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IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE
UNIVERSITY OF LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2002

MSc and EEE/ISE PART IV: M.Eng. and ACGI

DISCRETE-TIME SYSTEMS AND COMPUTER CONTROL

Thursday, 9 May 10:00 am

There are SIX questions on this paper.

Answer FOUR questions.

Time allowed: 3:00 hours

Examiners responsible:

First Marker(s): Allwright,J.C.

Second Marker(s): Vinter,R.B.

Special Information for Invigilators: **None**

Information for Candidates

Some useful transforms

f_k	$f^Z(z)$	$f^D(\gamma)$	
$i_k = 0^k$	1	T	
1^k	$\frac{z}{z-1}$	$\frac{1+\gamma T}{\gamma}$	
t_k	$\frac{Tz}{(z-1)^2}$	$\frac{1+\gamma T}{\gamma^2}$	
α^k	$\frac{z}{z-\alpha}$	$\frac{1+\gamma T}{\gamma-\bar{\alpha}}$	where $\bar{\alpha} = \frac{\alpha-1}{T}$
$k\alpha^k$	$\frac{z\alpha}{(z-\alpha)^2}$	$\frac{(1+\gamma T)(1+\bar{\alpha}T)}{T(\gamma-\bar{\alpha})^2}$	

$f(t)$	$f^L(s)$
$e^{\alpha t}$	$\frac{1}{s-\alpha}$

Some notation

$'$ denotes transposition of a vector or matrix

q is the forward shift operator

$f^Z(z)$, $f^D(\gamma)$, $f^F(j\omega)$, $f^W(w)$ denote the Z-, Delta-, discrete-time Fourier and W-transforms, respectively, of $\{f_k\}$

$f^L(s)$ denotes the Laplace transform of $f(t)$

$t_k = kT$.

The Routh Test

Every root of $a_0 w^n + a_1 w^{n-1} + \ldots. + a_n = 0$ has strictly negative real part iff all $n+1$ entries in the first column of the following Routh-table are non-zero and have the same sign:

1:	a_0	a_2	a_4	.....
2:	a_1	a_3	a_5	.....
3:	$\frac{a_1 a_2 - a_0 a_3}{a_1}$	$\frac{a_1 a_4 - a_0 a_5}{a_1}$	$\frac{a_1 a_6 - a_0 a_7}{a_1}$	.....
..:	....			
$n+1$:	....			

The Jury Test

Every root of $d(z) \triangleq \alpha_n z^n + \alpha_{n-1} z^{n-1} + \ldots. + \alpha_0 = 0$ has modulus strictly less than one iff

$$d(1) > 0,$$

and

$$d(-1) \quad \begin{cases} > 0 & \text{if } n \text{ is even} \\ < 0 & \text{if } n \text{ is odd} \end{cases}$$

and

$$|\, a_0 \,| < \; a_n, \; |b_0\,| > |\, b_{n-1}\,|, \; |\, c_0 \,| > |\; c_{n-2}|, \; \ldots,$$

where the b_i, c_i etc., are determined from the following Jury-table

1:	a_0	a_1	a_2	.....	a_n
2:	a_n	a_{n-1}	a_{n-2}	.....	a_0
3:	b_0	b_1	b_2	.....	b_{n-1}
	where $b_i = a_0 a_i - a_n a_{n-i}$				
4:	b_{n-1}	b_{n-2}	.....	b_0	
..:	....				
2n−3:	....				

Here, for all i,

$$a_i \;= \begin{cases} \alpha_i & \text{if } \alpha_n > 0 \\ -\alpha_i & \text{if } \alpha_n < 0. \end{cases}$$

The Questions

1. (a) Consider the system of Figure 1.1, where $G^L(s) = \frac{2-4s}{(s-1)(s-2)}$.
Using a step-response and a partial fraction expansion, determine the pulse Z-transfer function from $u^Z(z)$ to $y^Z(z)$ when the sample period is T. **[6]**

(b) Show that $Z\{kf_k\} = -z\frac{d}{dz}f^Z(z)$. **[2]**

(c) Assume that each pole of the transforms $f^Z(z)$ and $g^Z(z)$ has modulus smaller than one.

(i) Adapt the proof of the Z-transform version of Parseval's theorem to show that
$$\sum_{k=0}^{\infty} f_k g_k = \frac{1}{2\pi j}\oint_{\Gamma_1} f^Z(z)\, g^Z(z^{-1}) z^{-1} dz$$ **[6]**
where Γ_1 denotes the disc of unit radius in the complex plane that is centred on the origin. You may use without proof the fact that
$$f_k = \frac{1}{2\pi j}\oint_{\Gamma_1} f^Z(z) z^{k-1} dz.$$

(ii) For $\{f_k\} = \{1, 2, 0, 0, 0, \dots\}$, use residues to evaluate
$$\frac{1}{2\pi j}\oint_{\Gamma_1} f^Z(z)\, f^Z(z^{-1}) z^{-1} dz.$$
Check your result by carrying out a discrete-time summation. **[6]**

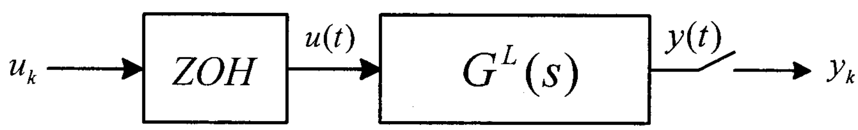


Figure 1.1

2. Consider the system of Figure 2.1 where the continuous-time system S_c is modelled by

$$\dot{x}(t) = Ax(t) + Bu(t),\, y(t) = c'x(t) \qquad (£)$$

and the sample period is T.

(a) (i) An approximation to $x(t+h)$ is given by

$$x(t+h) \approx x(t) + h\dot{x}(t).$$

Use this method of approximation to determine an approximation to $x(t_k+\frac{T}{2})$ and use it again to determine the matrices $\widetilde{A}$, $\widetilde{B}$ of an approximation x_{k+1} to $x(t_k+T)$ of the form

$$x_{k+1} = \widetilde{A}x_k + \widetilde{B}u_k,\; y_k = c'x_k. \qquad (\$)$$

[6]

(ii) State the connection between the eigenvalues of $\widetilde{A}$ of part (i) and BIBO-stability of (\$). **[1]**

(iii) Assuming that A of (£) has distinct eigenvalues, determine a formula for $\widetilde{A}$ of part (i) in terms of the spectral form for A and hence determine an inequality for each eigenvalue of A that guarantees BIBO-stability of (\$). **[4]**

(b) Consider the discrete-time model

$$x_{k+1} = \overline{A}x_k + \overline{B}u_k,\; y_k = c'x_k$$

of (£) that satisfies $x_k = x(t_k)$ for all $k \geq 0$.

Derive from $\overline{A}$ and $\overline{B}$ the approximation

$$x_{k+1} = (I + A\tfrac{T}{2})(I - A\tfrac{T}{2})^{-1}x_k + T(I - A\tfrac{T}{2})^{-1}\, Bu_k,\; y_k = c'x_k.$$

[3]

(c) Suppose the pulse Z-transfer function, from $u^Z(z)$ to $y^Z(z)$, for the system of Figure 2.1 is $G^Z(z) = \frac{z-1}{z^2-4}$.

Use residues to determine the corresponding pulse response sequence. **[6]**

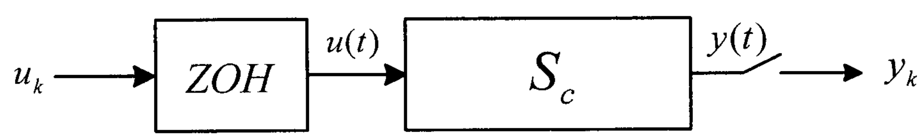


Figure 2.1

3. Consider the feedback system of Figure 3.1 where $K \geq 0$, the sample period is T and $G^Z(z)$ is the pulse Z-transfer function, from $u^Z(z)$ to $y^Z(z)$, of the system of Figure 3.2.

(a) Show that $y^L(j\omega) = \left(\frac{1-e^{-j\omega T}}{j\omega}\right) G^L(j\omega) u^F(\omega T)$. **[6]**

(b) Suppose $G^Z(z) = \frac{z+0.5}{z(z-2)}$.

(i) Determine the break-points of the root-locus for the closed-loop system of Figure 3.1 and hence draw accurately the root-locus for that system. **[6]**

(ii) Use the root-locus of part (i) to determine the set of values of the gain K for which the closed-loop sysem is BIBO-stable. **[4]**

(iii) Verify your set of part (ii) using the Jury test. **[4]**

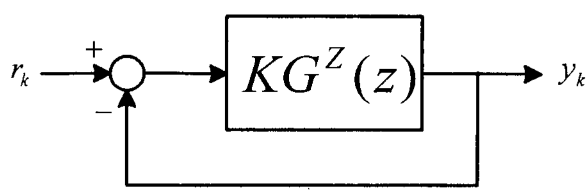


Figure 3.1

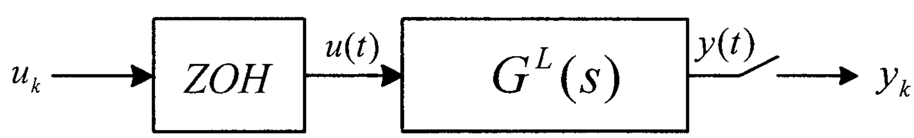


Figure 3.2

4. (a) Prove that if $z = \frac{1+w}{1-w}$ then $|z| < 1$ iff $\mathfrak{Re}(w) < 0$. Discuss very briefly (in, say, one or two sentences) the significance of this result. **[5]**

(b) Consider the system of Figure 4.1, for which $K \geq 0$ and

$$G^Z(z) = \frac{(z+10)}{(z-1)(z+0.3)}.$$

Let K_{max} be the largest value of K such that the closed-loop system is BIBO-stable for all $K \in [0.K_{max})$.

A plot of $G^Z(e^{j\Omega})$ for $\Omega \in (0,2\pi)$ is shown in Figure 4.2.

Draw the relevant discrete-time Nyquist path, sketch the corresponding discrete-time Nyquist locus and estimate K_{max} from your locus. Give enough explanation to make clear how you have obtained your locus and determined K_{max} from it. **[7]**

(c) Discuss the use of full-state observers in the feedback control of linear discrete-time single-input single-output systems described by

$$x_{k+1} = Ax_k + bu_k,\; y_k = c'x_k.$$

Your discussion should include: the observer equation, the associated eigenvalues and how to assign them using a standard eigenvalue assignment algorithm for choosing feedback gains, and properties of the observer that are relevant to feedback control. **[8]**

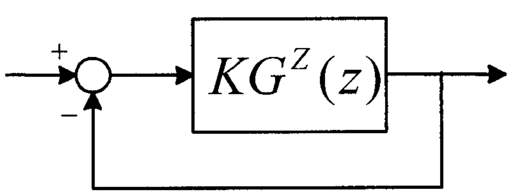


Figure 4.1

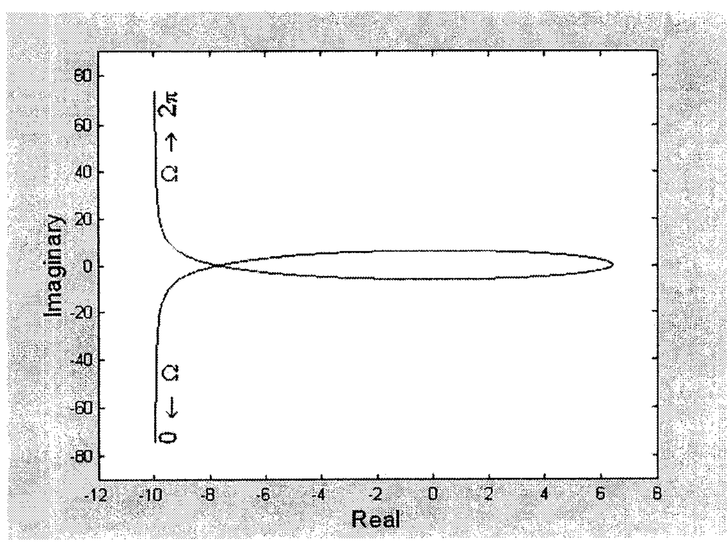


Figure 4.2

5. Consider the system of Figure 5.1 below, where the controller $C^Z(z)$ and plant $P^Z(z)$ are specified by

$[C^Z]$ $\qquad \overline{x}_{k+1} = \overline{A}\overline{x}_k + \overline{b}e_k,\ u_k = \overline{c}'\overline{x}_k$

$[P^Z]$ $\qquad x_{k+1} = Ax_k + bu_k,\ y_k = c'x_k.$

(i) Suppose

$$\overline{A} = \begin{bmatrix} 1 & -0.75 \\ 1 & -1 \end{bmatrix},\ \overline{b} = \begin{bmatrix} 1 \\ -1 \end{bmatrix},\ \overline{c}' = \begin{bmatrix} 1 & -1.5 \end{bmatrix}.$$

Determine $C^Z(z)$ and the decoupling zero(s). **[6]**

(ii) For a (different) controller $C^Z(z) = \frac{(z-2)(z+3)}{(z-1)(z+0.5)}$, determine a control canonical realisation and a series realisation. **[7]**

(iii) Let $\widetilde{x}_k = \begin{bmatrix} x_k \\ \overline{x}_k \end{bmatrix}$. Determine $\widetilde{A}, \widetilde{b}, \widetilde{c}$ of a model of the forward path having the form

$$\widetilde{x}_{k+1} = \widetilde{A}\,\widetilde{x}_k + \widetilde{b}e_k,\quad y_k = \widetilde{c}\,'\,\widetilde{x}_k.$$

Suppose the eigenvalues of A are λ_i, $i = 1, 2, \ldots, n$, and those of $\overline{A}$ are $\overline{\lambda}_i$, $i = 1, 2, \ldots, \overline{n}$. Prove, using the basic definition of an eigenvector, that the eigenvalues of A and $\overline{A}$ are also eigenvalues of $\widetilde{A}$.

Discuss very briefly the significance of this when an unstable pole of $P^Z(z)$ is cancelled by a zero of $C^Z(z)$. **[7]**

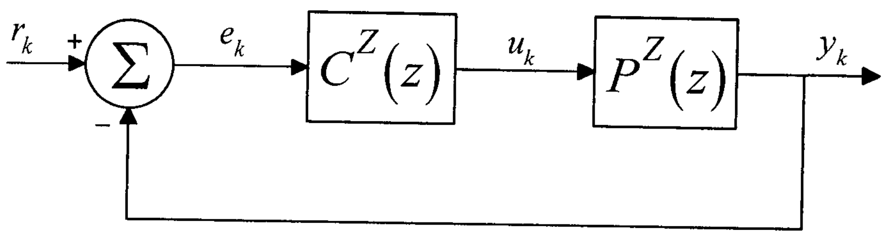


Figure 5.1

6. Consider the scalar-input scalar-output discrete-time system

$$x_{k+1} = Ax_k + bu_k; \; y_k = c'x_k$$

where $x_k \in \mathbb{R}^n$.

(a) Let M be the system's controllability matrix.
Use M to determine a sequence of controls that demonstrates that the system is reachable if M is non-singular. **[4]**

(b) Suppose $n = 2$ and

$$A = \begin{bmatrix} -3 & -5 \\ 5 & 3 \end{bmatrix}, b = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, c' = [1 \quad 1].$$

(i) The pulse Z-transfer function from $u^Z(z)$ to $y^Z(z)$ is $4/(z^2 + 16)$.
Show, using root-locus analysis, that the system cannot be stabilized by the control law $u_k = r_k - Ky_k$ for any positive gain K. **[3]**

(ii) Now consider control of the form $u_k = r_k - f'x_k$ for $f \in \mathbb{R}^2$.
Let p' be the bottom row of the inverse of the controllability matrix M and let

$$V = \begin{bmatrix} p' \\ p'A \\ \cdots \\ p'A^{n-1} \end{bmatrix}.$$

Use V to determine the control canonical form for the system and use it to choose f to stabilize the system by locating the closed-loop poles at the origin. **[11]**

Verify that your closed-loop system has the desired eigenvalues. **[2]**

Discrete - Time Systems - Solutions 2002

1. (a) Pulse Z-transfer function

$= \frac{(z-1)}{z} Z\{\mathcal{L}^{-1}(G^L(s)/s)(t_k)\} = \frac{(z-1)}{z} Z\{\mathcal{L}^{-1}(\frac{2-4s}{s(s-1)(s-2)})(t_k)\}$

$= \frac{(z-1)}{z} Z\{\mathcal{L}^{-1}(\frac{a}{s} + \frac{b}{s-1} + \frac{c}{s-2})(t_k)\}$

(where $a = (2-4s)(s-1)^{-1}(s-2)^{-1}|_{s=0} = 1$, $b = (2-4s)s^{-1}(s-2)^{-1}|_{s=1} = 2$,
$c = (2-4s)s^{-1}(s-1)^{-1}|_{s=2} = -3$)

$= \frac{(z-1)}{z} Z\{\mathcal{L}^{-1}(\frac{1}{s} + \frac{2}{s-1} - \frac{3}{s-2})(t_k)\} = \frac{(z-1)}{z} Z\{(1 + 2e^{+t} - 3e^{+2t})|_{t=kT}\}$

$= \frac{(z-1)}{z} Z\{1 + 2e^{+Tk} - 3e^{+2Tk}\} = \frac{(z-1)}{z}\{\frac{z}{z-1} + 2\frac{z}{z-e^{+T}} - 3\frac{z}{z-e^{+2T}}\}$

$= 1 + 2\frac{z-1}{z-e^{+T}} - 3\frac{z-1}{z-e^{+2T}}$. [6]

(b) $-z\frac{d}{dz}f^Z(z) = -z\frac{d}{dz}\sum_{k=0}^{\infty} f_k z^{-k} = -\sum_{k=0}^{\infty} f_k(-k)z^{-k} = \sum_{k=0}^{\infty} k f_k z^{-k} = Z\{kf_k\}(z)$. [2]

(c) (i) Now $f_k = \frac{1}{2\pi j}\oint_{\Gamma_1} f^Z(z)z^{k-1}dz$ because all the poles of $f^Z(z)$ are within Γ_1.

Hence $\sum_{k=0}^{\infty} f_k g_k = \sum_{k=0}^{\infty}\left\{\frac{1}{2\pi j}\oint_{\Gamma_1} f^Z(z)z^{k-1}dz\right\}g_k = \frac{1}{2\pi j}\oint_{\Gamma_1} f^Z(z)\sum_{k=0}^{\infty} g_k z^{k-1}dz$

$= \frac{1}{2\pi j}\oint_{\Gamma_1} f^Z(z)\left(\sum_{k=0}^{\infty} g_k z^k\right)z^{-1}dz = \frac{1}{2\pi j}\oint_{\Gamma_1} f^Z(z)\, g^Z(z^{-1})z^{-1}dz.$ [6]

(ii) For $\{f_k\} = \{1, 2, 0, 0, 0, \ldots\}$, we have $f^Z(z) = 1 + 2z^{-1} = \frac{z+2}{(z-0)}$ so
$f^Z(z^{-1}) = 1 + 2(z)^{-1}$.

Hence $\frac{1}{2\pi j}\oint_{\Gamma_1} f^Z(z^{-1})\, f^Z(z)z^{-1}dz$

$= \frac{1}{2\pi j}\oint_{\Gamma_1}\{1 + 2z\}\frac{z+2}{(z-0)}z^{-1}dz$

$= \frac{1}{2\pi j}\oint_{\Gamma_1}\{\frac{z+2+2z^2+4z}{(z-0)^2}\}\, dz$

$= \frac{1}{2\pi j}\oint_{\Gamma_1}\{\frac{2z^2+5z+2}{(z-0)^2}\}\, dz$

$=$ residue of $\{\frac{2z^2+5z+2}{(z-0)^2}\}$ at $z = 0$

$= \frac{d}{dz}\{\frac{2z^2+5z+2}{(z-0)^2}(z-0)^2\}\Big|_{z=0} = \frac{d}{dz}\{2z^2 + 5z + 2\}\Big|_{z=0} = \{4z + 5\}|_{z=0} = 5.$

According to Parseval's theorem,

$\frac{1}{2\pi j}\oint_{\Gamma_1} f^Z(z^{-1})\, f^Z(z)z^{-1}dz = \sum_{k=0}^{\infty} f_k^2 = 1^2 + 2^2 = 5$

confirming the above calculation using a residue. [6]

2. (a) Using $x(t+h) \approx x(t) + h\dot{x}(t)$, we obtain

$x(t_k + \frac{T}{2}) \approx x(t_k) + (\frac{T}{2})[Ax(t_k) + Bu(t_k)] = (I + A\frac{T}{2})x(t_k) + (\frac{T}{2})Bu(t_k)$

so

$$\begin{aligned} x(t_k + T) &\approx (I + A\tfrac{T}{2})x(t_k+\tfrac{T}{2}) + (\tfrac{T}{2})Bu(t_k+\tfrac{T}{2}) \\ &\approx (I + A\tfrac{T}{2})[(I + A\tfrac{T}{2})x(t_k) + (\tfrac{T}{2})Bu(t_k)] + (\tfrac{T}{2})Bu(t_k + \tfrac{T}{2}) \\ &= (I + A\tfrac{T}{2})[(I + A\tfrac{T}{2})x(t_k) + (\tfrac{T}{2})Bu_k] + (\tfrac{T}{2})Bu_k \\ &\qquad \text{(since } u(t_k) = u(t_k + \tfrac{T}{2}) = u_k) \\ &= (I + A\tfrac{T}{2})^2 x(t_k) + [(I + A\tfrac{T}{2})\tfrac{T}{2} + \tfrac{T}{2}]Bu_k \end{aligned}$$

giving rise to the approximation

$$x_{k+1} = \widetilde{A}\, x_k + \widetilde{B}u_k$$

where $\widetilde{A} = (I + A\frac{T}{2})^2$, $\widetilde{B} = [(I + A\frac{T}{2})\frac{T}{2} + \frac{T}{2}]B$. [6]

(i) the system is BIBO-stable if the eigenvalues of $\widetilde{A}$ each have modulus smaller than one. [1]

(ii) Since A has distinct eigenvalues, it has the spectral form $A = V\Lambda V^{-1}$. Therefore $\widetilde{A}$ can be written as $\widetilde{A} = (I + A\frac{T}{2})^2 = (I + V\Lambda V^{-1}\frac{T}{2})^2 = [V(I + \Lambda\frac{T}{2})V^{-1}]^2 = V(I + \Lambda\frac{T}{2})^2 V^{-1}$ so the eigenvalues of $\widetilde{A}$ are $(1 + \lambda_i\frac{T}{2})^2$. Hence the condition for BIBO-stability is that $|1 + \lambda_i\frac{T}{2}| < 1,\ \forall i$. [4]

(iii) $$\begin{aligned} x_{k+1} &= \exp(AT)x_k + \textstyle\int_0^T \exp(A\tau)d\tau Bu_k \approx e^{AT/2}(e^{-AT/2})^{-1}x_k + Te^{AT/2}Bu_k \\ &\approx e^{AT/2}(e^{-AT/2})^{-1}x_k + T[e^{-AT/2}]^{-1}Bu_k \\ &\approx (I + A\tfrac{T}{2})(I - A\tfrac{T}{2})^{-1}x_k + T(I - A\tfrac{T}{2})^{-1}Bu_k. \end{aligned}$$

Hence we have the approximation

$$x_{k+1} = (I + A\tfrac{T}{2})(I - A\tfrac{T}{2})^{-1}x_k + T(I - A\tfrac{T}{2})^{-1}Bu_k,\ y_k = c'x_k$$ [3]

(b) The pulse response sequence is $\{h_k\} = Z^{-1}\{G^Z(z)\} = Z^{-1}\{\frac{(z-1)}{z^2-4}\}$.

$h_0 = \underset{|z| \to \infty}{\text{Lt}} \frac{(z-1)}{z^2-4} = 0.$

For $k > 0$:

$$\begin{aligned} h_k &= \{\text{residue of } \tfrac{(z-1)z^{k-1}}{(z-2)(z+2)}(z-2) @ z = +2\} + \{\text{residue of } \tfrac{(z-1)z^{k-1}}{(z-2)(z+2)}(z+2) @ z = -2\} \\ &= \left.\tfrac{(z-1)z^{k-1}}{z+2}\right|_{z=2} + \left.\tfrac{(z-1)z^{k-1}}{z-2}\right|_{z=-2} = 0.25 \times 2^{k-1} + 0.75 \times (-2)^{k-1}. \end{aligned}$$ [6]

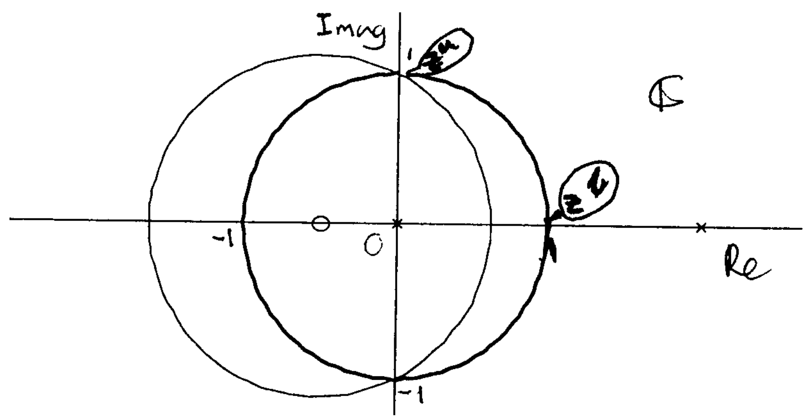
3. (a) Now $u(t) = \sum_{k=0}^{\infty} \alpha_k(t) u_k$ where $\alpha_k(t) = 1$ for $t \in [t_k, t_{k+1})$ and $\alpha_k(t) = 0$ for $t \notin [t_k, t_{k+1})$. And $a_k^L(s) = (\mathrm{e}^{-st_k} - \mathrm{e}^{-st_{k+1}})/s = (1 - \mathrm{e}^{-sT})\mathrm{e}^{-st_k}/s$. Hence $u^L(s) = \sum_{k=0}^{\infty} \alpha_k^L(s) u_k = \sum_{k=0}^{\infty} (1 - \mathrm{e}^{-sT})\mathrm{e}^{-st_k} u_k / s = [(1 - \mathrm{e}^{-sT})/s] \sum_{k=0}^{\infty} \mathrm{e}^{-st_k} u_k$
$= [(1 - \mathrm{e}^{-sT})/s] \sum_{k=0}^{\infty} (\mathrm{e}^{-sT})^k u_k = [(1 - \mathrm{e}^{-sT})/s]\, u^Z(\mathrm{e}^{sT})$.

Hence $y^L(s) = G^L(s) u(s) = [(1 - \mathrm{e}^{-sT})/s] G^L(s) u^Z(\mathrm{e}^{sT})$ so

$$y^L(j\omega) = (1 - \mathrm{e}^{-j\omega T})/(j\omega)] G^L(j\omega) u^Z(\mathrm{e}^{j\omega T})$$
$$= (1 - \mathrm{e}^{-j\omega T})/(j\omega)] G^L(j\omega) u^F(\omega T).$$

[6]

(b) (i) The break points σ_b are defined by $\frac{1}{\sigma_b + 0.5} = \frac{1}{\sigma_b} + \frac{1}{\sigma_b - 2}$,
i.e. by $\sigma_b(\sigma_b - 2) = (\sigma_b + 0.5)(\sigma_b - 2) + \sigma_b(\sigma_b + 0.5)$,
i.e. by $\sigma_b^2 - 2\sigma_b = \sigma_b^2 - 1.5\sigma_b - 1 + \sigma_b^2 + 0.5\sigma_b$
i.e. by $\sigma_b^2 + \sigma_b - 1 = 0$, i.e. by $\sigma_b = \frac{1}{2}(-1 \pm \sqrt{5}) = -1.618, 0.618$.
Hence the root-locus is as follows:



[6]

(ii) Consequently the range of values of K for which the closed-loop system is BIBO-stable is (K_{min}, K_{max}) where

$K_{min} = -1/G^Z(z_l) = -1/G^Z(1) = -1/(\frac{z+0.5}{z(z-2)})|_{z=1} \approx 0.6666$

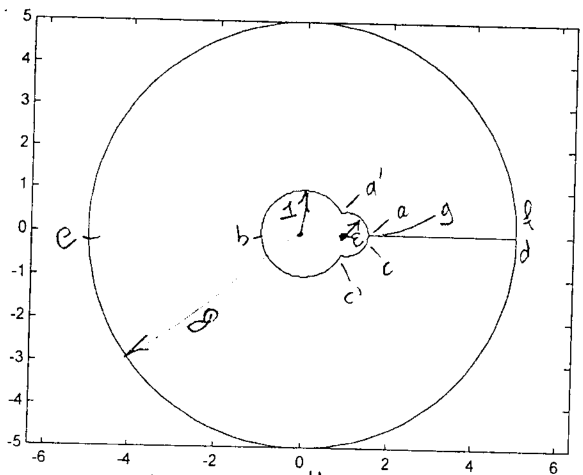
$K_{max} = -1/G^Z(z_u) = -1/G^Z(0+j) = -1/(\frac{z+0.5}{z(z-2)})\Big|_{z=j} = -\frac{j(j-2)}{(j+0.5)} = -\frac{j(j-2)(j-0.5)}{1.25}$
$= -j\frac{2.5j}{1.25} = 2.$

[4]

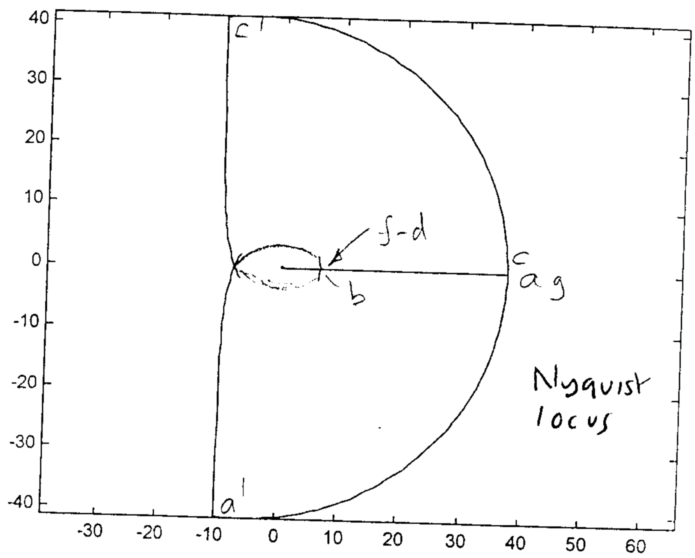
(iii) The pulse Z-transfer function of the closed-loop system is $\frac{KG^Z(z)}{1+KG^Z(z)} = \frac{K\frac{(z+0.5)}{z(z-2)}}{1+K\frac{(z+0.5)}{z(z-2)}}$
which has the denominator $d(z) = z(z-2) + K(z+0.5) = z^2 + (K-2)z + 0.5K$.
Now $d(1) = 1 + K - 2 + 0.5K = 1.5K - 1 > 0$ iff $K > 1/1.5$ i.e. iff $K > 2/3$
and
$d(-1) = 1 - K + 2 + 0.5K = 3 - 0.5K > 0$ iff $K < 6$.
The first row of the Jury table (the only relevant row in this second-order case) is
$0.5K \quad K-2 \quad 1$
so there is the extra condition that $|0.5\mathrm{K}| < 1$ i.e. $K < 2$.
Hence the closed-loop system is BIBO-stable iff $K > 2/3,\ K < \min\{6,2\} = 2$
which is consistent with the values obtained from the root-locus.

4. (a) For $z = \frac{1+w}{1-w}$ we have that $|z|^2 = |z^* z| = \left|\frac{(1+w)^*(1+w)}{(1-w)^*(1-w)}\right| = \left|\frac{1+(w+w^*)+w^*w}{1-(w+w^*)+w^*w}\right|$ $= \left|\frac{1+w^*w+2\mathfrak{Re}(w))}{1+w^*w-2\mathfrak{Re}(w))}\right| < 1$ iff $\mathfrak{Re}(w) < 0$. This is useful because it provides a connection between the BIBO-stability condition for poles for a discrete-time system (|pole|<1) and for the poles of a continuous-time system ($\mathfrak{Re}$(pole) < 0). [5]

(b) Since there is an open-loop pole at $z = 1$, the relevant Nyquist path is as shown



Nyquist path



Nyquist locus

For z in the path in the circle centred on $z = 1$ with radius ε,
$G^Z(z) = G^Z(1+\varepsilon e^{j\theta}) = \frac{(z+10)}{(z-1)(z+0.3)} = \frac{(1+\varepsilon e^{j\theta}+10)}{(1+\varepsilon e^{j\theta}-1)(1+\varepsilon e^{j\theta}+0.3)}$
$= \frac{(11+\varepsilon e^{j\theta})}{(\varepsilon e^{j\theta})(1.3+\varepsilon e^{j\theta})} \approx \frac{11e^{-j\theta}}{1.3\varepsilon} \approx 8.46e^{-j\theta}/\varepsilon$.
Hence we obtain parts a-a$'$ and c$'$-c of the Nyquist locus above.
The part a$'$-b-c$'$ follows from Figure 4.2.
Since $G^Z(z) \to 0$ as $z \to \infty$, the parts d-e-f and f-g are at the origin, as shown.
Since there are no open-loop poles in the region $E \cup L$, the closed-loop system is BIBO-stable if $-\frac{1}{K} <$ (approx.) -8, i.e. if $K < 0.125$. [7]

(c) The observer is $\widehat{x}_{k+1} = (A - lc')\widehat{x}_k + ly_k + bu_k : \widehat{x}_0 = \overline{x}_0$. For it: $\varepsilon_{k+1} = (A - lc')\varepsilon_k$ where $\varepsilon_k = x_k - \widehat{x}_k$. Therefore $\varepsilon_k \to 0$, so that (slightly abusing notation) $\widehat{x}_k \to x_k$, if $|\lambda_i(A - lc')| \leq 1$, $\forall i$. Since the eigenvalues of $A - lc'$ are those of $(A - lc')' = A' - cl'$, the eigenvalues of $A - lc'$ can be assigned arbitrarily by choosing l using a standard algorithm for assigning the eigenvalues of $A' - cl'$ provided (A', c) is a controllable pair.
The tranfer functions for the closed loop systems when $u_k = f'\widehat{x}_k$ and when $u_k = f'x_k$ are the same.The eigemvalues of the closed-loop system using feedback from $\widehat{x}_k$ instead of x_k consists of the eigenvalues of $A - bf'$ together with those of $A - lc'$. Hence the design of the feedback gain vector f can be decoupled from the design of l. [8]

5 (i) $C^Z(z) = \overline{c}'(zI - \overline{A})^{-1}\overline{b} = [1\ -1.5]\left(\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 1 & -0.75 \\ 1 & -1 \end{bmatrix}\right)^{-1}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$= [1\ -1.5]\begin{bmatrix} z-1 & 0.75 \\ -1 & z+1 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = [1\ -1.5]\dfrac{\begin{bmatrix} z+1 & -0.75 \\ 1 & z-1 \end{bmatrix}}{(z-1)(z+1)+0.75}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$= [1\ -1.5]\frac{1}{z^2-0.25}\begin{bmatrix} z+1.75 \\ 2-z \end{bmatrix} = \frac{2.5z-1.25}{z^2-0.25} = 2.5\frac{z-0.5}{(z-0.5)(z+0.5)} = \frac{2.5}{(z+0.5)}.$

The decoupling zero is the cancelled eigenvalue and so is 0.5. [6]

(ii) $u^Z(z) = C^Z(z)e^Z(z) = \frac{(z-2)(z+3)}{(z-1)(z+0.5)}e^Z(z) = \frac{z^2+z-6}{z^2-0.5z-0.5}e^Z(z) = \frac{1+z^{-1}-6z^{-2}}{1-0.5z^{-1}-0.5z^{-2}}e^Z(z)$

$= (1 + z^{-1} - 6z^{-2})w^Z(z)$ where $w^Z(z) = \frac{1}{1-0.5z^{-1}-0.5z^{-2}}e^Z(z)$.

Hence $(1 - 0.5z^{-1} - 0.5z^{-2})w^Z(z) = e^Z(z)$ so, taking Z^{-1},

$w_k - 0.5w_{k-1} - 0.5w_{k-2} = e_k$

i.e. $w_k = e_k + 0.5w_{k-1} + 0.5w_{k-2}$.

Similarly, since $u^Z(z) = (1 + z^{-1} - 6z^{-2})w^Z(z)$,

$u_k = w_k + w_{k-1} - 6w_{k-2}$.

Hence the canonical realisation is:

$u_k = w_k + w_{k-1} - 6w_{k-2}$, $w_k = e_k + 0.5w_{k-1} + 0.5w_{k-2}$.

For a series realisation, write $u^Z(z) = \frac{(z-2)}{(z-1)}\frac{(z+3)}{(z+0.5)}e^Z(z) = \frac{(1-2z^{-2})}{(1-z^{-1})}a^Z(z)$ where $a^Z(z) = \frac{(1+3z^{-1})}{(1+0.5z^{-1})}e^Z(z)$. Hence $(1 - z^{-1})u^Z(z) = (1 - 2z^{-2})a^Z(z)$ and $(1+0.5z^{-1})a^Z(z) = (1 + 3z^{-1})e^Z(z)$. Therefore the series realisation is:

$u_k = a_k - 2a_{k-2} + u_{k-1}$, $a_k = e_k + 3e_{k-1} - 0.5a_{k-1}$. [7]

(iii) $\widetilde{x}_{k+1} = \begin{bmatrix} x_{k+1} \\ \overline{x}_{k+1} \end{bmatrix} = \begin{bmatrix} Ax_k + bu_k \\ \overline{A}\overline{x}_k + \overline{b}e_k \end{bmatrix} = \begin{bmatrix} Ax_k + b\overline{c}'\overline{x}_k \\ \overline{A}\overline{x}_k + \overline{b}e_k \end{bmatrix} = \underbrace{\begin{bmatrix} A & b\overline{c}' \\ 0 & \overline{A} \end{bmatrix}}_{\widetilde{A}}\begin{bmatrix} x_k \\ \overline{x}_k \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \overline{b} \end{bmatrix}}_{\widetilde{b}}e_k$

Let v be an eigenvector of A associated with the eigenvalue λ of A and let $\widetilde{v} = [v'\ 0]'$.

Then $\widetilde{A}\widetilde{v} = \begin{bmatrix} A & b\overline{c}' \\ 0 & \overline{A} \end{bmatrix}\begin{bmatrix} v \\ 0 \end{bmatrix} = \begin{bmatrix} Av \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda v \\ 0 \end{bmatrix} = \lambda\begin{bmatrix} v \\ 0 \end{bmatrix} = \lambda\widetilde{v}$ so each eigenvalue of A is an eigenvalue of $\widetilde{A}$.

Now suppose w is an eigenvector of $\overline{A}'$ corresponding to the eigenvalue λ of $\overline{A}'$, which is automatically an eigevalue of $\overline{A}$. Then

$\widetilde{A}'\begin{bmatrix} 0 \\ w \end{bmatrix} = \begin{bmatrix} A' & 0' \\ \overline{c}b' & \overline{A}' \end{bmatrix}\begin{bmatrix} 0 \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ \overline{A}'w \end{bmatrix} = \begin{bmatrix} 0 \\ \lambda w \end{bmatrix} = \lambda\begin{bmatrix} 0 \\ w \end{bmatrix}$ so λ is an eigenvalue of $\widetilde{A}'$

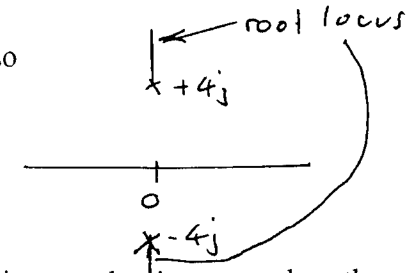
and hence is an eigenvalue of $\widetilde{A}$ since the eigenvalues of $\widetilde{A}'$ are those of $\widetilde{A}$.

Consequently the eigenvalues of A and $\overline{A}$ are eigenvalues of $\widetilde{A}$.

The cancellation mentioned causes the forward path's transfer function to have no unstable poles even though non-zero initial conditions might cause $y_k \to \infty$. The unstable eigevalue remains even if the feedback loop is closed, which shows the danger of choosing $C^Z(z)$ to cancel an unstable plant pole. [7]

6. (a) $x_1 = Ax_0 + bu_0$, $x_2 = A(Ax_0 + bu_0) + bu_1 = A^2x_0 + Abu_1 + bu_0$, etc., so $x_n = A^n x_0 + [b\ Ab\ \ldots A^{n-1}b]\underline{u} = A^n x_0 + M\underline{u}$ where $\underline{u} = [u_{n-1}\ u_{n-2}\ \ldots u_0]'$. If M is non-singular then x_n can be made equal to any given χ by choosing $\underline{u} = M^{-1}[\chi - A^n x_0]$ so there is a control sequence that transfers any initial condition to any given χ in finite time, so the system is reachable. [4]

(b) (i) The poles of the transfer function are at $\pm$ 4j so the root-locus is as shown.



The root-locus does not enter the disk with radius one that is centered on the origin so the system cannot be stabilised for any positive value of the gain K. [3]

(ii) $M = [b\ Ab] = \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}$. Then $M^{-1} = \begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix}/(2+2) = \begin{bmatrix} 0.5 & -0.5 \\ 0.25 & 0.25 \end{bmatrix}$

so $p' = [\,0.25 \quad 0.25\,]$ and consequently $V = \begin{bmatrix} 0.25 & 0.25 \\ 0.5 & -0.5 \end{bmatrix}$ so

$V^{-1} = \begin{bmatrix} -0.5 & -0.25 \\ -0.5 & 0.25 \end{bmatrix}/(-0.125-0.125) = \begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix}$.

Then

$$C_\alpha = VAV^{-1} = \begin{bmatrix} 0.25 & 0.25 \\ 0.5 & -0.5 \end{bmatrix}\begin{bmatrix} -3 & -5 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.25 & 0.25 \\ 0.5 & -0.5 \end{bmatrix}\begin{bmatrix} -16 & 2 \\ 16 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -16 & 0 \end{bmatrix}$$

which has the desired companion form with $\alpha = [\,-16 \quad 0\,.]$. The corresponding B-matrix is $Vb = \begin{bmatrix} 0.25 & 0.25 \\ 0.5 & -0.5 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Hence the transformed system has the controllable pair $\left(\begin{bmatrix} 0 & 1 \\ -16 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$.

The desired characteristic polynomial is $(\lambda - 0)(\lambda - 0) = \lambda^2 - 0.\lambda - 0$.

Hence the feedback vector required is $f = V'\left(\begin{bmatrix} -16 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)$

$= \begin{bmatrix} 0.25 & 0.5 \\ 0.25 & -0.5 \end{bmatrix}\begin{bmatrix} -16 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ -4 \end{bmatrix}$. [11]

Check of closed-loop eigenvalues: $A - bf' = \begin{bmatrix} -3 & -5 \\ 5 & 3 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \end{bmatrix}[\,-4 \quad -4\,.]$

$= \begin{bmatrix} -3 & -5 \\ 5 & 3 \end{bmatrix} - \begin{bmatrix} -4 & -4 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$.

$\det(\lambda I - \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}) = \det(\begin{bmatrix} \lambda-1 & 1 \\ -1 & \lambda+1 \end{bmatrix}) = (\lambda-1)(\lambda+1)+1 = \lambda^2 = (\lambda - 0)(\lambda - 0)$

so the closed-loop eigenvalues are indeed 0 and 0, as required. [2]