EE4-26

SOLUTIONS: ESTIMATION AND FAULT DETECTION

1. Solution

a) According to the theorems on convergence of the Riccati recursive equation, for every value of a such that $|a| < 1$, irrespective of the value taken on by b, the sequence of Riccati matrices $P(t)$ eventually converges for increasing values of t to a positive-definite steady-state matrix $\overline{P}$.

For every value of a such that $|a| > 1$, two different cases have to be considered:

- $b \neq 0$: the sequence of Riccati matrices $P(t)$ eventually converges for increasing values of t to a positive-definite steady-state matrix $\overline{P}$.
- $b = 0$: the sequence of Riccati matrices $P(t)$ diverges and a one-step ahead steady-state Kalman predictor does not exist.

[4 Marks]

b) i) The general recursive Riccati equation is

$$P(t+1) = F\left[P(t) - P(t)H^\top\left(V_2 + HP(t)H^\top\right)^{-1}HP(t)\right]F^\top + V_1, \quad t = 1,2,\dots$$

with the initialisation $P(1) = \text{var}[x(1)]$. Correspondingly, the time-varying predictor gain is

$$K(t) = FP(t)H^\top\left(V_2 + HP(t)H^\top\right)^{-1}, \quad t = 1,2,\dots$$

Letting $F = 1/3$, $H = 3$, $V_1 = 4$, $V_2 = 1$, and using $\text{var}[x(1)] = 4$, one gets:

$$P(1) = 4,\ P(2) = 4.012012,\ P(3) = 4.0120129$$

and

$$K(1) = 0.1081081,\ K(2) = 0.108116,\ K(3) = 0.108116.$$

As the dynamic system is asymptotically stable (its eigenvalue $\lambda = 1/3$ lies inside the unit circle), and from the answer to Question 1a), the sequence $P(t)$, $t = 1,\dots$ converges to a positive-definite steady-state matrix $\overline{P}$.

[3 Marks]

ii) The general algebraic Riccati equation is

$$P = F\left[P - PH^\top\left(V_2 + HPH^\top\right)^{-1}HP\right]F^\top + V_1.$$

Letting again $F = 1/3$, $H = 3$, $V_1 = 4$, $V_2 = 1$, one gets:

$$P = \frac{1}{9}\left(P - 9\frac{1}{1+9P}P^2\right) + 4 \Longrightarrow 9P^2 - \frac{316}{9}P - 4 = 0$$

thus obtaining the two solutions

$$\overline{P}_1 \simeq -0.11 \quad \text{and} \quad \overline{P}_2 \simeq 4.0120129$$

Clearly, the only admissible solution is the positive one, and therefore

$$\overline{P} = \overline{P}_2 \simeq 4.0120129$$

Accordingly:

$$\overline{K} = F\overline{P}H^\top \left(V_2 + H\overline{P}H^\top\right)^{-1} \simeq 0.108116$$

Comparing the time-behavior of the sequences $P(t), t = 1,2,3$ and $K(t), t = 1,2,3$, determined in the answer to Question 1b)i), with the above steady-state values $\overline{P}$ and $\overline{K}$, one obtains

$$P(t) \simeq \overline{P} \quad \text{and} \quad K(t) \simeq \overline{K} \quad \text{for} \quad t \geq 3$$

and

$$K(t) \simeq \overline{K} \text{ for } t \geq 3$$

Therefore, with the exception of the first two initial stages, the time-varying quantities $P(t)$ and $K(t)$ are nearly the same as the steady-state values $\overline{P}$ and $\overline{K}$ in accordance with the answer to Question 1b)i).

[3 Marks]

iii) The stochastic process $e(\cdot)$ is stationary because the steady-state one-step ahead Kalman predictor is asymptotically stable according to the theorems on the convergence of the Riccati recursive equation when the dynamic system is aymptotically stable (see also the answer to Question 1a)).

Moreover

$$\mathbb{E}(e) = 0$$

due to the unbiasedness of the Bayes estimation in the Gaussian case.

Finally

$$\text{var}(e) = \overline{P} \simeq 4.0120129$$

[3 Marks]

c) i) The general algebraic Riccati equation is

$$P = F\left[P - PH^\top \left(V_2 + HPH^\top\right)^{-1} HP\right] F^\top + V_1$$

For the values $F = 1/3$, $H = 0$, $V_1 = 4$, $V_2 = 1$, one gets:

$$P = \frac{1}{9}P + 4 \Longrightarrow \widetilde{P} = 9/2 = 4.5$$

and

$$\widetilde{K} = F\widetilde{P}H^\top \left(V_2 + H\widetilde{P}H^\top\right)^{-1} = 0.$$

[3 Marks]

ii) Clearly, from the answer to the Question 1c)-i), $\mathrm{var}(\tilde{e})$ is given by

$$\mathrm{var}(\tilde{e}) = \tilde{P} = 4.5$$

A comparison with $\mathrm{var}(e)$ calculated in the answer to Question 1b-iii) gives

$$\mathrm{var}(\tilde{e}) = \tilde{P} = 4.5 > \mathrm{var}(e) = \overline{P} \simeq 4.0120129$$

The larger variance $\mathrm{var}(\tilde{e})$ is caused by the value $b = 0$ which prevents the use of the measurements $y(t)$ by the predictor to reduce the uncertainty on the state estimate (in fact $\tilde{K} = 0$ according to the answer to Question 1c)-i)).

[4 Marks]

2. Solution

a) For $t < T_0$, the fault $f(t)$ is not acting on the system ($f(t) = 0, \forall t < T_0$).

i) Assigning to the block MS_1 a state variable x_1 and to the block MS_2 a state variable x_2, one immediately gets:

$$\begin{cases} \dot{x}_1 = -10x_1 + 10u \\ \dot{x}_2 = -5x_2 + 5w = -5x_2 + 5x_1 \\ y = x_2 \end{cases}$$

and in matrix form:

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -10 & 0 \\ 5 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 10 \\ 0 \end{bmatrix} u \\ \\ y = [0\ 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{cases}$$

[3 marks]

ii) The Luenberger observer takes on the form:

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}) \\ \hat{y} = C\hat{x} \end{cases}$$

where

$$A = \begin{bmatrix} -10 & 0 \\ 5 & -5 \end{bmatrix}; \quad B = \begin{bmatrix} 10 \\ 0 \end{bmatrix}; \quad C = [0\ 1]$$

are the matrices of the state-space description obtained in the answer to Question 2a)-i) which describe the dynamics of the system depicted in Fig. 2.1 of the text of the exam paper, before the occurrence of the fault $f(t)$.

L denotes the observer gain matrix to be designed. The pair (A, C) is observable as

$$\det \begin{bmatrix} 0 & 1 \\ 5 & -5 \end{bmatrix} = -5$$

We let

$$F = A - LC = \begin{bmatrix} -10 & -l_1 \\ 5 & -5 - l_2 \end{bmatrix} \quad \text{where} \quad L = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}$$

After some algebra, it is easy to see that, by selecting

$$L = \begin{bmatrix} 4 \\ -4 \end{bmatrix}$$

we have $\det(sI - F) = s^2 + 11s + 30$ and hence $\lambda_1 = -5, \lambda_2 = -6$.

Finally

$$e(t) = \mathscr{L}^{-1}[(sI - F)^{-1}]\tilde{e} = \mathscr{L}^{-1} \begin{bmatrix} \dfrac{s+1}{s^2+11s+30} & -\dfrac{6}{s^2+11s+30} \\ \dfrac{5}{s^2+11s+30} & \dfrac{s+10}{s^2+11s+30} \end{bmatrix} \begin{bmatrix} \tilde{e}_1 \\ \tilde{e}_2 \end{bmatrix}$$

and hence, after some algebra and the computation of the inverse Laplace transforms, we obtain

$$\begin{aligned}\varepsilon(t) &= Ce(t) = \mathscr{L}^{-1}\left[\frac{5}{s^2+11s+30}\right]\tilde{e}_1 + \mathscr{L}^{-1}\left[\frac{s+10}{s^2+11s+30}\right]\tilde{e}_2 \\ &= 5(\tilde{e}_1+\tilde{e}_2)e^{-5t} - (5\tilde{e}_1+4\tilde{e}_2)e^{-6t}, \ \forall t \in (0, T_0)\end{aligned}$$

[6 marks]

b) For $t \geq T_0$ the following vibration fault affects the dynamics of the system depicted in Fig. 2.1 in the text of the exam paper. The fault is given in equation (2.1) in the text of the exam paper:

$$f(t) = K\sin(t), \forall t \geq T_0$$

where $K > 0$ is a positive scalar.

i) The vibration fault $f(t)$ can described by the following state equation $\forall t \geq T_0$:

$$\begin{cases} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \\ \\ f = [0\ 1] \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \end{cases}$$

with $\begin{bmatrix} z_1(0^-) \\ z_2(0^-) \end{bmatrix} = \begin{bmatrix} K \\ 0 \end{bmatrix}$. Consider the state equation determined in the answer to Question 2a)i) and define the augmented state vector

$$x_a := \begin{bmatrix} x_1 \\ x_2 \\ z_1 \\ z_2 \end{bmatrix}$$

Thus, the following augmented state equations can be written:

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -10 & 0 & 0 & 0 \\ 5 & -5 & 0 & 5 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 10 \\ 0 \\ 0 \\ 0 \end{bmatrix} u \\ \\ y = [0\ 1\ 0\ 0] \begin{bmatrix} x_1 \\ x_2 \\ z_1 \\ z_2 \end{bmatrix} \end{cases}$$

If a full-order Luenberger observer scheme can be designed, then an asymptotic estimate $\begin{bmatrix} \widehat{x}_1 \\ \widehat{x}_2 \\ \widehat{z}_1 \\ \widehat{z}_2 \end{bmatrix}$ of the augmented state $\begin{bmatrix} x_1 \\ x_2 \\ z_1 \\ z_2 \end{bmatrix}$ can be computed and hence an estimate $\widehat{f}(t)$ of the fault $f(t)$ can be obtained as $\widehat{f}(t) = \widehat{z}_2(t)$.

[6 marks]

ii) Referring to the augmented state equation defined in the answer to Question 2b)-i), the observability matrix is given by

$$Q = \begin{bmatrix} C_a \\ C_a A_a \\ C_a A_a^2 \\ C_a A_a^3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 5 & -5 & 0 & 5 \\ -75 & 25 & 5 & -25 \\ 875 & -125 & -25 & 120 \end{bmatrix}$$

where

$$A_a = \begin{bmatrix} -10 & 0 & 0 & 0 \\ 5 & -5 & 0 & 5 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}; \quad C_a = [0\,1\,0\,0]$$

Since

$$\det Q = 12625$$

the augmented system is observable. Consequently, the initial value $\begin{bmatrix} x_1(0^-) \\ x_2(0^-) \\ z_1(0^-) \\ z_2(0^-) \end{bmatrix}$ of the augmented state can be determined from the knowledge of the output function $y(t), t \geq T_0$. The unknown amplitude K of the vibration fault is simply given by the first component $x_1(0^-)$ of the initial value of the augmented state.

[5 marks]

3. Solution

a) The following state variables can be defined:

$$z_1 := r_x; \quad z_2 := \dot{r}_x; \quad z_3 := r_y; \quad z_4 := \dot{r}_y$$

Moreover, from the text of Question 3a), the output variables are given by:

$$w_1 = z_1; \quad w_1 = z_3$$

Since the velocity of the vehicle is constant, it follows immediately that

$$\dot{z}_1 = z_2; \quad \dot{z}_2 = 0; \quad \dot{z}_3 = z_4; \quad \dot{z}_4 = 0$$

Then, a state-space description in matrix form of the movement of the vehicle M on the $x-y$ plane is given by:

$$\begin{cases} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = A \cdot \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} \\ \\ \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = C \cdot \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} \end{cases}$$

[5 marks]

b) From Fig. 3.1 reported in Question 3 in the text of the exam paper and from the assumptions $v \neq 0$, $r_x(t) > 0, \forall t \geq 0$ and $r_y(t) > 0, \forall t \geq 0$, it is immediately obtained

$$\vartheta = \arctan \frac{\dot{r}_y}{\dot{r}_x} = \arctan \frac{z_4}{z_2}$$

Therefore, referring to the state equations determined in the answer to Question 3a), a fourth-order Luenberger observer architecture can be devised which, under appropriate conditions, is able to provide an asymptotic estimate $\hat{z} = [\hat{z}_1 \; \hat{z}_2 \; \hat{z}_3 \; \hat{z}_4]^\top$ of the state $z = [z_1 \; z_2 \; z_3 \; z_4]^\top$.

The estimate $\hat{\vartheta}$ of the angle ϑ can then be calculated as

$$\hat{\vartheta} = \arctan \frac{\hat{z}_4}{\hat{z}_2}$$

The above state observer can be designed if

$$\text{rank}(Q) = 4$$

where $Q \in \Re^{8 \times 4}$ is the observability matrix

$$Q = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix}$$

Since

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad CA = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

it follows that $\text{rank}(Q) = 4$ and hence the asymptotic Luenberger observer can be designed.

[7 marks]

c) Since the positions r_x and r_y of the vehicle are independently measured by two different sensors, by inspecting the state and output equations determined in the answer to Question 3a), we can immediately conclude that two *independent* state and output equations can be written as:

$$\begin{cases} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = A_x \cdot \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \\ w_1 = C_x \cdot \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \end{cases}$$

and

$$\begin{cases} \begin{bmatrix} \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = A_y \cdot \begin{bmatrix} z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z_3 \\ z_4 \end{bmatrix} \\ w_2 = C_y \cdot \begin{bmatrix} z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} z_3 \\ z_4 \end{bmatrix} \end{cases}$$

Therefore, two separate Luenberger observers can be designed providing asymptotic estimates $\begin{bmatrix} \widehat{z}_1 \\ \widehat{z}_2 \end{bmatrix}$ and $\begin{bmatrix} \widehat{z}_3 \\ \widehat{z}_4 \end{bmatrix}$ of vectors $\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ and $\begin{bmatrix} z_3 \\ z_4 \end{bmatrix}$, respectively.

More specifically, the observer concerning the estimate of the state $\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ takes on the form:

$$\begin{cases} \begin{bmatrix} \dot{\widehat{z}}_1 \\ \dot{\widehat{z}}_2 \end{bmatrix} = A_x \begin{bmatrix} \widehat{z}_1 \\ \widehat{z}_2 \end{bmatrix} + L_x \left(w_1 - C_x \begin{bmatrix} \widehat{z}_1 \\ \widehat{z}_2 \end{bmatrix} \right) \\ \widehat{w}_1 = C_x \begin{bmatrix} \widehat{z}_1 \\ \widehat{z}_2 \end{bmatrix} \end{cases}$$

where

$$A_x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; \quad C_x = [1\ 0]$$

and L_x denotes the observer gain matrix to be designed. The pair (A_x, C_x) is completely observable as

$$\det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$$

which is, of course, consistent with the answer to Question 3b).

We let

$$F_x = A_x - L_x C_x = \begin{bmatrix} -l_x^{(1)} & 1 \\ -l_x^{(2)} & 0 \end{bmatrix} \quad \text{where} \quad L_x = \begin{bmatrix} l_x^{(1)} \\ l_x^{(2)} \end{bmatrix}$$

After some algebra, by selecting

$$L_x = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

we have $\det(sI - F_x) = s^2 + 4s + 4$ and hence $\lambda_1 = -2, \lambda_2 = -2$.

An analogous observer can be designed concerning the estimate of the state $\begin{bmatrix} z_3 \\ z_4 \end{bmatrix}$:

$$\begin{cases} \begin{bmatrix} \dot{\hat{z}}_3 \\ \dot{\hat{z}}_4 \end{bmatrix} = A_y \begin{bmatrix} \hat{z}_3 \\ \hat{z}_4 \end{bmatrix} + L_y \left(w_2 - C_y \begin{bmatrix} \hat{z}_3 \\ \hat{z}_4 \end{bmatrix} \right) \\ \hat{w}_2 = C_y \begin{bmatrix} \hat{z}_3 \\ \hat{z}_4 \end{bmatrix} \end{cases}$$

where

$$A_y = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; \quad C_y = [1\ 0]$$

and

$$L_y = \begin{bmatrix} 6 \\ 9 \end{bmatrix}$$

Clearly, $F_y = A_y - L_y C_y$ and $\det(sI - F_y) = s^2 + 6s + 9$ thus giving $\lambda_3 = -3, \lambda_4 = -3$. Finally, it immediately follows that

$$F = \text{blokdiag}\{F_x, F_y\}$$

[8 marks]

4. Solution

a) Referring to the block-diagram shown in Fig. 4.1 of the text of the exam paper, one assigns to the sub-system "Int. n. 1" a state variable denoted by x_1 and to the sub-system "Int. n. 2" a state variable denoted by x_2. Moreover, other two state variables x_3 and x_4 are assigned to the sub-system with second-order transfer function $\dfrac{s+1}{s^2+2s+2}$.

After inspection of the block-diagram shown in Fig. 4.1 of the text of the exam paper, we immediately have:

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = x_1 - x_2 \\ w = x_1 - x_2 \end{cases}$$

Moreover, using the observer canonical form for the block with transfer function $\dfrac{s+1}{s^2+2s+2}$, one obtains:

$$\begin{cases} \dot{x}_3 = -2x_4 + w \\ \dot{x}_4 = x_3 - 2x_4 + w \\ y = x_4 \end{cases}$$

Then, the matrix form of the state-space description of the whole interconnected system is given by:

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & -2 \\ 1 & -1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u \\ \\ y = [0\,0\,0\,1] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \end{cases}$$

[3 marks]

b) Using the standard rules for interconnected systems, and given that the transfer function of the integrator blocks is $\dfrac{1}{s}$, one immediately gets:

$$G_{uw}(s) = \frac{1}{s} \cdot \frac{1}{1+1/s} = \frac{1}{s+1}$$

and

$$G_{uy}(s) = G_{uw}(s) \cdot \frac{s+1}{s^2+2s+2} = \frac{1}{s^2+2s+2}$$

The order of the transfer function $G_{uy}(s)$ is 2 whereas the order of the state space realisation determined in the answer to Question 4a) is 4. This may be caused by a lack of observability from the output $y(t)$.

[3 marks]

c) The observability matrix is given by

$$Q = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & -1 & 1 & -2 \\ -2 & 2 & -2 & 2 \\ 2 & -2 & 2 & 0 \end{bmatrix}$$

Clearly

$$\text{rank}(Q) = 2.$$

Hence, we conclude that the system is not completely observable. Indeed, $\text{rank}(Q)$ coincides with the order of the transfer function $G_{uy}(s)$ computed in the answer to Question 4b).

[3 marks]

d) To answer Question 4d) of the exam paper, the observability canonical form has to be determined starting from the state-space description determined in the answer to Question 4a). In particular, recall that the observability matrix computed in the answer to Question 4c) is:

$$Q = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & -1 & 1 & -2 \\ -2 & 2 & -2 & 2 \\ 2 & -2 & 2 & 0 \end{bmatrix}$$

Since $\text{rank}(Q) = 2$, a basis $\{\alpha, \beta\}$ for $\ker(Q)$ can be determined computing a basis for the vector subspace of solutions of:

$$Qv = 0 \Longrightarrow \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & -1 & 1 & -2 \\ -2 & 2 & -2 & 2 \\ 2 & -2 & 2 & 0 \end{bmatrix} v = 0$$

A possible choice is:

$$\alpha = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \beta = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

A basis $\{\delta, \gamma\}$ for the orthogonal complement to $\ker(Q)$ has to be determined. After a simple algebra, we obtain, for example:

$$\delta = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}; \gamma = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -3 \end{bmatrix}$$

Now, select the matrix

$$T = [\gamma \,|\, \delta \,|\, \alpha \,|\, \beta] = \begin{bmatrix} 1 & 1 & -1 & 1 \\ -1 & -1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & -3 & 0 & 0 \end{bmatrix}$$

and compute the inverse

$$T^{-1} = \begin{bmatrix} 1/4 & -1/4 & 1/4 & 1/4 \\ 1/12 & -1/12 & 1/12 & -1/4 \\ -1/3 & 1/3 & 2/3 & 0 \\ 1/3 & 2/3 & 1/3 & 0 \end{bmatrix}$$

By setting $x = Tz$, the following equivalent observability canonical form is obtained:

$$\begin{cases} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = T^{-1}ATz + T^{-1}Bu = \begin{bmatrix} -1/4 & 15/4 & 0 & 0 \\ -5/12 & -7/4 & 0 & 0 \\ 2/3 & 6 & -1 & 0 \\ 4/3 & 4 & -1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} + \begin{bmatrix} 1/4 \\ 1/12 \\ -1/3 \\ 1/3 \end{bmatrix} u \\ \\ y = CTz = [1 \;\; -3 \; 0 \; 0] \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} \end{cases}$$

Upon the Inspection of the observability canonical form, it is immediately obvious that the sub-system described by the state variables z_1 and z_2 is observable whereas the sub-system described by the state variables z_3 and z_4 is not observable.

A basis for the non-observable subspace X_{no} is a basis for $\ker(Q)$, that is

$$\alpha = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \beta = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

[6 marks]

e) From the answer to Question 4d), it immediately follows that the eigenvalue $\lambda = 0$ (which corresponds to the dynamics of the block "Int. n. 1") is not observable. Therefore, the state x_1 does not influence the output $y(t)$ neither directly, nor indirectly. As a consequence, the initial state $x_1(0^-)$ cannot be determined by using the output function $y(t)$, $t \geq 0$.

[5 marks]