Imperial College London

MEng (Engineering) Examination 2017

Year 1

AE1-107 Mathematics Term II

Friday 9th June 2017: 14.00 to 16.00
[2 hours]

The paper is divided into Section A and Section B

Both sections carry the same weight

Candidates may obtain full marks for complete answers to ***ALL*** questions.

You must answer each section in a separate answer booklet

The use of lecture notes is NOT allowed.

Section A

1. Let

$$A = \begin{pmatrix} a & -1 \\ 1 & -1 \end{pmatrix}.$$

(a) Find the value of a such that $A^3 = I$. [25%]

(b) Find the value of a such that $A^3 = 0$. [25%]

(c) For these two values of a, either find A^{-1} or show that A^{-1} does not exist. [25%]

(d) Show that there is no value of a such that $A^3 = A$. [25%]

2. (a) Consider the linear system

$$
\begin{array}{ccccccc}
x & + & y & + & z & = & 1, \\
x & + & t^2 y & + & tz & = & 1, \\
x & + & y & + & 2z & = & 2,
\end{array}
$$

where t is a real number. Identify values of t for which there is

i) a unique solution, [20%]

ii) an infinite number of solutions, [20%]

iii) no solution. [20%]

(b) Find the LU decomposition of

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & -1 \\ 1 & -1 & 5 \end{pmatrix}.$$

[20%]

Hence solve the system

$$Ax = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

[20%]

Section B

3. Consider the ODE:
$$\left(\frac{\mathrm{d}^3y}{\mathrm{d}x^3}\right)^2 + y = \sin(2x).$$

(a) Is the ODE linear or non-linear? [5%]

(b) What is the order of the ODE? [5%]

Now consider the ODE:
$$\frac{\mathrm{d}y}{\mathrm{d}x} + 3y = f(x,y).$$
Find the general solution of the ODE if

(c) $f(x,y) = x.$ [20%]

(d) $f(x,y) = y^4.$ [20%]

Finally consider the ODE:
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{9x^2 - 2xy}{2y + x^2 + 1}.$$

(e) Show that the ODE is exact. [10%]

(f) Find the potential function for the ODE. [30%]

(g) Write down the general solution of the ODE. [5%]

(h) Find a solution of the ODE that satisfies $y(0) = 0$. [5%]

4. Consider the ODE:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 5\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = f(x).$$

Find the general solution of the ODE if

(a) $f(x) = 0$. [15%]

(b) $f(x) = e^{x}$. [15%]

(c) $f(x) = e^{-x}$. [25%]

Now consider the ODE:

$$ax^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + bx\frac{\mathrm{d}y}{\mathrm{d}x} + cy = 0 \qquad x > 0.$$

Find the general solution of the ODE if

(d) $a = 2, b = 3, c = -15$. [15%]

(e) $a = 1, b = -7, c = 16$. [15%]

The Laplace transform $F(s)$ of a function $f(t)$ is defined as:

$$F(s) = \int_0^\infty f(t)e^{-st}\,\mathrm{d}t$$

for some appropriate range of s.

(f) Using this expression or otherwise, derive an expression for the Laplace transform of $f(t) = \sinh(t)$. Show all your workings. [15%]

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Marks

1. Let

$$A = \begin{pmatrix} a & -1 \\ 1 & -1 \end{pmatrix}.$$

(a) Find the value of a such that $A^3 = I$. [25%]

(b) Find the value of a such that $A^3 = 0$. [25%]

(c) For these two values of a, either find A^{-1} or show that A^{-1} does not exist. [25%]

(d) Show that there is no value of a such that $A^3 = A$. [25%]

If $A = \begin{pmatrix} a & -1 \\ 1 & -1 \end{pmatrix}$ then

$$A^2 = \begin{pmatrix} a & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} a^2-1 & 1-a \\ a-1 & 0 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} a & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a^2-1 & 1-a \\ a-1 & 0 \end{pmatrix} = \begin{pmatrix} a(a^2-1)-(a-1) & a(1-a) \\ a^2-1+1-a & 1-a \end{pmatrix}$$

a)
$A^3 = I$ if $a = 0$
b)
$A^3 = 0$ if $a = 1$
c)
If $a = 1$ then $A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$, singular, A^{-1} does not exist
If $a = 0$ then $A = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$, $A^{-1} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$
d)
if $A^3 = A$ then (2,2) if $1 - a = -1$ and if $a = 2$ but (2,1) then $a^2 - a = 1$ but this is not if $a = 2$

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2. (a) Consider the linear system

$$\begin{array}{ccccccc} x & + & y & + & z & = & 1, \\ x & + & t^2y & + & tz & = & 1, \\ x & + & y & + & 2z & = & 2, \end{array}$$

where t is a real number. Find all solutions, and identify any values of t for which there is

i) a unique solution, [20%]

ii) an infinite number of solutions, [20%]

iii) no solution. [20%]

(b) Find the LU decomposition of [20%]

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & -1 \\ 1 & -1 & 5 \end{pmatrix}.$$

Hence solve the system [20%]

$$Ax = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

$$\begin{array}{ccccccc} x & + & y & + & z & = & 1, \\ x & + & t^2y & + & tz & = & 1, \\ x & + & y & + & 2z & = & 2, \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & t^2 & t & 1 \\ 1 & 1 & 2 & 2 \end{array}\right)$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & t^2-1 & t-1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

<u>$t = 1$</u>

$$\begin{array}{ccccccc} x & + & y & + & z & = & 1, \\ & & & & z & = & 1, \end{array}$$

then

$$\begin{array}{ccc} x & = & -y, \\ z & = & 1, \end{array}$$

Or

$$x = \begin{pmatrix} \propto \\ -\propto \\ 1 \end{pmatrix}$$

$t = -1$

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	Marks
3 ~~2~~) a.) Non linear	1
b.) 3rd	1

c.) $$\frac{dy}{dx} + 3y = x$$

$$I(x) = \exp\left[\int 3\,dx\right]$$

$$= \exp(3x)$$

$$\Rightarrow y(x) = e^{-3x}\int e^{3x} x\,dx$$

$$= e^{-3x}\left[\frac{1}{3}xe^{3x} - \frac{1}{3}\int e^{3x}dx\right]$$

$$= e^{-3x}\left[\frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C\right]$$

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$= \frac{1}{3}x - \frac{1}{9} + Ce^{-3x}$	4
d.) $\frac{dy}{dx} + 3y = y^4$	
$y = u^{-\frac{1}{3}} \Rightarrow \frac{dy}{dx} = -\frac{1}{3}u^{-\frac{4}{3}}\frac{du}{dx}$	
$\Rightarrow \frac{du}{dx} - 9u = -3$	
$\Rightarrow u = e^{9x}\left[-\int e^{-9x} 3\, dx\right]$	
$= \frac{1}{3} + Ce^{9x}$	

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Marks

$$\Rightarrow y = \frac{1}{\sqrt[3]{\frac{1}{3} + Ce^{9x}}}$$ 4

e.) $\frac{\partial}{\partial y}\left[-9x^2 + 2xy\right] =$ 2

$2x$

$\frac{\partial}{\partial x}\left[2y + x^2 + 1\right] =$

$2x$

$\Rightarrow$ <u>exact</u>.

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Marks

f.) $\frac{\partial u}{\partial x} = -9x^2 + 2xy$ (1)

$\frac{\partial u}{\partial y} = 2y + x^2 + 1$ (2)

(1) $\Rightarrow u = -3x^3 + x^2y + p(y)$

(2) $\Rightarrow u = y^2 + x^2y + y + q(x)$

$\Rightarrow p(y) = y^2 + y$

$q(x) = -3x^3$ $\Rightarrow$

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	Marks
$u = -3x^3 + x^2y + y^2 + y$	6
Potential function.	
g.) $-3x^3 + x^2y + y^2 + y = C$	1
h.) $0 + 0 + 0 + 0 = C$	
$\Rightarrow C = 0$	
$\Rightarrow -3x^3 + x^2y + y^2 + y = 0$	1

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4) a) $\alpha^2 + 5\alpha + 4 = 0$	
$\Rightarrow \alpha = -4$ or -1	
$\Rightarrow y = Ae^{-4x} + Be^{-x}$	3
b) $y = y_h + y_p$	
from (a)	
$y_p = Ce^{x} \Rightarrow$	
$Ce^{x} + 5Ce^{x} + 4Ce^{x} = e^{x}$	
$\Rightarrow 10C = 1 \Rightarrow$	
$C = \frac{1}{10} \Rightarrow$	

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$$y_p = \frac{1}{10}e^{x} \Rightarrow$$

$$y = Ae^{-4x} + Be^{-x} + \frac{1}{10}e^{x}$$ 3

c.) $y = y_h + y_p$

$\uparrow$ from (a)

$$y_p = Cxe^{-x}$$

$$y_p' = -Cxe^{-x} + Ce^{-x}$$

$$y_p'' = -2Ce^{-x} + Cxe^{-x}$$

$\Rightarrow$

N

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Marks

$$-2Ce^{-x} + \cancel{Cxe^{-x}} - 5\cancel{Cxe^{-x}}$$

$$+5Ce^{-x} + \cancel{4Cxe^{-x}} =$$

$$e^{-x}$$

$$\Rightarrow 3Ce^{-x} = e^{-x}$$

$$\Rightarrow C = \frac{1}{3} \Rightarrow$$

$$y = Ae^{-4x} + Be^{-x} + \frac{1}{3}xe^{-x}$$ 5

d) $2x^2y'' + x3y' - 15y = 0$

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$$\Rightarrow x^2 y'' + x\frac{3}{2}y' - \frac{15}{2}y = 0$$

$$\Rightarrow \alpha = \frac{\left(1-\frac{3}{2}\right) \pm \sqrt{\left(\frac{1}{2}\right)^2 + 30}}{2}$$

$$= \frac{-\frac{1}{2} \pm \frac{11}{2}}{2}$$

$$= \frac{5}{2}, -3$$

$$\Rightarrow y = Ax^{\frac{5}{2}} + Bx^{-3}$$ 3

e.) $x^2y'' - x7y' + 16y = 0$

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Marks

$$\Rightarrow \alpha = \frac{8 \pm \sqrt{(-8)^2 - 64}}{2}$$

$$= \frac{8 \pm 0}{2} = 4$$

$$\Rightarrow y = Ax^4 + B\ln|x|\, x^4$$ 3

5.) $f(t) = \sinh(t)$

$f''(t) = \sinh(t) \Rightarrow$

using $\mathcal{L}\{f''\} = s^2\mathcal{L}\{f\}$

$- s f(0) - f'(0)$

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$$L\{\sinh(t)\} = s^2 L\{\sinh(t)\} - s\sinh(0) - \cosh(0)$$

$$\Rightarrow L\{\sinh(t)\} = \frac{-1}{1-s^2}$$

$$= \frac{1}{s^2-1}$$

3