Paper Number(s): **E3.09**
ISE3.9

IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE
UNIVERSITY OF LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2002

EEE/ISE PART III/IV: M.Eng., B.Eng. and ACGI

CONTROL ENGINEERING

Wednesday, 8 May 10:00 am

There are SIX questions on this paper.

Answer FOUR questions.

Time allowed: 3:00 hours

Corrected Copy

Examiners responsible:

First Marker(s): Vinter,R.B.

Second Marker(s): Astolfi,A.

Special Instructions for Invigilator: None

Information for Students: None

1. What is the relationship between the Nyquist diagram of the forward path transfer function of a unity feedback control system and the number of 'unstable' open and closed poles of the system? [2]

Consider the unity feedback control system under proportional control, illustrated in Figure 1. The plant transfer function is

$$G(s) = \frac{100(s+1)}{s(s-2)(s+a)}.$$

The system parameter a is a positive constant. $K(>0)$ is the controller gain.

Find the least value $\bar{a}$ of a such that the Nyquist diagram of $G(s)$ intercepts the negative real axis. [4]

Sketch the Nyquist diagram of $G(s)$ in the two cases

(i) $a > \bar{a}$

(ii) $a \leq \bar{a}$. [10]

Predict from the Nyquist diagrams how closed loop stability is affected by increasing the gain K

$$0 < K < \infty,$$

in each of the two cases (i) and (ii). [4]

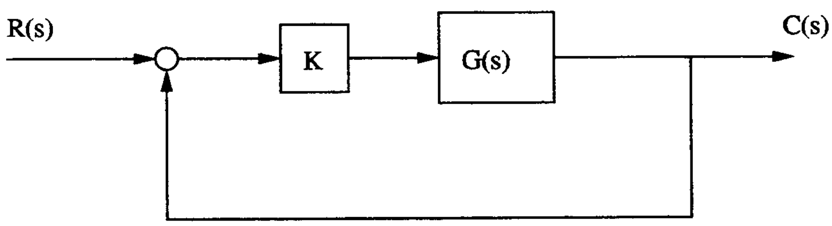


Figure 1

2. Two unit masses are attached to rigid supports, and to each other, by springs as indicated in *Figure 2*. Each spring has unit spring constant. Denote the displacements (from the left) of the masses, relative to their steady state positions, by z_1 and z_2.

The mechanism is controlled pneumatically: an equal and opposite force f is applied to both masses by means of a variable air jet, as indicated in the diagram.

Derive differential equations for z_1 and z_2. Hence derive a state space model, with input $u = f$ and state components $x_1 = z_1$, $x_2 = \dot{z}_1$, $x_3 = z_2$ and $x_4 = \dot{z}_2$. [10]

Show that the system is not controllable. [4]

By deriving a differential equation satisfied by $y(t) = z_1(t) + z_2(t)$, or otherwise, explain, qualitatively, why the system is uncontrollable. Show furthermore that whatever feedback control law

$$u = -k^T x$$

is implemented, the response of the closed loop system will have an undamped oscillatory component. What is its frequency? [6]

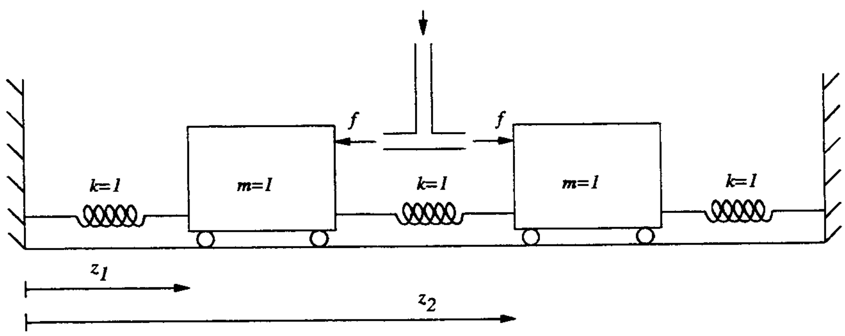


Figure 2

3 (a). Figure 3 shows the model of a spacecraft attitude control system, that takes account of a disturbance torque T_d and also the presence of a sensor lag (modelled as a first order transfer function). A PID compensator,

$$D(s) = K(1 + \frac{1}{T_I s})(1 + T_D s),$$

with design parameters the positive constants K, T_I and T_D, is to be used in the forward path. Write the spacecraft and sensor transfer function as

$$G(s) = \frac{1.8}{s^2(s+2)}.$$

Show that, provided the PID compensator is stabilizing, the control system has zero steady state output $\lim_{t\to\infty} \theta(t)$, when the the disturbance torque T_d is a step and the reference signal θ_{ref} is zero. [4]

Choose values of the compensator parameters to achieve the following specifications:

(i): The phase margin of $D(s)G(s)$ is 65°.

(ii): the value of T_D is the smallest possible for which the above phase margin specification can be achieved.

You are required to follow the following design procedure:

(a): For fixed T_D, $T_D > 0.5$, derive formulae for the maximum phase $\phi_{\max}$ of

$$\frac{(1 + T_D j\omega)}{(j\omega)^2(j\omega + 2)}$$

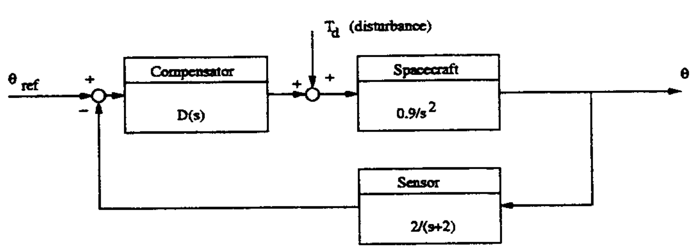
over ω values in the range $0 \le \omega < \infty$, and also for the frequency ω_{max} at which the maximum phase occurs. (See below.) [6]

(b). Choose T_D to have the minimum possible value such that $\phi_{max} = -180° + 65°$ and choose the gain cross-over frequency ω_c of $D(s)G(s)$ to be $\omega_c = \omega_{\max}$. Set $(1/T_I) = 0.05(1/T_D)$. (This ensures that $\angle(1 + 1/(T_I j\omega_c) \approx 0°$.) Determine K. [10]

In (a), you can use the information: for given constants $T > 0$, $1 > \alpha > 0$, the phase frequency response of $M(s) = \frac{Ts+1}{(\alpha Ts+1)}$ has maximum phase

$$90° - 2\tan^{-1}(\sqrt{\alpha})$$

and this is achieved at the frequency $1/(T\sqrt{\alpha})$ rs^{-1}.



4 (a). Consider a unity feedback system with plant transfer function

$$G(s) = \frac{\omega_n^2}{s(s+2\zeta\omega_n)}\,.$$

Here, $\omega_n > 0$ and $\zeta > 0$ are constants.

Show that the phase margin is

$$\phi = \tan^{-1}\left[\frac{2\zeta}{\sqrt{\sqrt{1+4\zeta^4}-2\zeta^2}}\right].$$

[6]

A standard formula, relating ϕ and ζ is

$$\zeta \approx \phi/100\,,$$

where ϕ is measured in degrees. To what extent is this justified? [2]

(b). A first order system has state space model

$$\dot{x}(t) = ax(t) + bu(t)\,,$$

in which a and b are constants.

A control strategy is required to track an exponential reference signal

$$r(t) = e^{-\beta t}\,,$$

in which β is a positive constant. This is to be achieved by choosing a control strategy to minimize

$$\int_0^\infty \left[|x(t) - r(t)|^2 + \alpha u^2(t)\right] dt\,, \qquad (1)$$

in which α is a positive constant.

By regarding $r(t)$ as an extra state variable,

$$\begin{cases} \dot{r}(t) = -\beta r(t) \\ r(0) = 1 \end{cases}$$

and by considering optimal controls for the optimization problem

$$\begin{cases} \text{Minimize } \int_0^\infty \left[\mathbf{x}^T(t)\mathbf{c}\mathbf{c}^T\mathbf{x}(t) + \alpha u^2(t)\right] dt \\ \text{subject to} \\ \dot{\mathbf{x}}(t) = A\mathbf{x}(t) + \mathbf{b}u(t) \\ x(0) = x_0, \end{cases} \qquad (2)$$

for suitably chosen matrices A, $\mathbf{b}$, $\mathbf{c}^T$ etc., derive equations for the time varying feedback control law

$$u(t) = -k_1 x(t) - k_2 e^{-\beta t}\,.$$

which minimizes the cost (1). [12]

You can use the fact that, for the matrices A, $\mathbf{b}$, $\mathbf{c}^T$ etc., satisfying suitable conditions, the solution to (2) is

$$u = -\mathbf{b}^T P\,\mathbf{x},$$

where P is a symmetric, positive definite solution of the Matrix Riccati equation:

$$A^T P + PA + \mathbf{c}\mathbf{c}^T - \alpha^{-1} P\mathbf{b}\mathbf{b}^T P = 0\,.$$

5 (a). A dynamic system, illustrated in Figure 5.1, has forward path transfer function

$$G(s) = \frac{1}{s(s+1)}.$$

What is the standard controllable state space representation

$$\begin{cases} \dot{x}(t) = Ax(t) + bu(t) \\ y(t) = c^T x(t) \end{cases} \qquad (3)$$

of this system? [2]

Design a dynamic output feedback control system for (3), choosing the control gain to give two closed loop poles with damping factor $\zeta = 1$ and undamped natural frequency $\omega_n = 2$, and choosing the observer gain to give two real closed loop poles at $s = -4 + 0j$. [10]

(b). A thermal control system, with plant modelled as a first order lag, is illustrated in Figure 5.2. To achieve zero steady state error for step inputs $r(t)$ and to increase the speed of response, a forward path compensator of the form

$$D(s) = \frac{1}{s}E(s),$$

incorporating integral control action, is required. By using the results of part (a), or otherwise, choose the transfer function $E(s)$ in the compensator to arrange that two closed loop poles have damping factor $\zeta = 1$ and undamped natural frequency $\omega_n = 2$ and two closed loop poles are located at $s = -4 + 0j$. [8]

Hint: consider the transfer function relating the output $y(s)$ to the control signal $u(s)$ in part (a).

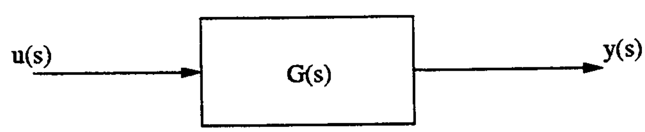


Figure 5.1

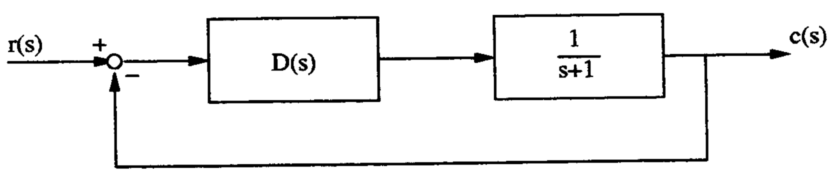


Figure 5.2

6. Figure 6.1 shows the characteristic of a 'relay with dead-space' nonlinearity. Show that the describing function is

$$N(A) = \frac{4b}{\pi A}\sqrt{1-(a/A)^2} \quad \text{for } A > a\,.$$

Here, a and b are positive constants. [7]

Consider now a velocity feedback control system with forward path transfer function $G(s)/s$, where

$$G(s) = \frac{(s+1)}{s^2}\,.$$

Suppose that the speed sensor fails, and, instead of providing a signal which is proportional to output velocity, provides a signal which is (approximately) the output of a ideal relay with dead-space. Figure 6.2 illustrates the control system after a failure of the speed sensor.

A limit cycle is observed. Determine its frequency. [10]

Suppose $a = 0$. (In this case $N(A)$ is a decreasing function). Briefly discuss whether you expect the limit cycle to be stable. [3]

Hint: For a control system with forward path transfer function $\frac{1}{s}G(s)$, and feedback path transfer function $1 + \tau_v s$ 'velocity feedback', assess whether increasing τ_v is stabilizing or de-stabilizing.

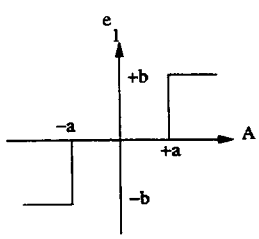


Figure 6.1

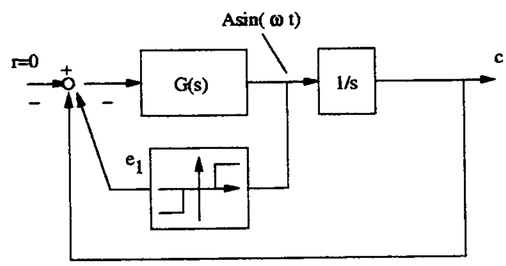


Figure 6.2

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1\. Write N = # clockwise encirclements of $-1+0j$,
P_o = # open-loop 'unstable' poles, P_c = # closed loop 'unstable' poles.

[2] Then $\underline{N = P_c - P_o}$.

$$\frac{1}{G(j\omega)} = \frac{j\omega(j\omega-2)(j\omega+a)}{100(j\omega+1)} = \frac{-j\omega(j\omega-1)(j\omega-2)(j\omega+a)}{100(\omega^2+1)}$$

$$= -j\omega\left[(2-\omega^2) - 3j\omega\right]\left[a+j\omega\right]/(\ldots) = -j\omega\left[(2-\omega^2)a + 3\omega^2 + j\omega(-3a+2-\omega^2)\right]/(\ldots)$$

$\mathrm{Im}\left\{\frac{1}{G(j\omega)}\right\} = 0 \Rightarrow 2a-(a-3)\omega^2.$

This has a solution $\omega^2 = \underline{2a/(a-3)}$ if $a > 3$

Then $\mathrm{Re}\left\{\frac{1}{G(j\omega)}\right\} = \frac{\omega^2}{(\ldots)}\left(-3a+2-\frac{2a}{a-3}\right) = \frac{\omega^2}{(\ldots)}\frac{(-3a^2+9a-6)}{(a-3)}$

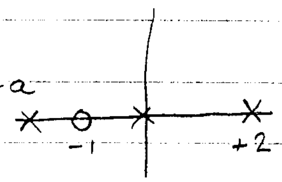
The right side is always negative for $a \geq 3$.

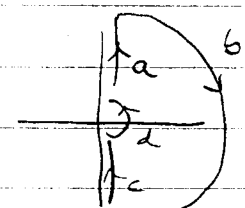
[6] Minimum value $\bar{a}$ of a for 'negative' intercept is $\underline{\bar{a} = 3}$

Nyquist Diagrams:

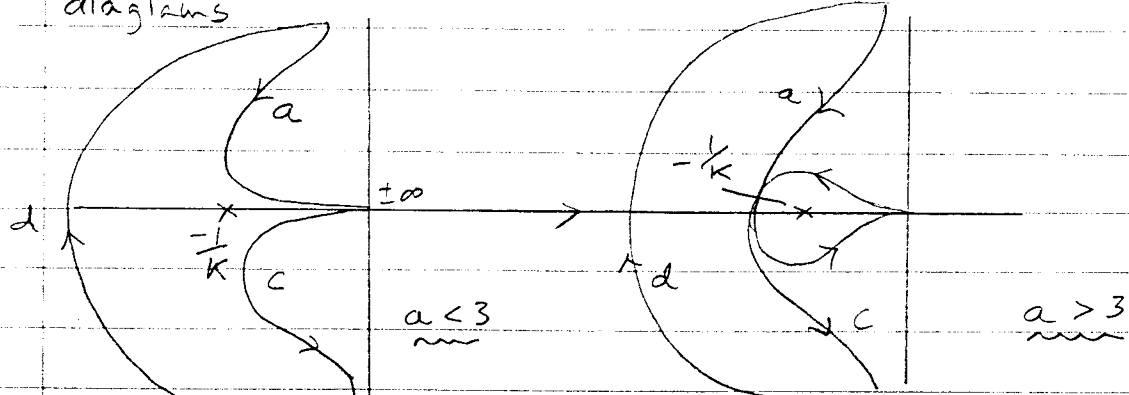
$|G(j0)| = +\infty$, $\measuredangle G(j0) = -270^\circ$

$|G(j\infty)| = 0$, $\measuredangle G(j\infty) = -180^\circ$





Using above intercept information, we can now sketch diagrams



[10] $\underline{\text{one clockwise encirclement}}$ One clockwise encirclement if K is small
" anticlockwise " " K is large

$P_0 = 1$. So $P_c = N+1$. $N = 1$ (for $a<3$), $N = \left.\begin{matrix}+1\\-1\end{matrix}\right\}$ (for $a>3$)

As K increases $0 < K < \infty$

∴ system is always unstable if $a < 3$

[4] ∴ system is unstable for small K and stable for large K, if $a > 3$.

2. Left hand mass : $\ddot{z}_1 = -z_1 + (z_2 - z_1) - f$ or $\ddot{z}_1 = -2z_1 + z_2 - f$.

Right hand mass : $\ddot{z}_2 = -z_2 + (z_1 - z_2) + f$ or $\ddot{z}_2 = -2z_2 + z_1 + f$

$$\ddot{z}_1 = -2z_1 + z_2 - f \quad , \quad \ddot{z}_2 = -2z_2 + z_1 + f \qquad — (2.1)$$

[6] Let $x_1 = z_1$, $x_2 = \dot{z}_1$, $x_3 = z_2$, $x_4 = \dot{z}_2$ and $u = f$. Then

$\dot{x}_1 = x_2$, $\dot{x}_2 = -2x_1 + x_3 - u$, $\dot{x}_3 = x_4$, $\dot{x}_4 = -2x_3 + x_1 + f$

Assemble as state space model :

$$\dot{\underline{x}} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 \\ -2 & 0 & +1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -2 & 0 \end{pmatrix}}_{A} \underline{x} + \underbrace{\begin{pmatrix} 0 \\ -1 \\ 0 \\ +1 \end{pmatrix}}_{b} u$$

state space model

[4] $b = \begin{pmatrix} 0 \\ -1 \\ 0 \\ +1 \end{pmatrix}$, $Ab = \begin{pmatrix} -1 \\ 0 \\ +1 \\ 0 \end{pmatrix}$, $A^2b = \begin{pmatrix} 0 \\ 3 \\ 0 \\ -3 \end{pmatrix}$, $A^3b = \begin{pmatrix} 3 \\ 0 \\ -3 \\ 0 \end{pmatrix}$

Controllability matrix is

$$W = [b \mid Ab \mid \ldots \mid A^3b] = \begin{bmatrix} 0 & -1 & 0 & 3 \\ -1 & 0 & 3 & 0 \\ 0 & +1 & 0 & -3 \\ +1 & 0 & -3 & 0 \end{bmatrix}$$

Since the 3rd column is a scaled version of the first column,

[4] $\det[W] = 0$, i.e. system is not controllable

Notice that, from (2.1), $y = z_1 + z_2$ satisfies the equation

$$\ddot{y} = \ddot{z}_1 + \ddot{z}_2 = -2(z_1 + z_2) + (z_1 + z_2) + 0$$

or $\ddot{y} = -y$. — (2.1)

The average displacement of the masses, $\frac{1}{2}y = \frac{z_1 + z_2}{2}$,

is not affected in any way by the actuator

From (2.1), y oscillates with a frequency

[6] $\omega = \sqrt{1} = 1$ rad s^{-1}.

3. The transfer function $\frac{\theta(s)}{T_d(s)} = \frac{0.9/s^2}{1+K(1+\frac{1}{T_I s})(1+T_D s)\cdot\frac{1.8}{s^2(s+2)}}$.

[4] For step disturbance, $\theta(t=\infty) = \lim_{s\to 0} \frac{\cancel{s}.\ 0.9.s}{s^3+K(s+\frac{1}{T_I})(1+T_D s)\frac{1.8}{(s+2)}}\ \frac{const.}{\cancel{s}} = \underset{\approx}{0}$.

("Integral control" term increases system "type" and eliminates disturbance error.)

(a) $\measuredangle \frac{(1+T_D j\omega)}{(j\omega)^2(j\omega+2)} = -180^\circ + \measuredangle \left.\frac{1+T_D s}{1+\alpha T_D s}\right|_{s=j\omega}$, where $\alpha = \frac{1}{2T_D}$.

From the given information

[6] $\underset{\sim\sim\sim}{\varphi_{max} = -180^\circ + 90^\circ - 2\tan^{-1}\sqrt{1/2T_D}}$ and $\underset{\sim\sim\sim}{\omega_{max} = \frac{1}{T_D\sqrt{\frac{1}{2T_D}}} = \sqrt{\frac{2}{T_d}}}$.

(b) Choose T_D to satisfy

$-180+65^\circ = -180^\circ + 90^\circ - 2\tan^{-1}\sqrt{1/2T_D}$

This gives $\frac{1}{2T_D} = (\tan(12.5^\circ))^2 = 0.0491$

whence $T_D = \frac{1}{2\times 0.0491} = \underset{\sim\sim}{10.183}$

The gain cross over should be

$\omega_c = \sqrt{\frac{2}{T_D}} = 0.4432\ \text{rsec}^{-1}$.

Choose $\frac{1}{T_I} = 0.05\cdot\frac{1}{T_D}$. This gives $\underset{\sim\sim}{T_I = 203.67\ s}$.

It remains to choose K, to arrange that $|DG(j\omega_c)| = 1$.

$1 = |D(j\omega_c)G(j\omega_c)| = K\cdot\left|1+\frac{1}{T_I j\omega_c}\right|\cdot\left|1+T_D j\omega_c\right|\cdot\frac{1.8}{|j\omega_c|^2|2+j\omega_c|}$

$\simeq K\times 1\times\sqrt{1+(T_D\omega_c)^2}\times\frac{1.8}{\omega_c^2\times\sqrt{4+\omega_c^2}}$

Hence

$K = \frac{1}{4.6205}\times\frac{0.1969\times 2.0485}{1.8} = \underset{\sim\sim}{0.0485}$

Note: T_D has been chosen to be the smallest possible value, when $\measuredangle(1+\frac{1}{T_I j\omega_c}) \simeq 0$. On the other hand, increasing $(1/T_I)$ reduces the phase of DG at ω_c and necessitates a larger T_D; so, considering this case also, T_D is smallest.

4(a). Let $\bar{\omega}$ be the cross-over frequency. Then

$$\omega_n^4 = \bar{\omega}^2(\bar{\omega}^2 + 4\zeta^2\omega_n^2) \quad \text{or} \quad \bar{\omega}^4 + 4\zeta^2\omega_n^2\bar{\omega}^2 - \omega_n^4 = 0$$

So, $\bar{\omega}^2 = -2\zeta^2\omega_n^2 \pm \sqrt{\{4\zeta^4\omega_n^4 + \omega_n^4\}}$

Choosing the positive root gives: $\bar{\omega}^2 = \omega_n^2\left(\sqrt{4\zeta^4+1} - 2\zeta^2\right)$

For this frequency

$$-180^\circ + \phi^\circ = \measuredangle G(j\bar{\omega}) = -90^\circ - \tan^{-1}\left(\frac{\bar{\omega}}{2\zeta\omega_n}\right) = -90^\circ - 90^\circ + \tan^{-1}\left(\frac{2\zeta\omega_n}{\bar{\omega}}\right)$$

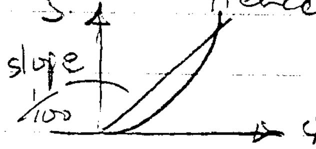
Hence $\frac{\pi}{180}\phi^\circ = \tan^{-1}\left(\frac{2\zeta\omega_n}{\bar{\omega}}\right) = \underbrace{\tan^{-1}\left[\frac{2\zeta}{\sqrt{\sqrt{4\zeta^4+1} - 2\zeta^2}}\right]}$

[6]

Notice that, for ζ small, ϕ is small and

$\frac{\pi}{180}\cdot\phi^\circ \simeq \tan\phi^\circ \simeq 2\zeta$ (+ higher order terms in ζ)

Hence $\zeta = \phi \Big/ \frac{360}{\pi} = \phi/114.6 \simeq \underbrace{\phi/100}$



[2] Approximation is good for small ϕ. (factor $1/100$ is used instead of $\frac{1}{114.6}$, because curve gradient is increasing.)

(b) Take $x_1 = x$ and $x_2 = r$. Then state equation is

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{bmatrix}a & 0\\ 0 & -\beta\end{bmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \begin{pmatrix}1\\ 0\end{pmatrix}u \text{ . Also}$$

$$\int_0^\infty [|x-r|^2 + \alpha u^2]\,dt = \int_0^\infty [\underline{x}^T cc^T x + \alpha u^2]\,dt$$

if $c^T x = (x_1 - x_2) = [1\ -1]\begin{bmatrix}x_1\\ x_2\end{bmatrix}$. So choose $c^T = [1\ -1]$

Solution to 'Linear Quadratic' problem now gives

$$u(t) = -k_1 x(t) - k_2 r(t) = \underbrace{-k_1 x(t) - k_2 e^{-\beta t}},$$

where

$$[k_1\ k_2] = b^T P = [1\ 0]\begin{bmatrix}P_{11} & P_{12}\\ P_{12} & P_{22}\end{bmatrix} = \underbrace{[P_{11}\ P_{12}]}$$

and P_{11}, P_{12} are obtained from matrix Riccati equation;

$$A^T P + PA + cc^T - \alpha^{-1}Pbb^T P = 0, \quad \text{or}$$

$$\begin{bmatrix}a & 0\\ 0 & -\beta\end{bmatrix}\begin{bmatrix}P_{11} & P_{12}\\ P_{12} & P_{22}\end{bmatrix} + \begin{bmatrix}P_{11} & P_{12}\\ P_{12} & P_{22}\end{bmatrix}\begin{bmatrix}a & 0\\ 0 & -\beta\end{bmatrix} + \begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} - \alpha^{-1}\begin{bmatrix}P_{11} & P_{12}\\ P_{12} & P_{22}\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}P_{11} & P_{12}\\ P_{12} & P_{22}\end{bmatrix} = 0$$

Hence $\underbrace{2aP_{11} + 1 - \alpha^{-1}P_{11}^2 = 0}$, (also, $P_{11} > 0$.)

$aP_{12} - \beta P_{12} - 1 - \alpha^{-1}P_{11}P_{12} = 0$

i.e. $\underbrace{P_{12} = (a - \beta - \alpha^{-1}P_{11})^{-1}}$

[12] (equating (2,2)'th component terms gives P_{22}, but this is not required)

(These equations for P_{11}, P_{12} have unique solutions but no comment is required, to this effect.)

5(a) Standard controllable representation of $G(s) = \dfrac{1}{s^2+s}$ is .

[2] $\dot{\underline{x}} = \underbrace{\begin{bmatrix}0 & 1\\ 0 & -1\end{bmatrix}}_{A} \underline{x} + \underbrace{\begin{bmatrix}0\\ 1\end{bmatrix}}_{b} u$ and $y = \underbrace{[1 \quad 0]}_{c^T} \underline{x}$

Desired ch. poly for controller gain design is $s^2+2\zeta\omega_n s+\omega_n^2 = \overbrace{s^2+4s+4}^{\gamma(s)}$

We require $\det\left[sI-(A-bk^T)\right] = s^2+4s+4$. Hence

$\det\left[sI - \begin{pmatrix}0 & 1\\ -k_1 & -1-k_2\end{pmatrix}\right] = s^2+4s+4$.

Hence $k_1 = 4$ and $1+k_2 = 4$, i.e. $\underset{\sim}{k_1 = 4}$, $\underset{\sim}{k_2 = 3}$

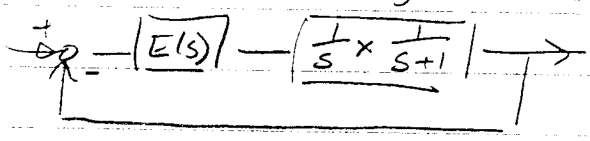
Desired ch. poly. for observer gain design is $(s+4)^2 = \overbrace{s^2+8s+16}^{\delta(s)}$.

We require $\det\left[sI-(A-gc^T)\right] = s^2+8s+16$

or $\det\begin{bmatrix}s+g_1 & -1\\ g_2 & s\end{bmatrix} = s^2+g_1 s+g_2 = s^2+8s+16$

[10] Hence $\underset{\sim}{g_1 = 8}$, $\underset{\sim}{g_2 = 16}$

(b) The thermal control system block diagram can be rearranged as:



. We must find E(s) to locate close loop poles, as in part (a).

E(s) is the transfer function $u(s)/y(s)$ for part (a)

But $u = -k^T\hat{x}$

and $\dot{\hat{x}} = A\hat{x} - bk^T\hat{x} + g(y-c^Tx)$

Hence $\left[sI - (A-bk^T-gc^T)\right]\hat{x} = gy$

So $\dfrac{u(s)}{y(s)} = -k^T\left[sI-(A-bk^T-gc^T)\right]^{-1}g$

$= -[k_1\ k_2]\begin{bmatrix}s+g_1 & -1\\ k_1+g_2 & s+k_2+1\end{bmatrix}^{-1}\begin{bmatrix}g_1\\ g_2\end{bmatrix} = -[4\ 4]\begin{bmatrix}s+8 & -1\\ 12 & s+4\end{bmatrix}^{-1}\begin{bmatrix}8\\ 16\end{bmatrix}$

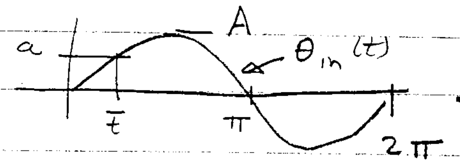
$= \left((s+8)(s+4)+12\right)^{-1}[4\ 4]\begin{bmatrix}s+4 & +1\\ -12 & s+8\end{bmatrix}\begin{bmatrix}8\\ 16\end{bmatrix}$

$= (96s+320)/(s^2+12s+44)$

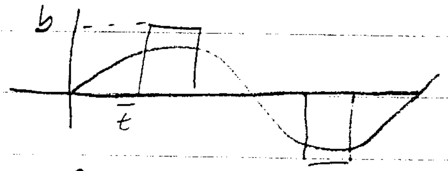
It follows that desired compensator is

[8] $D(s) = \dfrac{1}{s}E(s) = \dfrac{1}{s}\times\dfrac{(96s+320)}{(s^2+12s+44)}$

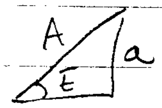
6. Take input $\theta_{in} = A \sin t \; (A \geq a)$



The output is



We calculate the first Fourier coefficient:



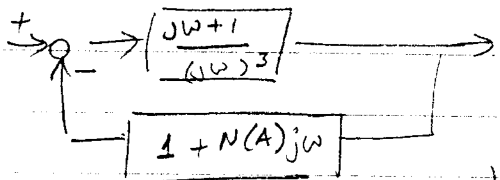
$$c_1 = 4 \times \frac{2}{2\pi} \int_{\bar{t}}^{\pi/2} b \sin t \, dt = \frac{4b}{\pi} \cdot -\cos t \Big|_{\bar{t}}^{\pi/2}$$

$$= \frac{4b}{\pi} \cos \bar{t} = \frac{4b}{\pi} \sqrt{\frac{A^2 - a^2}{A^2}} = \frac{4b}{\pi} \sqrt{1 - (a/A)^2}$$

The describing function is therefore

[7] $$N(A) = \frac{c_1}{A} = \frac{4b}{\pi A} \sqrt{1 - \left(\frac{a}{A}\right)^2} \qquad (A \geq a)$$

Using the describing function approximation, we can rewrite the block diagram as



A limit cycle (frequency $\bar{\omega}$, ampl. A at input to N/L) if

$$\frac{j\bar{\omega} + 1}{(j\bar{\omega})^3} \cdot (N(A) j\bar{\omega} + 1) = -1$$

Equate imag. parts:

$$-N(A)\bar{\omega}^2 + 1 = 0 \quad \text{or} \quad \bar{\omega}^2 = 1/N(A)$$

Equate real parts:

$$-\frac{(N(A) + 1)}{\bar{\omega}^2} = -1 \qquad \bar{\omega}^2 = 1 + N(A)$$

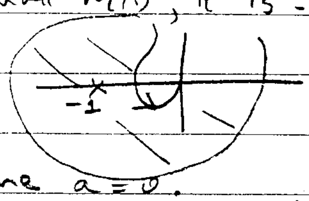
So $\frac{1}{N(A)} = 1 + N(A)$ or $N(A)^2 + N(A) - 1 = 0$,

giving $N(A) = (\sqrt{5} - 1)/2$

[10] We deduce: frequency of oscillations $= \sqrt{\dfrac{2}{\sqrt{5} - 1}}$.

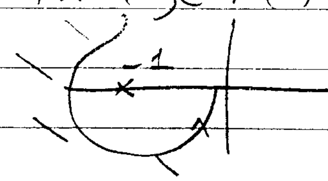
Look at Nyquist diagram of $\frac{(s+1)}{s^3} \times (1 + N(A)s)$

For small N(A), it is:



(unstable)

For large N(A) is it



stable

Assume $a = 0$.

Since N(A) is decreasing, increasing A drives system from

[3] stable to unstable region, i.e. limit cycle is unstable.