ISE2
Discrete Mathematics and Computational Complexity

Specimen Paper

George Constantinides (gac1)

1. Compulsory Question

(a) Which (if any) of these statements are propositions? Briefly justify your answers.

(i) $4x = 5$
(ii) $5x + 1 = 5$ if $x = 1$
(iii) $x + y + z = y + 2z$ if $x = z$

[3]

(b) Show that each of these implications is a tautology using an appropriate truth table.

(i) $p \wedge q \rightarrow p$
(ii) $p \rightarrow p \vee q$
(iii) $\neg p \rightarrow (p \rightarrow q)$

[3]

(c) Determine the truth of each of these propositions, where the universe of discourse is the set of integers. Briefly justify your answers.

(i) $\forall n(n^2 \geq n)$
(ii) $\exists n(n^2 = 2)$

[2]

(d) Find the power set of each of the following sets.

(i) $\{a\}$
(ii) $\{a,b\}$
(iii) $\{\varnothing,\{\varnothing\}\}$

[3]

(e) Let $f(n)$ be the function from the set of integers to the set of integers such that $f(n) = n^2 + 1$. What are the domain, codomain, and range of this function?

[3]

(f) Which (if any) of these functions are bijections from **R** to **R**? Briefly justify your answers.

(i) $f(x) = 2x + 1$
(ii) $f(x) = x^2 + 1$
(iii) $f(x) = x^3$

[3]

(g) Find an appropriate big-O expression for each of these recurrence relations.

(i) $f(n) = 2f(n\text{-}1)$, with $f(0) = 1$.
(ii) $f(n) = 2f(n\text{-}1) + 1$, with $f(0) = 0$.
(iii) $f(n) = 2f(n/3) + n^2$, with $f(0) = 0$.

[3]

Total: 20 Marks

2. Logic

Let P(x) denote the statement “x owns a computer”.
Let Q(x) denote the statement “x can program a computer”.
Let R(x) denote the statement “x has studied computing”.
Let S(x,y) denote the statement “x knows y”.
Let the universe of discourse be the set of all people.

(a) Write the following English statements using symbolic logic.
- (i) Everyone who owns a computer can program a computer.
- (ii) Someone can program a computer, but doesn’t own one.
- (iii) Someone who has studied computing can’t program a computer.
- (iv) Everyone who can program a computer has studied computing.
- (v) Steven has not studied computing.

[5]

(b) Use symbolic logic to construct a valid argument that results in the conclusion ¬ P(Steven), given the premises derived in part (a). At each step of your argument, state the rule of inference used.

[7]

(c) Write the following statements using symbolic logic.
- (i) Someone knows everyone who can program a computer.
- (ii) Everyone knows someone who can program a computer.
- (iii) Everyone who has studied computing knows someone who owns a computer.

[3]

Total: 15 Marks

3. Algorithm Analysis

a) Derive an expression for the number of multiplications performed by the code in Fig. 3.1, in terms of the input value n.

```
procedure p(n: integer)
begin
  total := 0
  for i := 1 to n
    for j := 1 to n
      total := total + i*j
  result := total
end
```

Figure 3.1

[2]

b) Hence derive a recurrence relation for the number of multiplications performed by the code in Fig. 3.2, in terms of the input value n.

```
procedure q(n: integer)
begin
  if n = 0 then
    result := 1
  else
    result := q(n/2) + p(n)
end
```

Figure 3.2

[3]

c) State the Master Theorem.

[3]

d) Derive a big-O expression for the number of multiplications performed by a call to **q**, in terms of n.

[1]

e) Prove that if the recurrence relation $f(n) = a\, f(n/b) + c$ is satisfied, where $a > 1$, then $f(n)$ is $O(n^{\log_b a})$. You need only consider the case where $n = b^k$ for some $k \in \mathbf{Z}^+$.

[6]

Total: 15 Marks

4. Computability

(a) Define what is meant if a problem is said to be *tractable*, and give an example of a tractable problem.

[2]

(b) Define what is meant if a problem is said to be *unsolvable*, and give an example of an unsolvable problem.

[2]

(c) Prove that the problem from part (b) is unsolvable.

[6]

(d) Let the set of ISE2 students be denoted S. Consider a symmetric relation D on S, such that s_1 D s_2 iff student s_1 dislikes student s_2.

The "student allocation" problem is defined as:

Can the set of students be partitioned into no more than *n* teams, such that no team contains any two students who dislike each other?

Prove that "student allocation" is at least as hard as *k*-colouring (defined below for convenience).

The k-colouring problem is defined as:

Given a set of nodes V, a set of edges E, and a positive integer *k*, does there exist a function $p : V \rightarrow \{1,2,\ldots,k\}$ such that $\forall v_1 \, \forall v_2 \, (\{v_1, v_2\} \in E \rightarrow p(w_1) \neq p(w_2))$, where the universe of discourse is the set V?

[5]

Total: 15 Marks.

Model Answers

Question 1.

(a)

(i) is not a proposition. It is neither true nor false, as x is undefined.
(ii) is a proposition, as it has a definite truth value (false).
(iii) is a proposition, as it has a definite truth value (true).

(b) The truth tables are shown below. In each case, the final column is always true, thus the expression is a tautology.

(i)

p	q	$p \wedge q$	$p \wedge q \rightarrow p$
F	F	F	T
F	T	F	T
T	F	F	T
T	T	T	T

(ii)

p	q	$p \vee q$	$p \rightarrow p \vee q$
F	F	F	T
F	T	T	T
T	F	T	T
T	T	T	T

(iii)

p	q	$\neg p$	$p \rightarrow q$	$\neg p \rightarrow (p \rightarrow q)$
F	F	T	T	T
F	T	T	T	T
T	F	F	F	T
T	T	F	T	T

(c)

(i) If n is negative, clearly $n^2 \geq 0 \geq n$. If n is zero, it is true. If n is positive, since n is an integer, $n \geq 1$, so multiplying both sides by n, we obtain $n^2 \geq n$. Thus this proposition is true.
(ii) There is no *integer* n with $n^2 = 2$, so the proposition is false.

(d)

(i) $\{\varnothing, \{a\}\}$
(ii) $\{\varnothing, \{a\}, \{b\}, \{a, b\}\}$
(iii) $\{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\}$

(e) Domain: the set of integers, Codomain: the set of integers,
Range: $\{n^2 + 1 \mid n \text{ is an integer}\}$.

(f)

(i) Yes. There is an inverse $g(x) = (x-1)/2$.
(ii) No, as the range is $\{x^2 + 1 \mid x \text{ is real}\}$, which is not the set of reals, so the function is not a surjection. Alternatively, we can simply note that $x = -1$ and $x = +1$ have the same value of $f(x)$, so the function is not an injection.
(iii) Yes. There is an inverse $g(x) = x^{1/3}$.

(g)
(i) $f(n) = 2^n$, which is $O(2^n)$.
(ii) $f(n) = 2^n - 1$, which is $O(2^n)$.
(iii) $f(n)$ is $O(n^2)$ from the Master Theorem.

Question 2.

(a)
(i) $\forall x(\ P(x) \rightarrow Q(x)\ )$
(ii) $\exists x(\ Q(x) \wedge \neg P(x)\ )$
(iii) $\exists x(\ R(x) \wedge \neg Q(x)\ )$
(iv) $\forall x(\ Q(x) \rightarrow R(x)\ )$
(v) $\neg$R(Steven)

(b)
1. Q(Steven) $\rightarrow$ R(Steven) [universal instantiation, from premise (iv)]
2. $\neg$ Q(Steven) [modus tollens, when combined with premise (v)]
3. P(Steven) $\rightarrow$ Q(Steven) [universal instantiation, from premise (i)]
4. $\neg$ P(Steven) [modus tollens, when combined with (3) above]

(c)
(i) $\exists x \forall y(\ Q(y) \rightarrow S(x,y)\ )$
(ii) $\forall x \exists y(\ Q(y) \wedge S(x,y)\ )$
(iii) $\forall x \exists y(\ R(x) \rightarrow S(x,y) \wedge P(y)\ )$

Question 3.

(a) Each outer loop executes n times. Each inner loop executes n times per iteration of the outer loop. There is one multiplication per inner loop iteration. Thus the number of multiplications is n^2.

(b) $f(0) = 0$: there are no multiplications in the base case. $f(n) = f(n/2) + n^2$.

(c) [see notes]

(d) $O(n^2)$. [A direct application of the Master Theorem]

(e) [see notes]

Question 4.

(a) [see notes]

(b) [see notes]. The only unsolvable example studied in lectures is the halting problem.

(c) [see notes]

(d) There is a direct correspondence between the two problems. “Dislikes” corresponds to edges, and students correspond to nodes. The relation D is symmetric, so the digraph of the relation is equivalent to the graph to be coloured. The following reduction is appropriate:
(i) Set S = V.
(ii) Set $D = \{(v_1,v_2) \mid \{v_1,v_2\} \in E\}$.
(iii) Set $n = k$.