

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2010

MSc and EEE/ISE PART III/IV: MEng, BEng and ACGI

Corrected Copy

MATHEMATICS FOR SIGNALS AND SYSTEMS

Wednesday, 5 May 10:00 am

Time allowed: 3:00 hours

There are FIVE questions on this paper.

Answer THREE questions.

*Q2 correction
announced at start of
exam*

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : M.M. Draief
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MATHEMATICS FOR SIGNAL AND SYSTEMS

1. a) Let P be the matrix defined as

$$P = \frac{1}{2} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

- i) Describe a basis of $\mathcal{N}(P)$ the null-space (kernel) of P and a basis of $\text{Range}(P)$ the range of P . Justify your answer. [3]
 - ii) Show that $\mathbb{R}^4 = \mathcal{N}(P) \oplus \text{Range}(P)$. [1]
 - iii) Show that for $x \in \mathcal{N}(P)$ and $y \in \text{Range}(P)$ then $x^T y = 0$. [2]
 - iv) Conclude that P is an orthogonal projection. [1]
- b) We assume that (e_1, \dots, e_n) is an orthonormal basis of \mathbb{R}^n and, for $k = 1, \dots, n-1$, we define $F_k = \text{Span}(e_1, \dots, e_k)$.
- i) Let $z \in \mathbb{R}^n$. Provide, without justification, the expression of Πz the orthogonal projection of z on F_k in terms of (e_1, \dots, e_k) . [1]
 - ii) Prove that for all $z \in \mathbb{R}^n$, we have $\|\Pi z\| \leq \|z\|$, $\|z\| = \sqrt{z^T z}$. [1]
 - iii) Express Π in terms (e_1, \dots, e_k) , and show that $\Pi^2 = \Pi \times \Pi = \Pi$ and $(\Pi x)^T y = x^T (\Pi y)$. In other words, we have $\Pi^T = \Pi$. [2]
 - iv) Suppose that Q is a projection (not necessarily orthogonal), such that $\|Qz\| \leq \|z\|$. Show that Q is an orthogonal projection. [2]
- Hint:* You have to show that $x^T y = 0$ for all $x \in \text{Range}(Q)$ and $y \in \mathcal{N}(Q)$. To this end, consider $z = \lambda x + y$ for all $\lambda \in \mathbb{R}$.
- c) Assume that we have two orthogonal projectors P and Q on the subspaces F and G respectively.
- We consider the matrix $R = PQ$, the product of the matrices P and Q , $\lambda \in \mathbb{R}$, $\lambda \neq 0$ an eigenvalue of R and $u \in \mathbb{R}^n$ an associated eigenvector, i.e. $Ru = \lambda u$.
- i) Show that $u \in \text{Range}(P)$ and that $Qu - \lambda u \in \mathcal{N}(P)$. [2]
 - ii) Using question 1.b) iii) and the previous question, prove that

$$\|Qu\|^2 = \lambda \|u\|^2.$$

[3]
 - iii) Using question 1.b) ii), conclude that the eigenvalues of R are in $[0, 1]$. [2]

2. Let m and n be two positive integers with $m \leq n$. We consider $A \in \mathbb{R}^{(n+1) \times (m+1)}$ the matrix defined by

$$A = \begin{pmatrix} 1 & x_0 & \dots & x_0^m \\ 1 & x_1 & \dots & x_1^m \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & \dots & x_n^m \end{pmatrix},$$

where x_0, \dots, x_n are $n+1$ distinct real numbers.

- a) Let $\mathbf{0}$ be the vector with all its entries equal to 0 (we will use the same notation for both the zero vector of \mathbb{R}^{m+1} and the one of \mathbb{R}^{n+1}).

$$\text{Let } v = \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_m \end{pmatrix} \in \mathbb{R}^{m+1}.$$

- i) Show that if $Av = \mathbf{0}$ then $v = \mathbf{0}$. [1]

Hint: Use the fact that if the polynomial $P(x) = v_0 + v_1x + \dots + v_mx^m$ has more than $m+1$ distinct zeros then $P(x) = 0$.

- ii) Using the previous question, show that if $A^T Av = \mathbf{0}$ then $v = \mathbf{0}$. [2]

- iii) Fix $y \in \mathbb{R}^{n+1}$. Justify the fact that the linear equation $A^T Ax = A^T y$ admits a unique solution. [2]

In the remainder of this problem, we will denote this solution by w , i.e.

$$A^T Aw = A^T y.$$

- b) For $v \in \mathbb{R}^{m+1}$ and $y \in \mathbb{R}^{n+1}$, define $g(v) = (y - Av)^T (y - Av)$.

- i) Show that $g(w) = y^T y - y^T Aw$, with w defined in 2. a) iii). [2]

- ii) Prove that $g(v) - g(w) = (w - v)^T A^T A (w - v)$. [3]

- iii) Show that for all $v \in \mathbb{R}^{m+1}$, we have $g(v) \geq g(w)$ and that $g(v) = g(w)$ if and only if $v = w$. [2]

- c) Let P be a polynomial such that $P(x) = \sum_{k=0}^m v_k x^k$. We define the quantity

$$\Phi_m(P) = \sum_{i=0}^n (y_i - P(x_i))^2.$$

$$\text{Let } v = \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_m \end{pmatrix} \in \mathbb{R}^{m+1} \text{ and } y = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^{n+1}.$$

- i) Show that $\Phi_m(P) = g(v)$. [2]

- ii) Using question 2.b), show that there exists a polynomial P_w such that $\Phi_m(P) \geq \Phi_m(P_w)$. [2]

- d) We now apply the analysis of question 2) c) to a numerical example. Let $n = m = 3$, $x_0 = -1$, $x_1 = 0$, $x_2 = 1$, $x_3 = 2$ and $y_0 = 1$, $y_1 = 2$, $y_2 = 1$, $y_3 = 0$.

- i) Solve $A^T Av = A^T y$. [2]

- ii) Derive the expression of the polynomial in $\mathbb{R}_3[X]$ that minimises Φ_3 and give the minimum value of Φ_3 on $\mathbb{R}_3[X]$. Justify your answer. [2]

3. Let $A \in \mathbb{R}^{n \times n}$ be a non-singular matrix, and consider three vectors $b, c, f \in \mathbb{R}^n$. Given two real numbers α and γ we want to solve the following linear system in $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$.

$$\begin{aligned} Ax + b\lambda &= f \\ c^T x + \alpha\lambda &= \gamma. \end{aligned} \quad (3.1)$$

- a) i) Write the system (3.1) in matrix form, i.e. $My = g$ with $M \in \mathbb{R}^{(n+1) \times (n+1)}$ and $y, g \in \mathbb{R}^{n+1}$. [2]
- ii) Give a necessary and sufficient condition for the system (3.1) to be solvable, i.e. to admit a unique solution. Justify your answer. [4]

In what follows we assume that $\alpha - c^T A^{-1} b \neq 0$.

- b) To solve (3.1), we will use the following algorithm.
Let z_0 be the solution of $Az = b$ and h_0 be the solution of $Ah = f$.

$$x = h_0 - \frac{\gamma - c^T h_0}{\alpha - c^T z_0} z_0, \quad \lambda = \frac{\gamma - c^T h_0}{\alpha - c^T z_0}.$$

- i) Show that the above algorithm gives the solution to (3.1). [2]
- ii) Assuming that we use one of the standard methods to solve $Az = b$ and $Ah = f$, how many additional operations are required to complete the algorithm? [5]
- c) We now solve (3.1) for

$$A = \begin{pmatrix} 25 & 15 & -5 \\ 15 & 18 & 0 \\ -5 & 0 & 11 \end{pmatrix}$$

$$b = \begin{pmatrix} 30 \\ 15 \\ -16 \end{pmatrix} \quad f = \begin{pmatrix} 35 \\ 33 \\ 6 \end{pmatrix}, \quad c = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

and $\gamma = 4$ and $\alpha = 1$

- i) Using Cholesky decomposition, solve $Az = b$ and $Ah = f$. [5]
- ii) Derive the solution to (3.1). [2]

4. Let $\mathbb{R}[X]$ be the vector space of polynomials with real coefficients, and $\mathbb{R}_n[X]$ be the subspace of polynomials with degree smaller or equal to n . Let w be a continuous function on $(-1, 1)$ taking positive real values. For P and Q in $\mathbb{R}[X]$, we define

$$\langle P, Q \rangle = \int_{-1}^1 P(x)Q(x)w(x)dx.$$

- a) First we assume that $w(x) = \frac{1}{\sqrt{1-x^2}}$, for $x \in (-1, 1)$ and define $T_k(x)$ the polynomials such that, for $k \geq 1$ and $\theta \in (0, \pi)$, we have

$$T_k(\cos(\theta)) = \cos(k\theta), \quad T_0 = 1,$$

known as *Chebyshev's polynomials*.

- i) Derive T_1, T_2 and T_3 . [1]

- ii) Show that, for $k \geq 1$, we have

$$T_{k+1} = 2XT_k - T_{k-1}.$$

[2]

- iii) Using the change of variable $\theta = \arccos(x)$, compute $\langle T_n, T_m \rangle$, when $n = m$ and $n \neq m$. [2]

- iv) Derive an orthonormal basis of $\mathbb{R}_3[X]$. Justify your answer. [1]

- b) For the remainder of the problem, we let w be a (general) given continuous function on $(-1, 1)$.

- i) Show that the application $(P, Q) \rightarrow \langle P, Q \rangle$ is an inner (scalar) product on $\mathbb{R}[X]$. [1]

- ii) Justify the existence of a family of orthogonal polynomials (P_0, P_1, P_2, \dots) , with respect to the above inner product on $\mathbb{R}[X]$, where the degree of P_k is equal to k . [3]

Hint: Use the Gram-Schmidt algorithm on $(1, X, X^2, X^3, \dots)$, the canonical basis of $\mathbb{R}[X]$.

- iii) For $k = 1, 2, \dots$, prove that $\langle P_k, Q \rangle = 0$, for all $Q \in \mathbb{R}_{k-1}[X]$. [1]

- iv) Show that, for $k \geq 2$ and $j \leq k-2$, we have $\langle XP_k, P_j \rangle = 0$. [2]

- c) For $k = 1, 2, \dots$, we write $P_k = \sum_{j=0}^k \alpha_{k,j} X^j$.

- i) Justify the fact that $XP_k = a_1 P_1 + b_0 P_0$, for some reals a_1 and b_0 . [1]

- ii) Show that $a_1 = \frac{\alpha_{0,0}}{\alpha_{1,1}}$ and $b_0 = -\frac{\alpha_{1,0}}{\alpha_{1,1}}$. [2]

- iii) Using similar arguments as in the previous two questions, show that, for $k \geq 1$, we have

$$XP_k = a_{k+1}P_{k+1} + b_k P_k + a_k P_{k-1},$$

where

$$a_k = \frac{\alpha_{k-1,k-1}}{\alpha_{k,k}} \quad \text{and} \quad b_k = \frac{\alpha_{k,k-1}}{\alpha_{k,k}} - \frac{\alpha_{k+1,k}}{\alpha_{k+1,k+1}}.$$

[4]

5. In this problem, we analyse the impact of perturbations on the solutions of linear equations.

- a) We will consider the standard Euclidean norm $\|x\| = \sqrt{x^T x}$, for $x \in \mathbb{R}^n$ and the associated matrix norm

$$\|A\| = \sup_{x: \|x\|=1} \|Ax\|.$$

- i) Show that the mapping $A \rightarrow \|A\|$ defines a norm on $\mathbb{R}^{n \times n}$. [3]
 ii) Let $x \in \mathbb{R}^n$, and A and B in $\mathbb{R}^{n \times n}$ show that $\|Ax\| \leq \|A\| \|x\|$ and that $\|AB\| \leq \|A\| \|B\|$. [3]

- b) In this question, we assume that A is a non-singular matrix in $\mathbb{R}^{n \times n}$ and y a non-zero vector in \mathbb{R}^n . Let $x_0 \in \mathbb{R}^n$ be the solution of $Ax = y$.

- i) Let $x_1 \in \mathbb{R}^n$ be the solution of $Ax = y + \delta y$, where $\delta y \in \mathbb{R}^n$. Prove that

$$\frac{\|x_0 - x_1\|}{\|x_0\|} \leq \|A\| \|A^{-1}\| \frac{\|\delta y\|}{\|y\|}.$$

[2]

- ii) Let $x_2 \in \mathbb{R}^n$ be a solution of $(A + \delta A)x = y$, where $\delta A \in \mathbb{R}^{n \times n}$. Prove that

$$\frac{\|x_0 - x_2\|}{\|x_0\|} \leq \|A\| \|A^{-1}\| \frac{\|\delta A\|}{\|A\|}.$$

[2]

- iii) The coefficient $\kappa(A) = \|A\| \|A^{-1}\|$ is known as the *condition number* of A .

Show that $\kappa(A) \geq 1$. Comment on the sensitivity of the solution of the equation $Ax = y$ to perturbations in terms of $\kappa(A)$. [3]

- c) Let $A \in \mathbb{R}^{n \times n}$ be a non-singular matrix with eigenvalues $\lambda_1, \dots, \lambda_n$.

- i) Derive the eigenvalues of A^{-1} . [1]
 ii) Show that $\|A\| \geq |\lambda_i|$ for all $i = 1, \dots, n$.
 iii) Derive a lower bound for $\kappa(A)$ in terms of the λ_i s. [3]
 iv) Show that if A is (non singular) symmetric then

$$\kappa(A) = \max_{i=1, \dots, n} |\lambda_i| \max_{i=1, \dots, n} \frac{1}{|\lambda_i|}.$$

[3]

Hint: Use the fact that if A is symmetric then there exists an orthonormal basis of eigenvectors of A .

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$$P = \frac{1}{2} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

(i) $x \in W(P)$ if $Px = 0$; $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4$

$$Px = \begin{bmatrix} \frac{1}{2} x_1 - \frac{1}{2} x_3 \\ \frac{1}{2} x_2 - \frac{1}{2} x_4 \\ -\frac{1}{2} x_1 + \frac{1}{2} x_3 \\ -\frac{1}{2} x_2 + \frac{1}{2} x_4 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x_1 = x_3 \\ x_2 = x_4 \end{cases}$$

Hence $W(P) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_1 \\ x_2 \end{pmatrix} ; (x_1, x_2) \in \mathbb{R}^2 \right\}$
 $= \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}.$

$y \in \text{Range}(P) \Rightarrow \exists x \in \mathbb{R}^4$ such that $Px = y.$

$$\begin{cases} y_1 = \frac{1}{2} x_1 - \frac{1}{2} x_3 \\ y_2 = \frac{1}{2} x_2 - \frac{1}{2} x_4 \\ y_3 = -\frac{1}{2} x_1 + \frac{1}{2} x_3 \\ y_4 = -\frac{1}{2} x_2 + \frac{1}{2} x_4 \end{cases} \Rightarrow \begin{cases} y_1 = -y_3 \\ y_2 = -y_4 \end{cases}$$

Hence $\text{Range}(P) \subset \left\{ \begin{pmatrix} y_1 \\ y_2 \\ -y_1 \\ -y_2 \end{pmatrix} ; (y_1, y_2) \in \mathbb{R}^2 \right\}.$ OK

1/a) Since $\text{Rank}(P) = 2 \Rightarrow \dim \text{Range}(P) = 2$

Hence $\text{Range}(P) = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \right\}$.

ii) It is not difficult to see that 2/23

$$W(P) \cap \text{Range}(P) = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}. (*)$$

and that $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$

is a basis of $\mathbb{R}^4 \Rightarrow W(P) + \text{Range}(P) = \mathbb{R}^4$
(**)

(*) & (**) $\Rightarrow W(P) \oplus \text{Range}(P) = \mathbb{R}^4$.

(iii) if $x \in W(P) \Rightarrow x = \begin{pmatrix} u_1 \\ u_2 \\ u_1 \\ u_2 \end{pmatrix}; u_1, u_2 \in \mathbb{R}^2$

if $y \in \text{Range}(P) \Rightarrow y = \begin{pmatrix} y_1 \\ y_2 \\ -y_1 \\ -y_2 \end{pmatrix}; y_1, y_2 \in \mathbb{R}^2$

$$x^T y = u_1 y_1 + u_2 y_2 - u_1 y_1 - u_2 y_2 = 0.$$

Hence $\text{Range}(P) = (W(P))^\perp$

(iv) For $z \in \mathbb{R}^4$ $z = x + y$

$x \in W(P); y \in \text{Range}(P)$.

$$Pz = Px + Py = y \Rightarrow$$

P is the orthogonal projection on $\text{Range}(P)$
(parallel to $W(P)$)

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1/b/

3/23

$$F_k \in \mathbb{R}^n$$

$F_k: \text{Span} \{e_1 \dots e_k\}$ $e_1 \dots e_k$ orthonormal

$$i) \quad z \in \mathbb{R}^n \quad \mathcal{P}z = \sum_{i=1}^k (e_i^T z) e_i$$

$$(ii) \quad \|\mathcal{P}z\|^2 = (\mathcal{P}z)^T \mathcal{P}z = \sum_{i=1}^k (e_i^T z)^2$$

$$\|z\|^2 = \sum_{i=1}^n (e_i^T z)^2$$

$$\Rightarrow \|\mathcal{P}z\|^2 \leq \|z\|^2 \quad \text{since } k < n.$$

(iii) $\left[\begin{array}{l} \text{for } i=1 \dots k \quad \mathcal{P}e_i = e_i \\ \Rightarrow \|\mathcal{P}\| = 1. \end{array} \right] \text{ 'Extra'}$

$$(iii) \quad \mathcal{P} = \sum_{i=1}^k e_i e_i^T$$

$$\begin{aligned} \mathcal{P}^2 &= \sum_{i=1}^k e_i e_i^T \sum_{j=1}^k e_j e_j^T \\ &= \sum_{i,j=1}^k e_i \underbrace{e_i^T e_j}_{\delta_{ij}} e_j^T = \sum_{i=1}^k e_i e_i^T = \mathcal{P}. \end{aligned}$$

$$\begin{aligned} (\mathcal{P}x)^T y &= \sum_{i=1}^k (e_i^T x) e_i^T \sum_{j=1}^n (e_j^T y) e_j \\ &= \sum_{i=1}^k (e_i^T x) (e_i^T y) \end{aligned}$$

$$\begin{aligned} x^T \mathcal{P}y &= \sum_{i=1}^n (e_i^T x) e_i^T \sum_{j=1}^k e_j y e_j^T \\ &= \sum_{i=1}^n (e_i^T x) (e_i^T y). \end{aligned}$$

1/b/

i))

$$z = \lambda x + y$$

$$x \in \text{Range}(d); y \in W(d)$$

$$\|dz\|^2 \leq \|z\|^2 \Rightarrow \lambda^2 \|x\|^2 \leq \lambda^2 \|x\|^2 + \|y\|^2 + 2\lambda \langle x, y \rangle.$$

$$\Rightarrow 0 \leq \|y\|^2 + 2\lambda \langle x, y \rangle.$$

This mly true for all $\lambda \in \mathbb{R}$ if $\langle x, y \rangle = 0$.

$\Rightarrow \text{Range}(d) \perp W(d) \Rightarrow d$ orthogonal projection.

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$\lambda \neq 0$.

T/23

$$\begin{aligned} \text{i)} \quad Pu &= P\left(\frac{1}{\lambda} Ru\right) = \frac{1}{\lambda} P P Q u = \frac{1}{\lambda} P^2 Q u \\ &= \frac{1}{\lambda} P Q u \quad \text{since } P \text{ projection} \\ &= \frac{1}{\lambda} Ru = \mu. \\ &\Rightarrow \mu \in \text{Range}(P) \quad (P \text{ being a projection}). \end{aligned}$$

$$\begin{aligned} P(Qu - \lambda u) &= P Q u - \lambda Pu \\ &= \lambda u - \lambda u \end{aligned}$$

Since $P Q u = Ru = \lambda u$
& $\mu \in \text{Range}(P)$.

$$\begin{aligned} \text{ii)} \quad \lambda u^T u &= u^T (\lambda u) = \mu^T P Q u \\ P \text{ orthogonal projection} &= (P\mu)^T Q u. \\ &= \mu^T Q u. \\ Q \text{ projection} &= \mu^T Q^2 u \\ Q \text{ orthogonal projection} &= (Q\mu)^T Q u \\ &= \|Q\mu\|^2 \\ \Rightarrow \lambda \|u\|^2 &= \|Q\mu\|^2. \end{aligned}$$

$$\begin{aligned} \text{iii)} \quad \cancel{\|Q\mu\|} \quad \|Q\mu\| &\leq \|u\| \\ \Rightarrow 0 \leq \lambda &= \frac{\|Q\mu\|^2}{\|u\|^2} \leq 1. \end{aligned}$$

2)

$$A = \begin{bmatrix} 1 & x_0 & \dots & x_0^m \\ \vdots & \vdots & & \vdots \\ 1 & x_m & \dots & x_m^m \end{bmatrix}$$

6/23

a) i) $Av = 0$

~~$$\sum_{i=0}^m v_i x_j^i$$~~

$$\sum_{i=0}^m v_i x_j^i = 0 \quad \text{for } j=1 \dots m.$$

The polynomial $p(x) = \sum_{i=0}^m v_i x^i$ has m distinct roots x_j ; $j=1 \dots m$ $\Rightarrow p(x) = 0$
 $\Rightarrow v_i = 0 \quad i=1 \dots m.$

ii)

$$A^T A v = 0 \Rightarrow$$

$$v^T A^T A v = 0$$

$$\Rightarrow$$

$$(Av)^T Av = 0 \Rightarrow Av = 0$$

$$\Rightarrow v = 0.$$

by previous question.

iii)

~~Many x~~

$$A^T A x = A^T y$$

By previous question $A^T A$ is non singular

hence $A^T A x = A^T y$ has a solution that is unique.

2/b/i/

$$g(w) = (y - Aw)^T (y - Aw).$$

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$$= y^T y - 2y^T A w + (Aw)^T A w$$

$$= y^T y - 2y^T A w + w^T \underbrace{A^T A w}_{A^T y}$$

$$= y^T y - 2y^T A w + (Aw)^T y$$

$$= y^T y - y^T A w.$$

$$\text{ii/ } g(v) - g(w) = y^T y - 2y^T A v + (Av)^T A v \\ - y^T y + y^T A w \quad (\text{from previous question})$$

$$\text{Since } A^T y = A^T A w \\ = -2(A^T y)^T v + v^T A^T A v \\ + (A^T y)^T w \\ = -2(A^T A w)^T v + v^T A^T A v \\ + (A^T A w)^T w \\ = -2w^T A^T A v + v^T A^T A v \\ + w^T A^T A w \\ = (w - v)^T A^T A (w - v).$$

2/ b/

iii/

$$g(u) - g(w) = (A(w-u))^T A(w-u) \\ = \|A(w-u)\|^2 \geq 0$$

$$\Rightarrow g(u) \geq g(w).$$

$$g(u) = g(w) \quad \text{if} \quad A(w-u) = 0$$

$$\Rightarrow \text{~~A(w-u)~~ } w-u = 0 \Rightarrow w=u.$$

By 2) a) i).

Q/13

2/c/.

$$i) \quad \phi_m(P) = \sum_{i=0}^m (y_i - P(x_i))^2.$$

i) The i -th component of $y - Av$ is given by

$$(y - Av)_i = y_i - \sum_{j=0}^m v_j x_i^j$$
$$= y_i - P(x_i).$$

Hence $(y - Av)^T (y - Av) = \sum_{i=0}^m (y_i - P(x_i))^2.$

ii). $\phi_m(P) = g(v) \geq g(w) = \phi_m(P_w).$

$$\text{Min}_{P \in \mathbb{R}_m[X]} \phi_m(P) = \phi_m(P_w).$$

where $P_w(u) = \sum_{i=0}^m w_i x^i$

$w = (w_i)$ is the ^{unique} solution of $A^T A x = A^T y.$

5/13

2/5/.

$$A = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \end{bmatrix}$$

$\lambda_0/1$

$$A^T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 4 \\ -1 & 0 & 1 & 8 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 4 \\ -1 & 0 & 1 & 8 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 & 6 & 8 \\ 2 & 6 & 8 & 18 \\ 6 & 8 & 18 & 32 \\ 8 & 18 & 32 & 66 \end{bmatrix}$$

$$A^T y = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 4 \\ -1 & 0 & 1 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

$$A^T A x = A^T y \Rightarrow$$

$$(ii) \Rightarrow x = \begin{pmatrix} 2 \\ -1/3 \\ -1 \\ 1/3 \end{pmatrix}$$

$$\begin{cases} 4x_0 + 2x_1 + 6x_2 + 8x_3 = 4 & (1) \\ 2x_0 + 6x_1 + 8x_2 + 18x_3 = 0 & (2) \\ 6x_0 + 8x_1 + 18x_2 + 32x_3 = 2 & (3) \\ 8x_0 + 18x_1 + 32x_2 + 66x_3 = 0 & (4) \end{cases}$$

$k. \phi_m(pw) = 0.$

3/

$$a/ \quad i/ \quad M = \begin{pmatrix} A & b \\ c^T & \alpha \end{pmatrix}$$

$$g = \begin{pmatrix} f \\ \gamma \end{pmatrix}$$

$$y = \begin{pmatrix} x \\ \lambda \end{pmatrix}$$

$$M y = g.$$

$$ib/ \quad x = A^{-1} (f - b\lambda)$$

$$\text{Then } c^T x + \alpha \lambda = \gamma$$

$$c^T A^{-1} (f - b\lambda) + \alpha \lambda = \gamma.$$

$$\Rightarrow \cancel{c^T A^{-1} b + \alpha} \lambda$$

$$(\alpha - c^T A^{-1} b) \lambda = \gamma - c^T A^{-1} f: \text{ simple linear equation}$$

$$\text{If } \alpha \neq \infty, \quad \text{a unique solution iff } \boxed{\alpha - c^T A^{-1} b \neq 0} \quad \text{in } \mathbb{R}.$$

$$\wedge \wedge / 23$$

3/ b/ if $\alpha - c^T A^{-1} b \neq 0$

we have

$$\lambda = \frac{\gamma - c^T A^{-1} f}{\alpha - c^T A^{-1} b}$$
$$= \frac{\gamma - c^T h_0}{\alpha - c^T z_0}$$

(2/2)

and

$$Ax = f - \frac{\gamma - c^T h_0}{\alpha - c^T z_0} b$$
$$= f - \lambda b.$$

ii/ $c^T h_0$ requires n multiplication
 $n-1$ summations.

Similarly for $c^T z_0$.

To compute λ we need $4n-2$ operations.

To compute x we have n multiplications
to compute λz_0 & n summations
to compute $h_0 - \lambda z_0$. Thus an additional
2n operations

In total, we need $6n-2$ extra operations, i.e. $6n$ flops

3/c/

i/

$$A = \begin{bmatrix} 25 & 15 & -5 \\ 15 & 18 & 0 \\ -5 & 0 & 11 \end{bmatrix}$$

By cholsky

$$= \begin{bmatrix} 5 & 0 & 0 \\ 3 & 3 & 0 \\ -1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 5 & 3 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

$$A Z = \begin{pmatrix} 30 \\ 15 \\ -16 \end{pmatrix} \Rightarrow Z = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$A h = \begin{pmatrix} 35 \\ 33 \\ 16 \end{pmatrix} \begin{pmatrix} 35 \\ 33 \\ 6 \end{pmatrix} \Rightarrow h = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Ans.

ii/

$$\lambda = \frac{\gamma - (c_1 + c_2 + c_3)}{2 - c_1 + c_3} = 1$$

$$x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

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4/

$$\langle f, g \rangle = \int_{-1}^1 f(x) g(x) w(x) dx.$$

a) Similar to pb in pb sheet / lecture

i/ $T_1(x) = x$

$$T_2(x) = 2x^2 - 1$$

$$\cos(2\theta) = 2\cos^2(\theta) - 1$$

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$$\cos(3\theta) = \dots$$

$$\begin{aligned} T_3(x) &= 2x(2x^2 - 1) - x \\ &= 4x^3 - 3x. \end{aligned}$$

ii/

$$\cos((k+1)\theta) + \cos((k-1)\theta)$$

$$= 2 \cos(k\theta) \cos(\theta).$$

$$T_{k+1}(x) = 2x T_k(x) - T_{k-1}(x).$$

iii)

$$\langle T_n, T_m \rangle = \int_{-1}^1 \frac{T_n(x) T_m(x)}{\sqrt{1-x^2}} dx$$

$$= + \int_0^\pi \cos(n\theta) \cos(m\theta) d\theta$$

$$= + \int_0^\pi \frac{1}{2} (\cos(n+m)\theta) d\theta$$

$$+ \frac{1}{2} \int_0^\pi \cos((n-m)\theta) d\theta.$$

$n \neq m$

$$\langle T_n, T_m \rangle = 0.$$

$$n=m \neq 0. \quad \langle T_n, T_n \rangle = + \pi/2.$$

$$4/a/iii/$$

$$n=m=0$$

$$\langle T_0, T_0 \rangle = \pi.$$

$$(iv) \quad \frac{T_0}{\sqrt{\pi}}, \quad \frac{T_1}{\sqrt{\pi/2}}, \quad \frac{T_2}{\sqrt{\pi/2}}, \quad \frac{T_3}{\sqrt{\pi/2}}$$

Since T_i orthogonal & $4a/iii/$ gives

the corresponding norms

T_0, T_1, T_2, T_3
are given in

$4/a/i/$.

$$\sqrt{\pi/2}$$

4/
b/.

i/ Trivial

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ii/ $p_n(x) = x^n + a_{n-1} x^{n-1} + \dots + a_0.$

$$p_0(x) = 1$$

if we construct $p_0 \dots p_{n-1}$

Let $q(x) = x^n.$

$$p_n(x) = q(x) - \sum_{k=0}^{n-1} \frac{\langle q, p_k \rangle}{\langle p_k, p_k \rangle} p_k.$$

we have

$\forall m < n, \langle p_n, p_m \rangle = 0$

$$\text{Since } \langle p_n, p_m \rangle = \langle p_m, q \rangle - \langle p_m, q \rangle \frac{\langle p_m, p_m \rangle}{\langle p_m, p_m \rangle} = 0.$$

iii/ $Q \in \mathbb{R}_{k-1}[x]$ & $p_0 \dots p_{k-1}$ basis

of $\mathbb{R}_{k-1}[x]$ then

$$Q = \sum_{i=0}^{k-1} \alpha_i p_i$$

$$\text{So } \langle Q, p_k \rangle = \sum_{i=0}^{k-1} \alpha_i \langle p_k, p_i \rangle = 0.$$

iv).

$$\langle X p_k, p_j \rangle = \int_{-1}^1 x p_k(x) p_j'(x) \omega(x) dx$$

$$= \langle p_k, X p_j \rangle$$

$$\text{since } X p_j \in \mathcal{P}_{k-1}(x)$$

$$= 0$$

by previous question.

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4/c/

i/. $X p_0 \in (R_1 [X])$.

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Hence $X p_0 = a_1 p_1 + b_0 p_0$

ii/ $x \alpha_{00} = a_1 [\alpha_{11} x + \alpha_{10}] + b_0 \alpha_{00}$.

$$\begin{cases} \alpha_{00} = a_1 \alpha_{11} \\ \alpha_{10} a_1 + b_0 \alpha_{00} = 0 \end{cases} \Rightarrow$$

$$a_1 = \frac{\alpha_{00}}{\alpha_{11}}$$

$$b_1 = - \frac{\alpha_{10}}{\alpha_{00}}.$$

iii/.

$$X p_k = \sum_{i=1}^{k+1} \beta_i p_i$$

$$\langle X p_k, p_i \rangle = \langle p_k, X p_i \rangle = 0 \quad \text{if } i \leq k-2.$$

Hence $X p_k = \beta_{k+1} p_{k+1} + \beta_k p_k + \beta_{k-1} p_{k-1}$.

Coefficient with x^{k+1}

$$\alpha_{k,k} = \beta_{k+1} \alpha_{k+1,k+1}$$

$$\Rightarrow \beta_{k+1} = a_{k+1} = \frac{\alpha_{k,k}}{\alpha_{k+1,k+1}}$$

~~$\beta_{k+1} + \beta_k + \beta_{k-1} = 0$ since.~~

Coefficient with x^k .

$$\alpha_{k,k-1} = \beta_{k+1} \alpha_{k+1,k} + \beta_k \alpha_{k,k}$$

~~$X p_k$ has no coefficient of constant.~~

4/c/iii/

$$\beta_{k+1} = a_{k+1} = \frac{\alpha_{k,k}}{\alpha_{k+1,k+1}}$$

$$\begin{aligned} \alpha_{k,k-1} &= \beta_{k+1} \alpha_{k+1,k} + \beta_k \alpha_{k,k} \\ &= \frac{\alpha_{k,k}}{\alpha_{k+1,k+1}} \alpha_{k+1,k} + \beta_k \alpha_{k,k} \end{aligned}$$

$$\Rightarrow b_k = \beta_k = \frac{\alpha_{k,k-1}}{\alpha_{k,k}} - \frac{\alpha_{k+1,k}}{\alpha_{k+1,k+1}}$$

~~coefficient with $k-1$.~~

$$\alpha_{k,k-2} = \beta_{k+1} \alpha_{k+1,k-1} + \beta_k \alpha_{k,k-1}$$

By induction,

$$\begin{aligned} \beta_{k-1} = \langle X p_k, p_{k-1} \rangle &= \langle p_k, X p_{k-1} \rangle \\ &= a_k. \end{aligned}$$

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5/ i)

$$+ \|A\| \geq 0$$

$$+ \|A\| = 0 \Rightarrow \frac{\|Ax\|}{\|x\|} = 0 \Rightarrow Ax = 0 \quad \forall x$$

$$\left\| \frac{x}{\|x\|} \right\| = 1$$

$$\Rightarrow A = 0$$

$$+ \|\lambda Ax\| = |\lambda| \|Ax\| \Rightarrow \|(\lambda A)x\| = |\lambda| \|Ax\|$$

$$+ \text{ii) } \| (A+B)x \| \leq \|Ax\| + \|Bx\|$$

$$\|x\|=1 \Rightarrow \| (A+B)x \| \leq \|Ax\| + \|Bx\|$$

$$\Rightarrow \|A+B\| \leq \|A\| + \|B\|$$

$$\text{ii) } \frac{\|Ax\|}{\|x\|} \leq \|A\| \Rightarrow \|Ax\| \leq \|A\| \|x\|$$

By above

$$\|ABx\| \leq \|A\| \|Bx\| \leq \|A\| (\|B\| \|x\|) = (\|A\| \|B\|) \|x\|$$

20/03

5/ b/.

21/23

i/.

$$Ax_1 = y + \delta y$$

$$Ax_0 = y$$

$$\Rightarrow A^{-1} \delta y = x_1 - x_0$$

$$\|x_1 - x_0\| \leq \|A^{-1}\| \|\delta y\| \quad \text{by 5a) ii)}$$

$$\|y\| = \|Ax_0\| \leq \|A\| \|x_0\|$$

$$\frac{\|x_1 - x_0\|}{\|x_0\|} \leq \|A^{-1}\| \|A\| \frac{\|\delta y\|}{\|y\|}$$

ii).

$$A(x_2 - x_0) = y - \delta A x_2 - y = -\delta A x_2.$$

$$\Rightarrow (x_2 - x_0) = -A^{-1} \delta A x_2.$$

$$\frac{\|x_2 - x_0\|}{\|x_2\|} \leq \|A^{-1}\| \|\delta A\| \|x_2\|$$

$$= \frac{\|A^{-1}\| \|A\| \|\delta A\|}{\|A\|} \|x_2\|$$

$$(iii) \kappa(A) = \|A\| \|A^{-1}\| \geq \|AA^{-1}\| = \|I\| = 1.$$

+ if $\kappa(A)$ is close to 1, then the relative error is not much larger than the perturbation due to y or A : WELL-CONDITIONED.

+ if $\kappa(A)$ large: the relative error ^{in x} ^{by} can far exceed the one in y or A . BADLY (ILL) CONDITIONED.

$$s/c/ \quad i/ \quad Ax_i = \lambda_i x_i \Rightarrow A^{-1}x_i = \frac{1}{\lambda_i} x_i$$

ii/ x_i eigenvector associated to λ_i

$$\frac{\|Ax_i\|}{\|x_i\|} = |\lambda_i| \quad \forall i = 1 \dots n$$

$$\|A\| \geq \max_{i=1 \dots n} |\lambda_i|$$

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$$i \quad ii/ \quad \|A^{-1}\| \geq \max_{i=1 \dots n} \frac{1}{|\lambda_i|}$$

$\lambda_i \neq 0$ since A non-singular

$$\kappa(A) \geq \max_i |\lambda_i| \max_i \frac{1}{|\lambda_i|}$$

(10) A symmetric $(x_i)_{i=1}^n$ basis of eigenvectors orthonormal.

$$x = \sum_{i=1}^n x_i x_i$$

$$\|Ax\|_2^2 = \sum x_i^2 \lambda_i^2$$

$$\|x\|_2^2 = \sum x_i^2$$

$$\text{if } |\lambda_1| = \max_{i=1}^n |\lambda_i|$$

$$\frac{\|Ax\|_2^2}{\|x\|_2^2} \leq \lambda_1^2$$

$$\text{or? } \|Ax_1\|^2 = \lambda_1^2$$

$$\Rightarrow \|A\| = |\lambda_1| = \max |\lambda_i|$$

Similarly we have $\|A^{-1}\| = \max_{i=1}^n \left| \frac{1}{\lambda_i} \right|$

$$\Rightarrow \quad \kappa(A) = \max_i |\lambda_i| \max_i \frac{1}{|\lambda_i|}.$$

$\left(\frac{23}{23} \right)$