UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 2002

BEng Honours Degree in Computing Part II
MEng Honours Degrees in Computing Part II
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the Associateship of the City and Guilds of London Institute

PAPER C233

COMPUTATIONAL TECHNIQUES

Thursday 9 May 2002, 10:00
Duration: 90 minutes
(Reading time 5 minutes)

Answer THREE questions

Paper contains 4 questions
Calculators required

1a Find a local minimum or maximum for

$$f(x, y) = x^2 + y - \frac{1}{3}y^3.$$

Explain your work.

b Explain the paradigm of iterative methods for solving $\boldsymbol{Ax} = \boldsymbol{b}$ system of linear equations, where $\boldsymbol{A}$ is a nonsingular $m \times m$ matrix. On what theorem is it based and how?

c The *harmonic series* is defined as the sum

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots .$$

Determine an upper bound for the number of terms after which the series would seem to converge if executed on a machine with single precision. Assume a single precision rounded binary arithmetic (`float`, 23 bit mantissa).

(The three parts carry, respectively, 50%, 20% and 30% of the marks).

2a Let I be the half-open interval $[1, 2)$ and $d(x, y) = |x - y|$ be the distance for any $x, y \in I$.

(i) Show that d is a metric on I.

(ii) Show that $x_n = \dfrac{2n-1}{n}$, for $n = 1, 2, \ldots$ is a Cauchy sequence in the metric d.

(iii) Is (I, d) complete? Justify your answer.

b Determine the condition number of the following matrix measured in the ℓ_∞ norm.

$$\boldsymbol{A} = \begin{bmatrix} 1 & & \\ -1 & 2 & \\ 2 & 0 & 1 \end{bmatrix}$$

c Find the minimum or maximum of $f(x) = \dfrac{1}{4}x^4 - \dfrac{1}{3}x^3 + \dfrac{1}{12}, \quad -\infty \leq x \leq +\infty.$

(The three parts carry, respectively, 40%, 40% and 20% of the marks).

3 Consider a permutation $\sigma \in \mathcal{S}_m$ of a finite set $\{1, 2, \ldots, m\}$, i.e. a bijective map

$$\sigma : \{1, 2, \ldots, m\} \mapsto \{1, 2, \ldots, m\}.$$

A permutation $\sigma \in \mathcal{S}_m$ can be represented by;

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & \ldots & m-1 & m \\ \sigma(1) & \sigma(2) & \sigma(3) & \ldots & \sigma(m-1) & \sigma(m) \end{pmatrix}.$$

Define for every permutation σ a "re-shuffling" $\mathbf{S}_\sigma$ of the components of m-vectors,

$$\mathbf{S}_\sigma([x_1, x_2, \ldots, x_m]^T) = [x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(m)}]^T$$

e.g. for $m = 3$ and $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ a vector $\mathbf{x} = [x_1, x_2, x_3]^T \in \mathbb{R}^3$ is transformed into $[x_3, x_2, x_1]^T$.

a For each $\sigma \in \mathcal{S}_m$ find a $m \times m$ matrix $\mathbf{S}_\sigma$ such that for all $\mathbf{x} \in \mathbb{R}^m$: $\mathbf{S}_\sigma \mathbf{x} = \mathbf{S}_\sigma(\mathbf{x})$.

(i) Write down the matrix $\mathbf{S}_\sigma$ for:

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \text{ and } \sigma_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 1 & 2 \end{pmatrix}.$$

(ii) How does the matrix $\mathbf{S}_\sigma$ look like for a general permutation

$$\sigma = \begin{pmatrix} 1 & 2 & \ldots & m-1 & m \\ \sigma(1) & \sigma(2) & \ldots & \sigma(m-1) & \sigma(m) \end{pmatrix}.$$

b Determine which of the following statements are true for permutations $\sigma \in \mathcal{S}_m$ (for a fixed m), and explain why:

(i) All matrices $\mathbf{S}_\sigma$ form a set of linearly independent set of matrices.

(ii) Any permutation matrix $\mathbf{S}_\sigma$ is non-singular.

(iii) Any permutation matrix $\mathbf{S}_\sigma$ is an orthogonal matrix.

(iv) No permutation matrix $\mathbf{S}_\sigma$ is a diagonal matrix.

Hint: Use the fact that for any two permutations σ_1 and σ_2:

$$\mathbf{S}_{\sigma_1 \circ \sigma_2} = \mathbf{S}_{\sigma_1} \mathbf{S}_{\sigma_2}$$

where '$\circ$' denotes the composition of permutations.

c Consider a permutation σ and its corresponding matrix $\mathbf{S}_\sigma$: What property of σ does the trace of $\mathbf{S}_\sigma$ represent?

(The three parts carry, respectively 40%, 40% and 20% of the marks).

4a What is the rank of the following matrix; justify your answer, show work:

$$\mathbf{U} = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 0 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$

b Consider the following matrix:

$$\mathbf{A} = \begin{bmatrix} 4 & -2 & 2 \\ -2 & 2 & 0 \\ 2 & 0 & 3 \end{bmatrix}$$

(i) Compute its Cholesky factorisation.

(ii) Determine if $\mathbf{A}$ is positive definite.

(iii) Check if the factorisation is correct.

(iv) Using the Cholesky factorisation of $\mathbf{A}$ solve $\mathbf{AX} = \mathbf{B}$, with $\mathbf{A}$ from before and:

$$\mathbf{B} = \begin{bmatrix} 6 & 2 \\ -4 & 2 \\ 2 & 6 \end{bmatrix}.$$

c Let $\mathbf{A}$ be an $m \times m$ positive definite matrix. Prove that all diagonal elements (a_{ii} for $i = 1, \ldots, m$) are strictly positive.

(The three parts carry, respectively 30%, 50% and 20% of the marks).