

Probability and Stochastic Processes 2003 Model Answers

1 (a) Let  $A$  be the event 'there is a closed path'. We have

$$A = A_4 \cup ((A_1 \cup A_2) \cap A_3)$$

$$\begin{aligned} \text{So } P(\bar{A}) &= P(\bar{A}_4 \cap \overline{(A_1 \cup A_2) \cap A_3}) \\ &= P(\bar{A}_4) \cdot P(\overline{(A_1 \cup A_2) \cap A_3}) \quad (\text{by independence}) \\ &= P(\bar{A}_4) \cdot [1 - P((A_1 \cup A_2) \cap A_3)] \\ &= P(\bar{A}_4) \cdot [1 - P(A_1 \cup A_2) \cdot P(A_3)] \quad (\text{by independence}) \\ &= P(\bar{A}_4) \cdot [1 - (1 - P(\bar{A}_1) \cdot P(\bar{A}_2)) \cdot P(A_3)] \\ &= P(\bar{A}_4) \cdot [1 - (1 - (1-p) \cdot (1-p)) \cdot p] \quad (\text{by independence}) \\ &= (1-p) [1 - (1 - (1-p)^2) p] \\ &= (1-p) [(1-p) + (1-p)^2 p] = (1-p)^2 (1 + (1-p)p) \end{aligned}$$

$$\text{Hence } P(A) (= 1 - P(\bar{A})) = \underline{1 - (1-p)^2 (1 + (1-p)p)}.$$

(b) We want

$P(A_1 | B_1)$  (the probability that the source is  $S_1$ , when the receiver indicates ' $S_1$ ')  
By Bayes' Rule

$$P(A_1 | B_1) = P(B_1 | A_1) P(A_1) / P(B_1). \quad (*)$$

But

$A_1, A_2, A_3$  are disjoint events and

$$A_1 \cup A_2 \cup A_3 = \Omega \quad (\text{'certain event'})$$

It follows  $B_1 \cap A_1, B_1 \cap A_2, B_1 \cap A_3$  are disjoint events and

$$B_1 = (B_1 \cap A_1) \cup (B_1 \cap A_2) \cup (B_1 \cap A_3)$$

$$\begin{aligned} \text{Hence } P(B_1) &= P(B_1 | A_1) P(A_1) + P(B_1 | A_2) P(A_2) + P(B_1 | A_3) P(A_3) \\ &= 0.8 \times 0.9 + 0.1 \times 0.05 + 0.1 \times 0.05 \\ &= 0.73 \end{aligned}$$

But then, from (\*),

$$P(A_1 | B_1) = \frac{0.8 \times 0.9}{0.73} = \underline{\underline{0.986}}$$

$$2 \quad F_{X|Y}(x|Y=0) = P[X(\omega) \leq x \text{ and } 0 \leq X(\omega) \leq 1] / P[0 \leq X(\omega) \leq 1] \\ = \frac{1}{2} \int_0^{\min\{x, 1\}} dx / \int_0^1 \frac{1}{2} dx = \begin{cases} x & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } 1 \leq x \\ 0 & \text{otherwise} \end{cases}$$

Also

$$F_{X|Y}(x|Y=1) = P[X(\omega) \leq x \text{ and } 1 \leq X(\omega) \leq 2] / P[1 \leq X(\omega) \leq 2] \\ = \frac{1}{2} \int_1^{\min\{x, 2\}} dx / \int_1^2 \frac{1}{2} dx = \begin{cases} x-1 & \text{if } 1 \leq x \leq 2 \\ 1 & \text{if } 2 \leq x \\ 0 & \text{otherwise} \end{cases}$$

These distributions have densities

$$f_{X|Y}(x|Y=0) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}, \quad f_{X|Y}(x|Y=1) = \begin{cases} 1 & 1 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

The conditional mean is

$$E[X|Y=0] = \int_0^1 x dx = \frac{1}{2} \quad \text{and} \quad E[X|Y=1] = \int_1^2 x dx = \frac{3}{2}$$

The error variance is

$$\int_{-\infty}^{\infty} |x - \hat{x}|^2 dx = \int_0^{\infty} |x - \hat{x}(0)|^2 P[Y=0] dx + \int_0^{\infty} |x - \hat{x}(1)|^2 P[Y=1] dx$$

$$\text{However, } P[Y=0] = P[0 \leq X(\omega) \leq 1] = \int_0^1 \frac{1}{2} dx = \frac{1}{2}$$

$$\text{and } P[Y=1] = P[1 \leq X(\omega) \leq 2] = \int_1^2 \frac{1}{2} dx = \frac{1}{2}$$

Hence

$$\int_{-\infty}^{\infty} |x - \hat{x}|^2 dx = \frac{1}{2} \int_0^1 (x - \frac{1}{2})^2 dx + \frac{1}{2} \int_1^2 (x - \frac{3}{2})^2 dx \\ = \frac{1}{2} \int_{-\frac{1}{2}}^{+\frac{1}{2}} x^2 dx + \frac{1}{2} \int_{-\frac{1}{2}}^{+\frac{1}{2}} x^2 dx = \int_{-\frac{1}{2}}^{+\frac{1}{2}} x^2 dx = \frac{1}{3} x^3 \Big|_{-\frac{1}{2}}^{+\frac{1}{2}} \\ = \frac{2}{3} \cdot \frac{1}{8} = \frac{1}{12}$$

$$\text{We have shown } E[|X(\omega) - \hat{x}(Y(\omega))|^2] = \frac{1}{12}$$

$$\text{The estimator } \hat{x}(y) = \begin{cases} \frac{1}{2} & \text{if } y=0 \\ \frac{3}{2} & \text{if } y=1 \end{cases}$$

This can be expressed as a linear estimator

$$\hat{x}(y) = y + 0.5$$

3 (a) Write  $Y(\omega) = T_1(\omega) + T_2(\omega)$ . Then

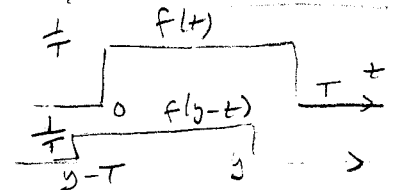
$$\begin{aligned} f_Y(y) \delta_y &= P[y \leq T_1(\omega) + T_2(\omega) \leq y + \delta_y] \\ &\approx \sum_i P[y - t_i \leq T_1(\omega) \leq y - t_i + \delta_y \text{ and } t_i \leq T_2(\omega) \leq t_i + \Delta t] \\ &= \sum_i P[y - t_i \leq T_1(\omega) \leq y - t_i + \delta_y] \cdot P[t_i \leq T_2(\omega) \leq t_i + \Delta t] \quad (\Delta t = t_{i+1} - t_i) \\ &= \int f_{T_1}(y-t) f_{T_2}(t) dt \cdot \delta_y \end{aligned}$$

Hence  $f_Y(y) = \int_{-\infty}^{+\infty} f(y-t) f(t) dt$

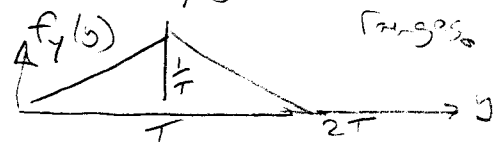
Evaluating the integral, with the help of the diagram, we see

$$f_Y(y) = \begin{cases} \frac{1}{T^2} y & 0 \leq y \leq T \\ \frac{1}{T^2} (2T - y) & T \leq y \leq 2T \end{cases}$$

and  $f_Y(y) = 0$  outside these ranges.



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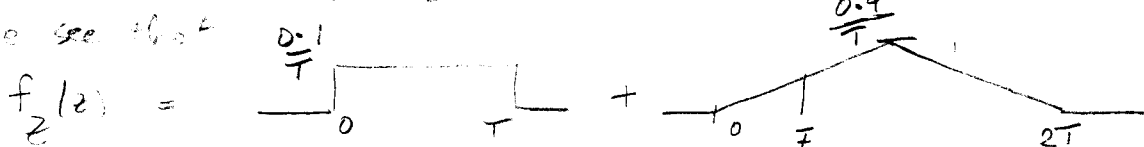
$$\begin{aligned} (b) \quad P[Z(\omega) \leq z] &= P[Z(\omega) \leq z \text{ and } F] + P[Z(\omega) \leq z \text{ and } \bar{F}] \\ &= P[T_1(\omega) \leq z \text{ and } F] + P[T_1(\omega) + T_2(\omega) \leq z \text{ and } \bar{F}] \\ &= P[T_1(\omega) \leq z] \cdot P[F] + P[T_1(\omega) + T_2(\omega) \leq z] \cdot P[\bar{F}] \end{aligned}$$

(by independence)

Hence  $f_Z(z) = f_{T_1}(z) \cdot P[F] + f_Y(z) (1 - P[F])$

where  $Y(\omega) = T_1(\omega) + T_2(\omega)$ .

We see that



$$f_Z(z) = \begin{cases} \frac{0.1}{T} + \frac{0.9}{T} z & 0 \leq z \leq T \\ \frac{0.9}{T} (2T - z) & T < z \leq 2T \\ 0 & \text{otherwise} \end{cases}$$

[4]

For  $\bar{E}$  such that  $0 \leq \bar{E} \leq T$

$$P[Z(\omega) \geq \bar{E}] = 1 - \frac{0.1}{T} \times \bar{E} - \frac{1}{2} \bar{E} \times \frac{\bar{E}}{T} \times \frac{0.9}{T}$$

$$= 1 - 0.1 (\bar{E}/T) - 0.45 (\bar{E}/T)^2$$

We require  $P[Z(\omega) \geq \bar{E}] = 0.95$  ('smallest'  $T$  case)

Solving  $0.95 = 1 - 0.1 (\bar{E}/T) - 0.45 (\bar{E}/T)^2$  gives  $\frac{\bar{E}}{T} = \frac{\sqrt{10}-1}{9}$

But  $\bar{E} = 10$  hrs, so

[6]

$$T = \frac{9}{\sqrt{10}-1} \times 10 = \underline{\underline{41.62 \text{ hrs}}}$$

4 (a)  $V(d) = E[D(\omega) - d]^2 = E[D^2(\omega)] - 2d E[D(\omega)] + d^2$

[2]  $V'(d) = 0$  gives  $-2E[D(\omega)] + 2d = 0$  hence minimizing  $d = E[D(\omega)]$ .

(b) A linear estimator has the form  $\hat{X} = aY + bZ + c$ . We want to choose  $a, b, c$  to minimize the least squares criterion

$$J(a, b, c) = E[|X(\omega) - aY(\omega) - bZ(\omega) - c|^2]$$

But, by (a), the minimizing  $c$  (for given  $a$  and  $b$ ) is

$$c(a, b) = m_X - a m_Y - b m_Z = E\left[\frac{1}{2}B + \frac{1}{2}F\right]_L = \frac{1}{2}L - aL = \left(\frac{1}{2} - a\right)L$$

Then  $J(a, b, c(a, b)) = \tilde{J}(a, b) = E[|X' - aY' - bZ'|^2]$ , where  $X' = X - m_X$ , etc.

$$\tilde{J}(a, b) = E X'^2 + a^2 E Y'^2 + b^2 E Z'^2$$

$$- 2a E(X'Y') - 2b E(X'Z') + 2ab E(Y'Z')$$

Setting gradients to zero ( $\frac{\partial}{\partial a}(\tilde{J}) = 0, \frac{\partial}{\partial b}(\tilde{J}) = 0$ ) to find minimum gives

$$E Y'^2 a^* - E(X'Y') + b^* E(Y'Z') = 0$$

$$E Z'^2 b^* - E(X'Z') + a^* E(Y'Z') = 0$$

But

$$E Y'^2 = \sigma^2, E(X'Y') = E\left(\frac{1}{2}(F+B)Y'\right) = \frac{1}{2}\alpha^2, E(X'Z') = \dots = \frac{1}{2}\alpha^2$$

$E(Y'Z') = r\sigma^2$ . Hence

$$\sigma^2 a^* + r\sigma^2 b^* = \frac{1}{2}\alpha^2 \quad \text{and} \quad \sigma^2 b^* + r\sigma^2 a^* = \frac{1}{2}\alpha^2$$

By symmetry,  $a^* = b^*$ . It follows

$$\sigma^2(1+r)a^* = \frac{1}{2}\alpha^2. \quad \text{Hence} \quad a^* = b^* = \frac{\alpha^2}{2\sigma^2(1+r)}$$

But then

$$c^* = c(a^*, b^*) = \left(\frac{1}{2} - a^*\right)L$$

Summary:  $\hat{X} = a^*Y + b^*Z$ , where

[16]  $a^* = b^* = \frac{\alpha^2}{2\sigma^2(1+r)} \quad \text{and} \quad c^* = \left(\frac{1}{2} - a^*\right)L.$

It is expected that the error variance is least when  $r=0$ . In this case the two measurements are uncorrelated and supply the most 'information' about  $X(\omega)$ . By contrast, if  $r=1$  (to take a different extreme)  $Y(\omega)$  and  $Z(\omega)$  are linearly related, and knowing  $Z(\omega)$

[2] does not add to the knowledge of knowing  $Y(\omega)$ , for example.

$$5(a) \quad x_{k+1} = Ax_k + be_k \quad - (1)$$

Post multiply right side by  $x_{k+1}^T$  and left side by  $(Ax_k + e_k)^T (= x_{k+1}^T)$ :

$$x_{k+1} x_{k+1}^T = Ax_k x_k^T A^T + Ax_k e_k^T b + be_k x_k^T A^T + be_k e_k^T b^T \quad - (2)$$

But  $x_k$  is a linear combination of  $e_{k-1}, e_{k-2}, \dots$ . Since the  $e_k$ 's are zero mean, uncorrelated, it follows  $A E\{x_k e_k^T\} + E\{e_k x_k^T\} A^T = 0$ .

Taking expectations across (2) gives

$$E\{x_{k+1} x_{k+1}^T\} = A E\{x_k x_k^T\} A^T + A E\{x_k e_k^T\} b + b E\{e_k x_k^T\} A^T + b E\{e_k e_k^T\} b^T$$

$$[8] \text{ Hence } R_x(0) = A R_x(0) A^T + \sigma^2 b b^T \quad - \text{Lyapunov equation}$$

(b) The coupled process can be expressed in terms of  $\underline{x} = \begin{pmatrix} y_k \\ w_k \end{pmatrix}$  as

$$\underline{x}_{k+1} = \begin{bmatrix} 0.5 & \alpha \\ 0 & 0.2 \end{bmatrix} \underline{x}_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e_k$$

Write  $R_x(0) = \begin{bmatrix} r_{00} & r_{01} \\ r_{01} & r_{11} \end{bmatrix}$ . Then the Lyapunov equation is

$$\begin{bmatrix} r_{00} & r_{01} \\ r_{01} & r_{11} \end{bmatrix} = \begin{bmatrix} 0.5 & \alpha \\ 0 & 0.2 \end{bmatrix} \begin{bmatrix} r_{00} & r_{01} \\ r_{01} & r_{11} \end{bmatrix} \begin{bmatrix} 0.5 & 0 \\ 0 & 0.2 \end{bmatrix} + \sigma^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.25\alpha & 0 \\ 0 & 0.2 \end{bmatrix} \begin{bmatrix} 0.5r_{00} + \alpha r_{01} & 0.2r_{01} \\ -0.5r_{01} + \alpha r_{11} & 0.2r_{11} \end{bmatrix} = \begin{bmatrix} 0.25r_{00} + \alpha r_{01} + \alpha^2 r_{11} & 0.1r_{01} + 0.2\alpha r_{11} \\ 0.1r_{01} + 0.2\alpha r_{11} & 0.04 + r_{11} \end{bmatrix}$$

$$\text{i.e. } \begin{bmatrix} r_{00} & r_{01} \\ r_{01} & r_{11} \end{bmatrix} = \begin{bmatrix} 0.25r_{00} + \alpha r_{01} + \alpha^2 r_{11} & 0.1r_{01} + 0.2\alpha r_{11} \\ 0.1r_{01} + 0.2\alpha r_{11} & 0.04 + r_{11} \end{bmatrix} + \sigma^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Equating entries, we obtain

$$r_{00} = 0.25r_{00} + \alpha r_{01} + \alpha^2 r_{11}, \quad r_{01} = 0.1r_{01} + 0.2\alpha r_{11}, \quad r_{11} = 0.04 + r_{11} + \sigma^2$$

We see  $r_{01} = \frac{2}{9}\alpha r_{11}$  and  $r_{11} = \frac{1}{9}\sigma^2$

$$\frac{3}{4}r_{00} = \frac{2}{9}\alpha^2 r_{11} + \alpha^2 r_{11} = \frac{11}{9}\alpha^2 r_{11}$$

Hence

$$r_{00}/r_{11} = \frac{E\{y_k^2\}}{E\{w_k^2\}} = \frac{4}{3} + \frac{11}{9}\alpha^2$$

$$\text{But } r_{00}/r_{11} = 2, \text{ so } \alpha^2 = \frac{2 \times \frac{3}{4} \times \frac{9}{11}}{\frac{11}{9}}, \text{ whence } \alpha = \sqrt{\frac{27}{22}}$$

[12]

6 (a) The spectral density of  $\{x_k\}$ ,  $\Phi_x(\omega) = \sum_{l=-\infty}^{+\infty} R(l) e^{-j\omega l}$ ,

[2] where  $R(l) = E\{x_k x_{k-l}\}$  for  $l = 0, \pm 1, \dots$

We have  $R_Y(l) = E\{(a_0 x_k + a_1 x_{k-1})(a_0 x_{k-l} + a_1 x_{k-l-1})\}$   
 $= (a_0^2 + a_1^2) R_x(l) + a_0 a_1 (R_x(l+1) + R_x(l-1))$ .

Hence  $\Phi_Y(\omega) = \sum_{l=-\infty}^{+\infty} R_Y(l) e^{-j\omega l}$   
 $= \sum_{l=-\infty}^{+\infty} [a_0^2 + a_1^2 + a_0 a_1 e^{j\omega} + a_0 a_1 e^{-j\omega}] e^{-j\omega l} R_x(l)$   
 $= (a_0^2 + a_1^2 + a_0 a_1 e^{j\omega} + a_0 a_1 e^{-j\omega}) \sum_{l=-\infty}^{+\infty} R_x(l) e^{-j\omega l}$   
 $= (a_0 + a_1 e^{-j\omega})(a_0 + a_1 e^{j\omega}) \Phi_x(\omega) = D(e^{j\omega}) D(e^{-j\omega}) \Phi_x(\omega)$

[6]

(b)  $\Phi_y(\omega) = \frac{(\frac{17}{16} + \frac{1}{4}[e^{-2j\omega} + e^{2j\omega}])}{(5/4 + \frac{1}{2}[e^{-j\omega} + e^{j\omega}]) (\frac{10}{9} + \frac{1}{3}[e^{-j\omega} + e^{j\omega}])}$   
 $= \frac{\frac{17}{16} + \frac{1}{4}(z^{-2} + z^2)}{(5/4 + \frac{1}{2}(z^{-1} + z)) (\frac{10}{9} + \frac{1}{3}(z^{-1} + z))} \Big|_{z=e^{j\omega}}$

But  $\frac{17}{16} + \frac{1}{4}(z^{-2} + z^2) = \frac{z^{-2}}{16} [4z^4 + 17z^2 + 4] = \frac{z^{-2}}{16} (4z^2 + 1)(z^2 + 4) = (1 + \frac{1}{4}z^{-2})(1 + \frac{1}{4}z^2)$

$\frac{5}{4} + \frac{1}{2}(z^{-1} + z) = \frac{z^{-1}}{4} [2z^2 + 5z + 2] = \frac{z^{-1}}{4} (2z + 1)(z + 2) = (1 + \frac{1}{2}z^{-1})(1 + \frac{1}{2}z)$

and

$\frac{10}{9} + \frac{1}{3}(z^{-1} + z) = \frac{z^{-1}}{9} [3z^2 + 10z + 3] = \frac{z^{-1}}{9} (3z + 1)(z + 3) = (1 + \frac{1}{3}z^{-1})(1 + \frac{1}{3}z)$

It follows that  $\Phi_y(\omega)$  can be factorized

$\Phi_y(\omega) = \frac{1 + \frac{1}{4}z^{-2}}{(1 + \frac{1}{2}z^{-1})(1 + \frac{1}{3}z^{-1})} \cdot \frac{1 + \frac{1}{4}z}{(1 + \frac{1}{2}z)(1 + \frac{1}{3}z)} \Big|_{z=e^{j\omega}}$

Hence the spectral density is 'realised' by the ARMA model

$y_k = \frac{1 + \frac{1}{4}z^{-2}}{(1 + \frac{1}{2}z^{-1})(1 + \frac{1}{3}z^{-1})} e_k$ , for some zero mean, unit variance, uncorrelated process  $\{e_k\}$ .

We see that

$y_k = (1 + \frac{1}{4}z^{-2}) x_k$ , where  $x_k = \frac{1}{(1 + \frac{1}{2}z^{-1})(1 + \frac{1}{3}z^{-1})} e_k$   
 and therefore  $a = \frac{1}{4}$  and  $l = 2$

Also

[12]

$x_k + \frac{5}{6}x_{k-1} + \frac{1}{6}x_{k-2} = e_k$