EE4-26

SOLUTIONS: ESTIMATION AND FAULT DETECTION

1. Solution

a) By inspection of the block-diagram shown in Fig. 1.1 of the text of the exam paper, we immediately obtain

$$W(s) = \frac{G(s)}{1+G(s)H(s)}E(s)$$

and

$$Y(s) = [1+C(s)]U(s)$$

Then

$$Y(s) = [1+C(s)]\frac{G(s)P(s)}{1+G(s)H(s)}[R(s)-Y(s)]$$

and, after a few algebraic manipulations, we obtain

$$Y(s) = \frac{[1+C(s)]G(s)P(s)}{1+G(s)H(s)+[1+C(s)]G(s)P(s)}$$

[3 marks]

b) The block given by the transfer function

$$\frac{W(s)}{E(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{1}{s+10}$$

can be described by the state equations

$$\begin{cases} \dot{x}_1 = -10x_1 + e \\ w = x_1 \end{cases}$$

whereas the block given by the transfer function

$$\frac{U(s)}{W(s)} = P(s) = \frac{s+10}{(s+3)^2} = \frac{s+10}{s^2+6s+9}$$

can be described by the state equations

$$\begin{cases} \begin{bmatrix} \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & -9 \\ 1 & -6 \end{bmatrix}\begin{bmatrix} x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 10 \\ 1 \end{bmatrix} w \\ u = [0\ 1]\begin{bmatrix} x_2 \\ x_3 \end{bmatrix} \end{cases}$$

Now, using

$$e = r - x_4 - u\ ;\ u = x_3\ ;\ w = x_1$$

we finally obtain

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} -10 & 0 & -1 & -1 \\ 10 & 0 & -9 & 0 \\ 1 & 1 & -6 & 0 \\ 0 & 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} r \\ u = [0\ 0\ 0\ 1]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \end{cases}$$

[3 marks]

c) The output feedback connection $e = r - y$ does not modify the observability properties of the overall system. Hence, we remove the feedback connection and focus on the open-loop system from the input e to the output y. More specifically, let us consider the state equations of the sub-system from the input e to the output u:

$$\begin{cases} \dot{x}_1 = -10x_1 + e \\ \dot{x}_2 = 10x_1 - 9x_3 \\ \dot{x}_3 = x_1 + x_2 - 6x_3 \\ u = x_3 \end{cases} \Longrightarrow \begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -10 & 0 & 0 \\ 10 & 0 & -9 \\ 1 & 1 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e \\ u = [0\,0\,1] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \end{cases}$$

Letting

$$A = \begin{bmatrix} -10 & 0 & 0 \\ 10 & 0 & -9 \\ 1 & 1 & -6 \end{bmatrix}; \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; \quad C = [0\,0\,1]$$

the observability matrix is given by

$$Q = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & -6 \\ -6 & -6 & 27 \end{bmatrix}$$

As

$$\det Q = 0$$

we conclude that the sub-system from the input e to the output u is not completely observable and hence the overall system is not completely observable as well.

[7 marks]

d) Exploiting the answer to Question 1c), let us first determine the non-observable poles of the sub-system from the input e to the output u (having removed the feedback connection $e = r - y$). To proceed, we determine the observability canonical form. To this end, we determine a basis for $\ker(Q)$:

$$Qv = 0 \Longrightarrow \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & -6 \\ -6 & -6 & 27 \end{bmatrix} v = 0 \Longrightarrow v = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Now, we look for a basis $\{\alpha, \beta\}$ for the orthogonal complement to $\ker(Q)$. For example:

$$\alpha = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}; \beta = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Selecting the matrix

$$T = [\alpha \,|\, \beta \,|\, v] = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

and computing the inverse

$$T^{-1} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1/2 & -1/2 & 0 \end{bmatrix}$$

By setting $x = Tz$, we obtain the following equivalent observability canonical form:

$$\begin{cases} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = T^{-1}ATz + T^{-1}Be = \begin{bmatrix} 0 & -9/2 & 0 \\ 2 & -6 & 0 \\ -10 & 9/2 & \mathbf{-10} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 1/2 \\ 0 \\ 0 \end{bmatrix} e \\ u = CTz = [0\ 1\ 0] \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \end{cases}$$

Hence, the non-observable pole of the sub-system from the input e to the output u (having removed the feedback connection $e = r - y$) is $p_{n.o.} = -10$ (it corresponds to the eigenvalue -10 highlighted in boldface in the above matrix $T^{-1}AT$).

As the feedback connection $e = r - y$ does not modify the observability properties of the whole system, the non-observable pole $p_{n.o.} = -10$ of the sub-system from the input e to the output u belongs to the set of non-observable poles of the whole system.

[7 marks]

2. Solution

a) The system can be written in as:

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = Ax + Bu = \begin{bmatrix} -1 & 2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\ u = Cx = [1\ 3] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{cases}$$

The observability matrix is given by

$$Q_x = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -4 & -1 \end{bmatrix}$$

As

$$\det Q_x \neq 0$$

we conclude that the system is completely observable. Hence a full-order state observer can be designed so that the estimation error $e(t) = x(t) - \hat{x}(t)$ satisfies $e(t) \to 0$ for $t \to \infty$.

[3 marks]

b) We have

$$\det(sI - A) = \det \begin{bmatrix} s+1 & -2 \\ 1 & s+1 \end{bmatrix} = s^2 + 2s + 3$$

Then, the matrices A_o and C_o of the observer canonical form are

$$A_o = \begin{bmatrix} 0 & -3 \\ 1 & -2 \end{bmatrix}; \quad C_o = [0\ 1]$$

The observability matrix Q_o computed on the basis of the pair (A_o, C_o) is

$$Q_o = \begin{bmatrix} C_o \\ C_o A_o \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$$

Using Q_x determined in the answer to Question 2a), the matrix T_o transforming the original state equations into the observer canonical form

$$\begin{cases} \dot{z} = A_o z + B_o u \\ y = C_o z \end{cases} \quad \text{with} \quad A_O = T_o^{-1} A T_o;\ B_o = T_o^{-1} B;\ C_o = C T_o$$

is given by

$$T_o = Q_x^{-1} Q_o = \begin{bmatrix} -3/11 & 5/11 \\ 1/11 & 2/11 \end{bmatrix} \quad \text{and} \quad T_o^{-1} = \begin{bmatrix} -2 & 5 \\ 1 & 3 \end{bmatrix}$$

Now considering

$$L_o = \begin{bmatrix} l_o^{(1)} \\ l_o^{(2)} \end{bmatrix}$$

we have

$$\det[sI - (A_o - L_o C_o)] = s^2 + (-2 + l_o^{(2)})s + (-3 + l_o^{(1)})$$

and by equating this polynomial with the polynomial having the desired observer eigenvalues as roots, that is

$$\alpha_d(s) = (s+2)(s+4) = s^2 + 6s + 8$$

we immediately obtain

$$l_o^{(1)} = 5; \quad l_o^{(2)} = 4$$

Finally, the observer gain L such that $F = A - LC$ has the desired eigenvalues $\lambda_1 = -2, \lambda_2 = -4$ is given by

$$L = T_o L_o = \begin{bmatrix} 5/11 \\ 13/11 \end{bmatrix}$$

[8 marks]

c) The reduced-order observer provides an estimate $\hat{x}_2$ of x_2. The estimate $\hat{x}_1$ of x_1 will be obtained through the output equation

$$\hat{x}_1 = y - 3\hat{x}_2$$

As

$$C = [1\ 3]$$

we immediately consider the following change of state vector

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = Tx = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

leading to the equivalent state equations

$$\begin{cases} \dot{z} = \tilde{A}z + \tilde{B}u \\ y = \tilde{C}z \end{cases}$$

with

$$\tilde{A} = TAT^{-1} = \begin{bmatrix} -4 & 11 \\ -1 & 2 \end{bmatrix}; \tilde{B} = TB = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \tilde{C} = CT^{-1} = [1\ 0]$$

The reduced order observer takes on the form

$$\begin{cases} \dot{\xi} = f\xi + hy + gu \\ \hat{z}_2 = \xi + ly \end{cases}$$

where $f = \bar{f} = -2$. The observer gain l is obtained as follows:

$$f = \tilde{a}_{22} - l\tilde{a}_{12} \Longrightarrow -2 = 2 - 11l \Longrightarrow l = \frac{4}{11}$$

The scalar h is computed as

$$fl = h + l\tilde{a}_{11} - \tilde{a}_{21} \Longrightarrow -\frac{8}{11} = h - \frac{16}{11} + 1 \Longrightarrow h = -\frac{3}{11}$$

Finally, the scalar g is computed as

$$g = \tilde{b}_2 - l\tilde{b}_1 = -\frac{4}{11}$$

[9 marks]

3. Solution

a) The general algebraic Riccati equation is

$$P = F\left[P - PH^\top\left(V_2 + HPH^\top\right)^{-1}HP\right]F^\top + V_1$$

Letting $F = 1/2, H = -1/3, V_1 = 1, V_2 = 1$, we have

$$P = \frac{1}{4}\left(P - \frac{\frac{1}{9}}{1+\frac{1}{9}P}P^2\right) + 1 \Longrightarrow P^2 + \frac{31}{5}P - 9 = 0$$

thus obtaining the two solutions

$$\bar{P}_1 \simeq -7.41 \quad \text{and} \quad \bar{P}_2 \simeq 1.21$$

Clearly, the only admissible solution is the positive one. Thus $\bar{P} = \bar{P}_2 \simeq 1.21$. Accordingly:

$$\bar{K} = F\bar{P}H^\top\left(V_2 + H\bar{P}H^\top\right)^{-1} \simeq -0.18$$

[4 marks]

b) We have

$$\text{var}[x(t) - \hat{x}(t|t-1)] = \bar{P} \simeq 1.21$$

Let us now compute $\text{var}[x(t)]$. The stochastic process $x(\cdot)$ generated by the system

$$x(t+1) = \frac{1}{2}x(t) + v_1(t)$$

is stationary because $v_1(\cdot) \sim WGN(0,1)$ and because the system is asymptotically stable.

Because of the stationarity of $x(\cdot)$, $\text{var}[x(t)] = \text{var}[x(t-1)]$ and hence

$$\text{var}[x(t)] = \frac{1}{4}\text{var}[x(t)] + 1 \Longrightarrow \text{var}[x(t)] = \frac{4}{3}$$

Now:

$$\text{var}[x(t) - \hat{x}(t|t-1)] \simeq 1.21 < \frac{4}{3} = \text{var}[x(t)]$$

As can be seen, the Kalman estimator allows to reduce the variance with respect the a-priori one thanks to the use of the measurements $y(t)$.

[5 marks]

c) The Kalman predictor obeys to the following equations:

$$\begin{cases} \hat{x}(t+1|t) = \frac{1}{2}\hat{x}(t|t-1) + \bar{K}e(t) \\ \hat{y}(t+1|t) = -\frac{1}{3}\hat{x}(t+1|t) \\ e(t) = y(t) + \frac{1}{3}\hat{x}(t|t-1) \end{cases}$$

and thus

$$\hat{x}(t+1|t) = \frac{1}{2}\hat{x}(t|t-1) + \bar{K}\left[y(t) + \frac{1}{3}\hat{x}(t|t-1)\right]$$

where $\bar{K} \simeq -0.18$. The block-diagram of the steady-state one-step ahead Kalman predictor is shown in Fig. 3.1.

[3 marks]

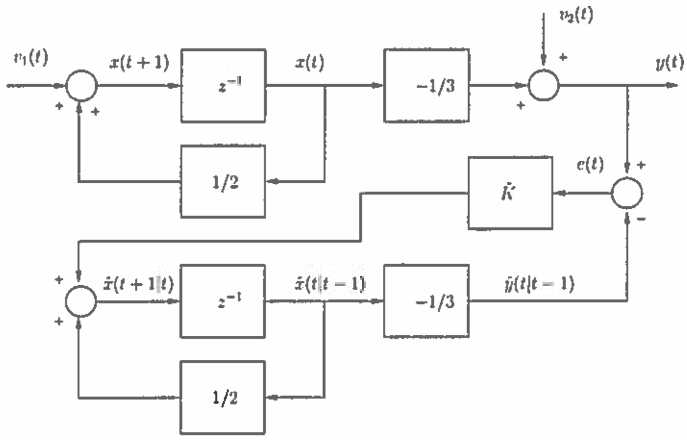

Figure 3.1 Block-diagram of the steady-state one-step ahead Kalman predictor.

d) Consider the system affected by an *unknown constant* fault $\bar{u}$:

$$\begin{cases} x(t+1) = \frac{1}{2}x(t) + v_1(t) + \bar{u} \\ y(t) = -\frac{1}{3}x(t) + v_2(t) \end{cases}$$

Clearly, the following equation holds:

$$u(t+1) = u(t), \quad \text{with} \quad u(0) = \bar{u}$$

We interpret this equation as an *additional* state equation thus getting:

$$\begin{cases} x(t+1) = \frac{1}{2}x(t) + v_1(t) + u(t) \\ u(t+1) = u(t) \\ y(t) = -\frac{1}{3}x(t) + v_2(t) \end{cases}$$

Introducing the new state vector

$$z = \begin{bmatrix} x \\ u \end{bmatrix}$$

we get:

$$\begin{cases} \begin{bmatrix} z_1(t+1) \\ z_1(t+1) \end{bmatrix} = A_z z(t) + B_z v_1(t) = \begin{bmatrix} 1/2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} v_1(t) \\ y(t) = C_z z(t) = [-1/3 \; 0] \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} \end{cases}$$

The observability matrix is given by

$$Q_z = \begin{bmatrix} C_z \\ C_z A_z \end{bmatrix} = \begin{bmatrix} -1/3 & 0 \\ -2/3 & -1/3 \end{bmatrix}$$

As

$$\det Q_z \neq 0$$

we conclude that the "augmented" system is completely observable. Hence a Kalman estimator can be designed so as to simultaneously provide an estimate of the state x and of the constant fault $\bar{u}$.

[8 marks]

4. Solution

a) The full-order observer takes on the form:

$$\begin{cases} \dot{\hat{x}} = A^{(0)}\hat{x} + B^{(0)}u + L^{(0)}(y - C\hat{x}) \\ \hat{y} = C^{(0)}\hat{x} \end{cases}$$

where

$$A^{(0)} = \begin{bmatrix} -1 & 2 \\ 0 & -3 \end{bmatrix}; \quad B^{(0)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad C^{(0)} = [1\ 0]$$

are the matrices of the system described by state equations (4.2) in the text of the exam paper. $L^{(0)}$ denotes the observer gain matrix to be designed. The pair $(A^{(0)}, C^{(0)})$ is completely observable as

$$\det \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} = 2$$

We let

$$F = A^{(0)} - L^{(0)}C^{(0)} = \begin{bmatrix} -1-l_1 & 2 \\ -l_2 & -3 \end{bmatrix} \quad \text{where} \quad L^{(0)} = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}$$

After some algebra, we obtain that by selecting

$$L^{(0)} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

we have $\det(sI - F) = s^2 + 2s + 1$ and hence $\lambda_1 = -1, \lambda_2 = -1$.

Finally

$$e(t) = \mathscr{L}^{-1}[(sI - F)^{-1}]\bar{e} = \mathscr{L}^{-1} \begin{bmatrix} \dfrac{s+3}{(s+1)^2} & \dfrac{2}{(s+1)^2} \\ -\dfrac{2}{(s+1)^2} & \dfrac{s-1}{(s+1)^2} \end{bmatrix} \begin{bmatrix} \bar{e}_1 \\ \bar{e}_2 \end{bmatrix}$$

and hence, after some algebra and computing the inverse Laplace transforms, we get

$$\begin{aligned} \varepsilon(t) &= Ce(t) = \mathscr{L}^{-1}\left[\tfrac{s+3}{(s+1)^2}\right]\bar{e}_1 - 2\mathscr{L}^{-1}\left[\tfrac{1}{(s+1)^2}\right]\bar{e}_2 \\ &= (2t+1)e^{-t}\bar{e}_1 - 2te^{-t}\bar{e}_2, \ \forall t \in (0, T_0) \end{aligned}$$

] 6 marks [

b) Consider a fault $f = f_1$ or $f = f_2$ occurring at some finite time T_0. Then, the dynamics of the state estimation error after the occurrence of the fault (that is, for $t \geq T_0$) can be written as

$$\dot{e}(t) = \dot{x}(t) - \dot{\hat{x}}(t) = Fe(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} f(t)$$

and thus, exploiting the answer to Question 4a), we have

$$\varepsilon(t) = (2t+1)e^{-t}\bar{e}_1 - 2te^{-t}\bar{e}_2 + 2\mathscr{L}^{-1}\left[\frac{1}{(s+1)^2}F(s)\right]$$

where $F(s)$ denotes the Laplace transform of the fault function $f(t)$. From this relationship, it follows immediately that the residual $\varepsilon(t)$ does not vanish asymptotically either under the action of fault f_1 or fault f_2.

Now, as $|x(0)| \leq 10$, for $t < T_0$ we can write

$$\begin{aligned} |\varepsilon(t)| = & \ |(2t+1)e^{-t}\tilde{e}_1 - 2te^{-t}\tilde{e}_2| \leq (2t+1)e^{-t}|\tilde{e}_1| + 2te^{-t}|\tilde{e}_2| \\ & \leq 10(4t+1)e^{-t} \end{aligned}$$

Then, letting

$$\bar{\varepsilon}(t) = 10(4t+1)e^{-t}$$

and observing that $\bar{\varepsilon}(t) \to 0$ for $t \to \infty$, it is immediate to conclude that for $t \geq T_0$

$$\exists T_d > T_0 : \varepsilon(T_d) > \bar{\varepsilon}(T_d)$$

[6 marks]

c) Fault f_1 can be rewritten as

$$f_1(t) = \sin(t) \cdot 1(t) = \mathscr{L}^{-1}\left(\frac{1}{s^2+1}\right)$$

A state-space description of the above dynamic system generating $f_1(t)$ is

$$\begin{cases} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \\ f_1 = [0\ 1] \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \end{cases}$$

Hence, we can write the state equation of the whole system affected by the fault f_1 as

$$\begin{cases} \dot{x}_1 = -x_1 + 2x_2 + u \\ \dot{x}_2 = -3x_2 + z_2 \\ \dot{z}_1 = -z_2 \\ \dot{z}_2 = z_1 \\ y = x_1 \end{cases}$$

The observability matrix Q_1 of the system affected by the fault f_1 can thus be computed:

$$Q_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 1 & -8 & 0 & 2 \\ -1 & 26 & 2 & -8 \end{bmatrix}$$

As $\det(Q_1) = -8$, the state space description of the faulty system is completely observable. Thus, an asymptotic observer can be designed providing the estimates of the state variables associated with the fault f_1.

Now, consider fault f_2. It can be rewritten as

$$f_2(t) = 1(t) = \mathscr{L}^{-1}\left(\frac{1}{s}\right)$$

A state-space description of the above dynamic system generating $f_1(t)$ is

$$\begin{cases} \dot{z}_1 = 0 \\ f_2 = z_1 \end{cases}$$

Hence, we can write the state equation of the whole system affected by the fault f_2 as

$$\begin{cases} \dot{x}_1 = -x_1 + 2x_2 + u \\ \dot{x}_2 = -3x_2 + z_1 \\ \dot{z}_1 = 0 \\ y = x_1 \end{cases}$$

The observability matrix Q_2 of the system affected by the fault f_2 can thus be computed:

$$Q_2 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 1 & -8 & 2 \end{bmatrix}$$

As $\det(Q_2) = 4$, the state space description of the faulty system is completely observable. Thus, an asymptotic observer can be designed providing the estimates of the state variables associated with the fault f_2.

[8 marks]