Imperial College London

M4/5S4

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May-June 2018

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Applied Probability

Date: Tuesday, 15 May 2018

Time: 2:00 PM - 4:30 PM

Time Allowed: 2.5 hours

This paper has 5 questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the relevant main book(s) are full.

All required additional material will be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Each question carries equal weight.
- Calculators may not be used.

1. (a) Define a discrete-time and time-homogeneous Markov chain.

   (b) Show that the transition matrix of a discrete-time time-homogeneous Markov chain is a stochastic matrix.

   (c) Consider a time-homogeneous Markov chain $(X_n)_{n\in\mathbb{N}_0}$ with state space $E = \{1,2,3,4,5,6\}$ and transition matrix given by

$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{9}{10} & \frac{1}{10} & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}.$$

      (i) Draw the transition diagram.

      (ii) Specify the communicating classes and determine whether they are transient, null recurrent or positive recurrent. *Please note that you need to justify your answers.*

      (iii) Find all stationary distributions.

2. Let $X = (X_n)_{n\in\mathbb{N}_0}$ denote a time-homogeneous Markov chain with countable state space E and transition matrix $\mathbf{P} = (p_{ij})$.

   (a) Define the first passage time as $f_{ij}(n) = \mathbb{P}(X_n = j, X_{n-1} \neq j, \ldots, X_1 \neq j | X_0 = i)$ for $i, j \in E$ and $n \in \mathbb{N}$. Show that the n-step transition probabilities $p_{ij}(n)$ satisfy the following equation

$$p_{ij}(n) = \sum_{l=0}^{n} f_{ij}(l) p_{jj}(n-l), \text{ for all } i, j \in E, n \in \mathbb{N},$$

   where we set $f_{ij}(0) = 0$ for all $i, j \in E$.

   (b) Show that if $j \in E$ is transient, then $\lim_{n\to\infty} p_{ij}(n) = 0$ for all $i \in E$.

   (c) Suppose that the Markov chain is irreducible and has a stationary distribution. Prove that all states are recurrent.

   Note that you need to state clearly any results from the lectures you use in your proofs of (a), (b), and (c).

3. (a) People arrive at a tube station according to a Poisson process of rate $\lambda = 3$ per hour. Assume that the tube station opens at 5am and we start counting people at that time.

   (i) What is the probability that exactly one person has arrived by 5:15am and a total of 10 people have arrived by 6:30am?

   (ii) Give an example of a class of stochastic processes which might be more suitable for describing the number of people arriving at a tube station than the Poisson process used in (i) and briefly justify your choice.

   (b) Consider random variables X, Y on a probability space $(\Omega, \mathcal{F}, P)$. Suppose that X follows the exponential distribution with parameter 1 and that the conditional distribution of Y given $X = x$ is given by the Poisson distribution with parameter x. Find the cumulative distribution function of Y.

   (c) Consider a non-homogeneous Poisson process $(N_t)_{t\geq 0}$ with intensity function $t \mapsto \lambda(t)$. For $0 < t_1 < t_2 < t_3$ and $n_1, n_2, n_3 \in \mathbb{N}_0$ with $0 \leq n_1 \leq n_2 \leq n_3$ find

$$P(N_{t_1} = n_1, N_{t_2} = n_2, N_{t_3} = n_3),$$

   expressing the answer in terms of appropriate integrals.

4. Consider a birth process $N = (N_t)_{t\geq 0}$ with rates $\lambda_0, \lambda_1, \ldots,$ such that $\lambda_i \neq \lambda_j$ for any $i \neq j$, and $N_0 = 0$. Define $p_n(t) = P(N_t = n)$ for $n \in \mathbb{N}_0$.

   (a) Derive an equation for $p_0'(t)$ in terms of $p_0(t)$.

   (b) Derive an equation for $p_n'(t)$ in terms of $p_n(t)$ and $p_{n-1}(t)$ valid for $n \in \mathbb{N}$.

   (c) Show that

$$p_0(t) = e^{-\lambda_0 t},$$
$$p_1(t) = \lambda_0 \left(\frac{1}{\lambda_1 - \lambda_0} e^{-\lambda_0 t} + \frac{1}{\lambda_0 - \lambda_1} e^{-\lambda_1 t} \right)$$

   are solutions to the differential equations derived in (a) and (b) (for $n = 1$).

   (d) Let T_n denoted the time of the nth birth. Show that

$$P(T_1 > t, T_2 > t + s) = p_0(t)[p_0(s) + p_1(s)] \text{ for } s, t > 0.$$

   (e) Use (c) and (d) to derive the joint density of (T_1, T_2). *Hint: You may use without proof that $p_0(t)$ and $p_1(t)$ given in part (c) are unique solutions to the differential equations derived in parts (a) and (b).*

5. Consider a probability space given by $(\Omega, \mathcal{F}, P)$ and denote by $\mathcal{B}$ the Borel σ-algebra.

(a) Let X be a random variable and $Y = f(X)$ for some Borel function $f : \mathbb{R} \mapsto \mathbb{R}$.

(i) Show that Y is a random variable.

(ii) Show that $\sigma(Y) \subset \sigma(X)$.

(b) Show that if X_n is a random variable for each $n \in \mathbb{N}$, then $\sup_{n\in\mathbb{N}} X_n$ is an extended random variable.

(c) Let X be a random variable. Define the probability measure induced by X as $P'(B) = P(X^{-1}(B))$ for every $B \in \mathcal{B}$. Show that P' is a probability measure on $(\mathbb{R}, \mathcal{B})$.

(d) Let $(\Omega, \mathcal{F}, P) = ([0,1], \mathcal{B}_{[0,1]}, \mu_L)$, where μ_L denotes the Lebesgue measure. Consider the random variable

$$X(\omega) = \frac{\omega}{1 + \mathbb{I}_{\{y:\ y > \frac{1}{2}\}}(\omega)}.$$

Note that $\mathbb{I}_{\{y:\ y>\frac{1}{2}\}}(\omega) = 1$ if $\omega > 1/2$ and $\mathbb{I}_{\{y:\ y>\frac{1}{2}\}}(\omega) = 0$ if $\omega \le 1/2$. Find $P'((1/4, 3/4))$, where P' is the probability measure induced by X.

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2018

This paper is also taken for the relevant examination for the Associateship.

M3/4/5 S4

Applied Probability (Solutions)

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1. (a) A discrete-time stochastic process $\{X_n\}_{n\in\mathbb{N}_0}$ on a finite or countably infinite state space E is a Markov chain if it satisfies the Markov condition seen ⇓

$$\mathbb{P}(X_n = x_n | X_{n-1} = x_{n-1}, \ldots, X_0 = x_0) = \mathbb{P}(X_n = x_n | X_{n-1} = x_{n-1})$$

for all $n \in \mathbb{N}$ and for all $x_0, \ldots, x_{n-1}, x_n \in E$. 3

The Markov chain $\{X_n\}_{n\in\mathbb{N}_0}$ is time-homogeneous if

$$\mathbb{P}(X_{n+1} = j | X_n = i) = \mathbb{P}(X_1 = j | X_0 = i)$$

for every $n \in \mathbb{N}_0$ and for all $i, j \in E$. 2

(b) As in (a) we denote by E the state space and set $K = |E|$. Then we denote by $\mathbf{P} = (p_{ij})_{i,j\in E}$ the $K \times K$ matrix of transition probabilities $p_{ij} = \mathbb{P}(X_1 = j | X_0 = i)$ for $i, j \in E$. $\mathbf{P}$ is a stochastic matrix since meth seen ⇓

1.) each element p_{ij} (for $i, j \in E$) is non-negative since it is a conditional probability, 2

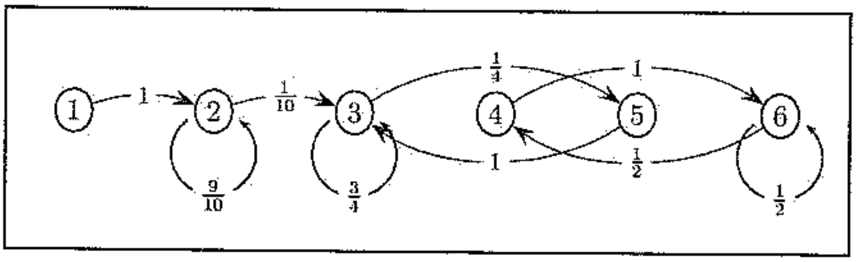
2.) the sum of the elements in each row is equal to one. To see this, note that for any $i \in E$ we have –using the law of total probability–

$$\sum_{j\in E} p_{ij} = \sum_{j\in E} \mathbb{P}(X_1 = j | X_0 = i) = \sum_{j\in E} \frac{\mathbb{P}(X_1 = j, X_0 = i)}{\mathbb{P}(X_0 = i)} = \frac{\mathbb{P}(X_0 = i)}{\mathbb{P}(X_0 = i)} = 1.$$

2

meth seen ⇓

(c) (i) The transition diagram is given by



2

(ii) We have a finite state space which can be divided into four communicating classes: The classes $T_1 = \{1\}, T_2 = \{2\}$ are not closed and hence transient. 2

The classes $C_1 = \{3, 5\}, C_2 = \{4, 6\}$ are finite and closed and hence positive recurrent. 2

(iii) This Markov chain does not have a unique stationary distribution π since we have two closed (essential) communicating classes. For the transient states we know from the lectures that $\pi_i = 0$ for $i = 1, 2$. For the two closed classes we need to solve the following system of equations: 1

Class C_1: $(\pi_3, \pi_5)\mathbf{P}_{C_1} = (\pi_3, \pi_5) \Leftrightarrow (\pi_3, \pi_5)\begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ 1 & 0 \end{pmatrix} = (\pi_3, \pi_5) \Leftrightarrow \frac{3}{4}\pi_3 + \pi_5 = \pi_3, \frac{1}{4}\pi_3 = \pi_5 \Leftrightarrow \frac{1}{4}\pi_3 = \pi_5$.

Class C_2: $(\pi_4, \pi_6)\mathbf{P}_{C_2} = (\pi_4, \pi_6) \Leftrightarrow (\pi_4, \pi_6)\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = (\pi_4, \pi_6) \Leftrightarrow \frac{1}{2}\pi_6 = \pi_4, \pi_4 + \frac{1}{2}\pi_6 = \pi_6 \Leftrightarrow \frac{1}{2}\pi_6 = \pi_4$. 2

Altogether, we conclude that the stationary distributions are given by $\pi = (0, 0, \pi_3, \frac{1}{2}\pi_6, \frac{1}{4}\pi_3, \pi_6)$ for all $\pi_3, \pi_6 \geq 0$ with $\frac{5}{4}\pi_3 + \frac{3}{2}\pi_6 = 1$. 2

2. (a) Let $i, j \in E$, $n \in \mathbb{N}$. We define the disjoint events $A_l := \{X_l = j, X_r \neq j, \text{for } 1 \leq r < l\}$ for $l = 1, \ldots, n$. $\boxed{\text{sim. seen} \Downarrow}$ $\boxed{2}$

Applying the law of total probability and the Markov property leads to

$$
\begin{aligned}
p_{ij}(n) &= \mathbb{P}(X_n = j | X_0 = i) = \sum_{l=1}^{n} \mathbb{P}(\{X_n = j\} \cap A_l | X_0 = i) \\
&= \sum_{l=1}^{n} \frac{\mathbb{P}(\{X_n = j\} \cap A_l \cap \{X_0 = i\})}{\mathbb{P}(X_0 = i)} \cdot \frac{\mathbb{P}(A_l \cap \{X_0 = i\})}{\mathbb{P}(A_l \cap \{X_0 = i\})} \text{ (law of total probability)} \\
&= \sum_{l=1}^{n} \mathbb{P}(\{X_n = j\} | A_l \cap \{X_0 = i\}) \mathbb{P}(A_l | X_0 = i) \\
&= \sum_{l=1}^{n} \mathbb{P}(X_n = j | X_l = j) \mathbb{P}(A_l | X_0 = i) \text{ (Markov property)} \\
&= \sum_{l=1}^{n} p_{jj}(n-l) f_{ij}(l) = \sum_{l=0}^{n} f_{ij}(l) p_{jj}(n-l), \text{ since } f_{ij}(0) = 0.
\end{aligned}
$$

$\boxed{5}$

$\boxed{\text{seen} \Downarrow}$

(b) Recall from lectures that $j \in E$ is transient if and only if $\sum_{n=1}^{\infty} p_{jj}(n) < \infty$. A necessary condition for the convergence of the infinite series is that $\lim_{n\to\infty} p_{jj}(n) = 0$. Further, we can deduce from (a) that $p_{ij}(n) = \sum_{l=0}^{n} f_{ij}(n-l) p_{jj}(l)$, for $i, j \in E, n \in \mathbb{N}$. Then for any $i \in E$ and $n \in \mathbb{N}$, we have $\boxed{1}$

$$
\begin{aligned}
\sum_{n=0}^{\infty} p_{ij}(n) &= \sum_{n=0}^{\infty} \sum_{l=0}^{n} f_{ij}(n-l) p_{jj}(l) = \sum_{l=0}^{\infty} p_{jj}(l) \sum_{n=l}^{\infty} f_{ij}(n-l) \\
&= \sum_{l=0}^{\infty} p_{jj}(l) \underbrace{\sum_{n=0}^{\infty} f_{ij}(n)}_{\leq 1} \leq \sum_{l=0}^{\infty} p_{jj}(l) < \infty.
\end{aligned}
$$

Hence $\sum_{n=0}^{\infty} p_{ij}(n) < \infty$, which implies that $\lim_{n\to\infty} p_{ij}(n) = 0$ for any $i \in E$. $\boxed{4}$

(c) Suppose that π is the stationary distribution of the Markov chain. Assume there exists a transient state. Then all states are transient since the chain is irreducible. If all states are transient then $\lim_{n\to\infty} p_{ij}(n) = 0$ for all i, j by (b). Since $\boldsymbol{\pi}\mathbf{P}^n = \boldsymbol{\pi}$, for any j, we have $\boxed{1}$ $\boxed{\text{seen} \Downarrow}$

$$
\pi_j = \sum_{i \in E} \pi_i p_{ij}(n) \to 0, \quad \text{as } n \to \infty,
$$

thus, π could not be a stationary vector, which is a contradiction; hence all states are recurrent. $\boxed{4}$

Since the state space is not necessarily finite, the switching of the order of summation and limits is justified by the **Dominated Convergence Theorem:** If $\sum_i a_i(n)$ is an absolutely convergent series for all $n \in \mathbb{N}$ such that (1) for all i the limit $\lim_{n\to\infty} a_i(n) = a_i$ exists, (2) there exists a sequence $(b_i)_i$, such that $b_i \geq 0$ for all i and $\sum_i b_i < \infty$ such that for all $n, i : |a_i(n)| \leq b_i$. Then $\sum_i |a_i| < \infty$ and $\sum_i a_i = \sum_i \lim_{n\to\infty} a_i(n) = \lim_{n\to\infty} \sum_i a_i(n)$.

Here we have $a_i(n) = \pi_i p_{ij}(n)$. Clearly, $\sum_i a_i(n)$ is absolutely convergent for all n since $\sum_i |\pi_i p_{ij}(n)| = \sum_i \pi_i p_{ij}(n) = \pi_j \leq 1 < \infty$. Also $\lim_{n\to\infty} a_i(n) = 0 =: a_i$ for all i. Next, $|a_i(n)| = \pi_i p_{ij}(n) \leq \pi_i =: b_i \geq 0$ and $\sum_i b_i = \sum_i \pi_i = 1 < \infty$. $\boxed{3}$

3. (a) (i) We denote the Poisson process with rate $\lambda = 3$ by N. Since time is measured in hours starting from 5am, we need to find the joint probability that $P(N_{1/4} = 1, N_{3/2} = 10)$. Using the independence and stationarity of the increments, we deduce that meth seen ⇓

$$P(N_{1/4} = 1, N_{3/2} = 10) = P(N_{1/4} = 1, N_{3/2} - N_{1/4} = 10 - 1)$$
$$= P(N_{1/4} = 1, N_{3/2} - N_{1/4} = 9) \overset{\text{indep. incr.}}{=} P(N_{1/4} = 1)P(N_{3/2} - N_{1/4} = 9)$$
$$\overset{\text{stat. incr.}}{=} P(N_{1/4} = 1)P(N_{5/4} = 9) = e^{-3/4}\frac{(3/4)^1}{1!}e^{-15/4}\frac{(15/4)^9}{9!} = e^{-9/2}\frac{3}{4}\frac{(15/4)^9}{9!},$$

since $N_{1/4} \sim \text{Poi}(\frac{3}{4}), N_{5/4} \sim \text{Poi}(\frac{15}{4})$. $\boxed{4}$

(ii) It would be better to work with a non-homogeneous Poisson process here since it would be able to allow for time varying arrival rates $\lambda(t)$ taking e.g. the beginning of the rush hour into account. $\boxed{2}$

(b) We note that the density of X is given by $f_X(x) = \exp(-x)$ for $x \geq 0$ and 0 otherwise. Also $F_{Y|X=x}(y|x) = P(Y \leq y|X = x) = \sum_{k=0}^{\lfloor y \rfloor} e^{-x}\frac{x^k}{k!}$ for $y \geq 0$ and 0 otherwise. meth seen ⇓ $\boxed{2}$

Here we work with the continuous version of the law of total probability to deduce that for $y \geq 0$, we have

$$P(Y \leq y) = \int_0^\infty P(Y \leq y|X = x)f_X(x)dx = \int_0^\infty \sum_{k=0}^{\lfloor y \rfloor} e^{-x}\frac{x^k}{k!}e^{-x}dx$$
$$= \sum_{k=0}^{\lfloor y \rfloor} \frac{1}{k!}\int_0^\infty x^k e^{-2x}dx \overset{z=2x}{=} \sum_{k=0}^{\lfloor y \rfloor} \frac{1}{k!}\int_0^\infty z^{(k+1)-1}e^{-z}dz2^{-(k+1)}$$
$$= \sum_{k=0}^{\lfloor y \rfloor} \frac{\Gamma(k+1)}{k!}2^{-(k+1)} = \frac{1}{2}\sum_{k=0}^{\lfloor y \rfloor}\frac{1}{2^k} \overset{\text{geom. series}}{=} \frac{1}{2}\frac{1-\frac{1}{2^{\lfloor y \rfloor+1}}}{1-\frac{1}{2}} = 1 - \frac{1}{2^{\lfloor y \rfloor+1}}.$$

Hence the cumulative distribution function of Y is given by $\boxed{4}$

$$F_Y(y) = \begin{cases} 1 - \frac{1}{2^{\lfloor y \rfloor+1}}, & \text{for } y \geq 0, \\ 0, & \text{for } y < 0. \end{cases}$$

$\boxed{2}$

(c) For $0 < t_1 < t_2 < t_3$ and $n_1, n_2, n_3 \in \mathbb{N}_0$ with $0 \leq n_1 \leq n_2 \leq n_3$ we have unseen ⇓

$$A := P(N_{t_1} = n_1, N_{t_2} = n_2, N_{t_3} = n_3)$$
$$= P(N_{t_1} = n_1, N_{t_2} - N_{t_1} = n_2 - n_1, N_{t_3} - N_{t_2} = n_3 - n_2)$$
$$= P(N_{t_1} = n_1)P(N_{t_2} - N_{t_1} = n_2 - n_1)P(N_{t_3} - N_{t_2} = n_3 - n_2),$$

by the independent increment property. Also, note that for $0 \leq s < t$, we have that $N_t - N_s \sim \text{Poi}\left(\int_s^t \lambda(u)du\right)$. Hence $\boxed{3}$

$$A = \frac{\left(\int_0^{t_1}\lambda(u)du\right)^{n_1}}{n_1!}\exp\left(-\int_0^{t_1}\lambda(u)du\right)$$
$$\cdot\frac{\left(\int_{t_1}^{t_2}\lambda(u)du\right)^{n_2-n_1}}{(n_2-n_1)!}\exp\left(-\int_{t_1}^{t_2}\lambda(u)du\right)\frac{\left(\int_{t_2}^{t_3}\lambda(u)du\right)^{n_3-n_2}}{(n_3-n_2)!}\exp\left(-\int_{t_2}^{t_3}\lambda(u)du\right)$$
$$= \frac{\left(\int_0^{t_1}\lambda(u)du\right)^{n_1}\left(\int_{t_1}^{t_2}\lambda(u)du\right)^{n_2-n_1}\left(\int_{t_2}^{t_3}\lambda(u)du\right)^{n_3-n_2}}{n_1!(n_2-n_1)!(n_3-n_2)!}\exp\left(-\int_0^{t_3}\lambda(u)du\right).$$

$\boxed{3}$

4. (a) Let $\delta > 0$. Then, using the initial condition, we have

meth seen ⇓

$$p_0(t+\delta) = \mathbb{P}(N_{t+\delta} = 0 | N_t = 0)\mathbb{P}(N_t = 0) = (1 - \lambda_0\delta)p_0(t) + o(\delta).$$

Then

2

$$\lim_{\delta \downarrow 0} \frac{p_0(t+\delta) - p_0(t)}{\delta} = p_0'(t) = -\lambda_0 p_0(t).$$

2

(b) Let $\delta > 0$ and $n \in \mathbb{N}$. Then, using the law of total probability, we have

meth seen ⇓

$$p_n(t+\delta) = \sum_{i=0}^{n} \mathbb{P}(N_{t+\delta} = n | N_t = i)\mathbb{P}(N_t = i) = \lambda_{n-1}\delta p_{n-1}(t) + (1 - \lambda_n\delta)p_n(t) + o(\delta),$$

where we used the single arrival property of a birth process. Then

2

$$\lim_{\delta \downarrow 0} \frac{p_n(t+\delta) - p_n(t)}{\delta} = p_n'(t) = \lambda_{n-1}p_{n-1}(t) - \lambda_n p_n(t).$$

2

(c) Differentiating $p_0(t)$ and $p_1(t)$ leads to

meth seen ⇓

$$p_0'(t) = -\lambda_0 e^{-\lambda_0 t} = -\lambda_0 p_0(t),$$

$$p_1'(t) = -\frac{\lambda_0^2}{\lambda_1 - \lambda_0}e^{-\lambda_0 t} - \frac{\lambda_0\lambda_1}{\lambda_0 - \lambda_1}e^{-\lambda_1 t},$$

also

$$\lambda_0 p_0(t) - \lambda_1 p_1(t) = \lambda_0 e^{-\lambda_0 t} - \lambda_0\lambda_1\left(\frac{1}{\lambda_1 - \lambda_0}e^{-\lambda_0 t} + \frac{1}{\lambda_0 - \lambda_1}e^{-\lambda_1 t}\right)$$

$$= \left(\lambda_0 - \frac{\lambda_0\lambda_1}{\lambda_1 - \lambda_0}\right)e^{-\lambda_0 t} - \frac{\lambda_0\lambda_1}{\lambda_0 - \lambda_1}e^{-\lambda_1 t} = -\frac{\lambda_0^2}{\lambda_1 - \lambda_0}e^{-\lambda_0 t} - \frac{\lambda_0\lambda_1}{\lambda_0 - \lambda_1}e^{-\lambda_1 t} = p_1'(t).$$

4

(d) We note that $T_1 > t \Leftrightarrow N_t = 0$. Also, if $T_1 > t, T_2 > t+s$, this implies that at time $t+s$, N_{t+s} can either take the value 0 or 1. Hence for $s, t > 0$ we have (using the definition of conditional probability, time homogeneity and the initial condition)

unseen ⇓

4

$$P(T_1 > t, T_2 > t+s) = P(N_t = 0, N_{t+s} = 0) + P(N_t = 0, N_{t+s} = 1)$$

$$= P(N_{t+s} = 0 | N_t = 0)P(N_t = 0) + P(N_{t+s} = 1 | N_t = 0)P(N_t = 0)$$

$$\overset{\text{time homogeneity}}{=} P(N_s = 0 | N_0 = 0)P(N_t = 0) + P(N_s = 1 | N_0 = 0)P(N_t = 0)$$

$$= p_0(t)[p_0(s) + p_1(s)].$$

(e) We denote by $f_{(T_1,T_2)}$ the joint density of (T_1, T_2). Using (c) and (d) we get for $t_2 > t_1 > 0$:

unseen ⇓

4

$$P(T_1 > t_1, T_2 > t_2) = p_0(t_1)[p_0(t_2 - t_1) + p_1(t_2 - t_1)]$$

$$= e^{-\lambda_0 t_1}\left[e^{-\lambda_0(t_2 - t_1)} + \lambda_0\left(\frac{1}{\lambda_1 - \lambda_0}e^{-\lambda_0(t_2 - t_1)} + \frac{1}{\lambda_0 - \lambda_1}e^{-\lambda_1(t_2 - t_1)}\right)\right]$$

$$= e^{-\lambda_0 t_2} + \frac{\lambda_0}{\lambda_1 - \lambda_0}e^{-\lambda_0 t_2} + \frac{\lambda_0}{\lambda_0 - \lambda_1}e^{-\lambda_1 t_2}e^{-(\lambda_0 - \lambda_1)t_1}$$

$$= \frac{\lambda_1}{\lambda_1 - \lambda_0}e^{-\lambda_0 t_2} + \frac{\lambda_0}{\lambda_0 - \lambda_1}e^{-\lambda_1 t_2}e^{-(\lambda_0 - \lambda_1)t_1} = \int_{t_1}^{\infty}\int_{t_2}^{\infty} f_{(T_1,T_2)}(u, v)\,du\,dv.$$

Hence the joint density of (T_1, T_2) is given by

$$f_{(T_1,T_2)}(t_1, t_2) = \frac{\partial^2}{\partial t_1 \partial t_2}P(T_1 > t_1, T_2 > t_2)$$

$$= \frac{\lambda_0}{\lambda_0 - \lambda_1}e^{-\lambda_1 t_2}e^{-(\lambda_0 - \lambda_1)t_1}(-\lambda_1)(-(\lambda_0 - \lambda_1)) = \lambda_0\lambda_1 e^{-\lambda_1 t_2}e^{-(\lambda_0 - \lambda_1)t_1},$$

for $t_2 > t_1 > 0$ and 0 otherwise.

5. (a) (i) Y is $\mathcal{F}$-measurable since for any $B \in \mathcal{B}$, $Y^{-1}(B) = \{\omega \in \Omega : Y(\omega) \in B\} = \{\omega \in \Omega : f(X(\omega)) \in B\} = \{\omega \in \Omega : X(\omega) \in f^{-1}(B)\} = X^{-1}(\underbrace{f^{-1}(B)}_{\in \mathcal{B}}) \in \mathcal{F}$, since f is Borel and X is $\mathcal{F}$-measurable. unseen ⇓ $\boxed{3}$

(ii) We show that for any $A \in \sigma(Y)$ it also holds that $A \in \sigma(X)$. For any $A \in \sigma(Y)$ there exists a $B \in \mathcal{B}$ such that $A = Y^{-1}(B) = X^{-1}(f^{-1}(B)) \in \sigma(X)$ since $f^{-1}(B) \in \mathcal{B}$. $\boxed{2}$

(b) Recall that a necessary and sufficient condition for $\mathcal{F}$-measurability of a random variable X is that $X^{-1}((-\infty, x]) \in \mathcal{F}$ for all $x \in \mathbb{R}$. Set $Y = \sup_{n\in\mathbb{N}} X_n$. For any $x \in \mathbb{R}$ we have that

$$\begin{aligned} Y^{-1}((-\infty, x]) &= \{\omega \in \Omega : Y(\omega) \leq x\} = \{\omega \in \Omega : \sup_{n\in\mathbb{N}} X_n(\omega) \leq x\} \\ &= \cap_{n\in\mathbb{N}}\{\omega \in \Omega : X_n(\omega) \leq x\} = \cap_{n\in\mathbb{N}} X_n^{-1}((-\infty, x]) \in \mathcal{F}, \end{aligned}$$

since for all $n \in \mathbb{N}$ we have that $X_n^{-1}((-\infty, x]) \in \mathcal{F}$ and $\mathcal{F}$ is closed under countable intersection. Note that in the above computation, one can also replace $(-\infty, x]$ by $[-\infty, x]$. Hence Y is an (extended) random variable. $\boxed{5}$

(c) P' is a measure since it

1. is a nonnegative set function since P is a probability measure; and it $\boxed{1}$
2. is countable additive since for any disjoint $A_n \in \mathcal{B}$

$$\begin{aligned} P'(\cup_n A_n) &= P(X^{-1}(\cup_n A_n)) = P(\cup_n X^{-1}(A_n)) = \sum_n P(X^{-1}(A_n)) \\ &= \sum_n P'(A_n), \end{aligned}$$

where we used the countable additivity of P and the fact that the $X^{-1}(A_n)$ are disjoint. $\boxed{3}$

The measure P' is a probability measure since it also satisfies $P'(\mathbb{R}) = P(X^{-1}(\mathbb{R})) = P(\Omega) = 1$. $\boxed{1}$

(d) Note that we can write X as

$$X(\omega) = \frac{\omega}{1 + \mathbb{I}_{\{y:\, y > \frac{1}{2}\}}(\omega)} = \begin{cases} \omega, & \text{for } 0 \leq \omega \leq \frac{1}{2}, \\ \frac{\omega}{2}, & \text{for } \frac{1}{2} < \omega \leq 1. \end{cases}$$

Then

$$X^{-1}\left(\left(\frac{1}{4}, \frac{3}{4}\right)\right) = \left\{\omega \in [0,1] : X(\omega) \in \left(\frac{1}{4}, \frac{3}{4}\right)\right\} = \left(\frac{1}{4}, \frac{1}{2}\right] \cup \left(\frac{1}{2}, 1\right] = \left(\frac{1}{4}, 1\right].$$

Hence $P'((1/4, 3/4)) = \mu_L\left(\left(\frac{1}{4}, 1\right]\right) = \frac{3}{4}$. $\boxed{5}$