EE4-23

SOLUTIONS: STABILITY AND CONTROL OF NONLINEAR SYSTEMS
MASTER IN CONTROL

1. Exercise

a) The vector field $f(x)$ is a polynomial in x_1 and x_2. It is therefore differentiable an infinite number of times, and in particular locally Lipschitz continuous. Therefore solutions exists and are unique.

b) The first nullcline, $\mathcal{N}_1$ is given by:

$$\mathcal{N}_1 = \{(x_1,x_2) : x_2 = x_1 - \frac{x_1^3}{3} + \frac{4}{3}\}.$$

This is the cubic shown in the Figure. In particular, it admits local minima and maxima in correspondence of the roots of the following equation:

$$\frac{d}{dx_1}\left(x_1 - \frac{x_1^3}{3} + \frac{4}{3}\right) = 1 - x_1^2 = 0,$$

that is for $x_1 = \pm 1$. The second nullcline is a straight line of equation:

$$\mathcal{N}_2 = \{(x_1,x_2) : x_2 = x_1 + 1\}.$$

c) A graphical sketch of the nullclines is shown in the Figure 1.1, with the 4 different regions labeled with the corresponding orientations of the vector-field.

d) The equilibria are the intersection of the Nullclines. In particular there is a single equilibrium, in $x_e = (1,2)$.

e) Differentiating $f(x)$ yields:

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 1-x_1^2 & -1 \\ 1 & -1 \end{bmatrix}.$$

Evaluating the previous expression at the equilibrium point $(1,2)$ yields the following linearized system:

$$\delta\dot{x} = \frac{\partial f}{\partial x}|_{x=x_e}\delta x = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \delta x.$$

The associated characteristic polynomial is given by:

$$\det\left(\begin{bmatrix} s & 1 \\ -1 & s+1 \end{bmatrix}\right) = s^2 + s + 1.$$

Its roots are given by:

$$\lambda_{1,2} = \frac{-1 \pm i\sqrt{3}}{2}$$

and therefore the local phase-portrait is that of a stable focus.

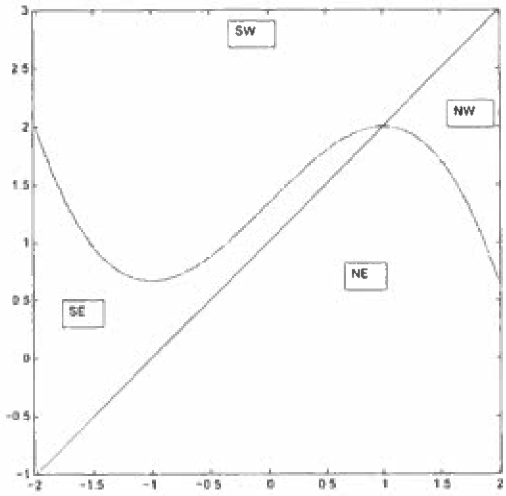

Figure 1.1 Nullclines and regions in state-space

f) Taking derivatives of $g(x) := x_1^2 + x_2^2$ along solutions of the system yields:

$$\frac{\partial g}{\partial x} f(x) = 2x_1^2 - 2x_1^4/3 + 8x_1/3 - 2x_2^2 + 2x_2.$$

Notice that, for all x fulfilling $x_1^2 + x_2^2 = R^2$ we have:

$$|x_1| \leq R, |x_2| \leq R$$

Moreover, either $|x_1| \geq R/2$ or $|x_2| \geq R/\sqrt{2}$ (or both). Let us consider the case $|x_1| \geq R/2$. We see that:

$$\frac{\partial g}{\partial x} f(x) = 2x_1^2 - 2x_1^4/3 + 8x_1/3 - 2x_2^2 + 2x_2 \leq 2R^2 - R^4/24 + 8R/3 + 2R$$

Since R^4 grows faster than all other terms, this is negative for all $R > \bar{R}$, where $\bar{R}$ is a sufficiently large positive real. If instead $|x_1| \leq R/2$ and $|x_2| \geq R/\sqrt{2}$ we see that:

$$\frac{\partial g}{\partial x} f(x) \leq R^2/2 + 8R/3 - R^2 + 2R = -R^2/2 + 8R/3 + 2R.$$

Since R^2 grows faster than all other terms, the expression is negative for sufficiently large $R > \hat{R}$. Overall then, combining the two cases, $\frac{\partial g}{\partial x} f(x) < 0$ for all x with $x_1^2 + x_2^2 = R^2$ for $R > \max\{\bar{R}, \hat{R}\}$. This implies forward invariance of the considered disc.

g) The phase-portrait of the system is shown in Fig. 1.2.

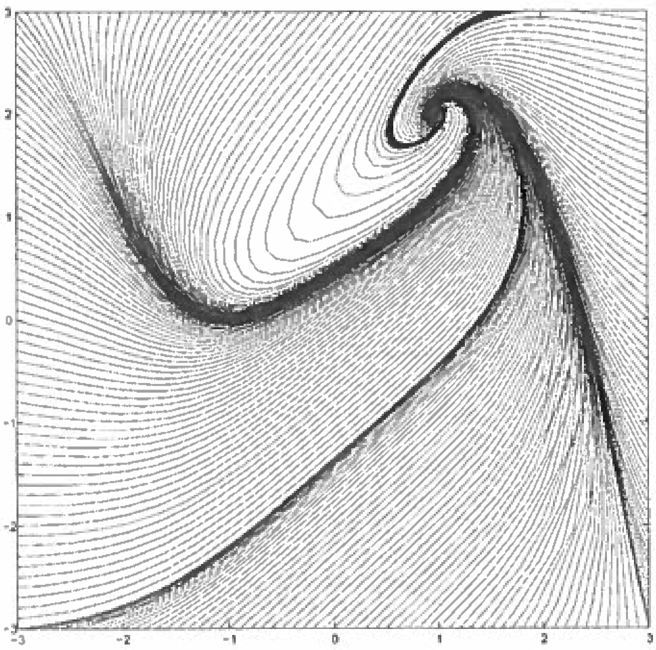

Figure 1.2 Phase-portrait

2. Exercise

a) Consider $V_1(x)$ first. Notice that the term $x_1x_2^2$ is not sign definite, and of degree 3, hence lower than the 2 sign definite terms x_1^4, x_2^4. This means that close to the origin there are points where it will dominate the other 2 terms. For instance taking $x_1 = x_2 < 0$ yields:

$$V_1(x_1,x_2) = 2x_1^4 + x_1^3$$

which is negative for all $x_1 \in (-2,0)$. Notice that for $x_2 = 0$ and $x_1 \neq 0$ it holds $V_1(x_1,x_2) = x_1^4 > 0$. Hence V_1 is not sign definite. Consider next $V_2(x_1,x_2)$. This can be written as:

$$V_2(x_1,x_2) = [x_1^2, x_1x_2, x_2^2] \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} x_1^2 \\ x_1x_2 \\ x_2^2 \end{bmatrix}.$$

Notice that the above symmetric matrix is positive definite. Indeed, the determinants of its principal minors are 1, 3/4 and 1/2.

b) The equilibrium points are all solutions of the following system of equations:

$$\begin{cases} -4x_1^3 + x_2^3 - 2x_1x_2^2 + 3x_1^2x_2 &= 0 \\ 3x_2^2x_1 - 2x_1^2x_2 + x_1^3 - 4x_2^3 &= 0 \end{cases}$$

Taking the sum of the two equations above yields:

$$-3x_1^3 - 3x_2^3 + x_1^2x_2 + x_1x_2^2 = 0$$

We factor the previous expression as follows:

$$(x_1 + x_2)(4x_1^2 - 3x_1x_2 + 4x_2^2) = 0$$

One equilibrium is clearly obtained for $(x_1,x_2) = 0$. However, if $(x_1,x_2) \neq 0$, then $(4x_1^2 - 3x_1x_2 + 4x_2^2) > 0$ and therefore, at equilibrium we necessarily have $(x_1 + x_2) = 0$. Substituting $x_1 = -x_2$ in the first equation yields $10x_2^3 = 0$ which in turn implies $0 = x_2 = x_1$. Therefore there exists only a single solution of the above system of equations and ultimately a unique equilibrium point for the considered dynamical system.

c) We use $V_2(x)$ as a candidate Lyapunov function. Notice that $V_2(x)$ is positive definite as shown in the first item of this exercise. Moreover, its derivative reads:

$$\begin{aligned} \dot{V}_2(x) &= \frac{\partial V_2}{\partial x}(x_1,x_2)\dot{x} \\ &= [4x_1^3 - x_2^3 + 2x_1x_2^2 - 3x_1^2x_2, -3x_2^2x_1 + 2x_1^2x_2 - x_1^3 + 4x_2^3]\dot{x} = -|\dot{x}|^2 \end{aligned}$$

Clearly $\dot{V}$ is at least negative semidefinite. Moreover, since $\dot{x}$ only vanishes for $x = 0$, negative definiteness follows. In order to prove global stability of the origin we only need to prove that $V_2(x)$ is radially unbounded. This follows for instance by homogeneity of V, by remarking that:

$$V_2(\lambda x) = \lambda^4 V_2(x).$$

In particular then, letting:

$$L := \min_{|x|=1} V_2(x) > 0$$

we see that for any $x \in \mathbb{R}^2$ it holds:

$$V_2(x) \leq M \Rightarrow L \leq V_2(x/|x|) = V_2(x)/|x|^4 \leq M/|x|^4.$$

Hence $V_2(x) \leq M \Rightarrow |x| \leq \sqrt[4]{M/L}$.

d) Taking derivatives of V_2 along solutions of the modified system yields:

$$\dot{V}_2(x) = \frac{\partial V_2}{\partial x}(x_1, x_2)\dot{x} = 0.$$

This implies that $V_2(\varphi(t, x_0)) = V_2(x_0)$ for all t. Moreover, since level sets of V_2 are closed curves and there is only a single equilibrium at the origin, each solution initiated at $x_0 \neq 0$ will have a periodic behaviour, with:

$$\omega(x_0) = \{x \in \mathbb{R}^2 : V_2(x) = V_2(x_0)\}.$$

3. Exercise

a) Consider the candidate Lyapunov function $V(x) = x^2/2$. This is clearly positive definite and radially unbounded. Notice that, for all $\lambda > 0$ the function $xg(\lambda x)$ is positive definite, while $xg(-\lambda x)$ is negative definite. Taking derivatives of V along solutions of the system yields:

$$\dot{V} = \frac{\partial V}{\partial x} f(x,u) = xg(u - \alpha x).$$

Let $|x| \geq 3|u|/2\alpha$. We prove that V is an ISS Lyapunov function by separately dealing with the case $x > 0$ and $x > 0$. Consider first $x > 0$. We have: $u - \alpha x \leq 2\alpha x/3 - \alpha x = -\alpha x/3$. Hence:

$$\dot{V} = xg(u - \alpha x) \leq xg(-\alpha x/3)$$

which is negative definite. A symmetric argument applies to the case $x < 0$. Hence:

$$|x| \geq 3|u|/2\alpha \Rightarrow \dot{V} < 0$$

which proves ISS of the system.

b) An upper of the ISS gain is therefore $\gamma(r) = 3r/2\alpha$.

c) Notice that the two-dimensional system can be interpreted as the feedback interconnection of two scalar systems:

$$\dot{x}_i = g_i(u_i - 2x_i) \qquad i = 1,2.$$

Their gain γ_i is less than $\gamma_i(r) = 3r/4$. Therefore $\gamma_1(\gamma_2(r)) = \frac{9}{16}r < r$ for all $r > 0$. Hence the small gain theorem applies and the system is GAS at the origin.

d) Let $V(x)$ be as before. For disturbances of absolute value $|d| \leq 1$ we have:

$$\dot{V} = x(2+d)g(-x) \leq xg(-x) < 0$$

Hence, $V(x)$ qualifies as a Lyapunov function for Uniform GAS of the system at the origin.

4. Exercise

a) We choose $x(t) = [\theta(t), \dot{\theta}(t)]' = [x_1(t), x_2(t)]$. Accordingly we see:

$$\dot{x} = \begin{bmatrix} x_2 \\ -\sin(x_1) + u \end{bmatrix}.$$

b) The system is input affine, in fact $\dot{x} = f(x) + g(x)u$ with

$$f(x) = \begin{bmatrix} x_2 \\ -\sin(x_1) \end{bmatrix} \qquad g(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

c) Letting $S_1(x) = \frac{x_2^2}{2} - \cos(x_1)$ we have:

$$\frac{\partial S_1}{\partial x} = [\sin(x_1), x_2].$$

Moreover:

$$\frac{\partial S_1}{\partial x} f(x) = [\sin(x_1), x_2] \begin{bmatrix} x_2 \\ -\sin(x_1) \end{bmatrix} = 0$$

Therefore, thanks to the nonlinear KYP lemma it is enough to choose $h(x)$ as follows:

$$\frac{\partial S_1}{\partial x} g(x) = x_2 = h(x).$$

for the system to be passive.

d) Let $x_3 = \int_0^t e(\tau) d\tau$. We have the following state-space equation:

$$\dot{x}_3(t) = e(t) \qquad v(t) = e(t) + x_3(t).$$

To show passivity notice that:

$$ve = e^2 + x_3 e \geq x_3 e = x_3 \dot{x}_3 = \frac{\partial S_2}{\partial x_3} \dot{x}_3$$

provided $S_2(x_3) = \frac{x_3^2}{2}$.

e) In closed-loop we have: $\chi = [x_1, x_2, x_3]'$ and:

$$\dot{\chi} = \begin{bmatrix} x_2 \\ -\sin(x_1) - x_3 - x_2 \\ x_2 \end{bmatrix}.$$

The sum of the two storage functions reads:

$$V(\chi) = \frac{x_2^2}{2} - \cos(x_1) + \frac{x_3^2}{2}.$$

Along solutions of the closed-loop system we have:

$$\dot{V} = [\sin(x_1), x_2, x_3] \cdot \begin{bmatrix} x_2 \\ -\sin(x_1) - x_3 - x_2 \\ x_2 \end{bmatrix} = -x_2^2.$$

Hence:

$$\mathrm{Ker}[\dot{V}] = \{x : x_2 = 0\}.$$

The largest invariant set contained in $\mathrm{Ker}[\dot{V}]$ must also belong to the following smaller set:

$$\{x : \dot{x}_2 = 0 \& x_2 = 0\} = \{x : x_2 = 0 \& \sin(x_1) + x_3 = 0\}$$

This is indeed an invariant set as:

$$x_2 = 0 \;\&\; \sin(x_1) + x_3 = 0 \;\Rightarrow\; \dot{x}_2 = 0 \;\&\; \frac{d}{dt}\sin(x_1) + x_3 = 0.$$