EE4-23
EE9-CS2-1

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2015

MSc and EEE/EIE PART IV: MEng and ACGI

# STABILITY AND CONTROL OF NON-LINEAR SYSTEMS

Friday, 8 May 10:00 am

Time allowed: 3:00 hours

Corrected Copy

**There are FOUR questions on this paper.**

**Answer FOUR questions.**

*All questions carry equal marks*

**Any special instructions for invigilators and information for candidates are on page 1.**

Examiners responsible First Marker(s) : D. Angeli
Second Marker(s) : E.C. Kerrigan

1. Consider the following second order differential equation of a single real unknown:

$$\ddot{y}(t) - (1 - y^2(t) - \dot{y}^2(t))\dot{y}(t) + y(t) = 0.$$

a) Choose the state variable $x$, and write a state-space realization, $\dot{x}(t) = f(x(t))$ for some suitable function $f$; [ 3 ]

b) Discuss existence and unicity of solutions; [ 2 ]

c) Find all equilibria of the system; [ 2 ]

d) Linearize the system around each linearizable equilibrium and discuss the local phase-portrait; [ 3 ]

e) Find $R$ so that the set $\{(y, \dot{y}) : y^2 + \dot{y}^2 \leq R^2\}$ is forward invariant for the system; [ 4 ]

f) Sketch the global phase portrait of the system. Do you expect periodic solutions to exist? Why? [ 4 ]

g) Show that the set $\{(y, \dot{y}) : y^2 + \dot{y}^2 = 1\}$ is an invariant set for the system. [ 2 ]

2. Consider the following second order differential equation in the real variable $y$ with external forcing input $d$:

$$\ddot{y}(t) = -\dot{y}(t) - \text{atan}(y(t)) + d(t).$$

a) Choose the state variable $x$, and write a state-space realization, $\dot{x}(t) = f(x(t), d(t))$ for some suitable function $f$; [ 3 ]

b) Consider first the unforced system, viz. $d(t) = 0$. Compute its equilibrium state and show that it is globally asymptotically stable. (*Hint: use the analog of mechanical energy as a candidate Lyapunov function.*) [ 7 ]

c) Assume now that $d(t)$ is specified by the following feedback law:

$$d(t) = \text{atan}(y(t)) + \dot{y}(t) + \frac{1}{1+t^2}.$$

i) Write equations describing the dynamics of the closed-loop system; [ 2 ]

ii) Argue that the new system can be seen as a linear system forced by an exogenous input signal and compute explicitly the solution $y(t)$ for an arbitrary initial condition $y(0)$, $\dot{y}(0)$; [ 2 ]

iii) Show that the signal $\dot{y}(t)$, $t \geq 0$, is bounded; viz. there exists $M > 0$ such that $|\dot{y}(t)| \leq M$ for all $t \geq 0$; [ 2 ]

iv) Show that $d(t)$, as specified from the previous formula, is also a bounded signal; [ 2 ]

v) Remark that for $\dot{y}(0) = 0$, $y(t)$ is a diverging signal, and explain why this proves that our system is 0-GAS but not Input-to-State Stable. [ 2 ]

3. Consider the following two-dimensional nonlinear control system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \sin(x_1)x_2 \\ -\sin(x_1)x_1 + u \end{bmatrix},$$

$$y = x_2.$$

a) Show that the system is passive from $u$ to $y$ (*Hint: use a quadratic storage function*); [ 4 ]

b) Consider the static output feedback law:

$$u(t) = -\mathrm{atan}(y(t)).$$

Prove that the point $x = [0,0]'$ is a locally asymptotically stable equilibrium of the closed-loop system. [ 6 ]

c) Consider next a linear controller:

$$u(t) = -x_2(t) - x_1(t) + d(t).$$

where $d(t)$ is an exogenous input disturbance. Prove that the closed-loop system is Input-to-State Stable. (*Hint: use a quadratic ISS candidate Lyapunov function. Try to dominate non-definite terms*). [ 6 ]

d) Show that $[0,0]'$ is a Lyapunov stable equilibrium of the open-loop system ($u(t) = 0$) but not an asymptotically stable one. [ 4 ]

4. Consider the following three-dimensional nonlinear control system:

$$\begin{aligned}\dot{x}_1 &= \sin(x_2) - x_1 \\ \dot{x}_2 &= \sin(x_3) - x_2 \\ \dot{x}_3 &= u + \sin(x_1),\end{aligned}$$

with state $x = [x_1, x_2, x_3]' \in \mathbb{R}^3$ and input $u \in \mathbb{R}$.

a) Assume that $y = x_1$; compute the relative degree at $x = 0$ of the system. Specify if this is local or global relative degree. [ 3 ]

b) Find a feedback $u = v + k(x)$, where $v$ denotes an auxiliary scalar control variable so that the resulting system is linear from input $v$ to output $y$. [ 3 ]

c) Design a feedback law that globally asymptotically stabilizes the system at some equilibrium with output $y = x_1 = 0$. [ 3 ]

d) Consider next the output equation $y = x_3$ and compute the relative degree at 0 of the system. [ 2 ]

e) Highlight the internal dynamics and the zero-dynamics of the system corresponding to the output choice $y = x_3$. [ 4 ]

f) Show that the internal dynamics are ISS with respect to the exogenous signal $y$. Argue how this result can be used to design an output feedback controller capable of globally stabilizing the system around the equilibrium corresponding to $y = x_3 = 0$. [ 5 ]