IMPERIAL COLLEGE LONDON

EE3-12
EE9SO13

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2012

MSc and EEE PART III/IV: MEng, BEng.and ACGI

OPTOELECTRONICS

Thursday, 10 May 2:30 pm

Time allowed: 3:00 hours

There are SIX questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s) :	R.R.A. Syms
	Second Marker(s) :	E. Shamonina

Fundamental constants

$e = 1.6 \times 10^{-19}$ C

$m_e = 9.1 \times 10^{-31}$ kg

$\varepsilon_0 = 8.85 \times 10^{-12}$ F/m

$\mu_0 = 4\pi \times 10^{-7}$ m kg/C^2

$c = 3 \times 10^8$ m/s

$h = 6.62 \times 10^{-34}$ Js

Maxwell's equations – integral form

$\int \int_A \underline{\mathbf{D}} \,.\, d\underline{a} = \int \int \int_V \rho \, dv$

$\int \int_A \underline{\mathbf{B}} \,.\, d\underline{a} = 0$

$\int_L \underline{\mathbf{E}} \,.\, d\underline{L} = - \int \int_A \partial \underline{\mathbf{B}}/\partial t \,.\, d\underline{a}$

$\int_L \underline{\mathbf{H}} \,.\, d\underline{L} = \int \int_A [\underline{\mathbf{J}} + \partial \underline{\mathbf{D}}/\partial t] \,.\, d\underline{a}$

Maxwell's equations – differential form

$\mathrm{div}(\underline{\mathbf{D}}) = \rho$

$\mathrm{div}(\underline{\mathbf{B}}) = 0$

$\mathrm{curl}(\underline{\mathbf{E}}) = -\partial \underline{\mathbf{B}}/\partial t$

$\mathrm{curl}(\underline{\mathbf{H}}) = \underline{\mathbf{J}} + \partial \underline{\mathbf{D}}/\partial t$

Material equations

$\underline{\mathbf{J}} = \sigma \underline{\mathbf{E}}$

$\underline{\mathbf{D}} = \varepsilon \underline{\mathbf{E}}$

$\underline{\mathbf{B}} = \mu \underline{\mathbf{H}}$

Vector calculus (Cartesian co-ordinates)

$\mathrm{grad}(\phi) = \partial\phi/\partial x \, \underline{i} + \partial\phi/\partial y \, \underline{j} + \partial\phi/\partial z \, \underline{k}$

$\mathrm{div}(\underline{\mathbf{F}}) = \partial \mathbf{F}_x/\partial x + \partial \mathbf{F}_y/\partial y + \partial \mathbf{F}_z/\partial z$

$\mathrm{curl}(\underline{\mathbf{F}}) = \underline{i} \, \{\partial \mathbf{F}_z/\partial y - \partial \mathbf{F}_y/\partial z\} + \underline{j} \, \{\partial \mathbf{F}_x/\partial z - \partial \mathbf{F}_z/\partial x\} + \underline{k} \, \{\partial \mathbf{F}_y/\partial x - \partial \mathbf{F}_x/\partial y\}$

$\mathrm{curl}\,\{\mathrm{curl}(\underline{\mathbf{F}})\} = \mathrm{grad}\,\{\mathrm{div}(\underline{\mathbf{F}})\} - \nabla^2 \underline{\mathbf{F}}$

$\int \int_A \underline{\mathbf{F}} \,.\, d\underline{a} = \int \int \int_V \mathrm{div}(\underline{\mathbf{F}}) \, dv$

$\int_L \underline{\mathbf{F}} \,.\, d\underline{L} = \int \int_A \mathrm{curl}(\underline{\mathbf{F}}) \,.\, d\underline{a}$

1. a) Prove that tangential components of the time-dependent electric field $\underline{\mathbf{E}}$ must be matched at the boundary between two media. What other boundary conditions must be satisfied? Which subset would you choose to solve a reflection problem?

[8]

b) An electromagnetic wave strikes the interface between two media with refractive indices n_1 and n_2 at normal incidence as shown in Figure 1.1. Assuming that the wave is polarized in the y-direction, write down the time-independent electric fields in each medium, and find the corresponding time-independent magnetic fields.

[6]

c) Calculate the transmission and reflection coefficients for electric fields. Assuming that the refractive indices of the two media are $n_1 = 1$ and $n_2 = 1.5$, calculate the power reflectivity.

[6]

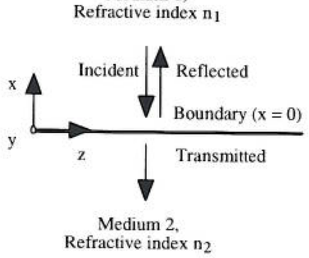

Figure 1.1.

2. a) Explain briefly the following terms: i) total internal reflection, ii) cutoff, iii) graded-index fibre.

[6]

b) A parabolic-index optical fibre has the radial variation of refractive index:

$$n = n_0\{1 - (r/r_0)^2\}.$$

Derive a waveguide equation for the fibre in cylindrical co-ordinates, and show that the Gaussian mode $E(r) = E_0 \exp(-r^2/a^2) \exp(-j\beta z)$ is a solution.

[8]

c) A parabolic-index optical fibre is illuminated on-axis by a Gaussian beam with radial amplitude variation $E(r) = E_I \exp(-r^2/b^2)$. Estimate the amplitude with which the mode of part b) is excited.

[6]

You may assume that the time-independent electric field E(x, y, z) in cartesian coordinates is governed by the scalar equation:

$$\nabla^2 E(x, y, z) + n^2(x, y)k_0^2\, E(x, y, z) = 0.$$

Here $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ and $k_0 = 2\pi/\lambda$.

3. Figure 3.1 shows the layout of a directional coupler, which consists of two parallel single-mode waveguides separated by a small distance. The two guides are assumed to be formed by refractive index distributions $n_i(x, y)$ and have eigenmode solutions $E_{yi}(x, y, z) = E_i(x, y) \exp(-j\beta_i z)$. The electric field in the complete coupler may be expanded as a sum of the modes in the two separate waveguides, as:

$$E_{yT}(x, y, z) = A_1(z)\, E_1(x, y) \exp(-j\beta_1 z) \; + \; A_2(z)\, E_2(x, y) \exp(-j\beta_2 z).$$

a) Using suitable approximations, show that the variation of the amplitudes A_1 and A_2 with distance z is governed by the coupled differential equations:

$$dA_1/dz + j\kappa\, A_2 \exp(-j\Delta\beta z) = 0$$
$$dA_2/dz + j\kappa\, A_1 \exp(+j\Delta\beta z) = 0$$

Here $\kappa = (k_0^2/2\beta_0) < (n_T^2 - n_2^2)\, E_2\, , E_1 > / < E_1\, , E_1 >$ is the coupling coefficient, n_T is the combined index distribution, $<E_A, E_B> = \int\int_A (E_A\, E_B^*)\, dx\, dy$ is the inner product of the fields E_A and E_B, and $\Delta\beta = \beta_2 - \beta_1$ is the difference in propagation constant.

[10]

b) Assuming that $A_1 = 1$ and $A_2 = 0$ at $z = 0$, explain how the mode amplitudes vary with distance z when the two guides are the same (so $\Delta\beta = 0$). Sketch the variation of the powers in the two guides as a function of normalised distance κz.

[10]

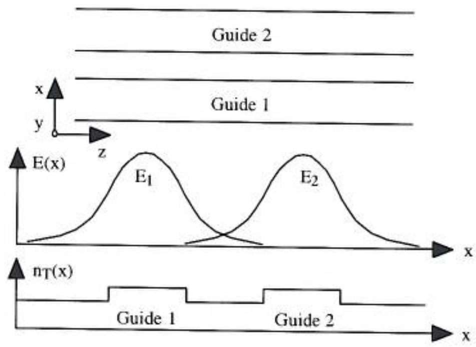

Figure 3.1

You may assume that the time-independent fields $E_{yi}(x, y, z)$ are governed by the scalar equations:

$$\nabla^2 E_{yi}(x, y, z) + n_i^2(x, y) k_0^2\, E_{yi}(x, y, z) = 0.$$

Here $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ and $k_0 = 2\pi/\lambda$. Similarly, you may assume that the total field $E_{yT}(x, y, z)$ is governed by the scalar equation:

$$\nabla^2 E_{yT}(x, y, z) + n_T^2(x, y) k_0^2\, E_{yT}(x, y, z) = 0.$$

4. Explain briefly the main attributes of the following and their particular applications in optical telecommunications systems:

i) Silica fibre

[5]

ii) Silica-on-silicon waveguide devices

[5]

iii) Ti:$LiNbO_3$ waveguide devices

[5]

iv) InGaAs:InP waveguide devices.

[5]

5. a) Describe operating principles of photoconductive detectors and p-n junction photodiodes, and explain the advantages of the latter.

[8]

b) Explain how the ideal responsivity of a photodiode is calculated. How is this result modified in a real device? How can the structure of a photodiode be altered to improve performance?

[8]

c) The response of a silicon photodiode to laser illumination at a number of discrete wavelengths is shown in Table 5.1. By plotting suitable graphs, compare the variations of the actual and the ideal responsivity with wavelength, and also the corresponding variations of the actual and the ideal quantum efficiency. The energy gap of Si is 1.14 eV.

[6]

Laser wavelength (μm)	Laser power (mW)	Measured Photocurrent (mA)
0.40	0.50	0.113
0.60	2.00	0.773
0.80	3.00	1.547
1.00	3.50	1.974
1.05	3.55	1.952
1.09	3.60	1.739

Table 5.1.

6. The lumped-element rate equations for a semiconductor laser are:

$$dn/dt = I/ev - n/\tau_e - G\phi(n - n_0)$$
$$d\phi/dt = \beta n/\tau_{rr} + G\phi(n - n_0) - \phi/\tau_p.$$

Here n and ϕ are the electron and photon densities and I is the injected current.

a) Explain why τ_e appears in the upper equation and τ_{rr} in the lower one. What is the relationship between τ_e and τ_{rr}? What is the meaning of the term $G\phi(n - n_0)$, why is it nonlinear, and why does it contain n_0?

[6]

b) Derive an expression for the photon lifetime τ_p in a laser cavity of length L, formed in a material with a group velocity v_g and with end-mirror reflectivities of R_1 and R_2. Estimate the photon lifetime in an InGaAsP/InP laser with a refractive index of 3.5 and a cavity length of 200 μm.

[8]

c) Explain how the rate equations should be modified to describe a light-emitting diode. How does the steady state, internally generated optical power vary with current I in a LED?

[6]

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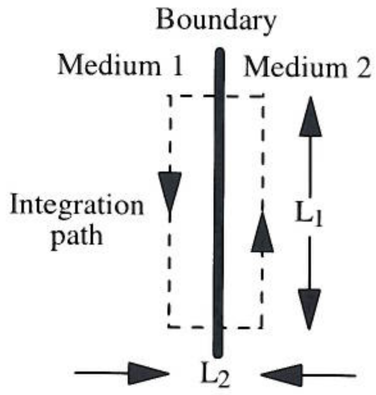
Optoelectronics 2012 – Solutions

1a) The figure below shows a junction between two different semi-infinite media, Medium 1 (where the time-dependent electric field is $\underline{\mathbf{E}}_1$), and Medium 2 (where it is $\underline{\mathbf{E}}_2$).



Faraday's law states that $\int_L \underline{\mathbf{E}} \cdot d\underline{L} = - \int\int_A \partial\underline{\mathbf{B}}/\partial t \cdot d\underline{a}$
Here $\underline{\mathbf{E}}$ and $\underline{\mathbf{B}}$ and the time-varying electric field and magnetic flux density. The LHS can be evaluated as a line integral round the rectangular loop, which crosses the boundary and has sides of length L_1 and L_2. Here $L_2 \ll L_1$, and L_1 is small enough for the fields to be approximately uniform along the long edges. The right hand side is a surface integral over the enclosed area.

If L_2 tends to zero, the RHS must tend to zero as well. In this case, we must have $\int_L \underline{\mathbf{E}} \cdot d\underline{L} = 0$
At this point, the integral is well approximated by $\int_L \underline{\mathbf{E}} \cdot d\underline{L} = (\mathbf{E}_{t1} - \mathbf{E}_{t2}) L_1$
Here $\mathbf{E}_{t1}$ and $\mathbf{E}_{t2}$ are the components of $\underline{\mathbf{E}}_1$ and $\underline{\mathbf{E}}_2$ tangential to the boundary.
Hence $\mathbf{E}_{t1} - \mathbf{E}_{t2} = 0$ and tangential components of $\underline{\mathbf{E}}$ must match across the boundary.

[4]

The result above can be written as $\underline{n} \times (\underline{\mathbf{E}}_2 - \underline{\mathbf{E}}_1) = \underline{0}$ where $\underline{n}$ is the local normal to the boundary.
The complete set of boundary conditions is then:
$\underline{n} \times (\underline{\mathbf{E}}_2 - \underline{\mathbf{E}}_1) = \underline{0}$ $\quad \underline{n} \times (\underline{\mathbf{H}}_2 - \underline{\mathbf{H}}_1) = \underline{\mathbf{K}}$
$\underline{n} \cdot (\underline{\mathbf{D}}_2 - \underline{\mathbf{D}}_1) = \rho_s$ $\quad \underline{n} \cdot (\underline{\mathbf{B}}_2 - \underline{\mathbf{B}}_1) = 0$
Here $\underline{\mathbf{K}}$ is the surface current (which can be ignored in dielectrics) and ρ_s is the surface charge density (again, zero in dielectrics).

[3]

For reflection problems the most useful boundary conditions are:
$\underline{n} \times (\underline{\mathbf{E}}_2 - \underline{\mathbf{E}}_1) = \underline{0}$ $\quad \underline{n} \times (\underline{\mathbf{H}}_2 - \underline{\mathbf{H}}_1) = 0$

[1]

b) For the geometry below, we can assume the time-independent electric fields:
$E_{y1} = E_I \exp(+jn_1k_0x) + E_R \exp(-jn_1k_0x)$
$E_{y2} = E_T \exp(+jn_2k_0x)$
Here $k_0 = 2\pi/\lambda$ is the propagation constant in vacuum.

[2]

Faraday's law states that $\text{curl}(\underline{E}) = -\partial\underline{\mathbf{B}}/\partial t$
Hence, for time independent fields, $\text{curl}(\underline{E}) = -j\omega\underline{B} = -j\omega\mu_0 \underline{H}$
Here $\underline{H}$ is the time independent magnetic field. In this case, we have:

$$\mathrm{curl}(\underline{E}) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & E_y & 0 \end{vmatrix} \qquad \text{Or } \mathrm{curl}(\underline{E}) = -\partial E_y/\partial z\ \underline{i} + \partial E_y/\partial x\ \underline{k}$$

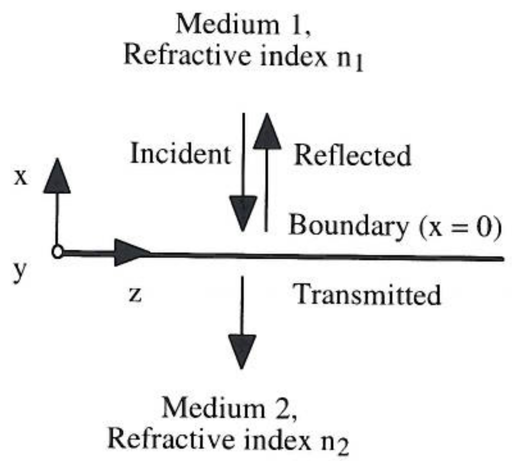
Since E_y is a function of x, the magnetic field only has a z-component, given by $H_z = (j/\omega\mu_0)\ \partial E_y/\partial x$

Hence, the magnetic fields in the two media are:

$H_{z1} = -E_I\ (n_1k_0/\omega\mu_0)\ \exp(+jn_1k_0x) + E_R\ (n_1k_0/\omega\mu_0)\ \exp(-jn_1k_0x)$

$H_{z2} = -E_T\ (n_2k_0/\omega\mu_0)\ \exp(+jn_2k_0x)$

[4]



c) The electric and magnetic fields are both wholly tangential.

Using the BCs $\underline{n} \times (\underline{E}_2 - \underline{E}_1) = \underline{0}$ and $\underline{n} \times (\underline{H}_2 - \underline{H}_1) = \underline{0}$ we have on x = 0:

$E_I + E_R = E_T$

$n_1(E_I - E_R) = n_2E_T$

Hence, the reflection and transmission coefficients are:

$E_R/E_I = (n_1 - n_2) / (n_1 + n_2)$

$E_T/E_I = 2n_1 / (n_1 + n_2)$

[4]

Assuming that $n_1 = 1$ and $n_2 = 1.5$, we obtain $E_R/E_I = -0.5/2.5 = -0.2$

Hence, the power reflectivity is $|E_R/E_I|^2 = 0.04$, or 4%

[2]

2a. i) Total internal reflection is the complete reflection of an electromagnetic wave when it is incident on the interface between two dielectric media with refractive indices n_1 and n_2, with $n_1 > n_2$, at an angle greater than the critical angle $\theta_c = \sin^{-1}(n_2/n_1)$.

[2]

ii) Cutoff is the condition at which a guided mode ceases to propagate along a waveguide. In a step-index guide, it corresponds to a failure of total internal reflection, because rays strike the core-cladding interface at angles less than the critical angle.

[2]

iii) A graded-index fibre has a core shape defined by a continuous variation in refractive index, and supports wave guiding by a gradual change in ray direction rather than by total internal reflection at an abrupt interface.

[2]

b) In cartesian coordinates, the scalar wave equation is $\nabla^2 E(x, y, z) + n^2(x, y)k_0^2\, E(x, y, z) = 0$
Here $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ and $k_0 = 2\pi/\lambda$.
In this case, we have $n^2 = n_0^2[1 - (r/r_0)^2]$, so that:
$\nabla^2 E(x, y, z) + n_0^2 k_0^2[1 - (r/r_0)^2]\, E(x, y, z) = 0$

Assuming the solution $E(x, y, z) = E_T(x, y) \exp(-j\beta z)$, we can write:
$\nabla^2 E(x, y, z) = \nabla^2_{xy} E_T(x, y) \exp(-j\beta z) - \beta^2\, E_T(x, y) \exp(-j\beta z)$

Substituting into the wave equation, we then obtain the waveguide equation:
$\nabla^2_{xy} E_T(x, y) + \{n^2 k_0^2[1 - (r/r_0)^2] - \beta^2\} E_T(x, y) = 0$
Here $\nabla^2_{xy} = \partial^2/\partial x^2 + \partial^2/\partial y^2$.

To convert to cylindrical coordinates we require the following:
$r = (x^2 + y^2)^{1/2}$
$\partial r/\partial x = 1/2\ .\ 2x\ .\ (x^2 + y^2)^{-1/2} = x/r$
$\partial r/\partial y = 1/2\ .\ 2y\ .\ (x^2 + y^2)^{-1/2} = y/r$
$\partial E/\partial x = \partial r/\partial/x\ dE/dr = (x/r)\ dE/dr$
$\partial E/\partial y = \partial r/\partial/y\ dE/d r = (y/r)\ dE/dr$

$\partial^2 E/\partial x^2 = (1/r)\ dE/dr + x\ \partial r/\partial x\ d(1/r)/dr\ dE/dr + (x/r)\ \partial r/\partial x\ d^2E/dr^2$
$\partial^2 E/\partial x^2 = (1/r)\ dE/dr + (x^2/r)\ (-1/r^2)\ dE/dr + (x^2/r^2)\ d^2E/dr^2$
$\partial^2 E/\partial x^2 = \{1/r - x^2/r^3\}\ dE/dr + (x^2/r^2)\ d^2E/dr^2$
$\partial^2 E/\partial y^2 = \{1/r - y^2/r^3\}\ dE/dr + (y^2/r^2)\ d^2E/dr^2$

$\partial^2 E/\partial x^2 + \partial^2 E/\partial y^2 = \{2/r - (x^2 + y^2)/r^3\}\ dE/dr + \{(x^2 + y^2)/r^2\}\ d^2E/dr^2$
$\partial^2 E/\partial x^2 + \partial^2 E/\partial y^2 = d^2E/dr^2 + 1/r\ dE/dr$
Hence, the waveguide equation becomes:
$d^2E_T/dr^2 + (1/r)\ dE_T/dr + \{n_0^2 k_0^2[1 - (r/r_0)^2] - \beta^2\}\ E_T = 0$

[4]

Assuming now that $E_T(r) = E_0 \exp(-r^2/a^2)$ we get:
$dE_T/dr = -(2r/a^2)\ E_0 \exp(-r^2/a^2)$
$d^2E_T/dr^2 = -(2/a^2)\ E_0 \exp(-r^2/a^2) + (4r^2/a^4)\ E_0 \exp(-r^2/a^2)$

Substituting into the waveguide equation and cancelling terms, we then get:
$-(2/a^2) + (4r^2/a^4) - (1/r)(2r/a^2) + \{n_0^2k_0^2[1 - (r/r_0)^2] - \beta^2\} = 0$
Regrouping terms, we then get:
$r^2\{4/a^4 - n_0^2k_0^2/r_0^2\} + \{n_0^2k_0^2 - 4/a^2 - \beta^2\} = 0$

Equating the two brackets separately with zero, we then get:
$a^4 = 4r_0^2/n_0^2k_0^2$ so that $a = \sqrt{(2r_0/n_0k_0)}$ and:
$\beta^2 = n_0^2k_0^2 - 4/a^2 = n_0^2k_0^2\{1 - 2/(n_0k_0r_0)\}$ so $\beta = n_0k_0 \sqrt{\{1 - 2/(n_0k_0r_0)\}}$
Because the coefficients can be uniquely determined, the solution satisfies the waveguide equation.

[4]

c) To calculate the coupling efficiency, we first assume that the incident field $E_{inc}(x, y)$ excites a weighted sum of all possible waveguide modes. Thus, we may write:
$E_{inc} = {}_{\nu}\Sigma\ A_{\nu}E_{\nu}$
where $E_{\nu}(x, y)$ is the transverse field distribution of the ν^{th} mode and A_{ν} is its amplitude.

To find the expansion coefficients, we multiply both sides of the equation above by the complex conjugate E_{μ}^* of a different modal field E_{μ}, and integrate over the waveguide cross section, to get:
$\int\int E_{inc}E_{\mu}^*\ dx\ dy = \int\int {}_{\nu}\Sigma\ A_{\nu}E_{\nu}E_{\mu}^*\ dx\ dy$

We now assume that the different modes are orthogonal, so that $\int\int E_{\nu}E_{\mu}^*\ dx\ dy = 0$ if $\mu \neq \nu$. Hence $\int\int E_{inc}E_{\mu}^*\ dx\ dy = A_{\mu} \int\int E_{\mu}E_{\mu}^*\ dx\ dy$, and the modal expansion coefficients can be found as:
$A_{\mu} = \int\int E_{inc}E_{\mu}^*\ dx\ dy / \int\int E_{\mu}E_{\mu}^*\ dx\ dy$

[3]

In cylindrical coordinates, $\int\int dx\ dy = {}_0\int^{\infty} 2\pi r\ dr$, so we the amplitude of the lowest order mode is:
$A_0 = I_1/I_2$, where
$I_1 = {}_0\int^{\infty} E_I \exp(-r^2/b^2)\ E_0 \exp(-r^2/a^2)\ 2\pi r\ dr = {}_0\int^{\infty} E_I\ E_0 \exp\{-r^2(a^2 + b^2)/a^2b^2\}\ 2\pi r\ dr$, or:
$I_1 = E_IE_0\ \pi a^2b^2/(a^2 + b^2)$
And:
$I_2 = {}_0\int^{\infty} E_0 \exp(-r^2/a^2)\ E_0 \exp(-r^2/a^2)\ 2\pi r\ dr = {}_0\int^{\infty} E_0^2 \exp(-2r^2/a^2)\ 2\pi r\ dr$, or:
$I_2 = E_0^2\ \pi a^2/2$
Hence $A_0 = (E_I/E_0)\ \{2b^2/(a^2 + b^2)\}$

[3]

3a) The modal solutions for the two guides in isolation are $E_{yi}(x, y, z) = E_i(x, y)\exp(-j\beta_i z)$
These fields satisfy the scalar wave equations $\nabla^2 E_{yi}(x, y, z) + n_i^2(x, y)k_0^2 E_{yi}(x, y, z) = 0$
Here $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ and $k_0 = 2\pi/\lambda$.
Hence $\nabla^2_{xy}E_i(x, y) + [n_i^2(x, y)k_0^2 - \beta_i^2] E_i(x, y) = 0$
Here $\nabla^2_{xy} = \partial^2/\partial x^2 + \partial^2/\partial y^2$.

The scalar wave equation for the complete system is $\nabla^2 E_{yT}(x, y, z) + n_T^2(x, y)k_0^2 E_{yT}(x, y, z) = 0$
Assuming that $E_{yT}(x, y, z) = A_1(z) E_1(x, y)\exp(-j\beta_1 z) + A_2(z) E_2(x, y)\exp(-j\beta_2 z)$
We obtain $\nabla^2 E_{yT} = \{A_1 \nabla^2_{xy}E_1 + (d^2A_1/dz^2 - 2j\beta_1 dA_1/dz - \beta_1^2 A_1)E_1\}\exp(-j\beta_1 z) +$
$\{A_2 \nabla^2_{xy}E_2 + (d^2A_2/dz^2 - 2j\beta_2 dA_2/dz - \beta_2^2 A_2)E_2\}\exp(-j\beta_2 z)$
Hence:
$\{A_1 \nabla^2_{xy}E_1 + (d^2A_1/dz^2 - 2j\beta_1 dA_1/dz - \beta_1^2 A_1)E_1\}\exp(-j\beta_1 z) +$
$\{A_2 \nabla^2_{xy}E_2 + (d^2A_2/dz^2 - 2j\beta_2 dA_2/dz - \beta_2^2 A_2)E_2\}\exp(-j\beta_2 z)$
$+ n_T^2(x, y)k_0^2\{A_1E_1\exp(-j\beta_1 z) + A_2E_2\exp(-j\beta_2 z)\} = 0$

Eliminating terms using the waveguide equations yields:
$(d^2A_1/dz^2 - 2j\beta_1 dA_1/dz) E_1 \exp(-j\beta_1 z) + (d^2A_2/dz^2 - 2j\beta_2 dA_2/dz) E_2 \exp(-j\beta_2 z)$
$+ (n_T^2 - n_1^2)k_0^2 A_1E_1\exp(-j\beta_1 z) + (n_T^2 - n_2^2)k_0^2 A_2E_2\exp(-j\beta_2 z)\} = 0$

Neglecting second derivatives of the wave amplitudes (since they are slowly-varying) then yields:
$-2j\beta_1 dA_1/dz\, E_1\exp(-j\beta_1 z) - 2j\beta_2 dA_2/dz\, E_2\exp(-j\beta_2 z)$
$+ (n_T^2 - n_1^2)k_0^2 A_1E_1\exp(-j\beta_1 z) + (n_T^2 - n_2^2)k_0^2 A_2E_2\exp(-j\beta_2 z)\} = 0$

[6]

Multiplying by E_1^* and integrating over the cross-section then yields:
$-2j\beta_1 dA_1/dz <E_1, E_1> \exp(-j\beta_1 z) - 2j\beta_2 dA_2/dz <E_2, E_1> \exp(-j\beta_2 z)$
$+ k_0^2 A_1<(n_T^2 - n_1^2)E_1, E_1> \exp(-j\beta_1 z) + k_0^2 A_2<(n_T^2 - n_2^2)E_2, E_1>\exp(-j\beta_2 z)\} = 0$
Here $<E_A, E_B> = \int\int_A (E_A E_B^*)\,dx\,dy$ is the inner product of the transverse fields E_A and E_B.

Neglecting the terms involving $<E_2, E_1>$ and $<(n_T^2 - n_1^2)E_1, E_1>$ as small we obtain:
$-2j\beta_1 dA_1/dz <E_1, E_1> \exp(-j\beta_1 z) + k_0^2 A_1<(n_T^2 - n_2^2)E_2, E_1> \exp(-j\beta_2 z) = 0$
Re-arranging, we then get:
$dA_1/dz + j\{k_0^2<(n_T^2 - n_2^2)E_2, E_1>/ 2\beta_1<E_1, E_1>\} A_2 \exp\{-j(\beta_2 - \beta_1)z\} = 0$
Or $dA_1/dz + j\kappa A_2 \exp(-j\Delta\beta z) = 0$
Where $\kappa = (k_0^2/2\beta_0) < (n_T^2 - n_2^2) E_2 , E_1 > / < E_1 , E_1 >$ and $\Delta\beta = \beta_2 - \beta_1$.

Repeating the process, but this time multiplying by E_2^* yields:
$dA_2/dz + j\{k_0^2<(n_T^2 - n_1^2)E_1, E_2>/ 2\beta_2<E_2, E_2>\} A_1 \exp\{-j(\beta_1 - \beta_2)z\} = 0$
Assuming that $\{k_0^2<(n_T^2 - n_2^2)E_2, E_1>/ 2\beta_1<E_1, E_1>\} \approx \{k_0^2<(n_T^2 - n_1^2)E_1, E_2>/ 2\beta_2<E_2, E_2>\}$ we get:
$dA_2/dz + j\kappa A_1 \exp(+j\Delta\beta z) = 0$

[4]

b) If $\Delta\beta = 0$ the two equations reduce to:
$dA_1/dz + j\kappa A_2 = 0$; $dA_2/dz + j\kappa A_1 = 0$
Differentiating the upper equation, we get:
$d^2A_1/dz^2 + j\kappa\, dA_2/dz = 0$

Substituting using the lower equation, we then get:
$d^2A_1/dz^2 + \kappa^2 A_1 = 0$
Hence the general solution for A_1 is $A_1 = C_1 \cos(\kappa z) + C_2 \sin(\kappa z)$

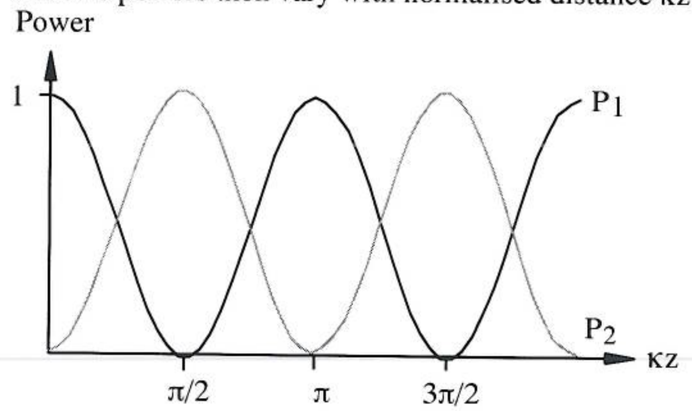
The boundary conditions are:
$A_1 = 0$ on $z = 0$, so $C_1 = 1$
$A_2 = 0$ on $z = 0$, so $dA_1/dz = -\kappa \sin(\kappa z) + C_2\kappa \cos(\kappa z) = 0$
Hence, $C_2 = 0$ and $A_1 = \cos(\kappa z)$
Hence $A_2 = (-1/j\kappa)\, dA_1/dz = (-1/j\kappa)\, \{-\kappa \sin(\kappa z)\} = -j \sin(\kappa z)$

[6]

The powers in the two guides are:
$P_1 = |A_1|^2 = \cos^2(\kappa z)$; $P_2 = |A_2|^2 = \sin^2(\kappa z)$

[2]

The two powers then vary with normalised distance κz as shown below.



[2]

4a) Silica fibres provide the transmission medium in high bit-rate, long-haul optical fibre telecoms systems. Silica is an amorphous, insulating material with excellent optical and mechanical properties. A typical fibre will be single-moded, and based on a germanium-doped graded-index core surrounded by a silica cladding. Because of the small core size (around 8 μm), optical fibres are manufactured using a two-step process. A relatively short, fat preform with the correct distribution of glass composition is first made, and the preform is then melted and stretched to form the fibre itself. The material combination is chosen to provide low propagation loss, while the core shape is chosen (in conjunction with the material dispersion) to provide low overall dispersion. Loss is primarily affected by Rayleigh scattering at short wavelengths (which rises as $1/\lambda^4$) and by bond stretching vibrations (which lie at around 10 μm wavelength) at long wavelengths. For silica fibre, the loss minimum lies at around 1.55 μm wavelength, and the minimum in material dispersion at around 1.3 μm wavelength. However, the dispersion minimum can be raised to 1.55 μm in dispersion-shifted fibre.

[5]

b) Silica-on-silicon waveguide devices provide passive splitting and filtering devices in optical fibre telecoms systems. The Si substrate is crystalline, and can be etched to form precision V-shaped mechanical alignment features for optical fibres and hence allow low cost interconnects. The silica layers are deposited by FHD, CVD or PECVD, and can be doped with a wide range of dopants including germanium. The materials and associated refractive index changes allow waveguides to have intrinsic compatibility with optical fibres. A thick buffer layer is used to isolate the core from the substrate. Rectangular cores are constructed by reactive ion etching and are then buried using a cladding layer. Typical passive components include star couplers, interferometric filters and arrayed waveguide grating multiplexers. Silica is an amorphous material, and it is difficult to provide high speed switching devices in this medium. However, slow-speed (1 kHz) phase modulators and switches may be constructed using the thermo-optic effect.

[5]

c) Ti:$LiNbO_3$ is used to provide high speed (50 GHz) modulators and switches in optical fibre telecoms systems. $LiNbO_3$ is a crystalline, insulating material with large electro-optic coefficients, and hence can provide a useful refractive index change in response to an external electric field. Channel waveguides can be formed by diffusion of strips of titanium metal into the lithium niobate. Electrodes can be provided either on top of or on either side of the waveguides to apply electric fields, using a silica buffer layer to space the electrodes away from the guided mode. Because the electro-optic effect involves tensor coefficients, different electrode configurations are used for different optical polarizations. The electro-optic effect causes changes in refractive index, which in turn alters the phase of a guided mode travelling in a nearby waveguide. Phase modulation can be converted into amplitude modulation in an interferometer structure such as a Mach-Zehnder interferometer. Switching can be carried out using directional coupler structures. Lithium niobate is relatively difficult to etch, and attachment of fibre pigtails usually requires additional micromachined silicon structures.

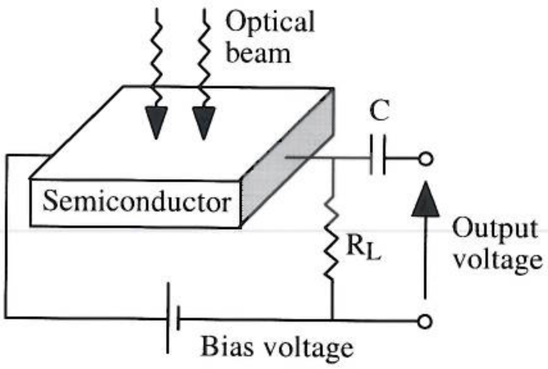
[5]

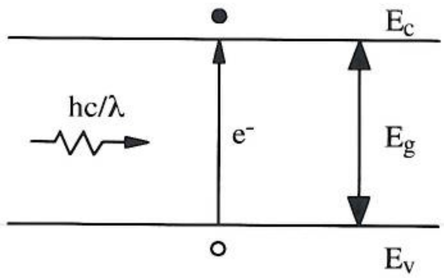
d) InGaAsP:InP used to provide lasers and detectors in optical fibre telecoms systems. The material is based on direct-gap III-V compounds, and consists of a layered arrangement of $In_{1-x}Ga_xAs_{1-y}P_y$ grown epitaxially on an InP substrate. InGaAsP is lattice-matched to inP over a wide range of x and y, along an approximate straight line from x = 0, y = 1 to x = 0.47, y = 0. Over this range, its energy gap varies from 1.35 eV (InP) to 0.74 eV ($In_{0.53}Ga_{0.47}As$). The corresponding optical wavelength ranges from 0.92 μm to 1.67 μm. Consequently, InGaAsP:InP can be used to construct sources and detectors operating at 1.55 μm wavelength. Waveguides are fabricated by etching of rib-shaped strip loading structures. However, because of the high refractive index (3.4) of InP, single moded waveguides have small cores and are consequently relatively difficult to connect to silica-based optical fibres. InGaAsP:InP is electro-optic, and consequently the material system can be used to construct phase shifters, interferometric modulators and switches. InGaAsP:InP is semiconducting, and can be used to construct PN junction photodiodes, PIN photodiodes, light-emitting diodes, ELEDs and semicondictor lasers. InGaAsP:InP can be cleaved to form reflective mirrors in Fabry-Perot laser structures, and etched to form Bragg gratings in distributed feedback lasers.

[5]

5. a) A photoconductive detector consists of a slab of semiconductor with a bias voltage and a load resistor as shown below. The surface of the semiconductor is simply illuminated. At a wavelength λ, each photon carries energy $E = hc/\lambda$ where h is Planck's constant and c is the velocity of light. If $E > E_c - E_v$, where E_c and E_v are the conduction and valence band energies, photons may be absorbed. Each absorbed photon will create an electron-hole pair, increasing the conductivity. Any change in the current through the circuit is then linearly proportional to the intensity of the illuminating beam. The output may be converted into a voltage, by measuring the voltage dropped across R_L.

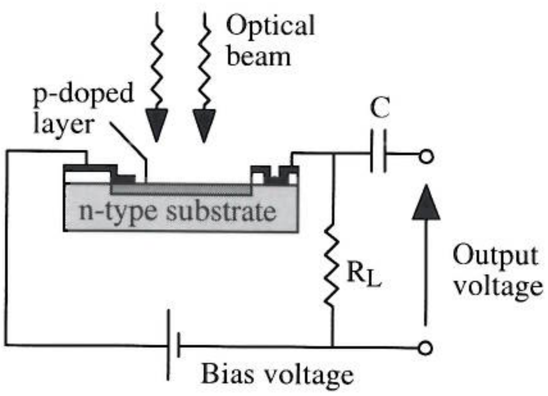
The photoconductive detector suffers from two important disadvantages. Firstly, the conductivity will not be zero, when the beam is switched off, due to the presence of thermally generated carriers. Consequently, there will be a DC voltage across R_L. Although this may be blocked by the capacitor C, thermal carriers will also be responsible for noise in the output. Secondly, the change in conductivity depends on the carrier lifetime so for high sensitivity we require a long lifetime. However, a long lifetime implies that the photo-generated carriers will persist after the beam is switched off. It is therefore difficult to combine high sensitivity with a high-speed response.

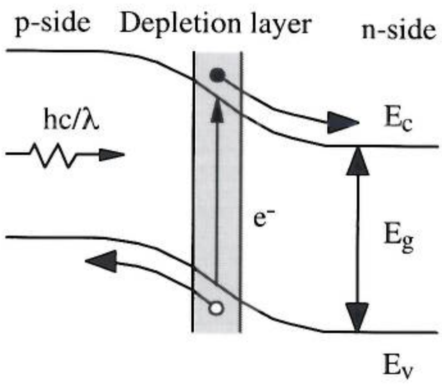




[4]

A p-n junction photodiode consists of a diode packaged with a transparent window to allow access for the optical beam as shown below. If the photons are absorbed in the depletion layer, the carriers will be separated by the built-in field in the diode and then diffuse to the contacts on either side. Recombination is dramatically reduced, and there is no longer a trade-off between efficiency and speed of response. As a result, photodiodes may be both fast and efficient.





[4]

b) The effect of each electron-hole pair reaching the contacts is equivalent to having one electron transit the entire device and gives rise to a photocurrent I_P, calculated as follows.

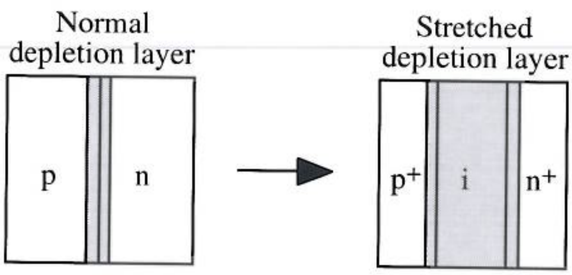
If the incident power is P, the number of photons per second is $n = P/E = P\lambda/hc$
If each photon generates an electron, the ideal photocurrent is $Ip_{ideal} = ne = Pe\lambda/hc$
The ideal responsivity is then $R_{ideal} = Ip_{ideal}/P = e\lambda/hc =$

[2]

Short-wavelength photons have a high probability of absorption, due to the availability of many empty states in the conduction band at high energies. Consequently, these photons may be absorbed before reaching the depletion layer. Long-wavelength photons have a low probability of absorption, due to the lack of empty states at low energies. Consequently, these photons may pass through the depletion layer before they are absorbed. In each case, the carrier pairs may well be lost through recombination before they reach the contacts. Both effects are accounted for by a quantum efficiency η, so that the actual responsivity is $R_{actual} = \eta e\lambda/hc$.

[2]

The limitation of a photodiode that the depletion layer is so thin that radiation of long wavelength is only weakly absorbed is overcome in the p-i-n diode. Here, a region of intrinsic or lightly doped material is introduced between two heavily doped p- and n-type regions. Because the doping is so low in this region, the depletion layer can then be arranged to extend right through it under a modest reverse bias. The effective depletion layer width may therefore be fixed at a value far greater than the 'natural' one, approximately the width of the intrinsic region.



[2]

The limitation that radiation of short wavelength is absorbed before it reaches the depletion layer can be overcome in the heterostructure p-i-n diode. Here, the front surface (which can now be the substrate) is formed from material with a much larger energy gap, so that radiation cannot be absorbed at all before it reaches the depletion layer.

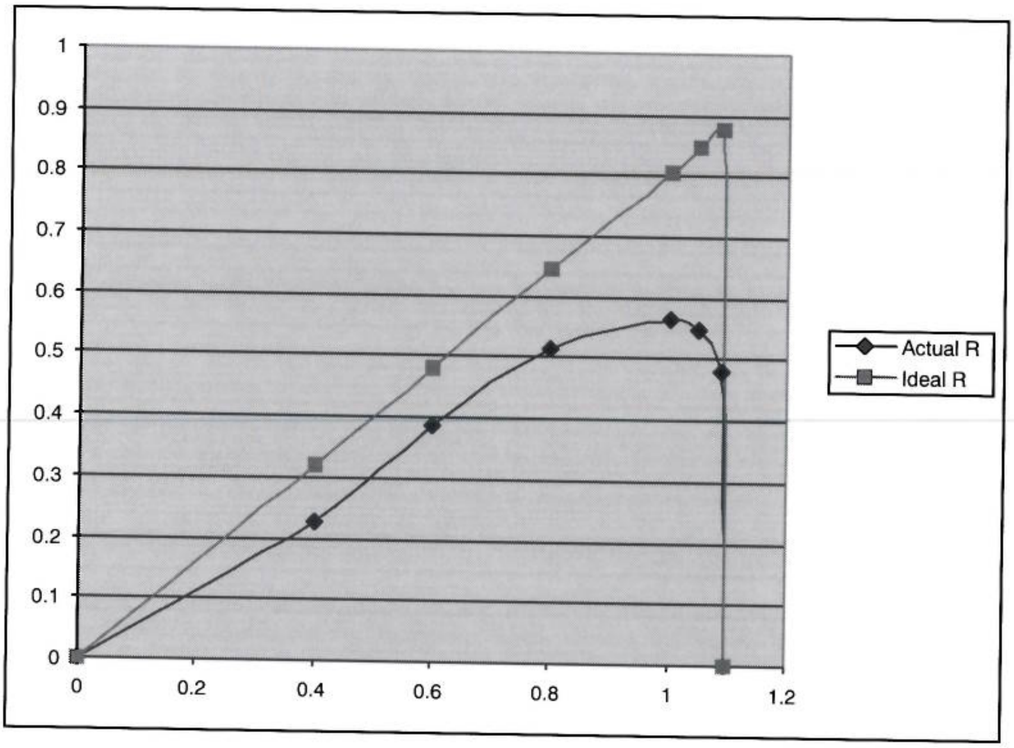
[2]

c) From the formula $R = Ip/P$ we can obtain the actual responsivity R_{act}
From the formula $R_{ideal} = e\lambda/hc = 0.8056\ \lambda$ (λ in μm) we can obtain the ideal responsivity R_{ideal}
From the ratio $\eta = R_{act}/R_{ideal}$ we can obtain the actual quantum efficiency η_{act}
The ideal quantum efficiency is $\eta_{act} = 1$ up to the wavelength at which light is no longer absorbed ($\lambda_{max} = hc/eE_g = 1.09\ \mu$m). The following results are then obtained:

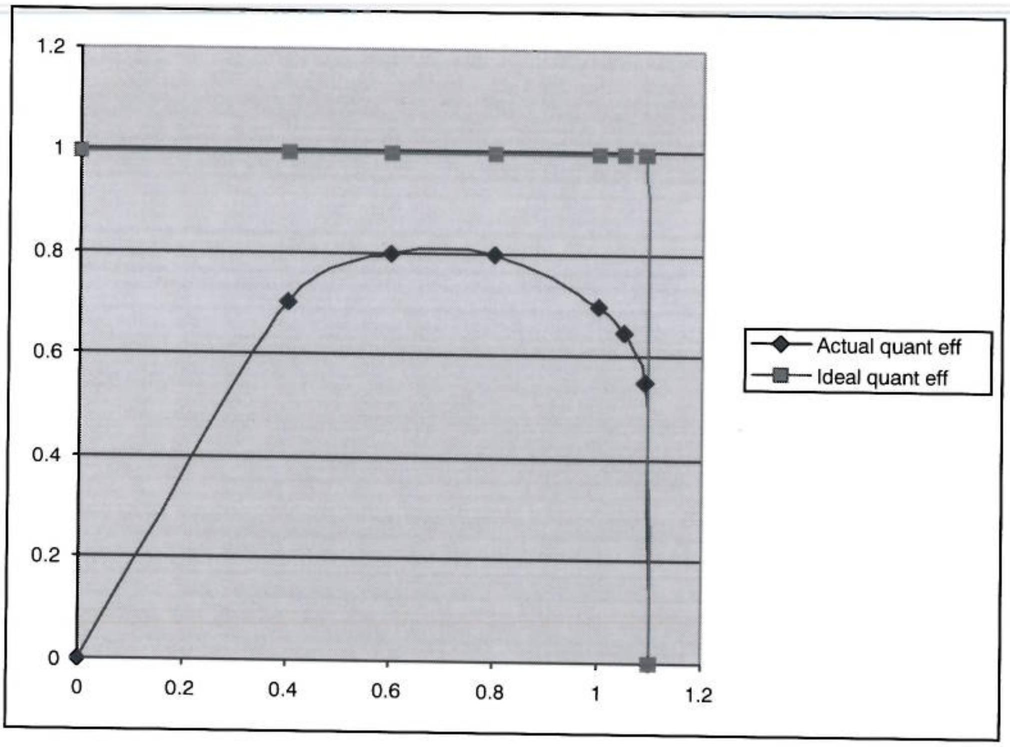
Wavelength (μm)	Power (mW)	Measured I_p (mA)	R_{act} (A/W)	R_{ideal} (A/W)	η_{act}	η_{ideal}
0.00	-	0.000	0.000	0.000	-	1

0.40	0.5	0.113	0.226	0.322	0.70	1
0.60	2.0	0.773	0.387	0.483	0.80	1
0.80	3.0	1.547	0.516	0.644	0.80	1
1.00	3.5	1.974	0.564	0.806	0.70	1
1.05	3.55	1.952	0.549	0.846	0.65	1
1.09	3.6	1.739	0.483	0.878	0.55	1
1.10	3.6	0.000	0.000	0.000	0.00	0

These data lead to the following graphs:



[3]



1.2
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1
1.2
Actual quant eff
Ideal quant eff

[3]

6. The rate equations for a semiconductor laser are:

$dn/dt = I/ev - n/\tau_e - G\phi(n - n_0)$
$d\phi/dt = \beta n/\tau_{rr} + G\phi(n - n_0) - \phi/\tau_p$

a) The term n/τ_e describes the rate of loss of electrons by both types of recombination, radiative and non-radiative. This term should be used in the upper equation, since the electron population is affected by the total rate of loss of electrons. The term n/τ_{rr} describes the rate of generation of photons by radiative recombination. This term should be used in the lower equation because the photon density is only affected by radiative recombination; non-radiative recombination generates heat rather than light.

[2]

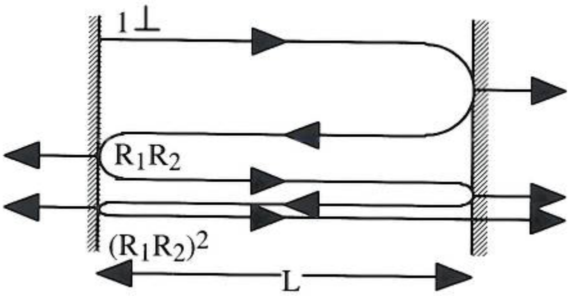
Time constants and rate constants have a reciprocal relation, so that:
$r_e = 1/\tau_e$
$r_{rr} = 1/\tau_{rr}$
$r_{nr} = 1/\tau_{nr}$
Since the total rate of recombination must be the sum of the rates of radiative and non-radiative recombination, $r_e = r_{rr} + r_{nr}$, Hence we must have $1/\tau_e = 1/\tau_{rr} + 1/\tau_{nr}$.

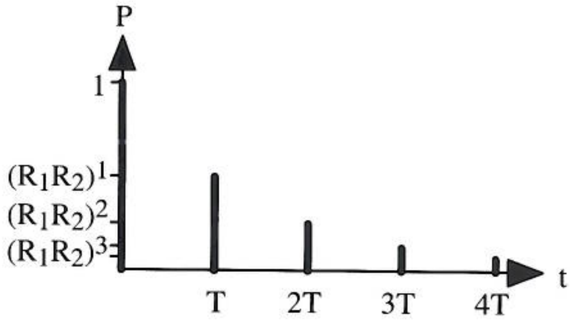
[2]

The term $G\phi(n - n_0)$ describes both stimulated emission and absorption. It contains the product of ϕ and n, because each process requires photons and electrons to interact together. The constant n_0 is the electron concentration at transparency, when the rates of stimulated emission and absorption just balance.

[2]

b) For a laser, the photon lifetime τ_p is found as follows:
Each transit of the cavity takes a time $T = 2L/v_g$where L is the cavity length
Assume initial unit power in a pulse propagating in the cavity
Ignoring amplification, the power P remaining after 1 transit is $(R_1R_2)^2$
The power P remaining in the cavity after N transits is $(R_1R_2)^{2N}$.
The number of transits needed to reduce the power to 1/e of an initial value is found from $(R_1R_2)^{2N} = 1/e$, or $N = 1/\{2 \log_e(1/R_1R_2)\}$.
These transits take a time $\tau_p = NT$
The photon lifetime is then $\tau_p = L/\{v_g \log_e(1/R_1R_2)\}$





[6]

For emission at normal incidence into air, the mirror reflectivities are $R_1 = R_2 = (n - 1) / (n + 1)$. For a refractive index of 3.5, we obtain $R_1 = R_2 = 2.5/4.5 = 0.556$.

Similarly, the group velocity is $v_g \approx c/n = 3 \times 10^8/3.5 = 8.571 \times 10^7$ m/s
Hence, the photon lifetime is $\tau_p = 200 \times 10^{-6} / \{8.571 \times 10^7 \log_e(1/0.556^2)\}$ s = 1.987 ps, ≈2 ps

[2]

c) For LED operation, the term $G\phi(n - n_0)$ should be omitted from the rate equations. The constant β (which describes the fraction of light that can couple into a lasing mode) should be replaced by unity, to account for all the emitted light. Finally, the appropriate photon lifetime $\tau_p = L/v_g$ (where L is now the distance from the active region to the surface of the semiconductor) should be used. The equations should therefore read:

$dn/dt = I/ev - n/\tau_e$
$d\phi/dt = n/\tau_{rr} - \phi/\tau_p$

[3]

In the steady state, these equations reduce to:
$I/ev - n/\tau_e = 0$ so $n = I\tau_e/ev$
$n/\tau_{rr} - \phi/\tau_p = 0$ so $\phi/\tau_p = n/\tau_{rr} = (\tau_e/\tau_{rr})\, I/ev$

The total photons flux is:
$\Phi = \phi v/\tau_p = (\tau_e/\tau_{rr})\, I/e$
Each photon carries energy hc/λ, where h is Planck's constant, c is the velocity of light and λ is the emission wavelength. The steady-state internal optical power is then:
$P = \Phi\, hc/\lambda = (\tau_e/\tau_{rr})\,(hc/\lambda)\, I/e$

[3]