Final

**E4.45**
**SO22**

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2010

MSc and EEE/ISE PART IV: MEng and ACGI

**WAVELETS AND APPLICATIONS**

Thursday, 20 May 10:00 am

Time allowed: 3:00 hours

**There are FOUR questions on this paper.**

**Answer THREE questions.**

*All questions carry equal marks.*

**Any special instructions for invigilators and information for candidates are on page 1.**

Examiners responsible First Marker(s) : P.L. Dragotti
Second Marker(s) : K.D. Harris

**Special Information for the Invigilators: NONE**

**Information for Candidates:**

*Sub-sampling by an integer N*:

$$x_{\downarrow N}[n] \longleftrightarrow \frac{1}{N}\sum_{k=0}^{N-1} X(e^{j(\omega-2\pi k)/N}) = \frac{1}{N}\sum_{k=0}^{N-1} X(W_N^k z^{1/N}),$$

where

$$W_N^k = e^{-j2\pi k/N}.$$

*Parseval's identity*:

$$\langle g(t), f^*(t)\rangle = \frac{1}{2\pi}\langle \hat{g}(\omega), \hat{f}^*(\omega)\rangle,$$

where $\hat{g}(\omega)$ and $\hat{f}(\omega)$ are the Fourier transforms of $g(t)$ and $f(t)$ respectively.

*Poisson summation formula*:

$$\sum_{n=-\infty}^{\infty} f(t-n) = \sum_{k=-\infty}^{\infty} \hat{f}(2\pi k)e^{j2\pi kt}.$$

*Fourier transform pair*:

$$\frac{\sin \pi t}{\pi t} \Longleftrightarrow \mathrm{rect}\left(\frac{\omega}{2\pi}\right),$$

where

$$\mathrm{rect}(x) = \begin{cases} 1 & |x| \leq 1/2 \\ 0 & \text{otherwise} \end{cases}$$

# The Questions

1. Consider the oversampled three-channel filter bank shown in Figure 1a. Note that the down-sampling as well as the up-sampling factor is two.

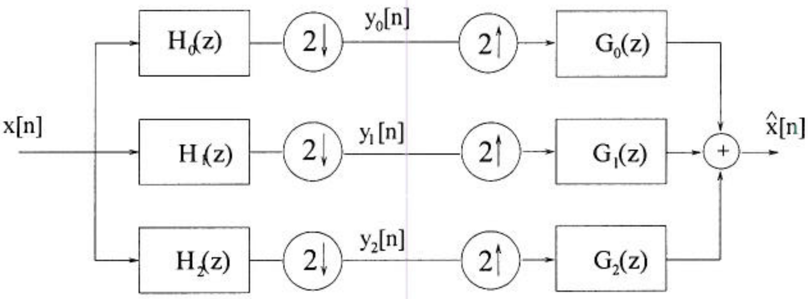


Figure 1a: Three-channel filter bank with down-sampling by 2.

(a) Express $\hat{X}(z)$ as a function of $X(z)$ and the filters. Then, derive the two perfect reconstruction conditions the filters have to satisfy.

[7]

(b) Assume that $G_0(z)$, $G_1(z)$ and $G_2(z)$ are the ideal filters shown in Figure 1b, (check carefully the cut-off frequencies). Assume that $H_i(z) = \alpha G_i(z^{-1})$, for $i = 0, 1, 2$. Here $\alpha$ is a constant. Sketch and dimension the Fourier transform of $y_0[n], y_1[n], y_2[n]$ and $\hat{x}[n]$ assuming that $x[n]$ has the spectrum shown in Figure 1c.

[5]

(c) Choose the constant $\alpha$, so that $\hat{X}(z) = X(z)$.

[2]

Question 1 continues on the next page

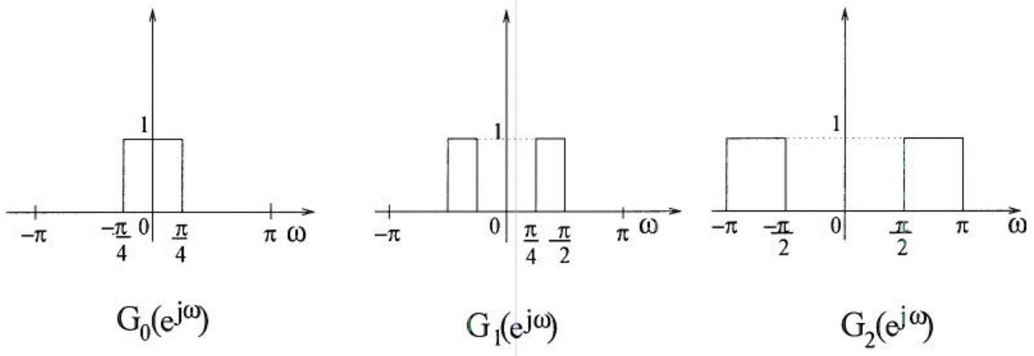


Figure 1b: Fourier transforms of the synthesis filters of Figure 1a.

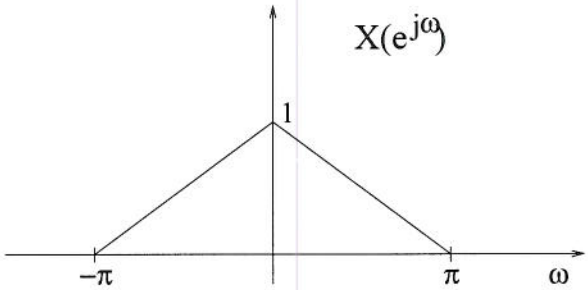


Figure 1c: Spectrum of $x[n]$.

(d) Now, the filter bank is iterated on the $H_0$ branch to form a 2-level decomposition.

i. Draw either the synthesis or the analysis filter bank of the equivalent 5-channel filter bank clearly specifying the transfer functions and downsampling factors.

[3]

ii. If the filters are those shown in Figure 1b, draw the Fourier transform of the equivalent filters of each branch before downsampling.

[3]

2. Consider the two-channel filter bank shown in Figure 2.

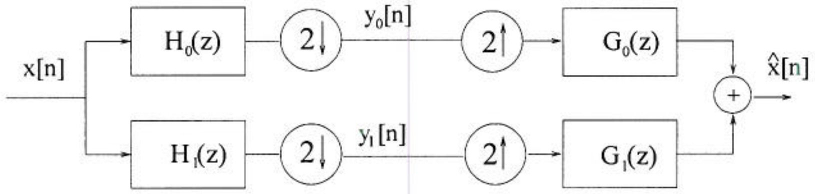


Figure 2: Two-channel filter bank.

(a) You are first asked to design an orthogonal filter-bank.

i. Start by designing the shortest possible filter $G_0(z)$ with a zero at $\omega = \pi$ and at least one zero at $\pi/3$. You want a filter with real-valued coefficients. [Hint: Recall that if the coefficients of a filter $G(z)$ are real then if $z_k$ is a complex root of $G(z)$ so is $z_k^*$, where * denotes the complex conjugate].

[4]

ii. Choose $H_0(z) = G_0(z^{-1})$ and make sure that $P(z) = H_0(z)G_0(z)$ satisfies the half-band condition: $P(z) + P(-z) = 2$. [Hint: you might have to multiply both filters $H_0(z)$ and $G_0(z)$ by a proper coefficient $\alpha$].

[4]

iii. Design the filters $H_1(z)$ and $G_1(z)$ in order to have a perfect reconstruction orthogonal filter-bank.

[4]

(b) You are now willing to design a symmetric/antisymmetric biorthogonal filter bank. Take

$$P(z) = (z + 1 + z^{-1}) \left( bz^2 + \frac{1}{4}z + \frac{1}{2} + \frac{1}{4}z^{-1} + bz^{-2} \right).$$

i. Find $b$ so that $P(z) + P(-z) = 2$.

[4]

ii. Choose $G_0(z) = (z + 1 + z^{-1})$. Design the filters $H_0(z), H_1(z)$ and $G_1(z)$ in order to have a perfect reconstruction bio-orthogonal filter-bank.

[4]

3. *Shannon Multiresolution Analysis.* For $j \in \mathbb{Z}$, let $V_j$ be the space of all finite energy signals $f$ for which the Fourier transform $\hat{f}(\omega) = 0$ outside of the interval $[-2^j\pi, 2^j\pi]$. We want to show that the function $\varphi(t) = \frac{\sin \pi t}{\pi t}$ with Fourier transform $\hat{\varphi}(\omega) = \mathrm{rect}(\omega/2\pi)$ is the scaling function of this multiresolution analysis. We need to show that $\varphi(t)$ satisfies the three criteria of a valid scaling function. More specifically:

   (a) Show that $\langle \varphi(t-n), \varphi(t-m) \rangle = \delta_{n,m}$. This is equivalent to showing that $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$ is an orthonormal basis of the space $V_0 = \mathrm{span}\{\varphi(t-n)\}_{n\in\mathbb{Z}}$. [Hint: use Parseval's identity].

   [5]

   (b) Show that $\varphi(t)$ satisfies partition of unity:

   $$\sum_{n=-\infty}^{\infty} \varphi(t-n) = 1.$$

   [5]

   (c) Finally, derive the coefficients $g_0[n]$ that lead to the two-scale equation

   $$\varphi(t) = \sqrt{2} \sum_{n=-\infty}^{\infty} g_0[n]\varphi(2t-n).$$

   [5]

   (d) Given $\varphi(t)$ and the two-scale equation, derive the corresponding wavelet $\psi(t)$.

   [5]

4. Let $\varphi(t)$ and $\psi(t)$ be the Haar scaling and wavelet functions, respectively. Let $V_j$ and $W_j$ be the spaces generated by $\varphi_{j,n}(t) = \sqrt{2^{-j}}\varphi(2^{-j}t-n)$, $n \in \mathbb{Z}$ and $\psi_{j,n}(t) = \sqrt{2^{-j}}\psi(2^{-j}t-n)$, $n \in \mathbb{Z}$, respectively. Consider the function defined on $0 \leq t < 2$ given by

$$f(t) = \begin{cases} 0 & 0 \leq t < 1/4 \\ 1 & 1/4 \leq t < 1/2 \\ 0 & 1/2 \leq t < 2. \end{cases}$$

(a) Decompose $f(t)$ into its component parts $W_{-1}$, $W_0$, and $V_0$. In other words, find the coefficients $c_{0,n}$, $d_{-1,n}$ and $d_{0,n}$, $n \in \mathbb{Z}$ that leads to the following decomposition

$$f(t) = \sum_{n=0}^{1} c_{0,n}\varphi_{0,n}(t) + \sum_{j=-1}^{0}\sum_{n=0}^{2^{-j+1}} d_{j,n}\psi_{j,n}(t).$$

[6]

(b) Verify the Parseval equality. That is, verify that:

$$\|f(t)\|^2 = \sum_{n} |c_{0,n}|^2 + \sum_{j=-1}^{0}\sum_{n} |d_{j,n}|^2.$$

[6]

(c) You now want to compress $f(t)$ using $R$ bits in total. Assume that $R$ is large.

i. Compute an approximated operational distortion-rate curve $D(R)$ that you obtain by scalar quantizing each of the 8 transform coefficient of the above decomposition and by allocating the same amount of bits to each coefficient. Justify your answer.

[4]

ii. Devise a non-linear strategy where only the non-zero coefficients are scalar quantized. Derive the new $D(R)$. Justify your answer.

[4]

# QUESTIONS

Wavelets & Applications
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## (a) QUESTION 1

$$Y_i(z) = \frac{1}{2} H_i(z^{1/2}) X(z^{1/2}) + \frac{1}{2} H_i(-z^{1/2}) X(-z^{1/2}) \qquad i=0,1,2$$

$$\hat{X}(z) = \frac{1}{2} X(z)\Big(G_0(z)H_0(z) + G_1(z)H_1(z) + G_2(z)H_2(z)\Big)$$

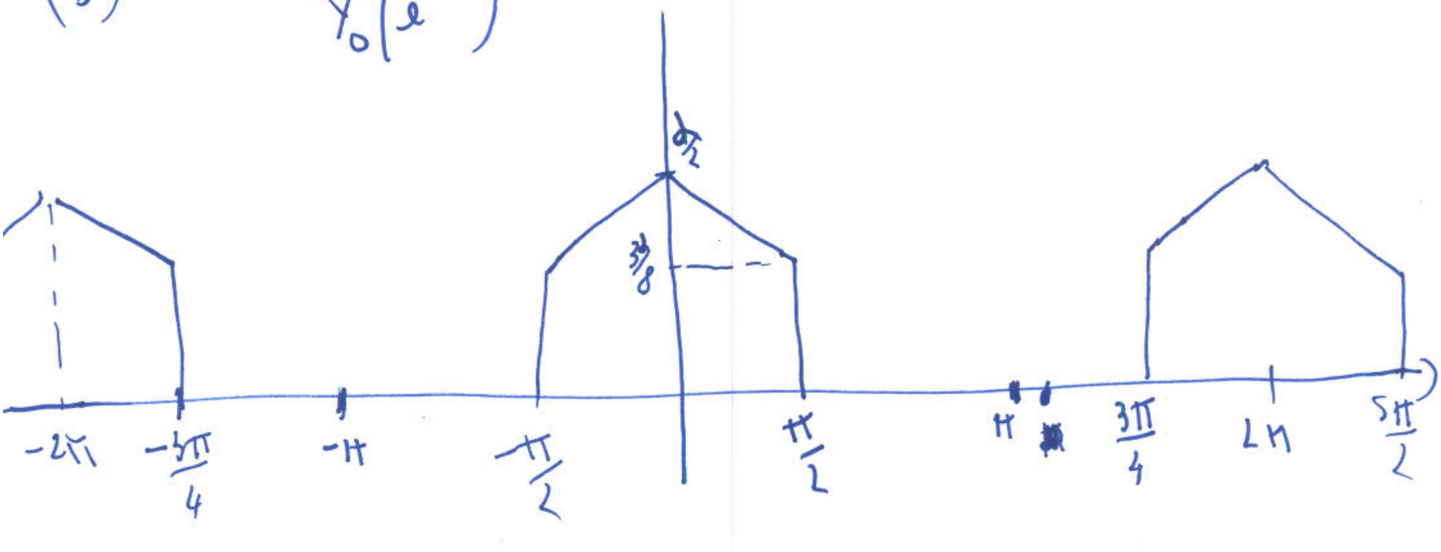
$$+ \frac{1}{2} X(-z)\Big(G_0(z)H_0(-z) + G_1(z)H_1(-z) + G_2(z)H_2(-z)\Big)$$

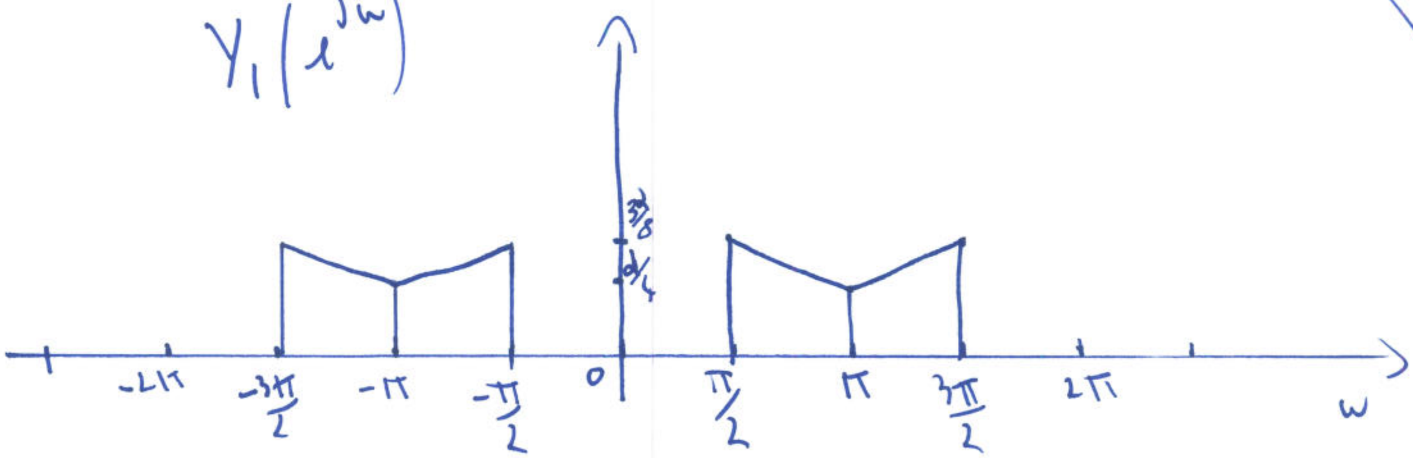
PR CONDITIONS

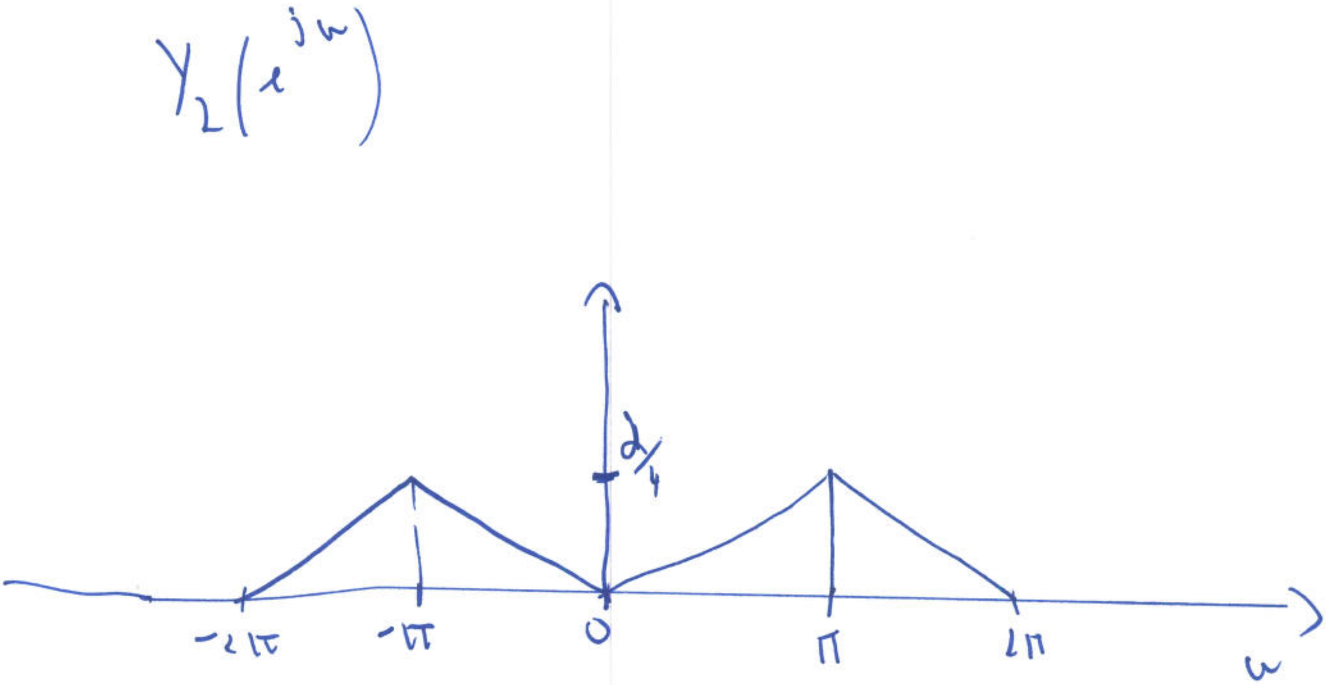
$$\begin{cases} G_0(z)H_0(z) + G_1(z)H_1(z) + G_2(z)H_2(z) = 2 \\ G_0(z)H_0(-z) + G_1(z)H_1(-z) + G_2(z)H_2(-z) = 0 \end{cases}$$

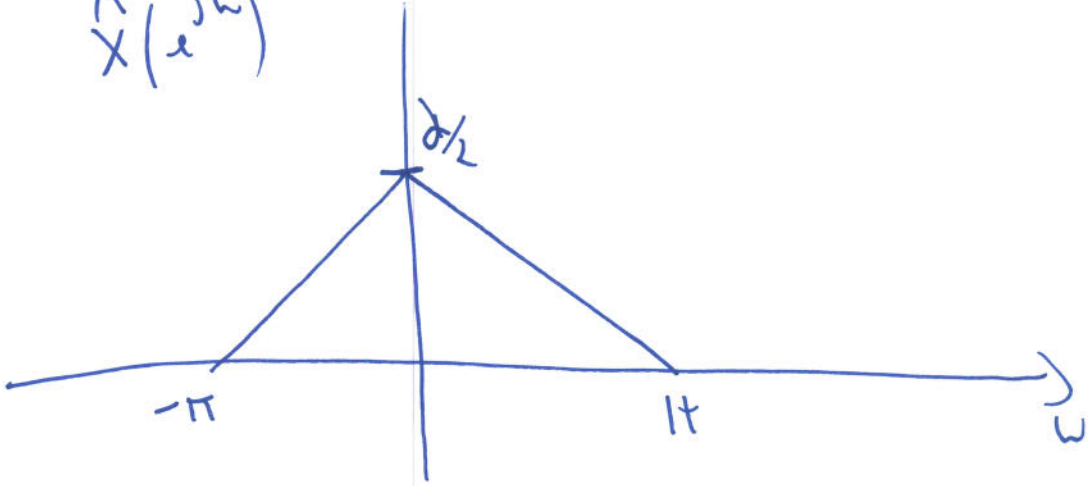
## (b)

$Y_0(e^{jw})$





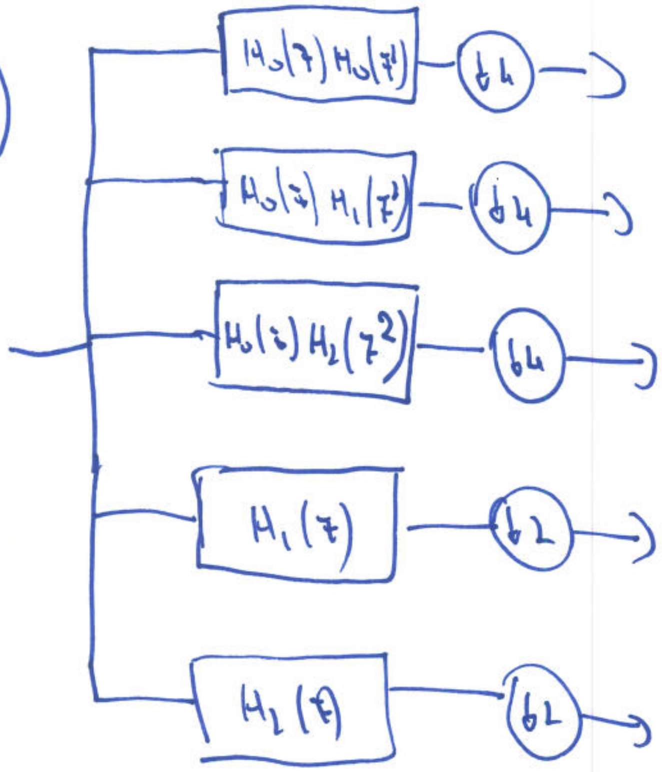



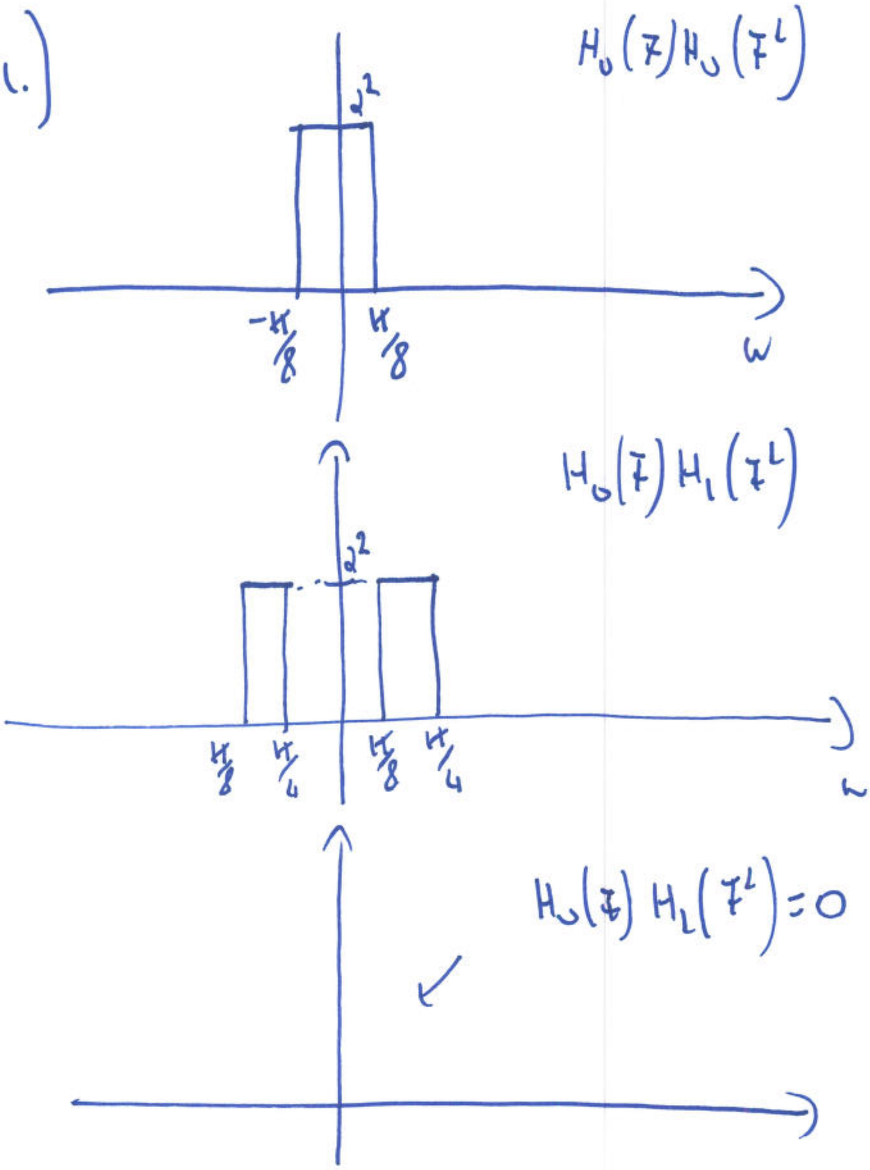

c) $\alpha = 2$ IN ORDER TO HAVE $X(f) = \hat{X}(f)$

d)

i.)




ii.)




H1(z) AND H2(z) AS BEFORE

QUESTION 2

a)

i.) ~~S~~HORTEST FILTER WITH REAL COEFFICIENTS:

$$G_0(z) = \alpha\left(z^{-1}+1\right)\left(z^{-1} - e^{j\frac{\pi}{3}}\right)\left(z^{-1} - e^{-j\frac{\pi}{3}}\right)$$

$$= \alpha\left(z^{-1}+1\right)\left(z^{-2} + 2\cos\frac{\pi}{3} \cdot z^{-1} + 1\right)$$

$$= \alpha\left(z^{-1}+1\right)\left(z^{-2} - z^{-1} + 1\right)$$

$$= \alpha\left(1 + z^{-3}\right)$$

$$H_0(z) = G_0\left(z^{-1}\right) = \alpha\left(1 + z^{3}\right)$$

ii.)

$$P(z) = \left(1 + z^{-3}\right)\left(1 + z^{3}\right) = \alpha^2\left(1 + z^3 + z^{-3} + 1\right)$$

$$P(z) = P(-z) = 4\alpha^2 = 1 \Rightarrow \alpha^2 = \frac{1}{2}. \Rightarrow \alpha = \frac{1}{\sqrt{2}}$$

iii.)

$$G_1(z) = -z^{-1} G_0\left(z^{-1}\right) = -\frac{z^{-1}}{\sqrt{2}}\left(1 - z^{3}\right) = \frac{z^{2}}{\sqrt{2}} - \frac{z^{-1}}{\sqrt{2}}$$

$$H_1(z) = G_1\left(z^{-1}\right) = \frac{z^{-2}}{\sqrt{2}} - \frac{z}{\sqrt{2}}$$

QUESTION 2

b)

i.) CLEARLY $P(z)+P(-z)=2$ WHEN

$b=-1/4$

i.i.)

$$H_0(z) = -\frac{1}{4}z^2 + \frac{1}{4}z + \frac{1}{2} + \frac{1}{4}z^{-1} - \frac{1}{4}z^2$$

$$H_1(z) = z\,G_0(-z) = z\left(-z+1-z^{-1}\right) = -z^2 + z - 1$$

$$G_1(z) = z^{-1}H_0(-z) = z^{-1}\left(-\frac{1}{4}z^2 - \frac{1}{4}z + \frac{1}{2} - \frac{1}{4}z^{-1} - \frac{1}{4}z^2\right)$$

$$= \left(-\frac{1}{4}z - \frac{1}{4} + \frac{1}{2}z^{-1} - \frac{1}{4}z^{-2} - \frac{1}{4}z^{-3}\right)$$

QUESTION 3

(a) PARSEVAL EQUALITY: $\langle g(t), f^*(t)\rangle = \frac{1}{2\pi}\langle \hat{g}(\omega), \hat{f}^*(\omega)\rangle$.

THEREFORE, WE HAVE THAT

$$\langle \varphi(t-n), \varphi(t-m)\rangle = \frac{1}{2\pi}\langle \mathrm{RECT}\left(\frac{\omega}{2\pi}\right)e^{-j\omega n}, \mathrm{RECT}\left(\frac{\omega}{2\pi}\right)e^{j\omega m}\rangle$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j\omega(n-m)}\, d\omega = \cancel{\frac{\sin \pi(n-m)}{\pi k}}$$

$$= \frac{\sin \pi(n-m)}{\pi(n-m)} = \begin{cases} 1 & \text{IF } n=m \\ 0 & \text{OTHERWISE} \end{cases}$$

(b)

USING POISSON SUM FORMULA, WE HAVE THAT

$$\sum_{m=-\infty}^{\infty}\varphi(t-m) = \sum_{k=-\infty}^{\infty}\hat{\varphi}(2\pi k)e^{j2\pi k t} = \sum_{k=-\infty}^{\infty}\mathrm{RECT}\left(\frac{2\pi k}{2\pi}\right)e^{j2\pi k t}$$

AND CLEARLY

$$\sum_{k=-\infty}^{\infty} \text{RECT}\left(\frac{2\pi k}{2H}\right) e^{j2\pi kt} = 1$$

(c) WE FIRST SHOW THAT

$$g_0[k] = \sqrt{2} \langle \varphi(t), \varphi(2t-k) \rangle .$$

SINCE $\varphi(t)$ SATISFIES THE TWO-SCALE RELATION $\varphi(t) = \sqrt{2} \sum_{n=-\infty}^{\infty} g_0[n] \varphi(2t-n)$, (1)

WE HAVE THAT

$$\langle \varphi(t), \varphi(2t-k) \rangle = \sqrt{2} \sum_{n=-\infty}^{\infty} g_0[n] \langle \varphi(2t-n), \varphi(2t-k) \rangle$$

WHERE I HAVE USED (1) AND THE LINEARITY OF INNER PRODUCT.

BY REPLACING $2t = x$ WE HAVE THAT

$$\langle \varphi(t), \varphi(2t-k) = \frac{\sqrt{2}}{2} \sum_{n=-\infty}^{\infty} g_0[n] \underbrace{\langle \varphi(x-n), \varphi(x-k) \rangle}_{\delta_{n,k}} =$$

$$= \frac{1}{\sqrt{2}} g_0[k],$$

WHERE I USED THE FACT THAT $\varphi(t)$ IS ORTHOGONAL TO ITS SHIFTS.

THEREFORE

$$g_0[k] = \sqrt{2}\langle \varphi(t), \varphi(2t-k)\rangle$$

USING PARSEVAL WE GET:

$$g_0[k] = \sqrt{2}\langle \varphi(t), \varphi(2t-k)\rangle = \frac{\sqrt{2}}{2\pi}\langle \hat{\varphi}(w), \frac{1}{2}\hat{\varphi}^*\left(\frac{w}{2}\right)e^{j\frac{kw}{2}}\rangle$$

$$= \frac{\sqrt{2}}{2\pi}\int_{-\pi}^{\pi}\frac{1}{2}e^{j\frac{kw}{2}}dw = \frac{\sqrt{2}}{2}\frac{\sin\frac{k\pi}{2}}{\frac{k\pi}{2}}$$

THEREFORE WE CAN WRITE

$$\varphi(t) = \sum_{m=-\infty}^{\infty}\frac{\sin\frac{m\pi}{2}}{\frac{m\pi}{2}}\varphi(2t-m).$$

(d)
$$\psi(t) = \sqrt{2}\sum_{n=-\infty}^{\infty}(-1)^n g_0[1-n]\varphi(2t-n)$$

$$= \sum_{m=-\infty}^{\infty}(-1)^m\frac{\sin(m-1)\frac{\pi}{2}}{\frac{(m-1)\pi}{2}}\varphi(2t-m).$$

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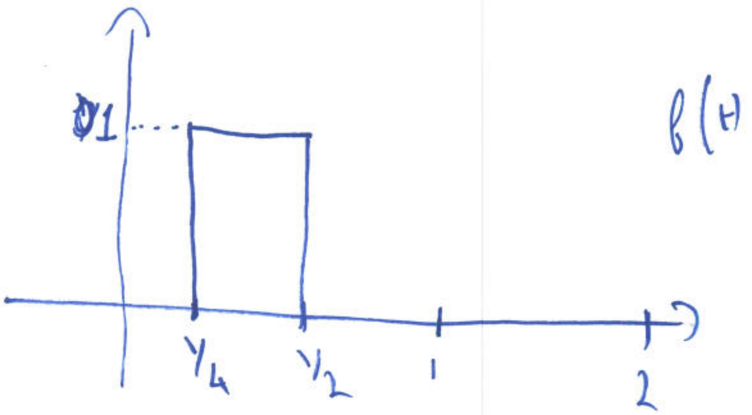
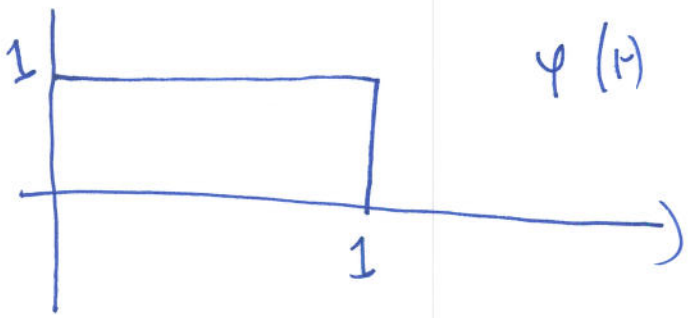
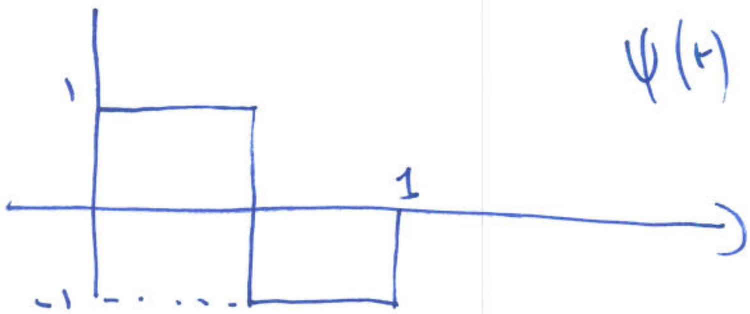
# QUESTION 4

(a)




CLEARLY

$$c_{0,m} = \langle f(t), \varphi(t-m) \rangle = \begin{cases} 1/4 & \text{FOR } m=0 \\ 0 & \text{OTHERWISE} \end{cases}$$

$$d_{0,m} = \langle f(t), \psi(t-m) \rangle = \begin{cases} 1/4 & \text{FOR } m=0 \\ 0 & \text{OTHERWISE} \end{cases}$$

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$$d_{-1,n} = \sqrt{2}\,\langle f(t), \psi(2t-n)\rangle = \begin{cases} -\frac{\sqrt{2}}{4} & n=0 \\ 0 & \text{OTHERWISE} \end{cases}$$

SO

$$f(t) = \frac{1}{4}\varphi(t) + \frac{1}{4}\psi(t) - \frac{\sqrt{2}}{4}\psi_{-1,0}(t).$$

(b)

$$\|f\|^2 = \int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{1/4}^{1/2} dt = 1/4$$

$$\|f\|^2 = |c_{0,0}|^2 + |d_{0,0}|^2 + |d_{-1,0}|^2 = \frac{1}{16} + \frac{1}{16} + \frac{2}{16} = 1/4.$$

PARSEVAL VERIFIED

(c) i) AT HIGH RATE THE MSE DISTORTION OF A SCALAR QUANTIZER IS

$$D(R) = c\, 2^{-2R}$$

WHERE $c$ DEPENDS ON THE STATISTICS OF THE SOURCE AND THE QUANTIZER USED.

~~BECAU~~ THE TOTAL BIT RATE IS R AND WE HAVE 8 COEFFICIENTS TO QUANTIZE. THEREFORE THE DISTORTION RELATED TO EACH OF THEM IS $D(R) = c\,2^{-2R/8}$

~~SINCE~~ BECAUSE OF PARSEVAL IDENTITY WE HAVE THAT:

$$D(R) = E\left[\|f - \hat{f}\|^2\right] = E\left[|c_{0,0} - \hat{c}_{0,0}|^2 + |d_{0,0} - \hat{d}_{0,0}|^2 + |d_{-1,0} - \hat{d}_{-1,0}|^2\right].$$

WHERE $\hat{c}_{0,0}$, $\hat{d}_{0,0}$, $\hat{d}_{-1,0}$ ARE THE QUANTIZED COEFFICIENTS.

WE THEREFORE HAVE THAT

$$D(R) \simeq (c_1 + c_2 + c_3)\,2^{-2R/8} = c\,2^{-2R/8}$$

ii. WE CAN ALTERNATIVELY QUANTIZE ONLY THE NON-ZERO COEFFICIENTS. WE NEED 3 BITS TO LOCATE EACH NON-ZERO COEFFICIENTS. THIS MEANS THAT WE NEED 9 BITS TO LOCATE THEM. WE CAN THEN USE THE REMAINING $R-9$ BITS TO QUANTIZE THEM. THIS LEADS TO THE FOLLOWING $D(R)$ CURVE : $D(R) \simeq c\,2^{-2(R-9)/3}$