MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

2002 - 2003

Please write on this side only, legibly and neatly, between the margins

PAPER ISE 1.6

QUESTION

SOLUTION 1

(i) (a) $\frac{1+i}{7-i} = \frac{(1+i)(7+i)}{7^2+1^2}$

$= \frac{3}{25} + \frac{4}{25} i$ 2

(b) $(1+3i)^3 = 1 + 9i - 27 - 27i$

$= -26 - 18i$ 2

(c) $(1-i)^{17} = \left[\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)\right]^{17}$

$= 2^{17/2}\left(\cos\left(-\frac{17\pi}{4}\right) + i\sin\left(-\frac{17\pi}{4}\right)\right)$ 4

$= 2^8\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$

$= 2^8(1-i)$.

(ii)
(a) The equation means that z is equidistant from ± 1; so it represents the Y-axis. 3

(b) $\mathrm{Re}(z^3) = \mathrm{Re}(z) \iff x^3 - 3xy^2 = x$

$\iff x(x^2 - 3y^2 - 1) = 0$. So the 4

Setter : WILSON
Setter's signature : J. Wilson

Checker : LIEBECK
Checker's signature : MWL

figure is the Y-axis together with the hyperbola $x^2 - 3y^2 = 1$.

(iii) (a) $\sinh z = 0 \iff e^{z} - e^{-z} = 0$

$\iff e^{2z} = 1 \iff 2z = \ln 1$. 3

So the solutions are $z = i\pi n$

(where n is an integer).

(b) $\sinh z + \cosh z = 0 \iff$

$e^{z} - e^{-z} + e^{z} + e^{-z} = 0$ 2

$\iff e^{z} = 0$.

So there are no solutions.

Setter : WILSON   Setter's signature : J. Wilson

Checker : LIEBECK   Checker's signature : MWL

(1) (a) $\dfrac{(x+2)^{1/2}-2}{x-2} = \dfrac{\left[(x+2)^{\frac{1}{2}}-2\right]\left[(x+2)^{\frac{1}{2}}+2\right]}{(x-2)\left((x+2)^{\frac{1}{2}}+2\right)}$

$= \dfrac{x-2}{(x-2)\left((x+2)^{\frac{1}{2}}+2\right)} = \dfrac{1}{(x+2)^{\frac{1}{2}}+2}.$ 4

As $x \to 2$, this $\to \dfrac{1}{2+2} = \dfrac{1}{4}$.

[Aliter: use l'Hôpital's rule]

(b) $|x \sin(\tan x)| \le |x|$, so the limit as $x \to 0$ is 0. 2

(c) $x^{-9}\left\{(x+3)^{10} - (x+1)^{10}\right\} =$

$x^{-9}\left\{x^{10} + 30x^9 + \dots - (x^{10} + 10x^9 + \dots)\right\}$

$= x^{-9}\left\{20x^9 + \text{lower terms}\right\}$ 5

$= 20 + \left(\text{terms in powers of } \frac{1}{x}\right)$

So the limit as $x \to \infty$ is 20.

Setter : WILSON Setter's signature : J. Wilson

Checker : LIEBECK Checker's signature : MWL

(ii) (a) let $y = \ln\left\{x + (1+x^2)^{\frac{1}{2}}\right\}$;

then $$\frac{dy}{dx} = \frac{1 + x(1+x^2)^{-\frac{1}{2}}}{x + (1+x^2)^{\frac{1}{2}}}$$

$$= \frac{(1+x^2)^{-\frac{1}{2}}\left[(1+x^2)^{\frac{1}{2}} + x\right]}{x + (1+x^2)^{\frac{1}{2}}}$$

$$= (1+x^2)^{-\frac{1}{2}}$$

4

(b) Let $y = (\sin x)^x$, then

$\ln y = x \ln(\sin x)$, so

$$\frac{1}{y}\frac{dy}{dx} = \ln(\sin x) + x\frac{\cos x}{\sin x}.$$

5

Hence $$\frac{dy}{dx} = (\sin x)^x\left\{\ln(\sin x) + x\cot x\right\}.$$

MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

2002 - 2003

Please write on this side only, legibly and neatly, between the margins

PAPER ISE 1.6

QUESTION

SOLUTION 3

(i) (a) $\dfrac{|u_{n+1}|}{|u_n|} = \dfrac{2^{n+1}\, n^7}{(n+1)^7\, 2^n} = \dfrac{2}{(1+\frac{1}{n})^7} \to 2 > 1$ as $n \to \infty$. By the ratio test, the series diverges. 3

(b) $u_n = \dfrac{1+\frac{1}{n}}{10+\frac{1}{n}} \to \dfrac{1}{10}$. Since $u_n \not\to 0$, the series diverges 3

(c) $\dfrac{|u_{n+1}|}{|u_n|} = \dfrac{e^{n+1}\, n!}{(n+1)!\, e^n} = \dfrac{e}{n+1}$, which $\to 0 < 1$ as $n \to \infty$. By the ratio test, the series converges. 3

(ii)(a) $\dfrac{|u_{n+1}|}{|u_n|} = \dfrac{(n+1)^3 |x|^{n+1}}{n^3 |x|^n} = (1+\frac{1}{n})^3 |x|$, which $\to |x|$ as $n \to \infty$. So by the ratio test $R = 1$. 3

Setter : WILSON

Setter's signature : J Wilson

Checker : LIEBECK

Checker's signature : MWL

PAPER ISE 1.6

QUESTION

SOLUTION 3 (cont)

(ii) $$\frac{|u_{n+1}|}{|u_n|} = \frac{(n+1)!\,(n+2)!\,|x|^{n+1}\,(2n+1)!}{(2n+3)!\;n!\;(n+1)!\;|x|^n}$$

$$= \frac{(n+2)(n+1)}{(2n+3)(2n+2)}|x| \to \frac{1}{4}|x| \text{ as } n \to \infty.$$ 4

So $R = 4$.

(iii) $$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots$$

$$\therefore \ln\frac{1+x}{1-x} = 2\left\{x + \frac{x^3}{3} + \dots\right\},$$ 4

and $$\ln\sqrt{\frac{1+x}{1-x}} = x + \frac{x^3}{3} + \frac{x^5}{5} + \dots$$

Setter : WILSON Setter's signature : J. Wilson

Checker : LIEBECK Checker's signature : MWL

PAPER ISE 1.6

QUESTION

SOLUTION 4

(i) Set $t = \tan\frac{x}{2}$; then

$$\int \frac{dx}{\sin x} = \int \frac{2\,dt/(1+t^2)}{2t/(1+t^2)} = \int \frac{dt}{t}$$

$$= \ln|t| + c = \ln\left|\tan\frac{x}{2}\right| + c,$$ 3

(ii) Set $u = x^2$, $du = 2x\,dx$. Then

$$\int \frac{x\,dx}{(1-x^2)^{3/2}} = \int (1-u)^{-3/2} \cdot \frac{1}{2}\,du$$

$$= (1-u)^{-\frac{1}{2}} + c = \frac{1}{\sqrt{1-x^2}} + c.$$ 4

(iii) Set $x = \sin u$, $dx = \cos u\,du$. Then

$$\int \frac{x^2\,dx}{(1-x^2)^{3/2}} = \int \frac{\sin^2 u \cos u\,du}{\cos^3 u}$$

$$= \int \tan^2 u\,du = \int (\sec^2 u - 1)\,du$$

$$= \tan u - u + c = \frac{x}{\sqrt{1-x^2}} - \sin^{-1} x + c.$$ 5

Setter: WILSON Setter's signature: J. Wilson

Checker: LIEBECK Checker's signature: MWL

(iv) Let $\dfrac{2x}{(x+1)(x^2+1)} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+1}$.

Then $2x = A(x^2+1) + (Bx+C)(x+1)$

$= (A+B)x^2 + (B+C)x + A+C,$

So $A+B=0$, $B+C=2$, $A+C=0$,

whence $A=-1$, $B=C=1$. 4

$\therefore \displaystyle\int \frac{2x\,dx}{(x+1)(x^2+1)} = \int \left(\frac{-1}{x+1} + \frac{x+1}{x^2+1} \right) dx$

$= -\ln|x+1| + \frac{1}{2}\ln|x^2+1| + \tan^{-1}x + c.$ 4

Setter : WILSON

Setter's signature : J. Wilson

Checker : LIEBECK

Checker's signature : MWL

(i) Separable : $\dfrac{dy}{1+y^2} = (1+x^2)\,dx$

$\therefore \tan^{-1} y = x + \dfrac{x^3}{3} + c$, i.e. 5

$y = \tan\left(x + \dfrac{x^3}{3} + c\right)$.

(ii) Linear with IF $e^{\int \frac{dx}{x}} = x$.

$x y' + y = x \sin x$, i.e.

$(xy)' = x \sin x$.

$\therefore xy = \int x \sin x \, dx = -x\cos x + \int \cos x \, dx$ 5

$= -x\cos x + \sin x + c$. So

$y = -\cos x + x^{-1}(\sin x + c)$.

(iii) The auxiliary equation is $r^2 + 2r - 3 = 0$,

i.e. $(r+3)(r-1) = 0$, so $r = 1$ or -3. 3

Thus the complementary function is

PAPER ISE1.6

QUESTION

SOLUTION 5 (cont.)

$c_1 e^x + c_2 e^{-3x}$. We seek a particular integral $y = Axe^x$, so $y' = A(x+1)e^x$, $y'' = A(x+2)e^x$, and $y'' + 3y' + y =$
$Ae^x(x + 2 + 2(x+1) - 3x) = 4Ae^x$.
To solve the given ODE we take $A = \frac{1}{4}$.
Thus the general solution is

$$y = c_1 e^x + c_2 e^{-3x} + \frac{1}{4} x e^x .$$

4

(iv)
Thus $y(0) = 0 \iff c_1 + c_2 = 0$.
And $y' = c_1 e^x - 3c_2 e^{-3x} + \frac{1}{4}(x+1)e^x$,
so $y'(0) = 0 \iff c_1 - 3c_2 + \frac{1}{4} = 0$.
To satisfy both conditions we need
$c_2 = -c_1 = \frac{1}{16}$, giving

$$y = -\frac{1}{16} e^x + \frac{1}{16} e^{-3x} + \frac{1}{4} x e^x .$$

3

Setter : WILSON — Setter's signature : J. Wilson
Checker : LIEBECK — Checker's signature : MWL

(i) Char. eqn. is $\begin{vmatrix} -10-\lambda & 9 \\ -18 & 17-\lambda \end{vmatrix} = 0$, i.e. $\lambda^2 - 7\lambda - 8 = 0$

3 roots $\underline{\lambda = -1, 8}$.

2 $\underline{\lambda = -1}$ Evectors are solns of $\begin{pmatrix} -9 & 9 \\ -18 & 18 \end{pmatrix} x = 0$ i.e. $\underline{a\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$ $(a \neq 0)$

2 $\underline{\lambda = 8}$ Evectors are solns of $\begin{pmatrix} -18 & 9 \\ -18 & 9 \end{pmatrix} x = 0$ i.e. $\underline{b\begin{pmatrix} 1 \\ 2 \end{pmatrix}}$ $(b \neq 0)$

5 (ii) Let $P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. Then $P^{-1}AP = \begin{pmatrix} -1 & 0 \\ 0 & 8 \end{pmatrix}$ diagonal.

(iii) Write $D = \begin{pmatrix} -1 & 0 \\ 0 & 8 \end{pmatrix}$, so $A = PDP^{-1}$.

If $C = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$ then $C^3 = D$, so

$$(PCP^{-1})^3 = PC\underbrace{P^{-1}.P}C\underbrace{P^{-1}.P}CP^{-1} = PC^3P^{-1}$$

$$= PDP^{-1}\cancel{8} = A.$$

So take

$$B = PCP^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

8
$$= \begin{pmatrix} -1 & 2 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -4 & 3 \\ -6 & 5 \end{pmatrix}.$$

MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

2002 - 2003

Please write on this side only, legibly and neatly, between the margins

PAPER ISE 1

QUESTION

SOLUTION 7

(i) $f_x = x^2+y^2-2+2x(x+y) = 3x^2+2xy+y^2-2$

$f_y = x^2+y^2-2+2y(x+y) = x^2+2xy+3y^2-2$

At stat. pts, $f_x = f_y = 0$, so subtracting, $x^2-y^2=0$, $\underline{x=\pm y}$

If $\underline{x=y}$, $6x^2=2 \rightarrow x = \pm\frac{1}{\sqrt{3}}$.

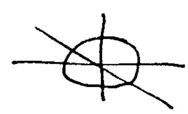
If $\underline{x=-y}$, $2x^2=2 \rightarrow x=\pm 1$.

So stat. pt $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, $\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$, $(1,-1)$, $(-1,1)$. 8

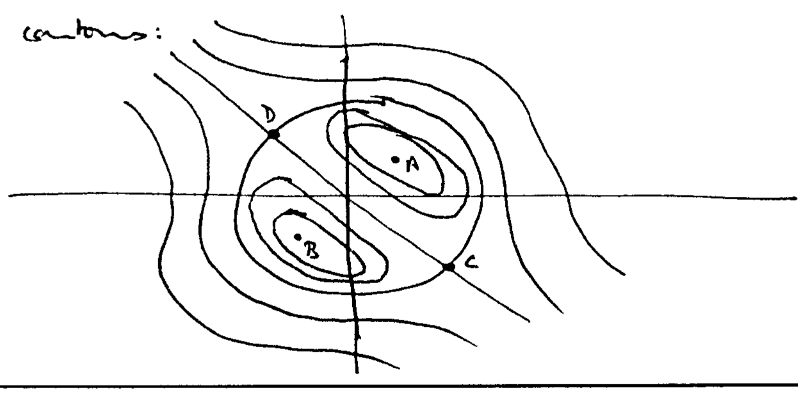
Nature: $f_{xx} = 6x+2y$, $f_{yy} = 2x+6y$, $f_{xy} = 2x+2y$.

At stat. pts

		f_{xx}	f_{yy}	f_{xy}	$D = f_{xx}f_{yy} - f_{xy}^2$		
A	$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$	$\frac{8}{\sqrt{3}}$	$\frac{8}{\sqrt{3}}$	$\frac{4}{\sqrt{3}}$	>0	$\underline{\text{MIN}}$	2
B	$\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$	$-\frac{8}{\sqrt{3}}$	$-\frac{8}{\sqrt{3}}$	$-\frac{4}{\sqrt{3}}$	>0	$\underline{\text{MAX}}$	2
C	$(1,-1)$	4	-4	0	<0	$\underline{\text{SADDLE}}$	3
D	$(-1,1)$	-4	4	0	<0	$\underline{\text{SADDLE}}$	

(ii) Contour $f=0$ is (line $y=-x$) $\cup$ (circle $x^2+y^2=2$) :  2

(iii) Further contours:



3

Setter : LIEBECK

Setter's signature : MLL

Checker : WILSON

Checker's signature : J.W.

MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

2002 - 2003

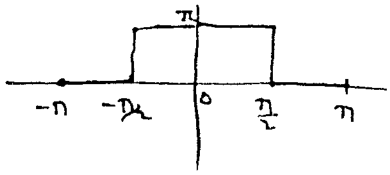
Please write on this side only, legibly and neatly, between the margins

PAPER ISE 1

QUESTION

SOLUTION 8

(a) Cosine series is for even function:



2

Series is $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ where

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\cdot\frac{\pi^2}{2} = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \pi \cos nx\,dx = 2\left[\frac{\sin nx}{n}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{2}{n}\sin\frac{n\pi}{2}$$

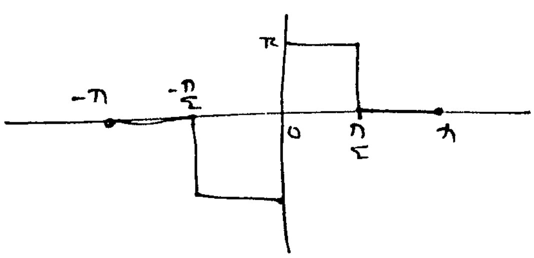
$$= \begin{cases} 0, & n \text{ even} \\ \frac{2}{2m+1}\cdot(-1)^m, & n = 2m+1 \text{ odd.} \end{cases}$$

So cosine series is

$$\frac{\pi}{2} + 2\sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1}\cos(2m+1)x.$$

6

(b) Sine series is for odd function:



2

Setter : LIEBECK

Setter's signature : MLL

Checker : WILSON

Checker's signature : J. W.

Series $\sum_{n=1}^{\infty} b_n \sin nx$ where

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi/2} \pi \sin nx\,dx = 2\left[-\frac{\cos nx}{n}\right]_0^{\pi/2}$$

$$= \frac{2}{n}\left(1-\cos\frac{n\pi}{2}\right)$$

$$= \begin{cases} \frac{2}{n}, & n \text{ odd} \\ \frac{1}{m}(1-(-1)^m), & n = 2m \text{ even.} \end{cases}$$

So sine series is

$$2\sum_{m=1}^{\infty} \frac{\sin(2m+1)x}{2m+1} + 2\sum_{m=0}^{\infty} \frac{\sin(4m+2)x}{2m+1}$$ 6

Putting $x = 0$ in the cosine series, we have

$$\pi = \frac{\pi}{2} + 2\sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1}$$

hence 4

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} = \frac{\pi}{4}.$$

MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

2002 - 2003

Please write on this side only, legibly and neatly, between the margins

PAPER ISE 1

QUESTION

SOLUTION 9

1 Sketch :

Note
$x \to t$
$t \to s$
in accordance with Homer Style.

3 $$L(H_0 - H_1) = \int_0^1 e^{-tx}\,dx = \frac{1}{t}(1 - e^{-t}).$$

Taking Laplace transforms,

$$-y'(0) - ty(0) + t^2 L(y) + 2\big(-y(0) + tL(y)\big) + L(y)$$

$$= \frac{1}{t}(1 - e^{-t})$$

$$\Rightarrow \quad (t^2 + 2t + 1)\,L(y) = \frac{1}{t}(1 - e^{-t})$$

6 $$\Rightarrow \quad L(y) = \frac{1}{t(t+1)^2} - \frac{e^{-t}}{t(t+1)^2}.$$

3 Partial fractions : $$\frac{1}{t(t+1)^2} = \frac{1}{t} - \frac{1}{t+1} - \frac{1}{(t+1)^2}$$

So

$$y = L^{-1}\left(\frac{1}{t} - \frac{1}{t+1} - \frac{1}{(t+1)^2} - \frac{e^{-t}}{t} + \frac{e^{-t}}{t+1} + \frac{e^{-t}}{(t+1)^2}\right)$$

$$= 1 - e^{-x} - xe^{-x} - H_1(x) + H_1(x)e^{-(x-1)} + H_1(x)(x-1)e^{-(x-1)}$$

7 $$= 1 - e^{-x} - xe^{-x} + H_1(x)\left(-1 + e^{-x+1} + (x-1)e^{-x+1}\right)$$

$$= 1 - e^{-x} - xe^{-x} - H_1(x) + H_1(x).\; xe^{1-x}$$

Setter : LIEBECK Setter's signature : MLL

Checker : WILSON Checker's signature : J.W.