UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 2001

BEng Honours Degree in Computing Part I
MEng Honours Degrees in Computing Part I
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the Associateship of the City and Guilds of London Institute

PAPER C142

DISCRETE MATHEMATICS

Wednesday 16 May 2001, 14:00
Duration: 90 minutes
(Reading time 5 minutes)

Answer THREE questions

Paper contains 4 questions
Calculators not required

1a Let A and B be arbitrary sets. Define $A \cup B$, $A \cap B$, $A - B$ and $A \times B$.

b Give $A \cup B$, $A \cap B$, $A - B$ and $A \times B$ for the following examples:

i) $A = \{a, c, d\}$ and $B = \{c, e\}$;

ii) $A = \{\emptyset\}$ and $B = \emptyset$.

c Let A, B, C and D be finite sets, and let $|A|$ denote the cardinality of A.

i) Express the cardinalities of $A \cup B$, $A - B$ and $A \times B$ using the cardinalities of A, B and $A \cap B$.

ii) Determine whether the following statements about cardinalities are true or false:

$$\begin{aligned} |(A \cup B) - (A \cap B)| &= |(A - B) \cup (B - A)| \\ |(A \times B) - (C \times D)| &= |(A - C) \times (B - D)| \end{aligned}$$

If true, give a proof. [The results given in part c(i) may be assumed.] If false, give a counter-example.

d Let A, B and C be sets. Prove the equivalence of the following statements:

i) $C \subseteq A \cup B$;

ii) $(C - A) \cap (C - B) = \emptyset$;

iii) $(C - A) \subseteq B$.

Hint: one method is to prove that (i) $\Rightarrow$ (iii), (iii) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (i). If you have difficulties, for partial marks explain why the implications hold in words.

The four parts are worth 10%, 20%, 35% and 35% of the marks respectively.

2a Let R be a binary relation on a set A. State what it means for R to be reflexive, transitive, anti-symmetric, a partial order and a total order.

b Consider the binary relation R on $A = \{a, b, c, d, e\}$ given by

$$R = \{(a, b), (a, d), (b, e), (c, b), (c, d)\}.$$

i) Give the smallest relation S such that $R \subseteq S$ and S is a partial order on A.

ii) What are the minimal and maximal elements of S?

iii) Define a relation T such that $S \subseteq T$ and T is a total order on A.

c Let A and B be sets. Give the definition of a partial function from A to B. State what it means for two such partial functions to be equal.

d Let A and B be sets, and let $A \rightharpoonup B$ denote the set of partial functions from A to B. Define a binary relation R on $A \rightharpoonup B$ by

$$f \, R \, g \text{ iff } \mathsf{dom}(f) \subseteq \mathsf{dom}(g), \text{ and } a \in \mathsf{dom}(f) \text{ implies } f(a) = g(a).$$

i) Prove in detail that R is a partial order on $A \rightharpoonup B$.

ii) Give the maximal and minimal elements of $A \rightharpoonup B$.

The four parts are worth 25%, 25%, 15% and 35% of the marks respectively.

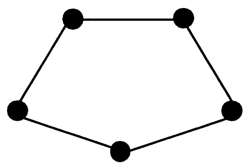
3a i) What does it mean to say that a graph is *planar*?

ii) Give an example of a non-planar graph.

iii) Show that any graph with four nodes is planar.
(NB You should not assume that the graph is simple or connected.)

b i) Let G_1, G_2 be two (undirected) graphs. What does it mean to say there is an *isomorphism* from G_1 to G_2?

ii) Let n>2. An *automorphism* is an isomorphism from a graph to itself. Let R_n be the graph which consists of n nodes connected in a ring. The diagram below illustrates R_5.



How many automorphisms does R_n have? Explain your answer.

c i) What does it mean for a graph to be *connected*?

ii) Show by induction that a connected graph with n nodes has at least n-1 arcs.

d i) What does it mean for a graph to be *2-colourable*?

ii) Show that a 2-colourable graph with n nodes has no more than $\lfloor n^2/4 \rfloor$ arcs.
You may assume that $\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x+y \rfloor$ for any non-negative numbers x,y.

4a i) Describe briefly the algorithm Insertion Sort (IS for short).

ii) What is the worst-case number of comparisons I(n) for sorting a list of n elements using IS? Give a brief explanation.

iii) Give an example for n=5 to show that the worst case can arise.

iv) Let L be a list consisting of n distinct numbers (n>1), which are in ascending order, except that the first and last elements are swapped. An example is the list [5,2,3,4,1]. Calculate how many comparisons IS takes to sort L.

b i) State, with brief justification, the recurrence relation for the worst case number of comparisons B(n) for Binary Search.

ii) Solve as exactly as possible your recurrence relation for B(n). You may assume that that $\lfloor \lfloor x/y \rfloor /z \rfloor = \lfloor x/(yz) \rfloor$ for any positive integers x,y,z.

c The Binary IS algorithm is a modification of IS, where elements are inserted using Binary Search.

i) Write down, with brief explanation, the recurrence relation for the worst-case number of comparisons W(n) for Binary IS.

ii) Explain when the worse case may arise.

iii) Do not solve the recurrence relation for W(n). Instead, obtain a suitable upper bound for W(n) and thereby deduce that W(n) is of strictly lower order than I(n) from Part a(ii) above.

The three parts carry, respectively, 40%, 25%, 35% of the marks.