E3.10
C1.5
ISE3.7

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2008

MSc and EEE/ISE PART III/IV: MEng, BEng and ACGI

MATHEMATICS FOR SIGNALS AND SYSTEMS

Wednesday, 30 April 10:00 am

Corrected Copy Q.4

Time allowed: 3:00 hours

There are FIVE questions on this paper.

Answer THREE questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s) :	M.M. Draief
	Second Marker(s) :	D. Angeli

MATHEMATICS FOR SIGNAL AND SYSTEMS

1. Consider the space $\mathcal{M}_3(\mathbb{C})$ of three-by-three matrices with complex entries. Let $A = (a_{ij})_{i,j=1,2,3} \in \mathcal{M}_3(\mathbb{C})$. We define the following functions

- for $k = 1,2,3$, let $l_k(A) = a_{k1} + a_{k2} + a_{k3}$,
- for $k = 1,2,3$, let $c_k(A) = a_{1k} + a_{2k} + a_{3k}$,
- let $\mathrm{tr}(A) = a_{11} + a_{22} + a_{33}$,
- and let $\mathrm{anti}(A) = a_{31} + a_{22} + a_{13}$,

a) Let $\mathcal{M}$ be the set of matrices $A \in \mathcal{M}_3(\mathbb{C})$, such that $l_k(A) = c_j(A)$ for $k, j = 1,2,3$. For $A \in \mathcal{M}$ we define

$$\alpha(A) = l_1(A) = l_2(A) = l_3(A) = c_1(A) = c_2(A) = c_3(A)$$

the common value.

(i) Give an example of a matrix $A \in \mathcal{M}$ such that $\alpha(A) = 0$ [2]

(ii) Give an example of a matrix $A \in \mathcal{M}$ such that $\alpha(A) = 1$ [2]

(iii) Show that $\mathcal{M}$ is a subspace of $\mathcal{M}_3(\mathbb{C})$ and that α is a linear operator on $\mathcal{M}$. [2]

(iv) Let $J = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. For $\lambda \in \mathbb{C}$, show that if A is such that

$$AJ = JA = \lambda J$$

then A is in $\mathcal{M}$ and that if $A \in \mathcal{M}$ then it satisfies $AJ = JA = \lambda J$. [6]

b) Let $\mathcal{M}^0 = \{A \in \mathcal{M}, \alpha(A) = \mathrm{tr}(A) = \mathrm{anti}(A)\}$.

(i) Prove that $\mathcal{M}^0$ is a vector space. [2]

(ii) Let $G = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ -1 & 2 & -1 \end{pmatrix}$ and G^T its transpose. Show that (G, G^T, J) is a basis of $\mathcal{M}^0$. What is the dimension of $\mathcal{M}^0$? [6]

2. Let $\mathbb{R}[X]$ be the vector space of all polynomials with real coefficients. We define the following function that, given two polynomials $P, Q \in \mathbb{R}[X]$, it associates the following number

$$\langle P, Q\rangle = \int_{-1}^{1} \frac{P(t)Q(t)}{\sqrt{1-t^2}} dt \, .$$

a) Show that the above function defines an inner product. [2]

b) Prove that, for any positive integer n, there exists a unique polynomial T_n such that: for every $\theta \in \mathbb{R}$, $T_n(\cos(\theta)) = \cos(n\theta)$. [2]

c) Show that the polynomials T_n, known as Chebychev polynomials, satisfy, for $n \geq 1$

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \, .$$

[4]

d) Prove that the sequence $(T_n)_n$ is orthogonal and compute $\langle T_n, T_n\rangle$. [6]

e) Show that T_n statisfies the following differential equation

$$(1-x)^2 y'' - xy' = -n^2 y$$

[6]

3. Consider the space $\mathcal{M}_n(\mathbb{R})$ of n-by-n matrices with real entries. We define the inner product $(A, B) = \mathrm{tr}(A^T B)$, where A^T is the transpose of A.

a) Check that the above product is indeed an inner product and give the expression of the corresponding norm. [4]

b) Let

$$\mathcal{S}_n = \{A \in \mathcal{M}_n(\mathbb{R}), A = A^T\}$$

the set of symmetric matrices and

$$\mathcal{A}_n = \{A \in \mathcal{M}_n(\mathbb{R}), A = -A^T\}$$

the set of anti-symmetric matrices.

(i) Show that $\mathcal{S}_n$ and $\mathcal{A}_n$ are two vector spaces and give their dimensions.

[4]

(ii) For $M \in \mathcal{M}_n(\mathbb{R})$ show that $M + M^T$ is an element of $\mathcal{S}_n$ and $M - M^T$ is an element of $\mathcal{A}_n$. [3]

(iii) Show that $\mathcal{S}_n$ is orthogonal to $\mathcal{A}_n$ and that any matrix $M \in \mathcal{M}_n(\mathbb{R})$ can be decomposed in a unique way as $M = M_S + M_A$ where $M_S \in \mathcal{S}_n$ and $M_A \in \mathcal{A}_n$. [6]

(iv) Determine the orthogonal projection on $\mathcal{S}_n$ and $\mathcal{A}_n$. [3]

4. Let H be *Hilbert (vector) space* with the inner product $(.,.)$ and the associated norm $||x||^2 = (x,x)$, $x \in H$.

a) Prove the *parallelogram identity*

$$||x+y||^2 + ||x-y||^2 = 2(||x||^2 + ||y||^2)$$

[5]

b) Let M be a closed subspace of H and x_0 a vector such that $x_0 \notin M$. We define $\delta = \inf\{||x_0 - y||, y \in M\}$, i.e. there exists a sequence $(y_n)_n$ of elements of M such that $\delta_n = ||x_0 - y_n||$ converges to δ when n goes to ∞.

Correction made at the start of the exam

(*i*) Using the fact that δ is an infimum over M show that, for any integers m and n, $||y_m + y_n - 2x|| \geq 2\delta$. [2] $\|y_m + y_n - 2x_0\| \geq 2\delta$

(*ii*) Using the parallelogram identity, show that $(y_n)_n$ is a Cauchy sequence.

[4]

(*iii*) Prove that there exists a unique vector $y \in M$ such that $||x_0 - y|| = d$. To this end, show the existence of the vector y in H and use the fact that M is closed to conclude. [6]

(*iv*) Justify that for any $z \in M$, $(x_0 - y, z) = 0$ (we do not require a detailed proof, you may give a graphical justification). [3]

5. The aim of this problem is to derive the minimum of

$$I(a,b) = \int_0^{\pi} [\sin(t) - (at^2 + bt)]^2 \, dt$$

over a, b in $\mathbb{R}$.

a) Introducing an appropriate setting restate the minimisation problem in terms of the distance between a vector and a closed vector space. [7]

b) Find α and β such that

$$\int_0^{\pi} [\sin(t) - (\alpha t^2 + \beta t)] t \, dt = 0$$

and

$$\int_0^{\pi} [\sin(t) - (\alpha t^2 + \beta t)] t^2 \, dt = 0$$

[6]

c) Using α and β from the previous question, compute

$$\int_0^{\pi} [\sin(t) - (\alpha t^2 + \beta t)]^2 \, dt$$

and conclude. [7]