IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2018

EEE/EIE PART I: MEng, BEng and ACGI

ANALYSIS OF CIRCUITS

Friday, 1 June 10:00 am

Time allowed: 2:00 hours

There are THREE questions on this paper.

Answer ALL questions.
Q1 carries 40% of the marks. Questions 2 and 3 carry equal marks (30% each).

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : D.M. Brookes

Second Marker(s) : P. Georgiou

ANALYSIS OF CIRCUITS

Information for Candidates:

- Numerical answers must be given as fully evaluated decimal values and not as unevaluated arithmetic expressions.

Notation

The following notation is used in this paper:

1. The voltage waveform at node X in a circuit is denoted by $x(t)$, the phasor voltage by X and the root-mean-square (or RMS) phasor voltage by $\widetilde{X} = \frac{X}{\sqrt{2}}$. The complex conjugate of X is X^*.
2. Component and source values in a circuit are normally given in Ohms, Farads, Henrys, Volts or Amps with the unit symbol omitted. Where an imaginary number is specified, it represents the complex impedance or phasor value.
3. Times are given in seconds unless otherwise stated.
4. Unless otherwise indicated, frequency response graphs should use a linear axis for phase and logarithmic axes for frequency and magnitude.
5. The real and imaginary parts of a complex number, X, are written $\Re(X)$ and $\Im(X)$ respectively.

1. a) Using nodal analysis, calculate the voltages at nodes X and Y of Figure 1.1. [5]

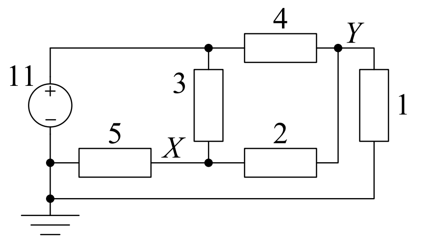

Figure 1.1

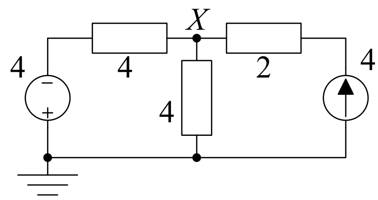

Figure 1.2

b) Use the principle of superposition to find the voltage X in Figure 1.2. [5]

c) Draw the Thévenin equivalent circuit of the two-terminal network in Figure 1.3 and find the values of its components. [5]

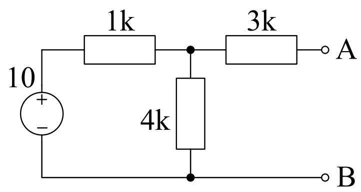

Figure 1.3

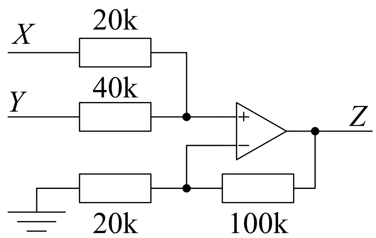

Figure 1.4

d) Assuming the opamp in the circuit of Figure 1.4 is ideal, give an expression for Z in terms of X and Y. [5]

e) Determine R_1 and R_2 in Figure 1.5 so that $Y = 0.25X$ and the parallel combination of R_1 and R_2 has an impedance of $75\,\Omega$. [5]

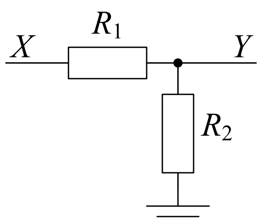

Figure 1.5

f) The circuit of Figure 1.6 shows a 50 Hz voltage source, with RMS voltage phasor $\widetilde{V} = 230$, driving a load of impedance $Z_L = 20 + 10j\,\Omega$ through a line of impedance $Z_T = 0.2 + 0.8j\,\Omega$. Calculate the complex power, $\widetilde{V} \times \widetilde{I}^*$, absorbed by (i) Z_T and (ii) Z_L. [5]

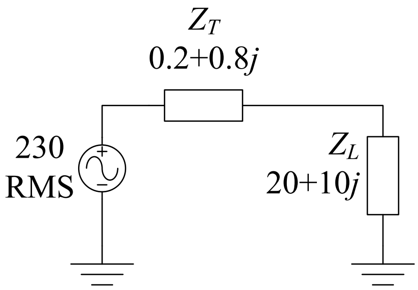

Figure 1.6

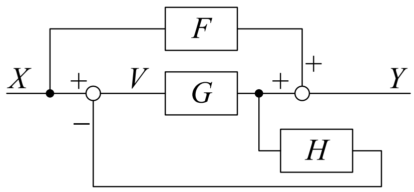

Figure 1.7

g) Determine the gain, $\frac{Y}{X}$, for the block diagram shown in Figure 1.7. The rectangular blocks are drawn with inputs at the left and outputs at the right and have gains of F, G and H respectively. The open circles represents adder/subtractors whose inputs have the signs indicated on the diagram and whose outputs are V and Y respectively. [5]

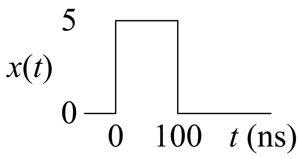

Figure 1.8

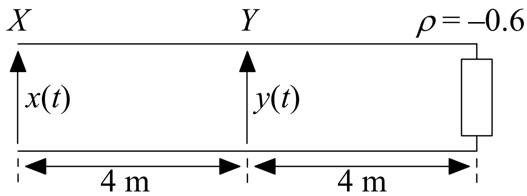

Figure 1.9

h) Figure 1.9 shows a transmission line of length 8 m that is terminated in a resistive load with reflection coefficient $\rho = -0.6$. The line has a propagation velocity of $u = 2 \times 10^8$ m/s. At time $t = 0$, a forward-travelling (i.e. left-to-right) pulse arrives at X with amplitude 5 V and duration 100 ns, as shown in Figure 1.8.

Draw a dimensioned sketch of the waveform at Y, a point 4 m from the end of the line, for $0 \le t \le 200$ ns. Assume that no reflections occur at point X. [5]

2. The frequency response of a highpass filter circuit is given by

$$H(j\omega) = \frac{k(j\omega)^2}{(j\omega)^2 + 2\zeta\omega_0 j\omega + \omega_0^2}$$

where k, ζ and ω_0 are positive real numbers and ω is in rad/s.

a) i) Give a simplified expression for the value of $H(j\omega)$ at the frequency $\omega = \omega_0$. [2]

ii) Determine the low and high frequency asymptotes of $H(j\omega)$. [2]

iii) By finding the squared magnitudes of the numerator and denominator expressions in $H(j\omega)$, show that [5]

$$|H(j\omega)|^2 = \frac{k^2}{\left(\frac{\omega_0^2}{\omega^2} - 1\right)^2 + 4\zeta^2\frac{\omega_0^2}{\omega^2}}.$$

iv) By writing the denominator of the previous expression in terms of $x = \frac{\omega_0^2}{\omega^2}$, show that the denominator has a minimum when $x = 1 - 2\zeta^2$.

Hence determine the value of ω at which $|H(j\omega)|$ is maximum and the value of $|H(j\omega)|$ at this frequency. [5]

b) Assuming $\omega_0 = k = 1$, draw a dimensioned sketch showing the magnitude response, $|H(j\omega)|$, in dB for the two cases: (A) $\zeta = 0.1$ and (B) $\zeta = 0.5$. Show both lines on the same set of axes. For each case, calculate the maximum value of $|H(j\omega)|$ in dB and the frequency, ω_p, at which it occurs. [6]

c) In the highpass filter circuit of Figure 2.1, the opamp is ideal, the capacitors have value C and the resistors have values P, Q, R and $(k-1)R$ respectively.

i) Explain why $Y = kV$. [1]

ii) By applying Kirchoff's current law at nodes U and V show that the transfer function $H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}$ is given by [6]

$$H(j\omega) = \frac{kPQC^2(j\omega)^2}{PQC^2(j\omega)^2 + (2P + (1-k)Q)Cj\omega + 1}.$$

iii) Determine simplified expressions for ζ and ω_0 when $H(j\omega)$ is written in the form given at the start of the question. [3]

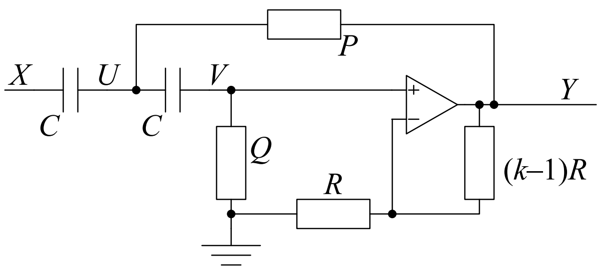

Figure 2.1

3. The diode in Figure 3.1 has a forward voltage of 0.7 V when it is conducting. The voltage waveforms at nodes X and Y are $x(t)$ and $y(t)$ respectively and the diode current is $i(t)$ as shown.

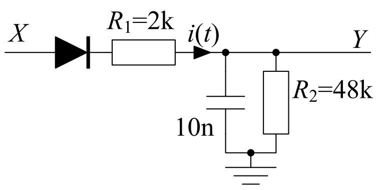

Figure 3.1

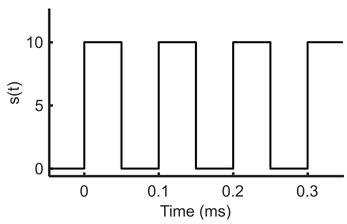

Figure 3.2

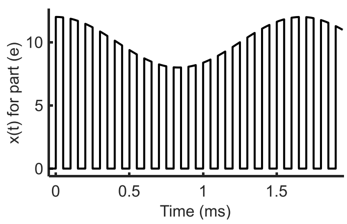

Figure 3.3

a) Assuming that node X is connected to a voltage source, calculate the time constant of the circuit when (a) the diode is conducting and (b) the diode is non-conducting. [4]

b) If $x(t)$ has a constant voltage of 10 V, determine the steady-state values of $i(t)$ and $y(t)$. [3]

c) Suppose $x(t) = \begin{cases} 0 & t < 0 \\ 10 & t \geq 0 \end{cases}$. Determine an expression for $y(t)$ for $t \geq 0$. [4]

d) Suppose now that $x(t) = s(t)$ as shown in Figure 3.2 where $s(t)$ is a positive-valued squarewave of period $T = 100\,\mu$s and amplitude 10 V.

i) Determine an expression for $y(t)$ for $0 \leq t < 0.5T$ assuming that the diode is conducting throughout this interval and that the value of y at the start of the interval is $y(0) = A$. Hence, derive and simplify an equation relating A and B where $B = y(0.5T)$ is the value of y at the end of the interval. [4]

ii) Determine an expression for $y(t)$ for $0.5T \leq t < T$ assuming that the diode is non-conducting throughout this interval and that the value of y at the start of the interval is $y(0.5T) = B$. Hence derive and simplify a second equation relating A and B assuming that the value of y at the end of the interval is $y(T) = A$. [3]

iii) By combining the equations determined in parts i) and ii), determine the numerical values of both A and B. [2]

iv) Sketch a dimensioned graph of $y(t)$ for $t \in [0, 200\,\mu\text{s}]$. [3]

e) Suppose now that $R_2 = 500\,\text{k}\Omega$ and that $x(t) = (1 + 0.2\cos(2\pi f t))\,s(t)$ is a modulated squarewave as illustrated in Figure 3.3 for $f = 600$ Hz.

i) Assuming that $y(t) \leq 11.3$, determine an upper bound on the current through R_2. [1]

ii) Explain why the average value of $i(t)$ must equal the average current through R_2. Hence find an upper bound on the average voltage across R_1 during the times that the diode is conducting. [3]

iii) Sketch the waveform $y(t)$ for a modulating frequency of $f = 20$ Hz. It is not necessary to calculate the value of $y(t)$ precisely. [3]

ANALYSIS OF CIRCUITS

**** Solutions 2018 ****

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Notation

The following notation is used in this paper:

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5. The real and imaginary parts of a complex number, X, are written $\Re(X)$ and $\Im(X)$ respectively.

Key: B=bookwork, U=unseen example

1. a) Using nodal analysis, calculate the voltages at nodes X and Y of Figure 1.1. [5]

[U] KCL at node X gives

$$\begin{aligned} \frac{X}{5}+\frac{X-11}{3}+\frac{X-Y}{2} &= 0 \\ \Rightarrow 6X+10X-110+15X-15Y &= 0 \\ \Rightarrow 31X-15Y &= 110 \end{aligned}$$

KCL at node Y gives

$$\begin{aligned} \frac{Y-X}{2}+\frac{Y}{1}+\frac{Y-11}{4} &= 0 \\ \Rightarrow -2X+7Y &= 11 \end{aligned}$$

Solving these simultaneous equations gives

$$X=5, \qquad Y=3.$$

Most got this right. A few thought that the left end of the $5\,\Omega$ resistor was connected to -11 instead of to 0 and wrote $\frac{X+11}{5}+\cdots$. Using the calculator to solve simultaneous equations is convenient but, in order to get the right answer, you need to enter them in the form $aX+bY=c$ rather than $aX+bY+c=0$. Another mistake when using the calculator was to write down the second equation as $7Y-2X=11$ and then enter the coefficients for X and Y in the wrong order.

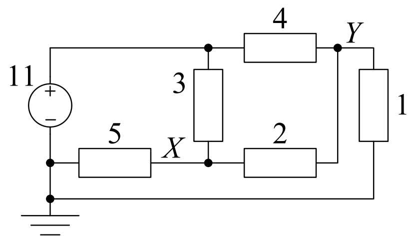

Figure 1.1

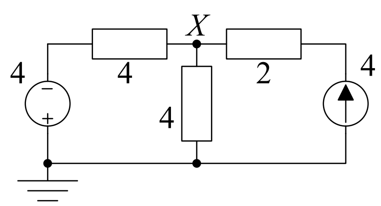

Figure 1.2

b) Use the principle of superposition to find the voltage X in Figure 1.2. [5]

[U] If we open-circuit the current source, the two $4\,\Omega$ resistors form a potential divider, so $X_1=-2\,\mathrm{V}$.

If we now short-circuit the voltage source, the two $4\,\Omega$ resistors are in parallel and are equivalent to a $2\,\Omega$ resistor. The current flowing through it is $4\,\mathrm{A}$ and so $X_2=+8\,\mathrm{V}$.

By superposition, the total voltage is therefore $X = X_1 + X_2 = -2 + 8 = 6\,\text{V}$.

Most got this right. Setting the unwanted source to zero turns a voltage source into a short circuit and a current source into an open circuit; a few people got it the other way around. Quite a few people had difficulty in working out the effect of the current source. You can either use KCL at node X or else just use ohm's law as above. Several people treated the current source as a $4\,\text{V}$ *voltage source and worked out X using the potential divider formula or using KCL. Several people got the polarity of either the voltage source or the current source reversed; a few re-drew the diagram but with the wrong polarity of voltage source.*

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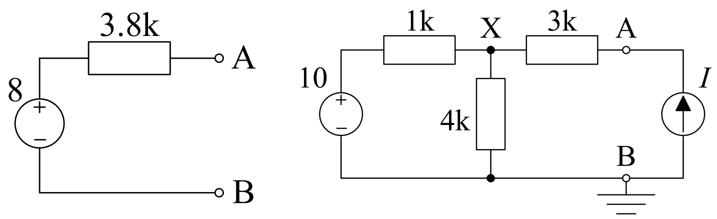
c) Draw the Thévenin equivalent circuit of the two-terminal network in Figure 1.3 and find the values of its components. [5]

[U] We can find the Thévenin resistance by short-circuiting the voltage source. This leaves two resistors in parallel with an equivalent resistance of $R_P = \frac{1\times 4}{1+4} = 0.8\,\text{k}\Omega$ *in series with a* $3\,\text{k}\Omega$ *resistor. The total resistoance is therefore* $R_{Thev} = 3.8\,\text{k}\Omega$.

To find the open circuit voltage, we note that there is no current through the $3\,\text{k}\Omega$ *resistor and hence no voltage across it. The other two resistors form a potential divider and the voltage across the* $4\,\text{k}\Omega$ *resistor is therefore* $V_{Thev} = 8V$. *Thus we get the diagram on the left below.*

An alternative method is to ground node B and append a current source, I, as shown in the rightmost diagram below. Now doing KCL at node A gives $\frac{A-X}{3} - I = 0$ *from which* $X = A - 3I$ *and KCL at node X gives* $\frac{X-10}{1} + \frac{X}{4} + \frac{X-A}{3} = 0$ *from which* $19X = 4A - 120$. *Eliminating X between these equations gives* $19A - 57I = 4A - 120$ *from which* $A = 8 + \frac{57}{15}I = 8 + 3.8I$ *which gives* V_{Thev} *and* R_{Thev} *directly.*



Most got this right. When calculating the Thévenin resistance, a few thought the $3k$ *and* $4k$ *resistors were either in series or parallel, neither of which is true. If the question says "Draw the Thévenin equivalent circuit ..." then you need to draw the circuit rather than just work out value for* V_{th} *and* R_{th}.

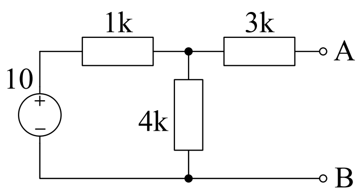


Figure 1.3

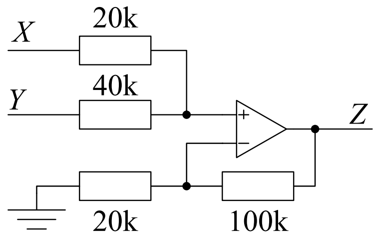


Figure 1.4

d) Assuming the opamp in the circuit of Figure 1.4 is ideal, give an expression for Z in terms of X and Y. [5]

[U] The upper two resistors form a weighted average circuit, so $V_+ = \frac{2X+Y}{3}$. The remainder of the circuit is a non-inverting amplifier with gain of $1 + \frac{100}{20} = 6$. Thus $Z = \frac{2X+Y}{3} \times 6 = 4X + 2Y$.

Most people got this right although some found it difficult. Some assumed that negative feedback meant that the op-amp inputs were both at 0; this is only true if one of them is connected to 0. Several misread 100k for 10k. Some got the potential divider formula wrong and said that $\frac{V_-}{Z} = \frac{20}{100}$ rather than $\frac{20}{120}$. Several people correctly wrote $\frac{Z}{2} = 2X + Y$ but then, surprisingly, simplified this to $Z = X + \frac{Y}{2}$.

---

e) Determine R_1 and R_2 in Figure 1.5 so that $Y = 0.25X$ and the parallel combination of R_1 and R_2 has an impedance of $75\,\Omega$. [5]

[U] The gain of the potential divider is $0.25 = \frac{R_2}{R_1+R_2}$ which implies that $0.25R_1 = (1-0.25)R_2 = 0.75R_2$ from which $R_1 = 3R_2$.

Substituting this relationship into the parallel impedance formula gives $75 = \frac{R_1R_2}{R_1+R_2} = \frac{3R_2^2}{4R_2} = 0.75R_2$ from which $R_2 = 100\,\Omega$ and $R_1 = 3R_2 = 300\,\Omega$.

Most got this right. Several correctly got $R_2 = 100\,\Omega$ but then wrongly said $R_1 = 3R_2 \Rightarrow R_1 = 33.3\,\Omega$.

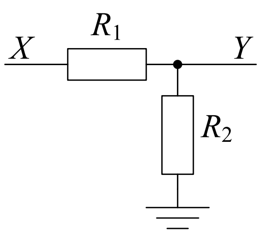


Figure 1.5

f) The circuit of Figure 1.6 shows a 50 Hz voltage source, with RMS voltage phasor $\widetilde{V} = 230$, driving a load of impedance $Z_L = 20 + 10j\,\Omega$ through a line of impedance $Z_T = 0.2 + 0.8j\,\Omega$. Calculate the complex power, $\widetilde{V} \times \widetilde{I}^*$, absorbed by (i) Z_T and (ii) Z_L. [5]

[U] The current phasor is $\widetilde{I}_L = \frac{230}{Z_L+Z_T} = \frac{230}{20.2+10.8j} = 8.855 - 4.734j$. *From this* $\left|\widetilde{I}_L\right| = 10.04$ *and* $\left|\widetilde{I}_L\right|^2 = 100.8$.

The complex power absorbed by Z_L *is*

$$\left|\widetilde{I}_L\right|^2 Z_L = 100.8\,(20+10j) = 2016 + 1008j\,\text{VA} = 2255\angle 26.6^\circ$$

.

The complex power absorbed by Z_T *is*

$$\left|\widetilde{I}_L\right|^2 Z_T = 100.8\,(0.2+0.8j) = 20.16 + 80.66j\,\text{VA} = 83.1\angle 76.0^\circ.$$

Alternatively, if you want to use the $\widetilde{V} \times \tilde{I}^*$ *formula directly, you must calculate*

$$\widetilde{V}_L = \widetilde{I}_L Z_L = (8.855 - 4.734j)(20+10j) = 224.4 - 6.1j$$

and

$$\widetilde{V}_T = \widetilde{I}_L Z_T = (8.855 - 4.734j)(0.2+0.8j) = 230 - \widetilde{V}_L = 5.6 + 6.1j.$$

From this we get the absorbed powers as

$$\widetilde{V}_L \widetilde{I}_L^* = (224.4 - 6.1j)(8.855 + 4.734j) = 2016 + 1008j\,\text{VA}$$

and

$$\widetilde{V}_T \widetilde{I}_L^* = (5.6 + 6.1j)(8.855 + 4.734j) = 20.16 + 80.66j\,\text{VA}.$$

Most people got this right. Several people wrongly included a square root and wrote $\left|\widetilde{I}_L\right|^2 = 10.04$ *others omitted the modulus signs and used* $S = \widetilde{I}_L^2 Z_L$ *instead of* $\left|\widetilde{I}_L\right|^2 Z_L$. *Some calculated the total power as* $230\widetilde{I}_L^*$ *and then subtracted the power in* Z_L *to obtain the power in* Z_T. *This is algebraically correct but, as always when two almost-equal quantities are subtracted, small percentage rounding errors in* Z_T *will result in large percentage errors in* Z_T. *Some people made use of the valid formula* $S = \frac{|\widetilde{V}|^2}{Z^*}$ *but often used* $\widetilde{V} = 230$ *rather than the voltage across* Z_T *or* Z_L. *Despite the question asking for the "complex" power, several people gave only the real part or else added the real and imaginary parts together (which is always a bad idea with complex numbers). One or two people left their answer as a complicated numerical expression rather than evaluating it as instructed on the front cover of the exam paper.*

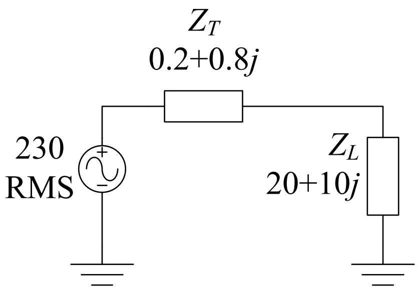


Figure 1.6

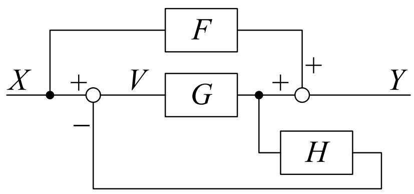


Figure 1.7

g) Determine the gain, $\frac{Y}{X}$, for the block diagram shown in Figure 1.7. The rectangular blocks are drawn with inputs at the left and outputs at the right and have gains of F, G and H respectively. The open circles represents adder/subtractors whose inputs have the signs indicated on the diagram and whose outputs are V and Y respectively. [5]

[U] We can write down the following equations from the block diagram:

$$\begin{aligned} V &= X - GHV \\ Y &= FX + GV \end{aligned}$$

We need to eliminate V from these equations:

$$\begin{aligned} V(1+GH) &= X \\ \Rightarrow V &= \frac{1}{1+GH}X \\ Y &= FX + GV \\ &= \left(F + \frac{G}{1+GH}\right)X \\ \Rightarrow \frac{Y}{X} &= F + \frac{G}{1+GH} \\ &= \frac{F+G+FGH}{1+GH} \end{aligned}$$

Most got this right. If the question asks for $\frac{Y}{X}$ then the last line of your answer should be $\frac{Y}{X} = \cdots$ rather than $Y = \cdots$. Surprisingly many people correctly wrote $FX + GV$ but then ignored the G when substituting for V.

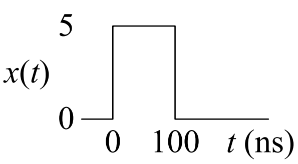


Figure 1.8

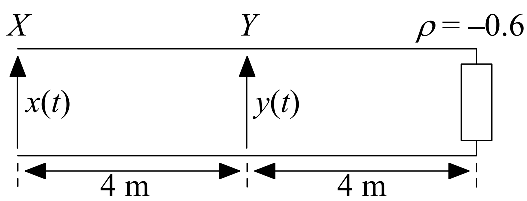
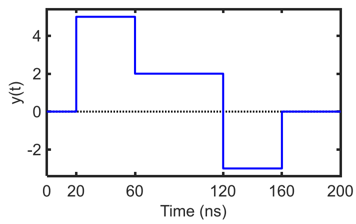


Figure 1.9

h) Figure 1.9 shows a transmission line of length 8 m that is terminated in a resistive load with reflection coefficient $\rho = -0.6$. The line has a propagation velocity of $u = 2 \times 10^8$ m/s. At time $t = 0$, a forward-travelling (i.e. left-to-right) pulse arrives at X with amplitude 5 V and duration 100 ns, as shown in Figure 1.8.

Draw a dimensioned sketch of the waveform at Y, a point 4 m from the end of the line, for $0 \le t \le 200$ ns. Assume that no reflections occur at point X. [5]

[U] The time taken to travel 4 m is $T = \frac{400}{20} = 20$ ns. So the forward wave (of amplitude 5 V) arrives at Y at $t = T = 20$ ns and ends at $t = 120$ ns. The reflected wave (of amplitude $5\rho = -3V$) has to travel an additional 8 m and so arrives at $t = 3T = 60$ ns and ends at $t = 160$ ns. Adding the two waves together gives the following graph where the horizontal values are $\{0, 5, 2, -3, 0\}$:



Most get this right. Several people did not indicate all the horizontal values on the graph (usually omitting the -3); if you do this, you need to make sure your plot is accurate enough that the values can be deduced from the graph (or else show working that calculates the horizontal values). A few people made the problem more complicated by including an additional reflection at point X even though the questions explicitly said there was no reflection at X. Several of the graphs were a bit vague (or even incorrect) about what happened for $t < 20$ and $t > 160$.

---

2. The frequency response of a highpass filter circuit is given by

$$H(j\omega) = \frac{k(j\omega)^2}{(j\omega)^2 + 2\zeta\omega_0 j\omega + \omega_0^2}$$

where k, ζ and ω_0 are positive real numbers and ω is in rad/s.

a) i) Give a simplified expression for the value of $H(j\omega)$ at the frequency $\omega = \omega_0$. [2]

[U] At $\omega = \omega_0$, $(j\omega)^2 + \omega_0^2 = -\omega^2 + \omega_0^2 = 0$ so these two terms cancel out in the denominator. So $H(j\omega) = \frac{-k\omega_0^2}{2\zeta\omega_0 j\omega_0} = \frac{jk}{2\zeta}$.

Most people got this right. Some did not simplify the answer as requested but gave it as, for example, $H(j\omega) = \frac{-k\omega_0^2}{2\zeta j\omega_0^2}$ which appears to depend on the value of ω_0. At the very least, "simplified" means cancelling out any false dependencies such as this. A few people did not realize that $j^2 = -1$.

---

ii) Determine the low and high frequency asymptotes of $H(j\omega)$. [2]

[U] LF asymptote is $H(j\omega) \to k\omega_0^{-2}(j\omega)^2 = -k\left(\frac{\omega}{\omega_0}\right)^2$.

HF asymptote is $H(j\omega) \to k$.

The asymptotes cross at $\omega = \omega_0$ [not requested].

Most people got this right. A few omitted the j and gave the LF asymptote of $|H(j\omega)|$ instead. A few people said the LF asmptote was 0; this is indeed the value of $H(j\omega)$ when $\omega = 0$ but it is not the LF asymptote which gives the asymptotic straight line on the magnitude response graph for values of ω near zero (rather than exactly at zero).

---

iii) By finding the squared magnitudes of the numerator and denominator expressions in $H(j\omega)$, show that [5]

$$|H(j\omega)|^2 = \frac{k^2}{\left(\frac{\omega_0^2}{\omega^2} - 1\right)^2 + 4\zeta^2\frac{\omega_0^2}{\omega^2}}.$$

[U] If $z = a + jb$, then $|z|^2 = (a+jb)(a-jb) = a^2 + b^2$. We can

write

$$\begin{aligned}
|H(j\omega)|^2 &= \frac{\left|k(j\omega)^2\right|^2}{\left|(j\omega)^2+2\zeta\omega_0 j\omega+\omega_0^2\right|^2} \\
&= \frac{k^2\omega^4}{\left|\left(\omega_0^2-\omega^2\right)+2\zeta\omega_0\omega j\right|^2} \\
&= \frac{k^2\omega^4}{\left(\omega_0^2-\omega^2\right)^2+\left(2\zeta\omega_0\omega\right)^2} \\
&= \frac{k^2\omega^4}{\left(\omega_0^2-\omega^2\right)^2+4\zeta^2\omega_0^2\omega^2} \\
&= \frac{k^2}{\left(\frac{\omega_0^2}{\omega^2}-1\right)^2+4\zeta^2\frac{\omega_0^2}{\omega^2}}
\end{aligned}$$

Most people did this OK although sometimes with a lot of algebra. A really terrible idea is to multiply the numerator and denominator of the original expression for $H(j\omega)$ by the complex conjugate of the denominator; this just makes everything more complicated and results in 8th order polynomials. Some thought the imaginary part of the denominator was $(j\omega)^2+2\zeta\omega_0 j\omega$ without realizing that the first of these terms is actually real-valued even though it includes a j.

---

iv) By writing the denominator of the previous expression in terms of $x=\frac{\omega_0^2}{\omega^2}$, show that the denominator has a minimum when $x=1-2\zeta^2$.

Hence determine the value of ω at which $|H(j\omega)|$ is maximum and the value of $|H(j\omega)|$ at this frequency. [5]

[U] Making the suggested substitution, the numerator of $|H(j\omega)|^2$ is positive and indpendent of x while the denominator becomes $(x-1)^2+4\zeta^2x$. The coefficient of x^2 in this quadratic expression is positive and so the expression has a minimum. Setting its derivative to zero gives $2(x-1)+4\zeta^2=0$ from which $x=1-2\zeta^2$. Note that, since we must have $x\geq 0$, this only has a solution in ω if $\zeta<\sqrt{0.5}=0.707$ [not requested]. You can also show that this is a minimum (in the denominator) by showing that the second derivative, $\frac{d}{dx}\left\{2(x-1)+4\zeta^2\right\}=2$, is positive.

Thus the maximum of $|H(j\omega)|^2$ and hence of $|H(j\omega)|$ occurs at when $x_p=\frac{\omega_0^2}{\omega^2}=1-2\zeta^2\Rightarrow\omega_p=\frac{\omega_0}{\sqrt{1-2\zeta^2}}$.

When $x=1-2\zeta^2$, the denominator of $|H(j\omega_p)|^2$ equals $(x-1)^2+4\zeta^2x=\left(2\zeta^2\right)^2+4\zeta^2\left(1-2\zeta^2\right)=4\zeta^2\left(1-\zeta^2\right)$. So $|H(j\omega_p)|^2=\frac{k^2}{4\zeta^2(1-\zeta^2)}$ and $|H(j\omega_p)|=\frac{k}{2\zeta\sqrt{1-\zeta^2}}$.

An alternative approach (which involves more work) is to substitute $\omega_p=\frac{\omega_0}{\sqrt{1-2\zeta^2}}\Rightarrow\frac{\omega_0}{\omega_p}=\sqrt{1-2\zeta^2}$ into the original expression for $H(j\omega_p)$ (with numerator and denominator divided by $(j\omega_p)^2=$

$-\omega_p^2$) to obtain:

$$\begin{aligned}
H(j\omega_p) &= \frac{k}{1-2j\zeta\frac{\omega_0}{\omega_p}-\frac{\omega_0^2}{\omega_p^2}} \\
&= \frac{k}{1-2j\zeta\sqrt{1-2\zeta^2}-(1-2\zeta^2)} \\
&= \frac{k}{2\zeta\left(\zeta-j\sqrt{1-2\zeta^2}\right)} \\
|H(j\omega_p)| &= \frac{k}{2\zeta\sqrt{\zeta^2+(1-2\zeta^2)}} \\
&= \frac{k}{2\zeta\sqrt{1-\zeta^2}}
\end{aligned}$$

Yet another approach is to complete the square of the denominator expression: $(x-1)^2+4\zeta^2x = x^2-2(1-2\zeta^2)x+1 = \left(x-\left(1-2\zeta^2\right)\right)^2 + 1-\left(1-2\zeta^2\right)^2$. This clearly has a minimum whan $x=1-2\zeta^2$ and the value of the minimum equals the constant term, $1-\left(1-2\zeta^2\right)^2 = 4\zeta^2\left(1-\zeta^2\right)$.

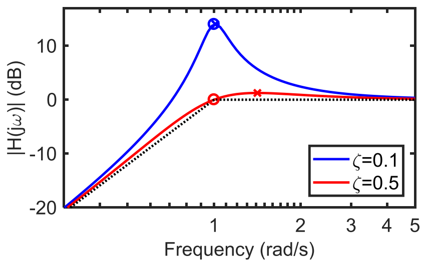
A few people with good memories gave the formula that was given in the notes for a low-pass filter, $\omega_p=\omega_0\sqrt{1-2\zeta^2}$, instead of calculating the correct formula for this high-pass filter of $\omega_p=\frac{\omega_0}{\sqrt{1-2\zeta^2}}$. Quite a few people failed to show that the denominator had a minimum rather then a maximum (by taking the second deritative or by noting that the coefficient of x^2 was positive). Several people thought that the denominator itself had to equal zero at a minimum (rather than its derivative equalling zero). Very many people obtained $4\zeta^2$ instead of $4\zeta^4$ when they substituted $x=1-2\zeta^2$ into $(x-1)^2$. Many people made other algebraic errors which resulted in the wrong formula for $|H(j\omega_p)|$. A few people assumed that the squared magnitude of a complex number was given by $|a+jb|^2=(a+b)^2$ instead of the correct formula a^2+b^2. A few people used the quotient rule to differentiate $|H(j\omega)|^2=\frac{k^2}{(x-1)^2+4\zeta^2x}$ and then used the quotient rule again to show that it was a minimum; this is quite a lot more effort than just differentiating the denominator (which is equivalent since the numerator is independent of x). To find the value of $|H(j\omega_p)|$, it is much easier to substitute into the expression given in part iii) and then take the square root rather than use the expression at the start of the question and then calculate the magnitude. If the question asks for the value of $|H(j\omega)|$ then the last line of your answer should be of the form $|H(j\omega)|=\cdots$ and not the equation of some related quantity such as $|H(j\omega)|^2=\cdots$.

b) Assuming $\omega_0=k=1$, draw a dimensioned sketch showing the magnitude response, $|H(j\omega)|$, in dB for the two cases: (A) $\zeta=0.1$ and (B) $\zeta=0.5$. Show both lines on the same set of axes. For each case, calculate the maximum value of $|H(j\omega)|$ in dB and the frequency, ω_p, at which it occurs. [6]

[U] Using the formulae from the answers to parts i) and iv) above, we can construct the following table (note that numerical values for $|H(j\omega_0)|$ were not requested):

ζ	ω_0	$\lvert H(j\omega_0)\rvert$	ω_p	$\lvert H(j\omega_p)\rvert$
0.1	1	$5 = 14\,\text{dB}$	$\frac{5\sqrt{2}}{7} = 1.01$	$5.03 = 14\,\text{dB}$
0.5	1	$1 = 0\,\text{dB}$	$\sqrt{2} = 1.41$	$1.15 = 1.25\,\text{dB}$

This results in the following graph where, in addition to the asymptotes (shown as dotted lines), we know the positions of two points on each curve: o and x denote $|H(j\omega_0)|$ and $|H(j\omega_p)|$ respectively:



Note that, from the expression given in part a)iii) of the question, increasing ζ always reduces $|H(j\omega)|$ at every ω. This means that the curves for two different values of ζ can never cross and the red curve ($\zeta = 0.5$) must lie entirely below the blue curve ($\zeta = 0.1$).

Not many people calculated the values at ω_0 (marked with circles above) even though the formula was the answer to part a)i). Strangely, several people thought that maximum of the graph was at ω_0 even though part a) was all about showing that the maximum was at $\omega_p = \frac{\omega_0}{\sqrt{1-2\zeta^2}}$. Also, many people had different asymptotes for the two values of ζ ($\zeta = 0.1$ and $\zeta = 0.5$) even though the formulae calculated in part a) didn't depend on ζ. Several people obtained dB values that were twice as large as they should be by using the incorrect formula $20\log_{10}\left(|H(j\omega)|^2\right)$; the 20 should be a 10 since $|H(j\omega)|^2$ is a power ratio. Some drew the graphs on separate axes even though the question asked for them on the same set of axes. Many people made the peaks too sharp and had curves that crossed. Quite a few people calculated the asymptotes correctly in part a) but then drew the graphs as lowpass filters.

c) In the highpass filter circuit of Figure 2.1, the opamp is ideal, the capacitors have value C and the resistors have values P, Q, R and $(k-1)R$ respectively.

i) Explain why $Y = kV$. [1]

[U] The resistors $(k-1)R$ and R form a potential divider and, assuming the opamp draws no input current, the voltage at its negative input is therefore $\frac{R}{R+(k-1)R}Y = k^{-1}Y$. Provided that the circuit has negative feedback, the opamp inputs will have the same voltage and so $V = k^{-1}Y$. Alternatively, just recognise that it is a standard non-inverting opamp configuration whose gain is $1 + \frac{(k-1)R}{R} = +k$. Note that, as well as the negative feedback path via $(k-1)R$, there is also a positive feedback path via P and C. We need to assume that the overall feedback is negative; this is certainly true at DC when the

capacitor acts as an open circuit. Although not requested, the condition for stability is actually that $\zeta > 0$; the expression for ζ given in the solution to part iii), $\zeta = \frac{2P+(1-k)Q}{2\sqrt{PQ}}$, maps this into $k < 1 + \frac{2P}{Q}$.

Several people said "Because of negative feedback." or something similar; this is not a proof or sufficient explanation. Surprisingly, some people "proved" that $Y = (k-1)V$ which shows a lot of self-conifidence. .

---

ii) By applying Kirchoff's current law at nodes U and V show that the transfer function $H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}$ is given by [6]

$$H(j\omega) = \frac{kPQC^2(j\omega)^2}{PQC^2(j\omega)^2 + (2P + (1-k)Q)Cj\omega + 1}.$$

[U] Applying KCL and U and $V = k^{-1}Y$ gives

$$(U - X)\,j\omega C + \left(U - k^{-1}Y\right)j\omega C + \frac{U-Y}{P} = 0 \qquad (2.1)$$

$$\left(k^{-1}Y - U\right)j\omega C + \frac{k^{-1}Y}{Q} = 0 \qquad (2.2)$$

from which

$$(2j\omega PC + 1)\,U = j\omega PCX + \left(j\omega k^{-1}PC + 1\right)Y \qquad (2.3)$$

$$\Rightarrow\ U = \frac{j\omega PCX + \left(j\omega k^{-1}PC + 1\right)Y}{2j\omega PC + 1} \qquad (2.4)$$

$$j\omega QCU = (j\omega QC + 1)\,k^{-1}Y$$

$$\Rightarrow\ U = \frac{(j\omega QC + 1)\,Y}{j\omega kQC} \qquad (2.5)$$

Equating equations (2.4) and (2.5) to eliminate U and then cross-multiplying by the denominators gives

$$(2j\omega PC + 1)(j\omega QC + 1)Y = j\omega kQC\left(j\omega PCX + \left(j\omega k^{-1}PC + 1\right)Y\right)$$

$$\Rightarrow \left(2PQC^2(j\omega)^2 + (2PC + QC)\,j\omega + 1\right)Y$$

$$= kPQC^2(j\omega)^2X + \left(PQC^2(j\omega)^2 + kQCj\omega\right)Y$$

$$\Rightarrow \left(PQC^2(j\omega)^2 + (2PC + (1-k)QC)\,j\omega + 1\right)Y = kPQC^2(j\omega)^2X$$

$$\Rightarrow \frac{Y}{X} = \frac{kPQC^2(j\omega)^2}{PQC^2(j\omega)^2 + (2P + (1-k)Q)Cj\omega + 1}.$$

Most wrote down the correct KCL equations correctly. However, rather than then systematically eliminating U between the two equations, many people then did some rather aimless algebra instead; this often took up several pages. Many people used equation 2.5 to substitute for U in equation 2.1 (or vice versa); this is a good approach but it is sensible to collect all the terms containing U together *(as in equation 2.3) before substituting for U. It is also sensible to get rid of the fractions as early as possible, especially fractions with other fractions in the numerator or denominator.*

---

iii) Determine simplified expressions for ζ and ω_0 when $H(j\omega)$ is written in the form given at the start of the question. [3]

[U] In order to make the coefficient of $(j\omega)^2$ equal to k (to match the equation for $H(j\omega)$), we divide numerator and denominator by PQC^2 to obtain

$$\frac{Y(j\omega)}{X(j\omega)} = \frac{k(j\omega)^2}{(j\omega)^2 + (2Q^{-1} + (1-k)P^{-1})C^{-1}j\omega + P^{-1}Q^{-1}C^{-2}}.$$

Matching coefficients gives

$$\omega_0 = \sqrt{P^{-1}Q^{-1}C^{-2}} = C^{-1}\sqrt{P^{-1}Q^{-1}} = \frac{1}{C\sqrt{PQ}}$$

$$2\zeta\omega_0 = \left(2Q^{-1} + (1-k)P^{-1}\right)C^{-1} = \frac{2P + (1-k)Q}{CPQ}$$

$$\Rightarrow \zeta = \sqrt{\frac{P}{Q}} + 0.5(1-k)\sqrt{\frac{Q}{P}} = \frac{2P + (1-k)Q}{2\sqrt{PQ}}$$

For the expression for ζ, several people included a C in the numerator and a $\sqrt{C^2}$ in the denominator and didn't notice that they cancelled out. Many people forgot to divide the denominator by PQC^2 in order to make the coefficient of $(j\omega)^2$ equal to unity. In the penultimate line above, a few people got confused when dividing by ω_0 and ended up multiplying the right-hand side of the equation by ω_0 instead. Several people assumed that $\omega_0 = 1$ as in part b) but this is clearly not true from the given transfer function.

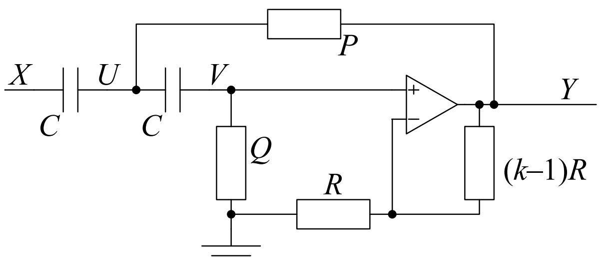


Figure 2.1

3. The diode in Figure 3.1 has a forward voltage of $0.7\,\mathrm{V}$ when it is conducting. The voltage waveforms at nodes X and Y are $x(t)$ and $y(t)$ respectively and the diode current is $i(t)$ as shown.

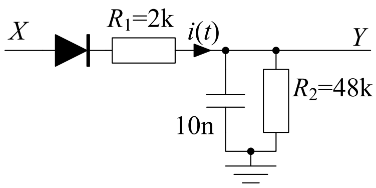


Figure 3.1

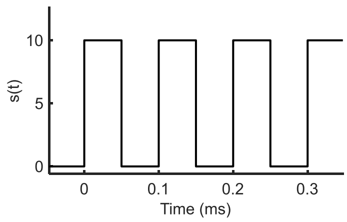


Figure 3.2

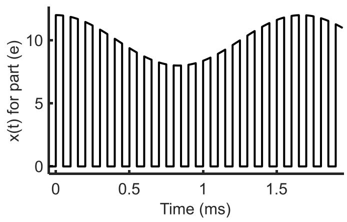


Figure 3.3

a) Assuming that node X is connected to a voltage source, calculate the time constant of the circuit when (a) the diode is conducting and (b) the diode is non-conducting. [4]

[U] (A) When the diode is conducting, the Thévenin resistance of the circuit driving the capacitance is $R_p = 2||48 = 1.92\,\mathrm{k}\Omega$. The time constant is therefore $\tau_{on} = R_pC = 19.2\,\mu\mathrm{s}$. A more complicated approach (not recommended) is to calculate the transfer function, $\frac{Y}{X}$, assuming that the diode voltage is zero (using superposition). This gives $\frac{Y}{X} = \frac{R_2}{R_1+R_2+j\omega R_1R_2C}$ from which the time constant is $\tau = \frac{R_1R_2}{R_1+R_2}C$ as before.

(B)When the diode is off, the Thevenin resistance of the circuit driving the capacitance is $R_2 = 48\,\mathrm{k}\Omega$. The time constant is therefore $\tau_{off} = R_2C = 480\,\mu\mathrm{s}$.

A common mistake was to omit a factor of 10^3 because the resistors are given in $\mathrm{k}\Omega$; factors of 1000 are generally quite important in engineering. If you use the transfer function method of calculating the time constant, you need to realize that, when the diode is conducting, there are two voltage sources: X and the diode voltage of $0.7\,\mathrm{V}$. In order to calculate the transfer function from X to Y, you need to set the other voltage source to zero (as in superposition); most people who used this method did not do this. A few people thought that the time constant did not exist when the diode was off because they were trying to calculate it from the transfer function, $\frac{Y}{X}$ which, of course, equals 0. You can still do it using this method by taking a limit as the resistance R_2 tends to infinity in $\tau = \frac{1}{R_1^{-1}+R_2^{-1}}C$ (thereby disconnecting the diode), but it is much easier to use the Thévenin method given above. When calculating the Thévenin resistance, quite a few people thought that setting a voltage source to zero makes it an open circuit whereas actually it makes it a short circuit (i.e. a voltage difference of zero). Conversely, several people thought that when the diode was non-conducting, the left end of the $2\,\mathrm{k}$ resistor was connected to ground whereas, in fact, it is not connected to anything.

b) If $x(t)$ has a constant voltage of $10\,\mathrm{V}$, determine the steady-state values of $i(t)$ and $y(t)$. [3]

[U] If the input voltage is constant, there is no current through the capacitor and so the circuit just consists of a diode and two resistors in series. The diode

voltage is $0.7V$ *and so the diode current is* $i = \frac{9.3}{R_1+R_2} = 186\,\mu\text{A}$*. Hence* $y = i_D R_2 = 8.928\,\text{V}$.

Most people got this right, but not everyone realised that this was the steady state for parts c) and d)i). Even though they often correctly calculated y, quite a lot of people calculated the current as $\frac{9.3}{R_1}$ *because they thought the capacitor acted as a short circuit at DC (whereas it actually acts as an open circuit). A few people got the polarity of the diode volatage wrong: since a diode does not generate energy, its voltage and current have the same sign when using the passive sign convention. Some gave the answers as exact fractions:* $i = \frac{93}{500}\,\text{mA}$ *and* $y = \frac{1116}{125}\,\text{V}$*; answers like this rarely make any sense in engineering and the rubric on the first page of the exam paper tells you to give answers as decimal numbers.*

---

c) Suppose $x(t) = \begin{cases} 0 & t < 0 \\ 10 & t \geq 0 \end{cases}$. Determine an expression for $y(t)$ for $t \geq 0$. [4]

[U] For $t < 0$*,* $y(t) = 0$ *and hence, since the capacitor voltage cannot change instantly,* $y(0+) = 0$*.*

For $t \geq 0$*, the diode will be on and the steady state value is* $y_{SS}(t) = 8.928$ *(from part b)) and so the full expression for* $y(t)$ *is* $y(t) = 8.928\left(1 - e^{-\frac{t}{\tau_{on}}}\right)$*.*

A few people assumed that the diode voltage was 0.7 *even when* $x = 0$*. However, this would make i negative which contradicts the assumption that the diode is on. It also makes no sense that current can flow when* $x = 0$ *since there is no energy source to suppy the power absorbed by the resistors. Since the capcitor is connected between Y and ground, the voltage at Y will not change instantly at* $t = 0$*; many people made it abruptly jump to* $10\,\text{V}$*. Several people instead assumed that* $x - y$ *wouldn't change at* $t = 0$ *; this would be true if the capacitor was connected beween X and Y instead of between Y and ground. Quite a lot of people did not say which time-constant,* τ*, they were talking about or give its value.*

---

d) Suppose now that $x(t) = s(t)$ as shown in Figure 3.2 where $s(t)$ is a positive-valued squarewave of period $T = 100\,\mu\text{s}$ and amplitude $10\,\text{V}$.

i) Determine an expression for $y(t)$ for $0 \leq t < 0.5T$ assuming that the diode is conducting throughout this interval and that the value of y at the start of the interval is $y(0) = A$. Hence, derive and simplify an equation relating A and B where $B = y(0.5T)$ is the value of y at the end of the interval. [4]

[U] This is the same as part c) except that $y(0+) = A$*. The formula is therefore* $y(t) = 8.928 + (A - 8.928)\,e^{-\frac{t}{\tau_{on}}}$*. Substituting* $t = 0.5T = 50\,\mu\text{s}$ *gives*

$$
\begin{aligned}
y(0.5T) = B &= 8.928 + (A - 8.928)\,e^{-\frac{0.5T}{\tau_{on}}} \\
&= 8.928 + (A - 8.928)\,0.074 \\
&\Rightarrow -0.074A + B - 8.2676 = 0
\end{aligned}
$$

.

Several people deduced the valid but inconvenient equation: $\ln\left(\frac{B-8.928}{A-8.928}\right) = -\frac{50}{19.2}$ *. Several people tried to solve this psrt from first principles by writing down a differential equation; this is a valid method but is much more complicated and most failed to do it correctly.*

---

ii) Determine an expression for $y(t)$ for $0.5T \le t < T$ assuming that the diode is non-conducting throughout this interval and that the value of y at the start of the interval is $y(0.5T) = B$. Hence derive and simplify a second equation relating A and B assuming that the value of y at the end of the interval is $y(T) = A$. [3]

[U] For the interval $0.5T \le t < T$*, the diode is off and so the steady state is* $y_{SS}(t) = 0$*. It follows that* $y(t) = Be^{-\frac{t-0.5T}{\tau_{off}}} = 1.11Be^{-\frac{t}{\tau_{off}}}$*. Substituting* $t = T = 100\,\mu\text{s}$ *gives* $y(T) = A = Be^{-\frac{0.5T}{\tau_{off}}} = 0.9011B \Rightarrow A = 0.9011B$*.*

Again, several gave the equation as $\ln\frac{A}{B} = -\frac{50}{480}$*. Several people omitted the* $0.5T$ *offset from the exponent and obtained* $A = Be^{-\frac{T}{\tau_{off}}} = 0.812B$*. Several people thought that the steady state value was* $y_{SS} = A$*; for DC inputs, the steady state value is the value at* $t = \infty$ *not the value at* $t = T$*.*

---

iii) By combining the equations determined in parts i) and ii), determine the numerical values of both A and B. [2]

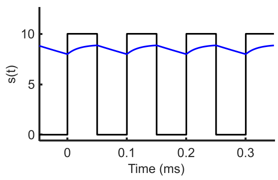
[U] Substituting $A = 0.9011B$ *into* $-0.074A + B - 8.2676 = 0$ *gives* $-0.0666B + B = 8.2676$ *from which* $B = \frac{8.2676}{0.9334} = 8.858$*. Hence* $A = 0.9011B = 7.9817$*.*

Most did this fine unless they had chosen equations containing $\ln(\cdot)$ *which made it much harder to solve.*

---

iv) Sketch a dimensioned graph of $y(t)$ for $t \in [0, 200\,\mu\text{s}]$. [3]

[U] The waveform of $y(t)$ *oscillates between* $A = 7.9817$ *and* $B = 8.858$ *with each segment a negative exponential having the appropriate time constant. Because of the difference in time constants, the rising portion of the waveform is much more curved than the falling part. The waveform is shown below:*



Quite a few people made the lines curve the wrong way (i.e. getting steeper over time instead of getting flatter over time). Very few *people realized that the falling waveform was almost straight; it is much straighter than the rising waveform because the time constant is much longer.*

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e) Suppose now that $R_2 = 500\,\text{k}\Omega$ and that $x(t) = (1 + 0.2\cos(2\pi f t))\,s(t)$ is a modulated squarewave as illustrated in Figure 3.3 for $f = 600\,\text{Hz}$.

i) Assuming that $y(t) \le 11.3$, determine an upper bound on the current through R_2. [1]

[U] Since $y(t) \le 11.3$, $i_{R2}(t) = \frac{y(t)}{R_2} \le \frac{11.3}{500\,\text{k}} = 22.6\,\mu\text{A}$.

Most got this right although quite a few used $R_2 = 48\,\text{k}$ *instead of* $R_2 = 500\,\text{k}$.

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ii) Explain why the average value of $i(t)$ must equal the average current through R_2. Hence find an upper bound on the average voltage across R_1 during the times that the diode is conducting. [3]

[U] The average current through the capacitor is zero (when averaged over one cycle) and so, by KCL, the average value of $i(t)$ *must equal the average current through* R_2. *Since* $i(t) = 0$ *when the diode is off (i.e. half the time), the average value of* $i(t)$ *when the diode is conducting must be* $\le 2 \times 22.6 = 45.2\,\mu\text{A}$. *Hence the average voltage across* R_1 *when the diode is conducting must be* $\le 2k \times 45.2\,\mu = 90.4\,\text{mV}$.

Very few people remembered to double the average voltage across R_1 *to obtain the average voltage "when the diode is conducting" (since the diode conducts only half the time and the volage across* R_1 *is zero the rest of the time).*

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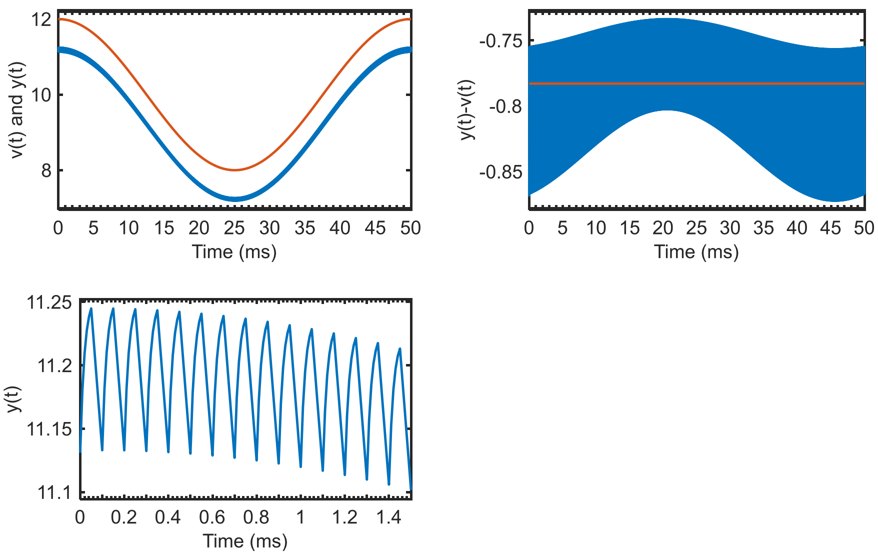
iii) Sketch the waveform $y(t)$ for a modulating frequency of $f = 20\,\text{Hz}$. It is not necessary to calculate the value of $y(t)$ precisely. [3]

[U] The circuit is an envelope detector or AM demodulator. When the diode is on, the capacitor will charge to slightly less than $x(t) - 0.7$ *(since we know that the average value of* $(x(t) - 0.7) - y(t) \le 90.4\,\text{mV}$*). When the diode turns off, the voltage will drop by* $\Delta V = \frac{\Delta Q}{C} \le \frac{22.6\,\mu\text{A} \times 50\,\mu\text{s}}{10\,\text{nF}} = 0.11\,\text{V}$. *So the waveform is approximately* $y(t) = 9.3 + 2\cos(2\pi f t) + r(t)$ *where* $r(t)$ *is a* $10\,kHz$ *negative-valued ripple of amplitude* $\le 0.11\,\text{V}$.

The plots below are much more detailed that were required in the exam but are included for interest. They show (a) $x(t)$ *and the modulation waveform,* $v(t) = 10 + 2\cos(2\pi f t)$*, for one full cycle, (b) the difference,* $y(t) - v(t)$*, and (c)* $y(t)$ *for the first part of a cycle.*



From graph (b) it can be seen that the minimum value of $y(t) - v(t)$ is $-0.87\,\mathrm{V}$ and the mean value (marked with a horizontal red line) is$-0.7829 = -0.7 - 0.0829$. So the average voltage across R_2 when the diode is conducting is $0.0829\,V$ which is only slightly less than the upper bound of 0.0904 calculated in part ii).

Many people got the shape of the curve approximately right but omitted any scale indications on either axis.