UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 1996

BEng Honours Degree in Computing Part I
MEng Honours Degrees in Computing Part I
BSc Honours Degree in Mathematics and Computer Science Part I
MSci Honours Degree in Mathematics and Computer Science Part I
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the
Associateship of the Royal College of Science
Associateship of the City and Guilds of London Institute

PAPER 1.1 / MC1.1

LOGIC

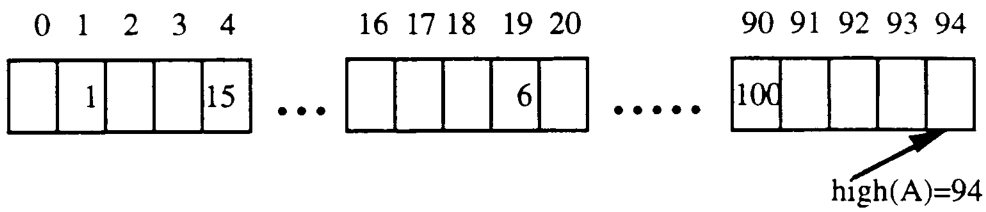
Monday, May 13th 1996, 4.00 - 5.30

Answer THREE questions

For admin. only: paper contains
4 questions
4 pages (excluding cover page)

1 A finite, but sparse, array of numbers is represented by the relation *is_in_at(x,y,z)*, which is read as "element *x* is the value in array *y* at index *z* ". Each array *y* is indexed from 0 up to a potential maximum given by the term *high(y)*.

a i Represent the array A (with 4 items present) using *is_in_at* :



We call this, together with the equation *high(A)* =94, the *is_in_at -style* representation of A.

ii In order for the *is_in_at -style* to be a valid representation, some constraints must be met.

Translate into *natural English* the constraints:

(1) $\forall x,y,z$ [is_in_at $(x,y,z) \rightarrow 0 \leq z \wedge z \leq$ high(y)]

(2) $\neg \exists$ x,y,z,w [is_in_at (z,y,x) $\wedge$ is_in_at (w,y,x) $\wedge$ $z \neq w$]

b Translate into logic the post-conditions of the following operations on valid arrays.

The type of a valid *is_in_at* represented array A is `is_in_at_style`.

i function IsIn (X: real; A: is_in_at_style): boolean
%post: result is true iff X occurs at some index in A.

ii function Upper (A: is_in_at_style): integer
%post: result is the maximum index currently used in A.
% Note: In the array of part a) Upper(A) is 90.

iii function IsCompact (A: is_in_at_style): boolean
%post: result is true iff the only positions in A that contain no value
% are those > Upper(A); i.e. there are no "gaps".

iv function MakeUnique (A: is_in_at_style) : is_in_at_style
%pre: high(A) <= high(result)
%post: result consists of exactly one occurrence of each value in A;
% no other value is in result.

The two parts carry, respectively, 20%, 80% of the marks.

$(A \to (B \to C)) \equiv ((A \wedge B) \to C)$

ii Show, using the equivalence

$$(x \in f(y , z)) \equiv ((x \in y) \wedge \neg (x \in z)) \text{ (for any x, y, z),}$$

the equivalence of part ai) and $A \to B \equiv \neg B \to \neg A$, that

$$(D \in f(A , B) \to D \in C) \equiv (D \in f(A , C) \to D \in B)$$

(the predicate $\in$ is infix).

iii Use Natural Deduction to show (without using equivalences)

$$\{\forall x [x \in A \to x \in C], \neg \exists x [x \in B \wedge x \in C]\} \vdash \forall x[x \in A \to \neg (x \in B)]$$

iv Using equivalences, rewrite $\neg \exists x [x \in B \wedge x \in C]$ into a sentence of the form $\forall x [\ldots \to \ldots]$ and show how the proof of part aiii) can be simplified by using the new form.

b Suppose the usual $\vee$E rule is replaced by the following rule, called alt-$\vee$E:

$$A \vee B , \neg B \vdash A.$$

Using this alt $\vee$E rule, PC and any other rules *except* the usual $\vee$E rule, use natural deduction to show

$$\{\forall x,y [R(x, y) \wedge R(y, x) \to x = y], \forall x[L(x) \to \forall y [R(x, y) \vee x = y]]\} \vdash \forall z,y [L(z) \wedge L(y) \to z = y]$$

(Hint: $\neg (Z = Y) \equiv \neg (Y = Z)$)

The two parts carry, respectively, 65%, 35% of the marks.

Turn over ...

E(x, y) is interpreted as $x \in y$ such that (1) is true and (2) is false.

$$(1)\ \exists z\, [\, E(a, z) \wedge \forall u\, [\, E(u,z) \rightarrow E(u,b)\,]\,]$$
$$(2)\ \forall z,w\, [\, E(z, b) \wedge E(w, b) \rightarrow z = w\,]$$

Justify your answer carefully.

(P(X) is the power set of X.)

(Hint: Remember to interpret "=" and find interpretations for the constants "a" and "b".)

ii What does this tell you about (1) $\vdash$ (2) (where $\vdash$ is proof by natural deduction) and why?

bi Translate into logic the following:

(a) There is something different from a.
(b) a makes contact only with itself.
(c) If x makes contact with y, then y makes contact with x.
(d) Something makes contact with everything.

ii Now consider the following outline proof of $\neg$ (d) from (a) - (c):

Suppose Z is an arbitrary thing that makes contact with everything.
Suppose also that $b \neq a$.
Hence Z makes contact with a and with b.
Hence a makes contact with Z and so Z = a.
But then b = a, a contradiction.
Therefore nothing makes contact with everything.

Translate the proof into natural deduction.

The two parts carry, respectively, 50%, 50% of the marks.

$$A \vee B,\ \neg C \rightarrow \neg A,\ \neg(B \wedge \neg C) \vdash C$$

b This part is about the relationship between $\equiv$, $\leftrightarrow$ and $\models$ for sentences A and B of *propositional* logic.

i What do $A \equiv B$ and $A \models B$ mean?

ii Explain carefully why $(A \equiv B)$ iff $(A \models B$ and $B \models A)$.

iii Explain why $(A \equiv B)$ iff $(A \leftrightarrow B$ is true$)$.

c Use natural deduction without using equivalences to show

$$\{P(0),\ \forall u\,[\,u\text{-}1 \geq 0 \wedge P(u\text{-}1) \rightarrow P(u)]\}$$
$$\vdash\ \forall z[\,\forall y\,(y < z \rightarrow P(y)) \rightarrow P(z)\,]$$

You may assume all values are of type Nat (integers ≥ 0) and that the following properties hold for any Nat x:

$x = 0 \vee x > 0$,
$x - 1 < x$, and
$x > 0 \rightarrow x\text{-}1 \geq 0$

The three parts carry, respectively, 20%, 45%, 35% of the marks.

End Of Paper