IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2018

EEE/EIE PART II: MEng, BEng and ACGI

SIGNALS AND LINEAR SYSTEMS

Corrected copy
(No corrections)

Thursday, 14 June 10:00 am

Time allowed: 2:00 hours

There are THREE questions on this paper.

Answer ALL questions.

Question One carries 40% of the marks. The other 2 questions each carry 30%.

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : P.T. Stathaki
Second Marker(s) : P.L. Dragotti

1. This question carries 40% of the mark.

(a) Consider each of the following continuous-time signals. For each case, specify if the signal is causal/non-causal, periodic/non-periodic, odd/even. If the signal is periodic specify its period.

(i) $x(t) = 2\sin(2\pi t)$ [2]

(ii) $x(t) = \begin{cases} 3e^{-2t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$ [2]

(b) Consider the signal

$$x(t) = \begin{cases} 1-t, & 0 \leq t \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Now sketch each of the following signals and describe briefly in words how each of the signals can be derived from the original signal $x(t)$.

(i) $x\left(\frac{t}{3}+1\right)$ [2]

(ii) $x(-2t+1)$ [2]

(c) Consider the continuous-time Linear Time-Invariant (LTI) system with input $x(t)$ and output $y(t)$. This system is called a moving average filter.

$$y(t) = \int_{t-1}^{t} x(s)ds$$

(i) Find the impulse response $h(t)$ of the system, expressing it compactly as a function. Sketch the impulse response. [2]

(ii) Find the output when $x(t) = u(t)$ (the continuous-time unit step function) by performing the continuous-time convolution $y(t) = x(t) * h(t)$. Check that the output is indeed the output expected from the moving average filter defined above. Sketch the output. [4]

(d) (i) Consider a continuous-time function $x(t)$. Show that if the Fourier Transform of $x(t)$ is $\mathcal{F}\{x(t)\} = X(\omega)$ then $\mathcal{F}\{x(t)e^{j\omega_0 t}\} = X(\omega - \omega_0)$. [2]

(ii) Show that $\mathcal{F}\{x(t)\cos(\omega_0 t)\} = \frac{1}{2}[X(\omega - \omega_0) + X(\omega + \omega_0)]$. [2]

(iii) Determine the Fourier Transform of $x(t) = e^{-at}\cos(\omega_0 t)u(t), a > 0$ and sketch its amplitude response. The function $u(t)$ is the unit step function. [2]

(e) The output $y(t)$ of a continuous-time LTI system is related to the input $x(t)$ by the differential equation

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 4y(t) = 4x(t)$$

Determine the frequency response of the system and sketch the asymptotic behavior of its Bode plots. [5]

(f) Consider the Laplace Transform of the impulse response of an LTI system $H(s)$ which is assumed to have one of its real zeros located to the right of the imaginary axis at $s = \gamma$. This zero is reflected through the $j\omega$-axis, whereas all poles and the rest of the zeros remain unchanged. This procedure results in a new system with transfer function $H_1(s) = H(s)H_0(s)$. Determine the function $H_0(s)$, its amplitude response and its phase response. [5]

(g) Two continuous-time signals $x_1(t)$ and $x_2(t)$ are multiplied and the product $x(t)$ is sampled by a periodic impulse train. Both $x_1(t)$ and $x_2(t)$ are band-limited so that

$$X_1(\omega) = 0, \omega \geq 2\pi B_1$$
$$X_2(\omega) = 0, \omega \geq 2\pi B_2$$

where $X_i(\omega), i = 1,2$ is the Fourier transform of $x_i(t)$. Determine the maximum sampling period T_s that will allow perfect reconstruction of $x(t)$ from its samples. [5]

(h) Consider the discrete-time, causal LTI system with input $x[n]$ and output $y[n]$ related with the difference equation:

$$y[n] - y[n-1] + \frac{1}{4}y[n-2] = x[n] - \frac{1}{2}x[n-1]$$

(i) Find the analytical expression and the Region of Convergence (ROC) of the z –transform of the impulse response of the above system.
[Hint: Use the fact that the z-transform $\frac{z}{z-a}$ corresponds to the function $a^n u[n]$ if $|z| > |a|$ and the function $-a^n u[-n-1]$ if $|z| < |a|$. The function $u[n]$ is the discrete-time unit step function.] [3]

(ii) Find the analytical expression and the Region of Convergence (ROC) of the z –transform of the output if $x[n] = \left(\frac{1}{2}\right)^n u[n]$. [2]

2. This question carries 30% of the mark.

(a) (i) Find the analytical expression and the Region of Convergence (ROC) of the Laplace transform of the continuous-time, causal signal $x(t) = e^{-at}u(t)$, with a real and positive and $u(t)$ the continuous-time unit step function. [3]

(ii) Find the analytical expression and the Region of Convergence (ROC) of the Laplace transform of the continuous-time, anti-causal signal $x(t) = -e^{-at}u(-t)$, with a real and positive and $u(t)$ the continuous-time unit step function. [3]

(iii) Is the analytical expression of the Laplace transform of a signal sufficient to determine the analytical expression of the signal in time? Justify your answer. [3]

(b) (i) Consider a continuous-time Linear Time-Invariant (LTI) system. Prove that the response of the system to a complex exponential input $e^{s_0 t}$ is the same complex exponential with only a change in amplitude; that is $H(s_0)e^{s_0 t}$. The function $H(s)$ is the Laplace transform of the impulse response of the system. [5]

(ii) A causal LTI system with impulse response $h(t)$ has the following properties:

1. The impulse response $h(t)$ satisfies the equation:

$$h(t) = ae^{-3t}u(t) + be^{-2t}u(t)$$

where a, b are unknown constants.

2. When the input to the system is $x(t) = e^{t}$ for all t, the output is $y(t) = \frac{11}{12}e^{t}$.
3. When the input to the system is $x(t) = e^{2t}$ for all t, the output is $y(t) = \frac{7}{10}e^{2t}$.

Determine the transfer function $H(s) = \mathcal{L}\{h(t)\}$ of the system, consistent with the information above. The constants a, b should not appear in your answer. [6]

(c) The output $y(t)$ of an LTI system is related to the input $x(t)$ by the differential equation

$$\frac{d^2y(t)}{dt^2} - \frac{dy(t)}{dt} - 2y(t) = x(t)$$

Let $X(s)$ and $Y(s)$ denote the Laplace transforms of $x(t)$ and $y(t)$, respectively, and let $H(s)$ denote the Laplace transform of the system's impulse response $h(t)$.

(i) Determine $H(s)$ as a ratio of two polynomials. [3]

(ii) Determine $h(t)$ for each of the following cases:

1. The system is stable.
2. The system is causal.
3. The system is neither stable nor causal. [7]

3. This question carries 30% of the mark.

(a) Consider a continuous-time, band-limited signal $x(t)$, limited to bandwidth $|\omega| \leq 2\pi \times 10^3$rad/sec. We sample $x(t)$ uniformly with sampling frequency $f_s = 1/T_s = 5 \times 10^3$Hz to obtain the discrete-time signal $x[n] = x(nT_s)$. In reconstructing the continuous-time signal from its samples, we use a Digital-to-Analogue Converter which outputs the waveform

$$x_{DA}(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\Pi\left(\frac{t - nT_s}{0.2 \times 10^{-3}}\right)$$

with

$$\Pi(t) = \begin{cases} 1 & |t| < 0.5 \\ 0.5 & |t| = 0.5 \\ 0 & \text{otherwise} \end{cases}$$

(i) Show that $x_{DA}(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\left[\delta(t - nT_s) * \Pi\left(\frac{t}{0.2\times10^{-3}}\right)\right]$ with $\delta(t)$ the Dirac function. The symbol "$*$" denotes the operation of convolution. [2]

(ii) Find the Fourier Transform of the signal $x_{DA}(t)$.
[Hint: Use the fact that the Fourier transform of the function $\sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$ is $\frac{1}{T_s}\sum_{n=-\infty}^{\infty} X\left(\omega - n\frac{2\pi}{T_s}\right)$.] [4]

(iii) Derive the frequency response, $H(\omega)$, of the filter (system) through which $x_{DA}(t)$ must be passed in order to perfectly reconstruct the signal $x(t)$. [6]

(b) (i) Show that the z −transform of the discrete causal signal $x[n + 1]u[n]$ is $z(X(z) - x(0))$, where $X(z)$ is the z −transform of the discrete causal signal $x[n]$. [5]

(ii) Consider the discrete signals $x_1(n) = 2^n$ and $x_2(n) = 3^n$ for $n \geq 0$. Find their convolution using their z −transforms and properties of convolution.
[Hint: Use the result of (b)(i) above and the fact that $x_1(0) = x_2(0)$.] [5]

(c) Consider a discrete LTI system with input $x[n]$ and output $y[n]$ related by the difference equation

$$2y[n] - 9y[n - 1] + 4y[n - 2] = -14x[n - 2]$$

Investigate whether the above system can be both stable and causal. Justify your answer.

[8]