Paper Number(s): **E3.08**
ISE3.17

IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE
UNIVERSITY OF LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2002

EEE/ISE PART III/IV: M.Eng., B.Eng. and ACGI

ADVANCED SIGNAL PROCESSING

Wednesday, 1 May 10:00 am

There are FIVE questions on this paper.

Answer TWO of the questions 1, 2, 3 and ONE of the questions 4, 5.

Corrected Copy

G.1.

Time allowed: 3:00 hours

Examiners responsible:

First Marker(s): Mandic,D.P.

Second Marker(s): Ward,D.B.

Special instructions for invigilators: None

Information for candidates: None

1. Consider a random process $Z(t) = X(t) \pm Y(t)$, where $X(t)$ and $Y(t)$ are also random processes. The $\pm$ sign means that $X(t)$ and $Y(t)$ can be either added or subtracted.

a) Define random variables Z_1 and Z_2 by $Z_1 = X(t_1) \pm Y(t_1)$ and $Z_2 = X(t_1+\tau) \pm Y(t_1+\tau)$. Find the autocorrelation function $R_Z(\tau) = E[Z_1 Z_2]$.

[8]

b) If the two random processes Z_1 and Z_2 are statistically independent, and zero mean, what is the resulting autocorrelation function $R_Z(\tau)$? An important consequence of this result arises in the extraction of periodic signals from random noise. Suppose the autocorrelation function of the desired signal $X(t)$ is $R_X(t) = \frac{1}{2}A^2 \cos \omega\tau$. Suppose also that there is a zero–mean random noise signal $Y(t)$ that is statistically independent of the signal and has $R_Y(\tau) = B^2 e^{-\alpha|\tau|}$. Find the autocorrelation function of $X(t) + Y(t)$.

10.15am

$R_X(\tau)$.

[4]

c) A radar system is transmitting a signal $X(t)$. The signal is then returned from the target attenuated and delayed in time, and can be represented as $Y(t) = aX(t-T) + N(t)$, where $a < 1$, T is the time delay and $N(t)$ is the noise which is statistically independent of the signal. Both the signal and noise have zero mean. Find the crosscorrelation function between the transmitted signal $X(t)$ and received signal $Y(t)$.

[8]

2. a) Define the likelihood function for a random signal x. State the likelihood function for the random variable $x(0) = A + w(0)$, where A is the DC level, and $w \sim \mathcal{N}(0, \sigma^2)$.

[4]

b) Define the curvature of the log–likelihood function. What is the curvature for the random process $x(0)$ of part a)? What does the curvature give information about?

[4]

c) Define the Cramer–Rao Lower Bound (CRLB) (scalar parameter).

[8]

d) What is the minimum attainable variance for the random process $x(0)$ of part a)?

[4]

3. a) State the equation of the second order autoregressive process $AR(2)$.

[2]

Derive the autocorrelation function of this process.

[4]

What is the stability condition for this process (stability triangle)?

[6]

b) Consider the process $z_t = 0.75 z_{t-1} - 0.5 z_{t-2} + a_t$, where a_t is white noise. Is the process z_t stable?

[2]

For z_t, state

i) the Yule-Walker equations.

[3]

ii) the spectrum.

[3]

4. The least mean square (LMS) algorithm for a linear finite impulse response (FIR) adaptive filter is given by

$$\begin{aligned} e(k) &= d(k) - y(k) \\ y(k) &= \mathbf{x}^T(k)\mathbf{w}(k) \\ \mathbf{w}(k+1) &= \mathbf{w}(k) + \eta e(k)\mathbf{x}(k) \end{aligned}$$

where $e(k)$ is the output error of the filter, $d(k)$ is the desired signal, $y(k)$ is the output of the filter, $\mathbf{x}(k)$ is the tap input vector to the filter, $\mathbf{w}(k)$ is the weight vector, η is the learning rate, and $(\cdot)^T$ denotes transposition.

a) Derive the learning rate of the normalised least mean square (NLMS) algorithm by expanding the error $e(k+1)$ using Taylor series around $e(k)$ and setting $e(k+1) = 0$.

[12]

b) Explain the difference between the learning rate η and the learning rate of the NLMS $\eta_{NLMS} = 1/ \| \mathbf{x}(k) \|_2^2$.

[4]

c) Why can we say that the NLMS algorithm minimises the so-called *a posteriori* output error?

[4]

where $e(k) = d(k) - y(k)$

5. A simple nonlinear finite impulse response (FIR) adaptive filter is shown in Figure 5.1. Φ is a saturation nonlinear function. The cost function for this filter is given by $E(k) = \frac{1}{2}e^2(k)$.

a) Give the reasons for the structure of Figure 5.1 also being called a dynamical neuron.

[4]

b) Derive the weight update equation $\Delta \mathbf{w}(k) = -\eta \nabla_{\mathbf{w}} E(k)$ for the filter above.

[8]

c) What is the difference between this filter and the standard adaptive FIR filter? Why does this structure perform generally better on filtering of nonlinear signals?

[4]

d) If the nonlinear function Φ is the arctan function, explain the effect of saturation in the output. What is the effect of saturation–type nonlinearity on the output magnitude range?

[4]

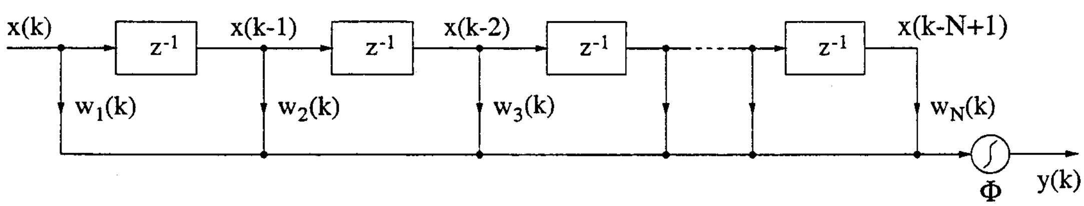

Figure 5.1. A nonlinear FIR filter

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1)

a) $Z(t) = X(t) \pm Y(t)$

$Z_1 = X(t_1) \pm Y(t_1)$ $\quad Z_2 = X(t_1+\tau) \pm Y(t_1+\tau)$

$$R_Z(\tau) = E[Z_1 Z_2] = E[(X_1 \pm Y_1)(X_2 \pm Y_2)]$$

$$= E[X_1X_2 + Y_1Y_2 \pm X_1Y_2 \pm Y_1X_2]$$

$$= R_X(\tau) + R_Y(\tau) \pm R_{XY}(\tau) \pm R_{YX}(\tau)$$

b) In the case of statistically independent and zero-mean random processes, both the cross-correlation functions $R_{XY}(\tau)$ and $R_{YX}(\tau)$ vanish and

$$R_Z(\tau) = R_X(\tau) + R_Y(\tau)$$

$$\Rightarrow R_Z(\tau) = R_{(X(t)+Y(t))}(\tau) = \frac{1}{2}A^2 \cos\omega\tau + B^2 e^{-\alpha|\tau|}$$

c) $Y(t) = \alpha X(t-T) + N(t)$

$$R_{XY}(\tau) = E[X(t)Y(t+\tau)] =$$

$$= E[aX(t)X(t+\tau-T) + X(t)N(t+\tau)]$$

$$= a R_X(\tau - T) + R_{XN}(\tau)$$

since signal and noise are statistically independent,

$$R_{XY}(\tau) = a R_X(\tau - T)$$

2. a) When the probability density function (PDF) is viewed as a function of the unknown parameter, it is termed likelihood function.

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$$p(x(0); A) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x(0)-A)^2\right)$$

b) The curvature of the log-likelihood function is the negative of the second derivative of the log-likelihood function at its peak.

The curvature gives information about the "sharpness" of the likelihood function, i.e. information on how accurately we can estimate the unknown parameter.

$$\ln(p(x(0); A)) = -\ln\sqrt{2\pi\sigma^2} - \frac{1}{2\sigma^2}(x(0)-A)^2$$

$$\frac{\partial \ln(p(x(0); A))}{\partial A} = \frac{1}{\sigma^2}(x(0)-A)$$

$$-\frac{\partial^2 \ln(p(x(0); A))}{\partial A^2} = \frac{1}{\sigma^2}$$

c) CRLB.

It is assumed that the PDF $p(\underline{x}; \theta)$ satisfies the regularity condition

$$E\left[\frac{\partial \ln p(\underline{x}; \theta)}{\partial \theta}\right] = 0, \quad \text{for all } \theta$$

then the variance of ~~the~~ any unbiased estimator $\hat{\theta}$ must satisfy

$$\operatorname{var}(\hat{\theta}) \geq \frac{1}{-E\left[\frac{\partial^2 \ln(p(\underline{x}; \theta))}{\partial \theta^2}\right]}$$

Furthermore, an unbiased estimator may be found that attains the bound for θ iff

$$\frac{\partial \ln p(\underline{x}; \theta)}{\partial \theta} = I(\theta)(g(x)-\theta),$$ for some functions I and g.

2) This is the MVU estimator, $\hat{\theta} = g(x)$ and the minimum variance is $1/I(\theta)$.

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d) From the CRLB

$\text{var}(\hat{A}) \geq \sigma^2, \quad \forall A.$

4/6

3). a) $z_t = \phi_1 z_{t-1} + \phi_2 z_{t-2} + a_t$

$\phi_1, \phi_2 \to$ coefficients

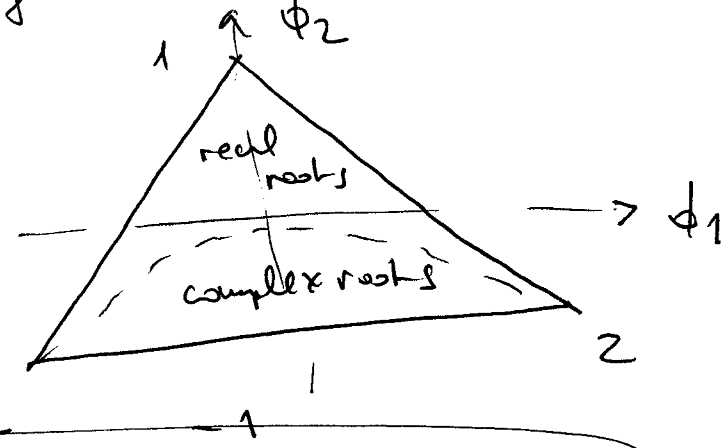
$a_t \to$ white noise

$\Rightarrow$ the stability triangle is defined by

$\phi_1 + \phi_2 < 1$

$\phi_2 - \phi_1 < 1$

$-1 < \phi_2 < 1$



$\boxed{\text{Process } z_t = 0.75 z_{t-1} - 0.5 z_{t-2} + a_t \text{ is stable.}}$

$\Rightarrow$ For the AR(2) process

$$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2} \quad , k > 0$$

Here $\cancel{\rho_k = 0.75 z_{t-1} - 0.5}$

$$\rho_k = 0.75 \rho_{k-1} - 0.5 \rho_{k-2}$$

b) Y-W equations for the AR(2) process are

$$\rho_1 = \phi_1 + \phi_2 \cdot \rho_1 \qquad \Rightarrow \qquad \rho_1 = 0.75 - 0.5 \rho_1$$
$$\rho_2 = \phi_1 \rho_1 + \phi_2 \qquad\qquad \rho_2 = 0.75 \rho_1 - 0.5$$

$$\Rightarrow p(f) = \frac{2\sigma_a^2}{|1 - \phi_1 e^{-j2\pi f} - \phi_2 e^{-j4\pi f}|^2}$$

$$= \frac{2\sigma_a^2}{1 + \phi_1^2 + \phi_2^2 - 2\phi_1(1-\phi_2)\cos 2\pi f - 2\phi_2 \cos(4\pi f)}$$

$0 \leq f \leq \frac{1}{2}$

4) $\frac{5}{6}$

a) $$e(k+1) = e(k) + \sum_{i=1}^{N} \frac{\partial e(k)}{\partial w_i(k)} \Delta w_i(k) + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\partial^2 e(k)}{\partial w_i(k) \partial w_j(k)} \cdot \Delta w_i(k) \Delta w_j(k) + \text{h.o.t.}$$

The second and higher-order terms (h.o.t) vanish due to the linearity of the filter.

From the LMS we have

$$\frac{\partial e(k)}{\partial w_i(k)} = \frac{\partial (d(k) - y(k))}{\partial w_i(k)} = -x_i(k)$$

$$\Delta w_i(k) = \eta\, e(k)\, x_i(k)$$

and hence

~~e(k) = e(~~ $e(k+1) = e(k)\left[1 - \eta \sum_{i=1}^{N} x_i^2(k)\right]$

$$e(k+1) = 0 \quad \text{for} \quad \eta = \frac{1}{\|\underline{x}(k)\|_2^2}$$

b) η_{NLMS} adapts dynamically according to the "tap input power" $\|\underline{X}(k)\|_2^2$, whereas η for the LMS is static

c) we minimise $e(k+1)$ using the variables from the time instant k. Therefore, we minimise the "a posteriori" error.

5)

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a) ~~It is~~ Has memory, nonlinearity and represents a "dynamical synapse"

b) $$\underline{w}(k+1) = \underline{w}(k) - \eta \nabla_{\underline{w}} E(k)$$

$$y(k) = \phi(\underline{x}^T(k)\underline{w}(k))$$

$$\Rightarrow \nabla_{\underline{w}} E(k) = -\phi'(\underline{x}^T(k)\underline{w}(k))\, e(k)\, \underline{x}(k)$$

and the weight update equation is

$$\underline{w}(k+1) = \underline{w}(k) + \eta\, e(k)\, \phi'(\underline{x}^T(k)\underline{w}(k))\, \underline{x}(k)$$

c) This algorithm is a variant of the LMS for the case of the nonlinear function at the output. Hence it is suitable for filtering of nonlinear inputs.

d) The input signals with the magnitude range greater or equal to the range of the arctan become "clipped". Hence the distortion of such signals