UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 2004

BEng Honours Degree in Computing Part III
MSc in Computing Science
MEng Honours Degree in Information Systems Engineering Part IV
MSci Honours Degree in Mathematics and Computer Science Part IV
BSc Honours Degree in Mathematics and Computer Science Part III
MSc in Advanced Computing
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the Associateship of the City and Guilds of London Institute
This paper is also taken for the relevant examinations for the Associateship of the Royal College of Science

PAPER C336=I4.50

PERFORMANCE ANALYSIS

Friday 30 April 2004, 10:00
Duration: 120 minutes

Answer THREE questions

Paper contains 4 questions
Calculators not required

1 a Define the term *renewal process* and explain why a *Poisson Process* is an instance. State and prove the *memoryless property* of the Poisson process.

b What is a *Markov modulated Poisson process (MMPP)* and why is it *not* a renewal process? In an MMPP with N phases, the arrival rate in phase i is $\lambda_i \ (1 \leq i \leq N)$. Given equilibrium probability vector $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_N)$ of the modulating Markov process, calculate the average arrival rate of the MMPP.

c A simple router handles two types of packetized traffic – data and video – and switches between the two modes. Data and video traffic arrive at rates λ_d and λ_v respectively. The router may be in data mode (transmitting data) or in video mode (transmitting video). When in data mode, it switches to video mode at rate q_d. Similarly, when in video mode, it switches to data mode at rate q_v.

i) What assumptions are required to model the overall arrival process by an MMPP?

ii) Suppose that packets of either type of traffic are processed by the router at the same rate, μ say. Under what conditions can the router's behaviour be described by a MMPP/M/1 queue?

iii) Under these conditions, derive a condition for the queue to have a steady state.

The three parts carry, respectively, 35%, 25%, and 40% of the marks.

2 In a closed queueing network of M nodes, with paths existing between any pair of nodes, and having population K, node i has constant service rate μ_i and relative load x_i, proportional to its utilisation ($1 \leq i \leq M$).

a i) Explain why this network has a steady state and write down its balance equations *in the case that* $M = 2, K \geq 1$.

ii) Write down the network's *product-form solution* for *arbitrary* $M, K \geq 1$.

b i) Define the normalising constant function $g(m, k)$ for this network's equilibrium state probabilities.

ii) Show that $g(m, k) = g(m-1, k) + x_m g(m, k-1)$ for $m \geq 2, k \geq 1$, that $g(1, k) = x_1^k$ for $k \geq 0$ and that $g(m, 0) = 1$ for $m \geq 1$.

iii) Prove that the utilisation of node $i = 1, 2, \ldots, M$ is $x_i g(M, K-1)/g(M, K)$.

iv) By considering the node b (which you may assume is unique) with the *maximum utilisation*, or otherwise, prove that $g(M, K-1)/g(M, K) \to 1/x_b$ as $K \to \infty$. Hence explain a sense in which the network behaves as a pseudo-open network as the population becomes very large.

The two parts carry, respectively, 40%, and 60% of the marks.

3 a i) State Jackson's Theorem for open queueing networks.

ii) State Norton's Decomposition Theorem for queueing networks.

iii) Suggest conditions under which Norton's Theorem provides a good approximation in networks which do not satisfy Jackson's Theorem, giving an example such as a multi-access system with paging.

b A client-server system consists of a set of independent, homogeneous workstations (the "clients"), a large computer-database installation (the "server") and two networks connecting the clients with the server: transactions are sent by the clients over one network to the server which processes them and returns a response via the other network.

The server has been the subject of extensive monitoring such that the rate at which it completes tasks has been tabulated reliably as $T(m)$ at each possible multiprogramming level $m = 1, 2, ..., M$ (here, m is the total number of tasks at the server). Similarly, the mean "think time" at a client workstation is estimated to be $1/\lambda$. Modelling each network as a single server queue with rate μ, apply Norton's Theorem to define a four-node queueing network model that approximates the performance characteristics of the client-server system. What is the throughput of the server in your model and how would you estimate its mean response time? Would you expect your model to provide a good approximation and why?

4 a Under what conditions does a *discrete time Markov chain* (DTMC) have a *steady state*? When these conditions are satisfied, what set of linear equations (for example, written in vector-matrix form) can be solved to obtain the chain's equilibrium state-probabilities?

b Consider an arrival counting process $C = \{A_s \mid s = 1, 2, \ldots\}$ in discrete time where, in any one time-slot, there is a single arrival with probability p and no arrivals with probability $1 - p$, independently of all other slots. The state A_s of the counting process C at time s slots is the total number of arrivals up to the end of slot s, with $A_0 = 0$.

i) Show that C is a Markov chain and define its one-step transition probability matrix, Q.

ii) Does C have a steady state? Justify your answer.

iii) Suppose that in state 2^{15} there is the possibility of an error occurring, which resets the state to 0 with probability r. For what values of r, if any, does a steady state now exist?

c i) In the arrival process C of part b, show that the probability of there being n arrivals in s slots is

$$\frac{s(s-1)\ldots(s-n+1)}{n!} p^n (1-p)^{s-n}$$

Hint: The number of heads n that occur in s tosses of a fair coin has a binomial probability distribution.

ii) Now suppose that the time-slot has size $1/k$ seconds and let $s = kt$ for some actual time t seconds. If p is inversely proportional to the time-slot size, say $p = \lambda/k$, show that the probability that the state of C (when there are no errors, i.e. $r = 0$) at actual time t is n, i.e. that $A_s = n$, may be written as

$$P(A_s = n) = \left(\frac{kt}{k}\right)\left(\frac{kt-1}{k}\right)\cdots\left(\frac{kt-n+1}{k}\right)\frac{\lambda^n}{n!}\frac{\left[(1-\lambda/k)^k\right]^t}{(1-\lambda/k)^n}$$

iii) In the limit that $k \longrightarrow \infty$, i.e. when the time-slot size approaches zero, with λ and t fixed so that $s \longrightarrow \infty$ and $p \longrightarrow 0$ jointly, show that the probability that there are n arrivals in time t approaches $\frac{(\lambda t)^n}{n!}e^{-\lambda t}$. What continuous time arrival process is this? **Hint**: Recall that $(1 - \lambda/k)^k \longrightarrow e^{-\lambda}$ as $k \longrightarrow \infty$.

The three parts carry, respectively, 25%, 40%, and 35% of the marks.