SOLUTIONS

EE3-07

DIGITAL SIGNAL PROCESSING

1. a) Given a discrete-time signal $x(n)$, write down the formula for the z-transform $X(z)$ and hence find the z-transform of

$$x(n) = [-10,\ 10,\ 5,\ 2].$$
$$\uparrow$$

. [2]

> *Solution:*
> $X(z) = \sum_{-\infty}^{\infty} x(n) z^{-n}$.
> For the particular finite duration sequence
> $X(z) = -10z^2 + 10z + 5 + 2z^{-1}$.

b) Consider a linear system with system function $H(z)$. Explain what is meant by the term *Region of Convergence* in the context of the z-transform and state the relationship between the *Region of Convergence* and the *stability* of $H(z)$. [2]

> *Solution:*
> The ROC of the z-transform is the region of values of z for which $H(z)$ is a convergent series. (The concept of 'series' must be included for full marks.) Stability of $H(z)$ is indicated when the ROC includes the unit circle in z.

c) Next consider

$$P(z) = \frac{1}{1 - pz^{-1}}$$

and the unit step function $u(n)$.

i) If the inverse z-transform of $P(z)$ corresponds to a causal signal, write an expression in the discrete-time domain for this causal signal and state the *Region of Convergence*. [1]

ii) If the inverse z-transform of $P(z)$ corresponds to an anticausal signal, write an expression in the discrete-time domain for this anticausal signal and state the *Region of Convergence*. [1]

> *Solution:*
> For the causal case, we obtain $p^n u(n)$ with ROC: $|z| > |p|$.
> For the anticausal case, we obtain $-p^n u(-n-1)$ with ROC: $|z| < |p|$.

d) For the causal system

$$Q(z) = \frac{5z + 15.4}{z^2 + 5.2z + 1}$$

find the inverse z-transform of $Q(z)$ and explain whether or not $Q(z)$ is stable. [5]

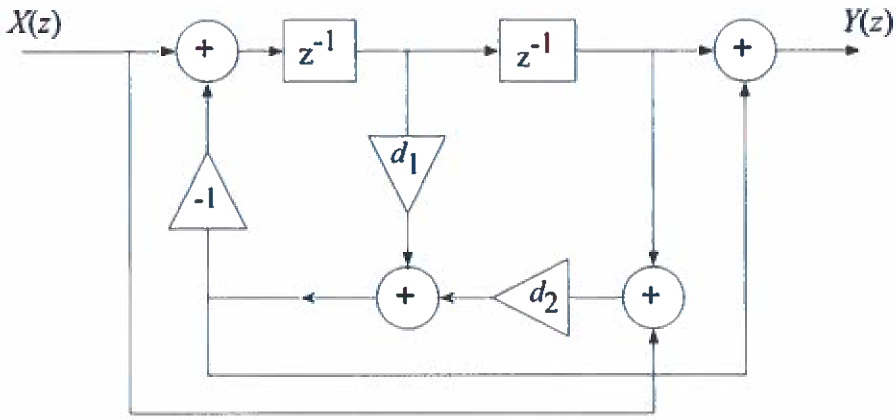


Figure 1.1 Signal flow graph

Solution:
First by partial fraction expansion:

$$Q(z) = 1 + \frac{3}{z+0.2} + \frac{2}{z+5}.$$

Then for a causal system we use ROC: $|z| > \max(|-0.2, -5|)$ so $|z| > 5$.
Hence $q(n) = \delta(n) + 3(-0.2)^n u(n) + 2(-5)^n u(n)$.
This is unstable because the ROC does not include the unit circle in z.

e) The block diagram of Figure 1.1 shows a discrete-time system with system function $H(z)$. The z-transforms of the input signal $x(n)$ and the output signal $y(n)$ are $X(z)$ and $Y(z)$ respectively. Evaluate the system function $H(z)$ in terms of the constant scalar coefficients d_1, d_2 and state the key property of $H(z)$. [9]

Solution:
Start by writing:

$$
\begin{aligned}
p &= x - q \\
q &= pz^{-1}d_1 + d_2 r \\
r &= x + pz^{-2} \\
y &= pz^2 + q.
\end{aligned}
$$

Then

$$
\begin{aligned}
q &= xd_1z^{-1} + xd_2z^{-2} - qz^{-1}d_1 - qz^{-2}d_2 \\
&= x\frac{d_1z^{-1} + d_2z^{-2} + d_2}{1 + d_1z^{-1} + d_2z^{-2}} \\
y &= (x-q)z^{-2} + q \\
&= xz^{-2} + x\frac{d_1z^{-1}d_2 + d_2}{1 + d_1z^{-1} + d_2z^{-2}}(1 - z^{-2})
\end{aligned}
$$

so

$$
H(z) = \frac{d_2 + d_1z_1 + z^{-2}}{1 + d_1z^{-1} + d_2z^{-2}}.
$$

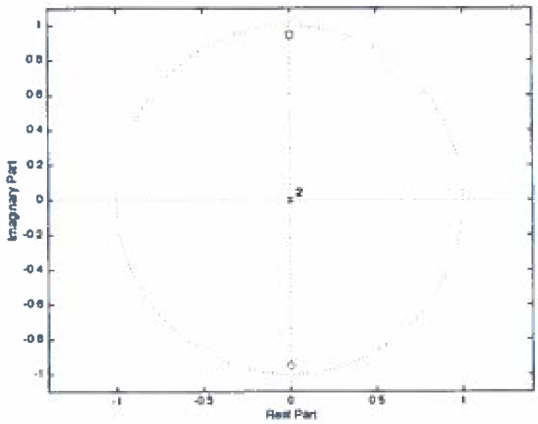
The reflected coefficients in the numerator and denominator tell us this is an allpass filter.

2. Consider a discrete-time filter with finite impulse response for which the input is denoted $x(n)$, the output is denoted $y(n)$, n is the discrete-time index and

$$y(n) = -0.5x(n) - 0.45x(n-2).$$

a) Draw a labelled sketch plot of the z-plane and indicate on the plot the positions of the poles and zeros of this filter. [4]

Solution:
This filter has zeros at $z_0 = \pm 0.95j$ with two trivial poles at the origin. The plot has the following form.



b) Write an expression for the transfer function of the filter. [2]

Solution:

$$H(z) = -0.5 - 0.45z^{-2}$$

c) Write an expression for the magnitude of the frequency response of this filter. [3]

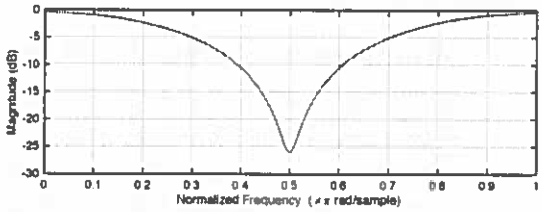
Solution:

$$\begin{aligned} H(e^{j\omega}) &= -0.5 - 0.45e^{-2j\omega} \\ &= -0.5 - 0.45\cos(2\omega) + 0.45j\sin(2\omega) \end{aligned}$$

$$\begin{aligned} |H(e^{j\omega})| &= \sqrt{(-0.5 - 0.45\cos(2\omega))^2 + (0.45\sin(2\omega))^2} \\ &= \sqrt{0.25 + 0.203\cos^2(2\omega) + 0.45\cos(2\omega) + 0.203\sin^2(2\omega)} \\ &= \sqrt{0.453 + 0.45\cos(2\omega)}. \end{aligned}$$

d) Draw a labelled sketch of the magnitude of the frequency response of this filter and mark on the sketch the values of the magnitude in dB at frequencies of 0, $\pi/2$, and π. [6]

Solution:



The magnitude response (gains) at 0, $\pi/2$, and π are 0.95= −0.45 dB, 0.05= −26 dB and 0.95= −0.45 dB respectively.

e) Define group delay for discrete-time filters and estimate the group delay of this filter in seconds at a frequency of 2 kHz given that the sampling frequency is 16 kHz. [5]

Solution:
The group delay is defined as the negative derivate of phase w.r.t. frequency.
The normalized frequency of interest corresponds to $\pi/4$.
The phase is found as

$$\angle = \arctan\left(\frac{0.45\sin(2\omega)}{-0.5-0.45\cos(2\omega)}\right).$$

To find the derivate we could, for example, use the first order difference around $\omega = \pi/4$ such as

$$\omega = 0.3\pi \ \angle H(e^{j\omega}) = -0.870$$
$$\omega = 0.2\pi \angle = -0.590$$

so that

$$-\frac{d\phi}{d\omega} \approx \frac{0.87-0.59}{0.1\pi}$$

The group delay is the found as approximately

$$\frac{0.28 \text{ (radians)}}{0.1\pi \text{ (radians per sample)} \times fs \text{ (samples per second)}} = 56\ \mu\text{s}.$$

3. a) Consider the discrete-time signal $x(n) = 2a^n u(n)$ where $u(n)$ is the unit step function and $|a| < 1$.

i) Write down an expression for the spectrum of this signal. [2]

ii) In an example of a multirate signal processing system, the signal $x(n)$ is decimated by a factor of 2. Explain, with an illustrative sketch, the effect that such decimation has in the frequency domain and hence determine the spectrum of the signal after downsampling. [3]

Solution:

$$X(\omega) = \frac{2}{1 - ae^{-j\omega}}.$$

A general description is expected but a description with decimation factor 2 is perfectly acceptable. The sketch should show an example spectrum before and after decimation. After downsampling the spectrum becomes

$$Y(\omega) = 0.5X(\omega/2) = 0.5\frac{2}{1 - ae^{-j\omega/2}}.$$

b) i) State and explain the Noble Identities. [3]

ii) Figure 3.1 shows two multirate signal processing systems.

Denoting the input samples as $x(n) = \{x0,\ x1,\ x2,\ \ldots\}$ and $y(n) = \{y0,\ y1,\ y2,\ \ldots\}$, find the corresponding samples at the points in the Figure marked A, C and, hence, the first 6 samples at each of the points marked B and D. Comment on the importance of the order of operation of multirate processing blocks such as those in Figure 3.1. [6]

Solution:
Noble identities - *bookwork.*
A: x0,x3,x6
B: x0,0,x3,0,x6,0
C: y0,0,y1,0,y2,0,y3,0,y5,0,y6,0
D: y0,0,y3,0,y6,0
The order of multirate blocks is significant except in the special case of decimation and expansion where the rate change factors are co-prime. This example is such a case so that the two systems are equivalent.

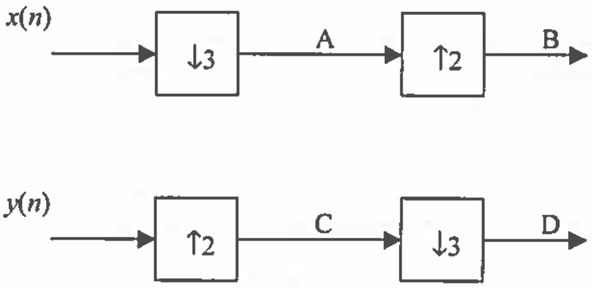


Figure 3.1 Multirate signal processing systems.

c) Consider a DSP system operating on an input signal $x(n)$ sampled at a rate of 1000 samples per second. Using multirate signal processing techniques, design a system to delay $x(n)$ by 300 μs. Show your design in terms of a labelled block diagram together with a detailed explanation of its operation. [6]

Solution:
The input sampling period is 1 ms. We require a delay of 0.3 samples. An approach is to upsample by a factor of 10, delay by 3 samples of the upsampled rate, and then downsample by a factor of 10. A suitable block diagram would show a sequence of processing blocks: expander by 10, lowpass filter, delay by 3 samples, decimation by 10.
50% of the available marks will be awarded for the overall concept and approach. The other 50% will be awarded for the details supplied in the explanation.

4. Consider a discrete-time signal $x(n)$ of length N samples and having N-point DFT $X(k)$.

a) If $X(k)$ is a real sequence, what conditions must be satisfied by $x(n)$, assuming $x(n)$ is real? Give an example for $x(n)$ of length 6 samples for which $X(k)$ is real. [4]

Solutions:
To have a real DFT, $x(n)$ must be an even sequence so that $x(n) = x(N-n)$, $n = 0,\ldots,N-1$. A simple example could be $x(n) = [1\ 0\ 0\ 1\ 0\ 0]$.

b) Let $x(n) = [1,\ -1,\ 0,\ 2]$. Calculate $X(k)$. [6]

Solutions:
Using the formula for the DFT we obtain $X(k) = [2,\ 1+3j,\ 0,\ 1-3j]$.

c) If $x(n)$ satisfies the condition $x(n) = x(N-1-n)$ and N is an even number, show that $X(N/2) = 0$. [3]

Solutions:

$$X(k) = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j2\pi nk/N}$$

$$\begin{aligned}
X(N/2) &= \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j\pi n} \\
&= \frac{1}{N}\sum_{n=0}^{N/2-1} x(n)(-1)^n + \frac{1}{N}\sum_{n=N/2}^{N-1} x(n)(-1)^n \\
&= \frac{1}{N}\sum_{n=0}^{N/2-1} x(n)(-1)^n + \frac{1}{N}\sum_{p=0}^{N/2-1} x(p)(-1)(-1)^n = 0
\end{aligned}$$

for N even and $x(n) = x(N-1-n)$.

d) If $x(n) = -x(N-1-n)$, show that $X(0) = 0$. [3]

Solutions:

$$X(k) = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j2\pi nk/N}$$

$$\begin{aligned}
X(0) &= \frac{1}{N}\sum_{n=0}^{N-1} x(n) \\
&= \frac{1}{N}\sum_{n=0}^{N/2-1} x(n) + \frac{1}{N}\sum_{n=0}^{N/2-1} -x(n) = 0
\end{aligned}$$

since $x(n) = -x(N-1-n)$.

e) Now consider the magnitude and phase of $X(k)$. If $x(n)$ is real, what conditions must be satisfied by the magnitude and phase of $X(k)$. Draw a labelled sketch of the magnitude and phase spectra of $X(k)$ for an illustrative example of a case for which $x(n)$ is real. [4]

Solutions:

For $x(n)$ to be a real sequence, $|X(k)|$ must be even symmetric and $\angle X(k)$ must be odd-symmetric.

Any reasonable example with correctly labelled sketch will be accepted provided it shows understanding of the concept being tested.