Exam 2018 Solutions

1.

a) It can be derived that

$$f_Y(y)=\begin{cases}\frac{1}{2\sqrt{y}}\left(f_X(\sqrt{y})+f_X(-\sqrt{y})\right), & y>0,\\ 0, & \text{otherwise.}\end{cases} \qquad [3]$$

Since the Gaussian pdf is even,

$$f_Y(y)=\frac{1}{\sqrt{y}}f_X\left(\sqrt{y}\right), \quad y>0. \qquad [3]$$

Substituting $f_X(x)=\frac{1}{\sqrt{2\pi}\sigma}e^{-x^2/2\sigma^2}$, we obtain

$$f_Y(y)=\frac{1}{\sqrt{2\pi y}\sigma}e^{-y/2\sigma^2} \quad y>0. \qquad [4]$$

b)

i) Markov

$$P(|X|>a)\leq\frac{E(|X|)}{a}=\frac{\sqrt{\frac{2}{\pi}}\sigma}{a}=\frac{\sqrt{\frac{2}{\pi}}\sigma}{3\sigma}=0.265 \qquad [5]$$

ii) Chebyshev

$$P(|X|>a)\leq\frac{\sigma^2}{a^2}=\frac{1}{9}=0.111 \qquad [5]$$

iii) Chernoff

$$P(|X|>a)=2P(X>a)\leq 2e^{-\frac{a^2}{2\sigma^2}}=2e^{-4.5}=0.022 \qquad [5]$$

Discussion: The accuracy of estimation improves since more moments are used from (i) to (iii).

EE 4-10 Probability.

2.

a)

For n samples we have

$$f(\underline{x}, c) = c^{4n}(x_1 \cdots x_n)^{3n} e^{-c(x_1+\cdots+x_n)}$$ [2]

$$\frac{\partial f(\underline{x}, c)}{\partial c} = 4n\, c^{4n-1}(x_1 \cdots x_n)^{3n} e^{-c(x_1+\cdots+x_n)}$$ [2]

$$- (x_1+\cdots+x_n)\, c^{4n}(x_1 \cdots x_n)^{3n} e^{-c(x_1+\cdots+x_n)}$$

$$= \left[\frac{4n}{c} - (x_1+\cdots+x_n)\right] f(\underline{x}, c)$$ [2]

$$= 0$$

$$c = \frac{4n}{x_1+\cdots+x_n}$$ [2]

In this problem, n = 4, so

$$c = \frac{4\times 4}{16} = 1.$$ [2]

b) Recall the Wiender-Hopf equation

$$\mathbf{c} = \mathbf{R}^{-1}\mathbf{r}$$

$$\sigma^2 = r_0 - \mathbf{r}^T\mathbf{R}^{-1}\mathbf{r}$$

i) If n=1, the Wiener-Hopf equation trivially reads

$$R_Y(0)c_1 = R_Y(1)$$

Therefore, [2]

$$c_1 = \frac{R_Y(1)}{R_Y(0)} = \frac{2}{3}$$ [3]

Mean-square error

$$\sigma^2 = 3 - 4/3 = 5/3$$

ii) When n = 2, we have

$$\mathbf{R} = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$$

$$\mathbf{r} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$ [3]

Thus the coefficient vector

$$\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \mathbf{R}^{-1}\mathbf{r} = \frac{1}{5}\begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1/5 \\ 4/5 \end{bmatrix}$$ [4]

Mean-square error

$$\sigma^2 = 3 - [1 \quad 2]\frac{1}{5}\begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 8/5$$ [3]

3. a) 1 mark each except those marked otherwise

(1) linearity

(2),(3), convolution

(4) definition of power

(5) due to definition of power spectral density

(6) spectrum of $y_s(t)$ is $S(\omega)H(\omega)$ from (2); power spectrum of n(t) is $S_{ww}(\omega)|H(\omega)|^2$ from (3) [2]

(7) $w(t)$ is white noise with power spectral density N_0

(8) because of Cauchy-Schwarz inequality

$$\left|\int_{-\infty}^{+\infty} S(\omega)H(\omega)e^{j\omega t_0}d\omega\right|^2 \leq \int_{-\infty}^{+\infty}|S(\omega)e^{j\omega t_0}|^2 d\omega \int_{-\infty}^{+\infty}|H(\omega)|^2 d\omega$$ [2]

(9) Parseval's theorem

(10) $\int_0^{+\infty} s(t)^2 dt = E_s$

b)

i) This is the same as the average time when the fourth student arrives.

$$E[T_4] = \frac{4}{\lambda} = \frac{4}{0.2} = 20\ minutes$$ [3]

ii) This means that the number of students arriving in the first half an hour is less than 4.

Let t = 30 minutes. Recall N(t) has Poisson distribution: [2]

$$P(N(t) = k) = e^{-\lambda t}\frac{(\lambda t)^k}{k!}, \qquad k = 0,1,2,\dots$$

Therefore,

$$P(N(t) < 4) = P(N(t) = 0) + P(N(t) = 1) + P(N(t) = 2) + P(N(t) = 3)$$ [3]

$$= e^{-6} + 6e^{-6} + 18e^{-6} + 36e^{-6} = 0.161$$

iii) Let t_1 = 10 minutes, t_2 = 20 minutes. Since a Poisson process is memoryless,

$$P(N(t_1) \geq 1)P(N(t_2) - N(t_1) \leq 2)$$ [1]

$$= P(N(t_1) \geq 1)P(N(t_2 - t_1) \leq 2)$$

$$= [1 - P(N(t_1) = 0)][P(N(t_2 - t_1) = 0) + P(N(t_2 - t_1) = 1) + P(N(t_2 - t_1) = 2)$$ [2]

$$= [1 - e^{-6}][e^{-6} + 6e^{-6} + 18e^{-6}] = 0.062$$

[2]

4. a) i) Using the identity $\cos(A+B)+\cos(A-B)=2\cos A\cos B$, we have

$$E[X_{n+1}] = \frac{\cos\left\{\lambda\left[S_n+1-\frac{1}{2}(b-a)\right]\right\}+\cos\left\{\lambda\left[S_n-1-\frac{1}{2}(b-a)\right]\right\}}{2(\cos\lambda)^{n+1}}$$ [3]

$$= \frac{\cos\left\{\lambda\left[S_n-\frac{1}{2}(b-a)\right]\right\}\cos\{\lambda\}}{(\cos\lambda)^{n+1}}$$

$$= \frac{\cos\{\lambda[S_n-\frac{1}{2}(b-a)]\}}{(\cos\lambda)^n} = X_n$$

Thus, X_n forms a martingale.

[2]

ii) Since X_n forms a martingale, we know

$$E[X_T] = X_0$$

Obviously

$$X_0 = \cos\{\frac{1}{2}\lambda(b-a)\}$$ [2]

Now

$$E[X_T] = E_T E_{S_T}[X_T|T]$$

$$= E_T\frac{E_{S_T}\cos\left\{\lambda\left[S_T-\frac{1}{2}(b-a)\right]\right\}}{(\cos\lambda)^T}$$ [2]

$$= E_T\frac{P(S_T=-a)\cos\left\{\lambda\left[-a-\frac{1}{2}(b-a)\right]\right\}+(S_T=b)\cos\left\{\lambda\left[b-\frac{1}{2}(b-a)\right]\right\}}{(\cos\lambda)^T}$$ [2]

$$= E_T\frac{P(S_T=-a)\cos\left\{\lambda\left[\frac{1}{2}(b+a)\right]\right\}+P(S_T=b)\cos\left\{\lambda\left[\frac{1}{2}(b+a)\right]\right\}}{(\cos\lambda)^T}$$ [2]

$$= E_T\frac{\cos\left\{\lambda\left[\frac{1}{2}(a+b)\right]\right\}}{(\cos\lambda)^T}$$

$$= \frac{\cos\left\{\lambda\left[\frac{1}{2}(a+b)\right]\right\}}{E[(\cos\lambda)^T]}$$ [2]

Note that $P(S_T=-a)+P(S_T=b)=1$.

(b) The stationary distribution satisfies

$$\pi_1=\pi_0, \pi_2=\frac{1}{2}\pi_1, \pi_3=\frac{1}{3}\pi_2, \pi_4=\frac{1}{4}\pi_3, \ldots$$ [2]

from which we get

[2]

$$\pi_i = \frac{1}{i!}\pi_0$$

Since

$$\sum_{i=0}^{\infty}\pi_i = \sum_{i=0}^{\infty}\frac{1}{i!}\pi_0 = e\pi_0 = 1,$$

we find $\pi_0 = e^{-1}$ and [3]

$$\pi_i = \frac{e^{-1}}{i!},\ \ i = 1,2,3\ldots$$

[3]