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IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2006

MSc and EEE PART IV: MEng and ACGI

LINEAR OPTIMAL CONTROL

Wednesday, 26 April 10:00 am

Time allowed: 3:00 hours

There are SIX questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

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LINEAR OPTIMAL CONTROL

1. Consider the system

$$\dot{x} = Ax + Bu = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

a) Show that the system is controllable. [4 marks]

b) Assume u_2=0. Show that the system with input u_1 is not controllable and not stabilizable. [4 marks]

c) Determine a state feedback control law

$$u = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \end{bmatrix} x$$

with $K_{22} = K_{23} = K_{24} = 0$ such that the closed-loop system has all eigenvalues equal to -1. Show that there are infinitely many selections of the gains K_{11}, K_{12}, K_{13}, K_{14}, K_{21} which achieve this objective. [8 marks]

d) Consider the feedback law

$$u_1 = v_1 \qquad u_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} x.$$

Show that the system with input v_1 is controllable. [4 marks]

2. Consider the system

$$\dot{x} = Ax, \qquad y = Cx,$$

with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -\alpha \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

a) Show that the system is observable for any real α. [2 marks]

b) Design an asymptotic observer for the system. Select the output injection gain L such that the matrix $A - LC$ has two eigenvalues equal to -3. [4 marks]

c) Suppose that one can measure $y(t)$ and a delayed copy of $y(t)$ given by $y(t-\tau)$, with $\tau > 0$. Assume (for simplicity) that $\alpha \neq 0$.

i) For $t \geq \tau$, express the vector

$$Y(t) = \begin{bmatrix} y(t) \\ y(t-\tau) \end{bmatrix}$$

from $x(0)$. [6 marks]

ii) Show that the relation determined in part c)i) can be used, for any $\tau > 0$, to compute $x(0)$ as a function of $Y(t)$, where $t \geq \tau$. [4 marks]

iii) Argue that the relation determined in part c)i) can be used to determine $x(t)$ from $Y(t)$, for $t \geq \tau$, exactly. [4 marks]

3. Consider the system

$$J\ddot{\theta} + \beta\dot{\theta} = Ju.$$

a) Let $x = (x_1, x_2)'$ with $x_1 = \theta$ and $x_2 = \dot{\theta}$. Write the equations of the system in the standard state space form

$$\dot{x} = Ax + Bu.$$

[2 marks]

b) The goal of the control is to drive the state x_1 to a reference value $\bar{x}_1$ and x_2 to zero, while minimizing the cost

$$J = \int_0^{\infty} ((x_1(\tau) - \bar{x}_1)^2 + ru^2(\tau))d\tau.$$

Show that this tracking problem can be transformed into a standard LQR problem. Write explicitly this LQR problem. [6 marks]

c) Write the algebraic Riccati equation associated with the LQR problem in part b). [2 marks]

d) For $\beta = 1$ and $J = 1$, find the positive definite solution of the algebraic Riccati equation determined in part c). [8 marks]

e) Let $\beta = 1$ and $J = 1$. Write the optimal control law, and the optimal closed-loop system, for the problem in part b). Show that the optimal closed-loop system is stable for any $r > 0$. [2 marks]

4. Consider a cart of unity mass moving along a straight line without friction. Suppose that at $t = 0$ its position is $s(0)$ and its velocity is $\dot{s}(0)$. In the time interval $[0, T]$, with T known, we want to apply a force u to minimize the cost

$$J = cs^2(T) + \int_0^T u^2(\tau)d\tau,$$

with $c \geq 0$.

a) Let $x = (x_1, x_2)$ with $x_1 = s$ and $x_2 = \dot{s}$ and determine matrices Q, R, M, A and B such that

$$J = \int_0^T [x(\tau)'Qx(\tau) + Ru^2(\tau)]d\tau + x(T)'Mx(T)$$

and

$$\dot{x} = Ax + Bu.$$

[2 marks]

b) Write the Hamiltonian matrix H and the differential Riccati equation associated with the considered optimal control problem. [4 marks]

c) The solution of the differential Riccati equation associated with the problem can be computed by integrating the system

$$\begin{bmatrix} \dot{X} \\ \dot{Y} \end{bmatrix} = H \begin{bmatrix} X \\ Y \end{bmatrix},$$

with appropriate final conditions $X(T)$ and $Y(T)$.

i) Choose $X(T)$ and $Y(T)$. [2 marks]

ii) Determine $X(t)$ and $Y(t)$. (Hint: use the fact that $H^4 = 0$.) [6 marks]

iii) Determine the solution $P(t)$ of the differential Riccati equation. [4 marks]

d) Determine the optimal control law. [2 marks]

5. Consider the problem of determining the optimal investment plan for a production unit. Denoting the rate of investment by u, the production level is described by

$$\dot{x} = -\alpha x + u$$

with $\alpha > 0$ and $x(0) > 0$, and the index to maximize is

$$J = \beta x(T) + \int_0^T (x(\tau) - u(\tau))d\tau$$

with $\beta > 0$ and $T > 0$ known.

Suppose $0 \leq u \leq \bar{u}$.

a) Write the necessary conditions of optimality for normal extremals. [4 marks]

b) Write the optimal control as a function of the optimal costate. [2 marks]

c) Integrate the differential equations of the costate with $\lambda^\star(T) = -\beta$. [2 marks]

d) Determine the optimal control as a function of t.

i) Show that if α and β are such that

$$D(t) = e^{\alpha(t-T)}(1/\alpha - \beta) - 1/\alpha + 1 \neq 0 \tag{5.1}$$

for all $t \in [0, T]$ then the optimal control law is constant. [4 marks]

ii) Show that $D(t)$ in equation (5.1) can change sign only once. If $D(t)$ changes sign at $t = t_s$ the optimal control has a jump at t_s. Compute t_s as a function of α and β. Show that if $D(t)$ changes sign and $\beta > 1$ then the optimal control is

$$u^\star = \begin{cases} 0 & \text{for } t \in [0, t_s) \\ \bar{u} & \text{for } t \in [t_s, T] \end{cases}$$

[8 marks]

6. Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u \end{aligned}$$

and the cost to minimize

$$J = -x_1(T) + \frac{1}{2}\int_0^T u^2(\tau)d\tau.$$

The cost represents a tradeoff between the maximization of $x_1(T)$ and the minimization of the control effort.

a) Write the necessary conditions of optimality for normal extremals. (Hint: use the condition $\frac{\partial H}{\partial u} = 0$ for minimizing H with respect to u.) [4 marks]

b) Write the optimal control as a function of the optimal costate. [2 marks]

c) Integrate the differential equations of the costate. Note that the optimal costate should be such that $\lambda_1^\star(T) = -1$ and $\lambda_2^\star(T) = 0$. [4 marks]

d) Determine the optimal control as a function of t. [2 marks]

e) Integrate the optimal state equations with initial conditions $x_1^\star(0) = 0$ and $x_2^\star(0) = 0$. Determine $x_1^\star(T)$. [4 marks]

f) Compute the optimal cost $J^\star$ corresponding to the initial conditions $x_1^\star(0) = 0$ and $x_2^\star(0) = 0$. [4 marks]

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Linear Optimal Control - Model answers 2006

Question 1

a) Consider the following submatrix of the controllability matrix

$$\bar{\mathcal{C}} = [B, AB, A^2B] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and note that the first, second, third and fifth columns are linearly independent. Hence the system is controllable.

b) If $u_2 = 0$ we have

$$\dot{x} = Ax + b_1 u_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_1.$$

The controllability pencil is

$$[sI - A | b_1] = \left[\begin{array}{cccc|c} s & 0 & 0 & 0 & 0 \\ 0 & s & -1 & 0 & 0 \\ 0 & 0 & s & -1 & 0 \\ 0 & 0 & -1 & s & 1 \end{array}\right]$$

and this loses rank for $s = 0$. Hence the system is neither controllable nor stabilizable.

c) Let

$$A_{cl} = A + BK = \begin{bmatrix} K_{21} & K_{22} & K_{23} & K_{24} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ K_{11} & K_{12} & K_{13}+1 & K_{14} \end{bmatrix} = \left[\begin{array}{c|ccc} K_{21} & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ K_{11} & K_{12} & K_{13}+1 & K_{14} \end{array}\right].$$

We have partitioned the last matrix to show that K_{21} is an eigenvalue of A_{cl} and the eigenvalues of the right lower 3×3 block are also eigenvalues of A_{cl}. Hence, setting $K_{21} = -1$, $K_{12} = -1$, $K_{13} = -4$, $K_{14} = -3$ yields

$$A_{cl} = \left[\begin{array}{c|ccc} -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ K_{11} & -1 & -3 & -3 \end{array}\right]$$

which has all eigenvalues equal to -1 for any K_{11}.

d) Setting $u_1 = v_1$ and $u_2 = [0\ 1\ 0\ 0]x$, we obtain

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} v_1.$$

The controllability matrix of this system is

$$\mathcal{C} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

This is full rank, hence the system is controllable.

Question 2

a) The observability matrix is

$$\mathcal{O} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

which does not depend upon α and has rank two. Hence the system is observable for all α.

b) An asymptotic observer is described by

$$\dot{\xi} = A\xi + L(y - C\xi) = (A - LC)\xi + Ly$$

for some L, where ξ is the asymptotic estimate of x provided the matrix $A - LC$ has all eigenvalues with negative real part. Note that

$$A - LC = \begin{bmatrix} -L_1 & 1 \\ -L_2 & -\alpha \end{bmatrix}$$

and its characteristic polynomial is

$$s^2 + s(\alpha + L_1) + \alpha L_1 + L_2.$$

This should be equal to $(s+3)^2 = s^2 + 6s + 9$, yielding

$$L_1 = -\alpha + 6 \qquad L_2 = 9 - (6 - \alpha)\alpha.$$

c) Note that

$$y(t) = Ce^{At}x(0)$$

and replacing t with $t - \tau$ one has

$$y(t - \tau) = Ce^{A(t-\tau)}x(0)$$

Then, for $t \geq \tau$,

$$Y(t) = \begin{bmatrix} y(t) \\ y(t-\tau) \end{bmatrix} = \begin{bmatrix} C \\ Ce^{-A\tau} \end{bmatrix} e^{At}x(0).$$

For the given A and C (using $\alpha \neq 0$) we have

$$\begin{bmatrix} C \\ Ce^{-A\tau} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -\frac{e^{\alpha\tau}-1}{\alpha} \end{bmatrix}$$

which is invertible for all $\alpha \neq 0$ and all $\tau > 0$. Hence

$$x(t) = e^{At}x(0) = \begin{bmatrix} 1 & 0 \\ 1 & -\frac{e^{\alpha\tau}-1}{\alpha} \end{bmatrix}^{-1} Y(t).$$

The above relation implies that, for all $t \geq \tau$ it is possible to obtain exactly $x(t)$.

Question 3

a) The equations of the system in standard state-space form are

$$\dot{x} = Ax + Bu = \begin{bmatrix} 0 & 1 \\ 0 & -\beta/J \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u.$$

b) The reference signal to be tracked is $w = [\bar{x}_1 \; 0]'$. Note that

$$Aw = 0,$$

hence the optimal tracking problem can be transformed into a standard LQR problem. To this end, set $\xi = x - w$ and note that the optimal tracking problem can be written as

$$\min_u \int_0^\infty (\xi_1^2(\tau) + ru^2(\tau))d\tau$$

with

$$\dot{\xi} = A\xi + Bu,$$

i.e. as a standard LQR regulator problem.

c) Set

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}$$

The ARE associated with the problem is

$$0 = A'P + PA + Q - \frac{1}{r}PBB'P = \begin{bmatrix} 1 - \dfrac{P_{12}^2}{r} & P_{11} - \dfrac{P_{12}\beta}{J} - \dfrac{P_{12}P_{22}}{r} \\ P_{11} - \dfrac{P_{12}\beta}{J} - \dfrac{P_{12}P_{22}}{r} & 2P_{12} - \dfrac{2\beta P_{22}}{J} - \dfrac{P_{22}^2}{r} \end{bmatrix}.$$

d) From the $(1,1)$ block we have $P_{12} = \pm\sqrt{r}$. Then, from the $(2,2)$ block and keeping in mind that P_{22} should be positive one has (one has to select $P_{12} = \sqrt{r}$)

$$P_{22} = -r + \sqrt{r^2 + 2r\sqrt{r}}).$$

Then, from the $(1,2)$ block one has (recall $P_{11} > 0$)

$$P_{11} = \sqrt{r + 2\sqrt{r}}.$$

Hence the matrix

$$P = \begin{bmatrix} \sqrt{r + 2\sqrt{r}} & \sqrt{r} \\ \sqrt{r} & -r + \sqrt{r^2 + 2r\sqrt{r}}) \end{bmatrix}$$

is the positive definite solution of the ARE.

e) The optimal control law is $u = -K\xi$, with

$$K = \begin{bmatrix} \frac{1}{\sqrt{r}} & -1 + \frac{\sqrt{r+2\sqrt{r}}}{\sqrt{r}} \end{bmatrix}.$$

The optimal closed loop system is

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1/\sqrt{r} & -\frac{\sqrt{r+2\sqrt{r}}}{\sqrt{r}} \end{bmatrix} x$$

The characteristic polynomial of the optimal closed-loop system is

$$s^2 + s\frac{\sqrt{r+2\sqrt{r}}}{\sqrt{r}} + 1/\sqrt{r},$$

hence the optimal closed loop system is stable for any $r > 0$.

Question 4

a) The matrices are

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$Q = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad R = 1 \qquad M = \begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix}.$$

b) The Hamiltonian matrix is

$$H = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right].$$

The DRE is

$$-\dot{P} = A'P + PA + Q - PBR^{-1}B'P.$$

c) The matrices are

$$X(T) = I, \qquad Y(T) = M.$$

To determine $X(t)$ and $Y(t)$ note that

$$\begin{bmatrix} X(t) \\ Y(t) \end{bmatrix} = e^{H(t-T)} \begin{bmatrix} X(T) \\ Y(T) \end{bmatrix} = e^{H(t-T)} \begin{bmatrix} I \\ M \end{bmatrix}$$

and that

$$\begin{aligned} e^{H(t-T)} &= I + H(t-T) + H^2 \frac{(t-T)^2}{2} + H^3 \frac{(t-T)^3}{3!} \\ &= \left[\begin{array}{cc|cc} 1 & t-T & -\frac{1}{6}(-t+T)^3 & -\frac{1}{2}(-t+T)^2 \\ 0 & 1 & \frac{1}{2}(-t+T)^2 & T-t \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & T-t & 1 \end{array}\right] \end{aligned}$$

As a result

$$X(t) = \begin{bmatrix} 1 + \frac{1}{6}c(t-T)^3 & t-T \\ \frac{1}{2}c(t-T)^2 & 1 \end{bmatrix} \qquad Y(t) = \begin{bmatrix} c & 0 \\ -c(t-T)^2 & 0 \end{bmatrix}.$$

Finally

$$P(t) = Y(t)X^{-1}(t) = \frac{1}{\det(X(t))} c \begin{bmatrix} 1 & -(t-T) \\ -(t-T) & (t-T)^2 \end{bmatrix}$$

with

$$\det(X(t)) = 1 + \frac{1}{3}c(T-t)^3.$$

d) The optimal control law is

$$u = -K(t)x = -R^{-1}B'P(t)x$$

with

$$K(t) = \frac{c}{\det(X(t))} \begin{bmatrix} (T-t) & (T-t)^2 \end{bmatrix}.$$

Question 5

a) Note that the cost function should be changed to $-J$ to have a minimization problem. Let

$$H = -x + u + \lambda(-\alpha x + u).$$

The necessary conditions of optimality for normal extremals are

$$\dot{x} = -\alpha x + u \qquad \dot{\lambda} = 1 + \alpha\lambda,$$

$$(1+\lambda)u \leq (1+\lambda)\omega \quad \forall \omega \in [0, \bar{u}].$$

b) The optimal control as a function of the costate is

$$u^\star(t) = \begin{cases} 0 \text{ if } 1 + \lambda^\star(t) > 0 \\ \bar{u} \text{ if } 1 + \lambda^\star(t) < 0 \end{cases}$$

If $1 + \lambda^\star(t) = 0$ we do not have information on the optimal control.

c) The optimal costate is

$$\lambda^\star(t) = e^{\alpha(t-T)}(\frac{1}{\alpha} - \beta) - \frac{1}{\alpha}.$$

d) The optimal control is (recall equation (5.1))

$$u^\star(t) = \begin{cases} 0 \text{ if } D(t) > 0 \\ \bar{u} \text{ if } D(t) < 0 \end{cases}$$

If $D(t) = 0$ we do not have information on the optimal control.
If $D(t)$ does not change sign then the optimal control is constant.
$D(t)$ changes sign if

$$e^{\alpha(t-T)} = \frac{1 - \frac{1}{\alpha}}{\beta - \frac{1}{\alpha}}.$$

This equation may have at most one solution

$$t_s = T + \frac{1}{\alpha} \log \frac{1 - \frac{1}{\alpha}}{\beta - \frac{1}{\alpha}}.$$

If $D(t)$ changes sign and $\beta > 1$ then $D(T) = 1 - \beta < 0$, hence the optimal control is such that $u^\star(T) = \bar{u}$. As a result the optimal control is equal to zero for $t \in [0, t_s)$ and equal to $\bar{u}$ otherwise.

Question 6

a) Let

$$H = \frac{u^2}{2} + \lambda_1 x_2 + \lambda_2 u.$$

The necessary conditions of optimality for normal extremals are

$$\dot{x}_1 = x_2 \qquad \dot{x}_2 = u$$

$$\dot{\lambda}_1 = 0 \qquad \dot{\lambda}_2 = -\lambda_1$$

$$\tfrac{\partial H}{\partial u} = u + \lambda_2 = 0.$$

b) The optimal control as a function of the optimal costate is

$$u^\star(t) = -\lambda_2^\star(t).$$

c) From the differential equations of the costate we obtain

$$\lambda_1^\star(t) = -1 \qquad \lambda_2^\star(t) = (t - T).$$

d) The optimal control as a function of time is

$$u^\star(t) = -(t - T).$$

e) From the differential equations of the state with the optimal control we obtain

$$x_1^\star(t) = \frac{T}{2}t^2 - \frac{t^3}{6} \qquad x_2^\star(t) = Tt - \frac{t^2}{2}.$$

Hence $x_1^\star(T) = \frac{T^3}{3}$.

f) The optimal cost is

$$J^\star = -\frac{T^3}{6}.$$