Imperial College London

M1S

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May-June 2019

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Probability and Statistics 1

Date: Friday 24 May 2019

Time: 14.00 - 16.00

Time Allowed: 2 Hours

This paper has 4 Questions.

Candidates should start their solutions to each question in a new main answer book.

Supplementary books may only be used after the relevant main book(s) are full.

All required additional material will be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Calculators may not be used.

1. (a) State the three axioms of probability for events defined on a sample space Ω.

   (b) Prove from the axioms that for events $E, F \subseteq \Omega$, $\mathsf{P}(E \cap F) \leq \mathsf{P}(E)$.

   (c) The conditional probability mass function of the discrete random variable Y, given that $\Theta = \theta$ follows a Bernoulli distribution with parameter θ.
   The random variable Θ is defined by

$$\Theta = \frac{X+1}{4},$$

   where $X \sim Binomial(2, 0.25)$.

   (i) Find the probability mass function of Θ.

   (ii) Find expressions for the mean and variance of Θ in terms of the mean and variance of X respectively.

   (iii) Determine $\mathsf{E}_{f_\Theta}(\Theta)$.

   (iv) What is $\mathsf{P}(\Theta > 0.5)$?

   (v) Determine $\mathsf{P}(Y = 0)$.

   (vi) Given that $Y = 0$ determine the probability that $\Theta > 0.5$.

2. A die is rolled three times with scores X_1, X_2 and X_3. Let Y_3 be the maximum score obtained and Z_3 the minimum score obtained.

   (a) Prove that $\mathsf{P}(Y_3 \leq i) = \mathsf{P}(X_1 \leq i)^3, i = 1, 2, \ldots 6$.

   (b) Show that the probability mass function of Y_3 is given by

$$f_{Y_3}(i) = \begin{cases} \left(\frac{i}{6}\right)^3 - \left(\frac{i-1}{6}\right)^3, & i = 1, 2, \ldots 6; \\ 0, & \text{otherwise.} \end{cases}$$

   (c) Find $\mathsf{E}_{f_{Y_3}}(Y_3)$.

   (d) Determine the probability mass function of Z_3.

   (e) Let Y_n be the maximum score obtained when n dice are rolled. Find the probability mass function of Y_n.

   (f) Let Z_n be the minimum score obtained when n dice are rolled and let $Q = Y_n - Z_n$. What is $\mathsf{P}(Q = 0)$?

3. (a) What properties must $f_X(x)$ have in order to be a valid probability density function (pdf)?

(b) Let the continuous random variables X_i, $i = 1, 2, \ldots, n$, be a sequence of independent exponential random variables with pdfs

$$f_{X_i}(x) = \lambda \mathrm{e}^{-\lambda x}, \quad x > 0.$$

With parameter $\lambda > 0$. Let $Y = \lambda X_1$.

(i) Show that the moment generating function of X_i is given by,

$$M_{X_i}(t) = \frac{\lambda}{\lambda - t}, \quad |t| < \lambda.$$

(ii) Prove that $\mathsf{E}_{f_{X_i}}(X_i) = \lambda^{-1}, i = 1, \ldots, n$.

(iii) Find the pdf of Y and prove that $\mathsf{E}_{f_{X_i}}(X_i) = \lambda^{-1}\mathsf{E}_{f_Y}(Y), i = 1, \ldots, n$.

(iv) Prove that $\mathsf{E}_{f_{X_i}}(X_i^k) = \lambda^{-k}\Gamma(k+1), i = 1, \ldots, n$.

(v) Find the pdf of

$$A = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

4. (a) For continuous random variables X and Y, prove that

$$\mathsf{E}_{f_Y}(Y) = \mathsf{E}_{f_X}\left[\mathsf{E}_{f_{Y|X}}(Y \mid X)\right].$$

(b) The continuous random variable X has pdf given by,

$$f_X(x) = \begin{cases} kx^2(1-x^2), & 0 < x < 1; \\ 0, & \text{otherwise.} \end{cases}$$

(i) Determine the value of k.

(ii) Determine $\mathsf{E}_{f_X}(X)$ and $\mathsf{var}_{f_X}(X)$.

The conditional pdf of Y given $X = x$ is given by

$$f_{Y|X}(y|x) = \begin{cases} \frac{3}{x^3}(2y - 3x)^2, & y \in (x, 2x); \\ 0, & \text{otherwise.} \end{cases}$$

(iii) Determine $\mathsf{E}_{f_{Y|X}}(Y \mid X = x)$.

(iv) Determine $\mathsf{E}_{f_Y}(Y)$.

(v) Find $f_{X,Y}(x, y)$.

(vi) Write $1 - \mathsf{P}(X + Y < 2)$ as an integral of the form

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2} f_{X,Y}(x, y)\; \mathrm{d}y\, \mathrm{d}x,$$

where x_1, x_2, y_1 and y_2 are to be determined and the limits y_1 and y_2 may depend on x.

DISCRETE DISTRIBUTIONS

	RANGE	PARAMETERS	MASS FUNCTION f_X	CDF F_X	$\mathsf{E}_{f_X}[X]$	$\mathsf{Var}_{f_X}[X]$	MGF M_X
$Bernoulli(\theta)$	$\{0,1\}$	$\theta \in (0,1)$	$\theta^x(1-\theta)^{1-x}$		θ	$\theta(1-\theta)$	$1-\theta+\theta e^t$
$Binomial(n,\theta)$	$\{0,1,...,n\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{n}{x}\theta^x(1-\theta)^{n-x}$		$n\theta$	$n\theta(1-\theta)$	$(1-\theta+\theta e^t)^n$
$Poisson(\lambda)$	$\{0,1,2,...\}$	$\lambda \in \mathbb{R}^+$	$\dfrac{e^{-\lambda}\lambda^x}{x!}$		λ	λ	$\exp\{\lambda(e^t-1)\}$
$Geometric(\theta)$	$\{1,2,...\}$	$\theta \in (0,1)$	$(1-\theta)^{x-1}\theta$	$1-(1-\theta)^x$	$\dfrac{1}{\theta}$	$\dfrac{(1-\theta)}{\theta^2}$	$\dfrac{\theta e^t}{1-e^t(1-\theta)}$
$NegBinomial(n,\theta)$	$\{n,n+1,...\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{x-1}{n-1}\theta^n(1-\theta)^{x-n}$		$\dfrac{n}{\theta}$	$\dfrac{n(1-\theta)}{\theta^2}$	$\left(\dfrac{\theta e^t}{1-e^t(1-\theta)}\right)^n$
or	$\{0,1,2,...\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{n+x-1}{x}\theta^n(1-\theta)^x$		$\dfrac{n(1-\theta)}{\theta}$	$\dfrac{n(1-\theta)}{\theta^2}$	$\left(\dfrac{\theta}{1-e^t(1-\theta)}\right)^n$

For **CONTINUOUS** distributions (see over), define the **GAMMA FUNCTION**

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx \qquad \Gamma(\alpha+1) = \alpha\Gamma(\alpha) \qquad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

and the LOCATION/SCALE transformation $Y = \mu + \sigma X$ gives

$$f_Y(y) = f_X\left(\frac{y-\mu}{\sigma}\right)\frac{1}{\sigma} \qquad F_Y(y) = F_X\left(\frac{y-\mu}{\sigma}\right) \qquad M_Y(t) = e^{\mu t}M_X(\sigma t) \qquad \mathsf{E}_{f_Y}[Y] = \mu+\sigma\mathsf{E}_{f_X}[X] \qquad \mathsf{Var}_{f_Y}[Y] = \sigma^2\mathsf{Var}_{f_X}[X]$$

CONTINUOUS DISTRIBUTIONS

	$\mathbb{X}$	PARAMS.	PDF f_X	CDF F_X	$\mathsf{E}_{f_X}[X]$	$\mathsf{Var}_{f_X}[X]$	MGF M_X
$Uniform(\alpha,\beta)$ (stand. model $\alpha=0,\beta=1$)	(α,β)	$\alpha<\beta\in\mathbb{R}$	$\dfrac{1}{\beta-\alpha}$	$\dfrac{x-\alpha}{\beta-\alpha}$	$\dfrac{(\alpha+\beta)}{2}$	$\dfrac{(\beta-\alpha)^2}{12}$	$\dfrac{e^{\beta t}-e^{\alpha t}}{t(\beta-\alpha)}$
$Exponential(\lambda)$ (stand. model $\lambda=1$)	$\mathbb{R}^+$	$\lambda\in\mathbb{R}^+$	$\lambda e^{-\lambda x}$	$1-e^{-\lambda x}$	$\dfrac{1}{\lambda}$	$\dfrac{1}{\lambda^2}$	$\left(\dfrac{\lambda}{\lambda-t}\right)$
$Gamma(\alpha,\beta)$ (stand. model $\beta=1$)	$\mathbb{R}^+$	$\alpha,\beta\in\mathbb{R}^+$	$\dfrac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}$		$\dfrac{\alpha}{\beta}$	$\dfrac{\alpha}{\beta^2}$	$\left(\dfrac{\beta}{\beta-t}\right)^\alpha$
$Weibull(\alpha,\beta)$ (stand. model $\beta=1$)	$\mathbb{R}^+$	$\alpha,\beta\in\mathbb{R}^+$	$\alpha\beta x^{\alpha-1}e^{-\beta x^\alpha}$	$1-e^{-\beta x^\alpha}$	$\dfrac{\Gamma(1+1/\alpha)}{\beta^{1/\alpha}}$	$\dfrac{\Gamma\left(1+\frac{2}{\alpha}\right)-\Gamma\left(1+\frac{1}{\alpha}\right)^2}{\beta^{2/\alpha}}$	
$Normal(\mu,\sigma^2)$ (stand. model $\mu=0,\sigma=1$)	$\mathbb{R}$	$\mu\in\mathbb{R}$ $\sigma\in\mathbb{R}^+$	$\dfrac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\dfrac{(x-\mu)^2}{2\sigma^2}\right\}$		μ	σ^2	$e^{\{\mu t+\sigma^2t^2/2\}}$
$Student(\nu)$	$\mathbb{R}$	$\nu\in\mathbb{R}^+$	$\dfrac{(\pi\nu)^{-\frac{1}{2}}\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\left\{1+\frac{x^2}{\nu}\right\}^{(\nu+1)/2}}$		0 (if $\nu>1$)	$\dfrac{\nu}{\nu-2}$ (if $\nu>2$)	
$Pareto(\theta,\alpha)$	$\mathbb{R}^+$	$\theta,\alpha\in\mathbb{R}^+$	$\dfrac{\alpha\theta^\alpha}{(\theta+x)^{\alpha+1}}$	$1-\left(\dfrac{\theta}{\theta+x}\right)^\alpha$	$\dfrac{\theta}{\alpha-1}$ (if $\alpha>1$)	$\dfrac{\alpha\theta^2}{(\alpha-1)(\alpha-2)}$ (if $\alpha>2$)	
$Beta(\alpha,\beta)$	$(0,1)$	$\alpha,\beta\in\mathbb{R}^+$	$\dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$		$\dfrac{\alpha}{\alpha+\beta}$	$\dfrac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	

M1S SOLUTIONS

seen ⇓

1. (a) **Axioms of Probability**

Given a $\sigma-$field, $\mathcal{F}$ (a set of subsets of the sample space Ω.) For events $E, E_1, E_2, \ldots \in \mathcal{F}$, then the probability function, $\mathsf{P}(\cdot)$, must satisfy:

(1) $\mathsf{P}(E) \geq 0$.

(2) $\mathsf{P}(\Omega) = 1$.

(3) If $E_1, E_2, \ldots$, are pairwise disjoint then $\mathsf{P}(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mathsf{P}(E_i)$ (Countable additivity).

(Do not need to specify $\sigma-$field, could instead say: for events $E, E_1, \ldots \subseteq \Omega$. Lose 1 mark if finite rather than countable additivity specified, but they do need to specify the meaning of finite/countable additivity).

3(A)

sim. seen ⇓

(b) Let $E_i = \phi$ in axiom (3), then $\mathsf{P}(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mathsf{P}(\phi)$ and for axiom (2) to hold we must have $\mathsf{P}(\cup_{i=1}^{n} E_i) \leq 1$, hence $\mathsf{P}(\phi) = 0$. We can then show that finite additivity: $(\mathsf{P}(\cup_{i=1}^{n} E_i) = \sum_{i=1}^{n} \mathsf{P}(E_i))$ follows from axiom (3) by setting $E_i = \phi, \forall i > n$.

$$E = (E \cap F) \cup (E \cap F^C)$$

$$\Rightarrow \mathsf{P}(E) = \mathsf{P}(E \cap F) + \mathsf{P}(E \cap F^C) \quad \text{axiom 3 as } (E \cap F) \text{ and } (E \cap F^C) \text{ disjoint}$$

$$\Rightarrow \mathsf{P}(E) \geq \mathsf{P}(E \cap F) \quad \text{axiom 1, as } \mathsf{P}(E \cap F^C) \geq 0$$

$$\Rightarrow \mathsf{P}(E \cap F) \leq \mathsf{P}(E),$$

3(A)

as required.

unseen ⇓

(c) (i) We have $\Theta = \frac{X+1}{4}$ and $X \sim Binomial(2, 0.25)$, so the range of X is $\{0, 1, 2\}$ $\Rightarrow$ range of Θ is $\left\{\frac{1}{4}, \frac{2}{4}, \frac{3}{4}\right\}$.

1(A)

$$\begin{aligned} f_\Theta(\theta) &= \mathsf{P}(\Theta = \theta) = \mathsf{P}\left(\frac{X+1}{4} = \theta\right) \\ &= \mathsf{P}(X = 4\theta - 1) = f_X(4\theta - 1) \\ &= \binom{2}{4\theta - 1} \left(\frac{1}{4}\right)^{4\theta-1} \left(\frac{3}{4}\right)^{3-4\theta}, \quad \theta \in \left\{\frac{1}{4}, \frac{2}{4}, \frac{3}{4}\right\}. \end{aligned}$$

meth seen ⇓

3(B)

(ii) Given $\Theta = \frac{X+1}{4}$,

$$\mathsf{E}_{f_\Theta}(\Theta) = \mathsf{E}_{f_X}\left(\frac{X+1}{4}\right) = \frac{1}{4}\left[\mathsf{E}_{f_X}(X) + 1\right],$$

$$\mathsf{var}_{f_\Theta}(\Theta) = \mathsf{var}_{f_X}\left(\frac{X+1}{4}\right) = \frac{1}{16}\mathsf{var}_{f_X}(X).$$

2(A)

(iii) $X \sim Binomial(2, 0.25)$ so, from formula sheet $\mathsf{E}_{f_X}(X) = 2 \times 0.25 = 0.5$. And,

$$\mathsf{E}_{f_\Theta}(\Theta) = \mathsf{E}_{f_X}\left(\frac{X+1}{4}\right) = \frac{1}{4}\left[\mathsf{E}_{f_X}(X) + 1\right] = \frac{3}{8}.$$

Could also determine from $\sum_\theta \theta f_\Theta(\theta)$. $\boxed{2(\mathrm{A})}$

(iv)

$$\mathsf{P}(\Theta > 0.5) = \mathsf{P}(\Theta = 3/4) = (0.25)^2 = \frac{1}{16}.$$

$\boxed{1(\mathrm{B})}$

(v)

$$\mathsf{P}(Y=0) = \sum_\theta \mathsf{P}(Y=0 \mid \Theta=\theta)\mathsf{P}(\Theta=\theta) = \sum_\theta (1-\theta) f_\Theta(\theta)$$
$$= \frac{3}{4}\left(\frac{3}{4}\right)^2 + 2\cdot\frac{2}{4}\left(\frac{1}{4}\right)\left(\frac{3}{4}\right) + \frac{1}{4}\left(\frac{1}{4}\right)^2 = \frac{40}{64} = \frac{5}{8}.$$

$\boxed{2(\mathrm{B})}$

$\boxed{\text{sim. seen} \Downarrow}$

(vi)

$$\mathsf{P}(\Theta > 0.5 \mid Y=0) = \frac{\mathsf{P}(Y=0 \mid \Theta > 0.5)\mathsf{P}(\Theta > 0.5)}{\mathsf{P}(Y=0)}$$
$$= \frac{\mathsf{P}(Y=0 \mid \Theta = 0.75)\mathsf{P}(\Theta = 0.75)}{\mathsf{P}(Y=0)}$$
$$= \frac{1/4 \times 1/16}{5/8} = \frac{1}{40}.$$

$\boxed{3(\mathrm{B})}$

Commentary: (a) and (b) are bookwork; developing the pmf in (c)(i) is straightforward, but may prove a little more challenging as it requires more abstraction. The rest of the question is relatively straightforward for those that have engaged with the material.

meth seen $\Downarrow$

2. (a) Prove that $\mathsf{P}(Y_3 \le i) = \mathsf{P}(X_1 \le i)^3, i = 1, 2, \dots 6$.

$$\begin{aligned}\mathsf{P}(Y_3 \le i) &= \mathsf{P}(\max\{X_1, X_2, X_3\} \le i) = \mathsf{P}((X_1 \le i) \cap (X_2 \le i) \cap (X_3 \le i))\\ &= \mathsf{P}(X_1 \le i)\mathsf{P}(X_2 \le i)\mathsf{P}(X_3 \le i) \quad \text{from independence}\\ &= \mathsf{P}(X_1 \le i)^3 \quad i = 1, 2, \dots, 6 \text{ as } X_1, X_2 \text{ and } X_3 \text{ are identically distributed.}\end{aligned}$$

4(A)

unseen $\Downarrow$

(b) Determine the probability mass function of Y_3. From (a) we have

$$\begin{aligned}\mathsf{P}(Y_3 \le i) &= \mathsf{P}(X_1 \le i)^3\\ \Rightarrow \mathsf{P}(Y_3 \le 1) &= \mathsf{P}(X_1 \le 1)^3 = \mathsf{P}(X_1 = 1)^3 = \frac{1}{6^3}\\ \mathsf{P}(Y_3 = i) &= \mathsf{P}(Y_3 \le i) - \mathsf{P}(Y_3 \le i-1), \; i = 2, \dots, 6\\ &= \mathsf{P}(X_1 \le i)^3 - \mathsf{P}(X_1 \le i-1)^3 = \left(\frac{i}{6}\right)^3 - \left(\frac{i-1}{6}\right)^3.\end{aligned}$$

So the pmf of Y_3 is

$$f_{Y_3}(i) = \begin{cases} \left(\frac{i}{6}\right)^3 - \left(\frac{i-1}{6}\right)^3, & i = 1, 2, \dots 6;\\ 0, & \text{otherwise.}\end{cases}$$

5(C)

meth seen $\Downarrow$

(c) Find $\mathsf{E}_{f_{Y_3}}(Y_3)$.

$$\begin{aligned}\mathsf{E}_{f_{Y_3}}(Y_3) &= \sum_{i=1}^{6} i f_{Y_3}(i)\\ &= \left(\frac{1}{6}\right)^3 + 2\left[\left(\frac{2}{6}\right)^3 - \left(\frac{1}{6}\right)^3\right] + 3\left[\left(\frac{3}{6}\right)^3 - \left(\frac{2}{6}\right)^3\right] + 4\left[\left(\frac{4}{6}\right)^3 - \left(\frac{3}{6}\right)^3\right]\\ &\quad + 5\left[\left(\frac{5}{6}\right)^3 - \left(\frac{4}{6}\right)^3\right] + 6\left[\left(\frac{6}{6}\right)^3 - \left(\frac{5}{6}\right)^3\right]\\ &= -\left(\frac{1}{6}\right)^3 - \left(\frac{2}{6}\right)^3 - \left(\frac{3}{6}\right)^3 - \left(\frac{4}{6}\right)^3 - \left(\frac{5}{6}\right)^3 + 6\\ &= 6 - \frac{1 + 2^3 + 3^3 + 4^3 + 5^3}{6^3} = 6 - \frac{225}{6^3} = 6 - \frac{225}{216}\\ &= 6 - \frac{25}{24} = \frac{119}{24}.\end{aligned}$$

3(B)

$\boxed{\text{sim. seen} \Downarrow}$

(d) Consider $\mathsf{P}(Z_3 \geq i)$,

$$\begin{aligned}
\mathsf{P}(Z_3 \geq i) &= \mathsf{P}(\min\{X_1, X_2, X_3\} \geq i\} = \mathsf{P}((X_1 \geq i) \cap (X_2 \geq i) \cap (X_3 \geq i)) \\
&= \mathsf{P}(X_1 \geq i)\mathsf{P}(X_2 \geq i)\mathsf{P}(X_3 \geq i) \quad \text{from independence} \\
&= \mathsf{P}(X_1 \geq i)^3 \quad i = 1, 2, \ldots, 6 \text{ as } X_1, X_2 \text{ and } X_3 \text{ are identically distributed.}
\end{aligned}$$

$$\Rightarrow \mathsf{P}(Z_3 \geq 6) = \mathsf{P}(X_1 \geq 6)^3 = \mathsf{P}(X_1 = 6)^3 = \frac{1}{6^3}$$

$$\begin{aligned}
\mathsf{P}(Z_3 = i) &= \mathsf{P}(Z_3 \geq i) - \mathsf{P}(Z_3 \geq i+1),\ i = 1, 2, \ldots, 5 \\
&= \mathsf{P}(X_1 \geq i)^3 - \mathsf{P}(X_1 \geq i+1)^3 = \left(\frac{7-i}{6}\right)^3 - \left(\frac{6-i}{6}\right)^3.
\end{aligned}$$

So the pmf of Z_3 is

$$f_{Z_3}(i) = \begin{cases} \left(\frac{7-i}{6}\right)^3 - \left(\frac{6-i}{6}\right)^3, & i = 1, \ldots, 6; \\ 0, & \text{otherwise.} \end{cases}$$

$\boxed{3(\mathrm{C})}$

Note, by symmetry that this is the same as $f_{Y_n}(7-i)$.

(e) Direct extension of (a) gives $\mathsf{P}(Y_n \leq i) = \mathsf{P}(X_1 \leq i)^n$, and pmf is given by

$$f_{Y_n}(i) = \begin{cases} \left(\frac{i}{6}\right)^n - \left(\frac{i-1}{6}\right)^n, & i = 1, 2, \ldots 6; \\ 0, & \text{otherwise.} \end{cases}$$

$\boxed{\text{unseen} \Downarrow}$

$\boxed{2(\mathrm{C})}$

(f)

$$\begin{aligned}
\mathsf{P}(Q = 0) &= \mathsf{P}(Y_n - Z_n = 0) = \mathsf{P}(Y_n = Z_n) = \mathsf{P}(\max\{X_1, \ldots, X_n\} = \min\{X_1, \ldots, X_n\}) \\
&= \sum_{i=1}^{6} \mathsf{P}((\max\{X_1, \ldots, X_n\} = i) \cap (\min\{X_1, \ldots, X_n\} = i)) \\
&= \sum_{i=1}^{6} \mathsf{P}((X_1 = i) \cap (X_2 = i) \cap \ldots \cap (X_n = i)) \\
&= \sum_{i=1}^{6} \left(\frac{1}{6}\right)^n = \left(\frac{1}{6}\right)^{n-1}.
\end{aligned}$$

$\boxed{3(\mathrm{D})}$

Commentary: (a) they have seen the continuous version of this, so this should be straightforward. The question requires a good understanding of the concepts and (f) in particular is more challenging.

seen ⇓

3. (a) Properties of a valid pdf are:

1. $f_X(x) \geq 0$ for all x in the range of X.
2. $\int_{-\infty}^{\infty} f_X(x)\,\mathrm{d}x = 1$.

2(A)

(b) (i) Let $X = X_i,\ i = 1, 2, \ldots, n,$

$$M_X(t) = \mathsf{E}_{f_X}(\mathrm{e}^{tX}) = \int_{-\infty}^{\infty} \mathrm{e}^{tx} f_X(x)\ \mathrm{d}x$$
$$= \int_0^{\infty} \mathrm{e}^{tx}\lambda\mathrm{e}^{-\lambda x}\ \mathrm{d}x = \int_0^{\infty} \lambda\mathrm{e}^{-x(\lambda-t)}\ \mathrm{d}x$$
$$= \left[\frac{\lambda\mathrm{e}^{-x(\lambda-t)}}{-(\lambda-t)}\right]_0^{\infty} = \frac{\lambda}{\lambda-t}, \quad |t| < \lambda,$$

as required.

3(A)

(ii)

$$\mathsf{E}_{f_{X_i}}(X_i) = \int_{-\infty}^{\infty} x f_X(x)\ \mathrm{d}x = \int_0^{\infty} \lambda x\mathrm{e}^{-\lambda x}\ \mathrm{d}x$$
$$= \left[-x\mathrm{e}^{-\lambda x}\right]_0^{\infty} + \int_0^{\infty} \mathrm{e}^{-\lambda x}\ \mathrm{d}x = \left[\frac{-\mathrm{e}^{-\lambda x}}{\lambda}\right]_0^{\infty} = \lambda^{-1}, i = 1, \ldots, n.$$

3(A)

(iii) Range of Y is $(0, \infty)$.

sim. seen ⇓

$$F_Y(y) = \mathsf{P}(Y \leq y) = \mathsf{P}(\lambda X_1 \leq y) = \mathsf{P}\left(X_1 \leq \frac{y}{\lambda}\right)$$
$$= F_{X_1}\left(\frac{y}{\lambda}\right)$$
$$\Rightarrow f_Y(y) = \frac{1}{\lambda} f_{X_1}\left(\frac{y}{\lambda}\right) = \mathrm{e}^{-y},\ y > 0.$$

3(A)

So $Y \sim \mathit{Exponential}(1)$ and $\mathsf{E}_{f_Y}(Y) = 1 (= \int_0^{\infty} y\mathrm{e}^{-y}\ \mathrm{d}y)$. We have already shown that $\mathsf{E}_{f_{X_i}}(X_i) = \frac{1}{\lambda} = \lambda^{-1}\mathsf{E}_{f_Y}(Y)$ as required.

2(A)

(iv) Note $Y = \lambda X_i$, as the X_i are identically distributed.

unseen ⇓

$$\mathsf{E}_{f_{X_i}}(X_i^k) = \mathsf{E}_{f_Y}\left(\frac{Y^k}{\lambda^k}\right) = \frac{1}{\lambda^k}\mathsf{E}_{f_Y}(Y^k)$$
$$= \frac{1}{\lambda^k}\int_0^{\infty} y^k f_Y(y)\ \mathrm{d}y = \frac{1}{\lambda^k}\int_0^{\infty} y^k\mathrm{e}^{-\lambda}\ \mathrm{d}y$$
$$= \lambda^{-k}\Gamma(k+1), i = 1, \ldots, n.$$

as required.

3(D)

meth seen $\Downarrow$

(v) $A = \frac{1}{n}\sum_{i=1}^{n} X_i$, so the range of A is $(0, \infty)$.
Let $S = \sum_{i=1}^{n} X_i$, and the X_i are independent, we have that

$$M_S(t) = \prod_{i=1}^{n} M_{X_i}(X_i) = \left(\frac{\lambda}{\lambda - t}\right)^n$$

Which, from the uniqueness of the MGF we identify from the formula sheet as a $Gamma(n, \lambda)$ distribution.
Now $A = \frac{1}{n}S$, so, for $x > 0$,

$$F_A(x) = \mathsf{P}(A \leq x) = \mathsf{P}\left(\frac{1}{n}S \leq x\right) = \mathsf{P}(S \leq nx) = F_S(nx)$$
$$\Rightarrow f_A(x) = nf_S(nx) = \frac{n\lambda^n}{\Gamma(n)}(xn)^{n-1}\mathrm{e}^{-\lambda nx} = \frac{(\lambda n)^n}{\Gamma(n)}x^{n-1}\mathrm{e}^{-\lambda nx}, \quad x > 0.$$

4(D)

Which we identify as a $Gamma(n, n\lambda)$ distribution.

Commentary: (a) and (b)(i), (ii) and (iii) are basic and should be easy for those that have engaged with the course; (b)(iv) and (v) require a deeper understanding.

4. (a) For continuous random variables X and Y, we have, $\boxed{\text{seen} \Downarrow}$

$$\begin{aligned}
\mathsf{E}_{f_X}\left[\mathsf{E}_{f_{Y|X}}(Y \mid X)\right] &= \int_{-\infty}^{\infty} \mathsf{E}_{f_{Y|X}}(Y \mid X = x) f_X(x)\, \mathrm{d}x \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f_{Y|X}(y|x) f_X(x)\, \mathrm{d}y\, \mathrm{d}x \\
&= \int_{-\infty}^{\infty} y \int_{-\infty}^{\infty} \frac{f_{X,Y}(x,y)}{f_X(x)} f_X(x)\, \mathrm{d}x\, \mathrm{d}y \\
&= \int_{-\infty}^{\infty} y \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, \mathrm{d}x\, \mathrm{d}y \\
&= \int_{-\infty}^{\infty} y f_Y(y)\, \mathrm{d}y = \mathsf{E}_{f_Y}(Y).
\end{aligned}$$

$\boxed{3(A)}$

(b) (i) $\boxed{\text{sim. seen} \Downarrow}$

$$\int_{\infty}^{\infty} f_X(x)\, \mathrm{d}x = 1 \Rightarrow \int_0^1 kx^2(1-x^2)\, \mathrm{d}x = 1 \Rightarrow k\int_0^1 (x^2 - x^4)\, \mathrm{d}x = 1$$

$$\Rightarrow k\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = 1 \Rightarrow k\frac{2}{15} = 1 \Rightarrow k = \frac{15}{2}.$$

$\boxed{1(B)}$

(ii) $\boxed{\text{meth seen} \Downarrow}$

$$\begin{aligned}
\mathsf{E}_{f_X}(X) &= \int_{-\infty}^{\infty} x f_X(x)\, \mathrm{d}x = \frac{15}{2}\int_0^1 x(x^2 - x^4)\, \mathrm{d}x \\
&= \frac{15}{2}\int_0^1 (x^3 - x^5)\, \mathrm{d}x = \frac{15}{2}\left[\frac{x^4}{4} - \frac{x^6}{6}\right]_0^1 = \frac{15}{2}\cdot\frac{1}{12} = \frac{5}{8}.
\end{aligned}$$

$\boxed{2(B)}$

$$\begin{aligned}
\mathsf{var}_{f_X}(X) &= \mathsf{E}_{f_X}(X^2) - \mathsf{E}^2_{f_X}(X) \\
\mathsf{E}_{f_X}(X^2) &= \frac{15}{2}\int_{-\infty}^{\infty} x^2 f_X(x)\, \mathrm{d}x = \frac{15}{2}\int_0^1 x^2(x^2 - x^4)\, \mathrm{d}x \\
&= \frac{15}{2}\int_0^1 (x^4 - x^6)\, \mathrm{d}x = \frac{15}{2}\left[\frac{x^5}{5} - \frac{x^7}{7}\right]_0^1 = \frac{15}{2}\cdot\frac{2}{35} = \frac{3}{7} \\
\Rightarrow \mathsf{var}_{f_X}(X) &= \frac{3}{7} - \frac{25}{64} = \frac{17}{448}.
\end{aligned}$$

$\boxed{3(B)}$

(iii)

$$\begin{aligned}
\mathsf{E}_{f_{Y|X}}(Y \mid X = x) &= \int_{-\infty}^{\infty} y f_{Y|X}(y|x)\, \mathrm{d}y = \int_x^{2x} y\frac{3}{x^3}(2y - 3x)^2\, \mathrm{d}y \\
&= \frac{3}{x^3}\int_x^{2x} (4y^3 - 12xy^2 + 9x^2y)\, \mathrm{d}y = \frac{3}{x^3}\left[y^4 - 4xy^3 + \frac{9x^2y^2}{2}\right]_x^{2x} \\
&= \frac{3}{x^3}\left[(16x^4 - 32x^4 + 18x^4) - \left(x^4 - 4x^4 + \frac{9}{2}x^4\right)\right] \\
&= 3x\left(5 - \frac{9}{2}\right) = \frac{3x}{2}.
\end{aligned}$$

As expected as the distribution is symmetric about $3x/2$. $\boxed{4(C)}$

$\boxed{\text{meth seen} \Downarrow}$

(iv)

$$\mathsf{E}_{f_Y}(Y) = \mathsf{E}_{f_X}\left[\mathsf{E}_{f_{Y|X}}(Y \mid X)\right] = \int_0^1 \mathsf{E}_{f_{Y|X}}(Y \mid X = x) f_X(x)\ \mathrm{d}x$$
$$= \int_0^1 \frac{3x}{2}\frac{15}{2}(x^2 - x^4)\ \mathrm{d}x = \frac{45}{4}\int_0^1 (x^3 - x^5)\ \mathrm{d}x = \frac{45}{4}\left[\frac{x^4}{4} - \frac{x^6}{6}\right]_0^1$$
$$= \frac{45}{4}\left(\frac{1}{4} - \frac{1}{6}\right) = \frac{45}{48} = \frac{15}{16}.$$

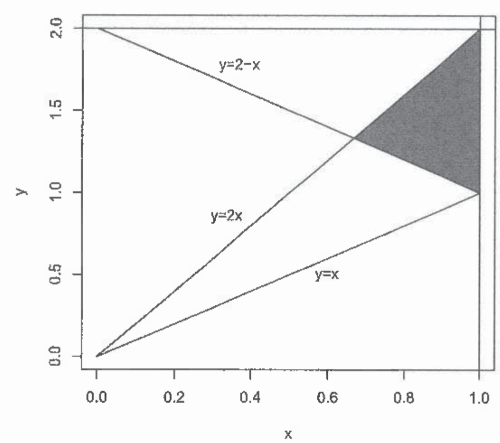
$\boxed{2(\text{D})}$

$\boxed{\text{unseen} \Downarrow}$

(v)

$$f_{X,Y}(x,y) = f_{Y|X}(y|x) f_X(x) = \frac{15}{2}x^2(1-x^2)\frac{3}{x^3}(2y-3x)^2$$
$$= \frac{45}{2x}(1-x^2)(2y-3x)^2, x \in (0,1), y \in (x, 2x).$$

$\boxed{2(\text{B})}$



(vi)

Shaded area shows $\mathsf{P}(X + Y \geq 2) = \mathsf{P}(Y \geq 2 - X)$. Note $y = 2 - x$ and $y = 2x$ intersect at $x = 2/3$.

$$\mathsf{P}(X + Y < 2) = 1 - \mathsf{P}(X + Y \geq 2) = 1 - \int_{2/3}^1 \int_{2-x}^{2x} f_{X,Y}(x,y)\ \mathrm{d}y\ \mathrm{d}x$$
$$\Rightarrow 1 - \mathsf{P}(X + Y < 2) = \int_{2/3}^1 \int_{2-x}^{2x} f_{X,Y}(x,y)\ \mathrm{d}y\ \mathrm{d}x$$

Hence $x_1 = 2/3, x_2 = 1, y_1 = 2 - x, y_2 = 2x$.

$\boxed{3(\text{D})}$

Commentary: (a) is bookwork; (b)(i), (ii) should be relatively straightforward for those that have engaged with the course; (b)(iii), (iv) requires an understanding of non-standard expectations; (b)(v) relies on basic definition; (b)(vi) is more challenging.