

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2008

MSc and EEE PART IV: MEng and ACGI

Corrected Copy

ESTIMATION AND FAULT DETECTION

Wednesday, 21 May 10:00 am

Time allowed: 3:00 hours

There are SIX questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

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Information for candidates:

Some formulae relevant to the questions.

The normal $N(m, \sigma^2)$ density:

$$p(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-m)^2}{2\sigma^2}\right)$$

System equations:

$$\begin{aligned}x_k &= Fx_{k-1} + u_k^s + w_k \\y_k &= Hx_k + u_k^o + v_k.\end{aligned}$$

Here, w_k and v_k are white noise sequences with covariances Q^s and Q^o respectively.

The Kalman filter equations are

$$\begin{aligned}P_{k|k-1} &= FP_{k-1}F^T + Q^s \\P_k &= P_{k|k-1} - P_{k|k-1}H^T(HP_{k|k-1}H^T + Q^o)^{-1}HP_{k|k-1}, \\K_k &= P_{k|k-1}H^T(HP_{k|k-1}H^T + Q^o)^{-1}, \\\hat{x}_k &= \hat{x}_{k|k-1} + K_k(y_k - \hat{y}_{k|k-1}), \\\text{in which } \hat{x}_{k|k-1} &= F\hat{x}_{k-1} + u_k^s \text{ and } \hat{y}_{k|k-1} = H\hat{x}_{k|k-1} + u_k^o\end{aligned}$$

1. Consider a stationary zero mean continuous 2-vector stochastic process $x(t) = [x^1(t), x^2(t)]^T$, governed by the stochastic differential equation

$$\frac{d}{dt} \begin{bmatrix} x^1(t) \\ x^2(t) \end{bmatrix} = \begin{bmatrix} -\alpha_1 & 0 \\ 0 & -\alpha_2 \end{bmatrix} \begin{bmatrix} x^1(t) \\ x^2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w(t)$$

in which $w(t)$ is a stationary scalar white noise process with unit variance; i.e. $E[w(t)w(s)] = \delta(t-s)$. α_1 and α_2 are positive constants.

The discrete time process $x_k = [x_k^1, x_k^2]^T$ is obtained by sampling the continuous time process at times $\dots, -2h, -h, 0, h, 2h, \dots$; thus

$$x_k = x(kh).$$

(h , the sampling period, is a given positive constant.)

- (a): Show that x_k satisfies a stochastic difference equation of the form

$$x_k = Fx_{k-1} + v_k,$$

in which v_k is a 2-vector white noise process with covariance Q . Evaluate F and Q . [8]

- (b): Show that covariance R of x_k

$$R := \text{cov}\{x_k\} (= E[x_k x_k^T]).$$

satisfies the Lyapunov Equation

$$R = FRF^T + Q$$

Solve the Lyapunov equation for R . [4]

Why is R independent of h ? [4]

- (c): Show that, in the case when $\alpha_1/\alpha_2 > 100$ then the correlation coefficient [2]

$$|\rho(x_k^1, x_k^2)| \leq 0.1$$

where $\rho(x_k^1, x_k^2)$ is the correlation coefficient of x_k^1 and x_k^2 ,

$$\rho(x_k^1, x_k^2) = \frac{E[x_k^1 x_k^2]}{(E(x_k^1)^2)^{\frac{1}{2}} (E(x_k^2)^2)^{\frac{1}{2}}}, \quad [2]$$

(This illustrates the fact that, if the time constants associated with the scalar processes x_k^1 and x_k^2 differ by an order of magnitude, then these processes are almost uncorrelated, even if they are generated by the same noise process.)

2. Denote the temperature of a reactor by x . A noisy measurement y is taken of x , using a thermometer. The thermometer is sometimes faulty; when it is faulty it introduces a bias of 1 unit in the measurement.

Model x as a normal random variable ($x \sim N(0, \sigma_x^2)$). Assume that y is governed by the equation

$$y = x + n + b,$$

in which the noise term n is a normal random variable ($n \sim N(0, \sigma_n^2)$) and the bias term b is a discrete random variable taking values 0 or 1, with probabilities

$$P[b = 0] = (1 - P) \quad \text{and} \quad P[b = 1] = P.$$

Here, σ_x^2 , σ_n^2 and P , $0 < P < 1$, are given positive constants. It is assumed that x and n and b are independent.

- (a): Show that the linear least squares estimate \hat{x}_L of x given y is of the form

$$\hat{x}_L = K(y - \alpha),$$

and evaluate the constants K and α .

[8]

- (b): Show that the nonlinear least squares estimate x_N of x given y is of the form

$$\hat{x}_{NL} = K_1(y - \alpha(y)),$$

for some constant K_1 , where

$$\alpha(y) = \frac{P \times N(1, \sigma_x^2 + \sigma_n^2)(y)}{(1 - P) \times N(0, \sigma_x^2 + \sigma_n^2)(y) + P \times N(1, \sigma_x^2 + \sigma_n^2)(y)}$$

and evaluate the constant K_1 .

[10]

Briefly comment on why \hat{x}_{NL} and \hat{x}_N differ.

[2]

Hint: Use Bayes' rules to show that $\alpha(y) = P[b = 1 | y]$. Then note that $p(x | y)$ is a weighted sum of normal random variables given by

$$p(x | y) = p(x | y, b = 0)p(b = 0) + p(x | y, b = 1)p(b = 1)$$

4. Denote by x_k the n -dimensional state of a deterministic system, with random initial state x_0 . Noisy measurements y_k of the state are taken at times $k = 1, 2, \dots$. Assume that the evolution of the state and the measurement process are modelled by the equations

$$\begin{cases} x_k = Fx_{k-1} \\ y_k = Hx_k + v_k. \end{cases}$$

Here, F and H are given $n \times n$ and $r \times n$ dimensional matrices. $\{v_k\}$ is Gaussian white noise sequence, with covariance the given $k \times k$ matrix Q . x_0 is a normal random variable ($x_0 \sim N(\hat{x}_{0|0}, P_{0|0})$) for a given n -vector $\hat{x}_{0|0}$ and given $n \times n$ matrix $P_{0|0}$. It is assumed that x_0 and $\{v_k\}$ are independent.

A recursive filter is required, to estimate the value of the *initial* state x_0 , based on measurement values $y_{0:k} := \{y_1, \dots, y_k\}$, $k = 1, 2, \dots$. Define the conditional means and covariances of x_0 given measurements up to time k :

$$\hat{x}_{0|k} = E[x_0 | y_{1:k}] \quad \text{and} \quad P_{0|k} = \text{cov}\{x_0 | y_{1:k}\}.$$

- (a): Show that

$$p(y_k | y_{1:k-1}, x_0) = N(HF^k x_0, Q)(y_k).$$

[4]

- (b): Using Bayes' rule in the form

$$\begin{aligned} \log p(x_0 | y_{1:k}) \\ = \log p(y_k | y_{1:k-1}, x_0) + \log p(x_0 | y_{1:k-1}) - \log p(y_k | y_{1:k-1}) \end{aligned}$$

derive the recursive equations for $P_{0|k}$ and $\hat{x}_{0|k}$:

$$P_{0|k}^{-1} = P_{0|k-1}^{-1} + (F^k)^T H^T Q^{-1} H F^k$$

and

$$P_{0|k}^{-1} \hat{x}_{0|k} = P_{0|k-1}^{-1} \hat{x}_{0|k-1} + (F^k)^T H^T Q^{-1} y_k.$$

[6]

[7]

- (c): Suppose that $n = r = 1$ (scalar state and observations). Suppose further that $P_{0|0} = 1$, $F = \sqrt{0.5}$ and $Q = 1$. Determine the limiting error covariance, $P_{0|\infty}$:

$$P_{0|\infty} := \lim_{k \rightarrow \infty} P_{0|k}.$$

[4]

5. Let X be a random variable with first and second moments

$$E[X] = 0 \quad \text{and} \quad E[X^2] = \sigma^2 \quad (1)$$

(for some known constant σ^2 .) A noisy ‘nonlinear’ measurement Y is taken of X . Assume that Y is a random variable satisfying the equation

$$Y = h(X) + V,$$

in which the noise term V is a zero mean random variable with known variance σ_n^2 and $h(x)$ is the cubic nonlinearity

$$h(x) = x^3.$$

Construct a linear estimator for X given Y of the form

$$\hat{X} = KY$$

using the following method:

Step 1.

Assume that, for purposes of constructing the filter, X has a discrete distribution

$$p_X(x) = \alpha\delta(x + \sigma) + (1 - 2\alpha)\delta(x) + \alpha\delta(x - \sigma),$$

for some α , $0 < \alpha < 1$; otherwise expressed, X is assumed to be a discrete random variable taking values $-\sigma$, 0 or σ , with probability weights α , $(1 - 2\alpha)$ and α respectively. Determine the value of α such that X has the correct first two moments (see (1)). [4]

Step 2.

Calculate $E[XY]$ and $E[Y^2]$, using the probability distribution for X that you have just calculated. Choose the filter gain K to be the gain of the linear least squares estimator of X given Y . [10]

(This estimate is a version of the widely used ‘unscented Kalman filter’)

An alternative approach to constructing a linear estimator (the ‘extended Kalman filter’) is to assume x is a Gaussian random variable and to approximate the nonlinear function $h(x)$ by the linear function $h(E[X]) + h_x(x - E[X])$, taking the correct value and slope at $x = E[X]$. Show that this gives an estimate:

$$\hat{x}_{EKS} = K_{EKS}Y$$

where [4]

$$H = h_x(E[X]) \quad \text{and} \quad K_{EKS} = \frac{\sigma^2 H}{H\sigma^2 H + \sigma_n^2}.$$

Why can we expect that the unscented Kalman filter performs far better than the extended Kalman filter in the case (1)? [2]

6. Consider the stationary, zero mean, Gaussian process y_k that satisfies the difference equation

$$y_k + 0.5y_{k-1} = e_k + de_{k-1} ,$$

in which e_k is a Gaussian white noise process with unit variance. d is a constant.

- (a): Calculate the variance $r_d(0)$ of y_k :

$$r_d(0) = E[y_k^2]$$

(it will depend on the constant d).

[10]

- (b): Now suppose that the value of d depends on whether a fault has occurred. We consider two hypotheses:

(H_0) : (a fault has not occurred) $d = 0$

(H_1) : (a fault has occurred) $d = 2$.

Write $P_i[A]$, $i = 0, 1$, for the probability of the event A under hypotheses (H_0) and (H_1) respectively.

For a single value of k , a perfect measurement of $z = y_k$ is taken. Design a Neyman Pearson-type decision rule $\delta(z)$ that takes values 0 (no fault) and 1 (fault), and which maximizes the power of test, namely

$$P_1[\delta(z) = 1]$$

(the probability that the rule will detect a fault if it has occurred) at the 0.05 significance level, i.e. under the following constraint on the probability of a false alarm:

$$P_0[\delta(z) = 1] = 0.05 .$$

[6]

Determine the power of the test.

[4]

You may use the following data about a normal random $x \sim N(0, 1)$:

c	:	0.039	0.01	1.28	3.84
$P[x^2 \geq c]$:	0.95	0.74	0.26	0.05