Imperial College
London

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2015

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Applied Probability

Date: Tuesday, 26 May 2015. Time: 10.00am – 12.00noon. Time allowed: 2 hours.

This paper has FOUR questions.

Candidates should use TWO main answer books (A & B) for their solutions as follows: book A - solutions to questions 1, 2 & 3; book B - solution to question 4.

Supplementary books may only be used after the relevant main book(s) are full.

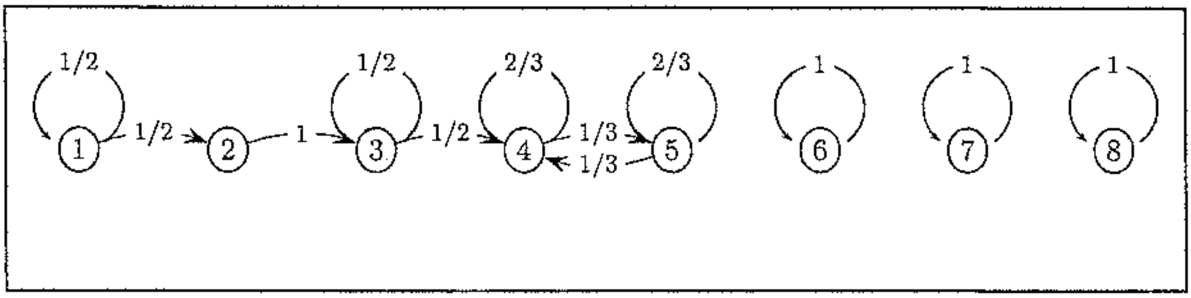
Statistical tables will not be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers to each question as per the table below.

Raw mark	up to 12	13	14	15	16	17	18	19	20
Extra credit	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4

- Each question carries equal weight.
- Calculators may not be used.

1. (a) Consider a homogeneous Markov chain $(X_n)_{n\in\mathbb{N}_0}$ with state space $E = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and transition diagram given by



(i) Find the transition matrix.

(ii) Specify the communicating classes and determine whether they are transient, null recurrent or positive recurrent.

(iii) Find all possible stationary distributions.

Please note that you need to justify your answers in (ii)-(iii).

(b) Suppose we have a Markov chain with finite state space E and transition matrix $\mathbf{P}$. Suppose for some $i \in E$ that

$$p_{ij}(n) \to \pi_j \text{ as } n \to \infty \text{ for all } j \in E.$$

Show that π (which is the row vector consisting of the elements π_j for $j \in E$) is a stationary distribution. *Please make sure that you justify all steps in your proof carefully.*

2. We define an irreducible, positive recurrent Markov chain $(X_n)_{n\in\{0,1,\ldots,N\}}$ for an $N \in \mathbb{N}$. We assume that π is the stationary distribution, and $\mathbf{P}$ is the transition matrix, and that for any $n \in \{0,1,\ldots,N\}$ the marginal distribution of X_n is also given by π.

(a) The reversed chain is defined as

$$Y_n = X_{N-n} \quad \text{for any } n \in \{0,1,\ldots,N\}.$$

Show that the sequence $Y = (Y_n)_{n\in\{0,1,\ldots,N\}}$ is a Markov chain which satisfies

$$\mathbb{P}(Y_{n+1} = j | Y_n = i) = \frac{\pi_j}{\pi_i} p_{ji}.$$

(b) Give the definition for $(X_n)_{n\in\{0,1,\ldots,N\}}$ to be *time-reversible.*

(c) Show that $(X_n)_{n\in\{0,1,\ldots,N\}}$ is time-reversible if and only if the detailed-balance equations hold.

(d) Suppose that π is given by the hypergeometric distribution HG(d), for $d \in \mathbb{N}$, i.e.

$$\pi_i = \frac{\binom{d}{i}^2}{\binom{2d}{d}}, \text{ for } i \in \{0,1,\ldots,d\}.$$

Further, suppose that

$$p_{i(i-1)} = \left(\frac{i}{d}\right)^2, \qquad p_{ii} = \frac{2i(d-i)}{d^2}, \qquad p_{i(i+1)} = \frac{(d-i)^2}{d^2},$$

and $p_{ij} = 0$ otherwise.

(i) Show that π given by the HG(d) distribution is indeed a stationary distribution for the Markov chain.

(ii) Show that the Markov chain is time-reversible.

Hint: You do NOT need to show that the HG(d) distribution is indeed a distribution.

3. Let $(N_t)_{t\geq 0}$ denote a Poisson process of rate $\lambda > 0$. Let $(X_i)_{i\in\mathbb{N}}$ denote its inter-arrival times and let $T_n = \sum_{i=1}^n X_i$ denote the time to the nth event for $n \in \mathbb{N}$ (also $T_0 = 0$).

(a) Show that $X_1 \sim \text{Exp}(\lambda)$.

(b) (i) Derive the Laplace transform of X_1, i.e. find $\mathbb{E}(e^{-uX_1})$ for $u > 0$.

(ii) Suppose that $Y \sim \text{Gamma}(n, \lambda)$, i.e. its probability density function is given by $f_Y(y) = \frac{\lambda^n}{\Gamma(n)} y^{n-1} e^{-\lambda y}$ for $y \geq 0$ and $\lambda > 0, n \in \mathbb{N}$. Derive the Laplace transform of Y.

(c) Show that $T_n \sim \text{Gamma}(n, \lambda)$.

(d) Define $Z_t = t - T_{N_t}$ for $t \geq 0$ and show that

$$\mathbb{P}(Z_t > x) = \begin{cases} e^{-\lambda x}, & 0 \leq x < t, \\ 0, & t \leq x. \end{cases}$$

4. (a) Define a birth-death process.

(b) Consider a simple birth, simple death process with immigration denoted by $(X_t)_{t\geq 0}$. I.e. the birth rates are given by $\lambda_n = \lambda n + \alpha$ and the death rates are given by $\mu_n = \mu n$ for constants $\lambda, \alpha, \mu > 0$ and for $n \in \mathbb{N}_0$. Let $\mathbf{P}_t = (p_{ij}(t))$ denote its stochastic semigroup.

(i) Derive the forward equations for $p_{i0}(t)$ and for $p_{ij}(t)$ for $i \in \mathbb{N}_0$ and $j \in \mathbb{N}$.

(ii) Suppose that the initial population at time 0 is of size $n_0 \in \mathbb{N}_0$. Show that

$$M(t) := \mathbb{E}(X_t) = \sum_{j=1}^{\infty} j p_{n_0 j}(t).$$

(iii) Show that $M(t)$ as defined in (ii) satisfies

$$M'(t) = \alpha + (\lambda - \mu)M(t), \quad t \geq 0.$$

1. (a) (i) The transition matrix is given by $\boxed{\text{meth seen} \Downarrow}$

$$\mathbf{P} = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2/3 & 1/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 2/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

$\boxed{2}$

(ii) We have a finite state space which can be divided into seven communicating classes: The classes $T_1 = \{1\}, T_2 = \{2\}, T_3 = \{3\}$ are not closed and hence transient. $\boxed{2}$

The classes $C_1 = \{4,5\}, C_2 = \{6\}, C_3 = \{7\}$ and $C_4 = \{8\}$ are finite and closed and hence positive recurrent. $\boxed{2}$

(iii) Note that we do not have a unique stationary distribution since we have four closed (essential) communicating classes.

Let π denote the vector of all stationary distributions. According to lectures, we know that $\pi_i = 0$ for all transient states i. I.e. $\pi_1 = \pi_2 = \pi_3$. $\boxed{2}$

We determine the remaining components by solving four systems of equations: We consider the transition matrices restricted to the essential communicating classes:

$$\mathbf{P}(C_1) := \begin{pmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{pmatrix}, \qquad \mathbf{P}(C_i) := 1, \text{ for } i = 2,3,4.$$

We need to solve $(\pi_4, \pi_5)\mathbf{P}(C_1) = (\pi_4, \pi_5)$ which results in $\pi_4 = \pi_5$. $\boxed{2}$

Also, we need to solve $\pi_{i+4}\mathbf{P}(C_i) = \pi_{i+4}$ for $i = 2,3,4$, which is trivial. $\boxed{2}$

Then all possible stationary distributions are given by $\boldsymbol{\pi} := (0,0,0,\pi_4,\pi_4,\pi_6,\pi_7,\pi_8)$ for constants $\pi_4, \pi_6, \pi_7, \pi_8 \geq 0$ such that $2\pi_4 + \pi_6 + \pi_7 + \pi_8 = 1$ since $\pi_i \geq 0$ for $i = 1, \ldots, 8$ and $\sum_{i=1}^{8} \pi_i = 1$, and also $\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P}$. $\boxed{2}$

(b) We show three properties. $\boxed{\text{seen} \Downarrow}$

1. Clearly for each $j \in E$ we have that $\pi_j \geq 0$ since it is a limit of a non-negative sequence. $\boxed{1}$
2. We get
$$\sum_{j\in E} \pi_j = \sum_{j \in E} \lim_{n\to\infty} p_{ij}(n) = \lim_{n\to\infty} \sum_{j\in E} p_{ij}(n) = 1,$$
since $\mathbf{P}_n$ is stochastic. $\boxed{2}$
3. Also, for all $j \in E$, we get
$$\begin{aligned} \pi_j &= \lim_{n\to\infty} p_{ij}(n) = \lim_{n\to\infty} \sum_{k\in E} p_{ik}(n-1)p_{kj} = \sum_{k\in E} \lim_{n\to\infty} p_{ik}(n-1)p_{kj} \\ &= \sum_{k\in E} \pi_k p_{kj}, \end{aligned}$$
where we used the Chapman-Kolmogorov equations. $\boxed{2}$

Note that we have used the finiteness of E to justify the interchange of summation and limit operations in (2.) and (3.). $\boxed{1}$

2. (a) For any $n \in \mathbb{N}$ and for any states $i_0, \dots, i_{n+1} \in E$ we have $\boxed{\text{seen} \Downarrow}$

$$\mathbb{P}(Y_{n+1} = i_{n+1} | Y_n = i_n, Y_{n-1} = i_{n-1}, \dots, Y_0 = i_0) = \frac{\mathbb{P}(Y_k = i_k, 0 \le k \le n+1)}{\mathbb{P}(Y_k = i_k, 0 \le k \le n)}$$
$$= \frac{\mathbb{P}(X_{N-k} = i_k, 0 \le k \le n+1)}{\mathbb{P}(X_{N-k} = i_k, 0 \le k \le n)}.$$

Now we apply Bayes theorem and the Markov property of (X_n) to deduce that

$$\begin{aligned}
&\mathbb{P}(X_{N-k} = i_k, 0 \le k \le n+1) \\
&= \mathbb{P}(X_N = i_0 | X_{N-k} = i_k, 1 \le k \le n+1)\mathbb{P}(X_{N-k} = i_k, 1 \le k \le n+1) \\
&= \mathbb{P}(X_N = i_0 | X_{N-1} = i_1)\mathbb{P}(X_{N-k} = i_k, 1 \le k \le n+1) \\
&= \mathbb{P}(X_N = i_0 | X_{N-1} = i_1)\mathbb{P}(X_{N-1} = i_1 | X_{N-2} = i_2) \cdots \mathbb{P}(X_{N-n} = i_n | X_{N-n-1} = i_{n+1}) \\
&\quad \mathbb{P}(X_{N-n-1} = i_{n+1}) \\
&= \pi_{i_{n+1}} p_{i_{n+1} i_n} \cdots p_{i_1 i_0}
\end{aligned}$$

Hence

$$\begin{aligned}
\mathbb{P}(Y_{n+1} = i_{n+1} | Y_n = i_n, Y_{n-1} = i_{n-1}, Y_0 = i_0) &= \frac{\pi_{i_{n+1}} p_{i_{n+1} i_n} \cdots p_{i_1 i_0}}{\pi_{i_n} p_{i_n i_{n-1}} \cdots p_{i_1 i_0}} \\
&= \frac{\pi_{i_{n+1}} p_{i_{n+1} i_n}}{\pi_{i_n}}.
\end{aligned}$$

Similarly, we get that $\boxed{4}$

$$\mathbb{P}(Y_{n+1} = i_{n+1} | Y_n = i_n) = \frac{\mathbb{P}(Y_{n+1} = i_{n+1}, Y_n = i_n)}{\mathbb{P}(Y_n = i_n)} = \frac{\mathbb{P}(X_{N-n-1} = i_{n+1}, X_{N-n} = i_n)}{\mathbb{P}(X_{N-n} = i_n)}$$
$$= \frac{\mathbb{P}(X_{N-n} = i_n | X_{N-n-1} = i_{n+1})\mathbb{P}(X_{N-n-1} = i_{n+1})}{\mathbb{P}(X_{N-n} = i_n)} = \frac{\pi_{i_{n+1}} p_{i_{n+1} i_n}}{\pi_{i_n}}.$$

So overall we have shown that for any $n \in \mathbb{N}$ and for any states $i_0, \dots, i_{n+1} \in E$ we have that

$$\mathbb{P}(Y_{n+1} = i_{n+1} | Y_n = i_n, Y_{n-1} = i_{n-1}, \dots, Y_0 = i_0) = \mathbb{P}(Y_{n+1} = i_{n+1} | Y_n = i_n)$$
$$= \frac{\pi_{i_{n+1}} p_{i_{n+1} i_n}}{\pi_{i_n}},$$

which completes the proof. $\boxed{2}$

(b) The Markov chain X is called **time–reversible** if the transition matrices of X and its time–reversal Y are the same. $\boxed{\text{seen} \Downarrow}$

$\boxed{2}$

(c) Let Q be the transition matrix of $\{Y_n\}_{n \in \{0,1,\dots,N\}}$. Then from (a) we have $\boxed{\text{seen} \Downarrow}$

$$q_{ij} = p_{ji} \frac{\pi_j}{\pi_i},$$

thus for any $i, j \in E$ we have that

$$q_{ij} = p_{ij} \Leftrightarrow p_{ij} = p_{ji} \frac{\pi_j}{\pi_i} \Leftrightarrow p_{ij}\pi_i = p_{ji}\pi_j,$$

which are the detailed-balance equations.

$\boxed{2}$

1. (a) (i) The transition matrix is given by meth seen ⇓

$$\mathbf{P} = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2/3 & 1/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 2/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

2

(ii) We have a finite state space which can be divided into seven communicating classes: The classes $T_1 = \{1\}, T_2 = \{2\}, T_3 = \{3\}$ are not closed and hence transient. 2

The classes $C_1 = \{4,5\}, C_2 = \{6\}, C_3 = \{7\}$ and $C_4 = \{8\}$ are finite and closed and hence positive recurrent. 2

(iii) Note that we do not have a unique stationary distribution since we have four closed (essential) communicating classes.

Let $\boldsymbol{\pi}$ denote the vector of all stationary distributions. According to lectures, we know that $\pi_i = 0$ for all transient states i. I.e. $\pi_1 = \pi_2 = \pi_3$. 2

We determine the remaining components by solving four systems of equations: We consider the transition matrices restricted to the essential communicating classes:

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We need to solve $(\pi_4, \pi_5)\mathbf{P}(C_1) = (\pi_4, \pi_5)$ which results in $\pi_4 = \pi_5$. 2

Also, we need to solve $\pi_{i+4}\mathbf{P}(C_i) = \pi_{i+4}$ for $i = 2,3,4$, which is trivial. 2

Then all possible stationary distributions are given by $\boldsymbol{\pi} := (0,0,0,\pi_4,\pi_4,\pi_6,\pi_7,\pi_8)$ for constants $\pi_4, \pi_6, \pi_7, \pi_8 \geq 0$ such that $2\pi_4 + \pi_6 + \pi_7 + \pi_8 = 1$ since $\pi_i \geq 0$ for $i = 1, \ldots, 8$ and $\sum_{i=1}^{8} \pi_i = 1$, and also $\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P}$. 2

(b) We show three properties. seen ⇓

1. Clearly for each $j \in E$ we have that $\pi_j \geq 0$ since it is a limit of a non-negative sequence. 1
2. We get
$$\sum_{j\in E} \pi_j = \sum_{j \in E} \lim_{n\to\infty} p_{ij}(n) = \lim_{n\to\infty} \sum_{j\in E} p_{ij}(n) = 1,$$
since $\mathbf{P}_n$ is stochastic. 2
3. Also, for all $j \in E$, we get
$$\begin{aligned} \pi_j &= \lim_{n\to\infty} p_{ij}(n) = \lim_{n\to\infty} \sum_{k\in E} p_{ik}(n-1)p_{kj} = \sum_{k\in E} \lim_{n\to\infty} p_{ik}(n-1)p_{kj} \\ &= \sum_{k\in E} \pi_k p_{kj}, \end{aligned}$$
where we used the Chapman-Kolmogorov equations. 2

Note that we have used the finiteness of E to justify the interchange of summation and limit operations in (2.) and (3.). 1

2. (a) For any $n \in \mathbb{N}$ and for any states $i_0, \dots, i_{n+1} \in E$ we have $\boxed{\text{seen} \Downarrow}$

$$\mathbb{P}(Y_{n+1} = i_{n+1} | Y_n = i_n, Y_{n-1} = i_{n-1}, \dots, Y_0 = i_0) = \frac{\mathbb{P}(Y_k = i_k, 0 \le k \le n+1)}{\mathbb{P}(Y_k = i_k, 0 \le k \le n)}$$
$$= \frac{\mathbb{P}(X_{N-k} = i_k, 0 \le k \le n+1)}{\mathbb{P}(X_{N-k} = i_k, 0 \le k \le n)}.$$

Now we apply Bayes theorem and the Markov property of (X_n) to deduce that

$$\begin{aligned}
&\mathbb{P}(X_{N-k} = i_k, 0 \le k \le n+1) \\
&= \mathbb{P}(X_N = i_0 | X_{N-k} = i_k, 1 \le k \le n+1)\mathbb{P}(X_{N-k} = i_k, 1 \le k \le n+1) \\
&= \mathbb{P}(X_N = i_0 | X_{N-1} = i_1)\mathbb{P}(X_{N-k} = i_k, 1 \le k \le n+1) \\
&= \mathbb{P}(X_N = i_0 | X_{N-1} = i_1)\mathbb{P}(X_{N-1} = i_1 | X_{N-2} = i_2) \cdots \mathbb{P}(X_{N-n} = i_n | X_{N-n-1} = i_{n+1}) \\
&\quad \mathbb{P}(X_{N-n-1} = i_{n+1}) \\
&= \pi_{i_{n+1}} p_{i_{n+1} i_n} \cdots p_{i_1 i_0}
\end{aligned}$$

Hence

$$\begin{aligned}
\mathbb{P}(Y_{n+1} = i_{n+1} | Y_n = i_n, Y_{n-1} = i_{n-1}, Y_0 = i_0) &= \frac{\pi_{i_{n+1}} p_{i_{n+1} i_n} \cdots p_{i_1 i_0}}{\pi_{i_n} p_{i_n i_{n-1}} \cdots p_{i_1 i_0}} \\
&= \frac{\pi_{i_{n+1}} p_{i_{n+1} i_n}}{\pi_{i_n}}.
\end{aligned}$$

Similarly, we get that $\boxed{4}$

$$\mathbb{P}(Y_{n+1} = i_{n+1} | Y_n = i_n) = \frac{\mathbb{P}(Y_{n+1} = i_{n+1}, Y_n = i_n)}{\mathbb{P}(Y_n = i_n)} = \frac{\mathbb{P}(X_{N-n-1} = i_{n+1}, X_{N-n} = i_n)}{\mathbb{P}(X_{N-n} = i_n)}$$
$$= \frac{\mathbb{P}(X_{N-n} = i_n | X_{N-n-1} = i_{n+1})\mathbb{P}(X_{N-n-1} = i_{n+1})}{\mathbb{P}(X_{N-n} = i_n)} = \frac{\pi_{i_{n+1}} p_{i_{n+1} i_n}}{\pi_{i_n}}.$$

So overall we have shown that for any $n \in \mathbb{N}$ and for any states $i_0, \dots, i_{n+1} \in E$ we have that

$$\mathbb{P}(Y_{n+1} = i_{n+1} | Y_n = i_n, Y_{n-1} = i_{n-1}, \dots, Y_0 = i_0) = \mathbb{P}(Y_{n+1} = i_{n+1} | Y_n = i_n)$$
$$= \frac{\pi_{i_{n+1}} p_{i_{n+1} i_n}}{\pi_{i_n}},$$

which completes the proof. $\boxed{2}$

(b) The Markov chain X is called **time–reversible** if the transition matrices of X and its time–reversal Y are the same. $\boxed{\text{seen} \Downarrow}$

$\boxed{2}$

(c) Let Q be the transition matrix of $\{Y_n\}_{n \in \{0,1,\dots,N\}}$. Then from (a) we have $\boxed{\text{seen} \Downarrow}$

$$q_{ij} = p_{ji} \frac{\pi_j}{\pi_i},$$

thus for any $i, j \in E$ we have that

$$q_{ij} = p_{ij} \Leftrightarrow p_{ij} = p_{ji} \frac{\pi_j}{\pi_i} \Leftrightarrow p_{ij} \pi_i = p_{ji} \pi_j,$$

which are the detailed-balance equations.

$\boxed{2}$

(d) $\boxed{\text{unseen} \Downarrow}$

(i) We already know that π is a distribution, so we only need to show that $\pi_j = \sum_{i\in E} \pi_i p_{ij}$, for all $j \in E$. We have

$$\begin{aligned}\sum_{i\in E} \pi_i p_{ij} &= \pi_{j-1}p_{(j-1)j} + \pi_j p_{jj} + \pi_{j+1}p_{(j+1)j}\\ &= \binom{2d}{d}^{-1} d^{-2} \underbrace{\left[\binom{d}{j-1}^2 (d-(j-1))^2 + \binom{d}{j}^2 2j(d-j) + \binom{d}{j+1}^2 (j+1)^2\right]}_{=:A}.\end{aligned}$$

Note that

$$\begin{aligned}\binom{d}{j-1}(d-j+1) &= \frac{d!(d-j+1)}{(d-j+1)!(j-1)!}\frac{j}{j} = j\binom{d}{j},\\ \binom{d}{j+1}(j+1) &= \frac{d!(j+1)}{(d-j-1)!(j+1)!}\frac{(d-j)}{(d-j)} = (d-j)\binom{d}{j}.\end{aligned}$$

Hence

$$A = \binom{d}{j}^2 \left[j^2 + 2jd - 2j^2 + d^2 - 2dj + j^2\right] = \binom{d}{j}^2 d^2.$$

Overall, we get that for all $j \in E$ we have that

$$\sum_{i\in E} \pi_i p_{ij} = \binom{d}{j}^2 \binom{2d}{d}^{-1} = \pi_j.$$

$\boxed{5}$

(ii) We only need to show that the Markov chain satisfies the detailed-balance equations

$$p_{ij}\pi_i = p_{ji}\pi_j \text{ for all } i, j \in E.$$

We consider the four cases:

Case: $j = i - 1$ We find that $p_{i(i-1)}\pi_i = p_{(i-1)i}\pi_{i-1}$ since

$$\begin{aligned}\binom{2d}{d} p_{i(i-1)}\pi_i &= \left(\frac{i}{d}\right)^2 \binom{d}{i}^2 = \frac{i^2(d!)^2}{d^2((d-i)!)^2(i!)^2} = \frac{((d-1)!)^2}{((d-i)!)^2((i-1)!)^2},\\ \binom{2d}{d} p_{(i-1)i}\pi_{i-1} &= \frac{(d-i+1)^2}{d^2}\binom{d}{(i-1)}^2 = \frac{(d-i+1)^2(d!)^2}{d^2((d-i+1)!)^2((i-1)!)^2}\\ &= \frac{((d-1)!)^2}{((d-i)!)^2((i-1)!)^2}.\end{aligned}$$

Case: $j = i$ We find that $p_{ii}\pi_i = p_{ii}\pi_i$ is trivially true.

Case: $j = i + 1$ We find that $p_{i(i+1)}\pi_i = p_{(i+1)i}\pi_{i+1}$ since

$$\begin{aligned}\binom{2d}{d} p_{i(i+1)}\pi_i &= \left(\frac{(d-i)}{d}\right)^2 \binom{d}{i}^2 = \frac{(d-i)^2(d!)^2}{d^2((d-i)!)^2(i!)^2} = \frac{((d-1)!)^2}{((d-i-1)!)^2(i!)^2},\\ \binom{2d}{d} p_{(i+1)i}\pi_{i+1} &= \frac{(i+1)^2}{d^2}\binom{d}{(i+1)}^2 = \frac{(i+1)^2(d!)^2}{d^2((d-i-1)!)^2((i+1)!)^2} = \frac{((d-1)!)^2}{((d-i-1)!)^2(i!)^2}.\end{aligned}$$

All other cases: are trivially true. $\boxed{5}$

3. (a) For $t > 0$ we have

$$\mathbb{P}(X_1 > t) = \mathbb{P}(\text{no events in}[0,t]) = \mathbb{P}(N_t = 0) = e^{-\lambda t},$$

which is the survival function of the Exp(λ) distribution.

seen ⇓

2

(b) (i) Let $u > 0$. Then

$$\mathbb{E}(e^{-uX_1}) = \int_0^\infty e^{-ux}\lambda e^{-\lambda x}dx = \int_0^\infty \lambda e^{-(\lambda+u)x}dx = \frac{\lambda}{\lambda+u}.$$

sim. seen ⇓

2

(ii) Let $u > 0$. Then

$$\mathbb{E}(e^{-uY}) = \int_0^\infty e^{-uy}\frac{\lambda^n}{\Gamma(n)}y^{n-1}e^{-\lambda y}dy = \frac{\lambda^n}{\Gamma(n)}\int_0^\infty y^{n-1}e^{-(\lambda+u)y}dy$$
$$= \frac{\lambda^n}{\Gamma(n)}\int_0^\infty z^{n-1}e^{-z}dz\frac{1}{(\lambda+u)^n} = \left(\frac{\lambda}{\lambda+u}\right)^n.$$

2

(c) For $u > 0$, we have

seen ⇓

$$\mathbb{E}[e^{-uT_n}] = \mathbb{E}[e^{-u(\sum_{i=1}^n X_i)}] = \prod_{i=1}^n \mathbb{E}[e^{-uX_i}] = (\mathbb{E}[e^{-uX_1}])^n = \left(\frac{\lambda}{\lambda+u}\right)^n$$

where we used the independence of the X_i in the second step and the identical distributions of the X_i in the third step. We notice (from (b)) that this is indeed the Laplace transform of a Gamma(n,λ) random variable, which concludes the proof.

3

(d) **Case** $t \le x \Leftrightarrow t - x \le 0$: We have that

meth seen ⇓

$$\mathbb{P}(Z_t > x) = \mathbb{P}(t - T_{N_t} > x) = \mathbb{P}(t - x > T_{N_t}) = 0,$$

since $T_n \ge 0$ for all $n \in \mathbb{N}_0$.

1

Case $0 \le x < t$: Here we have that

$$\mathbb{P}(Z_t > x) = \mathbb{P}(t - x > T_{N_t}) = \sum_{n=0}^\infty \mathbb{P}(t - x > T_{N_t}, N_t = n),$$

where we used the law of total probability. The summands can be simplified as follows (when noting that $t - x > 0$):

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$$\begin{aligned}
&\mathbb{P}(t - x > T_{N_t}, N_t = n) = \mathbb{P}(t - x > T_n, N_t = n) \\
&= \mathbb{P}(t - x > T_n, T_n \le t < T_{n+1}) = \mathbb{P}(t - x > T_n, T_n \le t < T_n + X_{n+1}) \\
&= \mathbb{P}(t - x > T_n, t < T_n + X_{n+1}) = \int_0^\infty \mathbb{P}(t - x > T_n, t < T_n + X_{n+1} | T_n = y) f_{T_n}(y)dy \\
&= \int_0^\infty \mathbb{P}(t - x > y, t - y < X_{n+1}) f_{T_n}(y)dy = \int_0^{t-x} \frac{\lambda^n}{\Gamma(n)}y^{n-1}e^{-\lambda y}e^{-\lambda(t-y)}dy \\
&= \frac{\lambda^n}{\Gamma(n)}e^{-\lambda t}\int_0^{t-x} y^{n-1}dy = \frac{\lambda^n}{n!}e^{-\lambda t}(t-x)^n,
\end{aligned}$$

where we used the fact that X_{n+1} is independent of T_n and that $T_n \sim$ Gamma(n,λ) and $X_{n+1} \sim$ Exp(λ). So, overall we have

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$$\mathbb{P}(Z_t > x) = \sum_{n=0}^\infty \frac{\lambda^n}{n!}e^{-\lambda t}(t-x)^n = e^{-\lambda t}e^{\lambda(t-x)} = e^{-\lambda x}.$$

2

4. (a) The stochastic process $\{X_t\}_{t\geq 0}$ is called a birth-death process if it satisfies the following properties: $\boxed{\text{seen} \Downarrow}$

1. $\{X_t\}_{t\geq 0}$ is Markov chain on $E = \{0, 1, \dots\}$
2. The infinitesimal transition probabilities are (for $t \geq 0$, $\delta > 0$, $n, m \in \mathbb{N}_0$):

$$\mathbb{P}(X_{t+\delta} = n+m | X_t = n) = \begin{cases} 1 - (\lambda_n + \mu_n)\delta + o(\delta), & \textit{if } m = 0, \\ \lambda_n \delta + o(\delta) & \textit{if } m = 1 \\ \mu_n \delta + o(\delta) & \textit{if } m = -1 \\ o(\delta) & \textit{if} |m| > 1 \end{cases}$$

3. The birth rates $\lambda_0, \lambda_1, \dots$ and the death rates $\mu_0, \mu_1, \dots$ satisfy

$$\lambda_i \geq 0 \quad \mu_i \geq 0 \quad \mu_0 = 0.$$

$\boxed{4}$

(b) (i) Let $t \geq 0$ and $\delta > 0$. Then $\boxed{\text{sim. seen} \Downarrow}$

$$\begin{aligned} p_{i0}(t+\delta) &= \mathbb{P}(X_{t+\delta} = 0 | X_0 = i) = \sum_{k\in\mathbb{N}_0} \mathbb{P}(X_{t+\delta} = 0 | X_t = k, X_0 = i)\mathbb{P}(X_t = k | X_0 = i) \\ &= \sum_{k\in\mathbb{N}_0} \mathbb{P}(X_{t+\delta} = 0 | X_t = k)\mathbb{P}(X_t = k | X_0 = i) = \sum_{k\in\mathbb{N}_0} p_{k0}(\delta) p_{ik}(t) \\ &= p_{00}(\delta)p_{i0}(t) + p_{10}(\delta)p_{i1}(t) + o(\delta) = (1 - \lambda_0\delta)p_{i0}(t) + \mu_1\delta p_{i1}(t) + o(\delta), \end{aligned}$$

where we used the law of total probability, the Markov property and the infinitesimal transition probabilities of the birth-death process. Subtracting $p_{i0}(t)$ on both sides, dividing by δ and sending $\delta \to 0$ leads to

$$p'_{i0}(t) = -\lambda_0 p_{i0}(t) + \mu_1 p_{i1}(t) = -\alpha p_{i0}(t) + \mu p_{i1}(t), \quad t \geq 0.$$

Similarly, for $j \in \mathbb{N}$, we get $\boxed{3}$

$$\begin{aligned} p_{ij}(t+\delta) &= \mathbb{P}(X_{t+\delta} = j | X_0 = i) = \sum_{k\in\mathbb{N}_0} \mathbb{P}(X_{t+\delta} = j | X_t = k)\mathbb{P}(X_t = k | X_0 = i) \\ &= \sum_{k\in\mathbb{N}_0} p_{kj}(\delta) p_{ik}(t) \\ &= p_{(j-1)j}(\delta)p_{i(j-1)}(t) + p_{jj}(\delta)p_{ij}(t) + p_{(j+1)j}(\delta)p_{i(j+1)}(t) + o(\delta) \\ &= \lambda_{j-1}\delta p_{i(j-1)}(t) + (1 - (\lambda_j + \mu_j)\delta)p_{ij}(t) + \mu_{j+1}\delta p_{i(j+1)}(t) + o(\delta). \end{aligned}$$

Hence

$$\begin{aligned} p'_{ij}(t) &= \lambda_{j-1} p_{i(j-1)}(t) - (\lambda_j + \mu_j)p_{ij}(t) + \mu_{j+1} p_{i(j+1)}(t) \\ &= (\lambda(j-1) + \alpha)p_{i(j-1)}(t) - ((\lambda + \mu)j + \alpha)p_{ij}(t) + \mu(j+1)p_{i(j+1)}(t), \quad t \geq 0. \end{aligned}$$

$\boxed{4}$

(ii) We know that the marginal distribution of X_t is given by $\nu^{(t)} = \nu^{(0)}\mathbf{P}_t$, where the row vector $\nu^{(0)}$ satisfies $\nu_j^{(0)} = 1$ if $j = n_0$ and $\nu_j^{(0)} = 0$ otherwise. Hence $\boxed{\text{unseen} \Downarrow}$

$$M(t) = \mathbb{E}(X_t) = \sum_{j=0}^{\infty} j\mathbb{P}(X_t = j) = \sum_{j=1}^{\infty} j\nu_j^{(t)} = \sum_{j=1}^{\infty} j p_{n_0 j}(t).$$

$\boxed{3}$

(iii) Let $t \geq 0$ and $\delta > 0$. Using properties of conditional expectation, we get that $M(t+\delta) = \mathbb{E}(\mathbb{E}(X_{t+\delta}|X_t))$. Note that

unseen $\Downarrow$

$\boxed{1}$

$$\begin{aligned}
\mathbb{E}(X_{t+\delta}|X_t = i) &= \sum_{k\in\mathbb{N}_0} k\mathbb{P}(X_{t+\delta} = k|X_t = i) = \sum_{k\in\mathbb{N}_0} kp_{ik}(\delta) \\
&= (i-1)p_{i(i-1)}(\delta) + ip_{ii}(\delta) + (i+1)p_{i(i+1)}(\delta) + o(\delta) \\
&= (i-1)\mu_i\delta + i(1 - (\lambda_i + \mu_i)\delta) + (i+1)\lambda_i\delta + o(\delta) \\
&= (i-1)i\mu\delta + i(1 - ((\lambda+\mu)i + \alpha)\delta) + (i+1)(i\lambda + \alpha)\delta + o(\delta) \\
&= (i^2 - i)\mu\delta + i - ((\lambda+\mu)i^2 + \alpha i)\delta + (i^2 + i)\lambda\delta + \alpha(i+1)\delta + o(\delta).
\end{aligned}$$

Hence $\boxed{3}$

$$\begin{aligned}
M(t+\delta) &= (\mathbb{E}(X_t^2) - M(t))\mu\delta + M(t) - ((\lambda+\mu)\mathbb{E}(X_t^2) + \alpha M(t))\delta \\
&\quad + (\mathbb{E}(X_t^2) + M(t))\lambda\delta + \alpha(M(t)+1)\delta + o(\delta) \\
&= -M(t)\mu\delta + M(t) + M(t)\lambda\delta + \alpha\delta + o(\delta).
\end{aligned}$$

Subtracting $M(t)$ on both sides, dividing by δ and sending $\delta \to 0$ leads to

$$M'(t) = \alpha + M(t)(\lambda - \mu), \quad t \geq 0.$$

$\boxed{2}$