UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 1999

BEng Honours Degree in Computing Part I
MEng Honours Degrees in Computing Part I
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the Associateship of the City and Guilds of London Institute

PAPER 1.3

MATHEMATICAL REASONING – DISCRETE MATHEMATICS

Monday, May 10th 1999, 4.00 – 5.30

Answer THREE questions

For admin. only:
paper contains 4 questions

1a i) Define the *Cartesian product* $A \times B$ of two sets A, B. Also, given $R \subseteq A \times B$, $S \subseteq B \times C$, define the *relational composition* $R \circ S$.

ii) Prove the following law of set algebra:

$$(A \times B) \cap (A' \times B') = (A \cap A') \times (B \cap B').$$

iii) State with reasons whether each of the following "laws" holds:

$$(A \times B) \cup (A' \times B') = (A \cup A') \times (B \cup B')$$

$$(R \cup S) \circ T = (R \circ T) \cup (S \circ T)$$

b i) Define the terms *reflexive, transitive,* and *anti-symmetric* as applied to a binary relation R on a set A.

ii) Give an example of a set A with three elements, and a relation R on A which is reflexive but not transitive. Can such an example still be given if the set A is required to have fewer than three elements? (Explain.)

iii) Determine (by a proof or counterexample) whether the following is necessarily true:

If a relation R on a set B is reflexive, symmetric and anti-symmetric, then R is the relation = (that is, the identity relation on B).

2a i) Let P be a set with a partial order $\leq$ defined on it. What is meant by saying that an element of P is (a) *least*, (b) *minimal*, with respect to $\leq$? Give a simple example to illustrate the difference between the two.

ii) Briefly explain the principle of the "topological sort" algorithm for arranging a finite partially ordered set in linear order.

iii) List the partial orders on the set $\{a,b\}$ (that is, the distinct relations on this set which are partial orders) which have b as a minimal element.

b Let a function $f{:}A \to B$ be given.

i) Explain what is meant by the *image* of a set $X \subseteq A$, and by the *pre-image* of a set $Y \subseteq B$, under f.

ii) Suppose that $|B| = n$, and that $|f^{-1}(b)| \leq 1$ for all b in B. What can we conclude about $|A|$?

iii) State the Pigeonhole Principle, with regard to the function f. How does this relate to your conclusion in part ii) ?

iv) Suppose instead that $|A| > 2|B|$. Show that there is an element of B such that at least three distinct elements of A are mapped to it.

v) Suppose that, in a network of k computers, each machine is directly linked to at least one, but not more than five, other machines. Show that if $k \geq 11$ there are (at least) three computers with exactly the same number of direct links to other computers.

The two parts carry, respectively, 40%, 60% of the marks.

Turn over ...

3a Define the following as applied to (undirected) graphs:

i) *connected*

ii) *simple*

iii) *Euler circuit*

iv) *degree* of a node

b For the purposes of this question, let an *RB graph* be defined as a graph where each arc (edge) is coloured either red or blue. An RB Euler circuit is an Euler circuit where successive arcs have different colours.

i) Draw two simple connected RB graphs G_1 and G_2, satisfying the following:

G_1 has an Euler circuit but no RB Euler circuit.
G_2 has an RB Euler circuit.

Label each graph clearly. For convenience, you can represent red arcs by solid lines and blue arcs by dotted lines.

ii) Suggest a condition on RB graphs which is necessary and sufficient for a connected RB graph to have an RB Euler circuit.

iii) Explain why your condition in (ii) is necessary.

iv) Give an informal description of an algorithm which, given a connected RB graph satisfying your condition in (ii), constructs an RB Euler circuit.

The two parts carry, respectively, 25%, 75% of the marks.

4a i) Describe the algorithm Insertion Sort.

ii) What is the worst case number of comparisons made by Insertion Sort when sorting a list of length n?

b i) Briefly describe the algorithm Quicksort.

ii) Give an example of a list for which Insertion Sort performs better than Quicksort, explaining your answer briefly.

iii) State a recurrence relation for the average number of comparisons made by Quicksort when sorting a list of length n. Do not solve the recurrence relation.

c A sorted list of length 2n is disarranged to create a new list L by placing the elements in the odd positions (still in order) before the elements in even positions (still in order). Thus e.g. the list [1,2,3,4,5,6] would become [1,3,5,2,4,6].

i) How many comparisons does Insertion Sort take when applied to L? Explain your working.

ii) How many comparisons does Quicksort take when applied to L? Assume that Quicksort always splits around the first element. Explain your working.

The three parts carry, respectively, 25%, 35%, 40% of the marks.

End of paper

