UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 2000

BEng Honours Degree in Computing Part I
MEng Honours Degrees in Computing Part I
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the Associateship of the City and Guilds of London Institute

PAPER C145

MATHEMATICAL METHODS AND GRAPHICS

Thursday 11 May 2000, 10:00
Duration: 120 minutes

Answer FOUR questions

Paper contains 6 questions

1. The vector equation of a plane is $\mathbf{n.P} = k$ where $\mathbf{n}$ is a normal vector to the plane, $\mathbf{P}$ is the position vector of a point on the plane and k is scalar constant.

The vector equation of a line segment is

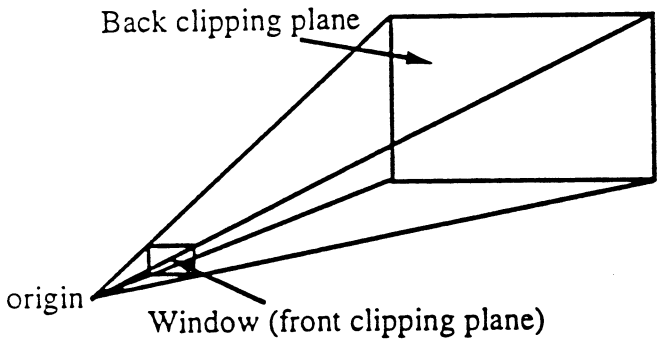
$$\mathbf{P} = \mu\mathbf{P1} + (1-\mu)\mathbf{P2}\ ,$$

where $\mathbf{P1}$ and $\mathbf{P2}$ are the position vectors of the end points and μ is a scalar parameter in the interval [0..1].

(i) Find an equation for the intersection of a plane and a line segment.

(ii) Use this equation to determine the point of intersection of the line joining the points $[8, 2, -3]$ to $[-6, 4, 5]$ with the plane $z = 3$.

(iii) In a flight simulator, the clipping is carried out in 3D (object) space. The objects to be drawn are planar polygons with uniform colour. Perspective projection is being used and the three dimensional viewing volume is a convex polyhedron bounded by the four planes through the viewpoint and the corners of the viewing window, and a front and back clipping plane.



Explain how, using the equations of the six planes that bound the viewing volume, it is possible to determine whether any vertex of a polygon is visible or not.

(iv) Describe what further extensions would be required to determine the part of the polygon that is visible.

(The four parts carry equal marks.)

2. Matrix Transformations

(i) In computer games and flight simulators, the scene data are normally expressed in an absolute $[x, y, z]$ co-ordinate system. Explain briefly why it is desirable to be able to transform the data to a new co-ordinate system.

(ii) Given a point $\mathbf{P} = [P_x, P_y]$ defined in the normal two dimensional Cartesian co-ordinate system $[x, y]$, and a new system which is defined by a point $\mathbf{C} = [C_x, C_y]$ and two orthogonal unit vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, show, with the aid of a suitable diagram that the co-ordinates of $\mathbf{P}$ expressed in the $[u, v]$ system are: $\boldsymbol{u}.(\mathbf{P} - \mathbf{C})$ and $\boldsymbol{v}.(\mathbf{P} - \mathbf{C})$.

(iii) Extend your result to three dimensions and derive a transformation matrix which will take co-ordinates with values expressed in an $[x, y, z]$ system to values expressed in a co-ordinate system whose origin is at point $\mathbf{C} = [C_x, C_y, C_z]$ and whose axis directions are expressed by the orthogonal unit vectors $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$.

(iv) A scene in a computer game is to be transformed into a new co-ordinate system with the following specification:

(a) The origin is at $(10, 15, 5)$.

(b) The view direction (equivalent to the w axis direction) is defined by the vector $(-6, 0, 8)$.

(c) The new u axis is to remain perpendicular to the old y axis, so a vector in the u direction may be written $[p_x, 0, p_z]$.

(d) The new v axis must have a positive y component, so a vector in the v direction may be written $[q_x, 1, q_z]$.

Find the three unit vectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ describing the new co-ordinate system and hence derive the transformation matrix.

(v) This transformation is called affine. What does the word affine mean in this context? Give an example of a non-affine transformation matrix.

(*The five parts carry equal marks.*)

3. (i) Find the equation of the plane P passing through the points $(1, 1, 1)$, $(1, 2, 1)$ and $(0, 0, -1)$.

   Hence find the line of intersection of P with the plane passing through $(1, 1, 0)$, $(1, 0, 1)$ and $(1, -1, 0)$.

   (ii) Solve the simultaneous equations

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & = & 10, \\ 2x_1 & + & x_2 & - & x_3 & = & 2, \\ 3x_1 & + & 4x_2 & - & 2x_3 & = & 8, \end{array}$$

   using Gaussian elimination, clearly stating the form of the triangular matrices L and U appearing in the LU decomposition. Hence find the determinant of $A = LU$.

(Parts (i) and (ii) carry equal marks.)

4. (i) The eigenvalues of a matrix A satisfy the characteristic equation

$$\lambda^n + b_{n-1}\,\lambda^{n-1} + \ldots + b_1\lambda + b_0 = 0\,.$$

   Defining a new matrix B by

$$B = A^n + b_{n-1}\,A^{n-1} + \ldots + b_1\,A + b_0\,I\,,$$

   show that $B\mathbf{e}_i = \mathbf{0}$ for any eigenvector $\mathbf{e}_i$ of A.

   (ii) Find the eigenvalues and normalized eigenvectors of the non-symmetrical matrix

$$\begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}.$$

   Hence evaluate the matrix product

$$\begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}^{100},$$

   leaving your answer in terms of 2^{100}, 3^{100}.

(Parts (i) and (ii) carry 25% and 75% of the marks respectively.)

5. (i) Find all roots of the polynomial equations:

   (a) $z^5 = 1 - i$;

   (b) $z^6 - z^3 + 1 = 0$,

   leaving your answers in the form $z = re^{i\theta}$.

   (ii) Two roots of a polynomial equation $P(z) = 0$ of degree three, with real coefficients, are 1 and $1 + i$.

   Find the form of the polynomial equation.

(Parts (i) and (ii) carry 75% and 25% of the marks respectively.)

6. (i) Find the general solution to the recurrence relation

$$u_n - 5u_{n-1} + 4u_{n-2} = F(n) ,$$

   when $F(n)$ takes the forms:

   (a) $F(n) = n$,

   (b) $F(n) = 2(4)^n$.

   (ii) Find the first and second partial derivatives of $f(x, y) = e^{y/x}$.

(Parts (i) and (ii) carry 75% and 25% of the marks respectively.)

END OF PAPER

DEPARTMENT OF MATHEMATICS

MATHEMATICAL FORMULAE

1. Vector Algebra

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1, a_2, a_3)$$

Scalar (dot) product:

$$\mathbf{a}\,.\,\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

Vector (cross) product:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Scalar triple product:

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a}\,.\,\mathbf{b} \times \mathbf{c} = \mathbf{b}\,.\,\mathbf{c} \times \mathbf{a} = \mathbf{c}\,.\,\mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c}\,.\,\mathbf{a})\mathbf{b} - (\mathbf{b}\,.\,\mathbf{a})\mathbf{c}$$

2. Series

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \ldots \quad (\alpha \text{ arbitrary, } |x| < 1)$$

$$e^x = 1 + x + x^2/2! + \ldots + x^n/n! + \ldots\,,$$

$$\cos x = 1 - x^2/2! + x^4/4! - \ldots + (-1)^n x^{2n}/(2n)! + \ldots\,,$$

$$\sin x = x - x^3/3! + x^5/5! - \ldots + (-1)^n x^{2n+1}/(2n+1)! + \ldots\,,$$

$$\ln(1+x) = x - x^2/2 + x^3/3 - \ldots + (-1)^n x^{n+1}/(n+1) + \ldots (-1 < x \leq 1)$$

3. Trigonometric identities and hyperbolic functions

$$\sin(a+b) = \sin a \cos b + \cos a \sin b\,;$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b\,.$$

$$\cos iz = \cosh z\,; \quad \cosh iz = \cos z\,; \quad \sin iz = i\sinh z\,; \quad \sinh iz = i \sin z\,.$$

4. Differential calculus:

(a) Leibniz's rule:

$$D^n(fg) = f\,D^n g + \tbinom{n}{1} Df\,D^{n-1}g + \ldots + \tbinom{n}{r} D^r f\,D^{n-r}g + \ldots + D^n f\,g\,.$$

(b) Taylor's expansion of $f(x)$ about $x = a$:

$$f(a+h) = f(a) + hf'(a) + h^2 f''(a)/2! + \ldots + h^n f^{(n)}(a)/n! + \epsilon_n(h)\,,$$

where $\quad \epsilon_n(h) = h^{n+1} f^{(n+1)}(a+\theta h)/(n+1)!, \quad 0 < \theta < 1$.

(c) Taylor's expansion of $f(x, y)$ about (a, b):

$$f(a+h, b+k) = f(a,b) + [hf_x + kf_y]_{a,b} + 1/2!\,\left[h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}\right]_{a,b} + \ldots$$

(d) Partial differentiation of $f(x, y)$:

i. If $y = y(x)$, then $f = F(x)$, and $\dfrac{dF}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dx}$.

ii. If $x = x(t)$, $y = y(t)$, then $f = F(t)$, and $\dfrac{dF}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$.

iii. If $x = x(u, v)$, $y = y(u, v)$, then $f = F(u, v)$, and

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}, \quad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$

(e) Stationary points of $f(x, y)$ occur where $f_x = 0$, $f_y = 0$ simultaneously.
Let (a, b) be a stationary point: examine $D = [f_{xx}f_{yy} - (f_{xy})^2]_{a,b}$.
If $D > 0$ and $f_{xx}(a, b) < 0$, then (a, b) is a maximum;
If $D > 0$ and $f_{xx}(a, b) > 0$, then (a, b) is a minimum;
If $D < 0$ then (a, b) is a saddle-point.

(f) Differential equations:

i. The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating factor $\quad I(x) = \exp[\int P(x)(dx]$, so that $\dfrac{d}{dx}(Iy) = IQ$.

ii. $P(x, y)dx + Q(x, y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y$.

5. Integral calculus

(a) An important substitution: $\tan(\theta/2) = t$:
$\sin\theta = 2t/(1+t^2)$, $\cos\theta = (1-t^2)/(1+t^2)$, $d\theta = 2\,dt/(1+t^2)$.

(b) Some indefinite integrals:

$$\int (a^2 - x^2)^{-1/2} dx = \sin^{-1}(x/a), \quad |x| < a.$$

$$\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1}(x/a) = \ln\left\{\frac{x}{a} + \left(1 + \frac{x^2}{a^2}\right)^{1/2}\right\}.$$

$$\int (x^2 - a^2)^{-1/2} dx = \cosh^{-1}(x/a) = \ln\left|\frac{x}{a} + \left(\frac{x^2}{a^2} - 1\right)^{1/2}\right|.$$

$$\int (a^2 + x^2)^{-1} dx = (1/a)\tan^{-1}(x/a).$$

6. Numerical methods

(a) Approximate solution of an algebraic equation:

If a root of $f(x) = 0$ occurs near $x = a$, take $x_0 = a$ and

$$x_{n+1} = x_n - [f(x_n)/f'(x_n)], \quad n = 0, 1, 2 \dots$$

(Newton-Raphson method).

(b) Formulae for numerical integration: Write $x_n = x_0 + nh$, $y_n = y(x_n)$.

i. Trapezium rule (1-strip): $\int_{x_0}^{x_1} y(x)dx \approx (h/2)[y_0 + y_1]$.

ii. Simpson's rule (2-strip): $\int_{x_0}^{x_2} y(x)dx \approx (h/3)[y_0 + 4y_1 + y_2]$.

(c) Richardson's extrapolation method: Let $I = \int_a^b f(x)dx$ and let I_1, I_2 be two estimates of I obtained by using Simpson's rule with intervals h and $h/2$. Then, provided h is small enough,

$$I_2 + (I_2 - I_1)/15,$$

is a better estimate of I.

7. Laplace transforms

Function	Transform	Function	Transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)dt$	$af(t) + bg(t)$	$aF(s) + bG(s)$
df/dt	$sF(s) - f(0)$	d^2f/dt^2	$s^2F(s) - sf(0) - f'(0)$
$e^{at}f(t)$	$F(s-a)$	$tf(t)$	$-dF(s)/ds$
$(\partial/\partial\alpha)f(t,\alpha)$	$(\partial/\partial\alpha)F(s,\alpha)$	$\int_0^t f(t)dt$	$F(s)/s$
$\int_0^t f(u)g(t-u)du$	$F(s)G(s)$		
1	$1/s$	$t^n (n = 1, 2 \dots)$	$n!/s^{n+1}$, $(s > 0)$
e^{at}	$1/(s-a)$, $(s > a)$	$\sin\omega t$	$\omega/(s^2+\omega^2)$, $(s > 0)$
$\cos\omega t$	$s/(s^2+\omega^2)$, $(s > 0)$	$H(t-T) = \begin{cases} 0, & t < T \\ 1, & t > T \end{cases}$	e^{-sT}/s, $(s, T > 0)$

8. Fourier series

If $f(x)$ is periodic of period $2L$, then $f(x + 2L) = f(x)$, and

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L},$$

where

$$a_n = (1/L)\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad n = 0, 1, 2, \dots,$$

and

$$b_n = (1/L)\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1, 2, 3, \dots$$

October 1998