Linear Optimal Control - Model answers 2003

Question 1

(a) Let i_1 and i_2 be the currents through L and C, respectively. Then $y = i = i_1 + i_2$. Moreover,

$$i_1 = x_2 = \frac{u - L\dot{x}_2}{R} \qquad i_2 = C\dot{x}_1 = \frac{u - x_1}{R}.$$

Hence,

$$A = \begin{bmatrix} -\frac{1}{RC} & 0 \\ 0 & -\frac{R}{L} \end{bmatrix} \qquad B = \begin{bmatrix} \frac{1}{RC} \\ \frac{1}{L} \end{bmatrix} \qquad C = \begin{bmatrix} -\frac{1}{R} & 1 \end{bmatrix} \qquad D = \frac{1}{R}.$$

(b) The controllability matrix is

$$\mathcal{C} = \begin{bmatrix} \frac{1}{RC} & -\frac{1}{R^2C^2} \\ \frac{1}{L} & -\frac{R}{L^2} \end{bmatrix}$$

and it is full rank if $R^2C \neq L$. If $R^2C = L$ the system is not controllable. However, as both the eigenvalues of A have negative real part, the system is stabilizable.

(c) The observability matrix is

$$\mathcal{O} = \begin{bmatrix} -\frac{1}{R} & 1 \\ \frac{1}{R_2C} & -\frac{R}{L} \end{bmatrix}$$

and it is full rank if $R^2C \neq L$. If $R^2C = L$ the system is not observable. However, as both the eigenvalues of A have negative real part, the system is detectable.

(d) The transfer function is

$$W(s) = C(sI - A)^{-1}B + D = \frac{LCs^2 + 2CRs + 1}{(Ls + R)(RCs + 1)}.$$

(e) If $R^2C = L$ then

$$W(s) = \frac{R^2C^2s^2 + 2CRs + 1}{(R^2Cs + R)(RCs + 1)} = \frac{(RCs + 1)^2}{R(RCs + 1)^2)} = \frac{1}{R}$$

which is the transfer function of the circuit in Figure 2.

Question 2

(a) Consider the following submatrix of the controllability matrix

$$\tilde{C} = [B\ AB] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2\omega_0^2 \\ 0 & 0 & 0 & 1/r_0 \\ 0 & 1/r_0 & -2\omega_0/r_0 & 0 \end{bmatrix}$$

and note that its determinant is $-1/r_0^2 \neq 0$. Hence the system is controllable.

(b) Consider the closed loop system

$$\dot{x} = (A + BK)x + BGv$$

and note that

$$A + BK = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega_0^2 + k_{11} & k_{12} & k_{13} & 2r_0\omega_0^2 + k_{14} \\ 0 & 0 & 0 & 1 \\ k_{21}/r_0 & -2\omega_0/r_0 + k_{22}/r_0 & k_{23}/r_0 & k_{24}/r_0 \end{bmatrix}.$$

Hence, selecting

$$k_{11} = -3\omega_0^2 - 1 \quad k_{12} = -2 \quad k_{13} = 0 \quad k_{14} = -2r_0\omega_0^2$$

$$k_{21} = 0 \quad k_{22} = 2\omega_0 \quad k_{23} = -r_0 \quad k_{24} = -2r_0$$

yields

$$A + BK = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -2 \end{bmatrix}$$

which shows that (b1) has been achieved.

To achieve (b2) note that

$$-C(A + BK)^{-1}BG = \begin{bmatrix} g_{11} & 0 \\ 0 & g_{22}/r_0 \end{bmatrix}.$$

Hence, it suffices to select

$$g_{11} = 1 \qquad g_{22} = r_0.$$

Question 3

(a) The controllability matrix is

$$\mathcal{C} = \begin{bmatrix} 1 & \alpha \\ 1 & 0 \end{bmatrix}$$

and the system is controllable if $\alpha \neq 0$.

(b) The observability matrix is

$$\mathcal{O} = \begin{bmatrix} 1 & 1 \\ \alpha & 0 \end{bmatrix}$$

and the system is observable if $\alpha \neq 0$.

(c) Let $K = [k_1 \; k_2]$ and note that

$$A - BK = \begin{bmatrix} \alpha - k_1 & -k_2 \\ -k_1 & -k_2 \end{bmatrix},$$

and that the characteristic polynomial of this matrix is $s^2 + (k_1 + k_2 - \alpha)s - \alpha k_2$. Hence the selection

$$k_1 = \frac{(\alpha+1)^2}{\alpha} \qquad k_2 = -\frac{1}{\alpha}$$

is such that the eigenvalues of $A - BK$ are equal to -1. Let $L = [l_1 \; l_2]^T$ and note that

$$A - LC = \begin{bmatrix} \alpha - l_1 & -l_1 \\ -l_2 & -l_2 \end{bmatrix},$$

and that the characteristic polynomial of this matrix is $s^2 + (l_1 + l_2 - \alpha)s - \alpha l_2$. Hence the selection

$$l_1 = \frac{(\alpha+3)^2}{\alpha} \qquad l_2 = -\frac{9}{\alpha}$$

is such that the eigenvalues of $A - LC$ are equal to -3. Finally, the controller is $\dot{\xi} = (A - BK - LC)\xi + Ly$, $u = -K\xi$.

(d) The limits for $\alpha \to 0$ of $\|L\|$ and $\|K\|$ are equal to $+\infty$, and this is in agreement with the loss of controllability and observability of the system for $\alpha = 0$.

(e) With K as selected in part (c), consider the return difference inequality

$$|1 + K(j\omega I - A)^{-1}B| \geq 1.$$

Note that

$$|1 + K(j\omega I - A)^{-1}B| = \sqrt{\frac{(1+\omega^2)^2}{\omega^2(\alpha^2+\omega^2)}}$$

and

$$\frac{(1+\omega^2)^2}{\omega^2(\alpha^2+\omega^2)} - 1 = \frac{1+\omega^2(2-\alpha^2)}{\omega^2(\alpha^2+\omega^2)}.$$

Hence, for all $|\alpha| \leq \sqrt{2}$ and $\alpha \neq 0$ the state feedback gain selected in part (c) is stabilizing and optimal in some sense. For $\alpha = 0$ it is not defined, and for $\alpha > \sqrt{2}$ it is not optimal in the sense stated in the question.

Question 4

(a) Let

$$\hat{x} = e^{\alpha t}x \qquad \hat{u} = e^{\alpha t}u$$

and note that

$$\dot{\hat{x}} = \begin{bmatrix} \alpha & -2 \\ 2 & \alpha \end{bmatrix}\hat{x} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}\hat{u} = (A+\alpha I)\hat{x} + B\hat{u}$$

and

$$J(x_0, u) = \int_0^\infty \left(\hat{x}_1^2 + q_{22}\hat{x}_2^2 + r\hat{u}^2\right) dt.$$

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(b) For any $\alpha \geq 0$ the pair $\{A, B\}$ is controllable, hence the pair $\{A+\alpha I, B\}$ is controllable, $r > 0$, $Q = \text{diag}(1, q_{22}) > 0$, and the pair $\{A, Q^{1/2}C\}$ is observable.

(c) The ARE is

$$(A+\alpha I)^T P + P(A+\alpha I) - \frac{PBB^TP}{r} + Q.$$

If we assume $P = \text{diag}(1, q_{22})$, then this becomes

$$\begin{bmatrix} 2\alpha - 1/r + 1 & 2p_{22} - 2 + p_{22}/r \\ 2p_{22} - 2 + p_{22}/r & 2\alpha p_{22} - p_{22}^2 + q_{22} \end{bmatrix} = 0.$$

Hence

$$r = \frac{1}{2\alpha+1} \qquad p_{22} = \frac{2}{3+2\alpha} \qquad q_{22} = 4\frac{1-\alpha-\alpha^2}{(3+2\alpha)^2}.$$

Note that $r > 0$ for all $\alpha \geq 0$, $p_{22} > 0$ for all $\alpha \geq 0$ and $q_{22} > 0$ for all $0 \leq \alpha < 1/2$, hence $\bar{\alpha} = 1/2$.

(d) The optimal control law for the original system is

$$u^\star = -\frac{1}{r}B^TPx = \frac{-x_1 + p_{22}x_2}{r},$$

with p_{22} as in part (c), and the optimal closed loop system is

$$\dot{x} = A_{cl}x = \begin{bmatrix} -2\alpha - 1 & -\dfrac{4}{3+2\alpha} \\ 3+2\alpha & -2\dfrac{2\alpha+1}{3+2\alpha} \end{bmatrix} x.$$

To prove that all eigenvalues of A_{cl} have real part less then $-\alpha$ consider the matrix $A_{cl} + \alpha I$. This matrix has the characteristic polynomial

$$s^2 + \frac{6\alpha+5}{2\alpha+3}s + \frac{14+16\alpha+8\alpha^2}{2\alpha+3}$$

and the coefficient are all positive for $\alpha \in [0, \bar{\alpha})$.

Question 5

(a) Let

$$H = 1 + \lambda_1 u + \lambda_2(-x_2 + u).$$

The necessary conditions of optimality, for normal extremals, are

$$\dot{x}_1 = x_2 \qquad \dot{x}_2 = -x_2 + u \qquad \dot{\lambda}_1 = 0 \qquad \dot{\lambda}_2 = \lambda_2$$

$$(\lambda_1 + \lambda_2)u \leq (\lambda_1 + \lambda_2)\omega, \ \forall \omega \in [-1, 1].$$

(b) The optimal control as a function of λ_1 and λ_2 is

$$u^\star = -\mathrm{sign}(\lambda_1^\star(t) + \lambda_2^\star(t)).$$

Hence, for any t such that $(\lambda_1^\star(t) + \lambda_2^\star(t)) \neq 0$, $|u^\star(t)| = 1$.

(c) From the necessary conditions in part (a) we obtain

$$\lambda_1^\star(t) = \lambda_1^\star(0) \qquad \lambda_2^\star(t) = \lambda_2^\star(0)e^t.$$

Hence, the equation

$$\lambda_1^\star(0) + \lambda_2^\star(0)e^t = 0$$

in the unknwon t may have at most one solution, say $\bar{t}$. This means that if $\bar{t} \in (0, T)$ then the optimal control will have a *switch* either from $+1$ to -1 or from -1 to $+1$ at $t = \bar{t}$. If $\bar{T} \notin (0, T)$ then the optimal control will be equal to $+1$ or to -1 for all t.

(d) If $u = 1$ then

$$x_1(t) = x_{10} + t \qquad x_2(t) = e^{-t}(x_{20} - 1) + 1.$$

(e) The trajectories in part (d) are optimal if they are such that $x_1(T) = x_2(T) = 0$, *i.e.*

$$0 = x_{10} + T = e^{-T}(x_{20} - 1) + 1.$$

Eliminating the variable $T > 0$ we obtain $T = -x_{10} > 0$ hence

$$e^{x_{10}}(x_{20} - 1) + 1 = 0.$$

Hence the set of initial conditions which are driven to the origin at time T by the control $u = 1$ consists of those $[x_{10}, x_{20}]^T$ for which

$$x_{20} = 1 - e^{-x_{10}},$$

with $x_{10} \leq 0$, and the time to reach the origin is $T = -x_{10} \geq 0$.

Question 6

(a) Let

$$H = \lambda(x + u).$$

The necessary conditions of optimality, for normal extremals, are

$$\dot{x} = x + u \qquad \dot{\lambda} = -\lambda$$

$$\lambda(x + u) \leq \lambda(x + \omega), \ \forall \omega \in [-1, 1].$$

The boundary conditions are

$$x(0) = x_0 \qquad \lambda(1) = x(1).$$

(b) The optimal control is

$$u^\star = -\text{sign}(\lambda^\star)),$$

hence $u = \pm 1$.

(c) If the optimal control is such that $x(1) = 0$ then $\lambda(1) = x(1) = 0$. However,

$$\lambda(t) = e^{-t}\lambda(0),$$

hence $\lambda(1) = 0$ implies that $\lambda(t) = 0$ for all t, hence H does not have a unique minimum. Thus the necessary conditions of optimality for normal extremals are not adequate to determine optimal controls yielding $x(1) = 0$, *i.e.* yielding the lowest possible value for $J(x_0, u)$.

(d) Assume $|x_0| \leq 1 - 1/e$ and $x_0 \neq 0$ and consider the control $(\star)$. Note that

$$x(t) = e^t(x_0 - \text{sign}(x_0)) + \text{sign}(x_0)$$

for $t \in [0, \bar{t}]$. At $t = \bar{t}$ one has $x(\bar{t}) = 0$,*i.e.*

$$e^{\bar{t}} = \frac{\text{sign}(x_0)}{\text{sign}(x_0) - x_0}.$$

However, $0 < \bar{t} < 1$, hence

$$1 < \frac{\text{sign}(x_0)}{\text{sign}(x_0) - x_0} < e,$$

and this holds for all $|x_0| < 1 - 1/e$. At time $\bar{t}$ one has $x(\bar{t}) = 0$ and $u(t) = 0$ for all $t > \bar{t}$. Hence, $x(t) = 0$ for all $\bar{t} \leq t \leq 1$.

(e) If $|x_0| \leq 1 - 1/e$ and $x_0 \neq 0$, then the control $(\star)$ yields $J(x_0, u) = 0$, which is the smallest acheivable value for the cost. As a result the control law $(\star)$ is optimal for all initial states such that $|x_0| \leq 1 - 1/e$ and $x_0 \neq 0$. Note, finally that this control law is not unique, *i.e.* if $|x_0| \leq 1 - 1/e$ and $x_0 \neq 0$ it is possible to construct other control signal yielding $x(1) = 0$.