UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 1998

BEng Honours Degree in Computing Part III
BSc Honours Degree in Mathematics and Computer Science Part III
MSci Honours Degree in Mathematics and Computer Science Part III
MSc Degree in Advanced Computing
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the
Diploma of Membership of Imperial College
Associateship of the Royal College of Science
Associateship of the City and Guilds of London Institute

PAPER 3.78

MATHEMATICAL STRUCTURES IN COMPUTING SCIENCE

Thursday, May 7th 1998, 2.00 - 4.00

Answer THREE questions

For admin. only: paper contains 4 questions

Section A *(Use a separate answer book for this Section)*

1a Define the terms *lattice, distributive* (as applied to a lattice), and *Boolean complement* (of an element of a lattice). Notions to do with partial order can be assumed.

i) Give an example of a latttice in which every element has a Boolean complement, but at least one element has more than one Boolean complement. What is the least number of elements that such an example can have? Explain.

ii) Briefly explain how a propositional logic of the usual kind gives rise to a Boolean algebra (as its "Lindenbaum algebra").

b Define the terms *monotonic* (in connection with a map between posets), and *lattice homomorphism*. Let L,M be lattices, and $h:L \rightarrow M$ a mapping. In the case of assertions i),ii) (to follow), you are asked to state with reasons whether the given assertion is necessarily true.

i) If h is monotonic, then it is a homomorphism.

ii) If h is a homomorphism, then it is monotonic.

iii) Assume now that L is a finite lattice, and that M is the two-element lattice $\{0,1\}$. Let A be the pre-image of 1, that is, $A = \{x \varepsilon A | \ h(x) = 1\}$. Show that, if h is a homomorphism, and A is non-empty, then A has a least element. Deduce that the number of homomorphisms from L to M is not greater than $|L| + 1$.

2a What is meant by the *left factor* relation of a monoid? Illustrate your definition with the example of the monoid of strings in a given alphabet.

i) Prove that the left factor relation is always a pre-order.

ii) Explain why there is always a least element with respect to the left factor order.

iii) Given a set S, define two distinct monoid structures on $\wp(S)$, each of which will induce either the usual (subset) ordering of $\wp(S)$ or its inverse (superset) as the left factor order.

iv) What simple condition on a monoid (M, o, e) would be enough to ensure that, for any elements x,y of M, xoy is an upper bound of x,y in the left factor order? (Explain.)

b What is meant by a *signature* for a class of algebras? Describe the signature Σ^1 of Hennessy's process algebras.

i) What advantage is gained by presenting the process language as the term algebra of Σ^1, rather than by a BNF-style syntax definition?

ii) Let RT and PS be the usual Σ^1-algebras of rooted trees and of sets of prefix-free strings, respectively. Define a Σ^1-algebra PS^+, differing from PS only in that it admits arbitrary sets of strings, rather than just the prefix-free sets.

iii) Let t be the term a(a + b) + abac. Find a term t' which has the same interpretation as t in PS, but a different interpretation in PS^+ ; and also a term t'' with the same interpretation as t in PS^+, but a different one in RT.

iv) Determine whether it is possible for two terms e,e' to have the same interpretation in RT, but different ones in PS^+.

turn over

Section B (Use a separate answer book for this Section)

3a i) Let Y be a set, and let M be the set of functions from Y to itself. Show that, with composition as the multiplication, M is a monoid. (Use "applicational" order for composition, $(f \circ g)(x) = f(g(x))$.)

ii) If X is a set, what does it mean to say that the monoid X* of lists over X is the *free monoid* over X?

b Let X and Y be sets, and let $\alpha: X \times Y \to Y$ be a function. Just by using the free monoid property of X*, show that there is a unique function $\alpha': X^* \times Y \to Y$ such that for all $x \in X$, $y \in Y$ and $xs1, xs2 \in X^*$ the following conditions hold. (We write $xs \cdot y$ as an abbreviation for $\alpha'(xs, y)$.)

$$[] \cdot y = y$$

$$(xs1{+}{+}xs2) \cdot y = xs1 \cdot (xs2 \cdot y)$$

$$[x] \cdot y = \alpha(x,y)$$

(*Hint:* consider the curried forms of α and α'.)

c Suppose X, Y and α are given as in part b. Show in detail how there is a category $\boldsymbol{C}$ for which the objects are the elements of Y, and the morphisms from y_1 to y_2 are in 1-1 correspondence with the elements xs of X* such that $y_1 = xs \cdot y_2$.

The three parts carry, respectively, 25%, 40%, 35% of the marks.

4a i) Let $\boldsymbol{C}$ be a category. What is meant by a *product* $X \times Y$ of two objects of $\boldsymbol{C}$?

ii) If M and N are two monoids, their set-theoretic product $M \times N$ can be given a monoid structure "componentwise" by

$$(x,y) \circ (x',y') = (x \circ x', y \circ y')$$

$$1_{M \times N} = (1_M, 1_N)$$

and the projection functions $p: M \times N \to M$ and $q: M \times N \to N$ are then monoid homomorphisms.

Show that $M \times N$ can be used as a product in the category **Mon** of monoids and monoid homomorphisms. Explain *briefly* why this argument works for any category of algebras described by operators and equational laws.

b i) In the category **ComMon** of *commutative* monoids and monoid homomorphisms, show that products are also coproducts.

ii) Let M and N be two commutative monoids. Is $M \times N$ necessarily a coproduct in **Mon**? Justify your answer.

End of paper