E4.24 / ISE 4.21 / C1.J

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The Answers

1. (a) {*unseen modification of standard result*}

The q-formulation of the differene equation is $qx_k = Ax_k + bq^2 u_k$.

By taking Z-transforms we obtain

$z(x^Z(z) - x_o) = Ax^Z(z) + bz^2(u^Z(z) - u_0 - u_1 z^{-1})$. Hence

$(zI - A)x^Z(z) = bz^2 u^Z(z) - bz^2 u_0 - bzu_1 + zx_0$, so

$x^Z(z) = (zI-A)^{-1}bz^2u^Z(z) - (zI-A)^{-1}bz^2u_0 - (zI-A)^{-1}bzu_1 + (zI-A)^{-1}zx_0$ and

$y^Z(z) = z^2c'(zI-A)^{-1}b + zc'(zI-A)^{-1}x_o - c'(zI-A)^{-1}b\{z^2u_0 + zu_1\}$, **[3]**

(b) {*unseen*}

(i) $y^Z(z) = c'(zI - A)^{-1}bu^Z(z) + zc'(zI-A)^{-1}x_o$. **[1]**

(ii) $$A^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & \alpha \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & \alpha \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 2\alpha \\ 0 & 0 & \alpha^2 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 2\alpha \\ 0 & 0 & \alpha^2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & \alpha \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2\alpha \\ 0 & 0 & 2\alpha^2 \\ 0 & 0 & \alpha^3 \end{bmatrix} = \alpha \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 2\alpha \\ 0 & 0 & \alpha^2 \end{bmatrix} = \alpha A^2$$

$A^4 = A^3 A = (\alpha A^2)A = \alpha A^3 = \alpha^2 A^2$

....

$A^k = \alpha^{k-2} A^2$

so $p(\alpha) = \alpha^{k-2}$. **[5]**

(iii) Since all the u_k are zero, $y_k = c'A^k x_o$ so

$$\begin{aligned} y^Z(z) &= z^0 c'x_o + z^{-1}c'Ax_o + z^{-2}c'A^2x_o + z^{-3}c'A^3x_o \quad + z^{-4}c'A^4x_o + \dots \\ &= z^0 c'x_o + z^{-1}c'Ax_o + z^{-2}c'A^2x_o + \alpha z^{-3}c'A^2x_o + \alpha^2 z^{-4}c'A^2x_o + \dots \\ &= z^0 c'x_o + z^{-1}c'Ax_o + z^{-2}c'A^2x_o(1 + \alpha z^{-1} + \alpha^2 z^{-2} + \dots) \\ &= z^0 c'x_o + z^{-1}c'Ax_o + \tfrac{z^{-2}}{(1-\alpha z^{-1})}\, c'A^2x_o. \end{aligned}$$

Hence $H^Z(z) = z^0c' + z^{-1}c'A + \frac{z^{-2}}{(1-\alpha z^{-1})}\, c'A^2$. **[5]**

(iv) From part (b-i): the pulse Z-transfer function is $G^Z(z) = c'(zI-A)^{-1}b$ and the contribution to $y^Z(z)$ from x_o is $zc'(zI-A)^{-1}x_o$ so $H^Z(z)$ of part (b-ii) equals $zc'(zI-A)^{-1}$. Therefore we obtain $G^Z(z) = z^{-1}H^Z(z)b$. **[3]**

(c) {*small modification of a standard result*}

The Variation of Constants formula gives the difference equation

$$\begin{aligned} x(t_{k+1}) &= \mathrm{e}^{A(t_{k+1}-t_k)}x(t_k) + \textstyle\int_{t_k}^{t_{k+1}} \mathrm{e}^{A(t_{k+1}-\tau)}bu(\tau)d\tau \\ &= \mathrm{e}^{AT}x(t_k) + \textstyle\int_{t_k}^{t_{k+1}} \mathrm{e}^{A(t_{k+1}-\tau)}b(\tau - t_k)u_k d\tau \\ &= \overline{A}x(t_k) + \overline{b}u_k \end{aligned}$$

where $\overline{A} = \mathrm{e}^{AT}$ and (after making the change of variable $\theta = \tau - t_k$)

$\overline{b} = \int_0^T \mathrm{e}^{A(T-\theta)}\theta d\theta b$. Hence the required difference equation is

$x_{k+1} = \overline{A}x_k + \overline{b}u_k$ where $x_k = x(t_k)$ for all k. **[3]**

2. (a) $\{bookwork\}$

Now $G^D(\gamma) \triangleq TG^Z(1+\gamma T)$ so the poles p_i^D of $G^D(\gamma)$ are the values of γ for which $G^Z(1+\gamma T)$ is infinite. Hence $1+p_i^D T = p_i^Z$ for all i, i.e. $p_i^D = \frac{1}{T}\left(p_i^Z - 1\right)$ for all i. **[3]**

The system is BIBO-stable iff $|p_i^Z| < 1$, i.e. iff $p_i^D \in \{\frac{1}{T}(z-1) : |z| < 1\}$ $= \{z - \frac{1}{T} : |z| < \frac{1}{T}\}$. Hence for BIBO-stability the poles of $G^D(\gamma)$ should all belong to the hatched area shown in Figure A2.1 below. **[2]**

(b) $\{unseen\}$

(i)Now $x^Z(z) = (zI - \{I+AT\})^{-1}\{Tbu^Z(z) + zx_o\}$ so

$$x^D(\gamma) = Tx^Z(1+\gamma T) = T(I+\gamma TI - I - AT)^{-1}\{Tbu^Z(1+\gamma T) + (1+\gamma T)x_o\}$$
$$= T(T(\gamma I - A))^{-1}\{Tbu^Z(1+\gamma T) + (1+\gamma T)x_o\}$$
$$= (\gamma I - A)^{-1}\{bTu^Z(1+\gamma T) + (1+\gamma T)x_o\}$$
$$= (\gamma I - A)^{-1}\{bu^D(\gamma) + (1+\gamma T)x_o\}$$
$$= (\gamma I - A)^{-1}bu^D(\gamma) + (1+\gamma T)(\gamma I - A)^{-1}x_o.$$

Hence $y^D(\gamma) = c'(\gamma I - A)^{-1}bu^D(\gamma) + (1+\gamma T)c'(\gamma I - A)^{-1}x_0$.

Therefore $G^D(\gamma) = c'(\gamma I - A)^{-1}b \;=\; c'(sI - A)^{-1}b|_{s=\gamma} = G^L(\gamma)$, which is the connection required between $G^D(\gamma)$ and $G^L(s)$. **[5]**

(ii)The system is BIBO-stable if each eigenvalue of $I+AT$ has modulus smaller than 1, Now $I+AT = VV^{-1} + V\Lambda V^{-1}T = V(I + \Lambda T)V^{-1}$ so the eigenvalues of $I + AT$ are $1+\lambda_i T$ and $|1+\lambda_i T| = \sqrt{(1 + \Re(\lambda_i)T)^2 + \Im(AT)^2}$ which is not necessarily smaller than one even if $\Re(\lambda_i) < 0$. Hence the discrete-time approximation is not necessarily BIBO-stable even if the original continuous-time system is (continuous-time) BIBO-stable. **[4]**

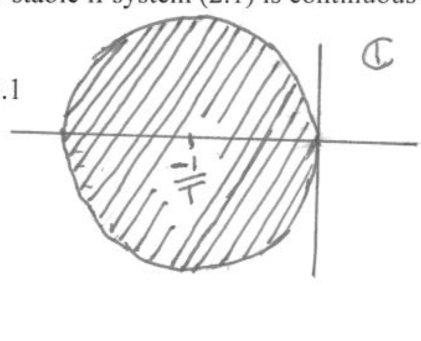
(iii)The standard exact relationship is that $x_{k+1} = \overline{A}x_k + \overline{B}u_k$ where $\overline{A} = \mathrm{e}^{AT}$ and $\overline{B} = \int_0^T \mathrm{e}^{A\tau}d\tau B$.

We can write e^{AT} as $\mathrm{e}^{AT/2}\mathrm{e}^{AT/2}$ i.e.as $\mathrm{e}^{AT/2}(\mathrm{e}^{-AT/2})^{-1}$. Approximating $\mathrm{e}^{AT/2}$ by $(I+AT/2)$and $\mathrm{e}^{-AT/2}$ by $(I-AT/2)$ gives the approximation $(I+AT/2)\,(I-AT/2)^{-1}$ to $\overline{A}$. Hence (2.3) is dt BIBO-stable iff each eigenvalue of $(I+AT/2)(I-AT/2)^{-1}$ has modulus smaller than one. Now

$$(I+AT/2)(I-AT/2)^{-1} = (VV^{-1} + V\Lambda V^{-1}T/2)(VV^{-1} - V\Lambda V^{-1}T/2)^{-1}$$
$$= V(I+\Lambda T/2)V^{-1}[V(I-\Lambda T/2)V^{-1}]^{-1} = V(I+\Lambda T/2)(I-\Lambda T/2)^{-1}V^{-1}$$

$= V\mathrm{diag}\{\ldots \frac{1+\lambda_i T/2}{1-\lambda_i T/2} \ldots\}V^{-1}$. Consequently system (2.3) is BIBO-stable iff for all i $|\frac{1+\lambda_i T/2}{1-\lambda_i T/2}| < 1$. Now $|\frac{1+\lambda_i T/2}{1-\lambda_i T/2}| = \frac{|1+\lambda_i T/2|}{|1-\lambda_i T/2|} = \frac{\sqrt{(1+\Re(\lambda_i)T)^2 + (\Im(\lambda_i)T/2)^2}}{\sqrt{(1-\Re(\lambda_i)T)^2 + (\Im(\lambda_i)T/2)^2}} < 1$ if $\Re\mathfrak{e}(\lambda_i) < 0$. Hence system (2.3) is discrete-time BIBO-stable if system (2.1) is continuous-time BIBO-stable. **[6]**

Figure A2.1



3. (a) {*modified bookwork*}

Now $u_k = \sum_{m=0}^{2} h_m u_{k-m} = \sum_{m=0}^{2} h_m \cos(\omega(k-m)T) = \sum_{m=0}^{2} h_m \, \frac{e^{j\omega(k-m)T} + e^{-j\omega(k-m)T}}{2}$

$= \sum_{m=0}^{2} h_m \, \frac{e^{j\omega(k-m)T}}{2} + \sum_{m=0}^{2} h_m \, \frac{e^{-j\omega(k-m)T}}{2}$

$= e^{j\omega kT} \sum_{m=0}^{2} h_m \, \frac{e^{-j\omega mT}}{2} + e^{-j\omega kT} \sum_{m=0}^{2} h_m \, \frac{e^{j\omega m)T}}{2}$

$= e^{j\omega kT} \sum_{m=0}^{\infty} h_m \, \frac{e^{-j\omega mT}}{2} + e^{-j\omega kT} \sum_{m=0}^{\infty} h_m \, \frac{e^{j\omega m)T}}{2}$ for $k \geq 3$

(since we have been told that $h_k = 0$ for $k > 2$)

$= \frac{1}{2}[e^{j\omega kT} h^F(\omega T) + e^{-j\omega kT} h^F(-\omega T)]$ **[5]**

$= \frac{1}{2}[e^{j\omega kT} h^F(\omega T) + e^{-j\omega kT} h^F(\omega T)^*]$ (since the h_k are real)

$= \frac{1}{2}[e^{j\omega kT} |h^F(\omega T)| e^{j\angle h^F(\omega T)} + e^{-j\omega kT} |h^F(\omega T)| e^{-j\angle h^F(\omega T)}]$

$= \frac{1}{2}|h^F(\omega T)| [e^{j\{\omega kT + \angle h^F(\omega T)\}} + e^{-j\{\omega kT + \angle h^F(\omega T)\}}]$

$= |h^F(\omega T)| \cos(\omega kT + \angle h^F(\omega T))$ (A3.1)

as required. **[4]**

(b) {*new example*}

Now $C^Z(z) = \frac{(z-j)(z+j)}{z^2} = \frac{z^2+1}{z^2} = 1 + 0.z + 1.z^{-2}$ so $h_0 = 1, \; h_1 = 0, \; h_2 = 1.$ **[2]**

The pole-zero configuration is shown in Figure A3.1 below and, since $T = 1$,

$C^F(\omega T) = C^Z(e^{j\omega T}) = 1 + e^{-2j\omega T}$ so $C^F(\omega T)|_{\omega=0} = 2$ and $C^F(\omega T)|_{\omega=\pi/2} = 1 - 1 = 0.$

This indicates that the frequency response $(|C^F(wT)| - v - \omega)$ is that shown in Figure A3.2.

Consequently (A3.1) indicates that $u_k = 0$ for all $k \geq 3$. **[3]**

In detail, $u^Z(z) = C^Z(z) e^Z(z)$ where $C^Z(z) = \frac{(z-j)(z+j)}{z^2} = \frac{z^2+1}{z^2}$ and

$e^Z(z) = \frac{z(z - \cos(\omega T))}{z^2 - 2z\cos(\omega T) + 1} = \frac{z^2}{z^2+1}$.

Hence $u^Z(z) = \frac{z^2+1}{z^2} \frac{z^2}{z^2+1} = 1$ and consequently $\{u_k\} = \{1, 0, 0, 0, \dots\}$,

which is consistent with the result predicted from the pole-zero pattern above. **[3]**

(c) {*new example* } Here

$$u^Z(z) = \frac{z^2+1}{z^2+z-2} e^Z(z) = \frac{1+z^{-2}}{1+z^{-1}-2z^{-2}} e^Z(z) = (1+z^{-2}) w^Z(z) \quad (A3.2)$$

where $w^Z(z) = \frac{e^Z(z)}{1+z^{-1}-2z^{-2}}$. Hence $(1+z^{-1}-2z^{-2}) w^Z(z) = e^Z(z)$ so we have

$$w^Z(z) = e^Z(z) - z^{-1} w^Z(z) + 2z^{-2} w^Z(z). \quad (A3.3)$$

Taking the inverse Z-transforms of (A3.2) and (A3.3), we obtain the canonical direct realization

$$u_k = w_k + w_{k-2}; \; w_k = e_k - w_{k-1} + 2w_{k-2}.$$ **[3]**

Figure A3.1

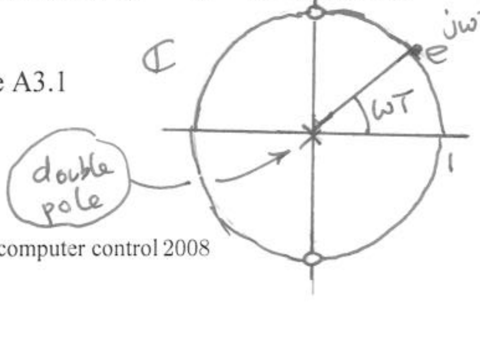
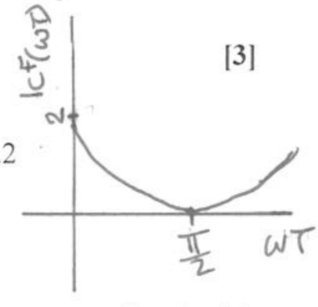


Figure A3.2



4. {*all parts unseen*}

(a) The pulse Z-transfer function for the minor loop in the forward path is $\frac{K}{1+0.75\frac{K}{z-1}} = \frac{K(z-1)}{z-1+0.75K}$.

The value of $H^Z(z)$ is $c'(zI-A)^{-1}b = [0.25 \quad 0.25]\begin{bmatrix} z-2.5 & 0.5 \\ -4.5 & z+0.5 \end{bmatrix}^{-1}\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$

$= [0.25 \quad 0.25]\begin{bmatrix} z+.5 & -0.5 \\ 4.5 & z-2.5 \end{bmatrix}\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} / \{(z+0.5)(z-2.5)+2.25\}$

$= [0.25 \quad 0.25]\begin{bmatrix} z+.5 & -0.5 \\ 4.5 & z-2.5 \end{bmatrix}\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} / \{z^2-2z+1\}$

$= [0.25 \quad 0.25]\begin{bmatrix} 0.5z \\ 2.25+0.5z\text{-}1.25 \end{bmatrix} / \{(z-1)^2\} = \frac{0.25(z+1)}{(x-1)^2}$.

The complete forward path is $\frac{K(z-1)}{z-1+0.75K} \times \frac{0.25(z+1)}{(z-1)^2} = \frac{0.25K(z+1)}{(z-1+0.75K)(z-1)}$.

Hence the closed-loop transfer function is $\dfrac{\frac{0.25K(z+1)}{(z-1+0.75K)(z-1)}}{1+\frac{0.25K(z+1)}{(z-1+0.75K)(z-1)}}$

$= \frac{0.25K(z+1)}{(z-1+0.75K)(z-1)+0.25K(z+1)} = \frac{0.25K(z+1)}{(z-1)^2+K(z-0.5)}$. **[4]**

(b) Let the closed-loop denominator be denoted $d(z) = (z-1)^2 + K(z-0.5)$

$= z^2 + z(K-2) + 1 - 0.5K$.

Then $d(1) = 0.5K > 0$ iff $K > 0$ and $d(-1) = 4-1.5K > 0$ iff $K < \frac{8}{3}$. The Jury array is:

$1-0.5K$	$K-2$	1

and the corresponding stability condition is $|1-0.5K| < 1$, i.e. $K \in (0,4)$.

Hence $K_{max} = \frac{8}{3}$. **[5]**

(c) Since the closed-loop denominator is $d(z) = (z-1)^2 + K(z-0.5)$ the corresponding root-locus is that for $G^Z(z) = \frac{z-0.5}{(z-1)^2}$. The break points σ_b for this root locus are amongst the solutions of $\frac{1}{\sigma_b-0.5} = \frac{1}{\sigma_b-1} + \frac{1}{\sigma_b-1}$ i.e. $\sigma_b-1 = 2(\sigma_b-0.5)$. Hence $\sigma_b = 0$. Therefore the root-locus is that shown in Figure A4.1 below. The root-locus shows there is a gain that locates both closed-loop poles at the origin and the gain required is $K = \frac{-1}{G^Z(0)} = 2$. **[5]**

(d) The Initial Value Theorem gives $e_0 = \lim_{|z|\to\infty} e^Z(z) = \lim_{|z|\to\infty} \frac{(z+0.5)}{z(z-1)} = 0$. The inversion integral involved gives $e_k = \frac{1}{2\pi j}\oint_{\Gamma_1} e^Z(z)z^{k-1}dz$ where Γ_1 is the circle in the complex plane that is centred on the origin and has radius 1. For $k \geq 2$, $\frac{1}{2\pi j}\oint_{\Gamma_1} e^Z(z)z^{k-1}dz = \frac{1}{2\pi j}\oint_{\Gamma_1}\frac{(z+0.5)}{z(z-1)}z^{k-1}dz$ $= \frac{1}{2\pi j}\oint_{\Gamma_1}\frac{(z+0.5)}{(z-1)}z^{k-2}dz =$ residue of $\frac{(z+0.5)}{(z-1)}z^{k-2}$ @ $1 = (z-1)\frac{(z+0.5)z^{k-2}}{(z-1)}\Big|_{z=1} = 1.5$. **[3]**

(e) The cancellation of the forward path pole at $z=1$ leaves the the A-matrix of the controlled system with 1 as an eigenvalue and this, in the presence of any noise and errors in parameter values, might well cause at least one state to diverge to ∞. **[3]**

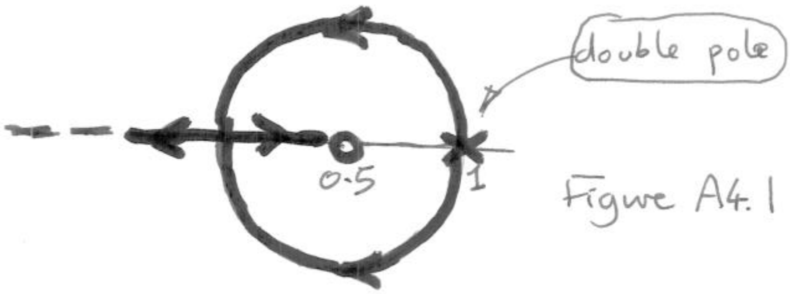


Figure A4.1

5. (a) *{bookwork}*

Let p' be the last row of the controllability matrix M and let $V = \begin{bmatrix} p' \\ p'A \\ \dots \\ p'A^{n-1} \end{bmatrix}$.

Then $VV^{-1} = I$ so $\begin{bmatrix} p' \\ p'A \\ \dots \\ p'A^{n-1} \end{bmatrix} V^{-1} = I$ and consequently $\begin{bmatrix} p'V^{-1} \\ p'AV^{-1} \\ \dots \\ p'A^{n-1}V^{-1} \end{bmatrix} = I$.

Therefore

$$\begin{aligned} p'V^{-1} &= [1 \quad 0 \quad 0 \quad \dots \quad 0] \\ p'AV^{-1} &= [0 \quad 1 \quad 0 \quad \dots \quad 0] \\ &\dots\dots\dots\dots\dots\dots \\ p'A^{n-1}V^{-1} &= [0 \quad 0 \quad 0 \quad \dots \quad 1]. \end{aligned}$$

Hence

$$VAV^{-1} = \begin{bmatrix} p' \\ p'A \\ \dots \\ p'A^{n-2} \\ p'A^{n-1} \end{bmatrix} (AV^{-1}) = \begin{bmatrix} p'AV^{-1} \\ p'A^2V^{-1} \\ \dots \\ p'A^{n-1}V^{-1} \\ p'A^nV \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & & 1 \\ * & * & * & * & * \end{bmatrix}$$

which is a companion matrix C. Consequently $A = V^{-1}CV$. **[8]**

(i) $\{$*unseen case*$\}$

The root-locus is that of Figure A5.1 so there is no gain K that locates the closed-loop poles (i.e. eigenvalues) at 0.2. **[2]**

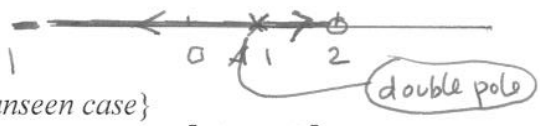


Figure A5.1

(ii) $\{$*Application of standard method to unseen case*$\}$

$Ab = \begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$ so $M = [b \ Ab] = \begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix}$.

$|M| = 4 \neq 0$ so the system is reachable.

$M^{-1} = \frac{1}{4}\begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 1 \end{bmatrix}$. Therefore $p' =$ last row of $M^{-1} = [0.5 \quad 1]$.

$p'A = [0.5 \quad 1]\begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix} = [0.5 \quad 0.5]$.

$V = \begin{bmatrix} p' \\ p'A \end{bmatrix} = \begin{bmatrix} 0.5 & 1 \\ 0.5 & 0.5 \end{bmatrix}$ so $V^{-1} = \frac{1}{(-0.25)}\begin{bmatrix} 0.5 & -1 \\ -0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ 2 & -2 \end{bmatrix}$.

The corresponding companion matrix is

$C_\alpha = VAV^{-1} = \begin{bmatrix} 0.5 & 1 \\ 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -2 & 4 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

where $\alpha' =$ last row of $C_\alpha = [1 \quad 0]$.

The desired characteristic polynomial $= (\lambda - 0.2)(\lambda - 0.2) = \lambda^2 - 0.4\lambda + 0.04$ $= \lambda^2 - \phi_2\lambda - \phi_1$ so the coefficient vector $\phi' = [-0.04 \quad 0.4]$.

The required feedback vector f is therefore

$f = V'(\alpha - \phi) = \begin{bmatrix} 0.5 & 0.5 \\ 1 & 0.5 \end{bmatrix}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} -0.04 \\ 0.4 \end{bmatrix}\right) = \begin{bmatrix} 0.32 \\ 0.84 \end{bmatrix}$. **[10]**

6. (a) {*bookwork*}

A and M are similar implies that $A = PMP^{-1}$ for a suitable P. Then $|\lambda I - A| = |\lambda I - PMP^{-1}| = |\lambda PP^{-1} - PMP^{-1}| = |P(\lambda I - M)P^{-1}| = |P||\lambda I - M||P^{-1}|$ $= |\lambda I - M|$. Therefore the eigenvalues of A (which are the values of λ such that $|\lambda I - A| = 0$) are the eigenvalues of M. **[3]**

(b) (i) {*a way of analysing the situation that is different from that used in the lectures*}

Now $\begin{bmatrix} x_{k+1} \\ \widehat{x}_{k+1} \end{bmatrix} = \begin{bmatrix} Ax_k + b(r_k - f'\widehat{x}_k) \\ (A - lc')\widehat{x}_k + ly_k + b(r_k - f'\widehat{x}_k) \end{bmatrix} = \begin{bmatrix} Ax_k - bf'\widehat{x}_k) \\ (A^{ob}\widehat{x}_k + lc'x_k - bf'\widehat{x}_k) \end{bmatrix} + \begin{bmatrix} br_k \\ br_k \end{bmatrix}$ $= \begin{bmatrix} A & -bf' \\ lc' & A^{ob} - bf' \end{bmatrix} \begin{bmatrix} x_k \\ \widehat{x}_k \end{bmatrix} + \begin{bmatrix} b \\ b \end{bmatrix} r_k = \overline{A}\,\overline{x}_k + \bar{b}r_k$ where $\overline{A} = \begin{bmatrix} A & -bf' \\ lc' & A^{ob} - bf' \end{bmatrix}$, $\bar{b} = \begin{bmatrix} b \\ b \end{bmatrix}$.

Further $P\overline{A}\,P^{-1} = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} A & -bf' \\ lc' & A^{ob} - bf' \end{bmatrix} \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix}$

$= \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} A - bf' & bf' \\ lc' + A^{ob} - bf' & -A^{ob} + bf' \end{bmatrix} = \begin{bmatrix} A^f & bf' \\ A^f - lc' - A^{ob} + bf' & A^{ob} \end{bmatrix} = \begin{bmatrix} A^f & bf' \\ 0 & A^{ob} \end{bmatrix}$

so, by a property of block upper-triangular matrices, the eigenvalues of $\overline{A}$ are the eigenvalues of $A^f = A - bf'$ together with the eigenvalues of $A^{ob} = A - lc'$. The eigenvalues of $\overline{A}$ determine overall performance and the above result reveals that f and l can be designed separately without interaction. This makes such design relatively simple. **[9]**

(ii) {*extension of bookwork*} If f and l are chosen so the eigenvalues of $A - bf$ and $A - lc'$ are all zero then it follows from above that all the eigenvalues of $\overline{A}$ are zero.

$\overline{A}$ is similar to a companion matrix for which the bottom row contains only zeros (since all its eigenvalues are zero).. Therefore $\overline{A} = \overline{P}\ \overline{C}\ \overline{P}^{-1}$ for an appropriate P

where $\overline{C} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Since $r_k \equiv 0$, it follows that $\overline{x}_k = \overline{A}^k \overline{x}_o$

$= (\overline{PCP}^{-1}) \ldots (\overline{PCP}^{-1})\overline{x}_o = \overline{P}\,\overline{C}^{\,k}\overline{P}^{-1}\overline{x}_o$. Note that

$\overline{C}^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\overline{C}^3 = \overline{C}^2\overline{C} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\overline{C}^4 = \overline{C}^3\,\overline{C} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and

$\overline{C}^5 = \overline{C}^4\,\overline{C} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Consequently $\overline{C}^k = 0$ for all $k \geq 5$. Therefore $\overline{x}_k = 0$, for all $k \geq 5$. Hence both x_k and $\widehat{x}_k$ become zero by time t_5, whatever the initial conditions of the plant and observer, which seems attractive.

A downside is that large control actions might be needed. **[8]**