EE4-27
EE9-CS5-2
EE9-FPN2-08

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2017

MSc and EEE/EIE PART IV: MEng and ACGI

Corrected copy

SYSTEMS IDENTIFICATION

Wednesday, 3 May 10:00 am

Time allowed: 3:00 hours

There are FOUR questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : T. Parisini
Second Marker(s) : S.A. Evangelou

1. Consider the electrical circuit depicted in Fig. 1.1.

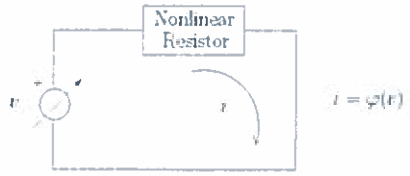

Figure 1.1 Electrical circuit with a nonlinear resistor and a DC power supply.

The model of the resistor is given by a static nonlinear function $i = \varphi(v)$ where v is the voltage and i is the current, and $\varphi(\cdot) : [2, 2.5] \mapsto \Re$ is an *unknown* nonlinear function.

An experiment to identify the static model of the resistor is set up in which the voltage v is assumed to be perfectly measurable. A set of M given values of the voltage $v(j)$, $j = 1, \ldots, M$ are imposed by the DC power supply. Correspondingly, a set of M measurements of the current are recorded by a current sensor:

$$m_c(j) = i(j) + \xi(j), \; j = 1, \ldots, M$$

where $i(j) = \varphi[v(j)]$ and $\xi(\cdot) \sim WN(0, \lambda^2)$.

a) Two approximate models $\mathcal{M}_1$ and $\mathcal{M}_2$ of the unknown function $\varphi(\cdot)$ are considered:

$$\mathcal{M}_1 : \; \widehat{\varphi}_1(v, \theta_1, \theta_2) = \theta_1 e^{-4(v-11/5)^2} + \theta_2 e^{-4(v-12/5)^2}, \quad v \in [2, 2.5]$$

and

$$\mathcal{M}_2 : \; \widehat{\varphi}_2(v, \alpha_1, \alpha_2) = \alpha_1 v + \alpha_2, \quad v \in [2, 2.5]$$

The parameters θ_1, θ_2 of model $\mathcal{M}_1$ and α_1, α_2 of model $\mathcal{M}_2$, respectively, have to be determined using the same set of measurements

$$\Theta = \{(v(j), m_c(j)), \; j = 1, \ldots, M\}.$$

Devise, if possible, a least-squares method the solution of which provides the optimal (in the least-squares sense) sets of parameters $\theta_1^\circ, \theta_2^\circ$ and $\alpha_1^\circ, \alpha_2^\circ$ of models $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively. [*Hint: Just describe the optimisation problem, but do not attempt to determine the general solution*].

[8 Marks]

b) Determine the general solution of the least-squares method devised in your answer to Question 1-a) and write the specific form that this general solution takes on for model $\mathcal{M}_1$ and for model $\mathcal{M}_2$.

[8 Marks]

c) Consider the following set Θ of 6 input voltages and measurements of current:

$$\Theta = \{(2, 0.03), (2.1, 0.77), (2.2, 0.79), (2.3, 0.9), (2.4, 0.84)(2.5, 0.91)\}$$

Using the solutions obtained in your answer to Question 1-b), compute the optimal parameters $\theta_1^\circ, \theta_2^\circ$ of model $\mathcal{M}_1$ and the optimal parameters $\alpha_1^\circ, \alpha_2^\circ$ of model $\mathcal{M}_2$ and compare the two models. Discuss your findings.

[4 Marks]

2. A stochastic process $w(\cdot)$ is generated by a linear dynamic system according to the block scheme shown in Fig. 2.1.

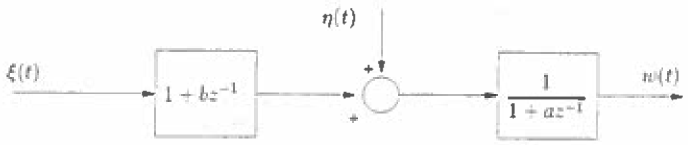

Figure 2.1 Dynamic system generating the stochastic process $w(\cdot)$.

where $a, b \in \Re$, $\eta(\cdot) \sim WN(0, \sigma_\eta^2)$ and $\xi(\cdot) \sim WN(0, \sigma_\xi^2)$ are two mutually independent white noise processes.

a) Suppose that $a = 0$, $b \neq 0$. Show that the stochastic process $w(\cdot)$ is stationary and determine its expected value $\mathbb{E}(w)$, its variance $\text{var}(w)$ and the expression of its correlation function $\gamma_w(\tau)$ for all $\tau \geq 0$.

[4 Marks]

b) Consider the moving-average stochastic process $v(\cdot)$ given by

$$MA(1): \; v(t) = e(t) + ce(t-1)$$

where $c \in \Re$, $c \neq 0$, $|c| < 1$, and $e(\cdot) \sim WN(0, \sigma_e^2)$.

Set $a = 0$, $b = -1/2$, $\sigma_\eta^2 = 1$, $\sigma_\xi^2 = 4$; compute suitable values of c and σ_e^2 such that the stochastic processes $w(\cdot)$ and $v(\cdot)$ have the same spectra, that is

$$\Gamma_w(\omega) = \Gamma_v(\omega), \quad \omega \in [-\pi, \pi]$$

Determine the analytical expression of the spectrum $\Gamma_w(\omega)$ and sketch its diagram in the interval $\omega \in [-\pi, \pi]$.

[7 Marks]

c) Set $a = 1/3$, $b = -1/2$, $\sigma_\eta^2 = 1$, $\sigma_\xi^2 = 4$; determine the spectrum $\Gamma_w'(\omega)$ of the process $w(\cdot)$ and sketch its diagram in the interval $\omega \in [-\pi, \pi]$.

[6 Marks]

d) Compare the spectra $\Gamma_w(\omega)$ and $\Gamma_w'(\omega)$ determined in your answers to Questions 2-b) and 2-c), respectively, and discuss your findings.

[3 Marks]

3. Consider the block scheme shown in Fig. 3.1.

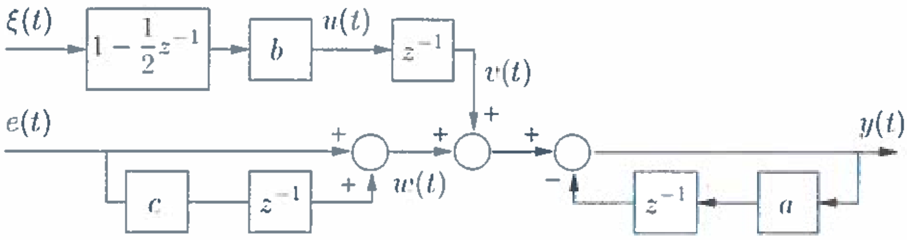

Figure 3.1 Dynamic system generating the stochastic process $y(\cdot)$.

where $e(\cdot) \sim WN(0, \sigma_e^2)$ and $\xi(\cdot) \sim WN(0, \sigma_\xi^2)$ are supposed to be mutually independent. Moreover, $a, b, c \in \Re$, with $a \neq 0$, $|a| < 1$, $b \neq 0$, $c \neq 0$.

a) Write the difference equation expressing $y(t)$ as a function of the inputs $e(t)$ and $\xi(t)$ and show that the stochastic processes $u(\cdot)$ and $y(\cdot)$ are stationary.

[3 Marks]

b) Compute the value of the correlation function of the output process $y(\cdot)$ for $\tau = 1$, that is, $\gamma_{yy}(1) = \mathbb{E}[y(t)y(t-1)]$. Moreover, compute the values for $\tau = 0$ and for $\tau = 1$ of the cross-correlation function between the output process $y(\cdot)$ and the process $u(\cdot)$, that is $\gamma_{yu}(0) = \mathbb{E}[y(t)u(t)]$ and $\gamma_{yu}(1) = \mathbb{E}[y(t)u(t-1)]$.

[7 Marks]

c) Suppose that $u(t)$ is measurable for any time-instant t and consider the prediction model

$$\widehat{\mathscr{M}}(\theta): \hat{y}(t|t-1) = \alpha y(t-1) + \beta u(t-1), \text{ where } \theta := [\alpha, \beta]^\top.$$

Denote with $\widehat{\theta}(N)$ the least squares estimate of θ based on the set of measurements $\{y(1), y(2), \ldots, y(N), u(1), u(2), \ldots, u(N)\}$ where N is arbitrarily large.

Exploiting the results in your answer to Question 3-b), devise the general formula giving the value $\overline{\theta}$ the estimate $\widehat{\theta}(N)$ approaches for large values of N (that is, $\overline{\theta} = \lim_{N\to\infty} \widehat{\theta}(N)$, a.s.) when the family $\widehat{\mathscr{M}}$ of prediction models is used.

[7 Marks]

d) Set $a = 1/2$, $b = 1/3$, $c = 1/4$, $\sigma_e^2 = 1$, $\sigma_\xi^2 = 9$. Compute the numerical value of $\overline{\theta}$ defined in Question 3-c) and also the variance of the prediction error $\varepsilon(t) = y(t) - \hat{y}(t|t-1)$. Comment on your findings.

[3 Marks]

4. A stochastic process $v(\cdot)$ is generated by a linear dynamic system according to the block scheme shown in Fig. 4.1, where $\xi(\cdot) \sim WN(0,4)$.

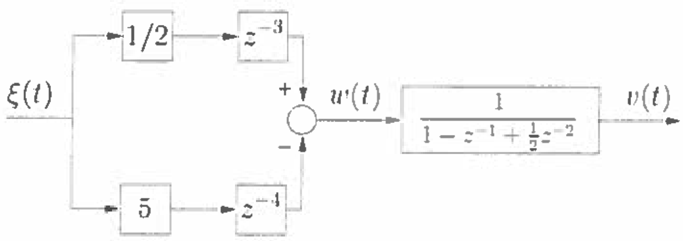

Figure 4.1 Dynamic system generating the stochastic process $v(\cdot)$.

a) Show that the stochastic process $v(\cdot)$ is stationary and compute its variance.

[5 Marks]

b) Determine the transfer function $G(z)$ from the input $\xi(t)$ and the output $v(t)$ and determine the spectral factor in canonical form $\widehat{G}(z)$.

[4 Marks]

c) Determine the difference equation yielding the optimal one-step ahead prediction $\hat{v}(t+1|t)$ of $v(t+1)$ on the basis of past values $v(t), v(t-1), v(t-2), \ldots$ of the process $v(\cdot)$ and compute the variance of the one-step ahead prediction error $\mathrm{var}[\varepsilon_1(t)] = \mathrm{var}[v(t+1) - \hat{v}(t+1|t)]$.

[4 Marks]

d) Determine the difference equation yielding the optimal two-steps ahead prediction $\hat{v}(t+2|t)$ of $v(t+2)$ on the basis of past values $v(t), v(t-1), v(t-2), \ldots$ of the process $v(\cdot)$ and compute the variance of the two-steps ahead prediction error $\mathrm{var}[\varepsilon_2(t)] = \mathrm{var}[v(t+2) - \hat{v}(t+2|t)]$.

[4 Marks]

e) Compare $\mathrm{var}[\varepsilon_1(t)]$ computed in your answer to Question 4-c) with $\mathrm{var}[\varepsilon_2(t)]$ computed in your answer to Question 4-d) and with the variance of $v(\cdot)$ computed in your answer to Question 4-a). Comment on your findings.

[3 Marks]