IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2005

EEE/ISE PART I: MEng, BEng and ACGI

COMMUNICATIONS 1

Friday, 27 May 10:00 am

Time allowed: 2:00 hours

None

Corrected Copy

There are FOUR questions on this paper.

Q1 is compulsory.
Answer Q1 and any two of questions 2-4.
Q1 carries 40% of the marks. Questions 2 to 4 carry equal marks (30% each).

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s) :	P.L. Dragotti, P.L. Dragotti
	Second Marker(s) :	M.K. Gurcan, M.K. Gurcan

Special Information for the Invigilators: none

Information for Candidates

The trigonometric Fourier series of a periodic signal $x(t)$ of period $T_0 = 2\pi/\omega_0$ is

$$x(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t),$$

with

$$a_0 = \frac{1}{T_0}\int_{T_0} x(t)dt, \quad a_n = \frac{2}{T_0}\int_{T_0} x(t)\cos n\omega_0 t dt, \quad b_n = \frac{2}{T_0}\int_{T_0} x(t)\sin n\omega_0 t dt.$$

Some Fourier Transforms

$$\cos\omega_0 t \iff \pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$$

$$\mathrm{rect}(\tfrac{t}{\tau}) \iff \tau\mathrm{sinc}(\tfrac{\omega\tau}{2})$$

$$\tfrac{W}{\pi}\mathrm{sinc}(Wt) \iff \mathrm{rect}(\tfrac{\omega}{2W})$$

Some useful trigonometric identities

$$\cos(x+y) = \cos x\cos y - \sin x\sin y$$

$$\sin x\cos y = \tfrac{1}{2}\sin(x-y) + \tfrac{1}{2}\sin(x+y)$$

$$\cos x\cos y = \tfrac{1}{2}\cos(x-y) + \tfrac{1}{2}\cos(x+y).$$

Euler's formula

$$e^{jx} = \cos x + j\sin x.$$

Steady-state impedance of a terminated transmission line

$$Z_{in} = Z_0\frac{Z_L + jZ_0\tan(kL)}{Z_0 + jZ_L\tan(kL)}$$

The Questions

1. This question is compulsory.

   (a) Consider the two signals $x_1(t) = \text{rect}(t)$ and $x_2(t) = \cos(4\pi t)\text{rect}(t - 0.5)$ shown in Figure 1a.

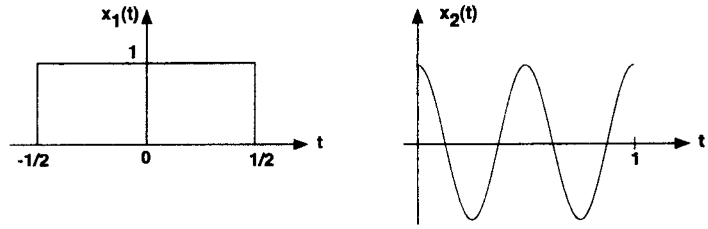


Figure 1a: The two energy signals $x_1(t)$ and $x_2(t)$.

   i. Determine the correlation between $x_1(t)$ and $x_2(t)$. Are $x_1(t)$ and $x_2(t)$ orthogonal?

[4]

   ii. Determine the energy of $z(t) = 4x_1(t) + 2x_2(t)$.

[4]

   (b) Consider the periodic signal $x(t)$ shown in Figure 1b. Compute the coefficients a_0 and a_1 of the trigonometric Fourier series of $x(t)$.

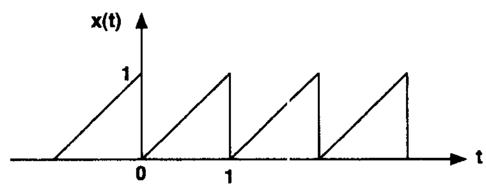


Figure 1b: The periodic signal $x(t)$.

[4]

   (c) From the definition of the Fourier transform, show that

$$g(t)e^{j\omega_0 t} \Longleftrightarrow G(\omega - \omega_0).$$

   Hence show that

$$g(t)\cos\omega_0 t \Longleftrightarrow \frac{1}{2}G(\omega - \omega_0) + \frac{1}{2}G(\omega + \omega_0)$$

[4]

(d) Consider the full AM signal $x(t) = [A + m(t)] \cos(\omega_c t)$ with $m(t) = 2 \cos 100t$ and $\omega_c = 10000$ rad/s.

i. Determine the minimum value of A that allows us to use an envelope detector.

[4]

ii. For $A = 4$, sketch and dimension the Fourier transform of $x(t)$.

[4]

iii. For $A = 4$, compute the power efficiency η.

[4]

(e) Develop a block diagram of an SSB-SC generator.

[4]

(f) Consider the PM signal

$$\varphi(t) = \cos[2\pi f_0 t + k_p m(t)]$$

where $m(t) = A \cos 2\pi f_m t$. Using Carson's rule, comment on the way the bandwidth of $\varphi(t)$ changes with the amplitude A, the frequency f_m and the frequency f_0.

[4]

(g) A 50 Ω transmission line is connected to a 100 Ω line with a matched termination. A sine wave of 10 V amplitude propagating in the former is incident on the junction. Find

i. The voltage reflection coefficient k_v.

[2]

ii. The current amplitude of the reflected wave.

[2]

2. Consider the FM signal

$$\varphi(t) = 10\cos[2\pi f_0 t + k_f \int_{-\infty}^{t} x(\alpha)d\alpha]$$

where $k_f = 10\pi$. The message $x(t)$ is given by

$$x(t) = \sum_{n=0}^{2} m_n(t)$$

with

$$m_n(t) = \frac{2^n}{\pi}\text{sinc}(t)\cos(2nt).$$

(a) Sketch and dimension the Fourier transform of $m_1(t)$.

[6]

(b) Sketch and dimension the Fourier transform of $x(t)$.

[6]

(c) Using Carson's rule, determine the bandwidth of $\varphi(t)$.

[6]

(d) Assume now that $x(t) = Ae^{-10t}u(t)$. Using Carson's rule, the bandwidth of $\varphi(t)$ is 50.4 Hz. Find the amplitude A of $x(t)$. Select the bandwidth, B, of the baseband message $x(t)$ so that it contains 95% of the signal energy.

[12]

3. Consider the system shown in Figure 3.

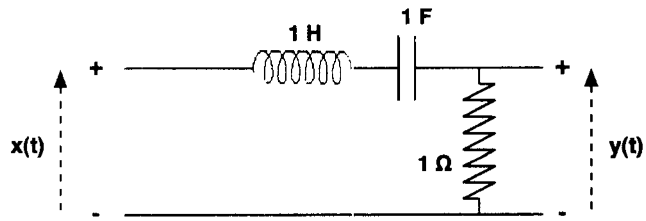


Figure 3: An RLC circuit.

(a) Determine the transfer function $H(\omega)$.

[6]

(b) Determine $|H(\omega)|^2$.

[6]

(c) Determine the frequency ω_0 at which $|H(\omega)|^2$ is maximum.

[6]

(d) The input voltage $x(t)$ has an autocorrelation $\mathcal{R}_x(\tau) = 5\cos(\omega_0\tau)$. Determine the maximum frequency ω_0 at which the ratio $P_y/P_x = 0.8$. Here, P_y and P_x are the power of the output and input signals respectively.

[12]

4. Three lines of identical length, characteristic impedance and phase velocity are connected in series as shown in Figure 4, one with an open circuit termination, one with a short circuit termination and the third with a matched termination. The three transmission lines have $L_0 = 0.25\ \mu\text{H/m}$ and $C_0 = 100\ \text{pF/m}$.

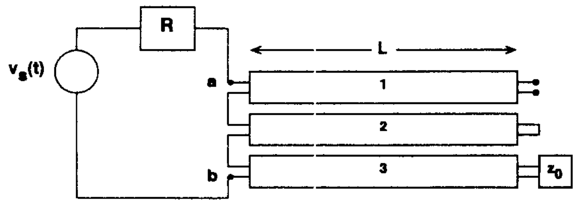


Figure 4: The circuit with three transmission lines.

(a) Determine the characteristic impedance and the phase velocity of the three lines.

[6]

(b) The circuit of Figure 4 is now driven by a signal $v_s(t) = V_0 \exp(j2\pi f_0 t)$ with $V_0 = 5$ V, $f_0 = 1$ MHz and internal resistance $R = 50\ \Omega$. Find the shortest length L for which the combined steady-state impedance of the three lines, as measured at terminals a-b, will be 50 Ω.

[12]

(c) If the length L satisfies the condition described in part (b) above, find the steady state voltage $v_1(x,t)$, along the first line. Hence, for this line, calculate the value of the largest voltage amplitude.

[12]

E1.6 Communications I 2005 **SOLUTIONS** 1

QUESTION 1 (ALL QUESTIONS IN QUESTION 1 ARE 'BOOKWORK')

a)

i) $$C_{x_1,x_2} = \int_0^{0.5} \cos 4\pi t \, dt = \frac{1}{4\pi} \sin 4\pi t \Big|_0^{0.5} = 0$$

$x_1 \perp x_2$

ii) $E_t = 16 E_{x_1} + 4 E_{x_2}$

$E_{x_1} = 1 \qquad E_{x_2} = \frac{1}{2}$

$E_t = 16 + 2 = 18$

b) $$a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)\,dt \qquad T_0 = 1 \Rightarrow a_0 = \frac{1}{2}$$

$$a_1 = \int_0^1 x(t)\cos 2\pi t\,dt = \int_0^1 t\cos 2\pi t\,dt =$$

$$= t \cdot \frac{\sin 2\pi t}{2\pi}\Big|_0^1 - \int_0^1 \frac{\sin 2\pi t}{2\pi}\,dt = 0$$

c) $$G(w) = \int_{-\infty}^{\infty} g(t)\, e^{-jwt}\,dt$$ WE HAVE

$$\int_{-\infty}^{\infty} g(t)\, e^{+jw_0 t}\, e^{-jwt}\,dt = \int_{-\infty}^{\infty} g(t)\, e^{-j(w-w_0)t}\,dt =$$

$$= G(w-w_0)$$

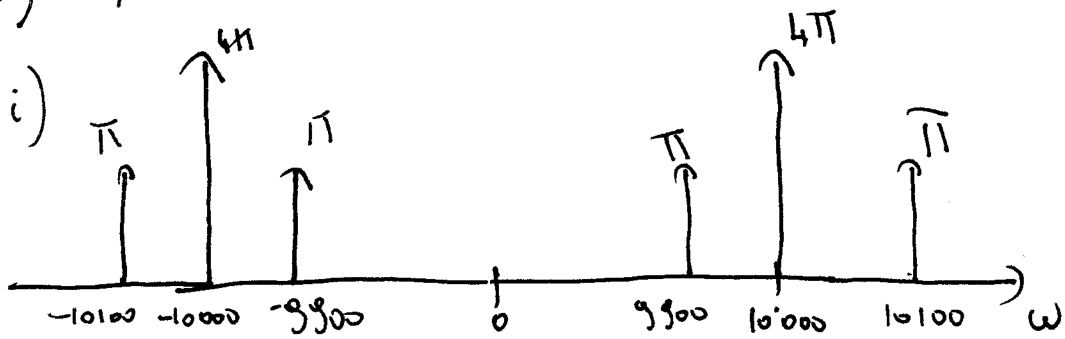
$\cos w_0 t = \frac{1}{2}\left(e^{jw_0 t} + e^{-jw_0 t}\right)$ THEREFORE USING THE LINEARITY PROPERTY OF THE FOURIER TRANSFORM AND THE RESULT ABOVE WE HAVE THAT

$$g(t)\cos w_0 t \iff \frac{1}{2}G(w-w_0) + \frac{1}{2}G(w+w_0)$$

d)

i) $A = 2$

ii)



iii) $P_c = \frac{A^2}{2} = \frac{16}{2} = 8$

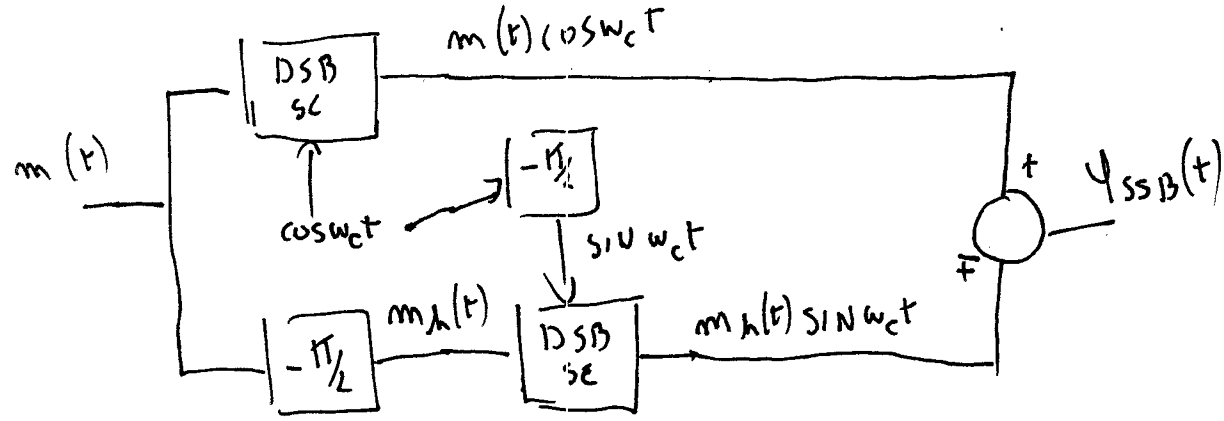
$$P_S = \frac{P_m}{2} = \frac{2}{2} = 1$$

$$\eta = \frac{P_S}{P_c + P_S} = \frac{1}{9}$$

3

e)

$$\varphi_{SSB} = m(t)\cos\omega_c t \mp m_h(t)\sin\omega_c t$$



f) THE BANDWIDTH INCREASES LINEARLY WITH A AND f_m AND IS NOT INFLUENCED BY f_0.

g)

i) $K_V = \dfrac{Z_L - Z_0}{Z_L + Z_0} = \dfrac{1}{3}$

ii) $I_- = -K_V I_+ = -K_V V_+/Z_0 = -\frac{1}{15}\ A =$

$= -0.67\ A$

2)

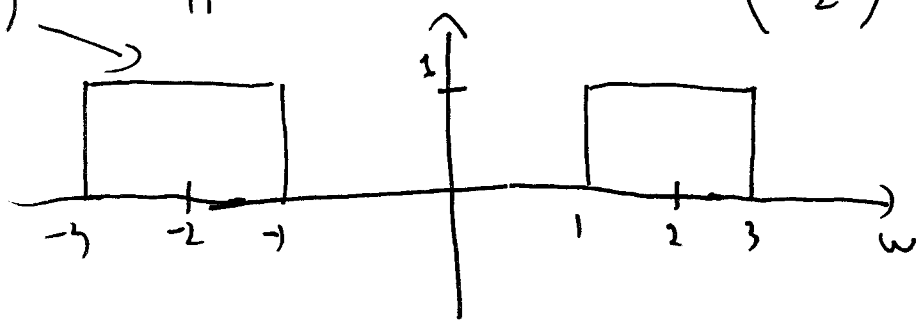
(a) $m_1(t) = \frac{2}{\pi} \operatorname{SINC}(t) \cos(2t)$

$$\frac{2}{\pi} \operatorname{SINC}(t) \Longleftrightarrow 2 \operatorname{RECT}\left(\frac{\omega}{2}\right)$$

THEREFORE

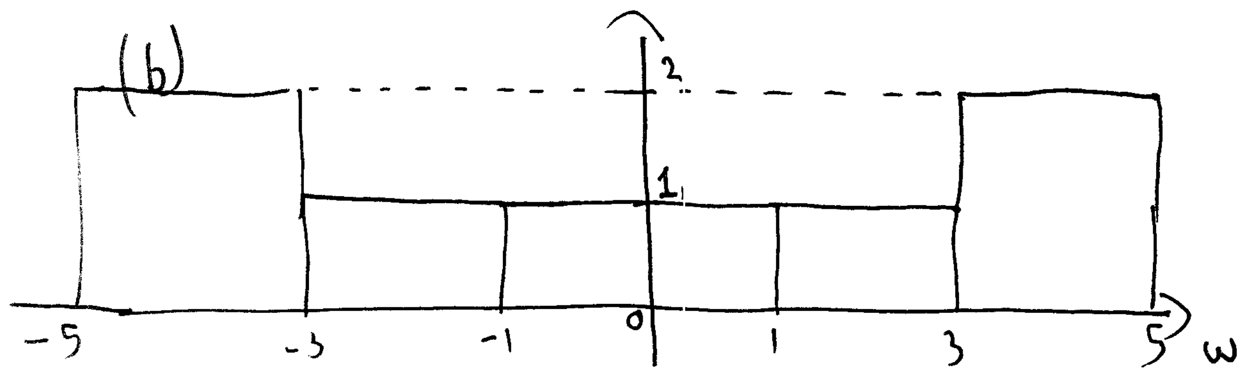
$$\frac{2}{\pi} \operatorname{SINC}(t) \cos 2t \Longleftrightarrow \operatorname{RECT}\left(\frac{\omega-2}{2}\right) + \operatorname{RECT}\left(\frac{\omega+2}{2}\right)$$

$M_1(\omega)$



(NEW COMPUTED EXAMPLE)

(b)



(NEW COMPUTED EXAMPLE)

(c) $B_{FM} = 2(\Delta f + B) = 2\left(\frac{K_f \cdot X_p}{2\pi} + B\right)$

$B = \frac{5}{2\pi}$ Hz $\quad X_p = \frac{4}{\pi}$

THUS $\quad B_{FM} = 2\left(\frac{10\pi \cdot \frac{4}{\pi}}{2\pi} + \frac{5}{2\pi}\right) = \frac{45}{\pi}$ Hz

(NEW COMPUTED EXAMPLE)

(d) $x(t) = Ae^{-10t}u(t)$

$$X(\omega) = A\int_{-\infty}^{\infty} e^{-10t}u(t)e^{-j\omega t}\,dt = \frac{A}{10+j\omega}$$

USE PARSEVAL'S THEOREM TO FIND THE BANDWIDTH OF $x(t)$:

$$E_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega = \frac{A^2}{20}$$

CALL $W = 2\pi B$, W MUST BE SUCH THAT

$$A^2\frac{0.95}{20} = \frac{1}{2\pi}\int_{-W}^{W} |X(\omega)|^2 d\omega = \frac{A^2}{2\pi}\int_{-W}^{W}\frac{d\omega}{\omega^2+100} =$$

$$= \frac{A^2}{10\pi}\tan^{-1}\frac{W}{10} \Rightarrow W = 127.61 \text{ RAD/s}$$

AND $B = 20.31$ Hz

$x_p = A$.

THUS

$$B_{FM} = 2\left(\frac{k_f \cdot A}{2\pi} + B\right) = 2\left(\frac{10\pi A}{2\pi} + 20.2\right) = 50.4\text{ Hz}$$

$\Rightarrow A = 1$

(NEW APPLICATION OF THEORY)

3)

(a) $$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{j\omega}{1 + j\omega - \omega^2}$$ ('NEW APPLICATION OF THE THEORY')

(b) $$|H(\omega)|^2 = \frac{\omega^2}{1 + \omega^4 - \omega^2}$$ ('NEW APPLICATION OF THE THEORY')

(c) $$\frac{d|H(\omega)|^2}{d\omega} = \frac{2\omega(1 + \omega^4 - \omega^2) - \omega^2(4\omega^3 - 2\omega)}{(1 + \omega^4 - \omega^2)^2} = 0$$

$$\Rightarrow \quad 2\omega(1 - \omega^4) = 0$$

$\omega = 0$ GIVES A MINIMUM

$\omega_0 = 1$ GIVES THE MAXIMUM

(d)

$$R_x(\tau) \Leftrightarrow S_x(\omega)$$

SINCE $$S_x(\omega) = 5\pi\left[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right]$$

AND $$S_y(\omega) = |H(\omega)|^2 \cdot S_x(\omega)$$

WE HAVE THAT

$$\frac{P_y}{P_x} = |H(\omega_0)|^2$$ ('NEW APPLICATION OF THE THEORY')

WE NEED TO FIND ω_0 SUCH THAT

$$|H(\omega_0)|^2 = 0.8 \Rightarrow$$

$$(*) \quad \frac{\omega^2}{1+\omega^4-\omega^2} = 0.8 \qquad \text{CALL} \quad x=\omega^2$$

WE HAVE

$$0.8x^2 - 1.8x + 0.8 = 0$$

$$x = \frac{0.9 \pm \sqrt{0.17}}{0.8} \Rightarrow$$

$$\omega = \pm\sqrt{\frac{0.9 \pm \sqrt{0.17}}{0.8}}$$

THE SOLUTION MUST BE POSITIVE, THUS THE MAXIMUM ω SATISFING $(*)$ IS

$$\omega_0 = \sqrt{\frac{0.9+\sqrt{0.17}}{0.8}} = 1.281 \text{ RAD/s}$$

4)

a) $Z_0 = 50\,\Omega$

$u = 2 \cdot 10^8$ m/sec

('BOOKWORK')

b) $$Z_{IN} = Z_0 \left[\frac{Z_L + j Z_0 \tan kl}{Z_0 + j Z_L \tan kl} \right]$$

LINE 1 $\quad Z_L = \infty$

THUS $\quad Z_{IN} = \dfrac{Z_0}{j \tan kl}$

LINE 2 $\quad Z_L = 0$

THUS $\quad Z_{IN} = j Z_0 \tan kl$

COMBINING IN SERIES

$$Z_{IN} = Z_0 \left(\frac{1}{j \tan kl} + j \tan kl \right) = j Z_0 \left(\tan kl - \frac{1}{\tan kl} \right) = 0$$

$$\Rightarrow \quad \tan kl = \frac{1}{\tan kl} = 1 \quad \Rightarrow \quad kl = \pi/4$$

$$L = \pi/4k = \frac{\pi}{4} \cdot \frac{2 \cdot 10^8}{2\pi \cdot 10^6} = 25\,m$$

('NEW COMPUTED EXAMPLE')

c) IN LINE 1 $\quad K_V = \dfrac{Z_L - Z_0}{Z_L + Z_0} = 1$

THUS $V_+ = V_-$

$$V_{ns}(x,t) = V_+ \exp(j\omega t - jkx) + V_+ \exp(j\omega t - jkx) =$$

$$= 2V_+ \cos(kx) \exp(j\omega t)$$

~~H~~ FOR $x = -L$ WE HAVE THAT

$$2V_+ \cos(kL) \cancel{\exp(j\omega t)} = t_{IN} \cdot \frac{V_0}{2t_0} \cancel{\exp(j\omega t)} =$$

$$= \frac{V_0}{j2 \tan kL}$$

$\tan kL = 1$ AND $\cos(kL) = \cos(\pi/4) = \frac{\sqrt{2}}{2}$

THUS $V_+ = \frac{V_0}{j2\sqrt{2}}$

THE MAXIMUM IS ACHIEVED FOR $x=0$ AND

$$|V_1(0,t)| = |2V_+| = \frac{V_0}{\sqrt{2}} = \frac{5}{\sqrt{2}} \text{ VOLTS}.$$

('NEW COMPUTED EXAMPLE')