Master copy - June 08

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SOLUTIONS (E2.6, Control Engineering, 2008)

1. (a) (i) Applying Newton's laws on the mass,

$$u(t) = M\ddot{y}(t) + D\dot{y}(t) + Ky(t).$$

(ii) Taking Laplace transforms,

$$\frac{y(s)}{u(s)} = \frac{1}{Ms^2 + Ds + K},$$

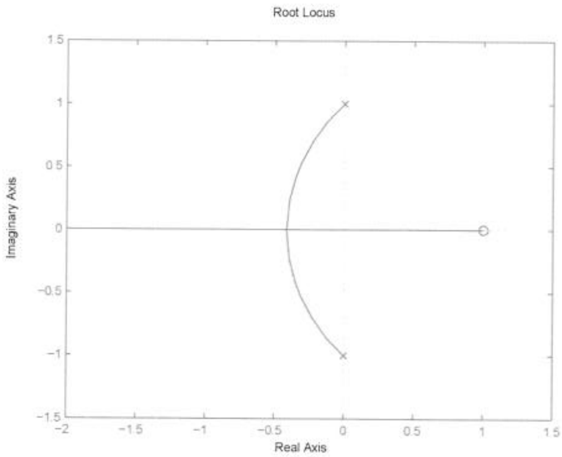
(iii) The response is critically damped when the two poles are equal. Thus $D = 2$.

(iv) When $u(s) = 1/s$, we have

$$y(s) = \frac{1}{s(s+1)(s+2)} = \frac{1}{2s} + \frac{1}{2(s+2)} - \frac{1}{s+1}$$

and so $y(t) = 0.5 + 0.5e^{-2t} - e^{-t}$.

(b) (i) The root locus is shown below. The breakaway point is at the negative root of $dG(s)/ds$ and is given by $\sigma_b = 1 - \sqrt{2} = -0.4142$. The angle of departure is given by the angle criterion: $135° - (\theta_d + 90°) = 180°$ and so $\theta_d = -135°$.



(ii) The Routh array gives the stability range as $0 < K < 1$.

(iii) For critical damping, the closed loop poles must be placed at the breakaway point σ_b. The corresponding gain is obtained from the gain criterion: $k = -G(s)^{-1}|_{s=\sigma_b} = 2(\sqrt{2} - 1) = 0.8284$.

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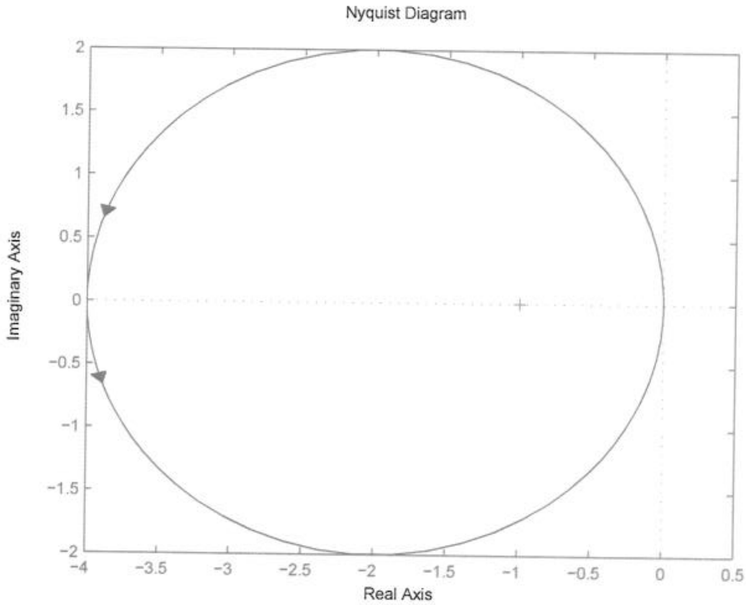
(c) (i) The Nyquist diagram is shown below:



(ii) From the Nyquist theorem, $N = Z - P$ where N $(= -1$ in this case) is the number of clockwise encirclement of the point -1, P $(= 1$ in this case) is the number of open loop unstable poles, and Z is the number of closed–loop unstable poles. Hence $Z = 0$ and the closed–loop is stable.

(iii) Since the negative real-axis intercept is at -4, the gain margin is 0.25.

2. (a) For an op-amp in the negative feedback mode, the transfer function is $-Z_f/Z_i$ where Z_f is the feedback impedance and Z_i is the input impedance. Thus the transfer function between $Vr(s)$ and $V(s)$ is

$$-\frac{s + 1/C1}{s}$$

and that between $Vo(s)$ and $V(s)$ is the same.

(b) The transfer function between $V(s)$ and $Vo(s)$ can be obtained in two stages. Let $V1$ be the voltage at the output of $O2$. Then $V1(s) = -\frac{1}{s+1}V(s)$. Also, $Vo(s) = -\frac{1}{s+2}V1(s)$. It follows that

$$Vo(s) = \frac{1}{(s+1)(s+2)}V(s).$$

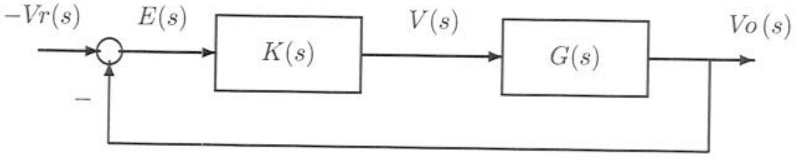
(c) The block diagram is given below, where

$$G(s) = \frac{1}{(s+1)(s+2)}$$

and

$$K(s) = \frac{s + 1/C1}{s}.$$

$K(s)$ is a proportional-plus-integral (PI) type compensator.



(d) Now,

$$E(s) = \frac{-Vr(s)}{1 + G(s)K(s)}.$$

Since $Vr(s) = 1/s$, it follows from the final value theorem that

$$E_{ss}^{step} = \lim_{s\to 0} sE(s) = 0.$$

(e) Here $Vr(s) = 1/s^2$. It follows from the final value theorem that

$$|E_{ss}^{ramp}| = |\lim_{s\to 0} sE(s)| = 2C1.$$

Thus, for $|E_{ss}^{ramp}| \leq \epsilon$, it follows that we require $C1 \leq \epsilon/2$, which is the maximum value.

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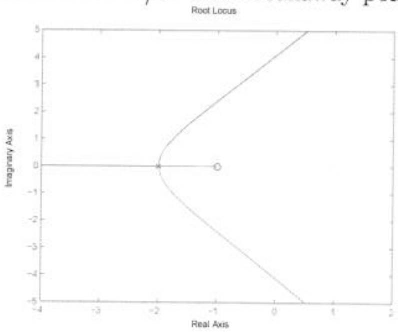
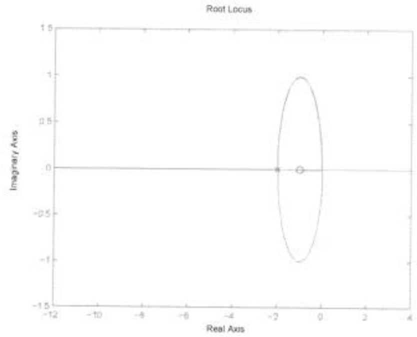
3. (a) The plot is shown below. The angles of the asymptotes are $\pm 60^\circ$, 180° and the centre is at $-7/3$. The breakaway point is at -2.





(b) The characteristic equation is $s^4+8s^3+24s^2+(32+k)s+16+k=0$.

The Routh array :

$$\begin{array}{c|ccc} s^4 & 1 & 24 & 16+k \\ s^3 & 8 & 32+k & \\ s^2 & \frac{160-k}{8} & 16+k & \\ s & \frac{4096+64k-k^2}{160-k} & & \\ 1 & 16+k & & \end{array}$$

The positive root of $4096+64k-k^2=0$ is $\bar{k}=103.5542$. Since we require no sign changes in the first column, $0<k<\bar{k}$.

(c) We have marginal stability when $k=\bar{k}$. Substituting in the array, the auxiliary polynomial is $\frac{160-\bar{k}}{8}s^2+16+\bar{k}=0$ which has roots at $\pm j\bar{\omega}$, $\bar{\omega}=16.9443$ rad/s. So the frequency of oscillations is $\bar{\omega}$.

(d) The compensator is $K(s)=k(s+z)$. To satisfy the specifications we place CL poles at $s_0=-1\pm j$. An inspection of the root locus shows that we must use negative k. Let the angle between s_0 and z be θ and that between s_0 and -1 be θ_0. Then $\theta_0=180^\circ-\tan^{-1}(2)$. The angle criterion gives $\theta+\theta_0-(4\times 90^\circ)=0^\circ$ or $\theta=\tan^{-1}(2)-180^\circ$. Thus $z=1$. The root locus is shown above. The gain criterion gives $k=-(s+2)^4/(s+1)^2|_{s=-2+j2}=-4$ so $K(s)=-4(s+1)$.

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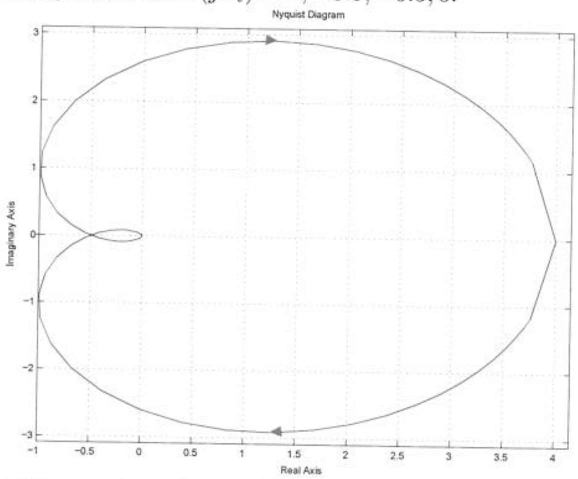
4. (a) The Nyquist plot is shown below. The real-axis intercepts can be found by evaluating the Routh array. This gives intercepts at $\omega_i = 0, \pm\sqrt{3}, \infty$ and so $G(j\omega_i) = 4, -0.5, -0.5, 0$.



(b) (i) The number of unstable closed-loop poles is determined by the number of encirclements by $G(s)$ of the point -1, which is zero. Thus the closed–loop is stable since $G(s)$ has no unstable poles.

(ii) Since the real–axis intercept is at -0.5, the gain margin is 2. The intercept with the unit circle centred on the origin occurs when $4 = \sqrt{(\omega^2+1)^3}$ or $\omega_p = \sqrt{16^{\frac{1}{3}} - 1} = 1.2328$. Then $\angle G(\omega_p) \sim -153^\circ$ and the phase margin is $180^\circ + \angle G(\omega_p) \sim 27^\circ$.

(c) (i) A PI compensator has the form $K(s) = K_p + K_i/s = K_p \frac{s + K_i/K_p}{s}$, with $K_p > 0,\ K_i > 0$. A PI compensator is a special form of phase-lag compensation and has high gain at low frequencies (infinite at DC) and close to K_p for high frequencies. The phase is close to -90° at low frequencies and tends to 0 at hight frequencies.

(ii) Thus PI compensation can increase low frequency gain and hence improve steady–state tracking since

$$|e(j\omega)| = |\frac{1}{1 + G(j\omega)K(j\omega)}|\ |r(j\omega)|$$

without increasing high frequency gain (thus degrading the gain margin). Care should be taken concerning the phase–margin since the phase lag may deteriorate this.

To reduce the destabilizing effect of the PI compensator we place the zero very near the origin, i.e. we choose $K_i << K_p$ so there is approximately a pole/zero cancellation at the origin.