E4.22
C1.2

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2003

MSc and EEE PART IV: M.Eng. and ACGI

LINEAR OPTIMAL CONTROL

Tuesday, 6 May 10:00 am

Time allowed: 3:00 hours

There are SIX questions on this paper.

Answer FOUR questions.

Corrected Copy

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s) :	A. Astolfi
	Second Marker(s) :	G. Weiss

Special instructions for invigilators: None

Information for candidates:

System: $\dot{x}(t) = Ax(t) + Bu(t),\ x(0) = x_0.$

Quadratic cost function:
$$J(x_0, u) = \int_0^\infty [x(t)'Qx(t) + u(t)'Ru(t)]\, dt,$$
$$Q = Q' \geq 0,\ R = R' > 0.$$

Riccati equation: $A'P + PA + Q - PBR^{-1}B'P = 0.$

Optimal control law: $u(t) = -R^{-1}B'Px(t) = -Kx(t).$

Minimum cost: $x_0'Px_0.$

Return difference inequality for scalar u: $|1 + K(j\omega I - A)^{-1}B| \geq 1,$

Minimum principle:
$$\dot{x} = f(x, u),\ u \in \mathcal{U}$$
$$J(x_0, u) = \int_0^{t_f} L(x(t), u(t))dt,$$
$$H(x, u, \lambda_0, \lambda) = \lambda_0 L(x, u) + \lambda^T f(x, u),$$
$$\dot{\lambda}^\star = -\left.\frac{\partial H}{\partial x}\right|^T_{(x^\star, u^\star, \lambda_0^\star, \lambda^\star)},$$
$$H(x^\star, \omega, \lambda_0^\star, \lambda^\star) \geq H(x^\star, u^\star, \lambda_0^\star, \lambda^\star),\ \forall \omega \in \mathcal{U},$$
$$H(x^\star, u^\star, \lambda_0^\star, \lambda^\star) = k.$$

1. Consider the linear electric network in Figure 1, with $R > 0$, $C > 0$ and $L > 0$. Denote by u the driving voltage, by x_1 the voltage across the capacitor C, by x_2 the current through the inductor L, and by y the current through the voltage source.

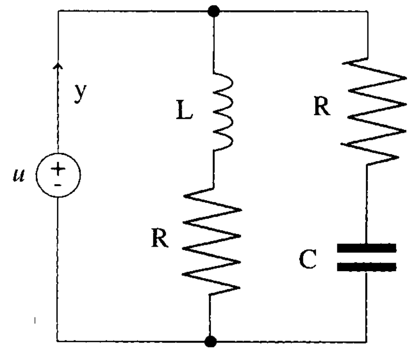

Figure 1.

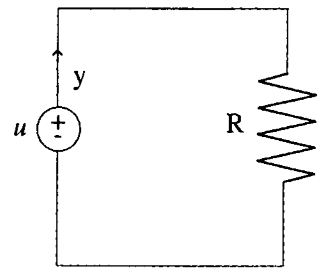

Figure 2.

(a) Using Kirchhoff's laws, or otherwise, express the dynamics of the circuit in the standard state-space form, regarding u as the input and y as the output. [4]

(b) Study the controllability/stabilizability of the dynamical system determined in part (a). [4]

(c) Study the observability/detectability of the dynamical system determined in part (a). [4]

(d) Compute the transfer function from the input u to the output y. [4]

(e) Show that if $R^2C = L$ then the transfer functions of the circuits in Figure 1 and Figure 2 are the same. [4]

2. The linearized model of an orbiting satellite about a circular orbit of radius $r_0 > 0$ and angular velocity $\omega_0 \neq 0$ is described by the equations

$$\dot{x} = Ax + Bu = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega_0^2 & 0 & 0 & 2r_0\omega_0^2 \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega_0/r_0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1/r_0 \end{bmatrix} u$$

$$y = Cx = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x.$$

The output components are variations in radius and angle of the orbit and the input components are radial and tangential forces.

(a) Show that the system is controllable. [6]

(b) Design a state feedback control law

$$u = Kx + Gv = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \end{bmatrix} x + \begin{bmatrix} g_{11} & 0 \\ 0 & g_{22} \end{bmatrix} v$$

such that

(b1) the matrix $A+BK$ has all eigenvalues equal to -1 and it is block diagonal, *i.e.*

$$A + BK = \begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix}$$

with $F_i \in I\!R^{2\times 2}$; [8]

(b2) the closed-loop system has unity DC gain, *i.e.*

$$-C(A+BK)^{-1}BG = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

[6]

3. A linear system is described by the differential equations

$$\begin{aligned} \dot{x}_1 &= \alpha x_1 + u \\ \dot{x}_2 &= u \\ y &= x_1 + x_2 \end{aligned}$$

where $u \in I\!R$ is the control input, y is the output variable and α is a constant parameter.

(a) Study the controllability property of the system as a function of α. [2]

(b) Study the observability property of the system as a function of α. [2]

(c) Assume $\alpha \neq 0$. Design an output feedback controller applying the separation principle. In particular, select the state feedback gain K such that the matrix $A - BK$ has two eigenvalues equal to -1 and the output injection gain L such that the matrix $A - LC$ has two eigenvalues equal to -3. Note that K and L will depend on α. [8]

(d) Compute

$$\lim_{\alpha \to 0} \|K\| \qquad \lim_{\alpha \to 0} \|L\|$$

and explain your results using the conclusions of parts (a) and (b). [2]

(e) Consider the state feedback control law designd in part (c). Verify if, for some α, this control law is *optimal* with respect to a cost of the form

$$\int_0^\infty \left[x(t)' Q x(t) + u^2(t) \right] dt,$$

with $Q \geq 0$. [6]

4. Consider the system

$$\dot{x} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u$$

with initial state x_0, with the quadratic cost to be minimised

$$J(x_0, u) = \int_0^\infty L(x_1, x_2, u, t)dt$$

where

$$L(x_1, x_2, u, t) = e^{2\alpha t}\left(x_1^2 + q_{22}x_2^2 + ru^2\right),$$

with $x = [x_1, x_2]'$, $\alpha \geq 0$, $q_{22} > 0$ and $r > 0$.

(a) Transform this optimal control problem into a standard problem, *i.e.* a problem in which $L(x_1, x_2, u, t)$ is replaced by a function of x_1, x_2 and u only. [4]

(b) Verify that, for any $\alpha \geq 0$, the conditions for the existence and uniqueness of an optimal feedback control law are met. [4]

(c) Write the ARE associated with the transformed optimal control problem defined in part (a). Find $q_{22} > 0$ and $r > 0$ such that the ARE is satisfied by a matrix of the form

$$P = \begin{bmatrix} 1 & 0 \\ 0 & p_{22} \end{bmatrix}.$$

Make sure that the resulting scalar r is positive and the resulting Q and P are positive definite for all $\alpha \in [0, \bar{\alpha})$. Determine the largest possible such $\bar{\alpha}$. [6]

(d) Suppose that q_{22}, r and $\bar{\alpha}$ are as required in part (c). Compute the optimal control law and the optimal closed-loop system for the original optimal control problem. Verify that the eigenvalues of the optimal closed-loop system have real part less than $-\alpha$ for all $\alpha \in [0, \bar{\alpha})$. [6]

5. Consider the system

$$\begin{aligned}\dot{x}_1 &= u \\ \dot{x}_2 &= -x_2 + u\end{aligned}$$

with $u \in [-1, 1]$, initial state $x(0) = [x_{10}, x_{20}]^T$, final state $x(T) = [0, 0]^T$ and the cost (to be minimized)

$$J(x_0, u) = \int_0^T 1\ dt.$$

(a) Write the necessary conditions of optimality in the case of normal extremals. [4]

(b) Compute the optimal control as a function of the costate $\lambda = [\lambda_1, \lambda_2]^T$ and show that $|u^\star(t)| = 1$ for all t such that $\lambda_1^\star(t) + \lambda_2^\star(t) \neq 0$. [2]

(c) Use the differential equations of the costate to show that the optimal control law has at most one switch, *i.e.* the optimal control law is one of the following:

- $u^\star(t) = 1$ for all $t \in [0, T]$;
- $u^\star(t) = -1$ for all $t \in [0, T]$;
- $u^\star(t) = 1$ for all $t \in [0, \bar{t})$ and $u^\star(t) = -1$ for all $t \in (\bar{t}, T]$, with $0 < \bar{t} < T$.
- $u^\star(t) = -1$ for all $t \in [0, \bar{t})$ and $u^\star(t) = 1$ for all $t \in (\bar{t}, T]$, with $0 < \bar{t} < T$.

(Note: the solution of the differential equation $\dot{x} = ax + b$, with constant $a \neq 0$ and constant b and initial condition x_0, is $x(t) = e^{at}(x_0 + b/a) - b/a$.) [6]

(d) Integrate the state equation with $u = 1$. [2]

(e) Determine the set of initial conditions for which the control $u(t) = 1$ for all $t \in [0, T]$ is optimal. For such initial conditions compute the time to reach the origin. [6]

6. Consider the system

$$\dot{x} = x + u$$

with $x(0) = x_0$, and the problem of finding a bounded control law $|u(t)| \leq 1$ that minimizes the cost

$$J(x_0, u) = \frac{x(1)^2}{2}.$$

(a) Write the necessary conditions of optimality for normal extremals and the boundary condition for the costate $\lambda(t)$ at $t = 1$. [6]

(b) Write the optimal control as a function of the optimal costate $\lambda^\star(t)$. [2]

(c) Assume there is an optimal control which yields the global minimum of $J(x_0, u)$, *i.e.* $J(x_0, u) = 0$. Show that such an optimal control cannot be computed using the necessary conditions derived in part (a). [4]

(d) Assume $|x_0| < 1 - \frac{1}{e}$ and $x_0 \neq 0$. Show that there exists a control $u(t)$ such that $x(t) = 0$ for all $t \in [\bar{t}, 1]$, for some $0 < \bar{t} < 1$. (Hint: try

$$u(t) = \begin{cases} -\text{sign}(x_0) & \text{for } t \in [0, \bar{t}] \\ 0 & \text{for } t \in (\bar{t}, 1] \end{cases} \qquad (\star)$$

for some $0 < \bar{t} < 1$.) (Note: the solution of the differential equation $\dot{x} = ax + b$, with constant $a \neq 0$ and constant b and initial condition x_0, is $x(t) = e^{at}(x_0 + b/a) - b/a$.) (Hint: you may use the inequality

$$1 < \frac{\text{sign}(x_0)}{\text{sign}(x_0) - x_0} < e,$$

for all $|x_0| < 1 - 1/e$.) [4]

(e) Assume $|x_0| < 1 - \frac{1}{e}$ and $x(0) \neq 0$. Using the result in part (c) discuss the optimality of the control law $(\star)$ and the uniqueness of the optimal control law. [4]