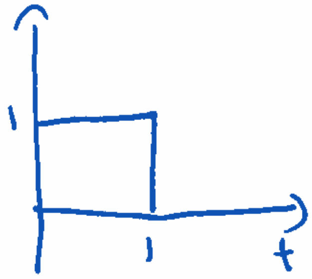
EE2-05 SOLUTIONS

Signals and Linear Systems

QUESTION 1

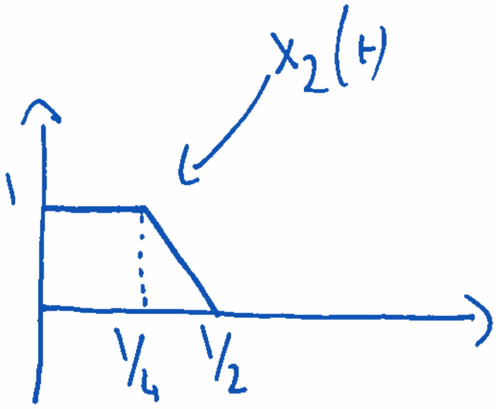
(a) i.

SINCE $u(t)-u(t-1) \rightarrow$ 

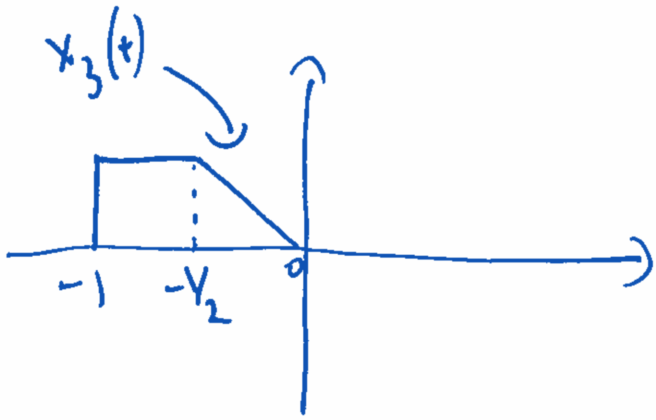
THEN

$x_1(t) = x(t)[u(t)-u(t-1)] \rightarrow$ 

ii.



iii.



(b)
$$x_{EVEN}(t) = \frac{1}{2}x(t) + \frac{1}{2}x(-t)$$
$$x_{odd}(t) = \frac{1}{2}x(t) - \frac{1}{2}x(-t)$$

THEREFORE
$$x_{EVEN}(t) = \frac{1}{2}e^{-t}u(t) + e^{t}u(-t)$$
$$x_{odd}(t) = e^{-t}u(t) - e^{t}u(-t)$$

(c) THE SYSTEMS ARE STABLE IF THEIR POLES ARE ON THE LEFT HAND SIDE OF THE S PLANE, THAT IS, IS THEY HAVE NEGATIVE REAL PART

SO $H_1(s)$ IS STABLE SINCE
$$(s^2+2s+1) = (s+1)^2 \Rightarrow s_0 = s_1 = -1$$

~~AND IS STABLE~~

AND $H_2(s)$ IS ALSO STABLE SINCE THE POLES ARE

$s_0 = -1 + j$ AND $s_1 = -1 - j$

(d) USING CONVOLUTION FORMULA

WE HAVE THAT THE OUTPUT

OF AN LTI SYSTEM IS:

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)\, x(\tau)\, d\tau$$

SO FOR $h_1(t)$ THE OUTPUT

AT INSTANT t_0 IS

$$y(t_0) = \int_{t_0-1}^{t_0} h_1(t-\tau)\, x(\tau)\, d\tau$$

THIS MEANS THAT THE OUTPUT DEPENDS ON THE PAST, THEREFORE THE SYSTEM HAS MEMORY AND IS CAUSAL.

USING A SIMILAR TYPE OF ARGUMENT WE SEE THAT $h_2(t)$ IS NOT CAUSAL

(a) FOR AN LTI SYSTEM WITH UNIT IMPULSE RESPONSE $h(t)$ WE HAVE THAT THE INPUT-OUTPUT RELATION IS

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)\, x(\tau)\, d\tau$$

IN OUR CASE

$$y(t) = \int_{t}^{\infty} e^{-2(t-\tau)}\, x(\tau)\, d\tau = \int_{-\infty}^{\infty} e^{-2(t-\tau)}\, u(\tau-t)\, x(\tau)\, d\tau$$

BY COMPARING THE TWO EXPRESSIONS WE CONCLUDE THAT $h(t) = e^{-2t}\, u(-t)$

(b) i. CHARACTERISTIC POLYNOMIAL:

$$s^2 + 6s + 5$$

CHARACTERISTIC ROOTS

$$\lambda_1 = -1 \qquad \lambda_2 = -5$$

CHARACTERISTIC MODES

$$e^{-t}, \; e^{-5t}$$

ii. ZERO INPUT COMPONENT MEANS $x(t) = 0$

$$y_0(t) = c_1 e^{-t} u(t) + c_2 e^{-5t} u(t)$$

$$\left.\begin{aligned} y_0(0) &= c_1 + c_2 = 1 \\ \dot{y}_0(0) &= -c_1 - 5c_2 = 1 \end{aligned}\right\} \Rightarrow \begin{aligned} c_2 &= -\frac{1}{2} \\ c_1 &= \frac{3}{2} \end{aligned}$$

SO $\quad y_0(t) = \left(\frac{3}{2} e^{-t} - \frac{1}{2} e^{-5t}\right) u(t)$

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IN THE LAPLACE DOMAIN
WE HAVE :

$$(s^2 + 6s + 5) Y(s) = s X(s)$$

SINCE

$$x(t) = e^{-5t} u(t) \Leftrightarrow \frac{1}{s+5}$$

THEN

$$Y(s) = \frac{s}{(s+5)(s+1)(s+5)}$$

$$= \frac{A}{s+1} + \frac{B}{s+5} + \frac{C}{(s+5)^2}$$

$$= -\frac{1}{16} \frac{1}{s+1} + \frac{1}{16} \frac{1}{s+5} + \frac{5}{4} \frac{1}{(s+5)^2}$$

$$y(t) = \left(-\frac{1}{16}e^{-t} + \frac{1}{16}e^{-5t} + \frac{5}{4}te^{-5t}\right)u(t)$$

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TOTAL RESPONSE $y_{TOT}(t) = y_0(t) + y(t)$

$$= \left(\frac{23}{16}e^{-t} - \frac{7}{16}e^{-5t} + \frac{5}{4}te^{-5t}\right)u(t)$$

(g)

$$X(s) = \frac{1}{(s+1)(s+2)(s+3)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+3}$$

$$= \frac{1}{2}\frac{1}{s+1} - \frac{1}{s+2} + \frac{1}{2}\frac{1}{s+3}$$

THEREFORE $x(t) = \left(\frac{1}{2}e^{-t} - e^{-2t} + \frac{1}{2}e^{-3t}\right)u(t)$

(h)

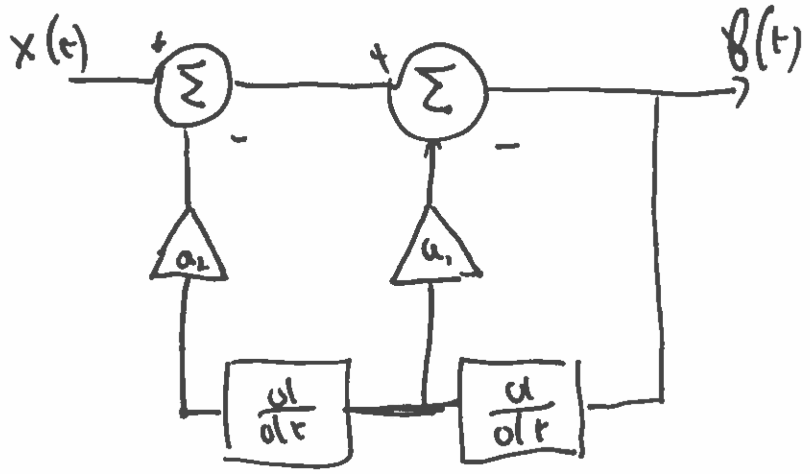
$$X(z) = \frac{5}{5z-1} + \frac{3z}{3z-1}$$

ROC $|z| > \frac{1}{3}$

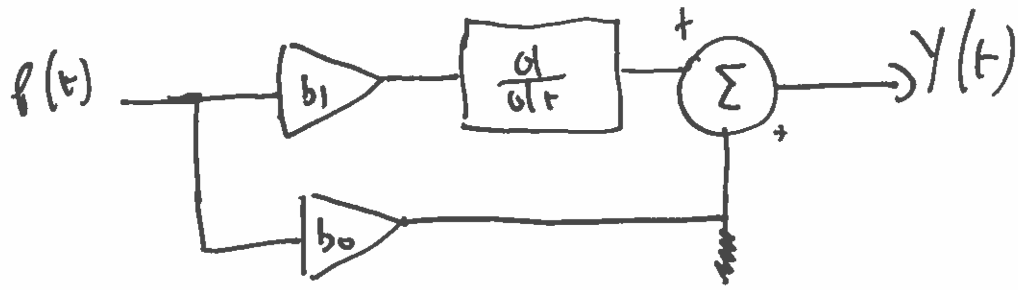
QUESTION 2

(a) THE OVERALL SYSTEM IS THE CASCADE OF TWO SYSTEMS:

SYSTEM A :



AND SYSTEM B :



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SO WE DETERMINE THE TWO TRANSFER FUNCTIONS FIRST

SYSTEM A :

$$f(t) = x(t) - a_2 f''(t) - a_1 f'(t)$$

$$a_2 f''(t) + a_1 f'(t) + f(t) = x(t) \Rightarrow$$

$$F(s) = \frac{1}{1 + a_1 s + a_2 s^2} X(s)$$

THEREFORE

$$H_A(s) = \frac{F(s)}{X(s)} = \frac{1}{a_2 s^2 + a_1 s + 1}$$

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FOR SYSTEM B:

$$y(t) = b_1 f'(t) + b_0 f(t)$$

$$\Rightarrow \quad H_B(s) = \frac{Y(s)}{F(s)} = b_1 s + b_0$$

THE TRASFER FUNCTION OF THE COMPLETE SYSTEM IS THUS GIVEN BY

$$H(s) = H_B(s) H_A(s) = \frac{b_1 s + b_0}{a_2 s^2 + a_1 s + 1}$$

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(b)

SINCE

$$\frac{Y(s)}{X(s)} = \frac{b_1 s + b_0}{a_2 s^2 + a_1 s + 1},$$

WE HAVE THAT $(a_2 s^2 + a_1 s + 1) Y(s) = (b_1 s + b_0) X(s)$

AND IN TIME DOMAIN:

$$a_2 \ddot{y}(t) + a_1 \dot{y}(t) + y(t) = b_1 \dot{x}(t) + b_0 x(t)$$

(c) WE USE THE FACT THAT WHEN THE INPUT TO AN LTI SYSTEM $H(s)$ IS $e^{j\omega_0 t}$ THEN THE OUTPUT IS

$$y(t) = H(j\omega_0)\, e^{j\omega_0 t}$$

IN OUR CASE

$$H(s) = \frac{b_0}{a_2 s^2 + 1}$$

AND THE INPUT IS $x(t) = e^{jt} + e^{j2t}$

SO THE OUTPUT IS

$$y(t) = H(j)\, e^{jt} + H(2j)\, e^{j2t}$$

$$= \frac{b_0}{1-a_2} e^{jt} + \frac{b_0}{1-4a_2} e^{j2t}$$

SINCE WE WANT $y(t) = 2e^{jt} + e^{j2t}$

WE NEED TO IMPOSE

$$\begin{cases} \dfrac{b_0}{1-a_2} = 2 \\ \dfrac{b_0}{1-4a_2} = 1 \end{cases} \Rightarrow \quad \begin{array}{l} a_2 = -\frac{1}{2} \\ b_0 = 3 \end{array}$$

(d) $H(s) = \dfrac{s+1}{2s^2+3s+1}$

$$x(t) = e^{-t}u(t) \iff X(s) = \frac{1}{s+1}$$

THEREFORE $Y(s) = H(s)X(s) = \dfrac{1}{2s^2+3s+1}$

THE ROOTS OF $s^2 + \frac{3}{2}s + \frac{1}{2}$ ARE

$s_0 = -1$, $s_1 = -1/2$, ~~WE~~ THEREFORE WE HAVE:

$$Y(s) = \frac{1}{2(s+1)(s+\frac{1}{2})} = \frac{A}{2(s+1)} + \frac{B}{(s+1/2)}$$

$$= \frac{1}{s+1/2} - \frac{1}{(s+1)}$$

BY USING $e^{-\lambda t}u(t) \iff \dfrac{1}{s+\lambda}$

WE OBTAIN

$$y(t) = \left(e^{-\frac{t}{2}} - e^{-t}\right)u(t)$$

QUESTION 3

(a) $$Y(s) = H_0(s)\left[X(s) - H_1(s)\,Y(s)\right];$$

$$Y(s) + H_0(s)H_1(s)Y(s) = H_0(s)X(s)$$

SINCE $H_1(s) = K$ WE HAVE THAT

$$Y(s) = \frac{H_0(s)}{1 + K H_0(s)} X(s)$$

AND THAT

$$H(s) = \frac{Y(s)}{X(s)} = \frac{H_0(s)}{1 + K H_0(s)}.$$

(b) UNDER THE ASSUMPTION $H_0(s) = \frac{2}{s-3}$,

WE HAVE THAT

$$H(s) = \frac{2}{s - 3 + 2K}$$

THE SYSTEM IS STABLE WHEN ITS POLES ARE IN THE LEFT HALF PLANE (i.e. REAL PART OF THE POLES ARE NEGATIVE)

IN THIS CASE THE POLE IS AT

$$s_0 = 3 - 2K$$

SO THE SYSTEM IS STABLE

WHEN $3-2K<0 \Rightarrow K>3/2$

(c) i. $H_0(s) = \frac{2}{s+2}$

$$H_0(\omega)\Big|_{\omega=0} = \frac{2}{2+j\omega}\Big|_{\omega=0} = 1$$

SO WE WANT ω_0 SUCH THAT

$|H_0(\omega_0)|^2 = \frac{1}{2}$. SINCE

$|H_0(\omega_0)|^2 = \frac{4}{4+\omega_0^2}$, WE WANT

$$\frac{4}{4+\omega_0^2} = \frac{1}{2} \Rightarrow \omega_0^2+4 = 8 \Rightarrow \omega_0 = 2$$

SO THE ESSENTIAL BANDWIDTH IS

$B = 2 \text{ rad/s}$

ii. $H(s) = \frac{2}{s+2+2K}$

(c) i.i.

$$H(s) = \frac{2}{s + 2 + 2K}$$

$$H(\omega)\Big|_{\omega = 0} = \frac{2}{2 + 2K}$$

SO THE ESSENTIAL BANDWIDTH IS GIVEN BY ω_0 SUCH THAT

$$|H(\omega_0)|^2 = \frac{4}{\omega^2 + 4(1+K)^2} = \frac{2}{4(1+K)^2}$$

$$\Rightarrow \quad \omega_0 = 2(1+K)$$

WE WANT $\omega_0 = 2B \Rightarrow 2(1+K) = 4$

CONSEQUENTLY $K = 1$.