

IMPERIAL COLLEGE LONDON

✓  
E3.10  
C1.5  
ISE3.7

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING  
EXAMINATIONS 2008

MSc and EEE/ISE PART III/IV: MEng, BEng and ACGI

**MATHEMATICS FOR SIGNALS AND SYSTEMS**

Wednesday, 30 April 10:00 am

**Corrected Copy**

Q.4

Time allowed: 3:00 hours

**There are FIVE questions on this paper.**

**Answer THREE questions.**

*All questions carry equal marks*

**Any special instructions for invigilators and information for candidates are on page 1.**

Examiners responsible      First Marker(s) :      M.M. Draief

Second Marker(s) :      D. Angeli



## MATHEMATICS FOR SIGNAL AND SYSTEMS

1. Consider the space  $\mathcal{M}_3(\mathbb{C})$  of three-by-three matrices with complex entries. Let  $A = (a_{ij})_{i,j=1,2,3} \in \mathcal{M}_3(\mathbb{C})$ . We define the following functions

- for  $k = 1, 2, 3$ , let  $l_k(A) = a_{k1} + a_{k2} + a_{k3}$ ,
- for  $k = 1, 2, 3$ , let  $c_k(A) = a_{1k} + a_{2k} + a_{3k}$ ,
- let  $\text{tr}(A) = a_{11} + a_{22} + a_{33}$ ,
- and let  $\text{anti}(A) = a_{31} + a_{22} + a_{13}$ ,

- a) Let  $\mathcal{M}$  be the set of matrices  $A \in \mathcal{M}_3(\mathbb{C})$ , such that  $l_k(A) = c_j(A)$  for  $k, j = 1, 2, 3$ . For  $A \in \mathcal{M}$  we define

$$\alpha(A) = l_1(A) = l_2(A) = l_3(A) = c_1(A) = c_2(A) = c_3(A)$$

the common value.

- (i) Give an example of a matrix  $A \in \mathcal{M}$  such that  $\alpha(A) = 0$  [ 2 ]  
 (ii) Give an example of a matrix  $A \in \mathcal{M}$  such that  $\alpha(A) = 1$  [ 2 ]  
 (iii) Show that  $\mathcal{M}$  is a subspace of  $\mathcal{M}_3(\mathbb{C})$  and that  $\alpha$  is a linear operator on  $\mathcal{M}$ . [ 2 ]

- (iv) Let  $J = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ . For  $\lambda \in \mathbb{C}$ , show that if  $A$  is such that

$$AJ = JA = \lambda J$$

then  $A$  is in  $\mathcal{M}$  and that if  $A \in \mathcal{M}$  then it satisfies  $AJ = JA = \lambda J$ . [ 6 ]

- b) Let  $\mathcal{M}^0 = \{A \in \mathcal{M}, \alpha(A) = \text{tr}(A) = \text{anti}(A)\}$ .

- (i) Prove that  $\mathcal{M}^0$  is a vector space. [ 2 ]  
 (ii) Let  $G = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ -1 & 2 & -1 \end{pmatrix}$  and  $G^T$  its transpose. Show that  $(G, G^T, J)$  is a basis of  $\mathcal{M}^0$ . What is the dimension of  $\mathcal{M}^0$ ? [ 6 ]

2. Let  $\mathbb{R}[X]$  be the vector space of all polynomials with real coefficients. We define the following function that, given two polynomials  $P, Q \in \mathbb{R}[X]$ , it associates the following number

$$\langle P, Q \rangle = \int_{-1}^1 \frac{P(t)Q(t)}{\sqrt{1-t^2}} dt.$$

- a) Show that the above function defines an inner product. [ 2 ]
- b) Prove that, for any positive integer  $n$ , there exists a unique polynomial  $T_n$  such that: for every  $\theta \in \mathbb{R}$ ,  $T_n(\cos(\theta)) = \cos(n\theta)$ . [ 2 ]

- c) Show that the polynomials  $T_n$ , known as Chebychev polynomials, satisfy, for  $n \geq 1$

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

[ 4 ]

- d) Prove that the sequence  $(T_n)_n$  is orthogonal and compute  $\langle T_n, T_n \rangle$ . [ 6 ]

- e) Show that  $T_n$  satisfies the following differential equation

$$(1-x)^2 y'' - xy' = -n^2 y$$

[ 6 ]

3. Consider the space  $\mathcal{M}_n(\mathbb{R})$  of  $n$ -by- $n$  matrices with real entries. We define the inner product  $\langle A, B \rangle = \text{tr}(A^T B)$ , where  $A^T$  is the transpose of  $A$ .

- a) Check that the above product is indeed an inner product and give the expression of the corresponding norm. [ 4 ]

- b) Let

$$\mathcal{S}_n = \{A \in \mathcal{M}_n(\mathbb{R}), A = A^T\}$$

the set of symmetric matrices and

$$\mathcal{A}_n = \{A \in \mathcal{M}_n(\mathbb{R}), A = -A^T\}$$

the set of anti-symmetric matrices.

- (i) Show that  $\mathcal{S}_n$  and  $\mathcal{A}_n$  are two vector spaces and give their dimensions.

[ 4 ]

- (ii) For  $M \in \mathcal{M}_n(\mathbb{R})$  show that  $M + M^T$  is an element of  $\mathcal{S}_n$  and  $M - M^T$  is an element of  $\mathcal{A}_n$ . [ 3 ]

- (iii) Show that  $\mathcal{S}_n$  is orthogonal to  $\mathcal{A}_n$  and that any matrix  $M \in \mathcal{M}_n(\mathbb{R})$  can be decomposed in a unique way as  $M = M_S + M_A$  where  $M_S \in \mathcal{S}_n$  and  $M_A \in \mathcal{A}_n$ .

[ 6 ]

- (iv) Determine the orthogonal projection on  $\mathcal{S}_n$  and  $\mathcal{A}_n$ . [ 3 ]

4. Let  $H$  be Hilbert (vector) space with the inner product  $(\cdot, \cdot)$  and the associated norm  $\|x\|^2 = (x, x)$ ,  $x \in H$ .

a) Prove the *parallelogram identity*

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

[ 5 ]

- b) Let  $M$  be a closed subspace of  $H$  and  $x_0$  a vector such that  $x_0 \notin M$ . We define  $\delta = \inf\{\|x_0 - y\|, y \in M\}$ , i.e. there exists a sequence  $(y_n)_n$  of elements of  $M$  such that  $\delta_n = \|x_0 - y_n\|$  converges to  $\delta$  when  $n$  goes to  $\infty$ .

$\propto$

- (i) Using the fact that  $\delta$  is an infimum over  $M$  show that, for any integers  $m$  and  $n$ ,  $\|y_m + y_n - 2x_0\| \geq 2\delta$ . [ 2 ]

- (ii) Using the parallelogram identity, show that  $(y_n)_n$  is a Cauchy sequence. [ 4 ]

- (iii) Prove that there exists a unique vector  $y \in M$  such that  $\|x_0 - y\| = \delta$ . To this end, show the existence of the vector  $y$  in  $H$  and use the fact that  $M$  is closed to conclude. [ 6 ]

- (iv) Justify that for any  $z \in M$ ,  $(x_0 - y, z) = 0$  (we do not require a detailed proof, you may give a graphical justification). [ 3 ]

5. The aim of this problem is to derive the minimum of

$$I(a, b) = \int_0^\pi [\sin(t) - (at^2 + bt)]^2 dt$$

over  $a, b$  in  $\mathbb{R}$ .

- a) Introducing an appropriate setting restate the minimisation problem in terms of the distance between a vector and a closed vector space. [ 7 ]

- b) Find  $\alpha$  and  $\beta$  such that

$$\int_0^\pi [\sin(t) - (\alpha t^2 + \beta t)] t dt = 0$$

and

$$\int_0^\pi [\sin(t) - (\alpha t^2 + \beta t)] t^2 dt = 0$$

[ 6 ]

- c) Using  $\alpha$  and  $\beta$  from the previous question, compute

$$\int_0^\pi [\sin(t) - (\alpha t^2 + \beta t)]^2 dt$$

and conclude.

[ 7 ]

Correction made at the start of the exam  
 $\|y_m + y_n - 2x_0\| \geq 2\delta$

