Final

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2010

MSc Control Systems

DISCRETE-TIME SYSTEMS AND COMPUTER CONTROL

Friday, 14 May 10:00 am

Time allowed: 3:00 hours

There are SIX questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s) :	A. Astolfi
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DTS AND COMPUTER CONTROL

Information for candidates:

- $Z\left(\frac{1}{s}\right) = \frac{z}{z-1} = \frac{1}{1-z^{-1}}$

- $Z\left(\frac{1}{s^2}\right) = T\frac{z}{(z-1)^2} = T\frac{z^{-1}}{(1-z^{-1})^2}$

- $Z\left(\frac{s}{s^2+\omega^2}\right) = \frac{z(z-\cos(\omega T))}{z^2-2z\cos(\omega T)+1}$

- $Z\left(\frac{\omega}{s^2+\omega^2}\right) = \frac{z\sin(\omega T)}{z^2-2z\cos(\omega T)+1}$

- Transfer function of the ZOH: $H_0(s) = \frac{1-e^{-sT}}{s}$

- Transfer function of the FOH: $H_1(s) = \frac{1+Ts}{T}\left(\frac{1-e^{-sT}}{s}\right)^2$

- Definition of the w-plane: $z = \frac{1+\frac{wT}{2}}{1-\frac{wT}{2}}$, $w = \frac{2}{T}\frac{z-1}{z+1}$

1. Consider the block diagram in Figure 1 where $P(s)$ admits a state space description with matrices A, B, C and D.

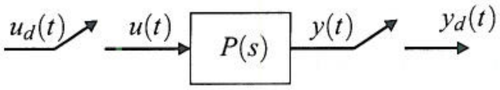


Figure 1: Block diagram for question 1.

a) The state $x(t)$ of P can be expressed as

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau. \tag{1.1}$$

Verify that the behaviour of the discrete-time system in Figure 1 is described by

$$\begin{aligned} x((k+1)T) &= e^{AT}x(kT) + \left(\int_0^T e^{A(T-\rho)}Bd\rho\right) u_d(kT), \\ y(kT) &= Cx(kT) + Du_d(kT). \end{aligned}$$

(Hint: compute $x((k+1)T)$ and $x(kT)$ from (1.1) and find a relation between them. Use the change of variables $\rho = \tau - kT$.) [4 marks]

b) Let

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = [0 \quad 1], \quad D = 0.$$

Compute the matrices A_d and B_d of the discrete time system. [4 marks]

c) Show that the pair (A, B) is reachable. [2 marks]

d) Study the reachability properties of the pair (A_d, B_d) and determine values of T such that (A_d, B_d) is unreachable. [4 marks]

e) Show that the pair (A, C) is observable. [2 marks]

f) Study the observability properties of the pair (A_d, C) and determine values of T such that (A_d, C) is unobservable. [4 marks]

2. Consider the digital control system in Figure 2.

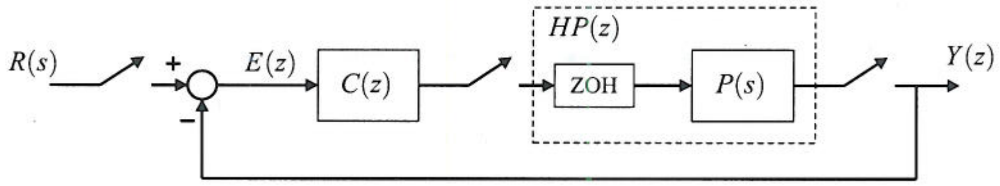


Figure 2: Block diagram for question 2.

Let

$$P(s) = \frac{2}{s-1}$$

and let the sampling period be $T = 0.02$ sec.

a) Compute the equivalent discrete-time model $HP(z)$ for the plant interconnected to the hold and the sampler. [4 marks]

b) Using the definition of the w-plane (see the "Information for candidates"), determine the transfer function $HP(w)$. [2 marks]

c) Design, in the w-plane, a controller

$$C(w) = k\frac{w+a}{w+b}$$

such that the closed-loop system is asymptotically stable, has a DC gain equal to 4 and all its poles are equal to -1. [8 marks]

d) Compute the transfer function $C(z)$ of the discrete-time controller. (Hint: apply the inverse of the definition of the w-plane reported in the "Information for candidates"). Verify that the closed-loop discrete-time system is asymptotically stable and has a DC gain equal to 4. [6 marks]

3. Consider the block diagram in Figure 3.

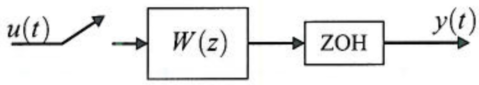


Figure 3: Block diagram for question 3.

a) Let

$$W(s) \triangleq \frac{Y(s)}{U(s)}.$$

Show that

$$W(s) = s\mathscr{L}\left\{\mathscr{Z}^{-1}\left(\frac{z}{z-1}W(z)\right)\right\}, \tag{3.1}$$

where $\mathscr{L}$ denotes the Laplace transform and $\mathscr{Z}^{-1}$ the inverse z-transform.

[6 marks]

b) Using Equation (3.1), find $W(z)$ such that

$$W(s) = \frac{10}{s+5}.$$

[4 marks]

c) Suppose that $W(z)$ is the discrete time closed-loop system represented in Figure 4 and let the sampling time be $T = 0.1$.

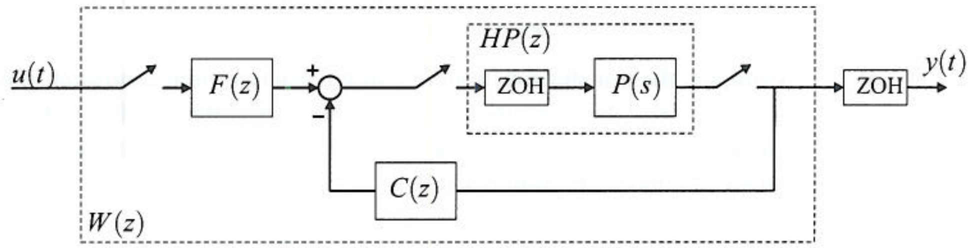


Figure 4: Block diagram for question 3.c).

i) Assume

$$P(s) = \frac{2}{s-2}$$

and determine $HP(z)$. [4 marks]

ii) Find the coefficients t_1, s_1 and s_0 such that setting

$$F(z) = \frac{t_1 z}{z-1}, \qquad C(z) = \frac{s_1 z + s_0}{z-1}$$

the closed-transfer function is equal to the transfer function $W(z)$ determined in part b). [6 marks]

4. Consider the digital control system in Figure 5.

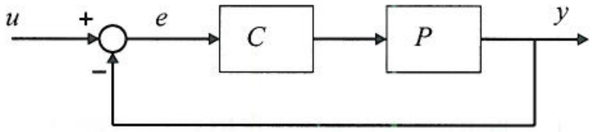


Figure 5: Block diagram for question 4.

Let P be a discrete-time plant described by

$$P(z) = \frac{z}{z-0.9}.$$

a) By using the definition of the w-plane (see the "Information for candidates"), compute $P(w)$ as a function of T. [4 marks]

b) Let $T = 0.2$. Find $C(w)$ such that the closed-loop transfer function is

$$G(w) = \frac{4}{w+5}.$$

(Hint: use a controller with a transfer function having one zero and two poles.) [8 marks]

c) Find the controller $C(z)$ by inverse transforming $C(w)$ (Hint: apply the inverse transformation of the function defining the w-plane reported in the "Information for candidates"). [4 marks]

d) Find the discrete-time closed-loop transfer function and compare its poles with the pole of $G(w)$. [4 marks]

5. Consider a unity (negative) feedback system with open-loop transfer function

$$P(s) = k\frac{1}{s+2},$$

with $k \in \mathbb{R}$.

a) Find the values of k for which the closed-loop system is asymptotically stable. [2 marks]

b) Assume that the input of $P(s)$ is connected to a sampler and to a ZOH and that the output is connected to a sampler. Let T be the sampling period.

i) Determine the discrete-time equivalent transfer function of the open-loop system. [5 marks]

ii) Study the stability of the closed-loop system as a function of $k > 0$. [4 marks]

iii) Show that as $T \to 0$ the stability condition on the parameter k tends to the condition found in part a). [2 marks]

c) Repeat part b) replacing the ZOH with a FOH and selecting $T = 0.5$.[5 marks]

d) Comment on the differences between the results obtained in parts b) and c) using in both cases $T = 0.5$. [2 marks]

6. Consider a continuous-time system described by the transfer function

$$G(s) = \frac{1}{s^2+4}.$$

a) Assume the system is connected to a ZOH and a sampler. Let T be the sampling period. Determine the discrete-time equivalent transfer function $HG(z)$. [4 marks]

b) Determine a value of T for which $HG(z) = 0$. [2 marks]

c) Find the analytic expression of the signal $g(t) = \mathscr{L}^{-1}\{G(s)\}$. [2 marks]

d) Find a value of a sampling time T_S allowing to sample $g(t)$ without aliasing. [2 marks]

e) Discuss the difference between T_S determined in part d) and the value of T determined in part b). [4 marks]

f) Repeat part a) using a FOH and a sampler and selecting $T = \pi$. [4 marks]

g) Repeat part f) selecting for T the value determined in part b). Discuss the result. [2 marks]

CS1.3

DTS and Computer Control

Model answers 2010

Question 1

a) From Equation (1.1) one obtains

$$x(kT) = e^{AkT}x(0) + \int_0^{kT} e^{A(kT-\tau)}Bu(\tau)d\tau$$

and

$$x((k+1)T) = e^{A(k+1)T}x(0) + \int_0^{(k+1)T} e^{A((k+1)T-\tau)}Bu(\tau)d\tau\,.$$

Hence

$$\begin{aligned} x((k+1)T) &= e^{AT}\left[e^{AkT}x(0) + \int_0^{kT} e^{A(kT-\tau)}Bu(\tau)d\tau\right] + \\ &\qquad + \int_{kT}^{(k+1)T} e^{A((k+1)T-\tau)}Bu(\tau)d\tau \\ &= e^{AT}x(kT) + \int_0^{T} e^{A(T-\rho)}Bu(\rho+kT)d\rho\,. \end{aligned}$$

Now, u is the output of a sampler, hence $u(\rho + kT)$ is constant for $\rho \in [0,T)$ and equal to $u_d(kT)$. Therefore it can be put outside the integral obtaining the desired equation

$$x((k+1)T) = e^{AT}x(kT) + \left[\int_0^{T} e^{A(T-\rho)}Bd\rho\right]u_d(kT)\,.$$

The continuous-time equation for y is:

$$y(t) = Cx(t) + Du(t)\,,$$

hence, for $t = kT$ we have

$$y(kT) = Cx(kT) + Du(kT) = Cx(kT) + Du_d(kT)\,,$$

where the last equality is due to the fact that in the sampling time-instants u and u_d have the same value.

b) The eigenvalues of A are the roots of the polynomial

$$p(s) = \det(sI - A) = \det\begin{bmatrix} s & -1 \\ 1 & s \end{bmatrix} = s^2 + 1\,,$$

namely $\lambda_1 = j$ and $\lambda_2 = -j$. The corresponding eigenvectors can be obtained from the equations

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = j\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix}, \qquad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = -j\begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix},$$

yielding, for instance,

$$\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 1 \\ j \end{bmatrix}, \qquad \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 1 \\ -j \end{bmatrix}.$$

Therefore

$$
\begin{aligned}
A_d = e^{AT} &= \begin{bmatrix} v_{11} & v_{21} \\ v_{12} & v_{22} \end{bmatrix} \begin{bmatrix} e^{jT} & 0 \\ 0 & e^{-jT} \end{bmatrix} \begin{bmatrix} v_{11} & v_{21} \\ v_{12} & v_{22} \end{bmatrix}^{-1} \\
&= \begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix} \begin{bmatrix} \cos(T) + j\sin(T) & 0 \\ 0 & \cos(T) - j\sin(T) \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & -j \\ 1 & j \end{bmatrix} \\
&= \begin{bmatrix} \cos(T) & \sin(T) \\ -\sin(T) & \cos(T) \end{bmatrix}.
\end{aligned}
$$

Moreover

$$
\begin{aligned}
B_d &= \int_0^T e^{A(T-\rho)} B d\rho = \int_0^T \begin{bmatrix} \cos(T-\rho) & \sin(T-\rho) \\ -\sin(T-\rho) & \cos(T-\rho) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} d\rho \\
&= \int_0^T \begin{bmatrix} \sin(T-\rho) \\ \cos(T-\rho) \end{bmatrix} d\rho = \int_0^T \begin{bmatrix} \sin(\sigma) \\ \cos(\sigma) \end{bmatrix} d\sigma = \begin{bmatrix} 1 - \cos(T) \\ \sin(T) \end{bmatrix}.
\end{aligned}
$$

c) The reachability matrix of the continuous-time system is

$$
Q = [B \quad AB] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},
$$

the rank of which is 2. Hence the system is reachable.

d) The reachability matrix of the discrete-time system is

$$
Q_d = [B_d \quad A_d B_d] = \begin{bmatrix} 1 - \cos(T) & \cos(T) - \cos^2(T) + \sin^2(T) \\ \sin(T) & -\sin(T) + 2\sin(T)\cos(T) \end{bmatrix},
$$

the rank of which is less than 2 if $T = k\pi$. For instance, for $T = \pi$ we have

$$
Q_d = \begin{bmatrix} 2 & -2 \\ 0 & 0 \end{bmatrix},
$$

the rank of which is 1. Therefore the discrete-time system with $T = \pi$ is not reachable. For any $T \neq kT$, $k \in \mathbb{Z}_+$, the system is reachable.

e) The observability matrix of the continuous-time system is

$$
O = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},
$$

the rank of which is 2. Hence the continuous-time system is observable.

f) The observability matrix of the discrete-time system is

$$
O_d = \begin{bmatrix} C \\ CA_d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\sin(T) & \cos(T) \end{bmatrix},
$$

the rank of which is 2 only if $T \neq k\pi$, for $k \in \mathbb{N}$. For instance, for $T = \pi$ we have

$$
O_d = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix},
$$

the rank of which is 1. Therefore the discrete-time system with $T = \pi$ is not observable. For any $T \neq kT$, $k \in \mathbb{Z}_+$, the system is observable.

Question 2

a) The equivalent discrete-time model is

$$\begin{aligned} HP(z) &= (1-z^{-1})\mathcal{Z}\left[\mathcal{L}^{-1}\left(\frac{2}{s(s-1)}\right)\right] = (1-z^{-1})\mathcal{Z}\left[\mathcal{L}^{-1}\left(-\frac{2}{s}+\frac{2}{s-1}\right)\right] \\ &= -2+2\frac{z-1}{z-e^T} = \frac{2(e^T-1)}{(z-e^T)}\,. \end{aligned}$$

b) The transfer function in the w-plane is

$$\begin{aligned} HP(w) = HP(z)|_{z=\frac{1+0.01w}{1-0.01w}} &= 2(e^{0.02}-1)\frac{1}{\frac{1+0.01w}{1-0.01w}-e^{0.02}} \\ &= 2(e^{0.02}-1)\frac{1-0.01w}{1+0.01w-e^{0.02}+0.01e^{0.02}w} \\ &\simeq 0.0404\frac{1-0.1w}{0.0202\,w-0.0202} \\ &\simeq -0.2\frac{w-10}{w-1}\,. \end{aligned}$$

c) The open-loop transfer function corresponding to the cascade of $C(w)$ and $HP(w)$ is

$$G(w) = C(w)HP(w) \simeq -0.2\,k\frac{(w-10)(w+a)}{(w-1)(w+b)}\,,$$

hence the closed-loop transfer function is

$$\frac{G(w)}{1+G(w)} \simeq \frac{-0.2\,k(w-10)(w+a)}{(w-1)(w+b)-0.2\,k(w-10)(w+a)}\,.$$

The denominator can be written as

$$(1-0.2\,k)w^2+(b-1-0.2\,k\,a+2\,k)w+(-b+2\,k\,a)$$

hence the coefficients k, a and b must be such that

$$\left\{\begin{aligned} \frac{2\,k\,a}{-b+2\,k\,a} &= 4\,, \\ \frac{b-1-0.2\,k\,a+2\,k}{1-0.2\,k} &= 2\,, \\ \frac{-b+2\,k\,a}{1-0.2\,k} &= 1\,. \end{aligned}\right.$$

For instance, $k \simeq 0.2128$ $a \simeq 8.908$ and $b \simeq 2.872$. The control system is

$$C(w) = 0.2128\frac{w+8.908}{w+2.872}\,.$$

d) The discrete-time controller is

$$C(z) = C(w)\Big|_{w=100\frac{z-1}{z+1}} = 0.2128\frac{100\frac{z-1}{z+1}+8.908}{100\frac{z-1}{z+1}+2.872} \simeq 0.2253\frac{z-0.8361}{z-0.9439}.$$

The transfer function of the closed-loop discrete-time system is

$$\frac{C(z)HP(z)}{1+C(z)HP(z)} \simeq \frac{0.0091\dfrac{z-0.8361}{(z-1.0202)(z-0.9439)}}{1+0.0091\dfrac{z-0.8361}{(z-1.0202)(z-0.9439)}} \simeq 0.0091\frac{z-0.8361}{z^2-1.954\,z+0.955}.$$

The closed-loop poles are

$$z_{1,2} \simeq \frac{1.954 \mp \sqrt{3.8181-3.82}}{2} \simeq \frac{1.954 \mp i\sqrt{0.0019}}{2}.$$

They are both inside the unity circle in fact

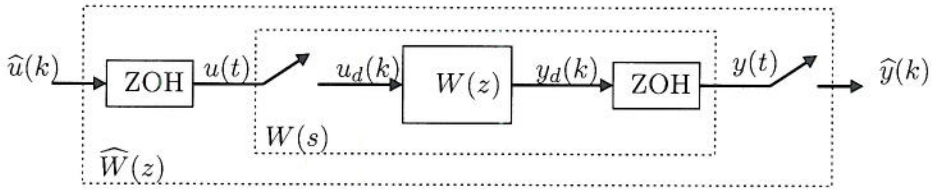
$$\|z_1\| = \|z_2\| \simeq 0.5\sqrt{3.82} \simeq 0.977\,,$$

hence the system is asymptotically stable. The DC gain of the closed-loop transfer function is

$$\lim_{z\to 1}\frac{C(z)HP(z)}{1+C(z)HP(z)} = \frac{(\lim_{z\to 1}C(z))\,(\lim_{z\to 1}HP(z))}{1+(\lim_{z\to 1}C(z))\,(\lim_{z\to 1}HP(z))} \simeq \frac{(0.6582)(-2)}{1+(0.6582)(-2)} \simeq 4.16\,.$$

Question 3

a) By adding a ZOH and an impulsive sampler as in the following figure



we obtain a system such that the transfer function $\widehat{W}(z)$ between the input $\widehat{u}(k)$ and the output $\widehat{y}(k)$ is

$$\widehat{W}(z) \triangleq \frac{\widehat{Y}(Z)}{\widehat{U}(z)} = \frac{z-1}{z}\mathcal{Z}\left\{\mathcal{L}^{-1}\left(\frac{W(s)}{s}\right)\right\}.$$

From the previous equation one obtains

$$W(s) = s\mathcal{L}\left\{\mathcal{Z}^{-1}\frac{z}{z-1}\widehat{W}(z)\right\}.$$

Since $\widehat{u}(k) = u(k)$ and $\widehat{y}(k) = y(k)$ for all k, one obtains $\widehat{W}(z) = W(z)$ and hence the required equivalence.

b) The discrete-time transfer function is

$$\begin{aligned} W(z) &= \frac{z-1}{z}\mathcal{Z}\left\{\mathcal{L}^{-1}\left(\frac{10}{s(s+5)}\right)\right\} = \frac{z-1}{z}\mathcal{Z}\left\{\mathcal{L}^{-1}\left(\frac{2}{s} - \frac{2}{s+5}\right)\right\} \\ &= \frac{z-1}{z}\mathcal{Z}\left\{2 - 2e^{-5t}\right\} = \frac{z-1}{z}\left(2\frac{z}{z-1} - 2\frac{z}{z-e^{-5T}}\right) \\ &= 2 - 2\frac{z-1}{z-e^{-5T}} = 2\frac{1-e^{-5T}}{z-e^{-5T}}. \end{aligned}$$

c) i) The equivalent discrete-time model of $P(s)$ is

$$\begin{aligned} HP(z) &= \frac{z-1}{z}\mathcal{Z}\left\{\mathcal{L}^{-1}\left(\frac{2}{s(s-2)}\right)\right\} = \frac{z-1}{z}\mathcal{Z}\left\{\mathcal{L}^{-1}\left(-\frac{1}{s} + \frac{1}{s-2}\right)\right\} \\ &= \frac{z-1}{z}\left(-\frac{z}{z-1} + \frac{z}{z-e^{2T}}\right) = -1 + \frac{z-1}{z-e^{2T}} \simeq \frac{0.221}{z-1.22}. \end{aligned}$$

ii) The closed-loop transfer function is

$$\begin{aligned} \frac{HP(z)F(z)}{1+HP(z)C(z)} &\simeq \frac{\left(\frac{0.221}{z-1.22}\right)\left(\frac{t_1 z}{z-1}\right)}{1 + \left(\frac{0.221}{z-1.22}\right)\left(\frac{s_1 z + s_0}{z-1}\right)} \\ &\simeq \frac{0.221\, t_1\, z}{z^2 - 2.22z + 1.22 + 0.221 s_1 z + 0.221 s_0}, \end{aligned}$$

which has to be equal to

$$W(z)|_{T=0.1} \simeq \frac{0.787}{z - 0.607}\,.$$

Hence the coefficients t_1, s_1 and s_0 have to be chosen to fulfill the following equations

$$\begin{cases} 0.221\, t_1 & = \; 0.787\,, \\ 1.22 + 0.221\, s_0 & = \; 0\,, \\ -2.22 + 0.221\, s_1 & = \; -0.607\,. \end{cases}$$

The solution of the previous system of equations is $t_1 \simeq 3.56$, $s_0 \simeq -5.52$ and $s_1 \simeq 7.3$.

Question 4

a)

$$P(w) = \frac{\dfrac{1+\frac{wT}{2}}{1-\frac{wT}{2}}}{\dfrac{1+\frac{wT}{2}}{1-\frac{wT}{2}} - 0.9} = \frac{1+\frac{wT}{2}}{0.1+1.9\frac{wT}{2}} = \frac{Tw+2}{1.9\,T\,w+0.2}\,.$$

b) Suppose that the control has the form

$$C(w) = K\frac{w+a}{w^2+bw+c}\,,$$

with $K, a, b, c \in \mathbb{R}$. The closed-loop transfer function is then

$$\begin{aligned}\frac{C(w)P(w)}{1+C(w)P(w)} &= \frac{K\dfrac{(w+a)}{(w^2+bw+c)}\dfrac{(0.2\,w+2)}{(0.38\,w+0.2)}}{1+K\dfrac{(w+a)}{(w^2+bw+c)}\dfrac{(0.2\,w+2)}{(0.38\,w+0.2)}} \\ &= \frac{K(w+a)(0.2\,w+2)}{0.38(w+0.5263)(w^2+bw+c)+K(0.2\,w+2)(w+a)}\,.\end{aligned}$$

If we choose $a = 0.5263$ the closed-loop transfer function becomes

$$\frac{0.2\,K(w+10)}{0.38(w^2+bw+c+\frac{0.2\,K}{0.38}w+\frac{2\,K}{0.38})}\,.$$

The value of the coefficients K, b and c can be computed as the solution of the system of equations

$$\begin{cases} 0.2\,K/0.38 &= 4\,, \\ b+0.2\,K/0.38 &= 15\,, \\ c+2\,K/0.38 &= 50\,, \end{cases}$$

yielding $K = 7.6$, $a = 0.5263$, $b = 11$ and $c = 10$.

c) To transform back the expression of the continuous-time controller

$$C(w) = 7.6\frac{w+0.5263}{(w+10)(w+1)}$$

into a discrete-time transfer function, we substitute w with $10\frac{z-1}{z+1}$, obtaining

$$
\begin{aligned}
C(z) &= 7.6\frac{10\dfrac{z-1}{z+1}+0.5263}{\left(10\dfrac{z-1}{z+1}+10\right)\left(10\dfrac{z-1}{z+1}+1\right)} \\
&= 7.6\frac{10(z-1)(z+1)+0.5263(z+1)^2}{(10(z-1)+10(z+1))(10(z-1)+(z+1))} \\
&= 7.6\frac{10(z^2-1)+0.5263(z^2+2\,z+1)}{(10\,z-10+10\,z+10)(10\,z-10+z+1)} \\
&= 7.6\frac{10.5263\,z^2+1.0526\,z-9.4737}{20\,z(11\,z-9)} \\
&= \frac{80\,z^2+8\,z-72}{20\,z(11\,z-9)} \\
&= \frac{4(z^2+0.1\,z-9)}{z(11\,z-0.9)} \\
&= \frac{4(z+1)(z-9)}{z(11\,z-0.9)}\,.
\end{aligned}
$$

d) The discrete-time transfer function is

$$
\frac{C(z)P(z)}{1+C(z)P(z)} = \frac{\dfrac{4(z+1)}{11\,z-9}}{1+\dfrac{4(z+1)}{11\,z-9}} = \frac{4(z+1)}{11\,z-9+4\,z+4} = \frac{4(z+1)}{15\,z-5}\,.
$$

The closed-loop pole in the discrete-time domain is $z_p = 0.333$, which is the transformation of the closed-loop continuous-time pole $w_p = -5$:

$$
\frac{1+\dfrac{0.2\,w_p}{2}}{1-\dfrac{0.2\,w_p}{2}} = \frac{1-0.5}{1+0.5} = 0.333 = z_p\,.
$$

Question 5

a) The closed-loop transfer function is

$$\frac{\dfrac{k}{s+2}}{1+\dfrac{k}{s+2}} = \frac{k}{s+2+k},$$

hence the closed-loop system is asymptotically stable for all $k > -2$.

b) i) The equivalent discrete-time model is given by

$$\begin{aligned} HP(z) &= \frac{z-1}{z}\mathcal{Z}\left\{\mathcal{L}^{-1}\left(\frac{k}{s(s+2)}\right)\right\} \\ &= \frac{k}{2}\frac{z-1}{z}\mathcal{Z}\left\{\mathcal{L}^{-1}\left(\frac{1}{s}-\frac{1}{s+2}\right)\right\} \\ &= \frac{k}{2}\frac{z-1}{z}\left(\frac{z}{z-1}-\frac{z}{z-e^{-2T}}\right) \\ &= \frac{k}{2}\left(1-\frac{z-1}{z-e^{-2T}}\right) \\ &= \frac{k}{2}\frac{1-e^{-2T}}{z-e^{-2T}}. \end{aligned}$$

ii) The closed-loop transfer function is

$$\frac{\dfrac{k}{2}\dfrac{1-e^{-2T}}{z-e^{-2T}}}{1+\dfrac{k}{2}\dfrac{1-e^{-2T}}{z-e^{-2T}}} = \frac{k(1-e^{-2T})}{2z-2e^{-2T}+k-ke^{-2T}}.$$

The closed-loop system is asymptotically stable if and only if

$$\left|e^{-2T}-\frac{k}{2}(1-e^{-2T})\right| < 1$$

which is equivalent to the two conditions

$$\left\{\begin{array}{l} 2e^{-2T}-k(1-e^{-2T}) > 0, \\ 2e^{-2T}-k(1-e^{-2T}) < 2, \end{array}\right. \qquad \left\{\begin{array}{r} 2e^{-2T}-k(1-e^{-2T}) \leq 0, \\ -2e^{-2T}+k(1-e^{-2T}) < 2. \end{array}\right.$$

The closed-loop system is asymptotically stable if

$$k \in \left(-2, \frac{2e^{-2T}}{1-e^{-2T}}\right) \cup \left[\frac{2e^{-2T}}{1-e^{-2T}}, 2\frac{1+e^{-2T}}{1-e^{-2T}}\right) = \left(-2, 2\frac{1+e^{-2T}}{1-e^{-2T}}\right).$$

iii) When $T \to 0$ we have

$$2\frac{1+e^{-2T}}{1-e^{-2T}} \to \infty,$$

hence the stability condition becomes $k \in (-2, \infty)$ which is the condition found in part a).

c) The equivalent discrete-time model is given by

$$
\begin{aligned}
HP_{FOH}(z) &= \frac{(z-1)^2}{z^2}\mathcal{Z}\left\{\mathcal{L}^{-1}\left(\frac{k(1+0.5\,s)}{0.5\,s^2(s+2)}\right)\right\} \\
&= k\frac{(z-1)^2}{z^2}\mathcal{Z}\left\{\mathcal{L}^{-1}\left(\frac{1}{s^2}\right)\right\} \\
&= k\frac{(z-1)^2}{z^2}0.5\frac{z}{(z-1)^2} = \frac{k}{2\,z}\,.
\end{aligned}
$$

The closed-loop transfer function is

$$
\frac{\frac{k}{2\,z}}{1+\frac{k}{2\,z}} = \frac{k}{2\,z+k}
$$

and is asymptotically stable if and only if $|-\frac{k}{2}| < 1$, namely if $k \in (-2, 2)$.

d) For $T = 0.5$ the condition found in part b,ii) becomes

$$
k \in \left(-2, 2\frac{1+e^{-1}}{1-e^{-1}}\right) \simeq (-2, 2.16)\,.
$$

The stability condition for the FOH is slightly more restrictive.

Question 6

a) The equivalent discrete-time transfer function is given by

$$\begin{aligned}
HG(z) &= \frac{z-1}{z}\mathcal{Z}\left\{\mathcal{L}^{-1}\left(\frac{1}{s(s^2+4)}\right)\right\} \\
&= \frac{z-1}{z}\mathcal{Z}\left\{\mathcal{L}^{-1}\left(\frac{1}{4\,s}-\frac{s}{4(s^2+4)}\right)\right\} \\
&= \frac{1}{4}\frac{z-1}{z}\left(\frac{z}{z-1}-\frac{z^2-z\cos(2T)}{z^2-2z\cos(2T)+1}\right) \\
&= \frac{1}{4}\left(1-\frac{(z-1)(z-\cos(2T))}{z^2-2z\cos(2T)+1}\right) \\
&= \frac{1}{4}\frac{z(1-\cos(2T))+1-\cos(2T)}{z^2-2z\cos(2T)+1}\,.
\end{aligned}$$

b) $HG(z)=0$ for $\cos(2T)=1$, namely for $T=2k\pi$, for all $k\in\mathbb{N}$, $k>0$.

c)

$$g(t)=\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}=\mathcal{L}^{-1}\left\{\frac{1}{2}\frac{2}{s^2+2^2}\right\}=\frac{1}{2}\sin(2t)\,.$$

d) Since the angular frequency of the signal is 2, in order not to have aliasing the sampling frequency must be at least $\omega_S=2\cdot 2=4$, hence the sampling period T_S must be less than

$$T_{S,max}=\frac{2\pi}{\omega_S}=\frac{\pi}{2}\simeq 1.57\,,$$

for instance $T_S=1$.

e) If the sampling time T is chosen in order to fulfill the Nyquist condition, it must be less than $\frac{\pi}{2}$ and, in particular, it cannot be equal to $2k\pi$ for any $k\in\mathbb{N}$, $k>0$. This implies that $HG(z)\neq 0$ for all $T<\frac{\pi}{2}$.

f) When using a FOH istead of a ZOH, the equivalent discrete-time transfer function is given by

$$\begin{aligned}
HG_{FOH}(z) &= \frac{(z-1)^2}{z^2}\mathcal{Z}\left\{\mathcal{L}^{-1}\left(\frac{1+Ts}{Ts^2(s^2+4)}\right)\right\} \\
&= \frac{1}{4T}\frac{(z-1)^2}{z^2}\mathcal{Z}\left\{\mathcal{L}^{-1}\left(\frac{T}{s}+\frac{1}{s^2}-\frac{Ts+1}{s^2+4}\right)\right\} \\
&= \frac{1}{4T}\frac{(z-1)^2}{z^2}\left(\frac{Tz}{z-1}+\frac{Tz}{(z-1)^2}-\right. \\
&\qquad \left. T\frac{z(z-\cos(2T))}{z^2-2\,z\cos(2T)+1}+\frac{1}{2}\frac{z\sin(2T)}{z^2-2\,z\cos(2T)+1}\right)\,.
\end{aligned}$$

For $T=\pi$ we have

$$HG_{FOH}(z)=\frac{1}{4}\left(\frac{z-1}{z}+\frac{1}{z}-\frac{z-1}{z}\right)=\frac{1}{4\,z}\,.$$

g) When $T = 2k\pi$, $k > 0$, we have

$$HG_{FOH}(z) = \frac{1}{4\,z}\,.$$

This means that the system behaves as a delay. Also when using a FOH, for $T = 2k\pi$ the information on the dynamics of the system is lost.