DTS AND COMPUTER CONTROL

1. a) The characteristic polynomial of the continuous-time closed-loop system is (note the pole-zero cancellation between $P(s)$ and $C(s)$)

$$s^2+6s+20.$$

By Routh test, all roots of the above polynomial are in the left half of the complex plane, hence the continuous-time system is asymptotically stable. The natural angular frequency is $\omega_n = \sqrt{20}$ and the damping coefficient is $\xi = \frac{3}{10}\sqrt{5}$. The angular frequency of the damped oscillations is $\omega_d = \sqrt{11}$, yielding $T_d = \frac{2\pi}{\omega_d} = 1.89$ [3 marks]

Typical mistakes include the incorrect computation of the characteristic polynomial or of the indicated parameters.

b) Select $T = 0.2$. [1 marks]

Hard to make a mistake here ...., although some managed to propose unreasonable sampling times.

c) The term approximating the hold is described by

$$H_a(s) = \frac{1}{\frac{T}{2}s+1} = \frac{1}{0.1s+1}.$$

The characteristic polynomial of the resulting continuous-time closed-loop system is

$$s^3+16s^2+60s+200.$$

Again, a direct application of Routh test shows that all roots of the above polynomial are in the left half of the complex plane: the closed-loop system is asymptotically stable. [4 marks]

Typical mistakes include the use of a delay to approximate the hold, the incorrect computation of the characteristic polynomial and/or the incorrect application of Routh test.

d) The equivalent discrete-time model is

$$HP(z) = \frac{z-1}{z}Z\left(\frac{P(s)}{s}\right) = \frac{z-1}{z}Z\left(\frac{1}{2}\frac{1}{s^2} - \frac{1}{4}\frac{1}{s} + \frac{1}{4}\frac{1}{s+2}\right) = \frac{0.0175z+0.0153}{(z-1)(z-0.6703)}.$$

[4 marks]

Typical mistakes include errors in the definition of the equivalent transfer function, in the computation of the residuals and/or of the Z-transforms.

e) Using the pole-zero correspondence method yields

$$C(z) = k\,\frac{z-0.6703}{z-0.3011},$$

with $k = 14.13$. Note that k has been selected to match the DC gain of the continuous-time controller. [2 marks]

Incorrect computation of the DC gain.

f) The closed-loop transfer function from $R(z)$ to $Y(z)$ is (note the pole-zero cancellation between $HP(z)$ and $C(z)$)

$$\frac{14.13(0.0175z + 0.0153)}{z^2 - 1.052z + 0.518}.$$

The roots of the denominator polynomial are

$$z_{1,2} = 0.5263 \pm 0.4914j$$

and their modulo is 0.7201, that is the roots are inside the unity disk. As a result, the closed-loop system is asymptotically stable. [6 marks]

Some students have made mistakes in the computation of the transfer function. Note that the resulting transfer function should have two poles, which simplifies the stability analysis. If higher order transfer functions are obtained, then the stability analysis is much more challenging.

2. a) The equivalent discrete-time model is

$$HP(z) = \frac{z-1}{z} Z\left(\frac{P(s)}{s}\right) = \frac{z-1}{z} Z\left(\frac{1}{s^2} - \frac{10}{s} + \frac{10}{s+0.1}\right) = \frac{0.0483z + 0.0467}{(z-1)(z-0.9048)}.$$

[4 marks]

Typical mistakes include errors in the definition of the equivalent transfer function, in the computation of the residuals and/or of the Z-transforms.

b) The transfer function of the controller is

$$C(z) = K\,\frac{z-0.88}{z+0.5}.$$

[4 marks]

Some students have been unable to obtain the controller transfer function from its time-domain equation.

c) The characteristic polynomial of the closed-loop system is

$$z^3 + (0.0483K - 1.404)z^2 + (0.0042K - 0.0475)z + (-0.0411K + 0.4524).$$

Using the transformation

$$z = \frac{1+w}{1-w}$$

yields the polynomial (after a change of sign)

$$(1.904 - 0.0029K)w^3 + (5.809 - 0.1761K)w^2 + (0.285 + 0.1676K)w + (0.0114).$$

Applying Routh test shows that the roots of the above polynomial are in the left half of the complex plane for all $K \in (0, 32.3236)$. Hence, the discrete-time system is asymtotically stable for all K in the same region. [6 marks]

Some students have made mistakes in the computation of the characteristic polynomial and/or in the bilinear transformation. Some may have also applied Routh test incorrectly.

d) To have a pole at $z = 0$ one has to select k such that $-0.0411K + 0.4524 = 0$, that is $K \approx 10.9$. For this value of K the characteristic polynomial of the closed-loop system is

$$z(z + 0.00139)(z - 0.87469).$$

The slowest mode is $\bar{z} = 0.87469$, which in the time-domain is $\bar{s} = -0.06$, yielding a time-constant of $1/\bar{s} = 14.96$ s (recall that the sampling time is $T = 1$).

[6 marks]

Some students have not recognized that to have a pole at zero it is sufficient (and necessary) that the term of degree zero of the characteristic polynomial be identically zero. This gives a simple equation for K. Time constants have also been erroneously evaluated.

3. a) The equivalent discrete-time model is

$$HP(z) = \frac{z-1}{z} Z\left(\frac{P(s)}{s}\right) = \frac{z-1}{z} Z\left(\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}\right) = \frac{0.0187z + 0.0175}{(z-1)(z-0.8187)}.$$

[4 marks]

Typical mistakes include errors in the definition of the equivalent transfer function, in the computation of the residuals and/or of the Z-transforms.

b) The transfer function in the w plane is (recall that $T = 0.2$)

$$HP(w) = HP(z)_{z=\left(\frac{1+0.1w}{1-0.1w}\right)} = -0.000332\frac{(w+300.19)(w-10)}{w(w+0.9966)}.$$

[4 marks]

Some students may have incorrectly applied the transformation to the w-plane and/or made mistakes in the computations.

c) One has to consider the unity feedback interconnection of the transfer function $C(w)HP(w)$.

i) Note that

$$K_v = k,$$

hence $k = 2$. The resulting closed-loop characteristic polynomial is

$$1.9933a + (0.8039a + 1.9933)w + (1.3315a - 0.1926)w^2 + (0.3333a - 0.00066)w^3.$$

Application of the Routh test shows that for all $a > 0.1499$ the roots of the above polynomial are in the left half of the complex plane. One could select, for example, $a = 1$ yielding the controller

$$C(w) = 2\frac{1+w}{1+w/3}.$$

[6 marks]

Typical mistakes include errors in the definition of the controller gain and/or in the computation of the characteristic polynomial. Selections of the parameter a have also been not always reasonable/motivated.

ii) The discrete-time controller is

$$C(z) = C(w)_{w=10\left(\frac{z-1}{z+1}\right)} = 2\frac{11z-9}{4.3333z-2.3333}.$$

The characteristic polynomial of the resulting discrete-time closed-loop system is

$$z^3 - 2.262z^2 + 1.8092z - 0.5136 = (z^2 - 1.444z + 0.6279)(z - 0.8179),$$

which has all roots inside the unity disk: the discrete-time closed-loop system is therefore asymptotically stable. [6 marks]

Typical mistakes include errors in the transformation into the z variable and/or in the computation of the characteristic polynomial, which may have rendered the stability analysis "messy" and time consuming.

Similar conclusions can be drawn for all values of $a > 0.1499$. Interestingly, for $a = 3$ the controller $C(z)$ simplifies to $C(z) = 2$ and for $a = 10$ the controller $C(z)$ has a zero at $z = 0$.

4. a) The equivalent discrete-time model is

$$HP(z) = \frac{z-1}{z} Z\left(\frac{P(s)}{s}\right) = \frac{z-1}{z} Z\left(\frac{1}{s-1} - \frac{1}{s}\right) = \frac{e-1}{(z-e)}.$$

[4 marks]

Typical mistakes include errors in the definition of the equivalent transfer function, in the computation of the residuals and/or of the Z-transforms.

b) The simplest possible controller is $C(z) = k$. The characteristic polynomial of the resulting closed-loop system is

$$z - e + k(e-1),$$

yielding (to assign the pole at $z = 0$)

$$k = \frac{e}{e-1} \approx 1.5819.$$

[4 marks]

Some students have not understood that a P-controller solves the problem.

c) To have a system of Type 1 one has to add a pole at $z = 1$ in the controller. The simplest controller achieving the desired objectives is described by

$$C(z) = \frac{az+b}{z-1},$$

yielding a closed-loop system with characteristic polynomial

$$z^2 + (-1 - e - a + ae)z + (e - b + eb).$$

Selecting

$$a = \frac{e+1}{e-1} \approx 2.163 \qquad b = -\frac{e}{e-1} \approx -1.5819$$

places all closed-loop poles at $z = 0$. [6 marks]

As above, some students have not recongnized that the simplest controller has to have a pole at $z = 1$ and a variable gain and zero.

d) The approximate controller is described by

$$C(z) = \frac{2z-1}{z-1}.$$

The characteristic polynomial of the resulting closed-loop system is

$$z^2 + (e-3)z + 1.$$

The roots of this polynomial are

$$z_{1,2} = 0.1408 \pm 0.99j,$$

and these have modulo equal to 1, that is the approximate controller places the closed-loop poles on the unity disk. One could alternatively use the bilinear transformation to check that the transformed roots are on the imaginary axis of the w-plane. This exercise highlights that approximating the coefficients of a discrete-time controller may significantly modify the behaviour of the closed-loop system. [6 marks]

All sort of approximations have been implemented, yielding erroneous conclusions.