E4.47 / C1.6

Modelling and control of multibody mechanical systems

Model answers 2008

Question 1

a) Three single-axis-rotation transformation matrices are needed.

$$D_\psi = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which is the rotation matrix by angle ψ about a z axis.

$$D_\theta = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which is the rotation matrix by angle θ also about a z axis.

$$B_\phi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix},$$

which is the rotation matrix by angle ϕ about an x axis.

The complete transformation from earth-fixed coordinates to body-fixed coordinates is $A = D_\theta B_\phi D_\psi$. The inverse transformation from body-fixed coordinates to earth-fixed coordinates is given by $A^T = D_\psi^T B_\phi^T D_\theta^T$ and it amounts to

$$\begin{bmatrix} \cos\psi\cos\theta - \sin\psi\cos\phi\sin\theta & -\cos\psi\sin\theta - \sin\psi\cos\phi\cos\theta & \sin\psi\sin\phi \\ \sin\psi\cos\theta + \cos\psi\cos\phi\sin\theta & -\sin\psi\sin\theta + \cos\psi\cos\phi\cos\theta & -\cos\psi\sin\phi \\ \sin\phi\sin\theta & \sin\phi\cos\theta & \cos\phi \end{bmatrix}$$

b)

$$\begin{aligned} \mathbf{\Omega} &= \begin{bmatrix} 0 \\ 0 \\ \dot\psi \end{bmatrix} + D_\psi^T \begin{bmatrix} \dot\phi \\ 0 \\ 0 \end{bmatrix} + D_\psi^T B_\phi^T \begin{bmatrix} 0 \\ 0 \\ \dot\theta \end{bmatrix} \\ &= \begin{bmatrix} \dot\phi\cos\psi + \dot\theta\sin\psi\sin\phi \\ \dot\phi\sin\psi - \dot\theta\cos\psi\sin\phi \\ \dot\psi + \dot\theta\cos\phi \end{bmatrix} \end{aligned}$$

Question 2

a) i) The constraints are related to the requirement that one point on the disc should always stay on the ground and should instantaneously be at rest.

b) i) $\boldsymbol{r} = \boldsymbol{R} + \boldsymbol{r}'$

ii)

$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} = \frac{d\boldsymbol{R}}{dt} + \frac{d\boldsymbol{r}'}{dt} = \frac{d\boldsymbol{R}}{dt} + \frac{d'\boldsymbol{r}'}{dt} + \boldsymbol{\Omega} \times \boldsymbol{r}' = \frac{d\boldsymbol{R}}{dt} + \boldsymbol{\Omega} \times \boldsymbol{r}'$$

iii)

$$\boldsymbol{n} = A^T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sin\psi \sin\phi \\ -\cos\psi \sin\phi \\ \cos\phi \end{bmatrix}.$$

$$\boldsymbol{p} = \boldsymbol{n} \times \boldsymbol{k} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \sin\psi \sin\phi & -\cos\psi \sin\phi & \cos\phi \\ 0 & 0 & 1 \end{vmatrix} = -\cos\psi \sin\phi \boldsymbol{i} - \sin\psi \sin\phi \boldsymbol{j}$$

iv)

$$\boldsymbol{p} \times \boldsymbol{n} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -\cos\psi \sin\phi & -\sin\psi \sin\phi & 0 \\ \sin\psi \sin\phi & -\cos\psi \sin\phi & \cos\phi \end{vmatrix} =$$

$$= -\sin\psi \sin\phi \cos\phi \boldsymbol{i} + \cos\psi \sin\phi \cos\phi \boldsymbol{j} + (\cos^2\psi \sin^2\phi + \sin^2\psi \sin^2\phi)\boldsymbol{k} =$$

$$= \sin\phi \ (-\sin\psi \cos\phi \boldsymbol{i} + \cos\psi \cos\phi \boldsymbol{j} + \sin\phi \boldsymbol{k}).$$

$$\boldsymbol{r}' = \frac{a\boldsymbol{p} \times \boldsymbol{n}}{|\boldsymbol{p}|} = \frac{a}{|\sin\phi|} \sin\phi \ (-\sin\psi \cos\phi \boldsymbol{i} + \cos\psi \cos\phi \boldsymbol{j} + \sin\phi \boldsymbol{k}) =$$

$$= a \sin\psi \cos\phi \boldsymbol{i} - a\cos\psi \cos\phi \boldsymbol{j} - a \sin\phi \boldsymbol{k},$$

since $\frac{\sin\phi}{|\sin\phi|} = -1$ for $\phi > 180^o$. When deciding on the order of the terms in the cross product calculations above, Figure "Side view of disc" was used in which it is assumed that $\phi > 180^o$.

c)

$$\boldsymbol{\Omega} \times \boldsymbol{r}' = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dot\phi \cos\psi + \dot\theta \sin\psi \sin\phi & \dot\phi \sin\psi - \dot\theta \cos\psi \sin\phi & \dot\psi + \dot\theta \cos\phi \\ a \sin\psi \cos\phi & -a\cos\psi \cos\phi & -a \sin\phi \end{vmatrix} =$$

$$(-a \sin\psi \sin\phi \dot\phi + a\cos\psi \dot\theta + a \cos\psi \cos\phi \dot\psi)\boldsymbol{i} +$$
$$+(a\cos\psi \sin\phi \dot\phi + a \sin\psi \dot\theta + a \sin\psi \cos\phi \dot\psi)\boldsymbol{j} - a\cos\phi \dot\phi \boldsymbol{k}.$$

But $\boldsymbol{v} = 0$ and therefore

$$\frac{d\boldsymbol{R}}{dt} + \boldsymbol{\Omega} \times \boldsymbol{r}' = 0,$$

or

$$\dot{x}\boldsymbol{i} + \dot{y}\boldsymbol{j} + \dot{z}\boldsymbol{k} + \boldsymbol{\Omega} \times \boldsymbol{r}' = 0.$$

Hence the three constraint equations are

$$\dot{x} - a\sin\psi\sin\phi\dot{\phi} + a\cos\psi\dot{\theta} + a\cos\psi\cos\phi\dot{\psi} = 0,$$

and

$$\dot{y} + a\cos\psi\sin\phi\dot{\phi} + a\sin\psi\dot{\theta} + a\sin\psi\cos\phi\dot{\psi} = 0,$$

and

$$\dot{z} - a\cos\phi\dot{\phi} = 0.$$

The last constraint is integrable and hence it is holonomic.It is equivalent to

$$z - a\sin\phi = 0,$$

which is the requirement that the disc always touches the ground plane at one point.

Question 3

a) The kinetic energy is

$$T = \frac{1}{2} I_1 \left(\dot{\psi} \sin\theta \sin\phi + \dot{\phi} \cos\theta \right)^2 + \frac{1}{2} I_2 \left(\dot{\psi} \cos\theta \sin\phi - \dot{\phi} \sin\theta \right)^2 + \frac{1}{2} I_3 \left(\dot{\psi} \cos\phi + \dot{\theta} \right)^2,$$

since the body-fixed axes used are principal. But $I_2 = I_1$ due to symmetry and hence

$$T = \frac{1}{2} I_1 \left(\dot{\psi}^2 \sin^2\phi + \dot{\phi}^2 \right) + \frac{1}{2} I_3 \left(\dot{\psi} \cos\phi + \dot{\theta} \right)^2$$

b) The Lagrangian is $L = T$ since there is no potential energy term. Therefore

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}} \right) - \frac{\partial T}{\partial \theta} = 0,$$

or

$$\frac{d}{dt}\left(I_3 (\dot{\psi} \cos\phi + \dot{\theta}) \right) = 0,$$

or

$$\dot{\psi} \cos\phi + \dot{\theta} = C_1. \tag{1}$$

Also

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\psi}} \right) - \frac{\partial T}{\partial \psi} = 0,$$

or

$$\frac{d}{dt}\left(I_1 \dot{\psi} \sin^2\phi + I_3 (\dot{\psi} \cos\phi + \dot{\theta}) \cos\phi \right) = 0,$$

or

$$I_1 \dot{\psi} \sin^2\phi + I_3 (\dot{\psi} \cos\phi + \dot{\theta}) \cos\phi = C_2,$$

or

$$I_1 \dot{\psi} \sin^2\phi + I_3 C_1 \cos\phi = C_2,$$

or

$$\dot{\psi} = \frac{C_2 - I_3 C_1 \cos\phi}{I_1 \sin^2\phi}. \tag{2}$$

c) If we substitute Equations 1 and 2 into the kinetic energy expression we get

$$T - \frac{1}{2} I_3 C_1^2 = \frac{1}{2} I_1 \left(\left(\frac{C_2 - I_3 C_1 \cos\phi}{I_1 \sin^2\phi} \right)^2 \sin^2\phi + \dot{\phi}^2 \right),$$

or

$$\dot{\phi}^2 = \frac{2T - I_3 C_1^2}{I_1} - \left(\frac{C_2 - I_3 C_1 \cos\phi}{I_1 \sin\phi} \right)^2.$$

This differential equation can be integrated using a substitution $u = \cos\phi$ say to give ϕ as a function of time. We omit this step and notice that ϕ can be chosen as a constant in such a way so that the above equation will give $\dot{\phi} = 0$. In that case ϕ will continue to have the same value at any time. In other words the axis of symmetry of the body always has the same angle from the vertical z axis. $\dot{\psi}$, which for the current choice of Euler angles is the angular velocity of precession, will also be constant according to Equation 2 (when ϕ is constant). Finally $\dot{\theta}$, which is the angular speed with which the body rotates about its symmetry axis, will remain constant in view of ϕ, $\dot{\psi}$ being constant and Equation 1. In other words the rigid body executes regular precession with its symmetry axis rotating about the z axis forming a cone.

Question 4

a) **Lagrangian approach:**

The velocity vector of the mass in spherical coordinates, as given in the handout, is given by

$$\boldsymbol{v} = r\dot{\theta}\cos\phi\boldsymbol{e_\theta} + r\dot{\phi}\boldsymbol{e_\phi},$$

and hence the kinetic energy is given by

$$T = \frac{1}{2}mr^2\left(\dot{\theta}^2\cos^2\phi + \dot{\phi}^2\right).$$

The potential energy is given by

$$V = -m\boldsymbol{r}\cdot\boldsymbol{g} = -mr\boldsymbol{e_r}\cdot(-g\boldsymbol{k}) = mgr\sin\phi.$$

the Lagrangian function is thus

$$L = T - V = \frac{1}{2}mr^2\left(\dot{\theta}^2\cos^2\phi + \dot{\phi}^2\right) - mgr\sin\phi.$$

The first Lagrangian equation gives

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0,$$

or

$$\frac{d}{dt}\left(mr^2\dot{\theta}\cos^2\phi\right) = 0,$$

$$mr^2\dot{\theta}\cos^2\phi = C_1, \tag{3}$$

where C_1 is a constant of the motion. One can continue with the second Lagrangian equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}}\right) - \frac{\partial L}{\partial\phi} = 0,$$

and derive that

$$r\ddot{\phi} + r\dot{\theta}^2\cos\phi\sin\phi + g\cos\phi = 0.$$

However, it is also possible to use the total energy of the system $E = T + V$ to find the other equation of motion, since E remains constant throughout the motion (no dissipative forces present). By rearranging Equation 3 to give $\dot{\theta}$ and substituting into the total energy expression we get that

$$\dot{\phi}^2 + \frac{(C_1/(mr^2))^2}{\cos^2\phi} + 2\frac{g}{r}\sin\phi - \frac{2E}{mr^2} = 0,$$

which is a differential equation involving ϕ alone.

The force of constraint can be found by allowing r to vary and then restricting its value by a holonomic constraint. The potential energy remains the same but the kinetic energy becomes

$$T = \frac{1}{2}mr^2\left(\dot{r}^2 + \dot{\theta}^2\cos^2\phi + \dot{\phi}^2\right),$$

and therefore

$$L = \frac{1}{2}mr^2\left(\dot{r}^2 + \dot{\theta}^2\cos^2\phi + \dot{\phi}^2\right) - mgr\sin\phi.$$

The constraint in r applies

$$r - a = 0,$$

where a is assumed to be the fixed length of the wire. The Lagrangian equation with respect to generalised coordinate r is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} - \lambda = 0,$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = \lambda,$$

where the Lagrange multiplier λ will furnish the force of constraint. Thus

$$\lambda = m\left(r\dot{r}^2 + r^2\ddot{r} - r\dot{\theta}^2\cos^2\phi - r\dot{\phi}^2 + g\sin\phi\right).$$

Use of the constraint gives $\dot{r} = \ddot{r} = 0$ and therefore the force of constraint is

$$\lambda = m(-a\dot{\theta}^2\cos^2\phi - a\dot{\phi}^2 + g\sin\phi).$$

b) **Newtonian approach:**

The acceleration vector of the mass in spherical coordinates is

$$\begin{aligned} \boldsymbol{a} &= (\ddot{r} - r\dot{\theta}^2\cos^2\phi - r\dot{\phi}^2)\boldsymbol{e_r} + \dots \\ \dots &\ (2\dot{r}\dot{\theta}\cos\phi + r\ddot{\theta}\cos\phi - 2r\dot{\theta}\dot{\phi}\sin\phi)\boldsymbol{e_\theta} + \dots \\ \dots &\ (2\dot{r}\dot{\phi} + r\dot{\theta}^2\sin\phi\cos\phi + r\ddot{\phi})\boldsymbol{e_\phi}. \end{aligned}$$

but since r is fixed it reduces to

$$\boldsymbol{a} = (-r\dot{\theta}^2\cos^2\phi - r\dot{\phi}^2)\boldsymbol{e_r} + (r\ddot{\theta}\cos\phi - 2r\dot{\theta}\dot{\phi}\sin\phi)\boldsymbol{e_\theta} + (r\dot{\theta}^2\sin\phi\cos\phi + r\ddot{\phi})\boldsymbol{e_\phi}.$$

One of the forces applied onto the mass comes from gravity and is given by

$$\boldsymbol{F_g} = -mg\sin\phi\boldsymbol{e_r} - mg\cos\phi\boldsymbol{e_\phi}.$$

There is also an unknown force provided by the wire, which keeps the wire length constant, and is given by

$$\boldsymbol{F_w} = \mu\boldsymbol{e_r}.$$

The total force is thus

$$\boldsymbol{F} = \boldsymbol{F_g} + \boldsymbol{F_w} = (-mg\sin\phi + \mu)\boldsymbol{e_r} - mg\cos\phi\boldsymbol{e_\phi}.$$

Newton's second law of motion gives

$$\boldsymbol{F} = m\boldsymbol{a},$$

and therefore equating the coefficients of $\boldsymbol{e_r}$ we get

$$m(-r\dot{\theta}^2\cos^2\phi - r\dot{\phi}^2) = -mg\sin\phi + \mu,$$

and hence the force of constraint is

$$\mu = m(-r\dot{\theta}^2\cos^2\phi - r\dot{\phi}^2 + g\sin\phi).$$

Equating the coefficients of $\boldsymbol{e_\theta}$ we get

$$m(r\ddot{\theta}\cos\phi - 2r\dot{\theta}\dot{\phi}\sin\phi) = 0,$$

which can be integrated to

$$\dot{\theta}\cos^2\phi = C_2,$$

where C_2 is a constant. Equating the coefficients of $\boldsymbol{e_\phi}$ we get

$$m(r\dot{\theta}^2\sin\phi\cos\phi + r\ddot{\phi}) = -mg\cos\phi,$$

and hence

$$r\ddot{\phi} + r\dot{\theta}^2\sin\phi\cos\phi + g\cos\phi = 0.$$

Question 5

a) The moment of inertia about the z axis is given by

$$I_{zz} = \int (x^2+y^2)dm = \rho \int_V (x^2+y^2)dV = \rho \int_0^a \int_0^a \int_0^a (x^2+y^2)dxdydz =$$

$$= \rho \int_0^a \int_0^a (\frac{x^3}{3}+y^2x)\bigg|_0^a dydz = \rho \int_0^a (\frac{a^3}{3}y+\frac{y^3}{3}a)\bigg|_0^a dz = \rho\,(\frac{2a^4}{3}z)\bigg|_0^a = \rho\frac{2a^5}{3}.$$

But the total mass is

$$m = \rho V = \rho a^3,$$

and therefore

$$I_{zz} = \frac{2ma^2}{3}.$$

The other diagonal elements of the inertia matrix, I_{xx} and I_{yy}, are obviously equal to I_{zz}. One of the off-diagonal terms (product of inertia) is given by

$$I_{xy} = -\int xydm = -\rho \int_V xydV = -\rho \int_0^a \int_0^a \int_0^a xydxdydz = -\rho \int_0^a \int_0^a (\frac{x^2}{2}y)\bigg|_0^a dydz =$$

$$= -\rho \int_0^a (\frac{a^2}{2}\frac{y^2}{2})\bigg|_0^a dz = -\rho\,(\frac{a^4}{4}z)\bigg|_0^a = -\rho\frac{a^5}{4}.$$

Using the mass expression the product of inertia becomes

$$I_{xy} = -\frac{ma^2}{4}.$$

It is straightforward to show that all other products of inertia are equal to I_{xy} and therefore the inertia matrix is

$$I_O = \begin{bmatrix} \frac{2ma^2}{3} & -\frac{ma^2}{4} & -\frac{ma^2}{4} \\ -\frac{ma^2}{4} & \frac{2ma^2}{3} & -\frac{ma^2}{4} \\ -\frac{ma^2}{4} & -\frac{ma^2}{4} & \frac{2ma^2}{3} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{2}{3} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{2}{3} \end{bmatrix} ma^2$$

b) Using the shift of origin formula from the notes where the distances shifted in the three direction $R_x = R_y = R_z = \frac{a}{2}$,

$$I_O = I_{CM} + \begin{bmatrix} \frac{ma^2}{2} & -\frac{ma^2}{4} & -\frac{ma^2}{4} \\ -\frac{ma^2}{4} & \frac{ma^2}{2} & -\frac{ma^2}{4} \\ -\frac{ma^2}{4} & -\frac{ma^2}{4} & \frac{ma^2}{2} \end{bmatrix}.$$

Therefore

$$I_{CM} = I_O - \begin{bmatrix} \frac{ma^2}{2} & -\frac{ma^2}{4} & -\frac{ma^2}{4} \\ -\frac{ma^2}{4} & \frac{ma^2}{2} & -\frac{ma^2}{4} \\ -\frac{ma^2}{4} & -\frac{ma^2}{4} & \frac{ma^2}{2} \end{bmatrix} = \left(\begin{bmatrix} \frac{2}{3} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{2}{3} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{2}{3} \end{bmatrix} - \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{2} \end{bmatrix} \right) ma^2.$$

Hence the inertia matrix about the centre of mass is

$$I_{CM} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \frac{ma^2}{6},$$

which has a diagonal form.

c) In order to find the moment of inertia about the diagonal of the cube we need to find the inertia matrix for a rotated reference frame,

$$I'_{CM} = AI_{CM}A^T,$$

where A is the orthogonal transformation matrix. Hence

$$I'_{CM} = A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A^T \frac{ma^2}{6} = AA^T \frac{ma^2}{6} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \frac{ma^2}{6}$$

since A is orthogonal and $A^T = A^{-1}$. Thus the inertia matrix for rotated axes passing through the centre of mass is exactly the same as the one corresponding to the original axes. Therefore, when we use the appropriate A to align one of the original axes with the cube diagonal the moment of inertia about that axis will still be $\frac{ma^2}{6}$.

Question 6

a) To compute the kinetic energy of the system note that the position P of the end of the pendulum on a Cartesian plane with the x-axis horizontal and the z-axis directed upward is given by (recall that $l = 1$)

$$P = (\sin q, \cos q).$$

As a result the kinetic energy is given by

$$\begin{aligned} K = & \frac{1}{2} M \|\dot{P}\|^2 \\ = & \frac{1}{2} M \dot{q}_2^2. \end{aligned}$$

The potential energy is given by (recall that $l = 1$)

$$V = Mg \cos q.$$

The internal Hamiltonian is

$$H_0(q, p) = \frac{1}{2M} p^2 + Mg \cos q.$$

b) The Hamiltonian equations of motion are

$$\dot{q} = \frac{1}{M} p$$

and

$$\dot{p} = Mg \sin q + u.$$

c) If u is constant, then the equilibrium points are the solution of the equation

$$Mg \sin q + u = 0.$$

If $|u| > Mg$ then this equation has no solution.
If $|u| = Mg$ then this equation has one solution, which is $\pi/2$ is $u = -Mg$ and $-\pi/2$ if $u = Mg$.
If $|u| < Mg$ then the equilibrium points are

$$q = -\arcsin \left(\frac{u}{Mg} \right).$$

Note that this gives two equilibrium points, *i.e.* ϕ and $\pi - \phi$, where ϕ is the smallest solution of the equation $Mg \sin q + u = 0$.

d) If u is constant then

$$Mg \sin q + u = -\frac{\partial (Mg \cos q - uq)}{\partial q},$$

hence the system can be written as a Hamiltonian system without input and with modified potential energy

$$Mg \cos q - uq.$$

e) Similarly to the previous point, setting $u = k(q)$ yields

$$Mg\sin q + k(q) = -\frac{\partial(Mg\cos q - \int_0^q k(x)dx)}{\partial q},$$

hence the system can be written as a Hamiltonian system without input and with modified potential energy

$$\tilde{V}(q) = Mg\cos q - \int_0^q k(x)dx.$$

Note that the modified internal Hamiltonian is

$$\tilde{H}_0(q,p) = \frac{1}{2M}p^2 + \tilde{V}(q).$$

Setting

$$\int_0^q k(x)dx = Mg\cos q - \frac{(q-q_\star)^2}{2}$$

yields

$$\tilde{V}(q) = \frac{(q-q_\star)^2}{2},$$

which has a (global) minimum at $q = q_\star$. Note, finally, that

$$k(q) = -Mg\sin q - (q - q_\star).$$

f) Note that

$$\dot{\tilde{H}}_0 = \frac{\partial \tilde{H}_0}{\partial q}\dot{q} + \frac{\partial \tilde{H}_0}{\partial p}\dot{p} = \frac{\partial \tilde{H}_0}{\partial q}\frac{\partial \tilde{H}_0}{\partial p} - \frac{\partial \tilde{H}_0}{\partial p}\frac{\partial \tilde{H}_0}{\partial q}.$$

Hence

$$\dot{\tilde{H}}_0 = \frac{\partial \tilde{H}_0}{\partial p}v = \frac{1}{M}pv.$$

For $v = 0$ we have $\dot{\tilde{H}}_0 = 0$, whereas setting $v = -\kappa p$ yields $\dot{\tilde{H}}_0 = -\frac{\kappa}{M}p^2$. Note that $\tilde{H}_0(q,p)$ is zero at the equilibrium $(q_\star, 0)$ positive around the equilibrium, and since $\dot{\tilde{H}}_0 \leq 0$, $\tilde{H}_0(q(t),p(t))$ decreases (along the trajectories of the system) if $p(t) \neq 0$. Hence, if $p(t)$ converges to zero then the system reaches an equilibrium, which is the desired one since this is the only equilibrium close to the initial condition. If $p(t)$ does not converge to zero, then $\tilde{H}_0(q(t),p(t))$ decreases for all t, which is not possible since $\tilde{H}_0(q,p)$ is bounded from below. (Note that this discussion is only qualitative, since the students may not have studied Lyapunov stability.)