Imperial College
London

MATH97040 MATH97149

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2020

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Measure and Integration

Date: 20th May 2020

Time: 09.00am - 11.30am (BST)

Time Allowed: 2 Hours 30 Minutes

Upload Time Allowed: 30 Minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS AS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD INCLUDING A COMPLETED COVERSHEET WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

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1. (a) Show that the function $f : \mathbf{R} \to \mathbf{R}$ given below is Borel measurable

$$f(x) = \begin{cases} \cos(|x|), & x \in Q \cap [0,1], \\ \frac{\tan x}{x^2+1}, & x \in (2,3], \\ 1, & x = 4, \\ 0, & \text{for all other values of } x \end{cases}$$

where Q denotes the set of rational numbers. (6 marks)

(b) Show that there exists AT MOST one probability measure on $\mathcal{B}(\mathbf{R})$ with the property that for any real numbers $a < b$,

$$\mu((a,b]) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-\frac{x^2}{2}} dx.$$

(7 marks)

(c) Let $F : \mathbf{R} \to \mathbf{R}$ be a non-decreasing and right-continuous function and μ_F the Lebesgue-Stieljes measure associated to F. Suppose that $F(t) = 1$ for $t < 0$, $F(0) = 3$, and $F(t) = 17$ for $t \geq 1$.

(i) Compute $\mu_F(\{0\})$. (3 marks)

(ii) Let $h : \mathbf{R} \to \mathbf{R}$ be integrable with respect to μ_F and such that $\int_{\mathbf{R}} h \, d\mu_F = \pi$. Compute $\int_{\mathbf{R}} (h+2) d\mu_F$. (4 marks)

(Total: 20 marks)

2. Let $(\mathcal{X}, \mathcal{F}, \mu)$ be a measure space.

(a) (i) If $f = \sum_{i=1}^n a_i 1_{B_i}$ is a simple function on $\mathcal{X}$, define $\int_{\mathcal{X}} f d\mu$. (2 marks)

(ii) In this question we take $\mathcal{X} = \mathbf{N} \times \mathbf{N}$ and μ to be the counting measure. Explain whether $f : \mathbf{N}^2 \to \mathbf{R}$ defined below is integrable.

$$f(n,m) = (-1)^m \frac{3^{-n}}{m+1}, \qquad (n,m) \in \mathbf{N}^2.$$

(8 marks)

(b) (i) Let f be a non-negative integrable function on $\mathcal{X}$. Let $\epsilon > 0$ be fixed. Setting $f_n = f \, \mathbf{1}_{\{f \geq n\}}$ for all $n \geq 1$, show that there exists a natural number n such that

$$\int_{\mathcal{X}} f_n \, d\mu \leq \frac{\epsilon}{2}.$$

(5 marks)

(ii) Deduce that there exists a $\delta > 0$ such that, for all $A \in \mathcal{F}$ with $\mu(A) \leq \delta$,

$$\int_A f \, d\mu \leq \epsilon.$$

(5 marks)

(Total: 20 marks)

3. (a) Let μ and ν be σ-finite measures on $(\mathcal{X}, \mathcal{F})$ and $(\mathcal{Y}, \mathcal{G})$ respectively. Suppose $f : \mathcal{X} \times \mathcal{Y} \to [0, \infty)$ is measurable. State the Fubini-Tonelli Theorem. (5 marks)

(b) Let $\mathbf{R}^2$ be given the Lebesgue measure λ and let

$$g(x,y) = \begin{cases} \frac{x^2-y^2}{(x^2+y^2)^2}, & (x,y) \in [0,1]^2 \setminus \{(0,0)\}, \\ 0, & (x,y) = (0,0), \end{cases}$$
$$f(x,y) = g(x,y) + x^5,$$

be two Borel measurable functions. Show that $f \notin L_1([0,1] \times [0,1])$.
($Hint.$ $\frac{\partial}{\partial y} \frac{y}{(x^2+y^2)} = \frac{x^2-y^2}{(x^2+y^2)^2}$, $\frac{\partial}{\partial x} \frac{x}{(x^2+y^2)} = \frac{y^2-x^2}{(x^2+y^2)^2}$.)

(9 marks)

(c) Answer YES or NO to the following questions (provide a justification).

(i) Let $(\mathcal{X}, \mathcal{A})$ be a measurable space. Let $g : \mathcal{X} \times \mathbf{R} \to [0, \infty)$ be a measurable function. Can we conclude that $\{x : g(x, 0) > 1\}$ belongs $\mathcal{A}$? (3 marks)

(ii) Define two Borel measures on $[0,1]$ by the following formulas:

$$\nu(A) = \int_A x\, \mathbf{1}_{[0,\frac{1}{2}]}(x)\, dx, \qquad \mu(A) = \int_A x^2 dx, \qquad A \in \mathcal{B}([0,1]),$$

where dx is the Lebesgue measure. Can one conclude that there exists a Lebesgue integrable function $D : [0,1] \to \mathbf{R}$ such that, for all $A \in \mathcal{B}([0,1])$, $\mu(A) = \int_A D d\nu$?

(3 marks)

(Total: 20 marks)

4. (a) Let $(A_i)_{i=1}^n$ be a partition of a non-empty set $\mathcal{X}$ and let $\mathcal{F} = \sigma(\{A_1, \dots, A_n\})$.

(i) Describe the elements of $\mathcal{F}$. *No proof is needed.*

(2 marks)

(ii) For $(\mathcal{X}, \mathcal{F})$ given above, show that if a function $g : \mathcal{X} \to \mathbf{R}$ is measurable, then it is constant on each A_i.

(4 marks)

(iii) Let $\mathcal{X} = \mathbf{R}$ with σ-algebra defined below.

$$A_1 = [0, 1], \quad A_2 = [\pi, 2\pi], \quad A_3 = \mathbf{R} \setminus (A_1 \cup A_2), \quad \mathcal{F} = \sigma(\{A_1, A_2, A_3\}).$$

Let $g(x) = x^2 + x$, for $x \in \mathbf{R}$. Find all functions h, measurable from $(\mathbf{R}, \mathcal{F})$ to $(\mathbf{R}, \mathcal{B}(\mathbf{R}))$, satisfying the following:

$$\int_0^1 h(x)dx = \int_0^1 g(x)dx, \qquad \int_\pi^{2\pi} h(x)dx = \int_\pi^{2\pi} g(x)dx.$$

(6 marks)

(b) Let $(\mathcal{X}, \mathcal{F})$ be a measurable space. Let μ be a finite signed measure and ν a positive finite measure on $(\mathcal{X}, \mathcal{F})$. Explain what is meant by μ to be absolutely continuous with respect to ν and what is meant by μ and ν to be singular. (4 marks)

(c) Let $(\mathcal{X}, \mathcal{F}, \mathbf{P})$ be a probability space and let $X : \mathcal{X} \to \mathbf{R}$ be in $L_1(\mathbf{P})$. Show that, if $\mathcal{G} \subset \mathcal{F}$ is a σ-algebra, there exists a $\mathcal{G}$-measurable random variable Y such that

$$\int_A X \, d\mathbf{P} = \int_A Y \, d\mathbf{P} \,, \quad \forall A \in \mathcal{G}.$$

In other words, prove the existence of the conditional expectation $\mathbf{E}(X|\mathcal{G})$, do not prove the uniqueness.

(4 marks)

(Total: 20 marks)

5. Let $(\mathcal{X}, \mathcal{F}, \mu)$ be a probability space and $T : \mathcal{X} \to \mathcal{X}$ a measurable map. Denote by $\mathcal{I} = \{A \in \mathcal{F} : T^{-1}(A) = A\}$ the invariant σ-algebra of T.

(a) (i) Define what is meant by T being a measure preserving transformation. (3 marks)

(ii) Suppose T is a measure preserving transformation. State what is meant by μ being ergodic.

(3 marks)

(b) Let $S^1 = \{e^{2i\pi\theta}, \theta \in [0,1)\}$ be the unit circle equipped with the Lebesgue measure $d\theta$. Consider the following measure preserving transformation on S^1:

$$T(e^{2i\pi\theta}) = e^{2i\pi(\theta+1/3)}, \qquad \theta \in [0,1).$$

Give an example of an invariant set which demonstrates that the Lebesgue measure is not ergodic for T. (7 marks)

(c) Prove the Poincaré Recurrence Theorem along the following lines. Suppose that T is a measure preserving transformation. Let $A \in \mathcal{F}$. We set

$$B = \{x \in A : \exists n \geq 1 \text{ such that } \forall k \geq n, T^k(x) \notin A\}.$$

For all $k \geq 1$, let $B_k = T^{-k}(B_0)$ where

$$B_0 = \{x \in A : \forall n \geq 1, T^n(x) \notin A\}.$$

Then $B \in \mathcal{F}$ and $B_k \in \mathcal{F}$ for all $k \geq 0$.

(i) Show that $B \subset \cup_{k=0}^{\infty} B_k$. (2 marks)

(ii) Show that the B_k are disjoint and all have same measure. (2 marks)

(iii) Deduce the value of $\mu(B)$. (3 marks)

(Total: 20 marks)

Module: MATH96031/MATH97040/MATH97149
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BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2020

MATH96031/MATH97040/MATH97149 Measure and Integration

The following information must be completed:

Is the paper suitable for resitting students from previous years: no.

Category A marks: available for basic, routine material (excluding any mastery question) (40 percent = 32/80 for 4 questions):
1(a) 6 marks; 1(c)(i) 3 marks; 1(c)(ii) 4 marks; 2(a)(i) 2 marks; 3(a) 5 marks; 4(a)(i) 2 marks; 4(a)(iii) 6 marks; 4(b) 4 marks;

Category B marks: Further 25 percent of marks (20/ 80 for 4 questions) for demonstration of a sound knowledge of a good part of the material and the solution of straightforward problems and examples with reasonable accuracy (excluding mastery question):
2(a)(ii) 8 marks; 2(b)(i) 5 marks; 3(c)(ii) 3 marks; 4(c) 4 marks

Category C marks: the next 15 percent of the marks (= 12/80 for 4 questions) for parts of questions at the high 2:1 or 1st class level (excluding mastery question):
3(b) 9 marks; 3(c)(i) 3 marks

Category D marks: Most challenging 20 percent (16/80 marks for 4 questions) of the paper (excluding mastery question):
1(b) 7 marks; 2(b)(ii) 5 marks; 4(a)(ii) 4 marks

Signatures are required for the final version:

Setter's signature	Checker's signature	Editor's signature
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BSc, MSc and MSci EXAMINATIONS (MATHEMATICS)

May – June 2020

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Measure and Integration

Date: ??
Time: ??
Time Allowed: 2 Hours for MATH96 paper; 2.5 Hours for MATH97 papers
This paper has *4 Questions (MATH96 version); 5 Questions (MATH97 versions)*.
Candidates should start their solutions to each question in a new main answer book.
Supplementary books may only be used after the relevant main book(s) are full.
Statistical tables will not be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted.
- Each question carries equal weight.
- Calculators may not be used.

1. (a) (SEEN SIMILAR) Show that the function $f : \mathbf{R} \to \mathbf{R}$ given below is Borel measurable

$$f(x) = \begin{cases} \cos(|x|), & x \in Q \cap [0,1], \\ \frac{\tan x}{x^2+1}, & x \in (2,3], \\ 1, & x = 4, \\ 0, & \text{for all other values of } x \end{cases}$$

where Q denotes the set of rational numbers. (6 marks)

Solution Let $A_1 = Q \cap [0,1]$, $A_2 = (2,3]$, $A_3 = \{4\}$. Since singletons, countable sets, and closed, open, semi-closed intervals are Borel measurable, so are A_i. Then $f(x) = \cos(|x|)\mathbf{1}_{A_1}(x) + \frac{\tan x}{x^2+1}\mathbf{1}_{A_2}(x) + \mathbf{1}_{A_3}(x)$. Since continuous functions are Borel measurable, products and sums of measurable functions are measurable, then f is measurable. (A)

(b) (UNSEEN) Show that there exists at most one probability measure on $\mathcal{B}(\mathbf{R})$ with the property that for any real numbers $a < b$,

$$\mu((a,b]) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-\frac{x^2}{2}} dx.$$

(7 marks)

Solution. The collection of sets of the form $(c,d]$ is a π-system, denote it by $\mathbf{C}$. Let ν be a probability measure with $\nu((a,b]) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-\frac{x^2}{2}} dx$. Let

$$\mathcal{G} = \{A \in \mathcal{B}(\mathbf{R}) : \mu(A) = \nu(A)\}.$$

We show $\mathcal{G}$ is a λ-system. Firstly $\nu(\mathbf{R}) = \mu(\mathbf{R})$, so $\mathbf{R} \in \mathcal{G}$. Secondly, if $A, B \in \mathcal{G}$ with $A \subset B$, then $\mu(B \setminus A) = \mu(B) - \mu(A) = \nu(B \setminus A)$. Thirdly, if $A_n \in \mathcal{G}$ and $(A_n)_{n \geq 1}$ increases to A, then

$$\mu(A) = \lim_{n\to\infty} \mu(A_n) = \lim_{n\to\infty} \nu(A_n) = \nu(A).$$

Thus $\mathcal{G}$ is a λ-system containing $\mathbf{C}$. This means, by the $\pi - \lambda$ theorem, $\mathcal{G} = \sigma(\mathbf{C}) = \mathcal{B}(\mathbf{R})$.

Alternative solution. Let $F(b) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^b e^{-\frac{x^2}{2}} dx$. Then F is a continuous increasing function and $\mu_F((a,b]) = F(b) - F(a)$. Let $\mathcal{A}$ be the algebra of finite disjoint union of half open intervals (i.e. sets of the form $(a,b]$ and (a,∞)). If $A = \bigcup_{j=1}^n (a_j, b_j] \in \mathcal{A}$, with $b_j < a_{j+1}$ for $j = 1, \dots, n$, we define $\varrho(A) = \sum_{j=1}^n (F(b_j) - F(a_j))$. Then ϱ is a pre-measure on $\mathcal{A}$ (shown in class for any right continuous increasing function F). Denote by $\varrho*$ the outer measure extending ϱ, it defines a Borel-measure. By the additive property of measures, μ agrees with ϱ on $\mathcal{A}$. However we have shown that there exists a unique Borel measure that extends the pre-measure μ on $\mathcal{A}$, and this proves the uniqueness. (D)

(c) Let $F : \mathbf{R} \to \mathbf{R}$ be a non-decreasing and right-continuous function and μ_F the Lebesgue-Stieljes measure associated to F. Suppose that $F(t) = 1$ for $t < 0$, $F(0) = 3$, and $F(t) = 17$ for $t \geq 1$.

(i) (SEEN SIMILAR) Compute $\mu_F(\{0\})$. (3 marks)

Solution. Seen

$$\mu_F(\{0\}) = \lim_{n\to\infty} \mu_F(\{(-\frac{1}{n}, 0]\}) = 3 - 1 = 2.$$

(A)

(ii) (UNSEEN) Let $h : \mathbf{R} \to \mathbf{R}$ be integrable with respect to μ_F and such that $\int_{\mathbf{R}} h\, d\mu_F = \pi$. Compute $\int_{\mathbf{R}}(h+2)d\mu_F$. (4 marks)

Solution. First note $\mu_F(\mathbf{R}) = 17 - 1 = 16$. Therefore

$$\int (h+2)d\mu_F = \int h\, d\mu_F + 2\mu_F(\mathbf{R}) = \pi + 32.$$

(A)

(Total: 20 marks)

2. Let $(\mathcal{X}, \mathcal{F}, \mu)$ be a measure space.

(a) (i) (SEEN) If $f = \sum_{i=1}^n a_i 1_{B_i}$ is a simple function on $\mathcal{X}$, define $\int_{\mathcal{X}} f d\mu$. (2 marks)

Solution. If $f = \sum_{i=1}^n a_i 1_{B_i}$ is a simple function on $\mathcal{X}$, then $\int_{\mathcal{X}} f\, d\mu = \sum_{i=1}^n a_i \mu(B_i)$. (A)

(ii) (UNSEEN) In this question we take $\mathcal{X} = \mathbf{N} \times \mathbf{N}$ and μ to be the counting measure. Explain whether $f : \mathbf{N}^2 \to \mathbf{R}$ defined below is integrable.

$$f(n,m) = (-1)^m \frac{3^{-n}}{m+1}, \qquad (n,m) \in \mathbf{N}^2.$$

(8 marks)

Solution. We compute $\int |f|\, d\mu$. For all $N \geq 1$, setting A_N to be the finite set $\{1, 2, \ldots, N\}^2$, then $|f|\mathbf{1}_{A_N} = \sum_{(n,m)\in A_N} |f(n,m)|\mathbf{1}_{\{(n,m)\}}$ is a simple function, the integral of which is given by $\sum_{(n,m)\in A_N} |f(n,m)|\mu(\{(n,m)\}) = \sum_{(n,m)\in A_N} |f(n,m)|$. Hence, invoking the Monotone Convergence Theorem,

$$\int |f|\, d\mu = \lim_{N\to\infty} \int |f|\mathbf{1}_{A_N}\, d\mu = \lim_{N\to\infty} \sum_{(n,m)\in A_N} |f(n,m)| = \sum_{(n,m)\in\mathbf{N}^2} |f(n,m)|.$$

Therefore

$$\int |f|\, d\mu = \sum_{m=1}^\infty \frac{1}{m+1} \sum_{n=1}^\infty 3^{-n} = \frac{1}{2} \sum_{m=1}^\infty \frac{1}{m+1} = \infty,$$

where we used Fubini-Tonelli for the first equality. Thus f is not integrable. (B)

(b) (i) (SEEN) Let f be a non-negative integrable function on $\mathcal{X}$. Let $\epsilon > 0$ be fixed. Setting $f_n = f\, \mathbf{1}_{\{f \geq n\}}$, show that there exists a natural number n such that

$$\int_{\mathcal{X}} f_n\, d\mu \leq \frac{\epsilon}{2}.$$

(5 marks)

Solution. It holds that $f_n = f\mathbf{1}_{f\geq n} \underset{n\to\infty}{\longrightarrow} 0$ μ a.e., and for all $n \geq 1$, we have $0 \leq f_n \leq f$. Since f is integrable, by dominated convergence

$$\int f\,\mathbf{1}_{f\geq n}\,d\mu \underset{n\to\infty}{\longrightarrow} 0.$$

In particular there exists a $n \geq 1$ such that

$$\int f_n\,d\mu \leq \frac{\epsilon}{2}.$$

(B)

(ii) (SEEN) Deduce that there exists a $\delta > 0$ such that, for all $A \in \mathcal{F}$ with $\mu(A) \leq \delta$,

$$\int_A f\,d\mu \leq \epsilon.$$

(5 marks)

Solution. With n as constructed above, for all $A \in \mathcal{A}$, we have

$$\int_A f\,d\mu = \int_A f\,\mathbf{1}_{f\geq n}\,d\mu + \int_A f\,\mathbf{1}_{f<n}\,d\mu.$$

Now the first integral on the right-hand side is bounded by $\epsilon/2$, while the second one is bounded by $\int_A n\,\mathbf{1}_{f<n}\,d\mu \leq n\mu(A)$. Hence

$$\int_A f\,d\mu \leq \frac{\epsilon}{2} + n\mu(A),$$

and, setting $\delta := \epsilon/(2n)$, we therefore have $\int_A f\,d\mu \leq \epsilon$ for all $A \in \mathcal{A}$ with $\mu(A) \leq \delta$. (D)

(Total: 20 marks)

3. (a) (SEEN) Let μ and ν be σ-finite measures on $(\mathcal{X}, \mathcal{F})$ and $(\mathcal{Y}, \mathcal{G})$ respectively. Suppose $f : \mathcal{X} \times \mathcal{Y} \to [0, \infty)$ is measurable. State the Fubini-Tonelli Theorem. (5 marks)

Solution. If f is non-negative, then the functions $g(x) = \int f_x(y)d\nu(y)$ and $h(y) = \int f^y(x)d\mu(x)$ are both non-negative and measurable. Furthermore,

$$\int f d(\mu \times \nu) = \int_{\mathcal{X}} \left(\int_{\mathcal{Y}} f(x,y)d\nu(y)\right) d\mu(x) = \int_{\mathcal{Y}} \left(\int_{\mathcal{X}} f(x,y)d\mu(x)\right) d\nu(y).$$

(A)

(b) (UNSEEN) Let $\mathbf{R}^2$ be given the Lebesgue measure λ and let

$$g(x,y) = \begin{cases} \frac{x^2-y^2}{(x^2+y^2)^2}, & (x,y) \in [0,1]^2 \setminus \{(0,0)\}, \\ 0, & (x,y) = (0,0), \end{cases}$$
$$f(x,y) = g(x,y) + x^5,$$

be two Borel measurable functions. Show that $f \notin L_1([0,1] \times [0,1])$.

Hint. $\frac{\partial}{\partial y}\frac{y}{(x^2+y^2)} = \frac{x^2-y^2}{(x^2+y^2)^2}$, $\frac{\partial}{\partial x}\frac{x}{(x^2+y^2)} = \frac{y^2-x^2}{(x^2+y^2)^2}$.

Solution. The function $h(x) = x^5$ is non-negative, measurable on $[0,1]^2$, and by Fubini's theorem $\int_{[0,1]^2} x^5 d\lambda = \int_0^1 (\int_0^1 x^5 dx)\, dy = \frac{1}{6}$ and h is therefore integrable.
Thus the integrability of f is equivalent to the integrability of g.

(3 marks)

Below we show f is not integrable (we show two possible solutions.)

(6 marks)

Solution a. Since $\lambda(\{0\} \times [0,1] \cup [0,1] \times \{0\}) = 0$, we can restrict the integration region to $(0,1]^2$. On the one hand,

$$\int_{(0,1]} \int_{(0,1]} f(x,y)\, dx\, dy = -\int_{(0,1]} \frac{x}{x^2+y^2}\Big|_0^1 dy = -\int_{(0,1]} \frac{1}{1+y^2} dy < 0.$$

On the other hand,

$$\int_{(0,1]} \int_{(0,1]} f(x,y)\, dy\, dx = \int_{(0,1]} \frac{y}{x^2+y^2}\Big|_0^1 dx = \int_{(0,1]} \frac{1}{x^2+1} dx > 0.$$

The iterated integrals are non-zero because the integrands are non-negative and not zero almost surely. If $f \in L_1$ then the two integrals must be equal, by Fubini's theorem, which is not the case.

Solution b. For all $(x,y) \in [0,1]^2$, $f^+(x,y) = f(x,y)\mathbf{1}_{\{x \geq y\}}$. To this non-negative function we apply Fubini's theorem

$$\begin{aligned}\int_{(0,1]^2} f^+ &= \int_{(0,1]} \int_{(0,1]} f^+(x,y)\, dy\, dx \\ &= \int_{(0,1]} \frac{y}{x^2+y^2}\Big|_{y=0}^{y=x} dx = \int_{(0,1]} \frac{1}{2x} = \infty.\end{aligned}$$

Therefore $f \notin L_1$. (C)

(c) Answer YES or NO to the following questions (provide a justification).

(i) (UNSEEN) Let $(\mathcal{X}, \mathcal{A})$ be a measurable space. Let $g : \mathcal{X} \times \mathbf{R} \to [0, \infty)$ be a measurable function. Can we conclude that $\{x : g(x,0) > 1\}$ belongs $\mathcal{A}$? (3 marks)

Solution. YES.
(Since $E = \{(x,y) : g(x,y) - 1 > 0\}$ is measurable, the sector

$$E^1 = \{x : (x,0) \in E\} = \{x : g(x,0) > 1\}$$

is measurable.) (C)

(ii) (UNSEEN) Define two Borel measures on $[0,1]$ by the following formulas:

$$\nu(A) = \int_A x\, \mathbf{1}_{[0,\frac{1}{2}]}(x)\, dx, \qquad \mu(A) = \int_A x^2 dx, \qquad A \in \mathcal{B}([0,1]),$$

where dx is the Lebesgue measure. Can one conclude that there exists a Lebesgue integrable function $D : [0,1] \to \mathbf{R}$ such that, for all $A \in \mathcal{B}([0,1])$, $\mu(A) = \int_A D d\nu$?

(3 marks)

Solution. NO. (if $A = (\frac{1}{2}, 1]$, then $\nu(A) = 0$ while $\mu(A) = \frac{7}{24} > 0$, contradicting $\mu \ll \nu$). (B)

(Total: 20 marks)

4. (a) Let $(A_i)_{i=1}^n$ be a partition of a non-empty set $\mathcal{X}$ and $\mathcal{F} = \sigma(\{A_1, \ldots, A_n\})$.

(i) (SEEN) Describe the elements of $\mathcal{F}$. *No proof is needed.*

(2 marks)

Solution. $\mathcal{F}$ is given by the collection of sets of the form

$$\phi, \qquad \text{and} \qquad A = \bigcup_{i \in I} A_i, \qquad \text{for } I \subset \{1, \ldots, n\}.$$

(A)

(ii) (UNSEEN) For $(\mathcal{X}, \mathcal{F})$ given above, show that if a function $g : \mathcal{X} \to \mathbf{R}$ is measurable, then it is constant on each A_i. (4 marks)

Solution. Let $i = 1, \ldots, n$, and let x be any element of A_i. Setting $y = g(x)$, we have $x \in g^{-1}(\{y\})$. Since $\{y\} \in \mathcal{B}(\mathbf{R})$ and g is measurable, $g^{-1}(\{y\})$ is an element of $\mathcal{F}$, so is of the form $\cup_{j \in J} A_j$ for some $J \subset \{1, \ldots, n\}$. Since moreover $g^{-1}(\{y\})$ contains x, it follows that $A_i \subset g^{-1}(\{y\})$, so g is identically equal to y on A_i. (D)

(iii) (UNSEEN) Let $\mathcal{X} = \mathbf{R}$ with σ-algebra defined below.

$$A_1 = [0, 1], \quad A_2 = [\pi, 2\pi], \quad A_3 = \mathbf{R} \setminus (A_1 \cup A_2), \quad \mathcal{F} = \sigma(\{A_1, A_2, A_3\}).$$

Let $g(x) = x^2 + x$, for $x \in \mathbf{R}$. Find all functions h, measurable from $(\mathbf{R}, \mathcal{F})$ to $(\mathbf{R}, \mathcal{B}(\mathbf{R}))$, satisfying the following:

$$\int_0^1 h(x)dx = \int_0^1 g(x)dx, \qquad \int_\pi^{2\pi} h(x)dx = \int_\pi^{2\pi} g(x)dx.$$

(6 marks)

Solution. If h is such a function, by the previous question, there exist a_i, $i = 1, 2, 3$, such that $h = \sum_{i=1}^3 a_i \mathbf{1}_{A_i}$. We must therefore have

$$\int_0^1 g(x)\, dx = \int_{A_1} h(x)\, dx = a_1,$$

so that $a_1 = \int_0^1 (x^2 + x)\, dx = \frac{1}{3} + \frac{1}{2} = \frac{1}{6}$ while

$$\int_\pi^{2\pi} g(x)\, dx = \int_{A_2} h(x)\, dx = \pi a_2,$$

so that $a_2 = \frac{1}{\pi} \int_\pi^{2\pi} (x^2 + x)\, dx = \frac{1}{\pi} \left(\frac{(2\pi)^3}{3} + \frac{(2\pi)^2}{2} - \frac{\pi^3}{3} - \frac{\pi^2}{2} \right) = \frac{7\pi^2}{3} + \frac{3\pi}{2}$. In conclusion the requested functions h are all functions of the form $\sum_{i=1}^3 a_i \mathbf{1}_{A_i}$ with $a_1 = \frac{1}{6}$, $a_2 = \frac{7\pi^2}{3} + \frac{3\pi}{2}$ and $a_3 \in \mathbf{R}$. (A)

(b) (SEEN) Let $(\mathcal{X}, \mathcal{F})$ be a measurable space. Let μ be a finite signed measure and ν a positive finite measure on $(\mathcal{X}, \mathcal{F})$. Explain what is meant by μ to be absolutely continuous with respect to ν and what is meant by μ and ν to be singular. (4 marks)

Solution. μ is said to be absolutely continuous w.r.t. ν if, for all $A \in \mathcal{A}$ such that $\nu(A) = 0$, $\mu(A) = 0$. μ and ν are said to be singular if there exists $A, B \in \mathcal{A}$ such that $A \cap B = \phi$ and $A \cup B = \mathcal{X}$ and satisfying $\mu(A) = 0$ and $\nu(B) = 0$. (A)

(c) (SEEN) Let $(\mathcal{X}, \mathcal{F}, \mathbf{P})$ be a probability space and let $X : \mathcal{X} \to \mathbf{R}$ be in $L_1(\mathbf{P})$. Show that, if $\mathcal{G} \subset \mathcal{F}$ is a σ-algebra, there exists a $\mathcal{G}$-measurable random variable Y such that

$$\int_A X \, d\mathbf{P} = \int_A Y \, d\mathbf{P} \,, \quad \forall A \in \mathcal{G}.$$

In other words, prove the existence of the conditional expectation $\mathbf{E}(X|\mathcal{G})$*, do not prove the uniqueness.*

(4 marks)

Solution. Define $Q(A) = \int_A X \, d\mathbf{P}$ for $A \in \mathcal{G}$. Then Q defines a signed measure on $(\mathcal{X}, \mathcal{G})$, as can be seen by linearity of the integral and by the Dominated Convergence Theorem. Moreover Q is absolutely continuous with respect to $\mathbf{P}$, so by the Radon-Nikodym Theorem there exists a $\mathcal{G}$-measurable random variable Y such that $Q(A) = \int_A Y \, d\mathbf{P}$ for all $A \in \mathcal{A}$. (B)

(Total: 20 marks)

5. Let $(\mathcal{X}, \mathcal{F}, \mu)$ be a probability measure space. Let $T : \mathcal{X} \to \mathcal{X}$ be a measurable map. Denote by $\mathcal{I} = \{A \in \mathcal{F} : T^{-1}(A) = A\}$ the invariant σ-algebra of T.

(a) (i) Define what is meant by T being a measure preserving transformation. (3 marks)

Solution. By T being a measure preserving transformation we mean that $\mu(T^{-1}(A)) = \mu(A)$ for all $A \in \mathcal{F}$.

(ii) Suppose T is a measure preserving transformation. State what is meant by μ being ergodic. (3 marks)

Solution. μ is ergodic means that $\mu(A) \in \{0, 1\}$ for all $A \in \mathcal{I}$.

(b) Let $S^1 = \{e^{2i\pi\theta}, \theta \in [0, 1)\}$ be the unit circle equipped with the Lebesgue measure $d\theta$. Consider the following measure preserving transformation on S^1:

$$T(e^{2i\pi\theta}) = e^{2i\pi(\theta+1/3)}, \qquad \theta \in [0, 1).$$

Give an example of an invariant set which demonstrates that the Lebesgue measure is not ergodic for T. (7 marks)

Solution. Identifying S^1 with $[0, 1)$ via the bijection $\theta \mapsto e^{2i\pi\theta}$, we define A to be the subset of S^1 given by $A = A_0 \cup T(A_0) \cup T^2(A_0)$, where $A_0 = [0, \frac{1}{6})$. Noting that T^3 is the identity map, it follows that $T^{-1}(A) = A$. However, noting that $A = [0, \frac{1}{6}) \cup [\frac{1}{3}, \frac{1}{2}) \cup [\frac{2}{3}, \frac{5}{6})$, it follows that A has Lebesgue measure $\frac{3}{6} = \frac{1}{2} \notin \{0, 1\}$, so T is not ergodic for the Lebesgue measure on S^1.

(c) Prove the Poincaré Recurrence Theorem along the following lines. Suppose that T is a measure preserving transformation. Let $A \in \mathcal{F}$. We set

$$B = \{x \in A : \exists n \geq 1 \text{ such that } \forall k \geq n, T^k(x) \notin A\}.$$

For all $k \geq 1$, let $B_k = T^{-k}(B_0)$ where

$$B_0 = \{x \in A : \forall n \geq 1, T^n(x) \notin A\}.$$

Then $B \in \mathcal{F}$ and $B_k \in \mathcal{F}$ for all $k \geq 0$.

(i) Show that $B \subset \cup_{k=0}^{\infty} B_k$. (2 marks)

Solution. Let $x \in B$, and let n be the smallest non-negative integer with the property that $T^k(x) \notin A$ for all $k \geq n$. Note that since $x = T^0(x) \in A$, therefore $n \geq 1$. Then $T^{n-1}(x) \in A$, hence $T^{n-1}(x) \in B_0$: if $n = 1$ this means that $x \in B_0$, while if $n \geq 2$ it means that $x \in T^{-(n-1)}(B_0) = B_{n-1}$. In all cases $x \in \cup_{k=0}^{\infty} B_k$ and the claim follows.

(ii) Show that the B_k are disjoint and all have same measure. (2 marks)

Solution. First note that, for all $k \geq 0$,

$$B_k = \{x \in \mathcal{X}, T^k(x) \in A, \text{and } \forall \ell > k, T^\ell(x) \notin A\}.$$

Let $1 \leq k < \ell$. Then if $x \in B_k$, since $\ell > k$, $T^\ell(x) \notin A$, while if $x \in B_\ell$, $T^\ell(x) \in A$. Therefore $B_k \cap B_\ell = \phi$. On the other hand, by definition, $B_\ell = T^{-(\ell-k)}(B_k)$, so, since T is measure preserving, $\mu(B_\ell) = \mu(B_k)$. Hence all the B_k are disjoint and have same measure.

(iii) Deduce the value of $\mu(B)$. (3 marks)

Solution. Since μ is a probability measure, and since the B_k are disjoint and have the same measure, we get

$$1 = \mu(\mathcal{X}) \geq \mu(\cup_{k=0}^{\infty} B_k) = \sum_{k=0}^{\infty} \mu(B_k) = \sum_{k=0}^{\infty} \mu(B_0).$$

If $\mu(B_0) > 0$, then the right-hand side is infinite yielding a contradiction, so $\mu(B_0) = 0$, and therefore $\mu(\cup_{k=0}^{\infty} B_k) = 0$. By (c) (i) this implies that $\mu(B) = 0$.

(Total: 20 marks)