✓ **E4.25**
C4.1
ISE4.23

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2008

MSc and EEE/ISE PART IV: MEng and ACGI

Corrected copy

DESIGN OF LINEAR MULTIVARIABLE CONTROL SYSTEMS

Tuesday, 6 May 10:00 am

Time allowed: 3:00 hours

There are SIX questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s) :	I.M. Jaimoukha
	Second Marker(s) :	D.J.N. Limebeer

Special Information for Invigilators :	None
Information for Candidates :	None

1. (a) Let the transfer matrix $G(s)$ have a state space realisation

$$G(s) \stackrel{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] := \left[\begin{array}{cccc|cc} 1 & 2 & 0 & 0 & 1 & 2 \\ 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 3 & 4 \\ 0 & 0 & 0 & -5 & 0 & 0 \\ \hline 2 & 3 & 0 & 0 & 1 & 0 \\ 1 & 4 & 0 & 0 & 0 & 1 \end{array}\right].$$

i. Find the uncontrollable and/or unobservable modes and determine whether the realisation is detectable and stabilisable. **[3]**

ii. Suppose that $K \in \mathcal{R}^{2\times 4}$ and $L \in \mathcal{R}^{4\times 2}$ are arbitrary matrices. Determine two of the eigenvalues of $A - BK$ and two of the eigenvalues of $A - LC$. Explain how you arrive at your answer. **[3]**

iii. Find a minimal realisation for $G(s)$. **[3]**

(b) Consider a state–variable model described by the dynamics

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t), \end{aligned}$$

and denote the corresponding transfer matrix by $H(s)$. Suppose that there exists $P = P' > 0$ such that

$$A'P + PA + C'C + \frac{1}{4}PBB'P < 0.$$

(i) Prove that A is stable. **[3]**

(ii) By defining the Lyapunov function

$$V(t) = x(t)'Px(t),$$

the cost function

$$J := \int_0^\infty [y(t)'y(t) - \gamma^2 u(t)'u(t)]dt,$$

and using a property of the integral $\int_0^\infty \dot{V}(t)dt$, or otherwise, prove that

$$\|H\|_\infty < 2.$$

State clearly the assumptions required on $u(t)$, $x(0)$ and $x(\infty)$. **[8]**

HINT: You may want to use the identity:

$$-(\gamma u - \gamma^{-1}B'Px)'(\gamma u - \gamma^{-1}B'Px) = x'PBu + u'B'Px - \gamma^2 u'u - \gamma^{-2}x'PBB'Px.$$

2. (a) Define internal stability for the feedback loop in Figure 2.1, and derive necessary and sufficient conditions for which this loop is internally stable. [4]

(b) Suppose that $G(s)$ is stable. Give a parameterisation of all internally stabilising controllers for $G(s)$ for the feedback loop in Figure 2.1. [4]

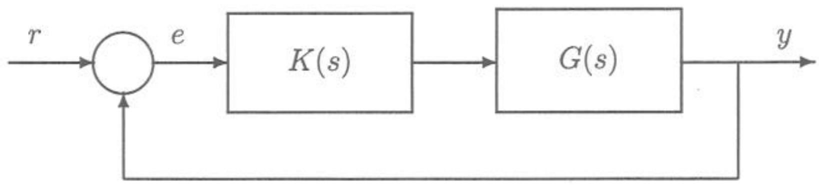

Figure 2.1

(c) Assume that $G(s)$ is minimum phase. Suppose now that an output multiplicative uncertainty on $G(s)$ is introduced as shown in Figure 2.2. Using the answer to part (b) above and the small gain theorem, which should be stated, design a family of internally stabilising controllers $K(s)$ that satisfy the following performance and robustness design specifications:

i. When $\Delta = 0$, the transfer matrix from r to e, $S(s)$, satisfies $\|S\|_\infty < 1/2$.

ii. The feedback loop is stable for all $\Delta \in \mathcal{RH}_\infty$ such that $\|\Delta\|_\infty < 1$.

Amongst the family of internally stabilising controllers $K(s)$ that satisfy the design specifications, what is the smallest achievable $\|KS\|_\infty$? [12]

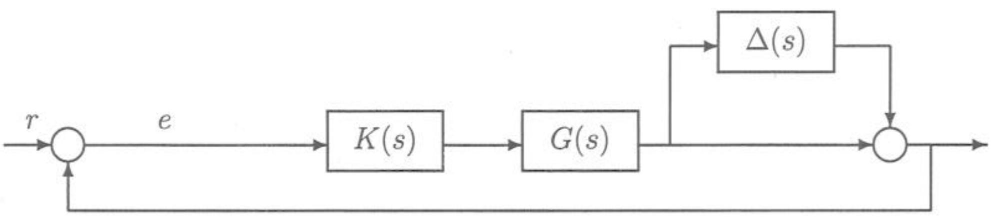

Figure 2.2

3. Consider the regulator in Figure 3.1 for which it is assumed that the triple (A, B, C) is minimal and $x(0) = x_0$. Let $z = \begin{bmatrix} z_1^T & z_2^T \end{bmatrix}^T$. A stabilizing state-feedback gain matrix F is to be designed such that the cost function $J := \int_0^\infty z(t)^T z(t) dt$ is minimized.

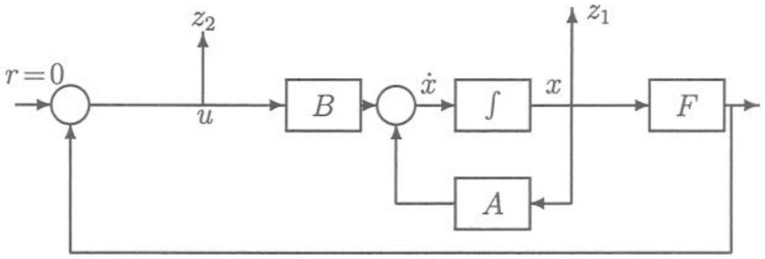

Figure 3.1

Let $V(t) = x(t)^T P x(t)$ where $P = P^T$ is the solution of an algebraic Riccati equation.

(a) Assuming the closed loop is asymptotically stable, obtain an expression for $\int_0^\infty \dot{V}(t)dt$ in terms of x_0. [4]

(b) Evaluate an expression for J using an appropriate completion of a square. Using this expression, find F that minimizes J. Give also the minimum value of J and the algebraic Riccati equation satisfied by P. [4]

(c) Prove that, for the value of F chosen in part (b), the closed loop system in Figure 3.1 is stable. State clearly the assumption on P required to guarantee stability. [4]

(d) Let $G(s) = (sI - A)^{-1}B$ and define $L(s) = I - FG(s)$. Using the algebraic Riccati equation show that

$$L(j\omega)' L(j\omega) = I + G(j\omega)' G(j\omega)$$ [4]

(e) Suppose that, due to uncertainties in the model, the actual system is given by Figure 3.2 where $G(s)$ is defined in part (d) and $\Delta(s)$ is a stable perturbation. Find the maximum value for $\|\Delta\|_\infty$ for which the closed loop in Figure 3.2 is guaranteed to be stable. [4]

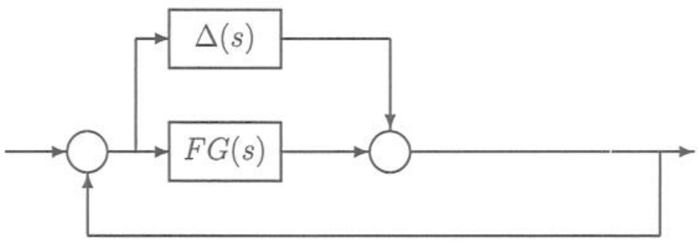

Figure 3.2

4. Consider the feedback configuration shown in Figure 4.1. Here, $G(s)$ represents a nominal plant model and $K(s)$ represents a compensator. The actual plant is given by $G_a(s) = (I + \Delta_1(s))(G(s) + \Delta_2(s))$ where $\Delta_1(s)$ and $\Delta_2(s)$ are stable transfer matrices that represent uncertainties. The design specification are to synthesize a compensator $K(s)$ such that the feedback loop is internally stable when:

(i) $\Delta_1 = 0$ and $\|\Delta_2(j\omega)\| \leq |w_2(j\omega)|, \forall\omega$, and,

(ii) $\Delta_2 = 0$ and $\|\Delta_1(j\omega)\| \leq |w_1(j\omega)|, \forall\omega$,

where $w_1(s)$ and $w_2(s)$ are appropriate weighting functions. The feedback loop is also required to have robust tracking specifications as follows:

(iii) If $y_o(s)$ is the output of the nominal loop (when $\Delta_1 = 0$ and $\Delta_2 = 0$), and $y_1(s)$ is the output when $\Delta_2(s) = 0$, then it is required that

$$\frac{\|y_o(j\omega) - y_1(j\omega)\|}{\|y_1(j\omega)\|} \leq \epsilon \ \forall\omega,$$

where $\epsilon > 0$ is given.

(a) Derive conditions, in terms of $G(s), K(s), w_1(s), w_2(s)$ and ϵ that are sufficient to achieve the design specifications. **[10]**

(*HINT:* For the robust tracking specification, carry out a manipulation to show that $y_o(s) - y_1(s) = H(s)\Delta_1(s)y_1(s)$, where $H(s)$ is a closed loop transfer matrix which you should calculate.)

(b) Derive a generalized regulator formulation of the design problem that captures the sufficient conditions in part (a). **[10]**

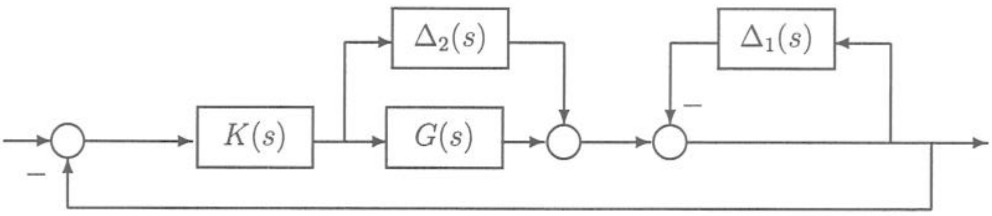

Figure 4.1

5. (a) Consider a state–variable model described by the dynamics

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t) + Du(t), \qquad x(0) = 0,$$

and let $H(s) = D + C(sI - A)^{-1}B$ denote the corresponding transfer matrix. Suppose there exists $P = P' > 0$ such that

$$\begin{bmatrix} A'P + PA + C'C & PB + C'D \\ B'P + D'C & D'D - \gamma^2 I \end{bmatrix} < 0.$$

(i) Prove that A is stable. [4]

(ii) By defining the Lyapunov function $V(t) = x(t)'Px(t)$, the cost function

$$J := \int_0^\infty [y(t)'y(t) - \gamma^2 u(t)'u(t)]dt,$$

and using a property of the integral $\int_0^\infty \dot{V}(t)dt$, or otherwise, prove that

$$\|H\|_\infty < \gamma.$$ [6]

(*HINT:* Express J in the form $J = \int_0^\infty \begin{bmatrix} x(t)^T & u(t)^T \end{bmatrix} M \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt$.)

(b) Consider the output injection problem shown in Figure 5.1 for which $x(0)=0$. Let $w = \begin{bmatrix} w_1^T & w_2^T \end{bmatrix}^T$ and let $T_{yw}(s)$ denote the transfer matrix from w to y. An internally stabilizing output injection gain matrix L is to be designed such that, for given $\gamma > 0$, $\|T_{yw}\|_\infty < \gamma$.

i. Derive a state space realization for $T_{yw}(s)$. (Take your state to be $x(t)$, the input to be $w(t)$ and the output to be $y(t)$). [4]

ii. By using the answer to part (a), or otherwise, derive sufficient conditions for the existence of a feasible L. Your conditions should be in the form of the existence of certain solutions to linear matrix inequalities. [6]

(*HINT:* Consider a simple change of variables to linearize any nonlinear matrix inequalities resulting from the use of part (a).)

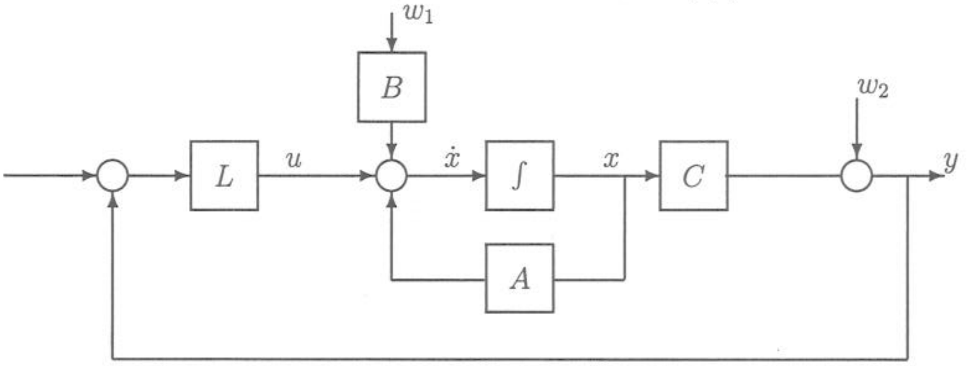

Figure 5.1

6. Consider the regulator shown in Figure 6.1 for which it is assumed that the triple (A, B, C) is minimal and $x(0)=0$. Note that this is a nonstandard problem in that $z_2 = u + w$, rather than $z_2 = u$.

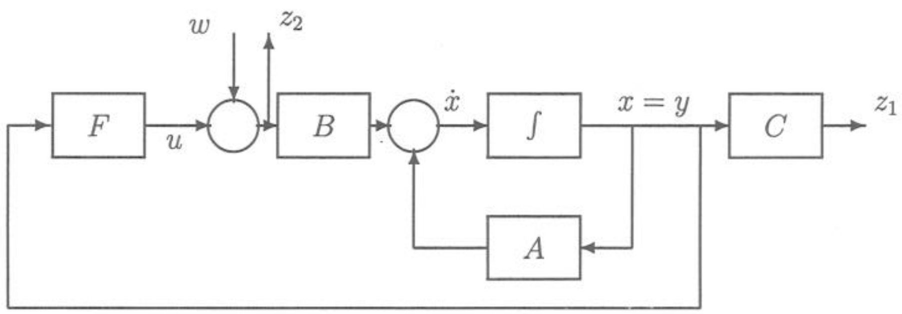

Figure 6.1

Let $z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ and let $H(s)$ denote the transfer matrix from w to z. A stabilizing state-feedback gain matrix F is to be designed such that, for $\gamma > 0$, $\|H\|_\infty < \gamma$. Assume that $\gamma > 1$.

(a) Write down the generalized regulator system for this design problem. **[6]**

(b) By using

(i) the Lyapunov function $V(t) = x(t)^T X x(t)$, where X is to be determined,

(ii) the cost function $J = \int_0^\infty [z(t)'z(t) - \gamma^2 w(t)'w(t)]\, dt$,

(iii) a property of the integral $\int_0^\infty \dot{V}(t)dt$,

(iv) two completion of squares procedures,

derive sufficient conditions for the solution of the design problem. Your conditions should be in the form of the existence of a certain solution to an algebraic Riccati equation. It should also include an expression for F and an expression for the worst-case disturbance w. [10]

HINT: Obtain an expression for J and separate it into three parts: one containing w in the form of a complete square, one containing F, also in the form of a complete square and a third term that involves a Riccati equation.

(c) What is the smallest γ for which your sufficient conditions guarantee the existence of F satisfying the design specifications. Justify your answer. [4]

[END]