EE1-10B MATHEMATICS II

1. a) Given the function

$$f(t) = e^{at}H(-t)\,,$$

where H is the Heaviside function, obtain $F(\omega)$, the Fourier transform of $f(t)$. State the condition on the constant a which is necessary for the existence of $F(\omega)$. [5]

b) Hence, or otherwise, obtain the inverse Fourier Transform of [5]

$$G(\omega) = \frac{1}{4 - 2i\omega - 3i}\,.$$

c) Given the plane with equation $\Pi\,:\,2x - 3y + 5z = -4$,

i) Find the minimum distance from the point $P(1,-1,2)$ to Π; obtain the point on Π nearest to P. [4]

ii) Another plane has equation $\Phi : x + \alpha y + \beta z = 0$. Give all values of α and β that make Π and Φ orthogonal. [3]

d) Given the vectors $\underline{\mathbf{u}} = (1,2,a)\,, \underline{\mathbf{v}} = (3,-4,b)$ and $\underline{\mathbf{w}} = (-5,6,c)$, find a condition on the scalars a,b,c so that $\underline{\mathbf{u}} \times \underline{\mathbf{v}} \cdot \underline{\mathbf{w}} = 0\,.$

Let this condition be satisfied. The vectors now form what kind of set? What is the determinant of the matrix whose columns are $\underline{\mathbf{u}}, \underline{\mathbf{v}}, \underline{\mathbf{w}}$? Finally, obtain scalars p,q such that $\underline{\mathbf{u}} = p\underline{\mathbf{v}} + q\underline{\mathbf{w}}\,.$ [8]

2. a) Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{pmatrix}.$$

i) Calculate A^2 and A^3 and find scalars ϕ and ψ such that [4]

$$A^3 + \phi A^2 + \psi A + I = \underline{\mathbf{0}}$$

where I is the identity matrix.

ii) Use the result from (i) to find the inverse of A. [4]

iii) Confirm your result in (ii) by calculating A^{-1} using Gaussian elimination. [4]

b) Given a matrix

$$A = \begin{pmatrix} 1 & -4 & 2 \\ -4 & 1 & -2 \\ 2 & -2 & -2 \end{pmatrix}$$

i) Show that $\lambda = -3$ is one of the eigenvalues of A and find the other two. [4]

ii) Find eigenvectors corresponding to the three eigenvalues of A. [4]

iii) Using projection, or otherwise, find a set of orthonormal eigenvectors for A, and hence obtain the orthogonal diagonalization of A. [5]

3. a) Find the general solution of the differential equation

$$(3t\cos x - 2x)\frac{dx}{dt} = 4t - 3\sin x\,.$$

Find also the particular solution satisfying the condition $x(1) = 0$. [6]

b) Given the Bernoulli equation

$$x\frac{dy}{dx} + y = x^2y^2\,,$$

use the substitution $v = y^{-1}$ to obtain a first order linear equation in v, and hence solve for y. [6]

c) Solve the following second order differential equation: [8]

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 26\cos(3x)\,.$$

d) The height h of a regular cone, with volume V and radius of the circular base r, is found using

$$V = \frac{1}{3}\pi r^2 h\,.$$

Given that the percentage errors in the measurements of r and V are at most 0.5% and 0.2%, respectively, give an estimate for the maximum percentage error in the calculation of h. [5]

4. a) A solution of the second order differential equation

$$\frac{d^2y}{dx^2} - 2xy = 0\,,$$

can be found in the form of a series, using the Leibnitz-Maclaurin method. Given the initial conditions $y(0) = 1$ and $y'(0) = 0$, differentiate the ODE n times to obtain the recurrence relation

$$y^{(n+2)}(0) = 2ny^{(n-1)}(0)\,, \quad (n \geq 1)\,,$$

where $y^{(k)}(0)$ is the k^{th} derivative of y, evaluated at zero.

Obtain the first three non-zero terms of the series. [8]

b) If $u = f(\phi)$ where f is not specified, and $\phi = \dfrac{2x - y}{3xy}$, show that [5]

$$y^2\frac{\partial u}{\partial y} + 2x^2\frac{\partial u}{\partial x} = 0\,,$$

c) A function of two variables is given as

$$f(x,y) = x(y+1)^2 - x^2 - x\,.$$

i) Find the stationary points of $f(x,y)$ and determine their nature using the Hessian determinant. [7]

ii) Sketch the contours of the surface $z = f(x,y)$. [5]