EE4-27
EE9CS5-2

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2012

MSc and EEE/ISE PART IV: MEng and ACGI

SYSTEMS IDENTIFICATION

Monday, 21 May 10:00 am

Time allowed: 3:00 hours

There are FIVE questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s) :	T. Parisini
	Second Marker(s) :	S. Evangelou

1. Given a stationary stochastic process $v(\cdot)$ with $\mathbb{E}(v) = 0$, consider its correlation function $\gamma_v(\tau)$ which takes on the following values:

$$\gamma_v(0) = 5; \quad \gamma_v(-1) = \gamma_v(1) = 2; \quad \gamma_v(\tau) = 0, \forall \tau \colon |\tau| \geq 2 \tag{1.1}$$

a) State which family of stochastic models (e.g., ARMA, ...) has a correlation function structure which is formally consistent with $\gamma_v(\tau)$ given in (1.1). Justify your answer.

[2 Marks]

b) Determine the parameters of a model in canonical form belonging to the family given in your answer to Question 1a) such that its correlation function is equal to $\gamma_v(\tau)$ given in (1.1).

[9 Marks]

c) Determine, in at least two different ways, the spectrum $\Gamma_v(\omega)$ of the process $v(\cdot)$ obtained in your answer to Question 1b) and sketch its behaviour in the interval $\omega \in [\pi, \pi]$.

[9 Marks]

2. Consider the stochastic process $y(\cdot)$ generated as shown in Fig. 2.1.

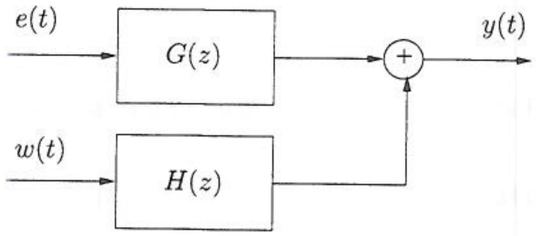

Figure 2.1 Dynamic system generating the stochastic process $y(\cdot)$.

where

$$G(z) = 1 - \frac{9}{10}z^{-1}; \quad H(z) = \frac{1}{1 + \frac{1}{2}z^{-1}}.$$

Moreover, the processes $e(\cdot)$ and $w(\cdot)$ are assumed to be independent, with $e(\cdot) \sim WN(0,1)$ and $w(\cdot) \sim WN(0,1)$.

a) Write the difference equation expressing the time-behaviour of $y(t)$.

[3 Marks]

b) Determine $\mathbb{E}[y(t)]$.

[2 Marks]

c) Determine $\text{var}[y(t)]$.

[6 Marks]

d) Determine the spectrum $\Gamma_y(\omega)$ of the process $y(\cdot)$ and sketch its behaviour in the interval $\omega \in [\pi, \pi]$.

[9 Marks]

3. Consider the stochastic process $v(\cdot)$ generated by the ARMA model

$$v(t) = \frac{1}{10}v(t-1) + e(t) + ce(t-1) \tag{3.1}$$

where $e(\cdot) \sim WN(0,1)$ and c denotes an *unknown* parameter such that (3.1) is in *canonical form.*

a) the stochastic process $v(\cdot)$ is stationary. Why? Justify your answer.

[1 Marks]

b) The normalized covariance of the stochastic process $v(\cdot)$

$$\rho(\tau) = \frac{\mathbb{E}[v(t)v(t-\tau)]}{\mathrm{var}[v(t)]} \tag{3.2}$$

for $\tau = 1$ takes on the value

$$\rho(1) = 1/10.$$

Determine the value of the parameter c.

[3 Marks]

c) For the value of c determined in your answer to Question 3b), write the difference equation representing the optimal two-steps ahead predictor $\hat{v}(t+2|t)$ of $v(t)$.

[5 Marks]

d) Now, assume that the normalized covariance given in (3.2), for $\tau = 1$ takes on the value

$$\rho(1) = 1/2.$$

Determine the value of the parameter c.

[7 Marks]

e) For the value of c determined in your answer to Question 3d), write the difference equation representing the optimal one-step ahead predictor of $v(t)$.

[4 Marks]

4. Consider a stochastic process $y(\cdot)$ and an arbitrarily large number N of measurements $\{y(1), y(2), \ldots, y(N)\}$.

Moreover, consider the family of AR(1) stochastic models

$$\mathscr{M}(\theta): \quad y(t) = ay(t-1) + e(t), \quad \theta := a$$

and consider $\hat{a}(N)$ as the least squares estimate of a based on N measurements $\{y(1), y(2), \ldots, y(N)\}$.

a) **Case 1.** If the process is generated as

$$y(t) = \frac{3}{10}y(t-1) + \xi(t), \quad \xi(\cdot) \sim WN(0,1) \tag{4.1}$$

determine the value $\bar{a}_1$ the estimate $\hat{a}(N)$ approaches for large values of N (that is, $\bar{a}_1 = \lim_{N\to\infty} \hat{a}(N)$, a.s.).

[4 Marks]

b) **Case 2.** If the process is generated as

$$y(t) = \frac{3}{10}y(t-1) + \xi(t) + \frac{1}{2}\xi(t-1), \quad \xi(\cdot) \sim WN(0,1) \tag{4.2}$$

determine the value $\bar{a}_2$ the estimate $\hat{a}(N)$ approaches for large values of N (that is, $\bar{a}_2 = \lim_{N\to\infty} \hat{a}(N)$, a.s.).

[8 Marks]

c) Denoting by $y_1(\cdot)$ and $y_2(\cdot)$ the stochastic processes generated by (4.1) and (4.2), respectively, compute and compare the variances of the prediction errors

$$\mathrm{var}[y_1(t) - \bar{a}_1 y_1(t-1)] \quad \text{and} \quad \mathrm{var}[y_2(t) - \bar{a}_2 y_2(t-1)]$$

Comment on your findings. Moreover, consider a different variance for the noise ξ, namely $\xi(\cdot) \sim WN(0,3)$. State whether or not the values of $\bar{a}_1$ and $\bar{a}_2$ computed in the answers to Questions 4a) and 4b) are different because of the different value taken on by the variance of ξ. Justify your answer.

[8 Marks]

5. Consider the following stochastic system:

$$\begin{cases} x(t+1) = \frac{3}{5}x(t) + v_1(t) \\ y(t) = \frac{4}{5}x(t) + v_2(t) \end{cases} \tag{5.1}$$

where $v_1(\cdot) \sim WGN(0,1)$, $v_2(\cdot) \sim WGN(0,1)$ and the stochastic processes $v_1(\cdot)$ and $v_2(\cdot)$ are supposed to be independent.

a) Referring to system (5.1), write the Algebraic Riccati Equation (ARE) of the one-step ahead optimal steady-state Kalman predictor and show that the ARE admits an admissible solution $\bar{P}$. Compute the corresponding constant gain $\bar{K}$.

[3 Marks]

b) Write the difference equation yielding the one-step ahead optimal steady-state Kalman prediction $\hat{x}(t+1|t)$ and draw the block-diagram of the predictor.

[3 Marks]

c) Determine the transfer function $G_{ye}(z)$ from the $y(t)$ to $e(t)$ (*Hint: exploit the block diagram drawn in answer to Question 5b)*). Analyze the stability properties of $G_{ye}(z)$.

[4 Marks]

d) Show briefly why the stochastic process $x(\cdot)$ is stationary and compute its variance $\text{var}[x(t)]$. Compare $\text{var}[x(t)]$ with $\text{var}[x(t) - \hat{x}(t|t-1)]$. Comment on your findings.

[4 Marks]

e) Show that the optimal steady-state Kalman filter yields the estimate

$$\hat{x}(t|t) = \frac{5}{3}\hat{x}(t+1|t)$$

Moreover, compute the variance of the error $\text{var}[x(t) - \hat{x}(t|t)]$ and compare it with $\text{var}[x(t) - \hat{x}(t|t-1)]$. Comment on your findings.

[6 Marks]

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1. Solution

a) Moving-average models of order n with $n \geq 1$ (typically denoted by $MA(n)$) have a correlation function γ such that $\gamma(\tau) = 0, \forall\tau\colon |\tau| > n$. Therefore, by noticing that $\gamma_v(\tau) = 0, \forall\tau\colon |\tau| \geq 2$ it is immediate to conclude that moving-average first-order models MA(1) have a correlation function whose form is in agreement with $\gamma_v(\tau)$.

b) Models MA(1) are characterised by

$$v(t) = c_0 e(t) + c_1 e(t-1); \quad e(\cdot) \sim WN(0, \lambda^2)$$

where c_0, c_1, λ^2 are the parameters to be determined on the basis of the information on the correlation function $\gamma_v(\tau)$. As two values of $\gamma_v(\tau)$ are available, we set $c_0 = 1$ to eliminate the parameter-redundancy and we consider the equivalent problem of determining the parameters c and λ^2 of the model

$$y(t) = e(t) + ce(t-1); \quad e(\cdot) \sim WN(0, \lambda^2)$$

such that $\gamma_y(0) = 5$ and $\gamma_y(1) = \gamma_y(-1) = 2$. It turns out that:

$$\gamma_v(0) = \text{var}[y(t)] = \mathbb{E}[e(t)^2] + c^2\mathbb{E}[e(t-1)^2] + 2\mathbb{E}[e(t)e(t-1)] = (1+c^2)\lambda^2$$

$$\gamma_v(1) = \mathbb{E}[y(t)y(t-1)] = \mathbb{E}\{[e(t) + ce(t-1)] \cdot [e(t-1) + ce(t-2)]\} = c\lambda^2$$

By imposing

$$\begin{cases} (1+c^2)\lambda^2 = \gamma_v(0) = 5 \\ c\lambda^2 = \gamma_v(1) = 2 \end{cases}$$

we obtain two solutions:

$$\text{sol}_1: \begin{cases} c = 2 \\ \lambda^2 = 1 \end{cases} \quad \text{and} \quad \text{sol}_2: \begin{cases} c = \frac{1}{2} \\ \lambda^2 = 4 \end{cases}$$

Clearly only sol_2 yields a MA(1) model in canonical form, that is:

$$y(t) = e(t) + \frac{1}{2}e(t-1); \quad e(\cdot) \sim WN(0,4)$$

c) As $\gamma_v(\tau) = 0, \forall\tau\colon |\tau| \geq 2$, we are able to compute the spectrum by direct use of the definition:

$$\Gamma_v(\omega) = \sum_{\tau=-\infty}^{+\infty} \gamma_v(\tau)e^{-j\omega\tau} = \gamma_v(-1)e^{j\omega} + \gamma_v(0) + \gamma_v(1)e^{-j\omega} = 5 + 4\cos\omega$$

Alternatively, by using the Z-transform notation and the complex spectrum properties, we have:

$$Y(z) = (1 + \frac{1}{2}z^{-1})E(z) \Longrightarrow \Gamma_v(\omega) = (1 + \frac{1}{2}z^{-1})(1 + \frac{1}{2}z)\bigg|_{z=e^{j\omega}} \cdot \lambda^2 = 5 + 4\cos\omega$$

The spectrum behaviour in the interval $\omega \in [\pi, \pi]$ is given in Fig. 1.1

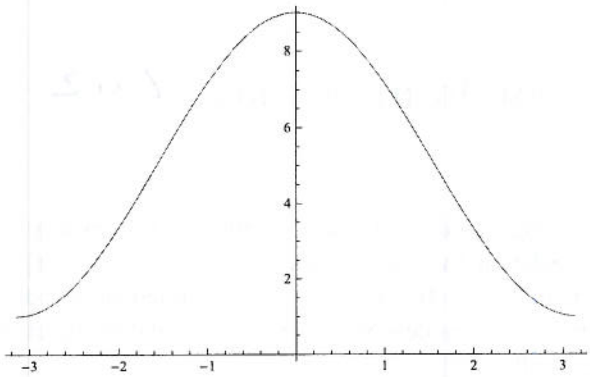

Figure 1.1 Plot of the spectrum $\Gamma_v(\omega) = 5 + 4\cos\omega$.

2. Solution

a) Clearly:

$$Y(z) = H(z)W(z) + G(z)E(z) = \frac{1}{1+\frac{1}{2}z^{-1}}W(z) + \left(1 - \frac{9}{10}z^{-1}\right)E(z)$$

After some algebra, it follows that:

$$y(t) = -\frac{1}{2}y(t-1) + w(t) + e(t) - \frac{2}{5}e(t-1) - \frac{9}{20}e(t-2) \qquad (2.1)$$

b) By applying the expected value operator on both sides of (2.1), we obtain

$$\mathbb{E}[y(t)] = -\frac{1}{2}\mathbb{E}[y(t-1)] + \mathbb{E}[w(t)] + \mathbb{E}[e(t)] - \frac{2}{5}\mathbb{E}[e(t-1)] - \frac{9}{20}\mathbb{E}[e(t-2)]$$

and, due to the stationarity of the process generated by (2.1), we can easily conclude that $\mathbb{E}[y(t)] = 0$.

c) Since we have shown that $\mathbb{E}[y(t)] = 0$, again due to the stationarity of the process generated by (2.1), and by exploiting the whiteness and the mutual independence of the processes $e(\cdot)$ and $w(\cdot)$, we obtain

$$\begin{aligned} \text{var}[y(t)] &= \mathbb{E}[y(t)^2] = \mathbb{E}\left\{\left[-\tfrac{1}{2}y(t-1) + w(t) + e(t) - \tfrac{2}{5}e(t-1) - \tfrac{9}{20}e(t-2)\right]^2\right\} \\ &= \tfrac{1}{4}\text{var}[y(t)] + \tfrac{45}{16} \Longrightarrow \text{var}[y(t)] = \tfrac{15}{4} \end{aligned}$$

d) Since

$$Y(z) = \frac{1}{1+\frac{1}{2}z^{-1}}W(z) + \left(1 - \frac{9}{10}z^{-1}\right)E(z)$$

and owing to the fact that $e(\cdot) \sim WN(0,1)$, $w(\cdot) \sim WN(0,1)$, and that $e(\cdot)$ and $w(\cdot)$ are independent, it follows that

$$\Gamma_y(\omega) = \left[\frac{1}{(1+\frac{1}{2}z^{-1})(1+\frac{1}{2}z)} + \left(1 - \frac{9}{10}z^{-1}\right)\left(1 - \frac{9}{10}z\right)\right]_{z=e^{j\omega}}$$

which, after some algebra, gives

$$\Gamma_y(\omega) = \frac{181}{100} - \frac{9}{5}\cos\omega + \frac{1}{\frac{5}{4} + \cos\omega}$$

As

$$Y(z) = \frac{1}{1+\frac{1}{2}z^{-1}}W(z) + \left(1 - \frac{9}{10}z^{-1}\right)E(z) = \frac{z}{z+\frac{1}{2}}W(z) + \frac{z - \frac{9}{10}}{z}E(z)$$

we have

$$\Gamma_y(\omega) = \frac{1}{\left|e^{j\omega} + \frac{1}{2}\right|^2} + \left|e^{j\omega} - \frac{9}{10}\right|^2$$

Thus, a few values taken on by the spectrum can be easily computed by geometric considerations:

$$\Gamma_y(0) = \frac{1}{(1+\frac{1}{2})^2} + \frac{1}{100} = \frac{409}{900}$$

$$\Gamma_y(\pi/2) = \frac{1}{1+\frac{1}{4}} + 1 + \frac{81}{100} = \frac{261}{100}$$

$$\Gamma_y(\pi) = \frac{1}{\frac{1}{4}} + \left(1 + \frac{9}{10}\right)^2 = \frac{761}{100}$$

The spectrum behaviour in the interval $\omega \in [\pi, \pi]$ is given in Fig. 2.1

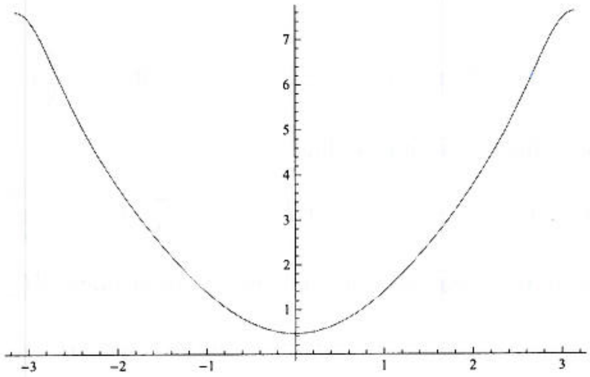

Figure 2.1 Plot of the spectrum $\Gamma_y(\omega) = \frac{181}{100} - \frac{9}{5}\cos\omega + \frac{1}{\frac{5}{4}+\cos\omega}$.

3. Solution

a) The transfer function associated with the difference equation $v(t) = \frac{1}{10}v(t-1) + e(t) + ce(t-1)$ is given by

$$V(z) = \frac{1+cz^{-1}}{1-\frac{1}{10}z^{-1}}E(z)$$

which clearly represents an asymptotically stable system. Therefore, the stochastic process $v(\cdot)$ is stationary.

b) We have:

$$\begin{aligned}
\gamma(0) &= \mathrm{var}[v(t)] \\
\gamma(1) &= \mathbb{E}[v(t)v(t-1)] = \mathbb{E}\left\{\left[\tfrac{1}{10}v(t-1) + e(t) + ce(t-1)\right] \cdot v(t-1)\right\} \\
&= \tfrac{1}{10}\mathbb{E}[v(t-1)^2] + c\mathbb{E}[v(t-1)e(t-1)] \\
&= \tfrac{1}{10}\gamma(0) + c
\end{aligned} \tag{3.1}$$

Thus

$$\rho(1) = \frac{1}{10} \Longrightarrow \frac{\gamma(1)}{\gamma(0)} = \frac{1}{10} \Longrightarrow c = 0$$

c) Imposing $c = 0$ into $v(t) = \frac{1}{10}v(t-1) + e(t) + ce(t-1)$ gives $v(t) = \frac{1}{10}v(t-1) + e(t)$ and hence

$$A(z)v(t) = C(z)e(t) \quad \text{with} \quad A(z) := 1 - \frac{1}{10}z^{-1}; C(z) := 1$$

By performing two steps of polynomial division of $C(z)$ by $A(z)$ we have

$$\widehat{W}(z) := \frac{C(z)}{A(z)} = 1 + \frac{1}{10}z^{-1} + z^{-2}\frac{\frac{1}{100}}{1-\frac{1}{10}z^{-1}}$$

thus obtaining

$$\hat{v}(t+2|t) = \frac{1}{100}v(t)$$

d) From (3.1) and taking into account that now $\rho(1) = \frac{1}{2}$, we have

$$\gamma(1) = \frac{1}{10}\gamma(0) + c \Longrightarrow \frac{1}{10}\gamma(0) + c = \frac{1}{2}\gamma(0) \Longrightarrow c = \frac{2}{5}\gamma(0)$$

Now:

$$\begin{aligned}
\gamma(0) &= \mathbb{E}\left\{\left[\tfrac{1}{10}v(t-1) + e(t) + ce(t-1)\right]^2\right\} \\
&= \tfrac{1}{100}\gamma(0) + 1 + c^2 + \tfrac{1}{5}c \Longrightarrow \gamma(0) = \tfrac{100}{99}(1 + c^2 + \tfrac{1}{5}c)
\end{aligned} \tag{3.2}$$

and replacing $c = \frac{2}{5}\gamma(0)$ into (3.2) we obtain the polynomial equation

$$16\gamma(0)^2 - 91\gamma(0) + 100 = 0 \tag{3.3}$$

Equation (3.3) has two solutions:

$$\gamma(0)_{(1)} = \frac{1}{32}(91 - 3\sqrt{209}); \quad \gamma(0)_{(2)} = \frac{1}{32}(91 + 3\sqrt{209})$$

yielding two possible values for the parameter c:

$$c_{(1)} = \frac{1}{80}(91 - 3\sqrt{209}); \quad c_{(2)} = \frac{1}{80}(91 - 3\sqrt{209});$$

As $c_{(1)} < 1$ and $c_{(2)} > 1$, only $c_{(1)} < 1$ is an admissible value for the parameter c because the ARMA model has to be in canonical form.

e) The ARMA model now is:

$$A(z)v(t) = C(z)e(t) \quad \text{with} \quad A(z) := 1 - \frac{1}{10}z^{-1}; C(z) := 1 + \frac{1}{80}(91 - 3\sqrt{209})z^{-1}$$

The one-step ahead optimal prediction $\hat{v}(t+1|t)$ can be obtained by using the general formula

$$\hat{v}(t+1|t) = \frac{C(z) - A(z)}{C(z)}v(t+1) = \frac{\frac{3}{80}(\sqrt{209} - 33)}{1 + \frac{1}{80}(91 - 3\sqrt{209})z^{-1}}z^{-1}v(t+1)$$

yielding

$$\hat{v}(t+1|t) = -\frac{1}{80}(91 - 3\sqrt{209})\hat{v}(t|t-1) + \frac{3}{80}(\sqrt{209} - 33)v(t)$$

4. Solution

a) The model in prediction form is

$$\widehat{\mathcal{M}}(\theta): \quad \hat{y}(t|t-1) = ay(t-1)$$

The estimate $\hat{a}(N)$ converges almost surely to the minima of

$$\begin{aligned}\bar{J}(a) &= \mathbb{E}\left\{[y(t)-\hat{y}(t|t-1)]^2\right\} = \mathbb{E}\left\{[y(t)-ay(t-1)]^2\right\} \\ &= \mathbb{E}[y(t)^2] - 2a\mathbb{E}[y(t)y(t-1)] + a^2\mathbb{E}[y(t-1)^2] = (1+a^2)\gamma_y(0) - 2a\gamma_y(1)\end{aligned}$$

where $\gamma_y(\tau)$ denotes the correlation function of the process $y(\cdot)$.

Therefore, $\bar{J}(a)$ has a single minimum attained for $\bar{a} = \gamma_y(1)/\gamma_y(0)$. Let us now compute $\gamma_y(1)$:

$$\gamma_y(1) = \mathbb{E}[y(t)y(t-1)] = \mathbb{E}\left\{\left[\frac{3}{10}y(t-1)+\xi(t)\right]y(t-1)\right\} = \frac{3}{10}\gamma_y(0)$$

Thus $\bar{a}_1 = \gamma_y(1)/\gamma_y(0) = \frac{3}{10}$.

b) Let us compute $\gamma_y(0)$ and $\gamma_y(1)$:

$$\begin{aligned}\gamma_y(0) &= \mathbb{E}[y(t)^2] = \mathbb{E}\left\{\left[\tfrac{3}{10}y(t-1)+\xi(t)+\tfrac{1}{2}\xi(t-1)\right]^2\right\} \\ &= \tfrac{9}{100}\gamma_y(0) + \text{var}[\xi(t)] + \tfrac{1}{4}\text{var}[\xi(t)] + \tfrac{3}{10}\mathbb{E}[y(t-1)\xi(t-1)]\end{aligned}$$

Hence $\gamma_y(0) = \frac{155}{91}\text{var}[\xi(t)] = \frac{155}{91}$.

$$\begin{aligned}\gamma_y(1) &= \mathbb{E}[y(t)y(t-1)] = \mathbb{E}\left\{\left[\tfrac{3}{10}y(t-1)+\xi(t)+\tfrac{1}{2}\xi(t-1)\right]y(t-1)\right\} \\ &= \tfrac{3}{10}\gamma_y(0) + \tfrac{1}{2}\text{var}[\xi(t)] = \tfrac{92}{91}\text{var}[\xi(t)] = \tfrac{92}{91}\end{aligned}$$

Thus $\bar{a}_2 = \gamma_y(1)/\gamma_y(0) = \frac{92}{155}$.

c) We have:

$$\text{var}[y_1(t)-\bar{a}_1y_1(t-1)] = \mathbb{E}\left\{\left[\frac{3}{10}y_1(t-1)+\xi(t)-\frac{3}{10}y_1(t-1)\right]^2\right\} = \text{var}[\xi(t)]$$

$$\begin{aligned}\text{var}[y_2(t)-\bar{a}_2y_2(t-1)] &= \mathbb{E}\left\{\left[-\tfrac{91}{310}y_2(t-1)+\xi(t)+\tfrac{1}{2}\xi(t-1)\right]^2\right\} \\ &= \tfrac{171}{55}\text{var}[\xi(t)]\end{aligned}$$

As should be expected because of the presence of the coloured noise in model generating $y_2(\cdot)$, the prediction error $y_2(t)-\bar{a}_2y_2(t-1)$ is not white and

$$\text{var}[y_2(t)-\bar{a}_2y_2(t-1)] = \frac{171}{55}\text{var}[\xi(t)] > \text{var}[\xi(t)] = \text{var}[y_1(t)-\bar{a}_1y_1(t-1)]$$

Finally, let us address the case $\xi(\cdot) \sim WN(0,3)$. As we have seen in the previous answers, $\gamma_y(0)$ and $\gamma_y(1)$ are always proportional to $\text{var}[\xi(t)]$ and hence the values $\bar{a}_1 = \gamma_y(1)/\gamma_y(0)$ and $\bar{a}_2 = \gamma_y(1)/\gamma_y(0)$ do not depend on $\text{var}[\xi(t)]$.

5. Solution

a) The general algebraic Riccati equation is

$$P = F\left[P - PH^{\top}\left(V_2 + HPH^{\top}\right)^{-1} HP\right]F^{\top} + V_1$$

Letting $F = 3/5$, $H + 4/5$, $V_1 = 1$, $V_2 = 1$, we have

$$P = \frac{9}{25}\left(P - \frac{\frac{16}{25}}{1+\frac{16}{25}P}P^2\right) + 1 \Longrightarrow P^2 = \frac{25}{16}$$

thus obtaining the two solutions

$$\bar{P}_1 = \frac{5}{4} \quad \text{and} \quad \bar{P}_2 = -\frac{5}{4}$$

Clearly, the only admissible solution is the positive one. Thus $\bar{P} = 5/4$. Accordingly:

$$\bar{K} = F\bar{P}H^{\top}\left(V_2 + H\bar{P}H^{\top}\right)^{-1} = \frac{1}{3}$$

b) We have

$$\begin{cases} \hat{x}(t+1|t) = \frac{3}{5}\hat{x}(t|t-1) + \frac{1}{3}e(t) \\ \hat{y}(t+1|t) = \frac{4}{5}\hat{x}(t+1|t) \\ e(t) = y(t) - \frac{4}{5}\hat{x}(t|t-1) \end{cases}$$

and thus

$$\hat{x}(t+1|t) = \frac{3}{5}\hat{x}(t|t-1) + \frac{1}{3}\left[y(t) - \frac{4}{5}\hat{x}(t|t-1)\right]$$

The block-diagram of the steady-state one-step ahead Kalman predictor is drawn in Fig. 5.1.

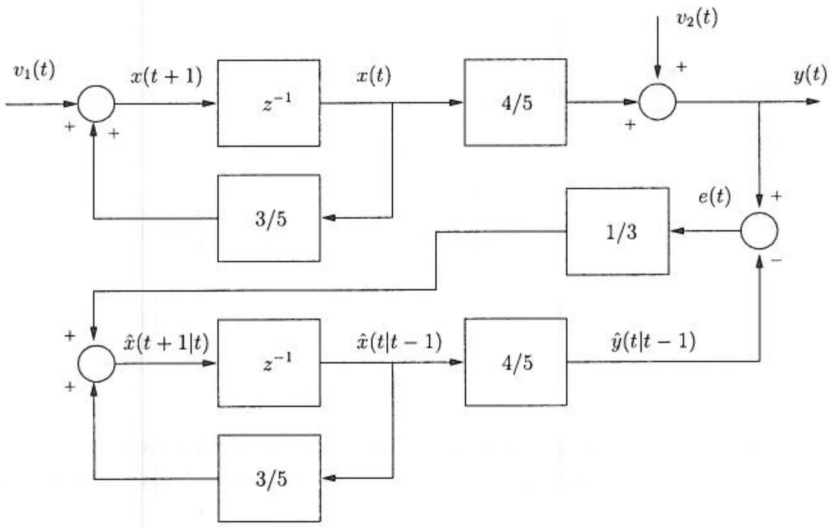

Figure 5.1 Block-diagram of the steady-state one-step ahead Kalman predictor.

c) Considering the block diagram depicted in Fig. 5.1, we can extract the "section" of the block diagram related to the input/output relationship between $y(t)$ and $e(t)$ shown in Fig. 5.2.

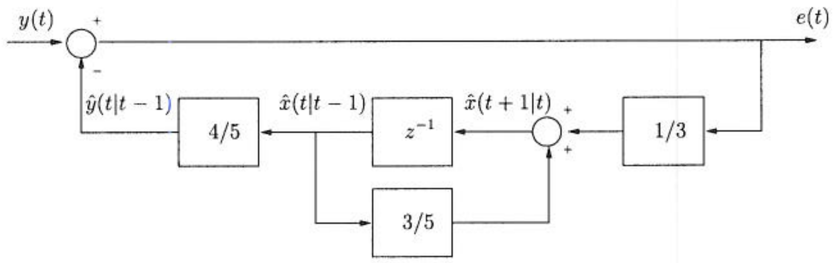

Figure 5.2 Block-diagram of the input/output relationship between $y(t)$ and $e(t)$ of the steady-state one-step ahead Kalman predictor.

Owing to the scheme in Fig. 5.2, it follows immediately that

$$G_{ye}(z) = \frac{1}{1 + \frac{1}{3} \cdot \frac{4}{5} \cdot \frac{z^{-1}}{1 - \frac{3}{5} z^{-1}}} = \frac{z - \frac{3}{5}}{z - \frac{1}{3}}$$

The transfer function $G_{ye}(z)$ represents an asymptotically stable system because the pole $\frac{1}{3}$ lies strictly inside the unit circle.

d) The stochastic process $x(\cdot)$ generated by the system

$$x(t+1) = \frac{3}{5} x(t) + v_1(t)$$

is stationary because $v_1(\cdot) \sim WGN(0,1)$ and because the system is asymptotically stable.

Because of the stationarity of $x(\cdot)$, $\text{var}[x(t)] = \text{var}[x(t-1)]$ and hence

$$\text{var}[x(t)] = \frac{9}{25} \text{var}[x(t)] + 1 \Longrightarrow \text{var}[x(t)] = \frac{25}{16}$$

We have

$$\text{var}[x(t) - \hat{x}(t|t-1)] = \bar{P} = \frac{5}{4} < \frac{25}{16} = \text{var}[x(t)]$$

and this reduction of the variance should be expected due to the exploitation of the measurements $y(t)$ in computing the steady-state one-step ahead Kalman prediction $\hat{x}(t+1|t)$.

e) We have:

$$\hat{x}(t+1|t) = \mathbb{E}\left[\tfrac{3}{5}x(t) + v_1(t)|y^t\right] = \tfrac{3}{5}\mathbb{E}\left[x(t)|y^t\right] + \mathbb{E}\left[v_1(t)|y^t\right]$$

$$= \tfrac{3}{5}\hat{x}(t|t) \Longrightarrow \hat{x}(t|t) = \tfrac{5}{3}\hat{x}(t+1|t)$$

Now, using the expression of the one-step ahead Kalman prediction $\hat{x}(t+1|t)$, it follows that:

$$\text{var}[x(t) - \hat{x}(t|t)] = \text{var}[x(t) - \tfrac{5}{3}\hat{x}(t+1|t)]$$

$$= \text{var}[x(t) - \tfrac{5}{9}\hat{x}(t|t-1) - \tfrac{5}{9}y(t)]$$

Exploiting the fact that $x(t)$ and $\hat{x}(t|t-1)$ are uncorrelated with $v_2(t)$, from $\text{var}[x(t) - \hat{x}(t|t-1)] = \bar{P} = \frac{5}{4}$, we obtain

$$\text{var}[x(t) - \hat{x}(t|t)] = \frac{25}{36} < \text{var}[x(t) - \hat{x}(t|t-1)]$$

This reduction in the variance of the state estimation error should be expected because in computing $\hat{x}(t|t)$ the measurement $y(t)$ is used whereas in the computation of $\hat{x}(t|t-1)$ this data-point was not yet available.