1. a) i) Since $\begin{bmatrix} A - sI & B \end{bmatrix}$ loses rank for $s = 3$, 3 is an uncontrollable mode, and since $\begin{bmatrix} A' - sI & C' \end{bmatrix}$ loses rank for $s = -4$, -4 is an unobservable mode. Since the uncontrollable mode is unstable, the realisation is not stabilisable and since the unobservable mode is stable, the realisation is detectable.

ii) By removing the uncontrollable and unobservable parts we get the minimal realisation

$$G(s) \stackrel{s}{=} \left[\begin{array}{c|cc} -1 & 1 & 2 \\ \hline 2 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right].$$

b) i) Suppose that λ is an eigenvalue of A and let $z \neq 0$ be the corresponding eigenvector. Then $Az = \lambda z$. Pre– and post–multiplying the matrix inequality by z' and z, respectively, we get

$$(\lambda + \bar{\lambda})z'Qz < 0.$$

Since $z \neq 0$ and $Q \succ 0$, this implies that $z'Qz > 0$ so that $\lambda + \bar{\lambda} < 0$ and so A is stable.

ii) The pair (A, C) is detectable if and only if there exists L such that $A + LC$ is stable. That is, the pair (A, C) is detectable if and only if there exist L and $Q = Q' \succ 0$ such that

$$(A + LC)'Q + Q(A + LC) \prec 0.$$

Comparing this with the inequality in the question, it follows that the pair (A, C) is detectable by identifying Y with QL. It follows that we can define L as $L = Q^{-1}Y$ since $Q \succ 0$ implies Q^{-1} exists.

iii) Using duality: Suppose there exist $P = P' \succ 0$ and Z such that

$$AP + PA' + BZ + Z'B' \prec 0.$$

Then the pair (A, B) is stabilisable. Furthermore, with $K = ZP^{-1}$, then $A + BK$ is stable.

2. a) i) Assume first A is stable. Then $\|H\|_\infty < \gamma$ if and only if, with $x(0)=0$, $J := \int_0^\infty [y'y - \gamma^2 u'u]dt < 0$, for all $u(t)$ such that $\|u\|_2 < \infty$. If $\|u\|_2$ is bounded, then $\lim_{t\to\infty} x(t) = 0$. Now, $\int_0^\infty \frac{d}{dt}[x'Px]dt = x(\infty)'Px(\infty) - x(0)'Px(0) = 0$. So,

$$0 = \int_0^\infty (\dot{x}'Px + x'P\dot{x})\,dt = \int_0^\infty [x'(A'P+PA)x + x'PBu + u'B'Px]dt.$$

Use $y = Cx + Du$ and add the last expression to J

$$\begin{aligned} J &= \int_0^\infty [x'(A'P+PA+C'C)x + 2x'(PB+C'D)u + u'(D'D-\gamma^2 I)u]dt \\ &= \int_0^\infty \begin{bmatrix} x' & u' \end{bmatrix} \overbrace{\begin{bmatrix} A'P+PA+C'C & PB+C'D \\ B'P+D'C & D'D-\gamma^2 I \end{bmatrix}}^{M} \begin{bmatrix} x \\ u \end{bmatrix} dt. \end{aligned}$$

It follows that $J < 0$, and therefore $\|H\|_\infty < \gamma$, if $M \prec 0$. Finally, the $(1,1)$ block of the inequality gives the inequality $A'P + PA \prec 0$. Since $P \succ 0$ this proves stability.

ii) Writing $M = M_1 + U'U$ where $U = \begin{bmatrix} C & D \end{bmatrix}$ and using a Schur complement, then

$$M \prec 0 \Leftrightarrow \begin{bmatrix} A'P+PA & PB & C' \\ B'P & -\gamma^2 I & D' \\ C & D & -I \end{bmatrix} \prec 0$$

b) i) Substituting $u = Lw_2 + Cx$, $e = w_2 + Cx$ into the state equation gives

$$\dot{x} = \underbrace{(A+LC)}_{A_c}x + \underbrace{\begin{bmatrix} B & L \end{bmatrix}}_{B_c}w, \qquad e = \underbrace{C}_{C_c}x + \underbrace{\begin{bmatrix} 0 & I \end{bmatrix}}_{D_c}w.$$

It follows that $T_{ew}(s) = D_c + C_c(sI - A_c)^{-1}B_c$.

ii) Using the results of part (a), by replacing A, B, C and D by A_c, B_c, C_c and D_c, we have that there exists a feasible L if there exists $P = P' \succ 0$ such that

$$\begin{bmatrix} (A+LC)'P + P(A+LC) + C'C & PB & PL + C' \\ B'P & -\gamma^2 I & 0 \\ L'P + C & 0 & (1-\gamma^2)I \end{bmatrix} \prec 0.$$

Noting that the only nonlinearity is due to the product PL, we define $Z = PL$ and so there exists a feasible L if there exists $P = P' \succ 0$ and Z such that

$$\begin{bmatrix} A'P + PA + ZC + C'Z' + C'C & PB & Z + C' \\ B'P & -\gamma^2 I & 0 \\ Z' + C & 0 & (1-\gamma^2)I \end{bmatrix} \prec 0.$$

3. a) It is clear that we require $K(s)$ to be internally stabilising.

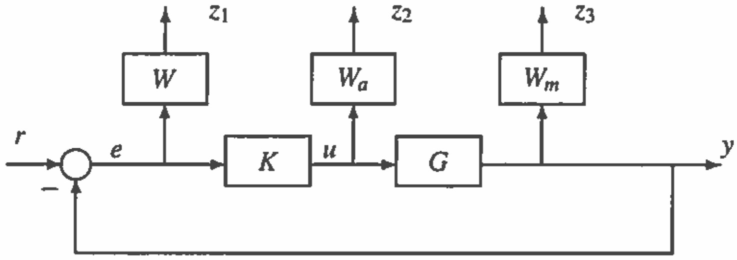
i) A calculation shows that, when $\Delta_a = 0$ and $\Delta_m = 0$, $e(s) = -S(s)r(s)$ where $S(s) = [I+G(s)K(s)]^{-1}$ is the sensitivity. Thus $\|e(j\omega)\| \le \|S(j\omega)\|\|r(j\omega)\|$. It follows that a sufficient condition to achieve the first design specification is $\|S(j\omega)\| < |w^{-1}(j\omega)|, \forall\omega$ or equivalently $\|WS\|_\infty < 1$, where $W = wI$.

ii) Suppose that $\Delta_a = 0$ and let the input to Δ_m be ε_2 while the output from Δ_m be δ_2. Then a calculation shows that $\varepsilon_2 = -GKS\delta_2$. Using the small gain theorem, to satisfy the first design requirement, it is sufficient that $\|\Delta_m(j\omega)G(j\omega)K(j\omega)S(j\omega)\| < 1, \forall\omega$. This can be satisfied if $\|W_mGKS\|_\infty < 1$, where $W_m = w_mI$.

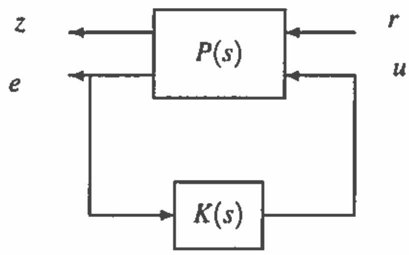
iii) An analogous procedure shows that to satisfy the second design requirement, it is sufficient that $\|\Delta_a(j\omega)K(j\omega)S(j\omega)\| < 1, \forall\omega$. This can be satisfied if $\|W_aKS\|_\infty < 1$, where $W_a = w_aI$.

To satisfy all design requirements, it is sufficient that $\left\| \begin{bmatrix} WS \\ W_aKS \\ W_mGKS \end{bmatrix} \right\|_\infty < 1.$

b) Since $e = -Sr$, the design specifications reduce to the requirement that the transfer matrix from r to $z = [z_1^T \; z_2^T \; z_3^T]^T$ in the following diagram has $\mathcal{H}_\infty$-norm less than 1.



c) The corresponding generalized regulator formulation is to find an internally stabilizing K such that $\|\mathcal{F}_l(P,K)\|_\infty < 1$:



$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \left[\begin{array}{c|c} W & -WG \\ 0 & W_a \\ 0 & W_mG \\ \hline I & -G \end{array}\right].$$

4. a) The generalized regulator formulation is given by

$$\begin{bmatrix} z(s) \\ y(s) \end{bmatrix} = P(s) \begin{bmatrix} w(s) \\ u(s) \end{bmatrix}, u(s) = Fy(s), P(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{c||c|c} A & B_1 & B_2 \\ \hline C_1 & 0 & 0 \\ 0 & 0 & I \\ \hline I & 0 & 0 \end{array}\right].$$

b) Let $V = x'Xx$ and set $u = Fx$. Provided that $X = X' \succ 0$ and $\dot{V} < 0$ along the closed-loop trajectory, we can assume $\lim_{t\to\infty} x(t) = 0$. Then

$$\dot{V} = \dot{x}'Xx + x'X\dot{x} = x'\left(A'X + XA + F'B_2'X + XB_2F\right)x + x'XB_1w_1 + w_1'B_1'Xx.$$

Integrating from 0 to ∞ and using $x(\infty) = 0$,

$$-x_0'Xx_0 = \int_0^\infty [x'\left(A'X + XA + F'B_2'X + XB_2F\right)x + x'XB_1w_1 + w_1'B_1'Xx]dt.$$

Using the definition of J, adding the last equation, and completing two squares,

$$J = x_0'Xx_0 + \int_0^\infty \{x'\left[A'X + XA + C_1'C_1 - X\left(B_2B_2' - \gamma^{-2}B_1B_1'\right)X\right]x + \|Zx\|^2 - \|v\|^2\}dt$$

where $Z = F + B_2'X$ and $v = \gamma w - \gamma^{-1}B_1'Xx$.

c) Setting $x_0 = 0$, the requirement $\|H\|_\infty \leq \gamma$ is equivalent to $J \leq 0$. Thus sufficient conditions for $J \leq 0$ are $Z = 0$ and the existence of X such that

$$A'X + XA + C_1'C_1 - X\left(B_2B_2' - \gamma^{-2}B_1B_1'\right)X = 0, \qquad X = X' \succ 0.$$

Setting $Z = 0$, the control policy is $F = -B'X$. Setting $v = 0$, the worst case disturbance is $w^* = \gamma^{-2}B_1'X$. The closed-loop with $u = Fx$ and $w = w^*$ is $\dot{x} = A_cx$ where $A_c = A - \left(B_2B_2' - \gamma^{-2}B_1B_1'\right)$ and a further condition is $Re\ \lambda_i(A_c) < 0,\ \forall\, i$.

d) Using the expression for J in Part b) and the solution in Part c), it follows that $J \leq x_0'Xx_0$, and so we can take $\gamma_2^2 = x_0'Xx_0$ as the tightest upper bound on the regulation cost.

e) Since $X(\gamma)$ is decreasing in γ and since $x_0'X(\gamma)x_0$ is an upper bound on the regulation cost, then we can use the value of γ as a trade-off in the design between maximum robustness (the minimum value of γ achieving the sufficient conditions in Part c)) and maximum regulation (the maximum value of γ, i.e., $\gamma \to \infty$).