IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2011

EEE/ISE PART II: MEng, BEng and ACGI

CONTROL ENGINEERING

Monday, 13 June 2:00 pm

Time allowed: 2:00 hours

There are THREE questions on this paper.

Answer ALL questions.
Q1 carries 40% of the marks. Questions 2 and 3 carry equal marks (30% each).

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : I.M. Jaimoukha
Second Marker(s) : S. Evangelou

1. a) Figure 1.1 illustrates a mechanical system where two masses are connected to each other by a spring with a spring constant K, but are otherwise free. Assume that the masses slide on a frictionless surface with a force $u(t)$ applied on one of the masses as shown. Take $M_1 = M_2 = 1$ kg and $K = 1$ N/m.

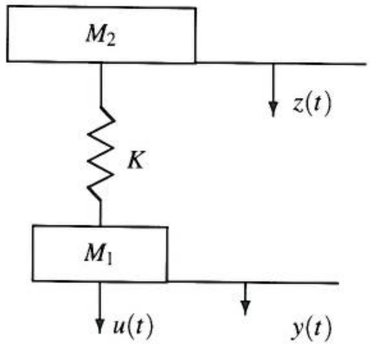


Figure 1.1

i) Derive the two balance of forces equations relating z, y, and u. [4]

ii) Determine the transfer function $G(s)$ relating $u(s)$ to $y(s)$. [4]

iii) Comment on the stability properties of $G(s)$. [4]

iv) Suppose that $u(t)$ is a unit step input. Determine the frequency of oscillation of the system. [4]

v) Derive a state-variable representation of the system $G(s)$. Take your states to be the displacements and their derivatives, the input to be the applied force and the output to be the displacement $y(t)$. [4]

b) In Figure 1.2 below, $G(s) = \dfrac{1}{s(s+1)^2}$ and $K > 0$ is a gain.

i) Determine the steady-state error for a unit step reference signal. [3]

ii) Use the Routh Hurwitz criterion to determine the range of values of K for closed-loop stability. [3]

iii) Determine the value of $K > 0$ for which the closed-loop is marginally stable. What is the resulting frequency of oscillations? [3]

iv) Sketch the locus of the closed–loop poles for $0 \leq K < \infty$. Determine the breakaway point and the imaginary-axis intercepts. [3]

v) Using the gain criterion, find the value of K for which the closed–loop is critically damped. What is the resulting time constant? [4]

vi) Give a brief qualitative description (in terms of stability and damping) of the closed-loop response of the dominant poles to a unit step reference signal as K tends from 0 to ∞. [4]

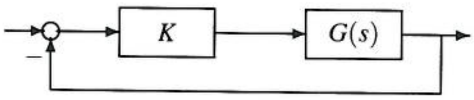


Figure 1.2

2. Let $G(s) = \dfrac{1}{s(s+4)^2}$ and consider the feedback loop shown in Figure 2.1 below.

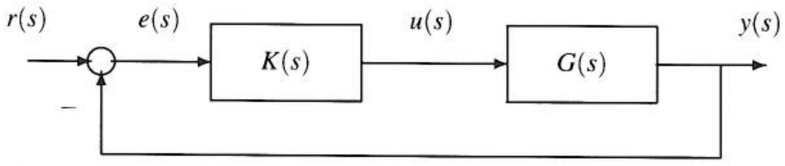


Figure 2.1

A PD compensator of the form $K(s) = k(s+z)$ where $k, z > 0$ is required such that the following specifications are satisfied

- The settling time (defined to be the first time beyond which the closed-loop step response is within 2% of its steady-state value) is 2 seconds.
- The damping ratio is given by $\zeta = 1/\sqrt{2}$.

a) Sketch the root locus of $G(s)$. Evaluate the breakaway point. [5]

b) Find the location of the closed-loop poles that achieves the design specifications above. [5]

c) Find the values of k and z that achieve the design specifications. Comment on the action of the compensator $K(s)$ on the system $G(s)$. [10]

d) Draw the root locus of the compensated system $G(s)K(s)$. [5]

e) Figure 2.2 illustrates an implementation of the PD compensator $K(s)$. Here, $C_i = 1\ \mu$F. Find the values of R_i and R_f. [5]

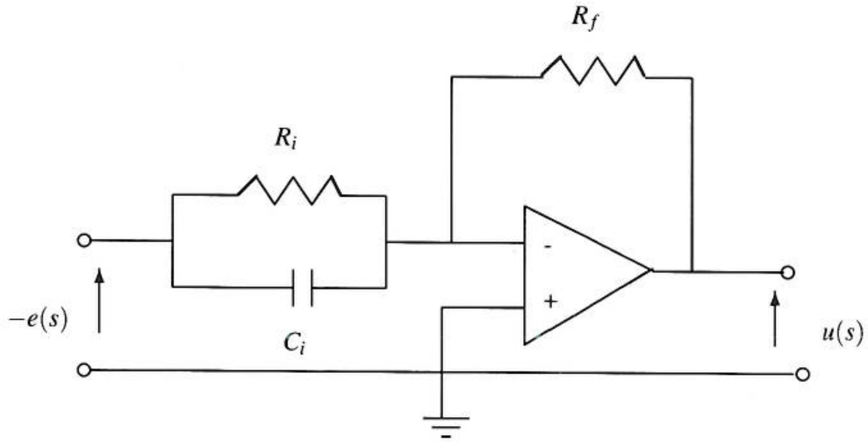


Figure 2.2

3. Consider the feedback control system in Figure 3.1 below.

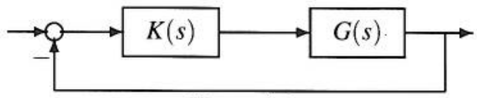


Figure 3.1

Here, $K(s)$ is the transfer function of a compensator while $G(s)$ is a stable transfer function with no finite zeros whose Bode plots are shown in Figure 3.2.

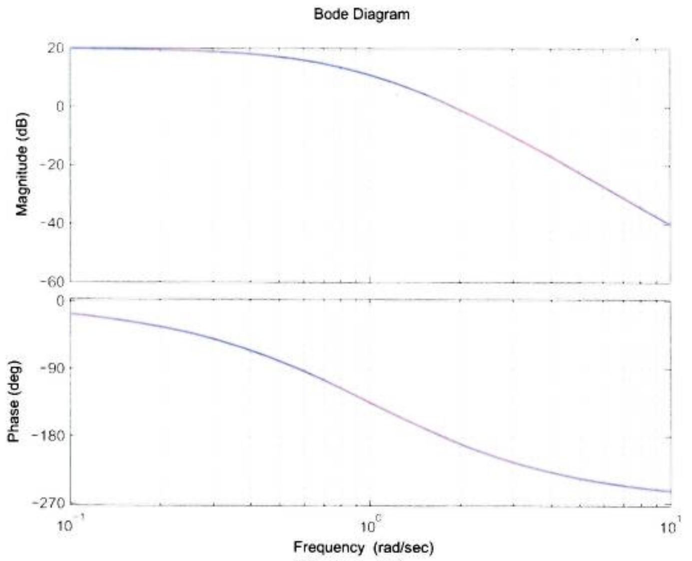


Figure 3.2

a) Use the Bode plots above to sketch a **rough** Nyquist diagram of $G(s)$, indicating the low and high frequency portions and the real–axis intercepts. [10]

b) Use the Nyquist stability criterion, which should be stated, to determine the number of unstable closed-loop poles when: (i)$K(s) = 1$, (ii) $K(s) = 0.1$.[10]

c) Let $K(s)$ have the Bode plots in Figure 3.3. Describe $K(s)$ briefly and indicate its effects on the performance and stability of the feedback loop. [10]

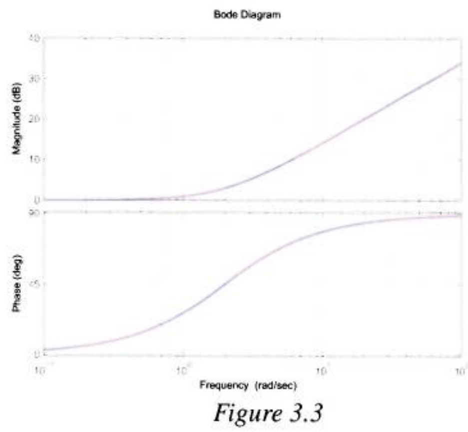


Figure 3.3

SOLUTIONS: Control Engineering 2011

1. a) i) The two balance of forces equations are given by

$$\ddot{z}(t) + (z(t) - y(t)) = 0, \qquad \ddot{y}(t) + (y(t) - z(t)) = u(t)$$

ii) By taking Laplace transforms and eliminating $z(s)$ we get

$$G(s) = \frac{s^2+1}{s^2(s^2+2)}$$

iii) Since the poles are on the imaginary-axis, $G(s)$ is marginally stable.

iv) Since $G(s)$ has poles at $\pm j\sqrt{2}$, the frequency of oscillations is $\sqrt{2}$ rad/s.

v) Let $x_1 = y(t)$, $x_2 = \dot{y}$, $x_3 = z$ and $x_4 = \dot{z}$. Then a state variable representation is given as

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u(t), \quad y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x(t)$$

b) i) Since the system is type 1, the steady-state value of the error signal for a unit step reference signal is zero.

ii) The characteristic equation for the closed-loop is

$$1 + KG(s) = 1 + \frac{K}{s(s+1)^2} = 0 \Rightarrow s^3 + 2s^2 + s + K = 0$$

The Routh array is:

$$\begin{array}{c|cc} s^3 & 1 & 1 \\ s^2 & 2 & K \\ s & 0.5(2-K) & \\ 1 & K & \end{array}$$

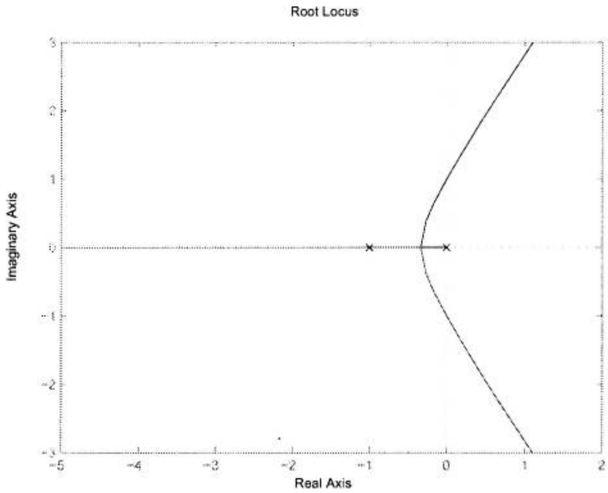
For stability we need the first column to be positive, so $0 < K < 2$.

iii) When $K = 2$ the third row is zero and so the closed-loop is marginally stable. The auxiliary equation is given by $2(s^2+1) = 0$ and so the resulting frequency of oscillations is 1 rad/s.

iv) The root-locus is shown below.

The breakaway point is $-1/3$ and the imaginary-axis intercepts are $\pm j$.

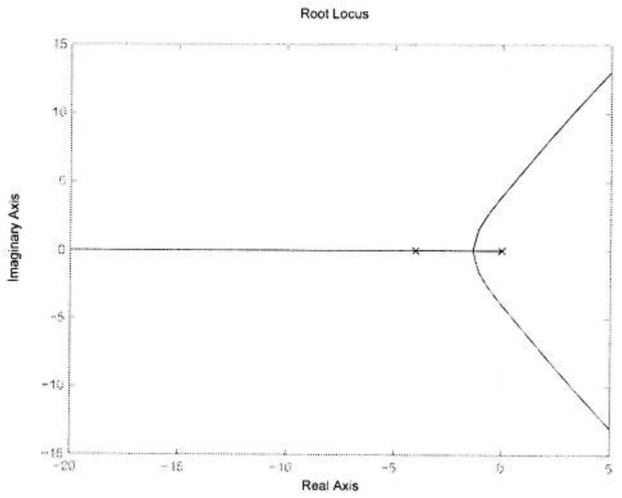
v) The closed–loop is critically damped when the poles are at the breakaway point. The gain criterion gives $K = 4/27$. The resulting time constant is the negative of the inverse of the pole and so is 3 s.



vi) The response can be characterised as follows:

- For $K = 0$, the response is marginally stable.
- For $0 < K < 4/27$, the response is stable and overdamped (non-oscillatory).
- When $K = 4/27$, the response is critically damped.
- For $4/27 < K < 2$ the response is underdamped (oscillatory).
- For $K = 2$, the response is marginally stable.
- For $K > 2$, the response is unstable.

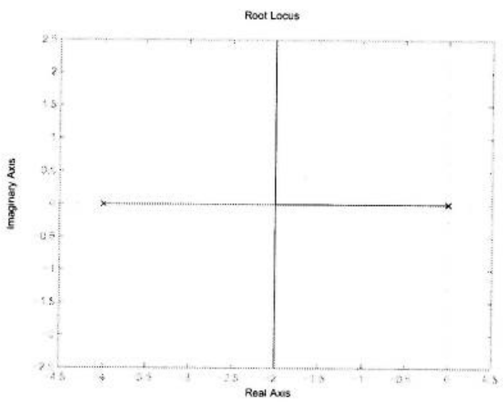
2. a) The root-locus is shown below. The breakaway point can be found by differentiating $G(s)$ and setting to zero and is given by -4/3.



b) Denote the pole as $p = \sigma + j\omega$. Then since $T_s = -4/\sigma$, $\sigma = -2$. Since a value of $\zeta = 1/\sqrt{2}$ implies that $\omega = -\sigma$ then the location of the pole is $p = -2 + j2$.

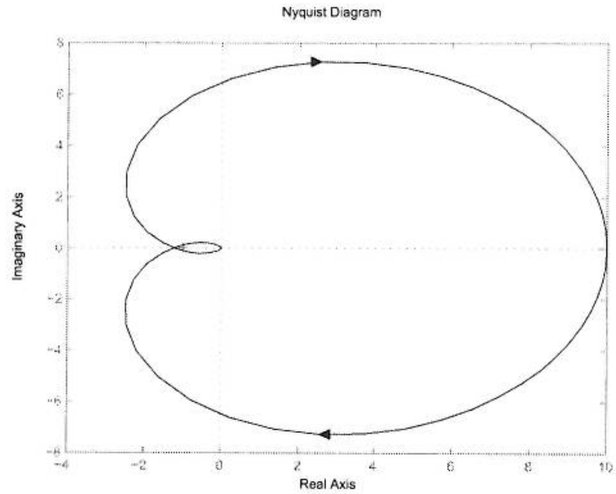
c) We use the angle criterion to obtain z. The angle from the required zero is given by $\theta = 135° + 2 \times 45° - 180° = 45°$. Therefore the required zero is at -4 and so $z = 4$. To find k we use the gain criterion which gives $k = 8$. Note that the compensator has cancelled one of the poles of $G(s)$.

d) The compensated system is $K(s)G(s) = 8/s(s+4)$. The root-locus is given below.



e) The transfer function from $-e(s)$ to $u(s)$ in the figure is given by $R_f C_i (s + \frac{1}{R_i C_i})$ which shows that $R_i C_i = 0.25$ and $R_f C_i = 8$. Since $C_i = 1\ \mu$F, it follows that $R_i = 0.25$ MΩ and $R_f = 8$ MΩ.

3. a) The Nyquist diagram is shown below. The low and high frequency real axis intercepts are at 10 (since $20\log_{10}(10) = 20$) and zero, respectively. The mid-frequency intercept is just to the left of the point $-1 + j0$ since the gain when the phase is 180° is just above 0db. The high frequency approach to the origin is at $-270°$ from the Bode plots.



b) When $K(s) = K$, we have $N = Z - P$, where N is the number of clockwise encirclements by the Nyquist diagram of the point $-K^{-1}$, P is the number of unstable open–loop poles and Z is the number of unstable closed–loop poles. Since $G(s)$ is assumed to be stable, $P = 0$.

i) When $K = 1$ then $N = 2$ from the Nyquist diagram and therefore $Z = P + N = 2$ so that the closed-loop has two unstable poles.

ii) When $K = 0.1$ then $N = 0$ from the Nyquist diagram and therefore $Z = P + N = 0$ so that the closed-loop is stable.

c) The bode plot is that of a proportional-plus-derivative compensator $K(s) = \frac{1}{\omega_0}(s + \omega_0)$ where $\omega_0 > 0$. It has gain close to unity for frequencies ω below ω_0 and increases as $\omega \to \infty$. The phase is positive and increases between 0 and 90° as ω increases. The increase in gain at frequencies above ω_0 tends to degrade the stability margins as well as the noise attenuation properties, while the phase-lead tends to increase the phase margin, which is stabilising.