CONTROL ENGINEERING

1. a) The state space description of the system, with the indicated state variables, is given by

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\omega_1^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\omega_2^2 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} u,$$

$$y = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x.$$

Incorrect dimensions of the matrices and/or incorrect entries.

[4 marks]

b) The controllability matrix of the system is

$$\mathscr{C} = \begin{bmatrix} 0 & 1 & 0 & -\omega_1^2 \\ 1 & 0 & -\omega_1^2 & 0 \\ 0 & 1 & 0 & -\omega_2^2 \\ 1 & 0 & -\omega_2^2 & 0 \end{bmatrix}.$$

Note that $\det\mathscr{C} = (\omega_1 - \omega_2)^2(\omega_1 + \omega_2)^2$, hence the system is controllable if and only if $\omega_1 \neq \omega_2$ (recall that $\omega_1 > 0$ and $\omega_2 > 0$).

Incorrect computation of the controllability matrix, or of its rank as a function of ω_i.

[4 marks]

c) The observability matrix of the system is (there is no need to compute CA^2 and CA^3)

$$\mathscr{O} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\omega_1^2 & 0 & 0 & 0 \\ 0 & 0 & -\omega_2^2 & 0 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

Note that the first four rows of the observability matrix are linearly independent (recall that $\omega_1 > 0$ and $\omega_2 > 0$).

Incorrect computation of the observability matrix, or of its rank. The condition $\omega_i > 0$ was used incorrectly. Some students failed to recognized that only the computation of a submatrix of the observability matrix is required to answer the question.

[2 marks]

d) Note that $u = -k\begin{bmatrix} 1 & 1 \end{bmatrix} Cx$. Hence the closed-loop "A" matrix is

$$A_{cl} = A - kB\begin{bmatrix} 1 & 1 \end{bmatrix} C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\omega_1^2 & -k & 0 & -k \\ 0 & 0 & 0 & 1 \\ 0 & -k & -\omega_2^2 & -k \end{bmatrix}.$$

Its characteristic polynomial is

$$p(s) = s^4 + 2ks^3 + (\omega_1^2 + \omega_2^2)s^2 + k(\omega_1^2 + \omega_2^2)s + \omega_1^2\omega_2^2.$$

Applying Routh test yields that the closed-loop system is asymptotically stable for any $k > 0$ provided $\omega_1 \neq \omega_2$.

Incorrect matrix representation of the feedback and mistakes in the computation of the closed-loop A matrix. Then, incorrect computation of the characteristic polynomial and incorrect application of Routh test.

[4 marks]

e) Differentiating E with respect to time yields

$$\dot{E} = \omega_1^2 x_1 \dot{x}_1 + x_2 \dot{x}_2 + \omega_2^2 x_3 \dot{x}_3 + x_4 \dot{x}_4,$$

hence

$$\dot{E} = (\dot{y}_1 + \dot{y}_2)u,$$

as indicated in the exam paper. Replacing the controller yields (recall that $k > 0$)

$$\dot{E} = -k(\dot{y}_1 + \dot{y}_2)^2 \leq 0.$$

As a result, E is a non-increasing function of time, and i) it decreases either to its minimum, that is $E = 0$ or ii) it stops decreasing when $(\dot{y}_1 + \dot{y}_2) = 0$. In case i) $x_1 = x_2 = x_3 = x_4 = 0$, hence the equilibrium is attractive (and asymptotically stable, since the system is linear). In case ii) $\dot{y}_1 + \dot{y}_2 = 0$ implies $u = 0$, hence y_1 and y_2 are described by two independent linear oscillators. If the frequencies of the oscillators are different, then the only solutions such that $\dot{y}_1 + \dot{y}_2 = 0$ for all $t \geq 0$ is the trivial solution $y_1 = y_2 = 0$, hence again the equilibrium of the system is attractive (and asymptotically stable, since the system is linear).

Incorrect evaluation of the time derivative of E along the trajectories of the system, in open-loop and/or in closed-loop. Incorrect, or lack of, interpretration of the condition $\dot{E} \leq 0$ and inability to draw conclusions from the condition $\dot{E} = 0$. The students should have also recalled the fact that for linear (time-invariant) systems attractivity implies exponential stability.

[6 marks]

2. a) The A and B matrices of the system are

$$A = \begin{bmatrix} -b & b \\ c & -c \end{bmatrix}, \qquad B = \begin{bmatrix} a \\ 0 \end{bmatrix}.$$

The controllability matrix is

$$\mathscr{C} = \begin{bmatrix} a & -ba \\ 0 & ca \end{bmatrix},$$

hence the system is controllable provided $ca \neq 0$. Since, as stated in the exam paper, a and c are positive the system is always controllable.

Incorrect matrices A and/or b and incorrect computation of the controllability matrix and its rank.

[4 marks]

b) Note that since $b+c>0$ the variables z are independent, that is they qualify as a valid change of coordinates. The inverse transformation is

$$x_1 = \frac{z_1 + bz_2}{b+c} \qquad x_2 = \frac{z_1 - cz_2}{b+c}.$$

In the variables z one has

$$\dot{z}_1 = acu, \qquad \dot{z}_2 = -(b+c)z_2 + au.$$

Some students failed to observe that $b+c>0$ (since $b>0$ and $c>0$), hence the indicated change of coordinates is always well-defined. Typical mistakes include the incorrect computation of the inverse change of coordinates and mistakes in computing the equations of the system in the new coordinates.

[4 marks]

c) i) The "A" matrix of the closed-loop system is

$$A_{cl} = \begin{bmatrix} k_1ac & k_2ac \\ k_1a & k_2a - b - c \end{bmatrix}$$

and its characteristic polynomial is

$$s^2 + (b+c-k_2a-k_1ac)s - k_1ac(b+c).$$

Hence the closed-loop system is asymptotically stable provided (recall that a, b and c are positive)

$$b+c-k_2a-k_1ac>0 \qquad k_1<0.$$

Incorrect computation of the closed-loop matrix and/or of its characteristic polynomial. Again, the positivity conditions on a, b and c should have been used.

[4 marks]

ii) The equilibrium of the closed-loop system is such that $\dot{z}_1 = \dot{z}_2 = 0$, yielding

$$z_{1,eq} = -\frac{u_\star}{k_1}, \qquad z_{2,eq} = 0.$$

Note that the equilibrium is well-defined provided $k_1 \neq 0$, condition which is implied by the asymptotic stability conditions in part c.ii).

Some students failed to use the definition of equilibrium points to compute the equilibria of the z system and to recognize the relation between existence of a unique equilibrium and asymptotic stability.

[4 marks]

iii) In the x_1 and x_2 variables the feedback is

$$u = u_\star + k_1 z_1 + k_2 z_2 = u_\star + (ck_1 + k_2)x_1 + (bk_1 - k_2)x_2.$$

To have a function of x_1 only one has to select $k_2 = bk_1$. To have a stabilizing feebback one needs (recall again that a, b and c are positive)

$$1 - ak_1 > 0 \qquad k_1 < 0.$$

Hence, selecting $k_1 < 0$ and $k_2 = bk_1 < 0$ yields an asymptotically stable closed-loop system and the set of assignable closed-loop equilibrim is the set

$$z_{1,eq} = -\frac{u_\star}{k_1}, \qquad z_{2,eq} = 0,$$

that is the whole z_1 positive semi-axis.

Typical mistakes: inability to rewrite the feedback controller in the x variables, mistakes in the computation of the equilibrium and in the stability analysis for this specific case (in which the control gains are constrained).

[4 marks]

3. a)

i) Reachability of the system implies that the reachability matrix

$$\mathscr{R} = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$$

is full rank. As a result, for any nonzero row vector v one has

$$v\mathscr{R} \neq 0,$$

that is the matrix $\mathscr{R}$ does not have a (left) kernel. Note now that when $p = 1$ the matrix C is a row vector: hence for any nonzero C

$$C\mathscr{R} = \begin{bmatrix} CB & CAB & \cdots & CA^{n-1}B \end{bmatrix} \neq 0,$$

which implies that the rank condition is satisfied, that is any reachable system is output controllable for any $C \neq 0$.

Most students were unable to exploit the definition of reachability matrix, and its properties, together with the notion of nullspace (kernel) to answer the question.

[4 marks]

ii) If the system is output controllable for any $C \neq 0$ then $C\mathscr{R} \neq 0$ for any nonzero C, that is the matrix $\mathscr{R}$ does not have any (left) kernel, hence it is full rank. From this we conclude reachability of the system.

Similarly to the previous question: this is a trivial exercise in linear algebra.

[4 marks]

iii) Note that (recall that $D = 0$)

$$\begin{aligned} y(0) &= Cx(0), \\ y(1) &= Cx(1) = C(Ax(0) + Bu(0)) = CAx(0) + CBu(0). \end{aligned}$$

Hence, if $CB \neq 0$ the selection $u(0) = -\dfrac{CAx(0)}{CB}$ is such that $y(1) = 0$, for any $x(0)$, which proves the claim.

Incorrect use of the equations of the discrete-time system to evaluate $y(1)$ and to show that the condition $CB \neq 0$ allows to select an input signal which zeroes the outout at $t = 1$, and also for all $t > 1$.

[2 marks]

iv) One has to generalize the derivation in part a.iii). Note that if $CB = CAB = \cdots = CA^{i-2}B = 0$ and $CA^{i-1}B \neq 0$ then

$$\begin{aligned} y(0) &= Cx(0), \\ y(1) &= Cx(1) = CAx(0), \\ y(2) &= Cx(2) = CA^2x(0), \\ &\vdots \\ y(i-2) &= Cx(i-2) = CA^{i-2}x(0), \\ y(i-1) &= Cx(i-1) = CA^{i-1}x(0) + CA^{i-2}Bu(0). \end{aligned}$$

Hence, selecting $u(0)$ one can zero $y(i-1)$, that is the i-th component of the output sequence (recall that the sequence starts at $t=0$).

This is a generalization of the construction in the previous question which requires some ability to manipulate the equations of a discrete-time system. The ideas/tools required were demonstrated in the lectures.

[8 marks]

b) If $D \neq 0$ then

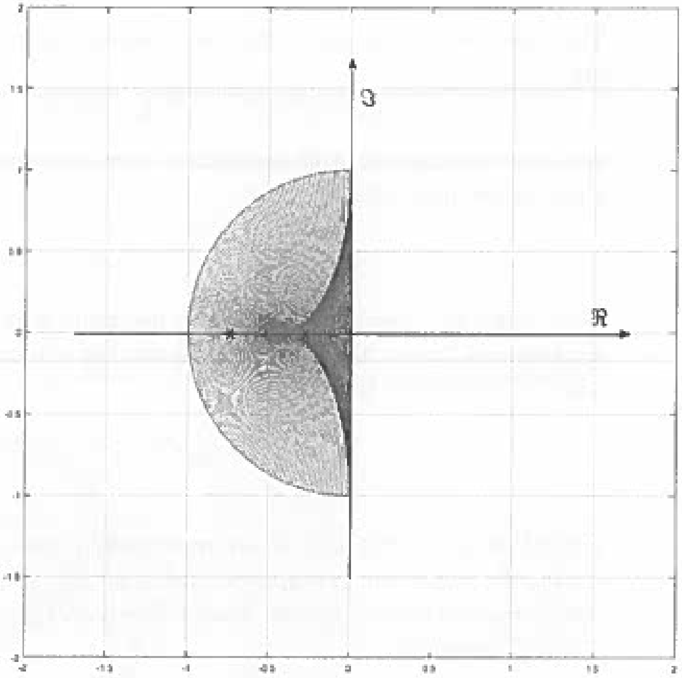
$$\begin{aligned}
y(0) &= Cx(0) && +Du(0),\\
y(1) &= CAx(0)+CBu(0) && +Du(1),\\
y(2) &= \cdots && +Du(2),\\
&\vdots\\
y(k) &= \cdots && +Du(k),
\end{aligned}$$

hence $y(i)$ can be set to zero for all i by a selection of $u(i)$.

This is a slightly more subtle exercise, since in the lectures we have almost always assumed $D=0$. Most students have failed to use the equations (in the Lecture Notes) expressing the output as a function of the input to provide the answer.

[2 marks]

4. a) The set of stability is the intersection of the (open) left half of the complex plane with the interior of the unity disk, that is the half disk indicated in the figure below. Note that the red points do not belong to the set.



Incorrect stability sets and incorrect intersections of the sets.

[4 marks]

b) The indicated points do belong to the admissible region in the figure above (they are indicated with x-marks). Note that

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}, \quad A_{cl} = \begin{bmatrix} 0 & 1 \\ k_1 & k_2 \end{bmatrix}.$$

The characteristic polynomial of A_{cl} is

$$p(s) = s^2 - k_2 s - k_1$$

and this is equal to the desired closed-loop characteristic polynomical $(s + 1/4)(s + 3/4)$ provided $k_1 = -3/16$ and $k_2 = -1$.

Incorrect location of the poles in the complex plane, incorrect feedback matrix (dimensionwise) and/or incorrect closed-loop matrix. Finally, some students computed wrong gains.

[4 marks]

c)

i) Since the system behaves like a continuous time system for $t \in [0, 1)$ one has

$$x(t) = e^{A_{cl} t} x(0) \qquad t \in [0, 1),$$

hence $x(1) = e^{A_{cl}}x(0)$. At $t = 1$ the system behaves like a discrete-time system, that is $x^+ = A_{cl}x$, hence, the state is updated to (note the abuse of notation for $x(1)$)

$$x(1) = A_{cl}x(1) = A_{cl}e^{A_{cl}}x(0).$$

The state then evolves as a continuous-time system for $t \in [1,2)$ yielding

$$x(2) = e^{A_{cl}}x(1) = e^{A_{cl}}A_{cl}e^{A_{cl}}x(0)$$

and, after one more step of the discrete-time iteration (note again the abuse of notation for $x(2)$)

$$x(2) = A_{cl}e^{A_{cl}}A_{cl}e^{A_{cl}}x(0) = (A_{cl}e^{A_{cl}})^2x(0),$$

from which the general expression in the exam paper follows. The sequence $x(k)$ can be thought of as generated by a discrete-time system with "A" matrix given by

$$A_d = A_{cl}e^{A_{cl}}.$$

Typical mistakes: inability to use recursively the notion of exponential matrix and of one step update, lack of abstraction to generate the equation of the hybrid *system. Most students also failed to recall that A and e^A commutes.*

[4 marks]

ii) Let v be an eigenvector of A_{cl} and recall that A_{cl} has eigenvalues $\lambda_1 = -1/4$ and $\lambda_2 = -3/4$. As a result

$$e^{A_{cl}}v = e^{\lambda_i}v$$

and

$$A_{cl}e^{A_{cl}}v = A_{cl}e^{\lambda_i}v = \lambda_i e^{\lambda_i}v,$$

that is the eigenvalues of $A_{cl}e^{A_{cl}}$ are

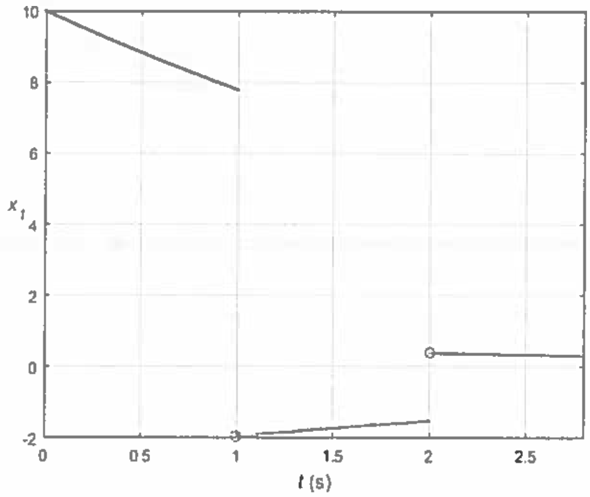
$$-\frac{1}{4}e^{-\frac{1}{4}} \approx= -0.19, \qquad -\frac{3}{4}e^{-\frac{3}{4}} \approx= -0.35.$$

Both eigenvalues are in the unity disk, hence the considered discrete-time system is asymptotically stable.

The student should have exploited the notion of eigenvalues/eigenvectors and recolled that A and e^A have the same eigenvectors. Typical mistakes include the incorrect computation of the eigenvalues of the discrete-time system associated to the matrix determined in part c.i).

[6 marks]

iii) A possible evolution of $x_1(t)$ is given in the figure below, in which the "o" sign indicate the *jumps* due to the discrete-time evolution. The change in sign at the *jumps* is due to the fact that both eigenvalues of $A_{cl}e^{A_{cl}}$ are negative.



Some students failed to recognize the jumps *in the state trajectories at $t = 1, 2, \cdots$ and the change in sign of the state at each jump.*

[2 marks]