UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 1996

BEng Honours Degree in Computing Part I
MEng Honours Degrees in Computing Part I
BSc Honours Degree in Mathematics and Computer Science Part I
MSci Honours Degree in Mathematics and Computer Science Part I
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the
Associateship of the Royal College of Science
Associateship of the City and Guilds of London Institute

PAPER 1.3 / MC1.3

DISCRETE MATHEMATICS
Friday, May 3rd 1996, 4.00 - 5.30

Answer THREE questions

For admin. only: paper contains
4 questions
3 pages (excluding cover page)

1a Let R, S be binary relations on a set W.

i) Define the relational composition $R \circ S$.

ii) What does it mean to say that relational composition is *associative* ?

iii) Define R^n, for $n \geq 1$.

iv) Write down the transitive closure R^+ of R in terms of R^n.

v) Suppose $R = \{(a,b), (b,c), (c,d), (b,e), (e,f), (g,h), (h,e), (h,i), (i,f)\}$.

Show both R and its transitive closure R^+ as a directed graph. (You may show both on the same diagram.)

b Let R, S, T be binary relations on a set W.

i) Show that the following distribution law holds, for any R, S, T:

$$(R \cup S) \circ T = (R \circ T) \cup (S \circ T)$$

ii) Let id be the identity relation on W. Using part (i) above, and the distribution law

$$R \circ (S \cup T) = (R \circ S) \cup (R \circ T)$$

(which you need not prove), simplify the expression

$$(R \cup id) \circ (R \cup id).$$

State clearly any other properties of composition that you use.

iii) Write down (without proof) an expression for $(R \cup id)^n$, for $n \geq 1$.

c Using parts (a) and (b) of the question, show:

i) $R^+ \circ R \subseteq R^+$.

ii) $(R \cup id)^+ = R^+ \cup id$.

2 Notation: $(A, \preccurlyeq)$ stands for any (non-strict) partial order (p.o.) on set A (i.e., $\preccurlyeq$ is a reflexive, anti-symmetric and transitive relation on A). $\prec$ is the associated strict (irreflexive and transitive) ordering, defined as

$$x \prec y \text{ iff } x \preccurlyeq y \text{ and } x \neq y.$$

a i) Define *transitive*, *anti-symmetric* and *irreflexive*.

ii) What additional property is required for the ordering $(A, \preccurlyeq)$ to be *total*?

iii) Define what is meant by *minimal* and *least* elements of a p.o. $(A, \preccurlyeq)$.

iv) Give an example of a partial order on a finite set which has minimal elements but no least element.

b Let W be a set. Pow(W) is the powerset of W.

i) Explain carefully why $(\text{Pow}(W), \subseteq)$ is a partial order.

ii) Is $(\text{Pow}(W), \subseteq)$ total? Justify your answer.

iii) What are the minimal element(s) of $(\text{Pow}(W), \subseteq)$?

c Let $(A, \prec)$ be a strict partial order. Let $X \subseteq A$. Then $x \in X$ is said to be $\prec$-minimal in X iff there is no element $y \in X$ such that $y \prec x$.

Let $\min_{\prec}(X)$ denote the set of elements that are $\prec$-minimal in X. $\min_{\prec}$ is thus a function $\text{Pow}(A) \rightarrow \text{Pow}(A)$.

i) What is $\min_{\subset}(\{a\}, \{b\}, \{a,b,c\})$?

ii) What does it mean to say that a function is *one-to-one* ?

ii) Is $\min_{\subset}$ a *one-to-one* function in general? Justify your answer.

d Let $(A, \prec)$ be a strict partial order. Show that the following holds for all subsets X and Y of A:

$$\min_{\prec}(X \cup Y) \subseteq \min_{\prec}(X) \cup \min_{\prec}(Y)$$

The four parts carry, respectively, 40%, 20% , 20% , 20% of the marks.

Turn over ...

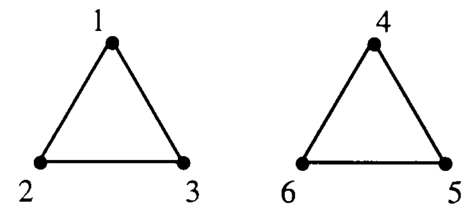
3a i) What does it mean for a graph to be *connected*?

ii) Define an *Euler circuit* in a graph.

iii) Draw two connected graphs, G1 and G2, each with 5 nodes, where G1 has an Euler circuit and G2 does not. Justify your answer briefly.

b i) Define *isomorphism* and *automorphism* as applied to graphs.

ii) How many automorphisms does the following graph with six nodes have? Explain your answer.



c Let a graph be called *k-robust* if it has the property that it is still connected if *any* k arcs (edges) are removed. (In other words, the graph is still a reliable network if up to k links are destroyed.)

i) Give an example of a graph with 5 nodes which is 2-robust but not 3-robust. Explain why your graph is 2-robust but not 3-robust.

ii) Let n be any natural number ≥ 1. Suggest a lower bound L(n) for the number of arcs in a 2-robust graph with n nodes. Explain your answer. [Hint: consider the degrees of the nodes.]

4a i) Let A be an algorithm to search for an element x in a list L of length n by comparing x with the entries of L. Explain how a *decision tree* can be associated with running A on L.

ii) Illustrate your answer to part (i) by giving a decision tree for the *Binary Search* algorithm applied to a list of length 9.

iii) Show the following by induction: if T is a binary tree with depth d, then T has no more than $2^{d+1}-1$ nodes.

iv) Explain how decision trees can be used to give *lower bounds* for searching by comparison, again illustrating your answer with the case n=9.

b i) Briefly describe an algorithm to merge two sorted lists, each of length m, by comparing their elements. What is the worst-case number of comparisons?

ii) State the recurrence relation for the worst-case number of comparisons W(n) made by Mergesort when sorting a list L of length n. You may assume that n is a power of 2. Explain your answer.

iii) Give an example of a list of length 8 where Mergesort actually uses W(8) comparisons. Explain your answer.

End of paper