Imperial College
London

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BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2014

M3S8/M4S8

Time Series

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Imperial College
London

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2014

This paper is also taken for the relevant examination for the Associateship.

M3S8/M4S8

Time Series

Date: ??day, May-June, 2014 Time: ?? 10 – 12 am

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

Note: Throughout this paper $\{\epsilon_t\}$ is a sequence of uncorrelated random variables (white noise) having zero mean and variance σ_ϵ^2, unless stated otherwise. The unqualified term "stationary" will always be taken to mean second-order stationary. All processes are real-valued unless stated otherwise. The sample interval is unity unless stated otherwise.

1. (a) (i) What is meant by saying that a stochastic process is stationary?

 (ii) Consider a process $\{X_t\}$ containing both linear trend $\alpha + \beta t$, (α, β are constants), and seasonality $\{\nu_t\}$ of period 2, such that $X_t = \alpha + \beta t + \nu_t + Y_t$, where $\{Y_t\}$ is a stationary zero-mean process. Find the values of the coefficients a, b, c such that the operator $(1 + aB + bB^2 + cB^3)$, when applied to X_t, completely eliminates the linear trend parameters, and also the seasonality. (Here B is the backward shift operator).

 (b) Use the relationship between the spectral density function and autocovariance sequence to show that the stationary process $\{X_t\}$ with spectral density function

$$S(f) = 4\sigma^2 \left[\frac{1}{2} - |f|\right], \qquad |f| \le 1/2$$

has variance, ($\mathrm{var}\{X_t\}$ or s_0), equal to σ^2.

 (c) The characteristic function for a *bivariate* AR(1) process is $\Phi(B) = I - \phi B$, where I is the (2×2) identity matrix, and ϕ is a (2×2) matrix of parameters. For the case

$$\phi = \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}$$

calculate the determinantal polynomial, $|\Phi(z)|$, and hence determine whether the corresponding bivariate AR(1) process is stationary.

 (d) The AR(1) process $X_t - \phi X_{t-1} = \epsilon_t$, $|\phi| < 1$, has general linear process form $X_t = \sum_{k=0}^{\infty} \psi_k \epsilon_{t-k}$. Its l-step ahead linear forecast is of the form $X_t(l) = \sum_{k=0}^{\infty} \delta_k \epsilon_{t-k}$. The mean square error (mse) $E\{(X_{t+l} - X_t(l))^2\}$ is minimized by the choice $\delta_k = \psi_{k+l}$. Show that in this case

$$X_t(l) = \phi^l X_t.$$

2. (a) Suppose $\{X_t\}$ is an MA(q) process with zero mean, i.e., X_t can be expressed in the form

$$X_t = -\theta_{0,q}\epsilon_t - \theta_{1,q}\epsilon_{t-1} - \ldots - \theta_{q,q}\epsilon_{t-q},$$

where the $\theta_{j,q}$'s are constants ($\theta_{0,q} \equiv -1, \theta_{q,q} \neq 0$). Show that its autocovariance sequence is given by

$$s_\tau = \begin{cases} \sigma_\epsilon^2 \sum_{j=0}^{q-|\tau|} \theta_{j,q}\theta_{j+|\tau|,q}, & \text{if } |\tau| \leq q, \\ 0, & \text{if } |\tau| > q. \end{cases}$$

(b) Let $\{X_t\}$ and $\{W_t\}$ be Gaussian/normal MA(1) processes of the form

$$X_t = \epsilon_{X,t} - \theta_X \epsilon_{X,t-1} \quad \text{and} \quad W_t = \epsilon_{W,t} - \theta_W \epsilon_{W,t-1},$$

where $\{\epsilon_{X,t}\}$ and $\{\epsilon_{W,t}\}$ are zero-mean white noise sequences which are also uncorrelated with each other at all times and have equal variance, σ_ϵ^2.

(i) Find the form of the autocovariance sequence of $Y_t = X_t + W_t$ in terms of the parameters of $\{X_t\}$ and $\{W_t\}$.

(ii) Assume without proof that $\{Y_t\}$ is itself a moving-average process. Using (i), find all possible value(s) of its θ parameter(s) when $\theta_X = 1, \theta_W = 2$.

(iii) Let $\overline{X} \equiv \frac{1}{N}\sum_{t=1}^{N} X_t$ be the sample mean based upon a portion $X_1, \ldots, X_N$ of the process $\{X_t\}$. Show that

$$\text{var}\{\overline{X}\} = \sigma_\epsilon^2 \left[\frac{(1-\theta_X)^2}{N} + \frac{2\theta_X}{N^2}\right].$$

3. (a) (i) State the three defining properties of a linear time-invariant digital filter.

(ii) Show that the spectral density function $S_X(f)$ for a p-th order autoregressive process,

$$X_t - \phi_{1,p}X_{t-1} - \ldots - \phi_{p,p}X_{t-p} = \epsilon_t,$$

is given by

$$S_X(f) = \frac{\sigma_\epsilon^2}{|1 - \phi_{1,p}e^{-i2\pi f} - \ldots - \phi_{p,p}e^{-i2\pi fp}|^2}.$$

(b) Consider the zero-mean AR(2) process $X_t - \phi_{2,2}X_{t-2} = \epsilon_t$ with $\phi_{1,2} = 0$ and $\phi_{2,2} > 0$.

(i) What condition must be satisfied by $\phi_{2,2}$ for the process to be stationary?

(ii) Show that the autocovariance sequence for this stationary process takes the form

$$s_\tau = \begin{cases} \sigma_\epsilon^2 \phi_{2,2}^{|\tau/2|}/[1 - \phi_{2,2}^2], & \tau = 0, \pm 2, \pm 4, \ldots \\ 0, & \text{otherwise.} \end{cases}$$

[Hint: Use the same approach as for producing the Yule-Walker equations to get a set of linear equations which may be easily solved.]

(iii) A stationary process with autocovariance sequence $\{s_\tau\}$ and sample interval Δt has spectral density function

$$S(f) = \Delta t \sum_{\tau=-\infty}^{\infty} s_\tau e^{-i2\pi f\tau\Delta t}, \qquad |f| \le 1/(2\Delta t).$$

Now let $Y_t = X_{2t}, t \in \mathbb{Z}$, i.e., the process $\{Y_t\}$ is formed by subsampling every other random variable from the AR(2) process $\{X_t\}$. With $\{X_t\}$ having $\Delta t = 1$ by default it follows that $\{Y_t\}$ has a sampling interval of $\Delta t = 2$. Given that $s_{Y,\tau} = s_{X,2\tau}$, show that

$$S_Y(f) = 2S_X(f), \quad |f| \le 1/4.$$

4. (a) Let $\widehat{S}^{(p)}(f)$ be the periodogram estimator of a spectrum $S(f)$. Standard statistical theory suggests that, for $0 < |f| < 1/2$, and for large N, the ratio

$$\frac{2\widehat{S}^{(p)}(f)}{S(f)}$$

is distributed as a χ_2^2 random variable, i.e., a chi-squared random variable with two degrees of freedom. (For the purposes of this question, assume this result holds for $f = 0, \pm 1/2$, also.) Hence argue that

$$\frac{1}{2}\int_{-1/2}^{1/2} [\widehat{S}^{(p)}(f)]^2 \, \mathrm{d}f$$

is, for large N, an unbiased estimator of

$$\int_{-1/2}^{1/2} S^2(f) \, \mathrm{d}f.$$

[Hint: You will need the fact that $E\{\chi_\nu^2\} = \nu$ and $\mathrm{var}\{\chi_\nu^2\} = 2\nu$.]

(b) Let the process $\{Y_t\}$ be defined as

$$Y_t = X_{t+a} + X_{t+b} + \epsilon_t,$$

where $\{X_t\}$ is a zero-mean stationary process and a and b are two different positive integer delays, and the processes $\{\epsilon_t\}$ and $\{X_t\}$ are uncorrelated at all times.

(i) What is meant by saying two real-valued discrete time stochastic processes $\{X_t\}$ and $\{Y_t\}$ are jointly stationary stochastic processes?

(ii) Find the cross-covariance sequence $s_{XY,\tau} = E\{X_t Y_{t+\tau}\}$ between $\{X_t\}$ and $\{Y_t\}$.

(iii) Derive the cross-spectrum $S_{XY}(f)$, and hence find the phase spectrum $\theta(f)$.

(iv) The quantity $-\frac{1}{2\pi}\frac{\mathrm{d}\theta(f)}{\mathrm{d}f}$ is called the group delay. When it is a constant, the group delay is said to measure where $s_{XY,\tau}$ is concentrated in terms of the lag τ. Compute the group delay and comment on its form.

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BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2014

M3S8/M4S8

Time Series (SOLUTIONS)

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1. (a) (i) $\{X_t\}$ is second-order stationary if $E\{X_t\}$ is a finite constant for all t, var$\{X_t\}$ is a finite constant for all t, and cov$\{X_t, X_{t+\tau}\}$, is a finite quantity depending only on τ and not on t. seen ⇓ 4

(ii) There are two possible ways to proceed. **Firstly** from first principles: sim. seen ⇓

$$\begin{aligned}(1 + aB + bB^2 + cB^3)X_t &= X_t + aX_{t-1} + bX_{t-2} + cX_{t-3}\\ &= \alpha + \beta t + \nu_t + Y_t\\ &+ a\alpha + a\beta(t-1) + a\nu_{t-1} + aY_{t-1}\\ &+ b\alpha + b\beta(t-2) + b\nu_{t-2} + bY_{t-2}\\ &+ c\alpha + c\beta(t-3) + c\nu_{t-3} + cY_{t-3}.\end{aligned}$$

For the constants we need $\alpha + \alpha(a+b+c) - \beta(a+2b+3c) = 0$ so for the deletion of terms in α and β we require $a+b+c = -1, a+2b+3c = 0$. For terms in βt to be eliminated we require $\beta + \beta(a+b+c) = 0$ so again $a+b+c = -1$. For the seasonal part, since the periodicity is 2, we have that $\nu_t = \nu_{t-2}$ and $\nu_{t-1} = \nu_{t-3}$, so that for elimination we require $b = -1$ and $c = -a$. Since $a + 2b + 3c = 0$ we then must have $a = -1$. The required coefficients are thus $(a, b, c) = (-1, -1, 1)$.

Alternatively, to eliminate both α and β parameters we need to difference twice, i.e., we need $(1-B)^2$. To remove the seasonality with period 2 we need the operator $1 - B^2$. Now $1 - B^2 = (1 + B)(1 - B)$ which already contains one of the required $(1-B)$ terms, so we just need another $(1-B)$ i.e., $(1-B^2)(1-B) = (1+B)(1-B)^2 = 1-B-B^2+B^3$ is what is needed so that we can take $(a, b, c) = (-1, -1, 1)$. 4

(b) unseen ⇓

$$\begin{aligned}s_0 &= \int_{-1/2}^{1/2} S(f)\mathrm{d}f = \int_{-1/2}^{1/2} 4\sigma^2\left[\frac{1}{2} - |f|\right]\mathrm{d}f = 8\sigma^2\int_0^{1/2}\left[\frac{1}{2} - f\right]\mathrm{d}f\\ &= 8\sigma^2\left[\frac{f}{2} - \frac{f^2}{2}\right]_0^{1/2} = 8\sigma^2\left[\frac{1}{4} - \frac{1}{8}\right] = \sigma^2.\end{aligned}$$

4

(c) The determinantal polynomial is sim. seen ⇓

$$\det\left\{\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} - \begin{bmatrix}4 & 3\\ 3 & 4\end{bmatrix}z\right\} = \det\left\{\begin{bmatrix}1-4z & -3z\\ -3z & 1-4z\end{bmatrix}\right\} = 1 - 8z + 7z^2.$$

2

So the roots are

$$z = \frac{8 \pm \sqrt{[64 - 4\cdot 7]}}{14} = \frac{8 \pm 6}{14} = 1, (1/7)$$

so one root is inside the unit circle and the process is non-stationary. 2

(d) Write $\{X_t\}$ in general linear process form $\sum_{k=0}^{\infty} \psi_k \epsilon_{t-k} = \Psi(B)\epsilon_t$: $\boxed{\text{seen} \Downarrow}$

$$X_t = (1 - \phi B)^{-1}\epsilon_t = \sum_{k=0}^{\infty} \phi^k \epsilon_{t-k} = \sum_{k=0}^{\infty} \psi_k \epsilon_{t-k} = \Psi(B)\epsilon_t$$

and so $\psi_k = \phi^k$. When the mse is minimized $\delta_k = \psi_{k+l}$ so

$$\begin{aligned} X_t(l) &= \sum_{k=0}^{\infty} \delta_k \epsilon_{t-k} = \sum_{k=0}^{\infty} \psi_{k+l} \epsilon_{t-k} \\ &= \phi^l \sum_{k=0}^{\infty} \phi^k \epsilon_{t-k} = \phi^l X_t. \end{aligned}$$

$\boxed{4}$

2. (a) Since $E\{\epsilon_t \epsilon_{t+\tau}\} = 0 \ \forall\, \tau \neq 0$ we have for $\tau \geq 0$. seen ⇓

$$s_\tau = \text{cov}\{X_t, X_{t+\tau}\} \;=\; \sum_{j=0}^{q} \sum_{k=0}^{q} \theta_{j,q}\theta_{k,q} E\{\epsilon_{t-j}\epsilon_{t+\tau-k}\}.$$

This is always identically zero if $\tau > q$. For $q \geq \tau \geq 0$, the double sum is only non-zero along the diagonal specified by $k = j+\tau$ so $s_\tau = \sigma_\epsilon^2 \sum_{j=0}^{q-\tau} \theta_{j,q}\theta_{j+\tau,q}$. Now, $s_\tau = s_{-\tau}$, and so the autocovariance sequence is given by

$$s_\tau = \begin{cases} \sigma_\epsilon^2 \sum_{j=0}^{q-|\tau|} \theta_{j,q}\theta_{j+|\tau|,q}, & \text{if } |\tau| \leq q, \\ 0, & \text{if } |\tau| > q. \end{cases}$$

$\boxed{4}$

(b) (i) Now $E\{Y_t\} = E\{X_t\} + E\{W_t\} = 0$. So unseen ⇓

$$\begin{aligned} s_{Y,\tau} = \text{cov}\{Y_t, Y_{t+\tau}\} &= E\{(X_t + W_t)(X_{t+\tau} + W_{t+\tau})\} \\ &= E\{X_t X_{t+\tau}\} + E\{W_t W_{t+\tau}\} = s_{X,\tau} + s_{W,\tau} \\ &= \begin{cases} \sigma_\epsilon^2(2 + \theta_X^2 + \theta_W^2), & \tau = 0, \\ -\sigma_\epsilon^2(\theta_X + \theta_W), & |\tau| = 1, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

where we have used the formula for the ACVS of a moving average process derived in part (a). $\boxed{4}$

(ii) From (i), given that $\{Y_t\}$ is a moving-average then it is an MA(1) also (the ACVS cuts-off at $q = 1$). Let $Y_t = \epsilon_{Y,t} - \theta_Y \epsilon_{Y,t-1}$ where $\text{var}\{\epsilon_{Y,t}\} = \sigma_{Y,\epsilon}^2$. Then its ACVS will take the form unseen ⇓

$$s_{Y,\tau} = \begin{cases} \sigma_{Y,\epsilon}^2(1 + \theta_Y^2) & \tau = 0, \\ -\sigma_{Y,\epsilon}^2 \theta_Y, & |\tau| = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then, with $\theta_X = 1, \theta_W = 2$,

$$\rho_{Y,1} = \frac{s_{Y,1}}{s_{Y,0}} = \frac{-\theta_Y}{(1 + \theta_Y^2)} = \frac{-(\theta_X + \theta_W)}{2 + \theta_X^2 + \theta_W^2} = -\frac{3}{7}$$

from which we get the quadratic equation

$$\theta_Y^2 - \frac{7}{3}\theta_Y + 1 = 0,$$

so that $\theta_Y = \frac{7}{6} \pm \frac{1}{6}\sqrt{13}$. $\boxed{6}$

(iii) Now $\boxed{\text{sim. seen} \Downarrow}$

$$
\begin{aligned}
\overline{X} &= \frac{1}{N}\sum_{t=1}^{N} X_t = \frac{1}{N}\left(\sum_{t=1}^{N}\epsilon_{X,t} - \theta_X \sum_{t=1}^{N}\epsilon_{X,t-1}\right) \\
&= \frac{1}{N}\left(-\theta_X\epsilon_{X,0} + \epsilon_{X,N} + (1-\theta_X)\sum_{t=1}^{N-1}\epsilon_{X,t}\right),
\end{aligned}
$$

Since the variance of a sum of uncorrelated random variables is the sum of the individual variances,

$$
\begin{aligned}
\mathrm{var}\left\{\overline{X}\right\} &= \frac{\sigma_\epsilon^2}{N^2}\left[\theta_X^2 + 1 + (1-\theta_X)^2(N-1)\right] \\
&= \frac{\sigma_\epsilon^2}{N^2}\left[1 + \theta_X^2 + N(1 - 2\theta_X + \theta_X^2) - (1+\theta_X^2) + 2\theta_X\right] \\
&= \frac{\sigma_\epsilon^2}{N^2}\left[N(1-\theta_X)^2 + 2\theta_X\right] \\
&= \sigma_\epsilon^2\left[\frac{(1-\theta_X)^2}{N} + \frac{2\theta_X}{N^2}\right].
\end{aligned}
$$

$\boxed{6}$

3. (a) (i) Let $\{x_t\}, \{y_t\}, \{x_{1,t}\}$ and $\{x_{2,t}\}$ be discrete-time sequences. $\boxed{\text{seen} \Downarrow}$

[1] Scale-preservation: Given a non-zero constant α,

$$L\{\alpha x_t\} = \alpha L\{x_t\}.$$

[2] Superposition:

$$L\{x_{1,t} + x_{2,t}\} = L\{x_{1,t}\} + L\{x_{2,t}\}.$$

[3] Time invariance: If $y_t = L\{x_t\}$ then

$$L\{x_{t+\tau}\} = y_{t+\tau}.$$

$\boxed{3}$

(ii) Define $L\{X_t\} = X_t - \phi_{1,p}X_{t-1} - \ldots - \phi_{p,p}X_{t-p}$, so that $L\{X_t\} = \epsilon_t$. Input a complex exponential:

$$\begin{aligned} L\left\{e^{i2\pi ft}\right\} &= e^{i2\pi ft} - \phi_{1,p}e^{i2\pi f(t-1)} - \ldots - \phi_{p,p}e^{i2\pi f(t-p)} \\ &= e^{i2\pi ft}[1 - \phi_{1,p}e^{-i2\pi f} - \ldots - \phi_{p,p}e^{-i2\pi fp}], \end{aligned}$$

Since $L\left\{e^{i2\pi ft}\right\} = e^{i2\pi ft}G(f)$,

$$G(f) = 1 - \phi_{1,p}e^{-i2\pi f} - \ldots - \phi_{p,p}e^{-i2\pi fp}.$$

Now, $|G(f)|^2 S_X(f) = S_\epsilon(f)$ and $S_\epsilon(f) = \sigma_\epsilon^2$, so

$$S_X(f) = \frac{\sigma_\epsilon^2}{|1 - \phi_{1,p}e^{-i2\pi f} - \ldots - \phi_{p,p}e^{-i2\pi fp}|^2}.$$

$\boxed{4}$

(b) **In the remainder of the solution we set $\phi_{2,2} = \phi$ for simplicity.** $\boxed{\text{unseen} \Downarrow}$

(i) We know that the roots of the characteristic polynomial $\Phi(z) = 1 - \phi z^2$ must lie outside the unit circle. Now $1 - \phi z^2 = (1 - \sqrt{\phi}z)(1 + \sqrt{\phi}z)$ so the roots are $\pm 1/\sqrt{\phi}$, both having magnitude $1/\sqrt{\phi}$, so we require $\sqrt{\phi} < 1$. $\boxed{4}$

(ii) As suggested in the hint, start with the defining equation and multiply through by $X_{t-\tau}$ for $\tau > 0$ and take expectations. We know $E\{\epsilon_t X_{t-\tau}\} = 0$ for $\tau > 0$, so $\boxed{\text{unseen} \Downarrow}$

$$\begin{aligned} E\{X_t X_{t-\tau}\} &= \phi E\{X_{t-2}X_{t-\tau}\} + E\{\epsilon_t X_{t-\tau}\} \\ \tau = 1 : s_1 &= \phi s_1 \Rightarrow s_1 = 0 \\ \tau = 2 : s_2 &= \phi s_0 \\ \tau = 3 : s_3 &= \phi s_1 \Rightarrow s_3 = 0 \\ \tau = 4 : s_4 &= \phi s_2 \Rightarrow s_4 = \phi^2 s_0 \text{ ...... etc} \end{aligned}$$

So for $\tau > 0$ and even
$$s_\tau = \phi^{\tau/2} s_0$$
and zero if odd. $\boxed{3}$

Also, as for the Yule-Walker equations multiply through by X_t and take expectation. Since $E\{\epsilon_t X_t\} = E\{\epsilon_t^2\} = \sigma_\epsilon^2$,
$$s_0 = \phi s_2 + \sigma_\epsilon^2 = \phi^2 s_0 + \sigma_\epsilon^2 \Rightarrow s_0 = \sigma_\epsilon^2/[1-\phi^2].$$
So, since $s_\tau = s_{-\tau}$, $\boxed{1}$
$$s_\tau = s_{X,\tau} = \begin{cases} \sigma_\epsilon^2 \phi^{|\tau/2|}/[1-\phi^2], & \tau = 0, \pm 2, \pm 4, \ldots \\ 0, & \text{otherwise.} \end{cases}$$
$\boxed{1}$

(iii)
$$\begin{aligned} S_Y(f) &= 2\sum_{\tau=-\infty}^{\infty} s_{Y,\tau} e^{-i2\pi f\tau\cdot 2} = 2\sum_{\tau=-\infty}^{\infty} s_{X,2\tau} e^{-i2\pi f(2\tau)} \\ &= 2\sum_{\tau=-\infty}^{\infty} s_{X,\tau} e^{-i2\pi f\tau} = 2S_X(f) \end{aligned}$$
where we have used the fact that $s_{X,\tau} = 0$ for odd τ as shown above in (ii). $\boxed{3}$

The Nyquist frequency is $1/(2\Delta t) = 1/4$, so $|f| \le 1/4$. $\boxed{1}$

4. (a) Since $E\{\chi_2^2\} = 2$ and $\text{var}\{\chi_2^2\} = 4$, we have, for large N, unseen ⇓

$$E\left\{\frac{2\widehat{S}^{(p)}(f)}{S(f)}\right\} = 2, \text{ i.e., } E\{\widehat{S}^{(p)}(f)\} = S(f),$$

and

$$\text{var}\left\{\frac{2\widehat{S}^{(p)}(f)}{S(f)}\right\} = 4, \text{ i.e., } \text{var}\{\widehat{S}^{(p)}(f)\} = S^2(f).$$

For any random variable U with mean value $E\{U\}$, we have

$$\text{var}\{U\} \equiv E\left\{(U - E\{U\})^2\right\} = E\{U^2\} - (E\{U\})^2,$$

so $E\{U^2\} = \text{var}\{U\} + (E\{U\})^2$. Thus

$$E\left\{\left[\widehat{S}^{(p)}(f)\right]^2\right\} = \text{var}\{\widehat{S}^{(p)}(f)\} + \left(E\{\widehat{S}^{(p)}(f)\}\right)^2 = S^2(f) + S^2(f) = 2S^2(f).$$

For large N we thus have

$$\begin{aligned} E\left\{\frac{1}{2}\int_{-1/2}^{1/2}[\widehat{S}^{(p)}(f)]^2\,\mathrm{d}f\right\} &= \frac{1}{2}\int_{-1/2}^{1/2} E\left\{\left[\widehat{S}^{(p)}(f)\right]^2\right\}\mathrm{d}f \\ &= \frac{1}{2}\int_{-1/2}^{1/2} 2S^2(f)\,\mathrm{d}f = \int_{-1/2}^{1/2} S^2(f)\,\mathrm{d}f, \end{aligned}$$

as required. 5

(b) (i) Two real-valued discrete time stochastic processes $\{X_t\}$ and $\{Y_t\}$ are said to be jointly stationary stochastic processes if $\{X_t\}$ and $\{Y_t\}$ are each, separately, second-order stationary processes, and $\text{cov}\{X_t, Y_{t+\tau}\}$ is a function of τ only. seen ⇓ 2

(ii) To calculate $s_{XY,\tau} = E\{X_t Y_{t+\tau}\}$ it is important to pay attention to the ordering of the processes. Pre-multiply through the defining equation for $Y_{t+\tau}$ by X_t sim. seen ⇓

$$X_t Y_{t+\tau} = X_t X_{t+\tau+a} + X_t X_{t+\tau+b} + X_t \epsilon_{t+\tau},$$

then taking expectation we get (using that all processes are zero-mean process and the processes $\{\epsilon_t\}$ and $\{X_t\}$ are uncorrelated)

$$s_{XY,\tau} = s_{X,\tau+a} + s_{X,\tau+b}.$$

3

(iii) Now, Fourier transforming $\{s_{X,\tau+a}\}$ we get $\boxed{\text{unseen} \Downarrow}$

$$\sum_{\tau=-\infty}^{\infty} s_{X,\tau+a} e^{-i2\pi f\tau} = e^{i2\pi fa} \sum_{\tau=-\infty}^{\infty} s_{X,\tau+a} e^{-i2\pi f(\tau+a)} = e^{i2\pi fa} S_X(f)$$

So $S_{XY}(f) = [e^{i2\pi fa} + e^{i2\pi fb}]S_X(f)$. $\boxed{3}$

If we write $S_{XY}(f) = |S_{XY}(f)|e^{i\theta(f)}$, then $\theta(f)$ is the phase spectrum. We know $S_X(f)$ is real so we have to write $[e^{i2\pi fa}+e^{i2\pi fb}] = r(f)e^{i\theta(f)}$, where $r(f)$ is real. There are at least two ways to do this:
Firstly,

$$\begin{aligned}
e^{i2\pi fa} + e^{i2\pi fb} &= e^{i2\pi fa/2}e^{i2\pi fb/2}\left[e^{i2\pi fa/2}e^{-i2\pi fb/2} + e^{-i2\pi fa/2}e^{i2\pi fb/2}\right] \\
&= e^{i2\pi f(a+b)/2}\left[e^{i2\pi f(a-b)/2} + e^{-i2\pi f(a-b)/2}\right] \\
&= e^{i\pi f(a+b)} \cdot 2\cos(\pi f(a-b)) \equiv e^{i\theta(f)} \cdot r(f),
\end{aligned}$$

or, alternatively, by expanding

$$\begin{aligned}
\cos(2\pi fa) &+ \cos(2\pi fb) + i[\sin(2\pi fa) + \sin(2\pi fb)] \\
&= r(f)\cos(\theta(f)) + i\, r(f)\sin(\theta(f)) \\
&\Rightarrow \cos(2\pi fa) + \cos(2\pi fb) = r(f)\cos(\theta(f)) \\
&\ \sin(2\pi fa) + \sin(2\pi fb) = r(f)\sin(\theta(f))
\end{aligned}$$

and then using standard trig identities we must have:

$$r(f) = 2\cos(2\pi f(a-b)/2) = 2\cos(\pi f(a-b));\ \theta(f) = 2\pi f(a+b)/2 = \pi f(a+b).$$

So in both cases, $\theta(f) = \pi f(a+b)$. $\boxed{4}$

(iv) The group delay is thus

$$-\frac{1}{2\pi}\frac{d\theta(f)}{df} = -(a+b)/2.$$

$\boxed{1}$

Now $\{s_{X,\tau+a}\}$ has a peak when $\tau = -a$ (since the maximum of $s_{X,\tau}$ is $s_{X,0}$) and likewise $\{s_{X,\tau+b}\}$ has a peak when $\tau = -b$ so the group delay corresponds to the average of the positions of the two individual peaks. $\boxed{2}$