1. a) Let the realisation be partitioned compatibly with P and Q as

$$G(s) \stackrel{s}{=} \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & 0 \end{array}\right].$$ Then

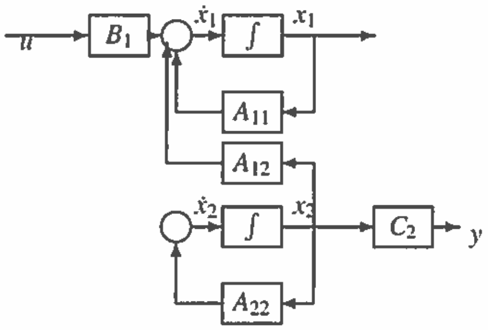
$$A^TQ+QA+C^TC = \begin{bmatrix} C_1^TC_1 & A_{21}^TQ_2+C_1^TC_2 \\ \star & A_{22}^TQ_2+Q_2A_{22}+C_2^TC_2 \end{bmatrix} = 0 \qquad (1.1)$$

$$AP+PA^T+BB^T = \begin{bmatrix} A_{11}P_1+P_1A_{11}^T+B_1B_1^T & P_1A_{21}+B_1B_2^T \\ \star & B_2B_2^T \end{bmatrix} = 0 \qquad (1.2)$$

It follows that $C_1=0$, $B_2=0$ and $A_{21}=0$. Thus the realisation decomposes into two subsystems $G_1: \dot{x}_1 = A_{11}x_1+B_1u+A_{12}x_2$ with n_1 modes and $G_2: \dot{x}_2 = A_{22}x_2$, $y=C_2x_2$ with n_2 modes. Since A is stable, both sub–systems are stable.

i) G_1 is unobservable since $C_1=0$. We prove controllability. Let $z'A_{11} = \lambda z'$, $z \neq 0$. It follows from the (1,1) entry of (1.2) that $(\lambda+\bar{\lambda})z'P_1z + z'B_1B_1'z = 0$. If $z'B_1=0$ then $\lambda+\bar{\lambda}=0$ which contradicts the assumption that A is stable. Thus the realisation is controllable.

ii) For the subsystem G_2 it is clear that all the modes are uncontrollable and observable using a dual argument to the above.



b) i) Using the result that A is stable if and only if there exists $P=P^T \succ 0$ such that $AP+PA^T \prec 0$ we have $\mathscr{S}$ is simultaneously stabilisable if and only if there exist $K \in \mathscr{R}^{m\times n}$ and $P_i = P_i^T \succ 0$ such that $(A_i+B_iK)P_i + P_i(A_i+B_iK)^T \prec 0$, $i=1,\dots,N$.

ii) The above conditions are nonlinear, but can be linearised by taking $P_i = P$, and defining a variable $\hat{K} := KP$ and so a set of sufficient conditions is: $\mathscr{S}$ is simultaneously stabilisable if there exist $\hat{K} \in \mathscr{R}^{m\times n}$ and $P=P^T \succ 0$ such that $A_iP+PA_i^T+B_i\hat{K}+\hat{K}^TB_i^T \prec 0$, $i=1,\dots,N$. If the conditions are satisfied, then $K = \hat{K}P^{-1}$.

2. a) i) The $(1,1)$ block of the inequality gives $A^TP + PA \prec 0$. Let $Az = \lambda z$, $z \neq 0$. Then multiplying the inequality from the left by z^T and from the right by z gives $(\lambda + \bar{\lambda})z^TPz < 0$. Since $P \succ 0$ it follows that $z^TPz > 0$ and it follows that $\lambda + \bar{\lambda} < 0$ so that A is stable.

ii) Since A is stable, $\|H\|_\infty < \gamma$ if and only if, with $x(0) = 0$, $J := \int_0^\infty [y^Ty - \gamma^2 u^Tu]dt < 0$, for all $u(t)$ such that $\|u\|_2 < \infty$. If $\|u\|_2$ is bounded, then $\lim_{t\to\infty} x(t) = 0$. Now, $\int_0^\infty \frac{d}{dt}[x^TPx]dt = x(\infty)^TPx(\infty) - x(0)^TPx(0) = 0$. So,

$$0 = \int_0^\infty (\dot{x}^TPx + x^TP\dot{x})\,dt = \int_0^\infty [x^T(A^TP+PA)x + x^TPBu + u^TB^TPx]dt.$$

Use $y = Cx + Du$ and add the last expression to J

$$\begin{aligned} J &= \int_0^\infty [x^T(A^TP+PA+C^TC)x + 2x^T(PB+C^TD)u + u^T(D^TD-\gamma^2I)u]dt \\ &= \int_0^\infty \begin{bmatrix} x^T & u^T \end{bmatrix} \overbrace{\begin{bmatrix} A^TP+PA+C^TC & PB+C^TD \\ B^TP+D^TC & D^TD-\gamma^2 I \end{bmatrix}}^{M} \begin{bmatrix} x \\ u \end{bmatrix} dt. \end{aligned}$$

It follows that $J < 0$, and so $\|H\|_\infty < \gamma$, if $M \prec 0$. This proves the result.

iii) Now, $H(s)^T = D^T + B^T(sI - A^T)^{-1}C^T$. It follows that the dual conditions are: there exists a $P = P^T \succ 0$ such that

$$\begin{bmatrix} AP + PA^T + BB^T & PC^T + BD^T \\ CP + DB^T & DD^T - \gamma^2 I \end{bmatrix} \prec 0.$$

b) i) Substituting $u(t) = Kx(t)$ into the dynamic equations gives

$$\dot{x}(t) = \underbrace{(A+BK)}_{A_c}x(t) + \underbrace{B_d}_{B_c}d(t), \qquad z(t) = \underbrace{(C+DK)}_{C_c}x(t) + \underbrace{D_d}_{D_c}d(t).$$

It follows that $T_{zd}(s) = D_c + C_c(sI - A_c)^{-1}B_c$.

ii) Using the results of Part (aiii), by replacing A, B, C and D by A_c, B_c, C_c and D_c, we have that there exists a feasible K if there exists $P = P^T \succ 0$ such that

$$\begin{bmatrix} (A+BK)P + P(A+BK)^T + B_dB_d^T & P(C+DK)^T + B_dD_d^T \\ (C+DK)P + D_dB_d^T & D_dD_d^T - \gamma^2 I \end{bmatrix} \prec 0.$$

Noting that the only nonlinearity is due to the product KP, we define $\hat{K} := KP$ and so there exists a feasible K if there exists $P = P^T \succ 0$ such that

$$\begin{bmatrix} AP + PA^T + B\hat{K} + \hat{K}^TB^T + B_dB_d^T & PC^T + \hat{K}^TD^T + B_dD_d^T \\ CP + D\hat{K} + D_dB_d^T & D_dD_d^T - \gamma^2 I \end{bmatrix} \prec 0.$$

3. a) It is assumed that the triple (A,B,C) is minimal and that D has full column rank (so that D^TD is nonsingular). A suitable Lyapunov function for regulating x is $V = x^TPx$ where $P = P^T$.

b) Set $u = -Fx$. Provided that $P = P^T \succ 0$ and $\dot{V} < 0$ along closed-loop trajectories, we can assume $\lim_{t\to\infty} x(t) = 0$. Then $\dot{V} = \dot{x}^TPx + x^TP\dot{x} = x^T\left(A^TP + PA - F^TB^TP - PBF\right)x$. Using $x(\infty) = 0$, $\int_0^\infty \dot{V}(t)dt = \int_0^\infty x^T\left(A^TP + PA - F^TB^TP - PBF\right)xdt = -x_0^TPx_0$.

c) Let $R = D^TD$. Adding the last equation to the expression for J and completing a square:

$$J = x_0^TPx_0 + \int_0^\infty \{x^T[A^TP + PA + C^TC - PBR^{-1}B^TP]x + \left\|D(F - R^{-1}B^TP)x\right\|^2\}dt.$$

Since the last term is always nonnegative, it follows that the minimizing value of F is given by $F = R^{-1}B^TP$. We can set the term in square brackets to zero provided P satisfies the Riccati equation $A^TP + PA + C^TC - PBR^{-1}B^TP = 0$. It follows that the minimum value of J is $x_0^TPx_0$.

d) We prove that $A_c := A - BR^{-1}B'P$ is stable. The Riccati equation can be written as

$$A_c'P + PA_c + C^TC + PBR^{-1}B'P = 0.$$

Let $\lambda \in \mathscr{C}$ be an eigenvalue of A_c and $y \neq 0$ be the corresponding eigenvector. Pre– and post–multiplying the Riccati equation be y' and y respectively gives $(\lambda + \bar{\lambda})y'Py + y'C'Cy + y'PBR^{-1}B'Py = 0$. Since $P \succ 0$ and $y \neq 0$, $y'Py > 0$, $y'C'Cy \geq 0$ and $y'PBR^{-1}B'Py \geq 0$. It follows that $\lambda + \bar{\lambda} \leq 0$ with $\lambda + \bar{\lambda} = 0$ if and only if $B'Py = 0$ and $Cy = 0$. But this implies $Ay = \lambda y$ and $Cy = 0$, which contradicts the observability of the pair (A,C). This proves that $\lambda + \bar{\lambda} < 0$ and so the closed loop is stable.

e) Now,

$$\begin{aligned} \dot{x} &= Ax + Bu = Ax + B(d - Fx) \\ &= (A - BF)x + Bd \\ z_1 &= Cx \end{aligned}$$

it follows that $T_{z_1d} \stackrel{s}{=} (A - BF, B, C, 0)$. It follows from the bounded real lemma that $||T_{z_1d}||_\infty < 1$ if there exists $P = P^T \succ 0$ such that

$$\begin{bmatrix} P(A - BF) + (A - BF)^TP + C^TC & PB \\ B^TP & -I \end{bmatrix} \prec 0$$

Using a Schur complement argument, this inequality is equivalent to

$$P(A - BF) + (A - BF)^TP + C^TC + PBB^TP \prec 0.$$

However, it follows from the Riccati equation in Part c above that $P(A - BF) + (A - BF)^TP + C^TC + PBB^TP = 0$. Ignoring issues relating to strict matrix inequalities, this proves that $||T_{z_1d}||_\infty \leq 1$.

4. a) The generalized regulator formulation is given by

$$\begin{bmatrix} z(s) \\ y(s) \end{bmatrix} = P(s) \begin{bmatrix} w(s) \\ u(s) \end{bmatrix}, \; u(s) = Fy(s), P(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{c||c|c} A & B_1 & B_2 \\ \hline C_1 & 0 & 0 \\ 0 & 0 & I \\ \hline I & 0 & 0 \end{array}\right].$$

b) Let $J = \|z\|_2^2 - \gamma^2 \|w\|_2^2$ denote the cost function. Let $V = x'Xx$ and set $u = Fx$. Provided that $X = X' \succ 0$ and $\dot{V} < 0$ along the closed-loop trajectory, we can assume $\lim_{t \to \infty} x(t) = 0$. Then

$$\dot{V} = \dot{x}'Xx + x'X\dot{x} = x'(A'X + XA + F'B_2'X + XB_2F)x + x'XB_1w_1 + w_1'B_1'Xx.$$

Integrating from 0 to ∞ and using $x(\infty) = 0$,

$$-x_0'Xx_0 = \int_0^\infty [x'(A'X + XA + F'B_2'X + XB_2F)x + x'XB_1w_1 + w_1'B_1'Xx]dt.$$

Using the definition of J, adding the last equation, and completing two squares,

$$J = x_0'Xx_0 + \int_0^\infty \{x'\left[A'X + XA + C_1'C_1 - X\left(B_2B_2' - \gamma^{-2}B_1B_1'\right)X\right]x + \|Zx\|^2 - \|v\|^2\}dt$$

where $Z = F + B_2'X$ and $v = \gamma w - \gamma^{-1}B_1'Xx$.

c) The requirement $\|H\|_\infty \leq \gamma$ is equivalent to $J \leq 0$ when $x_0 = 0$. Thus sufficient conditions for $J \leq 0$ are $Z = 0$ and the existence of X such that

$$A'X + XA + C_1'C_1 - X\left(B_2B_2' - \gamma^{-2}B_1B_1'\right)X = 0, \qquad X = X' \succ 0.$$

Setting $Z = 0$, the control policy is $F = -B'X$. Setting $v = 0$, the worst case disturbance is $w^* = \gamma^{-2}B_1'X$. The closed-loop with $u = Fx$ and $w = w^*$ is $\dot{x} = A_cx$ where $A_c = A - \left(B_2B_2' - \gamma^{-2}B_1B_1'\right)$ and a further condition is $Re\ \lambda_i(A_c) < 0, \ \forall\ i$.

d) Using the expression for J in Part b) and the solution in Part c) with $\gamma \to \infty$, it follows that $J \leq x_0^TXx_0$ and the Riccati equation becomes

$$A'X + XA + C_1'C_1 - XB_2B_2'X = 0, \qquad X = X' \succ 0.$$

Thus, as $\gamma \to \infty$, the solution of the $\mathscr{H}_\infty$ regulator converges to that of the LQR in Question 3 (with $C = C_1, B = B_2$ and $D = I$).

