Imperial College London

MATH97084 MATH97185

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2020

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Time Series

Date: 6th May 2020

Time: 09.00am - 11.30am (BST)

Time Allowed: 2 Hours 30 Minutes

Upload Time Allowed: 30 Minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS AS SEPARATE PDFs TO THE RELEVANT DROPBOXES ON BLACKBOARD (ONE FOR EACH QUESTION) WITH COMPLETED COVERSHEETS WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

Note: Throughout this paper $\{\epsilon_t\}$ is a sequence of uncorrelated random variables (white noise) having zero mean and variance σ_ϵ^2, unless stated otherwise. The term "stationary" will always be taken to mean second-order stationary. All processes are real-valued unless stated otherwise. The sample interval is unity unless stated otherwise. B denotes the backward shift operator.

1. (a) Consider the ARMA(1,1) model

$$X_t = \frac{1}{2}X_{t-1} + \epsilon_t + \frac{1}{8}\epsilon_{t-1}. \qquad (\dagger)$$

(i) Show $\{X_t\}$ is both stationary and invertible. (2 marks)

(ii) Express $\{X_t\}$ in general linear process form. (3 marks)

(iii) Show $\mathrm{Var}\{X_t\} = \frac{73}{48}\sigma_\epsilon^2$. (4 marks)

(b) Consider a stationary process $\{X_t\}$ that can be written as a general linear process,

$$X_t = \sum_{k=0}^{\infty} \psi_k \epsilon_{t-k} = \Psi(B)\epsilon_t.$$

We wish to construct the l-step ahead forecast of the form

$$X_t(l) = \sum_{k=0}^{\infty} \delta_k \varepsilon_{t-k}.$$

(i) Show that the l-step prediction variance $\sigma^2(l) = E\{(X_{t+l} - X_t(l))^2\}$ is minimized by setting $\delta_k = \psi_{k+l}$, $k \geq 0$.

(4 marks)

(ii) Show the l-step ahead forecast can be written in the form

$$X_t(l) = \Psi^{(l)}(B)\Psi^{-1}(B)X_t,$$

where $\Psi^{(l)}(z) = \sum_{k=0}^{\infty} \psi_{k+l} z^k$. (2 marks)

(iii) For the ARMA(1,1) model given in ($\dagger$) in part (a), express the 2-step ahead forecast $X_t(2)$ in the form

$$X_t(2) = \sum_{k=0}^{\infty} \pi_k X_{t-k}.$$

(5 marks)

(Total: 20 marks)

2. (a) Let $L\{\cdot\}$ be a linear time invariant (LTI) filter with frequency response function $G(f)$ defined as $L\{\mathrm{e}^{\mathrm{i}2\pi ft}\} = G(f)\mathrm{e}^{\mathrm{i}2\pi ft}$. You may take it as a fact that any LTI filter can be expressed in the form

$$L\{X_t\} = \sum_{u=-\infty}^{\infty} g_u X_{t-u} = Y_t.$$

Furthermore, if the spectral representations of $\{X_t\}$ and $\{Y_t\}$ are

$$X_t = \int_{-1/2}^{1/2} \mathrm{e}^{\mathrm{i}2\pi ft}\mathrm{d}Z_X(f) \qquad \text{and} \qquad Y_t = \int_{-1/2}^{1/2} \mathrm{e}^{\mathrm{i}2\pi ft}\mathrm{d}Z_Y(f),$$

respectively, then $\mathrm{d}Z_Y(f) = G(f)\mathrm{d}Z_X(f)$.

(i) Show that $G(f)$ and $\{g_u\}$ form a Fourier transform pair. (2 marks)

(ii) Show $S_Y(f) = |G(f)|^2 S_X(f)$, if the spectral density functions $S_X(f)$ and $S_Y(f)$ exist. (2 marks)

(iii) Consider the LTI filter $Y_t = L\{X_t\} = X_{t-1} + X_t + X_{t+1}$. Derive the spectral density function of the output $\{Y_t\}$ when the input is a white noise process with variance σ_ϵ^2. (3 marks)

(iv) Show that the spectral density function $S_X(f)$ for an AR(p) process

$$X_t - \phi_{1,p}X_{t-1} - \ldots - \phi_{p,p}X_{t-p} = \epsilon_t,$$

is given by

$$S_X(f) = \frac{\sigma_\epsilon^2}{|1 - \phi_{1,p}\mathrm{e}^{-\mathrm{i}2\pi f} - \ldots - \phi_{p,p}\mathrm{e}^{-\mathrm{i}2\pi fp}|^2}.$$

(4 marks)

(b) (i) Let $\Phi(B)X_t = \epsilon_t$ be an AR(2) process where $\Phi(z)$ has roots $z = 1/a$ and $z = 1/b$. Show

$$S_X(f) = \frac{\sigma_\epsilon^2}{|1 - a\mathrm{e}^{-\mathrm{i}2\pi f}|^2 |1 - b\mathrm{e}^{-\mathrm{i}2\pi f}|^2}.$$

(3 marks)

(ii) Let $\Phi(B)X_t = \epsilon_t$ be an AR(2) process where $\Phi(z)$ has complex conjugate roots and $\{X_t\}$ has has spectral density function

$$S(f) = \frac{\sigma_\epsilon^2}{[1 - \cos(2\pi(0.125 - f)) + 0.25][1 - \cos(2\pi(0.125 + f)) + 0.25]}.$$

Express $\{X_t\}$ in the form $X_t = \phi_{1,2}X_{t-1} + \phi_{2,2}X_{t-2} + \epsilon_t$, clearly stating the parameters $\phi_{1,2}$ and $\phi_{2,2}$.

HINT: Express the conjugate roots as

$$\frac{1}{r}\mathrm{e}^{\mathrm{i}2\pi f_0} \textit{ and } \frac{1}{r}\mathrm{e}^{-\mathrm{i}2\pi f_0}$$

and derive the spectral density function. (6 marks)

(Total: 20 marks)

3. (a) Let $X_1, ..., X_N$ be a realisation from a stationary process $\{X_t\}$. The following is an estimator for the autocovariance sequence

$$\hat{s}_\tau^{(p)} = \frac{1}{N} \sum_{t=1}^{N-|\tau|} (X_t - \bar{X})(X_{t+|\tau|} - \bar{X}) \qquad \text{for all } \tau \text{ with } |\tau| \le N-1,$$

where $\bar{X} = (1/N) \sum_{t=1}^{N} X_t$. When the mean is known, $\bar{X}$ is replaced by μ.

(i) When the mean of $\{X_t\}$ is known, show $\hat{s}_\tau^{(p)}$ is a biased estimator of the autocovariance sequence for $\{X_t\}$ when $\tau \neq 0$. Comment on the bias of the estimator as $N \to \infty$.

(3 marks)

(ii) Let $\{X_t\}$ be the MA(1) process $X_t = \epsilon_t - \theta \epsilon_{t-1}$. For some fixed constant $C > 0$, show that to obtain $|\text{bias}\{\hat{s}_\tau^{(p)}\}| < C$ for all $|\tau| < N - 1$, we require $N > \sigma_\epsilon^2 |\theta|/C$. You may assume the mean of $\{X_t\}$ is known to be zero.

(5 marks)

(b) Let $X_1, ..., X_N$ be a realisation from a stationary process $\{X_t\}$ with a known mean of zero. The direct spectral estimator is defined as

$$\hat{S}^{(d)}(f) = \left| \sum_{t=1}^{N} h_t X_t \mathrm{e}^{-\mathrm{i}2\pi f t} \right|^2 \qquad \text{for all } f \text{ with } |f| \le 1/2,$$

where $\{h_t\}$ is a data taper of length N normalised such that $\sum_{t=1}^{N} h_t^2 = 1$.

(i) Show

$$\hat{S}^{(d)}(f) = \sum_{\tau=-(N-1)}^{N-1} \hat{s}_\tau^{(d)} \mathrm{e}^{-\mathrm{i}2\pi f \tau},$$

where

$$\hat{s}_\tau^{(d)} = \sum_{t=1}^{N-|\tau|} h_t X_t h_{t+|\tau|} X_{t+|\tau|} \qquad \text{for all } \tau \text{ with } |\tau| \le N-1.$$

(4 marks)

(ii) Using (b)(i), show

$$\int_{-1/2}^{1/2} E\{\hat{S}^{(d)}(f)\} \mathrm{d}f = s_0.$$

(6 marks)

(iii) Let $S^{(p)}(f)$ denote the periodogram. Is $\int_{-1/2}^{1/2} E\{\hat{S}^{(p)}(f)\} \mathrm{d}f$ less than, greater than, or equal to s_0? Justify your answer. (2 marks)

(Total: 20 marks)

4. (a) Let $\{X_t\}$ be a stationary process whose autocovariance sequence is non-negative for all $\tau \in \mathbb{Z}$. Show that the spectral density function of $\{X_t\}$ attains its maximum value at $f = 0$.

(4 marks)

(b) Define what is means for a pair of processes $\{X_t\}$ and $\{Y_t\}$ to be jointly stationary.

(3 marks)

(c) Let $\{X_t\}$ be a zero mean stationary process with autocovariance sequence $\{s_{X,\tau}\}$ and spectral density function $S_X(f)$. Let $\{Y_t\}$ be defined as $Y_t = W_t X_t$, where $\{W_t\}$ is a sequence of independent and identically distributed Bernoulli(p) random variables and is independent of $\{X_t\}$.

(i) Show $\{X_t\}$ and $\{Y_t\}$ are jointly stationary, deriving both the autocovariance sequence $\{s_{Y,\tau}\}$ and cross covariance sequence $\{s_{XY,\tau}\}$ in terms of $s_{X,\tau}$ and p. (6 marks)

(ii) Show $\gamma^2_{XY}(f)$, the coherence between processes $\{X_t\}$ and $\{Y_t\}$, is given as

$$\gamma^2_{XY}(f) = \frac{1}{1 + \frac{(1-p)s_{X,0}}{pS_X(f)}}.$$

(4 marks)

(iii) Let $\{X_t\}$ be the MA(1) process

$$X_t = \epsilon_t + \frac{1}{2}\epsilon_{t-1}.$$

Show $\gamma^2_{XY}(f)$ attains its maximum value at $f = 0$. (3 marks)

(Total: 20 marks)

5. PRELIMINARY INFORMATION

- If $G_1, ..., G_N$ is a realisation from a stationary Gaussian zero mean process $\{G_t\}$, we define

$$J(f) = \sum_{t=1}^{N} h_t G_t \mathrm{e}^{-\mathrm{i}2\pi f t},$$

where $\{h_t\}$ is a data taper of length N normalised such that $\sum_{t=1}^{N} h_t^2 = 1$.

- In this question, you may assume the following results.

$$\sum_{t=1}^{N} \cos^2(2\pi f_j t) = \sum_{t=1}^{N} \sin^2(2\pi f_j t) = \frac{N}{2}$$
$$\sum_{t=1}^{N} \cos(2\pi f_j t)\sin(2\pi f_j t) = \sum_{t=1}^{N} \cos(2\pi f_j t)\sin(2\pi f_k t) = 0$$
$$\sum_{t=1}^{N} \cos(2\pi f_j t)\cos(2\pi f_k t) = \sum_{t=1}^{N} \sin(2\pi f_j t)\sin(2\pi f_k t) = 0,$$

where $f_j = j/N$ and $f_k = k/N$ with j and k both integers such that $j \neq k$ and $1 \leq j, k < N/2$.

- You may use the following version of Isserlis' Theorem. If Z_1, Z_2, Z_3 and Z_4 and four complex valued random variables with zero means, then

$$\mathrm{Cov}\{Z_1 Z_2, Z_3 Z_4\} = \mathrm{Cov}\{Z_1, Z_3\}\mathrm{Cov}\{Z_2, Z_4\} + \mathrm{Cov}\{Z_1, Z_4\}\mathrm{Cov}\{Z_2, Z_3\}.$$

Recall: for a pair of zero mean complex random variables S and T, $\mathrm{Cov}\{S, T\} = E\{S^* T\}$, where * denotes complex conjugation.

- Fejér's kernel is defined as

$$\mathcal{F}(f) = \left| \frac{1}{\sqrt{N}} \sum_{t=1}^{N} \mathrm{e}^{-\mathrm{i}2\pi f t} \right|^2 = \frac{\sin^2(N\pi f)}{N \sin^2(\pi f)}.$$

QUESTION BEGINS ON NEXT PAGE

(a) Consider the case of $\{G_t\}$ being Gaussian white noise with variance σ^2. Let $h_t = 1/\sqrt{N}$ for all $t = 1, ..., N$, and consider the decomposition of $J(f)$ into its real and imaginary parts, $J(f) = A(f) + \mathrm{i}B(f)$.

(i) Show $\mathrm{Var}\{A(f_k)\} = \mathrm{Var}\{B(f_k)\} = \sigma^2/2$ for $f_k \neq 0$ or $1/2$. (2 marks)

(ii) Show

$$\mathrm{Cov}\{A(f_j), A(f_k)\} = 0 \text{ for all } f_j \neq f_k.$$
$$\mathrm{Cov}\{B(f_j), B(f_k)\} = 0 \text{ for all } f_j \neq f_k.$$
$$\mathrm{Cov}\{A(f_j), B(f_k)\} = 0 \text{ for all } f_j \text{ and } f_k.$$

(3 marks)

(iii) Recall that, if $Y_1, Y_2, ..., Y_\nu$ are independent zero mean, unit variance Gaussian random variables, then $\chi^2_\nu \equiv Y_1^2 + Y_2^2 + ... + Y_\nu^2$ has a chi-square distribution with ν degrees of freedom. For $f_k \neq 0$ or $1/2$, show the periodogram is distributed

$$\hat{S}_G^{(p)}(f_k) \stackrel{\mathrm{d}}{=} \frac{\sigma^2}{2}\chi^2_2,$$

where $\stackrel{\mathrm{d}}{=}$ means *equal in distribution*. (4 marks)

(b) Consider now a general data taper $\{h_t\}$. We can write

$$J(f) = \int_{-1/2}^{1/2} H(f-u)\mathrm{d}Z(u),$$

where $H(f)$ is the Fourier transform of $\{h_t\}$ and $\{Z(\cdot)\}$ is the orthogonal increment process associated with a Gaussian zero mean stationary process $\{G_t\}$, with spectral density function $S_G(\cdot)$.

(i) Show

$$\mathrm{Cov}\{\hat{S}_G^{(d)}(f), \hat{S}_G^{(d)}(f+\eta)\} =$$
$$\left|\int_{-1/2}^{1/2} H^*(f-u)H(f+\eta-u)S_G(u)\mathrm{d}u\right|^2 + \left|\int_{-1/2}^{1/2} H(f+u)H(f+\eta-u)S_G(u)\mathrm{d}u\right|^2.$$

(7 marks)

(ii) For $\eta > 0$, it can be shown that the correlation between $\hat{S}_G^{(d)}(f)$ and $\hat{S}_G^{(d)}(f+\eta)$ is given approximately by

$$R(\eta) \equiv \frac{R(\eta, f)}{R(0, f)} \qquad \text{where} \qquad R(\eta, f) = S_G^2(f)\left|\sum_{t=1}^{N} h_t^2 \mathrm{e}^{-\mathrm{i}2\pi\eta t}\right|^2.$$

In the case of the rectangular taper $h_t = 1/\sqrt{N}$, express $R(\eta)$ in terms of Fejér's kernel and hence determine the values of η for which $R(\eta) = 0$. (4 marks)

(Total: 20 marks)

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2020

MATH96053/MATH97084/MATH97185

Time Series Analysis [SOLUTIONS]

1. (a) (i) Writing the process as $\Phi(B)X_t = \Theta(B)\epsilon_t$, we have $(1-\frac{1}{2}B)X_t = (1+\frac{1}{8}B)\epsilon_t$, i.e. $\Phi(z) = 1-\frac{1}{2}z$ and $\Theta(z) = 1+\frac{1}{8}z$. $\boxed{\text{sim. seen} \Downarrow}$
To show $\{X_t\}$ is stationary, we are required to show the roots of $\Phi(z)$ lie outside of the unit circle. The only root of $\Phi(z)$ is $z=2$, which lies outside of the unit circle, hence the process is stationary. To show $\{X_t\}$ is invertible, we are required to show the roots of $\Theta(z)$ lie outside of the unit circle. The only root of $\Theta(z)$ is $z=-8$, which lies outside the unit circle, hence the process is invertible. $\boxed{\text{2 marks (A)}}$

(ii) The general linear process form is $X_t = G(B)\epsilon_t$ where $G(z) = \frac{\Theta(z)}{\Phi(z)} = \frac{1+\frac{1}{8}z}{1-\frac{1}{2}z}$. Expanding gives

$$\begin{aligned}
G(z) &= \left(1+\frac{1}{8}z\right)\left(1+\frac{1}{2}z+\frac{1}{4}z^2+\frac{1}{8}z^3+\ldots\right)\\
&= 1+\left(\frac{1}{2}+\frac{1}{8}\right)z+\left(\frac{1}{4}+\frac{1}{16}\right)z^2+\left(\frac{1}{8}+\frac{1}{32}\right)z^3+\ldots\\
&= 1+\frac{5}{4}\sum_{k=1}^{\infty}\frac{1}{2^k}z^k
\end{aligned}$$

Therefore, the general linear process form is $X_t = \epsilon_t + \frac{5}{4}\sum_{k=1}^{\infty}\frac{1}{2^k}\epsilon_{t-k}$. $\boxed{\text{3 marks (A)}}$

(iii) For a process $\{X_t\}$ in general linear process form $X_t = \sum_{k=0}^{\infty} g_k\epsilon_{t-k}$, we have $\text{var}\{X_t\} = \sigma_\epsilon^2\sum_{k=0}^{\infty} g_k^2$. Therefore

$$\begin{aligned}
\text{var}\{X_t\} &= \sigma_\epsilon^2\left(1+\sum_{k=1}^{\infty}\left(\frac{5}{4}\right)^2\cdot\left(\frac{1}{2^k}\right)^2\right)\\
&= \sigma_\epsilon^2\left(1+\frac{25}{16}\sum_{k=1}^{\infty}\frac{1}{4^k}\right)\\
&= \sigma_\epsilon^2\left(1+\frac{25}{16}\cdot\frac{1}{4}\sum_{k=0}^{\infty}\frac{1}{4^k}\right)\\
&= \sigma_\epsilon^2\left(1+\frac{25}{64}\cdot\frac{1}{1-\frac{1}{4}}\right)\\
&= \sigma_\epsilon^2\left(1+\frac{25}{64}\cdot\frac{4}{3}\right)\\
&= \sigma_\epsilon^2\left(1+\frac{25}{48}\right) = \frac{73}{48}\sigma_\epsilon^2.
\end{aligned}$$

$\boxed{\text{4 marks (A)}}$

(b) (i) Using the GLP representation, we have $X_{t+l} = \sum_{k=0}^{\infty} \psi_k \epsilon_{t+l-k}$. We want to minimize, $\boxed{\text{seen} \Downarrow}$

$$\begin{aligned} E\{(X_{t+l} - X_t(l))^2\} &= E\left\{\left(\sum_{k=0}^{\infty} \psi_k \epsilon_{t+l-k} - \sum_{k=0}^{\infty} \delta_k \epsilon_{t-k}\right)^2\right\} \\ &= E\left\{\left(\sum_{k=0}^{l-1} \psi_k \epsilon_{t+l-k} + \sum_{k=0}^{\infty} [\psi_{k+l} - \delta_k] \epsilon_{t-k}\right)^2\right\} \\ &= \sigma_\epsilon^2 \left\{\left(\sum_{k=0}^{l-1} \psi_k^2\right) + \sum_{k=0}^{\infty} (\psi_{k+l} - \delta_k)^2\right\}. \end{aligned}$$

The first term is independent of the choice of $\{\delta_k\}$ and the second term is clearly minimized by choosing $\delta_k = \psi_{k+l}, k = 0, 1, 2, \ldots$. $\boxed{\text{4 marks (A)}}$

(ii) Part (i) means the l-step ahead forecast can be written $X_t(l) = \Psi^{(l)}(B)\epsilon_t$, where $\Psi^{(l)}(z)$ is as stated in the question. Given $X_t = \Psi(B)\epsilon_t$, we have $\epsilon_t = \Psi^{-1}(B)X_t$, giving $X_t(l) = \Psi^{(l)}(B)\Psi^{-1}(B)X_t$. $\boxed{\text{2 marks (A)}}$

(iii) Using (a)(ii), we have $X_t(2) = \Psi^{(2)}(B)\Psi^{-1}(B)X_t$, where $\Psi(z) = \frac{1+\frac{1}{8}z}{1-\frac{1}{2}z}$ and $\boxed{\text{sim. seen} \Downarrow}$

$$\Psi^{(2)}(z) = \frac{5}{4}\left(\frac{1}{4} + \frac{1}{8}z + \frac{1}{16}z^2 + \ldots\right) = \frac{5}{4} \cdot \frac{1}{4} \sum_{k=0}^{\infty} \frac{1}{2^k} z^k = \frac{5}{16} \cdot \frac{1}{1 - \frac{1}{2}z}.$$

Therefore,

$$\begin{aligned} X_t(2) &= \Psi^{(2)}(B)\Psi^{-1}(B)X_t \\ &= \frac{5}{16} \cdot \frac{1}{1 - \frac{1}{2}B} \cdot \frac{1 - \frac{1}{2}B}{1 + \frac{1}{8}B} X_t \\ &= \frac{5}{16} \cdot \frac{1}{1 + \frac{1}{8}B} X_t \\ &= \sum_{k=0}^{\infty} \frac{5}{16} \cdot \left(-\frac{1}{8}\right)^k X_{t-k} \\ &= \sum_{k=0}^{\infty} \frac{5}{16} \cdot \left(-\frac{1}{8}\right)^k X_{t-k} \end{aligned}$$

$\boxed{\text{5 marks (C)}}$

2. (a) (i) Using the given result, it follows that

seen ⇓

$$\begin{aligned} L\{e^{i2\pi ft}\} &= \sum_{u=-\infty}^{\infty} g_u e^{i2\pi f(t-u)} \\ &= e^{i2\pi ft} \sum_{u=-\infty}^{\infty} g_u e^{-i2\pi fu}. \end{aligned}$$

Therefore, given $L\{e^{i2\pi ft}\} = e^{i2\pi ft}G(f)$, it follows that $G(f) = \sum_{u=-\infty}^{\infty} g_u e^{-i2\pi fu}$, and hence $G(f)$ and $\{g_u\}$ are a Fourier transform pair.

2 marks (A)

(ii) Taking the given identity $dZ_Y(f) = G(f)dZ_X(f)$, we have $|dZ_Y(f)|^2 = |G(f)dZ_X(f)|^2 = |G(f)|^2|dZ_X(f)|^2$. Taking expectations gives

$$\begin{aligned} E\{|dZ_Y(f)|^2\} = |G(f)|^2 E\{|dZ_X(f)|^2\} &\iff \\ dS_Y^{(I)}(f) = |G(f)|^2 dS_X^{(I)}(f) &\iff \\ S_Y(f)df = |G(f)|^2 S_X(f)df &\iff \\ S_Y(f) = |G(f)|^2 S_X(f), & \end{aligned}$$

should the spectral density functions exist.

2 marks (A)

(iii) $L\{e^{i2\pi ft}\} = e^{i2\pi f(t-1)} + e^{i2\pi ft} + e^{i2\pi f(t+1)} = e^{i2\pi ft}(e^{-i2\pi f} + 1 + e^{i2\pi f}) = e^{i2\pi ft}(1 + 2\cos(2\pi f))$. Therefore $G(f) = 1 + 2\cos(2\pi f)$, and $S_Y(f) = \sigma_\epsilon^2(1+2\cos(2\pi f))^2$. The Fourier transform method for computing $G(f)$ can also be used here.

sim. seen ⇓

3 marks (A)

(iv) The Autoregressive Process can be written as $\epsilon_t = L\{X_t\}$ where $L\{X_t\} = X_t - \phi_{1,p}X_{t-1} - ... - \phi_{p,p}X_{t-p}$. To compute the frequency response function, consider

seen ⇓

$$\begin{aligned} L\{e^{i2\pi ft}\} &= e^{i2\pi ft} - \phi_{1,p}e^{i2\pi f(t-1)} - ... - \phi_{p,p}e^{i2\pi f(t-p)} \\ &= e^{i2\pi ft}(1 - \phi_{1,p}e^{-i2\pi f} - ... - \phi_{p,p}e^{-i2\pi fp}). \end{aligned}$$

Therefore $G(f) = 1 - \phi_{1,p}e^{-i2\pi f} - ... - \phi_{p,p}e^{-i2\pi fp}$. Using (a)(ii), it follows that $S_\epsilon(f) = |G(f)|^2 S_X(f)$, and hence

$$S_X(f) = \frac{S_\epsilon(f)}{|G(f)|^2} = \frac{\sigma_\epsilon^2}{|1 - \phi_{1,p}e^{-i2\pi f} - ... - \phi_{p,p}e^{-i2\pi fp}|^2}.$$

4 marks (A)

(b) (i) The spectrum can be written in terms of the complex roots, by substituting $z = \mathrm{e}^{-\mathrm{i}2\pi f}$ into the characteristic equation.

$$\begin{aligned} S_X(f) &= \frac{\sigma_\epsilon^2}{|1-\phi_{1,p}\mathrm{e}^{-\mathrm{i}2\pi f} - \phi_{p,p}\mathrm{e}^{-\mathrm{i}4\pi f}|^2} \\ &= \left.\frac{\sigma_\epsilon^2}{|1-\phi_{1,p}z - \phi_{p,p}z^2|^2}\right|_{z=\mathrm{e}^{-\mathrm{i}2\pi f}} \\ &= \left.\frac{\sigma_\epsilon^2}{|(1-az)(1-bz)|^2}\right|_{z=\mathrm{e}^{-\mathrm{i}2\pi f}} \\ &= \left.\frac{\sigma_\epsilon^2}{|(1-az)|^2|(1-bz)|^2}\right|_{z=\mathrm{e}^{-\mathrm{i}2\pi f}} \\ &= \frac{\sigma_\epsilon^2}{|(1-a\mathrm{e}^{-\mathrm{i}2\pi f})|^2|(1-b\mathrm{e}^{-\mathrm{i}2\pi f})|^2}. \end{aligned}$$

3 marks (B)

(ii) Using (i) and the hint, when have $a = r\mathrm{e}^{-\mathrm{i}2\pi f_0}$ and $b = r\mathrm{e}^{\mathrm{i}2\pi f_0}$, the spectral density function is therefore

sim. seen ⇓

$$S(f) = \frac{\sigma_\epsilon^2}{[1-2r\cos(2\pi(f_0-f))+r^2][1-2r\cos(2\pi(f_0+f))+r^2]}.$$

Comparing with the given spectral density function, we identify $r = 0.5$ and $f_0 = 0.125$.
To obtain the parameters $\phi_{1,2}$ and $\phi_{2,2}$ we recognise

$$\begin{aligned} \Phi(z) &= (1-r\mathrm{e}^{\mathrm{i}2\pi f_0}z)(1-r\mathrm{e}^{-\mathrm{i}2\pi f_0}z) \\ &= 1 - r(\mathrm{e}^{\mathrm{i}2\pi f_0} + \mathrm{e}^{-\mathrm{i}2\pi f_0})z + r^2z^2 \\ &= 1 - 2r\cos(2\pi f_0)z + r^2z^2. \end{aligned}$$

Therefore, $\phi_{1,2} = 2r\cos(2\pi f_0)$ and $\phi_{2,2} = -r^2$. Substituting the identified values of r and f_0, we get $\phi_{1,2} = \cos(\pi/4) = 1/\sqrt{2}$ and $\phi_{2,2} = -1/4$. Therefore $X_t = \frac{1}{\sqrt{2}}X_{t-1} - \frac{1}{4}X_{t-2} + \epsilon_t$.

6 marks (C)

3. (a) (i) When the mean is known, the estimator becomes $\boxed{\text{seen} \Downarrow}$

$$\widehat{s}_\tau^{(p)} = \frac{1}{N} \sum_{t=1}^{N-|\tau|} (X_t - \mu)(X_{t+|\tau|} - \mu) \qquad |\tau| \le N-1.$$

Taking expectations of both sides gives

$$\begin{aligned} E\{\widehat{s}_\tau^{(p)}\} &= \frac{1}{N} \sum_{t=1}^{N-|\tau|} E\{(X_t - \mu)(X_{t+|\tau|} - \mu)\} \\ &= \frac{1}{N} \sum_{t=1}^{N-|\tau|} s_\tau \\ &= \frac{N-|\tau|}{N} s_\tau \\ &= \left(1 - \frac{|\tau|}{N}\right) s_\tau, \end{aligned}$$

and therefore is biased ($\tau \neq 0$). The bias tends to zero as $N \to \infty$. $\boxed{\text{3 marks (A)}}$

$\boxed{\text{unseen} \Downarrow}$

(ii) With $\text{bias}\{\widehat{s}_\tau^{(p)}\} = E\{\widehat{s}_\tau^{(p)}\} - s_\tau$, we have $\text{bias}\{\widehat{s}_\tau^{(p)}\} = -\frac{|\tau|}{N} s_\tau$. Now, the MA(1) process has acvs

$$s_\tau = \begin{cases} \sigma_\epsilon^2(1+\theta^2) & \tau = 0 \\ -\theta\sigma_\epsilon^2 & |\tau| = 1 \\ 0 & |\tau| \neq 0 \end{cases}$$

Therefore $\text{bias}\{\widehat{s}_0^{(p)}\} = 0$, $\text{bias}\{\widehat{s}_1^{(p)}\} = \text{bias}\{\widehat{s}_{-1}^{(p)}\} = \frac{1}{N}\theta\sigma_\epsilon^2$, and $\text{bias}\{\widehat{s}_\tau^{(p)}\} = 0$ for all $|\tau| > 1$.
Therefore, for $|\text{bias}\{\widehat{s}_\tau^{(p)}\}| < C$, we require $\left|\frac{1}{N}\theta\sigma_\epsilon^2\right| < C$, which implies $N > |\theta|\sigma_\epsilon^2/C$. $\boxed{\text{5 marks (B)}}$

(b) (i) Substituting in the given form, we get $\boxed{\text{sim. seen} \Downarrow}$

$$\begin{aligned} \widehat{S}^{(d)}(f) = \sum_{\tau=-(N-1)}^{(N-1)} \widehat{s}_\tau^{(d)} \mathrm{e}^{-\mathrm{i}2\pi f\tau} &= \sum_{\tau=-(N-1)}^{(N-1)} \sum_{t=1}^{N-|\tau|} h_t X_t h_{t+\tau} X_{t+|\tau|} \mathrm{e}^{-\mathrm{i}2\pi f\tau} \\ &= \sum_{j=1}^{N} \sum_{k=1}^{N} h_j X_j h_k X_k \mathrm{e}^{-\mathrm{i}2\pi f(k-j)} \\ &= \left| \sum_{t=1}^{N} h_t X_t \mathrm{e}^{-\mathrm{i}2\pi ft} \right|^2, \end{aligned}$$

where the summation interchange has occurred by swapping diagonal sums for row sums. $\boxed{\text{4 marks (B)}}$

(ii) Using the form $\boxed{\text{unseen} \Downarrow}$

$$\widehat{S}^{(d)}(f) = \sum_{\tau=-(N-1)}^{(N-1)} \sum_{t=1}^{N-|\tau|} h_t X_t h_{t+\tau} X_{t+|\tau|} \mathrm{e}^{-\mathrm{i}2\pi f\tau}$$

and taking expectations gives

$$\begin{aligned} E\{\widehat{S}^{(p)}(f)\} &= \sum_{\tau=-(N-1)}^{(N-1)} \sum_{t=1}^{N-|\tau|} h_t h_{t+\tau} E\{X_t X_{t+|\tau|}\} \mathrm{e}^{-\mathrm{i}2\pi f\tau} \\ &= \sum_{\tau=-(N-1)}^{(N-1)} \sum_{t=1}^{N-|\tau|} h_t h_{t+\tau} s_\tau \mathrm{e}^{-\mathrm{i}2\pi f\tau} \end{aligned}$$

Therefore

$$\int_{-1/2}^{1/2} E\{\widehat{S}^{(p)}(f)\}\mathrm{d}f = \sum_{\tau=-(N-1)}^{(N-1)} \sum_{t=1}^{N-|\tau|} h_t h_{t+\tau} s_\tau \int_{-1/2}^{1/2} \mathrm{e}^{-\mathrm{i}2\pi f\tau}\mathrm{d}f$$

and with

$$\int_{-1/2}^{1/2} \mathrm{e}^{-\mathrm{i}2\pi f\tau}\mathrm{d}f = \begin{cases} 1 & \tau = 0 \\ 0 & \tau \neq 0 \end{cases}$$

we obtain

$$\int_{-1/2}^{1/2} E\{\widehat{S}^{(p)}(f)\}\mathrm{d}f = s_0 \sum_{t=1}^{N} h_t^2 = s_0$$

$\boxed{\text{6 marks (D)}}$

(iii) The periodogram can be considered a direct spectral estimator where $h_t = 1/\sqrt{N}$ for all t. Therefore, the result will also hold for the periodogram, i.e., it is equal to s_0. $\boxed{\text{seen} \Downarrow}$

$\boxed{\text{2 marks (B)}}$

4. (a) First note

seen ⇓

$$S(0) = \sum_{\tau=-\infty}^{\infty} s_\tau \mathrm{e}^{-\mathrm{i}2\pi 0\tau} = \sum_{\tau=-\infty}^{\infty} s_\tau.$$

Now, $S(f)$ is always non-negative, therefore

$$S(f) = |S(f)| = \left|\sum_{\tau=-\infty}^{\infty} s_\tau \mathrm{e}^{-\mathrm{i}2\pi f\tau}\right| \leq \sum_{\tau=-\infty}^{\infty} |s_\tau \mathrm{e}^{-\mathrm{i}2\pi f\tau}| = \sum_{\tau=-\infty}^{\infty} |s_\tau|.$$

If s_τ is positive for all τ, this gives

$$S(f) \leq \sum_{\tau=-\infty}^{\infty} s_\tau = S(0).$$

4 marks (B)

(b) Processes $\{X_t\}$ and $\{Y_t\}$ are said to be jointly stationary if the are both individually stationary and the cross-covariance $\text{cov}(X_t, Y_{t+\tau})$ depends only on τ.

3 marks (A)

(c) (i) To show joint stationarity, we must show $\{X_t\}$ and $\{Y_t\}$ are individually stationary, and that $\text{cov}\{X_t, Y_{t+\tau}\}$ depends only on τ. The question states $\{X_t\}$ is stationary. To show stationarity of $\{Y_t\}$, we first need to show it has a constant mean. By the independence of $\{X_t\}$ and $\{W_t\}$, it follows that

sim. seen ⇓

$$E\{Y_t\} = E\{W_t X_t\} = E\{W_t\}E\{X_t\} = p \cdot 0 = 0.$$

Next we need to show $\text{cov}\{Y_t, Y_{t+\tau}\}$ depends only on τ.

$$\begin{aligned}
\text{cov}\{Y_t, Y_{t+\tau}\} &= E\{Y_t Y_{t+\tau}\} \\
&= E\{W_t X_t W_{t+\tau} X_{t+\tau}\} \\
&= E\{W_t W_{t+\tau}\}E\{X_t X_{t+\tau}\} \\
&= E\{W_t W_{t+\tau}\} s_{X,\tau},
\end{aligned}$$

again, by the independence of $\{X_t\}$ and $\{W_t\}$.
When $\tau = 0$, we have $E\{W_t W_{t+\tau}\} = E\{W_t^2\} = p$. When $\tau \neq 0$, we have $E\{W_t W_{t+\tau}\} = E\{W_t\}E\{W_{t+\tau}\} = p^2$. Therefore

$$s_{Y,\tau} = \begin{cases} p s_{X,0} & \tau = 0 \\ p^2 s_{X,\tau} & \tau \neq 0 \end{cases}$$

which depends only on τ, hence $\{Y_t\}$ is stationary.

Finally, consider the cross-covariance

$$\begin{aligned}\operatorname{cov}\{X_t, Y_{t+\tau}\} &= E\{X_t Y_{t+\tau}\} \\ &= E\{X_t W_{t+\tau} X_{t+\tau}\} \\ &= E\{W_t\} E\{X_t X_{t+\tau}\} \\ &= p s_{X,\tau},\end{aligned}$$

which depends only on τ, and we have $s_{XY,\tau} = p s_{X,\tau}$. $\boxed{\text{6 marks (D)}}$

(ii) The cross-spectrum $S_{XY}(f)$ is given as $S_{XY}(f) = \sum_{\tau=-\infty}^{\infty} s_{XY,\tau} \mathrm{e}^{-\mathrm{i}2\pi f\tau} = p\sum_{\tau=-\infty}^{\infty} s_{X,\tau} \mathrm{e}^{-\mathrm{i}2\pi f\tau} = pS_X(f)$. The spectral density function of $\{Y_t\}$ is given as

$$\begin{aligned}S_Y(f) &= \sum_{\tau=-\infty}^{\infty} s_{Y,\tau} \mathrm{e}^{-\mathrm{i}2\pi f\tau} \\ &= p^2 \sum_{\tau=-\infty}^{\infty} s_{X,\tau} \mathrm{e}^{-\mathrm{i}2\pi f\tau} - p^2 s_{X,0} + p s_{X,0} \\ &= p^2 S_X(f) + p(1-p) s_{X,0}.\end{aligned}$$

Therefore

$$\begin{aligned}\gamma_{XY}^2(f) &= \frac{|S_{XY}(f)|^2}{S_X(f) S_Y(f)} \\ &= \frac{p^2 S_X^2(f)}{S_X(f)(p^2 S_X(f) + p(1-p)s_{X,0})} \\ &= \frac{1}{1 + \frac{(1-p)s_{X,0}}{pS_X(f)}}.\end{aligned}$$

$\boxed{\text{4 marks (D)}}$

(iii) The MA(1) process has a non-negative autocovariance sequence ($s_{X,0} = 5/4$, $s_{X,1} = s_{X,-1} = 1/2$, $s_{X,\tau} = 0$ for all $|\tau| > 1$). Therefore, from (a) $S_X(f)$ attains is maximum at $f = 0$, which means the denominator in the above expression for $\gamma_{XY}^2(f)$ will attain its minimum at $f = 0$, and hence $\gamma_{XY}^2(f)$ attains its maximum at $f = 0$. $\boxed{\text{3 marks (B)}}$

5. (a) (i) Representing $J(f) = A(f) + \mathrm{i}B(f)$, we have $A(f) = \sum_{t=1}^{N} h_t G_t \cos(2\pi f t)$ and $B(f) = \sum_{t=1}^{N} h_t G_t \sin(2\pi f t)$. It is immediate that $E\{A(f)\} = E\{B(f)\} = 0$, and therefore $\boxed{\text{seen} \Downarrow}$

$$\operatorname{var}\{A(f_k)\} = E\{A^2(f_k)\} = \sum_{t=1}^{N}\sum_{t'=1}^{N} h_t h_{t'} E\{G_t G_{t'}\} \cos(2\pi f_k t)\cos(2\pi f_k t').$$

With $\{G_t\}$ a white noise process and $\{h_t\}$ the rectangular kernel, it follows that

$$\operatorname{var}\{A(f_k)\} = \frac{\sigma^2}{N}\sum_{t=1}^{N} \cos^2(2\pi f_k t) = \frac{\sigma^2}{N}\cdot\frac{N}{2} = \frac{\sigma^2}{2}$$

using the given identities. The result for $\operatorname{var}\{B(f_k)\}$ follows in an identical way. $\boxed{\text{2 marks}}$

(ii) As above

$$\begin{aligned}
\operatorname{cov}\{A(f_j), A(f_k)\} &= E\{A(f_j)A(f_k)\} \\
&= \sum_{t=1}^{N}\sum_{t'=1}^{N} h_t h_{t'} E\{G_t G_{t'}\} \cos(2\pi f_j t)\cos(2\pi f_k t') \\
&= \frac{\sigma^2}{N}\sum_{t=1}^{N} \cos(2\pi f_j t)\cos(2\pi f_k t) = 0 \text{ for all } f_j \neq f_k. \\
\operatorname{cov}\{B(f_j), B(f_k)\} &= E\{B(f_j)B(f_k)\} \\
&= \sum_{t=1}^{N}\sum_{t'=1}^{N} h_t h_{t'} E\{G_t G_{t'}\} \sin(2\pi f_j t)\sin(2\pi f_k t') \\
&= \frac{\sigma^2}{N}\sum_{t=1}^{N} \sin(2\pi f_j t)\sin(2\pi f_k t) = 0 \text{ for all } f_j \neq f_k. \\
\operatorname{cov}\{A(f_j), B(f_k)\} &= E\{A(f_j)B(f_k)\} \\
&= \sum_{t=1}^{N}\sum_{t'=1}^{N} h_t h_{t'} E\{G_t G_{t'}\} \cos(2\pi f_j t)\sin(2\pi f_k t') \\
&= \frac{\sigma^2}{N}\sum_{t=1}^{N} \cos(2\pi f_j t)\sin(2\pi f_k t) = 0 \text{ for all } f_j \text{ and } f_k.
\end{aligned}$$

$\boxed{\text{3 marks}}$

(iii) We can write $S^{(p)}(f_k) = |J(f_k)|^2 = A^2(f_k) + B^2(f_k)$. Both $\sqrt{(2/\sigma^2)}A(f_k)$ and $\sqrt{(2/\sigma^2)}B(f_k)$ are unit variance zero mean Gaussian random variables, and furthermore are independent by part (ii) (uncorrelated Gaussian rvs $\Longrightarrow$ independence). Therefore, $(2/\sigma^2)(A^2(f_k) + B^2(f_k)) = (2/\sigma^2)S^{(p)}(f_k) \overset{\mathrm{d}}{=} \chi_2^2$, which gives $S^{(p)}(f_k) \overset{\mathrm{d}}{=} (\sigma^2/2)\chi_2^2$. $\boxed{\text{4 marks}}$

(b) (i) Let $f' = f + \eta$. From Isserlis theorem, we have

$$\begin{aligned}
\operatorname{cov}\{\widehat{S}_G^{(d)}(f), \widehat{S}_G^{(d)}(f')\} &= \operatorname{cov}\{J(f)J^*(f), J(f')J^*(f')\} \\
&= \operatorname{cov}\{J(f), J(f')\}\operatorname{cov}\{J^*(f), J^*(f')\} \\
&\qquad + \operatorname{cov}\{J(f), J^*(f')\}\operatorname{cov}\{J^*(f), J(f')\} \\
&= E\{J^*(f)J(f')\}E\{J(f)J^*(f')\} + E\{J^*(f)J^*(f')\}E\{J(f)J(f')\} \\
&= |E\{J^*(f)J(f')\}|^2 + |E\{J(f)J(f')\}|^2.
\end{aligned}$$

Using the stated identity, it follows that

$$\begin{aligned}
E\{J^*(f)J(f')\} &= \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} H^*(f-u)H(f'-u')E\{\mathrm{d}Z^*(u)\mathrm{d}Z(u')\} \\
&= \int_{-1/2}^{1/2} H^*(f-u)H(f'-u)S_G(u)\mathrm{d}u.
\end{aligned}$$

Since $\mathrm{d}Z(-u) = \mathrm{d}Z^*(u)$, it is also true that

$$J(f) = -\int_{-1/2}^{1/2} H(f+u)\mathrm{d}Z^*(u),$$

and hence

$$E\{J(f)J(f')\} = -\int_{-1/2}^{1/2} H(f+u)H(f'-u)S_G(u)\mathrm{d}u.$$

It follows that

$$\operatorname{cov}\{\widehat{S}_G^{(d)}(f), \widehat{S}_G^{(d)}(f+\eta)\} =$$
$$\left|\int_{-1/2}^{1/2} H^*(f-u)H(f+\eta-u)S_G(u)\mathrm{d}u\right|^2 + \left|\int_{-1/2}^{1/2} H(f+u)H(f+\eta-u)S_G(u)\mathrm{d}u\right|^2.$$

$\boxed{\text{7 marks}}$

(ii) Using the given result, for $\eta > 0$ $\boxed{\text{unseen} \Downarrow}$

$$R(\eta) = \frac{R(\eta, f)}{R(\eta, 0)} = \frac{S_G^2(f)\left|\sum_{t=1}^N h_t^2 \mathrm{e}^{-\mathrm{i}2\pi\eta t}\right|^2}{S_G^2(f)\left|\sum_{t=1}^N h_t^2\right|^2} = \left|\sum_{t=1}^N \frac{1}{N}\mathrm{e}^{-\mathrm{i}2\pi\eta t}\right|^2 = \frac{1}{N}\mathcal{F}(\eta).$$

Therefore $R(\eta) = 0$ when $\mathcal{F}(\eta) = 0$, which occurs at $\eta = k/N$ for $k = \pm 1, \pm 2, \pm 3, \ldots$. $\boxed{\text{4 marks}}$