EE1-10B

Solutions

1. See separate sheet.

2. See separate sheet.

3. a) i) The auxiliary polynomial is $a(\lambda) = \lambda^2 - \lambda = \lambda(\lambda - 1)$ and so the complementary function is $\boxed{y_c(x) = c_1 + c_2 e^x, \quad c_1, c_2 \text{ constants.}}$

ii) Use a trial solution $y_p(x) = b_1 + b_2 x e^x$ in the differential equation and equate coefficients to get $b_1 = b_2 = 1$ and so $\boxed{y_p(x) = xe^x + 1.}$

iii) The general solution is then $y(x) = c_1 + c_2 e^x + xe^x + 1$. Enforcing the initial conditions gives $c_1 = 0$ and $c_2 = -1$ and so $\boxed{y(x) = xe^x - e^x + 1.}$

b) i) Write the differential equation as $P(x,y)dx + Q(x,y)dy = 0$. The equation is exact if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \Leftrightarrow \quad \lambda_1 x - \cos x \sin y = 2x - \frac{1}{2}\lambda_2 \cos x \sin y \Leftrightarrow \boxed{\lambda_1 = \lambda_2 = 2.}$$

$$\Rightarrow \quad (2xy + \cos x \cos y)\, dx + (x^2 - \sin x \sin y)\, dy = 0.$$

ii) Since $df = f_x dx + f_y dy$ we need $f_y = Q$ and so

$$\frac{\partial f}{\partial y} = x^2 - \sin x \sin y \Rightarrow \boxed{f(x,y) = x^2 y + \sin x \cos y}$$

iii) Since $df = 0$ the solution is given implicitly by $f(x,y) = C$ or

$$\boxed{x^2 y + \sin x \cos y = C, \quad C \text{ constant.}}$$

c) i) The integrating factor $\mu(x)$ must be chosen to satisfy

$$\frac{d\mu(x)}{dx} = \mu(x)\frac{3}{x} \Rightarrow \boxed{\mu(x) = \exp\left(\int \tfrac{3}{x}\, dx\right) = e^{3\ln x} = x^3.}$$

ii) Multiplying the equation by $\mu(x)$:

$$x^3 \frac{dy}{dx} + 3x^2 y = 2x \quad \Rightarrow \quad \frac{d}{dx}(x^3 y) = 2x$$

$$\Rightarrow \quad y(x) = x^{-3}\int 2x dx + Cx^{-3}$$

$$\Rightarrow \quad \boxed{y(x) = x^{-1} + Cx^{-3}, \quad C \text{ constant}}$$

4. a) i) Evaluating the partial derivative for the chain rule

$$\begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{-y}{x^2+y^2} \\ \frac{y}{\sqrt{x^2+y^2}} & \frac{x}{x^2+y^2} \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial \rho} \\ \frac{\partial f}{\partial \phi} \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi/\rho \\ \sin\phi & \cos\phi/\rho \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial \rho} \\ \frac{\partial f}{\partial \phi} \end{bmatrix}$$

and so

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = \begin{bmatrix} \frac{\partial f}{\partial \rho} & \frac{\partial f}{\partial \phi} \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi/\rho & \cos\phi/\rho \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi/\rho \\ \sin\phi & \cos\phi/\rho \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial \rho} \\ \frac{\partial f}{\partial \phi} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial f}{\partial \rho} & \frac{\partial f}{\partial \phi} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/\rho \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial \rho} \\ \frac{\partial f}{\partial \phi} \end{bmatrix} = \left(\frac{\partial f}{\partial \rho}\right)^2 + \frac{1}{\rho^2}\left(\frac{\partial f}{\partial \phi}\right)^2$$

and so $\boxed{a = 1.}$

ii) Since f is radially symmetric, $f_\phi = 0$ and the equation transforms to

$$\left(\frac{\partial f}{\partial \rho}\right)^2 = \rho^2 \Rightarrow \frac{\partial f}{\partial \rho} = \pm\rho \Rightarrow f(\rho) = \pm\frac{1}{2}\rho^2 + C \Rightarrow \boxed{f(x,y) = \pm\tfrac{1}{2}(x^2+y^2)+C.}$$

b) i) Since $dF = 0$ we have that $\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz = 0$. It follows that

$$\frac{\partial z}{\partial x} := \frac{dz}{dx}\Big|_{y=\text{constant}} = \frac{dz}{dx}\Big|_{dy=0} = -\frac{\partial F}{\partial x}\Big/\frac{\partial F}{\partial z} \Rightarrow \boxed{\frac{\partial z}{\partial x} = \frac{2x}{z}.}$$

and since F is symmetric in x and y, $\boxed{\frac{\partial z}{\partial y} = \frac{2y}{z}.}$

ii) The stationary points are obtained by setting $z_x = z_y = 0$ and so $\boxed{x_0 = y_0 = 0}$ from Part (i). The same answer could be obtained by explicitly expressing z explicitly as a function of x and y. Note that $z(x_0, y_0) = 2$.

iii) Taking the second derivatives and evaluating at x_0, y_0.

$$\frac{\partial^2 z}{\partial x^2}\Big|_{x_0,y_0} = \frac{\partial}{\partial x}\left(\frac{2x}{z}\right)\Big|_{x_0,y_0} = \frac{2z - 2xz_x}{z^2}\Big|_{x_0,y_0} = 1 = \frac{\partial^2 z}{\partial y^2}\Big|_{x_0,y_0}$$

For the mixed derivatives,

$$\frac{\partial^2 z}{\partial x \partial y}\Big|_{x_0,y_0} = \frac{\partial}{\partial x}\left(\frac{2y}{z}\right)\Big|_{x_0,y_0} = \frac{-2yz_x}{z^2}\Big|_{x_0,y_0} = 0$$

The Hessian is then

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Since $m_{11} > 0$ and $\det M > 0$, the stationary point corresponds to a $\boxed{\text{local minimiser.}}$

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a) i) $\frac{1}{\sqrt{n^2-3}} \geq \frac{1}{n}$ for $n \geq 2$

$\sum \frac{1}{n}$ is divergent & so is

$$\sum_{n \geq 2} \frac{1}{\sqrt{n^2-3}}.$$

ii) $\sum (-1)^n \frac{3^n}{5^n}$ this an alternating series such that $\left(\frac{3}{5}\right)^n$ is decreasing

& $\lim_{n\to\infty} \left(\frac{3}{5}\right)^n = 0$ so it is convergent

iii) $U_n = \frac{5^n}{n^n}$ Using the root test

$\sqrt[n]{U_n} = \frac{5}{n} \xrightarrow[n\to\infty]{} 0$ Hence the series is convergent

b) $f(x) = \ln(1+x)$ (2)

$$f'(x) = \frac{1}{1+x}$$

$$f''(x) = -\frac{1}{(1+x)^2}$$

$$f'''(x) = \frac{2}{(1+x)^3}$$

Hence the Taylor series expansion of $\ln(1+x)$ about 0 is:

$$0 + 1x - \frac{1}{2}x^2 + \frac{2}{3!}x^3 = x - \frac{x^2}{2} + \frac{x^3}{3}.$$

c) $A = \begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 0 \\ 1 & -1 & 3 \end{bmatrix}$

Characteristic polynomial is:

$$-x^3 + 80x^2 - 20x + 16$$

this factorises

$$-(x-2)(x-2)(x-4)$$ ③

$Ax=4x \Rightarrow \begin{pmatrix}1\\0\\1\end{pmatrix}$ is an eigenvector

$Ax=2x \quad \Rightarrow \quad x_1 - x_2 + x_3 = 0.$

$\begin{pmatrix}-1\\0\\1\end{pmatrix}$ & $\begin{pmatrix}0\\1\\1\end{pmatrix}$ are eigenve

②

$L: \quad \begin{aligned} x &= -1+s \\ y &= -3 \\ z &= 4 \end{aligned}$

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a) $P \cap L = \{A\}$ where

$(-1+s) - 3 + 4 = 10 \Rightarrow s = 10.$

Hence $A = (9, -3, 4).$

b)

Orthogonal projection of $(1,0,0)$ on P.

i) $B=(x,y,z)$ such that

$$\overrightarrow{AB} = \begin{pmatrix} x-1 \\ y \\ z \end{pmatrix} \quad \text{is parall to} \quad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

& $x+y+z=10$.

$$\begin{pmatrix} x-1 \\ y \\ z \end{pmatrix} \wedge \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} +y-z \\ -(x-1)+z \\ (x-1)-y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} y = z \\ x-1 = z \\ x-1 = y \end{cases} \Rightarrow z+1+2z = 10 \Rightarrow z = 3$$

Hence $B = \begin{pmatrix} 4 \\ 3 \\ 3 \end{pmatrix}$

distance from $(1,1,0)$ to P is

$$\sqrt{3^2+3^2+3^2} = \sqrt{27}.$$

c) Q orthogonal to P

$Q \supset L$

$\Rightarrow$ The point $(-1,-3,4)$ is in Q

& Q has direction $(1,1,1)$ & $(1,0,0)$.

Let us consider the vector $(x,y,3)$ orthogonal to Q we have

$x+y+3=0$ & $x=0$ $\Rightarrow$ $y=-3$.

Hence $(0,1,-1)$ is orthogonal to P.

& so for a point in d we have its coordinates (x,y,z) satisfying

$$0\cdot(x+1)+(y+3)-(z-4)=0$$

$$\Rightarrow d: \boxed{y-z=-7}$$