SOLUTIONS

1. a) By using row reduction on the augmented matrix choose $a, b$ such that the system below

$$x + ay = 2$$
$$4x + 8y = b$$

has

(i) a unique solution. **[1]**

**SOLUTION**

**The system above has a unique solution if and only if the determinant of the system's matrix is non-zero. In that case $8 \neq 4a \Leftrightarrow a \neq 2$.**

(ii) many solutions. **[1]**

**SOLUTION**

**The system above has many solutions if the determinant of the system's matrix is zero and furthermore, the right-hand-side vector belong to the column space of the system's matrix. In that case $8 = 4a \Leftrightarrow a = 2$ and $b = 8$.**

(iii) no solution. **[1]**

**SOLUTION**

**The system above no solution if the determinant of the system's matrix is zero and furthermore, the right-hand-side vector does not belong to the column space of the system's matrix. In that case $8 = 4a \Leftrightarrow a = 2$ and $b \neq 8$.**

b) Three linear systems $Ax = \underline{d}$, $Bx = \underline{e}$, $Cx = f$ have the augmented matrices

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

respectively, given in reduced row echelon form.

(i) How many solutions does each system have? **[3]**

**SOLUTION**

**The first system has a unique solution because the system's matrix is invertible.**
**The second system has infinitely many solutions since the rank of system's matrix is 2 and therefore, there are 2 free variables and furthermore, the right-hand-side vector belong to the column space.**
**The third system has no solution since the right-hand-side vector does not belong in the column space.**

(ii) Which of the matrices $A$, $B$, $C$ are invertible? **[2]**

**SOLUTION**

**Only matrix $A$ is invertible.**

(iii) Find the rank of each matrix $A$, $B$, $C$. **[2]**

**SOLUTION**

**From the reduced row echelon forms, we observe that the rank of $A$ is 3, the rank of $B$ is 2 and the rank of $C$ is 2.**

c) Let $V$ be the subspace of $\Re^4$ consisting of all solutions to the system of equations written in matrix form as:

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Let $W$ be the orthogonal complement of $V$. Find row vectors $\underline{v} \in V$ and $\underline{w} \in W$ so that $\underline{v} + \underline{w} = [1 \quad 0 \quad 0 \quad 0]$. (Hint: Use the relationship $\underline{v} = [1 \quad 0 \quad 0 \quad 0]^T - \underline{w}$) **[6]**

**SOLUTION**

**$W$ is spanned by the rows of the system's matrix. Therefore, for some scalars $s,t$ we have $\underline{w}=s[0\quad 1\quad 2\quad 3]+t[3\quad 2\quad 1\quad 0]$**
**We know that $\underline{v}=[1\quad 0\quad 0\quad 0]-\underline{w}$ is in $V$. Therefore,**

$$\begin{bmatrix}0&1&2&3\\3&2&1&0\end{bmatrix}\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}-\begin{bmatrix}0&3\\1&2\\2&1\\3&0\end{bmatrix}\begin{bmatrix}s\\t\end{bmatrix}\right)=\begin{bmatrix}0\\0\end{bmatrix}$$

$$\begin{bmatrix}0&1&2&3\\3&2&1&0\end{bmatrix}\begin{bmatrix}0&3\\1&2\\2&1\\3&0\end{bmatrix}\begin{bmatrix}s\\t\end{bmatrix}=\begin{bmatrix}0&1&2&3\\3&2&1&0\end{bmatrix}\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\Rightarrow\begin{bmatrix}14&4\\4&14\end{bmatrix}\begin{bmatrix}s\\t\end{bmatrix}=\begin{bmatrix}0\\3\end{bmatrix}$$

**From the above system, we get**

$14s+4t=0\Rightarrow 28s+8t=0$

$4s+14t=3\Rightarrow 28s+98t=21$

$\Rightarrow t=\frac{21}{90}=\frac{7}{30}$

$4s+\frac{98}{30}=3\Rightarrow s=\frac{-2}{30}=\frac{-1}{15}$

$\underline{w}=\frac{1}{30}([0\quad -2\quad -4\quad -6]+[21\quad 14\quad 7\quad 0])=\frac{1}{30}[21\quad 12\quad 3\quad -6]$

$\underline{v}=\frac{1}{30}([30\quad 0\quad 0\quad 0]-[21\quad 12\quad 3\quad -6])=\frac{1}{30}[9\quad -12\quad -3\quad 6]$

d) If $A$ is a matrix of dimension $4\times 4$, describe all vectors in the nullspace of the $4\times 8$ matrix $B=[A\,A]$. **[4]**

**SOLUTION**

**The nullspace of $B=[A\,A]$ contains all vectors of the form $\underline{x}=\begin{bmatrix}\underline{y}\\-\underline{y}\end{bmatrix}$ where $\underline{y}\in\Re^4$.**

2. a) Consider a matrix $Q$ which is symmetric and orthogonal. Show that the absolute value of its eigenvalues is always 1. **[2]**

**SOLUTION**

$Qx = \lambda x$ **and** $(Qx)^T = x^T Q^T = \lambda x^T$ **. Therefore,**

$(Qx)^T Qx = \lambda^2 x^T x \Rightarrow x^T Q^T Qx = x^T Ix = x^T x = \lambda^2 x^T x \Rightarrow \lambda^2 = 1 \Rightarrow \lambda = \pm 1$

b) Consider a $3 \times 3$ matrix $Q$ which is symmetric and orthogonal. Mark each statement (i)-(v) **True** or **False**. Justify your answer.

(i) $Q$ does not have repeated eigenvalues. **[2]**

**SOLUTION**

**Matrix $Q$ has 3 eigenvalues which are 1 or -1. Therefore, it DOES have repeated eigenvalues. The statement is FALSE.**

(ii) $Q$ is always positive definite. **[2]**

**SOLUTION**

**Matrix $Q$ can have eigenvalues of value -1. Therefore, it cannot be always positive definite. A positive definite matrix has only positive eigenvalues. The statement is FALSE.**

(iii) $Q$ is diagonalizable. **[1]**

**SOLUTION**

**Yes, because all symmetric and all orthogonal matrices can be diagonalized. In fact, we can choose the eigenvectors of $Q$ to be orthogonal. The statement is TRUE.**

(iv) $Q$ is non-singular. **[1]**

**SOLUTION**

**Yes, orthogonal matrices are all nonsingular since their column vectors are orgononal to each other and also they are square matrices. The statement is TRUE.**

(v) The matrix $P = \frac{1}{2}(Q + I)$ is a projection matrix . **[2]**

**SOLUTION**

**We can check that** $P = \frac{1}{2}(Q + I)$ is symmetric and that $P^2 = P$ .

$P^T = \frac{1}{2}(Q^T + I^T) = P$ . Therefore, $P$ is symmetric.

$$P^2 = \frac{1}{4}(Q + I)(Q + I) = \frac{1}{4}(QQ + 2QI + II) = \frac{1}{4}(Q^T Q + 2QI + II)$$

$$= \frac{1}{4}(I + 2Q + I) = \frac{1}{4}(2Q + 2I) = P$$

Therefore, $P = \frac{1}{2}(Q + I)$ is a projection matrix. **The statement is TRUE.**

c) Find the projection matrix onto the subspace that is formed by the vectors $[1 \quad 2 \quad 1 \quad 1]^T$ and $[-1 \quad 1 \quad 0 \quad 1]^T$. Use it to compute the projection of the vector $[1 \quad 1 \quad 2 \quad 1]^T$ onto that subspace. **[3]**

**SOLUTION**

**Let** $A=\begin{bmatrix}1 & -1\\ 2 & 1\\ 1 & 0\\ 1 & 1\end{bmatrix}$. $A^T=\begin{bmatrix}1 & 2 & 1 & 1\\ -1 & 1 & 0 & 1\end{bmatrix}$. Therefore, $A^TA=\begin{bmatrix}7 & 2\\ 2 & 3\end{bmatrix}$ and

$\left(A^TA\right)^{-1}=\frac{1}{17}\begin{bmatrix}3 & -2\\ -2 & 7\end{bmatrix}$.

**We can easily show that the projection matrix is**

$$A\left(A^TA\right)^{-1}A^T=\frac{1}{17}\begin{bmatrix}14 & 1 & 5 & -4\\ 1 & 11 & 4 & 7\\ 5 & 4 & 3 & 1\\ -4 & 7 & 1 & 6\end{bmatrix}.$$

**Therefore, the projection of the given vector is** $\frac{1}{17}\begin{bmatrix}14 & 1 & 5 & -4\\ 1 & 11 & 4 & 7\\ 5 & 4 & 3 & 1\\ -4 & 7 & 1 & 6\end{bmatrix}\begin{bmatrix}1\\1\\2\\1\end{bmatrix}=\frac{1}{17}\begin{bmatrix}21\\27\\16\\11\end{bmatrix}$.

d) Consider the matrix

$$A=\begin{bmatrix}4 & -3 & 3\\ 3 & -2 & 3\\ 3 & -3 & 4\end{bmatrix}$$

with eigenvectors

$$v_1=\begin{bmatrix}\frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{3}}\end{bmatrix},\ v_2=\begin{bmatrix}\frac{1}{\sqrt{2}}\\ 0\\ -\frac{1}{\sqrt{2}}\end{bmatrix},\ v_3=\begin{bmatrix}0\\ \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix}.$$

The given order is the magnitude of the associated eigenvalue, with the first being the eigenvector that corresponds to the largest eigenvalue.

Give the diagonalized form of two distinct matrices $B, C$ such that $B^2=A=C^2$, without calculating the inverse matrix involved in the diagonalization. **[4]**

**SOLUTION**

$$A-\lambda I==\begin{bmatrix}4-\lambda & -3 & 3\\ 3 & -2-\lambda & 3\\ 3 & -3 & 4-\lambda\end{bmatrix}$$

$\det(A-\lambda I)=0\Rightarrow(4-\lambda)[-(\lambda+2)(4-\lambda)+9]+3[3(4-\lambda)-9]+3[-9+3(\lambda+2)]=0\Rightarrow$

$(4-\lambda)[-(4\lambda-\lambda^2+8-2\lambda)+9]+3(12-3\lambda-9)+3(-9+3\lambda+6)=0\Rightarrow$

$(4-\lambda)(-4\lambda+\lambda^2-8+2\lambda+9)+3(3-3\lambda)+3(-3+3\lambda)=0\Rightarrow(4-\lambda)(\lambda^2-2\lambda+1)=0$

**The eigenvalues of** $A$ **are 4 and 1 (with multiplicity 2).**

**If I set** $S=[v_1\quad v_2\quad v_3]$**, then** $B=S\begin{bmatrix}2 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}S^{-1}$ **and** $C=S\begin{bmatrix}-2 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}S^{-1}$.

e) Suppose that the matrices $A, B$ of dimension $m \times n$ have the same four subspaces. If they are both in reduced row echelon form, prove that $F$ must equal $G$.

$$A = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} B = \begin{bmatrix} I & G \\ 0 & 0 \end{bmatrix}$$

[3]

**SOLUTION**

**The first row of $A = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$ is a linear combination of the rows of $B = \begin{bmatrix} I & G \\ 0 & 0 \end{bmatrix}$. But since all first elements of rows of $B$ are 0 apart from $b_{11} = 1$, it is obvious that the first row of $A$ and $B$ are the same. The same observation is valid for all non zero rows. Therefore, $F$ must equal $G$.**

3. a) Consider a matrix $A$ of dimension $3 \times 3$ with eigenvalues $\lambda_1 = 0, \lambda_2 = c, \lambda_3 = 2$ where $c$ is a real scalar, and associated eigenvectors

$$x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\ x_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix},\ x_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

(i) For which values of $c$ is matrix $A$ diagonalizable? [2]

**SOLUTION**

**The matrix is diagonalizable if it has 3 independent eigenvectors. Not only are $x_1$, $x_2$ and $x_3$ independent, they're orthogonal. Thus, the matrix is diagonalizable for all values of $c$.**

(ii) For which values of $c$ is matrix $A$ symmetric? [2]

**SOLUTION**

**We see by diagonalizing $A = S\Lambda S^T$ that we get $A = \begin{bmatrix} c+2 & -c+2 & -4 \\ -c+2 & c+2 & -4 \\ -4 & -4 & 8 \end{bmatrix}$ which is symmetric for all values of $c$.**

(iii) For which values of $c$ is matrix $A$ positive definite? [1]

**SOLUTION**

**All positive definite matrices are symmetric, so $c$ must be real. The eigenvalues of a positive definite matrix must be positive. The eigenvalue 0 is not positive, so this matrix is not positive definite for any values of $c$.**

(iv) For which values of $c$ is matrix $A$ positive semi-definite? [1]

**SOLUTION**

**If $c = 0$ then the matrix is positive semi-definite.**

(v) Is $A$ a Markov matrix? [2]

**SOLUTION**

**In a Markov matrix, one eigenvalue is 1 and the other eigenvalues are smaller than 1. Because $\lambda_3 = 2$, this cannot be a Markov matrix for any value of $c$.**

(vi) Could $P = \frac{1}{2}A$ be a projection matrix? [2]

**SOLUTION**

**The eigenvalues of $\frac{1}{2}A$ are the halves of the eigenvalues of $A$, therefore they are $0, \frac{c}{2}, 1$. Projection matrices are real and symmetric so their eigenvalues are real. In addition, we know that their eigenvalues are 1 and 0 because $P^2 = P$ implies $\lambda^2 = \lambda$. Thus, $\frac{1}{2}A$ could be a projection matrix if $c = 0$ or $c = 1$.**

b) Consider the Singular Value Decomposition of matrix $A = U\Sigma V^T$. Suppose $\Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ and $U$ and $V$ each have two columns.

(i) What can we say about the dimension and rand of matrix $A$? [2]

**SOLUTION**

**We know $A$ is a two by two matrix, and because $U, \Sigma, V$ are all invertible we know that $A$ is nonsingular and therefore, its rank is 2.**

(ii) Show that if $\Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$, a basis for the null space of $A$ is the second column of $V$.

**[2]**

**SOLUTION**

**Then $A$ is a singular matrix of rank 1 and its null space has dimension 1. The four fundamental subspaces associated with $A$ are spanned by orthonormal bases made up of selected columns of $U$, and $V$. In this example, the second column of $V$ is a basis for the null space of $A$.**

c) Consider the matrix $A$:

$$A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$$

Find the Singular Value Decomposition of matrix $A = U\Sigma V^T$. It is given that a set of orthogonal but not orthonormal eigenvectors of $V$ is $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1/2 \end{bmatrix}$. The given order corresponds to the order of the eigenvalues of $V$ according to their magnitude with the first being the eigenvector that corresponds to the largest eigenvalue. **[6]**

**SOLUTION**

$AA^T = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}$. **Its eigenvalues are obtained by the following equation:**

$(17 - \lambda^2) - 64 = 0 \Rightarrow \lambda_1 = 25, \lambda_2 = 9$ **with corresponding eigenvectors** $\begin{bmatrix} x \\ x \end{bmatrix}$ **and** $\begin{bmatrix} x \\ -x \end{bmatrix}$.

**Therefore,** $U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$.

**The singular values are 5 and 3. Therefore,** $\Sigma = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}$. **From the given eigenvectors of $V$ we construct**

$V = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} & \frac{2}{3} \\ \frac{1}{\sqrt{2}} & \frac{-1}{3\sqrt{2}} & \frac{-2}{3} \\ 0 & \frac{4}{3\sqrt{2}} & \frac{1}{3} \end{bmatrix}$. **It can be verified that $A = U\Sigma V^T$.**