CONTROL ENGINEERING

1. Two types of algae evolve, in competition, in an aqueous solution. The equations describing the evolution of the two populations of algae are

$$\dot{x}_1 = x_1 \left(-x_1 + \frac{u}{1+x_2} \right), \qquad \dot{x}_2 = x_2(-x_2 + u),$$

where x_1 denotes the concentration of the first type, x_2 the concentration of the second type, and u the concentration of nutrient.

a) Assume $u > 0$ and constant. Determine all equilibrium points of the system. [4 marks]

b) Write the linearized models of the system around each of the equilibrium points determined in part a). [8 marks]

c) Using the linearized models determined in part b) determine (if possible) the stability properties of the equilibrium points computed in part a). [4 marks]

d) Show that all linearized models determined in part b) are not controllable. [4 marks]

2. A cart of mass $M = 1$ has two inverted pendulums attached to it of lengths l_1 and l_2 and both of mass m. Let θ_1 and θ_2 be the angles of the pendulums with respect to a vertical axis directed upward and let f be the force on the cart.

For small values of θ_1 and θ_2 the dynamic behaviour of the pendulums is described by the differential equations

$$m(f - mg\theta_1 - mg\theta_2 + l_1\ddot{\theta}_1) = mg\theta_1, \qquad m(f - mg\theta_1 - mg\theta_2 + l_2\ddot{\theta}_2) = mg\theta_2,$$

where g denotes the gravitational acceleration.

a) Let $x_1 = \theta_1$, $x_2 = \theta_2$, $x_3 = \dot{\theta}_1$, $x_4 = \dot{\theta}_2$, $u = f$, $y = x_1 - x_2$ and $x = [x_1\ x_2\ x_3\ x_4]'$. Write a state space representation of the considered system, i.e. determine matrices A, B and C such that

$$\dot{x} = Ax + Bu \qquad y = Cx.$$

[4 marks]

b) Study the controllability property of the system as a function of the physical parameters l_1 and l_2. [6 marks]

c) Study the observability property of the system as a function of the physical parameters l_1 and l_2. [6 marks]

d) Assume $l_1 = l_2$ and write a second order differential equation describing the behaviour of $\xi = \theta_1 - \theta_2$. Use this differential equation to assess the stabilizability property of the system. [4 marks]

3. Consider a herd of cattle composed of cows and calves. Let $x_1(t)$ be the number of calves in year t and $x_2(t)$ the number of cows in year t. The dynamical behaviour of the herd is described by the equation

$$x(t+1) = Ax(t) = \begin{bmatrix} \frac{1}{2} & \frac{2}{5} \\ \frac{1-k}{2} & \frac{4}{5} \end{bmatrix} x(t),$$

where $x(t) = [x_1(t), x_2(t)]'$ and $k \in [0,1]$ denotes the portion of calves slaugthered each year.

a) Compute the equilibrium points of the system as a function of $k \in [0,1]$. [4 marks]

b) Determine for which values of k the system is stable, asymptotically stable, unstable. [4 marks]

c) Show that for any initial condition $x(0)$ such that $x_1(0) \geq 0$ and $x_2(0) \geq 0$, the free response $x(t)$ of the system is such that $x_1(t) \geq 0$ and $x_2(t) \geq 0$, for all $t \geq 0$. [4 marks]

d) Assume $k = 1/2$.

   i) Show that the free-response of the system converges to the line

$$5x_1 - 4x_2 = 0.$$

   (Hint: write a difference equation for the variable $z(t) = 5x_1(t) - 4x_2(t)$ and show that $z(t)$ tends to zero as t tends to ∞.) [4 marks]

   ii) Suppose that for each slaughtered calf C_1 GBP are earned and that each cow costs C_2 GBP a year. The *revenue* of the herd in the year t is therefore

$$y(t) = C_1 k x_1(t) - C_2 x_2(t).$$

   Determine a condition on C_1 and C_2 so that the asymptotic revenue is non-negative for each $x_1(0) \geq 0$ and $x_2(0) \geq 0$. [4 marks]

4. The chemical reaction describing the production of water, namely

$$2H_2 + O_2 \leftrightarrow 2H_2O,$$

can be described by the nonlinear continuous-time system

$$\begin{aligned}
\dot{H} &= -2k_1H^2O + 2k_2W, \\
\dot{O} &= -k_1H^2O + k_2W, \\
\dot{W} &= -2k_2W + 2k_1H^2O,
\end{aligned}$$

where $H \geq 0$, $O \geq 0$ and $W \geq 0$ denote the concentrations of hydrogen, oxygen, and water, respectively, and $k_1 > 0$ and $k_2 > 0$ are positive constants which quantify the speed of the reaction.

To study the dynamical properties of the system consider the variables

$$x_1 = W, \qquad x_2 = W + 2O, \qquad x_3 = W + H.$$

a) Show that the variables (x_1, x_2, x_3) define a new set of coordinates for the system and determine (H, O, W) as a function of (x_1, x_2, x_3).
(Hint: show that there is a one-to-one relation between the variables (H, O, W) and the variables (x_1, x_2, x_3).) [4 marks]

b) Write differential equations for x_2 and x_3. Integrate the resulting differential equations and comment on the results. [4 marks]

c) Write a differential equation for x_1 and show that $\dot{x}_1$ can be written as a cubic polynomial in x_1 with coefficients that depend upon $x_2(0)$, $x_3(0)$, k_1 and k_2. In particular, show that

$$\dot{x}_1 = A - Bx_1 + Cx_1^2 - Dx_1^3, \qquad (\star)$$

where A, B, C and D are functions of $x_2(0)$, $x_3(0)$, k_1 and k_2 and take non-negative values. [4 marks]

d) Suppose that for all $x_2(0) > 0$ and $x_3(0) > 0$ the system $(\star)$ has only one equilibrium $x_1^\star$.

i) Sketch $\dot{x}_1$ as a function of x_1 and argue that the equilibrium $x_1 = x_1^\star$ is a globally asymptotically stable equilibrium for the x_1-system. [4 marks]

ii) Argue that the overal system with state (x_1, x_2, x_3) has infinitely many equilibria. Using the results of part d.i) determine the stability properties of these equilibria. [4 marks]

5. Consider a linear, time-varying, continuous-time system described by the equation

$$\dot{x} = A(t)x.$$

A common *belief* is the following.

> (C) If the matrix $A(t)$ has constant eigenvalues with negative real part then the linear, time-varying, system is asymptotically stable.

To disprove the claim (C) consider the matrix

$$A(t) = \begin{bmatrix} -1 & e^{2t} \\ 0 & -1 \end{bmatrix}.$$

Let $t_0 = 0$.

a) Show that the matrix $A(t)$ has constant eigenvalues with negative real part. [2 marks]

b) Determine the state transition matrix of the system, i.e. the matrix $\Phi(t,0)$ such that

$$\Phi(0,0) = I, \qquad \frac{d\Phi(t,0)}{dt} = A(t)\Phi(t,0).$$

(Hint: integrate the differential equations describing the system.) [8 marks]

c) Show that for almost any selection of the initial conditions $x(0)$

$$\lim_{t\to\infty} \|x(t)\| = \infty.$$

Determine the set of initial conditions such that

$$\lim_{t\to\infty} \|x(t)\| = 0.$$

[4 marks]

d) Using the results in part c) conclude that the considered linear, time-varying system, is not stable. [2 marks]

e) Show that the linear, time-varying, system

$$\dot{x} = B(t)x,$$

with

$$B(t) = \begin{bmatrix} -1 & b(t) \\ 0 & -1 \end{bmatrix}$$

and $|b(t)| \leq \bar{b}$, for some $\bar{b}$ positive, is asymptotically stable. [4 marks]

6. Consider a linear, single-input, single-output, system described by the equations

$$\sigma x = Ax + Bu, \qquad y = Cx,$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}$ is the input, and $y(t) \in \mathbb{R}$ is the output.

Consider the problem of studying the reachability and observability properties of the system using the PBH tests.

a) Show, using the PBH reachability test, that the system is reachable if and only if there is no left eigenvector of A which is orthogonal to B.
(Hint: recall that a left eigenvector of A is a row vector w such that $wA = \lambda w$, for some λ which is an eigenvalues of A.) [4 marks]

b) Show, using the PBH observability test, that the system is observable if and only if there is no right eigenvector of A which is orthogonal to C.
(Hint: recall that a right eigenvector of A is a column vector v such that $Av = \lambda v$, for some λ which is an eigenvalues of A.) [4 marks]

c) Consider the class of linear systems described by the equations

$$\begin{aligned}
\sigma x_1 &= \lambda_1 x_1 + B_1 u, \\
\sigma x_2 &= \lambda_2 x_2 + B_2 u, \\
&\vdots \\
\sigma x_n &= \lambda_n x_n + B_n u, \\
y &= C_1 x_1 + C_2 x_2 + \cdots + C_n x_n,
\end{aligned}$$

with $\lambda_i \neq \lambda_j$ for $i \neq j$.

i) Using the results in part a) determine conditions on the coefficients B_i such that the system is reachable. [4 marks]

ii) Using the results in part b) determine conditions on the coefficients C_i such that the system is observable. [2 marks]

d) Let

$$A = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}, \qquad B = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}.$$

Show, using the results in part a), that the system is not reachable (regardeless of the values of the coefficients B_1, B_2 and B_3). [6 marks]