EE208A MATHEMATICS

1. a) Show that the function $u(x,y) = 2\cos x\cosh y + \sin x\sinh y$ satisfies Laplace's equation and integrate the Cauchy-Riemann equations to find its harmonic conjugate $v(x,y)$. [5]

SOLUTION

Obtain the derivatives

$$u_x = -2\sin x\cosh y + \cos x\sinh y \Rightarrow u_{xx} = -2\cos x\cosh y - \sin x\sinh y$$

and

$$u_y = 2\cos x\sinh y + \sin x\cosh y \Rightarrow u_{yy} = 2\cos x\cosh y + \sin x\sinh y$$

so that

$$u_{xx} + u_{yy} = -2\cos x\cosh y - \sin x\sinh + 2\cos x\cosh y + \sin x\sinh y = 0$$

and so $u(x,y)$ is a solution of Laplace's equation and a complex conjugate $v(x,y)$ exists.

To find v, we solve the Cauchy-Riemann equations:

$$u_x = v_y \Rightarrow v = \int u_x\,dy = \int -2\sin x\cosh y + \cos x\sinh y\,dy = -2\sin x\sinh y + \cos x\cosh y + A(x)$$

and

$$u_y = -v_x \Rightarrow v = -\int u_y\,dx = -\int 2\cos x\sinh y + \sin x\cosh y\,dx = -2\sin x\sinh y + \cos x\cosh y + B(y)$$

so the arbitrary functions $A(x)$ and $B(y)$ are constants and

$$v(x,y) = -2\sin x\sinh y + \cos x\cosh y + C.$$

b) Hence obtain $f(z) = u(x,y) + iv(x,y)$ where f is an analytic function of $z = x + iy$, simplifying as much as possible. [4]

SOLUTION

Write down $f = u + iv$:

$$f(z) = 2\cos x\cosh y + \sin x\sinh y - 2i\sin x\sinh y + i\cos x\cosh y + iC$$

and noting the symmetry, we rearrange:

$$f(z) = (2+i)\cos x\cosh y + (1-2i)\sin x\sinh y = (2+i)\cos x\cosh y - (2+i)i\sin x\sinh y + iC$$

and the next steps are clear:

$$f(z) = (2+i)[\cos x\cos(iy) - \sin x\sin(iy)] + iC = (2+i)\cos(x+iy)\,,$$

so $f(z) = (2+i)\cos z + iC$.

2. a) The complex function

$$F(z) = \frac{1}{z(z^2+1)}$$

has three simple poles. Find the residues at the poles lying in the upper half of the complex plane and at the origin. [4]

SOLUTION Poles are at $z = 0, \pm i$.

Residue at $z = 0$:

$$\lim_{z\to 0} z\frac{1}{z(z^2+1)} = \lim_{z\to 0} \frac{1}{(z^2+1)} = 1$$

Residue at $z = i$:

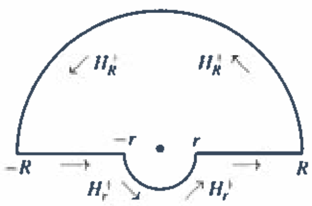
$$\lim_{z\to i} (z-i)\frac{1}{z(z^2+i)} = \lim_{z\to i} \frac{1}{z(z+i)} = \frac{1}{i(2i)} = -\frac{1}{2}$$

b) Consider the contour integral $I = \oint_C \frac{1}{z(z^2+1)}\,dz$,

where the closed contour C is taken to be the union of a semi-circle of radius R, lying in the upper half-plane, with a small semi-circle of radius r indented into the lower half-plane, both centred at $z = 0$ and the real intervals $[-R,-r]$ and $[r,R]$.

SOLUTION

Always useful to draw C:



i) Show that the contribution to I from the indented semi-circle of radius r, in the limit $r \to 0$, is $i\pi$.

SOLUTION

For the small semicircle we have

$$I_r = \int_{H_r} f(z)\, dz$$

Let $z = re^{i\theta}$, where $\pi \leq \theta \leq 2\pi$ - we are moving anticlockwise from $(-r,0)$ to $(r,0)$. Then $dz = ire^{i\theta}$. Substituting:

$$I_r = \int_{\pi}^{2\pi} \frac{i\, re^{i\theta}\, d\theta}{re^{i\theta}(1 + r^2 e^{i2\theta})}$$

$$\lim_{r \to 0} I_r = i \int_{\pi}^{2\pi} \frac{1}{1+0}\, d\theta = i\pi$$

The denominator simplifies given that $r^2 e^{i2\theta} \to 0$ as $r \to 0$. [3]

ii) Use Jordan's lemma to show that the contribution to I from the arc of the larger semi-circle, in the limit $R \to \infty$, is zero.

SOLUTION Since:

- the only singularities of the function are poles
- $m = 0$ but
- $\lim_{R \to \infty} |F(z)| = \left| \frac{1}{z(z^2+1)} \right| \to 0$ faster than $|1/z|$;

$\lim_{R \to \infty} \int_{H_R} f(z)\, dz = 0$ as conditions hold for Jordan's lemma. [3]

iii) Hence use your results from (a) and the Residue Theorem to obtain

$$\int_{-\infty}^{\infty} \frac{1}{x(x^2+1)}\, dx,$$

SOLUTION

Using the Residue Theorem, $I = 2\pi i(\text{sum of residues inside } C)$,

$$2\pi i\left(1 - \frac{1}{2}\right) = \oint_C f(z)\, dz = \int_{H_R} f(z)\, dz + \int_{-R}^{-r} f(z)\, dz + \int_{H_r} f(z)\, dz + \int_r^R f(z)\, dz$$

Taking the limits as $r \to 0$ and $R \to \infty$ does not affect the value of I obtained using the residue theorem, and the RHS simplifies to

$$\pi i = \pi i + 0 + \int_{-\infty}^{\infty} \frac{1}{x(x^2+1)} dx$$

and so the answer is zero. This can be confirmed by observing that we are taking a symmetric integral of an odd function. [4]

3. a) Given the integral of the real variable θ,

$$I = -\int_0^{2\pi} \sin[\cos(\theta) - \theta] e^{-\sin(\theta)} \, d\theta \, .$$

use the substitution $z = e^{i\theta}$ to show that I is equal to the real part of the complex contour integral

$$\oint_C \frac{e^{iz}}{z^2} \, dz,$$

where the contour C is the unit circle in the complex plane. [5]

SOLUTION

Using the substitution $z = e^{i\theta} \Rightarrow dz = ie^{i\theta} \, d\theta$ where $\theta = 0 \ldots 2\pi$ traces the unit circle, the contour C, so we can substitute as follows:

$$\oint_C \frac{e^{iz}}{z^2} \, dz = \int_0^{2\pi} \frac{e^{i(e^{i\theta})} ie^{i\theta} \, d\theta}{(e^{i\theta})^2} = i \int_0^{2\pi} e^{i(e^{i\theta} - \theta)} \, d\theta = i \int_0^{2\pi} e^{i[\cos(\theta) + i\sin(\theta) - \theta]} \, d\theta$$

$$= i \int_0^{2\pi} e^{i[\cos(\theta) - \theta]} e^{-\sin\theta} \, d\theta = i \int_0^{2\pi} e^{-\sin\theta} \{\cos[\cos(\theta) - \theta] + i\sin[\cos(\theta) - \theta]\} \, d\theta$$

and multiplying by i we take the real part to obtain the required result.

b) Using Cauchy's residue theorem, or otherwise, calculate I. [4]

SOLUTION The function $\frac{e^{iz}}{z^2}$ has a double pole at $z = 0$, so the residue is

$$\lim_{z \to 0} \frac{d}{dz} \left[(z-0)^2 f(z) \right] = \lim_{z \to 0} \frac{d}{dz} \left(e^{iz} \right) = \lim_{z \to 0} ie^{iz} = i \, .$$

So by Cauchy's Residue theorem,

$$\oint_C \frac{e^{iz}}{z^2} \, dz = 2\pi i (\text{Sum of Residues inside } C) = 2\pi i (i) = -2\pi \, .$$

Hence I is the real part of the above, which is real anyway, so $I = -2\pi$.

4. Consider the following second-order ODE

$$\frac{d^2 y}{dx^2} + 6\frac{dy}{dx} + 13y = f(x)$$

for some input function $f(x)$ and intial conditions $y(0) = y'(0) = 0$.

a) Take Laplace transforms to write the ODE in the form [3]

$$\bar{y}(s) = \bar{h}(s)\bar{f}(s)$$

SOLUTION Taking Laplace transforms of both sides, including the zero initial conditions we have

$$s^2\bar{y} + 6s\bar{y} + 13\bar{y} = \bar{f}(s) \Rightarrow \bar{y} = \frac{1}{s^2+6s+13}\bar{f}(s)$$

so that $\bar{h}(s) = \dfrac{1}{s^2+6s+13}$.

b) Hence use the Laplace convolution and shift theorems to write the solution in the form

$$y(x) = h(x) \star f(x)$$

,

where $h \star f$ is the convolution of $f(x)$ and $h(x)$, and $\mathscr{L}[h(x)] = \bar{h}(s)$. [5]

SOLUTION The convolution theorem states that $\mathscr{L}[h(x) * f(x)] = \bar{h}(s)\bar{f}(s)$, so we first find

$$h(x) = \mathscr{L}^{-1}[\bar{h}(s)] = \mathscr{L}^{-1}\left[\frac{1}{(s+3)^2+4}\right] = e^{-3x}\frac{1}{2}\mathscr{L}^{-1}\left[\frac{2}{s^2+2^2}\right] = \frac{1}{2}e^{-3x}\sin(2x)$$

using the table of transforms and the first shift-theorem.

Hence the solution $y(x)$ is the convolution of $h(x)$ and $f(x)$:

$$y(x) = \frac{1}{2}\int_0^x e^{-3u}\sin(2u)f(x-u)\,du.$$

[The other choice of convolution integral works equally well in (c).]

c) If $f(x) = e^{-3x}$, obtain the solution $y(x)$ by solving the integral found in part (b). [4]

SOLUTION Taking f as given, the integral simplifies:

$$y(x) = \frac{1}{2}\int_0^x e^{-3u}\sin(2u)e^{-3(x-u)}\,du = \frac{1}{2}e^{-3x}\int_0^x \sin(2u)\,du = \frac{1}{2}e^{-3x}\left[-\frac{\cos 2u}{2}\right]_0^x = \frac{1}{4}e^{-3x}(1-\cos 2x).$$

d) With $f(x) = e^{-3x}$, take Laplace transforms of the ODE and use partial fractions and the shift theorem to take the inverse Laplace transform and find $y(x)$, and thus confirm the result obtained in (c). [6]

SOLUTION

Taking Laplace transforms:

$$s^2\bar{y} + 6s\bar{y} + 13\bar{y} = \frac{1}{s+3} \Rightarrow \bar{y} = \frac{1}{(s+3)(s^2+6s+13)} = \frac{As+B}{s^2+6s+13} + \frac{C}{s+3}$$

Partical fractions gives $A = -1/4, B = -3/4, C = 1/4$, so

$$4\bar{y} = \frac{-s-3}{s^2+6s+13} + \frac{1}{s+3} = -\frac{s+3}{(s+3)^2+4} + \frac{1}{s+3}$$

and taking inverse Laplace transforms we get

$$4y(x) = -\mathscr{L}^{-1}\left[\frac{s+3}{(s+3)^2+4}\right] + \mathscr{L}^{-1}\left[\frac{1}{s+3}\right] = -e^{-3x}\mathscr{L}^{-1}\left[\frac{s}{s^2+2^2}\right] + e^{-3x}$$

where we have used the first shift-theorem. Finally, the last transform is a cosine and

$$4y(x) = e^{-3x} - e^{-3x}\cos(2x),$$

so $y = \frac{1}{4}e^{-3x}(1 - \cos 2x)$, as before.