}. \end{aligned}$$

Finally, since $(I-G(s)K(s))^{-1} = I + G(s)K(s)\,(I-G(s)K(s))^{-1}$, it follows that if $G(s)$ is stable, then the loop is internally stable if and only if $Q(s) := K(s)\,(I-G(s)K(s))^{-1}$ is stable. Rearranging terms shows that $K(s)$ is internally stabilizing if and only if $K(s) = Q(s)\,(I+G(s)Q(s))^{-1}$ for some stable $Q(s)$.

c) Since G is stable and K is required to be internally stabilising, $K = Q(I+GQ)^{-1}$ for some stable Q from Part (b). We search for a stable Q to satisfy the design requirements. Let the input to Δ be ε while the output from Δ be δ. Then a simple calculation shows that $\varepsilon = (I-KG)^{-1}\delta$. Now

$$(I-KG)^{-1} = I + QG.$$

The small gain theorem implies that for K to stabilise the loop in Figure 2.2 for all Δ such that $\|\Delta\|_\infty < 1$, we must have that $\|I+QG\|_\infty < 1$. We set $Q(s) = kG(s)^{-1}$ where k is chosen to be nondynamic to ensure $K(s)$ has the same order as $G(s)$. Thus we require $|1+k| \leq 1$ or

$$-2 \leq k \leq 0.$$

Also, $K(0)G(0) = kI/(1+k)$ so we require

$$|\frac{k}{1+k}| = 2.$$

It follows that $k = -2$ will satisfy both specifications, although other values of k will also satisfy the specifications. Thus

$$K(s) = -2G(s)^{-1}.$$

3. a) A simple calculation shows that $S(s) = L(s)^{-1}$. By direct evaluation, $L(-j\omega)^T L(j\omega) =$

$$I + K(j\omega I - A)^{-1}B + B^T(-j\omega I - A^T)^{-1}K^T + B^T(-j\omega I - A^T)^{-1}K^T K(j\omega I - A)^{-1}B.$$

But

$$K^T K = A^T P + PA + C^T C = -(-j\omega I - A^T)P - P(j\omega I - A) + C^T C$$

from the Riccati equation. So, $L(-j\omega)^T L(j\omega)$

$$\begin{aligned}
&= I + K(j\omega I - A)^{-1}B + B^T(-j\omega I - A^T)^{-1}K^T \\
&\quad + B^T(-j\omega I - A^T)^{-1}[-(-j\omega I - A^T)P - P(j\omega I - A) + C^T C](j\omega I - A)^{-1}B \\
&= I + [K - B^T P](j\omega I - A)^{-1}B + B^T(-j\omega I - A^T)^{-1}[K^T - PB] \\
&\quad + B^T(-j\omega I - A^T)^{-1}C^T C(j\omega I - A)^{-1}B = I + G(-j\omega)^T C^T C G(j\omega).
\end{aligned}$$

It follows that all the singular values of $L(j\omega)$ are greater than or equal to 1. Since $S = L^{-1}$ it follows that all the singular values of $S(j\omega)$ are less than or equal to 1 and so $\|S\|_\infty \leq 1$.

b) A minimal state–space realisation of $G(s)$ is given by

$$G(s) \stackrel{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cc|cc} -3 & 0 & 2 & 0 \\ 0 & -4 & 0 & \sqrt{3} \\ \hline 2 & 0 & 0 & 0 \\ 0 & \sqrt{3} & 0 & 0 \end{array}\right].$$

Setting $P = \mathrm{diag}(P_1, P_2)$ the Riccati equation implies

$$-3P_1 - 3P_1 - 4P_1^2 + 4 = 0, \qquad -4P_2 - 4P_2 - 3P_2^2 + 3 = 0$$

which has stabilising solutions $P_1 = 0.5$ and $P_2 = 1/3$. Hence $K = B^T P = \mathrm{diag}(1, 1/\sqrt{3})$.

c) Let ε be the input to Δ and δ be the output of Δ. Then

$$\varepsilon = -K(\delta + F\varepsilon) = -(I + KF)^{-1}K\delta.$$

Using the small gain theorem (since the regulator is stable and the perturbation is assumed stable), the loop is stable if $\|\Delta(I + KF)^{-1}K\|_\infty < 1$. But Part (a) implies that $\|(I + KF)^{-1}\|_\infty \leq 1$. Furthermore, the largest singular value of K is equal to 1 from Part (b). Hence the loop will tolerate perturbations of size (measured in the $\mathscr{H}_\infty$–norm) at least 1 without losing internal stability, since $\|\Delta\|_\infty < 1$ implies that

$$\|\Delta(I + KF)^{-1}K\|_\infty < 1.$$

4. a) It is clear that we require $K(s)$ to be internally stabilising.

- A simple calculation shows that $y(s) = T_{yd}(s)d(s)$ where

$$T_{yd}(s) = \left[\ (I+G(s)K(s))^{-1} \quad -(I+G(s)K(s))^{-1}G(s)\ \right].$$

It follows that a sufficient condition to achieve the first design specification is $\|T_{yd}(j\omega)\| < |w_1(j\omega)^{-1}|\ \forall\omega$ or, equivalently, $\|W_1T_{yd}\|_\infty < 1$, where $W_1(s) = w_1(s)I$.

- A similar calculation shows that $u(s) = T_{ud}(s)d(s)$ where

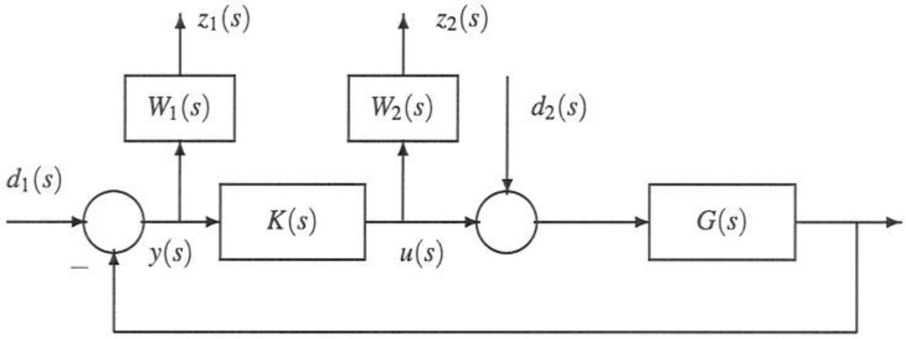
$$T_{ud}(s) = \left[\ K(s)(I+G(s)K(s))^{-1} \quad -K(s)(I+G(s)K(s))^{-1}G(s)\ \right].$$

It follows that a sufficient condition to achieve the second design specification is $\|T_{ud}(j\omega)\| < |w_2(j\omega)^{-1}|\ \forall\omega$ or, equivalently, $\|W_2T_{ud}\|_\infty < 1$, where $W_2(s) = w_2(s)I$.

Thus, to satisfy both design requirements, it is sufficient that

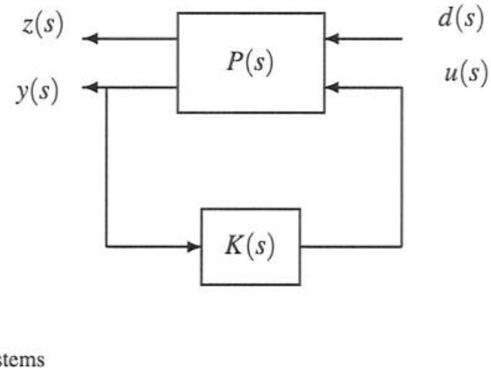
$$\left\| \begin{bmatrix} W_1T_{yd} \\ W_2T_{ud} \end{bmatrix} \right\|_\infty < 1.$$

b) The cost signals are given as $z_1(s) = W_1(s)y(s)$ and $z_2(s) = W_2(s)u(s)$. The block diagram incorporating $z_1(s)$ and $z_2(s)$ is shown below.



c) The corresponding generalised regulator formulation is to find an internally stabilising $K(s)$ such that $\|\mathscr{F}_l(P,K)\|_\infty < 1$ where

$$z(s) = \begin{bmatrix} z_1(s) \\ z_2(s) \end{bmatrix},\ P(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} = \left[\begin{array}{cc|c} W_1(s) & -W_1(s)G(s) & -W_1(s)G(s) \\ 0 & 0 & W_2(s) \\ \hline I & -G(s) & -G(s) \end{array}\right].$$



5. a) i) Suppose that λ is an eigenvalue of A and let $z \neq 0$ be the corresponding eigenvector. Then $Az = \lambda z$. Pre– and pos–multiplying the $(1,1)$ block of the matrix inequality by z' and z, respectively, we get $(\lambda + \bar{\lambda})z'Pz < 0$. Since $z \neq 0$ and $P \succ 0$, this implies that $z'Pz > 0$ so that $\lambda + \bar{\lambda} < 0$ and so A is stable. Let $x(t), u(t)$ and $y(t)$ be the state, input and output signals and assume that $x(0) = 0$. Since A is stable, $\lim_{t\to\infty} x(t) = 0$. Now $\|G\|_\infty < \gamma$ if and only if $J := \int_0^\infty \left(y^T y - \gamma^2 u^T u\right) dt < 0, |\|u\|_2 < \infty$. For $P = P^T$, $\int_0^\infty \frac{\mathrm{d}}{\mathrm{dt}}\left(x^T P x\right) dt = x(\infty)^T P x(\infty) - x(0)^T P x(0) = 0$. So

$$0 = \int_0^\infty \left(\dot{x}^T P x + x^T P \dot{x}\right) dt = \int_0^\infty \left(x^T (A^T P + PA)x + x^T PBu + u^T B^T P x\right) dt.$$

Use $y = Cx + Du$ and add the last expression to J,

$$J = \int_0^\infty \begin{bmatrix} x^T & u^T \end{bmatrix} \begin{bmatrix} A^T P + PA + C^T C & PB + C^T D \\ B^T P + D^T C & D^T D - \gamma^2 I \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} dt.$$

Thus $J < 0$ from the inequality (5.1) and so $\|G\|_\infty < \gamma$.

ii) We can write the matrix in (5.1) as

$$\begin{bmatrix} A^T P + PA & PB \\ B^T P & -\gamma^2 I \end{bmatrix} + \begin{bmatrix} C^T \\ D^T \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix}.$$

A Schur argument now shows that (5.1) is equivalent to (5.2).

iii) Pre– and post–multiplying (5.2) by $\mathrm{diag}(Q, I, I)$ where $Q = P^{-1}$ shows that (5.2) and (5.3) are equivalent and proves the result.

b) i) Now

$$\dot{x} = Ax + Bu + Bw = (A + BF)x + Bw, \qquad z = \begin{bmatrix} Cx \\ u \end{bmatrix} = \begin{bmatrix} C \\ F \end{bmatrix} x.$$

It follows that $H(s) = \begin{bmatrix} C \\ F \end{bmatrix} (sI - (A + BF))^{-1} B$.

ii) It follows from Part (a.iii) that $A + BF$ is stable and $\|H\|_\infty < \gamma$ if there exists $Q = Q^T \succ 0$ such that

$$\begin{bmatrix} (A+BF)Q + Q(A+BF)^T & B & QC^T & QF^T \\ B^T & -\gamma^2 I & 0 & 0 \\ CQ & 0 & -I & 0 \\ FQ & 0 & 0 & -I \end{bmatrix} \prec 0.$$

Defining $Y = FQ$ shows that $A + BF$ is stable and $\|H\|_\infty < \gamma$ if there exist $Q = Q^T \succ 0$ and Y such that

$$\begin{bmatrix} AQ + QA^T + BY + Y^T B^T & B & QC^T & Y^T \\ B^T & -\gamma^2 I & 0 & 0 \\ CQ & 0 & -I & 0 \\ Y & 0 & 0 & -I \end{bmatrix} \prec 0.$$

6. a) The generalized regulator formulation is given by

$$\begin{bmatrix} z(s) \\ y(s) \end{bmatrix} = P(s) \begin{bmatrix} w(s) \\ u(s) \end{bmatrix}, \; u(s) = Fy(s), \; P(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{c||c|c} A & B & B \\ \hline\hline 0 & 0 & I \\ \hline I & 0 & 0 \end{array}\right].$$

b) The requirement $\|H\|_\infty < \gamma$ is equivalent to $J := \|z\|_2^2 - \gamma^2\|w\|_2^2 < 0$. Let $V = x^T X x$ and set $u = Fx$. Provided that $X = X^T \succ 0$ and $\dot{V} < 0$ along the closed–loop trajectory, we can assume $\lim_{t\to\infty} x(t) = 0$. Then

$$\dot{V} = \dot{x}^T X x + x^T X \dot{x} = x^T(A^T X + XA + F^T B^T X + XBF)x + x^T XBw + w^T B^T X x.$$

Integrating from 0 to ∞ and using $x(0) = x(\infty) = 0$,

$$0 = \int_0^\infty \left(x^T(A^T X + XA + F^T B^T X + XBF)x + x^T XBw + w^T B^T X x\right) dt.$$

Using the definition of J and adding the last equation,

$$J = \int_0^\infty \left(x^T(A^T X + XA + F^T F + F^T B^T X + XBF)x - (\gamma^2 w^T w - x^T XBw - w^T B^T X x)\right) dt.$$

Let $Z = F + B^T X$. Completing the squares by using

$$\begin{aligned} Z^T Z &= F^T F + F^T B^T X + XBF + XBB^T X \\ \|\gamma w - \gamma^{-1} B^T X x\|^2 &= \gamma^2 w^T w - w^T B^T X x - x^T XBw + \gamma^{-2} x^T XBB^T X x, \end{aligned}$$

$$J = \int_0^\infty \left(x^T(A^T X + XA - (1-\gamma^{-2})XBB^T X)x + \|Zx\|^2 - \|\gamma w - \gamma^{-1} B^T X x\|^2\right) dt.$$

Thus two sufficient conditions for $J < 0$ are the existence of X such that

$$A^T X + XA - (1-\gamma^{-2})XBB^T X = 0, \qquad X = X^T \succ 0.$$

The feedback gain is $F = -B^T X$ and the worst case disturbance is $w^* = \gamma^{-2} B^T X x$. The closed–loop is $\dot{x} = \left(A - (1-\gamma^{-2})BB^T X\right)x$ and a third condition is therefore $Re\ \lambda_i\left(A - (1-\gamma^{-2})BB^T X\right) < 0\ \forall i$.

c) By inspecting Figure 6, it is clear that, provided A is stable, we can set $F = 0$ and w will have no effect on z and so the optimal value of γ is 0.

d) Recall that the closed–loop A–matrix is $A - (1-\gamma^{-2})BB^T X$. Setting $\gamma = 1$ shows that the closed–loop A–matrix is equal to A, which is unstable by assumption. Thus the optimal value of γ is greater than 1.

6 | 6