SOLUTIONS

QUESTION 1

(a) i. $Y(z) = H(z)X(z^2)$

$$\hat{X}(z) = \frac{X(z)}{2}\left[G(z)H(z^{\frac{1}{2}}) + G(-z^{\frac{1}{2}})H(-z^{\frac{1}{2}})\right]$$ [2/5]

THEREFORE

$$\hat{X}(z) = X(z) \iff G(z)H(z) + G(-z)H(-z) = 2$$

OR $P(z) + P(-z) = 2$ WITH $P(z) = G(z)H(z)$ [5/5]

ii.

WE REQUIRE $p[n] = \begin{cases} 1 & n = 0 \\ 0 & n \text{ EVEN} \end{cases}$ [1/5]

THEREFORE, SETTING $G(z) = a + bz$, YIELDS

$$P(z) = (1 + z^{-1})(a + bz) = a + az^{-1} + bz + b$$ [2/5]

AND THE PR CONDITION IS SATISFIED

WHEN $a + b = 1$. WE THUS PICK

$a = b = \frac{1}{2}$.

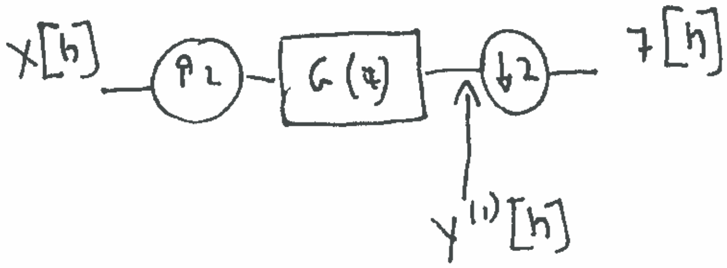
[5/5]

(b) $$Y(z) = \frac{1}{2}\left[G(z^{1/2}) + G(-z^{1/2})\right]X(z)$$ [2/5]

$$= \frac{1}{2}\left[E_0(z) + z^{1/2}E_1(z) + E_0(z) - z^{1/2}E_1(z)\right]X(z)$$ [4/5]

$$= E_0(z)\,X(z) \qquad \#$$ [5/5]

(c) i. CONSIDER THE FOLLOWING ALTERNATIVE SYSTEM



THE CONDITION $y^{(1)}[2n] = x[n]$ IS EQUIVALENT TO IMPOSING THAT $z[n] = x[n]$ AND [2/5]

THIS IMPLIES THAT

$$X(z) = Z(z) = \frac{1}{2}\left[G(z^{1/2}) + G(-z^{1/2})\right]X(z)$$

$$\Updownarrow$$

$$G(z) + G(-z) = 2$$ [5/5]

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ii

WE REQUIRE THAT $G(z) + G(-z) = 2$

IN TIME DOMAIN THIS MEANS:

$$g[n] = \begin{cases} 1, & n = 0 \\ 0, & n \text{ EVEN} \\ \text{ARBITRARY} & \text{FOR } n \text{ ODD} \end{cases}$$

[3/5]

SO A POSSIBLE SOLUTION IS:

$$G(z) = \left(a z^{-3} + b z^{-1} + 1 + b z + a z^{3}\right)$$

WITH a AND b ARBITRARY

[5/5]

QUESTION 2

(a)
$$Y_0(z) = \frac{1}{2}\left[H_0(z^{1/2})X(z^{1/2}) + H_0(-z^{1/2})X(-z^{1/2})\right]$$
$$Y_1(z) = \frac{1}{2}\left[H_1(z^{1/2})X(z^{1/2}) + H_1(-z^{1/2})X(-z^{1/2})\right]$$

THEREFORE

$$\hat{X}(z) = \frac{G_0(z)}{2}\left[H_0(z)X(z) + H_0(-z)X(-z)\right] + \frac{G_1(z)}{2}\left[H_1(z)X(z) + H_1(-z)X(-z)\right]$$

[3/5]

THIS IMPLIES THAT PR IS ACHIEVED WHEN

$$G_0(z)H_0(z) + G_1(z)H_1(z) = 2$$

AND

$$G_0(z)H_0(-z) + G_1(z)H_1(-z) = 0$$

[5/5]

(b) THE POLYNOMIAL $(z+2+z^{-1})$ HAS CLEARLY TWO ROOTS AT $z=-1$ SINCE $(z+2-z^{-1}) = (z+1)(z^{-1}+1)$.

[2/7]

WE KNOW THAT THE SECOND TERM $P_2(z) = (z^2 - 2z + 3 - 2z^{-1} + z^{-2})/2$ HAS A ROOT

$z_i = \frac{1}{2} + j\frac{\sqrt{3}}{2}$ AND SINCE THIS IS A POLYNOMIAL OF DEGREE 4, IT HAS 4 ROOTS IN TOTAL. SINCE $P_2(z)$ IS SYMMETRIC IF z_i IS A ROOT SO IS $\frac{1}{z_i} = \frac{1}{2} - j\frac{\sqrt{3}}{2}$

[5/7]

MOREOVER, SINCE THE COEFFICIENTS OF THE POLYNOMIAL ARE REAL IS z_i IS A ROOT SO IS ITS COMPLEX CONJUGATE $z_i^* = \frac{1}{2} - j\frac{\sqrt{3}}{2}$ AND SO IS $\frac{1}{z_i^*}$.

THIS MEANS THAT ROOTS COMES 4 AT TIME: z_i, z_i^*, $\frac{1}{z_i}$ AND $\frac{1}{z_i^*}$

IN ~~[illegible]~~ CONCLUSION $P(z)$ HAS THE FOLLOWING 6 ROOTS:

$$z_1 = -1, \quad z_2 = -1, \quad z_3 = \frac{1}{2} + j\frac{\sqrt{3}}{2}, \quad z_4 = \frac{1}{2} - j\frac{\sqrt{3}}{2}$$

$$z_5 = \frac{1}{2} + j\frac{\sqrt{3}}{2} \quad \text{AND} \quad z_6 = \frac{1}{2} - j\frac{\sqrt{3}}{2}$$

[7/7]

(c) FOR ORTHOGONALITY WE REQUIRE
$H_0(z) = G_0(z^{-1})$

[2/7]

CONSEQUENTLY, THROUGH SPECTRAL FACTORIZATION, WE OBTAIN:

$$G_0(z) = (z^{-1}+1)\left(z^{-1} - \frac{1}{2} + j\frac{\sqrt{3}}{2}\right)\left(z^{-1} - \frac{1}{2} - j\frac{\sqrt{3}}{2}\right)/\sqrt{2}$$

$$= (z^{-1}+1)(z^{-2} - z^{-1} + 1)/\sqrt{2} = (z^{-3}+1)/\sqrt{2}$$

$$H_0(z) = (z+1)\left(z - \frac{1}{2} + j\frac{\sqrt{3}}{2}\right)\left(z - \frac{1}{2} - j\frac{\sqrt{3}}{2}\right)/\sqrt{2}$$

$$= (z^3+1)/\sqrt{2}$$

[5/7]

FINALLY

$$G_1(z) = -z^{-1}G_0(-z^{-1}) = \frac{z^2 - z^{-1}}{\sqrt{2}}$$

$$H_1(z) = G_1(z^{-1}) = \frac{z^{-2} - z}{\sqrt{2}}$$

[7/7]

(d) A POSSIBLE SYMMETRIC BIORTHOGONAL FILTER BANK IS THE FOLLOWING

$$G_0(z) = (z + 2 + z^{-1})/\sqrt{2}$$

$$H_0(z) = (z^2 - 2z + 3 - 2z^{-1} + z^{-2})/\sqrt{2}$$

[3/6]

$$H_1(z) = z\, G_0(-z) = (-z^2 + 2z - 1)/\sqrt{2}$$

$$G_1(z) = z^{-1} H_0(-z) = (z + 2 + 3z^{-1} + 2z^{-2} + z^{-3})/\sqrt{2}$$

[6/6]

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QUESTION 3

(a) i. WE NOTE THAT THE SET $\{\varphi_1, \varphi_2, \varphi_3\}$ IS ORTHOGONAL BUT IT IS NOT ORTHO-NORMAL

CONSEQUENTLY THE DUAL BASIS IS SIMPLY GIVEN BY

$$\tilde{\varphi}_1(t) = \frac{1}{\|\varphi_1\|^2}\varphi_1(t)$$

$$\tilde{\varphi}_2(t) = \frac{1}{\|\varphi_2\|^2}\varphi_2(t)$$

$$\tilde{\varphi}_3(t) = \frac{1}{\|\varphi_3\|^2}\varphi_3(t)$$

[3/5]

SINCE $\|\varphi_1(t)\|^2 = \int_{-0.5}^{0.5} dt = 1,$

$$\|\varphi_2(t)\|^2 = \int_{-0.5}^{0.5} t^2 dt = \frac{1}{12},$$

[4/5]

$$\|\varphi_3(t)\|^2 = \int_{-0.5}^{0.5}\left(t^2-\frac{1}{12}\right)\left(t^2-\frac{1}{12}\right)dt = \frac{1}{180},$$

WE HAVE

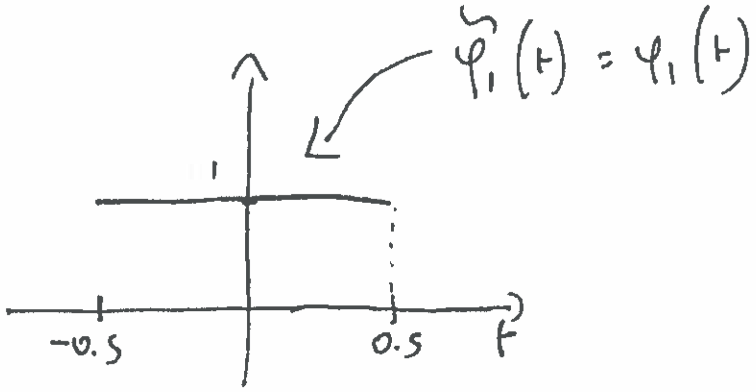
$$\tilde{\varphi}_1(t) = \varphi_1(t)$$

$$\tilde{\varphi}_2(t) = 12\varphi_2(t)$$

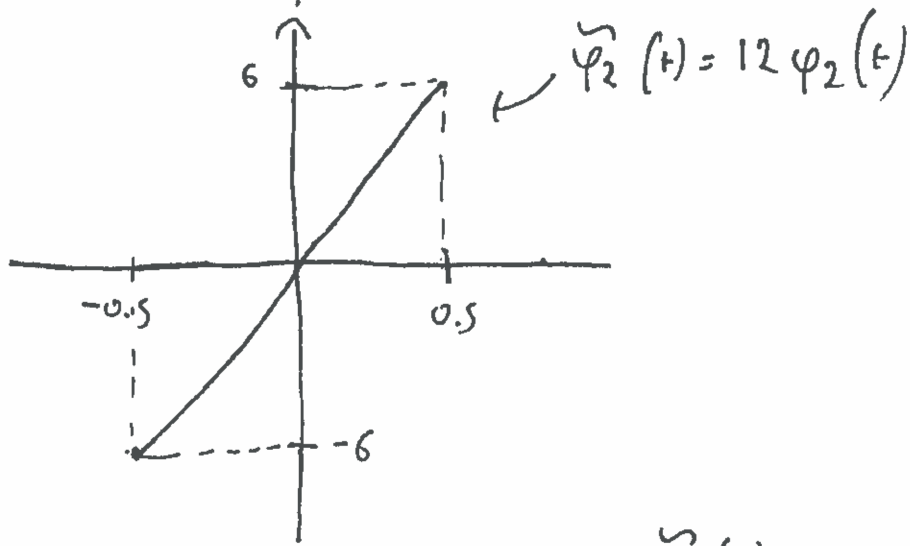
$$\tilde{\varphi}_3(t) = 180\,\varphi_3(t)$$

[5/5]

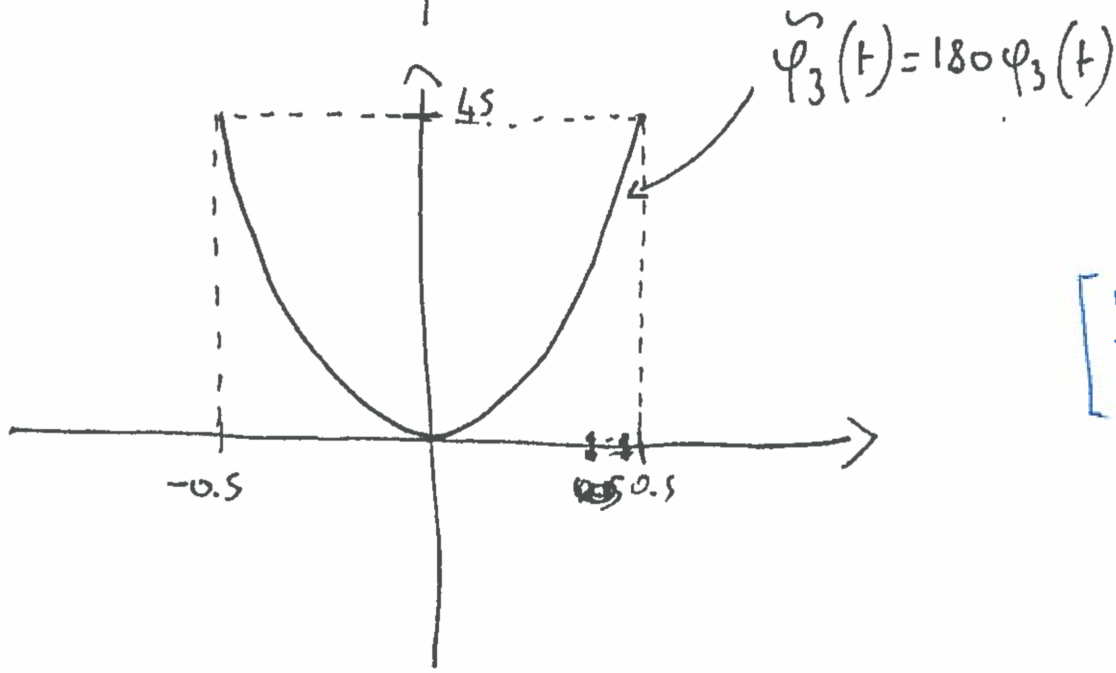
(a) i.i.


$\tilde{\varphi}_1(t) = \varphi_1(t)$
1
-0.5
0.5
t

[1/5]


$\tilde{\varphi}_2(t) = 12\varphi_2(t)$
6
-0.5
0.5
-6

[3/5]


$\tilde{\varphi}_3(t) = 180\varphi_3(t)$
45
-0.5
0.5

[5/5]

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(b) i. WE NOTE THAT $x(t)$ IS AN ODD FUNCTION WHILE $\tilde{\varphi}_1(t)$ AND $\tilde{\varphi}_3(t)$ ARE EVEN.

CONSEQUENTLY:

$$\begin{cases} \langle x(t), \tilde{\varphi}_1(t) \rangle = 0 \\ \langle x(t), \tilde{\varphi}_3(t) \rangle = 0 \end{cases}$$

[3/5]

WE ALSO HAVE:

$$\langle x(t), \tilde{\varphi}_2(t) \rangle = 12 \langle x(t), \varphi_2(t) \rangle = 24 \int_0^{1/2} t \sin \pi t \, dt$$

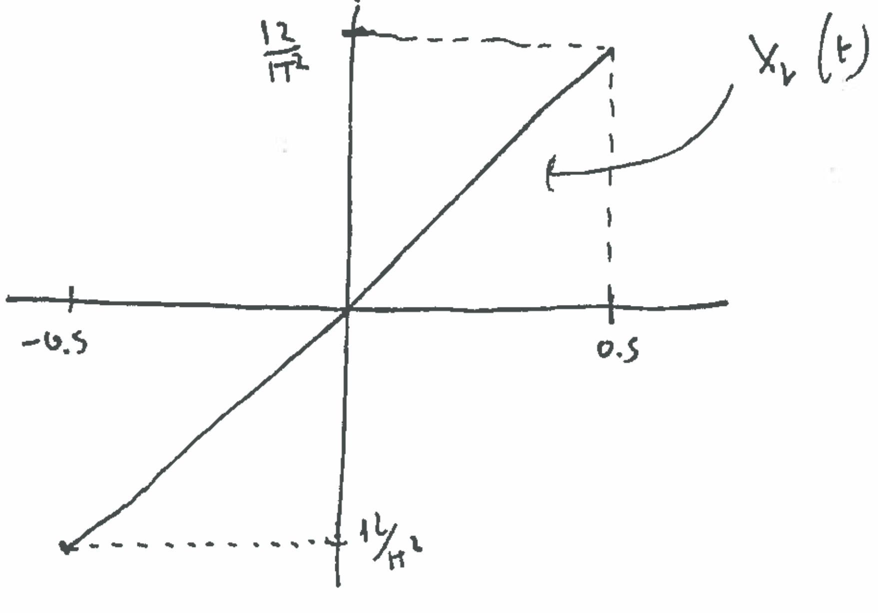
$$= 24 \left[-\frac{t}{\pi} \cos \pi t \Big|_0^{1/2} + \frac{1}{\pi} \int_0^{1/2} \cos \pi t \, dt \right] = \frac{24}{\pi^2}$$

[5/5]

ii. $x_V(t) = \frac{24}{\pi^2} \varphi_2(t)$

THEREFORE

$$x_V(t) = \frac{24}{\pi^2} t \qquad \text{FOR} \qquad t \in [-0.5, 0.5]$$



[5/5]

ii.i WE NEED TO SHOW THAT

$<\varepsilon(t), \varphi_i(t)> = 0 \qquad i = 1, 2, 3$ $\left[2/5\right]$

$\varepsilon(t) = \sin \pi t - \frac{24}{\pi^2} t$ FOR $t \in [-0.5, 0.5]$

$\varepsilon(t)$ IS AN ODD FUNCTION THEREFORE

$<\varepsilon(t), \varphi_i(t)> = 0$ FOR $i=1$ AND $i=3.$ $\left[3/5\right]$

FOR $i=2$, WE HAVE:

$$< \varepsilon(t), \varphi_2(t)> = <\sin \pi t, \varphi_2(t)> - \frac{24}{\pi^2}< t, \varphi_2(t)>$$

$$= 2\int_0^{1/2} t \sin \pi t - \frac{24}{\pi^2}\int_{-1/2}^{1/2} t^2 dt$$

$$= \frac{2}{\pi^2} - \frac{2}{\pi^2} = 0 \qquad \#$$ $\left[5/5\right]$

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QUESTION 4

(a) DEFINE

$$p(\tau) = \langle \varphi(t), \varphi(t-\tau) \rangle = \int_{-\infty}^{\infty} \varphi(t)\varphi(t-\tau)\,dt$$

THEN $a[n]$ CORRESPONDS TO

THE SAMPLED VERSION OF $p(\tau)$

WITH SAMPLING PERIOD $T=1$. [2/5]

SPECIFICALLY

$$a[n] = p(nT) \quad \text{WITH } T=1$$

BECAUSE OF THE SAMPLING THEOREM, WE CAN THEN SAY THAT

$$A(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \hat{p}(\omega + 2\pi k), \qquad (1)$$

WHERE $\hat{p}(\omega)$ IS THE FOURIER

TRANSFORM OF $p(\tau)$. [3/5]

WE NOW FIND THE EXPLICIT EXPRESSION FOR $\hat{p}(\omega)$.

BY DEFINITION:

$$\hat{p}(\omega) = \int_{-\infty}^{\infty} p(\tau) e^{-j\omega\tau}\,d\tau = \int_{-\infty}^{\infty} \varphi(t) \int_{-\infty}^{\infty} \varphi(t-\tau) e^{-j\omega\tau}\,dt\,d\tau$$

$$\stackrel{(a)}{=} \int_{-\infty}^{\infty} \varphi(t) e^{-j\omega t}\,dt \int_{-\infty}^{\infty} \varphi(x) e^{j\omega x}\,dx \stackrel{(b)}{=} |\hat{\varphi}(\omega)|^2$$

[4/5]

WHERE (a) FOLLOWS FROM THE CHANGE OF VARIABLE $x = t - \tau$ AND (b) FROM THE FACT THAT $\hat{\varphi}(-w) = \hat{\varphi}^*(w)$.

BY REPLACING $\hat{\rho}(w) = |\hat{\varphi}(w)|^2$ INTO EQ. (1) WE ARRIVE AT THE DESIRED RESULT

[5/5]

(b)

USING POISSON SUMMATION FORMULA WE HAVE THAT

$$\sum_k \varphi(t-k) = \sum_n \hat{\varphi}(2\pi n) e^{-j2\pi n t}$$

2/5

THEN

$$\sum_k \varphi(t-k) = 1 \iff \sum_n \hat{\varphi}(2\pi n) e^{-j2\pi n t} = 1$$

[2/5]

~~THE SECOND~~ THE EQUALITY

$$\sum_{n=-\infty}^{\infty} \hat{\varphi}(2\pi n) e^{-j2\pi n t} = 1$$

IS THAN SATISFIED ONLY WHEN $\hat{\varphi}(0) = 1$ AND $\hat{\varphi}(2\pi n) = 0$ $n \in \mathbb{Z}$ AND $n \neq 0$

[5/5]

(c) WE FIRST NOTE THAT SINCE

$\hat{\varphi}(0) = 1$ AND $\hat{\varphi}(2\pi n) = 0$ $n \in \mathbb{Z}$ AND $n \neq 0$,

THEN $\varphi(t)$ SATISFIES

$$\sum_k \varphi(t-k) = 1$$

[2/10]

WE THEN USE THE MOMENT PROPERTY WHICH SAYS THAT

$$\hat{\varphi}^{(1)}(\omega) = -j \int_{-\infty}^{\infty} t\,\varphi(t)\, e^{-j\omega t}\, dt \qquad (2)$$

AS FOLLOWS:

TAKE THE PERIODIC SIGNAL

$$\sum_k (t-k)\,\varphi(t-k)$$

AND APPLY THE POISSON SUMMATION FORMULA. WE OBTAIN:

$$\sum_k (t-k)\,\varphi(t-k) = j \sum_n \varphi^{(1)}(2\pi n)\, e^{-j2\pi n t}$$

WHERE WE HAVE ALSO USED EQ. (2).

SINCE $\varphi^{(1)}(2\pi n) = 0$ $n \neq 0$ $n \in \mathbb{Z}$

[6/10]

WE HAVE THAT:

$$\underbrace{t \sum_k \varphi(t-k)}_{=1 \text{ BECAUSE OF PARTITION OF UNITY.}} - \sum_k k \varphi(t-k) = j \varphi^{(1)}(0)$$

[9/10]

THEREFORE

$$\sum_k k \varphi(t-k) + j \varphi^{(1)}(0) = t$$

AND SINCE $\sum_k \varphi(t-k) = 1$ WE

HAVE

$$\sum_k \left[k + j \varphi^{(1)}(0)\right] \varphi(t-k) = t$$

THEREFORE

$$\sum_k c_k \varphi(t-k) = t$$

WHEN $c_k = k + j \varphi^{(1)}(0)$

[10/10]

#

(d)

$$\hat{\varphi}(\omega) = \int_{-\infty}^{\infty} \varphi(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} \sqrt{2} \sum_{n} g_0[n] \varphi(2t-n) e^{-j\omega t} dt \quad [2/5]$$

$$= \sqrt{2} \sum_{n} g_0[n] \int_{-\infty}^{\infty} \varphi(2t-n) e^{-j\omega t} dt$$

$$= \sqrt{2} \sum_{n} g_0[n] e^{-j\omega\frac{n}{2}} \int_{-\infty}^{\infty} \varphi(x) e^{-j\omega\frac{x}{2}} \frac{dx}{2} \quad [3/5]$$

$$= \frac{1}{\sqrt{2}} G_0\left(e^{-j\frac{\omega}{2}}\right) \hat{\varphi}\left(\frac{\omega}{2}\right) \quad [5/5]$$