EE4-25 ✓
EE9-CS1-2 ✓

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2014

MSc and EEE/EIE PART IV: MEng and ACGI

Corrected Copy

DESIGN OF LINEAR MULTIVARIABLE CONTROL SYSTEMS

11:35
Q5 (b)
change L → F

Monday, 19 May 10:00 am

Time allowed: 3:00 hours

There are SIX questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s) :	I.M. Jaimoukha
	Second Marker(s) :	E.C. Kerrigan

1. a) Let the transfer matrix $G(s)$ have a state space realisation

$$G(s) \stackrel{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \stackrel{s}{=} \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ 0 & A_{22} & B_2 \\ \hline 0 & C_2 & 0 \end{array}\right].$$

Assume that A_{22} is stable.

i) Show that the given realisation for $G(s)$ is unobservable. [2]

ii) What are the unobservable modes that can be deduced from the structure of the realisation for $G(s)$? [2]

iii) Give a necessary and sufficient condition for which the given realisation is detectable. [2]

iv) Draw a diagram involving two subsystems of $G(s)$ illustrating the unobservable part. [2]

b) Let the transfer matrix $G(s)$ have a state space realisation

$$G(s) \stackrel{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \stackrel{s}{=} \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & 0 \end{array}\right].$$

Assume that there exists a matrix X such that

$$A_{22}X - XA_{11} - XA_{12}X + A_{21} = 0, \qquad B_2 - XB_1 = 0, \qquad A_{11} + A_{12}X \text{ is stable.}$$

i) By applying a similarity transformation of the form

$$T = \begin{bmatrix} I & 0 \\ \star & I \end{bmatrix}$$

or otherwise, show that the realisation for $G(s)$ is uncontrollable. [3]

ii) What are the uncontrollable modes that can be deduced from the previous analysis? [3]

iii) Give a necessary and sufficient condition for which the given realisation is stabilisable. [3]

iv) Draw a diagram involving two subsystems of $G(s)$ illustrating the uncontrollable part. [3]

2. Consider the $\mathscr{H}_\infty$ filter for estimating $C_z x$ shown in Figure 2.

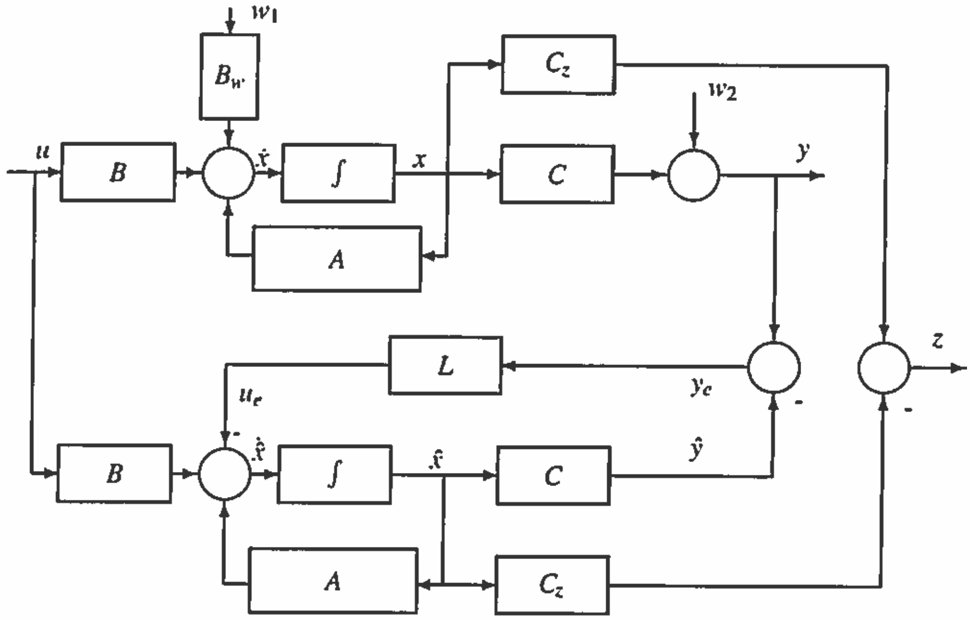

Figure 2

Let $T_{zw}(s)$ denote the transfer matrix from $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ to z. A stabilizing filter gain matrix L is to be designed such that, for $\gamma > 0$, $\|T_{zw}\|_\infty < \gamma$.

a) Derive a state-space realisation for the transfer matrix $T_{zw}(s)$. [5]

b) Use the Bounded Real Lemma, stated in Question 5 below, to derive necessary and sufficient conditions for $\|T_{zw}\|_\infty < \gamma$. These conditions should be in the form of matrix inequality constraints. [5]

c) Suggest a transformation of variables that turns the matrix inequality derived in Part (b) above into a linear matrix inequality. [5]

d) Suppose that $A = -1, B_w = C = 1, C_z = \sqrt{2}$. Find the optimal γ and the corresponding filter gain L. [5]

3. Consider the feedback configuration in Figure 3. Here, $G(s)$ is a plant model and $K(s)$ is a compensator. The signal $d(s)$ represent a disturbance signal. The design specifications are to synthesize a compensator $K(s)$ such that the feedback loop is internally stable and, for all real ω,

$$\begin{aligned}\|y(j\omega)\| &< |w_1(j\omega)^{-1}|\,\|d(j\omega)\| \\ \|u(j\omega)\| &< |w_2(j\omega)^{-1}|\,\|d(j\omega)\|\end{aligned}$$

where $w_1(s)$ and $w_2(s)$ are given filters.

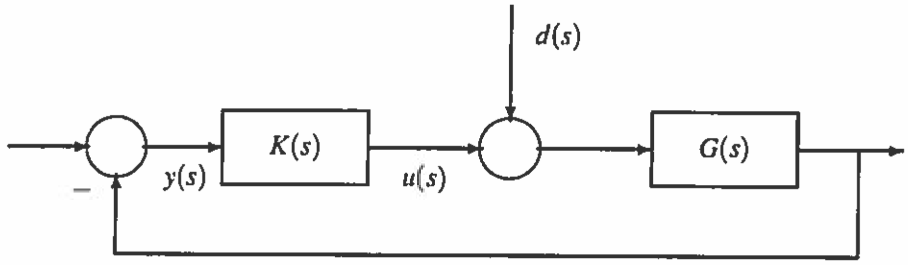

Figure 3

a) Derive $\mathscr{H}_\infty$–norm bounds that are sufficient to achieve the design specifications. [4]

b) Define cost signals $z_1(s)$ and $z_2(s)$ and draw a block diagram, of the same form as Figure 3, showing $z_1(s)$ and $z_2(s)$ as well as the weighting functions. [4]

c) Hence derive a generalised regulator formulation of the design problem that captures these sufficient conditions. [4]

d) State the small gain theorem concerning the internal stability of a feedback loop having a forward transfer matrix Δ and a feedback transfer matrix S. [4]

e) Assume that $K(s)$ achieves the design specifications in Parts (a)-(c) above. Suppose that the actual controller implemented is $K(s)+\Delta(s)$ where $\Delta(s)$ is a stable transfer matrix. Derive the maximal stability radius for $\|\Delta(j\omega)\|, \forall\omega$ that can be deduced from Parts (a)-(c) and the small gain theorem. [4]

4. Consider the regulator in Figure 4 for which it is assumed that the triple (A,B,C) is minimal and $x(0)=x_0$. A stabilizing state-feedback gain matrix F is to be designed such that the cost function

$$J = \left\| \begin{bmatrix} u \\ z \end{bmatrix} \right\|_2^2$$

is minimized. Take $w=0$ to begin with.

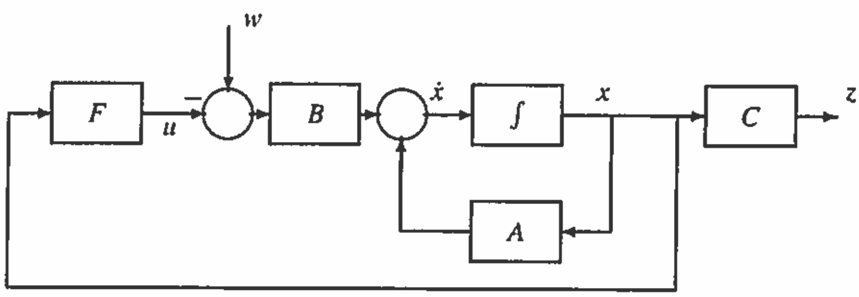

Figure 4

a) Define a suitable Lyapunov function in terms of $x(t)$ and a symmetric matrix P. [4]

b) Assuming the closed loop is asymptotically stable, obtain an expression for $\int_0^\infty \dot{V}(t)dt$. [4]

c) Find an expression for F that minimizes J. Give also the algebraic Riccati equation satisfied by P. What is the minimum value of J? [4]

d) Prove now that the closed loop system in Figure 4 is stable. State the assumption on P required to guarantee stability. [4]

e) Suppose now that $x_0 = 0$ and a signal w is applied such that that $\|w\|_2 < 1$. By evaluating an expression for $T_{zw}(s)$, the transfer matrix from w to z in Figure 4 above, give an upper bound on $\|z\|_2$.

You may wish to use

- the algebraic Riccati equation derived above and
- the Bounded Real Lemma (given in Question 5 below).

[4]

5. a) Consider a state–variable model

$$H(s) \stackrel{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

Suppose for a given $\gamma > 0$, there exists a $P = P'$ such that

$$\begin{bmatrix} A'P + PA + C'C & PB + C'D \\ B'P + D'C & D'D - \gamma^2 I \end{bmatrix} \prec 0$$
$$P \succ 0.$$

i) Prove that A is stable. [4]

ii) By defining suitable Lyapunov and cost functions and completing a square, prove that

$$\|H\|_\infty < \gamma.$$

[4]

b) Consider the static output feedback problem shown in Figure 5. Assume that the number of inputs is equal to the number of states and that the matrix B has full rank. Let $T_{zr}(s)$ denote the transfer matrix from r to z. An internally stabilizing gain matrix $\cancel{L}$ is to be designed such that, for a given $\gamma > 0$, $\|T_{zr}\|_\infty < \gamma$.
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i) Derive a state space realization for $T_{zr}(s)$. [4]

ii) Explain the significance of the requirement that $\|T_{zr}\|_\infty < \gamma$ in terms of performance specifications. [4]

iii) By using the answer to Part (a) above, or otherwise, derive sufficient conditions for the existence of a feasible F. Your conditions should be in the form of the existence of certain solutions to linear matrix inequalities. [4]

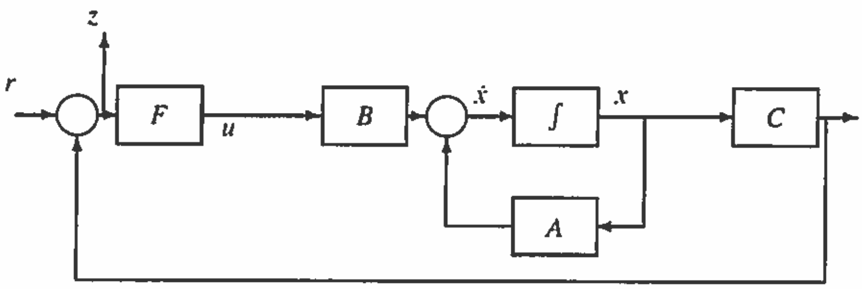

Figure 5

6. Consider the regulator shown in Figure 6 for which it is assumed that the triple (A, B, C) is minimal. Take $x(0) = 0$ to begin with.

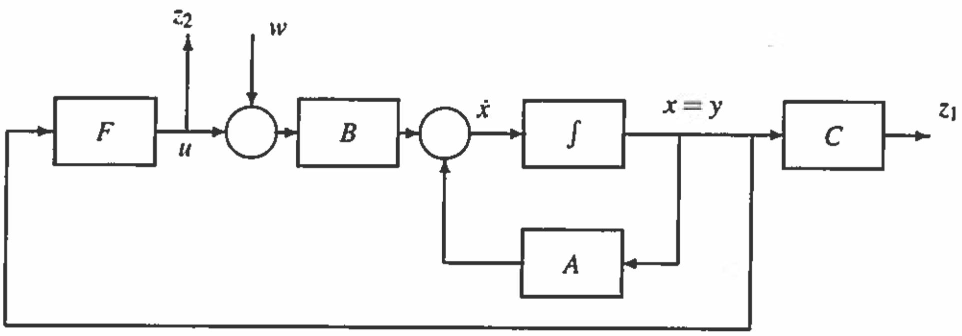

Figure 6

Let

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

and let H denote the transfer matrix from w to z. A stabilizing state-feedback gain matrix F is to be designed such that, for given $\gamma > 0$, $\|H\|_\infty < \gamma$.

a) Write down the generalized regulator system for this design problem. [5]

b) By using the Lyapunov function $V(t) = x(t)^T X x(t)$, where X is to be determined, derive sufficient conditions for the solution of the design problem. Your conditions should be in the form of the existence of a certain solution to an algebraic Riccati equation. It should also include an expression for F and an expression for the worst-case disturbance w. [10]

c) Suppose now that $w = 0$ and let $x(0) = x_0 \neq 0$. By letting $\gamma \to \infty$, write down the Riccati equation and cost function for the solution in Part (b). Comment on the answer. [5]

