E4.27
C2.2
ISE4.41

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2006

MSc and EEE/ISE PART IV: MEng and ACGI

Corrected Copy

SYSTEM IDENTIFICATION

Thursday, 11 May 10:00 am

Time allowed: 3:00 hours

There are FIVE questions on this paper.

Answer THREE questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : G. Weiss

Second Marker(s) : S. Evangelou

Special information for invigilators: none

Information for candidates:

$$C(\tau) = E[(u(t)-\mu)(u(t+\tau)-\mu)]$$

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{x^2}{2\sigma^2}} \qquad S_{yy} = |G|^2 S_{uu} \qquad Z_L = sL \qquad Z_c = \frac{1}{Cs}$$

$$\Phi^{\#} = (\Phi^*\Phi)^{-1}\Phi^* \qquad P = \Phi\Phi^{\#} \qquad S = \frac{1}{N-\rho}\|y - \Phi\widehat{\theta}\|^2$$

$$A^d = e^{Ah} \qquad B^d = (e^{Ah}-I)A^{-1}B \qquad G^d(z) \approx G(\tfrac{2}{h}\tfrac{z-1}{z+1}) \qquad G(s) \approx G^d(\tfrac{1+sh/2}{1-sh/2})$$

$$C_k^{uu} g_0 + C_{k-1}^{uu} g_1 + C_{k-2}^{uu} g_2 + \ldots = C_k^{uy}$$

$$\mathrm{Cov}(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$$

$$E(X\cdot Y) = E(X)\cdot E(Y) + \mathrm{Cov}(X,Y)$$

$$\mathrm{Cov}(TX) = T\mathrm{Cov}(X)T^*$$

$$\widehat{v}(z) = \sum_{k=0}^{\infty} v_k z^{-k}$$

$$[(\Delta v)_k = v_{k+1}] \quad \Rightarrow \Delta v(z) = z[\widehat{v}(z) - v_0]$$

$$[u_k = kv_k] \quad \Rightarrow \widehat{u}(z) = -z\frac{d}{dz}\widehat{v}(z)$$

$$[v_k = \sin k\nu] \quad \Rightarrow \widehat{v}(z) = \frac{z\sin\nu}{(z-e^{i\nu})(z-e^{-i\nu})}$$

$$[v_k = \rho^k] \quad \Rightarrow \widehat{v}(z) = \frac{z}{z-\rho}$$

$$\left[v_k = \frac{1}{\rho}k\rho^k\right] \quad \Rightarrow \widehat{v}(z) = \frac{z}{(z-\rho)^2}$$

$$P_n = \frac{1}{\lambda}\left[P_{n-1} - \frac{P_{n-1}\varphi_n^*\varphi_n P_{n-1}}{\lambda + \varphi_n P_{n-1}\varphi_n^*}\right]$$

$$\varepsilon_n = y_n - \varphi_n\hat{\theta}_{n-1}$$

$$\hat{\theta}_n = \hat{\theta}_{n-1} + P_n\varphi_n^*\varepsilon_n$$

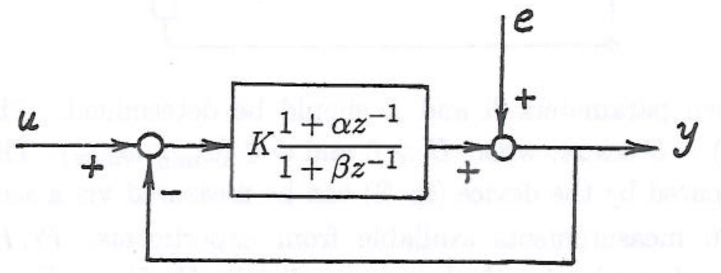
$$y_k + a_1 y_{k-1} \ldots + a_n y_{k-n} = b_0 u_k + b_1 u_{k-1} \ldots + b_n u_{k-n} + e_k + c_1 e_{k-1} \ldots + c_n e_{k-n}$$

$$C(z) = 1 + c_1 z^{-1} \ldots + c_n z^{-n}$$

$$\hat{u}^F = C^{-1}\hat{u}, \qquad \hat{y}^F = C^{-1}\hat{y}$$

$$\overline{y_k} = (c_1 - a_1)y_{k-1}^F + (c_2 - a_2)y_{k-2}^F \ldots + (c_n - a_n)y_{k-n}^F + b_0 u_k^F + b_1 u_{k-1}^F \ldots + b_n u_{k-n}^F$$

1. Consider the discrete-time feedback system shown in the figure below, with input signal u, output signal y and a white noise disturbance e with $E(e_k) = 0$. This disturbance e incorporates also the effects of measurement and modeling errors. It is known that $K > 0$, $|\alpha| < 1$ and $|\beta| < 1$, but otherwise K, α and β are unknown.



(a) Compute the transfer functions from u to y and from e to y. For which values of K, α and β are these transfer functions stable? Hint: show that the closed-loop pole is on a straight line segment connecting α and β. [4]

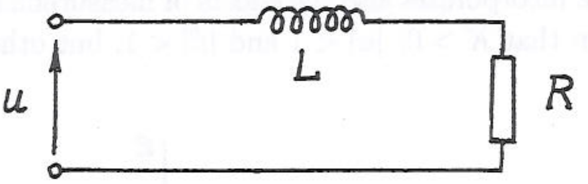
(b) Write an ARMAX difference equation which can be used to estimate K, α and β from the measurements of u_k and y_k for $k = 0, 1, 2, \ldots N$. Hint: you may introduce other variables instead of u, y or e if this is more convenient. [4]

(c) Describe a least squares based method for estimating K, α and β from the measurements described in part (b), using the ARMAX equation derived in part (b). Hint: it may be useful to work in several steps, and in each of these steps to transform the ARMAX equation into an ARX equation with the same unknown coefficients, by introducing new variables. [4]

(d) Briefly describe the method called *pseudolinear regression* to estimate the white noise signal e and then the unknown parameters K, α, β (based on the measurements of u_k and y_k for $k = 0, 1, 2, \ldots N$). [4]

(e) Assuming that K, α and β have been found, how can we approximate the transfer function from u to y by an FIR transfer function of order 15? Hint: introduce convenient notation which allows you to formulate a simple answer. [4]

2. A microwave device can be modeled in a certain frequency band $(\omega_{\min}, \omega_{\max})$ by the following simple circuit:



The unknown parameters R and L should be determined. The driving voltage is $u(t) = U \cos \omega t$, where $U > 0$ and $\omega \in (\omega_{\min}, \omega_{\max})$. The average power P dissipated by the device (in R) can be measured via a sensor.

We have 20 measurements available from experiments, $P_1, P_2, \ldots P_{20}$, which correspond to u having the known amplitudes $U_1, U_2, \ldots U_{20}$ (not zero) and the known frequencies $\omega_1, \omega_2, \ldots \omega_{20}$. Naturally, the data will not fit our simple model exactly, no matter how we choose R and L.

(a) By defining new variables if necessary, rewrite the model of the system in the form $y_k = \varphi_k \theta + e_k$, where y_k and φ_k are known, θ is the vector of unknown parameters and e_k are the equation errors. [4]

(b) Write the formula for the vector of estimated parameters $\hat{\theta}$ which minimizes $e_1^2 + e_2^2 \ldots + e_{20}^2$. Indicate how we can derive estimates for R and L from $\hat{\theta}$. [3]

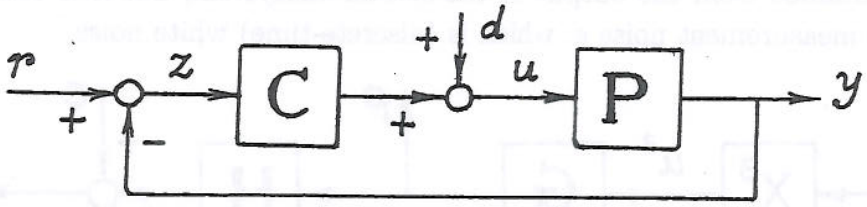
(c) Is it possible to estimate θ by the formula required in part (b), if the amplitudes are all equal: $U_1 = U_2 \ldots = U_{20}$ (but the frequencies are different)? Is it possible to estimate θ by the formula required in part (b), if the frequencies are all equal: $\omega_1 = \omega_2 \ldots = \omega_{20}$ (but the amplitudes are different)? [3]

(d) Assume that $e_1, e_2, \ldots e_{20}$ are independent and identically distributed. Assuming also that $E(e_k) = 0$, give a formula for an unbiased estimate of $Var(e_k)$. [3]

(e) Still assuming independent and identically distributed equation errors, give a formula for an unbiased estimate of $Cov(\hat{\theta})$, where $\hat{\theta}$ is the estimate from part (b). Note that $Var(e_k)$ is not known, but it can be estimated, as was required in part (d). [3]

(f) Now assume that the power sensor has an unknown gain $k > 0$, so that we are measuring kP (instead of P). We may connect a known resistor R_0 in series with the microwave device, for a part of our 20 measurements. (The voltage u is applied to this series connection.) Modify your answer to part (a) so that from the model we can find also the unknown parameter k (in addition to R and L). [4]

3. An unstable linear SISO plant with an unknown transfer function $\mathbf{P}$ is connected to a stabilizing controller as shown in the diagram below. The reference signal r is sinusoidal with a frequency of 50 Hz and d is a random disturbance. At frequencies higher than 1000 Hz the gain of the plant is practically zero. We assume that $\mathbf{P}(i100\pi) \neq 0$.



We use a controller with the transfer function

$$\mathbf{C}(s) = K_p + \frac{K_m s}{s^2 + (100\pi)^2},$$

where the controller gains K_p, K_m have to be tuned. It is known from experiments that for $K_p = 3$ and $K_m = 10$, this feedback system is stable.

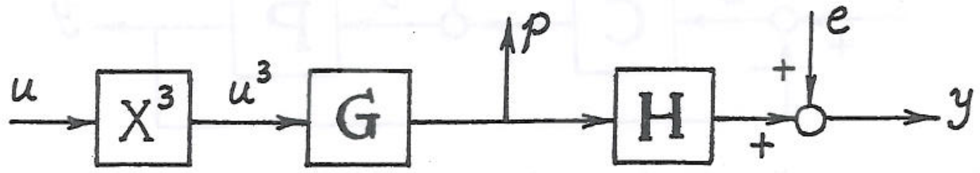
(a) Compute $\mathbf{S}$, the transfer function from r to z, in terms of K_p, K_m and $\mathbf{P}$. If K_p and K_m are such that the feedback system is stable, $d = 0$ and $r(t) = R\cos(\omega t)$, where $R \geq 0$, describe the behaviour of $z(t)$ for large $t > 0$. Comment in particular about the case when the frequency of r is 50 Hz. [5]

(b) In order to tune the controller, we would need an approximate Bode plot of $\mathbf{P}$. Describe identification experiments which can provide us with the necessary data for the Bode plot. For these experiments, we can make $d = 0$, we may use the controller and we can generate any bounded signal r. Describe briefly the computations necessary to process the measurement data. [5]

(c) Assume that the feedback system is stable, r and d are independent stationary random signals with expectations $E(r(t)) = E(d(t)) = 0$ and known power spectral densities S^{rr} and S^{dd}. Is z a stationary random signal? Compute $E(z(t))$ and write the formulas needed for computing $\mathrm{Var}\,(z(t))$ (the power of z), in terms of $\mathbf{P}$, $\mathbf{C}$, S^{rr} and S^{dd} (do not do any computation). [5]

(d) If the feedback system is stable, $r(t) = R\cos(100\pi t)$ for all $t \in \mathbb{R}$, where $R > 0$ and d is as in part (c), compute $E(y(t))$ and $\mathrm{Var}\,(y(t))$ in terms of $\mathbf{P}$, $\mathbf{C}$ and S^{dd}. Is y a stationary random signal? Give a brief reasoning. Hint: use your answer to part (a). [5]

4. We know that a nonlinear discrete-time system Σ is formed by the cascade connection of two stable LTI subsystems, and the input to the first linear subsystem is u^3, where u is the input signal of Σ. In the block diagram of Σ, shown below, the block marked X^3 is a static cubing block. The transfer functions of the LTI subsystems are denoted by **G** and **H**. The output y of Σ is obtained from the output of the second subsystem, but it is corrupted by the measurement noise e, which is (discrete-time) white noise.



Suppose that u is (discrete-time) white noise independent of e. The values u_k, p_k and y_k have been observed for $k = 0, 1, \ldots 10,000$. Based on these data, we would like to estimate the transfer functions **G** and **H**.

(a) Is u^3 stationary? Describe a method for estimating $E(u_k^3)$ and $C_\tau^{u^3u^3}$ (the autocorrelation function of u^3) for $\tau = 0, 1, \ldots 100$. [3]

(b) Describe a method for estimating the auto-correlation function C_τ^{pp} and the cross-correlation functions $C_\tau^{u^3p}$ and C_τ^{py} for $\tau = 0, 1, \ldots 100$. Explain very briefly how this problem is related to the concept of ergodicity. Is y ergodic? [3]

(c) Describe a method for estimating the terms $g_0, g_1, \ldots g_{100}$ in the impulse response of **G** from the results of parts (a) and (b). [3]

(d) Describe a method for estimating the terms $h_0, h_1 \ldots h_{100}$ in the impulse response of **H** from the results of part (b), and explain briefly how this method is derived from the properties of C_τ^{pp} and C_τ^{py}. [5]

(e) Having estimated $g_0, g_1, \ldots g_{100}$ from part (c), how can we build a FIR filter whose transfer function is a good approximation to **G**? Write the corresponding difference equation. [2]

(f) Express S^{yy}, the power spectral density of y, in terms of $E(u_k^3)$, $E(u_k^6)$, $E(e_k)$, $E(e_k^2)$, **G** and **H**. [4]

5. Consider the discrete-time LTI system with input u and output y described by the equations

$$5y_k - 8y_{k-1} + 3.15y_{k-2} = q_{k-1},$$
$$q_k = -3u_k + 5u_{k-1}.$$

(a) Compute the transfer function $\mathbf{G}$ of this system. [2]

(b) Determine if $\mathbf{G}$ is a stable transfer function, and compute its DC-gain. Is $\mathbf{G}$ proper? Is it strictly proper? Is it FIR? [2]

(c) Consider the signal u given by $u_0 = 0$ and

$$u_k = \frac{7}{10^k} - 6\delta_1 \quad (k = 1, 2, 3, \dots), \tag{2}$$

where δ_1 denotes the discrete-time unit pulse at $k = 1$. Compute the $\mathcal{Z}$-transform of u, denoted $\hat{u}(z)$. [2]

(d) Let y be the response of the system to the input signal u given in part (c). Assume that the initial state of the system is zero. Compute the $\mathcal{Z}$-transform $\hat{y}(z)$. [2]

(e) Explain why the signal y from (d) is of the form

$$y_k = c_1(0.7)^k + c_2(0.9)^k + c_3(0.1)^k \quad \text{for all} \quad k \geq 2.$$

Explain briefly how the constants c_1, c_2, c_3 can be computed, but do not compute them numerically. [5]

(f) Assume that the discrete-time transfer function $\mathbf{G}$ given above has been obtained by a discrete-time identification procedure applied to a continuous-time LTI system, via sample and hold blocks (i.e., D/A and A/D converters) with a sampling frequency of 10 kHz. Make an estimate of the transfer function $\mathbf{P}$ of the continuous-time system, which should be valid for frequencies that are significantly lower than the sampling frequency. [2]

(g) Suppose that the output measurements of the above system are subject to measurement errors, in that

$$\hat{y}(z) = \mathbf{G}(z)\hat{u}(z) + \hat{e}(z),$$

where $\hat{e}$ is the $\mathcal{Z}$-transform of the sequence e_k which is white noise with $E(e_k) = 2$ and $E(e_k^2) = 9$. Given measurements $y_1, y_2, \dots y_{300}$, how can you compute an unbiased prediction of y_{301}? How large is the variance of the prediction error? [5]

[END]

SYSTEMS IDENTIFICATION, May 2006
Solutions

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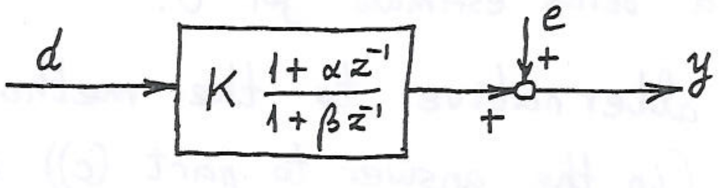
Question 1. (a) $$T_{yu}(z) = \frac{K\cdot\frac{1+\alpha z^{-1}}{1+\beta z^{-1}}}{1+K\cdot\frac{1+\alpha z^{-1}}{1+\beta z^{-1}}}$$

$$= \frac{K(1+\alpha z^{-1})}{(1+K)(1+\rho z^{-1})}, \text{ where } \rho = \frac{\beta + K\alpha}{1+K},$$

$$T_{ye}(z) = \frac{1}{1+K\cdot\frac{1+\alpha z^{-1}}{1+\beta z^{-1}}} = \frac{1+\beta z^{-1}}{(1+K)(1+\rho z^{-1})}.$$

Since $\rho = \frac{1}{1+K}\beta + \frac{K}{1+K}\alpha$ is a convex combination of α and β, which are both in the open unit disk $\mathbb{D}$, we have that also $\rho \in \mathbb{D}$. Hence, T_{yu} and T_{ye} are stable regardless of the choice of $K>0$, $\alpha,\beta\in\mathbb{D}$.

(b) Introduce $d = u - y$ (this is the signal at the output of the comparator), d is measurable (since u and y are). Then we have the output identification problem



which corresponds to the equation

$$(1+\beta z^{-1})\hat{y}(z) = K(1+\alpha z^{-1})\hat{d}(z) + (1+\beta z^{-1})\hat{e}(z).$$

If we denote $a_1 = \beta$, $b_0 = K$, $b_1 = K\alpha$, then we can write this as an ARMAX equation:

$$y_k + a_1 y_{k-1} = b_0 d_k + b_1 d_{k-1} + e_k + a_1 e_{k-1}.$$

(c) Denote $w_k = e_k + a_1 e_{k-1}$, then w is a (non-white) stationary ergodic random signal with $E(w_k) = 0$.

From

$$y_k = -a_1 y_{k-1} + b_0 d_k + b_1 d_{k-1} + w_k ,$$

denoting $\varphi_k = [-y_{k-1} \;\; d_k \;\; d_{k-1}]$, $\theta = \begin{bmatrix} a_1 \\ b_0 \\ b_1 \end{bmatrix}$,

we get $y_k = \varphi_k \theta + w_k$. From here, with $\Phi = \begin{bmatrix} \varphi_1 \\ \vdots \\ \varphi_N \end{bmatrix}$, $y = \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix}$

we construct the initial least squares estimate

$$\theta^e = (\Phi^* \Phi)^{-1} \Phi^* y = \begin{bmatrix} a_1^e \\ b_0^e \\ b_1^e \end{bmatrix},$$

which is unbiased, but may be inaccurate (i.e., with large variance of the prediction error).

We introduce the filtered signals

$$\hat{y}^F(z) = \frac{1}{1 + a_1^e z^{-1}} \hat{y}(z), \quad \hat{d}^F(z) = \frac{1}{1 + a_1^e z^{-1}} \hat{d}(z),$$

then the ARMAX equation becomes

$$y_k^F + a_1 y_{k-1}^F = b_0 d_k^F + b_1 d_{k-1}^F + v_k ,$$

where $\hat{v}(z) = \dfrac{1 + a_1 z^{-1}}{1 + a_1^e z^{-1}} \hat{e}(z)$. Since v is a good approximation to white noise, we may repeat the standard least squares algorithm for this equation to get a better estimate for θ.

(d) An alternative to the method described above (in the answer to part (c)) is to estimate the white noise signal e first. For this, divide the ARMAX equation on the bottom of p. 1 (written in the Z-transform domain) by $1 + a_1 z^{-1}$. Since $(1 + a_1 z^{-1})^{-1}$ is stable, we can approximate $(b_0 + b_1 z^{-1})/(1 + a_1 z^{-1})$ by a high order FIR transfer function (by truncating high powers of z^{-1}):

$$\frac{b_0 + b_1 z^{-1}}{1 + a_1 z^{-1}} \approx \beta_0 + \beta_1 z^{-1} + \beta_2 z^{-2} \ldots + \beta_n z^{-n}$$

(we remark that $\beta_0 = b_0$). Thus we get the X equation

$$y_k = \beta_0 d_k + \beta_1 d_{k-1} + \beta_2 d_{k-2} \ldots + \beta_n d_{k-n} + e_k .$$

Here we can estimate the coefficients β_j by the usual least squares method, based on the available data on d and y. After this, from the X equation we can compute the estimated values of $e_0, e_1, e_2, \ldots e_N$. (The first few terms will be corrupted by our lack of knowledge of the correct initial conditions for d.)

After having estimated e_k (as described), we go back to the original ARMAX equation, which we write as

$$y_k = \underbrace{\begin{bmatrix} e_{k-1} - y_{k-1} & d_k & d_{k-1} \end{bmatrix}}_{\varphi_k} \cdot \begin{bmatrix} a_1 \\ b_0 \\ b_1 \end{bmatrix} + \underbrace{e_k + \varepsilon_k}_{\delta_k} .$$

Here, ε_k is the equation error due to the various approximations that we made. We regard $\delta_k = e_k + \varepsilon_k$ as the total equation error that should be minimized (in norm). This will give us an estimate of a_1, b_0 and b_1.

(e) $T_{yu}(z) = \frac{K}{1+K}(1+\alpha z^{-1}) \cdot \frac{1}{1+pz^{-1}}$.

For $|z| > |p|$ we have

$$\frac{1}{1+pz^{-1}} = 1 - pz^{-1} + p^2 z^{-2} - p^3 z^{-3} + \ldots$$

Multiply this series with $\frac{K}{1+K}(1+\alpha z^{-1})$ and retain the first 16 terms only:

$$T_{yu}^a(z) = g_0 + g_1 z^{-1} + g_2 z^{-2} \ldots + g_{15} z^{-15}.$$

Then T_{yu}^a is a FIR approximation of T_{yu}.

<u>Question 2.</u> (a) The current through the device is

$$i(t) = \frac{U}{|R+i\omega L|}\cos(\omega t+\varphi), \quad \text{where } \varphi = -\arg(R+i\omega L)$$

(this φ is irrelevant for us), so that, denoting $T = \frac{2\pi}{\omega}$,

$$P = R\cdot \underbrace{\frac{1}{T}\int_0^T i(t)^2\,dt}_{i_{rms}^2} = \frac{RU^2}{2(R^2+\omega^2L^2)}.$$

Hence,
$$\frac{1}{2P} = \frac{R}{U^2} + \frac{\omega^2 L^2}{RU^2}.$$

Denoting $y_k = \frac{1}{2P_k}$, $\varphi_k = \begin{bmatrix} \frac{1}{U_k^2} & \frac{\omega_k^2}{U_k^2}\end{bmatrix}$, $\theta = \begin{bmatrix} R \\ \frac{L^2}{R}\end{bmatrix}$,

we get the usual $\quad y_k = \varphi_k\theta + e_k, \quad k = 1,2,\dots 20.$

(b) If $\Phi^*\Phi$ is invertible, then

$$\hat{\theta} = (\Phi^*\Phi)^{-1}\Phi^* y, \quad \text{where } \Phi = \begin{bmatrix}\varphi_1\\ \vdots \\ \varphi_{20}\end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ \vdots \\ y_{20}\end{bmatrix}.$$

Clearly $\hat{R} = \hat{\theta}_1$, while from $\frac{\hat{L}^2}{\hat{R}} = \hat{\theta}_2$ we get $\hat{L} = \sqrt{\hat{\theta}_1\hat{\theta}_2}$.

(c) With $U_1 = U_2 \dots = U_0$ but different frequencies, it will normally work. However, if $\omega_1 = \omega_2 \dots = \omega_{20}$, then the two columns of Φ are linearly dependent, so that $\Phi^*\Phi$ (having rank 1) cannot be invertible.

(d) $$\widehat{\mathrm{Var}}(e_k) = \frac{1}{18}\|y - \Phi\hat{\theta}\|^2 = \frac{1}{18}\, y^*\left[I - \Phi(\Phi^*\Phi)^{-1}\Phi^*\right]y.$$

$18 = 20-2$

(e) $\widehat{\mathrm{Cov}}(\hat{\theta}) = \widehat{\mathrm{Var}}(e_k)(\Phi^*\Phi)^{-1}$, where $\widehat{\mathrm{Var}}(e_k)$ has been computed in part (d).

(f) The current with R_0 in series is

$$i(t) = \frac{U}{|R+R_0+i\omega L|} \cos(\omega t + \varphi), \text{ whence}$$

(φ is irrelevant)

$$P = R \frac{1}{T} \int_0^T i(t)^2 \, dt = \frac{RU^2}{2((R+R_0)^2 + \omega^2 L^2)},$$

($T = 2\pi/\omega$)

$$\frac{1}{2P} = \frac{(R+R_0)^2}{RU^2} + \frac{\omega^2 L^2}{RU^2}.$$

(The measured quantity is kP, where $k>0$ is unknown.)

$$\frac{1}{2kP} = \frac{R}{k} \cdot \frac{1}{U^2} + \frac{1}{k} \cdot \frac{2R_0}{U^2} + \frac{L^2}{kR} \cdot \frac{\omega^2}{U^2} + \frac{R_0^2}{kRU^2}$$

$$= \begin{bmatrix} \frac{1}{U^2} & \frac{2R_0}{U^2} & \frac{\omega^2}{U^2} \end{bmatrix} \cdot \underbrace{\begin{bmatrix} \frac{R}{k} \\ \frac{1}{k} + \frac{R_0}{2kR} \\ \frac{L^2}{kR} \end{bmatrix}}_{\Theta}$$

(Note that k, R, L can be obtained from θ.)

We denote

$$\varphi_j = \begin{bmatrix} \frac{1}{U_j^2} & \frac{2R_{0j}}{U_j^2} & \frac{\omega_j^2}{U_j^2} \end{bmatrix} \quad (j = 1, 2, \dots 20),$$

where $R_{0j} = R_0$ if the additional resistor is connected in the j-th experiment, $R_{0j} = 0$ if it is not connected. Then, with $y_j = \frac{1}{2kP_j}$, we have $y_j = \varphi_j \theta + e_j$. Since R_0 is sometimes connected and sometimes not, the columns of Φ will be independent.

Question 3. (a) $S(s) = [1 + P(s) C(s)]^{-1}$

$$= \frac{s^2 + (100\pi)^2}{[s^2 + (100\pi)^2]\cdot[1 + K_p P(s)] + K_m P(s) s}.$$

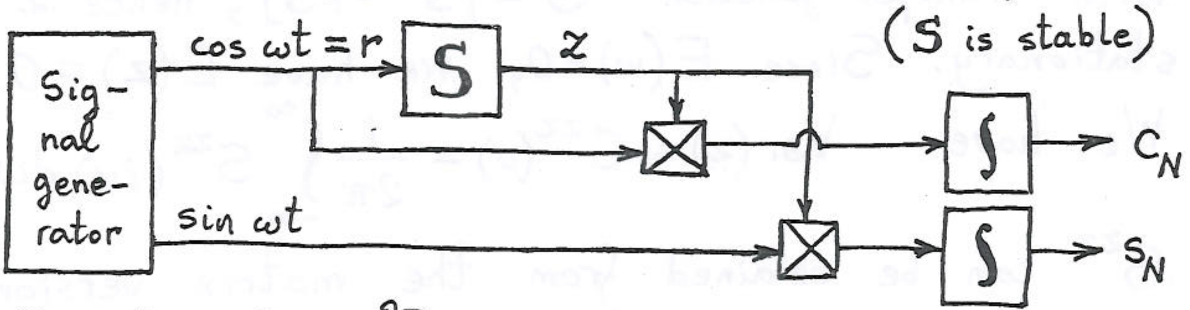
If $d = 0$ and $r(t) = R \cos \omega t$, then for large t we have $z(t) \approx Z \cos(\omega t + \varphi_\omega)$, where

$$Z = |S(i\omega)| \cdot R, \qquad \varphi_\omega = \arg S(i\omega).$$

For 50Hz we have $\omega = 100\pi$ so that (using $P(i100\pi) \neq 0$), $S(i\omega) = 0$. In this particular case $z(t) \to 0$, which is called asymptotic tracking.

(b) The identification experiments could be done separately for a number of different frequencies ω, as follows (making $d = 0$):

(S is stable)



Denoting $T = \frac{2\pi}{\omega}$, we compute for large $t_0 > 0$

$$C_N = \int_{t_0}^{t_0+NT} z(t) \cos \omega t \, dt = A_\omega \cos \varphi_\omega \frac{NT}{2},$$

$$S_N = \int_{t_0}^{t_0+NT} z(t) \sin \omega t \, dt = -A_\omega \sin \varphi_\omega \frac{NT}{2}.$$

From here we can compute A_ω and φ_ω, and hence $S(i\omega) = A_\omega e^{i\varphi_\omega}$.

Once we have estimated $\mathbf{S}(i\omega)$, we can obtain

$$\mathbf{P}(i\omega) = \frac{1}{\mathbf{C}(i\omega)} \left(\mathbf{S}^{-1}(i\omega) - 1 \right).$$

This formula will lead to large errors if $\mathbf{C}(i\omega)$ is very small (in which case $\mathbf{S}(i\omega) \approx 1$), i.e., a small relative error in estimating $\mathbf{S}(i\omega)$ will be amplified and become a large relative error in the estimate of $\mathbf{P}(i\omega)$. If $\mathbf{C}(i\omega)$ is very large, then the signal z will be very small, and this will increase the likelihood of errors corrupting the measurement of $\mathbf{S}(i\omega)$. The remarks on errors in this paragraph are not part of the solution.

(c) Denote $w = \begin{bmatrix} r \\ d \end{bmatrix}$. z is obtained by passing the stationary signal w through a stable LTI system with transfer function $G = [S \ \ -PS]$, hence z is stationary. Since $E(w) = 0$, we have $E(z) = 0$. We have $Var(z) = C^{zz}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S^{zz}(i\omega)\, d\omega$.

S^{zz} can be obtained from the matrix version of the Wiener-Lee formula: $S^{zz} = G S^{ww} G^*$, where $S^{ww} = \begin{bmatrix} S^{rr} & 0 \\ 0 & S^{dd} \end{bmatrix}$ (the zeros in S^{ww} appear because r, d are independ.).

(d) $\hat{z}(i\omega) = \mathbf{S}(i\omega)\hat{r}(i\omega) - \mathbf{P}(i\omega)\mathbf{S}(i\omega)\hat{d}(i\omega)$.
Since $\mathbf{S}(100\pi i) = 0$, the first term is zero. Indeed, this follows from what we said in our answer to part (a), since now $Z = |S(100\pi i)| \cdot R = 0$. Since

we now consider that time has started at $-\infty$, we are already in steady state, i.e., there is no transient component in $S(i\omega)\hat{r}(i\omega)$, so that $S(i\omega)\hat{r}(i\omega)=0$. (A direct verification of this is mathematically more problematic, since $S(\pm 100\pi i)=0$ and $\hat{r}$ is a distribution consisting of two "delta functions" located at $\pm 100\pi i$). We have got

$$\hat{z}(i\omega) = -P(i\omega)\, S(i\omega)\, \hat{d}(i\omega).$$

This implies $E(z(t)) = -P(0)\, S(0)\, \underbrace{E(d(t))}_{0} = 0.$

Since $z(t) = r(t) - y(t)$, it follows that $\boxed{E(y(t)) = r(t) = R\cos(100\pi t).}$

This depends on t, so that <u>y is not stationary</u>. We have

$$\begin{aligned} \operatorname{Var}(y(t)) &= E\left(\left[y(t) - E(y(t))\right]^2\right) \\ &= E\left(\left[y(t) - r(t)\right]^2\right) \\ &= E\left(\left[z(t)\right]^2\right) = \operatorname{Var}(z(t)) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} S^{zz}(i\omega)\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|P(i\omega)S(i\omega)\right|^2 \cdot S^{dd}(i\omega)\, d\omega \end{aligned}$$

so that

$$\boxed{\operatorname{Var}(y(t)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\frac{P(i\omega)}{1+P(i\omega)C(i\omega)}\right|^2 S^{dd}(i\omega)\, d\omega.}$$

Question 4. (a) Since $\dots u_{-1}, u_0, u_1, u_2, \dots$ are i.i.d. (independent and identically distributed), the same is true for $\dots u_{-1}^3, u_0^3, u_1^3, u_2^3, \dots$ (or any other function of u). Thus, u^3 is white noise, in particular, it is stationary and ergodic. Based on ergodicity, we can estimate ($\substack{\text{a hat denotes}\\ \text{an estimate}}$)

$$\hat{E}(u_k^3) = \frac{1}{N+1}\sum_{k=0}^{N} u_k^3, \quad \widehat{Var}(u_k^3) = \frac{1}{N}\sum_{k=0}^{N}\left(\widetilde{u_k^3}\right)^2,$$

where $N = 10{,}000$ and $\widetilde{u_k^3} = u_k^3 - \widehat{E(u_k^3)}$. Since u^3 is white noise, its autocorrelation function is estimated as $\widehat{C}_\tau^{u^3u^3} = \widehat{Var}(u_k^3)\cdot\delta_0(\tau)$,

where δ_0 is the discrete unit pulse, $\delta_0(\tau) = \begin{cases} 1 & \text{for } \tau = 0, \\ 0 & \text{else.} \end{cases}$

(b) Since u^3 is white noise, it is ergodic. Hence, the random signal $[u^3 \;\; p \;\; q]^T$, where q is the output signal of $\mathbf{H}$, is also ergodic. Since $y = q + e$, where e is again white noise, independent of u^3, we get that also $[u^3 \;\; p \;\; y]^T$ is (stationary and) ergodic. In particular, it follows that we can estimate $C_\tau^{u^3p}$ and C_τ^{py} (for $\tau = 0, 1, \dots 100$) by

$$\widehat{C}_\tau^{u^3p} = \frac{1}{N-\tau+1}\sum_{k=0}^{N-\tau} \widetilde{u_k^3}\,\tilde{p}_{k+\tau}, \quad \widehat{C}_\tau^{py} = \frac{1}{N-\tau+1}\sum_{k=0}^{N-\tau} \tilde{p}_k\,\tilde{y}_{k+\tau},$$

where $\widetilde{u_k^3}$ has been defined earlier, and similarly $\tilde{p}_k = p_k - \hat{E}(p_k)$, $\tilde{y}_k = y_k - \hat{E}(y_k)$. The same applies to $\hat{C}_\tau^{pp}$. (Obviously, y is ergodic, being a component of $[u^3 \;\; p \;\; y]^T$.)

(c) Since $C^{pu^3} = g * C^{u^3u^3}$ and

$C^{u^3u^3} = \sigma^2 \delta_0$, where $\sigma^2 = \mathrm{Var}(u_k^3)$,

we get $\boxed{C^{pu^3} = \sigma^2 \cdot g}$. We can estimate both C^{pu^3} and σ^2 (see the answer to parts (a) and (b)), and then we can obtain an estimate of g from the framed formula above.

(d) Denote again by q the output signal of the subsystem with transfer function $\mathbf{H}$. Then

$C^{qp} = h * C^{pp}$. Since $y = q + e$, we have $C^{yp} = C^{qp} + C^{ep}$. Since p is obtained from u and e is independent of u, we have that e is independent of p, hence $C^{ep} = 0$. Thus, $C^{yp} = h * C^{pp}$, or, written in matrix form,

$$\begin{bmatrix} C_0^{pp} & C_1^{pp} & C_2^{pp} & \cdots \\ C_1^{pp} & C_0^{pp} & C_1^{pp} & \cdots \\ C_2^{pp} & C_1^{pp} & C_0^{pp} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} \cdot \begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} C_0^{yp} \\ C_1^{yp} \\ C_2^{yp} \\ \vdots \end{bmatrix} \quad \left(\text{Remember that } C_k^{yp} = C_{-k}^{py}\right).$$

Since $\mathbf{H}$ is stable, we have $h_k \to 0$. We truncate the above infinite system of equations by keeping only the first 101 equations and the first 101 unknowns $h_0, h_1, \ldots h_{100}$. We replace C_τ^{pp} and C_τ^{yp} (which are not known) by their estimates, see part (b), and then we solve for $h_0, h_1, \ldots h_{100}$.

(e) The approximate transfer function is

$$G^a(z) = \hat{g}_0 + \hat{g}_1 z^{-1} + \hat{g}_2 z^{-2} \dots + \hat{g}_{100} z^{-100}.$$

This corresponds to the MA equation

$$p_k = \hat{g}_0 v_k + \hat{g}_1 v_{k-1} + \hat{g}_2 v_{k-2} \dots + \hat{g}_{100} v_{k-100}.$$

Here, v denotes the input signal of the FIR filter, p is the output signal, and $\hat{g}_k$ the estimate of g_k, $k = 0, 1, \dots 100$.

(f) We have $\hat{y} = \mathbf{H}\mathbf{G}\,\widehat{u^3} + \hat{e}$, where (for a change) a hat denotes the $\mathcal{Z}$ transformation. Thus,

$$\hat{y} = \underbrace{\begin{bmatrix} \mathbf{H}\mathbf{G} & \mathbf{I} \end{bmatrix}}_{\mathbf{L}} \hat{w}, \quad \text{where } w = \begin{bmatrix} u^3 \\ e \end{bmatrix}.$$

By the matrix version of the Wiener-Lee formula (discrete-time version), we have $S^{yy} = \mathbf{L}\, S^{ww} \mathbf{L}^*$. Since u^3 and e are independent white noises, we have

$$S^{ww} = \begin{bmatrix} \sigma^2 & 0 \\ 0 & \sigma_1^2 \end{bmatrix}, \quad \text{where}$$

$$\sigma^2 = \mathrm{Var}(u_k^3) = E(u_k^6) - E(u_k^3)^2,$$

$$\sigma_1^2 = \mathrm{Var}(e_k) = E(e_k^2) - E(e_k)^2.$$

Multiplying out, we obtain

$$S^{yy}(e^{i\nu}) = |\mathbf{H}(e^{i\nu})|^2 \cdot |\mathbf{G}(e^{i\nu})|^2 \sigma^2 + \sigma_1^2.$$

Question 5.

(a) $(5 - 8z^{-1} + 3.15\, z^{-2})\,\hat{y}(z) = z^{-1}\hat{q}(z)$

$\hat{q}(z) = (-3 + 5z^{-1})\,\hat{u}(z)$. Combining these, we get

$$G(z) = \frac{(-3+5z^{-1})z^{-1}}{5 - 8z^{-1} + 3.15\, z^{-2}} = \frac{-0.6\, z^{-1} + z^{-2}}{1 - 1.6\, z^{-1} + 0.63\, z^{-2}} .$$

(b) The poles of G are 0.7 and 0.9, so that it is stable. Its DC gain is $G(1) = \frac{0.4}{0.03} = 40/3$. G is strictly proper (i.e., $G(\infty) = 0$) and it is not FIR.

(c) If the formula for u_k were true also for $k=0$, then the $\mathcal{Z}$-transform would be $\frac{7z}{z-0.1} - 6z^{-1}$. Because $u_0 = 0$, we have to subtract $7\hat{\delta}_0 = 7$, so that

$$\hat{u}(z) = \frac{7z}{z-0.1} - 7 - 6z^{-1} = \frac{-5.3 + 0.6z^{-1}}{z-0.1}$$

(d) $$\hat{y}(z) = G(z)\,\hat{u}(z) = \frac{(-0.6z + 1)\cdot(-5.3z + 0.6)}{(z-0.7)(z-0.9)(z-0.1)} \cdot \frac{1}{z}$$

(e) We use the partial fractions decomposition

$$\hat{w}(z) = z\,\hat{y}(z) = \frac{d_1}{z-0.7} + \frac{d_2}{z-0.9} + \frac{d_3}{z-0.1} ,$$

new signal

where $d_1 = \lim_{z \to 0.7} (z-0.7)\, z\, \hat{y}(z)$ and similarly for d_2, d_3. Take inverse $\mathcal{Z}$ transforms:

$$w_k = d_1 (0.7)^{k-1} + d_2 (0.9)^{k-1} + d_3 (0.1)^{k-1}$$

for $k = 1, 2, 3, \ldots$ while $w_0 = 0$. Since $y_k = w_{k-1}$, for $k \geq 1$, for $c_1 = \frac{d_1}{(0.7)^2}$, $c_2 = \frac{d_2}{(0.9)^2}$, $c_3 = \frac{d_3}{(0.1)^2}$, we get the desired formula.

(f) $P(s) \approx G\left(\dfrac{1+\frac{hs}{2}}{1-\frac{hs}{2}}\right)$, where $h = 10^{-4}$, according to Tustin's formula.

(g) Introduce $\tilde{y} = y - 2$ and $\tilde{e} = e - 2$, then $E(\tilde{e}_k) = 0$ and $\hat{\tilde{y}}(z) = G(z)\,\hat{u}(z) + \hat{\tilde{e}}(z)$.

The general formula for the one step ahead prediction of $\tilde{y}_k$ in an ARMAX system described by

$$\tilde{y}_k + a_1 \tilde{y}_{k-1} + a_2 \tilde{y}_{k-2} \ldots = b_0 u_k + b_1 u_{k-1} + b_2 u_{k-2} \ldots + \tilde{e}_k + c_1 \tilde{e}_{k-1} + c_2 \tilde{e}_{k-2} \ldots$$

is

$$\overline{y}_k = (c_1 - a_1)\, y^F_{k-1} + (c_2 - a_2)\, y^F_{k-2} \ldots + b_0 u^F_k + b_1 u^F_{k-1} + b_2 u^F_{k-2} \ldots ,$$

where

$$y^F_k + c_1 y^F_{k-1} + c_2 y^F_{k-2} \ldots = \tilde{y}_k ,$$
$$u^F_k + c_1 u^F_{k-1} + c_2 u^F_{k-2} \ldots = u_k ,$$

and we have $\tilde{y}_k = \overline{y}_k + \tilde{e}_k$ (there is no need to write all this down in the exam).

In our case, e_k being an output error, we have $c_j = a_j$ $(j = 1, \ldots n)$, $b_0 = 0$, $n = 2$, so that

$$\overline{y}_k = b_1 u^F_{k-1} + b_2 u^F_{k-2} ,$$

whence

$$\hat{\overline{y}}(z) = (b_1 z^{-1} + b_2 z^{-2})\, \hat{u}^F(z) = \frac{b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}\, \hat{u}(z) = G(z)\,\hat{u}(z).$$

Thus, we have to filter u through G and the output obtained for $k = 301$ will be the unbiased prediction of $\tilde{y}_{301}$. In particular, the data $y_0, y_1, \ldots y_{300}$ are not needed. From $\tilde{y}_k = \overline{y}_k + \tilde{e}_k$ we see that the prediction error is $\tilde{e}_k$, so that its variance is 9.