Imperial College London

BSc/MSci EXAMINATION May 2012

This paper is also taken for the relevant Examination for the Associateship

ASTROPHYSICS

For 3rd-Year Physics Students

Wednesday, 23rd May 2012: 10:00 to 12:00

The paper consists of **two** *sections: A & B.*
Section A contains one question, comprising small parts. [20 marks total]
Section B will contain four questions on selected parts of the course. [15 marks each]

Candidates are required to:
Answer **ALL** *parts of Section A and* **TWO** *questions from Section B.*
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the 3 answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in 3 answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

CONSTANTS FOR ENTIRE PAPER:

$G = 6.7 \times 10^{-11}$ N m^2 kg^{-2}
$c = 3.0 \times 10^8$ m s^{-1}
$h = 6.6 \times 10^{-34}$ m^2 kg s^{-1}
$k = 1.4 \times 10^{-23}$ J K^{-1}
$\sigma = 5.7 \times 10^{-8}$ W m^{-2} T^{-4}
$m_e = 9.1 \times 10^{-31}$ kg (mass of electron)
$m_p = 1.7 \times 10^{-27}$ kg (mass of proton)
$M_\odot = 2.0 \times 10^{30}$ kg (solar mass)
$R_\odot = 7.0 \times 10^8$ m (solar radius)
$L_\odot = 3.9 \times 10^{26}$ W (solar luminosity)
1 day = 8.64×10^4 s
1 yr = 3.15×10^7 s
1 parsec (pc) = 3.1×10^{16} m
1 Astronomical Unit (AU) = 1.5×10^{11} m

SECTION A

1. (i) Explain, in words, the 3 conditions that must hold for a system to be in Thermodynamic Equilibrium. [3 marks]

(ii) **(a)** For a general potential energy (due to particle interactions) of the form $V[r] \propto r^{-n}$, where r is the inter-particle distance, what is the relationship between the kinetic and potential energies (as implied by the Virial Theorem)? **(b)** What does this relationship become for gravitational potential energy? [2 marks]

(iii) State the dependence, as a proportionality, of the Jeans mass for gravitational collapse (M_J) on the temperature (T) and density (ρ) of the region (assume magnetic fields are absent). [1 mark]

(iv) Briefly explain how *opacity limited fragmentation* arises within an initially isothermal gravitationally unstable core. [2 marks]

(v) Briefly explain, in words, the concepts **(a)** *magnetically sub-critical* and **(b)** *magnetically super-critical*, in the context of gravitational collapse in a molecular cloud in the presence of magnetic fields. [4 marks]

(vi) Briefly explain, in words, the concepts of **(a)** *radial drift* and **(b)** *runaway growth*, in the context of planet formation via core accretion in accretion disks.

[2 marks]

(vii) State the dependence, as a proportionality, of pressure P on density ρ for **(a)** non-relativistic degenerate electrons and **(b)** relativistic degenerate electrons.

[2 marks]

(viii) The "no hair" theorem states that black holes are completely specified by just 3 parameters. Name these 3 parameters. [1 mark]

(ix) Derive an expression for the Eddington luminosity of a black hole, by equating the radiation force on electrons (with a Thompson photon scattering cross-section σ_T) to the gravitational force on protons. [3 marks]

[Total 20 marks]

SECTION B

2. Imagine a clump of gas of mass M that is gravitationally collapsing, with an instantaneous radius given by R. Non-relativistic electron degeneracy pressure becomes important within this clump when the non-relativistic momentum of its electrons ($p_e = \sqrt{3m_e kT_e}$, where m_e and T_e are the electron mass and temperature respectively) becomes less than the Fermi momentum ($p_F = h(3n_e/8\pi)^{1/3}$, where n_e is the number density of electrons).

(i) Use the above condition to derive an expression for the number density of electrons, $n_{e,degen}$, at which non-relativistic degeneracy becomes important. [2 marks]

(ii) Assume charge neutrality (i.e., $n_e = n_p$, where the latter is the number density of protons), and that the mass density of the region is given by $\rho \approx m_p n_p \approx M/R^3$ (where m_p is the mass of a proton). Use your answer in (i) above to derive an expression for the temperature $T_{e,degen}$ at which non-relativistic degeneracy becomes important, in terms of M, R, m_e and m_p. [2 marks]

(iii) Write down expressions for **(a)** the total gravitational potential energy and **(b)** the total kinetic energy, for the collapsing clump. You may ignore numerical factors of order unity. [2 marks]

(iv) Assuming that the clump is collapsing very slowly, apply the Virial Theorem in equilibrium to your answer in (iii) above, to derive an expression for the instantaneous temperature T within the clump. [3 marks]

(v) The temperature required for Hydrogen fusion is $T_H \approx 5 \times 10^6$ K. Derive a numerical value for the critical clump mass, M_{crit}, at which the clump temperature T (derived in (iv) above) simultaneously equals the temperature required for degeneracy (T_{degen} derived in (ii) above) and the hydrogen ignition temperature. Express M_{crit} in units of solar masses. [4 marks]

(vi) What are objects with masses less than M_{crit} called? How are they supported against gravity? [2 marks]

[Total 15 marks]

3. Consider a patch of gas of size R within an accretion disk, at a radial distance r from a central star of mass M_*. Assume that the surface density of the disk at that location is $\Sigma(r)$, the sound speed there is $c_s(r)$ and the Keplerian angular velocity there is $\Omega(r)$.

(i) Using the equation of radial force balance in a Keplerian disk, express $\Omega(r)$ in terms of M_*, r and G. [1 mark]

(ii) Derive approximate expressions for: **(a)** the free-fall time t_{ff} of the patch (the timescale for the patch to dynamically collapse due to its self-gravity), **(b)** the sound-crossing time t_s across the patch (the timescale for thermal pressure to act against any compression), and **(c)** the rotational stabilization time t_{rot} (the timescale for rotation to act against any compression, via the conservation of angular momentum). [6 marks]

(iii) The Toomre criterion for the patch to gravitationally collapse is: $t_{ff}^2/(t_s t_{rot}) < 1$. Substituting in the expressions you derive for t_{ff}, t_s and t_{rot} in (ii), show that this criterion is equivalent to: $c_s(r)\Omega(r)/G\Sigma(r) < 1$. [2 marks]

(iv) Assume that the surface density has a power-law radial profile: $\Sigma(r) = \Sigma_0(r/r_0)^{-n}$, where Σ_0 is some constant specified surface density at some fixed disk radius r_0, and $n < 2$ for realistic disks.
Recall further that the total disk mass enclosed within a radius r is $M_d(r) = \int_0^r \Sigma(r')2\pi r' dr'$ (where we have assumed that the disk inner radius is ~ 0, and r' is a dummy integration variable for radius).
Remember finally that the density scale height of the disk at a radius r is given by $z_H(r) \approx c_s(r)/\Omega(r)$.
Using these and the expression for $\Omega(r)$ derived in (i), show that the Toomre criterion for gravitational collapse derived in (iii) reduces to the condition:

$$\frac{M_d(r)}{M_*} > \left(\frac{2\pi}{2-n}\right)\frac{z_H(r)}{r}$$

[3 marks]

(v) The planets in our Solar System all formed out of the accretion disk that existed around the young Sun. The surface density estimated for this disk is given by: $\Sigma(r) = \Sigma_0(r/r_0)^{-3/2}$, with $r_0 = 1$ AU and $\Sigma_0 \sim 1.7 \times 10^4$ kg m^{-2}. Using the equation for $M_d(r)$ in terms of $\Sigma(r)$ supplied in (iv), estimate the mass of this disk enclosed within a radius of $r = 5$ AU (the radius of Jupiter's orbit). [2 marks]

(vi) Inserting your answer into the form of the Toomre criterion derived in (iv), and noting that in standard disks $z_H(r)/r \sim 0.1$, comment on whether or not Jupiter is expected to have formed through gravitation collapse. [1 mark]

[Total 15 marks]

4. For degenerate electrons, the Fermi momentum is given by: $p_F = h(3n_e/8\pi)^{1/3}$, where n_e is the number density of electrons. The corresponding electron pressure is: $P = (8\pi/3h^3)\int_0^{p_F} \mathrm{v}\, p^3 dp$, where v is the electron velocity and p its momentum.

(i) Using the above, derive an expression for the pressure P due to ultra-relativistic (v $\sim$ c) degenerate electrons, in terms of the electron number density n_e. [2 marks]

(ii) Assuming charge neutrality ($n_e = n_p$, where n_p is the number density of protons), and that the mass density is given by $\rho \sim m_p n_p$ (where m_p is the mass of a proton), express the pressure derived in (i) in terms of ρ. [1 mark]

(iii) Consider a star of mass M and radius R, and thus density $\rho \sim M/R^3$, supported by ultra-relativistic degenerate electron pressure. Using the Virial Theorem and the expression for P derived in (ii), derive an expression for the critical mass at which such a star can remain in equilibrium. (You may drop numerical factors of order unity.) [6 marks]

(iv) **(a)** What are objects that exceed this critical mass called? **(b)** How are such objects supported against gravity? [2 marks]

(v) A non-rotating black hole of mass M is described by the Schwarzschild metric:

$$-c^2 d\tau^2 = -\left(1 - \frac{R_S}{r}\right) c^2 dt^2 + \frac{dr^2}{(1 - \frac{R_S}{r})} + r^2 d\theta^2 + r^2 \cos^2\theta\, d\phi^2$$

where $R_S \equiv 2GM/c^2$ is the event horizon (equal to the Schwarzschild radius) for such a black hole.

Show that a photon of any finite frequency emitted at the event horizon is red-shifted to zero frequency as measured by an observer at infinity. (Hint: since the start and end of the photon emission occur at the same location, you can assume that dr, $d\theta$ and $d\phi$ are all zero). [4 marks]

[Total 15 marks]

5. An object D of mass M, at a distance $D_d = 3\,\text{kpc}$ (kiloparsec) from the Earth, passes directly in front of a background star S located in the galactic Bulge, at a distance $D_s = 8\,\text{kpc}$ from the Earth (so that the distance between the object D and the star S is $D_{ds} = 5\,\text{kpc}$). See diagram below.

(i) Noting from the diagram that the segment AI equals the sum of the segments AS and SI, and using the small-angle approximation, derive the lens equation relating the angles β, θ and $\hat{\alpha}$ (where β is the angle subtended by the true position of the source, θ is the angle subtended by the observed position of the image, and $\hat{\alpha}$ is the deflection angle).
(NOTE: Be careful not to equate θ to $\hat{\alpha}$!). [3 marks]

(ii) Using the small-angle approximation $\theta = R/D_d$, and the fact that the deflection angle in general relativity is given by $\hat{\alpha} = 4GM/(c^2R)$, express the lens equation as a quadratic equation in θ. [3 marks]

(iii) Using the substitution:

$$\theta_E^2 \equiv \frac{D_{ds}}{D_d\,D_s}\frac{4GM}{c^2}$$

where θ_E is known as the angular Einstein radius, derive the two solutions $\theta_\pm$ for θ, which give the observed positions of the primary and secondary images.

[3 marks]

(iv) The physical diameter of the Einstein ring is given by $2D_d\theta_E$. Using this, and the definition of θ_E in (iii), derive an expression for the time t_E it takes for the Einstein ring to cross the source, if the lens D has a transverse velocity v. (The time t_E essentially expresses how long the lensing event lasts). [2 marks]

(v) Suppose we observe that the transverse velocity of the lens is $v = 100\,\text{km}\,\text{s}^{-1}$, and $t_E = 2\,\text{days}$. **(a)** Calculate the mass M of the lens. **(b)** Based on its mass, would you classify the lens object as a star, brown dwarf, or planet?
(NOTE: The mass of Jupiter is $M_{JUPITER} \sim 10^{-3}\,M_\odot$, and the dividing line between stars and brown dwarfs is at roughly $75\,M_{JUPITER}$.)

[4 marks]

[Total 15 marks]

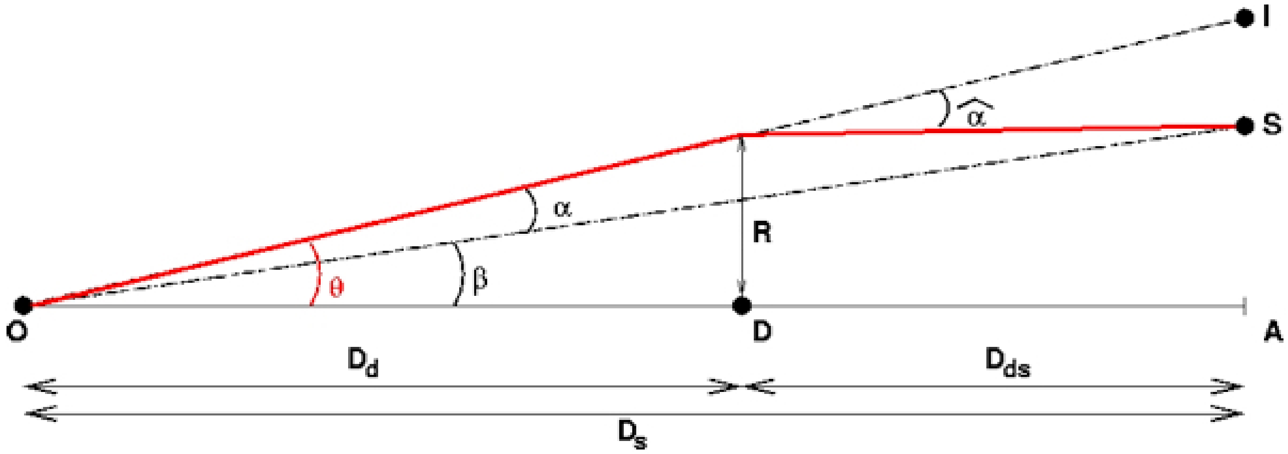

Figure 1: Figure for SECTION B, Question [5].