Modelling and control of multibody mechanical systems

Model answers

Question 1

a) i) $\boldsymbol{r}_1 = l\boldsymbol{e}_{r1}$ and $\boldsymbol{r}_2 = l\boldsymbol{e}_{r2}$. [2 marks]

ii) By differentiating each of the position vectors we obtain $\dot{\boldsymbol{r}}_1 = l\dot{\theta}_1\boldsymbol{e}_{\theta 1}$ and $\dot{\boldsymbol{r}}_2 = l\dot{\theta}_2\boldsymbol{e}_{\theta 2}$. [2 marks]

b) i) The total kinetic energy of the system is

$$T = \frac{1}{2}m_1 l^2\dot{\theta}_1^2 + \frac{1}{2}m_2 l^2\dot{\theta}_2^2.$$

[2 marks]

ii) The potential energy of the system, with zero potential energy at the level of the point O, is

$$V = -m_1 g l\cos\theta_1 - m_2 g l\cos\theta_2 + \frac{1}{2}k(\theta_1 - \theta_2)^2.$$

[3 marks]

iii) The Lagrangian function of the system is

$$L = T - V = \frac{1}{2}m_1 l^2\dot{\theta}_1^2 + \frac{1}{2}m_2 l^2\dot{\theta}_2^2 + m_1 g l\cos\theta_1 + m_2 g l\cos\theta_2 - \frac{1}{2}k(\theta_1 - \theta_2)^2$$

[1 mark]

c) i) The Lagrangian equation with respect to the generalised coordinate θ_1 is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\theta}_1}\right) - \frac{\partial L}{\partial\theta_1} = 0,$$

which yields the first equation of motion

$$m_1 l^2\ddot{\theta}_1 + m_1 g l\sin\theta_1 + k(\theta_1 - \theta_2) = 0.$$

The Lagrangian equation with respect to the generalised coordinate θ_2 is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\theta}_2}\right) - \frac{\partial L}{\partial\theta_2} = 0,$$

which yields the second equation of motion

$$m_2 l^2\ddot{\theta}_2 + m_2 g l\sin\theta_2 - k(\theta_1 - \theta_2) = 0.$$

[4 marks]

ii) Rewrite the Lagrangian function as

$$L = \frac{1}{2}m_1 r_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2 r_2^2\dot{\theta}_2^2 + m_1 g r_1\cos\theta_1 + m_2 g r_2\cos\theta_2 - \frac{1}{2}k(\theta_1 - \theta_2)^2,$$

where $r_1 = l$ and $r_2 = l$ are two constraints. Therefore, the force of constraint holding the first mass is

$$F_1 = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{r}_1}\right) - \frac{\partial L}{\partial r_1} = -\frac{\partial L}{\partial r_1} = -m_1 l\dot{\theta}_1^{\,2} - m_1 g\cos\theta_1.$$

The force of constraint holding the second mass is

$$F_2 = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{r_2}}\right) - \frac{\partial L}{\partial r_2} = -\frac{\partial L}{\partial r_2} = -m_2 l \dot{\theta_2}^2 - m_2 g \cos\theta_2.$$

The total external force of constraint in vector form is

$$\boldsymbol{F} = F_1 \boldsymbol{e_{r1}} + F_2 \boldsymbol{e_{r2}} = (-m_1 l \dot{\theta_1}^2 - m_1 g \cos\theta_1)\boldsymbol{e_{r1}} + (-m_2 l \dot{\theta_2}^2 - m_2 g \cos\theta_2)\boldsymbol{e_{r2}}.$$

[6 marks]

Question 2

a) L_1: $z = -\frac{2h}{b}x + h$ and L_2: $z = -\frac{2h}{b}y + h$, which is equivalent to L_1: $x = \frac{b}{2} - \frac{b}{2h}z$ and L_2: $y = \frac{b}{2} - \frac{b}{2h}z$ [3 marks]

b) i) The moment of inertia about the axis of symmetry (Z axis) is:

$$I_{zz} = \int (x^2 + y^2)dm = \rho \int_V (x^2 + y^2)dV.$$

If we consider an infinitesimal volume element given in Cartesian coordinates then I_{zz} becomes

$$I_{zz} = \rho \int_0^h \int_{-\frac{b}{2}+\frac{b}{2h}z}^{\frac{b}{2}-\frac{b}{2h}z} \int_{-\frac{b}{2}+\frac{b}{2h}z}^{\frac{b}{2}-\frac{b}{2h}z} (x^2 + y^2)dxdydz,$$

in which the limits for the integration with respect to x and y are variable with respect to z, according to the expressions derived in part a). Hence, after some effort,

$$I_{zz} = \frac{\rho h b^4}{30}.$$

Note that the integrations which involve the variable limits, which are functions of z, should be performed before the integration with respect to z. Finally, the mass of the pyramid is given by

$$m = \rho V = \frac{1}{3}\rho b^2 h,$$

and therefore I_{zz} becomes

$$I_{zz} = \frac{1}{10}mb^2.$$

[8 marks]

ii) The moment of inertia about the X axis can be found similarly via the equation

$$I_{xx} = \int (y^2 + z^2)dm = \rho \int_V (y^2 + z^2)dV.$$

By using a Cartesian volume element as above the volume integral becomes

$$I_{xx} = \rho \int_0^h \int_{-\frac{b}{2}+\frac{b}{2h}z}^{\frac{b}{2}-\frac{b}{2h}z} \int_{-\frac{b}{2}+\frac{b}{2h}z}^{\frac{b}{2}-\frac{b}{2h}z} (y^2 + z^2)dxdydz,$$

and by making use of the mass expression of the pyramid

$$I_{xx} = \frac{1}{20}m\left(b^2 + 2h^2\right).$$

[7 marks]

iii) Due to the symmetry of the pyramid, I_{yy} is the same as I_{xx}, i.e.

$$I_{yy} = I_{xx}.$$

[2 marks]

Question 3

a) $\boldsymbol{\Omega_a} = \dot{\psi}\boldsymbol{k}'$. [1 mark]

b) $\boldsymbol{\Omega_{wl}} = \dot{\theta}_l\boldsymbol{j}' + \dot{\psi}\boldsymbol{k}'$ and $\boldsymbol{\Omega_{wr}} = \dot{\theta}_r\boldsymbol{j}' + \dot{\psi}\boldsymbol{k}'$. [2 marks]

c) The moment of inertia of each wheel about the axis of rotation through A is found by the parallel axis theorem as $I_{xx} + m\left(\frac{l}{2}\right)^2$. Therefore the total moment of inertia is

$$I_{locked} = 2\left(I_{xx} + \frac{ml^2}{4}\right) + I_{axle}$$

[3 marks]

i) The velocity of the road contact point of the left wheel is

$$\frac{l}{2}\dot{\psi} + R\dot{\theta}_l = 0, \tag{1}$$

which represents the first constraint equation. The velocity of the road contact point of the right wheel is

$$\frac{l}{2}\dot{\psi} - R\dot{\theta}_r = 0, \tag{2}$$

which represents the second constraint equation. It is obvious that these equations are integrable, therefore they represent holonomic constraints.

[3 marks]

ii) The inertia matrix of the left wheel with respect to a set of axes parallel to the (unspun) axle-fixed axes and with origin the centre of mass of the wheel is

$$\boldsymbol{I_{COM}} = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{xx} \end{bmatrix},$$

due to the symmetry of the wheel. We can shift the origin of this set of axes by a distance $\frac{l}{2}$ along the axle, to the point A. We can then find the new inertia tensor with respect to the axle-fixed axes by adding to the inertia tensor about the centre of mass of the wheel a difference term as follows

$$\boldsymbol{I_{wlA}} = \boldsymbol{I_{COM}} + \begin{bmatrix} \frac{ml^2}{4} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{ml^2}{4} \end{bmatrix},$$

which amounts to

$$\boldsymbol{I_{wlA}} = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{xx} \end{bmatrix} + \begin{bmatrix} \frac{ml^2}{4} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{ml^2}{4} \end{bmatrix} = \begin{bmatrix} I_{xx} + \frac{ml^2}{4} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{xx} + \frac{ml^2}{4} \end{bmatrix}.$$

By following a similar procedure, the right wheel inertia matrix $\boldsymbol{I_{wrA}} = \boldsymbol{I_{wlA}}$.

[2 marks]

iii) The angular momentum vector of the left wheel is $\boldsymbol{H_{wl}} = \boldsymbol{I_{wlA}}\boldsymbol{\Omega_{wl}}$ and therefore

$$\boldsymbol{H_{wl}} = \begin{bmatrix} 0 \\ I_{yy}\dot{\theta}_l \\ \left(I_{xx} + \frac{ml^2}{4}\right)\dot{\psi} \end{bmatrix},$$

or in vector notation

$$\boldsymbol{H_{wl}} = I_{yy}\dot{\theta}_l\boldsymbol{j'} + \left(I_{xx} + \frac{ml^2}{4}\right)\dot{\psi}\boldsymbol{k'}.$$

By using equation (1)

$$\boldsymbol{H_{wl}} = -\frac{l}{2R}I_{yy}\dot{\psi}\boldsymbol{j'} + \left(I_{xx} + \frac{ml^2}{4}\right)\dot{\psi}\boldsymbol{k'}.$$

The angular momentum vector of the right wheel is $\boldsymbol{H_{wr}} = \boldsymbol{I_{wrA}}\boldsymbol{\Omega_{wr}}$ and therefore

$$\boldsymbol{H_{wr}} = \begin{bmatrix} 0 \\ I_{yy}\dot{\theta}_r \\ \left(I_{xx} + \frac{ml^2}{4}\right)\dot{\psi} \end{bmatrix},$$

or in vector notation

$$\boldsymbol{H_{wr}} = I_{yy}\dot{\theta}_r\boldsymbol{j'} + \left(I_{xx} + \frac{ml^2}{4}\right)\dot{\psi}\boldsymbol{k'}.$$

By using equation (2)

$$\boldsymbol{H_{wr}} = \frac{l}{2R}I_{yy}\dot{\psi}\boldsymbol{j'} + \left(I_{xx} + \frac{ml^2}{4}\right)\dot{\psi}\boldsymbol{k'}.$$

[3 marks]

iv) By considering the motion of the left wheel about point A

$$\frac{d\boldsymbol{H_{wl}}}{dt} = \frac{d'\boldsymbol{H_{wl}}}{dt} + \boldsymbol{\Omega_a} \times \boldsymbol{H_{wl}} = \boldsymbol{N_{wl}}.$$

Therefore

$$-\frac{l}{2R}I_{yy}\ddot{\psi}\boldsymbol{j'} + \left(I_{xx} + \frac{ml^2}{4}\right)\ddot{\psi}\boldsymbol{k'} + \dot{\psi}\boldsymbol{k'} \times \left(-\frac{l}{2R}I_{yy}\dot{\psi}\boldsymbol{j'} + \left(I_{xx} + \frac{ml^2}{4}\right)\dot{\psi}\boldsymbol{k'}\right)$$

$$= \frac{l}{2R}I_{yy}\dot{\psi}^2\boldsymbol{i'} - \frac{l}{2R}I_{yy}\ddot{\psi}\boldsymbol{j'} + \left(I_{xx} + \frac{ml^2}{4}\right)\ddot{\psi}\boldsymbol{k'} = M_{lx}\boldsymbol{i'} - FR\boldsymbol{j'} + \left(M_{lz} - F\frac{l}{2}\right)\boldsymbol{k'},$$

where M_{lx} and M_{lz} are the moments applied by the axle onto the wheel in the $\boldsymbol{i'}$ and $\boldsymbol{k'}$ directions respectively, and F is the force acting on the wheel contact point from the road. Therefore, $F = \frac{l}{2R^2}I_{yy}\ddot{\psi}$ and $M_{lz} = \left(I_{xx} + \frac{ml^2}{4}\right)\ddot{\psi} + \frac{l^2}{4R^2}I_{yy}\ddot{\psi}$.
By considering the motion of the right wheel about point A

$$\frac{d\boldsymbol{H_{wr}}}{dt} = \frac{d'\boldsymbol{H_{wr}}}{dt} + \boldsymbol{\Omega_a} \times \boldsymbol{H_{wr}} = \boldsymbol{N_{wr}}.$$

Therefore

$$\frac{l}{2R} I_{yy}\ddot{\psi}\boldsymbol{j}' + \left(I_{xx} + \frac{ml^2}{4}\right)\ddot{\psi}\boldsymbol{k}' + \dot{\psi}\boldsymbol{k}' \times \left(\frac{l}{2R} I_{yy}\dot{\psi}\boldsymbol{j}' + \left(I_{xx} + \frac{ml^2}{4}\right)\dot{\psi}\boldsymbol{k}'\right)$$

$$= -\frac{l}{2R} I_{yy}\dot{\psi}^2\boldsymbol{i}' + \frac{l}{2R} I_{yy}\ddot{\psi}\boldsymbol{j}' + \left(I_{xx} + \frac{ml^2}{4}\right)\ddot{\psi}\boldsymbol{k}' = M_{rx}\boldsymbol{i}' + FR\boldsymbol{j}' + \left(M_{rz} - F\frac{l}{2}\right)\boldsymbol{k}',$$

where M_{rx} and M_{rz} are the moments applied by the axle onto the wheel in the $\boldsymbol{i}'$ and $\boldsymbol{k}'$ directions respectively, and F is the force acting on the wheel contact point from the road. Therefore, $M_{rz} = \left(I_{xx} + \frac{ml^2}{4}\right)\ddot{\psi} + \frac{l^2}{4R^2} I_{yy}\ddot{\psi}$.
By considering the motion of the axle when a moment M about the vertical axis is acting on it

$$M - M_{lz} - M_{rz} = I_{axle}\ddot{\psi},$$

which yields

$$M = \left(2\left(I_{xx} + \frac{ml^2}{4}\right) + \frac{l^2}{2R^2} I_{yy} + I_{axle}\right)\ddot{\psi}.$$

Therefore the apparent inertia has increased by $\frac{l^2}{2R^2} I_{yy}$ compared to I_{locked} in part c).

[6 marks]

Question 4

a) 2 degrees of freedom, generalised coordinates are θ_1 and θ_2. [2 marks]

b) The moment acting on the first mass is

$$\boldsymbol{N_1} = \boldsymbol{r_1} \times m_1 g\boldsymbol{k} - k(\theta_1 - \theta_2)\boldsymbol{j},$$

where $\boldsymbol{r_1} = l\boldsymbol{e_{r1}}$ is the position vector of the first mass. Therefore,

$$\boldsymbol{N_1} = (-m_1 gl \sin\theta_1 - k(\theta_1 - \theta_2))\boldsymbol{j}.$$

The moment acting on the second mass is

$$\boldsymbol{N_2} = \boldsymbol{r_2} \times m_2 g\boldsymbol{k} + k(\theta_1 - \theta_2)\boldsymbol{j},$$

where $\boldsymbol{r_2} = l\boldsymbol{e_{r2}}$ is the position vector of the second mass. Therefore,

$$\boldsymbol{N_2} = (-m_2 gl \sin\theta_2 + k(\theta_1 - \theta_2))\boldsymbol{j}.$$

[3 marks]

c) The angular momentum vector of the first mass is $\boldsymbol{H_1} = \boldsymbol{r_1} \times m_1\dot{\boldsymbol{r}}_1$, where $\dot{\boldsymbol{r}}_1 = l\dot{\theta}_1\boldsymbol{e_{\theta 1}}$ is the velocity vector of the first mass. Therefore,

$$\boldsymbol{H_1} = l\boldsymbol{e_{r1}} \times ml\dot{\theta}_1\boldsymbol{e_{\theta 1}} = m_1 l^2\dot{\theta}_1\boldsymbol{j}.$$

The angular momentum vector of the second mass is $\boldsymbol{H_2} = \boldsymbol{r_2} \times m_2\dot{\boldsymbol{r}}_2$, where $\dot{\boldsymbol{r}}_2 = l\dot{\theta}_2\boldsymbol{e_{\theta 2}}$ is the velocity vector of the second mass. Therefore,

$$\boldsymbol{H_2} = l\boldsymbol{e_{r2}} \times ml\dot{\theta}_2\boldsymbol{e_{\theta 2}} = m_2 l^2\dot{\theta}_2\boldsymbol{j}.$$

[4 marks]

d) The motion of the first mass is given by $\frac{d\boldsymbol{H_1}}{dt} = \boldsymbol{N_1}$. Therefore the first equation of motion is

$$m_1 l^2\ddot{\theta}_1 = -m_1 gl \sin\theta_1 - k(\theta_1 - \theta_2).$$

The motion of the second mass is given by $\frac{d\boldsymbol{H_2}}{dt} = \boldsymbol{N_2}$. Therefore the second equation of motion is

$$m_2 l^2\ddot{\theta}_2 = -m_2 gl \sin\theta_2 + k(\theta_1 - \theta_2).$$

[4 marks]

e) By differentiating the velocity vector of each of the masses we obtain

$$\ddot{\boldsymbol{r}}_1 = -l\dot{\theta}_1^2\boldsymbol{e_{r1}} + l\ddot{\theta}_1\boldsymbol{e_{\theta 1}},$$

and

$$\ddot{\boldsymbol{r}}_2 = -l\dot{\theta}_2^2\boldsymbol{e_{r2}} + l\ddot{\theta}_2\boldsymbol{e_{\theta 2}}.$$

[3 marks]

f) By considering all the forces that act on the first mass (from the rod and due to gravity), we can write

$$m_1\ddot{\boldsymbol{r}}_1 = F_1\boldsymbol{e}_{r1} + F_{\theta 1}\boldsymbol{e}_{\theta 1} + mg\boldsymbol{k},$$

where F_1 is an external constraint force to the system and $F_{\theta 1}$ is an internal force to the system (since it produces the torsional spring moment between the two pendulums). Therefore,

$$F_1 = -m_1 l\dot{\theta}_1^2 - m_1 g\cos\theta_1.$$

By considering all the forces that act on the second mass (from the rod and due to gravity), we can write

$$m_2\ddot{\boldsymbol{r}}_2 = F_2\boldsymbol{e}_{r2} + F_{\theta 2}\boldsymbol{e}_{\theta 2} + mg\boldsymbol{k},$$

where F_2 is an external constraint force to the system and $F_{\theta 2}$ is an internal force to the system (since it produces the torsional spring moment between the two pendulums). Therefore,

$$F_2 = -m_2 l\dot{\theta}_2^2 - m_2 g\cos\theta_2.$$

Therefore, the overall external constraint force acting on the system at O is

$$\boldsymbol{F} = F_1\boldsymbol{e}_{r1} + F_2\boldsymbol{e}_{r2} = (-m_1 l\dot{\theta}_1{}^2 - m_1 g\cos\theta_1)\boldsymbol{e}_{r1} + (-m_2 l\dot{\theta}_2{}^2 - m_2 g\cos\theta_2)\boldsymbol{e}_{r2}.$$

[4 marks]