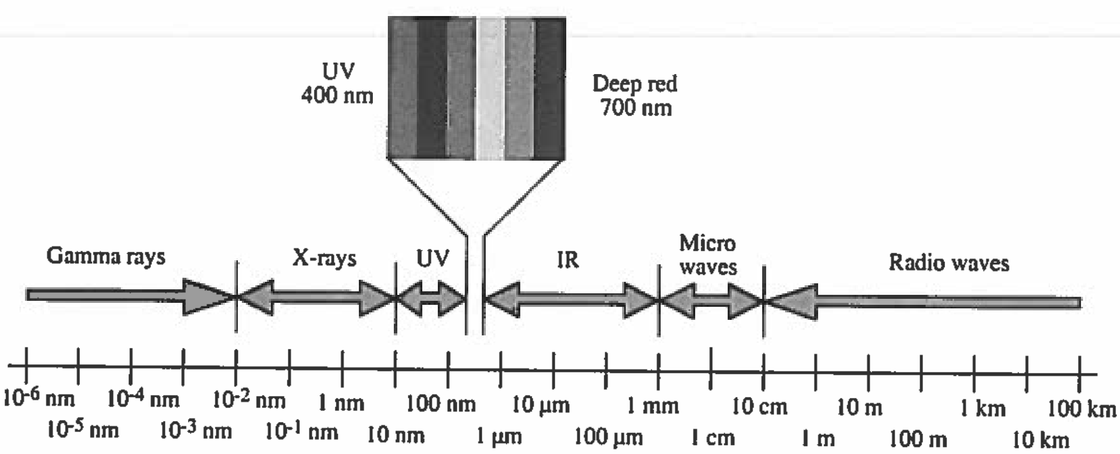
<u>Electromagnetic Fields 2015 – Solutions</u>

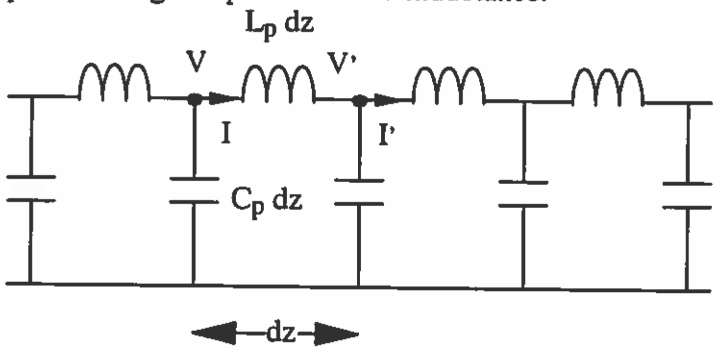
1. a) The electromagnetic spectrum has enormous range, and includes gamma rays (at the shortest wavelengths), X-rays, light waves, heat waves, microwaves and radio waves (at the longest wavelengths). Visible light is concentrated into a very short span between ≈ 0.4 μm and 0.8 μm.

[3]



[5]

b) The equivalent circuit of a transmission line is a ladder consisting of sections of length dz, where dz is very small. Each has inductance L_p dz and capacitance C_p dz, where C_p and L_p are the per-unit length capacitance and inductance.



[3]

Assigning nodal voltages V, V' and currents I, I' we obtain at angular frequency ω,

$V' = V - j\omega L_p I\, dz \qquad I' = I - j\omega C_p V\, dz$

However, if instead we write:

$V' = V + (dV/dz)\, dz \quad I' = I + (dI/dz)\, dz$

Then by comparison:

$dV/dz = -j\omega L_p I \qquad dI/dz = -j\omega C_p V$

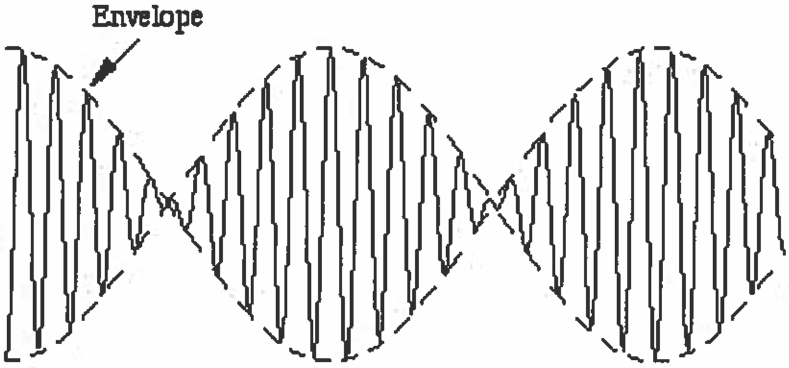
These are the transmission line equations

[5]

c) The phase velocity v_{ph} gives the speed of a single wave. Unfortunately a single wave cannot carry any information, since it never varies. To send some data, we need to modulate a carrier. The envelope (which contains the information) then travels at a slightly different speed, the group velocity v_g (which represents the velocity of a group of waves).

To calculate v_g, consider the simplest possible AM signal, formed by beating together two signals of different angular frequencies $\omega + d\omega$ and $\omega - d\omega$. The corresponding k-values at these frequencies are $k + dk$ and $k - dk$. For equal amplitudes, the combined voltage is:
$V = V_0 [\exp\{j((\omega + d\omega)t - (k + dk)z)\} + \exp\{j((\omega - d\omega)t - (k - dk)z)\}]$
This result can be written alternatively as $V = 2V_0 \exp\{j\omega t - kz)\} \cos\{d\omega t - dk z\}$.
Hence, the wave is an amplitude-modulated carrier as shown below. The velocity of the carrier is $v_p = \omega/k$ as before. The velocity of the envelope is $v_g = d\omega/dk$.

[5]

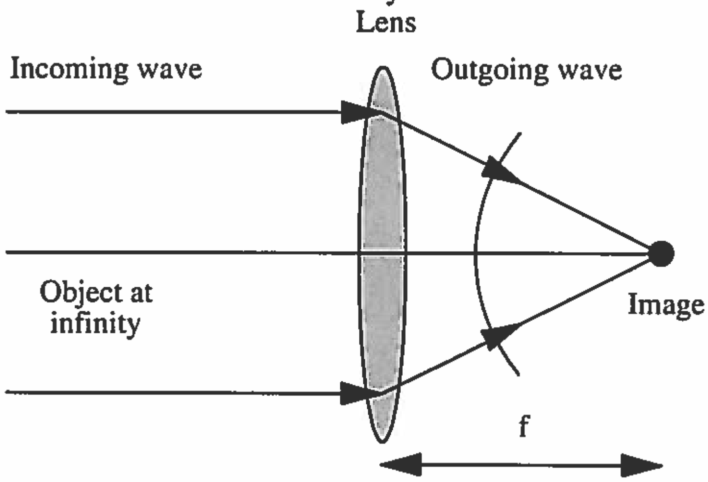


[3]

d) The imaging equation can be used to find the image position when a lens is used for imaging. The formula is $1/u + 1/v = 1/f$, where u is the object distance, v is the image distance and f is the focal length of the lens.

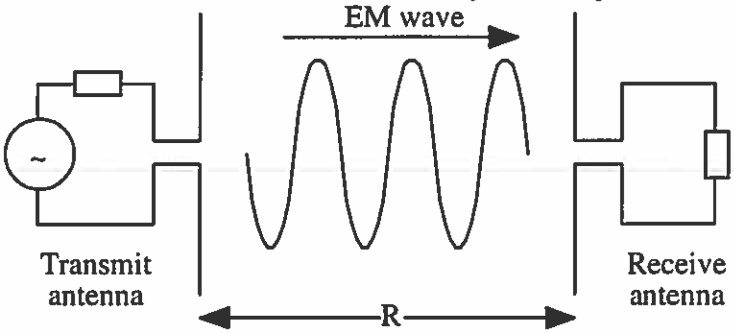
[3]

For example, if u is infinite (parallel beam incident) then $1/\text{infinity} + 1/v = 1/f$, so $v = f$ and the beam is focussed a distance f away from lens



[5]

e) The Friis transmission formula allows the received power to be calculated in a radio system. A transmitter generating power P_T is connected to a transmit antenna whose parameters are D_T, η_T, and A_T. The receive antenna is R away and has parameters D_R, η_R, and A_R.



[3]

Assuming initially that the transmit antenna is loss-less and isotropic, the power density at radius R is found by averaging the transmit power P_T over a spherical surface of radius R, as $S_{ISO} = P_T/4\pi R^2$

Real antennas are neither loss-less nor isotropic, so the real power density at radius R is:

$S_{REAL} = \eta_T D_T S_{ISO}$

Substituting for the directivity, we then get:

$S_{REAL} = \eta_T (4\pi A_T/\lambda^2) S_{ISO}$

Substituting for the isotropic power density, we then get:

$S_{REAL} = P_T (\eta_T A_T/R^2\lambda^2)$

The internal power in the receive antenna is found by multiplying by the effective area, as:

$P_{INT} = S_{REAL} A_R$

Substituting for S_{REAL} we then get:

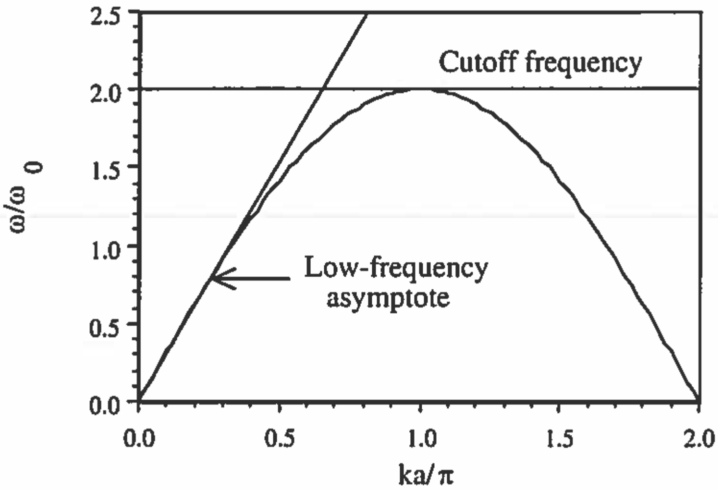
$P_{INT} = P_T (\eta_T A_T A_R/R^2\lambda^2)$

Taking into account the efficiency of the receive antenna, the power at the receiver is:

$P_R = \eta_R P_{INT} = P_T (\eta_T \eta_R A_T A_R/R^2\lambda^2)$ - the Friis formula.

[5]

2. a) The dispersion diagram is sinusoidal:



[3]

The cutoff frequency is $\omega = 2\omega_0$
Below cutoff, ka is real and propagating waves can exist.
Above cutoff, ka is imaginary so waves do not propagate but decay instead

[2]

At low frequency, $\omega \approx \omega_0 ka$.
Consequently, the phase velocity is $v_g = \omega/k = \sqrt{(a^2/LC)}$

[1]

b) A current wave travelling in the positive x-direction can be written as $I = (V_0/Z_0) \exp(-jkz)$
A voltage wave travelling in the negative x-direction can be written as $V = V_0 \exp(+jkz)$
A current wave travelling in the negative x-direction can be written as $I = -(V_0/Z_0) \exp(+jkz)$

[3]

Where k is the propagation constant and Z_0 is the characteristic impedance.

[2]

c) Assume the presence of incident and reflected waves in the first line and transmitted waves in the second line, and that the junction is at z = 0.

In the first line, the voltage and current waves can be written as:
$V_1 = V_I \exp(-jk_1z) + V_R \exp(+jk_1z)$
$I_1 = (V_I/Z_1) \exp(-jk_1z) - (V_R/Z_1) \exp(+jk_1z)$
In the second line, the voltage and current waves can be written as:
$V_2 = V_T \exp(-jk_2z)$
$I_2 = (V_T/Z_2) \exp(-jk_1z)$

[2]

Matching voltages and currents at the junction:
$V_I + V_R = V_T$
$(V_I/Z_1) - (V_R/Z_1) = (V_T/Z_2)$

[2]

Re-arranging:

$Z_2(V_I - V_R) = Z_1 V_T$

Eliminating V_T:

$Z_2(V_I - V_R) = Z_1(V_I + V_R)$

Re-arranging:

$V_I(Z_2 - Z_1) = V_R(Z_2 + Z_1)$

Hence, the voltage reflection coefficient is $R_V = V_R/V_I = (Z_2 - Z_1)/(Z_2 + Z_1)$

[3]

The transmission coefficient is $T_V = V_T/V_I = 1 + R_V = 2Z_2/(Z_2 + Z_1)$

[2]

When $Z_2 > Z_1$, T_V can be greater than unity; however this does not violate power conservation.

[1]

d) The power carried by the incident wave is:

$P_I = V_I^2/Z_1$

The power carried by the reflected wave is:

$P_R = V_R^2/Z_1 = (V_I^2/Z_1)\, R_V^2$

The power carried by the transmitted wave is:

$P_T = V_T^2/Z_2 = V_I^2\, T_V^2/Z_2 = (V_I^2/Z_1)\, T_V^2 Z_1/Z_2$

[3]

The power reflection and transmission coefficients are therefore:

$R_P = P_R/P_I = R_V^2 = (Z_2^2 - 2Z_2Z_1 + Z_1^2)/(Z_2^2 + 2Z_2Z_1 + Z_1^2)$

$T_P = P_T/P_I = T_V^2 Z_1/Z_2 = 4Z_2Z_1/(Z_2^2 + 2Z_2Z_1 + Z_1^2)$

[2]

Hence:

$R_P + T_P = (Z_2^2 - 2Z_2Z_1 + 4Z_2Z_1 + Z_1^2)/(Z_2^2 + 2Z_2Z_1 + Z_1^2) = 1$

And power is conserved

[2]

The minimum value of the power reflection coefficient is zero, obtained when $Z_1 = Z_2$. Hence, the maximum value of the power transmission coefficient is unity.

[2]

3 a) The time-averaged power flow is $\underline{S} = (1/T)\ {}_0\int^T \underline{S}\ dt$, or
$\underline{S} = (1/T)\ {}_0\int^T \underline{E} \times \underline{H}\ dt$, or
$\underline{S} = (1/T)\ {}_0\int^T \mathrm{Re}\{\underline{E}\exp(j\omega t)\} \times \mathrm{Re}\{\underline{H}\exp(j\omega t)\}\ dt$, or
$\underline{S} = (1/T)\ {}_0\int^T 1/4\{\underline{E}\exp(j\omega t) + \underline{E}^*\exp(-j\omega t)\} \times \{\underline{H}\exp(j\omega t) + \underline{H}^*\exp(-j\omega t)\}\ dt$

[3]

Product terms of the form $\underline{E} \times \underline{H}\exp(j2\omega t)$ and $\underline{E}^* \times \underline{H}^*\exp(-j2\omega t)$ will average to zero, leaving
$\underline{S} = (1/T)\ {}_0\int^T 1/4\{\underline{E} \times \underline{H}^* + \underline{E}^* \times \underline{H}\}\ dt$, or
$\underline{S} = (1/T)\ {}_0\int^T 1/2\ \mathrm{Re}\{\underline{E} \times \underline{H}^*\}\ dt$, or
$\underline{S} = 1/2\ \mathrm{Re}\{\underline{E} \times \underline{H}^*\}$

[3]

b) Assuming TE incidence from medium 1 at an angle θ_1, the amplitude reflection coefficient at an interface between two dielectric media with refractive indices n_1 and n_2 is:
$\Gamma_E = \{n_1\cos(\theta_1) - n_2\cos(\theta_2)\} / \{n_1\cos(\theta_1) + n_2\cos(\theta_2)\}$
Here, θ_2 is the angle of the transmitted wave in medium 2.

Now, Snell's law implies that $n_1\sin(\theta_1) = n_2\sin(\theta_2)$
Hence, $\sin(\theta_2) = (n_1/n_2)\sin(\theta_1)$
Total internal reflection starts to occur at the critical angle when $(n_1/n_2)\sin(\theta_1) = 1$
After this, there is no real solution for θ_2

[2]

Despite this, we may evaluate $\cos(\theta_2)$ as $\sqrt{\{1 - \sin^2(\theta_2)\}} = \sqrt{\{1 - (n_1/n_2)^2\sin^2(\theta_1)\}}$
Clearly, $\cos(\theta_2)$ is purely imaginary, and can be written as $\cos(\theta_2) = \pm j\sqrt{\{(n_1/n_2)^2\sin^2(\theta_1) - 1\}}$
For the +ve sign, $\Gamma_E = \{n_1\cos(\theta_1) - j\alpha\} / \{n_1\cos(\theta_1) + j\alpha\}$ where $\alpha = \sqrt{\{(n_1/n_2)^2\sin^2(\theta_1) - 1\}}$
This expression has the form $\Gamma_E = z/z^*$
Consequently the power reflectivity must be $\Gamma_E\Gamma_E^* = (z/z^*)(z^*/z) = 1$

[4]

c) The scalar wave equation for spherically symmetric electric fields $E(r)$ is:
$d^2E/dr^2 + (2/r)\ dE/dr + \omega^2\mu_0\varepsilon_0E = 0$
Hence $r\ d^2E/dr^2 + 2\ dE/dr + \omega^2\mu_0\varepsilon_0 rE = 0$

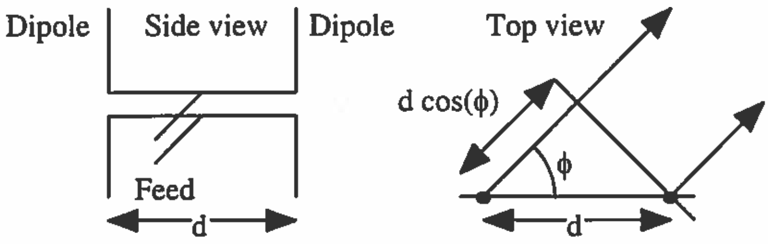
Define a new variable $F(r)$ such that $F = rE$
In this case, $dF/dr = r\ dE/dr + E$
And $d^2F/dr^2 = r\ d^2E/dr^2 + 2\ dE/dr$

[3]

Substituting into the wave equation, we get: $d^2F/dr^2 + \omega^2\mu_0\varepsilon_0F = 0$
This has the solution $F = E_0\exp(-jk_0r)$ where $k_0^2 = \omega^2\mu_0\varepsilon_0$
Hence, the solution for a spherical wave is $E = (E_0/r)\exp(-jk_0r)$

[3]

d) A two-element broadside antenna can be represented thus:



[1]

In a plane perpendicular to the conductors, the radiation pattern of a single dipole is isotropic. For both dipoles together, we must sum two similar contributions, taking account their relative phase.

At an angle ϕ from the axis, one wave travels d $\cos(\phi)$ further. Hence, if we define the electric field due to the RH dipole as $E_1 = E_0$, the field due to the LH dipole is $E_2 = E_0 \exp[-jk_0 d \cos(\phi)]$, where $k_0 = 2\pi/\lambda$ is the propagation constant.

The total field is then:
$E = E_1 + E_2 = E_0 \{1 + \exp[-jk_0 d \cos(\phi)]\}$
We can write this as
$E = 2E_0 \exp[-jk_0 d \cos(\phi)/2] \cos[k_0 d \cos(\phi)/2]$

The power density is $S = EE^*/2Z_0$, where Z_0 is the impedance of free space, so:
$S = (2E_0^2/Z_0) \cos^2[k_0 d \cos(\phi)/2]$
The normalised radiation pattern (S divided by its maximum value) is then:
$F = \cos^2[k_0 d \cos(\phi)/2]$

[4]

Clearly, this is maximum when $\phi = \pm\pi/2$, i.e. broadside on to the array.

[1]

e) The scalar wave equation for plane waves is:
$d^2E/dz^2 + \omega^2\mu_0\varepsilon_0 E = 0$
This has the solution $E = E_0 \exp(-jkz)$, where $k = \omega\sqrt{(\mu_0\varepsilon_0)}$

Assuming the material is a conductor, so $\varepsilon = \sigma/j\omega$, the propagation constant modifies to:
$k = \omega\sqrt{(\mu_0\sigma/j\omega)} = \sqrt{(-j\mu_0\sigma\omega)} = (1 - j)\sqrt{(\mu_0\sigma\omega/2)}$
We can write this as $k = k' - jk''$, where $k' = k'' = \sqrt{(\pi f\mu_0\sigma)}$

[3]

The wave propagates as $E = E_0 \exp(-jkz) = E_0 \exp(-jk'z) \exp(-k''z)$
The wave amplitude will fall to 1/e of its original value when $z = 1/k''$
This value of z is known as the skin depth, δ, and can be written as $\delta = 1/\sqrt{(\pi f\mu_0\sigma)}$

[3]