Paper Number(s): **E4.10**

C2.1

SC4

IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE
UNIVERSITY OF LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2001

MSc and EEE PART IV: M.Eng. and ACGI

PROBABILITY AND STOCHASTIC PROCESSES

Wednesday, 9 May 10:00 am

There are SIX questions on this paper.

Answer FOUR questions.

Time allowed: 3:00 hours

Corrected Copy

4, 3, 5, 6

Examiners: Vinter,R.B. and Clark,J.M.C.

Special instructions for invigilators: None

Information for candidates: None

1(a) A source signal X and a transmitted signal Z are modelled as binary random variables taking values 0 and 1.

A received signal Y is modelled as

$$Y = NZ$$

in which N is a binary random variable which takes values 0 or 1 and is independent of X and Z. ($N = 0$ corresponds to receiver failure.)

Assume that

$$\begin{aligned} P[X=0] &= P[X=1] = 1/2 \\ P[Z=0|X=0] &= P[Z=1|X=1] = \alpha \\ P[N=1] &= p\,. \end{aligned}$$

A signal $Y = 0$ is received. What is the probability that the source signal was $X = 0$?

1(b) A section of a communication link is modelled as indicated in *Figure 1.* The possible states of each switch $S_1, \ldots, S_5$ are *open* and *closed.* The events

$$S_i \textit{ is closed} \qquad i = 1, \ldots, 5$$

are independent and

$$P[S_i \textit{ is closed}\,] = q \quad \text{for each } i.$$

Calculate the probability that there is a closed path between points A and B.

Now suppose that a communication link is composed of identical sections, connected in series. Suppose that the states of all switches are independent,

$$q = (1 - 10^{-3})$$

and that the length of each section is 1 metre. What is the maximum length of the communication link such that

$$P[\textit{there is a closed path through the entire link}] \geq 0.99\,?$$

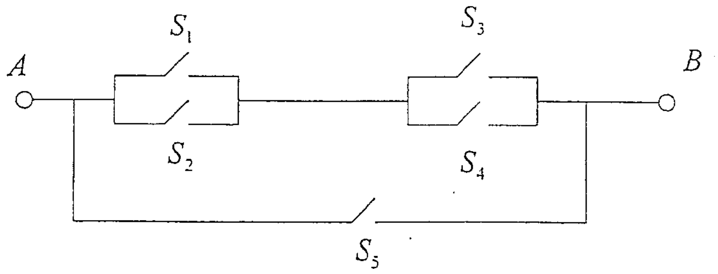

Figure 1

2 A signal $X(\omega)$, corrupted by a bias n, is rectified by a device with characteristic

$$R(x) = \begin{cases} x^2 & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \,. \end{cases}$$

$X(\omega)$ is modelled as a random variable with uniform probability density function

$$f_X(x) = \begin{cases} 1/2 & \text{for } -1 \leq x \leq +1 \\ 0 & \text{otherwise} \end{cases} \,.$$

The bias n is taken to be a constant $(0 \leq n \leq 1)$. See *Figure 2.*

Write $Z(\omega)$ and $Z^*(\omega)$ for the rectified noisy signal and the rectified noise-free signal respectively:

$$\begin{aligned} Z(\omega) &= R(X(\omega) + n) \\ Z^*(\omega) &= R(X(\omega)) \,. \end{aligned}$$

Calculate the average power of the rectifier output error, namely

$$E[(Z(\omega) - Z^*(\omega))^2] \,.$$

(Assume n is small, so that terms involving n^3, n^4, etc., can be ignored).

Hence determine the signal to noise ratios S_{in} and S_{out} at the input and output of the rectifier respectively

$$S_{in} = \frac{n^2}{E[X^2(\omega)]} \quad \text{and} \quad S_{out} = \frac{E[(Z(\omega) - Z^*(\omega))^2]}{E[Z^{*2}(\omega)]} \,.$$

Notice that $S_{in} \neq S_{out}$. Is this a nonlinear phenomenon, i.e., is it possible that $S_{in} \neq S_{out}$ if the nonlinear characteristic $R(x)$ were replaced by a pure gain Kx $(K \neq 0)$?

Hint: obtain formulae for $R(x+n) - R(x)$ in the regions

$$\textit{(i)}\ x < -n \quad \textit{(ii)}\ -n \leq x \leq 0 \quad \textit{(iii)}\ x \geq 0 \,.$$

Do not attempt to calculate the distribution function of $Z(\omega)$.

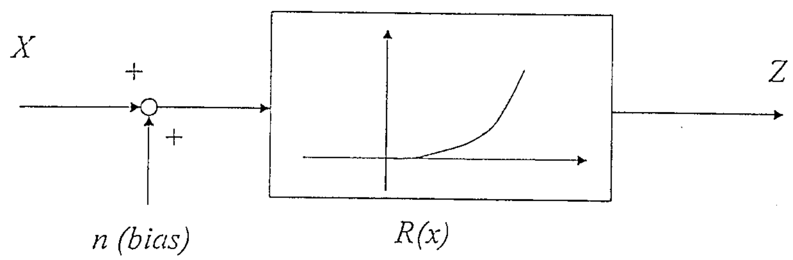

Figure 2

3(a) Let $X(\omega)$ and $Y(\omega)$ be jointly distributed, zero mean, normal random variables, with variance σ^2 and correlation coefficient r, $-1 < r < +1$:

$$f_{XY}(x, y) = \frac{1}{2\pi(1-r^2)^{\frac{1}{2}}\sigma^2} \exp\left\{-\frac{1}{2\sigma^2(1-r^2)}\left(|x - ry|^2 + (1-r^2)y^2\right)\right\}.$$

Derive the conditional probability density $f_{X|Y}(x; y)$ of $X(\omega)$ given $Y(\omega) = y$. Show that the mean and variance of the conditional probability density are

$$m_{X|Y}(y) = ry,$$
$$var_{X|Y}(y) = \sigma^2(1-r^2).$$

3(b) Now interpret $X(\omega)$ as a signal and $Y(\omega)$ as a measurement of the signal and

$$\hat{X}(y) = m_{X|Y}(y)$$

as an estimator of $X(\omega)$ given $Y(\omega) = y$.

The higher the quality of the sensor, the larger is the absolute value of the correlation coefficient. Assume

$$\text{sensor cost} = \frac{10}{1 - |r|^2} \quad (\$\text{'s})$$

What is the minimum cost of the sensor, if we require

$$\text{var}_{X|Y}(y) \leq 0.01E[|X(\omega)|^2] \quad ?$$

(In part (a), you can use the fact that

$$f(x) = \frac{1}{(2\pi\sigma^2)^{\frac{1}{2}}} \exp\left\{-\frac{1}{2\sigma^2}(x-m)^2\right\}$$

is the probability density function of a random variable with mean m and variance σ^2.)

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4 A transmitted signal $X(\omega)$ is modelled as a continuous random variable with probability density function $f_X(x)$. The received signal $Y(\omega)$ has a random offset $N(\omega)$:

$$Y(\omega) = X(\omega) + N(\omega).$$

The offset is modelled as a discrete random variable, independent of $X(\omega)$, with

$$P[N(\omega) = 0] = p \quad \text{and} \quad P[N(\omega) = \frac{1}{2}] = (1-p).$$

Show that the conditional probability mass function of $X(\omega)$ given $Y(\omega) = y$ is

$$\begin{aligned} P[X(\omega) = y \mid Y(\omega) = y] &= \frac{p f_X(y)}{p f_X(y) + (1-p) f_X(y - \frac{1}{2})}, \\ P[X(\omega) = y - \frac{1}{2} \mid Y(\omega) = y] &= \frac{(1-p) f_X(y - \frac{1}{2})}{p f_X(y) + (1-p) f_X(y - \frac{1}{2})}. \end{aligned}$$

Hence derive a formula for the least squares estimate

$$\hat{X} = g(y)$$

of $X(\omega)$ given $Y(\omega) = y$.

Evaluate $g(y)$ in the case when

$$f_X(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{otherwise}. \end{cases}$$

Comment on the results.

Hint: In the last part of the question, consider separately the cases

$$\textit{(i)}\ 0 \le y < \frac{1}{2}, \quad \textit{(ii)}\ \frac{1}{2} \le y \le 1, \quad \textit{(iii)}\ 1 < y \le 1\frac{1}{2}$$

5(a) An r-vector stationary stochastic process $\{y_k\}$ is modelled by the state space equations announced 10:43

$$x_{k+1} = Ax_k + be_k \qquad y_k = Cx_k$$

in which A is a 'stable' $n \times n$ matrix, b is an n-vector, C is an $r \times n$ matrix and $\{e_k\}$ is a sequence of zero mean, independent scalar random variables with common variance

$$E[e_k^2] = \sigma^2 .$$

Derive a set of equations, including the matrix Lyapunov equation, for the covariance function $R_y(l)$, $l = \ldots, -1, 0, +1 \ldots$

5(b) Consider the two stationary scalar processes $\{u_k\}$ and $\{v_k\}$ modelled by the difference equations:

$$\begin{aligned} u_{k+1} &= v_k + e_k \\ v_{k+1} &= -au_k + e_k . \end{aligned}$$

Here, a is a modelling parameter and $\{e_k\}$ is a sequence of zero mean, independent scalar random variables with common variance

$$E[e_k^2] = \sigma^2 .$$

It is known that the covariance functions $R_u(l)$ and $R_v(l)$ of $\{u_k\}$ and $\{v_k\}$ satisfy

$$R_v(0) = \frac{5}{8} R_u(0) .$$

By using the results of Part (a), or otherwise, determine the value of a.

6(a) Define the spectral density $\Phi_r(\omega)$ of a stationary, second order, zero mean, scalar stochastic process $\{r_k\}$.

Now suppose that

$$r_k = u_k + v_k, \quad k = \ldots, -1, 0, +1, \ldots$$

for stationary, second order, zero mean, scalar stochastic processes $\{u_k\}$ and $\{v_k\}$ such that

$$u_k \text{ and } v_j \text{ are independent for all } k, j.$$

Show that

$$\Phi_r(\omega) = \Phi_u(\omega) + \Phi_v(\omega)\,.$$

6(b) A transmitted signal $\{s_k\}$ is modelled as a stationary, zero mean, scalar stochastic process satisfying

$$(2 - z^{-1})s_k = e_k,$$

in which $\{e_k\}$ is a scalar, unit variance, white noise process.

Due to the presence of noise and channel distortion, the received signal $\{r_k\}$ is the solution to the difference equation

$$(3 - z^{-1})r_k = s_k + d_k.$$

Here, the disturbance process $\{d_k\}$ is generated by the equation

$$d_k = D(z)\tilde{e}_k,$$

in which $D(z)$ is an unknown transfer function and $\{\tilde{e}_k\}$ is a unit variance, white noise process, independent of $\{e_k\}$. See *Figure 3*.

The spectral density of the received signal $\{r_k\}$ is known to be

$$\Phi_r(\omega) = \frac{1 + (17 - 4e^{j\omega} - 4e^{-j\omega})(5 - 2e^{j\omega} - 2e^{-j\omega})}{(5 - 2e^{j\omega} - 2e^{-j\omega})(10 - 3e^{j\omega} - 3e^{-j\omega})}\,.$$

Find the transfer functions $G_1(z)$ and $G_2(z)$ such that

$$r_k = u_k + v_k, \text{ where } u_k = G_1 e_k,\ v_k = G_2 \tilde{e}_k.$$

By using the results of Part (a), or otherwise, determine the transfer function $D(z)$ which generates the disturbance process $\{d_k\}$.

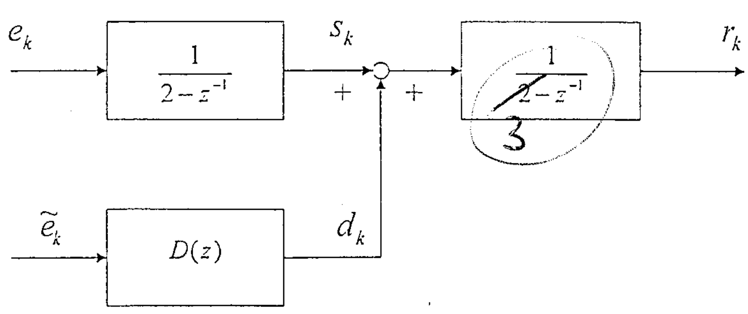

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Figure 3

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April 2001

Prob. & Stoch. Processes Exam, 2001. Model Answers

1 (i) $P[X=0|Y=0] = \dfrac{P[X=0 \text{ \& } Y=0]}{P[Y=0]} = \dfrac{P[X=0 \text{ \& } Y=0|N=0].P[N=0] + P[X=0 \text{ \& } Y=0|N=1]P[N=1]}{P[Y=0]}$

$= \dfrac{P[X=0]\,P[N=0] + P[X=0 \text{ \& } Z=0]\cdot P[N=1]}{P[Y=0|N=0]\,P[N=0] + P[Y=0|N=1]\,P[N=1]}$

$= \dfrac{P[X=0]P[N=0] + P[Z=0|X=0]\,P[X=0]\cdot P[N=1]}{P[N=0] + P[Z=0]\,P[N=1]} = \dfrac{\frac{1}{2}(1-p) + \frac{1}{2}\alpha p}{(1-p) + P[Z=0]\,p}$

But $P[Z=0] = P[Z=0|X=0]\,P[X=0] + P[Z=0|X=1]\,P[X=1]$

$= \alpha\frac{1}{2} + (1-\alpha)\frac{1}{2} = \frac{1}{2}$

8 Hence $P[X=0|Y=0] = \dfrac{\frac{1}{2}(1-p) + \frac{1}{2}\alpha p}{1-p+\frac{1}{2}p} = \underline{\dfrac{1-(1-\alpha)p}{2-p}}$

(ii) The event "there is a closed path between A and B" is
$Q = [(S_1 \cup S_2) \cap (S_3 \cup S_4)] \cup S_5$. Write $\bar{Q}$ = "complement of Q", etc.
Then $\bar{Q} = \bar{S}_5 \cap \overline{(S_1 \cup S_2) \cap (S_3 \cap S_4)}$ (Here $S_1 = \{S_1 \text{ is closed}\}$, etc.)
But
$P[(S_1 \cup S_2) \cap (S_3 \cap S_4)] = P[S_1 \cup S_2]\cdot P[S_3 \cup S_4]$ (by independence)
$= (1 - P[\bar{S}_1 \cap \bar{S}_2])(1 - P[\bar{S}_3 \cap \bar{S}_4])$
$= (1 - P[\bar{S}_1].P[\bar{S}_2])(1 - P[\bar{S}_3].P[\bar{S}_4])$ (by independence)
$= (1-(1-q)^2)^2$
Hence $P[\overline{(S_1 \cup S_2) \cap (S_3 \cap S_4)}] = 1 - (1-[1-q]^2)^2$. It follows that
$P[Q] = 1 - P[\bar{Q}] = 1 - P[\bar{S}_5]\left(1 - (1-(1-q)^2)^2\right)$
(by independence)

Then
$P(Q) = 1 - (1-q)(\cancel{1} - \cancel{1} + 2(1-q)^2 - (1-q)^4)$
9 $= \underline{1 - (1-q)^3(2-(1-q)^2)}$

If $1-q = 10^{-6}$, $P(Q) = 1 - 2\times 10^{-9}$
If N is length of link, we require
$(1 - 2\times 10^{-9})^N \geq 0.99$ (by independence)
or $1 - 2\times N\, 10^{-9} \geq 0.99$ (approximately)
3 or $N \geq \frac{1}{2}\times 10^{9}\times 10^{-2} = \underline{\frac{1}{2}\times 10^{7}}$ meters

2. $R(x+n) - R(x) = \begin{cases} (x+n)^2 - x^2 & x \geq 0 \\ (x+n)^2 & -n \leq x < 0 \\ 0 & x < -n \end{cases}$

The power of the rectified output is therefore

$$E[|R(X(\omega)+n) - R(X(\omega))|^2 f_X(x)\,dx = \tfrac{1}{2}\int_{-1}^{+1} (R(x+n) - R(x))^2 f_X(x)\,dx$$

$$= \tfrac{1}{2}\int_0^1 [(x+n)^4 - 2(x+n)^2x^2 + x^4]\,dx + \tfrac{1}{2}\int_{-n}^0 (x+n)^4\,dx.$$

$$= \tfrac{1}{2}\int_0^1 [(x+n)^4 - 2x^4 - 4x^3n - 2x^2n^2 + x^4]\,dx + \tfrac{1}{2}\int_{-n}^0 (x+n)^4\,dx$$

$$= \tfrac{1}{2}\Big\{ \tfrac{1}{5}(x+n)^5 - \tfrac{2}{5}x^5 - x^4n - \tfrac{2}{3}x^3n^2 + \tfrac{1}{5}x^5\Big|_0^1 + \tfrac{1}{5}n^5 \Big\}$$

$$= \tfrac{1}{2}\Big\{ \tfrac{1}{5}(1+n)^5 - \tfrac{1}{5}n^5 - \tfrac{2}{5} - n - \tfrac{2}{3}n^2 + \tfrac{1}{5} + \tfrac{1}{5}n^5 \Big\}$$

$$\simeq \tfrac{1}{2}\Big\{\tfrac{1}{5}[1+5n+10n^2] - \tfrac{2}{5} - n - \tfrac{2}{3}n^2 + \tfrac{1}{5} + 0 + 0 \ldots$$

0's for terms in $n^3, n^4, \ldots$ }

14 $= \tfrac{1}{2} \times (2 - 2/3)n^2 = 2/3\, n^2$

Also

$$E[X^2] = \tfrac{1}{2}\int_{-1}^{+1} x^2\,dx = \tfrac{1}{2}\cdot\tfrac{x^3}{3}\Big|_{-1}^{+1} = 1/3$$

$$E[Z^{*2}] = \tfrac{1}{2}\int_0^1 x^4\,dx = 1/10$$

It follows that (for n small)

$$S_{in} = \frac{n^2}{1/3} = 3n^2 \quad \text{and} \quad S_{out} \cong \frac{20}{3}n^2$$

2 . We see that $S_{in} \neq S_{out}$.

If $R(x)$ is replaced by a linear gain, $z - z^* = K(x+n) - Kx = Kn$ and $z^* = Kx$. So

$$S_{out} = \frac{K^2n^2}{K^2 E[|X|^2]} = \frac{n^2}{E[|X|^2]} = S_{in}$$

4 We have confirmed that $S_{in} \neq S_{out}$ is a nonlinear phenomenon.

3(i) $f_Y(y) = (\pi[1-r^2]^{\frac{1}{2}}\sigma^2)^{-1} \int_{-\infty}^{+\infty} \exp\left\{-\frac{|x-ry|^2}{2\sigma^2(1-r^2)}\right\} dx \,.\, \exp\left\{-y^2/2\sigma^2\right\}$

change variables $x' = x - ry$ (for fixed y), giving

$$\int_{-\infty}^{+\infty} \exp\{..\} = \int_{-\infty}^{+\infty} \exp\left\{-\frac{|x'|^2}{2\sigma^2(1-r^2)}\right\} dx'$$
$$= \pi^{\frac{1}{2}}\sqrt{\sigma^2(1-r^2)} \quad \text{(by properties of normal density)}$$

Hence $f_Y(y) = (\pi\sigma^2)^{-1/2} \exp\{-y^2/2\sigma^2\}$. So

$$f_{X|Y}(x;y) = \frac{(\pi\sigma^2)^{1/2}}{\pi(1-r^2)^{1/2}\sigma^2} \,.\, \exp\left\{-\frac{1}{2\sigma^2}a\right\} \quad \text{where}$$

$$a = \frac{1}{1-r^2}\left(|x-ry| + (1-r^2)y^2\right) - y^2$$
$$= \frac{1}{(1-r^2)}\left(x^2 - 2ry + \cancel{r^2y^2} + \cancel{y^2} - \cancel{r^2y^2} - \cancel{y^2} + r^2y^2\right) = \frac{|x-ry|^2}{1-r^2}$$

We have shown

$$f_{X/Y}(x;y) = \frac{1}{(\pi\bar{\sigma}^2)^{1/2}} \exp\left\{-\frac{1}{2\bar{\sigma}^2}|x-\bar{m}|^2\right\}$$

14 where $\bar{m}$ $(= m_{X/Y}(y)) = ry$ and $\bar{\sigma}^2$ $(= \mathrm{var}_{X/Y}(y)) = \sigma^2(1-r^2)$

(ii) We require that

$$\mathrm{var}_{X/Y}(y) \leq 0{\cdot}01\, E|X|^2$$

But $E|X|^2 = \sigma^2$ and $\mathrm{var}_{X/Y}(y) = \sigma^2(1-r^2)$

Hence

$$\sigma^2(1-r^2) \leq 0{\cdot}01\,\sigma^2.$$

It follows $\quad \text{cost} = \dfrac{10}{1-r^2} \geq 1000 \quad$ (£'s)

6 So minimum cost (to achieve specifications) is £1,000

4 If $Y(\omega) = y$ then $X(\omega)$ can only take values y and $y - \frac{1}{2}$.
For $\beta = 0$ or $\frac{1}{2}$

$$m_{X/Y}(X = y - \beta \mid Y = y) \simeq P[y \le X + \beta \le y + \delta y \mid y \le Y \le y + \delta y]$$

$$= \frac{P[y \le Y \le y + \delta y \mid y \le X + \beta \le y + \delta y] \cdot P[y \le X + \beta \le y + \delta y]}{P[y \le X + N \le y + \delta y \mid N = 0]\, P[N = 0] + P[y \le X + N \le y + \delta y \mid N = \frac{1}{2}]\, P[N = \frac{1}{2}]}$$

(by Bayes' Rule)

$$= \frac{P[N = \beta] \cdot f_X(y - \beta) \cancel{\delta y}}{[f_X(y)\, p + f_X(y - \frac{1}{2})(1 - p)] \cancel{\delta y}}$$

It follows

$$m_{X/Y}(X = x \mid Y = y) = \begin{cases} \left(\frac{\alpha_1}{\alpha_1 + \alpha_2}\right) & \text{if } x = y \\ \left(\frac{\alpha_2}{\alpha_1 + \alpha_2}\right) & \text{if } x = y - \frac{1}{2}. \end{cases}$$

where $\alpha_1 = p f_X(y)$ and $\alpha_2 = (1-p) f_X(y - \frac{1}{2})$

Hence, the least squares estimate, which coincides with the conditional mean, is

12 $$g(y) = \frac{\alpha_1}{\alpha_1 + \alpha_2} y + \frac{\alpha_2}{\alpha_1 + \alpha_2} (y - \tfrac{1}{2})$$

If $f_X(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$, then

for $0 \le y < \frac{1}{2}$, $\alpha_1 = p$, $\alpha_2 = 0$, so $g(y) = y$

for $\frac{1}{2} \le y \le 1$, $\alpha_1 = p$, $\alpha_2 = (1-p)$, so $g(y) = py + (1-p)(y - \frac{1}{2})$

6 for $1 < y \le 1\frac{1}{2}$, $\alpha_1 = 0$, $\alpha_2 = (1-p)$ so $g(y) = y - 1/2$

The point here is that, if $y < \frac{1}{2}$, then we must have $N = 0$, and $X(\omega) = Y(\omega)$ so $g(y) = y$ is the best estimate of $X(\omega)$ given $Y(\omega) = y$.
If $y > 1$, we must have $N = \frac{1}{2}$, so $X(\omega) = Y(\omega) - 1/2$.
Here $g(y) = y - 1/2$ is the best estimate of $X(\omega)$

2 For $\frac{1}{2} \le y \le 1$, $g(y)$ is close to y if $N(\omega) = 0$ with high probability
$g(y)$ is close to $y - 1$ if $N(\omega) = 1/2$ with high probability

5. (a) We have $x_{k+1} = Ax_k + be_k$. It follows that

$$x_{k+1}x_{k+1}^T = (Ax_k + be_k)(Ax_k + be_k)^T$$

But, from the state equations, x_k is independent of e_k. Hence

$$E\{be_k x_k^T b^T\} = 0$$

Hence $E\{x_{k+1}x_{k+1}^T\} = E\{A x_k x_k^T A^T\} + 0 + 0 + E\{b e_k e_k b^T\}$

It follows that

$$\underbrace{R_x(0) = A R_x(0) A^T + \sigma^2 b b^T}. \qquad \text{—— (1)}$$

Also $X_{k+1}x_k^T = Ax_k x_k^T + be_k x_k^T$. Taking expectations gives

$$\underbrace{R_x(1) = A R_x(0)}$$

Similarly $\underbrace{R_x(l) = A^l R_x(0)}$ for $l = 1, 2, ..$

and $\underbrace{R_x(-l) = R_x(0) A^{T|l|}}$ for $l = 1, 2, ..$

8 Then $\underbrace{R_y(l) = C E\{x_k x_{k-l}\} C^T} = C R_y(l) C^T$, for $l = 1, 2, ..$

(b) We can regard u_k and v_k as components of the 2-vector process

$$\begin{pmatrix} u_{k+1} \\ v_{k+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -a & 0 \end{pmatrix}\begin{pmatrix} u_k \\ v_k \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e_k .$$

Write $R_x(0) = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}$. Then, by (1),

$$\begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a & 0 \end{bmatrix}\begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}\begin{bmatrix} 0 & -a \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}[1 \ 1]\,\sigma^2$$

$$\Rightarrow \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} = \begin{bmatrix} r_{22} & 1 - ar_{21} \\ -ar_{12} & +a^2 r_{11} \end{bmatrix} + \sigma^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Equating matrix entries, we obtain

$$r_{11} = r_{22} + \sigma^2, \quad r_{12} = -ar_{21} + \sigma^2, \quad r_{21} = -ar_{12} + 1, \quad r_{22} = a^2 r_{11} + \sigma^2$$

Hence

$$r_{11} = a^2 r_{11} + 2\sigma^2 \Rightarrow \quad r_{11} = \frac{2\sigma^2}{1-a^2}$$

$$r_{22} = \sigma^2 \frac{1+a^2}{1-a^2}$$

Since $r_{11} = R_u(0)$ and $r_{22} = R_v(0)$, we have

$$5/8 = R_v(0)/R_u(0) = r_{22}/r_{11} = \tfrac{1}{2}(1+a^2)$$

12 Hence $a^2 = (5/4 - 1) \Rightarrow \underset{\sim\sim}{a = \pm\tfrac{1}{2}}$

6(i) The spectral density $\Phi_r(\omega)$ of $\{r_k\}$ is

$$\Phi_r(\omega) = \sum_{l=-\infty}^{+\infty} R(l)\, e^{-j\omega l}, \quad \text{where} \quad R_r(l) = E\{r_k r_{k-l}\}.$$

If $r_k = u_k + v_k$,

$$\begin{aligned} R_r(l) &= E\{(u_k+v_k)(u_{k-l}+v_{k-l})\} \\ &= E\{u_k u_{k-l}\} + 0 + 0 + E\{v_k v_{k-l}\} \\ &= R_u(l) + R_v(l). \end{aligned}$$

6 It follows that $\Phi_r(\omega) = \sum_{l=-\infty}^{+\infty} [R_u(l) + R_v(l)]\, e^{-j\omega l} = \Phi_u(\omega) + \Phi_v(\omega)$.

(ii) $(2 - z^{-1}) s_k = e_k$, $(3 - z^{-1}) r_k = s_k + d_k$, $d_k = N(z)\tilde{e}_k$ imply

$$r_k = \frac{1}{3-z^{-1}}\left(\frac{e_k}{(2-z^{-1})} + D(z)\tilde{e}_k\right) = G_1(z) e_k + G_2(z)\tilde{e}_k$$

in which $G_1(z) = \dfrac{1}{(2-z^{-1})(3-z^{-1})}$ and $G_2(z) = \dfrac{D(z)}{(3-z^{-1})}$

We know:

$$\Phi_r(\omega) = \frac{1 + (17 - 4e^{j\omega} - 4e^{-j\omega})(5 - 2e^{j\omega} - 2e^{-j\omega})}{(5 - 2e^{j\omega} - 2e^{-j\omega})(10 - 3e^{j\omega} - 3e^{-j\omega})}$$

But, writing $u_k = G_1(z) e_k$, we have

$$\Phi_u(\omega) = \frac{1}{(2-z^{-1})(2-z)(3-z^{-1})(3-z)}\Bigg|_{z=e^{j\omega}} = \frac{1}{(5-2e^{j\omega}-2e^{-j\omega})\cdot(10-3e^{j\omega}-3e^{-j\omega})}$$

So, writing $v_k = G_1(z) e_k$, we have

$$\Phi_v(\omega) = \Phi_r(\omega) - \Phi_u(\omega) = \frac{(17-4e^{j\omega}-4e^{-j\omega})\cancel{(5-2e^{j\omega}-2e^{-j\omega})}}{\cancel{(5-2e^{j\omega}-2e^{-j\omega})}(10-3e^{j\omega}-3e^{-j\omega})}$$

$$= \frac{(4-z^{-1})}{(3-z^{-1})} \cdot \frac{(4-z)}{(3-z)}\Bigg|_{z=e^{j\omega}} = G_2(z)\, G_2(z^{-1})\Big|_{z=e^{j\omega}}$$

It follows we can choose $G_2(z) = \dfrac{4-z^{-1}}{3-z^{-1}}$

Then $D(z) = (3-z^{-1})G_2(z) = 4 - z^{-1}$

14 (Note: we cannot choose $G_2(z) = \dfrac{1-4z^{-1}}{1-3z^{-1}}$, because then $D(z) = \dfrac{(3-z^{-1})(1-4z^{-1})}{(1-3z^{-1})}$ is an unstable transfer function.