IMPERIAL COLLEGE LONDON

B.Eng, M.Eng, and ACGI Examinations 2017 - 2018

Part 1

BE1-HMATH1 Mathematics I

31 May 2018 14:00-15:30
(duration: 90 minutes)

All three questions are compulsory.

Please answer each question in separate answer book.

A list of formulae is provided separately.

Each question is worth 100 marks.
Marks for questions and parts of questions are shown next to the question. The marks for questions (and parts thereof) are indicative, and they may be slightly moderated at the discretion of the Examiner.

Question 1 This question has two parts.

a) Find the complex Fourier series expansion of $f(x) = \cos x$ from $-\pi$ to π, to show that $\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$.

50 marks

See Appendix.

Marks: $\overline{\mathbf{50}}$

b) **i)** Find all the solutions to the complex equation

$$z^2 + \bar{z}^2 = 0 \ .$$

20 marks

Let $z = a + bi$, then the equation leads to $2a^2 - 2b^2 = 0$. The solutions are therefore $a = b$ or $a = -b$.

Marks: $\overline{\mathbf{20}}$

ii) Find all the solutions to the complex equation

$$z^6 + 7z^3 - 8 = 0 \ .$$

20 marks

Let $z = re^{i\theta}$ to find that the six solutions are $-2, 2e^{i\pi/3}, 2e^{-i\pi/3}, 1, e^{i2\pi/3}, e^{-i2\pi/3}$.

Marks: $\overline{\mathbf{20}}$

iii) Draw the solutions from i) and ii) on an Argand diagram.

10 marks

i) The lines $Im(z) = Re(z)$ and $Im(z) = -Re(z)$. ii) See the solutions in ii).

Marks: $\overline{\mathbf{10}}$

The two parts carry equal marks.

Question 2 This question has two parts.

a) Find the general solution $y(x)$ of the following differential equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + 3x^2 + \sin x = 0.$$

40 marks

See appendix.

Marks:

$\overline{40}$

b) **i)** Solve the following differential equation in the domain $D \equiv (-\infty, \infty)$:

$$\frac{2xy}{x^2+1} - x - \left[1 - \ln\left(x^2+1\right)\right] y' = 0$$

with the condition $y(0) = 1$.

30 marks

The equation is exact, so it can be obtained by integrating the $(\frac{2xy}{x^2+1} - x)$ with respect to x and $-[1 - \ln(x^2+1)]$ with respect to y. The answer is $y = C/(1 - \log(1+x^2))$. The constant C is set to be 1 given by condition $y(0) = 1$.

Marks:

$\overline{30}$

ii) Sketch the function

$$y(x) = \log(x^2) + x$$

Identify clearly the domain of the function, and whether the function intercepts with x-axis and the y-axis (no need to provide the exact expressions of the intercepts). Furthermore, identify, if exist, the local maxima/minima, intervals in which $y(x)$ is increasing and decreasing, intervals in which $y(x)$ is concave up and down, and asymptotes.

30 marks

The domain is the set of real numbers except $x = 0$. No y-intercept, but the x-intercept is given by $-x = \log x^2$. Local minimum at $x = -2$. y is decreasing in $[-2, 0)$ and increasing in $(-\infty, -2)$ and (0∞), and is always concave up. There is a vertical asymptote at $x = 0$ and a slanted asymptote at $y(x) = x$ (optional).

Marks:

$\overline{30}$

The two parts carry, respectively, 40%, and 60% of the marks.

Question 3a) Consider the three points, $O(0,0,0)$, $A(2,1,1)$ and $B(1,3,2)$.

i) Find the area of the parallelogram which is spanned by the vectors OA and OB.

20 marks

The area $= |\underline{OA} \times \underline{OB}| = \left| \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \right| = \left| \begin{pmatrix} 1\times 2 - 1\times 3 \\ -(2\times 2 - 1\times 1) \\ 2\times 3 - 1\times 1 \end{pmatrix} \right| = \left| \begin{pmatrix} -1 \\ -3 \\ 5 \end{pmatrix} \right| = \sqrt{1+9+25} = \sqrt{35}.$

Marks:

$\overline{\mathbf{20}}$

ii) Find an equation of the plane Π containing the points A, B and O.

10 marks

-x-3y+5z=0.

Marks:

$\overline{\mathbf{10}}$

b) Determine whether the set of three vectors $\left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$ is linearly independent or dependent.

20 marks

Independent because $\det \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix} = 1.$

Marks:

$\overline{\mathbf{20}}$

c) Find the derivative of x^x.

25 marks

Let $y = x^x$*. Taking the log on both sides,* $\ln y = x \ln x$*. By implicit differentiation,* $\frac{1}{y}\frac{dy}{dx} = \ln x + x \cdot \frac{1}{x} = \ln x + 1$*. Therefore,* $\frac{dy}{dx} = x^x(\ln x + 1)$*.*

Marks:

$\overline{\mathbf{25}}$

d) Consider a graph described by $xy + x^2y^2 = 6$. Obtain the two points on this graph whose x-coordinate is 1. For each of these two points, find an equation of the tangent line (a straight line that touches the graph at the point in question).

25 marks

Substituting $x = 1$ *in* $xy + x^2y^2 = 6$ *yields* $y^2 + y - 6 = (y+3)(y-2) = 0$. *Therefore the two points in question are* $(x, y) = (1, -3)$ *and* $(1, 2)$. *Taking the derivative of both sides of* $xy + x^2y^2 = 6$ *yields* $xy' + y + 2xy^2 + 2x^2yy' = 0$.

For $(x, y) = (1, -3)$, $y' = 3$ *and therefore the tangent line is* $y = 3(x-1) - 3 = 3x - 6$. *For* $(x, y) = (1, 2)$, $y' = -2$ *and therefore the tangent line is* $y = -2(x-1) + 2 = -2x + 4$.

Marks:

$\mathbf{25}$

The four parts carry, respectively, 30%, 20%, 25%, and 25% of the marks.

EXAM 2018

1a] Complex Fourier series expansion of $f(x)=\cos x$, $x\in(-\pi,\pi)$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos x\, e^{-inx}\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx}\, d(\sin x)$$

$$= \frac{1}{2\pi}\Big[e^{-inx}\sin x\Big]_{-\pi}^{\pi} - \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin x\, e^{-inx}\,(-in)\,dx$$

$$= \frac{1}{2\pi}\Big[e^{-in\pi}\sin\pi - \frac{1}{2\pi}e^{+in\pi}\sin(-\pi)\Big] + \frac{in}{2\pi}\int_{-\pi}^{\pi} -e^{-inx}\,d(\cos x)\,dx$$

$$= 0 - \frac{in}{2\pi}\Big[e^{-inx}\cos x\Big]_{-\pi}^{\pi} - \frac{in}{2\pi}(-in)\int_{-\pi}^{\pi} -e^{-inx}\cos x\,dx$$

$$= -\frac{in}{2\pi}\left(e^{-in\pi}\cos\pi - e^{in\pi}\cos(-\pi)\right) + i^2n^2c_n$$

$$c_n = -\frac{in}{2\pi}\left(-e^{-in\pi} + e^{in\pi}\right) + n^2c_n$$

$$\Rightarrow c_n(1-n^2) = \frac{in}{2\pi}\left(e^{-in\pi} - e^{in\pi}\right)$$

$$\Rightarrow c_n = \frac{in}{2\pi}\,\frac{e^{-in\pi}-e^{+in\pi}}{1-n^2} = \frac{in}{2\pi}\,\frac{e^{in\pi}-e^{-in\pi}}{n^2-1}, \quad n\neq\pm1 \qquad [30]$$

$\forall n\in\mathbb{Z}$, $e^{-in\pi} - e^{+in\pi} = 0$.

For $n=\pm1$ take the limit as $n\to\pm1$

$$c_1 = \lim_{n\to1}\frac{in}{2\pi}\,\frac{e^{+in\pi}-e^{-in\pi}}{n^2-1} = \frac{i}{2\pi}\lim_{n\to1}\frac{e^{+in\pi}-e^{-in\pi}}{n-1/n} =$$

$$\overset{(\frac{0}{0})}{=} \frac{i}{2\pi}\lim_{n\to1}\frac{i\pi e^{in\pi}+i\pi e^{-in\pi}}{1+1/n^2} = \frac{i^2\pi}{2\pi}\,\frac{e^{i\pi}+e^{-i\pi}}{2} = \frac{-1}{2}\,\frac{(-1)+(-1)}{2} = \frac{1}{2}$$

$$c_{-1} = \lim_{n\to-1}\frac{in}{2\pi}\,\frac{e^{in\pi}-e^{-in\pi}}{n^2-1} \overset{(\frac{0}{0})}{=} \frac{i}{2\pi}\lim_{n\to-1}\frac{i\pi e^{in\pi}+i\pi e^{-in\pi}}{1+1/n^2} =$$

$$= \frac{i}{2\pi}\,\frac{i\pi}{2}\left(e^{i\pi}+e^{-i\pi}\right) = \frac{i^2}{4}\left(-1+(-1)\right) = -\frac{1}{4}(-2) = \frac{1}{2} \qquad [20]$$

And so $f(x) = \cos x = \sum_{-\infty}^{\infty} c_n e^{-inx} = \frac{1}{2}e^{ix} + \frac{1}{2}e^{-ix} = \frac{1}{2}\left(e^{-ix}+e^{ix}\right)$

2a) $y'' + y' + 3x^2 + \sin x = 0 \Rightarrow y'' + y' = -3x^2 - \sin x$

Inhomogeneous 2nd order ODE.

Homog. solution: $(y'' + y' = 0)$

$\lambda^2 + \lambda = 0 \Rightarrow \lambda(\lambda+1) = 0 \Rightarrow \begin{matrix} \lambda_1 = 0 \\ \lambda_2 = -1 \end{matrix}$

and $y_0(x) = c_1 + c_2 e^{-x}$. [5]

RHS is superposition of 2 ODES, so 2 particular solutions [5]

1) $y''_{p_1}(x) + y'_{p_1}(x) = -3x^2$.

$p + iq = 0 = \lambda_1 \Rightarrow$ Try $y_{p_1}(x) = Ax^3 + Bx^2 + Cx + D$ [5]

Then $y'_{p_1} = 3Ax^2 + 2Bx + C$

$y''_{p_1} = 6Ax + 2B$

So $6Ax + 2B + 3Ax^2 + 2Bx + C = -3x^2$.

$$\Rightarrow \left\{ \begin{array}{l} 3A = -3 \\ 6A + 2B = 0 \\ 2B + C = 0 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} A = -1 \\ B = 3 \\ C = -6 \end{array} \right.$$

$\Rightarrow y_{p_1}(x) = -x^3 + 3x^2 - 6x + D$. [10]

2) $y''_{p_2}(x) + y'_{p_2}(x) = -\sin x$.

$p + iq = i \neq \lambda_{1,2} \Rightarrow$ TRY $y_{p_2}(x) = A\sin x + B\cos x$.

$y'_{p_2}(x) = A\cos x - B\sin x$

$y''_{p_2}(x) = -A\sin x - B\cos x$ [5]

So $-A\sin x - B\cos x + A\cos x - B\sin x = -\sin x$.

$$\Rightarrow \left\{ \begin{array}{l} -A - B = -1 \\ -B + A = 0 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} A + B = 1 \\ A = B \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} 2A = 1 \\ A = B \end{array} \right\} \Rightarrow A = B = 1/2.$$

$\Rightarrow y_{p_2}(x) = \frac{1}{2}(\sin x + \cos x)$. [5]

Therefore general solution of the ODE

$y(x) = y_0(x) + y_{p_1}(x) + y_{p_2}(x)$

$= c_1 + c_2 e^{-x} - x^3 + 3x^2 - 6x + D + \frac{1}{2}(\sin x + \cos x)$

$= c_3 + c_2 e^{-x} - x^3 + 3x^2 - 6x + \frac{1}{2}(\sin x + \cos x)$. [5]