UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 1996

BEng Honours Degree in Computing Part I
MEng Honours Degrees in Computing Part I
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the Associateship of the City and Guilds of London Institute

PAPER 1.9

MATHEMATICAL METHODS AND GRAPHICS

Wednesday, May 15th 1996, 2.00 - 4.00

Answer FOUR questions

For admin. only: paper contains
6 questions
6 pages (excluding cover page)

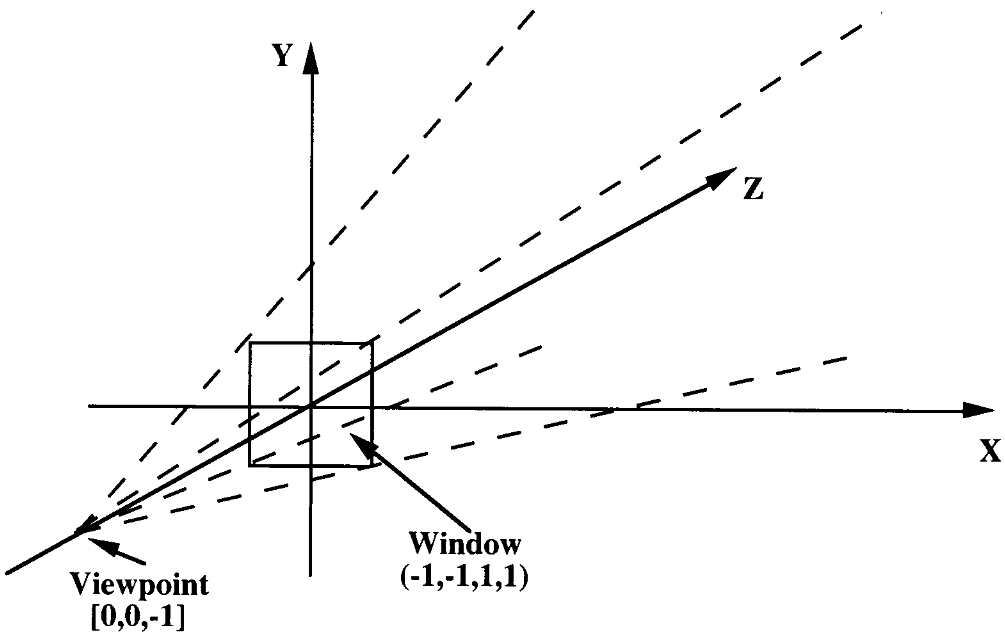
1. Clipping Algorithms

A plane can be completely defined by specifying the vector from the origin to the closest point on the plane (providing it does not go through the origin). This vector is of course normal to the plane.

a. For a plane defined by the vector $\mathbf{S}=[S_x,S_y,S_z]$ show that its vector equation can be written in the form $\mathbf{S.P}=|\mathbf{S}|^2$, where $P=[x,y,z]$ is a general point on the plane. If $S=[1,2,1]$ find the Cartesian equation of the plane.

b. Write a procedure in your favourite pseudocode (or programming language) which will determine whether a point **P** and the origin lie on the same side of a plane defined by a position vector **S** as above. Your procedure should return a boolean result, and take two points (**P** and **S**) as parameters

c. A graphics scene is to be viewed in perspective projection, but instead of the normal arrangement with the viewpoint at the origin, the viewpoint is placed at the point [0,0,-1]. The viewing direction is along the z-axis. The viewing window is on the x-y plane, bounded by the lines x=1,x=-1,y=1 and y=-1. The visible part of the scene is therefore bounded by four planes which pass through the point of projection and the four edges of the viewing window as shown in the figure. Determine the vector, equivalent to **S** in part a that defines each of these planes.



d. If the scene is made up of triangles, describe briefly how the procedure you wrote in part b could be used to determine whether a triangle is wholly visible, partly visible or totally invisible on the screen.

2. Anti-Aliasing

a. Explain, with a suitable diagram, what is meant by an alias frequency. In what way do alias frequencies manifest themselves in raster images.

b. Explain how the effect of alias frequencies can be reduced by means of a low pass convolution filter. Suggest a suitable filter for this purpose.

c. Explain how supersampling can be used to reduce alias effects in raster images.

d. What are the advantages and disadvantages of the antialiasing methods discussed in parts b and c

e. Suggest why alias effects can be particularly problematic when mapping texture onto polygons.

Turn Over

3a Show that if $\underline{x} = (x,\ y,\ z)^T$ is any 3-vector and A is the matrix

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix} \qquad \text{then} \quad |A\underline{x}|^2 = 9|\underline{x}|^2 \ .$$

b Deduce that all the eigenvalues of A have modulus 3, and by considering properties of the matrix A, such as its trace, $Tr(A)$, infer that the eigenvalues of A are 3, 3 and -3.

Demonstrate that $(1,\ 1,\ 0)^T$ and $(-1,\ 1,\ -2)^T$ are eigenvectors of A. Explain why the third eigenvector is perpendicular to these two, find it, and verify that it has the expected eigenvalue.

c Viewing A as a transformation matrix of 3-vectors, identify its action in terms of scalings, projections, reflections and rotations.

Parts a, b and c carry respectively 28%, 52% and 20% of the marks

4a Show that

$$(1+i)^n - (1-i)^n = 2^{\left(\frac{n+2}{2}\right)}\, i \sin\left(\tfrac{1}{4}n\pi\right) .$$

b The numbers $\{u_n\}$ for $n = 0, 1, 2 \ldots$ are defined by the recurrence relation

$$u_{n+1} - 2u_n + 2u_{n-1} = n \qquad \text{with} \quad u_0 = 1, \quad u_1 = 1 .$$

Obtain an explicit formula for u_n, and describe qualitatively the behaviour of u_n as $n \to \infty$.

Parts a and b carry respectively 24% and 76% of the marks

5a An iterative scheme is defined for a given smooth function $f(x)$ by

$$x_{n+1} = (1-k)x_n + kf(x_n) \qquad \text{where} \quad k > 0 \qquad \text{is constant.}$$

If the sequence $\{x_n\}$ converges to a limit X, infer an equation satisfied by X.

Defining $\varepsilon_n = x_n - X$ for every n, show that

$$\varepsilon_{n+1} = \varepsilon_n\Big(1 + k[f'(X) - 1]\Big) + O(\varepsilon_n^2)\ .$$

b Write down a necessary condition for the scheme to converge.

What value of k gives fastest convergence? Compare the scheme for this value with Newton's method, and explain why Newton's method is more practical.

c If $\varepsilon_n \to 0$ as $n \to \infty$, discuss whether $\sum_{n=1}^{\infty} x_n$ is convergent, stating which convergence test you use.

Parts a, b and c carry respectively 44%, 36% and 20% of the marks

6a A *tri-diagonal* matrix is one all of whose elements are zero apart from the leading diagonal and the diagonals on either side. That is, its $(i,\, j)$th element is zero unless $i = j$, $i = j - 1$ or $i = j + 1$.

Show that $(6N - 4)$ floating point operations (counting multiplications and reciprocals only) are necessary to solve $A\underline{x} = \underline{b}$, where A is an $N \times N$ tri-diagonal matrix.

b Write down 4 equations satisfied by the stationary point of the function

$$f(w,\, x,\, y,\, z) = w^2 + x^2 + y^2 + z^2 + wx + xy + yz - w + 2x + 2y - z$$

and solve them using Gaussian elimination.

c Show that the eigen-values of the matrix

$$H = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \qquad \text{satisfy} \quad (2 - \lambda)^4 - 3(2 - \lambda)^2 + 1 = 0\ .$$

Solve this equation and infer the nature of the stationary point.
[Hint: Consider $(\sqrt{5} \pm 1)^2$]

Parts a, b and c carry respectively 28%, 36% and 36% of the marks

