# EE1-10A MATHEMATICS I - SOLUTIONS

The below comments apply to common errors. Where there is no comment, the question was done well by most students and/or errors were arithmetic slip-ups, etc.

1. a) Express in the form $x+iy$:

$$(i) \quad \frac{3i+2}{2i-3}, \qquad (ii) \quad \left(\frac{\sqrt{3}-i}{2}\right)^{2018}.$$

SOLUTION

(i) Multiply by conjugate of denominator:

$$\frac{2+3i}{-3+2i}\left(\frac{-3-2i}{-3-2i}\right) = \frac{6-6-4i-9i}{4+9} = -i.$$

[ 2 ]

(ii) Express in complex exponential form and proceed:

$$\left(e^{-i\pi/6}\right)^{2018} = e^{-i\pi\ (2018/6)} = e^{-i\pi(336+\frac{1}{3})} = e^{-i\pi/3} = \frac{1}{2} - \frac{\sqrt{3}}{2}i.$$

[ 2 ]

*Many left it in exponential form or as* $\cos(*)+i\sin(*)$*, even with correct* $(*)$.

b) Obtain all complex $z$ such that $\sin(z^2)$ is purely imaginary. [ 5 ]

SOLUTION

Let $z^2 = (x+iy)^2 = x^2-y^2+2xyi$, then

$\sin(z^2) = \sin(x^2-y^2+2xyi) = \sin(x^2-y^2)\cos(2xyi)+\cos(x^2-y^2)\sin(2xyi)$

If this is to be purely imaginary, we require the real part to be equal to zero, hence

$$\sin(x^2-y^2) = 0 \quad \text{or} \quad \cos(2xyi) = \cosh(2xy) = 0.$$

The hyperbolic cosine is always positive, so the second equation has no solutions, leaving only the first which gives

$$x^2-y^2 = n\pi$$

where $n$ can be any integer, and the solution can be written as

$$z = x \pm i\sqrt{x^2 - n\pi}$$

where $x$ is any real number.

*Many people tried writing* $\sin z = \frac{1}{2i}\left(e^{iz^2} - e^{-iz^2}\right)$ *which almost always became unsolvable, especially after substituting* $z^2 = x^- y^2 + xyi$. *Lots of misunderstandings about what is real part/what is imaginary part*

c) (i) Show that if complex $z$ satisfies $\cot z = k$, where $k$ is real, then

$$e^{2iz} = \frac{k^2 - 1 + 2ki}{k^2 + 1}.$$

[ 4 ]

SOLUTION

Begin by writing in terms of complex exponentials:

$$\cot z = \frac{\cos z}{\sin z} = \frac{2i(e^{iz} + e^{-iz})}{2(e^{iz} - e^{-iz})} = k \Rightarrow \frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}} = -ki$$

now multiply top/bottom of lhs by $e^{iz}$:

$$\frac{e^{2iz} + 1}{e^{2iz} - 1} = -ki \Rightarrow e^{2iz} + 1 = -ki(e^{2iz} - 1) \Rightarrow e^{2iz} = \frac{-1 + ki}{1 + ki}\left(\frac{1 - ki}{1 - ki}\right) = \frac{k^2 - 1 + 2ki}{k^2 + 1},$$

as required.

*Many began by substituting* $\cot z$ *for* $k$ *in* $\frac{k^2-1+2ki}{k^2+1}$ *and obtaining* $e^{2iz}$*, perfectly valid.*

(ii) Hence, or otherwise, find all solutions of $\cot z = -1$. [ 3 ]

SOLUTION

With $k = -1$, the result above gives $e^{2iz} = -i$ and we write the rhs as a complex exponential:

$$e^{2iz} = e^{i(-\pi/2 + 2n\pi)} \Rightarrow 2z = -\frac{\pi}{2} + 2n\pi \Rightarrow z = -\frac{\pi}{4} + n\pi,$$

where $n$ is any integer.

d) Obtain the limits

$$(i)\ \lim_{x\to\pi/2}\left(x - \frac{\pi}{2}\right)\sin(\sec x), \quad (ii)\ \lim_{x\to\pi/4}\frac{1 - \tan x}{2\sin x - \sqrt{2}}.$$

SOLUTION

(i) Given that $-1 \leq \sin(\sec x) \leq 1$ wherever $\sec x$ is defined, we can write

$$-\left(x-\frac{\pi}{2}\right) \leq \left(x-\frac{\pi}{2}\right)\sin(\sec x) \leq \left(x-\frac{\pi}{2}\right)$$

and so

$$0 = \lim_{x\to\pi/2} -\left(x-\frac{\pi}{2}\right) \leq \lim_{x\to\pi/2}\left(x-\frac{\pi}{2}\right)\sin(\sec x) \leq \lim_{x\to\pi/2}\left(x-\frac{\pi}{2}\right) = 0$$

so by the sandwich theorem, the required limit is zero. [ 2 ]

(ii) Check that this is a case of "0/0", so we can apply l'Hopital's rule:

$$\lim_{x\to\pi/4} \frac{1-\tan x}{2\sin x - \sqrt{2}} = \lim_{x\to\pi/4} \frac{-\sec^2 x}{2\cos x} = \lim_{x\to\pi/4} -\frac{1}{2\cos^3 x} = -\frac{1}{2\cos^3(\pi/4)} = -\sqrt{2}\,.$$

*Lots of people left the answer as* $-\frac{2}{\sqrt{2}}$.

[ 3 ]

e) A function is defined as

$$f(x) = \begin{cases} \frac{\sqrt{2x+5}-\sqrt{x+7}}{x-2} & \text{for} \quad x \neq 2\,; \\ k & \text{for} \quad x = 2\,, \end{cases}$$

for $x \geq -2.5$. What value of $k$ makes $f(x)$ a continuous function? [ 4 ]

SOLUTION

We need the limit of $f(x)$ as $x \to 2$, but the form is "0/0". Begin by multiplying appropriately and cancelling:

$$f(x) = \frac{\sqrt{2x+5}-\sqrt{x+7}}{x-2}\left(\frac{\sqrt{2x+5}+\sqrt{x+7}}{\sqrt{2x+5}+\sqrt{x+7}}\right) = \frac{1}{\sqrt{2x+5}+\sqrt{x+7}}$$

Hence

$$\lim_{x\to 2} f(x) = \frac{1}{\sqrt{9}+\sqrt{9}} = \frac{1}{6}$$

and letting $k = 1/6$ will make the function continuous at $x = 2$. As this is the only discontinuity, the function is continuous for all $x \geq -2.5$.

*A large number of people left the answer in terms of* $\sqrt{9}$*!*

2. a) Differentiate to obtain $\dfrac{dy}{dx}$ : [ 3 ] each

$$(i) \quad y = x^{\ln x}\,, \qquad (ii) \quad y^2\sqrt{x} - \ln(x+y) = 1\,.$$

SOLUTION

(i) Take logarithms

$$y = x^{\ln x} \Rightarrow \ln y = \ln\left(x^{\ln x}\right) = (\ln x)^2$$

and differentiate implicitly:

$$\frac{1}{y}\frac{dy}{dx} = \frac{2\ln x}{x} \Rightarrow \frac{dy}{dx} = 2\ln x\left(x^{\ln x - 1}\right).$$

(ii) Differentiate implicitly:

$$\frac{d}{dx}\left(y^2\sqrt{x}\right) - \frac{d}{dx}\left(\ln(x+y)\right) = \frac{d}{dx}(1) \Rightarrow 2y\frac{dy}{dx}x^{1/2} + \frac{y^2}{2x^{1/2}} - \frac{1}{x+y}\left(1+\frac{dy}{dx}\right) = 0$$

collecting terms and simplifying we have

$$\frac{dy}{dx} = \frac{2x^{1/2} - xy^2 - y^3}{4yx^2 + 4y^2x - 2x^{1/2}},$$

or equivalent.

b) Differentiate from first principles to show that the derivative of $\sin x$ is $\cos x$. [You may quote the result for $\lim_{x\to 0}\frac{\sin x}{x}$.] [ 5 ]

SOLUTION

We require

$$\begin{aligned}
\frac{df}{dx} &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h} \\
&= \lim_{h\to 0}\frac{\sin x\cos h + \sin h\cos x - \sin(x)}{h} \\
&= \lim_{h\to 0}\sin x\frac{\cos h - 1}{h} + \cos x\frac{\sin h}{h}
\end{aligned}$$

For the second term we quote $\lim_{h\to 0}\frac{\sin h}{h} = 1$. The first term is zero:

$$\lim_{h\to 0}\frac{\cos h - 1}{h}\left(\frac{\cos h + 1}{\cos h + 1}\right) = \lim_{h\to 0}\frac{-\sin^2 h}{h} = -\lim_{h\to 0}(\sin h)\lim_{h\to 0}\frac{\sin h}{h} = 0,$$

Hence we have

$$\frac{df}{dx} = 0\sin x + 1\cos x = \cos x.$$

(Using l'Hopital's rule for the first term is acceptable as well, as it does not use the result being proved.)

*Many good answers, but lots of people ignored the term* $\frac{\cos h - 1}{h}$ *simply stating it is equal to zero - no mention of a limit! - or claimed that as* $\cos h \to 1$*, that would be enough.*

c) (i) Show that if $y = \left(\cos^{-1} x\right)^2$, then $\sqrt{1-x^2}\dfrac{dy}{dx} = -2\cos^{-1} x$. [ 3 ]

[Recall $\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}}$.]

SOLUTION

Given the hint,

$$\frac{d}{dx}\left[\left(\cos^{-1} x\right)^2\right] = -2\cos^{-1} x \frac{1}{\sqrt{1-x^2}}$$

and multiplying by $\sqrt{1-x^2}$ gives the desired result.

*Many began with $x = \cos(\sqrt{y})$ and then found $\frac{dx}{dy} = -\sin(\sqrt{y})\frac{1}{2\sqrt{y}}$ and got the same result, but with more work.*

(ii) Hence, or otherwise, deduce that $(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} - 2 = 0$. [ 3 ]

SOLUTION

Now differentiate both sides:

$$\frac{d}{dx}\left(\sqrt{1-x^2}\frac{dy}{dx}\right) = \frac{d}{dx}\left(-2\cos^{-1} x\right) \Rightarrow \frac{-2x}{2\sqrt{1-x^2}}\frac{dy}{dx} + \sqrt{1-x^2}\frac{d^2y}{dx^2} = \frac{2}{\sqrt{1-x^2}}$$

and multiplying by $\sqrt{1-x^2}$ gives the result.

d) For the function $f(x) = (x^2-3)e^x$, determine all stationary points and classify these using the second derivative test. Obtain all asymptotes. Sketch the graph of the function. You do not need to find points of inflection, but should indicate on your graph where other information allows you to deduce them. [ 8 ]

SOLUTION

Begin by differentiating $f(x) = (x^2-3)e^x$:

$$f'(x) = (x^2-3)e^x + 2xe^x = (x^2+2x-3)e^x = (x+3)(x-1)e^x$$

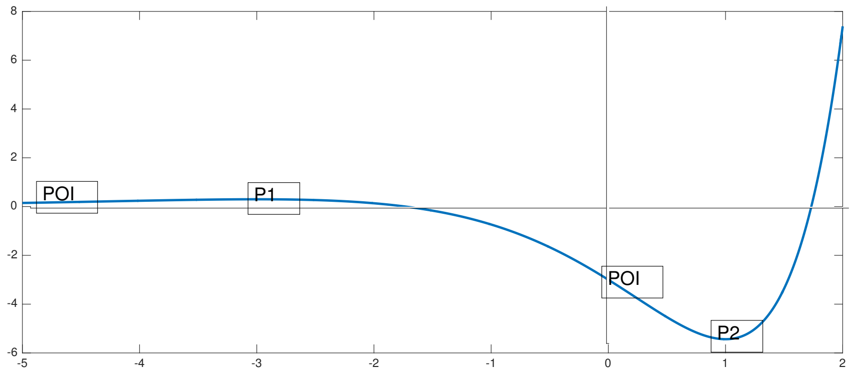
and as $e^x \neq 0$ we deduce that stationary points, given by $f'(x) = 0$ are at $x = 1$ and $x = -3$. Evaluating, we have stationary points $P_1(-3, 6e^{-3})$ and $P_2(1, -2e)$.

Differentiating again, we obtain $f''(x) = (x^2+4x-1)e^x$ and evaluating at the stationary points we get

$f''(1) = 3e > 0 \Rightarrow P_1$ is a local minimimum;

$f''(-3) = -4e^{-3} < 0 \Rightarrow P_2$ is a local maximum.

There is one asymptote: as $x \to -\infty$, the exponential goes to zero and so does $f(x)$. Intercepts are $(\pm\sqrt{3}, 0)$ and $(0, -3)$. This is sufficient to plot the function:




Stationary points and intercepts are shown as expected. Between max and min curvature changes, and we deduce a point of inflection; to the left of $P_1$, curvature must change for the asymptote and we infer another point of inflection, approximate position indicated on plot.

*Most people got the basics, but ignored the points of inflection. Some people indicated the POI between the max and min, but only 9 people noted the second POI. Quite a few had a vertical asymptote at the $y-$axis. Many seemed to think $f(x) \to -\infty$ as $x \to -\infty$. Others showed the curve concave down as $x \to +\infty$. Scale matters as well: though we don't obtain the $y-$values exactly at the extrema, we should attempt to keep the scale reasonable. You should be aware that $e \approx 2.7$ so that $6e^{-3} < 1$ and $-2e \approx -5.4$ but many students showed the $y-$value at the maximum much larger than at the minimum.*

3. a) Evaluate the following integrals.

i) $$\int \frac{18x+12}{\sqrt{3x^2+4x-7}}\,dx,$$ [ 3 ]

SOLUTION

Let $u = 3x^2+4x-7$, then $du = 6x+4\,dx$ and we have

$$\int \frac{18x+12}{\sqrt{3x^2+4x-7}}\,dx = \int \frac{3}{\sqrt{u}}\,du = 6\sqrt{u}+C = 6\sqrt{x^2+4x-7}+C.$$

ii) $$\int_3^4 \frac{2x+3}{x^2-x-2}\,dx,$$ [ 3 ]

SOLUTION

Partial fractions gives

$$\frac{2x+3}{x^2-x-2} = \frac{7}{3(x-2)} - \frac{1}{3(x+1)}$$

and so

$$\int_3^4 \frac{7}{3(x-2)} - \frac{1}{3(x+1)}\,dx = \left[\frac{7}{3}\ln(x-2) - \frac{1}{3}\ln(x+1)\right]_3^4 = 3\ln 2 - \frac{1}{3}\ln 5.$$

*Many students adopted another approach: first, split the numerator into a combination of terms* $2x-1$ *(derivative of the denominator) and* $4$ *and second, perform partial fractions on the remaining term. The additional steps that resulted led to arithmetical errors as multiple integration techniques were employed.*

iii) $$\int \frac{\cosh^{-1} x}{(x^2-1)^{1/2}}\, dx,$$ [ 3 ]

SOLUTION

Note that the derivative of $\cosh^{-1} x$ is present, so let

$$u = \cosh^{-1} x \Rightarrow du = \frac{1}{(x^2-1)^{1/2}} dx$$

and the integral becomes

$$\int u\, du = \frac{u^2}{2} + C = \frac{\left(\cosh^{-1} x\right)^2}{2} + C .$$

*No comment.*

iv) $$\int_{-2}^{2} \sqrt{4-x^2}\, dx, \quad \text{using a substitution.}$$ [ 4 ]

SOLUTION

We observe that the integrand is the upper semi-circle of a circle of radius two, centred at the origin, so the area should be $\frac{1}{2}(\pi 2^2) = 2\pi$, but we need to use a substitution. The usual argument suggest a trigonometric one: let

$$x = 2\sin u \Rightarrow 4 - x^2 = 4 - 4\sin^2 u = 4\cos^2 u$$

with limits $x = \pm 2 \Rightarrow \sin u = \pm 1 \Rightarrow u = \pm\pi/2$ and $dx = 2\cos u\, du$ so the integral becomes

$$\int_{-\pi/2}^{\pi/2} (\sqrt{4\cos^2 u})\, 2\cos u\, du = 4\int_{-\pi/2}^{\pi/2} \cos^2 u\, du = 4\int_{-\pi/2}^{\pi/2} \frac{\cos 2u + 1}{2}\, du = 2\left[\frac{\sin 2u}{2} + u\right]_{-\pi/2}^{\pi/2} = 2\pi .$$

*The errors observed fell into two categories: errors due to wrong limits due to the change of variable and arithmetic ones. Not many students observed that the integral was the area of an upper semicircle.*

b) Let

$$I_n = \int_0^\infty x^{2n} e^{-x}\, dx\,,$$

where $n$ is a positive integer. Find $I_0$ and show that $I_n = (2n)!I_0$. Hence, or otherwise, obtain $I_6$ in terms of a factorial. [ 7 ]

SOLUTION

Finding $I_0$ is trivial:

$$I_0 = \lim_{k\to\infty} \int_0^k e^{-x}\, dx = \lim_{k\to\infty} \left[-e^{-x}\right]_0^k = \lim_{k\to\infty} (1-e^{-k}) = 1\,.$$

For $I_n$ we integrate by parts:

$$I_n = \left[x^{2n} e^{-x}\right]_0^\infty + 2n \int_0^\infty x^{2n-1} e^{-x}\, dx$$

The first two terms vanish with the zero and infinite limits. As we need an even power of $x$, we integrate by parts again:

$$I_n = 2n \left\{ \left[-x^{2n-1} e^{-x}\right]_0^\infty + (2n-1) \int_0^\infty x^{2n-2} e^{-x}\, dx \right\}$$

The first two terms vanish as before, so

$$I_n = 2n(2n-1) \int_0^\infty x^{2(n-1)} e^{-x}\, dx = 2n(2n-1)I_{n-1}$$

The recursion formula gives

$$I_n = 2n(2n-1)\left[(2n-2)(2n-3)I_{n-2}\right]$$

and so on, with the last terms

$$I_2 = 4\cdot 3I_1 = 4\cdot 3\cdot 2\cdot 1I_0$$

so

$$I_n = (2n)!I_0$$

as required. Hence $I_6 = (12!)I_0 = 12!$ and we leave the solution as factorial.

*Students who made errors in this question had difficulty in formulating the logical flow of the question: show the relationship between $I_n$ and $I_{n-1}$ through integration by parts and then generalize to obtain the relationship between $I_n$ and $I_0$.*

c) Show that the harmonic series $\displaystyle\sum_{n=1}^\infty \frac{1}{n}$ diverges.

Hence, or otherwise, show that $\displaystyle\sum_{n=1}^\infty \frac{1}{\sqrt{n}}$ also diverges. [ 5 ]

SOLUTION

For the harmonic series we apply the integral test. Given

$$\int_1^\infty \frac{1}{x}\,dx < \sum_{n=1}^\infty \frac{1}{n}$$

we have

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{k\to\infty}[\ln x]_1^k = \lim_{k\to\infty}\ln k$$

which diverges, and so does the series which is greater than the integral.

Alternative: group terms as follows,

$$1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\left(\frac{1}{9}+\cdots+\frac{1}{16}\right)+\dots$$

$$>1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\left(\frac{1}{16}+\cdots+\frac{1}{16}\right)+\dots$$

As each of the brackets in the second expression is equal to $1/2$, the harmonic sum is greater than an infinite sum of terms of $1/2$ and so diverges.

We observe

$$n>\sqrt{n}\Rightarrow \frac{1}{n}<\frac{1}{\sqrt{n}}$$

for all $n>1$. Hence

$$\sum_{n=1}^\infty \frac{1}{n} < \sum_{n=1}^\infty \frac{1}{\sqrt{n}}$$

and the second series diverges by the comparison test.

*Many students tried the ratio test first and those who made an error for this question failed to move beyond the latter test. Several students employed the integral test as an alternative.*

4. a) Find the radius and interval of convergence of the infinite series [ 5 ]

$$\sum_{n=0}^\infty (-1)^n(n+1)x^n\,,$$

SOLUTION

We apply the ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1}(n+2)x^{n+1}}{(-1)^n(n+1)x^n}\right| = \frac{n+2}{n+1}|x|$$

and taking the limit, we get

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{n+2}{n+1}|x| = \lim_{n\to\infty}\frac{1+2/n}{1+1/n}|x| = \frac{1+0}{1+0}|x| < 1$$

for convergence, so the radius of convergence is 1 and the interval of convergence is $-1<x<1$. Testing at the endpoint $x=1$, we see

$$\sum_{n=0}^{\infty}(-1)^n(n+1) = 1-2+3-4+5-\cdots$$

which diverges. Similarly, letting $x=-1$

$$\sum_{n=0}^{\infty}(-1)^n(n+1)(-1)^n = \sum_{n=0}^{\infty}(-1)^{2n}(n+1) = 1+2+3+4+\cdots$$

which also diverges, so the interval of convergence cannot be extended.

*Most notable error is an incomplete solution: failure to consider the situation when $x=\pm 1$*

b) Obtain the $n^{\text{th}}$ derivative of $f(x)=(1+x)^{-2}$ and hence show that the Maclaurin series of $f(x)$ is given by the series in (a). [ 5 ]

SOLUTION

Differentiating a few times:

$$f(x)=\frac{1}{(1+x)^2} \Rightarrow f'(x)=-2\frac{1}{(1+x)^3} \Rightarrow f''(x)=6\frac{1}{(1+x)^2} \Rightarrow f'''(x)=-24\frac{1}{(1+x)^2}$$

until we can spot the pattern:

$$f^{(n)}(x)=(n+1)!(-1)^n\frac{1}{(1+x)^n} \Rightarrow f^{(n)}(0)=(n+1)!(-1)^n$$

and the Maclaurin series is

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n=\sum_{n=0}^{\infty}\frac{(n+1)!(-1)^n}{n!}x^n$$

and the factorials cancel, leaving only $n+1$ in the numerator, so that

$$f(x)=\sum_{n=0}^{\infty}(n+1)(-1)^n x^n\,,$$

matching the series from (a).

*No comment.*

c) i) Find the real Fourier Series for the functions with period 2, defined on $[-1,1]$ as $f(x)=|x|$ and $g(x)=x$. [ 7 ]

SOLUTION

For $f(x)$ this is an even function and $b_n = 0$ so that with $T = 2$

$$\begin{aligned}
a_n &= \frac{2}{T}\int_{-T/2}^{T/2} f(x)\cos(2n\pi x/T)\,dx = \int_{-1}^{1} |x|\cos(n\pi x)\,dx \\
&= 2\int_0^1 x\cos(n\pi x)\,dx = 2\left[\frac{x\sin(n\pi x)}{n\pi}\right]_0^1 - \frac{2}{n\pi}\int_0^1 \sin(n\pi x)\,dx \\
&= [0-0] + \frac{2}{n\pi}\left[\frac{\cos(n\pi x)}{n\pi}\right]_0^1 = \frac{2}{(n\pi)^2}[\cos(n\pi)-1] \\
&= \frac{2}{(n\pi)^2}[(-1)^n - 1]
\end{aligned}$$

so that $a_n = -4/(n\pi)^2$ for odd $n$ and zero for even $n$. Calculate $a_0$ separately: $a_0 = 2\int_0^1 x\,dx = 1$ and the series is

$$f(x) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{\substack{n=1 \\ n\ odd}}^{\infty} \frac{1}{n^2}\cos(n\pi x)\,.$$

For $g(x)$, an odd function, we have $b_n = 0$ and having done the set-up above, we proceed directly to

$$\begin{aligned}
b_n &= 2\int_0^1 x\sin(n\pi x)\,dx \\
&= 2\left[-x\frac{\cos(n\pi x)}{n\pi}\right]_0^1 + \frac{2}{n\pi}\int_0^1 \cos(n\pi x)\,dx \\
&= \frac{2}{n\pi}[-\cos(n\pi) - 0 + 0 - 0] = \frac{2}{n\pi}(-1)^{n+1}
\end{aligned}$$

where we omitted the detail on the integral vanishing, as we have done this for the first series already. Hence the Fourier series is

$$g(x) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin(n\pi x)\,.$$

ii) Choosing an appropriate value for $x$ in the Fourier series for $g(x)$ above, obtain the value of the infinite sum [ 4 ]

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots$$

SOLUTION

To find the correct value of $x$, we reason that the Fourier series does have the alternating pattern, but involves all fractions

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}\cdots$$

whereas we only want the odd denominators. We need a value of $x$ so that $\sin(n\pi x) = 0$ for even $n$: this must be $x = 1/2$. Trying it out we have

$$\frac{1}{2} = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(n\pi/2) = \frac{2}{\pi}\left(1 + 0 - \frac{1}{3} + 0 + \frac{1}{5} + 0 - \frac{1}{7}\cdots\right)$$

which fits, so the required series has sum $\pi/4$.

iii) Using Parseval's theorem on the Fourier series for $f(x)$ above, obtain the value of the infinite sum [ 4 ]

$$1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\frac{1}{6^4}+\cdots$$

SOLUTION Using Parseval's theorem for period 2 we have

$$\frac{1}{2}\int_{-1}^{1}[f(x)]^2\,dx = \frac{1}{2}a_0^2+\sum_{n=1}^{\infty}a_n^2$$

so

$$\frac{1}{2}\int_{-1}^{1}x^2\,dx = \frac{1}{2}\left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{1}{3} = \frac{1}{2}(1^2)+\sum_{\substack{n=1\\ n\ odd}}^{\infty}\left(\frac{4}{n^2\pi^2}\right)^2 \Rightarrow \frac{1}{6} = \frac{16}{\pi^4}\sum_{\substack{n=1\\ n\ odd}}^{\infty}\frac{1}{n^4}$$

so the series for odd $n$ has the value $\pi^4/96$. To get the full series, we reason:

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \sum_{\substack{n=1\\ n\ odd}}^{\infty}\frac{1}{n^4}+\sum_{\substack{n=1\\ n\ even}}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{96}+\sum_{m=1}^{\infty}\frac{1}{(2m)^4} = \frac{\pi^4}{96}+\frac{1}{16}\sum_{m=1}^{\infty}\frac{1}{m^4}$$

and since the index is irrelevant, we note that the first and last series above are the same, and so

$$\left(1-\frac{1}{16}\right)\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{96} \Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}.$$

*Many errors in spite of recognizing the geometric difference between the two functions. However, the geometric difference did not translate into the recognition of the difference in Fourier Series. This failure subsequently led to errors in the remaining two parts of the question.*