E3.09
C1.0
ISE3.9

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2005

EEE/ISE PART III/IV: MEng, BEng and ACGI

CONTROL ENGINEERING

Friday, 13 May 10:00 am

Time allowed: 3:00 hours

There are SIX questions on this paper.

Answer FOUR questions.

Corrected Copy

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : R.B. Vinter
Second Marker(s) : A. Astolfi

No special instructions for invigilators or instructions for candidates for this paper.

1. Figure 1 shows the block diagram of a control system in which k is an adjustable gain.

(a): Assume

$$G(s) = \frac{s+1}{(s-1)^2} .$$

Sketch the Nyquist diagram of $G(s)$, specifying points of intersection with the negative real axis, if they exist. Hence describe how the stability properties of the closed loop sytem are affected, as k increases in the range $0 < k < \infty$. [14]

(b): Now assume that

$$G(s) = \frac{s+1}{(s-1)^2(1+\alpha s)^2} ,$$

where α is a small positive parameter ($\alpha << 1$), i.e. we replace the plant transfer function of (a) by a more refined transfer function, including two extra lags, to model high frequency dynamic effects.

Briefly indicate how this change of $G(s)$ modifies the Nyquist diagram and comment on the way the closed loop stability properties of the system now change, as k varies in the range $0 < k < \infty$. [6]

Hint: The extra lags in $G(s)$ affect only the high frequency properties of $G(s)$.

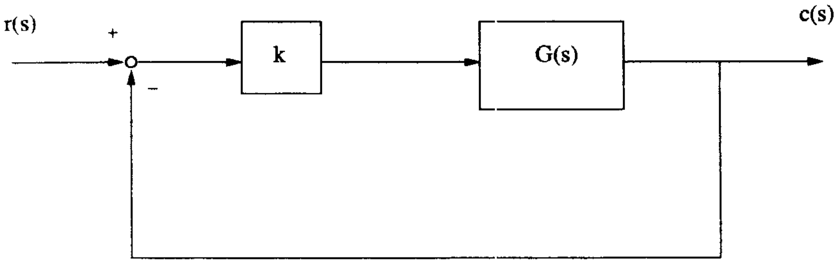


Figure 1

2(a). Consider the control system of Figure 2(a), in which

$$G(s) = \frac{K}{1 + s/\omega_0} .$$

K (> 1) is an adjustable gain and ω_0 is a positive constant. Calculate $\bar{\omega}$ and ω_b, where

$\bar{\omega}$ = the forward path gain cross-over frequency (*i.e.*, $\bar{\omega}$ satisfies $|G(j\bar{\omega})| = 1$)
ω_b = the closed loop bandwith (*i.e.*, ω_b satisfies $|\tilde{G}(j\omega_b)| = \frac{1}{\sqrt{2}}|\tilde{G}(j0)|$)

where $\tilde{G}$ is the closed loop transfer function. Show that

$$\bar{\omega}/\omega_b \to 1 \text{ as } K \to \infty ,$$

that is, for large K, the gain cross-over frequency is approximately the same as the closed loop bandwidth. [6]

2(b). Consider the control system of Fig. 2(b) with velocity feedback, the purpose of which is to increase the gain cross-over frequency (and hence closed loop bandwidth).

(i): Find the value of the velocity feedback gain K_d such that the compensated loop transfer function

$$G_c(s) = (1 + K_d s)\frac{1}{s(1 + 0.01s)^3}$$

has phase margin 45^0. [7]

(*You should assume that $K_d \bar{\omega} >> 1$, for $\bar{\omega}$ the gain cross-over frequency of $G_c(s)$.*)

(ii): As K_d is increased, the gain cross-over frequency $\bar{\omega}$ increases. What is the maximum achievable value of $\bar{\omega}$? What is the corresponding value of K_d? Why is this not a sensible value of K_d to choose? [7]

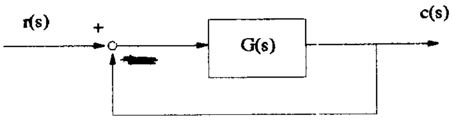


Figure 2(a)

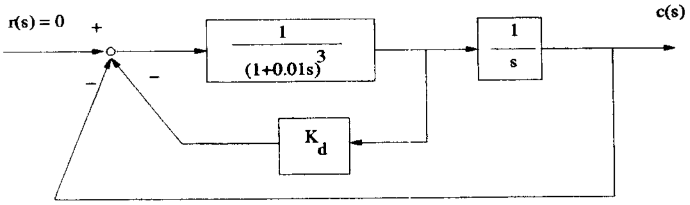


Figure 2(b)

3. Figure 3 shows a double inverted pendulum in the plane. The rods supporting the pendulum masses are rigid, of zero mass and of equal length, l. The pins connecting the two rods and also the lower rod to a rigid support are frictionless. The pendulum masses, m, are equal. A horizontal force F is applied to the lower mass. g is the gravitational constant.

Let θ_1, θ_2 be the angles of the upper and lower rods to the vertical. Let T_1, T_2 be the tensions in the upper and lower rods.

(i): Show that, for small values of the angles θ_1 and θ_2, the motion of the pendulums is approximately described by:

$$\begin{aligned} d^2\theta_1/dt^2 &= \left(\frac{2g}{l}\right)(\theta_1 - \theta_2) - \left(\frac{1}{ml}\right)F \\ d^2\theta_2/dt^2 &= \left(\frac{g}{l}\right)(2\theta_2 - \theta_1) + \left(\frac{1}{ml}\right)F \end{aligned}$$

[10]

Hint: Resolve forces at each mass vertically and horizontally and approximate $\sin(\theta) \approx \theta$, $\cos(\theta) \approx 1$. *Note that, since the vertical accelerations of the masses are small, resolving forces vertically gives*

$$0 \approx T_1\cos(\theta_1) + mg \quad \textit{and} \quad 0 \approx T_2\cos(\theta_2) - T_1\cos(\theta_1) + mg \ .$$

(ii): Let $m = l = g = 1$ (in dimensionless units). Develop a 'small signal' linear state space model, in which the state variables are $x_1 = \theta_1$, $x_2 = \dot{\theta}_1$, $x_3 = \theta_2$, $x_4 = \dot{\theta}_2$ and in which F is regarded as a control variable. [6]

(iii): Show that the system is controllable. [4]

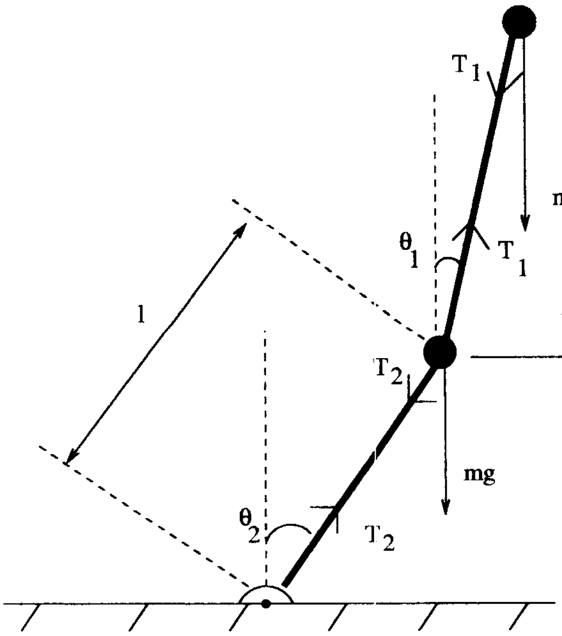


Figure 3

4. Consider the second order system (S):

$$\begin{cases} \dot{x}_1 = a_{11}x_1 + a_{12}x_2 \\ \dot{x}_2 = a_{21}x_1 + a_{22}x_2 + u \end{cases} \qquad \textit{(state equation)}$$

and

$$y(t) = x_1(t) \qquad \textit{(output equation)},$$

for some constants a_{11}, a_{12}, a_{21}, a_{22} ($a_{12} \neq 0$).

Note that the output y coincides with the first component of the state.

(a): Find $k^T = (k_1, k_2)$ such that, for the state feedback law

$$u = -k^T x\,,$$

the closed loop system has characteristic polynomial

$$\delta(s) = \delta_0 + \delta_1 s + s^2$$

for given constants δ_0 and δ_1. [10]

(b): A *reduced order observer* for $x_2(t)$ constructs an estimate $\hat{x}_2(t)$ of x_2 from a solution to the differential equation

$$d\hat{x}_2/dt = a_{21}y + a_{22}\hat{x}_2 + u + g(\dot{y} - a_{11}y - a_{12}\hat{x}_2) \tag{1}$$

in which g is a scalar gain. *(There is no need to estimate x_1 since it is measured directly.)*

(i): Show that the estimation error signal $e_2 = x_2 - \hat{x}_2$ satisfies

$$de_2/dt = (a_{22} - ga_{12})e_2\,,$$

rat
h

Hence choose g so that $e_2(t)$ decays with a time constant τ. [6]

(ii): The reduced order observer (1) requires differentiation of the output y. Show that $\hat{x}_2$ can alternatively be obtained, without differenting y, by solving the equations

$$\begin{cases} dz/dt = a_{21}y + a_{22}(z + gy) + u - g(a_{11}y + a_{12}(z + gy)) \\ \hat{x}_2 = z + gy\,. \end{cases}$$

[4]

5. A machine tool cuts a (time dependent) profile $y(t)$, that depends on the applied control action $u(t)$ according to the (Laplace transform) relationship

$$y(s) = \frac{1}{s-1} u(s)$$

The desired profile y(t) is an exponential curve

$$y_d(t) = e^{-t} \quad t \geq 0\,.$$

Find a feedback strategy of the form

$$u(t) = -k_1 y(t) - k_2 y_d(t)$$

to minimize the cost function

$$\int_0^\infty \left(|y(t) - y_d(t)|^2 + \alpha |u(t)|^2\right) dt$$

for an arbitrary initial value $y(0)$ of y.

You should follow the following procedure: reformulate the problem as a standard problem:

$$(OC) \begin{cases} \textit{Minimize} \int_0^\infty \left(x^T Q x + \alpha |u|^2\right) dt \\ \textit{subject to} \\ \dot{x} = Ax + bu \\ x(0) = x_0 \end{cases}$$

with state vector $(x_1, x_2) = (y(t), e^{-t})$, and use the data below. [14]

Hint: Note that $x_2(t) = e^{-t}$ *satisfies* $\dot{x}_2 = -x_2$, $x_2(0) = 1$.

Now suppose that $y(0) = 0$. Let $e(\alpha)$ be the minimum integrated tracking error:

$$e(\alpha) = min \int_0^\infty |y(t) - y_d(t)|^2 dt.$$

Show that $e(\alpha) \to 0$ as $\alpha \to 0$. [6]

Hint: use the formula (2) below for the minimum cost.

Data: The solution to (OC) *is given by:*

$$u = -\mathbf{b}^T P\, \mathbf{x},$$

where P *is a solution of the Algebraic Riccati Equation (ARE):*

$$\begin{cases} A^T P + PA + Q - \alpha^{-1} P\mathbf{b}\mathbf{b}^T P = 0\,. \\ P = P^T \quad \textit{and} \quad P > 0\,. \end{cases}$$

and

$$x_0^T P x_0 = min\,\{\int_0^\infty \left(x^T Q x + \alpha |u|^2\right) dt\} \qquad (2)$$

6. Consider the nonlinear 'step relay' device with input/output characteristic illustated in Figure 6(a), in which a and b are positive constants.

Show that the describing function of the device is

$$N(A) = \begin{cases} 4b/(\pi A)\left(1 + \frac{\sqrt{A^2-a^2}}{A}\right) & \text{if } A > a \\ 4b/(\pi A) & \text{if } A \leq a\,. \end{cases}$$

The device is incorporated into the control system of Figure 6(b), in which k is a gain parameter. [6]

A limit cycle oscillation is observed at the output of amplitute $A = \frac{5}{4}a$.

(i): What is the frequency of oscillations and what is the value of the gain k? [10]

(ii): Assess whether the limit cycle is stable. [4]

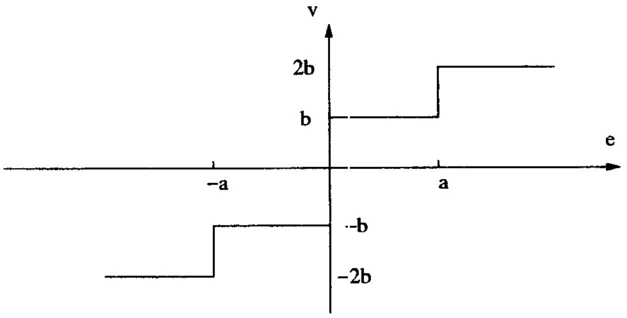


Figure 6(a)

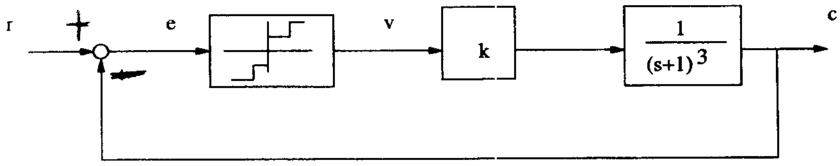


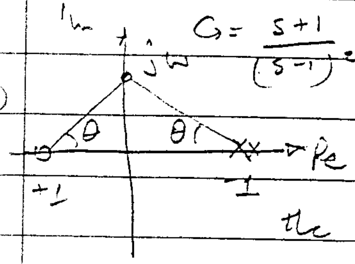
Figure 6(b)

Control Engineering Exam 2005 : Model answers (E3.09, ISE 3.9)

If N = # clockwise encirclements $-1+0j$, P_c = # "unstable" ℄ poles and P_o = # "unstable ⊘ poles, then

$$N = P_c - P_o$$

(a)

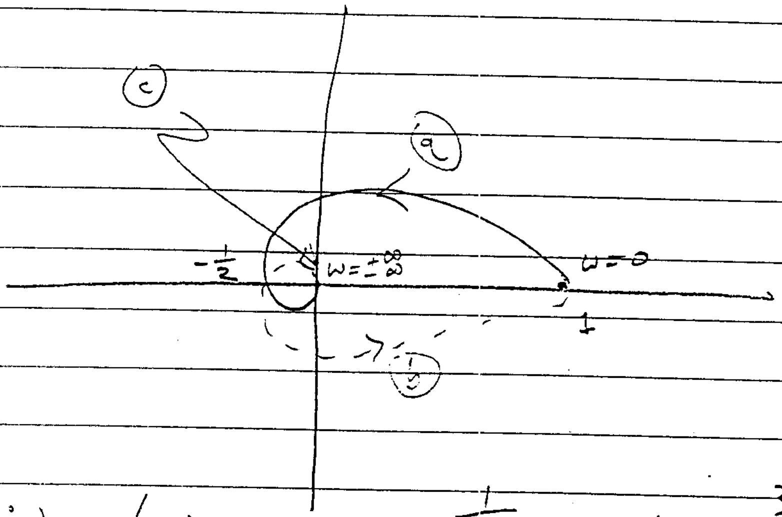


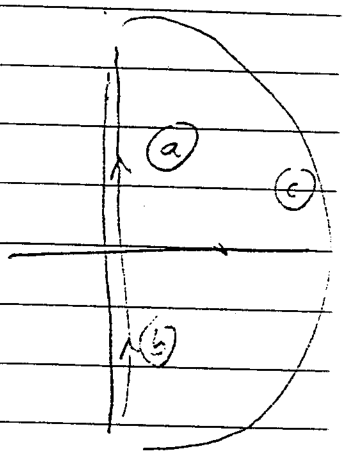
$G = \frac{s+1}{(s-1)^2}$

$$\measuredangle G(j\omega) = \theta - 2(180-\theta) = 3\theta - 360^\circ$$

$$|G(j\omega)| = 1/(1+\omega^2) \qquad 0<\omega<\infty$$

This information makes possible construction of the Nyquist diagram $G(j\omega)$, for $0<\omega<\infty$.





$$G(j\omega) = \frac{(1+j\omega)}{(1-j\omega)^2}, \text{ so } \frac{1}{G(j\omega)} = \frac{(1-j\omega)^3}{1+\omega^2} = \frac{1-3j\omega-3\omega^2+j\omega^3}{1+\omega^2}$$

$\text{Im}\{G(j\omega)\} = 0$ when $\omega = \sqrt{3}$.

Then $G(j\omega) = -\left(\frac{1+\omega^2}{3\omega^2-1}\right) = \frac{4}{8} = \frac{1}{2}$

For $k<2$, there are 0 encirclements of $-\frac{1}{k}+0j$ $(N=0.)$

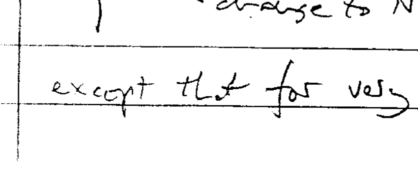
So $-0 = P_c - 2$ or $P_c = 2$ unstable

For $k>2$, there are two anti-clockwise encirclements $(N=-2)$

So $-2 = P_c - 2$ or $P_c = 0$ stable.

[14] unstable for $0 \le k < 2$, stable for $2 < k < \infty$

(b) Adding a double lag with small time constant leaves Nyquist diagram unaffected, but decreases phase by 180° for high frequencies. (It also further attenuates the gain at high frequencies.)



The closed loop stability properties are as before,

[6] except that for very large frequencies the system becomes unstable.

2 (a) The closed loop t.f. is $\tilde{G}(s) = G(1+G)^{-1} = \dfrac{K/(K+1)}{\left(1+s/(\omega_0 \times (1+K))\right)}$

$|\tilde{G}(j\omega_b)| = \frac{1}{\sqrt{2}}|\tilde{G}(j0)| \Longrightarrow$

$$\left[1+\left(\frac{\omega_b}{\omega_0(1+K)}\right)^2\right] = 2 \quad \text{or} \quad \underbrace{\omega_b = \omega_0 \times (1+K)}$$

$\bar{\omega}$ satisfies $|G(j\bar{\omega})| = 1 \Longrightarrow K^2 = 1+\left(\frac{\bar{\omega}}{\omega_0}\right)^2$ whence $\underbrace{\bar{\omega} = (K^2-1)^{1/2}\omega_0}$

We see $\dfrac{\bar{\omega}}{\omega_b} = \dfrac{\omega_0(1+K)}{\omega_0(K^2-1)^{\frac{1}{2}}} \simeq \dfrac{\omega_0 K}{\omega_0 K} = 1$ for K large,

[6] that is, $\bar{\omega}/\omega_b \to 1$ as $K \to \infty$.

(b) (i) Since we assume that $K_d\bar{\omega} >> 1$ ($\bar{\omega}$ = gain cross-over freq.),

$$\measuredangle G_c(j\bar{\omega}) = \measuredangle (1+K_d j\bar{\omega}) \times \frac{1}{j\bar{\omega}(1+0.01j\bar{\omega})^3}$$

$$\simeq +90^\circ - 90^\circ - 3\tan^{-1}\left(\frac{\bar{\omega}}{100}\right) = -180^\circ + 45^\circ$$

Hence $\bar{\omega} = 100\tan(45^\circ) = 100$.

We also require

$|G_c(j\bar{\omega})| = 1 \Longrightarrow \dfrac{K_d\cancel{\bar{\omega}}}{\cancel{\bar{\omega}}|1+j|^3} = 1$. Hence $\underbrace{K_d = 2\sqrt{2}}$

[7] (Check that $K_d\bar{\omega} = 2\sqrt{2} \times 100 >> 1$)

(b) (ii) The maximum gain cross-over frequency $\bar{\omega}$ is achieved when

$$\measuredangle G_c(j\bar{\omega}) = -180^\circ + 0^\circ, \quad \text{i.e.}$$

$$3\tan^{-1}\left(\frac{\bar{\omega}}{100}\right) = 180^\circ - 0 \Longrightarrow \underbrace{\bar{\omega} = 100\tan(60^\circ) = 100\sqrt{3}}$$

Also

$$|G_c(j\bar{\omega})| = 1 \Longrightarrow \frac{K_d\cancel{\bar{\omega}}}{\cancel{\bar{\omega}}|1+\sqrt{3}j|^3} = 1$$

$$\text{or } K_d = (1+3)^{3/2} \underbrace{= 8}$$

This is not a practical value of K_d, because it has a phase margin $\phi = 0^\circ$, so any small modelling errors introducing phase lag into the forward path will induce instability.

[7]

3. (a) Resolving forces vertically at each mass gives

$$0 = T_1 \cos\theta_1 + mg \quad \text{and} \quad 0 = T_2 \cos\theta_2 - T_1 \cos\theta_1 + mg. \quad \text{(see "hint")}$$

Resolving forces horizontally gives

$$-T_1 \sin\theta_1 = m \frac{d^2}{dt^2}(l \sin\theta_1 + l \sin\theta_2)$$

$$T_1 \sin\theta_1 - T_2 \sin\theta_2 + F = m \frac{d}{dt}(l \sin\theta_2)$$

Now approximate $\sin\theta_1 = \theta_1$, $\cos\theta_1 = 1$, etc. This gives

$$ml(\ddot{\theta}_1 + \ddot{\theta}_2) = -T_1\theta_1 = mg\theta_1$$

$$ml\ddot{\theta}_2 = T_1\theta_1 - T_2\theta_2 + F = -mg\theta_1 + 2mg\theta_2 + F$$

$T_1 = -mg$

$T_1 - T_2 = mg$

Subtracting equations gives

$$ml\ddot{\theta}_1 = 2mg(\theta_1 - \theta_2) - F$$

Hence

$$\ddot{\theta}_1 = (2g/l)(\theta_1 - \theta_2) - (1/ml)F$$

$$\ddot{\theta}_2 = (g/l)(2\theta_2 - \theta_1) + (1/ml)F$$

[10]

Setting $m = l = g = 1$ gives

$$\ddot{\theta}_1 = 2\theta_1 - 2\theta_2 - F \quad \text{and} \quad \ddot{\theta}_2 = (2\theta_2 - \theta_1) + F$$

Introducing state variables $x_1 = \theta_1$, $x_2 = \dot{\theta}_1$, $x_3 = \theta_2$, $x_4 = \dot{\theta}_2$ gives

$\dot{x}_1 = x_2$, $\dot{x}_2 = 2x_1 - 2x_3 - F$, $\dot{x}_3 = x_4$, $\dot{x}_4 = -x_1 + 2x_3 + F$.

Expressing these relationships as state space equations, we obtain

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 2 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & +2 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ 0 \\ +1 \end{pmatrix} F$$

[6]

(There is no output equation.)

The controllability matrix is

$$[b \mid Ab \mid \ldots \mid A^3 b] = \left[\begin{array}{c|c|c|c} 0 & -1 & 0 & -4 \\ -1 & 0 & -4 & 0 \\ 0 & +1 & 0 & 3 \\ +1 & 0 & 3 & 0 \end{array}\right]$$

$$\det[\ldots] = +1 \times \det\begin{bmatrix} -1 & -4 & 0 \\ 0 & 0 & 3 \\ 1 & 3 & 0 \end{bmatrix} = 1 \times (-1) \times \det\begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix} = 9$$

Since the controllability matrix is nonsingular, the system is controllable.

[4]

4(a) For state feedback $u = -k_1 x_1 - k_2 x_2$, the closed loop system matrix is

$$\tilde{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21}-k_1 & a_{22}-k_2 \end{bmatrix}$$

This has characteristic polynomial

$$p(s) = (a_{11}-s)(a_{22}-k_2-s) - a_{12}(a_{21}-k_1)$$

$$[a_{11}(a_{22}-k_2) - a_{12}(a_{21}-k_1)] + [k_2 - a_{22} - a_{11}]s + s^2$$

Match this to the desired ch. poly: $\delta_0 + \delta_1 s + s^2$. This gives $(k_2 - a_{22} - a_{11}) = \delta_1$ and $\delta_0 = a_{11}a_{22} - a_{11}k_2 - a_{12}a_{21} + a_{12}k_1$

Hence $\underline{k_2 = \delta_1 + a_{11} + a_{22}}$ and $\underline{k_1 = \dfrac{\delta_0 + a_{11}\delta_1 + a_{11}^2 + a_{12}a_{21}}{a_{12}}}$ 5

4(b) (i) Since $y(t) = x_1(t)$,

$$\dot{x}_2 = a_{21}x_1 + a_{22}x_2 + u$$

$$\dot{\hat{x}}_2 = a_{21}x_1 + a_{22}\hat{x}_2 + u + g(\overbrace{\dot{x}_1 - a_{11}x_1}^{a_{12}x_2} - a_{12}\hat{x}_2) \quad (*)$$

Subtracting, we obtain

$$\frac{d}{dt}(x_2 - \hat{x}_2) = a_{22}(x_2 - \hat{x}_2) - g a_{12}(x_2 - \hat{x}_2)$$

i.e. $\underline{\dot{e}_2(t) = (a_{22} - g a_{12}) e_2(t)}$

This has response $e_2(t) = \text{const.}\, e^{-t/\tau}$, where $\tau = \dfrac{1}{g a_{12} - a_{22}}$

We must choose the reduced order gain g to be

$$\underline{g = (\tau^{-1} + a_{22}) / a_{12}}$$

(ii) From $(*)$,

$$\frac{d}{dt}(\hat{x}_2 - g y) = a_{21}y + a_{22}(\hat{x}_2 - gy + gy) + u - g(a_{11}y + a_{12}(\hat{x}_2 - gy + gy))$$

Writing $z = \hat{x}_2 + gy$, we have

$$\underline{\frac{dz}{dt} = a_{21}y + a_{22}(z + gy) + u - g(a_{11}y + a_{12}(z + gy))}$$

4 and $\underline{\hat{x}_2 = z - gy}$.

5 Take $x_1 = y$ and $x_2 = e^{-t}$. Then $\dot{x}_2 = -x_2$ and $\dot{x}_1 = x_1 + u$

The problem can then be formulated as

$$\begin{cases} \text{Minimize } \int_0^\infty \left(|x_1 - x_2|^2 + \alpha |u|^2 \right) dt \\ \text{s.t. } (\dot{x}_1, \dot{x}_2) = (x_1 + u, -x_2) \\ x_1(0) = y(0), \; x_2(0) = 1. \end{cases}$$

This is a standard LQ optimal control problem with

$$Q = \begin{bmatrix} 1 \\ -1 \end{bmatrix} [1 \; -1] = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

The algebraic Riccati equation is

[4]
$$\begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - \alpha^{-1} \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} [1 \; 0] \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} = 0.$$

Equating entries gives:

$$2P_{11} + 1 = \alpha^{-1} P_{11}^2, \quad \cancel{-P_{12}} + \cancel{P_{12}} - 1 = \alpha^{-1} P_{11} P_{12}, \; -2P_{22} + 1 = \alpha^{-1} P_{12}^2.$$

or $P_{11}^2 - 2\alpha P_{11} - \alpha = 0 \implies P_{11} = \alpha + \sqrt{\alpha^2 + \alpha} = \alpha\left(1 + \sqrt{1 + \alpha^{-1}}\right)$

$$P_{12} = -\frac{\alpha}{P_{12}} = -\frac{1}{1 + \sqrt{1 + \alpha^{-1}}} \quad \text{and} \quad P_{22} = \frac{1}{2}\left(1 - \frac{\alpha^{-1}}{\left(1 + \sqrt{1 + \alpha^{-1}}\right)^2}\right)$$

According to general theory, the feedback solution to the optimal control problem is

$$u(t) = -[1 \; 0] \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -P_{11} x_1 - P_{12} x_2.$$

Since $x_2(t) = e^{-t}$, this gives

[10]
$$u(t) = -\alpha\left(1 + \sqrt{1 + \alpha^{-1}}\right) y(t) - \frac{1}{1 + \sqrt{1 + \alpha^{-1}}} e^{-t}$$

The minimum cost for the general optimal control problem is

$$(x_1(0), x_2(0)) \begin{pmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{pmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}.$$

For the 'machine tool' problem, the initial state $(x_1(0), x_2(0)) = (0, 1)$ since $y(0) = 0$. So, for the optimal policy

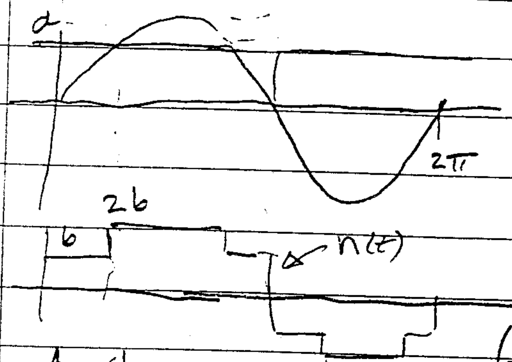
$$(0, 1) \begin{pmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = P_{22} = \int_0^\infty \left(|y(t) - e^{-t}|^2 + \alpha |u|^2 \right) dt$$

But $P_{22} = \frac{1}{2}\left(1 - \frac{\alpha^{-1}}{\left(1 + \sqrt{1 + \alpha^{-1}}\right)^2}\right) = \frac{1}{2}(1 - 1) = 0$, in limit as $\alpha \to 0$

In particular,

[6]
$$\int_0^\infty |y(t) - e^{-t}|^2 dt \to 0 \quad \text{as} \quad \alpha \to 0.$$

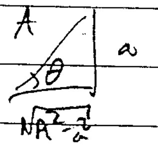
6.



For sinusoidal input signal A sin t, the first harmonic of output has amplitude

$$\frac{1}{2\pi}\int_0^{2\pi} n(t)\,dt = \frac{4}{\pi}\int_0^{\pi/2} n(t)\,dt$$

$$= \left(\frac{4}{\pi}\right)\left[b\int_0^{\sin^{-1}(\frac{a}{A})} \sin t\,dt + 2b\int_{\sin^{-1}(\frac{a}{A})}^{\pi/2} \sin t\,dt\right]$$



$$\left(\frac{4}{\pi}\right)\left[b\left(1-\frac{\sqrt{A^2-a^2}}{A}\right)+2b.\frac{\sqrt{A^2-a^2}}{A}\right) \quad (A>a)$$

$$= \frac{4b}{\pi}\left(1+\frac{\sqrt{A^2-a^2}}{A}\right)$$

If $a<A$, the second term is omitted, So

[6] $$N(A) = \frac{4b}{\pi A}\left(1+\begin{cases}\sqrt{A^2-a^2}/A & \text{if } A>a \\ 0 & \text{if } A<a\end{cases}\right)$$

~~Let ω, A, k be frequency, amplitude and system gain for limit cycle~~

The limit cycle equation can be written $\frac{1}{G(j\omega)} = -kN(A)$, i.e.

$$(1+j\omega)^3 = 1+3j\omega-3\omega^2-\omega^3 j = -kN(A)$$

It follows that $\omega^3-3\omega=0 \Rightarrow \omega=\sqrt{3}\ rs^{-1}$

and $-(3\omega^2-1) = -kN(A)$ or $8 = kN(A)$

But we know $A=\frac{5}{4}a$, so

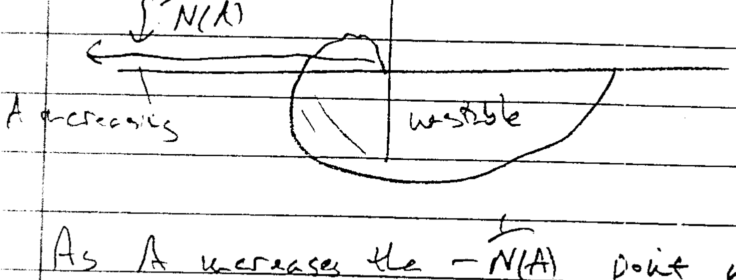
$$N(A) = \frac{16b}{5\pi a}\left(1+\frac{3}{4}\right) = \frac{28}{5}\times\frac{b}{\pi a}$$

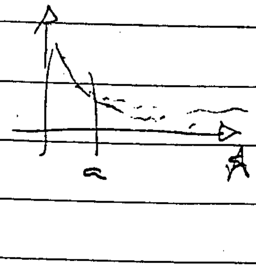
Hence: $8 = k\times\frac{28}{5}\times\frac{b}{\pi a}$ or $k = \frac{10}{7}\frac{\pi a}{b}$

Frequency of limit cycle $= \sqrt{3}\ rs^{-1}$

[5] System gain $k = \frac{10}{7}\frac{\pi a}{b}$

Superposing the Nyquist diagram and the locus of $-\frac{1}{N(A)}$, $0<A<\infty$ gives





As A increases the $-\frac{1}{N(A)}$ point moves from the unstable to the stable region of the complex plane (in relation to $G(j\omega)$, $0<\omega<\infty$)

[4] It follows that the limit cycle is stable