UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 2004

BEng Honours Degree in Computing Part I
MEng Honours Degrees in Computing Part I
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the Associateship of the City and Guilds of London Institute

PAPER C142

DISCRETE MATHEMATICS

Wednesday 12 May 2004, 10:00
Duration: 90 minutes
(Reading time 5 minutes)

Answer THREE questions

Paper contains 4 questions
Calculators not required

1 Let A be a set. Recall the definition of the powerset $\mathcal{P}(A)$ of A given by

$$\mathcal{P}(A) = \{B : B \subseteq A\}.$$

a Let $A = \{a, \{b\}\}$ and $B = \{a, b\}$. Determine the following sets:

$$A \cup B, \quad A \cap B, \quad A \times B, \quad \mathcal{P}(A), \quad \mathcal{P}(\emptyset), \quad \mathcal{P}(\{\emptyset\}).$$

b i) Give the definition of a binary relation between sets A and B.

ii) Let A and B be finite sets. Determine how many possible binary relations there are between A and B, in terms of the cardinality of A and B.

c Let A and B be arbitrary sets.

i) Prove that $\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B)$.

ii) Show that $\mathcal{P}(A) \cup \mathcal{P}(B)$ and $\mathcal{P}(A \cup B)$ are not equal by giving a simple counter-example. In fact, one set is contained in the other. Prove this.

iii) Show that $\mathcal{P}(A) = \mathcal{P}(B)$ implies $A = B$.

The three parts carry, respectively, 30%, 15%, and 55% of the marks.

2 a i) Define what it means for a binary relation R on set A to be reflexive, symmetric and transitive.

ii) Let R be an equivalence relation on A. Define the equivalence class $[a]$ of element $a \in A$ with respect to R.

iii) Let $\mathcal{Z}$ denote the set of integers. For $z \in \mathcal{Z}$, define $\lfloor z/2 \rfloor$ to be the integer part of $z/2$: for example, $\lfloor -3/2 \rfloor = -1$. Define a binary relation S on the integers by

$$\forall z_1, z_2 \in \mathcal{Z}.\ z_1 \mathrel{S} z_2 \text{ if and only if } \lfloor z_1/2 \rfloor = \lfloor z_2/2 \rfloor.$$

Show that S is an equivalence relation, and describe the equivalence classes of $\mathcal{Z}$ with respect to S.

b i) Let R be a binary relation between sets A and B. State the conditions required for R to be a function.

ii) Let $f : A \to B$ be a function. Define a binary relation R_f on A by

$$(a_1, a_2) \in R_f \text{ if and only if } f(a_1) = f(a_2).$$

Show that R_f is an equivalence relation.

iii) Let $f : A \to B$ be an onto function, and R_f be the equivalence relation defined in part 2b(ii). Let $[a]$ denote the equivalence class of a with respect to R_f, and let A/R_f denote the set of equivalence classes generated by the elements of A with respect to R_f.

Define a binary relation S between sets B and A/R_f by

$$b \mathrel{S} [a] \text{ if and only if } f(a) = b.$$

Prove that S is a function.

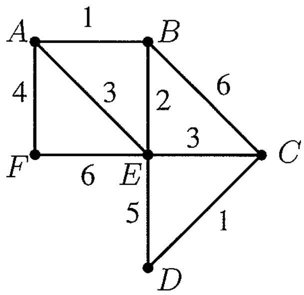
3 a i) For a graph G with m arcs, what is the relationship between m and the sum S of the degrees of the nodes of G? Explain your answer briefly.

ii) What does it mean for a graph to be a *tree*?

iii) State without proof the number of arcs in a tree with n nodes.

iv) Let T be a tree. Suppose that $d \geq 2$ is the maximum value of the degrees of nodes of T. Show that T has at least d nodes of degree one.

b Use Prim's algorithm (starting at node A) to obtain a minimum spanning tree for the following graph:



Give your answer in the form of a diagram. Also state the order in which nodes are added to the spanning tree.

c Let $G = (G, W)$ be a connected weighted graph, and let T be a minimum spanning tree for G. Let $nodes(G)$ be partitioned into two nonempty sets X and Y. Call an arc a *crossing arc* if it joins a node in X to a node in Y.

i) Explain why T must contain a crossing arc.

ii) Show that at least one crossing arc of minimum weight must belong to T.

The three parts carry, respectively, 45%, 20%, and 35% of the marks.

4 a i) What is the worst-case number of comparisons for Insertion Sort applied to a list of length n? Give a brief explanation.

ii) Suppose that Insertion Sort is applied to a list L of length $2n$ ($n \geq 1$), which is composed of n distinct numbers in ascending order followed by the same n numbers in the same order. An example for $n = 4$ is $[1, 2, 3, 4, 1, 2, 3, 4]$. How many comparisons does Insertion Sort take when applied to L? Your answer will be a formula which works for all values of $n \geq 1$.

b The sorting algorithm MaxSort finds the maximum element x in the unsorted portion of the list (initially the whole list), and then swaps x with the last element in the unsorted portion, at which point x is in its correct position and joins the sorted portion. This procedure is repeated until the list is sorted. Thus the unsorted portion lies to the left of the sorted portion.

i) Write out MaxSort in pseudocode.

ii) Use your answer to (i) to calculate the worst-case number of comparisons for MaxSort.

From the point of view of time complexity measured by comparisons, should we prefer Insertion Sort or MaxSort?

c You are given five coins of identical appearance. They all weigh the same, except for one which is counterfeit and which weighs a different amount from the others. You are given scales of the balance type, allowing you to compare the weights of the coins against each other. Each weighing has two outcomes: the weights are either equal or unequal.

i) Give an algorithm in the form of a decision tree to determine which is the counterfeit coin. The leaves of the tree will represent outcomes, and the internal nodes will represent weighings. Your algorithm should be *optimal*, in that it uses no more weighings in worst case than necessary.

ii) What is the worst-case number of comparisons for your algorithm?

Explain why your algorithm is optimal in worst-case.

The three parts carry, respectively, 35%, 35%, and 30% of the marks.