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**ISE3.7**

IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE  
UNIVERSITY OF LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING  
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EEE/ISE PART III/IV: M.Eng., B.Eng. and ACGI

**MATHEMATICS FOR SIGNALS AND SYSTEMS**

Friday, 3 May 10:00 am

There are FIVE questions on this paper.

Answer THREE questions.

**Corrected Copy**

Time allowed: 3:00 hours

**Examiners responsible:**

First Marker(s): Weiss, G.

Second Marker(s): Allwright, J.C.

**Special instructions to invigilators:** None

**Information for candidates:** None

1. On the real vector space  $\mathbb{R}^{3 \times 3}$  (which contains all the real  $3 \times 3$  matrices), we define an inner product by

$$\langle a, b \rangle = \frac{1}{2} \text{trace } a^T b,$$

where  $a^T$  is the transpose of  $a$ . We define the subspaces

$$S = \{a \in \mathbb{R}^{3 \times 3} \mid a^T = a\}$$

(these are the so-called symmetric, or self-adjoint real matrices), and

$$A = \{a \in \mathbb{R}^{3 \times 3} \mid a^T = -a\}$$

(these are the so-called anti-symmetric, or skew-adjoint real matrices).

- (a) What are the dimensions of  $\mathbb{R}^{3 \times 3}$ ,  $S$  and  $A$ ? [2]
- (b) Find an orthonormal basis in  $A$ . [4]
- (c) Show that  $A$  is orthogonal to  $S$ . From here, using your answer to part (a), conclude that in fact,  $A$  is the orthogonal complement of  $S$ . [4]
- (d) Show that if  $a \in A$ , then the eigenvalues of  $a$  are imaginary. (Hint: the complex matrix  $ia$  is self-adjoint.) [3]
- (e) Show that if  $a \in A$ , then  $\det a = 0$ . (Hint: use part (d) and a certain symmetry of the eigenvalues.) [3]
- (f) We denote the orthogonal projectors from  $\mathbb{R}^{3 \times 3}$  onto  $S$  and  $A$  by  $P_S$  and  $P_A$  (thus,  $P_S + P_A = I$ , the identity operator acting on  $\mathbb{R}^{3 \times 3}$ ). Check that these projectors are given by

$$P_S x = \frac{1}{2}(x + x^T), \quad P_A x = \frac{1}{2}(x - x^T).$$

(Hint: use the conclusion from part (c).) [4]

2. We denote by  $c_0$  is the space of sequences convergent to zero, and by  $c$  the space of convergent sequences. We consider the indices (i.e., the discrete time) to run from 0 to  $\infty$ .

- (a) Give an example of a sequence  $a \in l^1$  that has infinitely many nonzero terms, and also infinitely many zero terms. [3]
- (b) Which of the inclusions  $c_0 \subset l^2$  or  $l^2 \subset c_0$  is true? Give a very brief explanation of your answer, and show that  $c_0 \neq l^2$ . [3]
- (c) Give an example of a sequence  $b \in l^\infty$  such that  $b \notin c$ , and compute its norm in  $l^\infty$ . [2]
- (d) Compute the  $\mathcal{Z}$  transforms of the sequences  $u$  and  $y$  given by

$$u_k = k, \quad y_k = (-1)^k.$$

For each of these  $\mathcal{Z}$  transforms, indicate a domain (the largest domain that you can determine) where the series defining the  $\mathcal{Z}$  transform is convergent. [4]

- (e) If possible, find a linear system which, starting from initial state zero, if it receives the input  $u$ , it produces the output  $y$ . Here,  $u$  and  $y$  are the signals from part (d). If you think that this is impossible, then explain why you think so. [4]
- (f) Give an example of a sequence  $q = (q_k)$  such that the series defining its  $\mathcal{Z}$  transform does not converge for any value of the variable  $z$ . Hint: think of the  $\mathcal{Z}$  transform as a Taylor series in the variable  $\zeta = z^{-1}$ . How do you compute the radius of convergence of this series? Make this radius zero. [4]

3. In this question,  $S_\tau$  denotes the right shift operator by  $\tau$  on  $L^2[0, \infty)$  and  $*$  denotes the convolution product.

- (a) Define the natural inner product and the corresponding norm on the space  $L^2[0, \infty)$ . For  $s \in \mathbb{C}_+$  and  $\varphi \in L^2[0, \infty)$  defined by  $\varphi(t) = e^{-st}$ , compute  $\|\varphi\|_2$ . [3]
- (b) Let  $g \in L^2[0, \infty)$  and let  $\mathcal{L}g$  denote its Laplace transform. Show that

$$|(\mathcal{L}g)(s)| \leq \frac{\|g\|_2}{\sqrt{2\operatorname{Re} s}} \quad \text{for all } s \in \mathbb{C}_+.$$

Hint: use the result about  $\|\varphi\|_2$  from part (a) and the Cauchy-Schwarz inequality. [3]

- (c) In the sequel, consider  $f$  to be the characteristic function of the interval  $[0, 2]$  and  $g(t) = e^{-5t}$ ,  $t \geq 0$ . (Thus,  $f(t) = 1$  for  $t \in [0, 2]$  and  $f(t) = 0$  for  $t > 2$ .) Compute the Laplace transforms  $F = \mathcal{L}f$  and  $G = \mathcal{L}g$ . [3]
- (d) Compute  $\|f\|_2$ ,  $\langle f, g \rangle$  and  $\|g\|_2$  and check that the Cauchy-Schwarz inequality holds for them. [4]
- (e) Define  $h = S_3g$ , i.e.,  $h$  is obtained by delaying  $g$  by 3 time units. Compute

$$H = \mathcal{L}h, \quad \|h\|_2 \quad \text{and} \quad P = \mathcal{L}(h * g). \quad [4]$$

- (f) Compute

$$\|G\|_2, \quad \|H\|_2 \quad \text{and} \quad \langle F, G \rangle,$$

where the norms and the scalar products correspond to the Hardy space  $H^2(\mathbb{C}_+)$  and  $F, G, H$  are as defined above. [3]

4. Consider the system described by

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \alpha \\ \beta \end{bmatrix} u,$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

where  $u$  is the input signal,  $x$  is the state (with two components),  $y$  is the output signal and  $\alpha, \beta$  are real constants.

- (a) For which values of  $\alpha, \beta$  is the system stable? [2]
- (b) Compute the transfer function  $\mathbf{G}$  of this system. [3]
- (c) For  $\alpha = \beta = 1$ , compute the impulse response and the step response of this system, as functions of  $t \geq 0$ . [2]
- (d) Still considering  $\alpha = \beta = 1$ , compute  $\|\mathbf{G}\|_\infty$  and  $\|\mathbf{G}\|_2$  (i.e., the norms of  $\mathbf{G}$  in  $H^\infty(\mathbb{C}_+)$  and in  $H^2(\mathbb{C}_+)$ ). [3]
- (e) Still considering  $\alpha = \beta = 1$ , if  $u(t) = te^{-3t}$  and  $x(0) = 0$ , compute the output signal  $y$  as a function of  $t$ . [2]
- (f) For  $\alpha = 1$  and  $\beta = 0$  (be careful,  $\beta$  has changed), consider the cascade connection of the system with a delay line of 2 time units. Thus, if  $z$  is the output signal of the delay line, then  $z(t) = y(t - 2)$ . Compute the transfer function  $\mathbf{H}$  from  $u$  to  $z$ . [2]
- (g) Compute  $\|\mathbf{H}\|_\infty$ , where  $\mathbf{H}$  is the transfer function from part (f). [3]
- (h) Suppose now that  $\alpha$  and  $\beta$  are functions of  $t$ :  $\alpha(t) = \cos t$  and  $\beta(t) = \sin t$ . Is the the system with input  $u$  and output  $y$  still linear? Does this system have a transfer function? Explain very briefly your answer.

[3]

5. (a) Explain briefly what is meant by a time-invariant operator on  $l^2$ . [3]
- (b) State the discrete-time version of the Fourés-Segal theorem and discuss briefly its connections with systems theory. [7]
- (c) Define the space  $BL(\omega_b)$  of band-limited functions with angular frequencies not higher than  $\omega_b$ . Give two examples of functions in this space which are linearly independent. [4]
- (d) State the sampling theorem and discuss briefly its significance for the transmission and storage of signals. [6]

[ END ]

# Mathematics for Signals and Systems

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## SOLUTIONS

Question 1 (a) 9, 6 and 3.

(b) Matrices in  $A$  are of the form

$$m = \begin{bmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0 \end{bmatrix}. \text{ The scalar product of}$$

two matrices  $a, b \in \mathbb{R}^{3 \times 3}$  can also be written in the form  $\langle a, b \rangle = \frac{1}{2} \sum_{k=1}^3 \sum_{j=1}^3 a_{jk} b_{jk}$ .

Hence, the following is an orthonormal basis in  $A$ :

$$e_1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

(The matrix  $m$  above is  $m = \alpha e_1 + \beta e_2 + \gamma e_3$ .)

(c) If  $a \in S$ , then  $a_{jk} = a_{kj}$  ( $k, j = 1, 2, 3$ ). If  $b \in A$ , then  $b_{jk} = -b_{kj}$ , in particular,  $b_{jj} = 0$ . Hence,

$$\langle a, b \rangle = \sum_{k>j} a_{jk} b_{jk} + \sum_{k<j} a_{jk} b_{jk}$$



$$= \sum_{k>j} (a_{jk} b_{jk} + a_{kj} b_{kj}) = 0$$

(because  $a_{kj} b_{kj} = -a_{jk} b_{jk}$ ). (This argument has been written such that it remains valid for square matrices of arbitrary dimensions, in the  $3 \times 3$  case there are only three terms in the last sum:  $(k,j) = (3,1), (3,2), (2,1)$ .)

Since  $\dim S + \dim A = \dim \mathbb{R}^{3 \times 3}$  and  $S$  is orthogonal to  $A$ , it follows that  $A$  is the orthogonal complement of  $S$  (i.e., the space of all matrices orthogonal to  $S$ ).

(d) If  $a \in A$ , then  $(ia)^* = (\bar{i})a^T = (-i)(-a) = ia$ , so that  $ia$  (being self-adjoint) has only real eigenvalues. Hence, the eigenvalues of  $a$  are on  $i\mathbb{R}$ .

(e) Since  $a \in A$  is real, its eigenvalues are located symmetrically with respect to the real axis. Since  $a$  has 3 eigenvalues on  $i\mathbb{R}$ , one of them must be zero. Since  $\det a = \lambda_1 \lambda_2 \lambda_3$ , where  $\lambda_j$  are the eigenvalues of  $a$ , we get  $\det a = 0$ .

An entirely different way to see that  $\det a = 0$  is the following: using the structure from the answer to part (b), we have  $ax = 0$ , where  $x = \begin{bmatrix} \gamma \\ -\beta \\ \alpha \end{bmatrix}$ .

(f) If we denote  $a = \frac{1}{2}(x + x^T)$ ,  $b = \frac{1}{2}(x - x^T)$ ,

then it is easy to see that  $a^T = a$  and  $b^T = -b$ , i.e.,  $a \in S$  and  $b \in A$ . Moreover, we have  $x = a + b$ .

Since such a decomposition is unique, we must have  $a = P_S x$  and  $b = P_A x$ .

**Question 2** (a)  $a_k = \begin{cases} \frac{1}{k^2} & \text{if } k \text{ is odd,} \\ 0 & \text{if } k \text{ is even.} \end{cases}$

(b) We have  $\ell^2 \subset \rho_0$ . Indeed, if  $a \in \ell^2$ , i.e.,  $\sum_{k=0}^{\infty} |a_k|^2 < \infty$ , then  $\lim a_k = 0$ . We have  $\ell^2 \neq \rho_0$  because the sequence  $b_k = \frac{1}{\sqrt{k+1}}$  is in  $\rho_0$ , but not in  $\ell^2$ .

(c)  $d_k = (-1)^k$ ,  $d \notin \rho$ ,  $\|d\|_{\infty} = 1$ .

(d) If  $u_k = k$ ,  $y_k = (-1)^k$ ,  $k = 0, 1, 2, \dots$ , then

$$\hat{u}(z) = \frac{z}{(z-1)^2}, \quad \hat{y}(z) = \frac{z}{z+1}.$$

$\hat{u}$  has a singularity (a pole) at  $z=1$ , hence the series is convergent for  $|z| > 1$ .  $\hat{y}$  has a singularity at  $z=-1$ , hence its series is also convergent for  $|z| > 1$ .

(e) This is impossible, because any linear system is causal, i.e., at any time  $k$ , the response is caused by the past input only (up to  $k$ ). In our case, for  $k=0$ , we have  $u_0=0$ , which together with  $x_0=0$  (initial state zero) implies  $y_0=0$ , but we have  $y_0=1$ .

(f) Take  $q_k = 2^{2^k}$ . Denoting  $\xi = \frac{1}{z}$ , we have

$$\hat{q}(z) = q_0 + q_1 \xi + q_2 \xi^2 + q_3 \xi^3 + \dots,$$

for all  $\xi \in \mathbb{C}$  with  $|\xi| < R$ . The radius of convergence  $R$  of this Taylor series is given by

$$\frac{1}{R} = \limsup_{n \rightarrow \infty} |q_n|^{\frac{1}{n}}. \text{ We get } \frac{1}{R} = \infty, \text{ hence } R = 0.$$

**Question 3** (a) On the space  $L^2[0, \infty)$ ,

$$\langle f, g \rangle = \int_0^\infty f(t) \overline{g(t)} dt, \quad \|f\|_2^2 = \int_0^\infty |f(t)|^2 dt.$$

If  $\varphi(t) = e^{-st}$ , then  $\|\varphi\|_2^2 = \int_0^\infty e^{-2(\operatorname{Re} s)t} dt = \frac{1}{2\operatorname{Re} s}.$

Hence,  $\|\varphi\|_2 = \frac{1}{\sqrt{2\operatorname{Re} s}}.$

(b) We have, for any  $g \in L^2[0, \infty)$ ,

$$|(\mathcal{L}g)(s)| = \left| \int_0^\infty g(t) e^{-st} dt \right| = \langle g, \bar{\varphi} \rangle,$$

where  $\bar{\varphi}$  is the complex conjugate of  $\varphi$  introduced in part (a). By the Cauchy-Schwarz inequality, we get

$$\begin{aligned} |(\mathcal{L}g)(s)| &\leq \|g\|_2 \cdot \|\bar{\varphi}\|_2 = \|g\|_2 \cdot \|\varphi\|_2 \\ &= \|g\|_2 \cdot \frac{1}{\sqrt{2\operatorname{Re} s}}. \end{aligned}$$

(c)  $F(s) = \frac{1}{s} (1 - e^{-2s})$ ,  $G(s) = \frac{1}{s+5}$

(d)  $\|f\|_2^2 = \int_0^2 dt = 2$ , hence  $\|f\|_2 = \sqrt{2}.$

$$\|g\|_2^2 = \int_0^\infty e^{-10t} dt = \frac{1}{10}, \text{ so } \|g\|_2 = \frac{1}{\sqrt{10}}.$$

$$\langle f, g \rangle = \int_0^2 e^{-5t} dt = \frac{1}{5} (1 - e^{-10}) \approx \frac{1}{5}$$

Cauchy-Schwarz:

$$\frac{1}{5} < \frac{1}{\sqrt{10}} \cdot \sqrt{2} \left( = \frac{1}{\sqrt{5}} \right).$$

$$(e) \quad H(s) = e^{-3s} \frac{1}{s+5}, \quad \|h\|_2 = \|g\|_2 = \frac{1}{\sqrt{10}}$$

$$P(s) = H(s) G(s) = e^{-3s} \frac{1}{(s+5)^2}.$$

$$(f) \quad \|G\|_2 = \|g\|_2 = \frac{1}{\sqrt{10}}, \text{ by Paley-Wiener.}$$

$$\text{Similarly, } \|H\|_2 = \|h\|_2 = \frac{1}{\sqrt{10}}.$$

Using the Paley-Wiener theorem a third time, we have

$$\langle F, G \rangle = \langle f, g \rangle = \frac{1}{5} (1 - e^{-10}).$$

**Question 4** (a)  $\dot{x} = Ax + Bu$ ,  $B = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ ,

$A = \begin{bmatrix} 0 & -3 \\ 1 & -4 \end{bmatrix}$ ,  $\det(sI - A) = s^2 + 4s + 3$  (this is the characteristic polynomial),  $\sigma(A) = \{-1, -3\}$ . The system is stable regardless of  $\alpha, \beta$ .

(b)  $C = [0 \ 1]$ ,  $y = Cx$ ,  $G(s) = C(sI - A)^{-1}B$ .

$$C(sI - A)^{-1} = [0 \ 1] \frac{1}{s^2 + 4s + 3} \begin{bmatrix} s+4 & -3 \\ 1 & s \end{bmatrix} = \frac{1}{s^2 + 4s + 3} [1 \ s],$$

hence  $G(s) = \frac{\beta s + \alpha}{s^2 + 4s + 3} = \frac{\beta s + \alpha}{(s+1)(s+3)}$ .

(c) For  $\alpha = \beta = 1$ ,  $G(s) = \frac{1}{s+3}$ , hence the impulse response is  $g = \mathcal{L}^{-1}G$ ,  $g(t) = e^{-3t}$ . The step response is  $y_{\text{step}}(t) = \int_0^t g(\sigma) d\sigma = \frac{1}{3}(1 - e^{-3t})$ .

(d)  $\|G\|_{\infty} = \left\| \frac{1}{s+3} \right\|_{\infty} = \frac{1}{3}$ ,  $\|G\|_2 = \|g\|_2 = \frac{1}{\sqrt{6}}$ .

(e)  $u(t) = te^{-3t}$ ,  $\hat{u}(s) = \frac{1}{(s+3)^2}$ ,  $\hat{y}(s) = \frac{1}{(s+3)^3}$ ,  $y(t) = \frac{t^2}{2}e^{-3t}$ .

(f) For  $\alpha = 1, \beta = 0$ ,  $G(s) = \frac{1}{(s+1)(s+3)}$ , the delay transfer function is  $e^{-2s}$ , hence  $H(s) = e^{-2s} / (s+1)(s+3)$ .

(g)  $\|H\|_{\infty} = \|G\|_{\infty}$ , because  $|e^{-2i\omega}| = 1$  for  $\omega \in \mathbb{R}$ .

$G$  attains its sup at  $s=0$ , hence  $\|H\|_{\infty} = \frac{1}{3}$ .

(h) With  $\alpha, \beta$  functions of  $t$ , the system is still linear but it is not time-invariant. Hence, the system has no transfer function.

**Question 5** (2) We denote by  $S$  the operator of right shift (or delay) by one step on  $\ell^2$  (the indices are from 0 to  $\infty$ ). Thus,

$$S(u_0 u_1 u_2 \dots) = (0 u_0 u_1 \dots).$$

A bounded operator  $T$  from  $\ell^2$  to  $\ell^2$  is called time invariant if  $TS = ST$ .

(b) Theorem (Fourés-Segal) Let  $T$  be a bounded linear operator from  $\ell^2$  to  $\ell^2$ .  $T$  is time-invariant if and only if there exists  $G \in H^\infty(\mathbb{E})$  such that

$$T = \mathcal{Z}^{-1} G \mathcal{Z} \quad (\mathcal{Z} = \mathcal{Z}\text{-transform}).$$

If this is the case, then  $\|T\| = \|G\|_\infty$ .

Consider a linear system described by

$$\begin{cases} x_{k+1} = Ax_k + Bu_k \\ y_k = Cx_k + Du_k \end{cases} \quad \left| \begin{array}{l} u = \text{input signal} \\ x = \text{state} \\ y = \text{output signal} \end{array} \right.$$

where  $A, B, C, D$  are constant matrices and the eigenvalues of  $A$  are in  $\mathcal{D}$  (i.e.,  $A$  is stable). If  $x(0) = 0$ , then  $y = Tu$ , where  $T$  is the input-output operator of the system. This is bounded on  $\ell^2$  and time-invariant. According to the Fourés-Segal theorem,  $T = \mathcal{Z}^{-1} G \mathcal{Z}$ , with  $G \in H^\infty(\mathbb{E})$ . It can be checked that  $G(z) = C(zI - A)^{-1}B + D$ . The norm  $\|G\|_\infty$  can be seen from the magnitude Bode plot of  $G$ .

(c)  $BL(\omega_b)$  is the subspace of  $L^2(-\infty, \infty)$  consisting of those functions whose Fourier transform is in  $L^2[-i\omega_b, i\omega_b]$  (in other words,  $u \in BL(\omega_b)$  if  $(\mathcal{F}u)(i\omega) = 0$  for  $|\omega| > \omega_b$ ). Such functions are analytic on all  $\mathbb{C}$ , in particular, they are infinitely differentiable. In practice, signals will usually not belong to such a space, but they can be approximated very well by band-limited functions. The following functions form an orthonormal basis in  $BL(\omega_b)$ :

$$e_k(t) = \frac{\sin \omega_b(t - k\tau)}{\sqrt{\pi\omega_b}(t - k\tau)}, \quad k \in \mathbb{Z}, \quad \tau = \pi/\omega_b.$$

Notice that  $e_k$  is obtained by shifting  $e_0$  to the right by the amount  $k\tau$  (if  $k < 0$  then we are actually shifting to the left).

Thus, for example,  $e_0$  and  $e_\tau$  are linearly independent functions in  $BL(\omega_b)$ .

(d) Theorem (Whittaker - Kotelnikov - Shannon).

If  $u \in BL(\omega_b)$  and  $\tau \in (0, \frac{\pi}{\omega_b}]$ , then for all  $t \in \mathbb{R}$ ,

$$u(t) = \sum_{k \in \mathbb{Z}} u(k\tau) \frac{\sin \omega_b(t - k\tau)}{\omega_b(t - k\tau)}.$$

This shows that if we sample the signal at the time instants  $k\tau$ ,  $k \in \mathbb{Z}$ , where  $\tau$  is the sampling period, then  $u$  can be completely reconstructed from these samples. It is easier to store and/or transmit samples of a signal than the whole signal.

In practice, signals are not exactly bandlimited; just "almost" bandlimited. This means that  $u = v + e$ , where  $v \in BL(\omega_b)$  and  $e$  is a small error (deviation). Also, the samples  $u(k\tau)$  cannot be taken for all  $k \in \mathbb{Z}$ , only for a finite (but possibly very large) set of integers. Then, the formula will hold approximately, for values of  $t$  which are not close to the end of the time interval in which samples were taken. The condition  $\tau \leq \frac{\pi}{\omega_b}$  means that the sampling frequency  $\frac{1}{\tau} \geq 2$  times the highest frequency components of  $u$ .

[END]