

SOLUTIONS: Control Engineering

1. a) i) Let $z(t)$ be the position of the box. The force equations are

$$f = K_1 z + D \dot{z} + K_2(z - y), \quad M \ddot{y} + K_2(y - z) = 0.$$

Taking Laplace transforms, substituting and eliminating z gives

$$G(s) = \frac{1}{s^3 + (1 + K_1)s^2 + s + K_1}.$$

- ii) The Routh array is:

$$\begin{array}{c|cc} s^3 & 1 & 1 \\ s^2 & 1 + K_1 & K_1 \\ s & \frac{1}{1 + K_1} & \\ 1 & K_1 & \end{array}$$

So $K_1 > 0$ for positive signs for the first column and therefore stability.

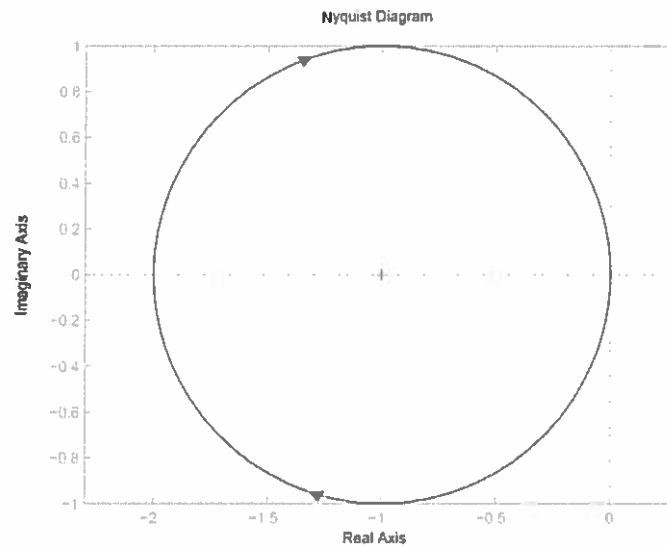
- iii) When $K_1 = 0$ the array has a zero entry in the first column corresponding to marginal stability. Substituting $K_1 = 0$ into $G(s)$ gives the poles as the roots of $s(s^2 + s + 1)$ which are $0, -0.5 \pm j0.5\sqrt{3}$.

- iv) Using the final value theorem and the fact that $f(s) = 1/s$,

$$y_{ss} := \lim_{t \rightarrow \infty} y(t) = \lim_{s \rightarrow 0} s y(s) = \lim_{s \rightarrow 0} s G(s) f(s) = \lim_{s \rightarrow 0} \frac{s G(s)}{s} = G(0) = \frac{1}{K_1}.$$

So for $y_{ss} = 2$, $K_1 = 0.5$.

- b) i) The Nyquist diagram is shown below.



- ii) The Nyquist criterion states that $N = Z - P$, where N is the number of clockwise encirclements by the Nyquist diagram of the point $-k^{-1}$, P is the number of unstable open-loop poles and Z is the number of unstable closed-loop poles. Since $G(s)$ has one unstable pole, $P = 1$.

- When $-\infty < k < 0.5$, $N = 0$ so $Z = 1$.
- When $0.5 < k < \infty$, $N = 1$ so $Z = 2$.
- When $k = 0.5$, the closed-loop is $\frac{0.5G(s)}{1 + 0.5G(s)} = \frac{1}{s^3}$ and so there are three closed-loop poles at the origin.

- iii) A PD compensator has the form $K(s) = k(s + z)$. The characteristic equation for the closed-loop is

$$1 + K(s)G(s) = 1 + \frac{2k(s+z)}{s^3-1} = 0 \Rightarrow s^3 + 2ks + 2kz - 1 = 0$$

Since the coefficient of s^2 is zero, the closed-loop cannot be stabilised.

- iv) Since $G(s) = \frac{2}{(s-1)(s^2+s+1)}$ then $G(s)K(s) = \frac{2k}{(s-1)(s^2+2s+3)}$. The characteristic equation for the closed-loop is

$$1 + K(s)G(s) = 1 + \frac{2k}{(s-1)(s^2+2s+3)} = 0 \Rightarrow s^3 + s^2 + s + 2k - 3 = 0$$

The Routh array:

s^3	1	1
s^2	1	$2k - 3$
s	$2(2 - k)$	
1	$2k - 3$	

For stability, we need $1.5 < k < 2$ which is clearly possible.

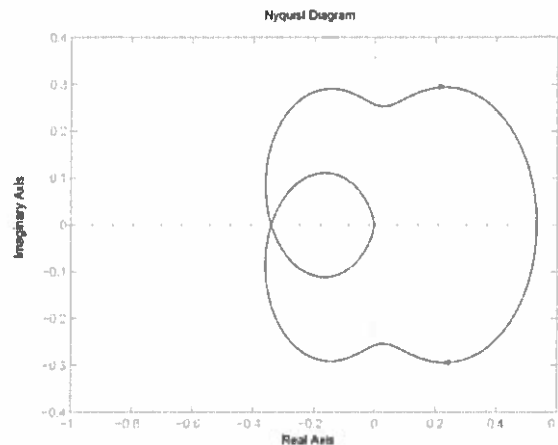
2. a) The characteristic equation for the closed-loop is

$$1 + KG(s) = 1 + \frac{K}{s^3 + as^2 + bs + c} = 0 \Rightarrow s^3 + as^2 + bs + c + K = 0$$

The Routh array:

$$\begin{array}{c|cc} s^3 & 1 & b \\ s^2 & a & c+K \\ s & b - \frac{c+K}{a} & \\ 1 & c+K & \end{array}$$

The real-axis intercepts: $0, -\frac{1}{ab-c}, 1/c$. A typical Nyquist diagram is:



- b) We have $N = Z - P$, where N is the number of clockwise encirclements by the Nyquist diagram of $-K^{-1}$, P is the number of unstable poles of G and Z is the number of unstable closed-loop poles. To find P , the Routh array for $G(s)$:

$$\begin{array}{c|cc} s^3 & 1 & b \\ s^2 & a & c \\ s & b - \frac{c}{a} & \\ 1 & c & \end{array}$$

Since $a > 0$, $c > 0$ and $ab > c$ then $G(s)$ is always stable and so $P = 0$.

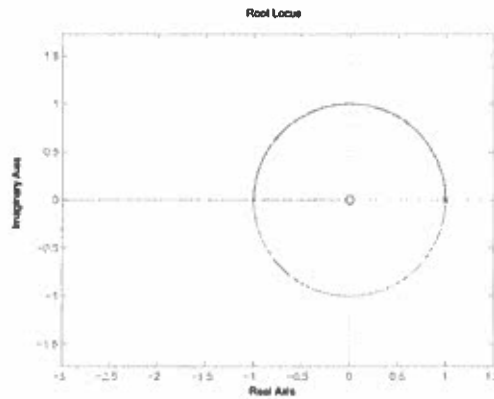
- When $-\infty < K < -c$, $N = 1$ so $Z = 1$.
 - When $-c < K < ab - c$, $N = 0$ so $Z = 0$.
 - When $ab - c < K < \infty$, $N = 2$ so $Z = 2$.
 - When $K = -c$, the closed-loop is marginally stable (a pole at 0)
 - When $K = ab - c$: the closed-loop is marginally stable (poles at $\pm j\sqrt{b}$)
- c) The gain margin is $ab - c$ since the real-axis intercept is at $\frac{1}{ab-c}$.
- d) Since $ab - c \geq 2$, the gain margin is at least 2 for all parameter values.
- e) The gain and phase margins are adequate and we expect good transient responses. However, the DC gain $G(0) = 1/c$ is less than 1 and so we need to improve the steady state performance. Since phase-lag compensation increases low frequency gain, and hence improve steady-state tracking it follows that the system requires phase-lag compensation.

3. a) For a maximum overshoot of 5% and a settling time of 4 seconds the closed-loop poles must be placed at $s_0, \bar{s}_0 = -1 \pm j$.

b) For $z=0$, the closed-loop characteristic equation is

$$1 + G(s) = 0 \Rightarrow 1 - \frac{2s}{s^2 + ks + 1} = 0 \Rightarrow s^2 - 2s + 1 + ks = 0 \Rightarrow 1 + k \frac{\overbrace{s}^{\hat{G}(s)}}{(s-1)^2} = 0$$

i) We plot the root locus of \hat{G} :



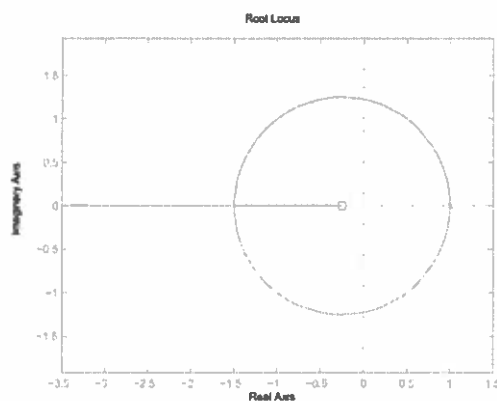
ii) Thus a settling time of 4s is only achievable with the closed-loop poles set at -1 and so the response is not oscillatory. The corresponding k is obtained from the gain criterion as $k = -1/\hat{G}(-1) = 4$.

c) For general z , proceeding as before, the closed-loop characteristic equation is

$$1 + G(s) = 0 \Rightarrow 1 - \frac{2s}{s^2 + k(s+z) + 1} = 0 \Rightarrow s^2 - 2s + 1 + k(s+z) = 0 \Rightarrow 1 + k \frac{\overbrace{s+z}^{\hat{G}(s)}}{(s-1)^2} = 0$$

i) $\hat{G}(s)$ has two poles at 1 and a zero at $-z$. Let the angle from $s_0 = -1 + j$ to $-z$ be θ and to 1 be θ_1 . The angle criterion requires $\theta = 2\theta_1 - 180^\circ$ and after some trigonometry this gives $z = 0.25$.

ii) For $z = 0.25$, the root-locus of $\hat{G}(s)$ is shown below.



iii) The gain criterion requires $k = -1/\hat{G}(s_0)$ and so $k = 4$.