IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2013

MSc and EEE/ISE PART IV: MEng and ACGI

DIGITAL SIGNAL PROCESSING AND DIGITAL FILTERS

Tuesday, 7 May 10:00 am

Time allowed: 3:00 hours

There are FOUR questions on this paper.

Answer Question 1 and any TWO other questions

Question 1 is worth 40% of the marks and other questions are worth 30%

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible   First Marker(s) :   D.M. Brookes

Second Marker(s) :   P.T. Stathaki

DIGITAL SIGNAL PROCESSING AND DIGITAL FILTERS

Information for Candidates:

Notation

- All signals and filter coefficients are real-valued unless explicitly noted otherwise.

- Unless otherwise specified, upper and lower case letters are used for sequences and their z-transforms. The signal at a block diagram node V is $v[n]$ and its z-transform is $V(z)$.

- $x[n] = [a, b, c, d, e, f]$ means that $x[0] = a,$ ... $,x[5] = f$ and that $x[n] = 0$ outside this range.

- $\Re(z)$, $\Im(z)$, z^*, $|z|$ and $\angle z$ denote respectively the real part, imaginary part, complex conjugate, magnitude and argument of a complex number z.

Abbreviations

BIBO	Bounded Input, Bounded Output
CTFT	Continuous-Time Fourier Transform
DCT	Discrete Cosine Transform
DFT	Discrete Fourier Transform
DTFT	Discrete-Time Fourier Transform
LTI	Linear Time-Invariant
MDCT	Modified Discrete Cosine Transform
SNR	Signal-to-Noise Ratio

Standard Sequences

- $\delta[n] = 1$ for $n = 0$ and 0 otherwise.

- $\delta_{\text{condition}}[n] = 1$ whenever "condition" is true and 0 otherwise.

- $u[n] = 1$ for $n \geq 0$ and 0 otherwise.

Geometric Progression

- $\sum_{n=0}^{r} \alpha^n z^{-n} = \frac{1 - \alpha^{r+1} z^{-r-1}}{1 - \alpha z^{-1}}$ or, more generally, $\sum_{n=q}^{r} \alpha^n z^{-n} = \frac{\alpha^q z^{-q} - \alpha^{r+1} z^{-r-1}}{1 - \alpha z^{-1}}$

Forward and Inverse Transforms

z: $\quad X(z) = \sum_{-\infty}^{\infty} x[n] z^{-n}$ $\qquad\qquad\qquad\qquad\qquad x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$

CTFT: $\quad X(j\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt$ $\qquad\qquad\qquad x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega) e^{j\Omega t} d\Omega$

DTFT: $\quad X(e^{j\omega}) = \sum_{-\infty}^{\infty} x[n] e^{-j\omega n}$ $\qquad\qquad\qquad x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$

DFT: $\quad X[k] = \sum_{0}^{N-1} x[n] e^{-j2\pi \frac{kn}{N}}$ $\qquad\qquad\qquad x[n] = \frac{1}{N} \sum_{0}^{N-1} X[k] e^{j2\pi \frac{kn}{N}}$

DCT: $\quad X[k] = \sum_{n=0}^{N-1} x[n] \cos \frac{2\pi(2n+1)k}{4N}$ $\qquad x[n] = \frac{X[0]}{N} + \frac{2}{N} \sum_{n=1}^{N-1} X[k] \cos \frac{2\pi(2n+1)k}{4N}$

MDCT: $\quad X[k] = \sum_{n=0}^{2N-1} x[n] \cos \frac{2\pi(2n+1+N)(2k+1)}{8N}$ $\qquad y[n] = \frac{1}{N} \sum_{0}^{N-1} X[k] \cos \frac{2\pi(2n+1+N)(2k+1)}{8N}$

Convolution

DTFT: $\quad v[n] = x[n] * y[n] = \sum_{r=-\infty}^{\infty} x[r] y[n-r]$ $\qquad \Leftrightarrow \qquad V(e^{j\omega}) = X(e^{j\omega}) Y(e^{j\omega})$

$\qquad v[n] = x[n] y[n] \qquad \Leftrightarrow \qquad V(e^{j\omega}) = \frac{1}{2\pi} X(e^{j\omega}) \circledast Y(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta}) Y(e^{j(\omega-\theta)}) d\theta$

DFT: $\quad v[n] = x[n] \circledast_N y[n] = \sum_{r=0}^{N-1} x[r] y[(n-r) \mod N]$ $\qquad \Leftrightarrow \qquad V[k] = X[k] Y[k]$

$\qquad v[n] = x[n] y[n] \qquad \Leftrightarrow \qquad V[k] = \frac{1}{N} X[k] \circledast_N Y[k] = \frac{1}{N} \sum_{r=0}^{N-1} X[r] Y[(k-r) \mod N]$

Group Delay

The group delay of a filter, $H(z)$, is $\tau_H(e^{j\omega}) = -\frac{d\angle H(e^{j\omega})}{d\omega} = \Re\left(\frac{-z}{H(z)} \frac{dH(z)}{dz}\right)\Big|_{z=e^{j\omega}} = \Re\left(\frac{\mathscr{F}(nh[n])}{\mathscr{F}(h[n])}\right)$ where $\mathscr{F}()$ denotes the DTFT.

Order Estimation for FIR Filters

Three increasingly sophisticated formulae for estimating the minimum order of an FIR filter with unity gain passbands:

1. $M \approx \frac{a}{3.5\Delta\omega}$

2. $M \approx \frac{a-8}{2.2\Delta\omega}$

3. $M \approx \frac{a-1.2-20\log_{10} b}{4.6\Delta\omega}$

where a = stop band attenuation in dB, b = peak-to-peak passband ripple in dB and $\Delta\omega$ = width of smallest transition band in normalized rad/s.

z-plane Transformations

A lowpass filter, $H(z)$, with cutoff frequency ω_0 may be transformed into the filter $H(\hat{z})$ as follows:

Target $H(\hat{z})$	Substitute	Parameters
Lowpass $\hat{\omega} < \hat{\omega}_1$	$z^{-1} = \frac{\hat{z}^{-1} - \lambda}{1 - \lambda \hat{z}^{-1}}$	$\lambda = \frac{\sin\left(\frac{\omega_0 - \hat{\omega}_1}{2}\right)}{\sin\left(\frac{\omega_0 + \hat{\omega}_1}{2}\right)}$
Highpass $\hat{\omega} > \hat{\omega}_1$	$z^{-1} = -\frac{\hat{z}^{-1} + \lambda}{1 + \lambda \hat{z}^{-1}}$	$\lambda = \frac{\cos\left(\frac{\omega_0 + \hat{\omega}_1}{2}\right)}{\cos\left(\frac{\omega_0 - \hat{\omega}_1}{2}\right)}$
Bandpass $\hat{\omega}_1 < \hat{\omega} < \hat{\omega}_2$	$z^{-1} = -\frac{(\rho - 1) - 2\lambda\rho\hat{z}^{-1} + (\rho + 1)\hat{z}^{-2}}{(\rho + 1) - 2\lambda\rho\hat{z}^{-1} + (\rho - 1)\hat{z}^{-2}}$	$\lambda = \frac{\cos\left(\frac{\hat{\omega}_2 + \hat{\omega}_1}{2}\right)}{\cos\left(\frac{\hat{\omega}_2 - \hat{\omega}_1}{2}\right)}, \rho = \cot\left(\frac{\hat{\omega}_2 - \hat{\omega}_1}{2}\right)\tan\left(\frac{\omega_0}{2}\right)$
Bandstop $\hat{\omega}_1 \not< \hat{\omega} \not< \hat{\omega}_2$	$z^{-1} = \frac{(1 - \rho) - 2\lambda\hat{z}^{-1} + (\rho + 1)\hat{z}^{-2}}{(\rho + 1) - 2\lambda\hat{z}^{-1} + (1 - \rho)\hat{z}^{-2}}$	$\lambda = \frac{\cos\left(\frac{\hat{\omega}_2 + \hat{\omega}_1}{2}\right)}{\cos\left(\frac{\hat{\omega}_2 - \hat{\omega}_1}{2}\right)}, \rho = \tan\left(\frac{\hat{\omega}_2 - \hat{\omega}_1}{2}\right)\tan\left(\frac{\omega_0}{2}\right)$

1. a) Figure 1.1 shows the magnitude response of a 2nd order all-pole filter, $H(z)$. Draw dimensioned sketches of the magnitude responses of (i) $H(-z)$, (ii) $H(z^2)$ and (iii) $H(1.2z)$. In each case state the approximate frequencies of any response peaks.   [5]
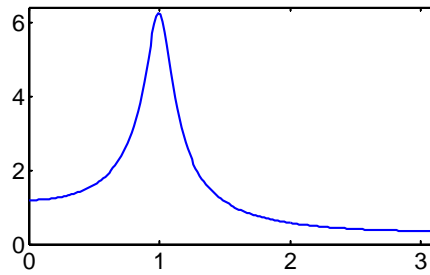

Figure 1.1

b) The impulse response, $h[n]$, of an FIR filter is obtained by multiplying that of an ideal lowpass filter by a window, $w[n]$, symmetric around $n = 0$. Fig. 1.2 shows the frequency response, $H(e^{j\omega})$, of the the FIR filter and also that of the ideal filter in the vicinity of $\omega = 1$. The scaling of the frequency axis has been omitted intentionally.

The DTFT of the window, $W(e^{j\omega})$, satisfies $\int_{\omega=0}^{\pi} W(e^{j\omega})d\omega = \pi$ and is plotted in Fig. 1.3.

For $\omega_0 = 0.0737$, $W(e^{j\omega_0}) = 0$ and $\int_{\omega=0}^{\omega_0} W(e^{j\omega})d\omega = 3.36$.

Estimate the maximum and minimum values of $H(e^{j\omega})$ in Fig. 1.2 and the transition width, $\Delta\omega$, that separates them.   [6]
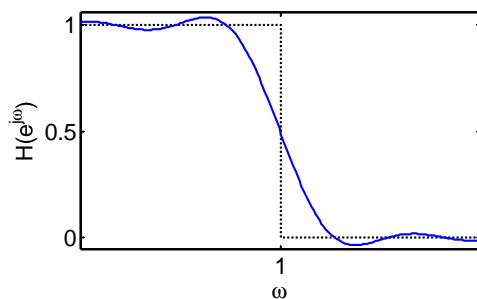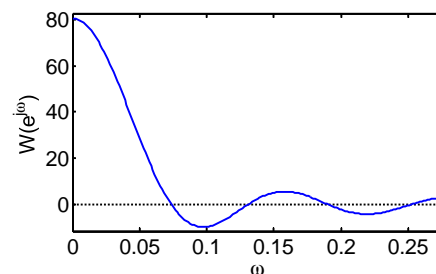

Figure 1.2

Figure 1.3

c)   Figure 1.4 shows the block diagram of a filter implementation comprising two delays, one multiplier with coefficient a and two adder/subtractor elements whose input polarities are as marked. All elements are drawn with their outputs on the right. Determine the transfer function $\frac{Y(z)}{X(z)}$.
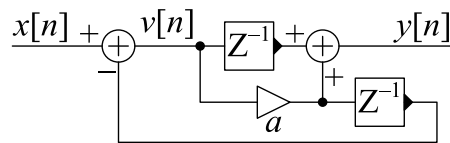
[5]

Figure 1.4

d)   Figure 1.5 shows a downsampler followed by an upsampler.

i)   Show that $V(z) = W\left(z^K\right)$.   [2]

ii)   Show that $c[n] = \frac{1}{K}\sum_{k=0}^{K-1} e^{\frac{j2\pi kn}{K}}$ is equal to 1 if n is an integer multiple of K and equal to 0 otherwise.   [2]

iii)   Hence show that $W(z) = \frac{1}{K}\sum_{k=0}^{K-1} X\left(e^{\frac{-j2\pi k}{K}} z^{\frac{1}{K}}\right)$.   [4]
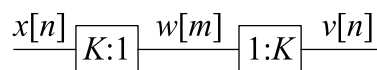

Figure 1.5

e)   Figure 1.6 shows a upsampler followed by a lowpass filter, $H(z)$.

i)   Explain why it is normally necessary to follow an upsampler by a lowpass filter.   [1]

ii)   If $H(z)$ is of order 10 with impulse response $[h[0]\,h[1]\cdots h[10]]$, draw a block diagram, functionally equivalent to Fig. 1.6, in which $H(z)$ is implemented as a polyphase filter operating at the sample rate of $x[n]$. State the impulse response of each filter block in your diagram.   [6]

iii)   Assuming that an FIR filter of order M requires $2M+1$ arithmetic operations (additions or multiplications) per output sample, estimate the number of operations per output sample required for Fig. 1.6 and for your polyphase implementation.
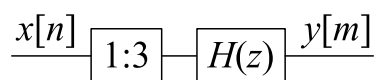
[2]

Figure 1.6

f) Figure 1.7 shows a circuit designed to extract a 19 kHz pilot tone from a baseband stereo FM signal. The bold signal lines denote complex-valued signals. The real-valued signal $x[n]$ is sampled at 200 kHz and its magnitude spectrum is shown in Fig. 1.8 using a frequency axis in unnormalized Hz. As well as the pilot tone at ± 19 kHz, the signal includes three blocks, each of bandwidth ± 15 kHz which are centred at -38, 0 and $+38$ kHz respectively.

The lowpass filters $F(z)$ and $G(z)$ both have passband widths of ± 1 kHz and have transition band widths of 18 and 2 kHz respectively in unnormalized Hz. Within the transition band, their gains (in dB) vary linearly with frequency from 0 dB to -50 dB.

i) Show that the signal $p[n] = e^{-0.2j\pi n}$ has an unnormalized frequency of -20 kHz. [1]

ii) Sketch graphs of the signal spectrum at each of $q[n]$, $r[n]$, $v[m]$ and $w[m]$. Each graph should cover the range ± 100 kHz (for q and r) or ± 10 kHz (for v and w) and should indicate the frequencies of important spectral features in unnormalized Hz. [6]
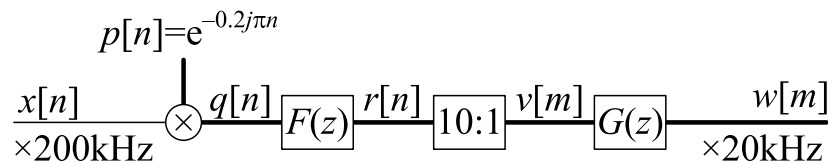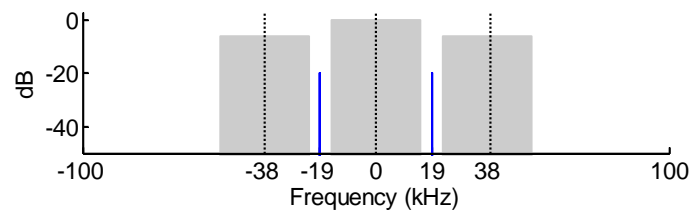

Figure 1.7

Figure 1.8

2. $H(z)$ is a causal FIR filter of order M whose real-valued impulse response satisfies the symmetry relation $h[n] = h[M - n]$. The sequence, $y[n] = h[n] * x[n]$ is the result of applying the filter to an input sequence $x[n]$. We define $x[n] = 0$ for $n < 0$.

   a)   Show that if $H(z_0) = 0$ for some $z_0 \neq 0$, then $H(z_0^*) = H(z_0^{-1}) = H\left((z_0^*)^{-1} \right) = 0$ where z_0^* denotes the complex conjugate of z_0. [3]

   b)   For the two cases (i) M odd and (ii) M even, express $H(e^{j\omega})$ as the product of a complex exponential and a sum of cosine terms. Hence determine the group delay of the filter, $\tau_H(e^{j\omega}) = -\frac{d\angle H(e^{j\omega})}{d\omega}$. [3]

   c)   i)   Determine the number of arithmetic operations (i.e. real-valued additions or multiplications) required per output sample to implement the filter directly in the form $y[n] = \sum_{r=0}^{M} h[r]x[n - r]$. You should exclude from your count the integer operations required for index calculations and assume that the input signal, $x[n]$, is very long. [2]

         ii)   Explain carefully, for both M odd and M even, how the number of arithmetic operations that are needed can be reduced by taking advantage of the symmetry of the filter coefficients and determine the number of arithmetic operations required per output sample when this is done. [4]

   d)   If the input sequence, $x[n]$ is of length $L > M$, explain how it is possible to perform the circular convolution $v[n] = h[n] \circledast_L x[n]$ using DFT and IDFT transforms of length L.

         Giving your reasons fully, identify the values of n for which it is necessarily true that $v[n] = y[n]$. [4]

   e)   In the "overlap-save" method of filtering, a long input sequence, $x[n]$, is divided into overlapping frames of length L which are each circularly convolved with $h[n]$. From each frame, $L - M$ samples are retained and concatenated to form the output sequence $y[n]$.

         If the frames are numbered $i = 0, 1, 2, \cdots$, determine as a function of i, the L sample numbers of $x[n]$ that form the ith input frame and the $L - M$ samples of $y[n]$ that are generated from them. [4]

   f)   If a complex multiply requires 6 arithmetic operations and an L-point DFT or IDFT requires $6L \log_2 L$ operations, estimate the total number of arithmetic operations required per output sample for the overlap-save method of filtering assuming that the input signal, $x[n]$, is very long. [5]

         Compare this with the number required for the direct implementations of part (c) for the cases (i) $M = 10$, $L = 64$ and (ii) $M = 1000$, $L = 8192$ and discuss the circumstances under which the use of the overlap-save method will reduce computation. [5]

3.   a)   Two alternative methods of transforming a continuous-time filter into a discrete-time filter are (i) the bilinear transformation and (ii) the impulse invariance transformation. Explain how the choice of method affects the frequency response of the resultant filter and outline the circumstances under which you would choose one method over the other. [5]

     b)   A continuous-time filter has the transfer function $H(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$.

         i)   Show that the gain of the filter, $|H(j\Omega)|$, equals $-3\,\mathrm{dB}$ at an angular frequency $\Omega = 1$. [1]

         ii)   Using the bilinear mapping, $s = 2f_s \frac{z-1}{z+1}$, determine the transfer function, $G(z)$, of the corresponding discrete-time filter. [5]

         iii)   Determine the sample frequency, f_s, so that the discrete-time filter has a gain of $-3\,\mathrm{dB}$ at an angular frequency $\omega = 1$. [3]

         iv)   For the sample frequency found in (iii) above, give the coefficients of $G(z)$ to 3 decimal places when the coefficients are normalized so that the leading coefficient of the denominator is unity. [4]

     c)   Derive the relationship between p_0 and q_0 so that the causal impulse response of the discrete-time filter $P(z) = \left(1 + p_0 z^{-1}\right)^{-1}$ is a sampled version of the impulse response, $e^{-q_0 t}$, of the continuous-time filter $Q(s) = (s + q_0)^{-1}$ at a sample frequency f_s. [4]

     d)   i)   Express $H(s)$ from part (b) above in the form $H(s) = a_1 \left(s + q_1\right)^{-1} + a_2 \left(s + q_2\right)^{-1}$. [4]

         ii)   Hence, if $F(z)$ is a discrete time filter whose impulse response is a sampled version of that of $H(s)$ at a sample frequency $f_s = 1$, determine the coefficients $F(z)$ to 3 decimal places. [4]

4.   a)   Outline the benefits of dividing a signal into subbands for processing.   [3]

b)   i)   Figure 4.1 shows a downsampler follower by an upsampler. Write down the sequences $s[r]$ and $w[n]$ when the input sequence is given by $v[n] = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix}$.   [1]

ii)   For this input sequence, give expressions for $V(z)$ and $W(z)$ and show that

$$W(z) = \frac{1}{2}\left(V(z) + V(-z)\right).$$

Explain why this relationship will hold for any input sequence $v[n]$.   [3]

c)   In the block diagram of Fig. 4.2, determine simplified expressions for $W_0(z)$, $W_1(z)$ and $Y(z)$ in terms of $X(z)$ and $H(z)$. In the block diagram, the adder input labelled "$-$" is negated before the addition.   [3]

Explain what is meant by saying that the output $y[n]$ is alias-free.   [1]

d)   Suppose that $H(z) = P(z^2) + z^{-1}Q(z^2)$. Show that the block diagram of Fig. 4.3 has the same transfer function, $T(z) = \frac{Y(z)}{X(z)}$, as that of Fig. 4.2 .   [3]

Give a simplified expression for $T(z)$ in terms of $P(z^2)$ and $Q(z^2)$.   [2]

e)   If $P(z^2) = \frac{p+z^{-2}}{1+pz^{-2}}$ and $Q(z^2) = \frac{q+z^{-2}}{1+qz^{-2}}$ are stable filters, determine the values of p and q so that the numerator of $H(z)$ is a scalar multiple of $\left(1+z^{-1}\right)^5$.   [5]

For these values of p and q, determine $H(e^{j\omega})$ and $T(e^{j\omega})$ in polar form for $\omega = 0, \frac{\pi}{2}$ and π .   [3]

Hence sketch dimensioned graphs of $\left|H(e^{j\omega})\right|$ and $\left|T(e^{j\omega})\right|$ versus ω using linear axes for both gain and frequency.   [2]

f)   Draw a block diagram, functionally equivalent to Fig. 4.3 and including $s_0[r]$ and $s_1[r]$ explicitly, showing how the Noble identities can be applied to improve its computational efficiency.   [4]
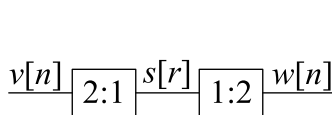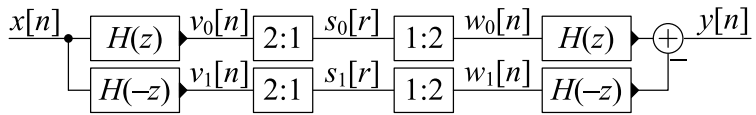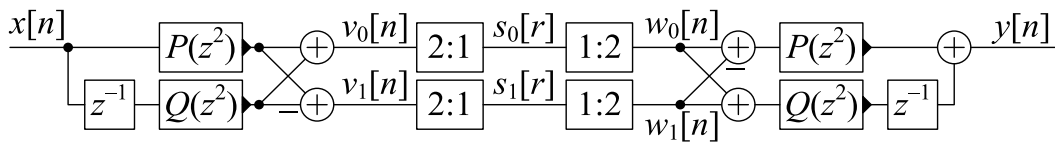

Figure 4.1

Figure 4.2

Figure 4.3

1. a) Figure 1.1 shows the magnitude response of a 2nd order all-pole filter, $H(z)$. Draw dimensioned sketches of the magnitude responses of (i) $H(-z)$, (ii) $H(z^2)$ and (iii) $H(1.2z)$. In each case state the approximate frequencies of any response peaks. [5]
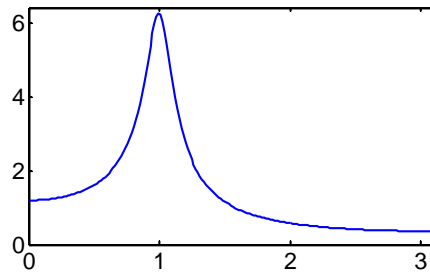

Figure 1.1

(i) The frequency axis is reflected so the peak is now at $\omega = \pi - 1 = 2.14$. Algebraically, $H_1(e^{j\omega}) = H(-e^{j\omega}) = H^*(e^{j(\pi-\omega)})$.

Most got this correct although a few thought the magnitude response would be unchanged. Several said the peak was now at $\omega = 2$ and it was difficult to tell if this was just low precision or an actual mistake. Several people calculated (or tried to calculate) the actual pole positions of $H(z)$ which was not asked for in the question.
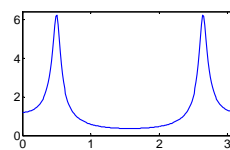
(ii) The frequency axis is compressed and replicated giving peaks at $\omega = 0.5$ and $\pi - 0.5 = 2.64$. Algebraically, $H_2(e^{j\omega}) = H(e^{j2\omega}) = H^*(e^{j2(\pi-\omega)})$.

Several left the peak at $\omega = 1$ and put another peak vaguely in the high frequency region. Others only drew half the spectrum (up to 0.5π).
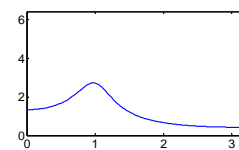
(iii) The pole radii are divided by 1.2 so the peak remains at $\omega = 1$ but has a lower amplitude and wider bandwidth.
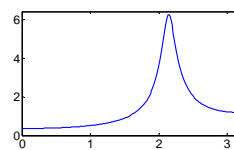
(i) $H(-z)$ *(ii) $H(z^2)$*   *(iii) $H(1.2z)$*

Quite a lot of people thought the peak frequency would change by a factor of 1.2. Very many thought the peak would become narrower instead of broader, perhaps because in the lecture notes I used $\frac{z}{\alpha}$.

b) The impulse response, $h[n]$, of an FIR filter is obtained by multiplying that of an ideal lowpass filter by a window, $w[n]$, symmetric around $n = 0$. Fig. 1.2 shows the frequency response, $H(e^{j\omega})$, of the the FIR filter and also that of the ideal filter in the vicinity of $\omega = 1$. The scaling of the frequency axis has been omitted intentionally.

The DTFT of the window, $W(e^{j\omega})$, satisfies $\int_{\omega=0}^{\pi} W(e^{j\omega}) d\omega = \pi$ and is plotted in Fig. 1.3.

For $\omega_0 = 0.0737$, $W(e^{j\omega_0}) = 0$ and $\int_{\omega=0}^{\omega_0} W(e^{j\omega}) d\omega = 3.36$.

Estimate the maximum and minimum values of $H(e^{j\omega})$ in Fig. 1.2 and the transition width, $\Delta\omega$, that separates them. [6]
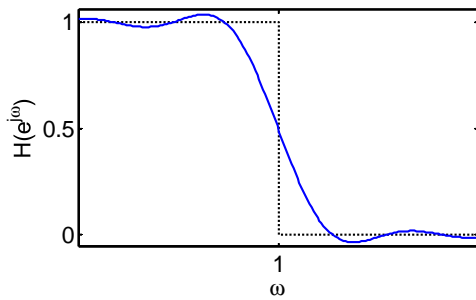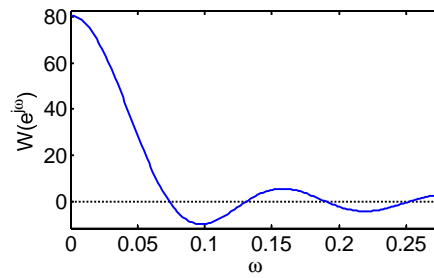

Figure 1.2

Figure 1.3

The filter response, $H\left(e^{j\omega}\right)$, is the circular convolution of $W\left(e^{j\omega}\right)$ with the ideal response, $G(e^{j\omega})$, i.e.

$$H\left(e^{j\omega}\right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} W\left(e^{j\theta}\right) G\left(e^{j(\omega-\theta)}\right) d\theta = \frac{1}{2\pi} \int_{\omega-1}^{\omega+1} W\left(e^{j\theta}\right) d\theta.$$

Provided that $W\left(e^{j\omega}\right)$ has decayed to a negligible value at $\omega = 2$, this integral can be approximated near $\omega = 1$ as

$$H\left(e^{j\omega}\right) = \frac{1}{2\pi} \int_{0}^{\omega+1} W\left(e^{j\theta}\right) d\theta - \frac{1}{2\pi} \int_{0}^{\omega-1} W\left(e^{j\theta}\right) d\theta \approx 0.5 - \frac{1}{2\pi} \int_{0}^{\omega-1} W\left(e^{j\theta}\right) d\theta$$

which reaches its extrema at the zero crossings of $W\left(e^{j\omega}\right)$. Thus the maximum is at $1 - \omega_0 = 0.9263$ and equals $0.5 + \frac{3.36}{2\pi} = 1.035$. The minimum is at $1 + \omega_0 = 1.0737$ and equals $0.5 - \frac{3.36}{2\pi} = 1.035 = -0.035$. The transition width is $\Delta\omega = 0.1474$. Note that the transition width depends on the length of the filter but the ripple does not.

For interest only: the window is a Kaiser window with $\beta = 2$.

Some people omitted the factor $\frac{1}{2\pi}$ in the integral (even though it is given in the formula sheet); this factor is present almost every time you integrate with respect to ω. A few assumed the window was rectangular even though the questions just said it was symmetric (actually it wasn't rectangular).

---

c) Figure 1.4 shows the block diagram of a filter implementation comprising two delays, one multiplier with coefficient a and two adder/subtractor elements whose input polarities are as marked. All elements are drawn with their outputs on the right. Determine the transfer function $\frac{Y(z)}{X(z)}$.
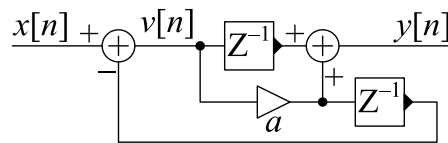
[5]

Figure 1.4

From the block diagram we can write $Y = \left(z^{-1} + a\right)V$ and $V = X - az^{-1}V$. From the second of these, we get $X = \left(1 + az^{-1}\right)V$ and combining the two equations (to eliminate V) then gives $\frac{Y}{X} = \frac{a + z^{-1}}{1 + az^{-1}}$ which is an allpass filter.

Most people found this straightforward. Several neglected the minus sign at the adder input. One or two took the + input to the second adder to be an output as well and the dot below it to be an adder. A few wrote the initial equations correctly but then got lost in a morass of algebra. The goal of the algebra needs to be to eliminate the intemediate variables (in this case V) from the simultaneous equations.

---

d) Figure 1.5 shows a downsampler followed by an upsampler.

i) Show that $V(z) = W\left(z^K\right)$. [2]

ii) Show that $c[n] = \frac{1}{K} \sum_{k=0}^{K-1} e^{\frac{j2\pi kn}{K}}$ is equal to 1 if n is an integer multiple of K and equal to 0 otherwise. [2]

iii) Hence show that $W(z) = \frac{1}{K} \sum_{k=0}^{K-1} X\left(e^{\frac{-j2\pi k}{K}} z^{\frac{1}{K}}\right)$. [4]
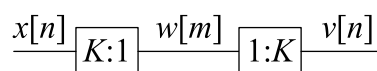

Figure 1.5

(i) From the upsampler, $v[Km] = w[m]$ and $v[n] = 0$ if $K \nmid n$. Hence $V(z) = \sum_n v[n]z^{-n} = \sum_m v[Km]z^{-Km} = \sum_m w[m]z^{-Km} = W\left(z^K\right)$.

Mostly OK but some people get quite confused about the summation indices. Several just wrote $\sum$ instead of $\sum_m$ or $\sum_n$ and seemd to be a bit vague about which they meant. Quite often people would write something like $\sum_n w\left[\frac{n}{K}\right]$ without specifying that the summation must be restricted to multiples of K. We can correctly write $w[m] = x[Km]$ but some instead wrote $w[m] = x[Kn]$ which is not only wrong but doesn't make any sense since the variables m and n are only on one side of the equation.

(ii) If $K \mid n$ then $e^{\frac{-j2\pi kn}{K}} = 1$ since the exponent is an integer multiple of 2π.

Hence, in this case, $c[n] = 1$. Otherwise, , using the geometric progression formula, we can write $c[n] = \frac{1 - e^{\frac{j2\pi Kn}{K}}}{1 - e^{\frac{-j2\pi n}{K}}} = 0$. The numerator is zero since the exponent is an integer multiple of 2π and the denominator is non-zero since $K \nmid n$.

(iii) We have $v[n] = x[n]c[n]$ so, $V(z) = \sum_n v[n]z^{-n} = \sum_n x[n]c[n]z^{-n} = \frac{1}{K}\sum_n \sum_k x[n]e^{\frac{j2\pi kn}{K}}z^{-n} = \frac{1}{K}\sum_k \sum_n x[n]\left(e^{\frac{-j2\pi k}{K}}z\right)^{-n} = \frac{1}{K}\sum_k X\left(e^{\frac{-j2\pi k}{K}}z\right)$.

From part (i), $W(z) = V\left(z^{\frac{1}{k}}\right)$ so $W(z) = \frac{1}{K}\sum_k X\left(e^{\frac{-j2\pi k}{K}}z^{\frac{1}{k}}\right)$.

e)   Figure 1.6 shows a upsampler followed by a lowpass filter, $H(z)$.

     i)   Explain why it is normally necessary to follow an upsampler by a low-pass filter.   [1]

     ii)   If $H(z)$ is of order 10 with impulse response $[h[0]\, h[1] \cdots h[10]]$, draw a block diagram, functionally equivalent to Fig. 1.6, in which $H(z)$ is implemented as a polyphase filter operating at the sample rate of $x[n]$. State the impulse response of each filter block in your diagram.   [6]

     iii)   Assuming that an FIR filter of order M requires $2M + 1$ arithmetic operations (additions or multiplications) per output sample, estimate the number of operations per output sample required for Fig. 1.6 and for your polyphase implementation.
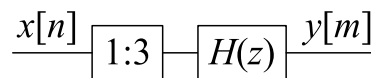
     [2]

Figure 1.6

(i) Upsampling by K introduces $K - 1$ replications of the spectrum which are normally unwanted and must therefore be removed by $H(z)$. Referring to Fig. 1.5, for any integer m, we can write

$$V\left(e^{j\left(\omega + m2\pi K^{-1}\right)}\right) = W\left(e^{jK\left(\omega + m2\pi K^{-1}\right)}\right) = W\left(e^{jK\omega}\right) = V\left(e^{j\omega}\right)$$

which demonstrates the replicated images explicitly.

(ii) We can derive the polyphase implementation by first splitting the filter $H(z)$ into three components with interleaved coefficients (first diagram) and then using the Noble identities to swap them with the upsampler which must now be replicated (second diagram). The filter coefficients are $h_0[n] = [h[0]\, h[3]\, h[6]\, h[9]]$, $h_1[n] = [h[1]\, h[4]\, h[7]\, h[10]]$ and $h_2[n] = [h[2]\, h[5]\, h[8]]$. We can represent the right hand diagram even more compactly by replacing the upsamplers, delays and adders with a commutator or even as a single filter with commutated coefficients.
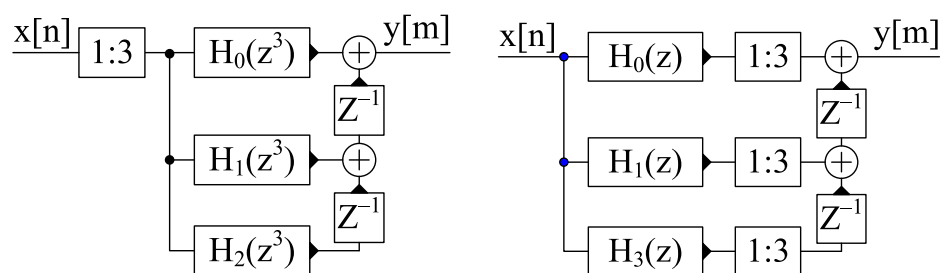
Several people put the had the delays going in the wrong direction (downwards instead of upwards in the figure above). Others put them on the left side of the filters which means they operate at the low sample rate of $x[n]$. Some people omitted the upsampler entirely meaning that y and x were at the same sample rate. Several people who implemented using a commutator had it going in the wrong direction: a commutator always moves in the opposite direction to the delays it replaces. A few put in a second delay line or commutator at the input.

(iii) The diagram in Fig. 1.6 needs 21 operations per output sample. The second diagram above needs $7+7+5+2 = 21$ per input sample which is 7 per output sample.

Many gave their answers in terms of an undefined variable M (presumably $M = 10$) instead of giving the answer 21. Quite a few people did not give the number of operations required for Fig. 1.6 as asked in the question. Quite a few people omitted from their count the two adders in the diagram above; this is especially easy to do if they are hidden inside a commutator.

f) Figure 1.7 shows a circuit designed to extract a 19 kHz pilot tone from a base-band stereo FM signal. The bold signal lines denote complex-valued signals. The real-valued signal $x[n]$ is sampled at 200 kHz and its magnitude spectrum is shown in Fig. 1.8 using a frequency axis in unnormalized Hz. As well as the pilot tone at ± 19 kHz, the signal includes three blocks, each of bandwidth ± 15 kHz which are centred at -38, 0 and $+38$ kHz respectively.

The lowpass filters $F(z)$ and $G(z)$ both have passband widths of ± 1 kHz and have transition band widths of 18 and 2 kHz respectively in unnormalized Hz. Within the transition band, their gains (in dB) vary linearly with frequency from 0 dB to -50 dB.

i) Show that the signal $p[n] = e^{-0.2j\pi n}$ has an unnormalized frequency of -20 kHz. [1]

p[n] has a normalized frequency of -0.2π. Therefore its unnormalized frequency is $-0.2\pi \times \frac{200\,kHz}{2\pi} = -20$ kHz. It follows that q[n] will be the same as x[n] but shifted down in frequency by 20 kHz.

ii) Sketch graphs of the signal spectrum at each of $q[n]$, $r[n]$, $v[m]$ and $w[m]$. Each graph should cover the range ± 100 kHz (for q and r) or ± 10 kHz (for v and w) and should indicate the frequencies of important spectral features in unnormalized Hz. [6]

$$p[n] = e^{-0.2j\pi n}$$

$$x[n] \xrightarrow{\times 200\text{kHz}} \otimes \xrightarrow{q[n]} \boxed{F(z)} \xrightarrow{r[n]} \boxed{10:1} \xrightarrow{v[m]} \boxed{G(z)} \xrightarrow{w[m]}_{\times 20\text{kHz}}$$
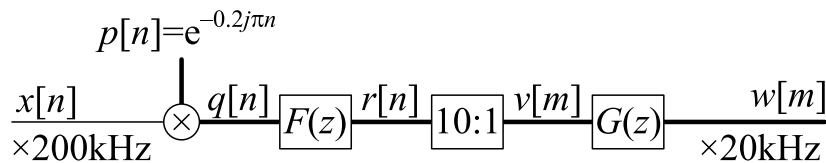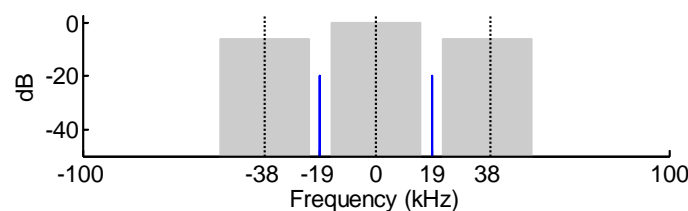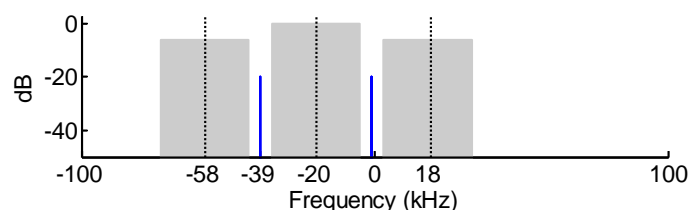
Figure 1.7

Figure 1.8

*q[n] is shifted down by 20 kHz so the pilot tone is now at -1 kHz.*

Quite a few people took p[n] to be a cosine wave rather than a complex exponential. This results in a conjugate symmetric spectrum consisting of the above spectrum added onto its reflection in $\omega = 0$.

r[n] is a filtered version of q[n]. The centre of the transition band is at ± 10 kHz at which point the attenuation is 25 dB. The triangle vertical edges are at -5 and $+3$ kHz respectively with attenuations of 11.1 and 5.6 dB respectively. The 50 dB attenuation frequencies are ± 19 kHz.
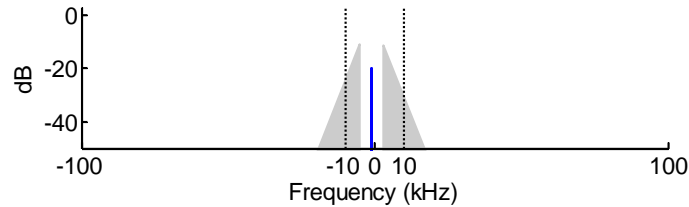
Quite a few people assumed that the filters were ideal lowpass filters at ± 1 kHz with infinitely steep transition regions. This makes the question very easy because all that is left after $F(z)$ is the pilot tone. A small number of people got the spectrum of q[n] correct but then applied the lowpass filter to the original spectrum of x[n] thereby missing out on the pilot tone entirely.

v[m] is decimated which causes the triangular regions outside ± 10 kHz to be aliased. Only the left triangle is large enough to appear on the plot:
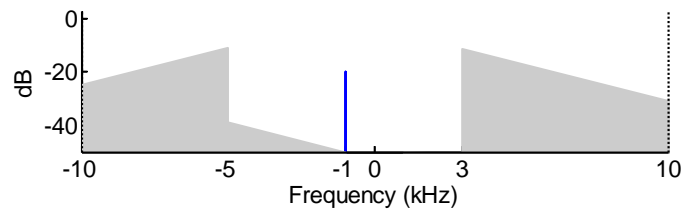
There will be some additional aliassing components but they are all attenuated by the stopband gain of -50 dB. The worst case is the region 5 to 10 kHz where two copies of the central block coincide. Assuming these add incoherently, the resultant interference will be at -47 dB (i.e. an additional $+3$ dB because we add two signals together).

Note that the unnormalized frequency of the pilot tone remains at 1 kHz (even though the normalized frequency is multiplied by 10). Several people wrongly divided the unnormalized frequency by 10 or even 100.

Finally, we apply another lowpass filter which removes everything bad
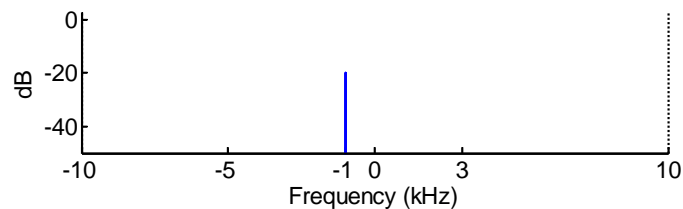
2. $H(z)$ is a causal FIR filter of order M whose real-valued impulse response satisfies the symmetry relation $h[n] = h[M - n]$. The sequence, $y[n] = h[n] * x[n]$ is the result of applying the filter to an input sequence $x[n]$. We define $x[n] = 0$ for $n < 0$.

a) Show that if $H(z_0) = 0$ for some $z_0 \neq 0$, then $H(z_0^*) = H(z_0^{-1}) = H\left((z_0^*)^{-1}\right) = 0$ where z_0^* denotes the complex conjugate of z_0. [3]

We know $H(z_0) = \sum_{m=0}^{M} h[m] z_0^{-m} = 0$.

Since $h[m]$ is real, $H(z_0^*) = \sum_{m=0}^{M} h[m] (z_0^*)^{-m} = \left(\sum_{m=0}^{M} h[m] z_0^{-m}\right)^* = H(z_0)^* = 0$.

We have therefore shown that the roots of a polynomial with real coefficients occur in complex conjugate pairs. Some just stated this as a fact without proving it. Many people had summation limits of $\pm\infty$ even though $h[n]$ is only of length $M + 1$.

$H(z_0^{-1}) = \sum_{m=0}^{M} h[m] z_0^{m} = z^M \sum_{r=0}^{M} h[M - r] z_0^{(M-r)}$ (substitute $m = M - r$)

$= z_0^M \sum_{r=0}^{M} h[r] z_0^{-r} = z^M H(z_0) = 0$.

We have therefore shown that the roots of a polynomial with symmetric coefficients occur in reciprocal pairs. You can also see that, for a symmetric polynomial, $H(z_0^{-1}) = z_0^M H(z_0)$ directly by writing out the polynomial in a more informal proof than the algebraic one given above.

We can obtain $H\left((z_0^*)^{-1}\right) = 0$ by applying the second result to the third.

So the combined result is that the roots of a polynomial with real symmetric coefficients occur in conjugate+reciprocal groups of four.

People found proving these results surprisingly difficult. Very very many people substituted $z = e^{j\omega}$ which not only makes the algebra messier but is also wrong since the zeros of $H(z)$ are not necessarily (or even usually) on the unit circle.

---

b) For the two cases (i) M odd and (ii) M even, express $H(e^{j\omega})$ as the product of a complex exponential and a sum of cosine terms. Hence determine the group delay of the filter, $\tau_H(e^{j\omega}) = -\frac{d\angle H(e^{j\omega})}{d\omega}$. [3]

(i) If M is odd, the impulse response, $h[m]$, has an even number of coefficients. We can pair each one with its partner that has the same value:

$$H\left(e^{j\omega}\right) = \sum_{m=0}^{M} h[m] e^{-j\omega m} = \sum_{m=0}^{0.5(M-1)} h[m] \left(e^{-j\omega m} + e^{-j\omega(M-m))}\right)$$

$$= e^{-0.5j\omega M} \sum_{m=0}^{0.5(M-1)} h[m] \left(e^{-j\omega(m-0.5M)} + e^{+j\omega(m-0.5M)}\right)$$

$$= 2e^{-0.5j\omega M} \sum_{m=0}^{0.5(M-1)} h[m] \cos \omega(0.5M - m)$$

When two complex exponentials are added together, it is often useful to take out an exponential factor whose exponent is the average of the exponents of the

original two terms. In this case, the average of $-j\omega m$ and $-j\omega(M-m)$ is $-0.5j\omega M$ so we take out a factor $e^{-0.5j\omega M}$ leaving behind exponential terms which combine to form a $\cos()$ term.

(ii) if M is even, we have an extra term without a pair (i.e. h[0.5M]) and so we take that out of the sum to leave an even number of terms

$$H\left(e^{j\omega}\right) = \sum_{m=0}^{M} h[m]e^{-j\omega m}$$

$$= e^{-0.5j\omega M}\left(h[0.5M] + \sum_{m=0}^{0.5M-1} h[m]\left(e^{-j\omega(m-0.5M)} + e^{+j\omega(m-0.5M)}\right)\right)$$

$$= e^{-0.5j\omega M}\left(h[0.5M] + 2\sum_{m=0}^{0.5M-1} h[m]\cos\omega(0.5M-m)\right)$$

In both cases, the only non-real term is $e^{-0.5j\omega M}$, so $\angle H(e^{j\omega}) = -0.5\omega M$ possibly $+\pi$ if the amplitude term is negative. The group delay is therefore $0.5M$.

Quite a lot of people got the summation ranges slightly wrong. For example, if M is odd, then 0.5M cannot be a summation limit because it is not an integer.

If you don't mind having more $\cos()$ terms than necessary, you can combine both odd M and even M cases into a single formula

$$H\left(e^{j\omega}\right) = 0.5\left(\sum_{m=0}^{M} h[m]e^{-j\omega m} + \sum_{m=0}^{M} h[m]e^{-j\omega(M-m)}\right)$$

$$= 0.5e^{-0.5j\omega M}\sum_{m=0}^{M} h[m]\left(e^{j\omega(0.5M-m)} + e^{-j\omega(0.5M-m)}\right)$$

$$= e^{-0.5j\omega M}\sum_{m=0}^{M} h[m]\cos\omega(0.5M-m)$$

Many people obtained a group delay of $-0.5M$; it is not possible to have a negative group delay from a causal filter..

---

c)   i)   Determine the number of arithmetic operations (i.e. real-valued additions or multiplications) required per output sample to implement the filter directly in the form $y[n] = \sum_{r=0}^{M} h[r]x[n-r]$. You should exclude from your count the integer operations required for index calculations and assume that the input signal, $x[n]$, is very long.   [2]

An order M filter has $M+1$ coefficients. For each output sample, we therefore require $M+1$ multiplications and M additions for a total of $2M+1$ operations.

The answer to this question was given in question 1(e)(iii) but even so, many people got it wrong. Many people were rather imprecise in counting operations: to add $M+1$ numbers together requires only M additions (e.g. to add two numbers together only requires one addition). Also, people quite often assumed that an order M filter had M coefficients rather than $M+1$. Some found the total number of operations for x[n] of length L or else the number of operations per second rather than the number per output sample as the question asked.

ii)   Explain carefully, for both M odd and M even, how the number of arithmetic operations that are needed can be reduced by taking advantage of the symmetry of the filter coefficients and determine the number of arithmetic operations required per output sample when this is done.   [4]

We can write for Modd:

$$y[n] = \sum_{m=0}^{M} h[r]x[n-r] = \sum_{m=0}^{0.5(M-1)} h[r]\left(x[n-r] + x[n-M+r]\right)$$

which then requires $0.5(M+1)$ additions of pairs of $x[\,]$ values, $0.5(M+1)$ multiplications and $0.5(M-1)$ additions for a total of $1.5M + 0.5$ operations.

When M is even, we have

$$y[n] = h[0.5M]x[n-0.5M] + \sum_{m=0}^{0.5M-1} h[r]\left(x[n-r] + x[n-M+r]\right)$$

which requires, 1 multiplication for the first term and $3 \times 0.5M$ for the summation for a total of $1.5M + 1$ operations.

d)   If the input sequence, $x[n]$ is of length $L > M$, explain how it is possible to perform the circular convolution $v[n] = h[n] \circledast_L x[n]$ using DFT and IDFT transforms of length L.

Giving your reasons fully, identify the values of n for which it is necessarily true that $v[n] = y[n]$.   [4]

From the formula sheet, $v[n] = h[n] \circledast_L x[n] \Leftrightarrow V[k] = X[k]Y[k]$. So we must (i) zero-pad $h[n]$ to length L, (ii) take the DFT of both $h[n]$ and $x[n]$, (iii) multiply them together and (iv) then take the IDFT. This yields $v[n]$ of length L.

Not many people mentioned that you must zero-pad $h[n]$ before doing the DFT to make it of length L. Some did not notice that the question stated that both $x[n]$ and the DFT were of length L; if you ignore this requirement and zero-pad both $x[n]$ and $h[n]$ to at least $M + L - 2$ you can obtain the whole of $y[n]$ from the circular convolution. Unfortunately, this was not the question that was asked and it is incorrect for overlap-save in the next part of the question. Several people lost marks because they omitted to mention steps (ii), (iii) and (iv) above which seemed surprising.

Because of the circular convolution, the first M values of $v[n]$ (i.e. $0 \leq n < M$) may be different from $y[n]$. This is because
$v[n] = h[n] \circledast_L x[n] = \sum_{r=0}^{L-1} h[n]x[(n-r) \mod L] = \sum_{r=0}^{M} h[n]x[(n-r) \mod L]$.

For $M \leq n \leq L-1$, $n-r$ is non-negative for all r, so $(n-r) \mod N = n-r$ and $v[n] = y[n] = \sum_{r=0}^{M} h[n]x[n-r]$.

Hardly anyone gave a logical argument for this result. The lower limit was sometimes given as $M-1$ or even $M+1$ and the upper limit was often given as L. It is quite easy to confuse the length of a sequence, L in this case, with its maximum index, $L-1$ in this case.

e)   In the "overlap-save" method of filtering, a long input sequence, $x[n]$, is divided into overlapping frames of length L which are each circularly convolved with $h[n]$. From each frame, $L-M$ samples are retained and concatenated to form the output sequence $y[n]$.

If the frames are numbered $i = 0, 1, 2, \cdots$, determine as a function of i, the L sample numbers of $x[n]$ that form the ith input frame and the $L-M$ samples of $y[n]$ that are generated from them.   [4]

Each frame generates $L-M$ values of $y[\,]$ beginning at $y[i(L-M)]$,

and therefore generates $y[i(L-M)], \cdots, y[(i+1)(L-M)-1]$.

To do this, M earlier values of $x[\,]$ are needed. The input frame therefore consists of the L samples $x[i(L-M)-M], \cdots, x[(i+1)(L-M)-1]$. For frame $i = 0$, this involves some $x[n]$ with negative n; these are defined to be zero.

Many people did not understand the procedure and, in several cases, confused it with the alternative approach of overlap add. The notation in the question used different symbols from those used in the notes which some found hard to cope with.

---

f)   If a complex multiply requires 6 arithmetic operations and an L-point DFT or IDFT requires $6L\log_2 L$ operations, estimate the total number of arithmetic operations required per output sample for the overlap-save method of filtering assuming that the input signal, $x[n]$, is very long.   [5]

Each frame generates $L-M$ samples and requires (i) an L-point DFT, (ii) L multiplies and (iii) an L-point IDFT for a total of $6L + 12L\log_2 L$ operations. We can actually reduce this by $3L$ by taking advantage of the conjugate symmetry of the DFT of a real signal (i.e. when we multiply together the outputs of the DFTs we only need to calculate the first half because the second half will be its reflected complex conjugate).

Thus the number of operations per output sample is $\frac{3L+12L\log_2 L}{L-M} = \frac{3L(1+4\log_2 L)}{L-M}$.

Note that we do not include the cost of the DFT on $h[n]$ because this need be performed only once.

From the IDFT we obtain $L-M$ output samples and so, to obtain the computation per output sample, we divide by $L-M$ in the above expression. Quite a few people forgot to do this and so obtained computation costs that were far too high.

Compare this with the number required for the direct implementations of part (c) for the cases (i) $M = 10$, $L = 64$ and (ii) $M = 1000$, $L = 8192$ and discuss the circumstances under which the use of the overlap-save method will reduce computation.   [5]

(i) If $M = 10$ then the two direct methods take 21 and 16 operations respectively. The above formula with $L = 64$ gives 89 operations which is significantly more expensive.

(ii) If $M = 1000$ then the direct methods take 2001 and 1501 operations respectively. The above formula with $L = 8192$ gives 181 operations.

The direct implementations have a cost that is approximately proportional to M, whereas the DFT implementation has a cost that varies very little with either M or L. Provided that $L \gg M$, the above expression becomes approximately $3 + 12\log_2 L$ which varies only slowly with L and is, for example, 123 for $L = 1024$. Thus for $M > 90$ or so, $1.5M$ will exceed this figure and so the DFT implementation is worthwhile.

3. a) Two alternative methods of transforming a continuous-time filter into a discrete-time filter are (i) the bilinear transformation and (ii) the impulse invariance transformation. Explain how the choice of method affects the frequency response of the resultant filter and outline the circumstances under which you would choose one method over the other. [5]

The bilinear transformation exactly preserves the frequency response of the continuous time filter but applies a non-linear frequency axis transformation that maps $[0, \infty]$ onto $[0, \pi]$. Thus properties such as equiripple are preserved. The mapping can be chosen so that one specific continuous time frequency maps to an arbitrarily chosen discrete time frequency but this fixes the mapping at all other frequencies. This makes them very suitable for frequency responses that are piecewise constant with only one gain transition (e.g. a lowpass or highpass filter). Use this method if it is important to preserve the gain characteristics exactly and the distortion of the frequency axis is unimportant.

The impulse invariance method does not distort the frequency axis at all, but the introduction of aliased copies of the response results in gain distortion if the original filter has significant gain at frequencies above the Nyquist frequency. The transformation is therefore not suitable for high pass or bandstop filters unless they are modified by adding an anti-aliasing lowpass filter. Use this method for filters with a lowpass response when it is important to preserve the frequency axis precisely.

Most people correctly described the non-linear frequency axis mapping of the bilinear transformation but were often much vaguer about the impulse invariance transformation. Only a few explicitly mentioned that the continuous-time response was aliased. Very few noted that the impulse invariance transformation is only suitable for filters whose gain is zero at high frequencies (i.e. an overall lowpass response).

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b) A continuous-time filter has the transfer function $H(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$.

i) Show that the gain of the filter, $|H(j\Omega)|$, equals $-3\,\mathrm{dB}$ at an angular frequency $\Omega = 1$. [1]

$H(j) = \left(-1 + \sqrt{2}j + 1\right)^{-1} = -j\sqrt{0.5}$ which equals $-3\,\mathrm{dB}$.

Some first calculated $|H(s)|^2$ which gets rid of the complex numbers but is a lot of effort.

---

ii) Using the bilinear mapping, $s = 2f_s \frac{z-1}{z+1}$, determine the transfer function, $G(z)$, of the corresponding discrete-time filter. [5]

$$G(z) = \left(4f_s^2 \frac{(z-1)^2}{(z+1)^2} + 2f_s\sqrt{2}\frac{z-1}{z+1} + 1\right)^{-1}$$

$$= (z+1)^2 \left(4f_s^2(z-1)^2 + 2f_s\sqrt{2}(z-1)(z+1) + (z+1)^2\right)$$

$$= (z+1)^2 \left(\left(4f_s^2 + 2f_s\sqrt{2} + 1\right)z^2 + \left(-8f_s^2 + 2\right)z + 4f_s^2 - 2f_s\sqrt{2} + 1\right)$$

$$= \frac{1 + 2z^{-1} + z^{-2}}{\left(4f_s^2 + 2f_s\sqrt{2} + 1\right) - (8f_s^2 - 2)z^{-1} + \left(4f_s^2 - 2f_s\sqrt{2} + 1\right)z^{-2}}$$

Most people did this OK although there were quite a few algebra errors such as forgetting to multiply 1 by $(z+1)^2$ when obtaining the second line above.

---

iii)   Determine the sample frequency, f_s, so that the discrete-time filter has a gain of $-3\,$dB at an angular frequency $\omega = 1$.   [3]

Substituting $s = j\Omega$ and $z = e^{j\omega}$ into the bilinear mapping, $s = 2f_s\frac{z-1}{z+1}$, gives $j\Omega = 2f_s\frac{e^{j\omega}-1}{e^{j\omega}+1}$. Substituting $\Omega = \omega = 1$ gives $f_s = 0.5j\frac{e^j+1}{e^j-1} = 0.5j\frac{e^{0.5j}+e^{-0.5j}}{e^{0.5j}-e^{-0.5j}} = 0.5j\frac{2\cos 0.5}{2j\sin 0.5} = 0.5\cot 0.5 = 0.9152\,$Hz.

Alternatively, write $e^j = \cos 1 + j\sin 1 = 0.5403 + 0.8415j$ from which $f_s = 0.5j\frac{e^j+1}{e^j-1} = \frac{0.5j(1.5403+0.8415j)}{-0.4597+0.8415j} = \frac{-0.4207+0.7702j}{-0.4597+0.8415j} = 0.9152\,$Hz.

You get the wrong answer of $57.29\,$Hz if your calculator is set to "degrees" rather than "radians". Several got a correct expression for f_s but lost a mark because they did not evaluate it, either because they chose not to or because they could not evaluate e^j. Some did not use the mapping function but tried to equate the modulus of $G(z)$ from part (iii) to $\sqrt{0.5}$ which gets very messy; the key property of the bilinear transform is that the frequency response maps exactly from the continuous-time domain onto the distorted discrete-time frequency axis.

---

iv)   For the sample frequency found in (iii) above, give the coefficients of $G(z)$ to 3 decimal places when the coefficients are normalized so that the leading coefficient of the denominator is unity.   [4]

From the part (ii),

$$G(z) = \frac{1 + 2z^{-1} + z^{-2}}{\left(4f_s^2 + 2f_s\sqrt{2} + 1\right) - (8f_s^2 - 2)z^{-1} + \left(4f_s^2 - 2f_s\sqrt{2} + 1\right)z^{-2}}$$

Substituting $f_s = 0.915$ into the denominator polynomial gives coefficients $[6.9394 \ -4.7014 \ 1.762]$ which we normalize by dividing by 6.9394 to give $[1 \ -0.677 \ 0.254]$. The scaled numerator then becomes $[0.144 \ 0.288 \ 0.144]$.

So $G(z) = \frac{0.144+0.288z^{-1}+0.144z^{-2}}{1-0.677z^{-1}+0.254z^{-2}}$.

Omitted by quite a lot of people, but mostly done correctly by those that did it.

---

c) Derive the relationship between p_0 and q_0 so that the causal impulse response of the discrete-time filter $P(z) = \left(1 + p_0 z^{-1}\right)^{-1}$ is a sampled version of the impulse response, $e^{-q_0 t}$, of the continuous-time filter $Q(s) = (s + q_0)^{-1}$ at a sample frequency f_s. [4]

The impulse response of $P(z) = \left(1 + p z^{-1}\right)^{-1}$ is, from the geometric progression formula, $p[n] = (-p_0)^n$.

Quite a few took the impulse response to be $(+p)^n$ instead.

The impulse response of $(s + q_0)^{-1}$ is $e^{-q_0 t}$. Equating these two responses gives $(-p_0)^n = e^{-q_0 n f_s^{-1}}$ from which $p_0 = -\exp\left(-q_0 f_s^{-1}\right)$ or, equivalently, $q_0 = -f_s \ln\left(-p_0\right)$.

Alternatively, we can work backwards and say that we require $p[n] = e^{-q_0 n f_s^{-1}}$ from which $P(z) = \sum_{n=0}^{\infty} e^{-q_0 n f_s^{-1}} z^{-n} = \left(1 - e^{-q_0 f_s^{-1}} z^{-1}\right)$ from which the value of p_0 may be read off.

Several people wrongly tried to do a bilinear transformation to map between the two domains.

---

d) i) Express $H(s)$ from part (b) above in the form $H(s) = a_1 \left(s + q_1\right)^{-1} + a_2 \left(s + q_2\right)^{-1}$. [4]

The roots of the denominator of $H(s)$ are $-q_{1,2} = \frac{-\sqrt{2} \pm \sqrt{2-4}}{2} = -\sqrt{0.5}\left(1 \pm j\right) = \exp \pm 0.75 \pi j$.

Some had the $q_{1,2}$ negated: the roots of $(s + q_1)(s + q_2)$ are $-q_{1,2}$.

We have $H(s) = \frac{1}{s^2 + \sqrt{2}s + 1} = \frac{1}{(s+q_1)(s+q_2)} = \frac{a_1}{s+q_1} + \frac{a_2}{s+q_2}$.

Multiplying out in order to equate coefficients gives

$$\frac{1}{s^2 + \sqrt{2}s + 1} = \frac{(a_1 + a_2)s + \sqrt{0.5}\left(a_1 + a_2 - (a_1 - a_2)j\right)}{s^2 + \sqrt{2}s + 1}$$

from which $a_1 = -a_2 = \frac{1}{q_2 - q_1} = \sqrt{0.5}j$. And so

$$H(s) = \frac{\sqrt{0.5}j}{s + \sqrt{0.5}(1+j)} - \frac{\sqrt{0.5}j}{s + \sqrt{0.5}(1-j)}$$

.

Alternatively a_1 and a_2 can be found by evaluating the residues:

$a_1 = \left.\frac{1}{s+q_2}\right|_{s=-q_1} = \sqrt{0.5}j$ and $a_2 = \left.\frac{1}{s+q_1}\right|_{s=-q_2} = -\sqrt{0.5}j$.

Mostly done correctly by those that attempted it.

---

ii) Hence, if $F(z)$ is a discrete time filter whose impulse response is a sampled version of that of $H(s)$ at a sample frequency $f_s = 1$, determine the coefficients $F(z)$ to 3 decimal places. [4]

Substituting $q_{1,2} = \sqrt{0.5}\,(1 \pm j)$ into the mapping from part (c), we obtain $p_{1,2} = -e^{-q_{1,2}} = -\exp\left(-\sqrt{0.5}\,(1 \pm j)\right) = -0.37485 \pm 0.32032\,j = 0.4931\angle \pm 2.4345 = 0.4931\angle \pm 139.5°$.

Hence $F(z) = \dfrac{a_1}{1+p_1 z^{-1}} + \dfrac{a_2}{1+p_2 z^{-1}} = \dfrac{a_1+a_2+(a_1 p_2 + a_2 p_1)z^{-1}}{1+(p_1+p_2)z^{-1}+p_1 p_2 z^{-2}} = \dfrac{0.453 z^{-1}}{1-0.750 z^{-1}+0.243 z^{-2}}.$

The poles use the mapping derived in part (c), but the numerator can not be deduced directly from the mapping; some people wrongly took the numerator to be 1.

---

4. a) Outline the benefits of dividing a signal into subbands for processing. [3]

> *i) If a signal is split into M equal subbands, the bandwidth of each is re-*
> *duced by M so the sample frequency can be reduced by this factor. Com-*
> *putation costs for filters of a given selectivity are $\propto f_s^2$ so the overall com-*
> *putation is decreased by M.*
>
> *ii) It is easy to apply different processing to different frequency bands*
>
> *iii) Variation in the signal spectrum within a subband is decreased which*
> *improves the convergence rate of adaptive algorithms.*
>
> *iv) There may be parallelism advantages in a hardware implementation.*
>
> *Most said that computation was reduced but a coherent reason for this was*
> *rare. A few people mentioned "perfect reconstruction" but this is a goal of*
> *subband processing rather than a benefit of it.*

b) i) Figure 4.1 shows a downsampler follower by an upsampler. Write down the sequences $s[r]$ and $w[n]$ when the input sequence is given by $v[n] = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix}$. [1]

> *$s[r] = [1\,3\,5]$ and $w[n] = [1\,0\,3\,0\,5]$*
>
> *Almost everyone got this right. A few said that $w[n] = [1\,1\,3\,3\,5]$.*

ii) For this input sequence, give expressions for $V(z)$ and $W(z)$ and show that
$$W(z) = \frac{1}{2}\left(V(z) + V(-z)\right).$$
Explain why this relationship will hold for any input sequence $v[n]$. [3]

> *For this input sequence, $V(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 5z^{-4}$ and so*
> *$V(-z) = 1 - 2z^{-1} + 3z^{-2} - 4z^{-3} + 5z^{-4}$.*
>
> *$W(z) = 1 + 3z^{-2} + 5z^{-4}$ which equals $\frac{1}{2}\left(V(z) + V(-z)\right)$.*
>
> *For any input sequence, $V(-z)$ is the same as $V(z)$ but with the sign of*
> *the odd samples reversed. Therefore when you form $\frac{1}{2}\left(V(z) + V(-z)\right)$,*
> *the even samples are unchanged from $V(z)$ but the odd samples are*
> *set to zero; this mirrors the effect of a downsampler followed by an*
> *upsampler. Alternatively, you can use the general formula for arbitrary*
> *sample ratios that was proved in Q1(d).*
>
> *Many people did not give explicit expressions for $V(z)$ and $W(z)$ even*
> *though the question asked them to.*

c) In the block diagram of Fig. 4.2, determine simplified expressions for $W_0(z)$, $W_1(z)$ and $Y(z)$ in terms of $X(z)$ and $H(z)$. In the block diagram, the adder input labelled "−" is negated before the addition. [3]

> *$V_0(z) = H(z)X(z)$*

$V_1(z) = H(-z)X(z)$

$W_0(z) = \frac{1}{2}\left(V_0(z) + V_0(-z)\right) = \frac{1}{2}\left(H(z)X(z) + H(-z)X(-z)\right)$

$W_1(z) = \frac{1}{2}\left(V_1(z) + V_1(-z)\right) = \frac{1}{2}\left(H(-z)X(z) + H(z)X(-z)\right)$

$Y(z) = H(z)W_0(z) - H(-z)W_1(z)$

$\qquad = \frac{1}{2}\left(H^2(z)X(z) + H(z)H(-z)X(-z)\right) - \frac{1}{2}\left(H^2(-z)X(z) + H(z)H(-z)X(-z)\right)$

$\qquad = \frac{1}{2}\left(H^2(z) - H^2(-z)\right)X(z)$

Several derived the result from scratch without using the result of part b(ii) which makes things easier. Some did not simplify the expression for $Y(z)$ which made it impossible to give a sensible answer to the next part about "alias-free".

Explain what is meant by saying that the output $y[n]$ is alias-free. [1]

"Alias-free" means that $Y(z)$ does not contain any component of $X(-z)$ which is an alias term introduced by the downsampling.

Quite a few people omitted to answer this completely. Many thought "alias-free" meant the same as "perfect reconstruction".

---

d) Suppose that $H(z) = P(z^2) + z^{-1}Q(z^2)$. Show that the block diagram of Fig. 4.3 has the same transfer function, $T(z) = \frac{Y(z)}{X(z)}$, as that of Fig. 4.2 . [3]

$H(-z) = P((-z)^2) + (-z)^{-1}Q((-z)^2) = P(z^2) - z^{-1}Q(z^2)$

Only a few people showed the relation for $H(-z)$ explicitly.

$V_0(z) = P(z^2)X(z) + z^{-1}Q(z^2)X(z) = H(z)X(z)$

$V_1(z) = P(z^2)X(z) - z^{-1}Q(z^2)X(z) = H(-z)X(z)$

$Y(z) = P(z^2)\left(W_0(z) - W_1(z)\right) + z^{-1}Q(z^2)\left(W_0(z) + W_1(z)\right)$

$\qquad = \left(P(z^2) + z^{-1}Q(z^2)\right)W_0(z) - \left(P(z^2) - z^{-1}Q(z^2)\right)W_1(z)$

$\qquad = H(z)W_0(z) - H(-z)W_1(z)$

We see that these relationships are identical to those in part (c) above. It follows that $T(z) = \frac{Y(z)}{X(z)}$ is also unchanged.

Give a simplified expression for $T(z)$ in terms of $P(z^2)$ and $Q(z^2)$. [2]

$T(z) = \frac{1}{2}\left(H^2(z) - H^2(-z)\right) = \frac{1}{2}\left(\left(P(z^2) + z^{-1}Q(z^2)\right)^2 - \left(P(z^2) - z^{-1}Q(z^2)\right)^2\right)$

$\qquad = 2z^{-1}P(z^2)Q(z^2)$

since the terms in $P^2(z^2)$ and $Q^2(z^2)$ cancel out.

---

e) If $P(z^2) = \frac{p + z^{-2}}{1 + pz^{-2}}$ and $Q(z^2) = \frac{q + z^{-2}}{1 + qz^{-2}}$ are stable filters, determine the values of p and q so that the numerator of $H(z)$ is a scalar multiple of $\left(1 + z^{-1}\right)^5$. [5]

$$H(z) = P(z^2) + z^{-1}Q(z^2)$$

$$= \frac{\left(p+z^{-2}\right)\left(1+qz^{-2}\right)+z^{-1}(q+z^{-2})(1+pz^{-2})}{(1+pz^{-2})(1+qz^{-2})}$$

$$= \frac{p+qz^{-1}+(pq+1)z^{-2}+(pq+1)z^{-3}+qz^{-4}+pz^{-5}}{(1+pz^{-2})(1+qz^{-2})}$$

We need the numerator to equal $k(1+z^{-1})^5 = k\left(1+5z^{-1}+10z^{-2}+10z^{-3}+5z^{-4}+z^{-5}\right)$.

Identifying coefficients gives, $p = k$, $q = 5k$, $pq+1 = 10k$ from which $5k^2 - 10k+1 = 0$ which has the solutions $k = \frac{10\pm\sqrt{80}}{10} = 1\pm\sqrt{0.8} = \{1.8944, 0.1056\}$. Since $P(z^2)$ has a pole at $z = \sqrt{-p}$ we require that $|p| = |k| < 1$ for stability. Hence $p = k = 1 - \sqrt{0.8} = 0.1056$ and $q = 5k = 5 - \sqrt{20} = 0.5279$.

Some didn't notice that the question said "a multiple" of $(1+z^{-1})^5$, and omitted k from the above (or equivalently assumed $k = 1$). Many did not realize that the "stable" requirement given in the question meant that p and q had to be less than 1. Surprisingly many tried to make the numerator of $T(z)$ equal to a multiple of $(1+z^{-1})^5$ rather than the numerator of $H(z)$. A few people had difficulty in expanding $(1+z^{-1})^5$ to get the correct target coefficients.

For these values of p and q, determine $H(e^{j\omega})$ and $T(e^{j\omega})$ in polar form for $\omega = 0$, $\frac{\pi}{2}$ and π . [3]

We need to evaluate $H(z) = P(z^2) + z^{-1}Q(z^2)$ and $T(z) = 2z^{-1}P(z^2)Q(z^2)$ for $z = \{1, j, -1\}$.

At $z = 1$, $H(1) = P(1) + Q(1) = 1 + 1 = 2\angle 0$. $T(1) = 2P(1)Q(1) = 2\angle 0$.
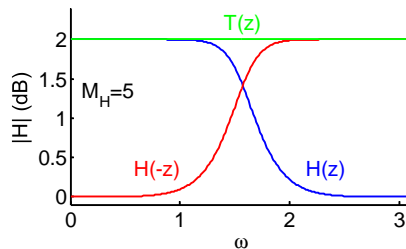
At $z = j$, $H(j) = P(-1) - jQ(-1) = -1 + j = 1.414\angle\frac{3\pi}{4}$. $T(j) = -2jP(-1)Q(-1) = -2j = 2\angle -\frac{\pi}{2}$.

At $z = -1$, $H(-1) = P(1) - Q(1) = 1 - 1 = 0$. $T(-1) = -2P(1)Q(1) = -2 = 2\angle\pi$.

Hence sketch dimensioned graphs of $\left|H(e^{j\omega})\right|$ and $\left|T(e^{j\omega})\right|$ versus ω using linear axes for both gain and frequency. [2]

$H(e^{j\omega})$ is a lowpass filter with a passband gain of 2, a -3dB frequency of $\omega = \frac{\pi}{2}$ and five zeros at $\omega = \pi$.

$T(z) = 2z^{-1}P(z^2)Q(z^2)$. Since $P(z^2)$ and $Q(z^2)$ are allpass filters, $\left|T(e^{j\omega})\right| = 2\forall\omega$.

The five zeros at $\omega = \pi$ mean that the response of $H(z)$ is very flat at this frequency..

f) Draw a block diagram, functionally equivalent to Fig. 4.3 and including $s_0[r]$ and $s_1[r]$ explicitly, showing how the Noble identities can be applied to improve its computational efficiency. [4]

Since $P(z^2)$ and $Q(z^2)$ depend only on z^2, we can use the Noble identities to interchange them with the resampling operations to give the following implementation which is more efficient since the filtering and addition is now performed at half the sampling rate.
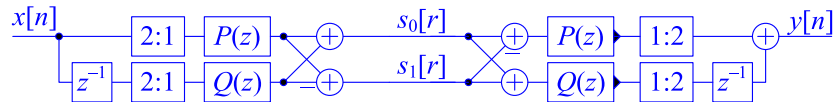
You can replace the input/output delays and down/up-samplers with commutators to make the drawing simpler.

Most people got this right. A few moved the z^{-1} blocks through the down/up-samplers as well and turned them into $z^{-\frac{1}{2}}$ blocks which are not realizable.
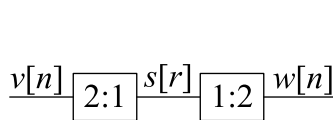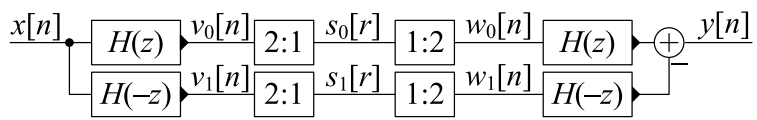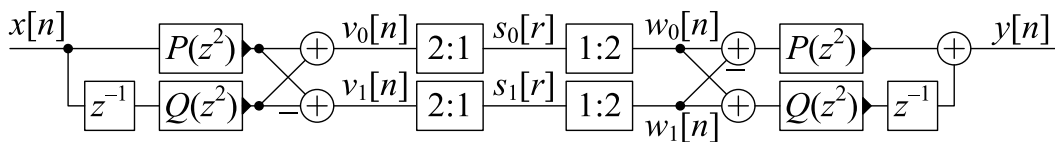

Figure 4.1

Figure 4.2

Figure 4.3