

MEng (Engineering) Examination 2017

Year 1

AE1-107 Mathematics Term II

**Friday 9th June 2017: 14.00 to 16.00
[2 hours]**

The paper is divided into Section A and Section B

Both sections carry the same weight

Candidates may obtain full marks for complete answers to **ALL** questions.

You must answer each section in a separate answer booklet

The use of lecture notes is NOT allowed.

Section A

1. Let

$$A = \begin{pmatrix} a & -1 \\ 1 & -1 \end{pmatrix}.$$

- (a) Find the value of a such that $A^3 = I$. [25%]
- (b) Find the value of a such that $A^3 = 0$. [25%]
- (c) For these two values of a , either find A^{-1} or show that A^{-1} does not exist. [25%]
- (d) Show that there is no value of a such that $A^3 = A$. [25%]

2. (a) Consider the linear system

$$\begin{aligned}x + y + z &= 1, \\x + t^2y + tz &= 1, \\x + y + 2z &= 2,\end{aligned}$$

where t is a real number. Identify values of t for which there is

- i) a unique solution, [20%]
- ii) an infinite number of solutions, [20%]
- iii) no solution. [20%]

(b) Find the LU decomposition of

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & -1 \\ 1 & -1 & 5 \end{pmatrix}.$$

[20%]

Hence solve the system

$$Ax = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

[20%]

Section B

3. Consider the ODE:

$$\left(\frac{d^3y}{dx^3}\right)^2 + y = \sin(2x).$$

- (a) Is the ODE linear or non-linear? [5%]
 (b) What is the order of the ODE? [5%]

Now consider the ODE:

$$\frac{dy}{dx} + 3y = f(x, y).$$

Find the general solution of the ODE if

- (c) $f(x, y) = x$. [20%]
 (d) $f(x, y) = y^4$. [20%]

Finally consider the ODE:

$$\frac{dy}{dx} = \frac{9x^2 - 2xy}{2y + x^2 + 1}.$$

- (e) Show that the ODE is exact. [10%]
 (f) Find the potential function for the ODE. [30%]
 (g) Write down the general solution of the ODE. [5%]
 (h) Find a solution of the ODE that satisfies $y(0) = 0$. [5%]

4. Consider the ODE:

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 4y = f(x).$$

Find the general solution of the ODE if

(a) $f(x) = 0$. [15%]

(b) $f(x) = e^x$. [15%]

(c) $f(x) = e^{-x}$. [25%]

Now consider the ODE:

$$ax^2 \frac{d^2y}{dx^2} + bx \frac{dy}{dx} + cy = 0 \quad x > 0.$$

Find the general solution of the ODE if

(d) $a = 2, b = 3, c = -15$. [15%]

(e) $a = 1, b = -7, c = 16$. [15%]

The Laplace transform $F(s)$ of a function $f(t)$ is defined as:

$$F(s) = \int_0^\infty f(t)e^{-st} dt$$

for some appropriate range of s .

- (f) Using this expression or otherwise, derive an expression for the Laplace transform of $f(t) = \sinh(t)$. Show all your workings. [15%]

Course Code and Title (Required): AE1-107 – Term II

Setter (Required): F. Montomoli

1

Write on this side only (in ink) between the margins, not more than one solution per sheet please. Solutions must be signed and dated by both exam setter and referee.

Marks

1. Let

$$A = \begin{pmatrix} a & -1 \\ 1 & -1 \end{pmatrix}.$$

(a) Find the value of a such that $A^3 = I$. [25%](b) Find the value of a such that $A^3 = 0$. [25%](c) For these two values of a , either find A^{-1} or show that A^{-1} does not exist. [25%](d) Show that there is no value of a such that $A^3 = A$. [25%]If $A = \begin{pmatrix} a & -1 \\ 1 & -1 \end{pmatrix}$ then

$$A^2 = \begin{pmatrix} a & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} a & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} a^2 - 1 & 1 - a \\ a - 1 & 0 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} a & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} a^2 - 1 & 1 - a \\ a - 1 & 0 \end{pmatrix} = \begin{pmatrix} a(a^2 - 1) - (a - 1) & a(1 - a) \\ a^2 - 1 + 1 - a & 1 - a \end{pmatrix}$$

a)

$$A^3 = I \text{ if } a = 0$$

b)

$$A^3 = 0 \text{ if } a = 1$$

c)

If $a = 1$ then $A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$, singular, A^{-1} does not existIf $a = 0$ then $A = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$, $A^{-1} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$

d)

if $A^3 = A$ then (2,2) if $1 - a = -1$ and if $a = 2$ but (2,1) then $a^2 - a = 1$ but this is not if $a = 2$

Course Code and Title (Required): AE1-107 – Term II

2

Setter (Required): F. Montomoli

2. (a) Consider the linear system

$$\begin{aligned}x + y + z &= 1, \\x + t^2y + tz &= 1, \\x + y + 2z &= 2,\end{aligned}$$

where t is a real number. Find all solutions, and identify any values of t for which there is

- i) a unique solution, [20%]
- ii) an infinite number of solutions, [20%]
- iii) no solution. [20%]

(b) Find the LU decomposition of [20%]

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & -1 \\ 1 & -1 & 5 \end{pmatrix}.$$

Hence solve the system [20%]

$$Ax = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

$$\begin{aligned}x + y + z &= 1, \\x + t^2y + tz &= 1, \\x + y + 2z &= 2,\end{aligned}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & t^2 & t & 1 \\ 1 & 1 & 2 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & t^2 - 1 & t - 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

$$t = 1$$

$$\begin{aligned}x + y + z &= 1, \\z &= 1,\end{aligned}$$

then

$$\begin{aligned}x &= -y, \\z &= 1,\end{aligned}$$

Or

$$x = \begin{pmatrix} \alpha \\ -\alpha \\ 1 \end{pmatrix}$$

Course Code and Title (Required): AE-107

Setter (Required): Peter Vincent

3

Write on this side only (in ink) between the margins, not more than one solution per sheet please. Solutions must be signed and dated by both exam setter and referee.

Marks

3 ~~4~~ a.) Non linear

1

b.) 3rd

1

c.) $\frac{dy}{dx} + 3y = x$

$$I(x) = \exp\left[\int 3dx\right]$$

$$= \exp(3x)$$

$$\Rightarrow y(x) = e^{-3x} \int e^{3x} x dx$$

$$= e^{-3x} \left[\frac{1}{3} x e^{3x} - \frac{1}{3} \int e^{3x} dx \right]$$

$$= e^{-3x} \left[\frac{1}{3} x e^{3x} - \frac{1}{9} e^{3x} + C \right]$$

W

Course Code and Title (Required): AE-107

④

Setter (Required): Peter Vincent

Write on this side only (in ink) between the margins, not more than one solution per sheet please. Solutions must be signed and dated by both exam setter and referee.

Marks

$$= \frac{1}{3}x - \frac{1}{9} + C e^{-3x}$$

4

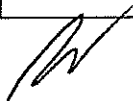
$$d.) \frac{dy}{dx} + 3y = y^4$$

$$y = u^{-\frac{1}{3}} \Rightarrow \frac{dy}{dx} = -\frac{1}{3}u^{-\frac{4}{3}} \frac{du}{dx}$$

$$\Rightarrow \frac{du}{dx} - 9u = -3$$

$$\Rightarrow u = e^{9x} \left[-\int e^{-9x} 3 dx \right]$$

$$= \frac{1}{3} + C e^{9x}$$



Course Code and Title (Required): AE-107

Setter (Required): Peter Vincent

⑤

Write on this side only (in ink) between the margins, not more than one solution per sheet please. Solutions must be signed and dated by both exam setter and referee.

Marks

$$\Rightarrow y = \frac{1}{\sqrt[3]{\frac{1}{3} + Ce^{9x}}}$$

4

$$e.) \frac{\partial}{\partial y} [-9x^2 + 2xy] =$$

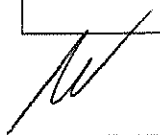
$$2x$$

2

$$\frac{\partial}{\partial x} [2y + x^2 + 1] =$$

$$2x$$

$$\Rightarrow \underline{\text{exact}}$$



Course Code and Title (Required): AE-107

Setter (Required): Peter Vincent

6

Write on this side only (in ink) between the margins, not more than one solution per sheet please. Solutions must be signed and dated by both exam setter and referee.

Marks

$$f.) \frac{\partial u}{\partial x} = -9x^2 + 2xy \quad (1)$$

$$\frac{\partial u}{\partial y} = 2y + x^2 + 1 \quad (2)$$

$$(1) \Rightarrow u = -3x^3 + x^2y + p(y)$$

$$(2) \Rightarrow u = y^2 + x^2y + y + q(x)$$

$$\Rightarrow p(y) = y^2 + y$$

$$q(x) = -3x^3 \Rightarrow$$

✓

Course Code and Title (Required): AE-107

Setter (Required): Peter Vincent

7

Write on this side only (in ink) between the margins, not more than one solution per sheet please. Solutions must be signed and dated by both exam setter and referee.

Marks

$$u = -3x^3 + x^2y + y^2 + y$$

↑
Potential function.

6

$$g.) -3x^3 + x^2y + y^2 + y = C \quad |$$

$$h.) 0 + 0 + 0 + 0 = C$$

$$\Rightarrow C = 0$$

$$\Rightarrow -3x^3 + x^2y + y^2 + y = 0 \quad |$$

Course Code and Title (Required): AE-107

Setter (Required): Peter Vincent

8

Write on this side only (in ink) between the margins, not more than one solution per sheet please. Solutions must be signed and dated by both exam setter and referee.

Marks

$$4/2) d.) \alpha^2 + 5\alpha + 4 = 0$$

$$\Rightarrow \alpha = -4 \text{ or } -1$$

$$\Rightarrow y = Ae^{-4x} + Be^{-x}$$

3

$$b) y = y_h + y_p$$

↑
from (a)

$$y_p = Ce^x \Rightarrow$$

$$Ce^x + 5Ce^x + 4Ce^x = e^x$$

$$\Rightarrow 10C = 1 \Rightarrow$$

$$C = \frac{1}{10} \Rightarrow$$

Course Code and Title (Required): AE-107

Setter (Required): Peter Vincent

9

Write on this side only (in ink) between the margins, not more than one solution per sheet please. Solutions must be signed and dated by both exam setter and referee.

Marks

$$y_p = \frac{1}{10} e^x \Rightarrow$$

$$y = Ae^{-4x} + Be^{-x} + \frac{1}{10} e^x \quad 3$$

$$c.) \quad y = y_h + y_p$$

↑
from (a)

$$y_p = Cx e^{-x}$$

$$y_p' = -Cx e^{-x} + C e^{-x}$$

$$y_p'' = -2C e^{-x} + Cx e^{-x}$$

$$\Rightarrow$$

Course Code and Title (Required): AE-107

Setter (Required): Peter Vincent

(10)

Write on this side only (in ink) between the margins, not more than one solution per sheet please. Solutions must be signed and dated by both exam setter and referee.

Marks

$$-2Ce^{-x} + \cancel{Cx e^{-x}} - \cancel{5Cx e^{-x}} + 5Ce^{-x} + \cancel{4Cx e^{-x}} =$$

$$e^{-x}$$

$$\Rightarrow 3Ce^{-x} = e^{-x}$$

$$\Rightarrow C = \frac{1}{3} \Rightarrow$$

$$y = Ae^{-4x} + Be^{-x} + \frac{1}{3}xe^{-x} \quad 5$$

$$d) 2x^2y'' + x^3y' - 15y = 0$$

M

Course Code and Title (Required): AE-107

Setter (Required): Peter Vincent

(11)

Write on this side only (in ink) between the margins, not more than one solution per sheet please. Solutions must be signed and dated by both exam setter and referee.

Marks

$$\Rightarrow x^2 y'' + x \frac{3}{2} y' - \frac{15}{2} y = 0$$

$$\Rightarrow \alpha = \frac{(1 - \frac{3}{2}) \pm \sqrt{(\frac{1}{2})^2 + 30}}{2}$$

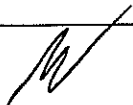
$$= \frac{-\frac{1}{2} \pm \frac{11}{2}}{2}$$

$$= \frac{5}{2}, -3$$

$$\Rightarrow y = Ax^{\frac{5}{2}} + Bx^{-3}$$

3

e.) $x^2 y'' - x^2 y' + 16y = 0$



Course Code and Title (Required): AE-107

Setter (Required): Peter Vincent

12

Write on this side only (in ink) between the margins, not more than one solution per sheet please. Solutions must be signed and dated by both exam setter and referee.

Marks

$$\Rightarrow \alpha = \frac{8 \pm \sqrt{(-8)^2 - 64}}{2}$$

$$= \frac{8 \pm 0}{2} = 4$$

$$\Rightarrow y = Ax^4 + B \ln|x| x^4 \quad 3$$

Ex. 1) $f(t) = \sinh(t)$

$$f''(t) = \sinh(t) \Rightarrow$$

$$\text{using } \mathcal{L}\{f''\} = s^2 \mathcal{L}\{f\} - s f(0) - f'(0)$$

Course Code and Title (Required): AE-107

LAST PAGE

(13)

Setter (Required): Peter Vincent

Write on this side only (in ink) between the margins, not more than one solution per sheet please. Solutions must be signed and dated by both exam setter and referee.

Marks

$$L\{\sinh(t)\} = s^2 L\{\sinh(t)\} - s \sinh(0) - \cosh(0)$$

$$- s \sinh(0) - \cosh(0)$$

$$\cosh(0)$$

$$\Rightarrow L\{\sinh(t)\} = \frac{-1}{1-s^2}$$

$$= \frac{1}{s^2 - 1}$$

3