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IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2012

MSc and EEE PART IV: MEng and ACGI

ESTIMATION AND FAULT DETECTION

Friday, 11 May 2:30 pm

Time allowed: 3:00 hours

There are FIVE questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s) :	R.B. Vinter
	Second Marker(s) :	D. Angeli

Information for candidates:

Some formulae relevant to the questions.

The normal $N(m, \sigma^2)$ density:

$$N(m, \sigma^2)(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-m)^2}{2\sigma^2}\right) .$$

System equations:

$$\begin{aligned} \mathbf{x}_t &= F\mathbf{x}_{t-1} + \mathbf{u}^s + \mathbf{w}_{t-1} \\ \mathbf{y}_t &= H\mathbf{x}_t + \mathbf{u}^o + \mathbf{v}_t . \end{aligned}$$

Here, $\{\mathbf{w}_t\}$ and $\{\mathbf{v}_t\}$ are white noise sequences with covariances Q^s and Q^o respectively.

The Kalman filter equations are

$$\begin{aligned} P_{t|t-1} &= FP_{t-1|t-1}F^T + Q^s \\ P_t &= P_{t|t-1} - P_{t|t-1}H^T(HP_{t|t-1}H^T + Q^o)^{-1}HP_{t|t-1} , \\ K_t &= P_{t|t-1}H^T(HP_{t|t-1}H^T + Q^o)^{-1} , \\ \hat{\mathbf{x}}_t &= \hat{\mathbf{x}}_{t|t-1} + K_t(\mathbf{y}_t - \hat{\mathbf{y}}_{t|t-1}) , \end{aligned}$$

in which $\hat{\mathbf{x}}_{t|t-1} = F\hat{\mathbf{x}}_{t-1} + \mathbf{u}^s$ and $\hat{\mathbf{y}}_{t|t-1} = H\hat{\mathbf{x}}_{t|t-1} + \mathbf{u}^o$.

1. (i): A stationary, Gaussian scalar process $\{y_k\}$ satisfies the equations

$$y_k + ay_{t-1} = e_t + 2e_{t-1}$$

in which $\{e_k\}$ is a sequence of zero mean, independent, Gaussian random variables such that $var\{e_k\} = 1$. Determine the covariance function of $\{y_k\}$. (It will depend on the value of the parameter a). [6]

It is known that the spectral density of $\{y_k\}$ satisfies

$$\Phi_{yy}(\omega) = 4 ,$$

when $\omega = \pi/3$ radians. Determine the value of a. [4]

(ii): A continuous time process $y(t)$ is modelled as

$$\frac{d^2y}{dt^2}(t) = n(t)$$

in which $n(t)$ is a unit variance, continuous time white noise process.

Take the state vector $\mathbf{x}(t)$ to be

$$\mathbf{x}(t) = \begin{bmatrix} y(t) \\ \frac{dy}{dt}(t) \end{bmatrix} .$$

For a time t and horizon $h > 0$, calculate the matrix A and the noise covariance $cov\{\mathbf{e}\}$ in the prediction model

$$\mathbf{x}(t+h) = A\mathbf{x}(t) + \mathbf{e} .$$

[6]

Hence calculate the conditional mean and variance of $y(t+h)$ given $y(t)$ and $\frac{dy}{dt}(t)$:

$$E[y(t+h) \,|\, y(t), \frac{dy}{dt}(t)] \quad \text{and} \quad cov\{y(t+h) \,|\, y(t), \frac{dy}{dt}(t)\}$$

[4]

2. The position x of an object on the line is modelled as a scalar random variable with probability density

$$x \sim N(0, \sigma^2)(x) \ .$$

in which σ^2 is a given positive constant. The measurement y from a simple sensing device yields some information about x. It is assumed that y is a discrete random variable taking values -1 or $+1$ and that the conditional probability mass function of y given x is as follows.

$$\begin{array}{llll} \text{For } x \geq 0: & p(y=-1|x) \ = \ \beta & \text{and} & p(y=+1|x) \ = \ 1-\beta \ , \\ \text{for } x < 0: & (y=-1|x) \ = \ 1-\beta & \text{and} & p(y=+1|x) \ = \ \beta \ , \end{array}$$

in which β is a parameter, $0 < \beta < 1$.

By applying Bayes' Rule, or otherwise, show that the conditional probability density $p(x|y=+1)$ of x given $y=+1$ is

$$p(x|y=+1) \ = \ \begin{cases} 2(1-\beta)N(0,\sigma^2)(x) & \text{for } x \geq 0 \\ 2\beta N(0,\sigma^2)(x) & \text{for } x < 0 \ . \end{cases}$$

[6]

For the cases $y=-1$ and $+1$, derive the least squares estimate $\hat{x}$ of x given y, and also the least squares estimation error

$$J(\beta) \ = \ E[|x-\hat{x}|^2] \ .$$

[8]

Sketch the plot of $J(\eta)$, $0 \leq \beta \leq 1$. Comment on the value of $J(\beta = 1/2)$. [6]

Data: you may use the information that $\int_0^\infty x p(x) dx = \sigma \times (2\pi)^{-\frac{1}{2}}$.

3. Let $\mathbf{x}$ and $\mathbf{y}$ be two independent Gaussian random variables with given means and covariances:

$$\mathbf{x} \sim N(\mathbf{m}_1, P_1)(\mathbf{x}), \quad \mathbf{y} \sim N(\mathbf{m}_2, P_2)(\mathbf{y}) .$$

Take a matrix K. By using the fact that the random variable $\mathbf{z} = \mathbf{x} + K\mathbf{y}$ has probability density

$$p(\mathbf{z}) = \int p(\mathbf{z}|\mathbf{y})p(\mathbf{y})d\mathbf{y} ,$$

or otherwise, show that, for every $\mathbf{z}$,

$$\int N(\mathbf{m}_1 + K\mathbf{y}, P_1)(\mathbf{z}) \times N(\mathbf{m}_2, P_2)(\mathbf{y})d\mathbf{y} = N(\mathbf{m}_1 + K\mathbf{m}_2, P_2 + KP_2K^T)(\mathbf{z}) . \quad (1)$$

[5]

Now let $\{\mathbf{x}_t\}$ and $\{\mathbf{y}_t\}$ be signal and measurement processes generated by the equations:

$$\begin{cases} \mathbf{x}_t = F\mathbf{x}_{t-1} + \mathbf{w}_{t-1} \\ \mathbf{y}_t = H\mathbf{x}_t + \mathbf{v}_t , \end{cases}$$

in which F and H are given matrices, and $\{\mathbf{w}_t\}$ and $\{\mathbf{v}_t\}$ are Gaussian white noise processes, independent of each other and of $\mathbf{x}_0$, and with covariances Q^s and Q^m respectively.

Denote by $\hat{\mathbf{x}}_t$ and P_t the filtered mean and covariance of the state at time t,

$$\hat{\mathbf{x}}_t = E[\mathbf{x}_t|\mathbf{y}_{1:t}] \quad \text{and} \quad P_t = \text{cov}\{\mathbf{x}_t|\mathbf{y}_{1:t}\} .$$

Show that the one-step-ahead *smoothed* mean and covariance of the state at time t,

$$\hat{\mathbf{x}}_{t|t+1} = E[\mathbf{x}_t|\mathbf{y}_{1:t+1}] \quad \text{and} \quad P_{t|t+1} = \text{cov}\{\mathbf{x}_t|\mathbf{y}_{1:t+1}\} ,$$

are related to the filtered means and covariances of the state at times t and $t+1$ as

$$\begin{aligned} \hat{\mathbf{x}}_{t|t+1} &= \hat{\mathbf{x}}_t + K_{t|t+1}(\hat{\mathbf{x}}_{t+1} - F\hat{\mathbf{x}}_t) \\ K_{t|t+1} &= P_tF^T[FP_tF^T + Q^s]^{-1} \\ P_{t|t+1} &= P_t - K_{t|t+1}FP_t + KP_{t+1}K^T . \end{aligned}$$

You should use the following steps:

(i) : Show that

$$p(\mathbf{x}_t|\mathbf{y}_{1:t+1}) = \int p(\mathbf{x}_t|\mathbf{x}_{t+1}, \mathbf{y}_{1:t})p(\mathbf{x}_{t+1}|\mathbf{y}_{1:t+1})d\mathbf{x}_{t+1} . \quad (2)$$

[5]

(ii): Regarding $\mathbf{x}_{t+1} = F\mathbf{x}_t + \mathbf{w}_t$ as a measurement of $\mathbf{x}_t$ with prior density $N(\hat{\mathbf{x}}_t, P_t)$, calculate

$$E[\mathbf{x}_t|\mathbf{x}_{t+1}, \mathbf{y}_{1:t}] \quad \text{and} \quad \text{cov}\{\mathbf{x}_t|\mathbf{x}_{t+1}, \mathbf{y}_{1:t}\} .$$

[5]

(iii): Obtain formulae for $\hat{\mathbf{x}}_{t|t+1}$ and $P_{t|t+1}$ from relations (1) and (2). [5]

4. (a): Consider signal and measurement processes, $\{\mathbf{x}_t\}$ and $\{\mathbf{y}_t\}$, modelled as

$$\begin{aligned} \mathbf{x}_t &= F\mathbf{x}_{t-1} + \mathbf{u}_t + \mathbf{w}_{t-1} \\ \mathbf{y}_t &= H\mathbf{x}_t + \mathbf{v}_t \, . \end{aligned}$$

Here, $\{\mathbf{w}_t\}$ and $\{\mathbf{v}_t\}$ are independent white noise sequences with covariances Q^s and Q^m. For each t, the control input $\mathbf{u}_t$ is deterministic (or , more generally, a deterministic function of current and past inputs $\mathbf{y}_t$, $\mathbf{y}_{t-1}$, ...).

State conditions on the matrices in these equations under which the estimation error covariance $P_t = \text{cov}\{x_t|y_t, y_{t-1}, \ldots\}$ and the Kalman filter gain K_t converge to steady-state values

$$P_t \to \bar{P} \quad \text{and} \quad K_t \to \bar{K} \quad \text{as } t \to \infty \, .$$

Derive a set of equations for the steady state values $\bar{P}$ and $\bar{K}$. [6]

(b): Consider a scalar stochastic control system with state equation

$$x_{t+1} = ax_t + u_t + v_t \, , \tag{3}$$

in which $a = \sqrt{\frac{11}{30}}$ and $\{v_t\}$ is Gaussian white noise with variance $\text{var}\{v_t\} = 1$. The purpose of the control u_t is to reduce the variance of the state x_t.

Case 1: (No control) Assume $u_t = 0$. Calculate the steady-state variance $\sigma_1^2 := \text{var}\{x_t\}$. [3]

Case 2: (Minimum variance control with perfect state measurements). Assume

$$u_t = -ax_t \, .$$

Calculate the steady-state variance $\sigma_2^2 := \text{var}\{x_t\}$. [3]

Case 3: (Minimum variance control with noisy state measurements). Assume now that only noisy measurements $\{y_t\}$ of the state are available:

$$y_t = x_t + w_t \, ,$$

in which $\{w_t\}$ is Gaussian white noise, $var\{w_t\} = 1$. In this case the same control law is applied as in Case 2, but the state x_t is replaced by the *estimate* of the state $\hat{x}_t = E[x_t|y_t, y_{t-1}, \ldots]$, thus

$$u_t = -a\hat{x}_t \, . \tag{4}$$

Derive the Kalman filter for calculating $\hat{x}_t$. Using the results of Part (a), or otherwise, derive equations for the steady-state estimation error variance $\bar{P}$.

Determine the steady-state variance $var\{x_t\}$,

$$\sigma_3^2 = var\{x_t\} \, .$$

[6]

Hint: in Case 3, use (3) and (4) to show that $cov\{x_t\} = a^2\bar{P} + 1$.

Comment on the relative magnitudes of σ_1^2, σ_2^2 and σ_3^2. [2]

5. (i): The acidity level of liquid in a polluted reservoir at time t is modelled as a scalar random variable x_t, $t = 0$ and 1. It is assumed, on the basis of earlier experiments, that

$$x_0 \sim N(m, \sigma^2) .$$

in which m and σ^2 are given numbers ($\sigma^2 > 0$).

An exact measurement is taken of the acidity level x_1 at time $t = 1$. Based on this measurement, we seek to classify the pollution as non-biological, in which case it is constant, or biological, in which case it time varying. Two hypotheses concerning x_1 need to be considered.

$(H0): \quad x_1 = x_0 \quad$ (non-biological pollution), and

$(H1): \quad x_1 = kx_0 \quad$ (biological pollution) .

in which k, the rate of the biological reaction, is a constant satisfying

$$k > 1.$$

Regarding $(H0)$ as the null-hypothesis, show that a Neyman-Pearson decision rule for accepting $(H1)$, at the significance level α, has the form: accept $(H1)$ if

$$|x - m| \geq d .$$

Here d is chosen so that

$$F(d/\sigma) - F(-d/\sigma) \;=\; \alpha,$$

[7]

in which $F(x)$ is the cummulative distribution function for the unit normal distribution $N(0,1)(x)$:

$$F(x) \;=\; \int_{-\infty}^{x} N(0,1)(x)dx .$$

Derive a formula for the power of the test. [5]

(ii): A feedback control system is modelled by the equations

$$\begin{cases} x_t &= Ax_{t-1} + bu_{t-1} + v_{t-1} \\ u_t &= \beta k^T x_t \\ y_t &= h^T x_t + w_t , \end{cases}$$

the data for which comprise: an $n \times n$ matrix A and n-vectors k and h. $\{v_t\}$ and $\{w_t\}$ are unit variance, white noise processes, independent of the initial state x_0. Here, β is a scalar parameter. Under normal operations, $\beta = 1$. If the control actuator fails however, $\beta = 0$.

Construct a scheme for detecting the occurence of an actuate fault, based on the extended Kalman filter. Briefly describe the principles behind the scheme. [8]

Estimation + Fault Detection, 2012. Model Answers

1 (i) $y_k + a y_{k-1} = e_t + 2e_{t-1}$.

$\times e_t$ and take $E\{..\}$ $\Rightarrow R_{ye}(0) = 1$

$y_k = -a y_{k-1} + e_t + 2e_{t-1}$. Square both sides and take $E\{..\}$ $\Rightarrow$

$$R_y(0) = a^2 R_y(0) - 4a R_{ye}(0) + 5 \Rightarrow (1-a^2) R_y(0) = 5 - 4a$$

Hence $\underline{R_y(0) = (5-4a)/(1-a^2)}$.

Then $\times y_{k-1}$ and take $E\{..\}$ $\Rightarrow R_y(1) + a R_y(0) = 2R_{ye}(0) = 2$

$\Rightarrow \underline{R_y(1) = 2 - a(5-4a)/(1-a^2)}$

Then $\times y_{k-\ell}$ and $E\{..\}$ $\Rightarrow R_y(k) = -a R_y(k-1)$, $k \geq 2$

So $\underline{R_y(k) = (-a)^{k-1} R_y(1)}$ for $k \geq 2$. Also $\underline{R_y(k) = R_y(-k)}$.

The spectral density is

$$\Phi_{yy}(\omega) = \frac{(1+2e^{-j\omega})(1+2e^{+j\omega})}{(1+ae^{-j\omega})(1+ae^{j\omega})} = \frac{5+4\cos\omega}{(1+a^2)+2a\cos\omega}.$$

We know $$\left.\frac{5+4\cos\omega}{(1+a^2)+2a\cos\omega}\right|_{\omega=60^\circ} = \frac{5+4/\sqrt{3}}{(1+a^2)+\frac{2}{\sqrt{3}}a} = 4$$

$\Rightarrow a^2 + \frac{2a}{\sqrt{3}} - \left(\frac{1}{4} + 4/\sqrt{3}\right) = 0 \quad \Rightarrow \quad \underline{a = \frac{1}{2}}$ (positive root)

(ii) Continuous state space model: $\dot{x}(t) = \overbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}^{\tilde{A}} x(t) + \overbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}^{\tilde{b}} w(t)$

But $\exp \tilde{A}h = I + \begin{bmatrix} 0 & h \\ 0 & 0 \end{bmatrix} + 0 + . = \underline{\begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix}} =: A$ — (1)

Also $\int_0^h (\exp \tilde{A}t)\tilde{b}\tilde{b}^T(\exp \tilde{A}t)^T dt$

$= \int_0^h \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} dt = \int_0^h \begin{bmatrix} t^2 & t \\ t & 1 \end{bmatrix} dt$

Discrete time description: $= \underline{\begin{bmatrix} h^3/3 & h^2/2 \\ h^2/2 & h \end{bmatrix}} =: Q$ (2)

Then

$$x(t+h) = A x(t) + n, \quad \operatorname{cov}\{n\} = Q$$

with A and Q given by (1) and (2).

Since $E\{n\} = 0$, and n is indep of $x(t)$

$E\{y(t+h) \mid x(t)\} = [1\ 0] A x(t) = \underline{y(t) + h\dot{y}(t)}$

and

$\operatorname{cov}\{x(t+h) \mid x(t)\} = [1\ 0]\, Q \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \underline{\frac{h^3}{3}}$.

2. Bayes' rule asserts $p(x \mid y=+1) = \dfrac{p(y=+1 \mid x)\, p(x)}{p(y=+1)}$. — (*)

But

$$p(y=+1) = \int_{-\infty}^{+\infty} p(y=+1 \mid x)\, p(x)\, dx$$
$$= \beta \int_{-\infty}^{0} p(x)\, dx + (1-\beta) \int_{0}^{\infty} p(x)\, dx = \tfrac{1}{2}\beta + \tfrac{1}{2}(1-\beta) = \tfrac{1}{2}$$

and $p(y=+1 \mid x) = \begin{cases} \beta & \text{if } x < 0 \\ (1-\beta) & \text{if } x \geq 0. \end{cases}$

It follows from (*) that $p(x \mid y=+1) = \begin{cases} 2\beta\, p(x) & \text{if } x < 0 \\ 2(1-\beta)\, p(x) & \text{if } x \geq 0 \end{cases}$

By symmetry $p(x \mid y=-1) = \begin{cases} 2(1-\beta)\, p(x) & \text{if } x < 0 \\ 2\beta\, p(x) & \text{if } x \geq 0 \end{cases}$

The least squares estimate $\hat{x}$ of x given $y=+1$ coincides with the conditional mean

$$E[x \mid y=+1] = 2\beta \int_{-\infty}^{0} x\, p(x)\, dx + 2(1-\beta) \int_{0}^{\infty} x\, p(x)\, dx$$
$$= \big(-2\beta + 2(1-\beta)\big) \int_{0}^{\infty} p(x)\, dx = \frac{2-4\beta}{(2\pi)^{1/2}} \cdot \sigma$$

By symmetry, $E\{x \mid y=-1] = -\dfrac{(2-4\beta)}{(2\pi)^{1/2}}\, \sigma$

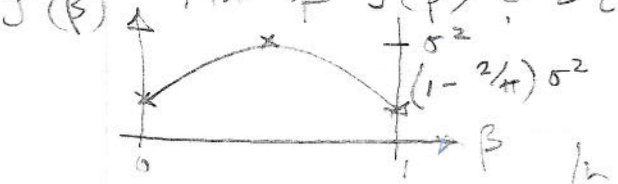
The mean square estimation error (when $y=+1$) coincides with the conditional variance of x given $y=+1$:

$$\operatorname{var}\{x \mid y=+1\} = E[x^2 \mid y=+1] - (E[x \mid y=+1])^2$$
$$= 2\beta \int_{-\infty}^{0} x^2 p(x)\, dx + 2(1-\beta) \int_{0}^{\infty} x^2 p(x)\, dx - \frac{(2-4\beta)^2}{2\pi}\sigma^2$$
$$= 2\int_{0}^{\infty} x^2 p(x)\, dx - \frac{(2-4\beta)^2}{2\pi}\sigma^2 = \operatorname{Var}\{x\} - \frac{(2-4\beta)^2}{2\pi}\sigma^2 = \left(1 - \frac{(2-4\beta)^2}{2\pi}\right)\sigma^2$$

By symmetry

$\operatorname{var}\{x \mid y=-1\} = \left(1 - \dfrac{(2-4\beta)^2}{2\pi}\right)\sigma^2$, also.

Plot of $J(\beta) = E[|x - \hat{x}|^2] = \operatorname{var}\{x \mid y\}$ $(y = -1 \text{ or } +1)$



Comment: when $\beta = \frac{1}{2}$

$p(x \mid y=-1) = p(x \mid y=+1) = p(x)$

In this case x is independent of y, so the measurement is useless and $J(\beta = \frac{1}{2}) = \operatorname{Var}\{x\} = \sigma^2$.

3. $z = x + Ky$ implies $E[z] = m_1 + Km_2$, $\mathrm{cov}\{z\} = P_1 + P_2$, so $p(z) = N(m_1 + Km_2, P_1 + KP_2K^T)$.

Also, $p(z) = \int p(z|y)\,p(y)\,dy$

$= \frac{1}{c}\int \exp\{-\frac{1}{2}\|(z-Ky)-m_1\|^2_{P_1^{-1}}\} \times \exp\{-\frac{1}{2}\|y-m_2\|^2_{P_2^{-1}}\}\,dy$

$= \frac{1}{c}\int \exp\{-\frac{1}{2}\|z-(m_1+Ky)\|^2_{P_1^{-1}}\} \times \exp\{\ldots\}\,dy = \int N(m_1+Ky, P_1)(z) \times N(m_2, P_2)(y)\,dy$

Equating the formula for $p(z)$ gives

$$\underbrace{\int N(m_1 + Ky, P_1)(z) \times N(m_2, P_2)(y)\,dy = N(m_1 + Km_2, P_1 + KP_2K^T)}_{(1)}$$

To derive smoothing equations:

<u>Step (i)</u> $p(x_t | y_{1:t+1}) = \int p(x_t | y_{1:t}, y_{t+1}, x_{t+1})\,p(x_{t+1} | y_{1:t+1})\,dx_{t+1}$

But $p(y_{t+1} | y_{1:t}, x_{t+1}, x_t) = p(y_{t+1} | x_{t+1})$ (by Markov properties)

So $p(x_t | y_{1:t}, x_{t+1}, y_{t+1}) = \dfrac{p(y_{t+1} | y_{1:t}, x_{t+1}, x_t)\,p(x_t | y_{1:t}, x_{t+1})}{p(y_{t+1} | y_{1:t}, x_{t+1})}$

$= p(x_t | y_{1:t}, x_{t+1})$. It follows

$$p(x_t | y_{1:t}, x_{t+1}, y_{t+1}) = p(x_t | y_{1:t}, x_{t+1}),$$ so

$$\underline{p(x_t | y_{1:t+1}) = \int p(x_t | y_{1:t}, x_{t+1})\,p(x_{t+1} | y_{1:t+1})\,dx_{t+1}}.$$

<u>Step (ii)</u> Consider $x_{t+1} = Fx_t + w_t$ as a measurement of $x_t \sim N(\hat{x}_t, P_t)$.

Then $E[x_t | y_{1:t}] = \hat{x}$, $\mathrm{cov}\{x_t | y_{1:t}\} = P_t$, $E[x_{t+1} | y_{1:t}] = F\hat{x}_t$

$\mathrm{cov}\{x_{t+1}\} = FP_tF + Q^s$, $\mathrm{cov}\{x_t, x_{t+1}\} = P_tF^T$.

Since the conditional mean and covariance of x_t, given x_{t+1} (and $y_{1:t}$) coincide with the linear least squares mean and covariance

$$\underline{E[x_t | y_{1:t}, x_{t+1}] = \hat{x}_t + P_tF^T[FP_tF^T + Q^s]^{-1}(x_{t+1} - F\hat{x}_t)}$$

and

$$\underline{\mathrm{cov}\{x_t | y_{1:t}, x_{t+1}\} = P_t - P_tF^T[FP_tF^T + Q^s]^{-1}FP_t}$$

<u>Step (iii)</u> From step (i) and (ii) and (1)

$N(\hat{x}_{t|t+1}, P_{t|t+1})(x_t) = \int N(E[x_t | y_{1:t}, x_{t+1}], \mathrm{cov}\{x_t | y_{1:t}, x_{t+1}\})(x_t) \times N(\hat{x}_{t+1}, P_{t+1})(x_{t+1})\,dx_{t+1}$

$= \int N(\hat{x}_t + K_{t|t+1}(x_{t+1} - F\hat{x}_t), P_t - K_{t|t+1}FP_t)(x_t) \times N(\hat{x}_{t+1}, P_{t+1})(x_{t+1})\,dx_{t+1}$

(in which $K_{t|t+1} = P_tF^T[FP_tF^T + Q^s]^{-1}$)

$= N(\hat{x}_t + K_{t|t+1}(\hat{x}_{t+1} - F\hat{x}_t), P_t - K_{t|t+1}FP_t + K_{t|t+1}P_{t+1}K^T_{t|t+1}$

4 (a) P_t and K_t converge to limiting values P_t and K_t is $\underbrace{(H, F)}$ is observable

The standard equations give

$S_t = F P_{t-1} F^T + Q^s$, $P_t = S_t - S_t H^T (H S_{t-1} H^T + Q^m)^{-1} H S_t$

and $K_t = S_t H^T (H S_t H^T + Q^m)^{-1}$.

Setting $S_t = \bar{S}$, $P_t = P_{t-1} = \bar{P}$, $K_t = \bar{K}$ gives

$\left\{ \begin{array}{l} \bar{P} = \bar{S} - \bar{S} H^T (H \bar{S} H^T + Q^m)^{-1} H \bar{S} \\ \text{where } \bar{S} = F \bar{P} F^T + Q^s \end{array} \right.$ and $\bar{K} = \bar{S} H^T (H \bar{S} H + Q^m)^{-1}$

(b) Case 1 : $u = 0$, $x_{t+1} = a x_t + v_t$. Hence $E\{x_{t+1}^2\} = \frac{1}{2} E |x_t + v_t|^2$

Hence $\underbrace{E\{x_t^2\}}_{\sigma_1^2} = a^2 \underbrace{E\{x_t^2\}}_{\sigma_1^2} + \underbrace{E\{v_t^2\}}_{1}$ or $\sigma_1^2 = 30/19$

Case 2 : $u = x$. Now $x_{t+1} = 0 + v_t$. Hence $E\{x_t^2\} = \text{var}\{v_t\}$

$\Rightarrow \sigma_2^2 = 1$

Case 3 : Standard theory gives

$\hat{x}_t = a \hat{x}_{t-1} - a \hat{x}_{t-1} + K_t (y_t - [a \hat{x}_{t-1} - \hat{a}_{t-1}])$

or $\hat{x}_t = K_t y_t$ and $\text{var}\{x_t | y_t, y_{t-1}, \} = P_t$

where $a = \sqrt{11/30}$

The system is observable. So $K_t \to K$, $P_t \to P$, where K, P (and S) satisfy :

$P = S - \frac{S^2}{S+1} = \frac{S}{S+1}$ $\quad S = a^2 P + 1$

So $\frac{S}{S+1} = \frac{1}{a^2}(S-1) \Rightarrow S^2 - 1 = a^2 S \Rightarrow S = \frac{6}{5}$

Then $P = 6/11$

and $K = S \frac{1}{(S+1)} = 6/11$

The system equations give : $x_{t+1} = a[x_t - \hat{x}_t] + v_t$

So $E\{x_t^2\} = a^2 E[|x_t - \hat{x}_t|^2] + 0 + 1 = a^2 P + 1$

We deduce that $\sigma_3^2 = E\{x_t^2\} = \frac{11}{30} \times \frac{6}{11} + 1 = \frac{6}{5}$

We see that $\sigma_2^2 < \sigma_3^2 < \sigma_1^2$

i.e. the minimum variance controller reduces the no-control state variance. If the minimum variance controller is implemente via state estimation, there is some improvement, but not as much as that obtained when the state is measured exactly.

5 (a) $(H_0): x_1 \sim N(m, \sigma^2)$, $(H_1): x_1 \sim N(km, (k\sigma)^2)$. So the log. likelihood ratio

$$LLR(x_1) = c_1 + \tfrac{1}{2}(x_1 - m)^2/\sigma^2 - \tfrac{1}{2}(x_1 - km)^2/(k\sigma)^2$$
$$= c_2[c_3 + x_1^2(k^2-1) - 2x_1 m(k^2-1)] \quad \text{(for some constants } c_1, c_2, \ldots$$
$$= c_4[\,|x_1 - m|^2 + c_5\,]$$

It follows $LLR(x_1) \le \text{constant} \iff |x_1 - m| \le \text{constant}$

The N-P test is then: accept (H_0) if $\underbrace{|x_1 - m| \le d}$

where d is adjusted s.t. $P[\,|x_1 - m| \le d \mid x_1 \sim N(m, \sigma^2)\,] = \alpha$ — (1)

But $|x_1 - m| \le d \iff |x'| \le d/\sigma$ where $x' = \frac{x_1 - m}{\sigma} \sim N(0,1)$

So (1) implies: $\underline{F(d/\sigma) - F(-d/\sigma) = \alpha}$

Power of test $= 1 - P[\,|x_1 - m| \le d \mid x_1 \sim N(km, (k\sigma)^2)\,]$

$$= 1 - P\left[\left|\frac{x_1 - km}{(k\sigma)} - \frac{m}{k\sigma} + \frac{m}{\sigma}\right| \le d/k\sigma\right]$$

$$= \underline{1 - \left(F\left(\frac{d}{k\sigma} - \frac{k-1}{k}\frac{m}{\sigma}\right) - F\left(-\frac{d}{k\sigma} - \frac{k-1}{k}\frac{m}{\sigma}\right)\right)}$$

(b) x_t and y_t are governed by the eqns: $x_t = (A - \beta k^T)x_{t-1} + v_{t-1}$, $y_t = h^T x_t + w_t$

Regard β as a state variable:

$$(x_t, \beta_t) = ((A - \beta_{t-1} k^T)x_{t-1}, \beta_{t-1}) + (v_{t-1}, 0); \quad y_t = h^T x_t + w_t.$$

Assume prior distribution:

$x_0 \sim N(\hat{x}_0, P_0)$ and $\beta_0 \sim N(1, \sigma^2)$ (x_0, β_0 indep) σ small.

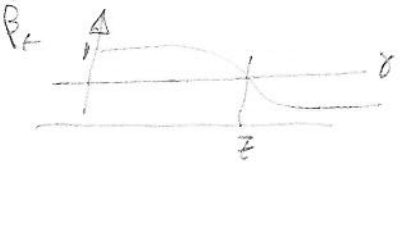
Now estimate x_t and β_t using the extended Kalman filter, based on the linearized equations

$$\begin{pmatrix} x_t \\ \beta_t \end{pmatrix} = \overbrace{\begin{bmatrix} A - \hat{\beta}_{t-1} k^T & -k^T \hat{x}_{t-1} \\ 0 & 1 \end{bmatrix}}^{\tilde{F}} \begin{pmatrix} x_{t-1} - \hat{x}_{t-1} \\ \beta_{t-1} - \hat{\beta}_{t-1} \end{pmatrix} + \begin{bmatrix} A - \hat{\beta}_{t-1} h^T \hat{x}_{t-1} \\ \hat{\beta}_{t-1} \end{bmatrix} + \overbrace{\begin{bmatrix} v_{t-1} \\ 0 \end{bmatrix}}^{\sim \tilde{Q}^S}$$

$$y_t = \underbrace{[h^T ; 0]}_{\tilde{h}^T} \begin{bmatrix} x_t - (A - \hat{\beta}_{t-1} k^T \hat{x}_{t-1}) \\ \beta_t - \hat{\beta}_{t-1} \end{bmatrix} + [h^T ; 0] \begin{bmatrix} (A - \hat{\beta}_{t-1}) k^T \hat{x}_{t-1} \\ \hat{\beta}_{t-1} \end{bmatrix} + \underbrace{w_t}_{\sim \sigma_0}$$

This gives $\begin{pmatrix} \hat{x}_t \\ \hat{\beta}_t \end{pmatrix} = \begin{pmatrix} A - \hat{\beta}_{t-1} h^T \hat{x}_{t-1} \\ \hat{\beta}_{t-1} \end{pmatrix} + \tilde{K}_t \left(y_t - h^T (A - \hat{\beta}_{t-1} k^T \hat{x}_{t-1}) \right)$

Here, the Kalman gain $\tilde{K}_t$ and error covariance are calculated from the standard formulae, with $\tilde{F}, \tilde{h}^T, \tilde{Q}^S, \sigma_0^2$



Assume a fault has occured when estimate $\hat{\beta}_t$ first falls beneath a threshold γ $(\in (0,1))$