EXAMINATION QUESTIONS/SOLUTIONS EE1 - Maths I. Solutions 2008 EE 1(1)

COURSE	I (1)
SETTER	Luzzatto/GW
QUESTION NO.	
SOLUTION NO.	1

Setters are advised that Checkers, Editors, Typists and External Examiners greatly appreciate the merits of accuracy, legibility and neatness.

Write on one side only, between the margins, double-spaced. Not more than one question or solution per sheet, please

MARKSCHEME

(i) f is even if $f(x) = f(-x)$ for all x;

f is odd if $f(x) = -f(-x)$ for all x. — 2

Examples: $f(x) = x^2$ is even ; $f(x) = x$ is odd

(ii) e^{-x} : neither

$x \sin x$: even — 4

$x^2 \sin x$: odd

$2x/(x^2-1)$: odd.

(iii) $f(g(x)) = e^{1/x^2}$, $g(f(x)) = e^{-2x}$. — 4

$f^{-1}(x) = \ln x$, $g^{-1}(x) = x^{-\frac{1}{2}}$. — 4

(iv) In general, we can write — 2

$$f(x) = \underbrace{\tfrac{1}{2}\left(f(x) + f(-x)\right)}_{\text{even}} + \underbrace{\tfrac{1}{2}\left(f(x) - f(-x)\right)}_{\text{odd}} .$$

When $f(x) = \dfrac{2x}{x+1}$, this gives

$$\frac{2x}{x+1} = \frac{-2x^2}{1-x^2} + \frac{2x}{1-x^2} .$$ — 4

EXAMINATION QUESTIONS/SOLUTIONS 2007-08

Course EE I (1)

2

Question Core 2 Solution

Marks & seen/unseen

Parts

(i) $\lim_{x\to\infty} \frac{(2x-1)(x+3)}{(x+5)(3x-2)} = \lim_{x\to\infty} \frac{(2-1/x)(1+3/x)}{(1+5/x)(3-2/x)}$ (2)

$= 2/3$ (2)

(ii) $|x \sin(\cot x)| \le |x|$

so $\lim_{x\to 0} x \sin(\cot x) = 0$ (4)

(iii) $\lim_{x\to 0} x^{-2}\ln(\cos x)$

$= \lim_{x\to 0} \frac{\ln(1 - x^2/2! \cdots)}{x^{+2}} = \lim_{x\to 0} \frac{-x^2/2! + \text{hot}}{x^2}$ (4)

$= -1/2$ (2)

or use L'hopital's rule

IV $\lim_{x\to\infty} x^{-9}\{(x+3)^{10} - (x+1)^{10}\}$

$= \lim_{x\to\infty} x^{-9}\{x^{10}(1+3/x)^{10} - x^{10}(1+1/x)^{10}\}$ (2)

$= \lim_{x\to\infty} x^{-9}\{x^{10}(1+\frac{30}{x}\cdots) - x^{10}(1+10/x\cdots)\}$ (2)

$= \lim_{x\to\infty} x^{-9}\{30x^9 - 10x^9 + \text{hot}\}$

$= 20$ (2)

Setter's initials

Checker's initials SL

Page number

	EXAMINATION QUESTIONS/SOLUTIONS 2007-08	Course EE I (1) 3
~~Question~~ Solution C3		Marks & seen/unseen

Parts

i) Use change of variable

$$\int (3-2x)^{-5}\,dx = \frac{(3-2x)^{-4}}{4} \times \frac{1}{(-2)} + C$$ 3

$$= \frac{(3-2x)^{-4}}{8} + C$$

ii) Partial fractions

$$\frac{5x+2}{(3x+4)(x-1)} = \frac{A}{3x+4} + \frac{B}{x-1}$$

$$5x+2 = A(x-1) + B(3x+4) = (A+3B)x + (-A+4B)$$ 3

So

$$\left.\begin{aligned} A+3B &= 5 \\ -A+4B &= 2 \end{aligned}\right\} \rightarrow \begin{aligned} B&=1 \\ A&=2 \end{aligned}$$

$$\int \frac{5x+2}{(3x+4)(x-1)}\,dx = \int \frac{2}{3x+4}\,dx + \int \frac{1}{x-1}\,dx$$ 2

$$= \frac{2}{3}\ln|3x+4| + \ln|x-1| + C$$

Setter's initials	Checker's initials	Page number
SL	JRC.	

EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2007-08

Course: EE I(1) 3 cont

~~Question~~ Solution C3 (cont)

Marks & seen/unseen

Parts

(iii) Integrate by parts

$$\int x \ln x \, dx = \ln x \left(\frac{x^2}{2}\right) - \frac{1}{2}\int x^2 \frac{1}{x} dx$$ 2

$$= \frac{x^2 \ln x}{2} - \frac{1}{2}\int x \, dx$$ 2

$$= \frac{x^2 \ln x}{2} - \frac{1}{2}\frac{x^2}{2} + c$$

$$= \frac{x^2}{2}\left[\ln x - \frac{1}{2}\right] + C$$ 1

$\therefore \int_1^2 x \ln x \, dx = 2\ln 2 - 3/4$

(iv) Integrating by parts 3

$$I_n = x^n e^x - n I_{n-1}$$ ← extremely unlikely anyone will do it like this.

$$I_0 = \int x^0 e^x dx = \int e^x dx = e^x + C$$

$$I_1 = xe^x - I_0 = xe^x - e^x + C$$ 4

$$I_2 = x^2 e^x - 2[xe^x - e^x] + C$$

$$I_3 = x^3 e^x - 3[x^2 e^x - 2xe^x + 2e^x] + C$$

$$= e^x[x^3 - 3x^2 + 6x - 6] + C$$

Alternative (more likely) method

$$I = \int x^3 e^x dx \overset{\text{parts}}{=} x^3 e^x - \int 3x^2 e^x dx$$

$$= x^3 e^x - 3x^2 e^x + \int 6x e^x dx$$

$$= x^3 e^x - 3x^2 e^x + 6x e^x - \int 6 e^x dx$$

$$= (x^3 - 3x^2 + 6x - 6)e^x + C$$

Setter's initials: SL

Checker's initials: J.R.C.

Page number

EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2007-08

Course EE I(1)

4

Question Solution C4

Marks & seen/unseen

Parts

i) a) $3+i5 = \sqrt{9+25}\,(\cos\theta + i\sin\theta)$

$\theta = \tan^{-1}(5/3)$

b) $r = \sqrt{6^2+3^2} = \sqrt{45}$, $\theta = \tan^{-1}(-3/6) + \pi$ 3

c) $r = \sqrt{4^2+5^2} = \sqrt{41}$, $\theta = \tan^{-1}(5/4) + \pi$

ii) By De Moivre 4

$\cos 3\theta + i\sin 3\theta = (\cos\theta + i\sin\theta)^3$

$= \cos^3\theta + i3\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$

Equating real and imaginary parts

$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

$= \cos^3\theta - 3\cos\theta(1-\cos^2\theta)$ 4

$= \cos^3\theta - 3\cos\theta + 3\cos^3\theta$

$= 4\cos^3\theta - 3\cos\theta$

other ways not using complex numbers are acceptable

e.g. $\cos 3\theta \equiv \cos(2\theta+\theta) = \cos 2\theta\cos\theta - \sin 2\theta\sin\theta$ etc.

iii) $z+1 = (x+1) + iy$ so 3

$\arg(z+1) = \tan^{-1}\left[\frac{y}{x+1}\right] = \pi/3$

implies

$$\frac{y}{x+1} = \tan\pi/3 = \sqrt{3}$$ 6

and so $y = \sqrt{3}(x+1)$ for $x \geq -1$

Setter's initials SL

Checker's initials JMC

Page number

Question 5 | Marks & seen/unseen

(i) $\frac{\partial f}{\partial x} = (y-2)^2 + 2x - 1 = 0$ — 2

$\frac{\partial f}{\partial y} = 2x(y-2) = 0.$ — 2

2nd $\Rightarrow$ $\boxed{x=0}$ or $\boxed{y=2}$

1st $\Rightarrow$ $\boxed{y-2 = \pm 1}$ or $\boxed{x = \frac{1}{2}}$ — 3

$\therefore$ S.P at $(0,1)$, $(0,3)$, $(\frac{1}{2}, 2)$.

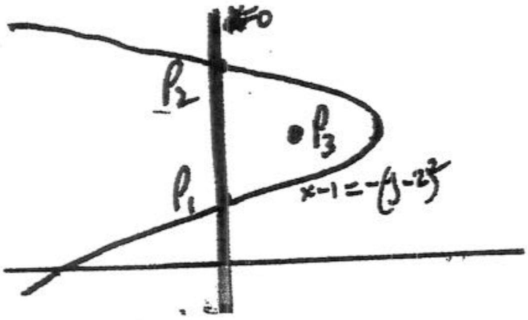
$\frac{\partial^2 f}{\partial x^2} = \underset{\equiv A}{2}$ $\quad \frac{\partial^2 f}{\partial x \partial y} = \underset{\equiv B}{2(y-2)}$ $\quad \frac{\partial^2 f}{\partial y^2} = 2x \equiv C.$

P_1 $(0,1)$: $A=2$, $B=-2$, $C=0$: $AC-B^2<0$ SADDLE P_1 — 3

P_2 $(0,3)$: $A=2$, $B=2$, $C=0$: $AC-B^2<0$ SADDLE PT. — 3

P_3 $(\frac{1}{2},2)$: $A=2$, $B=0$, $C=1$: $AC-B^2>0$, $A>0$ MINIMUM — 3

(ii)



ZERO CONTOUR $x=0$ AND $(x-1) = -(y-2)^2$ — 4

(20)

	EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2007-08	Course EE I(1) 6
Question C6		Marks & seen/unseen

Parts

(i) (a) eqns in std form are

$$\frac{(3,-5,-2)}{\sqrt{38}}\cdot \underline{r} = \frac{2}{\sqrt{38}} \text{ and } \frac{(1,1,6)}{\sqrt{38}}\cdot \underline{r} = \frac{-9}{\sqrt{38}}$$ 2

$\therefore$ DISTANCES are $\frac{2}{\sqrt{38}}$ and $\frac{9}{\sqrt{38}}$ respectively. 2

(b) N_1 given by $\left(\frac{2}{\sqrt{38}}\right)\frac{(3,-5,-2)}{\sqrt{38}} = \frac{(3,-5,-2)}{19}$. 2

N_2 given by $\left(\frac{-9}{\sqrt{38}}\right)\frac{(1,1,6)}{\sqrt{38}} = \frac{(-9,-9,-54)}{38}$.

$\overrightarrow{N_1N_2} = (-9-6,-9+10,-54+4)/38$ 2

$= (-15,1,-50)/38$.

$\therefore \underline{r} = \frac{(3,-5,-2)}{19} + \lambda\frac{(-15,1,-50)}{38}$. $(-\infty<\lambda<+\infty)$ 2

(OR ALTERNATIVE)

(ii) $|\underline{u}| = 1 \Rightarrow x^2+y^2+z^2=1$ 2

$|\underline{u}-\underline{k}|=1 \Rightarrow x^2+y^2+(z-1)^2=1$ 2

$\Rightarrow z=\frac{1}{2}$. 1

$\therefore \frac{1}{4}+x^2+y^2 = 1$. 2

$\Rightarrow \sqrt{x^2+y^2} = \frac{\sqrt{3}}{2}$. $(x^2+y^2=3/4)$

CIRCLE in the plane $z=\frac{1}{2}$ 1

Centre $(0,0,\frac{1}{2})$ and radius $\frac{\sqrt{3}}{2}$. 2

20

Setter's initials	Checker's initials	Page number
TTHB	gm	

	EXAMINATION QUESTIONS/SOLUTIONS 2007-08	Course EE I(1) 7
Question C7 solution		Marks & seen/unseen

Parts

$$LL^T = \begin{pmatrix} a & 0 & 0 \\ b & d & 0 \\ c & e & f \end{pmatrix}\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}$$

$$= \begin{pmatrix} a^2 & ab & ac \\ ab & b^2+d^2 & bc+de \\ ac & bc+de & c^2+e^2+f^2 \end{pmatrix} = A = \begin{pmatrix} 9 & 3 & -3 \\ 3 & 5 & 1 \\ -3 & 1 & 11 \end{pmatrix} \qquad 4$$

$\Rightarrow a = 3,\ b = 1,\ c = -1,\ d = 2,\ e = 1$ and $f = 3$ [Note that we could also have $a = -3$, $d = -2$ – this will affect answer for L^{-1} below] 2

$$\Rightarrow L = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{pmatrix}. \qquad 1$$

$A = LL^T \Rightarrow |A| = |L||L^T| = |L|^2 = (18)^2 = 324.$ 3

$$(L \mid I) = \left(\begin{array}{ccc|ccc} 3 & 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ -1 & 1 & 3 & 0 & 0 & 1 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 2 & 0 & -\frac{1}{3} & 1 & 0 \\ 0 & 1 & 3 & \frac{1}{3} & 0 & 1 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 1 & 0 & -1/6 & \frac{1}{2} & 0 \\ 0 & 0 & 3 & 1/2 & -1/2 & 1 \end{array}\right)$$

$$\therefore L^{-1} = \frac{1}{6}\begin{pmatrix} 2 & 0 & 0 \\ -1 & 3 & 0 \\ 1 & -1 & 2 \end{pmatrix} \qquad 5$$

$$A^{-1} = (LL^T)^{-1} = (L^T)^{-1}L^{-1} = (L^{-1})^T L^{-1} \qquad 2$$

$$A^{-1} = \frac{1}{36}\begin{pmatrix} 2 & -1 & 1 \\ 0 & 3 & -1 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 \\ -1 & 3 & 0 \\ 1 & -1 & 2 \end{pmatrix} = \frac{1}{36}\begin{pmatrix} 6 & -4 & 2 \\ -4 & 10 & -2 \\ 2 & -2 & 4 \end{pmatrix} \qquad 3$$

/20

Setter's initials	Checker's initials	Page number
JWB	JRC	

Parts

(i)

Substitute $y = xu$: $u + x\dfrac{du}{dx} = 2\left(\dfrac{2+y/x}{2-y/x}\right) = \dfrac{4+2u}{2-u}$ (1) — 2

$(1) \Leftrightarrow \quad x\dfrac{du}{dx} = \dfrac{4+2u}{2-u} - u = \dfrac{4+u^2}{2-u}$ — 2

$\Leftrightarrow \quad \displaystyle\int \frac{4-2u}{4+u^2}\,du = \int \frac{2}{x}\,dx$

$\Leftrightarrow \quad \dfrac{4}{2}\tan^{-1}\left(\dfrac{u}{2}\right) - \ln(4+u^2) = \ln(x^2) + c$ — 2

$\Leftrightarrow \quad 2\tan^{-1}\left(\dfrac{y}{2x}\right) = \ln\left(4 + \dfrac{y^2}{x^2}\right) + \ln(x^2) + c$

$= \underline{\ln(4x^2+y^2) + c}$ — 2

$y\left(\tfrac{1}{2}\right) = 0 \Leftrightarrow 0 = 0 + c \quad \therefore\ c = 0$ — 1

and we have $\underline{\underline{2\tan^{-1}\left(\dfrac{y}{2x}\right) = \ln(4x^2+y^2)}}$ — 1

(ii)

$\dfrac{d}{dx}\left(\ln(\sec x + \tan x)\right) = \dfrac{\sec x\tan x + \sec^2 x}{\sec x + \tan x}$

$= \dfrac{\sec x(\tan x + \sec x)}{\sec x + \tan x} = \underline{\sec x}$ — 2

$\cos x\dfrac{dy}{dx} + y = 1 - \sin x \Leftrightarrow \dfrac{dy}{dx} + y\sec x = \sec x - \tan x$ (ii) — 2

Now $\int \sec x\,dx = \ln|\sec x + \tan x| + \text{const}$ from the above

— integrating factor $\underline{\sec x + \tan x}$ — 1

(ii) $\Leftrightarrow (\sec x + \tan x)\dfrac{dy}{dx} + y(\sec^2 x + \sec x\tan x) = \sec^2 x - \tan^2 x = 1$ — 2

$\Leftrightarrow \dfrac{d}{dx}\left(y[\sec x + \tan x]\right) = 1$

$\Leftrightarrow y(\sec x + \tan x) = x + c$ — 1

$\Rightarrow \underline{\underline{y = \dfrac{x+c}{\sec x + \tan x}}}$ or equivalent — 2

(In particular, accept $(x+c)\dfrac{(1-\sin x)}{\cos x}$ or $\dfrac{(x+c)\cos x}{1+\sin x}$)

EXAMINATION QUESTIONS/SOLUTIONS 2007-08

Course: ANCILLARY MATHEMATICS EE I(1)

Question: C10

Marks & seen/unseen: 9

Parts

(i)

Aux. eqn $\mu^2+4\mu+5=0 \Leftrightarrow \mu=-2\pm i$

Comp. fn. $\underline{y=e^{-2x}(P\cos x+Q\sin x)}$ 2

Try PI $y=re^{-2x} \Rightarrow y'=-2re^{-2x} \Rightarrow y''=4re^{-2x}$ 1

substituting into the ODE:

$(4r-8r+5r)e^{-2x}=e^{-2x}$ $\therefore r=1$ 1

general solution $\underline{y=e^{-2x}(1+P\cos x+Q\sin x)}$ 1

whence $y'=e^{-2x}(-2+[Q-2P]\cos x-[P+2Q]\sin x)$ 1

$y(0)=0 \Leftrightarrow 1+P=0$ $\therefore P=-1$ 1

$y'(0)=0 \Leftrightarrow -2+Q-2P=0$ $\therefore Q=0$ 1

solution $\underline{\underline{y=e^{-2x}(1-\cos x)}}$ 1

(ii)

Aux. eqn $\mu^2+4\mu+4=0 \Leftrightarrow \mu=-2$, repeated

Comp. fn. $\underline{y=e^{-2x}(Ax+B)}$ 2

Try PI $y=rx^2e^{-2x} \Rightarrow y'=e^{-2x}(2rx-2rx^2)$

$\Rightarrow y''=e^{-2x}(2r-8rx+4rx^2)$ 2

substituting into the ODE:

$[(2r-8rx+4rx^2)+4(2rx-2rx^2)+4rx^2]e^{-2x}=e^{-2x}$ 2

$\therefore r=1/2$

general solution $\underline{y=e^{-2x}(1/2\,x^2+Ax+B)}$ 1

whence $y'=e^{-2x}([A-2B]+[1-2A]x-x^2)$ 1

$y(0)=0 \Leftrightarrow B=0$ 1

$y'(0)=0 \Leftrightarrow A-2B=0$ $\therefore A=0$ 1

solution $\underline{\underline{y=1/2\,x^2e^{-2x}}}$ 1

Setter's initials: PJR

Checker's initials: JRC.

	EXAMINATION QUESTIONS/SOLUTIONS 2007-08	~~Course~~ Solution C11
Question		Marks & seen/unseen
Parts		

$$y'' + xy = 0 \; ; \quad y(0) = 1, \; y'(0) = 0.$$

Differentiate n times using Leibnitz:

$$y^{(n+2)}(x) + x y^{(n)}(x) + n y^{(n-1)}(x) = 0.$$ (4)

Put $x = 0$

$$y^{(n+2)}(0) + n y^{(n-1)}(0) = 0 \qquad (*)$$ (2)

Put $n = 1$ in eq^n $(*)$ $\quad y'''(0) = -y(0) = -1$ (1)

$n = 2$ " " " $\quad y^{iv}(0) = -2y'(0) = 0$ (1)

From the original eq^n it follows $y''(0) = 0$

$\therefore n = 3$ in eq^n $(*)$ $\quad y^{(5)} - 3y''(0) = 0 \; \therefore y^{(5)}(0) = 0$ (2)

$n = 4$ $\quad y^{(6)}(0) + 4y'''(0) = 0 \quad y^{(6)}(0) = 4$ (2)

Clearly the only non-zero terms are $y(0), y'''(0), y^{(6)}(0)$ etc

$$\therefore y(x) = 1 - x^3/3! + \frac{4x^6}{6!} \ldots + \frac{y^{(3n)}(0)\, x^{3n}}{(3n)!} \ldots$$ (2)

By the ratio-test $L = \lim_{n\to\infty} \left| \frac{y^{(3n)}(0)\, x^{3n}}{(3n)!} \, \frac{(3n-3)!\, x}{y^{(3n-3)}(0)\, x^{3n-3}} \right|$ (2)

$$= \lim_{n\to\infty} \left| \frac{(3n-2)\, x^3}{(3n)(3n-1)(3n-2)} \right| \text{ from } (*)$$ (2)

$= 0 \quad \therefore$ Series converges $\forall x$. (2)

Setter's initials	Checker's initials	Page number
JRC.	SL	

	EXAMINATION QUESTIONS/SOLUTIONS 2007-08	Course
	EE1 - Maths 2 - 2008	EE I (2) 1
Question 1		Marks & seen/unseen

Parts

ii) Now $(\sinh(x))^2 = \left(\frac{e^x - e^{-x}}{2}\right)^2$

$$= \frac{e^{2x} + e^{-2x} - 2}{4} = \frac{\cosh 2x}{2} - \frac{1}{2}$$

so $A = 1/2$ and $B = -1/2$. 4

To evaluate $\lim_{x \to 0} \frac{\sinh x}{x}$ apply l'Hôpital's Rule as $\sinh 0 = 0$:

$$\lim_{x \to 0} \frac{\sinh x}{x} = \lim_{x \to 0} \cosh x = \cosh 0 = 1.$$ 4

Setter's initials	Checker's initials	Page number
RB	G. P.	

EXAMINATION QUESTIONS/SOLUTIONS 2007-08

Course EE I (2) 2

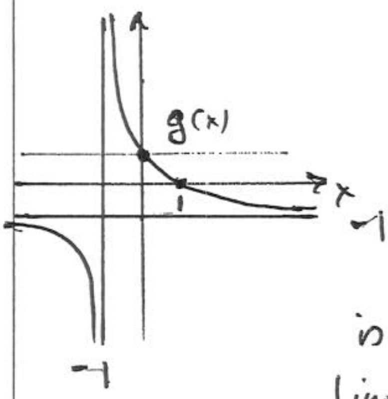
Question 2

Marks & seen/unseen

Parts

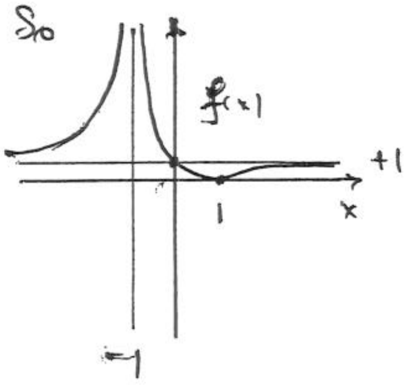
Consider $f(x) = \left(1 - \frac{2x}{x+1}\right)^2$; first sketch

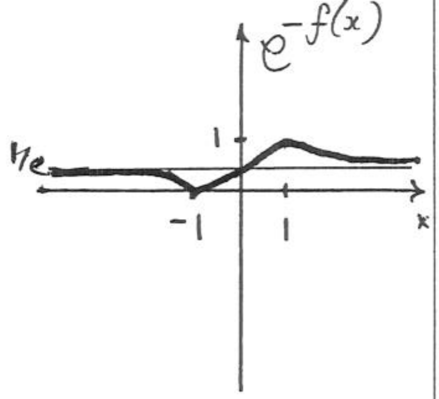
$g(x) := 1 - \frac{2x}{x+1} = \frac{x+1-2x}{x+1} = \frac{1-x}{1+x}$:



When sketching e^{-f}, the only difficulty is at $x=-1$ but $\lim_{x\to-1} e^{-f(x)} = 0$

So





6 / 4

f has a min at $x = 1$. $\Rightarrow$

$e^{-f(x)}$ has a max at $x = +1$.

$e^{-f(x)}$ has a min at $x = -1$

} 2

Setter's initials: RB

Checker's initials: G. P.

Page number

	EXAMINATION QUESTIONS/SOLUTIONS 2007-08	Course EE I(2) 2
Question 2		Marks & seen/unseen
Parts	This uses $f(x) = g(x)^2$ and so $f'(x) = 2g \cdot g'$ $f'' = 2(g'^2) + 2g\, g''$. As $g' = \frac{-1}{1+x} - \frac{1-x}{(1+x)^2}$ $= \frac{-(1+x) - (1-x)}{(1+x)^2} = \frac{-2}{(1+x)^2}$ $f' = 0$ iff $g = 0 \Rightarrow f$ has extrema only when $g=0$ & $x=1$. But then $g=0 \Rightarrow f'' = 2(g')^2 > 0$ so f has a local min at $x=1$. Then $\frac{d}{dx}(e^{-f}) = -f' e^{-f} \Rightarrow e^{-f}$ also has an extremum at $x=1$ as $f'=0$ there. & $\left(\frac{d}{dx}\right)^2 (e^{-f}) = \frac{d}{dx}\left(-f' e^{-f}\right) = -\left(f'' - f'^2 e^{-f}\right)$ $= -f'' \cdot e^{-f}$ if $f' = 0$ & $(-f'')e^{-f} > 0$ if $f'' < 0$ Hence $x=1$ is a local max for e^{-f}!	4 4
	Setter's initials RB	Checker's initials G. P.	Page number

	EXAMINATION QUESTIONS/SOLUTIONS 2007-08	Course EE I (2) 3
Question 3		Marks & seen/unseen

Parts

a) $\frac{d}{dx}\mathrm{sinc}(x) = \frac{\cos x}{x} - \frac{\sin x}{x^2} = \frac{x\cos x - \sin x}{x^2}$

and $\frac{d^2}{dx^2}\mathrm{sinc}\,x = \frac{-x^2\sin x - 2x\cos x + 2\sin x}{x^3}$ — 4

b) Now $\left.\frac{d}{dx}\mathrm{sinc}(x)\right|_{x=0} = \lim_{x\to 0} \frac{x\cos x - \sin x}{x}$

this is in part(c)

$= \lim_{x\to 0} \frac{\cos x - x\sin x - \cos x}{2x} = \lim_{x\to 0} -\frac{\sin x}{2} = 0.$ — 4

Also, $\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \cos x = 1$, so that

$\left.\mathrm{sinc}(x)\right|_{x=0} = \lim_{x\to 0} \frac{\sin x}{x} = 1.$ — 4

c) and $\left.\left(\frac{d}{dx}\right)^2 \mathrm{sinc}(x)\right|_{x=0} = \lim_{x\to 0} \left(\frac{d}{dx}\right)^2 \mathrm{sinc}(x)$

$= \lim_{x\to 0} \frac{-x^2\sin x - 2x\cos x + 2\sin x}{x^3}$

$= \lim_{x\to 0} \frac{-\cancel{2x\sin x} - x^2\cos x - \cancel{2\cos x} + \cancel{2x\sin x} + \cancel{2\cos x}}{3x^2}$

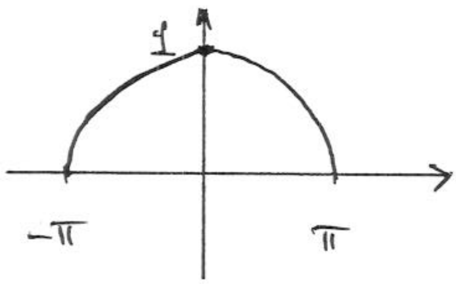
?!

$\cancel{= \lim_{x\to 0} \frac{-4\sin x + x\cos x}{3x} = -\frac{4}{3} + \frac{1}{3} = -1}$ — 4

Setter's initials RS	Checker's initials G. P.	Page number

$= \lim_{x\to 0} \frac{-\cancel{x^2}\cos x}{3\cancel{x^2}} = -\frac{1}{3}$

(Corrected by AGW)

	EXAMINATION QUESTIONS/SOLUTIONS 2007-08	Course EE I(2) 3
Question 3		Marks & seen/unseen
Parts		
d)	From a)–c) we find that sinc has a local max at $x=0$  and $\mathrm{sinc}(\pi) = \frac{\sin \pi}{\pi} = 0$ moreover sinc is an <u>even</u> function as the ratio of 2 odd functions.	4
	Setter's initials RB	Checker's initials G. P.	Page number

	EXAMINATION QUESTIONS/SOLUTIONS 2007-08	Course EE I(2) 4
Question 4		Marks & seen/unseen
Parts		
a)	The surface area of revolution of a function y is $$\int_a^b 2\pi y\sqrt{1+y'^2}\,dx$$ between limits a and b and using $y(x)=\sqrt{R^2-x^2}$ we get $y'=-x(R^2-x^2)^{-1/2}$ and so the req'd surface area is $2\times$ $$2\pi\int_0^R y\sqrt{1+y'^2}\,dx = 4\pi\int_0^R\sqrt{R^2-x^2}\left(\left(\frac{-x}{\sqrt{R^2-x^2}}\right)^2+1\right)^{1/2}dx$$ $$=4\pi\int_0^R\sqrt{R^2-x^2}\left(\frac{R^2}{R^2-x^2}\right)^{1/2}dx=4\pi R\int_0^R dx=\underline{4\pi R^2}.$$	4 8
b)	Now choose $y(x)=\frac{R}{h}(h-x)$ and then $$2\pi\int_0^h y\sqrt{1+y'^2}\,dx = 2\pi\int_0^h\frac{R}{h}(h-x)\sqrt{1+(-R/h)^2}\,dx$$ $$=\frac{2\pi R}{h}\int_0^h(h-x)\sqrt{1+R^2/h^2}\,dx$$ $$=\frac{2\pi R}{h}\sqrt{1+R^2/h^2}\left[hx-\frac{x^2}{2}\right]_0^h=\pi R\sqrt{h^2+R^2}.$$	2 6
	Setter's initials: RB · Checker's initials: G. P.	Page number

	EXAMINATION QUESTIONS/SOLUTIONS 2007-08	Course EE I(2) 5
Question 5		Marks & seen/unseen

Parts

i) a) $\sum_1^\infty \frac{1}{\sqrt{n}}$ diverges by the integral test as $\lim_{T\to\infty} \int_1^T x^{-1/2}dx$ diverges — 3

b) $\sum_1^\infty (-1)^n/\sqrt{2n+1}$ converges by the alternating series test. — 3

c) $\sum_1^\infty \frac{n}{n^2+1}$ diverges by comparison with $\sum_{n=m}^\infty \frac{1}{n}$ (for large enough m) — 3

$\left[n^2+1 \leq 2n^2 \text{ \& so } \frac{n}{n^2+1} \geq \frac{n}{2n^2} = \frac{1}{2n} \text{ \& } \sum \frac{1}{2n} \text{ diverges}\right]$

ii) The integral test states that

$$-f(1) + \sum_1^\infty f(n) \leq \int_1^\infty f(x)dx \leq \sum_1^\infty f(n)$$ — 2

and choosing $f(x) = \frac{1}{1+x^2}$ we get

$f(1)=\frac{1}{2}$?

$$-\frac{1}{2}\cancel{+} \sum_1^\infty \frac{1}{1+n^2} \leq \int_1^\infty \frac{dx}{1+x^2} \leq \sum_1^\infty \frac{1}{1+n^2}$$

But $\int_1^T \frac{dx}{1+x^2} = \tan^{-1}(T) - \tan^{-1}(1)$

$$\Rightarrow \int_1^\infty \frac{dx}{1+x^2} = \lim_{T\to\infty} \tan^{-1}(T) - \frac{\pi}{4} = \frac{\pi}{2} - \frac{\pi}{4} = \pi/4$$ — 8

Setter's initials RB	Checker's initials C. P.	Page number

	EXAMINATION QUESTIONS/SOLUTIONS 2007-08	Course EE $\hat{\underline{1}}$(2) 5
Question 5		Marks & seen/unseen
Parts ii)	~~and so~~ $$\cancel{\sum_1^\infty \frac{1}{1+n^2} \leq \frac{\pi}{4} + 1 < 1.79.}$$ $$\sum_1^\infty \frac{1}{1+n^2} \leq \frac{\pi}{4} + \frac{1}{2} < 1.29$$	~~8~~ 1
	Setter's initials RB	Checker's initials G. P.	Page number

	EXAMINATION QUESTIONS/SOLUTIONS 2007-08	Course EE I(2) 6
Question 6		Marks & seen/unseen

Parts

The given function is ODD (almost everywhere!), so there are NO cosine terms in the Fourier series. Now $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$

$$= \frac{1}{\pi}\int_0^{\pi}\sin nx\,dx + \frac{1}{\pi}\int_{-\pi}^{0}(-1)\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\sin nx\,dx = \frac{2}{\pi}\left[\frac{-\cos nx}{n}\right]_0^{\pi}$$

$$= \frac{-2}{n\pi}\left(\cos(n\pi)-1\right) = \frac{-2}{\pi n}\left((-1)^n-1\right)$$

$$= \begin{cases} 0 & ;\ n \text{ even} \\ 4/n\pi & ;\ n \text{ odd} \end{cases}$$

Hence $f(x) = \sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} \frac{4}{n\pi}\sin(nx)\ (=: F(x))$ **8**

at those points where f is CONTINUOUS. Otherwise,

$$F(x_0) = \lim_{\varepsilon\to 0}\frac{f(x_0+\varepsilon)+f(x_0-\varepsilon)}{2}$$

if f is discontinuous at x_0, provided these limits exist. **2**

Setter's initials RB	Checker's initials C. P.	Page number

EXAMINATION QUESTIONS/SOLUTIONS 2007-08

Course: EE I (2)

6

Question: 6

Marks & seen/unseen

Parts

Thus

$$F(\pi) = \lim_{\varepsilon \to 0} \frac{f(\pi+\varepsilon) + f(\pi-\varepsilon)}{2}$$

$$= \frac{1}{2}(-1+1) = 0$$

but $f(\pi) = 1 \neq F(\pi)$ so $x_0 = \pi$ will do. **4**

Setting $x = \pi/2$ into $F(x)$ gives

$$1 = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)\pi/2}{2n+1}$$

but $\sin((2n+1)\pi/2) = (-1)^n \implies$

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} (-1)^n/(2n+1).$$ **6**

Setter's initials: KB

Checker's initials: G. P.

Page number

	EXAMINATION QUESTIONS/SOLUTIONS 2007-08	Course EE I (2) 7
Question		Marks & seen/unseen

Parts

SOLUTIONS

Question 7

(i) Let $V(r) = \ln(r)$ and $r = \sqrt{x^2+y^2}$. We have that

$$V_x = \frac{1}{r} r_x, \quad V_{xx} = -\frac{1}{r^2} r_x^2 + \frac{1}{r} r_{xx}$$

and that

$$r_x = \frac{x}{r}, \quad r_{xx} = \frac{1}{r} - \frac{x^2}{r^3}.$$

We combine these calculations to obtain

$$\frac{\partial^2 V}{\partial x^2} = \frac{1}{r^2} - \frac{2x^2}{r^4}.$$

7

Similarly we have that

$$\frac{\partial^2 V}{\partial y^2} = \frac{1}{r^2} - \frac{2y^2}{r^4}.$$

Consequently

$$\begin{aligned} V_{xx} + V_{yy} &= \frac{1}{r^2} - \frac{2y^2}{r^4} + \frac{1}{r^2} - \frac{2y^2}{r^4} \\ &= \frac{2}{r^2} - 2\frac{x^2+y^2}{r^4} \\ &= \frac{2}{r^2} - \frac{2}{r^2} = 0. \end{aligned}$$

5

(ii) We have that

$$\begin{aligned} \frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} \\ &= y(-\sin(t)) + x\cos(t) + 1 \cdot 1 \\ &= -\sin^2(t) + \cos^2(t) + 1 \\ &= 1 + \cos(2t). \end{aligned}$$

8

	Setter's initials: G. P.	Checker's initials: RB	Page number

	EXAMINATION QUESTIONS/SOLUTIONS 2007-08	Course EE I (2) 8
Question		Marks & seen/unseen
Parts		

Question 8

(i) The trapezium rule with one interval gives:

$$\int_0^1 (x\cos(x)+1)\,dx \approx \frac{1}{2}(f(0)+f(1)) \cong 1.2\cancel{7}\cancel{1}5.$$

0 2

(1.2702 to 4 d.p.)

4

(ii) The trapezium rule gives:

$$\int_0^1 (x\cos(x)+1)\,dx \approx \frac{1}{4}(f(0)+2f(1/2)+f(1)) \cong 1.3545.$$

4dp

4

(iii) Simpson's rule gives:

$$\int_0^1 (x\cos(x)+1)\,dx \approx \frac{1}{6}(f(0)+4f(1/2)+f(1)) = 1.3826.$$

4 dp

4

(iv) The correct value is

$$\begin{aligned}\int_0^1 (\theta\cos(\theta)+1)\,d\theta &= \left(\theta\sin(\theta)+\cos(\theta)+\theta\right)\Big|_0^1 \\ &= 1.3818. \quad (4dp)\end{aligned}$$

4

The error when using the trapezium rule with 1 interval is ~~0.1103~~ 0.1116. The error when using the trapezium rule with 2 intervals is 0.0273. The error when using Simpson's rule is 0.0008.

4

	Setter's initials G.P.	Checker's initials RB	Page number

corrected by AGW

	EXAMINATION QUESTIONS/SOLUTIONS 2007-08	Course EE I (2) 9
Question		Marks & seen/unseen
Parts		

Question 9

(i) The characteristic polynomial is

$$p^2 - p - 2 = 0.$$

The roots are $p_{1,2} = -1, 2$. The solution of the homogeneous equation is

$$y_h(x) = c_1 e^{-x} + c_2 e^{2x}.$$

3

We look for a particular integral of the form

$$y_p(x) = Ae^{3x}.$$

We substitute this into the equation to obtain

$$4Ae^{3x} = e^{3x} \quad \Rightarrow \quad A = \frac{1}{4}.$$

3

The general solution of the differential equation is

$$y(x) = c_1 e^{-x} + c_2 e^{2x} + \frac{1}{4} e^{3x}.$$

2

(ii) The equation is of the form

$$N(x,y)\frac{dy}{dx} + M(x,y) = 0$$

with $N(x,y) = y^2 - x^2$ and $M(x,y) = 2xy$. We check that

$$\frac{\partial M}{\partial y} = 2x \neq \frac{\partial N}{\partial x} = -2x,$$

3

and, hence, the equation is not exact. We multiply it by a function $I = I(y)$:

$$\hat{N}(x,y)\frac{dy}{dx} + \hat{M}(x,y) = 0$$

where

$$\hat{N}(x,y) = I(y)(y^2 - x^2)$$

and

$$\hat{M}(x,y) = 2xyI(y).$$

We want to choose $I(y)$ in such a way that the equation becomes exact. We have that $\hat{N}_x = -2xI$ and $\hat{M}_y = 2xI + 2xyI'$. We choose I so that

$$\hat{N}_x = \hat{M}_y \quad \Rightarrow \quad -2xI = 2xI + 2xyI'$$

and consequently $yI' = -2I$. The solution of this equation is

$$I(y) = y^{-2}.$$

5

	Setter's initials G. P.	Checker's initials RB	Page number

	EXAMINATION QUESTIONS/SOLUTIONS 2007-08	Course EE I (2) 9
Question		Marks & seen/unseen
Parts		

To solve the equation in implicit form we look for $u(x,y)$ so that

$$u_x = \hat{M}, \quad u_y = \hat{N} \Rightarrow u = \frac{x^2}{y} + h(y) = y + \frac{x^2}{y} + H(x),$$

consequently

$$u = \frac{x^2}{y} + y + C.$$

The solution of the differential equation, in implicit form, is

$$\frac{x^2}{y} + y = C.$$

4

Setter's initials G. P.

Checker's initials RB

Page number

EXAMINATION QUESTIONS/SOLUTIONS 2007-08

Course: EE I (2) 10

Question: 10

Marks & seen/unseen

Parts

Let $f(x) = e^{-x}$ on $(-\pi,\pi)$

and compute

$$\int_{-\pi}^{\pi} e^{-x} e^{inx}\, dx$$

$$= \int_{-\pi}^{\pi} e^{(in-1)x} dx = \frac{e^{(in-1)x}}{in-1}\Big|_{-\pi}^{\pi} \quad (*) \qquad 4$$

$$= \frac{e^{(in-1)\pi} - e^{-(in-1)\pi}}{in-1}$$ but $e^{\pm in\pi} = (-1)^n$

and $(*) = (-1)^n \left(\frac{e^{-\pi} - e^{\pi}}{in-1}\right) \cdot \frac{-in-1}{-in-1}$

$$= \frac{(-1)^n (e^{-\pi} - e^{\pi})(-in-1)}{n^2+1} \Longrightarrow$$

$$a_n = \int_{-\pi}^{\pi} e^{-x} \cos nx\, dx = 2(-1)^n \frac{\sinh \pi}{n^2+1} \quad \&$$

$$b_n = \int_{-\pi}^{\pi} e^{-x} \sin nx\, dx = (-1)^n \frac{2n \sinh \pi}{n^2+1}. \qquad 8$$

and $C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-x} e^{-inx} dx$

$$= \frac{1}{2\pi} \frac{(-1)^n (e^{-\pi} - e^{\pi})(in-1)}{(n^2+1)} = \frac{(-1)^n}{n^2+1}(1-in)\frac{\sinh \pi}{\pi}$$

Setter's initials: RB

Checker's initials: C. P.

Page number

	EXAMINATION QUESTIONS/SOLUTIONS 2007-08	Course EE I(2) 10
Question 10		Marks & seen/unseen

Parts

So $|c_n|^2 = \dfrac{(\sinh\pi)^2(1+n^2)}{\pi^2(1+n^2)^2}$

$= \dfrac{(\sinh\pi)^2}{\pi^2(1+n^2)}$ and $c_0 = \dfrac{1}{2\pi}a_0$

$= \dfrac{1}{2\pi}2\sinh\pi = \dfrac{\sinh\pi}{\pi}$

$\therefore \quad \dfrac{1}{2\pi}\int_{-\pi}^{\pi} f^2\,dx = \dfrac{1}{2\pi}\int_{-\pi}^{\pi} e^{-2x}$

$= \dfrac{1}{2\pi}\,\dfrac{e^{2\pi}-e^{-2\pi}}{2} = \dfrac{\sinh 2\pi}{2\pi}$

(Parseval)

$= \underbrace{\dfrac{(\sinh\pi)^2}{\pi^2}}_{c_0^2} + \dfrac{2}{\pi^2}\sum_{n=1}^{\infty}\dfrac{(\sinh\pi)^2}{1+n^2}.$

Hence

$\sum_{1}^{\infty}\dfrac{1}{1+n^2} = \dfrac{\pi\sinh 2\pi}{4(\sinh\pi)^2} - \dfrac{1}{2}.$ 8

$\left(= \frac{1}{2}(\pi\coth\pi - 1)\right)$

Setter's initials: RB

Checker's initials: C. P.

Page number

Corrected by AGW