UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 1999

BEng Honours Degree in Computing Part I
MEng Honours Degrees in Computing Part I
BSc Honours Degree in Mathematics and Computer Science Part I
MSci Honours Degree in Mathematics and Computer Science Part I
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the
Associateship of the Royal College of Science
Associateship of the City and Guilds of London Institute

PAPER 1.1 / MC 1.1

MATHEMATICAL REASONING – LOGIC
Friday, May 14th 1999, 4.00 – 5.30

Answer THREE questions

For admin. only:
paper contains 4 questions

1 Complete in logic the definitions in (a) - (e), which are informally described in English.

None of the answers should be a recursive definition and you should use ***only*** the predicates *merge*, *in* and the standard $\leq, \geq, =, <, >$, where

merge(x,y,z) holds for lists x, y and z iff z is a permutation of x ++y, such that the relative order of elements in x and y is retained in z,
in(x,y) holds iff element x is in list y,

and the Haskell notations ++, !! , # and mod, with their usual meanings.
You need only use the types [Nat] and Nat, where Nat is the set of integers $\geq$0.

Example: ∀x,y:[Nat] [remove(x,y) ↔
y is the result of removing the first and last elements of x]

Answer: ∀x,y:[Nat] [remove(x,y) ↔ ∃ c,d:Nat [x = [c] ++ y ++ [d]]]

a ∀x,y:[Nat], n:Nat [rotate(x,y,n) ↔ y is the list x rotated n positions to the left]

e.g. rotate ([1,2,3,4,5],[3,4,5,1,2],7) is true.

b ∀x,y:[Nat] [bigger(x,y) ↔ for each element in x there is a larger element in y]

c ∀x,z:Nat, y:[Nat] [occurrences(x,y,z) ↔
z is the number of occurrences of x in y]

(**Hint:** Make use of merge.)

d ∀x,y [Nat], z:Nat [replace(x,y,z) ↔
y is the result of replacing all occurrences of z in x by 0]

(**Hint**: Make use of !!)

e ∀x:[Nat], y:[(Nat,Nat)] [makebag(x,y) ↔ for each element i in x there is exactly one pair (i,n) in y, and n is the number of occurrences of i in x]

(**Hint**: Make use of occurrences.)

The five parts carry, respectively, 20%, 10%, 25%, 20%, 25% of the marks.

2 a Consider the two derived rules *resolution* and *factor*:

$$\frac{A \vee B \qquad \neg A \vee C}{B \vee C}\ \text{(resolution)} \qquad \text{and} \qquad \frac{A \vee A}{A}\ \text{(factor)}$$

i) *Without rewriting using equivalences*, use natural deduction to prove the resolution derived rule.

ii) Using **only** the two rules *resolution and factor*, show

$$A \vee B,\ \neg B \vee C,\ \neg A \vee B \ \vdash\ C$$

b Use equivalences to prove

$$(\neg x \vee (x \wedge y)) \wedge (x \vee (\neg x \wedge z)) \equiv (x \rightarrow y) \wedge (\neg x \rightarrow z), \text{ and}$$
$$(\neg x \vee (x \wedge y)) \wedge (x \vee (\neg x \wedge z)) \equiv (x \wedge y) \vee (\neg x \wedge z)$$

c The IF connective can be defined as IF $(x,y,z) \equiv (x \rightarrow y) \wedge (\neg x \rightarrow z)$.
The following introduction and elimination rules for IF are based on the rules $\wedge$I, $\rightarrow$I, $\wedge$E and $\rightarrow$E .

$$\frac{\text{IF}(A, B, C) \quad A}{B}\ \text{(IF E)} \qquad \frac{\text{IF}(A, B, C) \quad \neg A}{C}\ \text{(IF E)}$$

A	¬ A
B	C

IF(A, B, C) (IF I)

Use these rules, together with ($\vee$E), to show

$$\text{IF}(a, \text{IF}(d, b, c), c) \ \vdash\ \text{IF}(d, \text{IF}(a, b, c), c)$$

The three parts carry, respectively, 40%, 30%, 30% of the marks.

Turn over ...

3 a By trying (and failing) to find a natural deduction proof to show that (1) $\vdash$ (2), or otherwise, find a counter-example structure with a domain of exactly 2 elements to show that (1) $\not\models$ (2), where

$$(1) \quad \forall x,y\ [P(x,y) \rightarrow E(y,x)]$$

$$(2) \quad \forall x,y\ [P(y,x) \rightarrow E(y,x)]$$

Justify *carefully* that your structure is a counter-example.

b Let S0 be a set of propositional sentences including the sentences ($b \vee c$) and b. Let S1 = S0 - {($b \vee c$)}.

The following is an informal proof of the statement

"Any arbitrary model of every sentence of the set S1,
is also a model of every sentence of the set S0".

(A model of a set of propositional sentences is a propositional assignment that makes every sentence in the set true.)

Proof: Given (3), (4) and (5):

(3) b is in S1.

(4) For any model m, if m is a model of b then m is a model of ($b \vee c$).

(5) For any sentence s, if s is in S0 then s = ($b \vee c$) or s is in S1.

Suppose N is an arbitrary model of every sentence in S1.
Let Z be an arbitrary sentence in S0.
Then either Z = ($b \vee c$) or Z is in S1.
In case 1, by (3), N is a model of b and hence, by (4), also of ($b \vee c$).
In case 2, N is a model of Z also, by assumption.
Hence N is a model of every sentence in S0.

Endproof

Using the above outline proof as a guide, translate (3), (4) and (5) into logic and give a correct natural deduction proof of

$$(3),(4),(5) \vdash \forall n(\ \forall y[in(y,S1) \rightarrow model(n,y)] \rightarrow \forall z[in(z,S0) \rightarrow model(n,z)]\ )$$

Use the predicates *in(x,y)*, which holds iff x is in y, *model(m,x)*, which holds iff m is a model of the sentence x, and the function symbol *or(x,y)*, meaning the term ($x \vee y$).

The two parts carry, respectively, 40%, 60% of the marks.

4 a) Given

$$
\begin{array}{ll}
(1) & \forall x,y\ [\mathrm{red}(x) \wedge \mathrm{red}(y) \rightarrow x=y\ ] \\
(2) & \forall y\ [\ (\exists x.\mathrm{on}(x,y)) \rightarrow \mathrm{red}(y)\ ] \\
(3) & \mathrm{on}(A,B) \\
(4) & \neg\ (A = B) \\
(5) & \forall x[\neg\ \mathrm{on}(x,A)]
\end{array}
$$

i) Translate sentences (1), (2) and (5) into *natural* English, where red(x) holds iff x is red and on(x,y) holds iff x is on y.

ii) *Without rewriting by equivalences* use natural deduction to show

$$(1), (2), (3), (4) \vdash (5).$$

b) *Without rewriting by equivalences* use natural deduction to show

$$\neg\ (P \wedge Q) \rightarrow C,\quad P \rightarrow (R \vee S),\quad \neg\ (T \vee R),\quad S \rightarrow T \quad \vdash \quad C$$

c) *Without rewriting by equivalences* use natural deduction to show

$$\exists x[P(x) \wedge a = x] \leftrightarrow P(a)$$

The three parts carry, respectively, 45%, 25%, 30% of the marks.

End of paper