EE4-25

SOLUTIONS: DESIGN OF LINEAR MULTIVARIABLE CONTROL SYSTEMS

1. a) Let the realisation be partitioned compatibly with P and Q as

$$G(s) \stackrel{s}{=} \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & 0 \end{array}\right].$$

Then

$$A'Q+QA+C'C = \begin{bmatrix} A'_{11}Q_1+Q_1A_{11}+C'_1C_1 & Q_1A_{12}+C'_1C_2 \\ \star & C'_2C_2 \end{bmatrix} = 0 \quad (1.1)$$

$$AP+PA'+BB' = \begin{bmatrix} A_{11}P_1+P_1A'_{11}+B_1B'_1 & P_1A_{21}+B_1B'_2 \\ \star & B_2B'_2 \end{bmatrix} = 0 \quad (1.2)$$

It follows from the (2,2) entry of (1.1) and (1.2) that $C_2=0$ and $B_2=0$. Also, it follows from the (1,2) entry of (1.1) and and (1.2) and the assumptions $Q_1 \succ 0$ and $P_1 \succ 0$ that $A_{21}=0$ and $A_{12}=0$. So, the realisation for $G(s)$ has the form

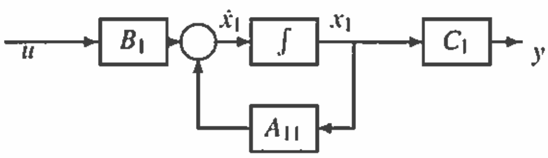
$$G(s) \stackrel{s}{=} \left[\begin{array}{cc|c} A_{11} & 0 & B_1 \\ 0 & A_{22} & 0 \\ \hline C_1 & 0 & 0 \end{array}\right]. \quad (1.3)$$

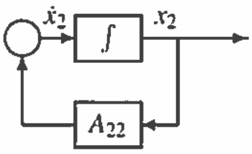
Thus we can decompose the realisation into two subsystems $G_1 : \dot{x}_1 = A_{11}x_1 + B_1u$, $y = C_1x_1$ with n_1 modes and $G_2 : \dot{x}_2 = A_{22}x_2$, with $n-n_1$ modes. For G_2 it is clear all the modes are uncontrollable and unobservable since the B and C-matrices are zero. We prove controllability for G_1. Let λ be an eigenvalue of A_{11} and $z \neq 0$ the corresponding left eigenvector. Then $z'A_{11} = \lambda z'$. Pre- and post-multiplying the the (1,1) entry of (1.2) by z' and z, respectively, we get $(\lambda+\bar{\lambda})z'P_1z+z'B_1B'_1z=0$. If $z'B_1=0$ then $\lambda+\bar{\lambda}=0$ which contradicts the assumption that A has no eigenvalues on the imaginary axis. Thus the realisation of G_1 is controllable. A dual argument shows that it is also observable.

b) In the proof of the previous part, since $z \neq 0$ and $P_1 \succ 0$, $z'P_1z > 0$ and $z'B_1B'_1z > 0$ then $\lambda+\bar{\lambda} < 0$ and A_{11} is stable.

c) Since the realisation for G_1 is both controllable and observable and that of G_2 is noncontrollable and nonobservable, the realisation for G is stabilisable and detectable if and only if all the eigenvalues of A_{22} are in the open left half plane.

d)





2. a) An inspection of Figure 2 shows that

$$\begin{aligned} \dot{x} - \dot{\hat{x}} &= (A + LC)(x - \hat{x}) + B_w w \\ z &= I(x - \hat{x}) \end{aligned}$$

It follows that

$$\begin{aligned} T_{zw}(s) &\stackrel{s}{=} \left[\begin{array}{c|c} A + LC & B_w \\ \hline I & 0 \end{array}\right] \\ &\stackrel{s}{=}: \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right] \end{aligned}$$

b) The Bounded Real Lemma states that A_c is stable and $\|T_{zw}\|_\infty < \gamma$ if there exists a $P = P'$ such that

$$\begin{aligned} \begin{bmatrix} A_c'P + PA_c + C_c'C_c & PB_c + C_c'D_c \\ B_c'P + D_c'C_c & D_c'D_c - \gamma^2 I \end{bmatrix} &\prec 0 \\ P = P' &\succ 0 \end{aligned}$$

By substituting the expressions for A_c, B_c, C_c and D_c, this becomes

$$\begin{aligned} \begin{bmatrix} (A + LC)'P + P(A + LC) + I & PB_w \\ \star & -\gamma^2 I \end{bmatrix} &\prec 0 \\ P = P' &\succ 0 \end{aligned}$$

where $\star$ denotes terms easily inferred from symmetry.

c) By defining $Y = PL$, the matrix inequalities become

$$\begin{aligned} \begin{bmatrix} PA + A'P + YC + C'Y' + I & PB_w \\ \star & -\gamma^2 I \end{bmatrix} &\prec 0 \\ P = P' &\succ 0 \end{aligned}$$

which are linear.

3. a) It is clear that we require $K(s)$ to be internally stabilising.

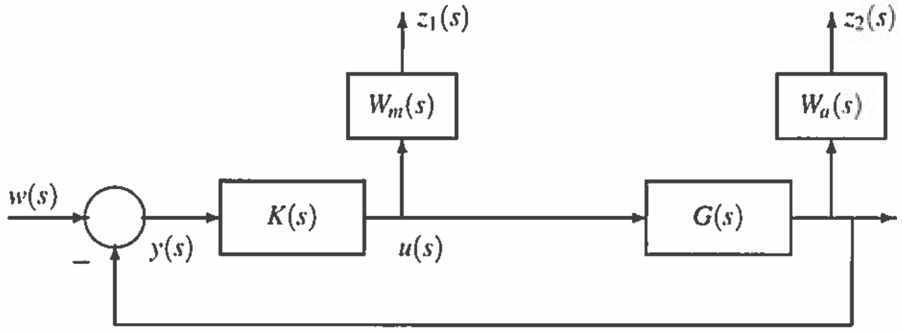
i) Suppose that $\Delta_a = 0$ and let the input to Δ_m be ε_2 while the output from Δ_m be δ_2. Then a calculation shows that $\varepsilon_2 = -K(I+GK)^{-1}\delta_2$. Using the small gain theorem, to satisfy the first design requirement, it is sufficient that $\left\|\Delta_m(j\omega)K(j\omega)\left(I+G(j\omega)K(j\omega)\right)^{-1}\right\| < 1, \forall\omega$. This can be satisfied if $\left\|W_mK(I+GK)^{-1}\right\|_\infty < 1$, where $W_m = w_mI$.

ii) An analogous procedure shows that to satisfy the second design requirement, it is sufficient that $\left\|\Delta_a(j\omega)G(j\omega)K(j\omega)\left(I+G(j\omega)K(j\omega)\right)^{-1}\right\| < 1, \forall\omega$. This can be satisfied if $\left\|W_aGK\left(I+GK\right)^{-1}\right\|_\infty < 1$, where $W_a = w_aI$.

Thus, to satisfy both design requirements, it is sufficient that

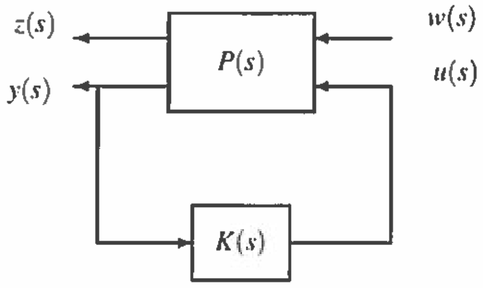
$$\left\| \begin{bmatrix} W_mK(I+GK)^{-1} \\ W_aGK\left(I+GK\right)^{-1} \end{bmatrix} \right\|_\infty < 1.$$

b) All the requested signals are shown in the block diagram shown below.



c) The corresponding generalised regulator formulation is to find an internally stabilising $K(s)$ such that $\|\mathscr{F}_l(P,K)\|_\infty < 1$ where

$$z(s) = \begin{bmatrix} z_1(s) \\ z_2(s) \end{bmatrix}, \; P(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} = \left[\begin{array}{c|c} 0 & W_m(s) \\ 0 & W_a(s)G(s) \\ \hline I & -G(s) \end{array}\right].$$



4. a) It is assumed that the triple (A,B,C) is minimal. A suitable Lyapunov function for regulating x is $V = x'Px$ where $P = P'$.

b) Set $u = -Fx$. Provided that $P = P' \succ 0$ and $\dot{V} < 0$ along closed-loop trajectories, we can assume $\lim_{t\to\infty} x(t) = 0$. Then $\dot{V} = \dot{x}'Px + x'P\dot{x} = x'(A'P + PA - F'B'P - PBF)x$. Using $x(\infty) = 0$, $\int_0^\infty x'(A'P + PA - F'B'P - PBF)x\,dt = -x_0'Px_0$.

c) Adding the last equation to the expression for J and completing a square:

$$J = x_0'Px_0 + \int_0^\infty \{x'[A'P + PA + C'C - PBB'P]x + \|(F - B'P)x\|^2\}dt.$$

Since the last term is always nonnegative, it follows that the minimizing value of F is given by $F = B'P$. We can set the term in square brackets to zero provided P satisfies the Riccati equation $A'P + PA + C'C - PBB'P = 0$. It follows that the minimum value of J is $x_0'Px_0$.

d) We prove that $A_c := A - BB'P$ is stable. The Riccati equation can be written as

$$A_c'P + PA_c + C'C + PBB'P = 0.$$

Let $\lambda \in \mathscr{C}$ be an eigenvalue of A_c and $y \neq 0$ be the corresponding eigenvector. Pre– and post–multiplying the Riccati equation be y' and y respectively gives $(\lambda + \bar{\lambda})y'Py + y'C'Cy + y'PBB'Py = 0$. Since $P \succ 0$ and $y \neq 0$, $y'Py > 0$. $y'C'Cy \geq 0$ and $y'PBB'Py \geq 0$. It follows that $\lambda + \bar{\lambda} \leq 0$ with $\lambda + \bar{\lambda} = 0$ if and only if $B'Py = 0$ and $Cy = 0$. But this implies $Ay = \lambda y$ and $Cy = 0$, which contradicts the observability of the pair (A,C). This proves that $\lambda + \bar{\lambda} < 0$ and so the closed loop is stable.

e) Now,

$$\begin{aligned} \dot{x} &= Ax + Bu = Ax + B(r - Fx) \\ &= (A - BF)x + Br \\ z &= Cx \end{aligned}$$

it follows that $T_{zr} \stackrel{s}{=} (A - BF, B, C, 0)$. It follows from the bounded real lemma that $||T_{zr}||_\infty < 1$ if there exists $P = P' \succ 0$ such that

$$\begin{bmatrix} P(A - BF) + (A - BF)'P + C'C & PB \\ B'P & -I \end{bmatrix} \prec 0$$

Using a Schur complement argument, this inequality is equivalent to

$$P(A - BF) + (A - BF)'P + C'C + PBB'P \prec 0.$$

However, it follows from the Riccati equation in Part b above that $P(A - BF) + (A - BF)'P + C'C + PBB'P = 0$. This proves that $||T_{zr}||_\infty < 1$.

5. a) i) The $(1,1)$ block of the inequality gives the inequality $A'P+PA \prec 0$. Let $z \neq 0$ be a right eigenvector of A and let λ be the corresponding eigenvalue. Then multiplying the inequality from the left by $\bar{z}'$ and from the right by z gives $(\lambda+\bar{\lambda})\bar{z}'Pz < 0$. Since $P \succ 0$ it follows that $\bar{z}'Pz > 0$ and it follows that $\lambda+\bar{\lambda} < 0$ so that A is stable.

ii) Since A is stable, $\|H\|_\infty < \gamma$ if and only if, with $x(0)=0$, $J := \int_0^\infty [y'y - \gamma^2 u'u]dt < 0$, for all $u(t)$ such that $\|u\|_2 < \infty$. If $\|u\|_2$ is bounded, then $\lim_{t\to\infty} x(t) = 0$. Now, $\int_0^\infty \frac{d}{dt}[x'Px]dt = x(\infty)'Px(\infty) - x(0)'Px(0) = 0$. So,

$$0 = \int_0^\infty (\dot{x}'Px + x'P\dot{x})\, dt = \int_0^\infty [x'(A'P+PA)x + x'PBu + u'B'Px]dt.$$

Use $y = Cx + Du$ and add the last expression to J

$$\begin{aligned} J &= \int_0^\infty [x'(A'P+PA+C'C)x + 2x'(PB+C'D)u + u'(D'D-\gamma^2 I)u]dt \\ &= \int_0^\infty \begin{bmatrix} x' & u' \end{bmatrix} \overbrace{\begin{bmatrix} A'P+PA+C'C & PB+C'D \\ B'P+D'C & D'D-\gamma^2 I \end{bmatrix}}^{M} \begin{bmatrix} x \\ u \end{bmatrix} dt. \end{aligned}$$

It follows that $J < 0$, and so $\|H\|_\infty < \gamma$, if $M \prec 0$. This proves the result.

b) i) The dynamics are described by

$$\dot{x}(t) = Ax(t) + Bw(t) + u(t), \qquad y(t) = Cx(t) + w(t).$$

Substituting $u(t) = Ly(t)$ into the state equation gives

$$\dot{x} = \underbrace{(A+LC)}_{A_c} x + \underbrace{(B+L)}_{B_c} w, \qquad y = \underbrace{C}_{C_c} x + \underbrace{I}_{D_c} w.$$

It follows that $T_{yw}(s) = D_c + C_c(sI - A_c)^{-1}B_c$.

ii) Using the results of part (a), by replacing A, B, C and D by A_c, B_c, C_c and D_c, we have that there exists a feasible L if there exists $P = P' \succ 0$ such that

$$\begin{bmatrix} (A+LC)'P + P(A+LC) + C'C & P(B+L) + C' \\ (B+L)'P + C & I - \gamma^2 I \end{bmatrix} \prec 0.$$

Noting that the only nonlinearity is due to the product PL, we define $Z = PL$ and so there exists a feasible L if there exists $P = P' \succ 0$ and Z such that

$$\begin{bmatrix} A'P + PA + ZC + C'Z' + C'C & PB + Z + C' \\ B'P + Z' + C & I - \gamma^2 I \end{bmatrix} \prec 0.$$

6. a) The generalized regulator formulation is given by

$$\begin{bmatrix} z(s) \\ y(s) \end{bmatrix} = P(s) \begin{bmatrix} w(s) \\ u(s) \end{bmatrix}, \; u(s) = Fy(s), \; P(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{c||c|c} A & B & B \\ \hline\hline I & 0 & 0 \\ 0 & 0 & I \\ \hline I & 0 & 0 \end{array}\right].$$

b) The requirement $\|H\|_\infty < \gamma$ is equivalent to $J := \|z\|_2^2 - \gamma^2 \|w\|_2^2 < 0$. Let $V = x'Xx$ and set $u = Fx$. Provided that $X = X' > 0$ and $\dot{V} < 0$ along the closed-loop trajectory, we can assume $\lim_{t \to \infty} x(t) = 0$. Then

$$\dot{V} = \dot{x}'Xx + x'X\dot{x} = x'\left(A'X + XA + F'B'X + XBFF\right)x + x'XBw + w'B'Xx.$$

Integrating from 0 to ∞ and using $x(0) = x(\infty) = 0$,

$$0 = \int_0^\infty [x'\left(A'X + XA + F'B'X + XBF\right)x + x'XBw + w'B'Xx]dt.$$

Using the definition of J and adding the last equation, $J =$

$$\int_0^\infty \{x'[A'X + XA + I + F'F + F'B'X + XBF]x - [\gamma^2 w'w - x'XBw - w'B'Xx]\}dt.$$

Let $Z = F + B'X$. Completing the squares by using

$$\begin{aligned} Z'Z &= F'F + F'B'X + XBF + XBB'X \\ \|(\gamma w - \gamma^{-1}B'Xx)\|^2 &= \gamma^2 w'w - w'B'Xx - x'XBw + \gamma^{-2}x'XBB'Xx, \end{aligned}$$

$$J = \int_0^\infty \{x'[A'X + XA + I - (1-\gamma^{-2})XBB'X]x + \|Zx\|^2 - \left\|\gamma w - \gamma^{-1}B'Xx\right\|^2\}dt.$$

Thus two sufficient conditions for $J < 0$ are the existence of X such that

$$A'X + XA + I - (1-\gamma^{-2})XBB'X = 0, \qquad X = X' > 0.$$

A feedback gain is $F = -B'X$, and the worst case disturbance is $w^* = \gamma^{-2}B'Xx$. The closed-loop is $\dot{x} = [A - (1-\gamma^{-2})BB'X]x$ and a third condition is therefore $Re\ \lambda_i[A - (1-\gamma^{-2})BB'X] < 0, \ \forall\, i$.

It remains to prove $\dot{V} < 0$ along state-trajectory with $u = Fx$ and $w = 0$. But

$$\dot{V} = x'\left(A'X + XA + F'B'X + XBF\right)x = -x'(I + (1+\gamma^{-2})XBB'X)x < 0$$

for all $x \neq 0$ (since (A, B) is assumed controllable) proving closed-loop stability.

c) Setting $\gamma = 1$, the Riccati equation reduces to the Lyapunov equation

$$A'X + XA + I = 0, \qquad X = X' > 0$$

and the closed–loop stability conditions become

$$Re\ \lambda_i[A] < 0, \ \forall\, i$$

$$-x'(I + 2XBB'X)x < 0$$

for all $x \neq 0$. It is immediate that these conditions are feasible when A is stable.

