EE4-10
EE9-CS5-1
EE9-SC3
EE9-FPN2-02

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2018

MSc and EEE PART IV: MEng and ACGI

Corrected copy

PROBABILITY AND STOCHASTIC PROCESSES

Friday, 18 May 10:00 am

Time allowed: 3:00 hours

There are FOUR questions on this paper.

Answer ALL questions. All questions carry equal marks.

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : C. Ling
Second Marker(s) : D. Angeli

Information for students

Each of the four questions has 25 marks.

The Questions

1. Random variables.

a) Let X be a Gaussian random variable with zero mean and variance σ^2. Find the probability density function of random variable $Y = X^2$. [10]

b) Again, let X be a Gaussian random variable with zero mean and variance σ^2. Estimate the tail probability $P(|X| > a)$ where $a = 3\sigma$ using

i) Markov inequality; [4]

ii) Chebyshev inequality; [4]

iii) Chernoff bound; [4]

iv) Discuss your findings. [3]

Hint: $E[|X|] = \sqrt{\frac{2}{\pi}}\sigma$ for a Gaussian random variable.

2. Estimation.

a) The random variable X has the density $f(x) \sim c^4 x^3 e^{-cx}, \ x > 0$. We observe the i.i.d. samples $x_i = 3.7, 4.4, 4.3, 3.6$. Find the maximum-likelihood estimate of parameter c.

[10]

b) Consider a random process $Y(n)$ with autocorrelation function

$$R_Y(m) = \begin{cases} 3 - |m|, & |m| < 3 \\ 0, & |m| \geq 3 \end{cases}$$

Suppose we wish to predict $Y(n+1)$ from $Y(n)$, $Y(n-1)$, …, $Y(1)$ using a linear MMSE estimator

$$Y(n+1) = \sum_{i=1}^{n} c_i Y(i).$$

i) Find the coefficient and mean-square error of the first-order MMSE estimator, i.e., $n = 1$. [5]

ii) Find the coefficients and mean-square error of the second-order MMSE estimator, i.e., $n = 2$. [10]

3. Random processes.

a) Justify each of the 10 steps labelled (1), (2), …, (10) in the following derivation of the matched filter. The input is given by

$$r(t) = s(t) + w(t), \qquad 0 < t < t_0$$

where $w(t)$ is white noise with power spectral density N_0.

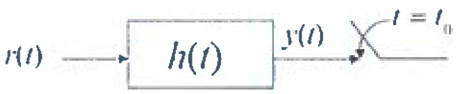

Fig. 3.1. Matched filter.

$$y(t) \overset{(1)}{=} y_s(t) + n(t) \quad \text{where } y_s(t) \overset{(2)}{=} s(t) * h(t), \qquad n(t) \overset{(3)}{=} w(t) * h(t),$$

$$(SNR)_0 = \frac{\text{Output signal power at } t = t_0}{\text{Average output noise power}} \overset{(4)}{=} \frac{|y_s(t_0)|^2}{E\{|n(t)|^2\}}$$

$$\overset{(5)}{=} \frac{|y_s(t_0)|^2}{\frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{nn}(\omega)d\omega} \overset{(6)}{=} \frac{\left|\frac{1}{2\pi}\int_{-\infty}^{+\infty} S(\omega)H(\omega)e^{j\omega t_0}d\omega\right|^2}{\frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{ww}(\omega)|H(\omega)|^2 d\omega}$$

$$\overset{(7)}{=} \frac{\left|\int_{-\infty}^{+\infty} S(\omega)H(\omega)e^{j\omega t_0}d\omega\right|^2}{2\pi N_0 \int_{-\infty}^{+\infty} |H(\omega)|^2 d\omega}$$

$$\overset{(8)}{\le} \frac{1}{2\pi N_0}\int_{-\infty}^{+\infty} |S(\omega)|^2 d\omega \overset{(9)}{=} \frac{\int_0^{+\infty} s(t)^2 dt}{N_0} \overset{(10)}{=} \frac{E_s}{N_0}$$

[12]

b) The number of tutees $N(t)$ arriving at a professor's office over the time interval $[0, t)$ can be modelled by a Poisson process $\{N(t), t \ge 0\}$. On average, there is a new tutee arriving after every 5 minutes, i.e., the intensity of the process is equal to $\lambda = 0.2$. The professor will not start the tutorial until at least 4 tutees are in the office.

i) Find the expected waiting time until the tutorial starts.

[3]

ii) What is the probability that the tutorial does not start in the first half an hour?

[5]

iii) What is the probability that at least one tutee arrives in the first 10 minutes while at most two tutees arrive in the second 10 minutes?

[5]

4. Markov chains and martingales.

a) Consider a symmetric random walk $S_0, S_1, S_2, \cdots, S_n, \cdots$ with $S_0 = 0$. There are two absorbing barriers $-a$ and b where a, b are positive integers. Suppose $0 < \lambda < \frac{\pi}{a+b}$ and $\cos\lambda \neq 0$.

i) Show that

$$X_n = \frac{\cos\{\lambda[S_n - \frac{1}{2}(b-a)]\}}{(\cos\lambda)^n}$$

forms a martingale. [5]

ii) Show that the stopping time T until absorption at one of the two barriers $-a$ and b satisfies

$$E[(\cos\lambda)^{-T}] = \frac{\cos\{\frac{1}{2}\lambda(b-a)\}}{\cos\{\frac{1}{2}\lambda(b+a)\}}$$

[10]

b) Calculate the stationary distribution for a Markov chain with state space $E = \{0, 1, 2, 3, \ldots\}$, whose only nonzero transitional probabilities are

$$p_{0,1} = 1$$

$$p_{i,0} = \frac{i}{i+1}, \qquad p_{i,i+1} = \frac{1}{i+1}, \quad i = 1,2,3, \ldots$$

[10]