Master.

E4.25
C4.1
ISE4.23

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2007

MSc and EEE/ISE PART IV: MEng and ACGI

DESIGN OF LINEAR MULTIVARIABLE CONTROL SYSTEMS

Wednesday, 2 May 10:00 am

Time allowed: 3:00 hours

There are SIX questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : I.M. Jaimoukha
Second Marker(s) : D.J.N. Limebeer

Special Information for Invigilators : None

Information for Candidates : None

1. (a) Let

$$G(s) = \frac{1}{s+1} \begin{bmatrix} 1 & s-2 \\ s-3 & 12 \end{bmatrix}$$

(i) Find the McMillan form of $G(s)$. **[2]**

(ii) Determine the pole and zero polynomials of $G(s)$. **[2]**

(iii) Find the poles and zeros of $G(s)$, specifying the multiplicity of each. **[2]**

(b) Consider a state–variable model described by the dynamics

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t), \end{aligned}$$

and denote the corresponding transfer matrix by $H(s)$. Suppose that there exists $P = P' > 0$ such that

$$\begin{bmatrix} A'P + PA & PB & C' \\ B'P & -I & 0 \\ C & 0 & -I \end{bmatrix} < 0.$$

(i) Prove that A is stable. **[3]**

(ii) Prove that

$$\begin{bmatrix} A'P + PA + C'C & PB \\ B'P & -I \end{bmatrix} < 0.$$

[3]

(iii) By defining the Lyapunov function

$$V(t) = x(t)'Px(t),$$

the cost function

$$J := \int_0^\infty [y(t)'y(t) - u(t)'u(t)]dt,$$

and using a property of the integral $\int_0^\infty \dot{V}(t)dt$, or otherwise, prove that

$$\|H\|_\infty < 1.$$

State clearly the assumptions required on $u(t)$, $x(0)$ and $x(\infty)$. **[8]**

2. Consider the feedback loop shown in Figure 2 below. Here $G(s)$ is a given system model and $K(s)$ is a compensator.

   (a) Define internal stability for the nominal loop, and derive necessary and sufficient conditions for which this feedback loop is internally stable.

   [6]

   (b) Suppose that the transfer matrix $G(s)$ in the nominal loop in Figure 2 is stable. Derive a parameterization of all internally stabilizing controllers for the feedback loop.

   [6]

   (c) Suppose that

$$G(s) = \frac{s-1}{s+1} G_o(s)$$

   where $G_o(s)$ is a stable and minimum phase transfer matrix (that is, $G_o(s)^{-1}$ is stable). Let $S(s)$ denote the transfer matrix from r to e in Figure 2. By using the answer to Part (b) above and the small gain theorem, or otherwise, find

$$\gamma = \min_{K \text{ is internally stabilizing}} \|S\|_\infty.$$

   [8]

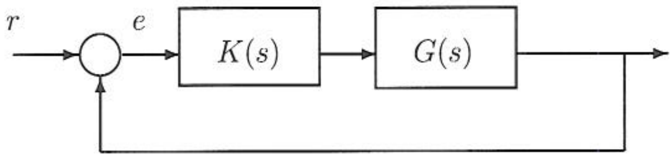

Figure 2

3. Consider the regulator shown in Figure 3 for which it is assumed that the triple (A, B, C) is minimal and $x(0) = x_0$. Take H initially to be equal to I.

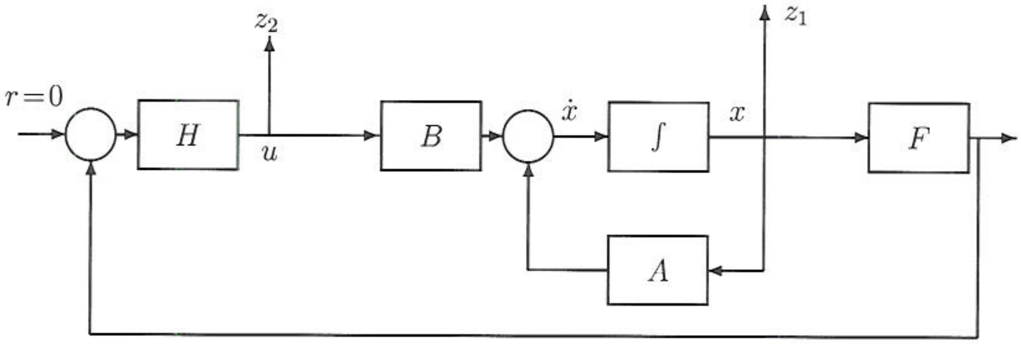

Figure 3

Let $z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$. A stabilizing state-feedback gain matrix F is to be designed such that the cost function $J := \int_0^\infty z(t)^T z(t) dt$ is minimized.

Let $V(t) = x(t)^T P x(t)$ where $P = P^T$ is the unique positive definite solution of the algebraic Riccati equation

$$A^T P + PA + I - PBB^T P = 0$$

(a) Assuming the closed loop is asymptotically stable, obtain an expression for $\int_0^\infty \dot{V}(t)dt$ in terms of x_0.

[5]

(b) Evaluate an expression for J using an appropriate completion of a square. Using this expression, find F that minimizes J. Give also the minimum value of J.

[5]

(c) Let $G(s) = (sI - A)^{-1}B$ and define $L(s) = I - FG(s)$. Using the algebraic Riccati equation show that

$$L(j\omega)' L(j\omega) = I + G(j\omega)' G(j\omega)$$

[5]

(d) Suppose that there is an uncertainty in modelling B so that the actual value of B is $B(I + \Delta)$, where Δ represents a perturbation. This perturbation is represented in Figure 3 by taking $H = I + \Delta$. Find the maximum value for $\|\Delta\|$ for which the closed loop in Figure 3 is stable.

[5]

4. Consider the feedback loop shown in Figure 4 where $G(s)$ represents a plant model and $K(s)$ represents an internally stabilizing compensator. Suppose that

$$G(s) \stackrel{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cc|cc} -1 & -1 & 1 & 1 \\ -1 & -1.25 & 0.6 & 0.8 \\ \hline 1 & 0.6 & 0 & 0 \\ 1 & 0.8 & 0 & 0 \end{array}\right] \in \mathcal{RH}_\infty.$$

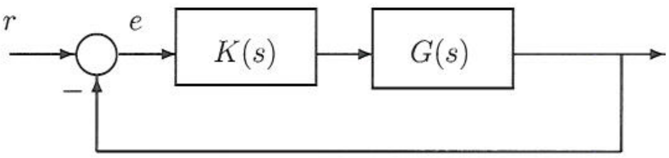

Figure 4

(a) Show that the given realization for $G(s)$ is balanced and evaluate the Hankel singular values of $G(s)$. **[6]**

(b) By using:

- the answer to Part (a),
- the small gain theorem (which should be stated),
- and a parameterization of the set of all internally stabilizing controllers,

derive a technique to design a first order internally stabilizing controller $K(s)$ for $G(s)$. **[8]**

(c) Design a non-dynamic internally stabilising controller K for $G(s)$ such that $\|K\| \geq 1$. **[6]**

(Hint: Use the procedure outlined in Part (b) and the fact that

$$G_1(s)\hat{Q} = 0$$

where $G_1(s)$ is a first order balanced truncation of $G(s)$, $\hat{Q} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and $\|\hat{Q}\| = 1$.)

5. Consider the feedback configuration in Figure 5.1. Here, $G(s)$ is a nominal plant model and $K(s)$ is a compensator. The signals $r(s)$ and $n(s)$ represent the reference and sensor noise, respectively. The design specifications are to synthesize a compensator $K(s)$ such that the feedback loop is internally stable and:

- For good tracking, it is required that, when $n(s) = 0$,
$$\|e(j\omega)\| < |w_1(j\omega)^{-1}|\|r(j\omega)\|, \forall\omega.$$
- To limit the control effort, it is required that when $n(s) = 0$,
$$\|u(j\omega)\| < |w_2(j\omega)^{-1}|\|r(j\omega)\|, \forall\omega.$$
- For good sensor noise attenuation it is required that, when $r(s) = 0$,
$$\|y(j\omega)\| < |w_3(j\omega)^{-1}|\|n(j\omega)\|, \forall\omega,$$

where $w_1(s)$, $w_2(s)$ and $w_3(s)$ are suitable filters.

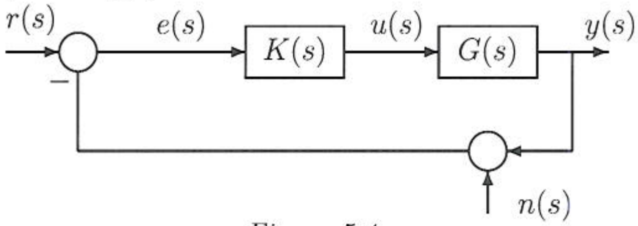

Figure 5.1

(a) Derive $\mathcal{H}_\infty$-norm bounds, in terms of $G(s), K(s), w_1(s)$, $w_2(s)$ and $w_3(s)$ that are sufficient to achieve the design specifications. **[6]**

(b) Derive a generalized regulator formulation of the design problem that captures the sufficient conditions in Part (a). **[7]**

(c) Assume that $K(s)$ achieves the design specifications in Part (a). Suppose that uncertainties $\Delta_1(s)$ and $\Delta_2(s)$ are introduced as in Figure 5.2 where $\Delta_1(s)$ and $\Delta_2(s)$ are stable transfer matrices.

i. Assume that $\Delta_2(s) = 0$. Derive an upper bound on $\|\Delta_1(j\omega)\|, \forall\omega$, for which robust stability is guaranteed.

ii. Assume that $\Delta_1(s) = 0$. Derive an upper bound on $\|\Delta_2(j\omega)\|, \forall\omega$, for which robust stability is guaranteed. **[7]**

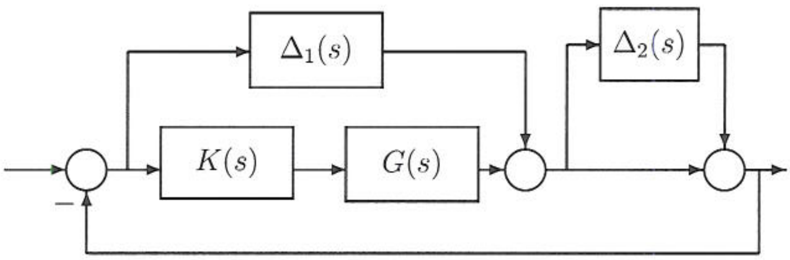

Figure 5.2

6. (a) Consider the regulator shown in Figure 6.1 for which it is assumed that the triple (A, B, C) is minimal and $x(0)=0$.

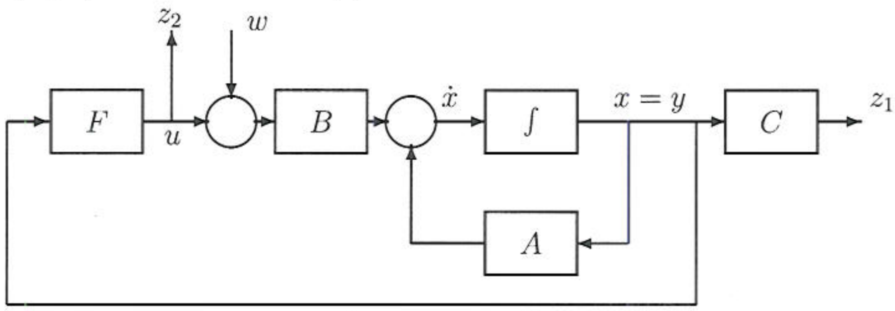

Figure 6.1

Let $z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ and let $H(s)$ denote the transfer matrix from w to z. A stabilizing state-feedback gain matrix F is to be designed such that, for $\gamma > 0$, $\|H\|_\infty < \gamma$.

i. Write down the generalized regulator system for this design problem. **[4]**

ii. By using the Lyapunov function $V(t) = x(t)^T X x(t)$, where X is to be determined, derive sufficient conditions for the solution of the design problem. Your conditions should be in the form of the existence of a certain solution to an algebraic Riccati equation. It should also include an expression for F and an expression for the worst-case disturbance w. **[8]**

(b) Consider the output injection problem shown in Figure 6.2 for which it is assumed that the triple (A, B, C) is minimal and $x(0)=0$.

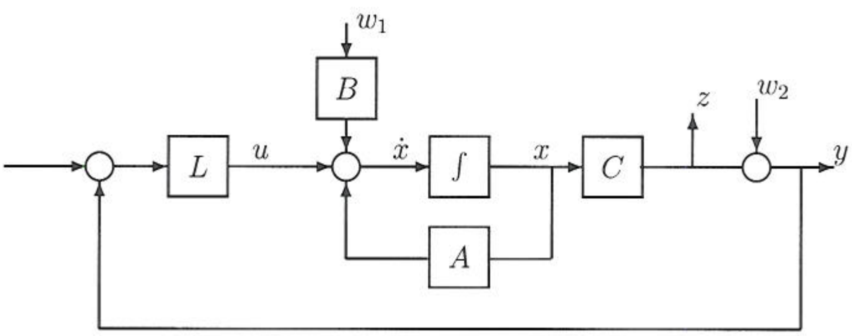

Figure 6.2

Let $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ and let $H(s)$ denote the transfer matrix from w to z. A stabilizing output injection gain matrix L is to be designed such that, for $\gamma > 0$, $\|H\|_\infty < \gamma$.

i. Write down the generalized regulator system for this design problem. **[4]**

ii. Use a duality argument to transform the output injection problem into the state-feedback problem of Part (a). **[4]**

[END]

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Design of Linear Multivariable Control Systems

Solutions 2007

1. (a) (i) By performing the following elementary operations: (A) $r_2 \leftrightarrow r_2-(s-3)r_1$, (B) $c_2 \leftrightarrow c_2-(s-2)\,c_1$, (C) $c_2 \leftrightarrow -c_2$, the McMillan form of $G(s)$ is

$$G(s) = \begin{bmatrix} 1 & 0 \\ s-3 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & s-6 \end{bmatrix} \begin{bmatrix} 1 & s-2 \\ 0 & -1 \end{bmatrix} = L(s)M(s)R(s)$$

where $L(s)$ and $R(s)$ are unimodular.

(ii) The pole and zero polynomials are $p(s) = s+1,\ z(s) = s-6$.

(iii) It follows that the system has a simple pole at -1 and a simple zeros at 6.

(b) (i) The $(1,1)$ block of the inequality gives the inequality $A'P + PA < 0$. Let $z \neq 0$ be a right eigenvector of A and let λ be the corresponding eigenvalue. Then multiplying the inequality from the left by z' and from the right by z gives $(\lambda + \bar{\lambda})z'Pz < 0$. Since $P > 0$ it follows that $z'Pz > 0$ and it follows that $\lambda + \bar{\lambda} < 0$ so that A is stable.

(ii) Call the matrix in Part (b) $\begin{bmatrix} X_{11} & X_{12} \\ X_{12}' & X_{22} \end{bmatrix}$ where $X_{22} = -I$ and call the matrix in (ii) S. Pre- and post-multiply the first matrix by T' and T where $T = \begin{bmatrix} I & 0 \\ -X_{22}^{-1}X_{12}' & I \end{bmatrix}$ gives $\begin{bmatrix} S & 0 \\ 0 & X_{22} \end{bmatrix}$ which proves the result.

(iii) Since A is stable, $\|H\|_\infty < 1$ if and only if, with $x(0) = 0$,

$$J := \int_0^\infty [y'y - u'u]dt < 0,$$

for all $u(t)$ such that $\|u\|_2 < \infty$. If $\|u\|_2$ is bounded, then $\lim_{t\to\infty} x(t) = 0$. Now,

$$\int_0^\infty \frac{\mathrm{d}}{\mathrm{dt}}[x'Px]dt = x(\infty)'Px(\infty) - x(0)'Px(0) = 0.$$

So,

$$\begin{aligned} 0 &= \int_0^\infty \dot{x}'Px + x'P\dot{x}dt = \int_0^\infty [(Ax+Bu)'Px + x'P(Ax+Bu)]dt \\ &= \int_0^\infty [x'(A'P+PA)x + x'PBu + u'B'Px]dt \end{aligned}$$

Use $y = Cx$ and add the last expression to J

$$\begin{aligned} J &= \int_0^\infty [x'(A'P+PA+C'C)x + x'PBu + u'B'Px - u'u]dt \\ &= \int_0^\infty \begin{bmatrix} x' & u' \end{bmatrix} \begin{bmatrix} A'P+PA+C'C & PB \\ B'P & -I \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} dt \\ &< 0 \end{aligned}$$

from the inequality in Part (ii). This proves the result.

2. (a) Inject a signal d in between $G(s)$ and $K(s)$ and call the input to $G(s)$, u and the input to $K(s)$, e. The loop is internally stable if and only if the transfer matrix from $\begin{bmatrix} d \\ r \end{bmatrix}$ to $\begin{bmatrix} u \\ e \end{bmatrix}$ is stable (no poles in the closed right half plane). Since

$$\begin{bmatrix} d \\ r \end{bmatrix} = \begin{bmatrix} I & -K \\ -G & I \end{bmatrix} \begin{bmatrix} u \\ e \end{bmatrix} =: T(s) \begin{bmatrix} u \\ e \end{bmatrix}$$

the loop is internally stable if and only if $T^{-1}(s)$ is stable.

(b) Since $G(s)$ is stable, we proceed as follows. Note that

$$\begin{bmatrix} I & -K \\ -G & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ -G & I \end{bmatrix} \begin{bmatrix} I & -K \\ 0 & I - GK \end{bmatrix}.$$

Hence

$$\begin{bmatrix} I & -K \\ -G & I \end{bmatrix}^{-1} = \begin{bmatrix} I & -K \\ 0 & I-GK \end{bmatrix}^{-1} \begin{bmatrix} I & 0 \\ -G & I \end{bmatrix}^{-1} = \begin{bmatrix} I & K(I-GK)^{-1} \\ 0 & (I-GK)^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ G & I \end{bmatrix}.$$

Finally, since $(I - GK)^{-1} = I + GK(I - GK)^{-1}$, it follows that if G is stable, then the loop is internally stable if and only if $Q := K(I - GK)^{-1}$ is stable. Rearranging terms shows that K is internally stabilizing if and only if $K = Q(I + GQ)^{-1}$ for some stable Q.

(c) Now, $e(s) = S(s)r(s)$ where $S(s) = (I - G(s)K(s))^{-1}$. Substituting the expression for stabilizing K from Part (b), and the expression for $G(s)$,

$$[I - G(s)K(s)]^{-1} = I + G(s)Q(s) = I + \frac{s-1}{s+1} G_o(s)Q(s).$$

Since $G_o(s)^{-1}$ is stable, we can set $Q(s) = G_o(s)^{-1}\hat{Q}(s)$ for some stable $\hat{Q}(s)$. It follows that

$$[I - G(s)K(s)]^{-1} = I + \frac{s-1}{s+1}\hat{Q}(s).$$

However, $\left\| I + \frac{s-1}{s+1}\hat{Q}(s) \right\|_\infty \geq 1$ for any $\hat{Q}(s)$ since $\hat{Q}(s)$ is stable. It follows that $\gamma = 1$.

3. (a) Let $V = x^T P x$ and set $u = Fx$. Provided that $P = P^T > 0$ and $\dot{V} < 0$ along closed-loop trajectories, we can assume $\lim_{t\to\infty} x(t) = 0$. Then

$$\dot{V} = \dot{x}^T P x + x^T P \dot{x} = x^T \left(A^T P + PA + F^T B^T P + PBF\right) x.$$

Integrating from 0 to ∞ and using $x(\infty) = 0$,

$$\int_0^\infty x^T \left(A^T P + PA + F^T B^T P + PBF\right) x dt = -x_0^T P x_0.$$

(b) Using the definition of J and adding the last equation,

$$J = x_0^T P x_0 + \int_0^\infty x^T [A^T P + PA + I + F^T F + F^T B^T P + PBF] x dt.$$

Completing the squares by using

$$(F + B^T P)^T (F + B^T P) = F^T F + F^T B^T P + PBF + PBB^T P,$$

$$J = x_0^T P x_0 + \int_0^\infty \{x^T [A^T P + PA + I - PBB^T P] x + \left\|(F + B^T P) x\right\|^2\} dt.$$

Since the last term is always nonnegative, it follows that the minimizing value of F is given by $F = -B^T P$. Since the term in square brackets is zero from the Riccati equation, it follows that the minimum value of J is $x_0^T P x_0$.

(c) By direct evaluation, $L(j\omega)' L(j\omega) =$

$$I - F(j\omega I - A)^{-1} B - B'(-j\omega I - A')^{-1} F' + B'(-j\omega I - A')^{-1} F' F (j\omega I - A)^{-1} B$$

But $F'F = A'P + PA + I = -(-j\omega I - A')P - P(j\omega I - A) + I$ from the Riccati equation. So, $L(j\omega)' L(j\omega)$

$$\begin{aligned}
&= I - F(j\omega I - A)^{-1} B - B'(-j\omega I - A')^{-1} F' \\
&\quad + B'(-j\omega I - A')^{-1} [-(-j\omega I - A')P - P(j\omega I - A) + I](j\omega I - A)^{-1} B \\
&= I - [F + B'P](j\omega I - A)^{-1} B - B'(-j\omega I - A')^{-1} [F' + PB] \\
&\quad + B'(-j\omega I - A')^{-1} (j\omega I - A)^{-1} B \\
&= I + G(j\omega)' G(j\omega)
\end{aligned}$$

(d) Let ϵ be the input to Δ and δ be the output of Δ. Then

$$\epsilon = FG(\delta + \epsilon) = (I - FG)^{-1} FG \delta = L^{-1}(I - L)\delta = (L^{-1} - I)\delta$$

Using the small gain theorem (since the regulator is stable and the perturbation is assumed nondynamic), the loop is stable if $\|\Delta(L^{-1} - I)\|_\infty < 1$. But part (c) implies that

$$\bar{\sigma}[L(j\omega)^{-1} - I] \le 1 + \bar{\sigma}[L(j\omega)^{-1}] \le 1 + \frac{1}{\underline{\sigma}[L(j\omega)]} \le 2$$

This shows that the loop will tolerate perturbations Δ of size $\|\Delta\| < 0.5$ without losing internal stability.

4. (a) The realization of $G(s)$ is balanced if

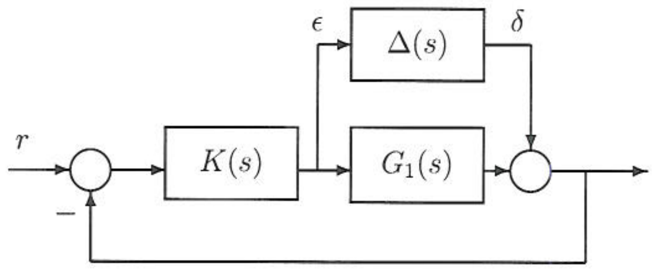
$$A\Sigma + \Sigma A' + BB' = A'\Sigma + \Sigma A + C'C = 0$$

for $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2) > 0$ and where the $\sigma_i's$ are the Hankel singular values of $G(s)$. A simple calculation gives $\Sigma = \mathrm{diag}(1, 0.4)$.

(b) Let $G_1(s)$ denote a first-order balanced truncation of $G(s)$. Then $G_1(s) = G(s) + \Delta(s)$ where

$$\|\Delta\|_\infty \leq 2\sum_{i=2}^{2} \sigma_i = 0.8.$$

Then replacing $G(s)$ by $G_1(s)$ in the loop of Figure 4 is equivalent to:



Now

$$\epsilon = -K(I + G_1 K)^{-1}\delta$$

and so the loop is stable if $\|K(I + G_1 K)^{-1}\|_\infty < 1.25$ from the small gain theorem and since $\|\Delta\|_\infty \leq 0.8$. However, the set of all internally stabilizing controllers for $G_1(s)$ is given by:

$$K = Q(I - G_1 Q)^{-1}$$

for stable Q. Furthermore,

$$K(I + G_1 K)^{-1} = Q.$$

Thus we can take $Q = qI_2$ where q is constant (to guarantee a first order controller) and $|q| < 1.25$ (to guarantee stabilization of G).

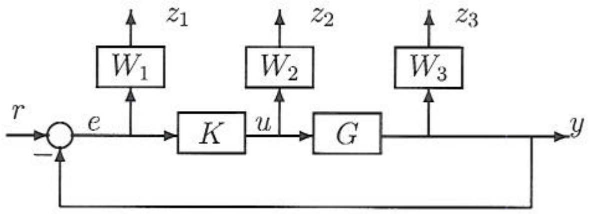
(c) Noting that the dynamic part for the expression for $K(s)$ in Part (b) comes from the product $G_1(s)Q$, we take the hint from the question and set $Q = q\hat{Q}$ so that $K = q\hat{Q}$. To satisfy $\|K\| \geq 1$, we need $|q| \geq 1$. Combining this with Part (b), which requires $|q| < 1.25$, we may take $q = 1$.

5. (a) It is clear that we require K to be internally stabilizing.

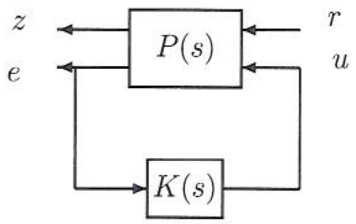
- A simple calculation shows that, when $n(s)=0$, $e(s) = -S(s)r(s)$ where $S(s) = [I+G(s)K(s)]^{-1}$ is the sensitivity. Thus $\|e(j\omega)\| \le \|S(j\omega)\|\|r(j\omega)\|$. It follows that a sufficient condition to achieve the first design specification is $\|S(j\omega)\| < |w_1^{-1}(j\omega)|, \forall\omega$ or equivalently $\|W_1 S\|_\infty < 1$, where $W_1 = w_1 I$.
- A similar calculation shows that, when $n(s) = 0$, $u(s) = -K(s)S(s)r(s)$. Thus $\|u(j\omega)\| \le \|K(j\omega)S(j\omega)\|\|r(j\omega)\|$. It follows that a sufficient condition to achieve the second design specification is $\|K(j\omega)S(j\omega)\| < |w_2^{-1}(j\omega)|$, $\forall\omega$ or equivalently $\|W_2 K S\|_\infty < 1$, where $W_2 = w_2 I$.
- When $r(s) = 0$, a similar calculation shows that $y(s) = -C(s)n(s)$ where $C(s) = G(s)K(s)[I+G(s)K(s)]^{-1}$ is the complementary sensitivity. Thus $\|y(j\omega)\| \le \|C(j\omega)\|\|n(j\omega)\|$. It follows that a sufficient condition to achieve the second design specification is $\|C(j\omega)\| < |w_3^{-1}(j\omega)|, \forall\omega$ or equivalently $\|W_3 C\|_\infty < 1$, where $W_3 = w_3 I$.

To satisfy all design requirements, it is sufficient that $\left\| \begin{bmatrix} W_1 S \\ W_2 K S \\ W_3 C \end{bmatrix} \right\|_\infty < 1.$

(b) The design specifications reduce to the requirement that the transfer matrix from r to $z = [z_1^T \ z_2^T \ z_3^T]^T$ in the following diagram has $\mathcal{H}_\infty$-norm less than 1.



The corresponding generalized regulator formulation is to find an internally stabilizing K such that $\|\mathcal{F}_l(P,K)\|_\infty < 1$:



$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \left[\begin{array}{c|c} W_1 & -W_1 G \\ 0 & W_2 \\ 0 & W_3 G \\ \hline I & -G \end{array}\right].$$

(c) (i) Set $\Delta_2 = 0$. Let ϵ be the input and δ be the output of Δ_1. Then $\epsilon = S\delta$. Using the small gain theorem the maximum stability radius is $|w_1(j\omega)|$.

(ii) Set $\Delta_1 = 0$. Let ϵ be the input and δ be the output of Δ_2. Then $\epsilon = GKS\delta$. Using the small gain theorem the maximum stability radius is $|w_3(j\omega)|$.

6. (a) i. The generalized regulator formulation is given by

$$\left[\begin{array}{c} z(s) \\ y(s) \end{array}\right] = P(s) \left[\begin{array}{c} w(s) \\ u(s) \end{array}\right],\ u(s) = Fy(s),\ P(s) = \left[\begin{array}{cc} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{array}\right] \stackrel{s}{=} \left[\begin{array}{c|c|c} A & B & B \\ \hline C & 0 & 0 \\ 0 & 0 & I \\ \hline I & 0 & 0 \end{array}\right].$$

ii. The requirement $\|H\|_\infty < \gamma$ is equivalent to $J := \|z\|_2^2 - \gamma^2 \|w\|_2^2 < 0$. Let $V = x^T X x$ and set $u = Fx$. Provided that $X = X^T > 0$ and $\dot{V} < 0$ along the closed-loop trajectory, we can assume $\lim_{t \to \infty} x(t) = 0$. Then

$$\dot{V} = \dot{x}^T X x + x^T X \dot{x} = x^T \left(A^T X + XA + F^T B^T X + XBF\right) x + x^T XBw + w^T B^T X x.$$

Integrating from 0 to ∞ and using $x(0) = x(\infty) = 0$,

$$0 = \int_0^\infty [x^T \left(A^T X + XA + F^T B^T X + XBF\right) x + x^T XBw + w^T B^T Xx] dt.$$

Using the definition of J and adding the last equation, $J =$

$$\int_0^\infty \{x^T [A^T X + XA + C^T C + F^T F + F^T B^T X + XBF] x - [\gamma^2 w^T w - x^T XBw - w^T B^T Xx]\} dt.$$

Let $Z = F + B^T X$. Completing the squares by using

$$Z^T Z = F^T F + F^T B^T X + XBF + XBB^T X$$

$$\|(\gamma w - \gamma^{-1} B^T Xx)\|^2 = \gamma^2 w^T w - w^T B^T Xx - x^T XBw + \gamma^{-2} x^T XBB^T Xx,$$

$$J = \int_0^\infty \{x^T [A^T X + XA + C^T C - (1 - \gamma^{-2}) XBB^T X] x + \|Zx\|^2 - \left\|\gamma w - \gamma^{-1} B^T Xx\right\|^2\} dt.$$

Thus two sufficient conditions for $J < 0$ are the existence of X such that

$$A^T X + XA + C^T C - (1 - \gamma^{-2}) XBB^T X = 0, \qquad X = X^T > 0.$$

The feedback gain is $F = -B^T X$ and the worst case disturbance is $w^* = \gamma^{-2} B^T Xx$. The closed-loop is $\dot{x} = [A - (1 - \gamma^{-2}) BB^T X] x$ and a third condition is therefore $Re\ \lambda_i [A - (1 - \gamma^{-2}) BB^T X] < 0,\ \forall\ i$.
It remains to prove $\dot{V} < 0$ along state-trajectory with $u = Fx$ and $w = 0$. But

$$\dot{V} = x^T \left(A^T X + XA + F^T B^T X + XBF\right) x = -x^T (C^T C + (1 + \gamma^{-2}) XBB^T X) x < 0$$

for all $x \neq 0$ (since (A, B, C) is assumed minimal) proving closed-loop stability.

(b) i. The generalized regulator formulation is given by

$$\left[\begin{array}{c} z(s) \\ y(s) \end{array}\right] = P(s) \left[\begin{array}{c} w(s) \\ u(s) \end{array}\right],\ u(s) = Ly(s),\ P(s) = \left[\begin{array}{cc} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{array}\right] \stackrel{s}{=} \left[\begin{array}{c|cc|c} A & B & 0 & I \\ \hline C & 0 & 0 & 0 \\ \hline C & 0 & I & 0 \end{array}\right].$$

ii. Taking the transpose of $P(s)$ in Part (a), redefining $A := A^T$, $B := C^T$, $C := B^T$, $F := L^T$ and exchanging w and z we get the state-feedback problem in Part (a).