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IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE
 UNIVERSITY OF LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
 EXAMINATIONS 2001

MSc and EEE/ISE PART IV: M.Eng. and ACGI

DISCRETE-TIME SYSTEMS AND COMPUTER CONTROL

Friday, 11 May 10:00 am

There are SIX questions on this paper.

Answer FOUR questions.

Time allowed: 3:00 hours

Special Information for Invigilators: **None**

Information for Candidates

Some notation

T is the sample period

q is the forward shift operator

$f^Z(z)$, $f^D(\gamma)$, $f^F(j\omega)$, $f^W(w)$ denote the Z -, Delta-, discrete-time Fourier and W -transforms, respectively, of $\{f_k\}$

$g^L(s)$ denotes the Laplace transform of $g(t)$

' denotes transposition of a vector or matrix

$t_k = kT$

Some useful transforms

f_k	$f^Z(z)$	$f^D(\gamma)$
$i_k = 0^k$	1	T
1^k	$\frac{z}{z-1}$	$\frac{1+\gamma T}{\gamma}$
t_k	$\frac{Tz}{(z-1)^2}$	$\frac{1+\gamma T}{\gamma^2}$
α^k	$\frac{z}{z-\alpha}$	$\frac{1+\gamma T}{\gamma-\bar{\alpha}}$
$k\alpha^k$	$\frac{z\alpha}{(z-\alpha)^2}$	$\frac{(1+\gamma T)(1+\bar{\alpha}T)}{T(\gamma-\bar{\alpha})^2}$

where $\bar{\alpha} = \frac{\alpha-1}{T}$

$$f^W(w) = f^Z\left(\frac{\mu+w}{\mu-w}\right) \text{ where } \mu = \frac{2}{T}.$$

Corrected Copy

Examiners: Allwright, J.C. and Vinter, R.B.

The Routh Test

Every root of $a_0 w^n + a_1 w^{n-1} + \dots + a_n = 0$ has strictly negative real part iff all $n + 1$ entries in the first column of the following Routh-table are non-zero and have the same sign:

1:	a_0	a_2	a_4
2:	a_1	a_3	a_5
3:	$\frac{a_1 a_2 - a_0 a_3}{a_1}$	$\frac{a_1 a_4 - a_0 a_5}{a_1}$	$\frac{a_1 a_6 - a_0 a_7}{a_1}$
..:			
$n+1$:			

The Jury Test

Every root of $d(z) \triangleq \alpha_n z^n + \alpha_{n-1} z^{n-1} + \dots + \alpha_0 = 0$ has modulus strictly less than one iff

$$d(1) > 0,$$

and

$$d(-1) \begin{cases} > 0 & \text{if } n \text{ is even} \\ < 0 & \text{if } n \text{ is odd} \end{cases}$$

and

$$|a_0| < a_n, |b_0| > |b_{n-1}|, |c_0| > |c_{n-2}|, \dots,$$

where the b_i, c_i etc., are determined from the following Jury-table

1:	a_0	a_1	a_2	a_n
2:	a_n	a_{n-1}	a_{n-2}	a_0
3:	b_0	b_1	b_2	b_{n-1}
	where $b_i = a_0 a_i - a_n a_{n-i}$				
4:	b_{n-1}	b_{n-2}		b_0
..:				
$2n-3$:				

Here, for all i ,

$$a_i = \begin{cases} \alpha_i & \text{if } \alpha_n > 0 \\ -\alpha_i & \text{if } \alpha_n < 0. \end{cases}$$

The Questions

1. (a) By considering the Z -transform of the sequence $\{x_k\}$ generated by the scalar system

$$x_{k+1} = \alpha x_k : x_0 = 1$$

for appropriate α , determine $Z\{(-1)^k\}$. [5]

- (b) Find $x^Z(z)$ for the system

$$x_{k+2} = -\beta^2 x_k : x_0 = 0, x_1 = 2\beta.$$

Use a partial fraction expansion to determine from $x^Z(z)$ a formula for x_k which involves a trigonometric function. [7]

- (c) Consider the discrete-time system S_d of Figure 1 below, with sample period T , input u_k and output y_k .

- (i) Determine a state-space model for S_c and suppose that the eigenvalues associated with it, denoted λ_i , are distinct. State a first-order vector difference equation that relates x_{k+1} to x_k , where $x_k \triangleq x(kT)$. [2]

- (ii) By considering spectral forms, determine the eigenvalues associated with the difference equation of part (i) in terms of the λ_i . Denote those eigenvalues by $\bar{\lambda}_i$. [2]

- (iii) State, without proof, the relationship between BIBO-stability of the complete system S_d of Figure 1, its poles and the eigenvalues $\bar{\lambda}_i$. Use the relationship to show that the discrete-time system S_d is BIBO-stable if the eigenvalues λ_i associated with S_c all belong to the set $\{s \in \mathbb{C} : \text{Re}(s) < 0\}$. [4]

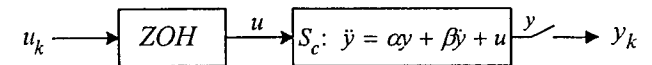


Figure 1

2. (a) Design the pole-zero pattern for a notch filter $G^Z(z)$ with 3 poles such that contributions at 0 Hz and 50 Hz in the continuous-time input $u(t)$ of Figure 2 do not appear (in discretized form) in the output signal y_k . The sample period is $T = (300)^{-1}$ second. The distance in the complex plane between any pole and any zero should be at least 0.1. [4]

- (b) Determine a canonical direct realization of your $G^Z(z)$ from part (a). [5]

- (c) Consider a system with zero initial conditions, transfer function

$$G^Z(z) = \frac{z-1}{(z-0.5)(z+0.5)},$$

input $u_k = \cos(\omega t_k)$ and output y_k . Note that a formula for $Z\{\cos(\omega t_k)\}$ is not needed here.

- (i) State a formula that provides information about the output y_k in terms of the value of $G^Z(z)$ at a specific z . [2]

- (ii) Use the integral inversion method to find a formula that predicts the output y_k when $\omega = 0$, exploiting the fact that $u_k = 1^k$ when $\omega = 0$. What are the numerical values of y_0, y_1, y_2, y_3 ?

Check your values for y_0, y_1, y_2, y_3 by long division.

Discuss very briefly the consistency, or otherwise, of these values with your information about y_k from part (i). [9]

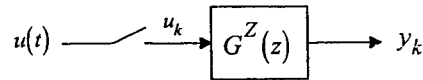


Figure 2

3. (a) Suppose $G^Z(z)$ is the pulse Z-transfer function from $u^Z(z)$ to $y^Z(z)$ of the system of Figure 3, and $G^D(\gamma)$ is the corresponding pulse Delta-transfer function. In Figure 3, S_c denotes a continuous-time linear system and the sample period is T .

- (i) Suppose S_c has the Laplace transfer function $\frac{1}{s(s+1)}$.

Find $G^Z(z)$ from the step response of S_c .

Determine $G^D(\gamma)$ from $G^Z(z)$. [6]

- (ii) Suppose S_c has the model $\dot{x} = Ax + Bu, y = Cx$ where

$$x \in \mathbb{R}^2, A = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = [1 \quad 0].$$

Use $I + \frac{1}{2}AT$ to approximate

$$\psi(AT) \triangleq I + \frac{1}{2!}AT + \frac{1}{3!}A^2T^2 + \dots$$

and hence approximate $G^D(\gamma)$ when $T = 0.1$ second. [6]

- (b) Consider the system of Figure 4 below, where $G^Z(z) = \frac{z-0.5}{z-1}$, $r_k = 2 \times 1^k$, $d_k = 1^k$ and f is a scalar gain. Use the Final Value Theorem to determine whether y_k converges to a constant for some f , and determine the constant if such convergence takes place. [8]

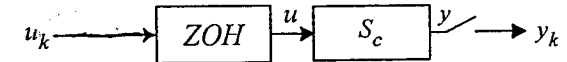


Figure 3

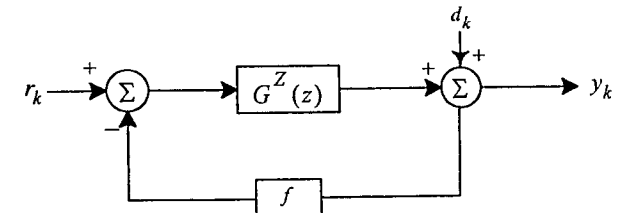


Figure 4

4. Consider the system of Figure 5 below, where $K > 0$.

(a) Suppose $G^Z(z) = \frac{z(z+1.5)}{(z-1.2)^2}$.

Draw the root-locus and determine from it the range of values of the gain K for which the closed-loop system is BIBO-stable, perhaps using the fact that $G^Z(0.604+0.797j) \approx -2.272$.

[6]

Confirm your results using the Jury test.

[4]

(b) Suppose $G^Z(z) = \frac{(z+1)^4}{z^4}$ and $T = 2$ seconds.

Apply the W -transform followed by continuous-time Nyquist analysis to determine the range of values of K for which the closed-loop system is BIBO-stable, perhaps making use of the fact that $(1+j)^4 = -4$.

[6]

Confirm your results using the Routh test.

[4]

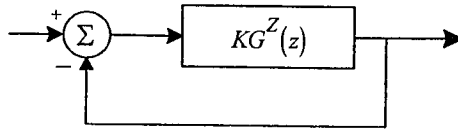


Figure 5

5. (a) Consider the system and observer defined below, where ' denotes transposition:

System: $x_{k+1} = Ax_k + bu_k, y_k = c'x_k$

Observer: $\hat{x}_{k+1} = (A - \ell c')\hat{x}_k + \ell y_k + bu_k$

$A = \begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, c' = [1 \quad 1], \ell \in \mathbb{R}^2.$

(i) Transform A' to companion form, using a transformation derived from the last row of the inverse of the relevant controllability matrix. [9]

(ii) Hence determine ℓ such that the eigenvalues associated with the observer are both zero. [4]

(b) Consider the system of Figure 6. Suppose $G^Z(z) = \frac{z-0.5}{(z-1)(z+0.5)}$ and $K > 0$.

A plot of $G^Z(e^{j\theta})$ as θ varies from 0.1 to 6.183 radians is shown in Figure 7, where the arrows indicate the movement of $G^Z(e^{j\theta})$ as θ increases from 0.1 radians.

Scale the real axis of the plot by evaluating $G^Z(-1)$. Use discrete-time Nyquist analysis and the plot to determine the range of values of $K > 0$ for which the closed-loop system is BIBO-stable. Give sufficient explanation to make your method clear. [7]

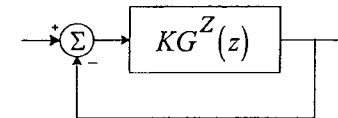


Figure 6

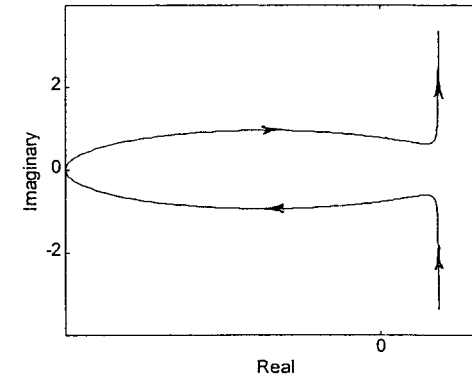


Figure 7

6. (a) Suppose it is required that $x_k \rightarrow 0$ for the control system consisting of

$$\text{Plant: } x_{k+1} = Ax_k + bu_k, \quad y_k = c'x_k$$

$$\text{Observer: } \hat{x}_{k+1} = (A - \ell c')\hat{x}_k + \ell y_k + bu_k$$

$$\text{Controller: } u_k = f'\hat{x}_k$$

where $x_k, \hat{x}_k, b, c, f, \ell \in \mathbb{R}^n$ and $'$ denotes transposition.

Suppose the eigenvalues of $A + bf'$, and of $A - \ell c'$, are all zero.

- (i) By obtaining a difference equation for $e_k = x_k - \hat{x}_k$ and using a companion form, show that $\hat{x}_k = x_k$ for $k \geq n$ [8]
- (ii) Use another companion form and the result of part (i) to show that $x_k = 0$ for all $k \geq 2n$. [4]

- (b) Consider the second-order system with output y_k ,

$$\begin{bmatrix} y_{k+1} \\ w_{k+1} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_k \\ w_k \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u_k$$

with the following reduced-order observer of w_k :

$$\hat{w}_k = v_k + h y_k$$

where

$$v_{k+1} = \ell v_k + m y_k + n u_k.$$

Here $h, \ell, m, n \in \mathbb{R}$ and

$$\ell = a_{22} - h a_{12} \text{ with } |\ell| < 1$$

$$m = a_{21} - h a_{11} + \ell h$$

$$n = b_2 - h b_1.$$

Let $e_k = w_k - \hat{w}_k \in \mathbb{R}$, $x_k = [y_k \ w_k]'$ and $\hat{x}_k = [y_k \ \hat{w}_k]'$.

- (i) Show that $e_k = \ell^k e_0$ and that, as $k \rightarrow \infty$, $e_k \rightarrow 0$. Show also that $\hat{x}_k - x_k \rightarrow 0$ as $k \rightarrow \infty$. [5]

- (ii) By considering w_k in terms of e_k and \hat{w}_k , show that the pulse Z -transfer function from $u^Z(z)$ to $\hat{w}^Z(z)$ is equal to that from $u^Z(z)$ to $w^Z(z)$. [3]

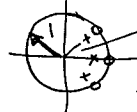
1 (a) $\{x_{k+1}\} = \alpha \{x_k\}$ so $z^{-1}x(z) - z^{-1}x_0 = \alpha x^2(z)$ so $x^2(z) = (z - \alpha)^{-1} z x_0$.
 Taking z^{-1} : $x_k = \alpha^k x_0 = \alpha^k$.
 Taking $\alpha = -1$ gives $x_k = (-1)^k$.
 Hence $z\{(-1)^k\} = \frac{z x_0}{(z - (-1))} = \frac{z}{z+1}$.

(b) $z^2(x^2(z) - x_0 - z^{-1}x_1) = -\beta^2 x^2(z)$
 $\Rightarrow (z^2 + \beta^2)x^2(z) = 2\beta z \Rightarrow x^2(z) = \frac{2\beta z}{z^2 + \beta^2} = \frac{2\beta z}{(z+j\beta)(z-j\beta)}$
 $\Rightarrow \frac{x^2(z)}{z} = \frac{2\beta}{(z+j\beta)(z-j\beta)} = \frac{2\beta}{-2j\beta} \cdot \frac{1}{z+j\beta} + \frac{2\beta}{2j\beta} \cdot \frac{1}{z-j\beta}$
 $= \frac{1}{j} \left(\frac{1}{z-j\beta} - \frac{1}{z+j\beta} \right)$
 $\Rightarrow x^2(z) = \frac{1}{j} \left(\frac{z}{z-j\beta} - \frac{z}{z+j\beta} \right) \Rightarrow x_k = \frac{1}{j} [(j\beta)^k - (-j\beta)^k]$
 $= \frac{1}{j} \beta^k [e^{j\frac{\pi}{2}k} - e^{-j\frac{\pi}{2}k}] = 2\beta^k \sin(k\frac{\pi}{2})$

(c) (i) For $x = \begin{pmatrix} y \\ y \end{pmatrix}$: $\dot{x} = \begin{pmatrix} 0 & 1 \\ \alpha & \beta \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$
 $A = V\Lambda V^{-1}$
Spectral form

Then $x_{k+1} = e^{AT} x_k + \int_0^T e^{A(T-\theta)} B u_k d\theta$ (*)
 (ii) $V \begin{pmatrix} e^{\lambda_1 T} & 0 \\ 0 & e^{\lambda_2 T} \end{pmatrix} V^{-1}$ so the eigenvalues associated with (*) are $e^{\lambda_i T} = \bar{\lambda}_i$.

(iii) The poles of S_d are a subset of the eigenvalues of e^{AT} so S_d is BIBO-stable if $|e^{\lambda_i T}| < 1, \forall i$.
 Now $\lambda_i \in \{\lambda \in \mathbb{C} : \text{Re}(\lambda) < 0\} \Rightarrow \lambda_i = \sigma_i + j\omega_i$ for some $\sigma_i < 0$
 $\Rightarrow |e^{\lambda_i T}| = |e^{\sigma_i T} e^{j\omega_i T}| = |e^{\sigma_i T}| |e^{j\omega_i T}| < 1, \forall i, \forall T > 0$
 $\Rightarrow S_d = \text{BIBO-stable}$

2 (a)  angle $\theta = \omega T = \frac{(2\pi 50) \cdot 0.0105}{2\pi} = 1.05 \text{ rad}$.
 for this filter, $G^2(z) = \frac{(z-1)(z-e^{j\theta})(z-e^{-j\theta})}{(z-0.9)(z-0.9e^{j\theta})(z-0.9e^{-j\theta})}$
 $= \frac{(z-1)(z^2 - z(e^{j\theta} + e^{-j\theta}) + 1)}{(z-0.9)(z^2 - 0.9(e^{j\theta} + e^{-j\theta})z + 0.81)} = \frac{(z-1)(z^2 - z + 1)}{(z-0.9)(z^2 - 0.92z + 0.81)}$
 $= \frac{z^3 - z^2 + z - z^2 + z - 1}{z^3 - 0.92z^2 + 0.81z - 0.92z^2 + 0.81z - 0.729} = \frac{z^3 - 2z^2 + 2z - 1}{z^3 - 1.8z^2 + 1.62z - 0.729}$

(b) $y^2(z) = (1 - 2z^{-1} + 2z^{-2} - z^{-3}) \left(\frac{u^2(z)}{z^3 - 1.8z^2 + 1.62z - 0.729} \right)$
 so $y_k = w_k - 2w_{k-1} + 2w_{k-2} - w_{k-3}$ (f)
 where $(1 - 1.8z^{-1} + 1.62z^{-2} - 0.729z^{-3}) w^2(z) = u^2(z)$
 i.e. $w_k = 1.8w_{k-1} - 1.62w_{k-2} + 0.729w_{k-3} + u_k$ (f)
 the canonical direct realization

(c) One would expect that after the transients have died away
 $y_k = |G^2(e^{j\omega T})| \cos[\omega t_k + \angle G^2(e^{j\omega T})]$

When $\omega = 0$: $G^2(e^{j\omega T}) = G^2(1) = 0$ so $y_k = 0$

(ii) Also $\cos(\omega t_k) = 1, \forall k \geq 0$, so $u^2(z) = \frac{z}{z-1}$.

Hence $y^2(z) = \frac{z-1}{z^2 - 0.25} \cdot \frac{z}{z-1} = \frac{z}{(z-0.5)(z+0.5)}$

so $y_0 = \lim_{|z| \rightarrow \infty} z y^2(z) = 0$. For $k > 0$: $y_k = \frac{1}{2\pi j} \oint \frac{G^2(z) z^{k-1}}{z} dz$
 $= \text{residue} \left[\frac{z^k}{(z-0.5)(z+0.5)} \right] @ z = -0.5$
 $+ \text{residue} \left[\frac{z^k}{(z-0.5)(z+0.5)} \right] @ z = 0.5$
 $= (0.5)^k - (-0.5)^k$

Hence $\{y_k\} = \{0, 0.5 + 0.5, 0, 0.125 + 0.125, 0, \dots\}$
 $= \{0, 1, 0, 0.25, \dots\}$

By long division: $\frac{z^3 - 0.25z}{z^3 - 0.25z^2 - 0.25z + 0.125} \Rightarrow \{y_k\} = \{0, 1, 0, 0.25, \dots\}$

Commentary: here we see the transient which dies to zero as $k \rightarrow \infty$.

3 (a) (i) Step response is $\mathcal{L}^{-1} \left[\frac{1}{s^2(1+s)} \right] (t) = \mathcal{L}^{-1} \left[-\frac{1}{s} + \frac{1}{s^2} + \frac{1}{s+1} \right] (t)$

The sampled step response is $-1 + t + e^{-t}$

which has the z-transform $\frac{-z}{z-1} + \frac{Tz}{(z-1)^2} + \frac{z}{z-e^{-T}}$

Hence the pulse z-transfer fn is $\left(\frac{z-1}{z} \right) \left[\frac{-z}{z-1} + \frac{Tz}{(z-1)^2} + \frac{z}{z-e^{-T}} \right]$

$= -1 + \frac{T}{z-1} + \frac{(z-1)}{(z-e^{-T})} = G(z)$

Then $G(p) = G(z)(1+e^{-T}) = -1 + \frac{T}{1+e^{-T}-1} + \frac{1+e^{-T}-1}{1+e^{-T}-e^{-T}}$

$= -1 + \frac{T}{e^{-T}} + \frac{e^{-T}}{1+e^{-T}-e^{-T}}$

(ii) $\Psi(AT) \approx I + \chi AT = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 0.05 \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0.05 \\ 0 & 0.9 \end{bmatrix}$

$G(p) = \begin{bmatrix} 1 & 0 \end{bmatrix} [rI - \Psi(AT)A]^{-1} \Psi(AT)B \quad \begin{pmatrix} 1 & 0.05 \\ 0 & 0.9 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.05 \\ 0.9 \end{pmatrix} = \tilde{B}$

$\begin{pmatrix} 1 & 0.05 \\ 0 & 0.9 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0.9 \\ 0 & -1.8 \end{pmatrix} = \tilde{A}$

$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0.9 \\ 0 & -1.8 \end{bmatrix}^{-1} \begin{bmatrix} 0.05 \\ 0.9 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} r+1.8 & 0.9 \\ 0 & r \end{bmatrix} \begin{bmatrix} 0.05 \\ 0.9 \end{bmatrix}$

$= \frac{[r+1.8 \quad 0.9] \begin{bmatrix} 0.05 \\ 0.9 \end{bmatrix}}{r(r+1.8)} = \frac{0.05r+0.9}{r(r+1.8)}$

(b) $y^z(z) = \frac{z}{z-1} + \left(\frac{z-0.5}{z-1} \right) \left[\frac{2z}{z-1} - 5y^z(z) \right]$

$\Rightarrow (1+f \frac{z-0.5}{z-1}) y^z(z) = \frac{z}{z-1} + \frac{2z(z-0.5)}{(z-1)^2}$

$\Rightarrow (z-1+f(z-0.5)) y^z(z) = z + 2z \frac{z-0.5}{(z-1)}$

$\Rightarrow y^z(z) = \frac{z + 2z \frac{z-0.5}{z-1}}{z-1+f(z-0.5)}$

$\Rightarrow (z-1) y^z(z) = \frac{z(z-1) + 2z(z-0.5)}{(1+f)z - (1+0.5f)} = \left(\frac{1}{1+f} \right) \frac{z(z-1) + 2z(z-0.5)}{z - \frac{(1+0.5f)}{(1+f)}}$

Here $|p_1(f)| < 1$ if $f > 0$.

Hence $y_k \rightarrow y_{\infty} = \left(\frac{1}{1+f} \right) \cdot \frac{1}{1-p_1(f)} = \frac{2}{f}$

4 (a) Breakpoints: $\frac{1}{\sigma} + \frac{1}{\sigma+1.5} = \frac{2}{\sigma-1.2}$ i.e. $(\sigma+1.5)(\sigma-1.2) + \sigma(\sigma-1.2) = 2\sigma(\sigma+1.5)$

i.e. $(\sigma-1.2)(2\sigma+1.5) = 2\sigma(\sigma+1.5)$ i.e. $2\sigma^2 - 2.4\sigma + 1.5\sigma - 1.8 = 2\sigma^2 + 3\sigma$

i.e. $-1.8 = 3.9\sigma$

i.e. $\sigma = -0.4615$

Hence the root-locus is \rightarrow

Therefore

$K_{min} = \frac{-1}{G^z(0.6+0.8j)} = 0.446$

$K_{max} = \frac{-1}{G^z(-1)} = \frac{-1}{\frac{-0.5}{(z-2)^2}} = 9.68$

Hence BIBO-stable for $0.446 < K < 9.68$.

Jury: Φ denominator is $d(z) = (z-1.2)^2 + Kz(z+1.5)$

$d(1) = 0.04 + 2.5K > 0$ for all $K > 0$

$d(-1) = (-2.2)^2 - K \times 0.5 = 4.84 - K/2 > 0$ if $K < 9.68$

Expanding $d(z)$: $d(z) = (1+K)z^2 + z(-2.4+1.5K) + 1.44$

Hence row 1 of Jury table is: $1.44 \quad -2.4+1.5K \quad 1+K$

Hence BIBO-stable if also $1.44 < 1+K$, i.e. if $K > 0.44$, i.e. (finally) if $0.44 < K < 9.68$

(b) $\bar{G}^w(w) = \left(\frac{z+1}{z} \right)^4 \Big|_{z=\frac{1+w}{1-w}} = \left(\frac{1+w}{1-w} + 1 \right)^4 = \left(\frac{2}{1-w} \right)^4 = \frac{16}{(1-w)^4}$

$G^w(j\omega) = \frac{16}{(1+j\omega)^4}$

$\frac{16}{(1-j)^4} = \frac{16}{-4} = -4 \Rightarrow$ BIBO-stable for $0 < K < 0.25$

Routh Test: Φ denom is $z^4 + K(z+1)^4$. Hence denom. for Φ w-transform

is $(1-w)^4 \left[\left(\frac{1+w}{1-w} \right)^4 + K \left(\frac{2}{1-w} \right)^4 \right] = (1+w)^4 + 16K$

$= w^4 + 4w^3 + 6w^2 + 4w + (16K+1)$

Hence Routh Table is:

1	1	6	16K+1
2	4	4	0
3	5	16K+1	0
4	20-4(16K+1)	5	0
5		(16K+1)	

\Rightarrow all entries in col 1 are > 0 if

$20 - 4(16K+1) > 0$

i.e. if $16 > 4 \times 16K$

i.e. if $K < \frac{1}{4} = 0.25$

Hence system is BIBO-stable if $0 < K < 0.25$, consistent with the above Nyquist analysis

5 (a) Since the eigenvalues of $A - cl'$ are those of $(A - cl')'$, they can be assigned to zero by assigning those of $A' - cl'$ to zero - which can be done using the following procedure for choosing l' .

$A' = \begin{bmatrix} 2 & -1 \\ 1 & -2 \end{bmatrix}$, $c = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The controllability matrix for (A', t) is $M = [c \ A'c] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Therefore $M^{-1} = \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} / (-2) = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

Let p' be the last row of M^{-1} , so $p' = [\frac{1}{2} \ -\frac{1}{2}]$.

Then $P'A' = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ so $V = \begin{bmatrix} P' \\ P'A' \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

and $V^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$.

Then A' is $\frac{x_4 + y_4}{2}$ similar to the comparison matrix

$$VA'V^{-1} = \begin{bmatrix} \frac{\lambda}{2} & -\frac{\lambda}{2} \\ \frac{\lambda}{2} & \frac{\lambda}{2} \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix}$$

Then $V(A' - c)V^{-1} = \begin{bmatrix} 3 & -1 \\ 0 & 1 \\ 3 & 0 \end{bmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} V^{-1}$ so choose $V^{-1} = \begin{bmatrix} 3 & 0 \end{bmatrix}$

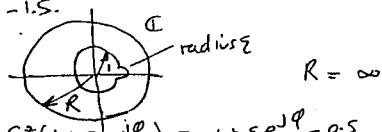
$$\text{So } U = [3 \ 0]V = [3 \ 0] \begin{bmatrix} x & -x \\ x & x \end{bmatrix} = \begin{bmatrix} 3x & -3x \end{bmatrix}$$

Therefore the l required is $l = \frac{1}{2} \begin{bmatrix} 3 \\ -3 \end{bmatrix}$.

Check $A - C^1 = \begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix} - \begin{bmatrix} \frac{3}{2} \\ -\frac{3}{2} \end{bmatrix} C^{-1} B = \begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix} + \begin{bmatrix} \frac{3}{2} & -\frac{3}{2} \\ \frac{3}{2} & \frac{3}{2} \end{bmatrix}$
 $= \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$ so $|\lambda - (A - C^1)| = \begin{vmatrix} \lambda - \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \lambda + \frac{1}{2} \end{vmatrix} = (\lambda - \frac{1}{2})(\lambda + \frac{1}{2}) + \frac{1}{4}$
 $= (\lambda - 0)(\lambda - 0)^2 = 0$ OK

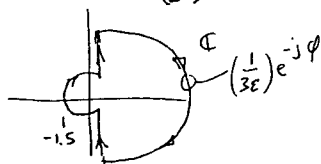
$$(b) \quad G^2(-1) = \frac{-1.5}{(-2)(-0.5)} = -1.5$$

Nyquist path:



$$\text{Fr. } z = 1 + \varepsilon e^{j\varphi}: G^*(1 + \varepsilon e^{j\varphi}) = \frac{1 + \varepsilon e^{j\varphi} - 0.5}{\varepsilon e^{j\varphi} (1.5 + \varepsilon e^{j\varphi})} \approx \frac{0.5}{1.5\varepsilon} e^{-j\varphi} = \frac{1}{3\varepsilon} e^{-j\varphi}$$

Hence Nyquist locus is



Since $\#(\text{poles in unit disc}) = 0$, system is BIBO-stable
for $-\gamma_K < -1.5$, i.e. for $K < 0.666$.

6. (a) Let $e_k = x_k - \hat{x}_k$.

$$\begin{aligned} \text{Then } e_{k+1} &= x_{k+1} - \hat{x}_{k+1} = A x_k + b - (A - (c')^T) \hat{x}_k - (b - b_0) \\ &= (A - (c')^T) x_k - (A - (c')^T) \hat{x}_k = (A - (c')^T) (x_k - \hat{x}_k) \\ &= (A - (c')^T) e_k \end{aligned}$$

$$e_k = (A - Lc')^k e_0.$$

Since the eigenvalues of $A - lc'$ are all 0, $A - lc'$ is similar to C_0 , which is the companion matrix that has every entry in its last row equal to 0. For $n=3$ \square

$$G = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, G^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$C_0^3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

for $C_0 \in \mathbb{R}^{n \times n}$: $C_0^k = 0, \forall k \geq n$.

Hence $A - (c'c) = VC_0V^{-1}$ and $(A - (c'c))^K = VC_0^KV^{-1}$.

Therefore $(A - \lambda C)^k = 0, \forall k \geq n$.

Hence $e_k = (A - C')^k e_0 = 0, \forall k \geq n$.

Consequently $x_k - \hat{x}_k = e_k = 0, \forall k \geq n$.

Therefore the controlled system $x_{k+1} = Ax_k + bf' \hat{x}_k$ acts like

Since $A + tB'$ has all its eigenvalues equal to zero, much as above we have $(A + tB')^m = 0$ for $m \geq n$. Hence $x_{n+m} = (A + tB')^m x_n = 0$.

Therefore $x_{n+1} = (A + bf')^n x_1$. Hence $(A + bf')^K = 0$, $\forall K \geq n$.

19. $x_k = 0, \forall k \geq 2, n$

$$e_{k+1} = w_{k+1} - \hat{w}_{k+1} = a_{21}y_k + a_{22}w_k + b_2v_k - v_{k+1} - h y_{k+1}$$

Therefore $x_k - \bar{x}_k \rightarrow 0$.

Now $w_k = \hat{w}_k + e_k$ indep. of u_k so the transfer fn from $u^i(z)$ to $w^i(z)$ is the same as that from $u^i(z)$ to $\hat{w}^i(z)$.