EE4-25
EE9CS1-2

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2012

Corrected Copy

MSc and EEE/ISE PART IV: MEng and ACGI

CA

DESIGN OF LINEAR MULTIVARIABLE CONTROL SYSTEMS

Thursday, 17 May 10:00 am

Q3 – Figure 3 @ 11:21

Time allowed: 3:00 hours

There are SIX questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : I.M. Jaimoukha
Second Marker(s) : E.C. Kerrigan

1. Let the transfer matrix $G(s)$ have a state space realisation

$$G(s) \stackrel{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$$

and let

$$A'Q + QA + C'C = 0$$

for some $Q = Q' \in \mathscr{R}^{n\times n}$.

Suppose that

$$Q = \begin{bmatrix} 0 & 0 \\ 0 & Q_2 \end{bmatrix}$$

where $\mathscr{R}^{n_2\times n_2} \ni Q_2 = Q_2' \succ 0$. Assume that A has no eigenvalues on the imaginary axis.

a) Prove that the state space realisation for the transfer matrix $G(s)$ is unobservable. [4]

b) Obtain a state space realisation for $G(s)$ of order n_2. [4]

c) Prove that the state space realisation in Part (b) is both stable and observable. [4]

d) State a condition on the unobservable part that guarantees that the realisation of $G(s)$ is detectable. [4]

e) Draw a diagram involving two subsystems of $G(s)$ illustrating the observable and unobservable parts. [4]

Hint: You may want to partition the realisation of $G(s)$ compatibly with the partitioning of Q.

2. Suppose that a state-space realisation of a transfer matrix $G(s)$ has the structure

$$G \stackrel{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ 0_2 & A_{22} & 0_2 \\ \hline C_1 & C_2 & 0_2 \end{array}\right]$$

where

- $A_{11} = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$ is stable and $a_2 < a_1$.
- $B_1 B_1' = C_1' C_1 = I_2$.
- $A_{22} = \begin{bmatrix} a_3 & 0 \\ 0 & a_4 \end{bmatrix}$.
- 0_2 (I_2) denotes the 2×2 null (identity) matrix.

a) What are the poles of $G(s)$? [1]

b) Use the PBH test to show that the realisation of $G(s)$ is uncontrollable. What are the uncontrollable modes? [3]

c) Find the input decoupling zeros of $G(s)$. [3]

d) Give necessary and sufficient conditions for the uncontrollable modes to be stabilizable. [2]

e) By removing the uncontrollable modes, obtain a second order realisation of $G(s)$. [2]

f) For the second order realisation of $G(s)$ in Part (e):

   i) Find the controllability and observability Grammians. [2]

   ii) Show that the realization is balanced. [2]

   iii) Evaluate the Hankel singular values. [2]

g) Write B_1 and C_1 as

$$B_1 = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \qquad C_1 = \begin{bmatrix} c_1 & c_2 \end{bmatrix}.$$

Obtain a first order balanced truncation of $G(s)$. [3]

3. Consider the feedback loop shown in Figure 3 where $G(s)$ represents a plant model and $K(s)$ represents an internally stabilizing compensator. Suppose that

$$G(s) \stackrel{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cc|cc} -1 & -1 & 1 & 1 \\ -1 & -1.125 & 0.75 & 0.75 \\ \hline 1 & 0.75 & 0 & 0 \\ 1 & 0.75 & 0 & 0 \end{array}\right] \in \mathcal{RH}_\infty.$$

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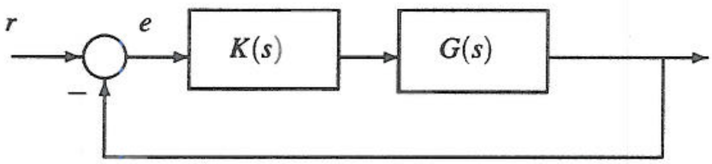

Figure 3

a) Show that the given realization for $G(s)$ is balanced and evaluate the Hankel singular values of $G(s)$. [6]

b) Give a first order balanced truncation of $G(s)$. [4]

c) By using:

- the answer to Part (a),
- the small gain theorem (which should be stated),
- and a parameterization of the set of all internally stabilizing controllers,

design a first order internally stabilizing controller $K(s)$ for $G(s)$ such that the DC gain of the nominal closed–loop is equal to 1. Use the balanced truncation of $G(s)$ in Part (b) above as your nominal plant. [10]

4. Consider the regulator in Figure 4 for which it is assumed that (A,B) is controllable and $x(0)=x_0$. A stabilizing state-feedback gain matrix F is to be designed such that the cost function $J := \int_0^\infty (u(t)'u(t) + z(t)'z(t))\,dt$ is minimized, where (A,C) is assumed to be observable.

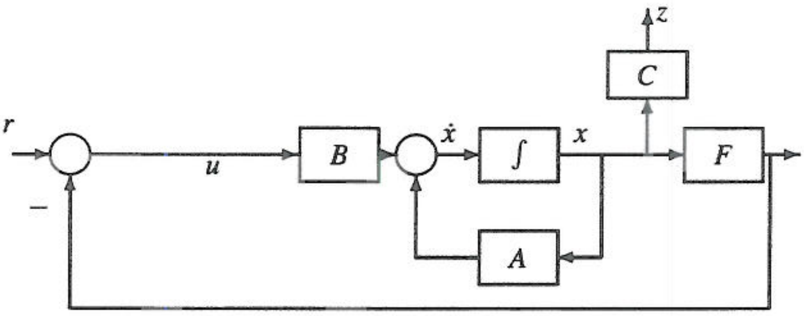

Figure 4

Let $V(t)=x(t)'Px(t)$ where $P=P'$ is the solution of an algebraic Riccati equation.

a) Assuming the closed loop is asymptotically stable, obtain an expression for $\int_0^\infty \dot{V}(t)dt$ in terms of x_0. [5]

b) Find an expression for F that minimizes J. Give also the minimum value of J and the algebraic Riccati equation satisfied by P. [5]

c) Prove that, for F chosen in part (b), the closed loop system in Figure 4 is stable. State clearly the assumption on P required to guarantee stability. [5]

d) Evaluate $T_{zr}(s)$, the transfer matrix from r to z in Figure 4 above. Use

- the Algebraic Riccati equation from Part (b) above,
- the Bounded Real Lemma (given in Question 5 below),
- a Schur complement argument,

to prove that $||T_{zr}||_\infty \leq 1$. [5]

5. Consider a state–variable model described by the dynamics

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t), \end{aligned}$$

and denote the corresponding transfer matrix by $H(s)$.

Assume that A has no unobservable modes on the imaginary axis.

Let $\gamma > 0$ be given and suppose that there exists a $P = P' \succ 0$ such that

$$\begin{bmatrix} A'P + PA + C'C & PB + C'D \\ B'P + D'C & D'D - \gamma^2 I \end{bmatrix} \preceq 0.$$

a) Prove that A is stable. [5]

b) By

- defining a suitable Lyapunov function,
- defining a suitable cost function,
- and using the matrix inequality above,

prove that

$$\|H\|_\infty \leq \gamma.$$

State clearly the assumptions required on $u(t)$, $x(0)$ and $x(\infty)$. [10]

c) Consider the allpass transfer function $G(s) = (s-1)/(s+1)$. Obtain a state-space realization of $G(s)$ and assume that there exists a P satisfying the matrix inequality above. Show that the allpass equations:

$$A'P + PA + C'C = 0, \qquad PB + C'D = 0, \qquad D'D = 1$$

are satisfied. [5]

(Hint: If $\begin{bmatrix} X_{11} & X_{12} \\ X_{12}' & 0 \end{bmatrix}$ *is semidefinite, then* $X_{12} = 0$.*)*

6. Consider the $\mathscr{H}_\infty$ filter for estimating $C_z x$ shown in Figure 6.

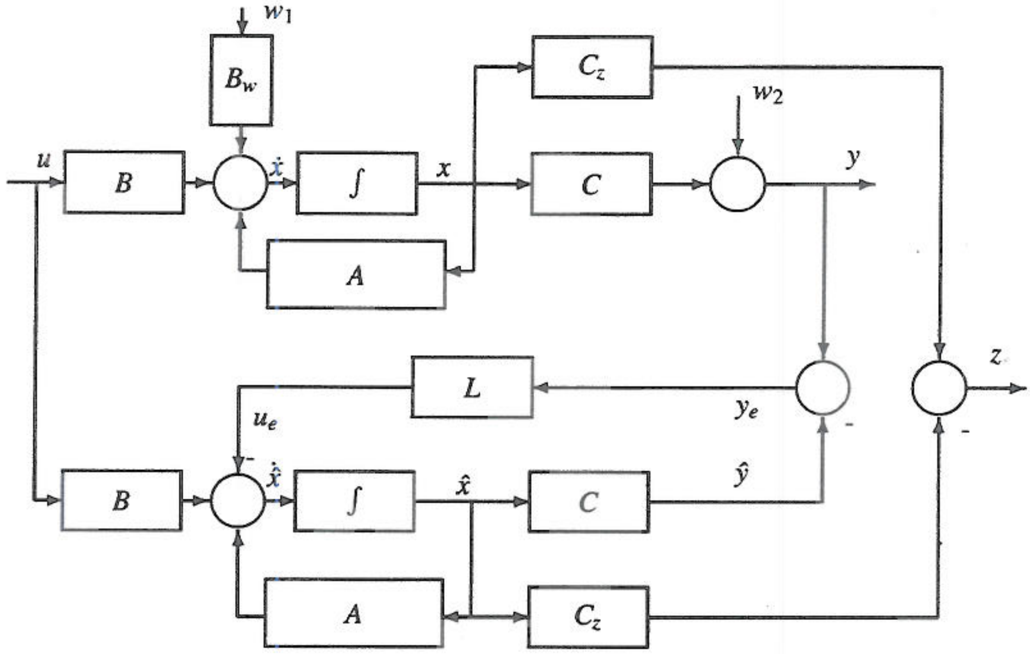

Figure 6

Let $T_{zw}(s)$ denote the transfer matrix from $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ to z. A stabilizing filter gain matrix L is to be designed such that, for $\gamma > 0$, $\|T_{zw}\|_\infty < \gamma$.

a) Write down the generalized regulator system for the filter. Take the state to be $x_e(t) = x(t) - \hat{x}(t)$, the external signal (to be rejected) to be $w(s)$, the actuator input to be $u_e(s)$, the measured output to be $y_e(s)$ and the cost signal to be $z\,(s)$. [4]

b) Use a duality argument to transform the filtering problem into a state-feedback problem. Write down the generalized regulator system for the dual design problem. [8]

c) Derive sufficient conditions for the solution of the design problem. Your conditions should be in the form of the existence of a certain solution to an algebraic Riccati equation. It should also include an expression for L and an expression for the worst-case disturbance w. [8]

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SOLUTIONS: DESIGN OF LINEAR MULTIVARIABLE CONTROL SYSTEMS 2012

1. Let the realisation of $G(s)$ be partitioned compatibly with the partitioning of Q as

$$G(s) \stackrel{s}{=} \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & 0 \end{array}\right].$$

Then

$$A'Q+QA+C'C = \begin{bmatrix} C_1'C_1 & A_{21}'Q_2+C_1'C_2 \\ \star & A_{22}'Q_2+Q_2A_{22}+C_2'C_2 \end{bmatrix} = 0 \qquad (1.1)$$

It follows from the (1,1) entry that $C_1 = 0$. Also, it follows from the (1,2) entry and the assumption that $Q_2 \succ 0$ that $A_{21} = 0$. So, the realisation for $G(s)$ has the form

$$G(s) \stackrel{s}{=} \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ 0 & A_{22} & B_2 \\ \hline 0 & C_2 & 0 \end{array}\right]. \qquad (1.2)$$

a) Applying the PBH test, it is clear that

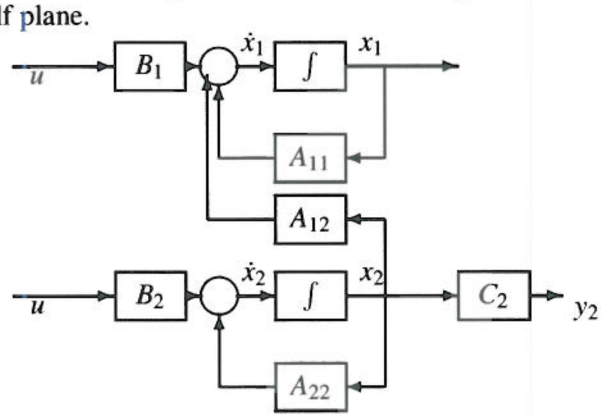
$$\left[\begin{array}{c} A-sI \\ \hline C \end{array}\right] = \left[\begin{array}{cc} A_{11}-sI & A_{12} \\ 0 & A_{22}-sI \\ \hline 0 & C_2 \end{array}\right]$$

loses rank when s is an eigenvalue of A_{11} so the realisation is unobservable. The number of observable states must be n_2.

b) By removing the unobservable part, an order n_2 state space realisation of $G(s)$ is given as $G(s) \stackrel{s}{=} (A_{22}, B_2, C_2, 0)$.

c) Suppose that λ is an eigenvalue of A_{22} and let $z \neq 0$ be the corresponding eigenvector. Then $A_{22}z = \lambda z$. Pre- and post-multiplying the the (2,2) entry of (1.1) by z' and z, respectively, we get $(\lambda+\bar{\lambda})z'Q_2z + z'C_2'C_2z = 0$. If $C_2z = 0$ then $\lambda+\bar{\lambda} = 0$ which contradicts the assumption that A has no eigenvalues on the imaginary axis. Thus the realisation is observable. Since $z \neq 0$ and $Q_2 \succ 0$, $z'Q_2z > 0$ and $z'C_2'C_2z > 0$ then $\lambda+\bar{\lambda} < 0$ and so A_{22} is stable.

d) The unobservable part must be stable in order for the realisation to be detectable, and so a necessary condition is that the eigenvalues of A_{11} are in the open left half plane.

e) 

2. a) Since A is upper block diagonal, the poles of $G(s)$ are a_1, a_2, a_3 and a_4.

b) Applying the PBH test, it is clear that

$$\left[\begin{array}{c|c} A - sI & B \end{array}\right] = \left[\begin{array}{cc|c} A_{11} - sI & A_{12} & B_1 \\ 0_2 & A_{22} - sI & 0_2 \end{array}\right]$$

loses rank when s is an eigenvalue of A_{22} so the realisation is uncontrollable. The uncontrollable modes are therefore the eigenvalues of A_{22} which are a_3 and a_4.

c) The input decoupling zeros are the uncontrollable modes and are therefore a_3 and a_4.

d) The realisation is stabilizable if and only if the uncontrollable modes are stable, equivalently, if and only if $a_3 < 0$ and $a_4 < 0$.

e) By removing the uncontrollable modes, a second order realisation of $G(s)$ is given as

$$G(s) \stackrel{s}{=} \left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & 0_2 \end{array}\right].$$

f) For the second order realisation of $G(s)$ in Part (d):

i) The controllability and observability Grammians are, respectively, the solutions P and Q of the Lyapunov equations

$$A_{11}P + PA_{11}' + B_1B_1' = A_{11}'Q + QA_{11} + C_1'C_1 = 0.$$

Since $A_{11} = \mathrm{diag}(a_1, a_2)$ and $B_1B_1' = C_1'C_1 = I_2$, it follows that $P = Q = \mathrm{diag}(-1/2a_1, -1/2a_2)$.

ii) Since $P = Q$ and is diagonal the realization is balanced.

iii) The Hankel singular values are the eigenvalues of PQ and are therefore $1/2|a_1|$ and $1/2|a_2|$.

g) Since $a_2 < a_1$ we keep the realization corresponding to a_1 and so a first order balanced truncation of $G(s)$ is given by

$$G_b(s) \stackrel{s}{=} \left[\begin{array}{c|c} a_1 & b_1 \\ \hline c_1 & 0_2 \end{array}\right].$$

3. a) The realization of $G(s)$ is balanced if

$$A\Sigma + \Sigma A' + BB' = A'\Sigma + \Sigma A + C'C = 0$$

for $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2) > 0$ and where the $\sigma_i's$ are the Hankel singular values of $G(s)$. A simple calculation gives $\Sigma = \mathrm{diag}(1, 0.5)$.

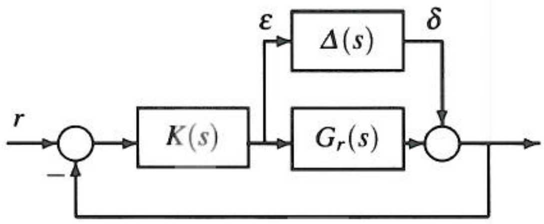
b) A first order balanced truncation of $G(s)$ is given by

$$G_r(s) \stackrel{s}{=} \left[\begin{array}{c|cc} -1 & 1 & 1 \\ \hline 1 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right] = \frac{1}{s+1}\left[\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array}\right].$$

c) Now $G_r(s) = G(s) + \Delta(s)$ where

$$\|\Delta\|_\infty \leq 2\sum_{i=2}^{2} \sigma_i = 1.$$

Then replacing $G(s)$ by $G_r(s)$ in the loop of Figure 3 is equivalent to:



Now

$$\varepsilon = -K(I + G_r K)^{-1}\delta$$

and so the loop is stable if $\|K(I + G_r K)^{-1}\|_\infty < 1$ from the small gain theorem since $\|\Delta\|_\infty \leq 1$. However, the set of all internally stabilizing controllers for $G_r(s)$ is given by:

$$K = Q(I - G_r Q)^{-1}$$

for stable Q. Furthermore,

$$K(I + G_r K)^{-1} = Q.$$

Thus we can take $Q = qI_2$ where q is constant (to guarantee a first order controller) and $|q| < 1$ (to guarantee stabilization of G).

The nominal closed–loop gain is $G_r(s)K(s)\,(I + G_r(s)K(s))^{-1} = G_r(s)Q = G_r(s)q$. For a DC gain of 1, we need

$$||G_r(0)q|| = |q|||\left[\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array}\right]|| = |q|||\left[\begin{array}{c} 1 \\ 1 \end{array}\right]\left[\begin{array}{cc} 1 & 1 \end{array}\right]|| = 2|q| = 1$$

So we choose $|q| = 0.5$. A computation gives

$$K(s) = \frac{1}{4s}\left[\begin{array}{cc} 2s+1 & 1 \\ 1 & 2s+1 \end{array}\right].$$

Note that $K(s)$ has increased the system type since it has a free integrator.

4. a) Let $V = x'Px$ and set $u = -Fx$. Provided that $P = P' \succ 0$ and $\dot{V} < 0$ along closed-loop trajectories, we can assume $\lim_{t\to\infty} x(t) = 0$. Then

$$\dot{V} = \dot{x}'Px + x'P\dot{x} = x'\left(A'P + PA - F'B'P - PBF\right)x.$$

Integrating from 0 to ∞ and using $x(\infty) = 0$,

$$\int_0^\infty x'\left(A'P + PA - F'B'P - PBF\right)x dt = -x_0'Px_0.$$

b) Using the definition of J, adding the last equation and completing a square:

$$J = x_0'Px_0 + \int_0^\infty \{x'[A'P + PA + C'C - PBB'P]x + \|(F - B'P)x\|^2\}dt.$$

Since the last term is always nonnegative, it follows that the minimizing value of F is given by $F = B'P$. We can set the term in square brackets to zero provided P satisfies the Riccati equation,

$$A'P + PA + C'C - PBB'P = 0.$$

It follows that the minimum value of J is $x_0'Px_0$.

c) We need to prove that $A_c := A - BB'P$ is stable. The Riccati equation can be written as $A_c'P + PA_c + C'C + PBB'P = 0$. Let $\lambda \in \mathscr{C}$ be an eigenvalue of A_c and $z \neq 0$ be the corresponding eigenvector. Pre– and post–multiplying the Riccati equation be z' and z respectively gives $(\lambda + \bar{\lambda})z'Pz + z'C'Cz + z'PBB'Pz = 0$. Since $P > 0$ and $z \neq 0$, $z'Pz > 0$, $z'z > 0$ and $z'PBB'Pz \geq 0$. It follows that $\lambda + \bar{\lambda} < 0$ and the closed loop is stable.

d) Since

$$\begin{aligned} \dot{x} &= Ax + Bu = Ax + B(r - Fx) \\ &= (A - BF)x + Br \\ z &= Cx \end{aligned}$$

it follows that $T_{zr} \stackrel{s}{=} (A - BF, B, C, 0)$. It follows from the bounded real lemma that $||T_{zr}||_\infty \leq 1$ if there exists $P = P' \succ 0$ such that

$$\begin{bmatrix} P(A - BF) + (A - BF)'P + C'C & PB \\ B'P & -I \end{bmatrix} \preceq 0$$

Using a Schur complement argument, this inequality is equivalent to

$$P(A - BF) + (A - BF)'P + C'C + PBB'P \preceq 0.$$

However, it follows from the Riccati equation in Part b above that $P(A - BF) + (A - BF)'P + C'C + PBB'P = 0$. This proves that $||T_{zr}||_\infty \leq 1$.

5. a) The (1,1) block of the inequality implies that $A'P+PA+C'C \preceq 0$. Let $z \neq 0$ be a right eigenvector of A and let λ be the corresponding eigenvalue. Then multiplying the inequality from the left by z' and from the right by z gives $(\lambda+\bar{\lambda})z'Pz + z'C'Cz \leq 0$. Since $P \succ 0$ and $z \neq 0$ then $z'Pz > 0$. It follows that $\lambda+\bar{\lambda} \leq 0$, with $\lambda+\bar{\lambda} = 0$ only if $Cz = 0$. However, $Az = \lambda z$, $Cz = 0$ and $\lambda+\bar{\lambda} = 0$ contradicts the assumption that A has no unobservable modes on the imaginary axis. It follows that $\lambda+\bar{\lambda} < 0$ and so A is stable.

b) Since A is stable, $\|H\|_\infty \leq \gamma$ if and only if, with $x(0) = 0$,

$$J := \int_0^\infty [y'y - \gamma^2 u'u]dt \leq 0,$$

for all $u(t)$ such that $\|u\|_2 < \infty$. If $\|u\|_2$ is bounded, then $\lim_{t\to\infty} x(t) = 0$. Define the Lyapunov function $x(t)'Px(t)$. Then,

$$\int_0^\infty \frac{\mathrm{d}}{\mathrm{dt}}[x'Px]dt = x(\infty)'Px(\infty) - x(0)'Px(0) = 0.$$

So,

$$\begin{aligned} 0 &= \int_0^\infty \dot{x}'Px + x'P\dot{x}dt = \int_0^\infty [(Ax+Bu)'Px + x'P(Ax+Bu)]dt \\ &= \int_0^\infty [x'(A'P+PA)x + x'PBu + u'B'Px]dt. \end{aligned}$$

Use $y = Cx+Du$ and add the last expression to J

$$\begin{aligned} J &= \int_0^\infty [x'(A'P+PA+C'C)x + x'(PB+C'D)u + u'(B'Px+D'C) + u'(D'D-\gamma^2 I)u]dt \\ &= \int_0^\infty \{ \begin{bmatrix} x' & u' \end{bmatrix} \begin{bmatrix} A'P+PA+C'C & PB+C'D \\ B'P+D'D & D'D-\gamma^2 I \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \} dt \\ &\leq 0 \end{aligned}$$

from the inequality. It follows that $\|H\|_\infty \leq \gamma$.

c) A state-space realization is given by $G(s) \stackrel{s}{=} (A,B,C,D) = (-1, -\sqrt{2}, \sqrt{2}, 1)$. Since $G(s)$ is allpass $||G||_\infty = 1$ and the matrix inequality becomes

$$\begin{bmatrix} 2(P-1) & \sqrt{2}(P-1) \\ \sqrt{2}(P-1) & 0 \end{bmatrix}$$

It follows that $P = 1$ and so $2(P-1) = 0$. This shows that the allpass equations are satisfied.

6. a) The generalized regulator formulation is given by

$$\begin{bmatrix} z_e(s) \\ y_e(s) \end{bmatrix} = P(s) \begin{bmatrix} w(s) \\ u_e(s) \end{bmatrix}, \; u_e(s) = L y_e(s), P(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{c||c|c|c} A & B_w & 0 & I \\ \hline\hline C_z & 0 & 0 & 0 \\ \hline C & 0 & I & 0 \end{array}\right].$$

b) Taking the transpose of $P(s)$ in Part (a), redefining $A := A'$, $B_w := C_z'$, $B = C'$, $C_z := B_w'$, $F := L^T$, $w := z_e$, $z := w$, $x := x_e$, $y := u_e$ and $u := y_e$ we get the state-feedback problem

$$\begin{bmatrix} z(s) \\ y(s) \end{bmatrix} = P(s) \begin{bmatrix} w(s) \\ u(s) \end{bmatrix}, \; u(s) = F y(s), P(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{c||c|c} A & B_w & B \\ \hline\hline C_z & 0 & 0 \\ 0 & 0 & I \\ \hline I & 0 & 0 \end{array}\right].$$

c) The requirement $\|T_{zw}\|_\infty < \gamma$ is equivalent to $J := \|z\|_2^2 - \gamma^2 \|w\|_2^2 < 0$. Let $V = x'Xx$. Provided that $X = X' > 0$ and $\dot{V} < 0$ along the closed-loop trajectory, we can assume $\lim_{t\to\infty} x(t) = 0$. Then

$$\dot{V} = \dot{x}'Xx + x'X\dot{x} = x'\left(A'X + XA + F'BX + XB'F\right)x + x'XB_w w + w'B_w'Xx.$$

Integrating from 0 to ∞ and using $x(0) = x(\infty) = 0$,

$$0 = \int_0^\infty [x'\left(A'X + XA + F'B'X + XBF\right)x + x'XB_w w + w'B_w'Xx]dt.$$

Using the definition of J and adding the last equation, $J =$

$$\int_0^\infty \{x'[A'X + XA + C_z'C_z + F'F + F'B'X + XBF]x - [\gamma^{-2}w'w - x'XB_w w - w'B_w'Xx]\}dt.$$

Completing the squares by using

$$\begin{aligned} (F + B'X)'(F + B'X) &= F'F + F'B'X + XBF + XBB'X \\ \|(\gamma w - \gamma^{-1}B_w'Xx)\|^2 &= \gamma^2 w'w - w'B_w'Xx - x'XB_w w + \gamma^{-2}x'XB_w B_w'Xx, \end{aligned}$$

$$J = \int_0^\infty \{x'[A'X + XA + C_z'C_z - X(BB' - \gamma^{-2}B_w B_w')X]x + \|(F + B'X)x\|^2 - \left\|\gamma w - \gamma^{-1}B_w'Xx\right\|^2\}dt.$$

Thus two sufficient conditions for $J < 0$ are the existence of X such that

$$A'X + XA + C_z'C_z - X(BB' - \gamma^{-2}B_w B_w')X = 0, \qquad X = X' > 0.$$

The feedback gain is obtained as $F = -B'X$. The worst case disturbance is $w^* = \gamma^{-2}B_w'Xx$. The closed-loop with $u = Fx$ and $w = w^*$ is $\dot{x} = [A - (BB' - \gamma^{-2}B_w B_w')X]x$ and a third condition is $Re\ \lambda_i[A - (BB' - \gamma^{-2}B_w B_w')X] < 0$, $\forall\, i$. It remains to prove $\dot{V} < 0$ for $u = Fx$ and $w = 0$. But

$$\dot{V} = -x'(C_z'C_z + X(BB' + \gamma^{-2}B_w B_w')X)x < 0$$

for all $x \neq 0$ proving closed-loop stability since (A, C_z) is assumed observable.