UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 2002

MSci Honours Degree in Mathematics and Computer Science Part IV
MEng Honours Degrees in Computing Part IV
MSc in Advanced Computing
for Internal Students of the Imperial College of Science, Technology and Medicine

*This paper is also taken for the relevant examinations for the Associateship of the City and Guilds of London Institute*
*This paper is also taken for the relevant examinations for the Associateship of the Royal College of Science*

# PAPER C473

# DOMAIN THEORY AND EXACT COMPUTATION

Wednesday 8 May 2002, 10:00
Duration: 120 minutes

*Answer THREE questions*

Paper contains 4 questions
Calculators required

1a Show that every continuous function on a cpo has a least fixed point.

b Let $f : D \to E$ be a monotone map between cpo's $D$ and $E$. Prove that if $f$ is onto (i.e. $\forall y \in E \, \exists x \in D.\ f(x) = y$) and if it reflects the order (i.e. $\forall x, x' \in D.\ f(x) \sqsubseteq f(x') \Rightarrow x \sqsubseteq x'$), then $f$ is continuous.

c Let $D$ be a poset and **2** the two-element poset $\{\bot, \top\}$ with $\bot \sqsubseteq \top$. Show that the poset of monotone maps from $D$ to **2** ordered pointwise (i.e., $f \sqsubseteq g \overset{\text{def}}{\iff} \forall x \in D.\ f(x) \sqsubseteq g(x)$) is isomorphic with the poset of upper subsets of $D$ ordered by subset inclusion. (A subset $A \subseteq D$ is an *upper subset* by definition if $\forall x, x' \in D.\ x \in A \;\&\; x \sqsubseteq x' \;\Rightarrow\; x' \in D$.)

*The three parts carry, respectively, 25%, 25%, 50% of the marks.*

 

2 Consider the two maps $f_1, f_2 : \mathbb{R}^2 \to \mathbb{R}^2$ given at a point $(x, y) \in \mathbb{R}^2$ by $f_1(z) = ce^{i\pi/2}z$ and $f_2(z) = ce^{-i\pi/2}z$, where $z = x + iy$ is the complex number representing $(x, y)$ and $c \in \mathbb{R}$ is a constant with $0 < c < 1$. Let $\mathcal{C}(\mathbb{R}^2)$ be the collection of all non-empty closed bounded subsets of $\mathbb{R}^2$ equipped with its Hausdorff distance function. We define the map $f : \mathcal{C}(\mathbb{R}^2) \to \mathcal{C}(\mathbb{R}^2)$ by

$$f(A) = K \cup f_1(A) \cup f_2(A),$$

where $K = \{(0, y) \in \mathbb{R}^2 \mid 0 \leq y \leq 1\}$ and $f_1(A)$ and $f_2(A)$ are the images of $A$ under $f_1$ and $f_2$ respectively.

a Show that $f$ is contracting with respect to the Hausdorff distance with a contracting factor of $c$.

b Obtain $A^*$ the attractor (fixed point) of $f$ and sketch its geometric structure.

c Develop a labelling scheme for the points of the attractor of $f$ using the unit interval as a reference point. Which points have more than one address?

d Find a rectangle with sides parallel to $x$ and $y$ axes which is mapped into itself by $f$.

*The four parts carry, respectively, 30%, 30%, 30%, 10% of the marks.*

3 Consider the framework for exact real number computation given by the action of linear fractional transformations (lft's) on the extended real line $\mathbb{R}^* = \mathbb{R} \cup \{\infty\}$ identified via the stereographic projection with the unit circle in the plane, with the closed interval $[0, \infty]$ as the base interval.

a Define the notions of signed and unsigned normal products, and the information of a matrix (lft with one argument) and that of a tensor (lft with two arguments).

b Assume that $f$ and $g$ are lft's with one argument and that $f$ has an inverse. Prove that $f[0, \infty] \supseteq g[0, \infty]$ if and only if $g = f \circ h$ where $h$ is an lft which has a representation with nonnegative coefficients. State clearly any result which you may use in your proof.

c The *golden ratio*, $\gamma$, is the positive root of $x^2 + x = 1$. By writing this equation as $x = 1/(1 + x)$ obtain a continued fraction and an unsigned normal product for $\gamma$. Evaluate, from the unsigned normal product, the first four finite rational intervals approximating $\gamma$.

d Describe the *emission* rule for extracting a matrix from a tensor.

e An expression tree for arctan is given by

$$\arctan(x) =$$
$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \left[ x, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 4 & 0 & 0 & 3 \end{pmatrix} \left[ x, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 9 & 0 & 0 & 5 \end{pmatrix} [x, \cdots] \right] \right]$$

Compute the sequence of emitted digit matrices from the expression tree for $\arctan(\gamma)$ when the first four matrices in the unsigned normal product for $\gamma$ are given as an input approximation for $\gamma$ and the emitted digit matrices are restricted to the set of three matrices:

$$D_{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} \quad D_0 = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \quad D_1 = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}.$$

*The five parts carry, respectively, 10%, 20%, 10%, 10%, 50%, of the marks.*

4a (i) Define the domain $\mathbf{S}([0,1]^n)$ of geometric (or solid) objects of the unit cube $[0,1]^n \subset \mathbb{R}^n$.

(ii) Define the continuous membership predicate of a given geometric object of this domain and prove that it is in fact continuous.

(iii) Define the binary Boolean operations for union and intersection on $\mathbf{S}([0,1]^n)$.

b Show that the extended membership predicate defined as:
$- \in - : \mathbf{I}[0,1]^n \times \mathbf{S}[0,1]^n \to \{\mathrm{tt}, \mathrm{ff}\}_\perp$ with:

$$C \in (A, B) = \begin{cases} \mathrm{tt} & \text{if } C \subseteq A \\ \mathrm{ff} & \text{if } C \subseteq B \\ \perp & \text{otherwise} \end{cases},$$

is continuous as a map of cpo's, stating clearly any result that you may use in your proof. (Note that $\mathbf{I}[0,1]^n$ is the cpo of the closed $n$-rectangles in $[0,1]^n$ ordered by reverse inclusion.)

c Consider the domain equation

$$D \cong F(D) = (\mathbb{N}_\perp \otimes D)_\perp$$

in the category **CPO**.
Find, up to isomorphism, the iterates $D_n = F^n(\{\perp\})$ for $n = 0, 1, 2, 3$ and the corresponding embedding-projection pairs $(e_n, p_n) : D_n \triangleleft D_{n+1}$ for $n = 0, 1, 2$.
Obtain the solution of the domain equation up to isomorphism.

*The three parts carry, respectively, 25%, 35%, 40% of the marks.*