Master

UNIVERSITY OF LONDON

E1.14 Mathematics 2

B.ENG. AND M.ENG. EXAMINATIONS 2006

For Internal Students of Imperial College London

This paper is also taken for the relevant examination for the Associateship of the City & Guilds of London Institute

PART I : MATHEMATICS 2 (ELECTRICAL ENGINEERING)

Wednesday 31st May 2006 10.00 am - 1.00 pm

Answer EIGHT questions.

Corrected Copy

[Before starting, please make sure that the paper is complete; there should be 6 pages, with a total of 10 questions. Ask the invigilator for a replacement if your copy is faulty.]

1. Find the stationary points of the function

$$f(x, y) = (y - 1)(y^2 - x^2)$$

and determine their nature.

Consider the surface $z = f(x, y)$.

Sketch the contour $z = 0$ and other representative contours of the surface.

Indicate the position of the stationary points.

2. (i) If $u = f(x, y)$ and x, y are defined parametrically in terms of s, t via

$$x(s, t) = s^2 + t^2, \qquad y(s, t) = 2st,$$

show that

$$s\frac{\partial u}{\partial s} - t\frac{\partial u}{\partial t} = 2(s^2 - t^2)\frac{\partial f}{\partial x}.$$

(ii) Furthermore show that

$$\frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial s} = 4(s^2 - t^2) = \left(\frac{\partial s}{\partial x}\frac{\partial t}{\partial y} - \frac{\partial t}{\partial x}\frac{\partial s}{\partial y}\right)^{-1}.$$

PLEASE TURN OVER

3. Show graphically that the equation

$$x^4 - x - 0.1 = 0$$

has one negative solution and one positive solution.

Consider the fixed point schemes

(i) $x_{n+1} = x_n^4 - 0.1$,

(ii) $x_{n+1} = (x_n + 0.1)^{1/4}$.

Show that (i) with $x_0 = 0$, and (ii) with $x_0 = 1$ are convergent schemes for computing the negative and positive solutions, respectively.

Compute the first two iterates for both schemes.

Write down the Newton-Raphson scheme for computing these solutions.

Starting with the same x_0 as above, compute the first two Newton-Raphson iterates for both solutions.

4. Two planes are defined by the equations

$$\begin{aligned} x - 4y + z &= 6 , \\ x - y + z &= 2 . \end{aligned}$$

(i) Find the perpendicular distance from the origin to each plane.

(ii) Find the vector equation of the straight line defined by the intersection of these planes.

5. Consider the $n \times n$ matrix

$$A = I - B .$$

where $B^m = 0$ for some positive integer m.

Show that

$$A^{-1} = I + B + B^2 + \ldots + B^{m-1} .$$

Given

$$B = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} ,$$

show that $B^3 = 0$.

Hence find A^{-1}. Use the Binomial Theorem to find A^p where $p \geq 2$ is an integer.

Calculate A^2 by direct multiplication and check your result.

6. (i) Using the substitution $v = y/x$, or otherwise, find an implicit solution of the differential equation

$$\frac{dy}{dx} = \frac{x}{2x - y}$$

that satisfies $y = 0$ at $x = 1$.

Give an expression for x in terms of y/x.

(ii) Find the general solution $y(x)$ of the differential equation

$$(x^2 + 1) \frac{dy}{dx} + xy = (x^2 + 1)^{1/2} .$$

Evaluate the arbitrary constant given that

$$\lim_{x \to \infty} x(y - 1) = 1 .$$

PLEASE TURN OVER

7. (i) Consider the ordinary differential equation

$$\frac{dy}{dx} = \frac{xy}{(x^2 + y^2)}$$

and, using the substitution $y = ux$, show that y and x are related via

$$x^2 = 2y^2 \ln [Ay] ,$$

where A is an arbitrary constant.

(ii) Solve the ordinary differential equation

$$\frac{dy}{dx} = \frac{x + y + 1}{x + y - 1} .$$

8. (i) Find the solution $y(x)$ of the differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 8y = 20 \sin 2x ,$$

that satisfies $y = 0$ and $\dfrac{dy}{dx} = 0$ at $x = 0$.

(ii) Find the general solution $y(x)$ of the differential equation

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 4y = e^{4x} .$$

9. Consider the differential equation

$$(1+x^2)\,\frac{d^2y}{dx^2} \;+\; (2x-1)\,\frac{dy}{dx} \;=\; 0\,. \tag{1}$$

Use Leibnitz's theorem to show by differentiating n times that

$$(1+x^2)\,y^{(n+2)}(x) \;+\; [2(n+1)x-1]\,y^{(n+1)}(x) \;+\; n(n+1)\,y^{(n)}(x) \;=\; 0\,.$$

Hence show that, if $y(0) \;=\; y'(0) \;=\; 1$, the Taylor-MacLaurin expansion of $y(x)$ is

$$y(x) \;=\; 1 \;+\; x \;+\; \frac{x^2}{2} \;-\; \frac{x^3}{6} \;+\; \ldots$$

Show also that the substitution if $v = y'$ reduces equation (1) to a first order equation and hence find a closed form solution for $y(x)$ subject to the same conditions

$$y(0) \;=\; y'(0) \;=\; 1\,.$$

10. Given the function $f(x) \;=\; \cos x$ for $0 < x \leq \pi$ and $f(x) \;=\; -\cos x$ in $-\pi < x \leq 0$, show that

$$\cos x \;=\; \frac{8}{\pi}\sum_{m=1}^{\infty}\frac{m}{4m^2-1}\;\sin 2mx$$

holds in $0 < x \leq \pi$.

Hence show that

$$\frac{1}{\sqrt{2}} \;=\; \frac{8}{\pi}\sum_{p=1}^{\infty}\frac{(2p-1)\,(-1)^{p+1}}{16p^2-16p+3}\,.$$

END OF PAPER

MATHEMATICS DEPARTMENT

MATHEMATICAL FORMULAE

1. VECTOR ALGEBRA

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1,\, a_2,\, a_3)$$

Scalar (dot) product: $\mathbf{a}\,.\,\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

Vector (cross) product:

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Scalar triple product:

$$[\mathbf{a},\,\mathbf{b},\,\mathbf{c}] = \mathbf{a}\,.\,\mathbf{b}\times\mathbf{c} = \mathbf{b}\,.\,\mathbf{c}\times\mathbf{a} = \mathbf{c}\,.\,\mathbf{a}\times\mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product: $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{c}\,.\,\mathbf{a})\mathbf{b} - (\mathbf{b}\,.\,\mathbf{a})\mathbf{c}$

2. SERIES

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \ldots \quad (\alpha \text{ arbitrary, } |x|<1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots\,,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n\frac{x^{2n}}{(2n)!} + \ldots\,,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + \ldots\,,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^n\frac{x^{n+1}}{(n+1)} + \ldots (-1 < x \le 1)$$

3. TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin(a+b) = \sin a\,\cos b + \cos a\,\sin b\,;$$

$$\cos(a+b) = \cos a\,\cos b - \sin a\,\sin b\,.$$

$$\cos iz = \cosh z\,; \quad \cosh iz = \cos z\,; \quad \sin iz = i\sinh z\,; \quad \sinh iz = i\sin z\,.$$

4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f\,D^n g + \tbinom{n}{1} Df\,D^{n-1}g + \ldots + \tbinom{n}{r} D^r f\,D^{n-r}g + \ldots + D^n f\;g\,.$$

(b) Taylor's expansion of $f(x)$ about $x = a$:

$$f(a+h) = f(a) + hf'(a) + h^2 f''(a)/2! + \ldots + h^n f^{(n)}(a)/n! + \epsilon_n(h)\,,$$

where $\epsilon_n(h) = h^{n+1}f^{(n+1)}(a+\theta h)/(n+1)!, \quad 0 < \theta < 1\,.$

(c) Taylor's expansion of $f(x,\, y)$ about $(a,\, b)$:

$$f(a+h,\, b+k) = f(a,\, b) + [hf_x + kf_y]_{a.b} + 1/2!\,\left[h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}\right]_{a.b} + \ldots$$

(d) Partial differentiation of $f(x,\, y)$:

i. If $y = y(x)$, then $f = F(x)$, and $\dfrac{dF}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dx}$.

ii. If $x = x(t)$, $y = y(t)$, then $f = F(t)$, and $\dfrac{dF}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$.

iii. If $x = x(u,\, v)$, $y = y(u,\, v)$, then $f = F(u,\, v)$, and

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}\,, \quad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}\,.$$

(e) Stationary points of $f(x,\, y)$ occur where $f_x = 0$, $f_y = 0$ simultaneously.
Let $(a,\, b)$ be a stationary point: examine $D = [f_{xx}f_{yy} - (f_{xy})^2]_{a.b}$.
If $D > 0$ and $f_{xx}(a,\, b) < 0$, then $(a,\, b)$ is a maximum;
If $D > 0$ and $f_{xx}(a,\, b) > 0$, then $(a,\, b)$ is a minimum;
If $D < 0$ then $(a,\, b)$ is a saddle-point.

(f) Differential equations:

i. The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating factor $I(x) = \exp[\int P(x)(dx)]$, so that $\dfrac{d}{dx}(Iy) = IQ$.

ii. $P(x,\, y)dx + Q(x,\, y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y$.

5. INTEGRAL CALCULUS

(a) An important substitution: $\tan(\theta/2) = t$:
$\sin\theta = 2t/(1+t^2)$, $\cos\theta = (1-t^2)/(1+t^2)$, $d\theta = 2\,dt/(1+t^2)$.

(b) Some indefinite integrals:

$$\int (a^2 - x^2)^{-1/2} dx = \sin^{-1}\left(\frac{x}{a}\right), \quad |x| < a .$$

$$\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1}\left(\frac{x}{a}\right) = \ln\left\{\frac{x}{a} + \left(1 + \frac{x^2}{a^2}\right)^{1/2}\right\} .$$

$$\int (x^2 - a^2)^{-1/2} dx = \cosh^{-1}\left(\frac{x}{a}\right) = \ln\left|\frac{x}{a} + \left(\frac{x^2}{a^2} - 1\right)^{1/2}\right| .$$

$$\int (a^2 + x^2)^{-1} dx = \left(\frac{1}{a}\right)\tan^{-1}\left(\frac{x}{a}\right) .$$

6. NUMERICAL METHODS

(a) Approximate solution of an algebraic equation:

If a root of $f(x) = 0$ occurs near $x = a$, take $x_0 = a$ and
$x_{n+1} = x_n - [f(x_n)/f'(x_n)]$, $n = 0, 1, 2 \ldots$

(Newton Raphson method).

(b) Formulae for numerical integration: Write $x_n = x_0 + nh$, $y_n = y(x_n)$.

i. Trapezium rule (1-strip): $\int_{x_0}^{x_1} y(x)dx \approx (h/2)[y_0 + y_1]$.

ii. Simpson's rule (2-strip): $\int_{x_0}^{x_2} y(x)dx \approx (h/3)[y_0 + 4y_1 + y_2]$.

(c) Richardson's extrapolation method: Let $I = \int_a^b f(x)dx$ and let I_1, I_2 be two estimates of I obtained by using Simpson's rule with intervals h and $h/2$. Then, provided h is small enough,

$$I_2 + (I_2 - I_1)/15 ,$$

is a better estimate of I .

7. LAPLACE TRANSFORMS

Function	Transform	Function	Transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)dt$	$af(t) + bg(t)$	$aF(s) + bG(s)$
df/dt	$sF(s) - f(0)$	d^2f/dt^2	$s^2F(s) - sf(0) - f'(0)$
$e^{at}f(t)$	$F(s-a)$	$tf(t)$	$-dF(s)/ds$
$(\partial/\partial\alpha)f(t,\alpha)$	$(\partial/\partial\alpha)F(s,\alpha)$	$\int_0^t f(t)dt$	$F(s)/s$
$\int_0^t f(u)g(t-u)du$	$F(s)G(s)$		
1	$1/s$	$t^n (n = 1, 2 \ldots)$	$n!/s^{n+1}$, $(s > 0)$
e^{at}	$1/(s-a)$, $(s > a)$	$\sin\omega t$	$\omega/(s^2 + \omega^2)$, $(s > 0)$
$\cos\omega t$	$s/(s^2 + \omega^2)$, $(s > 0)$	$H(t-T) = \begin{cases} 0, & t < T \\ 1, & t > T \end{cases}$	e^{-sT}/s, $(s, T > 0)$

8. FOURIER SERIES

If $f(x)$ is periodic of period $2L$, then $f(x + 2L) = f(x)$, and

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \text{ where}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad n = 0, 1, 2, \ldots, \text{ and}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1, 2, 3, \ldots$$

Parseval's theorem

$$\frac{1}{L}\int_{-L}^{L} [f(x)]^2\, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right) .$$

	EXAMINATION QUESTIONS/SOLUTIONS 2005-06	Course EE I (2) Q1 p1
~~Question~~ SOLN		Marks & seen/unseen

Parts

$$\frac{\partial f}{\partial x} = (y-1)(-2x) \quad \text{—— (1)}$$

$$\frac{\partial f}{\partial y} = y^2 - x^2 + 2y(y-1) \quad \text{—— (2)}$$

For stationary point we need

$$\frac{\partial f}{\partial x} = 0 \ \underline{\text{and}} \ \frac{\partial f}{\partial y} = 0.$$

2

From (1) $x = 0$ or $y = 1$

From (2) $y = 0$ or $\frac{2}{3}$ | $x = \pm 1$

Thus 4 st. pts

$P_1\,(0,0)$, $P_2\,(0,\frac{2}{3})$, $P_3\,(1,1)$, $P_4\,(-1,1)$ 4

Nature determined by

$$\Delta = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left[\frac{\partial^2 f}{\partial x \partial y}\right]^2$$

2

$$\frac{\partial^2 f}{\partial x^2} = -2(y-1), \quad \frac{\partial^2 f}{\partial y^2} = 6y - 2$$

$$\frac{\partial^2 f}{\partial x \partial y} = -2x$$

2

Setter's initials RLJ	Checker's initials RB	Page number

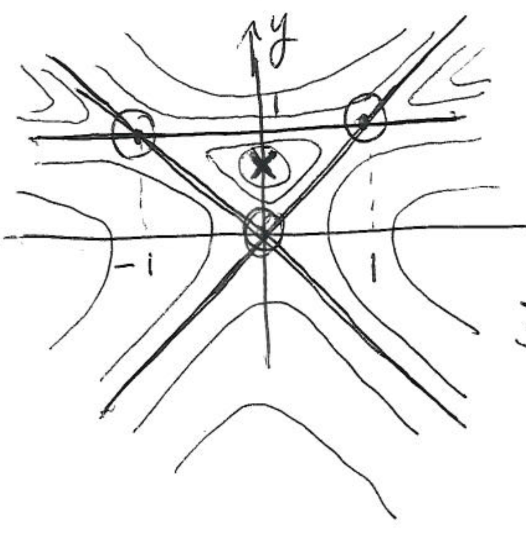
	EXAMINATION QUESTIONS/SOLUTIONS 2005-06	Course Q1 p2
~~Question~~ SOLN		Marks & seen/unseen

Parts

	P_1	P_2	P_3	P_4
$\frac{\partial^2 f}{\partial x^2}$	$+2$	$\frac{2}{3}$	0	0
$\frac{\partial^2 f}{\partial y^2}$	-2	2	4	$+\cancel{8}4$
$\frac{\partial^2 f}{\partial x \partial y}$	0	0	-2	$+2$
Δ	-4	$\frac{4}{3}$	-4	-4
$z=f(x,y)$	0	$-\frac{4}{27}$	0	0
NATURE	SADDLE	MINIMUM	SADDLE	SADDLE

~~formula sheet~~

4



$z=0$ if $x=\pm y$ or $y=1$

2

3 SADDLES AT ⊙
1 MINIMUM AT X

4

TOTAL 20

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RLJ	RB	

Solution 1

(a) $x = s^2+t^2$

$y = 2st$

$u = f$

$u_s = f_x x_s + f_y y_s = 2sf_x + 2tf_y$ $\times t$

$u_t = f_x x_t + f_y y_t = 2tf_x + 2sf_y$ $\times s$. and subtract.

$su_s - tu_t = 2(s^2-t^2)f_x$. 6 marks

(b)

$x_s = 2s$

$x_t = 2t$

$y_s = 2t$

$y_t = 2s$

$$x_s y_t - x_t y_s = 2s.2s - 4t^2 = 4(s^2-t^2).$$

4 marks

$x+y = (s+t)^2$

$x-y = (s-t)^2$

$2(s+t)[s_x + t_x] = 1$

$2(s-t)[s_x - t_x] = 1.$

$$s_x = \frac{1}{4(s+t)} + \frac{1}{4(s-t)} = \frac{s}{2(s^2-t^2)} = t_y$$ by interchanging variables.

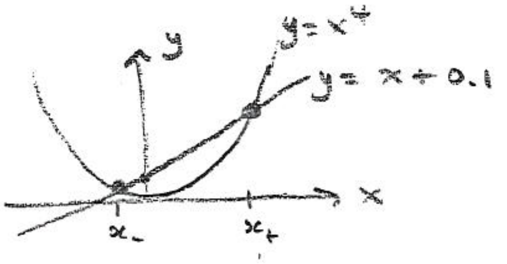
$$t_y = \frac{1}{4(s+t)} - \frac{1}{4(s-t)} = \frac{-t}{2(s^2-t^2)} = s_y$$ 5 marks

$$\therefore\ s_x t_y - t_x s_y = \frac{1}{4(s^2-t^2)}$$ as reqd 5 marks

R.B

	EXAMINATION QUESTIONS/SOLUTIONS 2005-06	Course Q3
Question Q7	Solution	Marks & seen/unseen

Parts



$x^4 = x + 0.1$

$-0.1 < x_- < 0$

$1 < x_+ < 1.1$ — 3

$x_{n+1} = g(x_n)$

(i) $g(x) = x^4 - 0.1 \implies g'(x) = 4x^3$

$|g'(x)| < 1$ if $|x| < \left(\frac{1}{4}\right)^{1/3}$ — 3

$\therefore x_0 = 0$, convergent to x_- as $|x_-| < \left(\frac{1}{4}\right)^{1/3}$

(ii) $g(x) = (x+0.1)^{1/4} \implies g'(x) = \frac{1}{4}(x+0.1)^{-\frac{3}{4}}$

$|g'(x)| < 1$ if $|x+0.1| \geq \left(\frac{1}{4}\right)^{4/3}$ — 3

$\therefore x_0 = 1$, convergent to $x_+ > 1$

(i) $x_0 = 0$, $x_1 = -0.1$, $x_2 = -0.0999$ — 2

(ii) $x_0 = 1$, $x_1 = (1.1)^{1/4} \approx 1.0241$

$x_2 \approx (1.1241)^{1/4} \approx 1.0297$ — 2

$f(x) = x^4 - x - 0.1 \implies f'(x) = 4x^3 - 1$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = \frac{3x_n^4 + 0.1}{4x_n^3 - 1}$$ — 3

$x_0 = 0$, $x_1 = -0.1$, $x_2 = \frac{-0.1003}{1.004} \approx -0.0999$ — 2

$x_0 = 1$, $x_1 = \frac{3.1}{3} = 1.0\dot{3}\dot{3}$, $x_2 \approx 1.0313$. — 2

/20

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JWB	JMC	

Parts

(i)

$x - 4y + z = 6$

can be written

$\underset{\sim}{r} \cdot \underset{\sim}{n}_1 = 6$ — 1

with $\underset{\sim}{n}_1 = (1, -4, 1)$

Normalize

$\underset{\sim}{r} \cdot \hat{\underset{\sim}{n}}_1 = 6/\sqrt{18}$ —— (1)

$\hat{\underset{\sim}{n}}_1 = \frac{1}{\sqrt{18}}(1, -4, 1) \leftarrow$ unit normal — 2

Similarly $x - y + z = 2$

can be written — 1

$\underset{\sim}{r} \cdot \hat{\underset{\sim}{n}}_2 = 2/\sqrt{3}$ —— (2)

$\hat{\underset{\sim}{n}}_2 = \frac{1}{\sqrt{3}}(1, -1, 1) \leftarrow$ unit normal — 2

Perp. dist from O to plane 1 is rhs. of eqn. (1) ie. $6/\sqrt{18} = \sqrt{2}$ — 1

Perp dist. from O to plane 2 is rhs. of eqn (2) ie. $2/\sqrt{3}$ — 1

	EXAMINATION QUESTIONS/SOLUTIONS 2005-06	Course EE1 P2 Q4
Question SOLN		Marks & seen/unseen

Parts

(ii) Vector equation of line is

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda \underset{\sim}{l}$$

where $\underset{\sim}{a}$ is a point in plane and $\underset{\sim}{l}$ is direction vector. **2**

$\underset{\sim}{l}$ is perp. to both $\underset{\sim}{n}_1$ and $\underset{\sim}{n}_2$

$$\therefore \underset{\sim}{l} = \underline{n}_1 \times \underline{n}_2 = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -4 & 1 \\ 1 & -1 & 1 \end{vmatrix}$$

determinant not needed for this computation

$$= \underline{i}(-3) - \underline{j}(0) + \underline{k}(3)$$ **2**

$$= (-3, 0, 3)$$

Choose $\underset{\sim}{a}$ to be any point in both planes. One choice is $z=0$

$x - 4y = 6$ and $x - y = 2$ **3**

i.e. $y = -4/3$, $x = 2/3$

i.e. $\underline{r} = (2/3, -4/3, 0) + \lambda(-3, 0, 3)$ **2**

Other valid choices possible

Total $\boxed{20}$

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	EXAMINATION QUESTIONS/SOLUTIONS 2005-06	Course EE1 p2
Question SOLN		Marks & seen/unseen Q5

Parts

The following multiplication verifies that A^{-1} is of the given form

$$(I-B)(I+B+B^2+\cdots+B^{m-1})$$

$$= I - B + B - B^2 + B^2 - B^3 + \cdots + B^{m-1} - B^m$$

$$= I \quad (\text{cancellation in pairs} + B^m = 0)$$ 4

$$B^2 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ 2

$$B^3 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ 2

Thus $A^{-1} = I + B + B^2$

$$= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$ 2

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	EXAMINATION QUESTIONS/SOLUTIONS 2005-06	Course EE1 P2
Question SOLN		Q5 Marks & seen/unseen
Parts		

$$A^p = (\underline{I} - B)^p = \underline{I} - pB + \frac{p(p-1)}{1.2} B^2$$

from binomial theorem and fact that $B^3 = 0$ — 5

$$A^2 = \underline{I} - 2B + B^2$$

$$= \begin{pmatrix} 1 & -2 & -3 \\ 0 & 1 & +2 \\ 0 & 0 & 1 \end{pmatrix}$$ — 3

Direct multiplication

$$A^2 = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$ — 2

CHECKS.

Total 20

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	EXAMINATION QUESTIONS/SOLUTIONS 2005-06	Course EE1 P2 (1) 6
~~Question~~ SOLN		Marks & seen/unseen
Parts		
(i)	Substitute $y = x\,v(x)$ in given equation to get $v + x\dfrac{dv}{dx} = \dfrac{1}{2-v}$	2
	Thus $x\dfrac{dv}{dx} = \dfrac{1-2v+v^2}{2-v}$	2
	and on separation $\int \dfrac{dx}{x} = \int \dfrac{2-v}{(1-v)^2}\,dv$ ie $\ln\lvert x\rvert = -\ln\lvert 1-v\rvert + \dfrac{1}{1-v} + C$	2
	$y=0$ when $x=1 \Rightarrow v=0$ and $\therefore C=-1$ Hence $\ln\lvert x\rvert = -\ln\left\lvert 1-\dfrac{y}{x}\right\rvert + \dfrac{x}{y-x} - 1$,	2
	Or equivalent form. No simplification needed. NB. 1 mark lost if modulus signs omitted.	
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Question SOLN		Marks & seen/unseen

Parts

(ii) Rewrite in standard form

$$\frac{dy}{dx} + \frac{x}{x^2+1} y = (x^2+1)^{-1/2}$$ 1

$$I.F. = \exp\left[\int \frac{x}{x^2+1} dx\right]$$ 2

$$= \exp\left[\frac{1}{2}\ln(x^2+1)\right] = (x^2+1)^{1/2}$$

$$\therefore (x^2+1)^{1/2}\frac{dy}{dx} + x(x^2+1)^{-1/2} y = 1$$ 1

i.e. $$\frac{d}{dx}\left[(x^2+1)^{1/2} y\right] = 1$$

i.e. $$(x^2+1)^{1/2} y = x + c$$ ✓ 2

or $$y = \frac{x}{(x^2+1)^{1/2}} + \frac{c}{(x^2+1)^{1/2}}$$

Now consider

$$\lim_{x\to\infty} x(y-1) = \lim_{x\to\infty}\left\{x\left[\frac{x}{(x^2+1)^{1/2}} - 1\right] + \frac{cx}{(x^2+1)^{1/2}}\right\}$$

1st term on r.h.s. $\to 0$ as can be seen from binomial th. or otherwise

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	EXAMINATION QUESTIONS/SOLUTIONS 2005-06	Course EE1 p2 Q6
~~Question~~ SOLN		Marks & seen/unseen
Parts	eg $x\left[\frac{x}{(x^2+1)^{1/2}}-1\right] = x\left[\left(1+\frac{1}{x^2}\right)^{-1/2}-1\right]$ $= x\left[1-\frac{1}{2}\frac{1}{x^2}+O\left(\frac{1}{x^4}\right)-1\right]$ $\to 0$	3
	2nd term on right $\to C$	2
	$\therefore\ C = 1$ ✓ and solution is $y = \frac{x}{(x^2+1)^{1/2}}+\frac{1}{(x^2+1)^{1/2}}$	1
		Total $\boxed{20}$
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2.

(a). $\frac{dy}{dx} = \frac{xy}{x^2+y^2} = \frac{1}{\frac{x}{y}+\frac{y}{x}}$

Let $y = ux$

$x\frac{du}{dx} + u = \frac{1}{u+\frac{1}{u}}$

$x\frac{du}{dx} = \frac{u}{u^2+1} - u = \frac{-u^3}{u^2+1}$

$\frac{dx}{x} = \frac{-(u^2+1)du}{u^3} = \left(-\frac{1}{u} - \frac{1}{u^3}\right)du$ 3 marks

$\log x = -\log u + \frac{1}{2u^2} + \text{Const.}$

$\log(xu) = \frac{1}{2u^2} + \text{Constant.}$

$\log y = \frac{1}{2\left(\frac{y^2}{x^2}\right)} \mp \log A$

$2y^2 \log Ay = x^2$ 7 marks.

(b) $\frac{dy}{dx} = \frac{x+y+1}{x+y-1}$

let $u = x+y$ 3 marks

$\frac{du}{dx} - 1 = \frac{dy}{dx}$

$\frac{du}{dx} - 1 = \frac{u+1}{u-1} \Rightarrow \frac{du}{dx} = \frac{2u}{u-1}$ $\quad 2dx = \left(1 - \frac{1}{u}\right)du$

$2x + c = u - \log u$

$\cancel{2}x + c = \cancel{x}+y - \log(x+y)$

$\cancel{y-c+}\; y - x + \text{const} = \log(x+y)$ 7 marks.

R.3

	EXAMINATION QUESTIONS/SOLUTIONS 2005-06	Course EE1 P2 Q8
~~Question~~ SOLN		Marks & seen/unseen

Parts

(i) Try complementary function $y = e^{\lambda x}$

This gives auxiliary equation

$$\lambda^2 - 4\lambda + 8 = 0$$ 2

i.e. $\lambda = 2 \pm \sqrt{4-8} = 2 \pm 2i$

So complementary function is

$$y = (A \sin 2x + B \cos 2x) e^{2x}$$ 1

For particular integral try

$$y = C \sin 2x + D \cos 2x$$ 1

Then equation gives

$$[-4C + 8D + 8C] \sin 2x + [-4D - 8C + 8D] \cos 2x = 20 \sin 2x$$

i.e. $+4C + 8D = 20$

$-8C + 4D = 0$ 2

$\Rightarrow \quad D = 2, \ C = 1$

Hence general solution is

$$y = (A \sin 2x + B \cos 2x) e^{2x} + \sin 2x + 2 \cos 2x$$

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	EXAMINATION QUESTIONS/SOLUTIONS 2005-06	Course EE1 P2
Question SOLN		Q 8 Marks & seen/unseen

Parts

$$y' = 2(A\sin 2x + B\cos 2x)e^{2x}$$
$$+ (2A\cos 2x - 2B\sin 2x)e^{2x}$$
$$+ 2\cos 2x - 4\sin 2x$$ 1

Conditions $\Rightarrow$ $B + 2 = 0$

and $2B + 2A + 2 = 0$

i.e. $B = -2,\ A = 1$

and $y = (\sin 2x - 2\cos 2x)e^{2x} + \sin 2x + 2\cos 2x$ 3

(ii) Try complementary function $y = e^{\lambda x}$, gives aux. eqn. $\lambda^2 - 5\lambda + 4 = 0$ 2

$\lambda = 4$ or 1

Complementary function is then

$$y = Ae^{4x} + Be^{x}$$ 2

Note inhomogeneous term, e^{4x}, appears in complementary function. Therefore try $y = Cxe^{4x}$ for particular integral 1

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	EXAMINATION QUESTIONS/SOLUTIONS 2005-06	Course EE1 P2 Q8
Question SOLN	.	Marks & seen/unseen

Parts

This gives $y' = Ce^{4x} + 4Cxe^{4x}$

$y'' = 8Ce^{4x} + 16Cxe^{4x}$

and subst. in given equ. we get

$8Ce^{4x} + 16Cxe^{4x} - 5Ce^{4x}$
$-20Cxe^{4x} + 4Cxe^{4x} = e^{4x}$ — 2

Collecting terms

$(8C - 5C - 1)e^{4x}$
$+ (16C - 20C + 4C)xe^{4x} = 0$ — 2

i.e. $3C - 1 = 0 \quad C = 1/3$

and general solution is

$y = Ae^{4x} + Be^{x} + \frac{1}{3}xe^{4x}.$ — 1

Total $\boxed{20}$

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	EXAMINATION QUESTIONS/SOLUTIONS 2005-06	Course EE1 P2 Q9
Question SOLN		Marks & seen/unseen

Parts

Differentiating n times by Leibnitz

$$(1+x^2)\, y^{(n+2)}(x) + n\,2x\, y^{(n+1)}(x) + \frac{2n(n-1)}{2} y^{(n)}(x)$$
$$+ (2x-1)\, y^{(n+1)}(x) + n\, 2\, y^{(n)}(x) = 0$$ 2

Thus
$$(1+x^2)\, y^{(n+2)}(x) + \left[2(n+1)x - 1\right] y^{(n+1)}(x)$$
$$+ n(n+1)\, y^{(n)}(x) = 0$$ 2

Now put $x = 0$ in this to get

$$y^{(n+2)}(0) - y^{(n+1)}(0) + n(n+1)\, y^{(n)}(0) = 0$$ 2

This is a recurrence relation which determines $y^{(n)}(0)$ for $n \geq 2$ given $y(0)$ and $y^{(1)}(0)$. Thus 1

$y(0) = 1$

$y'(0) = 1$

$n=0$ $y^{(2)}(0) = y^{(1)}(0) = 1$

$n=1$ $y^{(3)}(0) = y^{(2)}(0) - 2y^{(1)}(0) = -1$ 3

Enough!

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	EXAMINATION QUESTIONS/SOLUTIONS 2005-06	Course EE I(2) Q9
Question		Marks & seen/unseen
Parts	Taylor-MacLaurin expansion is $y(x) = y(0) + y'(0)x + y''(0)\frac{x^2}{2!} + \frac{y^{(3)}(0)}{3!}x^3 + \cdots$ $= 1 + x + \frac{x^2}{2} - \frac{x^3}{6} + \cdots$	2
	Substitution gives $(1+x^2)\frac{dv}{dx} + (2x-1)v = 0$	2
	Separate $\int \frac{dv}{v} = -\int \frac{2x-1}{1+x^2}\,dx$ $\ln\lvert v\rvert = -\ln(1+x^2) + \tan^{-1}x + C$	3
	$y' = 1$ when $x=0 \Rightarrow v=1 \Rightarrow C=0$ so $\frac{dy}{dx} = v = \frac{1}{1+x^2}\exp(\tan^{-1}x)$	2
	Integrate $\Rightarrow y = \exp(\tan^{-1}x)$ ✓	1
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3) $f(x) = \begin{cases} \cos x & 0 < x \le \pi \\ -\cos x & -\pi < x \le 0 \end{cases}$

EE I (2) Q10

Use a Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx.$$ 3 marks

$$b_n = \frac{2}{\pi}\int_0^{\pi} \cos x \sin nx \, dx$$

$$= \frac{1}{\pi}\int_0^{\pi}\left[\sin(n+1)x + \sin(n-1)x\right]dx$$

$$= -\frac{1}{\pi}\left\{\frac{1}{(n+1)}\cos(n+1)x + \frac{\cos(n-1)x}{(n-1)}\right\}\Big|_0^{\pi}$$ 3 marks.

$$= -\frac{1}{\pi}\left\{\frac{(-1)^{n+1}-1}{(n+1)} + \frac{(-1)^{n-1}-1}{(n-1)}\right\}$$

$= 0$ n odd

$$= \frac{2}{\pi}\left(\frac{\cancel{2}1}{(n+1)} + \frac{\cancel{2}1}{(n-1)}\right) = \frac{4n}{\pi(n^2-1)}$$

$$\therefore f = \sum_{m=1}^{\infty} \frac{\cancel{8}}{\cancel{\pi}} \cdot \frac{4.2m}{\pi((2m)^2-1)} \sin 2mx$$ 4 marks

Set $x = \frac{\pi}{4}$ $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ 3 marks

$$\therefore \frac{1}{\sqrt{2}} = \sum_{m=1}^{\infty} \frac{8m}{\pi[4m^2-1)} \sin\left[\frac{m\pi}{2}\right]$$

" 0 if m even

set $m = 2p-1$ so odd for integer p. if $p=1,3.. = +$, $p=2.. = -$

$$= \sum_{p=1}^{\infty} \frac{8[2p-1]}{\pi[4(4p^2+1-4p)-1]} \sin\left[(2p-1)\frac{\pi}{2}\right] = (-1)^{p+1}$$

$$\frac{1}{\sqrt{2}} = \sum_{p=1}^{\infty} \frac{8(2p-1)}{\pi[16p^2+3-16p)} (-1)^{p+1}.$$ 7 marks.

RB