EE4-27

EE9-CS5-2

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2013

MSc and EEE/EIE PART IV: MEng and ACGI

SYSTEMS IDENTIFICATION

Thursday, 23 May 10:00 am

Time allowed: 3:00 hours

There are FOUR questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible   First Marker(s) : T. Parisini

Second Marker(s) : S. Evangelou

1. Consider a stationary stochastic process $v(\cdot)$ with correlation function $\gamma(\tau)$ of which the first samples ($\tau = 0, 1, \ldots, 10$) are drawn in Fig. 1.1.

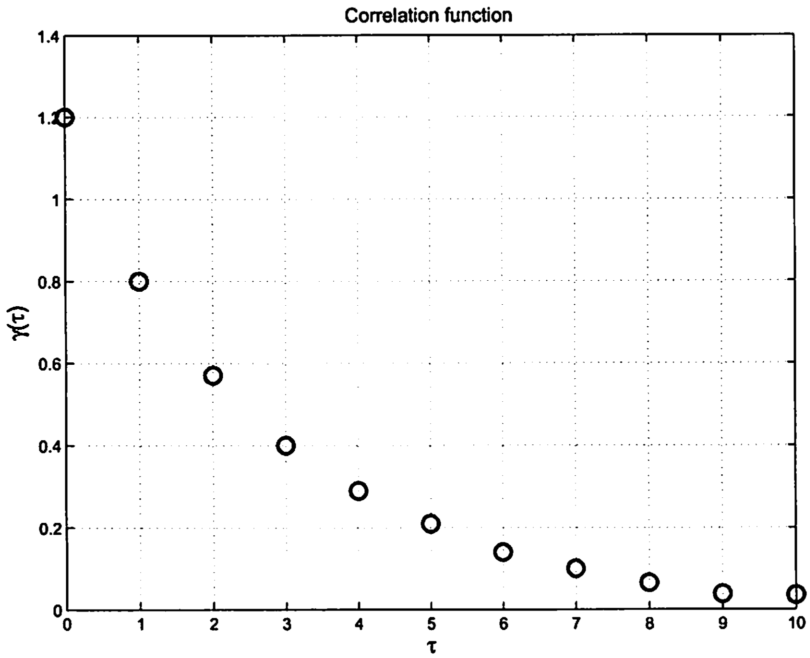

Figure 1.1 Plot of the correlation function $\gamma(\tau)$, $\tau = 0, 1, \ldots, 10$.

a) Among the following three models, (1), (2) and (3), state which one has generated the process $v(\cdot)$, where $\xi(\cdot) \sim WN(0, \lambda^2)$. Justify your answer.

$$(1) \quad v(t) = \xi(t) - \xi(t-1) + \frac{1}{2}\xi(t-2)$$

$$(2) \quad v(t) = \frac{7}{10}v(t-1) + \xi(t) + \xi(t-1)$$

$$(3) \quad v(t) = -\frac{7}{10}v(t-1) + \xi(t) + \xi(t-1)$$

[8 Marks]

b) Determine the analytical expressions of the spectra $\Gamma_v(\omega)$ as functions of the angular frequency ω of the stochastic processes generated by models (1), (2), and (3) given in Question 1a).

[7 Marks]

c) Sketch the behaviours of the three spectra determined in your answer to Question 1b) in the interval $\omega \in [-\pi, \pi]$.

[5 Marks]

2. Consider the stochastic system expressed through the following state equations:

$$\begin{cases} x_1(t+1) = \frac{4}{5}x_1(t) + u(t) + e(t) \\ x_2(t+1) = \frac{1}{2}x_2(t) + x_1(t) + 4e(t) \\ y(t) = x_2(t) \end{cases}$$

where $e(t)$ is a stochastic process, with $e(\cdot) \sim WN(0,1)$ and $u(t)$ is a deterministic and perfectly known input variable.

a) Determine the corresponding ARMAX model of the form

$$A(z)y(t) = B(z)u(t) + C(z)e(t)$$

[7 Marks]

b) Show that the stochastic residual term $C(z)/A(z)$ is not in canonical spectral form. Find the equivalent canonical spectral factor. Justify your answer.

[3 Marks]

c) Determine the difference equation by which the optimal two-steps ahead prediction $\hat{y}(t+2|t)$ can be computed using available information till time-instant t.

[8 Marks]

d) Compute the variance of the prediction error $\varepsilon(t) = y(t+2) - \hat{y}(t+2|t)$ associated with the optimal two-steps ahead predictor determined in your answer to Question 1c).

[2 Marks]

3. Consider the stochastic process $v(\cdot)$ generated by the ARMA model

$$v(t) = \frac{3}{10}v(t-2) + e(t) + \frac{7}{10}e(t-1) \tag{3.1}$$

where $e(\cdot) \sim WN(2,1)$.

a) The stochastic process $v(\cdot)$ is stationary. Why? Justify your answer.

[1 Marks]

b) Compute the expected value of $v(\cdot)$.

[3 Marks]

c) Compute the variance of $v(\cdot)$.

[3 Marks]

d) Compute $\gamma_v(\tau)$, for $\tau = 1,2$, where $\gamma_v(\tau)$ denotes the correlation function of $v(\cdot)$.

[5 Marks]

e) Consider the stochastic process generated by the ARMA model (3.1) with $e(\cdot) \sim WN(0,1)$ (that is we now consider a zero-mean white process $e(\cdot)$). Moreover, consider an arbitrarily large number N of measurements $\{v(1), v(2), \ldots, v(N)\}$. Finally, consider the following two families of AR stochastic models:

$$\begin{aligned} \mathscr{M}_1(\theta_1): &\quad v(t) = av(t-1) + \xi(t), \quad \theta_1 = a \\ \mathscr{M}_2(\theta_2): &\quad v(t) = a_1 v(t-1) + a_2 v(t-2) + \xi(t), \quad \theta_2 = [a_1 \; a_2]^{\mathsf{T}}. \end{aligned}$$

Denote with $\hat{\theta}_1(N) = \hat{a}(N)$ the least squares estimate of θ_1 based on N measurements $\{v(1), v(2), \ldots, v(N)\}$.

Likewise denote with $\hat{\theta}_2(N) = [\hat{a}_1(N), \hat{a}_2(N)]^{\mathsf{T}}$ the least squares estimate of θ_2 based on N measurements $\{v(1), v(2), \ldots, v(N)\}$.

Determine the value $\bar{\theta}_1$ the estimate $\hat{\theta}_1(N)$ approaches for large values of N (that is, $\bar{\theta}_1 = \lim_{N\to\infty} \hat{\theta}_1(N)$, a.s.) when the family $\mathscr{M}_1$ of models is used.

Moreover, determine the value $\bar{\theta}_2$ the estimate $\hat{\theta}_2(N)$ approaches for large values of N (that is, $\bar{\theta}_2 = \lim_{N\to\infty} \hat{\theta}_2(N)$, a.s.) when the family $\mathscr{M}_2$ of models is used.

Compute the variances of the prediction errors when using the models $\mathscr{M}_1(\bar{\theta}_1)$ and $\mathscr{M}_2(\bar{\theta}_2)$ and compare them. Discuss your findings.

[8 Marks]

4. Consider the following stochastic system:

$$\begin{cases} x(t+1) = \frac{1}{2}x(t) + v_1(t) \\ y(t) = 2x(t) + v_2(t) \end{cases}$$

where $v_1(\cdot) \sim WGN(0, 19/20)$, $v_2(\cdot) \sim WGN(0,1)$ and the stochastic processes $v_1(\cdot)$ and $v_2(\cdot)$ are supposed to be independent.

a) Discuss the existence of the one-step ahead steady-state Kalman predictor.

[1 Marks]

b) Determine the one-step ahead steady-state Kalman predictor.

[3 Marks]

c) Determine the one-step ahead steady-state Kalman filter.

[3 Marks]

d) Compute the steady-state variance of the filtering error $V_{\text{filt}} = \text{var}[x(t) - \hat{x}(t|t)]$ and compare it with the variance of the process $x(t)$. Comment on your findings.

[5 Marks]

e) Assume that the state noise $v_1(\cdot)$ is not zero-mean, that is $v_1(\cdot) \sim WGN(\bar{v}_1, 19/20)$, with $\bar{v}_1$ *constant and unknown*. Modify the one-step ahead steady-state Kalman predictor determined in your answer to Question 4b) so as to estimate $\hat{x}(t+1|t)$ and $\bar{v}_1$ simultaneously.

[8 Marks]

SOLUTIONS: SYSTEMS IDENTIFICATION

1. Solution

a) Consider model (1):

$$v(t) = \xi(t) - \xi(t-1) + \frac{1}{2}\xi(t-2)$$

This model has the structure of a MA(2) model. Recall that moving-average models of order n with $n \geq 1$ (typically denoted by $MA(n)$) have a correlation function γ such that $\gamma(\tau) = 0, \forall \tau\colon |\tau| > n$. Therefore, by noticing that $\gamma_v(\tau) \neq 0, \forall \tau \in [0, 10]$ it is immediate to conclude that model (1) cannot be characterised in terms of a correlation function behaving as in Fig. 1.1.

Hence, we now focus on models (2) and (3). Let's consider model (3):

$$v(t) = -\frac{7}{10}v(t-1) + \xi(t) + \xi(t-1)$$

and let's compute a few samples of the correlation function $\gamma_v(\tau)$:

$$\begin{aligned}
\gamma_v(0) &= \mathrm{var}[y(t)] = \mathbb{E}\left[\left(-\tfrac{7}{10}v(t-1) + \xi(t) + \xi(t-1)\right)^2\right] \\
&= \tfrac{49}{100}\mathrm{var}[v(t-1)] + \mathrm{var}[\xi(t)] + \mathrm{var}[\xi(t-1)] + \tfrac{14}{10}\mathbb{E}[v(t-1)\xi(t)] \\
&-\tfrac{14}{10}\mathbb{E}[v(t-1)\xi(t-1)] + 2\mathbb{E}[\xi(t)\xi(t-1)] \\
&= \tfrac{49}{100}\gamma_v(0) + \tfrac{3}{5}\mathrm{var}[\xi(t)]
\end{aligned}$$

and hence

$$\gamma_v(0) = \frac{20}{17}\lambda^2.$$

$$\begin{aligned}
\gamma_v(1) &= \mathbb{E}[v(t)v(t-1)] = \mathbb{E}\left[\left(-\tfrac{7}{10}v(t-1) + \xi(t) + \xi(t-1)\right)v(t-1)\right] \\
&= -\tfrac{7}{10}\mathrm{var}[v(t-1)] + \mathbb{E}[\xi(t)v(t-1)] + \mathbb{E}[\xi(t-1)v(t-1)] \\
&= -\tfrac{7}{10}\gamma_v(0) + \lambda^2
\end{aligned}$$

and hence

$$\gamma_v(1) = \frac{3}{17}\lambda^2.$$

$$\begin{aligned}
\gamma_v(2) &= \mathbb{E}[v(t)v(t-2)] = \mathbb{E}\left[\left(-\tfrac{7}{10}v(t-1) + \xi(t) + \xi(t-1)\right)v(t-2)\right] \\
&= -\tfrac{7}{10}\gamma_v(1)
\end{aligned}$$

and hence

$$\gamma_v(2) = -\frac{21}{170}\lambda^2.$$

$$\begin{aligned}
\gamma_v(3) &= \mathbb{E}[v(t)v(t-3)] = \mathbb{E}\left[\left(-\tfrac{7}{10}v(t-1) + \xi(t) + \xi(t-1)\right)v(t-3)\right] \\
&= -\tfrac{7}{10}\gamma_v(2)
\end{aligned}$$

and hence

$$\gamma_v(3) = \frac{147}{1700}\lambda^2.$$

Thus, we notice that

$$\gamma_v(0) > 0;\ \gamma_v(1) > 0;\ \gamma_v(2) < 0;\ \gamma_v(3) > 0$$

which is not consistent with the positive values of $\gamma(\tau)$, $\tau = 0, 1, \dots, 10$ shown in Fig. 1.1. Then, it can be concluded that the model that generated the process $v(\cdot)$ is model (2).

[8 Marks]

b) Concerning model (1), we have:

$$v(t) = \left(1 - z^{-1} + \frac{1}{2}z^{-2}\right)\xi(t)$$

Hence

$$\begin{aligned}\Gamma_1(\omega) &= \left|e^{2j\omega} - e^{j\omega} + \tfrac{1}{2}\right|^2 \lambda^2 \\ &= \left[\cos(2\omega) - \cos(\omega) + \tfrac{1}{2}\right]^2 + \left[\sin(2\omega) - \sin(\omega)\right]^2 \\ &= \tfrac{9}{4} - 3\cos(\omega) + \cos(2\omega)\end{aligned}$$

Concerning model (2), we have:

$$\left(1 - \frac{7}{10}z^{-1}\right)v(t) = \left(1 + z^{-1}\right)\xi(t)$$

Hence

$$\begin{aligned}\Gamma_2(\omega) &= \frac{\left|e^{j\omega} + 1\right|^2}{\left|e^{j\omega} - \frac{7}{10}\right|^2}\lambda^2 = \frac{\left|\cos(\omega) + j\sin(\omega) + 1\right|^2}{\left|\cos(\omega) + j\sin(\omega) - \frac{7}{10}\right|^2}\lambda^2 \\ &= \frac{2 + 2\cos(\omega)}{\frac{149}{100} - \frac{7}{5}\cos(\omega)}\end{aligned}$$

Finally, concerning model (3), we have:

$$\left(1 + \frac{7}{10}z^{-1}\right)v(t) = \left(1 + z^{-1}\right)\xi(t)$$

Hence

$$\begin{aligned}\Gamma_3(\omega) &= \frac{\left|e^{j\omega} + 1\right|^2}{\left|e^{j\omega} + \frac{7}{10}\right|^2}\lambda^2 = \frac{\left|\cos(\omega) + j\sin(\omega) + 1\right|^2}{\left|\cos(\omega) + j\sin(\omega) + \frac{7}{10}\right|^2}\lambda^2 \\ &= \frac{2 + 2\cos(\omega)}{\frac{149}{100} + \frac{7}{5}\cos(\omega)}\end{aligned}$$

[7 Marks]

c) The spectra behaviours in the interval $\omega \in [-\pi, \pi]$ are given in Figs. 1.1, 1.2, 1.3

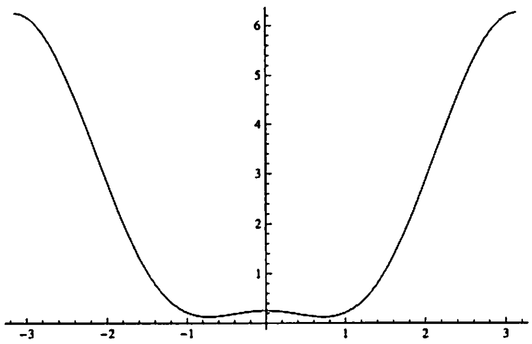

Figure 1.1 Plot of the spectrum $\Gamma_1(\omega) = \frac{9}{4} - 3\cos(\omega) + \cos(2\omega)$.

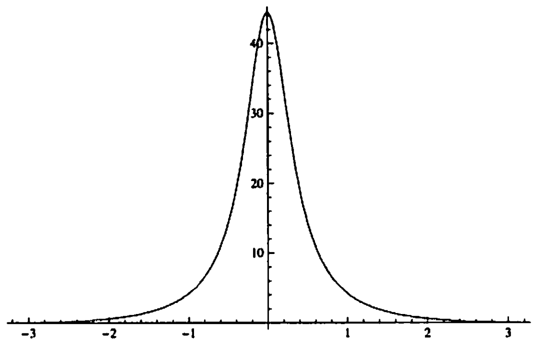

Figure 1.2 Plot of the spectrum $\Gamma_2(\omega) = \frac{2+2\cos(\omega)}{\frac{149}{100} - \frac{7}{5}\cos(\omega)}$.

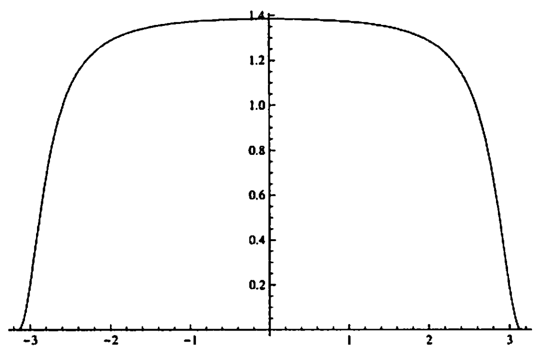

Figure 1.3 Plot of the spectrum $\Gamma_3(\omega) = \frac{2+2\cos(\omega)}{\frac{149}{100} + \frac{7}{5}\cos(\omega)}$.

To sketch the plots in Figs. 1.1, 1.2, 1.3, a few values of the spectra can be computed directly from the analytical expressions of $\Gamma_1(\omega), \Gamma_2(\omega), \Gamma_3(\omega)$ provided in your answer to Question 1b). Alternatively, these values can be easily computed by geometric considerations:

$$\Gamma_1 : \begin{cases} \Gamma_1(0) = \left(\frac{1}{4} + \frac{1}{4}\right)\left(\frac{1}{4} + \frac{1}{4}\right) = \frac{1}{4} \\ \Gamma_1(\pi/2) = \left(\left(1 + \frac{1}{2}\right)^2 + \frac{1}{4}\right)\frac{1}{2} = \frac{5}{4} \\ \Gamma_1(\pi) = \left(\frac{9}{4} + \frac{1}{4}\right)\left(\frac{9}{4} + \frac{1}{4}\right) = \frac{25}{4} \end{cases}$$

$$
\Gamma_2: \begin{cases} \Gamma_2(0) = \frac{2^2}{\left(\frac{3}{10}\right)^2} = \frac{400}{9} \\ \Gamma_2(\pi/2) = \frac{1^2+1^2}{1^2+\left(\frac{7}{10}\right)^2} = \frac{200}{149} \\ \Gamma_2(\pi) = 0 \end{cases}
$$

$$
\Gamma_3: \begin{cases} \Gamma_3(0) = \frac{2^2}{\left(1+\frac{7}{10}\right)^2} = \frac{400}{289} \\ \Gamma_3(\pi/2) = \frac{1^2+1^2}{1^2+\left(\frac{7}{10}\right)^2} = \frac{200}{149} \\ \Gamma_3(\pi) = 0 \end{cases}
$$

[5 Marks]

2. Solution

a) Using the operator z, the discrete-time state equations

$$\begin{cases} x_1(t+1) = \frac{4}{5}x_1(t) + u(t) + e(t) \\ x_2(t+1) = \frac{1}{2}x_2(t) + x_1(t) + 4e(t) \\ y(t) = x_2(t) \end{cases}$$

can be rewritten as follows:

$$\begin{cases} (z - \frac{4}{5})x_1(t) = u(t) + e(t) \\ (z - \frac{1}{2})x_2(t) = x_1(t) + 4e(t) \\ y(t) = x_2(t) \end{cases}$$

Then

$$x_1(t) = \frac{1}{z - \frac{4}{5}}u(t) + \frac{1}{z - \frac{4}{5}}e(t)$$

and hence

$$\begin{aligned} x_2(t) &= \tfrac{1}{z-\frac{1}{2}}\left(\tfrac{1}{z-\frac{4}{5}}u(t) + \tfrac{1}{z-\frac{4}{5}}e(t)\right) + \tfrac{4}{z-\frac{1}{2}}e(t) \\ &= \tfrac{1}{z^2-\frac{13}{10}z+\frac{2}{5}}u(t) + \tfrac{4z-\frac{11}{5}}{z^2-\frac{13}{10}z+\frac{2}{5}}e(t) \end{aligned}$$

which finally gives

$$A(z)y(t) = B(z)u(t) + C(z)e(t)$$

with

$$\begin{aligned} A(z) &= 1 - \tfrac{13}{10}z^{-1} + \tfrac{2}{5}z^{-2} \\ B(z) &= z^{-2} \\ C(z) &= 4z^{-1} - \tfrac{11}{5}z^{-2} \end{aligned}$$

[7 Marks]

b) From the expressions of $A(z)$ and $C(z)$ we note that these two polynomials have not the same degree; moreover $C(z)$ is not monic:

$$C(z) = 4z^{-1} - \frac{11}{5}z^{-2}$$

To obtain the spectral factor equivalent to $C(z)/A(z)$ and in canonical form, we first introduce a "delayed version" of the stochastic process $e(\cdot)$, that is

$$\xi(t) := e(t-1)$$

thus obtaining

$$A(z)y(t) = B(z)u(t) + \tilde{C}(z)e(t)$$

with $\tilde{C}(z) := 4 - \frac{11}{5}z^{-1}$. Introducing

$$\bar{C}(z) := \frac{1}{4}\tilde{C}(z) = 1 - \frac{11}{20}z^{-1}; \quad \eta(\cdot) \sim WN(0, 16)$$

the equivalent model in canonical form is given by

$$A(z)y(t) = B(z)u(t) + \bar{C}(z)\eta(t)$$

[3 Marks]

c) Let's carry out two iterations of polynomial division of $\bar{C}(z)$ by $A(z)$:

$$
\begin{array}{llll|l}
1 & -\frac{11}{20}z^{-1} & & & 1 \;\; -\frac{13}{10}z^{-1} \;\; +\frac{2}{5}z^{-2} \\
-1 & +\frac{13}{10}z^{-1} & -\frac{2}{5}z^{-2} & & \mathbf{1} \;\; +\frac{3}{4}\mathbf{z}^{-1} \\
// & +\frac{3}{4}z^{-1} & -\frac{2}{5}z^{-2} & & \\
 & -\frac{3}{4}z^{-1} & +\frac{39}{40}z^{-2} & -\frac{3}{10}z^{-3} & \\
 & // & +\frac{23}{40}\mathbf{z}^{-2} & -\frac{3}{10}\mathbf{z}^{-3} &
\end{array}
$$

Then, it follows that

$$\bar{C}(z) = A(z)E(z) + z^{-2}F(z) \tag{2.1}$$

with

$$E(z) := \left(1 + \frac{3}{4}z^{-1}\right); \quad F(z) := \left(\frac{23}{40} - \frac{3}{10}z^{-1}\right)$$

Let's consider the expression of the ARMAX model

$$A(z)y(t) = B(z)u(t) + \bar{C}(z)\eta(t)$$

and multiply both sides of this expression by $E(z)$, getting

$$E(z)A(z)y(t) = E(z)B(z)u(t) + E(z)\bar{C}(z)\eta(t)$$

From (2.1) it follows that

$$\bar{C}(z)y(t) = z^{-2}F(z)y(t) + E(z)B(z)u(t) + E(z)\bar{C}(z)\eta(t)$$

and hence

$$y(t) = z^{-2}\frac{F(z)}{\bar{C}(z)}y(t) + \frac{E(z)B(z)}{\bar{C}(z)}u(t) + \underbrace{E(z)\eta(t)}_{\text{unpredictable from data till time } t-2}$$

We conclude that (recall that $B(z) = z^{-2}$)

$$\hat{y}(t|t-2) = \frac{F(z)}{\bar{C}(z)}y(t-2) + \frac{E(z)}{\bar{C}(z)}u(t-2)$$

thus yielding the following difference equation for the optimal two-steps ahead prediction $\hat{y}(t|t-2)$:

$$\hat{y}(t|t-2) = \frac{11}{20}\hat{y}(t-1|t-3) + \frac{23}{40}y(t-2) - \frac{3}{10}y(t-3) + u(t-2) + \frac{3}{4}u(t-3)$$

[8 Marks]

d) The prediction error is given by

$$\varepsilon(t) := y(t+2) - \hat{y}(t+2|t)$$

The variance can be easily computed (the prediction error is a MA(1) process):

$$\text{var}[\varepsilon(t)] = \text{var}[y(t+2) - \hat{y}(t+2|t)] = \text{var}[\eta(t) + \frac{3}{4}\eta(t-1)] = 25$$

[2 Marks]

3. Solution

a) We can write

$$v(t)\left(1-\frac{3}{10}z^{-2}\right)=\left(1+\frac{7}{10}z^{-1}\right)e(t)$$

Hence

$$v(t)=\frac{z^2+\frac{7}{10}}{z^2-\frac{3}{10}}e(t)$$

The poles are: $z_{1,2}=\pm\sqrt{3/10}$. Since $|z_1|<1$ and $|z_2|<1$, the dynamic model generating the stochastic process is asymptotically stable which implies that the steady state stochastic process $v(\cdot)$ is stationary.

[1 Marks]

b) We have

$$\begin{aligned}\mathbb{E}[v(t)] &= \tfrac{3}{10}\mathbb{E}[v(t-2)]+\mathbb{E}[e(t)]+\tfrac{7}{10}\mathbb{E}[e(t-1)]\\ &= \tfrac{3}{10}\mathbb{E}[v(t)]+\tfrac{17}{5}\end{aligned}$$

and thus $\mathbb{E}[v(t)]=\frac{34}{7}$.

[3 Marks]

c) Let

$$\bar{v}:=\mathbb{E}[v(t)]=\frac{34}{7},\quad \tilde{v}(t):=v(t)-\bar{v},\quad \bar{e}:=\mathbb{E}[v(t)]=2,\quad \tilde{e}(t):=e(t)-\bar{e}$$

Then the ARMA model can be equivalently rewritten as:

$$\tilde{v}(t)+\bar{v}=\frac{3}{10}(\tilde{v}(t-2)+\bar{v})+\tilde{e}(t)+\bar{e}+\frac{7}{10}(\tilde{e}(t-1)+\bar{e})$$

which, after some algebra, gives

$$\tilde{v}(t)=\frac{3}{10}\tilde{v}(t-2)+\tilde{e}(t)+\frac{7}{10}\tilde{e}(t-1)$$

where, of course, $\mathbb{E}[\tilde{v}(t)]=0$ and $\tilde{e}(\cdot)\sim WN(0,1)$.

Now:

$$\begin{aligned}\mathrm{var}[v(t)] &= \mathrm{var}[\tilde{v}(t)]=\mathbb{E}[\tilde{v}(t)^2]=\mathbb{E}\left[\left(\tfrac{3}{10}\tilde{v}(t-2)+\tilde{e}(t)+\tfrac{7}{10}\tilde{e}(t-1)\right)^2\right]\\ &= \tfrac{9}{100}\mathbb{E}\left[\tilde{v}(t-2)^2\right]+1+\tfrac{49}{100}\\ &\quad +\tfrac{3}{5}\underbrace{\mathbb{E}[\tilde{v}(t-2)\tilde{e}(t)]}_{=0}+\tfrac{21}{100}\underbrace{\mathbb{E}[\tilde{v}(t-2)\tilde{e}(t-1)]}_{=0}+\tfrac{7}{5}\underbrace{\mathbb{E}[\tilde{e}(t)\tilde{e}(t-1)]}_{=0}\\ &= \tfrac{9}{100}\mathrm{var}[\tilde{v}(t)]+\tfrac{149}{100}\end{aligned}$$

and hence $\mathrm{var}[v(t)]=\frac{149}{91}\simeq 1.64$.

[3 Marks]

d) We have:

$$\begin{aligned}
\gamma_v(1) &= \gamma_{\tilde{v}}(1) = \mathbb{E}[\tilde{v}(t)\tilde{v}(t-1)] = \mathbb{E}\left[\left(\tfrac{3}{10}\tilde{v}(t-2)+\tilde{e}(t)+\tfrac{7}{10}\tilde{e}(t-1)\right)\tilde{v}(t-1)\right] \\
&= \tfrac{3}{10}\mathbb{E}\left[\tilde{v}(t-2)\tilde{v}(t-1)\right] + \underbrace{\mathbb{E}[\tilde{e}(t)\tilde{v}(t-1)]}_{=0} + \tfrac{7}{10}\mathbb{E}\left[\tilde{e}(t-1)\tilde{v}(t-1)\right] \\
&= \tfrac{3}{10}\gamma_v(1) + \tfrac{7}{10}
\end{aligned}$$

Thus $\gamma_{\tilde{v}}(1) = 1$. Analogously:

$$\begin{aligned}
\gamma_v(2) &= \gamma_{\tilde{v}}(2) = \mathbb{E}[\tilde{v}(t)\tilde{v}(t-2)] = \mathbb{E}\left[\left(\tfrac{3}{10}\tilde{v}(t-2)+\tilde{e}(t)+\tfrac{7}{10}\tilde{e}(t-1)\right)\tilde{v}(t-2)\right] \\
&= \tfrac{3}{10}\mathbb{E}\left[\tilde{v}(t-2)^2\right] + \underbrace{\mathbb{E}[\tilde{e}(t)\tilde{v}(t-2)]}_{=0} + \tfrac{7}{10}\underbrace{\mathbb{E}\left[\tilde{e}(t-1)\tilde{v}(t-2)\right]}_{=0} \\
&= \tfrac{3}{10}\gamma_{\tilde{v}}(0)
\end{aligned}$$

which gives $\gamma_v(2) = \frac{447}{910} \simeq 0.49$.

[5 Marks]

e) Model $\mathcal{M}_1$ in prediction form is

$$\widehat{\mathcal{M}_1}(\theta_1): \quad \hat{v}(t|t-1) = av(t-1)$$

where $\theta_1 = a$. The estimate $\hat{\theta}_1(N)$ converges almost surely to the minima of

$$\begin{aligned}
\bar{J}(a) &= \mathbb{E}\left\{[v(t)-\hat{v}(t|t-1)]^2\right\} = \mathbb{E}\left\{[v(t)-av(t-1)]^2\right\} \\
&= \mathbb{E}[v(t)^2] - 2a\mathbb{E}[v(t)v(t-1)] + a^2\mathbb{E}[v(t-1)^2] = (1+a^2)\gamma_v(0) - 2a\gamma_v(1)
\end{aligned}$$

Now

$$\frac{d\bar{J}}{da} = 0 \iff 2a\gamma_v(0) - 2\gamma_v(1) = 0$$

Therefore, $\bar{J}(a)$ has a single minimum attained for $\bar{a} = \gamma_v(1)/\gamma_v(0)$ and hence, using the values of the correlation function computed in the answers to Questions 3c) and 3d), we obtain $\bar{\theta}_1 = \bar{a} = \gamma_v(1)/\gamma_v(0) = \frac{91}{149} \simeq 0.61$.

Model $\mathcal{M}_2$ in prediction form is

$$\widehat{\mathcal{M}_2}(\theta_2): \quad \hat{v}(t|t-1) = a_1v(t-1) + a_2v(t-2)$$

where $\theta_2 = [a_1\ a_2]^\top$. The estimate $\hat{\theta}_2(N)$ converges almost surely to the minima of

$$\begin{aligned}
\bar{J}(a_1,a_2) &= \mathbb{E}\left\{[v(t)-\hat{v}(t|t-1)]^2\right\} = \mathbb{E}\left\{[v(t)-a_1v(t-1)-a_2v(t-2)]^2\right\} \\
&= (1+a_1^2+a_2^2)\gamma_v(0) - 2a_1\gamma_v(1) - 2a_2\gamma_v(2) + 2a_1a_2\gamma_v(1)
\end{aligned}$$

Then:

$$\begin{cases}
\dfrac{\partial \bar{J}}{\partial a_1} = 0 \iff 2a_1\gamma_v(0) - 2\gamma_v(1) + 2a_2\gamma_v(1) = 0 \\[2ex]
\dfrac{\partial \bar{J}}{\partial a_2} = 0 \iff 2a_2\gamma_v(0) - 2\gamma_v(2) + 2a_1\gamma_v(1) = 0
\end{cases}$$

Hence, using the values of the correlation function computed in the answers to Questions 3c) and 3d), after some algebra $\bar{\theta}_2 = [\bar{a}_1\ \bar{a}_2]^\top = [\frac{94913}{139200}, -\frac{16207}{139200}]^\top \simeq [0.68, -0.12]^\top$.

Finally, the variances of the prediction errors are given by:

$$\bar{J}(\bar{a}) \simeq 1.41\,; \qquad \bar{J}(\bar{a}_1, \bar{a}_2) \simeq 1.01$$

Both prediction error processes are not white because $\mathscr{M}_1$ and $\mathscr{M}_2$ do not represent ARMA processes. However, as expected, model $\mathscr{M}_2$ is better than $\mathscr{M}_1$ in terms of a smaller variance of the prediction error.

[8 Marks]

4. Solution

a) The stochastic processes $v_1(\cdot)$ and $v_2(\cdot)$ are assumed to be independent. Moreover, the systems' dynamics matrix (a scalar in this case) has its single eigenvalue strictly inside the unit circle, thus implying that the dynamics is asymptotically stable. Therefore, the first theorem of the Kalman estimation asymptotic theory allows to conclude that the one-step ahead steady-state Kalman predictor does exist and it is asymptotically stable.

[1 Marks]

b) To determine the one-step ahead steady-state Kalman predictor we first need to solve the corresponding algebraic Riccati equation. Its general form in case of un-correlated system and measurement noises is given by:

$$P = F\left[P - PH^{\top}\left(V_2 + HPH^{\top}\right)^{-1}HP\right]F^{\top} + V_1$$

Letting $F = 1/2$, $H = 2$, $V_1 = 19/20$, $V_2 = 1$, we have

$$P = \frac{1}{4}\left(P - \frac{4}{1+4P}P^2\right) + \frac{19}{20} \Longrightarrow 80P^2 - 61P - 19 = 0$$

thus obtaining the two solutions

$$\bar{P}_1 = 1 \quad \text{and} \quad \bar{P}_2 = -\frac{19}{80}$$

Clearly, the only admissible solution is the positive one. Thus $\bar{P} = \bar{P}_1 = 1$. Accordingly:

$$\bar{K} = F\bar{P}H^{\top}\left(V_2 + H\bar{P}H^{\top}\right)^{-1} = \frac{1}{5}$$

Then we have

$$\begin{cases} \hat{x}(t+1|t) = \frac{1}{2}\hat{x}(t|t-1) + \frac{1}{5}e(t) \\ \hat{y}(t+1|t) = 2\hat{x}(t+1|t) \\ e(t) = y(t) - 2\hat{x}(t|t-1) \end{cases}$$

and thus the one-step ahead steady-state Kalman predictor is given by:

$$\hat{x}(t+1|t) = \frac{1}{2}\hat{x}(t|t-1) + \frac{1}{5}\left[y(t) - 2\hat{x}(t|t-1)\right] \qquad (4.1)$$

[3 Marks]

c) To determine the one-step ahead steady-state Kalman filter, let's first observe that (matrix F is invertible in this specific case):

$$\hat{x}(t+1|t) = F\hat{x}(t|t) \Longrightarrow \hat{x}(t|t) = F^{-1}\hat{x}(t+1|t) = 2\hat{x}(t+1|t)$$

Now, equations (4.1) in the answer to Question 4b) can be exploited thus obtaining:

$$\begin{aligned} \hat{x}(t|t) &= \hat{x}(t|t-1) + \tfrac{2}{5}\left[y(t) - 2\hat{x}(t|t-1)\right] \\ &= \tfrac{1}{2}\hat{x}(t-1|t-1) + \tfrac{2}{5}\left[y(t) - \hat{x}(t-1|t-1)\right] \end{aligned}$$

[3 Marks]

d) The stochastic process $x(\cdot)$ generated by the system

$$x(t+1) = \frac{1}{2}x(t) + v_1(t)$$

is stationary because $v_1(\cdot) \sim WGN(0, 19/20)$ and because the system is asymptotically stable.

Because of the stationarity of $x(\cdot)$, $\text{var}[x(t)] = \text{var}[x(t-1)]$ and hence

$$\text{var}[x(t)] = \frac{1}{4}\text{var}[x(t)] + \frac{19}{20} \Longrightarrow \text{var}[x(t)] = \frac{19}{25}$$

Let's now compute $V_{\text{filt}} = \text{var}[x(t) - \hat{x}(t|t)]$. As $\hat{x}(t|t) = 2\hat{x}(t+1|t)$ (see your answer to Question 4c)), it follows that:

$$\begin{aligned}
\text{var}[x(t) - \hat{x}(t|t)] &= \text{var}[x(t) - 2\hat{x}(t+1|t)] \\
&= \text{var}\left[x(t) - \tfrac{1}{5}\hat{x}(t|t-1) - \tfrac{2}{5}y(t))\right] \\
&= \text{var}\left[x(t) - \tfrac{1}{5}\hat{x}(t|t-1) - \tfrac{2}{5}(2x(t) + v_2(t))\right] \\
&= \text{var}\left[\tfrac{1}{5}x(t) - \tfrac{1}{5}\hat{x}(t|t-1) - \tfrac{2}{5}v_2(t)\right] \\
&= \tfrac{1}{25}\text{var}\left[x(t) - \hat{x}(t|t-1)\right] + \tfrac{4}{25}
\end{aligned}$$

where we have exploited the fact that $x(t)$ and $\hat{x}(t|t-1)$ are uncorrelated with $v_2(t)$. Then, from $\text{var}[x(t) - \hat{x}(t|t-1)] = \bar{P} = 1$, we obtain

$$V_{\text{filt}} = \text{var}[x(t) - \hat{x}(t|t)] = \frac{1}{5} < \text{var}[x(t) - \hat{x}(t|t-1)]$$

This reduction in the variance of the state estimation error should be expected because in computing $\hat{x}(t|t)$ the measurement $y(t)$ is used whereas in the computation of $\hat{x}(t|t-1)$ this data-point was not yet available.

[5 Marks]

e) As $\bar{v}_1$ is constant, we can introduce the discrete-time equation

$$\begin{cases} z(t+1) = z(t), & t = 0, 1, \ldots \\ z(0) = \bar{v}_1 \end{cases}$$

Now, introducing the zero-mean noise process $\tilde{v}_1(t) := v_1(t) - \bar{v}_1$, the original state equations can be rewritten as

$$\begin{cases} x(t+1) = \dfrac{1}{2}x(t) + \tilde{v}_1(t) + z(t) \\ z(t+1) = z(t) \\ y(t) = 2x(t) + v_2(t) \end{cases} \tag{4.2}$$

Then, by defining the "augmented" state vector $w(t) := [x(t), z(t)]^\top$, equations (4.2) take on the form

$$\begin{cases} w(t+1) = \begin{bmatrix} 1/2 & 1 \\ 0 & 1 \end{bmatrix} w(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tilde{v}_1(t) \\ y(t) = [2\ 0]\, w(t) + v_2(t) \end{cases}$$

Now the standard one-step ahead steady-state Kalman predictor can be designed to predict the whole state vector $w(t)$ (and hence $x(t)$ and $\bar{v}_1$ simultaneously) using the following data:

$$F = \begin{bmatrix} 1/2 & 1 \\ 0 & 1 \end{bmatrix}, H = [2\ 0], V_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

[8 Marks]