E4.26 | C2.3   Master – April 08

Estimation + Fault Detection 2008 Exam: Model Answers

1. (a) The discrete time process satisfies

$$x_k = x(kh) = e^{\tilde{F}h} x([k-1]h) + \underbrace{\int_{(k-1)h}^{kh} e^{\tilde{F}(kh-\sigma')} \tilde{b}\, w(\sigma')\, d\sigma'}_{v_k}$$

where $\tilde{F} = \begin{bmatrix} -\alpha_1 & 0 \\ 0 & -\alpha_2 \end{bmatrix}$, $\tilde{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

By properties of the stochastic integral, the v_k's are zero mean + indep.

Also, $\operatorname{cov}\{v_k\} = \int_0^h \left(e^{\sigma \tilde{F}} \tilde{b}\right)\left(e^{\sigma\tilde{F}}\tilde{b}\right)^T d\sigma \qquad (\sigma = kh - \sigma')$

We have

$$F = \begin{bmatrix} e^{-\alpha_1 h} & 0 \\ 0 & e^{-\alpha_2 h} \end{bmatrix} \text{ and}$$

$$Q = \int_0^h \begin{bmatrix} e^{-\alpha_1\sigma} \\ e^{-\alpha_2\sigma} \end{bmatrix} \begin{bmatrix} e^{-\alpha_1\sigma} & e^{-\alpha_2\sigma} \end{bmatrix} d\sigma = \begin{bmatrix} \frac{1}{2\alpha_1}\left(1-e^{-2\alpha_1 h}\right) & \frac{1}{(\alpha_1+\alpha_2)}\left(1-e^{-(\alpha_1+\alpha_2)h}\right) \\ \frac{1}{(\alpha_1+\alpha_2)}\left(1-e^{-(\alpha_1+\alpha_2)h}\right) & \frac{1}{2\alpha_2}\left(1-e^{-2\alpha_2 h}\right) \end{bmatrix}$$

[8] and Q is as given by (*).

(b) $x_k = F x_{k-1} + v_k$, so $E\{x_k x_k^T\} = E\{(F x_{k-1} + v_k)(F x_{k-1} + v_k)^T\}$

Since x_{k-1} depends on past v_k's, x_{k-1} and v_k are indep., so

$$R = E\{x_k x_k^T\} = F E\{x_k x_k^T\} F^T + 0 + Q = FRF^T + Q$$

[4] So $\quad \underline{R = FRF^T + Q}$

Inserting calculated values for F and Q gives

$$\begin{bmatrix} r_{11} & r_{12} \\ r_{12} & r_{22} \end{bmatrix} = \begin{bmatrix} e^{-\alpha_1 h} & 0 \\ 0 & e^{-\alpha_2 h} \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} \\ r_{12} & r_{22} \end{bmatrix} \begin{bmatrix} e^{-\alpha_1 h} & 0 \\ 0 & e^{-\alpha_2 h} \end{bmatrix} + \begin{bmatrix} \frac{1}{2\alpha_1}\left(1-e^{-2\alpha_1 h}\right) & \frac{1}{\alpha_1+\alpha_2}(\ldots) \\ \frac{1}{\alpha_1+\alpha_2}(\ldots) & \frac{1}{2\alpha_2}\left(1-e^{-2\alpha_2 h}\right) \end{bmatrix}$$

So

$$\left(1-e^{-2\alpha_1 h}\right) r_{11} = \frac{1}{2\alpha_1}\left(1-e^{-2\alpha_1 h}\right) \qquad \Rightarrow \qquad r_{11} = \frac{1}{2\alpha_1}$$

$$\left(1-e^{-(\alpha_1+\alpha_2)h}\right) r_{12} = \frac{1}{\alpha_1+\alpha_2}\left(1-e^{-(\alpha_1+\alpha_2)h}\right) \qquad r_{12} = 1/\alpha_1+\alpha_2$$

$$\left(1-e^{-2\alpha_2 h}\right) r_{22} = 1/2\alpha_2\left(1-e^{-2\alpha_2 h}\right) \qquad r_{22} = 1/2\alpha_2$$

[4] So

$$\underline{R = \begin{bmatrix} 1/2\alpha_1 & 1/\alpha_1+\alpha_2 \\ \frac{1}{\alpha_1+\alpha_2} & 1/2\alpha_2 \end{bmatrix}}$$

covariance of unsampled process

[2] R is independent of h because $E\{x_k x_k^T\} = E\{x(t) x^T(t)\}$

(c) The square of the cor. $f^{h} = \rho(x_k^1, x_k^2)^2 = \dfrac{r_{12}^2}{r_{11} \times r_{22}}$

[2]
$$= \frac{\left(1/\alpha_1+\alpha_2\right)^2}{1/\alpha_1 \times 1/\alpha_2} = \frac{\alpha_2/\alpha_1}{\left(1+\alpha_2/\alpha_1\right)^2} = \frac{\gamma}{(1+\gamma)^2} = \frac{100}{(101)^2} < 0.01$$

So $\quad |\rho(x_k^1, x_k^2)| \le 0.1$

2(a) $m_x = 0$ and $m_y = E[x+n+b] = 0 + 0 + P \times 1 = P$

$E[x(y-m_y)] = E[x^2 + xn + x(b-P)] = \sigma_x^2 + 0 + 0$

$E[(y-m_y)^2] = E([x+n+(b-P)]^2) = \sigma_x^2 + \sigma_n^2 + (1-P)P$

It follows

$$\hat{x}_{LLS} = K(y-\alpha) \quad \text{where } K = \frac{\sigma_x^2}{\sigma_x^2+\sigma_n^2+(1-P)P}, \ \alpha = P.$$

and

[8] $$E[\|x-\hat{x}_{LS}\|^2] = \sigma_x^2\left[1 - \frac{\sigma_x^2}{\sigma_x^2+\sigma_n^2+(1-P)P}\right]$$

(ii) By Bayes Rule, $$p(b=1|y) = \frac{p(y|b=1)\,p(b=1)}{p(y|b=0)\,p(b=0) + p(y|b=1)\,p(b=1)}$$

But $p(y|b=1) = N(1, \sigma_x^2+\sigma_n^2)(y)$, $p(y|b=0) = N(0, \sigma_x^2+\sigma_n^2)$,

(since n and b are independent).

So $$p(b=1|y) = \frac{P\,N(1,\sigma_x^2+\sigma_n^2)y}{(1-P)N(0,\sigma_x^2+\sigma_n^2)(y) + P\,N(1,\sigma_x^2+\sigma_n^2)} = \alpha(y)$$

Note: $$\alpha(y) = \frac{P}{P + \exp\left\{-\frac{1}{2}\frac{(+2y-1)}{\sigma_x^2+\sigma_y^2}\right\}(1-P)} \qquad (1)$$

Then

$$p(x|y) = p(x|y,b=0)\,P(b=0|y) + p(x|y,b=1)\,P(b=1|y)$$

But $p(x|y,b=0)$ and $p(x|y,b=1)$ are obtained simply by solving the standard linear least squares problem with b a constant ($b=0$ or 1

i.e.

$$P(x|y,b=0) = N\left(K_1 y,\ \sigma_x^2\left(1 - \frac{\sigma_x^2}{\sigma_x^2+\sigma_n^2}\right)\right)$$

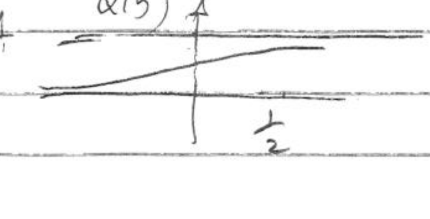
and

$$p(x|y,b=1) = N\left(K_1(y-1),\ \sigma_x^2\left(1 - \frac{\sigma_x^2}{\sigma_x^2+\sigma_n^2}\right)\right) \quad \text{where } K_1 = \frac{\sigma_x^2}{\sigma_x^2+\sigma_n^2}$$

It follows from (1)

[8] $$\hat{x}_{NLS} = (1-\alpha(y))K_1 y + \alpha(y)K_1(y-1) = K_1(y-\alpha(y))$$

Since $K_1 > K_{LS}$, $\hat{x}_{NLS} \begin{cases} > x_{LLS} & \text{for } y \text{ large } (+ve) \\ < x_{LLS} & \text{for } y \text{ large } (-ve) \end{cases}$



The 'offset' monotonically increases for 0 to 1, and is $\frac{P}{1+P}$ when $y = \frac{1}{2}$.

[2]

3.

Denote by $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}$ the covariance of $\begin{bmatrix} x \\ y \end{bmatrix}$.

We calculate, using $y = x\cos\theta + n$ (write $\cos\theta = c$)

$E[x] = E[y] = 0$, and $P_{22} = c^2 \operatorname{var}\{x\} + \operatorname{var}\{n\} = 1 + c^2$. Also

$P_{12} = c$ and $P_{11} = 1$. $\left(P = \begin{bmatrix} 1 & c \\ c & 1+c^2 \end{bmatrix} \right)$

[8] So, $p(x,y) = \dfrac{1}{1 \times (2\pi)} \times \exp\left\{ -\frac{1}{2} \left[[x\ y] \begin{bmatrix} 1 & c \\ c & 1+c^2 \end{bmatrix}^{-1} \begin{bmatrix} x \\ y \end{bmatrix} \right] \right\}$

But
$$\begin{bmatrix} 1 & c \\ c & 1+c^2 \end{bmatrix}^{-1} = \begin{bmatrix} 1+c^2 & -c \\ -c & 1 \end{bmatrix} \quad (\text{note: } \det[..] = 1)$$

Also
$$p(y) = \frac{1}{(2\pi)^{\frac{1}{2}}} \frac{1}{(1+c^2)^{\frac{1}{2}}} \exp\left\{ -\frac{1}{2} y^2 / 1+c^2 \right\}$$

So
$$p(x|y) = \frac{\sqrt{1+c^2}}{(2\pi)^{\frac{1}{2}}} \exp\left\{ -\frac{1}{2} \left[(1+c^2)x^2 - 2cxy + y^2 - \frac{y^2}{(1+c^2)} \right] \right.$$

Factorize:
$$(1+c^2)\left[x^2 - \frac{2c}{(1+c^2)} xy + \frac{c^2}{(1+c^2)^2} \cdot y^2 \right]$$
$$= (1+c^2) \left[x - \frac{c}{1+c^2} y \right]^2$$

[6] So, $p(x|y) = \dfrac{1}{(2\pi\sigma^2)^{1/2}} \exp\left\{ -\dfrac{1}{2} \dfrac{|x - m|^2}{\sigma^2} \right\}$

where $m = \dfrac{\cos\alpha}{1+\cos^2\alpha} y$ and $\sigma^2 = \dfrac{1}{1+\cos^2\alpha}$

[4] So $E[x|y] = \dfrac{\cos\alpha}{1+\cos^2\alpha} \times y$ and $\operatorname{cov}\{x|y\} = \dfrac{1}{1+\cos^2\alpha}$ $(=: \sigma^2)$

We require that $P[|x - E[x|y]| > 0.8] \leq 0.2$

Since (from data) $P[|x - E[x|y]| > 0.9\sigma] = 0.2$

We must have $0.9\sigma \leq 0.8$,

i.e. $\sqrt{\dfrac{1}{1+\cos^2\alpha}} \leq \dfrac{0.8}{0.9}$ or $\cos\alpha^2 > 0.2656$

[2] or $\alpha \leq \cos^{-1}\sqrt{0.2656}$

4 (a) We know $x_k = F^k x_0$, so $y_k = x_k + v_k = F^k x_0 + v_k$
Since x_0 and v_k are independent, and v_k is independent of $y_{1:k-1}$, and finally $v_k \sim N(0, Q)$, we have

[3] $$p(y_k \mid y_{1:k-1}, x_0) = p(y_k \mid x_0) = N(F^k x_0, Q)(y_k)$$

(b) Insert $p(x_0 \mid y_{1:k}) = N(\hat{x}_{0|k}, P_{0|k})$ etc into Bayes rule gives

$$-\frac{1}{2}(x_0 - \hat{x}_{0|k})^T P_{0|k}^{-1}(x_0 - \hat{x}_{0|k}) = -\frac{1}{2}(y_k - F^k x_0)^T Q^{-1}(y_k - F^k x_0)$$

$$-\frac{1}{2}(x_0 - \hat{x}_{0|k-1})^T P_{0|k-1}^{-1}(x_0 - \hat{x}_{0|k-1}) + (\text{term that depends only on } y_{1:k})$$

Expand this equation (and multiply by $-\frac{1}{2}$) $\Rightarrow$

$$x_0^T \left[P_{0|k}^{-1} - (F^k)^T Q^{-1} F^k - P_{0|k-1} \right] x_0$$

$$- 2 x_0^T \left[P_{0|k}^{-1} \hat{x}_{0|k} - (F^k)^T Q^{-1} y_k - P_{0|k-1}^{-1} \hat{x}_{0|k-1} \right]$$

$$+ (\text{term independent of } x_0) = 0$$

Since this equation is valid for all x_0,

[6] $$P_{0|k}^{-1} = P_{0|k-1}^{-1} + (F^k)^T Q^{-1} F^k \qquad — (*)$$

and

$$P_{0|k}^{-1} \hat{x}_{0|k} = P_{0|k-1}^{-1} \hat{x}_{0|k-1} + (F^k)^T Q^{-1} y_k .$$

[7] [8]

(c) Specialize to scalar case, and set $Q = 1$, $F = \sqrt{0.5}$. Then, by $(*)$,

$$P_{0|k}^{-1} = P_{0|0}^{-1} + \sum_{j=1}^{k} a^k, \quad \text{where } a = 0.5$$

$$= P_{0|0}^{-1} + a\left[\sum_{j=0}^{k} a^k\right] \to P_{0|0}^{-1} + \frac{a}{1-a} \quad \text{as } k \to \infty$$

[4] So $P_{0|\infty}^{-1} = 1 + 0.5/0.5 = 2$,

whence $P_{0|\infty} = \frac{1}{2}$

5 (a) Assume $X \sim \tilde{p}(x) = \alpha\delta(x+\sigma) + (1-2\alpha)\delta(x) + \alpha\delta(x-\sigma)$

Then $E[X] = \alpha \times (-\sigma) + (1-2\alpha)\times 0 + \alpha \times (+\sigma) = 0$ $\quad m_x = 0 \quad (\checkmark)$

and $\mathrm{var}[X] = \sigma^2 \times \alpha + (1-2\alpha)\times 0^2 + \alpha \times \sigma^2 = 2\alpha\sigma^2$

[4] We require then $\underline{\alpha = \frac{1}{2}}$ for $E[X^2] = \sigma^2$.

(b) $Y = X^3 + V$. Clearly Y has zero mean. So, to construct the least squares estimate we must calculate

$$E[XY] = E[X^4 + XV] = E[X^4] + E[X]E[V]\,. \quad (E[V] = 0)$$

$$= \sigma^4 \times \tfrac{1}{2} + 0^2 \times 0 + \sigma^4 \times \tfrac{1}{2} = \sigma^4$$

and $E[Y^2] = E[(X^3+V)(X^3+V)] = E[X^6] + 2E[V]\underbrace{E[X^3]}_{} + E[V^2]$

$$= \sigma^6 \times \tfrac{1}{2} + \sigma^6 \times \tfrac{1}{2} + 0 + \sigma_n^2 = \underset{\sim}{\sigma^6} + \sigma_n^2$$

It follows that the linear least squares estimate is

$$\hat{x} = k y$$

where

[10] $$k = E[XY]\,E[Y^2]^{-1} = \underline{\frac{\sigma^4}{\sigma^6 + \sigma_n^2}}$$

(c) The approximate linear least squares estimate based on linearizing $h(x)$ about m_x is

$$\hat{x}' = k' y\,, \quad \text{where } k' = \frac{\sigma^2 H}{H\sigma^2 H + \sigma_n^2}$$

in which $H = h_x(m_x)$.

[4] But $h_x(m_x) = \frac{d}{dx}x^3\Big|_{x = m_x = 0} = 3x^2\Big|_{x=0} = 0$.

So the approximate linear least squares estimate is simply

$$\hat{x}' = 0 \quad (= m_x)\,.$$

[2] It is a bad estimate because it does not make use of the measurement

6 (a) $y_k + 0.5\,y_{k-1} = e_k + d\,e_k$ — (1)

So $E[(y_k + 0.5\,y_{k-1})^2] = E\{(e_k + d\,e_{k-1})^2\}$. This implies

$r_d(0) + r_d(1) + 0.25\,r_d(0) = 1 + d^2$ — (2)

Multiply across (1) by y_{k-1} and take expectations. This gives

$E\{y_k y_{k-1} + 0.5\,y_{k-1}^2\} = E\{e_k y_{k-1} + d\,y_{k-1} e_{k-1}\}$

whence $r_d(1) + 0.5\,r_d(0) = 0 + d\,E\{y_k e_k\}$.

Also, multiply across (1) by e_k and take $E\{..\}$. This gives

$E\{y_k e_k\} + 0 = 1 + 0$. So

$r_d(1) + 0.5\,r_d(0) = d$ — (3)

From (2) and (3), $0.75\,r_d(0) = 1 - d + d^2$. Hence

$$r_d(0) = \frac{4}{3}(1 - d + d^2)$$ [10]

(b) The likelihood function is $LR(z) = \dfrac{p(z \mid d=2)}{p(z \mid d=0)}$

where $p(z \mid d=2) = N(0,4)$ and $p(z \mid d=0) = N(0, 4/3)$

The log-likelihood function is

$LLF(z) = \frac{1}{2}\left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right) z^2 + \text{constant} \quad (\sigma_0^2 = 4/3,\ \sigma_1^2 = 4)$

So, $P_0[LLF(z) > c] = P_0\left[z^2 > c' := \left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right)^{-1}(c - \text{const.})\right]$

We must choose c' (and hence c) such that

$\text{Prob}[z^2 > c'] = 0.05 \quad (z \sim N(0, 3/4))$

$= \text{Prob}[z'^2 \geq \frac{3}{4}c'] = 0.05 \quad (z \sim N(0,1))$

From tables, $3/4\,c' = 3.84 \implies c' = \dfrac{4 \times 3.84}{3} = 5.12$

N-P test: $\begin{cases} \text{If } z^2 \leq 5.12 & \text{choose } H_0 \\ \text{If } z^2 > 5.12 & \text{choose } H_1. \end{cases}$ [6]

Also

$P_1[z^2 > 5.12] \quad (z \sim N(0,4))$

$= \text{Prob}[(z')^2 > 5.12/4 = 1.28] = 0.26$

We conclude that the power of the test is 0.26. (Poor test — will fail to detect many 'faults'!) [4]