Master.

UNIVERSITY OF LONDON [E1.11 (Maths) ISE 2007]

B.ENG. AND M.ENG. EXAMINATIONS 2007

For Internal Students of Imperial College London

This paper is also taken for the relevant examination for the Associateship of the City & Guilds of London Institute

INFORMATION SYSTEMS ENGINEERING E1.11

MATHEMATICS

Date Wednesday 30th May 2007 10.00 am - 1.00 pm

Answer ANY SEVEN questions

Answers to Section A *questions must be written in a different answer book from answers to* Section B *questions.*

[Before starting, please make sure that the paper is complete. There should be SEVEN pages, with a total of NINE questions. Ask the invigilator for a replacement if this copy is faulty.]

1. (i) Find all possible values of the following complex numbers. Give your answers in the form $x + iy$ (with x and y real):

   (a) i^{12};

   (b) $e^{-i\pi/6}$;

   (c) $1^{1/3}$;

   (ii) Find all the solutions of the equation $\cos^2 z = 4$.

   Give your answer in the form $x + iy$ (with x and y real).

2. (i) Using l'Hôpital's Rule, evaluate the limit

$$\lim_{x \to 0} \frac{\sinh^2 x}{x} .$$

   (ii) Differentiate:

   (a) $y = x^{\tan x}$,

   (b) $y = (\sin^{-1} x)^2$.

   (iii) Find the interval of convergence of

$$\sum_{n=1}^{\infty} \frac{(1 + x^4)^n}{2^n n^4} .$$

   Investigate the endpoints of the interval.

PLEASE TURN OVER

3. Evaluate the following integrals :

(i) $$\int e^x \cosh x \, dx \; ;$$

(ii) $$\int \frac{2 \ln x}{x} \, e^{(\ln x)^2} \, dx \; ;$$

(iii) $$\int_{-\pi}^{\pi} x \sin^4 x \, dx \; ;$$

(iv) $$\int_0^{\pi/2} (\cos x \; + \; \sin x)^2 \, dx \; ;$$

(v) $$\int_1^{\infty} \frac{\ln x}{x^2} \, dx \; .$$

4. Find the general solution of the following differential equations:

(i) $$\frac{dy}{dx} = \frac{\tan x}{(1+y)^3} \; ;$$

(ii) $$(1-x^2) \frac{dy}{dx} + (1+x) \, y = (1-x) \; ;$$

(iii) $$\frac{dy}{dx} = \frac{x^4 + x^3 y - y^4}{x^4} \; .$$

PLEASE TURN OVER

5. Find the general solution of the following differential equations:

(i)
$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 5y = 3\cos x \; ;$$

(ii)
$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = 5x^2 + 2x \; ;$$

(iii)
$$\frac{d^2y}{dx^2} - 10\frac{dy}{dx} + 25y = e^{5x} \; .$$

For (iii) find also the solution subject to the conditions

$$y = 1 \quad \text{and} \quad \frac{dy}{dx} = 0 \quad \text{at} \quad x = 0 \; .$$

PLEASE TURN OVER

SECTION B

6. (i) Find the four stationary points of the function

$$f(x,\, y) \;=\; 2\,x^3 \;+\; 6\,xy^2 \;-\; 3\,y^3 \;-\; 150\,x$$

and determine their nature.

(ii) Show that the differential equation

$$xy^2 \;+\; y \;+\; (x^2 y + x)\,\frac{dy}{dx} \;=\; 0$$

is exact, and solve it.

7. Define $f(x)$ to be the periodic function with period 2π such that

$$f(x) \;=\; \frac{x}{\pi} \qquad \text{for} \qquad -\,\pi \;\le\; x \;<\; \pi\,.$$

Find the Fourier series of $f(x)$.

By substituting a suitable value of x, deduce that

$$1 \;-\; \frac{1}{3} \;+\; \frac{1}{5} \;-\; \frac{1}{7} \;+\; \ldots \;=\; \frac{\pi}{4}\,.$$

Use Parseval's formula to deduce that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \;=\; \frac{\pi^2}{6}\,.$$

PLEASE TURN OVER

8. The Laplace transform of a function $f(t)$ is defined by

$$\mathcal{L}(f(t)) = F(s) = \int_0^\infty e^{-st} f(t)\,dt\ .$$

For $a > 0$, the Heaviside step function $H_a(t)$ is defined by

$$H_a(t) = \begin{cases} 1 & \text{if} \quad t > a\ , \\ 0 & \text{if} \quad t \leq a\ . \end{cases}$$

(i) Show that $\mathcal{L}(H_a(t)) = \dfrac{e^{-as}}{s} \quad (s > 0)$.

(ii) Prove the shift rule

$$\mathcal{L}(H_a(t)\,f\,(t-a)) = e^{-as}\,\mathcal{L}(f(t))\ .$$

(iii) Find the function $y = y(t)$ satisfying the differential equation

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} - 2y = 1 - H_1(t)$$

with $y(0) = y'(0) = 0$.

You may assume that

$$\mathcal{L}(f'(t)) = -f(0) + s\mathcal{L}(f(t))$$

and

$$\mathcal{L}(f''(t)) = -f'(0) - sf(0) + s^2\mathcal{L}(f(t))\ .$$

PLEASE TURN OVER

9. (i) For which values of λ does the following system of linear equations have no solutions :

$$\begin{array}{rcrcrcrcl} x & + & y & + & z & + & t & = & \lambda\,, \\ x & - & y & + & z & - & t & = & 0\,, \\ x & - & 3y & + & z & + & \lambda t & = & 2\,. \end{array}$$

(ii) Let

$$A = \begin{pmatrix} 5 & -2 \\ 12 & -5 \end{pmatrix} .$$

Find a 2×2 matrix P such that $P^{-1} A P$ is diagonal.

Find A^{101} .

END OF PAPER

MATHEMATICS DEPARTMENT

MATHEMATICAL FORMULAE

1. VECTOR ALGEBRA

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1,\, a_2,\, a_3)$$

Scalar (dot) product: $\mathbf{a}\,.\,\mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

Vector (cross) product:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Scalar triple product:

$$[\mathbf{a},\, \mathbf{b},\, \mathbf{c}] = \mathbf{a}\,.\,\mathbf{b} \times \mathbf{c} = \mathbf{b}\,.\,\mathbf{c} \times \mathbf{a} = \mathbf{c}\,.\,\mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product: $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c}\,.\,\mathbf{a})\mathbf{b} - (\mathbf{b}\,.\,\mathbf{a})\mathbf{c}$

2. SERIES

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \ldots \quad (\alpha \text{ arbitrary, } |x| < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots\,,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots\,,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \ldots\,,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^n \frac{x^{n+1}}{(n+1)} + \ldots (-1 < x \le 1)$$

3. TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin(a+b) = \sin a\, \cos b + \cos a\, \sin b\,;$$

$$\cos(a+b) = \cos a\, \cos b - \sin a\, \sin b\,.$$

$$\cos iz = \cosh z\,; \quad \cosh iz = \cos z\,; \quad \sin iz = i \sinh z\,; \quad \sinh iz = i \sin z\,.$$

4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f\,D^n g + \tbinom{n}{1} Df\,D^{n-1}g + \ldots + \tbinom{n}{r} D^r f\,D^{n-r} g + \ldots + D^n f\; g\,.$$

(b) Taylor's expansion of $f(x)$ about $x = a$:

$$f(a+h) = f(a) + hf'(a) + h^2 f''(a)/2! + \ldots + h^n f^{(n)}(a)/n! + \epsilon_n(h)\,,$$

where $\epsilon_n(h) = h^{n+1} f^{(n+1)}(a + \theta h)/(n+1)!, \quad 0 < \theta < 1\,.$

(c) Taylor's expansion of $f(x,\, y)$ about $(a,\, b)$:

$$f(a+h,\, b+k) = f(a,\, b) + [hf_x + kf_y]_{a.b} + 1/2!\, \left[h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}\right]_{a.b} + \ldots$$

(d) Partial differentiation of $f(x,\, y)$:

i. If $y = y(x)$, then $f = F(x)$, and $\dfrac{dF}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dx}\,.$

ii. If $x = x(t)$, $y = y(t)$, then $f = F(t)$, and $\dfrac{dF}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}\,.$

iii. If $x = x(u,\, v)$, $y = y(u,\, v)$, then $f = F(u,\, v)$, and

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}\,, \quad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}\,.$$

(e) Stationary points of $f(x,\, y)$ occur where $f_x = 0,\ f_y = 0$ simultaneously.
Let $(a,\, b)$ be a stationary point: examine $D = [f_{xx} f_{yy} - (f_{xy})^2]_{a.b}\,.$
If $D > 0$ and $f_{xx}(a,\, b) < 0$, then $(a,\, b)$ is a maximum;
If $D > 0$ and $f_{xx}(a,\, b) > 0$, then $(a,\, b)$ is a minimum;
If $D < 0$ then $(a,\, b)$ is a saddle-point.

(f) Differential equations:

i. The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating factor $I(x) = \exp[\int P(x)(dx]$, so that $\dfrac{d}{dx}(Iy) = IQ$.

ii. $P(x,\, y)dx + Q(x,\, y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y\,.$

5. INTEGRAL CALCULUS

(a) An important substitution: $\tan(\theta/2) = t$:
$\sin\theta = 2t/(1+t^2)\,,\quad \cos\theta = (1-t^2)/(1+t^2)\,,\quad d\theta = 2\,dt/(1+t^2)\,.$

(b) Some indefinite integrals:

$$\int (a^2-x^2)^{-1/2}dx = \sin^{-1}\left(\frac{x}{a}\right),\quad |x|<a\,.$$

$$\int (a^2+x^2)^{-1/2}dx = \sinh^{-1}\left(\frac{x}{a}\right) = \ln\left\{\frac{x}{a}+\left(1+\frac{x^2}{a^2}\right)^{1/2}\right\}.$$

$$\int (x^2-a^2)^{-1/2}dx = \cosh^{-1}\left(\frac{x}{a}\right) = \ln\left|\frac{x}{a}+\left(\frac{x^2}{a^2}-1\right)^{1/2}\right|.$$

$$\int (a^2+x^2)^{-1}dx = \left(\frac{1}{a}\right)\tan^{-1}\left(\frac{x}{a}\right).$$

6. NUMERICAL METHODS

(a) Approximate solution of an algebraic equation:

If a root of $f(x)=0$ occurs near $x=a$, take $x_0=a$ and
$x_{n+1} = x_n - [f(x_n)/f'(x_n)]\,,\quad n=0,1,2\ldots$

(Newton Raphson method).

(b) Formulae for numerical integration: Write $x_n = x_0 + nh\,,\quad y_n = y(x_n)\,.$

i. Trapezium rule (1-strip): $\int_{x_0}^{x_1} y(x)dx \approx (h/2)\,[y_0+y_1]\,.$

ii. Simpson's rule (2-strip): $\int_{x_0}^{x_2} y(x)dx \approx (h/3)\,[y_0+4y_1+y_2]\,.$

(c) Richardson's extrapolation method: Let $I=\int_a^b f(x)dx$ and let $I_1,\ I_2$ be two

estimates of I obtained by using Simpson's rule with intervals h and $h/2$.

Then, provided h is small enough,

$$I_2 + (I_2 - I_1)/15\,,$$

is a better estimate of I.

7. LAPLACE TRANSFORMS

Function	Transform	Function	Transform
$f(t)$	$F(s)=\int_0^\infty e^{-st}f(t)dt$	$af(t)+bg(t)$	$aF(s)+bG(s)$
df/dt	$sF(s)-f(0)$	d^2f/dt^2	$s^2F(s)-sf(0)-f'(0)$
$e^{at}f(t)$	$F(s-a)$	$tf(t)$	$-dF(s)/ds$
$(\partial/\partial\alpha)f(t,\alpha)$	$(\partial/\partial\alpha)F(s,\alpha)$	$\int_0^t f(t)dt$	$F(s)/s$
$\int_0^t f(u)g(t-u)du$	$F(s)G(s)$		
1	$1/s$	$t^n (n=1,2\ldots)$	$n!/s^{n+1}$, $(s>0)$
e^{at}	$1/(s-a)$, $(s>a)$	$\sin\omega t$	$\omega/(s^2+\omega^2)$, $(s>0)$
$\cos\omega t$	$s/(s^2+\omega^2)$, $(s>0)$	$H(t-T)=\begin{cases}0, & t<T\\ 1, & t>T\end{cases}$	e^{-sT}/s, $(s,T>0)$

8. FOURIER SERIES

If $f(x)$ is periodic of period $2L$, then $f(x+2L)=f(x)$, and

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{L}\,,\ \text{where}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx,\quad n=0,1,2,\ldots,\ \text{and}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx,\quad n=1,2,3,\ldots$$

Parseval's theorem

$$\frac{1}{L}\int_{-L}^{L}[f(x)]^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2+b_n^2\right).$$

	EXAMINATION QUESTIONS/SOLUTIONS 2006-07	Course ISE 1.6
Question 1		Marks & seen/unseen

Parts

(i)

(a) $i^{12} = (i^2)^6 = (-1)^6 = 1$ — 2

(b) $e^{-i\pi/6} = \cos\left(\frac{-\pi}{6}\right) + i\sin\left(\frac{-\pi}{6}\right) = \frac{\sqrt{3}}{2} - \frac{i}{2}$ — 2

(c) $1^{1/3} = \left(e^{i2n\pi}\right)^{1/3} = 1 \quad (n=0)$

$= e^{i2\pi/3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i \quad (n=1)$ — 4

$= e^{i4\pi/3} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i \quad (n=2)$

(ii) $\cos^2 z = 4 \Rightarrow \cos z = \pm 2$

$\Rightarrow \frac{e^{iz}+e^{-iz}}{2} = \pm 2 \Rightarrow e^{iz} \pm 4 + e^{-iz} = 0$

$\Rightarrow e^{2iz} \pm 4e^{iz} + 1 = 0$

Let $u = e^{iz}$ $\Rightarrow u^2 \pm 4u + 1 = 0$

$\Rightarrow u = \frac{\pm 4 \pm \sqrt{16-4}}{2} = \pm 2 \pm \sqrt{3}$ — 12

$\Rightarrow e^{iz} = \pm 2 \pm \sqrt{3}$

$e^{iz} = (2\pm\sqrt{3})e^{i2n\pi}$ or $e^{iz} = -2\pm\sqrt{3} = (2\pm\sqrt{3})e^{i\pi}e^{i2n\pi}$

$\Rightarrow iz = \ln(2\pm\sqrt{3}) + i2n\pi$ $\quad\Rightarrow iz = \ln(2\pm\sqrt{3}) + i\pi + i2n\pi$

$\Rightarrow z = -i\ln(2\pm\sqrt{3}) + 2n\pi$ $\quad\Rightarrow z = -i\ln(2\pm\sqrt{3}) + (2n+1)\pi$.

	Setter's initials MJM	Checker's initials mwr	Page number 1

Question 2

Marks & seen/unseen

Parts

(i) $\lim_{x\to 0} \frac{\sinh^2 x}{x} = \lim_{x\to 0} \frac{2\sinh x \cosh x}{1} = 0$ 3

(ii)

(a) $y = x^{\tan x}$

$\Rightarrow \ln y = \tan x \ln x$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \sec^2 x \ln x + \frac{\tan x}{x} \Rightarrow \frac{dy}{dx} = x^{\tan x}\left(\sec^2 x \ln x + \frac{\tan x}{x}\right)$ 4

(b) $y = (\sin^{-1} x)^2$

$u = \sin^{-1} x \Rightarrow y = u^2$

$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 2u \frac{d}{dx}(\sin^{-1} x)$ 5

$\sin u = x \Rightarrow \cos u \frac{du}{dx} = 1 \Rightarrow \frac{du}{dx} = \frac{1}{\sqrt{1-\sin^2 u}} = \frac{1}{\sqrt{1-x^2}}$

$\Rightarrow \frac{dy}{dx} = \frac{2\sin^{-1} x}{\sqrt{1-x^2}}$

(iii) $\rho = \lim_{n\to\infty} \left|\frac{\rho_{n+1}}{\rho_n}\right| = \lim_{n\to\infty} \left|\frac{(1+x^4)^{n+1}}{2^{n+1}(n+1)^4} \frac{2^n n^4}{(1+x^4)^n}\right|$

$= \lim_{n\to\infty} \left|\frac{1+x^4}{2}\frac{n^4}{(1+n)^4}\right| = \lim_{n\to\infty}\left|\frac{1+x^4}{2}\frac{1}{(1+\frac{1}{n})^4}\right| = \left|\frac{1+x^4}{2}\right|$ 6

Converges if $\left|\frac{1+x^4}{2}\right| < 1 \Rightarrow -1 < x < 1$

	EXAMINATION QUESTIONS/SOLUTIONS 2006-07	Course ISE 1.6
Question 2		Marks & seen/unseen
Parts (iii) (cont.)	At endpoints $x = \pm 1$ Series becomes $\sum_{n=1}^{\infty} \frac{1}{n^4}$ which is convergent	2
	Setter's initials MJH Checker's initials [illegible]	Page number 2

	EXAMINATION QUESTIONS/SOLUTIONS 2006-07	Course ISE 1.6
Question 3		Marks & seen/unseen

Parts

(i) $\int e^{x}\cosh x\,dx = \frac{1}{2}\int e^{x}(e^{x}+e^{-x})\,dx = \frac{1}{2}\int (e^{2x}+1)\,dx$ — 3

$= \frac{1}{4}e^{2x} + \frac{1}{2}x + c$

(ii) $\int \frac{2\ln x}{x} e^{(\ln x)^2}\,dx \qquad u = (\ln x)^2 \Rightarrow \frac{du}{dx} = \frac{2\ln x}{x}$ — 4

$= \int e^{u}\,du = e^{u} + c = e^{(\ln x)^2} + c$

(iii) $\int_{-\pi}^{\pi} x\sin^4 x\,dx = 0 \quad$ (symmetry) — 2

(iv) $\int_0^{\pi/2} (\cos x + \sin x)^2\,dx = \int_0^{\pi/2} (\cos^2 x + \sin^2 x + 2\cos x\sin x)\,dx$ — 5

$= \int_0^{\pi/2} (1 + \sin 2x)\,dx = \left[x - \frac{1}{2}\cos 2x\right]_0^{\pi/2} = \frac{\pi}{2} + \frac{1}{2} + \frac{1}{2}$

$= 1 + \frac{\pi}{2}.$

(v) $\int_1^{\infty} \frac{\ln x}{x^2}\,dx = \lim_{a\to\infty} \int_1^{a} \frac{\ln x}{x^2}\,dx \qquad u = \ln x \quad \frac{dv}{dx} = \frac{1}{x^2}$ — 6

$\frac{du}{dx} = \frac{1}{x} \quad v = -\frac{1}{x}$

$= \lim_{a\to\infty}\left\{\left[-\frac{\ln x}{x}\right]_1^a + \int_1^a \frac{1}{x^2}\,dx\right\} = \lim_{a\to\infty}\left\{-\frac{\ln a}{a} - \left[\frac{1}{x}\right]_1^a\right\}$

$= \lim_{a\to\infty}\left\{-\frac{\ln a}{a} - \frac{1}{a} + 1\right\} = 1$

Setter's initials MJA	Checker's initials MLR	Page number 1

	EXAMINATION QUESTIONS/SOLUTIONS 2006-07	Course ISE 1.6
Question 4		Marks & seen/unseen

Parts

(i) $\dfrac{dy}{dx} = \dfrac{\tan x}{(1+y)^3} \Rightarrow \int (1+y)^3\, dy = \int \tan x\, dx$

$\Rightarrow \frac{1}{4}(1+y)^4 = -\ln(\cos x) + c$ — 4

(ii) $(1-x^2)\dfrac{dy}{dx} + (1+x)y = (1-x)$

$\Rightarrow \dfrac{dy}{dx} + \dfrac{(1+x)}{(1+x)(1-x)}\, y = \dfrac{(1-x)}{(1+x)(1-x)}$

$\Rightarrow \dfrac{dy}{dx} + \dfrac{y}{1-x} = \dfrac{1}{1+x}$ — 7

Integrating factor $e^{\int \frac{dx}{1-x}} = e^{-\ln(x-1)} = \dfrac{1}{x-1}$

$\Rightarrow \dfrac{1}{x-1}\dfrac{dy}{dx} - \dfrac{y}{(x-1)^2} = \dfrac{1}{(x+1)(x-1)} \Rightarrow \dfrac{d}{dx}\left(\dfrac{y}{x-1}\right) = \dfrac{\frac{1}{2}}{x-1} - \dfrac{\frac{1}{2}}{x+1}$

$\Rightarrow \dfrac{y}{x-1} = \frac{1}{2}\ln(x-1) - \frac{1}{2}\ln(x+1) + c = \frac{1}{2}\ln\dfrac{x-1}{x+1} + c$

$\Rightarrow y = (x-1)\left[\frac{1}{2}\ln\dfrac{x-1}{x+1} + c\right]$

(iii) $\dfrac{dy}{dx} = \dfrac{x^4 + x^3 y - y^4}{x^4}$

Let $y = vx \Rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx} = 1 + v - v^4 \Rightarrow x\dfrac{dv}{dx} = 1 - v^4$

$\Rightarrow x\dfrac{dv}{dx} = (1+v^2)(1-v^2)$ — 9

$\Rightarrow \int \dfrac{dx}{x} = \int \dfrac{dv}{(1+v^2)(1-v^2)} = \int dv \left\{ \dfrac{\frac{1}{2}}{1-v^2} + \dfrac{\frac{1}{2}}{1+v^2} \right\} = \int dv \left\{ \dfrac{\frac{1}{2}}{1+v^2} + \dfrac{\frac{1}{2}}{(1+v)(1-v)} \right\}$

$= \int dv \left\{ \dfrac{\frac{1}{2}}{1+v^2} + \dfrac{\frac{1}{4}}{1-v} + \dfrac{\frac{1}{4}}{1+v} \right\} = \frac{1}{2}\tan^{-1} v + \frac{1}{4}\ln(v+1) - \frac{1}{4}\ln(v-1) + c$

Setter's initials	Checker's initials	Page number
MJG	[illegible]	1

Parts

(iii) cont.

$$\Rightarrow \ln x = \frac{1}{2}\tan^{-1} v + \frac{1}{4}\ln \frac{v+1}{v-1} + c$$

$$\Rightarrow x = e^{\frac{1}{2}\tan^{-1} v}\left(\frac{v+1}{v-1}\right)^{1/4} A$$

$$\Rightarrow x = A\left(\frac{y+x}{y-x}\right)^{1/4} e^{\frac{1}{2}\tan^{-1}\frac{y}{x}}.$$

EXAMINATION QUESTIONS/SOLUTIONS 2006-07

Course: ISE 1.6

Question 5

Marks & seen/unseen

Parts

(i) $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 5y = 3\cos x$

Auxiliary equation $\lambda^2 - 6\lambda + 5 = 0 \Rightarrow (\lambda-5)(\lambda-1) = 0 \Rightarrow \lambda = 5$ or 1

C.F. $y = Ae^{5x} + Be^{x}$

P.I. $y = a\cos x + b\sin x$

$\Rightarrow 5(a\cos x + b\sin x) - 6(-a\sin x + b\cos x) + (-a\cos x - b\sin x) = 3\cos x$

$\Rightarrow 5a - 6b - a = 4a - 6b = 3$

$5b + 6a - b = 4b + 6a = 0 \quad \Rightarrow a = -\frac{2}{3}b$

$\Rightarrow -\frac{8}{3}b - 6b = -\frac{26b}{3} = 3 \quad \Rightarrow b = -\frac{9}{26} \quad a = \frac{3}{13}$

$\Rightarrow y = Ae^{5x} + Be^{x} + \frac{3}{13}\cos x - \frac{9}{26}\sin x$.

6

(ii) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = 5x^2 + 2x$

Auxiliary equation $\lambda^2 - 4\lambda + 5 = 0 \Rightarrow \lambda = \frac{4 \pm \sqrt{16-20}}{2} = 2 \pm i$

C.F. $y = e^{2x}(A\cos x + B\sin x)$

P.I. $y = ax^2 + bx + c$

$\Rightarrow 5(ax^2 + bx + c) - 4(2ax + b) + 2a = 5x^2 + 2x$

$\Rightarrow 5a = 5, \ 5b - 8a = 2, \ 5c - 4b + 2a = 0$

$\Rightarrow a = 1, \ b = 2, \ c = 6/5$

$\Rightarrow y = e^{2x}(A\cos x + B\sin x) + x^2 + 2x + \frac{6}{5}$

6

(iii) $\frac{d^2y}{dx^2} - 10\frac{dy}{dx} + 25y = e^{5x}$

Auxiliary equation $\lambda^2 - 10\lambda + 25 = 0 \Rightarrow (\lambda-5)(\lambda-5) = 0 \Rightarrow \lambda = 5$ (repeated)

Setter's initials: MJA

Checker's initials: MM

Question 5

Marks & seen/unseen

Parts

(iii) cont.

C.F. $y = (Ax+B)e^{5x}$

P.I. $y = ax^2e^{5x}$

$\Rightarrow 25ax^2e^{5x} - 10(2axe^{5x} + 5ax^2e^{5x}) + (2ae^{5x} + 20axe^{5x} + 25ax^2e^{5x}) = e^{5x}$

$\Rightarrow e^{5x}(\cancel{25ax^2} - \cancel{20ax} - \cancel{50ax^2} + 2a + \cancel{20ax} + \cancel{25ax^2}) = e^{5x}$ 6

$\Rightarrow a = \frac{1}{2}$

$\Rightarrow y = (Ax+B)e^{5x} + \frac{1}{2}x^2e^{5x}$

$y=1$ at $x=0 \Rightarrow 1 = B$

$\frac{dy}{dx} = 0$ at $x=0 \Rightarrow Ae^{5x} + 5(Ax+B)e^{5x} + xe^{5x} + \frac{5}{2}x^2e^{5x} = y'$ 2

$\Rightarrow y'(0) = A + 5B = A + 5 = 0$

$\Rightarrow A = -5$

$\Rightarrow y = (-5x+1)e^{5x} + \frac{1}{2}x^2e^{5x}$

Setter's initials MJH

Checker's initials mm

	EXAMINATION QUESTIONS/SOLUTIONS 2006-07	Course ISE 1
Question 6 (solution)		Marks & seen/unseen

Parts

a) $f_x = 6x^2 + 6y^2 - 150, \quad f_y = 12xy - 9y^2$
$\qquad\qquad\qquad\qquad = 3y(4x - 3y).$

For stationary point,

$f_x = 0 \Rightarrow x^2 + y^2 = 25$

$f_y = 0 \Rightarrow y = 0, \; x = \pm 5$

$\quad$ or $y = \frac{4x}{3}, \quad x^2 + \frac{16x^2}{9} = 25$

$\rightarrow x^2\left(\frac{25}{9}\right) = 25 \rightarrow x^2 = 9$

$\rightarrow x = \pm 3, \; y = \cancel{\pm}\,\frac{4x}{3} = \pm 4.$

Similar seen

So four stationary points are

$(5, 0), \; (-5, 0), \; (3, 4), \; (-3, -4).$

6

$f_{xx} = 12x, \; f_{xy} = 12y, \; f_{yy} = 12x - 18y$

So :

Point	$f_{xx}f_{yy} - f_{xy}^2$	f_{xx}	Nature
$(5,0)$	$60.60 > 0$	>0	minimum
$(-5,0)$	$(-60).(-60) > 0$	<0	maximum
$(3,4)$	$36.(-36) - 48^2 < 0$		saddle point
$(-3,-4)$	$(-36).(36) - 48^2 < 0$		saddle point

6

	Setter's initials MLL	Checker's initials MJN	Page number 2

	EXAMINATION QUESTIONS/SOLUTIONS 2006-07	Course ISE 1
Question 6 solution		Marks & seen/unseen

Parts

b) Eqn. is $P + Q\frac{dy}{dx} = 0$

where $P = xy^2+y$, $Q = x^2y + x$.

As $P_y = Q_x = 2xy+1$, it is exact. — 2

To solve, look for $u(x,y)$ such that

$$u_x = P = xy^2+y \quad (1)$$

$$u_y = Q = x^2y+x \quad (2)$$

Similar seen.

From (1), $u = \frac{x^2}{2}y^2 + xy + f(y)$

From (2), $u_y = x^2y + x + f'(y) \Rightarrow f'(y) = 0$.

~~So solution is~~

Take $f(y) = 0$. So solution is $u(x,y) = c$, constant, i.e.

$$x^2y^2 + 2xy = c.$$

6

Setter's initials	Checker's initials	Page number
MLR	MJH	3

	EXAMINATION QUESTIONS/SOLUTIONS 2006-07	Course ISE 1
Question 7. (solution)		Marks & seen/unseen

Parts

As $f(x)$ is an odd function, it has a Fourier sine series $\sum_{n=1}^{\infty} b_n \sin nx$, where

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{x}{\pi} \sin nx \, dx$$

$$= \frac{2}{\pi^2}\int_0^{\pi} x \sin nx .$$

Integrate by parts:

$$b_n = \frac{2}{\pi^2}\left(\left[-x\frac{\cos nx}{n}\right]_0^{\pi} + \int_0^{\pi}\frac{\cos nx}{n}dx\right)$$ Similar seen.

$$= \frac{2}{\pi^2}\left(-\pi\frac{\cos n\pi}{n} + 0\right)$$

$$= \frac{2.(-1)^{n+1}}{n\pi}.$$ 9

So Fourier series is

$$\underline{\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx} = \frac{2}{\pi}\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots\right)$$

Substitute $x = \frac{\pi}{2}$: gives

$$f\left(\tfrac{\pi}{2}\right) = \frac{1}{2} = \frac{2}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\right)$$

Hence

$$\underline{1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}}$$ 5

Setter's initials	Checker's initials	Page number
MLR	MJH	5

	EXAMINATION QUESTIONS/SOLUTIONS 2006-07	Course ISE 1
Question 7 (solution)		Marks & seen/unseen
Parts	Parseval's formula says $$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2\,dx = \sum b_n^2$$ i.e. $$\frac{1}{\pi}\int_{-\pi}^{\pi} \frac{x^2}{\pi^2}\,dx = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$ LHS is $\frac{1}{\pi^3}\left(\frac{2\pi^3}{3}\right) = \frac{2}{3}$. So $$\frac{2}{3} = \frac{4}{\pi^2}\sum\frac{1}{n^2}$$ $$\therefore \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$	6
	Setter's initials MLR	Checker's initials MJH.	Page number 6

Question 8 Solution | Marks & seen/unseen

Parts

a) $\mathcal{L}(H_a(t)) = \int_a^\infty e^{-st}\,dt = \left[-\frac{e^{-st}}{s}\right]_a^\infty = \frac{e^{-as}}{s}$ $(s>0)$. — 2 seen

b) $\mathcal{L}(H_a(t) f(t-a)) = \int_a^\infty e^{-st} f(t-a)\,dt$

Put $u = t-a$: $\quad = \int_0^\infty e^{-s(u+a)} f(u)\,du$ — 4 seen

$= e^{-as}\mathcal{L}(f(t))$.

c) Take Laplace transforms: — Similar seen

$$s^2\mathcal{L}(y) + s\mathcal{L}(y) - 2\mathcal{L}(y) = \frac{1}{s} - \frac{e^{-s}}{s}$$

So

$$\mathcal{L}(y) = \frac{1-e^{-s}}{s(s-1)(s+2)}$$

By Partial Fractions,

$$\frac{1}{s(s-1)(s+2)} = \frac{1}{6}\left(-\frac{3}{s} + \frac{2}{s-1} + \frac{1}{s+2}\right)$$

So — 14

$$\mathcal{L}(y) = \frac{1-e^{-s}}{6}\left(-\frac{3}{s} + \frac{2}{s-1} + \frac{1}{s+2}\right)$$

Invert using shift rule in (b):

$$y = \frac{1}{6}\left(-3 + 2e^{t} + e^{-2t}\right)$$
$$- \frac{1}{6}H_1(t)\left(-3 + 2e^{t-1} + e^{-2(t-1)}\right)$$

	EXAMINATION QUESTIONS/SOLUTIONS 2006-07	Course ISE 1
Question 9 Solution		Marks & seen/unseen

Parts

(a) Use Gaussian elimination:

$$\left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & \lambda \\ 1 & -1 & 1 & -1 & 0 \\ 1 & -3 & 1 & \lambda & 2 \end{array}\right) \longrightarrow \left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & \lambda \\ 0 & -2 & 0 & -2 & -\lambda \\ 0 & -4 & 0 & \lambda-1 & 2-\lambda \end{array}\right)$$

Similar seen

$$\longrightarrow \left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & \lambda \\ 0 & -2 & 0 & -2 & -\lambda \\ 0 & 0 & 0 & \lambda+3 & 2+\lambda \end{array}\right)$$

This is echelon form. Last equation is

$$(\lambda+3)\,t = 2+\lambda.$$

8

So no solutions if $\underline{\underline{\lambda = -3}}$.

(b) Characteristic poly of A is $\begin{vmatrix} 5-\lambda & -2 \\ 12 & -5-\lambda \end{vmatrix}$

Similar seen

$= \lambda^2 - 1$

So eigenvalues are $1, -1$.

Evectors: $\underline{\lambda = 1}$, solve $\begin{pmatrix} 4 & -2 & 0 \\ 12 & -6 & 0 \end{pmatrix} \rightarrow$ evectors $a\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\underline{\lambda = -1}$: solve $\begin{pmatrix} 6 & -2 & 0 \\ 12 & -4 & 0 \end{pmatrix} \rightarrow$ evectors $a\begin{pmatrix} 1 \\ 3 \end{pmatrix}$

So take $P = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$. Then

7

$$\underline{P^{-1}AP = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = D.}$$

Then $A = PDP^{-1} \Rightarrow A^n = PD^nP^{-1}$.

So $A^{101} = PD^{101}P^{-1} = PDP^{-1} = A = \begin{pmatrix} 5 & -2 \\ 12 & -5 \end{pmatrix}$. 5

Setter's initials	Checker's initials	Page number
MLL	MJA	10