Imperial College
London

M3S4

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May-June 2016

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Applied Probability

Date: **Tuesday 24th May 2016**

Time: **09.30 – 11.30**

Time Allowed: **2 Hours**

This paper has Four Questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the relevant main book(s) are full.

Statistical tables will not be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers to each question as per the table below.

Raw Mark	Up to 12	13	14	15	16	17	18	19	20
Extra Credit	0	½	1	1 ½	2	2 ½	3	3 ½	4

- Each question carries equal weight.
- Calculators may not be used.

1. Consider a homogeneous Markov chain $(X_n)_{n\in\mathbb{N}_0}$ with state space $E = \{1, \cdots, 7\}$ and transition matrix given by

$$\mathbf{P} = \begin{pmatrix} 1/2 & 1/4 & 0 & 1/4 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 1/8 & 0 & 7/8 & 0 & 0 \\ 1/4 & 0 & 0 & 0 & 0 & 0 & 3/4 \\ 0 & 1/9 & 7/9 & 0 & 0 & 1/9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

(1) Draw the transition diagram.

(2) Determine the communicating classes.

(3) For each class, specify whether the class is transient or recurrent. *(Justify your answer)*

(4) Is the chain irreducible? *(Justify your answer)*

(5) Find $\mathbb{P}_3(X_2 = 6)$ and $\mathbb{P}_1(X_2 = 7)$.
([Note]. $\mathbb{P}_i(X_n = j)$ is the probability of X_n reaches state j starting from i)

2. Let $N = (N_t)_{t\geq 0}$ denote a Poisson process of rate $\lambda > 0$. Define the stochastic process $X = (X_t)_{t\geq 0}$ by

$$X_t = N_t - \lambda t.$$

(1) Prove that, for $s \leq t$, $X_s = \mathbb{E}(X_t | \mathcal{F}_s)$ where $\mathcal{F}_s = \sigma(N_u,\ u \leq s)$.

(2) Find the Laplace transform ϕ of X where $\phi(u) = \mathbb{E}(e^{-uX_t})$.

(3) Denote $S_t = \sum_{i=1}^{N_t} Y_i$ where $(Y_i)_{i\in\mathbb{N}_0}$ is a sequence of independent and identically random variables, independent of N.

(i) Prove that the Laplace transform of S_t is

$$\psi(u) = \mathbb{E}(e^{-uS_t}) = \exp(-\lambda t + \lambda t G(u)),$$

where G is the Laplace transform of Y.

(ii) Using (i), compute $\mathbb{E}(S_t)$ and $\mathrm{Var}(S_t)$.

3. (1) Define the generator of a continuous-time Markov chain.

(2) Suppose the generator of a continuous-time Markov chain with state space $E = \{1, 2, 3, 4\}$ is given by

$$\mathbf{G} = \begin{pmatrix} -3 & 1 & 1 & 1 \\ 0 & -1 & 1 & 0 \\ 2 & 0 & -5 & 3 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

What are the transition probabilities of the corresponding jump-chain?

(3) Consider a birth-death process with birth rates $\lambda_n = (n+1)\lambda$ and death rates $\mu_n = \mu n^2$, for $n \in \mathbb{N}_0$ and $0 < \lambda < \mu$.

(i) Write down the generator.

(ii) Find the stationary distribution.

4. Let $(B_t)_{t\geq 0}$ be a standard Brownian motion.

(1) For $0 \leq s \leq t$, give the law of the random variable $B_t - B_s$.

(2) Prove that, for $t, s \geq 0$, $\mathrm{Cov}(B_t, B_s) = \min(s, t)$.

(3) Let $a > 0$ be a deterministic constant.
Prove that $(W_t)_{t\geq 0}$ with $W_t = aB_{t/a^2}$ is a standard Brownian motion.

(4) Let Z be a standard normal random variable [i.e. $Z \sim N(0, 1)$]. For $t \geq 0$, denote, $X_t = \sqrt{t}Z$.
Show that X is not a Brownian motion.

Imperial College London

M4S4

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May-June 2016

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Applied Probability

Date: **Tuesday 24th May 2016**

Time: **09.30 – 12.00**

Time Allowed: **2 Hours 30 Mins**

This paper has Five Questions.

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(1) Draw the transition diagram.

(2) Determine the communicating classes.

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(5) Find $\mathbb{P}_3(X_2 = 6)$ and $\mathbb{P}_1(X_2 = 7)$.
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2. Let $N = (N_t)_{t\geq 0}$ denote a Poisson process of rate $\lambda > 0$. Define the stochastic process $X = (X_t)_{t\geq 0}$ by

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(1) Prove that, for $s \leq t$, $X_s = \mathbb{E}(X_t|\mathcal{F}_s)$ where $\mathcal{F}_s = \sigma(N_u,\ u \leq s)$.

(2) Find the Laplace transform ϕ of X where $\phi(u) = \mathbb{E}(e^{-uX_t})$.

(3) Denote $S_t = \sum_{i=1}^{N_t} Y_i$ where $(Y_i)_{i\in\mathbb{N}_0}$ is a sequence of independent and identically random variables, independent of N.

(i) Prove that the Laplace transform of S_t is

$$\psi(u) = \mathbb{E}(e^{-uS_t}) = \exp(-\lambda t + \lambda t G(u)),$$

where G is the Laplace transform of Y.

(ii) Using (i), compute $\mathbb{E}(S_t)$ and $\mathrm{Var}(S_t)$.

3. (1) Define the generator of a continuous-time Markov chain.

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4. Let $(B_t)_{t \geq 0}$ be a standard Brownian motion.

(1) For $0 \leq s \leq t$, give the law of the random variable $B_t - B_s$.

(2) Prove that, for $t, s \geq 0$, $\mathrm{Cov}(B_t, B_s) = \min(s, t)$.

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(4) Let Z be a standard normal random variable [i.e. $Z \sim N(0, 1)$]. For $t \geq 0$, denote, $X_t = \sqrt{t}Z$.
Show that X is not a Brownian motion.

Mastery Question

5. Let W and $\tilde{W}$ be two independent standard Brownian motions. For $t \geq 0$, denote $X_t = \rho W_t + \sqrt{1-\rho^2}\tilde{W}_t$.

(1) Prove that X is a Brownian motion.

(2) Denote $X_t = \exp(\sigma W_t - \frac{\sigma^2}{2}t)$ where $\sigma > 0$. Show that $\mathbb{E}(X_t) = 1$.

Imperial College
London

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2016

This paper is also taken for the relevant examination for the Associateship.

M3/4/5 S4

Applied Probability (Solutions)

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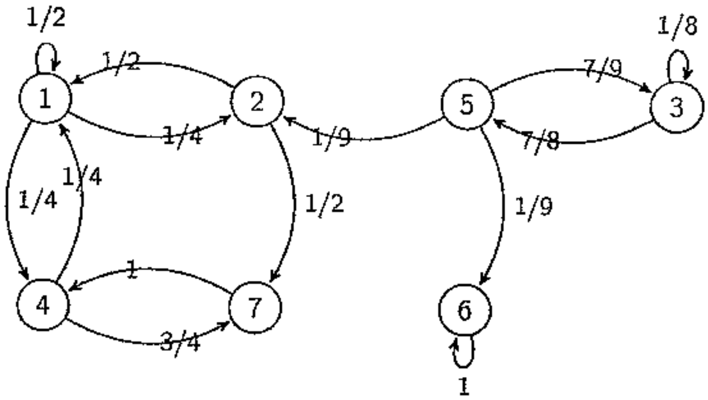

meth seen $\Downarrow$

1. (1) $\boxed{3}$

   (2) We have three communicating classes: $\{1, 2, 4, 7\}$, $\{6\}$, and $\{3, 5\}$. $\boxed{3}$

   (3) From the graph, we have two recurrent classes: $\{1, 2, 4, 7\}$ and $\{6\}$, and one transient class $\{3, 5\}$. $\boxed{3}$

   (4) It's a reducible chain: the chain has two recurrent classes. $\boxed{2}$

   (5) Using the formula: $\mathbb{P}_i(X_2 = j) = Q^2(i, j) = \sum_k Q(i, k)Q(k, j)$, we have

$$\mathbb{P}_3(X_2 = 6) = Q(3, 5)Q(5, 6) = \frac{7}{8} \cdot \frac{1}{9} = \frac{7}{72}.$$

$$\mathbb{P}_1(X_2 = 7) = Q(1, 2)Q(2, 7) + Q(1, 4)Q(4, 7) = \frac{1}{4} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{3}{4} = \frac{5}{16}.$$

$\boxed{4}$

2. (1) $\mathbb{E}(X_t - X_s|\mathcal{F}_s) = \mathbb{E}(N_t - N_s - \lambda(t-s)|\mathcal{F}_s) = \mathbb{E}(N_t - N_s|\mathcal{F}_s) - \lambda(t-s) = \lambda(t-s) - \lambda(t-s) = 0$. $\boxed{\text{seen} \Downarrow}$
Therefore $\mathbb{E}(X_t|\mathcal{F}_s) = X_s$. $\boxed{3}$

(2) By definition,

$$\mathbb{E}[e^{-uN_t}] = \sum_{n=0}^{\infty} e^{-un}\mathbb{P}(N_t = n) = e^{-\lambda t}\sum_{n=0}^{\infty}\frac{(\lambda t e^{-u})^n}{n!} = \exp\{\lambda t[e^{-u} - 1]\}.$$

Therefore,

$$\phi(u) = \mathbb{E}[e^{-uX_t}] = \exp\{\lambda t[u + e^{-u} - 1]\}.$$

$\boxed{3}$

(3) The Laplace transform of S_t is $\boxed{\text{part seen} \Downarrow}$

$$\begin{aligned}
\mathbb{E}(e^{-uS_t}) &= \mathbb{E}(e^{-u\sum_{i=1}^{N_t} Y_i}) \\
&= \sum_{n=0}^{\infty}\mathbb{E}(e^{-u\sum_{i=1}^{n} Y_i})\mathbb{P}(N_t = n) \\
&= \sum_{n=0}^{\infty}\prod_{i=1}^{n}\mathbb{E}(e^{-uY_i})\mathbb{P}(N_t = n) \quad \textit{(the r.v } Y_i \textit{ are iid)} \\
&= \sum_{n=0}^{\infty} G(u)^n\mathbb{P}(N_t = n) \\
&= e^{-\lambda t}\sum_{n=0}^{\infty} G(u)^n\frac{(\lambda t)^n}{n!} \\
&= e^{-\lambda t}e^{\lambda t G(u)}
\end{aligned}$$

$\boxed{3}$

Since $\mathbb{E}(S_t) = \psi'(0)$ and $\mathbb{E}(Y_t) = G'(0)$, we have $\boxed{\text{unseen} \Downarrow}$

$$\begin{aligned}
\mathbb{E}(S_t) &= \psi'(0) \\
&= \lambda t G'(0)\underbrace{\exp(-\lambda t + \lambda t G(0))}_{=1} \\
&= \lambda t\mathbb{E}(Y_t)
\end{aligned}$$

$\boxed{3}$

Also, we have $\mathbb{E}(Y_t^2) = G''(0)$

$$\begin{aligned}
\mathrm{Var(S_t)} &= \mathbb{E}(S_t^2) - \mathbb{E}(S_t)^2 \\
&= [\lambda t G''(0) - \lambda t G'(0)^2]\exp(-\lambda t + \lambda t G(0)) \\
&= \lambda t G''(0) - \lambda t G'(0)^2 \\
&= \lambda t\mathbb{E}(Y_t^2) - \lambda t\mathbb{E}(Y_t)^2 \\
&= \lambda t\mathrm{Var(Y_t)}.
\end{aligned}$$

$\boxed{3}$

3. (1) Let $\{P_t\}_{t\geq 0}$ denote the standard stochastic semigroup associated with the Markov chain. The generator is defined as

$$G = \lim_{h\to 0}\frac{1}{h}(P_h - I).$$

$\boxed{3}$

(2) From the lectures, we know that the transition probabilities , denoted by p_{ij} for $i, j \in E$, of the corresponding jump chain are given by $p_{ij} = g_{ij}/(-g_{ii})$ for $i \neq j$ if $-g_{ii} > 0$. If $-g_{ii} = 0$, then the state i is absorbing. Noting that the row elements in the transition have to sum to one we get

$$\mathbf{P} = \begin{pmatrix} 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 1 & 0 \\ 2/5 & 0 & 0 & 3/5 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

$\boxed{4}$

(3) (i) The generator is

$$\mathbf{P} = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & 0 & \cdots \\ \mu & -(2\lambda+\mu) & 2\lambda & 0 & 0 & \cdots \\ 0 & 4\mu & -(3\lambda+4\mu) & 3\lambda & 0 & \cdots \\ 0 & 0 & 9\mu & -(4\lambda+9\mu) & 4\lambda & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

$\boxed{4}$

(ii) As in the lectures, we use the result that $\pi G = 0$, then

$$\begin{aligned} -\lambda_0\pi_0 + \mu_1\pi_1 &= 0 \\ -\lambda_{n-1}\pi_{n-1} - (\lambda_n + \mu_n)\pi_n + \mu_{n+1}\pi_{n+1} &= 0 \quad n \geq 1. \end{aligned}$$

This leads to

$$\pi_n = \frac{\lambda_0 \times \cdots \times \lambda_{n-1}}{\mu_1 \times \cdots \times \mu_n}\pi_0$$

for any $n \in \mathbb{N}$. Such a vector π is stationary distribution if and only if $\sum_n \pi_n = 1$; that is

$$\sum_{n=0}^{\infty}\frac{\lambda_0 \times \cdots \times \lambda_{n-1}}{\mu_1 \times \cdots \times \mu_n} < \infty,$$

with the term $(n = 0)$ defined to be 1 (i.e. $\lambda_0\lambda_{-1}/\mu_1\mu_0 = 1$). Given this condition, it follows that

$$\pi_0 = \left(\sum_{n=0}^{\infty}\frac{\lambda_0 \times \cdots \times \lambda_{n-1}}{\mu_1 \times \cdots \times \mu_n}\right)^{-1}.$$

Here, we get

$$\pi_0 = \left(\sum_{n=0}^{\infty}\frac{\lambda_0 \times \cdots \times \lambda_{n-1}}{\mu_1 \times \cdots \times \mu_n}\right)^{-1} = \left(\sum_{n=0}^{\infty}\frac{\lambda^n n!}{\mu^n (n!)^2}\right)^{-1} = \left(\sum_{n=0}^{\infty}\frac{\lambda^n}{\mu^n n!}\right)^{-1} = e^{\lambda/\mu},$$

and for $n \in \mathbb{N}$, we have

$$\pi_n = \left(\frac{\lambda}{\mu}\right)^n \frac{1}{n!}e^{-\lambda/\mu}.$$

$\boxed{4}$

4. (1) For $0 \le s < t$, from the lectures, we have $\boxed{\text{seen} \Downarrow}$

$$B_t - B_s \sim N(0, t-s).$$

$\boxed{2}$

(2) For $s < t$, and using the fact that $B_t - B_s$ and B_s are independent, we have $\boxed{\text{seen} \Downarrow}$

$$\begin{aligned}\mathrm{Cov}(B_t, B_s) &= \mathbb{E}(B_t B_s) \\ &= \mathbb{E}((B_t - B_s)B_s) + \mathbb{E}(B_s^2) \\ &= 0 + s = s\end{aligned}$$

For $t < s$, and using the fact that $B_s - B_t$ and B_t are independent, we have $\boxed{3}$

$$\begin{aligned}\mathrm{Cov}(B_t, B_s) &= \mathbb{E}(B_t B_s) \\ &= \mathbb{E}((B_s - B_t)B_t) + \mathbb{E}(B_t^2) \\ &= 0 + t = t\end{aligned}$$

Therefore ,

$$\mathrm{Cov}(B_t, B_s) = \min(t, s).$$

$\boxed{3}$

$\boxed{\text{part seen} \Downarrow}$

(3) The stochastic process W is a Gaussian process with zero mean $\mathbb{E}(W_t) = a\mathbb{E}(B_{t/a^2}) = 0$, and

$$\mathrm{Cov}(W_t, W_s) = a^2 \mathbb{E}(B_{t/a^2} B_{s/a^2}) = a^2 \min(t/a^2, s/a^2) = \min(t, s).$$

The process W has also continuous sample paths. Therefore, from the lectures, W is a standard Brownian motion.

$\boxed{4}$

(4) No, since for $0 \le s \le t$, $\boxed{\text{unseen} \Downarrow}$

$$\begin{aligned}\mathrm{Var}(X_t - X_s) &= \mathrm{Var}(\sqrt{t}Z_t - \sqrt{s}Z) \\ &= (\sqrt{t} - \sqrt{s})^2 \mathrm{Var}(Z) \\ &= (\sqrt{t} - \sqrt{s})^2 \neq t - s.\end{aligned}$$

$\boxed{3}$

Mastery Question $\boxed{\text{part seen} \Downarrow}$

5. (1) The process X is the sum of two Gaussian processes. Therefore it is a Gaussian process with continuous sample paths. We have

$$\mathbb{E}(X_t) = \rho\mathbb{E}(W_t) + \sqrt{1-\rho^2}\mathbb{E}(\widetilde{W}_t) = 0.$$

Moreover,

$$\begin{aligned}
\mathrm{Cov}(X_t, X_s) &= \mathbb{E}(X_t X_s) \\
&= \mathbb{E}\left[(\rho W_t + \sqrt{1-\rho^2}\widetilde{W}_t)(\rho W_s + \sqrt{1-\rho^2}\widetilde{W}_s)\right] \\
&= \mathbb{E}\left[\rho^2 W_t W_s + \rho\sqrt{1-\rho^2} W_t \widetilde{W}_s + \rho\sqrt{1-\rho^2} W_s \widetilde{W}_t + (1-\rho^2)\widetilde{W}_t \widetilde{W}_s\right] \\
&= \rho^2(\min(s,t)) + 0 + 0 + (1-\rho^2)(\min(s,t)) \\
&= \min(s,t)
\end{aligned}$$

Therefore, X is a Brownian motion. $\boxed{5}$

(2) $\boxed{\text{unseen} \Downarrow}$

$$\begin{aligned}
\mathbb{E}(X_t) &= \mathbb{E}(e^{\sigma W_t - \sigma^2 t/2}) \\
&= e^{-\sigma^2 t/2}\mathbb{E}(e^{\sigma W_t}) \\
&= e^{-\sigma^2 t/2}\int_{\mathbb{R}} e^{\sigma x} \cdot \frac{1}{\sqrt{2\pi t}} e^{\frac{-x^2}{2t}} dx \\
&= e^{-\sigma^2 t/2}\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-\sigma\sqrt{t})^2} e^{\sigma^2 t/2} dz \\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-\frac{1}{2}(z-\sigma\sqrt{t})^2} dz = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-z^2/2} dz \\
&= 1
\end{aligned}$$

$\boxed{5}$