

Paper Number(s): **E3.08**
ISE3.17

IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE
UNIVERSITY OF LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2001

EEE/ISE PART III/IV: M.Eng., B.Eng. and ACGI

ADVANCED SIGNAL PROCESSING

Wednesday, 16 May 10:00 am

There are FIVE questions on this paper.

Answer ONE question from Section A, and TWO from Section B.

Use the same answer book for each section.

Time allowed: 3:00 hours

Examiners: Ward, D.B. and Constantinides, A.G.

Corrected Copy

N.~

Special instructions for invigilators:

One main answer book only is needed on each desk (not one each for Sections A and B).

Information for candidates:

Write your answers for Sections A and B in the same answer book.

Section A

1.

(a) Explain what is meant by the following terms used to describe a parameter estimator

- (i) unbiased (ii) minimum variance (iii) efficient

[5 marks]

(b) A linear model is given by

$$\underline{x} = H\underline{\theta} + \underline{w}$$

where \underline{x} is an $N \times 1$ column vector of data observations, H is a known $N \times p$ observation matrix, with $N > p$ and full column rank, $\underline{\theta}$ is an unknown $p \times 1$ parameter vector and \underline{w} is an $N \times 1$ vector with multivariate normal distribution $N(\underline{0}, \sigma^2 \underline{I})$.

- (i) Determine the minimum variance unbiased (MVU) estimator $\hat{\underline{\theta}}$, given the observation \underline{x} . **[10 marks]**

- (ii) Calculate the covariance matrix of $\hat{\underline{\theta}}$. **[2 marks]**

(c) The observations from a seismic sensor are assumed to satisfy

$$x[n] = \sum_{i=1}^p A_i (r_i)^n + w[n] \quad n = 0, 1, \dots, N-1$$

where $w[n]$ is zero mean white normally distributed noise with variance σ^2 .

- (i) Show how these can be put in the form of the linear model in (b). **[3 marks]**

- (ii) Evaluate the MVU estimator of the amplitudes, A_i , when $p = 2$, $r_1 = 1$, $r_2 = -1$, and N is even.

[5 marks]

2.

(a) Define the power spectral density of a wide sense stationary discrete time random signal and comment upon its properties. **[3 marks]**

(b) A linear shift-invariant discrete time system with z-domain transfer function

$$H(z) = \frac{1 - \frac{1}{2} z^{-1}}{1 - \frac{1}{3} z^{-1}}$$

is excited by zero mean exponentially correlated noise $x[n]$, which has autocorrelation sequence

$$r_{xx}(k) = \left(\frac{1}{2}\right)^{|k|}.$$

Denoting the output of the system by $y[n]$, which is calculated from $y[n] = x[n] * h[n]$, where $*$ represents discrete time convolution, then

(i) Calculate the z-domain power spectrum, $P_{yy}(z)$, of $y[n]$. **[3 marks]**

(ii) Evaluate the autocorrelation sequence, $r_{yy}(k)$, of $y[n]$. **[3 marks]**

(iii) Determine the cross-correlation sequence, $r_{xy}(k)$, between $x[n]$ and $y[n]$. **[10 marks]**

(iv) Evaluate the z-domain cross-power spectral density, $P_{xy}(z)$. **[2 marks]**

(c) Design a whitening filter for $y[n]$. **[4 marks]**

Section B

3.

(a) Summarize the difference between the problems of filtering, smoothing and prediction. **[5 marks]**

(b) The output of a forward prediction error filter at discrete time N is given by

$$e[N] = x[N] + \hat{x}[N] = x[N] + \sum_{k=1}^N a[k]x[N-k].$$

(i) Formulate, in matrix form, the parameter vector

$\underline{a}_{\text{opt}} = [a_{\text{opt}}[1], a_{\text{opt}}[2], \dots, a_{\text{opt}}[N]]^T$ which minimizes the mean square error $E\{e[N]^2\}$.

The discrete time random input signal $x[n]$ is zero mean and wide sense stationary.

[8 marks]

(ii) Evaluate the corresponding minimum mean squared error.

[2 marks]

(c) Given that the autocorrelation function of $x[n]$ is

$$r_{xx}[k] = \frac{\sigma^2}{1 - \rho^2} (-\rho)^{|k|} \quad \forall k, \quad \rho \in (-1, 1)$$

find the optimal in the minimum mean square error sense parameters and the corresponding mean square error of the forward prediction error filter for $N = 1$ and 2 , and comment upon the results. **[5 marks]**

(d) Show how the elements of the optimal parameter vector for a forward predictor are related to those of a backward prediction error filter with output error of the form

$$e_b[N] = x[0] + \hat{x}[N] = x[0] + \sum_{k=1}^N a[k]x[k].$$

[5 marks]

4.

(a) Explain the difference between the term estimate and estimator of an unknown parameter θ . **[3 marks]**

(b) Given that the joint probability density function of the measurement vector $\underline{x} = [x[0], x[1], \dots, x[N-1]]^T$, parameterised by the unknown scalar parameter θ , i.e. $p(\underline{x}; \theta)$, satisfies the regularity condition

$$E\left[\frac{\partial \ln p(\underline{x}; \theta)}{\partial \theta}\right] = 0 \quad \forall \theta$$

and the Cauchy-Schwartz inequality

$$\left[\int w(\underline{x})g(\underline{x})h(\underline{x})d\underline{x}\right]^2 \leq \int w(\underline{x})g^2(\underline{x})d\underline{x} \int w(\underline{x})h^2(\underline{x})d\underline{x}$$

where $g(\cdot)$ and $h(\cdot)$ are arbitrary functions, and $w(\underline{x}) \geq 0$ for all \underline{x} ,

prove that the variance of any unbiased estimator of θ , i.e. $\hat{\theta}$, must satisfy

$$\text{var}(\hat{\theta}) \geq \frac{1}{-E\left[\frac{\partial^2 \ln p(\underline{x}; \theta)}{\partial \theta^2}\right]}.$$

[13 marks]

(c) Determine the Cramer Rao Lower Bound for an unbiased estimator of the phase, ϕ , within the model $x[n] = A\cos(2\pi f_0 n + \phi) + w[n]$, $n = 0, 1, \dots, N-1$, where $w[n]$ is zero mean white Gaussian noise with variance σ^2 . The amplitude A and frequency f_0 are assumed to be fixed.

[7 marks]

(d) Comment upon the existence of an efficient estimator of the phase in the model in (c). **[2 marks]**

5.

(a) Describe the difference between the linear and nonlinear least squares problems.

[3 marks]

(b) Find the least squares estimator and the corresponding minimum least squares error for the parameter A in the signal model

$$s[n] = \begin{cases} A & 0 \leq n \leq M-1 \\ -A & M \leq n \leq N-1 \end{cases}$$

given the observation $x[n] = s[n] + w[n]$ for $n = 0, 1, \dots, N-1$.

[10 marks]

(c) Calculate the probability density function of the estimator in (b) if $w[n]$ is zero mean white Gaussian noise with variance σ^2 .

[5 marks]

(d) Formulate a least squares estimator for θ given the observations

$$x[n] = \exp(\theta) + w[n] \quad n = 0, 1, \dots, N-1$$

[7 marks]

- 1 a) (i) Unbiased - $E\{\hat{\theta}\} = \theta$; $\theta \in [a, b]$ (Entire parameter range)
 $\hat{\theta}$ - Estimator, θ - True value
- (ii) Minimum mean square error: $\min E\{(\hat{\theta} - \theta)^2\}$; generally leads to an estimator which would be a function of the true value - impractical. Hence, constrain estimator to be unbiased, then $MSE = VAR$, and therefore minimum variance = minimum mean square error.
- (iii) Efficient - The minimum variance solution attains the CRLB for $\theta \in [a, b]$ - no other unbiased estimator has a lower error variance. (5)

b) (i) $p(\underline{x}; \underline{\theta}) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left(-\frac{1}{2\sigma^2} (\underline{x} - H\underline{\theta})^T (\underline{x} - H\underline{\theta})\right)$

$$\frac{\partial \ln p(\underline{x}; \underline{\theta})}{\partial \underline{\theta}} = \frac{\partial}{\partial \underline{\theta}} \left[-\ln(2\pi\sigma^2)^{N/2} - \frac{1}{2\sigma^2} (\underline{x} - H\underline{\theta})^T (\underline{x} - H\underline{\theta}) \right] = \frac{1}{\sigma^2} [H^T \underline{x} - H^T H \underline{\theta}]$$

$$I(\underline{\theta}) \Rightarrow \frac{\partial}{\partial \underline{\theta}} \left(\frac{\partial \ln p(\underline{x}; \underline{\theta})}{\partial \underline{\theta}} \right)^T = -\frac{H^T H}{\sigma^2} \Rightarrow I(\underline{\theta}) = \frac{H^T H}{\sigma^2}$$

Therefore, using CRLB theory

$$\frac{\partial \ln p(\underline{x}; \underline{\theta})}{\partial \underline{\theta}} = \frac{H^T H}{\sigma^2} \left[\frac{(H^T H)^{-1} H^T \underline{x}}{g(\underline{x})} - \underline{\theta} \right] \Rightarrow \hat{\underline{\theta}}_{MML} = \underbrace{(H^T H)^{-1} H^T \underline{x}}_{\text{Invertible, since } H \text{ has full column rank}} \quad (10)$$

(ii) $\underline{C}_{\hat{\underline{\theta}}} = I^{-1}(\underline{\theta}) = \sigma^2 (H^T H)^{-1} \quad (2)$

c) (i) $\begin{bmatrix} \underline{x}[0] \\ \underline{x}[1] \\ \vdots \\ \underline{x}[N-1] \end{bmatrix} = \begin{bmatrix} (r_1)^0 & (r_2)^0 & \dots & (r_p)^0 \\ (r_1)^1 & (r_2)^1 & \dots & (r_p)^1 \\ \vdots & \vdots & \ddots & \vdots \\ (r_1)^{N-1} & (r_2)^{N-1} & \dots & (r_p)^{N-1} \end{bmatrix} \begin{bmatrix} A_1 \\ \vdots \\ A_p \end{bmatrix} + \begin{bmatrix} w[0] \\ w[1] \\ \vdots \\ w[N-1] \end{bmatrix}$
 $\underline{x} = H \underline{\theta} + \underline{w} \quad \underline{w} \sim \mathcal{N}(\underline{0}, \sigma^2 \underline{I}) \quad (3)$

(ii) $\hat{\underline{\theta}} = \begin{bmatrix} \hat{A}_1 \\ \hat{A}_2 \end{bmatrix} = (H^T H)^{-1} H^T \underline{x}$ where $H = \begin{bmatrix} 1 & -1 \\ \vdots & \vdots \\ 1 & -1 \end{bmatrix} \Rightarrow H^T H = \begin{bmatrix} N & 0 \\ 0 & N \end{bmatrix}$
 $\hat{\underline{\theta}} = \frac{1}{N} \begin{bmatrix} \sum_{i=0}^{N-1} x[i] \\ \sum_{i=0}^{N-1} (-1)^i x[i] \end{bmatrix} \quad (5)$

$$2) (a) P(f) = \sum_{k=-\infty}^{\infty} r_{xx}[k] e^{-j2\pi f k} \quad f \in [-\frac{1}{2}, \frac{1}{2}) \quad (3)$$

Properties - Real, even and non negative

$$(b) (i) \underbrace{\frac{\frac{3}{4}}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{2}z)}}_{\text{Driving, correlated process}} \cdot \frac{(1-\frac{1}{2}z^{-1})(1-\frac{1}{2}z)}{(1-\frac{1}{3}z^{-1})(1-\frac{1}{3}z)} = \frac{\frac{3}{4}}{(1-\frac{1}{3}z^{-1})(1-\frac{1}{3}z)} = P_{xx}(z) \cdot |H(z)|^2 = P_{yy}(z) \quad (3)$$

[First order AR process]

$$(ii) r_{yy}(k) = \frac{\frac{3}{4}}{1-\frac{1}{9}} \left[\frac{1}{3}\right]^{|k|} \quad (3)$$

$$(iii) r_{xy}(k) = E\{x[n]y[n+k]\} = h[k] * r_{xx}[k]$$

$$h[k] = \frac{1}{3}^k u[k] - \frac{3}{2} \left(\frac{1}{3}\right)^k u[k-1]$$

$$r_{xx}(k) = \left(\frac{1}{2}\right)^{|k|}$$

$$\frac{1}{3}^k u[k] * \frac{1}{2}^{|k|}$$

$$\text{For } k \leq 0 \quad r_{xy}(k) = \sum_{l=-\infty}^k \left(\frac{1}{3}\right)^{(k-l)-l} \frac{1}{2} = \frac{1}{3} \frac{1}{6} \frac{1}{1-\frac{1}{6}} = \frac{6}{5} 2^k$$

$$\text{For } k > 0 \quad r_{xy}(k) = \sum_{l=-\infty}^0 \left(\frac{1}{3}\right)^{k-l-l} \frac{1}{2} + \sum_{l=1}^k \frac{1}{3} \frac{1}{2}^l$$

$$= \frac{1}{3} (k-1) \left[\left(\frac{3}{2}\right)^k - \frac{3}{5} \right]$$

$$\frac{1}{3}^k u[k-1] * \frac{1}{2}^{|k|}$$

$$\text{For } k \leq 0 \quad r_{xy}(k) = \sum_{l=-\infty}^{k-1} \left(\frac{1}{3}\right)^{k-l-1-l} \frac{1}{2}$$

$$= \frac{1}{5} 2^k$$

$$\text{For } k \geq 1 \quad r_{xy}(k) = \frac{1}{3} \left(\frac{6}{5}\right)^k + \frac{1}{3} \left(\frac{3}{2}\right)^{k-1} \frac{1}{2}$$

$$\text{Combining: } r_{xy}(k) = \frac{9}{10} 2^k u[-k-1] + \frac{9}{10} \left(\frac{1}{3}\right)^k u[k] \quad (10)$$

$$(iv) P_{xy}(z) = \frac{\frac{9}{10}}{1-2z^{-1}} + \frac{\frac{9}{10}}{1-\frac{1}{3}z^{-1}} = \frac{\frac{3}{10} [6-7z^{-1}]}{(1-2z^{-1})(1-\frac{1}{3}z^{-1})} \quad (2)$$

$$(c) H_w(z) = \sqrt{\frac{4}{3}} \left(1 - \frac{1}{3}z^{-1}\right) \quad (4)$$

(25/25)

3) a) Filtering - Given $x[m] = \underset{\substack{\uparrow \\ \text{Signal}}}{s[m]} + \underset{\substack{\uparrow \\ \text{Noise}}}{w[m]}$ $m=0, 1, \dots, n$

filter the signal from the noise using present and past data only

Smoothing - As filtering, except that future data is also used to estimate the signal - generally block-based.

Prediction - To estimate data outside of the observation interval - 1-step forward prediction uses $\{x[0], x[1], \dots, x[N-1]\}$ to predict $x[N-1+1]$ (5)

$$b) (i) J \triangleq E\{e[N]^2\} = E\left\{\left(x[N] + \sum_{k=1}^N a[k]x[N-k]\right)^2\right\}$$

$$\frac{\partial J}{\partial a[m]} \quad m=1, 2, \dots, N = 2 E\left\{\left(x[N] + \sum_{k=1}^N a[k]x[N-k]\right)x[N-m]\right\} = 0 \quad \uparrow \text{For solution.}$$

$$\text{i.e.} \quad r_{xx}[-m] = - \sum_{k=1}^N a[k] r_{xx}[k-m]$$

In matrix form

$$\begin{bmatrix} r_{xx}[-1] \\ r_{xx}[-2] \\ \vdots \\ r_{xx}[-N] \end{bmatrix} = - \begin{bmatrix} r_{xx}[0] & r_{xx}[1] & \dots & r_{xx}[N-1] \\ r_{xx}[1] & r_{xx}[0] & \dots & r_{xx}[N-2] \\ \vdots & \vdots & \ddots & \vdots \\ r_{xx}[N-1] & r_{xx}[N-2] & \dots & r_{xx}[0] \end{bmatrix} \begin{bmatrix} a[1] \\ a[2] \\ \vdots \\ a[N] \end{bmatrix} \quad \underline{a}_{\text{opt}} = -\underline{R}_{xx}^{-1} \underline{r}_{xx}$$

$$\underline{r}_{xx} = - \underline{R}_{xx} \underline{a}_{\text{opt}} \quad (8)$$

$$(ii) J_{\min} = r_{xx}[0] + \underline{a}_{\text{opt}}^T \underline{r}_{xx} \quad (2)$$

$$c) \quad r_{xx}[0] = \sigma^2 / (1-p^2), \quad r_{xx}[1] = -\sigma^2 p / (1-p^2), \quad r_{xx}[2] = \sigma^2 p^2 / (1-p^2)$$

$$N=1 \quad r_{xx}[-1] = -r_{xx}[0]a[1] \Rightarrow a[1] = p \quad J_{\min}^{(1)} = r_{xx}[0] + p r_{xx}[-1] = r_{xx}[0] \left(1 - \left(\frac{r_{xx}[-1]}{r_{xx}[0]}\right)^2\right)$$

$$N=2 \quad \begin{bmatrix} r_{xx}[-1] \\ r_{xx}[-2] \end{bmatrix} = - \begin{bmatrix} r_{xx}[0] & r_{xx}[1] \\ r_{xx}[1] & r_{xx}[0] \end{bmatrix} \begin{bmatrix} a[1] \\ a[2] \end{bmatrix} \Rightarrow \begin{bmatrix} a[1] \\ a[2] \end{bmatrix} = \frac{-p}{1-p^2} \begin{bmatrix} 1 & p \\ p & 1 \end{bmatrix} \begin{bmatrix} -1 \\ p \end{bmatrix} = \begin{bmatrix} p \\ 0 \end{bmatrix}$$

$$\therefore a[i] = a'[i], \quad J_{\min}^{(2)} = J_{\min}^{(1)} \quad - r_{xx}[k] \text{ is ACF for an AR(1) process} \quad (5)$$

$$d) J_b \triangleq E\{e_b^2[n]\} \text{ as in b), } E\left\{\left(x[0] + \sum_{k=1}^N a[k]x[k]\right)x[m]\right\} = 0 \quad m=1, \dots, N$$

$$r_{xx}[m] + \sum_{k=1}^N a[k] r_{xx}[m-k] = 0$$

Due to even symmetry in $r_{xx}[k]$, a consequence of the WSS property, (5)

4) a) Estimate is the value of the parameters for a given observation of the data.

Estimator is a rule that assigns a value to the parameter for any observation of the data.

(3)

b) Given a scalar parameter $\alpha = g(\theta)$, and any estimators which are unbiased $E\{\hat{\alpha}\} = \alpha = g(\theta)$,

thus

$$\int \hat{\alpha} p(x; \theta) dx = g(\theta)$$

Differentiating w.r.t. θ and assuming regularity conditions are satisfied

$$\int \hat{\alpha} \frac{\partial p(x; \theta)}{\partial \theta} dx = \frac{\partial g(\theta)}{\partial \theta}$$

$$\text{or } \int \hat{\alpha} \frac{\partial \ln p(x; \theta)}{\partial \theta} p(x; \theta) dx = \frac{\partial g(\theta)}{\partial \theta}$$

Adding zero to both sides

$$\int \underbrace{(\hat{\alpha} - \alpha)}_{g(x)} \underbrace{\frac{\partial \ln p(x; \theta)}{\partial \theta}}_{h(x)} \underbrace{p(x; \theta)}_{w(x)} dx = \frac{\partial g(\theta)}{\partial \theta}$$

$$\text{Thus } \left(\frac{\partial g(\theta)}{\partial \theta} \right)^2 \leq \int (\hat{\alpha} - \alpha)^2 p(x; \theta) dx \int \left(\frac{\partial \ln p(x; \theta)}{\partial \theta} \right)^2 p(x; \theta) dx$$

$$\text{or } \text{var}(\hat{\alpha}) \geq \frac{\left(\frac{\partial g(\theta)}{\partial \theta} \right)^2}{E \left[\left(\frac{\partial \ln p(x; \theta)}{\partial \theta} \right)^2 \right]}$$

$$\text{and if } \alpha = g(\theta) = \theta \quad \frac{\partial g(\theta)}{\partial \theta} = 1$$

$$\text{hence } \text{var}(\hat{\theta}) \geq \frac{1}{-E \left\{ \frac{\partial^2 \ln p(x; \theta)}{\partial \theta^2} \right\}}$$

(13)

$$\begin{aligned} &\text{Identity} \\ &E \left[\left(\frac{\partial \ln p(x; \theta)}{\partial \theta} \right)^2 \right] = \\ &-E \left[\frac{\partial^2 \ln p(x; \theta)}{\partial \theta^2} \right] \end{aligned}$$

$$c) \quad p(\underline{x}; \phi) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp \left\{ -\frac{1}{2\sigma^2} \sum_{n=0}^{N-1} [x[n] - A \cos(2\pi f_0 n + \phi)]^2 \right\}$$

$$\frac{\partial}{\partial \phi} \ln p(\underline{x}; \phi) = -\frac{A}{\sigma^2} \sum_{n=0}^{N-1} [x[n] \sin(2\pi f_0 n + \phi) - \frac{A}{2} \sin(4\pi f_0 n + 2\phi)]$$

and

$$\frac{\partial^2}{\partial \phi^2} \ln p(\underline{x}; \phi) = -\frac{A}{\sigma^2} \sum_{n=0}^{N-1} [x[n] \cos(2\pi f_0 n + \phi) - A \cos(4\pi f_0 n + 2\phi)]$$

$$-E \left[\frac{\partial^2 \ln p(\underline{x}; \phi)}{\partial \phi^2} \right] = \frac{A^2}{\sigma^2} \sum_{n=0}^{N-1} \left[\frac{1}{2} + \frac{1}{2} \cos(4\pi f_0 n + 2\phi) - \cos(4\pi f_0 n + 2\phi) \right]$$

$$\approx \frac{NA^2}{2\sigma^2} \quad \text{since } \frac{1}{N} \sum_{n=0}^{N-1} \cos(4\pi f_0 n + 2\phi) \approx 0 \quad \text{for } f_0 \neq 0/2$$

$$\therefore \text{var}(\hat{\phi}) \geq \frac{2\sigma^2}{NA^2} \quad - \text{CRLB}$$

(7)

d) A phase estimator does not exist which is unbiased and attains the CRLB (with equality) - no efficient estimator

(2)

$$\frac{25}{25}$$

5 (a) Least squares problem

$$\min_{\Theta} J(\Theta) = \sum_{n=0}^{N-1} (x[n] - s[n])^2$$

\uparrow Observation \downarrow Model

- If the model is linear in Θ , linear least squares and easy to solve analytically, e.g. Θn , $\Theta \sin(2\pi f n)$
- If the model is non linear in Θ , non linear least squares and difficult to solve analytically, save for special case when substitution is possible, e.g. $\exp(\Theta n)$, $\sin(2\pi \Theta n)$

(3)

$$(b) J(A) = \sum_{n=0}^{M-1} (x[n] - A)^2 + \sum_{n=M}^{N-1} (x[n] + A)^2$$

$$\frac{\partial J(A)}{\partial A} = -2 \sum_{n=0}^{M-1} (x[n] - A) + 2 \sum_{n=M}^{N-1} (x[n] + A) = 0 \quad (*)$$

$$\Rightarrow -2 \sum_{n=0}^{M-1} x[n] + 2MA + 2 \sum_{n=M}^{N-1} x[n] + 2(N-M)A = 0$$

$$\Rightarrow \hat{A} = \frac{1}{N} \left(\sum_{n=0}^{M-1} x[n] - \sum_{n=M}^{N-1} x[n] \right)$$

$$J_{\min} = \sum_{n=0}^{M-1} (x[n] - \hat{A})(x[n] - \hat{A}) + \sum_{n=M}^{N-1} (x[n] + \hat{A})(x[n] + \hat{A})$$

$$= \sum_{n=0}^{M-1} x[n](x[n] - \hat{A}) + \sum_{n=M}^{N-1} x[n](x[n] + \hat{A}) \text{ using } (*)$$

$$= \sum_{n=0}^{N-1} x^2[n] - \hat{A} \left(\sum_{n=0}^{M-1} x[n] - \sum_{n=M}^{N-1} x[n] \right)$$

$$= \sum_{n=0}^{N-1} x^2[n] - N\hat{A}^2$$

(10)

$$(c) E[\hat{A}] = \frac{1}{N} [MA - (N-M)A] = \frac{2MA - NA}{N} = A'$$

$$\text{VAR}(\hat{A}) = \frac{1}{N^2} \left[\text{VAR} \left(\sum_{n=0}^{M-1} x[n] \right) + \text{VAR} \left(\sum_{n=M}^{N-1} x[n] \right) \right] = \frac{1}{N^2} [M\sigma^2 + (N-M)\sigma^2] = \frac{\sigma^2}{N}$$

$\Rightarrow \hat{A} \sim \mathcal{N}(A', \frac{\sigma^2}{N})$ - follows from \hat{A} being a linear function of the $x[n]$'s

(5)

d) Quick method - $\alpha = e^{\theta} \Rightarrow \hat{\alpha} = \bar{x}$ (sample mean)
and hence $\hat{\theta} = \ln(\bar{x})$

Iteratively - Newton Rapsan

$$\theta_{k+1} = \theta_k + \left(H^T(\theta_k) H(\theta_k) - \sum_{n=0}^{N-1} G_n(\theta_k) (x[n] - e^{\theta_k}) \right)^{-1} H^T(\theta_k) (x - e^{\theta_k} \mathbf{1})$$

$$[H(\theta)]_i = \frac{\partial s[i]}{\partial \theta} = e^{\theta} \quad i = 0, 1, \dots, N-1$$

$$[G_n(\theta)]_{ii} = \frac{\partial^2 s[n]}{\partial \theta^2} = e^{\theta}$$

$$\Rightarrow H(\theta) = e^{\theta} \mathbf{1} \quad G_n(\theta) = e^{\theta}$$

$$\theta_{k+1} = \theta_k + \left(N e^{2\theta_k} - \sum_{n=0}^{N-1} e^{\theta_k} (x[n] - e^{\theta_k}) \right)^{-1} e^{\theta_k} \mathbf{1}^T (x - e^{\theta_k} \mathbf{1})$$

$$= \theta_k + \frac{e^{\theta_k} (N\bar{x} - N e^{\theta_k})}{N e^{2\theta_k} - N\bar{x} e^{\theta_k} + N e^{2\theta_k}}$$

$$= \theta_k + \frac{\bar{x} - e^{\theta_k}}{2e^{\theta_k} - \bar{x}} \quad (7)$$

Jonathan Chambers
20-01-01

(25/25)