UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 2001

MSci Honours Degree in Mathematics and Computer Science Part IV
MEng Honours Degrees in Computing Part IV
MSc in Advanced Computing
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the Associateship of the City and Guilds of London Institute
This paper is also taken for the relevant examinations for the Associateship of the Royal College of Science

PAPER C477=I4.20

COMPUTING FOR OPTIMAL DECISIONS

Friday 18 May 2001, 10:00
Duration: 120 minutes

Answer THREE questions

Paper contains 4 questions
Calculators not required

1 a An electricity generating company needs to determine its pricing structure according to demand during peak and off-peak times. The price and peak/off-peak demand relations are as follows:

	price(£)	electricity demand (kwh)
peak-time	$p^{(1)}$/kwh	$(80 - 0.7\, p^{(1)})$
off-peak time	$p^{(2)}$/kwh	$(50 - p^{(2)})$

where kwh denotes kilowatt-hour. The company must have sufficient capacity to meet demand during both peak and off-peak times and capacity must not be less than 5 kwh at any time. It costs £12 to maintain each kwh capacity during the period in question. The company wishes to maximise daily revenues, less operating costs. Formulate this problem. Do not solve the nonlinear programming problem. [Hint: In addition to $p^{(1)}$, $p^{(2)}$, introduce a variable for capacity.]

b A mobile telephone manufacturer produces brands A and B. If the charge per unit is £p^A/brand A and £p^B/brand B, it can sell q^A of A and q^B of B, where

$$q^A = 5500 - 10\, p^A + p^B; \quad q^B = 3700 - 9\, p^A + 0.8\, p^B.$$

The manufacturing requirements are

	labour (minutes)	chips
Brand A	20	3
Brand B	30	1

At present, 1000 hours of labour and 5000 chips are available. The company wants to maximise its revenue. Formulate this problem. Do not solve the nonlinear programming problem.

c The steepest descent algorithm for minimising the function f(x) generates a direction of search d at x_k, $k = 0, 1, 2, \ldots$, solving

$$\min_d \left\{ \nabla f(x_k)^T d \;\middle|\; \|d\|_2^2 = 1 \right\},$$

where $x, d \in \Re^n$, $\|d\|_2^2 \equiv d^T d$. The same solution can also be obtained by solving

$$\min_d \left\{ \nabla f(x_k)^T d + c\, \|d\|_2^2 \right\}$$

by an appropriate choice of $c > 0$. Evaluate d by solving both problems and establish the appropriate choice of c which would make the solution of the second identical to that of the first.

(All parts carry equal marks)

2 a Consider the problem

$$\min \left\{ x_1^4 + 2\, x_1^2 + 2\, x_1 x_2 + 4\, x_2^2 \,\middle|\, 2\, x_1 + x_2 = 10;\ x_1 + 2\, x_2 \geq 10;\ x_1,\, x_2 \geq 0 \right\}.$$

Write the Karush-Kuhn-Tucker optimality conditions for this problem.

b If the SUMT algorithm were to be applied directly to this problem, what would be the unconstrained function $\ell(x,\, \eta)$ to be minimised at each iteration? ($\eta \geq 0$ is the penalty parameter).

c Consider the Frank-Wolfe algorithm for solving the nonlinear program

$$\min_{x} \left\{ f(x) \;\middle|\; A\, x \leq b \right\},$$

$x \in \Re^n$, $b \in \Re^m$, using the following linear programming (LP) subproblem at every iterate x_k, $A\, x_k \leq b$,

$$\min_{x} \left\{ \nabla f(x_k)^T x \;\middle|\; A\, x \leq b \right\}.$$

Let x_{LP} denote the solution of this LP. Suggest an amendment to the above subproblem that will ensure the following descent condition

$$\nabla f(x_k)^T (x_{QP} - x_k) \leq -\, c\, (x_{QP} - x_k)^T (x_{LP} - x_k),$$

where x_{QP} denotes the solution of the amended subproblem and c a nonnegative scalar ($\infty \gg c > 0$). Establish this condition for the suggested amendment.

[Hint: as a result of this amendment, the LP subproblem should be replaced by a quadratic program.]

(All parts carry equal marks)

3 a Consider the convex function g(x). Show that, for any constant value c, the set

$$\left\{ x \;\middle|\; g(x) = c \right\}$$

is not necessarily convex, unless g is linear.

b Show that if $h^i(x)$, i = 1, ..., m, are convex functions, the set

$$\left\{ x \;\middle|\; h^i(x) \leq 0, i = 1, ..., m \right\}$$

is convex.

c Let f(x) be a convex function defined over a convex set $\mathfrak{C}$. Show that any local minimum of f(x) is also a global minimum.

(All parts carry equal marks)

4 a We are considering investing in three stocks. The random variable s^i represents the annual return on £1 invested in stock i. We are given the expected value of these returns to be $E(s^1) = .15$, $E(s^2) = .21$, $E(s^3) = .09$. The uncertainties are as follows: $\text{var}(s^1) = .09$, $\text{var}(s^2) = .04$, $\text{var}(s^3) = .01$, $\text{cov}(s^1 s^2) = .06$, $\text{cov}(s^1 s^3) = -.04$, $\text{cov}(s^2 s^3) = .05$. We have £100 to invest and wish to have an expected return of at least 15% on our portfolio at the end of the year. Formulate a quadratic programming problem to find the portfolio of minimum variance that attains an expected return of at least 15%.

b Consider the positive definite quadratic programming problem

$$\min \left\{ a^T x + \tfrac{1}{2} x^T \mathfrak{Q} x \;\middle|\; \mathfrak{H}^T x = h \right\}$$

Where x, a and h are vectors, $\mathfrak{Q}$ is a symmetric positive definite matrix, $\mathfrak{H}$ has linearly independent columns and T denotes the transpose of a vector, or a matrix. Derive the optimal solution and the corresponding multipliers to this problem. If you can think of more than one way of solving this, use only one method.

(All parts carry equal marks)