E1.5
Solutions 2008

Special instructions for students

Fundamental constants

Permittivity of free space, ε_0 = 8.85 x 10^{-12} F/m
Permeability of free space, μ_0 = 4π x 10^{-7} H/m
Planck's constant, h = 6.6 x 10^{-34} Js
Boltzmann's constant, k = 1.38 x 10^{-23} J/K
Electron charge, e = 1.6 x 10^{-19} C
Electron mass, m = 9.1 x 10^{-31} kg
Speed of light, c = 3.0 x 10^{8} ms^{-1}

Schrödinger's equation

General form:

$$-\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r}) + V(\mathbf{r})\psi(\mathbf{r}) = E\psi(\mathbf{r})$$

In one dimension:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x)$$

In spherical coordinates:

$$\nabla^2 = \frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}$$

Free-electron theory

Density of states (3D):

$$g(E) = \frac{1}{\pi^2\hbar^3}(m)^{3/2}\sqrt{2E}$$

Fermi energy

$$E_f = \frac{\hbar^2\pi^2}{2m}\left(\frac{3n}{\pi}\right)^{2/3}$$

Fermi distribution

$$f(E) = \frac{1}{1+\exp\left(\frac{E-E_f}{kT}\right)}$$

Electrons in semiconductors

Effective mass:

$$m_e^* = \frac{\hbar^2}{d^2E(k)/dk^2}$$

Concentration of electrons in a semiconductor of bandgap E_g:

$$n = \frac{1}{\sqrt{2}\hbar^3}\left(\frac{m_e^* kT}{\pi}\right)^{3/2} e^{-\frac{(E_g - E_f)}{kT}}$$

$$= N_c e^{-\frac{(E_g - E_f)}{kT}}$$

Concentration of holes

$$p = \frac{1}{\sqrt{2}\hbar^3}\left(\frac{m_h^* kT}{\pi}\right)^{3/2} e^{-\frac{E_f}{kT}}$$

$$= N_v e^{-\frac{E_f}{kT}}$$

Polarization

Lorentz correction for local field:

$$\mathbf{E}_{loc} = \mathbf{E} + \frac{\mathbf{P}}{3\varepsilon_0}$$

Electronic polarization:

$$P_0 = \frac{\varepsilon_0 \omega_p^2 E_0}{\omega_m^2 - \omega^2 + j\omega\gamma}$$

where

$$\gamma = \frac{r}{m},$$

$$\omega_m^2 = \omega_0^2 - \frac{\omega_p^2}{3},$$

$$\omega_0^2 = k/m,$$

$$\omega_p^2 = \frac{ne^2}{m\varepsilon_0}.$$

Orientational Polarization:

Static:

$\mathrm{P} = n\mu\, \mathrm{L}(\mu E/kT)$ where $\mathrm{L}(x) = \coth(x) - 1/x$

Dynamic:

$$P_0 = \frac{P_s}{1 + \mathrm{j}\omega\tau},$$

Magnetism

Magnet dipole due to electron angular momentum:

$$\boldsymbol{\mu}_m = -\frac{e\mathbf{L}}{2m}$$

Magnet dipole due to electron spin:

$$\boldsymbol{\mu}_m = -\frac{e\mathbf{S}}{m}$$

Paramagnetism:

$$M = n\mu_m \mathrm{L}\left(\frac{\mu_m \mu_0 H}{kT}\right)$$

The Answers

1. [Compulsory]

a) *A device contains a large resistor fabricated from a block of conductive material. with two contacts on opposite faces. To miniaturise the resistor, its height, width and length are halved. How does its resistance change?* [4]

As the resistance of a block of material is given by $R = \dfrac{\rho l}{A}$, where r is the resistivity, A the cross-sectional area and l the length, if all the dimensions are halved, l is halved and A is quartered. Hence the resistance is doubled.

[new application of theory]

b) *Explain how the Hall effect can produce a voltage whose sign depends on the charge of the current carrier.* [4]

To produce the Hall effect, a magnetic field **B** is applied perpendicular to the current flow. This produces a Lorentz force, $q\,\mathbf{v} \times \mathbf{B}$ on the carriers. If the current is made of electrons or holes, q and **v** will swap signs to produce the same current, and hence both types of carriers will feel a force in the same direction. Hence depending on the carrier either positive or negative carriers will accumulate on the same side of the conductor, producing a Hall voltage of opposite sign.

[bookwork]

c) *What is the angle between the (100) and the (111) planes of silicon?* [4]

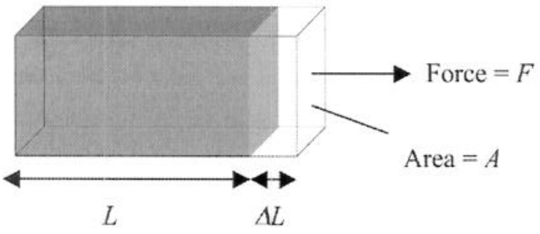
$\mathbf{a}.\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta$. Hence in this case $\cos\theta = 1/\sqrt{(3)}$ and so $\theta = 54.7°$.

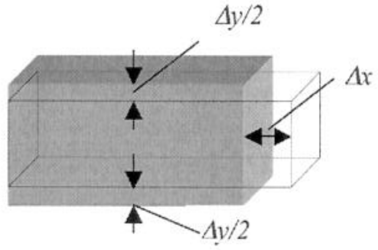
[new application of theory]

d) *For a block of elastic material. using diagrams define Young's modulus, the shear modulus and Poisson's ratio* [8]

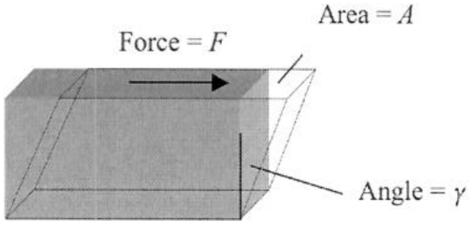
Young's modulus: $E = \dfrac{F/A}{\Delta L/L}$

Poisson's Ratio: $\sigma = \dfrac{\Delta y}{\Delta x}$





Shear Modulus: $G = \dfrac{F/A}{\gamma}$



[bookwork]

e) *The equation for the vertical deflection v(x) of a cantilever at distance x from its support for a load P at its end can be expressed by* $v = \dfrac{P}{EI}\left(\dfrac{lx^2}{2} - \dfrac{x^3}{6}\right)$. *Define E and I and derive an expression for the angular deflection from the horizontal of the end of a cantilever of length l.* [8]

E is Young's modulus for the material, I is the second moment of the area. The angular slope will be given by dv/dx at $x = l$:

$$\frac{dv}{dx} = \frac{P(lx - x^2/2)}{EI}$$

which for $x = l$ gives

$$\frac{dv}{dx} = \frac{3Pl^2}{4EI}$$

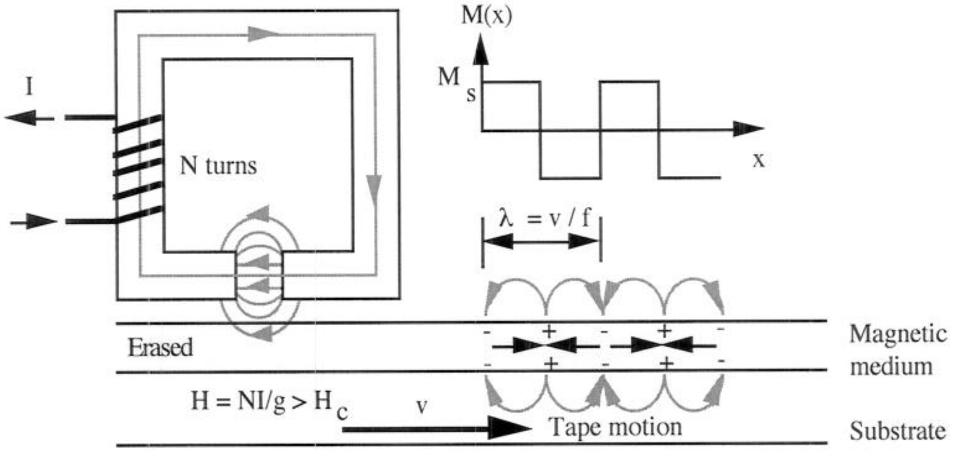
[new application of theory]

f) *Explain how polarisation at the atomic scale of a dielectric material between the plates of a capacitor can increase the capacitance.* [4]

When a dielectric is put into the field between the plates of a capacitor the electrons will be displaced from their equilibrium position. Although overall the dielectric remains neutral, a negative surface charge is produced on the surface of the dielectric facing the positive plate with an equal and opposite negative surface charge on the surface of the dielectric facing the positive plate. These surface charges attract additional positive and negative charges to the plates of the capacitor, increasing the capacitance.

[bookwork]

g) *Draw a diagram of a magnetic tape write head, illustrating its principle of operation.* *[8]*



[bookwork] [8]

2.

a) *Show that a solution of the one-dimensional Schroedinger equation for wavefunction $\psi(x)$ for a particle of mass m and total energy E in a constant potential V,*

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2}+V\psi(x)=E\psi(x).$$

is $\psi(x) = A\exp(jkx)$ (where A is a constant) with the wavenumber, k, of the particle given by:

$$k=\pm\frac{\sqrt{2m(E-V)}}{\hbar}.$$ *[10]*

$$\frac{d^2}{dx^2}A\exp(jkx)=-Ak^2\exp(jkx)$$ [2]

Substituting into

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2}+V\psi(x)=E\psi(x)$$

and dividing through by A exp(jkx) gives:

$$\frac{\hbar^2}{2m}k^2+V=E$$ [4]

and so

$$k=\pm\frac{\sqrt{2m(E-V)}}{\hbar}.$$ [2]

Hence $\psi(x) = A$ exp(jkx) is a solution with

$$k=\pm\frac{\sqrt{2m(E-V)}}{\hbar}.$$ [2]

[bookwork]

b) *A particle of energy E is travelling in a potential which is 0 for $x < 0$ and V, where $V > E$ for $x > 0$. Show that the probability of finding the particle at any value of x is independent of x for $x < 0$ but decays exponentially with distance for $x > 0$ with the probability of the particle reaching a distance $x = L$ given by*

$$\exp\left(-2L\frac{\sqrt{2m(V-E)}}{\hbar}\right).$$ *[10]*

The probability of finding a particle at any point is proportional $\psi\,\psi^*$ or $|\psi|^2$.

Hence when $x < 0$, $E > V$, k is a real number and

A exp(jkx) x [A exp(jkx)]* = A exp(jkx) x A exp(-jkx) = A^2

and so is a constant. [4]

When $x > 0$, $V > E$, k is an imaginary and can be written as jκ, where κ is real and given by

$$\kappa = \pm \frac{\sqrt{2m(V-E)}}{\hbar}$$ [2]

Substituing in to ψ:

$A \exp(\pm\kappa x) \text{ x } [A \exp(\pm\kappa x)]^* = A^2 \exp(\pm 2\kappa x)$ [2]

For an electron travelling in the positive x direction, only the negative exponent gives a physical solution, and the probability falls off exponentially with distance L. Dividing $|\psi(x = \tilde{L})|^2$ by $|\psi(x = 0)|^2$ gives the required probability:

$$\exp\left(-2L\frac{\sqrt{2m(V-E)}}{\hbar}\right)$$ [3]

[new application of theory/bookwork]

c) *Electrons accelerated by 2 V are incident on a potential energy barrier 2.1 eV high. How thick must the barrier be to ensure that only half the electrons tunnel through?* *[8]*

The origin can be set to be at the start of the barrier.

From

$$\Pr = \exp\left(-2L\frac{\sqrt{2m(V-E)}}{\hbar}\right)$$

if Pr = 0.5,

$$\begin{aligned} L &= \ln 2 \frac{\hbar}{2\sqrt{2m(V-E)}} \\ &= \frac{\ln 2 \times 1.03 \times 10^{-34}}{2 \times \sqrt{2 \times 9.1 \times 10^{-31} \times 1.6 \times 10^{-19} \times 0.1}} \\ &= 0.94 \text{ nm} \end{aligned}$$ [8]

[new computed example]

3.

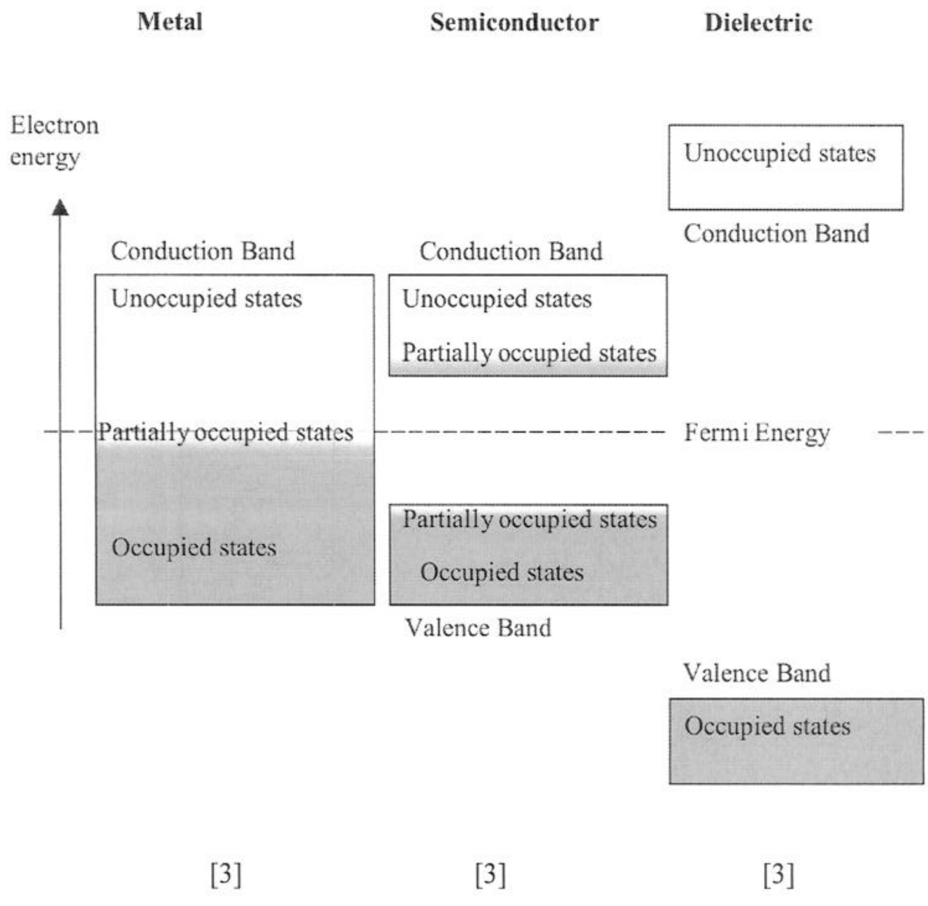
a) *Draw labelled band diagrams for*

(i) a typical conductor,

(ii) an intrinsic semiconductor and

(iii) a dielectric.

All have the same Fermi energy. The diagrams should label the bands and indicate the what regions of the bands have occupied, unoccupied or partially occupied states at room temperature. *[9]*



[bookwork/new application of theory]

b) *In a metal , what is the occupancy of an electron state located at the Fermi energy? For an intrinsic semiconductor of bandgap 1 eV. calculate the occupancy of the states at the bottom of the conduction band. Show that a 10 C rise in temperature will increase the concentration of electrons in the conduction band of the semiconductor by a factor of nearly two.* *[9]*

The occupancy of the states is given by the Fermi distribution:

$$f(E) = \frac{1}{1+\exp\left(\frac{E-E_f}{kT}\right)}.$$

When $E = E_f$, $f(E) = 0.5$ [2]

For an intrinsic semiconductor, the Fermi level is mid gap, so for a bandgap of 1 eV, $E_c - E_f = 0.5$ eV for states at the bottom of the conduction band. [2]

Hence the occupancy will be given by:

$$f(E_c) = \frac{1}{1+\exp\left(\frac{0.5}{0.025}\right)}$$

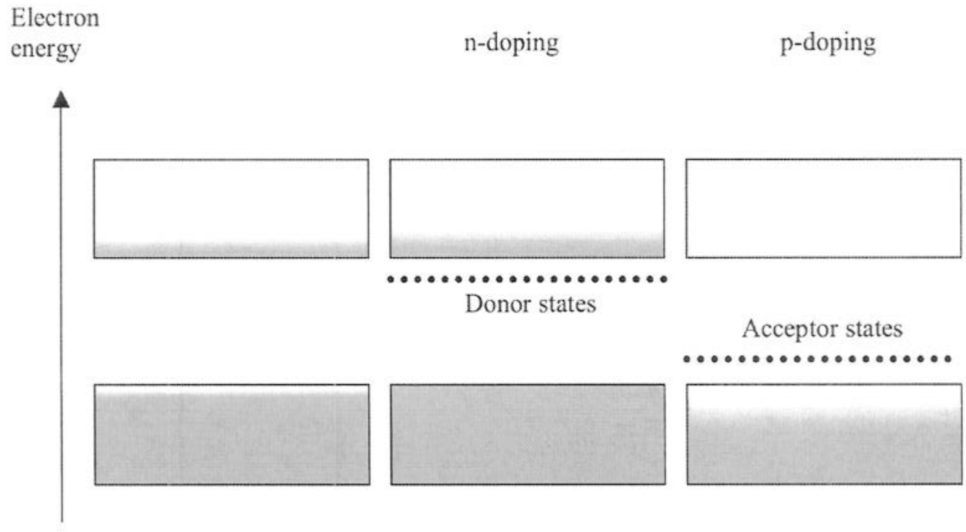
$$\approx e^{-20}$$ [2]

$$= 2 \times 10^{-9}$$

Increasing the temperature by 10 C increases the thermal energy, kT by 1/30. Hence the new occupancy will be increased by exp(2/3) = 1.95 or nearly a factor of two. [3]

[new application of theory]

c) *Using band diagram explain how doping can be used to alter the occupancy of states in a semiconductor and therefore affect the concentration of electrons and holes.* *[12]*



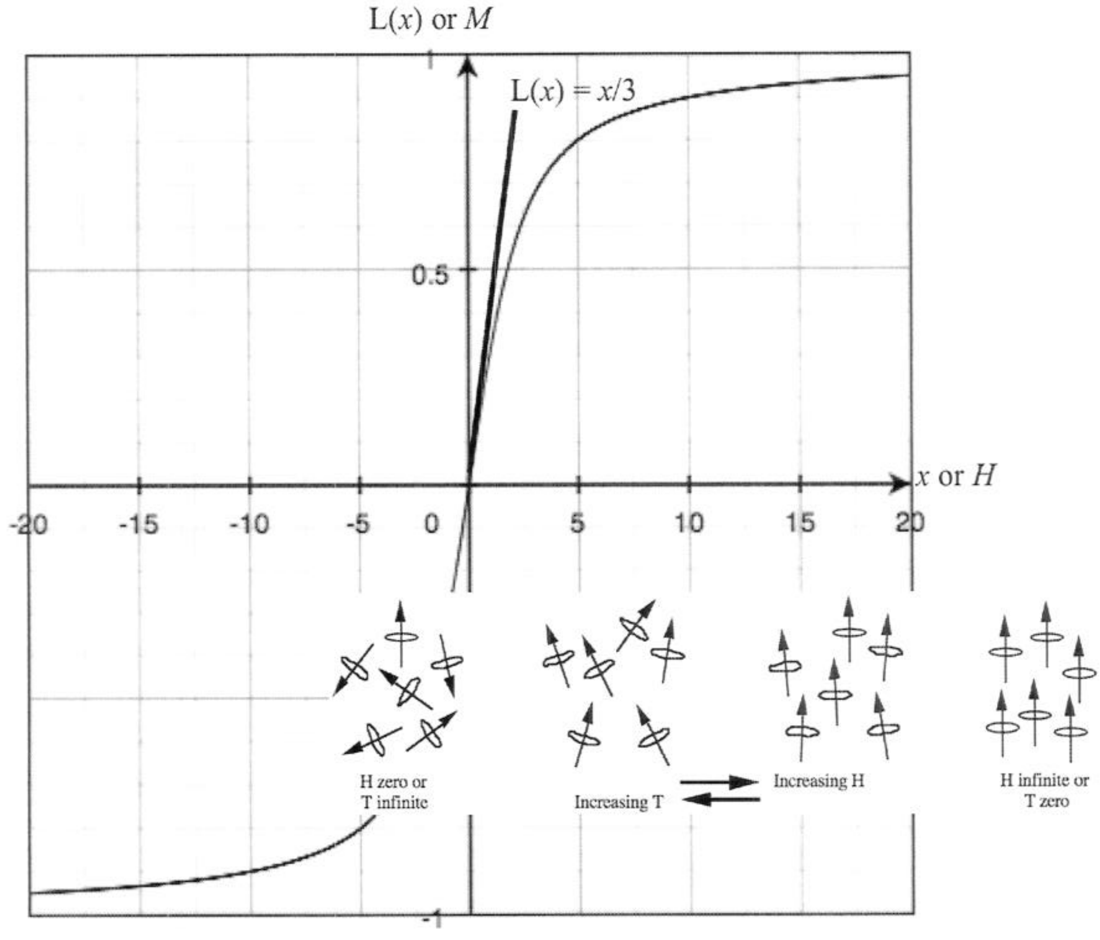
[6]

Doping introduces additional states into the band gap of the semiconductor, either occupied, or donor states just below the conduction band or unoccupied or acceptor states just above the valence band. Due to the much smaller energy differences between the doping states and the conduction bands compared to the band gap, at room temperature it is movement of electrons to and from these states that dominates the carrier densities. In particular, for n-doped semiconductors nearly all of the donor electrons will be thermalised into the conduction band, and by the law of mass action there will be very few empty states in the valence band; for p-doped semiconductors nearly all of the acceptor states will be occupied by valence electrons producing an equal number of holes, with in this case very few electrons remaining in the conduction band, again by the law of mass action. Hence by suitable doping it is possible to control the concentration of electrons and holes in a semiconductor. [6]

[bookwork]

4.

a) *Sketch the Langevin function L(x), noting that for small x, L(x) ~ x/3. Using the formula given for paramagnetic materials describe the magnetisation of a paramagnet in response to an external magnetic field. Indicate how the orientation of the atomic magnetic dipoles corresponds to the observed response.* [12]



[9]

$M = n\mu_m \mathrm{L}\left(\frac{\mu_m\mu_0 H}{kT}\right)$ so with suitable scaling the Langevin function also represent the response of a paramagnet. [3]

[bookwork]

b) *In Weiss theory the response of a ferromagnet to an external magnetic field can be modelled by replacing H by (H + λ M), where λ is known as the Weiss constant. Show by making a suitable substitution that the ferromagnetic response can be rewritten as*

$\alpha x - \beta H = L(x)$, (4.1)

and find expressions for α and β. *[8]*

From $M = n\mu_m \mathrm{L}\left(\frac{\mu_m \mu_0 H}{kT}\right)$

Replacing replacing H by $H + \lambda$ M gives

$$M = n\mu_m \mathrm{L}\left(\frac{\mu_m \mu_0 (H + \lambda M)}{kT}\right)$$ [2]

Substitute

$$x = \frac{\mu_0 \mu_m (H + \lambda M)}{kT}$$

which implies that

$$M = \frac{kTx}{\mu_0 \mu_m \lambda} - \frac{H}{\lambda}$$ [2]

so we can rewrite the first expression as

αx - $\beta H = \mathrm{L}(x)$

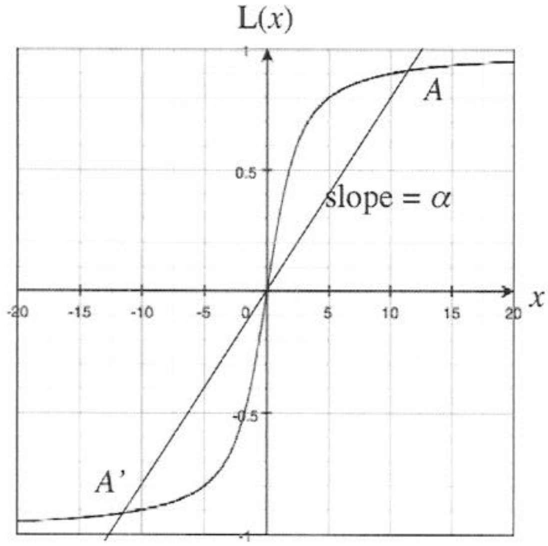
where

$$\alpha = \frac{kT}{\lambda n \mu_0 \mu_m^2} \text{ and } \beta = \frac{1}{\lambda n \mu_m}$$ [4]

[bookwork]

c) *Hence by graphically solving 4.1 show that in the absence of a magnetic field, a ferromagnet can exhibit permanent magnetism if the temperature of the magnet is less than a critical temperature given by $T_C = \frac{\lambda n \mu_0 \mu_m^2}{3k}$.* [10]

If $H = 0$, we have to solve $\alpha x = \mathrm{L}(x)$, with the solution representing the intersection of a straight line through the origin of slope α with the Langevin function:



[4]

If α is not too large, there are three possible solutions: one at the origin, which is unstable, and two marked at A and A'. These represent magnetization in the absence of a magnet field, or permanent magnetism. [2]

When α is greater than 1/3, the only solution is at the origin and no permanent magnetism occurs. Magnetism will disappear when $\alpha = 1/3$ giving

$$\frac{kT}{\lambda n \mu_0 \mu_m^2} = \frac{1}{3} \quad [2]$$

or

$$T_C = \frac{\lambda n \mu_0 \mu_m^2}{3k}. \quad [2]$$

[bookwork/new application of theory]