EE4-23
EE9-CS2-1
EE9-FPN2-05

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2016

MSc and EEE/EIE PART IV: MEng and ACGI

STABILITY AND CONTROL OF NON-LINEAR SYSTEMS

Corrected copy

Friday, 6 May 10:00 am

Time allowed: 3:00 hours

There are FOUR questions on this paper.

Answer ALL questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : D. Angeli
Second Marker(s) : E.C. Kerrigan

1. Consider the following two-dimensional time-invariant nonlinear system:

$$\begin{aligned} \dot{x}_1 &= x_1 - \frac{x_1^3}{3} - x_2 + \frac{4}{3} \\ \dot{x}_2 &= x_1 - x_2 + 1 \end{aligned}$$

a) Discuss existence and uniqueness of solutions. [2]

b) Find the nullclines of the system and sketch them graphically. Find out, in particular, the points where the first nullcline achieves local maxima and minima when regarded as the graph of a function from x_1 to x_2. [3]

c) Identify the regions of phase-plane in which the nullclines partition $\mathbb{R}^2$ and the orientation of the vector-field in each one of them. [3]

d) Find all equilibria of the system. [2]

e) Linearize the system around each linearizable equilibrium and discuss the local phase-portrait. [3]

f) Show that for sufficiently large R the set $\{(x_1, x_2) : x_1^2 + x_2^2 \leq R^2\}$ is forward invariant for the system. [4]

g) Sketch the global phase portrait of the system. [3]

2. Consider the following polynomials of two real variables:

$$V_1(x_1,x_2) = x_1^4 + x_1x_2^2 + x_2^4,$$

$$V_2(x_1,x_2) = x_1^4 - x_1x_2^3 + x_1^2x_2^2 - x_1^3x_2 + x_2^4.$$

a) For each polynomial, show whether this is positive definite, negative definite or not sign definite. [6]

b) Consider next the following time-invariant bidimensional system:

$$\dot{x} = \begin{bmatrix} -4x_1^3 + x_2^3 - 2x_1x_2^2 + 3x_1^2x_2 \\ 3x_2^2x_1 - 2x_1^2x_2 + x_1^3 - 4x_2^3 \end{bmatrix}$$

Show that this only admits a single equilibrium at the origin. (*Hint: take the sum of the 2 equations defining the equilibrium condition and factor* $(x_1 + x_2)$ *from the resulting polynomial expression. All solutions can then be obtained by considering several cases.*) [4]

c) Using one of the two polynomials V_1 or V_2 as a Lyapunov function (and justifying your choice) prove Global Asymptotic Stability of the origin. [4]

d) Consider the following modified system:

$$\dot{x} = \begin{bmatrix} 3x_2^2x_1 - 2x_1^2x_2 + x_1^3 - 4x_2^3 \\ 4x_1^3 - x_2^3 + 2x_1x_2^2 - 3x_1^2x_2 \end{bmatrix}.$$

Exploit the polynomial V_2 in order to characterize the ω-limit set of the system's solutions for all initial conditions $x_0 \in \mathbb{R}^2$. [6]

3. Let $\mathscr{G}$ denote the set of Lipschitz continuous and strictly increasing functions $g : \mathbb{R} \to \mathbb{R}$ fulfilling $g(0) = 0$. Consider the following scalar nonlinear system:

$$\dot{x} = g(u - \alpha x),$$

where g belongs to $\mathscr{G}$ and α is a positive scalar.

a) Show that the system in question is Input-to-State Stable. [5]

b) Find an upper-bound of the Input-to-State gain γ (as a function of the parameter α). [4]

c) Consider next the following two-dimensional system:

$$\begin{aligned} \dot{x}_1 &= g_1(x_2 - 2x_1) \\ \dot{x}_2 &= g_2(x_1 - 2x_2) \end{aligned}$$

where both g_1 and g_2 belong to $\mathscr{G}$. Prove that the system is Globally Asymptotically Stable at the origin. (*Hint: use the small gain theorem and regard* $[x_1, x_2]^T$ *as the state vector of a suitable feedback system.*) [6]

d) Let now $d(t)$ denote a bounded disturbance, taking values in $[-1, 1]$. Consider the system:

$$\dot{x} = (2 + d(t))g(-x).$$

Is this system UGAS at the origin ? (Justify your answer). [5]

4. Consider the following normalized equation of a mathematical pendulum:

$$\ddot{\theta}(t) = -\sin(\theta(t)) + u(t),$$

where θ denotes the oriented angle of the pendulum with respect to the vertical line and u the torque applied to the pendulum.

a) Choose a state vector x and derive a state space representation of the system. [3]

b) Show that this is an input-affine system. [3]

c) Choose the scalar output function $y = h(x)$ and an appropriate storage function $S_1(x)$ so as to have a SISO passive system with input u and output y. (*Hint: how about the mechanical energy as a storage function?*) [5]

d) Consider the PI controller of equations $v(t) = e(t) + \int_0^t e(\tau)d\tau$. Find a state space realization of the controller and show that this is a passive system from input e to output v (for a suitable choice of a storage function S_2). [4]

e) Consider the negative feedback interconnection of the pendulum and the controller $u = -v$ and $y = e$. Show that the total energy $V = S_1 + S_2$ is dissipated along solutions of the closed-loop system and find the largest invariant set in $\mathrm{Ker}[\dot{V}]$. [5]