Final

E4.47
CS6.1

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2010

MSc and EEE PART IV: MEng and ACGI

MODELLING AND CONTROL OF MULTI-BODY MECHANICAL SYSTEMS

Tuesday, 27 April 2:30 pm

Time allowed: 3:00 hours

There are SIX questions on this paper.

Answer FOUR questions.

All questions carry equal marks.

This is an OPEN BOOK examination.

Any special instructions for invigilators and information for candidates are on page 1.

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MODELLING AND CONTROL OF MULTIBODY MECHANICAL SYSTEMS

1. Consider a nonlinear Hamiltonian system with internal Hamiltonian ($q = (q_1, q_2)$, $p = (p_1, p_2)$)

$$H_0(q,p) = \frac{1}{2}\left((1+q_2^2)p_1^2 + p_1p_2 + (1+q_1^2)p_2^2\right) + \cos(q_1+2q_2) - \cos(q_1),$$

and interaction Hamiltonians

$$H_1(q) = q_1 \qquad H_2(q) = q_2.$$

Note that the functions H_0, H_1 and H_2 define a Hamiltonian system with state (q, p), input $u = (u_1, u_2)$ and output $y = (y_1, y_2)$.

All parts of this question must be answered without writing the differential equations describing the Hamiltonian system.

a) Write the kinetic energy of the system in the form

$$K = \frac{1}{2}\dot{q}' M(q)\dot{q},$$

and show that the intertia matrix $M(q)$ is positive definite for all q. [2 marks]

b) Write the potential energy $U(q)$ of the system and compute its stationary points. [2 marks]

c) Assume that the input u is such that $u_1 = u_2 = 0$. Determine the equilibria of the system. [2 marks]

d) Assume that the input u is such that u_1 is constant and $u_2 = 0$. Determine the equilibria of the system as a function of u_1. Plot in the (q_1, q_2)-plane the set of equilibria as a function of u. Show that there exists a constant $\bar{u}_1$ such that if $|u_1| > \bar{u}_1$ the system does not have any equilibrium. Determine the value of $\bar{u}_1$. [4 marks]

e) Determine a function $S(q)$ such that the function $U(q) + S(q)$ is positive definite around $q = (0,0)$, that is $U(0) + S(0) = 0$ and $U(q) + S(q)$ is positive for all q close to $(0,0)$. [2 marks]

f) Determine a feedback control law $u = u(q) + v$ such that the resulting closed-loop system is a Hamiltonian system with the same kinetic energy as the open-loop system and with potential energy equal to $U(q) + S(q)$. [6 marks]

g) Consider the Hamiltonian system in closed-loop with the control law determined in part f). Discuss the stability properties of the equilibrium $q = (0,0)$. [2 marks]

2. A sphere of radius a has area $4\pi a^2$ and volume $\frac{4}{3}\pi a^3$.

a) Consider a uniform sphere of radius a and mass m.

i) Use spherical coordinates to find the moment of inertia of the sphere about an axis passing through its centre. (Hint: $\int \cos^3 x \, dx = \frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\sin x$). [8]

ii) Write the principal moments of inertia (through the centre) of the sphere and state their direction. [2]

b) Consider a uniform thin spherical shell of radius a and mass m.

i) Use spherical coordinates to find the moment of inertia of the shell about an axis passing through its centre. [6]

ii) Compute the inertia tensor of the shell with respect to a body fixed reference frame whose origin is on the surface of the shell and which has one axis passing through the centre of the shell. [4]

3. A uniform sphere of radius a and mass m travels on a flat horizontal surface as shown in Figure 3.1.

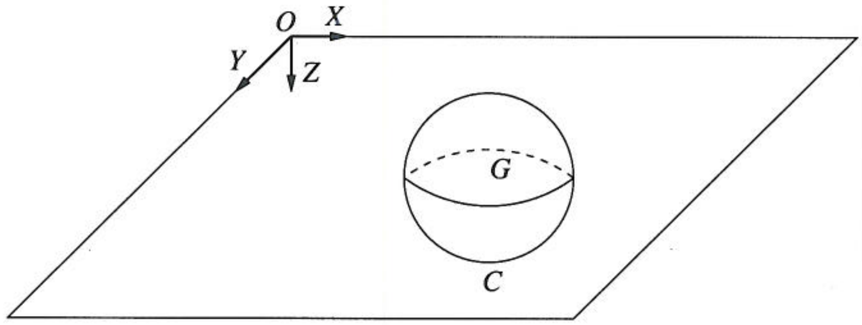

Figure 3.1 Rolling sphere.

Unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are associated with the earth-fixed axes X, Y and Z. The centre of the sphere is G and the instantaneous point of ground contact is C. The sphere has instantaneous angular velocity $\boldsymbol{\omega}$ and the translation of G is given by the position vector $\boldsymbol{r}_G$.

a) Show that the angular momentum vector, $\boldsymbol{H}$, of the sphere, about its centre, is given by

$$\boldsymbol{H}_G = \frac{2}{5} m a^2 \boldsymbol{\omega}.$$

[3]

b) The sphere travels on the plane under the influence of an unknown horizontal force $\boldsymbol{F}$ at the ground contact point.

i) Write in vector form the equation of translational motion of the centre of the sphere. [2]

ii) Determine the moment vector about the centre of the sphere due to force $\boldsymbol{F}$. [2]

iii) Show that the equation that describes the rotational motion of the sphere about its centre is

$$a\boldsymbol{\omega} = \frac{5}{2} \boldsymbol{k} \times (\dot{\boldsymbol{r}}_G - \dot{\boldsymbol{r}}_G(0)) + a\boldsymbol{\omega}(0),$$

where $\dot{\boldsymbol{r}}_G(0)$ and $\boldsymbol{\omega}(0)$ are the initial velocity and angular velocity vectors respectively. [5]

iv) Derive an expression for the unknown force vector $\boldsymbol{F}$. [3]

c) The plane and sphere surfaces are made rough so that the sphere rolls on the plane without slipping.

i) Write the vector equation in terms of $\dot{\boldsymbol{r}}_G$ and $\boldsymbol{\omega}$ which describes the velocity constraint. [2]

ii) Use the velocity constraint equation to derive and discuss the motion of the sphere. [3]

4. A car of mass m travels on a horizontal plane under the influence of a longitudinal force $\boldsymbol{F}_\mathrm{l}$, a lateral force $\boldsymbol{F}_\mathrm{t}$ and a normal moment $\boldsymbol{N}$, as shown in Figure 4.1.

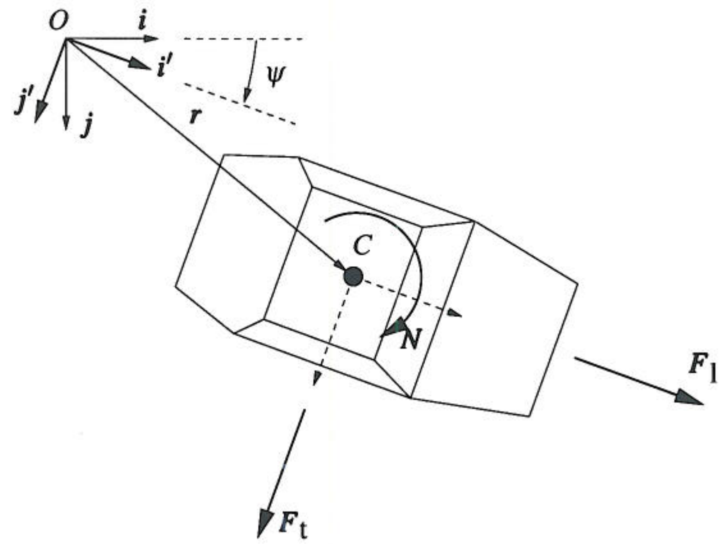

Figure 4.1 Simple car.

The moment of inertia of the car about a normal axis through its centre of mass C is I. Unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ are associated with an earth-fixed axis system, and unit vectors $\boldsymbol{i}'$ and $\boldsymbol{j}'$ are associated with a rotating axis system about the same origin. The latter axes rotate by angle ψ so that they have the same orientation as the car fixed axes, which are shown with dashed lines on the car.

a) The position vector of C is given by

$$\boldsymbol{r} = x'\boldsymbol{i}' + y'\boldsymbol{j}'.$$

i) Compute the velocity vector of the centre of mass. [2]

ii) Determine the kinetic energy of the car. [3]

iii) Write the longitudinal and lateral velocities, v_x and v_y, of the car, in terms of the generalised coordinates. [1]

b) Assume that the car does not slip laterally.

i) State the velocity constraint and discuss its type. [2]

ii) Use the Lagrangian approach to derive the equations of motion of the car. [10]

iii) Determine the force that maintains the lateral constraint. [2]

5. a) Investigate the integrability of the following differential relation:

$$\frac{8}{r}\,\mathrm{d}r - 8\tan\phi\,\mathrm{d}\phi + r^2\cos^2\phi\sin\theta\cos\theta\,\mathrm{d}\theta = 0.$$

[7]

b) i) Show that the differential relation

$$zx\,\mathrm{d}x + yz\,\mathrm{d}y + (c - x^2 - y^2)\,\mathrm{d}z = 0,$$

where c is a constant, is integrable. [7]

ii) Integrate the expression and obtain a finite relation. [6]

6. A uniform sphere of radius a and mass m travels on a flat horizontal surface under the influence of an unknown horizontal force at the ground contact point, as shown in Figure 6.1.

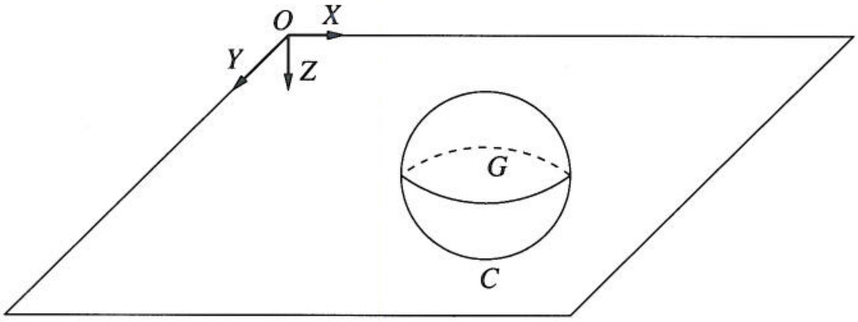

Figure 6.1 Rolling sphere.

Unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are associated with the earth-fixed axes X, Y and Z. The centre of the sphere is G and the instantaneous point of ground contact is C. The sphere has instantaneous angular velocity $\boldsymbol{\omega}$.

It can be shown that the equation that describes the rotational motion of the sphere about its centre is

$$a\boldsymbol{\omega} = \frac{5}{2}\boldsymbol{k} \times (\dot{\boldsymbol{r}}_G - \dot{\boldsymbol{r}}_G(0)) + a\boldsymbol{\omega}(0),$$

where $\boldsymbol{r}_G$ is the position vector of G, and $\dot{\boldsymbol{r}}_G(0)$ and $\boldsymbol{\omega}(0)$ are the initial velocity and angular velocity vectors respectively.

Assume that the horizontal plane rotates at constant angular velocity $\boldsymbol{\Omega}$ about the Z axis. and that the sphere rolls without slipping. It follows that any material contact point on the surface of the sphere must have the same velocity as the corresponding material contact point on the plane.

a) Write the vector equation in terms of $\boldsymbol{r}_G$, $\dot{\boldsymbol{r}}_G$, $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}$ which describes the velocity constraint. [5]

b) Use the velocity constraint equation to eliminate $\boldsymbol{\omega}$ in the rotational motion vector equation above, and hence show that

$$\dot{\boldsymbol{r}}_G = \frac{2}{7}\boldsymbol{\Omega} \times (\boldsymbol{r}_G - \boldsymbol{r}_G(0)) + \dot{\boldsymbol{r}}_G(0),$$

in which $\boldsymbol{r}_G(0)$ is the initial position of the sphere. [7]

c) Argue that the sphere will execute circular motion and determine the angular velocity of the motion. [2]

d) Hence determine the position vector of the centre of the motion in terms of $\boldsymbol{r}_G(0)$ and $\dot{\boldsymbol{r}}_G(0)$. [3]

e) Compute the radius of the circular motion. [3]

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Modelling and control of multibody mechanical systems

Model answers 2010

Question 1

a) The kinetic energy is

$$\frac{1}{2}\left((1+q_2^2)p_1^2 + p_1p_2 + (1+q_1^2)p_2^2\right) = \frac{1}{2}\begin{bmatrix} \dot{q}_1 & \dot{q}_2 \end{bmatrix}\begin{bmatrix} 1+q_2^2 & 1/2 \\ 1/2 & 1+q_1^2 \end{bmatrix}\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}.$$

The inertia matrix $M(q)$ is positive definite since

$$M_1 1(q) = 1+q_2^2 \qquad \det M(q) = (1+q_1^2)(1+q_2^2) - \frac{1}{4} > 0.$$

b) The potential energy is

$$U(q) = \cos(q_1+2q_2) - \cos(q_1).$$

The stationary points are the solution of the equations

$$0 = \frac{\partial U}{\partial q_1} = -\sin(q_1+2q_2) + \sin(q_1), \qquad 0 = \frac{\partial U}{\partial q_2} = -2\sin(q_1+2q_2)$$

which are given by

$$q_1 = k\pi \qquad q_2 = \frac{h-k}{2}\pi = \frac{l}{2}\pi,$$

with k and h integers.

c) The equilibria of the system are the stationary points of the potential energy, as calculated in part b).

d) The equilibria of the system, with $u_2 = 0$ and u_1 constant are the solutions of the equations

$$u_1 = \frac{\partial U}{\partial q_1} = -\sin(q_1+2q_2) + \sin(q_1), \qquad 0 = \frac{\partial U}{\partial q_2} = -2\sin(q_1+2q_2).$$

Note that if $|u_1| \leq 1$ these equations have the solutions

$$q_1 = \arcsin u_1 \qquad q_2 = \frac{h}{2}\pi - \frac{1}{2}\arcsin u_1,$$

with h integer, whereas if $|u_1| > 1$ they have no solution.

e) The function S can be selected, for example, as

$$S(q) = -U(q) + \frac{1}{2}q'q.$$

f) The feedback that yields the required result is

$$u = -\frac{\partial S'}{\partial q} + v = -\frac{\partial U'}{\partial q} - q' + v.$$

Note that the feedback cancels the potential energy and replaces it with the term $-q'$.

g) The time-derivative of the shaped Hamiltonian $H_s = K + U + S$ along the trajectories of the closed-loop system with $v = 0$ is $\dot{H}_s = 0$. Hence, since the shaped Hamiltonian is (locally) positive definite around the origin, the origin is a stable, but not attractive, equilibrium.

Question 2

a) i) We consider the moment of inertia about the z axis (x or y axes can also be used):

$$I_{zz} = \int r^2 \cos^2\phi \; \mathrm{d}m = \rho \int_V r^2 \cos^2\phi \; \mathrm{d}V = \rho \int\int\int r^2 \cos^2\phi \; r\cos\phi \mathrm{d}\theta \; r\mathrm{d}\phi \; \mathrm{d}r =$$

$$= \rho \int_0^a \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} r^4 \cos^3\phi \; \mathrm{d}\theta \; \mathrm{d}\phi \; \mathrm{d}r = 2\pi\rho \int_0^a \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r^4 \cos^3\phi \; \mathrm{d}\phi \; \mathrm{d}r =$$

$$= 2\pi\rho \int_0^a r^4 \left(\frac{1}{3}\cos^2\phi \sin\phi + \frac{2}{3}\sin\phi\right)\Big|_{-\pi/2}^{\pi/2} \; \mathrm{d}r = \frac{8}{3}\pi\rho \int_0^a r^4 \; \mathrm{d}r = \frac{8}{15}\pi\rho a^5.$$

But

$$m = \rho V = \rho \cdot \frac{4}{3}\pi a^3,$$

and therefore

$$I_{zz} = \frac{2}{5} m a^2.$$

ii) The principal moments of inertia are all the same due to symmetry and have value $\frac{2}{5}ma^2$. Any set of three mutually orthogonal axes passing through the centre of the sphere are principal axes.

b) i) We consider the moment of inertia about the z axis (x or y axes can also be used):

$$I_{zz} = \int a^2 \cos^2\phi \; \mathrm{d}m = \rho \int_A a^2 \cos^2\phi \; \mathrm{d}A,$$

where ρ is mass per unit area of the shell surface. Hence

$$I_{zz} = \rho a^2 \int\int \cos^2\phi \; a\cos\phi \mathrm{d}\theta \; a\mathrm{d}\phi =$$

$$= \rho a^4 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} \cos^3\phi \; \mathrm{d}\theta \; \mathrm{d}\phi = 2\pi\rho a^4 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3\phi \; \mathrm{d}\phi =$$

$$= 2\pi\rho a^4 \left(\frac{1}{3}\cos^2\phi \sin\phi + \frac{2}{3}\sin\phi\right)\Big|_{-\pi/2}^{\pi/2} = \frac{8}{3}\pi\rho a^4.$$

But

$$m = \rho A = \rho \cdot 4\pi a^2,$$

and therefore

$$I_{zz} = \frac{2}{3} m a^2.$$

ii) We can shift the origin of the body fixed reference frame from the centre to the surface of the shell by moving it along one axis, say x, by a distance equal to the radius. We can then find the new inertia tensor by adding to the inertia tensor about the centre of the shell a difference term as follows,

$$I_{\text{surface}} = I_{\text{centre}} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & ma^2 & 0 \\ 0 & 0 & ma^2 \end{bmatrix},$$

or

$$I_{\text{surface}} = \begin{bmatrix} \frac{2}{3}ma^2 & 0 & 0 \\ 0 & \frac{2}{3}ma^2 & 0 \\ 0 & 0 & \frac{2}{3}ma^2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & ma^2 & 0 \\ 0 & 0 & ma^2 \end{bmatrix},$$

and hence

$$I_{\text{surface}} = \begin{bmatrix} \frac{2}{3}ma^2 & 0 & 0 \\ 0 & \frac{5}{3}ma^2 & 0 \\ 0 & 0 & \frac{5}{3}ma^2 \end{bmatrix}.$$

Question 3

a) The moment of inertia of the sphere about any axis through its centre is $\frac{2}{5}ma^2$. Therefore the inertia matrix is

$$\boldsymbol{I} = \begin{bmatrix} \frac{2}{5}ma^2 & 0 & 0 \\ 0 & \frac{2}{5}ma^2 & 0 \\ 0 & 0 & \frac{2}{5}ma^2 \end{bmatrix} = \frac{2}{5}ma^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The angular momentum is given by

$$\boldsymbol{H}_G = \boldsymbol{I}\boldsymbol{\omega},$$

and hence

$$\boldsymbol{H}_G = \frac{2}{5}ma^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{\omega} = \frac{2}{5}ma^2\boldsymbol{\omega}.$$

b) i) The translational motion of the centre of the sphere is described by

$$\boldsymbol{F} = m\ddot{\boldsymbol{r}}_G.$$

ii) The moment is simply

$$\boldsymbol{N}_G = \boldsymbol{r}_{GC} \times \boldsymbol{F},$$

which upon substitution of $\boldsymbol{r}_{GC}$ and $\boldsymbol{F}$ gives

$$\boldsymbol{N}_G = a\boldsymbol{k} \times m\ddot{\boldsymbol{r}}_G = ma\boldsymbol{k} \times \ddot{\boldsymbol{r}}_G.$$

iii) The equation of motion about the centre of mass is

$$\boldsymbol{N}_G = \frac{\mathrm{d}\boldsymbol{H}_G}{\mathrm{d}t},$$

or

$$ma\boldsymbol{k} \times \ddot{\boldsymbol{r}}_G = \frac{2}{5}ma^2\dot{\boldsymbol{\omega}},$$

or

$$\boldsymbol{k} \times \ddot{\boldsymbol{r}}_G = \frac{2}{5}a\dot{\boldsymbol{\omega}}. \tag{1}$$

Straight forward integration of the above equation yields

$$\boldsymbol{k} \times (\dot{\boldsymbol{r}}_G - \dot{\boldsymbol{r}}_G(0)) = \frac{2}{5}a(\boldsymbol{\omega} - \boldsymbol{\omega}(0)),$$

which can be rearranged into

$$a\boldsymbol{\omega} = \frac{5}{2}\boldsymbol{k} \times (\dot{\boldsymbol{r}}_G - \dot{\boldsymbol{r}}_G(0)) + a\boldsymbol{\omega}(0). \tag{2}$$

iv) By taking the right cross product with $\boldsymbol{k}$ in Equation 1 we get

$$(\boldsymbol{k} \times \ddot{\boldsymbol{r}}_G) \times \boldsymbol{k} = \frac{2}{5}a\dot{\boldsymbol{\omega}} \times \boldsymbol{k}.$$

But $\ddot{\boldsymbol{r}}_G$ is horizontal and therefore $(\boldsymbol{k} \times \ddot{\boldsymbol{r}}_G) \times \boldsymbol{k} = \ddot{\boldsymbol{r}}_G$. The above equation becomes

$$\ddot{\boldsymbol{r}}_G = \frac{2}{5}a\dot{\boldsymbol{\omega}} \times \boldsymbol{k}, \tag{3}$$

and therefore

$$\boldsymbol{F} = \frac{2}{5}ma\dot{\boldsymbol{\omega}} \times \boldsymbol{k}.$$

c) i) The velocity of the material contact point on the surface of the sphere is zero. Therefore

$$\dot{\boldsymbol{r}}_G + \boldsymbol{\omega} \times a\boldsymbol{k} = 0.$$

ii) We can take the right cross product of Equation 2 with $\boldsymbol{k}$ to obtain

$$\boldsymbol{\omega} \times a\boldsymbol{k} = \frac{5}{2}\boldsymbol{k} \times (\dot{\boldsymbol{r}}_G - \dot{\boldsymbol{r}}_G(0)) \times \boldsymbol{k} + \boldsymbol{\omega}(0) \times a\boldsymbol{k},$$

which yields

$$\boldsymbol{\omega} \times a\boldsymbol{k} = \frac{5}{2}(\dot{\boldsymbol{r}}_G - \dot{\boldsymbol{r}}_G(0)) + \boldsymbol{\omega}(0) \times a\boldsymbol{k},$$

by making use of the idea that $\dot{\boldsymbol{r}}_G$ and $\boldsymbol{k}$ are always perpendicular to each other. The constraint equation can be used to substitute the left hand side term and the second term on the right hand side of the above equation. Hence,

$$-\dot{\boldsymbol{r}}_G = \frac{5}{2}(\dot{\boldsymbol{r}}_G - \dot{\boldsymbol{r}}_G(0)) - \dot{\boldsymbol{r}}_G(0),$$

and therefore

$$\dot{\boldsymbol{r}}_G = \dot{\boldsymbol{r}}_G(0).$$

In other words the sphere will travel along a straight line at constant velocity. If we use the constraint equation together with the above equation we can see that

$$\boldsymbol{\omega} \times a\boldsymbol{k} = -\dot{\boldsymbol{r}}_G(0).$$

This equation implies that the sphere will also roll with a constant angular velocity of magnitude $-\frac{|\dot{\boldsymbol{r}}_G(0)|}{a}$, with the possibility of having a nonzero twist component in the $\boldsymbol{k}$ direction (normal to the plane).

Question 4

a) i) The velocity vector of the centre of mass C is found by differentiating the position vector. Therefore

$$\dot{\boldsymbol{r}} = (\dot{x}' - y'\dot{\psi})\boldsymbol{i}' + (\dot{y}' + x'\dot{\psi})\boldsymbol{j}'.$$

ii) The kinetic energy is given by

$$T = \frac{1}{2}m\dot{\boldsymbol{r}}\cdot\dot{\boldsymbol{r}} + \frac{1}{2}I\dot{\psi}^2 = \frac{1}{2}m\left((\dot{x}' - y'\dot{\psi})^2 + (\dot{y}' + x'\dot{\psi})^2\right) + \frac{1}{2}I\dot{\psi}^2$$

iii)

$$\boldsymbol{v}_x = \dot{x}' - y'\dot{\psi},$$

$$\boldsymbol{v}_y = \dot{y}' + x'\dot{\psi}.$$

b) i) The constraint is nonholonomic since it is a velocity constraint which cannot be integrated. It is given by

$$\boldsymbol{v}_y = 0,$$

or

$$\dot{y}' + x'\dot{\psi} = 0.$$

ii) The Lagrangian is $L = T - V = T$. The Lagrangian equation with respect to the generalised coordinate x' is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}'}\right) - \frac{\partial L}{\partial x'} = F_1,$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m(\dot{x}' - y'\dot{\psi})\right) - m(\dot{y}' + x'\dot{\psi})\dot{\psi} = F_1,$$

and by making use of the velocity constraint equation it becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m(\dot{x}' - y'\dot{\psi})\right) = F_1,$$

or

$$m\dot{v}_x = F_1, \tag{4}$$

which is the first equation of motion. The Lagrangian equation with respect to the generalised coordinate y' is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{y}'}\right) - \frac{\partial L}{\partial y'} + \lambda = 0,$$

where λ is the Lagrange multiplier corresponding to the equation of constraint. From the above equation we get

$$\frac{d}{dt}\left(m(\dot{y}' + x'\dot{\psi})\right) + m(\dot{x}' - y'\dot{\psi})\dot{\psi} + \lambda = 0.$$

By substituting the equation of the nonholonomic constraint in the above equation we obtain

$$m(\dot{x}' - y'\dot{\psi})\dot{\psi} + \lambda = 0,$$

or

$$\lambda = -mv_x\dot{\psi}.$$

The Lagrangian equation with respect to ψ is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\psi}}\right) - \frac{\partial L}{\partial \psi} + \lambda x' = N - F_{\mathrm{l}}y',$$

or

$$\frac{d}{dt}\left(-m(\dot{x}' - y'\dot{\psi})y' + m(\dot{y}' + x'\dot{\psi})x' + I\dot{\psi}\right) + \lambda x' = N - F_{\mathrm{l}}y'.$$

We can substitute the constraint equation and v_x in the above equation to obtain

$$\frac{d}{dt}\left(-mv_x y'\right) + I\ddot{\psi} + \lambda x' = N - F_{\mathrm{l}}y',$$

which upon substitution of λ gives

$$-m\dot{v}_x y' - mv_x\dot{y}' + I\ddot{\psi} - mv_x\dot{\psi}x' = N - F_{\mathrm{l}}y'.$$

If we further substitute the constraint equation and Equation 4 we obtain the second equation of motion

$$I\ddot{\psi} = N.$$

iii) The force that maintains the lateral constraint is

$$F_{\mathrm{t}} = -\lambda = mv_x\dot{\psi}.$$

Question 5

a) Rewrite the given relation in the form

$$\mathrm{d}r = B_1 \mathrm{d}\phi + B_2 \mathrm{d}\theta,$$

or

$$\mathrm{d}r = r \tan\phi \; \mathrm{d}\phi - \frac{1}{8} r^3 \cos^2\phi \sin\theta \cos\theta \; \mathrm{d}\theta.$$

The relation is integrable if

$$\frac{\partial B_1}{\partial \theta} + \frac{\partial B_1}{\partial r} B_2 = \frac{\partial B_2}{\partial \phi} + \frac{\partial B_2}{\partial r} B_1.$$

Upon application of the above expression we obtain

$$0 - \tan\phi \cdot \frac{1}{8} r^3 \cos^2\phi \sin\theta \cos\theta = \frac{1}{4} r^3 \cos\phi \sin\phi \sin\theta \cos\theta - \frac{3}{8} r^2 \cos^2\phi \sin\theta \cos\theta \cdot r \tan\phi,$$

or

$$0 = 0,$$

and therefore the integrability condition is satisfied. (Hint: the differential relation comes from the finite expression

$$r^2 \cos^2\phi \left(\frac{\cos^2\theta}{2} + \sin^2\theta \right) = 4.)$$

b) i) Rewrite the expression in the form

$$\mathrm{d}z = B_1 \mathrm{d}x + B_2 \mathrm{d}y,$$

or

$$\mathrm{d}z = \frac{xz}{x^2 + y^2 - c} \mathrm{d}x + \frac{yz}{x^2 + y^2 - c} \mathrm{d}y.$$

The relation is integrable if

$$\frac{\partial B_1}{\partial y} + \frac{\partial B_1}{\partial z} B_2 = \frac{\partial B_2}{\partial x} + \frac{\partial B_2}{\partial z} B_1.$$

The above equation gives

$$-\frac{2xzy}{(x^2 + y^2 - c)^2} + \frac{xyz}{(x^2 + y^2 - c)^2} = -\frac{2yzx}{(x^2 + y^2 - c)^2} + \frac{yxz}{(x^2 + y^2 - c)^2},$$

or,

$$0 = 0,$$

and therefore the expression is integrable.

ii) Use the substitution

$$u = c - x^2 - y^2,$$

to integrate the differential relation. By differentiation

$$\mathrm{d}u = -2x\mathrm{d}x - 2y\mathrm{d}y.$$

By substitution in the differential relation we obtain

$$zx \, \mathrm{d}x + yz \, \mathrm{d}y + u \, \mathrm{d}z = 0,$$

or

$$z\left(-\frac{1}{2}\mathrm{d}u\right) + u \, \mathrm{d}z = 0,$$

or

$$2\frac{\mathrm{d}z}{z} = \frac{\mathrm{d}u}{u},$$

or

$$2\ln(bz) = \ln u,$$

where b is a constant, which yields

$$u = (bz)^2,$$

and therefore the finite relation is

$$b^2 z^2 + x^2 + y^2 = c.$$

Question 6

a) The velocity of the material contact point on the surface of the sphere is

$$\dot{\boldsymbol{r}}_G + \boldsymbol{\omega} \times a\boldsymbol{k}.$$

The velocity of the material contact point on the plane is

$$\boldsymbol{\Omega} \times (\boldsymbol{r}_G + a\boldsymbol{k}),$$

or, because $\boldsymbol{\Omega}$ and $\boldsymbol{k}$ are parallel,

$$\boldsymbol{\Omega} \times \boldsymbol{r}_G.$$

Hence the velocity constraint is

$$\dot{\boldsymbol{r}}_G + \boldsymbol{\omega} \times a\boldsymbol{k} = \boldsymbol{\Omega} \times \boldsymbol{r}_G. \tag{5}$$

b) By substituting Equation 2 in Equation 5 we obtain

$$\dot{\boldsymbol{r}}_G + \left(\frac{5}{2}\boldsymbol{k} \times (\dot{\boldsymbol{r}}_G - \dot{\boldsymbol{r}}_G(0)) + a\boldsymbol{\omega}(0)\right) \times \boldsymbol{k} = \boldsymbol{\Omega} \times \boldsymbol{r}_G,$$

or

$$\dot{\boldsymbol{r}}_G + \frac{5}{2}\dot{\boldsymbol{r}}_G - \frac{5}{2}\dot{\boldsymbol{r}}_G(0) + a\boldsymbol{\omega}(0) \times \boldsymbol{k} = \boldsymbol{\Omega} \times \boldsymbol{r}_G,$$

or

$$\frac{7}{2}\dot{\boldsymbol{r}}_G - \frac{7}{2}\dot{\boldsymbol{r}}_G(0) + \dot{\boldsymbol{r}}_G(0) + \boldsymbol{\omega}(0) \times a\boldsymbol{k} = \boldsymbol{\Omega} \times \boldsymbol{r}_G,$$

or

$$\frac{7}{2}\dot{\boldsymbol{r}}_G - \frac{7}{2}\dot{\boldsymbol{r}}_G(0) + \boldsymbol{\Omega} \times \boldsymbol{r}_G(0) = \boldsymbol{\Omega} \times \boldsymbol{r}_G,$$

or

$$\dot{\boldsymbol{r}}_G = \frac{2}{7}\boldsymbol{\Omega} \times (\boldsymbol{r}_G - \boldsymbol{r}_G(0)) + \dot{\boldsymbol{r}}_G(0). \tag{6}$$

c) $\boldsymbol{\Omega}$ and $\dot{\boldsymbol{r}}_G(0)$ are perpendicular to each other and therefore Equation 6 can be rearranged as

$$\dot{\boldsymbol{r}}_G = \frac{2}{7}\boldsymbol{\Omega} \times \left(\boldsymbol{r}_G - \boldsymbol{r}_G(0) + \frac{7}{2}\frac{\dot{\boldsymbol{r}}_G(0) \times \boldsymbol{\Omega}}{|\boldsymbol{\Omega}|^2}\right). \tag{7}$$

Define

$$\boldsymbol{w} = \boldsymbol{r}_G - \boldsymbol{r}_G(0) + \frac{7}{2}\frac{\dot{\boldsymbol{r}}_G(0) \times \boldsymbol{\Omega}}{|\boldsymbol{\Omega}|^2}.$$

Then Equation 7 can be rewritten as

$$\dot{\boldsymbol{w}} = \frac{2}{7}\boldsymbol{\Omega} \times \boldsymbol{w},$$

which implies that $\boldsymbol{w}$ is a fixed length vector that rotates with an angular velocity of $\frac{2}{7}\boldsymbol{\Omega}$.

d) The origin of the $\boldsymbol{w}$ vector is at $\boldsymbol{w} = 0$. Therefore the position vector of the centre of the circular motion, $\boldsymbol{r}_0$, is given by

$$\boldsymbol{r}_0 - \boldsymbol{r}_G(0) + \frac{7}{2}\frac{\dot{\boldsymbol{r}}_G(0) \times \boldsymbol{\Omega}}{|\boldsymbol{\Omega}|^2} = 0,$$

or

$$\boldsymbol{r}_0 = \boldsymbol{r}_G(0) - \frac{7}{2}\frac{\dot{\boldsymbol{r}}_G(0) \times \boldsymbol{\Omega}}{|\boldsymbol{\Omega}|^2}.$$

e) The radius of the circular motion can be found be considering the initial (at $t = 0$) configuration. The radius $|\boldsymbol{r}'|$ is going to be given by the magnitude of

$$\boldsymbol{r}_G(0) - \boldsymbol{r}_0.$$

Therefore

$$|\boldsymbol{r}'| = \left|\boldsymbol{r}_G(0) - \boldsymbol{r}_G(0) + \frac{7}{2}\frac{\dot{\boldsymbol{r}}_G(0) \times \boldsymbol{\Omega}}{|\boldsymbol{\Omega}|^2}\right|,$$

or

$$|\boldsymbol{r}'| = \left|\frac{7}{2}\frac{\dot{\boldsymbol{r}}_G(0) \times \boldsymbol{\Omega}}{|\boldsymbol{\Omega}|^2}\right|,$$

or

$$|\boldsymbol{r}'| = \frac{7}{2}\frac{|\dot{\boldsymbol{r}}_G(0)|}{|\boldsymbol{\Omega}|}.$$