UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 2000

BEng Honours Degree in Computing Part III
BEng Honours Degree in Mathematics and Computer Science Part III
MEng Honours Degree in Mathematics and Computer Science Part III
MSc in Advanced Computing
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the Associateship of the City and Guilds of London Institute
This paper is also taken for the relevant examinations for the Associateship of the Royal College of Science

PAPER C378

MATHEMATICAL STRUCTURES IN COMPUTER SCIENCE

Monday 15 May 2000, 14:30
Duration: 120 minutes

Answer THREE questions

Paper contains 4 questions

1a i) Define the terms *lattice*, *distributive lattice* (terms to do with partial order can be assumed). From your definition give an outline proof that any chain (totally ordered set) is distributive.

ii) Give Hasse diagrams for the non-distributive lattices L for which the cardinality |L| is least. Explain briefly why all other non-distributive lattices have greater cardinality than those you have listed.

iii) Let L be a lattice having a least element and a greatest element. Define *Boolean complement*, with respect to an element of L. Prove that, if L is distributive, each element of L has at most one Boolean complement.

b i) Define: *homomorphism* of lattices.

An element c of a lattice will be called *irreducible* if c cannot be expressed as a join of two elements distinct from c. Let B be the (usual) two-element lattice $\{0,1\}$.

ii) Show that, if L is a finite lattice and $h: L \to B$ an onto homomorphism, then there is a *least* element b of L such that $h(b) = 1$; also show that this least element b is irreducible.

iii) Hence or otherwise determine the number of onto homomorphisms from $\wp(S_n)$ to B, where S_n is a set with n elements. ($\wp(S_n)$ is the power set of S_n, viewed as a lattice in the usual way.)

2a What, briefly, is meant by (i) the *term algebra* T_Σ for a given signature Σ, and (ii) *initiality* of a given algebra A of signature Σ ? State, with an indication of the main steps (only) of the proof, the initiality theorem for term algebras.

b If x,y are elements of a monoid M, what is meant by saying that x is a *left factor* of y? Show that, in any monoid, the relation "left factor" is a pre-order. What is this pre-order in the case of the monoids $(\mathbb{N}, \times)$ and $(A^*, \circ)$ (that is, the finite words in some alphabet A, with concatenation)? Are there any monoids for which this pre-order is not a partial order?

c Let S be a set of finite words in some fixed alphabet A. We say that S is *prefix-free* if x is not a prefix of y for any two distinct words x,y of S. Also we say that S is a *code* if, for any $x_1,\ldots,x_m, y_1,\ldots,y_n \in S$ such that

$$x_1 \ldots x_m = y_1 \ldots y_n$$

we have $m = n$ and $x_1 = y_1, x_2 = y_2, \ldots, x_m = y_m$ (informally, this says that any word of A^* can be "decoded", or parsed, into words of S in at most one way). Show that if S is prefix-free then S is a code. Deduce that, with A as $\{a,b\}$, the set $\{b,abb,aa,aba\}$ is a code. Give an example of a code having at least four words that is *not* prefix-free, explaining why it is a code. (Hint: consider *suffixes*.)

3a Define *hereditary set system* and *matroid*.
Let G be a graph. State with reasons whether each of the following collections of sets is a matroid (for arbitrary G):

i) The cliques of G. (A *clique* is a set of mutually adjacent vertices. The empty set and singletons are considered as cliques.)

ii) The sets S of vertices of G such that $|S| \leq 3$.

iii) The matchings of G. (A *matching* is a set of edges, no two of which have a vertex in common.)

b Write out (in pseudo-code form) a general "greedy algorithm" for finding a maximum weight set in a (weighted) hereditary set system (E,I).

c State a necessary and sufficient condition on (E,I) for your algorithm to succeed. Explain briefly how these ideas can be used to justify a suitable form of Kruskal's algorithm.

d Let the bipartite graph G have bipartition $\{A,B\}$. (That is, $\{A,B\}$ is a partition of the vertices, and every edge has one vertex in A and the other vertex in B.) It is known that, for arbitrary such G, the subsets of A which have a matching (that is, can be matched with subsets of B) form a matroid. Does this provide the basis for a greedy algorithm for finding a maximum (greatest cardinality) matching in G? Explain (preferably with the aid of an example).

4a i) Explain briefly how an arbitrary monoid may be viewed as a category.

ii) Define the terms *initial* and *terminal*, as applied to objects of a category. Is it possible for a monoid, when considered as a category, to have a terminal object? Explain.

iii) Show that, for any $k > 0$, there is a directed graph G with k vertices such that the path category Path(G) has an initial object. Show also that, if $k > 1$, Path(G) cannot have both an initial and a terminal object.

b i) Let C be a category, and X and Y two objects in it. What is meant by a (binary) *product* $X \times Y$, and by a *coproduct* $X + Y$? Also, what is meant by saying that X and Y are *isomorphic*?

ii) Let C be a category in which any two objects X,Y have both a product $X \times Y$ and a coproduct $X + Y$. Show how to define, for any objects X,Y,Z a morphism:

$$m_{XYZ}\colon (X \times Y) + (X \times Z) \to X \times (Y + Z)$$

iii) Give an example of a category C and objects X,Y,Z of C for which m_{XYZ} is *not* an isomorphism.

