Solutions to Question 1:

(a) First of all, electricity is inextricably linked with a physical delivery system which operates much faster than any market. Generation and load must be balanced at all times because failure to balance leads to collapse of the system and blackouts, which have enormous economic consequences. As a result physical balance of the system cannot be left entirely to a market.
Secondly, electricity cannot be stored in large quantities as most storage technologies are characterised by high capital costs and low efficiency. As a result, the total production needs to be equal to the total consumption at any given time.
Thirdly, the demand curve is almost vertical, meaning that the price elasticity of the demand is very low. In other words, the consumers do not often respond to high market prices, making the exercise of market power by large generators more probable.
Finally, electricity is inextricably linked with a network characterised by certain physical laws. Therefore, the electricity produced by different generators and electricity consumed by different consumers is indistinguishable and thus pooled. In other words, a generator cannot physically direct its produced electricity to a specific consumer and a consumer cannot choose which specific generator produces its load.

(b) In the optimal solution of the problem, the marginal cost of each generator is equal to the system marginal cost at each hour λ_t yielding:

$$\frac{\partial C_{1,t}(P_{1,t})}{\partial P_{1,t}} = 1 + 0.09 * P_{1,t} = \lambda_t \rightarrow P_{1,t} = \frac{\lambda_t - 1}{0.09} \tag{1}$$

$$\frac{\partial C_{2,t}(P_{2,t})}{\partial P_{2}} = 5 + 0.3 * P_{2,t} = \lambda_t \rightarrow P_{2,t} = \frac{\lambda_t - 5}{0.3} \tag{2}$$

$$\frac{\partial C_{3,t}(P_{3,t})}{\partial P_{3,t}} = 16 + 1 * P_{3,t} = \lambda_t \rightarrow P_{3,t} = \frac{\lambda_t - 16}{1} \tag{3}$$

The satisfaction of the system demand implies that:

$$P_{1,t} + P_{2,t} + P_{3,t} = D_t \tag{4}$$

Substituting (1)-(3) in (4) gives for the system marginal cost:

$$\frac{\lambda_t - 1}{0.09} + \frac{\lambda_t - 5}{0.3} + \frac{\lambda_t - 16}{1} = D_t \rightarrow \lambda_1 = 22.259 \left[\frac{£}{MWh}\right] \text{ and } \lambda_2 = 67.583 \left[\frac{£}{MWh}\right] \tag{5}$$

Hour 2 is characterised by higher system marginal cost, since it exhibits a higher level of demand.
Substituting (5) in (1)-(3) yields for the optimal production of each generator:

$$P_{1,1} = 236.211\ [MW] \text{ and } P_{1,2} = 739.808\ [MW] \tag{6}$$

$P_{2,1} = 57.530\ [MW]$ and $P_{2,2} = 208.609\ [MW]$

(7)

$P_{3,1} = 6.259\ [MW]$ and $P_{3,2} = 51.583\ [MW]$

(8)

Substituting (6)-(8) in the cost functions of the generators yields for the total cost of operating the system over the 2-hour period:"

$$C_{tot} = \sum_i \sum_t C_{i,t}(P_{i,t}) = 40846.307[£]$$

(9)

The total profit for each generator Ω_i is given by the difference between its respective revenue and cost:

$$\Omega_1 = \sum_t[\lambda_t P_{1,t} - C_{1,t}(P_{1,t})] = 25940.029\ [£]$$

(10)

$$\Omega_2 = \sum_t[\lambda_t P_{1,t} - C_{2,t}(P_{1,t})] = 6724.119\ [£]$$

(11)

$$\Omega_3 = \sum_t[\lambda_t P_{1,t} - C_{3,t}(P_{1,t})] = 749.977\ [£]$$

(12)

(c) At hour 1, given that $P_{3,1} < P_3^{min}$ the solution calculated in **(a)** is not feasible when taking into account the given output limits of the generators. The optimal solution for hour 1 can now be calculated by fixing the output of generator 3 to:

$$P'_{3,1} = P_3^{min} = 50[MW]$$

(13)

and satisfying the remaining demand:

$$D'_1 = D_1 - P'_{3,1} = 250\ [MW]$$

(14)

through the remaining generators 1 and 2. Following the same method as in **(a)**, we get:

$$\frac{\lambda'_1 - 1}{0.09} + \frac{\lambda'_1 - 5}{0.3} = 250 \rightarrow \lambda'_1 = 19.231\ [\frac{£}{MWh}]$$

(15)

Substituting (15) in (1)-(2) yields for the optimal production of each generator:

$$P'_{1,1} = 202.564\ [MW]$$

(16)

$$P'_{2,1} = 47.436\ [MW]$$

(17)

Similarly, at hour 2, given that $P_{2,2} > P_2^{max}$ the solution calculated in **(a)** is not feasible when taking into account the given output limits of the generators. The optimal solution for hour 2 can now be calculated by fixing the output of generator 2 to:

$$P'_{2,2} = P_2^{max} = 160\ [MW] \tag{18}$$

and satisfying the remaining demand:

$$D'_2 = D_2 - P'_{2,2} = 840\ [MW] \tag{19}$$

through the remaining generators 1 and 3. Following the same method as in **(a)**, we get:

$$\frac{\lambda'_2 - 1}{0.09} + \frac{\lambda'_2 - 16}{1} = 840 \rightarrow \lambda'_2 = 71.596\ [\frac{£}{MWh}] \tag{20}$$

Substituting (20) in (1) and (3) yields for the optimal production of each generator:

$$P'_{1,2} = 784.404\ [MW] \tag{21}$$

$$P'_{3,2} = 55.596\ [MW] \tag{22}$$

Substituting (13), (16)-(17), (18), and (21)-(22) in the cost functions of the generators yields for the total cost of operating the system:

$$C'_{tot} = \sum_i \sum_t C_{i,t}(P'_{i,t}) = 42321.149\ [£] \tag{23}$$

The profit for each generator is given by the difference between its respective revenue and cost:

$$\Omega'_1 = \sum_t [\lambda'_t P'_{1,t} - C_{1,t}(P'_{1,t})] = 28334.460\ [£] \tag{24}$$

$$\Omega'_2 = \sum_t [\lambda'_t P'_{2,t} - C_{2,t}(P'_{2,t})] = 6852.937\ [£] \tag{25}$$

$$\Omega'_3 = \sum_t [\lambda'_t P'_{3,t} - C_{3,t}(P'_{3,t})] = -142.986\ [£] \tag{26}$$

(d) The negative profit of generator 3 means that the operating cost of the latter is larger than its revenue in the optimal solution and thus it experiences economic losses at the optimal solution (does not recover its operating costs). This is happening due to the combination of the high variable costs of generator 3 and its high minimum output limit with respect to the initial unconstrained dispatch calculated in **(b)**. Since the generators are free to participate in the market based on their profitability, generator 3 can avoid the negative profits (economic losses) by decommitting from the market.

Solution to Question 2:

(a) In imperfect markets, large producers can exert market power and manipulate the prices either by withholding quantity (physical withholding) or by raising its offered price (economic withholding).
Factors facilitating the exercise of market power include: i) the existence of a small number of generation companies, each controlling a large share of the market, ii) the low price elasticity of the demand (consumers do not often respond to high market prices), iii) congestion effects in the transmission network, which limit the number of effective competitors in supplying the demand at particular areas and iv) the fact that demand exhibits predictable cyclical variations (daily, weekly, seasonal) which allow the producers to optimise their strategic actions at peak demand periods (where the system operates close to its maximum capacity).

(b) Under perfect competition, the price is equal to the marginal cost of each generator. Therefore, the following two equations hold:

$$\frac{\partial C_A(P_A)}{\partial P_A} = \pi \qquad (1)$$

$$\frac{\partial C_B(P_B)}{\partial P_B} = \pi \qquad (2)$$

Since $\pi = 210 - 1.3 \cdot D$ and $D = P_A + P_B$ (demand-supply balance condition), equations (1) and (2) yield respectively:

$$25 + 0.8 \cdot P_A = 210 - 1.3 \cdot (P_A + P_B) \rightarrow P_A = 88.0952 - 0.6190 \cdot P_B \qquad (3)$$

$$27 + 0.6 \cdot P_B = 210 - 1.3 \cdot (P_A + P_B) \rightarrow P_B = 96.3158 - 0.6842 \cdot P_A \qquad (4)$$

The combination of (3) and (4) yields:

$$P_A = 49.3913\ [MW] \qquad (5)$$

$$P_B = 62.5217\ [MW] \qquad (6)$$

The total demand is given by:

$$D = P_A + P_B \rightarrow D = 111.9130\ [MW] \qquad (7)$$

The electricity price is given by:

$$\pi = 210 - 1.3 \cdot D \rightarrow \pi = 64.5130\ [\tfrac{£}{MWh}] \qquad (8)$$

The profits made by the two generators are given by:

$$\Omega_A = P_A \cdot \pi - 25 \cdot P_A - 0.4 \cdot P_A^2 \rightarrow \Omega_A = 975.8004\ [\tfrac{£}{h}] \qquad (9)$$

$$\Omega_B = P_B \cdot \pi - 27 \cdot P_B - 0.3 \cdot P_B^2 \rightarrow \Omega_B = 1172.6904[\tfrac{£}{h}] \quad (10)$$

(c) (i) In the Cournot model of competition the state of the market is determined by the production decisions made by each firm. We summarize the possible outcomes using a table where all the cells in a column correspond to a given production by company A and the cells in a row correspond to a given production by company B. Each cell contains four pieces of information arranged in the following format:

D	Ω_A
Ω_B	π

where:

π price $[\frac{£}{MWh}]$

D demand $[MW]$

Ω_A profit made by company A $[\frac{£}{h}]$

Ω_B profit made by company B $[\frac{£}{h}]$

Given the productions P_A and P_B of the two companies, the other quantities are calculated as follows:

$$D = P_A + P_B \quad (11)$$

$$\pi = 210 - 1.3 \cdot D \quad (12)$$

$$\Omega_A = P_A * \pi - 25 \cdot P_A - 0.40 \cdot P_A^2 \quad (13)$$

$$\Omega_B = P_B * \pi - 27 \cdot P_B - 0.30 \cdot P_B^2 \quad (14)$$

Based on the above, the table expressing the Cournot model of competition for the conditions of the problem is shown below. A Nash equilibrium state implies that given the production level of either of the two companies in this state, the other company cannot achieve a higher profit by following a production level different than the one at the equilibrium. It can thus be observed from the table below that the cell corresponding to $P_A = 39MW$ and $P_B = 41MW$ is an equilibrium point. In other words, given that $P_A = 39MW$, company B achieves its highest profit by producing $P_B = 41MW$ AND given that $P_B = 41MW$, company A achieves its highest profit by producing $P_A = 39MW$.

P_B/P_A	37		39		41	
37	74	2738	76	2753.4	78	2755.2
	2800.9	113.8	2704.7	111.2	2608.5	108.6
39	76	2641.8	78	2652	80	2648.6
	2827.5	111.2	2726.1	108.6	2624.7	106
41	78	2545.6	80	2550.6	82	2542
	2841.3	108.6	2734.7	106	2628.1	103.4

(ii) We can observe that under imperfect competition the market price and the generation profits are significantly higher than under perfect competition, since producers exercise market power and manipulate the prices to strategically increase their profits. As a result of the price elasticity of the demand, the increased price leads to lower demand (and consequently to lower generation levels).

Solutions to Question 3

(a) A transmission system is required to efficiently and securely transport electric power from generating plants to distribution systems and large consumers so as to: a) minimize fuel costs in the production of electricity by allowing the utilization of those available generating sources having the lowest marginal costs, b) interconnect systems and generating plants to reduce overall generating requirements by taking advantage of the diversity of peak loads (peak loads occur at different times in different systems) and diversity of generation outages and reserve requirements and c) to enable a competitive trading of electric energy in the marketplace.
Transmission business is characterised by capital intensity, economies of scale, long-lived assets with small re-sale value and long-lead times of construction. The capital intensity means that it is practically unrealistic to construct multiple transmission networks and thus the nature of the transmission business is physically monopolistic. Therefore, the transmission business should be subject to regulation. The regulation fulfil their responsibilities by determining the maximum revenue of the transmission network company and certain quality of supply standards the company has to respect.

(b) The cost of network constraints is equal to the difference between the optimal total cost of operating the system with and without taking into account the network constraints. When the network constraints are not taken into account, the optimal solution for the operation of the system is given by the solution of the system of equations (1)-(2):

$$\frac{\partial C_1(P_1^{unc})}{\partial P_1} = \frac{\partial C_2(P_2^{unc})}{\partial P_2} \quad (1)$$

$$P_1^{unc} + P_2^{unc} = D_1 + D_2 \rightarrow P_1^{unc} + P_2^{unc} = 2400 \text{ [MW]} \quad (2)$$

which yields:

$$P_1^{unc} = 907.963 \text{ [MW]} \quad (3)$$

$$P_2^{unc} = 1492.037 \text{ [MW]} \quad (4)$$

$$C_{tot}^{unc} = C_1(P_1^{unc}) + C_2(P_2^{unc}) = 116233.576 \text{ [£]} \quad (5)$$

When the network constraints are taken into account, the optimal solution in the unconstrained case is not feasible since the flow from busbar 2 to busbar 1 is:

$$F = D_1 - P_1^{unc} = P_2^{unc} - D_2 = 1042.037 \text{ [MW]} > F^{max} \quad (6)$$

Since the flow cannot be larger than F^{max}, the total cost of operating the system in this case is calculated by substituting (7) and (8):

$$P_1^{con} + F^{max} = D_1 \rightarrow P_1^{con} = 1150 \text{ [MW]} \quad (7)$$

$$P_2^{con} - F^{max} = D_2 \rightarrow P_2^{con} = 1250 \text{ [MW]} \quad (8)$$

into the given cost functions of the two generators yielding:

$$C_{tot}^{con} = C_1(P_1^{con}) + C_2(P_2^{con}) = 119397 \text{ [£]} \quad (9)$$

Therefore, the cost of network constraints is calculated as (through (5) and (9)):

$$C_{net} = C_{tot}^{con} - C_{tot}^{unc} = 3163.424 \text{ [£/}h\text{]} \quad (10)$$

Since the network line is congested, the marginal prices in the two buses are different. An additional unit of demand in Bus 1 will be satisfied by generator 1 and thus the locational marginal price at bus 1 is:

$$\pi_1 = \frac{\partial C_1(P_1^{con})}{\partial P_1} = 95.070 \text{ [£/MWh]} \quad (11)$$

An additional unit of demand in Bus 2 will be satisfied by generator 2 and thus the locational marginal price at bus 2 is:

$$\pi_2 = \frac{\partial C_2(P_2^{con})}{\partial P_2} = 68.930 \text{ [£/MWh]} \quad (12)$$

(c) The total cost of operating the system as a function of ΔF when network constraints are taken into account is calculated by substituting (13) and (14):

$$P_1^{con} + F^{max} + \Delta F = D_1 \rightarrow P_1^{con} = 1150 - \Delta F \quad (13)$$

$$P_2^{con} - F^{max} - \Delta F = D_2 \rightarrow P_2^{con} = 1250 + \Delta F \quad (14)$$

into the given cost functions of the two generators yielding:

$$C_{tot}^{con} = C_1(P_1^{con}) + C_2(P_2^{con}) = 119397 - 26.14\,\Delta F + 0.054\,(\Delta F)^2 \quad (15)$$

Therefore, the cost of network constraints is calculated as (through (5) and (15)):

$$C_{net} = C_{tot}^{con} - C_{tot}^{unc} = 3163.424 - 26.14\,\Delta F + 0.054\,(\Delta F)^2 \quad (16)$$

(d) The optimal level of network capacity that should be built corresponds to the minimum of the total system costs C_{totsys} composed of cost of network constraints C_{net} and cost of network investment C_{inv}:

$$\min_{\Delta F} C_{totsys} = C_{net} + C_{inv} \quad (17)$$

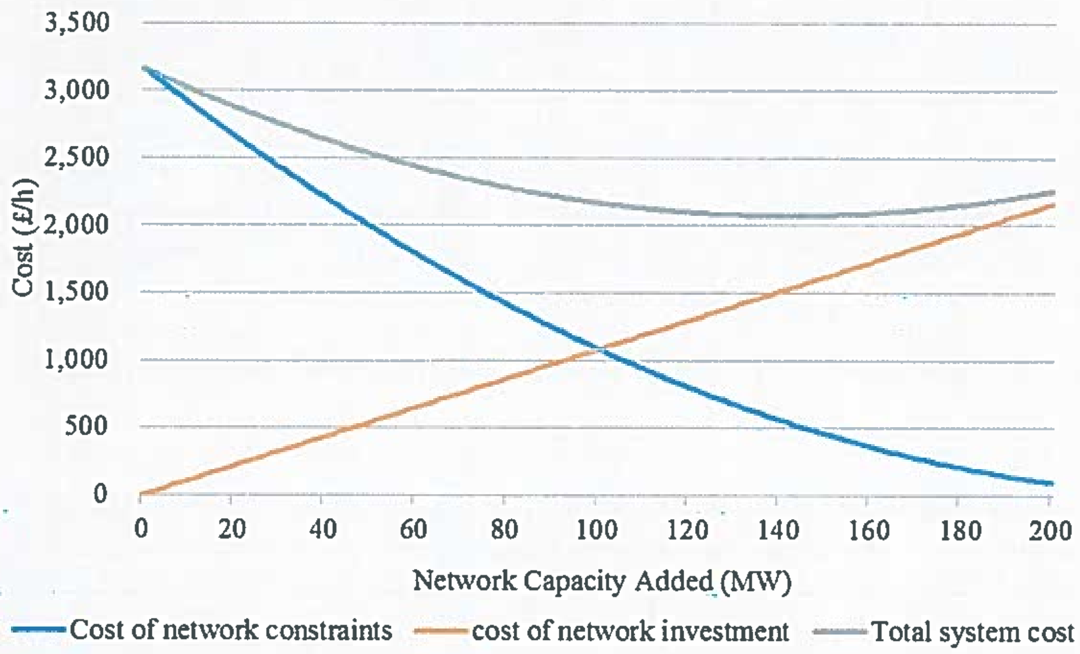
In order to express these two components in the same temporal basis, the cost of network investment –which is given per year- is expressed per hour by dividing it by the number of hours (8760) in a year. Based on this modification, the given data regarding k and L, the cost of network investment C_{inv} can be expressed as:

$$C_{inv} = \frac{205.67 * 460}{8760} \Delta F \tag{18}$$

Based on (16) and (18) the total system cost C_{totsys} can be expressed as:

$$C_{totsys} = C_{net} + C_{inv} = 3163.424 - 15.34\,\Delta F + 0.054\,(\Delta F\,)^2 \tag{19}$$

According to (16), (18) and (19), the cost of network constraints C_{net}, the cost of network investment C_{inv} and the total system cost C_{totsys} are plotted in the figure below as a function of network capacity added ΔF.



The optimal capacity of the line that should be added between the two areas are determined by minimizing the total system cost C_{totsys} as follows:

$$\min_{\Delta F} C_{totsys} = 3163.424 - 15.34\,\Delta F + 0.054\,(\Delta F\,)^2 \tag{20}$$

whose solution is given by:

$$\frac{\partial C_{totsys}}{\partial \Delta F} = 0 \rightarrow \Delta F = 142.037\ [\text{MW}] \tag{21}$$

(e) When the optimal network capacity is built, the new line capacity will be:

$$F^{max'} = F^{max} + \Delta F = 942.037\ [MW]$$

(22)

The dispatch of the two generators in this case is given by:

$$P_1^{con'} + F^{max'} = D_1 \rightarrow P_1^{con'} = 1007.963\ [MW]$$

(23)

$$P_2^{con'} - F^{max'} = D_2 \rightarrow P_2^{con'} = 1392.037\ [MW]$$

(24)

Thus, the locational marginal prices at the two buses are:

$$\pi_1' = \frac{\partial C_1\left(P_1^{con'}\right)}{\partial P_1^{con'}} = 85.696\ [£/MWh]$$

(25)

$$\pi_2' = \frac{\partial C_2\left(P_2^{con'}\right)}{\partial P_2^{con'}} = 74.896\ [£/MWh]$$

(26)

On a merchant basis, the revenue of operating this transmission link is equal to the congestion surplus, given by:

$$CS = (\pi_1' - \pi_2')F^{max'} = 10174.019\ [£/h]$$

(27)

The investment cost of this link in hourly basis is according to the given data:

$$C_{inv} = \frac{k * L}{8760} * F^{max'} = 10174.019\ [£/h]$$

(28)

The profit of investing and operating this transmission link is therefore zero.

Solutions to Question 4

a) Since the generators' marginal costs are fixed, the minimum cost dispatch can be calculated by dispatching the generators in an ascending marginal cost order until the total demand of $D_1 + D_2 + D_3 = 500\ [MW]$ is satisfied. We thus get for the minimum cost dispatch:

$$P_a = 155\ [MW] \tag{1}$$

$$P_b = 345\ [MW] \tag{2}$$

$$P_c = 0\ [MW] \tag{3}$$

$$P_d = 0\ [MW] \tag{4}$$

The line flow problem can be solved using the superposition principle and also directly. To this effect, we write the power balance equation at two buses and KVL around the loop.

Bus 1: $$P_a + P_b - 80 = F_{12} + F_{13} \tag{5}$$

Bus 2: $$P_c - 90 = -F_{12} + F_{23} \tag{6}$$

Bus 3: $$P_d - 330 = -F_{13} - F_{23} \tag{7}$$

Loop equation: $$0.3F_{12} + 0.2F_{23} - 0.3F_{13} = 0 \tag{8}$$

Putting these equations in the matrix form gives:

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & -1 \\ 0.3 & -0.3 & 0.2 \end{bmatrix} \begin{bmatrix} F_{12} \\ F_{13} \\ F_{23} \end{bmatrix} = \begin{bmatrix} P_c - 90 \\ P_d - 330 \\ 0 \end{bmatrix} \tag{9}$$

Substituting (1)-(4) in (9), we get:

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & -1 \\ 0.3 & -0.3 & 0.2 \end{bmatrix} \begin{bmatrix} F_{12} \\ F_{13} \\ F_{23} \end{bmatrix} = \begin{bmatrix} -90 \\ -330 \\ 0 \end{bmatrix} \tag{10}$$

Solving these equations, we get:

$$F_{12} = 180\ [MW] \tag{11}$$

$$F_{13} = 240\ [MW] \tag{12}$$

$$F_{23} = 90[MW] \tag{13}$$

Since $F_{12} > F_{12}^{max}$ ($F_{12}^{max} = 160\ [MW]$ according to the data) the branch 1-2 is overloaded.

b) In order to eliminate this overload the output of generator G_d is increased and the output of generator G_a is decreased by the same amount. Decreasing the output of generator G_b is not desirable as it is cheaper than generator G_a.
To calculate how big this increase should be to remove the violation of the flow limit on line 1-2, consider an injection of $+1[MW]$ at bus 3 and an injection of $-1[MW]$ at bus 1. This pair of injection causes a flow in the network that divides itself as follows:

$$\frac{0.3}{(0.3+0.2)+0.3} \cdot 1 = 0.375[MW] \quad \text{along the path 3-2-1} \tag{14}$$

$$\frac{(0.3+0.2)}{(0.3+0.2)+0.3} \cdot 1 = 0.625[MW] \quad \text{along the path 3-1} \tag{15}$$

Since we use a linear (dc) model, we can say that to remove $F_{12} - F_{12}^{max} = 20\ [MW]$ overload on line 1-2, we therefore need to increase the output of G_d by:

$$\frac{20}{0.375} = 53.3333[MW] \tag{16}$$

Thus, the new generators' dispatch becomes:

$$P_a' = 101.6667\ [MW] \tag{17}$$

$$P_b' = 345\ [MW] \tag{18}$$

$$P_c' = 0\ [MW] \tag{19}$$

$$P_d' = 53.3333[MW] \tag{20}$$

Using the nodal and loop equations, the flows are calculated solving the following linear system:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 0.32 & -0.32 & 0.17 \end{bmatrix} \begin{bmatrix} F_{12}' \\ F_{13}' \\ F_{23}' \end{bmatrix} = \begin{bmatrix} P_a' + P_b' - 80 \\ P_d' - 330 \\ 0 \end{bmatrix} \tag{21}$$

We get:

$$F_{12}' = 160\ [MW] \tag{22}$$

$$F_{13}' = 206.6667\ [MW] \tag{23}$$

$$F_{23}' = 70[MW] \tag{24}$$

which satisfy the lines' capacities constraints.

The price at bus 2 is the cost of supplying an additional MW of load at this bus without violating the lines' capacities' constraints. The output of generator G_b cannot be increased since the latter runs at full output according to (18) and the given data. Increasing the output of generator G_a or generator G_d per 1 MW will lead to a violation of line 1-2 capacity constraint. Increasing the output of generator G_c will not lead to such a violation but seems as a costly alternative since in that case the price at bus 2 is:

$$\pi_2 = MC_c = 35\ [£/\mathrm{MWh}] \tag{25}$$

Our goal is to check whether a combination of modifications in the outputs of generators G_a and G_d can lead to a lower price than the one in (25).

Extracting an additional $1[MW]$ at bus 2 and generating it at bus 1 with G_a causes the following change in the flow on line 1-2:

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0.3 & -0.3 & 0.2 \end{bmatrix} \begin{bmatrix} \Delta F_{12} \\ \Delta F_{13} \\ \Delta F_{23} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \Delta F_{12} \\ \Delta F_{13} \\ \Delta F_{23} \end{bmatrix} = \begin{bmatrix} 0.625 \\ 0.375 \\ -0.375 \end{bmatrix} \tag{26}$$

Similarly, extracting an additional $1[MW]$ at bus 2 and generating it at bus 3 with G_d causes the following change in the flow on line 1-2:

$$\begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & 1 \\ 0.3 & -0.3 & 0.2 \end{bmatrix} \begin{bmatrix} \Delta F_{12} \\ \Delta F_{13} \\ \Delta F_{23} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \Delta F_{12} \\ \Delta F_{13} \\ \Delta F_{23} \end{bmatrix} = \begin{bmatrix} 0.25 \\ -0.25 \\ -0.75 \end{bmatrix} \tag{27}$$

Therefore, increase of 1 MW of the output of G_a and G_d will increase the flow in line 1-2 by 0.625 MW and 0.25 MW respectively.

We need a total increase of 1 MW in the outputs of G_a and G_d (28) but without increasing the power flow in line 1-2 (29):

$$\Delta P_a + \Delta P_d = 1\ [MW] \tag{28}$$

$$0.625 * \Delta P_a + 0.25 * \Delta P_d = 0 \tag{29}$$

which yields:

$$\Delta P_a = -0.6667[MW] \tag{30}$$

$$\Delta P_d = 1.6667\ [MW] \tag{31}$$

When these outputs' modifications are carried out the price at bus 2 is:

$$\pi_2' = -0.6667 * MC_A + 1.6667 * MC_D = 29[£/\mathrm{MWh}] \tag{32}$$

Which is accepted as the final value of price at bus 2 since it is lower than the value in (25).

c) Although the price at bus 2 is 29 [£/MWh] (32), the contract for difference that Load at bus 2 has made at the price of 26.5 [£/MWh] means that it will pay for satisfying its demand:

$$C_2 = 29 * D_2 + (26.5 - 29) * D_2 = 2385 \text{ [£/h]} \quad (33)$$

Its income from the transmission congestion contract (regarding line 1-2) it has made is given by the product of the volume of the flow in the contract (90 $[MW]$) and the price differential in the two buses where line 1-2 is connected. In other words:

$$I_2 = 90 * (\pi_2' - \pi_1) \quad (34)$$

Since generator G_a is the cheapest generator whose output has not reached its maximum limit (17)-(20), the price at bus 1 can be easily shown to be:

$$\pi_1 = MC_a = 19 \text{ [£/MWh]} \quad (35)$$

Substituting (32) and (35) in (34) gives:

$$I_2 = 900 \text{ [£/h]} \quad (36)$$

Therefore, the net cost that load at bus 2 has to pay is (from (33) and (36)):

$$C_2^{net} = C_2 - I_2 = 1485 \text{ [£/h]} \quad (37)$$

and the unit price that demand at bus 2 will pay is:

$$\pi_2^{net} = \frac{C_2^{net}}{D_2} = 16.5 \text{ [£/MWh]} \quad (38)$$

d) The flows calculated in **b)** now lead to an overload of line 2-3. In order to eliminate this overload the output of generator G_d is increased and the output of generator G_a is reduced by the same amount. Increase of 1 MW of the output of G_d has been shown in equation (14) to cause a 0.375 MW reduction in F_{23}. Therefore in order to cause a reduction of $F_{23}' - F_{23}^{max} = 3\ [MW]$, we therefore need to increase the output of G_d by:

$$\frac{3}{0.375} = 8\ [MW] \quad (39)$$

Therefore the new dispatch is:

$$P_a'' = 93.6667\ [MW] \quad (40)$$

$$P_b'' = 345\ [MW] \quad (41)$$

$$P_c'' = 0\ [MW] \quad (42)$$

$$P_d'' = 61.3333\ [MW] \quad (43)$$

Using the nodal and loop equations, the flows are calculated solving the following linear system:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 0.3 & -0.3 & 0.2 \end{bmatrix} \begin{bmatrix} F''_{12} \\ F''_{13} \\ F''_{23} \end{bmatrix} = \begin{bmatrix} P''_a + P''_b - 80 \\ P''_d - 330 \\ 0 \end{bmatrix} \quad (44)$$

the new network flows are:

$$F''_{12} = 157\ [MW] \quad (45)$$

$$F''_{13} = 201.6667\ [MW] \quad (46)$$

$$F''_{23} = 67\ [MW] \quad (47)$$

Price at bus 2 is calculated by following the same methodology as in **b)**. Increase of 1 MW of the output of G_a and G_d can be easily shown to reduce the flow in line 2-3 per 0.375 MW and 0.75 MW respectively, as shown in (26)-(27).

We need a total increase of 1 MW in the outputs of G_a and G_d (28) but without increasing the power flow in line 2-3 (29):

$$\Delta P'_a + \Delta P'_d = 1\ [MW] \quad (48)$$

$$-0.375\Delta P'_a - 0.75 * \Delta P'_d = 0 \quad (49)$$

which yields:

$$\Delta P'_a = 2\ [MW] \quad (50)$$

$$\Delta P'_d = -1\ [MW] \quad (51)$$

When these outputs' modifications are carried out the flow in line 1-2 becomes (according to (26)-(27)):

$$F'''_{12} = F''_{12} + \Delta P'_a * 0.625 + \Delta P'_d * 0.25 = 158\ [MW] \quad (52)$$

and thus capacity constraint of line 1-2 is not violated. The price at bus 2 is:

$$\pi''_2 = 2 * MC_A - 1 * MC_D = 13\ [£/\text{MWh}] \quad (53)$$

Based on its contract for differences, load at bus 2 will now pay for satisfying its demand

$$C'_2 = 13 * D_2 + (26.5 - 13) * D_2 = 2385\ [£/\text{h}] \quad (54)$$

Its income from the transmission congestion contract is now:

$$I'_2 = 90 * (\pi''_2 - \pi_1) = -540\ [£/\text{h}] \quad (55)$$

Therefore, the net cost that load at bus 2 has to pay is (from (54) and (55)):

$$C_2^{net\prime} = C_2' - I_2' = 2925 \text{ [£/h]} \tag{56}$$

and the unit price that demand at bus 2 will pay is:

$$\pi_2^{net\prime} = \frac{C_2^{net\prime}}{D_2} = 32.5 \text{ [£/MWh]} \tag{57}$$