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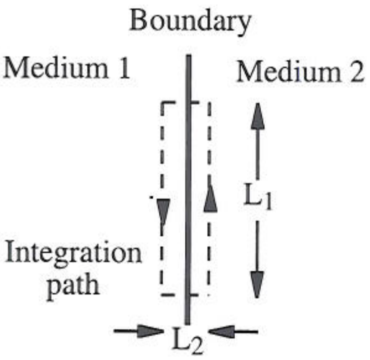
Optoelectronics 2008 - Solutions

1a) To prove that tangential electric fields must match at a boundary, start with Faraday's law $\int_L \underline{\mathbf{E}} \,.\, d\underline{\mathrm{L}} = - \iint_A \partial\underline{\mathbf{B}}/\partial t \,.\, d\underline{a}$. Evaluate both sides of the equation for the rectangular loop shown below, which has sides of length L_1 and L_2. If L_2 to tends to zero, the area enclosed must also tend to zero, so that $\int_L \underline{\mathbf{E}} \,.\, d\underline{\mathrm{L}} = 0$. If L_1 is small enough that the fields are approximately constant, the left-hand side is given by $\int_L \underline{\mathbf{E}} \,.\, d\underline{\mathrm{L}} \approx (E_{t1} - E_{t2})L_2$, where E_{t1} and E_{t2} are the components of the electric fields E_1 and E_2 in the two media that are tangential to the boundary. Hence $E_{t1} - E_{t2} = 0$, and tangential fields must match.

[2]



[2]

b) The time averaged Poynting vector is $\underline{S} = (1/T)\, {}_0\!\int^T (\underline{\mathbf{E}} \times \underline{\mathbf{H}})\, dt$

Put $\underline{\mathbf{E}} = \mathrm{Re}\{\underline{E} \exp(j\omega t)\}$ and $\underline{\mathbf{H}} = \mathrm{Re}\{\underline{H} \exp(j\omega t)\}$. Now $\mathrm{Re}(z) = 1/2\ \{z + z^*\}$

Hence $\underline{S} = (1/4T)\, {}_0\!\int^T \{\underline{E} \exp(j\omega t) + \underline{E}^* \exp(-j\omega t)\} \times \{\underline{H} \exp(j\omega t) + \underline{H}^* \exp(-j\omega t)\} dt$

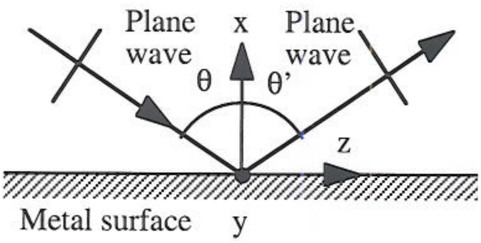
So $\underline{S} = (1/4T)\, {}_0\!\int^T \{(\underline{E} \times \underline{H}) \exp(2j\omega t) + (\underline{E}^* \times \underline{H}^*) \exp(-2j\omega t) + (\underline{E} \times \underline{H}^*) + (\underline{E}^* \times \underline{H})\}\, dt$

Or $\underline{S} = (1/2T)\, \mathrm{Re}[{}_0\!\int^T \{(\underline{E} \times \underline{H}) \cos(2\omega t) + (\underline{E} \times \underline{H}^*)\} dt]$

The first term in the integral averages to zero, if T is large, leaving $\underline{S} = (1/2)\, \mathrm{Re}(\underline{E} \times \underline{H}^*)$.

[4]

c) Assume first that a reflected wave is generated at an arbitrary angle θ' as shown below.



The incident and reflected waves are y-polarized and in time-independent form are:

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$\underline{E}_I = E_I \exp\{-jk_0(z\ \sin(\theta) - x \cos(\theta))\}\ \underline{j}$ and $\underline{E}_R = E_R \exp\{-jk_0(z\ \sin(\theta') + x \cos(\theta'))\}\ \underline{j}$

Here $k_0 = 2\pi/\lambda_0$. The total electric field outside the metal is therefore:

$\underline{E} = [E_I \exp\{-jk_0(z\ \sin(\theta) - x \cos(\theta))\} + E_R \exp\{-jk_0(z\ \sin(\theta') + x \cos(\theta'))\}]\ \underline{j}$

This field is entirely tangential to the metal surface.

[2]

Inside the metal, the electric field must be zero, if it is a perfect conductor. The boundary conditions are that tangential components must match on the boundary ($x = 0$).

Hence: $E_I \exp\{-jk_0 z\ \sin(\theta)\} + E_R \exp\{-jk_0 z\ \sin(\theta')\} = 0$

The only way this equation can be satisfied is if $\theta' = \theta$ and if $E_R = - E_I$. Hence:

$\underline{E} = E_I \exp\{-jk_0 z\ \sin(\theta)\}\ [\exp\{+jk_0 x \cos(\theta)\} - \exp\{-jk_0 x \cos(\theta)\}]\ \underline{j}$ or:

$\underline{E} = 2jE_I \exp\{-jk_0 z \sin(\theta)\} \sin\{k_0 x \cos(\theta)\}\ \underline{j}$

[4]

The associated magnetic field can be found from Faraday's law $\mathrm{curl}(\underline{E}) = -\partial\underline{\mathbf{B}}/\partial t$

In time-independent form this becomes $\mathrm{curl}(\underline{E}) = -j\omega\underline{B} = -j\omega\mu_0\underline{H}$ so $\underline{H} = (j/\omega\mu_0)\ \mathrm{curl}(\underline{E})$

$$\mathrm{curl}(\underline{E}) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & E_y & 0 \end{vmatrix} = \{-\partial E_y/\partial z\}\ \underline{i} + \{\partial E_y/\partial x\}\ \underline{k}$$

H_x is then given by $H_x = -j\{2E_I k_0 \sin(\theta)/\omega\mu_0\} \exp\{-jk_0 z \sin(\theta)\} \sin\{k_0 x \cos(\theta)\}$

[2]

The irradiance $\underline{S} = 1/2\ \mathrm{Re}\{\underline{E} \times \underline{H}^*\}$ is then given by:

$$\underline{S} = 1/2\ \mathrm{Re}\left\{ \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & E_y & 0 \\ H_x^* & 0 & H_z^* \end{vmatrix} \right\} = 1/2\ \mathrm{Re}\ (E_y H_z^*\ \underline{i} - E_y H_x^*\ \underline{k})$$

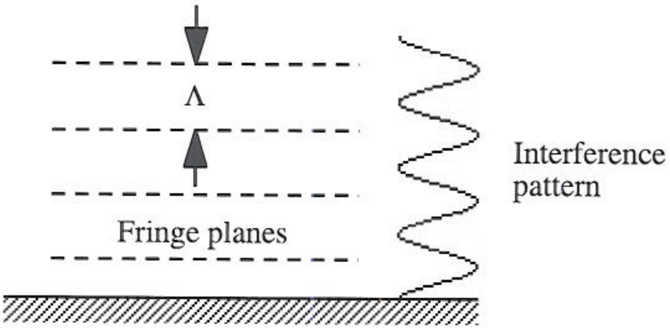
The z-component of the irradiance is then:

$S_z = 1/2\ \mathrm{Re}\ (-E_y H_x^*) = \{2E_I^2 k_0 \sin(\theta)/\omega\mu_0\} \sin^2\{k_0 x \cos(\theta)\}$

[2]

The irradiance therefore varies periodically in the x-direction, with period $\Lambda = \lambda_0/\cos(\theta)$.

The result is a set of parallel fringe planes with the first zero on the metal as shown below.



[2]

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2.a) Phase velocity is $v_{ph} = \omega/k$; group velocity is $v_g = d\omega/dk$.
Since $\omega = 2\pi f$ and $k = 2\pi/\lambda$, $\omega/k = f\lambda = c/n$ and $k = n\omega/c$
Hence, $1/v_p = k/\omega = n/c$. Similarly, $1/v_g = dk/d\omega = \{n + \omega\, dn/d\omega\}/c$

Now, $f\lambda_0 = c$, where λ_0 is the free space wavelength, so $\lambda_0 = 2\pi c/\omega$
Hence $d\lambda_0/d\omega = -2\pi c/\omega^2 = -\lambda_0/\omega$
We can then write $dn/d\omega = (dn/d\lambda_0)\,(d\lambda_0/d\omega) = -\lambda_0/\omega\ \ dn/d\lambda_0$
So that $1/v_g = \{n - \lambda_0\, dn/d\lambda_0\}/c$

[4]

Intermodal dispersion is the time spread caused by components of the signal carried by different modes in a multi-mode system travelling at different group velocities because of their different effective indices. Intramodal dispersion is the spread caused by components in a single-mode system travelling at different group velocities because of their different frequencies.

[4]

b) For multimode fibre, the slowest modes have their energy mainly in the core and propagate with a group velocity $v_{g1} \approx v_{ph1} = c/n_1$. Similarly, the fastest modes have their energy mainly in the cladding and propagate with a group velocity $v_{g2} \approx v_{ph2} = c/n_2$ The difference in time for the two sets of modes to travel a distance L is then:
$\Delta T = L\{1/v_{g1} - 1/v_{g2}\} = (L/c)\,\{n_1 - n_2\}$
In a digital system, intersymbol interference will start to occur when the pulse spread is around half the symbol duration T, so that $\Delta T = T/2$. The bit rate $B = 1/T$ is then limited to:
$B = 1/2\Delta T = c/\{2L(n_1 - n_2)\}$
The bit-rate length product is therefore limited to $BL = c/\{2(n_1 - n_2)\}$

[4]

c) In a single mode fibre, the time spread is now:
$\Delta T = L\{1/v_{g1} - 1/v_{g2}\} = L\, d(1/v_g)/d\lambda_0\, \Delta\lambda_0$, where $\Delta\lambda_0$ is the wavelength spread of signal.

[4]

From part a) we have $1/v_g = \{n - \lambda_0\,(dn/d\lambda_0)\}/c$
So that $d(1/v_g)/d\lambda_0 = \{dn/d\lambda_0 - dn/d\lambda_0 - \lambda_0\,(d^2n/d\lambda_0^2)\}/c = -(\lambda_0/c)\, d^2n/d\lambda_0^2$
Hence $\Delta T = -(L\lambda_0\Delta\lambda_0/c)\, d^2n/d\lambda_0^2$
Dispersion can therefore be minimised by operating in a spectral region when $d^2n/d\lambda_0^2 \approx 0$

[4]

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3. a) Start with Maxwell's equations and the material equations as given:

$\mathrm{div}(\underline{\mathbf{D}}) = \rho$

$\mathrm{div}(\underline{\mathbf{B}}) = 0$

$\mathrm{curl}(\underline{\mathbf{E}}) = -\partial\underline{\mathbf{B}}/\partial t$

$\mathrm{curl}(\underline{\mathbf{H}}) = \underline{\mathbf{J}} + \partial\underline{\mathbf{D}}/\partial t$

$\underline{\mathbf{J}} = \sigma\,\underline{\mathbf{E}}$

$\underline{\mathbf{D}} = \varepsilon\,\underline{\mathbf{E}}$

$\underline{\mathbf{B}} = \mu\,\underline{\mathbf{H}}$

Assume that $\rho = 0$; $\sigma = 0$ (so $\underline{\mathbf{J}} = 0$), $\varepsilon = \varepsilon_0\varepsilon_r$ and $\mu = \mu_0$ for optical materials.

[2]

Hence:

$\mathrm{div}(\varepsilon\underline{\mathbf{E}}) = 0$

$\mathrm{div}(\mu_0\underline{\mathbf{H}}) = 0$

$\mathrm{curl}(\underline{\mathbf{E}}) = -\mu_0\partial\underline{\mathbf{H}}/\partial t$

$\mathrm{curl}(\underline{\mathbf{H}}) = \varepsilon_0\varepsilon_r\partial\underline{\mathbf{E}}/\partial t$

[2]

Assume that each field component varies harmonically at a single angular frequency ω, so that $E(x, y, z, t) = E(x, y, z)\exp(j\omega t)$ etc. Assume also that ε_r is constant. Hence:

$\mathrm{div}(\underline{E}) = 0$ 1)

$\mathrm{div}(\underline{H}) = 0$ 2)

$\mathrm{curl}(\underline{E}) = -j\omega\mu_0\underline{H}$ 3)

$\mathrm{curl}(\underline{H}) = j\omega\varepsilon_0\varepsilon_r\underline{E}$ 4)

[2]

Take the curl of Equation 3, and combine the result with Equation 4 to get:

$\mathrm{curl}\,\{\mathrm{curl}(\underline{E})\} = -j\omega\mu_0\,\mathrm{curl}(\underline{H}) = \omega^2\mu_0\varepsilon_0\varepsilon_r\underline{E}$

Use the expression for curl {curl(E)} given to obtain:

$\mathrm{curl}\,\{\mathrm{curl}(\underline{E})\} = \mathrm{grad}\,\{\mathrm{div}(\underline{E})\} - \nabla^2\underline{E} = \omega^2\mu_0\,\varepsilon_0\varepsilon_r\underline{E}$

Use Equation 1 to obtain the time-independent vector wave equation:

$\nabla^2\underline{E} + \omega^2\mu_0\varepsilon_0\varepsilon_r\underline{E} = 0$

Since $\omega^2\mu_0\varepsilon_0 = k_0^{\,2}$ where $k_0 = 2\pi/\lambda$, and $\varepsilon_r = n^2$, we can write:

$\nabla^2\underline{E} + n^2k_0^{\,2}\underline{E} = 0$

Assuming that $\underline{E} = E_y\,\underline{j}$ for TE waves we obtain the time-independent scalar wave equation:

$\nabla^2E_y + n^2k_0^{\,2}E_y = 0$

[4]

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If we now assume further that the field is propagating in the z-direction, and that there is no variation in the y-direction, the solution can be written as:

$E_y = E_T(x) \exp(-j\beta z)$

Here $E_T(x)$ is the transverse field variation and β is the propagation constant.

In this case, the Laplacian is:

$\nabla^2 E_y = \partial^2 E_y/\partial x^2 + \partial^2 E_y/\partial y^2 + \partial^2 E_y/\partial z^2 = \{d^2E_T/dx^2 - \beta^2 E_T\} \exp(-j\beta z)$

Substituting into the scalar wave equation we obtain the scalar waveguide equation:

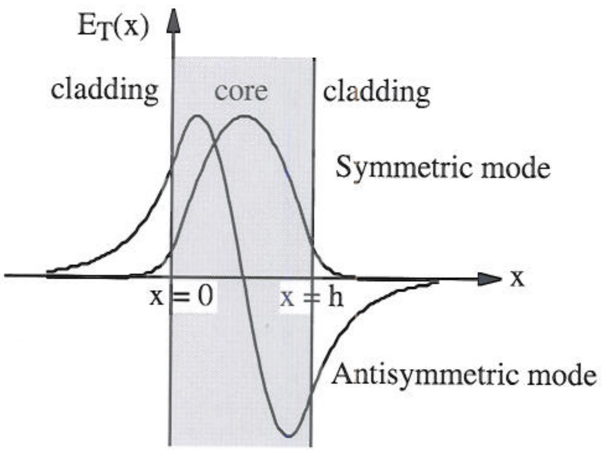
$d^2E_T/dx^2 + \{n^2k_0^{\ 2} - \beta^2\}E_T = 0$

This equation can be used to describe the fields in all three layers of a slab guide if we write $d^2E_T/dx^2 + \{n_i^{\ 2}k_0^{\ 2} - \beta^2\}E_T = 0$ for the i^{th} layer.

[4]

b) The waveguide equation has the general form $d^2E_T/dx^2 + C^2E_T = 0$ where $C^2 = n_i^{\ 2}k_0^{\ 2} - \beta^2$ Hence, the only possible solutions for E_T are complex exponentials or trigonometric functions (if $C^2 > 0$) and exponentials or hyperbolic functions (if $C^2 < 0$). Since $n_1 > n_2, n_3$ for total internal reflection to occur, the solutions for guided modes are trigonometric functions in layer 1 and exponentials in layers 2 and 3. The solutions are found by assuming transverse fields as above, and matching the tangential component of the electric field (E_y) and the magnetic field (H_z, which is proportional to $\partial E_y/\partial x$) on the two boundaries ($x = 0$ and $x = h$). The four equations obtained can be reduced to a single equation, which is solved for β. The fields may then be reconstructed in each of the three layers. For a symmetric guide, the transverse fields for the two lowest-order modes are:

[3]



[3]

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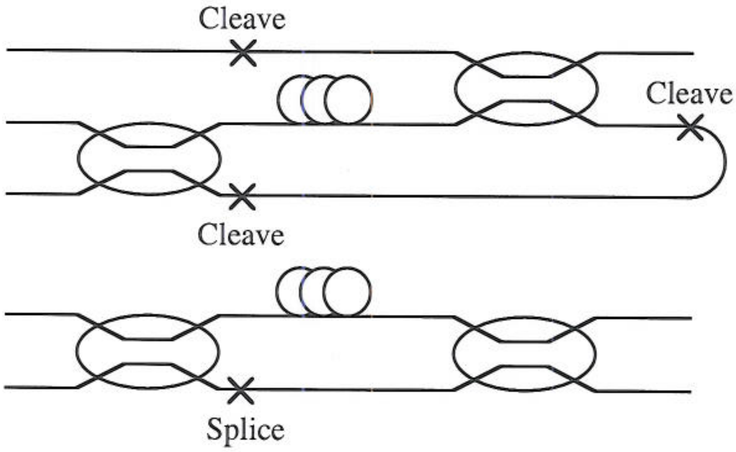
5. a) Up to three cleaving (NB: minimum is one) and one splicing operations are required to tidy up the student's fibre optical Mach-Zehnder interferometric sensor, as shown below.



[4]

b) For unity input amplitude, the transmitted and cross-coupled outputs from a lossless coupler are $\cos(\kappa L)$ and $-j \sin(\kappa L)$, where κL is the coupling length, leading to transmitted and cross-coupled powers of $\cos^2(\kappa L)$ and $\sin^2(\kappa L)$. These outputs will be proportionally reduced if there is loss. A coupling length of $\kappa L = \pi/4$ is required for equal power split, which will give maximum contrast in an interferometer.

[2]

For coupler 1, $P_A = 0.54$ and $P_B = 0.36$ so the total throughput is $P = P_A + P_B = 0.90$
The loss is $-10 \log_{10}(P) = 0.46$ dB
The coupling length is $\kappa L_1 = \tan^{-1}\{(P_B/P_A)^{1/2}\} = 0.685 = 0.87 \times \pi/4$.

[3]

For coupler 2, $P_C = 0.55$ and $P_D = 0.30$ so the total throughput is $P' = P_C + P_D = 0.85$
The loss is $-10 \log_{10}(P') = 0.70$ dB
The coupling length is $\kappa L_2 = \tan^{-1}\{(P_D/P_C)^{1/2}\} = 0.636 = 0.81 \times \pi/4$.

[3]

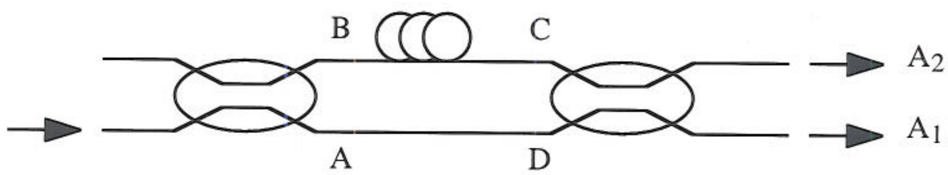
c) The outputs can be found as the sum of the two possible paths between input and output. For an input to the lower branch, ignoring common mode losses and phase shifts:

$A_1 = \cos(\kappa L_1)\cos(\kappa L_2) + (-j)^2 \sin(\kappa L_1)\sin(\kappa L_2)\exp(-j\beta\Delta L)$

$A_2 = -j\{\sin(\kappa L_1)\cos(\kappa L_2) + \cos(\kappa L_1)\sin(\kappa L_2)\exp(-j\beta\Delta L)\}$

Here ΔL is the excess length of fibre in the upper arm and β is the propagation constant.

[4]



From part b) we know that:

$\cos(\kappa L_1) = 0.774, \qquad \sin(\kappa L_1) = 0.632$

$\cos(\kappa L_2) = 0.804, \qquad \sin(\kappa L_2) = 0.594$

The main effect of temperature is to alter $\beta\Delta L$ through thermal expansion. The maximum and minimum outputs from port 1 are found when $\beta\Delta L$ is 0 or π. Hence:

$A_{1min} = \cos(\kappa L_1)\cos(\kappa L_2) - \sin(\kappa L_1)\sin(\kappa L_2) = 0.774 \text{ x } 0.804 - 0.632 \text{ x } 0.594 = 0.247$

$A_{1max} = \cos(\kappa L_1)\cos(\kappa L_2) + \sin(\kappa L_1)\sin(\kappa L_2) = 0.774 \text{ x } 0.804 + 0.632 \text{ x } 0.594 = 0.998$

Hence, the corresponding powers are:

$P_{1min} = 0.247^2 = 0.061$

$P_{1max} = 0.998^2 = 0.997$

[2]

Similarly, for port 2 we obtain:

$A_{2min} = -j\{0.632 \text{ x } 0.804 - 0.774 \text{ x } 0.594\} = -j0.048$

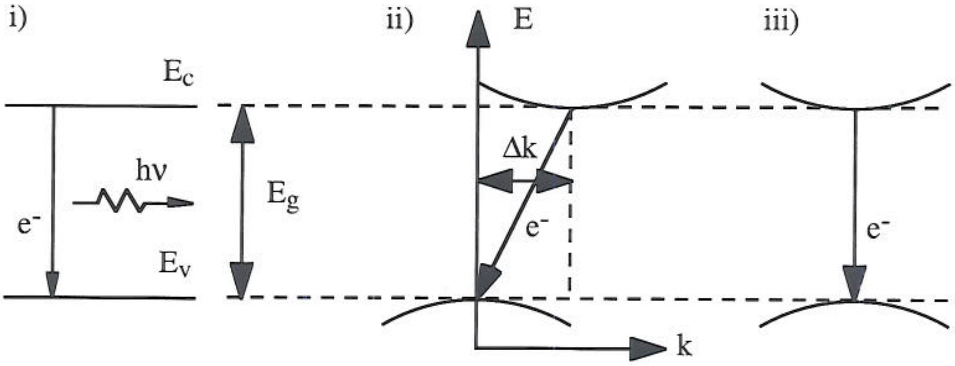
$A_{2min} = -j\{0.632 \text{ x } 0.804 + 0.774 \text{ x } 0.594\} = -j0.968$

$P_{2min} = 0.048^2 = 0.002$

$P_{2max} = 0.968^2 = 0.937$

[2]

5. a) Emission of light in semiconductors is often represented as shown in i) below. Here a photon of energy $h\nu = E_g = E_c - E_v$ is generated by the recombination of an electron from the conduction band with a hole in the valence band. This simple analysis ignores the momenta of electrons and holes. According to de Broglie, the momentum of a single electron wave is $P = h/\lambda = hk/2\pi$, where $k = 2p/\lambda$ and λ is the wavelength. The relation between momentum and energy can be shown on an E-k diagram. Near the bottom of the conduction band and the top of the valence band the E-k diagram varies parabolically, as shown in ii) and iii). In the former (indirect gap material) the conduction band minimum and the valence band maximum are offset by Δk, whereas in the latter (for a direct gap material) they coincide. A photon has energy but little momentum. Energy and momentum can theforefore only be conserved simultaneously in ii) if an additional quantity (normally, a phonon) can provide the missing momentum, without significantly disturbing the energy balance. However, the rate of such a process is much lower in ii) than in iii), since it depends on the supply of phonons. Direct gap materials are essential for room temperature emission.



[5]

b) III-V semiconductors are used for light emission and detection in optoelectronic systems. Epitaxial growth, which involves the ordered deposition of material on a crystalline template, is used to form the complex multilayer structures involved. The layers can have different doping (for homostructures) or different bandgap (for heterostructures). However, hetero-structures require lattice-matching to avoid strain. Examples of epitaxially grown materials include $In_{1-x}Ga_xAs_{1-y}P_y$ on InP and $Ga_{1-x}Al_xAs$ on GaAs. The former is more important in optical communications, since it allows sources and detectors in the low loss window of silica fibre near 1.5 μm wavelength. There are two common variants: liquid-phase and vapour phase expitaxy. The older method, LPE, results in interdiffusion between the layers. The more modern method, VPE, uses rapid gas switching to achieve abrupt interfaces.

[5]

c) The double heterostructure laser uses three layers of material with different bandgap and different doping to achieve separate control of the confinement of electrons, holes and

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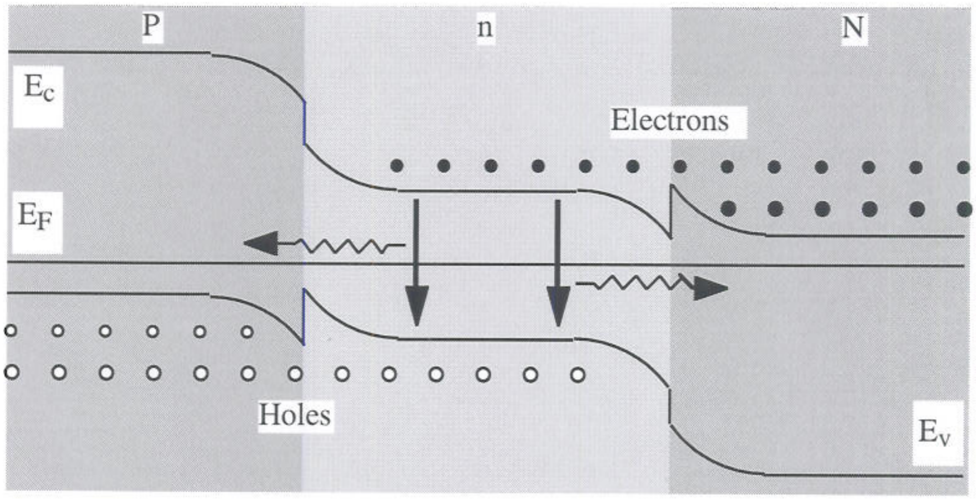
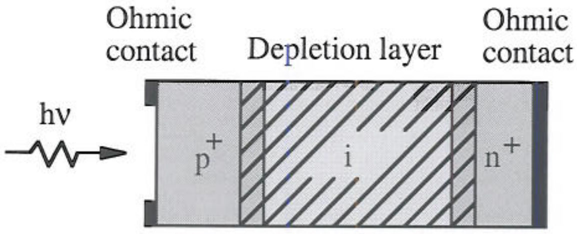
photons. The figure below shows a P-n-N heterostructure. The large step in the conduction-band prevents electrons injected into the n-layer from diffusing into the P-layer. Similarly, the large step in the valence band prevents the holes injected into the n-layer from diffusing into the N-layer. Consequently, the population in the n-layer can be strongly inverted and available for light amplification by stimulated emission. This process is enhanced if the local population of photons is high. A high photon density can be achieved, by confining the light in a waveguide formed by a large step in refractive index between the P- and n-layers and between the n- and N-layers, and by forming the waveguide into a resonant cavity.



[5]

d) A P-N junction photodiode ideally converts photons to electrons via band-to-band transitions in its depletion region. In this region, the electric field is so strong that the photo-generated carrier pairs are separated before they can recombine, and then diffuse to the contacts. However, the depletion layer is normally so thin that short wavelength (high energy) photons are absorbed before they reach the depletion layer, while long wavelength (low energy) photons are only absorbed after they have passed through. Consequently the quantum efficiency of a P-N junction photodiode can be variable. The PIN photodiode uses an additional layer of intrinsic material between heavily doped P- and N-layers to created a stretched depletion layer, which is more effective at capturing low energy photons. Heterostructure construction can also be used to ensure that incoming photons initially pass through a wide-bandgap layer, so that short wavelength photons reach the deplection layer.



[5]

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6. a) The five terms in the rate equations are:

I/ev	rate of electron injection per unit volume
n/τ_e	rate of electron recombination ditto
$\beta n/\tau_{rr}$	rate of spontaneous emission into the cavity ditto
$G\phi(n - n_0)$	rate of stimulated emission ditto
ϕ/τ_p	rate of escape of photons from the cavity ditto

[3]

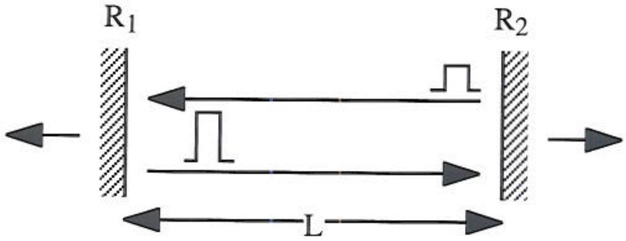
τ_e appears in the upper equation because electrons are lost by both radiative and non-radiative transitions. τ_{rr} appears in the lower one because photons are only generated by radiative transitions.

[2]

Rates (which are the reciprocal of time constants) are additive, hence $1/\tau_e = 1/\tau_{rr} + 1/\tau_{nr}$.

[1]

b) The photon lifetime can be calculated as follows. Consider a pulse of light propagating up and down a cavity of length L as shown below.



The round trip time is $T = 2L/v_g$, where $v_g \approx c/n_{eff}$ is the group velocity. In that time, the pulse energy has reduced from E to E x $(R_1R_2)^2$ where R_1 and R_2 are the amplitude reflectivities of the facets. After N transits, the pulse energy has reduced to E x $(R_1R_2)^{2N}$. In a time t, the number of transits is $N = t/T$, so the fractional power remaining is $(R_1R_2)^{2t/T}$. The time t taken for the power to reduce to 1/e of its initial value (the photon lifetime) is found by setting $(R_1R_2)^{2t/T} = 1/e$. Rearranging this we obtain:

$(2t/T) \log_e(R_1R_2) = -1$, or $t = T/\{2 \log_e(1/R_1R_2)\} = L/\{v_g \log_e(1/R_1R_2)\} = \tau_P$.

[4]

c) In the steady state, $d/t = 0$, so that:

$I/ev - n/\tau_e - G\phi(n - n_0) = 0$

$\beta n/\tau_{rr} + G\phi(n - n_0) - \phi/\tau_p = 0$

During lasing, spontaneous emission is negligible, so the lower equation approximates to:

$G\phi(n - n_0) - \phi/\tau_p = 0$ and hence $\phi/\tau_p = G\phi(n - n_0)$ and $n = n_0 + 1/G\tau_p$

This result implies that the electron concentration is clamped during lasing.

R.R.A. [signature] [signature]

[2]

The upper equation then approximates to:

$\phi/\tau_p = I/ev - n/\tau_e = \{I - I_{th}\}/ev$ where the threshold current is $I_{th} = nev/\tau_e$.

[2]

For semiconductor-air interfaces, $R_1 = R_2 = (n_{eff} - 1)\ (n_{eff} + 1)$.

If $n_{eff} = 3.5$, the facet reflectivity is $R_1 = R_2 = (3.5 - 1) / (3.5 + 1) = 0.56$.

[2]

If L = 250 μm, the photon lifetime is

$\tau_p = 250 \text{ x } 10^{-6} \text{ x } 3.5/\{3 \text{ x } 10^8 \log_e(1/0.56^2)\} = 2.48 \text{ x } 10^{-12}$ s

[1]

If $n_0 = 10^{24}$ m^{-3} and $G = 10^{-12}$ m^3/sec, the electron density during lasing is:

$n = 10^{24} + 1/(10^{-12} \text{ x } 2.48 \text{ x } 10^{-12}) = 1.4 \text{ x } 10^{24}$ m^{-3}

[1]

The threshold current is then

$I = 1.4 \text{ x } 10^{24} \text{ x } 1.6 \text{ x } 10^{-19} \text{ x } 250 \text{ x } 3 \text{ x } 0.1 \text{ x } 10^{-18}/ 10^{-9} = 0.0168$ A or 16.8 mA

[2]