Paper Number(s): **E4.25**
C4.1
ISE4.23

IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE
UNIVERSITY OF LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2002

MSc and EEE/ISE PART IV: M.Eng. and ACGI

DESIGN OF LINEAR MULTIVARIABLE CONTROL SYSTEMS

Tuesday, 30 April 10:00 am

There are SIX questions on this paper.

Answer FOUR questions.

Corrected Copy

Time allowed: 3:00 hours

Examiners responsible:

First Marker(s): Jaimoukha,I.M.

Second Marker(s): Clark,J.M.C.

Special Information for Invigilators :	None
Information for Candidates :	None

1. Let the transfer matrix $G(s)$ have a state space realisation

$$G(s) \stackrel{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] := \left[\begin{array}{ccc|cc} 1 & 2 & 0 & 1 & 2 \\ 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & 4 & 3 & 4 \\ \hline 2 & 3 & 0 & 1 & 0 \\ 1 & 4 & 0 & 0 & 1 \end{array}\right].$$

(a) Find the uncontrollable and/or unobservable modes and determine whether the realisation is detectable and stabilisable. [4]

(b) Determine whether there exist matrices

$$K \in \mathcal{R}^{2\times 3},$$

and

$$L \in \mathcal{R}^{3\times 2},$$

such that $A - BK$ and $A - LC$ are stable. Justify your answer. [4]

(c) Find a minimal realisation for $G(s)$. [4]

(d) Find the McMillan form of $G(s)$ and determine the pole and zero polynomials. What is the McMillan degree of $G(s)$? [4]

(e) Determine the system zeros, indicating the type of each zero. [4]

2. (a) Define internal stability for the feedback loop in Figure 2.1, and derive necessary and sufficient conditions for which this loop is internally stable. [6]

(b) Suppose that $G(s)$ is stable. Give a parameterisation of all internally stabilising controllers for $G(s)$ for the feedback loop in Figure 2.1. [4]

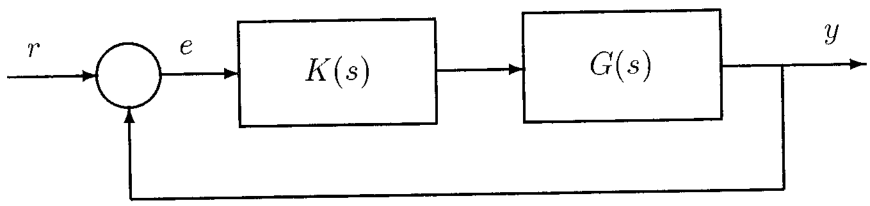


Figure 2.1

(c) Let $G(s)$ be given by

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+2} \\ 0 & \frac{1}{s+1} \end{bmatrix}.$$

Suppose now that an output multiplicative uncertainty on $G(s)$ is introduced as shown in Figure 2.2. Design an internally stabilising controller $K(s)$ that satisfies the following performance and robustness design specifications:

i. When $\Delta = 0$, the transfer matrix from r to e, $S(s)$, satisfies $\|S(0)\| < 1/2$.

ii. The feedback loop is stable for all $\Delta \in \mathcal{RH}_\infty$ such that $\|\Delta\|_\infty < 1$. [10]

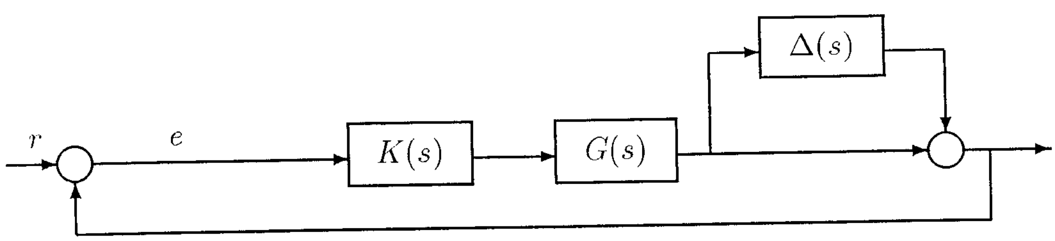


Figure 2.2

3. (a) Let $A \in \mathcal{R}^{n \times n}$ and $B \in \mathcal{R}^{n \times p}$ be given. Suppose that $AP + PA^T + BB^T = 0$ where

$$P = \begin{bmatrix} 6 & 3 & -2 \\ 3 & 12 & 6 \\ -2 & 6 & 18 \end{bmatrix}$$

By using Gershgorin's theorem, show that A is stable and that the pair (A, B) is controllable. [4]

(b) For the feedback loop in Figure 3.1, state a Nyquist type stability criterion in terms of the direct Nyquist array of a transfer matrix $G(s)$. [6]

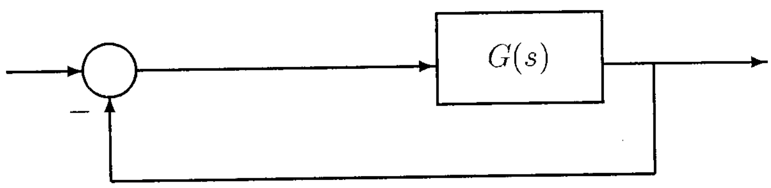


Figure 3.1

(c) Consider the feedback loop in Figure 3.2. Here

$$G(s) = \begin{bmatrix} 5/(s+1) & 1/(s+4) \\ 1/(s+4) & 5/(s+1) \end{bmatrix},$$

and $\Delta(s)$ is a transfer matrix representing a stable additive structured uncertainty of the form

$$\Delta(s) = \begin{bmatrix} 0 & \delta_{12}(s) \\ \delta_{21}(s) & 0 \end{bmatrix}.$$

Use the answer to Part (b) to derive the maximal stability radius (using the $\mathcal{L}_\infty$-norm as a measure) guaranteed by Gershgorin's theorem for the feedback loop in Figure 3.2 below. [10]

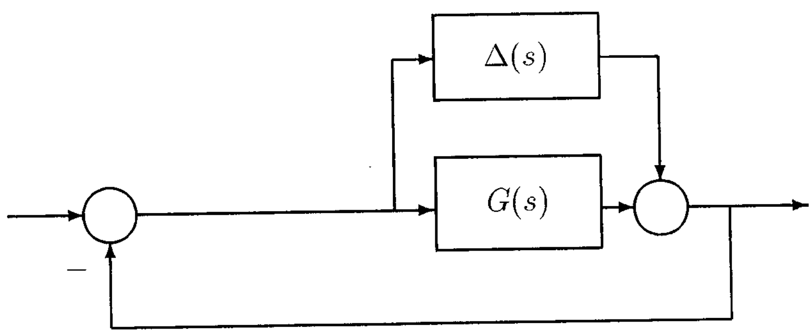


Figure 3.2

4. Figure 4.1 illustrates the implementation of the control law $u = -Kx$ which minimises

$$J(x_0, u) = \int_0^\infty \|Cx(t)\|^2 + \|u(t)\|^2 \, dt$$

subject to $\dot{x} = Ax(t) + Bu(t)$, $x(0) = x_0$. Here $K = B'P$ and $P = P'$ is the unique positive definite solution of $A'P + PA - PBB'P + C'C = 0$. Assume that the triple (A, B, C) is minimal. Define $G(s) = (sI - A)^{-1}B$.

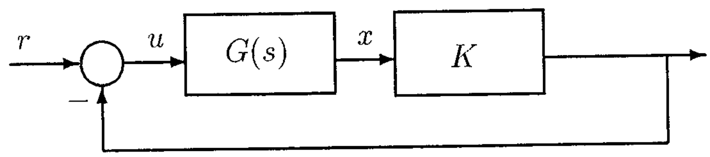


Figure 4.1

(a) Let $L(s) = I + KG(s)$. Show that

$$L(j\omega)'L(j\omega) = I + G(j\omega)'C'CG(j\omega), \ \forall\, \omega \in \mathcal{R}.$$ [5]

(b) Derive the smallest upper bounds on $\|(I + KG)^{-1}\|_\infty$ and $\|(I + KG)^{-1}KG\|_\infty$ guaranteed by Part (a). [5]

(c) Suppose that stable perturbations Δ_1 and Δ_2 are introduced as shown in Figure 4.2. using the answer to Part (b), derive the maximal stability radius (using the $\mathcal{L}_\infty$-norm as a measure):

(i) for Δ_1 when $\Delta_2 = 0$, [5]

(ii) for Δ_2 when $\Delta_1 = 0$. [5]

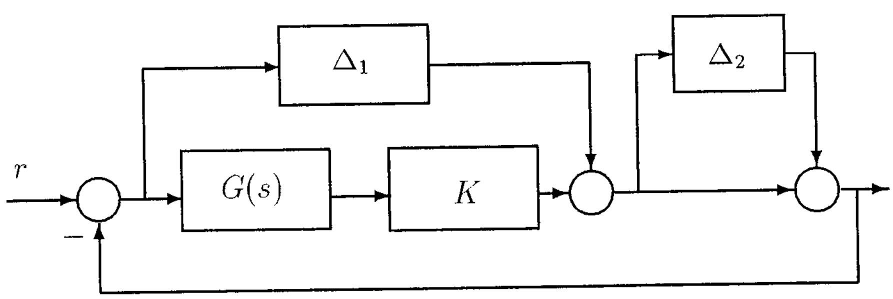


Figure 4.2

5. Consider the feedback configuration in Figure 5.1. Here, $G(s)$ is a nominal plant model and $K(s)$ is a compensator. The transfer matrices $\Delta_a(s)$ and $\Delta_m(s)$ represent stable additive and multiplicative uncertainties on $G(s)$. The uncertainties are described as follows:

$$
\begin{aligned}
\|\Delta_a(j\omega)\| &< |w_a(j\omega)^{-1}|, \ \forall\, \omega \\
\|\Delta_m(j\omega)\| &< |w_m(j\omega)^{-1}|, \ \forall\, \omega
\end{aligned}
$$

where $w_a(s)$ and $w_m(s)$ are high pass filters.

The design specification is to synthesise a controller $K(s)$ such that the closed-loop is stable

(a) for all Δ_a when $\Delta_m = 0$, and,

(b) for all Δ_m when $\Delta_a = 0$.

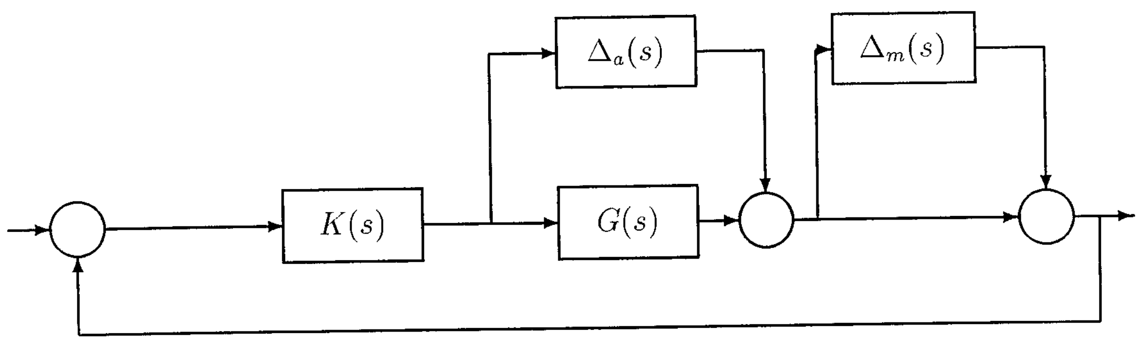


Figure 5.1

(a) Derive $\mathcal{H}_\infty$-norm bounds, in terms of $G(s), K(s), w_a(s)$ and $w_m(s)$ that are sufficient to achieve the design specifications. [6]

(b) Derive a generalised regulator formulation of the design problem that captures the sufficient conditions in Part (a). [10]

(c) Assume that a compensator $K(s)$ achieves the design specifications in Part (a). Let $n(s)$ denote sensor noise in the feedback-loop in Figure 5.2 below. Comment on the noise attenuation properties of this loop. [4]

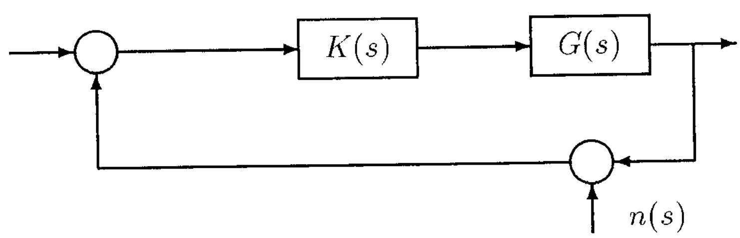
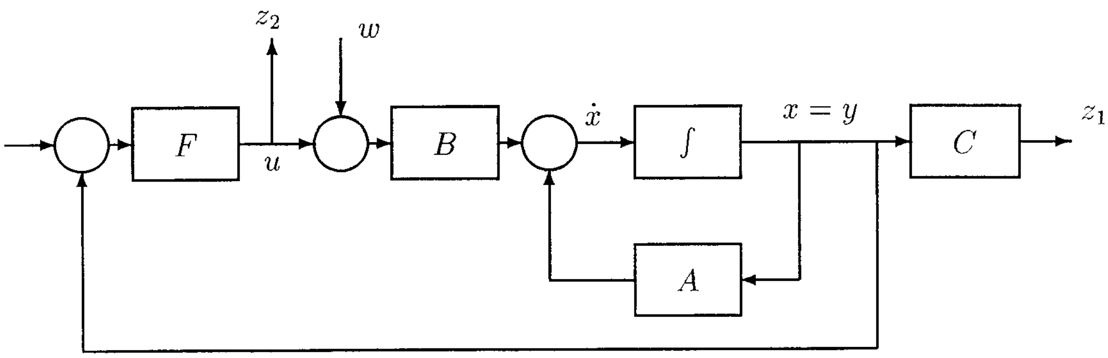


Figure 5.2

6. Consider the simplified generalised regulator shown in the figure below.



Assume that $x(0)=0$ and that (A,B,C) is minimal. The design objective is, for a given $\gamma>0$, to find a stabilising state-feedback gain matrix F, if it exists, such that

$$J := \|z\|_2^2 - \gamma^2\|w\|_2^2 \leq 0, \ \forall w \text{ such that } \|w\|_2^2 < \infty,$$

where $z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ and with $\|v\|_2^2 := \int_0^\infty \|v(t)\|^2 dt$ and $\|v(t)\|^2 := v(t)^T v(t)$.

(a) Write down the generalised regulator system for this design problem. [8]

(b) By using the Lyapunov function $V(t) = x(t)^T X x(t)$, where X is to be determined, derive sufficient conditions for the solution of the design problem. Your conditions should be in the form of the existence of a certain solution to an algebraic Riccati equation. It should also include an expression for F and an expression for the worst-case disturbance w.
Use the identity

$$(\alpha R - \alpha^{-1}S)^T(\alpha R - \alpha^{-1}S) = \alpha^2 R^T R + \alpha^{-2} S^T S - R^T S - S^T R,$$

for scalar $\alpha \neq 0$ and matrices R and S to complete the squares. [8]

(c) Comment on the sufficient conditions in the limit as $\gamma \to \infty$. (Hint: Read Question 4.) [4]

Final copy

1/6

Design of Linear Multivariable Control Systems

Solutions 2001/2002

1\. (a) Since $[A - sI \quad B]$ loses rank for $s = -3$, $\boxed{-3 \text{ is an uncontrollable mode,}}$ and since $[A^T - sI \quad C^T]$ loses rank for $s = 4$, $\boxed{4 \text{ is an unobservable mode.}}$ Since the uncontrollable mode is stable, $\boxed{\text{the realisation is stabilisable}}$ and since the unobservable mode is unstable, $\boxed{\text{the realisation is not detectable.}}$ [4]

(b) Since the mode $\lambda = -3$ is uncontrollable, it cannot be assigned via state feedback. However, since it is stable, the matrix $\boxed{K \text{ exists.}}$ Since $\lambda = 4$ is unobservable, it cannot be assigned via output injection and since it is unstable, $\boxed{L \text{ does not exist.}}$ [4]

(c) By removing the uncontrollable and unobservable modes we get the minimal realisation

$$\boxed{G(s) \stackrel{s}{=} \left[\begin{array}{c|cc} 1 & 1 & 2 \\ \hline 2 & 1 & 0 \\ 1 & 0 & 1 \end{array}\right] = \begin{bmatrix} \frac{s+1}{s-1} & \frac{4}{s-1} \\ \frac{1}{s-1} & \frac{s+1}{s-1} \end{bmatrix} = \frac{1}{s-1}\begin{bmatrix} s+1 & 4 \\ 1 & s+1 \end{bmatrix}.}$$

[4]

(d) By performing the following elementary operations: (1) $r_1 \leftrightarrow r_2$, (2) $r_2 := r_2 - (s+1)r_1$, (3) $c_2 := c_2 - (s+1)c_1$, (4) $c_2 = -c_2$, the McMillan form of $G(s)$ is given by,

$$\boxed{G(s) = \begin{bmatrix} s+1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{s-1} & 0 \\ 0 & s+3 \end{bmatrix} \begin{bmatrix} 1 & s+1 \\ 0 & -1 \end{bmatrix} =: L(s)M(s)R(s),}$$

where $L(s)$ and $R(s)$ are unimodular.
The pole and zero polynomials are given by

$$\boxed{p(s) = s-1,} \quad \& \quad \boxed{z(s) = s+3}$$

respectively. $\boxed{\text{The McMillan degree is 1}}$ since it is equal to the degree of the pole polynomial. [4]

(e) Since $s = -3$ is an uncontrollable mode, $\boxed{-3 \text{ is an input decoupling zero.}}$ Since $s = 4$ is an unobservable mode, $\boxed{4 \text{ is an output decoupling zero.}}$ It follows from Part (d) that the system has a $\boxed{\text{transmission zero at } s = -3.}$ [4]

2. (a) Inject a signal d in between $G(s)$ and $K(s)$ and call the input to $G(s)$ u. The loop is internally stable if and only if the transfer matrix from $\begin{bmatrix} d \\ r \end{bmatrix}$ to $\begin{bmatrix} u \\ \epsilon \end{bmatrix}$ is stable (no poles in the closed right half plane). Since

$$\begin{bmatrix} d \\ r \end{bmatrix} = \begin{bmatrix} I & -K \\ -G & I \end{bmatrix} \begin{bmatrix} u \\ e \end{bmatrix} =: T(s) \begin{bmatrix} u \\ e \end{bmatrix}$$

$\boxed{\text{the loop is internally stable if and only if } T^{-1}(s) \text{ is stable.}}$ [6]

(b) Since $G(s)$ is stable, we proceed as follows. Note that

$$\begin{bmatrix} I & -K \\ -G & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ -G & I \end{bmatrix} \begin{bmatrix} I & -K \\ 0 & I - GK \end{bmatrix}.$$

Hence

$$\begin{bmatrix} I & -K \\ -G & I \end{bmatrix}^{-1} = \begin{bmatrix} I & -K \\ 0 & I-GK \end{bmatrix}^{-1} \begin{bmatrix} I & 0 \\ -G & I \end{bmatrix}^{-1} = \begin{bmatrix} I & K(I-GK)^{-1} \\ 0 & (I-GK)^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ G & I \end{bmatrix}.$$

Finally, since $(I - GK)^{-1} = I + GK(I - GK)^{-1}$, it follows that if G is stable, then the loop is internally stable if and only if $Q := K(I - GK)^{-1}$ is stable. Rearranging terms shows that K internally stabilising if and only if $\boxed{K = Q(I + GQ)^{-1} \text{ for some stable } Q.}$ [4]

(c) Since K is required to be internally stabilising, $\boxed{K = Q(I + GQ)^{-1}}$ for some stable Q from Part (b). We search for a stable Q to satisfy the design requirements. Let the input to Δ be ϵ while the output from Δ be δ. Then a simple calculation shows that $\epsilon = C\delta$ where $C = (I-GK)^{-1}GK$ is the complementary sensitivity which is stable. Now

$$C = GK(I - GK)^{-1} = GQ.$$

The small gain theorem implies that for K to stabilise the loop in Figure 2.2 for all Δ such that $\|\Delta\|_\infty < 1$, we must have $\|GQ\|_\infty < 1$, so we choose

$$\boxed{Q(s) = h(s)G^{-1}(s)} = h(s) \begin{bmatrix} s+1 & \frac{-(s+1)^2}{s+2} \\ 0 & s+1 \end{bmatrix}$$

where $h(s)$ must satisfy $\|h\|_\infty < 1$. To ensure that Q is stable and proper, we may choose

$$\boxed{h(s) = h_0/(s+1)^2}$$

with $-1 < h_0 < 1$ to satisfy the infinity norm constraint.
Since the transfer matrix from r to e is

$$S(s) = (I - G(s)K(s))^{-1} = I + G(s)Q(s) = [1 + h(s)]I = [1 + h0/(s+1)^2]I$$

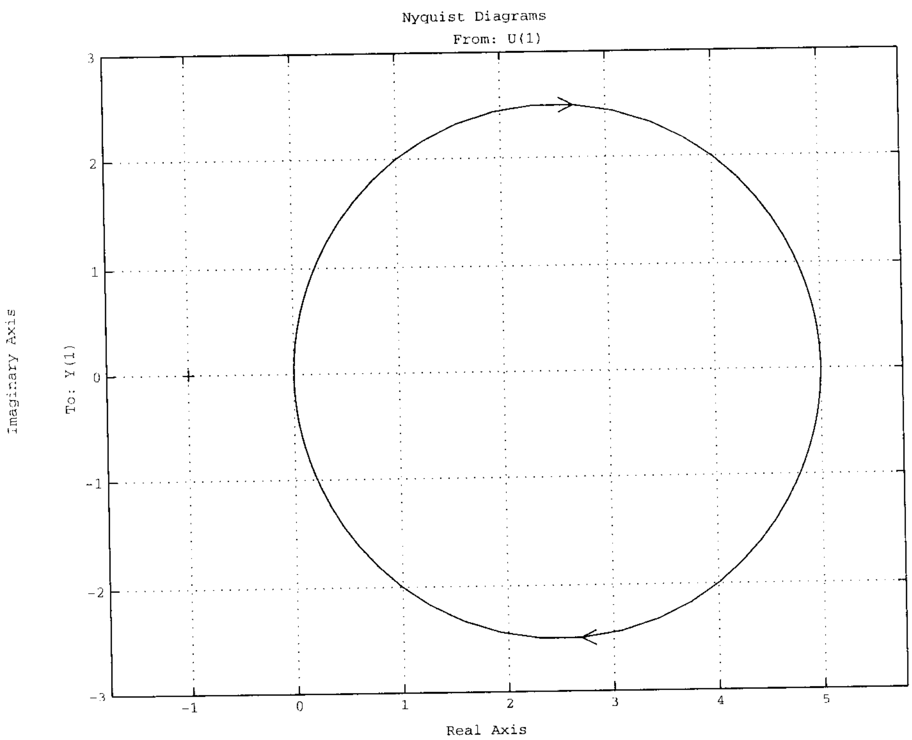
we also need $|1 + h_0| < 1/2$. It follows that any $\boxed{-1 < h_0 < -0.5}$ will satisfy the design specifications. **[10]**

3. (a) The matrix A will be stable and the pair (A, B) controllable if $P > 0$. Using Gershgorin's theorem, the eigenvalues of P lie in the union of the discs,

$$\begin{aligned}|\lambda - 6| &\leq 5, \\ |\lambda - 12| &\leq 9, \\ |\lambda - 18| &\leq 8.\end{aligned}$$

It follows that $\boxed{\text{the eigenvalues are positive and so } P > 0.}$ [4]

(b) Let $G(s)$ have P closed right half plane poles. Assume that $I + G(s)$ is diagonally dominant, that is, $|1 + G_{ii}(s)| \geq \sum_{j \neq i} |G_{ji}(s)|$, for all i and for all s on the Nyquist contour. Here I denotes the identity matrix. Let the ith Gershgorin band of $G(s)$ encircle the point -1 a total of N_i times anticlockwise. Then $\boxed{\text{the loop is internally stable if and only if } \sum_i N_i = P.}$ [6]



(c) For the given $G(s)$, $P = 0$. The Nyquist plots for G_{11} and G_{22}, which coincide, are shown above. Note that the closest distance from the Nyquist diagrams to the point $-1 + j0$ is 1. Since $\|G_{12}\|_\infty = \|G_{21}\|_\infty = 1/4$, it follows that we can tolerate δ_{12} and δ_{21} such that $\max\{\|\delta_{12}\|_\infty, \|\delta_{21}\|_\infty\} < 3/4$. It follows that $\boxed{\text{the maximal stability radius is } 3/4.}$ [10]

4. (a) By direct evaluation, $L(j\omega)'L(j\omega) =$

$$I + K(j\omega I - A)^{-1}B + B'(-j\omega I - A')^{-1}K' + B'(-j\omega I - A')^{-1}K'K(j\omega I - A)^{-1}B$$

But

$$K'K = A'P + PA + C'C = -(-j\omega I - A')P - P(j\omega I - A) + C'C$$

from the Riccati equation. So, $L(j\omega)'L(j\omega)$

$$\begin{aligned} &= I + K(j\omega I - A)^{-1}B + B'(-j\omega I - A')^{-1}K' \\ &\quad + B'(-j\omega I - A')^{-1}[-(-j\omega I - A')P - P(j\omega I - A) + C'C](j\omega I - A)^{-1}B \\ &= I + [K - B'P](j\omega I - A)^{-1}B + B'(-j\omega I - A')^{-1}[K' - PB] \\ &\quad + B'(-j\omega I - A')^{-1}C'C(j\omega I - A)^{-1}B = I + G(j\omega)'C'CG(j\omega) \end{aligned}$$

[5]

(b) Part (a) implies that $\underline{\sigma}[I + KG(j\omega)] \geq 1,\ \forall\, \omega \in \mathcal{R}$. It follows that

$$\boxed{\|(I + KG)^{-1}\|_\infty \leq 1.}$$

Now, $(I + KG)^{-1}KG = L(L^{-1} - I) = I - L^{-1}$. Thus, Part (a) implies that

$$\bar{\sigma}[L(j\omega)^{-1} - I] \leq 1 + \bar{\sigma}[L(j\omega)^{-1}] \leq 1 + \frac{1}{\underline{\sigma}[L(j\omega)]} \leq 2,$$

so that

$$\boxed{\|(I + KG)^{-1}KG\|_\infty \leq 2.}$$

[5]

(c) (i) Set $\Delta_2 = 0$. Let ϵ be the input to Δ_1 and δ be the output of Δ_1. Then

$$\epsilon = -(\delta + KG\epsilon) = -(I + KG)^{-1}\delta$$

Using the small gain theorem (since the regulator is stable and the perturbation is assumed stable), the loop is stable if $\|\Delta_1(I + KG)^{-1}\|_\infty < 1$. But Part (b) implies that $\|(I + KG)^{-1}\|_\infty \leq 1$. This shows that the loop will tolerate perturbations of size

$$\boxed{\|\Delta_1\|_\infty < 1}$$

without losing internal stability. [5]

(ii) Set $\Delta_1 = 0$. Let ϵ be the input to Δ_2 and δ be the output of Δ_2. Then

$$\epsilon = -KG(\delta + \epsilon) = -(I + KG)^{-1}KG\delta.$$

Using the small gain theorem (since the regulator is stable and the perturbation is assumed stable), the loop is stable if $\|\Delta_2(I + KG)^{-1}KG\|_\infty < 1$. But Part (b) implies that $\|(I + KG)^{-1}KG\|_\infty < 2$. This shows that the loop will tolerate perturbations Δ_2 of size

$$\boxed{\|\Delta_2\|_\infty < 0.5}$$

without losing internal stability. [5]

5\. (a) It is clear that we require K to be internally stabilising. Let the inputs to Δ_a and Δ_m be ϵ_a and ϵ_m while the outputs from Δ_a and Δ_m be δ_a and δ_m respectively.

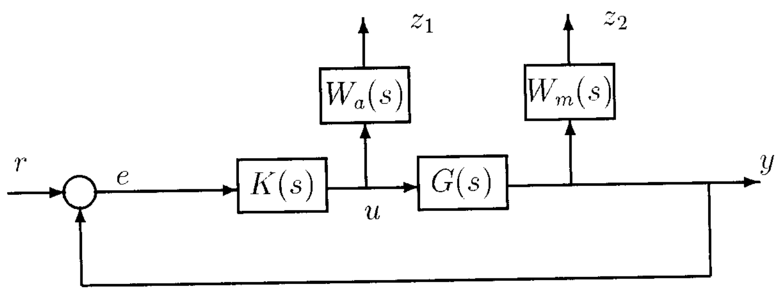
- A simple calculation shows that, when $\Delta_m = 0$, $\epsilon_a = K(I - GK)^{-1}\delta_a$. It follows from the small gain theorem that a sufficient condition to achieve the first design specification is $\|K(j\omega)[I - G(j\omega)K(j\omega)\| < |w_a^{-1}(j\omega)|, \forall\omega$ or equivalently $\boxed{\|W_a K(I - GK)^{-1}\|_\infty < 1,}$ where $W_a = w_a I$.
- When $\Delta_a = 0$, a similar calculation shows that $\epsilon_m = GK(I - GK)^{-1}\delta_m$. It follows that a sufficient condition to achieve the second design specification is $\|G(j\omega)K(j\omega)[I - G(j\omega)K(j\omega)\| < |w_m^{-1}(j\omega)|, \forall\omega$ or equivalently $\boxed{\|W_m GK(I - GK)^{-1}\|_\infty < 1,}$ where $W_m = w_m I$.

Thus, to satisfy both design requirements, it is sufficient that

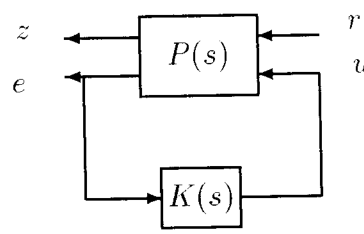
$$\boxed{\left\| \begin{bmatrix} W_a K(I-GK)^{-1} \\ W_m GK(I-GK)^{-1} \end{bmatrix} \right\|_\infty < 1.}$$

[6]

(b) The design specifications reduce to the requirement that the transfer matrix from r to $z = [z_1^T \ \ z_2^T]^T$ in the following diagram has $\mathcal{H}_\infty$-norm less than 1.



The corresponding generalised regulator formulation is to find an internally stabilising K such that $\|\mathcal{F}_l(P, K)\|_\infty < 1$:



$$\boxed{P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \left[\begin{array}{c|c} 0 & W_a \\ 0 & W_m G \\ \hline I & G \end{array}\right].}$$

[10]

(c) The transfer matrix from $n(s)$ to $y(s)$ is the same as that between r and y. Thus $\boxed{\text{the noise attenuation properties are satisfactory since } w_m \text{ is high pass.}}$

[4]

6. (a) The generalised regulator formulation is given by

$$\boxed{\begin{bmatrix} z(s) \\ y(s) \end{bmatrix} = P(s)\begin{bmatrix} w(s) \\ u(s) \end{bmatrix},\ u(s) = Fy(s),\ P(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{c||c|c} A & B & B \\ \hline\hline C & 0 & 0 \\ 0 & 0 & I \\ \hline I & 0 & 0 \end{array}\right].}$$

[8]

(b) Let $V = x^T X x$ and set $u = Fx$. Provided that $X = X^T > 0$ and we show that $\dot{V} < 0$ along closed loop trajectory, we can assume $\lim_{t\to\infty} x(t) = 0$. Then

$$\dot{V} = \dot{x}^T X x + x^T X \dot{x} = x^T\left(A^T X + XA + F^T B^T X + XBF\right)x + x^T XBw + w^T B^T Xx.$$

Integrating from 0 to ∞ and using $x(0) = x(\infty) = 0$,

$$0 = \int_0^\infty [x^T\left(A^T X + XA + F^T B^T X + XBF\right)x + x^T XBw + w^T B^T Xx]dt.$$

Using the definition of J and adding the last equation,

$$J = \int_0^\infty \{x^T\left(C^T C + F^T F\right)x - \gamma^2 w^T w\}dt$$

$$= \int_0^\infty \{x^T[A^T X + XA + C^T C + F^T F + F^T B^T X + XBF]x - [\gamma^2 w^T w - x^T XBw - w^T B^T Xx]\}dt.$$

Completing the squares by using

$$(F + B^T X)^T (F + B^T X) = F^T F + F^T B^T X + XBF + XBB^T X$$

$$(\gamma w - \gamma^{-1} B^T Xx)^T (\gamma w - \gamma^{-1} B^T Xx) = \gamma^2 w^T w - w^T B^T Xx - x^T XBw + \gamma^{-2} x^T XBB^T Xx,$$

$$J = \int_0^\infty \{x^T[A^T X + XA + C^T C - (1-\gamma^{-2})XBB^T X]x + \left\|(F + B^T X)x\right\|^2 - \left\|\gamma w - \gamma^{-1} B^T Xx\right\|^2\}dt.$$

Thus two sufficient conditions for $J \le 0$ are the existence of X such that

$$\boxed{A^T X + XA + C^T C - (1-\gamma^{-2})XBB^T X = 0,} \qquad \boxed{X = X^T > 0.}$$

The state feedback gain is $F = -B^T X$ and the worst case disturbance is $w^* = \gamma^{-2} B^T Xx$. The closed-loop with these feedback laws is $\dot{x} = [A-(1-\gamma^{-2})BB^T X]x$ and a third condition is therefore $\boxed{Re\ \lambda_i[A-(1-\gamma^{-2})BB^T X] < 0,\ \forall\, i.}$

It remains to show that $\dot{V} < 0$ along state-trajectory with $u = Fx$ and $w = 0$. Using the Riccati equation in the expression for $\dot{V}$

$$\dot{V} = x^T\left(A^T X + XA + F^T B^T X + XBF\right)x = -x^T(C^T C + (1+\gamma^{-2})XBB^T X)x < 0,$$

for all $x \neq 0$ (since (A, B, C) is assumed minimal) proving closed-loop stability.

[8]

(c) In the limit as $\gamma \to \infty$, the sufficiency conditions above give the solution of the $\boxed{\text{LQR problem of minimising } J_2 = \|z\|_2^2 \text{ with } w = 0 \text{ and starting at } x(0).}$ [4]