SOLUTIONS: Control Engineering

1. a) The transfer function for the circuit in Figure 1.2 is given by

$$\frac{Z_f(s)}{Z_i(s)} = -\frac{C_i(s+1/R_iC_i)}{C_f(s+1/R_fC_f)} = -\frac{C_is+1/R_i}{C_fs+1/R_f}$$

Putting in the values: $\boxed{G(s) = G_1(s)G_2(s) = \frac{s+3}{(s+1)(s+2)}}$. [5]

b) Since $y(s) = G(s)u(s)$ we have $\boxed{\ddot{y}(t)+3\dot{y}(t)+2y(t) = \dot{u}(t)+3u(t)}$. [5]

c) If $z(t)$ solves $\ddot{z}(t)+3\dot{z}(t)+2z(t) = u(t)$ then $y(t) = 3z(t)+\dot{z}(t)$. Let $x_1(t) = z(t), x_2(t) = \dot{z}(t)$ and $x(t) = [\, x_1(t) \;\; x_2(t)\,]^T$. Then a state–space realisation is

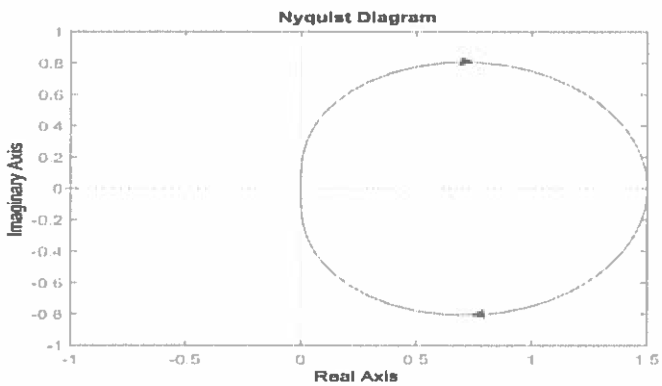
$$\boxed{\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \quad y(t) = [\; 3 \quad 1 \;] x(t)}$$ [5]

d) $\boxed{y_{ss} := \lim_{t\to\infty} y(t) = \lim_{s\to 0} sy(s) = \lim_{s\to 0} sG(s)u(s) = G(0) = 1.5.}$ [5]

e) $\lim_{t\to\infty} e(t) = \lim_{s\to 0} se(s) = \lim_{s\to 0} \frac{sr(s)}{1+K_pG(s)} = \frac{1}{1+K_pG(0)} \leq .01 \Rightarrow \boxed{K_p \geq 66.}$ [5]

f) $\boxed{\lim_{t\to\infty} e(t) = \lim_{s\to 0} se(s) = \lim_{s\to 0} \frac{sr(s)}{1+K_pG(s)} = \lim_{s\to 0} \frac{1}{s+sK_pG(s)} = \infty}$. [5]

g) The Nyquist diagram is shown below. [5]

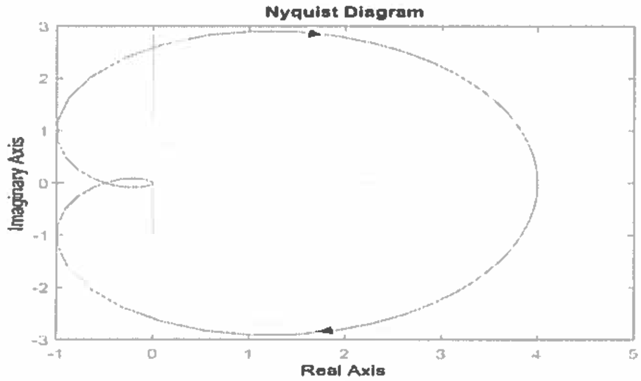


h) The Nyquist criterion: $N = Z - P$ where N is the number of clockwise encirclements of $-1/K_p$, Z is the number of unstable closed loop poles and P is the number of unstable open loop poles ($=0$). So for:

- $\boxed{-2/3 < K_p < \infty \Rightarrow N = 0 \Rightarrow Z = 0}$,
- $\boxed{-\infty < K_p < -2/3 \Rightarrow N = 1 \Rightarrow Z = 1}$,
- $\boxed{K_p = -2/3 \Rightarrow \text{ the closed loop is marginally stable}}$.

[5]

2. a) The Nyquist diagram is shown below. The real–axis intercepts can be found from the Routh array with $K(s) = K$. The characteristic equation is $1 + KG(s) = 0$ or $s^3 + 3s^2 + 3s + 1 + 4K = 0$. The Routh array is



$$\begin{array}{c|cc} s^3 & 1 & 3 \\ s^2 & 3 & 1+4K \\ s^1 & (8-4K)/3 & \\ s^0 & 1+4K & \end{array}$$

The values $K = 2$ and $K = -1/4$ result in a zero row, and so the real–axis intercepts $-1/K$ are obtained as $\boxed{-0.5, 4}$ and the corresponding frequencies are obtained from the auxiliary polynomials as $\boxed{\sqrt{3}, 0}$, respectively [6]

b) The closed loop is stable $\boxed{\text{since the elements of the first column of the Routh array have positive signs.}}$ The $\boxed{\text{gain margin} = 2}$ from the Routh array. At the cross–over frequency ω_c, $|G(j\omega_c)| = 1$, or $4 = |j\omega_c + 1|^3$ or $\boxed{\omega_c \sim 1.23}$. The angle of $G(j\omega_c)$ is $\sim -152.9°$ and so the $\boxed{\text{phase margin is} \sim 27.1°}$. [6]

c) Phase–lead compensation introduces positive phase in the cross–over frequency range and so tends to $\boxed{\text{improve the phase margin}}$. However, it introduces high gain in that frequency range, and may therefore $\boxed{\text{reduce the gain margin}}$. [6]

d) i) Here, $\varepsilon(s) = -(\delta(s) + Q(s)\varepsilon(s))$ where $\boxed{Q(s) = G(s)K(s)}$. [3]

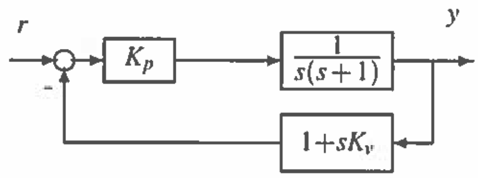
ii) Solving for $\varepsilon(s)$, $\varepsilon(s) = -S(s)\delta(s)$ where $\boxed{S(s) = (I + G(s)K(s))^{-1}}$, so $\boxed{\text{the loops are equivalent with this value of } S(s)}$. [3]

iii) Since $\Delta(s)$ and the closed loop of Figure 2.1 (and therefore $S(s)$) are stable, the Nyquist stability criterion states that the loop in Figure 2.3 is stable if there are no encirclements by $S(j\omega)\Delta(j\omega)$ of the point -1. The given condition implies that $|S(j\omega)\Delta(j\omega)| < 1$ so there are $\boxed{\text{no encirclements by } S(j\omega)\Delta(j\omega) \text{ of } -1 \text{ so the loop is stable}}$. [3]

iv) It follows that the smaller $|S(j\omega)|$ is, the larger the allowed $|\Delta(j\omega)|$ for closed loop stability, and since $|S(j\omega)| = 1/|1 + G(j\omega)K(j\omega)|$, $\boxed{\text{the larger the loop gain } |G(j\omega)K(j\omega)| \text{ is, the more robust the closed–loop will be against additive uncertainties}}$. [3]

3. a) The closed loop poles should be located at $\boxed{s_1, \bar{s}_1 = -2 \pm j2}$. [5]

b) The block diagram is shown below. [5]



c) The closed loop transfer function is given by

$$H(s) = \frac{K_p G(s)}{1 + KG(s)(s+z)}$$

where

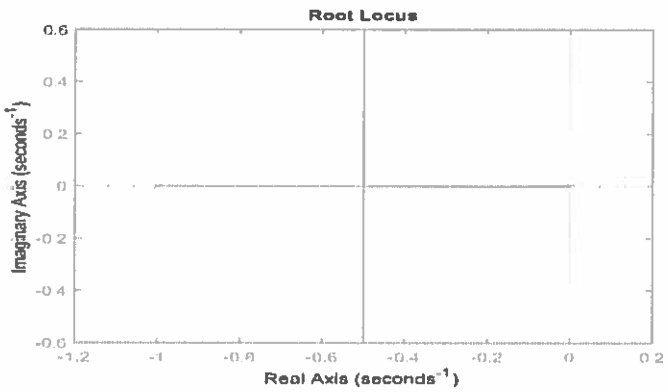
$$\boxed{G(s) = \frac{1}{s(s+1)},\ K = K_p K_v,\ z = 1/K_v}.$$

It follows that the characteristic equation is given by

$$\boxed{1 + K(s+z)G(s) = 0}.$$

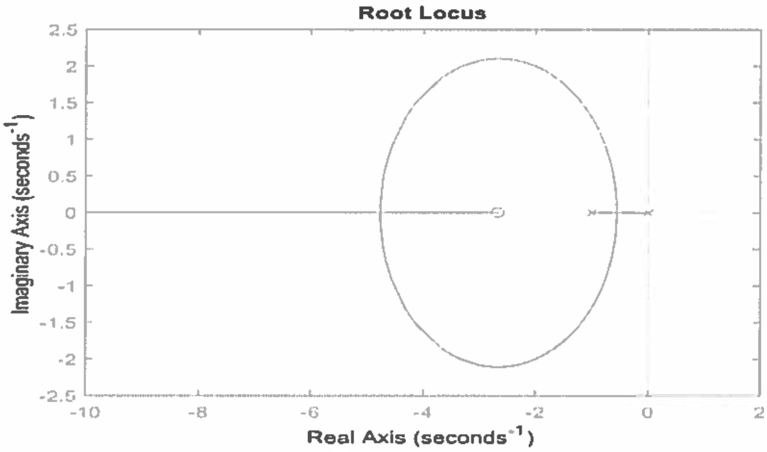
[5]

d) The root-locus is shown below. The specifications cannot be satisfied if $K_v = 0$ $\boxed{\text{since the root locus does not pass through the required pole locations}}$. [5]



e) The location of the zero $-z$ is determined from the angle criterion: $\theta = 116° + 135° - 180° \sim 71.6°$ which is satisfied by $\boxed{K_v = 3/8}$. The gain is obtained from the gain criterion $K = -s_1(s_1+1)/(s_1+8/3) = 3$ so $\boxed{K_p = 8}$. [5]

f) The root-locus is shown below. [5]


Root Locus
Imaginary Axis (seconds^-1)
Real Axis (seconds^-1)
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-10
-8
-6
-4
-2
0
2