EE1-05

ENERGY CONVERSION: ANSWERS

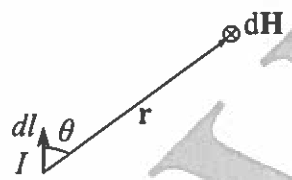
Question 1. [(a) = 5], [(b) = 5], [(c) = 30]

a) (bookwork) The expression for the magnetic flux density is given by the Biot-Savart law. The expression (listed in the formula sheet available to the students during the exam) is

$$\mathbf{dB} = \frac{\mu_0}{4\pi} I \frac{[\mathbf{dl} \times \mathbf{r}]}{r^3} \qquad [1]$$

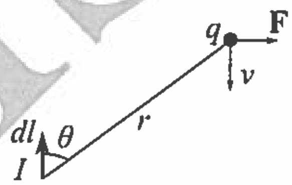
The magnetic field strength is given by $\mathbf{dH} = \mathbf{dB}/\mu_0$ [1]. We use the righ-hand rule to find the direction of the field. It will be into the paper, as shown in the figure below [2]. The magnitude will be

$$dH = \frac{I}{4\pi} \frac{dl \sin\theta}{r^2} \qquad [1]$$



b) (bookwork) The expression for the force on a charge moving in a magnetic field is given in the available formula sheet as $\mathbf{F} = q[\mathbf{v} \times \mathbf{B}]$ [1]. From the right-hand rule and the solution of the previous problem, which showed the direction of the field, the force will be directed to the right, as in the figure below [1]. The magnetic field density and the velocity are perpendicular to each other [1], so that

$$F = qvB = \frac{qvI\mu_0}{4\pi} \frac{dl \sin\theta}{r^2} \qquad [2]$$



c) (calculated problem) We will use the superposition principle and Biot-Savart's law [1]. The total flux density will be the sum of the densities created separately by both lines and by the half-ring [3]. We denote them by 1, 2, and 3.

We first look at the flux density created by one of the lines. According to Biot-Savart's law [1]

$$\mathbf{dB} = \frac{\mu_0 I}{4\pi} \frac{[\mathbf{dl} \times \mathbf{r}]}{r^3}$$

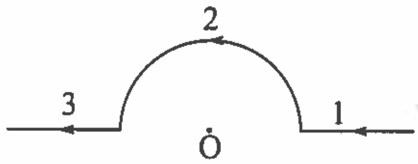
For the lines, we see that $\mathbf{dl} \parallel \mathbf{r}$ for every small element $\mathbf{dl}$ [1]. Therefore $[\mathbf{dl} \times \mathbf{r}] = 0$ [1], and B created by the lines at O is zero [3].

We now look at the half-circle. We have $\mathrm{d}\mathbf{l} \perp \mathbf{r}$ for every small element $\mathrm{d}\mathbf{l}$ on the circle [3]. Also $r = R$ [2]. Therefore,

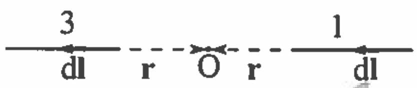
$$\mathrm{d}B = \frac{\mu_0 I}{4\pi}\frac{\mathrm{d}l\, r}{r^3} = \frac{\mu_0 I}{4\pi}\frac{\mathrm{d}l}{R^2} \qquad [3]$$

According to the right-hand rule, B is directed out of the plane of paper for any element, and hence the whole half-ring [5]. The magnitude of B is given by integration. Since all quantities apart from $\mathrm{d}l$ are constant, integration gives $\int \mathrm{d}l = \pi R$ [4] and

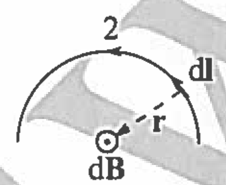
$$B = \frac{\mu_0 I}{4\pi}\frac{\pi R}{R^2} = \frac{\mu_0 I}{4R} \qquad [3]$$



(a)



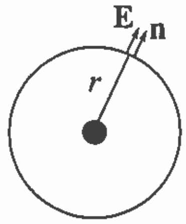
(b)



(c)

Question 2. bookwork [a) = 5], [b) = 10], [c) = 15]

a) (bookwork) Gauss's law has the form $\oint(\mathbf{D}\cdot d\mathbf{S}) = q$. The electric field of a point charge is $\mathbf{E} = q\mathbf{r}/(4\pi\varepsilon_0 r^3)$. The electric field is pointing radially from the charge. When the Gauss surface is the sphere specified in the problem, the electric field of the point charge is parallel to the normal to the sphere. That is why we have $(\mathbf{D}\cdot d\mathbf{S}) = \varepsilon_0(\mathbf{E}\cdot d\mathbf{S}) = \varepsilon_0(\mathbf{E}\cdot\mathbf{n})dS = \varepsilon_0 E dS$ [3]. Also on the sphere, we have, due to the symmetry, $E = \text{const}$ [1]. Therefore, we get for the flux through the sphere $\oint(\mathbf{D}\cdot d\mathbf{S}) = \varepsilon_0\oint E dS = \varepsilon_0 E\oint dS = \varepsilon_0 E 4\pi r^2$. Substituting the expression for the field gives, as required, $\oint(\mathbf{D}\cdot d\mathbf{S}) = q$ [1].



b) (calculated problem) The field of the point charge is $\mathbf{E} = q\mathbf{r}/(4\pi\varepsilon_0 r^3)$. Taking zero of the potential at infinity, the potential at point A is $\varphi(A) = q/(4\pi\varepsilon_0 r_A)$ [3]. The potential at point B is given by taking r_B instead of r_A [1]. The voltage U_{AB} is given by

$$U_{AB} = \varphi(A) - \varphi(B) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_A} - \frac{1}{r_B}\right) \qquad [4]$$

The voltage is positive [2].

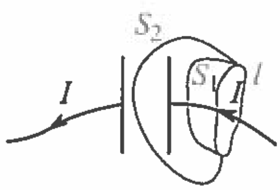
c) (bookwork) Let us enclose the wire by a closed path l. We also choose two open surfaces S_1 and S_2 so that l defines the border of both of them. However, S_1 lies entirely at one side of the capacitor, whereas S_2 passes between the capacitor plates [2]. The current passing through S_1 is I. Ampere's law then gives

$$\oint_l(\mathbf{H}\cdot d\mathbf{l}) = \int_{S_1}(\mathbf{J}\cdot d\mathbf{S}) = I \qquad [2]$$

On the other hand, the current passing through S_2 is zero, so that Ampere's law gives

$$\oint_l(\mathbf{H}\cdot d\mathbf{l}) = \int_{S_2}(\mathbf{J}\cdot d\mathbf{S}) = 0 \qquad [2]$$

The left-hand sides of both equations are the same, but the right-hand sides are different. We have a contradiction [2].



The contradiction arises because Ampere's law does not work for ac currents [1]. It is resolved by introducing the displacement current. The corresponding Maxwell's equation (it is given in the formula sheet) is

$$\oint_l(\mathbf{H}\cdot d\mathbf{l}) = \int_S(\mathbf{J}\cdot d\mathbf{S}) + \frac{d}{dt}\int_S(\mathbf{D}\cdot d\mathbf{S}) = 0 \quad [1]$$

Then for S_1, we have $\int_{S_1}(\mathbf{J}\cdot d\mathbf{S}) = I$ and $\frac{d}{dt}\int_{S_1}(\mathbf{D}\cdot d\mathbf{S}) = 0$ [1]. For S_2 we have $\int_{S_1}(\mathbf{J}\cdot d\mathbf{S}) = 0$ and $\frac{d}{dt}\int_{S_2}(\mathbf{D}\cdot d\mathbf{S}) = \varepsilon_0 UA/l$, where A is the area of the plates, l is the distance and $U$is the voltage between them [3]. The capacitance is $C = \varepsilon_0 A/l$, so that equating the expressions for S_1 and S_2, we get

$$I = C\frac{dU}{dt} \qquad [1]$$

which is a known expression for the ac current flowing through a capacitor.

Question 3. [a) = 10], [b) = 10], [c) = 10]

a) (bookwork) The physical mechanism responsible is electromagnetic induction [1]. A time-varying current in the first loop creates a time-varying magnetic field [1]. This field passes through the surface of the second loop, and the resulting flux creates an emf in it [1]. The emf affects the current flowing in the second loop; this current creates a time-varying magnetic flux through the first loop [1]. Therefore, the currents in both loops create an emf in the other loop [1].

For two loops, the definition of mutual inductance is

$$M = \frac{\Phi_{12}}{I_2} = \frac{\Phi_{21}}{I_1} \qquad [3]$$

Here I_1 and I_2 are the currents in the first and second loop, respectively; Φ_{12} is the flux through the first loop caused by the current in the second loop (I_2), and Φ_{21} is the flux through the second loop caused by the current in the first loop [2].

b) (calculated example) By definition of the mutual inductance

$$M = \frac{\Phi_{12}}{I_2} = \frac{\Phi_{21}}{I_1} \qquad [1]$$

from which we get

$$M^2 = \frac{\Phi_{12}\Phi_{21}}{I_2 I_1} \qquad [3]$$

Total flux linkage means $\Phi_{12} = \Phi_{12} = \Phi$ [1]. On the other hand, according to the definition of the self-inductance, $L = \Phi/I_1$ and $L = \Phi/I_2$ [2]. Therefore, we have

$$M^2 = \frac{\Phi_{12}\Phi_{21}}{I_2 I_1} = \frac{\Phi^2}{I_2 I_1} = L^2 \qquad [3]$$

and $M = L$.

c) (calculated example) We denote the amplitude of the current in the first loop (with the source) as I_1. The amplitude of the current in the second loop is I_2. Kirchhoff's equation for the first loop will take the form

$$U = j\omega L I_1 + j\omega M I_2 \qquad [3]$$

For the second loop, the equation is

$$0 = j\omega L I_2 + j\omega M I_1 \qquad [2]$$

This is a linear system of equations for I_1 and I_2. Solution gives

$$I_1 = \frac{UL}{j\omega(L^2 - M^2)} \qquad [3]$$

and

$$I_2 = -\frac{M}{L}I_1 = -\frac{UM}{j\omega(L^2 - M^2)} \qquad [2]$$