EE3-09

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2011

EEE/ISE PART III/IV: MEng, BEng and ACGI

Corrected Copy Q4.
Correction made A.A.

CONTROL ENGINEERING

Tuesday
Monday, ~~16~~ 17 May 10:00 am

Time allowed: 3:00 hours

There are SIX questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : A. Astolfi
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CONTROL ENGINEERING

1. Consider a linear, multiple-input, discrete-time, system described by the equations

$$x(k+1) = Ax(k) + Bu(k) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} u(k).$$

a) Study the reachability and controllability properties of the system. Show that the system is reachable and controllable in two steps. [4 marks]

b) Consider the initial state $x(0) = 0$ and the final state

$$x_f = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

Show that the state x_f cannot be reached in one step. Determine all input sequences which are such that $x(2) = x_f$. [6 marks]

c) Exploiting the results in part b) determine the input sequence which transfers the state from $x(0) = 0$ to $x(2) = x_f$ and which minimizes the input energy

$$\|u(0)\|^2 + \|u(1)\|^2.$$

(Hint: $\|v\|^2 = v'v$.). [4 marks]

d) Exploiting the results in part b) determine the input sequence which transfers the state from $x(0) = 0$ to $x(2) = x_f$ and which minimizes the input amplitude

$$\max(\|u(0)\|_\infty, \|u(1)\|_\infty).$$

(Hint: $\|v\|_\infty = \max_i(|v_i|)$.) [4 marks]

e) Compare, briefly, the results in parts c) and d). [2 marks]

2. A (normalized) synchronous generator connected to an infinite bus is described by the equations

$$\begin{aligned} \ddot{\delta} &= 1 - \dot{\delta} - E \sin \delta, \\ \dot{E} &= -E + \cos \delta + u, \end{aligned}$$

where $\delta(t)$ is an angle in radians, $E(t)$ is a voltage, and $u(t)$ is an input signal.

a) Let $x_1 = \delta$, $x_2 = \dot{\delta}$ and $x_3 = E$. Write the system in the standard state space representation. [2 marks]

b) Assume that u is constant. Determine for which values of u the system has no equilibrium point, one equilibrium point or two equilibrium points, respectively.
(Do not compute explicitly the equilibrium points!). [8 marks]

c) Show that the point $x_1 = \pi/2$, $x_2 = 0$ and $x_3 = 1$ is an equilibrium point for $u = 1$. Write the equations describing the linearized system around this equilibrium point. [6 marks]

d) Study, using the principle of stability in the first approximation, the stability properties of the equilibrium point given in part c). [4 marks]

3. A nonlinear, single-input, single-ouput, continuous-time, system

$$
\begin{aligned}
\dot{x} &= f(x) + g(x)u, \\
y &= h(x),
\end{aligned}
$$

with $x(t) \in I\!R^n$, is said to be input-output linearizable if there exists an integer $p \leq n$ such that

$$
\frac{d^p y}{dt^p} = \eta(x_1, x_2, x_3) + \theta(x_1, x_2, x_3)u.
$$

Consider the system

$$
\begin{aligned}
\dot{x}_1 &= -x_1 + x_2^2 x_1, \\
\dot{x}_2 &= x_3 + x_1 x_2, \\
\dot{x}_3 &= x_1 + (1 + x_1^2)u, \\
y &= x_2.
\end{aligned}
$$

a) Determine an integer p such that

$$
\frac{d^p y}{dt^p} = \frac{d^p x_2}{dt^p} = \eta(x_1, x_2, x_3) + \theta(x_1, x_2, x_3)u.
$$

Hence argue that the system is input-output linearizable. Determine explicitly the functions η and θ. [6 marks]

b) Consider the new variables

$$
z = x_1 \qquad \xi_1 = y \qquad \xi_2 = \dot{y}.
$$

Show that these variables define new coordinates, that is there is a one-to-one relation between the variables (x_1, x_2, x_3) and the variables (z, ξ_1, ξ_2). [4 marks]

c) Let

$$
u = \frac{v - \eta(x_1, x_2, x_3)}{\theta(x_1, x_2, x_3)},
$$

with $\eta(x_1, x_2, x_3)$ and $\theta(x_1, x_2, x_3)$ as determined in part a).

i) Write the equations of the system in the variables z, ξ_1 and ξ_2. [4 marks]

ii) Show that the system is described by a linear subsystem, with state (ξ_1, ξ_2), which is affected by the new input v and *contributes* to the output y and by a nonlinear subsystem, with state z, which is affected by the state of the linear subsystem and does not *contribute* to the output y. Argue that the system is not observable and that the nonlinear subsystem is the unobservable subsystem. [6 marks]

4. Consider a linear, continuous-time, system described by the equations

$$\dot{x} = Ax + Bu = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -8 & 0 \\ 0 & 0 & 2 \end{bmatrix} x + \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} u,$$

where $x = [x_1, x_2, x_3]'$ and $u = [u_1, u_2]'$.

a) Compute the eigenvalues of the matrix A. [2 marks]

b) Show that the system is controllable. [4 marks]

c) Let

$$u = Fx + v,$$

with

$$F = \begin{bmatrix} F_{11} & 0 & F_{13} \\ F_{21} & 0 & F_{23} \end{bmatrix}.$$

i) Determine F such that the closed-loop system has eigenvalues at -6, -7 and -8. Show that there are infinitely many matrices achieving this objective. [6 marks]

ii) Show that in the set of matrices F determined in part c.i) there is one matrix such that the closed-loop system is composed of the parallel interconnection of two systems, one with state (x_1, x_3) and input v_1, and one with state x_2 and input v_2. [4 marks]

iii) Explain why the result in part b) implies that the two subsystems determined in part c.ii) are controllable. [4 marks]

5. A simple model describing the attitude of a satellite around one axis is given by

$$J\ddot{\theta} = u,$$

where J is the moment of inertia of the satellite around the considered axis, $\theta(t) \in I\!R$ is an angle describing the attitude, and $u(t) \in I\!R$ is a control torque. Suppose that two sensors are used to determine the satellite attitude: a star tracker, measuring the angular position $\theta(t)$, and a rate gyro, measuring the angular velocity $\dot{\theta}(t)$.

a) Write a state space representation of the system with states $x_1 = \theta$ and $x_2 = \dot{\theta}$, input u, and outputs $y_1 = \theta$ and $y_2 = \dot{\theta}$. [2 marks]

b) Assume that the rate gyro has a constant bias b_g, that is $y_2(t) = \dot{\theta}(t) + b_g$. Write a state space representation of the satellite and the sensor with states θ, $\dot{\theta}$, and b_g. [4 marks]

c) Study the controllability and observability properties of the system determined in part b). [4 marks]

d) Assume, now, that the star tracker has a constant bias b_s, that is $y_1(t) = \theta(t) + b_s$. Write a state space representation of the satellite and the sensor with states θ, $\dot{\theta}$, and b_s. [4 marks]

e) Study the controllability and observability properties of the system determined in part d). [4 marks]

f) Compare briefly the results obtained in parts c) and e). [2 marks]

6. Consider a linear, *periodic*, discrete-time, system described by the equations

$$x(k+1) = A_e x(k) \qquad \text{if } k \text{ is even}$$
$$x(k+1) = A_o x(k) \qquad \text{if } k \text{ is odd}$$

where $x(k) \in I\!R^2$,

$$A_e = \begin{bmatrix} 0 & \frac{1}{2} \\ -3 & 0 \end{bmatrix} \qquad A_o = \begin{bmatrix} 2 & 0 \\ 0 & \beta \end{bmatrix}$$

and $\beta \in I\!R$ is a constant parameter.

a) Show that, for any β, the system has the unique equilibrium $x = 0$. [6 marks]

b) Show that, for any β, both matrices A_e and A_o are unstable, that is they both have at least one eigenvalue with modulo larger than one. [2 marks]

c) Show that

$$x(1) = A_e x(0) \qquad x(2) = A_o x(1) = (A_o A_e) x(0).$$

Hence show that for any non-negative even integer ℓ

$$x(\ell+2) = (A_o A_e) x(\ell),$$

and that for any non-negative odd integer ℓ

$$x(\ell+2) = (A_e A_o) x(\ell).$$

Finally show that, for any non-negative integer ℓ,

$$x(2\ell) = (A_o A_e)^\ell x(0) \qquad x(2\ell+1) = A_e (A_o A_e)^\ell x(0).$$

[6 marks]

d) The equilibrium $x = 0$ of the periodic system is asymptotically stable if and only if the discrete time systems

$$z(k+1) = (A_o A_e) z(k) \qquad \zeta(k+1) = (A_e A_o) \zeta(k)$$

are simultaneously asymptotically stable. Determine for which values of β these systems are asymptotically stable, hence discuss for which values of β the equilibrium $x = 0$ of the periodic system is asymptotically stable.

[6 marks]

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Control engineering exam paper - Model answers

Question 1

a) The reachability matrix is

$$\mathcal{C} = \left[\begin{array}{ccc} B & AB & A^2B \end{array} \right] \left[\begin{array}{cc|cc|cc} 1 & 0 & 0 & -1 & -1 & -2 \\ 0 & 1 & 1 & 2 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{array} \right].$$

Note that the first three columns of $\mathcal{C}$ are independent, hence rank $\mathcal{C} = 3$ and the system is reachable in two steps. Note now that

$$A^2 = \left[\begin{array}{ccc} -1 & 0 & -2 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right],$$

hence

$$\mathrm{Im} A^2 \subset \left[\begin{array}{cc} B & AB \end{array} \right],$$

which shows that the system is controllable in two steps.

b) Let

$$u(0) = \left[\begin{array}{c} u_1(0) \\ u_2(0) \end{array} \right] \qquad u(1) = \left[\begin{array}{c} u_1(1) \\ u_2(1) \end{array} \right].$$

Note that

$$x(1) = Ax(0) + Bu(0) = \left[\begin{array}{c} u_1(0) \\ u_2(0) \\ u_2(0) \end{array} \right] \qquad x(2) = Ax(1) + Bu(1) = \left[\begin{array}{c} -u_2(0) + u_1(1) \\ u_1(0) + 2u_2(0) \\ u_2(1) \end{array} \right]$$

To reach x_f in one step we need to solve the equation

$$x(1) = x_f \Rightarrow \left[\begin{array}{c} u_1(0) \\ u_2(0) \\ u_2(0) \end{array} \right] = \left[\begin{array}{c} 1 \\ -1 \\ 0 \end{array} \right],$$

which clearly has no solution. Hence x_f is not reachable in one step. Similarly, to reach x_f in two steps we need to solve the equation

$$x(2) = x_f \Rightarrow \left[\begin{array}{c} -u_2(0) + u_1(1) \\ u_1(0) + 2u_2(0) \\ u_2(1) \end{array} \right] = \left[\begin{array}{c} 1 \\ -1 \\ 0 \end{array} \right].$$

This equation has infinitely many solutions that can be written as

$$u(0) = \left[\begin{array}{c} -1 - 2\alpha \\ \alpha \end{array} \right] \qquad u(1) = \left[\begin{array}{c} \alpha + 1 \\ 0 \end{array} \right]$$

where α is a free parameter.

c) Note that

$$\|u(0)\|^2 + \|u(1)\|^2 = 6\alpha^2 + 6\alpha + 2,$$

which is minimized selecting $\alpha = -1/2$.

d) Note that

$$\max\{|u_1(0)|, |u_2(0)|, |u_1(1)|, |u_2(1)|\} = \max\{|1 + 2\alpha|, |\alpha|, |\alpha + 1|\}.$$

which is minimized, again, selecting $\alpha = -1/2$.

e) The control minimizing energy and minimizing amplitude is the same, namely

$$u(0) = \begin{bmatrix} 0 \\ -1/2 \end{bmatrix} \qquad u(1) = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}.$$

Question 2

a) The state space representation of the system is given by the equations

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= 1 - x_2 - x_3 \sin x_1,\\ \dot{x}_3 &= -x_3 + \cos x_1 + u.\end{aligned}$$

b) The equilibrium points are the solutions of the equations

$$x_2 = 0 \qquad 1 - x_2 - x_3 \sin x_1 = 0 \qquad -x_3 + \cos x_1 + u = 0.$$

These imply that at the equilibrium $x_2 = 0$ and

$$1 - x_3 \sin x_1 = 0 \qquad -x_3 + \cos x_1 + u = 0.$$

Solving the first equation for x_3 yields (note that at any equilibrium $\sin x_1 \neq 0$)

$$x_3 = \frac{1}{\sin x_1},$$

which replaced in the second equation gives

$$\frac{1}{\sin x_1} - \cos x_1 = u.$$

This equation, for $x_1 \in (-\pi, \pi)$ has two solutions for all u such that

$$|u| > \min_{x_1 \in (-\pi,\pi)} \left| \frac{1}{\sin x_1} - \cos x_1 \right| \approx 0.6469,$$

has one solution for all u such that

$$|u| = \min_{x_1 \in (-\pi,\pi)} \left| \frac{1}{\sin x_1} - \cos x_1 \right| \approx 0.6469,$$

and has no solution otherwise.

c) Replacing $x_1 = \pi/2$ and $u = 1$ in the equation characterizing the equilibrium points, namely

$$\frac{1}{\sin x_1} - \cos x_1 = u,$$

yields an identity, hence the point

$$(x_1, x_2, x_3) = (\pi/2, 0, 1)$$

is an equilibrium for $u = 1$. The linearized system is described by the equations

$$\dot{\delta}_x = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & -1 \\ -1 & 0 & -1 \end{bmatrix} \delta_x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \delta_u.$$

d) The characteristic polynomial of the matrix A determined in part c) is

$$\det(sI - A) = s^3 + 2s^2 + s - 1.$$

By Routh's test, this polynomial has one root with positive real part, hence the given equilibrium of the nonlinear system is unstable.

Question 3

a) Differentiating $y = x_2$ with respect to time yields

$$\dot{y} = x_3 + x_1 x_2 \qquad \ddot{y} = x_2(-x_1 + x_2^2 x_1) + x_1(x_3 + x_1 x_2) + x_1 + (1 + x_1^2)u.$$

Hence, the system is input-output linearizable, with $p = 2$ and

$$\eta(x_1, x_2, x_3) = x_2(-x_1 + x_2^2 x_1) + x_1(x_3 + x_1 x_2) \qquad \theta(x_1, x_2, x_3) = 1 + x_1^2.$$

b) By definition

$$z = x_1 \qquad \xi_1 = y = x_2 \qquad \xi_2 = \dot{y} = x_3 + x_1 x_2,$$

hence

$$x_1 = z \qquad x_2 = \xi_1 \qquad z_3 = -z\xi_1 + \xi_2,$$

which shows that there is a one-to-one relation between the variables (x_1, x_2, x_3) and the variables (z, ξ_1, ξ_2). It is therefore possible to use the latter as coordinates for the system.

c) Setting u as indicated in the text of the exam yields

$$\ddot{y} = v.$$

i) A direct computation yields

$$\dot{z} = -z + \xi_1^2 z \qquad \dot{\xi}_1 = \xi_2 \qquad \dot{\xi}_2 = v.$$

ii) The system determined in part c.i) can be considered as composed of the two subsystems

$$\left\{ \begin{array}{rcl} \dot{\xi}_1 & = & \xi_2 \\ \dot{\xi}_2 & = & v \\ y & = & \xi_1 \end{array} \right. \qquad \left\{ \dot{z} = -z + \xi_1^2 z. \right.$$

The former is a linear system with state (ξ_1, ξ_2), input v and output y, and the latter is a nonlinear system, driven by ξ_1, but which does not contribute to the output. Recalling the decomposition of a system into observable and unobservable part, we conclude that the overall system is not observable and that the nonlinear system is the unobservable component.

Question 4

a) By a direct inspection, the eigenvalues of A are $\sigma(A) = \{2, -1, -8\}$.

b) The controllability matrix is

$$\mathcal{C} = \left[\begin{array}{cc|cc|cc} 2 & 0 & -1 & 0 & \star & \star \\ 0 & 1 & 2 & -8 & \star & \star \\ 1 & 0 & 2 & 0 & \star & \star \end{array}\right].$$

This matrix has rank three (the first three columns are linearly independent), hence the system is controllable.

c) Note that

$$A + BF = \begin{bmatrix} -1 + 2F_{11} & 0 & 1 + 2F_{13} \\ 1 + F_{21} & -8 & F_{23} \\ F_{11} & 0 & 2 + F_{13} \end{bmatrix}.$$

i) The characteristic polynomial of the matrix $A + BF$ is

$$\det(sI - (A + BF)) = (s + 8)(s^2 - (F_{13} + 2F_{11} + 1)s + (3F_{11} - F_{13} - 2))$$

and this should be equal to

$$(s + 8)(s + 6)(s + 7).$$

As a result, $F_{11} = 6$, $F_{13} = -26$, and F_{21} and F_{23} can be assigned arbitrarily, that is there are infinitely many matrices F yielding the desired eigenvalues.

ii) Selecting $F_{21} = -1$ and $F_{23} = 0$ yields a closed loop system described by

$$\dot{x} = (A + BF)x + Bv = \begin{bmatrix} 11 & 0 & -51 \\ 0 & -8 & 0 \\ 6 & 0 & -24 \end{bmatrix} x + \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} v$$

which shows the fact that the system is composed of a parallel interconnection of the two subsystems

$$\begin{aligned} \dot{x}_1 &= 11x_1 - 52x_3 + v_1 \\ \dot{x}_2 &= 6x_1 - 24x_3 + v_1 \end{aligned} \qquad\qquad \dot{x}_2 = -8x_2 + v_2.$$

iii) The two subsystems are obtained applying state feedback to a controllable system. The controllability property is not modified by state feedback, hence the two subsystems are controllable.

Question 5

a) The state space representation is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{J} \end{bmatrix} u$$

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

b) Since the bias b_g is constant then $\dot{b}_g = 0$. Hence, a state space representation is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{b}_g \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ b_g \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{J} \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ b_g \end{bmatrix}$$

c) The controllability and reachability matrices are

$$\mathcal{C} = \begin{bmatrix} 0 & 1/J & 0 \\ 1/J & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \mathcal{O} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

hence the system is not controllable and observable.

d) Similarly to the case in part b), $\dot{b}_s = 0$. Hence, a state space representation is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{b}_s \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ b_s \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{J} \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ b_s \end{bmatrix}$$

e) The controllability matrix is the same as in part c) and the observability matrix is

$$\mathcal{O} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

hence the system is not controllable and not observable.

f) A bias in the gyro can be estimated, since the system is observable, whereas a bias in the star sensor cannot be estimated.

Question 6

a) A point $\bar{x}$ is an equilibrium if $x(k) = \bar{x}$ for all $k \geq 0$. For the considered system this is equivalent to

$$\bar{x} = A_e \bar{x} \qquad \bar{x} = A_o \bar{x}.$$

These equations, for any β, have only the solution $\bar{x} = 0$, which is therefore the unique equilibrium point of the system.

b) The eigenvalues of the matrices A_o and A_e are

$$\sigma(A_e) = \{i\sqrt{3/2}, -i\sqrt{3/2}\} \qquad \sigma(A_o) = \{2, \beta\}$$

hence, for any β both matrices have at least one eigenvalue outside the unity disk, that is they are unstable.

c) Setting $k = 0$ and $k = 1$ in the equations describing the system yields

$$x(1) = A_e x(0) \qquad x(2) = A_o x(1).$$

Note now that, for any non-negative even integer ℓ, $x(\ell+1) = A_e x(\ell)$, hence $x(\ell+2) = A_o A_e x(\ell)$. Similarly, for any non-negative odd integer ℓ, $x(\ell+1) = A_o x(\ell)$, hence $x(\ell+2) = A_e A_o x(\ell)$. Applying the above relation recursively yields (note that 2ℓ is an even number, and $2\ell+1$ is an odd number)

$$x(2\ell) = (A_o A_e)^\ell x(0) \qquad x(2\ell+1) = A_e (A_o A_e)^\ell x(0).$$

d) Note that

$$A_e A_0 = \begin{bmatrix} 0 & 1 \\ -3\beta & 0 \end{bmatrix} \qquad A_o A_e = \begin{bmatrix} 0 & \beta/2 \\ -6 & 0 \end{bmatrix},$$

hence

$$\sigma(A_e A_0) = \sigma(A_o A_e) = \{\sqrt{-3\beta}, -\sqrt{-3\beta}\}.$$

The matrices $A_e A_0$ and $A_o A_e$ have all eigenvalues with modulo less than one if and only if

$$|\beta| < \frac{1}{3},$$

hence the equilibrium $x = 0$ of the periodic system is asymptotically stable if and only if $|\beta| < \frac{1}{3}$.