Imperial College
London

[E1.11 (Maths) ISE 2008]

B.ENG. AND M.ENG. EXAMINATIONS 2008

MATHEMATICS (INFORMATION SYSTEMS ENGINEERING E1.11)

Date Wednesday 4th June 2008 10.00 am - 1.00 pm

Answer ANY SEVEN questions.

Answers to questions from Section A and Section B should be written in different answer books.

A mathematical formulae sheet is provided.

[Before starting, please make sure that the paper is complete; there should be 7 pages, with a total of 9 questions. Ask the invigilator for a replacement if your copy is faulty.]

1. Show that

$$\int_0^{2\pi} \exp(in\theta)\,\mathrm{d}\theta = 0,$$

if n is any non-zero integer.

Hence, by expressing $\cos(\theta)$ in terms of complex exponentials, evaluate

$$\int_0^{2\pi} \cos^{2n}\theta\,\mathrm{d}\theta.$$

2. (i) State whether the following series converge or diverge, explaining your reasons:

(a)
$$\sum_{n=1}^{\infty} \frac{n+3}{(n+2)(n+1)} \ ,$$

(b)
$$\sum_{n=1}^{\infty} (-1)^n \frac{n+1}{(n+2)(n+3)} \ .$$

(ii) Explain what is meant by the *radius of convergence* of a power series. Calculate the radii of convergence of the following two power series:

(a)
$$\sum_{n=0}^{\infty} \frac{2n+1}{\sqrt{n^2+1}} z^n \ ,$$

(b)
$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} z^n.$$

PLEASE TURN OVER

3. (i) Evaluate the limits

(a) $$\lim_{x \to 0} \frac{\exp(2x)}{\cosh x},$$

(b) $$\lim_{x \to \pi/4} \frac{2\sin^2 x - 1}{\tan x - 1},$$

(c) $$\lim_{n \to \infty} \left[n \left\{ (n^2+3)^{1/2} - (n^3+n)^{1/3} \right\} \right].$$

(ii) Write down the Maclaurin series for the two functions $\exp(x^2)$ and $\cos x$.

Hence calculate the first three non-zero terms of the Maclaurin series for the product

$$\exp(x^2)\, \cos x.$$

4. Evaluate the definite integrals

(i)
$$\int_0^{\frac{\pi}{2}} x \cos x \, dx,$$

(ii)
$$\int_0^1 x^2 \ln x \, dx,$$

(iii)
$$\int_1^\infty \frac{1}{x(x+1)(x+2)} \, dx.$$

PLEASE TURN OVER

5. Solve the ordinary differential equations

(i)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x+2y}{2x+y};$$

(ii)

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 3x^2 y = \exp(-x^3), \quad \text{with} \quad y(0) = 0;$$

(iii)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = \exp(-x), \text{with} \quad y(0) = 1, \text{and} \quad y'(0) = 1.$$

In each case, find the most general solution possible.

PLEASE TURN OVER

SECTION B

6. (i) Find $\dfrac{dy}{dx}$ if $xy^2 - 2x\sin y = 5$.

(ii) If $z = x^2 + y^2$ where $x = r^2\cos\theta$ and $y = r\sin 2\theta$,

find $\dfrac{\partial z}{\partial r}$ and $\dfrac{\partial z}{\partial \theta}$, in terms of r and θ.

(iii) Find the stationary points of the function

$$f(x, y) = \frac{1}{3}x^3 + y^2 - 2xy$$

and determine their nature.

7. Find the Fourier cosine series for the function

$$f(x) = x, \qquad 0 \le x \le \pi .$$

Use this to evaluate the sum

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$$

8. (i) Use the method of Laplace transforms to solve the differential equation

$$\frac{dy}{dx} + 2y = 5\cos x$$

with $y(0) = 1$.

No credit will be given if you use a different method.

(ii) Use Laplace transforms to find functions $y = y(x)$, $z = z(x)$ satisfying the simultaneous differential equations

$$\begin{aligned} \frac{dy}{dx} + \frac{dz}{dx} + y &= 0 , \\ \frac{dy}{dx} + 2\frac{dz}{dx} - y &= e^{-x} , \end{aligned}$$

where $y(0) = 0$, $z(0) = 0$.

The Laplace transform of $f(x)$ *is defined as*

$$\mathcal{L}(f(x)) = F(t) = \int_0^\infty e^{-tx} f(x)\, dx .$$

You may assume that

$$\mathcal{L}(f'(x)) = -f(0) + t\mathcal{L}(f(x)) .$$

9. (i) Consider the three planes

$$\begin{aligned} \boldsymbol{r} \cdot (1,\, 1,\, 1) &= 1\,, \\ \boldsymbol{r} \cdot (2,\, 1,\, a) &= -1\,, \\ \boldsymbol{r} \cdot (1,\, -1,\, 1) &= b\,, \end{aligned}$$

where $\boldsymbol{r} = (x,\, y,\, z)$. For which values of a and b do the three planes

(a) meet in exactly one point ,

(b) meet in a line ,

(c) not meet at all?

(ii) Let

$$A = \begin{pmatrix} 1 & -7 \\ 0 & 8 \end{pmatrix}.$$

Find a 2 x 2 matrix P such that $P^{-1} A P$ is a diagonal matrix.

Find a 2 x 2 matrix B such that $B^3 = A$.

END OF PAPER

MATHEMATICS DEPARTMENT

MATHEMATICAL FORMULAE

1. VECTOR ALGEBRA

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1, a_2, a_3)$$

Scalar (dot) product: $\mathbf{a}\,.\,\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

Vector (cross) product:

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Scalar triple product:

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a}\,.\,\mathbf{b}\times\mathbf{c} = \mathbf{b}\,.\,\mathbf{c}\times\mathbf{a} = \mathbf{c}\,.\,\mathbf{a}\times\mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product: $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{c}\,.\,\mathbf{a})\mathbf{b} - (\mathbf{b}\,.\,\mathbf{a})\mathbf{c}$

2. SERIES

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \ldots \quad (\alpha \text{ arbitrary, } |x| < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots ,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n\frac{x^{2n}}{(2n)!} + \ldots ,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + \ldots ,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^n\frac{x^{n+1}}{(n+1)} + \ldots (-1 < x \le 1)$$

3. TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin(a+b) = \sin a\cos b + \cos a\sin b\,;$$

$$\cos(a+b) = \cos a\cos b - \sin a\sin b\,.$$

$$\cos iz = \cosh z\,;\quad \cosh iz = \cos z\,;\quad \sin iz = i\sinh z\,;\quad \sinh iz = i\sin z\,.$$

4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f\,D^n g + \binom{n}{1} Df\,D^{n-1}g + \ldots + \binom{n}{r} D^r f\,D^{n-r} g + \ldots + D^n f\,g\,.$$

(b) Taylor's expansion of $f(x)$ about $x = a$:

$$f(a+h) = f(a) + hf'(a) + h^2 f''(a)/2! + \ldots + h^n f^{(n)}(a)/n! + \epsilon_n(h)\,,$$

where $\epsilon_n(h) = h^{n+1}f^{(n+1)}(a+\theta h)/(n+1)!,\quad 0 < \theta < 1\,.$

(c) Taylor's expansion of $f(x, y)$ about (a, b):

$$f(a+h, b+k) = f(a,b) + [hf_x + kf_y]_{a,b} + 1/2!\left[h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}\right]_{a,b} + \ldots$$

(d) Partial differentiation of $f(x, y)$:

i. If $y = y(x)$, then $f = F(x)$, and $\dfrac{dF}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dx}$.

ii. If $x = x(t)$, $y = y(t)$, then $f = F(t)$, and $\dfrac{dF}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$.

iii. If $x = x(u, v)$, $y = y(u, v)$, then $f = F(u, v)$, and

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}, \quad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$

(e) Stationary points of $f(x, y)$ occur where $f_x = 0$, $f_y = 0$ simultaneously.
Let (a, b) be a stationary point: examine $D = [f_{xx}f_{yy} - (f_{xy})^2]_{a,b}$.
If $D > 0$ and $f_{xx}(a, b) < 0$, then (a, b) is a maximum;
If $D > 0$ and $f_{xx}(a, b) > 0$, then (a, b) is a minimum;
If $D < 0$ then (a, b) is a saddle-point.

(f) Differential equations:

i. The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating factor $I(x) = \exp[\int P(x)(dx]$, so that $\dfrac{d}{dx}(Iy) = IQ$.

ii. $P(x, y)dx + Q(x, y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y$.

5. INTEGRAL CALCULUS

(a) An important substitution: $\tan(\theta/2) = t$:
$\sin\theta = 2t/(1+t^2)$, $\cos\theta = (1-t^2)/(1+t^2)$, $d\theta = 2\,dt/(1+t^2)$.

(b) Some indefinite integrals:

$$\int (a^2 - x^2)^{-1/2} dx = \sin^{-1}\left(\frac{x}{a}\right), \quad |x| < a .$$

$$\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1}\left(\frac{x}{a}\right) = \ln\left\{\frac{x}{a} + \left(1 + \frac{x^2}{a^2}\right)^{1/2}\right\} .$$

$$\int (x^2 - a^2)^{-1/2} dx = \cosh^{-1}\left(\frac{x}{a}\right) = \ln\left|\frac{x}{a} + \left(\frac{x^2}{a^2} - 1\right)^{1/2}\right| .$$

$$\int (a^2 + x^2)^{-1} dx = \left(\frac{1}{a}\right)\tan^{-1}\left(\frac{x}{a}\right) .$$

6. NUMERICAL METHODS

(a) Approximate solution of an algebraic equation:

If a root of $f(x) = 0$ occurs near $x = a$, take $x_0 = a$ and
$x_{n+1} = x_n - [f(x_n)/f'(x_n)]$, $n = 0, 1, 2 \ldots$

(Newton Raphson method).

(b) Formulae for numerical integration: Write $x_n = x_0 + nh$, $y_n = y(x_n)$.

i. Trapezium rule (1-strip): $\int_{x_0}^{x_1} y(x)dx \approx (h/2)\,[y_0 + y_1]$.

ii. Simpson's rule (2-strip): $\int_{x_0}^{x_2} y(x)dx \approx (h/3)\,[y_0 + 4y_1 + y_2]$.

(c) Richardson's extrapolation method: Let $I = \int_a^b f(x)dx$ and let I_1, I_2 be two estimates of I obtained by using Simpson's rule with intervals h and $h/2$. Then, provided h is small enough,

$$I_2 + (I_2 - I_1)/15 ,$$

is a better estimate of I .

7. LAPLACE TRANSFORMS

Function	Transform	Function	Transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)dt$	$af(t) + bg(t)$	$aF(s) + bG(s)$
df/dt	$sF(s) - f(0)$	d^2f/dt^2	$s^2F(s) - sf(0) - f'(0)$
$e^{at}f(t)$	$F(s-a)$	$tf(t)$	$-dF(s)/ds$
$(\partial/\partial\alpha)f(t,\alpha)$	$(\partial/\partial\alpha)F(s,\alpha)$	$\int_0^t f(t)dt$	$F(s)/s$
$\int_0^t f(u)g(t-u)du$	$F(s)G(s)$		
1	$1/s$	$t^n (n = 1, 2 \ldots)$	$n!/s^{n+1}$, $(s > 0)$
e^{at}	$1/(s-a)$, $(s > a)$	$\sin\omega t$	$\omega/(s^2+\omega^2)$, $(s > 0)$
$\cos\omega t$	$s/(s^2+\omega^2)$, $(s > 0)$	$H(t-T) = \begin{cases} 0, & t < T \\ 1, & t > T \end{cases}$	e^{-sT}/s, $(s, T > 0)$

8. FOURIER SERIES

If $f(x)$ is periodic of period $2L$, then $f(x + 2L) = f(x)$, and

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \text{ where}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad n = 0, 1, 2, \ldots, \text{ and}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1, 2, 3, \ldots$$

Parseval's theorem

$$\frac{1}{L}\int_{-L}^{L} [f(x)]^2\, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right) .$$

September 2000