EE4-25
EE9-CS1-2

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2013

MSc and EEE/EIE PART IV: MEng and ACGI

DESIGN OF LINEAR MULTIVARIABLE CONTROL SYSTEMS

Monday, 20 May 10:00 am

Time allowed: 3:00 hours

There are SIX questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s) :	I.M. Jaimoukha
	Second Marker(s) :	E.C. Kerrigan

1. Let the n-th order transfer matrix $G(s)$ have a state space realisation

$$G(s) \stackrel{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$$

and let

$$A'Q + QA + C'C = 0,$$

and

$$AP + PA' + BB' = 0$$

for some $Q = Q'$ and $P = P'$.

Suppose that

$$Q = \begin{bmatrix} 0 & 0 \\ 0 & Q_2 \end{bmatrix},$$

and

$$P = \begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix}$$

where $\mathscr{R}^{n_1 \times n_1} \ni P_1 \succ 0$ and $\mathscr{R}^{n_2 \times n_2} \ni Q_2 \succ 0$ and where $n_1 + n_2 = n$. Assume that A has no eigenvalues on the imaginary axis.

a) By partitioning the realisation for $G(s)$ compatibly with P and Q, prove that the realisation can be decomposed into two subsystems:

i) A subsystem with n_1 modes that are stable, controllable and unobservable. [7]

ii) A subsystem with n_2 modes that are stable, uncontrollable and observable. [7]

b) Draw a diagram illustrating the two subsystems of $G(s)$. [6]

2. Consider the feedback loop shown in Figure 2 where $G(s)$ represents a plant model and $K(s)$ represents an internally stabilizing compensator. Suppose that

$$G(s) \stackrel{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cc|c} -1 & 1 & 2 \\ 1 & -1.125 & -1.5 \\ \hline 2 & -1.5 & 0 \end{array}\right] \in \mathcal{RH}_\infty.$$

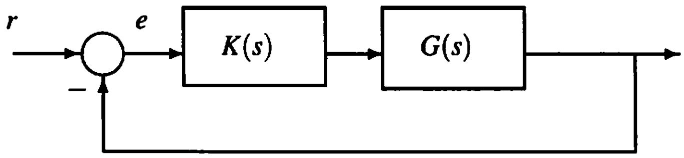


Figure 2

a) Show that the given realization for $G(s)$ is balanced and evaluate the Hankel singular values of $G(s)$. [6]

b) Give a first order balanced truncation of $G(s)$. [4]

c) By using:

- the answer to Part (a),
- the small gain theorem (which should be stated),
- and a parameterization of the set of all internally stabilizing controllers,

design a first order internally stabilizing controller $K(s)$ for $G(s)$ such that the DC gain of the nominal loop gain is as large as possible. Use the balanced truncation of $G(s)$ in Part (b) above as your nominal plant. Comment on the controller type and the maximum DC value of the loop gain. [10]

3. Consider the feedback configuration in Figure 3. Here, $G(s)$ is a nominal plant model and $K(s)$ is a compensator. The design specifications are to synthesize a compensator $K(s)$ such that the feedback loop is internally stable and such that

- when $n(s) = 0$,

$$\begin{aligned} \|e(j\omega)\| &< |w_1(j\omega)^{-1}|\|r(j\omega)\|, \forall\omega. \\ \|u(j\omega)\| &< |w_2(j\omega)^{-1}|\|r(j\omega)\|, \forall\omega, \end{aligned}$$

- when $r(s) = 0$,

$$\|y(j\omega)\| < |w_3(j\omega)^{-1}|\|n(j\omega)\|, \forall\omega,$$

where $w_1(s)$, $w_2(s)$ and $w_3(s)$ are suitable filters.

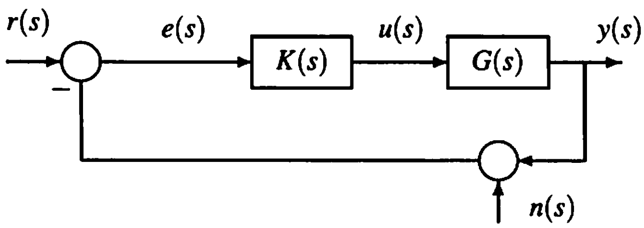


Figure 3

a) Give an interpretation of the signals in Figure 3 and a brief explanation of the significance of the design specifications for the performance of the feedback loop in Figure 3. [3]

b) Derive $\mathscr{H}_\infty$-norm bounds, in terms of $G(s), K(s), w_1(s)$, $w_2(s)$ and $w_3(s)$ that are sufficient to achieve the design specifications. [6]

c) Derive a generalized regulator formulation of the design problem that captures the sufficient conditions in Part (b). [6]

d) Assume that $K(s)$ achieves the design specifications. Suppose that the actual system is $G(s) + \Delta(s)$ where $\Delta(s)$ is a stable transfer matrix. Derive the maximal stability radius for $\|\Delta(j\omega)\|, \forall\omega$ that can be deduced from Part (b) and the small gain theorem. [5]

4. Consider the regulator in Figure 4 for which it is assumed that B has full column rank, the pair (A, B) is controllable and $x(0) = x_0$. A stabilizing state-feedback gain matrix F is to be designed such that the cost function

$$J = \left\| \begin{bmatrix} u \\ x \end{bmatrix} \right\|_2^2$$

is minimized.

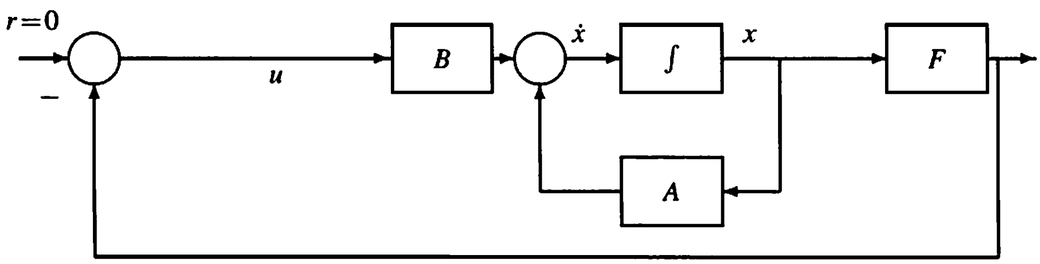


Figure 4

a) By defining suitable cost and Lyapunov functions and completing a square, find an expression for F that achieves the design specifications in terms of the solution of an algebraic Riccati equation. Give the minimum value of the cost function as well as the algebraic Riccati equation. [10]

b) Prove that, for F chosen in part (a), the closed loop system in Figure 4 is stable. State clearly the assumptions required to guarantee stability. [5]

c) Prove that, for F chosen in part (a), the loop gain in Figure 4 is minimum-phase. [5]

5. a) Consider a state–variable model described by the dynamics

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t) + Du(t),$$

and let $H(s) = D + C(sI - A)^{-1}B$ denote the corresponding transfer matrix. Suppose there exists a $P = P' \succ 0$ such that

$$\begin{bmatrix} A'P + PA + C'C & PB + C'D \\ B'P + D'C & D'D - \gamma^2 I \end{bmatrix} < 0.$$

i) Prove that A is stable. [4]

ii) By defining suitable Lyapunov and cost functions and completing a square, prove that

$$\|H\|_\infty < \gamma.$$

[6]

b) Consider the output injection problem shown in Figure 5. Let $w = \begin{bmatrix} w_1^T & w_2^T \end{bmatrix}^T$ and let $T_{yw}(s)$ denote the transfer matrix from w to y. An internally stabilizing output injection gain matrix L is to be designed such that, for a given $\gamma > 0$, $\|T_{yw}\|_\infty < \gamma$.

i) Derive a state space realization for $T_{yw}(s)$. [4]

ii) By using the answer to Part (a) above, or otherwise, derive sufficient conditions for the existence of a feasible L. Your conditions should be in the form of the existence of certain solutions to linear matrix inequalities. [6]

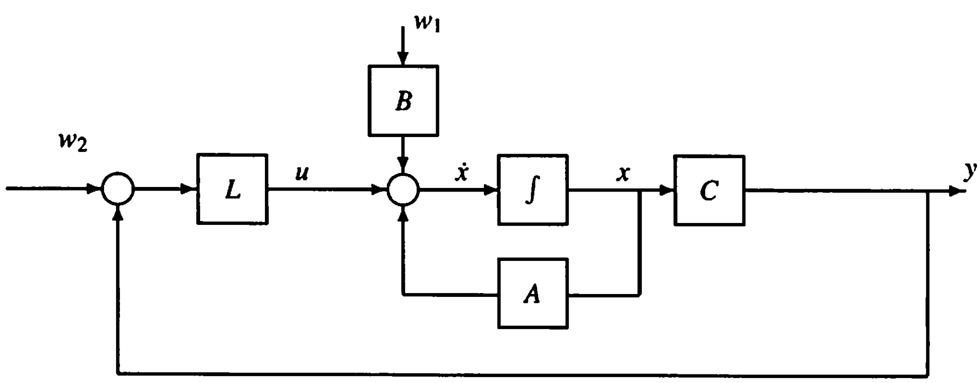


Figure 5

6. Consider the regulator shown in Figure 6 for which it is assumed that the pair (A, B) is controllable and $x(0) = 0$.

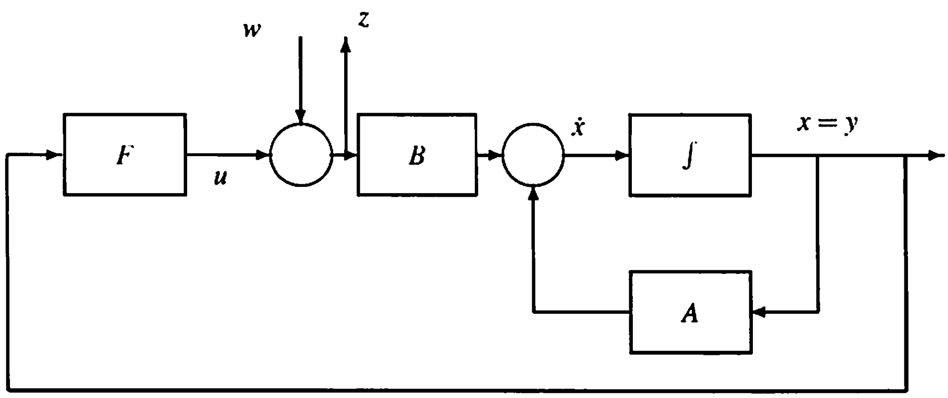


Figure 6

Let $H(s)$ denote the transfer matrix from w to z. A stabilizing state-feedback gain matrix F is to be designed such that, for $\gamma > 0$, $\|H\|_\infty < \gamma$. Assume that $\gamma > 1$.

a) Write down the generalized regulator system for this design problem. [4]

b) By defining suitable Lyapunov and cost functions as well as two completion of squares procedures, derive sufficient conditions for the solution of the design problem. Your conditions should be in the form of the existence of a certain solution to an algebraic Riccati equation. It should also include an expression for F and an expression for the worst-case disturbance w. [8]

c) What is the smallest γ for which your sufficient conditions guarantee the existence of F satisfying the design specifications. Justify your answer. [4]

d) Suppose that A is stable. Show that $F = 0$ is a solution. [2]

e) Suppose that $-A$ is stable. Show that the solution of the Riccati equation reduces to the solution of a Lyapunov equation. [2]

1. a) Let the realisation be partitioned compatibly with P and Q as

$$G(s) \stackrel{s}{=} \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & 0 \end{array}\right].$$

Then

$$A'Q+QA+C'C = \begin{bmatrix} C_1'C_1 & A_{21}'Q_2+C_1'C_2 \\ \star & A_{22}'Q_2+Q_2A_{22}+C_2'C_2 \end{bmatrix} = 0 \quad (1.1)$$

$$AP+PA^T+BB^T = \begin{bmatrix} A_{11}P_1+P_1A_{11}^T+B_1B_1^T & P_1A_{21}+B_1B_2^T \\ \star & B_2B_2^T \end{bmatrix} = 0 \quad (1.2)$$

It follows from the (1,1) entry of (1.1) and the (2,2) entry of (1.2) that $C_1 = 0$ and $B_2 = 0$. Also, it follows from the (1,2) entry of (1.1) and the assumption that $Q_2 \succ 0$ that $A_{21} = 0$. So, the realisation for $G(s)$ has the form

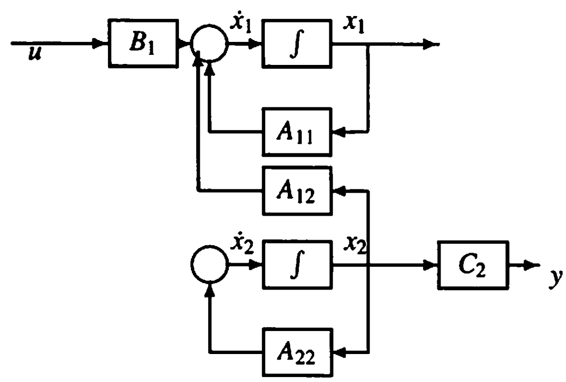
$$G(s) \stackrel{s}{=} \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ 0 & A_{22} & 0 \\ \hline 0 & C_2 & 0 \end{array}\right]. \quad (1.3)$$

Thus we can decompose the realisation into two subsystems $G_1 : \dot{x}_1 = A_{11}x_1 + B_1u + A_{12}x_2$ with n_1 modes and $G_2 : \dot{x}_2 = A_{22}x_2$, $y = C_2x_2$ with n_2 modes.

i) For G_1 it is clear all the modes are unobservable since the C-matrix is zero. We prove controllability and stability. Let λ be an eigenvalue of A_{11} and $z \neq 0$ the corresponding left eigenvector. Then $z'A_{11} = \lambda z'$. Pre- and post-multiplying the the (1,1) entry of (1.2) by z' and z, respectively, we get $(\lambda+\bar{\lambda})z'P_1z + z'B_1B_1'z = 0$. If $z'B_1 = 0$ then $\lambda+\bar{\lambda} = 0$ which contradicts the assumption that A has no eigenvalues on the imaginary axis. Thus the realisation is controllable. Since $z \neq 0$ and $P_1 \succ 0$, $z'P_1z > 0$ and $z'B_1B_1'z > 0$ then $\lambda+\bar{\lambda} < 0$ and A_{11} is stable.

ii) For the subsystem G_2 it is clear that all the modes are uncontrollable since the B-matrix is zero. We prove observability and stability. Using a duality argument, G_2 is stable and observable if and only if the pair A_{22}^T is stable and (A_{22}^T, C_2^T) is controllable. But this follows from an argument dual to that used above.

b)



2. a) The realization of $G(s)$ is balanced if

$$A\Sigma + \Sigma A' + BB' = A'\Sigma + \Sigma A + C'C = 0$$

for $\Sigma = \text{diag}(\sigma_1, \sigma_2) > 0$ and where the $\sigma_i's$ are the Hankel singular values of $G(s)$. A simple calculation gives $\Sigma = \text{diag}(2, 1)$.

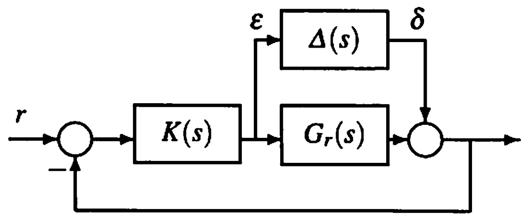
b) A first order balanced truncation of $G(s)$ is given by

$$G_r(s) \doteq \left[\begin{array}{c|c} -1 & 2 \\ \hline 2 & 0 \end{array}\right] = \frac{4}{s+1}.$$

c) Now $G_r(s) = G(s) + \Delta(s)$ where

$$\|\Delta\|_\infty \le 2\sum_{i=2}^{2} \sigma_i = 2.$$

Then replacing $G(s)$ by $G_r(s)$ in the loop of Figure 2 is equivalent to:



Now

$$\varepsilon = -K(I + G_r K)^{-1}\delta$$

and so the loop is stable if $\left\|K(I + G_r K)^{-1}\right\|_\infty < 0.5$ from the small gain theorem since $\|\Delta\|_\infty \le 2$. However, the set of all internally stabilizing controllers for $G_r(s)$ is given by:

$$K = Q(I - G_r Q)^{-1}$$

for stable Q. Furthermore,

$$K(I + G_r K)^{-1} = Q.$$

Thus we can take Q to be constant (to guarantee a first order controller) and $|Q| < 0.5$ (to guarantee stabilization of G).

The nominal loop gain is $G_r(s)K(s) = G_r(s)Q(s)\,(I - G_r(s)Q(s))^{-1}$ and so the DC value is $4Q(1 - 4Q)^{-1}$. This is maximised when $Q = 0.25$ to give an infinite DC gain. This value of Q is internally stabilising. A computation gives

$$K(s) = \frac{s+1}{4s}.$$

Note that $K(s)$ has increased the system type since it has a free integrator.

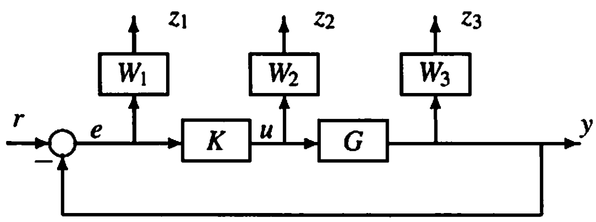
3. a) Signal interpretation: $r(s)$, $e(s)$, $u(s)$, $y(s)$ and $n(s)$ are the reference, error, control input, output and sensor noise signals, respectively. Specifications: ensure good tracking, limit control effort and attenuate noise, respectively.

b) It is clear that we require K to be internally stabilizing.

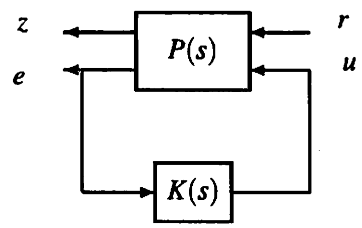
- A calculation shows that, when $n(s)=0$, $e(s)=-S(s)r(s)$ where $S(s)=[I+G(s)K(s)]^{-1}$ is the sensitivity. Thus $\|e(j\omega)\| \leq \|S(j\omega)\|\|r(j\omega)\|$. It follows that a sufficient condition to achieve the first design specification is $\|S(j\omega)\| < |w_1^{-1}(j\omega)|, \forall\omega$ or equivalently $\|W_1S\|_\infty < 1$, where $W_1 = w_1 I$. A similar calculation shows that, when $n(s) = 0$, $u(s) = -K(s)S(s)r(s)$. Thus $\|u(j\omega)\| \leq \|K(j\omega)S(j\omega)\|\|r(j\omega)\|$. It follows that a sufficient condition to achieve the second design specification is $\|K(j\omega)S(j\omega)\| < |w_2^{-1}(j\omega)|$, $\forall\omega$ or equivalently $\|W_2KS\|_\infty < 1$, where $W_2 = w_2 I$.
- When $r(s) = 0$, a similar calculation shows that $y(s) = -C(s)n(s)$ where $C(s) = G(s)K(s)[I+G(s)K(s)]^{-1}$ is the complementary sensitivity. Thus $\|y(j\omega)\| \leq \|C(j\omega)\|\|n(j\omega)\|$. It follows that a sufficient condition to achieve the second design specification is $\|C(j\omega)\| < |w_3^{-1}(j\omega)|, \forall\omega$ or equivalently $\|W_3C\|_\infty < 1$, where $W_3 = w_3 I$.

To satisfy all design requirements, it is sufficient that $\left\| \begin{bmatrix} W_1S \\ W_2KS \\ W_3C \end{bmatrix} \right\|_\infty < 1.$

c) The design specifications reduce to the requirement that the transfer matrix from r to $z = [z_1^T \ z_2^T \ z_3^T]^T$ in the following diagram has $\mathscr{H}_\infty$-norm less than 1.



The corresponding generalized regulator formulation is to find an internally stabilizing K such that $\|\mathscr{F}_l(P,K)\|_\infty < 1$:



$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \left[\begin{array}{c|c} W_1 & -W_1G \\ 0 & W_2 \\ 0 & W_3G \\ \hline I & -G \end{array}\right].$$

d) Let $G(s)$ be replaced by $G(s)+\Delta(s)$ in Figure 3 and let ε be the input and δ be the output of Δ. Then $\varepsilon = -KS\delta$. Using the small gain theorem the maximum stability radius is $|w_2(j\omega)|$.

4. a) Let $V = x'Px$ and set $u = -Fx$. Provided that $P = P' \succ 0$ and $\dot{V} < 0$ along closed-loop trajectories, we can assume $\lim_{t\to\infty} x(t) = 0$. Then

$$\dot{V} = \dot{x}'Px + x'P\dot{x} = x'\left(A'P + PA - F'B'P - PBF\right)x.$$

Integrating from 0 to ∞ and using $x(\infty) = 0$,

$$\int_0^\infty x'\left(A'P + PA - F'B'P - PBF\right)x\,dt = -x_0'Px_0.$$

Using the definition of J, adding the last equation and completing a square:

$$J = x_0'Px_0 + \int_0^\infty \{x'[A'P + PA + I - PBB'P]x + \|(F - B'P)x\|^2\}dt.$$

Since the last term is always nonnegative, it follows that the minimizing value of F is given by $F = B'P$. We can set the term in square brackets to zero provided P satisfies the Riccati equation,

$$A'P + PA + I - PBB'P = 0.$$

It follows that the minimum value of J is $x_0'Px_0$.

b) We need to prove that $A_c := A - BB'P$ is stable. The Riccati equation can be written as $A_c'P + PA_c + I + PBB'P = 0$. Let $\lambda \in \mathscr{C}$ be an eigenvalue of A_c and $z \neq 0$ be the corresponding eigenvector. Pre– and post–multiplying the Riccati equation be z' and z respectively gives $(\lambda + \bar{\lambda})z'Pz + z'z + z'PBB'Pz = 0$. Since $P \succ 0$ and $z \neq 0$, $z'Pz > 0$, $z'z > 0$ and $z'PBB'Pz \geq 0$. It follows that $\lambda + \bar{\lambda} < 0$ and the closed loop is stable.

c) The loop-gain is given by $L(s) = F(sI - A)^{-1}B$.. Let λ be a zero of $L(s)$ so that,

$$\begin{bmatrix} A - \lambda I & B \\ F & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

for some v_1, v_2 with $v_1 \neq 0$ since B has full column rank. Then,

$$Av_1 = \lambda v_1 - Bv_2,$$

and,

$$Fv_1 = 0 \Rightarrow B'Pv_1 = 0.$$

Consider the Riccati equation. Then,

$$\begin{aligned} 0 &= v_1'[A'P + PA - PBB'P + I]v_1 \\ &= (\bar{\lambda}v_1' - v_2'B')Pv_1 + v_1'P(\lambda v_1 - Bv_2) + v_1'v_1 \\ &= (\bar{\lambda} + \lambda)\underbrace{v_1'Pv_1}_{>0} + \underbrace{v_1'v_1}_{>0} \end{aligned}$$

Hence $\lambda + \bar{\lambda} < 0$ and $L(s)$ is minimum-phase.

5. a) i) The $(1,1)$ block of the inequality gives the inequality $A'P+PA \prec 0$. Let $z \neq 0$ be a right eigenvector of A and let λ be the corresponding eigenvalue. Then multiplying the inequality from the left by $\bar{z}'$ and from the right by z gives $(\lambda+\bar{\lambda})\bar{z}'Pz < 0$. Since $P \succ 0$ it follows that $\bar{z}'Pz > 0$ and it follows that $\lambda+\bar{\lambda} < 0$ so that A is stable.

ii) Since A is stable, $\|H\|_\infty < \gamma$ if and only if, with $x(0)=0$, $J := \int_0^\infty [y'y - \gamma^2 u'u]dt < 0$, for all $u(t)$ such that $\|u\|_2 < \infty$. If $\|u\|_2$ is bounded, then $\lim_{t\to\infty} x(t) = 0$. Now, $\int_0^\infty \frac{d}{dt}[x'Px]dt = x(\infty)'Px(\infty) - x(0)'Px(0) = 0$. So,

$$0 = \int_0^\infty (\dot{x}'Px + x'P\dot{x})\,dt = \int_0^\infty [x'(A'P+PA)x + x'PBu + u'B'Px]dt.$$

Use $y = Cx+Du$ and add the last expression to J

$$\begin{aligned} J &= \int_0^\infty [x'(A'P+PA+C'C)x + 2x'(PB+C'D)u + u'(D'D-\gamma^2 I)u]dt \\ &= \int_0^\infty \begin{bmatrix} x' & u' \end{bmatrix} \overbrace{\begin{bmatrix} A'P+PA+C'C & PB+C'D \\ B'P+D'C & D'D-\gamma^2 I \end{bmatrix}}^{M} \begin{bmatrix} x \\ u \end{bmatrix} dt. \end{aligned}$$

It follows that $J < 0$, and so $\|H\|_\infty < \gamma$, if $M \prec 0$. This proves the result.

b) i) Substituting $u = Lw_2 + Cx$, $y = Cx$ into the state equation gives

$$\dot{x} = \underbrace{(A+LC)}_{A_c}x + \underbrace{\begin{bmatrix} B & L \end{bmatrix}}_{B_c}w, \qquad y = \underbrace{C}_{C_c}x + \underbrace{0}_{D_c}w.$$

It follows that $T_{yw}(s) = D_c + C_c(sI - A_c)^{-1}B_c$.

ii) Using the results of part (a), by replacing A, B, C and D by A_c, B_c, C_c and D_c, we have that there exists a feasible L if there exists $P = P^T \succ 0$ such that

$$\begin{bmatrix} (A+LC)'P + P(A+LC) + C'C & PB & PL \\ B'P & -\gamma^2 I & 0 \\ L'P & 0 & -\gamma^2 I \end{bmatrix} \prec 0.$$

Noting that the only nonlinearity is due to the product PL, we define $Z = PL$ and so there exists a feasible L if there exists $P = P^T \succ 0$ and Z such that

$$\begin{bmatrix} A'P + PA + ZC + C'Z' + C'C & PB & Z \\ B'P & -\gamma^2 I & 0 \\ Z' & 0 & -\gamma^2 I \end{bmatrix} \prec 0.$$

6. a) The generalized regulator formulation is given by

$$\begin{bmatrix} z(s) \\ y(s) \end{bmatrix} = P(s) \begin{bmatrix} w(s) \\ u(s) \end{bmatrix}, u(s) = Fy(s), P(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{c||c|c} A & B & B \\ \hline 0 & I & I \\ \hline I & 0 & 0 \end{array}\right].$$

b) The requirement $\|H\|_\infty < \gamma$ is equivalent to $J := \|z\|_2^2 - \gamma^2 \|w\|_2^2 < 0$. Let $V = x^T X x$ and set $u = Fx$. Provided that $X = X^T \succ 0$ and $\dot{V} < 0$ along the closed-loop trajectory, we can assume $\lim\limits_{t \to \infty} x(t) = 0$. Then

$$\dot{V} = \dot{x}^T X x + x^T X \dot{x} = x^T \left(A^T X + XA + F^T B^T X + XBF\right) x + x^T XBw + w^T B^T X x.$$

Integrating from 0 to ∞ and using $x(0) = x(\infty) = 0$,

$$0 = \int_0^\infty [x^T \left(A^T X + XA + F^T B^T X + XBF\right) x + x^T XBw + w^T B^T X x] dt.$$

Using the definition of J and adding the last equation, $J =$

$$J = \int_0^\infty \{x^T [A^T X + XA + F^T F + F^T B^T X + XBF] x - [\beta^2 w^T w - x^T Z^T w - w^T Z x]\} dt$$

where $Z = F + B^T X$ and $\beta^2 = \gamma^2 - 1 > 0$ since $\gamma > 1$ by assumption. Now

$$\begin{aligned} Z^T Z &= F^T F + F^T B^T X + XBF + XBB^T X \\ \|(\beta w - \beta^{-1} Z x)\|^2 &= \beta^2 w^T w - w^T Z x - x^T Z^T w + \beta^{-2} x^T Z^T Z x, \end{aligned}$$

$$J = \int_0^\infty \{x^T [A^T X + XA - XBB^T X] x + (1 + \beta^{-2}) \|Zx\|^2 - \left\|\beta w - \beta^{-1} Z x\right\|^2\} dt.$$

Thus two sufficient conditions for $J < 0$ are the existence of X such that

$$A^T X + XA - XBB^T X = 0, \qquad X = X^T \succ 0.$$

Setting $Z = 0$, $F = -B^T X$. The worst case disturbance is $w^* = \beta^{-2} Z x = 0$. The closed-loop with $u = Fx$ and $w = w^*$ is $\dot{x} = [A - BB^T X] x$ and a third condition is $Re\ \lambda_i [A - BB^T X] < 0$, $\forall\, i$. To prove $\dot{V} < 0$ for $u = Fx$ and $w = 0$,

$$\dot{V} = x^T \left(A^T X + XA + F^T B^T X + XBF\right) x = -x^T (XBB^T X) x < 0$$

for all $x \neq 0$ (since (A, B) is assumed controllable) proving closed-loop stability.

c) It is clear that our procedure breaks down if $\gamma \leq 1$ since in that case $\beta^2 \leq 0$. Thus the smallest value of γ is 1.

d) If A is stable, in the limit $X = 0$ and hence $F = 0$ is a solution.

e) If $-A$ is stable, pre- and post-multiplying the Riccati equation by X^{-1},

$$AX^{-1} + X^{-1} A^T - BB^T = 0 \Leftrightarrow (A - BB^T X) X^{-1} + X^{-1} (A - BB^T X) + BB^T = 0$$

which has a unique solution $X^{-1} \succ 0$ if $-A$ is stable and so $A - BB^T X$ is stable.

1. a) Let the realisation be partitioned compatibly with P and Q as

$$G(s) \stackrel{s}{=} \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & 0 \end{array}\right].$$

Then

$$\begin{aligned} A'Q+QA+C'C &= \begin{bmatrix} C_1'C_1 & A_{21}'Q_2+C_1'C_2 \\ \star & A_{22}'Q_2+Q_2A_{22}+C_2'C_2 \end{bmatrix} = 0 \quad (1.1) \\ AP+PA^T+BB^T &= \begin{bmatrix} A_{11}P_1+P_1A_{11}^T+B_1B_1^T & P_1A_{21}+B_1B_2^T \\ \star & B_2B_2^T \end{bmatrix} = 0 (1.2) \end{aligned}$$

It follows from the (1,1) entry of (1.1) and the (2,2) entry of (1.2) that $C_1 = 0$ and $B_2 = 0$. Also, it follows from the (1,2) entry of (1.1) and the assumption that $Q_2 \succ 0$ that $A_{21} = 0$. So, the realisation for $G(s)$ has the form

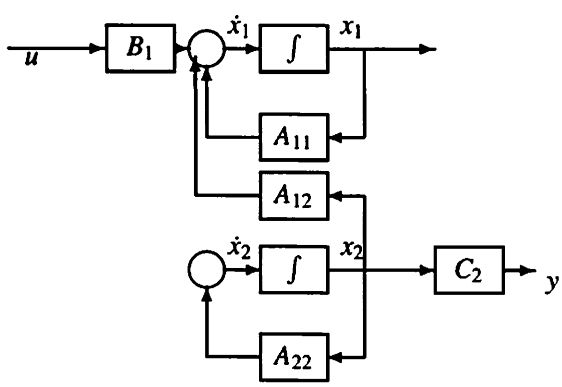
$$G(s) \stackrel{s}{=} \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ 0 & A_{22} & 0 \\ \hline 0 & C_2 & 0 \end{array}\right]. \quad (1.3)$$

Thus we can decompose the realisation into two subsystems $G_1 : \dot{x}_1 = A_{11}x_1 + B_1u + A_{12}x_2$ with n_1 modes and $G_2 : \dot{x}_2 = A_{22}x_2$, $y = C_2x_2$ with n_2 modes.

i) For G_1 it is clear all the modes are unobservable since the C-matrix is zero. We prove controllability and stability. Let λ be an eigenvalue of A_{11} and $z \neq 0$ the corresponding left eigenvector. Then $z'A_{11} = \lambda z'$. Pre- and post-multiplying the the (1,1) entry of (1.2) by z' and z, respectively, we get $(\lambda + \bar{\lambda})z'P_1z + z'B_1B_1'z = 0$. If $z'B_1 = 0$ then $\lambda + \bar{\lambda} = 0$ which contradicts the assumption that A has no eigenvalues on the imaginary axis. Thus the realisation is controllable. Since $z \neq 0$ and $P_1 \succ 0$, $z'P_1z > 0$ and $z'B_1B_1'z > 0$ then $\lambda + \bar{\lambda} < 0$ and A_{11} is stable.

ii) For the subsystem G_2 it is clear that all the modes are uncontrollable since the B-matrix is zero. We prove observability and stability. Using a duality argument, G_2 is stable and observable if and only if the pair A_{22}^T is stable and (A_{22}^T, C_2^T) is controllable. But this follows from an argument dual to that used above.

b)



2. a) The realization of $G(s)$ is balanced if

$$A\Sigma + \Sigma A' + BB' = A'\Sigma + \Sigma A + C'C = 0$$

for $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2) > 0$ and where the $\sigma_i' s$ are the Hankel singular values of $G(s)$. A simple calculation gives $\Sigma = \mathrm{diag}(2,1)$.

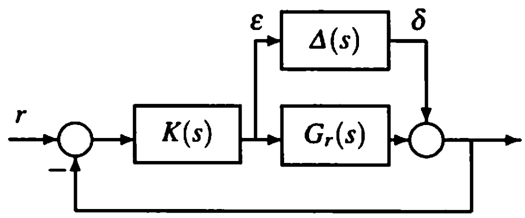
b) A first order balanced truncation of $G(s)$ is given by

$$G_r(s) \stackrel{s}{=} \left[\begin{array}{c|c} -1 & 2 \\ \hline 2 & 0 \end{array}\right] = \frac{4}{s+1}.$$

c) Now $G_r(s) = G(s) + \Delta(s)$ where

$$\|\Delta\|_\infty \leq 2\sum_{i=2}^{2} \sigma_i = 2.$$

Then replacing $G(s)$ by $G_r(s)$ in the loop of Figure 2 is equivalent to:



Now

$$\varepsilon = -K(I + G_rK)^{-1}\delta$$

and so the loop is stable if $\|K(I + G_rK)^{-1}\|_\infty < 0.5$ from the small gain theorem since $\|\Delta\|_\infty \leq 2$. However, the set of all internally stabilizing controllers for $G_r(s)$ is given by:

$$K = Q(I - G_rQ)^{-1}$$

for stable Q. Furthermore,

$$K(I + G_rK)^{-1} = Q.$$

Thus we can take Q to be constant (to guarantee a first order controller) and $|Q| < 0.5$ (to guarantee stabilization of G).

The nominal loop gain is $G_r(s)K(s) = G_r(s)Q(s)\,(I - G_r(s)Q(s))^{-1}$ and so the DC value is $4Q(1-4Q)^{-1}$. This is maximised when $Q = 0.25$ to give an infinite DC gain. This value of Q is internally stabilising. A computation gives

$$K(s) = \frac{s+1}{4s}.$$

Note that $K(s)$ has increased the system type since it has a free integrator.

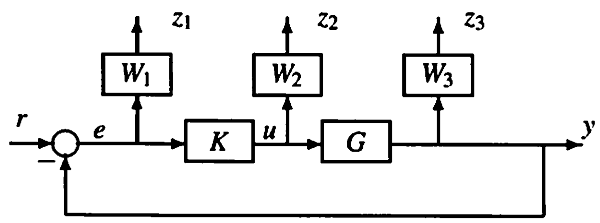
3. a) Signal interpretation: $r(s)$, $e(s)$, $u(s)$, $y(s)$ and $n(s)$ are the reference, error, control input, output and sensor noise signals, respectively. Specifications: ensure good tracking, limit control effort and attenuate noise, respectively.

b) It is clear that we require K to be internally stabilizing.

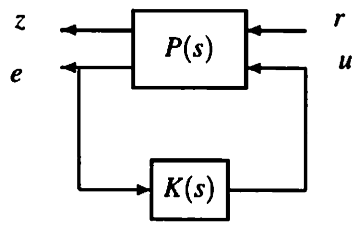
- A calculation shows that, when $n(s) = 0$, $e(s) = -S(s)r(s)$ where $S(s) = [I+G(s)K(s)]^{-1}$ is the sensitivity. Thus $\|e(j\omega)\| \le \|S(j\omega)\|\|r(j\omega)\|$. It follows that a sufficient condition to achieve the first design specification is $\|S(j\omega)\| < |w_1^{-1}(j\omega)|, \forall\omega$ or equivalently $\|W_1 S\|_\infty < 1$, where $W_1 = w_1 I$. A similar calculation shows that, when $n(s) = 0$, $u(s) = -K(s)S(s)r(s)$. Thus $\|u(j\omega)\| \le \|K(j\omega)S(j\omega)\|\|r(j\omega)\|$. It follows that a sufficient condition to achieve the second design specification is $\|K(j\omega)S(j\omega)\| < |w_2^{-1}(j\omega)|$, $\forall\omega$ or equivalently $\|W_2 KS\|_\infty < 1$, where $W_2 = w_2 I$.
- When $r(s) = 0$, a similar calculation shows that $y(s) = -C(s)n(s)$ where $C(s) = G(s)K(s)[I + G(s)K(s)]^{-1}$ is the complementary sensitivity. Thus $\|y(j\omega)\| \le \|C(j\omega)\|\|n(j\omega)\|$. It follows that a sufficient condition to achieve the second design specification is $\|C(j\omega)\| < |w_3^{-1}(j\omega)|, \forall\omega$ or equivalently $\|W_3 C\|_\infty < 1$, where $W_3 = w_3 I$.

To satisfy all design requirements, it is sufficient that $\left\| \begin{bmatrix} W_1 S \\ W_2 KS \\ W_3 C \end{bmatrix} \right\|_\infty < 1.$

c) The design specifications reduce to the requirement that the transfer matrix from r to $z = [z_1^T \; z_2^T \; z_3^T]^T$ in the following diagram has $\mathcal{H}_\infty$-norm less than 1.



The corresponding generalized regulator formulation is to find an internally stabilizing K such that $\|\mathcal{F}_l(P,K)\|_\infty < 1$:



$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \left[\begin{array}{c|c} W_1 & -W_1 G \\ 0 & W_2 \\ 0 & W_3 G \\ \hline I & -G \end{array}\right].$$

d) Let $G(s)$ be replaced by $G(s) + \Delta(s)$ in Figure 3 and let ε be the input and δ be the output of Δ. Then $\varepsilon = -KS\delta$. Using the small gain theorem the maximum stability radius is $|w_2(j\omega)|$.

4. a) Let $V = x'Px$ and set $u = -Fx$. Provided that $P = P' \succ 0$ and $\dot{V} < 0$ along closed-loop trajectories, we can assume $\lim_{t\to\infty} x(t) = 0$. Then

$$\dot{V} = \dot{x}'Px + x'P\dot{x} = x'\left(A'P + PA - F'B'P - PBF\right)x.$$

Integrating from 0 to ∞ and using $x(\infty) = 0$,

$$\int_0^\infty x'\left(A'P + PA - F'B'P - PBF\right)x dt = -x_0'Px_0.$$

Using the definition of J, adding the last equation and completing a square:

$$J = x_0'Px_0 + \int_0^\infty \{x'[A'P + PA + I - PBB'P]x + \|(F - B'P)x\|^2\}dt.$$

Since the last term is always nonnegative, it follows that the minimizing value of F is given by $F = B'P$. We can set the term in square brackets to zero provided P satisfies the Riccati equation,

$$A'P + PA + I - PBB'P = 0.$$

It follows that the minimum value of J is $x_0'Px_0$.

b) We need to prove that $A_c := A - BB'P$ is stable. The Riccati equation can be written as $A_c'P + PA_c + I + PBB'P = 0$. Let $\lambda \in \mathscr{C}$ be an eigenvalue of A_c and $z \neq 0$ be the corresponding eigenvector. Pre– and post–multiplying the Riccati equation be z' and z respectively gives $(\lambda + \bar{\lambda})z'Pz + z'z + z'PBB'Pz = 0$. Since $P \succ 0$ and $z \neq 0$, $z'Pz > 0$, $z'z > 0$ and $z'PBB'Pz \geq 0$. It follows that $\lambda + \bar{\lambda} < 0$ and the closed loop is stable.

c) The loop-gain is given by $L(s) = F(sI - A)^{-1}B$.. Let λ be a zero of $L(s)$ so that,

$$\begin{bmatrix} A - \lambda I & B \\ F & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

for some v_1, v_2 with $v_1 \neq 0$ since B has full column rank. Then,

$$Av_1 = \lambda v_1 - Bv_2,$$

and,

$$Fv_1 = 0 \Rightarrow B'Pv_1 = 0.$$

Consider the Riccati equation. Then,

$$\begin{aligned} 0 &= v_1'[A'P + PA - PBB'P + I]v_1 \\ &= (\bar{\lambda}v_1' - v_2'B')Pv_1 + v_1'P(\lambda v_1 - Bv_2) + v_1'v_1 \\ &= (\bar{\lambda} + \lambda)\underbrace{v_1'Pv_1}_{>0} + \underbrace{v_1'v_1}_{>0} \end{aligned}$$

Hence $\lambda + \bar{\lambda} < 0$ and $L(s)$ is minimum-phase.

5. a) i) The $(1,1)$ block of the inequality gives the inequality $A'P+PA \prec 0$. Let $z \neq 0$ be a right eigenvector of A and let λ be the corresponding eigenvalue. Then multiplying the inequality from the left by z' and from the right by z gives $(\lambda+\bar{\lambda})z'Pz < 0$. Since $P \succ 0$ it follows that $z'Pz > 0$ and it follows that $\lambda+\bar{\lambda} < 0$ so that A is stable.

ii) Since A is stable, $\|H\|_\infty < \gamma$ if and only if, with $x(0)=0$, $J := \int_0^\infty [y'y - \gamma^2 u'u]dt < 0$, for all $u(t)$ such that $\|u\|_2 < \infty$. If $\|u\|_2$ is bounded, then $\lim_{t\to\infty} x(t) = 0$. Now, $\int_0^\infty \frac{d}{dt}[x'Px]dt = x(\infty)'Px(\infty) - x(0)'Px(0) = 0$. So,

$$0 = \int_0^\infty (\dot{x}'Px + x'P\dot{x})\,dt = \int_0^\infty [x'(A'P+PA)x + x'PBu + u'B'Px]dt.$$

Use $y = Cx+Du$ and add the last expression to J

$$\begin{aligned} J &= \int_0^\infty [x'(A'P+PA+C'C)x + 2x'(PB+C'D)u + u'(D'D-\gamma^2 I)u]dt \\ &= \int_0^\infty \begin{bmatrix} x' & u' \end{bmatrix} \overbrace{\begin{bmatrix} A'P+PA+C'C & PB+C'D \\ B'P+D'C & D'D-\gamma^2 I \end{bmatrix}}^{M} \begin{bmatrix} x \\ u \end{bmatrix} dt. \end{aligned}$$

It follows that $J < 0$, and so $\|H\|_\infty < \gamma$, if $M \prec 0$. This proves the result.

b) i) Substituting $u = Lw_2 + Cx$, $y = Cx$ into the state equation gives

$$\dot{x} = \underbrace{(A+LC)}_{A_c}x + \underbrace{\begin{bmatrix} B & L \end{bmatrix}}_{B_c}w, \qquad y = \underbrace{C}_{C_c}x + \underbrace{0}_{D_c}w.$$

It follows that $T_{yw}(s) = D_c + C_c(sI - A_c)^{-1}B_c$.

ii) Using the results of part (a), by replacing A, B, C and D by A_c, B_c, C_c and D_c, we have that there exists a feasible L if there exists $P = P^T \succ 0$ such that

$$\begin{bmatrix} (A+LC)'P + P(A+LC) + C'C & PB & PL \\ B'P & -\gamma^2 I & 0 \\ L'P & 0 & -\gamma^2 I \end{bmatrix} \prec 0.$$

Noting that the only nonlinearity is due to the product PL, we define $Z = PL$ and so there exists a feasible L if there exists $P = P^T \succ 0$ and Z such that

$$\begin{bmatrix} A'P + PA + ZC + C'Z' + C'C & PB & Z \\ B'P & -\gamma^2 I & 0 \\ Z' & 0 & -\gamma^2 I \end{bmatrix} \prec 0.$$

6. a) The generalized regulator formulation is given by

$$\begin{bmatrix} z(s) \\ y(s) \end{bmatrix} = P(s)\begin{bmatrix} w(s) \\ u(s) \end{bmatrix}, u(s)=Fy(s), P(s)=\begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{c||c|c} A & B & B \\ \hline 0 & I & I \\ \hline I & 0 & 0 \end{array}\right].$$

b) The requirement $\|H\|_\infty < \gamma$ is equivalent to $J := \|z\|_2^2 - \gamma^2\|w\|_2^2 < 0$. Let $V = x^T X x$ and set $u = Fx$. Provided that $X = X^T \succ 0$ and $\dot{V} < 0$ along the closed-loop trajectory, we can assume $\lim_{t\to\infty} x(t) = 0$. Then

$$\dot{V} = \dot{x}^T X x + x^T X \dot{x} = x^T\left(A^T X + XA + F^T B^T X + XBF\right)x + x^T XBw + w^T B^T X x.$$

Integrating from 0 to ∞ and using $x(0) = x(\infty) = 0$,

$$0 = \int_0^\infty [x^T\left(A^T X + XA + F^T B^T X + XBF\right)x + x^T XBw + w^T B^T X x]dt.$$

Using the definition of J and adding the last equation, $J =$

$$J = \int_0^\infty \{x^T[A^T X + XA + F^T F + F^T B^T X + XBF]x - [\beta^2 w^T w - x^T Z^T w - w^T Z x]\}dt$$

where $Z = F + B^T X$ and $\beta^2 = \gamma^2 - 1 > 0$ since $\gamma > 1$ by assumption. Now

$$\begin{aligned} Z^T Z &= F^T F + F^T B^T X + XBF + XBB^T X \\ \|(\beta w - \beta^{-1} Z x)\|^2 &= \beta^2 w^T w - w^T Z x - x^T Z^T w + \beta^{-2} x^T Z^T Z x, \end{aligned}$$

$$J = \int_0^\infty \{x^T[A^T X + XA - XBB^T X]x + (1+\beta^{-2})\|Zx\|^2 - \|\beta w - \beta^{-1} Z x\|^2\}dt.$$

Thus two sufficient conditions for $J < 0$ are the existence of X such that

$$A^T X + XA - XBB^T X = 0, \qquad X = X^T \succ 0.$$

Setting $Z = 0$, $F = -B^T X$. The worst case disturbance is $w^* = \beta^{-2} Z x = 0$. The closed-loop with $u = Fx$ and $w = w^*$ is $\dot{x} = [A - BB^T X]x$ and a third condition is $Re\ \lambda_i[A - BB^T X] < 0$, $\forall\ i$. To prove $\dot{V} < 0$ for $u = Fx$ and $w = 0$,

$$\dot{V} = x^T\left(A^T X + XA + F^T B^T X + XBF\right)x = -x^T(XBB^T X)x < 0$$

for all $x \neq 0$ (since (A, B) is assumed controllable) proving closed-loop stability.

c) It is clear that our procedure breaks down if $\gamma \leq 1$ since in that case $\beta^2 \leq 0$. Thus the smallest value of γ is 1.

d) If A is stable, in the limit $X = 0$ and hence $F = 0$ is a solution.

e) If $-A$ is stable, pre- and post-multiplying the Riccati equation by X^{-1},

$$AX^{-1} + X^{-1}A^T - BB^T = 0 \Leftrightarrow (A - BB^T X)X^{-1} + X^{-1}(A - BB^T X) + BB^T = 0$$

which has a unique solution $X^{-1} \succ 0$ if $-A$ is stable and so $A - BB^T X$ is stable.

Examination Paper Submission document for 2012-2013 academic year.

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Code: EE4_25

Title: Design of linear multivariable control systems

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1. There is no full or partial reuse of questions.
2. This examination yields an appropriate range of marks that is well balanced, reflecting the quality of student (with weak students failing, capable students getting at least 40% and bright industrious students obtaining more than 70%)
3. The model answers give a fair indication of the amount of work needed to answer the questions. Each part has a comment indicating to the external examiners the nature of the question; i.e. whether it is bookwork, new theory, a new theoretical application, a calculation for a new example, etc.
4. The exam paper does not contain any grammar and spelling mistakes.
5. The marking schedule is shown in the answers document and the resolution of each allocated mark is better than 3/20 for each question.
6. The examination paper can be completed by the students within time allowed.

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