IMPERIAL COLLEGE LONDON

E3.10
CS3.1
ISE3.7

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2010

MSc and EEE/ISE PART III/IV: MEng, BEng and ACGI

Corrected Copy

MATHEMATICS FOR SIGNALS AND SYSTEMS

Wednesday, 5 May 10:00 am

Time allowed: 3:00 hours

There are FIVE questions on this paper.

Answer THREE questions.

Q2 correction announced at start of exam

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : M.M. Draief
Second Marker(s) : D. Angeli

MATHEMATICS FOR SIGNAL AND SYSTEMS

1. a) Let P be the matrix defined as

$$P = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

i) Describe a basis of $\mathcal{N}(P)$ the null-space (kernel) of P and a basis of Range(P) the range of P. Justify your answer. [3]

ii) Show that $\mathbb{R}^4 = \mathcal{N}(P) \oplus \mathrm{Range}(P)$. [1]

iii) Show that for $x \in \mathcal{N}(P)$ and $y \in \mathrm{Range}(P)$ then $x^T y = 0$. [2]

iv) Conclude that P is an orthogonal projection. [1]

b) We assume that $(e_1, \ldots, e_n)$ is an orthonormal basis of $\mathbb{R}^n$ and, for $k = 1, \ldots, n-1$, we define $F_k = \mathrm{Span}(e_1, \ldots, e_k)$.

i) Let $z \in \mathbb{R}^n$. Provide, without justification, the expression of Πz the orthogonal projection of z on F_k in terms of $(e_1, \ldots, e_k)$. [1]

ii) Prove that for all $z \in \mathbb{R}^n$, we have $||\Pi z|| \leq ||z||$, $||z|| = \sqrt{z^T z}$. [1]

iii) Express Π in terms $(e_1, \ldots, e_k)$, and show that $\Pi^2 = \Pi \times \Pi = \Pi$ and $(\Pi x)^T y = x^T(\Pi y)$. In other words, we have $\Pi^T = \Pi$. [2]

iv) Suppose that Q is a projection (not necessarily orthogonal), such that $||Qz|| \leq ||z||$. Show that Q is an orthogonal projection. [2]

Hint: You have to show that $x^T y = 0$ for all $x \in \mathrm{Range}(Q)$ and $y \in \mathcal{N}(Q)$. To this end, consider $z = \lambda x + y$ for all $\lambda \in \mathbb{R}$.

c) Assume that we have two orthogonal projectors P and Q on the subspaces F and G respectively.

We consider the matrix $R = PQ$, the product of the matrices P and Q, $\lambda \in \mathbb{R}, \lambda \neq 0$ an eigenvalue of R and $u \in \mathbb{R}^n$ an associated eigenvector, i.e. $Ru = \lambda u$.

i) Show that $u \in \mathrm{Range}(P)$ and that $Qu - \lambda u \in \mathcal{N}(P)$. [2]

ii) Using question 1.b) iii) and the previous question, prove that

$$||Qu||^2 = \lambda ||u||^2.$$

[3]

iii) Using question 1.b) ii), conclude that the eigenvalues of R are in $[0, 1]$. [2]

2. Let m and n be two positive integers with $m \leq n$. We consider $A \in \mathbb{R}^{(n+1)\times(m+1)}$ the matrix defined by

$$A = \begin{pmatrix} 1 & x_0 & \dots & x_0^m \\ 1 & x_1 & \dots & x_1^m \\ \vdots & \vdots & & \vdots \\ 1 & x_n & \dots & x_n^m \end{pmatrix},$$

n+1

where $x_0, \dots, x_n$ are ~~n~~ distinct real numbers.

a) Let $\mathbf{0}$ be the vector with all its entries equal to 0 (we will use the same notation for both the zero vector of $\mathbb{R}^{m+1}$ and the one of $\mathbb{R}^{n+1}$).

Let $v = \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_m \end{pmatrix} \in \mathbb{R}^{m+1}$.

i) Show that if $Av = \mathbf{0}$ then $v = \mathbf{0}$. [1]

Hint: Use the fact that if the polynomial $P(x) = v_0 + v_1 x + \dots + v_m x^m$ has more than $m+1$ distinct zeros then $P(x) = 0$.

ii) Using the previous question, show that if $A^T A v = \mathbf{0}$ then $v = \mathbf{0}$. [2]

iii) Fix $y \in \mathbb{R}^{n+1}$. Justify the fact that the linear equation $A^T A x = A^T y$ admits a unique solution. [2]

In the remainder of this problem, we will denote this solution by w, i.e.

$$A^T A w = A^T y.$$

b) For $v \in \mathbb{R}^{m+1}$ and $y \in \mathbb{R}^{n+1}$, define $g(v) = (y - Av)^T (y - Av)$.

i) Show that $g(w) = y^T y - y^T A w$, with w defined in 2. a) iii). [2]

ii) Prove that $g(v) - g(w) = (w - v)^T A^T A (w - v)$. [3]

iii) Show that for all $v \in \mathbb{R}^{m+1}$, we have $g(v) \geq g(w)$ and that $g(v) = g(w)$ if and only if $v = w$. [2]

c) Let P be a polynomial such that $P(x) = \sum_{k=0}^{m} v_k x^k$. We define the quantity

$$\Phi_m(P) = \sum_{i=0}^{n} (y_i - P(x_i))^2 .$$

Let $v = \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_m \end{pmatrix} \in \mathbb{R}^{m+1}$ and $y = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^{n+1}$.

i) Show that $\Phi_m(P) = g(v)$. [2]

ii) Using question 2.b), show that there exists a polynomial P_w such that $\Phi_m(P) \geq \Phi_m(P_w)$. [2]

d) We now apply the analysis of question 2) c) to a numerical example. Let $n = m = 3$, $x_0 = -1$, $x_1 = 0$, $x_2 = 1$, $x_3 = 2$ and $y_0 = 1$, $y_1 = 2$, $, y_2 = 1$, $y_3 = 0$.

i) Solve $A^T A v = A^T y$. [2]

ii) Derive the expression of the polynomial in $\mathbb{R}_3[X]$ that minimises Φ_3 and give the minimum value of Φ_3 on $\mathbb{R}_3[X]$. Justify your answer. [2]

3. Let $A \in \mathbb{R}^{n\times n}$ be a non-singular matrix, and consider three vectors $b, c, f \in \mathbb{R}^n$. Given two real numbers α and γ we want to solve the following linear system in $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$.

$$\begin{aligned} Ax + b\lambda &= f \\ c^T x + \alpha\lambda &= \gamma. \end{aligned} \tag{3.1}$$

a) i) Write the system (3.1) in matrix form, i.e. $My = g$ with $M \in \mathbb{R}^{(n+1)\times(n+1)}$ and $y, g \in \mathbb{R}^{n+1}$. [2]

ii) Give a necessary and sufficient condition for the system (3.1) to be solvable, i.e. to admit a unique solution. Justify your answer. [4]

In what follows we assume that $\alpha - c^T A^{-1} b \neq 0$.

b) To solve (3.1), we will use the following algorithm.

Let z_0 be the solution of $Az = b$ and h_0 be the solution of $Ah = f$.

$$x = h_0 - \frac{\gamma - c^T h_0}{\alpha - c^T z_0} z_0, \quad \lambda = \frac{\gamma - c^T h_0}{\alpha - c^T z_0}.$$

i) Show that the above algorithm gives the solution to (3.1). [2]

ii) Assuming that we use one of the standard methods to solve $Az = b$ and $Ah = f$, how many additional operations are required to complete the algorithm? [5]

c) We now solve (3.1) for

$$A = \begin{pmatrix} 25 & 15 & -5 \\ 15 & 18 & 0 \\ -5 & 0 & 11 \end{pmatrix}$$

$$b = \begin{pmatrix} 30 \\ 15 \\ -16 \end{pmatrix} \quad f = \begin{pmatrix} 35 \\ 33 \\ 6 \end{pmatrix}, \quad c = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

and $\gamma = 4$ and $\alpha = 1$

i) Using Cholesky decompostion, solve $Az = b$ and $Ah = f$. [5]

ii) Derive the solution to (3.1). [2]

4. Let $\mathbb{R}[X]$ be the vector space of polynomials with real coefficients, and $\mathbb{R}_n[X]$ be the subspace of polynomials with degree smaller or equal to n. Let w be a continuous function on $(-1,1)$ taking positive real values. For P and Q in $\mathbb{R}[X]$, we define

$$\langle P,Q\rangle = \int_{-1}^{1} P(x)Q(x)w(x)dx\,.$$

a) First we assume that $w(x) = \frac{1}{\sqrt{1-x^2}}$, for $x \in (-1,1)$ and define $T_k(x)$ the polynomials such that, for $k \geq 1$ and $\theta \in (0,\pi)$, we have

$$T_k(\cos(\theta)) = \cos(k\theta)\,, \quad T_0 = 1\,,$$

known as *Chebyshev's polynomials*.

i) Derive T_1, T_2 and T_3. [1]

ii) Show that, for $k \geq 1$, we have

$$T_{k+1} = 2XT_k - T_{k-1}\,.$$

[2]

iii) Using the change of variable $\theta = \arccos(x)$, compute $\langle T_n, T_m\rangle$, when $n = m$ and $n \neq m$. [2]

iv) Derive an orthonormal basis of $\mathbb{R}_3[X]$. Justify your answer. [1]

b) For the remainder of the problem, we let w be a (general) given continuous function on $(-1,1)$.

i) Show that the application $(P,Q) \to \langle P,Q\rangle$ is an inner (scalar) product on $\mathbb{R}[X]$. [1]

ii) Justify the existence of a family of orthogonal polynomials $(P_0, P_1, P_2, \dots)$, with respect to the above inner product on $\mathbb{R}[X]$, where the degree of P_k is equal to k. [3]

Hint: Use the Gram-Schmidt algorithm on $(1, X, X^2, X^3, \dots)$, the canonical basis of $\mathbb{R}[X]$.

iii) For $k = 1, 2, \dots$, prove that $\langle P_k, Q\rangle = 0$, for all $Q \in \mathbb{R}_{k-1}[X]$. [1]

iv) Show that, for $k \geq 2$ and $j \leq k-2$, we have $\langle XP_k, P_j\rangle = 0$. [2]

c) For $k = 1, 2, \dots$, we write $P_k = \sum_{j=0}^{k} \alpha_{k,j} X^j$.

i) Justify the fact that $XP_0 = a_1P_1 + b_0P_0$, for some reals a_1 and b_0. [1]

ii) Show that $a_1 = \frac{\alpha_{0,0}}{\alpha_{1,1}}$ and $b_0 = -\frac{\alpha_{1,0}}{\alpha_{1,1}}$. [2]

iii) Using similar arguments as in the previous two questions, show that, for $k \geq 1$, we have

$$XP_k = a_{k+1}P_{k+1} + b_kP_k + a_kP_{k-1}\,,$$

where

$$a_k = \frac{\alpha_{k-1,k-1}}{\alpha_{k,k}} \quad \text{and} \quad b_k = \frac{\alpha_{k,k-1}}{\alpha_{k,k}} - \frac{\alpha_{k+1,k}}{\alpha_{k+1,k+1}}\,.$$

[4]

5. In this problem, we analyse the impact of perturbations on the solutions of linear equations.

a) We will consider the standard Euclidean norm $||x|| = \sqrt{x^T x}$, for $x \in \mathbb{R}^n$ and the associated matrix norm

$$|||A||| = \sup_{x:||x||=1} ||Ax|| .$$

i) Show that the mapping $A \to |||A|||$ defines a norm on $\mathbb{R}^{n\times n}$. [3]

ii) Let $x \in \mathbb{R}^n$, and A and B in $\mathbb{R}^{n\times n}$ show that $||Ax|| \leq |||A|||\,||x||$ and that $|||AB||| \leq |||A|||\,|||B|||$. [3]

b) In this question, we assume that A is a non-singular matrix in $\mathbb{R}^{n\times n}$ and y a non-zero vector in $\mathbb{R}^n$. Let $x_0 \in \mathbb{R}^n$ be the solution of $Ax = y$.

i) Let $x_1 \in \mathbb{R}^n$ be the solution of $Ax = y + \delta y$, where $\delta y \in \mathbb{R}^n$. Prove that

$$\frac{||x_0 - x_1||}{||x_0||} \leq |||A|||\,|||A^{-1}|||\frac{||\delta y||}{||y||} .$$

[2]

ii) Let $x_2 \in \mathbb{R}^n$ be a solution of $(A + \delta A)x = y$, where $\delta A \in \mathbb{R}^{n\times n}$. Prove that

$$\frac{||x_0 - x_1||}{||x_0||} \leq |||A|||\,|||A^{-1}|||\frac{|||\delta A|||}{|||A|||} .$$

[2]

iii) The coefficient $\kappa(A) = |||A|||\,|||A^{-1}|||$ is known as the *condition number* of A.

Show that $\kappa(A) \geq 1$. Comment on the sensitivity of the solution of the equation $Ax = y$ to perturbations in terms of $\kappa(A)$. [3]

c) Let $A \in \mathbb{R}^{n\times n}$ be a non-singular matrix with eigenvalues $\lambda_1, \dots, \lambda_n$.

i) Derive the eigenvalues of A^{-1}. [1]

ii) Show that $|||A||| \geq |\lambda_i|$ for all $i = 1, \dots, n$.

iii) Derive a lower bound for $\kappa(A)$ in terms of the λ_is. [3]

iv) Show that if A is (non singular) symmetric then

$$\kappa(A) = \max_{i=1\dots,n} |\lambda_i| \max_{i=1\dots,n} \frac{1}{|\lambda_i|} .$$

[3]

Hint: Use the fact that if A is symmetric then there exists an orthonormal basis of eigenvectors of A.

1/a/

$$P = \frac{1}{2}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

(i) $x \in \mathcal{N}(P)$ if $P(x) = 0$; $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4$

$$Px = \begin{bmatrix} \frac{1}{2}x_1 - \frac{1}{2}x_3 \\ \frac{1}{2}x_2 - \frac{1}{2}x_4 \\ -\frac{1}{2}x_1 + \frac{1}{2}x_3 \\ -\frac{1}{2}x_2 + \frac{1}{2}x_4 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x_1 = x_3 \\ x_2 = x_4 \end{cases}$$

Hence $\mathcal{N}(P) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_1 \\ x_2 \end{pmatrix} ; (x_1, x_2) \in \mathbb{R}^2 \right\}$

$$= \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

$y \in \text{Range}(P) \Rightarrow \exists x \in \mathbb{R}^4$ such that $Px = y$.

$$\begin{cases} y_1 = \frac{1}{2}x_1 - \frac{1}{2}x_3 \\ y_2 = \frac{1}{2}x_2 - \frac{1}{2}x_4 \\ y_3 = -\frac{1}{2}x_1 + \frac{1}{2}x_3 \\ y_4 = -\frac{1}{2}x_2 + \frac{1}{2}x_4 \end{cases} \Rightarrow \begin{cases} y_1 = -y_3 \\ y_2 = -y_4 \end{cases}$$

Hence $\text{Range}(P) \subset \left\{ \begin{pmatrix} y_1 \\ y_2 \\ -y_1 \\ y_4 \end{pmatrix} ; (y_1, y_2) \in \mathbb{R}^2 \right\}$. OK

1/a) Since $\mathrm{Rank}(P)=2 \Rightarrow \dim \mathrm{Range}(P)=2$

Hence $\mathrm{Range}(P) = \mathrm{Span}\left\{ \begin{pmatrix}1\\0\\-1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\-1\end{pmatrix} \right\}$.

ii) It is not difficult to see that 2/23

$N(P) \cap \mathrm{Range}(P) = \left\{ \begin{pmatrix}0\\0\\0\\0\end{pmatrix} \right\}$. $(*)$

and that $\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\-1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\-1\end{pmatrix}$

is a basis of $\mathbb{R}^4 \Rightarrow N(P) + \mathrm{Range}(P) = \mathbb{R}^4$

$(**)$

$(*)$ & $(**)$ $\Rightarrow$ $N(P) \oplus \mathrm{Range}(P) = \mathbb{R}^4$.

(iii) if $x \in N(P) \Rightarrow x = \begin{pmatrix}x_1\\x_2\\x_1\\x_2\end{pmatrix}$; $x_1, x_2 \in \mathbb{R}^2$

if $y \in \mathrm{Range}(P) \Rightarrow y = \begin{pmatrix}y_1\\y_2\\-y_1\\-y_2\end{pmatrix}$; $y_1, y_2 \in \mathbb{R}^2$

$x^T y = x_1 y_1 + x_2 y_2 - x_1 y_1 - x_2 y_2 = 0$.

Hence $\mathrm{Range}(P) = (N(P))^{\perp}$

OK

(iv) For $z \in \mathbb{R}^4$ $z = x + y$

$x \in N(P)$; $y \in \mathrm{Range}(P)$.

$Pz = Px + Py = y \Rightarrow$

P is the orthogonal projection on Range(P)

(parallel to ~~the~~ $N(P)$)

OK

MATHEMATICS FOR Signal & Systems.

1/6/ 3/23

$F_k \in \mathbb{R}^n$ $\quad F_k = \text{Span}\{e_1 \dots e_k\}$ $\quad e_1 \dots e_k$ orthonormal

i) $z \in \mathbb{R}^n \quad Pz = \sum_{i=1}^{k} (e_i^T z)\, e_i$

(ii) $\|Pz\|^2 = (Pz)^T Pz = \sum_{i=1}^{k} (e_i^T z)^2$

$\|z\|^2 = \sum_{i=1}^{n} (e_i^T z)^2$

$\Rightarrow \quad \|Pz\|^2 \leq \|z\|^2$ since $k < n$.

$\left[\begin{array}{l} \text{for } i = 1 \dots k \qquad Pe_i = e_i \\ \qquad \Rightarrow \quad |||P||| = 1. \end{array}\right]$ "Extra"

(iii) $P = \sum_{i=1}^{k} e_i e_i^T$

$P^2 = \sum_{i=1}^{k} e_i e_i^T \sum_{j=1}^{k} e_j e_j^T$

OK

$= \sum_{i,j=1}^{k} e_i \underbrace{e_i^T e_j}_{\delta_{ij}} e_j^T = \sum_{i=1}^{k} e_i e_i^T$

$= P.$

$(Px)^T y = \sum_{i=1}^{k} (e_i^T x)\, e_i^T \sum_{j=1}^{n} (e_j^T y)\, e_j$

$= \sum_{i=1}^{k} (e_i^T x)(e_i^T y)$

$x^T P y = \sum_{i=1}^{n} (e_i^T x)\, e_i^T \sum_{j=1}^{k} e_j^T y\, e_j$

$= \sum_{i=1}^{k} (e_i^T x)\, (e_i^T y).$

1/b/

iv) $z = \lambda x + y \qquad x \in Range(Q); y \in W(Q)$

$$\|Qz\|^2 \leq \|z\|^2 \Rightarrow \lambda^2 \|x\|^2 \leq \lambda^2 \|x\|^2 + \|y\|^2 + 2\lambda \langle x, y\rangle.$$

$$\Rightarrow \quad 0 \leq \|y\|^2 + 2\lambda \langle x, y\rangle.$$

This only true for all $\lambda \in \mathbb{R}$ if $\langle x, y\rangle = 0$.

$\Rightarrow \quad Range(Q) \perp W(Q) \Rightarrow Q$ orthogonal projection.

4/4

1/c/. $\quad$ T/23

i) $Pu = P\left(\frac{1}{\lambda} Ru\right) = \frac{1}{\lambda} P P Q u = \frac{1}{\lambda} P^2 Q u$

$= \frac{1}{\lambda} P Q u$ since P projection

$= \frac{1}{\lambda} Ru = u.$

$\Rightarrow u \in \text{Range}(P)$ (P being a projector).

$P(Qu - \lambda u) = PQu - \lambda Pu$ since $PQu = Ru = \lambda u$

$= \lambda u - \lambda u$ & $u \in \text{Range}(P)$.

ii) $\lambda u^T u = u^T(\lambda u) = u^T PQ u$

P orthogonal projector $= (Pu)^T Qu.$

$= u^T Q u.$

Q projection $= u^T Q^2 u$

Q orthogonal projection $= (Qu)^T Qu$

$= \|Qu\|^2$

$\Rightarrow \lambda \|u\|^2 = \|Qu\|^2.$

iii) $\|Qu\| \leq \|u\|$

$\Rightarrow 0 \leq \lambda = \dfrac{\|Qu\|^2}{\|u\|^2} \leq 1.$

2) $A = \begin{bmatrix} 1 & x_0 & \cdots & x_0^m \\ \vdots & & & \\ 1 & x_m & \cdots & x_m^m \end{bmatrix}$ 6/23

a) i) $Av = 0$

$$\sum_{i=0}^{m} v_i x_j^i = 0 \quad \text{for } j = 1 .. m.$$

The polynomial $P(x) = \sum_{i=1}^{m} v_i x^i$ has m distinct roots the x_j; $j = 1..m$ $\Rightarrow$ $P(x) = 0$

$\Rightarrow$ $v_i = 0$ $i = 1..m$.

OK

ii) $A^T A v = 0$ $\Rightarrow$ $v^T A^T A v = 0$

$\Rightarrow$ $(Av)^T Av = 0$ $\Rightarrow$ $Av = 0$

$\Rightarrow$ $v = 0$.

by previous question.

OK

iii) $A^T A x = A^T y$

By previous question $A^T A$ is non singular hence $A^T A x = A^T y$ has a solution that is unique.

2/b/i/

$$7/23$$

$$g(w) = (y - Aw)^T (y - Aw).$$

$$= y^T y - 2y^T Aw + (Aw)^T Aw$$

$$= y^T y - 2y^T Aw + w^T \underbrace{A^T Aw}_{A^T y}$$

$$= y^T y - 2y^T Aw + (Aw)^T y$$

$$= y^T y - y^T Aw.$$

ii/ $g(v) - g(w) = y^T y - 2 y^T Av + (Av)^T Av$

$- y^T y + y^T Aw$ (from previous question)

since $A^T y = A^T Aw$

$$= -2 (A^T y)^T v + v^T A^T A v$$

$$+ (A^T y)^T w$$

$$= -2 (A^T Aw)^T v + v^T A^T A v$$

$$+ (A^T A w)^T w$$

$$= -2 w^T A^T A v + v^T A^T A v$$

$$+ w^T A^T A w$$

$$= (w - v)^T A^T A (w - v).$$

2/ b/

iii/ $g(v) - g(w) = (A(w-v))^T A(w-v)$

$= \| A(w-v) \|^2 \geq 0$

$\Rightarrow g(v) \geq g(w).$

$g(v) = g(w)$ if $A(w-v) = 0$

$\Rightarrow$ ~~Aw~~ $w - v = 0 \Rightarrow w = v.$

By 2) a) i).

4/13

2/ c/.

i) $$\phi_m(P) = \sum_{i=0}^{n} \left(y_i - P(x_i)\right)^2 .$$

i) The i-th component of $y - Av$ is given by

$$(y - Av)_i = y_i - \sum_{j=0}^{m} v_j \, x_i^{\,j}$$

$$= y_i - P(x_i).$$

Hence $$(y - Av)^T (y - Av) = \sum_{i=0}^{n} \left(y_i - P(x_i)\right)^2 .$$

ii). $$\phi_m(P) = g(v) \geq g(w) = \phi_m(P_w) .$$

$$\min_{P \in \mathbb{R}_m[X]} \phi_m(P) = \phi_m(P_w).$$

where $P_w(x) = \sum_{i=0}^{m} w_i \, x^i$

$w = (w_i)$ is the unique solution of $A^T A x = A^T y$.

2/ d/.

$$A = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \end{bmatrix}$$ 10/21

$$A^T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 4 \\ -1 & 0 & 1 & 8 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 4 \\ -1 & 0 & 1 & 8 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 & 6 & 8 \\ 2 & 6 & 8 & 18 \\ 6 & 8 & 18 & 32 \\ 8 & 18 & 32 & 66 \end{bmatrix}$$

$$A^T y = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 4 \\ -1 & 0 & 1 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

$$A^T A w = A^T y \Rightarrow \begin{cases} 4x_0 + 2x_1 + 6x_2 + 8x_3 = 4 & (1) \\ 2x_0 + 6x_1 + 8x_2 + 18x_3 = 0 & (2) \\ 6x_0 + 8x_1 + 18x_2 + 32x_3 = 2 & (3) \\ 8x_0 + 18x_1 + 32x_2 + 66x_3 = 0 & (4) \end{cases}$$

(ii) $\Rightarrow w = \begin{pmatrix} 2 \\ -1/3 \\ -1 \\ 1/3 \end{pmatrix}$

X. $\Phi_m(Pw) = 0$.

3/

a/ i/ $M = \begin{pmatrix} A & b \\ c^T & \alpha \end{pmatrix}$

$g = \begin{pmatrix} f \\ \gamma \end{pmatrix}$ $My = g.$

$y = \begin{pmatrix} x \\ \lambda \end{pmatrix}$

ii/ $x = A^{-1}(f - b\lambda)$

Moreover $c^T x + \alpha\lambda = \gamma$

$c^T A^{-1}(f - b\lambda) + \alpha\lambda = \gamma.$

$\Rightarrow$ $(\alpha - c^T A^{-1} b)\lambda = \gamma - c^T A^{-1} f$: simple linear equation in $\mathbb{R}$.

It admits a unique solution iff $\boxed{\alpha - c^T A^{-1} b \neq 0}$

11/23

3/ b/

i/ if $\alpha - c^T A^{-1} b \neq 0$

we have $\lambda = \dfrac{\gamma - c^T A^{-1} f}{\alpha - c^T A^{-1} b}$ (12/2)

$= \dfrac{\gamma - c^T h_0}{\alpha - c^T \cancel{A^{-1}} z_0}$

and $A x = f - \dfrac{\gamma - c^T h_0}{\alpha - c^T z_0} b$

$= f - \lambda b.$

ii/ $c^T h_0$ requires n multiplication
$n-1$ summations.

similarly for $c^T z_0$.

To compute λ we need $4n-2$ operations.

To compute x we have n multiplications to compute λz_0 & n summations to compute $h_0 - \lambda z_0$. Thus an additional $2n$ operat

In total, we need $\boxed{6n-2 \text{ extra operations}}$, i.e. $\underline{6n \text{ flops}}$

3/ c/

i/ $$A = \begin{bmatrix} 25 & 15 & -5 \\ 15 & 18 & 0 \\ -5 & 0 & 11 \end{bmatrix}$$

By Cholesky

$$= \begin{bmatrix} 5 & 0 & 0 \\ 3 & 3 & 0 \\ -1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 5 & 3 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

$$A\,z = \begin{pmatrix} 30 \\ 15 \\ -16 \end{pmatrix} \Rightarrow z_0 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$A\,h = \begin{pmatrix} 35 \\ 33 \\ 10 \end{pmatrix} \begin{pmatrix} 35 \\ 33 \\ 6 \end{pmatrix} \Rightarrow h_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

ii/ $$\lambda = \frac{\gamma - (c_1 + c_2 + c_3)}{\alpha - c_1 + c_3} = 1$$

$$x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$$

13/23

4/ $\langle P, Q \rangle = \int_{-1}^{1} P(x)\, Q(x)\, w(x)\, dx.$

a) Similar to Pb in Pb sheet / lecture

i/ $T_1(x) = x$

$T_2(x) = 2x^2 - 1$

$\cos(2\theta) = 2\cos^2(\theta) - 1$ 14/23

$\cos(3\theta) = \ldots$

$T_3(x) = 2x\,(2x^2 - 1) - x$

$= 4x^3 - 3x.$

ii/ $\cos((k+1)\theta) + \cos((k-1)\theta)$

$= 2\cos(k\theta)\cos(\theta).$

$T_{k+1}(X) = 2X\, T_k(X) - T_{k-1}(X).$

iii) $\langle T_n, T_m \rangle = \int_{-1}^{1} \frac{T_n(x)\, T_m(x)}{\sqrt{1-x^2}}\, dx$

$= + \int_0^{\pi} \cos(n\theta) \cos(m\theta)\, d\theta$

$= + \int_0^{\pi} \frac{1}{2} \left(\cos((n+m)\theta) \right) d\theta$

$+ 1/2 \int_0^{\pi} \cos((n-m)\theta)\, d\theta.$

$n \neq m$ $\langle T_n, T_m \rangle = 0.$

$n = m \neq 0.$ $\langle T_n, T_m \rangle = + \pi/2.$

4/ a/ iii/ $n = m = 0$ $\langle T_0, T_0 \rangle = \pi$.

(iv) $\frac{T_0}{\sqrt{\pi}}, \frac{T_1}{\sqrt{\pi/2}}, T_2/\sqrt{\pi/2}, T_3/\sqrt{\pi/2}$

Since T_i orthogonal & 4 a/ iii/ gives the corresponding norms T_0, T_1, T_2, T_3 are given in 4/ a/ i/.

4/

b/.

i/ Trivial 16/27

ii/ $p_n(x) = x^n + a_{n-1} x^{n-1} + \dots + a_0.$

$p_0(x) = 1$

if we constructed $p_0 \dots p_{n-1}$

Let $q(x) = x^n$.

$$p_n(x) = q(x) - \sum_{k=0}^{n-1} \frac{\langle q, p_k \rangle}{\langle p_k, p_k \rangle} p_k .$$

we have

$m < n$, $\langle p_n, p_m \rangle = 0$

Since $\langle p_n, p_m \rangle = \langle p_m, q \rangle - \langle p_m, q \rangle \frac{\langle p_m, p_m \rangle}{\langle p_m, p_m \rangle}$

$= 0 .$

iii/ $Q \in \mathbb{R}_{k-1}[X]$ & $p_0 \dots p_{k-1}$ basis

of $\mathbb{R}_{k-1}[X]$ hence.

$$Q = \sum_{i=0}^{k-1} \alpha_i p_i$$

so $\langle Q, p_k \rangle = \sum_{i=0}^{k-1} \alpha_i \langle p_k, p_i \rangle = 0 .$

iv).

$$\langle X P_k , P_j \rangle = \int_{-1}^{1} x P_k(x) P_j(x) w(x) dx$$

$$= \langle P_k , X P_j \rangle$$

since $X P_j \in \mathbb{R}_{k-1}[X]$ $= 0$

by previous question.

17/29

4/ c/

18/23

i/. $XP_0 \in \mathbb{R}_1[X]$.

Hence $XP_0 = a_1 P_1 + b_0 P_0$

ii/ $x\,\alpha_{00} = a_1\left[\alpha_{11} x + \alpha_{10}\right] + b_0 \alpha_{00}$.

$$\begin{cases} \alpha_{00} = a_1 \alpha_{11} \\ \alpha_{10} a_1 + b_0 \alpha_{00} = 0 \end{cases} \Rightarrow \quad a_1 = \frac{\alpha_{00}}{\alpha_{11}} \qquad b_1 = -\frac{\alpha_{10}}{\alpha_{00}}.$$

iii/.

$$XP_k = \sum_{i=1}^{k+1} \beta_i P_i$$

$\langle XP_k, P_i\rangle = \langle P_k, XP_i\rangle = 0$ if $i \leq k-2$.

Hence $XP_k = \beta_{k+1} P_{k+1} + \beta_k P_k + \beta_{k-1} P_{k-1}$.

Coefficient with x^{k+1}

$$\alpha_{kk} = \beta_{k+1}\,\alpha_{k+1\,k+1} \Rightarrow \boxed{\beta_{k+1} = a_{k+1} = \frac{\alpha_{kk}}{\alpha_{k+1,k+1}}}$$

$\cancel{\boxed{\beta_{k+1} + \beta_k + \beta_{k-1} = 0}}$ ~~since XP_k has no coefficient constant.~~

Coefficient with x^k.

$$\alpha_{k,k-1} = \beta_{k+1}\,\alpha_{k+1,k} + \beta_k\,\alpha_{k,k}$$

4/ c/ iii/

$$\beta_{k+1} = a_{k+1} = \frac{\alpha_{kk}}{\alpha_{k+1,k+1}}$$

$$\alpha_{k,k-1} = \beta_{k+1} \alpha_{k+1,k} + \beta_k \alpha_{k,k}$$

$$= \frac{\alpha_{k,k}}{\alpha_{k+1,k+1}} \alpha_{k+1,k} + \beta_k \alpha_{k,k}.$$

$$\Rightarrow b_k = \beta_k = \frac{\alpha_{k,k-1}}{\alpha_{k,k}} - \frac{\alpha_{k+1,k}}{\alpha_{k+1,k+1}}$$

~~coefficient with $k-1$~~.

~~$\alpha_{k,k-2} = \beta_{k+1} \alpha_{k+1,k-1} + \beta_k \alpha_{k,k-1}$~~.

By induction,

$$\beta_{k-1} = \langle X p_k, p_{k-1} \rangle = \langle p_k, X p_{k-1} \rangle$$

$$= a_k.$$

13/23

5/ a)

i/ + $|||A||| \geq 0$

+ $|||A||| = 0 \Rightarrow \left\|A\frac{x}{\|x\|}\right\| = 0 \quad \forall x \neq 0 \Rightarrow Ax = 0 \quad \forall x$

$\Rightarrow A = 0$

$\left\|\frac{x}{\|x\|}\right\| = 1$

+ $\|\lambda A x\| = |\lambda| \, \|Ax\| \Rightarrow |||\lambda A||| = |\lambda| \, ||A||.$

+ $\|(A+B)x\| \leq \|Ax\| + \|Bx\|$

$\|x\| = 1 \Rightarrow \|(A+B)x\| \leq |||A||| + |||B|||$

$\Rightarrow |||A+B||| \leq |||A||| + |||B|||.$

ii/ $\forall x \quad \left\|A\frac{x}{\|x\|}\right\| \leq |||A||| \Rightarrow \|Ax\| \leq |||A||| \, \|x\|$

By above

$\|ABx\| \leq |||A||| \, \|Bx\| \leq |||A||| \, |||B||| \, \|x\|$

10/23

5/ b/. 21/23

i/.

$$A x_1 = y + \delta y$$
$$A x_0 = y \quad \Rightarrow \quad A^{-1} \delta y = x_1 - x_0$$

$$\| x_1 - x_0 \| \leq |||A^{-1}||| \, \|\delta y\| \quad \text{by 5a) ii)}$$

$$\|y\| = \|A x_0\| \leq |||A||| \, \|x_0\|$$

$$\frac{\|x_1 - x_0\|}{\|x_0\|} \leq |||A^{-1}||| \, |||A||| \, \frac{\|\delta y\|}{\|y\|}$$

ii).

$$A(x_2 - x_0) = y - \delta A x_2 - y = -\delta A x_2.$$

$$\Rightarrow (x_2 - x_0) = -A^{-1} \delta A x_2.$$

$$\frac{\|x_2 - x_0\|}{\|x_2\|} \leq |||A^{-1}||| \, |||\delta A||| \, \cancel{\|x_2\|}.$$
$$= \|A^{-1}\| \, \|A\| \, \frac{\|\delta A\|}{\|A\|}$$

(iii) $K(A) = |||A||| \, |||A^{-1}||| \geq \|A A^{-1}\| = \|I\| = 1$.

+ if $K(A)$ is close to 1, then the relative error is not much larger that the perturbation due to y or A: WELL_CONDITIONED.

+ if $K(A)$ large: the relative error in x can by far exceed the one in y or A: BADLY (ILL) CONDITIONED.

5/ c/ i/ $Ax_i = \lambda_i x_i \Rightarrow A^{-1} x_i = \frac{1}{\lambda_i} x_i$

ii/ X_i eigenvector associated to λ_i

$$\frac{\|AX_i\|}{\|X_i\|} = |\lambda_i| \quad \forall i = 1..n$$

$$|||A||| \geq \max_{i=1..n} |\lambda_i|$$

22/23

iii/ $$|||A^{-1}||| \geq \max_{i=1..n} \frac{1}{|\lambda_i|}.$$

$\lambda_i \neq 0$ since A non-singular

$$K(A) \geq \max_i |\lambda_i| \max_i \frac{1}{|\lambda_i|}.$$

(iv) A symmetric $(X_i)_{i=1}^{n}$ basis of eigenvector orthonormal.

$$X = \sum_{i=1}^{n} x_i X_i$$

$$\|AX\|_2^2 = \sum x_i^2 \lambda_i^2$$

$$\|X\|^2 = \sum_i x_i^2.$$

if $|\lambda_1| = \max_{i=1}^{n} |\lambda_i|$. $\quad \frac{\|AX\|^2}{\|X\|^2} \leq \lambda_1^2$

or! $\|AX_1\|^2 = \lambda_1 \Rightarrow |||A||| = |\lambda_1|$
$= \max |\lambda_i|$

Similarly we have $|||A^{-1}||| = \max\limits_{i=1} \left|\frac{1}{\lambda_i}\right|$

$$\Rightarrow \quad k_2(A) = \max_i |\lambda_i| \max_i \frac{1}{|\lambda_i|}.$$

23/23