Solutions EE2-05

1. (a) Consider each of the following continuous-time signals. For each case, specify if the signal is causal/non-causal, periodic/non-periodic, odd/even. If the signal is periodic specify its period.

(i) $x(t) = 2\sin(2\pi t)$ **[2]**

(ii) $x(t) = \begin{cases} 3e^{-2t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$ **[2]**

Answer

(i) Non-causal, because it takes non-zero values for $-\infty < t < \infty$. Periodic with period 1. Odd because $x(-t) = -x(t)$.

(ii) Causal, because it takes non-zero values for $0 \leq t < \infty$. Non-periodic. Neither odd nor even.

(b) Consider the signal:

$$x(t) = \begin{cases} 1-t, & 0 \leq t \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Now sketch each of the following signals and describe briefly in words how each of the signals can be derived from the original signal $x(t)$.

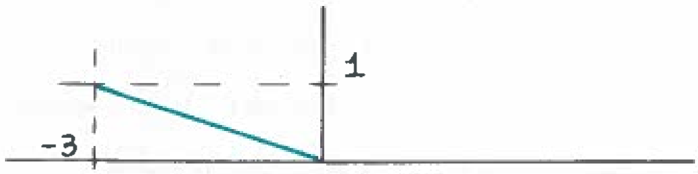
(i) $y(t) = x\left(\frac{t}{3}+1\right)$ **[2]**

(ii) $y(t) = x(-2t+1)$ **[2]**

Answer

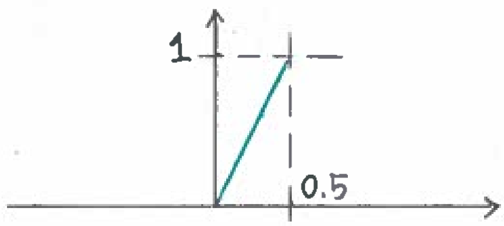
(i) $y(t) = \begin{cases} 1-\left(\frac{t}{3}+1\right), & 0 \leq \left(\frac{t}{3}+1\right) \leq 1 \\ 0, & \text{otherwise} \end{cases} = \begin{cases} -\frac{t}{3}, & -3 \leq t \leq 0 \\ 0, & \text{otherwise} \end{cases}$

Linearly stress (expand) by factor of 3 and shift left by 1.



(ii) $y(t) = \begin{cases} 1-(-2t+1), & 0 \leq (-2t+1) \leq 1 \\ 0, & \text{otherwise} \end{cases} = \begin{cases} 2t, & 0 \leq t \leq 1/2 \\ 0, & \text{otherwise} \end{cases}$

Time reverse, linearly compress by factor of 2 and shift right by ½.



(c) Consider the continuous-time LTI system with input $x(t)$ and output $y(t)$. This system is called a moving average filter.

$$y(t) = \int_{t-1}^{t} x(s)ds$$

(i) Find the impulse response of the system $h(t)$, expressing it compactly as a function. Sketch the impulse response. **[2]**

(ii) Find the output when $x(t) = u(t)$ (the unit step function) by performing the continuous time convolution $y(t) = x(t) * h(t)$. Check that the output is indeed the output expected from the moving average filter defined above. Sketch the output.

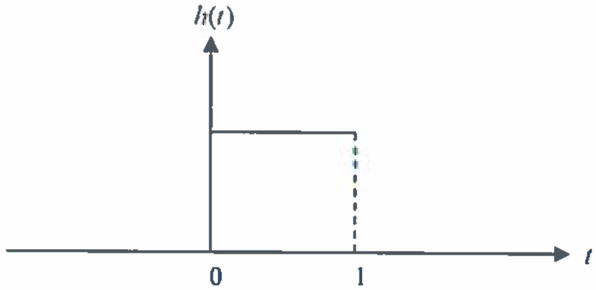
[4]

Answer

(i) The impulse response of the system $h(t)$ is defined as the output of the system when the input is the impulse function $\delta(t)$. Therefore,

$$h(t)=\int_{t-1}^{t}\delta(s)ds=\begin{cases}0, & t<0 \text{ or } t-1>0\Rightarrow t>1\\ 1, & 0\le t\le 1\end{cases}.$$

This function is shown below:

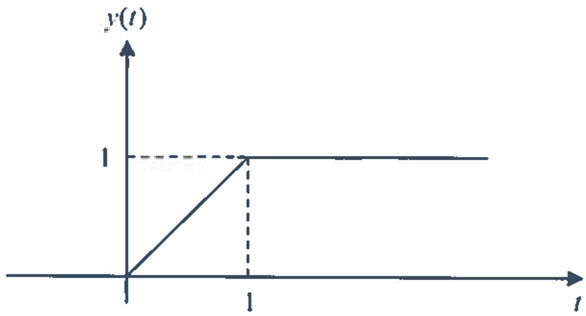


(ii) This is defined as $y(t)=\int_{-\infty}^{+\infty}x(\tau)h(t-\tau)d\tau$. In that case $x(\tau)=u(\tau)$ is non-zero if $0\le\tau\le\infty$ and $h(t-\tau)$ is non-zero if $0\le t-\tau\le 1\Rightarrow -1\le -t+\tau\le 0\Rightarrow t-1\le\tau\le t$. We may find two separate cases for which the two intervals overlap, and therefore the convolution is non-zero.

1. The lower bound of the function $u(\tau)$ lies within the bounds of the function $h(t-\tau)$, $t-1\le 0\le t\Rightarrow 0\le t\le 1$. In that case $y(t)=\int_{-\infty}^{+\infty}x(\tau)h(t-\tau)d\tau=\int_{0}^{t}1\cdot 1d\tau=t$.
2. The lower bound of the function $u(\tau)$ is smaller that the lower bound of the function $h(t-\tau)$, $0\le t-1\Rightarrow t\ge 1$. In that case $y(t)=\int_{t-1}^{t}1\cdot 1d\tau=t-(t-1)=1$.

Thus,

$$y(t)=\begin{cases}t & 0\le t\le 1\\ 1 & t\ge 1\\ 0 & t<0\end{cases}$$



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(d) (i) Consider a continuous-time function $x(t)$. Show that if the Fourier Transform of $x(t)$ is $\mathcal{F}\{x(t)\} = X(j\omega)$ then $\mathcal{F}\{x(t)e^{j\omega_0 t}\} = X(j\omega - j\omega_0)$. **[2]**

(ii) Show that $\mathcal{F}\{x(t)\cos(\omega_0 t)\} = \frac{1}{2}X(j\omega - j\omega_0) + \frac{1}{2}X(j\omega + j\omega_0)$. **[2]**

(iii) Determine the Fourier Transform of $x(t) = e^{-at}\cos(\omega_0 t), a > 0$ and sketch its amplitude response. **[2]**

Answer

(i) The Fourier transform of the function $x(t)e^{j\omega_0 t}$ is given by

$$\int_{-\infty}^{\infty} x(t)e^{j\omega_0 t}e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(t)e^{-j(\omega-\omega_0)t}dt = X(j(\omega-\omega_0))$$

with $X(j\omega)$ the Fourier transform of the signal $x(t)$.

(ii) The function $x(t)\cos(\omega_0 t)$ can be written as $x(t)(e^{j\omega_0 t} + e^{-j\omega_0 t})/2$.

In view of the results of (c) the Fourier transform of the signal $x(t)\cos(\omega_0 t)$ is $(X(j(\omega-\omega_0)) + X(j(\omega+\omega_0)))/2$.

(iii) The Fourier transform of the function $e^{-at}, t \geq 0, a > 0$ is given by

$$\int_{0}^{\infty} e^{-at}e^{-j\omega t}dt = \int_{0}^{\infty} e^{-(a+j\omega)t}dt = \frac{1}{-(a+j\omega)}e^{-(a+j\omega)t}\Big|_0^{\infty} = \frac{1}{a+j\omega}.$$

According to the result of Problem 1(d) the Fourier transform of $x(t)$ is $\frac{1}{2}\left(\frac{1}{a+j(\omega-\omega_0)} + \frac{1}{a+j(\omega+\omega_0)}\right)$.

(e) The output $y(t)$ of a causal LTI system is related to the input $x(t)$ by the differential equation:

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 4y(t) = 4x(t)$$

Determine the frequency response of the system and sketch its Bode plots. **[5]**

Answer

From $\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 4y(t) = 4x(t)$ we see that $(s^2 + 4s + 4)Y(s) = 4X(s)$.

The transfer function is $H(s) = \frac{Y(s)}{X(s)} = \frac{1}{(1+\frac{s}{2})^2}$

Let $s = j\omega$.

The amplitude response $|H(j\omega)|$ is:

$$|H(j\omega)| = \frac{1}{\left|1+\frac{j\omega}{2}\right|\left|1+\frac{j\omega}{2}\right|}$$

We express the above in decibel (i.e., $20\log(\cdot)$):

$$20\log|H(j\omega)| = -20\log\left|1+\frac{j\omega}{2}\right|\left|1+\frac{j\omega}{2}\right| = -40\log\left|1+\frac{j\omega}{2}\right|$$

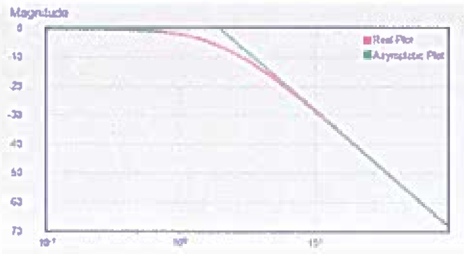
- $\omega \ll 2 \Rightarrow -40\log\left|1+\frac{j\omega}{2}\right| \approx -40\log 1 = 0$
- $\omega \gg 2 \Rightarrow -40\log\left|1+\frac{j\omega}{2}\right| \approx -40\log(\frac{\omega}{2}) = -40\log\omega + 40\log 2 = -40\log\omega + 40$
- $\omega = 2 \Rightarrow -40\log|1+j| \approx -40\log\sqrt{2} = -20 \Rightarrow 20\log|H(j\omega)| = -100$

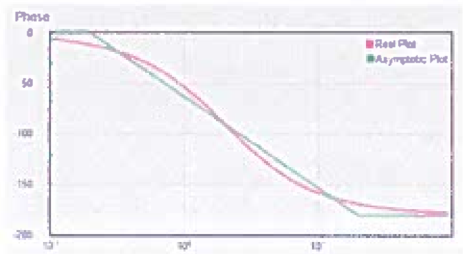
The phase response $\angle H(j\omega)$ is:

$$\angle H(j\omega) = -2\angle\left(1+\frac{j\omega}{2}\right) = -2\tan^{-1}\left(\frac{\omega}{2}\right)$$

- $\omega \ll 2 \Rightarrow -2\tan^{-1}\left(\frac{\omega}{2}\right) \approx 0$

- $\omega \gg 2 \Rightarrow -2\tan^{-1}\left(\frac{\omega}{2}\right) \approx -180^{\circ}$





(f) Consider the Laplace Transform of the impulse response of an LTI system $H(s)$ which is assumed to have one of its real zeros located to the right of the imaginary axis at $s = \gamma$. This zero is reflected through the $j\omega$-axis whereas all poles and the rest of the zeros remain unchanged. This procedure results into the function $H_1(s) = H(s)H_0(s)$. Determine the function $H_0(s)$, its amplitude response and its phase response. **[5]**

Answer

$$H_0(s)\big|_{s=j\omega} = H_0(j\omega) = \frac{j\omega+\gamma}{j\omega-\gamma}$$

$$|H_0(j\omega)| = \frac{|j\omega+\gamma|}{|j\omega-\gamma|} = 1$$

$$\text{phase}(H_0(j\omega)) = \text{phase}(j\omega+\gamma) - \text{phase}(j\omega-\gamma) = \tan^{-1}\left(\frac{\omega}{\gamma}\right) - \tan^{-1}\left(-\frac{\omega}{\gamma}\right) = 2\tan^{-1}\left(\frac{\omega}{\gamma}\right)$$

(g) Two continuous-time signals $x_1(t)$ and $x_2(t)$ are multiplied and the product $x(t)$ is sampled by a periodic impulse train. Both $x_1(t)$ and $x_2(t)$ are band-limited so that

$$X_1(\omega) = 0, \omega \geq 2\pi B_1$$
$$X_2(\omega) = 0, \omega \geq 2\pi B_2$$

where $X_i(\omega), i = 1,2$ is the Fourier transform of $x_i(t)$. Determine the maximum sampling period T_s that will allow perfect reconstruction of $x(t)$ from its samples. **[5]**

Answer

Multiplication in time is equivalent to convolution in frequency. In continuous domains the size of the convolution equals the summation of the sizes of the signals that are convolved. Therefore, $x(t)$ will be band-limited with bandwidth limited to $2\pi(B_1+B_2)$. For the Nyquist criterion to be satisfied the sampling frequency and period must satisfy the following:

$$\Omega_s = 2\pi f_s \geq 2[2\pi(B_1+B_2)] \Rightarrow f_s \geq 2(B_1+B_2) \Rightarrow T_s = \frac{1}{f_s} \leq \frac{1}{2(B_1+B_2)}$$

(h) Consider the causal LTI system with input $x[n]$ and output $y[n]$ related with the difference equation:

$$y[n] - y[n-1] + \frac{1}{4}y[n-2] = x[n] - \frac{1}{2}x[n-1]$$

(i) Find the analytical expression and the region of convergence (ROC) of z −transform of the impulse response. **[2]**

(ii) Find the analytical expression and the region of convergence (ROC) of z −transform of the output if $x[n] = \left(\frac{1}{2}\right)^n u[n]$. Use the relationship $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, |x| < 1$. **[3]**

Answer

(i) By taking the z-transform in both sides of the input-output relationship we end up with the following expression for the z-transform of the system.

$$Y(z)-z^{-1}Y(z)+\frac{1}{4}z^{-2}Y(z)=X(z)-\frac{1}{2}z^{-1}X(z)\Rightarrow\frac{Y(z)}{X(z)}\Rightarrow H(z)=\frac{1}{1-\frac{1}{2}z^{-1}},\ |\frac{1}{2}z^{-1}|\leq 1$$

(ii) $Y(z)=H(z)X(z)=\dfrac{1}{(1-\frac{1}{2}z^{-1})^2},\ |\frac{1}{2}z^{-1}|\leq 1$

2. (a) (i) Find the analytical expression and the region of convergence (ROC) of the Laplace transform of the continuous causal signal $x(t) = e^{-at}u(t)$, with a real and positive and $u(t)$ the unit step function. **[3]**

(ii) Find the analytical expression and the region of convergence (ROC) of the Laplace transform of the continuous anti-causal signal $x(t) = -e^{-at}u(-t)$, with a real and positive and $u(t)$ the unit step function. **[3]**

(iii) Is the analytical expression $X(s)$ of the Laplace transform of a signal sufficient, in order to determine the analytical expression $x(t)$ of the signal in time? Justify your answer.

[3]

Answer

(i) Consider the signal $x(t) = e^{-at}u(t)$. The Fourier transform $X(j\omega)$ converges only for $a > 0$ as shown in the following.

$$X(j\omega) = \int_{-\infty}^{+\infty} e^{-at}u(t)e^{-j\omega t}dt = \int_{0}^{+\infty} e^{-at}e^{-j\omega t}dt = \frac{1}{-(j\omega + a)}e^{-(j\omega+a)t}\Big|_0^{+\infty} = \frac{1}{j\omega + a}, a > 0$$

The Laplace transform is

$$X(s) = \int_{-\infty}^{+\infty} e^{-at}u(t)e^{-st}dt = \int_{0}^{+\infty} e^{-(s+a)t}dt$$

With $s = \sigma + j\omega$ we have

$$X(\sigma + j\omega) = \int_{0}^{+\infty} e^{-(\sigma+a)t}e^{-j\omega t}dt$$

The above is the Fourier transform of $e^{-(\sigma+a)t}u(t)$, and as shown above

$$X(\sigma + j\omega) = \frac{1}{(\sigma + a) + j\omega}, \sigma + a > 0$$

or since $s = \sigma + j\omega$ and $\sigma = \mathrm{Re}\{s\}$, we have

$$X(s) = \frac{1}{s+a}, \text{ ROC: } \mathrm{Re}\{s\} > -a$$

(ii) Consider the signal $x(t) = -e^{-at}u(-t)$. The Fourier transform $X(j\omega)$ converges only for $a < 0$ as shown in the following.

$$X(j\omega) = \int_{-\infty}^{+\infty} -e^{-at}u(-t)e^{-j\omega t}dt = \int_{-\infty}^{0} -e^{-at}e^{-j\omega t}dt = \frac{1}{j\omega + a}e^{-(j\omega+a)t}\Big|_{-\infty}^{0} = \frac{1}{j\omega + a}, a < 0$$

The Laplace transform is

$$X(s) = \int_{-\infty}^{+\infty} -e^{-at}u(-t)e^{-st}dt = -\int_{-\infty}^{0} e^{-(s+a)t}dt$$

With $s = \sigma + j\omega$ we have

$$X(\sigma + j\omega) = -\int_{-\infty}^{0} e^{-(\sigma+a)t}e^{-j\omega t}dt$$

The above is the Fourier transform of $-e^{-(\sigma+a)t}u(-t)$, and thus,

$$X(\sigma + j\omega) = \frac{1}{(\sigma + a) + j\omega}, \sigma + a < 0$$

or since $s = \sigma + j\omega$ and $\sigma = \mathrm{Re}\{s\}$, we have

$$X(s) = \frac{1}{s+a}, \text{ ROC: } \mathrm{Re}\{s\} < -a$$

(iii) From (i) and (ii) it is obvious that the analytical expression $X(s)$ of the Laplace transform of a signal is NOT sufficient in order to determine the analytical expression $x(t)$ of the signal in time. The ROC is also necessary.

(b) (i) Consider a continuous time Linear Time Invariant (LTI) system. Prove that the response of the system to a complex exponential input $e^{s_0 t}$ is the same complex exponential with only a change in amplitude; that is $H(s_0)e^{s_0 t}$. The function $H(s)$ is the Laplace transform of the impulse response of the system.

[5]

(ii) A causal LTI system with impulse response $h(t)$ has the following properties:

1. When the input to the system is $x(t) = e^t$ for all t, the output is $y(t) = \frac{11}{12}e^t$.
2. When the input to the system is $x(t) = e^{2t}$ for all t, the output is $y(t) = \frac{7}{10}e^{2t}$.
3. The impulse response $h(t)$ satisfies the equation:

$$h(t) = ae^{-3t}u(t) + be^{-2t}u(t)$$

where a, b are unknown constants.

Determine the response $H(s)$ of the system, consistent with the information above. The constants a, b should not appear in your answer. **[6]**

Answer

(i) The output of the system $y(t)$ is given as the convolution between the input of the system $x(t) = e^{s_0 t}$ and the impulse response $h(t)$. This will be

$$y(t) = \int_{-\infty}^{+\infty} x(t-\tau)h(\tau)d\tau = \int_{-\infty}^{+\infty} e^{s_0(t-\tau)}h(\tau)d\tau = e^{s_0 t}\int_{-\infty}^{+\infty} e^{-s_0\tau}h(\tau)d\tau = e^{s_0 t}H(s_0)$$ where $H(s)$

is the Laplace transform of the impulse response given by $H(s) = \int_{-\infty}^{+\infty} e^{-s\tau}h(\tau)d\tau$

evaluated at $s = s_0$.

(ii) The Laplace transform of the impulse response $h(t) = ae^{-3t}u(t) + be^{-2t}u(t)$ is $H(s) = \frac{a}{s+3} + \frac{b}{s+2}$, $\mathrm{Re}\{s\} > -2$. According to the information provided we have that $H(1) = \frac{11}{12}$ and $H(2) = \frac{7}{10}$. We form the system of equations:

1. $H(1) = \frac{a}{4} + \frac{b}{3} = \frac{11}{12}$
2. $H(2) = \frac{a}{5} + \frac{b}{4} = \frac{7}{10}$

From (1) and (2) we have $a = 1$, $b = 2$. Thus,

$$H(s) = \frac{1}{s+3} + \frac{2}{s+2} = \frac{3s+8}{(s+3)(s+2)}, \quad \mathrm{Re}\{s\} > -2$$

(c) The output $y(t)$ of a causal LTI system is related to the input $x(t)$ by the differential equation

$$\frac{d^2y(t)}{dt^2} - \frac{dy(t)}{dt} - 2y(t) = x(t)$$

Let $X(s)$ and $Y(s)$ denote Laplace transforms of $x(t)$ and $y(t)$, respectively, and let $H(s)$ denote the Laplace transform of $h(t)$, the system's impulse response.

(i) Determine $H(s)$ as a ratio of two polynomials. **[5]**

(ii) Determine $h(t)$ for each of the following cases:

1. The system is stable.
2. The system is causal.
3. The system is neither stable nor causal. **[5]**

Answer

(i) By taking the Laplace transform in both sides we get:

$$s^2Y(s) - sY(s) - 2Y(s) = X(s) \Rightarrow \frac{Y(s)}{X(s)} = H(s) = \frac{1}{(s-2)(s+1)}$$ or

$$H(s) = \frac{1}{3}\frac{1}{s-2} - \frac{1}{3}\frac{1}{s+1}$$

(ii) Since we have no information about the ROC's, the factor $\frac{1}{3}\frac{1}{s-2}$ in time is either the function $\frac{1}{3}e^{2t}u(t)$ or the function $-\frac{1}{3}e^{2t}u(-t)$. Also, the factor $\frac{1}{3}\frac{1}{s+1}$ in time is either the function $\frac{1}{3}e^{-t}u(t)$ or the function $-\frac{1}{3}e^{-t}u(-t)$.

1. The system is stable. In that case $h(t) = -\frac{1}{3}e^{2t}u(-t) + \frac{1}{3}e^{-t}u(t)$
2. The system is causal. In that case $h(t) = \frac{1}{3}e^{2t}u(t) + \frac{1}{3}e^{-t}u(t)$
3. The system is neither stable nor causal. In that case $h(t) = -\frac{1}{3}e^{2t}u(-t) - \frac{1}{3}e^{-t}u(-t)$

   or $h(t) = \frac{1}{3}e^{2t}u(t) - \frac{1}{3}e^{-t}u(-t)$.

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3. (a) Consider a continuous-time, band-limited signal $x(t)$, limited to bandwidth $|\omega| \leq 2\pi \times 10^{-3}$rad/sec. We sample $x(t)$ uniformly with sampling frequency $f_s = 3 \times 10^3$Hz to obtain the discrete-time signal $x[n]$. In reconstructing the continuous-time signal from its samples, the D/A converter outputs the waveform (this operation is often called **sample and hold** for obvious reasons):

$$x_{DA}(t) = \begin{cases} x[n] & \frac{n}{f_s} - 0.1 \times 10^{-3} < t < \frac{n}{f_s} + 0.1 \times 10^{-3} \\ x[n]/2 & t = \frac{n}{f_s} \pm 0.1 \times 10^{-3} \\ 0 & \text{otherwise} \end{cases}$$

(i) Taking into consideration that:

$$x_{DA}(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\Pi\left(\frac{t - \frac{n}{f_s}}{0.2 \times 10^{-3}}\right)$$

show that $x_{DA}(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\left[\delta(t - nT_s) * \Pi\left(\frac{t}{0.2\times 10^{-3}}\right)\right]$ with

$$\Pi(t) = \begin{cases} 1 & |t| < 0.5 \\ 0.5 & |t| = 0.5 \\ 0 & \text{otherwise} \end{cases}$$

[6]

(ii) Derive the frequency response, $H(\omega)$, of the filter (system) through which $x_{DA}(t)$ must be passed in order to perfectly reconstruct the signal $x(t)$. Use the fact that the Fourier transform of the function $\sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$ is $\frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(\omega + k\frac{2\pi}{T_s})$. **[6]**

Answer

(i)

$$x_{DA}(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\Pi\left(\frac{t-\frac{n}{f_s}}{0.2\times 10^{-3}}\right) = \sum_{n=-\infty}^{\infty} x(nT_s)\Pi\left(\frac{t-nT_s}{0.2\times 10^{-3}}\right) = \sum_{n=-\infty}^{\infty} x(nT_s)[\delta(t-nT_s)*\Pi(\frac{t}{0.2\times 10^{-3}})] =$$

$$x_{DA}(t) = \left(\sum_{n=-\infty}^{\infty} x(nT_s)\delta(t-nT_s)\right)*\Pi(\frac{t}{0.2\times 10^{-3}}) = \left(\sum_{n=-\infty}^{\infty} x(nT_s)\delta(t-\frac{n}{f_s})\right)*\Pi(\frac{t}{0.2\times 10^{-3}})$$

The function $\Pi(\frac{t}{0.2\times 10^{-3}})$ is equal to 1 if $-\frac{1}{2} < \frac{t}{0.2\times 10^{-3}} < \frac{1}{2} \Rightarrow -0.1\times 10^{-3} < t < 0.1\times 10^{-3}$, equal to $\frac{1}{2}$ if $t = \pm 0.1\times 10^{-3}$ and 0 otherwise. Therefore, its Fourier transform is found as follows:

$$\int_{-\infty}^{+\infty} \Pi(\frac{t}{0.2\times 10^{-3}})e^{-j\omega t}dt = \int_{-0.1\times 10^{-3}}^{0.1\times 10^{-3}} 1\; e^{-j\omega t}dt = \left.\frac{e^{-j\omega t}}{-j\omega}\right|_{-0.1\times 10^{-3}}^{0.1\times 10^{-3}} = \frac{-2j\sin(0.1\times 10^{-3}\omega)}{-j\omega} = \frac{2\times 10^{-4}\sin(10^{-4}\omega)}{10^{-4}\omega}$$

$$= 0.2\times 10^{-3}\,\text{sinc}(10^{-4}\omega)$$

The Fourier transform of the function $\sum_{n=-\infty}^{\infty} x(nT_s)\delta(t-nT_s)$ is $\frac{1}{T_s}\sum_{k=-\infty}^{k=+\infty} X(\omega + k\omega_s), \omega_s = \frac{2\pi}{T_s} = \frac{2\pi}{\frac{1}{3}\times 10^{-3}}$.

Using the property that convolution in time domain becomes multiplication in frequency domain we see that the Fourier transform of the function $x_{DA}(t)$ is

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$$X_{DA}(\omega)=\left(\frac{1}{T_s}\sum_{k=-\infty}^{k=+\infty}X(\omega+k\omega_s)\right)0.2\times10^{-3}\text{sinc}(10^{-4}\omega)$$

(ii) We now wish to pass $X_{DA}(\omega)$ through a system $H(\omega)$ and get $X(\omega)$ at the output, i.e., we are looking for a function $H(\omega)$ that satisfies the relation $X(\omega)=H(\omega)X_{DA}(\omega)$. Definitely, in order to remove the replications $X(\omega+k\omega_s)$, $H(\omega)$ must be zero for $|\omega|>2\pi\times10^3\text{rad/sec}$ or $\frac{|\omega|}{4\pi\times10^3}>\frac{1}{2}\text{rad/sec}$. Taking into consideration the term $\frac{1}{T_s}0.2\times10^{-3}\text{sinc}(10^{-4}\omega)$ that must be removed from $X_{DA}(\omega)$ we see that

$$H(\omega)=T_s\frac{\Pi(\frac{|\omega|}{4\pi\times10^3})}{0.2\times10^{-3}\text{sinc}(10^{-4}\omega)}=\frac{1}{3}\times10^{-3}\frac{\Pi(\frac{|\omega|}{4\pi\times10^3})}{0.2\times10^{-3}\text{sinc}(10^{-4}\omega)}=\frac{5}{3}\frac{\Pi(\frac{|\omega|}{4\pi\times10^3})}{\text{sinc}(10^{-4}\omega)}$$

(b) (i) Show that the z –transform of the discrete causal signal $x[n+1]u[n]$ is $z(X(z)-x(0))$, where $X(z)$ is the z –transform of the discrete causal signal $x[n]$. **[5]**

(ii) Consider the discrete signals $x_1(n)=2^n$ and $x_2(n)=3^n$ for $n\geq 0$. Find their convolution using their z –transforms and properties of convolution. **[5]**

For part (b) (ii) you may wish to use the relationship $\sum_{n=0}^{\infty}x^n=\frac{1}{1-x}$, if $|x|<1$.

Answer

(i) $z(x[0]+x[1]z^{-1}+x[2]z^{-2}+\cdots+)-zx[0]=x[1]+x[2]z^{-1}+\cdots$

The inverse z –transform is $x[n+1]u[n]$

(ii) In the z-domain we have multiplication so that the convolution becomes

$\frac{z}{z-2}\frac{z}{z-3}=z^2\frac{(z-2)-(z-3)}{(z-2)(z-3)}=z^2\left(\frac{1}{z-3}-\frac{1}{z-2}\right)=z\left(\frac{z}{z-3}-\frac{z}{z-2}\right)=z\left(\frac{z}{z-3}-x_1(0)-\frac{z}{z-2}+x_2(0)\right)=z\left[\left(\frac{z}{z-3}-x_1(0)\right)-\left(\frac{z}{z-2}-x_2(0)\right)\right]$ with $|z|>1$.

The inverse z –transform is $(3^{n+1}-2^{n+1})u[n]$ and the ROC is $|z|>1$.

(c) Consider a LTI system with input $x[n]$ and output $y[n]$ related by the difference equation

$$2y[n]-9y[n-1]+4y[n-2]=-14x[n-2]$$

Investigate whether the above system can be both stable and causal. Justify your answer. Determine the impulse response and its z-transform in the following three cases:

Use the fact that the z-transform $\frac{z}{z-a}$ corresponds to the function $a^n u[n]$ in discrete time if $|z|>|a|$ and the function $-a^n u[-n-1]$ if $|z|<|a|$. **[8]**

Answer

$$Y(z)[1-\frac{9}{2}z^{-1}+2z^{-2}]=-7X(z)\Rightarrow H(z)=\frac{-7}{(1-\frac{1}{2}z^{-1})(1-4z^{-1})}=\frac{1}{(1-\frac{1}{2}z^{-1})}-\frac{8}{(1-4z^{-1})}$$

$$=\frac{z}{(z-\frac{1}{2})}-\frac{8z}{(z-4)}=\frac{z}{(z-\frac{1}{2})}-8\frac{z}{(z-4)}$$

(i) The system is causal.

$$h[n]=(\frac{1}{2})^n u[n]-8(4)^n u[n], |z|>4$$

(ii) The system is stable.

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$$h[n] = (\frac{1}{2})^n u[n] + 8(4)^n u[-n-1], |z| > \frac{1}{2} \cap |z| < 4$$

(iii) The system is neither stable nor causal.

$$h[n] = -(\frac{1}{2})^n u[-n-1] + 8(4)^n u[-n-1], |z| < \frac{1}{2}$$

It is obvious that there isn't any combination of factors which yield a system that is both stable and causal.