SOLUTIONS: ESTIMATION AND FAULT DETECTION

1. Solution

a) With reference to the block-diagram shown in Fig. 1.1 of the text of the exam paper, one assigns to "Int. n. 1" a state variable denoted as x_1, to "Int. n. 2" a state variable denoted as x_2, and to "Int. n. 3" a state variable denoted as x_3.

After inspection of the block-diagram shown in Fig. 1.1 of the text of the exam paper, one gets immediately:

$$\begin{cases} \dot{x}_1 = -(K_1+1)x_1 + r - K_5 x_3 \\ \dot{x}_2 = K_2 x_1 - K_3 x_3 \\ \dot{x}_3 = x_1 - K_4 x_3 + x_2 \\ y = x_3 \end{cases}$$

and in matrix form:

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -(K_1+1) & 0 & -K_5 \\ K_2 & 0 & -K_3 \\ 1 & 1 & -K_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} r \\ y = [0\,0\,1] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \end{cases}$$

[3 marks]

b) Setting $K_1 = 5$, $K_2 = 6$, $K_3 = 4$, $K_4 = 4$, and $K_5 = 0$ the state equations determined in the answer to Question 1a) become:

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -6 & 0 & 0 \\ 6 & 0 & -4 \\ 1 & 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u \\ y = [0\,0\,1] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \end{cases}$$

i) the observability matrix is given by

$$Q = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & -4 \\ -4 & -4 & 12 \end{bmatrix}$$

As

$$\det Q = 0$$

we conclude that the system is not completely observable.

[3 marks]

ii) Inspecting the row vector C, one immediately recognises that only the third row of the matrix $(sI-A)^{-1}$ has to be computed. After computing

$$\det(sI-A) = (s+6)[s(s+4)+4] = (s+6)(s+2)^2$$

one gets the values of the three eigenvalues of matrix A: $\lambda_1 = -6$, $\lambda_2 = -2$, $\lambda_3 = -2$. Moreover, after some algebra, it follows that

$$C(sI-A)^{-1} = \left[\frac{1}{(s+2)^2}, \frac{1}{(s+2)^2}, \frac{s}{(s+2)^2}\right]$$

As the eigenvalue $\lambda_1 = -6$ does not appear among the poles of $C(sI-A)^{-1}$, it follows that the system is not completely observable.

[3 marks]

c) To answer Question 1c) of the exam paper, the observability canonical form has to be determined starting from the state-space description determined in the answer to Question 1b. In particular, one has to consider again the observability matrix computed in the answer to Question 1b-i)

$$Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & -4 \\ -4 & -4 & 12 \end{bmatrix}$$

and has to determine a basis for $\ker(Q)$:

$$Qv = 0 \Longrightarrow \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & -4 \\ -4 & -4 & 12 \end{bmatrix} v = 0 \Longrightarrow v = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Now, a basis $\{\alpha, \beta\}$ for the orthogonal complement to $\ker(Q)$ has to be determined. For example:

$$\alpha = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}; \beta = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Selecting the matrix

$$T = [\alpha \,|\, \beta \,|\, v] = \begin{bmatrix} -1 & 0 & -1 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

and computing the inverse

$$T^{-1} = \begin{bmatrix} -1/2 & -1/2 & 0 \\ 0 & 0 & 1 \\ -1/2 & 1/2 & 0 \end{bmatrix}$$

By setting $x = Tz$, we obtain the following equivalent observability canonical form:

$$\begin{cases} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = T^{-1}ATz + T^{-1}Be = \begin{bmatrix} 0 & 2 & 0 \\ -2 & -4 & 0 \\ -6 & -2 & -6 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} -1/2 \\ 0 \\ -1/2 \end{bmatrix} u \\ y = CTz = [0\ 1\ 0] \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \end{cases}$$

Inspecting the observability canonical form, one immediately recognises that the sub-system described by the state variables z_1 and z_2 is observable whereas the sub-system described by the state variable z_3 is not observable. Moreover, the eigenvalue associated with the non-observable system (highlighted in boldface in the above matrix $T^{-1}AT$) is $\lambda_{z_3} = -6$ which does not appear among the poles of $C(sI-A)^{-1}$ computed in the answer to Question 1b)-ii).

Finally, a basis for the non-observable subspace X_{no} is a basis for $\ker(Q)$, that is

$$v = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

[8 marks]

d) The output feedback connection $u = r - y$ does not modify the observability properties of the overall system. Thus, for the values $K_1 = 5$, $K_2 = 6$, $K_3 = 4$, $K_4 = 4$, and for every $K_5 \neq 0$, the overall system is not completely observable.

[3 marks]

2. Solution

a) The overall system depicted in Fig. 2.1 in the text of the exam paper is made by the cascade interconnection of two sub-systems each fully described by a first-order transfer function. Hence, to describe each sub-system in state-space form, a single state variable suffices.

More specifically, associating to the sub-system with transfer function $\frac{1}{s+1}$ a state variable denoted as x_1 and associating to the sub-system with transfer function $\frac{1}{s+2}$ a state variable denoted as x_2, from the assumption $d(t) = 0$, it follows that the following state-space description can be devised:

$$\begin{cases} \dot{x}_1 = -x_1 + u \\ \dot{x}_2 = -2x_2 + x_1 \\ y = x_2 \end{cases}$$

and in matrix form:

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\ y = [0\ 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{cases}$$

[3 marks]

b) Consider the additional input $d(t) = K \cdot 1(t)$ where $K > 0$ is an *unknown* scalar with $1(t)$ denoting the unit-step function.

i) The additional input $d(t)$ can be generated as follows:

$$\begin{cases} \dot{z}(t) = 0 \\ d(t) = z(t) \end{cases}$$

with $z(0^-) = K$. Therefore, introducing the augmented state vector

$$x_a := \begin{bmatrix} x_1 \\ x_2 \\ z \end{bmatrix}$$

the following augmented state equations can be written:

$$\begin{cases} \dot{x}_1 = -x_1 + u \\ \dot{x}_2 = -2x_2 + x_1 + z \\ \dot{z} = 0 \\ y = x_2 \end{cases}$$

and in matrix form:

$$\begin{cases} \dot{x}_a = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ z \end{bmatrix} = Ax_a + Bu = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ z \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u \\ y = Cx = [0\ 1\ 0] \begin{bmatrix} x_1 \\ x_2 \\ z \end{bmatrix} \end{cases}$$

Therefore, a third-order Luenberger observer architecture can be devised that, under appropriate conditions, is able to provide an asymptotic estimate of the augmented state x_a, hence also providing an estimate of the unknown constant input $d(t) = K \cdot 1(t)$ (which coincides with the third component of x_a).

[6 marks]

ii) With reference to the third-order Luenberger observer mentioned in the answer to Question 2b-i), a necessary and sufficient condition for this observer to be an asymptotic estimator of the augmented state x is the complete observability of the pair (A,C) of the augmented state-space realization determined in the answer to Question 2b-i).

After some easy algebra, the observability matrix is given by

$$Q = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 1 \\ -3 & 4 & -2 \end{bmatrix}$$

As

$$\det Q \neq 0$$

we conclude that the the pair (A,C) is completely observable.

[3 marks]

c) Referring to the the augmented state-space realization determined in the answer to Question 2b-i), it is immediate to see that the estimation error $e(t)$ defined in Question 2c) satisfies

$$e(t) = \begin{bmatrix} e_x(t) \\ e_K(t) \end{bmatrix} = x_a(t) - \widehat{x}_a(t)$$

where x_a is the augmented state vector introduced in the answer to Question 2b-i) and $\widehat{x}_a$ is its estimate provided by the observer. Therefore, the design of the observer consists in determining a matrix L such that the eigenvalues of $F = A - LC$ are:

$$\lambda_1 = -5, \lambda_2 = -5, \lambda_3 = -5.$$

Considering the matrix A of the augmented state-space realization determined in the answer to Question 2b-i), one gets:

$$\det(sI - A) = \det \begin{bmatrix} s+1 & 0 & 0 \\ -1 & s+2 & -1 \\ 0 & 0 & s \end{bmatrix} = s(s+1)(s+2) = s^3 + 3s^2 + 2s$$

Then, the matrices A_o and C_o of the observer canonical form are

$$A_o = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -2 \\ 0 & 1 & -3 \end{bmatrix}; \quad C_o = [0\,0\,1]$$

The observability matrix Q_o computed on the basis of the pair (A_o, C_o) is

$$Q_o = \begin{bmatrix} C_o \\ C_o A_o \\ C_o A_o^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & -3 & 7 \end{bmatrix}$$

Using Q determined in the answer to Question 2b)-ii), the matrix T_o transforming the original augmented state equations into the observer canonical form

$$\begin{cases} \dot{w} = A_o w + B_o u \\ y = C_o w \end{cases} \quad \text{with} \quad A_O = T_o^{-1} A T_o; B_o = T_o^{-1} B; C_o = C T_o$$

is given by

$$T_o = Q^{-1} Q_o = \begin{bmatrix} -1 & 1 & -1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad T_o^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Now considering

$$L_o = \begin{bmatrix} l_o^{(1)} \\ l_o^{(2)} \\ l_o^{(3)} \end{bmatrix}$$

after some algebra one gets

$$\det[sI - (A_o - L_o C_o)] = s^3 + (3 + l_o^{(3)}) s^2 + (2 + l_o^{(2)}) s + l_o^{(1)}$$

and by equating this polynomial with the polynomial having the desired observer eigenvalues as roots, that is

$$\alpha_d(s) = (s+5)^3 = s^3 + 15s^2 + 75s + 125$$

one immediately obtains

$$l_o^{(1)} = 125; \quad l_o^{(2)} = 73; \quad l_o^{(3)} = 12$$

Finally, the observer gain L such that $F = A - LC$ has the desired eigenvalues $\lambda_1 = -5, \lambda_2 = -5, \lambda_3 = -5$ is given by

$$L = T_o L_o = \begin{bmatrix} -64 \\ 12 \\ 125 \end{bmatrix}$$

[8 marks]

3. Solution

a) The general recursive Riccati equation is

$$P(t+1) = F\left[P(t) - P(t)H^\top \left(V_2 + HP(t)H^\top\right)^{-1} HP(t)\right] F^\top + V_1, \quad t = 1,2,\dots$$

with the initialisation $P(1) = \mathrm{var}[x(1)]$. Correspondingly, the time-varying predictor gain is

$$K(t) = FP(t)H^\top \left(V_2 + HP(t)H^\top\right)^{-1}, \quad t = 1,2,\dots$$

Letting $F = -1/3, H = 2, V_1 = 4, V_2 = 1$, and using $\mathrm{var}[x(1)] = 9$, one gets:

$$P(1) = 9,\ P(2) = 4.02702,\ P(3) = 4.02615,\ P(4) = 4.02615$$

and

$$K(1) = -0.16216,\ K(2) = -0.15692,\ K(3) = -0.15692,\ K(4) = -0.15692$$

As the dynamic system is asymptotically stable (its eigenvalue $\lambda = -1/3$ lies inside the unit circle), thanks to the first convergence theorem, the sequence $P(t), t = 1,\dots$ converges to a positive-definite steady-state matrix $\overline{P}$.

[4 Marks]

b) The general algebraic Riccati equation is

$$P = F\left[P - PH^\top \left(V_2 + HPH^\top\right)^{-1} HP\right] F^\top + V_1$$

Letting again $F = -1/3, H = 2, V_1 = 4, V_2 = 1$, one gets:

$$P = \frac{1}{9}\left(P - 4\frac{1}{1+4P}P^2\right) + 4 \Longrightarrow P^2 - \frac{34}{9}P - 1 = 0$$

thus obtaining the two solutions

$$\overline{P}_1 \simeq -0.2483 \quad \text{and} \quad \overline{P}_2 \simeq 4.02615$$

Clearly, the only admissible solution is the positive one. Thus

$$\overline{P} = \overline{P}_2 \simeq 4.02615$$

Accordingly:

$$\overline{K} = F\overline{P}H^\top \left(V_2 + H\overline{P}H^\top\right)^{-1} \simeq -0.15692$$

Comparing the time-behaviors of the sequences $P(t), t = 1,\dots,4$ and $K(t), t = 1,\dots,4$ determined in the answer to Question 3a) with the above steady-state values $\overline{P}$ and $\overline{K}$, respectively, it turns out that

$$P(t) \simeq \overline{P} \quad \text{and} \quad K(t) \simeq \overline{K} \quad \text{for} \quad t \geq 3$$

and

$$K(t) \simeq \overline{K} \text{ for } t \geq 3$$

Therefore, with the exception of the first two initial stages, the time-varying quantities $P(t)$ and $K(t)$ are nearly the same as the steady-state values $\overline{P}$ and $\overline{K}$ in accordance with the answer to Question 3a). This fact could be exploited for a more computationally efficient (though suboptimal) implementation of Kalman one-step ahead predictor.

[5 marks]

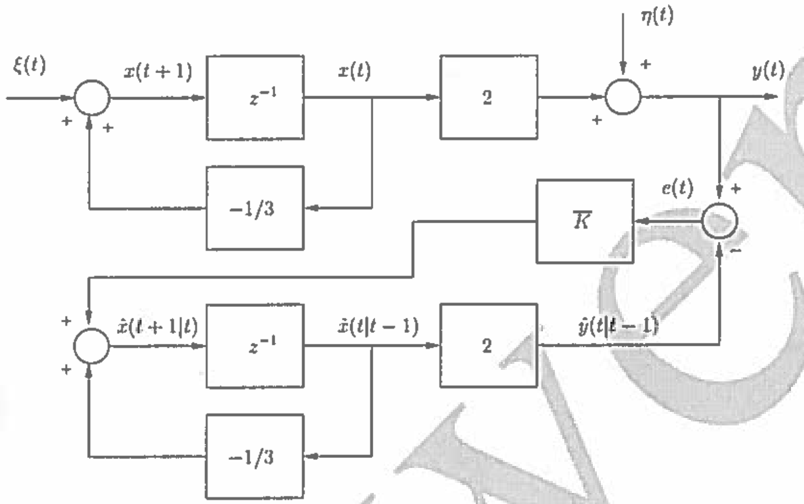
c) The Kalman predictor obeys to the following equations:

$$\begin{cases} \hat{x}(t+1|t) = -\frac{1}{3}\hat{x}(t|t-1) + \overline{K}e(t) \\ \hat{y}(t+1|t) = 2\hat{x}(t+1|t) \\ e(t) = y(t) - 2\hat{x}(t|t-1) \end{cases}$$

and thus

$$\hat{x}(t+1|t) = -\frac{1}{3}\hat{x}(t|t-1) + \overline{K}\left[y(t) - 2\hat{x}(t|t-1)\right]$$

where $\overline{K} \simeq -0.15692$. The block-diagram of the steady-state one-step ahead Kalman predictor is shown in the figure below.



[3 marks]

d) i) To determine the discrete-time transfer function $G_{\xi e}$ from the noise ξ to the output prediction error e, one inspects the block-diagram in the answer to Question 3c) setting $\eta = 0$. After some easy algebra, it is immediate to get

$$G_{\xi e} = \frac{2}{z+1/3} \cdot \frac{1}{1 + \dfrac{2\overline{K}}{z+1/3}} = \frac{2}{z+1/3+2\overline{K}}$$

[4 marks]

ii) Following the same procedure of the answer to Question 3d), to determine the discrete-time transfer function $G_{\eta e}$ from the noise η to the output prediction error e, one inspects the block-diagram in the answer to Question 3c) setting $\xi = 0$. After some easy algebra, it is immediate to get

$$G_{\eta e} = \frac{1}{1 + \dfrac{2\overline{K}}{z+1/3}} = \frac{z+1/3}{z+1/3+2\overline{K}}$$

[4 marks]

4. Solution

a) For $t < T_0$, the input $f(t)$ is not acting on the system ($f(t) = 0, \forall t < T_0$).

i) Using the observer canonical form, the closed-loop control system shown in Fig. 4.1 in the text of the exam paper can be described by the following state-equations

$$\begin{cases} \dot{x}_1 = -14x_2 + 10u \\ \dot{x}_2 = x_1 - 14x_2 + 10u \\ y = x_2 \end{cases}$$

and in matrix form:

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -14 \\ 1 & -14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 10 \\ 10 \end{bmatrix} u \\ \\ y = [0\ 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{cases}$$

[3 marks]

ii) The full-order observer takes on the form:

$$\begin{cases} \dot{\hat{x}} = A^{(0)}\hat{x} + B^{(0)}u + L^{(0)}(y - C\hat{x}) \\ \hat{y} = C^{(0)}\hat{x} \end{cases}$$

where

$$A^{(0)} = \begin{bmatrix} 0 & -14 \\ 1 & -14 \end{bmatrix}; \quad B^{(0)} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}; \quad C^{(0)} = [0\ 1]$$

are the matrices of the state-space description obtained in the answer to Question 4a)-i). The superscript "(0)" enhances the fact that this state-space description holds before the occurrence of the fault.

$L^{(0)}$ denotes the observer gain matrix to be designed. The pair $(A^{(0)}, C^{(0)})$ is completely observable as

$$\det \begin{bmatrix} 0 & 1 \\ 1 & -14 \end{bmatrix} = -1$$

We let

$$F = A^{(0)} - L^{(0)}C^{(0)} = \begin{bmatrix} 0 & -14 - l_1 \\ 1 & -14 - l_2 \end{bmatrix} \quad \text{where} \quad L^{(0)} = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}$$

After some algebra, we obtain that by selecting

$$L^{(0)} = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$$

we have $\det(sI - F) = s^2 + 8s + 16$ and hence $\lambda_1 = -4, \lambda_2 = -4$.

Finally

$$e(t) = \mathscr{L}^{-1}[(sI - F)^{-1}]\bar{e} = \mathscr{L}^{-1} \begin{bmatrix} \dfrac{s+8}{(s+4)^2} & -\dfrac{16}{(s+4)^2} \\ \dfrac{1}{(s+4)^2} & \dfrac{s+8}{(s+4)^2} \end{bmatrix} \begin{bmatrix} \bar{e}_1 \\ \bar{e}_2 \end{bmatrix}$$

and hence, after some algebra and computing the inverse Laplace transforms, we get

$$\begin{aligned}\varepsilon(t) &= Ce(t) = \mathscr{L}^{-1}\left[\frac{1}{(s+4)^2}\right]\tilde{e}_1 + \mathscr{L}^{-1}\left[\frac{s}{(s+4)^2}\right]\tilde{e}_2 \\ &= (\tilde{e}_1 - 4\tilde{e}_2)te^{-4t} + \tilde{e}_2 e^{-4t}\,,\ \forall t \in (0, T_0)\end{aligned}$$

[7 marks]

b) Now, one considers the presence of the actuator fault of the form given in equation (4.1) in the text of the exam paper, that is:

$$f(t) = Ke^{\alpha t}, \forall t \geq T_0$$

where $K > 0$ and $\alpha > 0$ are positive scalars.

i) The actuator fault $f(t)$ can be generated as follows:

$$\begin{cases} \dot{\phi}(t) = -\alpha\phi(t) \\ f(t) = \phi(t) \end{cases}$$

with $\phi(T_0^-) = K$. Consider the state equation determined in the Answer to Question 4a). Introducing the augmented state vector

$$x_a := \begin{bmatrix} x_1 \\ x_2 \\ \phi \end{bmatrix}$$

the following augmented state equations can be written:

$$\begin{cases} \dot{x}_1 = -14x_2 + \phi + 10u \\ \dot{x}_2 = x_1 - 14x_2 + \phi + 10u \\ \dot{\phi} = -\alpha\phi \\ y = x_2 \end{cases}$$

and in matrix form:

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & -14 & 1 \\ 1 & -14 & 1 \\ 0 & 0 & -\alpha \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \phi \end{bmatrix} + \begin{bmatrix} 10 \\ 10 \\ 0 \end{bmatrix} u \\ \\ y = [0\ 1\ 0] \begin{bmatrix} x_1 \\ x_2 \\ \phi \end{bmatrix} \end{cases}$$

Therefore, owing to the fact that the scalar α is known, if a full-order Luenberger observer can be designed for the augmented system, an estimate $\widehat{f}(t)$ of the actuator fault can be provided by such an observer thus enabling the construction of a fault estimation scheme for this specific kind of faults.

[7 marks]

ii) After some easy algebra, the observability matrix is given by

$$Q = \begin{bmatrix} C_a \\ C_aA_a \\ C_aA_a^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -14 & 1 \\ -14 & 182 & -13-\alpha \end{bmatrix}$$

where A_a and C_a are the state and output matrices of the augmented system determined in the answer to Question 4b)-i), respectively.

It can be easily seen that

$$\det Q \neq 0 \quad \text{if and only if} \quad \alpha \neq 1$$

hence concluding that the Luenberger observer mentioned in the answer to Question 4b)-i) cannot be designed when $\alpha = 1$.

[3 marks]