UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 2001

BEng Honours Degree in Computing Part I
MEng Honours Degrees in Computing Part I
BSc Honours Degree in Mathematics and Computer Science Part I
MSci Honours Degree in Mathematics and Computer Science Part I
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the Associateship of the City and Guilds of London Institute
This paper is also taken for the relevant examinations for the Associateship of the Royal College of Science

PAPER C140=MC140

LOGIC

Wednesday 9 May 2001, 16:00
Duration: 90 minutes
(Reading time 5 minutes)

Answer THREE questions

Paper contains 4 questions
Calculators not required

1a Explain what it means to say that a propositional formula is

i) a clause,

ii) in disjunctive normal form,

iii) in conjunctive normal form.

Give an illustrative example in each case.

b Let p, q, r be propositional atoms.

i) Use propositional equivalences to show that the formulas $(p \rightarrow r) \wedge (q \rightarrow r)$ and $(p \vee q) \rightarrow r$ are logically equivalent. In each step of your argument, state the rule you use.

ii) Write down a formula in disjunctive normal form that is logically equivalent to $(p \rightarrow r) \wedge (q \rightarrow r)$.

iii) Write down a formula in conjunctive normal form that is logically equivalent to $(p \vee q) \rightarrow r$.

c The Senior Tutor suspects two students, Amir and Beth, of cheating on coursework. Under questioning,

Amir says "One or both of us cheated",
Beth says "If I cheated then so did Amir."

i) Using propositional atoms A, meaning "Amir cheated" and B, meaning "Beth cheated", express the statements of the students in logic.

ii) Assuming that both students told the truth, who must definitely have cheated? Explain your answer.

iii) Could both have lied? If so, who cheated? Explain your answer.

iv) Could the innocent have told the truth and the guilty lied? If so, who cheated? Explain your answer.

The three parts carry, respectively, 30%, 30%, 40% of the marks.

2a Let A, B be first-order formulas.

i) Explain what it means to say that A and B are logically equivalent.

ii) A is valid if and only if it is logically equivalent to $\top$. In a similar way, explain how to define "A is satisfiable" and "$A \vDash B$" in terms of logical equivalence.

b Let C(x) be a first-order formula and let D, F be first-order sentences.

i) Draw the formation trees of the four sentences

1. $(\exists x(C(x) \vee D)) \rightarrow F$
2. $\exists x((C(x) \vee D) \rightarrow F)$
3. $((\exists x C(x)) \vee D) \rightarrow F$
4. $(\exists x C(x)) \vee (D \rightarrow F)$

ii) Which of your trees in part b(i) is the formation tree of $\exists x C(x) \vee D \rightarrow F$?

c Let G(x) be a first-order formula, and let H be a first-order sentence. Using either direct argument or natural deduction, but not using equivalences, show that $\forall x(G(x) \rightarrow H)$ is logically equivalent to $\exists x G(x) \rightarrow H$.

[If you use natural deduction, you will have to show that $\forall x(G(x) \rightarrow H) \vdash \exists x G(x) \rightarrow H$ *and* that $\exists x G(x) \rightarrow H \vdash \forall x(G(x) \rightarrow H)$.]

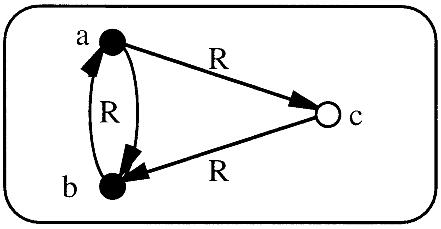
d Let P be a unary relation symbol, and let Q be a nullary relation symbol (a propositional atom). By describing appropriate structures, show that

i) $\forall x(P(x) \rightarrow Q)$ is not logically equivalent to $\forall x P(x) \rightarrow Q$,

ii) $\exists x P(x) \rightarrow Q$ is not logically equivalent to $\exists x(P(x) \rightarrow Q)$.

3a Let L be a signature consisting of a unary relation symbol P, a binary relation symbol R, and constants a, b, c. Let N be the L-structure shown below (only the black dots satisfy P; R(a,c) is true, R(b,c) false, etc).

N



Which of the following sentences are true in N? Give brief justification in each case.

i) $\exists x(P(x) \wedge \neg(x=a))$

ii) $\forall x \forall y(P(x) \wedge P(y)) \rightarrow \forall x \forall y R(x,y)$

iii) $\forall x \forall y(P(x) \wedge P(y) \rightarrow R(x,y))$

iv) $\forall x(P(x) \leftrightarrow \exists y(R(x,y) \wedge R(y,x)))$

v) $\forall x \forall y(P(x) \wedge P(y) \leftrightarrow \exists z(R(x,z) \wedge R(z,y)))$

b Now let L be the 2-sorted signature with sorts $\mathtt{Nat}$ and $[\mathtt{Nat}]$, containing constants $\underline{\mathbf{0}}, \underline{\mathbf{1}}, \underline{\mathbf{2}}, \ldots : \mathtt{Nat}$ and $[]:[\mathtt{Nat}]$, function symbols +, –, ×, :, ++, !!, #, and relation symbols <, ≤, and $\mathtt{merge}$, of appropriate sorts. The symbols have the usual meanings: for example, $\mathtt{merge}$(ys, zs, xs) holds iff xs is a permutation of ys++zs and the relative order of entries in ys and in zs is retained in xs.

Let xs, ys, zs be variables of sort $[\mathtt{Nat}]$, let i, j, k, n, x, y, z be variables of sort $\mathtt{Nat}$, and let in(x,xs) be a formula expressing that n occurs as an entry in xs.

Write down, in a short simple English sentence (not just a literal translation), what each of the following formulas says about the list xs. In each case, give an example of a non-empty list (xs) that satisfies the formula, and a list that does not.

i) $\forall i \forall j(i<j \wedge j<\#xs \rightarrow \neg\exists x \exists y \exists z(x>\underline{\mathbf{1}} \wedge xs!!i = x \times y \wedge xs!!j = x \times z))$.

ii) $\forall ys \forall zs(\mathtt{merge}(ys,zs,xs) \wedge ys \neq [] \wedge zs \neq [] \rightarrow \exists n(in(n,yz) \wedge in(n,zs)))$.

iii) $\exists ys \exists zs(\mathtt{merge}(ys,zs,xs) \wedge \exists x(\underline{\mathbf{2}} \times x=\#zs)$
$\wedge \forall y(in(y,ys) \rightarrow \exists x(\underline{\mathbf{2}} \times x=y)) \wedge \forall z(in(z,zs) \rightarrow \neg\exists x(\underline{\mathbf{2}} \times x=z)))$.

c In the notation of part b, write down L-formulas expressing each of the following properties of xs, ys, k, and n.

i) n occurs as an entry in xs (that is, in(n,xs), as in part b)

ii) xs is ordered (sorted)

iii) xs contains exactly k entries that are equal to n [hint: use $\mathtt{merge}$]

iv) some number occurs in xs exactly twice

v) xs is non-empty and ys contains precisely the minimal entries in xs (eg: xs = [4,6,4,5], ys = [4,4]).

4a State clearly the natural deduction rules for introduction and elimination of $\vee$ ("or"). Justify the soundness of each rule briefly.

b Let A, B be arbitrary propositional formulas. Prove by natural deduction that:

i) $A \vee B,\ \neg A \ \vdash B$,

ii) $\neg A \rightarrow B \ \vdash \ A \vee B$.

You may employ the lemma $X \vee \neg X$, for suitable X, without proving it.

c The following rule $\vee E^*$ has been proposed as an alternative to the rule $\vee E$:

	…	(formulas)
1.	$A \vee B$	we proved this somehow …
	...	(more formulas)
2.	$\neg A$	… we proved this too …
3.	B	$\vee E^*(1,2)$

That is, once we have derived $A \vee B$ and $\neg A$, we can derive B from them by the rule $\vee E^*$.

For formulas X, Y, we write $X \vdash^* Y$ if Y can be proved from X by natural deduction but with the rule $\vee E$ replaced by $\vee E^*$.

Show that for any formulas A, B, C, if $A \vdash^* C$ and $B \vdash^* C$ then $A \vee B \vdash^* C$.
Hint: it may help to use the rule PC ("proof by contradiction").

The three parts carry, respectively, 20%, 45%, 35% of the marks.