EE3-09

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2017

EEE/EIE PART III/IV: MEng, BEng and ACGI

Corrected copy

CONTROL ENGINEERING

Friday, 15 December 9:00 am

Time allowed: 3:00 hours

NO ERRORS

There are FOUR questions on this paper.

Answer ALL questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

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CONTROL ENGINEERING

1. The *bipendulum* consists of a horizontal rod with a pendulum attached to each end. If the rod is moved horizontally, the pendula begin to swing. After some idealizations, this system is described by the equations

$$\ddot{z}_1 + \omega_1^2 z_1 = u, \qquad \ddot{z}_2 + \omega_2^2 z_2 = u,$$

in which, for $i = 1, 2$, z_i is the angle between the i-th pendulum and the vertical direction, $\omega_i > 0$ is the characteristic frequency of the i-th pendulum (determined by its length and the gravitational acceleration) and the input u is proportional to the acceleration of the rod.

Let $\dot{z}_1$ and $\dot{z}_2$ be the output of the system.

a) Write a state space description of the system with state $x = [z_1, \dot{z}_1, z_2, \dot{z}_2]'$, input u and output $y = [\dot{z}_1, \dot{z}_2]'$. [4 marks]

b) Determine a condition on ω_1 and ω_2 such that the system is controllable. [4 marks]

c) Show that the system is observable. [2 marks]

d) Let $u = -k(\dot{z}_1 + \dot{z}_2)$. Write the state space equations of the closed-loop system and show that the closed-loop system is asymptotically stable for all $k > 0$. (Hint: Use Routh test.) [4 marks]

e) The *energy* of the system is given by

$$E(x) = \frac{1}{2}\left(\omega_1^2 x_1^2 + x_2^2 + \omega_2^2 x_3^2 + x_4^2\right).$$

Note that E is always non-negative and it is zero only if $x = 0$. Show that

$$\dot{E} = (x_2 + x_4)u = (\dot{z}_1 + \dot{z}_2)u.$$

Show that the equation of $\dot{E}$ in closed-loop with the controller in part d) is such that (recall that $k > 0$)

$$\dot{E} = -k(\dot{z}_1 + \dot{z}_2)^2 \leq 0.$$

Hence argue that the energy of the closed-loop system is always non-increasing and that it stops decreasing when $\dot{z}_1(t) = \dot{z}_2(t) = 0$. Conclude, by observability of the system, that all trajectories of the closed-loop system have to converge to zero, that is the zero equilibrium is attractive. Explain why the above argument can be used to prove asymptotic stability of the system.
(Hint: note that when $u = 0$ one has two independent harmonic oscillators.) [6 marks]

2. Let V be the voltage applied to an electric water kettle and x_1 and x_2 the temperature of the heater coil and of the water, respectively. The rate of change of x_1 is proportional to the electric power fed into the system minus the heat loss to the water. The electric power is proportional to V^2 and the heat loss is proportional to the temperature difference $x_1 - x_2$. The rate of change of x_2 is proportional to the heat loss of the coil. As a result, we obtain the state-space equations

$$\dot{x}_1 = aV^2 - b(x_1 - x_2), \qquad \dot{x}_2 = c(x_1 - x_2),$$

with state $x(t) = [x_1(t), x_2(t)]' \in \mathbb{R}^2$ and input $V(t)^2 = u(t) \in \mathbb{R}$, and a, b and c positive constants. In what follows disregard the fact that $u(t)$ has to be non-negative to retain its physical meaning.

a) Study the controllability of the system as a function of a, b and c. [4 marks]

b) Consider the new variables

$$z_1 = cx_1 + bx_2 \qquad z_2 = x_1 - x_2.$$

Show that the system can be described by the pair of state variables (z_1, z_2) and determine a state-space description in the variables (z_1, z_2). [4 marks]

c) Let $u = u_\star + k_1 z_1 + k_2 z_2$, with $u_\star > 0$, and k_1 and k_2 constants to be determined.

i) Determine conditions on k_1 and k_2 such that the system is asymptotically stable. [4 marks]

ii) Show that the equilibrium of the closed-loop system is described by

$$(z_{1,eq}, z_{2,eq}) = (\star, 0),$$

with $\star$ indicating a function of k_1, k_2 and $u_\star$. Determine this function explicitly. [4 marks]

iii) Select k_1 and k_2 such that the considered state feedback is only a function of x_1. Determine, using the results in parts c.i) and c.ii), if it is possible to have a stabilizing feedback which is only a function of x_1 and determine the set of equilibria that can be achieved using this feedback.

[4 marks]

3. Consider a linear, discrete-time, system with state $x(k) \in \mathbb{R}^n$, input $u(k) \in \mathbb{R}$ and output $y(k) \in \mathbb{R}^p$, described by the equations

$$x(k+1) = Ax(k) + Bu(k), \qquad y(k) = Cx(k) + Du(k).$$

The system is said to be output controllable if

$$\text{rank}\begin{bmatrix} CB & CAB & CA^2B & \cdots & CA^{n-1}B & D \end{bmatrix} = p$$

(recall that p is the number of output signals).
In what follows, let $p = 1$.

a) Assume $D = 0$.

i) Show that if the system is reachable then it is output controllable for all $C \neq 0$. [4 marks]

ii) Conversely, show that if the system is output controllable for all $C \neq 0$ then it is reachable. [4 marks]

iii) Show that if $CB \neq 0$ then one can control the output of the system to 0 in one step regardless of the initial state. [2 marks]

iv) Show that the number of steps required to control the output of the system to zero is equal to the smallest integer i such that $CB = 0$, $CAB = 0$, $\dots$, $CA^{i-2}B = 0$, $CA^{i-1}B \neq 0$. [8 marks]

b) Assume $D \neq 0$. Show that the system is output controllable to zero in zero steps, that is one can render $y(k) = 0$ for all $k \geq 0$, for all A, B and C. [2 marks]

4. Consider a linear system described by the equations

$$\begin{aligned} \sigma x_1 &= x_2, \\ \sigma x_2 &= u, \end{aligned}$$

with state $x = [x_1, x_2]'$ and input u. Note that one does not know if the system is continuous-time or discrete-time.

a) Suppose one wishes to design a state feedback $u = Kx$ stabilizing the closed-loop system. Determine in the complex plane the set to which the eigenvalues of the closed-loop system should belong to have asymptotic stability of the closed-loop system (recall that we do not know if the system is continuous-time or discrete-time). [4 marks]

b) Show that the points $s = -1/4$ and $s = -3/4$ belong to the stability region determined in part a) and determine the state feedback gain K such that the closed-loop system, with matrix $A_{cl} = A + BK$, has eigenvalues at $s = -1/4$ and $s = -3/4$.
(Note that the variable s has been used for convenience, I could have used z instead!) [4 marks]

c) Suppose that the system is a continuous time system for 1 second, then a discrete-time system for one step, and that this behaviour repeats itself indefinitely. Assume that the discrete-time behaviour is instantaneous, that is the discrete-time evolution occurs at $t = 1$, $t = 2$ and so on.

i) Show that

$$x(1) = A_{cl}e^{A_{cl}}x(0), \quad x(2) = (A_{cl}e^{A_{cl}})^2 x(0), \quad \cdots \quad x(k) = (A_{cl}e^{A_{cl}})^k x(0).$$

Hence argue that the state of the system at time $t = k$, with k a non-negative integer, is such that

$$x(k+1) = A_{cl}e^{A_{cl}}x(k) \qquad (\star).$$

[4 marks]

ii) Determine the eigenvalues of the discrete-time system $(\star)$ determined in part c.i) and show that the discrete-time system is asymptotically stable.
(Hint: do not compute the matrix $e^{A_{cl}}$, but exploit the definition of eigenvalue, that is the fact that if λ is an eigenvalue of A then $Av = \lambda v$ for some nonzero vector v!) [6 marks]

iii) Sketch the time evolution of the variable x_1 for all $t \geq 0$.
(Hint: recall that the system behaves like a continuous-time system for all times t which do not have an integer value, and as a discrete-time system when t is an integer. Note that one does not have to select any initial condition: it is sufficient to show a possible evolution.)
[2 marks]