IMPERIAL COLLEGE LONDON

EE4-10
EE9-CS5-1
EE9-SO26

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2013

MSc and EEE PART IV: MEng and ACGI

PROBABILITY AND STOCHASTIC PROCESSES

Tuesday, 21 May 10:00 am

Time allowed: 3:00 hours

There are THREE questions on this paper.

Answer ALL questions. All questions carry equal marks.

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s) :	M.M. Draief
	Second Marker(s) :	D. Angeli

PROBABILITY AND STOCHASTIC PROCESSES

1. We consider a setting where we have m indistinguishable balls that are placed one by one uniformly at random in one of n bins.

   a) Using the inequality $1-x \leq e^{-x}$, for $x \geq 0$, show that the probability that bin number i remains empty is smaller than $e^{-m/n}$. [2]

   b) We now let $m = n$. Show that the probability that every bin gets a ball goes to 0 as n gets to infinity.

      Hint: Use the fact that $n! \leq \frac{n^n}{2^{n/2}}$ [2]

   c) We now let $m = 2n\ln n$ where $\ln(e) = 1$. Let A_i be the event that the i-th bin is empty.

      i) Show that $\mathbf{P}(A_i) \leq 1/n^2$. [2]

      ii) Using an induction show the following inequality known as the union bound

      $$\mathbf{P}(A_1 \cup A_2 \cup \cdots \cup A_n) \leq \mathbf{P}(A_1) + \mathbf{P}(A_2) + \cdots + \mathbf{P}(A_n).$$

      [2]

      iii) Conclude that the probability that some bin is empty is smaller than $1/n$. [3]

   d) We are now back to the general setting with m balls and n bins where $m \leq n$.

      i) Show that the probability that every bin gets 0 or 1 ball is smaller than

      $$\exp\left(-\frac{m(m-1)}{2n}\right).$$

      [3]

      ii) Assume that we are in a room containing m individuals. How large should m be so that the probability that two individuals are born on the same day (of the year) is greater than $1/2$? [2]

      We will assume that there are 365 days in a year and that each individual is equally likeeitly to be born on any of them and that the birthdays of individuals are independent.

   e) We say that there is a collision between ball i and ball j if they happen to fall in the same bin. Let $X_{ij} = 1$ if there is a collision between i and j and $X_{ij} = 0$ otherwise and let $X = \sum_{1 \leq i < j \leq n} X_{ij}$ be the total number of collisions.

      i) Show that the expected number of collisions

      $$\mathbf{E}(X) = \frac{m(m-1)}{2n}.$$

      [2]

      ii) Show that

      $$\mathbf{P}\left(X \geq \frac{m(m-1)}{n}\right) \geq \frac{1}{2}.$$

      [2]

2. Consider a stock that has correlation in its market performance. More precisely, if the stock has been up in the last two days, then it will be up today with probability 0.7. If it has been down in the last two days, then it will be up today with probability 0.3. If it was up yesterday and down two days ago, it will be up today with probability 0,6. If it was down yesterday and up two days ago, it will be up today with probability 0.4.

Assume that the stock has been up in the past two days.

a) Describe the stock's performance as a Markov chain. Describe the state space, the transition matrix and draw the diagram of the chain.

Hint: Consider the state of two consecutive days as your Markov chain. [2]

b) Is the chain irreducible? Justify your answer. [1]

c) Let T be the first day that the stock drops (we number today as day 0).

i) Find the probability that T will immediately be followed by another day in which the stock drops. [1]

ii) Find $\mathbf{P}(T = k)$, for $k = 0, 1, 2 \ldots$ [1]

iii) Compute its expectation $\mathbf{E}(T)$. [2]

d) Let π be the invariant distribution of the chain described in 2.a).

i) Derive π and explain why you find a unique such invariant distribution. [3]

ii) Find the average fraction of time that the stock goes up. [2]

iii) Find a good approximation to the probability that the stock will go up on the 10000-th day from now, given that it moves in the same direction on both days 10000 and 10001 from now. [3]

e) Let S be the number of days until the stock drops two days in a row (including those two days). Write a system of equations that can be used to calculate S.

You do **not** need to compute S. [2]

f) Assume that you will sell the stock if it falls for three days in a row. Let R be the number of days you will hold the stock. Write a system of equations that can be used to calculate R.

You do **not** need to compute R. [3]

3. A store has a parking lot with N spaces, which are numbered $1,2,\ldots,N$. The number of the parking spot denotes the distance from the front door of the store. Cars arrive according to a Poisson process at rate λ. Upon arrival, car parks in the *lowest numbered parking spot* that is available. If the parking lot is full, the car leaves immediately. Assume that each car park spends an exponentially distributed amount of time with mean $1/\mu$ in the parking lot, independently of other cars.

Let $\rho = \lambda/\mu$. In what follows we will use the following notation, for $n \geq 1$,

$$E(\rho,n) = \frac{\frac{\rho^n}{n!}}{1+\rho+\frac{\rho^2}{2!}+\cdots+\frac{\rho^n}{n!}}.$$

a) For this question only we assume that $N=1$. Describe the chain thus obtained and find its stationary distribution. [2]

b) We now assume that $N=2$.

i) What is the long run fraction of time that there is at least one car in the parking lot? [2]

ii) Assume that we have two cars in the parking lot. How long before either of them leaves the parking? [1]

iii) Describe the state of the parking lot as a Markov chain and show that its stationary distribution is given by, $\pi(i) = \frac{\rho^i/i!}{1+\rho+\frac{\rho^2}{2!}}$, $i = 0,1,2$. [1]

iv) Assume that the parking lot is in equilibrium as given by 3.b)iii), what is the mean distance from a car to the front of the store in terms of $E(\rho,1)$ and $E(\rho,2)$? [3]

c) In this question, we assume N to be some positive integer. It is clear that the number of cars in the parking lot constitutes a continuous-time Markov chain that this chain is ergodic and that its stationary distribution is given by

$$\pi(i) = \frac{\frac{\rho^i}{i!}}{1+\rho+\frac{\rho^2}{2!}+\cdots+\frac{\rho^N}{N!}}, \quad i = 0,\ldots N.$$

i) Derive the long run proportion of arriving cars that are turned away in terms of $E(\rho,N)$. [2]

ii) For $1 \leq n \leq N$, let $X_n(t)$ denote the number of free parking spaces at time t among the spaces numbered $1,2,\ldots,n$. For example, $X_4(t) = 2$ implies that two of the four spaces $1,2,3,4$ are free. For each n, is $\{X_n(t), t \geq 0\}$ a continuous-time Markov chain? [3]

d) Now assume that $N = \infty$, i.e., the parking lot has infinitely many spots. Assume that the parking lot is in equilibrium.

i) Assume that a new car arrives at time t, and let F be the distance from where it parks to the the front door of the store. Show that

$$\mathbf{P}(F > n) = \mathbf{P}(X_n(t) = 0) = E(\rho,n).$$

[3]

ii) Prove that an arriving car parks at an average distance $\mathbf{E}(F)$ from the front door of the store where $\mathbf{E}(F) = 1 + \sum_{n=1}^{\infty} E(\rho,n)$. [3]

PROBABILITY & STOCHASTIC PROCESSES
2012/2013.

Q1 1/3

1)

a) Prob. that 1^{st} ball miss ~~ball~~ bin i is $\frac{n-1}{n} = 1 - \frac{1}{n}$ which is the same for each of the subsequent balls and all these events are independent.

Hence, $\mathbb{P}(\text{bin remains empty}) = \left(1 - \frac{1}{n}\right)^{m}$ (1)

by the given inequality $\leq \left(e^{-1/n}\right)^{m}$ (1)

$= e^{-m/n}$.

$\boxed{2}$

b) $m = n$

$\mathbb{P}(\text{every bin gets a ball}) = \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \cdots \left(\frac{1}{n}\right)$. (1)

$= \frac{(n-1)}{n} \frac{(n-2)}{n} \cdots \frac{1}{n}$

$= \frac{(n-1)!}{n^{n-1}} = \frac{n!}{n^{n}}$ (1)

by the inequality provided $\leq \frac{1}{\sqrt{2}^{n}} \xrightarrow[n \uparrow \infty]{} 0$.

$\boxed{2}$

Q1 2/3

c) $m = 2n \log n$.

$A_i = \{ \text{i-th bin empty} \}$.

i) By qustion 1.a)

$$\mathbb{P}(A_i) \le e^{-m/n} = e^{-2\log n} \quad [1]$$

$$= \frac{1}{n^2} \quad [1]$$

$\boxed{2}$

ii) $\mathbb{P}(A_1 \cup A_2) = \mathbb{P}(A_1) + \mathbb{P}(A_2) - \mathbb{P}(A_1 \cap A_2)$

$\le \mathbb{P}(A_1) + \mathbb{P}(A_2)$. $(*)$ [1]

Suppose

$$\mathbb{P}(A_1 \cup .. \cup A_{n-1}) \le \sum_{i=1}^{n-1} \mathbb{P}(A_i) \quad (**)$$

$\mathbb{P}\left((A_1 \cup .. \cup A_{n-1}) \cup A_n \right)$ [1]

$\le \mathbb{P}(A_1 \cup .. \cup A_{n-1}) + \mathbb{P}(A_n)$ by $(*)$

$\le \sum_{i=1}^{n} \mathbb{P}(A_i)$ by $(**)$.

$\boxed{2}$.

iii)

$\mathbb{P}(\text{some bin is empty}) = \mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right)$ [1]

by 1.c. ii) $\le \sum_{i=1}^{n} \mathbb{P}(A_i)$ [1]

by 1.c. i) $\le n \frac{1}{n^2}$ [1]

$= 1/n$.

$\boxed{3}$

Q1 3/3

d)

i) $\mathbb{P}$(every bin gets on 1 ball) $= \left(1-\frac{1}{n}\right)\cdots\left(1-\frac{m-1}{n}\right)$. [1]

using $1-x \leq e^{-x}$ $\leq \exp\left\{-\frac{1}{n}\left(1+\cdots+(m-1)\right)\right\}$ [1]

$= \exp\left\{-\frac{m(m-1)}{2n}\right\}$. [1] $\boxed{3}$

ii) $\mathbb{P}$(2 individuals share the same birthday).

$= 1-\mathbb{P}$(every bin has at most one ball)

where bins = days of year.

throw ball: select a birthday of a person in the room.

bin with 2 balls: two people with the same birthday. [1]

$\mathbb{P}$(2 ind. share same birthday) $\geq 1 - e^{-\frac{m(m-1)}{2n}}$.

for $n = 365$ choose $m = 23$.

$\geq 1-e^{-\frac{m(m-1)}{2n}} \geq 1/2$ [1] $\boxed{2}$

e) i) $\mathbb{P}(X_{ij}=1) = \frac{1}{n}$ fix position of i & j has to fall in same bin. [1]

$\mathbb{E}(X_{ij}) = 1/n \Rightarrow \mathbb{E}\left(\sum_{i<j} X_{ij}\right) = \binom{m}{2} 1/n$ [1] $\boxed{2}$

$= m(m-1)/2n$.

ii) By Markov's inequality [1]

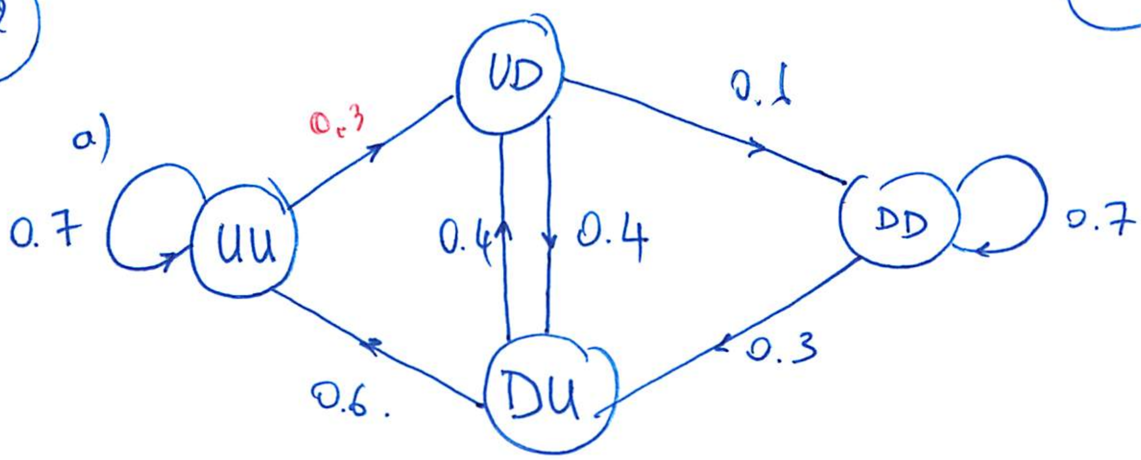
$\mathbb{P}\left(X \geq \frac{m(m-1)}{n}\right) \leq \frac{\mathbb{E}(X)}{m(m-1)/n} = \frac{\frac{m(m-1)}{2n}}{m(m-1)/n} = 1/2$ [1] $\boxed{2}$

PROBABILITY & STOCHASTIC PROCESSES

Q2 1/4

(2)

a)



[1] point for the right transitions

[1] " " correct probabilities.

$\boxed{2}$

b) The chain is irreducible since in the directed graph above there is a path (with non-zero probability) between any pair of states.

$\boxed{1}$

c)

i) T: 1st day stock drops.

U . . . . UDD

$\uparrow$ T

$\boxed{1}$

$\mathbb{P}$(T immediately followed by another drop)

$= \mathbb{P}(X_{T+1} = DD \mid X_T = UD) = 0.6.$

ii) $\{T = k\} = \{$ stock goes up k-times then down $\}$.

$\mathbb{P}(T = k) = (0.7)^k \cdot 0.3$

$\boxed{1}$

$k = 0, 1, 2 \ldots$

Q2 2/4

d)

i) The chain is irreducible, aperiodic (self loops in UU & DD).

$\Rightarrow$ it is ergodic so it has a unique stationary distribution. [1]

[1] Note by symmetry we expect

$\alpha = \pi(UU) = \pi(DD)$ &

$\beta = \pi(UD) = \pi(DU)$.

solving $\pi P = \pi$.

Looking at $\pi(UU)$: $\alpha = 0.7\alpha + 0.6\beta$

& $\pi(UD)$: $\beta = 0.3\alpha + 0.4\beta$.

Equations for DD & DU are redundant.

$\Rightarrow \alpha = 2\beta$.

Recall that $2\alpha + 2\beta = 1$ (π is a probability distribution).

$\Rightarrow \alpha = 1/3, \ \beta = 1/6$.

Hence $\boxed{\begin{array}{l}\pi(UU) = \pi(DD) = 1/3 \\ \pi(UD) = \pi(DU) = 1/6\end{array}}$ [1]

3

Q2 3/4

ii)

+ 1st method.

Fraction time the stock goes up

$= 0.7\,\pi(UU) + 0.4\,\pi(UD) + 0.6\,\pi(DU)$ [1]

$+ 0.3\,\pi(DD) = 1/2.$ [1]

+ 2nd method:

or

= Fraction time last state was U [1]

$= \pi(UU) + \pi(DU) = 1/2.$ [1]

$\boxed{2}$.

iii) Chain is ergodic so.

$\lim_{n\to\infty} \mathbb{P}(X_n = j \mid X_0 = UU) = \pi(j).$ [1]

& $\mathbb{P}(X_{10001} = UU \text{ or } X_{10001} = DD \mid X_0 = UU).$

$= \pi(UU) + \pi(DD).$ [1]

$\mathbb{P}(X_{10001} = UU \mid X_0 = UU) = \pi(U,U)$

$\mathbb{P}(X_{10001} = UU \mid X_{10001} = UU \text{ or } DD ; X_0 = UU)$

$$= \frac{\mathbb{P}(X_{10001} = UU \mid X_0 = UU)}{\mathbb{P}(X_{10001} = UU \text{ or } DD \mid X_0 = UU)}$$ [1]

$$= \frac{\pi(UU)}{\pi(UU) + \pi(DD)} = 1/2.$$

$\boxed{3}$

e) Let $K_{DD}(i)$ = the expected time to hit DD starting from state i

d2 4/4

$$K_{DD}(DD)=0$$

$$K_{DD}(UD)=1+0.4\,K_{DD}(DU)$$

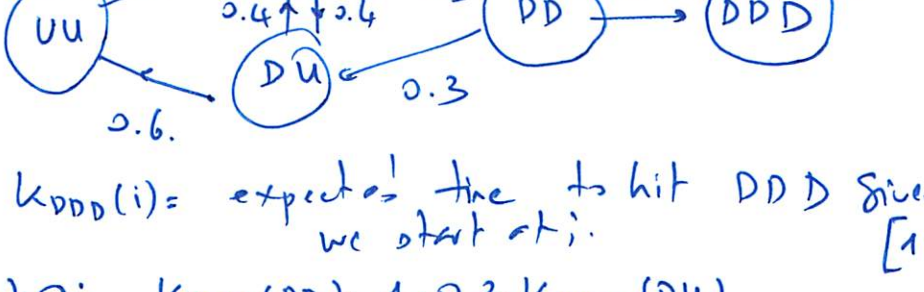
$$K_{DD}(DU)=1+0.4\,K_{DD}(UD)+0.6\,K_{DD}(UU).$$

$$DDD(UU)=1+0.7\,K_{DD}(UU)+0.3\,K_{DD}(UD).$$

[1] point for the use of hitting time results.
[1] point for the complete system equations. [2]

f) Note that the only way to have 3 down days in a row is to 1st have two down days. [1]

Add DDD a new state.



Let $K_{DDD}(i)$ = expected time to hit DDD given we start at i. [1]

$K_{DDD}(DDD)=0$; $K_{DDD}(DD)=1+0.3\,K_{DDD}(DU)$

$K_{DDD}(UD)=1+0.4\,K_{DDD}(DU)+0.6\,K_{DDD}(DD)$ [1] [3]

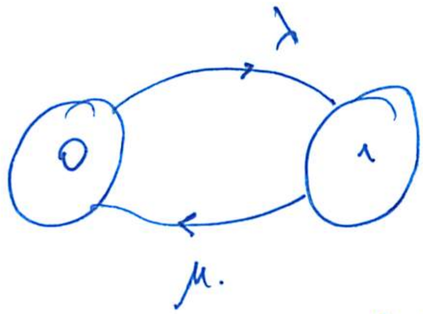
$K_{DDD}(DU)=1+0.4\,K_{DDD}(UD)+0.6\,K_{DDD}(UU).$

$K_{DDD}(UU)=1+0.7\,K_{DDD}(UU)+0.3\,K_{DDD}(UD)$

3)

3 1/4

a)



$$\begin{bmatrix} \pi_0 & \pi_1 \end{bmatrix} \begin{bmatrix} -\lambda & \lambda \\ +\mu & -\mu \end{bmatrix} = 0 \quad \Rightarrow \begin{cases} \pi_0 = \mu/_{\lambda+\mu} \\ \pi_1 = \lambda/_{\lambda+\mu} \end{cases}$$

[1] [1]

$\boxed{2}$

b) i) This is similar to question 1

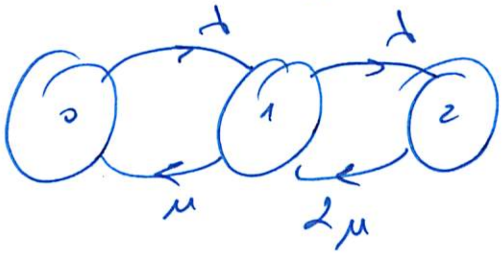
as we do not worry about state 2

Hence the answer is $\lambda/_{\lambda+\mu}$. [1]

as the chain for N=2 is ergodic so it is long run

behaviour is given by its stationary distribution [1]

$\boxed{2}$



? ~~Long run fraction that~~ more than 1 car is in parking lot.

ii) This is the minimum of two

independent exponential distribution

of parameter μ ~ Exp(2μ). [1]

1 Bonus point if they prove this

in detail [+1]

$\boxed{1}$

iii)

Q 3 2/4

$$Q = \begin{bmatrix} -\lambda & \lambda & 0 \\ \mu & -(\lambda+\mu) & \lambda \\ 0 & 2\mu & -2\mu \end{bmatrix}$$

and $[\pi_0 \ \pi_1 \ \pi_2]\, Q = 0$. with $\begin{cases} \pi_0 = \dfrac{1}{1+\rho+\rho^2/2} \\ \pi_1 = \dfrac{\rho}{1+\rho+\rho^2/2} \\ \pi_2 = \dfrac{\rho^2/2}{1+\rho+\rho^2/2} \end{cases}$ $\boxed{1}$

iv) Observe that an arriving car parks in plot 1 as long as that plot is free

whith probability; i.e.

$1 - E(\rho, 1)$ by question 3 b) i).

[1]

An arriving car parks in plot 1 or 2 if at least one of those is free i.e.

with probability

$1 - \pi_2 = 1 - E(\rho, 2)$. [1]

and the probability of parking in plot 2 is.

$[1 - E(\rho,2)] - [1 - E(\rho,1)]$.

$\boxed{3}$

So the expectation is.

$1 - E(\rho,1) + 2\,[E(\rho,1) - E(\rho,2)]$ [1].

$= 1 + E(\rho,1) - E(\rho,2)$

d) 4/4

c)

i) A car is turned away if there are N cars already in the parking lot

Since the car is ergodic the fraction of time this happens is given by [1]

$$\pi_N = E(\rho, N).$$

Hence $E(\rho, N)$ of the cars are turned away.

~~OK?~~ NOT SURE. sh- [1]

2

ii) Let $Y_n(t)$: be the number of full parking spaces among the first n.

if Y_n is a M.C. then $X_n = n - Y_n$ is also a M.C. (1)

Claim: Y_n is a continuous-time MC.

Proof:

if $Y_n(t) < n$ the next arrival will take a space in $\{1, .., n\}$ and Y_n increases by 1 [1]

if $Y_n(t) = n$ the next arrival will take a space in $\{n, .. N\}$ and Y_n is unchanged. [1] 3

Furthermore the service times are independent [3]

Q4 4/4

d)

i)

$$\mathbb{P}(F > n) = \mathbb{P}(X_n(t) = 0). \quad [1]$$

$$= E(\rho, n)$$

Since the system is ergodic and when we focus on the first n slots we can completely ignore the other ns and the problem reduces to a parking with n plots and so this is reminiscent of the result in 3c) i) where N is replace by n. [2]

$\boxed{3}$

ii).

$$\mathbb{E}(F) = \sum_{n=0}^{\infty} \mathbb{P}(F > n). \quad [2]$$

$$= 1 + \sum_{n \geq 0} E(\rho, n). \quad [1].$$

$\boxed{3}$

PROBABILITY AND STOCHASTIC PROCESSES

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   b) We now let $m = n$. Show that the probability that every bin gets a ball goes to 0 as n gets to infinity.

      Hint: Use the fact that $n! \le \frac{n^n}{2^{n/2}}$ [2]

   c) We now let $m = 2n\ln n$ where $\ln(e) = 1$. Let A_i be the event that the i-th bin is empty.

      i) Show that $\mathbf{P}(A_i) \le 1/n^2$. [2]

      ii) Using an induction show the following inequality known as the union bound

      $$\mathbf{P}(A_1 \cup A_2 \cup \cdots \cup A_n) \le \mathbf{P}(A_1) + \mathbf{P}(A_2) + \cdots + \mathbf{P}(A_n).$$

      [2]

      iii) Conclude that the probability that some bin is empty is smaller than $1/n$. [3]

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      i) Show that the expected number of collisions

      $$\mathbf{E}(X) = \frac{m(m-1)}{2n}.$$

      [2]

      ii) Show that

      $$\mathbf{P}\left(X \ge \frac{m(m-1)}{n}\right) \ge \frac{1}{2}.$$

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Hint: Consider the state of two consecutive days as your Markov chain. [2]

b) Is the chain irreducible? Justify your answer. [1]

c) Let T be the first day that the stock drops (we number today as day 0).

i) Find the probability that T will immediately be followed by another day in which the stock drops. [1]

ii) Find $\mathbf{P}(T = k)$, for $k = 0, 1, 2 \ldots$ [1]

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e) Let S be the number of days until the stock drops two days in a row (including those two days). Write a system of equations that can be used to calculate S.

You do **not** need to compute S. [2]

f) Assume that you will sell the stock if it falls for three days in a row. Let R be the number of days you will hold the stock. Write a system of equations that can be used to calculate R.

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3. A store has a parking lot with N spaces, which are numbered $1,2,\ldots,N$. The number of the parking spot denotes the distance from the front door of the store. Cars arrive according to a Poisson process at rate λ. Upon arrival, car parks in the *lowest numbered parking spot* that is available. If the parking lot is full, the car leaves immediately. Assume that each car park spends an exponentially distributed amount of time with mean $1/\mu$ in the parking lot, independently of other cars.

Let $\rho = \lambda/\mu$. In what follows we will use the following notation, for $n \geq 1$,

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a) For this question only we assume that $N = 1$. Describe the chain thus obtained and find its stationary distribution. [2]

b) We now assume that $N = 2$.

i) What is the long run fraction of time that there is at least one car in the parking lot? [2]

ii) Assume that we have two cars in the parking lot. How long before either of them leaves the parking? [1]

iii) Describe the state of the parking lot as a Markov chain and show that its stationary distribution is given by, $\pi(i) = \frac{\rho^i/i!}{1+\rho+\frac{\rho^2}{2!}}$, $i = 0,1,2$. [1]

iv) Assume that the parking lot is in equilibrium as given by 3.b)iii), what is the mean distance from a car to the front of the store in terms of $E(\rho,1)$ and $E(\rho,2)$? [3]

c) In this question, we assume N to be some positive integer. It is clear that the number of cars in the parking lot constitutes a continuous-time Markov chain that this chain is ergodic and that its stationary distribution is given by

$$\pi(i) = \frac{\frac{\rho^i}{i!}}{1+\rho+\frac{\rho^2}{2!}+\cdots+\frac{\rho^N}{N!}}, \quad i = 0,\ldots N.$$

i) Derive the long run proportion of arriving cars that are turned away in terms of $E(\rho,N)$. [2]

ii) For $1 \leq n \leq N$, let $X_n(t)$ denote the number of free parking spaces at time t among the spaces numbered $1,2,\ldots,n$. For example, $X_4(t) = 2$ implies that two of the four spaces $1,2,3,4$ are free. For each n, is $\{X_n(t), t \geq 0\}$ a continuous-time Markov chain? [3]

d) Now assume that $N = \infty$, i.e., the parking lot has infinitely many spots. Assume that the parking lot is in equilibrium.

i) Assume that a new car arrives at time t, and let F be the distance from where it parks to the the front door of the store. Show that

$$\mathbf{P}(F > n) = \mathbf{P}(X_n(t) = 0) = E(\rho,n).$$

[3]

ii) Prove that an arriving car parks at an average distance $\mathbf{E}(F)$ from the front door of the store where $\mathbf{E}(F) = 1 + \sum_{n=1}^{\infty} E(\rho,n)$. [3]