Imperial College London

M3P6

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May-June 2019

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Probability

Date: Friday 24 May 2019

Time: 14.00 - 16.00

Time Allowed: 2 Hours

This paper has 4 Questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the relevant main book(s) are full.

All required additional material will be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Calculators may not be used.

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M4/5P6

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Probability

Date: Friday 24 May 2019

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Time Allowed: 2 Hours 30 Minutes

This paper has 5 Questions.

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1. (1.a) Give the definition of a random variable on a probability space and the corresponding probability distribution explaining carefully all notions involved.

(1.b) Prove that the distribution function is continuous from the right.

(1.c) Let λ denote the Lebesgue measure on $(\mathbb{R}, \Sigma_L)$, where Σ_L denotes the σ-algebra of Lebesgue measurable sets. Let $d\mu = e^{-2|x|}d\lambda$ be a probability measure on $(\mathbb{R}, \Sigma_L)$.

(i) Find the distribution function of the random variable

$$X \equiv -\frac{1}{2}\chi_{\mathbb{R}^-} + \frac{1}{2}\chi_{\mathbb{R}^+}$$

where χ_A denotes the characteristic function of a set A.

(ii) Let F_{γ_0} denote the distribution function of the Gaussian $N(0,1)$ random variable. Let η denote the distribution functions of the variable given by $Z(x) = x$ on $(\mathbb{R}, \Sigma_L, \mu)$. Let $m \in \mathbb{R}$ and $\sigma \in (0, \infty)$. Find the distribution function of the random variable

$$X \equiv \frac{1}{\sigma}(F_{\gamma_0}^{-1} \circ \eta(Z)) - m$$

on (Ω, Σ, μ).

2. (2.a) Give the definition of two mutually independent random variables, explaining carefully all notions involved.

(2.b) Decide which of the following random variables are mutually independent and give a brief justification :

(i) Gaussian random variables $\{X_j\}_{j\in\mathbb{N}}$ with diagonal covariance matrix ?

(ii) Monomials on the interval $[0,1]$ with Lebesgue measure ?

(2.c) Prove the Weak Law of Large Numbers for the family of Gaussian random variables $\{X_j\}_{j\in\mathbb{N}}$ on a probability space (Ω, Σ, μ) such that

$$E_\mu(X_j) = m, \qquad |Cov_\mu(X_j, X_k)| \le (1 + |j-k|)^{-\beta}$$

for $m \in \mathbb{R}$ and $\beta \in (1, \infty)$.

3. (3.a) (i) Let A_n , $n \in \mathbb{N}$, be a sequence of events in a probability space. Define the corresponding $\limsup_n A_n$ and $\liminf_n A_n$ events.

(ii) For a family of mutually independent random variables X_n, $n \in \mathbb{N}$, on a probability space (Ω, Σ, μ), define the tail σ-algebra Σ_∞.

(3.b) Let Z_k, $k \in \mathbb{N}$, be a family of mutually independent random variables on a probability space (Ω, Σ, μ). Let $A_n \equiv \bigcap_{k=n,..,n+L(n)}\{Z_k^{-1}([a_k, b_k)\}$, for $a_k, b_k \in \mathbb{R}$, $a_k < b_k$.

Prove or disprove that

$$\mu(\limsup_n A_n) = \frac{1}{2}.$$

(3.c) Let $(\Omega, \Sigma, \mu) \equiv (\mathbb{R}, \Sigma_L, \nu_0)^{\mathbb{N}}$, with the student's distribution on the real line

$$d\nu_0 \equiv \frac{\Gamma(\frac{1+\alpha}{2})}{\sqrt{\alpha\pi}\Gamma(\frac{\alpha}{2})} \left(1 + \frac{x^2}{\alpha}\right)^{-\frac{1+\alpha}{2}} \lambda(dx),$$

defined with $\alpha \in (0, \infty)$. For $j \in \mathbb{N}$, let $\Omega \ni \omega \mapsto \pi_j(\omega) := \omega_j$ and set

$$X_j := \pi_j \pi_{j+1}.$$

Prove that for $\alpha > 2$ the following quantity

$$S_n \equiv \frac{1}{n} \sum_{j=1}^{n} X_j$$

converges to zero almost everywhere.

4. (4.a) Give a definition of convergence in distribution.

(4.b) State and prove the Central Limit Theorem for the Rademacher functions R_k on the unit interval with the Lebesgue measure.

(4.c) With the setup as in (4.b), prove that for any bounded continuous real function f we have

$$\int f^2 \left(\frac{1}{\sqrt{n}} \sum_{k=1,..,n} R_k \right) d\lambda - \left(\int f \left(\frac{1}{\sqrt{n}} \sum_{k=1,..,n} R_k \right) d\lambda \right)^2 \to_{n\to\infty} E_\gamma \left(f(X) - E_\gamma(f(X))\right)^2$$

for any random variable X with a Gaussian distribution γ.

5. (5.a) State the Birkhoff Ergodic Theorem explaining carefully all notions involved.

(5.b) Prove that the Birkhoff Ergodic Theorem implies the Strong Law of Large numbers for i.i.d. random variables.

(5.c) Let $(\Omega, \Sigma, \mu) \equiv (\mathbb{R}, \Sigma_L, \nu_0)^{\mathbb{N}}$, where ν_0 is the student's distribution on the real line given by

$$d\nu_0 \equiv \frac{\Gamma(\frac{1+\alpha}{2})}{\sqrt{\alpha\pi}\Gamma(\frac{\alpha}{2})} \left(1 + \frac{x^2}{\alpha}\right)^{-\frac{1+\alpha}{2}} \lambda(dx),$$

with $\alpha \in (1, \infty)$. Let $m_1 \equiv \int |x| d\nu_0$.

With $\pi_i(\omega) := \omega_i$, $i \in \mathbb{N}$, for $\varepsilon \in (0, \frac{1}{m_1})$ define

$$f_k(\omega) = \sum_{n \in \mathbb{N}} \varepsilon^n \prod_{j=1}^{n} \pi_{j+k}$$

Using the ergodic theorem prove that

$$\frac{1}{n} \sum_{k=1}^{n} f_k \to_{n \to \infty} 0$$

almost everywhere and in $\mathbb{L}_1$.

SOLUTIONS MP6/2019

1. (1.a) Let (Ω, Σ, μ) be a probability space, i.e. a triple consisting of a nonempty set Ω, a σ-algebra Σ which by definition is a family of subsets of Ω containing this set and closed with respect of the operations of taking complement and countable unions, and a function $\mu : \Sigma \to [0, 1]$ such that $\mu(\Omega) = 1$ and for any sequence $(A_n \in \Sigma)_{n\in\mathbb{N}}$ of pairwise disjoint sets we have

$\boxed{5 \text{ pts}}$ $\boxed{seen}$

$$\mu(\bigcup_{n\in\mathbb{N}} A_n) = \sum_{n\in\mathbb{N}} \mu(A_n).$$

Let $\mathcal{B}_{\mathbb{R}}$ denote the Borel σ-algebra in $\mathbb{R}$.
A function $X : \Omega \to \mathbb{R}$ is called a random variable iff

$$\forall B \in \mathcal{B}_{\mathbb{R}} \quad X^{-1}(B) \in \Sigma.$$

A function

$$F_X(z) \equiv \mu\{X^{-1}((-\infty, z])\}$$

is called the distribution function of the random variable X.

$\boxed{6 \text{ pts}}$ $\boxed{seen}$

(1.b) We need to show that for any $z \in \mathbb{R}$

$$\lim_{0<\delta\to 0} F_X(z + \delta) = F_X(z)$$

or equivalently for any monotone sequence $0 < \delta_n \to_{n\to\infty} 0$

$$lim_{n\to\infty}\mu\{X^{-1}((-\infty, z + \delta_n])\} = \mu\{X^{-1}((-\infty, z])\}.$$

We have for $\delta_n > \delta_{n+1}$

$$X^{-1}((-\infty, z + \delta_{n+1}]) \subset X^{-1}((-\infty, z + \delta_n])$$

so the sets $X^{-1}((z + \delta_{k+1}, z + \delta_k])$ are pairwise disjoint and we have

$$\begin{aligned} X^{-1}((-\infty, z + \delta_n]) &= \bigcup_{k=n}^{\infty} X^{-1}((z + \delta_{k+1}, z + \delta_k]) \cup \bigcap_{m\in\mathbb{N}} X^{-1}((-\infty, z + \delta_m]) \\ &= \bigcup_{k=n}^{\infty} X^{-1}((z + \delta_{k+1}, z + \delta_k]) \cup X^{-1}((-\infty, z]) \end{aligned}$$

Hence by σ-additivity of the measure we get

$$\mu(X^{-1}((-\infty, z + \delta_n])) = \sum_{k=n}^{\infty} \mu(X^{-1}((z + \delta_{k+1}, z + \delta_k])) + \mu(X^{-1}((-\infty, z])).$$

Since the series on the right hand side is convergent, this implies the desired property

$$lim_{n\to\infty}\mu\{X^{-1}((-\infty, z + \delta_n])\} = \mu\{X^{-1}((-\infty, z])\}.$$

(1.c) (i) Let $\boxed{\textbf{4 pts}}$ $\boxed{\textit{unseen}}$

$$X \equiv -\frac{1}{2}\chi_{\mathbb{R}^-} + \frac{1}{2}\chi_{\mathbb{R}^+}$$

Then we have

$$\{-\frac{1}{2}\chi_{\mathbb{R}^-} + \frac{1}{2}\chi_{\mathbb{R}^+} \leq t\} = \begin{cases} \emptyset & if \quad t < -\frac{1}{2} \\ \mathbb{R}^- & for \quad -\frac{1}{2} \leq t < \frac{1}{2} \\ \mathbb{R} & for \quad \frac{1}{2} \leq t \end{cases}$$

Hence

$$\mu(X^{-1}(-\infty, t)) = \begin{cases} \mu(\emptyset) = 0 & if \quad t < -\frac{1}{2} \\ \mu(\mathbb{R}^-) = \frac{1}{2} & for \quad -\frac{1}{2} \leq t < \frac{1}{2} \\ \mu(\mathbb{R} = 1 & for \quad \frac{1}{2} \leq t \end{cases}$$

(ii) Let F_{γ_0} denotes the distribution function of the Gaussian $N(0,1)$ random variable. Let η denotes the distribution functions of the variable given by $Z(x) = x$ on $(\mathbb{R}, \Sigma_L, \mu)$. Define on (Ω, Σ, μ) a random variable $\boxed{\textbf{5 pts}}$ $\boxed{\textit{unseen}}$

$$X \equiv \sigma^{-1}(F_{\gamma_0}^{-1} \circ \eta(Z)) - m$$

Hence we have

$$F_X(t) \equiv \mu\left(\sigma^{-1}(F_{\gamma_0}^{-1} \circ \eta(Z)) - m \leq t\right) = \mu\left(x \leq \eta^{-1}(F_{\gamma_0}\sigma(t+m))\right)$$
$$= \eta(\eta^{-1}(F_{\gamma_0}\sigma(t+m))) = F_{\gamma_0}(\sigma(t+m)) = F_{\gamma_{\sigma,m}}(t)$$

the last being the distribution function of a Gaussian variable with variance σ^2 and mean value m.

2. (2.a) Two real valued random variables X and Y on a probability space (Ω, Σ, μ) are called mutually independent iff the σ-algebras $\Sigma_X \equiv X^{-1}(\mathcal{B}_{\mathbb{R}})$ and $\Sigma_Y \equiv Y^{-1}(\mathcal{B}_{\mathbb{R}})$ are mutually independent, i.e. $\boxed{\textbf{5 pts}}$ $\boxed{\textit{seen}}$

$$\forall A \in \Sigma_X, B \in \Sigma_Y \qquad \mu(A \cap B) = \mu(A) \cdot \mu(B).$$

(2.b) (i) If the Gaussian random variables $\{X_j\}_{j\in\mathbb{N}}$ have diagonal covariance matrix this implies that any finite product of monomials in variables $X_{i_1}, .., X_{i_n}$, $i_k \in \mathbb{N}$, $k = 1, .., n$, has expectation equal to the product of expectations of monomials in each particular random variable. This can be shown e.g. using integration by parts formula for Gaussian random variables. Once this is established, one can get similar result for all compactly supported continuous function (using Weierstrass approximation theorem proven in the course as an application of WLLN). $\boxed{\textbf{4 pts}}$ $\boxed{\textit{unseen}}$

(ii) Monomials on interval $[0,1]$ with Lebesgue measure are not mutually independent. For example for any $n, m \in \mathbb{N}$ by direct calculation we see that **3 pts** *unseen*

$$\int_{[0,1]} (x^n)^2 (x^m)^2 d\lambda \neq \int_{[0,1]} (x^n)^2 d\lambda \cdot \int_{[0,1]} (x^m)^2 d\lambda$$

(2.c) Let $\{X_j\}_{j\in\mathbb{N}}$ be the family of Gaussian random variables on a probability space (Ω, Σ, μ) such that **8 pts** *unseen*

$$E_\mu(X_j) = m, \qquad |Cov_\mu(X_j, X_k)| \leq (1 + |j-k|)^{-\beta}$$

for $m \in \mathbb{R}$ and $\beta \in (1, \infty)$. Define

$$s_n \equiv \frac{1}{n} \sum_{j=1,..,n} X_j.$$

For $\varepsilon > 0$ we have

$$\mu\{|s_n - m| \geq \varepsilon\} \leq \frac{1}{\varepsilon^2} E_\mu((s_n - m)^2)$$

and **3 pts**

$$E_\mu((s_n - m)^2) = \frac{1}{n^2} \sum_{i,j=1,..,n} E_\mu((X_j - m)(X_i - m)) \equiv \frac{1}{n^2} \sum_{i,j=1,..,n} C_{ij}.$$

From our assumption about the covariance matrix C_{ij}, we have

$$\sum_{i,j=1,..,n} C_{ij} \leq \sum_{i,j=1,..,n} (1 + |j-i|)^{-\beta} \leq nA$$

with a constant

$$A \equiv \sum_{k=0}^{\infty} (1 + |k|)^{-\beta} < \infty$$

for $\beta > 1$. Hence

$$E_\mu((s_n - m)^2) \leq \frac{A}{n}$$

which implies

$$\mu\{|s_n - m| \geq \varepsilon\} \leq \frac{A}{n\varepsilon^2},$$

that is $s_n \to_{n\to\infty} m$ in probability, i.e. the Weak Law of Large Numbers for the sequence of random variables in question is satisfied. **5 pts**

3. (3.a) $\boxed{\textbf{7 pts}}$ $\boxed{\textit{seen}}$

(i) Let (Ω, Σ, μ) be a probability space. For a sequence of events $A_n \in \Sigma$, $n \in \mathbb{N}$, we define $\boxed{\textbf{3 pts}}$

$$\limsup_n A_n := \bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} A_k$$

and

$$\liminf_n A_n := \bigcup_{n\in\mathbb{N}} \bigcap_{k\geq n} A_k$$

$\boxed{\textbf{4 pts}}$

(ii) Let X_n, $n \in \mathbb{N}$, be a sequence of mutually independent random variables. The tail sigma algebra associated to this sequence is by definition given by

$$\Sigma_\infty \equiv \bigcap_{n=1}^{\infty} \sigma(X_n, X_{n+1}, ...)$$

where $\sigma(X_n, X_{n+1}, ...)$ denotes the sigma algebra generated by X_j, $j \geq n$.

$\boxed{\textbf{5 pts}}$

(3.b) Let Z_k, $k \in \mathbb{N}$, be a family of mutually independent random variables on a probability space (Ω, Σ, μ). Let $A_n \equiv \bigcap_{k=n,..,n+L(n)}\{Z_k^{-1}([a_k, b_k))\}$, for $a_k, b_k \in \mathbb{R}$, $a_k < b_k$. $\boxed{\textit{unseen}}$
By definition of

$$\limsup_n A_n := \bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} A_k$$

and

$$\liminf_n A_n := \bigcup_{n\in\mathbb{N}} \bigcap_{k\geq n} A_k$$

Both of the events belong to the tail σ-algebra Σ_∞. This is because in particular for any $j \in \mathbb{N}$ we have

$$\limsup_n A_n \subset \bigcup_{k\geq j+1} A_k$$

which is independent of the smallest σ-algebra for which $Z_1, .., Z_j$ are measurable.
On the other hand each of the sets $\bigcap_{k\geq n} A_k \in \Sigma_\infty$, hence their countable union also belongs to Σ_∞.

For the last part of the question we note that Kolmogorov 0-1 law says that any event in Σ_∞ can have a probability equal either to zero or to one.
In particular we have

$$\mu(\limsup_n A_n) \in \{0, 1\}.$$

Hence the statement in the question is not true.

(3.c) Let $(\Omega, \Sigma, \mu) \equiv (\mathbb{R}, \Sigma_L, \nu_0)^{\mathbb{N}}$, with the student's distribution on the real line **8 pts** *unseen*

$$d\nu_0 \equiv C_\alpha \left(1 + \frac{x^2}{\alpha}\right)^{-\frac{1+\alpha}{2}} \lambda(dx),$$

defined with $\alpha \in (0, \infty)$ and the normalisation constant

$$C_\alpha \equiv \frac{\Gamma(\frac{1+\alpha}{2})}{\sqrt{\alpha\pi}\Gamma(\frac{\alpha}{2})}.$$

For $j \in \mathbb{N}$, let $\Omega \ni \omega \mapsto \pi_j(\omega) := \omega_j$ and set

$$X_j(\omega) := \pi_j(\omega)\pi_{j+1}(\omega) = \omega_j \cdot \omega_{j+1}.$$

For every $j \in \mathbb{N}$, using the definition of the product measure and symmetry of the measure ν_0, we have

$$E_\mu X_j = \int \omega_j \cdot \omega_{j+1} \nu_0(d\omega_j)\nu_0(d\omega_{j+1}) = 0$$

and

$$E_\mu |X_j| = \int |\omega_j| \cdot |\omega_{j+1}| \nu_0(d\omega_j)\nu_0(d\omega_{j+1})$$
$$= C_\alpha^2 \left(\int |x| \left(1 + \frac{x^2}{\alpha}\right)^{-\frac{1+\alpha}{2}} dx \right)^2 < \infty$$

for $\alpha > 1$.

The random variables X_j, $j \in \mathbb{N}$, are identically distributed, but not mutually independent, so we cannot apply directly the Strong Law of Large Numbers to the empirical mean **4 pts**

$$S_n \equiv \frac{1}{n} \sum_{j=1}^{n} X_j$$

However we note that

$$S_n = s_e(n) + s_0(n)$$

where

$$s_e(n) \equiv \frac{1}{n} \sum_{2k \leq n} X_{2k}$$

and

$$s_o(n) \equiv \frac{1}{n} \sum_{2k-1 \leq n} X_{2k-1}.$$

Each of the sums in s_e and s_o contains mutually independent random variable (with mean zero and finite first moment) and we can apply SLLN to each of them. This gives

$$s_e(n), s_0(n) \rightarrow_{n \to \infty} 0$$

almost everywhere. Hence also

$$S_n \rightarrow_{n \to \infty} 0$$

almost everywhere. **4 pts**

4. (4.a) A sequence of random variables $(X_n)_{n\in\mathbb{N}}$ with probability distributions F_n is said to converge to a random variable with the distribution function F iff $F_n(x) \to_{n\in\mathbb{N}} F(x)$ at every point x at which F is continuous. $\boxed{5 \text{ pts}}$ $\boxed{\textit{seen}}$

$\boxed{7 \text{ pts}}$

(4.b) Theorem : (Central Limit Theorem for the Rademacher functions) $\boxed{\textit{unseen}}$

Let R_k, $k \in \mathbb{N}$, be the Rademacher functions on $([0,1], \Sigma_L \cap [0,1], \lambda)$. Then

$$\frac{1}{\sqrt{n}} \sum_{k=1}^{n} R_k$$

converges in distribution to the Gaussian random variable with mean zero and variance equal to 1. $\boxed{3 \text{ pts}}$

Proof: First we recall that R_k's are mutually independent with the same symmetric Bernoulli distribution $\lambda(R_k = \pm 1) = \frac{1}{2}$. Hence, for

$$G_n \equiv \frac{1}{\sqrt{n}} \sum_{k=1,..,n} R_k$$

has a characteristic function

$$\varphi_{G_n}(t) \equiv \left(\cos(t/\sqrt{n})\right)^n$$

for which we have

$$\left(\cos(t/\sqrt{n})\right)^n = \left(1 - \frac{t^2}{n} + \mathcal{O}(n^{-3/2})\right)^n \to_{n\to\infty} e^{-\frac{t^2}{2}}$$

which is the characteristic function of the Gaussian random variable with mean zero and variance one. Hence by Levy continuity theorem we have also the convergence of the corresponding distributions. $\boxed{4 \text{ pts}}$

$\boxed{8 \text{ pts}}$

(4.c) It is enough to show that for any bounded continuous function g, we have $\boxed{\textit{unseen}}$

$$\int g\left(\frac{1}{\sqrt{n}} \sum_{k=1,..,n} R_k\right) d\lambda \to_{n\to\infty} E_\gamma g(X)$$

where the r.v. X has the standard Gaussian distribution γ. Under the current conditions the limiting Gaussian distribution is continuous and the central limit theorem implies pointwise convergence of distributions. Hence the convergence of integrals follows using the definition of Lebesgue integral which for a bounded continuous function g of a random variable Y is approximated by $\boxed{3 \text{ pts}}$

$$\sum_k g(\varepsilon_k)\left(F_Y(\varepsilon_{k+1}) - F_Y(\varepsilon_k)\right).$$

$\boxed{5 \text{ pts}}$

5. (5.a) Let (Ω, Σ, μ) be a probability space and let $T : \Omega \to \Omega$ be a measure preserving map on (Ω, Σ, μ), i.e. for any $A \in \Sigma$ $\boxed{\mathbf{5\ pts}}$ $\boxed{\textit{seen}}$

$$\mu(T^{-1}A) = \mu(A).$$

A set $A \in \Sigma$ is called T-invariant iff the symmetric difference of A and $T^{-1}A$ has probability zero. Let $\mathcal{I}$ be the σ-algebra of T-invariant sets and denote by $E_\mu(\cdot|\mathcal{I})$ the corresponding conditional expectation associated to the probability measure μ.

Birkhoff's Ergodic Theorem : Suppose $X \in \mathbb{L}_1(\Omega, \Sigma, \mu)$. Then

$$\frac{1}{n}\sum_{j=0}^{n} X(T^j\omega) \to_{n\to\infty} E_\mu(X|\mathcal{I})$$

almost surely and in $\mathbb{L}_1(\Omega, \Sigma, \mu)$.

(5.b) Birkhoff's Ergodic Theorem implies the Strong Law of Large numbers for i.i.d.s . $\boxed{\mathbf{7\ pts}}$ $\boxed{\textit{unseen}}$
Suppose $(X_k)_{k\in\mathbb{N}}$ be i.i.d. real valued random variables on a probability space (Ω, Σ, μ) which are integrable with respect to μ. Without loss of generality we can assume that they have mean zero. Let $(\mathbb{R}, \mathcal{B}, \mu_0)^{\mathbb{N}}$ be the product probability space defined with $\mu_0(A) := \mu(X_1^{-1}(A))$. Let

$$\mathbb{R}^{\mathbb{N}} \ni \omega \equiv (\omega_k \in \mathbb{R})_{k\in\mathbb{N}} \to \pi_j(\omega) \equiv \omega_j$$

and define a map $T : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ by

$$(T\omega)_j = \omega_{j+1}$$

Such map is measure preserving for the product measure $\mu_0^{\otimes\mathbb{N}}$ and for $j \in \mathbb{N}$ we have $\pi_{j+1} = \pi_j \circ T$. Moreover

$$E|X_j| = \int |\pi_j| d\mu_0^{\otimes\mathbb{N}} = \int |\omega_1| d\mu_0 < \infty,$$

i.e. each π_j is integrable. Applying Birkhoff's Ergodic Theorem to π_j on $(\mathbb{R}, \mathcal{B}, \mu_0)^{\mathbb{N}}$, together with Kolmogorov 0-1 law we get a.e. and $\mathbb{L}_1((\mathbb{R}, \mathcal{B}, \mu_0)^{\mathbb{N}})$ convergence to zero of

$$\frac{1}{n}\sum_{j=1}^{n} \pi_j.$$

Since

$$E\left|\frac{1}{n}\sum_{j=1}^{n} X_k\right| = \int \left|\frac{1}{n}\sum_{j=1}^{n} \pi_j\right| d\mu_0^{\otimes\mathbb{N}}$$

we get

$$\frac{1}{n}\sum_{j=1}^{n} X_k \to_{n\to\infty} 0$$

in $\mathbb{L}_1((\Omega, \Sigma, \mu))$. The a.e. convergence of $\frac{1}{n}\sum_{j=1}^{n} X_k$ is a consequence of the a.e. convergence of $\frac{1}{n}\sum_{j=1}^{n} \pi_j$ and the fact that both families of random variables have the same finite dimensional distribution.

8 pts

unseen

(5.c) We note first that for the student's distribution with $\alpha > 1$ we have $m_1 \equiv \int |x| d\nu_0 < \infty$. This implies that for the product probability measure $\nu_0^{\mathbb{N}}$, we have

$$\int |\prod_{j=1}^{n} \pi_{j+k}| d\nu_0^{\mathbb{N}} = m_1^n$$

Hence

$$\int |\sum_{n\in\mathbb{N}} \varepsilon^n \prod_{j=1}^{n} \pi_{j+k}| d\nu_0^{\mathbb{N}} \leq \int \sum_{n\in\mathbb{N}} \varepsilon^n \prod_{j=1}^{n} |\pi_{j+k}| d\nu_0^{\mathbb{N}} \leq \sum_{n\in\mathbb{N}} \varepsilon^n m_1^n$$

which converges for $\varepsilon \in (0, \frac{1}{m_1})$. That means that $f_k \in \mathbb{L}_1(\nu_0^{\mathbb{N}})$. A map $T : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ defined by

$$(T\omega)_j = \omega_{j+1}$$

is preserving the product measure. Since we have $\pi_{j+1} = \pi_j \circ T$, we get

$$f_k = f_1 \circ T^{k-1}$$

Using the ergodic theorem we conclude that

$$\frac{1}{n} \sum_{k=1}^{n} f_k = \frac{1}{n} \sum_{k=1}^{n} f_1 \circ T^{k-1} \to_{n\to\infty} 0$$

almost everywhere and in $\mathbb{L}_1$.