IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2018

EEE PART II: MEng, BEng and ACGI

MATHEMATICS 2B (E-STREAM AND I-STREAM)

Corrected copy

Friday, 25 May 2:00 pm

Time allowed: 1:30 hours

There are TWO questions on this paper.

Answer TWO questions.

~~All questions carry equal marks~~ Q1 carries 60 marks. Q2 carries 40 marks.

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : B. Clerckx
Second Marker(s) : D. Nucinkis

THE QUESTIONS

1. Consider two continuous random variables X and Y characterized by the following joint probability density function [30]

$$f_{X,Y}(x,y) = \frac{2}{\pi} e^{-2(x^2+y^2)}, \quad -\infty < x, y < +\infty,$$

a) Compute the probability that X is smaller than or equal to 0.5 and Y is smaller than or equal to 0.7, i.e. $P(X \leq 0.5 \cap Y \leq 0.7)$.

[2]

b) Compute the marginal probability density function of X.

[2]

c) Compute the expectation of X, i.e. $E(X)$, and the variance of X, i.e. $\mathrm{Var}(X)$.

[4]

d) Compute the marginal probability density function of Y.

[2]

e) Compute the expectation of Y, i.e. $E(Y)$, and the variance of Y, i.e. $\mathrm{Var}(Y)$.

[4]

f) Compute the covariance between X and Y, i.e. $\mathrm{Cov}(X,Y)$, and the correlation coefficient between X and Y, i.e. $\mathrm{Corr}(X,Y)$.

[2]

g) Are X and Y uncorrelated? Independent? Provide your reasoning.

[2]

h) Make the change of variables $U = \sqrt{X^2+Y^2}$, $V = \tan^{-1}\left(\frac{Y}{X}\right)$ and compute the joint probability density function $f_{U,V}(u,v)$.

[4]

i) Compute the marginal probability density function of U and V, i.e. $f_U(u)$ and $f_V(v)$.

[2]

j) Are U and V independent? Provide your reasoning.

[2]

k) Compute the conditional probability density function of U given V, i.e. $f_{U|V}(u|v)$.

[2]

l) Compute the conditional expectation of U given V, i.e. $E(U|V)$.

[2]

2. a) Consider a communication system with one transmitter and two receivers. The power of the signal received at receiver i is denoted as P_i, $i = 1, 2$, and is modeled as an exponentially distributed random variable with parameter $\lambda > 0$. The transmitter transmits a message intended to both receivers. For the message to be correctly decoded at both receivers, the message is transmitted at a rate proportional to the power level P given by the minimum among the received signal power at the two receivers. Hence the power level P is given by $P = \min_{i=1,2} P_i$. We assume the receivers are deployed far apart from each other such that P_1 and P_2 are assumed independent. [20]

i) Find the probability that the power level P is larger than a certain level S. Provide your reasoning. [4]

ii) Find the probability density function of P. Provide your reasoning. [3]

iii) Compute the moment generating function of P. Provide your reasoning. [3]

iv) Making use of iii), find the expected value of P. Provide your reasoning. [2]

b) i) State the three axioms of probability. [3]

ii) Making use of i), prove the following relationship on the union of two arbitrary events A and B:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

[5]