Master.

E4.24
CS1.3
ISE4.21

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2009

MSc and EEE/ISE PART IV: MEng and ACGI

DISCRETE-TIME SYSTEMS AND COMPUTER CONTROL

Friday, 8 May 10:00 am

Time allowed: 3:00 hours

There are SIX questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible   First Marker(s) :   A. Astolfi

Second Marker(s) :   R.B. Vinter

DTS and Computer Control

Information for candidates:

- $Z\left(\frac{1}{s}\right) = \frac{z}{z-1} = \frac{1}{1-z^{-1}}$

- $Z\left(\frac{1}{s^2}\right) = T\frac{z}{(z-1)^2} = T\frac{z^{-1}}{(1-z^{-1})^2}$

- $Z\left(\frac{1}{s^3}\right) = \frac{T^2}{2}\frac{z(z+1)}{(z-1)^3} = \frac{T^2}{2}\frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3}$

- Transfer function of the ZOH: $H_0(s) = \frac{1-e^{-sT}}{s}$

- Transfer function of the FOH: $H_1(s) = \frac{1+Ts}{T}\left(\frac{1-e^{-sT}}{s}\right)^2$

- Definition of the w-plane: $z = \frac{1+\frac{wT}{2}}{1-\frac{wT}{2}}$, $w = \frac{2}{T}\frac{z-1}{z+1}$

1. Consider the block diagram in Figure 1 with input $u(t)$ and output $y(t)$. Let $T > 0$ be the sampling period.

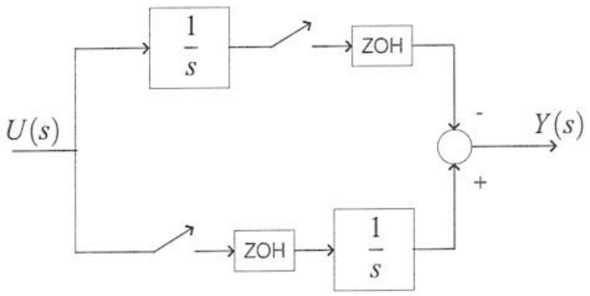

Figure 1: Block diagram for question 1.

a) Using the notion of pulse transfer function write the relation between the Laplace transform $U(s)$ of the input and the Laplace transform $Y(s)$ of the output. [4 marks]

b) Assume that the output is connected to a sampler. Discuss why it is not possible to obtain an equivalent discrete-time transfer function for the considered system. [2 marks]

c) Assume that $u(t)$ is a unity step. Sketch graphs for the output $y(t)$ and the sampled output $y(kT)$. (Assume that the outputs of the integrators are zero at $t = 0$.) [6 marks]

d) Repeat part c) assuming that the sampler and hold in the upper path are removed. [4 marks]

e) Explain why the results in parts c) and d) are different. [4 marks]

2. Consider the digital control system in Figure 2.

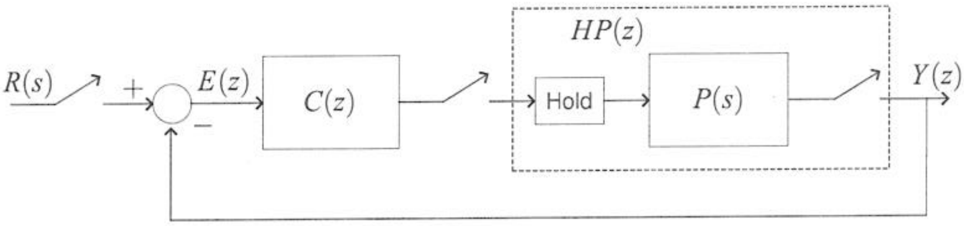

Figure 2: Block diagram for question 2.

Let

$$P(s) = \frac{e^{-s}}{s-1},$$

assume the hold is a ZOH and let the sampling period be $T = 1$ sec.

a) Compute the equivalent discrete-time model $HP(z)$ for the plant interconnected to the hold and the sampler. [4 marks]

b) Using the definition of the w-plane, determine the transfer function $HP(w)$. [2 marks]

c) Design, in the w-plane, a controller $C(w)$ such that the closed-loop system is asymptotically stable. [10 marks]

d) Compute the transfer function $C(z)$ of the discrete-time controller. Verify that the discrete-time closed-loop system resulting from the use of $C(z)$ is asymptotically stable. [4 marks]

3. The transfer function of a simple antenna angle-tracker system is given by

$$P(s) = \frac{1}{s(10s+1)}.$$

Consider the problem of designing a unity feedback controller, using the indirect method, achieving the following specifications:

- closed-loop stability;
- tracking error for a ramp input of slope 1 rad/sec less than 1 rad;
- 2% settling time not larger than 10 sec.
  (For a continuous-time system with characteristic polynomial $s^2 + 2\delta\omega_n s + \omega_n^2$, the 2% settling time is given by $T_s = \frac{4}{\delta\omega_n}$.)

a) Show that the controller

$$C(s) = \frac{10s+1}{s+1}$$

achieves the desired specifications. [6 marks]

b) The controller in part a) has to be implemented digitally and connected to the antenna angle-tracker by means of a ZOH and a sampler.

i) Select a sampling time consistent with the required closed-loop specifications. Explain your selection. [2 marks]

ii) Compute a discrete-time controller $C(z)$ using the pole-zero correspondance method and such that the DC-gains of $C(s)$ and $C(z)$ are the same. [2 marks]

iii) Compute the discrete-time equivalent transfer function of $P(s)$ connected to the sampler and hold.

[4 marks]

iv) Determine the location of the poles of the closed-loop system and discuss if the desired specifications are achieved. (An estimate of the value of the transient response after k samples is given by $|p|^k$, where p is the dominant pole of the closed-loop system.) [6 marks]

4. Consider the digital control system in Figure 4.

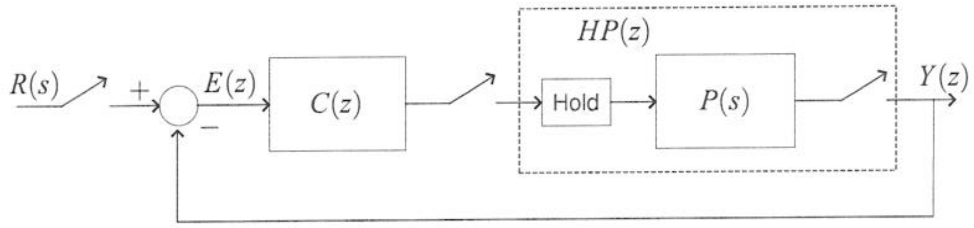

Figure 4: Block diagram for question 4.

Let

$$P(s) = \frac{e^{-Ts}}{s+1},$$

assume the hold is a ZOH and let $T > 0$ be the sampling period.

a) Compute the equivalent discrete-time model $HP(z)$ for the plant with the hold and sampler. [4 marks]

b) Design the controller $C(z)$ such that the closed-loop system has all poles at $z = 0$. [6 marks]

c) Let $R(s) = \frac{1}{s}$, i.e. the reference signal is a step. Evaluate the steady-state output of the closed-loop system resulting from the implementation of the controller in part b). [2 marks]

d) Redesign the controller $C(z)$ such that the closed-loop system has all poles at $z = 0$ and the system is of type one. Let $R(s) = \frac{1}{s}$, i.e. the reference signal is a step. Evaluate the steady-state output of the closed-loop system resulting from the implementation of this controller. [8 marks]

5. Consider a unity feedback system with open-loop transfer function

$$P(s) = k\frac{1}{s},$$

with $k > 0$.

a) Show that the closed-loop system is stable for all $k > 0$. [1 marks]

b) Assume that the input of the system $P(s)$ is connected to a sampler and to a FOH and that the output is connected to a sampler. Let $T > 0$ be the sampling period.

i) Determine the discrete-time equivalent transfer function of the open-loop system. [8 marks]

ii) Study the stability of the closed-loop system as a function of $k > 0$. [4 marks]

c) Repeat part b) replacing the FOH with a ZOH. [6 marks]

d) Comment on the differences between the results obtained in parts b) and c). [1 marks]

6. Consider a continuous-time system described by the transfer function

$$P(s) = \frac{1}{s^2}.$$

a) Assume the system is connected to a ZOH and a sampler. Let T be the sampling period. Determine the discrete-time equivalent transfer function $HP(z)$.

[4 marks]

b) The system can be considered as the cascaded interconnection of two integrators. Write the discrete-time equivalent transfer function $P_I(z)$ of an integrator, connected to a ZOH and a sampler. An approximate discrete-time model for the system is described by

$$HP_a(z) = P_I^2(z).$$

Determine $HP_a(z)$. [2 marks]

c) Discuss why $HP(z)$ and $HP_a(z)$ are not equal. [4 marks]

d) Consider the PD-type discrete-time controller

$$C(z) = K\left(1 + \alpha\frac{z-1}{z-\beta}\right).$$

Determine K, α and β such that the closed-loop system composed of the controller and the approximate discrete-time model has all poles at $z = 0$.

[6 marks]

e) Consider the closed-loop system obtained by interconnecting the controller in part d) with the system in part a). Study the stability of this closed-loop system and comment on the suitability of the approximate model for the controller design. [4 marks]

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DTS and Computer Control

Model answers 2009

Question 1

a) Following the procedure in the lecture notes, let $X_u(s)$ and $X_u^\star(s)$ be the input and output variables for the sampler in the upper path, and $X_l(s)$ and $X_l^\star(s)$ be the input and output variables for the sampler in the lower path. Then

$$X_u(s) = \frac{U(s)}{s} \qquad X_l(s) = U(s) \qquad Y(s) = \frac{ZOH(s)}{s}X_l^\star(s) - ZOH(s)X_u^\star(s).$$

Transforming these equations in terms of sampled quantities yields

$$X_u^\star(s) = \left(\frac{U(s)}{s}\right)^\star \qquad X_l^\star(s) = U^\star(s) \qquad Y^\star(s) = \left(\frac{ZOH(s)}{s}\right)^\star X_l^\star(s) - ZOH(s)^\star X_u^\star(s),$$

hence (recall that $(ZOH(s))^\star = 1$)

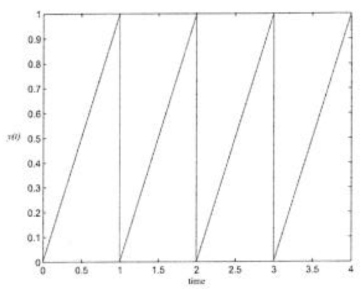
$$Y^\star(s) = \left(\frac{ZOH(s)}{s}\right)^\star U^\star(s) - \left(\frac{U(s)}{s}\right)^\star$$

and

$$Y(z) = Z\left(\frac{ZOH(s)}{s}\right)U(z) - Z\left(\frac{U(s)}{s}\right) = \frac{T}{z-1}U(z) - Z\left(\frac{U(s)}{s}\right). \qquad (1)$$

b) It is not possible to obtain a discrete-time equivalent transfer function, i.e. an expression of the form $Y(z) = P(z)U(z)$, for some $P(z)$, because the input $U(s)$ goes through the integrator in the upper part before being sampled. This is also reflected in the presence of the term $Z\left(\frac{U(s)}{s}\right)$ in equation (1).

c) The graph of $y(t)$, with a sampling time $T = 1$, is as in the figure.



Note that $y(kT) = 0$, for all k.

d) If the sampler and hold in the upper part are removed then $y(t) = 0$ for all t. This is consistent with the fact that if the input is a step the the sampler and hold in the lower path do not modify the signal, i.e. the input of the integrator in the lower path is a step.

e) The difference between the results in parts c) and d) is due to the presence, or otherwise, of the sampler and hold in the upper path. In fact, if the input is a step, as already noted, the sampler and hold in the lower path can be removed without affecting the signals.

Question 2

a) Note that

$$\begin{aligned} HP(z) &= (1-z^{-1})Z\left(\frac{e^{-s}}{s(s-1)}\right) \\ &= (1-z^{-1})z^{-1}Z\left(\frac{1}{s(s-1)}\right) \\ &= (1-z^{-1})z^{-1}Z\left(-\frac{1}{s}+\frac{1}{s-1}\right) \\ &= \frac{e-1}{z(z-e)} \end{aligned}$$

b) The transfer function in the w-plane is

$$HP(w) = HP(z)\Big|_{z=\frac{1+w/2}{1-w/2}} = (e-1)\frac{(w-2)^2}{(2-2e+w+ew)(w+2)}.$$

c) A stabilizing control law can be obtained cancelling the pole at $w=-2$ and adding a pole, i.e. selecting

$$C(w) = \frac{w+2}{aw+b}.$$

The coefficients a and b can be selected to assign the poles of the closed-loop system. For example, selecting

$$a \approx -0.3176 \qquad\qquad b \approx 1.8437$$

yields two poles at $w=-1$. The controller is thus

$$C(w) = \frac{w+2}{-0.3176w+1.8437}.$$

d) The discrete-time controller is

$$C(z) = C(w)\Big|_{w=2\frac{z-1}{z+1}} = \frac{4z}{1.2083z+2.479}.$$

The characteristic polynomial of the closed-loop discrete-time system is (note the pole-zero cancellation at $z=0$)

$$p(z) = 1.208z^2 - 0.805z + 0.1342,$$

and its roots are both approximately equal to 0.333. As a result, the controller yields closed-loop stability.

Question 3

a) The open-loop transfer function of the continuous-time closed-loop system with the given controller is (note the pole-zero cancellation)

$$C(s)P(s) = L(s) = \frac{1}{s(s+1)}.$$

The system is of type one, and the velocity constant is $K_v = \lim_{s\to 0} sL(s) = 1$, which is sufficient to satisfy the condition on the steady-state response. In addition, the closed-loop characteristic polynomial is $p(s) = s^2 + s + 1$, hence the closed-loop system is stable. Note that

$$p(s) = s^2 + 2\delta\omega_n s + \omega_n^2,$$

with $\delta = 1/2$ and $\omega_n = 1$. Hence, $T_s = \frac{4}{\delta\omega_n} = 8$, which is within the specifications.

b) i) The sampling time has to be selected as a fraction of the settling time, for example any $T \in [8/10, 8/4]$ is reasonable. In what follows we select $T = 1$.

ii) The discrete-time controller is

$$C(z) = k\frac{1 - e^{-\frac{1}{10}}z^{-1}}{1 - e^{-1}z^{-1}},$$

where k has to be selected such that

$$1 = \lim_{s\to 0} C(s) = \lim_{z\to 1} C(z) = k\frac{1 - e^{-\frac{1}{10}}}{1 - e^{-1}},$$

yielding the controller

$$C(z) \approx 6.64\frac{z - 0.905}{z - 0.367}.$$

iii) The discrete-time equivalent model of the plant $P(s)$ is given by

$$\begin{aligned}
HP(z) &= (1 - z^{-1})Z\left(\frac{P(s)}{s}\right) \\
&= (1 - z^{-1})Z\left(\frac{1}{s^2} - \frac{10}{s} + \frac{10}{s + 1/10}\right) \\
&\approx \frac{0.05z + 0.046}{(z-1)(z-0.905)}.
\end{aligned}$$

iv) The open-loop transfer function of the closed-loop system is (note the pole-zero cancellation)

$$C(z)HP(z) \approx \frac{0.32z + 0.31}{(z - 0.367)(z-1)},$$

and the characteristic polynomial is

$$z^2 - 1.04z + 0.678,$$

with roots $0.52 \pm j0.63$. A crude estimate of the value of the error transient response after ten samples, i.e. after 10 sec, is given by

$$|0.52 + j0.678|^{10} \approx 0.13 > 0.02,$$

which shows that the performance of the sampled-data system, in terms of settling time, are not within the specifications. Note that the stability and steady-state accuracy specifications are achieved.

Question 4

a) The equivalent discrete-time model is given by

$$\begin{aligned} HP(z) &= (1-z^{-1})Z\left(\tfrac{e^{-Ts}}{s(s+1)}\right) \\ &= (1-z^{-1})z^{-1}Z\left(\tfrac{1}{s(s+1)}\right) \\ &= (1-z^{-1})z^{-1}Z\left(\tfrac{1}{s}-\tfrac{1}{s+1}\right) \\ &= (1-z^{-1})z^{-1}\left(\tfrac{1}{1-z^{-1}}-\tfrac{1}{1-e^{-T}z^{-1}}\right) \\ &= \frac{1-e^{-T}}{z(z-e^{-T})}. \end{aligned}$$

b) The order m of the controller can be selected equal to one and the controller can be parameterized as

$$C(z) = \frac{a_1 z + a_0}{z + b_0}.$$

The closed-loop characteristic polynomial is

$$p(z) = (1-e^{-T})(a_1 z + a_0) + z(z-e^{-T})(z+b_0).$$

Selecting

$$a_0 = 0 \qquad a_1 = \frac{e^{-T}}{e^{T}-1} \qquad b_0 = e^{-T}$$

yields $p(z) = z^3$ as required.

c) The system, with the controller in part b), is of type zero. The position constant is

$$K_p = \lim_{z\to 1} C(z)HP(z) = \frac{1}{(e^T+1)(e^T-1)}.$$

Note that $0 < K_p < 1$, for all $T > 0$, hence, in steady-state, the output does not track exactly the reference signal.

d) Since the system has to be of type one, and $HP(z)$ does not have a pole at $z = 1$, this pole should be added to the controller. As a result, the order m of the controller can be selected equal to two and the controller can be parameterized as

$$C(z) = \frac{a_2 z^2 + a_1 z + a_0}{(z+b_0)(z-1)}.$$

The closed-loop characteristic polynomial is

$$p(z) = (1-e^{-T})(a_2 z^2 + a_1 z + a_0) + z(z-1)(z-e^{-T})(z+b_0).$$

Selecting

$$a_0 = 0 \qquad a_1 = -e^{-T}\frac{e^T+1}{e^T-1} \qquad a_2 = e^{-T}\frac{e^T+e^{2T}+1}{e^T-1} \qquad b_0 = e^{-T}(1+e^T)$$

yields $p(z) = z^4$ as required. Since the system of of type one, the position constant is $K_p = \infty$, hence the steady-state output is equal to one, i.e. the output tracks, asymptotically, the step reference input.

Question 5

a) The closed-loop characteristic polynomial is $s + k$, hence the closed-loop system is asymptotically stable for all $k > 0$.

b) i) The equivalent discrete-time model is given by

$$\begin{aligned} HP(z) &= kZ\left(\frac{1+Ts}{T}\left(\frac{1-e^{-sT}}{s}\right)^2 \frac{1}{s}\right) \\ &= \frac{k}{T}(1-z^{-1})^2 Z\left(\frac{1+Ts}{s^3}\right) \\ &= \frac{k}{T}(1-z^{-1})^2 Z\left(\frac{1}{s^3} + T\frac{1}{s^2}\right) \\ &= \frac{k}{T}(1-z^{-1})^2\left(\frac{T^2}{2}\frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3} + T\frac{Tz^{-1}}{(1-z^{-1})^2}\right) \\ &= kT\frac{3z-1}{2z(z-1)} \end{aligned}$$

ii) The characteristic polynomial of the closed-loop system is

$$2z^2 + (3Tk-2)z - Tk.$$

The roots of this polynomial are all inside the unity circle if

$$Tk > 0 \qquad 2Tk + 4 > 0 \qquad 4 - 4Tk > 0$$

which are equivalent to (recall that $T > 0$ and $k > 0$) $k < \frac{1}{T}$. As a result, the closed-loop system is asymptotically stable for $k < \frac{1}{T}$ and unstable for $k > \frac{1}{T}$.

c) i) The equivalent discrete-time model is given by

$$\begin{aligned} HP(z) &= kZ\left(\frac{1-e^{-sT}}{s}\frac{1}{s}\right) \\ &= k(1-z^{-1})Z\left(\frac{1}{s^2}\right) \\ &= k(1-z^{-1})\frac{Tz^{-1}}{(1-z^{-1})^2} \\ &= k\frac{T}{z-1} \end{aligned}$$

ii) The characteristic polynomial of the closed-loop system is

$$z - 1 + Tk$$

The roots of this polynomial are all inside the unity circle if (recall that $T > 0$ and $k > 0$) $k < \frac{2}{T}$. As a result, the closed-loop system is asymptotically stable for $k < \frac{2}{T}$ and unstable for $k > \frac{2}{T}$.

d) The stability condition for the case in which the ZOH is used is weaker than for the case of FHO. Note that in both cases the continuous-time condition, i.e. $k > 0$ is recovered as $T \to 0$.

Question 6

a) The discrete-time equivalent transfer function is

$$HP(z) = Z\left(\frac{1-e^{-sT}}{s}\frac{1}{s^2}\right) = (1-z^{-1})Z\left(\frac{1}{s^3}\right) = (1-z^{-1})\frac{T^2}{2}\frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3} = \frac{T^2}{2}\frac{z+1}{(z-1)^2}.$$

b) Note that

$$P_I(z) = Z\left(\frac{1-e^{-sT}}{s}\frac{1}{s}\right) = (1-z^{-1})Z\left(\frac{1}{s^2}\right) = (1-z^{-1})T\frac{z^{-1}}{(1-z^{-1})^2} = \frac{T}{z-1},$$

hence

$$HP_a(z) = \frac{T^2}{(z-1)^2}.$$

c) $HP(z)$ is the transfer function of a system with transfer function $1/s^2$ with a sampler and ZOH at the input and a sampler at the output. $HP_a(z)$ is the transfer function of a system composed of the cascade of two integrators with both sampler and ZOH at the input and sampler at the output. In summary

$$HP(z) = Z\left(\frac{1}{s^2}\right) \neq Z\left(\frac{1}{s}\right)Z\left(\frac{1}{s}\right) = HP_a(z).$$

d) The characteristic polynomial of the closed-loop system obtained interconnecting $HP_a(z)$ and $C(z)$ is

$$p_a(z) = z^3 + (-2-\beta)z^2 + (1+T^2K+T^2K\alpha+2\beta)z + (-\beta - T^2K\beta - T^2K\alpha),$$

hence selecting

$$K = \frac{1}{3T^2} \qquad \alpha = 8 \qquad \beta = -2$$

yields $P_a(z) = z^3$ as requested.

e) The characteristic polynomial of the closed-loop system obtained interconnecting $HP(z)$ with $C(z)$ in part d) is

$$p(z) = 2z^3 + 3z^2 - 5z + 2.$$

Using the bilinear transformation

$$z = \frac{1+w}{1-w}$$

yields

$$\frac{8w^3 - 14w^2 - 8w - 2}{(1-w)^3}.$$

Since the coefficients of the numerator polynomial do not have all the same sign the closed-loop system is unstable. In summary, the proposed approximate model is unsuitable for control design. (Note that the result does not depend upon the selection of the sampling time.)