

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2011

EEE/ISE PART II: MEng, BEng and ACGI

SIGNALS AND LINEAR SYSTEMS

Time allowed: 2:00 hours

There are THIRTEEN questions on this paper.

This paper is accompanied by four tables of formulae.

ANSWER ALL QUESTIONS.

**Any special instructions for invigilators and information
for candidates are on page 1.**

Examiners responsible

First marker: P.Y.K. Cheung
Second marker: A. Manikas

Special instructions for invigilators: None

Information for candidates: None

1. The unit step sequence $u[n]$ is defined as

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}.$$

A discrete-time signal $x[n]$ is shown in Figure 1.1.

- a) Sketch the functions $u[-n]$ and $u[1-n]$.
b) Sketch and label each of the following signals:

- i) $x[n]u[1-n]$
ii) $x[n](u[n+2] - u[n])$
iii) $x[n]\delta[n-1]$

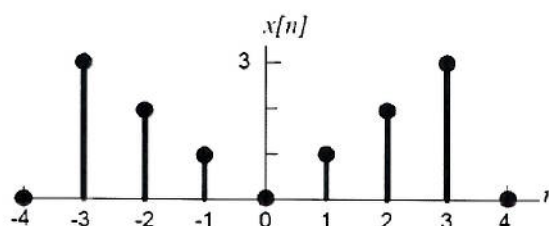


Figure 1.1

2. Sketch the even and odd components of the signal $x(t) = 4e^{-0.5t}u(t)$ as shown in Figure 2.1.

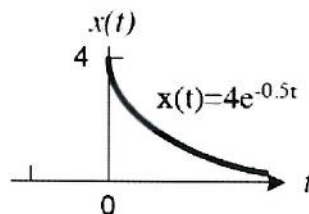


Figure 2.1

3. Compute the output $y(t)$ for a continuous-time LTI system whose impulse response $h(t)$ and the input $x(t)$ are given by

$$h(t) = e^{-\alpha t}u(t) \quad x(t) = e^{\alpha t}u(-t) \quad \alpha > 0.$$

[6]

4. Consider the capacitor shown in Figure 4.1. Let input $x(t) = i(t)$ and output $y(t) = v_c(t)$.
- a) Find the input-output relationship in the form of a differential-integral equation. [2]
- b) Determine with justifications whether the system is:
- i) memoryless;
 - ii) casual;
 - iii) linear;
 - iv) time-invariant. [4]

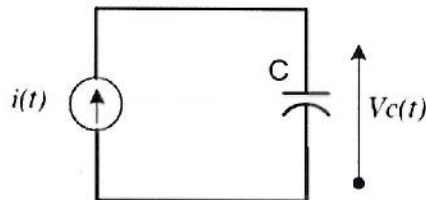


Figure 4.1

5. Consider a continuous-time system whose input $x(t)$ and output $y(t)$ are related by the following differential equation:
- $$\frac{dy(t)}{dt} + ay(t) = x(t)$$
- where a is a constant.
- a) Given the initial condition $y(0) = y_0$, find the zero-input response $y_{zi}(t)$ of the system. [3]
- b) Derive the impulse response $h(t)$. If the input $x(t) = Ke^{-bt}u(t)$, derive the zero-state response of the system. You may use the formula tables provided. [4]
- c) Hence or otherwise, derive the general expression for $y(t)$. [3]
6. The output $y(t)$ of a continuous-time LTI system is found to be $2e^{-3t}u(t)$ when the input $x(t)$ is $u(t)$.
- a) Derive the transfer function $H(s)$ of the system. Hence or otherwise, find the impulse response $h(t)$ of the system. [5]
- b) Find the output $y(t)$ when the input $x(t)$ is $e^{-t}u(t)$. [5]
7. Find the inverse Laplace transform of
- $$X(s) = \frac{5s + 13}{s(s^2 + 4s + 13)}.$$
- [6]

8. The frequency response of a system is given by:

$$H(\omega) = \frac{10^4(1 + j\omega)}{(10 + j\omega)(100 + j\omega)}.$$

Sketch the Bode amplitude plot for this system.

[8]

9. Plot on the s-plane the pole and zero locations for a system with the transfer function:

$$H(s) = \frac{2s + 4}{s^2 + 4s + 20}.$$

[4]

10. It is known that $g(t) = x(t)\cos t$ and that the Fourier transform of $g(t)$ is:

$$G(\omega) = \begin{cases} 1, & -2 \leq \omega \leq +2 \\ 0, & \text{otherwise} \end{cases}.$$

- a) Derive an expression for $G(\omega)$ in terms of the Fourier transform $X(\omega)$ of $x(t)$. Sketch $G(\omega)$ and $X(\omega)$.

[6]

- b) Hence determine $x(t)$.

[3]

11. A continuous-time signal

$$f(t) = 10 \cos 2000\pi t + \sqrt{2} \sin 3000\pi t + 2 \cos(5000\pi t + \frac{\pi}{4})$$

is sampled at a rate of 4000 samples/second.

- a) Derive an expression for the discrete-time signal $f[k]$ in terms of k , where k is the sample number.

[5]

- b) Explain with justification why this sampling rate causes aliasing.

[3]

- c) Determine the maximum sampling period T that can be used to sample this signal without aliasing.

[2]

12. A discrete-time shift-invariant system is described by the difference equation

$$y[n] = 0.5y[n-1] + bx[n].$$

- a) Derive the frequency response $H(\omega)$ of the system. [4]
- b) Find the value of b so that $|H(\omega)|$ is equal to 1 at $\omega = 0$. [3]
- c) Find the frequency ω at which the output power is half that of its peak value. [3]

13. Find the z-transform of each of the following sequences:

- a) $x[n] = 2^n u[n] + 3\left(\frac{1}{2}\right)^n u[n]$ [3]
- b) $x[n] = \cos[n\omega_0] u[n]$ [3]

[THE END]

Table of formulae for E2.5 Signals and Linear Systems
(For use during examination only.)

Convolution Table

No.	$x_1(t)$	$x_2(t)$	$x_1(t) * x_2(t) = x_2(t) * x_1(t)$
1	$x(t)$	$\delta(t - T)$	$x(t - T)$
2	$e^{\lambda_1 t} u(t)$	$u(t)$	$\frac{1 - e^{\lambda_1 t}}{-\lambda_1} u(t)$
3	$u(t)$	$u(t)$	$t u(t)$
4	$e^{\lambda_1 t} u(t)$	$e^{\lambda_2 t} u(t)$	$\frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} u(t) \quad \lambda_1 \neq \lambda_2$
5	$e^{\lambda_1 t} u(t)$	$e^{\lambda_1 t} u(t)$	$t e^{\lambda_1 t} u(t)$
6	$t e^{\lambda_1 t} u(t)$	$e^{\lambda_1 t} u(t)$	$\frac{1}{2} t^2 e^{\lambda_1 t} u(t)$
7	$t^N u(t)$	$e^{\lambda_1 t} u(t)$	$\frac{N! e^{\lambda_1 t}}{\lambda_1^{N+1}} u(t) - \sum_{k=0}^N \frac{N! t^{N-k}}{\lambda_1^{k+1} (N-k)!} u(t)$
8	$t^M u(t)$	$t^N u(t)$	$\frac{M! N!}{(M+N+1)!} t^{M+N+1} u(t)$
9	$t e^{\lambda_1 t} u(t)$	$e^{\lambda_2 t} u(t)$	$\frac{e^{\lambda_2 t} - e^{\lambda_1 t} + (\lambda_1 - \lambda_2) t e^{\lambda_1 t}}{(\lambda_1 - \lambda_2)^2} u(t)$
10	$t^M e^{\lambda_1 t} u(t)$	$t^N e^{\lambda_2 t} u(t)$	$\frac{M! N!}{(N+M+1)!} t^{M+N+1} e^{\lambda_2 t} u(t)$
11	$t^M e^{\lambda_1 t} u(t)$	$t^N e^{\lambda_2 t} u(t)$	$\sum_{k=0}^M \frac{(-1)^k M! (N+k)! t^{M-k} e^{\lambda_1 t}}{k! (M-k)! (\lambda_1 - \lambda_2)^{N+k+1}} u(t)$ $+ \sum_{k=0}^N \frac{(-1)^k N! (M+k)! t^{N-k} e^{\lambda_2 t}}{k! (N-k)! (\lambda_2 - \lambda_1)^{M+k+1}} u(t)$
	$\lambda_1 \neq \lambda_2$		
12	$e^{-\alpha t} \cos(\beta t + \theta) u(t)$	$e^{\lambda_1 t} u(t)$	$\frac{\cos(\theta - \phi) e^{\lambda_1 t} - e^{-\alpha t} \cos(\beta t + \theta - \phi)}{\sqrt{(\alpha + \lambda_1)^2 + \beta^2}} u(t)$ $\phi = \tan^{-1}[-\beta/(\alpha + \lambda_1)]$
13	$e^{\lambda_1 t} u(t)$	$e^{\lambda_2 t} u(-t)$	$\frac{e^{\lambda_1 t} u(t) + e^{\lambda_2 t} u(-t)}{\lambda_2 - \lambda_1} \quad \text{Re } \lambda_2 > \text{Re } \lambda_1$
14	$e^{\lambda_1 t} u(-t)$	$e^{\lambda_2 t} u(-t)$	$\frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_2 - \lambda_1} u(-t)$

Laplace Transform Table

No.	$x(t)$	$X(s)$
1	$\delta(t)$	1
2	$u(t)$	$\frac{1}{s}$
3	$tu(t)$	$\frac{1}{s^2}$
4	$t^n u(t)$	$\frac{n!}{s^{n+1}}$
5	$e^{\lambda t} u(t)$	$\frac{1}{s - \lambda}$
6	$t e^{\lambda t} u(t)$	$\frac{1}{(s - \lambda)^2}$
7	$t^n e^{\lambda t} u(t)$	$\frac{n!}{(s - \lambda)^{n+1}}$
8a	$\cos bt u(t)$	$\frac{s}{s^2 + b^2}$
8b	$\sin bt u(t)$	$\frac{b}{s^2 + b^2}$
9a	$e^{-at} \cos bt u(t)$	$\frac{s + a}{(s + a)^2 + b^2}$
9b	$e^{-at} \sin bt u(t)$	$\frac{b}{(s + a)^2 + b^2}$
10a	$r e^{-at} \cos (bt + \theta) u(t)$	$\frac{(r \cos \theta)s + (ar \cos \theta - br \sin \theta)}{s^2 + 2as + (a^2 + b^2)}$
10b	$r e^{-at} \cos (bt + \theta) u(t)$	$\frac{0.5re^{j\theta}}{s + a - jb} + \frac{0.5re^{-j\theta}}{s + a + jb}$
10c	$r e^{-at} \cos (bt + \theta) u(t)$	$\frac{As + B}{s^2 + 2as + c}$
	$r = \sqrt{\frac{A^2 c + B^2 - 2ABa}{c - a^2}}$	
	$\theta = \tan^{-1} \left(\frac{Aa - B}{A\sqrt{c - a^2}} \right)$	
	$b = \sqrt{c - a^2}$	
10d	$e^{-at} \left[A \cos bt + \frac{B - Aa}{b} \sin bt \right] u(t)$	$\frac{As + B}{s^2 + 2as + c}$
	$b = \sqrt{c - a^2}$	

Fourier Transform Table

No.	$x(t)$	$X(\omega)$	
1	$e^{-at}u(t)$	$\frac{1}{a + j\omega}$	$a > 0$
2	$e^{at}u(-t)$	$\frac{1}{a - j\omega}$	$a > 0$
3	$e^{-a t }$	$\frac{2a}{a^2 + \omega^2}$	$a > 0$
4	$te^{-at}u(t)$	$\frac{1}{(a + j\omega)^2}$	$a > 0$
5	$t^n e^{-at}u(t)$	$\frac{n!}{(a + j\omega)^{n+1}}$	$a > 0$
6	$\delta(t)$	1	
7	1	$2\pi\delta(\omega)$	
8	$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	
9	$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	
10	$\sin \omega_0 t$	$j\pi[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]$	
11	$u(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$	
12	$\text{sgn } t$	$\frac{2}{j\omega}$	
13	$\cos \omega_0 t u(t)$	$\frac{\pi}{2}[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] + \frac{j\omega}{\omega_0^2 - \omega^2}$	
14	$\sin \omega_0 t u(t)$	$\frac{\pi}{2j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)] + \frac{\omega_0}{\omega_0^2 - \omega^2}$	
15	$e^{-at} \sin \omega_0 t u(t)$	$\frac{\omega_0}{(a + j\omega)^2 + \omega_0^2}$	$a > 0$
16	$e^{-at} \cos \omega_0 t u(t)$	$\frac{a + j\omega}{(a + j\omega)^2 + \omega_0^2}$	$a > 0$
17	$\text{rect}\left(\frac{t}{\tau}\right)$	$\tau \text{sinc}\left(\frac{\omega\tau}{2}\right)$	
18	$\frac{W}{\pi} \text{sinc}(Wt)$	$\text{rect}\left(\frac{\omega}{2W}\right)$	
19	$\Delta\left(\frac{t}{\tau}\right)$	$\frac{\tau}{2} \text{sinc}^2\left(\frac{\omega\tau}{4}\right)$	
20	$\frac{W}{2\pi} \text{sinc}^2\left(\frac{Wt}{2}\right)$	$\Delta\left(\frac{\omega}{2W}\right)$	
21	$\sum_{n=-\infty}^{\infty} \delta(t - nT)$	$\omega_0 \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0)$	$\omega_0 = \frac{2\pi}{T}$
22	$e^{-t^2/2\sigma^2}$	$\sigma\sqrt{2\pi}e^{-\sigma^2\omega^2/2}$	

z-transform Table

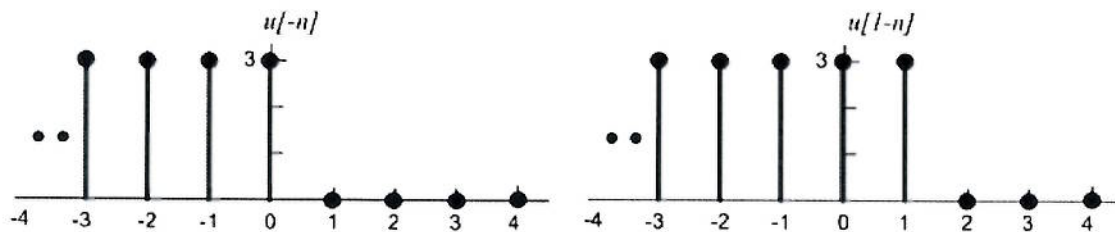
No.	$x[n]$	$X[z]$
1	$\delta[n - n]$	z^{-k}
2	$u[n]$	$\frac{z}{z - 1}$
3	$nu[n]$	$\frac{z}{(z - 1)^2}$
4	$n^2u[n]$	$\frac{z(z + 1)}{(z - 1)^3}$
5	$n^3u[n]$	$\frac{z(z^2 + 4z + 1)}{(z - 1)^4}$
6	$\gamma^n u[n]$	$\frac{z}{z - \gamma}$
7	$\gamma^{n-1} u[n - 1]$	$\frac{1}{z - \gamma}$
8	$n\gamma^n u[n]$	$\frac{\gamma z}{(z - \gamma)^2}$
10	$\frac{n(n - 1)(n - 2) \cdots (n - m + 1)}{\gamma^m m!} \gamma^n u[n]$	$\frac{z}{(z - \gamma)^{m+1}}$
11a	$ \gamma ^n \cos \beta n u[n]$	$\frac{z(z - \gamma \cos \beta)}{z^2 - (2 \gamma \cos \beta)z + \gamma ^2}$
11b	$ \gamma ^n \sin \beta n u[n]$	$\frac{z \gamma \sin \beta}{z^2 - (2 \gamma \cos \beta)z + \gamma ^2}$
12a	$r \gamma ^n \cos (\beta n + \theta) u[n]$	$\frac{rz[z \cos \theta - \gamma \cos (\beta - \theta)]}{z^2 - (2 \gamma \cos \beta)z + \gamma ^2}$
12b	$r \gamma ^n \cos (\beta n + \theta) u[n] \quad \gamma = \gamma e^{j\theta}$	$\frac{(0.5re^{j\theta})z}{z - \gamma} + \frac{(0.5re^{-j\theta})z}{z - \gamma^*}$
12c	$r \gamma ^n \cos (\beta n + \theta) u[n]$	$\frac{z(Az + B)}{z^2 + 2az + \gamma ^2}$
$r = \sqrt{\frac{A^2 \gamma ^2 + B^2 - 2AaB}{ \gamma ^2 - a^2}} \quad \beta = \cos^{-1} \frac{-a}{ \gamma } \quad \theta = \tan^{-1} \frac{Aa - B}{A\sqrt{ \gamma ^2 - a^2}}$		

E2.5 Signals and Linear Systems Solutions 2011

All questions are UNSEEN and covers the whole syllabus of this course.

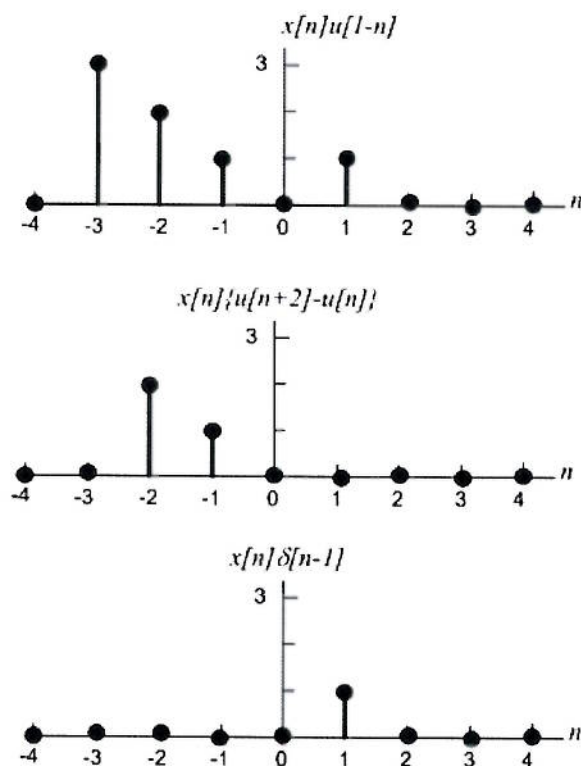
Answer to Question 1 (Topic: Signal modelling)

a)



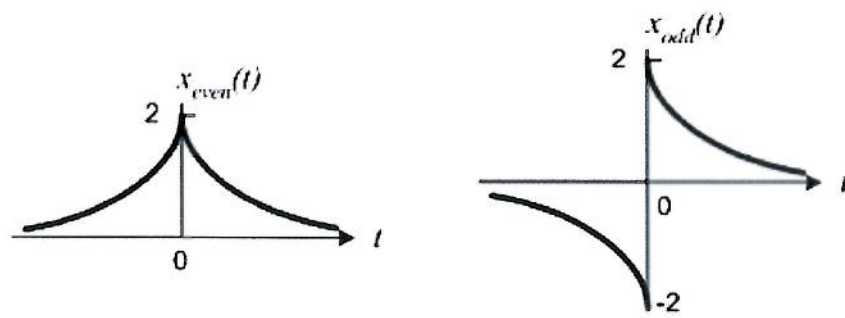
[4]

b)



[6]

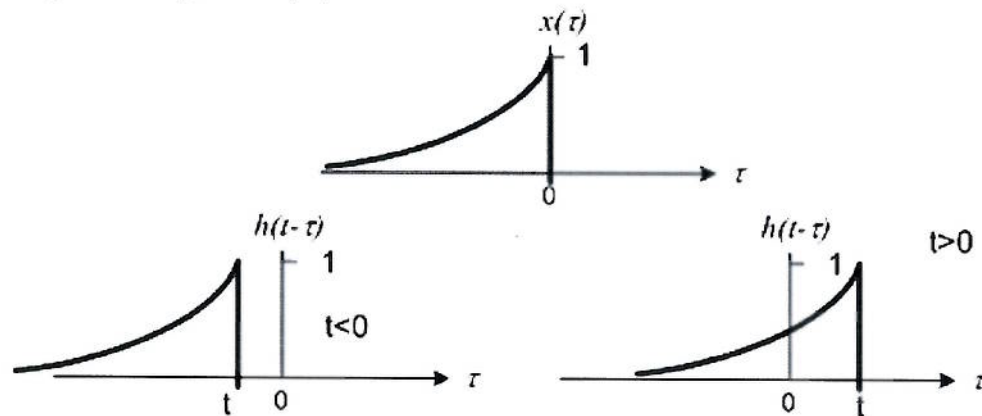
Answer to Question 2 (Topic: Signal modelling & classification)



[5]

Answer to Question 3 (Topic: Convolution)

Here are the plots for $x(\tau)$ and $h(t-\tau)$:

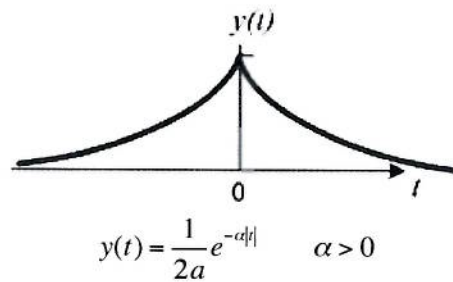


We can see from this diagram that for $t < 0$, $x(\tau)$ and $h(t-\tau)$ overlap from $\tau = -\infty$ to $\tau = t$. For $t > 0$, they overlap from $\tau = -\infty$ to $\tau = 0$. Hence:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$= \begin{cases} \int_{-\infty}^t e^{\alpha\tau} e^{-\alpha(t-\tau)} d\tau = e^{-\alpha t} \int_{-\infty}^t e^{2\alpha\tau} d\tau = \frac{1}{2\alpha} e^{\alpha t} & \text{for } t < 0 \\ \int_{-\infty}^0 e^{\alpha\tau} e^{-\alpha(t-\tau)} d\tau = e^{-\alpha t} \int_{-\infty}^0 e^{2\alpha\tau} d\tau = \frac{1}{2\alpha} e^{-\alpha t} & \text{for } t > 0 \end{cases}$$

Therefore the shape of $y(t)$ is:



Alternatively an easier solution is to use Table 1, Pair 13. This yield an equivalent solution:

$$y(t) = \frac{e^{-\alpha t} u(t) + e^{\alpha t} u(-t)}{2\alpha}$$

[6]

Answer to Question 4 (Topic: System classification & time-domain analysis)

a)

This is a simple circuit where the capacitor C is charged by a constant current $x(t)$. The output voltage $y(t)$ across the capacitor and the input current $x(t)$ are related by:

$$y(t) = \frac{1}{C} \int_{-\infty}^t x(\tau) d\tau$$

[2]

b)

- i) From the above equation, the output depends on both the past and the present input, therefore the system is not memoryless.
- ii) Since the output $y(t)$ does not depend on the future values of the input, the system is causal.
- iii) Since, the integration operation obeys the principle of superposition, therefore the system is linear.
- iv) Let $x_1(t) = x(t - t_0)$, then

$$y_1(t) = \frac{1}{C} \int_{-\infty}^t x(\tau - t_0) d\tau = \frac{1}{C} \int_{-\infty}^{t-t_0} x(\lambda) d\lambda = y(t - t_0)$$

Therefore the system is time-invariant.

[4]

Answer to Question 5 (Topic: Time-domain analysis)

a)

$$\frac{dy(t)}{dt} + ay(t) = x(t)$$

Characteristic equation: $\lambda + a = 0$

Therefore, characteristic root is $\lambda = -a$

Characteristic mode is e^{-at}

Hence $y_{zi}(t) = c_1 e^{-at}$.

Since $y(0) = y_0$, therefore $c_1 = y_0$.

Therefore the zero-input response is:

$$y_{zi}(t) = y_0 e^{-at}$$

b)

$$h(t) = [P(D)y_n(t)]u(t), \quad P(D) = 1 \text{ in this case.}$$

From a), $y_n(t) = k e^{-at}$, where k is a constant.

Now, we know that $y_n(0) = 1$. Therefore $k = 1$.

Therefore the impulse response is:

$$h(t) = e^{-at} u(t).$$

Given that $x(t) = K e^{-bt} u(t)$,

We also know that the zero-state response is:

$$y_{zs}(t) = h(t) * x(t)$$

Use the convolution pair:

$$e^{\lambda_1 t} u(t) \quad e^{\lambda_2 t} u(t) \quad \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} u(t) \quad \lambda_1 \neq \lambda_2$$

$$\begin{aligned} y_{zs}(t) &= e^{-at} u(t) * K e^{-bt} u(t) \\ &= \frac{K}{a-b} (e^{-bt} - e^{-at}) u(t) \end{aligned}$$

c)

$$y(t) = y_0 e^{-at} + \frac{K}{a-b} (e^{-bt} - e^{-at}) u(t)$$

[3]

[4]

[3]

Answer to Question 6 (Topic: Transfer function and Laplace Transform)

a)

The step response of the system is: $2e^{-3t}u(t)$, i.e.

$$y(t) = 2e^{-3t}u(t), \quad x(t) = u(t).$$

Take the Laplace Transform of these:

$$X(s) = \frac{1}{s}, \quad Y(s) = \frac{2}{s+3}.$$

Hence, the system transfer function is:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{2s}{s+3} = 2 - \frac{6}{s+3}.$$

Taking the inverse Laplace transform of $H(s)$:

$$h(t) = 2\delta(t) - 6e^{-3t}u(t).$$

Alternatively, one could calculate $h(t)$ by differentiating the step response:

$$h(t) = \frac{d}{dt}(2e^{-3t}u(t)) = 2\delta - 6e^{-3t}u(t).$$

[5]

b)

$$x(t) = e^{-t}u(t) \leftrightarrow \frac{1}{s+1}.$$

Thus,

$$Y(s) = X(s)H(s) = \frac{2s}{(s+1)(s+3)}.$$

Use partial fraction expansions:

$$Y(s) = \frac{2s}{(s+1)(s+3)} = -\frac{1}{s+1} + \frac{3}{s+3}.$$

Take inverse Laplace transform of $Y(s)$, we get:

$$y(t) = (-e^{-t} + 3e^{-3t})u(t).$$

[5]

Answer to Question 7 (Topic: Inverse Laplace Transform)

Using partial fraction expansion gives:

$$\begin{aligned} X(s) &= \frac{5s+13}{s(s^2+4s+13)} = \frac{5s+13}{s(s+2+j3)(s+2-j3)} \\ &= \frac{c_1}{s} + \frac{c_2}{s-(-2+j3)} + \frac{c_3}{s-(-2-j3)}. \end{aligned}$$

$$c_1 = \left. \frac{5s+13}{(s^2+4s+13)} \right|_{s=0} = 1$$

$$c_2 = \left. \frac{5s+13}{s(s+2+j3)} \right|_{s=-2+j3} = -\frac{1}{2}(1+j)$$

$$c_3 = \left. \frac{5s+13}{s(s+2-j3)} \right|_{s=-2-j3} = -\frac{1}{2}(1-j).$$

Therefore,

$$X(s) = \frac{1}{s} + \frac{-\frac{1}{2}(1+j)}{s-(-2+j3)} + \frac{-\frac{1}{2}(1-j)}{s-(-2-j3)}.$$

Use the Laplace Transform table:

$$x(t) = u(t) - \frac{1}{2}(1+j)e^{(-2+j3)t}u(t) - \frac{1}{2}(1-j)e^{(-2-j3)t}u(t).$$

Therefore,

$$x(t) = [1 - e^{-2t}(\cos 3t - \sin 3t)]u(t).$$

An alternative and equivalent solution is:

$$X(s) = \frac{5s+13}{s(s^2+4s+13)} = \frac{P}{s} + \frac{Qs+R}{(s^2+4s+13)}$$

Equating coefficients of the numerator yields: $P = 1$, $Q = -1$, $R = 1$. Therefore:

$$X(s) = \frac{1}{s} + \frac{1-s}{(s^2+4s+13)}$$

Using transform table pairs 2 and 10c gives:

$$x(t) = [1 + \sqrt{2} e^{-2t} \cos(3t + \pi/4)]u(t)$$

[6]

Answer to Question 8 (Topic: Frequency Response and Bode plot)

$$H(\omega) = \frac{10(1+j\omega)}{(1+j\omega/10)(1+j\omega/100)}.$$

Therefore,

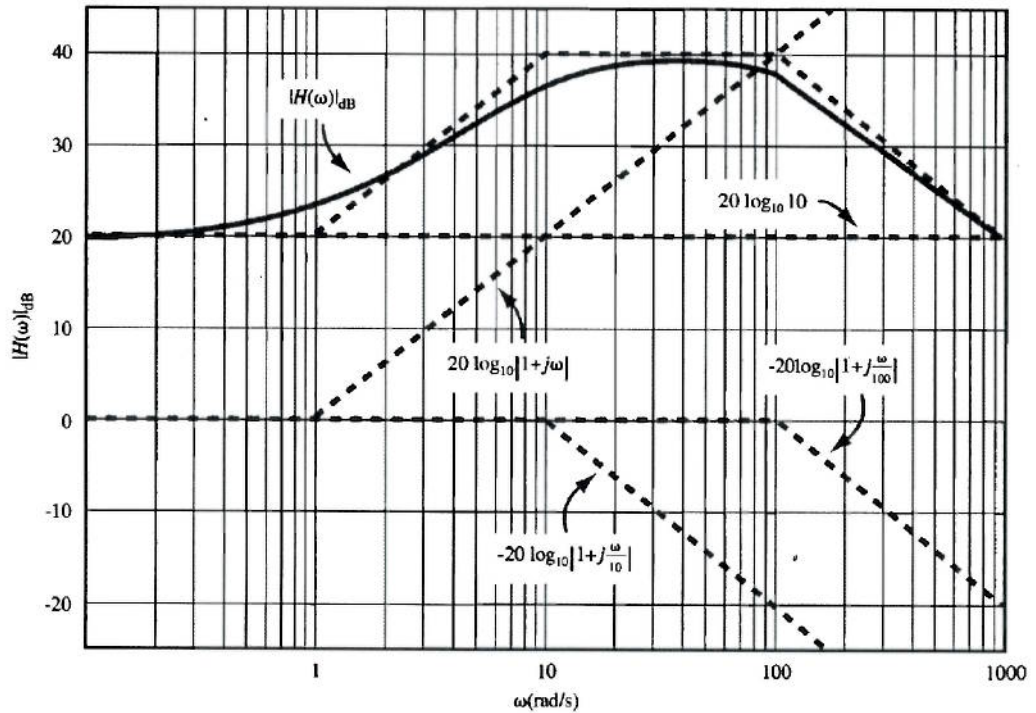
$$|H(\omega)|_{dB} = 20\log_{10} 10 + 20\log_{10} |1+j\omega| - 20\log_{10} |1+j\omega/10| - 20\log_{10} |1+j\omega/100|$$

Therefore the corner frequencies are: $\omega = 1, 10$ and 100 , and the gain at each corner frequencies are:

$$H(1)|_{dB} = 20 + 20\log_{10} \sqrt{2} - 20\log_{10} \sqrt{1.01} - 20\log_{10} \sqrt{1.0001} = 23dB$$

$$H(10)|_{dB} = 20 + 20\log_{10} \sqrt{101} - 20\log_{10} \sqrt{2} - 20\log_{10} \sqrt{1.01} = 37dB$$

$$H(100)|_{dB} = 20 + 20\log_{10} \sqrt{10001} - 20\log_{10} \sqrt{101} - 20\log_{10} \sqrt{2} = 37dB$$

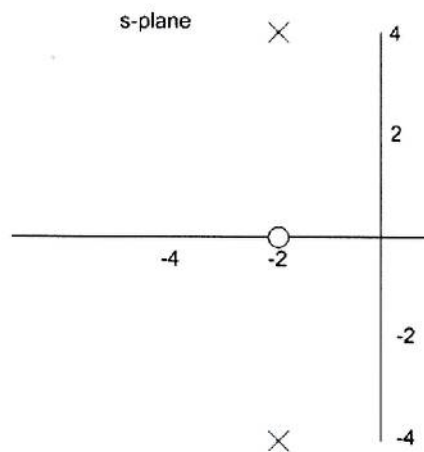


[8]

Answer to Question 9 (Topic: Poles & Zeroes)

$$H(s) = \frac{2s+4}{s^2+4s+20} = \frac{2(s+2)}{(s-(-2+4j))(s-(-2-4j))}$$

The zero is therefore at $s = -2$, the conjugate poles are at $s = -2 \pm 4j$.



[4]

Answer to Question 10 (Topic: Fourier Transform)

a)

$x(t)$	$X(\omega)$
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From the Fourier Transform table, we know that $\cos \omega_0 t$ $\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$.

Therefore

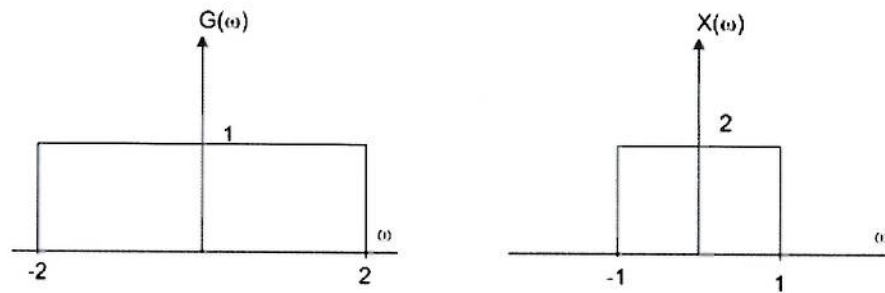
$$w(t) = \cos(t) \leftrightarrow W(\omega) = \pi[\delta(\omega - 1) + \delta(\omega + 1)]$$

Further more we know:

$$g(t) = x(t)\cos(t) \leftrightarrow G(\omega) = \frac{1}{2\pi}\{X(\omega) * W(\omega)\}$$

Therefore,

$$G(\omega) = \frac{1}{2}X((\omega - 1)) + \frac{1}{2}X((\omega + 1)).$$



[6]

b)

Inverse Fourier transform of $X(j\omega)$ gives a sinc-function:

$$x(t) = \frac{2 \sin t}{\pi t}.$$

[3]

Answer to Question 11 (Topic: Sampling)

a) The sampling period $T = 1/4000$. Hence we can replace t with $kT = k/4000$. Hence

$$\begin{aligned}f[k] &= 10 \cos \frac{\pi}{2} k + \sqrt{2} \sin \frac{3\pi}{4} k + 2 \cos \left(\frac{5\pi}{4} k + \frac{\pi}{4} \right) \\&= 10 \cos \frac{\pi}{2} k + \sqrt{2} \sin \frac{3\pi}{4} k + 2 \cos \left(\frac{-3\pi}{4} k + \frac{\pi}{4} \right) \\&= 10 \cos \frac{\pi}{2} k + \sqrt{2} \sin \frac{3\pi}{4} k + 2 \cos \left(\frac{3\pi}{4} k - \frac{\pi}{4} \right)\end{aligned}$$

This can be simplified further using trigonometric identity, but is not compulsory:

$$\begin{aligned}f[k] &= 10 \cos \frac{\pi}{2} k + \sqrt{2} \sin \frac{3\pi}{4} k + 2 \cos \left(\frac{3\pi}{4} k - \frac{\pi}{4} \right) \\&= 10 \cos \frac{\pi}{2} k + \sqrt{2} \sin \frac{3\pi}{4} k + 2 \cos \frac{3\pi}{4} k \cos \frac{\pi}{4} + 2 \sin \frac{3\pi}{4} k \sin \frac{\pi}{4} \\&= 10 \cos \frac{\pi}{2} k + \sqrt{2} \sin \frac{3\pi}{4} k + \sqrt{2} \cos \frac{3\pi}{4} k + \sqrt{2} \sin \frac{3\pi}{4} k \\&= 10 \cos \frac{\pi}{2} k + 2\sqrt{2} \sin \frac{3\pi}{4} k + \sqrt{2} \cos \frac{3\pi}{4} k \\&= 10 \cos \frac{\pi}{2} k + \sqrt{10} \cos \left(\frac{3\pi}{4} k - 1.107 \right)\end{aligned}$$

[5]

b)

Since the frequency $5\pi/4$ has been reduced to $3\pi/4$, there is aliasing.

[3]

c)

The highest frequency in the signal is $\omega = 5000\pi$ or 2500Hz. Therefore to avoid aliasing, the signal must be sampled at a minimum of 5000Hz, or $T = 0.2\text{ms}$.

[2]

Answer to Question 12 (Topic: DFT)

a)

$$y[n] = 0.5y[n-1] + bx[n]$$

Therefore the transfer function is found by taking the z-transform of both sides:

$$\begin{aligned} Y(z) &= 0.5z^{-1}Y(z) + bX(z) \\ H(z) &= \frac{Y(z)}{X(z)} = \frac{b}{1 - 0.5z^{-1}} \end{aligned}$$

To find the frequency response, substitute $z = e^{j\omega}$,

$$H(z)\big|_{z=e^{j\omega}} = \frac{b}{1 - 0.5e^{-j\omega}}.$$

[4]

b)

$$|H(\omega)|^2 = \frac{b^2}{(1 - 0.5e^{-j\omega})(1 - 0.5e^{+j\omega})} = \frac{b^2}{1.25 - \cos\omega}$$

When $\omega = 0$, $|H(\omega)|^2 = 1$.

Therefore

$$\frac{b^2}{1.25 - 1} = 1 \Rightarrow b = \pm 0.5.$$

[3]

c) To find the half-power point,

$$\begin{aligned} |H(\omega)|^2 &= \frac{0.25}{1.25 - \cos\omega} = 0.5 \\ \Rightarrow \cos\omega &= 0.75 \\ \Rightarrow \omega &= 0.23\pi. \end{aligned}$$

[3]

Answer to Question 13 (Topic: z-transform)

a)

Use the z-transform pair:

$$\gamma^n u[n] \quad \frac{z}{z - \gamma}$$

we have:

$$\begin{aligned} X[z] &= \frac{1}{1 - 2z^{-1}} + \frac{3}{1 - 0.5z^{-1}} \\ &= \frac{4 - \frac{13}{2}z^{-1}}{(1 - 2z^{-1})(1 - \frac{1}{2}z^{-1})} \end{aligned}$$

[3]

b)

$$x[n] = \cos[n\omega_0] u[n] = \frac{1}{2} [e^{jn\omega_0} + e^{-jn\omega_0}] u[n].$$

Therefore

$$\begin{aligned} X[z] &= \frac{1}{2} \frac{1}{1 - e^{j\omega_0} z^{-1}} + \frac{1}{2} \frac{1}{1 - e^{-j\omega_0} z^{-1}} \\ &= \frac{1 - \cos \omega_0 z^{-1}}{1 - 2 \cos \omega_0 z^{-1} + z^{-2}}. \end{aligned}$$

[3]