UNIVERSITY OF LONDON

IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 1997

BEng Honours Degree in Computing Part I
MEng Honours Degrees in Computing Part I
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the Associateship of the City and Guilds of London Institute

PAPER 1.9

MATHEMATICAL METHODS AND GRAPHICS

Tuesday, May 6th 1997, 10.00 - 12.00

Answer FOUR questions

For admin. only: paper contains 6 questions

1 A three dimensional graphics scene made up of polygons is to be drawn in perspective projection viewed from the origin, with the direction of view along the z-axis.

The viewplane has equation z=10, and the viewing window defining the world coordinate system has corners given by the points:

$\{10,10,10\}$, $\{10,-10,10\}$, $\{-10,10,10\}$ and $\{-10,-10,10\}$.

One of the polygons that makes up the scene has corners at the following three dimensional points:

P0=$\{10,40,50\}$, **P1**=$\{10,-5,50\}$, **P2**=$\{160,40,80\}$ and **P3**=$\{60,54,60\}$.

The scene is to be drawn in a window whose pixel position on the screen is defined by [128..255] in the x direction and [0..127] in the y direction.

a What are the x and y coordinates of the projections of the four points **P0**, **P1**, **P2**, and **P3** onto the viewplane (in world coordinates)?

b Sketch what would be seen in the window on the screen.

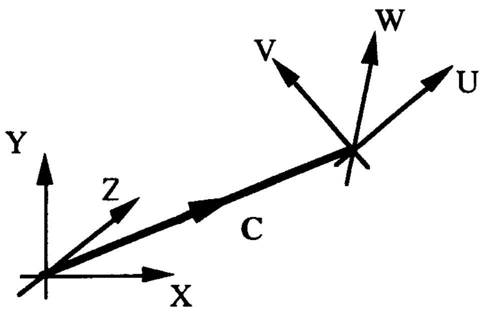
c What is the matrix that calculates the projection, using homogeneous coordinates?

d Calculate the equation pair that carry out the 2D normalisation transformation between the world coordinate system defined by the window, and the actual pixel addresses:

$$\text{Xpix} = A\,x + B$$
$$\text{Ypix} = C\,y + D$$

e Express the transformation of part d as a four by four matrix using homogeneous coordinates.

2 A graphics scene is made up of points defined in an absolute coordinate system denoted {X,Y,Z}. As part of an animation sequence it is to be viewed from a point **C** = {Cx,Cy,Cz} (in the absolute coordinate system). The viewing coordinate system is defined by three unit vectors, **u**,**v** and **w** defined in the absolute coordinate system as indicated in the diagram.



a. Using dot products determine the coordinates of the point P in the {U,V,W} axis system.

b. By expanding the dot products of patr a using the notation **u** = {ux,uy,uz} etc. derive the transformation matrix that will transform the points of the scene from the {X,Y,Z} axis system to the {U,V,W} axis system.

c. Each row of the matrix you have found in part b can be treated as a vector. Explain the meaning of each of these four vectors in terms of the two coordinate systems.

d. After transformation the scene is to be drawn in orthographic projection on the plane W=0. Find the matrix that will first transform the points and then project them.

The four parts carry, respectively, 25%,20%, 40%, 15% of the marks.

Turn over

3a Use Gaussian Elimination to find all solutions to the system of equations

$$\begin{aligned} x + \quad\quad ay \quad\quad &= 0 \\ x + (a+2)y + az &= 5 \\ 2x + \quad\quad ay + az &= 3 - a \end{aligned}$$

in terms of the parameter a. Pay particular attention to any special values of a which may arise.

3b Find the inverse of the matrix

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 2 & -1 & -1 \end{pmatrix}$$

Use this inverse to solve $A\underline{x} = (0, 5, 4)^T$, and compare the solution with part (3a) when $a = -1$.

Parts a and b carry respectively 60% and 40% of the marks

4a Define the eigenvectors and eigenvalues of a matrix A. What can be said about the eigenvalues and eigenvectors of a real, symmetric matrix?

4b Show directly from the definition that $(1, 1, 1)^T$ and $(1, 0, -1)^T$ are eigenvectors of the matrix

$$A = \begin{pmatrix} 3 & 4 & 1 \\ 4 & 0 & 4 \\ 1 & 4 & 3 \end{pmatrix}$$

and give the corresponding eigenvalues.

4c Use the results of (4a) to find the third eigenvector. What is its eigenvalue?

Verify the relation between eigenvalues and determinant for A.

4d For A as above, describe the behaviour as $n \to \infty$ of the vector $\underline{x}_n$ defined by

$$\underline{x}_{n+1} = A\underline{x}_n \qquad \text{with} \quad \underline{x}_0 = (2, 0, -2) + \underline{\varepsilon}_0 \ .$$

where $\underline{\varepsilon}_0$ denotes the small ($|\underline{\varepsilon}_0| \simeq 10^{-16}$) rounding error on a machine of finite precision.

Parts a, b, c, d carry respectively 15%, 15%, 45%, 25% of the marks.

5a If $X = f(X)$ for a differentiable function $f(x)$, prove that a sufficient condition for the iterative scheme

$$x_{n+1} = f(x_n)$$

to converge to X for a sufficiently close initial estimate x_0, is that

$$|f'(X)| < 1 .$$

5b Show that the equation $x = e^{-x}$ is equivalent to $x = -\ln x$, and draw rough graphs to show that this equation has a unique root X with $0 < X < 1$.

5c Discuss whether the two sequences $\{x_n\}$ and $\{y_n\}$ defined by

$$x_{n+1} = e^{-x_n} \qquad \text{and} \qquad y_{n+1} = -\ln y_n \; .$$

with $x_0 = y_0 = 0.5$, are likely to converge to X.

5d Write down a Newton scheme for this problem.

Parts a, b, c, d carry respectively 40%, 15%, 30%, 15% of the marks.

6a The price, P, of hiring a computer depends on c, the amount of CPU time, and s, the amount of storage required, according to the formula

$$P(c, s) = cs(c + 2s)$$

when P, c and s are measured in suitable units.

If P, c and s depend on a parameter N, obtain a relation between $\frac{dP}{dN}$, $\frac{dc}{dN}$ and $\frac{ds}{dN}$.

6b Currently, a program requires $c = 1$ and $s = 1$. It would be possible to rewrite the code to use slightly less storage but more CPU. Obtain a relation between the small changes δc and δs to determine when such a change would be worthwhile.

6c The major part of the program involves the storing and inverting of $N \times N$ matrices. If $c = 1$ and $s = 1$ for $N = 100$, deduce formulae for c and s in terms of N.

Show that currently $\frac{dP}{dN} = 0.22$.

6d Find a stationary point of the function $P(c, s)$, and evaluate the matrix of second derivatives at that point.

Parts a, b, c, d carry respectively 15%, 25%, 35%, 25% of the marks.

