UNIVERSITY OF LONDON

IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 1997

BEng Honours Degree in Computing Part III
BSc Honours Degree in Mathematics and Computer Science Part III
MSci Honours Degree in Mathematics and Computer Science Part III
MSc Degree in Advanced Computing
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the
Diploma of Membership of Imperial College
Associateship of the Royal College of Science
Associateship of the City and Guilds of London Institute

PAPER 3.78

MATHEMATICAL STRUCTURES IN COMPUTING SCIENCE

Wednesday, April 30th 1997, 10.00 - 12.00

Answer THREE questions

For admin. only: paper contains 4 questions

Section A *(Use a separate answer book for this Section)*

1a Define the terms *monotonic* and *continuous* as applied to functions between cpo's (complete partial orders).

Suppose that $f:D \rightarrow D$ is a monotonic function, where D is the cpo of finite and infinite unary strings ("Tapes(1)"). State, with reasons, which of the following is necessarily true:

i) If $f(D)$ is finite (that is, D is mapped into a finite subset of D), then f is continuous.

ii) If f is surjective (onto) , then it is strict.

iii) If f is injective (one-one), then it is continuous.

b How does an (unordered) set give rise to a "flat cpo"? What is meant by the *natural extension* $f_\perp$ (or f^+) of a function $f: S_1 \times \ldots \times S_n \rightarrow V$, where $S_1,\ldots,V$ are sets? Illustrate with the example of equality (as a predicate).

Explain briefly why, in a typical recursive definition (such as that for **factorial**), it is appropriate to take the equality predicate as naturally extended, but not the conditional.

c A collection $\mathbb{C}$ of subsets of $\mathbb{N}$ will be called *separating* if, for any distinct $x,y \in \mathbb{N}$, there exist disjoint members A,B of $\mathbb{C}$ (that is, $A \cap B = \emptyset$) such that $x \in A$, $y \in B$. *Even* is the set of even numbers; likewise for *Odd*. In each of i) - iii) below, can we conclude that, if $\mathbb{C}$ satisfies the condition stated, $\mathbb{C}$ must be separating? Explain.

i) $Odd \in \mathbb{C}$ and $Even \in \mathbb{C}$.

ii) Every set (of natural numbers) with at least three elements is a member of $\mathbb{C}$.

iii) $\mathbb{C}$ is a Boolean algebra of sets (as a subalgebra of $\wp(\mathbb{N})$).

2a i) What is meant by saying that a given poset (partially ordered set) is (1) a *lattice*, (2) a *complete* lattice?

ii) Explain briefly why it is always true, in any lattice, that:

$$x \vee (y \wedge z) \le (x \vee y) \wedge (x \vee z).$$

iii) Suppose that, in a lattice L, for every $x,y,z \in L$:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

Prove that, for all x,y,z:

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

What can we conclude about *distributivity* in a lattice?

iv) Give an example of a lattice L, and three elements a,b,c of L, such that

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c),$$

yet

$$a \vee (b \wedge c) \neq (a \vee b) \wedge (a \vee c).$$

b Let M be a monoid that is both commutative and idempotent (that is, $x \circ x = x$ for all $x \in M$), and let $\le$ be the left factor pre-order of M, given by

$$x \le y \iff \exists a.\ x \circ a = y.$$

i) Prove that $\le$ is actually a partial order (you are required to verify only the anti-symmetry).

ii) Prove that, in the poset $(M,\le)$, $x \circ y$ is the least upper bound of x,y.

iii) Deduce that, if M is finite, then $(M,\le)$ is a complete lattice. (Any general result about posets or latttices that you use should be stated, but need not be proved.)

The two parts carry, respectively, 60%, 40% of the marks.

Turn over ...

Section B *(Use a separate answer book for this Section)*

3a i) What is a monoid?

ii) If Y is a set, describe how Y^Y (or, in Miranda notation, $Y \to Y$), the set of functions from Y to itself, is a monoid.

iii) If X is a set, what does it mean to say that the list monoid X^* (in Miranda, [X]) is the *free monoid* over X?

b Suppose X and Y are sets. For $e \in Y$, $f: X \times Y \to Y$ and $xs \in X^*$, we define foldr and foldl recursively by

i) foldr (e, f, []) = e

foldr (e, f, x:xs) = f(x, foldr(e, f, xs))

ii) foldl (e, f, []) = e

foldl (e, f, x:xs) = foldl(f(x, e), f, xs)

For each function foldr and foldl, show how it may be defined using the free monoid property of X^* instead of the recursion above, and verify from your definition that it satisfies the recursive equations.

Hint: For foldr, use f in its curried form $f': X \to Y^Y$ and use part a ii).

For foldl, define $g: X \times Y^Y \to Y^Y$ by

$$g(x, h)(y) = h(f(x, y))$$

and again curry it.

The two parts carry, respectively, 40%, 60% of the marks.

4a What is meant by the following?

i) A *functor* from a category $\mathcal{C}$ to a category $\mathcal{D}$.

ii) A *product* $X \times Y$ of two objects X and Y in a category $\mathcal{C}$.

iii) The *diagonal morphism* $\Delta: X \to X \times X$.

b Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. Describe their product $\mathcal{C} \times \mathcal{D}$ (including the two projections) in the category **cat** of categories.

c Let $G: \mathcal{D} \to \mathcal{C}$ be a functor, and X an object of $\mathcal{C}$. A *universal arrow* from X to G is a pair (A, η) where A is an object of $\mathcal{D}$ and $\eta: X \to G(A)$ is a morphism in $\mathcal{C}$ such that for any similar pair (B, f) (B an object of $\mathcal{D}$, $f: X \to G(B)$ in $\mathcal{C}$) there is a unique morphism $f': A \to B$ in $\mathcal{D}$ such that $f = \eta;G(f')$.

i) Show how free monoids can be expressed through the notion of universal arrows.

In the dual notion, of a universal arrow *to* X *from* G, the morphisms η, f and f' above are reversed – for instance, we have $\eta: G(A) \to X$. Show how the following can be expressed through these dual universal arrows.

ii) Binary products in a category. (Hint: use a diagonal functor in b.)

iii) Powersets $\wp X$. (Hint: a *function* from B to $\wp X$ is equivalent to a *relation* from B to X. Use the category **rel** whose objects are sets but whose morphisms are relations.)

End of paper

