

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING  
EXAMINATIONS 2008

MSc and EEE/ISE PART IV: MEng and ACGI

**STABILITY AND CONTROL OF NON-LINEAR SYSTEMS**

Monday, 12 May 10:00 am

Time allowed: 3:00 hours

Corrected Copy

**There are SIX questions on this paper.**

**Answer FOUR questions.**

*All questions carry equal marks*

**Any special instructions for invigilators and information for candidates are on page 1.**

Examiners responsible	First Marker(s) :	J.C. Allwright
	Second Marker(s) :	E.C. Kerrigan

**Information for invigilators:** none

**Information for candidates:**

$\|x\|$  denotes  $\sqrt{x^T x}$  for  $x \in \mathbb{R}^n$

$\|x\|_{\mathcal{L}_2}$  denotes  $\sqrt{\int_0^\infty x(t)^T x(t) dt}$  for  $x \in \mathcal{L}_2$

$x_\tau$  denotes the truncation of  $x$  onto  $[0, \tau]$

$\mathcal{L}_{2e}$  denotes the extension of  $\mathcal{L}_2$

Unless clear from the context:  $\varepsilon(t)$ ,  $y(t)$ ,  $u(t)$  and  $r(t)$  are all scalar valued

For a symmetric matrix  $P$ :  $\lambda_{\min}(P)$  and  $\lambda_{\max}(P)$  denote the minimum and maximum eigenvalues of  $P$ , respectively

$*$  denotes complex conjugate

$\triangleq$  means 'is defined equal to'

$I_n$  is the  $n \times n$  identity matrix.

## The Questions

1. (a) Here  $x \in \mathbb{R}$ .

(i) Consider the systems

$$\ddot{x} = f_1(x, \dot{x}) \triangleq -x + 1; \quad \ddot{x} = f_2(x, \dot{x}) \triangleq -x.$$

By adapting the general form of solution for such systems, and without using isoclines, sketch typical trajectories for these systems in the  $(x, \dot{x})$  phase-space. [3]

(ii) Consider also the systems

$$\ddot{x} = f_3(x, \dot{x}) \triangleq -\dot{x}; \quad \ddot{x} = f_4(x, \dot{x}) \triangleq \dot{x}.$$

Determine the corresponding phase-plane differential equations and use them to sketch typical trajectories for each system in the  $(x, \dot{x})$  phase-plane. [3]

(iii) Use the results of parts (a-i) and (a-ii) above to sketch the complete closed trajectory (starting and returning later to the starting point) that has the initial condition

$$x(0) = 0, \dot{x}(0) = 3 \text{ for the system}$$

$$\ddot{x} = \begin{cases} f_1(x, \dot{x}) & \text{if } x > 1 \\ f_2(x, \dot{x}) & \text{if } x < 0. \\ f_3(x, \dot{x}) & \text{if } x \in [0, 1] \text{ and } \dot{x} \geq 0 \\ f_4(x, \dot{x}) & \text{if } x \in [0, 1] \text{ and } \dot{x} < 0. \end{cases}$$

Here the  $f_i$  are those of parts (a-i) and (a-ii) above. [4]

What is the value of  $t$  at which  $x$  becomes equal to 1 for the second time?.

**Hint:**  $\int \frac{1}{\gamma x + \delta} dx = \frac{1}{\gamma} \ln(\gamma x + \delta)$ . [3]

(b) In Figures 1.1 and 1.2 below:  $n$  is a skew-symmetric function.

(i) Consider Figure 1.1. Suppose  $e(t) = a \sin(\omega t)$  for scalar  $a \geq 0$  and all  $t \geq 0$ . Outline the way in which a Fourier series representation of  $u$  over one period of  $\sin(\omega t)$  can be used to obtain an approximation to  $u$  that has the form  $\bar{a} \sin(\omega t + \phi)$  for values of  $\bar{a}$  and  $\phi$  which should be specified. Hence derive briefly a general formula for the describing-function  $N(a)$  of  $n$ . [4]

(ii) Consider Figure 1.2. State the harmonic balance equation. Outline a graphical method for predicting whether  $e$  will oscillate. How can you predict the amplitude and frequency of such an oscillation? [3]

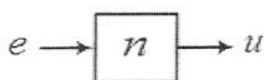


Figure 1.1

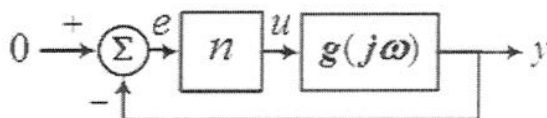


Figure 1.2

2. (a) Consider the function  $v(x, t) = x^T P x$  where  $x \in \mathbb{R}^n$  and  $P^T = P > 0$ .  
Show that  $v$  is radially-unbounded positive-definite and decrescent on  $\mathbb{R}^n$ . [4]

- (b) (i) Consider the system

$$\begin{aligned}\dot{x}_1 &= 2x_2 - x_1 \\ \dot{x}_2 &= -x_1 - 3x_2\end{aligned}\quad (2.1)$$

and the function

$$v(x, t) = x_1^2 + 2x_2^2. \quad (2.2)$$

Why is the origin an equilibrium state for this system?

Write the strongest stability result for the origin that can be obtained using  $v$ , the result of part (a) and Lyapunov theory. Justify your application of the relevant theorem. [4]

- (ii) Consider  $v$  of (2.2) and the following modified version of system (2.1):

$$\begin{aligned}\dot{x}_1 &= 2x_2 - x_1^3 \\ \dot{x}_2 &= -x_1 - 3x_2^5.\end{aligned}\quad (2.3)$$

What can be said about the stability properties of the origin for (2.3) using the Lyapunov Linearization Theorem? Justify your answer. [2]

For (2.3), show that  $-\dot{v}$  is positive definite on the set

$$G_1 = \{x \in \mathbb{R}^2 : \|x\|_2 \leq 1\}. \quad [2]$$

Obtain the strongest stability result for the origin regarding system (2.3) that can be proved using part (a) and the above property of  $-\dot{v}$ . [1]

- (c) This part concerns the application of sliding mode control to the system

$$\dot{x}(t) = Ax(t) + bu(t) + d : x(0) = x_0$$

where  $d, x \in \mathbb{R}^2$  and  $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ ,  $b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$  with constant  $d$ .

Suppose it is desired that  $x_1(t) = \exp(-2t)$  when  $t \geq 0$ .

Let

$$e(t) = x_1(t) - \exp(-2t)$$

and

$$s(t) = \dot{e}(t) + 3e(t).$$

What useful outcome is obtained by arranging that  $s(t) = 0$  for all  $t$  greater than a particular time  $\tau$ ? Justify your answer. [3]

Show that there are control laws that will reduce  $s$  to zero and keep it at zero. [4]

3. Here  $A \in \mathbb{R}^{n \times n}$  is a stability matrix if the real part of each eigenvalue of  $A$  is strictly negative.

(a) Consider the model of a system given by

$$\dot{x}(t) = Ax(t) + bu(t) : x(0) = x_o. \quad (3.1)$$

Here  $A$  is a stability matrix and  $b, x \in \mathbb{R}^n$ . The control  $u$  is scalar-valued.

(i) Suppose  $P \in \mathbb{R}^{n \times n}$  satisfies the equation

$$A^T P + PA + Q - Pbb^T P = 0 \quad (3.2)$$

where  $P = P^T > 0$  and  $Q \in \mathbb{R}^{n \times n}$  with  $Q = Q^T > 0$ .

Use the radially-unbounded positive-definite decrescent function

$$v(x, t) = x^T P x \quad (3.3)$$

to show that, when the control law

$$u(t) = -b^T P x(t) \quad (3.4)$$

is applied, the origin is globally asymptotically stable for system (3.1). [5]

(ii) Now suppose that, owing to a modelling error, the system is actually

$$\dot{x}(t) = (A + \delta A)x(t) + bu(t) : x(0) = x_o \quad (3.5)$$

where  $\delta A \in \mathbb{R}^{n \times n}$ .

Use the  $P$ ,  $v$  and  $u$  defined in part (a-i) to show that the origin is

globally asymptotically stable for the actual system of (3.5) if  $\|\delta A\| < \Delta$  for an

appropriate strictly positive  $\Delta$ , which should be specified. [8]

(b) Suppose the matrix  $A \in \mathbb{R}^{n \times n}$  has the real Schur factorization  $A = ZMZ^T$  where  $Z$  is orthogonal.

Apply the method used by Bartels and Stuart to exploit this factorization in order to transform the matrix Lyapunov equation

$$A^T P + PA = -Q \quad (3.6)$$

into an equation which is more easy to solve. Here  $A, P, Q \in \mathbb{R}^{n \times n}$  with  $P^T = P > 0$  and  $Q^T = Q > 0$ .

State the way in which you would determine the solution  $P$  of (3.6) from the solution of the transformed equation. [5]

Use this method to determine  $P$  when  $M$  is a non-singular diagonal stability matrix and

$Q = qI_n$  with  $q$  a strictly positive scalar and  $I_n$  the identity matrix from  $\mathbb{R}^{n \times n}$ . [2]

4. Consider the system

$$x_{k+1} = Ax_k + bu_k : x_0 = x_o$$

where  $b, x_k \in \mathbb{R}^n$  and where  $A \in \mathbb{R}^{n \times n}$  is a discrete-time stability matrix in that the modulus of each eigenvalue of  $A$  is strictly less than 1.

For symmetric positive-definite  $Q \in \mathbb{R}^{n \times n}$ , let

$$P = Q + A^T Q A + (A^T)^2 Q A^2 + (A^T)^3 Q A^3 + \dots$$

and

$$v(x) = x^T P x.$$

- (a) Suppose  $n > 2$  and  $Q = I_n$  and  $A^2 = 0$ , where  $I_n$  is the  $n \times n$  identity matrix. You may assume without proof that  $v$  is radially-unbounded positive-definite and decrescent on  $\mathbb{R}^n$ .

- (i) Determine the value of  $P$ . Show that if the  $u_k$  are all zero then

$$v(x_{k+1}) - v(x_k) \leq -\|x_k\|^2, \forall k \geq 0. \quad [4]$$

Hence prove global asymptotic stability of the origin using a Lyapunov theorem. [1]

- (ii) Now suppose that each  $u_k$  is chosen to be  $\hat{u}_k(x_k)$  where this minimizes  $v(Ax_k + bu_k)$  with respect to  $u_k \in [-1, 2]$ .

Why might one wish to use such an optimal control? [1]

Derive a simple method for finding  $\hat{u}_k(x_k)$  that takes advantage of the simple form of  $v$ . [4]

Modify the analysis of part (a-i) to show that the origin is globally asymptotically stable for the optimally controlled system. [5]

- (b) Suppose  $n = 2$  and the matrix Lyapunov equation

$$A^T P A - P = -Q$$

has a unique symmetric solution  $P$ .

Define the vec of a matrix  $M \in \mathbb{R}^{2 \times 2}$  and the Kronecker product of two matrices from  $\mathbb{R}^{2 \times 2}$ .

Use vecs and a Kronecker product to determine  $P$  in terms of the solution of a linear equation  $Mf = g$  for appropriate  $M$ ,  $f$  and  $g$ , which should be specified.

Why is your  $M$  non-singular? [5]

5. (a) Consider the system  $H$  of Figure 5.1 where  $u$  and  $y$  are scalar valued.

(i) Define strict input passivity for  $H$ .

Suppose the output  $y$  of  $H$  is given by  $y(t) = \phi(u(t), t)$  where  $\phi \in \text{sector}[\alpha, \beta]$  with  $0 < \alpha < \beta < \infty$ .

Show that  $H$  is strictly input passive and that

$$\|y_T\|_{\mathcal{L}_2} \leq \beta \|u_T\|_{\mathcal{L}_2}, \forall T \in [0, \infty), \forall u \in \mathcal{L}_{2e}. \quad [5]$$

(ii) Define strict output passivity for  $H$ .

Suppose  $H$  consists of a time-invariant system

$$\begin{aligned} \dot{x} &= Ax + bu : x(0) = x_0 \\ y &= c^T x \end{aligned}$$

where the real part of each eigenvalue of  $A$  is strictly negative.

Suppose  $P \in \mathbb{R}^{n \times n}$  is symmetric and positive definite and solves

$$A^T P + PA = -cc^T$$

and suppose that

$$c = Pb.$$

By considering  $\int_0^T \frac{d}{dt} x(t)^T P x(t) dt$ , show that  $H$  is strictly output passive. [7]

(b) Consider the system of Figure 5.2 below, where  $H_1$  is passive and  $H_2$  is strictly output passive and the system equations have a unique solution whenever  $r \in \mathcal{L}_{2e}$ .

The subsystem  $H_1$  does not contain any dynamics and  $x_o$  represents the initial condition for  $H_2$ .

Suppose  $r(t) = 0$  for all  $t$ .

Show that  $\|y_T\|_{\mathcal{L}_2} < \infty$  for all finite  $T$ .

**Hint:** start by considering  $\int_0^T u(t)r(t)dt$  for general  $r$ . [8]

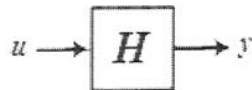


Figure 5.1

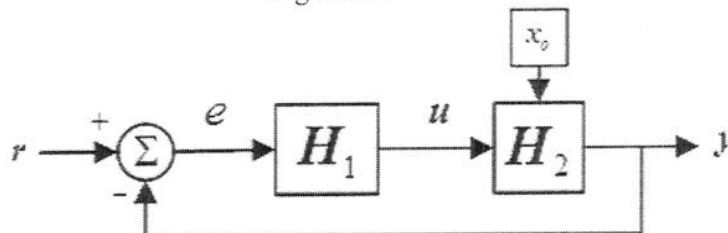


Figure 5.2

6. (a) Consider a SISO system with input  $u(t)$  and output  $y(t)$ .

Show that the system is passive if  $y(t)u(t) = \alpha f(t)\dot{f}(t)$  for scalar  $\alpha > 0$ .

Now suppose  $y(t)u(t) = (1 - \alpha\gamma(t))p(t)$  for scalar  $\alpha, \gamma(t), p(t)$  with  $\alpha > 0$ .

Write  $(1 - \alpha\gamma(t))p(t)$  as  $\alpha f(t)\dot{f}(t)$  with  $f(t) = \frac{\alpha^{-1} - \gamma(t)}{g}$  for scalar  $g > 0$  and use the above

to show that the system is passive if  $\gamma$  is chosen so that  $\dot{\gamma}(t) = -g^2 p(t)$ . [5]

- (b) Consider a plant modelled by

$$\dot{x}(t) = Ax(t) + \alpha bu(t)$$

where  $\alpha$  is a strictly positive scalar with unknown value.

Suppose you would like the plant to behave like the reference model

$$\dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + br(t)$$

where  $\bar{A} \triangleq A - bf^T$  and  $f$  has been chosen so that each eigenvalue of  $\bar{A}$  has real part that is strictly negative.

The control law

$$u(t) = \gamma(t)[r(t) - f^T x(t)]$$

is to be applied to the plant with  $\gamma$  chosen adaptively.

Show that perfect model following is possible by suitable choice of  $\gamma$ . [2]

Let

$$e = \bar{x} - x.$$

Show that

$$\dot{e}(t) = \bar{A}e(t) + w(t)$$

where

$$w(t) = \{\alpha\gamma(t) - 1\}b\{f^T x(t) - r(t)\}.$$

Determine an error subsystem, with a suitable output  $z$ , that is strictly output passive.

There is no need to solve any Lyapunov equations that are involved. [6]

Then, possibly using a result from part (a) above, derive an adaptation law for  $\gamma(t)$  such that

$$\|\bar{x} - x\|_{\mathcal{L}_2} < \infty.$$

Give sufficient detail to make your method clear. [7]