

**Imperial College  
London**

[E2.8 (Maths 3) 2009]

**B.ENG. AND M.ENG. EXAMINATIONS 2009**

**PART II Paper 3 : MATHEMATICS (ELECTRICAL ENGINEERING)**

**Date    Wednesday 3rd June 2009    2.00 - 5.00 pm**

***DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.***

**Answer EIGHT questions.**

**Please answer questions from Section A and Section B in separate answer-books.**

*A mathematical formulae sheet is provided.*

*Statistical data sheets are provided.*

*[Before starting, please make sure that the paper is complete; there should be EIGHT pages, with a total of TWELVE questions. Ask the invigilator for a replacement if your copy is faulty.]*

## SECTION A

[E2.8 (Maths 3) 2009]

1. The complex variable  $w$  is related to the complex variable  $z = x + iy$  by

$$w = \frac{e^z - i}{e^z + i}.$$

Find  $|w|$  as a function of  $x$  and  $y$  and deduce that  $|w| < 1$  if  $0 < y < \pi$ .

Illustrate in a diagram in the  $w$ -plane the curves corresponding to the lines  $y = 0$ ,  $y = \frac{1}{2}\pi$  and  $y = \pi$ .

Find  $z$  as a function of  $w$ .

The curves  $\operatorname{Im}(z) = \text{constant}$  are known as streamlines. Sketch these in the  $z$ -plane, and infer the streamlines in the  $w$ -plane.

2. Use the Residue Theorem to show that:

(i)

$$\oint_{C_1} \frac{z^3 dz}{(z-i)^4} = 2\pi i,$$

(ii)

$$\oint_{C_1} \frac{z dz}{(z-1)(z-i)^2} = 0,$$

where  $C_1$  is the circle centred at  $z = 0$  of radius 2.

- (iii) If the circle  $C_1$  is changed into a new circle  $C_2$  centred at  $z = 1$  of radius  $1/2$ , what is the value of the contour integral

$$\oint_{C_2} \frac{z dz}{(z-1)(z-i)^2} ?$$

*The residue of a complex function  $f(z)$  at a pole at  $z = a$  of multiplicity  $m$  is given by*

$$\lim_{z \rightarrow a} \frac{1}{(m-1)!} \left[ \frac{d^{m-1}}{dz^{m-1}} \{(z-a)^m f(z)\} \right].$$

PLEASE TURN OVER

[E2.8 (Maths 3) 2009]

3. The complex function

$$\frac{e^{iz}}{z(z^2 + 1)^2}$$

has a single pole at  $z = 0$  and a double pole at  $z = i$  in the upper half-plane.

- (i) Show that the residue at  $z = 0$  is 1.
- (ii) Calculate the residue at  $z = i$ .

Now consider the contour integral

$$\oint_C \frac{e^{iz} dz}{z(z^2 + 1)^2},$$

where  $C$  is taken to be a closed semi-circle of radius  $R$  lying in the upper half-plane centred at  $z = 0$ , but with a small semi-circle of radius  $r$  ( $r < R$ ), also centred at  $z = 0$ , indented into the lower half-plane.

- (iii) Show that the contribution to the above integral from this indentation, in the limit  $r \rightarrow 0$ , is  $i\pi$ .
- (iv) In the limit  $R \rightarrow \infty$ , show why the contribution from the arc of the larger semi-circle is zero.
- (v) Hence deduce the value of

$$\int_{-\infty}^{\infty} \frac{\sin x dx}{x(x^2 + 1)^2}.$$

The residue of a complex function  $f(z)$  at a pole at  $z = a$  of multiplicity  $m$  is given by

$$\lim_{z \rightarrow a} \frac{1}{(m-1)!} \left[ \frac{d^{m-1}}{dz^{m-1}} \{(z-a)^m f(z)\} \right].$$

[E2.8 (Maths 3) 2009]

4. Two functions of time  $f(t)$  and  $g(t)$  have Laplace Transforms  $\mathcal{L}\{f(t)\} = \bar{f}(s)$  and  $\mathcal{L}\{g(t)\} = \bar{g}(s)$  respectively. The convolution product between the two is defined by

$$f(t) * g(t) = \int_0^t f(u)g(t-u)du .$$

(i) Prove the Laplace convolution theorem

$$\mathcal{L}(f * g) = \bar{f}(s)\bar{g}(s) .$$

(ii) Find the inverse transforms of the following in terms of convolution integrals :

$$(a) \quad \mathcal{L}^{-1} \left\{ \frac{\bar{f}(s)}{s^2 - 3s + 2} \right\} ,$$

$$(b) \quad \mathcal{L}^{-1} \left\{ \frac{\bar{f}(s)}{s^2 + 2s + 5} \right\} .$$

5. Consider the ordinary differential equation

$$\frac{d^2x}{dt^2} + 4 \frac{dx}{dt} + 5x = F(t)$$

where  $F(t)$  has a finite Laplace Transform  $\bar{F}(s)$ .

The initial condition for  $\frac{dx}{dt}$  is

$$\frac{dx}{dt}(0) = -2x(0) .$$

Show that

$$(i) \quad \bar{x}(s) = \frac{(s+2)}{s^2 + 4s + 5} x(0) + \frac{\bar{F}(s)}{s^2 + 4s + 5} ,$$

$$(ii) \quad x(t) = x(0)p(t) + \int_0^t F(t-u)q(u)du ,$$

where  $p$  and  $q$  are functions to be determined.

**PLEASE TURN OVER**

[E2.8 (Maths 3) 2009]

6. You are given a function  $f(t)$  that is periodic in time with a period  $T$  such that  $f(t) = f(t \pm nT)$  for  $n = 1, 2, 3, \dots$

Show that the Laplace transform  $\bar{f}(s)$ , is given by

$$\bar{f}(s) = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t) dt, \quad (s > 0).$$

If  $f(t)$  is the periodic function of period  $T$

$$f(t) = \begin{cases} e^{at} & 0 \leq t \leq \frac{1}{2}T, \\ 0 & \frac{1}{2}T \leq t \leq T, \end{cases}$$

find its Laplace transform for  $s > 0$ .

7. Consider a two-dimensional region  $R$  bounded by a closed piecewise smooth curve  $C$ . Using Green's Theorem in a plane (given below), choose the components of a vector field  $\mathbf{u}(x, y)$  in terms of two arbitrary differentiable functions  $P(x, y)$  and  $Q(x, y)$  to prove the two-dimensional form of the Divergence Theorem

$$\oint_C \mathbf{u} \cdot \hat{\mathbf{n}} ds = \iint_R \operatorname{div} \mathbf{u} dx dy \quad (1)$$

where  $\hat{\mathbf{n}}$  is a unit vector normal to  $C$ .

Suppose  $R$  is the wedge-shaped region lying between the lines  $y = ax$  and  $y = bx$  ( $0 < a < b$ ) and bounded by the vertical line  $x = 1$ .

Sketch the region  $R$ .

If

$$\mathbf{u} = i x^2 \sin y + j x \cos y$$

show that the double integral on the right hand side of (1) is

$$\iint_R \operatorname{div} \mathbf{u} dx dy = \frac{\sin a}{a} - \frac{\sin b}{b} + \frac{\cos a - 1}{a^2} - \frac{\cos b - 1}{b^2}.$$

*Green's Theorem in a plane states that for a two-dimensional region  $R$  bounded by a closed, piecewise smooth curve  $C$ , then*

$$\oint_C \{P(x, y) dx + Q(x, y) dy\} = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy.$$

[E2.8 (Maths 3) 2009]

8. (i) The region  $S$  is the sector of a disc lying between the lines  $y = x$ ,  $y = -x$  and the circle  $x^2 + y^2 = a^2$  with  $y \geq 0$ .  
 Find the coordinates  $(\bar{x}, \bar{y})$  of the centroid of  $S$  and the value  $I_{0z}$  of the moment of inertia about the  $z$ -axis of  $S$ .

*Note: The coordinates of the centroid are given by*

$$\begin{aligned}\bar{x} &= \frac{1}{A} \iint_S x \, dx \, dy, \\ \bar{y} &= \frac{1}{A} \iint_S y \, dx \, dy,\end{aligned}$$

*where  $A$  is the area of  $S$ .*

*The moment of inertia is given by*

$$I_{0z} = \iint_S (x^2 + y^2) \, dx \, dy.$$

- (ii) The integral  $\iint_R (x^4 - y^4) \, dx \, dy$  is to be evaluated over the region  $R$  bounded by the four hyperbolae  $xy = 1$ ,  $xy = 2$ ,  $x^2 - y^2 = 1$  and  $x^2 - y^2 = -1$  with  $x > 0$  and  $y > 0$ .

Sketch the region  $R$  and find the values of the new variables  $u = xy$  and  $v = x^2 - y^2$  on each part of the boundary of  $R$ .

Evaluate the integral by first changing to the new variables, and using the Jacobian for the transformation.

9. Show for any twice differentiable scalar field  $\phi(x, y, z)$  and any twice differentiable vector field  $\mathbf{E}(x, y, z)$  that

$$\operatorname{curl} \operatorname{grad} \phi = 0 \quad \text{and} \quad \operatorname{div} \operatorname{curl} \mathbf{E} = 0.$$

Suppose that the vector field  $\mathbf{E}(x, y, z)$  is given by

$$\mathbf{E} = (x^2 - y^2)\mathbf{i} + (axy + z^2)\mathbf{j} + (byz + z)\mathbf{k}.$$

Find the values of the constants  $a$  and  $b$  that make  $\operatorname{curl} \mathbf{E} = 0$ .

For these values of  $a$  and  $b$  find the potential  $\phi(x, y, z)$  that satisfies

$$\mathbf{E} = \operatorname{grad} \phi.$$

PLEASE TURN OVER

[E2.8 (Maths 3) 2009]

10. Suppose  $R$  is a bounded simply-connected region in the  $x$ - $y$  plane. Green's theorem in the plane states that

$$\oint_C (f \, dx + g \, dy) = \iint_R \left( \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx \, dy$$

where  $C$  is the counter-clockwise boundary of  $R$ .

- (i) Suppose there is a bounded simply-connected region  $R^*$  where the condition  $\partial g / \partial x = \partial f / \partial y$  holds, and let  $C_1$  be a path joining the points  $A$  and  $B$ . Use Green's theorem to show that the line integral

$$I = \int_{C_1} (f \, dx + g \, dy)$$

is independent of the path  $C_1$ , provided that the points  $A$  and  $B$  and the path lie in the region  $R^*$ .

Suppose now that the functions  $f$  and  $g$  are given by

$$f = \frac{x-y}{x^2+y^2}, \quad g = \frac{x+y}{x^2+y^2}$$

and the points  $A$  and  $B$  are  $(1, 0)$  and  $(0, 1)$  respectively.

- (ii) Show that  $\partial g / \partial x = \partial f / \partial y$  everywhere except at the origin  $(0, 0)$ .
- (iii) Calculate  $I$  in the following cases :
  - (a)  $C_1$  is the counter-clockwise quarter circle joining  $A$  to  $B$  along the curve  $x^2 + y^2 = 1$  ;
  - (b)  $C_1$  is the clockwise three quarter circle joining  $A$  to  $B$  along the curve  $x^2 + y^2 = 1$ .
- (iv) Why are the answers to (a) and (b) different ?

## SECTION B

[E 2.8 (Maths 3) 2009]

11. A fair six-sided die is thrown twice.

- (i) Write down a suitable sample space  $\Omega$  for this experiment that takes the order of the throws into account.
- (ii) Let  $A$  be the event that the first number thrown is even.  
Let  $B$  be the event that the sum of the numbers thrown is odd.
  - (a) Write down  $A$  and  $B$  in set notation.
  - (b) Are  $A$  and  $B$  independent? Justify your answer.
- (iii) Let  $C$  be the event that the sum of the numbers thrown equals 4.
  - (a) Write down  $C$  in set notation.
  - (b) Find the conditional probability of  $A$  given  $C$ .
- (iv) Using only the definition of conditional probability, prove Bayes' theorem

$$P(D|E) = \frac{P(E|D) P(D)}{P(E)}$$

for events  $D, E$  with  $P(E) > 0$  and  $P(D) > 0$ .

12. Consider the time series  $\{y_t, t = 0, 1, \dots\}$  given by

$$y_t = 1 + \sum_{s=0}^t e_s, \quad t = 0, 1, 2, \dots$$

where  $\{e_t\}$  is white noise with  $E(e_t) = 0$  and  $\text{Var}(e_t) = 1$ .

- (i) Find  $E(y_t)$  for  $t = 0, 1, 2, \dots$
- (ii) Find  $\text{Var}(y_t)$  for  $t = 0, 1, 2, \dots$
- (iii) Find the covariance  $\text{cov}(y_t, y_{t+s})$  of  $\{y_t\}$  for  $s = 1, 2, 3, \dots$
- (iv) Define when a time series is called weakly stationary.
- (v) Is  $\{y_t\}$  weakly stationary? Justify your answer.
- (vi) Define when a time series is called strictly stationary.
- (vii) Is  $\{y_t\}$  strictly stationary? Justify your answer.

END OF PAPER

## MATHEMATICS DEPARTMENT

### 3. TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

#### MATHEMATICAL FORMULAE

$$\begin{aligned}\sin(a+b) &= \sin a \cos b + \cos a \sin b; \\ \cos(a+b) &= \cos a \cos b - \sin a \sin b.\end{aligned}$$

#### 1. VECTOR ALGEBRA

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1, a_2, a_3)$$

Scalar (dot) product:  $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

Vector (cross) product:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix};$$

Scalar triple product:

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = \mathbf{b} \cdot \mathbf{c} \times \mathbf{a} = \mathbf{c} \cdot \mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product:  $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c} \cdot \mathbf{a})\mathbf{b} - (\mathbf{b} \cdot \mathbf{a})\mathbf{c}$

#### 2. SERIES

$$(1+x)^{\alpha} = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \dots \quad (\alpha \text{ arbitrary, } |x| < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^n \frac{x^{2n}}{(2n)!} + \dots,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \dots,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^n \frac{x^{n+1}}{(n+1)} + \dots \quad (-1 < x \leq 1)$$

### 4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f D^n g + \binom{n}{1} Df D^{n-1} g + \dots + \binom{n}{r} D^r f D^{n-r} g + \dots + D^n f g.$$

(b) Taylor's expansion of  $f(x)$  about  $x=a$ :

$$f(a+h) = f(a) + hf'(a) + h^2 f''(a)/2! + \dots + h^n f^{(n)}(a)/n! + \epsilon_n(h),$$

where  $\epsilon_n(h) = h^{n+1} f^{(n+1)}(a+\theta h)/(n+1)!$ ,  $0 < \theta < 1$ .

(c) Taylor's expansion of  $f(x, y)$  about  $(a, b)$ :

$$f(a+h, b+k) = f(a, b) + [hf_x + kf_y]_{a,b} + 1/2! [h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}]_{a,b} + \dots$$

(d) Partial differentiation of  $f(x, y)$ :

- i. If  $y = y(x)$ , then  $f = F(x)$ , and  $\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx}$ .
- ii. If  $x = x(t)$ ,  $y = y(t)$ , then  $f = F(t)$ , and  $\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$ .
- iii. If  $x = x(u, v)$ ,  $y = y(u, v)$ , then  $f = F(u, v)$ , and

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u}, \quad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial v}.$$

(e) Stationary points of  $f(x, y)$  occur where  $f_x = 0$ ,  $f_y = 0$  simultaneously.  
Let  $(a, b)$  be a stationary point: examine  $D = [f_{xx} f_{yy} - (f_{xy})^2]_{a,b}$ .

If  $D > 0$  and  $f_{xx}(a, b) < 0$ , then  $(a, b)$  is a maximum;

If  $D > 0$  and  $f_{xx}(a, b) > 0$ , then  $(a, b)$  is a minimum;

If  $D < 0$  then  $(a, b)$  is a saddle-point.

(f) Differential equations:

- i. The first order linear equation  $\frac{dy}{dx} + P(x)y = Q(x)$  has an integrating factor  $I(x) = \exp[\int P(x)(dx)]$ , so that  $\frac{d}{dx}(Iy) = IQ$ .
- ii.  $P(x, y)dx + Q(x, y)dy = 0$  is exact if  $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ .

## 5. INTEGRAL CALCULUS

- (a) An important substitution:  $\tan(\theta/2) = t$ :  
 $\sin \theta = 2t/(1+t^2)$ ,  $\cos \theta = (1-t^2)/(1+t^2)$ ,  $d\theta = 2dt/(1+t^2)$ .

- (b) Some indefinite integrals:

$$\int (a^2 - x^2)^{-1/2} dx = \sin^{-1}\left(\frac{x}{a}\right), \quad |x| < a.$$

$$\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1}\left(\frac{x}{a}\right) = \ln\left\{\frac{x}{a} + \left(1 + \frac{x^2}{a^2}\right)^{1/2}\right\}.$$

$$\int (x^2 - a^2)^{-1/2} dx = \cosh^{-1}\left(\frac{x}{a}\right) = \ln\left|\frac{x}{a} + \left(\frac{x^2}{a^2} - 1\right)^{1/2}\right|.$$

$$\int (a^2 + x^2)^{-1} dx = \left(\frac{1}{a}\right) \tan^{-1}\left(\frac{x}{a}\right).$$

## 7. LAPLACE TRANSFORMS

	Function	Transform	Function	Transform	Function	Transform
	$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t) dt$	$a f(t) + b g(t)$	$a F(s) + b G(s)$		
	$df/dt$	$sF(s) - f(0)$	$d^2 f/dt^2$	$s^2 F(s) - s f(0) - f'(0)$		
	$e^{at} f(t)$	$F(s-a)$	$t f(t)$	$-dF(s)/ds$		
	$(\partial/\partial \alpha) f(t, \alpha)$	$(\partial/\partial \alpha) F(s, \alpha)$	$\int_0^t f(u) du$	$F(s)/s$		
	$\int_0^t f(u)g(t-u)du$	$F(s)G(s)$				
	1	$1/s$	$t^n (n = 1, 2, \dots)$		$n!/s^{n+1}$ , ( $n > 0$ )	
	$e^{at}$	$1/(s-a)$ , ( $s > a$ )	$\sin \omega t$		$\omega/(s^2 + \omega^2)$ , ( $s > 0$ )	
	$\cos \omega t$	$s/(s^2 + \omega^2)$ , ( $s > 0$ )	$H(t-T) = \begin{cases} 0, & t < T \\ 1, & t > T \end{cases}$		$e^{-sT}/s$ , ( $s, T > 0$ )	

## 6. NUMERICAL METHODS

- (a) Approximate solution of an algebraic equation:

If a root of  $f(x) = 0$  occurs near  $x = a$ , take  $x_0 = a$  and  
 $x_{n+1} = x_n - [f(x_n)/f'(x_n)]$ ,  $n = 0, 1, 2, \dots$

(Newton Raphson method).

- (b) Formulae for numerical integration: Write  $x_n = x_0 + nh$ ,  $y_n = y(x_n)$ .

i. Trapezium rule (1-strip):  $\int_{x_0}^{x_1} y(x) dx \approx (h/2) [y_0 + y_1]$ .

ii. Simpson's rule (2-strip):  $\int_{x_0}^{x_2} y(x) dx \approx (h/3) [y_0 + 4y_1 + y_2]$ .

- (c) Richardson's extrapolation method: Let  $I = \int_a^b f(x) dx$  and let  $I_1, I_2$  be two estimates of  $I$  obtained by using Simpson's rule with intervals  $h$  and  $h/2$ . Then, provided  $h$  is small enough,

$$I_2 + (I_2 - I_1)/15,$$

is a better estimate of  $I$ .

## 7. FOURIER SERIES

If  $f(x)$  is periodic of period  $2L$ , then  $f(x+2L) = f(x)$ , and

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \text{ where}$$

(Parseval's theorem)

- $a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx, \quad n = 0, 1, 2, \dots$ , and

i. Trapezium rule (1-strip):  $\int_{x_0}^{x_1} y(x) dx \approx (h/2) [y_0 + y_1]$ .

ii. Simpson's rule (2-strip):  $\int_{x_0}^{x_2} y(x) dx \approx (h/3) [y_0 + 4y_1 + y_2]$ .

- (c) Richardson's extrapolation method: Let  $I = \int_a^b f(x) dx$  and let  $I_1, I_2$  be two estimates of  $I$  obtained by using Simpson's rule with intervals  $h$  and  $h/2$ . Then, provided  $h$  is small enough,

Parseval's theorem

$$\frac{1}{L} \int_{-L}^L [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

## 1. Probabilities for events

For events  $A$ ,  $B$ , and  $C$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

More generally  $P(\bigcup A_i) = \sum P(A_i) - \sum P(A_i \cap A_j) + \sum P(A_i \cap A_j \cap A_k) - \dots$

The odds in favour of  $A$

$$P(A) / P(\bar{A})$$

Conditional probability

$$P(A | B) = \frac{P(A \cap B)}{P(B)} \quad \text{provided that } P(B) > 0$$

Chain rule

$$P(A \cap B \cap C) = P(A) P(B | A) P(C | A \cap B)$$

Bayes' rule

$$P(A | B) = \frac{P(A) P(B | A)}{P(A) P(B | A) + P(\bar{A}) P(B | \bar{A})}$$

$A$  and  $B$  are independent if

$$P(B | A) = P(B)$$

$A$ ,  $B$ , and  $C$  are independent if

$$P(A \cap B \cap C) = P(A)P(B)P(C), \text{ and}$$

$$P(A \cap B) = P(A)P(B), \quad P(B \cap C) = P(B)P(C), \quad P(C \cap A) = P(C)P(A)$$

## 2. Probability distribution, expectation and variance

The probability distribution for a discrete random variable  $X$  is called the probability mass function (pmf) and is the complete set of probabilities  $\{p_x\} = \{P(X = x)\}$

Expectation  $E(X) = \mu = \sum_x x p_x$

For function  $g(x)$  of  $x$ ,  $E\{g(X)\} = \sum_x g(x)p_x$ , so  $E(X^2) = \sum_x x^2 p_x$

Sample mean  $\bar{x} = \frac{1}{n} \sum_k x_k$  estimates  $\mu$  from random sample  $x_1, x_2, \dots, x_n$

Variance  $\text{var}(X) = \sigma^2 = E\{(X - \mu)^2\} = E(X^2) - \mu^2$

Sample variance  $s^2 = \frac{1}{n-1} \left\{ \sum_k x_k^2 - \frac{1}{n} \left( \sum_j x_j \right)^2 \right\}$  estimates  $\sigma^2$

Standard deviation  $\text{sd}(X) = \sigma$

If value  $y$  is observed with frequency  $n_y$

$$n = \sum_y n_y, \quad \sum_k x_k = \sum_y y n_y, \quad \sum_k x_k^2 = \sum_y y^2 n_y$$

Skewness  $\beta_1 = E\left(\frac{X - \mu}{\sigma}\right)^3$  is estimated by  $\frac{1}{n-1} \sum \left(\frac{x_i - \bar{x}}{s}\right)^3$

Kurtosis  $\beta_2 = E\left(\frac{X - \mu}{\sigma}\right)^4 - 3$  is estimated by  $\frac{1}{n-1} \sum \left(\frac{x_i - \bar{x}}{s}\right)^4 - 3$

Sample median  $\tilde{x}$  or  $x_{\text{med}}$ . Half the sample values are smaller and half larger

If the sample values  $x_1, \dots, x_n$  are ordered as  $x_{(1)} \leq x_{(2)} \leq \dots \leq x_{(n)}$ , then  $\tilde{x} = x_{(\frac{n+1}{2})}$  if  $n$  is odd, and  $\tilde{x} = \frac{1}{2}(x_{(\frac{n}{2})} + x_{(\frac{n+2}{2})})$  if  $n$  is even

$\alpha$ -quantile  $Q(\alpha)$  is such that  $P(X \leq Q(\alpha)) = \alpha$

Sample  $\alpha$ -quantile  $\widehat{Q}(\alpha)$  Proportion  $\alpha$  of the data values are smaller

Lower quartile  $Q1 = \widehat{Q}(0.25)$  one quarter are smaller

Upper quartile  $Q3 = \widehat{Q}(0.75)$  three quarters are smaller

Sample median  $\tilde{x} = \widehat{Q}(0.5)$  estimates the population median  $Q(0.5)$

### 3. Probability distribution for a continuous random variable

The cumulative distribution function (cdf)  $F(x) = P(X \leq x) = \int_{x_0=-\infty}^x f(x_0)dx_0$

The probability density function (pdf)  $f(x) = \frac{dF(x)}{dx}$

$E(X) = \mu = \int_{-\infty}^{\infty} x f(x)dx$ ,  $\text{var}(X) = \sigma^2 = E(X^2) - \mu^2$ , where  $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx$

### 4. Discrete probability distributions

Discrete Uniform  $Uniform(n)$

$$p_x = \frac{1}{n} \quad (x = 1, 2, \dots, n) \quad \mu = (n+1)/2, \quad \sigma^2 = (n^2 - 1)/12$$

Binomial distribution  $Binomial(n, \theta)$

$$p_x = \binom{n}{x} \theta^x (1-\theta)^{n-x} \quad (x = 0, 1, 2, \dots, n) \quad \mu = n\theta, \quad \sigma^2 = n\theta(1-\theta)$$

Poisson distribution  $Poisson(\lambda)$

$$p_x = \frac{\lambda^x e^{-\lambda}}{x!} \quad (x = 0, 1, 2, \dots) \quad (\text{with } \lambda > 0) \quad \mu = \lambda, \quad \sigma^2 = \lambda$$

Geometric distribution  $Geometric(\theta)$

$$p_x = (1-\theta)^{x-1}\theta \quad (x = 1, 2, 3, \dots) \quad \mu = \frac{1}{\theta}, \quad \sigma^2 = \frac{1-\theta}{\theta^2}$$

### 5. Continuous probability distributions

Uniform distribution  $Uniform(\alpha, \beta)$

$$f(x) = \begin{cases} \frac{1}{\beta - \alpha} & (\alpha < x < \beta), \\ 0 & (\text{otherwise}). \end{cases} \quad \mu = (\alpha + \beta)/2, \quad \sigma^2 = (\beta - \alpha)^2/12$$

Exponential distribution  $Exponential(\lambda)$

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & (0 < x < \infty), \\ 0 & (-\infty < x \leq 0). \end{cases} \quad \mu = 1/\lambda, \quad \sigma^2 = 1/\lambda^2$$

Normal distribution  $N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\} \quad (-\infty < x < \infty), \quad E(X) = \mu, \quad \text{var}(X) = \sigma^2$$

Standard normal distribution  $N(0,1)$

If  $X$  is  $N(\mu, \sigma^2)$ , then  $Y = \frac{X-\mu}{\sigma}$  is  $N(0,1)$

## 6. Reliability

For a device in continuous operation with failure time random variable  $T$  having pdf  $f(t)$  ( $t > 0$ )

The reliability function at time  $t$   $R(t) = P(T > t)$

The failure rate or hazard function  $h(t) = f(t)/R(t)$

The cumulative hazard function  $H(t) = \int_0^t h(t_0) dt_0 = -\ln\{R(t)\}$

The Weibull( $\alpha, \beta$ ) distribution has  $H(t) = \beta t^\alpha$

## 7. System reliability

For a system of  $k$  devices, which operate independently, let

$$R_i = P(D_i) = P(\text{"device } i \text{ operates"})$$

The system reliability,  $R$ , is the probability of a path of operating devices

A system of devices in series operates only if every device operates

$$R = P(D_1 \cap D_2 \cap \dots \cap D_k) = R_1 R_2 \dots R_k$$

A system of devices in parallel operates if any device operates

$$R = P(D_1 \cup D_2 \cup \dots \cup D_k) = 1 - (1 - R_1)(1 - R_2) \dots (1 - R_k)$$

## 8. Covariance and correlation

The covariance of  $X$  and  $Y$   $\text{cov}(X, Y) = E(XY) - \{E(X)\}\{E(Y)\}$

From pairs of observations  $(x_1, y_1), \dots, (x_n, y_n)$   $S_{xy} = \sum_k x_k y_k - \frac{1}{n} (\sum_i x_i)(\sum_j y_j)$

$$S_{xx} = \sum_k x_k^2 - \frac{1}{n} (\sum_i x_i)^2, \quad S_{yy} = \sum_k y_k^2 - \frac{1}{n} (\sum_j y_j)^2$$

Sample covariance  $s_{xy} = \frac{1}{n-1} S_{xy}$  estimates  $\text{cov}(X, Y)$

Correlation coefficient  $\rho = \text{corr}(X, Y) = \frac{\text{cov}(X, Y)}{\text{sd}(X) \cdot \text{sd}(Y)}$

Sample correlation coefficient  $r = \frac{S_{xy}}{\sqrt{S_{xx}S_{yy}}}$  estimates  $\rho$

9. Sums of random variables

$$E(X + Y) = E(X) + E(Y)$$

$$\text{var}(X + Y) = \text{var}(X) + \text{var}(Y) + 2\text{cov}(X, Y)$$

$$\text{cov}(aX + bY, cX + dY) = (ac)\text{var}(X) + (bd)\text{var}(Y) + (ad + bc)\text{cov}(X, Y)$$

If  $X$  is  $N(\mu_1, \sigma_1^2)$ ,  $Y$  is  $N(\mu_2, \sigma_2^2)$ , and  $\text{cov}(X, Y) = c$ , then  $X + Y$  is  $N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2 + 2c)$

10. Bias, standard error, mean square error

If  $t$  estimates  $\theta$  (with random variable  $T$  giving  $t$ )

$$\underline{\text{Bias}} \text{ of } t \quad \text{bias}(t) = E(T) - \theta$$

$$\underline{\text{Standard error}} \text{ of } t \quad \text{se}(t) = \text{sd}(T)$$

$$\underline{\text{Mean square error}} \text{ of } t \quad \text{MSE}(t) = E\{(T - \theta)^2\} = \{\text{se}(t)\}^2 + \{\text{bias}(t)\}^2$$

If  $\bar{x}$  estimates  $\mu$ , then  $\text{bias}(\bar{x}) = 0$ ,  $\text{se}(\bar{x}) = \sigma/\sqrt{n}$ ,  $\text{MSE}(\bar{x}) = \sigma^2/n$ ,  $\widehat{\text{se}}(\bar{x}) = s/\sqrt{n}$

Central limit property If  $n$  is fairly large,  $\bar{x}$  is from  $N(\mu, \sigma^2/n)$  approximately

11. Likelihood

The likelihood is the joint probability as a function of the unknown parameter  $\theta$ .

For a random sample  $x_1, x_2, \dots, x_n$

$$\ell(\theta; x_1, x_2, \dots, x_n) = P(X_1 = x_1 | \theta) \cdots P(X_n = x_n | \theta) \quad (\text{discrete distribution})$$

$$\ell(\theta; x_1, x_2, \dots, x_n) = f(x_1 | \theta) f(x_2 | \theta) \cdots f(x_n | \theta) \quad (\text{continuous distribution})$$

The maximum likelihood estimator (MLE) is  $\hat{\theta}$  for which the likelihood is a maximum

12. Confidence intervals

If  $x_1, x_2, \dots, x_n$  are a random sample from  $N(\mu, \sigma^2)$  and  $\sigma^2$  is known, then

the 95% confidence interval for  $\mu$  is  $(\bar{x} - 1.96 \frac{\sigma}{\sqrt{n}}, \bar{x} + 1.96 \frac{\sigma}{\sqrt{n}})$

If  $\sigma^2$  is estimated, then from the Student t table for  $t_{n-1}$  we find  $t_0 = t_{n-1, 0.05}$

The 95% confidence interval for  $\mu$  is  $(\bar{x} - t_0 \frac{s}{\sqrt{n}}, \bar{x} + t_0 \frac{s}{\sqrt{n}})$

13. Standard normal table      Values of pdf  $\phi(y) = f(y)$  and cdf  $\Phi(y) = F(y)$

$y$	$\phi(y)$	$\Phi(y)$	$y$	$\phi(y)$	$\Phi(y)$	$y$	$\phi(y)$	$\Phi(y)$	$y$	$\Phi(y)$
0	.399	.5	.9	.266	.816	1.8	.079	.964	2.8	.997
.1	.397	.540	1.0	.242	.841	1.9	.066	.971	3.0	.999
.2	.391	.579	1.1	.218	.864	2.0	.054	.977	0.841	.8
.3	.381	.618	1.2	.194	.885	2.1	.044	.982	1.282	.9
.4	.368	.655	1.3	.171	.903	2.2	.035	.986	1.645	.95
.5	.352	.691	1.4	.150	.919	2.3	.028	.989	1.96	.975
.6	.333	.726	1.5	.130	.933	2.4	.022	.992	2.326	.99
.7	.312	.758	1.6	.111	.945	2.5	.018	.994	2.576	.995
.8	.290	.788	1.7	.094	.955	2.6	.014	.995	3.09	.999

14. Student t table      Values  $t_{m,p}$  of  $x$  for which  $P(|X| > x) = p$ , when  $X$  is  $t_m$

$m$	$p=0.10$	$0.05$	$0.02$	$0.01$	$m$	$p=0.10$	$0.05$	$0.02$	$0.01$
1	6.31	12.71	31.82	63.66	9	1.83	2.26	2.82	3.25
2	2.92	4.30	6.96	9.92	10	1.81	2.23	2.76	3.17
3	2.35	3.18	4.54	5.84	12	1.78	2.18	2.68	3.05
4	2.13	2.78	3.75	4.60	15	1.75	2.13	2.60	2.95
5	2.02	2.57	3.36	4.03	20	1.72	2.09	2.53	2.85
6	1.94	2.45	3.14	3.71	25	1.71	2.06	2.48	2.78
7	1.89	2.36	3.00	3.50	40	1.68	2.02	2.42	2.70
8	1.86	2.31	2.90	3.36	$\infty$	1.645	1.96	2.326	2.576

15. Chi-squared table      Values  $\chi^2_{k,p}$  of  $x$  for which  $P(X > x) = p$ , when  $X$  is  $\chi^2_k$  and  $p = .995, .975, \text{etc}$

$k$	.995	.975	.05	.025	.01	.005	$k$	.995	.975	.05	.025	.01	.005
1	.000	.001	3.84	5.02	6.63	7.88	18	6.26	8.23	28.87	31.53	34.81	37.16
2	.010	.051	5.99	7.38	9.21	10.60	20	7.43	9.59	31.42	34.17	37.57	40.00
3	.072	.216	7.81	9.35	11.34	12.84	22	8.64	10.98	33.92	36.78	40.29	42.80
4	.207	.484	9.49	11.14	13.28	14.86	24	9.89	12.40	36.42	39.36	42.98	45.56
5	.412	.831	11.07	12.83	15.09	16.75	26	11.16	13.84	38.89	41.92	45.64	48.29
6	.676	1.24	12.59	14.45	16.81	18.55	28	12.46	15.31	41.34	44.46	48.28	50.99
7	.990	1.69	14.07	16.01	18.48	20.28	30	13.79	16.79	43.77	46.98	50.89	53.67
8	1.34	2.18	15.51	17.53	20.09	21.95	40	20.71	24.43	55.76	59.34	63.69	66.77
9	1.73	2.70	16.92	19.02	21.67	23.59	50	27.99	32.36	67.50	71.41	76.15	79.49
10	2.16	3.25	13.31	20.48	23.21	25.19	60	35.53	40.48	79.08	83.30	88.38	91.95
12	3.07	4.40	21.03	23.34	26.22	28.30	70	43.28	48.76	90.53	95.02	100.4	104.2
14	4.07	5.63	23.68	26.12	29.14	31.32	80	51.17	57.15	101.9	106.6	112.3	116.3
16	5.14	6.91	26.30	28.85	32.00	34.27	100	67.33	74.22	124.3	129.6	135.8	140.2

16. The chi-squared goodness-of-fit test

The frequencies  $n_y$  are grouped so that the fitted frequency  $\hat{n}_y$  for every group exceeds about 5.

$X^2 = \sum_y \frac{(n_y - \hat{n}_y)^2}{\hat{n}_y}$  is referred to the table of  $\chi_k^2$  with significance point  $p$ ,

where  $k$  is the number of terms summed, less one for each constraint, eg matching total frequency, and matching  $\bar{x}$  with  $\mu$

17. Joint probability distributions

Discrete distribution  $\{p_{xy}\}$ , where  $p_{xy} = P(\{X = x\} \cap \{Y = y\})$ .

Let  $p_{x0} = P(X = x)$ , and  $p_{0y} = P(Y = y)$ , then

$$p_{x0} = \sum_y p_{xy} \quad \text{and} \quad P(X = x \mid Y = y) = \frac{p_{xy}}{p_{0y}}$$

Continuous distribution

$$\text{Joint cdf} \quad F(x, y) = P(\{X \leq x\} \cap \{Y \leq y\}) = \int_{x_0=-\infty}^x \int_{y_0=-\infty}^y f(x_0, y_0) dx_0 dy_0$$

$$\text{Joint pdf} \quad f(x, y) = \frac{\partial^2 F(x, y)}{\partial x \partial y}$$

$$\text{Marginal pdf of } X \quad f_X(x) = \int_{-\infty}^{\infty} f(x, y_0) dy_0$$

$$\text{Conditional pdf of } X \text{ given } Y = y \quad f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(y)} \quad (\text{provided } f_Y(y) > 0)$$

18. Linear regression

To fit the linear regression model  $y = \alpha + \beta x$  by  $\hat{y}_x = \hat{\alpha} + \hat{\beta}x$  from observations

$$(x_1, y_1), \dots, (x_n, y_n), \text{ the least squares fit is} \quad \hat{\alpha} = \bar{y} - \bar{x}\hat{\beta}, \quad \hat{\beta} = \frac{S_{xy}}{S_{xx}}$$

$$\text{The residual sum of squares} \quad \text{RSS} = S_{yy} - \frac{S_{xy}^2}{S_{xx}}$$

$$\widehat{\sigma^2} = \frac{\text{RSS}}{n-2} \quad \frac{n-2}{\sigma^2} \widehat{\sigma^2} \text{ is from } \chi_{n-2}^2$$

$$E(\hat{\alpha}) = \alpha, \quad E(\hat{\beta}) = \beta,$$

$$\text{var}(\hat{\alpha}) = \frac{\sum x_i^2}{n S_{xx}} \sigma^2, \quad \text{var}(\hat{\beta}) = \frac{\sigma^2}{S_{xx}}, \quad \text{cov}(\hat{\alpha}, \hat{\beta}) = -\frac{\bar{x}}{S_{xx}} \sigma^2$$

$$\hat{y}_x = \hat{\alpha} + \hat{\beta}x, \quad E(\hat{y}_x) = \alpha + \beta x, \quad \text{var}(\hat{y}_x) = \left\{ \frac{1}{n} + \frac{(x - \bar{x})^2}{S_{xx}} \right\} \sigma^2$$

$$\frac{\hat{\alpha} - \alpha}{\text{se}(\hat{\alpha})}, \quad \frac{\hat{\beta} - \beta}{\text{se}(\hat{\beta})}, \quad \frac{\hat{y}_x - \alpha - \beta x}{\text{se}(\hat{y}_x)} \text{ are each from } t_{n-2}$$

## Engineering 2 Core, Solutions:

C8. We have  $e^z = e^x(\cos y + i \sin y)$  and hence

$$|W|^2 = \frac{e^{2x} \cos^2 y + (e^x \sin y - 1)^2}{e^{2x} \cos^2 y + (e^x \sin y + 1)^2} = \frac{e^{2x} + 1 - 2e^x \sin y}{e^{2x} + 1 + 2e^x \sin y}.$$

Now if  $0 < y < \pi$  then  $0 < \sin y < 1$ . The denominator is thus the sum of two positive numbers and the numerator is their (smaller) difference. So we have  $|W| < 1$ .

If  $y = \frac{1}{2}\pi$  then  $e^z = ie^x$  and  $W = (e^x - 1)/(e^x + 1) = \tanh(x/2)$  which is real. As  $x \rightarrow \pm\infty, W \rightarrow \pm 1$ . So we have the line segment with  $|W| \leq 1$  ( $W$  real). ← 3

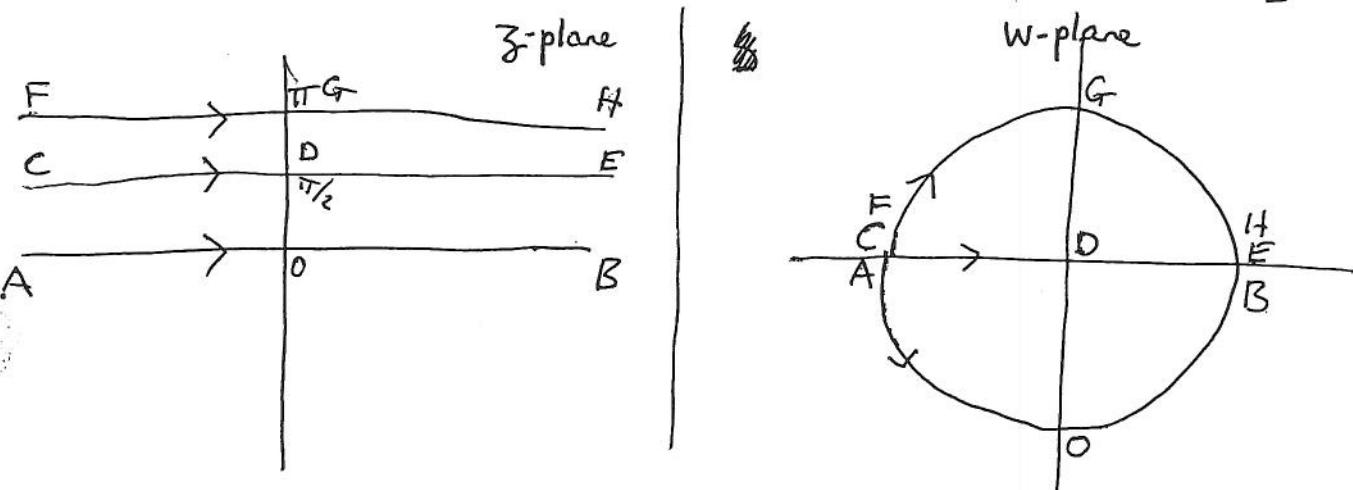
i.e. If  $y = 0$ ,  $W = (e^x - i)/(e^x + i) = (e^x - i)^2/(e^{2x} + 1)$ . Thus  $|W| = 1$  with  $\operatorname{Im}(W) < 0$ , (lower) half of the unit circle. If  $y = \pi$  then we have the other half of the unit circle. See figure. Now (upper) } 4

$$W = \frac{e^z - i}{e^z + i} \implies e^z = i \frac{1 + W}{1 - W}$$

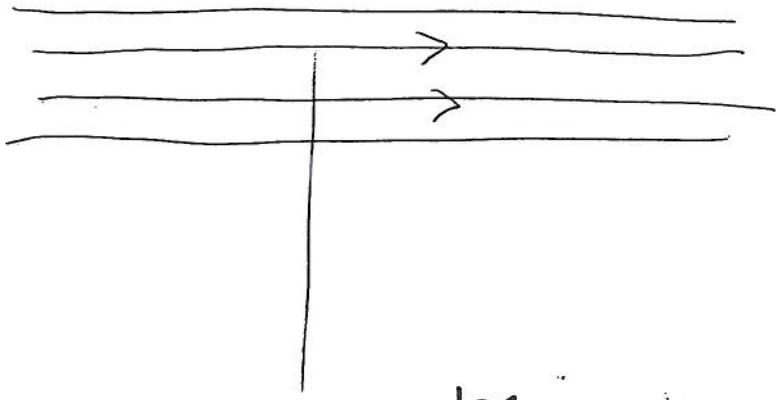
Thus

$$z = \log i + \log(1 + W) - \log(1 - W) \implies z = \frac{1}{2}i\pi + \log \frac{1 + W}{1 - W} \quad \text{Im}(z) = \text{const}$$

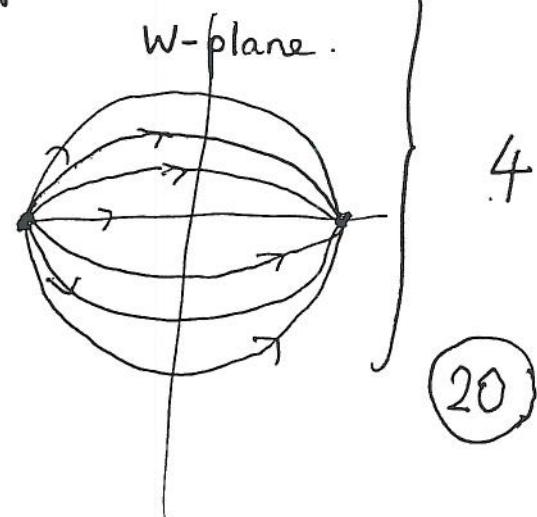
Now in the  $z$ -plane there is uniform flow in the  $x$ -direction. In the  $W$ -plane, all the streamlines emerge from the point  $W = -1$ , which corresponds to  $x = -\infty$  and head into the point  $W = 1$  ( $x = +\infty$ ). [Thus the flow is of a source/sink pair, as illustrated in the diagram]



$z$ -plane      Streamlines for final part



AOG



20

(2)

Soln	EXAMINATION QUESTIONS/SOLUTIONS 2008-09	Course (3) EE2
Question 2		Marks & seen/unseen
Parts		
(i)	<p>The residue at the pole of multiplicity 4  at <math>z=i</math> is <math>\lim_{z \rightarrow i} \left\{ \frac{1}{3!} \frac{d^3}{dz^3} [(z-i)^4 \frac{z^3}{(z-i)^4}] \right\}</math>  <math>= \lim_{z \rightarrow i} \frac{1}{3!} \frac{d^3}{dz^3} z^3 = 1</math></p> <p><math>\therefore</math> By R.T. <math>\oint_{C_1} = 2\pi i \times 1 = 2\pi i</math></p>	4
(ii)	<p>Residue at the simple pole at <math>z=1</math> is  <math>\lim_{z \rightarrow 1} \left\{ (z-1) \frac{z}{(z-1)(z-i)^2} \right\} = \frac{1}{(1-i)^2} = \frac{1}{-2i} = i/2</math></p> <p>Residue at the double pole at <math>z=i</math> is  <math>\lim_{z \rightarrow i} \frac{d}{dz} \left\{ (z-i)^2 \frac{z}{(z-1)(z-i)^2} \right\} = \lim_{z \rightarrow i} \frac{d}{dz} \left\{ \frac{z}{z-1} \right\}</math>  <math>= \lim_{z \rightarrow i} \left\{ \frac{1 \cdot (z-1) - z}{(z-1)^2} \right\} = -\frac{1}{(i-1)^2}</math>  <math>= \frac{1}{2i} = -i/2</math></p> <p><math>\therefore \oint_{C_1} = 2\pi i \left\{ i/2 - i/2 \right\} = 0</math></p> <p>by the Residue Theorem.</p>	3
(iii)	<p>For the new contour, the pole at <math>z=1</math> is included by the double pole at <math>z=i</math> is excluded so</p> $\oint_{C_1} = 2\pi i \times i/2 = -\pi$	4
		(20) ok
	Setter's initials J D G	Checker's initials AOG
		Page number 1

(3)

	EXAMINATION QUESTIONS/SOLUTIONS 2008-09	Course (3) EE2
John.		
Question	3	Marks & seen/unseen
Parts		
(i)	Residue at $z=0$ is $\lim_{z \rightarrow 0} \left\{ \frac{z \cdot e^{iz}}{z(z^2+1)^2} \right\} = 1$	3
(ii)	" " $z=i$ is $\lim_{z \rightarrow i} \frac{(z-i)^2 e^{iz}}{z(z^2+1)^2}$	
	$= \lim_{z \rightarrow i} \frac{d}{dz} \left\{ \frac{e^{iz}}{z(z^2+1)^2} \right\} = e^{-1} \left\{ \frac{4+4i+4}{(4i)^2} \right\} = -\frac{3}{4e}$	5
(iii)	$\lim_{r \rightarrow 0} \int_U \frac{e^{ir e^{i\theta}} i r e^{i\theta} d\theta}{r e^{i\theta} (r^2 e^{2i\theta} + 1)} = \cancel{i} \int_{\pi}^{2\pi} d\theta = \pi i$	3
(iv)	Use Jordan's Lemma: conditions for its use are i) Only singularities are poles: yes ✓ ii) $m=1$ ✓ iii) $\left  \frac{e^{iz}}{z(z^2+1)^2} \right  \rightarrow 0$ as $R \rightarrow \infty$ fast enough. Yes. ✓	4
	Then $\lim_{R \rightarrow \infty} \int_{-R}^R \frac{e^{iz}}{z(z^2+1)^2} dz = 0$	
(v)	By the Residue Thm $\oint_C \frac{e^{iz} dz}{z(z^2+1)^2} = 2\pi i \left( 1 - \frac{3}{4e} \right)$ $= \int_{-\infty}^{\infty} \frac{e^{ix} dx}{x(x^2+1)^2} + 0 + \pi i$ $\therefore \pi i \left( 1 - \frac{3}{4e} \right) = \int_{-\infty}^{\infty} \frac{(\cos x + i \sin x) dx}{x(x^2+1)^2}$	2 2
	The $\cos$ -integral is zero as integrand is odd.	
	$\therefore \int_{-\infty}^{\infty} \frac{\sin x dx}{x(x^2+1)^2} = \frac{\pi}{2e} (2e-3)$	1 (20) OK
	Setter's initials JDG	Checker's initials AOG
		Page number 1

(4)

EXAMINATION QUESTIONS/SOLUTIONS 2008-09		Course (3) EE2
Soln	Question	Marks & seen/unseen
4		
Parts	<p>a) <math>\mathcal{L}(f*g) = \int_0^\infty e^{-st} \left\{ \int_0^t f(u)g(t-u)du \right\} dt</math></p> <p>Re-arranging order of integr. <math>\begin{array}{c} u \uparrow \\ u=t \\ \diagdown \end{array}</math></p> $= \int_0^\infty f(u) \left( \int_u^\infty e^{-st} g(t-u)dt \right) du$ <p>Substitute <math>t-u=\tau</math></p> $\begin{array}{c} \text{Diagram: A triangle in the } t-u \text{ plane. The vertical axis is } u, \text{ the horizontal axis is } t. \text{ The vertices are at } (0,0), (\infty, \infty), \text{ and } (u, u). \text{ The region } R \text{ is shaded in the first quadrant.} \\ \text{The region } R \text{ is bounded by } u=0, u=\infty, \text{ and } t=u. \end{array}$ $\mathcal{L}(f*g) = \int_0^\infty f(u) \left( \int_0^\infty e^{-s(u+\tau)} g(\tau) d\tau \right) du$ $= \left( \int_0^\infty f(u) e^{-su} du \right) \left( \int_0^\infty e^{-s\tau} g(\tau) d\tau \right) = \bar{f}(s) \bar{g}(s)$	2 2 (per)
b)	<p>(i) <math>\bar{g}(s) = \frac{1}{s^2 - 3s + 2} = \frac{1}{(s-1)(s-2)} = \frac{1}{s-2} - \frac{1}{s-1}</math></p> $\therefore g(t) = e^{2t} - e^t$ $\therefore \mathcal{L}^{-1}\{\bar{f}(s)\bar{g}(s)\} = f(t)*g(t)$ $= \int_0^t f(t-u) (e^{2u} - e^u) du.$	4 6
	<p>(ii) <math>\bar{g}(s) = \frac{1}{s^2 + 2s + 5} = \frac{1}{(s+1)^2 + 2^2} = \frac{1}{2} \left\{ \frac{2}{(s+1)^2 + 2^2} \right\}</math></p> $\therefore g(t) = \frac{1}{2} e^{-t} \sin 2t \text{ by Shift Thm.}$ $\therefore \mathcal{L}^{-1}\{\cdot\} = \frac{1}{2} \int_0^t f(t-u) e^{-u} \sin 2u du.$	6
	Setter's initials JDE	Checker's initials AOG
		Page number 1

(20)

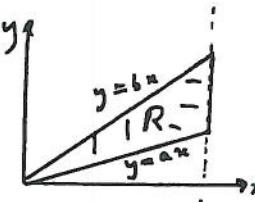
(5)

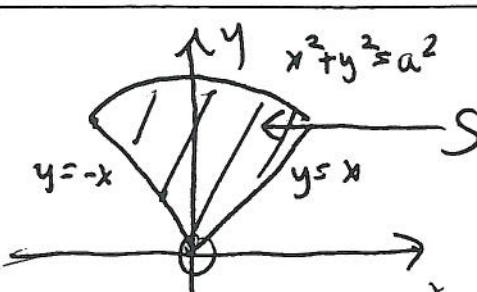
	EXAMINATION QUESTIONS/SOLUTIONS 2008-09	Course (3) EE 2
Question 5	Marks & seen/unseen	
Parts		
$L.T. \ddot{x} + 4\dot{x} + 5x = F(t) \text{ to get}$ $[s^2 \bar{x}(s) - s x(0) - \dot{x}(0)] + 4[s \bar{x}(s) - x(0)] + 5\bar{x} = \bar{F}(s)$ 3 $\therefore \bar{x}(s)(s^2 + 4s + 5) = (s+4)x(0) + \dot{x}(0) \neq \bar{F}(s)$ $= (s+2)x(0) + \bar{F}(s)$ 3 Use Ies to get: $\therefore \bar{x}(s) = \frac{s+2}{s^2 + 4s + 5} x(0) + \frac{\bar{F}(s)}{s^2 + 4s + 5} \text{ as requested.}$ 4 Re-write this as $\bar{x}(s) = \frac{s+2}{(s+2)^2 + 1} x(0) + \frac{\bar{F}(s)}{(s+2)^2 + 1} \quad (\text{which is the answer to (i)})$ 3 Now $\mathcal{Z}(\sin t) = \frac{1}{s^2 + 1} ; \mathcal{Z}(\cos t) = \frac{s}{s^2 + 1}$ $\therefore \text{Using the shift + convolution thms, we have}$ 3 + 3 $x(t) = x(0)e^{-2t} \cos t + \dots; e^{-2t} \sin t * F(t)$ $\text{or } x(t) = x(0)e^{-2t} \cos t + \int_0^t F(t-u) e^{-2u} \sin u du$ 1  <i>i.e.</i> $p(t) = e^{-2t} \cos t$ $\& q(u) = e^{-2u} \sin u$ <i>answer to (ii)</i>		
Setter's initials TDG	Checker's initials AOG	Page number

(20)

6

	EXAMINATION QUESTIONS/SOLUTIONS 2008-09	Course (3) EE2
Question 6	Marks & seen/unseen	
Parts		
$\int_0^\infty = \int_0^T + \sum_{n=1}^{\infty} \int_{nT}^{(n+1)T}$ and, with $\tau_n = t - nT$ $\int_{nT}^{(n+1)T} e^{-st} f(t) dt = \int_0^T e^{-s(\tau_n + nT)} f(\tau_n + nT) d\tau_n$ periodic $= e^{-snT} \int_0^T e^{-s\tau_n} f(\tau_n) d\tau_n$	4	
$\therefore \int_0^\infty = \left[ \sum_{n=0}^{\infty} e^{-snT} \right] \int_0^T$ $= \frac{1}{1 - e^{-sT}} \int_0^T$ Only for $s > 0$	4	
If $f(t) = \begin{cases} e^{at} & 0 \leq t \leq \frac{1}{2}T \\ 0 & \frac{1}{2}T \leq t \leq T \end{cases}$		
then $\int_0^T f(t) e^{-st} dt = \int_0^{\frac{1}{2}T} e^{-(s-a)t} dt$ $= -\frac{1}{s-a} [e^{-(s-a)t}]_0^{\frac{1}{2}T}$ $= \frac{1 - e^{-\frac{1}{2}(s-a)T}}{s-a}$	6	
$\therefore \int_0^\infty f(t) e^{-st} dt = \frac{1}{s-a} \left( \frac{1 - e^{-\frac{1}{2}(s-a)T}}{1 - e^{-sT}} \right)$	2	
		(20)
Setter's initials JDR	Checker's initials AOG	Page number 1

	EXAMINATION QUESTIONS/SOLUTIONS 2008-09	Course (3) EE 2
Question 7		Marks & seen/unseen
Parts	<p>Arc length <math>s_s</math>; chord vector <math>\underline{s}_c</math></p> <p>Unit tangent vector <math>\hat{\underline{t}} = d\underline{s}/ds</math></p> <p><math>\therefore \hat{\underline{t}} = \hat{i} \frac{dx}{ds} + \hat{j} \frac{dy}{ds} \therefore</math> construct <math>\hat{\underline{n}}</math></p> <p>from <math>\hat{\underline{n}} \cdot \hat{\underline{t}} = 0 \Rightarrow \hat{\underline{n}} = \pm \left( \hat{i} \frac{dy}{ds} - \hat{j} \frac{dx}{ds} \right)</math> + outer normal</p> <p>Now try <math>\hat{\underline{u}} = \hat{i} Q_x - \hat{j} P_y \Rightarrow \operatorname{div} \underline{u} = Q_x - P_y</math></p> <p>and <math>\underline{u} \cdot \hat{\underline{n}} = Q \frac{dy}{ds} + P \frac{dx}{ds}</math>. Thus, from G.T.</p> $\oint_C P dx + Q dy = \oint_C \underline{u} \cdot \hat{\underline{n}} ds = \iint_R [Q_x - P_y] dx dy = \iint_R \operatorname{div} \underline{u} dx dy$ <hr/> <p>If <math>\underline{u} = \hat{i} x^2 \sin y + x \cos y \hat{j}</math></p> <p><math>\therefore \operatorname{div} \underline{u} = 2x \sin y - x \sin y = x \sin y</math></p> <p><math>\therefore \iint_R \operatorname{div} \underline{u} dx dy = \int_0^1 x \left( \int_{a^x}^{b^x} x \sin y dy \right) dx</math></p> $= \int_0^1 x (\cos a^x - \cos b^x) dx$ <p>Now <math>\int_0^1 x \cos a^x dx = \frac{1}{a} \int x d(\sin a^x)</math></p> $= \frac{1}{a} \left\{ [x \sin a^x]_0^1 + \frac{1}{a} \left[ \sin a^x \right]_0^1 \right\} = \frac{1}{a} \left\{ \sin a + \frac{1}{a^2} (\cos a - 1) \right\}$ <p><math>\therefore \iint_R \operatorname{div} \underline{u} dx dy = \frac{\sin a}{a} - \frac{b-a}{b^2} + \frac{\cos a - 1}{a^2} - \frac{\cos b - 1}{b^2}</math></p> 	2 (fig) 2 2 2 2 2 2 2 3 20
	Setter's initials JDL	Checker's initials ADG
		Page number 1

	EXAMINATION QUESTIONS/SOLUTIONS 2008-09	Course C3
SOLN		
Question C3		Marks & seen/unseen
Parts (i)	 <p><math>S</math> is a <math>\frac{1}{4}</math> circle! <math>A = \frac{1}{4}\pi a^2</math> ✓ 2</p> $\bar{x} = \frac{1}{A} \iint_S x \, dx \, dy = 0$ <p>by symmetry or by (longer) calculation ✓ 2</p> $\begin{aligned} \bar{y} &= \frac{1}{A} \iint_S y \, dx \, dy \\ &= \frac{1}{A} \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{a \sin \theta} r \sin \theta \, r \, dr \, d\theta \quad (\text{polar}) \\ &= \frac{1}{A} \frac{a^3}{3} \left[ -\cos \theta \right]_{0}^{\frac{\pi}{2}} = \frac{4\sqrt{2}}{3\pi} a^3 \quad \checkmark \end{aligned}$ <p style="margin-left: 200px;">3</p> $\begin{aligned} I_{0z} &= \iint_S (x^2 + y^2) \, dx \, dy \\ &= \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{a \sin \theta} r^2 \, r \, dr \, d\theta \quad (\text{polar}) \\ &= \frac{\pi a^4}{24} = \frac{\pi a^4}{8} \quad \checkmark \end{aligned}$ <p style="margin-left: 200px;">3</p>	
	Setter's initials RIS	Checker's initials AOG
		Page number 1

	EXAMINATION QUESTIONS/SOLUTIONS 2008-09	Course CORE
SOLN		
Question C3		Marks & seen/unseen
Parts (ii)		2
On 1 $u=1$		
On 3 $u=2$	2	
On 2 $v=1$		
On 4 $v=-1$		
$J' = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} y & x \\ 2x & -2y \end{pmatrix} \Rightarrow -2(y^2 + x^2)$	2	
$J = 1/J' = -\frac{1}{2(x^2+y^2)}$	1	
Int $\int \int_{R^*} (x^4 - y^4)  J  du dv$	1	
$= \int \int_{R^*} \frac{1}{2} (x^2 - y^2)^2 du dv$	1	
$= \int_{v=-1}^1 \int_{u=1}^2 \frac{1}{2} v^2 du dv = 0$	1	
	Total: 20	
Setter's initials	PW	Checker's initials
		Page number 2

	EXAMINATION QUESTIONS/SOLUTIONS 2008-09	Course CORE
SOLN		
Question C4		Marks & seen/unseen
Parts	$\text{grad } \varphi = \hat{i} \frac{\partial \varphi}{\partial x} + \hat{j} \frac{\partial \varphi}{\partial y} + \hat{k} \frac{\partial \varphi}{\partial z}$ $\text{curl grad } \varphi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} & \frac{\partial \varphi}{\partial z} \end{vmatrix}$ $= \hat{i} \left[ \frac{\partial}{\partial y} \left( \frac{\partial \varphi}{\partial z} \right) - \frac{\partial}{\partial z} \left( \frac{\partial \varphi}{\partial y} \right) \right] - \hat{j} \left[ \frac{\partial}{\partial x} \left( \frac{\partial \varphi}{\partial z} \right) - \frac{\partial}{\partial z} \left( \frac{\partial \varphi}{\partial x} \right) \right]$ $+ \hat{k} \left[ \frac{\partial}{\partial x} \left( \frac{\partial \varphi}{\partial y} \right) - \frac{\partial}{\partial y} \left( \frac{\partial \varphi}{\partial x} \right) \right]$ <p>Each square bracket vanishes because</p> $\frac{\partial}{\partial y} \left( \frac{\partial \varphi}{\partial z} \right) = \frac{\partial}{\partial z} \left( \frac{\partial \varphi}{\partial y} \right)$ and two other similar relations <u>Hence result</u>	
	$\text{div curl } \vec{E} = \frac{\partial}{\partial x} (\text{curl } \vec{E})_x + \frac{\partial}{\partial y} (\text{curl } \vec{E})_y + \frac{\partial}{\partial z} (\text{curl } \vec{E})_z$ $= \frac{\partial}{\partial x} \left( \frac{\partial E_3}{\partial y} - \frac{\partial E_2}{\partial z} \right) + \frac{\partial}{\partial y} \left( \frac{\partial E_1}{\partial z} - \frac{\partial E_3}{\partial x} \right)$ $+ \frac{\partial}{\partial z} \left( \frac{\partial E_2}{\partial x} - \frac{\partial E_1}{\partial y} \right)$ <p>Terms cancel in pairs such as <math>\frac{\partial}{\partial x} \frac{\partial E_3}{\partial y} - \frac{\partial}{\partial y} \frac{\partial E_3}{\partial x}</math></p> <p>Hence result</p>	5 5
	Setter's initials R.L.J	Checker's initials AOG
		Page number 1

	EXAMINATION QUESTIONS/SOLUTIONS 2008-09	Course CORE
SOLN		
Question C4		Marks & seen/unseen
Parts	$\text{curl } \vec{E} = i(bz - 2z) + j(-ay - 0) + k(a y - (-2y))$ $= 0$ $\text{if } b=2, a=-2$ <hr/> $\vec{E} = \text{grad } \varphi$ $\Rightarrow x^2 - y^2 = \frac{\partial \varphi}{\partial x} \quad \text{--- (1)}$ $-2xy + z^2 = \frac{\partial \varphi}{\partial y} \quad \text{--- (2)}$ $2yz + z = \frac{\partial \varphi}{\partial z} \quad \text{--- (3)}$ $(1) \Rightarrow \varphi = \frac{x^3}{3} - xy^2 + h(y, z) \quad \text{--- (4)}$ $(2) + (4) \Rightarrow -2xy + z^2 = -2xy + \frac{\partial h}{\partial y} \quad \text{partial der}$ $\Rightarrow h(y, z) = yz^2 + g(z) \quad \text{--- (5)}$ $(3) + (4) + (5) \Rightarrow 2yz + z = 2yz + \frac{dg}{dz} \quad \text{ord. der.}$ $\Rightarrow g = \frac{z^2}{2} + C \quad \text{arb. const.}$ $\Rightarrow \varphi(x, y, z) = \frac{x^3}{3} - xy^2 + yz^2 + \frac{z^2}{2} + C$	2
	Setter's initials R.L	Checker's initials AOG
		Page number 2

	EXAMINATION QUESTIONS/SOLUTIONS 2008-09	Course CORE
SOLN Question CS		Marks & seen/unseen
Parts		
(i)	<p>If <math>\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y}</math> everywhere inside <math>R</math> in GT</p> <p>then <math>\oint_C g = 0</math>. If <math>C</math> is made up of two bits <math>C_1</math> and <math>C_2</math> this implies</p> $\int_{C_1, A}^B = \int_{C_2, B}^A$ <p>but note that <math>R</math> is the whole region enclosed by <math>C</math> so it is <u>simply connected</u>.</p> <p><math>R^*</math> is a larger region surrounding <math>C_1, C_2, A</math> and <math>B</math>.</p> <p>Thus if <math>\frac{\partial f}{\partial y} = \frac{\partial g}{\partial y}</math> everywhere inside <math>R^*</math></p> <p>and <math>R^*</math> is simply connected</p> <p>then <math>\int_{C_1, A}^B = \int_{C_2, B}^A</math> for any pair of curves joining <math>A</math> to <math>B</math>.</p> <p>[Students must state <math>\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}</math> and <math>R^*</math> is simply connected and present some fairly logical argument.]</p>	5 for part (i)
	Setter's initials R.L.J	Checker's initials AOG
		Page number 1

	EXAMINATION QUESTIONS/SOLUTIONS 2008-09	Course
SOLN		CORR
Question		Marks & seen/unseen
C6		
Parts		
(ii)	<p>If <math>f = \frac{x-y}{x^2+y^2} \Rightarrow g = \frac{x+y}{x^2+y^2}</math></p> <p>then <math>\frac{\partial f}{\partial y} = \frac{-1}{x^2+y^2} - \frac{(x-y)}{(x^2+y^2)^2} 2y</math></p> $\Rightarrow \frac{y^2-x^2-2xy}{(x^2+y^2)^2} \quad \textcircled{1}$	
	$\frac{\partial g}{\partial x} = \frac{1}{x^2+y^2} - \frac{x+y}{(x^2+y^2)^2} 2x$ $\Rightarrow \frac{y^2-x^2-2xy}{(x^2+y^2)^2} \quad \textcircled{2}$	4
	<p><math>\textcircled{1} + \textcircled{2}</math> are the same everywhere except possibly at <math>x=0, y=0</math>. <math>\leftarrow</math></p> <p>Thus we have <del>part</del> independence for <math>\int_C f dx - g dy</math> in any simply connected region excluding <math>0</math>.</p>	2
(iii)	<p>(a) <math>\int_{C_1} = \int_{C_1} \left( \frac{x-y}{x^2+y^2} dx + \frac{x+y}{x^2+y^2} dy \right)</math> use polar</p> $\begin{aligned} x &= \cos \theta \\ y &= \sin \theta \\ dx &= -\sin \theta d\theta \\ dy &= \cos \theta d\theta \end{aligned}$ $\begin{aligned} &\leq \int_{\theta=0}^{\pi/2} \left[ \frac{\cos \theta - \sin \theta}{\cos^2 \theta + \sin^2 \theta} (-\sin \theta) + \frac{\cos \theta + \sin \theta}{\cos^2 \theta + \sin^2 \theta} \cos \theta \right] d\theta \end{aligned}$	
	Setter's initials	Checker's initials
	AOG	Page number 2

	EXAMINATION QUESTIONS/SOLUTIONS 2008-09	Course
Soln		
Question	C5	Marks & seen/unseen
Parts	$  \begin{aligned}  &= \int_0^{\pi/2} (\sin \theta \cos \theta + \sum \theta + \omega^2 \theta + \sin \theta \cos \theta) d\theta \\  &= \int_0^{\pi/2} d\theta \\  &= \frac{\pi}{2}  \end{aligned}  $	7 marks for part (iii)
(b)	$  \begin{aligned}  \int_{C_1} &= \int_{\theta=0}^{-\pi/2} ( \dots ) d\theta \\  &= -\frac{3\pi}{2}  \end{aligned}  $ <p>Answers differ because <math>C_1</math> and <math>C_2</math> are not in a simply connected region in which <math>\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}</math> because 0 is excluded</p>	$\boxed{5}$ for showing that integrand equals unity $+1$ each for the two sets of limits 2
		20
Setter's initials	PWJ	Checker's initials
	AOB	Page number
		3

	EXAMINATION SOLUTIONS 2008-09	Course EE2(3)
Question 11		Marks & seen/unseen
Parts		sim. seen ↓
(i)	$\Omega = \{(i, j) : i = 1, \dots, 6, j = 1, \dots, 6\}$	2
(ii)	<p>(a) <math>A = \{(i, j) : i = 2, 4, 6, j = 1, \dots, 6\}</math>  <math>B = \{(i, j) : i + j \text{ is odd}\}</math></p> <p>(b) <math>P(A) = \frac{1}{2}, P(B) = \frac{1}{2},</math>  <math>P(A \cap B) = P(\{(i, j) : i = 2, 4, 6, j = 1, 3, 5\}) = 9/36 = 1/4.</math>      Thus <math>P(A)P(B) = P(A \cap B)</math>, showing that <math>A</math> and <math>B</math> are independent.</p>	4
(iii)	<p>(a) <math>C = \{(1, 3), (2, 2), (3, 1)\}</math></p> <p>(b)</p> $P(A C) = \frac{P(A \cap C)}{P(C)} = \frac{P(\{(2, 2)\})}{3/36} = \frac{1}{3}$	4
(iv)	<p>By the definition of conditional probability,</p> $P(D E) = \frac{P(D \cap E)}{P(E)} \quad (*)$ <p>Also by the definition of conditional probability, <math>P(E D) = \frac{P(D \cap E)}{P(D)}</math> and thus <math>P(D \cap E) = P(E D)P(D)</math>.</p> <p>Plugging this into (*) yields</p> $P(D E) = \frac{P(E D)P(D)}{P(E)}$	3
		seen ↓
	Setter's initials AG	Checker's initials SM
		Page number

	EXAMINATION SOLUTIONS 2008-09	Course EE2(3)
Question 12		Marks & seen/unseen
Parts		sim. seen ↓
(i)	$E(y_t) = E(1 + \sum_{s=0}^t e_s) = 1 + \sum_{s=0}^t E(e_s) = 1 + 0 = 1$	3
(ii)	$\text{Var}(y_t) = \text{Var}(1 + \sum_{s=0}^t e_s) = 0 + \sum_{s=0}^t \text{Var}(e_s) = t + 1$	4
(iii)	<p>For <math>s &gt; 0</math>,</p> $\begin{aligned}\text{cov}(y_t, y_{t+s}) &= \text{cov}(y_t, (y_{t+s} - y_t) + y_t) = \text{cov}(y_t, y_{t+s} - y_t) + \text{cov}(y_t, y_t) \\ &= \text{cov}\left(\sum_{u=0}^t e_u, \sum_{v=t+1}^{t+s} e_v\right) + \text{Var}(y_t) \\ &= \sum_{u=0}^t \sum_{v=t+1}^{t+s} \text{cov}(e_u, e_v) + t + 1 = t + 1.\end{aligned}$	5
(iv)	A time series $y_t$ is called weakly stationary if $E(y_t)$ and $\text{cov}(y_t, y_{t+s})$ , $s = 0, 1, 2, \dots$ are independent of $t$ .	3
(v)	No, since $\text{cov}(y_t, y_{t+s})$ depends on $t$ .	2
(vi)	A time series $\{y_t\}$ is called strictly stationary if for all $m = 1, 2, \dots$ and all indices $t_1, \dots, t_m$ , the joint distribution $(y_{t_1+s}, \dots, y_{t_m+s})$ is the same as that of $(y_{t_1}, \dots, y_{t_m})$ for any $s$ .	3
(vii)	No. Strict stationarity would imply weak stationarity. Since $y_t$ is not weakly stationary it cannot be strictly stationary.	2
	Setter's initials AG	Checker's initials GM
		Page number