

The Questions

1

- 1.1. Define the root moments $\{S_m\}$ of the real polynomial $f(z) = K \prod_{i=1}^n (1 - r_i z^{-1})$ where m is the degree of the moment, and comment on their dependence on r_i $i = 1, 2, \dots, n$ as $m \rightarrow \infty$.

[2]

- 1.2. A Finite Impulse Response transfer function is of the form

$$H(z) = K \prod_{i=1}^{n_1} (1 - \alpha_i z^{-1}) \prod_{i=1}^{n_2} (1 - \beta_i z^{-1}),$$

where K is a constant, α_i are the zeros inside the unit circle and β_i are the zeros outside the unit circle.

$$\text{Set } N_1(z) = \prod_{i=1}^{n_1} (1 - \alpha_i z^{-1}) \text{ and } N_2(z) = \prod_{i=1}^{n_2} (1 - \beta_i z^{-1}),$$

Show that if $H(z)$ is real then the root moments of both $N_1(z)$ and $N_2(z)$ are also real.

[2]

- 1.3. Given the amplitude and phase responses are $A(\theta)$ and $\phi(\theta)$ of $H(z)$ derive the *Fundamental Relationships*

$$\ln(A(\theta)) = \ln(K_1) - \sum_{m=1}^{\infty} \frac{S_m^{N_1} + S_{-m}^{N_2}}{m} \cos(m\theta),$$

$$\phi(\theta) = -n_2 \theta + \sum_{m=1}^{\infty} \frac{S_m^{N_1} - S_{-m}^{N_2}}{m} \sin(m\theta)$$

where K_1 is an appropriate real constant, $S_m^{N_1}$ are the root moments of the minimum phase factor and $S_{-m}^{N_2}$ the inverse root moments of the maximum phase factor of $H(z)$.

[7]

- 1.4. Hence show that if the transfer function $H(z)$ is linear phase then it must have zeros located outside the unit circle, and determine their number in relation to the number of zeros located inside the unit circle.

[4]

- 1.5. Determine the *Fundamental Relationships* for the allpass transfer function

$$H(z) = \prod_{i=1}^m A_i(z) \text{ where } A_i(z) = \left(\frac{z^{-1} - \alpha_i^*}{1 - \alpha_i z^{-1}} \right).$$

[5]

2.

2.1. Define the normalised group delay $\tau(\theta)$ of a discrete time system of transfer function $H(z)$ and show that if on the unit circle $H(e^{j\theta}) = A(\theta)e^{j\phi(\theta)}$ then we may write

$$\tau(\theta) = -\text{Im} \left[\frac{d}{d\theta} (\ln H(e^{j\theta})) \right]. \quad [2]$$

2.2. Let the transfer function of a real allpass system of order m that has no real zeros be

given by $H(z) = \prod_{i=1}^m A_i(z)$ where $A_i(z) = \left(\frac{1 - \alpha_i^* z}{z - \alpha_i} \right)$, $\alpha_i = \rho_i e^{j\psi_i}$ and $|\rho_i| < 1$. Show that

$$\text{the phase response of } A_i(z) \text{ is given by } \arg(A_i(e^{j\theta})) = -\theta - 2 \arctan \frac{\rho_i \sin(\theta - \psi_i)}{1 - \rho_i \cos(\theta - \psi_i)}. \quad [2]$$

2.3. Determine an expression for the overall group delay $\tau(\theta)$ of the real allpass $H(z)$ defined as above. [4]

2.4. Show that for ρ_i as above and for any ψ_i , $\int_0^{2\pi} \frac{d}{d\theta} \frac{\rho_i \sin(\theta - \psi_i)}{1 - \rho_i \cos(\theta - \psi_i)} d\theta = 0$ [2]

2.5. Hence determine the average group delay $\tau_{av} = \frac{1}{2\pi} \int_0^{2\pi} \tau(\theta) d\theta$ and explain the significance of this result. [5]

2.6. Show that the group delay $\tau(\theta)$ of the real allpass is always positive. [5]

3.

3.1. A real digital filter transfer function $H_N(z)$ is given by

$$H_N(z) = \frac{p_0 + p_1 z^{-1} + \dots + p_{N-1} z^{-(N-1)} + p_N z^{-N}}{1 + d_1 z^{-1} + \dots + d_{N-1} z^{-(N-1)} + d_N z^{-N}}.$$

It is proposed to realise this transfer function as in Figure 1 where

$$H_N(z) = Y_1 / X_1.$$

The subsystem S in the figure is linear and is characterised by the relationships $Y_1 = AX_2 + BY_2$ and $X_1 = CX_2 + DY_2$. Express $H_N(z)$ as a function of $H_{N-1}(z)$ and A, B, C, D . Determine an expression for $H_{N-1}(z)$ in terms of $H_N(z)$ and the parameters A, B, C, D . [5]

3.2. By examining $\left[H_N(z) - \frac{B}{D} \right]$, or otherwise, determine the condition under which $H_N(z)$ is independent of $H_{N-1}(z)$. [2]

3.3. The parameters of S are chosen so as to make $H_{N-1}(z)$ of degree $(N-1)$. Verify that the following choice satisfies the requirements: $A = p_N z^{-1}$, $B = p_0$, $C = d_N z^{-1}$, $D = 1$. [3]

3.4. Discuss other possible and alternative non-trivial values for these parameters. [3]

3.5. For the given selection above derive the coefficients of $H_{N-1}(z)$ in terms of the coefficients of $H_N(z)$. Explain how such a procedure may be used iteratively to realise a given transfer function, assuming no terms become infinite. For the given selection of parameters as in 3.3 above, produce a minimal component realisable signal flow graph in terms of appropriate adders, multipliers and delays, indicating the first step of the iteration [7]

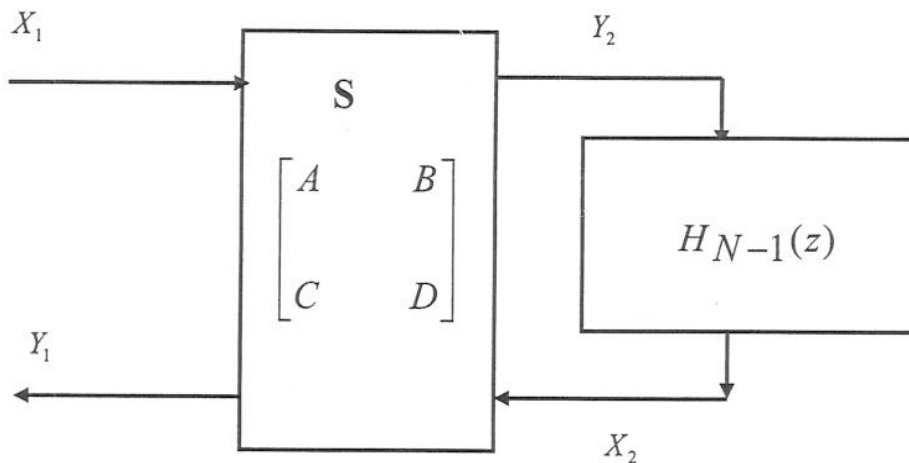


Figure 1

4.

4.1. Consider an ideal linear phase lowpass digital filter transfer function $H(z)$. On the unit

circle $z = e^{j\theta}$, the function $H(z)$ takes the values $H(e^{j\theta}) = \begin{cases} e^{-jM\theta} & -\frac{\pi}{M} \leq \theta \leq \frac{\pi}{M} \\ 0 & \text{elsewhere} \end{cases}$

where M is a positive integer. Sketch the amplitude response of $H(e^{j(\theta - \frac{2\pi}{M}r)})$ for $r = 0$, $r = 1$ and $r = 2$. [3]

4.2. Show that the frequency response shown below is allpass and determine its phase response [4]

$$G(e^{j\theta}) = \sum_{r=0}^{M-1} H(e^{j(\theta - \frac{2\pi}{M}r)}).$$

4.3. Let $H(z)$ be expressed as $H(z) = \sum_{r=0}^{M-1} z^{-r} H_r(z^M)$ where $H_r(z)$ are some appropriate

subfilter transfer functions. By replacing z by $ze^{-j\frac{2\pi}{M}k}$ in the expression above for $H(z)$ and summing over k , or otherwise, show that the subfilter transfer function

$$H_0(z^M) \text{ is given by the expression } H_0(e^{jM\theta}) = \frac{1}{M} \sum_{r=0}^{M-1} H(e^{j(\theta - \frac{2\pi}{M}r)}). [10]$$

4.4. What is the amplitude response of $H_0(z^M)$? [3]

5.

5.1. Explain what is meant by terms computational complexity and twiddle factors in the context of evaluating the Discrete Fourier Transform (DFT), and derive the computational complexity of a N-point DFT. [4]

5.2. It is given that $N = N_1 N_2$ with N_1 and N_2 co-prime. It is proposed to carry out on the data array $\{x(n)\}$, $0 \leq n \leq N - 1$, the following 1-D to 2-D mapping

$$n = \langle An_1 + Bn_2 \rangle_N \quad \begin{cases} 0 \leq n_1 \leq N_1 - 1 \\ 0 \leq n_2 \leq N_2 - 1 \end{cases} \quad k = \langle Ck_1 + Dk_2 \rangle_N \quad \begin{cases} 0 \leq k_1 \leq N_1 - 1 \\ 0 \leq k_2 \leq N_2 - 1 \end{cases} \quad \text{where}$$

$\langle M \rangle_N$ means a reduction of the number M modulo N . Derive the conditions that must prevail on the products AC , BD , AD , and BC in order that all possible twiddle factors in the 2-D DFT computation are eliminated. [10]

5.3. Show that the following set of parameters satisfies these conditions $A = N_2$,

$B = N_1$, $C = N_2 \langle N_2^{-1} \rangle_{N_1}$, $D = N_1 \langle N_1^{-1} \rangle_{N_2}$ where $\langle L^{-1} \rangle_P$ denotes the multiplicative inverse of L evaluated modulo P . [3]

5.4. Hence outline the algorithm for the computation of the N-point DFT. [3]

Question 1

SOLUTIONS

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1.1 The root moments S_m are defined as the sum of powers of the roots

$$S_m = \sum_{i=1}^n r_i^m$$

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If $|r_i| < 1$ $|S_m| \rightarrow 0$ as $m \rightarrow \infty$

If $|r_i| > 1$ $|S_m| \rightarrow \infty$ as $m \rightarrow \infty$

2

1.2 If $H(z)$ is real then for every complex α_i is in the RHS, \exists another factor containing α_i^* . Similarly with β_i .

$$\text{Hence } \sum_{i=1}^n \alpha_i^m = \sum_{i=1}^{n/2} \alpha_i^m + (\alpha_i^*)^m \rightarrow \text{real}$$

Similarly with β_i

2

1.3 The Fundamental Relationships involve taking logarithms and thus

$$\ln H(z) = \ln K + \sum_{i=1}^{n_1} \ln(1 - \alpha_i z^{-1}) + \sum_{i=1}^{n_2} \ln(1 - \beta_i z^{-1})$$

The infinite power series involve Taylor expansions but the last term needs to be re-expressed for convergence as $\sum_{i=1}^{n_2} \ln \left[(-\beta_i z^{-1}) \left(1 - \frac{1}{\beta_i} z^{-1}\right) \right]$

and hence

$$\ln H(z) = \ln K + \sum_{i=1}^{n_1} \ln(-\alpha_i) - n_2 \ln z - \sum_{m=1}^{\infty} \frac{S_m^{N_1}}{m} z^{-m} + \frac{S_m^{N_2}}{m} z^{-m}$$

with $z = e^{j\theta}$ and $H(e^{j\theta}) = A(\theta) \exp(j\phi(\theta))$

we have after equating real with real and imaginary with imaginary

$$\ln A(\theta) = \ln K + \sum_{m=1}^{\infty} \frac{S_m^{N_1} + S_m^{N_2}}{m} \cos m\theta$$

$$\text{and } \phi(\theta) = -n_2\theta + \sum_{m=1}^{\infty} \frac{S_m^{N_1} - S_m^{N_2}}{m} \cdot \sin m\theta$$

$$\text{where } k_1 = \ln k + \sum_{i=1}^{N_2} \ln(-\beta_i)$$

1.4 From the phase expression it is seen that if $S_m^{N_1} = -S_m^{N_2}$ then $\phi(\theta) = -n_2\theta$

i.e. the phase is precisely linear.

The root moments of N_1 and N_2 must have the above relationship $\forall m$. i.e. there are roots located outside $|z|=1$ when there are roots inside $|z|=1$. There are as many roots in one region as there are in the other in a reciprocal pairing

1.5 If $H(z)$ is allpass then $A(\theta) = 1$ since $|A_i(z)| = 1$

$$\text{For any allpass } \ln A_i(z) = \ln(z^{-1} - \alpha_i) - \ln(1 - \alpha_i z^{-1})$$

$$\text{or } \ln A_i(z) = \ln[z^{-1}(1 - \alpha_i z)] - \ln(1 - \alpha_i z^{-1})$$

$$\begin{aligned} &= -\ln z + \ln(1 - \alpha_i z) - \ln(1 - \alpha_i z^{-1}) \\ &= -\ln z - \left(\alpha_i z + \frac{\alpha_i^2 z^2}{2} + \frac{\alpha_i^3 z^3}{3} + \dots \right) \\ &\quad + \left(\alpha_i z^{-1} + \frac{\alpha_i^2 z^{-2}}{2} + \frac{\alpha_i^3 z^{-3}}{3} + \dots \right) \end{aligned}$$

$$= -\ln z - \left[\alpha_i (z - z^{-1}) + \frac{\alpha_i^2}{2} (z^2 - z^{-2}) + \frac{\alpha_i^3}{3} (z^3 - z^{-3}) + \dots \right]$$

On $z = e^{j\theta}$

$$\ln A_i(e^{j\theta}) = -j\theta - 2j \left[\alpha_i \sin \theta + \frac{\alpha_i^2}{2} \sin 2\theta + \frac{\alpha_i^3}{3} \sin 3\theta + \dots \right]$$

i.e. as expected completely imaginary and thus $\phi(\theta) = -\theta - 2 \sum_{m=1}^{\infty} \frac{S_m^{N_1}}{m} \sin m\theta$

where $s_{\mu}^{A_i} = \alpha_i^{\mu}$ and for $H(z)$

$$s_{\mu}^H = \sum_{i=1}^m \alpha_i^{\mu}$$

, the root moments of the denominator.

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Question 2.

2.1 The group delay $\tau(\theta)$ is defined as

$$\tau(\theta) = - \frac{d\phi(\theta)}{d\theta}$$

where $\phi(\theta)$ is the unwrapped phase response

from $H(e^{j\theta}) = A(\theta) \cdot e^{j\phi(\theta)}$ we have

$$\ln H(e^{j\theta}) = \ln A(\theta) + j\phi(\theta)$$

and hence

$$\tau(\theta) = - \operatorname{Im} \frac{d \ln H(e^{j\theta})}{d\theta}$$

2

2.2

Set $z = e^{j\theta}$ $\alpha_i = p_i e^{j\psi_i}$ in $A_i(z)$
so that

$$\begin{aligned} A_i(e^{j\theta}) &= \frac{1 - p_i e^{-j\psi_i} e^{j\theta}}{e^{j\theta} - p_i e^{j\psi_i}} \\ &= e^{-j\theta} \cdot \frac{1 - p_i e^{+j(\theta - \psi_i)}}{1 - p_i e^{-j(\theta - \psi_i)}} \\ &= e^{-j\theta} \cdot \frac{B_i(\theta) \cdot e^{-j\mu_i(\theta)}}{B_i(\theta) \cdot e^{+j\mu_i(\theta)}} \end{aligned}$$

where

$$B_i(\theta) = |1 - p_i e^{+j(\theta - \psi_i)}| = |1 - p_i e^{-j(\theta - \psi_i)}|$$

$$\mu_i(\theta) = \tan^{-1} \frac{p_i \sin(\theta - \psi_i)}{1 - p_i \cos(\theta - \psi_i)}$$

Hence
$$\phi_i(\theta) = -\theta - 2 \tan^{-1} \frac{p_i \sin(\theta - \psi_i)}{1 - p_i \cos(\theta - \psi_i)}$$

2

2.3 The group delay associated with $\phi_i(\theta)$ is

$$\tau_i(\theta) = 1 + 2 \cdot \frac{d}{d\theta} \cdot \tan^{-1} \frac{p_i \sin}{1 - p_i \cos}$$

where $s = \sin(\theta - \psi_i)$ $c = \cos(\theta - \psi_i)$

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$$\begin{aligned} \tau_i(\theta) &= 1 + 2 \frac{1}{1 + \left(\frac{p_i s}{1 - p_i c}\right)^2} \cdot \frac{p_i [c(1 - p_i c) - s p_i s]}{(1 - p_i c)^2} \\ &= 1 + \frac{2 p_i (c - p_i)}{(1 - p_i c)^2 + (p_i s)^2} \end{aligned}$$

Hence

$$\tau(\theta) = \sum_{i=1}^m \tau_i(\theta) = m + 2 \sum_{i=1}^m \frac{p_i (c - p_i)}{(1 - p_i c)^2 + (p_i s)^2}$$

4

2.4

Since

$$\begin{aligned} I &= \int_0^{2\pi} \frac{d}{d\theta} \cdot \frac{p_i \sin(\theta - \psi_i)}{1 - p_i \cos(\theta - \psi_i)} d\theta \\ &= \int_0^{2\pi} d \left[\frac{p_i \sin(\theta - \psi_i)}{1 - p_i \cos(\theta - \psi_i)} \right] \end{aligned}$$

C

It follows that

$$I = \left. \frac{p_i \sin(\theta - \psi_i)}{1 - p_i \cos(\theta - \psi_i)} \right|_0^{2\pi} = 0.$$

2

2.5

$$\tau_{av} = \frac{1}{2\pi} \int_0^{2\pi} \tau(\theta) \cdot d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left(m + 2 \frac{d}{d\theta} \left(\frac{p_i s}{1 - p_i c} \right) \right) \cdot d\theta$$

C

and in view of 2.4

$$\tau_{av} = m$$

The average group delay is equal to the number of zeros or degree of the allpass

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2.6 From 2.3 we have

$$\begin{aligned}T_i(\theta) &= 1 + \frac{2p_i(c - p_i)}{(1 - p_i c)^2 + (p_i s)^2} \\&= \frac{(1 - p_i c)^2 + (p_i s)^2 + 2p_i c - 2p_i^2}{(1 - p_i c)^2 + (p_i s)^2} \\&= \frac{1 - p_i^2}{(1 - p_i c)^2 + (p_i s)^2}\end{aligned}$$

and since $0 < p_i < 1$ it follows that both numerator $[1 - p_i^2]$ and denominator (a sum of squares) are positive
i.e. $T_i(\theta) \geq 0$

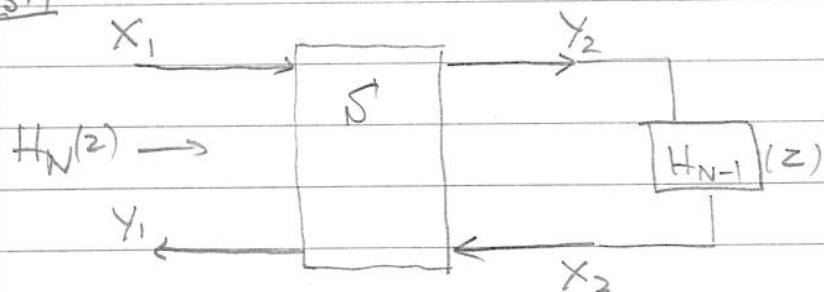
$$\text{Hence } T(\theta) = \sum_{i=1}^M T_i(\theta) \geq 0$$

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Question 3

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3.1



$$H_N(z) = Y_1 / X_1 \quad \begin{aligned} Y_1 &= AX_2 + BY_2 \\ X_1 &= CX_2 + DY_2 \end{aligned}$$

Use $X_2 = Y_2 \cdot H_{N-1}$ so that

$$Y_1 = A Y_2 H_{N-1} + B Y_2$$

$$X_1 = C Y_2 H_{N-1} + D Y_2$$

and hence

$$\frac{Y_1}{X_1} = \frac{A H_{N-1} + B}{C H_{N-1} + D}$$

$$\text{i.e. } H_N(z) = \frac{A H_{N-1}(z) + B}{C H_{N-1}(z) + D} \quad (1)$$

5

3.2

Examine $H_N(z) - \frac{B}{D} = T(z)$ say

$$T(z) = \frac{A H_{N-1}(z) + B}{C H_{N-1}(z) + D} - \frac{B}{D} = \frac{(AD - BC) H_{N-1}(z)}{D (C H_{N-1}(z) + D)}$$

Thus if $AD - BC = 0$ then $T(z) = 0$ and hence

$$H_N(z) = B/D$$

ie independent of $H_{N-1}(z)$.

2

3.3

Write equ(1) (3.1) in terms of $H_{N-1}(z)$ on LHS.

$$\text{i.e. } H_N \cdot C \cdot H_{N-1} + H_N D = A H_{N-1} + B$$

$$\text{or } H_{N-1} = \frac{(B - D H_N)}{(C H_N - A)}$$

then with the given expression for $H_N(z)$ we have

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$$H_{N-1}(z) = \frac{B(1+d_1 z^{-1} + \dots + d_N z^{-N}) - D(p_0 + p_1 z^{-1} + \dots + p_N z^{-N})}{C(p_0 + p_1 z^{-1} + \dots + p_N z^{-N}) - A(1+d_1 z^{-1} + \dots + d_N z^{-N})}$$

For $A = p_N z^{-1}$, $B = p_0$, $C = d_N z^{-1}$ $D = 1$ we have

$$H_{N-1}(z) = \frac{0 + (p_0 d_1 - p_1) z^{-1} + \dots + (p_0 d_N - p_N) z^{-N}}{(d_N - p_N) z^{-1} + (d_N p_1 - p_N d_1) z^{-2} + \dots + (d_N p_{N-1} - p_N d_{N-1}) z^{-N} + 0}$$

There is a common factor of z^{-1} between the numerator and denominator, which upon cancellation makes $H_{N-1}(z)$ of degree N .

3

3.4

The selection of $[A, B, C, D]$ must be such that

a) $AD - BC \neq 0$ as seen in 3.2. This ensures the dependence of $H_N(z)$ on $H_{N-1}(z)$ and hence the possibility of selecting $H_{N-1}(z)$ appropriately for a given $H_N(z)$.

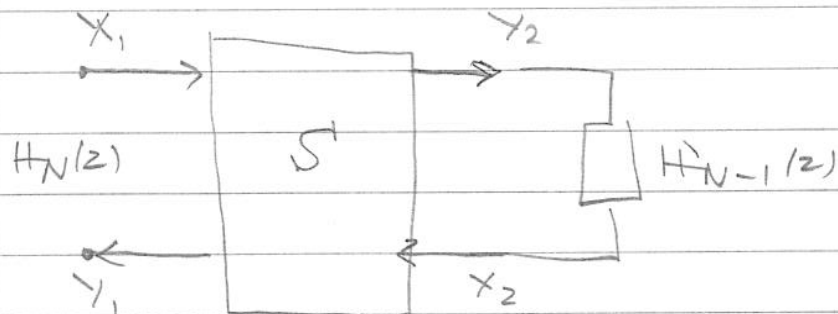
b) There needs to be a common factor for cancellation as in 3.3. The selection given is not unique, but it is one that makes the common factor very simple i.e. z^{-1} . Other selections are possible for example by making these parameters second order thereby reducing the degree of $H_{N-1}(z)$ by 2 less than the degree of $H_N(z)$.

3

3.5. The coefficients of $H_{N-1}(z)$ are given already above.

The procedure may be iterated now

with respect to $H_{N-1}(z)$ thereby producing



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$$Y_1 = AX_2 + BY_2 = AX_2 + \frac{B}{D} (X_1 - CX_2)$$

$$Y_2 = \frac{1}{D} (X_1 - CX_2)$$

$$\text{or } Y_1 = \left(A - \frac{BC}{D}\right) \cdot X_2 + \frac{B}{D} X_1$$

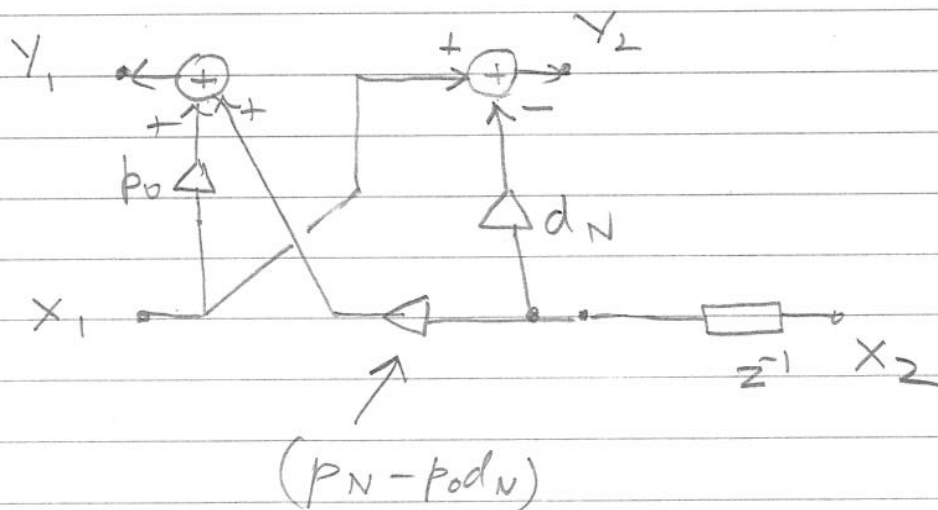
$$Y_2 = \frac{1}{D} (X_1 - CX_2)$$

For the given set

$$Y_1 = (p_N \bar{z}^{-1} - p_0 d_N \bar{z}^{-1}) X_2 + p_0 X_1$$

$$= p_0 X_1 + (p_N - p_0 d_N) \cdot \bar{z}^{-1} \cdot X_2$$

$$Y_2 = X_1 - d_N \bar{z}^{-1} \cdot X_2$$



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Question 4

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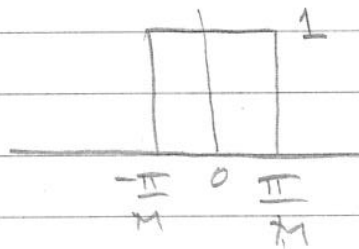
4.1 Since $|H(e^{j\theta})| = 1$ in the range $-\frac{\pi}{M} \leq \theta \leq \frac{\pi}{M}$

the function $H(e^{j(\theta - \frac{2\pi}{M}r)})$

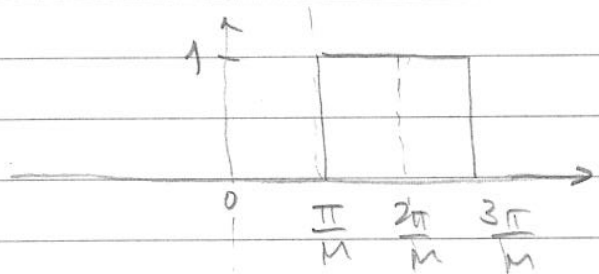
will be unity in the shifted range

$$\left(-\frac{\pi}{M} + \frac{2\pi}{M}r, \frac{\pi}{M} + \frac{2\pi}{M}r\right)$$

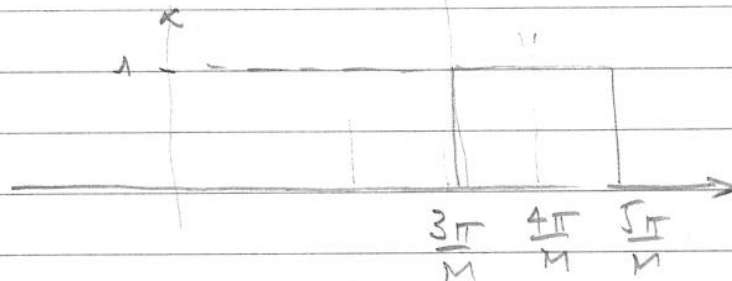
For $r=0$



For $r=1$



$r=2$



3.

4.2 From the given form for $H(z)$ we have

$$\begin{aligned} G(e^{j\theta}) &= \sum_{r=0}^{M-1} e^{-jM\left(\theta - \frac{2\pi}{M}r\right)} = e^{jM\theta} \sum_{r=0}^{M-1} e^{-jM \cdot \frac{2\pi}{M}r} \\ &= M \cdot e^{jM\theta} \end{aligned}$$

i.e. $|G(e^{j\theta})| = M$ constant for all frequencies.

Its phase response $\phi(\theta) = M\theta$.

4

4.3

Set

 $n-1$

$$H(z) = \sum_{r=0}^{n-1} z^{-r} \cdot H_r(z^M)$$

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Now replace z by $z e^{-j \frac{2\pi}{M} \cdot k}$

$$H(z e^{-j \frac{2\pi}{M} \cdot k}) = \sum_{r=0}^{M-1} z^{-r} \cdot e^{j \frac{2\pi}{M} \cdot kr} \cdot H_r(z^M)$$

Note that $H_r(z^M)$ remain the same.

Now sum over $k = 0, 1, \dots, M-1$

$$\sum_{k=0}^{M-1} H(z e^{j \frac{2\pi}{M} \cdot k}) = \sum_{r=0}^{M-1} z^{-r} \cdot H_r(z^M) \cdot \sum_{k=0}^{M-1} e^{j \frac{2\pi}{M} \cdot kr}$$

$$\text{But } \sum_{k=0}^{M-1} e^{j \frac{2\pi}{M} \cdot kr} = \frac{1 - e^{j \frac{2\pi}{M} \cdot r \cdot M}}{1 - e^{j \frac{2\pi}{M} \cdot r}} = 0$$

for any $r \neq 0$.

$$\text{For } r=0 \quad \sum_{k=0}^{M-1} e^{j \frac{2\pi}{M} \cdot kr} = \underbrace{1+1+\dots+1}_{M \text{ times}} = M$$

$$\text{i.e. } \sum_{k=0}^{M-1} H(z e^{-j \frac{2\pi}{M} \cdot k}) = M \cdot H_0(z^M)$$

$$\text{or } H_0(z^M) = \frac{1}{M} \sum_{k=0}^{M-1} H(z e^{-j \frac{2\pi}{M} \cdot k})$$

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4.4. Refer to 4.2. It is seen that the expressions are the same and hence $H_0(z^M)$ is allpass

Question 5

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5.1 The DFT requires complex multiplications and additions for its evaluation. Thus for

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j \frac{2\pi}{N} k \cdot n} \quad (k=0,1,\dots,N-1)$$

we require N complex multiplications for each value of k . The twiddle factors are the necessary phases for the implementation of the computational scheme.

For a length N we therefore need N times as many multiplications thereby producing a computational complexity of $O(N^2)$.

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5.2 It is observed from 5.1 that n and k & nk need only be taken modulo N .

For $n = An_1 + Bn_2$ and $k = Ck_1 + Dk_2$ we have

$$nk = (An_1 + Bn_2)(Ck_1 + Dk_2)$$

$$= \underbrace{ACn_1k_1}_{\text{possible DFT in } n_1} + \underbrace{ADn_1k_2 + BCn_2k_1}_{\text{twiddle factors}} + \underbrace{BDn_2k_2}_{\text{possible DFT in } n_2}$$

twiddle factors

$$\left\langle \frac{ACn_1k_1}{N} \right\rangle_N \equiv \frac{n_1k_1}{N_1} \quad \text{ie. } \left\langle \frac{AC}{N} \right\rangle_N = N_1^{-1}$$

$$\text{or } AC = N_2$$

$$\langle AD \rangle_N = 0$$

$$\langle BC \rangle_N = 0$$

$$\text{and } \left\langle \frac{BD}{N} \right\rangle_N \equiv \frac{n_2 k_2}{N_2} \quad \text{or } \left\langle \frac{BD}{N} \right\rangle_N = N_2^{-1} \quad \frac{13}{13}$$

$$\text{or } BD = N_1$$

5.3 Form $AC = N_2 \cdot N_2 \langle N_2^{-1} \rangle_{N_1} = N_2 \pmod{N}$ 10

$$AD = N_1 N_2 \langle N_1^{-1} \rangle_{N_2} \equiv 0 \quad -u-$$

$$BC = N_1 N_2 \langle N_2^{-1} \rangle_{N_1} \equiv 0 \quad -u-$$

$$BD = N_1 N_1 \langle N_1^{-1} \rangle_{N_2} \equiv N_1 \quad -n-$$

Hence the given values satisfy the required condition 3

5.4

The algorithm proceeds as follows.

- The data is sectioned into lengths of N_2 and placed as consecutive rows (columns) in a 2-D array.
- The 1-D DFT of each row (column) is carried out and placed in the same location.
- The 1-D DFT of each column (row) is carried out and placed in the same location.
- The 1-D DFT is read out from the 2-D array according to $k = Ck_1 + Dk_2$ where now the rows and columns are labelled as k_1 and k_2 .