Imperial College
London

M4/5S8

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May-June 2017

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Date: Tuesday 30 May 2017

Time: 10:00 - 12:30

Time Allowed: 2.5 Hours

This paper has 5 Questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the relevant main book(s) are full.

All required additional material will be provided.

Time Series

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers to each question as per the table below.

Raw Mark	Up to 12	13	14	15	16	17	18	19	20
Extra Credit	0	½	1	1 ½	2	2 ½	3	3 ½	4

- Each question carries equal weight.
- Calculators may not be used.

Note: Throughout this paper $\{\epsilon_t\}$ is a sequence of uncorrelated random variables (white noise) having zero mean and variance σ_ϵ^2, unless stated otherwise. The unqualified term "stationary" will always be taken to mean second-order stationary. All processes are real-valued unless stated otherwise. The sample interval is unity unless stated otherwise.

1. (a) What is meant by saying that a stochastic process is stationary?

   (b) A continuous-time stationary process $\{X(t)\}$, with t in seconds (s), has spectral density function

$$S_{X(t)}(f) = \begin{cases} 1, & -6 < |f| < 6, \\ 0, & \text{otherwise}, \end{cases}$$

   with f in cycles/s. It is sampled with a sample interval $\Delta t = 1/8$ s to produce the discrete-time process $\{X_t\}$.

   What is the spectral density function $S_{X_t}(f)$ of $\{X_t\}$ for $|f| < f_{\mathcal{N}}$, where $f_{\mathcal{N}}$ is the Nyquist frequency? *A solution via a graphical argument is perfectly acceptable.*

   (c) Write down the missing material $\boxed{?}$ to complete the following:

      (i) a second-order stationary time series $\{X_t\}$ which is Gaussian/normal is completely stationary because $\boxed{?}$;

      (ii) the sequence of uncorrelated random variables $\{\epsilon_t\}$ is called *white* noise because $\boxed{?}$;

      (iii) a stationary process $\{X_t\}$ with mean $\mu \neq 0$ has spectral representation $\boxed{?}$;

      (iv) the (magnitude squared) coherence $\gamma_{XY}^2(f)$ between two jointly stationary processes $\{X_t\}$ and $\{Y_t\}$ measures $\boxed{?}$;

      (v) the bivariate AR(1) process $\boldsymbol{X}_t = \boldsymbol{\phi}\boldsymbol{X}_{t-1} + \boldsymbol{\epsilon}_t$, where $\boldsymbol{\phi} = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$ and $\{\boldsymbol{\epsilon}_t\}$ is bivariate white noise, is stationary because $\boxed{?}$.

   (d) Let * denote complex conjugation. A *complex-valued* stationary series $\{Z_t\}$ with mean zero has an autocovariance sequence $\{s_\tau\}$ defined as

$$s_\tau = \text{cov}\{Z_t, Z_{t+\tau}\} = E\{Z_t^* Z_{t+\tau}\}, \quad \tau \in \mathbb{Z}.$$

      (i) Express $s_{-\tau}$ in terms of s_τ.

      (ii) Derive the autocovariance sequence for the MA(1) process $Z_t = \epsilon_t - \theta\epsilon_{t-1}$, where θ is a complex-valued parameter and $\{\epsilon_t\}$ has variance unity.

      (iii) Show that the spectral density function $S(f)$ of $\{Z_t\}$ in (d)(ii) is *not* symmetric about $f = 0$ when the imaginary part of θ is non-null.

2. Let $X_1, ..., X_N$ be a sample of size N from a real-valued stationary process $\{X_t\}$ with unknown mean μ and unknown autocovariance sequence $\{s_\tau\}$. With $\overline{X} = (1/N)\sum_{t=1}^{N} X_t$, the so-called 'unbiased' autocovariance estimator is given by

$$\widehat{s}_\tau^{(u)} = \frac{1}{N-|\tau|}\sum_{t=1}^{N-|\tau|}(X_t - \overline{X})(X_{t+|\tau|} - \overline{X}),$$

and the so-called 'biased' autocovariance estimator is given by

$$\widehat{s}_\tau^{(p)} = \frac{1}{N}\sum_{t=1}^{N-|\tau|}(X_t - \overline{X})(X_{t+|\tau|} - \overline{X}).$$

(a) Give two reasons why, in practice, $\widehat{s}_\tau^{(p)}$ is often preferred to $\widehat{s}_\tau^{(u)}$ for $0 < |\tau| < N-1$.

(b) Suppose that the random variables X_1 and X_2 are part of a real-valued Gaussian/normal stationary process $\{X_t\}$ with unknown mean μ, and unknown autocovariance sequence $\{s_\tau\}$ which is arbitrary except for the mild restriction $s_1 < s_0$. Based upon *just these two random variables*, we will consider $\widehat{s}_1^{(u)}$ and $\widehat{s}_1^{(p)}$, estimators of s_1 defined by:

$$\widehat{s}_1^{(u)} = (X_1 - \overline{X})(X_2 - \overline{X}) \quad \text{and} \quad \widehat{s}_1^{(p)} = \frac{1}{2}(X_1 - \overline{X})(X_2 - \overline{X}), \quad \text{where } \overline{X} = \frac{X_1 + X_2}{2}.$$

(i) Express

$$g(M) = E\left\{\left[\frac{(X_1 - X_2)^2}{M} + s_1\right]^2\right\}.$$

as a quadratic in $1/M$, (with M a positive integer). Hence express $g(M)/s_0^2$ as a quadratic in $1/M$ with coefficients involving ρ_1, the lag-1 autocorrelation.

Hints: Consider the mean of $X_1 - X_2$. You will need the following result: Suppose that Z_0, Z_1, Z_2 and Z_3 are any four real-valued Gaussian random variables with zero means. Then Isserlis' theorem says that

$$E\{Z_0Z_1Z_2Z_3\} = E\{Z_0Z_1\}E\{Z_2Z_3\} + E\{Z_0Z_2\}E\{Z_1Z_3\} + E\{Z_0Z_3\}E\{Z_1Z_2\}.$$

(ii) Making use of the result in (b)(i), show that $g(4) = \text{mse}\{\widehat{s}_1^{(u)}\} = E\{(\widehat{s}_1^{(u)} - s_1)^2\}$ and $g(8) = \text{mse}\{\widehat{s}_1^{(p)}\} = E\{(\widehat{s}_1^{(p)} - s_1)^2\}$, and hence verify that the following inequality always holds for the mean squared error (mse) relationship: $\text{mse}\{\widehat{s}_1^{(p)}\} < \text{mse}\{\widehat{s}_1^{(u)}\}$.

(iii) When $\text{var}\{X_1\} = 4/5$ and $\rho_1 = 0.5$ show that $\Pr(\widehat{s}_1^{(u)} \leq -1) = \Pr(\chi_1^2 \geq a)$ for an integer a which you should find. (χ_1^2 denotes a chi-squared random variable with one degree of freedom).

3. (a) Consider the real-valued time series $Z_t = X_t \, \mathrm{e}^{Y_t^2/8}$ where $\{X_t\}$ is a zero-mean real-valued stationary process with autocovariance sequence $\{s_{X,\tau}\}$, variance unity, and spectral density function $S_X(f)$. $\{Y_t\}$ is a sequence of independent random variables drawn from the Gaussian/normal distribution with mean zero and variance unity. $\{X_t\}$ and $\{Y_t\}$ are assumed independent of each other (i.e., the random variables $X_{t_1}, \ldots, X_{t_n}$ and $Y_{t'_1}, \ldots, Y_{t'_n}$ are mutually independent for any $n \geq 1$). Define constants $c_0 = \sqrt{2}$ and $c_1 = 4/3$.

(i) Show that

$$\operatorname{cov}\{Z_t, Z_{t+\tau}\} = \begin{cases} c_0, & \tau = 0; \\ c_1 s_{X,\tau}, & \tau \neq 0. \end{cases}$$

Justify that $\{Z_t\}$ is stationary.

(ii) Express the spectral density function $S_Z(f)$ of $\{Z_t\}$ in terms of $S_X(f)$ and c_0 and c_1.

(b) Let $S(f)$ denote the spectral density function of a stationary process, and let $\widehat{S}^{(p)}(f)$ denote the periodogram derived from a finite realization of the process. We know

$$E\{\widehat{S}^{(p)}(f)\} = \int_{-1/2}^{1/2} \mathcal{F}(f - f') S(f') \, \mathrm{d}f',$$

where $\mathcal{F}$ is Féjer's kernel defined by

$$\mathcal{F}(f) = \left| \sum_{t=1}^{N} \tfrac{1}{\sqrt{N}} \mathrm{e}^{-\mathrm{i}2\pi f t} \right|^2 = \frac{\sin^2(N\pi f)}{N \sin^2(\pi f)}.$$

The quantity

$$b(f) = E\{\widehat{S}^{(p)}(f)\} - S(f)$$

is the bias in the periodogram at frequency f. If $g(\cdot)$ is a function defined over the interval $[a, b]$, then, by definition,

$$\frac{1}{b-a} \int_a^b g(x) \, \mathrm{d}x$$

is the average value of $g(x)$ over $[a, b]$.

(i) Show that the average value of the bias in the periodogram over the frequency interval $[-1/2, 1/2]$ is zero.

(ii) Show that $E\{\widehat{S}^{(p)}(f)\}$ is an even function about zero.

(iii) Hence show that the average value of the bias in the periodogram over the frequency interval $[0, 1/2]$ is also zero.

4. (a) (i) What is meant by saying two discrete time stochastic processes $\{X_t\}$ and $\{Y_t\}$ are jointly second-order stationary stochastic processes?

(ii) Suppose $\{X_t\}$ is a zero mean stationary process and that $\{Y_t\}$ is the result of a linear filtering of $\{X_t\}$ with a filter having real-valued coefficients $\{g_u,\, u = -K, \ldots, K\}$, i.e.,

$$Y_t = \sum_{u=-K}^{K} g_u X_{t-u}.$$

Find $\mathrm{cov}\{X_t, Y_{t+\tau}\}$ and show that $\{X_t\}$ and $\{Y_t\}$ are jointly second-order stationary stochastic processes.

(iii) Show that the cross-spectrum $S_{XY}(f)$ for $\{X_t\}$ and $\{Y_t\}$ can be written as $G(f)S_X(f)$ where $G(f)$ is the frequency response function for $\{g_u\}$ and $S_X(f)$ is the spectral density function for $\{X_t\}$. Hence find the phase spectrum $\theta(f)$ for the particular filter $g_0 = 1/4, g_1 = 1/2, g_2 = 1/4$, with all other g_u's being zero.

(iv) The quantity $-\frac{1}{2\pi}\frac{\mathrm{d}\theta(f)}{\mathrm{d}f}$ is called the group delay. When it is a constant, the group delay is said to measure where $\mathrm{cov}\{X_t, Y_{t+\tau}\} = s_{XY,\tau}$ is concentrated in terms of the lag τ. Compute the group delay when using the particular filter in (a)(iii) and comment on why this is a reasonable value.

(b) Consider a zero mean AR(p) process,

$$X_t = \phi_{1,p}X_{t-1} + \ldots + \phi_{p,p}X_{t-p} + \epsilon_t.$$

From the course notes on forecasting we know that the l-step ahead forecast $X_t(l)$ of X_{t+l} depends only on the last p observed values of $\{X_t\}$ and may be obtained by setting future innovations to zero; in particular

$$X_t(1) = \phi_{1,p}X_t + \ldots + \phi_{p,p}X_{t-p+1}$$

which is X_{t+1} with the future innovation ϵ_{t+1} set to zero.

(i) Show that, for $p \geq 2$,

$$X_{t+2} = \phi_{1,p}\left[X_t(1) + \epsilon_{t+1}\right] + \sum_{j=2}^{p} \phi_{j,p}X_{t+2-j} + \epsilon_{t+2}$$

and hence that

$$X_{t+2} = \sum_{j=1}^{p} [\phi_{1,p}\phi_{j,p} + \phi_{j+1,p}]X_{t+1-j} + \phi_{1,p}\epsilon_{t+1} + \epsilon_{t+2}, \qquad (*)$$

where $\phi_{j,p} = 0$ if $j > p$.

(ii) Using equation $(*)$ of (b)(i) find the general form for $X_t(2)$ for AR(p) processes, $(p \geq 2)$, for which $\phi_{j,p} = 0, j = 1, \ldots, p-1$. Why might you have expected this result?

5. Consider the zero-mean stationary time series

$$X_t = D\cos(2\pi f_0 t + \phi) + \epsilon_t,$$

where $D > 0$ and $f_0 > 0$ are real-valued constants, and ϕ is a real-valued random phase angle having a uniform distribution on $[-\pi, \pi]$, independent of the white noise $\{\epsilon_t\}$.

(a) Given a sample $X_1, \ldots, X_N$ show that $E\{\widehat{S}^{(p)}(f)\}$ may be written as

$$E\{\widehat{S}^{(p)}(f)\} = \frac{1}{N}\sum_{t=1}^{N}\sum_{u=1}^{N} E\{X_t X_u^*\}\mathrm{e}^{-\mathrm{i}2\pi f(t-u)},$$

where $\widehat{S}^{(p)}(f)$ is the periodogram.

(b) (i) We know from the assigned reading that X_t can be written in the form

$$X_t = C_1\mathrm{e}^{\mathrm{i}2\pi f_0 t} + C_{-1}\mathrm{e}^{-\mathrm{i}2\pi f_0 t} + \epsilon_t,$$

where C_1 and C_{-1} are uncorrelated random variables. Give the forms of C_1 and C_{-1} in terms of D and ϕ.

(ii) Using X_t as in (b)(i), with summation terms in the same order, the spectral representation for X_t may be written

$$X_t = \int_{-1/2}^{1/2} \mathrm{e}^{\mathrm{i}2\pi f t}\mathrm{d}Z_1(f) + \int_{-1/2}^{1/2} \mathrm{e}^{\mathrm{i}2\pi f t}\mathrm{d}Z_2(f) + \int_{-1/2}^{1/2} \mathrm{e}^{\mathrm{i}2\pi f t}\mathrm{d}Z_3(f).$$

Write down the form of $E\{|\mathrm{d}Z_j(f)|^2\}$ for each of $j = 1, 2, 3$.

(c) (i) Using the representation for X_t in (b)(ii) show that

$$E\{X_t X_u^*\} = \frac{D^2}{4}[\mathrm{e}^{\mathrm{i}2\pi f_0(t-u)} + \mathrm{e}^{-\mathrm{i}2\pi f_0(t-u)}] + \sigma_\epsilon^2 \int_{-1/2}^{1/2} \mathrm{e}^{\mathrm{i}2\pi f'(t-u)}\mathrm{d}f'.$$

(ii) Using the fact that Fejér's kernel can be written as

$$\mathcal{F}(f) = \frac{1}{N}\sum_{t=1}^{N}\sum_{u=1}^{N} \mathrm{e}^{\mathrm{i}2\pi f(t-u)},$$

show that

$$E\{\widehat{S}^{(p)}(f)\} = \frac{D^2}{4}[\mathcal{F}(f-f_0) + \mathcal{F}(f+f_0)] + \sigma_\epsilon^2.$$

Course: M3S8/M4S8/M5S8
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BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2017

M3S8/M4S8/M5S8

Time Series [SOLUTIONS]

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1. (a) $\{X_t\}$ is second-order stationary if $E\{X_t\}$ is a finite constant for all t, var$\{X_t\}$ is a finite constant for all t, and cov$\{X_t, X_{t+\tau}\}$, is a finite quantity depending only on τ and not on t. $\boxed{\text{seen} \Downarrow}$ $\boxed{4}$

(b) With $\Delta_t = 1/8$s the Nyquist frequency is $f_{\mathcal{N}} = 1/(2\Delta_t) = 4$Hz. From the aliasing formula the spectrum for the continuous-time process is folded around $f = \pm 4$Hz and added. The result is $\boxed{\text{sim. seen} \Downarrow}$

$$S_{X_t}(f) = \begin{cases} 1, & |f| \leq 2; \\ 2, & 2 < |f| < 4. \end{cases}$$

$\boxed{3}$

(c) (i) the multivariate Gaussian is completely characterized by its 1st and 2nd-order joint moments, and since these are time invariant under (second-order) stationarity, then all moments will be time-invariant; complete stationarity follows. $\boxed{\text{seen} \Downarrow}$

(ii) the spectral density function is constant over $f \in [-1/2, 1/2]$ so that all frequencies (colours) contribute equally to the variance.

(iii)

$$X_t - \mu = \int_{-1/2}^{1/2} e^{i2\pi ft} dZ(f),$$

where $Z(f)$ is the orthogonal process.

(iv) the (squared) linear correlation between the components of $\{X_t\}$ and $\{Y_t\}$ at frequency f.

(v) the determinantal polynomial which is

$$\det\{\boldsymbol{I} - \boldsymbol{\phi} z\} = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right] - \left[\begin{smallmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{3} \end{smallmatrix}\right] z = (1 - \frac{1}{4}z)(1 - \frac{1}{3}z),$$

has roots of 3 and 4, both outside the unit circle. $\boxed{5}$

(d) (i) Choose $\tau > 0$. Then $s_\tau = \text{cov}\{Z_t, Z_{t+\tau}\} \stackrel{\text{def}}{=} E\{Z_t^* Z_{t+\tau}\}$, where we notice that $t + \tau$ is later in time than t. Then $\boxed{\text{sim. seen} \Downarrow}$

$$s_{-\tau} = \text{cov}\{Z_t, Z_{t-\tau}\} \stackrel{\text{def}}{=} E\{Z_t^* Z_{t-\tau}\} = E\{Z_{t-\tau} Z_t^*\} = (E\{Z_{t-\tau}^* Z_t\})^* \stackrel{\text{def}}{=} s_\tau^*,$$

since t is later in time than $t - \tau$. $\boxed{2}$

(ii) For the complex-valued MA(1) series:

$$s_0 = \text{var}\{Z_t\} = E\{|Z_t|^2\} = E\{(\epsilon_t - \theta^* \epsilon_{t-1})(\epsilon_t - \theta \epsilon_{t-1})\} = E\{\epsilon_t^2\} + |\theta|^2 E\{\epsilon_{t-1}^2\} = 1 + |\theta|^2,$$

since $\{\epsilon_t\}$ has variance unity. Then for $\tau = 1$,

$$s_1 = \text{cov}\{Z_t, Z_{t+1}\} = E\{Z_t^* Z_{t+1}\} = E\{(\epsilon_t - \theta^* \epsilon_{t-1})(\epsilon_{t+1} - \theta \epsilon_t)\} = -\theta E\{\epsilon_t^2\} = -\theta.$$

We then know that $s_{-1} = s_1^* = -\theta^*$. So,

$$s_\tau = \begin{cases} 1 + |\theta|^2, & \tau = 0; \\ -\theta, & \tau = 1; \\ -\theta^*, & \tau = -1; \\ 0, & \text{o/w}. \end{cases}$$

[Terms are zero for $|\tau| > 1$ since the process is an MA(1).] $\boxed{3}$

(iii) Now $S(f) = \sum_{\tau=-\infty}^{\infty} s_\tau \mathrm{e}^{-\mathrm{i}2\pi f\tau}$. For the MA(1), with $\theta = x + \mathrm{i}y$,

$$\begin{aligned} S(f) &= -\theta^* \mathrm{e}^{\mathrm{i}2\pi f} + 1 + |\theta|^2 - \theta \mathrm{e}^{-\mathrm{i}2\pi f} \\ &= -(x - \mathrm{i}y)\mathrm{e}^{\mathrm{i}2\pi f} + 1 + |\theta|^2 - (x + \mathrm{i}y)\mathrm{e}^{-\mathrm{i}2\pi f} \\ &= -2x\cos(2\pi f) + 1 + |\theta|^2 - 2y\sin(2\pi f) \neq S(-f), \end{aligned}$$

since $\sin$ is an odd function. $\boxed{3}$

2. (a) Any *two* of the following three: $\boxed{\text{seen} \Downarrow}$

[1] For many stationary processes of practical interest mse$\{\widehat{s}_\tau^{(p)}\}$ $<$ mse$\{\widehat{s}_\tau^{(u)}\}$, where mse $=$ variance $+$ (bias)^2.

[2] If $\{X_t\}$ has a purely continuous spectrum we know that $s_\tau \rightarrow 0$ as $|\tau| \rightarrow \infty$. It therefore makes sense to choose an estimator that decreases nicely as $|\tau| \rightarrow N-1$ (i.e. choose $\widehat{s}_\tau^{(p)}$).

[3] We know that the autocovariance sequence must be positive semidefinite; sequence $\{\widehat{s}_\tau^{(p)}\}$ has this property, whereas the sequence $\{\widehat{s}_\tau^{(u)}\}$ may not. $\boxed{4}$

(b) (i) Now $\boxed{\text{unseen} \Downarrow}$

$$\begin{aligned} g(M) &= E\left\{\left[\frac{(X_1-X_2)^2}{M}+s_1\right]^2\right\} \\ &= E\left\{\frac{(X_1-X_2)^4}{M^2}+2\frac{(X_1-X_2)^2 s_1}{M}+s_1^2\right\} \\ &= \frac{1}{M^2}E\{(X_1-X_2)^4\}+\frac{2s_1}{M}E\{(X_1-X_2)^2\}+s_1^2. \quad (1) \end{aligned}$$

For the expectation in the first term of (1), we have that $X_1 - X_2$ is zero mean and normal so we can use Isserlis' result to write

$$E\{(X_1-X_2)^4\} = 3E\{(X_1-X_2)^2\}E\{(X_1-X_2)^2\} = 3E^2\{(X_1-X_2)^2\}.$$

But $E\{(X_1-X_2)^2\} = 2s_0 - 2s_1$. So

$$E\{(X_1-X_2)^4\} = 3\cdot(2s_0-2s_1)^2 = 12s_0^2 - 24s_0s_1 + 12s_1^2.$$

OR, since the distribution of $X_1 - X_2$ does not depend upon μ, we could assume $\mu = 0$ for this calculation. So assuming $\mu = 0$ an alternative derivation is

$$E\{(X_1-X_2)^4\} = E\{X_1^4 - 4X_1^3X_2 + 6X_1^2X_2^2 - 4X_1X_2^3 + X_2^4\}.$$

Three applications of Isserlis' result yield

$E\{X_1^4\} = E\{X_2^4\} = 3s_0^2$, $E\{X_1^3X_2\} = E\{X_1X_2^3\} = 3s_0s_1$ and $E\{X_1^2X_2^2\} = s_0^2 + 2s_1^2$.

Hence

$$E\{(X_1-X_2)^4\} = 12s_0^2 - 24s_0s_1 + 12s_1^2,$$

as before. EITHER DERIVATION OF THIS EXPECTATION IS FINE. $\boxed{4}$

Turning now to the second term of $(*)$, we have

$$E\{(X_1 - X_2)^2\} = 2s_0 - 2s_1.$$

Hence the quadratic in $1/M$ is

$$g(M) = \frac{12s_0^2 - 24s_0s_1 + 12s_1^2}{M^2} + \frac{4s_1(s_0 - s_1)}{M} + s_1^2.$$

and further, the desired quadratic in $1/M$ involving ρ_1, is

$$\frac{g(M)}{s_0^2} = \frac{12(1 - 2\rho_1 + \rho_1^2)}{M^2} + \frac{4\rho_1(1 - \rho_1)}{M} + \rho_1^2,$$

where $\rho_1 \stackrel{\text{def}}{=} s_1/s_0$. $\boxed{4}$

(ii) Now $X_1 - \overline{X} = (X_1 - X_2)/2$ and $X_2 - \overline{X} = -(X_1 - X_2)/2$, from which it follows that

$$\widehat{s}_1^{(u)} = -\frac{(X_1 - X_2)^2}{4} \text{ and } \widehat{s}_1^{(p)} = -\frac{(X_1 - X_2)^2}{8}.$$

So $g(4) = \text{mse}\{\widehat{s}_1^{(u)}\}$ and $g(8) = \text{mse}\{\widehat{s}_1^{(p)}\}$. The desired result follows if we can show that

$$\frac{g(8)}{s_0^2} < \frac{g(4)}{s_0^2}, \text{ i.e., } \frac{3(1 - 2\rho_1 + \rho_1^2)}{16} + \frac{\rho_1(1 - \rho_1)}{2} < \frac{3(1 - 2\rho_1 + \rho_1^2)}{4} + \rho_1(1 - \rho_1);$$

however this inequality is equivalent to $0 < (9 - \rho_1)(1 - \rho_1)$. Now $-1 \le \rho_1 < 1$ (any correlation satisfies $-1 \le \rho_1 \le 1$ and we are told in the question that $s_1 < s_0$ i.e., $\rho_1 < 1$ here). So both $(1 - \rho_1)$ and $(9 - \rho_1)$ are positive, so the above terms are positive thus establishing the desired result. $\boxed{4}$

(iii) To calculate $\Pr(\widehat{s}_1^{(u)} \le -1)$ we must first derive the distribution of $\widehat{s}_1^{(u)} = -\frac{(X_1 - X_2)^2}{4}$. Concentrate on $X_1 - X_2$. From the standard formula

$$\text{var}\{X_1 - X_2\} = \text{var}\{X_1\} + \text{var}\{X_2\} - 2\,\text{cov}\{X_1, X_2\} = 2s_0(1 - \rho_1).$$

So

$$X_1 - X_2 \stackrel{\text{d}}{=} N(0, 2s_0(1 - \rho_1)).$$

Then we know that

$$\frac{(X_1 - X_2)^2}{2s_0(1 - \rho_1)} \stackrel{\text{d}}{=} \chi_1^2 \Rightarrow \frac{-4\widehat{s}_1^{(u)}}{2s_0(1 - \rho_1)} \stackrel{\text{d}}{=} \chi_1^2.$$

So

$$\Pr(\widehat{s}_1^{(u)} \le -1) = \Pr\left(\frac{-4\widehat{s}_1^{(u)}}{2s_0(1 - \rho_1)} \ge \frac{4}{2s_0(1 - \rho_1)}\right) = \Pr\left(\chi_1^2 \ge \frac{4}{2s_0(1 - \rho_1)}\right).$$

When $s_0 = 4/5$ and $\rho_1 = 0.5$ the required probability is $\Pr(\chi_1^2 \ge 5)$. $\boxed{4}$

3. (a) (i) Here $Z_t = X_t e^{Y_t^2/8}$ and $Y_t \stackrel{d}{=} N(0,1)$. $\{X_t\}, \{Y_t\}$ are independent of each other. Then, by independence of $\{X_t\}, \{Y_t\}$ and zero mean of X_t, $\boxed{\text{sim. seen} \Downarrow}$

$$E\{Z_t\} = E\{X_t\}E\{e^{Y_t^2/8}\} = 0.$$

$\boxed{1}$

Then, again by independence of $\{X_t\}, \{Y_t\}$,

$$\text{cov}\{Z_t, Z_{t+\tau}\} \stackrel{\text{def}}{=} E\{Z_t Z_{t+\tau}\} = E\{X_t e^{Y_t^2/8} \cdot X_{t+\tau} e^{Y_{t+\tau}^2/8}\} = s_{X,\tau} E\{e^{Y_t^2/8} e^{Y_{t+\tau}^2/8}\}.$$

When $\tau = 0$, using that $\{Y_t\}$ is identically distributed, we get

$$\text{var}\{Z_t\} = s_{X,0} E\{e^{Y_t^2/4}\} = 1 \cdot E\{e^{Y^2/4}\},$$

$\boxed{2}$

and when $\tau \neq 0$, using that $\{Y_t\}$ is iid,

$$\text{cov}\{Z_t, Z_{t+\tau}\} = s_{X,\tau} E\{e^{Y_t^2/8}\} E\{e^{Y_{t+\tau}^2/8}\} = s_{X,\tau} E^2\{e^{Y^2/8}\}.$$

$\boxed{2}$

We thus need to work out $E\{e^{uY^2}\}$ where $Y \stackrel{d}{=} N(0,1)$, for some constant u. This is straightforward:

$$I \stackrel{\text{def}}{=} E\{e^{uY^2}\} = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} e^{(uy^2 - \frac{y^2}{2})} dy = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}(1-2u)} dy.$$

But $\frac{1}{\sigma\sqrt{(2\pi)}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2\sigma^2}} dy = 1$, so $I = (1-2u)^{-1/2}$. Clearly we require $u < 1/2$. So $E\{e^{Y^2/4}\} = \sqrt{2}$ and $E^2\{e^{Y^2/8}\} = 4/3$. Thus,

$$\text{cov}\{Z_t, Z_{t+\tau}\} = \begin{cases} c_0, & \tau = 0; \\ c_1 s_{X,\tau}, & \tau \neq 0. \end{cases}$$

$\boxed{3}$

We see that the covariance does not depend on t and since the mean is zero, the process is stationary. $\boxed{1}$

(We shall now denote: $s_{Z,\tau} = \text{cov}\{Z_t, Z_{t+\tau}\}$.)

(ii) The spectral density function $S_Z(f)$ of $\{Z_t\}$ is $S_Z(f) = \sum_{\tau=-\infty}^{\infty} s_{Z,\tau} e^{-i2\pi f\tau}$. To express the spectral density function $S_Z(f)$ of $\{Z_t\}$ in terms of $S_X(f)$: $\boxed{\text{unseen} \Downarrow}$

$$\begin{aligned} S_Z(f) &= s_{Z,0} + \sum_{\tau=-\infty}^{-1} s_{Z,\tau} e^{-i2\pi f\tau} + \sum_{\tau=1}^{\infty} s_{Z,\tau} e^{-i2\pi f\tau} \\ &= \sqrt{2} + \tfrac{4}{3} \left[\sum_{\tau=-\infty}^{-1} + \sum_{\tau=1}^{\infty} \right] s_{X,\tau} e^{-i2\pi f\tau} \\ &= \sqrt{2} - \tfrac{4}{3} s_{X,0} + \tfrac{4}{3} S_X(f) = [c_0 - c_1] + c_1 S_X(f). \end{aligned}$$

$\boxed{3}$

(b) (i) $\boxed{\text{unseen} \Downarrow}$

$$
\begin{aligned}
\int_{-1/2}^{1/2} b(f)\,\mathrm{d}f &= \int_{-1/2}^{1/2} E\{\widehat{S}^{(p)}(f)\} - S(f)\,\mathrm{d}f \\
&= \int_{-1/2}^{1/2} E\{\widehat{S}^{(p)}(f)\}\,\mathrm{d}f - \int_{-1/2}^{1/2} S(f)\,\mathrm{d}f \\
&= \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} \mathcal{F}(f-f')S(f')\,\mathrm{d}f'\,\mathrm{d}f - \int_{-1/2}^{1/2} S(f)\,\mathrm{d}f \\
&= \int_{-1/2}^{1/2} S(f')\left[\int_{-1/2}^{1/2} \mathcal{F}(f-f')\,\mathrm{d}f\right]\mathrm{d}f' - \int_{-1/2}^{1/2} S(f)\,\mathrm{d}f \\
&= \int_{-1/2}^{1/2} S(f')\,\mathrm{d}f' - \int_{-1/2}^{1/2} S(f)\,\mathrm{d}f = 0,
\end{aligned}
$$

since $\mathcal{F}(\cdot)$ is a periodic function with a period of unity, and therefore $\int_{-1/2}^{1/2}\mathcal{F}(f-f')\,\mathrm{d}f = 1$ for any f' because $\int_{-1/2}^{1/2}\mathcal{F}(f)\,\mathrm{d}f = 1$. $\boxed{4}$

(ii) To show $E\{\widehat{S}^{(p)}(f)\}$ is an even function,

$$
\begin{aligned}
E\{\widehat{S}^{(p)}(-f)\} &= \int_{-1/2}^{1/2} \mathcal{F}(-f-f')S(f')\,\mathrm{d}f' \overset{\mathcal{F}\,\text{even}}{=} \int_{-1/2}^{1/2} \mathcal{F}(f+f')S(f')\,\mathrm{d}f' \\
&= -\int_{1/2}^{-1/2} \mathcal{F}(f-f'')S(-f'')\,\mathrm{d}f'' \overset{S\,\text{even}}{=} \int_{-1/2}^{1/2} \mathcal{F}(f-f'')S(f'')\,\mathrm{d}f'' \\
&= E\{\widehat{S}^{(p)}(f)\},
\end{aligned}
$$

where $\mathcal{F}$ and S being even is pointed out in the course. $\boxed{2}$

(iii) As both $E\{\widehat{S}^{(p)}(f)\}$ and $S(f)$ are even functions about $f = 0$, so is $b(f)$. Then,

$$
0 = \int_{-1/2}^{1/2} b(f)\,\mathrm{d}f = 2\int_0^{1/2} b(f)\,\mathrm{d}f \Rightarrow \int_0^{1/2} b(f)\,\mathrm{d}f = 0.
$$

$\boxed{2}$

4. (a) (i) Two real-valued discrete time stochastic processes $\{X_t\}$ and $\{Y_t\}$ are said to be jointly stationary stochastic processes if $\{X_t\}$ and $\{Y_t\}$ are each, separately, second-order stationary processes, and cov$\{X_t, Y_{t+\tau}\}$ is a function of τ only. seen ⇓ 2

(ii) Now we know that $E\{X_t\} = 0$, and $E\{Y_t\} = 0$ by linearity. So

$$\begin{aligned}\text{cov}\{X_t, Y_{t+\tau}\} &= E\{X_t Y_{t+\tau}\} = E\{X_t \sum_{u=-K}^{K} g_u X_{t+\tau-u}\} \\ &= \sum_{u=-K}^{K} g_u E\{X_t X_{t+\tau-u}\} = \sum_{u=-K}^{K} g_u s_{X,\tau-u}.\end{aligned}$$

The process $\{X_t\}$ is stationary and we have just shown that cov$\{X_t, Y_{t+\tau}\}$ is a function of τ only. We need to show that $\{Y_t\}$ is stationary. 2

$$\begin{aligned}\text{cov}\{Y_t, Y_{t+\tau}\} &= E\{Y_t Y_{t+\tau}\} = E\{\sum_{u=-K}^{K} g_u X_{t-u} \sum_{v=-K}^{K} g_v X_{t+\tau-v}\} \\ &= \sum_{u=-K}^{K}\sum_{v=-K}^{K} g_u g_v E\{X_{t-u} X_{t+\tau-v}\} = \sum_{u=-K}^{K}\sum_{v=-K}^{K} g_u g_v s_{X,\tau-v+u},\end{aligned}$$

which is a function of τ only so $\{Y_t\}$ is stationary, and the processes are jointly stationary. 2

(iii)

$$\begin{aligned}S_{XY}(f) &= \sum_{\tau=-\infty}^{\infty} s_{XY,\tau} e^{-i2\pi f\tau} = \sum_{\tau=-\infty}^{\infty}\sum_{u=-K}^{K} g_u s_{X,\tau-u} e^{-i2\pi f\tau} \\ &= \sum_{u=-K}^{K} g_u \sum_{\tau=-\infty}^{\infty} s_{X,\tau-u} e^{-i2\pi f\tau} = \sum_{u=-K}^{K} g_u e^{-i2\pi f u} \sum_{\tau=-\infty}^{\infty} s_{X,\tau-u} e^{-i2\pi f(\tau-u)} \\ &= G(f) S_X(f).\end{aligned}$$

2

Now $S_{XY}(f) = |S_{XY}(f)| e^{i\theta(f)} = |G(f)| S_X(f) e^{i\theta(f)}$ and unseen ⇓

$$G(f) = \frac{1}{4} + \frac{1}{2} e^{-i2\pi f} + \frac{1}{4} e^{-i4\pi f} = e^{-i2\pi f}\left[\frac{1}{4} e^{i2\pi f} + \frac{1}{2} + \frac{1}{4} e^{-i2\pi f}\right] = e^{-i2\pi f}\frac{1}{2}\left[1 + \cos(2\pi f)\right].$$

So

$$\begin{aligned}S_{XY}(f) &= |G(f)| S_X(f) e^{i\theta(f)} = \left[\frac{1}{2}\left[1 + \cos(2\pi f)\right] S_X(f)\right] e^{-i2\pi f} = |S_{XY}(f)| e^{i\theta(f)} \\ \Rightarrow \theta(f) &= -2\pi f.\end{aligned}$$

2

(iv) The group delay is hence $-\frac{1}{2\pi}\frac{d\theta(f)}{df} = 1$. So $s_{XY,\tau}$ is concentrated about $s_{XY,1} = \text{cov}\{X_t Y_{t+1}\}$; this makes sense since Y_t is a symmetric averaging of X_t's about $t-1$ and thus Y_t needs to be advanced in time by 1 unit to match with X_t. 2

(b) (i) $\boxed{\text{unseen} \Downarrow}$

$$
\begin{aligned}
X_{t+2} &= \phi_{1,p}X_{t+1} + \ldots + \phi_{p,p}X_{t-p+2} + \epsilon_{t+2} \\
&= \phi_{1,p}\left[\sum_{j=1}^{p}\phi_{j,p}X_{t+1-j} + \epsilon_{t+1}\right] + \sum_{j=2}^{p}\phi_{j,p}X_{t+2-j} + \epsilon_{t+2} \\
&= \phi_{1,p}\left[X_t(1) + \epsilon_{t+1}\right] + \sum_{j=2}^{p}\phi_{j,p}X_{t+2-j} + \epsilon_{t+2}.
\end{aligned}
$$

So $\boxed{2}$

$$
\begin{aligned}
X_{t+2} &= \sum_{j=1}^{p}\phi_{1,p}\phi_{j,p}X_{t+1-j} + \phi_{1,p}\epsilon_{t+1} + \sum_{j=2}^{p}\phi_{j,p}X_{t+2-j} + \epsilon_{t+2} \\
&= \sum_{j=1}^{p}[\phi_{1,p}\phi_{j,p} + \phi_{j+1,p}]X_{t+1-j} + \phi_{1,p}\epsilon_{t+1} + \epsilon_{t+2}. \qquad (2)
\end{aligned}
$$

$\boxed{2}$

(ii) For $X_t(2)$ set future innovations to zero in (2):

$$X_t(2) = \sum_{j=1}^{p}[\phi_{1,p}\phi_{j,p} + \phi_{j+1,p}]X_{t+1-j}.$$

$\boxed{1}$

The particular form of AR(p) processes, $(p \geq 2)$, to consider are those for which $\phi_{j,p} = 0, j = 1, \ldots, p-1$. We know from the course notes that $\phi_{p,p} \neq 0$ in order that the process is AR(p).
The first term in the sum in $X_t(2)$ is always zero $(\phi_{1,p} = 0)$ and the second term will only be non-zero when $j = p-1$, $(\phi_{j,p} = 0$ if $j > p)$, so we get the general form

$$X_t(2) = \phi_{p,p}X_{t-p+2.}$$

$\boxed{2}$

Starting from the defining AR equation, with the parameter values given, we have

$$X_{t+2} = \phi_{1,1}X_{t+1} + \cdots \phi_{p,p}X_{t+2-p} + \epsilon_{t+2} = \phi_{p,p}X_{t+2-p} + \epsilon_{t+2},$$

so setting the future innovation ϵ_{t+2} to zero gives $X_t(2) = \phi_{p,p}X_{t-p+2}$. $\boxed{1}$

5. (a) By definition, seen ⇓

$$
\begin{aligned}
E\{\widehat{S}^{(p)}(f)\} &= \frac{1}{N}E\left\{\left|\sum_{t=1}^{N} X_t e^{-i2\pi ft}\right|^2\right\} = \frac{1}{N}E\left\{\sum_{t=1}^{N} X_t e^{-i2\pi ft}\sum_{u=1}^{N} X_u^* e^{i2\pi fu}\right\} \\
&= \frac{1}{N}\sum_{t=1}^{N}\sum_{u=1}^{N} E\{X_t X_u^*\}e^{-i2\pi f(t-u)}.
\end{aligned}
$$

2

(b) (i) Now unseen ⇓

$$
D\cos(2\pi f_0 t+\phi) = \frac{D}{2}\left[e^{i\phi}e^{i2\pi f_0 t} + e^{-i\phi}e^{-i2\pi f_0 t}\right],
$$

so $C_1 = (D/2)e^{i\phi}$ and $C_{-1} = (D/2)e^{-i\phi}$. 2

(ii) Using the assigned reading for the first two results, and standard theory for the third

$$
\begin{aligned}
E\{|dZ_1(f)|^2\} &= \begin{cases} E\{|C_1|^2\} = D^2/4, & f = f_0 \\ 0, & \text{otherwise.} \end{cases} \\
E\{|dZ_2(f)|^2\} &= \begin{cases} E\{|C_{-1}|^2\} = D^2/4, & f = -f_0 \\ 0, & \text{otherwise.} \end{cases} \\
E\{|dZ_3(f)|^2\} &= S_\epsilon(f)df = \sigma_\epsilon^2 df,
\end{aligned}
$$

where $S_\epsilon(f)$ is the spectrum of the white noise. 3

(c) (i) As we will be looking at products we use a different dummy frequency index in each integral (emphasized in the course):

$$
\begin{aligned}
E\{X_t X_u^*\} = E\Bigg\{&\left[\int_{-1/2}^{1/2} e^{i2\pi f_1 t}dZ_1(f_1) + \int_{-1/2}^{1/2} e^{i2\pi f_2 t}dZ_2(f_2) + \int_{-1/2}^{1/2} e^{i2\pi f_3 t}dZ_3(f_3)\right] \\
&\times\left[\int_{-1/2}^{1/2} e^{-i2\pi f_4 u}dZ_1^*(f_4) + \int_{-1/2}^{1/2} e^{-i2\pi f_5 u}dZ_2^*(f_5) + \int_{-1/2}^{1/2} e^{-i2\pi f_6 u}dZ_3^*(f_6)\right]\Bigg\}
\end{aligned}
$$

Now C_1 and C_{-1} are stated to be uncorrelated, and both are functions of ϕ which is independent of $\{\epsilon_t\}$. Hence $\text{cov}\{dZ_j(f), dZ_k(f)\} = 0$ for $j \neq k$. So we need only evaluate the (co)variance terms for $j = k, j = 1, 2, 3$. 3

Using the orthogonality of the increments, the first such term is

$$
\begin{aligned}
&E\left\{\int_{-1/2}^{1/2} e^{i2\pi f_1 t}dZ_1(f_1)\int_{-1/2}^{1/2} e^{-i2\pi f_4 u}dZ_1^*(f_4)\right\} \\
&= \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} e^{i2\pi f_1 t}e^{-i2\pi f_4 u}E\{dZ_1(f_1)dZ_1^*(f_4)\} \\
&= \int_{-1/2}^{1/2} e^{i2\pi f_1(t-u)}E\{|dZ_1(f_1)|^2\} = (D^2/4)e^{i2\pi f_0(t-u)}.
\end{aligned}
$$

The second is, analogously, $(D^2/4)\mathrm{e}^{-\mathrm{i}2\pi f_0(t-u)}$. $\boxed{3}$
The third is

$$\int_{-1/2}^{1/2} \mathrm{e}^{\mathrm{i}2\pi f_3(t-u)} E\{|\mathrm{d}Z_3(f_3)|^2\} = \sigma_\epsilon^2 \int_{-1/2}^{1/2} \mathrm{e}^{\mathrm{i}2\pi f_3(t-u)} \mathrm{d}f_3.$$

$\boxed{2}$

So altogether we get (with $f_3 \to f'$)

$$E\{X_t X_u^*\} = \frac{D^2}{4}[\mathrm{e}^{\mathrm{i}2\pi f_0(t-u)} + \mathrm{e}^{-\mathrm{i}2\pi f_0(t-u)}] + \sigma_\epsilon^2 \int_{-1/2}^{1/2} \mathrm{e}^{\mathrm{i}2\pi f'(t-u)} \mathrm{d}f'.$$

(The dummy variable f' has been used rather than f to assist with the next part of the question).

(ii) Now

$$E\{\widehat{S}^{(p)}(f)\} = \frac{1}{N}\sum_{t=1}^{N}\sum_{u=1}^{N} E\{X_t X_u^*\}\mathrm{e}^{-\mathrm{i}2\pi f(t-u)}$$

$$= \frac{1}{N}\sum_{t=1}^{N}\sum_{u=1}^{N}\left[\frac{D^2}{4}[\mathrm{e}^{\mathrm{i}2\pi f_0(t-u)} + \mathrm{e}^{-\mathrm{i}2\pi f_0(t-u)}] + \sigma_\epsilon^2 \int_{-1/2}^{1/2} \mathrm{e}^{\mathrm{i}2\pi f'(t-u)} \mathrm{d}f'\right]\mathrm{e}^{-\mathrm{i}2\pi f(t-u)}.$$

The first term is

$$\frac{1}{N}\sum_{t=1}^{N}\sum_{u=1}^{N}\frac{D^2}{4}\mathrm{e}^{\mathrm{i}2\pi f_0(t-u)}\mathrm{e}^{-\mathrm{i}2\pi f(t-u)} = \frac{D^2}{4}\left[\frac{1}{N}\sum_{t=1}^{N}\sum_{u=1}^{N}\mathrm{e}^{\mathrm{i}2\pi (f_0-f)(t-u)}\right] = \frac{D^2}{4}\mathcal{F}(f-f_0),$$

$\boxed{2}$

where we have also used the symmetry of $\mathcal{F}$ about zero (lectures). The second term follows directly by negating f_0 in the previous result:

$$\frac{1}{N}\sum_{t=1}^{N}\sum_{u=1}^{N}\frac{D^2}{4}\mathrm{e}^{-\mathrm{i}2\pi f_0(t-u)}\mathrm{e}^{-\mathrm{i}2\pi f(t-u)} = \frac{D^2}{4}\mathcal{F}(f+f_0).$$

$\boxed{1}$

Finally

$$\sigma_\epsilon^2 \int_{-1/2}^{1/2} \frac{1}{N}\sum_{t=1}^{N}\sum_{u=1}^{N} \mathrm{e}^{\mathrm{i}2\pi (f'-f)(t-u)}\mathrm{d}f' = \sigma_\epsilon^2 \int_{-1/2}^{1/2} \mathcal{F}(f'-f)\mathrm{d}f' = \sigma_\epsilon^2,$$

since $\mathcal{F}$ is periodic with period 1 and integrates to unity over a period (lectures). $\boxed{2}$
So

$$E\{\widehat{S}^{(p)}(f)\} = \frac{D^2}{4}\left[\mathcal{F}(f-f_0) + \mathcal{F}(f+f_0)\right] + \sigma_\epsilon^2.$$