Paper Number(s): **E3.09**
ISE3.9

IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE
UNIVERSITY OF LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2000

EEE/ISE PART III/IV: M.Eng., B.Eng. and ACGI

CONTROL ENGINEERING

Tuesday, May 2 2000, 10:00 am

There are SIX questions on this paper.

Answer FOUR questions.

All questions carry equal marks.

Time allowed: 3:00 hours

Corrected Copy

xaminers: Prof R.B. Vinter, Dr A. Astolfi

Special instructions for invigilators: None

Information for candidates: None

1. Consider the unity feedback control system of Figure 1, in which the plant transfer function is

$$G(s) = \frac{(s-1)^2}{s(s+1)^2}.$$

Sketch the extended Nyquist diagram for $G(s)$, showing the intercepts of the Nyquist diagram with the negative real axis and with the positive imaginary axis.

(Hint: to evaluate the intercepts, use the fact that, because of the symmetric location of the poles and zeros of $G(s)$, $\angle G(j\omega)$ can be simply expressed in terms of $\tan^{-1}\omega$ at relevant frequencies.)

Deduce that the closed loop system is unstable without compensation; i.e., when $D(s) = 1$.

By indicating how the Nyquist diagram is modified, show that the system can be stabilized by a phase lag compensator

$$D(s) = \frac{s/a+1}{s/b+1}, \quad 0 < b < a,$$

(with unity DC gain) for suitable choices of the design parameters a and b. (You do not have to choose values for a and b.)

Show from the Nyquist diagram for $G(s)$ that the system *cannot* be stabilized by a phase advance compensator

$$D(s) = \frac{s/a+1}{s/b+1}, \quad 0 < a < b,$$

(with unity DC gain), for any choice of the design parameters a and b.

(Hint: in the last part, use the earlier calculated intercept of the positive imaginary axis by the Nyquist diagram for $G(s)$.)

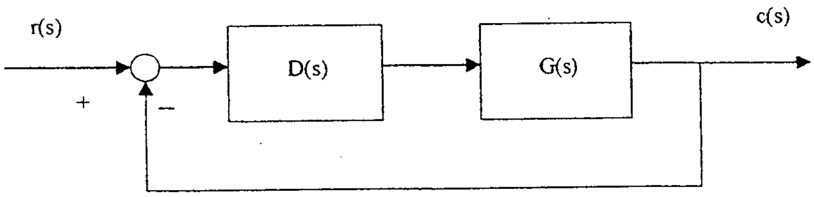

Figure 1

2. Figure 2 shows a control system to improve the transient response of an under-damped large space structure, subject to a disturbance $d(s)$. The plant transfer function is

$$G(s) = \frac{0.1}{s(s^2+0.2s+1)}.$$

The compensator is a double phase advance compensator, with transfer function

$$D(s) = \frac{K(s/a+1)^2}{(s/b+1)^2},$$

in the design parameters K, a and b satisfy $K>0$, $0<a<b$. Show that

$$\angle G(s) = \begin{cases} -270^\circ + \tan^{-1}(0.2\omega/(\omega^2-1)) & \text{for } \omega > 1 \\ -90^\circ - \tan^{-1}(0.2\omega/(1-\omega^2)) & \text{for } \omega < 1\,. \end{cases}$$

Choose values of the design parameters K, a and b to meet the specifications:

(i) (disturbance attenuation) When $r=0$ and the disturbance is a unit step, the steady-state output $c(t=\infty)$ satisfies

$$|c(t=\infty)| < 0.5\,.$$

(ii) (bandwidth) the gain cross-over frequency ω^* is $\omega^* = 2$ rads^{-1}.

(iii) (phase margin) The compensated system has phase margin 45°.

You should use the following design procedure:

Step 1. Choose K, a and b to meet specifications (ii) and (iii) and also to ensure $\omega_{\max} = \omega^*$. Here $\omega_{\max}$ is the frequency for which the phase advance of the compensator $G(s)$ is maximized.

(i) Step 2. Check that specification ~~(a)~~ is met. 11:40

You can quote the facts that the maximum phase advance of

$$(j\omega/a+1)/(j\omega/b+1) \quad (0<a<b)$$

is $90^\circ - 2\times \tan^{-1}\sqrt{a/b}$, *and occurs at frequency* $\omega = \sqrt{ab}$.

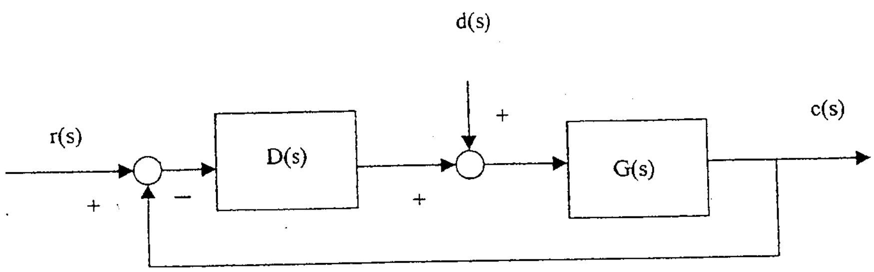

Figure 2

3. Figure 3 shows a control system in which a force u is applied to a trolley, on which is mounted an inverted pendulum. The trolley and pendulum have masses M and m respectively. The pendulum rod, which is assumed weightless, has length L. g denotes the gravitational constant.

Assume that the rod angle from the vertical θ radians remains small, so that

$$\theta \simeq \sin\theta = (y - x)/L \quad \text{and} \quad T = mg\cos\theta \simeq mg\,.$$

($-T$ denotes the tension of the rod.) Take as state vector $x = (x_1, x_2, x_3, x_4)^T$

$$x_1 = y/g,\ x_2 = \dot{y}/g,\ x_3 = \theta,\ x_4 = \dot{\theta}\,.$$

Show that the system is governed by a state space model of the form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & d_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ d_2 \end{bmatrix} u$$

and evaluate the constants d_1 and d_2.

Determine coefficients $k_1, \ldots, k_4$ in the state feedback law

$$u = -k_1 y - k_2 \dot{y} - k_3 \theta - k_4 \dot{\theta}$$

to ensure that the closed loop system has the characteristic polynomial

$\alpha_1 s$

Announced at 10.24 am after consultation with examiner Richard Vinter

$$\tau_d(s) = s^4 + \alpha_3 s^3 + \alpha_2 s^2 + \alpha_1 s^2 + \alpha_0\,,$$

for specified coefficients $(\cancel{\alpha_1}^{\,\alpha_0}, \ldots, \alpha_3)$.

(Hint: Note that the state space model, expressed in terms of the scaled control $d_2 u$, is in control canonical form.)

α_0, not α_1

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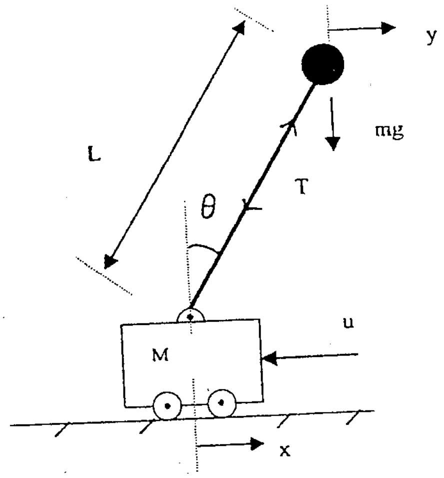

Figure 3

4. Consider the control-free single output, state space model

$$\begin{aligned} \dot{x} &= Ax \\ y &= c^T x\,. \end{aligned}$$

Describe the structure of a state observer with gain g, which provides an estimate $\hat{x}(t)$ of the state $x(t)$, given $y(s)$, $s \le t$. Derive the differential equation governing the estimation error $e(t) = x(t) - \hat{x}(t)$.

The linear system of Figure 4, relating the scalar input d and the scalar output y, has transfer function

$$G(s) = 1/s^2\,.$$

Develop a state space model, in which the state variables are $x_1 = y$ and $x_2 = \dot{y}$.

Now regard d as a constant unknown disturbance of magnitude D, which we wish to estimate:

$$d(t) = D \quad t \ge 0\,.$$

Develop a third order state space model, in which the state variables are $x_1 = y$, $x_2 = \dot{y}$ and $x_3 = d(t)$.

(Note that x_3 satisfies $\dot{x}_3 = 0$ and $x_3(0) = D$.)

Using the third order model, design an observer providing an estimate $\hat{d}(t)$ of D, given $y(s)$, $s \le t$, such that the estimation error

$$|D - \hat{d}(t)|$$

decays exponentially with a time constant of 0.5 seconds.

(Hint: arrange that all observer eigenvalues are located at $-2 + 0j$. Take $\hat{d}(t)$ to be the third component of an estimate $\hat{x}(t)$ of the state of the third order model.)

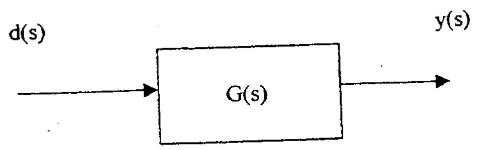

Figure 4

5. A state feedback controller is required so that the state x_1 of the single input/single output control system

$$\dot{x}_1 = a_1 x_1 + b_1 u$$

tracks a decaying reference signal $z(t)$:

$$z(t) = \xi e^{-\lambda t}.$$

Here a_1, b_1, ξ and $\lambda (> 0)$ are constants.

This is achieved by finding a feedback solution to the optimal control problem:

$$\begin{cases} \text{Minimize } \int_0^\infty [|x_1(t) - z(t)|^2 + \alpha |u(t)|^2] dt \\ \text{subject to } \dot{x}_1 = a_1 x_1 + b_1 u(t), \quad x_1(0) = \eta, \end{cases}$$

in which η and α are given positive constants.

Show that the above problem can be expressed as a standard optimal control problem, in which the state variables are x_1 and $x_2 = z$:

$$(P) \begin{cases} \text{Minimize } \int_0^\infty [x^T(t) Q x(t) + \alpha |u(t)|^2] dt \\ \text{subject to } \dot{x} = Ax + bu(t), \quad x(0) = (\eta, \xi). \end{cases}$$

What are A, b and the symmetric matrix Q?

(Hints: note that $z(t)$ satisfies $\dot{z} = -\lambda z$ and $z(0) = \xi$. To determine the entries q_{11}, q_{12} and q_{22} in the symmetric matrix Q, match

$$|x_1 - x_2|^2 = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.)$$

By quoting the solution to the standard problem, and by solving the 2×2 Riccati equation for the p_{11} and p_{12} entries of the solution P, show that the tracking problem has the feedback solution

$$u(t) = -\alpha^{-1} b_1 (p_{11} x_1(t) + p_{12} \xi e^{-\lambda t}),$$

where p_{11} and p_{12} are solutions of the equations

$$2a_1 p_{11} + 1 - \alpha^{-1} p_{11}^2 b_1^2 = 0,$$
$$p_{12} = 1/(a_1 - \alpha^{-1} p_{11} b_1^2) - \lambda$$

The solution to (P) is

$$u(t) = -\alpha^{-1} b^T P x(t),$$

where P is the positive, symmetric matrix satisfying

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$$A^T P + PA + Q - \alpha^{-1} P b b^T P = 0.$$

$$p_{12} = \frac{1}{(a_1 - \alpha^{-1} p_{11} b_1^2) - \lambda}$$

6. A sensing device (NL) has a cubic characteristic $g(a)$:

$$g(a) = a^3 .$$

By quoting the trigonometric identity

$$\sin^3(\omega t) = (1/4)[3\sin(\omega t) - \sin(3\omega t)]) ,$$

or otherwise, determine the describing function of the device.

(NL) is present in the feedback loop of the control system of Figure 6.

(a) Assume $r(t) = 0$ and

$$G(s) = \frac{1}{s(s+1)^3} .$$

Show that describing function analysis predicts a limit cycle. Determine the frequency of limit cycle oscillations and the amplitude of the output signal $c(t)$. Assess whether the limit cycle is stable or unstable.

(b) Now assume that

$$G(s) = 1/s .$$

When a sinusoidal signal $r(t) = R\sin(\omega t)$ is applied at the input, the output is approximately sinusoidal, with amplitude A. Show that, according to describing function analysis, A is related to R and ω according to

$$(9/16)A^6 + \omega^2 A^2 = R^2 .$$

Hint: carry out a 'linear' steady state frequency response analysis, in which $N(A)$ *replaces the gain of the nonlinearity.*

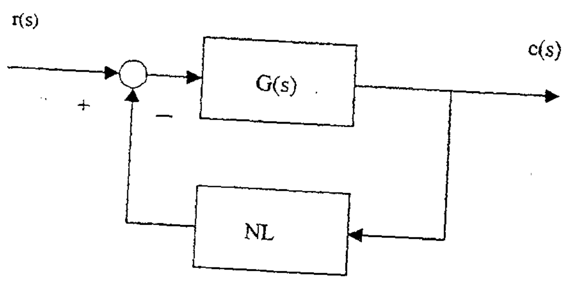

Figure 6

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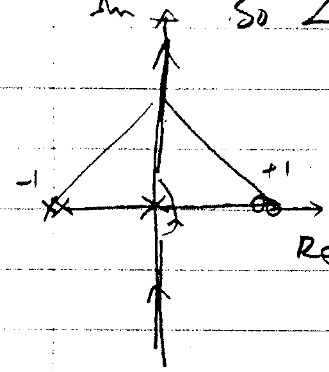
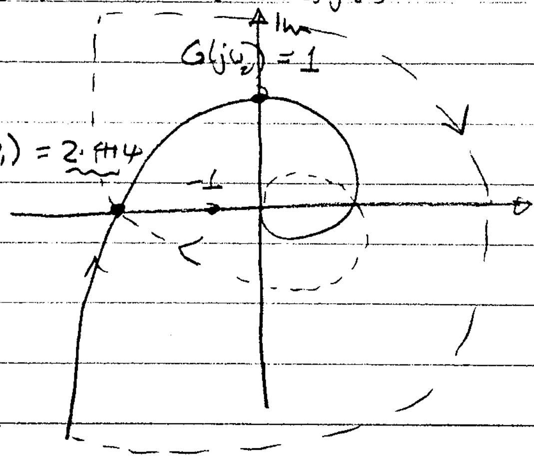
MASTER - with mark scheme

Control Engineering Model Answers 2000

1. $G(s) = (s-1)^2 / s(s+1)$. From "graphical interpretation" of $G(j\omega)$ and $\angle G(j\omega)$ we see that $\angle G(j\omega) = 270^\circ - 4\tan^{-1}\omega$ and $|G(j\omega)| = 1/\omega$, $0<\omega<\infty$

So $\angle G(j\omega) \to 270^\circ$ as $\omega \to 0$ and $\angle G(j\omega) \to -90^\circ$ as $\omega \to \infty$





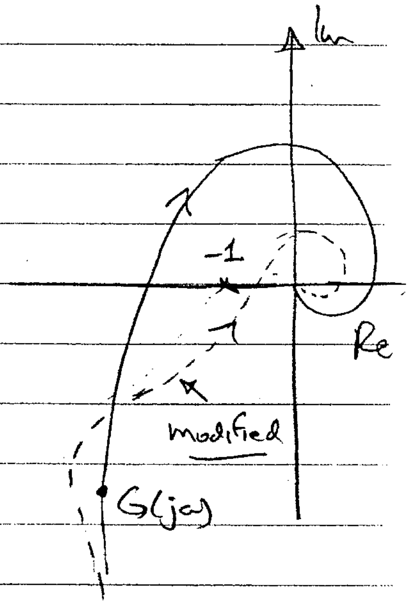
Intercept frequencies ω_1 and ω_2 are given by:

$270^\circ - 4\tan^{-1}\omega_1 = 180 \Rightarrow \omega_1 = \tan 22.5^\circ = 0.4142135 \Rightarrow |G(j\omega_1)| = 2.414$

$270^\circ - 4\tan^{-1}\omega_2 = 90 \Rightarrow \omega_2 = \tan 45^\circ = 1 \Rightarrow |G(j\omega_2)| = 1.$

There are two encirclements of $-1+0j$. So the closed loop

14 system is unstable.



Phase lag compensation can be used to stabilize the system, by reducing the high frequency gain, while not significantly increasing the high frequency phase lag. This can be achieved by choosing $b/a << 2.414$

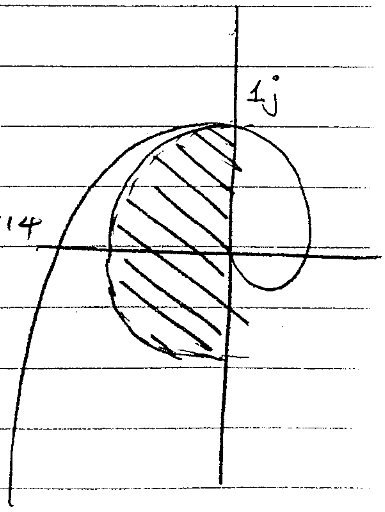
3 and $a << 0.4142135$

Phase advance compensation increases forward path gain at all frequencies, and ~~can~~ increase phase by at most 90°. But from intercept calculation,

$$|G(j\omega)| > 1$$

for any frequency ω such that $\measuredangle G(j\omega) < -180^\circ + 90^\circ$. Hence phase advance compensation cannot shift

3 the Nyquist diagram into the shaded half disc, and so cannot be stabilizing.

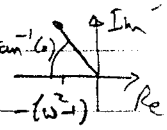


2 $G(j\omega) = \dfrac{0.1}{j\omega(0.2j\omega + (1-\omega^2))}$. For $\omega < 1$, $\measuredangle G^{-1} = +90^\circ + \tan^{-1}(0.2\omega/(1-\omega^2))$

Hence $\measuredangle G_2 = -\measuredangle G^{-1} = \underset{\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim}{-90^\circ - \tan^{-1}(0.2\omega/(1-\omega^2))}$. For $\omega > 1$,

$\measuredangle G^{-1}(j\omega) = \angle j\omega + \angle\{-(\omega^2-1) + 0.2j\omega\} = 90^\circ + 180^\circ - \tan^{-1}(0.2/(\omega^2-1))$

Then $\measuredangle G(j\omega) = -\measuredangle G^{-1} = \underset{\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim}{-270^\circ + \tan^{-1}(0.2\omega/(\omega^2-1))}$



$D(s)$ has maximum phase advance at $\omega = 2$ if

$\sqrt{ab} = 2$ — (1)

$\omega = 2$ is cross-over freq. and $\phi = 45^\circ$ implies

$\underbrace{\left(90 - 2\tan^{-1}\sqrt{\tfrac{a}{b}}\right)}_{\text{maximum phase adv.}} \times 2 + \measuredangle D(j2) = -180^\circ + 45^\circ$.

$G(j2) = \dfrac{0.1}{2j(0.4j-3)}$. So $\measuredangle G(j2) = -270 + \tan^{-1}\dfrac{0.4}{3} = -262.40566$

and $|G_p(j2)| = \dfrac{0.1}{2\sqrt{0.4^2+3^2}} = 0.01652046$

Hence

$90 - 2\tan^{-1}\sqrt{\tfrac{a}{b}} = \tfrac{1}{2}\left[262.40536 - 180^\circ + 45^\circ\right] = 63.70268^\circ$

$\Rightarrow a/b = \tan^2(13.14866) \Rightarrow a/b = 0.054570237 6$

From (1), $0.0545702376\, b^2 = 4 \Rightarrow b = 8.561536553$ & $a = 0.4672050839$

To arrange that $\omega = 2$ is the gain crossover frequency, we require:

$1 = G(j2)D(j2) = 0.01652046 \cdot K \dfrac{\left|\left(\frac{2}{0.4672}\right)^2 + 1\right|}{\left|\left(\frac{2}{8.5654}\right)^2 + 1\right|}$

$\Rightarrow K = \dfrac{1}{0.01652046} \times \dfrac{1.054570284}{19.32503748} = 3.303983323$

We should choose

16 $\underset{\sim\sim\sim\sim\sim\sim\sim\sim}{K = 3.303983323}$, $\underset{\sim\sim\sim\sim\sim\sim\sim\sim}{a = 0.4672050839}$, $\underset{\sim\sim\sim\sim\sim\sim\sim\sim}{b = 8.561536553}$

For $r = 0$ and $d(s) = 1/s$,

$c(s) = \dfrac{1}{1 + \dfrac{K(s/a+1)^2}{(s/b+1)^2} \cdot \dfrac{0.1}{s(s^2+0.2s+1)}} \cdot \dfrac{0.1}{s(s^2+0.2s+1)} \cdot \dfrac{1}{s}$

$\lim_{t\to\infty} c(t) = \lim_{s\downarrow 0} sC(s) = \lim_{s\downarrow 0} \cancel{s} \cdot \dfrac{(s/b+1)^2 \cancel{s}(s^2+0.2s+1)}{(s/b+1)^2 s(s^2+0.2s+1) + K(s/a+1)^2 \times 0.1} \cdot \dfrac{0.1}{\cancel{s}(s^2+0.2s+1)} \cdot \cancel{\dfrac{1}{s}}$

$= 1/K$

We require $c(t=\infty) < 0.5$, hence $K > 2$

4 We have shown that our design also satisfies the first specification

3. Consider trolley. Resolving horizontally gives

$$M\ddot{x} = -T\sin\theta - u \cong -mg\theta - u,$$

since $T \simeq mg$ and $\theta \simeq \sin\theta$. Also from "geometry" of mechanism

$$(y - x) \cong \theta L \quad \text{—— } (*)$$

Consider pendulum. Resolving horizontally gives

$$m\ddot{y} = T\sin\theta \simeq mg\theta \Rightarrow \ddot{y} = g\theta$$

But $\ddot{\theta} = \frac{1}{L}(\ddot{y} - \ddot{x})$ (from $(*)$). So

$$\ddot{\theta} = \tfrac{1}{L}\left[+g\theta + \left(\tfrac{m}{M}\right) g\theta \right] + \tfrac{1}{LM} u$$

$$= \frac{g(1+m/M)}{L}\theta + \frac{1}{LM} u.$$

Set $x_1 = y/g$, $x_2 = \dot{y}/g$, $x_3 = \theta$ and $x_4 = \dot{\theta}$. Then these equations can be expressed as

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & d_1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ d_2 \end{pmatrix} u$$

14 in which $d_1 = (g/L)(1 + m/M)$ and $d_2 = (1/LM)$

If $u = -k_1 y - k_2\dot{y} - k_3\theta - k_4\dot{\theta}$

$$= -k_1 g x_1 - k_2 g x_2 - k_3 x_3 - k_4 x_4,$$

the closed loop characteristic polynomial becomes

$$\tau(s) = s^4 + d_2 k_1 g s^3 + d_2 k_2 g s^2 + (d_2 k_3 - d_1)s + d_2 k_4$$

To obtain desired characteristic polynomial, choose

$$k_1 = \frac{\alpha_3}{d_2 g} = \frac{LM\alpha_3}{g}, \quad k_2 = \frac{\alpha_2}{d_2 g} = \frac{LM\alpha_2}{g}$$

$$k_3 = \frac{d_1 + \alpha_1}{d_2} = [(g/L)(1 + m/M) + \alpha_1]LM$$

$$k_4 = \frac{\alpha_0}{d_2} = LM\alpha_0$$

6

4/6

4. A state observer with gain g has the structure

$$\underbrace{\dot{\hat{x}} = A\hat{x} + g[y - c^T\hat{x}]}, \quad \underbrace{\hat{x}(0) = \hat{x}_0}$$

Since $\dot{x} = Ax$, the error equation (for $e = x - \hat{x}$) is

$$\frac{d}{dt}(x - \hat{x}) = A(x - \hat{x}) - g[c^T x - c^T\hat{x}]$$

or $\frac{d}{dt} e = Ae + gc^T e \Rightarrow \underbrace{\begin{cases} \frac{d}{dt}e = (A - gc^T)e. \\ e(0) = x(0) - \hat{x}_0 \end{cases}}$

4

$$d(s) \longrightarrow \boxed{1/s^2} \longrightarrow y(s)$$

A state space model is $\underbrace{\begin{cases} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} d, \\ y = [1 \;\; 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \end{cases}}$

Here, $x_1 = y$ and $x_2 = \dot{y}$.

Regarding d as a state variable $x_3 = d$, we have
$\dot{x}_1 = x_2$, $\dot{x}_2 = x_3$, $\dot{x}_3 = 0$ and $y = x_1$. i.e.

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad y = (1 \;\; 0 \;\; 0) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

A state observer $\hat{d}$ for D is

$$\begin{pmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \\ \dot{\hat{x}}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{pmatrix} - \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} [1 \;\; 0 \;\; 0], \quad \hat{d}(t) = (0 \, 0 \, 1) \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{pmatrix}$$

A

The observer "poles" are the roots of

$$\det[sI - A + gc^T] = \det \begin{bmatrix} s+g_1 & -1 & 0 \\ g_2 & s & -1 \\ g_3 & 0 & s \end{bmatrix}$$

$$= (s + g_1)s^2 + (g_2 s + g_3) = s^3 + g_1 s^2 + g_2 s + g_3$$

We require all poles are at -1 i.e.

$$\det[sI - A + gc^T] = (s+1)^3 = s^3 + 3s^2 + 3s + 1$$

Matching coefficients, we require

16

$$\underbrace{g_1 = 3, \quad g_2 = 3 \text{ and } g_3 = 1}$$

5 The optimal control problem is

$$\text{Minimize} \left\{ \int_0^\infty [|x_1 - z|^2 + \lambda |u|^2]\,dt : \dot{x}_1 = a x_1 + b_1 u,\ x_1(0) = \bar{x}_1 \right\}$$

Let $x_2 = z$. Then the state vector $x = (x_1, x_2)^T$ is governed by

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_1 & 0 \\ 0 & -\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ 0 \end{bmatrix} u$$

In terms of x_1 and x_2, the cost is

$$x_1^2 - 2x_1x_2 + x_2^2 = (x_1, x_2)^T \begin{bmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = q_{11}x_1^2 + 2q_{12}x_1x_2 + q_{11}x_2^2$$

Matching terms $\Rightarrow q_{11} = 1,\ q_{12} = -1,\ q_{22} = 1.$

We have reformulated the problem as a standard optimal control problem with

6 $$A = \begin{bmatrix} a_1 & 0 \\ 0 & \lambda \end{bmatrix},\ b = \begin{bmatrix} b_1 \\ 0 \end{bmatrix},\ Q = \begin{bmatrix} 1 & -1 \\ -1 & +1 \end{bmatrix}$$

The Riccati equation for $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}$ is

$$A^T P + PA + Q - \alpha^{-1} P b b^T P = 0 \Rightarrow$$

$$\begin{bmatrix} a_1 & 0 \\ 0 & -\lambda \end{bmatrix}\begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} + \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}\begin{bmatrix} a_1 & 0 \\ 0 & -\lambda \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ -1 & +1 \end{bmatrix} - \alpha^{-1}\begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}\begin{bmatrix} P_{11}b_1^2 + P_{12}b_1^2 \\ 0 & 0 \end{bmatrix}$$

Matching (1,1) coeffs $\Rightarrow$

$$a_1 P_{11} + P_{11} a_1 + 1 - \alpha^{-1} P_{11}^2 b_1^2 = 0 \qquad \text{— (1)}$$

Matching (1,2) coeffs $\Rightarrow$

$$(a_1 - \lambda) P_{12} - 1 - \alpha^{-1} P_{11} P_{12} b_1^2 = 0$$

We see that

$$P_{12} = [(a - \alpha^{-1} P_{11} b_1^2) - \lambda]^{-1} \qquad \text{— (2)}$$

Relevant entries of P are given by (1) and (2) (choose +ve sol$^{\underline{n}}$ of (1))

By standard theory, the optimal control is given by

$$u = -\lambda^{-1} b^T P x \quad \text{or}$$

$$u = -[b_1\ 0]\begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ z \end{bmatrix}$$

4 $\Rightarrow$ $$u(t) = -\alpha^{-1} b_1 \left(P_{11} x_1(t) + P_{12} \bar{z} e^{-\lambda t} \right).$$

6. $V_{in} \to \boxed{NL} \to V_{out}$. When $V_{in} = A\sin\omega t$,

$$V_{out} = A^3 \sin^3 \omega t = \frac{A^3}{4}\left[3\sin\omega t - \sin 3\omega t\right]$$

On the right side, we can interpret $\frac{A^3}{4} \cdot 3\sin\omega t$ as the fundamental oscillation and $-A^3/4 \sin 3\omega t$ as the third harmonic. In describing function analysis, we discard the higher harmonics. So

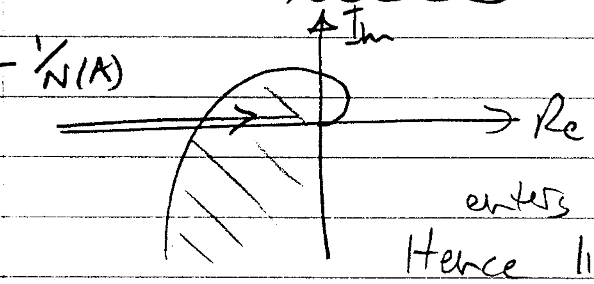
6 describing function $N(A) = \frac{3}{4}A^3/A = 3A^2/4$.

(a) The limit cycle condition is $G(j\bar{\omega}) = -1/N(\bar{A})$, where $\bar{\omega}$ and $\bar{A}$ are the freq. and amplitude (at input to NL). Hence

$$\frac{1}{G(j\bar{\omega})} = s^4 + 3s^3 + 3s^2 + s\,\Big|_{s=j\bar{\omega}} = (\bar{\omega}^4 - 3\bar{\omega}^2) + j(-3\bar{\omega}^3 + \bar{\omega}) = -N(\bar{A})$$

$$\Rightarrow \bar{\omega}\left[1 - 3\bar{\omega}^2\right] = 0 \quad \text{and} \quad 3\bar{\omega}^2 - \omega^4 = 3\bar{A}^2/4$$

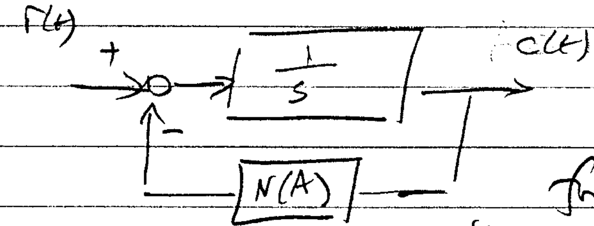
We have $\bar{\omega} = 1/\sqrt{3}$ and $\bar{A} = \sqrt{4/3 \cdot 8/9} = \left(\frac{2}{3}\right)^{3/2}$



$-1/N(A) = -\frac{4}{3A^2}$. As A increases $-1/N(A)$ enters unstable (shaded) region.

8 Hence limit cycle is unstable

(c)



According to describing function analysis we regard NL as having an (amplitude dependent) gain. "Linear analysis" tells us, if

$$r(t) = R\sin\omega t$$

then $c(t) = A\sin(\omega t + \phi)$, where

$$A = \left|\frac{1/s}{1 + N(A)\cdot 1/s}\right|_{s=j\omega} R$$

$$= \left|\frac{1}{j\omega + N(A)}\right| R = \frac{1}{(\omega^2 + N^2(A))^{1/2}} R$$

6 Hence $A^2\omega^2 + (9/16)A^6 = R^2$

The graph of A against ω is: