Imperial College London

M3S8

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May-June 2019

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Time Series

Date: Thursday 09 May 2019

Time: 10.00 - 12.00

Time Allowed: 2 Hours

This paper has 4 Questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the relevant main book(s) are full.

All required additional material will be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Calculators may not be used.

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M4/5S8

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May-June 2019

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Time Series

Date: Thursday 09 May 2019

Time: 10.00 - 12.30

Time Allowed: 2 Hours 30 Minutes

This paper has 5 Questions.

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Note: Throughout this paper $\{\epsilon_t\}$ is a sequence of uncorrelated random variables (white noise) having zero mean and variance σ_ϵ^2, unless stated otherwise. The unqualified term "stationary" will always be taken to mean second-order stationary. All processes are real-valued unless stated otherwise. The sample interval is unity unless stated otherwise. B denotes the backward shift operator.

1. (a) What is meant by saying that a stochastic process is stationary?

   (b) Determine whether each of the following models for a random process $\{X_t\}$ is stationary, justifying your answer. For those that are stationary, specify their mean and autocovariance sequence.

   (i) $X_t = \epsilon_t \cos(ct)$, where $c \neq 0$ is a fixed constant.

   (ii) $X_t = \epsilon_t \epsilon_{t-1}$.

   (iii) $X_t = W y_t$, where W is a random variable with distribution $N(0,1)$ and

$$y_t = \begin{cases} +1 & t \text{ even} \\ -1 & t \text{ odd.} \end{cases}$$

   (c) An ARMA(p,q) process $\{X_t\}$ can be represented by the equation

$$\Phi(B)X_t = \Theta(B)\epsilon_t$$

   where $\Phi(z)$ and $\Theta(z)$ are p and q order z-polynomials, respectively. State conditions on $\Phi(z)$ and $\Theta(z)$ for $\{X_t\}$ to be

   (i) stationary,

   (ii) invertible.

   (d) Consider the process $\{X_t\}$ defined through the model

$$X_t - \epsilon_t = \frac{1}{2}X_{t-1} + \frac{1}{4}\epsilon_{t-1}.$$

   (i) Show that $\{X_t\}$ is both stationary and invertible.

   (ii) Express $\{X_t\}$ in its *general linear process* form and hence show $\text{var}\{X_t\} = \frac{7}{4}\sigma_\epsilon^2$.

2. (a) Let $\{X_t\}$ be the MA(1) process

$$X_t = \epsilon_t - \theta\epsilon_{t-1}.$$

(i) Derive the form of the autocorrelation sequence $\{\rho_\tau\}$ for $\{X_t\}$.

(ii) Show that $|\rho_1| \leq 1/2$ for any value of θ. For which values of θ does ρ_1 attain its maximum and minimum?

(b) The difference operator is defined as $\Delta = 1 - B$. You may use without proof:

$$X_t^{(d)} \equiv \Delta^d X_t = \sum_{k=0}^{d} \binom{d}{k} (-1)^k X_{t-k}.$$

Consider the process $\{X_t\}$ defined as

$$X_t = \alpha + \beta t + Y_t,$$

where $\{Y_t\}$ is a zero-mean stationary process.

(i) Show $\{X_t^{(2)}\}$ is a zero-mean stationary process.

(ii) State the three conditions that must be satisfied by a linear time invariant (LTI) digital filter and hence show that Δ^d is an LTI filter.

(iii) Find the frequency response function $G(f)$ associated with the LTI filter Δ^d.

(iv) If $\{Y_t\}$ is the MA(1) process $Y_t = \epsilon_t - \theta\epsilon_{t-1}$, show the spectral density function of $X_t^{(2)}$ is

$$S_{X^{(2)}}(f) = \sigma_\epsilon^2(6 - 8\cos(2\pi f) + 2\cos(4\pi f))(1 + \theta^2 - 2\theta\cos(2\pi f)).$$

3. (a) Let $X_1, ..., X_N$ be a realisation from a stationary process $\{X_t\}$ with mean μ and autocovariance sequence $\{s_\tau\}$.

Show that the sample mean

$$\bar{X} = \frac{1}{N}\sum_{t=1}^{N} X_t$$

is an unbiased estimator for μ and

$$\mathrm{var}\{\bar{X}\} = \frac{1}{N}\sum_{\tau=-(N-1)}^{(N-1)} \left(1 - \frac{|\tau|}{N}\right) s_\tau.$$

HINT: when confronted with a double sum, instead of summing across "rows", sum across "diagonals".

(b) The *biased* estimator $\hat{s}_\tau^{(p)}$ of $\{s_\tau\}$ is defined as

$$\hat{s}_\tau^{(p)} = \frac{1}{N}\sum_{t=1}^{N-|\tau|} (X_t - \bar{X})(X_{t+|\tau|} - \bar{X}) \qquad \tau = 0, \pm 1, \pm 2, ..., \pm(N-1).$$

When the mean is known, $\bar{X}$ is replaced by μ.

The periodogram $\hat{S}^{(p)}(\cdot)$ can be defined as the Fourier transform of $\{\hat{s}_\tau^{(p)}\}$. Show

$$\frac{1}{N}\sum_{k=0}^{N-1} \hat{S}^{(p)}(f_k) = \hat{s}_0^{(p)},$$

where $f_k = k/N$. *HINT: you may use without proof that*

$$\sum_{k=0}^{N-1} \mathrm{e}^{\mathrm{i}kx} = \frac{1 - \mathrm{e}^{\mathrm{i}Nx}}{1 - \mathrm{e}^{\mathrm{i}x}}$$

for $x \neq 0, \pm 2\pi, \pm 4\pi, ...$

[Question 3 continues on the next page]

(c) A random process $\{X_t\}$ with non-zero mean μ has the spectral representation

$$X_t - \mu = \int_{-1/2}^{1/2} e^{i2\pi ft} dZ(f).$$

If a time series has a non-zero mean, centring it (removing the mean) before performing spectral estimation is crucial. Failure to do so can have consequences.

Let $X_1, ..., X_N$ be a realisation of a white noise process with mean and variance both equal to 1. Consider the direct spectral estimator

$$\widehat{S}^{(d)}(f) \equiv \left|\sum_{t=1}^{N} h_t X_t e^{-i2\pi ft}\right|^2, \qquad |f| \leq \frac{1}{2}$$

where $\{h_t\}$ is a data taper of length N normalised such that $\sum_{t=1}^{N} h_t^2 = 1$.
Show

$$E\{\widehat{S}^{(d)}(f)\} = 1 + \mathcal{H}(f),$$

where

$$\mathcal{H}(f) = \left|\sum_{t=1}^{N} h_t e^{-i2\pi ft}\right|^2.$$

You may use without proof that $\int_{-1/2}^{1/2} \mathcal{H}(f - f')df' = 1$ for all $f \in [-\frac{1}{2}, \frac{1}{2}]$.

4. (a) What is meant by saying a pair of real-valued discrete-time stochastic processes are jointly stationary?

(b) Let $\{X_t\}$ and $\{Y_t\}$ be a pair of zero-mean real-valued jointly stationary processes with spectral density functions $S_X(\cdot)$ and $S_Y(\cdot)$, respectively, and cross-spectrum $S_{XY}(\cdot)$. Considering their individual spectral representations

$$X_t = \int_{-1/2}^{1/2} e^{i2\pi ft} dZ_X(f) \qquad Y_t = \int_{-1/2}^{1/2} e^{i2\pi ft} dZ_Y(f),$$

show the coherence

$$\gamma_{XY}^2(f) = \frac{|S_{XY}(f)|^2}{S_X(f)S_Y(f)}$$

at frequency $f \in [-1/2, 1/2]$ is the magnitude square of the correlation between $dZ_X(f)$ and $dZ_Y(f)$. *HINT: for a pair of zero mean complex random variables S and T, $\text{cov}(S,T) = E\{S^*T\}$, where * denotes complex conjugation.*

(c) Let $\{X_t\}$ and $\{Y_t\}$ be a pair of zero-mean real-valued stationary processes that are independent of each other. They have autocovariance sequences $\{s_{X,\tau}\}$ and $\{s_{Y,\tau}\}$ and spectral density functions $S_X(\cdot)$ and $S_Y(\cdot)$, respectively. Consider the processes $\{V_t\}$ and $\{W_t\}$ defined as

$$V_t = AX_t + BY_t$$
$$W_t = CX_t + DY_t,$$

where A, B, C and D are each unit variance, zero mean real-valued random variables that are all independent of $\{X_t\}$ and $\{Y_t\}$. The vector $(A, B, C, D)^T$ has covariance matrix

$$\Sigma = \begin{pmatrix} 1 & 0 & \alpha & 0 \\ 0 & 1 & 0 & \beta \\ \alpha & 0 & 1 & 0 \\ 0 & \beta & 0 & 1 \end{pmatrix}.$$

(i) Show that $\{V_t\}$ and $\{W_t\}$ are jointly stationary and determine their autocovariance sequences $\{s_{V,\tau}\}$ and $\{s_{W,\tau}\}$, and the cross-covariance sequence $\{s_{VW,\tau}\}$ in terms of $s_{X,\tau}$, $s_{Y,\tau}$, α and β.

(ii) Give an expression for $\gamma_{VW}^2(f)$, the coherence between $\{V_t\}$ and $\{W_t\}$ at frequency f, in terms of $S_X(f)$, $S_Y(f)$, α and β.

(iii) Suppose the autocovariance sequences for $\{X_t\}$ and $\{Y_t\}$ are given as

$$s_{X,\tau} = \begin{cases} 1 & \tau = 0 \\ \frac{1}{2} & |\tau| = 1 \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad s_{Y,\tau} = \begin{cases} 2 & \tau = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Show that

$$\gamma_{VW}^2(0) = \frac{(\alpha+\beta)^2}{4}.$$

5. Let $X_1, ..., X_N$ be a realisation of a zero mean stationary process $\{X_t\}$ with autocovariance sequence $\{s_\tau\}$ and spectral density function $S(\cdot)$. The direct spectral estimator $\widehat{S}^{(d)}(\cdot)$ is as defined in 3(c).

(a) What are the reasons for smoothing the periodogram and direct spectral estimators? Justify smoothing in the frequency domain by considering the average of the periodogram over a finite set of adjacent Fourier frequencies.

(b) The lag-window estimator of $S(f)$ is defined as

$$\widehat{S}^{(lw)}(f) = \int_{-1/2}^{1/2} W_m(f-\phi)\widehat{S}^{(d)}(\phi)\mathrm{d}\phi$$

where $W_m(\cdot)$ is a symmetric real-valued periodic (period 1) function which is square integrable over $[-\frac{1}{2}, \frac{1}{2}]$ and whose smoothing properties can be controlled by a parameter m. Furthermore, $W_m(\cdot)$ is normalised such that $\int_{-1/2}^{1/2} W_m(f)\mathrm{d}f = 1$.
Using a data taper $\{h_t\}$, it is true that

$$E\{\widehat{S}^{(lw)}(f)\} = \int_{-1/2}^{1/2} \mathcal{U}_m(f-\phi)S(\phi)\mathrm{d}\phi,$$

where $\mathcal{U}_m(f) = \int_{-1/2}^{1/2} W_m(f-f')\mathcal{H}(f')\mathrm{d}f'$ and $\mathcal{H}(\cdot)$ is as defined in Question 3(c).

(i) Show that

$$\mathcal{U}_m(f) = \sum_{\tau-(N-1)}^{N-1} w_{\tau,m}\left(\sum_{t=1}^{N-|\tau|} h_t h_{t+|\tau|}\right)\mathrm{e}^{-\mathrm{i}2\pi f\tau},$$

where $W_m(\cdot)$ and $\{w_{\tau,m}\}$ form the Fourier transform pair

$$w_{\tau,m} = \int_{-1/2}^{1/2} W_m(f)\mathrm{e}^{\mathrm{i}2\pi f\tau}\mathrm{d}\phi \qquad W_m(f) = \sum_{\tau=-(N-1)}^{N-1} w_{\tau,m}\mathrm{e}^{-\mathrm{i}2\pi f\tau}.$$

HINT: it is true that

$$\left|\sum_{t=1}^{N} h_t\mathrm{e}^{-\mathrm{i}2\pi ft}\right|^2 = \sum_{\tau=-(N-1)}^{N-1}\sum_{t=1}^{N-|\tau|} h_t h_{t+|\tau|}\mathrm{e}^{-\mathrm{i}2\pi f\tau}.$$

(ii) Show that

$$E\{\widehat{S}^{(lw)}(f)\} = \sum_{\tau=-(N-1)}^{N-1}\left(w_{\tau,m}s_\tau\sum_{t=1}^{N-|\tau|} h_t h_{t+|\tau|}\right)\mathrm{e}^{-\mathrm{i}2\pi f\tau}.$$

(iii) Let $X_1, ..., X_N$ be a realisation of the zero mean process $\{X_t\}$ given in Question 4(c)(iii). Using the rectangular data-taper $h_t = N^{-1/2}$, $t = 1, ..., N$, show that

$$E\{\widehat{S}^{(lw)}(f)\} - S(f) = \left(w_{1,m}\frac{N-1}{N} - 1\right)\cos(2\pi f).$$

HINT: $\{w_{m,\tau}\}$ is symmetric about $\tau = 0$.

M345S8 SOLUTIONS

1. (a) $\{X_t\}$ is second-order stationary if $E\{X_t\}$ is a finite constant for all t, var$\{X_t\}$ is a finite constant for all t, and cov$\{X_t, X_{t+\tau}\}$, is a finite quantity depending only on τ and not on t. seen $\Downarrow$ 3(A)

(b) (i) This is non-stationary. The mean is constant as $E\{X_t\} = 0$ for all t, however $E\{X_t^2\} = \sigma_\epsilon^2 \cos^2(ct)$ which depends on t. 2(A)

(ii) This is stationary and is shown by considering the mean and autocovariance. sim. seen $\Downarrow$
Mean: $E\{X_t\} = E\{\epsilon_t \epsilon_{t-1}\} = E\{\epsilon_t\}E\{\epsilon_{t-1}\} = 0$.
Autocovariance: $E\{X_t X_{t+\tau}\} = E\{\epsilon_t \epsilon_{t-1} \epsilon_{t+\tau} \epsilon_{t-1+\tau}\}$. First consider $\tau = 0$, then $E\{\epsilon_t \epsilon_{t-1} \epsilon_t \epsilon_{t-1}\} = E\{\epsilon_t^2\}E\{\epsilon_{t-1}^2\} = \sigma_\epsilon^4$. When $\tau = 1$ (and similarly for $\tau = -1$), $E\{\epsilon_t \epsilon_{t-1} \epsilon_{t+1} \epsilon_t\} = E\{\epsilon_t^2\}E\{\epsilon_{t+1}\}E\{\epsilon_{t-1}\} = 0$. Likewise, for all $|\tau| > 1$, $E\{X_t X_{t+\tau}\} = 0$. Therefore

$$E\{X_t X_{t+\tau}\} = \begin{cases} \sigma_\epsilon^4 & \tau = 0 \\ 0 & \text{otherwise} \end{cases}$$

and the process is stationary. 3(B)

(iii) This is stationary and is shown by considering the mean and autocovariance. unseen $\Downarrow$
Mean: $E\{X_t\} = y_t E\{W\} = 0$.
Autocovariance: $E\{X_t X_{t+\tau}\} = E\{W^2\} y_t y_{t+\tau} = y_t y_{t+\tau}$. For any fixed t, when τ is even $y_t y_{t+\tau} = 1$ and when τ is odd $y_t y_{t+\tau} = -1$, therefore

$$E\{X_t X_{t+\tau}\} = \begin{cases} 1 & \tau \text{ even} \\ -1 & \tau \text{ odd} \end{cases}$$

and it is not dependent t. Therefore it is stationary. 3(B)

(c) (i) Roots of $\Phi(z)$ lie outside the unit circle. seen $\Downarrow$

(ii) Roots of $\Theta(z)$ lie outside the unit circle. 2(A)

(d) (i) The process can be formulated as $\boxed{\text{sim. seen} \Downarrow}$

$$X_t - \frac{1}{2}X_{t-1} = \epsilon_t + \frac{1}{4}\epsilon_{t-1}$$

which is in ARMA(1,1) form where $\Phi(z) = 1 - \frac{1}{2}z$ and $\Theta(z) = 1 + \frac{1}{4}z$. The roots of $\Phi(z)$ and $\Theta(z)$ are 2 and -4, respectively, which both lie outside the unit circle. Process $\{X_t\}$ is therefore stationary and invertible. $\boxed{2(A)}$

(ii) We are required to put it in general linear form $X_t = G(B)\epsilon_t$. Here

$$\begin{aligned}
G(z) &= \Phi^{-1}(z)\Theta(z) \\
&= (1 + \frac{1}{2}z + \frac{1}{4}z^2 + ...)(1 + \frac{1}{4}z) \\
&= (1 + \frac{1}{2}z + \frac{1}{4}z^2 + ...) + (\frac{1}{4}z + \frac{1}{8}z^2 + ...) \\
&= 1 + \sum_{k=1}^{\infty} \frac{3}{2^{k+1}}z^k.
\end{aligned}$$

Therefore, general linear process form is

$$X_t = \epsilon_t + \sum_{k=1}^{\infty} \frac{3}{2^{k+1}}\epsilon_{t-k}.$$

$\boxed{2(B)}$

For a general linear process $X_t = \sum_{k=0}^{\infty} g_k\epsilon_{t-k}$, we have $\mathrm{var}\{X_t\} = \sigma_\epsilon^2 \sum_{k=0}^{\infty} g_k^2$. Therefore, $\mathrm{var}\{X_t\} = \sigma_\epsilon^2(1+\sum_{k=1}^{\infty} \frac{9}{4^{k+1}}) = \sigma_\epsilon^2(1+9(\frac{1}{1-\frac{1}{4}} - 1 - \frac{1}{4})) = \frac{7}{4}\sigma_\epsilon^2$. $\boxed{3(B)}$

2. (a) We immediately have $E\{X_t\} = 0$, therefore $\boxed{\text{seen} \Downarrow}$

$$\begin{aligned} s_\tau &= E\{X_t X_{t+\tau}\} = E\{(\epsilon_t - \theta\epsilon_{t-1})(\epsilon_{t+\tau} - \theta\epsilon_{t-1+\tau})\} \\ &= E\{\epsilon_t\epsilon_{t+\tau}\} - \theta(E\{\epsilon_t\epsilon_{t-1+\tau}\} + E\{\epsilon_{t-1}\epsilon_{t+\tau}\}) + \theta^2 E\{\epsilon_{t-1}\epsilon_{t-1+\tau}\} \end{aligned}$$

This gives

$$\begin{aligned} s_0 &= \sigma_\epsilon^2(1+\theta^2), \\ s_1 &= -\sigma_\epsilon^2\theta = s_{-1} \end{aligned}$$

and $s_\tau = 0$ for all $|\tau| > 1$. Therefore the autocorrelation sequence, defined as $\rho_\tau = s_\tau/s_0$ is

$$\rho_\tau = \begin{cases} 1 & \tau = 0 \\ -\frac{\theta}{1+\theta^2} & |\tau| = 1 \\ 0 & |\tau| > 1 \end{cases}$$

$\boxed{3(\text{A})}$

It follows that $\frac{\mathrm{d}\rho_1}{\mathrm{d}\theta} = \frac{\mathrm{d}}{\mathrm{d}\theta}\frac{-\theta}{1+\theta^2} = \frac{-(1+\theta)(1-\theta)}{(1+\theta^2)^2}$. This equals zero when $\theta = 1$ (minimum, $\rho_1 = -1/2$) and -1 (maximum, $\rho_1 = 1/2$.) $\boxed{\text{unseen} \Downarrow}$ $\boxed{1(\text{A})}$

(b) (i) $\boxed{\text{seen} \Downarrow}$

$$\begin{aligned} X_t^{(2)} &= \Delta^2 X_t \\ &= \Delta(\alpha - \alpha + \beta - \beta(t-1) + \Delta Y_t) \\ &= \Delta^2 Y_t \\ &= Y_t - 2Y_{t-1} + Y_{t-2}. \end{aligned}$$

This is stationary because, by the stationarity of $\{Y_t\}$,

$$E\{X_t^{(2)}\} = E\{Y_t - 2Y_{t-1} + Y_{t-2}\} = E\{Y_t\} - 2E\{Y_{t-1}\} + E\{Y_{t-2}\} = 0$$

and

$$\begin{aligned} E\{X_t^{(2)}X_{t+\tau}^{(2)}\} &= E\{(Y_t - 2Y_{t-1} + Y_{t-2})(Y_{t+\tau} - 2Y_{t-1+\tau} + Y_{t-2+\tau})\} \\ &= s_{Y,\tau} - 2s_{Y,\tau-1} + s_{Y,\tau-2} - 2s_{Y,\tau+1} + 4s_{Y,\tau} - 2s_{Y,\tau-1} + s_{Y,\tau+2} - 2s_{Y,\tau+1} + s_{Y,\tau} \\ &= s_{Y,\tau-2} - 4s_{Y,\tau-1} + 6s_{Y,\tau} - 4s_{Y,\tau+1} + s_{Y,\tau+2} \end{aligned}$$

which depends only on τ. $\boxed{3(\text{B})}$

(ii) The conditions for an LTI digital filter are: $\boxed{\text{sim. seen} \Downarrow}$

1. Scale preservation:

$$L\{\alpha x_t\} = \alpha L\{x_t\}.$$

This is satisfied by Δ^d because

$$\Delta^d \alpha x_t = \sum_{k=0}^{d} \binom{d}{k} (-1)^k \alpha x_{t-k} = \alpha \sum_{k=0}^{d} \binom{d}{k} (-1)^k x_{t-k} = \alpha \Delta^d x_t.$$

2. Superposition:

$$L\{x_{1,t} + x_{2,t}\} = L\{x_{1,t}\} + L\{x_{2,t}\}.$$

This is satisfied by Δ^d because

$$\Delta^d (x_{1,t} + x_{2,t}) = \Delta^d x_{1,t} + \Delta^d x_{2,t}.$$

3. Time invariance: if $L\{x_t\} = y_t$ then $L\{x_{t+\tau}\} = y_{t+\tau}$.
This is satisfied by Δ^d because if $\Delta^d x_t = y_t$

$$\Delta^d x_{t+\tau} = \sum_{k=0}^{d} \binom{d}{k} (-1)^k x_{t+\tau-k} = \sum_{k=0}^{d} \binom{d}{k} (-1)^k x_{t'-k} = y_{t'} = y_{t+\tau}.$$

$\boxed{5(\mathrm{A})}$

(iii) Frequency response function is computed by considering

$$\Delta^d e^{i2\pi ft} = \sum_{k=0}^{d} \binom{d}{k} (-1)^k e^{i2\pi f(t-k)} = e^{i2\pi ft} \sum_{k=0}^{d} \binom{d}{k} (-1)^k e^{-i2\pi fk}.$$

Therefore, $G(f) = \sum_{k=0}^{d} \binom{d}{k} (-1)^k e^{-i2\pi fk} = (1 - e^{-i2\pi f})^d.$ $\boxed{2(\mathrm{A})}$

(iv) We have that $X_t^{(2)} = Y_t^{(2)} = \Delta^2 Y_t$. Therefore, $S_{X^{(2)}}(f) = |G(f)|^2 S_Y(f)$, where from 2b(iii) the frequency response function of LTI filter Δ^2 is given as $G(f) = 1 - 2\mathrm{e}^{-\mathrm{i}2\pi f} + \mathrm{e}^{-\mathrm{i}4\pi f}$. unseen ⇓

We first have to compute the spectral density function of $\{Y_t\}$. This can either be done via a Fourier transform of the autocovariance sequence (see 2(a)), or by considering an MA(1) process to be a linear filter on a white noise process. Here is the Fourier transform method:

$$\begin{aligned}
S_Y(f) &= \sum_{\tau=-\infty}^{\infty} s_{Y,\tau} \mathrm{e}^{-\mathrm{i}2\pi f\tau} \\
&= \sigma_\epsilon^2(-\theta \mathrm{e}^{\mathrm{i}2\pi f} + 1 + \theta^2 - \theta \mathrm{e}^{-\mathrm{i}2\pi f}) \\
&= \sigma_\epsilon^2(1 + \theta^2 - 2\theta\cos(2\pi f t)).
\end{aligned}$$

With

$$\begin{aligned}
|G(f)|^2 &= (1 - 2\mathrm{e}^{-\mathrm{i}2\pi f} + \mathrm{e}^{-\mathrm{i}4\pi f})(1 - 2\mathrm{e}^{\mathrm{i}2\pi f} + \mathrm{e}^{\mathrm{i}4\pi f}) \\
&= 6 - 8\cos(2\pi f) + 2\cos(4\pi f),
\end{aligned}$$

it follows that

$$S_{X^{(2)}}(f) = \sigma_\epsilon^2(6 - 8\cos(2\pi f) + 2\cos(4\pi f))(1 + \theta^2 - 2\theta\cos(2\pi f t)).$$

6(D)

3. (a) To show unbiasedness: $\boxed{\text{seen} \Downarrow}$

$$\mathsf{E}\{\bar{X}\} = \frac{1}{N}\sum_{t=1}^{n}\mathsf{E}\{X_t\} = \frac{1}{N}N\mu = \mu,$$

therefore $\bar{X}$ is an unbiased estimator of μ. $\boxed{2(\text{A})}$

Now,

$$\begin{aligned}
\text{var}\{\bar{X}\} &= \mathsf{E}\{(\bar{X}-\mu)^2\} \\
&= \mathsf{E}\left\{\left(\frac{1}{N}\sum_{i=1}^{N}(X_i-\mu)\right)^2\right\} \\
&= \frac{1}{N^2}\sum_{t=1}^{N}\sum_{u=1}^{N}\mathsf{E}\{(X_t-\mu)(X_u-\mu)\} \\
&= \frac{1}{N^2}\sum_{t=1}^{N}\sum_{u=1}^{N}s_{u-t} \\
&= \frac{1}{N^2}\sum_{\tau=-(N-1)}^{N-1}\sum_{k=1}^{N-|\tau|}s_\tau \\
&= \frac{1}{N^2}\sum_{\tau=-(N-1)}^{N-1}(N-|\tau|)s_\tau \\
&= \frac{1}{N}\sum_{\tau=-(N-1)}^{N-1}\left(1-\frac{|\tau|}{N}\right)s_\tau .
\end{aligned}$$

The summation interchange merely swaps row sums for diagonal sums. $\boxed{5(\text{A})}$

(b) Using the Fourier relationship $\boxed{\text{unseen} \Downarrow}$

$$\widehat{S}^{(p)}(f) = \sum_{\tau=-(N-1)}^{(N-1)}\widehat{s}_\tau^{(p)}\mathrm{e}^{-\mathrm{i}2\pi f\tau},$$

it follows that

$$\begin{aligned}
\frac{1}{N}\sum_{k=0}^{N-1}\widehat{S}^{(p)}(f_k) &= \frac{1}{N}\sum_{k=0}^{N-1}\sum_{\tau=-(N-1)}^{(N-1)}\widehat{s}_\tau^{(p)}\mathrm{e}^{-\mathrm{i}2\pi f_k\tau} \\
&= \frac{1}{N}\sum_{\tau=-(N-1)}^{(N-1)}\widehat{s}_\tau^{(p)}\sum_{k=0}^{N-1}\mathrm{e}^{-\mathrm{i}2\pi f_k\tau} \\
&= \frac{1}{N}\sum_{\tau=-(N-1)}^{(N-1)}\widehat{s}_\tau^{(p)}\sum_{k=0}^{N-1}\mathrm{e}^{-\mathrm{i}k\frac{2\pi\tau}{N}}
\end{aligned}$$

Using the given hint,

$$\sum_{k=0}^{N-1} e^{-ik\frac{2\pi\tau}{N}} = \frac{1-e^{i2\pi\tau}}{1-e^{-i\frac{2\pi\tau}{N}}} = 0 \qquad \text{for } \tau = \pm 1, \pm 2, \ldots \pm (N-1)$$

and clearly for $\tau = 0$

$$\sum_{k=0}^{N-1} e^{-ik\frac{2\pi\tau}{N}} = \sum_{k=0}^{N-1} 1 = N.$$

Therefore

$$\frac{1}{N}\sum_{\tau=-(N-1)}^{(N-1)} \widehat{s}_\tau^{(p)} \sum_{k=0}^{N-1} e^{-ik\frac{2\pi\tau}{N}} = \frac{1}{N} N \widehat{s}_0^{(p)} = \widehat{s}_0^{(p)}.$$

$\boxed{5(\mathrm{C})}$

(c) Considering:

$\boxed{\text{sim. seen} \Downarrow}$

$\boxed{\text{unseen} \Downarrow}$

$$\mathrm{E}\{|J(f)|^2\} \quad \text{where} \quad J(f) = \sum_{t=1}^{N} h_t X_t e^{-i2\pi f t}, \quad |f| \le \frac{1}{2}.$$

$[\widehat{S}^{(d)}(f) = |J(f)|^2.]$

We know from the spectral representation theorem that,

$$X_t = 1 + \int_{-1/2}^{1/2} e^{i2\pi f' t}\, dZ(f'),$$

so that,

$$\begin{aligned} J(f) &= \sum_{t=1}^{N}\left(\int_{-1/2}^{1/2} h_t e^{i2\pi f' t}\, dZ(f')\right) e^{-i2\pi f t} + \sum_{t=1}^{N} h_t e^{-i2\pi f t} \\ &= \int_{-1/2}^{1/2} \sum_{t=1}^{N} h_t e^{-i2\pi (f-f')t}\, dZ(f') + \sum_{t=1}^{N} h_t e^{-i2\pi f t} \end{aligned}$$

Then

$$\begin{aligned}
\mathsf{E}\{\widehat{S}^{(p)}(f)\} &= \mathsf{E}\{|J(f)|^2\} = \mathsf{E}\{J^*(f)J(f)\} \\
&= \mathsf{E}\left\{\left(\int_{-1/2}^{1/2}\sum_{t=1}^{N} h_t e^{i2\pi(f-f')t}\,\mathrm{d}Z(f') + \sum_{t=1}^{N} h_t e^{i2\pi ft}\right)\right. \\
&\qquad \left.\times\left(\int_{-1/2}^{1/2}\sum_{t=1}^{N} h_t e^{-i2\pi(f-f')t}\,\mathrm{d}Z(f') + \sum_{t=1}^{N} h_t e^{-i2\pi ft}\right)\right\} \\
&= \int_{-1/2}^{1/2}\int_{-1/2}^{1/2}\sum_{t=1}^{N} h_t e^{i2\pi(f-f')t}\sum_{s=1}^{N} h_s e^{-i2\pi(f-f'')s}\mathsf{E}\{\mathrm{d}Z^*(f')\,\mathrm{d}Z(f'')\} \\
&\qquad + \sum_{t=1}^{N} h_t e^{-i2\pi ft}\int_{-1/2}^{1/2}\sum_{s=1}^{N} h_s e^{i2\pi(f-f')s}\,E\{\mathrm{d}Z(f')\} \\
&\qquad + \sum_{t=1}^{N} h_t e^{i2\pi ft}\int_{-1/2}^{1/2}\sum_{s=1}^{N} h_s e^{-i2\pi(f-f')s}\,E\{\mathrm{d}Z(f')\} \\
&\qquad + \left|\sum_{t=1}^{N} h_t e^{-i2\pi ft}\right|^2 \\
&= \int_{-1/2}^{1/2}\mathcal{H}(f-f')\cdot 1\,\mathrm{d}f' + \mathcal{H}(f) \\
&= 1 + \mathcal{H}(f)
\end{aligned}$$

using properties of orthogonal increment process that $E\{\mathrm{d}Z(f')\} = 0$ and $E\{\mathrm{d}Z^*(f')\mathrm{d}Z(f'')\} = 0$ if $f' \neq f''$ and equals $S(f) = 1$ (unit variance white noise) if $f' = f''$. $\boxed{8(\mathrm{D})}$

4. (a) Two real-valued discrete-time processes $\{X_t\}$ and $\{Y_t\}$ are said to be jointly stationary stochastic processes if each are separately second-order stationary processes, and $\text{cov}\{X_t, Y_{t+\tau}\}$ is a function of τ only. $\boxed{\text{seen} \Downarrow}$ $\boxed{3(A)}$

(b) The cross-spectrum is defined as

$$S_{XY}(f)\mathrm{d}f = E\{\mathrm{d}Z_X^*(f)\mathrm{d}Z_Y(f)\},$$

and the sdfs as $S_X(f)\mathrm{d}f = E\{|\mathrm{d}Z_X(f)|^2\}$ and $S_Y(f)\mathrm{d}f = E\{|\mathrm{d}Z_Y(f)|^2\}$, so

$$\begin{aligned}\gamma_{XY}^2(f) &= \frac{E\{\mathrm{d}Z_X^*(f)\mathrm{d}Z_Y(f)\}}{E\{|\mathrm{d}Z_X(f)|^2\}E\{|\mathrm{d}Z_Y(f)|^2\}} \\ &= \frac{\text{cov}\{\mathrm{d}Z_X(f), \mathrm{d}Z_Y(f)\}}{\text{var}\{\mathrm{d}Z_X(f)\}\,\text{var}\{\mathrm{d}Z_Y(f)\}}\end{aligned}$$

which is the magnitude square of the correlation between $\mathrm{d}Z_X(f)$ and $\mathrm{d}Z_Y(f)$. $\boxed{3(A)}$

(c) (i) We are first required to check that $\{V_t\}$ and $\{W_t\}$ are individually stationary. Considering $\{V_t\}$: $\boxed{\text{sim. seen} \Downarrow}$
Mean: $E\{V_t\} = E\{AX_t + BY_t\} = E\{AX_t\} + E\{BY_t\} = E\{A\}E\{X_t\} + E\{B\}E\{Y_t\} = 0$.
Autocovariance: $s_{V,\tau} = \text{cov}\{V_t, V_{t+\tau}\} = E\{V_tV_{t+\tau}\} = E\{(AX_t + BY_t)(AX_{t+\tau} + BY_{t+\tau})\} = E\{A^2\}E\{X_tX_{t+\tau}\} + E\{AB\}E\{X_tY_{t+\tau}\} + E\{AB\}E\{X_{t+\tau}Y_t\} + E\{B^2\}E\{Y_tY_{t+\tau}\} = 1 \cdot s_{X,\tau} + 0 + 0 + 1 \cdot s_{Y,\tau} = s_{X,\tau} + s_{Y,\tau}$. Therefore $\{V_t\}$ is stationary, as is $\{W_t\}$ by an identical argument with also $s_{W,\tau} = s_{X,\tau} + s_{Y,\tau}$.
It is next required that $\text{cov}\{V_t, W_{t+\tau}\}$ depends only on τ.
$\text{cov}\{V_t, W_{t+\tau}\} = E\{V_tW_{t+\tau}\} = E\{(AX_t + BY_t)(CX_{t+\tau} + DY_{t+\tau})\} = E\{AC\}E\{X_tX_{t+\tau}\} + E\{AD\}E\{X_tY_{t+\tau}\} + E\{BC\}E\{X_{t+\tau}Y_t\} + E\{BD\}E\{Y_tY_{t+\tau}\} = \alpha s_{X,\tau} + 0 + 0 + \beta s_{Y,\tau} = \alpha s_{X,\tau} + \beta s_{Y,\tau}$. $\boxed{5(B)}$

(ii) The cross spectrum for $\{V_t\}$ and $\{W_t\}$ is the Fourier transform of the $s_{VW,\tau}$. It is shown in the (c)(i) that $s_{VW,\tau} = \alpha s_{X,\tau} + \beta s_{Y,\tau}$, therefore, taking the Fourier transform, we have $S_{VW}(f) = \alpha S_X(f) + \beta S_Y(f)$, which we recognise as being real valued. Using an analogous argument, we have $S_V(f) = S_W(f) = S_X(f) + S_Y(f)$. Therefore the coherence is

$$\gamma_{VW}^2(f) = \left(\frac{\alpha S_X(f) + \beta S_{Y(f)}}{S_X(f) + S_Y(f)}\right)^2.$$

$\boxed{5(C)}$

(iii) The spectral density functions are computed by taking the Fourier transform of the respective autocovariance sequence. Specifically, $\boxed{\text{unseen} \Downarrow}$

$$
\begin{aligned}
S_X(f) &= \sum_{\tau=-\infty}^{\infty} s_{X,\tau} e^{-i2\pi f\tau} \\
&= \frac{1}{2}e^{i2\pi f} + 1 + \frac{1}{2}e^{-i2\pi f} \\
&= 1 + \cos(2\pi f)
\end{aligned}
$$

and

$$
\begin{aligned}
S_Y(f) &= \sum_{\tau=-\infty}^{\infty} s_{Y,\tau} e^{-i2\pi f\tau} \\
&= 2.
\end{aligned}
$$

Therefore

$$
\gamma^2_{VW}(f) = \left(\frac{\alpha(1+\cos(2\pi f)) + 2\beta}{3+\cos(2\pi f)} \right)^2.
$$

When $f = 0$, this becomes

$$
\gamma^2_{VW}(0) = ((2\alpha + 2\beta)/4)^2 = (\alpha + \beta)^2/4.
$$

$\boxed{4(\mathrm{C})}$

5. (a) - The periodogram has poor bias and variance properties. seen ⇓

- For spectra with large dynamic range, the bias can be significantly reduced by tapering. However, variance problems still persist.
- Traditional approaches to this problem look to smooth $\widehat{S}^{(d)}(\cdot)$ across frequencies.
- Suppose N is large enough so that the periodogram $\widehat{S}^{(p)}(\cdot)$ can reasonably be considered an unbiased estimator of $S(\cdot)$ and is pair-wise uncorrelated at the Fourier frequencies $f_k = k/N$. If $S(\cdot)$ is slowly varying in the neighbourhood of, for example, f_k, then

$$S(f_{k-M}) \approx \cdots \approx S(f_k) \approx \cdots \approx S(f_{k+M})$$

are a set of $2M+1$ unbiased and uncorrelated estimators of $S(f_k)$. Therefore the average of these, namely

$$\bar{S}(f_k) \equiv \frac{1}{2M+1} \sum_{j=-M}^{M} \widehat{S}^{(p)}(f_{k-j})$$

will have

$$E\{\bar{S}(f_k)\} \approx S(f_k)$$

and

$$\operatorname{var} \bar{S}(f_k) \approx \frac{\operatorname{var}\{\widehat{S}^{(p)}(f_k)\}}{2M+1}.$$

- This concept can be extended to averaging over any discrete set of frequencies, or over a continuous range of frequencies through a convolution of the type

$$\widehat{S}^{(lw)}(f) = \int_{-1/2}^{1/2} V_m(f-\phi)\widehat{S}^{(d)}(\phi)\mathrm{d}\phi.$$

8(B)

(b) (i) unseen ⇓

$$\begin{aligned}
\mathcal{U}_m(f) &= \int_{-1/2}^{1/2} W_m(f-f')\mathcal{H}(f')\mathrm{d}f' \\
&= \int_{-1/2}^{1/2} \sum_{\tau=-(N-1)}^{(N-1)} w_{\tau,m}\mathrm{e}^{-\mathrm{i}2\pi(f-f')\tau} \left|\sum_{t=1}^{N} h_t \mathrm{e}^{-\mathrm{i}2\pi f' t}\right|^2 \mathrm{d}f' \\
&= \int_{-1/2}^{1/2} \sum_{\tau=-(N-1)}^{(N-1)} w_{\tau,m}\mathrm{e}^{-\mathrm{i}2\pi(f-f')\tau} \sum_{\tau'=-(N-1)}^{N-1} \sum_{t=1}^{N-|\tau'|} h_t h_{t+|\tau'|}\mathrm{e}^{-\mathrm{i}2\pi f'\tau'}\mathrm{d}f' \\
&= \sum_{\tau=-(N-1)}^{(N-1)} w_{\tau,m}\mathrm{e}^{-\mathrm{i}2\pi f\tau} \sum_{\tau'=-(N-1)}^{N-1} \sum_{t=1}^{N-|\tau'|} h_t h_{t+|\tau'|} \int_{-1/2}^{1/2} \mathrm{e}^{-\mathrm{i}2\pi f'(\tau'-\tau)}\mathrm{d}f'
\end{aligned}$$

Considering the integral, we have

$$\int_{-1/2}^{1/2} \mathrm{e}^{-\mathrm{i}2\pi f'(\tau'-\tau)}\mathrm{d}f' = \begin{cases} 1 & \tau = \tau' \\ 0 & \tau \neq \tau' \end{cases},$$

and it follows that

$$\mathcal{U}_m(f) = \sum_{\tau-(N-1)}^{(N-1)} w_{\tau,m} \left(\sum_{t=1}^{N-|\tau|} h_t h_{t+|\tau|} \right) \mathrm{e}^{-\mathrm{i}2\pi f\tau}.$$

$\boxed{5(\mathrm{D})}$

(ii) Using the result from (i), we have

$$\begin{aligned} E\{\widehat{S}^{(lw)}(f)\} &= \int_{-1/2}^{1/2} \mathcal{U}_m(f-\phi)S(\phi)\mathrm{d}\phi \\ &= \int_{-1/2}^{1/2} \sum_{\tau-(N-1)}^{(N-1)} w_{\tau,m} \left(\sum_{t=1}^{N-|\tau|} h_t h_{t+|\tau|} \right) \mathrm{e}^{-\mathrm{i}2\pi(f-\phi)\tau} S(\phi)\mathrm{d}\phi \\ &= \sum_{\tau-(N-1)}^{(N-1)} w_{\tau,m} \left(\sum_{t=1}^{N-|\tau|} h_t h_{t+|\tau|} \right) \mathrm{e}^{-\mathrm{i}2\pi f\tau} \int_{-1/2}^{1/2} S(\phi)\mathrm{e}^{\mathrm{i}2\pi\phi\tau}\mathrm{d}\phi \\ &= \sum_{\tau=-(N-1)}^{N-1} \left(w_{\tau,m} s_\tau \sum_{t=1}^{N-|\tau|} h_t h_{t+|\tau|} \right) \mathrm{e}^{-\mathrm{i}2\pi f\tau} \end{aligned}$$

due to $s_\tau = \int_{-1/2}^{1/2} S(\phi)\mathrm{e}^{\mathrm{i}2\pi\phi\tau}\mathrm{d}\phi$. $\boxed{4(\mathrm{D})}$

(iii) For $\{X_t\}$, $s_0 = 1$, $s_{-1} = s_1 = \frac{1}{2}$ and $s_\tau = 0$ for all $|\tau| > 1$. Therefore (with $w_{0,m} = \int_{-1/2}^{1/2} W_m(f)\mathrm{d}f = 1$),

$$\begin{aligned} E\{\widehat{S}^{(lw)}(f)\} &= \frac{1}{2}w_{-1,m}\frac{N-1}{N}\mathrm{e}^{\mathrm{i}2\pi f} + w_{0,m} + \frac{1}{2}w_{1,m}\frac{N-1}{N}\mathrm{e}^{-\mathrm{i}2\pi f} \\ &= 1 + w_{1,m}\frac{N-1}{N}\cos(2\pi f). \\ E\{\widehat{S}^{(lw)}(f)\} - S(f) &= 1 + w_{1,m}\frac{N-1}{N}\cos(2\pi f) - (1+\cos(2\pi f)) \\ &= \left(w_{1,m}\frac{N-1}{N} - 1 \right) \cos(2\pi f). \end{aligned}$$

$\boxed{3(\mathrm{D})}$