UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 1998

MEng Honours Degrees in Computing Part IV
MSci Honours Degree in Mathematics and Computer Science Part IV
MSc Degree in Advanced Computing
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the
Diploma of Membership of Imperial College
Associateship of the Royal College of Science
Associateship of the City and Guilds of London Institute

PAPER 4.38

COMPLEXITY

Friday, May 8th 1998, 10.00 - 12.00

Answer THREE questions

For admin. only: paper contains 4 questions

1a Let L, L' be languages.

i) What does it mean for L to be decided in non-deterministic polynomial (NP) time?

ii) What does it mean for L to be polynomial-time reducible to L' ($L \leq L'$)?

iii) What does it mean for L to be NP-complete (NPC)?

b i) Define the problems SAT and 3SAT.

ii) Sketch a proof that SAT and 3SAT are reducible to each other.

c The vertex (node) cover (VC) problem is the following: Given a graph G and a natural number k, is there a set $X \subseteq$ nodes(G) with $|X| \leq k$ which *covers* G, in the sense that every edge of G has at least one endpoint in X?

Two variant problems are defined as follows:

VCN: Given G, k and a node x of G, is there $X \subseteq$ nodes(G) which covers G with $|X| \leq k$ and $x \notin X$?

VCS: Given G, k and a set of nodes $S \subseteq$ nodes(G), is there $X \subseteq$ nodes(G) which covers G with $|X| \leq k$ and $S \subseteq X$?

Assuming that VC is NP-complete, show that VCN and VCS are both NP-complete.

2a i) Define the class LOGSPACE, explaining space usage carefully.

ii) Show that LOGSPACE $\subseteq$ P.

b Let $LR \subseteq \{0,1\}^*$ be the language of all "repeated" words:

$$LR = \{xx : x \in \{0,1\}^*\}$$

Show that LR is in LOGSPACE.

c Given a language L, let the language L* be defined by L* = {w : w is the concatenation of 0 or more words in L}. Show that if L is in NLOGSPACE then so is L*.

The three parts carry, respectively, 40%, 30%, 30% of the marks.

3a Design a boolean circuit, with two input gates and one output gate, whose output is true if and only if one of the input gates is made true and the other input gate made false.

b Explain the meaning of the following.

i) L has uniform polynomial circuits (for a language $L \subseteq \{0,1\}^*$)

ii) NC_j, for $j \geq 1$

iii) NC

c Let $L \subseteq \{0,1\}^*$ be the language consisting of all words with an even number of 1's. Show that $L \in NC_1$.

d Show that NLOGSPACE $\subseteq NC_2$. You may assume without proof that graph reachability, RCH, is in NC_2.

4a i) What is a *polynomially decidable, polynomially balanced* relation on the set Σ^* of strings of the alphabet Σ?

ii) Outline a proof that a language $L \subseteq \Sigma^*$ is in NP if and only if $L = \{x \in \Sigma^* : R(x,y) \text{ for some } y \in \Sigma^*\}$ for some polynomially decidable, polynomially balanced relation R on Σ^*.

iii) Define the classes FNP and FP of functional problems.

iv) What are the functional problems FSAT and FHAM?

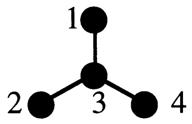
b Recall that a *3-colouring* of a graph is an assignment of each node of the graph to one of three colours, in such a way that if (x,y) is any edge of the graph then x and y are given different colours. A graph is *3-colourable* if it has a 3-colouring. 3COL is the problem of deciding whether a given graph is 3-colourable.

Let G be a 3-colourable graph. We form a new graph G* with the same nodes as G, by adding edges to G as follows. First, we enumerate all the non-edges of G in an arbitrary order: say, $e_1, e_2, \ldots, e_k$. We consider each non-edge of G in turn, and add it as an edge if and only if the resulting graph is still 3-colourable. Then we pass to the next non-edge of G, and so on. G* is the result of this process.

More precisely, we let $G_1 = G+e_1$ if $G+e_1$ is 3-colourable, and $G_1=G$ otherwise; $G_2 = G_1+e_2$ if G_1+e_2 is 3-colourable, and $G_2=G_1$ otherwise; ...; finally, $G^* = G_k$.

i) Choose an enumeration of non-edges and calculate G* for the following graph G. Give a 3-colouring of G*.

G:



ii) Show that G* has a 3-colouring with the following property: any two distinct nodes x, y of G* are given different colours if *and only if* (x,y) is an edge of G*.

iii) Deduce that if 3COL $\in$ P then F3COL $\in$ FP.

End of paper