Imperial College London

M3B

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May-June 2017

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Mathematics of Business & Economics

Date: Tuesday 23 May 2017

Time: 14:00 - 16:00

Time Allowed: 2 Hours

This paper has 4 Questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the relevant main book(s) are full.

All required additional material will be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers to each question as per the table below.

Raw Mark	Up to 12	13	14	15	16	17	18	19	20
Extra Credit	0	½	1	1 ½	2	2 ½	3	3 ½	4

- Each question carries equal weight.
- Calculators may not be used.

1. Throughout this question, we consider firms that use two inputs of production to produce a single output. We will denote the quantities of the inputs to production by $\underline{x} = (x_1, x_2)$, and the quantity of the output by y.

(a) The following production functions all describe the technological capabilities of different firms. For each, state whether the production function is homogeneous or not; if it is homogeneous, give its degree of homogeneity.

(i) $f(x_1, x_2) = \min\{2x_1, 3x_2\}$

(ii) $f(x_1, x_2) = Ax_1/x_2$, for $A > 0$

(iii) $f(x_1, x_2) = \left(x_1^{1/2} + x_2^{1/4}\right)^4$

(iv) $f(x_1, x_2) = [g(x_1, x_2)]^{1/\alpha}$, where $g(x_1, x_2)$ is homogeneous of degree β.

(b) Consider the competitive firm with production function given by

$$f(x_1, x_2) = (x_1^r + x_2^r)^{1/r} \qquad r \in (0, 1].$$

In this part of the question, $\underline{w} = (w_1, w_2)$ denotes the vector of unit factor prices and p denotes the price at which the output is sold. Assume that these are fixed, and that $w_1, w_2, p > 0$.

(i) Derive an expression for the firm's output elasticity with respect to x_1.

(ii) Describe the returns to scale for this firm.

(iii) What conditions must the Hessian matrix $\nabla^2 f(\underline{x})$ satisfy, in order for a profit-maximising position to exist?

For parts (iv) and (v), consider the case where $r = 1/2$.

(iv) Assuming the conditions in (iii) to be satisfied, state the maximised profit for this firm. Provide brief justification.

(v) Hence, derive expressions for the firm's profit-maximising choice of input factors; if this choice is dependent on the price vector $(p, \underline{w})$, you should explain this dependence.

(c) Consider the profit-maximising firm acting in a competitive market for which the unit price of the firm's output is a random quantity: at any given time, the price is p_1 with probability q, and it is p_2 with probability $1 - q$. Suppose that the changes in price are infrequent enough for the firm to recalibrate its operations immediately and without any penalty. Suppose also that the unit cost of any input to production remains fixed.

One suggested approach is for the firm to calibrate its operations towards the expected value of the output price. Explain whether this is a preferable approach for the firm, when compared with the alternative of recalibrating each time the price changes.

2. Greig, a consumer, is faced with the problem of choosing a combination of three goods, with prices $\underline{p}_G = (p_1, p_2, p_3)$. Denote Greig's consumption set by $X_G \subseteq \mathbb{R}^3_{\geq 0}$ and suppose that his preferences may be represented by the utility function

$$u_G(\underline{x}_G) = x_{G,1}^{1/2} x_{G,2}^{1/3} x_{G,3}^{1/6},$$

for $\underline{x}_G = (x_{G,1}, x_{G,2}, x_{G,3}) \in X$. Denote Greig's budget for all three goods by m_G.

(a) Derive expressions for Greig's Marshallian demand for each good.

Note: you do not have to check the second-order conditions for any optimisation you perform.

(b) Hence, find expressions for Greig's indirect utility function and expenditure function.

(c) State two properties that are shared by all valid indirect utility functions.

For parts (d) and (e), we consider another consumer with an $n-$good consumption bundle $\underline{x} = (x_1, \ldots, x_n)$ with corresponding price vector $\underline{p} = (p_1, \ldots, p_n)$. We use $u(\underline{x})$ to denote the consumer's utility function, and we use the notation $x_i^*(\underline{p}, m)$ to denote the consumer's Marshallian demand for good i, $i = 1, \ldots, n$, where m is their budget for all n goods.

(d) This part of the question concerns the change to the consumer's Marshallian demand for good i, which comes as a result of a change in the price of good j, $i, j \in \{1, \ldots, n\}$.

(i) In this context, explain briefly what is meant by the terms 'substitution effect' and 'income effect'.

(ii) State the Slutsky equation, indicating clearly the components that correspond to the substitution and income effects.

(iii) The 'substitution matrix' contains, in its (i, j)-th element, the component of the Slutsky decomposition of $\partial x_i^*(\underline{p}, m)/\partial p_j$ that corresponds to the substitution effect. State, with brief justification, whether the substitution matrix is positive definite, positive semidefinite, negative definite or negative semidefinite.

(e) By noting the identities $v(\underline{p}, m) \equiv u(\underline{x}^*(\underline{p}, m))$ and $\underline{p}\underline{x}^{*T}(\underline{p}, m) \equiv m$, prove 'Roy's identity':

$$x_i^*(\underline{p}, m) = -\frac{\partial v(\underline{p}, m)}{\partial p_i} \Bigg/ \frac{\partial v(\underline{p}, m)}{\partial m}.$$

3. Consider the long-run behaviour of a firm with two inputs to production, denoted $\underline{x} = (x_1, x_2)$, and a single output, denoted y. Denote the unit cost of the inputs to production by $\underline{w} = (w_1, w_2)$, and the unit price at which the output is sold by p. Assume throughout that $w_1, w_2, p > 0$.

Suppose the firm is a cost-minimising monopolist, with cost function given by

$$c^*(\underline{w}, y) = \sqrt{w_1 w_2}\, y^2,$$

for $y \geq 0$.

(a) By explicitly treating the firm's output as a function of its price, i.e. by considering $y = y(p)$:

(i) state the monopolist's profit maximisation problem.

(ii) derive and simplify first- and second-order conditions for finding a solution to this problem.

Now suppose that the monopolist is acting in a market in which the overall demand for their good, denoted Q_D, is described by

$$Q_D = Ap^{-b}.$$

(b) Derive a restriction on b that must be satisfied for a profit-maximising position for the monopolist to exist.

For the remainder of the question, set $A = 1$, $b = 2$.

(c) Derive an expression for the profit-maximising price at which this monopolist will sell their good.

(d) (i) Give an expression, in terms of y, for the social marginal cost of this good.

(ii) Hence, derive an expression for the level of the monopolist's output at which the contribution of this good to social welfare would be maximised.

(iii) Provide a sketch of the social marginal benefit and social marginal cost curves for this market. On your diagram, outline clearly the regions that correspond to the producer surplus and the consumer surplus.

4. The analysis of a national economy largely focuses on the interaction of aggregate demand and aggregate supply. One factor that contributes to aggregate demand is total consumer spending across the nation; suppose we model this using a linear consumption function:

$$C = a + bY_d,$$

where a, b are constants, with $a > 0$, and $b \in (0, 1)$, and where Y_d is disposable income, aggregated across all households.

(a) Describe the four remaining components of aggregate demand.

(b) Draw a diagram to show the circular flow of income between households and firms in an economy. In this context, state the conditions required for the economy to be in equilibrium.

Suppose that a government decides to invest £2bn in public infrastructure. Suppose that prior to this investment, the nation's GDP was £1.756tn, and that as a direct result of the investment, the absolute change to the nation's GDP is £5bn (£1tn=£1,000bn).

(c) (i) Calculate the new value of the nation's GDP.

(ii) Calculate the value of b for this particular economy.

Suppose that the government also levies a tax on the income of its people. The mathematically-minded Chancellor announces that each individual with annual income £Y must pay £$\sqrt{Y}$ in annual income tax.

(d) Establish whether this is a progressive, proportional or regressive tax; provide justification.

(e) Provide a brief description of each of the following examples of governmental policy; explain clearly how it will impact the national economy.

(i) An increase in the national minimum wage.

(ii) An increase in the nation's banks' reserve ratio.

	EXAMINATION SOLUTIONS 2016-17	Course M3B
Question 1		Marks & seen/unseen
Part (a) (i)	$f(\underline{x}) = \min\{2x_1, 3x_2\}$ is homogeneous of degree 1	Seen 2
Part (a) (ii)	$f(\underline{x}) = Ax_1/x_2$ is homogeneous of degree 0	2
Part (a) (iii)	$f(x_1, x_2) = \left(x_1^{1/2} + x_2^{1/4}\right)^4$ is not homogeneous	1
Part (a) (iv)	$f(x_1, x_2) = [g(x_1, x_2)]^{1/\alpha}$, where $g(x_1, x_2)$ is homogeneous of deg. β $\implies$ f is homogeneous of degree β/α.	2
Part (b) (i)	Elasticity of the output with respect to x_1 is given by $\epsilon_1 = \dfrac{\partial f(\underline{x})}{\partial x_1}\dfrac{x_1}{f(\underline{x})} = x_1^{r-1}\left(x_1^r + x_2^r\right)^{1/r-1}\dfrac{x_1}{f(\underline{x})} = \dfrac{x_1^r}{x_1^r + x_2^r}$	Seen sim 3
Part (b) (ii)	This firm's production process displays constant returns to scale.	Seen 1
Part (b) (iii)	In order for a profit-maximising position to exist, the Hessian matrix $\nabla^2 f(\underline{x})$ must be negative semi-definite.	Seen 1
Part (b) (iv)	The maximised profit for this firm is zero. This is the case as the production function displays constant returns to scale.	Seen 1 1
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	EXAMINATION SOLUTIONS 2016-17	Course M3B
Question 1		Marks & seen/unseen
Part (b) (v)	Denote the profit-maximising choice of inputs $\underline{x}^* = (x_1^*, x_2^*)$	Unseen
	From (b)(iv), the maximised profit, $\pi(\underline{x}^*) = pf(\underline{x}^*) - \underline{w}\ \underline{x}^{*T} = 0$ $\implies p\left(\sqrt{x_1^*} + \sqrt{x_2^*}\right)^2 - w_1 x_1^* - w_2 x_2^* = 0$ $\implies (p - w_1)x_1^* + 2p\sqrt{x_1^* x_2^*} + (p - w_2)x_2^* = 0 \qquad (\star)$	1
	Also, since $\underline{x}^*$ is profit-maximising, it satisfies the first-order conditions for profit-maximisation: $\left.\frac{\partial f(\underline{x})}{\partial x_i}\right\rvert_{x_i = x_i^*} = \frac{w_i}{p}, \quad i = 1, 2 \implies \left\lvert MRTS\Big(x_1^*, x_2^*\Big)\right\rvert = \frac{w_1}{w_2}$ $\implies \left(\frac{x_1^*}{x_2^*}\right)^{-1/2} = \frac{w_1}{w_2} \implies x_2^{*1/2} = x_1^{*1/2}\left(\frac{w_1}{w_2}\right) \qquad (\star\star)$	1
	Substituting into $(\star)$, $\implies x_1^*\left[(p - w_1) + 2p\frac{w_1}{w_2} + (p - w_2)\frac{w_1^2}{w_2^2}\right] = 0$	
	So either $x_1^* = 0$ (and $x_2^* = 0$ by $(\star\star)$), or $(p - w_1)w_2^2 + 2pw_1w_2 + (p - w_2)w_1^2 = 0,$	1
	in which case $\underline{x}^*$ can be any combination of x_1^*, x_2^* that satisfies $(\star\star)$ i.e. $\underline{x}^* = \left(k, k\frac{w_1^2}{w_2^2}\right)$, for any $k \geq 0$	1
Part (c)	We need to compare the firm's expected maximised profits when operations are recalibrated with each change of prices: $q\pi^*(p_1, \underline{w}) + (1 - q)\pi^*(p_2, \underline{w}) \qquad (\star)$ with the maximised profits when operations are calibrated to the expected price: $\pi^*(qp_1 + (1 - q)p_2, \underline{w}) \qquad (\star\star)$	Unseen
	Since the maximised profit function is convex in $(p, \underline{w})$, it is convex in p when $\underline{w}$ is fixed...	1
	... so $(\star) \geq (\star\star)$, and therefore the firm should recalibrate whenever the price changes.	1
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	EXAMINATION SOLUTIONS 2016-17	Course M3B
Question 2		Marks & seen/unseen

For notational ease in the solutions to parts (a)-(c), we drop the subscript G, used in the question to distinguish the utility function and its associated variables from those used in parts (d)-(e) .

Part (a) — Seen sim

$$u(\underline{x}) = x_1^{1/2} x_2^{1/3} x_3^{1/6}$$

We look to find $\max_{\underline{x}\geq 0} u(\underline{x})$ such that $\underline{p}\,\underline{x}^T = m$.

Define the Lagrangian:

$$\mathcal{L} = x_1^{1/2} x_2^{1/3} x_3^{1/6} - \lambda(p_1x_1 + p_2x_2 + p_3x_3 - m)$$ 1

We can maximise this without constraint:

First-order conditions:

$$\frac{\partial u(\underline{x})}{\partial x_i} = \lambda p_i, \quad i = 1, 2, 3, \qquad \text{and} \qquad \underline{p}\,\underline{x}^T = m$$ 1

Now,

$$\frac{\partial u(\underline{x})}{\partial x_1} = \frac{1}{2}\frac{u(\underline{x})}{x_1}, \qquad \frac{\partial u(\underline{x})}{\partial x_2} = \frac{1}{3}\frac{u(\underline{x})}{x_2}, \qquad \frac{\partial u(\underline{x})}{\partial x_3} = \frac{1}{6}\frac{u(\underline{x})}{x_3}$$

$$\Longrightarrow \begin{cases} \dfrac{3x_2}{2x_1} = \dfrac{p_1}{p_2} & \Longrightarrow\ p_2x_2 = \dfrac{2p_1}{3}x_1 \quad (\star) \\ \dfrac{6x_2}{2x_1} = \dfrac{p_1}{p_3} & \Longrightarrow\ p_3x_3 = \dfrac{p_1}{3}x_1 \quad (\star\star) \end{cases}$$

which we can substitute into the budget constraint to get:

$$x_1\left(p_1 + \frac{2p_1}{3} + \frac{p_1}{3}\right) = m \ \Longrightarrow\ x_1^*(\underline{p}, m) = \frac{m}{2p_1}$$ 1

and we can get the remaining cond. factor demands from $(\star)$ and $(\star\star)$:

$$x_2^*(\underline{p}, m) = \frac{m}{3p_2} \qquad x_3^*(\underline{p}, m) = \frac{m}{6p_3}$$ 1

Part (b) — Seen

The indirect utility function is given by $v(\underline{p}, m) = u(\underline{p}, \underline{x}^*(\underline{p}, m))$

So, $$v(\underline{p}, m) = m\,(2p_1)^{-1/2}\,(3p_2)^{-1/3}\,(6p_3)^{-1/6}.$$ 1

To get the expenditure function, we set $v(\underline{p}, m)$ to some fixed value, $\bar{u}$, say, and invert to find the corresponding m:

So, $$e(\underline{p}, \bar{u}) = \bar{u}\,(2p_1)^{1/2}\,(3p_2)^{1/3}\,(6p_3)^{1/6}.$$ 1

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	EXAMINATION SOLUTIONS 2016-17	Course M3B
Question 2		Marks & seen/unseen
Part (c)	All valid indirect utility functions $v(\underline{p}, m)$ are: • nonincreasing in $\underline{p}$; • nondecreasing in m; • homogeneous of degree 0 in $(\underline{p}, m)$ *[must include both $\underline{p}$ and m for mark]* • quasiconvex in $\underline{p}$ • continuous in $\underline{p}$ *[1 mark available for each, maximum of 2 marks available]*	Seen 2
Part (d)(i)	The substitution effect and income effect describe the two components of the change in good i's demand that results from a change in the price of good j. The 'substitution effect' is the resulting change to the optimal balance of goods; this is an expenditure-optimising change for a fixed level of utility. The income effect is the change in the magnitude of the optimally-balanced bundle, which can be made due to the change in purchasing power that comes from the substitution effect. *[1 mark for mentioning each of the underlined points; maximum 3 marks]*	Seen 3
Part (d)(ii)	The Slutsky Equation is $$\frac{\partial x_i^*(\underline{p}, m)}{\partial p_j} \equiv \underbrace{\frac{\partial x_{H,i}^*(\underline{p}, v)}{\partial p_j}}_{\text{sub effect}} - \underbrace{\frac{\partial x_i^*(\underline{p}, m)}{\partial m} x_j^*(\underline{p}, m)}_{\text{income effect}}$$ where $x_{H,i}^*(\underline{p}, v)$ is the Hicksian demand for good j (i.e. the expenditure-minimising demand at fixed utility $v = v(\underline{p}, m)$) *[2 marks available for equation (1 mark if given without definition of $x_{H,j}^*$) 1 further mark for correct labelling of substitution & income effects.]*	Seen 2 1
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	EXAMINATION SOLUTIONS 2016-17	Course M3B
Question 2		Marks & seen/unseen
Part (d)(iii)	The $(i,j)^{\text{th}}$ element of the substitution matrix is $S_{ij} = \dfrac{\partial x^*_{H,i}(\underline{p},v)}{\partial p_j}$.	Unseen
	But the Hicksian demand is defined as $x^*_{H,i}(\underline{p},v) = \dfrac{\partial e(\underline{p},v)}{\partial p_i}$, so we can write this as the matrix of second derivatives of the expenditure function:	
	$$S_{ij} = \frac{\partial^2 e(\underline{p},v)}{\partial p_i \partial p_j}$$	1
	since the expenditure function is concave in $\underline{p}$, we can conclude that the substitution matrix S is negative semidefinite.	1
Part (e)	By the chain rule, we have $$\frac{\partial v(\underline{p},m)}{\partial p_i} = \sum_{j=1}^{n} \frac{\partial u(\underline{x}^*)}{\partial x_j^*}\frac{\partial x_j^*}{\partial p_i}$$ and since $\underline{x}^*$ is utility-maximising, it satisfies the first-order conditions of utility maximisation, namely $$\frac{\partial u(\underline{x}^*)}{\partial x_j^*} = \lambda p_j, \quad j = 1,\dots,n, \quad \text{some } \lambda \in \mathbb{R}.$$	Unseen
	Hence $$\frac{\partial v(\underline{p},m)}{\partial p_i} = \lambda \sum_{j=1}^{n} p_j \frac{\partial x_j^*}{\partial p_i}. \qquad (\star)$$	1
	Similarly, we can use the chain rule and the first-order conditions of utility maximisation to show $$\frac{\partial v(\underline{p},m)}{\partial m} = \lambda \sum_{j=1}^{n} p_j \frac{\partial x_j^*}{\partial m}. \qquad (\star\star)$$	1
	Now, differentiating the budget identity wrt p_i and wrt m gives, resp.: $$x_i^*(\underline{p},m) + \sum_{j=1}^{n} p_j \frac{\partial x_j^*}{\partial p_i} = \frac{\partial m}{\partial p_i} = 0$$ $$\implies \sum_{j=1}^{n} p_j \frac{\partial x_j^*}{\partial p_i} = -x_i^*(\underline{p},m),$$ and $$\sum_{j=1}^{n} p_j \frac{\partial x_j^*}{\partial m} = \frac{\partial m}{\partial m} = 1.$$ Using these to simplify $(\star)$ and $(\star\star)$, we obtain $$\frac{\partial v(\underline{p},m)}{\partial p_i} = -\lambda x_i^*(\underline{p},m), \quad \text{and} \quad \frac{\partial v(\underline{p},m)}{\partial m} = \lambda,$$ which we can straightforwardly combine to obtain:	1
	$$x_i^*(\underline{p},m) = -\frac{\partial v(\underline{p},m)}{\partial p_i} \Bigg/ \frac{\partial v(\underline{p},m)}{\partial m}$$	1
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	EXAMINATION SOLUTIONS 2016-17	Course M3B
Question 3		Marks & seen/unseen
Part (a) (i)	The firm's profit maximisation problem is to find $$\max_{p>0}\left\{p\,y(p)-c^*\Big(\underline{w},y(p)\Big)\right\}$$	Seen sim 2
Part (a) (ii)	The first-order condition for finding a solution to this problem is $$\frac{\partial}{\partial p}\left\{p\,y(p)-c^*\Big(\underline{w},y(p)\Big)\right\}=0$$ $$\implies y(p)\left[1+\frac{\partial y}{\partial p}\frac{p}{y}\right]=\frac{\partial c^*(\underline{w},y)}{\partial y}\frac{\partial y}{\partial p}$$ $$\implies p\left[1+\left(\frac{\partial y}{\partial p}\right)^{-1}\frac{y}{p}\right]=\frac{\partial c^*(\underline{w},y)}{\partial y}=MC(y),$$ where $MC(y)$ are the (long-run) marginal costs for the firm. The second-order condition is $$\frac{\partial^2}{\partial p^2}\left\{p\,y(p)-c^*\Big(\underline{w},y(p)\Big)\right\}\le 0$$ $$\implies \frac{\partial}{\partial p}\left\{y(p)+p\frac{\partial y}{\partial p}-\frac{\partial c^*(\underline{w},y)}{\partial y}\frac{\partial y}{\partial p}\right\}\le 0$$ $$\implies \frac{\partial y}{\partial p}+\frac{\partial y}{\partial p}+p\frac{\partial^2 y}{\partial p^2}\le\frac{\partial y}{\partial p}\frac{\partial}{\partial p}\left(\frac{\partial c^*(\underline{w},y)}{\partial y}\right)+\frac{\partial c^*(\underline{w},y)}{\partial y}\frac{\partial^2 y}{\partial p^2}$$ $$\implies 2\frac{\partial y}{\partial p}-\frac{\partial^2 c^*(\underline{w},y)}{\partial y^2}\left[\frac{\partial y}{\partial p}\right]^2\le\frac{\partial^2 y}{\partial p^2}\left[\frac{\partial c^*(\underline{w},y)}{\partial y}-p\right]$$ *[Some simplification & collection of terms is required for both FOC & SOC; exact form given may differ from the final form given here]*	Seen sim 1 1 1 1
Part (b)	For a monopolist, a profit-maximising position can only exist if they are facing a demand curve with absolute price-elasticity $\lvert\epsilon_D\rvert\ge 1$. *[This can be derived from the FOC in Part (a)(ii), but may alternatively be stated without derivation/justification]* i.e. $$\lvert\epsilon_D\rvert=-\epsilon_D=-\frac{\partial Q_d}{\partial p}\frac{p}{Q_D}\ge 1$$ $$\implies -A\left(-bp^{-b-1}\right)\frac{p}{Ap^{-b}}\ge 1 \qquad \implies b\ge 1.$$	Seen 1 2
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	EXAMINATION SOLUTIONS 2016-17	Course M3B
Question 3		Marks & seen/unseen
Part (c)	We solve the FOC, and check the SOC, for the monopolist's profit-maximisation problem; we can use the FOC & SOC derived in Part (a)(ii). *[Errors from part (a) may be carried forward without penalty. The student may alternatively use the following FOC & SOC, obtained in lectures by writing $p = p(y)$ and maximising wrt y:* *FOC:* $p(y)\left[1 + \frac{\partial y}{\partial p}\frac{y}{p}\right] = MC(y)$ *SOC:* $\frac{\partial^2 c^*(\underline{w}, y)}{\partial y^2} \geq \frac{\partial^2 p}{\partial y^2} y + 2\frac{\partial p}{\partial y}$ *]*	Seen sim
	We solve the FOC for p, setting output equal to demand, i.e. setting $y = p^{-2}$, and using $c^*(\underline{w}, y) = \sqrt{w_1 w_2} y^2$:	1
	$p\left[1 + \left(\frac{\partial y}{\partial p}\right)^{-1}\frac{y}{p}\right] = MC(y),$ $\implies p\left[1 - \frac{p^3}{2}\frac{p^{-2}}{p}\right] = 2\sqrt{w_1 w_2} y = 2\sqrt{w_1 w_2} p^{-2}$ $\implies p^* = 4^{1/3}(w_1 w_2)^{1/6}$	2
	Check the SOC at this value of p^*: $2\frac{\partial y}{\partial p} - \frac{\partial^2 c^*(\underline{w}, y)}{\partial y^2}\left[\frac{\partial y}{\partial p}\right]^2 \leq \frac{\partial^2 y}{\partial p^2}\left[\frac{\partial c^*(\underline{w}, y)}{\partial y} - p\right]$ $LHS\vert_{p=p^*} = -4p^{*-3} - 2\sqrt{w_1 w_2}\left[-2p^{*-3}\right]^2 = -4p^{*-3} - 8\sqrt{w_1 w_2} p^{*-6} = -(w_1 w_2)^{-1/2} - 2(w_1 w_2)^{1/2}\left[-\tfrac{1}{2}(w_1 w_2)^{-1/2}\right]^2 = -\tfrac{3}{2}(w_1 w_2)^{-1/2}$ $RHS\vert_{p=p^*} = 6p^{*-4}\left[2\sqrt{w_1 w_2} p^{*-2} - p^*\right] = 12\sqrt{w_1 w_2} p^{*-6} - 6p^{*-3} = \frac{12}{16}(w_1 w_2)^{-1/2} - \frac{6}{4}(w_1 w_2)^{-1/2} = -\frac{3}{4}(w_1 w_2)^{-1/2}$	1
	Since $LHS \leq RHS$, the SOC is satisfied, and we conclude that $p^* = 4^{1/3}(w_1 w_2)^{1/6}$ is the profit-maximising price for the monopolist to sell their good at.	1
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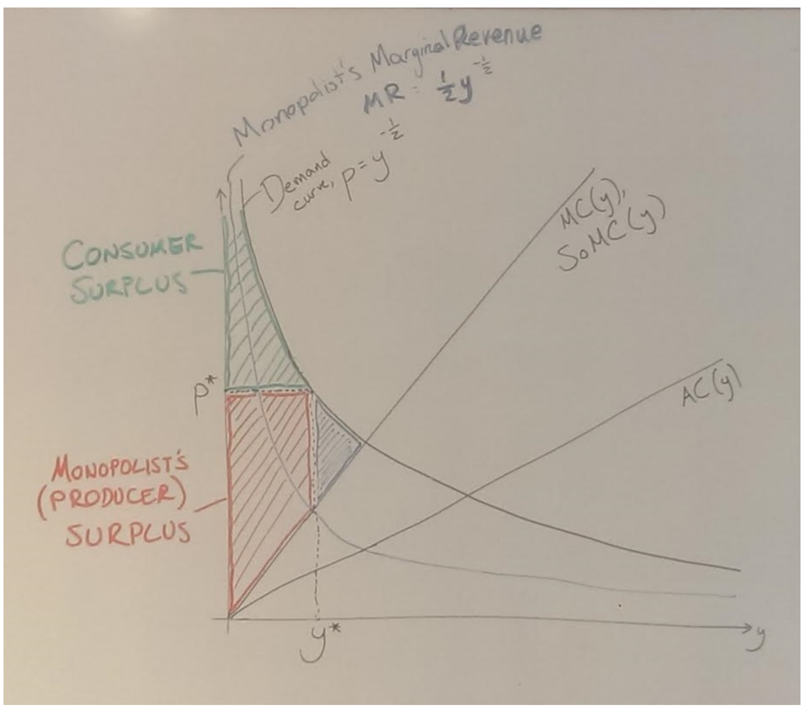
	EXAMINATION SOLUTIONS 2016-17	Course M3B
Question 3		Marks & seen/unseen
Part (d)(i)	The monopolist is the only producer of the good in the market, so the social marginal costs will be equal to the monopolists marginal costs. Thus, the social marginal cost curve $SoMC(y)$ is given by $SoMC(y) = 2\sqrt{w_1 w_2} y.$	Unseen 1 1
Part (d)(ii)	The contribution to social welfare is maximised when social marginal costs equal social marginal benefits; the latter is obtained by inverting the demand for the good: $y = p^{-2} \implies p = y^{-1/2} \implies SoMB(y) = y^{-1/2}$ Hence, the contribution to social welfare is maximised when $y^{-1/2} = 2\sqrt{w_1 w_2} y \quad$ i.e. when $\quad y = (4w_1 w_2)^{-1/3}$	Unseen 1 1
Part (d)(iii)	 *[The sketch must demonstrate allocative inefficiency (i.e. the blue triangle above must be present, though not necessarily highlighted)* *The sketch must have (y^*, p^*) on the demand curve (the SoMB curve), leading to the correct split of consumer & producer surplus]*	Unseen 1 1
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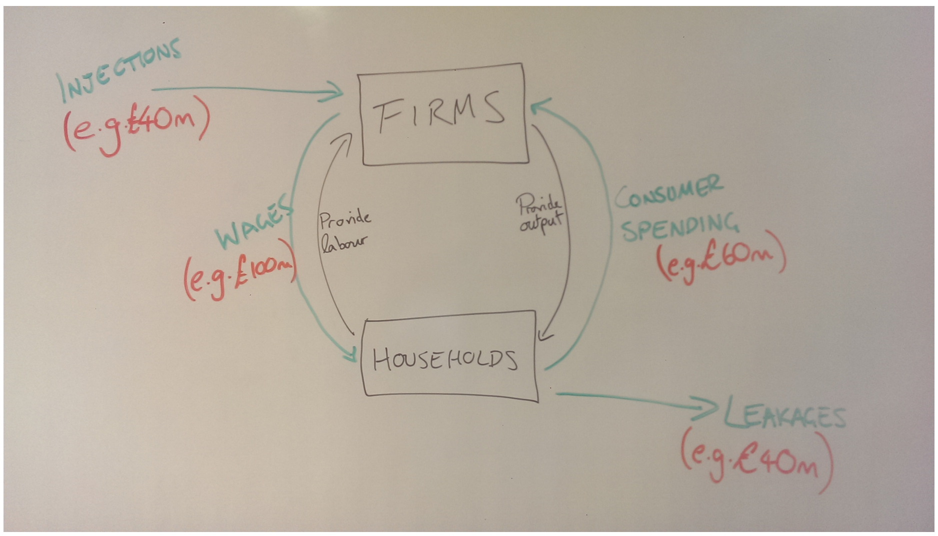
Question 4		Marks & seen/unseen

Part (a) — Seen

The four remaining components of Aggregate Demand are:

- I - Planned investment spending by firms; 1
- G - Government spending on final goods and services 1
- X - Exports, i.e. non-domestic spending on domestic output 1
- M - Imports, i.e. domestic spending on foreign output; 1

Part (b) — Seen



One mark (max 3) for each of the following points being evident from diagram: 3

- *firms and households exchange both money and goods/services, i.e. both pairs of arrows are present between firms & households;*
- *There will, in general, be leakages from, and injections to, the economy;*
- *leakages + consumer spending = wages*
- *leakages comprise: savings*
  taxation
  imports

  and injections comprise: investment by firms
  government spending
  exports

In this context, the economy will be in equilibrium when injections equal leakages. 1

Setter's initials	Checker's initials	Page number

	EXAMINATION SOLUTIONS 2016-17	Course M3B
Question 4		Marks & seen/unseen
Part (c)(i)	Through considering the circular flow of income, we can treat government spending as an injection into the economy. The investment will therefore increase GDP, so new GDP = original GDP + £5bn = £1.761tn	Seen sim 1 1
Part (c)(ii)	b is the marginal propensity to consume. An injection of $£a$ will increase the economy's output by $£\frac{a}{1-b}$; this is the multiplier effect. For our example, we therefore have $$£5bn = \frac{£2bn}{1-b} \implies b = 0.6$$	Seen sim 1 1
Part (d)	To establish the progressivity of the tax, we must find the average tax rate, $$ATR(Y) = T(Y)/Y \ ,$$ where $T(Y)$ is the tax paid and Y is the amount on which the tax is levied. For our example, we have that $ATR = Y^{-1/2}$, where Y is the individual's income. Since this decreases with respect to income, we conclude that this is a regressive tax.	Seen sim 1 1
Part (e)(i)	Increasing the national minimum wage is an example of <u>contractionary fiscal</u> policy. This will increase the cost of labour for producers in the economy, and so will primarily decrease aggregate supply; there may also be a secondary impact on aggregate demand as consumer spending increases as a result of increased income. *[The final mark requires some sensible discussion of the <u>mechanism</u> by which AS or AD is affected]*	Unseen 2 1
Part (e)(ii)	Increasing the nation's banks' reserve ratio is an example of <u>contractionary monetary</u> policy. This will decrease the money supply, reducing both consumer spending and investment by firms, hence reducing aggregate demand. This will likely have a knock-on impact on aggregate supply. *[The final mark requires some sensible discussion of the <u>mechanism</u> by which AS or AD is affected]*	 2 1
	Setter's initials Checker's initials	Page number

Imperial College London
Department of Mathematics

Examiner's Comments

Exam: M3B, Mathematics of Business and Economics   Session: 2016-2107

Question 1

Please use the space below to comment on the candidates' overall performance in the exam. A brief paragraph highlighting common mistakes and parts of questions done badly (or well) is sufficient. Do not refer to individual candidates. The purpose of this exercise is to provide guidance to the external examiners, and to the candidates themselves, on how you feel the cohort faired. Your comments will be available to students online.

This question was, in general, reasonably well-answered. Parts (a) and (b)(i)-(iii) were answered correctly by most candidates. In part (b)(iv), several candidates missed the fact that, due to constant returns to scale, the maximized profit was zero. This led to marks being missed in part (v) also, where setting the profit function equal to zero was a key step in obtaining the factor demand functions. Part (c) was well-answered by most of the candidates that attempted it.

Marker: J S Martin

Signature: ______________________ **Date:** 30/05/2017

Please return with exam marks (one report per marker)

Imperial College London
Department of Mathematics

Examiner's Comments

Exam: M3B, Mathematics of Business and Economics Session: 2016-2107

Question 2

Please use the space below to comment on the candidates' overall performance in the exam. A brief paragraph highlighting common mistakes and parts of questions done badly (or well) is sufficient. Do not refer to individual candidates. The purpose of this exercise is to provide guidance to the external examiners, and to the candidates themselves, on how you feel the cohort faired. Your comments will be available to students online.

Performance in this question was surprisingly poor. Parts (a) (b) and (c) were generally well answered, with the majority of candidates claiming most of the marks on offer. In part (d)(i), a common mistake was to explain the substitution effect in terms of substitute goods; this missed the context of the question, and many candidates missed the key point that the substitution effect is the portion of the Slutsky equation that assumes utility to be constant. Some candidates correctly linked the income effect to the purchasing power of the consumer, but some incorrectly attributed the income effect to a change in the consumer's budget. Most candidates got the form of the Slutsky equation correct, though some dropped marks for index errors. Part (d) proved challenging, with only a few students using the link between the concave expenditure function and the Hicksian demand. Part (e) proved especially challenging; several students started along the right path of repeatedly using the chain rule, but none completed the question without errors.

Marker: J S Martin

Signature: ________________ **Date:** 30/5/17

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Department of Mathematics

Examiner's Comments

Exam: M3B, Mathematics of Business and Economics Session: 2016-2107

Question 3

Please use the space below to comment on the candidates' overall performance in the exam. A brief paragraph highlighting common mistakes and parts of questions done badly (or well) is sufficient. Do not refer to individual candidates. The purpose of this exercise is to provide guidance to the external examiners, and to the candidates themselves, on how you feel the cohort faired. Your comments will be available to students online.

Some candidates performed reasonably in this question, however in general, performance was lower than expected.

In part (a), a number of candidates treated p as a function of y, and recited the first and second-order conditions for profit maximization derived in the lecture notes; this was contrary to the clear instructions in the question to treat y as a function of p, and so gained very few marks. Part (b) was generally successfully completed. In part (c), most of the cohort solved the FOC as required, but few checked the SOC as was expected.

Part (d) comprised the unseen material in the question, and proved more challenging than expected. Few candidates realised that the social marginal costs for an industry with a monopolist correspond to the monopolist's marginal costs, and not their supply curve. Most students provided a sketch that correctly labelled the consumers' surplus and the producer's surplus, but no candidates provided a sketch that correctly indicated allocative inefficiency (i.e. the monopolist's profit-maximising price/output combination is not at the point at which SoMC=SoMB).

Marker: J S Martin

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Department of Mathematics

Examiner's Comments

Exam: M3B, Mathematics of Business and Economics   Session: 2016-2107

Question 4

Please use the space below to comment on the candidates' overall performance in the exam. A brief paragraph highlighting common mistakes and parts of questions done badly (or well) is sufficient. Do not refer to individual candidates. The purpose of this exercise is to provide guidance to the external examiners, and to the candidates themselves, on how you feel the cohort faired. Your comments will be available to students online.

In general, candidates performed very well in Q4. Most students managed to answer parts (a)-(d) with little trouble; occasional mathematical errors surfaced, and some students presented answers to part (a) with too little detail, but for the majority, these questions were well answered. In part (e), most students discussed the suggested policies well; the most common dropped marks here were for simply neglecting to mention whether the policy was fiscal/monetary or contractionary/expansionary.

Marker: J S Martin

Signature: ____________ **Date:** 30/5/17

Please return with exam marks (one report per marker)