Imperial College
London

[E1.10 (Maths 1) 2009]

B.ENG. and M.ENG. EXAMINATIONS 2009

PART I : MATHEMATICS 1 (ELECTRICAL ENGINEERING)

Date Wednesday 3rd June 2009 10.00 am - 1.00 pm

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.

Answer EIGHT questions.

CALCULATORS MAY **NOT** BE USED.

A mathematical formulae sheet is provided.

[Before starting, please make sure that the paper is complete; there should be 6 pages, with a total of 10 questions. Ask the invigilator for a replacement if your copy is faulty.]

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function.

   (i) Give the formal definition of differentiability of f at the point x.

   (ii) Use this definition to calculate from first principles the derivative of the function
   $$f(x) = e^x .$$

   (iii) Using any valid method, differentiate
   $$f(x) = \frac{5x^2 e^{3x}}{3 - 2x^3} .$$
   and show that
   $$f'(x) = \frac{5x e^{3x}}{(3-2x^3)^2} p(x) ,$$
   where $p(x)$ is a polynomial to be found.

2. Evaluate the following limits:

   (i) $$\lim_{x \to 0} \frac{x^2 - 1}{x^2 + x - 2} ;$$

   (ii) $$\lim_{x \to 1} \frac{x^2 - 1}{x^2 + x - 2} ;$$

   (iii) $$\lim_{x \to \pi/2} (\sec^2 x)(1 - \sin x) ;$$

   (iv) $$\lim_{x \to \infty} x \left(\sqrt{x^2 + 4} - x \right) .$$

PLEASE TURN OVER

3. Let

$$I_r = \int \frac{1}{(1+x^2)^r}\, dx\,.$$

(i) Show that

$$I_r - I_{r-1} = -\int \frac{x^2}{(1+x^2)^r}\, dx\,.$$

(ii) Using integration by parts, show that for $r > 1$

$$\int \frac{x^2}{(1+x^2)^r}\, dx = \frac{1}{2r-2}\left(I_{r-1} - \frac{x}{(1+x^2)^{r-1}}\right)\,.$$

(iii) Hence calculate

$$\int \frac{1}{(1+x^2)^3}\, dx\,.$$

4. Let $z = x + iy$ be a complex number.

(i) Define the following terms:

(a) the modulus $|z|$ of z;

(b) the argument $\arg(z)$ of z.

(ii) Write the following complex numbers in the form $r(\cos\theta + i\sin\theta)$:

(a) $-3i$;

(b) $4-4i$;

(c) -1.

(iii) Use De Moivre's formula

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

to calculate

$$(-3 + 3i)^4\,.$$

Express your answer as a complex number of the form $x + iy$.

(iv) Find all complex numbers such that $|z-i| > |z+i|$.

Describe geometrically the set of points that satisfies this condition.

5. Consider the function

$$f(x) = \frac{2x^2 - 5x - 25}{x^2 + x - 2} ;$$

(i) Find the points where $f(x) = 0$.

(ii) Find any vertical and horizontal asymptotes.

(iii) Use (i) and (ii) to determine where $f(x)$ is positive.

(iv) Find the points where $f'(x) = 0$.

(v) Determine any local minima and maxima of f.

(vi) Sketch the graph of f.

6. (i) Consider the planes $3x + 6z = 1$ and $2x + 2y - z = 3$.

(a) Show that the planes are orthogonal.

(b) Find the equation of a plane through the origin which is perpendicular to the line of intersection of these two planes.

(ii) Find a vector parallel to the plane $2x - y - z = 4$ and orthogonal to the vector $\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$.

(iii) Show that $\boldsymbol{c} = |\,\boldsymbol{b}\,|\,\boldsymbol{a} + |\,\boldsymbol{a}\,|\,\boldsymbol{b}$ bisects the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.

7. (i) Factorise the matrix

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 3 & 3 & 9 \\ 3 & 3 & 5 \end{pmatrix}$$

into a product $L\,U$, where L and U are lower and upper triangular matrices, respectively, with ones down the main diagonal of L.

(ii) Find L^{-1} and U^{-1}, and hence A^{-1}.

(iii) Hence, or otherwise, solve

$$A\,\boldsymbol{x} = \boldsymbol{b}, \quad \text{where} \quad \boldsymbol{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

8. (i) Show that the differential equation

$$(6y - x^2 y)\,\frac{dy}{dx} + (x - x\,y^2) = 0$$

is exact.

Hence find an explicit solution satisfying $y(2) = -2$.

State the range of x for which this solution is valid.

(ii) Find the solution of

$$(x+1)\,\frac{dy}{dx} - 2y = (x+1)^3$$

satisfying $y(-3) = 1$.

9. For the differential equation

$$\frac{d^2y}{dx^2} - 3\,\frac{dy}{dx} + 2y = 3e^{2x} + k\,\sin 2x\,,$$

where k is a constant, find the solution that satisfies

$$y(0) = 1\,, \qquad \frac{dy}{dx}(0) = 2\,,$$

in the two cases:

(i) $k = 0$;

(ii) $k = 1$.

10. Show that

$$\frac{d}{dx}\,\left[\sin^{-1} x\right] = (1-x^2)^{-1/2}\,.$$

If $y(x) = \sin(\alpha \sin^{-1} x)$, where α is a constant, then show that

$$(1-x^2)^{1/2}\,\frac{d}{dx}\,\left[(1-x^2)^{1/2}\,\frac{dy}{dx}\right] = -\,\alpha^2\,y\,;$$

and hence that

$$(1-x^2)\,\frac{d^2y}{dx^2} - x\,\frac{dy}{dx} + \alpha^2 y = 0\,.$$

Use the Leibnitz formula to differentiate this equation n times, and show that

$$\frac{d^{n+2}y}{dx^{n+2}}(0) = (n^2 - \alpha^2)\,\frac{d^n y}{dx^n}(0) \qquad \text{for} \qquad n \geq 0\,.$$

Hence write down the Maclaurin expansion for $y(x)$ by stating formulae for the general even and odd terms.

Use the ratio test to find the radius of convergence of this expansion.

END OF PAPER

MATHEMATICS DEPARTMENT

MATHEMATICAL FORMULAE

1. VECTOR ALGEBRA

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1, a_2, a_3)$$

Scalar (dot) product: $\mathbf{a}\,.\,\mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

Vector (cross) product:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Scalar triple product:

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a}\,.\,\mathbf{b} \times \mathbf{c} = \mathbf{b}\,.\,\mathbf{c} \times \mathbf{a} = \mathbf{c}\,.\,\mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product: $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c}\,.\,\mathbf{a})\mathbf{b} - (\mathbf{b}\,.\,\mathbf{a})\mathbf{c}$

2. SERIES

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \ldots \quad (\alpha \text{ arbitrary, } |x| < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \ldots,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^n \frac{x^{n+1}}{(n+1)} + \ldots (-1 < x \le 1)$$

3. TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin(a+b) = \sin a \cos b + \cos a \sin b\,;$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b\,.$$

$$\cos iz = \cosh z\,; \quad \cosh iz = \cos z\,; \quad \sin iz = i \sinh z\,; \quad \sinh iz = i \sin z\,.$$

4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f\,D^n g + \binom{n}{1} Df\,D^{n-1}g + \ldots + \binom{n}{r} D^r f\,D^{n-r}g + \ldots + D^n f\,g\,.$$

(b) Taylor's expansion of $f(x)$ about $x = a$:

$$f(a+h) = f(a) + hf'(a) + h^2 f''(a)/2! + \ldots + h^n f^{(n)}(a)/n! + \epsilon_n(h)\,,$$

where $\epsilon_n(h) = h^{n+1} f^{(n+1)}(a+\theta h)/(n+1)!, \quad 0 < \theta < 1\,.$

(c) Taylor's expansion of $f(x, y)$ about (a, b):

$$f(a+h, b+k) = f(a,b) + [hf_x + kf_y]_{a,b} + 1/2!\left[h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}\right]_{a,b} + \ldots$$

(d) Partial differentiation of $f(x, y)$:

i. If $y = y(x)$, then $f = F(x)$, and $\dfrac{dF}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dx}$.

ii. If $x = x(t)$, $y = y(t)$, then $f = F(t)$, and $\dfrac{dF}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$.

iii. If $x = x(u, v)$, $y = y(u, v)$, then $f = F(u, v)$, and

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}, \quad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$

(e) Stationary points of $f(x, y)$ occur where $f_x = 0$, $f_y = 0$ simultaneously.
Let (a, b) be a stationary point: examine $D = [f_{xx}f_{yy} - (f_{xy})^2]_{a,b}$.
If $D > 0$ and $f_{xx}(a, b) < 0$, then (a, b) is a maximum;
If $D > 0$ and $f_{xx}(a, b) > 0$, then (a, b) is a minimum;
If $D < 0$ then (a, b) is a saddle-point.

(f) Differential equations:

i. The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating factor $I(x) = \exp[\int P(x)(dx)]$, so that $\dfrac{d}{dx}(Iy) = IQ$.

ii. $P(x, y)dx + Q(x, y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y$.

5. INTEGRAL CALCULUS

(a) An important substitution: $\tan(\theta/2) = t$:
$\sin\theta = 2t/(1+t^2)$, $\cos\theta = (1-t^2)/(1+t^2)$, $d\theta = 2\,dt/(1+t^2)$.

(b) Some indefinite integrals:

$$\int (a^2 - x^2)^{-1/2} dx = \sin^{-1}\left(\frac{x}{a}\right), \quad |x| < a.$$

$$\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1}\left(\frac{x}{a}\right) = \ln\left\{\frac{x}{a} + \left(1 + \frac{x^2}{a^2}\right)^{1/2}\right\}.$$

$$\int (x^2 - a^2)^{-1/2} dx = \cosh^{-1}\left(\frac{x}{a}\right) = \ln\left|\frac{x}{a} + \left(\frac{x^2}{a^2} - 1\right)^{1/2}\right|.$$

$$\int (a^2 + x^2)^{-1} dx = \left(\frac{1}{a}\right)\tan^{-1}\left(\frac{x}{a}\right).$$

6. NUMERICAL METHODS

(a) Approximate solution of an algebraic equation:

If a root of $f(x) = 0$ occurs near $x = a$, take $x_0 = a$ and
$x_{n+1} = x_n - [f(x_n)/f'(x_n)]$, $n = 0, 1, 2 \ldots$

(Newton Raphson method).

(b) Formulae for numerical integration: Write $x_n = x_0 + nh$, $y_n = y(x_n)$.

i. Trapezium rule (1-strip): $\int_{x_0}^{x_1} y(x)dx \approx (h/2)\,[y_0 + y_1]$.

ii. Simpson's rule (2-strip): $\int_{x_0}^{x_2} y(x)dx \approx (h/3)\,[y_0 + 4y_1 + y_2]$.

(c) Richardson's extrapolation method: Let $I = \int_a^b f(x)dx$ and let I_1, I_2 be two estimates of I obtained by using Simpson's rule with intervals h and $h/2$. Then, provided h is small enough,

$$I_2 + (I_2 - I_1)/15,$$

is a better estimate of I.

7. LAPLACE TRANSFORMS

Function	Transform	Function	Transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)dt$	$af(t) + bg(t)$	$aF(s) + bG(s)$
df/dt	$sF(s) - f(0)$	d^2f/dt^2	$s^2F(s) - sf(0) - f'(0)$
$e^{at}f(t)$	$F(s-a)$	$tf(t)$	$-dF(s)/ds$
$(\partial/\partial\alpha)f(t,\alpha)$	$(\partial/\partial\alpha)F(s,\alpha)$	$\int_0^t f(t)dt$	$F(s)/s$
$\int_0^t f(u)g(t-u)du$	$F(s)G(s)$		
1	$1/s$	$t^n (n = 1, 2 \ldots)$	$n!/s^{n+1}$, $(s > 0)$
e^{at}	$1/(s-a)$, $(s > a)$	$\sin\omega t$	$\omega/(s^2+\omega^2)$, $(s > 0)$
$\cos\omega t$	$s/(s^2+\omega^2)$, $(s > 0)$	$H(t-T) = \begin{cases} 0, & t < T \\ 1, & t > T \end{cases}$	e^{-sT}/s, $(s, T > 0)$

8. FOURIER SERIES

If $f(x)$ is periodic of period $2L$, then $f(x + 2L) = f(x)$, and

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \text{ where}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad n = 0, 1, 2, \ldots, \text{ and}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1, 2, 3, \ldots$$

Parseval's theorem

$$\frac{1}{L}\int_{-L}^{L} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right).$$

	EXAMINATION QUESTIONS/SOLUTIONS 2008-09 SOLUTION Page 1 of 2	Course
Question C1	DIFFERENTIATION	Marks & seen/unseen
Parts	(a) f is differentiable at x if $\lim_{\delta \to 0} \frac{f(x+\delta)-f(x)}{\delta}$ exists.	4
	(b) For $f(x)=e^x$ we have $\frac{e^{x+\delta}-e^x}{\delta} = \frac{e^x(e^\delta-1)}{\delta} = e^x \frac{(1+\delta+\frac{\delta^2}{2!}\cdots -1)}{\delta}$ $= e^x(1+\delta/2! + \frac{\delta^2}{3!}\cdots)$	4
	and so $\lim_{\delta\to 0} \frac{e^{x+\delta}-e^x}{\delta} = \lim_{\delta \to 0} e^x(1+\delta/2!+\delta^2/3!\cdots) = e^x.$	4
	(c) Quotient rule $f'(x) = \frac{(5x^2e^{3x})'(3-2x^3)-(3-2x^3)'(5x^2e^{3x})}{(3-2x^3)^2}$	3
	Product rule $(5x^2e^{3x})' = (5x^2)'e^{3x} + 5x^2(e^{3x})'$ $= 10x e^{3x} + 15x^2 \cdot e^{3x}$ $(3-2x^3)' = -6x^2$	3
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	EXAMINATION QUESTIONS/SOLUTIONS 2008-09	Course
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Parts

$$f'(x) = \frac{(10xe^{3x} + 15x^2e^{3x})(3-2x^3)}{(3-2x^3)^2}$$

$$+ \frac{6x^2(5x^2e^{3x})}{(3-2x^3)^2}$$

$$= e^{3x}\frac{[(10x+15x^2)(3-2x^3)+30x^4]}{(3-2x^3)^2}$$

$$= e^{3x}\frac{[-30x^5+10x^4+45x^2+30x]}{(3-2x^3)^2}$$

$$= \frac{5xe^{3x}}{(3-2x^3)^2}\underbrace{\{-6x^4+2x^3+9x+6\}}_{p(x)}$$

2

Total 20

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EXAMINATION QUESTIONS/SOLUTIONS 2008-09

Course: FIRST YEAR ENGINEERING

Question C2

Marks &

(i) $\lim_{x\to 0} \frac{x^2-1}{x^2+x-2} = \frac{0^2-1}{0^2+0-2} = \frac{1}{2}$ // — 5

(ii) $\lim_{x\to 1} \frac{x^2-1}{x^2+x-2} = \lim_{x\to 1} \frac{(x+1)\cancel{(x-1)}}{\cancel{(x-1)}(x+2)} = \frac{2}{3}$ // — 5

[Alternatively: "$\frac{0}{0}$" limit, so use L'Hôpital's rule
$\lim_{x\to 1} \frac{x^2-1}{x^2+x-2} = \lim_{x\to 1} \frac{2x}{2x+1} = \frac{2}{3}$.]

(iii) $\lim_{x\to\pi/2} \sec^2 x(1-\sin x) = \lim_{y\to 0} \operatorname{cosec}^2 y\,(1-\cos y)$ (where $y = \pi/2 - x$)

$= \lim_{y\to 0} \frac{1-\{1-y^2/2+O(y^4)\}}{\{y+O(y^3)\}^2}$

$= \lim_{y\to 0} \frac{y^2/2+O(y^4)}{y^2+O(y^4)}$

$= \underline{\underline{\tfrac{1}{2}}}$ (accept any valid method) — 5

[Alternatively, by L'Hôpital:
$\lim_{x\to\frac{\pi}{2}} \frac{(1-\sin x)}{\cos^2 x}$
$= \lim_{x\to\frac{\pi}{2}} \frac{-\cancel{\cos x}}{-2\cancel{\cos x}\sin x}$
$= \frac{1}{2}$ //]

(iv) $\lim_{x\to\infty} x\left(\sqrt{x^2+4}-x\right) = \lim_{x\to\infty} \frac{x\,(x^2+4-x^2)}{\sqrt{x^2+4}+x}$

$= \lim_{x\to\infty} \frac{4x}{\sqrt{x^2+4}+x}$

$= \lim_{x\to\infty} \frac{4}{\sqrt{1+4/x^2}+1}$

$= \underline{\underline{2}}$ (accept any valid method) — 5

[One possible alternative
Write as $x^2\left(\left(1+\frac{4}{x^2}\right)^{1/2}-1\right) \simeq x^2\left(1+\frac{2}{x^2}+\cdots-1\right)$
$\to 2$ as $x\to\infty$]

Total 20

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	EXAMINATION QUESTIONS/SOLUTIONS 2008-09	Course
	SOLUTION (page 1 of 2)	
Question C3	INTEGRATION	Marks & seen/unseen

Parts

(i) Use recursive formula.

$$I_r = \int \frac{1}{(1+x^2)^r}\,dx$$

$$= \int \frac{1}{(1+x^2)^{r-1}} - \frac{x^2}{(1+x^2)^r}\,dx$$

$$= I_{r-1} - \int \frac{x^2}{(1+x^2)^r}\,dx$$

5

(ii) Letting $u(x) = x$, $v(x) = \frac{1}{(1+x^2)^{r-1}}$

$v'(x) = \frac{(1-r)2x}{(1+x^2)^r}$ gives

$$\int \frac{x^2}{(1+x^2)^r}\,dx = \frac{1}{2(1-r)} \int u(x)\,v'(x)\,dx$$

$$= \frac{1}{2(1-r)} \left[u(x)v(x) - \int v(x)\,u'(x)\,dx \right]$$

4

$$= \frac{1}{2(1-r)} \left[\frac{x}{(1+x^2)^{r-1}} - \int \frac{1}{(1+x^2)^{r-1}}\,dx \right]$$

$$= \frac{1}{2(1-r)} \left[\frac{x}{(1+x^2)^{r-1}} - I_{r-1} \right]$$

4

8 for part (ii)

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EXAMINATION QUESTIONS/SOLUTIONS 2008-09

Course

SOLUTION (page 2 of 2)

Question C3

INTEGRATION CONTINUED

Marks & seen/unseen

Parts

Substituting back

(iii)

$$I_r = I_{r-1} + \frac{1}{2(1-r)} I_{r-1} - \frac{x}{2(1-r)(1+x^2)^{r-1}}$$

$$= \frac{2r-3}{2r-2} I_{r-1} - \frac{x}{2(1-r)(1+x^2)^{r-1}}$$

2

$$r = 1 \rightarrow \int \frac{1}{1+x^2}\,dx = \tan^{-1} x + c$$

1

$$r = 2 \rightarrow I_2 = \frac{1}{2}\tan^{-1} x + \frac{x}{2(1+x^2)}$$

2

$$r = 3 \rightarrow I_3 = \frac{3}{4} I_2 + \frac{x}{4(1+x^2)^2}$$

$$= \boxed{\frac{3}{8}\tan^{-1} x + \frac{3x}{8(1+x^2)} + \frac{x}{4(1+x^2)^2} + c}$$

2

7 for part (iii)

Total 20

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EXAMINATION ~~QUESTION~~S/SOLUTIONS 2007-08

Course

SOLUTIONS

Question C4

COMPLEX NUMBERS

Marks & seen/unseen

Parts

(i) a) $|z| = \sqrt{x^2+y^2}$ — 2

b) $Arg(z)$ = direction of the vector from the origin to z, measured in radians, anticlockwise from the positive horizontal axis. — 2

4 for (i)

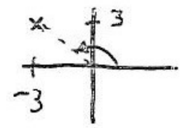
(ii) a) $z = -3i$, $|z| =$, $Arg(z) = -\pi/2$, $z \quad (\cos(-\pi/2) + i\sin(-\pi/2))$ — 2

b) $z = 4-4i$ $|z| = \sqrt{32}$, $Arg(z) = -\pi/4$, $z = \sqrt{32}\left(\cos\left(\frac{-\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$ — 2

c) $z = -1$, $|z| = 1$, $Arg(z) = \pi$, $z = \cos\pi + i\sin\pi$ — 2

6 for (ii)

(iii) $z = -3+3i$, $|z| = \sqrt{18}$, $Arg(z) = \frac{3\pi}{4}$



$z = \sqrt{18}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$ — 2

$z^4 = \sqrt{18}^4 (\cos 3\pi + i\sin 3\pi)$ — 3

$= 324(\cos\pi + i\sin\pi)$

$= -324$

[N.B. EE students will not have a calculator so allow them to leave it as $-(18)^2$ (!)]

5 for (iii)

(iv) Write $z = x+iy$. Then

$z - i = x + (y-1)i$

$z + i = x + (y+1)i$

So

$|z-i| = \sqrt{x^2+(y-1)^2}$ — 2

$|z+i| = \sqrt{x^2+(y+1)^2}$

So

$|z-i| > |z+i| \iff (y-1)^2 > (y+1)^2 \iff y < 0$ — 3

Thus $|z-i| > |z+i|$ for all points in lower half plane.

5 for (iv)

Total 20

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Page number

EEI(1) 5

EXAMINATION QUESTIONS/SOLUTIONS 2008-09

Course: YEAR 1

~~Question~~ Answer: C5

Marks & seen/unseen

Parts

$$f(x) = \frac{(2x+5)(x-5)}{(x+2)(x-1)}$$ 2

(i) $x = -\frac{5}{2}$ and 5 2

(ii) vertical asymptotes $x = -2$ and 1 2

horizontal asymptote $f(x) \to 2$ as $x \to \pm\infty$ 2

(iii) + − | + | − + 2

$-\frac{5}{2}$ -2 1 5

(iv) $$f'(x) = \frac{(4x-5)(x^2+x-2) - (2x+1)(2x^2-5x-25)}{(x^2+x-2)^2}$$

$f'(x) = 0 \iff (4x-5)(x^2+x-2) = (2x+1)(2x^2-5x-25)$

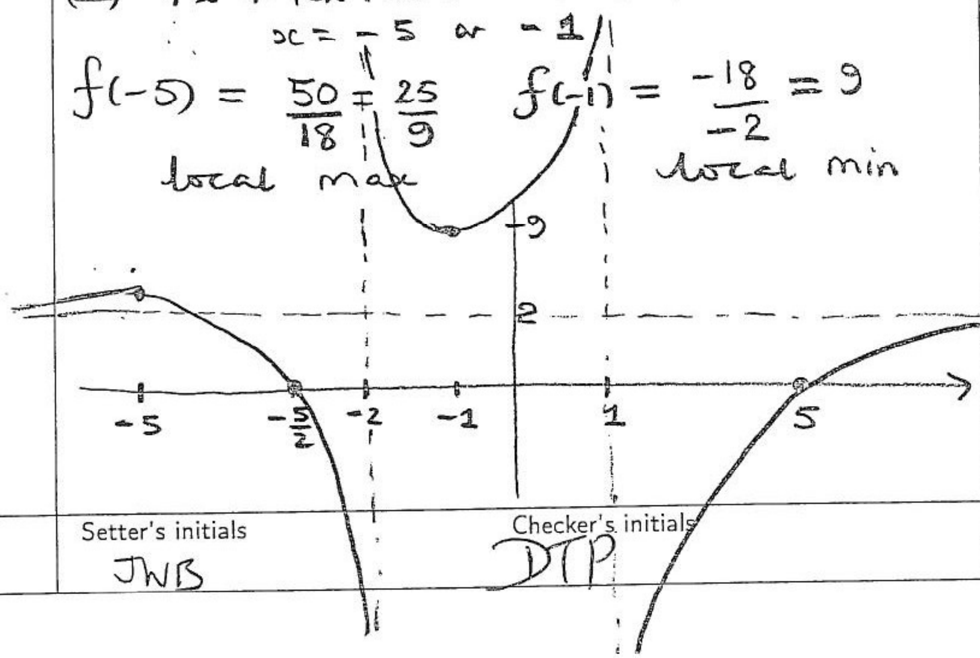
$\iff 4x^3 + 4x^2 - 8x - 5x^2 - 5x + 10 = 4x^3 - 10x^2 - 50x + 2x^2 - 5x - 25$

$\iff 7x^2 + 42x + 35 = 0 \iff x^2 + 6x + 5 = 0$ 3

$x = -5$ or -1

$f(-5) = \frac{50}{18} = \frac{25}{9}$ $\quad f(-1) = \frac{-18}{-2} = 9$ 2

local max $\quad$ local min



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Parts

(i) (a) Normals to the planes

$\underline{n}_1 = (3, 0, 6)$ — 1

$\underline{n}_2 = (2, 2, -1)$ — 1

$\underline{n}_1 \cdot \underline{n}_2 = (3,0,6)\cdot(2,2,-1) = 0$ — 2

$\Rightarrow \underline{n}_1 \perp \underline{n}_2$ planes orthogonal.

(b) The vector $\underline{n}_1 \times \underline{n}_2$ is in the direction of the line of intersection

$$\underline{n}_1 \times \underline{n}_2 = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & 0 & 6 \\ 2 & 2 & -1 \end{vmatrix} = (-12, 15, 6)$$ — 2

Normal to the required plane is in the direction of $\underline{n}_1 \times \underline{n}_2$.

Plane is

$$-12x + 15y + 6z = K \quad , k \text{ const.}$$

Plane passes through $(0,0,0) \Rightarrow K = 0$

$$-12x + 15y + 6z = 0$$ — 2

is the required equation.

8 for (i)

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Parts

(ii) Let the required vector be $\underline{a} = (a_1, a_2, a_3)$

$\underline{n} = (2, -1, -1)$ is normal to the plane, so $\underline{a}$ is parallel to the plane if $\underline{a} \cdot \underline{n} = 0$

$\Rightarrow \quad 2a_1 - a_2 - a_3 = 0 \quad -①$ — 2

$\underline{a}$ is orthogonal to $\underline{i} + \underline{j} + \underline{k}$ if

$a_1 + a_2 + a_3 = 0 \quad -②$ — 2

Solve ① & ② to find

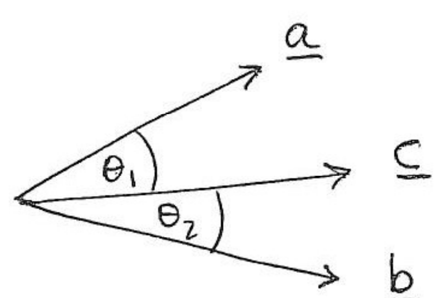
$a_1 = 0$

$a_2 = -a_3$

$\Rightarrow \underline{a} = t\,\underline{j} - t\,\underline{k}$ for any real t. — 2

6 for (ii)

(iii)



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Parts

(iii) cont.

$$\underline{a} \cdot \underline{c} = |\underline{a}|\,|\underline{c}| \cos\theta_1 \qquad 1$$

$$\underline{b} \cdot \underline{c} = |\underline{b}|\,|\underline{c}| \cos\theta_2 \qquad 1$$

So $\theta_1 = \theta_2$ if $\dfrac{\underline{a}\cdot\underline{c}}{|\underline{a}|} = \dfrac{\underline{b}\cdot\underline{c}}{|\underline{b}|}$ 2

Verify this for $\underline{c} = |\underline{b}|\,\underline{a} + |\underline{a}|\,\underline{b}$

$$\frac{\underline{a}\cdot\underline{c}}{|\underline{a}|} = \frac{|\underline{b}|\,|\underline{a}|^2 + |\underline{a}|\,\underline{a}\cdot\underline{b}}{|\underline{a}|}$$

$$= |\underline{a}|\,|\underline{b}| + \underline{a}\cdot\underline{b} \qquad 1$$

$$\frac{\underline{b}\cdot\underline{c}}{|\underline{b}|} = \frac{|\underline{b}|^2|\underline{a}| + |\underline{b}|\,\underline{a}\cdot\underline{b}}{|\underline{b}|}$$

$$= |\underline{a}|\,|\underline{b}| + \underline{a}\cdot\underline{b} \qquad 1$$

6 for (iii)

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Parts

(i)

$$\begin{pmatrix} 2 & -1 & 1 \\ 3 & 3 & 9 \\ 3 & 3 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}$$

<u>1st row</u>

$2 = u_{11}$ $\qquad$ $\underline{u_{11} = 2}$

$-1 = u_{12}$ $\qquad$ $\underline{u_{12} = -1}$

$1 = u_{13}$ $\qquad$ $\underline{u_{13} = 1}$

<u>1st column</u>

$3 = \ell_{21} u_{11} \Rightarrow \underline{\ell_{21} = 3/2}$

$3 = \ell_{31} u_{11} \Rightarrow \underline{\ell_{31} = 3/2}$

<u>Element a_{22}</u>: $\ell_{21} u_{12} + u_{22} = 3 \Rightarrow \underline{u_{22} = 9/2}$

<u>Element a_{23}</u>: $9 = \ell_{21} u_{13} + u_{23} \Rightarrow \underline{u_{23} = \frac{15}{2}}$

<u>Element a_{32}</u>: $3 = \ell_{31} u_{12} + \ell_{32} u_{22} \Rightarrow \underline{\ell_{32} = 1}$

<u>Element a_{33}</u>: $5 = \ell_{31} u_{13} + \ell_{32} u_{23} + u_{33} \Rightarrow$

$\underline{u_{33} = -4}$

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Parts

(i) cont.

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 3/2 & 1 & 0 \\ 3/2 & 1 & 1 \end{pmatrix} \qquad U = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 9/2 & 15/2 \\ 0 & 0 & -4 \end{pmatrix}$$

(Any valid method to get L & U is acceptable)

8 for part (i)

(ii)

$$L^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ l_1 & 1 & 0 \\ l_2 & l_3 & 1 \end{pmatrix}. \qquad LL^{-1} = I$$

$$\Rightarrow \quad \frac{3}{2} + l_1 = 0 \Rightarrow l_1 = -3/2$$

$$\frac{3}{2} + l_1 + l_2 = 0 \Rightarrow l_2 = 0$$

$$1 + l_3 = 0 \Rightarrow l_3 = -1$$

$$L^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -3/2 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

3

$$U^{-1} = \begin{pmatrix} 1/2 & u_1 & u_2 \\ 0 & 2/9 & u_3 \\ 0 & 0 & -1/4 \end{pmatrix}$$

Since $UU^{-1} = I$ we get

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Parts

(ii) cont

$$\left.\begin{aligned} 2u_1 - \tfrac{2}{9} &= 0 \\ 2u_2 - u_3 - \tfrac{1}{4} &= 0 \\ \tfrac{9}{2}u_3 - \tfrac{15}{8} &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} u_1 &= 1/9 \\ u_2 &= 1/3 \\ u_3 &= 5/12 \end{aligned}$$

$$U^{-1} = \begin{pmatrix} 1/2 & 1/9 & 1/3 \\ 0 & 2/9 & 5/12 \\ 0 & 0 & -1/4 \end{pmatrix}$$ 3

$$A = LU \Rightarrow A^{-1} = U^{-1}L^{-1}$$

$$\Rightarrow A^{-1} = \begin{pmatrix} 1/3 & -2/9 & 1/3 \\ -1/3 & -7/36 & 5/12 \\ 0 & 1/4 & -1/4 \end{pmatrix}$$ 2

[8 for part (ii)]

(iii) $$\underline{x} = A^{-1}\underline{b} = \begin{pmatrix} 1/3 & -2/9 & 1/3 \\ -1/3 & -7/36 & 5/12 \\ 0 & 1/4 & -1/4 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4/9 \\ -1/9 \\ 0 \end{pmatrix}$$ 4

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Answer	EXAMINATION QUESTIONS/SOLUTIONS 2008-09	Course YEAR 1
Question C9		Marks & seen/unseen

Parts

(i) $Q(x,y)\frac{dy}{dx} + P(x,y) = 0$ exact if and only if $P_y = Q_x$

$P = x - xy^2,\quad Q = 6y - x^2y \Rightarrow P_y = -2xy = Q_x$ — 4

$\therefore$ exact, solution $u(x,y)=0$ s.t. $u_x = P,\ u_y = Q$

$u_x = x - xy^2 \Rightarrow u(x,y) = \frac{1}{2}x^2(1-y^2) + g(y)$

$u_y = -x^2y + g'(y) = 6y - x^2y \Rightarrow g'(y) = 6y$.

$\Rightarrow g(y) = 3y^2 + c \Rightarrow u(x,y) = \frac{1}{2}x^2(1-y^2) + 3y^2 + c$ — 3

$y(2) = -2 \Rightarrow 2(-3) + 12 + c = 0 \Rightarrow c = -6$

$\therefore x^2(1-y^2) + 6y^2 = 12 \Rightarrow y(x) = -\left(\frac{12-x^2}{6-x^2}\right)^{1/2}$ — 2

valid if $x^2 < 6$ i.e. $-\sqrt{6} < x < \sqrt{6}$. — 2

11 for (i)

(ii) $y' - \frac{2}{x+1}y = (x+1)^2$

I.F. $e^{-\int\frac{2}{x+1}dx} = e^{-2\ln(x+1)} = (x+1)^{-2}$ — 4

$\therefore (x+1)^{-2}y' - 2(x+1)^{-3} = 1$

$\Rightarrow ((x+1)^{-2}y)' = 1$ — 2

$\Rightarrow (x+1)^{-2}y = x + c$

$\Rightarrow y(x) = (x+c)(x+1)^2$ — 2

$y(-3) = 4(c-3) = 1 \Rightarrow c = \frac{13}{4}$

$y(x) = \left(x + \frac{13}{4}\right)(x+1)^2$. — 1

9 for (ii)

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EXAMINATION QUESTIONS/SOLUTIONS 2008-09	Course
Answer	YEAR 1
~~Question~~ C10	Marks & seen/unseen

Parts

(i) $\lambda^2 - 3\lambda + 2 = 0 \Rightarrow (\lambda-2)(\lambda-1) = 0$

$\Rightarrow y_{CF}(x) = Ae^{2x} + Be^{x}$ — 3

$f_1(x) = 3e^{2x} \Rightarrow y_1(x) = Cxe^{2x}$ — 2

$y_1'(x) = C(1+2x)e^{2x}, \quad y_1''(x) = C(4+4x)e^{2x}$

$Ce^{2x}\left[(4+4x) - 3(1+2x) + 2x\right] = 3e^{2x} \Rightarrow C = 3$

$\therefore$ (i) $y(x) = Ae^{2x} + Be^{x} + 3xe^{2x}$ — 2

$y'(x) = 2Ae^{2x} + Be^{x} + 3(1+2x)e^{2x}$

$y(0) = A+B = 1, \quad y'(0) = 2A+B+3 = 2 \Rightarrow \begin{matrix} A=-2 \\ B=3 \end{matrix}$

$\Rightarrow y(x) = 3e^{x} + (3x-2)e^{2x}$. ← Answer to (i) — 2

9 for part (i)

(ii) $f_2(x) = \sin 2x \Rightarrow y_2(x) = D\sin 2x + E\cos 2x$ — 3

$y_2'(x) = 2D\cos 2x - 2E\sin 2x, \quad y_2''(x) = -4y_2(x) \Rightarrow$

$(-2D+6E)\sin 2x + (-2E-6D)\cos 2x = \sin 2x$

$\Rightarrow -2D+6E = 1, \quad E = -3D \Rightarrow \begin{matrix} D = -\frac{1}{20} \\ E = \frac{3}{20} \end{matrix}$ — 2

$\therefore$ (ii)

$y(x) = (A+3x)e^{2x} + Be^{x} + \frac{1}{20}\left[3\cos 2x - \sin 2x\right]$ — 2

$y'(x) = (2A+3+6x)e^{2x} + Be^{x} - \frac{1}{10}\left[3\sin 2x + \cos 2x\right]$

$y(0) = A + B + \frac{3}{20} = 1$

$y'(0) = (2A+3) + B - \frac{1}{10} = 2$

$\Big\} \quad A+B = \frac{17}{20}, \quad 2A+B = -\frac{9}{10}$

$\Rightarrow A = -\frac{7}{4} \quad B = \frac{13}{5}$

$\Rightarrow y(x) = \left(3x - \frac{7}{4}\right)e^{2x} + \frac{13}{5}e^{x} + \frac{1}{20}\left[3\cos 2x - \sin 2x\right]$ — 4

11 for part (ii)

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	EXAMINATION QUESTIONS/SOLUTIONS 2008-09	Course Year 1
~~Question~~ Answer C11		Marks & seen/unseen
Parts	$z = \sin^{-1} x \Rightarrow \sin z = x \Rightarrow$	
	$\frac{d}{dz}(\sin z)\frac{dz}{dx} = \frac{dx}{dx} \Rightarrow \frac{dz}{dx} = \frac{1}{\cos z} = (1-x^2)^{-1/2}.$	3
	$y(x) = \sin(\alpha \sin^{-1} x) \Rightarrow$	
	$\frac{dy}{dx} = \alpha \cos(\alpha \sin^{-1} x)(1-x^2)^{-1/2} \Rightarrow$	2
	$\frac{d}{dx}\left[(1-x^2)^{1/2}\frac{dy}{dx}\right] = -\alpha^2 \sin(\alpha \sin^{-1} x)(1-x^2)^{-1/2}$	
	$\therefore (1-x^2)^{1/2}\frac{d}{dx}\left[(1-x^2)^{1/2}\frac{dy}{dx}\right] = -\alpha^2 y$	3
	$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + \alpha^2 y = 0.$	1
	$(1-x^2)y^{(n+2)}(x) + n(-2x)y^{(n+1)}(x) + \frac{n(n-1)}{2}(-2)y^{(n)}(x)$	
	$-xy^{(n+1)}(x) - ny^{(n)}(x) + \alpha^2 y^{(n)}(x) = 0$	
	$\Rightarrow y^{(n+2)}(0) = (n^2-\alpha^2)y^{(n)}(0)$ for $n \geq 0$.	3
	$y(0) = 0, \; y'(0) = \alpha \Rightarrow$	1
	$y^{(2r)}(0) = 0$ for all integers $r \geq 0$	1
	$y^{(2r+1)}(0) = ((2r-1)^2 - \alpha^2)y^{(2r-1)}(0)$	
	$= \alpha \prod_{k=1}^{r}((2k-1)^2 - \alpha^2)$ for all integers $r \geq 1$	2
	[N.B. Not all depts may have seen this notation]	
	$y(x) = a_1 x + a_3 x^3 + \ldots a_{2r+1} x^{2r+1} + \ldots$	
	$a_1 = \alpha, \quad a_{2r+1} = \alpha \prod_{k=1}^{r}((2k-1)^2 - \alpha^2)/(2r+1)!$, $r \geq 1$	2
	Radius of convergence $R^2 = \lim_{r\to\infty}\left\lvert\frac{a_{2r-1}}{a_{2r+1}}\right\rvert = 1$	2
	$\Rightarrow \lvert x \rvert < 1$.	
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