

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING  
EXAMINATIONS 2007

EEE/ISE PART III/IV: MEng, BEng and ACGI

**CONTROL ENGINEERING**

Friday, 27 April 2:30 pm

Time allowed: 3:00 hours

**There are SIX questions on this paper.**

**Answer FOUR questions.**

*All questions carry equal marks*

**Any special instructions for invigilators and information for candidates are on page 1.**

Examiners responsible      First Marker(s) :      A. Astolfi  
   Second Marker(s) :      C. Ling

## CONTROL ENGINEERING

1. Consider a linear, single-input, single-output, continuous-time system described by the equations

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ \beta \end{bmatrix} u \quad y = \begin{bmatrix} 1 & \alpha \end{bmatrix} x.$$

- a) Study the reachability and stabilizability properties of the system as a function of  $\beta$ . [ 4 marks ]
- b) Study the observability and detectability properties of the system as a function of  $\alpha$ . [ 4 marks ]
- c) Design, using the separation principle, an output feedback control law such that all eigenvalues of the closed-loop system are at  $-2$ . Discuss for which values of  $\alpha$  and  $\beta$  it is possible to design such a control law. [ 10 marks ]
- d) Consider a static output feedback control law

$$u = Ky.$$

Assume  $\beta = 0$  and  $\alpha \geq 0$ . Determine for which values of  $K$  the closed-loop system is asymptotically stable. [ 2 marks ]

2. Consider an inverted pendulum described by the equation

$$Ml^2 \ddot{\theta} = Mgl \sin \theta + u,$$

where  $\theta$  describes the angle of the pendulum with respect to a vertical axis directed upward,  $M$  is the mass of the pendulum,  $l$  is the length of the pendulum,  $g$  is the gravitational acceleration, and  $u$  is an external torque. (Obviously  $M > 0$ ,  $l > 0$  and  $g > 0$ !)

- a) Write the system in state space form. [ 2 marks ]
- b) Assume  $u$  is constant and compute all equilibrium points of the system. [ 4 marks ]
- c) Compute the linearized system around the equilibrium point corresponding to  $u = 0$  and  $\theta = 0$ . [ 4 marks ]
- d) Show that the equilibrium point in part c) is unstable. [ 2 marks ]
- e) Assume  $M = 1$ ,  $l = 1$  and  $g = 10$ . Design a state feedback control law  $u = Kx$  which asymptotically stabilizes the linearized system determined in part c). [ 4 marks ]
- f) Assume  $l = 1$ ,  $g = 10$  and  $u = Kx$  as determined in part e). Determine for which values of  $M$  the closed-loop linearized system is asymptotically stable. [ 4 marks ]

3. Consider a linear, single-input, single output, discrete-time system described by the equations

$$x^+ = Ax + Bu + Pd \quad y = Cx,$$

where  $x \in X = \mathbb{R}^n$  is the state,  $u(t) \in \mathbb{R}$  is the control input,  $y(t) \in \mathbb{R}$  is the output and  $d(t) \in \mathbb{R}$  is a disturbance. The effect of the disturbance on the output  $y(t)$  has to be cancelled by means of a suitably designed control action.

Assume that the disturbance  $d(t)$  is such that

$$d^+ = Sd.$$

The problem of cancelling the effect of the disturbance  $d$  on the output  $y$  can be solved selecting a control law of the form

$$u = Kx + Ld,$$

where  $K$  is such that the system

$$x^+ = (A + BK)x$$

is asymptotically stable, and

$$L = \Gamma - K\Pi,$$

with  $\Pi \in \mathbb{R}^{n \times 1}$  and  $\Gamma \in \mathbb{R}$  solutions of the equations (known as the FBI equations)

$$\Pi S = A\Pi + B\Gamma + P \quad 0 = C\Pi.$$

Assume

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad P = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$C = [C_1 \quad C_2 \quad C_3] \quad S = [1],$$

with  $C_i \in \mathbb{R}$ , for  $i = 1, 2, 3$ .

- Find  $K$  such that the matrix  $A + BK$  has all eigenvalues at zero. [ 4 marks ]
- Show that, for the selected matrices, the FBI equations have solutions  $\Pi$  and  $\Gamma$  if and only if

$$C_3 = 0$$

or

$$C_1 - C_2 \neq 0.$$

[ 8 marks ]

- Using the results in parts a) and b) write a control law which solves the considered disturbance cancellation problem. [ 2 marks ]
- The control law determined in part c) requires the knowledge of the state  $x$  of the system and of the disturbance  $d$ . It is possible to circumvent this problem by constructing an observer for the system with state  $x$  and  $d$  and output  $y$ . Assume that  $C_1 \neq 0$ ,  $C_2 = C_3 = 0$ , and show that it is possible to design an asymptotic observer for such a system. (Do not design the observer!) [ 6 marks ]

4. Consider a nonlinear, continuous-time system described by the equation

$$\dot{x} = f(x),$$

with  $x \in X = \mathbb{R}^n$ . Suppose  $x = 0$  is an equilibrium.

To study global asymptotic stability of the equilibrium  $x = 0$  of the system the following condition is often used.

**Krasowsky condition.** *The equilibrium  $x = 0$  of the system  $\dot{x} = f(x)$  is globally asymptotically stable if the matrix*

$$\frac{\partial f(x)}{\partial x} + \left( \frac{\partial f(x)}{\partial x} \right)'$$

*has all its eigenvalues with negative real part for all  $x \in \mathbb{R}^n$ .*

Consider the system

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -x_1 \\ -x_2 - x_2^3 + hx_3 \\ kx_2 - x_3 - x_3^3 \end{bmatrix},$$

with  $h$  and  $k$  constant.

- Show that the system has an equilibrium for  $x = 0$  and compute the linearized model of the system around the equilibrium  $x = 0$ . [ 4 marks ]
  - Study the stability properties of the linearized system as a function of  $h$  and  $k$ . In particular show that the linearized system is asymptotically stable if  $1 - kh > 0$ , it is stable if  $1 - kh = 0$  and it is unstable if  $1 - kh < 0$  [ 5 marks ]
  - Using the principle of stability in the first approximation discuss the stability properties of the zero equilibrium of the nonlinear system as a function of  $h$  and  $k$ . [ 5 marks ]
  - Using Krasowsky condition for global asymptotic stability show the following. If  $h + k = 0$  then the equilibrium  $x = 0$  of the nonlinear system is globally asymptotically stable. Hence argue that  $x = 0$  is the only equilibrium of the system. [ 6 marks ]
5. Consider a linear, single-input, single-output, discrete-time system described by the equations

$$x(k+1) = Ax(k) + Bu(k) \quad y(k) = Cx(k),$$

where  $x \in X = \mathbb{R}^n$  is the state,  $u(k) \in \mathbb{R}$  is the input and  $y(k) \in \mathbb{R}$  is the output.

Suppose that the initial state is  $x(0) = 0$ . The matrices  $A$ ,  $B$  and  $C$  are unknown and also the dimension  $n$  of the state space is unknown.

The unknown  $n$ ,  $A$ ,  $B$  and  $C$  can be determined by performing the following experiment. Set

$$u(k) = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{if } k \geq 1, \end{cases}$$

and let  $y(0), y(1), y(2), \dots$ , be the corresponding output sequence. The Hankel matrix associated with this output sequence is defined as

$$H = \begin{bmatrix} y(1) & y(2) & y(3) & \dots \\ y(2) & y(3) & y(4) & \dots \\ y(3) & y(4) & y(5) & \dots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

The dimension of the state space is equal to the rank of  $H$ , i.e.

$$n = \text{rank} H.$$

The matrices  $A$ ,  $B$  and  $C$  can be selected with the following form

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & \cdots & -\alpha_{n-1} \end{bmatrix} \quad B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{bmatrix} \quad C = [1 \ 0 \ 0 \ \cdots \ 0].$$

- a) Let  $H_n$  be the matrix composed of the first  $n$  rows and the first  $n+1$  columns of  $H$ . This matrix has rank  $n$ . Show that

$$H_n = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} [B \ AB \ \cdots \ A^{n-1}B \ A^nB]. \quad (5.1)$$

[ 4 marks ]

- b) Show that the observability matrix

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

coincides with the identity matrix.

[ 2 marks ]

- c) Using equation (5.1) show that

$$B = \begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(n) \end{bmatrix}$$

[ 4 marks ]

- d) Assume  $n = 2$ . Using equation (5.1) show that the coefficients  $\alpha_0$  and  $\alpha_1$  are the solutions of a linear system of equations.

[ 6 marks ]

- e) Assume that  $y(1) = 0$ ,  $y(2) = 1$  and  $y(k) = 0$  for all  $k \geq 3$ . Construct the Hankel matrix associated with this output sequence and compute its rank. Hence compute matrices  $A$ ,  $B$  and  $C$  such that equation (5.1) holds.

[ 4 marks ]

6. Consider a linear, single-input, single-output, continuous-time system described by the equation

$$\dot{x} = Ax + Bu = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u,$$

with initial state  $x(0)$ , and the problem of determining a state feedback control law which stabilizes the system and minimises the cost

$$J(x_0, u) = \int_0^\infty (x'Qx + R^2u^2(t))dt$$

with

$$Q = C'C,$$

$$C = \begin{bmatrix} C_1 & 0 \end{bmatrix},$$

$C_1 > 0$  and  $R > 0$ . The sought after control law can be determined by means of the following steps.

- Verify that the system with output  $y = Cx$  is reachable and observable. [ 4 marks ]
- Consider the Hamiltonian matrix

$$H = \begin{bmatrix} A & -\frac{BB'}{R^2} \\ -Q & -A' \end{bmatrix}$$

Show that the characteristic polynomial of  $H$  is  $p(s) = s^4 + \frac{C_1^2}{R^2}$  and compute the eigenvalues of  $H$  as a function of  $C_1$  and  $R$ . [ 8 marks ]

- Let  $u = Kx = K_1x_1 + K_2x_2$ . Find  $K_1$  and  $K_2$  such that the eigenvalues of the resulting closed-loop system coincide with the eigenvalues of  $H$  having negative real part. Such a feedback control law solves the considered problem. [ 4 marks ]
- Finally, the optimal cost associated with the initial state  $x(0)$  is

$$x'(0)Px(0),$$

where  $P$  is a symmetric matrix such that

$$K = -\frac{B'P}{R^2}.$$

Assume  $x(0) = [0, x_2(0)]'$  and determine the optimal cost associated to this initial state. [ 4 marks ]

## Control engineering exam paper - Model answers - 2007

## Question 1

- a) The reachability matrix is

$$R = \begin{bmatrix} 1 & -1 \\ \beta & 1 - 2\beta \end{bmatrix},$$

and  $\det(R) = 1 - \beta$ . Hence the system is reachable for all  $\beta \neq 1$ . For  $\beta = 1$ , consider the reachability pencil

$$\left[ sI - A \mid B \right] = \left[ \begin{array}{cc|c} s+1 & 0 & 1 \\ -1 & s+2 & 1 \end{array} \right],$$

and note that it loses rank for  $s = -2$ . Hence, the unreachable mode is  $s = -2$ , and the system is stabilizable.

- b) The observability matrix is

$$O = \begin{bmatrix} 1 & \alpha \\ \alpha - 1 & -2\alpha \end{bmatrix},$$

and  $\det(O) = -\alpha - \alpha^2$ . Hence the system is observable for all  $\alpha \neq 0$  and  $\alpha \neq -1$ . For  $\alpha = 0$ , consider the observability pencil

$$\left[ \frac{sI - A}{C} \right] = \left[ \begin{array}{cc|c} s+1 & 0 & 1 \\ -1 & s+2 & 0 \\ 1 & 0 & 0 \end{array} \right],$$

and note that it loses rank for  $s = -2$ . Hence, the unobservable mode is  $s = -2$ , and the system is detectable. For  $\alpha = -1$ , the observability pencil is

$$\left[ \frac{sI - A}{C} \right] = \left[ \begin{array}{cc|c} s+1 & 0 & 1 \\ -1 & s+2 & 0 \\ 1 & -1 & 0 \end{array} \right],$$

and note that it loses rank for  $s = -1$ . Hence, the unobservable mode is  $s = -1$ , and the system is detectable.

- c) To design an output feedback control law with the separation principle and with the given requirement on the eigenvalues we have to find matrices
- $K = [K_1 \ K_2]$
- and
- $L = [L_1 \ L_2]$
- such that the eigenvalues of
- $A + BK$
- and
- $A + LC$
- are all equal to
- $-2$
- .

Note that

$$A + BK = \begin{bmatrix} -1 + K_1 & K_2 \\ 1 + \beta K_1 & -2 + \beta K_2 \end{bmatrix}.$$

Its characteristic polynomial is

$$s^2 + (3 - K_1 - \beta K_2)s + (-\beta K_2 - 2K_1 + 2 - K_2)$$

and this should be equal to  $(s + 2)^2$ . As a result we have to solve the equations

$$3 - K_1 - \beta K_2 = 4 \qquad -\beta K_2 - 2K_1 + 2 - K_2 = 4,$$

yielding

$$K_1 = -1 \quad K_2 = 0.$$

This implies that the state feedback problem is solvable for any  $\beta$ . In fact, it is solvable for  $\beta \neq 1$ , by reachability of the system, and for  $\beta = 1$  because the unreachable mode coincides with one of the desired closed-loop eigenvalues.

Note now that

$$A + LC = \begin{bmatrix} -1 + L_1 & \alpha L_1 \\ 1 + L_2 & -2 + \alpha L_2 \end{bmatrix}.$$

Its characteristic polynomial is

$$s^2 + (3 - L_1 - L_2\alpha)s + (-L_2\alpha - 2L_1 + 2 - L_1\alpha)$$

and this should be equal to  $(s + 2)^2$ . As a result we have to solve the equations

$$3 - L_1 - L_2\alpha = 4 \quad -L_2\alpha - 2L_1 + 2 - L_1\alpha = 4,$$

yielding

$$L_1 = L_2 = -\frac{1}{1 + \alpha}.$$

This implies that the output injection problem is solvable for any  $\alpha \neq -1$ . In fact, it is solvable for  $\alpha \neq -1$  and  $\alpha \neq 0$ , by observability of the system, and for  $\alpha = 0$  because the unobservable mode coincides with one of the desired closed-loop eigenvalues. For  $\alpha = -1$  it is not solvable because the unobservable mode is  $s = -1$ .

Finally, the output feedback control law is described by

$$\dot{\xi} = (A + BK + LC)\xi - Ly \quad u = K\xi.$$

- d) If a static output feedback control law is used then the closed-loop system is described by

$$\dot{x} = (A + BKC)x = \begin{bmatrix} -1 + K & K\alpha \\ 1 + \beta K & K\beta\alpha - 2 \end{bmatrix}$$

If  $\beta = 0$  the characteristic polynomial of  $A + BKC$  is

$$s^2 + (3 - K)s + (2 - 2K - \alpha K).$$

To have asymptotic stability all coefficients of this polynomial have to be positive (by Routh test), hence

$$K < \frac{2}{2 + \alpha} \quad K < 3.$$

Note that the first inequality implies the second (by positivity of  $\alpha$ ), and the admissible values of  $K$  include  $K = 0$ .



## Question 2

- a) To write the system in state space form define the state variables  $x_1 = \theta$  and  $x_2 = \dot{\theta}$ . As a result we have the equations

$$\dot{x}_1 = x_2 \quad \dot{x}_2 = \frac{g}{l} \sin x_1 + \frac{1}{Ml^2} u.$$

- b) The equilibrium points of the system are the solutions of the equations  $\dot{x}_1 = \dot{x}_2 = 0$ . This implies  $x_2 = 0$  and

$$0 = g \sin x_1 + \frac{1}{Ml} u.$$

Therefore if

$$|u| \leq gMl$$

we have infinitely many equilibria, i.e. all solutions of the equation

$$\sin x_1 = -\frac{1}{gMl} u.$$

Note that, from a physical point of view, only two of these solutions are distinct, i.e. describe different positions of the pendulum.

If

$$|u| > gMl$$

the system does not have any equilibrium.

(The above result has a very simple physical interpretation. If the input torque  $u$  is constant and smaller, in absolute value, than the torque generated by the gravity then the pendulum can be in equilibrium, otherwise the pendulum will rotate indefinitely.)

- c) The linearized system around the equilibrium point  $x_1 = 0$ ,  $x_2 = 0$ ,  $u = 0$  is

$$\dot{\delta}_x = A\delta_x + B\delta_u = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & 0 \end{bmatrix} \delta_x + \begin{bmatrix} 0 \\ \frac{1}{Ml^2} \end{bmatrix} \delta_u.$$

- d) The characteristic polynomial of the matrix  $A$  of the linearized system is

$$s^2 - \frac{g}{l},$$

which has the roots

$$s_1 = -\sqrt{\frac{g}{l}} < 0 \quad s_2 = \sqrt{\frac{g}{l}} > 0.$$

Therefore, by the principle of stability in the first approximation, the equilibrium is unstable.

- e) Setting  $M = 1$ ,  $l = 1$ ,  $g = 10$  and  $K = [K_1 \ K_2]$  yields

$$A + BK = \begin{bmatrix} 0 & 1 \\ 10 + K_1 & K_2 \end{bmatrix}.$$

By Routh test, this matrix has all eigenvalues with negative real part if  $K_2 < 0$  and  $10 + K_1 < 0$ . We can, for example, select  $K_1 = -11$  and  $K_2 = -1$ .

f) If  $M$  varies we have

$$A + BK = \begin{bmatrix} 0 & 1 \\ 10 - \frac{11}{M} & -\frac{1}{M} \end{bmatrix},$$

and this matrix has all eigenvalues with negative real part if

$$10 - \frac{11}{M} < 0$$

or equivalently if

$$M < \frac{11}{10}.$$

(It is interesting to note that

- the selection of  $K_2 < 0$  does not affect the values of  $M$  for which we have asymptotic stability;
- a reduction in  $M$  does not yield an unstable closed-loop system;
- to cope with large values of  $M$  it is necessary to select a large (in absolute value) and negative  $K_1$ .)

### Question 3

a) Note that (set  $K = [K_1 \ K_2 \ K_3]$ )

$$A + BK = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ K_1 & K_2 & K_3 + 1 \end{bmatrix}$$

and

$$\det(sI - (A + BK)) = s^3 + (-K_3 - 4)s^2 + (3K_3 - K_2 + 5)s + (2K_2 - K_1 - 2 - 2K_3).$$

This polynomial should be equal to  $s^3$ , hence

$$K_1 = -8 \quad K_2 = -7 \quad K_3 = -4.$$

b) Note that

$$\begin{bmatrix} \Pi_1 \\ \Pi_2 \\ \Pi_3 \end{bmatrix} = \Pi S = A\Pi + B\Gamma + P = \begin{bmatrix} 2\Pi_1 + \Pi_2 \\ \Pi_2 + \Pi_3 + 1 \\ \Pi_3 + \Gamma + 1 \end{bmatrix}$$

and

$$0 = C\Pi = C_1\Pi_1 + C_2\Pi_2 + C_3\Pi_3.$$

From the first equation we obtain

$$\Pi_2 = -\Pi_1 \quad \Pi_3 = -1 \quad \Gamma = -1,$$

which, replaced in the second equation, yields

$$0 = (C_1 - C_2)\Pi_1 - C_3.$$

This equation, in the unknown  $\Pi_1$ , has a solution  $\Pi_1$  if  $C_3 = 0$ , yielding  $\Pi_1 = 0$ , or if  $C_1 - C_2 \neq 0$ , yielding  $\Pi_1 = \frac{C_3}{C_1 - C_2}$ .

c) The required control law is

$$u = Kx + Ld = Kx + (\Gamma - K\Pi)d = \begin{bmatrix} -8 & -7 & -4 \end{bmatrix} x + (-5 + \Pi_1)d,$$

where  $\Pi_1$  is as computed above.

d) The extended system, with state  $d$  and  $x$ , and  $C_2 = C_3 = 0$  is described by the equations

$$\begin{bmatrix} d^+ \\ x^+ \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} d \\ x \end{bmatrix}$$

$$y = \begin{bmatrix} 0 & C_1 & 0 & 0 \end{bmatrix} \begin{bmatrix} d \\ x \end{bmatrix}.$$

The observability matrix of this system is

$$O = \begin{bmatrix} 0 & C_1 & 0 & 0 \\ 0 & 2C_1 & C_1 & 0 \\ C_1 & 4C_1 & 3C_1 & C_1 \\ 5C_1 & 8C_1 & 7C_1 & 4C_1 \end{bmatrix},$$

which has rank four for all  $C_1 \neq 0$ . Hence, the system is observable and it is possible to reconstruct the states  $x$  and  $d$  from measurements of  $y$  (and  $u$ ).

## Question 4

- a) Replacing  $x_1 = x_2 = x_3 = 0$  in the differential equations yields  $\dot{x}_1 = \dot{x}_2 = \dot{x}_3 = 0$ . Hence  $x = 0$  is an equilibrium. The linearized system around this equilibrium is described by

$$\dot{\delta}_x = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & h \\ 0 & k & -1 \end{bmatrix} \delta_x.$$

- b) The characteristic polynomial of the linearized system is

$$(s + 1)(s^2 + 2s + (1 - kh)).$$

Hence, by Routh test, if  $1 - kh > 0$  the system is asymptotically stable, if  $1 - kh < 0$  the system is unstable, if  $1 - kh = 0$  the system is stable (not asymptotically).

- c) The zero equilibrium of the nonlinear system is locally asymptotically stable if  $1 - kh > 0$ , and unstable if  $1 - kh < 0$ . If  $1 - kh = 0$  the principle of stability in the first approximation does not allow to draw any conclusion on the stability properties of such equilibrium.

- d) Note that

$$\frac{\partial f(x)}{\partial x} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 - 3x_2^2 & h \\ 0 & k & -1 - 3x_3^2 \end{bmatrix}$$

hence

$$\frac{\partial f(x)}{\partial x} + \left( \frac{\partial f(x)}{\partial x} \right)' = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 - 6x_2^2 & h + k \\ 0 & h + k & -2 - 6x_3^2 \end{bmatrix}.$$

If  $h + k = 0$  the matrix

$$\frac{\partial f(x)}{\partial x} + \left( \frac{\partial f(x)}{\partial x} \right)'$$

is diagonal and has all eigenvalues negative. Therefore, by Krasowsky condition, the zero equilibrium of the nonlinear system is globally asymptotically stable. This implies that, for any initial condition  $x(0)$ , we have  $\lim_{t \rightarrow \infty} x(t) = 0$ , hence the system cannot have any other equilibrium point.

## Question 5

a) Note that

$$y(1) = CB \quad y(2) = CAB \quad \dots \quad y(i) = CA^{i-1}B \quad \dots,$$

hence

$$H_n = \begin{bmatrix} CB & CAB & \dots & CA^n B \\ CAB & CA^2 B & \dots & CA^{n+1} B \\ \vdots & \vdots & \vdots & \vdots \\ CA^{n-1} B & CA^n B & \dots & CA^{2n-1} B \end{bmatrix}$$

and this coincides with

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \begin{bmatrix} B & AB & \dots & A^n B \end{bmatrix}.$$

b) By a direct computation we have

$$CA = [0 \ 1 \ 0 \ 0 \ \dots \ 0], \quad CA^2 = [0 \ 0 \ 1 \ 0 \ \dots \ 0], \quad \dots$$

which proves the claim.

c) Since the observability matrix is the identity, equation (5.1) is

$$H_n = \begin{bmatrix} B & AB & \dots & A^n B \end{bmatrix}.$$

Therefore,  $B$  is equal to the first column of  $H_n$ , i.e.

$$B = \begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(n) \end{bmatrix}.$$

d) Note that (recall that  $n = 2$ )

$$H_2 = \begin{bmatrix} y(1) & y(2) & y(3) \\ y(2) & y(3) & y(4) \end{bmatrix} = \begin{bmatrix} B & AB & A^2 B \end{bmatrix}$$

where

$$AB = \begin{bmatrix} B_2 \\ -\alpha_0 B_1 - \alpha_1 B_2 \end{bmatrix} \quad A^2 B = \begin{bmatrix} -\alpha_0 B_1 - \alpha_1 B_2 \\ -\alpha_0 B_2 - \alpha_1 (-\alpha_0 B_1 - \alpha_1 B_2) \end{bmatrix}.$$

Therefore,

$$\begin{aligned} B_1 &= y(1) & B_2 &= y(2), \\ -\alpha_0 y(1) - \alpha_1 y(2) &= y(3) \end{aligned}$$

and

$$-\alpha_0 y(2) - \alpha_1 (-\alpha_0 B_1 - \alpha_1 B_2) = -\alpha_0 y(2) - \alpha_1 (y(3)) = y(4).$$

Therefore, to determine  $\alpha_0$  and  $\alpha_1$  we have to solve the last two linear equations.

e) The Hankel matrix associated to the given output sequence is

$$H = \begin{bmatrix} 0 & 1 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix},$$

which has rank equal to two.

Exploiting the results of the previous points we have

$$B_1 = 0 \quad B_2 = 1$$

and the equations

$$-\alpha_1 y(2) = 0 \quad -\alpha_0 y(2) = 0,$$

yielding  $\alpha_0 = \alpha_1 = 0$ . Therefore,

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

## Question 6

- a) The system is in reachability canonical form, hence it is reachable. The observability matrix is

$$O = C_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

hence the system is observable.

- b) The Hamiltonian matrix is

$$H = \left[ \begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{R^2} \\ \hline -C_1^2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array} \right].$$

To determine the characteristic polynomial  $p(s)$  of  $H$  compute the determinant of  $sI - H$  using the 'expansion by minors method' starting from the first row. This yields  $p(s) = s(s^3 + 1(\frac{C_1^2}{R^2}))$ . Note now that

$$p(s) = s^4 + \frac{C_1^2}{R^2} = \left( s^2 + \sqrt{2}\sqrt{\frac{C_1}{R}}s + \frac{C_1}{R} \right) \left( s^2 - \sqrt{2}\sqrt{\frac{C_1}{R}}s + \frac{C_1}{R} \right)$$

The eigenvalues of  $H$  are the roots of  $p(s)$ , namely

$$\sqrt{\frac{C_1}{R}} \left( \pm \frac{\sqrt{2}}{2} \pm i \frac{\sqrt{2}}{2} \right).$$

- c)  $K_1$  and  $K_2$  have to be such that the characteristic polynomial of

$$A + BK = \begin{bmatrix} 0 & 1 \\ K_1 & K_2 \end{bmatrix},$$

namely

$$s^2 - K_2s - K_1,$$

equals

$$s^2 + \sqrt{2}\sqrt{\frac{C_1}{R}}s + \frac{C_1}{R}.$$

As a result,

$$K_1 = -\frac{C_1}{R} \quad K_2 = -\sqrt{2}\sqrt{\frac{C_1}{R}}$$

- d) Let

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix},$$

and note that

$$x(0)'Px(0) = [0 \ x_2(0)]P \begin{bmatrix} 0 \\ x_2(0) \end{bmatrix} = x_2(0)^2 P_{22}.$$

Finally,

$$K = [K_1 \ K_2] = -\frac{B'P}{R^2} = -\frac{1}{R^2}[P_{12} \ P_{22}],$$

hence

$$P_{22} = \sqrt{2}\sqrt{\frac{C_1}{R}}R^2,$$

and the optimal cost is  $x_2(0)^2\sqrt{2}\sqrt{\frac{C_1}{R}}R^2$ .