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IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE
UNIVERSITY OF LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2002

MSc and EEE/ISE PART IV: M.Eng. and ACGI

SYSTEM IDENTIFICATION

Tuesday, 7 May 10:00 am

There are SIX questions on this paper.

Answer FOUR questions.

Corrected Copy

Time allowed: 3:00 hours

Examiners responsible:

First Marker(s): Weiss,G.

Second Marker(s): Allwright,J.C.

Special information for invigilators: none

Information for candidates:

$$C(\tau) = E[(u(t) - \mu)(u(t+\tau) - \mu)]$$

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{x^2}{2\sigma^2}} \qquad S_{yy} = |G|^2 S_{uu}$$

$$Z_L = sL \qquad Z_c = \frac{1}{Cs}$$

$$\Phi^{\#} = (\Phi^*\Phi)^{-1}\Phi^* \qquad P = \Phi\Phi^{\#} \qquad S = \frac{1}{N-\rho}\|y - \Phi\widehat{\theta}\|^2$$

$$A^d = e^{Ah} \qquad B^d = \left(e^{Ah} - I\right)A^{-1}B \qquad G^d(z) \approx G\left(\frac{2}{h}\frac{z-1}{z+1}\right) \qquad G(s) \approx G^d\left(\frac{1+sh/2}{1-sh/2}\right)$$

$$C_k^{uu} g_0 + C_{k-1}^{uu} g_1 + C_{k-2}^{uu} g_2 + \ldots = C_k^{uy}$$

$$\text{Cov}(X,Y) = E[(X - \mu_X)(Y - \mu_Y)]$$

$$E(X \cdot Y) = E(X) \cdot E(Y) + \text{Cov}(X,Y)$$

$$\text{Cov}(TX) = T\text{Cov}(X)T^*$$

$$\widehat{v}(z) = \sum_{k=0}^{\infty} v_k z^{-k}$$

$$[(\Delta v)_k = v_{k+1}] \quad \Rightarrow \Delta v\,(z) = z[\widehat{v}(z) - v_0]$$

$$[u_k = k v_k] \quad \Rightarrow \widehat{u}(z) = -\,z\frac{d}{dz}\widehat{v}(z)$$

$$[v_k = \sin k\nu] \quad \Rightarrow \widehat{v}(z) = \frac{z\sin\nu}{(z-e^{i\nu})(z-e^{-i\nu})}$$

$$\left[v_k = \rho^k\right] \quad \Rightarrow \widehat{v}(z) = \frac{z}{z-\rho}$$

$$\left[v_k = \frac{1}{\rho} k\rho^k\right] \quad \Rightarrow \widehat{v}(z) = \frac{z}{(z-\rho)^2}$$

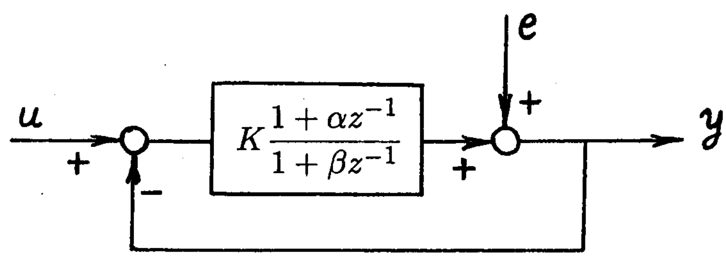
$$\begin{aligned} y_k + a_1 y_{k-1} \cdots + a_n y_{k-n} &= b_0 u_k + b_1 u_{k-1} \cdots + b_n u_{k-n} \\ &\quad + e_k + c_1 e_{k-1} \cdots + c_n e_{k-n} \end{aligned}$$

$$C(z) = 1 + c_1 z^{-1} \ldots + c_n z^{-n}$$

$$\hat{u}^F = C^{-1}\hat{u}, \qquad \hat{y}^F = C^{-1}\hat{y}$$

$$\begin{aligned} \overline{y_k} &= (c_1 - a_1) y_{k-1}^F + (c_2 - a_2) y_{k-2}^F \cdots + (c_n - a_n) y_{k-n}^F \\ &\quad + b_0 u_k^F + b_1 u_{k-1}^F \cdots + b_n u_{k-n}^F \end{aligned}$$

1. Consider the discrete-time feedback system shown in the figure below, with input signal u, output signal y and a white noise disturbance e with $E(e_k) = 0$. This disturbance e incorporates also the effects of measuring and modeling errors. It is known that $K > 0$, $|\alpha| < 1$ and $|\beta| < 1$, but otherwise K, α and β are unknown.



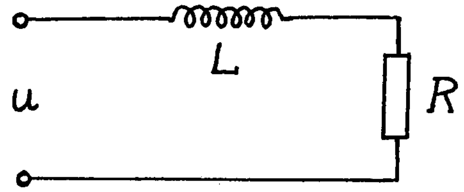
(a) Compute the transfer functions from u to y and from e to y. For which values of K, α and β are these transfer functions stable? [5]

(b) Write an ARMAX difference equation which can be used to estimate K, α and β from the measurements of u_k and y_k for $k = 0, 1, 2, \dots N$. Hint: you may introduce other variables instead of u, y or e if this is more convenient. [5]

(c) Describe a least squares based method for estimating K, α and β from the measurements described in part (b), using the ARMAX equation derived in part (b). Hint: it may be useful to transform the ARMAX equation into an ARX equation by introducing new variables. [5]

(d) Assuming that K, α and β have been found, how could we approximate the transfer function from u to y by a FIR transfer function of order 12? Hint: introduce convenient notation which allows you to formulate a simple answer. [5]

2. A microwave device can be modeled in a certain frequency band $(\omega_{\min}, \omega_{\max})$ by the following simple circuit:



The unknown parameters L and R should be determined. The driving voltage is $u(t) = U \cos \omega t$, where the amplitude U and the angular frequency ω are known. U can be varied in a certain range and ω can be varied in the range $(\omega_{\min}, \omega_{\max})$ mentioned earlier. The average power P dissipated by the device (in the resistor R) can be measured.

We have 100 measurements available from experiments, $P_1, P_2, \ldots P_{100}$, which correspond to the known amplitudes $U_1, U_2, \ldots U_{100}$ (not zero) and the known frequencies $\omega_1, \omega_2, \ldots \omega_{100}$. Naturally, the data will not fit our simple model exactly, no matter how we choose L and R.

(a) By defining new variables if necessary, rewrite the model of the system in the form $y_k = \varphi_k \theta + e_k$, where y_k and φ_k are known, θ is the vector of unknown parameters and e_k are the equation errors. [5]

(b) Write the formula for the vector of estimated parameters $\hat{\theta}$ which minimizes $e_1^2 + e_2^2 \ldots + e_{100}^2$. Indicate how we can derive estimates for R and L from $\hat{\theta}$. [4]

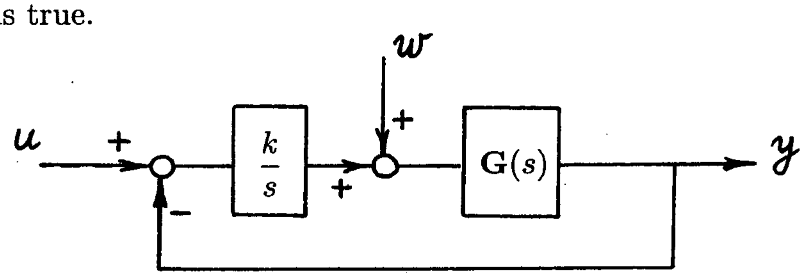
(c) Is it possible to estimate $\hat{\theta}$ by the formula required in part (b), if the amplitudes are all equal: $U_1 = U_2 \ldots = U_{100}$ (but the frequencies are different)? Is it possible to estimate $\hat{\theta}$ by the formula required in part (b), if the frequencies are all equal: $\omega_1 = \omega_2 \ldots = \omega_{100}$ (but the amplitudes are different)? [4]

(d) Assume that $e_1, e_2, \ldots e_{100}$ are independent and identically distributed. Assuming also that $E(e_k) = 0$, give a formula for an unbiased estimate of $Var(e_k)$. [3]

(e) Still assuming independent and identically distributed equation errors, give a formula for an unbiased estimate of $Cov(\hat{\theta})$, where $\hat{\theta}$ is the estimate from part (b). Note that $Var(e_k)$ is not known, but it can be estimated, as was required in part (c). [4]

3. We have a stable linear SISO system with an unknown transfer function $\mathbf{G}$. From physical considerations we know that $\mathbf{G}$ is minimum phase, i.e., all its zeros are in the open left half-plane. The relevant frequency range on which this system operates is from 0 to 2000 Hz. At higher frequencies we expect the transfer function of the system to be practically zero.

We want to connect this system to an integral controller, as shown in the block diagram below. We want to find a controller gain $k > 0$ such that the feedback system is stable, and k should be close to the largest value for which this is true.



(a) In order to choose a good value for k (as described above), we would like to plot an approximate Nyquist plot of $\mathbf{G}$. What sort of identification experiments could provide us with the necessary data for the Nyquist plot? Describe these experiments very briefly (assuming that looking at an oscilloscope is not accurate enough), and also describe briefly the computations necessary to process the data from these experiments. [6]

(b) Suppose that $k > 0$ has been chosen such that the feedback system is stable. Suppose that the input signal u is

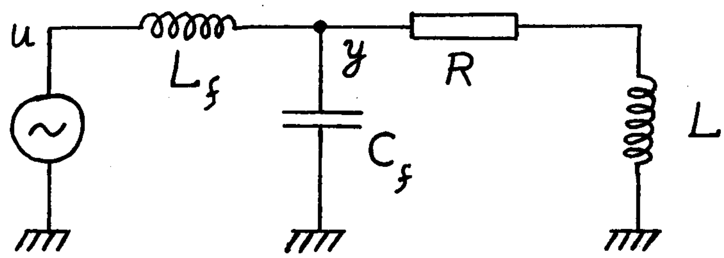
$$u(t) = 10(1 - e^{-7t}) + \cos 300t \tag{1}$$

and the corresponding output signal is denoted by y, as shown in the block diagram. Assume that $w = 0$. Describe the structure of $y(t)$ for large t (i.e., in steady state) and explain how such $y(t)$ can be computed in terms of the data ($\mathbf{G}$, k and u). [5]

(c) Assume that $u = 0$ and w is a stationary ergodic random signal with expectation $E(w) = 13$ and a certain known power spectral density S_{ww}. Is y a stationary random signal? Is y ergodic? Compute $E(y)$ and write a formula for computing $Var(y)$ (the power of y), in terms of $\mathbf{G}$, k and S_{ww}. [6]

(d) If u is given in (1) and w is as in part (c), is y a stationary random signal? Give a very brief reasoning. [3]

4. We want to model the output circuit of a power converter by the simplified circuit shown below, where the filter inductor L_f and the filter capacitor C_f are known, and the load resistor R and load inductance L are unknown. We can choose the waveform of u (which is the output voltage of the converter) and we can measure the load voltage y. R and L are to be determined (they should be positive). We cannot expect a perfect match between our true circuit and this model, but we would like to get a close match in a certain frequency range. Warning: this problem may look similar to one in last year's exam, but the solution is in fact rather different.



(a) Compute the transfer function $\mathbf{G}$ of the model circuit (from u to y), in terms of L_f, C_f, R and L. Is $\mathbf{G}$ stable? [6]

(b) Suppose that by measurements that use sinusoidal u, we have obtained estimates for $\mathbf{G}$ at 30 angular frequencies $\omega_1, \ldots \omega_{30}$, in the frequency range of interest. Using these data, how could we estimate R and L using a least squares based algorithm? Write down the formulas which give the estimated R and L, taking care to define all the symbols that you use. Hint: do not use directly the transfer function $\mathbf{G}$, because it gets you into a complicated problem (four unknown transfer function coefficients which depend on the two unknowns R and L, hence an overdetermined system). Instead, compute the load impedance $Z(i\omega) = R + i\omega L$ in terms of $\mathbf{G}(i\omega)$ and then use the experimentally determined values of $\mathbf{G}$ to compute the corresponding values for Z. [6]

(c) Construct a realization of the transfer function $\mathbf{G}$, of the form $\dot{x} = Ax + Bu$, $y = Cx + Du$, where A, B, C and D are matrices. [3]

(d) We connect a hold device (D/A converter) at the input of our system (i.e., we use a digitally controlled converter) and we connect a sampler (A/D converter) at its output (e.g., a digital voltmeter), both converters working with the sampling period T. How can we compute the transfer function of the resulting discrete-time LTI system? There is no need to perform any computations to answer this part. [5]

5. Consider the discrete-time LTI system with input u and output y described by the equations

$$5y_k - 8.5y_{k-1} + 3.6y_{k-2} = q_{k-1},$$

$$q_k = 3u_k - 2u_{k-1}.$$

(a) Compute the transfer function $\mathbf{G}$ of this system. [2]

(b) Determine if $\mathbf{G}$ is a stable transfer function, and compute its DC-gain. Is $\mathbf{G}$ proper? Is it strictly proper? Is it FIR? [2]

(c) Consider the signal u given by $u_0 = 0$ and

$$u_k = \frac{3}{10^k} \quad (k = 1, 2, 3, \ldots). \tag{2}$$

Compute the $\mathcal{Z}$-transform $\hat{u}(z)$. [2]

(d) Let y be the response of the system to the input signal u given in part (c). Assume that the initial state of the system is zero. Compute the $\mathcal{Z}$-transform $\hat{y}(z)$. [2]

(e) Explain why the signal y from (d) is of the form

$$y_k = c_1(0.8)^k + c_2(0.9)^k + c_3(0.1)^k \quad \text{for all} \quad k \geq 1.$$

Explain briefly how the constants c_1, c_2, c_3 can be computed, but do not compute them numerically. [5]

(f) Assume that the discrete-time transfer function $\mathbf{G}$ given above has been obtained by a discrete-time identification procedure applied to a continuous-time LTI system, via sample and hold blocks (i.e., D/A and A/D converters) with a sampling frequency of 10 kHz. Make an estimate of the transfer function $\mathbf{P}$ of the continuous-time system, which should be valid for frequencies that are significantly lower than the sampling frequency. [2]

(g) Suppose that the output measurements of the above system are subject to measurement errors, in that

$$\hat{y}(z) = \mathbf{G}(z)\hat{u}(z) + \hat{e}(z),$$

where $\hat{e}$ is the $\mathcal{Z}$-transform of the sequence e_k which is normalized white noise (so that $E(e_k) = 0$ and $E(e_k^2) = 1$). Given measurements $y_1, y_2, \ldots y_{300}$, how can you compute an unbiased prediction of y_{301}? How large is the variance of the prediction error? [5]

6. Consider a stable discrete-time LTI plant with a stationary random input signal u_k and the corresponding output signal y_k, which have been observed for some finite time interval.

(a) Describe a method for estimating the auto-correlation function C_τ^{uu} and the cross-correlation function C_τ^{uy} for $0 \leq \tau \leq N-1$, where N is much smaller than the length of the time interval of our observations. Explain very briefly how this problem is related to the concept of ergodicity. [3]

(b) Describe a method for estimating the first N terms $g_0, g_1, \cdots g_{N-1}$ in the impulse response of the plant from the results of part (a), and explain briefly how this method is derived from the properties of C_τ^{uu} and C_τ^{uy}. [3]

(c) What is the meaning of a random signal being "persistent of order N"? What is the significance of this concept in the context of part (b) above? [5]

(d) After having estimated the first N terms of the impulse response, $g_0, g_1, \cdots g_{N-1}$, how can we build a FIR filter whose transfer function is a good approximation to the true transfer function? Write the corresponding difference equation. [4]

(e) Suppose that a discrete-time stationary random signal u_k has power spectral density S^{uu} such that $S^{uu}(\zeta) \geq \varepsilon$ for all ζ on the unit circle, where $\varepsilon > 0$. Note that then the infinite Toeplitz matrix T with entries $T_{jk} = C_{j-k}^{uu}$ satisfies

$$\sum_{k=0}^{\infty} (Tx)_k x_k \geq \varepsilon \sum_{k=0}^{\infty} x_k^2,$$

for every real sequence x_k for which $\sum_{k=0}^{\infty} x_k^2 < \infty$. Explain why this implies that u is persistent of order N for any $N = 1, 2, 3, \ldots$. [5]

[END]

SYSTEMS IDENTIFICATION, May 2002 Solutions

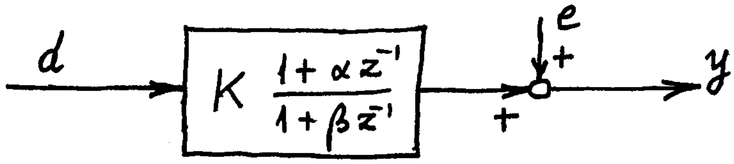
Question 1. (a) $T_{yu}(z) = \dfrac{K \cdot \dfrac{1+\alpha z^{-1}}{1+\beta z^{-1}}}{1 + K \cdot \dfrac{1+\alpha z^{-1}}{1+\beta z^{-1}}}$

$$= \frac{K(1+\alpha z^{-1})}{(1+K)(1+\rho z^{-1})}, \quad \text{where} \quad \rho = \frac{\beta + K\alpha}{1+K},$$

$$T_{ye}(z) = \frac{1}{1 + K \cdot \dfrac{1+\alpha z^{-1}}{1+\beta z^{-1}}} = \frac{1+\beta z^{-1}}{(1+K)(1+\rho z^{-1})}.$$

Since $\rho = \frac{1}{1+K}\beta + \frac{K}{1+K}\alpha$ is a convex combination of α and β, which are both in the open unit disk $\mathbb{D}$, we have that also $\rho \in \mathbb{D}$. Hence, T_{yu} and T_{ye} are stable regardless of the choice of $K > 0$, $\alpha, \beta \in \mathbb{D}$.

(b) Introduce $d = u - y$ (this is the signal at the output of the comparator), d is measurable (since u and y are). Then we have the output identification problem



which corresponds to the equation

$$(1+\beta z^{-1})\,\hat{y}(z) = K(1+\alpha z^{-1})\,\hat{d}(z) + (1+\beta z^{-1})\,\hat{e}(z).$$

If we denote $a_1 = \beta$, $b_0 = K$, $b_1 = K\alpha$, then we can write this as an ARMAX equation:

$$y_k + a_1 y_{k-1} = b_0 d_k + b_1 d_{k-1} + e_k + a_1 e_{k-1}.$$

(c) Denote $w_k = e_k + a_1 e_{k-1}$, then w is a (non-white) stationary ergodic random signal with $E(w_k) = 0$. From

$$y_k = -a_1 y_{k-1} + b_0 d_k + b_1 d_{k-1} + w_k,$$

denoting $\varphi_k = [-y_{k-1} \ \ d_k \ \ d_{k-1}]$, $\theta = \begin{bmatrix} a_1 \\ b_0 \\ b_1 \end{bmatrix}$,

we get $y_k = \varphi_k \theta + w_k$. From here, with $\phi = \begin{bmatrix} \varphi_1 \\ \vdots \\ \varphi_N \end{bmatrix}$, $y = \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix}$

we construct the initial least squares estimate

$$\theta^e = (\phi^* \phi)^{-1} \phi^* y = \begin{bmatrix} a_1^e \\ b_0^e \\ b_1^e \end{bmatrix},$$

which is unbiased, but may be inaccurate (i.e., with large covariance).

We introduce the filtered signals

$$\hat{y}^F(z) = \frac{1}{1 + a_1^e z^{-1}} \hat{y}(z), \quad \hat{d}^F(z) = \frac{1}{1 + a_1^e z^{-1}} \hat{d}(z),$$

then the ARMAX equation becomes

$$y_k^F + a_1 y_{k-1}^F = b_0 d_k^F + b_1 d_{k-1}^F + v_k,$$

where $\hat{v}(z) = \dfrac{1 + a_1 z^{-1}}{1 + a_1^e z^{-1}} \hat{e}(z)$. Since v is a good approximation to white noise, we may repeat the standard least squares algorithm for this equation to get a better estimate for θ.

(d) $T_{yu}(z) = \dfrac{K}{1+K}(1 + \alpha z^{-1}) \dfrac{1}{1 + p z^{-1}}$. For $|z| > |p|$,

$$1/(1 + p z^{-1}) = 1 - p z^{-1} + p^2 z^{-2} - p^3 z^{-3} + \ldots$$

Multiply this series with $\frac{K}{1+K}(1 + \alpha z^{-1})$ and retain the first 13 terms only:

$$T_{yu}^a = g_0 + g_1 z^{-1} + g_2 z^{-2} \ldots + g_{12} z^{-12}.$$

Then T_{yu}^a is a FIR approximation of T_{yu}.

<u>Question 2</u>. (a) The current through the device is

$$i(t) = \frac{U}{|R+i\omega L|}\cos(\omega t+\varphi), \quad \text{where } \varphi = -\arg(R+i\omega L)$$

(this φ is irrelevant for us), so that, denoting $T = \frac{2\pi}{\omega}$

$$P = R \cdot \underbrace{\frac{1}{T}\int_0^T i(t)^2\,dt}_{i_{rms}^2} = \frac{RU^2}{2(R^2+\omega^2 L^2)}.$$

Hence,

$$\frac{1}{2P} = \frac{R}{U^2} + \frac{\omega^2 L^2}{RU^2}.$$

Denoting $y_k = \frac{1}{2P_k}$, $\varphi_k = \begin{bmatrix} \frac{1}{U_k^2} & \frac{\omega_k^2}{U_k^2} \end{bmatrix}$, $\theta = \begin{bmatrix} R \\ \frac{L^2}{R} \end{bmatrix}$,

we get the usual $y_k = \varphi_k \theta + e_k$.

(b) If $\Phi^*\Phi$ is invertible, then

$$\hat{\theta} = (\Phi^*\Phi)^{-1}\Phi^* y, \quad \text{where } \Phi = \begin{bmatrix} \varphi_1 \\ \vdots \\ \varphi_{100} \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ \vdots \\ y_{100} \end{bmatrix}.$$

Clearly $\hat{R} = \hat{\theta}_1$, while from $\frac{\hat{L}^2}{\hat{R}} = \hat{\theta}_2$ we get $\hat{L} = \sqrt{\hat{\theta}_1\hat{\theta}_2}$.

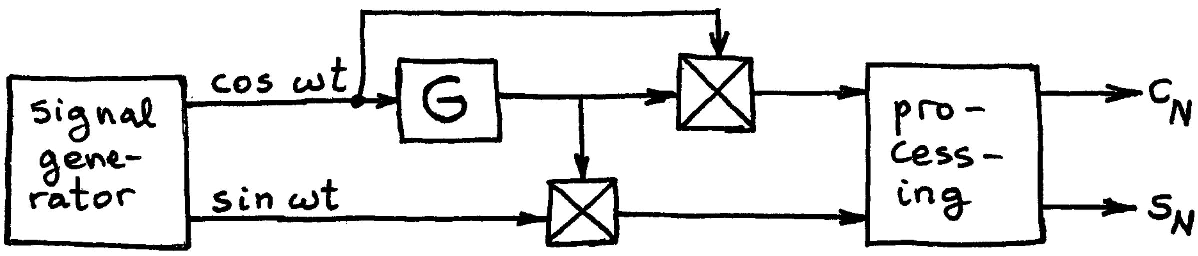
(c) With $U_1 = U_2 \dots = U_{100}$ but different frequencies, it will normally work. However, if $\omega_1 = \omega_2 \dots = \omega_{100}$, then the two columns of Φ are linearly dependent, so that $\Phi^*\Phi$ (having rank 1) cannot be invertible.

(d) $\widehat{Var}(e_k) = \frac{1}{98}\|y - \Phi\hat{\theta}\|^2 = \frac{1}{98}\, y^*\left[I - \Phi(\Phi^*\Phi)^{-1}\Phi^*\right]y.$

$98 = 100-2$

(e) $\widehat{Cov}(\hat{\theta}) = \widehat{Var}(e_k)(\Phi^*\Phi)^{-1}$, where $\widehat{Var}(e_k)$ has been computed in part (d).

Question 3. (a) Identification experiment:



For several frequencies ω in the relevant range, we compute in steady state, denoting $T = \frac{2\pi}{\omega}$,

$$c_N = \int_0^{NT} y(t)\cos\omega t\, dt = A_\omega \cdot \cos\varphi_\omega \cdot \frac{NT}{2}$$

$$s_N = \int_0^{NT} y(t)\sin\omega t\, dt = -A_\omega \sin\varphi_\omega \cdot \frac{NT}{2}$$

and from these we can easily determine A_ω and φ_ω. This gives $G(i\omega) = A_\omega e^{i\varphi_\omega}$, and if we do this for several ω, we get the approximate shape of the Nyquist plot.

(b) The transfer function from u to y is

$$T_{yu}(s) = \frac{\frac{k}{s}G(s)}{1 + \frac{k}{s}G(s)}, \quad T_{yu}(0) = \lim_{s\to 0} T_{yu}(s) = 1$$

(for $T_{yu}(0)$ we have used that $G(0) \neq 0$, because G is minimum phase). We have

$$u(t) = 10 + \cos 300t + \underbrace{e(t)}_{\to 0},$$

hence

$$y(t) = T_{yu}(0)\cdot 10 + K\cos(300t + \psi) + \underbrace{\varepsilon(t)}_{\to 0},$$

where $K = |T_{yu}(300i)|$, $\psi = \arg T_{yu}(300i)$.

Hence, in steady state,

$$y(t) = 10 + K\cos(300t + \psi).$$

(c) The transfer function from w to y is

$$T_{yw}(s) = \frac{G(s)}{1 + \frac{k}{s}G(s)} = \frac{s\,G(s)}{s + kG(s)},$$

in particular, $T_{yw}(0) = 0$. y is stationary, ergodic and $E(y) = T_{yw}(0) \cdot E(w) = 0$. Finally,

$$\mathrm{Var}(y) = C_{yy}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{yy}(i\omega)\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |T_{yw}(i\omega)|^2 S_{ww}(i\omega)\,d\omega.$$

(d) Now $y = y_1 + y_2$, where y_1 is the function computed in part (b) and y_2 is the random signal described in part (c). We have

$$E(y(t)) = y_1(t) + E(y_2(t)) = y_1(t),$$

which is not constant, so that y is not stationary.

Question 4. (a) The impedance of the sub-circuit consisting of the capacitor and the load, Q is given by

$$\frac{1}{Q(s)} = C_f s + \frac{1}{R+Ls} = \frac{LC_f s^2 + RC_f s + 1}{Ls+R}.$$

We have

$$G(s) = \frac{Q(s)}{L_f s + Q(s)} = \frac{Ls+R}{L_f s\left(LC_f s^2 + RC_f s + 1\right) + Ls + R}$$

$$= \frac{\overbrace{\frac{1}{L_f C_f}}^{b_1} \cdot s + \overbrace{\frac{R}{LL_f C_f}}^{b_0}}{s^3 + \underbrace{\frac{R}{L}}_{a_2} s^2 + \underbrace{\frac{L+L_f}{LL_f C_f}}_{a_1} \cdot s + \underbrace{\frac{R}{LL_f C_f}}_{a_0}}.$$

To check the stability of G, first we note that a_0, a_1 and a_2 are positive (this is a necessary condition). Now, according to the Hurwitz test, G is stable if and only if $a_2 a_1 > a_0$. Multiplying both sides of this inequality by $LL_f C_f$, the condition becomes

$$\frac{R(L+L_f)}{L} > R,$$

which is clearly true (so that G is stable).

(b) We have (from the solution of part (a))

$$G(s) L_f s + G(s) Q(s) = Q(s),$$

whence

$$Q(s) = \frac{G(s) L_f s}{1 - G(s)},$$

so that $\dfrac{1}{Z(s)} = \dfrac{1}{Q(s)} - C_f s = \dfrac{1-G(s)}{G(s) L_f s} - C_f s .$

Thus, for each of the 30 angular frequencies, we can compute an estimated value of Z, denoted $Z^e(i\omega_k)$, from the estimated value $G^e(i\omega_k)$. These estimated values are subject to modeling and measurement errors denoted e_k:

$$Z^e(i\omega_k) = R + i\omega_k L + e_k .$$

Denoting $y_k = Z^e(i\omega_k)$, $\varphi_k = [\,1 \;\; i\omega_k\,]$, $\theta = \begin{bmatrix} R \\ L \end{bmatrix}$, we obtain $y_k = \varphi_k \theta + e_k$. From here, R and L can be estimated using the standard least squares algorithm:

$$\hat{\theta} = \Phi^{\#} y, \quad y = \begin{bmatrix} y_1 \\ \vdots \\ y_{30} \end{bmatrix}, \quad \Phi = \begin{bmatrix} \varphi_1 \\ \vdots \\ \varphi_{30} \end{bmatrix}, \quad \Phi^{\#} = (\Phi^* \Phi)^{-1} \Phi^* .$$

(c) The realization of $G(s) = \dfrac{b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0}$ is

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$C = [\, b_0 \;\; b_1 \;\; 0 \,], \qquad D = [\,0\,] .$$

(d) Discretization in state space:

$$A^d = e^{AT}, \qquad B^d = (e^{AT} - I) A^{-1} B ,$$

$$G^d(z) = C(zI - A^d)^{-1} B^d + D .$$

Alternatively, by Tustin's formula,

$$G^d(z) \approx G\left(\frac{2}{T} \cdot \frac{z-1}{z+1} \right) .$$

Question 5. (a) $(5 - 8.5z^{-1} + 3.6z^{-2})\hat{y}(z) = z^{-1}\hat{q}(z)$

$\hat{q}(z) = (3 - 2z^{-1})\hat{u}(z)$. Combining these, we get

$$G(z) = \frac{(3-2z^{-1})z^{-1}}{5-8.5z^{-1}+3.6z^{-2}} = \frac{0.6z^{-1} - 0.4z^{-2}}{1 - 1.7z^{-1} + 0.72z^{-2}}.$$

(b) The poles of G are 0.8 and 0.9, so that it is stable. Its DC gain is $G(1) = \frac{0.2}{0.02} = 10$. G is strictly proper (i.e., $G(\infty) = 0$) and it is not FIR.

(c) If the formula for u_k were true also for $k=0$, then the $\mathcal{Z}$-transform would be $\frac{3z}{z-0.1}$. From this we have to subtract $3\hat{\delta}_0 = 3$, so that

$$\hat{u}(z) = \frac{3z}{z-0.1} - 3 = \frac{0.3}{z-0.1}.$$

(d) $$\hat{y}(z) = G(z)\,\hat{u}(z) = \frac{0.3\,(0.6z-0.4)}{(z-0.8)(z-0.9)(z-0.1)}.$$

(e) We use the partial fractions decomposition

$$\hat{y}(z) = \frac{d_1}{z-0.8} + \frac{d_2}{z-0.9} + \frac{d_3}{z-0.1},$$

where $d_1 = \lim_{z\to 0.8} (z-0.8)\,\hat{y}(z)$, and similarly for d_2, d_3. Take inverse $\mathcal{Z}$ transforms:

$$y_k = d_1(0.8)^{k-1} + d_2(0.9)^{k-1} + d_3(0.1)^{k-1}$$

for $k = 1, 2, 3, \ldots$ while $y_0 = 0$. Denoting

$$c_1 = \frac{d_1}{0.8}, \quad c_2 = \frac{d_2}{0.9}, \quad c_3 = \frac{d_3}{0.1},$$ we get the desired formula.

(f) $P(s) \approx G\left(\dfrac{1+\frac{hs}{2}}{1-\frac{hs}{2}}\right)$, where $h = 10^{-4}$,

according to Tustin's formula.

(g) The general formula for the one step ahead prediction of an ARMAX system described by

$$y_k + a_1 y_{k-1} + a_2 y_{k-2} \cdots = b_0 u_k + b_1 u_{k-1} + b_2 u_{k-2} \cdots + e_k + c_1 e_{k-1} + c_2 e_{k-2} \cdots$$

is $$\bar{y}_k = (c_1 - a_1) y^F_{k-1} + (c_2 - a_2) y^F_{k-2} \cdots + b_0 u^F_k + b_1 u^F_{k-1} + b_2 u^F_{k-2} \cdots ,$$

where $$u^F_k + c_1 y^F_{k-1} + c_2 y^F_{k-2} \cdots = y_k ,$$
$$u^F_k + c_1 u^F_{k-1} + c_2 u^F_{k-2} \cdots = u_k ,$$

and we have $y_k = \bar{y}_k + e_k$ (there is no need to write all this down in the exam).

In our case, e_k being an output error, we have $c_j = a_j$ $(j = 1, \dots n)$, $b_0 = 0$, $n = 2$, so that

$$\bar{y}_k = b_1 u^F_{k-1} + b_2 u^F_{k-2} ,$$

whence $$\hat{\bar{y}}(z) = (b_1 z^{-1} + b_2 z^{-2})\, \hat{u}^F(z) = \frac{b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}} \hat{u}(z) = G(z)\hat{u}(z).$$

Thus, we have to filter u through G and the output obtained for $k = 301$ will be the unbiased prediction of y_{301}. In particular, the data $y_0, y_1, \dots y_{300}$ are not needed. From $y_k = \bar{y}_k + e_k$ we see that the prediction error is e_k, so that its variance is 1.

Question 6. (a) If u is ergodic then so is also y, hence time averages (of any expression) converge to expectations (of those expressions). In particular, with probability 1,

$$\lim_{n\to\infty} \frac{1}{n+1}\sum_{k=0}^{n} u_k u_{k+\tau} = E(u_k u_{k+\tau}) = C_\tau^{uu},$$

$$\lim_{n\to\infty} \frac{1}{n+1}\sum_{k=0}^{n} u_k y_{k+\tau} = E(u_k y_{k+\tau}) = C_\tau^{uy}.$$

Thus, we can estimate C_τ^{uu} and C_τ^{uy} by taking the expressions on the left for large n.

(b) We have $C^{uy} = g * C^{uu}$, i.e.,

$$C_\tau^{uy} = \sum_{k=0}^{\infty} g_k C_{\tau-k}^{uu}.$$

Taking estimates of C_τ^{uu} and C_τ^{uy} for $0 \le \tau \le N-1$, as explained in part (a), denoting these estimates by $\widehat{C_\tau^{uu}}$ and $\widehat{C_\tau^{uy}}$, and assuming that $g_k \approx 0$ for $k \ge N$ (by stability), we get the equation

$$\begin{bmatrix} \widehat{C_0^{uu}} & \widehat{C_1^{uu}} & \cdots & \widehat{C_{N-1}^{uu}} \\ \widehat{C_1^{uu}} & \widehat{C_0^{uu}} & \cdots & \widehat{C_{N-2}^{uu}} \\ \vdots & \vdots & & \vdots \\ \widehat{C_{N-1}^{uu}} & \widehat{C_{N-2}^{uu}} & \cdots & \widehat{C_0^{uu}} \end{bmatrix} \cdot \begin{bmatrix} \hat{g}_0 \\ \hat{g}_1 \\ \vdots \\ \hat{g}_{N-1} \end{bmatrix} = \begin{bmatrix} \widehat{C_0^{uy}} \\ \widehat{C_1^{uy}} \\ \vdots \\ \widehat{C_{N-1}^{uy}} \end{bmatrix},$$

where $\hat{g}_0, \dots \hat{g}_{N-1}$ are estimates of $g_0, \dots g_{N-1}$.

(c) The signal u is called <u>persistent of order N</u> if the square matrix appearing above, but with the true values instead of the estimates, is invertible. If this is the case, then there is a good chance for the estimated matrix to be invertible as well, so we can solve the equation from part (b) for $\hat{g}_0, \dots \hat{g}_{N-1}$.

(d) The FIR filter with impulse response

$$\hat{g} = (\hat{g}_0, \hat{g}_1, \dots \hat{g}_{N-1}, 0, 0, 0, \dots)$$

is described by the MA equation

$$y_k = \hat{g}_0 u_k + \hat{g}_1 u_{k-1} + \hat{g}_2 u_{k-2} \dots + \hat{g}_{N-1} u_{k-N+1}.$$

(e) In the inequality given in part (e), take

$$x = (x_0, x_1, x_2, \dots x_{N-1}, 0, 0, 0, \dots).$$

Then the inequality becomes

$$\xi^* T_N \xi \geqslant \varepsilon \|\xi\|^2,$$

where $\xi = [x_0 \ x_1 \ \dots \ x_{N-1}]^T \in \mathbb{R}^N$,

and

$$T_N = \begin{bmatrix} c_0^{uu} & c_1^{uu} & \dots & c_{N-1}^{uu} \\ c_1^{uu} & c_0^{uu} & \dots & c_{N-2}^{uu} \\ \vdots & & & \vdots \\ c_{N-1}^{uu} & c_{N-2}^{uu} & \dots & c_0^{uu} \end{bmatrix}.$$

It follows that $T_N \xi \neq 0$ for all $\xi \in \mathbb{R}^N$, so that T_N is invertible. This means that u is persistent of order N. This argument is valid for any $N = 1, 2, 3, \dots$.