UNIVERSITY OF LONDON [I(1) 2001]

B.ENG. AND M.ENG. EXAMINATIONS 2001

For Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examination for the Associateship.

PART I : MATHEMATICS 1

Wednesday 6th June 2001 10.00 am - 1.00 pm

Answer EIGHT questions.

[*Before starting, please make sure that the paper is complete; there should be 6 pages, with a total of 10 questions. Ask the invigilator for a replacement if your copy is faulty.*]

1. Let

$$f(x) \;=\; \frac{x+3}{2x+1}\;.$$

   (i) Find the inverse function $f^{-1}(x)$ of $f(x)$.

   (ii) Write $f(x)$ as the sum of an even and an odd function.

   (iii) Find all solutions of the equation

$$f(f(x)) \;=\; 0\,.$$

   (iv) Find all solutions of the equation

$$\frac{1}{f(\cos\theta)} \;=\; 0\,.$$

2. Consider the curve defined by the equation

$$y^2 \;=\; x^2 \;-\; \frac{x^4}{4}\;.$$

   (i) Find the coordinates of all stationary points of the curve.

   (ii) Find the coordinates of all points at which $\dfrac{dy}{dx}$ becomes infinite.

   (iii) Sketch the curve.

PLEASE TURN OVER

3. Find $\dfrac{dy}{dx}$ in each of the following cases.

In case (v) *you may express your answer in terms of x and y.*

(i) $$y = e^{\sin x}.$$

(ii) $$y = \ln(\ln x).$$

(iii) $$y = x^2 e^x \cos x.$$

(iv) $$y = x^{\ln x}.$$

(v) $$xy + \ln(xy) = 1.$$

4. (i) Show that if $y = (\sin^{-1} x)^2$, then

$$(1-x^2)^{1/2} \frac{dy}{dx} = 2 \sin^{-1} x.$$

Hence or otherwise show that y satisfies the equation

$$(1-x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} - 2 = 0\,.$$

(ii) Find the n^{th} derivatives of the functions $f(x) = e^{3x}$ and $h(x) = x^2 e^{3x}$.

(iii) Two sides of a triangle are of unit length and meet at angle θ. The length of the third side is given by $l(\theta) = (2 - 2\cos\theta)^{1/2}$. Find $dl/d\theta$.

By using the formula

$$\frac{dl}{d\theta} = \lim_{h \to 0} \frac{l(\theta+h) - l(\theta)}{h}\,,$$

find the approximate change in l if θ changes from $\dfrac{\pi}{3}$ to $\dfrac{\pi}{3} + 0.01$ (in radians).

5. Evaluate the following limits:

(i) $$\lim_{x \to 5} \frac{3 \; - \; \sqrt{x+4}}{x-5} \; ;$$

(ii) $$\lim_{x \to 0} \; x^{-3} \, \tan^3 (3x) \, ;$$

(iii) $$\lim_{x \to 0} \frac{\ln (1+3x^2)}{1+x-e^x} \; ;$$

(iv) $$\lim_{x \to \pi/3} \frac{1+\cos 3x}{\sqrt{3} - \tan x} \; .$$

6. Evaluate the following integrals :

(i) $$\int_1^e \frac{(\ln x)^2}{x} \, dx \, ;$$

(ii) $$\int_0^1 \sqrt{1-x^2} \; dx \; ;$$

(iii) $$\int \frac{x \, dx}{(1+x^2)^2} \; ;$$

(iv) $$\int \frac{x^2 \, dx}{(1+x^2)^2} \; .$$

PLEASE TURN OVER

7. (i) Express the function

$$\frac{2x}{(x^2+1)\,(x-1)}$$

in partial fraction form, and hence find

$$\int \frac{2x\,dx}{(x^2+1)\,(x-1)}\ .$$

(ii) Let

$$I_n \;=\; \int_0^{\pi} \sin^n x\,dx.$$

By integrating by parts, prove that for $n \geq 2$,

$$I_n \;=\; \frac{n-1}{n}\, I_{n-2}\,.$$

Hence find

$$\int_0^{\pi} \sin^6 x\,dx\,.$$

8. (i) Find which of the following series converge:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}\,;$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{2^n}\,;$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+10}\,.$

(ii) Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} (n+1)\,x^n\,.$$

(iii) Find $\dfrac{d^n}{dx^n}\,(1-x)^{-2}$ and hence show that the Maclaurin expansion of $(1-x)^{-2}$ is given by the series in part (ii).

9. (i) Express each of the following in the form $a + ib$:

$$\text{(a) } (1+i)^2\,, \qquad \text{(b) } \frac{1+i}{1-i}\,, \qquad \text{(c) } \left(\frac{\sqrt{3}+i}{2}\right)^{101}.$$

(ii) Find all complex roots z of the equation

$$z^4 = \frac{1}{4}\,(1+i)^4\,.$$

Show on a diagram where these roots lie.

What is the sum of all the roots?

(iii) If $z = x + iy$, express the equation

$$z + \bar{z} = \frac{1}{z} + \frac{1}{\bar{z}}$$

in terms of x and y. Hence sketch the solution curves of this equation in the complex plane.

10. (i) (a) Define the functions $\sin z$, $\cos z$, $\sinh z$, $\cosh z$ (where z is a complex number) in terms of the exponential function.

(b) Find all complex roots z of the equation $\tanh z = i$.

(c) Hence or otherwise find all roots of the equation $\tan^2(iz) = 1$.

(ii) If $z = x + iy$, find the real and imaginary parts of $\cos(z^2)$ in terms of trigonometric and hyperbolic functions of x and y.

Hence, find all complex numbers such that $\cos(z^2)$ is real.

END OF PAPER

PART I - MATHS I - 2001

EXAMINATION QUESTIONS/SOLUTIONS | SESSION 2000-2001 | COURSE B. Eng.

Setters are advised that Checkers, Editors, Typists and External Examiners greatly appreciate the merits of accuracy, legibility and neatness.

Write on one side only, between the margins, double-spaced. Not more than one question or solution per sheet, please

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QUESTION NO.

SOLUTION NO. 1

Mark Scheme

(a) $f = \dfrac{x+3}{2x+1} \Rightarrow (2x+1)f = x+3$

$\Rightarrow (2f-1)x = 3-f$

$\Rightarrow x(f) = \dfrac{3-f}{2f-1}$

$\therefore f^{-1}(x) = \dfrac{3-x}{2x-1}$ — 3

(b) $f(x) = \left[\dfrac{f(x)+f(-x)}{2}\right] + \left[\dfrac{f(x)-f(-x)}{2}\right]$

$= \underbrace{\dfrac{2x^2-3}{4x^2-1}}_{\text{even}} + \underbrace{\dfrac{5x}{4x^2-1}}_{\text{odd}}$ — 3

(c) $f(f(x)) = \dfrac{f(x)+3}{2f(x)+1} = \dfrac{\dfrac{x+3}{2x+1}+3}{\dfrac{2(x+3)}{2x+1}+1}$

$= \dfrac{7x+6}{4x+7}$ — 5

$\therefore f(f(x)) = 0 \Leftrightarrow x = -\dfrac{6}{7}$

(d) $\dfrac{1}{f(\cos\theta)} = \dfrac{2\cos\theta+1}{\cos\theta+3} = 0 \Leftrightarrow \cos\theta = -\tfrac{1}{2}$ — 4

$\therefore \theta = \begin{cases} 2\pi/3 + 2k\pi \\ 4\pi/3 + 2k\pi \end{cases}$ k any integer

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MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

SESSION : 2000 – 2001

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PAPER 1

QUESTION

SOLUTION 2

(a) $y^2 = x^2 - x^4/4$

Differentiating wrt x:

$$2y\frac{dy}{dx} = 2x - x^3$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2x - x^3}{2y} = \frac{x(2-x^2)}{2y} \quad \text{Ⓐ}$$

Stat. pts $\longleftrightarrow$ $\frac{dy}{dx} = 0$

Only stat pts are at $x = \pm\sqrt{2}$

i.e $(\sqrt{2}, 1)$; $(\sqrt{2}, -1)$; $(-\sqrt{2}, 1)$; $(-\sqrt{2}, -1)$ 6

__N.B__ __no__ stat pt at $(0,0)$ – look at limit as $x \to 0$

(b) From Ⓐ, $\frac{dy}{dx} \to \infty$ when $\frac{dy}{dx} = \pm\frac{x(2-x^2)}{2x\sqrt{1-x^2/4}} \to \infty$ 4

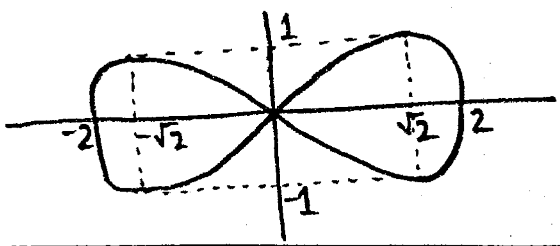
i.e when $x = \pm 2$

(c) Note curve invariant under transformations

$x \leftrightarrow -x$

$y \leftrightarrow -y$

$\Rightarrow$ reflectionally-symmetric in x & y axes 5



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PAPER I.1

QUESTION

SOLUTION 3

i) $y' = \frac{d}{dx} e^u$, where $u = \sin x$,

$= \frac{du}{dx} \frac{d}{du} e^u = \cos x \, e^u = \cos x \, e^{\sin x}$. 3

ii) $y' = \frac{d}{dx} \ln u$, where $u = \ln x$,

$= \frac{du}{dx} \frac{d}{du} \ln u = \frac{1}{x} \frac{1}{u} = \frac{1}{x \ln x}$. 3

iii) $y' = (x^2 v)'$ where $v = e^x \cos x$

$= 2xv + x^2 v'$

$= 2x e^x \cos x + x^2 e^x \cos x - x^2 e^x \sin x$,

(or use logarithmic differentiation). 3

iv) $\ln y = (\ln x)^2$.

$\therefore \frac{y'}{y} = \frac{2}{x} \ln x$, $y' = 2x^{-1} y \ln x = 2x^{\ln x - 1} \ln x$. 3

v) $xy + \ln x + \ln y = 1$,

$\therefore y + xy' + \frac{1}{x} + \frac{y'}{y} = 0$, $y' \frac{xy+1}{y} + \frac{xy+1}{x} = 0$. 3

$y' = -y/x$.

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i) $y' = 2(1-x^2)^{-1/2}\sin^{-1}x$ so $(1-x^2)^{1/2}y' = 2\sin^{-1}x.$ 2

$\therefore -2x\frac{1}{2}(1-x^2)^{-1/2}y' + (1-x^2)^{1/2}y'' = 2(1-x^2)^{-1/2}$,

so $(1-x^2)y'' - xy' - 2 = 0.$ 3

ii) $y' = 3e^{3x}, y'' = 3^2e^{3x}, \ldots, y^{(n)} = 3^n e^{3x}.$ 2

With $f(x) = x^2$, $g(x) = e^{3x}$,

$h^{(n)} = (fg)^{(n)} = fg^{(n)} + {}^nC_1 f' g^{(n-1)} + {}^nC_2 f'' g^{(n-2)} + \ldots$

$= x^2 3^n e^{3x} + n2x3^{n-1}e^{3x} + n(n-1)3^{n-2}e^{3x}.$ 2

iii) $\frac{df}{d\theta} = \sin\theta\,(2-2\cos\theta)^{-1/2}$ 2

$f(\theta+h) - f(\theta) \simeq h\frac{df}{d\theta}$ for h small. 2

$\therefore$ For $\theta = \pi/3$, $h = 0.01$,

the change in f is $\simeq 0.01.\frac{\sqrt{3}}{2}\cdot 1 = \frac{1.732}{200}$

$= 0.00866.$ 2

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EXAMINATION QUESTION / SOLUTION

SESSION : 2000 - 2001

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	PAPER	I (i
	QUESTIO	
	SOLUTIO	5
(i) $\dfrac{3-\sqrt{x+4}}{x-5} = \dfrac{5-x}{(x-5)(3+\sqrt{x+4})} = \dfrac{-1}{3+\sqrt{x+4}}$. As $x \to 5$, this $\to \dfrac{-1}{3+\sqrt{9}} = -\tfrac{1}{6}$		3
(ii) $x^{-3}\tan^3(3x) = \left(\dfrac{\sin 3x}{3x}\right)^3 \cdot \dfrac{27}{\cos^3(3x)}$. As $x \to 0$, this $\to 1^3 \cdot \dfrac{27}{1^3} = 27$		3
(iii) $\dfrac{\ln(1+3x^2)}{1+x-e^x} = \dfrac{+3x^2+\ldots}{-x^2/2+\ldots}$. As $x \to 0$, this $\to \dfrac{3}{-1/2} = -6$.		4
(iv) By l'Hôpital's rule, $\lim\limits_{x\to\pi/3} \dfrac{1+\cos 3x}{\sqrt{3}-\tan x} = \lim\limits_{x\to\pi/3} \dfrac{-3\sin 3x}{-\sec^2 x} = \lim\limits_{x\to\pi/3} \left(+3\sin 3x \cos^2 x\right) = 0$		5

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EXAMINATION QUESTION / SOLUTION

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PAPER I(i)

QUESTION

SOLUTION 6

(i) Set $u = \ln x$, so $du = dx/x$. The integral becomes $\int_0^1 u^2\,du = \left[\frac{1}{3}u^3\right]_0^1 = \frac{1}{3}$. 3

(ii) Set $x = \sin u$. The integral becomes

$$\int_0^{\pi/2} \cos^2 u\,du = \frac{1}{2}\int_0^{\pi/2}(1+\cos 2u)\,du$$ 4

$$= \frac{1}{2}\left[u + \frac{1}{2}\sin 2u\right]_0^{\pi/2} = \pi/4.$$

(iii) Set $x^2 = u$. The integral becomes

$$\int \frac{\frac{1}{2}\,du}{(1+u)^2} = \frac{-1}{2(1+u)} = -\frac{1}{2(1+x^2)} \;(+c).$$ 3

(iv) By (iii), the integrand is $x\frac{d}{dx}\left(-\frac{1}{2(1+x^2)}\right)$.

Integrating by parts, we get that the given integral is $= -\frac{x}{2(1+x^2)} + \frac{1}{2}\int\frac{dx}{1+x^2}$ 5

$$= +\frac{1}{2}\left[\tan^{-1}x - \frac{x}{1+x^2}\right] \;(+c).$$

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EXAMINATION QUESTION / SOLUTION

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PAPER I (i)

QUESTION

SOLUTION 7

(i) $$\frac{2x}{(x^2+1)(x-1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1}\ ;\ \text{so}$$

$$2x = (Ax+B)(x-1) + C(x^2+1).$$

Comparing coefficients $\Rightarrow$ $A = -1$, $B = C = 1$, so

$$\frac{2x}{(x^2+1)(x-1)} = \frac{-x+1}{x^2+1} + \frac{1}{x-1}$$ 5

$$\therefore \int \frac{2x\,dx}{(x^2+1)(x-1)} = -\tfrac{1}{2}\ln(x^2+1) + \tan^{-1}x + \ln|x-1|,\ (+c).$$ 3

(ii) $$I_n = \int_0^{\pi} \sin^n x\,dx = -\int_0^{\pi} \sin^{n-1}x \frac{d}{dx}(\cos x)\,dx$$

$$= -\Big[\sin^{n-1}x\cos x\Big]_0^{\pi} + \int_0^{\pi} (n-1)\sin^{n-2}x\cos^2 x\,dx.$$ 5

$$= 0 + (n-1)\int_0^{\pi} \sin^{n-2}x\,(1-\sin^2 x)\,dx$$

$$= (n-1)\left(I_{n-2} - I_n\right).$$

Hence $nI_n = (n-1)I_{n-2}$, i.e. $I_n = \frac{n-1}{n} I_{n-2}$

$$\therefore I_6 = \frac{5}{6}I_4 = \frac{5}{6}\frac{3}{4}I_2 = \frac{5}{6}\frac{3}{4}\frac{1}{2}I_0$$ 2

$$= \frac{5}{16}I_0 = \frac{5\pi}{16}.$$

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MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

SESSION : 2000 - 2001

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PAPER I.1

QUESTION

SOLUTION 8

i) a) Using the Ratio Test

$$\left|\frac{(n+1)^{th}\text{ term}}{n^{th}\text{ term}}\right| = \frac{n+1}{2^{n+1}}\frac{2^n}{n} = \frac{n+1}{n}\frac{1}{2} \to \frac{1}{2} \text{ as } n\to\infty.$$

Limit is < 1. $\therefore$ Series converges. 2

b) Using the Ratio Test

$$\left|\frac{(n+1)^{th}\text{ term}}{n^{th}\text{ term}}\right| = \frac{(n+1)!}{2^{n+1}}\frac{2^n}{n!} = \frac{n+1}{2} \to \infty \text{ as } n\to\infty.$$

Limit is > 1 $\therefore$ Series diverges. 2

c) n^{th} term $\to 1$ as $n\to\infty$. $\therefore$ Divergent since n^{th} term $\not\to 0$. 2

ii) Fix $x \neq 0$. $\left|\frac{(n+1)^{th}\text{ term}}{n^{th}\text{ term}}\right| = \frac{n+2}{n+1}|x| \to |x|$ as $n\to\infty$.

$\therefore$ By Ratio Test, the series converges if limit $|x| < 1$ and diverges if limit $|x| > 1$.

So radius of convergence $= 1$. 4

iii) Put $f(x) = (1-x)^{-2}$. $f' = 2(1-x)^{-3}$, $f'' = 2.3(1-x)^{-4}$,

$f''' = 2.3.4\,(1-x)^{-5}$, ... , $f^{(n)} = (n+1)!\,(1-x)^{-n-2}$. 3

Maclaurin series has n^{th} term

$$\frac{f^{(n)}(0)}{n!}x^n = \frac{(n+1)!}{n!}x^n = (n+1)\,x^n$$

giving the series in ii). 2

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MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

SESSION : 2000 – 2001

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PAPER 1

QUESTION

SOLUTION 9

(i) (a) $(1+i)^2 = 2i$ 1

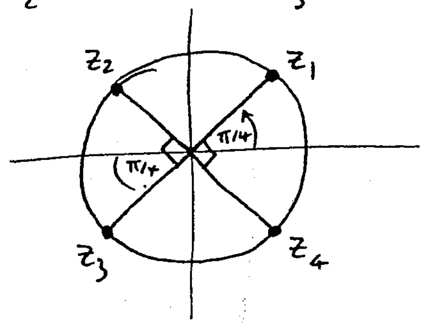
(b) $\dfrac{1+i}{1-i} = \dfrac{(1+i)(1+i)}{(1-i)(1+i)} = i$ 2

(c) $\left(\dfrac{\sqrt{3}+i}{2}\right)^{101} = \left(e^{i\pi/6}\right)^{101} = e^{16i\pi + \frac{5i\pi}{6}} = \dfrac{(-\sqrt{3}+i)}{2}$ 2

(ii) $z^4 = \frac{1}{4}(1+i)^4 = \left(\dfrac{1+i}{\sqrt{2}}\right)^4 = \left(e^{i\pi/4}\right)^4 = e^{i\pi + 2in\pi}$

$\Rightarrow \quad z = e^{i\pi/4 + in\pi/2} \qquad n = 0, 1, 2, 3$

$z_1 = e^{i\pi/4}, \quad z_2 = e^{3i\pi/4}, \quad z_3 = e^{5i\pi/4}, \quad z_4 = e^{7i\pi/4}$ 3



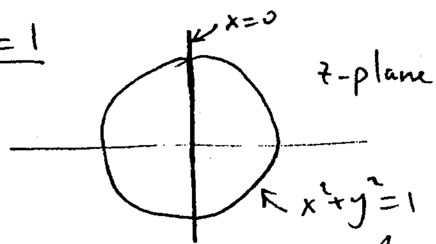
2

$\sum z_i = 0$ from the diagram 1

(iii) $x + iy + x - iy = \dfrac{1}{x+iy} + \dfrac{1}{x-iy}$

$2x = \dfrac{x-iy}{x^2+y^2} + \dfrac{x+iy}{x^2+y^2} \Rightarrow 2x = \dfrac{2x}{x^2+y^2}$ 3

$\underline{x = 0}$ OR $\underline{x^2+y^2 = 1}$



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MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

SESSION : 2000 - 2001

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PAPER 1

QUESTION

SOLUTION 10

(i) (a) $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$, $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$

$\sinh z = \dfrac{e^{z} - e^{-z}}{2}$, $\cosh z = \dfrac{e^{z} + e^{-z}}{2}$ 2

(b) $\dfrac{e^{z} - e^{-z}}{e^{z} + e^{-z}} = i \Rightarrow e^{2z} = \dfrac{1+i}{1-i} = i$

$|i| = 1$ $\arg(i) = \frac{\pi}{2} + 2n\pi \Rightarrow 2z = \ln 1 + i\left(\frac{\pi}{2} + 2n\pi\right)$

$z = i\left(\frac{\pi}{4} + n\pi\right)$ any integer n. 3

(c) $\tan^2 iz = 1 \Rightarrow \tan iz = 1 \Rightarrow i\tanh z = 1 \Rightarrow \tanh z = i$

$\tan iz = -1 \Rightarrow i\tanh z = -1 \Rightarrow \tanh z = -i$ 2

$\tanh z = i$ as in part (b)

$\tanh z = -i \Rightarrow \tanh(-z) = i \Rightarrow -z = i\left(\frac{\pi}{4} + n\pi\right)$

$z = i\left(-\frac{\pi}{4} + n\pi\right)$ 2

n any integer

(ii) $\cos(z^2) = \cos(x^2 - y^2 + 2ixy) = \cos(x^2 - y^2)\cos(2ixy)$

$- \sin(x^2 - y^2)\sin(2ixy)$

$= \cos(x^2 - y^2)\cosh 2xy - i\sin(x^2 - y^2)\sinh 2xy$ 3

$\cos(z^2)$ real $\Rightarrow \sin(x^2 - y^2)\sinh 2xy = 0$ 3

$\Rightarrow \underline{x = 0}$ or $\underline{y = 0}$ or $\underline{x^2 - y^2 = n\pi}$

n any integer

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