

Paper Number(s): **E4.40**  
**SO20**

IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE  
UNIVERSITY OF LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING  
EXAMINATIONS 2002

MSc and EEE PART IV: M.Eng. and ACGI

### **INFORMATION THEORY**

Monday, 22 April 10:00 am

There are SIX questions on this paper.

Answer FOUR questions.

Time allowed: 3:00 hours

**Corrected Copy**

#### **Examiners responsible:**

First Marker(s): Turner, L.F.

Second Marker(s): Barria, J.A.

**Special instructions for invigilators:**      Grid to be provided.

**Information for candidates:**      If you answer Question 3, please  
attach your grid securely to your main  
answer book.

1. Prove that for two probability sets

$$P_1, P_2, \dots, P_M \quad (P_1 + \dots + P_M = 1)$$

and  $Q_1, Q_2, \dots, Q_M \quad (Q_1 + \dots + Q_M = 1)$

$$\sum_{i=1}^M P_i \log \frac{1}{P_i} \leq \sum_{i=1}^M P_i \log \frac{1}{Q_i}$$

with equality if, and only if,  $P_i = Q_i$  for all  $i$ .

Hence, or otherwise, prove that the entropy function  $H$  satisfies the condition

$$H(P_1, \dots, P_M) \leq \log M$$

where  $H(P_1, \dots, P_M) = \sum_{i=1}^M P_i \log \frac{1}{P_i}$ . [4]

If  $H(x)$  denotes the entropy of a discrete random variable  $x$ ,  $H(y)$  is the entropy of a discrete random variable  $y$ , and  $H(x/y)$  and  $H(y/x)$  are the associated conditional entropies, prove that:

(i)  $H(x, y) \leq H(x) + H(y)$

(ii)  $H(xy) = H(x) + H(y/x) = H(y) + H(x/y)$

(iii)  $H(x) - H(x/y) \geq 0$ ,

and explain the significance of the results. [9]

For the case in which the elements of the channel matrix are such that a) each of the rows of the matrix is a permutation of a basic set of numbers and b) each of the columns of the matrix is a permutation of a basic set of numbers, determine the relationship between the probabilities associated with the channel inputs if channel capacity is to be achieved.

[7]

2. Data is transmitted over a discrete memoryless noisy binary channel using pulses of amplitude  $\pm V$  Volts. The channel is corrupted by zero-mean additive white Gaussian noise and attenuation can be neglected. If two decision thresholds at  $\pm k.V$  are employed in the channel decision marking system and the received signal levels between the decision thresholds are considered to be 'ambiguous'; determine the capacity of the channel as a function of the various decision probabilities when

(i)  $k < 1$

(ii)  $k \gg 1$ .

[14]

Explain the practical significance of the two approaches to decoding and discuss the associated advantages and disadvantages. [6]

If, in deriving your results, you employ any special arguments then these should be proved.

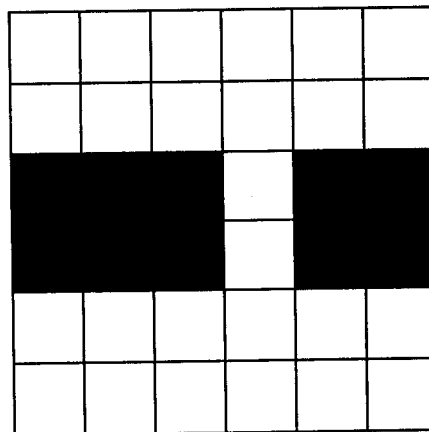
3. Figure 3.1 shows an image that is comprised of black and white pixels. The first-order statistics are fully representative of a class of images that are to be transmitted and it may be assumed that the pixels are statistically independent of each other.

The image is to be transmitted using a binary channel after it has been scanned on a line-by-line basis.

Determine the entropy of the image source and compare this with the image compression that can be achieved through the use of block encoding, with blocks of length 2. [10]

If the information was to be transmitted using your encoding scheme, and an error was to occur in the image in the seventh transmitted digit, what would be the output image obtained? Sketch the image on the grid provided. [7]

What general conclusion can you draw from your answer? [3]



*Figure 3.1*

4. Suppose the pixels of an image can be represented by  $K$  binary digits. Suppose further that the image is to be transmitted over a binary communication channel using the well-known method of run-length coding.

If runs of lengths  $l_1, l_2, l_3, \dots, l_n$  can occur with associated probabilities  $P_1, P_2, \dots, P_n$ , and successive runs are statistically independent, determine the data compression that can be achieved when the following encoding schemes are used:

- a) At the start of each run a  $(K + 1)$ -binary digit word (a zero plus a  $K$ -bit word representing the amplitude of the pixels in the run) is used, and a binary 1 is used to represent each other pixel of the run. [6]
- b) Optimum Run-Length encoding based on entropy considerations. [5]
- c) Run-Length encoding in which a Shannon-Fano code is employed and a code-word of length  $L_i$ , is used to represent a run whose probability is  $P_i, i=1, \dots, n$ . [5]

Compare the achievable compression ratios and draw conclusions as to the desirable properties of the run-lengths. [4]

**Note:** Compression Ratio, CR, is defined to be

$$CR = \frac{\text{Number of Data Digits Before Encoding}}{\text{Number of Data Digits After Encoding}}$$

5. Discuss the statement; “A restriction on the rate at which data pulses can be transmitted over a communication channel does not in itself place a fundamental restriction on the rate at which data can be transmitted over the channel”. [2]

A binary symmetric channel has a cross-over probability of  $P$ . Determine **from first principles** the capacity of the channel. [6]

The channel is to be used to transmit information from an information source whose outputs are the integers 0,1 and 2, which occur with equal probabilities. Explain what has to be done in order to transmit data at, or close to, the channel capacity. [12]

6. Two modems whose constellation (signal-point) diagrams are shown in *Figure 6.1* are to be used to communicate data over a channel having a bandwidth of  $W$  Hz.

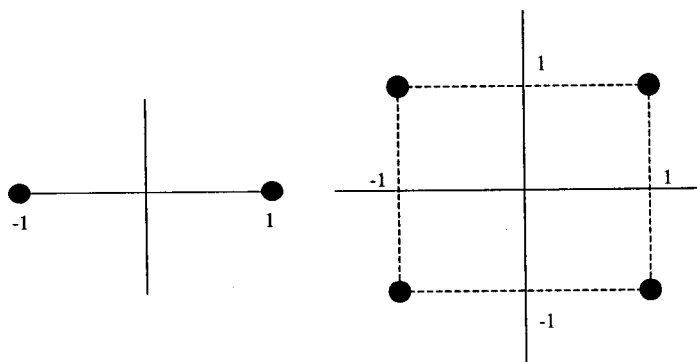
The noise which affects each dimension of the constellation identically and independently has an amplitude  $x$  whose probability density function  $P(x)$  is

$$P(x) = \text{constant for } -q \leq x \leq q$$

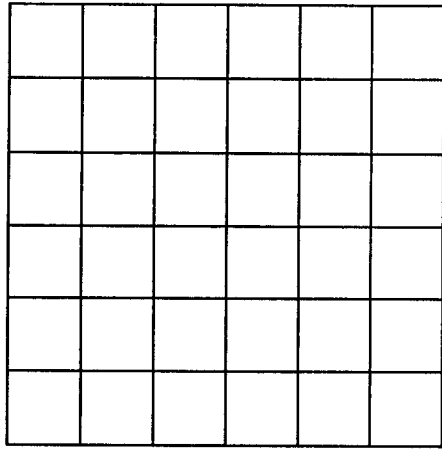
$$= 0 \text{ elsewhere.}$$

Calculate and plot the capacity of the two modem systems as a function of  $q$ . [18]

What important practical conclusions can you draw from your plots? [2]



*Figure 6.1*



*Grid*



Q1. Solution

## SOLUTIONS

marks

Part 1 Now  $\log x \leq x-1$ , with equality at  $x=1$

$$\therefore \log \frac{Q_i}{P_i} \leq \frac{Q_i}{P_i} - 1, \text{ with equality iff } P_i = Q_i$$

and  $P_i \log \frac{Q_i}{P_i} \leq Q_i - P_i$

$$\sum_{i=1}^M P_i \log \frac{Q_i}{P_i} \leq \sum_{i=1}^M (Q_i - P_i) = 0$$

$$\therefore \sum_{i=1}^M P_i \log \frac{Q_i}{P_i} \leq 0 \text{ with equality iff } P_i = Q_i \text{ for all } i.$$

It thus follows directly that

$$\sum_{i=1}^M P_i \log \frac{1}{P_i} \leq \sum_{i=1}^M P_i \log \frac{1}{Q_i} \quad (1)$$

with equality iff  $P_i = Q_i$  for all  $i$

Now let  $Q_i = \frac{1}{M}$  for all  $i$

$$\therefore \sum_{i=1}^M P_i \log \frac{1}{P_i} \leq \log M. \quad \underline{QED}$$

Part 2

$$(i) H(X,Y) = - \sum_{i=1}^N \sum_{j=1}^M P(x_i, y_j) \log P(x_i, y_j)$$

$$H(X) = - \sum_{i=1}^N P(x_i) \log P(x_i)$$

$$H(Y) = - \sum_{j=1}^M P(y_j) \log P(y_j)$$

$$\therefore H(X) + H(Y) = -c \sum_{i=1}^N \sum_{j=1}^M P(x_i, y_j) \log P(x_i) P(y_j)$$

Let  $P(x_i) P(y_j) = Q(x_i, y_j)$  then it follows from ① above that

$$-c \sum_{i=1}^N \sum_{j=1}^M P(x_i, y_j) \log(P(x_i, y_j)) \leq -c \sum_{i=1}^N \sum_{j=1}^M P(x_i, y_j) \log Q(x_i, y_j)$$

$$\therefore H(X, Y) \leq H(X) + H(Y) \quad (2)$$

with equality iff  $P(x_i, y_j) = P(x_i) P(y_j)$

Significance: The entropy of the joint distribution is equal to the sum of the entropies iff  $X$  &  $Y$  are statistically independent, i.e. on learning  $x$  we learn nothing of  $Y$ , and vice versa.

$$(ii) \quad H(X, Y) \leq H(X) + H(Y/X) = H(Y) + H(X/Y)$$

$$\begin{aligned} \text{Now } H(Y/X) &= -c \sum_{i=1}^N P(x_i) \sum_{j=1}^M P(y_j/x_i) \log P(y_j/x_i) \\ &= -c \sum_{i=1}^N \sum_{j=1}^M P(x_i, y_j) \log P(y_j/x_i) \end{aligned}$$

$$\text{But } H(X, Y) = -c \sum_{i=1}^N \sum_{j=1}^M P(x_i, y_j) \log P(x_i, y_j)$$

$$\begin{aligned} &= -c \sum_{i=1}^N P(x_i) \log P(x_i) - c \sum_{i=1}^N \sum_{j=1}^M P(x_i, y_j) \log P(y_j/x_i) \\ &= H(X) + H(Y/X) \quad (3) \end{aligned}$$

Similarly by using  $P(x_i, y_j) = P(y_j) P(x_i/y_j)$

we get  $H(XY) = H(Y) + H(X/Y)$

Significance: uncertainty about  $X$  &  $Y$  is equal to uncertainty about  $Y$  plus the uncertainty about  $X$ , given that  $Y$  has been observed/learned.

(iii)  $H(X/Y) \leq H(X)$ .

This result follows directly from (2) and (3).

Significance: If  $X$  is dependent on  $Y$ , then having received  $Y$  we know something about  $X$ , i.e. our uncertainty is reduced as compared with our initial uncertainty, but if  $X$  &  $Y$  are statistically independent then  $H(X/Y) = H(X)$ , which is the most uncertain we can be about  $X$ .

### Part 3

For the noisy channel

$$I = H(Y) - H(Y/X)$$

$$= H(Y) - \sum_{j=1}^N P(x_j) \sum_{i=1}^K P(y_i/x_j) \log \frac{1}{P(y_i/x_j)}$$

and  $C = \frac{I}{\max[A(x)]}$

If the channel matrix is such that all rows are permutations of a set of numbers  $a_1, \dots, a_K$ , then

$$\sum_{i=1}^K P(y_i/x_j) \log \frac{1}{P(y_i/x_j)} = \sum_{i=1}^K a_i \log \frac{1}{a_i} = \text{constant}$$

for all rows.

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$$\text{Thus } H(Y/X) = \sum_{j=1}^N P(x_j) \sum_{i=1}^K a_i \log(1/a_i) = \sum_{i=1}^K a_i \log 1/a_i$$

$$\text{Hence } C = I_{\max(P(X))} = H(Y) + \sum_{i=1}^K a_i \log a_i$$

Now if all columns are permutations of a set of numbers  $b_1, \dots, b_N$  (may be same set as set  $a_i$ 's, or subset thereof)

$$\begin{aligned} \text{Then } P(y_i) &= \sum_{j=1}^N P(x_j) P(y_i/x_j) \\ &= P(x_1) b_i + \dots + P(x_N) b_N \end{aligned}$$

If we put  $P(x_1) = P(x_2) = \dots = P(x_N) = 1/N$  then

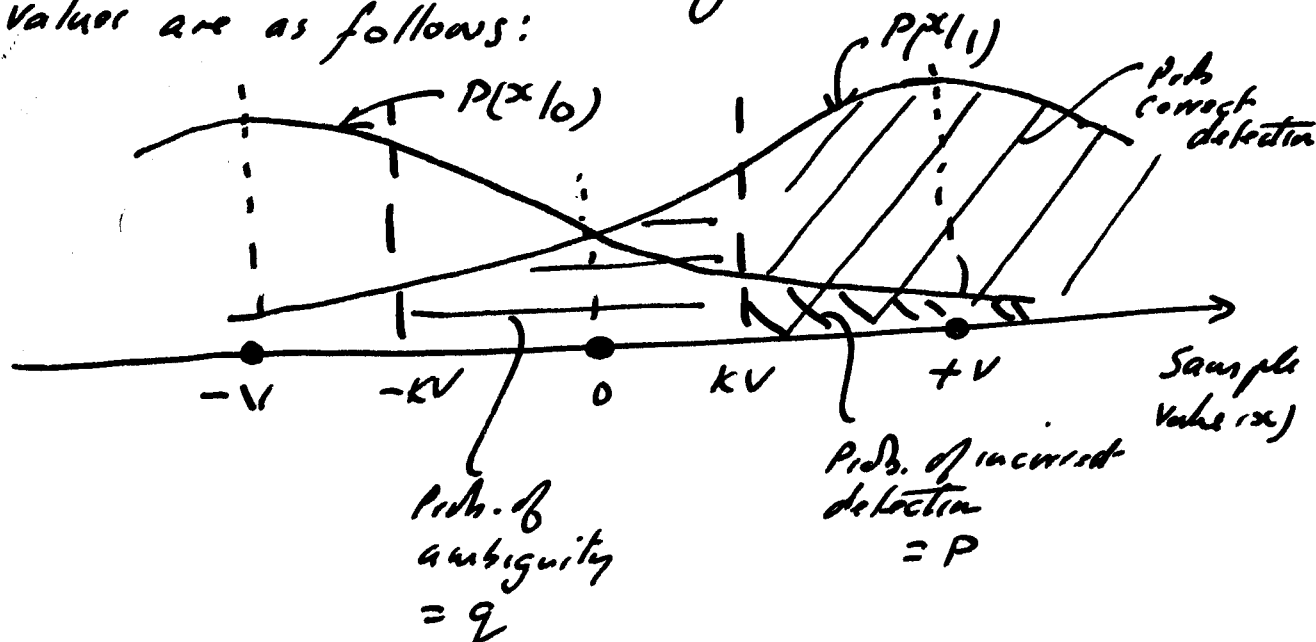
$$P(y_i) = 1/K \text{ for all } i$$

and hence  ~~$P(y_i)$~~   $H(Y)$  is maximised at  $\log K$  and hence capacity is achieved.

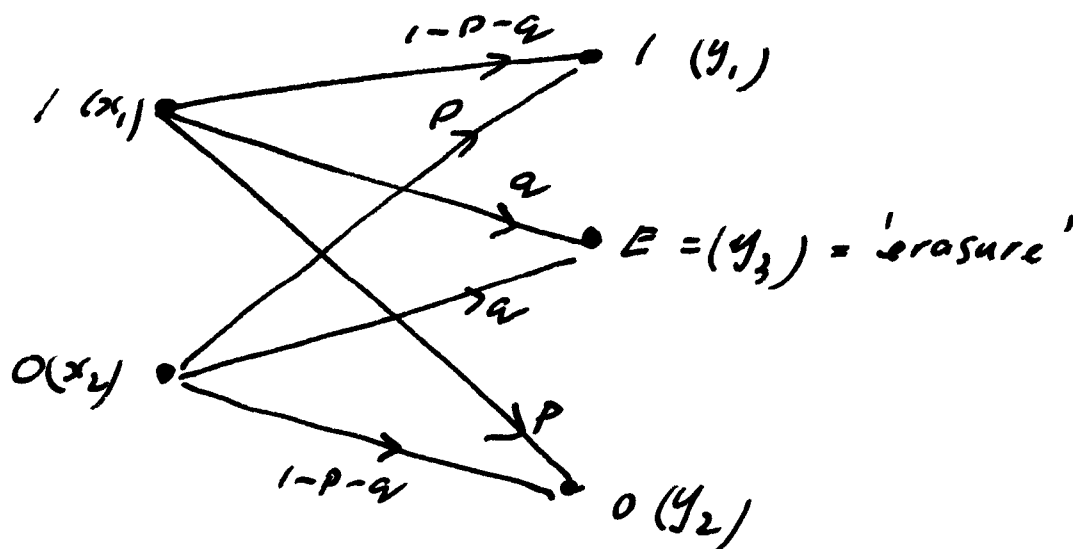
The relationship is that for ~~equally probable~~ equally probable input symbols we get equally probable outputs — hence capacity is achieved.

Q2 Solution

Assume that attenuation in cable is neglected. In this case the conditional pdf's relating to the received sample values are as follows:



Thus the channel transition diagram is as follows:



and the associated matrix is

$$\begin{array}{c} \text{input} \begin{array}{c} x_1 \\ x_2 \end{array} \begin{array}{c} y_1 \quad y_2 \quad y_3 \text{ - outputs} \\ \left[ \begin{array}{ccc} 1-p-q & p & q \\ p & 1-p-q & q \end{array} \right] \end{array} \end{array}$$

Thus the channel matrix is 'uniform' from the input

$$\therefore I = H(Y) + (1-q-p) \log(1-q-p) + q \log q + p \log p$$

↖ expect most of this

Now although the channel appears to be symmetrical it is NOT 'uniform' from the output

But it is easy to show that in this case  $I_{\max} = C$   
is obtained when  $P(x_1) = P(x_2) = \frac{1}{2}$

↖ expect this to be proved.

This condition makes  $H(Y)$  maximum.

With these probabilities

$$P(y_1) = P(x_1)(1-p-q) + P(x_2)p = (1-q)\frac{1}{2}$$

$$P(y_2) = qP(x_1) + qP(x_2) = q$$

$$P(y_3) = P(x_1)p + P(x_2)(1-p-q) = (1-q)\frac{1}{2}$$

and hence the capacity is

$$\begin{aligned}
 C &= \overbrace{-q \log q - \frac{1}{2}(1-q) \log \frac{1}{2}(1-q) - \frac{1}{2}(1-q) \log \frac{1}{2}(1-q)}^{H(Y)} \\
 &\quad + (1-p-q) \log (1-p-q) + q \log q + p \log p \\
 &= (1-q) \left[ 1 - \log_2 (1-q) \right] + (1-p-q) \left[ \log_2 (1-p-q) \right] \\
 &\quad + p \log_2 p \quad \text{bits/input}^{\text{Sym}} \quad (1)
 \end{aligned}$$

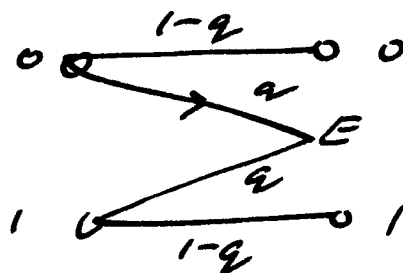
Note: (i) If  $q=0$  then this reduces to

$1 + p \log p + (1-p) \log (1-p)$  which is the capacity of a B.S.C

(ii) the capacity in (1) is changed from that of BSC since we are saying if signals are somewhat uncertain (are in ambiguous region) then this will be indicated (soft decision)

If  $K$  becomes large, then  $p \rightarrow 0$  i.e. we are attempting to rule out errors, at the expense of an increasing number of 'ambiguous' / erasure outputs.

The transition diagram becomes



### Q 3 Solution

Part 1

$$P_{\text{white}} = P_1 = \frac{26}{36} = 0.72$$

$$P_{\text{black}} = P_2 = \frac{10}{36} = 0.28$$

} these are to be taken as representative probabilities.

The entropy of the source is

$$H = - \left\{ 0.72 \log_2 0.72 + 0.28 \log_2 0.28 \right\}$$

$$\approx 0.855 \text{ bits/pixel}$$

Now if we encode using blocks of length 2, we have

$X_1$ (WW)	$P_1 P_1 = 0.52$	0
$X_2$ (WB)	$P_1 P_2 = 0.20$	1 0
$X_3$ (BW)	$P_2 P_1 = 0.20$	1 1 0
$X_4$ (BB)	$P_2 P_2 = 0.08$	1 1 1

Therefore: the average codeword length

$$= 0.52 \times 1 + 2 \times 0.20 + 3 \times 0.20 + 3 \times 0.08$$

$$= 0.88 \text{ binary digits}$$

Using the block encoding the image is represented by the sequence

$X_1 X_1 X_1 X_1 X_1 X_1 X_4 X_3 X_4 X_4 X_3 X_4 X_1 X_1 X_1 X_1 X_1 X_1$   
(sequence 1)



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and the corresponding sequence of digits for transmission is

0000001111011111110111000000

= 30 digits

The compression ratio is thus  $\frac{36}{30} = 1.2$ .

Note that  $\frac{30}{36} = 0.833$  bits/pixel. This is less than

the entropy and I explain them to explain why.

(The image is not large enough for all black/white, white/black possibilities to occur — note that  $x_2 = WB$  does not occur.

## Part 2

Suppose an error occurs in the 7<sup>th</sup> digit as indicated in the question. Then the received binary sequence is

↓  
0000001111011111110111000000

and this will be decoded as

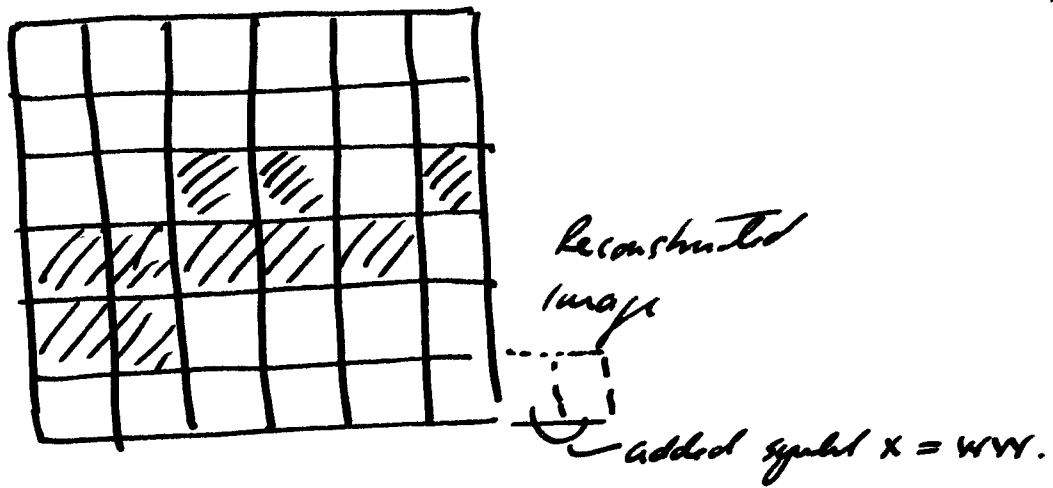
$x_1, x_1, x_1, x_1, x_1, x_1, x_1, x_1, x_4, x_2, x_4, x_4, x_3, x_4, x_1, x_1, x_1, x_1, x_1,$

note symbol (sequence 2)  
recovery from  
this point on

I expect them to note that after the error the sequence 'recovers' i.e. code is self-synchronising in respect of letters/symbols.

But!

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I expect the following points to be made:

- (1) The major point to note is that although symbol synchronisation is recovered, spatial positioning is lost and the image would be of little value.
- (2) This type of effect is not unusual since we know that removal of redundancy makes messages more sensitive to errors.
- (3) We have to have low error rates for image transmission and 'end-of-line' signals are usually employed.

#### Q4 Solution

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Suppose the source generates a long sequence of  $N$  runs, containing runs of the various lengths  $l_1, l_2, \dots, l_n$ .

Thus the number of runs of length  $l_i$  will be  $Np_i, i=1, \dots, n$

Thus the number of symbols generated will be

$$S = \sum_{i=1}^n Np_i l_i = N \cdot l_{ave}, \text{ where } l_{ave} = \text{average run length}$$

Thus the number of bits generated is  $Q$ , where

$$Q = KN \cdot \sum_{i=1}^n p_i l_i \quad \text{--- this is data generated in uncompressed form.}$$

With coding scheme A

The number of binary digits generated by the encoding with respect to the  $N$  runs is

$$R = S + KN = N \sum_{i=1}^n p_i l_i + KN$$

and the compression ratio is

$$CR_1 = \frac{KN \sum_{i=1}^n p_i l_i}{KN + \sum_{i=1}^n p_i l_i} = \frac{K \sum_{i=1}^n p_i l_i}{K + \sum_{i=1}^n p_i l_i} \quad (1)$$

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With coding scheme B

With optimum run-length encoding, the average number of binary digits used to encode a run-length is

$$H = - \sum_{i=1}^n P_i \log_2 P_i \text{ bits (binary digits)}$$

Thus the number of digits used to represent the  $N$  runs is

$$KN + N \sum_{i=1}^n P_i \log_2 1/P_i$$

to indicate  
'amplitude'

and the compression ratio in this case is

$$CR_2 = \frac{KN \sum_{i=1}^n P_i l_i}{KN + N \sum_{i=1}^n P_i \log_2 1/P_i} = \frac{K \sum_{i=1}^n P_i l_i}{K + \sum_{i=1}^n P_i \log_2 1/P_i} \quad (2)$$

With coding scheme C

The average code-word length is  $\sum_{i=1}^n P_i L_i$  (per run)

and hence the average number of digits used is

$$KN + N \sum_{i=1}^n P_i L_i$$

and the compression ratio is

$$CR_3 = \frac{KN \sum_{i=1}^n p_i l_i}{KN + N \sum_{i=1}^n p_i l_i}$$

$$= \frac{K \sum_{i=1}^n p_i l_i}{K + \sum_{i=1}^n p_i l_i} \quad (3)$$

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I want the following points to be made in respect of discussion and comparison

(i) No note from ① that

$$CR_1 = \frac{K}{1 + K / \sum_{i=1}^n p_i l_i}$$

So the larger the average run-length the better (i.e. higher the compression ratio)

(ii) Limiting value of  $CR_1 = 1$

(iii) The same general conclusion can be drawn with respect to ② since  $\sum p_i \log 1/p_i$  is fixed and it is in a sense independent of the run-length, so we would like long runs so that

$$\frac{\sum p_i \log 1/p_i}{\sum p_i l_i} \rightarrow 0 \quad \text{and} \quad CR \rightarrow \text{large.}$$

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ie the entropy/average run length becomes small.

iv, The Shannon coding technique is reasonably efficient - and hence

$$\sum_i p_i L_i \approx \sum_i p_i \log_2 1/p_i$$

and hence we would expect (3) to be ~~greater~~ slightly worse than (2), but not too different



Part 1

The statement in itself does not constitute a restriction since we can send as many data bits/pulse as we like, by increasing the size of the symbol set. But if we are power constrained then the pulses get 'closer together' and hence if noise is present then the system error probability increases.

Part 2

$$C = I = \max_{P[X]} \{ H(X) - H(X/Y) \} = \max_{P[X]} \{ H(X) - H(Y/X) \}$$

I will accept a solution in which  $I$  is expressed as a function of  $P(x_1)$ ,  $P(x_2)$  (the input probabilities) and  $P$  (error probability), and the function is then maximized by obtaining  $\frac{\partial I}{\partial P(x_1)} = 0$ ;  $\frac{\partial I}{\partial P(x_2)} = 0$  with a solution being obtained for  $P(x_1)$ ,  $P(x_2)$ .

The result is  $P(x_1) = P(x_2) = 1/2$

This is a difficult approach and I will accept the following alternative:

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$$I = H(Y) - H(Y/X)$$

and the channel matrix is

$$\begin{matrix} & \begin{matrix} y_1 & y_2 \end{matrix} \\ \begin{matrix} x_1 \\ x_2 \end{matrix} & \begin{bmatrix} (1-p) & p \\ p & (1-p) \end{bmatrix} \end{matrix}$$

$$\begin{aligned} \text{Now } H(Y/X) &= \sum_{j=1}^2 P(x_j) [p \log p + (1-p) \log(1-p)] \\ &= p \log p + (1-p) \log(1-p) \end{aligned}$$

$$\text{and } H(Y) = - \{ P(y_1) \log P(y_1) + P(y_2) \log P(y_2) \}$$

$$\begin{aligned} \text{But } P(y_1) &= P(x_1) P(y_1/x_1) + P(x_2) P(y_1/x_2) \\ &= P(x_1)(1-p) + p \cdot P(x_2) \end{aligned}$$

$$\begin{aligned} P(y_2) &= P(x_1) P(y_2/x_1) + P(x_2) P(y_2/x_2) \\ &= P(x_1) \cdot p + (1-p) P(x_2) \end{aligned}$$

Now if  $P(y_1) = P(y_2)$  then  $H(Y)$  is maximised

hence if we set  $P(x_1) = P(x_2) = 1/2$

then  $P(y_1) = P(y_2)$  and hence

$$C = 1 + p \log_2 p + (1-p) \log_2 (1-p)$$



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Part 3

In order to transmit at capacity it is necessary that the input binary digits to the channel be used with equal probability.

Now the source is

$x_1$	0	$P(x_1)$
$x_2$	1	$1/3$
$x_3$	2	$1/3$

Consider the following coding in which  $x$ 's are mapped into binary digits.

Case I

$x_1$	00
$x_2$	10
$x_3$	11

$\therefore P(0) = P(1) = 1/2$   
 $\therefore$  looks to be perfect

But the entropy of the source  $= \log_2 3 = 1.59 \text{ bits/symbol}$

We are using 2 bits/symbol  $\therefore$  wasteful

Case II

Using a Shannon/Fano code we obtain

$x_1$	0	$P(0) = 0.4$
$x_2$	10	$P(1) = 0.6$
$x_3$	11	$\therefore$ Prob match is required $P(0) = P(1) = 0.5$

With this code average code word length  $= \bar{S}_3 = 1.67$  bits/symbol.

Case III Apply SF code to extended symbol set

Prob $1/9$	$x_1 x_1$	0 0 0
	$x_1 x_2$	0 0 1
	$x_1 x_3$	0 1 0
	$x_2 x_1$	0 1 1
	$x_2 x_2$	1 0 0
	$x_2 x_3$	1 0 1
	$x_3 x_1$	1 1 0
	$x_3 x_2$	1 1 1 0
$1/9$	$x_3 x_3$	1 1 1 1

$$\therefore P(0) = 0.45 \quad \left. \begin{array}{l} \\ P(1) = 0.55 \end{array} \right\} \text{quite close to } 0.5$$

Average code word length

$$= \frac{29}{9} \text{ / pair / symbols}$$

$$= \frac{29}{18} = 1.61 \text{ bits/symbol.}$$

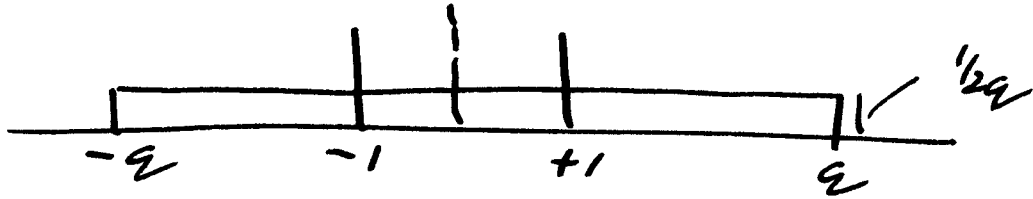
If we repeat by coding longer blocks of source symbols then  $P(0) \rightarrow 0.5$  and  $\bar{n} \rightarrow H(x)$

The final requirement is for a proper channel code which is inserted between the source code and the channel to achieve reliable communication — care needed to ensure that  $P(0) = P(1) = 0.5$  requirement is retained at coder output.

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Q6 Solution

Consider the given noise pdf



From this we see that the probability of the noise taking the signal level across a decision boundary in each dimension is  $\frac{1}{2q}(q-1) = P$ .

Thus for the two systems we have channel matrices as follows:

Binary

$$\begin{matrix} & y_1 & y_2 \\ x_1 & \begin{bmatrix} (1-P) & P \end{bmatrix} \\ x_2 & \begin{bmatrix} P & (1-P) \end{bmatrix} \end{matrix}$$

4-phase

$$\begin{matrix} & y_1 & y_2 & y_3 & y_4 \\ x_1 & \begin{bmatrix} (1-P)^2 & P(1-P) & P^2 & P(1-P) \\ P(1-P) & (1-P)^2 & P(1-P) & P^2 \\ P^2 & P(1-P) & (1-P)^2 & P(1-P) \\ P(1-P) & P^2 & P(1-P) & (1-P)^2 \end{bmatrix} \\ x_2 & \\ x_3 & \\ x_4 & \end{matrix}$$

where

$$P = \frac{1}{2q}(q-1)$$

Both channels are doubly symmetric and hence the respective capacities are

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Binary  $C = 2W \left[ 1 + p \log_2 p + (1-p) \log_2 (1-p) \right]$  ①

bits/sec.

4-phase

$C = 2W \left[ 2 + p^2 \log_2 p^2 + 2(1-p)p \log_2 (1-p) + (1-p)^2 \log_2 (1-p)^2 \right]$  ②

bits/sec.

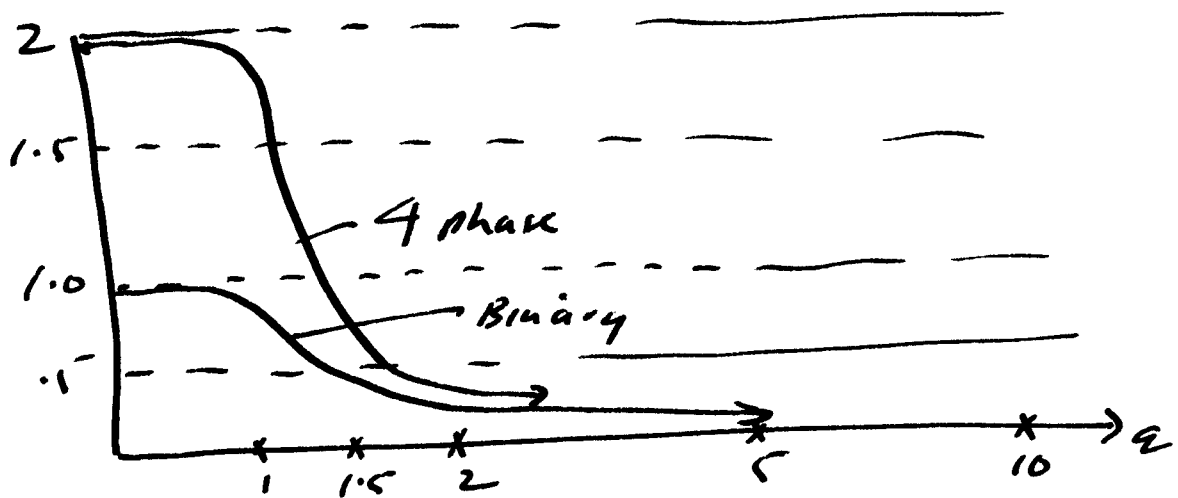
Now we can evaluate the capacity expressions as function of  $q$ .

$q$	$p$	$p^2$	$(1-p)$	$p(1-p)$	$(1-p)^2$	$C_{\text{binary}}$	$C_{\text{4phase}}$
$\leq 1$	0	0	1	0	.1	1	2
1.5	.167	.028	.83	.139	.694	.35	.69
2	.25	.063	.75	.188	.563	.19	.38
5	.4	.16	.60	.240	.360	.03	.07
10	.45	.20	.55	.248	.303	.01	.02
100	.495	.25	.5	.25	.25	0	0

$p = \frac{1}{2q} (q-1)$

and on plotting we obtain

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The main points are

- (i) at high SN we get main from using more symbol points
- (ii) at low SNR although  $C_{4P} > C_B$  the difference becomes small and gets less
- (iii) at very low SN  $C \rightarrow 0$

