UNIVERSITY OF LONDON [ISE 2.6 2001]

B.ENG. AND M.ENG. EXAMINATIONS 2001

For Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examination for the Associateship of the City and Guilds of London Institute.

INFORMATION SYSTEMS ENGINEERING 2.6

MATHEMATICS

Date Wednesday 2nd May 2001 2.00 - 4.00 pm

Answer FOUR questions, to include at least one from Section B

Answers to Section A *questions must be written in a different answer book from answers to* Section B *questions.*

[Before starting, please make sure that the paper is complete. There should be SIX pages, with a total of SIX questions. Ask the invigilator for a replacement if this copy is faulty.]

Section A

1. (i) The Fourier transform of $f(t)$ is given by

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt\,.$$

Show that the Fourier transform of $f(-t)$ is $\hat{f}(-\omega)$.

Now let

$$f(t) = \begin{cases} e^{-at} & \text{for } t \geq 0\,, \\ 0 & \text{for } t < 0\,, \end{cases}$$

where a is a positive constant, and

$$g(t) = e^{-|t|} \text{ for all } t\,.$$

Show that the Fourier transform of $f(t)$ is $\dfrac{1}{a+i\omega}$.

Hence, or otherwise, show that the Fourier transform of $g(t)$ is $\dfrac{2}{1+\omega^2}$.

Using $\hat{g}(\omega)$, deduce the value of $\displaystyle\int_0^{\infty} \frac{\cos\omega}{1+\omega^2}\,d\omega$.

(ii) Find the function $h(t)$ whose Fourier transform is

$$\hat{h}(\omega) = \frac{1}{(2+i\omega)(3+i\omega)}\,,$$

by using partial fractions.

Check your answer by using the convolution property.

You may assume

(a) $f(t) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \hat{f}(\omega)e^{+i\omega t}\,d\omega,$

(b) the convolution property that if $F(t)$ and $G(t)$ have Fourier transforms

$\widehat{F}(\omega)$ and $\widehat{G}(\omega)$ then $\displaystyle\int_{-\infty}^{\infty} F(u)\,G(t-u)\,du$ has Fourier transform $\widehat{F}(\omega)\,\widehat{G}(\omega)$.

PLEASE TURN OVER

2. (i) Use Laplace transforms to find the function $f(t)$, defined for $t \geq 0$, which satisfies

$$f(t) = 6t - 4 \int_{u=0}^{t} f(t-u)\, u\, du \qquad \text{for } t \geq 0\,.$$

(ii) Prove that

$$I = \int_C \left\{ \left(3x^2y^2 + y\, e^{xy}\right) dx + \left(2x^3y + x\, e^{xy} + \pi \cos(\pi y)\right) dy \right\}$$

is path-independent, i.e. depends only on the starting and finishing points of the path C in the xy plane.

Find a suitable potential function and hence evaluate I in the case where C starts at (0, 0) and finishes at (1, 2).

Check your last answer by finding a suitable parameterisation when C is the straight line segment leading from (0, 0) to (1, 2) and evaluating I using this parameterisation.

3. (i) Make a sketch of the region in the xy plane, over which the following integral is taken:

$$\int_0^1 dx \int_x^1 x(1+3y^3)^{1/2}\, dy\,.$$

Using this sketch as necessary, change the order of integration and hence evaluate this integral.

(ii) Using polar coordinates, show that

$$\iint_A \frac{2xy}{(x^2+y^2)^{1/2}}\, dx\, dy = \frac{2^{3/2}}{15}\,,$$

where A is the region in the xy plane, which lies above the line $y = x$ and inside the circle centre (1, 0) and radius 1.

PLEASE TURN OVER

4. Let

$$f(z) = \frac{1}{(z^2+1)(z^2+4)}.$$

Show that the residues of $f(z)$ at its poles at $z = i$ and $z = 2i$ are respectively $\frac{1}{6i}$ and $-\frac{1}{12i}$.

Consider the closed path $C = C_1 + C_2$ in the complex plane, where C_1 leads along the real axis from $-R$ to $+R$ and C_2 is the upper semicircular path leading from $+R$ to $-R$. Assume that $R > 2$.

Prove that

$$\int_C f(z)\,dz = \frac{\pi}{6}.$$

Hence find

$$\int_0^\infty \frac{dx}{(x^2+1)(x^2+4)}.$$

PLEASE TURN OVER

Section B **[ISE 2.6 2001]**

1. A company houses its computer server in a machine room cooled by an air-conditioning system. The air-conditioning system breaks down if a certain component fails.

The probability that this component lasts longer than t hours is $e^{-\frac{t}{\mu}}$, where μ is a constant.

(i) Derive expressions for

(a) the probability density function (pdf) for the lifetime of the component (in hours);

(b) the hazard function.

(ii) Show that the mean lifetime is μ.

Suppose the mean lifetime $\mu = 1500$ hours.

(iii) What is the probability that the component will last more than twice the mean lifetime?

(iv) If the component is still functioning after 1500 hours, what is the probability it will last another 1500 hours?

The maintenance contract for the air-conditioning system provides a free repair service if the system breaks down within 1500 hours of the last breakdown. If the system breaks down after 1500 hours, there is a £200 call out charge. If the air-conditioning is still functioning after 1500 hours, the computer manager decides that it is more economical to replace the existing component with a new one (which only costs £50) in the belief that this will reduce the risk of the system failing and hence incurring the £200 call out charge.

(v) Suppose that the first component functions for 1500 hours and is then replaced by a new component. What is the probability that the air-conditioner is still functioning after 3000 hours (assume the air-conditioner functions so long as it is fitted with a functioning component)?

(vi) Comment on the appropriateness of the system manager's strategy in the light of your answers to (iv) and (v), and with reference to the hazard function in (i)(b).

The manufacturer brings out an upgraded model of the air-conditioner which has two components connected in parallel. The air-conditioner works if at least one of the components operates.

(vii) How much better is the reliability of the new system after 3000 hours compared to the old system (assume that the components used in the new system operate independently and are of the same type as used in the old system)?

PLEASE TURN OVER

2. A company mass-produces electrical circuit boards with a nominal resistance of 10 ohm. The company's quality control procedure involves randomly selecting 24 circuit boards from the current batch (which comprises a large number of boards) on the production line and testing the actual resistance of each board. The batch is accepted if fewer than 2 circuit boards in the sample are found to have a resistance less than 7.5 ohm; otherwise the batch is rejected and the production line equipment is checked for faults.

Let π denote the probability that any given circuit board has resistance less then 7.5 ohm.

(i) Write down an expression for the probability, as a function of π, that the batch is rejected.

An alternative quality control procedure is proposed in which a random sample of 12 circuit boards from the current production batch is selected and tested. The batch is accepted if all 12 boards have resistance of at least 7.5 ohm; the batch is rejected if 2 or more boards have resistance less than 7.5 ohm; if 1 of the boards has resistance less than 7.5 ohm, a second sample of 12 boards is randomly selected from the same batch. If this sample also contains at least 1 board with resistance less than 7.5 ohm then the batch is rejected.

(ii) Taking $\pi = 0.06$, calculate the probability that:

(a) the second sample of 12 boards is needed;

(b) the batch is rejected.

(iii) The company produces a large number of batches per week and decides to adopt the second quality control scheme to test each batch. What is the average number of circuit boards the company can expect to sample per batch?

The manufacturer decides to change the way the circuit boards are produced. In each circuit board the nominal resistance of 10 ohm is now obtained by connecting in series two resistors each nominally 5 ohm, each operating independently. The true resistance of the resistors can be approximated by a normal distribution with mean $\mu = 5$ ohm and standard deviation $\sigma = 1.3$ ohm.

(iv) What is the distribution of the combined resistances of two resistors connected in series?

(v) Evaluate π, the probability that the combined resistance of two resistors in a given circuit board is less than 7.5 ohm.

END OF PAPER

MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

SESSION : 2000 - 2001

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PAPER ISE 2.6

QUESTION

SOLUTION 1

$$\int_{-\infty}^{\infty} f(-t)\, e^{-i\omega t}\, dt = \int_{\infty}^{-\infty} f(s)\, e^{-i\omega(-s)}(-ds), \quad (s=-t)$$

$$= \int_{-\infty}^{\infty} f(s)\, e^{-i(-\omega)s}\, ds = \hat{f}(-s).$$ 2

With $f(t)$ as given $\hat{f}(\omega) = \int_0^{\infty} e^{-at} e^{-i\omega t}\, dt$

$$= \int_0^{\infty} e^{-(a+i\omega)t}\, dt = \left[-(a+i\omega)^{-1} e^{-(a+i\omega)t}\right]_0^{\infty} = \frac{1}{a+i\omega}$$

where $\lim_{t\to+\infty} e^{-(a+i\omega)t} = 0$ since $|e^{-(a+i\omega)t}| = e^{-at}$. 2

$$\hat{g}(\omega) = \int_{-\infty}^{\infty} (f(t) + f(-t))\, e^{-i\omega t}\, dt$$

$$= \hat{f} + \widehat{f(-t)} = \hat{f}(\omega) + \hat{f}(-\omega) \text{ using first part}$$

$$= \frac{2}{1+\omega^2}$$ 2

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{+i\omega t}\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2e^{i\omega t}}{1+\omega^2}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big(\underbrace{\frac{2\cos\omega t}{1+\omega^2}}_{\text{even}} + i\underbrace{\frac{2\sin\omega t}{1+\omega^2}}_{\text{odd}}\Big)\, d\omega$$

$$= \frac{2}{\pi}\int_0^{\infty} \frac{\cos\omega t}{1+\omega^2}\, d\omega$$

$$\therefore \int_0^{\infty} \frac{\cos\omega}{1+\omega^2}\, d\omega = \frac{\pi}{2} f(1) = \pi/2e.$$ 4

$\hat{h}(\omega) = \dfrac{1}{2+i\omega} - \dfrac{1}{3+i\omega}$, by partial fractions 2

So using second part of question, $h(t) = \begin{cases} e^{-2t} - e^{-3t}, & t \geq 0, \\ 0, & t < 0. \end{cases}$ 2

<u>Check:</u> $\hat{h}(\omega) = \hat{h}_1(\omega)\,\hat{h}_2(\omega)$

where $h_1(t) = e^{-2t}$, $h_2(t) = e^{-3t}$ for $t \geq 0$

and $h_1(t) = h_2(t) = 0$ for $t < 0$.

$$\therefore\ h(t) = (h_1 * h_2)(t) = \int_{-\infty}^{\infty} h_1(u)\, h_2(t-u)\, du$$ 3

so $h(t) = 0$ for $t < 0$

$$\left.\begin{aligned} \text{and } h(t) &= \int_0^t e^{-2u} e^{-3(t-u)}\, du \\ &= e^{-3t}\int_0^t e^{u}\, du = e^{-2t} - e^{-3t} \end{aligned}\right\} \text{for } t \geq 0$$ 3

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(i) Putting $\bar{f}(p) = \int_0^\infty f(t) e^{-pt}\, dt$ and using the convolution result

$$\bar{f} = 6p^{-2} - 4\bar{f}\, p^{-2} \quad \text{(Tables for L.T. of } t\text{)}$$ 3

$$\therefore \quad \bar{f}(p) = \frac{6}{4+p^2}.$$ 2

Inverting (using tables), $f(t) = 3\sin 2t$. 2
(7)

(ii) Writing $I = \int_C P\,dx + Q\,dy$,

$$\frac{\partial P}{\partial y} = 6x^2y + e^{xy} + xye^{xy} = \frac{\partial Q}{\partial x}$$

so path independence follows. 2

Hence there exists a potential $F(x,y)$.

$\frac{\partial F}{\partial x} = P$ gives $F = x^3y^2 + e^{xy} + g(y)$ 2

$\frac{\partial F}{\partial y} = Q$ implies $2x^3y + xe^{xy} + \frac{dg}{dy} = Q$

so $\frac{dg}{dy} = \pi\cos(\pi y)$ $\quad\therefore$ Take $g = \sin(\pi y)$

$$F = x^3y^2 + e^{xy} + \sin(\pi y).$$ 3

In given case $I = \left[F\right]_{(0,0)}^{(1,2)} = 3 + e^2.$ 2
(9)

Parameterise by $x(t) = t,\ y(t) = 2t,\ 0 \le t \le 1$. 1

$$\therefore I = \int_{t=0}^{1} \left\{ P(x(t),y(t))\frac{dx}{dt} + Q(x(t),y(t))\frac{dy}{dt} \right\} dt$$

$$= \int_0^1 \left\{ (12t^4 + 2te^{2t^2})\,1 + (4t^4 + te^{2t^2} + \pi\cos(2\pi t))\,2 \right\} dt$$ 2

$$= \int_0^1 \left(20t^4 + 4te^{2t^2} + 2\pi\cos(2\pi t)\right) dt$$

$$= \left[4t^5 + e^{2t^2} + \sin 2\pi t\right]_0^1$$

$$= 3 + e^2.$$ 1
(4)

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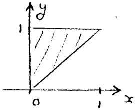
(i) Integral

$$= \int_0^1 dy \int_0^y x\,(1+3y^3)^{1/2}\,dx$$ 2, for diagram 2

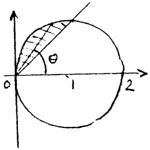


$$= \int_0^1 \left[\frac{x^2}{2}(1+3y^3)^{1/2} \right]_{x=0}^{y} dy$$

$$= \int_0^1 \frac{1}{2} y^2 (1+3y^3)^{1/2}\,dy$$ 2

$$= \left[\frac{1}{2}\,\frac{1}{9}\,\frac{2}{3}\,(1+3y^3)^{3/2} \right]_0^1 = \frac{7}{27}.$$ 2 (8)

(ii) $\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$. 1 for θ



Equation of circle : $(x-1)^2+y^2=1$.

Point (r,θ) lies on circle if

$(r\cos\theta-1)^2+r^2\sin^2\theta = 1$

giving $r^2-2r\cos\theta=0$ so $r=0$ or $2\cos\theta$. 3

$$\therefore \text{ Integral } = \iint_A \frac{2r^2\cos\theta\sin\theta}{r}\,r\,dr\,d\theta$$ 1

$$= \int_{\pi/4}^{\pi/2} d\theta \int_0^{2\cos\theta} 2r^2\cos\theta\sin\theta\,dr$$ 2

$$= \int_{\pi/4}^{\pi/2} \left[\frac{2}{3} r^3 \cos\theta\sin\theta \right]_{r=0}^{2\cos\theta}$$

$$= \frac{16}{3}\int_{\pi/4}^{\pi/2} \cos^4\theta\sin\theta\,d\theta$$ 2

$$= \frac{16}{3}\left[-\frac{\cos^5\theta}{5} \right]_{\pi/4}^{\pi/2}$$ 2

$$= \frac{16}{3}\,\frac{1}{5 \cdot 2^{5/2}} = \frac{2^{3/2}}{15}.$$ 1

(12)

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MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

SESSION : 2000 - 2001

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PAPER 15E26

QUESTION

SOLUTION 4

$(z^2+1)(z^2+4) = (z-i)(z+i)(z-2i)(z+2i)$ has simple zeros at i and $2i$, so that

$$\mathrm{Res}(f, i) = \left.\frac{1}{\frac{d}{dz}(z^2+1)(z^2+4)}\right|_{z=i} = \left.\frac{1}{2z(z^2+4)}\right|_{z=i} = \frac{1}{6i},$$

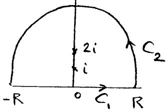
and similarly $\mathrm{Res}(f, 2i) = \left.\dfrac{1}{(z^2+1)2z}\right|_{z=2i} = -\dfrac{1}{12i}$. 5

(or consider coefficients of $(z-i)^{-1}$ and $(z-2i)^{-1}$ in $f(z)$).

By the Residue Theorem

$\int_C f(z)dz = 2\pi i \times \{$sum of residues of f inside $C\}$

$$= 2\pi i \left\{\frac{1}{6i} - \frac{1}{12i}\right\} = \frac{\pi}{6}.$$ 3



Also $\int_C f(z)\,dz = I_1 + I_2$ where

$$I_1 = \int_{C_1} f(z)dz = \int_{-R}^{R} f(x)\,dx = \int_{-R}^{R} \frac{1}{(x^2+1)(x^2+4)}\,dx.$$ 3

For z on C_2, $|z^2+1| \geq |z^2| - 1 = R^2-1$

and similarly $|z^2+4| \geq R^2-4$

so $\left|\dfrac{1}{(z^2+1)(z^2+4)}\right| \leq \dfrac{1}{(R^2-1)(R^2-4)}$, $(R>2)$.

Hence by the M-L Inequality

$$|I_2| \leq \frac{1}{(R^2-1)(R^2-4)}\pi R$$

so $I_2 \to 0$ as $R \to +\infty$. 5

Hence $\displaystyle\int_{-R}^{R} \frac{dx}{(x^2+1)(x^2+4)} = I_1 = \int_C f(z)\,dz - I_2$

$$= \frac{\pi}{6} - I_2$$ 2

and letting $R \to +\infty$ obtain

$\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+4)} = \frac{\pi}{6}$. Integrand is even so required $\int$ is $\frac{\pi}{12}$. 2

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MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

SESSION : 2000 - 2001

PAPER ISE2.6

QUESTION

SOLUTION B1

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1. (a) (i)

$$\text{Reliability, } R(t) = e^{-\frac{t}{\mu}}$$
$$F(t) = 1 - e^{-\frac{t}{\mu}}$$
$$\text{pdf, } f(t) = F'(t) = \frac{1}{\mu}e^{-\frac{t}{\mu}}$$ 3

(ii)

$$\text{Hazard, } h(t) = \frac{\frac{1}{\mu}e^{-\frac{t}{\mu}}}{e^{-\frac{t}{\mu}}} = \frac{1}{\mu}$$ 2

(b)

$$\text{Mean } = E(t) = \int_0^\infty \frac{t}{\mu}e^{-\frac{t}{\mu}}dt$$

Integration by parts:

$$\int u\,dv = uv - \int v\,du$$
$$u = -t \quad \frac{dv}{dt} = -\frac{1}{\mu}e^{-\frac{t}{\mu}}$$
$$\frac{du}{dt} = -1 \quad v = e^{-\frac{t}{\mu}}$$

$$= \left[-te^{-\frac{t}{\mu}}\right]_0^\infty + \int_0^\infty e^{-\frac{t}{\mu}}dt = 0 + \left[- \quad e^{-\frac{t}{\mu}}\right]_0^\infty = \mu$$ 3

(c)

$$R(3000) = e^{-\frac{3000}{1500}} = e^{-2} = 0.1353$$ 2

(d)

$$P(t > 3000|t > 1500) = \frac{R(3000)}{R(1500)} = 0.3679$$ 2

$$OR\ P(t > 3000|t > 1500) = R(1500) = 0.3679 \text{ (by memoryless property)}$$

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(e)

$$R(1500) = e^{-\frac{1500}{1500}} = e^{-1} = 0.3679$$

2

(f) If the original component is still functioning after 1500 hours, then the probability of the air-conditioner subsequently breaking down are not affected by whether the original component is retained or replaced by a new one. This is because the hazard function for each component is constant and so the hazard (risk of failure) doesn't depend on how long the component has already operated for (i.e. it is memoryless). The system manager's strategy is therefore a false economy. 3

(g) Let C_1 denote event that component 1 fails before 3000 hours
Let C_2 denote event that component 2 fails before 3000 hours
Let A denote event that air-conditioner fails before 3000 hours
So $\overline{A}$ denotes event that air-conditioner still functions after 3000 hours 3

$$\begin{aligned} \mathrm{P}(A) &= \mathrm{P}(C_1 \cap C_2) = (1 - 0.1353)^2 = 0.7477 \\ \mathrm{P}(\overline{A}) &= 1 - 0.7477 = 0.2523 \end{aligned}$$

So reliability at 3000 hours has increased by nearly 12% from 13.53% to 25.23%.

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2. (a) Let r be number of circuit boards with resistance < 7.5 ohm. Number of circuit boards with a resistance < 7.5 ohm in a random sample of 24 has binomial(24, π) distribution.

$$\begin{aligned}
\text{P(batch rejected)} &= 1 - \text{P(batch accepted)} \\
&= 1 - (\text{P}(r=0) + \text{P}(r=1)) \\
&= 1 - \sum_{x=0}^{1} \binom{24}{x} \pi^x (1-\pi)^{24-x}.
\end{aligned}$$

OR

$$\begin{aligned}
\text{P(batch rejected)} &= 1 - ((1-\pi)^{24} + 24\pi(1-\pi)^{23}) \\
&= 1 - (1-\pi)^{23}(1+23\pi)
\end{aligned}$$

3

(b) (i) P(2nd sample needed) = P(1st sample contains 1 defective board) $= 12 \times 0.06(1-0.06)^{11} = 0.3645$ 3

(ii) Batch is rejected if either the 1st sample contains 2 or more defective boards or the second sample is needed and contains at least 1 defective board.

$$\begin{aligned}
\text{P}(r \geq 2 \text{ in sample 1}) &= 1 - \sum_{x=0}^{1} \binom{12}{x} 0.06^x (1-0.06)^{12-x} \\
&= 1 - 0.8405 = 0.1595 \\
\text{P}(r \geq 1 \text{ in sample 2}) &= 1 - (1-0.06)^{12} \\
&= 1 - 0.4759 = 0.5241 \\
\text{P(batch rejected)} &= 0.1595 + 0.3645 \times 0.5241 = 0.3505
\end{aligned}$$

5

(c) Probability of needing second sample is 0.3645, so quality control process uses samples of size 12 with probability $1 - 0.3645 = 0.6355$ and samples of size 24 with probability 0.3645. Average sample size is therefore $12 \times 0.6355 + 24 \times 0.3645 = 16.37$. 3

(d) Let Y be a random variable denoting the combined resistance of two resistors connected in series. Then Y is normally distributed with mean $5+5 = 10$ ohm and standard deviation $\sqrt{1.3^2 + 1.3^2} = 1.838$. 2

(e)

$$\begin{aligned}
\pi &= \text{P}(Y < 7.5) = \text{P}\left(\frac{Y-10}{1.838} < \frac{7.5-10}{1.838}\right) \\
&= \text{P}\left(\frac{Y-10}{1.838} < -1.36\right) = \Phi(-1.36) = 1 - \Phi(1.36) = 1 - 0.9131 \\
&= 0.0869
\end{aligned}$$

4

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