Electromagnetic Fields 2017 – Solutions

1. a) Consider a time-varying magnetic flux $\underline{\mathbf{B}}$ passing through a closed loop L, defining the rim of an open surface as shown below: The total flux of magnetic induction ψ_B through the surface can be found by integration as $\psi_B = \iint_A \underline{\mathbf{B}} \cdot d\underline{a}$

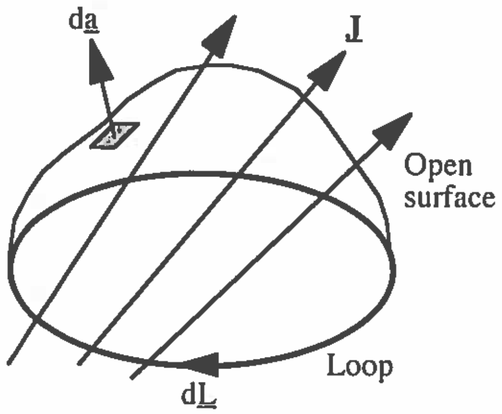
Faraday discovered that the electromotive force E induced round the loop is $E = -\partial\psi_B/\partial t$

Clearly, the induced voltage is related to the electric field by $E = \oint_L \underline{\mathbf{E}} \cdot d\underline{L}$

Combining the above together we obtain Faraday's law:

$\oint_L \underline{\mathbf{E}} \cdot d\underline{L} = -\iint_A \partial\underline{\mathbf{B}}/\partial t \cdot d\underline{a}$

[5]



[3]

b) A quarter-wave transformer can be used to match two different impedances Z_1 and Z_2, for example in two different transmission lines. The solution is to insert a short section of line of length $d = \lambda/4$ (so $k_3d = \pi/2$) as shown below.

The input impedance of Line 3 and Line 2 together is:

$Z_{in} = Z_3 \{Z_2 + jZ_3 \tan(k_3d)\} / \{Z_3 + jZ_2 \tan(k_3d)\}$

If $k_3d = \pi/2$, the tan functions are both infinite. Consequently, we can approximate Z_{in} as:

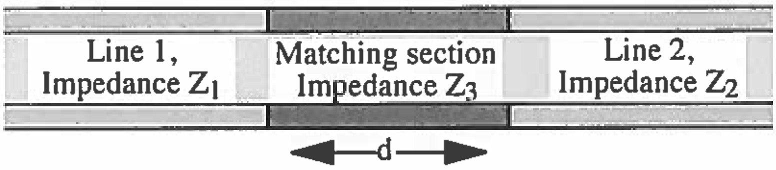
$Z_{in} = Z_3 \{jZ_3 \tan(\pi/2)\} / \{jZ_2 \tan(\pi/2)\} = Z_3^2/ Z_2$

Clearly, Z_{in} is a real impedance, and can present a matched load to Line 1 if $Z_1 = Z_3^2/Z_2$

Hence Z_3 should be the geometric mean of Z_1 and Z_2, namely $Z_3 = (Z_1Z_2)^{1/2}$

In this case, exact matching will be obtained and there can be no reflection.

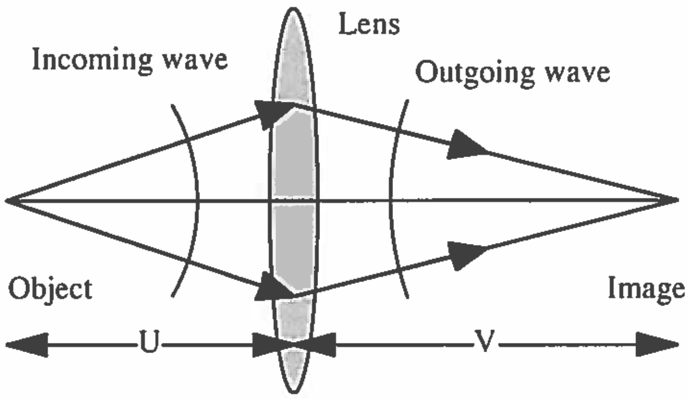
[5]



[3]

c) The imaging equation relates the image distance v to the object distance u and the focal length of the lens used in optical imaging. It can be written as $1/u + 1/v = 1/f$. For example, if the object is at infinity, $1/u = 0$ so $v = f$. A parallel beam will therefore be focused at the focal point.

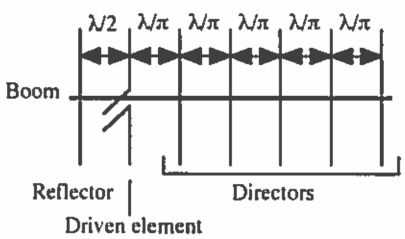
[5]



[3]

d) Because radio signals falls off as $1/r^2$, they are very weak at a large distance r. Simple dipoles are isotropic and radiate equally well in all directions. The main function of a well-designed antenna is to overcome this problem by introducing signal gain in a particular direction in both transmission and reception. Directivity may be introduced using arrays of dipoles. High directivity can be obtained easy if many elements are driven; however, such arrays are too expensive to use outside military applications. The Yagi array achieves the performance of a multi-element array using only one electrical feed. Although there are many elements, most act as passive directors or reflectors, which pass their separate signals to the single connected element.

[5]



[3]

e) Microstrip is a planar transmission line whose layout can be defined by surface patterning. It consists of a metal strip conductor spaced by a dielectric above a rear ground plane.

The electric field passing between the two the strip conductor and ground gives rise to a capacitance.

Including fringing fields (which spread h on either side), the capacitance per-unit-length is:

$C_p \approx \varepsilon(w + 2h)/h$

The magnetic field trapped between the conductors gives rise an inductance. Using Ampere's law, the magnetic field can be related to the current by:

$Hw \approx I$

The magnetic flux density is therefore:

$B = \mu_0 I/w$

The total flux per-unit-length linked between the conductors is therefore:

$\Phi_p = Bh = \mu_0 Ih/w$

Hence the inductance per-unit-length is:

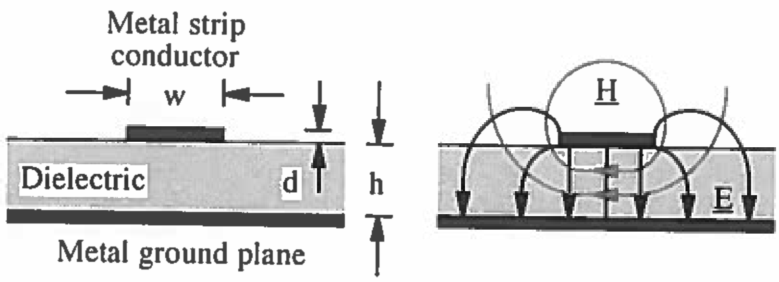
$L_p = \Phi_p/I, = \mu_0 h/w$

Hence the characteristic impedance is:

$Z_0 = (L_p/C_p)^{1/2} \approx \{\mu_0 h/\varepsilon w(2 + w/h)\}^{1/2}$

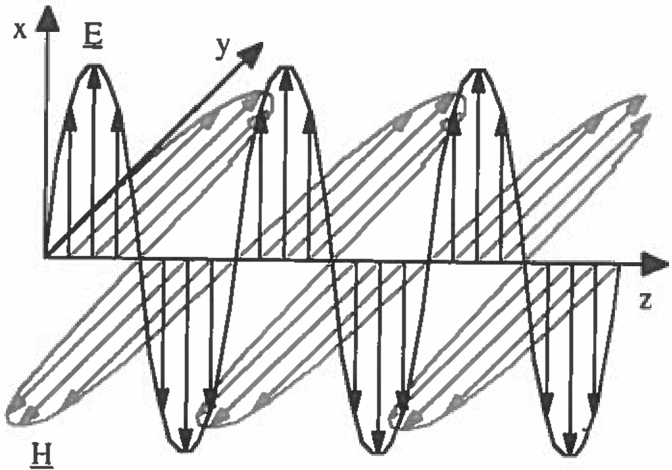
Z_0 is clearly adjustable to any desired value (say, 50 Ω) by careful choice of dimensions.

[5]



[3]

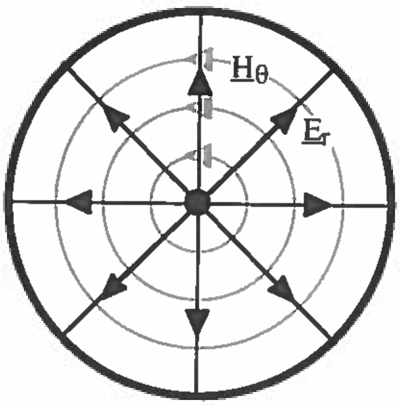
2a) For an x-polarised wave travelling in the z-direction, the electric field oscillates up and down in the x-direction, while the magnetic field oscillates in the y-direction as shown below.



[2]

The electric and magnetic fields inside a coaxial cable are radial and circumferential, respectively, as shown below. Outside the cable, they are both zero (as shown from Gauss' and Ampere's laws).

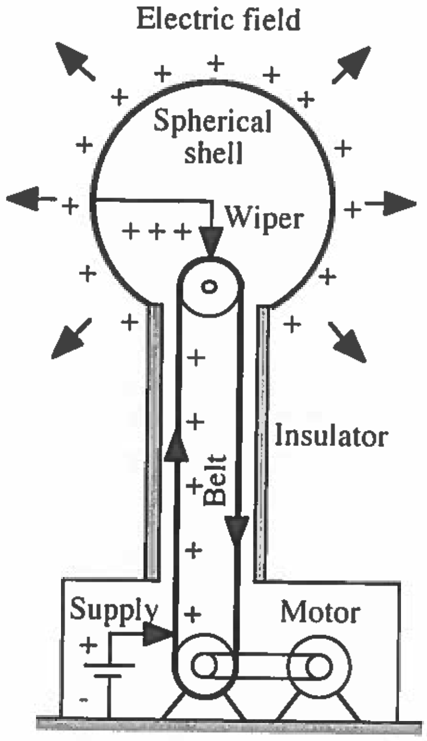
[2]



[2]

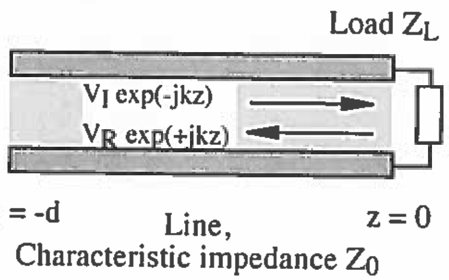
b) In integral form, Gauss' Law states that $\iint_A \underline{\mathbf{D}} \cdot d\underline{a} = \iiint_V \rho \, dv$. This relation implies that the electric flux out of a surface is equal to the charge enclosed. If the amount of charge can be made high, the electric field generated will therefore also be high. The Van de Graaf generator achieves a high field by using a motor driven belt to transfer a large quantity of charge from a battery to a spherical shell as shown below.

[3]



[3]

c) To find the input impedance of a line of length d, impedance Z_0 and propagation constant k, with a terminating load Z_L, first assume that an input voltage wave $V_I \exp(-jkz)$ will give rise to a reflected wave $V_R \exp(+jkz)$ as shown below (and similarly for the current wave).



[2]

At the input (z = -d), the combined voltage and current are therefore:

$V(-d) = V_I \exp(+jkd) + V_R \exp(-jkd)$

$I(-d) = (V_I/Z_0) \exp(+jkd) - (V_R/Z_0) \exp(-jkd)$

[2]

Now, the input impedance is $Z_{in} = V(-d)/I(-d)$, or:

$Z_{in} = Z_0 \{V_I \exp(+jkd) + V_R \exp(-jkd)\} / \{V_I \exp(+jkd) - V_R \exp(-jkd)\}$

Since V_R/V_I must equal the reflection coefficient $R_V = (Z_L - Z_0)/(Z_L + Z_0)$, we get:

$Z_{in} = Z_0\{(Z_L + Z_0)\exp(+jkd) + (Z_L - Z_0)\exp(-jkd)\}/\{(Z_L - Z_0)\exp(+jkd) - (Z_L - Z_0 \exp(-jkd)\}$

[2]

Expanding the exponential terms and rearranging we then get:

$Z_{in} = Z_0\{Z_L \cos(kd) + jZ_0 \sin(kd)\} / \{Z_0 \cos(kd) + jZ_L \sin(kd)\}$, or:

$Z_{in} = Z_0\{Z_L + jZ_0 \tan(kd)\} / \{Z_0 + jZ_L \tan(kd)\}$

[2]

d) The phase velocity of a coaxial cable is $v_{ph} = c/\sqrt{\varepsilon_r}$ where ε_r is the relative dielectric constant of the plastic fill. In this case, $\varepsilon_r = 2.25$ so $v_{ph} = 3 \times 10^8/1.5 = 2 \times 10^8$ m/s.

[2]

In any medium, the phase velocity is $v_{ph} = \omega/k$ so the propagation constant is $k = \omega/v_{ph}$. At 10 MHz frequency, this gives $k = 2\pi \times 10^7/(2 \times 10^8) = \pi/10 = 0.314\ m^{-1}$.

[2]

The general expression for input impedance is:

$Z_{in} = Z_0\{Z_L + jZ_0 \tan(kd)\} / \{Z_0 + jZ_L \tan(kd)\}$

If $k = \pi/10\ m^{-1}$ and $d = 5$ m, $kd = 5 \times \pi/10 = \pi/2$, so $\tan(kd)$ is infinite.

Hence $Z_{in} = Z_0^2/Z_L = 25\ \Omega$

[2]

e) If the end of the line is short-circuited, $Z_L = 0$. In this case, $Z_{in} = jZ_0 \tan(kd)\}$, which corresponds to an inductive impedance if $0 \leq kd \leq \pi/2$.

[2]

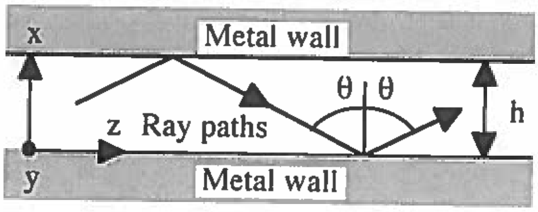
At 10 MHz, an inductor $L = 1\ \mu H$ has a reactance $X = \omega L = 2\pi \times 10^7 \times 10^{-6} = 20\pi$. To achieve this value, we require $kd = \tan^{-1}(X/Z_0) = \tan^{-1}(0.4\pi) = 0.8986$. Hence $d = 0.8986/0.314 = 2.8605$ m.

[2]

3. a) The field in the waveguide can be written as the sum of two travelling waves in the form:

$E_y = E_+ \exp\{-jk_0[z \sin(\theta) + x \cos(\theta)]\} + E_- \exp\{-jk_0[z \sin(\theta) - x \cos(\theta)]\}$

Here, $k_0 = 2\pi/\lambda$ is the propagation constant, λ is the wavelength and θ is the ray direction.



[2]

Since the walls are conducting, the electric field must vanish on $x = 0$ and $x = h$.

The first boundary condition $E_y = 0$ on $x = 0$ is satisfied if $E_- = -E_+$ so

$E_y = E_+\{\exp\{-jk_0[z \sin(\theta) + x \cos(\theta)]\} - \exp\{-jk_0[z \sin(\theta) - x \cos(\theta)]\}\}$, or

$E_y = E \sin[k_0 x \cos(\theta)] \exp[-jk_0 z \sin(\theta)]$ where E is a new constant.

[2]

This field is the product of a transverse field $E(x) = E \sin[k_0 x \cos(\theta)]$ and a complex exponential term $\exp[-jk_0 z \sin(\theta)]$ that describes propagation. It can be written as $E_y = E(x) \exp(-j\beta z)$ where $\beta = k_0 \sin(\theta)$ is the propagation constant.

[2]

The second boundary condition requires $\sin[k_0 h \cos(\theta)] = 0$, so:

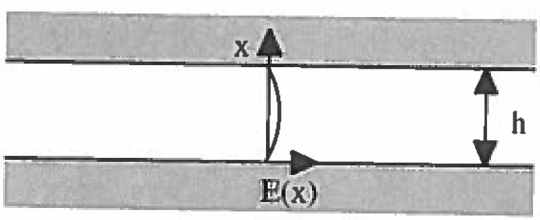
$k_0 h \cos(\theta) = \nu\pi$ where $\nu = 1, 2$ .... is the mode number

Hence, the propagation constant is:

$\beta = \sqrt{\{1 - \cos^2(\theta)\}} = k_0\sqrt{\{1 - (\nu\pi/k_0 h)^2\}}$

[2]

The transverse field can be written in terms of the mode number as $E(x) = E \sin(\nu\pi x/h)$. Modal fields are therefore sinusoidal standing wave patterns. The lowest-order transverse field consists of a single half-period, as shown below:



[2]

b) In three dimensions, the scalar wave equation is $\partial^2E/\partial x^2 + \partial^2E/\partial y^2 + \partial^2E/\partial z^2 + \omega^2\mu_0\varepsilon_0E = 0$

Spherically symmetric solutions will have $E(x, y, z) = E(r)$ so that:

$\partial E/\partial x = dE/dr\ \partial r/\partial x$ and

$\partial^2E/\partial x^2 = d^2E/dr^2\ (\partial r/\partial x)^2 + dE/dr\ \partial^2r/\partial x^2$

[2]

Since $r^2 = x^2 + y^2 + z^2$, derivatives may be found as:

$\partial r/\partial x = x/r$

$\partial^2r/\partial x^2 = (1/r) + x\ \partial(1/r)/\partial x = (1/r)\ (1 - x^2/r^2)$

$\partial^2E/\partial x^2 = (x^2/r^2)\ d^2E/dr^2 + (1/r)\ (1 - x^2/r^2)\ dE/dr$

Similar expressions can be obtained for $\partial^2E/\partial y^2$ and $\partial^2E/\partial z^2$

[3]

Using these derivatives, the scalar wave equation reduces to:

$(x^2/r^2)\ d^2E/dr^2 + (1/r)\ (1 - x^2/r^2)\ dE/dr +$

$(y^2/r^2)\ d^2E/dr^2 + (1/r)\ (1 - y^2/r^2)\ dE/dr +$

$(z^2/r^2)\ d^2E/dr^2 + (1/r)\ (1 - z^2/r^2)\ dE/dr + \omega^2\mu_0\varepsilon_0E = 0$

[3]

Collecting terms together we get:

$\{(x^2 + y^2 + z^2)/r^2\}\ d^2E/dr^2 + (1/r)\ \{3 - (x^2 + y^2 + z^2)/r^2\}\ dE/dr\ + \omega^2\mu_0\varepsilon_0E = 0$, or:

$d^2E/dr^2 + (2/r)\ dE/dr + \omega^2\mu_0\varepsilon_0\ E = 0$

[2]

c) Assuming that $E(r) = (E_0/r)\exp(-jk_0r)$

$dE/dr = E_0(-1/r^2 - jk_0/r)\exp(-jk_0r)$

$d^2E/dr^2 = E_0(2/r^3 + 2jk_0/r^2 - k_0^2/r)\exp(-jk_0r)$

[3]

Substituting into the wave equation:

$E_0(2/r^3 + j2k_0/r^2 - k_0^2/r)\exp(-jk_0r) + E_0(-2/r^3 - j2k_0/r^2)\exp(-jk_0r) + E_0(\omega^2\mu_0\varepsilon_0/r)\exp(-jk_0r) = 0$

Cancelling terms:

$E_0(-k_0^2/r)\exp(-jk_0r) + E_0(\omega^2\mu_0\varepsilon_0/r)\exp(-jk_0r) = 0$

The solution is therefore valid if $-k_0^2 + \omega^2\mu_0\varepsilon_0 = 0$

[2]

The time-averaged power carried by a spherical wave in the radial direction is:

$S_r = |E^2|/2Z_0 = E_0^2/2r^2Z_0$

[2]

The total power passing through a spherical surface of radius r is:

$P = 4\pi r^2 S_r = 2\pi E_0^2/Z_0$

Clearly this expression is independent of radius.

[3]