Master

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2009

EEE/ISE PART I: MEng, BEng and ACGI

COMMUNICATIONS 1

Corrected Copy

Friday, 12 June 10:00 am

Time allowed: 2:00 hours

There are FOUR questions on this paper.

Q1 is compulsory.
Answer Q1 and any two of questions 2-4.
Q1 carries 40% of the marks. Questions 2 to 4 carry equal marks (30% each).

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : P.L. Dragotti, P.L. Dragotti

Second Marker(s) : M.K. Gurcan, M.K. Gurcan

Special Information for the Invigilators: none

Information for Candidates

Some Fourier Transforms

$$\cos\omega_0 t \iff \pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$$

$$\mathrm{rect}(\tfrac{t}{\tau}) \iff \tau\mathrm{sinc}(\tfrac{\omega\tau}{2})$$

$$\tfrac{W}{\pi}\mathrm{sinc}(Wt) \iff \mathrm{rect}(\tfrac{\omega}{2W})$$

$$\tfrac{\alpha^2}{2\pi}\mathrm{sinc}^2\left(\tfrac{\alpha t}{2}\right) \iff \Delta\left(\tfrac{\omega}{\alpha}\right)$$

where

$$\Delta(\omega)=\begin{cases}1-|\omega|, & |\omega|\le 1\\ 0, & \text{otherwise.}\end{cases}$$

The Scaling Property of the Fourier Transform

$$g(\alpha t)\iff \frac{1}{|\alpha|}G\left(\frac{\omega}{\alpha}\right).$$

Time-Shifting Property of the Fourier Transform

$$g(t-t_0)\iff G(\omega)e^{-j\omega t_0}$$

Time differentiation

$$\frac{d^n g}{dt^n}\iff (j\omega)^n G(\omega)$$

Some useful trigonometric identities

$$\cos x\cos y=\tfrac{1}{2}\cos(x-y)+\tfrac{1}{2}\cos(x+y).$$

$$\sin x\sin y=\tfrac{1}{2}\cos(x-y)-\tfrac{1}{2}\cos(x+y)$$

$$\sin x\cos y=\tfrac{1}{2}\sin(x-y)+\tfrac{1}{2}\sin(x+y)$$

The Questions

1. This question is compulsory.

    (a) Consider the following two signals: $x_1(t) = \Delta(t-1)$ and $x_2(t) = \Delta(t-2)$, where $\Delta(t)$ is given by

    $$\Delta(t) = \begin{cases} 1 - |t| & \text{for } |t| \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

    Notice that the signals are also sketched in Figure 1.

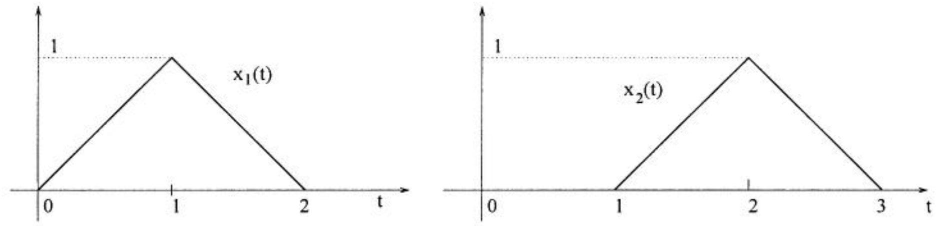


Figure 1: The two signals $x_1(t)$ and $x_2(t)$.

i. Compute the energy of $x_1(t)$.

[4]

ii. Compute the energy of $x_2(t)$.

[4]

iii. Compute the energy of $x_1(t) + x_2(t)$.

[4]

Question 1 continues on next page

(b) The Fourier transform of the triangular pulse $x(t)$ in Figure 2(a) is

$$X(\omega) = \frac{1}{\omega^2}(e^{j\omega} - j\omega e^{j\omega} - 1).$$

Using this information, the scaling property and the time-shifting property, find the Fourier transform of the signal $y(t)$ shown in Figure 2(b). Notice that $y(t)$ is real and even, so you expect $Y(\omega)$ to be real and even as well.

[4]

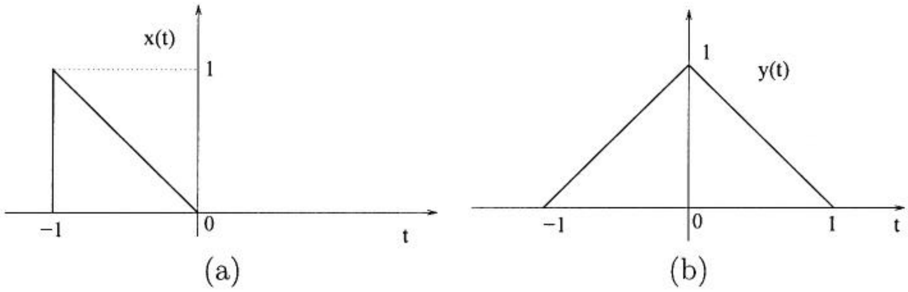


Figure 2: The two signals $x(t)$ and $y(t)$.

(c) Consider the linear time-invariant system $h(t)$ where the input $x(t)$ and output $y(t)$ are related by the following linear differential equation:

$$\frac{dy(t)}{dt} + y(t) = x(t).$$

i. Find the transfer function of $h(t)$. Recall that the transfer function is defined as $H(\omega) = Y(\omega)/X(\omega)$.

[4]

ii. Assume that the Power Spectral Density (PSD) of the input is $S_x(\omega) = \text{rect}(\omega/2)$. Compute the power of $x(t)$.

[4]

iii. Compute the power of the output signal $y(t)$.

[4]

Question 1 continues on next page

(d) Consider the following full-AM signal:

$$\varphi(t) = (A + m(t)) \cos \omega_c t,$$

where $m(t) = \frac{t}{1+t^2}$. The modulation index is $\mu = 0.5$. Find the amplitude A.

[4]

(e) Sketch the PM and FM waves for the modulating signal shown in Figure 3. The constants k_f and k_p are $2\pi 10^5$ and 10π respectively and the carrier frequency is $f_c = 100\text{MHz}$

[4]

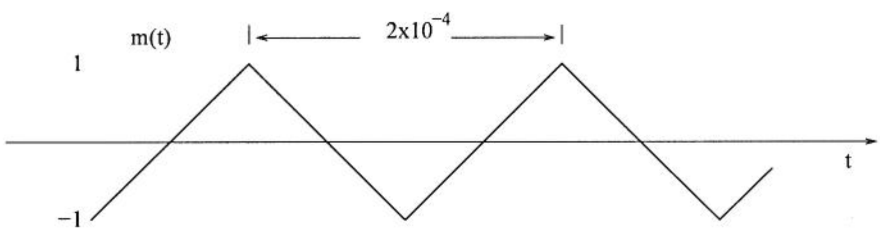


Figure 3: The modulating signal $m(t)$.

(f) A 50 Ω transmission line is connected to a 100 Ω line with a matched termination. A sine wave of 1 V amplitude propagating in the former is incident on the junction. Find

i. The voltage reflection coefficient k_v.

[2]

ii. The fraction of the incident power which is transmitted into the second line.

[2]

2. Consider the FM signal

$$\varphi(t) = 10\cos[2\pi f_c t + k_f \int_{-\infty}^{t} x(\alpha)d\alpha].$$

Assume that $k_f = 100\pi$, $f_c = 1\text{MHz}$ and that the modulating signal is given by

$$x(t) = 100\text{sinc}(1000\pi t).$$

(a) Determine the frequency deviation Δf.

[7]

(b) Using Carson's rule, determine the bandwidth of $\varphi(t)$.

[7]

(c) The signal $\varphi(t)$ is fed to the 'Armstrong Modulator' shown in Figure 4. The output of the non-linear device is $g(t) = \varphi(t) + \varphi^2(t) + \varphi^3(t)$. The bandpass filter $H(\omega)$ is shown in Figure 5.

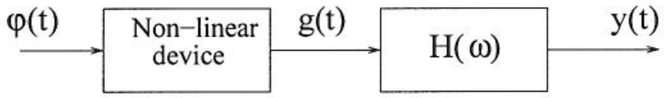


Figure 4: The 'Armstrong modulator'.

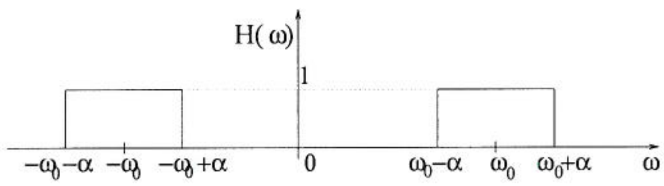


Figure 5: The band-pass filter $H(\omega)$.

Question 2 continues on next page

Determine the bandwidth α and centre frequency ω_0 of the bandpass filter $H(\omega)$ so that the resulting FM signal $y(t)$ has standard deviation $\Delta f = 10000$Hz. Use Carson's rule to calculate the bandwidth of the FM signals.

[8]

(d) Assume now that the signal $\varphi(t)$ is fed to the modulator shown in Figure 6. Find the frequency ω_3, the bandwidth and centre frequency of the two bandpass filters $H_1(\omega)$ and $H_2(\omega)$ that would give an FM signal $y(t)$ with carrier $f_c = 4.5$MHz and frequency deviation $\Delta f = 15000$Hz. The output of the non-linear device is $g(t) = z(t) + z^2(t) + z^3(t)$. Use Carson's rule to calculate the bandwidth of the FM signals.

[8]

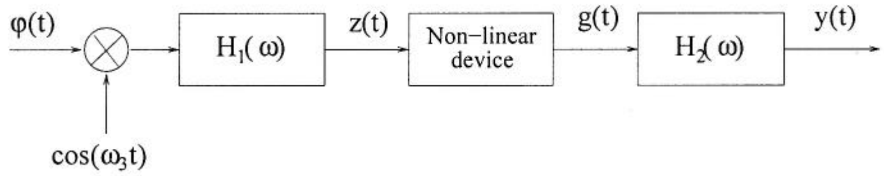


Figure 6: FM modulator.

3. The signals $x_1(t) = \frac{300}{\pi}\text{sinc}(300t)$ and $x_2(t) = 3\cos 60t + 2\cos 100t$ are applied at the input of the two ideal low-pass filters $H_1(\omega) = \text{rect}(\omega/100)$ and $H_2(\omega) = \text{rect}(\omega/140)$ (see also Figure 7). The outputs of these two filters are multiplied to obtain the signal $y(t) = y_1(t)y_2(t)$.

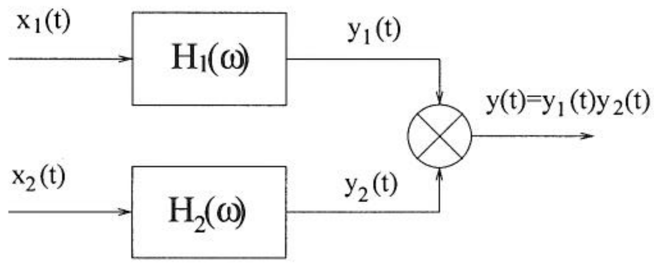


Figure 7: Filtering of $x_1(t)$ and $x_2(t)$.

(a) Sketch and dimension the Fourier transform of $x_1(t)$

[6]

(b) Sketch and dimension the Fourier transform of $x_2(t)$

[6]

(c) Sketch and dimension the Fourier transform of $y_1(t)$

[6]

(d) Sketch and dimension the Fourier transform of $y_2(t)$

[6]

(e) Sketch and dimension the Fourier transform of $y(t)$

[3]

(f) Write the exact time-domain expression of $y(t)$.

[3]

4. Consider the demodulator shown in Figure 8, where $y_1(t) = x^2(t)$, $y_3(t) = \sqrt{y_2(t)}$ and the frequency response of the filter $H(\omega)$ is

$$H(\omega) = \begin{cases} 1 & \text{for } \ |\omega| \leq 6 \text{ rad/s} \\ 0 & \text{otherwise} \end{cases}$$

Assume that $x(t) = m(t)\cos(\omega_c t) + m(t)\sin(\omega_c t)$, where $m(t) = A - \text{sinc}(3t)$ and $\omega_c = 100$.

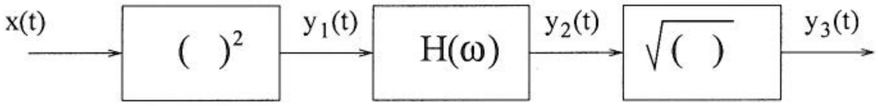


Figure 8: Demodulator.

(a) Write the exact expression of $y_1(t)$.

[6]

(b) Write the exact expression of $y_2(t)$.

[6]

(c) Write the exact expression of $y_3(t)$.

[6]

(d) Find the minimum value of A that guarantees that $y_3(t) = m(t)$.

[6]

(e) Now assume that $A = 2$, determine the minimum value of ω_c that guarantees that $y_3(t) = m(t)$. Justify your answer.

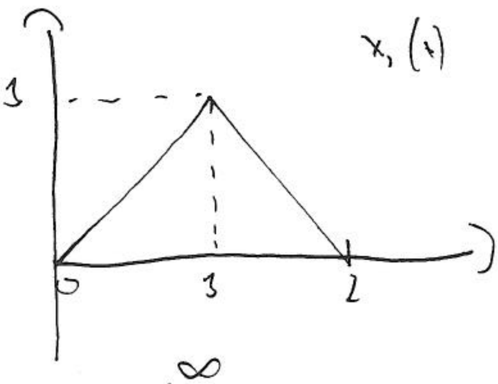
[6]

SOLUTIONS

E1.6 Communications I

2009

QUESTION 1



i) $$E_{x_1} = \int_{-\infty}^{\infty} x_1^2(t)\,dt = \int_0^2 x_1^2(t)\,dt = 2\int_0^1 t^2\,dt = 2\frac{t^3}{3}\Big|_0^1 =$$

$$= \frac{2}{3}$$

ii) $E_{x_2} = E_{x_1} = \frac{2}{3}$ SINCE A SHIFT IN TIME ~~HA~~ DOES NOT CHANGE THE ENERGY OF A SIGNAL

iii. $$E_{x_1+x_2} = E_{x_1} + E_{x_2} + 2\int_{-\infty}^{\infty} x_1(t)\,x_2(t)\,dt\ ;$$

$$\int_{-\infty}^{\infty} x_1(t)\,x_2(t)\,dt = \int_1^2 x_1(t)\,x_2(t)\,dt = \int_0^1 t(1-t)\,dt =$$

$$\int_0^1 t \, dt - \int_0^1 t^2 \, dt = \frac{t^2}{2}\Big|_0^1 - \frac{t^3}{3}\Big|_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

$$E_{x_1+x_2} = E_{x_1} + E_{x_2} + 2\int_{-\infty}^{\infty} x_1(t)\, x_2(t)\, dt = \frac{2}{3} + \frac{2}{3} + 2 \cdot \frac{1}{6} =$$

$$= \frac{5}{3}$$

b)

$$y(t) = x(t-1) + x(-(t-1))$$

$$x(t-1) \iff X(\omega)\, e^{-j\omega}$$

$$x(-(t-1)) \iff X(-\omega)\, e^{j\omega}$$

$$Y(\omega) = X(\omega)\, e^{-j\omega} + X(-\omega)\, e^{j\omega} =$$

$$= \frac{1}{\omega^2}\left(1 - j\omega - e^{-j\omega} + 1 + j\omega - e^{j\omega}\right)$$

$$= \frac{1}{\omega^2}\left(2 - e^{j\omega} - e^{-j\omega}\right) = 4\,\frac{\left(e^{\frac{j\omega}{2}} - e^{\frac{-j\omega}{2}}\right)^2}{\left(\frac{j\omega}{2}\right)^2} =$$

$$= \left(\mathrm{SINC}\ \frac{\omega}{2}\right)^2$$

c) i. WE TAKE THE FOURIER TRANSFORM ON BOTH SIDES:

$$j\omega Y(\omega) + Y(\omega) = X(\omega)$$

$$\Downarrow$$

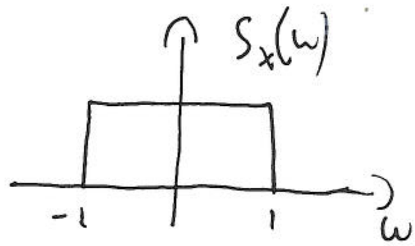
$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{1+j\omega}$$

ii.

BY DEFINITION OF PSD WE KNOW THAT:

$$P_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_x(\omega)\,d\omega$$

$$S_x(\omega) = \mathrm{RECT}\left(\frac{\omega}{2}\right) \rightarrow$$



THUS

$$P_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{RECT}\left(\frac{\omega}{2}\right)d\omega = \frac{1}{2\pi}\int_{-1}^{1} d\omega = \frac{1}{\pi}$$

iii.

$$P_Y = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_Y(\omega)\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |H(\omega)|^2 S_x(\omega)\,d\omega$$

$$= \frac{1}{2\pi}\int_{-1}^{1}\frac{1}{1+\omega^2}\,d\omega = \frac{1}{2\pi}\,\mathrm{TAN}^{-1}\omega\Big|_{-1}^{1} = \cancel{4}\,\frac{1}{2\pi}\cdot\frac{\pi}{2} = \frac{1}{4}$$

d)

$\mu = \frac{m_p}{A}$. WE NEED TO FIND $m_p = \max |m(t)|$

$$\frac{d\,m(t)}{dt} = \frac{1+t^2 - t(2t)}{(1+t^2)^2} = 0 \Rightarrow \quad t = \pm 1$$

THUS $\quad m_p = \frac{1}{2}$

$$A = \frac{1/2}{1/2} = 1$$

e) IN BOTH CASES THE MINIMUM AND MAXIMUM INSTANTANEOUS FREQUENCIES ARE :

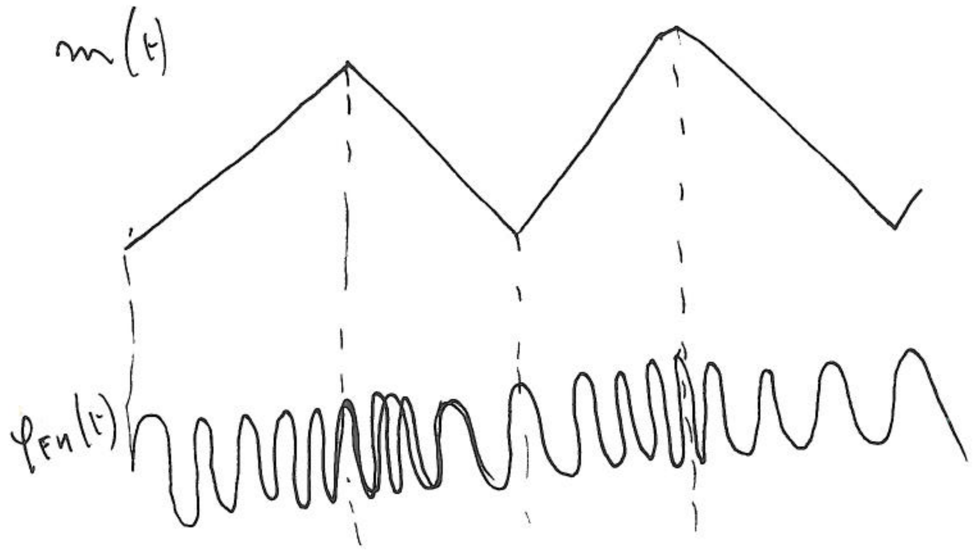
$$(f_i)_{MIN} = 99.9\,MHz$$

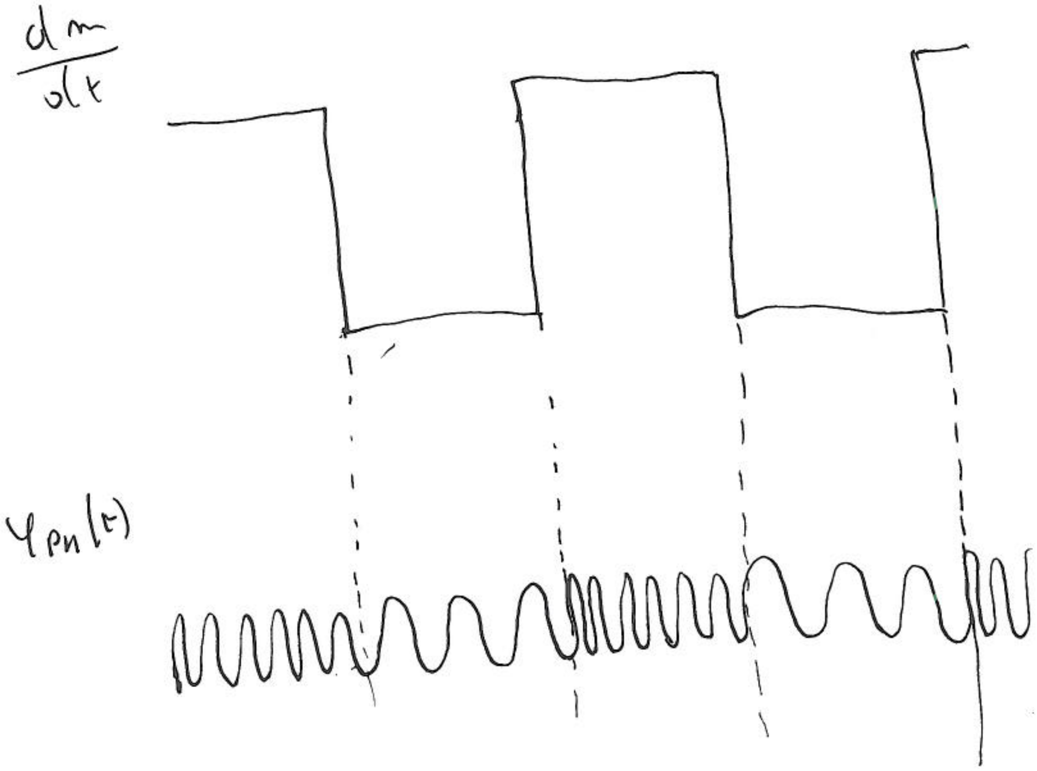
$$(f_i)_{MAX} = 100.1\,MHz$$

- IN THE FM CASE THE INSTANTANEOUS FREQUENCY DEPENDS LINEARLY ON $m(t)$.

- IN THE PM CASE IT DEPENDS ON $\frac{d\,m(t)}{dt}$

THUS :





b) i. $K_V = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{100-50}{100+50} = \frac{1}{3}$

ii. $W_p = 1 - |K_V|^2 = 1 - \frac{1}{9} = \frac{8}{9}$

$T_p = \frac{V_L I_L}{V_t I_t} = 1 - W_p = \frac{1}{9}$

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2.

(a) $\Delta f = \frac{K_f x_p}{2\pi}$

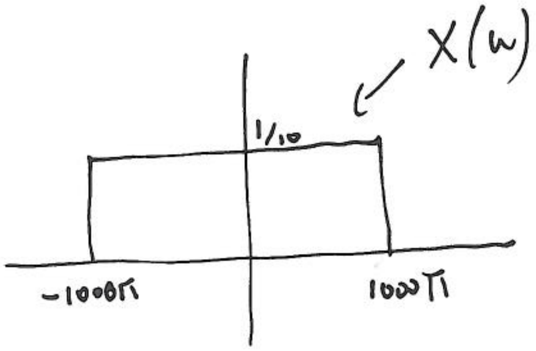
$x_p = 100$, $K_f = 100\pi$

THUS WE HAVE

$$\Delta f = \frac{100\pi \cdot 100}{2\pi} = 5000 Hz$$

(b) $B_{FM} = 2(\Delta f + B)$

$100 \text{ SINC } 1000\pi t \Longleftrightarrow \frac{1}{10} \text{ RECT}\left(\frac{w}{2000\pi}\right)$



$$B = \frac{1000\pi}{2\pi} = 500 Hz$$

$$B_{FM} = 2(5000 + 500) = 11000 Hz$$

(c)

$$g(t) = \varphi(t) + \varphi^2(t) + \varphi^3(t) =$$

$$= 10\Big[a_1 \cos\left(2\pi f_c t + k_f \int x(\lambda)\,d\lambda\right) +$$

$$a_2 \cos\left(4\pi f_c t + 2k_f \int x(\lambda)\,d\lambda\right) + \qquad (1)$$

$$a_3 \cos\left(6\pi f_c t + 3k_f \int x(\lambda)\,d\lambda\right)\Big]$$

WE WANT $\Delta f = 10000\,Hz$, WE THUS NEED TO KEEP THE SECOND TERM OF (1). THE BANDWIDTH OF THIS SECOND TERM IS

$$B_{FM} = 2(10000 + 500) = 21000\,Hz$$

THUS

$$\frac{\omega_0}{2\pi} = 2 \cdot \frac{\omega_c}{2\pi} = 2\,MHz$$

$$\frac{\lambda}{2\pi} = 21.000\,\cancel{Hz}/2\ Hz = 10500\,Hz$$

(d) WE NOW NEED TO KEEP THE THIRD TERM IN (1) TO OBTAIN

$\Delta f = 15000 Hz$.

HOWEVER THIS GIVE US AN $f_c = 3MHz$.

WE THUS CHOOSE $\omega_3 = 2\pi \cdot 5 \cdot 10^5$ rad/s

$\Rightarrow f_3 = 5 \cdot 10^5 Hz$

$$\varphi(t) \cos \omega_1 t = \frac{10}{2} \cos\left((\omega_c - \omega_3)t + k_f \int x(\lambda) d\lambda\right) +$$

$$\frac{10}{2} \cos\left((\omega_c + \omega_1)t + k_f \int x(\lambda) d\lambda\right)$$

NOW $f_c + f_3 = 1.5 MHz$ WHICH IS THE FREQUENCY WE NEED.

$B_{\varphi(t)} = 11000 Hz$

SO $H_1(\omega)$ HAS CENTRAL FREQUENCY

$f_1 = 1.5 MHz$

AND BANDWIDTH $B = 11000 Hz$ (BANDWIDTH OF $\varphi(t)$)

$H_2(\omega)$ HAS CENTRAL FREQUENCY $f_2 = 4.5 MHz$

AND BANDWIDTH $B_y = 2(15000 + 500) = 46000 Hz$

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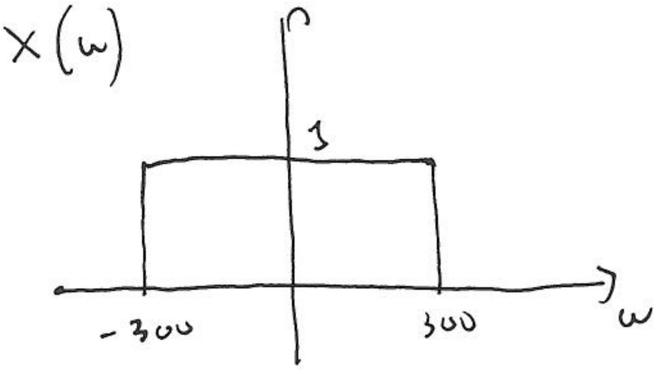
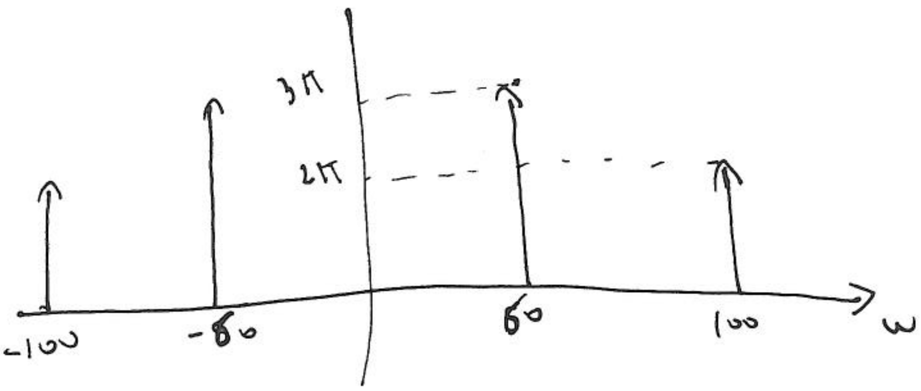
3.

(a) $\frac{a}{\pi} \operatorname{SINC} at \Longleftrightarrow \operatorname{RECT}\left(\frac{\omega}{2a}\right)$

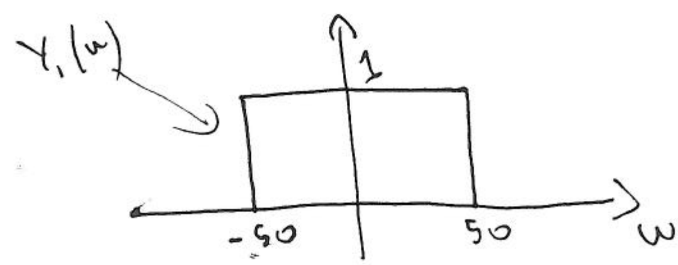
$\frac{300}{\pi} \operatorname{SINC} 300t \Longleftrightarrow \operatorname{RECT}\left(\frac{\omega}{600}\right)$



(b)



(c) $Y_1(\omega) = H_1(\omega) = \operatorname{RECT}\left(\frac{\omega}{100}\right)$



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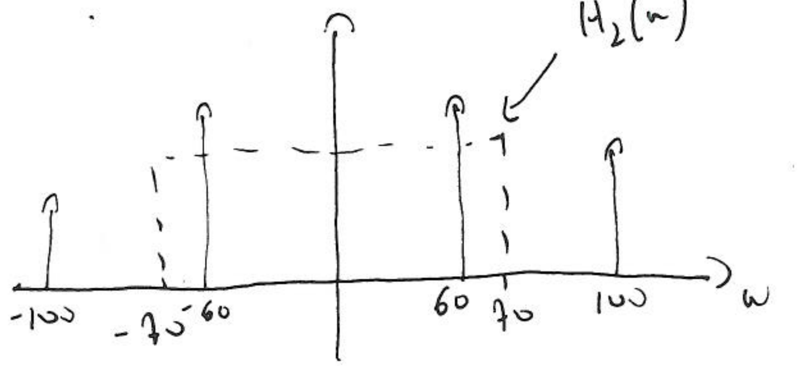
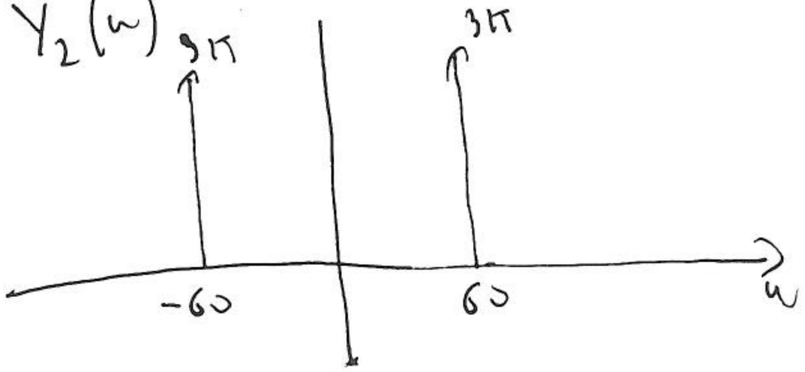
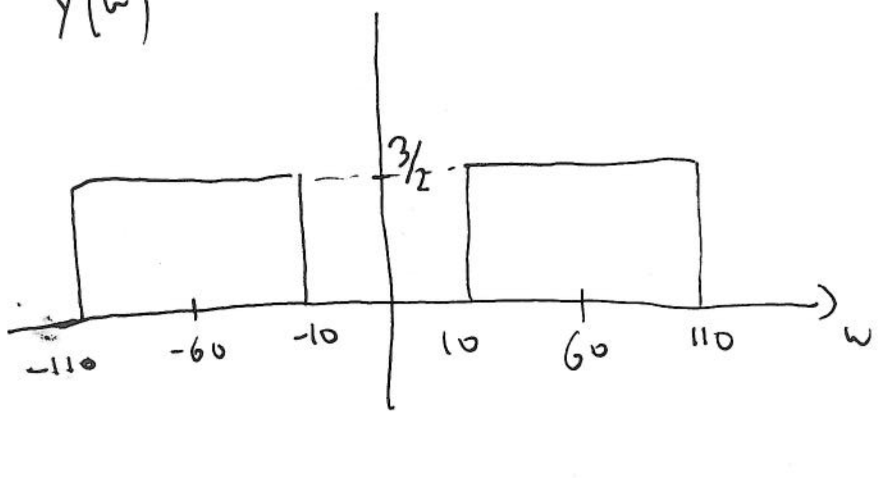
(d)





(e)

i.) $Y(\omega)$



ii. $y(t) = 3 \cdot \frac{50}{\pi} \text{ sinc } 50t \cdot \cos 60t = \frac{150}{\pi} \text{ sinc } 50t \cdot \cos 60t$

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4.

(a) $Y_1(t) = m^2(t)\left(\cos^2 \omega_c t + \sin^2 \omega_c t + 2\sin\omega_c t \cos\omega_c t\right) =$

$= m^2(t) + 2m^2(t)\sin\omega_c t\cos\omega_c t$

$= m^2(t) + m^2(t)\sin 2\omega_c t$

(b) $Y_2(t) = m^2(t)$

(c) $y_3(t) = \sqrt{m^2(t)} = |m(t)|$

(d) NOTICE THE SYSTEM BEHAVES LIKE ~~AS~~ AN ENVELOPE DETECTOR.

THUS $y_3(t) = m(t)$ WHEN $m(t) \geq 0 \ \forall t$,

THIS HAPPENS WHEN $A \geq 1$. MINIMUM VALUE ~~OF~~ $A = 1$.

(e) THE BANDWIDTH OF $m^2(t)$ IS 6 RAD/S

THE MINIMUM VALUE OF $\omega_c = 12$ RAD/S.

IN THIS WAY THE TWO SPECTRA $m^2(t)$ AND $m^2(t)\cos\omega_c t$ DO NOT OVERLAP