E4.04
SC6
ISE4.9

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2006

MSc and EEE/ISE PART IV: MEng and ACGI

ADVANCED DATA COMMUNICATIONS

Thursday, 27 April 2:30 pm

Time allowed: 3:00 hours

Corrected Copy
Q5, Q7

There are EIGHT questions on this paper.

Answer FIVE questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s) :	M.K. Gurcan
	Second Marker(s) :	P.T. Stathaki

Special Instructions for Invigilators: None

Information for candidates:

Notation

BSC	Binary Symmetric Channel
NNUB	Nearest Neighbour Union Bound Probability of Error
PAM	Pulse Amplitude Modulation
QAM	Quadrature Amplitude Modulation
QPSK	Quadrature Phase Shift Keying
MMSE	Minimum Mean Square Error
DFE	Decision Feedback Equaliser
ZFE	Zero Forcing Equaliser
AWGN	Additive White Gaussian Noise
MFB	Matched Filter Bound

Useful equations

$$\sum_{k=0}^{\infty}(a)^{2k}=\frac{1}{1-a^2} \quad \textit{for} \quad |a|<1$$

If $g(t)$ and $G(f)$ are Fourier transform pairs such that

$$g(t)\Leftrightarrow G(f)=\mathcal{F}(g(t))$$

where

$$G(f)=\mathcal{F}\{g(t)\}=\int_{-\infty}^{\infty}g(t)\exp(-j2\pi f t)\,dt \text{ and}$$

$$g(t)=\mathcal{F}^{-1}\{G(f)\}=\int_{-\infty}^{\infty}G(f)\exp(j2\pi f t)\,df.$$

then the following Fourier transform relationships might be useful

$$p(kT)=p_k=(-a)^k\Leftrightarrow P\left(e^{-j2\pi fT}\right)=\frac{1}{1+a\,e^{-j2\pi f}} \qquad \textit{for} \quad T=1$$

$$\frac{1}{\sqrt{T}}\mathrm{sinc}\left(\frac{\overset{t}{\cancel{1}}}{T}\right)\Leftrightarrow\sqrt{T}\,\mathrm{rect}(T\,f)$$

$$g(t)=\mathrm{rect}\left(\frac{t}{T}\right) \quad\Leftrightarrow\quad G(f)=T\,\mathrm{sinc}(f\,T)$$

$$g(t)=\delta(t) \quad\Leftrightarrow\quad G(f)=1$$

$$Q(x) = 0.5\ \mathrm{erfc}\left(\frac{x}{\sqrt{2}}\right)$$

x	$erfc(x)$
0.1	8.88×10^{-1}
0.2	7.77×10^{-1}
0.3	6.71×10^{-1}
0.4	5.72×10^{-1}
0.5	4.80×10^{-1}
0.6	3.96×10^{-1}
0.7	3.22×10^{-1}
0.8	2.58×10^{-1}
0.9	2.03×10^{-1}
1	1.57×10^{-1}
1.1	1.20×10^{-1}
1.2	8.97×10^{-2}
1.3	6.60×10^{-2}
1.4	4.77×10^{-2}
1.5	3.39×10^{-2}
1.6	2.37×10^{-2}
1.7	1.62×10^{-2}
1.8	1.09×10^{-2}
1.9	7.21×10^{-3}
2	4.68×10^{-3}
2.1	2.98×10^{-3}
2.2	1.86×10^{-3}
2.3	1.14×10^{-3}
2.4	6.89×10^{-4}
2.5	4.07×10^{-4}
2.6	2.36×10^{-4}
2.7	1.34×10^{-4}
2.8	7.50×10^{-5}
2.9	4.11×10^{-5}
3	2.21×10^{-5}
3.1	1.16×10^{-5}
3.2	6.03×10^{-6}
3.3	3.06×10^{-6}
3.4	1.52×10^{-6}
3.5	7.43×10^{-7}
3.6	3.56×10^{-7}
3.7	1.67×10^{-7}
3.8	7.70×10^{-8}
3.9	3.48×10^{-8}
4	1.54×10^{-8}
4.1	6.70×10^{-9}
4.2	2.86×10^{-9}
4.3	1.19×10^{-9}

1. Consider the following signals

$$x_0(t) = \begin{cases} \dfrac{2}{\sqrt{T}} \cos\left(\dfrac{2\pi t}{T} + \dfrac{\pi}{6}\right) & \text{for} \quad 0 \le t \le T \\ 0 & \text{otherwise} \end{cases}$$

$$x_1(t) = \begin{cases} \dfrac{2}{\sqrt{T}} \cos\left(\dfrac{2\pi t}{T} + \dfrac{5\pi}{6}\right) & \text{for} \quad 0 \le t \le T \\ 0 & \text{otherwise} \end{cases}$$

$$x_2(t) = \begin{cases} \dfrac{2}{\sqrt{T}} \cos\left(\dfrac{2\pi t}{T} + \dfrac{3\pi}{2}\right) & \text{for} \quad 0 \le t \le T \\ 0 & \text{otherwise} \end{cases}$$

a) Find two orthonormal basis functions for this signal set and show that they are orthonormal. [7]

b) Find the data signal corresponding to the signals above for the basis functions you found in (a). [6]

c) Find the following inner products

i) $\langle x_0(t), x_0(t) \rangle$. [3]

ii) $\langle x_0(t), x_1(t) \rangle$. [2]

iii) $\langle x_0(t), x_2(t) \rangle$. [2]

2. Consider the following signal constellation that is to be used over an AWGN channel having a noise variance $\sigma^2 = 0.05$.

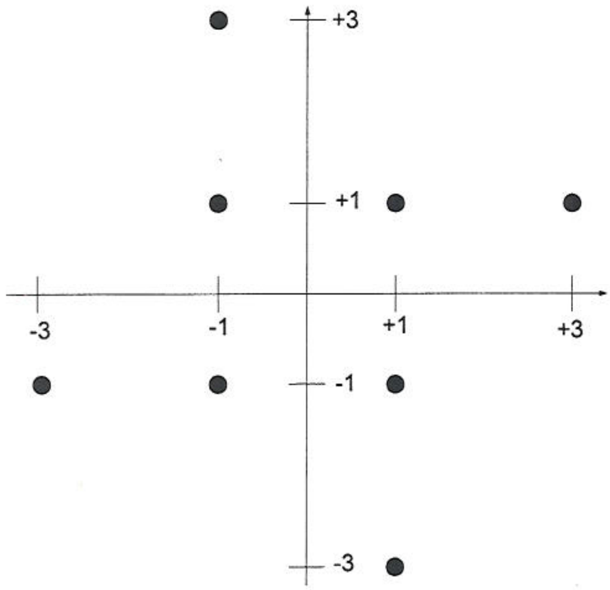

Figure 1 Constellation for 8 points

Assuming that the signal points are to be used with equal probability,

a) Find the total energy, ε_x, and the average energy, $\bar{\varepsilon}_x$, per dimension for this constellation. [7]

b) Find the average number of bits, $\bar{b}$, per dimension the minimum distance, $d_{\min}$, and the average number of neighbours, N_e, for this constellation. [8]

c) Find the union bound for the probability of error, P_e, when a maximum likelihood (ML) detector is employed at the receiver. [5]

3. Either a square or a cross-QAM constellation can be used when transmitting information over an AWGN channel having a $SNR = 30.2\,dB$ and symbol rate $1/T = 10^6$ per second.

a) Select one of the two QAM constellations and specify a corresponding integer number, b, of bits per symbol for a modem which will have the highest ***data rate*** such that the probability of error, $P_e \leq 10^{-6}$. [7]

b) Compute the data rate for part a. [3]

c) Repeat part (a) when $P_e \leq 2 \times 10^{-7}$. [7]

d) Compute the data rate for part c. [3]

4) Consider a communication system having an average constellation energy, $\bar{\varepsilon}_x$, per dimension, and a given pulse response $p(t)$. Assume that one symbol drawn from the constellation is transmitted at a time over an AWGN channel, and that the matched filter bound, SNR_{MFB}, is given by

$$SNR_{MFB} = \frac{\bar{\varepsilon}_x \|p\|^2}{\frac{N_0}{2}}$$

where $\|p\|^2$ is the pulse energy and $\frac{N_0}{2}$ is the noise variance.

The *matched filter bound, MFB,* denotes the square of the argument of the Q-function that arises in the equivalent AWGN analysis of *M-ary* modulation schemes. Given that , $\bar{b}$, is the average number of bits per dimension, show that the *MFB* is given by

a) $MFB = SNR_{MFB}$ and $P_e \geq Q\left(\sqrt{SNR_{MFB}}\right)$ for binary PAM. [5]

b) $MFB = \frac{3}{M^2-1} SNR_{MFB}$ and $P_e \geq 2\left(1-\frac{1}{M}\right) Q\left(\sqrt{\frac{3\ SNR_{MFB}}{M^2-1}}\right)$ for *M-ary* PAM. [5]

c) $MFB = SNR_{MFB}$ and $P_e \geq Q\left(\sqrt{SNR_{MFB}}\right)$ for QPSK. [5]

d) $MFB = \frac{3}{M-1} SNR_{MFB}$ and $P_e \geq 2\left(1-\frac{1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{3\ SNR_{MFB}}{M-1}}\right)$ for *M-ary* QAM. [5]

6) Suppose the Fourier transform of the pulse response of a strictly bandlimited channel using binary PAM is

$$P(\omega)=\begin{cases}\sqrt{T}\left(1+0.9e^{j\omega T}\right) & |\omega|\le\dfrac{\pi}{T}\\[2ex] 0 & |\omega|>\dfrac{\pi}{T}\end{cases}$$

and that the pulse energy, $\|p\|^2$, for the channel is $\|p\|^2=\dfrac{1}{2\pi}\displaystyle\int_{-\frac{\pi}{T}}^{\frac{\pi}{T}}|P(\omega)|^2\,d\omega=1.81$.

The function characterizing the Inter-Symbol-Interference (ISI) is

$$Q(D)=\frac{0.9D^{-1}+1.81+0.9D}{1.81}=\frac{1}{1.81}(1+0.9D)\left(1+0.9D^{-1}\right).$$

Given that the matched filter signal-to-noise ratio bound, SNR_{MFB}, is 10 dB :

a) Find the feedforward filter transfer function, $W(D)$, for the zero forcing decision feedback equaliser (ZF-DFE). [6]

b) Find the feedback filter section transfer function, $B(D)$, for the ZF-DFE. [3]

c) Find the transfer function, $W(D)$, for the minimum-mean-square-error, MMSE-DFE. [4]

d) Find the coefficients $B(D)$ for the MMSE-DFE. [3]

e) Find the mean-square-error values, σ^2_{ZF-DFE} and $\sigma^2_{MMSE-DFE}$ for the ZF-DFE and MMSE-DFE respectively. [4]

7) Assume that the channel gain for a four-channel multi-tone modulation scheme is given by

$$H_n = \begin{cases} 1 + 0.9 \bullet \exp\left(\dfrac{j(n-1)\pi}{4}\right) & \text{for } n = 1, \cdots 4 \\ 0 & \text{otherwise} \end{cases}$$

giving a system having a total of $N = 8$ dimensions. The first sub-channel uses one-dimensional PAM and each of the remaining three sub-channels uses a two - dimensional QAM signal. Assume that the sub-channels are loaded such that the average number of bits per dimension, as a function of the signal-to-noise ratio, is given by

$$\bar{b}_n = \frac{1}{2}\log_2\left(1 + \frac{SNR_n}{2.65}\right)$$

Given that the energy per dimension, $\bar{\varepsilon}_n$, is $\bar{\varepsilon}_n = \frac{8}{7}$ for $n = 1 \cdots \overset{4}{\cancel{7}}$, and the noise variance is $\sigma^2 = 0.181$:

a) Find the channel gain and hence the channel-SNR for each sub-channel. [7]

b) Find the SNR for each sub channel. [3]

c) Find the number of bits per dimension and hence [7]

d) Find the average number of bits per dimension (hint: 7 out of 8 dimensions are used). [3]

8) An $N = 8$ dimensional multi-tone modulation signal is transmitted over a channel having a gain of

$$H_n = \begin{cases} 1 + 0.5 \bullet \exp\left(\dfrac{j\,(n-1)\,\pi}{4}\right) & \textit{for } n = 1, \cdots 4 \\ 0 & \text{otherwise} \end{cases}$$

for each multi-tone sub-channel n. The first sub-channel uses one-dimensional PAM and each of the remaining three sub-channels uses a two-dimensional QAM signal. The Levin-Campello algorithm is used to maximise the total number of transmitted bits per symbol. For each sub-channel n, the energy function, ε_n, is given by

$$\varepsilon_n\left(b_n\right) = k \times \frac{\Gamma}{g_n}\left(2^{2\bar{b}_n} - 1\right),$$

where $k = 1$ for PAM and $k = 2$ for QAM. The term g_n is the channel-*SNR* and $\bar{b}_n$ is the average number of bits per dimension in the sub-channel n. Given that the gap value is $\Gamma = 8.8\,dB$, the channel noise variance, σ_n^2, is 0.125 and the average energy, $\bar{\varepsilon}_n$, per dimension is 1, and also that the granularity, β, of the bit loading algorithm is 1:

a) Produce a table of incremental energies $e(n)$. [7]

b) Use the ***Efficientizing*** (EF) algorithm to make the total number of bits equal to 8. [5]

c) Use the ***E-tightening*** algorithm to find the largest total number of bits. [4]

d) If the total number of transmitted bits, calculated in part (c), is reduced by 2 bits, use the ***B-tightening*** algorithm to maximize the margin. What is the maximum margin? [4]

1. By using the following trigonometric identity

$\cos(a+b) = \cos(a)\cos b - \sin(a)\sin(b)$

the signals $x_0(t)$, $x_1(t)$, $x_2(t)$ can be written as

$$x_0(t) = \sqrt{2}\left[\phi_1(t)\cos\left(\frac{\pi}{6}\right) - \phi_2(t)\sin\left(\frac{\pi}{6}\right)\right]$$

$$x_1(t) = \sqrt{2}\left[\phi_1(t)\cos\left(\frac{5\pi}{6}\right) - \phi_2(t)\sin\left(\frac{5\pi}{6}\right)\right]$$

$$x_2(t) = \sqrt{2}\left[\phi_1(t)\cos\left(\frac{\pi}{2}\right) + \phi_2(t)\sin\left(\frac{\pi}{2}\right)\right]$$

where

$$\phi_1(t) = \begin{cases} \sqrt{\frac{2}{T}}\left(\cos\left(\frac{2\pi t}{T}\right)\right) & \text{for } 0<t<T \\ 0 & \text{otherwise} \end{cases}$$

$$\phi_2(t) = \begin{cases} \sqrt{\frac{2}{T}}\left(\sin\left(\frac{2\pi t}{T}\right)\right) & \text{for } 0<t<T \\ 0 & \text{otherwise} \end{cases}$$

are orthonormal

$$\int_0^T \phi_1(t)\phi_2(t)\,dt = \int_0^T \frac{2}{T}\sin\left(\frac{2\pi t}{T}\right)\cos\left(\frac{2\pi t}{T}\right)dt = \int_0^T \sin\left(\frac{4\pi t}{T}\right)dt = 0$$

$$\int_0^T \phi_1^2(t)\,dt = \int_0^T \frac{2}{T}\cos^2\left(\frac{2\pi t}{T}\right)dt = \int_0^T \frac{1}{T}\left[1+\cos\left(\frac{4\pi t}{T}\right)\right]dt = 1$$

$$\int_0^T \phi_2^2(t)\,dt = \int_0^T \frac{2}{T}\sin^2\left(\frac{2\pi t}{T}\right)dt = \int_0^T \frac{1}{T}\left[1-\cos\left(\frac{4\pi t}{T}\right)\right]dt = 1$$

—///—

$$x_0 = \left[\sqrt{\frac{3}{2}}, -\frac{\sqrt{2}}{2}\right],$$

$$x_1 = \left[-\sqrt{\frac{3}{2}}, -\frac{\sqrt{2}}{2}\right]$$

$$x_3 = \left[0, \sqrt{2}\right]$$

—///—

$$\langle x_0(t), x_0(t)\rangle = \frac{3}{2} + \frac{1}{2} = 2$$

$$\langle x_0(t), x_1(t)\rangle = -\frac{3}{2} + \frac{1}{2} = -1$$

$$\langle x_0(t), x_1(t)\rangle = -1$$

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MODEL ANSWERs and MARKING SCHEME

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2

$$\mathcal{E}_x = \frac{1}{2}(2+10) = 6$$

$$\bar{\mathcal{E}}_x = \frac{\mathcal{E}_x}{N} = 3$$

$$\bar{b} = \frac{\log_2 M}{N} = 1.5$$

$$d_{min} = 2$$

—//—

$$N_e = 4$$

$$\bar{N}_e = 2$$

—//—

$$P_e \leq N_e \, Q\left(\frac{d_{min}}{2\sigma}\right)$$

$$\leq 4 \, Q(4.47)$$

$$\leq 1.6 \times 10^{-5}$$

$$P_e \leq 8 \times 10^{-6}$$

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3

$$P_e \lessapprox 4\, Q\left(\sqrt{\frac{3\,SNR}{M-1}}\right)$$

$$P_e \lessapprox 4\, Q\left(\sqrt{\frac{3\,SNR}{\frac{31}{32}M-1}}\right)$$

—///—

$$\frac{3\,SNR}{M-1} = 10^{1.39} \quad \text{for SQ-QAM} \qquad P_e = 10^{-6} \Rightarrow Q(x)$$
$$x = 10^{1.39}$$

$$\frac{3\,SNR}{\frac{31}{32}M-1} = 10^{1.39} \quad \text{for CR-QAM} \qquad SNR = 10^{3.02}$$

$M = 129 \Rightarrow b = \log_2 M = 7.0$ for SQ-QAM

$M = 139 \Rightarrow b = \log_2 M = 7.1$ for CR-QAM

SQ-QAM requires even $b \Rightarrow b = 6$ SQ-QAM

CR-QAM requires odd $b \Rightarrow b = 7$ CR-QAM

we would use 128 point CR-QAM

—///—

Data rate $R = \frac{b}{T} = 7 \times 10^6 = 7$ Mbps.

—///—

$P_e < 2 \cdot 10^{-7}$ we need $\Rightarrow 2 \times 10^{-7} = 4\,Q(\sqrt{x}) \Rightarrow x = 10^{1.45}$

$$\frac{3\,SNR}{M-1} = 10^{1.45} \quad \text{for SQ-QAM} \qquad SNR = 10^{3.02}$$

$$\frac{3\,SNR}{\frac{31}{32}M-1} = 10^{1.45} \quad \text{for CR-QAM} \qquad SNR = 10^{3.02}$$

$M = 112 \Rightarrow b = \log_2 112 = 6.81$ for SQ-QAM

$M = 133 \Rightarrow b = \log_2 133 = 6.86$ for CR-QAM

highest SQ-QAM is obtained when $b = 6$

highest CR-QAM is obtained when $b = 5$

Therefore we select 64 QAM

Data rate $R = \frac{b}{T} = 6 \times 10^6 = 6$ Mbps.

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4 Given pulse with pulse response $p(t)$ and isolated input x_0.

The maximum output sample of the matched filter is $\|p\| x_0$. The normalised average energy $\|p\|^2 \bar{\mathcal{E}}_x$ while corresponding noise sample energy $\frac{N_0}{2}$

$$SNR_{MFB} = \frac{\bar{\mathcal{E}}_x \|p\|^2}{\frac{N_0}{2}}.$$

—///—

4.a For binary PAM

$$x_p(t) = \sum_k x_k \, p(t-kT)$$

$x_k = \pm\sqrt{\mathcal{E}_x}$. The minimum distance at matched filter output is

$$\|p\| d_{min} = \|p\| d = 2\|p\|\sqrt{\mathcal{E}_x}$$

$$\mathcal{E}_x = \frac{d_{min}^2}{4} \qquad k=1 \qquad MFB = SNR_{MFB} = \frac{\|p\|^2}{\sigma^2} \quad \text{for } \mathcal{E}_x = 1$$

where $\sigma^2 = \frac{N_0}{2}$

$$P_e \geq Q\left(\sqrt{SNR_{MFB}}\right)$$

—///—

4.b The data symbol amplitudes are $\pm\frac{d}{2}, \pm\frac{3d}{2}, \pm\frac{5d}{2}, \ldots \pm\frac{(M-1)d}{2}$

$$d_{min} = d$$

$$\mathcal{E} = \bar{\mathcal{E}}_x = \frac{2}{M}\sum_{k=1}^{M/2}\left(\frac{2k-1}{2}\right)^2 d^2$$

$$= \frac{d^2}{2M}\sum_{k=1}^{M/2}\left(4k^2 - 4k + 1\right)$$

$$= \frac{d^2}{2M}\left[4\left(\frac{(M/2)^3}{3} + \frac{(M/2)^2}{2} + \frac{M/2}{6}\right) - 4\left(\frac{(M/2)^2}{2} + \frac{M/2}{2}\right) + \frac{M}{2}\right]$$

$$\mathcal{E}_x = \frac{d^2}{2M}\left[\frac{M^3}{6} - \frac{M}{6}\right]$$

$$\therefore \quad \mathcal{E}_x = \bar{\mathcal{E}}_x = \frac{d^2}{12}\left[M^2 - 1\right] \Rightarrow d = \sqrt{\frac{12\,\mathcal{E}_x}{M^2-1}}$$

$$k = \frac{3}{M^2-1} \Rightarrow MFB = \frac{3}{M^2-1} SNR_{MFB}$$

—///—

Probability of correct detection

$$P_C = \sum_{i=0}^{M-1} P_{c/i}\, P_x(i) = \frac{M-2}{M}\left(1 - 2Q\left[\frac{d_{min}}{2\sigma}\right]\right) + \frac{2}{M}\left(1 - Q\left[\frac{d_{min}}{2\sigma}\right]\right)$$

$$= 1 - \frac{2M-4+2}{M} Q\left[\frac{d_{min}}{2\sigma}\right]$$

$$P_e = \bar{P}_e = 2\left(1 - \frac{1}{2^b}\right) Q\left[\frac{d_{min}}{2\sigma}\right] < 2Q\left[\frac{d_{min}}{2\sigma}\right]$$

$$P_e = 2\left(1 - \frac{1}{M}\right) Q\left(\sqrt{\frac{3}{M^2-1} SNR}\right)$$

—//—

4-c For QPSK $x_k = \pm\frac{d}{2} \pm j\frac{d}{2}$, $d = 2\sqrt{\mathcal{E}_x}$ so $k=1$

$$MFB = SNR_{MFB} = \frac{\|P\|^2}{\sigma^2}$$

$$P_e \simeq Q\left(\sqrt{SNR_{MFB}}\right)$$

—///

4d square QAM constellation

$$\mathcal{E}_x = 2\bar{\mathcal{E}}_x = \frac{1}{M}\sum_{i,j=1}^{\sqrt{M}} (x_i^2 + x_j^2) = \frac{1}{M}\left[\sqrt{M}\sum_{i=1}^{\sqrt{M}} x_i^2 + \sqrt{M}\sum_{J=1}^{\sqrt{M}} x_j^2\right] = \frac{2}{\sqrt{M}}\sum_{i=1}^{\sqrt{M}} x_i^2$$

$$= \frac{2}{\sqrt{M}}\sum_{i=1}^{\sqrt{M}} x_i^2 = d^2\,\frac{M-1}{6}$$

$$\bar{\mathcal{E}}_x = d^2\,\frac{M-1}{12}$$

Probability of error calculation

corner points

$$P_{c|corner} = \left(1 - Q\left(\frac{d}{2\sigma}\right)\right)^2$$

inner points

$$P_{c|inner} = \left(1 - 2Q\left[\frac{d}{2\sigma}\right]\right)^2$$

3 edge points

$$P_{c|edge} = \left(1 - Q\left[\frac{d}{2\sigma}\right]\right)\left(1 - 2Q\left[\frac{d}{2\sigma}\right]\right)$$

Probability of correct detection

$$P_c = \sum_{i=0}^{M-1} P_{c|i} P_x(i)$$

$$= \frac{4}{M}(1-Q)^2 + \frac{(\sqrt{M}-2)^2}{M}(1-2Q)^2 + \frac{4(\sqrt{M}-2)}{M}(1-2Q)(1-Q)$$

$$= \frac{1}{M}\left[(4 - 8Q + 4Q^2) + (4\sqrt{M} - 8)(1 - 3Q + 2Q^2)\right.$$
$$\left. + (M - 4\sqrt{M} - 4M)Q + (4 - 8\sqrt{M} + 4M)Q^2\right.$$

$$= 1 + 4\left(\frac{1}{\sqrt{M}} - 1\right)Q + 4\left(\frac{1}{\sqrt{M}} - 1\right)^2 Q^2$$

$$\bar{P}_e \leq 2\left(1 - \frac{1}{2^b}\right) Q\left(\frac{d}{2\sigma}\right) = 2\left(1 - \frac{1}{2^b}\right) Q\left[\sqrt{\frac{3}{M-1} SNR}\right]$$

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5 $p(t) = \phi(t) * h(t)$

$$\phi(t) = \frac{1}{\sqrt{T}} \operatorname{sinc}\left(\frac{t}{T}\right) \Rightarrow \Phi(f) = \sqrt{T}\, \operatorname{rect}(tf)$$

$$P(f) = \Phi(f)\, H(f) = (\sqrt{T}\, \operatorname{rect}(tf)\, \frac{1}{1 + a e^{-j2\pi f}} \operatorname{rect}(f)$$

$$= \frac{1}{1 + a e^{-j2\pi f}} \operatorname{rect}(f) \qquad \text{since} \quad T = 1$$

In terms ω

$$P(\omega) = \begin{cases} \frac{1}{1 + a e^{-j2\pi f}} & |\omega| \leq \pi \\ 0 & |\omega| \geq \pi \end{cases}$$

Sampled response

$$P(e^{-j\omega T}) = \frac{1}{T} \sum_{n=-\infty}^{\infty} P\left(\omega + \frac{2\pi n}{T}\right)$$

Since $T=1$ $\quad P(\omega) = 0$ for $|\omega| > \pi$

$$P(e^{-j\omega T}) = \frac{1}{1 + a e^{-j2\pi f}} \quad \text{Then} \quad |a| < 1$$

inverse Fourier transform relationship

$$P(kT) = P_k = (-a)^k \overset{FT}{\Longleftrightarrow} P(e^{-j\omega T}) = \frac{1}{1 + a e^{-2\pi f}}$$

as the pulse is causal

$$\|P\|^2 = T \sum_{0}^{\infty} P_k^2 = \sum_{k=0}^{\infty} (-a)^{2k} = \frac{1}{1 - a^2}$$

—///—

5-C By substituting $D = e^{-j2\pi f T}$

$$P(D) = \frac{1}{1 + aD} \Rightarrow Q(D) = \frac{T}{\|P\|^2} P(D)\, P^*(D^{-*}) = \frac{1 - a^2}{(1 + aD)(1 + aD^{-1})}$$

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	MODEL ANSWERs and MARKING SCHEME		
First Examiner:	**Gurcan, M.K.**	**Paper Code :**	**E4.04, SC6, ISE4.9**
Second Examiner:	**Stathaki, T.**	**Question** 5	**Page** 8 *out of* 12
Question labels in left margin			**Marks allocations in right margin**

5-d

ZFE

$$W_{ZFE}(D) = \frac{1}{\|P\|\, Q(D)} = \frac{(1+aD)(1+aD^{-1})}{\sqrt{1-a^2}}$$

—///

For MMSE linear equaliser we need

$$SNR_{MFB} = \frac{\|P\|^2 \bar{\mathcal{E}}_x}{\sigma^2} \quad \text{as} \quad \frac{\bar{\mathcal{E}}_x}{\sigma^2} = 10^{1.5}$$

$$SNR_{MFB} = \frac{10^{1.5}}{1-a^2}$$

MMSE Linear equaliser

$$W_{MMSE-LE} = \frac{1}{\|P\|\left(Q(D) + \frac{1}{SNR_{MFB}}\right)}$$

$$= \frac{\sqrt{1-a^2}}{\frac{1-a^2}{(1+aD)(1+aD^{-1})} + \frac{1+a^2}{10^{1.5}}}$$

$$= \frac{(1+aD)(1+aD^{-1})}{\sqrt{1-a^2}\left[1+(1+aD)(1+aD^{-1})\,10^{-1.5}\right]}$$

———///———

5-e

when $a=0$ $Q(D)=1$ and $\|P\|^2=1$ since $SNR = 15dB$ and $\Gamma = 8.8\,dB$

$$b = \frac{1}{2}\log_2\left(1+\frac{10^{1.5}}{10^{0.88}}\right) = 1.18$$

Rate achievable

$$R = \frac{b}{T} = \frac{1.18}{1} = 1.18 \text{ bits/sec}$$

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6a

$$\bar{Q}(D) \triangleq Q(D) + \frac{1}{SNR}$$

$$SNR = 10\ dB \Rightarrow SNR = 10^1 = 10$$

$$\bar{Q}(D) = \frac{1.81 + 0.9D + 0.9D^{-1}}{1.81} + \frac{1}{10} = \frac{1.81 + 0.9D + 0.9D^{-1} + \frac{1.81}{10}}{1.81}$$

$$= \frac{1}{1.81}\left(0.9D + 1.991 + 0.9D^{-1}\right) = 0.785\,(1 + 0.633D)(1 + 0.633D^{-1})$$

$$\gamma_0 = 0.785$$

—///

$$Q(D) = \frac{1}{1.81}(1 + 0.9D)(1 + 0.9D^{-1})$$

ZF DFE

$$W(D) = \frac{1}{\|P\| \frac{1}{1.81}(1 + 0.9D^{-1})} = \frac{1}{(1 + 0.9D^{-1})}$$

—//

6-b ZF DFE Feedback section

$$B(D) = (1 + 0.9D)$$

—///—

6-c MMSE DFE

$$W(D) = \frac{1}{\|P\|\,\gamma_0\,P_c^*(D^{-*})} = \frac{1}{\sqrt{1.81}\ 0.785\,(1 + 0.633D^{-1})} = \frac{0.9469}{1 + 0.633D^{-1}}$$

6-d MMSE DFE B(D)

$$B(D) = 1 + 0.633D$$

—///

6-e

$$\sigma^2_{MMSE} = \frac{N_0}{2}\frac{1}{\|P\|^2\gamma_0} = \frac{0.181}{1.81 \quad 0.785} = 0.1274$$

7- $\sigma_n^2 = 0.181$ g_n = Channel gain

7a & 7b

n	$\mathcal{E}_n$	$\|H_n\|$	g_n	SNR_n	b_n
1	8/7	1.9	19.95	22.8	1.63
2	16/7	1.76	17.11	19.5	2x1.53
3	16/7	1.35	10.06	11.49	2x1.2
4	16/7	0.733	2.96	3.38	2x0.6
			7-a	7-b	

c $b_T = 1.63 + 2\times 1.53 + 2\times 1.2 + 2\times 0.6$

$= 8.29$ bits/symbol

d $\bar{b} = \underline{8.29}$

8

$$g_n = \frac{|H_n|^2}{\sigma_n^2} \qquad \sigma_n^2 = 0.125$$

$K=1$ for PAM

$K=2$ for QAM

$\bar{b}_n = \frac{b_n}{2}$ for QAM

$\bar{b}_n = b_n$ for PAM

n	$\lvert H_n\rvert^2$	g_n
1	2.25	18
2	1.957	15.66
3	1.25	10
4	0.543	4.34

$$\mathcal{E}_n(b_n) = K \frac{10^{0.88}}{g_n}\left(2^{2\bar{b}_n} - 1\right)$$

$$e_n(b_n) = c \times \frac{10^{0.88}}{g_n} 2^{2\times\bar{b}_n}$$

$c = 1$ for QAM

$c = 0.75$ for PAM

sub channel	1	2	3	4
$e_n(1)$	1.2463	0.969	1.517	3.49
$e_n(2)$	5.05	1.938	3.04	6.986
$e_n(3)$	20.22	3.876	6.06	13.97
$e_n(4)$	80.91	7.75	12.137	27.94

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Bit allocation

sub channel	1	2	3	4
b_n	2	3	2	1
$\mathcal{E}_n(b_n)$	6.32	6.78	4.55	3.49

bits were chosen 1, 0, 2, 1, 2, 3, 1, 0

c

$N * \bar{\mathcal{E}}_x = 8$

Sub-Channel	1	2	3	4
b_n	1	2	1	0
$\mathcal{E}_n(b_n)$	1.246	2.9	1.517	0

8.d

Sub-Channel	1	2	3	4
b_n	1	1	0	0
$\mathcal{E}_n(b_n)$	1.246	0.969		

$$\text{margin} = 10 * \log\left(\frac{8}{1.2463 + 0.969}\right)$$

$$= 5.577 \text{ dB}$$