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UNIVERSITY OF LONDON

[II(3)E 2003]

B.ENG. AND M.ENG. EXAMINATIONS 2003

For Internal Students of Imperial College London

This paper is also taken for the relevant examination for the Associateship of the City & Guilds of London Institute

PART II : MATHEMATICS 3 (ELECTRICAL ENGINEERING)

Wednesday 4th June 2003 2.00 - 5.00 pm

Answer EIGHT questions.
Answers to Section A *questions must be written in a different answer book from answers to* Section B *questions.*

Corrected Copy

[Before starting, please make sure that the paper is complete; there should be 7 pages, with a total of 12 questions. Ask the invigilator for a replacement if your copy is faulty.]

1. Consider the system of equations

$$\frac{dx_1}{dt} = x_1 + 2x_2 \ ,$$

$$\frac{dx_2}{dt} = 2x_1 + x_2 \ .$$

Show that the system can be written in matrix form

$$\frac{d}{dt}\boldsymbol{x} = A\boldsymbol{x} \ ,$$

where $\boldsymbol{x}$ is a 2×1 column vector and A is a 2×2 symmetric matrix. Find the eigenvalues and eigenvectors of A and hence find the general solution of the system.

Find the particular solution corresponding to the initial conditions $x_1(0) = 1$ and $x_2(0) = 0$.

Show that $x_1(t) \sim x_2(t) \sim \frac{1}{2}e^{3t}$ as $t \to \infty$.

2. (i) Consider the mapping

$$w = \frac{1}{(z-1)^2}$$

from the z-plane to the w-plane where $w = u + iv$. Show that the circle

$$(x-1)^2 + y^2 = R^2$$

in the z-plane, maps to the circle

$$u^2 + v^2 = R^{-4}$$

in the w-plane.

(ii) Now consider the mapping

$$w = \frac{1}{z} \ .$$

For the class of circles

$$(x-1)^2 + (y-1)^2 = r^2 \ ,$$

show that when $r = \sqrt{2}$ the circle maps to the straight line $v = u - \frac{1}{2}$ in the w-plane, whereas when $r = 1$ it maps to the circle $(u-1)^2 + (v+1)^2 = 1$.

PLEASE TURN OVER

3. The complex function

$$\frac{e^{iz}}{z(z^2+1)(z^2+4)},$$

has simple poles in the upper half-plane at $z = i$ and $z = 2i$, with another at $z = 0$ and two simple poles in the lower half-plane at $z = -i$ and $z = -2i$. Show that

(i) the residue at $z = 0$ is $\frac{1}{4}$,

(ii) the residue at $z = i$ is $-\frac{1}{6e}$,

(iii) the residue at $z = 2i$ is $\frac{1}{24e^2}$.

Now consider the contour integral

$$\oint_C \frac{e^{iz}\,dz}{z(z^2+1)(z^2+4)}$$

where C is taken to be a semi-circle in the upper half of the complex plane, with an additional small semi-circular indentation below the pole at $z = 0$. Show that the contribution to the above integral from this indentation, in the limit when its radius goes to zero, is $\pi i/4$. Altogether, show that

$$\int_{-\infty}^{\infty} \frac{\sin x\,dx}{x(x^2+1)(x^2+4)} = \frac{\pi(3e-1)(e-1)}{12e^2}.$$

4. Using the unit circle $z = e^{i\theta}$ as your contour C, convert the integral

$$I = \int_0^{2\pi} \frac{d\theta}{2+\sin\theta}$$

to a complex integral over C and hence show that

$$I = \frac{2\pi}{\sqrt{3}}.$$

5. If $\overline{f}(\omega)$ is the Fourier transform of $f(t)$, prove Parseval's equality

$$\int_{-\infty}^{\infty} |f(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\overline{f}(\omega)|^2 \, d\omega.$$

The sinc-function $\operatorname{sinc}(t)$ and the tent function $\Lambda(t)$ are defined respectively by

$$\operatorname{sinc}(t) = \frac{\sin(t/2)}{(t/2)},$$

$$\Lambda(t) = \begin{cases} 1+t, & -1 \le t \le 0, \\ 1-t, & 0 \le t \le 1. \end{cases}$$

Show that

(i) $$\overline{\Lambda}(\omega) = \operatorname{sinc}^2(\omega)$$

(ii) $$\int_{-\infty}^{\infty} \operatorname{sinc}^4(\omega) \, d\omega = \frac{4\pi}{3}.$$

6. $\overline{f}(s) = \mathcal{L}\{f(t)\}$ and $\overline{g}(s) = \mathcal{L}\{g(t)\}$ are the Laplace transforms of two functions $f(t)$ and $g(t)$ respectively. The convolution of $f(t)$ with $g(t)$ is defined as

$$f * g = \int_0^t f(u) g(t-u) \, du.$$

Use double integration to prove the Laplace convolution theorem

$$\mathcal{L}\{f * g\} = \overline{f}(s)\overline{g}(s)$$

providing a sketch of the region over which the integration takes place.

Hence, or otherwise, show that

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \frac{1}{2} t \sin t.$$

You may assume that $\mathcal{L}\{\cos \omega t\} = \frac{s}{s^2+\omega^2}$ *and* $\mathcal{L}\{\sin \omega t\} = \frac{\omega}{s^2+\omega^2}$.

PLEASE TURN OVER

7. A second order ordinary differential equation, with initial values, takes the form

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = f(t)\,, \qquad x = \frac{dx}{dt} = 0 \text{ when } t = 0\,,$$

for some function $f(t)$.

Use the Laplace convolution theorem and the Shift Theorem to show that

$$x(t) = \frac{1}{2}\int_0^t e^{-u}\sin(2u)\,f(t-u)\,du$$

satisfies the differential equation and its initial conditions.

You may assume that $\mathcal{L}\{\sin\omega t\} = \frac{\omega}{s^2+\omega^2}$.

8. (i) The vector field $\boldsymbol{F}$ is given by

$$\boldsymbol{F} = (x+3y^2)\boldsymbol{i} + (y-2z)\boldsymbol{j} + (x+\alpha z)\boldsymbol{k}\,,$$

where α is a constant scalar.

Calculate

(a) $\mathrm{div}\boldsymbol{F}$, (b) $\mathrm{curl}\,\boldsymbol{F}$, (c) $\mathrm{div}(\mathrm{curl}\,\boldsymbol{F})$.

(ii) If $\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r}$, with ω a constant vector and $\boldsymbol{r} = x\boldsymbol{i} \overset{+}{\cancel{=}} y\boldsymbol{j} + z\boldsymbol{k}$, show that

$$\boldsymbol{\omega} = \frac{1}{2}\,\mathrm{curl}\,\boldsymbol{v}\,.$$

(iii) Calculate

$$\mathrm{curl}(f(r)\boldsymbol{r})\,,$$

where $r = |\boldsymbol{r}|$, with $\boldsymbol{r}$ defined as in part (ii) and $f(r)$ is an arbitrary differentiable function of r.

9. A closed curve C is defined parametrically by the equations

$$x = \rho \cos t, \quad y = 4\rho \sin t, \quad 0 \le t \le 2\pi,$$

where ρ is a positive constant.

A vector function $\boldsymbol{F}$ is given by

$$\boldsymbol{F} = F_1 \boldsymbol{i} + F_2 \boldsymbol{j}, \quad \text{with} \quad F_1 = 2x + 3y, \quad F_2 = x + y\,.$$

(i) Sketch the curve C and calculate the area A enclosed within it.

(ii) Calculate div$\boldsymbol{F}$.

(iii) Compute the quantity

$$Q \;=\; \int_C (F_1\, dy - F_2\, dx)\,,$$

where C is traversed in an anti-clockwise sense.

(iv) From your calculations demonstrate that

$$\mathrm{div}\boldsymbol{F} \;=\; \frac{Q}{A}\,.$$

10. Consider a two-dimensional region R bounded by a closed piecewise smooth curve C. Using Green's Theorem in a plane, choose the components of a vector field $\boldsymbol{u}(x, y)$ in terms of $P(x, y)$ and $Q(x, y)$ to prove the two-dimensional form of the Divergence Theorem

$$\oint_C (\boldsymbol{u}.\hat{\boldsymbol{n}})\, ds = \int\int_R \mathrm{div}\, \boldsymbol{u}\, dxdy \qquad (1)$$

where $\hat{\boldsymbol{n}}$ is a unit vector normal to C.

If

$$\boldsymbol{u} = \left(\frac{x^2 y}{1 + y^2}\right) \boldsymbol{i} + \left[x \ln(1 + y^2)\right] \boldsymbol{j}$$

and R is the region in the first quadrant bounded by the x-axis and the lines $y = x$ and $x = 1$, sketch the region R and then evaluate the double integral on the right hand side of (1) to show that

$$\oint_C (\boldsymbol{u}.\hat{\boldsymbol{n}})\, ds = 2 \ln 2 - 1\,.$$

Green's Theorem in a plane states that for a two-dimensional region R bounded by a closed, piecewise smooth curve C

$$\oint_C \{P(x, y)dx + Q(x, y)dy\} = \int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy.$$

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SECTION B

11. (i) Given

$$P(A) = \frac{1}{2},\ P(B) = \frac{1}{3} \text{ and } P(A \cap B) = \frac{1}{4},$$

calculate

$$P(A \cup B), \quad P(A \mid B)/P(B \mid A) \quad \text{and} \quad P\,(\text{exactly one of } A,\ B \text{ occurs}).$$

(ii) A bag contains six components of which two are defective. Two components are selected at random from the bag (without replacement). What is the probability that (a) both of them are defective, (b) just one is defective, (c) neither is defective?

(iii) Three electrical components in line are subject to failure independently, each with probability p. Calculate the probability that at least two adjacent components fail. Extend the calculation to the case of two adjacent failures among four such components.

12. In a computer network X and Y are two performance measures that vary randomly with bivariate density

$$f(x,\, y) = kx(x - y) \text{ on } 0 < y < x^2 < 1\,.$$

Calculate k and the marginal densities, $f_X(x)$ and $f_Y(y)$. Is the criterion for independence of X and Y satisfied here?

Evaluate

$$E(X^2 - Y), \quad P\left(Y < \frac{1}{2} \mid X < \frac{1}{2}\right) \quad \text{and} \quad P\left(Y < \frac{1}{2} \mid X = 0.9\right)\,.$$

END OF PAPER

MATHEMATICS DEPARTMENT

MATHEMATICAL FORMULAE

1. VECTOR ALGEBRA

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1, a_2, a_3)$$

Scalar (dot) product: $\mathbf{a}\,.\,\mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

Vector (cross) product:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Scalar triple product:

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a}\,.\,\mathbf{b} \times \mathbf{c} = \mathbf{b}\,.\,\mathbf{c} \times \mathbf{a} = \mathbf{c}\,.\,\mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product: $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c}\,.\,\mathbf{a})\mathbf{b} - (\mathbf{b}\,.\,\mathbf{a})\mathbf{c}$

2. SERIES

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \ldots \quad (\alpha \text{ arbitrary}, |x| < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \ldots,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^n \frac{x^{n+1}}{(n+1)} + \ldots (-1 < x \le 1)$$

3. TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin(a+b) = \sin a \cos b + \cos a \sin b\,;$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b\,.$$

$$\cos iz = \cosh z\,;\quad \cosh iz = \cos z\,;\quad \sin iz = i \sinh z\,;\quad \sinh iz = i \sin z\,.$$

4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f\,D^n g + \tbinom{n}{1} Df\, D^{n-1} g + \ldots + \tbinom{n}{r} D^r f\, D^{n-r} g + \ldots + D^n f\, g\,.$$

(b) Taylor's expansion of $f(x)$ about $x = a$:

$$f(a+h) = f(a) + hf'(a) + h^2 f''(a)/2! + \ldots + h^n f^{(n)}(a)/n! + \epsilon_n(h)\,,$$

where $\epsilon_n(h) = h^{n+1} f^{(n+1)}(a + \theta h)/(n+1)!$, $0 < \theta < 1$.

(c) Taylor's expansion of $f(x, y)$ about (a, b):

$$f(a+h, b+k) = f(a, b) + [hf_x + kf_y]_{a,b} + 1/2!\left[h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}\right]_{a,b} + \ldots$$

(d) Partial differentiation of $f(x, y)$:

i. If $y = y(x)$, then $f = F(x)$, and $\dfrac{dF}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dx}$.

ii. If $x = x(t)$, $y = y(t)$, then $f = F(t)$, and $\dfrac{dF}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$.

iii. If $x = x(u, v)$, $y = y(u, v)$, then $f = F(u, v)$, and

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}, \quad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$

(e) Stationary points of $f(x, y)$ occur where $f_x = 0$, $f_y = 0$ simultaneously.
Let (a, b) be a stationary point: examine $D = [f_{xx}f_{yy} - (f_{xy})^2]_{a,b}$.
If $D > 0$ and $f_{xx}(a, b) < 0$, then (a, b) is a maximum;
If $D > 0$ and $f_{xx}(a, b) > 0$, then (a, b) is a minimum;
If $D < 0$ then (a, b) is a saddle-point.

(f) Differential equations:

i. The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating factor $I(x) = \exp[\int P(x)(dx]$, so that $\dfrac{d}{dx}(Iy) = IQ$.

ii. $P(x, y)dx + Q(x, y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y$.

5. INTEGRAL CALCULUS

(a) An important substitution: $\tan(\theta/2) = t$:
$\sin\theta = 2t/(1+t^2)$, $\cos\theta = (1-t^2)/(1+t^2)$, $d\theta = 2\,dt/(1+t^2)$.

(b) Some indefinite integrals:

$$\int (a^2 - x^2)^{-1/2} dx = \sin^{-1}\left(\frac{x}{a}\right), \quad |x| < a.$$

$$\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1}\left(\frac{x}{a}\right) = \ln\left\{\frac{x}{a} + \left(1 + \frac{x^2}{a^2}\right)^{1/2}\right\}.$$

$$\int (x^2 - a^2)^{-1/2} dx = \cosh^{-1}\left(\frac{x}{a}\right) = \ln\left|\frac{x}{a} + \left(\frac{x^2}{a^2} - 1\right)^{1/2}\right|.$$

$$\int (a^2 + x^2)^{-1} dx = \left(\frac{1}{a}\right)\tan^{-1}\left(\frac{x}{a}\right).$$

6. NUMERICAL METHODS

(a) Approximate solution of an algebraic equation:

If a root of $f(x) = 0$ occurs near $x = a$, take $x_0 = a$ and
$x_{n+1} = x_n - [f(x_n)/f'(x_n)]$, $n = 0, 1, 2 \ldots$

(Newton Raphson method).

(b) Formulae for numerical integration: Write $x_n = x_0 + nh$, $y_n = y(x_n)$.

i. Trapezium rule (1-strip): $\int_{x_0}^{x_1} y(x)dx \approx (h/2)[y_0 + y_1]$.

ii. Simpson's rule (2-strip): $\int_{x_0}^{x_2} y(x)dx \approx (h/3)[y_0 + 4y_1 + y_2]$.

(c) Richardson's extrapolation method: Let $I = \int_a^b f(x)dx$ and let I_1, I_2 be two estimates of I obtained by using Simpson's rule with intervals h and $h/2$.

Then, provided h is small enough,

$$I_2 + (I_2 - I_1)/15,$$

is a better estimate of I.

7. LAPLACE TRANSFORMS

Function	Transform	Function	Transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)dt$	$af(t) + bg(t)$	$aF(s) + bG(s)$
df/dt	$sF(s) - f(0)$	d^2f/dt^2	$s^2F(s) - sf(0) - f'(0)$
$e^{at}f(t)$	$F(s-a)$	$tf(t)$	$-dF(s)/ds$
$(\partial/\partial\alpha)f(t,\alpha)$	$(\partial/\partial\alpha)F(s,\alpha)$	$\int_0^t f(t)dt$	$F(s)/s$
$\int_0^t f(u)g(t-u)du$	$F(s)G(s)$		
1	$1/s$	$t^n (n = 1, 2 \ldots)$	$n!/s^{n+1}$, $(s > 0)$
e^{at}	$1/(s-a)$, $(s > a)$	$\sin\omega t$	$\omega/(s^2+\omega^2)$, $(s > 0)$
$\cos\omega t$	$s/(s^2+\omega^2)$, $(s > 0)$	$H(t-T) = \begin{cases} 0, & t < T \\ 1, & t > T \end{cases}$	e^{-sT}/s, $(s, T > 0)$

8. FOURIER SERIES

If $f(x)$ is periodic of period $2L$, then $f(x + 2L) = f(x)$, and

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \text{ where}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad n = 0, 1, 2, \ldots, \text{ and}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1, 2, 3, \ldots$$

Parseval's theorem

$$\frac{1}{L}\int_{-L}^{L} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right).$$