EE4-45
EE9CS7-21
EE9SO22

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2012

MSc and EEE/ISE PART IV: MEng and ACGI

WAVELETS AND APPLICATIONS

Tuesday, 15 May 10:00 am

Time allowed: 3:00 hours

There are FOUR questions on this paper.

Answer ALL questions.

All questions carry equal marks.

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s) :	P.L. Dragotti
	Second Marker(s) :	K.D. Harris

Special Information for the Invigilators: NONE

Information for Candidates:

Poisson summation formula:

$$\sum_{n=-\infty}^{\infty} f(t-n) = \sum_{k=-\infty}^{\infty} \hat{f}(2\pi k) e^{j2\pi kt}.$$

The Questions

1. Consider a set of linearly independent functions $\{\varphi_i\}_{i=1,2,..,N}$ that covers the sub-space V, that is, $V = span\{\varphi_i\}_{i=1,2,...N}$. Consider a function $f(t) \in L_2(\mathbb{R})$, the orthogonal projection of $f(t)$ onto V is:

$$\hat{f}(t) = \sum_{i=1}^{N} c_i \varphi_i(t),$$

with $c_i = \langle f(t), \tilde{\varphi}_i(t) \rangle$, $i = 1, 2, ...N$. Here $\{\tilde{\varphi}_i\}_{i=1,2,..,N}$ is the dual basis of $\{\varphi_i\}_{i=1,2,..,N}$. Based on the above formula and assuming

$$f(t) = \begin{cases} t & \text{for} \quad 0 \leq t < 1 \\ 0 & \qquad \text{otherwise} \end{cases}$$

(a) compute the coefficients c_i of the orthogonal projection of $f(t)$ onto the space spanned by $\varphi(t), \psi(t), \sqrt{2}\psi(2t), \sqrt{2}\psi(2t-1)$ with

$$\varphi(t) = \begin{cases} 1 & \text{for} \quad 0 \leq t < 1 \\ 0 & \qquad \text{otherwise} \end{cases}$$

and

$$\psi(t) = \begin{cases} 1 & \text{for} \quad 0 \leq t < 1/2 \\ -1 & \text{for} \quad 1/2 \leq t < 1 \\ 0 & \qquad \text{otherwise,} \end{cases}$$

[7]

(b) sketch and dimension the function $\hat{f}(t)$,

[6]

(c) compute the energy of the error function $\epsilon(t) = f(t) - \hat{f}(t)$,

[6]

(d) verify that $\|f(t)\|^2 = \|\hat{f}(t)\|^2 + \|\epsilon(t)\|^2$, where $\|x(t)\|^2 = \int_{-\infty}^{\infty} x^2(t)dt$.

[6]

2. Consider a filter bank specified by the following signal equations:

$$\begin{aligned} y_0 &= D_2 G D_2 G x \\ y_1 &= D_2 G D_2 H D_2 G x \\ y_2 &= D_2 H D_2 H D_2 G x \\ y_3 &= D_2 G D_2 G D_2 H x \\ y_4 &= D_2 H D_2 G D_2 H x \\ y_5 &= D_2 H D_2 H x, \end{aligned}$$

where G and H are the infinite matrix representations for filtering with a lowpass filter g_n and a highpass filter h_n, respectively, and D_2 is the matrix representation of down-sampling by 2.

(a) Draw a block diagram of the system using two-channel filter banks.

[8]

(b) Draw the equivalent single-level six-channel filter bank clearly specifying the down-sampling factors and transfer functions of the filters in each branch.

[8]

(c) Consider now a filter bank specified by the following signal equations:

$$\begin{aligned} y_0 &= D_2 G D_2 G x \\ y_1 &= D_2 H D_2 G x \\ y_2 &= D_2 H x. \end{aligned}$$

Draw the equivalent single-level three-channel filter bank and derive the exact transfer functions of the equivalent filters assuming that g_n and h_n are the low-pass and highpass Haar filters respectively. Specifically, the z-transform of g_n is $G(z) = (1+z)/\sqrt{2}$ and $H(z) = (1-z)/\sqrt{2}$.

[9]

3. Consider the tree-structured filter bank shown in Fig. 2.

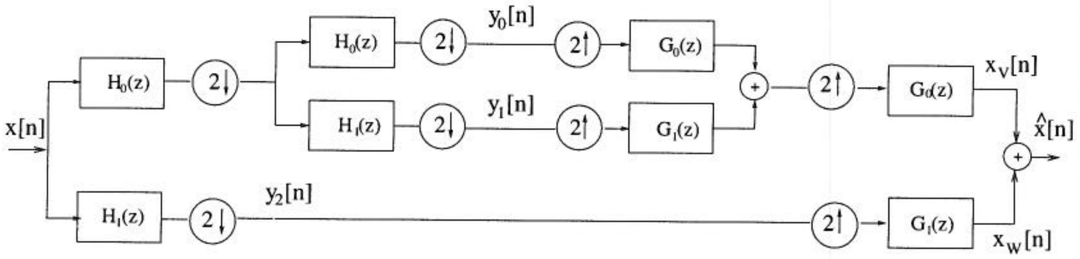


Figure 2: Tree-structured filter bank.

(a) Let $G_1(z) = \frac{3\sqrt{2}}{5}\left(\frac{1}{2} + \frac{1}{6}z^{-1} + \frac{1}{3}z^{-2} - z^{-3}\right)$. Design $G_0(z), H_0(z), H_1(z)$ in order to obtain an orthogonal perfect reconstruction filter bank.

[8]

(b) Find the zeros of $G_1(z)$ [Hint: if you correctly guess one of the zeros, you will be left with an easy factorization].

[7]

(c) Assume $x[n] = 1$ and ignore any boundary effect. Which of the signals $y_0[n], y_1[n], y_2[n]$ is nonzero? (Justify your answer).

[5]

(d) Assume now that $x[n] = n$ and again ignore any boundary effect. Which of the signals $y_0[n], y_1[n], y_2[n]$ is nonzero? (Justify your answer).

[5]

4. Suppose you are given a two-channel FIR filter bank with real coefficients and synthesis lowpass filter

$$g_0[n] = \frac{1}{2\sqrt{2}}(\delta_n + 2\delta_{n-1} + \delta_{n-2}).$$

(a) Is it possible that you were given an orthogonal filter bank? Justify your answer.

[6]

(b) Consider the equivalent filter

$$G_0^{(i)}(z) = \prod_{k=0}^{i-1} G_0(z^{2^k})$$

obtained by iterating the filter bank decomposition i times. Consider the function

$$\varphi^{(i)}(t) = 2^{i/2} g_0^{(i)}[n], \qquad n/2^i \leq t < (n+1)/2^i.$$

Can you say anything about the convergence of $\lim_{i\to\infty} \varphi^{(i)}(t)$?

[6]

(c) Assume that $\varphi(t) = \lim_{i\to\infty} \varphi^{(i)}(t)$ exists. We know that, in the case of convergence, $\varphi(t)$ is a valid scaling function. Therefore, by operating the frequency domain, show that $\varphi(t)$ satisfies partition of unity:

$$\sum_{n=-\infty}^{\infty} \varphi(t-n) = 1.$$

[6]

(d) Show that $\varphi(t)$ satisfy the two scale equation

$$\varphi(t) = \sqrt{2} \sum_{n=-\infty}^{\infty} g_0[n]\varphi(2t-n).$$

[6]

ANSWERS Wavelets - 2011-12

1.

(a) SINCE THE FUNCTIONS $\varphi(t)$, $\psi(t)$, $\sqrt{2}\psi(2t)$, $\sqrt{2}\psi(2t-1)$ ARE ORTHONORMAL, WE DO NOT NEED TO FIND THE DUAL BASIS: $\varphi_i(t) = \tilde{\varphi}_i(t)$.

THEREFORE

$$c_1 = \langle f(t), \varphi(t) \rangle = \int_0^1 t\,dt = \left.\frac{t^2}{2}\right|_0^1 = \frac{1}{2}$$

$$c_2 = \langle f(t), \psi(t) \rangle = \int_0^1 t\,\psi(t)\,dt = \int_0^{1/2} t\,dt - \int_{1/2}^1 t\,dt$$

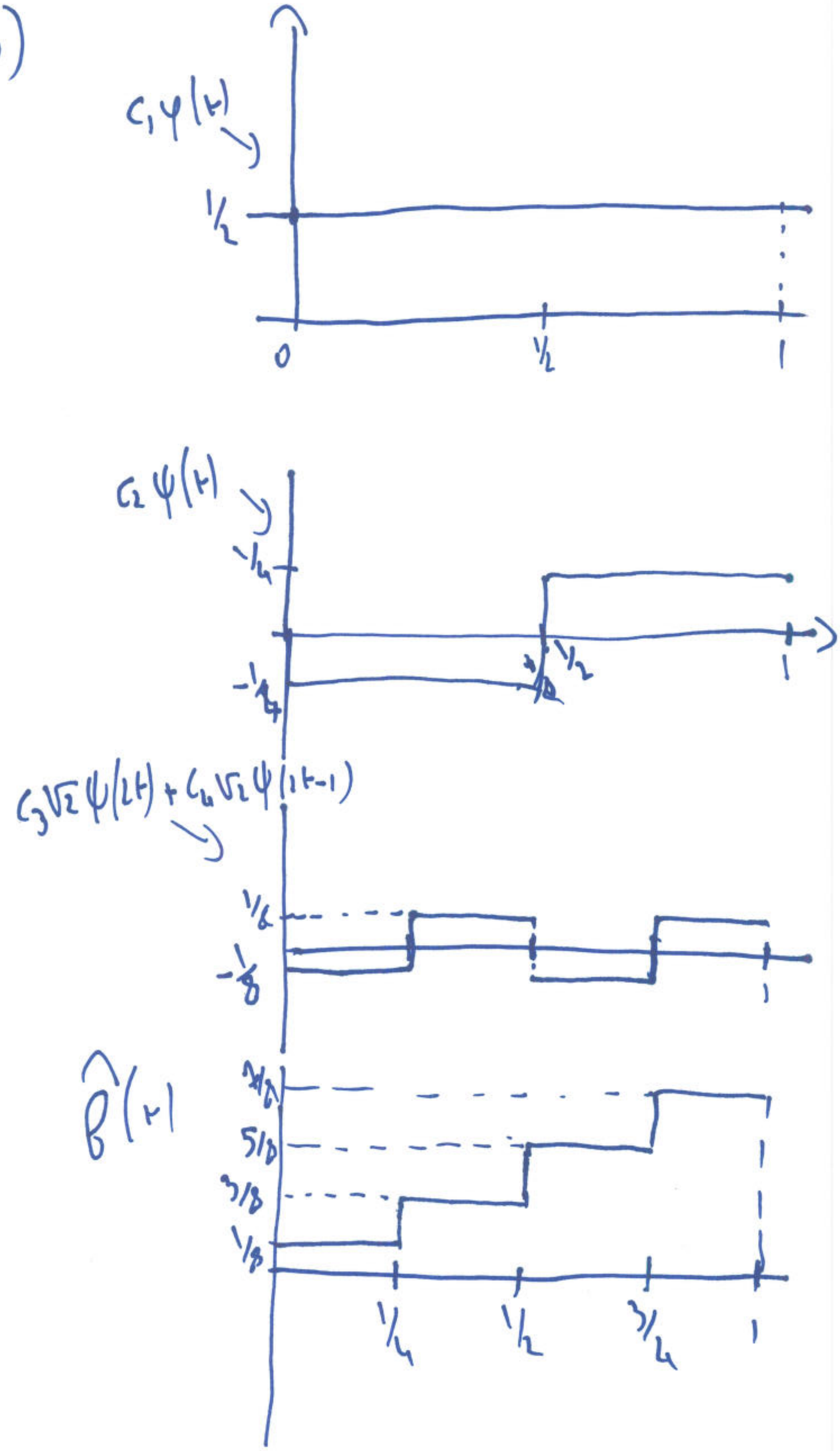
$$= \frac{1}{2}\left(\frac{1}{4} - 1 + \frac{1}{4}\right) = -\frac{1}{4}$$

$$c_3 = \langle f(t), \sqrt{2}\,\psi(2t) \rangle = \sqrt{2}\int_0^{1/4} t\,dt - \sqrt{2}\int_{1/4}^{1/2} t\,dt$$

$$= -\frac{\sqrt{2}}{16}$$

$$c_4 = c_3 = -\frac{\sqrt{2}}{16}$$

(b)



$c_1\psi(t)$
1/2
0
1/2
1
$c_2\psi(t)$
1/4
-1/4
1/2
1
$c_3\sqrt{2}\psi(2t)+c_4\sqrt{2}\psi(2t-1)$
1/8
-1/8
1
$\hat{\beta}(t)$
7/8
5/8
3/8
1/8
1/4
1/2
3/4
1

2

(c)

$$\|\varepsilon(t)\|^2 = 4\int_0^{1/4}\left(t-1/8\right)^2 dt$$

$$= \int_0^{1/4}\left(t^2+\frac{1}{16}-\frac{t}{4}\right)dt$$

$$= \frac{t^3}{3}\Big|_0^{1/4} + \frac{t}{64}\Big|_0^{1/4} - \frac{1}{4}\frac{t^2}{2}\Big|_0^{1/4}$$

$$= \frac{1}{3}\cdot\frac{1}{64}$$

(d)

USING PARSEVAL, WE HAVE THAT:

$$\|\hat{f}\|^2 = c_1^2+c_2^2+c_3^2+c_4^2 = \frac{1}{4}+\frac{1}{16}+\frac{4}{(16)^2} = \frac{1}{4}+\frac{1}{16}+\frac{1}{64} = \frac{21}{64}$$

MOREOVER

$$\|f\|^2 = \int_0^1 t^2 dt = \frac{1}{3}$$

THUS

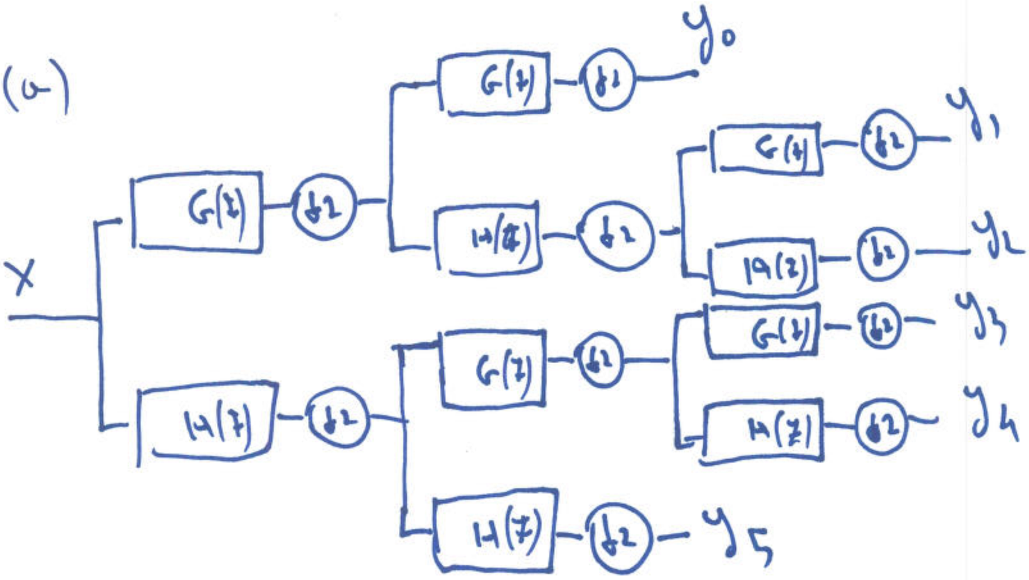
$$\|\hat{f}\|^2+\|\varepsilon\|^2 = \frac{1}{3}\cdot\frac{1}{64}+\frac{21}{64} = \frac{1}{3} = \|f\|^2$$

$\square$

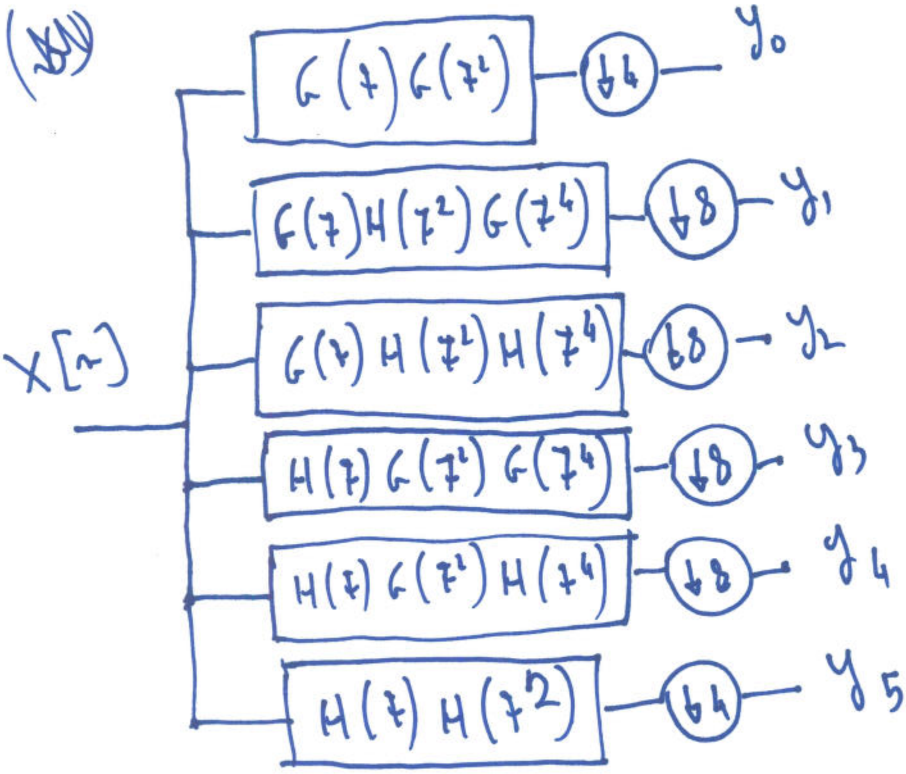
4

2.

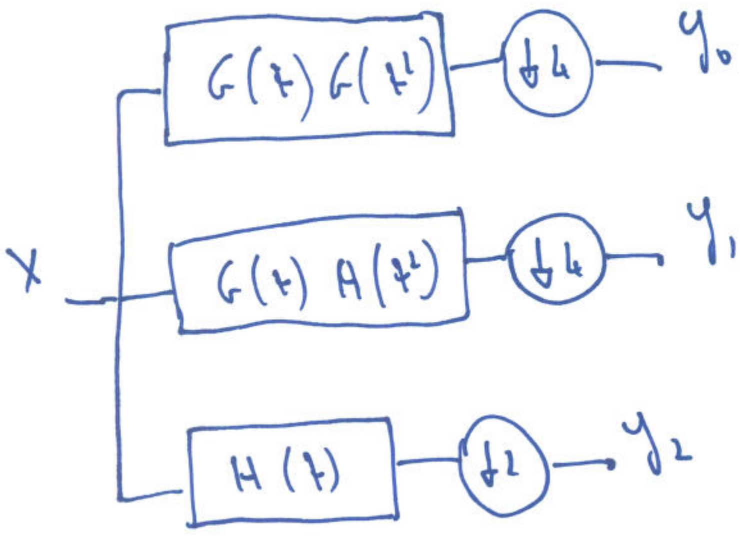
(a)



(b) USING NOBLE IDENTITY WE GET:



(c)



$$G(z)\,G(z^2) = \left(\frac{1+z}{\sqrt{2}}\right)\left(\frac{1+z^2}{\sqrt{2}}\right) = \frac{1}{2}\left(1+z+z^2+z^3\right)$$

$$G(z)\,H(z^2) = \left(\frac{1+z}{\sqrt{2}}\right)\left(\frac{1-z^2}{\sqrt{2}}\right) = \frac{1}{2}\left(1+z-z^2-z^3\right)$$

$$H(z) = \frac{1-z}{\sqrt{2}}$$

3 6

(a) USING SHIFT AND MODULATION WE OBTAIN

$$G_0(z) = -z^{-1} G_1(-z^{-1})$$

$$= \left(\frac{1}{2} z^{-1} - \frac{1}{6} + \frac{1}{3} z + z^{2}\right) \cdot \frac{3\sqrt{2}}{5}$$

THE OTHER TWO FILTERS ARE

$$H_0(z) = G_0(z^{-1}) = \frac{3\sqrt{2}}{5}\left(\frac{1}{2} z - \frac{1}{6} + \frac{1}{3} z^{-1} + z^{-2}\right)$$

$$H_1(z) = G_1(z^{-1}) = \frac{3\sqrt{2}}{5}\left(\frac{1}{2} + \frac{1}{6} z + \frac{1}{3} z^{2} - z^{3}\right)$$

(b)

WE TRY $z=1$

$$G_1(1) = \left(\frac{1}{2} + \frac{1}{6} + \frac{1}{3} - 1\right) \cdot \frac{3\sqrt{2}}{5} = 0 \checkmark$$

THUS $G_1(z)$ HAS A FACTOR $(1-z^{-1})$ SO

$$G_1(z) = (-z^{-1}+1)\left(\frac{1}{2} + \frac{2}{3} z^{-1} + z^{-2}\right)$$

THE OTHER TWO ROOTS ARE

$$z_{1,2} = \left(-\frac{1}{3} \pm j\frac{\sqrt{14}}{6}\right)^{-1} = -\frac{2}{3} \pm j\frac{\sqrt{14}}{3}$$

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(c) SINCE $G_1(z)$ HAS ONLY ONE ZERO AT $z=1$, ALSO $H_1(z)$ HAS ONLY ONE ZERO AT $z=1$

THUS THE HIGHPASS FILTER ANNIHILATES CONSTANTS BUT NOT HIGHER DEGREE POLYNOMIALS.

THUS $y_1[n]$ AND $y_2[n]$ ARE ZERO, BUT $y_0[n] \neq 0$

(d) AS DESCRIBED ABOVE $H(z)$ DOES NOT ANNIHILATES POLYNOMIALS WITH DEGREE GREATER THAN ZERO. THEREFORE

$y_0[n] \neq 0$, $y_1[n] \neq 0$, $y_2[n] \neq 0$.

4. 8

(a) THE ANSWER IS NO, BECAUSE THE GIVEN $g_0[n]$ IS SYMMETRIC AND WE KNOW THAT WITH THE ONLY EXCEPTION OF THE HAAR FILTER, IT IS NOT POSSIBLE TO DESIGN PERFECT-RECONSTRUCTION REAL-VALUED LINEAR-PHASE ORTHOGONAL FILTER BANKS

(b)

$$g_0[n] = \frac{1}{2\sqrt{2}}\left(\delta_n + 2\delta_{n-1} + \delta_{n-2}\right)$$

THUS

$$G_0(z) = \frac{1}{2\sqrt{2}}\left(1 + 2z^{-1} + z^{-2}\right)$$

$$= \frac{1}{2\sqrt{2}}\left(1 + z^{-1}\right)^2$$

$$G_0(e^{j\omega})\Big|_{\omega=0} = \frac{1}{\sqrt{2}} \quad \text{AND} \quad G_0(e^{j\omega})\Big|_{\omega=\pi} = 0$$

THE TWO NECESSARY CONDITIONS FOR THE LIMIT TO EXIST ARE SATISFIED.

MOREOVER,

IF WE DENOTE WITH $M_0(\omega) = \frac{G_0(e^{j\omega})}{\sqrt{2}}$

WE HAVE,

THAT $M_0(\omega) = \left(\frac{1+e^{-j\omega}}{2}\right)^N \cdot R(\omega)$

WITH $R(\omega)=1$ AND $N=2$.

WE DENOTE WITH $B = \sup_\omega R(\omega) = 1$

AND SINCE

$$B < 2^{N-1} = 2^{2-1} = 2$$

WE KNOW THAT ~~THE~~ A SUFFICIENT CONDITION FOR $\varphi(t)$ TO CONVERGE TO A CONTINUOUS FUNCTION IS SATISFIED.

(c)

10

[redacted] y using Poisson summation formula, we can show that $\varphi(t)$ satisfies partition of unity. The Poisson summation formula says that:

$$\sum_{n=-\infty}^{\infty} \varphi(t-nT) = \frac{1}{T}\sum_{k=-\infty}^{\infty} \hat{\varphi}\left(\frac{2\pi k}{T}\right) e^{j2\pi kt/T}$$

and we want to verify that:

$$\sum_{n=-\infty}^{\infty} \varphi(t-n) = 1.$$

Thus by combining the two equations and for $T = 1$, we obtain the following:

$$\sum_{n=-\infty}^{\infty} \varphi(t-n) = \sum_{k=-\infty}^{\infty} \hat{\varphi}(2\pi k) e^{j2\pi kt} = 1.$$

The condition $\sum_{k=-\infty}^{\infty} \hat{\varphi}(2\pi k) e^{j2\pi kt} = 1$ is then clearly satisfied. Indeed, by using the infinite product formula [redacted] and since $G(e^{j\omega}) = \sqrt{2}$ for $\omega = 0$ and $G(e^{j\omega}) = 0$ for $\omega = \pi$, we have that $\hat{\varphi}(2\pi k) = 1$ for $k = 0$ and $\hat{\varphi}(2\pi k) = 0$ otherwise. $\square$

(d) IN FREQUENCY DOMAIN THE TWO-SCALE EQUATION CAN BE WRITTEN AS FOLLOWS:

$$\varphi(t) = \sqrt{2}\sum_m g_0[m]\,\varphi(2t-m) \Longleftrightarrow \frac{1}{\sqrt{2}} G_0\left(e^{j\omega/2}\right)\hat{\varphi}\left(\frac{\omega}{2}\right)$$

MOREOVER WE KNOW THAT

$$\hat{\varphi}(\omega) = \lim_{i\to\infty} \hat{\varphi}^{(i)}(\omega) = \prod_{k=1}^{\infty} M_0\left(\frac{\omega}{2^k}\right)$$

WITH $M_0(\omega) \triangleq \dfrac{G_0\left(e^{j\omega}\right)}{\sqrt{2}}$

THEREFORE

$$\hat{\varphi}(\omega) = \prod_{k=1}^{\infty} M_0\left(\frac{\omega}{2^k}\right) = M_0\left(\frac{\omega}{2}\right)\prod_{k=2}^{\infty} M_0\left(\frac{\omega}{2^k}\right)$$

$$= M_0\left(\frac{\omega}{2}\right)\hat{\varphi}\left(\frac{\omega}{2}\right) = \frac{1}{\sqrt{2}} G_0\left(e^{j\omega/2}\right)\hat{\varphi}\left(\frac{\omega}{2}\right)$$

$\square$