



DEPARTMENT of ELECTRICAL and ELECTRONIC ENGINEERING
EXAMINATIONS 2001
M.Sc in Communications and Signal Processing
M.Eng. Part IV

Solutions 2001
ADVANCED COMMUNICATION THEORY

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ANSWER to Q1

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|-----|---|---|---|---|---|
| 1) | A | B | C | D | E |
| 2) | A | B | C | D | E |
| 3) | A | B | C | D | E |
| 4) | A | B | C | D | E |
| 5) | A | B | C | D | E |
| 6) | A | B | C | D | E |
| 7) | A | B | C | D | E |
| 8) | A | B | C | D | E |
| 9) | A | B | C | D | E |
| 10) | A | B | C | D | E |
| 11) | A | B | C | D | E |
| 12) | A | B | C | D | E |
| 13) | A | B | C | D | E |
| 14) | A | B | C | D | E |
| 15) | A | B | C | D | E |
| 16) | A | B | C | D | E |
| 17) | A | B | C | D | E |
| 18) | A | B | C | D | E |
| 19) | A | B | C | D | E |
| 20) | A | B | C | D | E |

Solutions

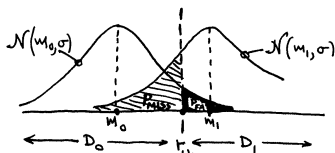
ANSWER to Q2

The correlation receiver implements the following optimum decision rule:

choose H_1 iff $G \geq r_{th}$

where $G \triangleq \int_0^{T_{cs}} r(t) \cdot s_1(t) dt - \int_0^{T_{cs}} r(t) \cdot s_0(t) dt$

$$r_{th} \triangleq \frac{N_0}{2} \ln(\gamma_0) + \frac{1}{2} \int_0^{T_{cs}} (s_1(t)^2 - s_0(t)^2) dt$$



where $w_0 = E\{G|H_0\} = \dots = \int_0^{T_{cs}} (s_0(t) s_1(t) - s_0(t)^2) dt$

$$m_1 = E\{G | H_1\} = \dots = \int_0^{\tau_{cs}} (s_1(t)^2 - s_0(t)s_1(t)) dt$$

$$\begin{aligned}\sigma^2 &= \text{Var}\{G\} \leftarrow \text{for any Hypothesis} \\ &= \dots = \frac{N_0}{2} \int_0^{T_s} (s_1(t)^2 + s_0(t)^2 - 2s_1(t)s_0(t)) dt \\ &= N_0 E(1-\rho)\end{aligned}$$

$$\therefore P_E = \underbrace{\Pr(H_0) \cdot \Pr(D_1 | H_0)}_{P_{FA} = T\left\{\frac{r_{th} - \mu_0}{\sigma}\right\}} + \underbrace{\Pr(H_1) \cdot \Pr(D_0 | H_1)}_{P_{MSJ} = T\left\{\frac{r_{th} - \mu_1}{\sigma}\right\}}$$

Solutions

However $\frac{r_{th}-m_0}{\sigma}$ can be express as a function of

γ_0, EVE, p as follows:

$$\begin{aligned} \frac{r_{ph} - m_0}{\sigma} &= \frac{\frac{N_0}{2} \mathcal{Q}_4(\lambda_0) + \frac{1}{2} \int_{\tau_{ph}}^{\tau_{ph}} (s_1(\tau)^2 - s_0(\tau)^2) d\tau}{\sqrt{N_0 E(1-p)}} - \frac{\int_{\tau_0}^{\tau_0} (s_0(\tau) s_1(\tau) - s_0^2(\tau)) d\tau}{\sqrt{N_0 E(1-p)}} \\ &= \frac{\frac{N_0}{2} \mathcal{Q}_4(\lambda_0) + \frac{1}{2} \int_{\tau_{ph}}^{\tau_{ph}} (s_1^2(\tau) + s_0^2(\tau)) d\tau}{\sqrt{N_0 E(1-p)}} - \frac{\int_{\tau_0}^{\tau_0} s_0(\tau) \cdot s_1(\tau) d\tau}{\sqrt{N_0 E(1-p)}} \\ &= \frac{\frac{N_0}{2} \mathcal{Q}_4(\lambda_0) + E(1-p)}{\sqrt{N_0 E(1-p)}} = \frac{\frac{1}{2} \mathcal{Q}_4(\lambda_0)}{\sqrt{(1-p) E U E}} + \sqrt{(1-p) E U E} \end{aligned}$$

Similarly, it can be found that

$$\frac{r_{fk} - w_1}{\sigma} = \frac{\frac{1}{2} \ln(\lambda_0)}{\sqrt{(1-\phi) EUE}} - \sqrt{(1-\phi) EUE}$$

\therefore Equ-1 (above) becomes

$$P_e = P_r(u_0) \cdot \Gamma \left(\frac{\frac{1}{2} \rho_u(\lambda_0)}{A} + A \right) + P_r(u_1) \cdot \Gamma \left(\frac{\frac{1}{2} \rho_u(\lambda_0)}{A} - A \right)$$

where $A = \sqrt{(1-p)EVE}$

Solutions

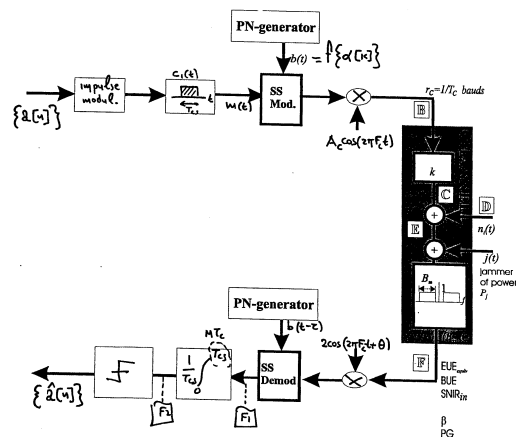
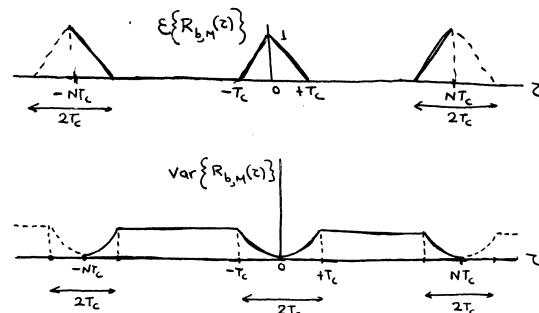
* Bayes' decision criterion: if $q(r) > q_0$ then choose H_1 otherwise H_0
 where $q_0 = \frac{\Pr(H_0) \cdot C_{10} - C_{00}}{\Pr(H_1) \cdot C_{01} - C_{11}}$
 $\Rightarrow q_0 = \frac{1/3}{2/3} \cdot \frac{1.958}{0.5} \Rightarrow q_0 = 1.858$

* $P_{Miss} = \Pr(D_0|H_1) = \Pr\left\{\frac{\frac{1}{2}b_1A_0}{A} - A\right\} = 0.04$
 \Rightarrow (from Tail function graph) $\Rightarrow \frac{\frac{1}{2}b_1A_0}{A} - A = 1.75$
 $\Rightarrow A^2 + 1.75A - \frac{1}{2}b_1A_0 = 0$ with $A > 0$
 $\Rightarrow A = \frac{-1.75 \pm \sqrt{1.75^2 + 4 \cdot \frac{1}{2}b_1A_0}}{2} = 0.162$
 $\Rightarrow \sqrt{(1-p)EUE} = 0.162 \Rightarrow (1-p)EUE = 0.162^2$
 $\Rightarrow p = \frac{1}{2}$

* $P_{FA} = \Pr(D_1|H_0) = \Pr\left\{\frac{\frac{1}{2}b_1A_0}{A} + A\right\} = \Pr(2.0740) \approx 1.7 \times 10^{-2}$

$P_e = \Pr(H_0) \cdot P_{FA} + \Pr(H_1) \cdot P_{Miss} = 3.23 \times 10^{-2}$

ANSWER to Q3



[B]: $s(t) = A_c m(t) \cdot b(t) \cos(2\pi F_c t)$

[F]: desired signal term $= K s(t) = K A_c m(t) b(t) \cos(2\pi F_c t)$
 $\sqrt{2P_s}$

[F1]: desired signal term $= K s(t) \cdot b(t-z) 2\cos(2\pi F_c t + \theta)$
 $= \sqrt{2P_s} m(t) b(t) \cdot b(t-z) 2\cos(2\pi F_c t) \cos(2\pi F_c t + \theta)$

[F2]: desired signal term $= w_0(t) =$
 $= \frac{\sqrt{2P_s}}{T_{cs}} \int_{-1}^{1} m(t) \cdot b(t) \cdot b(t-z) \cos \theta \cdot dt$
 $= \pm \frac{\sqrt{2P_s}}{T_{cs}} \cos \theta \int_0^{MT_c} b(t) b(t-z) dt$
 $= \pm \sqrt{2P_s} \cos \theta R_{b,M}(z)$

Power of $w_0(t) = E\{w_0^2(t)\} = 2P_s \cos^2 \theta \cdot E\{R_{b,M}^2(z)\}$
 $= 2P_s \cos^2 \theta (\text{var}\{R_{b,M}(z)\} + E\{R_{b,M}(z)\}^2)$
 $= 2P_s \cos^2 \theta \text{var}\{R_{b,M}(z)\} + 2P_s \cos^2 \theta E\{R_{b,M}(z)\}^2$
 (code noise power) (desired term)

1. If $0 \leq z \leq T_c$ then power of code noise $= 2P_s \cos^2 \theta \text{var}\{R_{b,M}(z)\}$
 $= 2P_s \cos^2 \theta \cdot \frac{1}{M} \left(\frac{T_c}{T_c}\right)^2$
 $= 2P_s \cos^2 \theta \cdot \frac{1}{M} \cdot \frac{T_c^2}{T_c^2}$ (note: $M = \frac{T_{cs}}{T_c}$)
 $= 2P_s \cos^2 \theta \cdot \frac{1}{T_{cs} T_c} \cdot T_c^2$ [1]

2. If $z > T_c$ then power of code noise $= 2P_s \cos^2 \theta \text{var}\{R_{b,M}(z)\}$
 $= 2P_s \cos^2 \theta \cdot \frac{1}{M} \cdot \frac{T_c^2}{T_{cs} T_c}$ [2]

$T_c = 10 \text{ M chips/sec} \Rightarrow T_c = 10^{-7}$

$T_b = 1000 \text{ bits/sec} \Rightarrow T_{cs} = 10^{-3}$

$N_0 = 0.5 \times 10^{-8} \Rightarrow N_0 = 10^{-8}$

$EUE = 100 \Rightarrow \frac{E_b}{N_0} = 10^2 \Rightarrow \frac{P_s T_{cs}}{N_0} = 10^2 \Rightarrow P_s = 10^2 \cdot \frac{N_0}{T_{cs}} = 10^3$

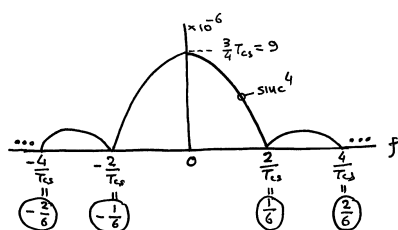
$P_{\text{code noise}} = 1.5 \times 10^{-7} \Rightarrow 1.5 \times 10^{-7} = 2P_s \cos^2 \theta \cdot \frac{T_c}{T_{cs}}$
 $\Rightarrow \cos^2 \theta = \frac{1.5 \times 10^{-7} \cdot T_{cs}}{2P_s T_c} = \frac{1.5 \times 10^{-7} \cdot 10^{-3}}{2 \cdot 10^3 \cdot 10^{-7}} = 0.75$
 $\Rightarrow \cos \theta = \sqrt{0.75} = 0.866 \Rightarrow \theta = 30^\circ$

$P_{\text{code noise}} = 3.75 \times 10^{-8} \Rightarrow 3.75 \times 10^{-8} = 2P_s \cos^2 \theta \cdot \frac{1}{T_{cs} T_c} \cdot T_c^2$
 $\Rightarrow T_c = \frac{3.75 \times 10^{-8} \cdot T_{cs}}{2 \cdot 0.75 \cdot P_s} = \frac{3.75 \times 10^{-8} \cdot 10^{-3}}{2 \cdot 0.75 \cdot 10^3} = 0.5 \times 10^{-7} \text{ le } T_c = 0.5 T_c$

ANSWER to Q4

$$\begin{array}{ll} N_0 = 2 \times 10^{-6} & A_1 = -3 \text{ mV} \\ T_{cs} = 12 & A_2 = -1 \text{ mV} \\ D = 1 & A_3 = 1 \text{ mV} \\ M = 4 & A_4 = 3 \text{ mV} \end{array}$$

$$\begin{aligned}
 P_{SD_s}(f) &= \frac{1}{T_{cs}} \mathbb{E} \left\{ \left| \text{FT} \left(A_L \cdot \Lambda \left(\frac{t}{T_{cs}/2} \right) \right) \right|^2 \right\} \\
 &= \frac{1}{T_{cs}} \mathbb{E} \left\{ \left| A_L \frac{T_{cs}}{2} \text{sinc}^2 \left(f \frac{T_{cs}}{2} \right) \right|^2 \right\} \\
 &= \frac{1}{T_{cs}} \mathbb{E} \left\{ A_L^2 \frac{T_{cs}^2}{4} \text{sinc}^4 \left(f \frac{T_{cs}}{2} \right) \right\} \\
 &\quad \quad \quad \uparrow \\
 &\quad \quad \quad \text{random variable} \\
 &= \frac{1}{T_{cs}} \cdot \frac{T_{cs}^2}{4} \cdot \text{sinc}^4 \left(f \cdot \frac{T_{cs}}{2} \right) \cdot \mathbb{E} \{ A_L^2 \} \\
 &= \frac{T_{cs}}{4} \text{sinc}^4 \left(f \cdot \frac{T_{cs}}{2} \right) \left[(-3)^2 \frac{1}{8} + (-1)^2 \frac{3}{8} + (1)^2 \frac{3}{8} + 3^2 \frac{1}{8} \right] \times 10^{-6} \\
 &= \frac{3}{4} T_{cs} \text{sinc}^4 \left(f \cdot \frac{T_{cs}}{2} \right) \times 10^{-6} = 9 \text{sinc}^4 (f \cdot 6) \times 10^{-6} \\
 &\quad \quad \quad \uparrow \\
 &\quad \quad \quad 12
 \end{aligned}$$

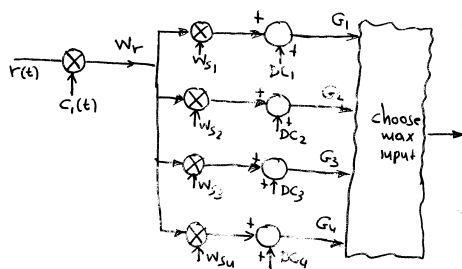


$$\begin{aligned}
 E_L &= \int_{-T_{cs}/2}^{T_{cs}/2} A_L^2 \lambda^2 \left(\frac{t}{T_{cs}/2} \right) dt \\
 &= 2 \int_{-T_{cs}/2}^0 A_L^2 \left(\frac{t + T_{cs}/2}{T_{cs}/2} \right)^2 dt \\
 &= 2 A_L^2 \int_{-T_{cs}/2}^0 \frac{t^2 + T_{cs}t/2 + 2t \frac{T_{cs}}{2}}{T_{cs}^2/4} dt \\
 &= \frac{8 A_L^2}{T_{cs}^2} \int_{-T_{cs}/2}^0 \left(t^2 + \frac{T_{cs}t}{2} + \frac{T_{cs}^2}{4} \right) dt \\
 &= \frac{8 A_L^2}{T_{cs}^2} \left(\frac{t^3}{3} \Big|_{-T_{cs}/2}^0 + T_{cs} \frac{t^2}{2} \Big|_{-T_{cs}/2}^0 + \frac{T_{cs}^2}{4} t \Big|_{-T_{cs}/2}^0 \right) \\
 &= \frac{8 A_L^2}{T_{cs}^2} \left(\frac{T_{cs}^3}{3 \cdot 8} - \frac{T_{cs}^3}{8} + \frac{T_{cs}^3}{8} \right) \\
 &= \frac{1}{3} T_{cs} \cdot A_L^2 = 4 \cdot A_L^2 \quad \uparrow \\
 &\quad \quad \quad 12
 \end{aligned}$$

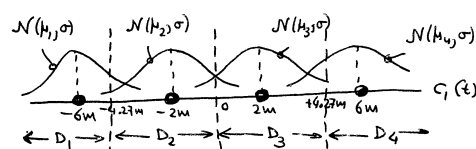
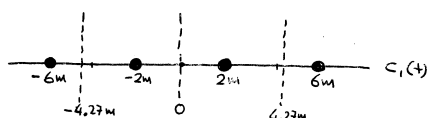
$$\therefore C_1(t) = A_1 \wedge \left(\frac{2t}{T_{cs}} \right) / \sqrt{E_1} = \frac{A_1}{2 A_1} \wedge \left(\frac{2t}{T_{cs}} \right)$$

$$\therefore C_1(t) = \frac{1}{2} \wedge \left(\frac{t}{6} \right)$$

$$\left. \begin{aligned} \underline{W}_{S_1} &= -\sqrt{E_1} = -6 \text{ mV} \\ \underline{W}_{S_2} &= -\sqrt{E_2} = -2 \text{ mV} \\ \underline{W}_{S_3} &= \sqrt{E_3} = 2 \text{ mV} \\ \underline{W}_{S_4} &= \sqrt{E_4} = 6 \text{ mV} \end{aligned} \right\} \begin{aligned} \text{DC}_1 &= \frac{N_0}{2} \varrho_1(P_1) - \frac{1}{2} E_1 = -20.079 \times 10^{-6} \\ \text{DC}_2 &= \frac{N_0}{2} \varrho_2(P_2) - \frac{1}{2} E_2 = -2.98 \times 10^{-6} \\ \text{DC}_3 &= \frac{N_0}{2} \varrho_3(P_3) - \frac{1}{2} E_3 = -2.98 \times 10^{-6} \\ \text{DC}_4 &= \frac{N_0}{2} \varrho_4(P_4) - \frac{1}{2} E_4 = -20.079 \times 10^{-6} \end{aligned}$$



$$\begin{aligned} G_1 &= W_1 W_{S_1} + D C_1 \\ G_1 = G_2 &\Rightarrow W_1 W_{S_1} + D C_1 = W_1 W_{S_2} + D C_2 \Rightarrow W_1 = \frac{D C_2 - D C_1}{W_{S_1} - W_{S_2}} = -4.27 \text{ m} \\ G_2 = G_3 &\Rightarrow \dots \Rightarrow W_2 = \frac{D C_3 - D C_2}{W_{S_2} - W_{S_3}} = 0 \\ G_3 = G_4 &\Rightarrow \dots \Rightarrow W_3 = \frac{D C_4 - D C_3}{W_{S_3} - W_{S_4}} = +4.27 \text{ m} \end{aligned}$$



$$\mu_1 = -6\text{ mV} \quad \sigma^2 = \frac{N_0}{2} \cdot \frac{2R}{2T_0} T_0 = \frac{N_0}{2} = 10^{-6} \Rightarrow \sigma = 10^{-3}$$

$$\left. \begin{aligned} \Pr(D_1|H_1) &= \Pr(D_4|H_4) \\ \Pr(D_2|H_2) &= \Pr(D_3|H_3) \end{aligned} \right\} \text{due to symmetry}$$

$$\begin{aligned} P_{B,CS} &= 1 - 2 \Pr(D_1 | H_1) P_1 - 2 \Pr(D_2 | H_2) P_2 \\ &= 1 - 2 \left(1 - \Gamma \left(\frac{6 \cdot 4.27 \cdot 10^{-3}}{10^{-3}} \right) \right) \frac{1}{8} - 2 \left(1 - \Gamma(2) - \Gamma(2.27) \right) \frac{3}{8} \\ &= \frac{2}{8} \Gamma(1.73) + \frac{6}{8} \Gamma(2) + \frac{6}{8} \Gamma(2.27) \end{aligned}$$

$$= 2/8 \cdot 0.0418 + 6/8 \cdot 0.0228 + 6/8 \cdot 0.0116$$

$$= 0.0362$$