E2.17

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2005

**DISCRETE MATHEMATICS AND COMPUTATIONAL COMPLEXITY**

Tuesday, 7 June 2:00 pm

Time allowed: 2:00 hours

Corrected Copy

**There are FOUR questions on this paper.**

**Q1 is compulsory.**
**Answer Q1 and any two of questions 2-4.**
**Q1 carries 40% of the marks. Questions 2 to 4 carry equal marks (30% each).**

**Any special instructions for invigilators and information for candidates are on page 1.**

Examiners responsible First Marker(s) : G.A. Constantinides, G.A. Constantinides
Second Marker(s) : T.J.W. Clarke, T.J.W. Clarke

1 **[Compulsory]**

a) Prove the following statements.

(i) For arbitrary sets $A$ and $B$, $A = A \cap (A \cup B)$.

(ii) The function $f : \boldsymbol{Z}^- \to \boldsymbol{Z}^+$ defined by $f(x) = |x|$ is a bijection.
(where $\boldsymbol{Z}^-$ is the set of negative integers and $\boldsymbol{Z}^+$ is the set of positive integers).

[10]

b) State the rule of inference or common fallacy corresponding to each of these statements.

(i) $\neg q \wedge (p \to q) \to \neg p$.

(ii) If there is an exam, I am nervous. I am nervous, therefore there is an exam.

(iii) I am quiet. Therefore I am either quiet or nervous.

(iv) $\neg p \wedge (p \vee q) \to q$.

(v) I am both quiet and nervous. Therefore I am nervous.

[10]

c) Using the Master Theorem, provide a big-O expression for each function $f_i(n)$ below.

(i) $f_1(n) = f_1(n/2) + 3$.

(ii) $f_2(n) = 2f_2(n/2) + 3$.

(iii) $f_3(n) = 2f_3(n/2) + 3n^2$.

[10]

d) State an example problem for each of these categories.

(i) The problem is known to be solvable, but not known to be tractable.

(ii) The problem is known to be tractable.

(iii) The problem is known to be unsolvable.

[10]

a) For a relation $R$, define

(i) transitivity,
(ii) symmetry,
(iii) reflexivity.

[6]

b) Prove that a relation $R$ on a set $A$ is transitive iff $R^n \subseteq R$ for all $n \in \mathbf{Z}^+$.

[10]

An *equivalence relation* is a reflexive, symmetric, and transitive relation.
Let $x$ mod $b$ denote the remainder of $x$ when divided by $b$.
Let $M$ be the relation on the set $A \subseteq \mathbf{Z}^+$ where $(x,y) \in M$ iff $x$ mod $3 = y$ mod 3.

c) Prove that $M$ is an equivalence relation when $X = \mathbf{Z}^+$.

[8]

d) Construct the digraph of the relation $M$ when $X = \{1,2,3,4,5\}$.

[6]

3

Let $f : \mathbf{R} \to \mathbf{R}$ be the function given by $f(x) = x^3 + x^2 + x$.
Let $g : \mathbf{R} \to \mathbf{R}$ be the function given by $g(x) = -x^2 - 6x - 8$.
Let $h : \mathbf{R} \to \mathbf{R}$ be the function given by $h(x) = -x^2 - 6x - 5$.

Let $P(x)$ be the predicate $x < 0$.
Let $Q(x)$ be the predicate $f(x) < 0$.
Let $R(x)$ be the predicate $g(x) < 0$.
Let $S(x)$ be the predicate $h(x) < 0$.

$X$ is an arbitrary subset of $\mathbf{R}$.

a) Express each of these propositions using the predicates above and appropriate symbolic logic connectives and quantification. You may take the universe of discourse as $X$.

(i) "For all real numbers in $X$, whenever $h(x)$ is negative, so is $g(x)$".
(ii) "$x$ is negative whenever $f(x)$ is negative, when x is a real number in $X$".
(iii) "For every real number $x$ in $X$, either $f(x)$ is negative or $h(x)$ is negative".

[6]

b) Given that $1 + x + x^2$ is positive for all real $x$, show by factorising $f$, $g$, and $h$, or otherwise, that the three propositions in part (a) are true.

[14]

c) Given as premises your propositions from part (a) together with the proposition $\exists x\ \neg P(x)$, construct a valid argument leading to the conclusion $\exists x\ R(x)$. At each step of your argument, state the rule of inference used.

[10]

4.

a) Define what is meant by the statement $f(x)$ is $O(g(x))$.

[4]

b) Prove that $f(x) = c_0 + c_1x + \ldots + c_nx^n$ is $O(x^n)$ if $\forall i\ (\ c_i \in \boldsymbol{R}\ )$.

[6]

c) Derive an expression for the number of multiplications performed by a call to `f1`($n$), shown in Figure 4.1.

[4]

d) Using the result from part (b), derive a big-O expression of the form $O(n^k)$ for the number of multiplications performed by a call to f1($n$).

[2]

e) Consider the increasing function $f(n)$, which satisfies Equation 4.1 whenever $n$ is a multiple of $b$. Prove that for $b > 1$ and integer and $c > 0$ and real, $f(n)$ is $O(\log n)$.

FROM FIRST PRINCIPLES

$$f(n) = f(n/b) + c \qquad \text{(Equation 4.1)}$$

[10]

f) Derive a recurrence relation and initial condition(s), and hence a big-O expression, for the number of multiplications performed by a call to `f2`($n$), shown in Figure 4.2.

[4]

```
function f1(n)
begin
  total := 1;
  for i = 1 to n
    for j = i to n
      total := total * i * j;
  result := total;
end
```

Figure 4.1

```
function f2(n)
begin
  if( n = 0 ) then
    result := 1;
  else
    result := 3*f2(n/2);
end
```

Figure 4.2

2.17

1 a) (i) $A \cap (A \cup B) = A \cap \{x \mid x \in A \text{ or } x \in B\}$

$= \{x \mid x \in A \text{ and } (x \in A \text{ or } x \in B)\}$

$= \{x \mid x \in A\}$

$= A$

(ii) Bijection $\rightarrow$ Injection & Surjection

(a) Injection: $|x| = |y|$

But $|x| = -x$, $|y| = -y$

so $-y = -x \quad \therefore \quad x = y$

(b) Surjection

For an arbitrary $y \in Z^+$, choose $x = -y \in Z^-$ to give $f(x) = y$.

[10]

b) (i) Modus tollens

(ii) Affirming the conclusion

(iii) Addition

(iv) Disjunctive syllogism

(v) Simplification

[10]

c) (i) $O(\log n)$

(ii) $O(n^{\log_2 2}) = O(n)$

(iii) $O(n^2)$

[10]

d) (i) Graph colouring

(ii) Matrix multiplication

(iii) The halting problem

[10]

2 a) (i) $(a,b) \in R \wedge (b,c) \in R \rightarrow (a,c) \in R$

(ii) $(a,b) \in R \rightarrow (b,a) \in R$

(iii) $\forall a \ (a,a) \in R$ [6]

b) (i) $R^n \subseteq R$ for all $n \in Z^+ \longrightarrow R$ is transitive

$R^2 \subseteq R$.

Also if $(a,b) \in R \wedge (b,c) \in R$ then $(a,c) \in R^2$.

But since $R^2 \subseteq R$, $(a,c) \in R$.

(ii) $R$ is transitive $\rightarrow R^n \subseteq R$ for all $n \in Z^+$

True for $n = 1$. Induction for $n > 1$.

From $R^n \subseteq R$, show $R^{n+1} \subseteq R$

Consider $(a,b) \in R^{n+1}$.

Then $\exists x \left( (a,x) \in R \wedge (x,b) \in R^n \right)$

But $R^n \subseteq R$ so $(x,b) \in R$.

Thus $\exists x \left( (a,x) \in R \wedge (x,b) \in R \right)$.

But $R$ is transitive, so $(a,b) \in R$.

[10]

c) (i) reflexivity:

$(x,y) \in M$ iff $x \bmod 3 = y \bmod 3$.

Now $x \bmod 3 = x \bmod 3$, thus $(x,x) \in M$ for all $x \in Z$.

(ii) symmetry:

$(x,y) \in M \rightarrow x \bmod 3 = y \bmod 3 \rightarrow$

$y \bmod 3 = x \bmod 3 \rightarrow (y,x) \in M$

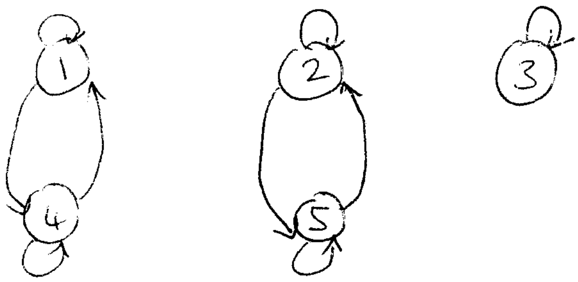
(iii) transitivity

$(x, y) \in M \wedge (y, z) \in M$

$\Rightarrow x \bmod 3 = y \bmod 3 \wedge y \bmod 3 = z \bmod 3$

$\Rightarrow x \bmod 3 = z \bmod 3$

$\Rightarrow (x, z) \in M$ [8]

d) 

[6]

3. a) (i) $\forall x\,(S(x) \rightarrow R(x))$

(ii) $\forall x\,(Q(x) \rightarrow P(x))$

(iii) $\forall x\,(Q(x) \vee S(x))$ [6]

b) $f(x) = x^3 + x^2 + x = x(x^2 + x + 1)$

$g(x) = -x^2 - 6x - 8 = -(x+2)(x+4)$

$h(x) = -x^2 - 6x - 5 = -(x+1)(x+5)$

(i) False if $\exists x\,(S(x) \wedge \neg R(x))$

$S(x) \Leftrightarrow (x < -5) \vee (x > -1)$

$\neg R(x) \Leftrightarrow -4 \leq x \leq -2$

No such $x$.

(ii) False if $\exists x\,(Q(x) \wedge \neg P(x))$

$Q(x) \Leftrightarrow x < 0$

$P(x) \Leftrightarrow x < 0$ so $\neg P(x) \Leftrightarrow x \geq 0$

No such $x$.

(iii) False if $\exists x\,(\neg Q(x) \wedge \neg S(x))$

$\neg Q(x) \Leftrightarrow x \geq 0$

$\neg S(x) \Leftrightarrow -5 \leq x \leq -1$

No such $x$.

[14]

c) $\exists x \neg P(x) \wedge \forall x (Q(x) \rightarrow P(x))$

$\exists x (\neg P(x) \wedge (Q(x) \rightarrow P(x)))$ [Universal instantiation]

$\exists x \neg Q(x)$ [Modus tollens]

$\exists x \neg Q(x) \wedge \forall x (Q(x) \vee S(x))$

$\exists x (\neg Q(x) \wedge (Q(x) \vee S(x)))$ [Universal instantiation]

$\exists x\ S(x)$ (Disjunctive syllogism)

$\exists x\ S(x) \wedge \forall x (S(x) \rightarrow R(x))$

$\exists x (S(x) \wedge (S(x) \rightarrow R(x)))$ (Universal instantiation)

$\exists x\ R(x)$ [Modus ponens]

[10]

4 a) $f(x)$ is $O(g(x))$

$\Leftrightarrow \exists c \in R^+ \ \exists k \in R^+ \ \forall x \left( (x > k) \Rightarrow (|f(x)| \leq c\,|g(x)|) \right)$ [4]

b) $f(x) = c_0 + c_1 x + \dots + c_n x^n$

$|f(x)| \leq |c_0| + |c_1 x| + \dots + |c_n x^n|$

$= |x^n| \left( |c_n| + |c_{n-1}|/|x| + \dots + |c_0|/|x^n| \right)$

$\leq |x^n| \left( |c_n| + \dots + |c_0| \right)$ for $x > 1$

So with $c = |c_n| + \dots + |c_0|$ and $k = 1$,

$f(x)$ is $O(x^n)$. [6]

c) $2\,(n + (n-1) + \dots + 1) = \dfrac{2n(n+1)}{2} = n(n+1)$ [4]

d) $O(n^2)$ [2]

e) First consider $n = b^k$

Then $f(n) = f(n/b) + c$

$= f(n/b^2) + 2c$

$\vdots$

$= f(1) + kc$

$= f(1) + c \log_b n$

for $n \neq b^k$, choose $k$ s.t.

$b^k < n < b^{k+1}$

$f(n) \leq f(b^{k+1}) = f(1) + c(k+1)$

$= (f(1) + c) + ck$

$f(n) \leq (f(1) + c) + c \log_b n$. [10]

f) $f(n) = f(n/2) + 1$ with $f(0) = 0$

Hence $f(n)$ is $O(\log n)$. [4]