Modelling and control of multibody mechanical systems
Sample exam paper - Model answers

Question 1

a) Setting $v_1 = v_2 = 0$ yields

$$\ddot{x} = 0 \qquad \ddot{z} = -g \qquad \theta = 0.$$

Let $q = (x, z, \theta)$ and $M = I$, hence $p = (\dot{x}, \dot{z}, \dot{\theta})$. Then

$$\dot{q} = p \qquad \dot{p} = \begin{bmatrix} 0 \\ -g \\ 0 \end{bmatrix}.$$

The above system is a simple Hamiltonian system with internal Hamiltonian

$$H_0(q, p) = \frac{1}{2}p'p + gz.$$

b) Note that

$$\dot{H}_0 = p'\dot{p} + g\dot{z} = -\dot{z}g + g\dot{z} = 0.$$

c) The system can be written as

$$\dot{q} = p \qquad \dot{p} = \begin{bmatrix} 0 \\ -g \\ 0 \end{bmatrix} + \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix} v_1 + \begin{bmatrix} \epsilon\cos\theta \\ \epsilon\sin\theta \\ 1 \end{bmatrix} v_2.$$

The system can be written as a simple Hamiltonian system provided there exist $H_1(q)$ and $H_2(q)$ such that

$$\left(\frac{\partial H_1}{\partial q}\right)' = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix}$$

and

$$\left(\frac{\partial H_2}{\partial q}\right)' = \begin{bmatrix} \epsilon\cos\theta \\ \epsilon\sin\theta \\ 1 \end{bmatrix}.$$

From the first equation we have that

$$\frac{\partial H_1}{\partial \theta} = 0$$

implying that $H_1(q)$ is not a function of θ. However, this contradicts the equation

$$\frac{\partial H_1}{\partial x} = \sin\theta.$$

As a result the system cannot be written as a simple Hamiltonian system.

d) Setting $\dot{q} = \dot{p} = 0$ yields $p = 0$ and

$$0 = -\sin\theta v_1 + \epsilon\cos\theta v_2 \qquad 0 = \cos\theta v_1 + \epsilon\sin\theta v_2 - g \qquad 0 = v_2.$$

This implies $v_2 = 0$ and

$$0 = -\sin\theta v_1 \qquad 0 = \cos\theta v_1 - g.$$

Note that we do not have any constraint on x and z. As a result, $\theta = 0 + 2k\pi$, yielding $v_1 = g$, or $\theta = \pi + 2k\pi$, yielding $v_1 = -g$, with k integer. In summary we have two types of equilibria:

$$q = (\star, \star, 0) \qquad p = (0, 0, 0) \qquad (v_1, v_2) = (g, 0)$$

and

$$q = (\star, \star, \pi) \qquad p = (0, 0, 0) \qquad (v_1, v_2) = (-g, 0).$$

Both equilibria correspond to the aircraft 'floating' at some fixed location with zero roll angle or with roll angle equal to π, *i.e.* the aircraft is upside down. The corresponding input signals are such that the aircraft does not roll: $v_2 = 0$, and the effect of gravity is compensated: $v_1 = \pm g$.

e) From the above discussion we have that $q = p = 0$ is an equilibrium associated to $v_1 = g$ and $v_2 = 0$.

f) The system linearised around the zero equilibrium is described by

$$\begin{bmatrix} \dot{\delta}_q \\ \dot{\delta}_q \end{bmatrix} = \begin{bmatrix} 0 & I \\ -G & 0 \end{bmatrix} \begin{bmatrix} \delta_q \\ \delta_q \end{bmatrix} + \begin{bmatrix} 0 \\ B \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

with

$$G = \begin{bmatrix} 0 & 0 & -g \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 0 & \epsilon \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The zero equilibrium may be stabilised by a state feedback control law if the reduced system

$$\dot{x} = Gx + Bu$$

is controllable. The controllability matrix of this system is

$$C = \begin{bmatrix} B & GB & G^2B \end{bmatrix} = \begin{bmatrix} 0 & \epsilon & 0 & -g & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and this has rank three, hence the system is controllable.
Finally, the zero equilibrium may be stabilised by an output feedback control law if the system with output $y = q$ is observable. A submatrix of the observability matrix is

$$\tilde{O} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

which has rank six, hence the system is observable.

Question 2

a) The Hamiltonian equations of motion are

$$\dot{q} = p(1 + \alpha q^2) \qquad\qquad \dot{p} = -\alpha q p^2 - q + q^{n-1} + u.$$

b) The equilibria of the system for $u = 0$ are obtained solving the equation

$$-q + q^{n-1} = q(-1 + q^{n-2}) = 0.$$

Since $n > 2$, and even, we have three solutions: $q = 0$, $q = 1$ and $q = -1$. Therefore, for $u = 0$ we have three equilibria

$$(q, p) = (0, 0) \qquad\qquad (q, p) = (1, 0) \qquad\qquad (q, p) = (-1, 0).$$

c) The equilibrium $(0, 0)$ is associated to a local minimum of the potential energy, hence it is a stable equilibrium. The other two equilibria are associated to local maxima of the potential energy, hence they are unstable equilibria.

d) The linearised system around the $(0, 0)$ equilibrium is described by the equation

$$\dot{\delta}_x = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \delta_x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \delta_u.$$

Setting $\delta = K\delta_x$, with $K = [K_1, K_2]$ yields a closed-loop system which is described by

$$\dot{\delta}_x = \begin{bmatrix} 0 & 1 \\ -1 + K_1 & K_2 \end{bmatrix} \delta_x.$$

This system is asymptotically stable if $K_2 < 0$ and $-1 + K_1 < 0$.

e) To obtain a control law which globally asymptotically stabilises the zero equilibrium using the shaping function method we first have to modify the potential energy of the system. To this end, consider a modified Hamiltonian

$$H_m(q, p) = H_0(q, p) - qu + V_d(q),$$

with $V_d(q)$ such that

$$\frac{1}{2}q^2 - \frac{1}{n}q^n + V_d(q)$$

has a global minimum at $q = 0$. For example, select $V_d(q) = \frac{1}{n}q^n$. Note now that

$$\dot{H}_0 + \dot{V}_d = p(1 + \alpha q^2)(u + q^{n-1}).$$

Hence, selecting

$$u = -q^{n-1} - p$$

yields

$$\dot{H}_0 + \dot{V}_d = -p^2.$$

This implies that p converges to zero. The system restricted to the set $p = 0$ is described by

$$\dot{q} = 0 \qquad\qquad 0 = \dot{p} = -q,$$

which implies that the equilibrium $(0, 0)$ is globally asymptotically stable.

Question 3

a) The position vector of the centre of mass is given by

$$\boldsymbol{r} = x'\boldsymbol{i'} + y'\boldsymbol{j'},$$

and therefore the velocity vector by differentiation is

$$\dot{\boldsymbol{r}} = (\dot{x}' - y'\dot{\psi})\boldsymbol{i'} + (\dot{y}' + x'\dot{\psi})\boldsymbol{j'}.$$

Hence the kinetic energy of the system is

$$T = \frac{1}{2}m\left((\dot{x}' - y'\dot{\psi})^2 + (\dot{y}' + x'\dot{\psi})^2\right) + \frac{1}{2}I_{zz}\dot{\psi}^2 + \frac{1}{2}I_{yy}\dot{\theta}^2.$$

b) The equations of the rolling constraint are

$$\dot{x}' - y'\dot{\psi} + a\dot{\theta} \quad = \quad 0, \tag{1}$$
$$\dot{y}' + x'\dot{\psi} = 0. \tag{2}$$

c) The Lagrangian function is $L = T - V = T$. The Lagrangian equation for the generalised coordinate x' is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}'}\right) - \frac{\partial L}{\partial x'} + \lambda_1 = 0,$$

or

$$\frac{d}{dt}\left(m(\dot{x}' - y'\dot{\psi})\right) - m(\dot{y}' + x'\dot{\psi})\dot{\psi} + \lambda_1 = 0,$$

or by making use of Equations 1, 2,

$$\frac{d}{dt}\left(-ma\dot{\theta}\right) + \lambda_1 = 0,$$

or

$$\lambda_1 = ma\ddot{\theta}.$$

The Lagrangian equation for the generalised coordinate y' is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}'}\right) - \frac{\partial L}{\partial y'} + \lambda_2 = 0,$$

or

$$\frac{d}{dt}\left(m(\dot{y}' + x'\dot{\psi})\right) + m(\dot{x}' - y'\dot{\psi})\dot{\psi} + \lambda_2 = 0,$$

or by making use of Equations 1, 2,

$$\lambda_2 = ma\dot{\theta}\dot{\psi}.$$

The Lagrangian equation for the generalised coordinate ψ is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\psi}}\right) - \frac{\partial L}{\partial \psi} - \lambda_1 y' + \lambda_2 x' = 0,$$

or

$$\frac{d}{dt}\left(m(-(\dot{x}' - y'\dot{\psi})y' + (\dot{y}' + x'\dot{\psi})x') + I_{zz}\dot{\psi}\right) - \lambda_1 y' + \lambda_2 x' = 0,$$

or by making use of Equations 1, 2 and substituting the λ's from above we get

$$\frac{d}{dt}\left(ma\dot{\theta}y' + I_{zz}\dot{\psi}\right) - ma\ddot{\theta}y' + ma\dot{\theta}\dot{\psi}x' = 0,$$

or

$$ma\ddot{\theta}y' + ma\dot{\theta}(\dot{y}' + \dot{\psi}x') + I_{zz}\ddot{\psi} - ma\ddot{\theta}y' = 0,$$

or by making use of Equation 2

$$I_{zz}\ddot{\psi} = 0.$$

The Lagrangian equation for the generalised coordinate θ is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} + \lambda_1 a = 0,$$

or

$$\frac{d}{dt}\left(I_{yy}\dot{\theta}\right) + \lambda_1 a = 0,$$

or by substituting λ_1 by $ma\ddot{\theta}$ we get

$$(I_{yy} + ma^2)\ddot{\theta} = 0.$$

d) The force which maintains the longitudinal rolling constraint is

$$-\lambda_1 = -ma\ddot{\theta} = 0.$$

The force which prevents side-slipping of the object is

$$-\lambda_2 = -ma\dot{\theta}\dot{\psi}.$$

Question 4

a) We attach two rotating Cartesian coordinate systems, one on the rotor and one on the blade as shown in Figure 1.

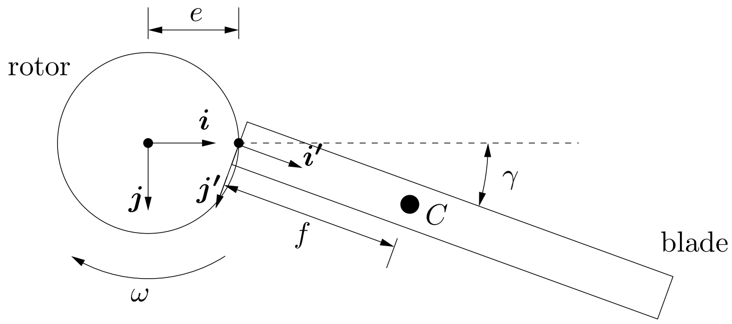

Figure 1: Plan view of a helicopter rotor with one blade.

i) The position vector of the blade centre of mass is

$$\boldsymbol{r} = e\boldsymbol{i} + f\boldsymbol{i'},$$

and therefore the velocity vector is

$$\dot{\boldsymbol{r}} = e\omega\boldsymbol{j} + f(\omega+\dot\gamma)\boldsymbol{j'} = -f(\omega+\dot\gamma)\sin\gamma\boldsymbol{i} + (e\omega + f(\omega+\dot\gamma)\cos\gamma)\boldsymbol{j}.$$

The kinetic energy is thus

$$T = \frac{1}{2}m\left(f^2(\omega+\dot\gamma)^2\sin^2\gamma + (e\omega + f(\omega+\dot\gamma)\cos\gamma)^2\right) + \frac{1}{2}I_{zz}(\omega+\dot\gamma)^2,$$

or

$$T = \frac{1}{2}m\left(e^2\omega^2 + 2ef\omega(\omega+\dot\gamma)\cos\gamma + f^2(\omega+\dot\gamma)^2\right) + \frac{1}{2}I_{zz}(\omega+\dot\gamma)^2.$$

ii) We find the lagging equation of motion using the Lagrangian approach. The Lagrangian function is $L = T - V = T$ and therefore,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\gamma}\right) - \frac{\partial L}{\partial\gamma} = -D\dot\gamma,$$

or

$$\frac{d}{dt}\left(m(ef\omega\cos\gamma + f^2(\omega+\dot\gamma)) + I_{zz}(\omega+\dot\gamma)\right) + mef\omega(\omega+\dot\gamma)\sin\gamma = -D\dot\gamma,$$

or

$$(mf^2 + I_{zz})\ddot\gamma + D\dot\gamma + m\omega^2 ef\sin\gamma = 0.$$

b) The kinetic energy is given by

$$T = \frac{1}{2}\boldsymbol{\Omega}\cdot\boldsymbol{H}.$$

The rate of change of T is

$$\frac{dT}{dt} = \frac{d}{dt}\left(\frac{1}{2}\boldsymbol{\Omega}\cdot\boldsymbol{H}\right) = \frac{1}{2}\left(\frac{d\boldsymbol{\Omega}}{dt}\cdot\boldsymbol{H} + \boldsymbol{\Omega}\cdot\frac{d\boldsymbol{H}}{dt}\right)$$

We can make use of a rectangular coordinate system that is fixed onto the rigid body and rotates with it. The derivatives of the vectors appearing in the above expression can be expressed in terms of the derivatives observed inside the rotating reference frame. Thus,

$$\frac{1}{2}\left(\frac{d\boldsymbol{\Omega}}{dt}\cdot\boldsymbol{H}+\boldsymbol{\Omega}\cdot\frac{d\boldsymbol{H}}{dt}\right)=\frac{1}{2}\left(\left(\boldsymbol{\Omega}\times\boldsymbol{\Omega}+\frac{d'\boldsymbol{\Omega}}{dt}\right)\cdot\boldsymbol{H}+\boldsymbol{\Omega}\cdot\left(\boldsymbol{\Omega}\times\boldsymbol{H}+\frac{d'\boldsymbol{H}}{dt}\right)\right)$$

The scalar triple product identity gives us

$$(\boldsymbol{\Omega}\times\boldsymbol{\Omega})\cdot\boldsymbol{H}=(\boldsymbol{\Omega}\times\boldsymbol{H})\cdot\boldsymbol{\Omega}.$$

If we express $\boldsymbol{\Omega}$ in terms of its components along the body fixed axes then

$$\boldsymbol{\Omega}=\begin{bmatrix}\Omega_x\\ \Omega_y\\ \Omega_z\end{bmatrix}.$$

The vector $\boldsymbol{H}$ in similarly given by

$$\boldsymbol{H}=\boldsymbol{I}\begin{bmatrix}\Omega_x\\ \Omega_y\\ \Omega_z\end{bmatrix},$$

where $\boldsymbol{I}$ is the inertia tensor of the rigid body. The derivatives of $\boldsymbol{\Omega}$ and $\boldsymbol{H}$ observed in the moving reference frame are

$$\frac{d'\boldsymbol{\Omega}}{dt}=\begin{bmatrix}\dot{\Omega}_x\\ \dot{\Omega}_y\\ \dot{\Omega}_z\end{bmatrix},$$

and

$$\frac{d'\boldsymbol{H}}{dt}=\boldsymbol{I}\begin{bmatrix}\dot{\Omega}_x\\ \dot{\Omega}_y\\ \dot{\Omega}_z\end{bmatrix}.$$

The term

$$\frac{d'\boldsymbol{\Omega}}{dt}\cdot\boldsymbol{H}=\begin{bmatrix}\dot{\Omega}_x & \dot{\Omega}_y & \dot{\Omega}_z\end{bmatrix}\boldsymbol{I}\boldsymbol{\Omega},$$

and the term

$$\boldsymbol{\Omega}\cdot\frac{d'\boldsymbol{H}}{dt}=\frac{d'\boldsymbol{H}}{dt}\cdot\boldsymbol{\Omega}=\begin{bmatrix}\dot{\Omega}_x & \dot{\Omega}_y & \dot{\Omega}_z\end{bmatrix}\boldsymbol{I}^{\boldsymbol{T}}\boldsymbol{\Omega}=\frac{d'\boldsymbol{\Omega}}{dt}\cdot\boldsymbol{H},$$

since $\boldsymbol{I}^{\boldsymbol{T}}=\boldsymbol{I}$. The rate of change of kinetic energy becomes

$$\frac{dT}{dt}=\frac{1}{2}\left(2(\boldsymbol{\Omega}\times\boldsymbol{H})\cdot\boldsymbol{\Omega}+2\frac{d'\boldsymbol{H}}{dt}\cdot\boldsymbol{\Omega}\right)=\left(\boldsymbol{\Omega}\times\boldsymbol{H}+\frac{d'\boldsymbol{H}}{dt}\right)\cdot\boldsymbol{\Omega}=\frac{d\boldsymbol{H}}{dt}\cdot\boldsymbol{\Omega}=\boldsymbol{N}\cdot\boldsymbol{\Omega}.$$

Question 5

a) The angular velocity vector of the bar in body fixed axes is

$$\boldsymbol{\Omega} = -\omega \sin\theta \boldsymbol{i} + \omega \cos\theta \boldsymbol{k}.$$

b) The angular momentum vector in body fixed axes is

$$\boldsymbol{H} = -I_{xx}\omega \sin\theta \boldsymbol{i} + I_{zz}\omega \cos\theta \boldsymbol{k},$$

since the chosen axes are principal. I_{zz} is the moment of inertia about the axis of symmetry and I_{xx} is the moment of inertia about a diameter passing through the centre of mass.

c) The torque driving the bar is given by the rate of change of the angular momentum:

$$\boldsymbol{N} = \frac{d\boldsymbol{H}}{dt} = \boldsymbol{\Omega} \times \boldsymbol{H} + \frac{d'\boldsymbol{H}}{dt},$$

when the derivative is observed in the body fixed reference frame. This gives

$$\boldsymbol{\Omega} \times \boldsymbol{H} + \frac{d'\boldsymbol{H}}{dt} = (-\omega \sin\theta \boldsymbol{i} + \omega \cos\theta \boldsymbol{k}) \times (-I_{xx}\omega \sin\theta \boldsymbol{i} + I_{zz}\omega \cos\theta \boldsymbol{k}) + 0,$$

since θ and ω are not changing. We therefore get

$$\boldsymbol{N} = (I_{zz} - I_{xx})\omega^2 \sin\theta \cos\theta \boldsymbol{j}.$$

The direction $\boldsymbol{j}$ is always perpendicular to both the axis of rotation and the axis of symmetry of the bar.

d) i) The new angular velocity vector is given by

$$\boldsymbol{\Omega} = -\omega \sin\theta \cos\phi \boldsymbol{i} + \omega \sin\theta \sin\phi \boldsymbol{j} + (\omega \cos\theta + \dot{\phi})\boldsymbol{k}.$$

The new angular momentum vector is given by

$$\boldsymbol{H} = -I_{xx}\omega \sin\theta \cos\phi \boldsymbol{i} + I_{yy}\omega \sin\theta \sin\phi \boldsymbol{j} + I_{zz}(\omega \cos\theta + \dot{\phi})\boldsymbol{k},$$

where I_{yy} is the moment of inertia about the direction of $\boldsymbol{j}$ and is equal to I_{xx} due to symmetry.

ii) The torque that is needed to drive the bar is

$$\boldsymbol{N} = \frac{d\boldsymbol{H}}{dt} = \boldsymbol{\Omega} \times \boldsymbol{H} + \frac{d'\boldsymbol{H}}{dt}.$$

The first term on the right-hand side is

$$\boldsymbol{\Omega} \times \boldsymbol{H} = (I_{zz} - I_{yy})\omega \sin\theta \sin\phi(\omega \cos\theta + \dot{\phi})\boldsymbol{i} + (I_{zz} - I_{xx})\omega \sin\theta \cos\phi(\omega \cos\theta + \dot{\phi})\boldsymbol{j} + \\ +(I_{xx} - I_{yy})\omega^2 \sin^2\theta \sin\phi \cos\phi \boldsymbol{k},$$

or

$$\boldsymbol{\Omega} \times \boldsymbol{H} = (I_{zz} - I_{xx})\omega \sin\theta(\omega \cos\theta + \dot{\phi})(\sin\phi \boldsymbol{i} + \cos\phi \boldsymbol{j}),$$

since $I_{xx} = I_{yy}$. The last term on the right-hand side is

$$\frac{d'\boldsymbol{H}}{dt} = I_{xx}\omega\dot{\phi}\sin\theta\sin\phi\boldsymbol{i} + I_{yy}\omega\dot{\phi}\sin\theta\cos\phi\boldsymbol{j} = I_{xx}\omega\dot{\phi}\sin\theta(\sin\phi\boldsymbol{i} + \cos\phi\boldsymbol{j}).$$

The torque is therefore

$$\boldsymbol{N} = \left((I_{zz} - I_{xx})\omega\cos\theta + I_{zz}\dot{\phi}\right)\omega\sin\theta(\sin\phi\boldsymbol{i} + \cos\phi\boldsymbol{j}).$$

This torque is always in the direction normal to the axis of symmetry of the bar and the axis of rotation with speed ω (same as in part c) above). The magnitude of the additional torque needed due to the extra rotation with speed $\dot{\phi}$ is

$$I_{zz}\dot{\phi}\omega\sin\theta.$$

Question 6

a) We make use of a fixed Cartesian coordinate system as shown in Figure 2.

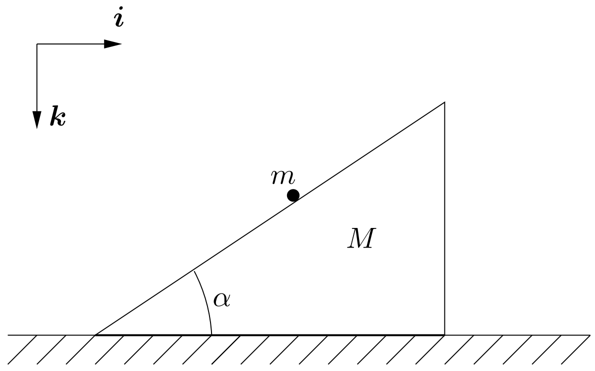

Figure 2: A particle slides on a wedge. The wedge slides on the horizontal surface.

Let's assume that

$$\boldsymbol{r_m} = x_m\boldsymbol{i} + z_m\boldsymbol{k}$$

and

$$\boldsymbol{r_M} = x_M\boldsymbol{i} + z_M\boldsymbol{k}$$

are the position vectors of the particle and the highest vertex of the wedge respectively. The system configuration can be characterised by three generalised coordinates x_M, x_m and z_m. z_M is a constant. The constraint that the particle stays on the wedge can be expressed by a dot product as follows:

$$f = (\boldsymbol{r_m} - \boldsymbol{r_M}) \cdot (\sin\alpha\boldsymbol{i} + \cos\alpha\boldsymbol{k}) = 0,$$

or

$$f = (x_m - x_M)\sin\alpha + (z_m - z_M)\cos\alpha = 0. \tag{3}$$

The kinetic energy of the system is

$$T = \frac{1}{2}M\dot{x}_M^2 + \frac{1}{2}m(\dot{x}_m^2 + \dot{z}_m^2),$$

and the potential energy is

$$V = -mgz_m.$$

The Lagrangian function is

$$L = T - V = \frac{1}{2}M\dot{x}_M^2 + \frac{1}{2}m(\dot{x}_m^2 + \dot{z}_m^2) + mgz_m.$$

The Lagrangian equation corresponding to the generalised coordinate x_M is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_M}\right) - \frac{\partial L}{\partial x_M} + \lambda\frac{\partial f}{\partial x_M} = 0,$$

or

$$\frac{d}{dt}(M\dot{x}_M) - \lambda\sin\alpha = 0,$$

or

$$M\ddot{x}_M - \lambda \sin\alpha = 0. \tag{4}$$

The Lagrangian equation corresponding to the generalised coordinate x_m is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_m}\right) - \frac{\partial L}{\partial x_m} + \lambda \frac{\partial f}{\partial x_m} = 0,$$

or

$$\frac{d}{dt}\left(m\dot{x}_m\right) + \lambda \sin\alpha = 0,$$

or

$$m\ddot{x}_m + \lambda \sin\alpha = 0. \tag{5}$$

The Lagrangian equation corresponding to the generalised coordinate z_m is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}_m}\right) - \frac{\partial L}{\partial z_m} + \lambda \frac{\partial f}{\partial z_m} = 0,$$

or

$$\frac{d}{dt}\left(m\dot{z}_m\right) - mg + \lambda \cos\alpha = 0,$$

or

$$m\ddot{z}_M - mg + \lambda \cos\alpha = 0. \tag{6}$$

From Equations 4, 5 we get

$$\ddot{x}_M = -\frac{m}{M}\ddot{x}_m.$$

From Equation 5, 6 we get

$$m\ddot{x}_m \cos\alpha - m\ddot{z}_m \sin\alpha + mg \sin\alpha = 0, \tag{7}$$

and by differentiating Equation 3 and substituting $\ddot{x}_M$ we get

$$\ddot{z}_m = -\left(1 + \frac{m}{M}\right) \tan\alpha \ddot{x}_m.$$

If we substitute the value of $\ddot{z}_m$ into Equation 7 we get

$$\ddot{x}_m = -\frac{g \sin\alpha \cos\alpha}{1 + \frac{m}{M}\sin^2\alpha}.$$

By substitution

$$\ddot{z}_m = \frac{(1 + \frac{m}{M}) g \sin^2\alpha}{1 + \frac{m}{M}\sin^2\alpha},$$

and

$$\ddot{x}_M = \frac{\frac{m}{M} g \sin\alpha \cos\alpha}{1 + \frac{m}{M}\sin^2\alpha},$$

and finally

$$\lambda = -\frac{m\ddot{x}_m}{\sin\alpha} = \frac{mg\cos\alpha}{1 + \frac{m}{M}\sin^2\alpha}.$$

b) The force of constraint on the particle in the horizontal direction is $-\lambda \sin\alpha$ or

$$-\frac{mg\cos\alpha\sin\alpha}{1+\frac{m}{M}\sin^2\alpha},$$

and in the vertical direction it is $-\lambda\cos\alpha$ or

$$-\frac{mg\cos^2\alpha}{1+\frac{m}{M}\sin^2\alpha}.$$

The force of constraint on the wedge exerted by the particle is given by $\lambda\sin\alpha$ or

$$\frac{mg\cos\alpha\sin\alpha}{1+\frac{m}{M}\sin^2\alpha}.$$