E3.20

1. a)(i) $P(S_1) = \{ \phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\} \}$.

(ii) $S_1 \cup S_2 = \{a,b,c\}$

(iii) $S_1 \cap S_2 = \{a,b\}$

(iv) $S_2 - S_1 = \cancel{\{} \phi$

(v) $|S_1 \cup S_2| = 3$ [6 MARKS]

b) Finite : S_1 from (a)
Infinite – Countable : $\mathbb{N}$
Infinite – Uncountable : $\mathbb{R}$ [3 MARKS]

c) (i) No, e.g. $(2,1) \in R$ but $(1,2) \notin R$.

(ii) No, e.g. $(2,1) \in R$ and $(4,2) \in R$ but $(4,1) \notin R$.

(iii) No, e.g. $(1,1) \notin R$.

(iv) No, e.g. there is no element $(1,x)$ for any x.

(v) No — (ii) implies this. [6 MARKS]

d) (i) $\forall x (J(x) \rightarrow L(x))$

(ii) $\exists x (L(x) \wedge \neg J(x))$

(iii) $\forall x (J(x) \rightarrow A(x, Jones))$

(iv) $\forall x \forall y (A(x,y) \wedge J(x) \rightarrow J(y))$. [5 MARKS]

e) Simplify (i) : b

Modus ~~Ponens~~ Tollens w/ (ii) : $\neg c$ [2 MARKS]

1. f) (i) From the theorem, $f(x)$ is $O(x^3)$.

$$|f(x)| = |x^3 + 2x^2 + 1| = |x^3| + |2x^2| + 1 \quad \text{for } x > 0$$
$$\geq |x^3|$$

So with any k, e.g. $k = 1$ & with $c = 1$,

$$\forall x \left((x > k) \rightarrow |f(x)| \geq c|x^3| \right) \quad \square.$$

(ii)

```
proc1 (int x) {
    for i = 1 to x*x*x + 2*x*x + 1 - 4
        avar = avar * 2;
}
```

(iii)

```
proc2(int x) {
    if x = 1
        return 2*x*x;
    else
        return proc2(x-1) * proc2(x-1);
}
```

[9 MARKS]

g) Let $a \geq 1$ be a real number, $b > 1$ be an integer, $c > 0$ be a real number and $d \geq 0$ be a real number.

Let f be an increasing function s.t.

$$f(n) = a f(n/b) + cn^d \quad \text{whenever } n = b^k$$

for positive integer k.

(i) If $a < b^d$, $f(n)$ is $O(n^d)$

(ii) If $a = b^d$, $f(n)$ is $O(n^d \log n)$

(iii) If $a > b^d$, $f(n)$ is $O(n^{\log_b a})$.

[9 MARKS]

2 a) $\text{ratapprox}(0, \sqrt{2}) = (1,1)$

$\text{ratapprox}(1, \sqrt{2}) = (3,2)$

$\text{ratapprox}(2, \sqrt{2}) = (7,5)$ [6 MARKS]

b) $a_n = 1 + a_{n-1}, \quad n \geq 1$

$a_n = n$

This is $O(n)$, as are all other factors, so exec time is $\underline{\underline{O(n)}}$. [4 MARKS]

c) $\forall x\,(\neg R(x) \rightarrow \forall y\,(R(y) \rightarrow (y < x) \vee (y > x)))$.

This is true. [6 MARKS]

d) $r_d = \{p \mid p \in Q \wedge p <_Q r\}$

(i) $[r_d \subset Q]$

$r_d \subseteq Q$ from defn

$r_d \neq Q$ as, for example, $(r+1) \in Q$ but $r+1 \not<_Q r \Rightarrow (r+1) \notin r_d$.

(ii) $r_d \neq \emptyset$ as, for example, $(r-1) \in Q$

Also $(r-1) <_Q r \Rightarrow (r-1) \in r_d$.

(iii) $q \in S \Rightarrow q \in Q \wedge q <_Q r$

$p <_Q q \Rightarrow p <_Q q <_Q r$

$\Rightarrow p <_Q r$

Also $p \in Q$. Thus $p \in S$.

$\mathcal{E}$

2. (d) (iv) Take $q = \frac{1}{2}(p+r)$

Then $q \in Q$ and $p <_Q q <_Q r$

$\Rightarrow p <_Q q$. [~~8~~6 MARKS]

e) (i) For any set x, $x \subseteq x \Rightarrow x \leq_d x$.

(ii) $(x \leq_d y) \wedge (y \leq_d x)$

$\Rightarrow (x \subseteq y) \wedge (y \subseteq x)$

$\equiv x = y$ (by definition of set equality)

(iii) $(x \leq_d y) \wedge (y \leq_d z)$

$\Rightarrow (x \subseteq y) \wedge (y \subseteq z)$

$\Rightarrow x \subseteq z$

$\Rightarrow x \leq_d z$.

(iv) We wish to show $(x \leq_d y) \vee (y \leq_d x)$

i.e. $(x \subseteq y) \vee (y \subseteq x)$.

If $x = y$, this is true.

Otherwise $\exists q \in Q$ s.t. $q \in x$ but $q \notin y$ (*)

or $q \in y$ but $q \notin x$.

Take (*), w.l.o.g.

Then $\forall p \in y \, (p <_Q q)$ (from Dedekind defn).

$\Rightarrow \forall p \in y \, (p \in x)$ i.e. $y \subseteq x$.

[8 MARKS]

2. f) $T = \{ p \mid p \in \mathbb{Q} \wedge (p^2 <_{\mathbb{Q}} 2) \vee (p < 0) \}$.

(i) $T \subseteq \mathbb{Q}$ by defn.

$T \neq \mathbb{Q}$ as, for example, $2 \in \mathbb{Q}$ but $2^2 \not<_{\mathbb{Q}} 2$

$\Rightarrow \quad 2 \notin T$.

(ii) $T \neq \emptyset$ as, for example, $1 \in T$ since $1 \in \mathbb{Q}$ & $1^2 = 1 <_{\mathbb{Q}} 2$.

(iii) $q \in T \Rightarrow q \in \mathbb{Q}$

and $q^2 <_{\mathbb{Q}} 2$ or $q <_{\mathbb{Q}} 0$.

Let $q <_{\mathbb{Q}} 0$. Then $p <_{\mathbb{Q}} q$

$\Rightarrow p <_{\mathbb{Q}} q <_{\mathbb{Q}} 0$ so $p <_{\mathbb{Q}} 0$

so $p \in T$.

Let $0 <_{\mathbb{Q}} q^2 <_{\mathbb{Q}} 2$. Then $p <_{\mathbb{Q}} q$

~~$\Rightarrow pq$~~

$p <_{\mathbb{Q}} 0 \Rightarrow p \in T$, or $p = 0 \Rightarrow p \in T$

or $0 <_{\mathbb{Q}} p$.

Now $p^2 <_{\mathbb{Q}} pq$ (since $0 <_{\mathbb{Q}} p$)

$<_{\mathbb{Q}} q^2$ (since $p <_{\mathbb{Q}} q$)

$<_{\mathbb{Q}} 2$ (since $q^2 <_{\mathbb{Q}} 2$).

$\Rightarrow p \in T$.

2. ~~(e)~~ (f) (iv) Choose $q = p + \frac{1}{n}$ with n as defined in "hint".

Then q is rational, since p is rational.

Now $q^2 = p^2 + \frac{1}{n}\left(2p + \frac{1}{n}\right) <_{\mathbb{Q}} p^2 + \frac{1}{n}(2p+1)$

as $n >_{\mathbb{Q}} 1$.

$$n >_{\mathbb{Q}} \frac{2p+1}{2-p^2} \Rightarrow \frac{1}{n} <_{\mathbb{Q}} \frac{2-p^2}{2p+1}$$

$$\Rightarrow q^2 <_{\mathbb{Q}} p^2 + 2 - p^2$$
$$= 2 \qquad (*)$$

Also $n >_{\mathbb{Q}} 1 \Rightarrow q = p + \frac{1}{n} >_{\mathbb{Q}} p \quad (+)$

So (i) $q \in S$ from $(*)$

(ii) $p <_{\mathbb{Q}} q$ from $(+)$ $\square$.

[10 MARKS]

~~As a result, q~~

3. a) (i) Let $x \in B = \mathbb{R}$.

Then $2^x \in \mathbb{R}$ and $2^x > 0 \Rightarrow 2^x \in R_+$.

So $f(2^x, 0) = x$ $\square$. [3 MARKS]

(ii) Choose $A = R_+ \times \{0\}$

Proof above holds for surjectivity.

For injectivity,

$$f(x_1, y_1) = f(x_2, y_2)$$

But $y_1 = y_2 = 0$.

Also $\log_2(x_1 + 0) = \log_2(x_2 + 0)$

$\Rightarrow x_1 = x_2$ $\square$. [6 MARKS]

(iii) It is possible to obtain any ~~positive~~ non-negative value $v \in R_+ \cup \{0\}$ from

$$f(2^v, 0) = v \quad \& \quad 2^v \geq 1$$

~~Also zero is possible if~~

Negative values are not possible - we would require $f(x,y) < 0$

$\Rightarrow \log_2(x+y) < 0$

So either $\log_2 x < 0 \Rightarrow x < 1$ ×

or $\log_2(x+1) < 0 \Rightarrow x + 1 < 1$

i.e. $x < 0$ ×.

So image is $\underline{R_+ \cup \{0\}}$. [6 MARKS]

3. (b)(i) $R^n \subseteq R \Rightarrow R$ is transitive:

$R^n \subseteq R \Rightarrow R^2 \subseteq R$. $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R^2$. But $R^2 \subseteq R \Rightarrow (a,c) \in R$.
So R is transitive.

R is transitive $\Rightarrow R^n \subseteq R$

True for $n=1$. Use induction to show $R^{n+1} \subseteq R$ assuming $R^n \subseteq R$.

Consider $(a,b) \in R^{n+1} = R \cdot R^n$

$\Rightarrow \exists x \left((a,x) \in R \wedge (x,b) \in R^n \right)$

$R^n \subseteq R \Rightarrow (x,b) \in R$. R is transitive

$\Rightarrow (a,b) \in R$. So $R^{n+1} \subseteq R$ $\square$. [6 MARKS]

(ii) $\forall a \in A \; \exists b \in B \left((a,b) \in R \right)$

$\wedge \; \forall a \in A \; \forall b \in B \; \forall c \in B \big((a,b) \in R \wedge (a,c) \in R$

$\Rightarrow b = c \big)$. [6 MARKS]

(iii) Trivially, $f : \{0\} \to \{0\}$, $f(0) = 0$ is transitive.

[3 MARKS]

4. a) (i) $T(\text{Steven}) \wedge I(\text{Steven})$

(ii) $G(\text{Steven}) \wedge \forall x\,(I(x) \rightarrow G(x))$

(iii) $\exists x\,(T(x) \wedge I(x))$

$\wedge\ \forall x\, \forall y\,(T(x) \wedge I(x) \wedge T(y) \wedge I(y) \rightarrow x = y)$

(iv) $\forall x\,(T(x) \wedge I(x) \rightarrow x = \text{Steven})$

(v) $T(\text{Amanda}) \wedge \neg T(\text{James})$

(vi) $\neg I(\text{Amanda})$ [12 MARKS]

b) Simplify (iii) $\Rightarrow \forall x\, \forall y\,(T(x) \wedge I(x) \wedge T(y) \wedge I(y) \rightarrow x = y)$

Universal Instantiation

$$\forall x\,(T(x) \wedge I(x) \wedge T(\text{Steven}) \wedge I(\text{Steven}) \rightarrow x = \text{Steven})$$

Hypothesis $\Rightarrow \forall x\,(T(x) \wedge I(x) \rightarrow x = \text{Steven})$

[9 MARKS]

c) Universal Instantiation on (iv)

$$T(\text{Amanda}) \wedge I(\text{Amanda}) \rightarrow \text{Amanda} = \text{Steven}$$

Modus Ponens

$$\neg(T(\text{Amanda}) \wedge I(\text{Amanda}))$$

$$\equiv \neg T(\text{Amanda}) \vee \neg I(\text{Amanda}) \quad (*)$$

Simplify (v) $T(\text{Amanda})$ (+)

Disjunctive Syllogism (*) & (+)

$\Rightarrow \neg I(\text{Amanda})$

[9 MARKS]

5. a) $f(x)$ is $O(g(x)) \equiv \exists c \in R^+ \exists k \in R^+ \forall x \left((x > k) \to (|f(x)| \leq c\,|g(x)|) \right)$

$f(x)$ is $\Omega(g(x)) \equiv \exists c \in R^+ \exists k \in R^+ \forall x \left((x > k) \to (|f(x)| \geq c\,|g(x)|) \right)$

$f(x)$ is $\Theta(g(x)) \equiv [f(x) \text{ is } O(g(x))] \wedge [f(x) \text{ is } \Omega(g(x))]$.

[6 MARKS]

b) $\exists k_1, k_2, c_1, c_2$ s.t.

$|f_1(x)| \leq c_1 |g_1(x)|, \quad x > k_1$

& $|f_2(x)| \leq c_2 |g_2(x)|, \quad x > k_2$

By the triangle inequality,

$$|f_1(x) + f_2(x)| \leq |f_1(x)| + |f_2(x)|$$
$$\leq c_1|g_1(x)| + c_2|g_2(x)|, \quad x > \max(k_1, k_2)$$
$$\leq c_1 \max(|g_1(x)|, |g_2(x)|) + c_2 \max(|g_1(x)|, |g_2(x)|)$$
$$= (c_1 + c_2)\left|\max(|g_1(x)|, |g_2(x)|)\right|.$$

So with $c = c_1 + c_2$, $k = \max(k_1, k_2)$,

$f_1(x) + f_2(x)$ is $O(\max(|g_1(x)|, |g_2(x)|))$.

[6 MARKS]

c)
```
proc1(int n) {
  total := 1;
  for i = 1 to 2*n
    total := total * n;
}
```

(Assuming loop term condition evaluated once)

[6 MARKS]

d)
```
proc2(int n) {
  if n=1
    return 3*n;
  else if n=2
    return n*n*n;
  else
    return n * proc2(n div 3);
}
```

[6 MARKS]

5\. (e) Proc1 has $O(n)$ exec time (#mults)

Proc2 has $O(\log n)$ exec time (#mults)

Proc therefore has $O(\max(n, \log n))$

$= O(n)$

[6 MARKS]

$\underline{\underline{K = 1}}$.