UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 2004

MEng Honours Degree in Information Systems Engineering Part IV
MSci Honours Degree in Mathematics and Computer Science Part IV
MEng Honours Degrees in Computing Part IV
MSc in Advanced Computing
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the Associateship of the City and Guilds of London Institute
This paper is also taken for the relevant examinations for the Associateship of the Royal College of Science

PAPER C477=I4.20

COMPUTING FOR OPTIMAL DECISIONS

Friday 14 May 2004, 10:00
Duration: 120 minutes

Answer THREE questions

Paper contains 4 questions
Calculators not required

1 a Consider the inequality constrained quadratic programming problem

$$\underset{x \in \Re^2}{\text{minimise}} \left\{ \left(x_1 - 1\right)^2 + \left(x_2 + 1\right)^2 \;\middle|\; x_1 \geq 0 \right\},$$

where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Suppose you only have access to linear programming (LP) software to solve this optimisation problem. Describe fully an algorithm that uses LP iteratively to solve the problem. [Do not solve the problem at this stage.]

b The solution of the optimisation problem in (a) above is $x_1 = 1$, $x_2 = -1$. We can reformulate this inequality constrained optimisation problem as an unconstrained optimisation problem by using the transformation $x_1 = \omega_1^2$, $x_2 = \omega_2$ and optimise with respect to ω_1, ω_2. Since $x_1 = \omega_1^2$, this ensures $x_1 \geq 0$. Write the transformed optimisation problem and its first order optimality conditions.

c Consider possible solutions of the first order conditions of the transformed problem in (b). Examine if these may be different from the optimal solution $(1, -1)$ of the original problem in (a) and if so discuss the reason in view of the Hessian of the transformed problem.

(All parts carry equal marks)

2 a Consider the general nonlinear programming problem

$$\underset{x \in \Re^n}{\text{minimise}} \left\{ f(x) \;\middle|\; g(x) = 0;\ h(x) \leq \delta \right\},$$

where f, h, g are differentiable convex functions of x, g is an m_1-dimensional vector, 0 is the m_1-dimensional vector of all zeros, h is an m_2-dimensional vector and δ is a given constant m_2-dimensional vector. Write the first-order necessary conditions for optimality for this problem.

b Consider the optimisation problem in part (a) with no equality constraints (for simplicity). Hence, we have

$$\underset{x \in \Re^n}{\text{minimise}} \left\{ f(x) \;\middle|\; h(x) \leq \delta \right\},$$

Let δ be replaced by $\delta + \Delta$. Show that the sensitivity of the optimal value of f(x) to any change in δ, say when δ be replaced by $\delta + \Delta$, is given by the multiplier corresponding to the inequality constraints.

(Both parts carry equal marks)

3 a Parent company (P) owns a portfolio of four factories producing its products. P is considering reinvesting its profits to improve the productivity of these factories. The budget (B) for the total amount available for reinvestment is fixed at £B million. It has been estimated that investment of £ω_i million in factory i (i = 1, 2, 3, 4) will yield a return of r_i *per £ invested*, and

$$r_i = \alpha_i \left(\omega_i\right)^{\gamma_i} + \beta_i \, \epsilon_i \, ; \quad i = 1, \ldots, 4,$$

where α_i, γ_i, β_i are given nonzero constants, ϵ_i is a zero mean, normally distributed random variable with given constant covariance $\mathcal{E}[\epsilon_i \epsilon_j] = q_{ij}$, i, j = 1, ..., 4. Let Q denote the covariance matrix which is assumed to be positive definite. Assume the production of i does not interfere with the production of j, except through Q. Write the expected return, $\mathcal{E}\,(r_i)$, i = 1, ..., 4; the covariance of the returns; and the expected return *and* expected variance of this portfolio (assuming the company decides to adopt an investment portfolio of $\bar{\omega}_i$, i = 1, ..., 4).

b Suppose that in part (a), instead of $\bar{\omega}_i$, i = 1, ..., 4, it is desired to determine ω_i, i=1, ..., 4, to optimise investment policy by maximising the expected return and minimising the expected risk for P. The investment in i = 1 and 4 should be at least double the amount of the investment in i = 2 and 3. Formulate a robust mean-variance optimisation investment problem for P, to determine ω_i, i = 1, ..., 4.

(Both parts carry equal marks)

4 a Consider a non-cooperative game for two players with objectives to maximise

$$\mathcal{F}_1(u_1, u_2) = \tfrac{1}{2}\, u_1^2 + u_1 u_2 \; - 3\, u_1$$

$$\mathcal{F}_2(u_1, u_2) = u_2\, u_1^2 + \tfrac{1}{3}\, u_2^3 \; - 9\, u_2$$

where subscripts 1 and 2 refer to the two players and u_i; i = 1, 2, are the decision variables for each player. State and perform one step of the best replay algorithm to solve the problem using $u_1 = 1$, $u_2 = 5$ as the initial point. Do not continue after the first step.

b State the necessary condition for a Nash strategy for the game in part (a) above and use the initial points $u_1 = 1$, $u_2 = 5$ to perform one step of the Newton algorithm to compute the Nash strategy. Describe the Newton algorithm in full. Explain but do not compute any stepsize search. Do not continue the computation after the first step.

(Both parts carry equal marks)