

Stability and control of nonlinear systems: Model Answers 2008

1. (a) {unseen}

(i) The general case of these systems is $\ddot{x} + \kappa \dot{x} = d$ with $\kappa > 0$ (#), for which the solution has the form $x(t) = \alpha + \beta \cos(\omega t + \phi)$ with $\dot{x}(t) = -\beta \omega \sin(\omega t + \phi)$. Substituting this into (#) gives the values $\alpha = d/\kappa$, $\omega = \sqrt{k}$ with β and ϕ determined by the initial conditions. For f_1 we obtain $\alpha = 1$, $\omega = \sqrt{k}$ corresponding to a circular trajectory centred on $(1, 0)$ and with radius β . For f_2 we have $\alpha = 0$ and $\omega = 1$, corresponding to circles centred on $(0, 0)$ with radius β (in general not the same as the previous β). Hence we obtain the trajectories shown in Figure A1.1. [3]

(ii) Consider $\ddot{x} = f_3(x, \dot{x}) = -\dot{x}$, i.e. $\frac{d\dot{x}}{dx} \dot{x} = -\dot{x}$ giving $\frac{d\dot{x}}{dx} = -1$ or $\dot{x} = 0$. Similarly for $\ddot{x} = f_4(x, \dot{x}) = \dot{x}$, we have $\frac{d\dot{x}}{dx} = 1$ or $\dot{x} = 0$. Hence we obtain the trajectories shown in Figures A1.2-3. [3]

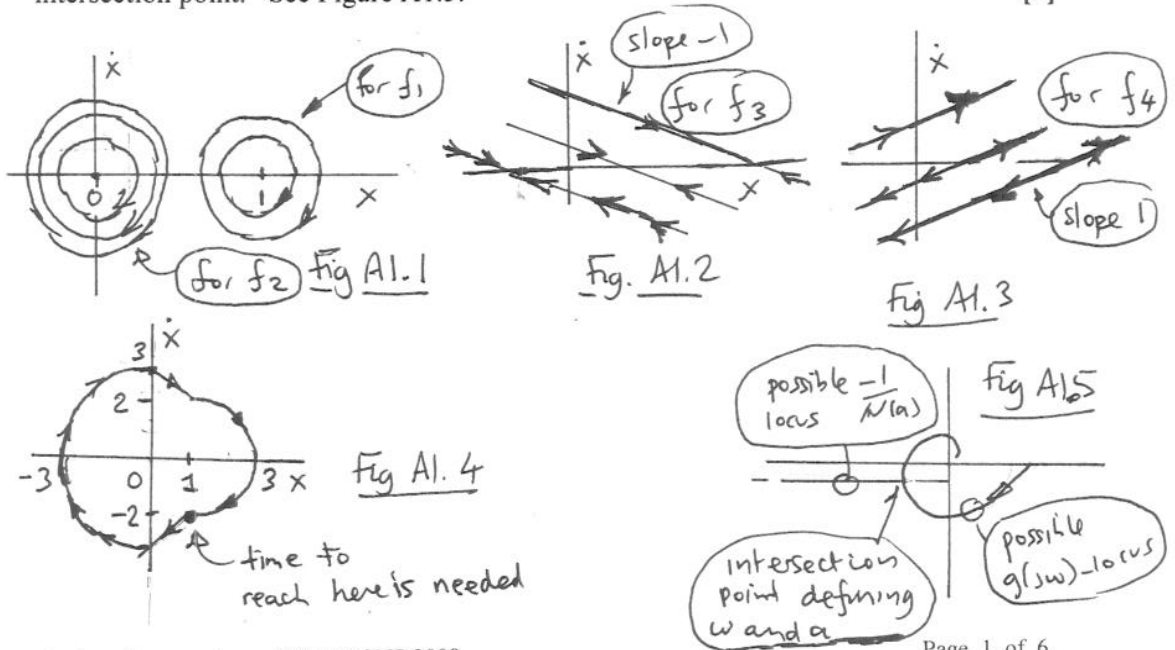
(iii) By making use of the trajectories of Figures A1.1-3 we obtain the trajectory of Figure A1.4. [4]
The time taken for \dot{x} to decrease from 3 to 2, corresponding to x moving from 0 to 1, is $\int_0^1 \frac{1}{\dot{x}} dx = (\text{from Figure A1.1}) \int_0^1 \frac{1}{3-x} dx = -\ln(3-x)|_0^1 = \ln(3) - \ln(2)$ s. The time taken to move from $(1, 2)$ to $(1, -2)$ is $\pi/\omega = \pi$. Hence the total time needed is $\pi + \ln(3) - \ln(2)$ s. [3]

(b) {bookwork}

(i) The Fourier series for u is $u(t) = \sum_{k=0}^{\infty} a_k(a) \sin(k\omega t) + \sum_{k=0}^{\infty} b_k(a) \cos(k\omega t)$
 $\approx b_0(a) + a_1(a) \sin(\omega t) + b_1(a) \cos(\omega t)$ (keeping only the contributions at freq. ω)
where $a_1(a) = \frac{\omega}{\pi} \int_0^T n(a \sin(\omega t)) \sin(\omega t) dt$, $b_1(a) = \frac{\omega}{\pi} \int_0^T n(a \sin(\omega t)) \cos(\omega t) dt$
and $T = \frac{2\pi}{\omega}$. Hence, and since the skew-symmetry of n yields $b_0(a) = 0$, we obtain $u(t) \approx a_1(a) \sin(\omega t) + b_1(a) \cos(\omega t) = \sqrt{a_1(a)^2 + b_1(a)^2} \sin(\omega t + \phi)$
where $\phi = \text{atan}(\frac{b_1(a)}{a_1(a)})$. So the \bar{a} required is $\sqrt{a_1(a)^2 + b_1(a)^2}$. Further, $a_1(a)$ and $b_1(a)$ are independent of ω .

Hence the result of n operating on e can be approximated by the describing function $N(a) = \frac{\sqrt{a_1(a)^2 + b_1(a)^2}}{a} e^{j\psi} = \frac{a_1(a)}{a} + j \frac{b_1(a)}{a}$. [4]

(ii) The harmonic balance equation is $1 = N(a)g(j\omega)$ i.e. $g(j\omega) = -\frac{1}{N(a)}$. Hence if the locus of $g(j\omega)$, as ω varies, and the locus of $-\frac{1}{N(a)}$, as a varies, intersect then an oscillation is predicted with the amplitude a and frequency ω corresponding to the intersection point. See Figure A1.5. [3]



2. (a) $\{bookwork\}$

Now for all $x \in \mathbb{R}^n$, $\lambda_{\min}(P)\|x\|^2 \leq x^T Px = v(x, t)$ where $\lambda_{\min}(P)$ is the smallest eigenvalue of P and is strictly positive since $P > 0$. Hence $\psi(\|x\|) \triangleq \lambda_{\min}(P)\|x\|^2$ is a class- K function that satisfies $\psi(\|x\|) \leq v(x, t)$ for all $x \in \mathbb{R}^n$ and all t . In addition, $\psi(\|x\|) \rightarrow \infty$ as $\|x\| \rightarrow \infty$. Hence $v(x, t)$ is radially-unbounded positive-definite on \mathbb{R}^n . Similarly, $x^T Px \leq \lambda_{\max}(P)\|x\|^2$ so we can define $\phi(\|x\|) = \lambda_{\max}(P)\|x\|^2$ where $0 < \lambda_{\max}(P) < \infty$. Then $v(x, t) \leq \phi(\|x\|)$ for all $x \in \mathbb{R}^n$ and all t so $v(x, t)$ is decrescent on \mathbb{R}^n .

[4]

(b) $\{unseen examples\}$

(i) The origin is an equilibrium state of (2.1) since if we regard (2.1) as $\dot{x} = f(x, t)$ then $f(0, t) = 0$ for all t .

Since $x_1^2 + 2x_2^2 = x^T Px$ for $P = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} > 0$, it follows from part (a) that

$v(x, t)$ is radially-unbounded positive-definite and decrescent on \mathbb{R}^2 .

Further, $\dot{v}(x, t) = 2x_1\dot{x}_1 + 4x_2\dot{x}_2 = 2x_1(2x_2 - x_1) + 4x_2(-x_1 - 3x_2)$
 $= -2x_1^2 - 12x_2^2 = -x^T \begin{bmatrix} 2 & 0 \\ 0 & 12 \end{bmatrix} x$ so, by part (a), $-\dot{v}(x, t)$ is positive-

definite on \mathbb{R}^2 . Consequently, by the Lyapunov Global Asymptotic Stability Theorem, the origin is globally asymptotically stable.

[4]

(ii) If we regard (2.3) as $\dot{x} = f(x)$ then $f_x(0) = \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}$ for which the eigenvalues

are the solutions of $\det \begin{bmatrix} \lambda & -2 \\ 1 & \lambda \end{bmatrix} = \lambda^2 + 2$. Therefore the eigenvalues are

$\pm j\sqrt{2}$ and the Lyapunov Linearization Theorem does not allow us to claim anything about the stability properties of the origin for (2.3) since the eigenvalues are exactly in the imaginary axis.

[2]

The only difference here from part (b-i) concerns \dot{v} . It is now

$\dot{v}(x, t) = 2x_1\dot{x}_1 + 4x_2\dot{x}_2 = 2x_1(2x_2 - x_1^3) + 4x_2(-x_1 - 3x_2^5)$
 $= -2x_1^4 - 12x_2^6$. So $-\dot{v}(x, t) = 2x_1^4 + 12x_2^6$ and clearly this is strictly positive for all non-zero $x \in G_1$. Therefore $-\dot{v}$ is positive definite on G_1 .

[2]

Consequently, by the Lyapunov Asymptotic Stability Theorem, the origin is asymptotically stable.

[1]

(c) $\{application of bookwork to a new example\}$

Now $s(t) = 0, \forall t \geq \tau$, implies that $\dot{e}(t) = -3e(t), \forall t \geq \tau$, i.e. that

$e(t) = \exp(-3t)e(\tau)$. Hence $e(t) \rightarrow 0$ as $t \rightarrow \infty$ so the aim $x_1(t) = \exp(-2t)$ is achieved asymptotically.

[3]

Further, $\dot{s}(t) = \ddot{e}(t) + 3\dot{e}(t) = \ddot{x}_1(t) - 4\exp(-2t) + 3(\dot{x}_1(t) + 2\exp(-2t))$
 $= -2x_1(t) - 3x_2(t) + u(t) + d_2 - 4\exp(-2t) + 3x_2(t) + 3d_1 + 6\exp(-2t)$
 $= -2x_1(t) + u(t) + d_2 + 3d_1 + 2\exp(-2t)$.

Hence the control u can be chosen so \dot{s} is equal to any desired value ρ . Denote such u by $u_\rho(x(t))$. Usually $s(0)$ will not be zero. If $s(0)$ is positive we apply u_ρ for a negative ρ until s becomes 0. If $s(0) < 0$ we apply u_ρ for a positive ρ until s becomes zero. Once s is zero, we apply u_0 to keep s zero forever, thereby causing the desired behaviour of x_1 to be achieved asymptotically.

[4]

3. (a) (i) *{The heart of this involves a somewhat different approach to a part of a question set last year - and is here to lead into the unseen part (ii)}*

The closed-loop system is

$$\dot{x}(t) = Ax(t) - bb^T Px(t) = [A - bb^T P]x(t)$$

so

$$\begin{aligned}\dot{v}(t) &= \frac{d}{dt}x(t)^T Px(t) \\ &= \dot{x}(t)^T Px(t) + x(t)^T P\dot{x}(t) \\ &= x(t)^T [A - bb^T P]^T Px(t) + x(t)^T P[A - bb^T P]x(t) \\ &= x(t)^T [A^T P + PA]x(t) - 2x(t)^T Pbb^T Px(t) \\ &= x(t)^T [A^T P + PA - 2Pbb^T P]x(t) \quad (\#) \\ &= x(t)^T [-Q + Pbb^T P - 2Pbb^T P]x(t) \\ &= x(t)^T [-Q - Pbb^T P]x(t) \\ &= -x(t)^T Qx(t) - \|b^T Px(t)\|^2 \\ &\leq -x(t)^T Qx(t).\end{aligned}$$

Since Q is positive-definite, this shows that $-\dot{v}$ is positive definite. Hence, by the Lyapunov Global Asymptotic Stability Theorem, the origin is globally asymptotically stable. [5]

- (ii) *{new problem}*

From (#) in part (a-i), but with A replaced by $A + \delta A$, we have

$$\begin{aligned}\dot{v}(t) &= x(t)^T [(A + \delta A)^T P + P(A + \delta A) - 2Pbb^T P]x(t) \\ &= x(t)^T [A^T P + PA + \{(\delta A)^T P + P\delta A\} - 2Pbb^T P]x(t) \\ &= x(t)^T [A^T P + PA - 2Pbb^T P]x(t) + x(t)^T \{(\delta A)^T P + P\delta A\}x(t) \\ &= -x(t)^T Qx(t) - \|b^T Px(t)\|^2 + x(t)^T \{(\delta A)^T P + P\delta A\}x(t) \\ &\leq -x(t)^T Qx(t) + \|(\delta A)^T P + P\delta A\| \|x(t)\|^2 \\ &\leq -x(t)^T Qx(t) + 2\|\delta A\| \|P\| \|x(t)\|^2 \\ &\leq -\|x(t)\|^2 \lambda_{\min}(Q) + 2\|\delta A\| \|P\| \|x(t)\|^2 \\ &= -\|x(t)\|^2 \{\lambda_{\min}(Q) - 2\|\delta A\| \|P\|\}\end{aligned}$$

so $-\dot{v}$ is positive definite on \mathbb{R}^n if $\lambda_{\min}(Q) - 2\|\delta A\| \|P\| > 0$,

i.e. if $2\|\delta A\| \|P\| < \lambda_{\min}(Q)$

i.e. if $\|\delta A\| < \frac{1}{2}\lambda_{\min}(Q)/\|P\|$. Hence, by the Lyapunov Global Asymptotic Stability theorem, the origin is globally asymptotically stable for system (3.5) if

$$\|\delta A\| < \Delta \triangleq \frac{1}{2}\lambda_{\min}(Q)/\|P\|, \quad [8]$$

- (b) *{mostly bookwork with the unseen special case $M=I$ considered at the end}*

Substitution of $A = ZMZ^T$ into $A^T P + PA = -Q$ yields

$ZM^T Z^T P + PZMZ^T = -Q$. Pre-multiplying by Z^T and post-multiplying by Z gives $Z^T ZM^T Z^T PZ + Z^T PZMZ^T Z = -Z^T QZ$. Since $Z^T Z = I$ owing to the orthogonality of Z , this gives $M^T Z^T PZ + Z^T PZM = -Z^T QZ$.

Let $S = Z^T PZ$. Then S is symmetric and is defined by

$$M^T S + SM = -W \quad (A3.1)$$

where $W \triangleq Z^T QZ$. This is generally easier to solve for S than it is to solve

$A^T P + PA = -Q$ for P because M is block-upper-triangular with blocks which are either 1×1 or 2×2 and usually A does not have a special structure.

Since $S = Z^T PZ$ and Z is orthogonal, we can find P from S using $P = ZSZ^T$. [5]

For the special case $M = D$ with D diagonal and $Q = qI$, (A3.1) gives

$$DS + SD = -Z^T qI_n Z = -qI$$

which has the solution $S = -\frac{1}{2}qD^{-1}$. So $P = -\frac{1}{2}qZZ^{-1}Z^T$. [2]

4. (a) {Modification of bookwork}

(i) Since $A^2 = 0$ and $Q = I_n$,

$$P = Q + A^T Q A + (A^T)^2 Q A^2 + (A^T)^3 Q A^3 + \dots = I + A^T A.$$

Hence

$$\begin{aligned} v(x_{k+1}) - v(x_k) &= v(Ax_k) - v(x_k) = (Ax_k)^T P (Ax_k) - x_k^T P x_k \\ &= x_k^T [A^T P A - P] x_k = x_k^T [A^T (I + A^T A) A - (I + A^T A)] x_k \\ &= -x_k^T x_k. \end{aligned} \quad (\#)$$

Hence $v(x_{k+1}) - v(x_k) < 0$ whenever $x_k \neq 0$. [4]

Therefore, accepting the given fact that our v is positive-definite etc. on \mathbb{R}^n , the discrete-time Lyapunov Global Asymptotic Stability Theorem reveals that the origin is globally asymptotically stable. [1]

(ii) Since $v(x_{k+1}) = (Ax_k + bu_k)^T P (Ax_k + bu_k)$ and $P > 0$, choosing u_k to minimize $v(x_{k+1})$ will tend to reduce $\|x_{k+1}\|$ which will tend to increase the rate of convergence of the sequence $\{x_k\}$ to zero. [1]

Now

$$\begin{aligned} v(x_{k+1}) &= (Ax_k + bu_k)^T P (Ax_k + bu_k) \\ &= x_k^T A^T P A x_k + 2x_k^T A^T P b u_k + u_k^T b^T P b. \end{aligned}$$

Since $\partial^2 v(Ax_k + bu_k) / \partial (u_k)^2 = b^T P b > 0$ (assuming $b \neq 0$) since $P > 0$, choosing u_k so $\partial v(Ax_k + bu_k) / \partial (u_k) = 0$ yields the unconstrained minimizer u_k , which will be denoted here by \tilde{u}_k . Hence $\tilde{u}_k = -x_k^T A^T P b / (b^T P b)$.

Then it is clear that the constrained minimizer $\hat{u}_k(x_k)$ is \tilde{u}_k if $\tilde{u}_k \in [-1, 2]$, is -1 if $\tilde{u}_k < -1$ and is 2 if $\tilde{u}_k > 2$. [4]

Now, for each $k \geq 0$,

$$v(Ax_k + b\hat{u}_k(x_k)) \leq v(Ax_k + b0) = v(Ax_k) \quad (\pounds)$$

since $0 \in [-1, 2]$ but $u_k = 0$ does not necessarily minimize $v(Ax_k + bu_k)$ with respect to $u_k \in [-1, 2]$.

Hence, from (#) and (\pounds) above, for all $k \geq 0$, for the optimally-controlled system:

$$\begin{aligned} v(x_{k+1}) - v(x_k) &= v(Ax_k + b\hat{u}_k(x_k)) - v(x_k) \leq v(Ax_k) - v(x_k) \\ &\leq -\|x_k\|^2 < 0 \text{ for all } x_k \neq 0. \end{aligned}$$

Therefore, by the Lyapunov Global Asymptotic Stability Theorem, the origin is globally asymptotically stable for the optimally-controlled system. [5]

(b) {Unseen application of a method for solving the continuous-time Lyapunov equation to solution of the discrete-time Lyapunov equation}

Now, for a matrix $M \in \mathbb{R}^{2 \times 2}$, $\text{vec}(M) = [m_{11} \ m_{12} \ m_{21} \ m_{22}]^T$ and the

Kronecker product of $L \in \mathbb{R}^{2 \times 2}$ and $N \in \mathbb{R}^{2 \times 2}$ is $L \otimes N = \begin{bmatrix} l_{11}N & l_{12}N \\ l_{21}N & l_{22}N \end{bmatrix} \in \mathbb{R}^{4 \times 4}$.

Taking vecs, the equation $A^T P A - P = -Q$ becomes $\text{vec}(A^T P A) - \text{vec}(P) = -\text{vec}(Q)$, i.e. $(A^T \otimes A^T - I_{n^2})p = -q$ where $p = \text{vec}(P)$ and $q = \text{vec}(Q)$.

Since the solution P of $A^T P A - P = -Q$ is unique, the solution p of

$(A^T \otimes A^T - I_{n^2})p = -q$ is also unique. Hence, since $A^T \otimes A^T - I_{n^2}$ is square, the null-space of $A^T \otimes A^T - I_{n^2}$ must equal just $\{0\}$ so $A^T \otimes A^T - I_{n^2}$ is non-singular.

Therefore, conceptually at least, there is no problem in solving $(A^T \otimes A^T - I_{n^2})p = -q$ for p and the required solution P of $A^T P A - P = -Q$ is then $\text{vec}^{-1}(p)$. [5]

5. (a) (i) *{Apart from the initial definition, this is a bit new for them}*

A subsystem with initial condition x_o , input e and output u is strictly input passive if there is a scalar $\beta(x_o) \leq 0$ and a scalar $\delta > 0$ such that $\int_0^\tau y(t)u(t)dt \geq \beta(x_o) + \delta \int_0^\tau u(t)^2 dt, \forall \tau \geq 0, \forall u \in \mathcal{L}_{2e}$.

Now consider H when $y(t) = \phi(u(t), t)$ for all t . For $\phi \in \text{sector}[\alpha, \beta]$:

$$\alpha u^2 \leq \phi(u, t)u \leq \beta u^2.$$

$$\text{Hence } \int_0^\tau y(t)u(t)dt = \int_0^\tau \phi(u(t), t)u(t)dt \geq \int_0^\tau \alpha u(t)^2 dt = \delta \int_0^\tau u(t)^2 dt + \beta(x_o)$$

where $\delta = \alpha$ and $\beta(x_o) = 0$.

Hence H is strictly input passive.

Similarly, since $\phi \in \text{sector}[\alpha, \beta]$, we have $\alpha \leq \frac{\phi(u(t), t)}{u(t)} \leq \beta$ whenever $u(t) \neq 0$ so

$$|\phi(u, t)| \leq \beta |u|. \text{ Consequently } \int_0^\tau y(t)^2 dt = \int_0^\tau \phi(u(t), t)^2 dt \leq \int_0^\tau \beta^2 u(t)^2 dt$$

$$\text{so } \|y_T\|_{\mathcal{L}_2} \leq \beta \|u_T\|_{\mathcal{L}_2}.$$

[5]

- (ii) *{apart from the initial definition, this is a bit new but since it is quite complicated they will not find it straightforward unless they know well what they are doing}*

H is strictly output passive if there is a scalar $\beta(x_o) \leq 0$ and a scalar $\delta > 0$

such that $\int_0^\tau y(t)u(t)dt \geq \beta(x_o) + \delta \int_0^\tau y(t)^2 dt, \forall \tau \geq 0, \forall u \in \mathcal{L}_{2e}$.

For our particular H :

$$\int_0^\tau \frac{d}{dt} x(t)^T P x(t) dt = x(t)^T P x(t)|_0^\tau = x(\tau)^T P x(\tau) - x_o^T P x_o \geq -x_o^T P x_o \quad (\$)$$

since $x(\tau)^T P x(\tau) \geq 0$ owing to the fact that P is positive-definite.

Also:

$$\begin{aligned} \int_0^\tau \frac{d}{dt} x(t)^T P x(t) dt &= \int_0^\tau \{ \dot{x}^T P x + x^T P \dot{x} \} dt \\ &= \int_0^\tau \{ [Ax + bu]^T P x + x^T P [Ax + bu] \} dt \\ &= \int_0^\tau \{ x^T [A^T P + P A] x + 2ub^T P x \} dt = \int_0^\tau \{ -x^T [cc^T] x + 2ub^T P x \} dt \\ &= \int_0^\tau \{ -(c^T x)^2 + 2uc^T x \} dt = \int_0^\tau \{ -y^2 + 2uy \} dt. \end{aligned} \quad (\pounds)$$

From (\$) and (\pounds),

$$\int_0^\tau uy dt \geq \frac{1}{2} \int_0^\tau y^2 dt - \frac{1}{2} x_o^T P x_o = \delta_2 \int_0^\tau y^2 dt + \beta_2(x_o)$$

where $\delta_2 = \frac{1}{2} > 0$ and $\beta_2(x_o) = -\frac{1}{2} x_o^T P x_o \leq 0$.

Hence H is strictly output passive.

[7]

- (b) *{repackaged tutorial question which they will not recognise}*

Now

$$\begin{aligned} \|u_T\|_{\mathcal{L}_2} \|r_T\|_{\mathcal{L}_2} &\geq (\text{by Cauchy-Schwartz}) \int_0^T u(t)r(t)dt \\ &= \int_0^T u(t)\{e(t)+y(t)\}dt = \int_0^T u(t)e(t)dt + \int_0^T u(t)y(t)dt. \end{aligned} \quad (\pounds)$$

Here $\int_0^T u(t)e(t)dt \geq \beta_1$ for $\beta_1 \leq 0$ since H_1 is passive, and

$\int_0^T u(t)y(t)dt \geq \beta_2(x_o) + \delta_2 \int_0^T y(t)^2 dt$ where $\beta_2 \leq 0$ and $\delta_2 > 0$ since H_2 is strictly output passive.

Use of these in (\pounds) yields:

$$\|u_T\|_{\mathcal{L}_2} \|r_T\|_{\mathcal{L}_2} \geq \int_0^T u(t)e(t)dt + \int_0^T u(t)y(t)dt \geq \beta_1 + \beta_2(x_o) + \delta_2 \int_0^T y(t)^2 dt.$$

For our case with $r \equiv 0$, this gives

$$0 \geq \beta + \delta_2 \int_0^T y(t)^2 dt \text{ where } \beta = \beta_1 + \beta_2(x_o) \leq 0 \text{ and } \delta_2 > 0, \text{ i.e. } \|y_T\|_{\mathcal{L}_2}^2 \leq -\frac{\beta}{\delta} < \infty,$$

$\forall T < \infty$, as required.

[8]

6. (a) *{this is an application of material they know for the case $y = \dot{f}$ to a more complicated case that will help with the solution of part (b) below}*

The system is passive if there is a $\beta \leq 0$ such that

$$\int_0^\tau y(t)u(t)dt \geq \beta \text{ for all } \tau \geq 0 \text{ and for all input functions } u.$$

If $y(t)u(t)$ can be written as $\alpha f(t)\dot{f}(t)$ with $\alpha > 0$, then

$$\begin{aligned} \int_0^\tau y(t)u(t)dt &= \alpha \int_0^\tau f(t)\dot{f}(t)dt = \frac{1}{2}\alpha \int_0^\tau \frac{d}{dt}\{f(t)^2\}dt = \frac{1}{2}\alpha\{f(\tau)^2 - f(0)^2\} \\ &\geq -\frac{1}{2}\alpha f(0)^2 \triangleq \beta \text{ with } \beta \leq 0, \text{ so the system is passive.} \end{aligned}$$

If $y(t)u(t) = (1 - \alpha\gamma(t))p(t)$ then this can be written as $y(t)u(t) = \alpha \frac{(\alpha^{-1} - \gamma(t))}{g} gp(t)$

and this can be written as $\alpha f(t)\dot{f}(t)$ with $f(t) = \frac{(\alpha^{-1} - \gamma(t))}{g}$ and $\dot{f}(t) = gp(t)$. Then we obtain passivity if $-\dot{\gamma} = gp(t)$, i.e. if $\dot{\gamma}(t) = -g^2 p(t)$. [5]

- (b) *{new case - probably they are expecting a question of this type however it is still a searching question that cannot be done without understanding}*

The Controlled plant is $\dot{x} = Ax + \alpha b\gamma(t)\{r(t) - f^T x\}$.

Perfect model following is possible since the choice $\gamma(t) = \alpha^{-1}$ causes the equation of the controlled plant to become $\dot{x} = Ax + b\{r(t) - f^T x\}$ which is the equation for the reference model. [2]

The error subsystem: Define $e = \bar{x} - x$. Then

$$\begin{aligned} \dot{e} &= \dot{\bar{x}} - \dot{x} = \bar{A}\bar{x} + br - \{Ax + \alpha b\gamma(r - f^T x)\} \\ &= \bar{A}(\bar{x} - x) + \bar{A}x + br - \{Ax + \alpha b\gamma(r - f^T x)\} \\ &= \bar{A}e + Iw \end{aligned}$$

where

$$\begin{aligned} w &= \bar{A}x + br - \{Ax + \alpha b\gamma(r(t) - f^T x)\} \\ &= -bf^T x + br - \alpha\gamma br(t) + \alpha\gamma bf^T x \\ &= (\alpha\gamma - 1)b(f^T x - r). \end{aligned}$$

Solve $\bar{A}^T P + P\bar{A} = -I$ for P , giving a positive definite P since \bar{A} is a stability matrix. Choose the output matrix to be $b^T P$. Then, from the Kalman-Popov-Yakubovic Lemma, the error subsystem of Figure A6.1 is strictly output passive. [6]

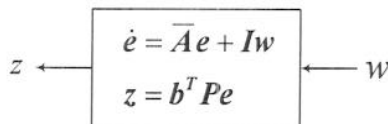


Figure A6.1

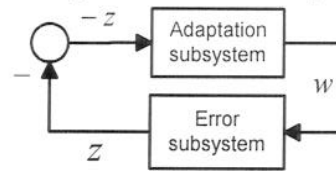


Figure A6.2

Design of adaptation law

We view the situation as in Figure A6.2 and require the connection between $-z$ and w to be passive so we consider

$$\begin{aligned} \int_0^\tau (-z(t)^T w(t))dt &= \int_0^\tau -z(t)^T \{\alpha\gamma(t) - 1\} b \{f^T x(t) - r(t)\} dt \\ &= \int_0^\tau \{1 - \alpha\gamma(t)\} z(t)^T b \{f^T x(t) - r(t)\} dt \\ &= \int_0^\tau \{1 - \alpha\gamma(t)\} p(t) dt \end{aligned}$$

where

$$p(t) = z(t)^T b \{f^T x(t) - r(t)\}.$$

From part (a), this is passive if

$$\dot{\gamma}(t) = -g^2 p(t)$$

where $g > 0$.

So we just need to use the adaptation law

$$\gamma(t) = -g^2 \int_0^t z(t)^T b \{f^T x(t) - r(t)\} dt.$$

Then, by a Passivity Theorem, $\|\bar{x} - x\|_{\mathcal{L}_2} < \infty$, as required. [7]