UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE TECHNOLOGY AND MEDICINE [MSc-C2, E401, I411]

DEPARTMENT of ELECTRICAL and ELECTRONIC ENGINEERING
EXAMINATIONS 2001
M.Sc in Communications and Signal Processing
M.Eng. Part IV

Solutions 2001
ADVANCED COMMUNICATION THEORY

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ANSWER to Q1

1)	A	B	C	D	E
2)	A	B	C	D	E
3)	A	B	C	D	E
4)	A	B	C	D	E
5)	A	B	C	D	E
6)	A	B	C	D	E
7)	A	B	C	D	E
8)	A	B	C	D	E
9)	A	B	C	D	E
10)	A	B	C	D	E
11)	A	B	C	D	E
12)	A	B	C	D	E
13)	A	B	C	D	E
14)	A	B	C	D	E
15)	A	B	C	D	E
16)	A	B	C	D	E
17)	A	B	C	D	E
18)	A	B	C	D	E
19)	A	B	C	D	E
20)	A	B	C	D	E

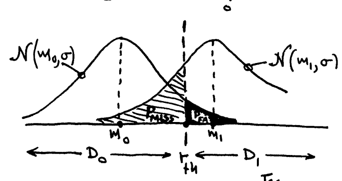
ANSWER to Q2

The correlation receiver implements the following optimum decision rule:

$$\boxed{\text{choose } H_1 \ \underline{\text{iff}} \ G \geq r_{th}}$$

where $G \triangleq \int_0^{T_{cs}} r(t).s_1(t)\,dt - \int_0^{T_{cs}} r(t).s_0(t)\,dt$

$$r_{th} \triangleq \frac{N_0}{2}\ln(\lambda_0) + \frac{1}{2}\int_0^{T_{cs}} (s_1(t)^2 - s_0(t)^2)\,dt$$



where $m_0 = \mathcal{E}\{G|H_0\} = \dots = \int_0^{T_{cs}} (s_0(t)s_1(t) - s_0(t)^2)\,dt$

$$m_1 = \mathcal{E}\{G|H_1\} = \dots = \int_0^{T_{cs}} (s_1(t)^2 - s_0(t)s_1(t))\,dt$$

$\sigma^2 = Var\{G\}$ ← for any Hypothesis

$$= \dots = \frac{N_0}{2}\int_0^{T_{cs}} (s_1(t)^2 + s_0(t)^2 - 2s_1(t)s_0(t))\,dt$$

$$= N_0 E(1-\rho)$$

$$\therefore P_e = Pr(H_0).\underbrace{Pr(D_1|H_0)}_{=P_{FA} = \mathcal{T}\left\{\frac{r_{th}-m_0}{\sigma}\right\}} + Pr(H_1).\underbrace{Pr(D_0|H_1)}_{=P_{MISS} = \mathcal{T}\left\{\frac{r_{th}-m_1}{\sigma}\right\}}$$

However $\frac{r_{th}-m_0}{\sigma}$ can be express as a function of λ_0, EUE, ρ as follows:

$$\frac{r_{th}-m_0}{\sigma} = \frac{\overbrace{\frac{N_0}{2}\ln(\lambda_0) + \frac{1}{2}\int_0^{T_{cs}} (s_1(t)^2 - s_0(t)^2)\,dt}^{=r_{th}} - \overbrace{\int_0^{T_{cs}} (s_0(t)s_1(t) - s_0(t)^2)\,dt}^{=m_0}}{\underbrace{\sqrt{N_0 E(1-\rho)}}_{=\sigma}}$$

$$= \frac{\frac{N_0}{2}\ln(\lambda_0) + \frac{1}{2}\overbrace{\int_0^{T_{cs}} (s_1(t)^2 + s_0^2(t))\,dt}^{=E} - \overbrace{\int_0^{T_{cs}} s_0(t).s_1(t)\,dt}^{=\rho E}}{\sqrt{N_0 E(1-\rho)}}$$

$$= \frac{\frac{N_0}{2}\ln(\lambda_0) + E(1-\rho)}{\sqrt{N_0 E(1-\rho)}} = \frac{\frac{1}{2}\ln(\lambda_0)}{\sqrt{(1-\rho)EUE}} + \sqrt{(1-\rho)EUE}$$

similarly, it can be found that

$$\frac{r_{th}-m_1}{\sigma} = \frac{\frac{1}{2}\ln(\lambda_0)}{\sqrt{(1-\rho)EUE}} - \sqrt{(1-\rho)EUE}$$

$\therefore$ Equ-1 (above) becomes

$$\boxed{P_e = Pr(H_0).\mathcal{T}\left(\frac{\frac{1}{2}\ln(\lambda_0)}{A} + A\right) + Pr(H_1)\,\mathcal{T}\left(\frac{\frac{1}{2}\ln(\lambda_0)}{A} - A\right) \quad \text{Where } A = \sqrt{(1-\rho)EUE}}$$

* Bayes' decision criterion: if $\Lambda(r) \gtrless \Lambda_0$ $\begin{cases}\text{then choose } H_1\\ \text{otherwise } H_0\end{cases}$

where $\Lambda_0 = \frac{Pr(H_0)}{Pr(H_1)}\cdot\frac{C_{10}-C_{00}}{C_{01}-C_{11}}$

$$\Rightarrow \Lambda_0 = \frac{1/3}{2/3}\cdot\frac{1.858}{0.5} \Rightarrow \boxed{\Lambda_0 = 1.858}$$

* $P_{Miss} = Pr(D_0|H_1) = T\left\{\frac{\frac{1}{2}\ln\Lambda_0}{A} - A\right\} = 0.04$

$\Rightarrow$ (from Tail function graph) $\Rightarrow \frac{\frac{1}{2}\ln\Lambda_0}{A} - A = 1.75$

$\Rightarrow A^2 + 1.75A - \frac{1}{2}\ln\Lambda_0 = 0$ with $A > 0$

$\Rightarrow A = \frac{-1.75 \pm \sqrt{1.75^2 + 4\frac{1}{2}\ln\Lambda_0}}{2} = 0.162$

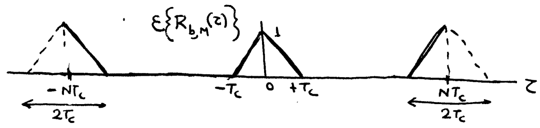
$\Rightarrow \sqrt{(1-\rho)EUE} = 0.162 \Rightarrow (1-\rho)\underbrace{EUE}_{5.25\times10^{-2}} = 0.162^2$

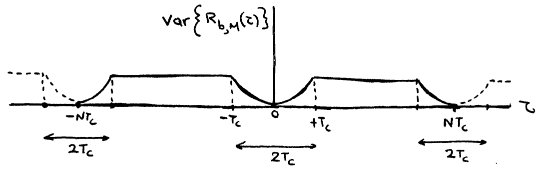
$\Rightarrow \boxed{\rho = \frac{1}{2}}$

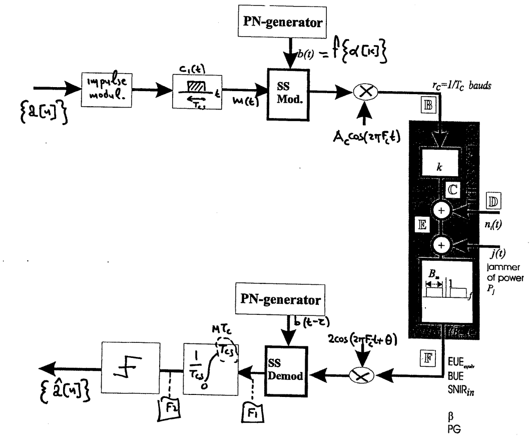
* $P_{FA} = Pr(D_1|H_0) = T\left(\frac{\frac{1}{2}\ln\overset{1.858}{\Lambda_0}}{\underset{0.162}{A}} + A\right) = T(2.0740) \cong 1.7\times10^{-2}.$

$$P_e = \underbrace{Pr(H_0)}_{1/3}\cdot\underbrace{P_{FA}}_{1.7\times10^{-2}} + \underbrace{Pr(H_1)}_{\frac{2}{3}}\cdot\underbrace{P_{Miss}}_{4\times10^{-2}} = 3.23\times10^{-2}$$

ANSWER to Q3







$\boxed{B}$: $s(t) = A_c m(t)\cdot b(t)\cos(2\pi F_c t)$

$\boxed{F}$: desired signal term $= ks(t) = \underbrace{kA_c}_{\sqrt{2P_s}} m(t)b(t)\cos(2\pi F_c t)$

$\boxed{F1}$: desired signal term $= ks(t).\,b(t-\tau)\,2\cos(2\pi F_c t+\theta)$
$= \sqrt{2P_s}\,m(t)\,b(t).b(t-\tau)\;2\cos(2\pi F_c t)\cos(2\pi F_c t+\theta)$

$\boxed{F2}$: desired signal term $= m_0(t) =$

$$= \frac{\sqrt{2P_s}}{T_{cs}}\int_0^{T_{cs}} \underbrace{m(t)}_{\pm1}.b(t).b(t-\tau)\cos\theta.dt$$

$$= \pm\frac{\sqrt{2P_s}}{T_{cs}}\cos\theta\int_0^{MT_c} b(t)b(t-\tau)\,dt$$

$$= \pm\sqrt{2P_s}\cos\theta\; R_{b,M}(\tau)$$

Power of $m_0(t) = \mathcal{E}\{m_0^2(t)\} = 2P_s\cos^2\theta.\mathcal{E}\{R_{b,M}^2(\tau)\}$
$= 2P_s\cos^2\theta\left(var\{R_{b,M}(\tau)\} + \mathcal{E}^2\{R_{b,M}(\tau)\}\right)$
$= \underbrace{2P_s\cos^2\theta\, var\{R_{b,M}(\tau)\}}_{\text{code noise power}} + \underbrace{2P_s\cos^2\theta\,\mathcal{E}^2\{R_{b,M}(\tau)\}}_{\text{desired term}}$

1. If $0 \le \tau \le T_c$ then $\underline{\text{power of code noise}} = 2P_s\cos^2\theta\, var\{R_{b,M}(\tau)\}$
$= 2P_s\cos^2\theta.\frac{1}{M}\left(\frac{\tau}{T_c}\right)^2$
$= 2P_s\cos^2\theta\,\frac{1}{M}\,\frac{\tau^2}{T_c^2}$ (note: $M = \frac{T_{cs}}{T_c}$)
$= 2P_s\cos^2\theta\,\frac{1}{T_{cs}T_c}.\tau^2$ $\boxed{1}$

2. If $\tau > T_c$ then $\underline{\text{power of code noise}} = 2P_s\cos^2\theta\, var\{R_{b,M}(\tau)\}$
$= 2P_s\cos^2\theta\,\frac{1}{M}$
$= 2P_s\cos^2\theta\,\frac{T_c}{T_{cs}}$ $\boxed{2}$

$r_c = 10M\frac{\text{chips}}{\text{sec}} \Rightarrow T_c = 10^{-7}$

$r_b = 1000\frac{\text{bits}}{\text{sec}} \Rightarrow T_{cs} = 10^{-3}$

$\frac{N_0}{2} = 0.5\times10^{-8} \Rightarrow N_0 = 10^{-8}$

$EUE = 100 \Rightarrow \frac{E_b}{N_0} = 10^2 \Rightarrow \frac{P_s T_{cs}}{N_0} = 10^2 \Rightarrow P_s = 10^2\frac{\overset{10^{-8}}{N_0}}{\underset{10^{-3}}{T_{cs}}} \Rightarrow \boxed{P_s = 10^{-3}}$

$P_{\substack{\text{code noise}\\ \tau>T_c}} = 1.5\times10^{-7} \overset{②}{\Longrightarrow} 1.5\times10^{-7} = 2P_s\cos^2\theta\,\frac{T_c}{T_{cs}}$

$\Rightarrow \cos^2\theta = \frac{1.5\times10^{-7}.\overset{10^{-3}}{T_{cs}}}{2\underset{10^{-3}}{P_s}\underset{10^{-7}}{T_c}} \Rightarrow$

$\Rightarrow \cos^2\theta = \frac{1.5}{2} = 0.75 \Rightarrow \cos\theta = 0.866$

$\Rightarrow \boxed{\theta = 30^\circ}$

$P_{\substack{\text{code noise}\\ \tau}} = 3.75\times10^{-8} \overset{①}{\Longrightarrow} 3.75\times10^{-8} = 2\underset{10^{-3}}{P_s}\underset{0.75}{\cos^2\theta}\,\frac{1}{\underset{10^{-3}}{T_{cs}}\underset{10^{-7}}{T_c}}\tau^2$

$\Rightarrow \tau^2 = \frac{3.75\times10^{-15}}{2\times0.75} \Rightarrow \boxed{\tau = 0.5\times10^{-7}}$ i.e. $\boxed{\tau = 0.5\,T_c}$

ANSWER to Q4

$N_0 = 2\times10^{-6}$ $\qquad A_1 = -3mV$
$T_{cs} = 12$ $\qquad A_2 = -1mV$
$D = 1$ $\qquad A_3 = 1mV$
$M = 4$ $\qquad A_4 = 3mV$

$$PSD_s(f) = \frac{1}{T_{cs}}\,\mathcal{E}\left\{\left|FT\left(A_L\cdot\Lambda\left(\frac{t}{T_{cs}/2}\right)\right)\right|^2\right\}$$

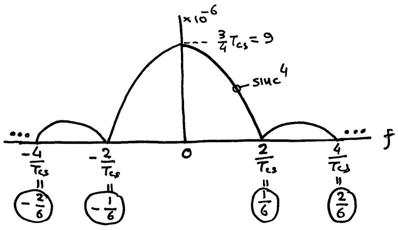
$$= \frac{1}{T_{cs}}\,\mathcal{E}\left\{\left|A_L\frac{T_{cs}}{2}\,\mathrm{sinc}^2\left(f\frac{T_{cs}}{2}\right)\right|^2\right\}$$

$$= \frac{1}{T_{cs}}\,\mathcal{E}\left\{\underset{\substack{\uparrow\\ \text{random}\\ \text{variable}}}{A_L^2}\frac{T_{cs}^2}{4}\,\mathrm{sinc}^4\left(f\frac{T_{cs}}{2}\right)\right\}$$

$$= \frac{1}{T_{cs}}\cdot\frac{T_{cs}^2}{4}\cdot\mathrm{sinc}^4\left(f\cdot\frac{T_{cs}}{2}\right)\cdot\mathcal{E}\left\{A_L^2\right\}$$

$$= \frac{T_{cs}}{4}\,\mathrm{sinc}^4\left(f\cdot\frac{T_{cs}}{2}\right)\left[(-3)^2\frac{1}{8}+(-1)^2\frac{3}{8}+(1)^2\frac{3}{8}+3^2\frac{1}{8}\right]\times10^{-6}$$

$$= \frac{3}{4}\underset{\substack{\uparrow\\ 12}}{T_{cs}}\,\mathrm{sinc}^4\left(f\cdot\frac{T_{cs}}{2}\right)\times10^{-6} = 9\,\mathrm{sinc}^4(f6)\times10^{-6}$$



$$E_L = \int_{-T_{cs}/2}^{T_{cs}/2} A_L^2\,\Lambda^2\left(\frac{t}{T_{cs}/2}\right)dt$$

$$= 2\int_{-T_{cs}/2}^{0} A_L^2\left(\frac{t+T_{cs}/2}{T_{cs}/2}\right)^2 dt$$

$$= 2A_L^2\int_{-T_{cs}/2}^{0}\frac{t^2+T_{cs}^2/4+2t\frac{T_{cs}}{2}}{T_{cs}^2/4}\,dt$$

$$= \frac{8A_L^2}{T_{cs}^2}\int_{-T_{cs}/2}^{0}\left(t^2+\frac{\cancel{2}T_{cs}t}{\cancel{2}}+\frac{T_{cs}^2}{4}\right)dt$$

$$= \frac{8A_L^2}{T_{cs}^2}\left(\frac{t^3}{3}\Big|_{-T_{cs}/2}^{0}+T_{cs}\frac{t^2}{2}\Big|_{-T_{cs}/2}^{0}+\frac{T_{cs}^2}{4}t\Big|_{-T_{cs}/2}^{0}\right)$$

$$= \frac{8A_L^2}{T_{cs}^2}\left(\frac{T_{cs}^3}{3\times8}-\cancel{\frac{T_{cs}^3}{8}}+\cancel{\frac{T_{cs}^3}{8}}\right)$$

$$= \frac{1}{3}\underset{\substack{\uparrow\\ 12}}{T_{cs}}\cdot A_L^2 = 4\cdot A_L^2$$

$$\therefore\ c_1(t) = +A_1\,\Lambda\left(\frac{2t}{T_{cs}}\right)\Big/\sqrt{E_1} = \frac{\cancel{A_1}}{2\cancel{A_1}}\Lambda\left(\frac{2t}{T_{cs}}\right)$$

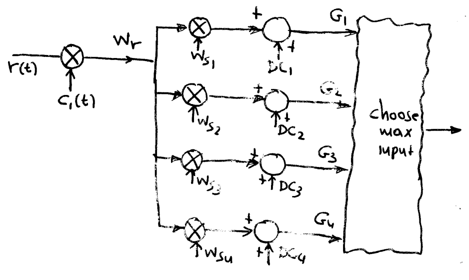
$$\therefore\ \boxed{c_1(t) = \frac{1}{2}\,\Lambda\left(\frac{t}{6}\right)}$$

$\underline{w}_{s_1} = -\sqrt{E_1} = -6mV$ $\qquad DC_1 = \frac{N_0}{2}\ln(P_1)-\frac{1}{2}E_1 = -20.079\times10^{-6}$

$\underline{w}_{s_2} = -\sqrt{E_2} = -2mV$ $\qquad DC_2 = \frac{N_0}{2}\ln(P_2)-\frac{1}{2}E_2 = -2.98\times10^{-6}$

$\underline{w}_{s_3} = \sqrt{E_3} = 2mV$ $\qquad DC_3 = \frac{N_0}{2}\ln(P_3)-\frac{1}{2}E_3 = -2.98\times10^{-6}$

$\underline{w}_{s_4} = \sqrt{E_4} = 6mV$ $\qquad DC_4 = \frac{N_0}{2}\ln(P_4)-\frac{1}{2}E_4 = -20.079\times10^{-6}$

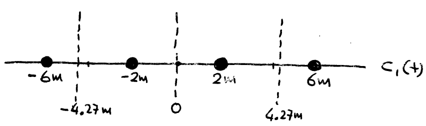


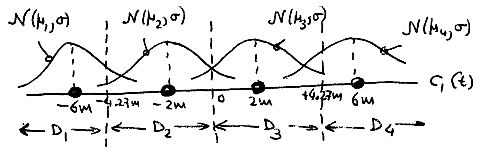
$G_i = w_r w_{s_i} + DC_i$

$G_1 = G_2 \Rightarrow w_r w_{s_1} + DC_1 = w_r w_{s_2} + DC_2 \Rightarrow w_{r_1} = \dfrac{DC_2 - DC_1}{w_{s_1} - w_{s_2}} = -4.27m$

$G_2 = G_3 \Rightarrow \cdots \Rightarrow w_{r_2} = \dfrac{DC_3 - DC_2}{w_{s_2} - w_{s_3}} = 0$

$G_3 = G_4 \Rightarrow \cdots \Rightarrow w_{r_3} = \dfrac{DC_4 - DC_3}{w_{s_3} - w_{s_4}} = +4.27m$





$\mu_1 = -6m$
$\mu_2 = -2m$
$\mu_3 = 2m$
$\mu_4 = 6m$

$$\sigma^2 = \frac{N_0}{2}\,2\underset{\substack{\uparrow\\ \frac{1}{2T_{cs}}}}{B}\,T_{cs} = \frac{N_0}{2} = 10^{-6} \Rightarrow \boxed{\sigma = 10^{-3}}$$

$\left.\begin{array}{l} Pr(D_1|H_1) = Pr(D_4|H_4) \\ Pr(D_2|H_2) = Pr(D_3|H_3) \end{array}\right\}$ due to symmetry

$\therefore$

$$P_{e,cs} = 1 - 2\,Pr(D_1|H_1)\,P_1 - 2\,Pr(D_2|H_2)\,P_2$$

$$= 1 - 2\left(1 - T\left(\frac{6-4.27}{10^{-3}}\times10^{-3}\right)\right)\frac{1}{8} - 2\left(1 - T(2) - T(2.27)\right)\frac{3}{8}$$

$$= \frac{2}{8}T(1.73) + \frac{6}{8}T(2) + \frac{6}{8}T(2.27)$$

$= 2/8*0.0418 + 6/8*0.0228 + 6/8*0.0116$
$= 0.0362$