

OPTIMIZATION -

SOLUTIONS - 2003

Ex 1

$$(a) \quad \frac{\partial f}{\partial x_1} = 6x_1^2 - 6x_1 - 12x_1x_2 + 6x_2^2 + 12x_2$$

$$\frac{\partial f}{\partial x_2} = -6x_1^2 + 12x_1x_2 - 6x_1 = -6x_1[x_1 - 2x_2 - 2]$$

$$\text{Stationary points: } \frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = 0$$

$$\begin{aligned} \frac{\partial f}{\partial x_2} = 0 & \begin{cases} \rightarrow x_1 = 0 \rightarrow 6x_2(x_2 + 2) = 0 \begin{cases} P_1 = (0, 0) \\ P_2 = (0, -2) \end{cases} \\ \rightarrow x_1 - 2x_2 - 2 = 0 \end{cases} \\ & \downarrow \\ & x_1 = 2x_2 + 2 \\ & \downarrow \\ & (2x_2 + 2)^2 - (2x_2 + 2) - 2(2x_2 + 2)x_2 + x_2(x_2 + 2) = 0 \\ & \downarrow \\ & x_2^2 + 4x_2 + 3 = 0 \begin{cases} \rightarrow x_2 = -2 + \sqrt{2} \\ \rightarrow x_2 = -2 - \sqrt{2} \end{cases} \\ & \downarrow \\ & P_3 = (-2 + 2\sqrt{2}, -2 + \sqrt{2}) \\ & P_4 = (-2 - 2\sqrt{2}, -2 - \sqrt{2}) \end{aligned}$$

(b)

$$\nabla^2 f(P_1) = \begin{bmatrix} -6 & 12 \\ 12 & 0 \end{bmatrix}$$

$$\nabla^2 f(P_2) = \begin{bmatrix} 18 & -12 \\ -12 & 0 \end{bmatrix}$$

 P_1 is a saddle point P_2 is a saddle point

(2)

$$\nabla' f(p_3) \approx \begin{bmatrix} 10.9 & -4.9 \\ -4.9 & 9.9 \end{bmatrix} > 0 \quad p_3 \text{ is a local MIN}$$

$$\nabla' f(p_4) \approx \begin{bmatrix} -22.9 & 28.9 \\ 28.9 & -57.9 \end{bmatrix} < 0 \quad p_4 \text{ is a local MAX}$$

(c)

$$(*) \quad p_{k+1} = \begin{bmatrix} x_1^{k+1} \\ x_2^{k+1} \end{bmatrix} = \begin{bmatrix} x_1^k \\ x_2^k \end{bmatrix} - \alpha \begin{bmatrix} 6(x_1^k)^2 - 6x_1^k - 12x_1^k x_2^k + 6(x_2^k)^2 + 12x_2^k \\ -6x_1^k(x_1^k - 2x_2^k - 2) \end{bmatrix}$$

(d) Linear approx of (*) close to p_3

$$p_{k+1} = \begin{bmatrix} 1 + 6\alpha - 12\alpha\sqrt{2} & 12\alpha(\sqrt{2}-1) \\ 12\alpha(\sqrt{2}-1) & 1 + 24\alpha - 24\alpha\sqrt{2} \end{bmatrix} p_k$$

Try, e.g. $\alpha = \frac{1}{10}$,

$$p_{k+1} \approx \begin{bmatrix} -0.1 & 0.5 \\ 0.5 & 0.06 \end{bmatrix} p_k$$



$$\text{Eig} = \{-0.54, 0.45\}$$

The gradient algorithm defines a stable iteration for $\alpha > 0$ small and locally around a MIN.

(a) Consider the system of eqs

$$F(x) = 0, \quad x \in \mathbb{R}^n, \quad F: \mathbb{R}^n \rightarrow \mathbb{R}^n$$

If the Jacobian of F exists and it is continuous then

$$F(x+s) = F(x) + \frac{\partial F}{\partial x}(x)s + \delta(x,s)$$

With

$$\lim_{\|s\| \rightarrow 0} \frac{\delta(x,s)}{\|s\|} = 0.$$

Hence, given x_k we compute $x_{k+1} = x_k + s$

with $s = -\left[\frac{\partial F}{\partial x}(x_k)\right]^{-1} F(x_k)$, if the inverse exists.

This yields the Newton iteration

$$x_{k+1} = x_k - \left[\frac{\partial F}{\partial x}(x_k)\right]^{-1} F(x_k)$$

(b) $F(x) = x^2 + 2bx + c$

$$\frac{\partial F}{\partial x} = 2x + 2b$$

$$\Rightarrow x_{k+1} = x_k - \frac{x_k^2 + 2bx_k + c}{2(x_k + b)}$$

↓

$$x_{k+1} = \frac{x_k^2 - c}{2(x_k + b)} \quad (*)$$

(4)

Note that

$$x_{k+1} + b = \frac{x_k^2 - c}{2(x_k + b)} + b = \frac{x_k^2 - c + 2bx_k + 2b^2}{2(x_k + b)}$$

hence

$$x_{k+1} + b = \frac{\overset{>0}{(x_k + b)^2} + \overset{>0}{b^2 - c}}{2(x_k + b)}$$

So

$$x_k + b > 0 \longrightarrow x_{k+1} + b > 0 \quad \text{for all } k$$

$$x_k + b < 0 \longrightarrow x_{k+1} + b < 0 \quad \text{for all } k$$

(c) Set $b = 0$, $c = -3$ in (a):

$$x_{k+1} = \frac{x_k^2 + 3}{2x_k}$$

$$x_0 = 1$$

$$x_1 = 2$$

$$x_2 = 7/4$$

$$x_3 = 1.73214$$

$$x_4 = 1.732050810 \quad (\sqrt{3} = 1.732050808)$$

$$\frac{x_4 - \sqrt{3}}{\sqrt{3}} = 0.1 \cdot 10^{-8}$$

$$(a) \quad \mathcal{L} = x'x + 2d'x + \lambda(x'x - a^2)$$

$$\frac{\partial \mathcal{L}}{\partial x} = 2x + 2d + 2\lambda x = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = x'x - a^2 = 0$$

$$(b) \quad \frac{\partial \mathcal{L}}{\partial x} = 0 \rightarrow x = -\frac{d}{1+\lambda}$$

↓

$$x'x = a^2 \rightarrow \frac{d'd}{(1+\lambda)^2} = a^2$$

$$\rightarrow 1+\lambda = \pm \frac{1}{a} \sqrt{d'd}$$

$$\lambda = -1 \pm \frac{1}{a} \sqrt{d'd}$$

The candidate optimal solutions are

$$x_1^* = -\frac{a d}{\sqrt{d'd}}$$

$$x_2^* = \frac{a d}{\sqrt{d'd}}$$

$$(\lambda_1^* = -1 + \sqrt{d'd}/a)$$

$$(\lambda_2^* = -1 - \sqrt{d'd}/a)$$

$$\nabla_{xx}' \mathcal{L} = 2(\lambda + 1)I$$

hence x_1^* is a local min

x_2^* is a local max

(c) If $x_1 = a \cos \vartheta$
 $x_2 = a \sin \vartheta$

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then $x'x = x_1^2 + x_2^2 = a^2$

$$x'x + 2d_1'x = a^2 + 2d_1 \cos \vartheta + 2d_2 \sin \vartheta$$

Hence the constrained optimization problem is now

$$\min_{\vartheta} a^2 + 2d_1 \cos \vartheta + 2d_2 \sin \vartheta = f(\vartheta)$$

The stationary points are such that

$$\frac{\partial f}{\partial \vartheta} = -2d_1 \sin \vartheta + 2d_2 \cos \vartheta = 0$$

↓

$$d_1 \sin \vartheta = d_2 \cos \vartheta$$

$$\vartheta = \arctan(d_2/d_1)$$

and we have two candidate solutions as in (b).

Ex 4

(7)

$$(0) \quad L = -x_1 - x_2 + p(x_1^2 + x_2^2 - 1)$$

$$\frac{\partial L}{\partial x_1} = -1 + 2p x_1 = 0$$

$$\frac{\partial L}{\partial x_2} = -1 + 2p x_2 = 0$$

$$-1 + x_1^2 + x_2^2 \leq 0$$

$$p \geq 0$$

$$p \cdot (-1 + x_1^2 + x_2^2) = 0$$

Nec. Cond

$$(*) \quad s' \begin{bmatrix} 2p & 0 \\ 0 & 2p \end{bmatrix} s > 0 \quad \forall s \neq 0$$

such that

$$\begin{bmatrix} 2x_1 & 2x_2 \end{bmatrix} s = 0$$

Suff. Cond

Note that if $p > 0$, then $(*)$ holds for any s , if $p = 0$ then $(*)$ does not hold for all s .

$$\text{Note that } \nabla_x (x_1^2 + x_2^2 - 1) = [2x_1, 2x_2]$$

and this is always non-zero when $x_1^2 + x_2^2 - 1 = 0 \Rightarrow$ All points are regular.

From the Nec Cond.

(8)

either $\rho = 0 \longrightarrow -1 = 0$ False

or $\rho > 0 \longrightarrow -1 + x_1^2 + x_2^2 = 0$, i.e. all
candidate solutions are
on the boundary.

Note now that

$$x_1 = x_2 = \frac{1}{\sqrt{\rho}} \longrightarrow x_1^2 + x_2^2 = 1 = \frac{1}{\rho} + \frac{1}{\rho} = 1$$

↓

$$\rho = \pm \sqrt{\frac{1}{2}} \quad \text{but}$$

$$\rho > 0 \implies \rho = \sqrt{\frac{1}{2}}$$

↓

$$p_1 = (x_1, x_2) = \left(\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}\right)$$

p_1 is the optimal solution as also the
Suff. Cond. are satisfied.

(b) Define the sequential penalty function as

$$F_\varepsilon = -x_1 - x_2 + \frac{1}{\varepsilon} \left[\max(0, -1 + x_1^2 + x_2^2) \right]^2$$

(c)

$$\text{If } x_1^2 + x_2^2 - 1 \leq 0 \rightarrow F_\epsilon = -x_1 - x_2$$

$$\downarrow$$

$$\nabla F_\epsilon = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \neq 0 \quad \text{No stationary points}$$

$$\text{If } x_1^2 + x_2^2 - 1 > 0 \rightarrow F_\epsilon = -x_1 - x_2 + \frac{(x_1^2 + x_2^2 - 1)^2}{\epsilon}$$

$$\downarrow$$

$$\nabla F_\epsilon = \begin{bmatrix} -1 + \frac{4(x_1^2 + x_2^2 - 1)x_1}{\epsilon} \\ -1 + \frac{4(x_1^2 + x_2^2 - 1)x_2}{\epsilon} \end{bmatrix}$$

$$\nabla F_\epsilon = 0 \Rightarrow \begin{aligned} (x_1^2 + x_2^2 - 1)x_1 &= \frac{\epsilon}{4} \\ (x_1^2 + x_2^2 - 1)x_2 &= \frac{\epsilon}{4} \end{aligned} \Rightarrow x_1 = x_2 = \xi$$

$$\Downarrow$$

$$(2\xi^2 - 1)\xi = \frac{\epsilon}{4}$$

if ϵ is small

$$\xi \approx 0 \quad \xi \approx \pm \sqrt{\frac{1}{2}}$$

Not in
 $x_1^2 + x_2^2 - 1 > 0$

We obtain two candidate optimal solutions

$$P_2 = \left(\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}} \right)$$

$$P_3 = \left(-\sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}} \right)$$

Note that as $\epsilon \rightarrow 0$ P_2 approaches the exact optimal solution.

(a) Note that

$$x_1 = Ax_0 + Bu_0$$

$$x_2 = Ax_1 + Bu_1 = A^2x_0 + ABu_0 + Bu_1$$

$$x_n = A^n x_0 + A^{n-1}Bu_0 + A^{n-2}Bu_1 + \dots + Bu_{n-1}$$

$$\begin{cases} \min \frac{1}{2} (u_0^2 + \dots + u_{n-1}^2) \\ A^n x_0 + \dots + Bu_{n-1} = 0 \end{cases}$$

(b) If $n = n$ then

$$x_n = x_n = A^n x_0 + [A^{n-1}B, \dots, B] \begin{bmatrix} u_0 \\ \vdots \\ u_{n-1} \end{bmatrix} = 0$$

$$= A^n x_0 + C U = 0$$

Hence the unique solution is

$$U = -C^{-1}A^n x_0.$$

(c) If $n = n+1$ then

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$$\begin{aligned} x_{n+1} = x_n &= A^{n+1} x_0 + A^n B u_0 + A^{n-1} B u_1 + \dots + B u_n \\ &= A^{n+1} x_0 + A^n B u_0 + G \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \end{aligned}$$

hence $x_{n+1} = 0$ implies

$$\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = -G^{-1} [A^{n+1} x_0 + A^n B u_0] = F x_0 + G u_0$$

This means that u_1, \dots, u_n are functions of x_0 and u_0 . The problem is not recast as

$$\min_{u_0} \frac{1}{2} \left(u_0^2 + [u_1 \dots u_n] \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \right)$$

\Downarrow

$$\min_{u_0} \frac{1}{2} \left(u_0^2 + (F x_0 + G u_0)' (F x_0 + G u_0) \right)$$

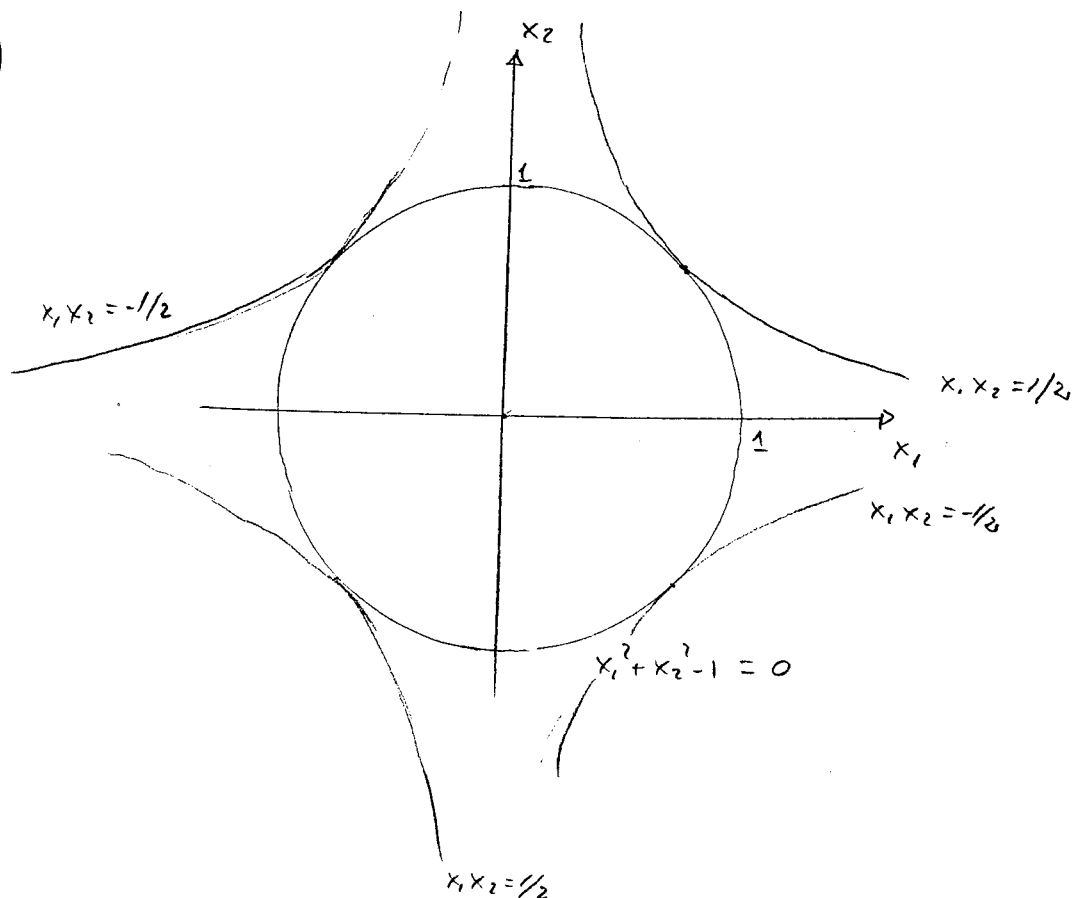
\Downarrow

$$\min_{u_0} \frac{1}{2} \left[u_0^2 (1 + G'G) + 2u_0 G'F x_0 + x_0^T F'F x_0 \right]$$

(d)
$$J = \frac{1}{2} (u_0^2 (1 + G'G) + 2u_0 G'F x_0 + x_0^T F'F x_0)$$

$$\frac{\partial J}{\partial u_0} = u_0 (1 + G'G) + G'F x_0 \Rightarrow u_0^* = - \frac{G'F x_0}{1 + G'G}$$

(a)



$$(b) \quad \mathcal{L}_a = x_1 x_2 + \lambda (x_1^2 + x_2^2 - 1) + \frac{1}{\varepsilon} (x_1^2 + x_2^2 - 1)^2$$

$$\text{with } \lambda = -\frac{x_1 x_2}{x_1^2 + x_2^2}$$

$$(c) \quad \frac{\partial \mathcal{L}_a}{\partial x_1} = \frac{\partial \mathcal{L}_a}{\partial x_2} = 0 \quad \begin{array}{l} \rightarrow p_1 = \left(\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}} \right) \\ \rightarrow p_2 = \left(-\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}} \right) \\ \rightarrow p_3 = \left(\sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}} \right) \\ \rightarrow p_4 = \left(-\sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}} \right) \end{array}$$

$$f(p_1) = \frac{1}{2}$$

$$f(p_2) = -\frac{1}{2}$$

$$f(p_3) = -\frac{1}{2}$$

$$f(p_4) = \frac{1}{2}$$

$$\nabla^2 L_0(p_2) = \begin{bmatrix} 1 + 4/\epsilon & 1 - 4/\epsilon \\ 1 - 4/\epsilon & 1 + 4/\epsilon \end{bmatrix} > 0$$

$$\nabla^2 L_0(p_3) = \nabla^2 L_0(p_2) > 0$$

p_2 and p_3 are local minima.

$$(d) \quad L = x_1 x_2 + \lambda (x_1^2 + x_2^2 - 1)$$

$$\frac{\partial L}{\partial x_1} = x_2 + 2\lambda x_1 \implies \lambda^* = -\frac{x_2}{2x_1} = \frac{1}{2}$$