IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2016

EEE PART I: MEng, BEng and ACGI

Corrected Copy

MATHEMATICS 1B (E-STREAM AND I-STREAM)

Tuesday, 24 May 10:00 am

Time allowed: 2:00 hours

There are FOUR questions on this paper.

Answer ALL questions. All questions carry equal marks (25% each)

NO CALCULATORS ALLOWED

Any special instructions for invigilators and information for candidates are on page 1.

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EE1-10B MATHEMATICS II

1. a) Obtain the Fourier transform of the rectangular pulse function

$$f(t) = \begin{cases} 1, & |t| < T/2, \\ 0, & |t| \geq T/2, \end{cases}$$

where T is a positive constant. [5]

b) Given that a function $g(t)$ has Fourier transform $G(\omega)$, prove the frequency shift property, that is,

$$\mathscr{F}\left[e^{i\omega_0 t} g(t)\right] = G(\omega - \omega_0),$$

where $\mathscr{F}$ denotes the Fourier transform. [5]

c) Two planes, Π_1 and Π_2, have cartesian equations

$$-2x + y + 2z = 1, \text{ and } x - 2y + z = 3.$$

Find the intersection of Π_1 and Π_2 and give a geometric interpretaton. [5]

d) Let

$$A = \begin{pmatrix} \alpha & -1 \\ 1 & 0 \end{pmatrix}.$$

i) Find the values of α such that $A^n = I$, where I is the identity matrix, for $n = 3$ and $n = 4$. [5]

ii) Prove that A^{-1} exists for all α, but that there is no value of α such that $A = A^{-1}$. [5]

2. a) The vector $\underline{y} \in \mathbb{R}^3$ satisfies

$$\underline{y} = \underline{a} + \underline{b} \times \underline{y},$$

where $\underline{a}$ and $\underline{b}$ are given vectors. By taking the scalar and vector products of this equation with $\underline{b}$, or otherwise, solve for $\underline{y}$ in terms of $\underline{a}$ and $\underline{b}$. [6]

b) Given a matrix and two vectors:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ -1 & 1 & \alpha \end{pmatrix} \quad \text{and} \quad \underline{b} = \begin{pmatrix} 1 \\ \beta \\ 2 \end{pmatrix} \quad \text{and} \quad \underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

i) Find $\det(A)$ and hence, or otherwise, give conditions on α and β so that the linear system of equation with matrix

$$A\underline{x} = \underline{b}$$

has (I) no solutions; (II) a unique solution; (III) infinitely many solutions; [5]

ii) For $\alpha = 2$, use Gaussian elimination to obtain the inverse of A. [4]

iii) Solve the system of equations

$$\begin{aligned} -x+y+2z &= 1 \\ x+y+z &= 2 \\ x+2y+3z &= 5. \end{aligned}$$

[2]

c) Given the symmetric matrix

$$B = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

i) Obtain the eigenvalues and normalized eigenvectors of B. [4]

ii) Find an orthogonal diagonalizing matrix P and a diagonal matrix D satisfying

$$B = PDP^T,$$

and without further calculation, write down P^{-1}. [4]

3. a) Find the general solution of the differential equation

$$\left(2xte^{t^2}+t^2\sin x+\sec^2 x\right)\frac{dx}{dt}=2t\cos x+\cos t-e^{t^2}. \qquad [\,6\,]$$

b) Consider the following second order differential equation:

$$\frac{d^2y}{dx^2}-2\frac{dy}{dx}+2y=85\cos(3x). \qquad (1)$$

i) Obtain the complementary function for (1). [2]

ii) Obtain a particular integral for (1). [4]

iii) Hence obtain the general solution of the ODE in (1). [1]

c) Find the solution of the first-order ordinary differential equation

$$t^2\frac{dx}{dt}=x^2+xt+t^2,$$

which satisfies the initial condition $x=1$ when $t=1$. [6]

d) Consider the following first order differential equation:

$$\cos y\frac{dy}{dx}=3x+\sin y. \qquad (2)$$

i) Use the transformation $u=3x+\sin y$ to rewrite the differential equation in terms of x and u. [2]

ii) Hence, or otherwise, obtain the solution u to the differerential equation in part (i). [2]

iii) Hence find the solution y of the differential equation (2) which satisfies the initial condition $y=0$ when $x=0$. [2]

4. a) The function $f(x,t)$ satisfies the wave equation:

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}.$$

i) By considering the change of coordinates

$$\rho = x - ct, \qquad \phi = x + ct,$$

and using the chain rule, obtain

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial t}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial t^2}$$

in terms of derivatives with respect to ρ and ϕ. [6]

ii) Use the results of (i) to show that the wave equation in the new variables is

$$\frac{\partial^2 f}{\partial \rho \partial \phi} = 0.$$

[2]

iii) Hence find the general solution of the wave equation for f in terms of x and t. [4]

b) Let $g(x,y) = (x-y)(xy-1)$.

i) Evaluate the gradient of g and derive its stationary points. [4]

ii) Evaluate the Hessian of g and use it to classify the stationary points of g. Justify your classification. [4]

iii) Evaluate the solutions of $g(x,y) = 0$ and hence, or otherwise, sketch the contours of this function. [5]