1. This question carries 40% of the mark.

(a) Consider each of the following continuous-time signals. For each case, specify if the signal is causal/non-causal, periodic/non-periodic, odd/even. If the signal is periodic specify its period.

(i) $x(t) = 2\sin(2\pi t)$ **[2]**

Most students answered most of the question correctly.
Typical mistakes were:
-To answer that the signal is causal instead of non-causal.
-To answer that $T = 2\pi$ instead of $T = 1$.

(ii) $x(t) = \begin{cases} 3e^{-2t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$ **[2]**

Most students got this answer correct. There wasn't any repeated pattern in the mistakes.

(b) Consider the signal

$$x(t) = \begin{cases} 1-t, & 0 \leq t \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Now sketch each of the following signals and describe briefly in words how each of the signals can be derived from the original signal $x(t)$.

(i) $x\left(\frac{t}{3}+1\right)$ **[2]**

(ii) $x(-2t+1)$ **[2]**

Most students got these answers correct. There wasn't any repeated pattern in the mistakes.

(c) Consider the continuous-time Linear Time-Invariant (LTI) system with input $x(t)$ and output $y(t)$. This system is called a moving average filter.

$$y(t) = \int_{t-1}^{t} x(s)ds$$

(i) Find the impulse response $h(t)$ of the system, expressing it compactly as a function. Sketch the impulse response. **[2]**

Approximately half of the students answered this question correctly. A common mistake was to give the Delta function as the answer to this question.

(ii) Find the output when $x(t) = u(t)$ (the continuous-time unit step function) by performing the continuous-time convolution $y(t) = x(t) * h(t)$. Check that the output is indeed the output expected from the moving average filter defined above. Sketch the output. **[4]**

Approximately half of the students answered this question correctly. The students who gave the Delta function as the answer to the question above, gave a rectangular pulse or the unit step function as the answer to this question.

(d) (i) Consider a continuous-time function $x(t)$. Show that if the Fourier Transform of $x(t)$ is $\mathcal{F}\{x(t)\} = X(\omega)$ then $\mathcal{F}\{x(t)e^{j\omega_0 t}\} = X(\omega - \omega_0)$. **[2]**

(ii) Show that $\mathcal{F}\{x(t)\cos(\omega_0 t)\} = \frac{1}{2}[X(\omega - \omega_0) + X(\omega + \omega_0)]$. **[2]**

Most students got the above answers correct. There wasn't any repeated pattern in the mistakes.

(iii) Determine the Fourier Transform of $x(t) = e^{-at}\cos(\omega_0 t)u(t)$, $a > 0$ and sketch its amplitude response. The function $u(t)$ is the unit step function. **[2]**

A substantial amount of students got this answer correct. There wasn't any repeated pattern in the mistakes.

(e) The output $y(t)$ of a continuous-time LTI system is related to the input $x(t)$ by the differential equation

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 4y(t) = 4x(t)$$

Determine the frequency response of the system and sketch the asymptotic behavior of its Bode plots. **[5]**

Most students provided correct figures for the Bode plots. However, very few students provided a formal theoretical analysis related to the derivation of these plots.

(f) Consider the Laplace Transform of the impulse response of an LTI system $H(s)$ which is assumed to have one of its real zeros located to the right of the imaginary axis at $s = \gamma$. This zero is reflected through the $j\omega$-axis, whereas all poles and the rest of the zeros remain unchanged. This procedure results to a new system with transfer function $H_1(s) = H(s)H_0(s)$. Determine the function $H_0(s)$, its amplitude response and its phase response. **[5]**

Not many students answered fully this question.
Some students found the correct form of the function $H_0(s)$, but they did not provide the forms for the amplitude and the phase of it.

(g) Two continuous-time signals $x_1(t)$ and $x_2(t)$ are multiplied and the product $x(t)$ is sampled by a periodic impulse train. Both $x_1(t)$ and $x_2(t)$ are band-limited so that

$$X_1(\omega) = 0, \omega \geq 2\pi B_1$$
$$X_2(\omega) = 0, \omega \geq 2\pi B_2$$

where $X_i(\omega), i = 1,2$ is the Fourier transform of $x_i(t)$. Determine the maximum sampling period T_s that will allow perfect reconstruction of $x(t)$ from its samples. **[5]**

A substantial amount of students answered this question correctly.
Typical mistakes were:
-To give an answer with reversed inequality.
-To miss the fact that $T_s = \frac{1}{f_s}$ and consider that $T_s = \frac{1}{\omega_s}$.

(h) Consider the discrete-time, causal LTI system with input $x[n]$ and output $y[n]$ related with the difference equation:

$$y[n] - y[n-1] + \frac{1}{4}y[n-2] = x[n] - \frac{1}{2}x[n-1]$$

(i) Find the analytical expression and the Region of Convergence (ROC) of the z −transform of the impulse response of the above system.
[Hint: Use the fact that the z-transform $\frac{z}{z-a}$ corresponds to the function $a^n u[n]$ if $|z| > |a|$ and the function $-a^n u[-n-1]$ if $|z| < |a|$. The function $u[n]$ is the discrete-time unit step function.] **[3]**

Most students got this answer correct. There wasn't any repeated pattern in the mistakes. However, a substantial amount of students did not reach the simplest version of the transfer function because they didn't simplify the fractional transfer function.

(ii) Find the analytical expression and the Region of Convergence (ROC) of the z −transform of the output if $x[n] = \left(\frac{1}{2}\right)^n u[n]$. **[2]**

Strangely, a large amount of students provided the z −transform of $x[n]$ instead.

2. This question carries 30% of the mark.

(a) (i) Find the analytical expression and the Region of Convergence (ROC) of the Laplace transform of the continuous-time, causal signal $x(t) = e^{-at}u(t)$, with a real and positive and $u(t)$ the continuous-time unit step function. **[3]**

Most students got the answer for the analytical expression correct. There wasn't any repeated pattern in the mistakes.

A large amount of students got the answer for the ROC correct.

However, a substantial amount of students did not prove the analytical expression and the ROC; instead they just provided those.

(ii) Find the analytical expression and the Region of Convergence (ROC) of the Laplace transform of the continuous-time, anti-causal signal $x(t) = -e^{-at}u(-t)$, with a real and positive and $u(t)$ the continuous-time unit step function. **[3]**

Same comments as above are valid here.

(iii) Is the analytical expression of the Laplace transform of a signal sufficient to determine the analytical expression of the signal in time? Justify your answer. **[3]**

Most students got the answer for this question correct. There wasn't any repeated pattern in the mistakes.

(b) (i) Consider a continuous-time Linear Time-Invariant (LTI) system. Prove that the response of the system to a complex exponential input $e^{s_0 t}$ is the same complex exponential with only a change in amplitude; that is $H(s_0)e^{s_0 t}$. The function $H(s)$ is the Laplace transform of the impulse response of the system. **[5]**

Most students got the answer for this question correct. There wasn't any repeated pattern in the mistakes.

(ii) A causal LTI system with impulse response $h(t)$ has the following properties:

1. The impulse response $h(t)$ satisfies the equation:
$$h(t) = ae^{-3t}u(t) + be^{-2t}u(t)$$
where a, b are unknown constants.
2. When the input to the system is $x(t) = e^{t}$ for all t, the output is $y(t) = \frac{11}{12}e^{t}$.
3. When the input to the system is $x(t) = e^{2t}$ for all t, the output is $y(t) = \frac{7}{10}e^{2t}$.

Determine the transfer function $H(s) = \mathcal{L}\{h(t)\}$ of the system, consistent with the information above. The constants a, b should not appear in your answer. **[6]**

Most students got the answer for this question correct. There wasn't any repeated pattern in the mistakes.

(c) The output $y(t)$ of an LTI system is related to the input $x(t)$ by the differential equation
$$\frac{d^2y(t)}{dt^2} - \frac{dy(t)}{dt} - 2y(t) = x(t)$$
Let $X(s)$ and $Y(s)$ denote the Laplace transforms of $x(t)$ and $y(t)$, respectively, and let $H(s)$ denote the Laplace transform of the system's impulse response $h(t)$.

(i) Determine $H(s)$ as a ratio of two polynomials. **[3]**

Most students got this answer correct. There wasn't any repeated pattern in the mistakes. However, a substantial amount of students did not reach the simplest version of the transfer function because they didn't simplify the fractional transfer function.

(ii) Determine $h(t)$ for each of the following cases:

1. The system is stable.

2. The system is causal.
3. The system is neither stable nor causal. **[7]**

Not many students got this question completely correct. There was a lot of confusion in the attempt to find the various ROCs. Furthermore, very few students showed that one of the two cases where the system is both non-stable and non-causal is not valid due to the non-existence of a ROC.

3. This question carries 30% of the mark.

(a) Consider a continuous-time, band-limited signal $x(t)$, limited to bandwidth $|\omega| \leq 2\pi \times 10^3$rad/sec. We sample $x(t)$ uniformly with sampling frequency $f_s = 1/T_s = 5 \times 10^3$Hz to obtain the discrete-time signal $x[n] = x(nT_s)$. In reconstructing the continuous-time signal from its samples, we use a Digital-to-Analogue Converter which outputs the waveform

$$x_{DA}(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\Pi\left(\frac{t - nT_s}{0.2 \times 10^{-3}}\right)$$

with

$$\Pi(t) = \begin{cases} 1 & |t| < 0.5 \\ 0.5 & |t| = 0.5 \\ 0 & \text{otherwise} \end{cases}$$

(i) Show that $x_{DA}(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\left[\delta(t - nT_s) * \Pi\left(\frac{t}{0.2\times10^{-3}}\right)\right]$ with $\delta(t)$ the Dirac function. The symbol "$*$" denotes the operation of convolution. **[2]**

Most students got this answer correct. There wasn't any repeated pattern in the mistakes.

(ii) Find the Fourier Transform of the signal $x_{DA}(t)$.

[Hint: Use the fact that the Fourier transform of the function $\sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$ is $\frac{1}{T_s}\sum_{n=-\infty}^{\infty} X\left(\omega - n\frac{2\pi}{T_s}\right)$.] **[4]**

A substantial amount of students got this answer almost correct. A common mistake was to give wrong parameters for the rectangular function involved.

(iii) Derive the frequency response, $H(\omega)$, of the filter (system) through which $x_{DA}(t)$ must be passed in order to perfectly reconstruct the signal $x(t)$. **[6]**

Very few students got this answer correct. A common mistake was to give an answer where the rectangular pulse involved was in the numerator instead of the denominator.

(b) (i) Show that the z −transform of the discrete causal signal $x[n + 1]u[n]$ is $z(X(z) - x(0))$, where $X(z)$ is the z −transform of the discrete causal signal $x[n]$. **[5]**

Most students got this answer correct. There wasn't any repeated pattern in the mistakes.

(ii) Consider the discrete signals $x_1(n) = 2^n$ and $x_2(n) = 3^n$ for $n \geq 0$. Find their convolution using their z −transforms and properties of convolution.

[Hint: Use the result of (b)(i) above and the fact that $x_1(0) = x_2(0)$.] **[5]**

Most students got this answer correct but without providing a formal proof.

(c) Consider a discrete LTI system with input $x[n]$ and output $y[n]$ related by the difference equation

$$2y[n] - 9y[n-1] + 4y[n-2] = -14x[n-2]$$

Investigate whether the above system can be both stable and causal. Justify your answer. **[8]**

Most students got this answer correct. There wasn't any repeated pattern in the mistakes. However, a substantial amount of students did not reach the simplest version of the transfer function because they didn't simplify the fractional transfer function.

