

Solutions

1. See separate sheet.

2. See separate sheet.

3. a) i) The auxiliary polynomial is $a(\lambda) = \lambda^2 - \lambda = \lambda(\lambda - 1)$ and so the complementary function is $y_c(x) = c_1 + c_2 e^x$, c_1, c_2 constants.ii) Use a trial solution $y_p(x) = b_1 + b_2 x e^x$ in the differential equation and equate coefficients to get $b_1 = b_2 = 1$ and so $y_p(x) = x e^x + 1$.iii) The general solution is then $y(x) = c_1 + c_2 e^x + x e^x + 1$. Enforcing the initial conditions gives $c_1 = 0$ and $c_2 = -1$ and so $y(x) = x e^x - e^x + 1$.b) i) Write the differential equation as $P(x, y)dx + Q(x, y)dy = 0$. The equation is exact if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \Leftrightarrow \lambda_1 x - \cos x \sin y = 2x - \frac{1}{2} \lambda_2 \cos x \sin y \Leftrightarrow \lambda_1 = \lambda_2 = 2.$$

$$\Rightarrow (2xy + \cos x \cos y)dx + (x^2 - \sin x \sin y)dy = 0.$$

ii) Since $df = f_x dx + f_y dy$ we need $f_y = Q$ and so

$$\frac{\partial f}{\partial y} = x^2 - \sin x \sin y \Rightarrow f(x, y) = x^2 y + \sin x \cos y$$

iii) Since $df = 0$ the solution is given implicitly by $f(x, y) = C$ or

$$x^2 y + \sin x \cos y = C, \quad C \text{ constant.}$$

c) i) The integrating factor $\mu(x)$ must be chosen to satisfy

$$\frac{d\mu(x)}{dx} = \mu(x) \frac{3}{x} \Rightarrow \mu(x) = \exp\left(\int \frac{3}{x} dx\right) = e^{3 \ln x} = x^3.$$

ii) Multiplying the equation by $\mu(x)$:

$$x^3 \frac{dy}{dx} + 3x^2 y = 2x \Rightarrow \frac{d}{dx}(x^3 y) = 2x$$

$$\Rightarrow y(x) = x^{-3} \int 2x dx + Cx^{-3}$$

$$\Rightarrow y(x) = x^{-1} + Cx^{-3}, \quad C \text{ constant}$$

4. a) i) Evaluating the partial derivative for the chain rule

$$\begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{-y}{x^2+y^2} \\ \frac{y}{\sqrt{x^2+y^2}} & \frac{x}{x^2+y^2} \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial \rho} \\ \frac{\partial f}{\partial \phi} \end{bmatrix} = \begin{bmatrix} \cos \phi & -\sin \phi / \rho \\ \sin \phi & \cos \phi / \rho \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial \rho} \\ \frac{\partial f}{\partial \phi} \end{bmatrix}$$

and so

$$\begin{aligned} \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 &= \begin{bmatrix} \frac{\partial f}{\partial \rho} & \frac{\partial f}{\partial \phi} \end{bmatrix} \begin{bmatrix} \cos \phi & \sin \phi \\ -\sin \phi / \rho & \cos \phi / \rho \end{bmatrix} \begin{bmatrix} \cos \phi & -\sin \phi / \rho \\ \sin \phi & \cos \phi / \rho \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial \rho} \\ \frac{\partial f}{\partial \phi} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial f}{\partial \rho} & \frac{\partial f}{\partial \phi} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/\rho^2 \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial \rho} \\ \frac{\partial f}{\partial \phi} \end{bmatrix} = \left(\frac{\partial f}{\partial \rho}\right)^2 + \frac{1}{\rho^2} \left(\frac{\partial f}{\partial \phi}\right)^2 \end{aligned}$$

and so $a = 1$.

ii) Since f is radially symmetric, $f_\phi = 0$ and the equation transforms to

$$\left(\frac{\partial f}{\partial \rho}\right)^2 = \rho^2 \Rightarrow \frac{\partial f}{\partial \rho} = \pm \rho \Rightarrow f(\rho) = \pm \frac{1}{2} \rho^2 + C \Rightarrow f(x, y) = \pm \frac{1}{2} (x^2 + y^2) + C.$$

b) i) Since $dF = 0$ we have that $\frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial z} dz = 0$. It follows that

$$\frac{\partial z}{\partial x} := \frac{dz}{dx} \Big|_{y=\text{constant}} = \frac{dz}{dx} \Big|_{dy=0} = -\frac{\partial F / \partial x}{\partial F / \partial z} \Rightarrow \frac{\partial z}{\partial x} = \frac{2x}{z}.$$

and since F is symmetric in x and y , $\frac{\partial z}{\partial y} = \frac{2y}{z}$.

ii) The stationary points are obtained by setting $z_x = z_y = 0$ and so $x_0 = y_0 = 0$ from Part (i). The same answer could be obtained by explicitly expressing z as a function of x and y . Note that $z(x_0, y_0) = 2$.

iii) Taking the second derivatives and evaluating at x_0, y_0 ,

$$\frac{\partial^2 z}{\partial x^2} \Big|_{x_0, y_0} = \frac{\partial}{\partial x} \left(\frac{2x}{z} \right) \Big|_{x_0, y_0} = \frac{2z - 2xz_x}{z^2} \Big|_{x_0, y_0} = 1 = \frac{\partial^2 z}{\partial y^2} \Big|_{x_0, y_0}$$

For the mixed derivatives,

$$\frac{\partial^2 z}{\partial x \partial y} \Big|_{x_0, y_0} = \frac{\partial}{\partial x} \left(\frac{2y}{z} \right) \Big|_{x_0, y_0} = \frac{-2yz_x}{z^2} \Big|_{x_0, y_0} = 0$$

The Hessian is then

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Since $m_{11} > 0$ and $\det M > 0$, the stationary point corresponds to a local minimiser.

①

①

a) i) $\frac{1}{\sqrt{n^2-3}} \sim \frac{1}{n}$ for $n \geq 2$

$\sum \frac{1}{n}$ is divergent & so is

$$\sum_{n \geq 2} \frac{1}{\sqrt{n^2-3}}.$$

ii) $\sum (-1)^n \frac{3^n}{5^n}$ this is an alternating series such that $\left(\frac{3}{5}\right)^n$ is decreasing

& $\lim_{n \rightarrow \infty} \left(\frac{3}{5}\right)^n = 0$ so it is convergent

iii) $u_n = \frac{5^n}{n^n}$ using the root test

$\sqrt[n]{u_n} = \frac{5}{n} \rightarrow 0$ hence the series is convergent

$$b) f(x) = \ln(1+x)$$

(2)

$$f'(x) = \frac{1}{1+x}$$

$$f''(x) = -\frac{1}{(1+x)^2}$$

$$f'''(x) = \frac{2}{(1+x)^3}$$

Hence the Taylor series expansion of $\ln(1+x)$ about 0 is.

$$0 + 1x - \frac{1}{2}x^2 + \frac{2}{3!}x^3 = x - \frac{x^2}{2} + \frac{x^3}{3}.$$

$$c) A = \begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 0 \\ 1 & -1 & 3 \end{bmatrix}$$

Characteristic polynomial is.

$$-x^3 + 8x^2 - 20x + 16$$

this factor is a

$$- (n-2) (n-2) (n-4)$$

③

$Av = 4v \Rightarrow \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ is an eigenvector

$$Av = 2x \Rightarrow x_1 - x_2 + x_3 = 0.$$

$\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ & $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ are eigenvectors

②

$L:$

$$x = -1 + b$$

$$y = -3$$

$$z = 4$$

$$a) P \cap L = \{A\} \text{ where}$$

$$(-1 + \lambda) - 3 + 4 = 10 \Rightarrow \lambda = 10.$$

$$\text{Hence } A = (9, -3, 4).$$

b)

④

orthogonal projection of $(1, 0, 0)$ on Γ .

i) $B = (x, y, z)$ such that

$$\overrightarrow{AB} = \begin{pmatrix} x-1 \\ y \\ z \end{pmatrix} \text{ is parallel to } \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\& \quad x + y + z = 10.$$

$$\begin{pmatrix} x-1 \\ y \\ z \end{pmatrix} \wedge \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} y \neq z \\ -(x-1) + z \\ (x-1) - y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} y = z \\ x-1 = z \\ x-1 = y \end{cases}$$

$$\Rightarrow z + 1 + z = 10$$

$$\Rightarrow z = 3$$

Hence $B = \begin{pmatrix} 4 \\ 3 \\ 3 \end{pmatrix}$

5

distance from $(1, 1, 1)$ to P is

$$\sqrt{3^2 + 3^2 + 3^2} = \sqrt{27}.$$

c) Q orthogonal to P

$$Q \supset L$$

\Rightarrow The point $(-1, -3, 4)$ is
in Q

& Q has direction $(1, 1, 1)$
& $(1, 0, 0)$.

Let us consider the vector (x, y, z)

Orthogonal to Q we have

$$x + y + z = 0 \quad \& \quad x = 0 \quad \Rightarrow \quad y = -3.$$

Here $(0, 1, -1)$ is orthogonal to P . ⑥

& so in a point A we have
its coordinates (x, y, z) satisfying

$$0 \cdot (x+1) + (y+3) - (z-4) = 0$$

$$\Rightarrow Q: \boxed{y - z = -7}$$