UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 2004

MSci Honours Degree in Mathematics and Computer Science Part IV
MEng Honours Degrees in Computing Part IV
MSc in Advanced Computing
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the Associateship of the City and Guilds of London Institute
This paper is also taken for the relevant examinations for the Associateship of the Royal College of Science

PAPER C499

MODAL AND TEMPORAL LOGIC

Thursday 29 April 2004, 14:30
Duration: 120 minutes

Answer THREE questions

Paper contains 4 questions
Calculators not required

1 a Let A be a modal formula, and let C be a class of Kripke frames. Define the following italicised terms:

i) A is *valid over* C,

ii) A is *satisfiable,*

iii) A is *valid.*

State and prove

iv) a relationship between the satisfiability of A and the validity of $\neg A$.

b Let p, q be propositional atoms. Which of the following formulas are valid? Give reasons in each case.

i) $\Box p \to \Diamond p$

ii) $\Box(p \vee q) \to \Box p \vee \Box q$

iii) $\Box p \wedge \Diamond q \to \Diamond(p \wedge q)$

c Briefly outline a 'bulldozing' argument to show that for any modal formula A, if A is valid over the class of *irreflexive frames* (those frames (W, R) such that $R(w, w)$ is false for all $w \in W$) then A is valid. You may use facts about p-morphisms without proof, so long as you state them clearly.

d Let A, B be modal formulas.

i) Show that if A is satisfiable, then so is $\Diamond A$.

ii) Show that if $\Diamond A$ and $\Diamond B$ are satisfiable then so is $\Diamond A \wedge \Diamond B$.

The four parts carry, respectively, 20%, 30%, 25%, and 25% of the marks.

2 a Express the following in temporal logic. Use suitable atoms and the connectives U, S, F, P, G, H as you wish. If you feel the English is ambiguous then explain what additional assumptions your logic translation makes.

i) The UK joined the European Monetary Union, but later left it.

ii) Whenever the yen fell in the past, it always rose again later.

iii) As soon as the UK joins the Euro (if ever), the Euro will fall and will continue to fall ever after.

iv) While B. remains in office, the UK will never join the Euro.

b The connective W ('weak until') is defined as follows. For a temporal model $\mathcal{M} = (T, <, h)$ and a time $t \in T$, we define $\mathcal{M}, t \models (X\ W\ Y)(t)$ iff either $\mathcal{M} \models X(u)$ for all $u > t$, or $\mathcal{M} \models Y(u)$ for some $u > t$ such that $\mathcal{M} \models X(v)$ for all v with $t < v < u$.

i) Using a similar style, define a connective B ('before'), where $X\ B\ Y$ means 'every future time when Y is true is after a future time when X is true'.

ii) Show that each of W, B, and U (Until) can express the other two. For example, you could show that U can express W by writing a formula using only U that is equivalent to $X\ W\ Y$ (you need not prove the equivalence).

c Let Z be the formula $F(q \wedge Gq) \wedge F\neg q \rightarrow F(Gq \wedge H\neg Gq)$.

i) Show that Z is valid in the flow of time $(\mathbb{N}, <)$ (where $\mathbb{N}$ is the natural numbers).

ii) Give an example of a linear flow of time in which Z is not valid. Explain your answer.

The three parts carry, respectively, 30%, 30%, and 40% of the marks.

3 a A set Σ of formulas is a system of modal logic iff it contains all propositional tautologies (PL) and is closed under the rules of modus ponens and uniform substitution. Define each of the following terms:

i) A formula A is a *theorem* of Σ (denoted $\vdash_{\Sigma} A$).

ii) A formula A is Σ-*deducible* from a set Γ of formulas (denoted $\Gamma \vdash_{\Sigma} A$).

iii) A set Γ of formulas is Σ-*inconsistent.*

iv) Σ is a *normal* system of modal logic.

b KD is the smallest normal system of modal logic containing the schema D ($\Box A \rightarrow \Diamond A$). Show that the following set of formulas is KD-inconsistent.

$$\{\Box p, \quad \Box(p \rightarrow q), \quad \neg p \rightarrow \Box\neg q, \quad \neg p\}$$

c What is a *canonical model* for a *normal* system of modal logic Σ? How can such a model be used to prove completeness of Σ with respect to a given class of Kripke models? Identify the key properties clearly. It is not necessary to provide proofs.

d $KD45$ is the smallest normal system of modal logic containing the schemas D, 4 ($\Box A \rightarrow \Box\Box A$) and 5 ($\Diamond A \rightarrow \Box\Diamond A$).

The 'proper canonical model' for $KD45$ is the canonical model whose accessibility relation R_c^{KD45} is defined as follows:

$$w \, R_c^{KD45} \, w' \quad \text{iff} \quad \{A \mid \Box A \in w\} \subseteq w'$$

Equivalently: $w \, R_c^{KD45} \, w'$ iff $\{\Diamond A \mid A \in w'\} \subseteq w$.

Prove *completeness* of $KD45$ with respect to the class of serial, transitive, and euclidean Kripke models. (You do not need to prove that the proper canonical model for $KD45$ is a canonical model for $KD45$.)

For the purposes of the question, assume without proof that the relation R_c^{KD45} is serial.

A relation R is *euclidean* iff for all w, w', and w'', $w \, R \, w'$ and $w \, R \, w''$ implies $w' \, R \, w''$.

You may find it helpful to note that schema 5 is equivalent to the schema $\Diamond\Box A \rightarrow \Box A$.

The four parts carry, respectively, 20%, 20%, 20%, and 40% of the marks.

4a A set Σ of formulas is a system of modal logic iff it contains all propositional tautologies (PL) and is closed under the rules of modus ponens and uniform substitution.

i) What is a *classical* system of modal logic?

ii) What is the structure of *neighbourhood models* for classical systems? State the associated truth conditions for $\Box A$ and $\Diamond A$ both in terms of the neighbourhood function ν and its 'inverse' f.

b i) Identify suitable model conditions for a neighbourhood model to validate schemas M, D, and P, respectively.

(M is $\Box(A \wedge B) \rightarrow (\Box A \wedge \Box B)$. D is $\Box A \rightarrow \Diamond A$. P is $\Diamond \top$.)

ii) Construct a model containing two worlds such that schemas M and P are valid but D is not.

c Suppose $\mathcal{M}$ is a canonical model for a classical system Σ of modal logic.

Explain briefly, but carefully, why each of the following statements is true, for all worlds w in $\mathcal{M}$ and all formulas A. The last part requires an outline of the proof by induction.

i) $\bot \notin w$

ii) if $\vdash_\Sigma A$ then $A \in w$

iii) $\top \in w$

iv) $w \in |A|_\Sigma$ iff $A \in w$

v) $\|A\|^{\mathcal{M}} = |A|_\Sigma$, given that in a canonical model $|\Box B|_\Sigma = \{w : |B|_\Sigma \in \nu(w)\}$ for all formulas B.

$\|A\|^{\mathcal{M}}$ denotes the set of worlds of $\mathcal{M}$ at which A is true. $|A|_\Sigma$ is the 'proof set' of A: the set of Σ-maxiconsistent sets that contain A.

You may assume without proof that $\|\neg A\|^{\mathcal{M}} = W - \|A\|^{\mathcal{M}}$, $\|A \wedge B\|^{\mathcal{M}} = \|A\|^{\mathcal{M}} \cap \|B\|^{\mathcal{M}}$, $|\neg A|_\Sigma = |\top|_\Sigma - |A|_\Sigma$, $|A \wedge B|_\Sigma = |A|_\Sigma \cap |B|_\Sigma$, and similarly for other truth-functional connectives.

The three parts carry, respectively, 15%, 45%, and 40% of the marks.