EXAM QUESTIONS

Information for Students

Fourier Transform Pairs

Pair Number	$x(t)$	$X(f)$
1.	$\Pi\left(\frac{t}{\tau}\right)$	$\tau \operatorname{sinc} \tau f$
2.	$2W \operatorname{sinc} 2Wt$	$\Pi\left(\frac{f}{2W}\right)$
3.	$\Lambda\left(\frac{t}{\tau}\right)$	$\tau \operatorname{sinc}^2 \tau f$
4.	$\exp(-\alpha t)u(t),\ \alpha > 0$	$\frac{1}{\alpha + j2\pi f}$
5.	$t\exp(-\alpha t)u(t),\ \alpha > 0$	$\frac{1}{(\alpha + j2\pi f)^2}$
6.	$\exp(-\alpha\lvert t\rvert),\ \alpha > 0$	$\frac{2\alpha}{\alpha^2 + (2\pi f)^2}$
7.	$e^{-\pi(t/\tau)^2}$	$\tau e^{-\pi(f\tau)^2}$
8.	$\delta(t)$	1
9.	1	$\delta(f)$
10.	$\delta(t - t_0)$	$\exp(-j2\pi f t_0)$
11.	$\exp(j2\pi f_0 t)$	$\delta(f - f_0)$
12.	$\cos 2\pi f_0 t$	$\frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)$
13.	$\sin 2\pi f_0 t$	$\frac{1}{2j}\delta(f - f_0) - \frac{1}{2j}\delta(f + f_0)$
14.	$u(t)$	$(j2\pi f)^{-1} + \frac{1}{2}\delta(f)$
15.	$\operatorname{sgn} t$	$(j\pi f)^{-1}$
16.	$\frac{1}{\pi t}$	$-j\operatorname{sgn}(f)$
17.	$\hat{x}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(\lambda)}{t - \lambda}\, d\lambda$	$-j\operatorname{sgn}(f)X(f)$
18.	$\sum_{m=-\infty}^{\infty} \delta(t - mT_s)$	$f_s \sum_{m=-\infty}^{\infty} \delta(f - mf_s)$, $f_s = T_s^{-1}$

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

$$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

$$\tan(\frac{\pi}{2} - x) = \cot x$$

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x} = \frac{2\cot x}{\cot^2 x - 1} = \frac{2}{\cot x - \tan x}$$

$$\tan\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}} = \frac{\sin x}{1+\cos x} = \frac{1-\cos x}{\sin x}$$

$$\sin x \cos y = \frac{1}{2}[\sin(x+y) + \sin(x-y)]$$

$$\cos x \sin y = \frac{1}{2}[\sin(x+y) - \sin(x-y)]$$

$$\sin x \sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$$

$$\cos x \cos y = \frac{1}{2}[\cos(x+y) + \cos(x-y)]$$

$$\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$$

$$\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$$

$$\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$$

$$\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$$

Fourier Transform Theorems[a]

Name of Theorem		
1. Superposition (a_1 and a_2 arbitrary constants)	$a_1x_1(t) + a_2x_2(t)$	$a_1X_1(f) + a_2X_2(f)$
2. Time delay	$x(t - t_0)$	$X(f)e^{-j2\pi f t_0}$
3a. Scale change	$x(at)$	$\lvert a\rvert^{-1}X\left(\frac{f}{a}\right)$
b. Time reversal	$x(-t)$	$X(-f) = X*(f)$
4. Duality	$X(t)$	$x(-f)$
5a. Frequency translation	$x(t)e^{j\omega_0 t}$	$X(f - f_0)$
b. Modulation	$x(t)\cos\omega_0 t$	$\frac{1}{2}X(f - f_0) + \frac{1}{2}X(f + f_0)$
6. Differentiation	$\frac{d^n x(t)}{dt^n}$	$(j2\pi f)^n X(f)$
7. Integration	$\int_{-\infty}^{t} x(t')\,dt'$	$(j2\pi f)^{-1}X(f) + \frac{1}{2}X(0)\delta(f)$
8. Convolution	$\int_{-\infty}^{\infty} x_1(t - t')x_2(t')\,dt' = \int_{-\infty}^{\infty} x_1(t')x_2(t - t')\,dt'$	$X_1(f)X_2(f)$
9. Multiplication	$x_1(t)x_2(t)$	$\int_{-\infty}^{\infty} X_1(f - f')X_2(f')\,df' = \int_{-\infty}^{\infty} X_1(f')X_2(f - f')\,df'$

Differentiation Rule of Leibnitz

Let $F(z) = \int_{a(z)}^{b(z)} f(x,z)dx$. Then we have

$$\frac{dF(z)}{dz} = \frac{db(z)}{dz}f(b(z),z) - \frac{da(z)}{dz}f(a(z),z) + \int_{a(z)}^{b(z)} \frac{\partial f(x,z)}{\partial z}dx$$

Joint Gaussian density

The joint probability density function (pdf) of two correlated Gaussian random variables X and Y is given by

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}e^{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_X)^2}{\sigma_X^2} - \frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} + \frac{(y-\mu_Y)^2}{\sigma_Y^2}\right]}.$$

where $\mu_X = E[X]$, $\mu_Y = E[Y]$ are the mean values, σ_X and σ_Y are the standard deviation of X and Y, respectively, and ρ is the correlation coefficient defined as

$$\rho = \frac{\text{Cov}(X,Y)}{\sigma_X\sigma_Y}.$$

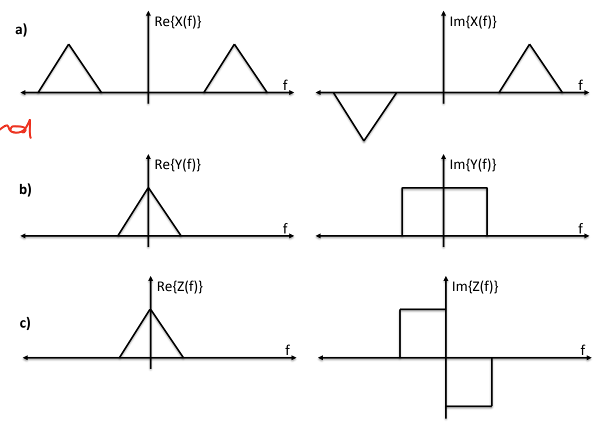
1. a) i) Explain how you can convert a continuous signal into sequences of bits that can be transmitted over a digital communication system. [2]

Most students answered the first two questions correctly.

ii) Write down the definitions of baseband and passband signals. [2]

iii) The spectrum of three signals $x(t)$, $y(t)$, and $z(t)$ are depicted below. Which of these (there may be multiple) may represent a real baseband signal? Explain your answer.

Almost all students were able to distinguish baseband and passband signals, but not all very able to recognize the conjugate symmetry



[3]

iv) Consider a binary frequency shift keying (FSK) communication system, in which bit 0 is transmitted with signal $A\cos(2\pi f_0 t)$ and bit 1 is transmitted with signal $A\cos(2\pi f_1 t)$.

Draw the diagram of a coherent FSK receiver for this system, and explain the function of each component of the receiver. [5]

Most students were able to answer this correctly. The part that was missing in many answers was the subtraction of the signals, and the decision device at the end.

b) State whether each of the following statements are true or false, and discuss your answer:

Many students confused this with the autocorrelation function of a w.s.s. process.

i) $R_X(t,s) = \sin(t+s)$ cannot be the autocorrelation function of a random process. [2]

The last two were answered by a majority of the students correctly.

ii) If the input to a linear time-invariant (LTI) system is wide sense stationary, so is the output. [2]

iii) Let $\Phi(x)$ denote the cumulative distribution function of a standard Gaussian random variable, and $Q(x) \triangleq \frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-t^2/2}dt$. The following relation between these two functions hold for any real x:

$$\Phi(x) + Q(x) = 1.$$

[2]

c) Consider two independent Gaussian random variables X and Y. Suppose X has a mean value of -2 and variance 2, i.e., $X \sim \mathcal{N}(-2,2)$, while Y has a mean value of 1 and variance 3, i.e., $Y \sim \mathcal{N}(1,3)$.

Express the following probabilities in terms of the Q function.

i) $\mathrm{P}\{-5 < X < 2\}$. [2]

ii) $\mathrm{P}\{X^2 - 2X > 3\}$. [3]

iii) $\mathrm{P}\{X \cdot Y - Y + 3X < 3\}$. [4]

iv) $\mathrm{P}\left\{\frac{X+2}{\sqrt{2}} > \frac{Y-1}{\sqrt{3}}\right\}$. [3]

While most students got the first part correctly, many of them tried to find the mean and variance of the left hand side in ii and iii. However, they did not notice that these are not Gaussian random variables due to the non-linear operations involved.

A good number of students answered iv correctly.

d) Assume that $X(t)$ is a real wide sense stationary (WSS) random process whose power spectral density (PSD) is given as follows:

$$S_X(f) = \frac{2\alpha}{\alpha^2 + (2\pi f)^2} + \frac{1}{2}\left[\delta(f - \frac{1}{4}) + \delta(f + \frac{1}{4})\right] + 3\delta(f)$$

for some $\alpha > 0$.

i) What is the second moment of the random variable $X(1) - X(-1)$? (*Hint: Second moment of a random variable Y is given by $E[Y^2]$.*) [5]

ii) What is the second moment of the following random variable?

$$X(2) + X(0) - X(-2)$$

[5]

Many students answered this question correctly, and almost all got partial credits at least.

This was the hardest question for the students. It seems that the main difficulty was the way the problem formulated.

2. a) Consider the random process

$$X(t) = Y(t)\cos(2\pi f_c t) - Z(t)\sin(2\pi f_c t),$$

where $Y(t)$ and $Z(t)$ are two independent random processes.

i) Find the conditions on $Y(t)$ and $Z(t)$ under which the mean of $X(t)$ is shift-invariant, i.e., $E[X(t)]$ does not depend on t. [4]

ii) Assume that both $Y(t)$ and $Z(t)$ are zero-mean processes, i.e., $E[Y(t)] = E[Z(t)] = 0,\ \forall t$.

Find the conditions on $Y(t)$ and $Z(t)$ under which $X(t)$ is a wide sense stationary (WSS) process. [6]

iii) Assume that both $Y(t)$ and $Z(t)$ are zero-mean WSS processes, and their autocorrelation functions are identical, i.e., $E[Y(t)] = E[Z(t)] = 0,\ \forall t$, and $R_Y(\tau) = R_Z(\tau),\ \forall \tau$.

If the power spectral density (PSD) of $Y(t)$ is $S_Y(f)$, find the PSD of $X(t)$ in terms of $S_Y(f)$. [3]

iv) Assume that both $Y(t)$ and $Z(t)$ are zero-mean white Gaussian noise processes with PSD $N_o/2$.

What is the PSD of $X(t)$? Is $X(t)$ strict sense stationary? [3]

b) Consider a binary communication system. When a 0 is transmitted, probability of error is p_0; while when a 1 is transmitted, probability of error is p_1.

i) Assuming that bit 0 is transmitted with probability q_0, if the decoder outputs 1, what is the probability of the input being 0? Express this probability in terms of q_0, p_0 and p_1. [3]

ii) Consider using $(7,4)$ Hamming code to communicate over this channel. The generator matrix of this code is given by

Some students did not attempt this problem. They were probably put off by the Hamming code, although it requires only very basic knowledge.

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Remember that the codeword for any 4-bit message $\mathbf{u} = [u_1\ u_2\ u_3\ u_4]$ is generated by $\mathbf{x} = \mathbf{u} \cdot \mathbf{G}$.

If we transmit message $\mathbf{u} = [0\,0\,0\,0]$ using this code, what is the probability of an error at the receiver? If $p_0 = 10^{-3}$ and $p_1 = 10^{-1}$, find the approximate value of this block error probability. [6]

iii) Assume that $p_0 = p_1 = 10^{-3}$. What is the average error probability if the 4-bit message sequences are generated as independent outcomes of a Bernoulli distribution, such that $\Pr(u_i = 1) = 0.3$, for $i = 1, \ldots, 4$. For example, $\Pr\{\mathbf{u} = [0111]\} = 0.7 \times 0.3^3$. [5]

3. a) Consider a binary communication system, where bit "0" is transmitted with a pulse of amplitude 0, and bit "1" is transmitted with a pulse of amplitude A. The channel is an additive Laplacian noise channel: For an input signal X, where $X \in \{0, A\}$, the output Y is given by

$$Y = X + W,$$

Many students answered this question correctly, but a few were confused to see a non-Gaussian noise distribution. Quite a few had some minor mistake in the derivation.

where W is the zero-mean additive noise component, which is independent of X and has the following probability density function (pdf):

$$f_W(w) = \frac{1}{2b} e^{-\frac{|w|}{b}}.$$

Assume that the detection threshold at the receiver is $T \in (0, A)$; that is, if $Y \geq T$, the transmitted bit is estimated as 1, while if $Y < T$, it is estimated as 0.

i) Given that a bit 0 was sent, derive the error probability P_{e0} in terms of T and b. [3]

ii) Given that a bit 1 was sent, derive the error probability P_{e1} in terms of A, T and b. [3]

iii) If a bit 0 is sent with probability p_0 and a bit 1 is sent with probability p_1, write down the total error probability P_e in terms of p_1, P_{e0} and P_{e1}. [2]

iv) Assume $p_1 = 2/3, A = 2$ and $b = 1/\ln 2$. Find the detection threshold T that minimizes P_e. What is the corresponding error probability? [10]

b) Consider a language which has only 3 letters in its alphabet: $\{x, y, z\}$. This language has 6 words in total: $\{xxx, xyz, yyy, yzx, zzz, zxy\}$. We take a sufficiently long book written in this language, and choose a random word. The probabilities of different words are given as follows:

Word	xxx	xyz	yyy	yzx	zzz	zxy
Probability	0.3	0.25	0.2	0.15	0.05	0.05

While quite a few students got the first two parts correctly, many had difficulty in answering the last two.

i) What is the word entropy of this language; that is, if W is the random variable denoting the randomly chosen word, what is $H(W)$? [2]

ii) If you choose a random letter from the book, what is the probability of encountering each letter (Note that each word consists of three letters)? What is the entropy of the randomly chosen letter? [3]

iii) Among i) and ii) above, which one has a higher entropy per letter? Explain why the two numbers are different? [3]

iv) Consider removing the last letter of each word. What is the word entropy of this new language? What would be the advantages/ disadvantages of this new language compared to the original one? [4]