

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2005

EEE/ISE PART II: MEng, BEng and ACGI

Corrected Copy

SIGNALS AND LINEAR SYSTEMS

Monday, 23 May 2:00 pm

Time allowed: 2:00 hours

There are FOUR questions on this paper.

Q1 is compulsory.

Answer Q1 and any two of questions 2-4.

Q1 carries 40% of the marks. Questions 2 to 4 carry equal marks (30% each).

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible

First Marker(s) : R.U. Nabar, R.U. Nabar

Second Marker(s) : P.T. Stathaki, P.T. Stathaki

Formulae

Convolution: $x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$

Continuous Time Fourier Transform: $x(t) \xleftrightarrow{F} X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt$

Useful Properties & Relations

1. $\delta(t-a) \xleftrightarrow{F} e^{-j\omega a}$

2. $x(t-a) \xleftrightarrow{F} e^{-j\omega a} X(\omega)$

3. $\text{sinc}\left(\frac{t-a}{b}\right) \xleftrightarrow{F} |b| e^{-j\omega a} \Pi\left(\frac{b\omega}{2\pi}\right)$

4. $e^{-at}u(t) \xleftrightarrow{F} \frac{1}{a+j\omega}, a > 0$

5. $x(-t) \xleftrightarrow{F} X(-\omega)$

6. $e^{jat} \xleftrightarrow{F} 2\pi\delta(\omega-a)$

Parseval's Relation: $\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$

Discrete Time Fourier Transform: $x[n] \xleftrightarrow{F} X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$

Useful Properties & Relations

1. $\delta[n-a] \xleftrightarrow{F} e^{-j\omega a}$

2. $e^{jna}x[n] \xleftrightarrow{F} X(e^{j(\omega-a)})$

Laplace Transform: $x(t) \xleftrightarrow{L} X(s) = \int_{-\infty}^{\infty} x(t)e^{-st} dt, \text{ ROC}$

Useful Properties & Relations

1. $\delta(t-a) \xleftrightarrow{L} e^{-sa}$ ROC: entire s plane

2. $x'(t) \xleftrightarrow{L} sX(s)$, same ROC

3. $te^{-at}u(t) \xleftrightarrow{L} \frac{1}{(s+a)^2}$, ROC: $\text{Re}\{s\} > -a$

4. $u(t) \xleftrightarrow{L} \frac{1}{s}$, ROC: $\text{Re}\{s\} > 0$

Useful Identities:

1. $\cos(t) = \frac{e^{jt} + e^{-jt}}{2}$

2. $\sin(t) = \frac{e^{jt} - e^{-jt}}{2j}$

3. $(-1)^n = e^{jn\pi}$

The Questions

[Compulsory]

Consider the wireless communication system shown in Figure 1.1 below. The signal $x(t)$ is transmitted and $y(t)$ is the received signal. There is no direct path between transmitter and receiver. The receiver receives two copies of the transmitted signal, through two reflections that we name Reflection 1 and Reflection 2. The first copy (Reflection 1) arrives at the receiver with a delay of 1 second and is attenuated in amplitude by a factor of 0.5. The second copy (Reflection 2) arrives with a delay of 2 seconds, also attenuated in amplitude by a factor of 0.5.

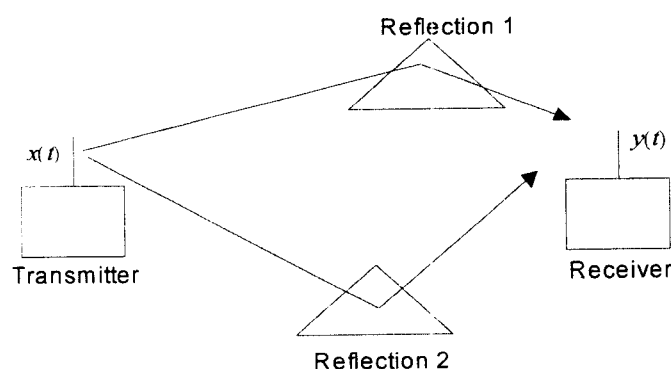


Figure 1.1

- (a) Is the system LTI? Is the system causal? [6]
- (b) Determine the impulse response, $h(t)$, of the system. [6]
- (c) Show that the output of the system to the input $x(t) = \sin(t)$ is

$$y(t) = \cos(0.5) \sin(t-1.5)$$
 [6]
- (d) Sketch the frequency response $H(\omega)$ (magnitude and phase) of the system. Which transmission frequencies, if any, should the transmitter avoid using? [8]
- (e) Evaluate the magnitude of the frequency response of the system if Reflection 1 is delayed by 3 seconds (instead of 1 second) and Reflection 2 is unchanged. [4]
- (f) Determine the Laplace transform of the impulse response of the system and the associated region of convergence. Reflections 1 and 2 are as in the original description. [4]
- (g) Sketch the step response of the system, i.e., the output of the system when the input is the unit-step function $u(t)$? Reflections 1 and 2 are as in the original description. [6]

2 Evaluate the following

(a) $\int_{-\infty}^{\infty} \text{sinc}\left(\frac{\tau-t}{2}\right) \text{sinc}\left(\frac{\tau}{3}\right) d\tau$ [8]

(b) $\int_{-\infty}^{\infty} \text{sinc}(t) \text{sinc}(t) dt$ [7]

(c) $\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{j\omega x}}{6 + j\omega + \omega^2} d\omega$ [8]

(d) $u(t) * e^t u(-t)$, where $u(t)$ is the unit-step function and $*$ denotes convolution [7]

Consider a continuous time bandwidth limited signal $x(t)$ that is uniformly sampled with period T to produce the discrete time signal $x[n]$. The sampling period **satisfies** the Nyquist criterion. The Discrete Time Fourier Transform (DTFT) of $x[n]$, denoted by $X(e^{j\omega})$, is shown below (Assume that the phase response is 0 for all frequencies).

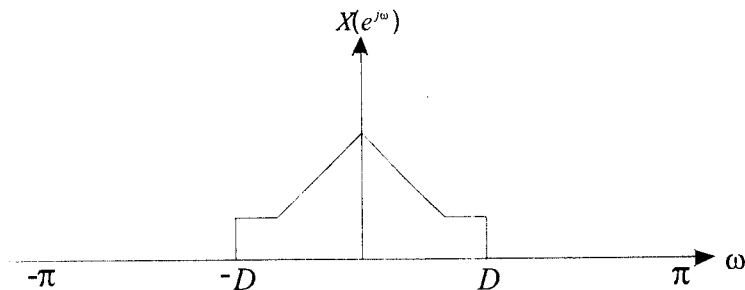


Figure 3.1

- (a) What is the bandwidth of the signal $x(t)$? Express your answer in terms of D , T and constants. [3]

- (b) Now, suppose we form the new discrete time signal $y[n]$ such that

$$y[n] = x[n], \text{ if } n \text{ is even, and } 0 \text{ otherwise.}$$

Express the DTFT of $y[n]$ in terms of $X(e^{j\omega})$. [6]

- (c) Derive a condition on D that will allow a digital-to-analog converter to perfectly reconstruct $x(t)$ from the samples $y[n]$. [6]

- (d) We now linearly interpolate the zeroed out samples of $x[n]$ to produce $z[n]$ such that

$$z[n] = x[n] \text{ if } n \text{ is even, and } (x[n+1] + x[n-1])/2 \text{ otherwise}$$

Express the DTFT of $z[n]$ as a function of $X(e^{j\omega})$. [15]

Consider the RLC circuit shown below

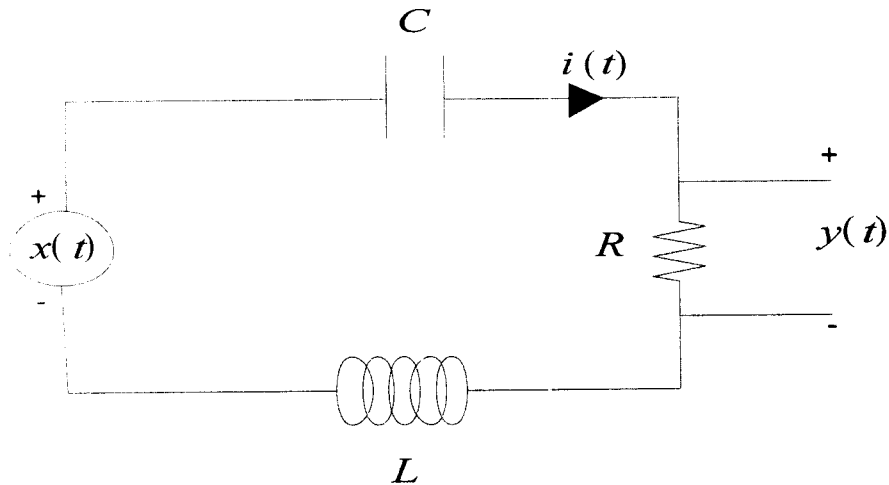


Figure 4.1

The source voltage is $x(t)$ and is the input to the system. The output of the system is the voltage $y(t)$ across the resistance. The circuit equation is

$$C x'(t) = i(t) + RC i'(t) + LC i''(t),$$

where $i(t)$ is the current in the circuit and ' and '' denote the derivative and double derivative respectively. In the following assume $C = 2$ F, $R = 1 \Omega$ and $L = 0.5$ H.

- (a) Find the Laplace transform of the system impulse response, $H(s)$. Sketch its region of convergence (ROC). [6]
- (b) Find the step response, $w(t)$, of the system. [12]
- (c) Now assume $y(t)$ is the input to an LTI system with impulse response $g(t)$. Determine $g(t)$ in order to recover $x(t)$ perfectly at the output of the system. (Assume $x(t)$ has no DC component). [12]

1 [Compulsory]

$$(a) \quad y(t) = \frac{x(t-1)}{2} + \frac{x(t-2)}{2} \quad (\text{Bookwork})$$

The system is LTI. The system is causal.
[6]

$$(b) \quad h(t) = \frac{\delta(t-1)}{2} + \frac{\delta(t-2)}{2} \quad (\text{Bookwork})$$

[6]

$$(c) \quad \sin(t) = \frac{e^{jt} - e^{-jt}}{2j} \quad (\text{New Application of theory})$$

Frequency response of system is

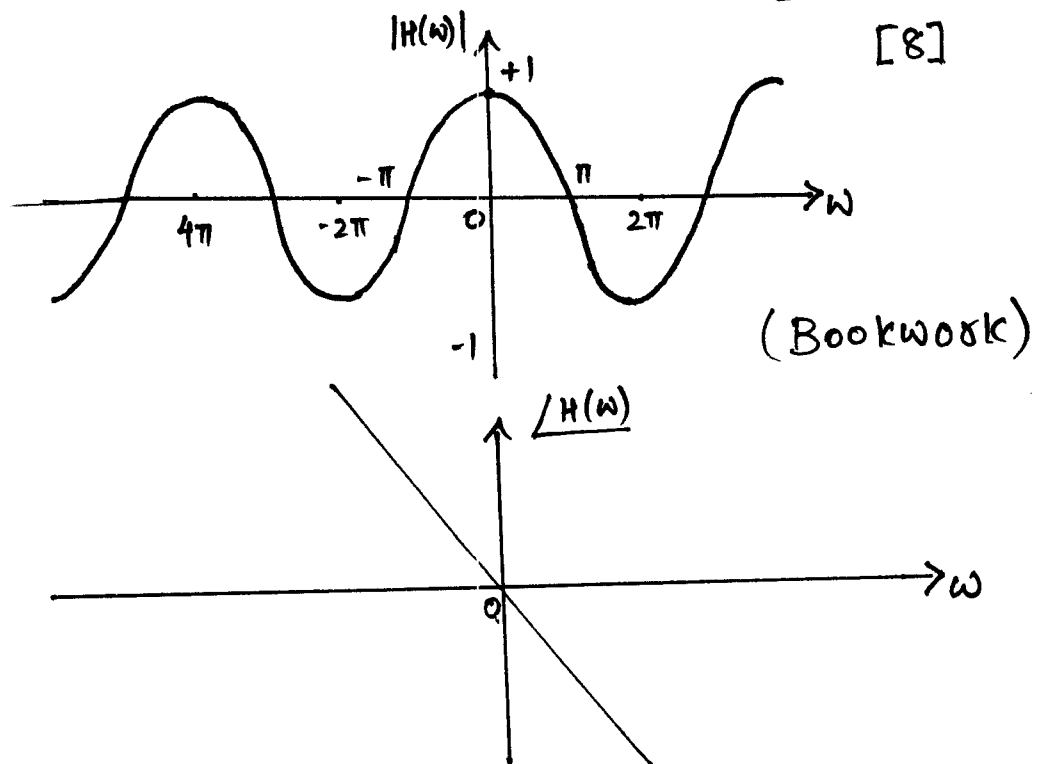
$$\begin{aligned} H(\omega) &= \frac{1}{2} (e^{-j\omega} + e^{-2j\omega}) \quad [6] \\ &= \frac{e^{-j3\omega/2}}{2} (e^{j\omega/2} + e^{-j\omega/2}) \\ &= e^{-j3\omega/2} \cos(\omega/2) \end{aligned}$$

Complex exponentials are eigenfunctions of LTI systems.

$$e^{j\omega_0 t} \longrightarrow \boxed{H(\omega)} \longrightarrow H(\omega_0) e^{j\omega_0 t}$$

$$\begin{aligned}
 \text{Hence, } y(t) &= \frac{e^{-j3/2} \cos(1/2) e^{jt} - e^{j3/2} \cos(1/2) e^{-jt}}{2j} \\
 &= \cos(1/2) \frac{e^{j(t-3/2)} - e^{-j(t-3/2)}}{2j} \\
 &= \cos(1/2) \sin(t-3/2)
 \end{aligned}$$

$$(d) |H(\omega)| = \cos(\omega/2) \quad \angle H(\omega) = -\frac{3\omega}{2}$$



The transmitter should avoid transmitting at frequencies where $|H(\omega)| = 0$

(New Application) i.e. $\frac{\omega}{2} = (2n+1)\frac{\pi}{2} \Rightarrow \omega = (2n+1)\pi$

(e) If Reflection 1 is delayed by 3 seconds

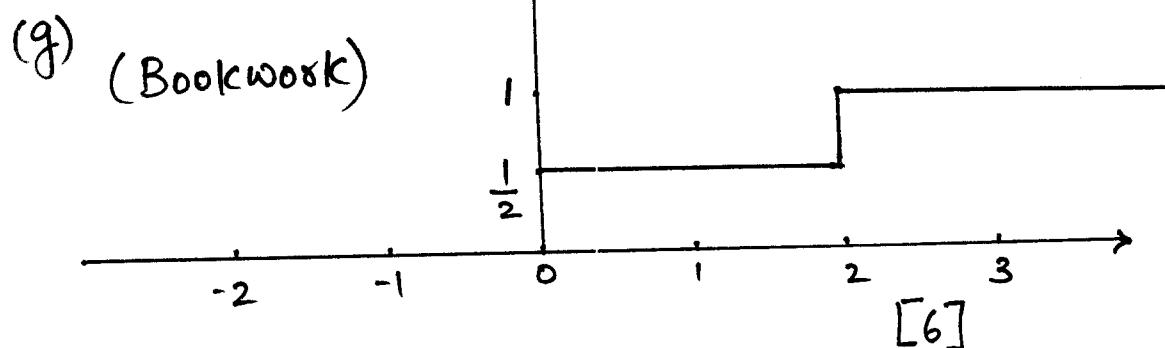
$$h_{\text{new}}(t) = \frac{\delta(t-2)}{2} + \frac{\delta(t-3)}{2},$$

which is the new impulse response.

Hence, $h_{\text{new}}(t) = h(t-1)$ [4]

(New Application) $\Rightarrow H_{\text{new}}(\omega) = H(\omega) e^{-j\omega}$
 $\Rightarrow |H_{\text{new}}(\omega)| = \cos(\omega/2)$

(f) $H(s) = e^{-s} + e^{-2s}$ ROC: entire s plane [4] (Bookwork)



2. (a) $\int_{-\infty}^{\infty} \text{sinc}\left(\frac{\tau-t}{2}\right) \text{sinc}(\tau/3) d\tau$ (New Computed Example)

$$= \text{sinc}\left(\frac{t}{2}\right) * \text{sinc}(t/3)$$

[8]

$$= F^{-1}\left[2\pi\left(\frac{\omega}{\pi}\right) \cdot 3\pi\left(\frac{3\omega}{2\pi}\right)\right]$$

$$= F^{-1} \left[6 \Pi \left(\frac{3\omega}{2\pi} \right) \right]$$

$$= 2 \operatorname{sinc} \left(\frac{t}{3} \right)$$

$$(b) \int_{-\infty}^{\infty} \operatorname{sinc}(t) \operatorname{sinc}(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Pi^2 \left(\frac{\omega}{2\pi} \right) d\omega \quad [7]$$

$$(\text{New Computed Example}) = 1 \quad (\text{Parseval's relation})$$

$$(c) \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{j\omega t}}{6+j\omega+\omega^2} d\omega = F^{-1} \left[\frac{1}{6+j\omega+\omega^2} \right] \quad (\text{Bookwork})$$

$$\frac{1}{6+j\omega+\omega^2} = \frac{1}{6+j\omega-(j\omega)^2} = \frac{-1}{(j\omega+2)(j\omega-3)}$$

Using partial fraction expansion

$$\frac{-1}{(j\omega+2)(j\omega-3)} = \frac{1/5}{j\omega+2} + \frac{-1/5}{j\omega-3} \quad [8]$$

$$= \frac{1/5}{j\omega+2} + \frac{1/5}{-j\omega+3}$$

$$\Rightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{j\omega t}}{6+j\omega+\omega^2} d\omega = \frac{1}{5} e^{-2t} u(t) + \frac{1}{5} e^{3t} u(-t)$$

(d) for $t < 0$ (New computed example)

$$u(t) * e^t u(-t) = \int_{-\infty}^t e^{\tau} d\tau = e^t \quad [7]$$

For $t \geq 0$

$$u(t) * e^t u(-t) = \int_{-\infty}^0 e^{\tau} d\tau = 1$$

$$\text{Hence, } u(t) * e^t u(-t) = \begin{cases} e^t, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

3.

(a) $B = \frac{D}{2\pi T}$ (Bookwork) [3]

(b) $y[n] = \frac{x[n] + (-1)^n x[n]}{2} = \frac{x[n] + e^{jn\pi} x[n]}{2}$

$$\Rightarrow Y(e^{j\omega}) = \frac{X(e^{j\omega}) + X(e^{j(\omega-\pi)})}{2} \quad [6]$$

(New Application of Theory)

(c) Retaining even samples is equivalent to increasing sampling period by 2.

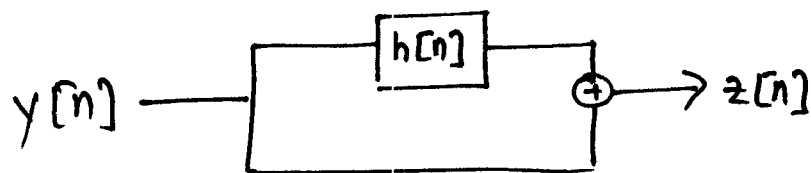
Hence, to satisfy Nyquist criterion-

(New Application of Theory) $2T < \frac{1}{2B} \Rightarrow T < \frac{\pi T}{2D} \quad [6]$

$$\Rightarrow D < \pi/2$$

(d) Note that $z[n]$ can be obtained by filtering $y[n]$ through a filter with impulse response $h[n] = \frac{\delta[n+1] + \delta[n-1]}{2}$

and summing the result with $y[n]$.



$$\Rightarrow z(e^{j\omega}) = Y(e^{j\omega}) H(e^{j\omega}) + Y(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{e^{j\omega} + e^{-j\omega}}{2} = \cos(\omega)$$

$$\text{Hence } z(e^{j\omega}) = Y(e^{j\omega}) (1 + \cos(\omega))$$

$$= \frac{X(e^{j\omega}) + X(e^{j(\omega-\pi)})}{2} (1 + \cos(\omega))$$

(New Application
of Theory)

[15]

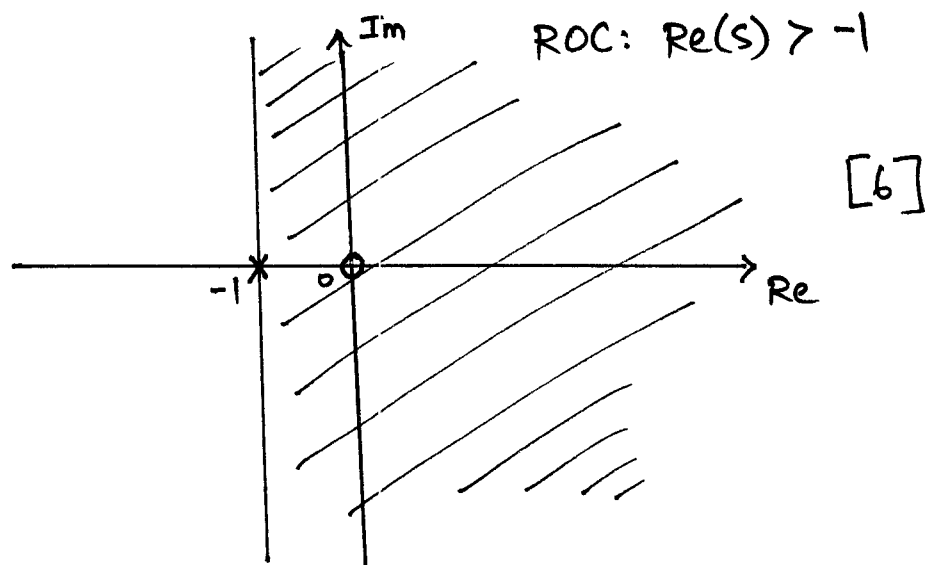
4.

Since $R=1\Omega$, $y(t) = i(t)$

$$(a) \quad CsX(s) = Y(s) + R(sY(s) + L(s^2Y(s)))$$

$$2sX(s) = Y(s) + 2sY(s) + s^2Y(s)$$

$$\Rightarrow H(s) = \frac{2s}{(s+1)^2} \quad (\text{Bookwork})$$



$$(b) \quad \frac{1}{(s+1)^2} \xleftrightarrow{\mathcal{L}^{-1}} t e^{-t} u(t) \quad (\text{Bookwork})$$

$$w(t) = u(t) * h(t) \quad [12]$$

$$\Rightarrow W(s) = U(s) H(s) \\ = 2/(s+1)^2$$

$$\Rightarrow w(t) = 2t e^{-t} u(t)$$

$$(c): G(s) = \frac{1}{H(s)} \quad \left(\text{New application of theory} \right)$$

$$= \frac{(s+1)^2}{2s}$$

$$= \frac{s^2 + 1 + 2s}{2s}$$

[12]

$$= \frac{s}{2} + \frac{1}{2s} + 1$$

$$\Rightarrow g(t) = \frac{1}{2} \delta'(t) + \frac{1}{2} u(t) + \delta(t)$$