UNIVERSITY OF LONDON [II(4)E 2000]

B.ENG. AND M.ENG. EXAMINATIONS 2000

For Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examination for the Associateship.

PART II : MATHEMATICS 4 (ELECTRICAL ENGINEERING)

Thursday 8th June 2000 2.00 - 4.00 pm

Answer FOUR questions.

[*Before starting, please make sure that the paper is complete; there should be 4 pages, with a total of 6 questions. Ask the invigilator for a replacement if your copy is faulty.*]

1. Find the eigenvalues of

$$A = \begin{pmatrix} 3 & -3 & 2 \\ -1 & 5 & -2 \\ -1 & 3 & 0 \end{pmatrix} .$$

Determine the general form of the eigenvector corresponding to the repeated eigenvalue of A.

Obtain three independent eigenvectors and, hence, determine a matrix C such that

$$C^{-1} AC = D$$

where D is a diagonal matrix.
Show that D is the spectral matrix of A, i.e. the diagonal elements of D are the eigenvalues of A.

2. Reduce the quadratic form

$$Q = \mathbf{x}^T A\mathbf{x} = 2x_1^2 + 5x_2^2 + 3x_3^2 + 4x_1x_2$$

to the form

$$Q = \mathbf{y}^T D\mathbf{y} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2$$

where $\mathbf{x} = (x_1, x_2, x_3)^T$, $\mathbf{y} = (y_1, y_2, y_3)^T$, $\mathbf{x} = P\,\mathbf{y}$ and $D = P^T AP$.

The scalars $\lambda_1, \lambda_2, \lambda_3$ and the matrices P and D are to be determined.

Hence, find y_1, y_2 and y_3 in terms of x_1, x_2 and x_3.

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3. (i) For any two events A and B which are subsets of a sample space Ω, use a Venn diagram to show that

$$A = (A \cap B) \cup (A \cap \overline{B}) \quad \text{and} \quad B = (A \cap B) \cup (\overline{A} \cap B).$$

Hence show that for events A, B the probability of *one and only one* of them occurring is

$$P(A) + P(B) - 2P(A \cap B).$$

(ii) If A and B are independent events, prove that $\overline{A}$ and $\overline{B}$ are independent events, and also that $\overline{A}$ and B are independent events.

[*Hint: You may use the fact that* $\overline{A} \cap \overline{B} = \overline{A \cup B}$.]

(iii) The probability that generating equipment will still be in use in 10 years time is 1/4 at power plant 1 and 1/3 at plant 2. Given that the equipment at the two sites behaves independently, find the probability that in 10 years time

(a) equipment at both plants will be functioning,

(b) equipment at at least one plant will be functioning,

(c) equipment at neither plant will be functioning,

(d) only the equipment at plant 2 will be functioning,

(e) equipment at plant 2 will be functioning, given that equipment at plant 1 is no longer functioning.

4. Printed circuit boards (PCBs) leave a production line randomly and independently and each is routinely subject to a quality check. It has been found over a long period of time that a proportion q of PCBs will fail the quality check. Let X be the number of PCBs tested up to and including the first rejection, and let $p = 1 - q$.

(i) Show that $P(X = j) = p^{j-1}q$, $j = 1, 2, \ldots,$ i.e., X has a geometric distribution, and find $P(X > j)$.

(ii) Demonstrate that X has the lack-of-memory property by showing that for j, k positive integers,

$$P(X > j + k \mid X > j) = P(X > k).$$

Why should it be no surprise that the geometric distribution enjoys the lack-of-memory property?

(iii) Let Y be the total number of PCBs tested up to and including the second rejection. Determine the joint probability distribution $P(X = j, Y = k)$ and the marginal probability distribution $P(Y = k)$, where $k > j \geq 1$.

5. Let the positive random variable T represent the lifetime of an electrical system. Carefully define the hazard rate $z(t)$ of the system.

A system consists of two components connected in parallel so that $T = \max\{T_1, T_2\}$ where T_1 and T_2 are independent lifetimes of component 1 and component 2, respectively, both having exponential failure time distributions, i.e.,

$$f_{T_1}(t) = \lambda_1 e^{-\lambda_1 t},\ t \geq 0,\ \text{and}\ f_{T_2}(t) = \lambda_2 e^{-\lambda_2 t},\ t \geq 0,\ \text{with}\ \lambda_1, \lambda_2 > 0.$$

(i) Find the cumulative distribution function, $F_T(t)$, of T.

(ii) Show that the probability density function $f_T(t)$ of T is given by

$$f_T(t) = \lambda_1 e^{-\lambda_1 t} + \lambda_2 e^{-\lambda_2 t} - (\lambda_1 + \lambda_2)\, e^{-(\lambda_1+\lambda_2)\, t}, \quad t \geq 0,$$

and demonstrate that it integrates to unity.

(iii) When $\lambda_1 = \lambda_2 = \lambda > 0$, show that the hazard rate is given by

$$z(t) = \frac{2\lambda}{\frac{1}{1-e^{-\lambda t}} + 1},$$

and comment on its form as $t \to \infty$.

6. Given two random variables X and Y with joint probability density function $f_{X,Y}(x, y)$, the minimum mean square error estimate of the unobserved value $Y = y$ in terms of the observed value $X = x$ is given by $E\{Y \mid X = x\}$.

(i) Carefully explain the meaning of the above statement.

The random variables X and Y which represent the amplitudes of two signals have joint probability density function

$$f_{X,Y}(x, y) = \begin{cases} x^{-1}, & 0 \leq y \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

(ii) Show that the double integral of $f_{X,Y}(x, y)$ over the specified ranges of x and y takes the value unity, as required for such a joint probability density function.

(iii) Find $f_{Y|X=x}(y|x)$, the conditional probability density function of Y given $X = x$. What type of continuous distribution is this ?

(iv) Hence derive the minimum mean square error estimate of $Y = y$ in terms of $X = x$.

END OF PAPER

$$\begin{vmatrix} 3-\lambda & -3 & 2 \\ -1 & 5-\lambda & -2 \\ -1 & 3 & -\lambda \end{vmatrix} \quad \begin{matrix} R_1 = R_1 + R_2 \\ \\ R_3 = R_3 - R_2 \end{matrix} \rightarrow \begin{vmatrix} 2-\lambda & 2-\lambda & 0 \\ -1 & 5-\lambda & -2 \\ 0 & \lambda-2 & 2-\lambda \end{vmatrix}$$

$$= (2-\lambda)^2 \begin{vmatrix} 1 & 1 & 0 \\ -1 & 5-\lambda & -2 \\ 0 & -1 & 1 \end{vmatrix} = (2-\lambda)^2 (5-\lambda-2+1) = (2-\lambda)^2(4-\lambda)$$

$\therefore$ E-values : 2, 2, 4. 3

$\lambda = 4$ $\begin{pmatrix} -1 & -3 & 2 \\ -1 & 1 & -2 \\ -1 & 3 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \rightarrow \begin{pmatrix} -1 & -3 & 2 \\ 0 & 4 & -4 \\ 0 & 6 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$

$\therefore \underset{\sim}{e}_4 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ 1

$\lambda = 2$ $\begin{pmatrix} 1 & -3 & 2 \\ -1 & 3 & -2 \\ -1 & 3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underset{\sim}{0} \rightarrow x - 3y + 2z = 0$

$\therefore \underset{\sim}{e}_2 = \begin{pmatrix} 3\alpha - 2\beta \\ \alpha \\ \beta \end{pmatrix} = \alpha \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$. 2

$\therefore \underset{\sim}{e}_4 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \underset{\sim}{e}_{21} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \underset{\sim}{e}_{22} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$ 2

and $C = \begin{pmatrix} -1 & 3 & -2 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ 1

To find C^{-1}: $\left.\begin{matrix} -1 & 3 & -2 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{matrix}\right| \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \quad \begin{matrix} R_1 = -R_1 \\ R_2 = R_2 + R_1 \\ R_3 = R_3 + R_1 \end{matrix} \rightarrow \left.\begin{matrix} +1 & -3 & +2 \\ 0 & 4 & -2 \\ 0 & 3 & -1 \end{matrix}\right| \begin{matrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{matrix}$

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$R_1 = R_1 + R_3$

$R_3 = R_3 - \frac{3}{4}R_2$

$$\rightarrow \left.\begin{matrix} 1 & 0 & 1 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{matrix}\right| \begin{matrix} 0 & 0 & 1 \\ \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & -\frac{3}{4} & 1 \end{matrix}$$

$R_1 = R_1 - 2R_3$

$R_2 = R_2 + R_3$

$R_3 = 2R_3$

$$\rightarrow \left.\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right| \begin{matrix} -\frac{1}{2} & \frac{3}{2} & -1 \\ \frac{1}{2} & -\frac{1}{2} & 1 \\ \frac{1}{2} & -\frac{3}{2} & 2 \end{matrix} \qquad \therefore C^{-1} = \frac{1}{2}\begin{pmatrix} -1 & 3 & -2 \\ 1 & -1 & 2 \\ 1 & -3 & 4 \end{pmatrix}$$

6

$$\therefore C^{-1}AC = \frac{1}{2}\begin{pmatrix} -1 & 3 & -2 \\ 1 & -1 & 2 \\ 1 & -3 & 4 \end{pmatrix}\begin{pmatrix} 3 & -3 & 2 \\ -1 & 5 & -2 \\ -1 & 3 & 0 \end{pmatrix}\begin{pmatrix} -1 & 3 & -2 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} -1 & 3 & -2 \\ 1 & -1 & 2 \\ 1 & -3 & 4 \end{pmatrix}\begin{pmatrix} -4 & 6 & -4 \\ 4 & 2 & 0 \\ 4 & 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 3 & -2 \\ 1 & -1 & 2 \\ 1 & -3 & 4 \end{pmatrix}\begin{pmatrix} -2 & 3 & -2 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

5

2

$$Q = (x_1, x_2, x_3)\begin{pmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Eigenvalues of A: $\begin{vmatrix} 2-\lambda & 2 & 0 \\ 2 & 5-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$

$\therefore (3-\lambda)\left((2-\lambda)(5-\lambda)-4\right) = (3-\lambda)(\lambda^2-7\lambda+6) = 0$

ie $(3-\lambda)(\lambda-6)(\lambda-1) = 0$ $\quad\therefore$ E-values: $1, 3, 6$. 3

Eigenvectors: $\lambda = 1$ $\begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underset{\sim}{0}$ $\quad\therefore \underset{\sim}{e}_1 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$

$\lambda = 3$: $\begin{pmatrix} -1 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underset{\sim}{0}$ $\quad\therefore \underset{\sim}{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ 3

$\lambda = 6$: $\begin{pmatrix} -4 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underset{\sim}{0}$ $\quad\therefore \underset{\sim}{e}_6 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

$\therefore$ Normalised e-vectors are $\begin{pmatrix} \frac{2}{\sqrt5} \\ -\frac{1}{\sqrt5} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt5} \\ \frac{2}{\sqrt5} \\ 0 \end{pmatrix}$

$\therefore P = \begin{pmatrix} \frac{2}{\sqrt5} & 0 & \frac{1}{\sqrt5} \\ -\frac{1}{\sqrt5} & 0 & \frac{2}{\sqrt5} \\ 0 & 1 & 0 \end{pmatrix}$ 4

$\underset{\sim}{x} = P\underset{\sim}{y}$ $\quad\therefore Q = \underset{\sim}{x}^T A \underset{\sim}{x} = \underset{\sim}{y}^T P^T A P \underset{\sim}{y}$ 1

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$$P^TAP = \begin{pmatrix} \frac{2}{\sqrt5} & -\frac{1}{\sqrt5} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt5} & \frac{2}{\sqrt5} & 0 \end{pmatrix}\begin{pmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} \frac{2}{\sqrt5} & 0 & \frac{1}{\sqrt5} \\ -\frac{1}{\sqrt5} & 0 & \frac{2}{\sqrt5} \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{\sqrt5} & -\frac{1}{\sqrt5} & 0 \\ 0 & 0 & 3 \\ \frac{6}{\sqrt5} & \frac{12}{\sqrt5} & 0 \end{pmatrix}\begin{pmatrix} \frac{2}{\sqrt5} & 0 & \frac{1}{\sqrt5} \\ -\frac{1}{\sqrt5} & 0 & \frac{2}{\sqrt5} \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix} \qquad \therefore Q = y_1^2 + 3y_2^2 + 6y_3^2$$ 5

$$\underset{\sim}{y} = P^{-1}\underset{\sim}{x} = P^T\underset{\sim}{x} = \begin{pmatrix} \frac{2}{\sqrt5} & -\frac{1}{\sqrt5} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt5} & \frac{2}{\sqrt5} & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\therefore y_1 = \frac{1}{\sqrt5}(2x_1 - x_2)$$

$$y_2 = x_3$$ 4

$$y_3 = \frac{1}{\sqrt5}(x_1 + 2x_2).$$

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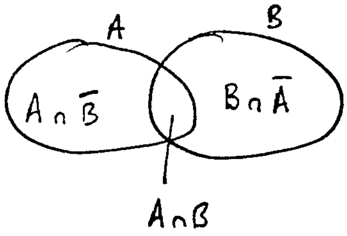
PAPER II (4) EE
QUESTION Stats
SOLUTION 3

(i)



$A = (A \cap B) \cup (A \cap \bar{B})$ [disjoint]
$B = (A \cap B) \cup (\bar{A} \cap B)$

2

Hence $P(A \cap \bar{B}) = P(A) - P(A \cap B)$; $P(B \cap \bar{A}) = P(B) - P(A \cap B)$

4 $P(\text{one and only one}) = P\left((A \cap \bar{B}) \cup (B \cap \bar{A})\right) = P(A \cap \bar{B}) + P(B \cap \bar{A})$
$= P(A) + P(B) - 2P(A \cap B)$.

(ii) $P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B) = 1 - P(A) - P(B) + P(A \cap B)$
$= 1 - P(A)P(B) + P(A)P(B)$ [indep]

3 $= [1 - P(A)][1 - P(B)] = P(\bar{A})P(\bar{B})$. $\bar{A}, \bar{B}$ ind.

$P(\bar{A} \cap B) = P(A \cup B) - P(A)$ [$A, \bar{A} \cap B$ disjoint]

3 $= P(A) + P(B) - P(A)P(B) - P(A) = P(B)[1 - P(A)] = P(B)P(\bar{A})$
$\bar{A}, B$ ind.

(iii) Let A = equip at plant 1 still functions in 10 yrs time
B = 2

$P(A) = 1/4$; $P(B) = 1/3$; $P(\bar{A}) = 3/4$; $P(\bar{B}) = 2/3$

1 (a) $P(A \cap B) = P(A)P(B) = \frac{1}{4} \cdot \frac{1}{3} = \frac{1}{12}$

1 (b) $P(A \cup B) = P(A) + P(B) - P(A)P(B) = \frac{1}{4} + \frac{1}{3} - \frac{1}{12} = \frac{1}{2}$

2 (c) $P(\bar{A} \cap \bar{B}) = P(\bar{A})P(\bar{B})$ [by part (ii)] $= \frac{3}{4} \cdot \frac{2}{3} = \frac{1}{2}$

2 (d) $P(\bar{A} \cap B) = P(B)P(\bar{A})$ [by part (ii)] $= \frac{1}{3} \cdot \frac{3}{4} = \frac{1}{4}$

2 (e) $P(B \mid \bar{A}) = P(\bar{A} \cap B) / P(\bar{A}) = P(\bar{A})P(B) / P(\bar{A}) = P(B) = \frac{1}{3}$

or, since $\bar{A}$ and B are indep, $P(B \mid \bar{A}) = P(B) = 1/3$.

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MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

SESSION : 1999/2000

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PAPER EE II(4) IEE

QUESTION Stats

SOLUTION 4

4 (i) $P(X=j) = P(\text{first } (j-1) \text{ PCBs ok, } j\text{th is faulty})$

$$= (1-q)^{j-1} q = p^{j-1} q, \quad j = 1, 2, \ldots$$

$P(X>j) = P(\text{first } j \text{ are all o.k.}) = p^j$ <u>or</u>

4 $$p^j q + p^{j+1} q + \ldots = p^j q (1 + p + \ldots)$$

$$= p^j q / (1-p) = p^j.$$

(ii) $P(X > j+k \mid X > j) = P(X > j+k \cap X > j) / P(X > j)$

$$= P(X > j+k) / P(X > j) \qquad j, k \text{ +ve int}$$

4 $$= p^{j+k} / p^j \qquad \text{from (i)}$$

$$= p^k = P(X > k).$$

2 We know (class) that the exponential dist^n has lack-of-memory, and geometric is discrete analogue of exponential.

(iii) $P(X=j, Y=k) = p^{j-1} q \, P(\text{next } k-j-1 \text{ o.k., } k\text{th is faulty})$

$$= p^{j-1} q \, p^{k-j-1} q$$

3 $$= p^{k-2} q^2 \qquad k > j \geq 1.$$

$$P(Y=k) = \sum_{j=1}^{k-1} P(X=j, Y=k) = \sum_{j=1}^{k-1} p^{k-2} q^2$$

3 $$= (k-1) p^{k-2} q^2.$$

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MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

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PAPER EE II (4

QUESTION Stats

SOLUTION 5

The hazard rate is the conditional probability intensity that a t-unit-old system will fail. It is defined as $z(t) = f(t) / 1-F(t) = f(t) / R(t)$

5 where $f(t)$ is p.d.f., $F(t)$ is c.d.f. and $R(t)$ is reliability.

(i) $F_T(t) = P(T \le t) = P(\max\{T_1, T_2\} \le t)$

$= P(T_1 \le t)\, P(T_2 \le t) = F_{T_1}(t) F_{T_2}(t)$

4 $= (1-e^{-\lambda_1 t})(1-e^{-\lambda_2 t}), \quad t \ge 0.$

(ii) $f_T(t) = \frac{d}{dt} F_T(t) = \frac{d}{dt}\left(1 - e^{-\lambda_1 t} - e^{-\lambda_2 t} + e^{-(\lambda_1+\lambda_2)t}\right)$

3 $= \lambda_1 e^{-\lambda_1 t} + \lambda_2 e^{-\lambda_2 t} - (\lambda_1+\lambda_2) e^{-(\lambda_1+\lambda_2)t} \quad t \ge 0.$

All three components are p.d.fs (of exponentials with parameters λ_1, λ_2 and $\lambda_1+\lambda_2$). Hence when integrated from 0 to ∞ we get

2 $\underline{1} + \underline{1} - \underline{1} = 1.$

(iii) Put $\lambda_1 = \lambda_2 = \lambda$. Then

$$z(t) = \frac{f(t)}{1-F(t)} = \frac{2\lambda e^{-\lambda t} - 2\lambda e^{-2\lambda t}}{2e^{-\lambda t} - e^{-2\lambda t}} = \frac{2\lambda e^{-\lambda t}(1-e^{-\lambda t})}{e^{-\lambda t}(2-e^{-\lambda t})}$$

4 $$= \frac{2\lambda(1-e^{-\lambda t})}{(2-e^{-\lambda t})} = \frac{2\lambda(1-e^{-\lambda t})}{1+(1-e^{-\lambda t})}$$

$$= \frac{2\lambda}{\frac{1}{1-e^{-\lambda t}} + 1}.$$

2 As $t \to \infty$, $z(t) \to \lambda$, which is the same hazard rate as obtained with a single component with $\mathrm{Exp}(\lambda)$ lifetime.

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5 (i) The minimum mean square error estimate of $Y=y$ in terms of $X=x$ is got by finding that function $c(x)$ which minimizes

$$E\{(Y-c(x))^2\} = \iint (y-c(x))^2 f_{X,Y}(x,y)\,dx\,dy.$$

It turns out that $c(x) = E\{Y|X=x\}$ is the minimizing function.

Given $f_{X,Y}(x,y) = \begin{cases} x^{-1} & 0\le y\le x\le 1 \\ 0 & \text{otherwise.}\end{cases}$

4 (ii) $$\int_0^1 dx \int_0^x \frac{1}{x}\,dy = \int_0^1 \frac{1}{x}\left\{ y\Big|_0^x \right\} dx = \int_0^1 dx = x\Big|_0^1 = 1.$$

(iii) $$f_{Y|X=x}(y|x) \equiv f_{X,Y}(x,y)\,/\,f_X(x).$$

4 But $$f_X(x) = \int_0^x f_{X,Y}(x,y)\,dy = \int_0^x \frac{1}{x}\,dy = 1, \quad 0\le x\le 1.$$

So $$f_{Y|X=x}(y|x) = \frac{1}{x}, \quad 0\le y\le x.$$

2 Hence, conditional on $X=x$, $0\le x\le 1$, Y is uniform on $[0,x]$.

(iv) $$E\{Y|X=x\} = \int_0^x y\, f_{Y|X=x}(y|x)\,dy$$

5 $$= \int_0^x \frac{y}{x}\,dy = \frac{1}{x}\left\{\frac{y^2}{2}\Big|_0^x\right\} = \frac{x}{2}.$$

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