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IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2016

MSc and EEE/EIE PART IV: MEng and ACGI

Corrected copy

DIGITAL SIGNAL PROCESSING AND DIGITAL FILTERS

Friday, 29 April 10:00 am

Time allowed: 3:00 hours

There are FOUR questions on this paper.

Answer Question 1 and any TWO other questions

Question 1 is worth 40% of the marks and other questions are worth 30%

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : D.M. Brookes
Second Marker(s) : P.T. Stathaki

DIGITAL SIGNAL PROCESSING AND DIGITAL FILTERS

Information for Candidates:

Notation

- All signals and filter coefficients are real-valued unless explicitly noted otherwise.
- Unless otherwise specified, upper and lower case letters are used for sequences and their z-transforms respectively. The signal at a block diagram node V is $v[n]$ and its z-transform is $V(z)$.
- $x[n] = [a, b, c, d, e, f]$ means that $x[0] = a, \ldots x[5] = f$ and that $x[n] = 0$ outside this range.
- $\Re(z)$, $\Im(z)$, z^*, $|z|$ and $\angle z$ denote respectively the real part, imaginary part, complex conjugate, magnitude and argument of a complex number z.
- Where necessary, the sample rate of a signal in a block diagram is indicated in the form "@f".

Abbreviations

BIBO	Bounded Input, Bounded Output
CTFT	Continuous-Time Fourier Transform
DCT	Discrete Cosine Transform
DFT	Discrete Fourier Transform
DTFT	Discrete-Time Fourier Transform
FIR	Finite Impulse Response

IIR	Infinite Impulse Response
LTI	Linear Time-Invariant
MDCT	Modified Discrete Cosine Transform
PSD	Power Spectral Density
SNR	Signal-to-Noise Ratio

A datasheet is included at the end of the examination paper.

1. a) The signals $x[n]$ and $y[n]$ are defined as

$$x[n] = \begin{cases} 2^{-n} & n > 0 \\ 0 & n \leq 0 \end{cases}, \quad \text{and} \quad y[n] = \begin{cases} 0 & n > 0 \\ 5^{-n} & n \leq 0 \end{cases}.$$

i) Determine the z-transform of $x[n]$ and its region of convergence. [3]

ii) Determine the z-transform of $y[n]$ and its region of convergence. [3]

You may assume without proof that $\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}$ provided that $|\alpha| < 1$.

b) Consider the convolution $y[n] = h[n] * x[n] = \sum_{m=0}^{M} h[m]x[n-m]$ where $h[n]$ is the impulse response of an FIR filter of order M (i.e. $n \in [0, M]$) and $x[n]$ is a signal defined for $n \in [-\infty, \infty]$.

In the overlap-save method of convolution, $y[n]$ is divided into blocks of length K and a circular convolution of length $K+M$ is used to calculate each block. To calculate block b, the circular convolution evaluates

$$y[bK+n] = \sum_{m=0}^{M} h[m]x[bK - M + (n-m+M)_{\mathrm{mod}(K+M)}]$$

for $n \in [0, K-1]$. The notation $P_{\mathrm{mod}\,Q}$ denotes the remainder when P is divided by Q and satisfies $0 \leq P_{\mathrm{mod}\,Q} < Q$.

i) Show that the expression given above is equivalent to the direct convolution given by $y[bK+n] = \sum_{m=0}^{M} h[m]x[bK+n-m]$. [3]

ii) Suppose that a circular convolution of length R requires approximately $5R\log_2 R$ multiplications. If $M = 200$, estimate the number of multiplications per output sample required by the overlap-save method when $K = 20$, 1500 and 10^4 and compare these results with the number of multiplications required for implementing the direct convolution. [4]

c) The filter $H(z)$ is given by

$$H(z) = \frac{1+z^{-1}}{1-z^{-1}+0.8z^{-2}}.$$

The pole-zero diagram and magnitude response (in dB) of $H(z)$ are shown in Figures 1.1 and 1.2 respectively.

i) Determine the transfer function $F(z) = H(z^2)$ and sketch its pole-zero diagram and magnitude response in dB. It is not necessary to determine exact values of the magnitude response. [4]

ii) Determine the transfer function $G(z) = H(1.25z)$ and sketch its pole-zero diagram and magnitude response in dB. It is not necessary to determine exact values of the magnitude response. [4]

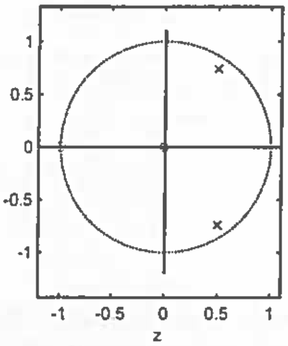

Figure 1.1

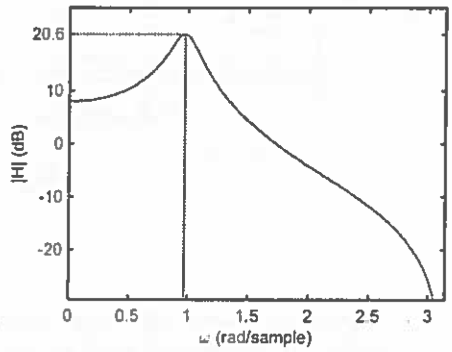

Figure 1.2

d) A bilinear transformation, $s = \alpha\frac{z-1}{z+1}$, is used to convert a continuous-time filter into a discrete-time filter.

i) Show that if $\alpha = \frac{\Omega_0}{\tan(0.5\omega_0)}$ then $z = e^{j\omega_0} \Leftrightarrow s = j\Omega_0$. [3]

ii) A continuous-time highpass filter with a cutoff frequency of 1 kHz is given by $H(s) = \frac{s}{s+\Omega_0}$ where $\Omega_0 = 2000\pi$ rad/s. Using the bilinear transformation given above, determine the coefficients (to 3 decimal places) of a discrete-time filter having an unnormalized cutoff frequency of 1 kHz. The sample frequency is 8 kHz. [4]

e) i) Explain why the average power of a discrete time signal (i.e. the average energy per sample) is always decreased by upsampling but is normally unchanged by downsampling. Give an example of a signal for which the latter statement is untrue. [3]

ii) Figure 1.3 shows the power spectral density (PSD) of a real-valued signal, $x[n]$; the horizontal portions of the PSD have values 1 or 4. The signal $y[m]$ is obtained by upampling $x[n]$ by a factor of 3 as shown. Draw a dimensioned sketch of the PSD of $y[m]$ giving the values of all horizontal portions of the graph and the values of all frequencies at which there is a discontinuity in the PSD. [3]

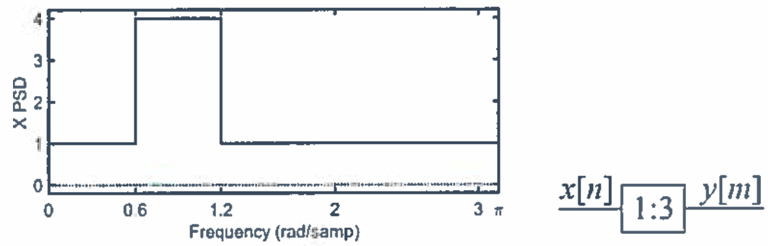

Figure 1.3

f) Figure 1.4 shows the block diagram of a two-band analysis processor. The inputs to the adder/subtractor blocks, ⊕, are additive unless labelled with a minus sign in which case they are subtractive.

i) By using the Noble identities or otherwise, determine $H_0(z)$ and $H_1(z)$ so that Figure 1.5 is equivalent to Figure 1.4. [3]

ii) Assuming that $P_0(z)$ and $P_1(z)$ are FIR filters with real-valued coefficients, show that $\left|H_1\left(e^{j\omega}\right)\right| = \left|H_0\left(e^{j(\pi-\omega)}\right)\right|$ and explain the significance of this relationship. [3]

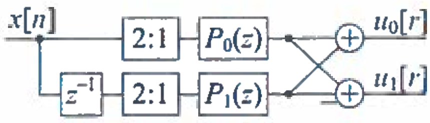

Figure 1.4

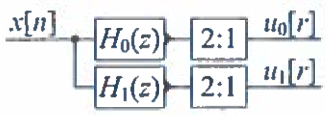

Figure 1.5

2. In the block diagram of Figure 2.1 the outputs of all adders and delay elements are on the right and solid arrows indicate the direction of information flow. The real-valued gain of each multiplier is written adjacent to its triangular symbol.

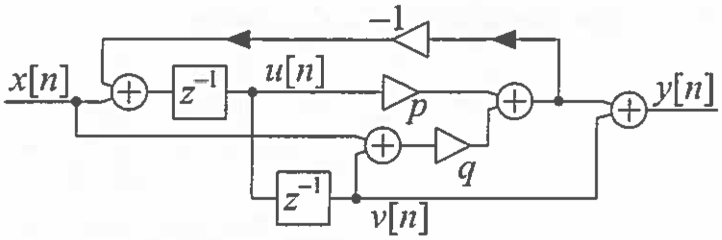

Figure 2.1

a) Show that [8]

$$G(z) = \frac{Y(z)}{X(z)} = \frac{q + pz^{-1} + z^{-2}}{1 + pz^{-1} + qz^{-2}}.$$

b) Prove that $|G(e^{j\omega})| = 1$ for all ω. [6]

c) Figure 2.2 shows a graph of $\angle G(e^{j\omega})$ when $p = -1.2$ and $q = 0.8$. The dotted lines indicate the frequencies, $\omega = \{a, b, c\}$, at which $G(e^{j\omega}) = \{-j, -1, +j\}$ respectively. Given that $G(e^{jb}) = -1$, show that $\cos b = \frac{-p}{q+1}$ and find the numerical value of b for the given values of p and q. [6]

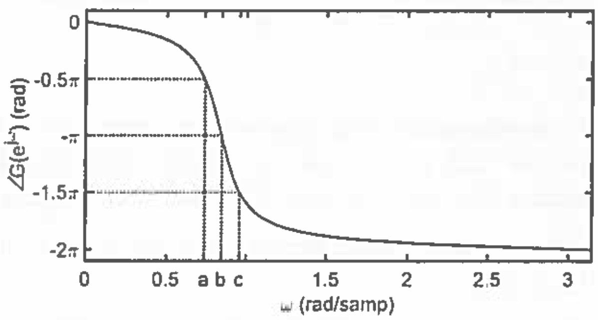

Figure 2.2

d) The filter $H(z)$ is defined as $H(z) = \frac{1}{2}(1 + G(z))$.

i) Determine the value of $H(e^{j\omega})$ for each of $\omega = \{a, b, c\}$ defined above.
Hence, for $p = -1.2$ and $q = 0.8$, sketch a graph of $|H(e^{j\omega})|$ for $\omega \in [0, \pi]$ using linear scales for both axes. [6]

ii) For $p = -1.2$ and $q = 0.8$, determine the poles and zeros of $H(z)$ in polar form and sketch a diagram of the complex plane that includes the unit circle and the poles and zeros of $H(z)$ (indicated by × and ○ respectively). [4]

3. a) A symmetric Hanning window of odd length $M+1$ is defined as

$$w[n] = 0.5 + 0.5\cos\omega_M n,$$

where $\omega_M = \frac{2\pi}{(M+1)}$ and $-0.5M \le n \le 0.5M$.

i) Show that the DTFT of $w[n]$ is given by

$$\begin{aligned} W(e^{j\omega}) &= 0.5\frac{\sin 0.5(M+1)\omega}{\sin 0.5\omega} \\ &\quad +0.25\frac{\sin 0.5(M+1)(\omega-\omega_M)}{\sin 0.5(\omega-\omega_M)} \\ &\quad +0.25\frac{\sin 0.5(M+1)(\omega+\omega_M)}{\sin 0.5(\omega+\omega_M)}. \end{aligned}$$

You may assume without proof that $\sum_{n=-0.5M}^{0.5M} e^{j\alpha n} = \frac{\sin 0.5(M+1)\alpha}{\sin 0.5\alpha}$ provided that $\alpha \neq 0$. [5]

ii) We define $S(\omega)$ to be the integrated spectrum

$$S(\omega) = \frac{1}{2\pi}\int_{\theta=0}^{\omega} W(e^{j\theta})d\theta.$$

Using the inverse DTFT formula or otherwise, show that $S(\pi) = 0.5$. [3]

b) i) Show that, if $G(z)$ is an ideal lowpss filters with

$$G(e^{j\omega}) = \begin{cases} 1 & |\omega| \le \omega_0 \\ 0 & \omega_0 < |\omega| \le \pi \end{cases}, \quad \text{then} \quad g[n] = \frac{\sin\omega_0 n}{\pi n}.$$

[4]

ii) By combining the ideal response from part b) i) with the Hanning window from part a), use the window method to design a causal lowpass filter, $H(z)$, of order M with a cutoff frequency of ω_0.

Give a formula for the impulse response, $h[n]$, of the filter where $0 \le n \le M$. [3]

iii) Show that $H(e^{j\omega}) = e^{-0.5j\omega M}\left(S(\omega+\omega_0) - S(\omega-\omega_0)\right)$. [5]

[This question is continued on the next page]

c) Figures 3.1 and 3.2 show $W(e^{j\omega})$ and $S(\omega)$ for a Hanning window of length $M+1=41$. The first few values of ω for which $S(\omega)$ either equals 0.5 or has a turning point are listed in the following table:

ω	0.2565	0.3065	0.4003	0.4598	0.5490
$S(\omega)$	0.5	0.5064	0.5	0.4981	0.5

For the case $\omega_0 = 1$, the magnitude response, $\left|H(e^{j\omega})\right|$, of the resultant lowpass FIR filter from part b) is shown in Figure 3.3 plotted in dB. The ideal response, $G(e^{j\omega})$, is shown on the graph as a dashed line. Using appropriate values from the table given above,

i) estimate the smallest positive ω (marked "a" in Figure 3.3) for which $H(e^{j\omega}) = 0$; [3]

ii) estimate the magnitude in dB (marked "b" in Figure 3.3) of the first peak in the stopband; [3]

iii) estimate the peak passband gain in dB. [4]

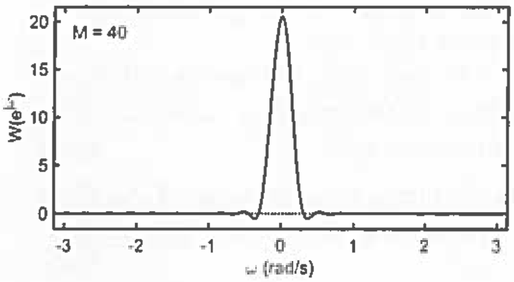

Figure 3.1

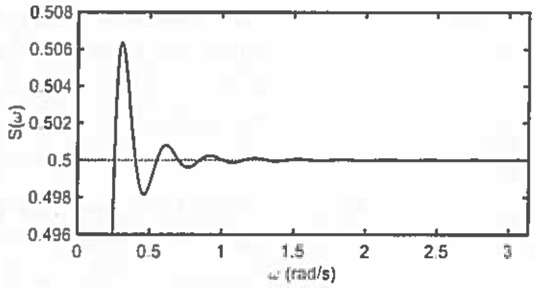

Figure 3.2

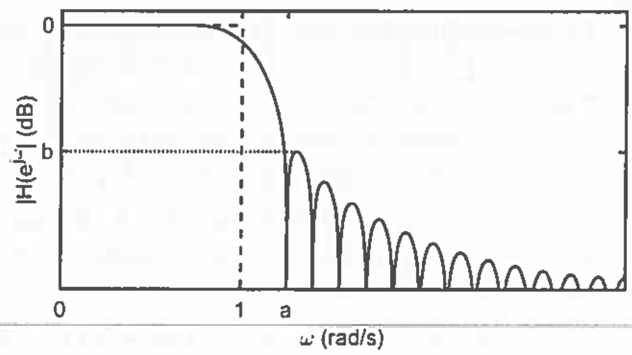

Figure 3.3

4. Figure 4.1 shows a system intended to apply a fractional-sample delay to its input signal, $x[n]$, where the delay is an integer multiple of $\frac{1}{P}$ samples.

Figure 4.1

Figure 4.2

a) i) Explain the purpose of the lowpass filter, $H(z)$, in Figure 4.1. [2]

ii) The input signal, $x[n]$, contains frequency components in the range $0 \leq \omega \leq 0.8\pi$. Using the datasheet formula, $M \approx \frac{a}{3.5\Delta\omega}$, estimate the order required for $H(z)$ to give a stopband attenuation of 60 dB. [3]

iii) For a direct implementation of Figure 4.1, estimate as a function of P the number of multiplications required per input sample, $x[n]$. [3]

iv) If $H(z)$ is a causal symmetric FIR filter of order M, determine the delay of $y[n]$ relative to $x[n]$ as a function of M and d. [3]

b) i) The signal $w[m]$ in Figure 4.1 is given by $w[m] = \sum_{s=0}^{M} h[s]u[m-s]$ where $h[s]$ is the impulse response of $H(z)$.
If $m = Pn + p$ where $0 \leq p < P$, show that $w[m]$ may be written in the form $w[m] = \sum_{r=0}^{R} h_p[r]x[n-r]$. Determine the value of R and give an expression for $h_p[r]$ in terms of $h[s]$. [6]

ii) Derive expressions for q and k in Figure 4.2 as functions of d in Figure 4.1 so that that the two figures are equivalent. You may assume that $y[n] = w[Pn-d]$ in Figure 4.1. [4]

c) Suppose now that, for each r, the coefficients $h_q[r]$ may be closely approximated using a polynomial of order T as $h_q[r] \approx \sum_{t=0}^{T} f_t[r] \left(\frac{q}{P}\right)^t$ where the polynomial argument, $\frac{q}{P}$, lies in the range $0 \leq \frac{q}{P} < 1$.

i) The Farrow filter shown in Figure 4.3 calculates its output, $y[n]$, from $y[n+k] = \sum_{t=0}^{T} \left(\frac{q}{P}\right)^t v_t[n]$ where each of the signals $v_t[n]$ is obtained from $x[n]$ by applying a filter, $G_t(z)$, whose coefficients do not depend on q. Derive an expression for the coefficients of $G_t(z)$ so that Figure 4.3 is approximately equivalent to Figure 4.2. [4]

ii) Suppose that $P = 20$, $M = 199$ and $T = 4$. For each of Figure 4.2 and Figure 4.3, estimate the number of multiplications required per input sample, $x[n]$. [3]

iii) Explain why the implementation of Figure 4.3 may be preferable to that of Figure 4.2 under some circumstances. [2]

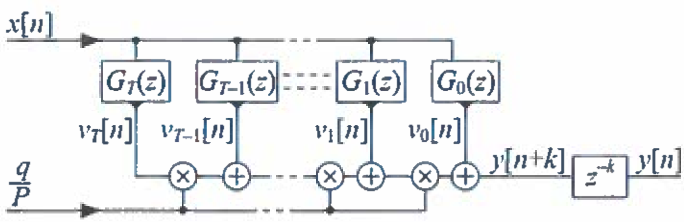

Figure 4.3

Datasheet:

Standard Sequences

- $\delta[n] = 1$ for $n = 0$ and 0 otherwise.
- $\delta_{\text{condition}}[n] = 1$ whenever "condition" is true and 0 otherwise.
- $u[n] = 1$ for $n \geq 0$ and 0 otherwise.

Geometric Progression

- $\sum_{n=0}^{r} \alpha^n z^{-n} = \frac{1-\alpha^{r+1} z^{-r-1}}{1-\alpha z^{-1}}$ provided that $\alpha z^{-1} \neq 1$.
- $\sum_{n=0}^{\infty} \alpha^n z^{-n} = \frac{1}{1-\alpha z^{-1}}$ provided that $\left|\alpha z^{-1}\right| < 1$.

Forward and Inverse Transforms

z:	$X(z) = \sum_{-\infty}^{\infty} x[n] z^{-n}$	$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$
CTFT:	$X(j\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt$	$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega) e^{j\Omega t} d\Omega$
DTFT:	$X(e^{j\omega}) = \sum_{-\infty}^{\infty} x[n] e^{-j\omega n}$	$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$
DFT:	$X[k] = \sum_{0}^{N-1} x[n] e^{-j2\pi \frac{kn}{N}}$	$x[n] = \frac{1}{N} \sum_{0}^{N-1} X[k] e^{j2\pi \frac{kn}{N}}$
DCT:	$X[k] = \sum_{n=0}^{N-1} x[n] \cos \frac{2\pi(2n+1)k}{4N}$	$x[n] = \frac{X[0]}{N} + \frac{2}{N} \sum_{n=1}^{N-1} X[k] \cos \frac{2\pi(2n+1)k}{4N}$
MDCT:	$X[k] = \sum_{n=0}^{2N-1} x[n] \cos \frac{2\pi(2n+1+N)(2k+1)}{8N}$	$y[n] = \frac{1}{N} \sum_{0}^{N-1} X[k] \cos \frac{2\pi(2n+1+N)(2k+1)}{8N}$

Convolution

DTFT: $v[n] = x[n] * y[n] \triangleq \sum_{r=-\infty}^{\infty} x[r] y[n-r] \quad \Leftrightarrow \quad V\left(e^{j\omega}\right) = X\left(e^{j\omega}\right) Y\left(e^{j\omega}\right)$

$v[n] = x[n] y[n] \quad \Leftrightarrow \quad V\left(e^{j\omega}\right) = \frac{1}{2\pi} X\left(e^{j\omega}\right) \circledast Y\left(e^{j\omega}\right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X\left(e^{j\theta}\right) Y\left(e^{j(\omega-\theta)}\right) d\theta$

DFT: $v[n] = x[n] \circledast_N y[n] \triangleq \sum_{r=0}^{N-1} x[r] y[(n-r)_{\bmod N}] \quad \Leftrightarrow \quad V[k] = X[k] Y[k]$

$v[n] = x[n] y[n] \quad \Leftrightarrow \quad V[k] = \frac{1}{N} X[k] \circledast_N Y[k] \triangleq \frac{1}{N} \sum_{r=0}^{N-1} X[r] Y[(k-r)_{\bmod N}]$

Group Delay

The group delay of a filter, $H(z)$, is $\tau_H(e^{j\omega}) = -\frac{d\angle H(e^{j\omega})}{d\omega} = \Re\left(\frac{-z}{H(z)} \frac{dH(z)}{dz}\right)\Big|_{z=e^{j\omega}} = \Re\left(\frac{\mathscr{F}(nh[n])}{\mathscr{F}(h[n])}\right)$ where $\mathscr{F}()$ denotes the DTFT.

Order Estimation for FIR Filters

Three increasingly sophisticated formulae for estimating the minimum order of an FIR filter with unity gain passbands:

1. $M \approx \frac{a}{3.5\Delta\omega}$
2. $M \approx \frac{a-8}{2.2\Delta\omega}$
3. $M \approx \frac{a-1.2-20\log_{10} b}{4.6\Delta\omega}$

where a =stop band attenuation in dB, b = peak-to-peak passband ripple in dB and $\Delta\omega$ = width of smallest transition band in radians per sample.

z-plane Transformations

A lowpass filter, $H(z)$, with cutoff frequency ω_0 may be transformed into the filter $H(\hat{z})$ as follows:

Target $H(\hat{z})$	Substitute	Parameters
Lowpass $\hat{\omega} < \hat{\omega}_1$	$z^{-1} = \frac{\hat{z}^{-1}-\lambda}{1-\lambda\hat{z}^{-1}}$	$\lambda = \frac{\sin\left(\frac{\omega_0-\hat{\omega}_1}{2}\right)}{\sin\left(\frac{\omega_0+\hat{\omega}_1}{2}\right)}$
Highpass $\hat{\omega} > \hat{\omega}_1$	$z^{-1} = -\frac{\hat{z}^{-1}+\lambda}{1+\lambda\hat{z}^{-1}}$	$\lambda = \frac{\cos\left(\frac{\omega_0+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\omega_0-\hat{\omega}_1}{2}\right)}$
Bandpass $\hat{\omega}_1 < \hat{\omega} < \hat{\omega}_2$	$z^{-1} = -\frac{(\rho-1)-2\lambda\rho\hat{z}^{-1}+(\rho+1)\hat{z}^{-2}}{(\rho+1)-2\lambda\rho\hat{z}^{-1}+(\rho-1)\hat{z}^{-2}}$	$\lambda = \frac{\cos\left(\frac{\hat{\omega}_2+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)}, \rho = \cot\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)\tan\left(\frac{\omega_0}{2}\right)$
Bandstop $\hat{\omega}_1 \not< \hat{\omega} \not< \hat{\omega}_2$	$z^{-1} = \frac{(1-\rho)-2\lambda\hat{z}^{-1}+(\rho+1)\hat{z}^{-2}}{(\rho+1)-2\lambda\hat{z}^{-1}+(1-\rho)\hat{z}^{-2}}$	$\lambda = \frac{\cos\left(\frac{\hat{\omega}_2+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)}, \rho = \tan\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)\tan\left(\frac{\omega_0}{2}\right)$

Noble Identities

$$-\boxed{Q:1}-\boxed{H(z)}- \quad = \quad -\boxed{H(z^Q)}-\boxed{Q:1}-$$

$$-\boxed{H(z)}-\boxed{1:Q}- \quad = \quad -\boxed{1:Q}-\boxed{H(z^Q)}-$$

Multirate Spectra

Upsample: $\underline{v[n]}\ \boxed{1:Q}\ \underline{x[r]}$ $\quad x[r] = \begin{cases} v\left[\frac{r}{Q}\right] & \text{if } Q \mid r \\ 0 & \text{if } Q \nmid r \end{cases} \quad \Rightarrow \quad X(z) = V(z^Q)$

Downsample: $\underline{v[n]}\ \boxed{Q:1}\ \underline{y[m]}$ $\quad y[m] = v[Qm] \quad \Rightarrow \quad Y(z) = \frac{1}{Q}\sum_{k=0}^{Q-1} V\left(e^{\frac{-j2\pi k}{Q}} z^{\frac{1}{Q}}\right)$

Multirate Commutators

Input Commutator	Output Commutator