E3.10
C1.6
ISE3.7

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2005

MSc and EEE/ISE PART III/IV: MEng, BEng and ACGI

Corrected Copy

MATHEMATICS FOR SIGNALS AND SYSTEMS

Monday, 9 May 10:00 am

Time allowed: 3:00 hours

There are FIVE questions on this paper.

Answer THREE questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : G. Weiss

Second Marker(s) : J.C. Allwright

1. Consider the space $H = \mathbb{C}^{4\times 2}$ of matrices with four rows and two columns. We define an inner product on H by $< A, B > \; = \text{trace} B^* A$, where B^* is the complex conjugate of the transpose of B, and we define $\|A\|_H^2 = \; < A, A >$.

   (a) What is the dimension of H? [1]

   (b) If

$$A = \alpha \begin{bmatrix} 1 & -1 \\ 0 & 7 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

   find a number α and two matrices B and C in H such that $\{A, B, C\}$ is an orthonormal set. [4]

   (c) Give an example of a subspace $V \subset H$ with $V \neq H$ which contains A, B and C. [2]

   (d) Give an example of a subspace $W \subset H$ with $W \neq \{0\}$ which does not contain any of A, B and C. [2]

   (e) If

$$D = \begin{bmatrix} 3 & 0 \\ 0 & 0 \\ 0 & -5 \\ 0 & 0 \end{bmatrix},$$

   compute $\|D\|_H$. Compute also the norm $\|D\|$ when D is regarded as an operator from $\mathbb{C}^2$ to $\mathbb{C}^4$. (Hint: be careful, the norm of D as an operator is not the same as $\|D\|_H$.) [4]

   (f) We define on $\mathbb{C}^2$ the functions p, q and r by

$$p(x) = |x_1 + x_2|, \qquad q(x) = \|Dx\|, \qquad r(x) = \|Dx\|^2,$$

   where D is the matrix from part (e) and $\|\cdot\|$ is the usual Euclidean norm on $\mathbb{C}^4$. State, without proof, which of the functions p, q and r is a norm on $\mathbb{C}^2$? For each function (if any) that is not a norm, explain briefly why it is not a norm. [4]

   (g) Which of the norms that you found in part (f) is derived from an inner product? If there is such a norm, indicate the corresponding inner product. [3]

2. For $1 \le p < \infty$, we denote by l^p the space of all sequences u indexed by $k \in \{0, 1, 2, 3, ...\}$ for which $\Sigma_{k=0}^{\infty}|u_k|^p < \infty$. For such sequences u, we use the notation $\|u\|_p = (\Sigma_{k=0}^{\infty}|u_k|^p)^{\frac{1}{p}}$. We denote by l^∞ the space of all bounded sequences, and let $\|u\|_\infty = \sup |u_k|$.

We define the sequence v by

$$v = \left(1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{5}, \dots\right),$$

so that the value $\frac{1}{k}$ is repeated k times.

(a) Estimate $\|v\|_p$ for $p = 1000$ (an estimation error of $\pm 5\%$ is acceptable). Briefly explain your reasoning. [2]

(b) For which values of p do we have $v \in l^p$? [3]

(c) Let $\hat{v}$ denote the $\mathcal{Z}$-transform of v. Find the largest domain where the series defining $\hat{v}(z)$ is convergent. Hint: notice that the series for $\hat{v}(z)$ cannot be convergent for $z = 1$. [3]

(d) Consider the discrete-time linear system with transfer function

$$\mathbf{G}(z) = \frac{1 - z^{-1}}{2 + z^{-1}}.$$

Is $\mathbf{G}$ rational? Is $\mathbf{G}$ stable? Is $\mathbf{G}$ strictly proper? Is $\mathbf{G}$ a FIR filter? If y is the step response of $\mathbf{G}$, compute $\lim_{k\to\infty} y_k$. [2]

(e) If the input signal of $\mathbf{G}$ from part (d) is the sequence v defined earlier, and the output signal is denoted by w, show that $w \in l^1$. Hint: First show that the signal whose $\mathcal{Z}$-transform is $(1 - z^{-1})\hat{v}$ is in l^1. Then think about the impulse response of the transfer function $\frac{1}{2+z^{-1}}$. [4]

(f) If $a \in l^1$ and $b \in l^2$, is it true that the series $\sum_{k=1}^{\infty} a_k b_k$ converges to a finite sum? If yes, then say why, if no, then give a counterexample. [3]

(g) For a and b as in part (f), show that the sequence $a * b$ (the convolution of a and b) is in l^2. [3]

3. In this question, $\mathbf{S}_\tau$ denotes the right shift operator by τ on $L^2[0,\infty)$ and $*$ denotes the convolution product.

(a) Define the inner product and the norm on the space $L^2[0,\infty)$. Give an example of a closed sub-space $V \subset L^2[0,\infty)$, other than $\{0\}$ and the whole space $L^2[0,\infty)$. [2]

(b) Give an example of a continuous function $\varphi : [0,\infty) \to \mathbb{R}$ such that $\lim_{t\to\infty} \varphi(t) = 0$ and $\varphi \notin L^2[0,\infty)$. [3]

(c) In the sequel, consider $f(t) = te^{-t}$ and $g(t) = e^{-5t}$, $t \geq 0$. Compute the Laplace transforms $F = \mathcal{L}f$ and $G = \mathcal{L}g$. [2]

(d) Compute the dimension of span$\{f, \dot{f}, \ddot{f}, \dddot{f}\}$. (Here, a dot denotes a derivative.) Give a very brief explanation for your result. [3]

(e) Compute $\|f\|_2$, $< f, g >$ and $\|g\|_2$ and check that the Cauchy-Schwarz inequality holds for them. [4]

(f) Define $h = \mathbf{S}_{13}g$, i.e., h is obtained by delaying g by 13 time units. Compute

$$H = \mathcal{L}h, \qquad \|h\|_2 \qquad \text{and} \quad P = \mathcal{L}(h * g). \qquad [3]$$

(g) With H as in part (f), compute

$$\|G\|_2, \qquad \|H\|_2 \quad \text{and} \quad < F, G > ,$$

where the norms and the scalar product correspond to the Hardy space $H^2(\mathbb{C}_+)$. [3]

4. Consider the system described by

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ K \end{bmatrix} u,$$

$$y = [0 \quad 1]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

where u is the input signal, x is the state (with two components), y is the output signal and K is a real constant.

(a) For which values of K is the system stable? [2]

(b) Compute the transfer function $\mathbf{G}$ of this system. [2]

(c) Determine a value of K for which $\mathbf{G}$ becomes a first order transfer function. [2]

(d) For $K = 0$, compute $\|\mathbf{G}\|_\infty$. [3]

(e) If $u(t) = e^{-t} - e^{-3t}$ and $x(0) = 0$, compute the Laplace transform of the output signal, $\hat{y} = \mathcal{L}y$, and determine whether $\hat{y} \in H^2(\mathbb{C}_+)$. [3]

(f) Consider the cascade connection of the system with a delay line of 0.7 time units. Thus, if z is the output signal of the delay line, then $z(t) = y(t - 0.7)$. Compute the transfer function $\mathbf{H}$ from u to z. [2]

(g) For $\mathbf{H}$ as in part (f) and $K = 1$, compute $\|\mathbf{H}\|_\infty$. [3]

(h) Suppose now that K is a function of t: $K(t) = \cos t$. Does the system with input u and output y have a transfer function? Explain very briefly your answer. [3]

5. In parts (a) and (c) of this question you are asked to explain a concept and in (b) and (d) you are asked to state a theorem and to comment on it. You may state the two theorems in your own words. Try to add comments about the significance and the applications of the two theorems, but do not exceed one page per theorem (including the comments).

   (a) Explain briefly what is a time-invariant operator on $L^2[0,\infty)$. [4]

   (b) State the Fourés-Segal theorem (continuous-time version) and make some comments about its connections with systems theory. [6]

   (c) Define the space $BL(\omega_b)$ of band-limited functions with angular frequencies not higher than ω_b. [4]

   (d) State the sampling theorem and, if possible, make some comments about its significance for the transmission and storage of signals. [6]

[END]

Mathematics for Signals & Systems

Exam of May 2005

SOLUTIONS

Question 1 (a) 8. (b) $\alpha = \frac{1}{\sqrt{51}}$,

$$B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$

(there are many other solutions, of course).

(c) $V = \operatorname{span}\{A, B, C\}$ (d) $W = V^{\perp}$

(e)

$$\|D\|_H = \sqrt{3^2 + 5^2} = \sqrt{34}.$$

$$\|D\|^2 = \max \sigma(D^* D) = \max \sigma \begin{bmatrix} 9 & 0 \\ 0 & 25 \end{bmatrix}$$

$$\Longrightarrow \|D\| = 5.$$

(f) q is a norm. p is not a norm because $p\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = 0$, r is not a norm because $r(\lambda x) \neq \lambda r(x)$.

(g) $q(x) = \sqrt{\langle\langle x, x \rangle\rangle}$, where $\langle\langle \cdot, \cdot \rangle\rangle$ is an inner product, $\langle\langle x, y \rangle\rangle = \langle Dx, Dy \rangle$.

$\boxed{\text{Question 2}}$ (a) $\|v\|_{1000}^{1000} = 1 + \underbrace{\frac{2}{2^{1000}} + \frac{3}{3^{1000}} + \dots}_{\text{very small}}$

≈ 1, so that $\|v\|_{1000} \approx 1$. A more sophisticated reasoning uses $\lim_{p\to\infty} \|v\|_p = \|v\|_\infty = 1$.

(b) We have $\|v\|_p^p = 1 + \frac{1}{2^{p-1}} + \frac{1}{3^{p-1}} + \frac{1}{4^{p-1}} + \dots$

and this converges for $p-1>1$, i.e., for $p>2$. (The convergence follows for these values of p from

$$\int_1^\infty \frac{1}{x^\alpha}\,dx < \sum_{k=1}^\infty \frac{1}{k^\alpha} < 1 + \int_1^\infty \frac{1}{x^\alpha}\,dx \quad \forall \alpha > 0.)$$

(c) From $\frac{1}{R} = \limsup |v_k|^{\frac{1}{k}} \leq 1$ (actually, $=1$) we see that the Taylor series in z^{-1} $\hat{v}(z) = \sum v_k z^{-k}$ is convergent for $|z^{-1}| < R$, i.e., $|z| > \frac{1}{R}$. Since it is obviously not convergent for $z=1$, it follows that the domain of convergence is $\mathcal{E} = \{z \in \mathbb{C} \mid |z| > 1\}$.

(d) $G(z) = \frac{z-1}{2z+1}$ is rational, stable, not strictly proper and not FIR. $\lim_{k\to\infty} y_k = G(1) = 0$ (this is the DC gain).

(e) $\hat{w}(z) = \frac{1}{2+z^{-1}} (1-z^{-1})\hat{v}(z) + \hat{\varphi}(z)$, where φ is the component of the output caused by the initial state. If $\hat{\eta}(z) = (1-z^{-1})\hat{v}(z)$ then $\eta_k = v_k - v_{k-1}$, and it is easy to see that $\eta \in \ell^1$. Let q be the impulse response corresponding to $1/(2+z^{-1})$, so that $q \in \ell^1$. Then $w = q * \eta + \varphi$. We have $q, \eta, \varphi \in \ell^1$, so that $w \in \ell^1$.

(f) $a \in \ell^1 \Rightarrow a \in \ell^2$ and $a_k b_k \leq \frac{1}{2}(|a_k|^2 + |b_k|^2)$.

(g) Denote S = right shift (delay) by one step. Then $a*b = a_0 b + a_1 S b + a_2 S^2 b + \dots$, hence $\|a*b\|_2 \leq |a_0| \cdot \|b\|_2 + |a_1| \cdot \|b\|_2 + |a_2| \cdot \|b\|_2 + \dots = \|a\|_1 \cdot \|b\|_2$.

Question 3 (a) $\langle f,g\rangle = \int_0^\infty f(t)\overline{g(t)}\,dt$, $\|f\| = \sqrt{\langle f,f\rangle}$. Here, f and g are determined up to equality almost everywhere (they are equivalence classes of functions). $V = \{ f \in L^2[0,\infty) \mid f(t)=0 \text{ for } t\in[0,3]\}$. Another example: $V = \{ f\in L^2[0,\infty) \mid \hat{f}(7)=0\}$. A wrong example would be: $V=\{f\in L^2[0,\infty) \mid f(8)=0\}$, because $f(8)$ is not determined (it may vary within an equivalence class). (Here, $\hat{f}$ denotes the Laplace transform of f.)

(b) $\varphi(t) = 1/\sqrt{1+t}$. (c) $F(s) = \frac{1}{(s+1)^2}$, $G(s) = \frac{1}{s+5}$.

(d) $\dim \operatorname{span}\{f,\dot{f},\ddot{f},\dddot{f}\} = 2$, because all the derivatives of f are linear combinations of e^{-t} and te^{-t}.

(e) $\|f\|_2^2 = \int_0^\infty e^{-2t}t^2\,dt = \widehat{t^2}(2) = \frac{2}{s^3}\Big|_{s=2} = \frac{1}{4}$, so that $\|f\|_2 = \frac{1}{2}$. $\langle f,g\rangle = \int_0^\infty e^{-6t}t\,dt = \hat{t}(6) = \frac{1}{s^2}\Big|_{s=6} = \frac{1}{36}$.
$\|g\|_2^2 = \int_0^\infty e^{-10t}\,dt = \frac{1}{10}$. Cauchy-Schwarz: $\frac{1}{36} \le \frac{1}{2}\cdot\frac{1}{\sqrt{10}}$.

(f) $H(s) = e^{-13s}G(s)$, $\|h\|_2 = \|g\|_2 = \frac{1}{\sqrt{10}}$, $P = H\cdot G$, so that $P(s) = e^{-13s}\frac{1}{(s+5)^2}$.

(g) According to the Paley-Wiener theorem (continuous-time version) the Laplace transformation is a unitary operator, so that it preserves norms and inner products. Thus, $\|G\|_2 = \|g\|_2 = \frac{1}{\sqrt{10}}$, $\|H\|_2 = \|h\|_2 = \frac{1}{\sqrt{10}}$, $\langle F,G\rangle = \langle f,g\rangle = \frac{1}{36}$.

Question 4

(a) $\dot{x} = Ax + Bu$, $B = \begin{bmatrix} 1 \\ K \end{bmatrix}$,

$A = \begin{bmatrix} 0 & -3 \\ 1 & -4 \end{bmatrix}$, $\det(sI - A) = s^2 + 4s + 3$ (characteristic polynomial)

$= (s+1)(s+3)$, so that the eigenvalues of A are -1 and -3. The system is stable regardless of K.

(b) $G(s) = [0 \ 1](sI-A)^{-1}B = \dfrac{Ks+1}{s^2+4s+3}$.

(c) For $K = 1$, $G(s) = \dfrac{s+1}{(s+1)\cdot(s+3)}$

$= \dfrac{1}{s+3}$. Another possibility: $K = \frac{1}{3}$.

(d) $\left\| \dfrac{1}{s^2+4s+3} \right\|_\infty = \dfrac{1}{3}$, because the maximum is attained at $s=0$.

(e) $(\mathcal{L}u)(s) = \dfrac{1}{s+1} - \dfrac{1}{s+3} = \dfrac{2}{s^2+4s+3}$,

$\underbrace{(\mathcal{L}y)}_{\hat{y}}(s) = G(s)(\mathcal{L}u)(s) = \dfrac{(Ks+1)\cdot 2}{(s^2+4s+3)^2}$, $\hat{y} \in H^2(\mathbb{C}_+)$.

(f) $H(s) = e^{-0.7s} \dfrac{Ks+1}{s^2+4s+3}$.

(g) $\|H\|_\infty = \|G\|_\infty = \left\| \dfrac{1}{s+3} \right\|_\infty = \dfrac{1}{3}$ (maximum at $s=0$).

(h) If $K(t) = \cos t$, then the system (its input-output operator) is not time-invariant, hence it has no transfer function.

Question 5 (a) We denote by S_τ the operator of right shift by τ on $L^2[0,\infty)$ (delay by τ time units). A **bounded** operator T from $L^2[0,\infty)$ to $L^2[0,\infty)$ is called <u>time-invariant</u> if $TS_\tau = S_\tau T$ for all $\tau>0$.

(b) <u>Theorem (Fourés-Segal)</u>. Let T be a bounded linear operator from $L^2[0,\infty)$ to $L^2[0,\infty)$. T is time-invariant iff there exists $G \in H^\infty(\mathbb{C}_+)$ such that

$$T = \mathcal{L}^{-1} G \mathcal{L} \quad (\mathcal{L} = \text{Laplace tr.}).$$

If this is the case, then $\|T\| = \|G\|_\infty$.

Consider a linear system described by

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases} \quad \begin{array}{|l} u = \text{input signal} \\ x = \text{state} \\ y = \text{output signal} \end{array}$$

where A,B,C,D are constant matrices and A is stable, i.e., its eigenvalues are in $\mathbb{C}_-$. If $x(0) = 0$, then $y = Tu$, where T is the <u>input-output operator</u> of the system. This is a bounded and time-invariant operator from $L^2[0,\infty)$ to $L^2[0,\infty)$. It is of the form indicated in the theorem, with $G(s) = C(sI-A)^{-1}B + D$. Such a system is called a finite-dimensional LTI system, and for this situation, $T = \mathcal{L}^{-1}G\mathcal{L}$ is easy to prove directly from the equations, without using the Fourés-Segal theorem. For more complicated systems (such as those involving propagating waves), a direct proof becomes more difficult. Even for finite-dimensional systems, it is not easy to prove directly that $\|T\| = \|G\|_\infty$.

(c) $BL(\omega_b)$ is the subspace of $L^2(-\infty,\infty)$ consisting of those functions whose Fourier transform is in $L^2[-i\omega_b, i\omega_b]$ (in other words, $u \in BL(\omega_b)$ if $(\mathcal{F}u)(i\omega) = 0$ for $|\omega| > \omega_b$). Such functions u are infinitely differentiable (actually, analytic).

(d) Theorem (Whittaker–Kotelnikov–Shannon).

If $u \in BL(\omega_b)$ and $\tau \in (0, \frac{\pi}{\omega_b}]$, then for all $t \in \mathbb{R}$

$$u(t) = \sum_{k\in\mathbb{Z}} u(k\tau) \frac{\sin \omega_b(t-k\tau)}{\omega_b(t-k\tau)} .$$

This shows that if we sample the signal at the time instants $k\tau$, $k \in \mathbb{Z}$, where τ is the sampling period, then u can be completely reconstructed from these samples. It is easier to store and/or transmit samples of a signal than the whole signal.

In practice, signals are not exactly bandlimited, just "almost" bandlimited. This means that $u = v + e$, where $v \in BL(\omega_b)$ and e is a small error (deviation). Also, the samples $u(k\tau)$ cannot be taken for all $k \in \mathbb{Z}$, only for a finite (but possibly very large) set of integers. Then, the formula will hold approximately, for values of t which are **not close** to the end of the time interval in which samples were taken. The condition $\tau \leq \frac{\pi}{\omega_b}$ means that the sampling frequency $\frac{1}{\tau} \geqslant 2$ times the highest frequency components of u.

 END