[E2.8 (Maths 3) 2012]

B.ENG. AND M.ENG. EXAMINATIONS 2012

PART II Paper 3 : MATHEMATICS (ELECTRICAL AND INFORMATION SYSTEMS ENGINEERING)

Date Thursday 7th June 2012 2.00 - 4.00 pm

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.

Answer FOUR questions.

A mathematical formulae sheet is provided.

[*Before starting, please make sure that the paper is complete; there should be 6 pages, with a total of SIX questions. Ask the invigilator for a replacement if your copy is faulty.*]

1. (i) Consider the function

$$f(x,\, y) \;=\; x^3 \;-\; 6x^2 \;+\; 7x \;-\; y^2 \;+\; 2yx \;-\; 3\;.$$

Find the gradient vector $\nabla f \;=\; \left(\frac{\partial f}{\partial x}\,,\;\frac{\partial f}{\partial y}\right).$

Use this to identify the direction of the constant contour at $(0,\,0)$. Identify the locations of the stationary points of $f(x,\,y)$. Classify each stationary point as a maximum, minimum or saddle point.

(ii) Sketch the function $g(x,\,y) = x^6 + y^6$.

Write down the location of its stationary point.

Explain, in words, why classifying this stationary point is difficult and explain what would be required in order to provide a classification.

2. (i) Consider the function u below:

$$u \;=\; x^3\,/\,6 \;+\; 4x^2 \;-\; xy^2\,/\,2 \;-\; 4y^2\;.$$

Show that it satisfies Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} \;+\; \frac{\partial^2 u}{\partial y^2} \;=\; 0\;.$$

Use the Cauchy-Riemann equations to construct a conjugate function v.

Construct from u and v an analytic function $f(z)$.

Show that $f(z)$ can be put in the form

$$f(z) \;=\; az^3 \;+\; bz^2 \;+\; c$$

and find a and b (c is an arbitrary constant).

(ii) Consider the map $w = z^2$ (which is conformal everywhere except at $z = 0$). The straight line $u = 1$ (for all v) in the w-plane meets the straight line $v = 0$ (for all u) at right angles.

Explain what the angle of intersection of these curves is when they are mapped to the z-plane.

PLEASE TURN OVER

3. (i) Find the residue of

$$F(z) = \frac{1}{(1+z^2)} \quad \text{at} \quad z = i \ .$$

(ii) Consider the function

$$\frac{1}{(z^3-8)\,(z-1)^2} \ .$$

State the location of all of its poles in the z-plane and state if they are single or multiple poles.

Evaluate the contour integral

$$\oint \frac{1}{(z^3-8)\,(z-1)^2} \, dz \ ,$$

when the contour is a counter-clockwise circle of radius 1 with centre at $z = \frac{3}{2}$.

Sketch the contour and the location of the poles.

Evaluate the integral when the contour is a counter-clockwise circle of radius 1 with centre $z = 100$.

Hint: *The residue of a complex function $f(z)$ at a pole $z = a$ of multiplicity m is given by the expression*

$$\lim_{z \to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \ \{(z-a)^m f(z)\} \ .$$

4. (i) Defining the convolution of $f(t)$ with $g(t)$ as $\int_{-\infty}^{\infty} f(t')\, g\,(t - t')dt'$, find the convolution of $f(t) = \sum_{n=-\infty}^{\infty} \delta(t - t_n)$ (the Shannon sampling function) with a function $g(t)$.

Show that the Fourier transform of this convolution is $\sum_{n=-\infty}^{\infty} e^{-i\omega t_n}\, \overline{g}(\omega)$.

(ii) It is the case that:

$$\int_{-\infty}^{\infty} \frac{e^{ip\omega}}{\omega}\, d\omega \;=\; \begin{cases} +i\pi & p > 0\ ; \\ -i\pi & p < 0\ . \end{cases}$$

Consider a counter-clockwise contour integral in the z-plane $\oint_C \frac{e^{ipz}}{z}\, dz$ where $p > 0$ and where C is a semi-circular contour in the upper half-plane of radius R, with a suitable semi-circular indentation into the upper half-plane of radius r around $z = 0$. Use Jordan's Lemma to show that in the limits $R \to \infty$ and $r \to 0$

$$\int_{-\infty}^{\infty} \frac{e^{ip\omega}}{\omega}\, d\omega \;=\; +\,i\,\pi \quad \text{where } \ w,\, p \ \text{ are real and } \ p > 0.$$

Explain briefly why a different contour must be used for $p < 0$.

Use the above integral to show that the inverse Fourier transform of $F(\omega) = \frac{\cos\omega}{\omega}$ is:

$$F(t) \;=\; \begin{cases} +i/2 & t > 1\ ; \\ 0 & -1 \le t \le 1\ ; \\ -i/2 & t < -1\ . \end{cases}$$

It might be useful to consider the three cases when $t > 1$, $-1 \le t \le 1$ and $t < -1$ separately.

PLEASE TURN OVER

5. (i) Show that the Fourier transform of $f(t+a)$ is $e^{ia\omega}\,\overline{f}(\omega)$ and that the Fourier transform of $e^{\alpha t}f(t)$ is $\overline{f}(\omega+i\alpha)$ (where α can be real, complex or imaginary).

(ii) Using Plancherel's integral relation between $f(t)$ and $g(t)$:

$$\int_{-\infty}^{\infty} f(t)g^*(t)dt \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{f}(\omega)\overline{g}(\omega)d\omega$$

and using Fourier transforms, show that

$$\int_{-\infty}^{\infty} e^{-|t|}\cos(\omega_0 t \;+\; \phi)dt \;=\; \frac{e^{i\phi}+e^{-i\phi}}{1+\omega_0^2}\;, \tag{1}$$

where ω_0 and ϕ are real constants. You may need to use the definition

$$\delta(\Omega) \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\pm i\Omega\tau}\;d\tau\;.$$

Find the right hand side of equation (1) when $\phi=-\pi/2$.

Why, when $\phi=-\pi/2$, is this result obvious from inspection of the integrand of the left hand side of equation (1)?

6. Show that the Laplace transform of $\dfrac{d^3x}{dt^3}$ is

$$s^3\overline{x}(s) \;-\; s^2x(0) \;-\; s\frac{dx(0)}{dt} \;-\; \frac{d^2x(0)}{dt^2}\;,$$

where you may use the result that the Laplace transform of $\dfrac{d^2x}{dt^2}$ is

$$s^2\,\overline{x}(s) \;-\; sx(0) \;-\; \frac{dx(0)}{dt}\;.$$

Recall the notation that $\overline{x}(s)$ *is the Laplace transform of* $x(t)$ *and*

$$x(0) = x(t=0), \quad \frac{dx(0)}{dt} = \frac{dx(t=0)}{dt}, \quad \frac{d^2x(0)}{dt^2} = \frac{d^2x(t=0)}{dt^2}\;.$$

Prove that the Laplace transform of t^2 is $2/s^3$ (for $Re(s) > 0$).

Find the Laplace transform of $e^{-2t}\,t^2$.

Consider the third order differential equation

$$\frac{d^3x}{dt^3} \;+\; 6\,\frac{d^2x}{dt^2} \;+\; 12\,\frac{dx}{dt} \;+\; 8x \;=\; f(t)\;,$$

with $x = \dfrac{dx}{dt} = \dfrac{d^2x}{dt^2} = 0$ at $t = 0$.

Show that

$$x(t) \;=\; \frac{1}{2}\int_0^t e^{-2v}v^2 f(t-v)dv\;.$$

END OF PAPER

	EXAMINATION QUESTIONS/SOLUTIONS 2011-2012	Course EE II (3)
Question 1.	TOPIC Real Valued Functions	Marks & seen/unseen

Parts

a) - $\frac{\partial f}{\partial x} = 3x^2 - 12x + 7 + 2y$; $\frac{\partial f}{\partial y} = -2y + 2x$. — 2

- At $(0,0)$ $\nabla f = (7,0)$. ∇f is perpendicular to the constant contours. — 2

- ∇f is parallel to x-axis so constant contour is parallel to the y-axis at $(0,0)$. — 2] 4

- stationary points have $\nabla f = \underline{0}$ it follows that $x=y$ and so $3x^2 - 10x + 7 = (3x-7)(x-1) = 0$. — 2

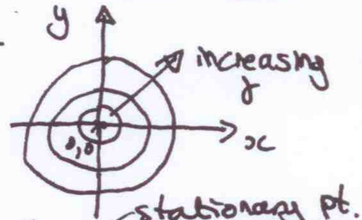
- Therefore $(7/3, 7/3)$ and $(1,1)$ are stationary pts. — 2] 4

- $f_{xx} = 6x - 12$; $f_{xy} = 2$; $f_{yy} = -2$

- at $(1,1)$ $f_{xx} = -6 < 0$ and $f^2_{xy} - f_{xx}f_{yy} = 4 - 12 = -8 < 0$
$\Rightarrow (1,1)$ is a Maximum. — 2

- at $(7/3, 7/3)$ $f^2_{xy} - f_{xx}f_{yy} = 4 - 2 \cdot 2 = 8 > 0$
$\Rightarrow$ saddle point. — 2] 4

b) -



minimum at $(0,0)$

Good sketch + statement of minimum — 2

- This is hard to classify as all of its second differentials are zero at the origin, so the classification tests — 2

- for stationarity are unclear. In order to classify this one needs to ~~also~~ consider higher order terms in a Taylor expansion about $(0,0)$. — 2

Show test fails

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	EXAMINATION QUESTIONS/SOLUTIONS 2011-2012	Course EE II (3)
Question 2	TOPIC Complex Variables	Marks & seen/unseen

Parts

a) – $u_x = \frac{3x^2}{6} + 8x - \frac{y^2}{2}$; $u_y = -\frac{2yx}{2} - 8y$ — 4

$u_{xx} = x+8$; $u_{yy} = -x-8$

$u_{xx} + u_{yy} = 0.$

– $v_y = u_x = \frac{x^2}{2} - \frac{y^2}{2} + 8x$ ① ; $v_x = -u_y = yx + 8y$ ② — 7

$V = \frac{x^2y}{2} - \frac{y^3}{6} + 8xy + A(x)$ integrating ① w.r.t. y

$V = \frac{yx^2}{2} + 8yx + B(y)$ integrating ② w.r.t. x.

$\Rightarrow A(x) = c$; $B(y) = -y^3/6 + c$.

$\Rightarrow V = x^2y/2 - y^3/6 + 8xy + c.$

– $f(z) = \frac{x^3}{6} - \frac{y^2x}{2} + 4x^2 - 4y^2 + i\left(\frac{x^2y}{2} - \frac{y^3}{6} + 8xy + c\right)$ — 2

+1 intermediate step

– $f(z) = \frac{1}{6}z^3 + 4z^2 + c$; $a = 1/6$; $b = 4$. — 3] 5

b) – Since the map is conformal at the intersection of $u=1$ and $v=0$ and conformal maps are angle preserving then since the curves meet at right-angles in the w-plane the same is true in the z-plane. — 2, 2] 4

or – $u=1$ and $v=0$ are "curves of constant u and v" and it is true that such curves meet at right angles in z (in regions of analyticity).

or – $x^2-y^2+2ixy = u+iv$; $v=0 \Rightarrow x=0$ and $y=0$ curves in z. $u=1 \Rightarrow x^2-1 = y^2$. These curves intersect at $z=(1,0)$. Accept any reasonable argument that they meet at right angles.

(including a sketch)

(if candidates find a circle/ellipse $\Rightarrow$ 2/4)

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	EXAMINATION QUESTIONS/SOLUTIONS 2011-2012	Course EE II (3)
Question 3	TOPIC Contour Integrals	Marks & seen/unseen

Parts

a) $f(z) = \frac{1}{1+z^2} = \frac{1}{(z-i)(z+i)}$ so $\lim_{z\to i} \frac{z-i}{(z-i)(z+i)} = \frac{1}{2i}$ — 2

(simple pole at $z=i$)

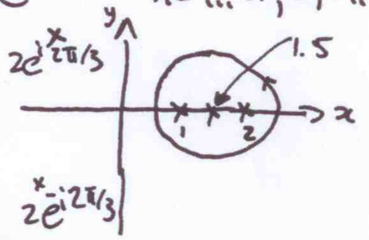
or $f(z) = \frac{h(z)}{g(z)} = \frac{1}{1+z^2}$ so $\frac{h(z)}{g'(z)} = \frac{1}{2z} = \frac{1}{2i}$ at $z=i$.

simple pole.

b) Pole of multiplicity 2 at $z=1$ — 2

Pole of multiplicity 1 at $z^3=8$ — must get all three poles correct to score — 1

$z^3 = 8e^{i\,2\pi n}$ $n = \ldots -1, 0, 1 \ldots$ $z = 2, 2e^{\pm i 2\pi/3}$ — 2 (3)



- if the calculated poles, even wrong ones, are displayed then award - but need either contour sense or axes labels. — 1

Residue at 1 : $\frac{1}{1!}\frac{d}{dz}\left[\frac{(z-1)^2}{(z-1)^2}\frac{1}{(z^3-8)}\right] = \frac{-3z^2}{(z^3-8)^2} = \frac{-3}{49}$ — 3

Residue at 2 : write $(z^3-8) = (z-2)(z^2+2z+4)$ — 3

$\Rightarrow$ Residue $\lim_{z\to 2} \frac{(z-2)}{(z-2)(z^2+2z+4)(z-1)^2} = \frac{1}{(4+4+4).1} = \frac{1}{12}$.

Residue theorem states that the value of the integral is $2\pi i \times$ (sum of residues enclosed) — if not stated clearly but used give 1/2 — 2

$2\pi i \times \left(\frac{1}{12} - \frac{3}{49}\right) = i\pi \frac{13}{294}$ — 1 mark correct use, 1 mark correct answer — 2

No poles are enclosed by this contour. Cauchy's theorem tells us this integral is zero. — 2

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	EXAMINATION QUESTIONS/SOLUTIONS 2011-2012	Course EE II (3)
Question 4	TOPIC Fourier Transforms	Marks & seen/unseen

Parts

a) $\int_{-\infty}^{\infty} f(t')g(t-t')dt' = \int_{-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta(t'-t_n)g(t-t')dt' = \sum_{n=-\infty}^{\infty} g(t-t_n)$ — 2

$\int_{-\infty}^{\infty} e^{-i\omega t}\sum_{n=-\infty}^{\infty} g(t-t_n)dt = \sum_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\omega(\tau_n+t_n)} g(\tau_n)d\tau_n = \sum_{n=-\infty}^{\infty} e^{-i\omega t_n}\bar{g}(\omega)$ — 2

b) $p>0$ consider contour

$Re(z)=\omega$

Cauchy's theorem

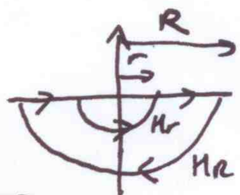
$\oint_C \frac{e^{ipz}}{z}dz = 0 = \lim_{\substack{R\to\infty \\ r\to 0}}\left[\int_{H_R}\frac{e^{ipz}}{z}dz + \int_{H_r}\frac{e^{ipz}}{z}dz + \int_{-\infty}^{\infty}\frac{e^{ip\omega}}{\omega}d\omega\right]$ ⊛ — 2, 3 (5)

By Jordan's Lemma (since $p>0$; $\frac{1}{z}\to 0$ as $R\to\infty$; no singularities) $\int_{H_R}\to 0$. — 2

On contour $\int_{H_r}$ substitute $z=re^{i\theta}$
as $r\to 0$ $e^{ipz} = e^{ipre^{i\theta}} \to \boxed{e^0}$ so $\int_{H_r}\frac{e^{ipz}}{z}dz = \int_{\pi}^{0}\frac{1\, rie^{i\theta}d\theta}{re^{i\theta}}$ — 2
$= -i\pi$.

From ⊛
$\int_{-\infty}^{\infty}\frac{e^{ip\omega}}{\omega}d\omega = i\pi$. $p>0$.

For $p<0$ consider contour



~~Same arguments show~~ $\int_{H_R}\to 0$ — 2
~~and $\int_{-\pi}^{0}\frac{rie^{i\theta}d\theta}{re^{i\theta}} = \int_{H_r H_R} = i\pi$~~

or For $P<0$ cannot use Jordan's Lemma with a contour in the upper half plane. ~~Need $e^{ip\omega}$~~

or Need $e^{ipz}\to 0$ so contour in lower half plane.

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Question 4 cont

TOPIC Fourier Transforms.

Marks & seen/unseen

Parts

b) $F(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\frac{\cos\omega}{\omega}\,d\omega$ — 2

$$= \frac{1}{2\pi}\Big[\underbrace{\int_{-\infty}^{\infty}\frac{e^{i(t+1)\omega}}{2\omega}\,d\omega}_{I_1} + \underbrace{\int_{-\infty}^{\infty}\frac{e^{i(t-1)\omega}}{2\omega}\,d\omega}_{I_2}\Big]$$

From above integral. when $t>1$

$t+1>0$ and $t-1>0$ so $I_1 = I_2 = \frac{i\pi}{2}$ $F(t)$ — 2

$\Rightarrow F(t) = \frac{1}{2\pi}\times\frac{2i\pi}{2} = i/2$. — 3

When $-1\le t\le 1$ $I_1 = \frac{i\pi}{2}$; $I_2 = -i\pi/2$ $\Rightarrow F(t) = 0$ — 1

When $t<-1$ $I_1 = I_2 = -i\pi/2$ $\Rightarrow F(t) = -i/2$.

EXAMINATION QUESTIONS/SOLUTIONS 2011-2012	Course EE II (3)

Question 5 — TOPIC: Fourier Transforms

Marks & seen/unseen

Parts

a) – $\int_{-\infty}^{\infty} e^{-i\omega t} f(t+a)\,dt = \int_{-\infty}^{\infty} e^{-i\omega(\tau - a)} f(\tau)\,d\tau = \bar{f}(\omega)\, e^{i\omega a}$ **3**

change of variables $t + a = \tau$.

– $\int_{-\infty}^{\infty} e^{-i\omega t} e^{\alpha t} f(t)\,dt = \int_{-\infty}^{\infty} e^{-i\omega' t} f(t)\,dt = \bar{f}(\omega') = \bar{f}(\omega + i\alpha)$ **3**

require this line (or equiv) for full marks?

writing $+i(\omega + i\alpha) = +i\omega' = i\omega - \alpha$

$\omega' = \omega + i\alpha$

$-i\omega' = -i\omega + \alpha$

b) – $FT[e^{-|t|}] = \int_{0}^{\infty} e^{-t} e^{-i\omega t}\,dt + \int_{-\infty}^{0} e^{-|t|} e^{-i\omega t}\,dt$ **4**

$= I_1 + I_2$.

defining $t' = -t$; $I_2 = -\int_{\infty}^{0} e^{-t'} e^{i\omega t'}\,dt' = \int_{0}^{\infty} e^{-t'} e^{i\omega t'}\,dt'$

$= \left[\frac{1\, e^{-t' + i\omega t'}}{i\omega - 1}\right]_0^{\infty} = \frac{1}{1 - i\omega}$

$I_1 = \frac{1}{i\omega + 1}$; $I_1 + I_2 = \frac{2}{1+\omega^2}$.

– $FT[\cos(\omega_0 t + \phi)] = \int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{2} e^{i(\omega_0 t + \phi)}\,dt + \int_{-\infty}^{\infty} \frac{e^{-i\omega t}\, e^{-i(\omega_0 t + \phi)}}{2}\,dt$ **4**

$= \pi[e^{i\phi}\delta(\omega - \omega_0) + e^{-i\phi}\delta(\omega + \omega_0)]$.

(using def^n of δ-function).

so – $\int_{-\infty}^{\infty} e^{-|t|}\cos(\omega_0 t + \phi)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\pi}{1+\omega^2}[e^{i\phi}\delta(\omega - \omega_0) + e^{-i\phi}\delta(\omega + \omega_0)]\,d\omega$ **3**

$= \frac{e^{i\phi} + e^{-i\phi}}{1+\omega_0^2}$

– If $\phi = -\pi/2$ RHS is zero. In this case the integrand is an odd function and so must integrate to zero [$e^{-|t|}$ even and $\sin\omega_0 t$ odd] **3**

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	EXAMINATION QUESTIONS/SOLUTIONS 2011-2012	Course EE II (3)
Question 6	TOPIC Laplace Transforms	Marks & seen/unseen

Parts

$-\; LT\left[\frac{d^3x(t)}{dt^3}\right] = \int_0^\infty e^{-st}\dddot{x}\,dt = \left[\ddot{x}\,e^{-st}\right]_0^\infty - \int_0^\infty -se^{-st}\,\ddot{x}\,dt$ 4

$= -\ddot{x}(t=0) + s \times LT[\ddot{x}]$

$= s^3\bar{x}(s) - s^2x(0) - s\dot{x}(0) - \ddot{x}(0).$

$-\; LT[t^2] = \int_0^\infty e^{-st}t^2\,dt = \left[-\frac{e^{-st}}{s}t^2\right]_0^\infty + \frac{1}{s}\int_0^\infty 2te^{-st}\,dt$ 5

$= \frac{2}{s}\left[\left[\frac{e^{-st}}{s}t\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\,dt\right] = 2/s^3$

For full marks need to state in full the shift theorem.

$-\; LT[e^{-2t}t^2]$ well $LT[e^{at}f(t)] = F(s-a)$ so 4

$LT[e^{-2t}f(t)] = \bar{f}(s+2)$ so $LT[e^{-2t}t^2] = 2/(s+2)^3.$

$-\; \cancel{\frac{3}{s^3+6s}}\; s^3\bar{x}(s) + 6s^2\bar{x}(s) + 12s\bar{x}(s) + 8\bar{x}(s) = \bar{f}(s)$ 7

$\bar{x}(s) = \frac{\bar{f}(s)}{(s+2)^3}.$

Treat $\bar{x}(s) = \bar{g}(s)\bar{f}(s) \Rightarrow \bar{g}(s) = 1/(s+2)^3.$

From the Laplace convolution theorem and noting $g(t) = e^{-2t}t^2/2$

$x(t) = \int_0^t \frac{e^{-2v}v^2}{2}f(t-v)\,dv.$

$\int t^2 e^{-(s+2)t}\,dt = \frac{2}{(s+2)^3}$ ~2/4

one more line of argument required somewhere for full marks.

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