Maths for Signals and Systems Exam 2014-Solutions

1. a) (i) I is the identity matrix, O are zero matrices and F is a matrix that is related to the special solutions of the system.
The dimensions of the individual matrices are given in the subscripts $R=\begin{bmatrix} I_{r\times r} & F_{r\times(n-r)} \\ 0_{(m-r)\times r} & 0_{(m-r)\times(n-r)} \end{bmatrix}$ The subscripts in the individual matrices reveal their corresponding sizes. [2]

(ii) Due to the special column rearrangement of R the special solution vectors contain the pivot variables in their first r elements and the free variables in their the last $n-r$ elements. **As already mentioned above**, each special solution has one free variable equal to 1 and the other free variables are all zero. Therefore, the null space matrix N is given by $N=\begin{bmatrix} X_{r\times(n-r)} \\ I_{(n-r)\times(n-r)} \end{bmatrix}$ where $X_{r\times(n-r)}$ is an unknown matrix of size $r\times(n-r)$.

Knowing that $RN=\mathrm{O}$ we get:

$$RN=\begin{bmatrix} I_{r\times r} & F_{r\times(n-r)} \\ 0_{(m-r)\times r} & 0_{(m-r)\times(n-r)} \end{bmatrix}\begin{bmatrix} X_{r\times(n-r)} \\ I_{r\times(n-r)} \end{bmatrix}$$

$$=\begin{bmatrix} I_{r\times r}\times X_{r\times(n-r)}+F_{r\times(n-r)}\times I_{r\times(n-r)} \\ \mathrm{O}_{(m-r)\times(n-r)} \end{bmatrix}=\begin{bmatrix} X_{r\times(n-r)}+F_{r\times(n-r)} \\ \mathrm{O}_{(m-r)\times(n-r)} \end{bmatrix}$$

$$X_{r\times(n-r)}+F_{r\times(n-r)}=\mathrm{O}_{r\times(n-r)} \Rightarrow X_{r\times(n-r)}=-F_{r\times(n-r)}$$

Therefore, $N=\begin{bmatrix} -F_{r\times(n-r)} \\ I_{(n-r)\times(n-r)} \end{bmatrix}$. [2]

(iii) $R=\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ [2]

(iv) We know that $EA=R$ where $E=\prod_{ij} E_{ij}$ is the product of all elimination matrices used in the procedure. If the rank of matrix A is r then the last $m-r$ rows of R are zero rows. Therefore, from the equation $EA=R$ we see that each of the last $m-r$ rows of E multiplied with A from the left gives a zero row vector. This verifies the fact that the last $m-r$ rows of E belong to the left null space, since they satisfy the relationship $x^T A=0^T$. Due to the method that we use to construct E, it can be shown easily that E is a full rank matrix (rank is m) and therefore its last $m-r$ rows are independent. Since these rows belong to the left null space and knowing that the left null space has dimension $m-r$, we can say that the last $m-r$ rows of E form a basis of the left null space. [2]

b) (i) $\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ [2]

(ii) 5(row1)+4(row2) [2]

(iii) A has rank 2 and A^T is 4 by 3 so its null space has dimension 3-2=1. [2]

c) (i) The pivots of A^{-1} are equal to 1/(pivots of A) because $\det A^{-1}=1/(\det A)$. [2]

(ii) Multiply row 1 by A^{-1} and add to row 2 to obtain $\begin{bmatrix} A & I \\ O & A^{-1} \end{bmatrix}$ [2]

(iii) The determinant is +1. Exchange the first n columns with the last n. This produces a factor $(-1)^n$ and leaves $\begin{bmatrix} I & A \\ O & -I \end{bmatrix}$ which is triangular with determinant $(-1)^n$. Then $(-1)^n(-1)^n = +1$. [2]

2. a) (i) $P = A(A^T A)^{-1} A^T$ [2]

(ii) $A^T A$ is symmetric and therefore $(A^T A)^{-1}$ is symmetric. (To prove this we use the property $(A^{-1})^T = (A^T)^{-1}$.)

$P^T = [A(A^T A)^{-1} A^T]^T = (A^T)^T [(A^T A)^{-1}]^T A^T = A(A^T A)^{-1} A^T = P$

$P^2 = [A(A^T A)^{-1} A^T][A(A^T A)^{-1} A^T] = [A(A^T A)^{-1}(A^T A)(A^T A)^{-1} A^T] = P$

If A is square and invertible its column space is the entire n-dimensional space and therefore the projection of b onto A should be b. In that case $P = AA^{-1}(A^T)^{-1}A^T = I$. [2]

(iii) If b is perpendicular to the column space of A then $Pb = AA^{-1}(A^T)^{-1}A^T b = 0$. [2]

(iv) $e = b - Pb$, $A^T e = A^T b - A^T p = A^T b - A^T Pb = 0$ [2]

b) (i) The projection matrix P is of the form $P = A(A^T A)^{-1} A^T$ with A being the column vector $[1 \quad 2 \quad -4]^T$. Therefore, it projects onto the column space of A which is the line $c[1 \quad 2 \quad -4]^T$. [2]

(ii) The error is $e = b - Pb = \frac{1}{21}\begin{bmatrix} 22 \\ 23 \\ 17 \end{bmatrix}$ and the distance is $\|e\| = \frac{\sqrt{1302}}{21}$. [2]

(iii) Since P projects onto a line, its three eigenvalues are 0,0,1. Since P is symmetric, it has a full set of (orthogonal) eigenvectors, and is then diagonalizable. [2]

c) (i) We have a set of equations

$$C - 2D = 0$$
$$C - D = 0$$
$$C = 1$$
$$C + D = 1$$
$$C + 2D = 1$$

and therefore the system is

$$\begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

The system doesn't have a solution since the solutions that is obtained from 2 of the equations doesn't satisfy the rest. [2]

(ii) The projection matrix is

$$\begin{bmatrix} 3/5 & 2/5 & 1/5 & 0 & -1/5 \\ 2/5 & 3/10 & 1/5 & 1/10 & 0 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 0 & 1/10 & 1/5 & 3/10 & 2/5 \\ -1/5 & 0 & 1/5 & 2/5 & 3/5 \end{bmatrix}$$

and the projection vector is

$$\begin{bmatrix} 0 \\ 3/10 \\ 3/5 \\ 9/10 \\ 6/5 \end{bmatrix}$$

Approximate solution is $C=6/10$ and $D=3/10$. Straight line is $6/10+3t/10$. [2]

(iii) error vector is

$$e=b-p=\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}-\begin{bmatrix} 0 \\ 3/10 \\ 3/5 \\ 9/10 \\ 6/5 \end{bmatrix}=\begin{bmatrix} 0 \\ -3/10 \\ 2/5 \\ 1/10 \\ -1/5 \end{bmatrix}$$

[2]

3. a) (i) [2]

$$\mathcal{N}(A) = span\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right\}$$ (Recall that applying A to a vector of potentials gives the potential drops along edges, so in order for a vector of potentials to be in the null space, all the potentials within one connected component must be the same.)

(ii)-(iii) [2], [2]

$B = \vec{1}\cdot\vec{1}^T = 4(\vec{1}/2)(\vec{1}/2)^T$, where $\vec{1}$ is the all-ones vector in $\mathbb{R}^4$. So B has eigenvalues $4, 0, 0, 0$.

I and B diagonalize in the same eigenbasis, so $\lambda_i(4I - B) = \lambda_i(4I) - \lambda_i(B) = 4\lambda_i(I) - \lambda_i(B)$ for all i. So the eigenvalues of A^TA are $0, 4, 4, 4$.

(iv)

$\sigma_i = \sqrt{\lambda_i(A^TA)}$, so the nonzero singular values are $2, 2, 2$. We only need to find one eigenvector of A^TA. An obvious one is $\vec{1}/2$, since all the rows sum up to 0. [2]

b) Really easy bookwork

(i) [2]

(ii) [2]

Let v_1, v_2, v_3, v_4 be the column vectors of A. Set

$$w_1 = v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Then $v_1 = w_1$. Set

$$\begin{aligned} w_2 &= v_2 - \frac{\langle w_1, v_2 \rangle}{\langle w_1, w_1 \rangle} w_1 \\ &= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \\ -1/2 \end{bmatrix}. \end{aligned}$$

Then $v_2 = (1/2)w_1 + w_2$. Set

$$\begin{aligned} w_3 &= v_3 - \frac{\langle w_1, v_3 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle w_2, v_3 \rangle}{\langle w_2, w_2 \rangle} w_2 \\ &= \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 1 \\ -1/2 \end{bmatrix} = 0. \end{aligned}$$

Then $v_3 = (3/2)w_1 + w_2 + w_3$. Set

$$\begin{aligned} w_4 &= v_4 - \frac{\langle w_1, v_4 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle w_2, v_4 \rangle}{\langle w_2, w_2 \rangle} w_2 \\ &= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1/2 \\ 1 \\ -1/2 \end{bmatrix} \\ &= \begin{bmatrix} -2/3 \\ 2/3 \\ 2/3 \end{bmatrix}. \end{aligned}$$

[8]

Then $v_4 = (1/2)w_1 + (1/3)w_2 + w_4$. Thus matrixes Q and R for QR-decomposition of A are as follows:

$$Q = \begin{bmatrix} 1 & 1/2 & 0 & -2/3 \\ 0 & 1 & 0 & 2/3 \\ 1 & -1/2 & 0 & 2/3 \end{bmatrix},$$

$$R = \begin{bmatrix} 1 & 1/2 & 3/2 & 1/2 \\ 0 & 1 & 1 & 1/3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$