Imperial College London

[MP2 2012]

B.Sc. and M.Sci. EXAMINATIONS 2012

SECOND YEAR STUDENTS OF PHYSICS

MATHEMATICS - M.PHYS 2

Date Wednesday 6th June 2012   2 - 4 pm

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.

Do not attempt more than FOUR questions.

A mathematical formulae sheet is provided

[*Before starting, please make sure that the paper is complete; there should be 4 pages, with a total of SIX questions. Ask the invigilator for a replacement if your copy is faulty.*]

1. (i) Define $\cos(z)$ and $\cosh(z)$ for a complex variable $z = x + iy \in \mathbb{C}$.

   (ii) Show that $\cos(iz) = \cosh(z)$.

   Let

$$f(z) \;=\; \frac{1}{e^z - 1} \;.$$

   (iii) Identify all the poles of $f(z)$ and determine their nature.

   (iv) Determine the first three terms of the Laurent series of $f(z)$ about the point $z = 0$.

2. (i) Let

$$f(z) \;=\; \sum_{n=-\infty}^{\infty} K_n \, z^n \;,$$

   where $K_n \in \mathbb{C}$ are constants.

   Show by direct calculation that the integral

$$I \;=\; \int_C f(z)\, dz \;,$$

   around the unit circle C with centre at the origin and positive orientation, is given by

$$I \;=\; 2\pi i K_{-1} \;.$$

   (ii) Let $f(z)$ be a complex function. Consider the integral

$$I(R) \;=\; \int_{\cap} e^{ikz}\, f(z)\, dz$$

   around the semicircle in the upper half-plane from $(R, 0)$ to $(-R, 0)$. Assume $k > 0$ and that $|f(z)| \;\propto\; R^{\alpha}$ for $|z| = R$.

   Show that $\lim_{R\to\infty} I\,(R) = 0$ when $\alpha < 0$.

   You may use, without proof, Jordan's lemma $\displaystyle\int_0^{\pi/2} e^{-R\sin\theta}\, d\theta \;<\; \frac{\pi}{2R}$.

   (iii) Use contour integration to compute the integral

$$J \;=\; \int_0^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\, dx \;.$$

PLEASE TURN OVER

3. Dirac's δ-function is defined as follows

$$\forall f \in C^0([a, b], \mathbb{C}) \wedge \forall x_0 \in (a, b) : \int_a^b \delta(x - x_0)\, f(x)\, dx = f(x_0) .$$

(i) Show that

$$\delta(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ikx} .$$

(ii) Calculate the Fourier transform of $\sin x$.

(iii) Show that if the Fourier transforms satisfy

$$\widehat{f}_1(k) = \widehat{f}_2(k)\widehat{f}_3(k)$$

then the functions $f_1(x)$, $f_2(x)$ and $f_3(x)$ are related as follows

$$f_1(x) = \int_{-\infty}^{\infty} f_2(x - y)\, f_3(y)\, dy .$$

(iv) Assume the function $f(x)$ satisfies the equation

$$\frac{d^2 f}{dx^2} + m f(x) + \int_{-\infty}^{\infty} dy\, e^{-|x-y|}\, f(y) = \delta(x) .$$

Express the Fourier transform of $f(x)$ as a rational function of k.

4. (i) Show that any strongly convergent sequence is also a Cauchy sequence.

(ii) Let $f_n(x) = \tanh(nx) \in C^0(\mathbb{R})$ with $n \in \mathbb{N}$.

Use the l_2 norm to show that

$$\lim_{n\to\infty} \|f_n - \phi\| = 0$$

where

$$\phi(x) = \begin{cases} 1 & \text{for } x \geq 0 , \\ -1 & \text{for } x < 0 . \end{cases}$$

Hint: Consider the behaviour of $f_n(x)$ for n large. Consider the two cases $x > 0$ and $x < 0$ separately.

(iii) Explain why the result in (ii) proves that $C^0(\mathbb{R})$ with the l_2 norm is not a complete space.

(iv) Let $\langle x, y\rangle$ denote a scalar product on a complex vector space S.
Show that $\|x\| \equiv \langle x, x\rangle^{1/2}$ defines a norm on S.

5. (i) Use calculus of variations to find the shortest distance between two points in the plane.

   (ii) Consider a function $y(x)$ which satisfies

$$y(x) \;>\; 0 \;\text{ for }\; -a < x < a \;,$$

$$y(x) \;=\; 0 \;\text{ for }\; |x| \geq a \;.$$

   Assume also that for a given specified length of the curve between $(-a,\, 0)$ and $(a,\, 0)$ the function $y(x)$ maximizes the area between the x-axis and the graph of $y(x)$. Use calculus of variation under the constraint to derive an equation for $y(x)$ and thereby show that for $|x| \leq a$ the function $y(x)$ is part of a circle through $(-a,\, 0)$ and $(a,\, 0)$.

6. (i) Derive the Newton-Raphson algorithm.

   (ii) Find to two significant digits the roots of the equation

$$\cosh(x) \;+\; 2x \;=\; \cos(x) \;.$$

   (iii) Derive the trapezium rule.

   (iv) Derive the first order Runge-Kutta iteration scheme for the equation

$$\frac{dy}{dx} \;=\; f(x,\, y\,(x)) \;.$$

END OF PAPER