Final

EE3-09

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2012

EEE/ISE PART III/IV: MEng, BEng and ACGI

CONTROL ENGINEERING

Friday, 11 May 2:30 pm

Time allowed: 3:00 hours

There are SIX questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

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CONTROL ENGINEERING

1. Consider the linear, continuous-time, system described by the equation

$$\dot{x} = rx,$$

with $x(t) \in \mathbb{R}$ and $r > 0$, and the nonlinear system

$$\dot{\xi} = r\xi\left(1 - \frac{\xi}{K}\right),$$

with $\xi(t) \in \mathbb{R}$, and $K > 0$.

The linear system models the evolution of a population when there are unlimited resources, whereas the nonlinear system models the evolution of a population which grows when *small* and decreases when *large*, that is, when the resources are limited.

a) Let $x(0) = x_0 > 0$. Compute the solution $x(t)$, for all $t \geq 0$, of the linear system. [2 marks]

b) Let $\xi(0) = \xi_0 > 0$. Show that the solution of the nonlinear system is given by the equation

$$\xi(t) = \frac{K}{K + \xi_0(e^{rt} - 1)}\xi_0 e^{rt},$$

for all $t \geq 0$. [4 marks]

c) Show that for $0 < x_0 = \xi_0 \ll K$, and $t \geq 0$ and sufficiently small, $x(t) \approx \xi(t)$. [4 marks]

d) Assume $x_0 > 0$ and $\xi_0 > 0$. Evaluate $\lim_{t \to \infty} x(t)$ and $\lim_{t \to \infty} \xi(t)$. [2 marks]

e) Exploiting the results of parts c) and d), explain why the solution of the linear equation provides a good approximation of the solution of the nonlinear equation, for $t \geq 0$ and sufficiently small, and $0 < x_0 = \xi_0 \ll K$.
Similarly, explain why the solution of the linear equation does not provide a good approximation of the solution of the nonlinear equation for $t \geq 0$ and *large*. [2 marks]

f) Compute the equilibrium points of the nonlinear system and determine their stability properties. [4 marks]

g) Show that, for $x_0 = \xi_0 > 0$,

$$\lim_{K \to \infty} (x(t) - \xi(t)) = 0,$$

for all finite $t > 0$. [2 marks]

2. The equation describing the dynamic behaviour of a pendulum on a moving cart, in which the input signal is the acceleration of the cart, is

$$\ddot{\phi} = a\sin\phi - b\cos\phi u,$$

where $\phi(t)$ is an angle in radians, which is zero when the pendulum is vertical and upright, $a > 0$ and $b > 0$ are physical parameters, and $u(t)$ is the input signal.

a) Write a state-space realization of the system with state $(x_1, x_2) = (\phi, \dot{\phi})$ and control input u. [2 marks]

b) Assume that u is constant. Determine the equilibria of the system. [6 marks]

c) Compute the matrices A and B of the system linearized around an equilibrium point $(\bar{x_1}, 0)$. Note that the matrices A and B are functions of $\bar{x_1}$. [4 marks]

d) Study, using the principle of stability in the first approximation, the stability properties of all equilibrium points of the system. [4 marks]

e) Study the controllability properties of the linearized system as a function of $\bar{x_1}$. [4 marks]

3. The simplified model of a system composed of an electromagnet and an iron ball both moving in a vertical plane is described by the equations

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -g + k\frac{x_3^2}{x_1^2}, \\ \dot{x}_3 &= -x_3 + u, \end{aligned}$$

where $x_1(t)$ is the distance between the iron ball and the magnet, $x_2(t)$ is the relative velocity, $x_3(t)$ is the current in the winding of the electromagnet, and $u(t)$ is the voltage applied to the electromagnet. The constant $k > 0$ describes the strenght of the magnetic force exerted by the magnet on the ball and the constant $g > 0$ describes the effect of gravity.

a) Assume $u = \bar{u}$, with $\bar{u}$ constant. Show that for any $\bar{u} \neq 0$ the system has two equilibria. Compute all equilibria of the system as a function of $\bar{u}$. [4 marks]

b) Suppose that the only measured variable is $y = x_1 + \beta x_3$, where β is a constant. Compute the linearized system around the equilibrium points with the x_1 component positive.
(Note that the matrix A depends upon $\bar{u}$.) [6 marks]

c) Assume $\beta = 0$. Study the observability properties of the system determined in part b) as a function of $\bar{u}$. [4 marks]

d) Let $k = g = \bar{u} = 1$. Study the observability and detectability properties of the system determined in part b) as a function of β. [6 marks]

4. Consider a linear, discrete-time, system described by the equation

$$x^+ = Ax + Bu = \begin{bmatrix} 1 & 0 \\ 2 & a \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u,$$

where a is a constant parameter.

a) Study the reachability, controllability and stabilizability properties of the system as a function of a. [6 marks]

b) Let $a = 0$.

i) Show that it is not possible to steer the state of the system from $x(0) = [1\ 1]'$ to $x_f = [2\ 2]'$ in one step. [2 marks]

ii) Show that it is possible to steer the state of the system from $x(0) = [1\ 1]'$ to $x_f = [2\ 2]'$ in two steps, and compute an input sequence which steers $x(0)$ to x_f in two steps. [4 marks]

iii) Design a state feedback control law $u = Kx$ placing all closed-loop eigenvalues at $\frac{1}{2}$. [4 marks]

iv) Suppose that the amplifier of the actuator of the system undergoes a fault, hence provides a control signal which is only a fration of u, that is, $u = \alpha Kx$, with $\alpha \in (0, 1]$, and K as computed in part b.iii). Study the stability properties of the closed-loop system as a function of α. [4 marks]

5. Consider the problem of controlling a mechanical system described by the equation

$$\ddot{q} + D\dot{q} = u + w,$$

where $q(t) \in \mathbb{R}$ is the position of the mass, $u(t) \in \mathbb{R}$ is the control force, $w(t) \in \mathbb{R}$ is the disturbance force, and $D > 0$.

The controller has a PI structure, that is, it has the transfer function $C(s) = K_P + \dfrac{K_I}{s}$, where K_P is the proportional gain, and K_I is the integral gain. The controller has the state-space realization

$$\dot{\xi} = v, \qquad \eta = K_I\xi + K_P v,$$

with state $\xi(t) \in \mathbb{R}$, input $v(t) \in \mathbb{R}$, and output $\eta(t) \in \mathbb{R}$.

a) Write a state-space realization of the equation describing the mechanical system with state $(x_1, x_2) = (q, \dot{q})$, control input u, and disturbance input w. [2 marks]

b) The mechanical system and the controller are interconnected by means of the equations $u = -\eta$ and $v = x_1$. Write the equations of the closed-loop system, with state (x, ξ), input w and output x_1. [4 marks]

c) Show that the closed-loop system is observable for any $K_P > 0$ and $K_I > 0$. [4 marks]

d) Determine values of $K_P > 0$ and $K_I > 0$ such that the closed-loop system is asymptotically stable. Is it possible to arbitrarily assign the eigenvalues of the closed-loop system selecting K_P and K_I? [4 marks]

e) Suppose the disturbance w is constant, that is, it satisfies the linear differential equation $\dot{w} = 0$. Write the equations of the overall system with state (w, x, ξ). Compute the equilibrium points of such system. Show that all equilibria are of the form $(\bar{w}, \bar{x}, \bar{\xi})$ with $\bar{x} = [0, \bar{x}_2]'$. Hence, argue that the constant disturbance does not affect the *asymptotic* position of the mechanical system.
(Hint: assume that all trajectories of the overall system converge to an equilibrium.)
[6 marks]

6. Consider the statement in the box.

> A linear, time-invariant, system described by the equations
>
> $$\dot{x} = Ax + Rw, \qquad z = Cx,$$
>
> where $x(t) \in I\!R^n$ is the state, $w(t) \in I\!R$ is a disturbance, and $z(t) \in I\!R$ is a performance variable, is said to have L_2-gain smaller or equal to $\gamma > 0$ if there exists a matrix $P = P' > 0$ such that
>
> $$A'P + PA + \frac{PRR'P}{4\gamma^2} + C'C = 0. \qquad (\star)$$

Consider the system

$$\dot{x} = Ax + Bu + Rw = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u + \begin{bmatrix} 2 \\ 1 \end{bmatrix} w,$$

$$z = Cx = \begin{bmatrix} 1 & 1 \end{bmatrix} x.$$

a) Let $u = 0$. Show that, for any $\gamma > 0$, the system does not have L_2-gain less than or equal to γ.
(Hint: show that there is no matrix $P = P' > 0$ such that condition $(\star)$ holds. Recall that a matrix

$$P = P' = \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}$$

is positive definite if, and only if, $P_{11} > 0$ and $P_{11}P_{22} - P_{12}^2 > 0$.) [6 marks]

b) Consider the problem of designing a state feedback control law $u = Kx$ such that the closed-loop system has L_2-gain smaller or equal than some $\gamma > 0$. This problem can be solved in steps.

i) Find a matrix K such that the eignevalues of the closed-loop system, that is, of the matrix $A + BK$, are both equal to -1. [4 marks]

ii) Write the equations of the closed-loop system, with state x, input w, and performance variable z. [2 marks]

iii) Using the statement in the box, and the equations in part b.ii), show that there exists a value $\gamma^\star > 0$ such that the system in part b.ii) has L_2-gain smaller or equal to $\gamma^\star$.
(Hint: use a matrix P with $P_{12} = 0$.)

[8 marks]

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Control engineering exam paper - Model answers

Question 1

a) The solution $x(t)$ of the linear differential equation is

$$x(t) = x_x e^{rt},$$

for all $t \geq 0$.

b) To begin with note that

$$\xi(t)_{t=0} = \left.\frac{K}{K + \xi_0(e^{rt} - 1)}\xi_0 e^{rt}\right|_{t=0} = \xi_0.$$

Note now that

$$\frac{d}{dt}\frac{K}{K + \xi_0(e^{rt} - 1)}\xi_0 e^{rt} = r\frac{K}{K + \xi_0(e^{rt} - 1)}\xi_0 e^{rt} - r\frac{K}{(K + \xi_0(e^{rt} - 1))^2}(\xi_0 e^{rt})^2 = r\xi(t) - \frac{r}{K}\xi^2(t),$$

which shows that the given function of time is indeed a solution of the differential equation.

c) For $t > 0$ and sufficiently small

$$x(t) \approx x(0)(1 + rt).$$

Similarly, for $t > 0$ and sufficiently small, and $\xi(0) \ll K$,

$$\xi(t) =\approx \xi(0)(1 + r\frac{K - \xi(0)}{K}t) =\approx \xi(0)(1 + rt),$$

hence the claim.

d) Since $r > 0$ and $x(0) > 0$, $\lim_{t\to\infty} x(t) = +\infty$. On the other hand, for all $\xi(0) > 0$,

$$\lim_{t\to\infty} \xi(t) = K.$$

e) The solution of the linear equation is approximately the same as the solution of the nonlinear equation, under the stated conditions. Hence $x(t)$ can approximate $\xi(t)$ for $t \geq 0$ and small. For $t \geq 0$ and large the solution of the nonlinear equation differs substantially from the solution of the linear equation: the former is bounded and converges to K, whereas the latter is unbounded.

f) The equilibrium points of the nonlinear system are the solution of the equation

$$0 = r\xi\left(1 - \frac{\xi}{K}\right),$$

that is $\xi = 0$ and $\xi = K$. A simple plot of $\dot{\xi}$ as a function of ξ reveals that $\xi = 0$ is an unstable equilibrium, whereas $\xi = K$ is (locally) asymptotically stable. (Similar conclusions can be obtained computing the linearization of the nonlinear system around the two equilibrium points.)

g) Note that, for $x(0) = \xi(0)$,

$$x(t) - \xi(t) = \frac{e^{rt} - e^{2rt}}{K + x(0)e^{rt} - x(0)}x(0)^2.$$

Hence, for any finite $t > 0$, $\lim_{K\to\infty}(x(t) - \xi(t)) = 0$.

Question 2

a) The state space representation of the system is

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = a \sin x_1 - b \cos x_1 u.$$

b) The equilibrium points of the system are the given by the points $(x_1, x_2) = (\bar{x}_1, 0)$, with $\bar{x}_1$ solutions of the equation

$$0 = a \sin \bar{x}_1 - b \cos \bar{x}_1 u,$$

that is

$$\tan \bar{x}_1 = \frac{b}{a} u.$$

This equation has, for any value of u infinitely many solutions, given by

$$\bar{x}_1 = \arctan\left(\frac{bu}{a}\right) + k\pi,$$

with k integer.

c) The linearization of the system is described by the matrices

$$A(\bar{x}_1) = \begin{bmatrix} 0 & 1 \\ a \cos \bar{x}_1 + b \sin \bar{x}_1 u & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ -b \cos \bar{x}_1 \end{bmatrix}$$

d) Note that

$$A\left(\frac{bu}{a}\right) = \begin{bmatrix} 0 & 1 \\ \sqrt{a^2 + b^2u^2} & 0 \end{bmatrix} \qquad A\left(\frac{bu}{a} + \pi\right) = \begin{bmatrix} 0 & 1 \\ -\sqrt{a^2 + b^2u^2} & 0 \end{bmatrix}.$$

Hence, by the principle of stability in the first approximation, the equilibrium $(\bar{x}_1, 0) = \left(\frac{bu}{a}, 0\right)$ is unstable, whereas it is not possible to decide the stability properties of the equilibrium $(\bar{x}_1, 0) = \left(\frac{bu}{a} + \pi, 0\right)$. The same conclusions hold for the equilibria

$$(\bar{x}_1, 0) = \left(\frac{bu}{a} + 2k\pi, 0\right) \qquad (\bar{x}_1, 0) = \left(\frac{bu}{a} + (2k+1)\pi, 0\right)$$

e) The reachability matrix is

$$\mathcal{R} = \begin{bmatrix} B(\bar{x}_1), A(\bar{x}_1)B(\bar{x}_1) \end{bmatrix} = \begin{bmatrix} 0 & -b \cos \bar{x}_1 \\ -b \cos \bar{x}_1 & 0 \end{bmatrix}.$$

Note that $\det \mathcal{C} = -b^2 \cos^2 \bar{x}_1$. Hence the linear approximation of the system is controllable for all $\bar{x}_1 \neq \frac{\pi}{2} + k\pi$.

Question 3

a) The equilibria of the system are the solution of the equations

$$x_2 = 0 \qquad -g + k\frac{x_3^2}{x_1^2} = 0 \qquad -x_3 + u = 0$$

For $\bar{u} = 0$ the system does not have any equilibrium. For $\bar{u} \neq 0$ the system has two equilibria, that is

$$\left(\pm\sqrt{\frac{k}{g}}|\bar{u}|, 0, \bar{u}\right).$$

b) The linearized system is described by the matrices

$$A(\bar{u}) = \begin{bmatrix} 0 & 1 & 0 \\ -2\frac{g}{|\bar{u}|}\sqrt{\frac{g}{k}} & 0 & 2\frac{g}{|\bar{u}|} \\ 0 & 0 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & \beta \end{bmatrix}$$

c) For $\beta = 0$ the observability matrix of the linearized system is

$$\mathcal{O} = \begin{bmatrix} C \\ CA(\bar{u}) \\ CA^2(\bar{u}) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \star & 0 & 2\frac{g}{|\bar{u}|} \end{bmatrix},$$

where $\star$ is a function of k, g and $\bar{u}$, hence the system is observable for every $\bar{u} \neq 0$.

d) For $k = g = u = 1$ the observability matrix of the linearized system is

$$\mathcal{O} = \begin{bmatrix} C \\ CA(\bar{u}) \\ CA^2(\bar{u}) \end{bmatrix} = \begin{bmatrix} 1 & 0 & \beta \\ 0 & 1 & -\beta \\ -2 & 0 & 2+\beta \end{bmatrix}.$$

The determinant of the observability matrix is $2 + 3\beta$, hence the system is observable for $\beta \neq -2/3$. For $\beta = -2/3$ the rank of the observability matrix is two, hence there is one unobservable mode. Theobservability pencil is

$$\begin{bmatrix} C \\ sI - A \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2/3 \\ s & -1 & 0 \\ 2 & s & -2 \\ 0 & 0 & s+1 \end{bmatrix}$$

and this loses rank for $s = -1$. The system is therefore detectable.

Question 4

a) The reachability matrix of the system is

$$\mathcal{R} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2+a \end{bmatrix}$$

Note that $\det\mathcal{R} = 1 + a$, hence the system is reachable for all $a \neq -1$. For $a = -1$ the eigenvalues of the matrix A are -1 and 1, hence the system is not controllable, nor stabilizable.

b)
- Note that

  $$x(1) = Ax(0) + Bu(0) = \begin{bmatrix} 1 + u(0) \\ 2 + u(0) \end{bmatrix},$$

  hence there is no selection of $u(0)$ such that $x_f = x(1)$.
- Note that

  $$x(2) = Ax(1) + Bu(1) = \begin{bmatrix} 1 + u(0) + u(1) \\ 2 + 2u(0) + u(1) \end{bmatrix},$$

  hence the selection $u(0) = -1$ and $u(1) = 2$ is such that $x_f = x(2)$.
- Let $K = [k_1 \ k_2]$ and note that

  $$A + BK = \begin{bmatrix} 1 + k_1 & k_2 \\ 2 + k_1 & k_2 \end{bmatrix}$$

  and $\det(\lambda I - (A + BK)) = \lambda^2 - (k_1 + k_2 + 1)\lambda - k_2$. Hence, the selection $k_1 = 1/4$ and $k_2 = -1/4$ assigns the eigenvalues of $A + BK$ as requested.
- Selecting $u = \alpha K x$ yields

  $$x^+ = (A + \alpha BK)x = \begin{bmatrix} 1 + \frac{\alpha}{4} & -\frac{\alpha}{4} \\ 2 + \frac{\alpha}{4} & -\frac{\alpha}{4} \end{bmatrix} x$$

  The characteristic polynomial of $A + \alpha BK$ is $\lambda^2 - \lambda + \frac{\alpha}{4}$. This polynomial has all roots inside the unity disk for $\alpha \in (0, 4]$. Hence the faulty actuator does not affect the stability of the closed-loop system.

Question 5

a) The state space realization is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -D \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (u + w)$$

b) The equation describing the closed-loop system is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -K_P & -D & -K_I \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \xi \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} w$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \xi \end{bmatrix}$$

c) The observability matrix is

$$\mathcal{O} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -K_P & -D & -K_I \end{bmatrix}$$

hence the system is observable for all $K_P > 0$ and $K_I > 0$.

d) The characteristic polynomial of the closed-loop matrix is

$$\lambda^3 + D\lambda^2 + K_P\lambda + K_I,$$

hence it is possible to obtain an asymptotically stable closed-loop system selecting the design parameters K_P and K_I. For example, selecting $K_P = D^2/3$ and $K_I = D^3/27$ yields a closed-loop system with all eigenvalues at $-D/3$. Note however that it is not possible to assign the eigenvalues of the closed-loop system using K_P and K_I.

e) The equations describing the overall system are

$$\dot{w} = 0 \qquad \dot{x}_1 = x_2 \qquad \dot{x}_2 = -K_P x_1 - D x_2 - K_i \xi + w \qquad \dot{\xi} = x_1.$$

The system has infinitely many equilibria given by

$$(w, x_1, x_2, \xi) = (\bar{w}, 0, 0, \bar{w}/K_I)$$

with $\bar{w} \in I\!R$. Since all trajectories converge to an equilibrium (as stated in part e)), then

$$\lim_{t \to \infty} x_1(t) = 0,$$

that is the constant disturbance does not affect the asymptotic value of x_1.

Question 6

a) Note that for the considered system

$$A'P + PA + \frac{PRR'P}{4\gamma^2} + C'C = \begin{bmatrix} \star & \star \\ \star & 2P_{22} + \frac{P_{22}^2}{4\gamma^2} + 1 \end{bmatrix},$$

where the $\star$'s indicate functions of P_{11}, P_{12} and P_{22}. Note that the $(2,2)$ element of the above matrix is positive (since $P_{22} > 0$), hence the matrix cannot be negative semi-definite.

b) Let $K = [K_1 \; K_2]$ and note that

$$A + BK = \begin{bmatrix} K_1 + 1 & K_2 \\ 2K_1^1 + 2K_2 & \end{bmatrix}.$$

The characteristic polynomial of $A + BK$ is

$$s^2 - (K_1 + 2K_2)s + K_1 - 2K_2 - 1,$$

and the selection $K_1 = 0$ and $K_2 = -1$ yields the requested closed-loop eigenvalues.

c) The equations of the closed-loop system are

$$\dot{x} = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w \qquad z = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

d) The statement in the box, with $P_{12} = 0$, yields

$$(A + BK)'P + P(A + BK) + \frac{PRR'P}{4\gamma^2} + C'C = \begin{bmatrix} 1 - 2P_{11} & 1 - P_{11} \\ 1 - P_{11} & 1 - 2P_{22} + \frac{P_{22}^2}{4\gamma^2} \end{bmatrix}.$$

Selecting $P_{11} = 1$ and $P_{22} = 4\gamma^2$ (which is the value of P_{22} which minimizes the $(2,2)$ element of the above matrix) yields $P > 0$ and

$$(A + BK)'P + P(A + BK) + \frac{PRR'P}{4\gamma^2} + C'C = \begin{bmatrix} -1 & 0 \\ 0 & 1 - 4\gamma^2 \end{bmatrix},$$

which is negative semi-definite for all $\gamma \geq 1/2 = \gamma^\star$.