UNIVERSITY OF LONDON [ISE 2.6 2002]

B.ENG. AND M.ENG. EXAMINATIONS 2002

For Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examination for the Associateship of the City and Guilds of London Institute.

INFORMATION SYSTEMS ENGINEERING 2.6

MATHEMATICS

Date Thursday 30th May 2002 2.00 - 4.00 pm

Answer FOUR questions, to include at least one from Section B

Answers to Section A *questions must be written in a different answer book from answers to* Section B *questions.*

[*Before starting, please make sure that the paper is complete. There should be SIX pages, with a total of SIX questions. Ask the invigilator for a replacement if this copy is faulty.*]

Section A

1. (i) We define the Fourier transform $\hat{f}(\omega)$ of a function $f(t)$ as

$$\hat{f}(\omega) \;=\; \int_{-\infty}^{\infty} e^{-i\omega t} f(t)dt\,.$$

Show that, if f is smooth and $|tf(t)| \to 0$ sufficiently fast as $t \to \pm\infty$,

then the Fourier transforms of $f'(t)$ and $tf(t)$ are given by the formulae

$$\widehat{f'(t)}(\omega) \;=\; i\omega\hat{f}(\omega), \qquad i\frac{d}{d\omega}\hat{f}(\omega) \;=\; \widehat{tf(t)}\,(\omega), \text{ respectively.}$$

(ii) Show that $ie^{-\omega^2/2a}\dfrac{d}{d\omega}\left[e^{\omega^2/2a}\hat{f}(\omega)\right]$ is the Fourier transform of $\dfrac{1}{a}f'(t) \;+\; tf(t)$.

Here a, > 0, is a constant.

(iii) Apply this result to find the Fourier Transform of $f(t) \;=\; e^{-at^2/2}$.

You will need to determine an arbitrary constant; do this by computing

$\hat{f}(0)$ *directly, using the fact that* $\displaystyle\int_{-\infty}^{\infty} e^{-u^2} du \;=\; \sqrt{\pi}$.

(iv) What is $\displaystyle\int_{-\infty}^{\infty} \cos\omega t\; e^{-at^2/2}\,dt\,$?

PLEASE TURN OVER

2. (i) Take the Laplace transform of

$$y(t) \;=\; e^{2t} \;+\; \int_0^t y(u)\,du, \qquad t \geq 0\,, \qquad (*)$$

to find the solution $y(t)$.

(ii) Differentiate the above equation $(*)$ once to find the differential equation satisfied by y. What is $y(0)$?

(iii) Show that $\displaystyle\int_{(0,0)}^{(u,v)} (2xye^{x^2} + 1)\,dx \;+\; e^{x^2}dy$ is independent of the path of integration, running from (0, 0) to $(u,\, v)$. Choose a suitable path and *integrate along it* to evaluate the integral.

(iv) Determine a function $\Phi(x,\, y)$ such that

$$\frac{\partial \Phi}{\partial x} \;=\; 2xye^{x^2} \;+\; 1\,,$$

$$\frac{\partial \Phi}{\partial y} \;=\; e^{x^2}\,.$$

3. (i) By switching to polar coordinates, show that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{-(x^2+y^2)}}{\sqrt{x^2+y^2}}\; dxdy \;=\; \pi^{3/2}\,.$$

You may use the fact that $\displaystyle\int_0^\infty e^{-u^2} du \;=\; \frac{\sqrt{\pi}}{2}$.

(ii) Sketch the region in the xy-plane over which the integral

$$\int_0^1 \int_{y^2}^{y^{2/3}} \frac{e^x}{\sqrt{x}}\; dxdy$$

is taken, and hence reverse the order of integration.

(iii) Evaluate the integral.

PLEASE TURN OVER

4. (i) At what points z does $f(z) = \dfrac{e^{iz}}{(z^2+1)(z^2+4)}$ have poles, and what are their residues?

(ii) By integrating $f(z)$ around a large semicircle (centred at the origin) in the upper half plane $Im\, z \geq 0$, show that

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)(x^2+4)}\, dx = \frac{\pi}{6e^2}(2e-1)\ .$$

PLEASE TURN OVER

1. A stochastic search algorithm is implemented so that it will attempt for up to 1 minute to match a search term. After one minute, if it has been unsuccessful, a 'no match' is returned. The length of time, in minutes, taken by the algorithm to match a term is a continuous random variable X with probability density function

$$\begin{aligned} f(x) &= cx^2 + x, \qquad 0 < x \le 1\,, \\ &= 0 \qquad\qquad\quad \text{elsewhere}\,. \end{aligned}$$

(i) Find the value of the constant c.

(ii) Find the cumulative distribution function of X.

(iii) What is the probability that the algorithm returns a result in less than 30 seconds?

(iv) Given that the search takes more than 15 seconds, what is the probability that it takes more than thirty seconds?

Two hundred independent searches are conducted. Let Y be a random variable that denotes the number of searches that return a result within 10 seconds.

(v) What distribution is an appropriate model for Y?

(vi) What is the probability that at most 2 matches are completed within 10 seconds?

PLEASE TURN OVER

2. (i) The discrete random variable X has a distribution given by

$$\begin{array}{ccccc} x & 0 & 1 & 2 & 3 \\ f(x) & \frac{1}{2} & \frac{1}{4} & \frac{3}{16} & \frac{1}{16} \end{array}.$$

Compute $E(X)$ and var (X).

(ii) In computer interface design, it is known that 70% of females react positively to a given stimulus, whereas only 40% of males react positively to the same stimulus. In an experiment, 20 people, 15 female and 5 male, were exposed to the stimulus. They then completed a questionnaire describing whether they felt positive or negative about the stimulus.

(a) What is the probability of a positive reaction?

(b) A questionnaire picked at random from the 20 was positive. What is the probability that this questionnaire was completed by a male?

(c) Now suppose that the probability of a female responding positively is p, and the probability of a male responding positively is r. An expert in interface design states that, for the same experimental group of 20 people, the probability of a positive response is 0.2 and the probability of a female responding positively is one and a half times the probability of a male responding positively. In this case, what is the probability of a negative response by a female?

END OF PAPER

BOOKWORK

a) $\widehat{f'}(\omega)=\int_{-\infty}^{\infty} e^{-i\omega t} f'(t)\,dt = \left[e^{-i\omega t} f(t)\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}(-i\omega e^{-i\omega t}) f(t)\,dt$

by parts (with $u=e^{-i\omega t}$, $\frac{dv}{dt}=f'(t)$).

$|e^{-i\omega t}f(t)| = |f(t)| \to 0$ as $t\to\pm\infty$ (since $|tf(t)|\to 0$).

So term in [ ] is zero, and we are left with

$\widehat{f'}(\omega) = i\omega\int_{-\infty}^{\infty} e^{-i\omega t} f(t)\,dt = \underline{i\omega \hat{f}(\omega)}.$ 3

$i\frac{d}{d\omega}\hat{f}(\omega) = i\int_{-\infty}^{\infty}\frac{d}{d\omega}e^{-i\omega t}f(t)\,dt = i\int_{-\infty}^{\infty}(-ite^{-i\omega t})f(t)\,dt$

$=\int_{-\infty}^{\infty} e^{-i\omega t}(tf(t))\,dt = \underline{\widehat{tf(t)}(\omega)}.$ 3

b) $ie^{-\omega^2/2a}\frac{d}{d\omega}\left(e^{\omega^2/2a}\hat{f}(\omega)\right) = ie^{-\frac{\omega^2}{2a}}\left[\frac{2\omega}{2a}e^{\omega^2/2a}\hat{f}(\omega)+e^{\omega^2/2a}\frac{d}{d\omega}\hat{f}(\omega)\right]$

$=\frac{i\omega}{a}\hat{f}(\omega)+i\frac{d}{d\omega}\hat{f}(\omega)$ which from 1a) gives

$=\frac{1}{a}\widehat{f'}(\omega)+\widehat{tf(t)}(\omega)$ so this is the fourier 4

transform of $\underline{\frac{1}{a}f'(t)+tf(t)}.$

c) For $f(t)=e^{-at^2/2}$, $f'(t)=-ate^{-at^2/2}=-atf(t)$

$\Rightarrow \frac{1}{a}f'(t)+tf(t)=0.$

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c) ctd $\Rightarrow i e^{-\omega^2/2a} \frac{d}{d\omega}\left(e^{\omega^2/2a}\hat{f}(\omega)\right) = 0$

$$\Rightarrow e^{\omega^2/2a}\hat{f}(\omega) = \text{constant } C. \quad (*)$$ 4

Set $\omega = 0$ to give $C = \hat{f}(0) = \int_{-\infty}^{\infty} f(t)\,dt$ (by definition of $\hat{f}(0)$).

$\Rightarrow C = \int_{-\infty}^{\infty} e^{-at^2/2}\,dt$, substitute $u = \sqrt{\frac{a}{2}}\,t$ to give

$$= \int_{-\infty}^{\infty} e^{-u^2}\sqrt{\frac{2}{a}}\,du = \sqrt{\frac{2}{a}}\sqrt{\pi} = \sqrt{\frac{2\pi}{a}}.$$

So from $(*)$ $\hat{f}(\omega) = \sqrt{\frac{2\pi}{a}}\, e^{-\omega^2/2a}$. 2

d) $\int_{-\infty}^{\infty} \cos\omega t\, e^{-at^2/2}\,dt = \mathrm{Re}\left(\int_{-\infty}^{\infty} e^{-i\omega t} e^{-at^2/2}\,dt\right)$

$$= \mathrm{Re}\,\hat{f}(\omega) = \mathrm{Re}\sqrt{\frac{2\pi}{a}}\, e^{-\omega^2/2a}$$ 4

$$= \sqrt{\frac{2\pi}{a}}\, e^{-\omega^2/2a}$$

MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

SESSION : 2001-2002

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PAPER ISE 2.6

QUESTION

SOLUTION 2

a) $y(t) = e^{2t} + \int_0^t y(u)\,du \,, \quad t \geq 0 .$

$\Rightarrow \bar{y}(p) = \dfrac{1}{p-2} + \dfrac{\bar{y}(p)}{p}$ by tables (or 2nd term by noting $\int_0^t y = (y * 1)(t)$) 2

$\Rightarrow \dfrac{p-1}{p}\,\bar{y}(p) = \dfrac{1}{p-2}$

$\Rightarrow \bar{y} = \dfrac{p}{(p-1)(p-2)} = \dfrac{2}{p-2} - \dfrac{1}{p-1}$ 3

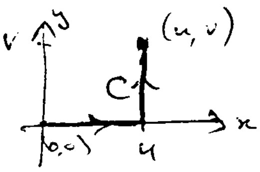
$\Rightarrow$ by tables $\underline{y(t) = 2e^{2t} - e^{t}}$ 2

b) $\underline{y'(t) = 2e^{2t} + y(t)}$, $\quad y(0) = e^{2.0} + \int_0^0 y(u)\,du = \underline{1}$ 3

c) $\displaystyle\int_{(0,0)}^{(u,v)} P\,dx + Q\,dy \quad \begin{pmatrix} P = 2xye^{x^2} + 1 \\ Q = e^{x^2} \end{pmatrix}$ is independent of

path since P, Q smooth and $\underline{\dfrac{\partial P}{\partial y} = 2xe^{x^2} = \dfrac{\partial Q}{\partial x}}$. 3

Pick path C :



to give $\displaystyle\int_0^u (0+1)\,dx + \int_0^v e^{u^2}\,dy$

$= \underline{u + ve^{u^2}}$ 4

d) Either take $\Phi(u,v)$ to be the above integral, or solve for Φ by integrating the PDEs $\dfrac{\partial \Phi}{\partial x} = P, \ \dfrac{\partial \Phi}{\partial y} = Q$.

Either way get (to within a constant) $\underline{\Phi(x,y) = x + ye^{x^2}}$. 3

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PAPER ISE 2.6

QUESTION

SOLUTION 3

a) $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{-(x^2+y^2)}}{\sqrt{x^2+y^2}}\,dx\,dy = \int_0^{2\pi}\int_0^{\infty} \frac{e^{-r^2}}{r}\, r\,dr\,d\theta$$

$$= 2\pi \int_0^{\infty} e^{-r^2}\,dr = 2\pi \frac{\sqrt{\pi}}{2} = \underline{\pi^{3/2}}.$$

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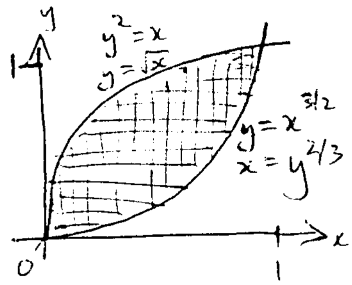
b)



3 for sketch

So reversing gives $\int_0^1 \int_{x^{3/2}}^{\sqrt{x}} \frac{e^x}{\sqrt{x}}\,dy\,dx$

5 for reversing

c) $$= \int_0^1 (e^x - xe^x)\,dx \quad + \text{do } 2^{\text{nd}} \text{ term by parts}$$

$$= \left[e^x\right]_0^1 - \left[xe^x\right]_0^1 + \int_0^1 e^x\,dx$$

$$= (e-1) - (e-0) + (e-1) = \underline{e-2}$$

5 for evaluating

PAPER ISE 2.6
QUESTION
SOLUTION 4

a) Poles where $z^2+1=0$ or $z^2+4=0$

ie at $z=\pm i, \pm 2i$.

Since $f(z)=\dfrac{e^{iz}}{(z+i)(z-i)(z+2i)(z-2i)}$, its residue at:

$z=-i$ is $\left.\dfrac{e^{iz}}{(z-i)(z+2i)(z-2i)}\right|_{z=-i} = \dfrac{e}{-6i} = \dfrac{ie}{6}$

$z=i$ is $\left.\dfrac{e^{iz}}{(z+i)(z+2i)(z-2i)}\right|_{z=i} = \dfrac{e^{-1}}{6i} = \dfrac{-i}{6e}$.

$z=2i$ is $\left.\dfrac{e^{iz}}{(z^2+1)(z+2i)}\right|_{z=2i} = \dfrac{e^{-2}}{-12i} = \dfrac{i}{12e^2}$.

$z=-2i$ is $\left.\dfrac{e^{iz}}{(z^2+1)(z-2i)}\right|_{z=-2i} = \dfrac{e^2}{12i} = \dfrac{-ie^2}{12}$.

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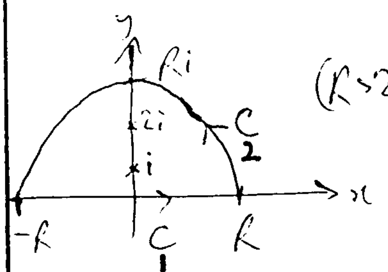

$(R>2)$

$C = C_1 + C_2$

and $\displaystyle\int_C \frac{e^{iz}}{(z^2+1)(z^2+4)}\,dz = 2\pi i\left(\mathrm{Res}_{z=i} f(z) + \mathrm{Res}_{z=2i} f(z)\right)$

$= 2\pi i\left(\dfrac{-i}{6e} + \dfrac{i}{12e^2}\right)$

$= \dfrac{\pi}{6e^2}(2e-1)$.

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So $\displaystyle\int_{C_1} + \int_{C_2} = \frac{\pi}{6e^2}(2e-1)$; now tend $R \to \infty$.

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PAPER

QUESTION

SOLUTION 4 ctd

On C_2, $|f(z)| \quad \frac{|e^{iz}|}{|z^2+1||z^2+4|} \le \frac{|e^{-\operatorname{Im} z}|}{(|z|^2-1)(|z|^2-4)} \le \frac{e^{-0}}{(R^2-1)(R^2-4)}$

So by ML inequality, $\left|\int_{C_2} f(z)\,dz\right| \le \pi R \cdot \frac{1}{(R^2-1)(R^2-4)} \to 0$ as $R\to\infty$ 4

So taking $R\to\infty$, as $\int_{C_1} f(z)\,dz \to \int_{-\infty}^{\infty} f(x)\,dx$ we get

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+1)(x^2+4)}\,dx = \frac{\pi}{6e^2}(2e-1).$$

Taking real parts gives $\underline{\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)(x^2+4)}\,dx = \frac{\pi}{6e^2}(2e-1)}$ 4

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EXAMINATION QUESTION / SOLUTION

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PAPER ISE 2.6

QUESTION

SOLUTION 1 B

i) $\int_0^1 (cx^2 + x)\,dx = 1$ 1

$\left[\frac{cx^3}{3} + \frac{x^2}{2}\right]_0^1 = 1$ 1

$\frac{c}{3} + \frac{1}{2} = 1$

$c = 3/2$ 1

ii) $F(x_0) = \int_0^{x_0} f(x)\,dx$ 1

$= \left[\frac{\cancel{3}x^3}{2\cancel{3}} + \frac{x^2}{2}\right]_0^{x_0}$ for $0 < x \le 1$

0 for $x < 0$

1 for $x > 1$

$= \frac{x_0^3 + x_0^2}{2}$ 2

iii) $P(X < \frac{1}{2}) = F(\frac{1}{2}) = \frac{(\frac{1}{2})^3 + (\frac{1}{2})^2}{2} = \frac{\frac{1}{8} + \frac{1}{4}}{2} = \frac{3}{16}$ 2

iv) $P(X > \frac{1}{2} \mid X > \frac{1}{4}) = \frac{P(X > \frac{1}{2} \cap X > \frac{1}{4})}{P(X > \frac{1}{4})}$ 1

$= \frac{P(X > \frac{1}{2})}{P(X > \frac{1}{4})}$ 1

$= \frac{1 - F(\frac{1}{2})}{1 - F(\frac{1}{4})}$ 1

$= \frac{104}{123} \doteq 0.9455$ 2

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Checker : WHITE Checker's signature : LWW

PAPER

QUESTION

SOLUTION 1 (CONT)

v). $Y \sim \text{Binomial}(200, \theta)$

$\theta = P(X < \tfrac{1}{6}) = F(\tfrac{1}{6}) = \frac{7}{432} \doteq 0.016,$

hence $Y \sim \text{Binomial}(200, \frac{7}{432})$ 3

vi). $P(Y \leq 2) = P(Y=0) + P(Y=1) + P(Y=2)$

$$= \binom{200}{0}\left(1-\frac{7}{432}\right)^{200} + \binom{200}{1}\left(\frac{7}{432}\right)\left(1-\frac{7}{432}\right)^{199} + \binom{200}{2}\left(\frac{7}{432}\right)^{2}\left(1-\frac{7}{432}\right)^{198}$$

$\doteq 0.369.$ 4

a) $E(X) = \sum_{x=0}^{3} x f(x)$

$= 0\left(\tfrac{1}{2}\right) + 1\left(\tfrac{1}{4}\right) + 2\left(\tfrac{3}{16}\right) + 3\left(\tfrac{1}{16}\right)$

$= \underline{13/16}$ 3

$var(X) = E(X^2) - [E(X)]^2$

$E(X^2) = \sum_{x=0}^{3} x^2 P(x)$

$= 0\left(\tfrac{1}{2}\right) + 1^2\left(\tfrac{1}{4}\right) + 2^2\left(\tfrac{3}{16}\right) + 3^2\left(\tfrac{1}{16}\right)$

$= \tfrac{1}{4} + \tfrac{3}{4} + \tfrac{9}{16}$

so $var(X) = \frac{25}{16} - \left(\frac{13}{16}\right)^2$

$= \underline{\dfrac{231}{256}}$ 4

b)

i). Denote a positive response as R, a Male as M and a female as F. 1

The question gives

$P(F) = 3/4 \quad P(M) = 1/4 \quad P(R|F) = 7/10 \quad P(R|M) = 2/5$ 1

Compute $P(R)$ using total probability

$P(R) = P(R|F)P(F) + P(R|M)P(M)$ 1

$= \frac{7}{10}\left(\frac{3}{4}\right) + \frac{2}{5}\left(\frac{1}{4}\right)$

$= \underline{5/8 = 0.625}$ 1

ii) Bayes theorem

$P(M|R) = \dfrac{P(R|M)P(M)}{P(R)} = \dfrac{\left(\frac{2}{5}\right)\frac{1}{4}}{5/8} = \underline{\dfrac{4}{25}}$ 3

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iii) Now we have

$P(R) = 0.2 \quad P(R|F) = p \quad P(R|M) = r \quad p = \frac{3}{2}r$ 1

$$P(R) = P(R|F)P(F) + P(R|M)P(M)$$

$$0.2 = \frac{3p}{4} + \frac{r}{4}$$ 1

combining equations for p and r we have

$$0.2 = \frac{3}{4}\left(\frac{3}{2}r\right) + \frac{r}{4}$$

$$0.2 = \frac{9}{8}r + \frac{r}{4}$$

$$\underline{r = \frac{8}{55}}$$

so $p = \frac{3}{2}\left(\frac{8}{55}\right) = \underline{\frac{12}{55}}$ 2

Then $P(R|F) = 1 - P(\bar{R}|F)$ 1

$$\frac{12}{55} = 1 - P(\bar{R}|F)$$

$$P(\bar{R}|F) = \underline{\frac{43}{55}}$$ 1