

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2005

EEE/ISE PART I: MEng, BEng and ACGI

COMMUNICATIONS 1

Friday, 27 May 10:00 am

Time allowed: 2:00 hours

None

Corrected Copy

There are FOUR questions on this paper.

Q1 is compulsory.

Answer Q1 and any two of questions 2-4.

Q1 carries 40% of the marks. Questions 2 to 4 carry equal marks (30% each).

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible

First Marker(s) : P.L. Dragotti, P.L. Dragotti

Second Marker(s) : M.K. Gurcan, M.K. Gurcan

Special Information for the Invigilators: none

Information for Candidates

The trigonometric Fourier series of a periodic signal $x(t)$ of period $T_0 = 2\pi/\omega_0$ is

$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t),$$

with

$$a_0 = \frac{1}{T_0} \int_{T_0} x(t) dt, \quad a_n = \frac{2}{T_0} \int_{T_0} x(t) \cos n\omega_0 t dt, \quad b_n = \frac{2}{T_0} \int_{T_0} x(t) \sin n\omega_0 t dt.$$

Some Fourier Transforms

$$\cos \omega_0 t \quad \Longleftrightarrow \quad \pi [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$

$$\text{rect}\left(\frac{t}{\tau}\right) \quad \Longleftrightarrow \quad \tau \text{sinc}\left(\frac{\omega\tau}{2}\right)$$

$$\frac{W}{\pi} \text{sinc}(Wt) \quad \Longleftrightarrow \quad \text{rect}\left(\frac{\omega}{W}\right)$$

Some useful trigonometric identities

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

$$\sin x \cos y = \frac{1}{2} \sin(x - y) + \frac{1}{2} \sin(x + y)$$

$$\cos x \cos y = \frac{1}{2} \cos(x - y) + \frac{1}{2} \cos(x + y).$$

Euler's formula

$$e^{jx} = \cos x + j \sin x.$$

Steady-state impedance of a terminated transmission line

$$Z_{in} = Z_0 \frac{Z_L + jZ_0 \tan(kL)}{Z_0 + jZ_L \tan(kL)}$$

The Questions

1. This question is compulsory.

- (a) Consider the two signals $x_1(t) = \text{rect}(t)$ and $x_2(t) = \cos(4\pi t)\text{rect}(t - 0.5)$ shown in Figure 1a.

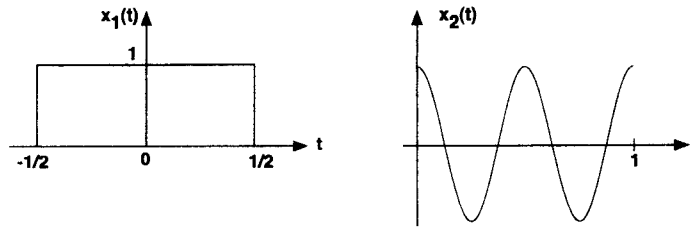


Figure 1a: The two energy signals $x_1(t)$ and $x_2(t)$.

- i. Determine the correlation between $x_1(t)$ and $x_2(t)$. Are $x_1(t)$ and $x_2(t)$ orthogonal? [4]
 - ii. Determine the energy of $z(t) = 4x_1(t) + 2x_2(t)$. [4]
- (b) Consider the periodic signal $x(t)$ shown in Figure 1b. Compute the coefficients a_0 and a_1 of the trigonometric Fourier series of $x(t)$.

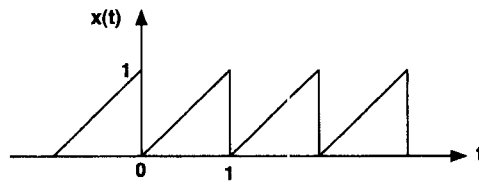


Figure 1b: The periodic signal $x(t)$.

- (c) From the definition of the Fourier transform, show that

$$g(t)e^{j\omega_0 t} \iff G(\omega - \omega_0).$$

Hence show that

$$g(t) \cos \omega_0 t \iff \frac{1}{2}G(\omega - \omega_0) + \frac{1}{2}G(\omega + \omega_0)$$

[4]

(d) Consider the full AM signal $x(t) = [A + m(t)] \cos(\omega_c t)$ with $m(t) = 2 \cos 100t$ and $\omega_c = 10000 \text{ rad/s}$.

i. Determine the minimum value of A that allows us to use an envelope detector.

[4]

ii. For $A = 4$, sketch and dimension the Fourier transform of $x(t)$.

[4]

iii. For $A = 4$, compute the power efficiency η .

[4]

(e) Develop a block diagram of an SSB-SC generator.

[4]

(f) Consider the PM signal

$$\varphi(t) = \cos[2\pi f_0 t + k_p m(t)]$$

where $m(t) = A \cos 2\pi f_m t$. Using Carson's rule, comment on the way the bandwidth of $\varphi(t)$ changes with the amplitude A , the frequency f_m and the frequency f_0 .

[4]

(g) A 50Ω transmission line is connected to a 100Ω line with a matched termination. A sine wave of 10 V amplitude propagating in the former is incident on the junction. Find

i. The voltage reflection coefficient k_v .

[2]

ii. The current amplitude of the reflected wave.

[2]

2. Consider the FM signal

$$\varphi(t) = 10 \cos[2\pi f_0 t + k_f \int_{-\infty}^t x(\alpha) d\alpha]$$

where $k_f = 10\pi$. The message $x(t)$ is given by

$$x(t) = \sum_{n=0}^2 m_n(t)$$

with

$$m_n(t) = \frac{2^n}{\pi} \text{sinc}(t) \cos(2nt).$$

(a) Sketch and dimension the Fourier transform of $m_1(t)$.

[6]

(b) Sketch and dimension the Fourier transform of $x(t)$.

[6]

(c) Using Carson's rule, determine the bandwidth of $\varphi(t)$.

[6]

(d) Assume now that $x(t) = Ae^{-10t}u(t)$. Using Carson's rule, the bandwidth of $\varphi(t)$ is 50.4 Hz. Find the amplitude A of $x(t)$. Select the bandwidth, B , of the baseband message $x(t)$ so that it contains 95% of the signal energy.

[12]

3. Consider the system shown in Figure 3.

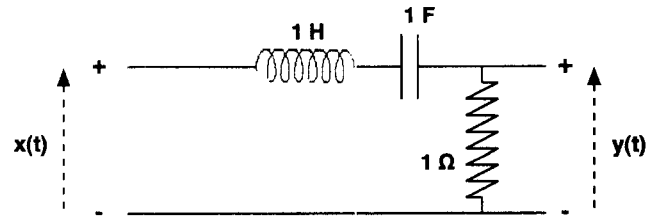


Figure 3: An RLC circuit.

- (a) Determine the transfer function $H(\omega)$. [6]
- (b) Determine $|H(\omega)|^2$. [6]
- (c) Determine the frequency ω_0 at which $|H(\omega)|^2$ is maximum. [6]
- (d) The input voltage $x(t)$ has an autocorrelation $\mathcal{R}_x(\tau) = 5 \cos(\omega_0 \tau)$. Determine the maximum frequency ω_0 at which the ratio $P_y/P_x = 0.8$. Here, P_y and P_x are the power of the output and input signals respectively. [12]

4. Three lines of identical length, characteristic impedance and phase velocity are connected in series as shown in Figure 4, one with an open circuit termination, one with a short circuit termination and the third with a matched termination. The three transmission lines have $L_0 = 0.25 \mu\text{H/m}$ and $C_0 = 100 \text{ pF/m}$.

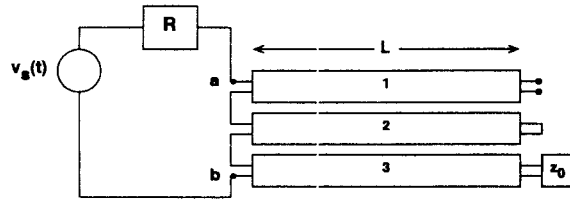


Figure 4: The circuit with three transmission lines.

- (a) Determine the characteristic impedance and the phase velocity of the three lines.

[6]

- (b) The circuit of Figure 4 is now driven by a signal $v_s(t) = V_0 \exp(j2\pi f_0 t)$ with $V_0 = 5 \text{ V}$, $f_0 = 1 \text{ MHz}$ and internal resistance $R = 50 \Omega$. Find the shortest length L for which the combined steady-state impedance of the three lines, as measured at terminals a-b, will be 50Ω .

[12]

- (c) If the length L satisfies the condition described in part (b) above, find the steady state voltage $v_1(x, t)$, along the first line. Hence, for this line, calculate the value of the largest voltage amplitude.

[12]

E1.6 Communications 1 SOLUTIONS

1

QUESTION 1 (ALL QUESTIONS IN QUESTION 1 ARE 'BOOK WORK')

a)

$$i) C_{x_1, x_2} = \int_0^{0.5} \cos 4\pi t \, dt = \frac{1}{4\pi} \sin 4\pi t \Big|_0^{0.5} = 0$$

$$x_1 \perp x_2$$

ii) $E_T = 16 E_{x_1} + 4 E_{x_2}$

$$E_{x_1} = 1 \quad E_{x_2} = \frac{1}{2}$$

$$E_T = 16 + 2 = 18$$

b) $a_0 = \frac{1}{T_0} \int_0^{T_0} x(t) \, dt$ $T_0 = 1 \Rightarrow a_0 = \frac{1}{2}$

$$\begin{aligned} a_1 &= \int_0^1 x(t) \cos 2\pi t \, dt = \int_0^1 t \cos 2\pi t \, dt = \\ &= t \cdot \frac{\sin 2\pi t}{2\pi} \Big|_0^1 - \int_0^1 \frac{\sin 2\pi t}{2\pi} \, dt = 0 \end{aligned}$$

c) $G(\omega) = \int_{-\infty}^{\infty} g(t) e^{-j\omega t} \, dt$ WE HAVE

$$\int_{-\infty}^{\infty} g(t) e^{+j\omega_0 t} e^{-j\omega t} \, dt = \int_{-\infty}^{\infty} g(t) e^{-j(\omega - \omega_0)t} \, dt = G(\omega - \omega_0)$$

$$\cos \omega_0 t = \frac{1}{2} \left(e^{j\omega_0 t} + e^{-j\omega_0 t} \right)$$

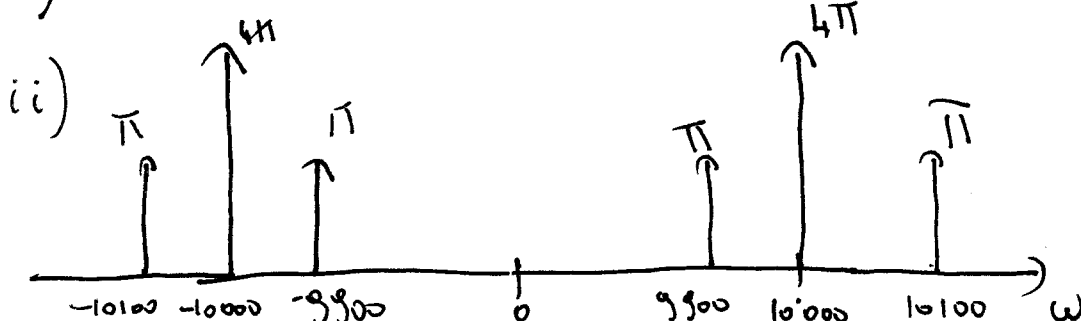
THEREFORE USING

THE LINEARITY PROPERTY OF THE FOURIER TRANSFORM AND THE RESULT ABOVE WE HAVE THAT

$$g(t) \cos \omega_0 t \Leftrightarrow \frac{1}{2} G(\omega - \omega_0) + \frac{1}{2} G(\omega + \omega_0)$$

d)

i) $A = 2$



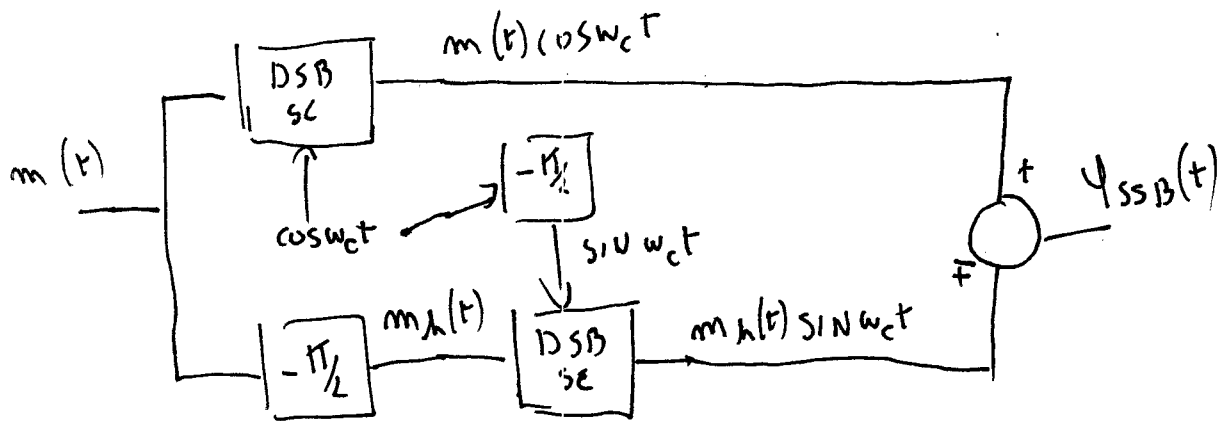
iii) $P_c = \frac{A^2}{2} = \frac{16}{2} = 8$

$$P_s = \frac{P_m}{2} = \frac{2}{2} = 1$$

$$\eta = \frac{P_s}{P_c + P_s} = \frac{1}{9}$$

e)

$$\psi_{SSB} = m(t) \cos \omega_c t \mp m_h(t) \sin \omega_c t$$



f) THE BANDWIDTH INCREASES LINEARLY WITH Δ AND f_m AND IS NOT INFLUENCED BY f_o .

g)

$$(i) \quad K_V = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{1}{3}$$

$$(ii) \quad I_- = -K_V I_+ = -K_V V_+ / Z_0 = -\frac{1}{15} A = -0.67 A$$

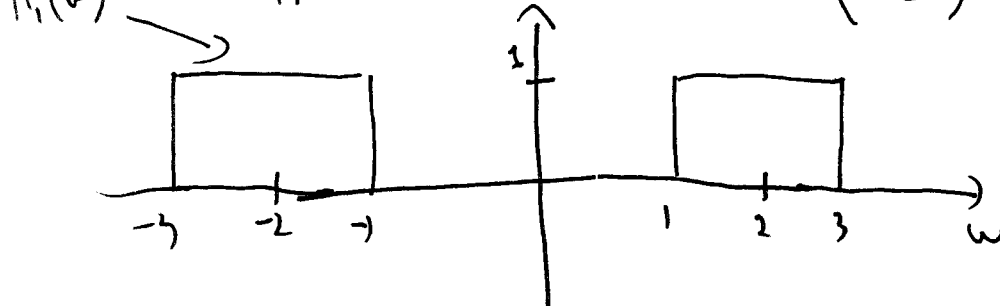
2)

$$(a) \quad m_1(t) = \frac{2}{\pi} \text{SINC}(t) \cos(2t)$$

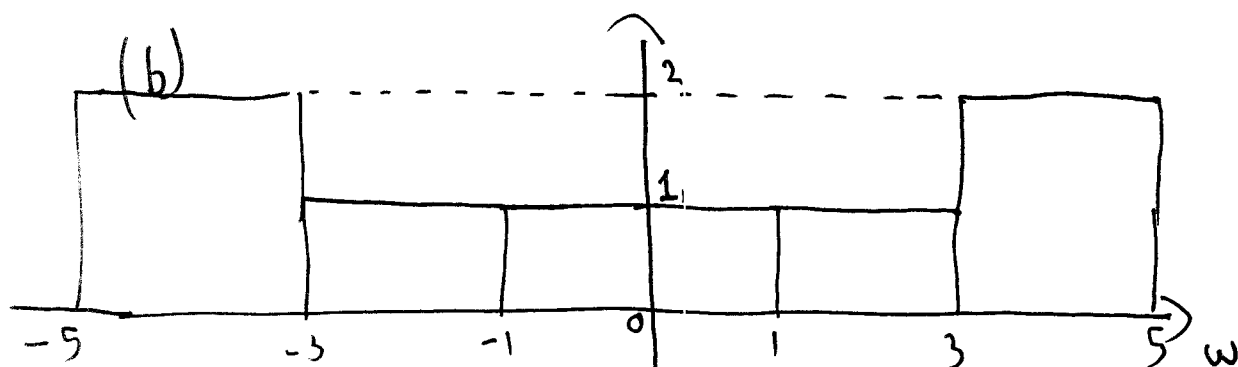
$$\frac{2}{\pi} \text{SINC}(t) \Leftrightarrow 2 \text{RECT}\left(\frac{\omega}{2}\right)$$

THEREFORE

$$\frac{2}{\pi} \text{SINC}(t) \cos 2t \Leftrightarrow \text{RECT}\left(\frac{\omega-2}{2}\right) + \text{RECT}\left(\frac{\omega+2}{2}\right)$$



(NEW COMPUTED EXAMPLE)



(NEW COMPUTED EXAMPLE)

$$(c) \quad B_{FH} = 2(Bf + B) = 2\left(\frac{K_f \cdot X_p}{2\pi} + B\right)$$

$$B = \frac{5}{2\pi} \text{ Hz} \quad X_p = \frac{4}{\pi}$$

$$\text{THUS} \quad B_{FH} = 2\left(\frac{10\pi \cdot \frac{4}{\pi}}{2\pi} + \frac{5}{2\pi}\right) = \frac{45}{\pi} \text{ Hz}$$

(NEW COMPUTED EXAMPLE)

$$(d) \quad x(t) = A e^{-10t} u(t)$$

$$X(\omega) = A \int_{-\infty}^{\infty} e^{-10t} u(t) e^{-j\omega t} dt = \frac{A}{10 + j\omega}$$

USE PARSEVAL'S THEOREM TO FIND THE BANDWIDTH OF $x(t)$:

$$E_x = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega = \frac{A^2}{20}$$

CALL $W = 2\pi B$, W MUST BE SUCH THAT

$$A^2 \frac{0.95}{20} = \frac{1}{2\pi} \int_{-W}^W |X(\omega)|^2 d\omega = \frac{A^2}{2\pi} \int_{-W}^W \frac{d\omega}{\omega^2 + 100} =$$

$$= \frac{A^2}{10\pi} \tan^{-1} \frac{W}{10} \Rightarrow W = 124.6 \text{ RAD/s}$$

AND $B = 20.2 \text{ Hz}$

$x_p = A$

THUS

$$B_{FH} = 2 \left(\frac{124.6 \cdot A}{2\pi} + B \right) = 2 \left(\frac{10\pi A}{2\pi} + 20.2 \right) = 50.4 \text{ Hz}$$

$\Rightarrow A = 1$

(NEAR APPLICATION OF THEORY)

3)

$$(a) \quad H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{j\omega}{1+j\omega-\omega^2} \quad (\text{'NEW APPLICATION OF THE THEORY'})$$

$$(b) \quad |H(\omega)|^2 = \frac{\omega^2}{1+\omega^4-\omega^2} \quad (\text{'NEW APPLICATION OF THE THEORY'})$$

$$(c) \quad \frac{d|H(\omega)|^2}{d\omega} = \frac{2\omega(1+\omega^4-\omega^2) - \omega^2(4\omega^3-2\omega)}{(1+\omega^4-\omega^2)^2} = 0$$

$$\Rightarrow 2\omega(1-\omega^4) = 0$$

$\omega = 0$ GIVES A MINIMUM

$\omega_0 = 1$ GIVES THE MAXIMUM

(d)

$$R_x(\tau) \Leftrightarrow S_x(\omega)$$

$$\text{SINCE} \quad S_x(\omega) = 5\pi [\delta(\omega-\omega_0) + \delta(\omega+\omega_0)]$$

$$\text{AND} \quad S_y(\omega) = |H(\omega)|^2 \cdot S_x(\omega)$$

WE HAVE THAT

$$\frac{P_y}{P_x} = |H(\omega_0)|^2 \quad (\text{'NEW APPLICATION OF THE THEORY'})$$

7

WE NEED TO FIND ω_0 SUCH THAT

$$|H(\omega_0)|^2 = 0.8 \Rightarrow$$

$$(*) \quad \frac{\omega^2}{1 + \omega^4 - \omega^2} = 0.8 \quad \text{CALL } x = \omega^2$$

WE HAVE

$$0.8x^2 - 1.8x + 0.8 = 0$$

$$x = \frac{0.9 \pm \sqrt{0.17}}{0.8} \quad \Rightarrow$$

$$\omega = \pm \sqrt{\frac{0.9 \pm \sqrt{0.17}}{0.8}}$$

THE SOLUTION MUST BE POSITIVE, THUS
THE MAXIMUM ω SATISFYING $(*)$ IS

$$\omega_0 = \sqrt{\frac{0.9 + \sqrt{0.17}}{0.8}} = 1.281 \text{ rad/s}$$

4)

$$a) \quad \Gamma_0 = 50 \Omega$$

$$u = 2 \cdot 10^8 \text{ m/SEC}$$

('BOOK WORK')

$$b) \quad \Gamma_{IN} = \Gamma_0 \left[\frac{\Gamma_L + j \Gamma_0 \tan \beta L}{\Gamma_0 + j \Gamma_L \tan \beta L} \right]$$

$$\text{LINE 1} \quad \Gamma_L = \infty$$

$$\text{THUS} \quad \Gamma_{IN} = \frac{\Gamma_0}{j \tan \beta L}$$

$$\text{LINE 2} \quad \Gamma_L = 0$$

$$\text{THUS} \quad \Gamma_{IN} = j \Gamma_0 \tan \beta L$$

COMBINING IN SERIES

$$Z_{IN} = \Gamma_0 \left(\frac{1}{j \tan \beta L} + j \tan \beta L \right) = j \Gamma_0 \left(\tan \beta L - \frac{1}{\tan \beta L} \right) = 0$$

$$\Rightarrow \tan \beta L = \frac{1}{\tan \beta L} = 1 \Rightarrow \beta L = \frac{\pi}{4}$$

$$L = \frac{\pi}{4 \beta} = \frac{\pi}{4} \cdot \frac{2 \cdot 10^8}{2\pi \cdot 10^6} = 25 \text{ m}$$

('NEW COMPUTED
EXAMPLE')

$$c) \quad \text{IN LINE 1} \quad \Gamma_V = \frac{\Gamma_L - \Gamma_0}{\Gamma_L + \Gamma_0} = 1$$

THUS $V_+ = V_-$

$$\begin{aligned} V_{\text{as}}(x, t) &= V_+ \exp(j\omega t - jkx) + V_+ \exp(j\omega t + jkx) = \\ &= 2V_+ \cos(kx) \exp(j\omega t) \end{aligned}$$

FOR $x = -L$ WE HAVE THAT

$$\begin{aligned} 2V_+ \cos(kL) \exp(j\omega t) &= \frac{1}{2} \cdot \frac{V_0}{t_0} \exp(j\omega t) = \\ &= \frac{V_0}{j2 \tan kL} \end{aligned}$$

$$\tan kL = 1 \quad \text{AND} \quad \cos(kL) = \cos(\pi/4) = \frac{\sqrt{2}}{2}$$

$$\text{THUS} \quad V_+ = \frac{V_0}{j2\sqrt{2}}$$

THE MAXIMUM IS ACHIEVED FOR $x=0$
AND

$$|V_1(0, t)| = |2V_+| = \frac{V_0}{\sqrt{2}} = \frac{5}{\sqrt{2}} \text{ VOLTS}$$

(NEW COMPUTED
EXAMPLES)