UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 2003

BEng Honours Degree in Computing Part III
MSc in Computing Science
MSci Honours Degree in Mathematics and Computer Science Part IV
MSc in Advanced Computing
PhD
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the Associateship of the City and Guilds of London Institute
This paper is also taken for the relevant examinations for the Associateship of the Royal College of Science

PAPER C336=I4.50

PERFORMANCE ANALYSIS

Wednesday 7 May 2003, 14:00
Duration: 120 minutes

Answer THREE questions

Paper contains 4 questions
Calculators not required

1 a Define the term *Markov property* as applied to continuous time stochastic processes. State and prove the *decomposition property* of Poisson processes.

b i) Define the *one-step transition probability matrix* Q of a time homogeneous *discrete time Markov chain* (DTMC);

ii) State the *Steady State Theorem* for a DTMC, paying attention to its conditions;

iii) Give an example of a Markov chain that does *not* satisfy these conditions;

iv) Prove that the *n-step transition probability matrix* of the DTMC is Q^n for $n \geq 1$.

c A router switches, in round robin order, amongst input ports, $A_1, \ldots, A_k$, processing packets and passing them to output ports. During any time-slot (where time is considered as a countable set of contiguous discrete slots) the router switches to the next input port with constant probability p, otherwise continues processing the packets at the current port with probability $1-p$.

i) Describe the evolution of the process describing the current port as a DTMC by defining its one-step transition matrix.

ii) For the two-port case (k=2), if the router is at port A_1 at time slot 0, what is the probability that it is at port A_1 at time slot 3? What is the probability that it is at port A_1 at time slot 3 *without having left* that port.

The three parts carry, respectively, 30%, 40% and 30% of the marks.

2 An M/M/1 queue has constant arrival rate λ, constant service rate μ and first-come-first-served (FCFS) queueing discipline.

a Write down the condition that this queue has a steady state and, assuming it holds, prove that the equilibrium probability that the queue has length n (including any customer in service) is $(1-\rho)\rho^n$ where $\rho = \lambda/\mu$ $(n \geq 0)$.

b Show that, if the queue length existing just before a customer arrives is n, this customer's waiting time distribution is the convolution of n+1 negative exponential distributions, each with parameter μ. State any additional assumption you make.

c State the *random observer property* of Poisson processes and use it to show that a customer's unconditional waiting time distribution is negative exponential with parameter $\mu-\lambda$. You may assume that the convolution of part b has probability density function $\mu\frac{(\mu t)^n}{n!}e^{-\mu t}$.

d Let C_1 and C_2 be two customers that are served consecutively with time D (a random variable) between their service completion instants. Show that, stating clearly any assumptions you make:

i) At the *arrival* instant of C_2, the queue was empty with probability $1-\rho$.

ii) D is distributed as a service time with probability ρ, and as the sum of an interarrival time and an independent service time with probability $1-\rho$.

iii)The mean interdeparture time is $1/\lambda$.

e Suppose now there is also an independent Poisson arrival stream of *negative tasks* to the queue, each of which *deletes* a task in the queue (if any is present), or has no effect if the queue is empty. Show that the steady state queue length probability distribution is that of a similar M/M/1 queue and give its service rate. Why does it not matter which task is chosen for deletion by a negative arrival?

3 a Give an informal proof of Little's result "$L = \lambda W$", based on a charging argument or otherwise.

b A queueing network has M nodes $\{1, 2, ..., M\}$ and population of N stochastically identical tasks. At node i, service times have mean value m_i and a task that completes service at node i proceeds immediately to node j with probability p_{ij} $(1 \leq i, j \leq M)$. For $1 \leq i \leq M$, at equilibrium:

i) Show that, in a *closed* network, between every arrival of a given task at node 1, the average number of visits the task makes to node i is e_i defined by $\mathbf{e} = \mathbf{e}.\mathbf{p}$ where $\mathbf{e} = \{e_1,...,e_M\}$ and $\mathbf{p}$ is the routing probability matrix $(p_{ij} \mid 1 \leq i, j \leq M\}$. What is the value of e_1?

ii) Write down an expression for the network throughput, T, measured on a link between two specified nodes, as a function of N, $\mathbf{e}$ and $\{W_i \mid 1 \leq i \leq M\}$, where W_i is the mean waiting time of a task at node i.

iii) Find W_i in terms of L_i, the mean queue length at node i. What is $\sum_{i=1}^{M} L_i$?

iv) Suppose that node i has *ample servers* so there is no queueing, i.e. "infinite server" (IS) queueing discipline. What is the value of W_i? If *every* node has ample servers, calculate the network throughput T at population N.

The two parts carry, respectively, 25%, and 75% of the marks.

4 a In a closed product-form queueing network, the steady state probability distribution for the queue lengths vector $\mathbf{n} = (n_1,...,n_M)$ is

$$\pi(\mathbf{n}) = \frac{1}{g(M,N)} \prod_{i=1}^{M} x_i^{n_i}$$

for appropriate fixed visitation rate : service rate ratios x_i $(1 \leq i \leq M)$.

i) Define the state space of this network and the normalising constant $g(M, N)$.

ii) Show that, for $M, N > 0$,

$$g(M, N) = g(M-1, N) + x_M\, g(M, N-1)$$

b i) Find the *marginal probability* that the equilibrium queue length at node i is k, in terms of a normalising constant function G_i, defined for a smaller network.

ii) Show that the equilibrium probability that the queue at node i has length k, given that there are at least h customers at queue $j \neq i$ is

$$\frac{G_i(N - h - k)}{G(N - h)} x_i^k$$

c Denoting the utilisation of node i at network population n by $U_i(n)$ and its mean queue length by $L_i(n)$, show that $L_i(n) = [1 + L_i(n-1)]\, U_i(n)$. Hence show that asymptotically, at large populations, *assuming the limits exist*, mean queue length $L_i(n) \rightarrow L_i = U_i/(1-U_i)$ at unsaturated nodes i, where $U_i(n) \rightarrow U_i < 1$ as $n \rightarrow \infty$.

The three parts carry, respectively, 35%, 35% and 30% of the marks.

End of paper