MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

2002 - 2003

PAPER 1SE 2.6

QUESTION

Please write on this side only, legibly and neatly, between the margins

SOLUTION 1

(i) $$\int_{-\infty}^{\infty} f'(t) e^{-i\omega t} dt = \left[f(t) e^{-i\omega t}\right]_{-\infty}^{+\infty} + i\omega \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

$$= i\omega \hat{f}(\omega) \qquad \boxed{a_1(\omega) = i\omega}$$ (2)

(ii) $$\int_{-\infty}^{\infty} f''(t) e^{-i\omega t} dt = \left[f'(t) e^{-i\omega t}\right]_{-\infty}^{\infty} + i\omega \int_{-\infty}^{\infty} f'(t) e^{-i\omega t} dt = i\omega \widehat{f'(t)}(\omega)$$

$$= -\omega^2 \hat{f}(\omega) \qquad \boxed{a_2(\omega) = -\omega^2}$$ (2)

(iii) $$\frac{d\hat{f}(\omega)}{d\omega} = -i\int_{-\infty}^{\infty} t e^{-i\omega t} f(t) dt \quad \text{so} \quad \widehat{tf(t)}(\omega) = i\, d\hat{f}(\omega)/d\omega$$ (2)

$$\int_{-\infty}^{\infty} t f'(t) e^{-i\omega t} dt \qquad \frac{d}{d\omega}\left(\widehat{f'(t)}(\omega)\right) = -i\int_{-\infty}^{\infty} t f'(t) e^{-i\omega t} dt$$

$$\text{so} \int_{-\infty}^{\infty} t f'(t) e^{-i\omega t} dt = i \frac{d}{d\omega}\left(\widehat{f'(t)}(\omega)\right) = i\frac{d}{d\omega}\left(i\omega \hat{f}(\omega)\right)$$

$$= -\hat{f}(\omega) - \omega \frac{d\hat{f}}{d\omega}(\omega)$$ (4)

$$= -\hat{f}(\omega) + i a_1(\omega) \frac{d\hat{f}}{d\omega}$$

$$y'' + 2t y' + 2y = 0$$

F.T. w.r.t.

$$\Rightarrow \left\{-\omega^2 \hat{y}(\omega) + 2\left(-\cancel{\hat{y}(\omega)} - \omega \hat{y}'(\omega)\right) + \cancel{2\hat{y}} = 0\right.$$

$$\Rightarrow \omega \hat{y} + 2\hat{y}'(\omega) = 0$$ (3)

$$\hat{y}' = -\frac{\omega}{2}\hat{y}$$ (2)

$$\Rightarrow \hat{y} = A e^{-\omega^2/4} \qquad (A \text{ constant})$$

$$\Rightarrow y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{y}(\omega) e^{i\omega t} d\omega =$$

$$= \frac{A}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{\omega^2}{4} + i\omega t} d\omega = \frac{A}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{1}{4}(\omega - 2it)^2 - t^2} d\omega$$ (3)

$$= \frac{A}{2\pi} e^{-t^2}\int_{-\infty}^{\infty} e^{-u^2} 2\, du \qquad \omega - it = 2u$$

$$= \frac{A}{\sqrt{\pi}} e^{-t^2} = e^{-t^2} \quad \text{if } y(0) = 1$$ (2)

Direct Calculation: $y' = -2t e^{-t^2}$, $y'' = 4t^2 e^{-t^2} - 2e^{-t^2}$ o.k.

Setter : ATKINSON Setter's signature : C. Atkinson

Checker : C J RIDLER-ROWE Checker's signature : CJ Ridler Rowe

MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

2002 - 2003

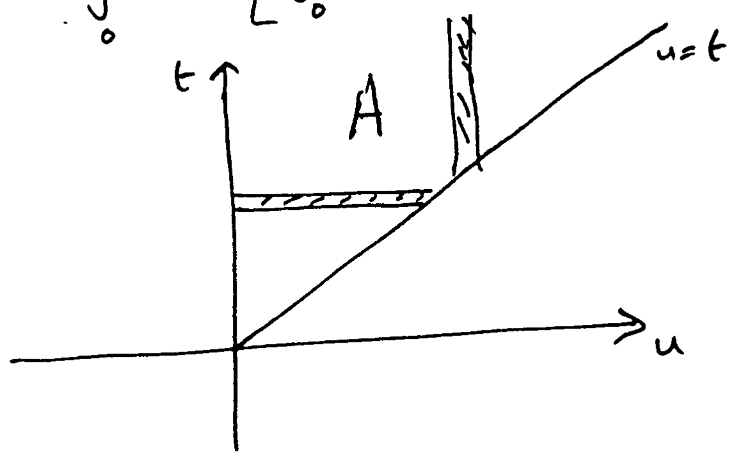
Please write on this side only, legibly and neatly, between the margins

PAPER ISE 2.6

QUESTION

SOLUTION 2

Treat $I = \int_0^\infty e^{-st}\left[\int_0^t f(t-u)\,g(u)\,du\right]dt$ as a double integral.



$$= \iint_A e^{-st} f(t-u)\,g(u)\,du\,dt$$ 4

$$= \int_0^\infty g(u)\,du\left[\int_u^\infty e^{-st} f(t-u)\,dt\right]$$

$$= \int_0^\infty g(u)\,du\left[\int_0^\infty e^{-s(u+t_1)} f(t_1)\,dt_1\right] \qquad t-u=t_1$$ 4

$$= \int_0^\infty g(u)\,e^{-su}\,du \int_0^\infty e^{-st_1} f(t_1)\,dt_1 = \bar{g}(s)\bar{f}(s)$$

L.T. of integral eqn using Convolution Thm 4

$$\Rightarrow \bar{y}(s) = \frac{1}{s+3} - \bar{y}(s)\,\frac{1}{(s+1)}$$

$$\Rightarrow \bar{y}(s)\,\frac{(s+2)}{(s+1)} = \frac{1}{(s+3)} \Rightarrow \bar{y}(s) = \frac{(s+1)}{(s+2)(s+3)}$$ 3

$$= \frac{-1}{(s+2)} + \frac{2}{(s+3)}$$ Partial Fractions 2

(Invert) $\Rightarrow y(t) = 2e^{-3t} - e^{-2t}$

Direct Substn R.H.S. $= e^{-3t} - 2\int_0^t e^{-3(t-u)-u}\,du + \int_0^t e^{-2(t-u)-u}\,du$

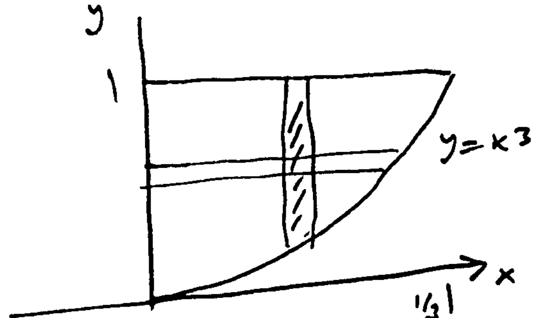
$$= e^{-3t} - 2e^{-3t}\left[\frac{e^{2u}}{2}\right]_0^t + e^{-2t}\left[e^{+u}\right]_0^t = e^{-3t} - e^{-t} + e^{-3t} + e^{-t} - e^{-2t}$$ 3

$$= 2e^{-3t} - e^{-2t} \qquad \text{Q.E.D}$$

Setter : C. ATKINSON Setter's signature : C. Atkinson

Checker : C J RIDLER-ROWE Checker's signature : C J Ridler Rowe

(i)



5

$$I = \int_0^1 dy\, e^{y^2} \int_0^{y^{1/3}} x^2\, dx = \int_0^1 e^{y^2} dy \left[\frac{x^3}{3}\right]_0^{y^{1/3}}$$

$$= \int_0^1 \frac{y}{3} e^{y^2} dy = \frac{1}{6}\left[e^{y^2}\right]_0^1$$

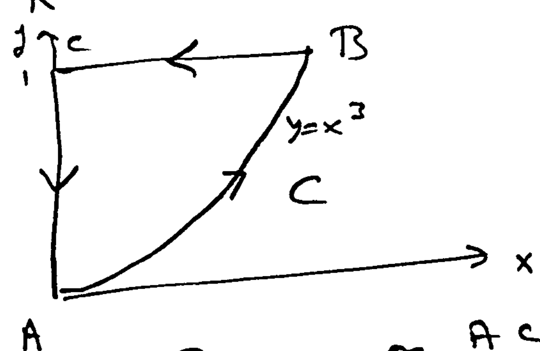
5

$$= \frac{1}{6}\left[e - 1\right]$$

5

(ii) Choose $\phi = \dfrac{x^3}{3} e^{y^2}$

Then $\displaystyle\iint_R x^2 e^{y^2} dx\,dy = \int_C \frac{x^3}{3} e^{y^2} dy$



C : curve ABC

on AC $x=0$

on BC $dy=0$

5

But on AB $x^3=y$

so $\displaystyle\int_C = \int_A^B \frac{x^3}{3} e^{y^2} dy$

$$= \int_0^1 \frac{y}{3} e^{y^2} dy = \frac{1}{6}\left[e-1\right]$$ as above

Setter : C. ATKINSON Setter's signature : C. Atkinson

Checker : CJ RIDLER-ROWE Checker's signature : CJ Ridler-Rowe

MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

2002 - 2003

Please write on this side only, legibly and neatly, between the margins

PAPER ISE 2.6

QUESTION

SOLUTION 4

(i) $\dfrac{1}{(az^2-(a^2+1)z+a)} = \dfrac{1}{(az-1)(z-a)} = \dfrac{1}{(a^2-1)(z-a)} - \dfrac{1}{(a^2-1)(z-1/a)}$

Poles at $z = a,\ 1/a$ (2)

Residues $\dfrac{1}{(a^2-1)}$ at $z = a$ (2)

$\dfrac{-1}{(a^2-1)}$ at $z = 1/a$

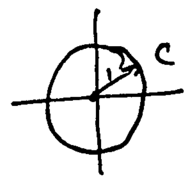
Integral around arc at ∞ $z = Re^{i\theta}$

$$\int \frac{Rie^{i\theta}\,d\theta}{R^2\left[ae^{2i\theta} - \frac{(a^2+1)e^{i\theta}}{R} + \frac{a}{R^2}\right]} \to 0 \text{ as } R \to \infty.$$ (3)

close to the left $I = 0$

close to the right $I = 0$

! Residues cancel out. (3)



(ii) on $z = e^{i\theta}$ $dz = iz\,d\theta$ (2)

$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}(z + 1/z)$

so $$J = -i\int_C \frac{dz}{z\left[a(z+1/z) - (a^2+1)\right]}$$

$$= -i\int_C \frac{dz}{(az^2 - (a^2+1)z + a)}$$ (4)

$\Rightarrow \boxed{\alpha = a \quad \beta = -(a^2+1) \quad \gamma = a}$

poles at $z = a,\ 1/a$, $a = 2$ $\quad$ $1/a$ is inside unit circle (4)

$$J = (-i)\ 2\pi i \operatorname*{Res}_{z=1/2}(\ \ )$$

$$= 2\pi\left\{\frac{-1}{3}\right\} = -\frac{2\pi}{3} \ ✓$$

Setter : ATKINSON $\qquad$ Setter's signature : C. Atkinson

Checker : P J RIDLER-ROWE $\qquad$ Checker's signature : [signature]

PAPER ISE 2.6

QUESTION

SOLUTION 5i

i). μ the mean, describes the location of the expected value. σ^2, the variance, describes the amount of dispersion about μ. 2

ii). a).

$$\begin{aligned} P(17 < X < 24) &= P\left(\frac{17-20}{9} < \frac{X-20}{9} < \frac{24-20}{9}\right) \\ &= P\left(-\frac{1}{3} < Z < \frac{4}{9}\right) \\ &= \Phi(4/9) - \Phi(-1/3) \\ &= 0.6700 - (1 - 0.6293) \approx \underline{0.2993} \end{aligned}$$

3

b).

$$\begin{aligned} P(X > 20 | X > 19) &= \frac{P(X > 20 \cap X > 19)}{P(X > 19)} \\ &= \frac{P(X > 20)}{P(X > 19)} \\ &= \frac{P(Z > 0)}{P(Z > -1/9)} \\ &= \frac{1 - \Phi(0)}{1 - (1 - \Phi(1/9))} \\ &= \frac{0.5}{0.5438} \approx \underline{0.9195} \end{aligned}$$

4

iii). $Y \sim N(0, \sigma_1^2 + \sigma_2^2)$ 2

Setter : NA

Setter's signature : NMA

Checker : LVW

Checker's signature : LVW

PAPER

QUESTION

SOLUTION 5ii

iv). a).

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \approx \underline{19.375}$$ 1

The sorted values are

4.11, 6.21, 12.28, 13.39, 15.48, 20.42, 21.04, 22.95, 26.01, 27.59, 43.64

so the median is $\underline{20.42}$. 1

The sample standard deviation

$$s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2} \approx \underline{11.066}$$ 1

b). Use small sample confidence interval

$$\bar{x} \pm t_{\alpha/2}\frac{s}{\sqrt{n}}$$

where we use a t distribution with $n-1 = 10$ degrees of freedom. Since a 90% confidence interval is required, $1-\alpha = 0.9$, so $\alpha = 0.1$ and $\alpha/2 = 0.05$. The required value from the t distribution table is 1.8125, so the interval is

$$19.375 \pm 1.8125\left(\frac{11.066}{\sqrt{11}}\right)$$

and so a 90% confidence interval for μ is $(13.328, 25.422)$. 4

c). This claim should be regarded with suspicion. The confidence interval, which has high probability of containing the population mean, does not contain the specific value. 2

Setter : N ADAMS

Setter's signature : N M Adams

Checker : Lynda V. White

Checker's signature :

PAPER ISE 2.6

QUESTION

SOLUTION 6

i).

$$
\begin{aligned}
F(t_0) &= \int_0^{t_0} \lambda e^{-\lambda t}\, dt \\
&= \left[\frac{-\lambda e^{-\lambda t}}{\lambda}\right]_0^{t_0} \\
&= -e^{-\lambda t_0} - (-1) \\
&= \underline{1 - e^{-\lambda t_0}} \quad t_0 > 0, \qquad F(t_0) = 0 \quad t_0 \le 0
\end{aligned}
$$

3

ii).

$$
\begin{aligned}
P(T > t + s | T > s) &= \frac{P(T > t + s \cap T > s)}{P(T > s)} \\
&= \frac{P(T > t + s)}{P(T > s)} \\
&= \frac{1 - F(t + s)}{1 - F(s)} \\
&= \frac{e^{-\lambda(t+s)}}{e^{-\lambda(s)}} \\
&= \underline{e^{-\lambda t}}
\end{aligned}
$$

4

This is the *memoryless* property of the exponential distribution. 1

iii).

$$
E(T) = \int_{-\infty}^{\infty} t f(t)\, dt = \int_0^{\infty} t\lambda e^{-\lambda t}\, dt
$$

using integration by parts

$$
\begin{aligned}
u &= \lambda t & \frac{dv}{dt} &= e^{-\lambda t} \\
\frac{du}{dt} &= \lambda & v &= \frac{-e^{-\lambda t}}{\lambda}
\end{aligned}
$$

so,

$$
\begin{aligned}
E(X) &= -te^{-\lambda t}\Big|_0^{\infty} - \int_0^{\infty} \lambda \left(\frac{-e^{-\lambda t}}{\lambda}\right) dt \\
&= 0 - \frac{e^{-\lambda t}}{\lambda}\Big|_0^{\infty} \\
&= \underline{\frac{1}{\lambda}}
\end{aligned}
$$

5

Setter : NA

Setter's signature : N M A

Checker : LWW

Checker's signature : LWW

iv).

$$L(\lambda) = \prod_{i=1}^{n} \lambda e^{-\lambda x_i}$$
$$= \lambda^n e^{-\lambda \sum_{i=1}^{n} x_i}$$

taking logs

$$\log(L(\lambda)) = n\log(\lambda) - \lambda \sum_{i=1}^{n} x_i$$

we seek a maximum

$$\frac{d\log(L(\lambda))}{d\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i$$

turning points occur when the derivative is 0, so

$$0 = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i$$

so the maximum likelihood estimator is

$$\hat{\lambda} = \frac{n}{\sum_{i=1}^{n} x_i}$$

and examine the second derivative

$$\frac{d^2\log(L(\lambda))}{d\lambda^2} = -\frac{n}{\lambda^2} < 0$$

to verify that this solution is a maximum.

7