

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING  
EXAMINATIONS 2009

MSc and EEE PART IV: MEng and ACGI

*Corrections.*

*Q1 (a)*

*Q2 (b)(iii)*

*Q5 (b)(iv)*

**PROBABILITY AND STOCHASTIC PROCESSES**

Thursday, 7 May 10:00 am

Time allowed: 3:00 hours

**There are SIX questions on this paper.**

**Answer FOUR questions.**

*All questions carry equal marks*

**Any special instructions for invigilators and information for candidates are on page 1.**

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## PROBABILITY AND STOCHASTIC PROCESSES

1. Let  $X$  and  $Y$  be two random variables, where  $m_X$  and  $m_Y$  denote their respective means,  $\sigma_X$  and  $\sigma_Y$  denote their respective variances.

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- a) Give the definition of the ~~covariance~~  $\rho_{XY}$  and prove that  $\rho_{XY} \in [-1, 1]$ . [ 5 ]

Hint: Let  $U = \frac{X - m_X}{\sigma_X}$  and  $V = \frac{Y - m_Y}{\sigma_Y}$ . Use the fact that

$$\mathbf{E}((U + V)^2) \geq 0 \quad \text{and} \quad \mathbf{E}((U - V)^2) \geq 0.$$

- b) We say that  $X$  and  $Y$  are linearly dependent if there exist two constants  $a \neq 0$  and  $b$  such that  $Y = aX + b$ .

- (i) Show that if  $X$  and  $Y$  are linearly dependent then  $|\rho_{XY}| = 1$ . [ 2 ]  
(ii) Show that if  $\rho_{XY} = 1$  then  $X$  and  $Y$  are linearly dependent. [ 3 ]

Hint: In (ii), show that  $\mathbf{E}((U - V)^2) = 0$ , for  $U = \frac{X - m_X}{\sigma_X}$  and  $V = \frac{Y - m_Y}{\sigma_Y}$ .

- c) We consider two dependent (correlated) random variables  $X$  and  $Y$ . The best linear estimator of  $Y$  given  $X$  is given by  $\hat{Y} = aX + b$  which minimises  $\mathbf{E}[(Y - \hat{Y})^2]$ .

- (i) Show that

$$\hat{Y} = \rho_{XY} \frac{\sigma_Y}{\sigma_X} (X - m_X) + m_Y.$$

[ 5 ]

- (ii) Prove that

$$\mathbf{E}[(Y - \hat{Y})^2] = (1 - \rho_{XY}^2) \sigma_Y^2.$$

[ 3 ]

- (iii) For which value of  $\rho_{XY}$  is the estimation exact?

[ 2 ]

Correlation  
Variance of  $X$  is  $\sigma_X^2$   
Variance of  $Y$  is  $\sigma_Y^2$

2. Let  $(X_t, t \geq 0)$  be a random telegraph process with parameter  $\lambda$ , i.e. a  $\{-1, 1\}$ -valued continuous time process such that the number of 0 crossings in the interval  $(0, t)$  is described by a Poisson process with parameter  $\lambda t$ . Assume that  $X_0$  is such that

$$\mathbf{P}(X_0 = 1) = \mathbf{P}(X_0 = -1) = \frac{1}{2}$$

- a) Let  $t, \tau \geq 0$

- (i) Show that

$$\mathbf{P}(\text{There are } n \text{ crossings between } t \text{ and } t + \tau) = e^{-\lambda\tau} \frac{(\lambda\tau)^n}{n!}.$$

[ 1 ]

- (ii) Prove that

$$\begin{aligned} \mathbf{P}(X_{t+\tau} = 1 \mid X_t = 1) &= \mathbf{P}(X_{t+\tau} = -1 \mid X_t = -1) \\ &= e^{-\lambda\tau} \left[ 1 + \frac{(\lambda\tau)^2}{2!} + \frac{(\lambda\tau)^4}{4!} + \dots \right]. \end{aligned}$$

[ 3 ]

- (iii) Prove that

$$\begin{aligned} \mathbf{P}(X_{t+\tau} = 1 \mid X_t = -1) &= \mathbf{P}(X_{t+\tau} = -1 \mid X_t = 1) \\ &= e^{-\lambda\tau} \left[ \lambda\tau + \frac{(\lambda\tau)^3}{3!} + \frac{(\lambda\tau)^5}{5!} + \dots \right]. \end{aligned}$$

[ 3 ]

- (iv) Show that  $\mathbf{E}(X_\tau) = 0$  for all  $\tau \geq 0$ .

[ 3 ]

*Hint: Compute  $\mathbf{E}(X_{t+\tau} \mid X_t = 1)$  and  $\mathbf{E}(X_{t+\tau} \mid X_t = -1)$  then use Bayes's rule.*

- b) We now focus on the autocorrelation function  $R_X(\tau)$  of the process  $(X_t, t \geq 0)$ .

- (i) Give the definition of  $R_X(\tau)$ .

[ 1 ]

- (ii) Show that

$$R_X(\tau) = \frac{1}{2} \mathbf{E}(X_{t+\tau} \mid X_t = 1) - \frac{1}{2} \mathbf{E}(X_{t+\tau} \mid X_t = -1).$$

[ 3 ]

- (iii) Prove that  $R_X(\tau) = e^{-2\lambda\tau}$ .

[ 4 ]

- c) Conclude that  $X_t$  is wide-sense stationary process.

[ 2 ]

3. We perform  $n$  tosses of a fair coin. The variable  $X_i$  describes the outcome of the  $i$ -th toss:  $X_i = 1$  if Heads shows and  $X_i = 0$  if Tail shows. Let  $X = \sum_{i=1}^n X_i$ .

a) State the distribution of  $X$ , and compute its expectation and its variance. [ 2 ]

b) We now examine the probability that  $X$  deviates from its mean.

(i) Show that

$$\mathbf{P}\left(X \geq \frac{3n}{4}\right) \leq \mathbf{P}\left(\left|X - \frac{n}{2}\right| \geq \frac{n}{4}\right).$$

[ 2 ]

(ii) Using Chebyshev's inequality, prove that

$$\mathbf{P}\left(X \geq \frac{3n}{4}\right) \leq \frac{4}{n}.$$

[ 4 ]

(iii) Show that  $\lim_{n \rightarrow \infty} \mathbf{P}(X \geq \frac{3n}{4}) = 0$  and comment.

[ 2 ]

c) We now derive a tighter bound for the convergence of  $\mathbf{P}(X \geq \frac{3n}{4})$  to 0 as  $n$  goes to  $\infty$ .

(i) Let  $x, \theta \geq 0$ . Combining Markov's inequality and the fact that

$$\{X \geq x\} = \{e^{\theta X} \geq e^{\theta x}\},$$

prove that

$$\mathbf{P}(X \geq x) \leq \exp\left(\frac{n}{2}(e^\theta - 1) - \theta x\right).$$

[ 5 ]

*Hint: Use the following inequality  $1 + \alpha \leq e^\alpha$ , for  $\alpha \geq 0$ .*

(ii) Choose  $\theta$  so that  $\frac{1}{2}(e^\theta - 1) - \frac{3}{4}\theta \leq -0.01$ .

[ 1 ]

(iii) Prove that for the choice of  $\theta$  in the previous question

$$\mathbf{P}\left(X \geq \frac{3n}{4}\right) \leq e^{-0.01n}.$$

[ 4 ]

4. Let  $Y_1, Y_3, Y_5, \dots$  be a sequence of independent and identically distributed random variables such that  $\mathbf{P}(Y_{2k+1} = -1) = \mathbf{P}(Y_{2k+1} = 1) = \frac{1}{2}$ , for  $k = 0, 1, 2, \dots$ . Let  $Y_{2k} = Y_{2k-1}Y_{2k+1}$ , for  $k = 1, 2, \dots$ .
- a) Show that  $\mathbf{P}(Y_{2k} = \alpha, Y_{2k+2} = \beta) = 1/4$ , for  $\alpha, \beta \in \{-1, 1\}$ . Conclude that  $Y_2, Y_4, Y_6, \dots$  is a sequence of independent and identically distributed random variables and give their joint distribution. [ 3 ]
- b) Show that  $\mathbf{P}(Y_{2k} = \alpha, Y_{2k+1} = \beta) = 1/4$ , for  $\alpha, \beta \in \{-1, 1\}$ . Is the sequence  $Y_1, Y_2, Y_3, \dots$  independent and identically distributed? [ 3 ]
- c) Compute  $\mathbf{P}(Y_{2k+1} = 1 \mid Y_{2k} = -1)$  and  $\mathbf{P}(Y_{2k+1} = 1 \mid Y_{2k} = -1, Y_{2k-1} = 1)$ . Is the process  $Y_1, Y_2, Y_3, \dots$  a Markov chain? [ 4 ]
- d) Let  $Z_n = (Y_n, Y_{n+1})$  be a process in  $\{0, 1\}^2$ .
- (i) Show that
- $$\mathbf{P}(Z_{n+1} = (1, 1) \mid Z_n = (1, 1)) = \begin{cases} \frac{1}{2}, & \text{if } n \text{ even,} \\ 1, & \text{if } n \text{ odd,} \end{cases}$$
- [ 4 ]
- (ii) Show that  $(Z_n, n \geq 0)$  is a (non-homogeneous) Markov chain and give its transition probabilities. [ 6 ]

5. Consider the weather chain  $(X_n, n \geq 0)$  set to  $X_n = 1$  if it rains on day  $n$  and  $X_n = 0$  otherwise. Suppose that  $X_n$  evolves as a Markov chain with transition matrix

$$P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}.$$

where  $\alpha, \beta \in [0, 1]$ .

- a) Sketch the diagram of the evolution of the above Markov chain and briefly describe the dynamics in the following cases (i)  $\alpha = \beta = 0$ , (ii)  $\alpha = \beta = 1$ , (iii)  $\alpha = 1, \beta = 0$  and (iv)  $\alpha = 0, \beta = 1$ . [ 2 ]

- b) In what follows we suppose that  $\alpha, \beta \in (0, 1)$ .

- (i) Let  $p_{00}(n) = \mathbf{P}(X_n = 0 \mid X_0 = 0)$ . Show that

$$p_{00}(n+1) = (1 - \alpha - \beta)p_{00}(n) + \beta, \quad \text{for } n \geq 0.$$

[ 4 ]

- (ii) Prove that

$$p_{00}(n) = \frac{\beta}{\alpha + \beta} + \frac{\alpha}{\alpha + \beta}(1 - \alpha - \beta)^n.$$

[ 2 ]

- (iii) Derive the expressions for  $p_{01}(n) = \mathbf{P}(X_n = 1 \mid X_0 = 0)$ ,  $p_{10}(n) = \mathbf{P}(X_n = 0 \mid X_0 = 1)$  and  $p_{11}(n) = \mathbf{P}(X_n = 1 \mid X_0 = 1)$ . [ 4 ]

- (iv) Compute the limits  $p_{11}(n), p_{10}(n), p_{01}(n), p_{00}(n)$  when  $n$  goes to infinity. [ 2 ]

- c) Using two different methods, compute the stationary distribution of the weather chain. [ 6 ]

6. Consider the Markov chain on  $\{1, 2, 3, 4\}$  with the following transition matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We define the probability of absorption by

$$h_i = \mathbf{P}(X_n \text{ is absorbed in state 4} \mid X_0 = i)$$

and the expected time to absorption by

$$k_i = \mathbf{E}(\text{time for } X_n \text{ to be absorbed in state 1 or 4} \mid X_0 = i)$$

where  $i = 1, 2, 3, 4$ .

In what follows, **carefully justify** your results.

- a) First compute the probabilities of absorption  $h_i$  :

- (i) Show that

$$h_2 = \frac{1}{2}h_1 + \frac{1}{2}h_3 \quad \text{and} \quad h_3 = \frac{1}{2}h_2 + \frac{1}{2}h_4.$$

[ 6 ]

- (ii) After deriving the values of  $h_1$  and  $h_4$ , show that

$$h_2 = \frac{1}{3} \quad \text{and} \quad h_3 = \frac{2}{3}.$$

[ 4 ]

- b) Now compute the expected times to absorption  $k_i$  :

- (i) Show that

$$k_2 = 1 + \frac{1}{2}k_1 + \frac{1}{2}k_3 \quad \text{and} \quad k_3 = 1 + \frac{1}{2}k_2 + \frac{1}{2}k_4$$

[ 6 ]

- (ii) After deriving the values of  $k_1$  and  $k_4$ , show that

$$k_2 = 2 \quad \text{and} \quad k_3 = 2.$$

[ 4 ]

Q1

$$a) \rho_{xy} = \frac{IE((X-m_x)(Y-m_y))}{\sigma_x \sigma_y} = \frac{IE(XY) - m_x m_y}{\sigma_x \sigma_y}.$$

We immediately check that

$$IE(U) = IE(V) = 0, \quad \sigma_U = \sigma_V = 1 \quad \& \quad IE(UV) = \rho_{xy}.$$

$(U+V)^2$  &  $(U-V)^2$  are non-negative so

$$(1) IE((U+V)^2) \geq 0 \quad \& \quad (2) IE((U-V)^2) \geq 0.$$

$$(1) \Rightarrow \rho_{xy} \geq -1 \quad ; \quad (2) \Rightarrow \rho_{xy} \leq 1.$$

$$b/ (i) \quad Y = aX + b.$$

$$\sigma_Y^2 = IE((Y-m_Y)^2) = a^2 \sigma_X^2 \Rightarrow \sigma_Y = |a| \sigma_X$$

$$\rho_{xy} = \frac{IE((X-m_x)(aX+b-a m_x-b))}{\sigma_x \sigma_y}$$

$$= \frac{a IE((X-m_x)^2)}{|a| \sigma_x} = \frac{a}{|a|}$$

$$|\rho_{xy}| = 1.$$

$$(ii) \quad \rho_{xy} = 1. \quad IE((U-V)^2) = IE(U^2) - 2IE(UV) + IE(V^2) = 0$$

$$\Rightarrow U = V \quad (\text{with probability } 1).$$

$$\Rightarrow \frac{X-m_x}{\sigma_x} = \frac{Y-m_y}{\sigma_y} \quad \& \quad X \& Y \text{ are linearly dependent}$$



(2)

Q1)

(c) (i)  $\hat{y} = aX + b$  let  $a^*$  &  $b^*$  be  $a$  &  $b$  to minimize

$$E((Y - \hat{y})^2) = f(a, b) = E((Y - aX - b)^2).$$

$$\frac{\partial f(a, b)}{\partial b} = -2 E(Y - aX - b)$$

$$\Rightarrow b^* = m_Y - a m_X.$$

$$f(a, b^*) = E\left(\left[(Y - m_Y) - a(X - m_X)\right]^2\right)$$

$$\frac{\partial f(a, b^*)}{\partial a} = -2 E[(Y - m_Y)(X - m_X)] + 2a E((X - m_X)^2).$$

$$\Rightarrow a^* = \frac{E((Y - m_Y)(X - m_X))}{E((X - m_X)^2)} = \rho_{XY} \frac{\sigma_Y}{\sigma_X}.$$

$$\boxed{\hat{y} = \rho_{XY} \frac{\sigma_Y}{\sigma_X} (X - m_X) + m_Y}$$

$$\begin{aligned} \text{(ii)} \quad E((Y - \hat{y})^2) &= E\left(\left[(Y - m_Y) - \rho_{XY} \frac{\sigma_Y}{\sigma_X} (X - m_X)\right]^2\right) \\ &= (1 - \rho_{XY}^2) \sigma_Y^2 \end{aligned}$$

(iii) The above implies that the estimator  $\hat{y}$  determines  $Y$  exactly if  $\rho_{XY}^2 = 1$ .

(3)

(Q2) Similar to problem solved in lecture.

(i) a) Direct translation of the definitions

(ii)

$$P(X_{t+\tau} = 1 | X_t = 1) = P(X_{t+\tau} = -1 | X_t = -1)$$

=  $P(\exists \text{ even number of crossings between } t \text{ and } t+\tau)$

$$= e^{-\lambda\tau} \sum_{k \geq 0} \frac{(\lambda\tau)^{2k}}{(2k)!} = e^{-\lambda\tau} \cosh(\lambda\tau)$$

(iii)

Similarly

$$P(X_{t+\tau} = 1 | X_t = -1) = P(X_{t+\tau} = -1 | X_t = 1)$$

=  $P(\exists \text{ odd number of crossings in } (t, t+\tau))$

$$= e^{-\lambda\tau} \sum_{k \geq 0} \frac{(\lambda\tau)^{2k+1}}{(2k+1)!} = e^{-\lambda\tau} \sinh(\lambda\tau)$$

$$(iv) E(X_t) = E(X_t | X_0 = 1) P(X_0 = 1) + E(X_t | X_0 = -1) P(X_0 = -1)$$

$$= \frac{1}{2} (E(X_t | X_0 = 1) + E(X_t | X_0 = -1))$$

$$E(X_t | X_0 = 1) = P(X_t = 1 | X_0 = 1) - P(X_t = -1 | X_0 = 1)$$

$$= e^{-\lambda t} \sum_{k \geq 0} \frac{(\lambda t)^{2k}}{(2k)!} - e^{-\lambda t} \sum_{k \geq 0} \frac{(\lambda t)^{2k+1}}{(2k+1)!}$$

$$= e^{-\lambda t} \left( \sum_{k \geq 0} \frac{(-1)^k (\lambda t)^k}{k!} \right)$$

$$= e^{-2\lambda t}$$

Q2

a) (iv)

(4)

$$\begin{aligned}
 E(X_t | X_0 = -1) &= IP(X_t = 1 | X_0 = -1) - IP(X_t = -1 | X_0 = -1) \\
 &= e^{-\lambda t} \sum_{k \geq 0} \frac{(\lambda t)^{2k+1}}{(2k+1)!} - e^{-\lambda t} \sum_{k \geq 0} \frac{(\lambda t)^{2k}}{(2k)!} \\
 &= -e^{-2\lambda t}.
 \end{aligned}$$

$$\Rightarrow E(X_t) = 0 \quad \forall t$$

b)

$$(i) \quad R_X(\tau) = E(X_{t+\tau} X_t).$$

$$\begin{aligned}
 (ii) \quad R_X(\tau) &= E(X_{t+\tau} X_t | X_t = 1) IP(X_t = 1) \\
 &\quad + E(X_{t+\tau} X_t | X_t = -1) IP(X_t = -1)
 \end{aligned}$$

$$= E(X_{t+\tau} | X_t = 1) IP(X_t = 1) - E(X_{t+\tau} | X_t = -1) IP(X_t = -1),$$

As previously

$$= \cancel{e^{-2\lambda\tau}} e^{2\lambda\tau} IP(X_t = 1) + e^{-2\lambda\tau} IP(X_t = -1)$$

$$(iii) \quad X_t \text{ is } \{1, -1\} \text{-valued} \Rightarrow IP(X_t = 1) + IP(X_t = -1) = 1$$

$$\Rightarrow R_X(\tau) = e^{-2\lambda\tau}.$$

$$\begin{aligned}
 c) \quad X_t \text{ is such that } E(X_t) &= 0 \text{ \& } \\
 E(X_{t+\tau} X_t) &= e^{-2\lambda\tau} \quad \forall t, \tau
 \end{aligned}$$

$$\Rightarrow X_t \text{ is a wide sense stationary process.}$$

Q3

a)  $X$  is Binomial with parameters  $n$  &  $1/2$

$$IP(X=k) = \binom{n}{k} \left(\frac{1}{2}\right)^n$$

$$E(X) = E\left(\sum_i X_i\right) = n E(X_1) = np = n/2$$

$$Var(X) = n Var(X_1) = np(1-p) \quad (X_i \text{ independent}) \\ = n/4$$

b).

(i) The event  $\{ |X - n/2| \geq n/4 \} = \{ X \geq \frac{3n}{4} \} \cup \{ X \leq \frac{n}{4} \}$

which contains the event  $\{ X \geq \frac{3n}{4} \}$

$$\Rightarrow IP(|X - n/2| \geq n/4) \geq IP(X \geq \frac{3n}{4})$$

(ii) By Chebyshev,

$$IP(X \geq \frac{3n}{4}) \leq IP(|X - n/2| \geq \frac{n}{4}) \\ \leq \frac{Var(X)}{(n/4)^2}$$

$$= \frac{4}{n}$$

$$(iii) \lim_{n \rightarrow \infty} IP(X \geq \frac{3n}{4}) \leq \lim_{n \rightarrow \infty} 4/n \rightarrow 0$$

(6)

c) By Markov

$$P(X \geq n) = P(e^{\theta X} \geq e^{\theta n}) \leq E(e^{\theta X}) e^{-\theta n}.$$

$$E(e^{\theta X}) = \left( \frac{1}{2} + \frac{1}{2} e^{\theta} \right)^n = \left( 1 + \frac{1}{2} (e^{\theta} - 1) \right)^n \\ \leq \exp \left\{ \frac{n}{2} (e^{\theta} - 1) \right\}.$$

Hence,  $P(X \geq n) \leq \exp \left( \frac{n}{2} (e^{\theta} - 1) - \theta n \right).$

In our example  $n = 3n/4$ .

$$P(X \geq \frac{3n}{4}) \leq \exp \left\{ \frac{n}{2} (e^{\theta} - 1) - \frac{3n}{4} \theta \right\}.$$

(ii) For  $\theta = \log 2$   $\frac{1}{2} (e^{\theta} - 1) - \frac{3}{4} \theta \approx -0.019$   
 $\leq -0.01.$

(iii) Putting everything together we get

$$P(X \geq \frac{3n}{4}) \leq \exp \{ -0.01 n \}. \xrightarrow{n \rightarrow \infty} 0$$

Q4

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a)  
Joint distribution.

$Y_{2k} \backslash Y_{2k+2}$	1	-1
1	$1/4$	$1/4$
-1	$1/4$	$1/4$

Marginal distribution.

$$P(Y_{2k}=1) = P(Y_{2k}=-1) = 1/2$$

$$\& P(Y_{2k+2}=1) = P(Y_{2k+2}=-1) = 1/2.$$

$(Y_{2k})$  is i.i.d.

b)  
Joint distribution

$Y_{2k} \backslash Y_{2k+1}$	1	-1
1	$1/4$	$1/4$
-1	$1/4$	$1/4$

$$\& P(Y_{2k}=1) = P(Y_{2k}=-1) = 1/2$$

$$\text{an! } P(\sum_{2k+1} = 1) = P(Y_{2k+1}=1) = 1/2$$

The sequence is pairwise independent but not i.i.d.

as  $Y_{2k}, Y_{2k+1}$  &  $Y_{2k-1}$  are not independent by construction.

d4

(8)

c) ~~IP~~  $IP(Y_{2k+1} = 1 \mid Y_{2k} = -1) = 1/2$

$$IP(Y_{2k+1} \mid Y_{2k} = 1 \text{ \& } Y_{2k-1} = -1) = 0.$$

Which ensures that  $Y_k$  is not a Markov chain.

(d)  $Z_n = (Y_n, Y_{n+1})$  ;  $S = \{-1, +1\}^2$

We need to distinguish between  $n$  even and  $n$  odd.

(1)

$n = 2k$

$$Z_{2k} = (Y_{2k}, Y_{2k+1})$$

$$Z_{2k+1} = (Y_{2k+1}, Y_{2k+2}).$$

\*  $Z_{2k} = (-1, -1) \Rightarrow Y_{2k} = -1 ; Y_{2k+1} = -1.$

Hence  $Z_{2k+1}$  must be of the form  $(-1, Y_{2k+2}).$

&  $Y_{2k+2} = Y_{2k+1} Y_{2k+3} = Y_{2k+3} = \begin{cases} 1 & \text{w. p. } 1/2 \\ -1 & \text{w. p. } 1/2. \end{cases}$

•  $IP(Z_{2k+1} = (-1, 1) \mid Z_{2k} = (-1, -1)) = IP(Z_{2k+1} = (-1, -1) \mid Z_k = (-1, -1)) = 1/2.$

Q4

(9)

$$* Z_{2k} = (-1, 1) \Rightarrow Y_{2k} = -1, Y_{2k+1} = 1.$$

$$Z_{2k+1} = (1, Y_{2k+2})$$

$$Y_{2k+2} = Y_{2k+1} Y_{2k+3} = Y_{2k+3} = \begin{cases} 1 & \text{w.p. } 1/2 \\ -1 & \text{w.p. } 1/2. \end{cases}$$

$$\bullet \text{IP}(Z_{2k+1} = (1, 1) \mid Z_{2k} = (-1, 1)) = \text{IP}(Z_{2k+1} = (1, -1) \mid Z_{2k} = (-1, 1)) = 1/2.$$

$$* Z_{2k} = (1, -1) \Rightarrow Y_{2k} = 1, Y_{2k+1} = -1.$$

$$Z_{2k+1} = (-1, Y_{2k+2}).$$

$$\text{IP}(Z_{2k+1} = (-1, 1) \mid Z_{2k} = (1, -1)) = \text{IP}(Z_{2k+1} = (-1, -1) \mid Z_{2k} = (1, -1)) = 1/2.$$

$$* Z_{2k} = (1, 1) \Rightarrow Y_{2k} = 1, Y_{2k+1} = 1.$$

$$Z_{2k+1} = (1, Y_{2k+2}).$$

$$\text{IP}(Z_{2k+1} = (1, 1) \mid Z_{2k} = (1, 1)) = \text{IP}(Z_{2k+1} = (1, -1) \mid Z_{2k} = (1, 1)) = 1/2.$$



Q4  
 $n = 2k+1$

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$$Z_{2k+1} = (Y_{2k+1}, Y_{2k+2}) \quad Z_{2k+2} = (Y_{2k+2}, Y_{2k+3})$$

Recall  $Y_{2k+2} = Y_{2k+1} Y_{2k+3}$ .

given  $Z_{2k+1}$ ,  $Z_{2k+2}$  is completely determined.

$$IP(Z_{2k+2} = (-1, 1) \mid Z_{2k+1} = (-1, 1)) = 1$$

$$IP(Z_{2k+2} = (1, -1) \mid Z_{2k+1} = (-1, 1)) = 1$$

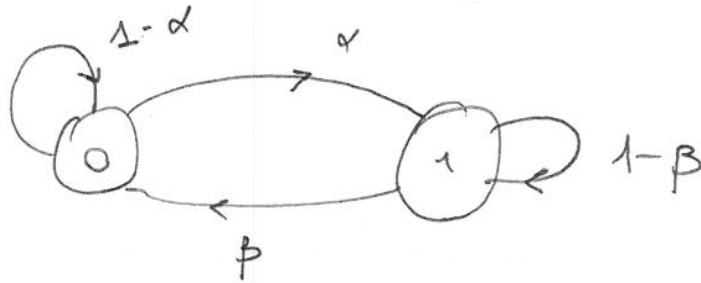
$$IP(Z_{2k+2} = (-1, -1) \mid Z_{2k+1} = (1, -1)) = 1$$

$$IP(Z_{2k+2} = (1, 1) \mid Z_{2k+1} = (1, 1)) = 1.$$

This completely determines the transition probabilities of  $Z_n$ .

Q5.

a)



b)

$$\begin{aligned}
 (i) \quad p_{00}^{(n+1)} &= P(X_{n+1}=0 \mid X_0=0) \\
 &= p_{01}^{(n)} p_{10} + p_{00}^{(n)} p_{00} \\
 &= \beta p_{01}^{(n)} + (1-\alpha) p_{00}^{(n)}
 \end{aligned}$$

Since  $p_{00}^{(n)} + p_{01}^{(n)} = 1.$

$$\begin{aligned}
 p_{00}^{(n+1)} &= (1 - p_{00}^{(n)}) \beta + (1-\alpha) p_{00}^{(n)} \\
 &= (1-\alpha-\beta) p_{00}^{(n)} + \beta
 \end{aligned}$$

(ii) By induction

$$(iii) \quad p_{01}^{(n)} = 1 - p_{00}^{(n)} = \frac{\alpha}{\alpha+\beta} + \frac{\alpha}{\alpha+\beta} (1-\alpha-\beta)^n.$$

By symmetry

$$\begin{aligned}
 p_{11}^{(n)} &= \frac{\beta}{\alpha+\beta} + \frac{\beta}{\alpha+\beta} (1-\alpha-\beta)^n \\
 p_{10}^{(n)} &= \frac{\beta}{\alpha+\beta} + \frac{\beta}{\alpha+\beta} (1-\alpha-\beta)^n.
 \end{aligned}$$

(12)

$$\alpha / (1 - (\alpha + \beta)) / 5$$

Q5

b) (15)

$$\lim_n p_{00}(n) = \lim_{n \rightarrow \infty} p_{10}(n) = \frac{\beta}{\alpha + \beta}$$

$$\lim_n p_{11}(n) = \lim_{n \rightarrow \infty} p_{01}(n) = \frac{\alpha}{\alpha + \beta}.$$

c)

Use the result  
stationary distribution

b) (iv) which gives

$$\Rightarrow \pi_1 = \beta / (\alpha + \beta), \pi_2 = \alpha / (\alpha + \beta)$$

or

solve the invariant distribution eqn,

$$\pi P = \pi$$

$\Rightarrow$

$$\pi_1 + \pi_2 = 1$$

$$\begin{cases} \pi_1 (1 - \alpha) + \beta \pi_2 = \pi_1 \\ \alpha \pi_1 + (1 - \beta) \pi_2 = \pi_2 \end{cases}$$

$$\pi_1 + \pi_2 = 1$$

$$\Rightarrow \begin{cases} \pi_2 = \frac{\alpha}{\beta} \pi_1 \\ \pi_1 + \pi_2 = 1 \end{cases}$$

$$\Rightarrow \begin{cases} \pi_1 = \beta / (\alpha + \beta) \\ \pi_2 = \alpha / (\alpha + \beta) \end{cases}$$

Q6

Similar Problem solved in lecture.

a)

(i) Starting from 2 we jump to 1 w.p.  $1/2$  and to 3 w.p.  $1/2$ .

$$h_2 = 1/2 h_1 + 1/2 h_3.$$

Similarly for a start at 3

$$h_3 = 1/2 h_2 + 1/2 h_4.$$

(ii) as 1 is an absorbing state  
if we start from 1 we stay there  
&  $h_1 = 0$ .

It is clear that  $h_4 = 1$ .

$$h_2 = 1/2 h_3 = 1/2 (1/2 h_2 + 1/2)$$

$$\Rightarrow h_2 = 1/3.$$

$$\& h_3 = 2/3.$$

Q6

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b)

(i) Starting from 2 after one jump we jump to 1 w.p.  $\frac{1}{2}$  or to 3 w.p.  $\frac{1}{2}$

$k_2$  The average time to be absorbed at 1 or 4

starting from 2 is therefore given by

$$k_2 = 1 + \frac{1}{2} k_3 + \frac{1}{2} k_1$$

Similarly

$$k_3 = 1 + \frac{1}{2} k_2 + \frac{1}{2} k_4$$

(ii) It is easily seen that  $k_1 = k_4 = 0$ .

$$k_2 = (1 + \frac{1}{2} k_3) = 1 + \frac{1}{2} (1 + \frac{1}{2} k_2)$$

$$\Rightarrow k_2 = 2$$

$$\& \text{ so } k_3 = 2$$