UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 1996

BEng Honours Degree in Computing Part III
BEng Honours Degree in Information Systems Engineering Part III
MEng Honours Degree in Information Systems Engineering Part III
BSc Honours Degree in Mathematics and Computer Science Part III
MSc Degree in Computing Science
MSc Degree in Foundations of Advanced Information Technology
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the
Diploma of Membership of Imperial College
Associateship of the City and Guilds of London Institute
Associateship of the Royal College of Science

PAPER 3.36 / I3.6

PERFORMANCE ANALYSIS

Tuesday, May 7th 1996, 10.00 - 12.00

Answer THREE questions

For admin. only: paper contains
4 questions
2 pages (excluding cover page)

1 a i) State the conditions for a discrete-time stochastic process $X = \{X_n \mid n=0,1,2,...\}$ to be a *Markov Chain* and define its *one-step transition probability matrix* Q;

ii) Give an example of each of a *reducible* and *periodic* Markov chain;

iii) State the *steady state theorem* for Markov chains, paying particular attention to the conditions under which it is valid. Give an informal justification of the theorem using the Law of Total Probability, *assuming* the steady state exists.

b In a *write-through* cache coherency protocol for a k-processor shared memory system with a total of N memory blocks, a memory block may be in any one of the states 0 (uncached) or i (cached by i of the processors), $1 \le i \le k$. Each cache has n lines and several memory blocks may map to the same line. Memory blocks are accessed uniformly at rate τ by each processor and can be regarded as independent. Transitions occur between states $i-1$ and i when any one of the $k-i+1$processors that does not have a copy in its cache reads the block. Transitions occur between states i and $i-1$ when any one of the i processors with a cached copy misses on the block. Moreover, whenever *any* processor *writes* to a block, it invalidates all cached copies in other processors and becomes the exclusive owner. Assuming that memory accesses are reads with probability α and (independently) cache-hits with probability β, derive a *continuous-time* Markov model for the state of a memory block. *Only* write down the balance equations. Be careful to distinguish accesses to a cache line holding a block and the block itself.

2 a i) Derive an expression for the *number of states* in a *closed* queueing network with M servers and population of N customers, where the state of a network is an M-tuple comprising the queue lengths at each server;

ii) State Jackson's Theorem for *closed* queueing networks with M *fixed-rate* servers and population of N customers;

iii) Define the *normalising constant* $g(M,N)$ for this network's equilibrium state probabilities in terms of the ratio, x_i, of *visitation rate* to *service rate* at server i, $1 \le i \le M$

b Show that, for $M, N > 0$, $g(M,N) = g(M-1,N) + x_M\, g(M,N-1)$.

c Find an expression for the *utilisation* of node i at equilibrium and show that the equilibrium probability that the queue at node i has length k, given that there are exactly h customers at queue j is

$$\frac{G_{ij}(N-h-k)}{G_j(N-h)}\, x_i^k$$

where G_i and G_{ij} are normalising constant functions for smaller networks which you must define.

The three parts carry, respectively, 40%, 30% and 30% of the marks.

3 a Explain briefly why the Markov property holds at departure instants in an M/G/1 queue. Hence outline the basic principles of the *embedded Markov chain method* for deriving the equilibrium queue length probability distribution in an M/G/1 queue at departure instants.

b State Little's result and apply it to write down an equation for the mean number of customers waiting to commence service. Hence show that the *mean queueing time* in an M/G/1 queue, with arrival rate λ and service time with mean $1/\mu$ and variance v, is

$$Q = \frac{\lambda(v+\mu^{-2})}{2(1-(\lambda/\mu))}$$

You may assume that a customer is being served just before an arrival instant with probability λ/μ and that the mean of its remaining service time is equal to $(v+\mu^{-2})\mu/2$.

c In a two-level bus architecture, requests on either bus also require the use of the other bus. The service times on both buses are the constant value $1/\mu$ (i.e. have zero variance). Requests of type A first queue for bus 1 and then for bus 2 whilst holding bus 1. Requests of type B first queue for bus 2.

i) Show that if type B requests queue for bus 1 whilst also holding bus 2, the system can deadlock;

ii) Suppose there is a buffer between the buses and that type B requests release bus 2 after being served before queueing for bus 1. Give expressions for the mean service times of type A requests at bus 2 and at bus 1, i.e. for the expected *total* time that they hold each bus. You may make the approximating assumption that the net arrival process at each bus is Poisson with rate 2λ.

4 a Derive the equations of *Mean Value Analysis,* given below, for a closed Markovian queueing network of M nodes and k customers, with *throughput* $T(k)$ along a specially chosen arc:

$$L_i(k) = v_i T(k) W_i(k); \quad W_i(k) = (1+L_i(k-1))/\mu_i\,; \quad k = T(k) \sum_{i=1}^{M} v_i W_i(k)$$

where v_i is the *visitation rate* of server i and μ_i is its service rate $(1 \le i \le M)$.

b i) By substituting the second equation into the third, or otherwise, show that $\dfrac{k}{M+k-1} \min\limits_{1\le i\le M} (\mu_i/v_i) \le T(k) \le \dfrac{k}{M+k-1} \max\limits_{1\le i\le M} (\mu_i/v_i)$

ii) Derive a recurrence relation for $L_i(k)$ in terms of $L_i(k-1)$, $(1 \le i \le M)$.

iii) Suppose there is exactly one *bottleneck server, b,* such that $\mu_b/v_b = \min\limits_{1\le i\le M} (\mu_i/v_i)$. Use a steady state argument to show that $T(k) < \mu_b/v_b$ for all $k>0$. Hence show that $T(k) \to \mu_b/v_b$ as $k \to \infty$ and that at a non-bottleneck server $i \ne b$, the mean queue length approaches $\dfrac{\mu_b v_i}{v_b\mu_i - \mu_b v_i}$.

End of paper