

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2009

ISE PART II: MEng, BEng and ACGI

Corrected Copy

DISCRETE MATHEMATICS AND COMPUTATIONAL COMPLEXITY

Tuesday, 26 May 2:30 pm

Time allowed: 2:00 hours

There are FOUR questions on this paper.

Q1 is compulsory.

Answer Q1 and any two of questions 2-4.

Q1 carries 40% of the marks. Questions 2 to 4 carry equal marks (30% each).

Any special instructions for invigilators and information for candidates are on page 1.

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NOTATION

The following notation may be used throughout this paper:

\mathbb{R} : The set of real numbers.

\mathbb{R}_+ : The set of positive real numbers.

\mathbb{Z} : The set of integers.

\mathbb{Z}_+ : The set of positive integers.

\mathbb{N} : The set of natural numbers.

\mathbb{Q} : The set of rational numbers.

\mathbb{Q}_+ : The set of positive rational numbers.

$\mathcal{P}(S)$: The power set of set S .

The Questions

1. [Compulsory]

- a) Show that $|\mathbb{Z}_+| = |\mathbb{N}|$.

[2]

- b) State whether each of the following relations are (i) reflexive, (ii) symmetric, (iii) transitive.

- i) "is a sibling of" on the set of all people. (Note: sibling means brother or sister).
- ii) "is the son of" on the set of all people.
- iii) "is the same sex as" on the set of all people.
- iv) "is greater than" on the set of all integers.

[8]

- c) Which of the following functions are (i) injective, (ii) surjective? Prove your answer in each case.

- i) $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x, y) = x + y$.
- ii) $f : \mathbb{N} \rightarrow \mathbb{N}$ defined by $f(n) = n + 1$.
- iii) $f : \mathcal{P}(X) \rightarrow X \cup \{0\}$ defined by $f(x) = |x|$, where $X = \{x | x \in \mathbb{Z}_+ \wedge x \leq 10\}$.

[8]

- d) State the truth value of each of the following statements, first using \mathbb{R} as the universe of discourse, and then using \mathbb{Z} as the universe of discourse.

- i) $\exists x \forall y (y + (-y) = x)$.
- ii) $\forall x (x \leq 0 \vee x \geq 1)$.
- iii) $\forall x \exists y (xy = 1)$.

[8]

- e) Let $R(p, b)$ denote the predicate 'Person p has borrowed book b from the library'. Let $O(b)$ denote the predicate 'Book b is overdue'. Let the set of people be P and the set of books be B . Write the following sentences in predicate logic, using predicates $R(p, b)$ and $O(b)$.

- i) Steven has borrowed a book.
- ii) "Crime and Punishment" has been borrowed.
- iii) No book has been borrowed by more than one person.
- iv) If a book is overdue, then it must have been borrowed.

[6]

- f) State the predicate logic definitions of ' $f(x)$ is $O(g(x))$ ' and ' $f(x)$ is $\Omega(g(x))$ '. Use these definitions to show that $x^2 + 1$ is $\Theta(x^2)$ from first principles.

[8]

2. Let $X = \{x | x \in \mathbb{Z}_+ \wedge x \leq 20\}$, and let Y be the set of non-empty character strings of length up to twenty characters. Define $f : X \rightarrow Y$ where $f(x)$ is the string representing the English word for x , e.g. $f(1) = \text{"one"}$.

Let $g : Y \rightarrow X$ be a function where $g(y)$ is the number of characters in the string y , e.g. $g(\text{"one"}) = 3$. Let R be a relation on the set X defined by $R = g \cdot f$.

a) State, with justification, whether:

- i) f is injective,
- ii) f is surjective,
- iii) f^{-1} exists,
- iv) g is injective,
- v) g is surjective,
- vi) g^{-1} exists.

[12]

b) Evaluate $f(X)$ and $g(Y)$.

[4]

c) i) Compute the elements of R .

ii) State, with justification, whether R is reflexive, R is symmetric, and R is transitive.

iii) Show that $\exists y \in X \exists k \in \mathbb{Z}_+ (R^k = \{(x, y) | x \in X\})$ is true.

[14]

3. Consider the function $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ defined by $f(n, a, b) = a \cdot b^n$.
- a) Show that $f(n, 10^3, 10^{-1}) \leq 1$ for all n greater than some value N .
[4]
- b) Show further that for any positive ε , $f(n, 10^3, 10^{-1}) \leq \varepsilon$ for all n greater than some value N (which may depend on ε).
[5]
- c) Express as a predicate $P(b)$ the English statement “for every positive real number a there is another real number N such that for all values of n greater than N , $f(n, a, b)$ can be bounded from above by as small a positive number as you wish” using predicate logic. You should use \mathbb{R} as the universe of discourse.
[8]
- d) Show that $P(0.5)$ is true.
[9]
- e) The proposition $\forall b \in \mathbb{R}(b \in X \rightarrow P(b))$ is true for a variety of sets X . Find the largest set X for which this proposition is true, in the sense that if the proposition is also true for X' , it follows that $X' \subseteq X$.
[4]

4. This question concerns two C/C++ methods to evaluate a polynomial $c[0] + c[1]x + c[2]x^2 + \dots + c[n]x^n$. Algorithm 1 is shown in Fig. 4.1, and Algorithm 2 is shown in Fig. 4.2. In both algorithms, the first argument is a pointer to an array of length $n + 1$ of real coefficients, the second argument is the value of n , and the third argument is a real value of x .
- a) Let p_i denote the value of p at entry to iteration i of the i loop in Algorithm 1.
- What is p_1 ?
 - Express p_{i+1} in terms of p_i as a first order homogeneous recurrence relation. Solve the recurrence relation, and hence find the value of p at the exit of the i loop in Algorithm 1 as a function of k and x .
- [6]
- b) Explain why Algorithm 2 correctly evaluates the polynomial at x .
- [6]
- c) Let $f(n)$ denote the number of multiplications executed by a call to **evalpoly1**(c, n, x).
- Find a formula for $f(n)$ in terms of n only.
 - Hence show that $f(n)$ is $\Omega(n^2)$.
- [9]
- d) Let $g(n)$ denote the number of multiplications executed by a call to **evalpoly2**(c, n, x).
- Find a recurrence relation for $g(n)$.
 - Hence find a suitable big-O expression for $g(n)$.
- [6]
- e) Contrast the efficiency of the two algorithms.
- [3]

```

float evalpoly1( float *c, int n, float x ) {
    float f = 0.0;
    float p;
    int k,i;

    for(k = 0; k <= n; k++) {
        p = 1.0;
        for(i = 1; i <= k; i++) {
            p *= x;
        }
        f += c[k]*p;
    }
    return(f);
}

```

Figure 4.1 Algorithm 1.

```

float evalpoly2( float *c, int n, float x ) {
    if(n == 0)
        return(c[0]);
    else
        return(c[0] + x*evalpoly2(c+1, n-1, x));
}

```

Figure 4.2 Algorithm 2.

1. a) A suitable bijection is $f(x) = x - 1$, $f: \mathbb{Z}_+ \rightarrow \mathbb{N}$.

[2]

b) (i) symmetric, transitive (excluding half-siblings)

(ii) none

(iii) reflexive, symmetric & transitive

(iv) transitive

~~SA~~

[5]

c) (i) Not injective, e.g. $f(0,1) = f(1,0)$ but $(1,0) \neq (0,1)$.
 Surjective, e.g. $f(x,0) = x$ for any $x \in \mathbb{R}$.

(ii) Injective: $f(n) = f(m)$

$$\Rightarrow n+1 = m+1 \Rightarrow n = m$$

Not surjective, as $0 \notin f(\mathbb{N})$.

(iii) Not injective, e.g. $f(\{1,2\}) = f(\{2,3\})$ but $\{1,2\} \neq \{2,3\}$.

Surjective, as we may choose any $x \in X$.

Since $|X| = 10$, $|x|$ may be any integer $0, 1, \dots, 10$
 $= X \cup \{0\}$.

[8]

d) (i) True in both cases ($x = 0$).

(ii) True for \mathbb{Z} but not for \mathbb{R} (e.g. $x = \frac{1}{2}$)

(iii) True for \mathbb{R} but not for \mathbb{Z} (e.g. $x = 2$).

[8]

1. e) (i) $\exists b \in B \quad R(\text{Steven}, b)$

(ii) $\exists p \in P \quad R(p, \text{"Crime and Punishment"})$

(iii) $\forall b_1 \in B \quad \forall p_1 \in P \quad \forall p_2 \in P \quad (R(p_1, b) \wedge R(p_2, b) \rightarrow p_1 = p_2)$

(iv) $\forall b \in B \quad (O(b) \rightarrow \exists p \in P \quad R(p, b))$

[6]

f) $f(x)$ is $O(g(x)) \equiv$

$$\exists c \in \mathbb{R}_+ \exists \kappa \in \mathbb{R}_+ \forall x ((x > \kappa) \rightarrow (|f(x)| \leq c |g(x)|))$$

$f(x)$ is $\Omega(g(x)) \equiv$

$$\exists c \in \mathbb{R}_+ \exists \kappa \in \mathbb{R}_+ \forall x ((x > \kappa) \rightarrow (|f(x)| \geq c |g(x)|))$$

Let $f(x) = x^2 + 1$

For $x > 1$, $f(x) \leq x^2 + x^2$
 $= 2x^2$

So $f(x)$ is $O(x^2)$

For $x > 1$ (say),

$$f(x) \geq x^2$$

So $f(x)$ is $\Omega(x^2)$

$\therefore f(x)$ is $\Theta(x^2)$.

[8]

23. a) (i) Yes - no two integers have the same English word.
 (ii) No, e.g. there is no integer with English word "xxx".
 (iii) No, from (ii). Inverse only exists for bijections.
 (iv) No, e.g. $g(\text{"one"}) = g(\text{"xxx"})$ but $\text{"one"} \neq \text{"xxx"}$.
 (v) Yes, one can construct a string consisting of x "a"s, for any $1 \leq x \leq 20$. This is a non-empty character string of length ≤ 20 .
 (vi) No, from (iv). Inverse only exists for bijections.
- [12]

b) $f(X) = \{\text{"one"}, \text{"two"}, \text{"three"}, \text{"four"}, \dots, \text{"twenty"}\}$
 $g(Y) = \{1, 2, \dots, 20\} = X$.

[4]

c) $R = g \circ f$
 $= \{(1, 3), (2, 3), (3, 5), (4, 4), (5, 4), (6, 3), (7, 5), (8, 5), (9, 4), (10, 3), (11, 6), (12, 6), (13, 8), (14, 8), (15, 7), (16, 7), (17, 9), (18, 8), (19, 7), (20, 6)\}$

R is not reflexive: $(1, 1) \notin R$

R is not symmetric: $(1, 3) \in R$ but $(3, 1) \notin R$

R is not transitive: $(1, 3) \in R$, $(3, 5) \in R$, but $(1, 5) \notin R$.

$$R^2 = \{(1, 5), (2, 5), (3, 4), (4, 4), (5, 4), (6, 5), (7, 4), (8, 4), (9, 4), (10, 5), (11, 3), (12, 3), (13, 5), (14, 5), (15, 5), (16, 5), (17, 4), (18, 5), (19, 5), (20, 3)\}$$

$$R^3 = \{(1, 4), (2, 4), (3, 4), (4, 4), (5, 4), (6, 4), (7, 4), (8, 4), (9, 4), (10, 4), (11, 5), (12, 5), (13, 4), (14, 4), (15, 4), (16, 4), (17, 4), (18, 4), (19, 4), (20, 5)\}$$

28.c) $R^4 = \{(x, 4) \mid x \in X\}$.
[contd]

So with $y=4 (\in X)$ and $\kappa=4 (\in \mathbb{Z}_+)$,
we have our existential quantifier,

$$\exists y \exists \kappa (R^\kappa = \{(x, y) \mid x \in X\}) \quad [14]$$

34. a) $f(n, 10^3, 10^{-1}) = 10^{3-n}$

$$10^{3-n} \leq 1 \quad \text{for } n \geq 3$$

[4]

b) Choose $3-n \leq \log_{10} \epsilon$

i.e.

$$n \geq 3 - \log_{10} \epsilon.$$

[5]

c) $\forall a \forall \epsilon \exists N \forall n (\epsilon > 0 \wedge a > 0 \wedge n \geq N \rightarrow f(n, a, b) \leq \epsilon)$
 $\equiv P(b)$

[8]

d) $f(n, a, 0.5) = a(0.5)^n$

Since we may take $\epsilon > 0$, $a > 0$, choose

$$N = \frac{\log \epsilon - \log a}{\log(0.5)}$$

In the case $a \leq 0$ or $\epsilon \leq 0$, take arbitrary N .

Then $n \geq N \Rightarrow n \log(0.5) \leq \log \epsilon - \log a$

$$\Rightarrow a(0.5)^n < \epsilon$$

$$\Rightarrow f(n, a, b) < \epsilon.$$

[9]

e) $X = \{x \mid x \in \mathbb{R} \wedge -1 < x < 1\}.$

[4]

48. a) (i) $p_1 = 1.0$
 (ii) $p_{i+1} = x p_i$

$$\Rightarrow p_i = x^{i-1}$$

Value at exit is $p_{k+1} = x^k$.

[6]

- b) If $n=0$ then evalpoly2 returns $c[0]$, which is the zero-order (constant) polynomial.

Induction hypothesis: evalpoly2($c, n-1, x$) returns the desired result, i.e. $c[1] + c[2]x + \dots + c[n]x^{n-1}$.

Then evalpoly2(c, n, x) returns

$$c[0] + x(c[1] + c[2]x + \dots + c[n]x^{n-1})$$

= the desired polynomial.

[6]

- c) (i) One mult per iteration of i loop + one extra per iteration of k loop

$$\Rightarrow \#mults = n+1 + \sum_{k=0}^n k$$

$$= n+1 + \frac{1}{2}n(n+1)$$

$$= \frac{1}{2}(n+1)(n+2)$$

[6]

(ii) $\#mults = \frac{1}{2}n^2 + \frac{3}{2}n + 1$

$$\geq \frac{1}{2}n^2 \quad \text{for } n > 0$$

$$\Rightarrow \#mults \text{ is } \Omega(n^2) \quad [c = \frac{1}{2}, k = 1, \text{ say}]$$

[9]

48. d) (i) $g(0) = 0$
 $g(n) = 1 + g(n-1)$ for $n > 0$.

(ii) This is an A.P.

$g(n) = n$
 $g(n)$ is a $O(n)$ function

[6]

e) One is $\Omega(n^2)$ and the other is $O(n)$

\Rightarrow quadratic versus linear time. Alg 2 is preferable.

[3]