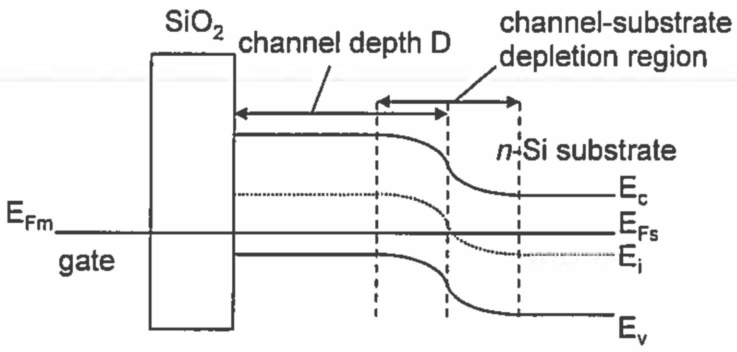
EE4-55

Solution 1:

a)

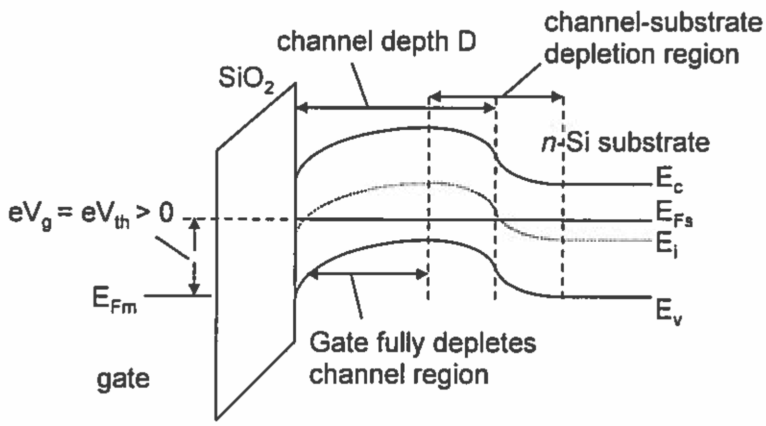
(i) For 'flat-band' conditions at zero applied bias, the energy band diagram along the line *y-y'* from the gate to the substrate region is as follows:



The diagram should show the *n*-substrate, *p*-implanted channel, oxide and gate regions, and reflect flat-band conditions.

[2]

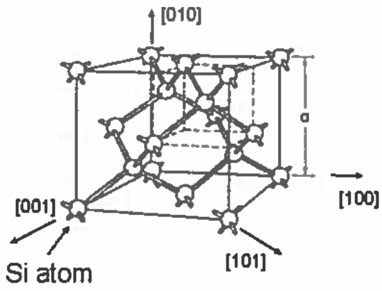
(ii) At the threshold voltage, $V_{gs} = V_{th}$, the band diagram along the line *y-y'* from the gate to the substrate region is as follows:



The diagram should show the correct bending of the band, the full depletion of the channel region due to both the gate, and the substrate.

[3]

b) Inspecting the unit cell:



Total number of atoms in the unit cell $N_t = 18$.

[1]

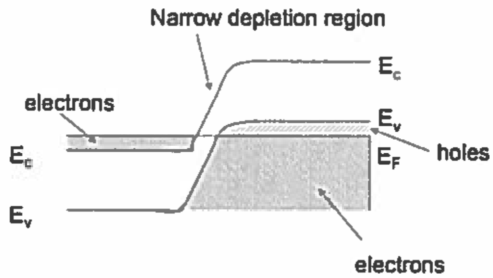
However, in the crystal a group of only $n = 8$ atoms repeats, the remaining 10 atoms lie in the next unit cells in the x, y and z directions.

[2]

Volume of unit cell $V = a^3 = 1.57 \times 10^{-28}$ m^3. Therefore, atomic concentration $D = n/V = 8/1.57 \times 10^{-28} = 5 \times 10^{28}$ atoms/m^3.

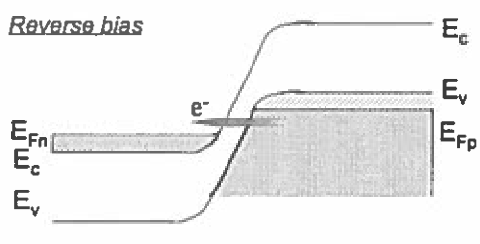
[2]

c) The Esaki tunnelling diode uses degenerately-doped n and p semiconductor regions to create conduction band – valance band tunnelling of electrons and holes across a narrow depletion region. Here, the heavy doping in the n and p regions restricts the depletion region width. The band diagram of the device at zero bias is shown below:



[2]

When the diode is reverse biased, electrons tunnel through the barrier formed by the energy gap in the depletion region, leading to a linear increase in current for reverse bias, electron flowing from the p (valance band) to n (conduction band) region:

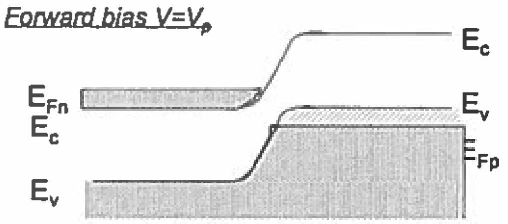


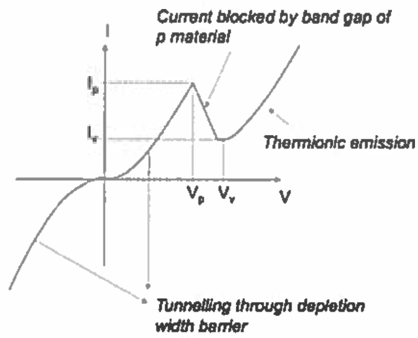
When the diode is forward biased, electrons tunnel from the conduction band of the *n* region to the valance band of the *p* region only as long as filled states in the conduction band overlap with empty states in the valance band (see the band diagram below, left).

[2]

As the bias increases further, the overlap ends and current falls, creating a current peak at V_p and a negative differential resistance region (see in the *I-V* characteristic below, right). Further increases in forward bias lead to increasing current flow due to electrons overcoming the *n-p* built-in potential barrier in the conduction band.

[1]





d) In the Deal-Grove equation, τ is the 'time' corresponding to the initial oxide thickness X_i = 20 nm. Working in Angstroms, we first calculate τ:

$$\tau = (X_i^2 + AX_i)/B = (500^2 + 0.33 \times 10^4 \times 500) / 2.86 \times 10^4 = 66.4 \text{ min.}$$

[2]

Therefore, at t = 120 minutes, we have:

$$X^2 + AX = B(t + \tau)$$

$$\Rightarrow \quad X^2 + (0.33 \times 10^4)X = (2.86 \times 10^4)(120 + 66.4) = 5.33 \times 10^6$$
$$\Rightarrow \quad X = (-3300 \pm \sqrt{(3300^2 - 4(-5.33 \times 10^6)})/2$$
$$\Rightarrow \quad X = 118.7 \text{ nm}$$

Here we have used the positive root only.

[3]

e) λ is the optical wavelength of the light source.

NA is the numerical aperture, which is a measure of the maximum off axis angle for rays forming the image. θ_m is the maximum ray angle, which is set by the lens, while n is the refractive index of the medium between the lens and the wafer.

k_l is a dimensionless parameter of order unity which depends primarily on the illumination conditions, but also on the resist contrast and the type of mask used (normal/phase-shift/proximity-corrected).

Typical values would be λ = 193 nm, NA = 1.3 (immersion), k_l = 0.4, giving R = 60 nm.

No - R sets the minimum half-pitch, but smaller features can be defined because the line/space ratio is not necessarily 1:1. [6]

f) The cavity etched from the back side will have {111} sidewalls which will be inclined at an angle of $\tan^{-1}(\sqrt{2})$ to the wafer surface. The opening at the back side will therefore need to be larger than the membrane, the difference in side length being $\sqrt{2}D$ where D is the etch depth. Also, the mask on the back side will be undercut by an amount $\sqrt{6}D/S$ where S is the anisotropy. The mask opening should therefore be a square with side given by:

$$W_m = W + \sqrt{2}D - \sqrt{6}D/S$$

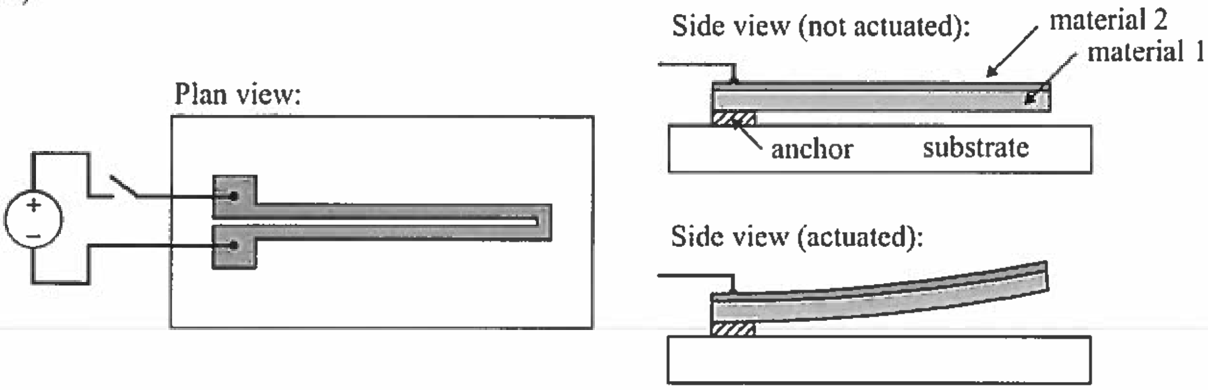
With W = 1 mm, D = 490 μm (wafer thickness – membrane thickness), and S = 30, this gives W_m = 1.653 mm.

With the above values of D and W_m, the extreme values of S give W = 1.008 mm (S = 25) and W = 0.994 mm (S = 35), so the uncertainty in the membrane side will be +8/-6 μm. [5]

g) The force on a current-carrying conductor in a magnetic field is $F = BIl$. (This is for a straight conductor of length l, aligned perpendicular to a uniform magnetic field of flux density B. However, these constraints will not restrict the validity of any scaling law we derive.)
Case (i): if the current density J is constant, then the current JA will be proportional to the conductor cross-section A, and hence will scale as L^2. The force will then scale as $F \propto L^3$.
Case (ii): the heat dissipated per unit length is $\rho J^2 A$ where ρ is the resistivity. For this to be constant we require $J \propto L^{-1}$ and hence $I \propto L$. In this case the force will scale as $F \propto L^2$.

In both cases B is assumed to be invariant which is reasonable (until things get very small).
The second scenario is more reasonable; it leads to invariance of the temperature rise as the device is scaled down. With the first scenario, smaller devices will run cooler. [5]

h)



This kind of actuator relies on differential thermal expansion between the two materials in a bi-layer. Typically it comprises a beam with a thin film coating of a different material. When the structure is heated, differential thermal expansion leads to a bending moment, causing the structure to deflect upwards (if $\alpha_{beam} > \alpha_{film}$) or downwards ($\alpha_{beam} < \alpha_{film}$). Heating is generally effected by passing a current through either the beam or the thin film. A typical material system might be polyimide beam + metal film or silicon beam with metal film. [4]

Solution 2:

a) General forms of the wave-functions ψ_1, ψ_2 and ψ_3 in the regions $x < -a/2$, $-a/2 \leq x \leq a/2$, and $x > +a/2$ respectively, are as follows:

Within the well: $\psi_2 = A\sin k_1 x + B\cos k_1 x$

[2]

Outside the well, the wave-function has the form:

$\psi = C\exp(k_2 x) + D\exp(-k_2 x)$

As the wave-function must be normalisable as x tends to $\pm\infty$, this implies:

$\psi_1 = C\exp(k_2 x)$, for $x < -a/2$

$\psi_3 = D\exp(-k_2 x)$, for $x > a/2$

[2 + 2]

b) The wave-vectors k_1 and k_2 within and outside the well respectively are as follows:

$$k_1 = \sqrt{\frac{2m\left(E+V_0\right)}{\hbar^2}} \text{ and } k_2 = \sqrt{\frac{2m\left(-E\right)}{\hbar^2}}$$

where $-E > 0$. This implies that k_2 is real.

[2 + 2]

c) Applying boundary conditions to match the wave-functions, and the derivative of the wave-functions at $x = a/2$ and $x = -a/2$:

At $x = a/2$:

$$A\sin(k_1 a/2) + B\cos(k_1 a/2) = D\exp(-k_2 a/2) \qquad \text{Eq. 1}$$
$$k_1 A\cos(k_1 a/2) - k_1 B\sin(k_1 a/2) = -k_2 D\exp(-k_2 a/2) \qquad \text{Eq. 2}$$

[2]

At $x = -a/2$:

$$-A\sin(k_1 a/2) + B\cos(k_1 a/2) = C\exp(-k_2 a/2) \qquad \text{Eq. 3}$$
$$k_1 A\cos(k_1 a/2) + k_1 B\sin(k_1 a/2) = k_2 C\exp(-k_2 a/2) \qquad \text{Eq. 4}$$

[2]

Simplifying Eqs. 1 – 4 as follows:

$$\text{Eq. 1} - \text{Eq. 3} \Rightarrow \; 2A\sin(k_1 a/2) = \left(D - C\right)\exp(-k_2 a/2) \qquad \text{Eq. 5}$$

Eq. 1 + Eq. 3 $\Rightarrow$ $2B\cos(k_1 a/2) = \left(D + C\right)\exp(-k_2 a/2)$ Eq. 6

[2]

Eq. 4 – Eq. 2 $\Rightarrow$ $2k_1 B\sin(k_1 a/2) = k_2\left(D + C\right)\exp(-k_2 a/2)$ Eq. 7

Eq. 4 + Eq. 2 $\Rightarrow$ $2k_1 A\cos(k_1 a/2) = -k_2\left(D - C\right)\exp(-k_2 a/2)$ Eq. 8

[2]

Equations 5 – 8 give us the relationships between the wave-vectors:

Eq. 8 / Eq. 5 $\Rightarrow$ $k_1\cot(k_1 a/2) = -k_2$ Eq. 9

Unless $C = D$ and $A = 0$

Eq. 7 / Eq. 6 $\Rightarrow$ $k_1\tan(k_1 a/2) = k_2$ Eq. 10

Unless $C = -D$ and $B = 0$

[2]

Equations 9 and 10 cannot be satisfied simultaneously, implying two types of solutions.

Finally, we may substitute $\alpha = k_1\frac{a}{2}$ and $\beta = k_2\frac{a}{2}$ into Eqs. 9 and 10 to give the form required by the question:

$\alpha\tan(\alpha) = \beta$ (Class 1 solutions)

and $-\alpha\cot(\alpha) = \beta$ (Class 2 solutions)

[2]

d) The Class 1 and 2 solutions are restricted by the following conditions:

For solutions of Class 1 to exist, we have $C = D$ and $A = 0$
For solutions of Class 2 to exist, we have $C = -D$ and $B = 0$

[2 + 2]

These restrictions give us the following forms for the wave-functions within the well:

Class 1: $A = 0 \Rightarrow \psi_2 = B\cos\left(k_1 x\right)$ Even solution

Class 2: $B = 0 \Rightarrow \psi_2 = A\sin\left(k_1 x\right)$ Odd solution

[2 + 2]

Solution 3:

a) (i) The energy levels E_n in a 1-D quantum well of width D, measured from the bottom of the well E_w, are given by $E_n = \frac{n^2\pi^2\hbar^2}{2mD^2}$ where $n \geq 1$. These levels are shown in Fig. A3.1 below:

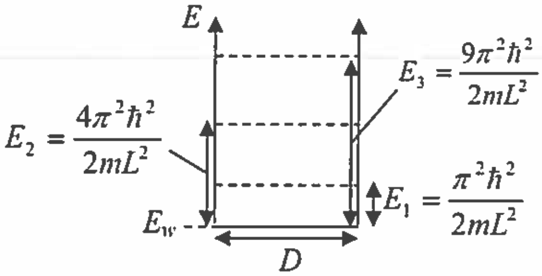


Figure A3.1

Using n = 1, 2 and 3, this gives E_1 = 3.5 meV, E_2 = 14 meV and E_3 = 31.5 meV.

[1+1+1]

To find the Fermi energy E_F relative to the bottom of the well E_F, we write the Fermi-Dirac distribution using the probability of occupation of the first energy level at $E_1 + E_w$:

$$f(E_1 + E_w) = \frac{1}{1+\exp\left(\frac{E_1 + E_w - E_F}{k_B T}\right)} = 0.01$$

[1]

$$\Rightarrow 1+\exp\left(\frac{E_1 + E_w - E_F}{k_B T}\right) = 100$$

$$\Rightarrow \exp\left(\frac{E_1 + E_w - E_F}{k_B T}\right) = 99$$

$$\Rightarrow E_1 + E_w - E_F = k_B T \ln 99$$

$$\Rightarrow E_1 + E_w - E_F = 1.38\times10^{-23}\times4.2\times4.59/1.6\times10^{-19} = 1.67eV$$

$$\Rightarrow E_F - E_w = E_1 - 1.67eV = 3.5eV - 1.67eV = 1.83eV$$

This implies that $E_1 > E_F > E_w$.

[2]

(ii) The probability of occupation of energy levels E_2 and E_3 is given by:

$$\therefore f(E_2 + E_w) = \frac{1}{1+\exp\left(\frac{E_2 + E_w - E_F}{k_B T}\right)} = \frac{1}{1+\exp\left(\frac{14-1.83}{0.361}\right)} = 2.3\times10^{-15}$$

$$f(E_3+E_w)=\frac{1}{1+\exp\left(\frac{E_3+E_w-E_F}{k_BT}\right)}=\frac{1}{1+\exp\left(\frac{31.5-1.83}{0.361}\right)}=1.6\times10^{-36}$$

This implies that both levels E_2 and E_3 are likely to remain unoccupied.

[2 +2]

b)

$$\psi = A\exp\left(\frac{-Zr}{a_0}\right)$$

Normalising in spherical coordinates:

$$\int_{r=0}^{\infty}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\psi_{100}^2\, r^2 \sin\theta\, dr\, d\theta\, d\phi = 1$$

$$\Rightarrow A^2\int_0^{\infty}\int_0^{\pi}\int_0^{2\pi} e^{\frac{-2Zr}{a_0}} r^2 \sin\theta\, dr\, d\theta\, d\phi$$ [4]

$$= A^2\int_0^{\infty} e^{\frac{-2Zr}{a_0}} r^2 dr \int_0^{\pi}\sin\theta\, d\theta \int_0^{2\pi} d\phi$$

$$= A^2 \left|-\cos\theta\right|_0^{\pi} \cdot \left|\phi\right|_0^{2\pi} \cdot \left(\underbrace{\left|\frac{r^2 e^{-2Zr/a_0}}{\frac{-2Z}{a_0}}\right|_0^{\infty}}_{0} - \int_0^{\infty}\frac{2r e^{\frac{-2Zr}{a_0}}}{\frac{-2Z}{a_0}}dr\right)$$ [2]

$$= -A^2(-1-1)\cdot(2\pi-0)\cdot\left(+\frac{2a_0}{2Z}\int_0^{\infty} r e^{\frac{-2Zr}{a_0}}dr\right)$$

$$= +8\pi A^2\left(\frac{a_0}{2Z}\left(\underbrace{\left|\frac{r e^{-2Zr/a_0}}{\frac{-2Z}{a_0}}\right|_0^{\infty}}_{0} - \int_0^{\infty}\frac{e^{-2Zr/a_0}}{-2Z/a_0}dr\right)\right)$$ [2]

$$= +8\pi A^2\left(\frac{a_0}{2Z}\left(+\frac{a_0}{2Z}\left|\frac{e^{\frac{-2Zr}{a_0}}}{\frac{-2Z}{a_0}}\right|_0^{\infty}\right)\right)$$

$$= +8\pi A^2\left(\frac{a_0}{2Z}\right)^2\cdot\left(-\frac{a_0}{2Z}(0-1)\right)$$

$$= +8\pi A^2\left(\frac{a_0}{2Z}\right)^3 = 1$$

$$\Rightarrow A = \sqrt{\frac{1}{\pi}}\left(\frac{Z}{a_0}\right)^{3/2}$$ [2]

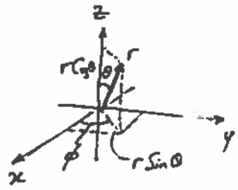
c) (i) The hybridised $|2p_x\rangle$, $|2p_y\rangle$ and $|2p_z\rangle$ states may be constructed as follows:

$$|2s\rangle = \psi_{200} \sim \left(1 - \frac{Zr}{2a_0}\right) e^{-Zr/2a_0}$$

$$|2p_0\rangle = \psi_{210} \sim e^{-Zr/2a_0} \cdot r\cos\theta$$

$$|2p_{\pm1}\rangle = \psi_{21,\pm1} \sim e^{-Zr/2a_0} \cdot e^{\pm i\phi} r\sin\theta$$

The $|2p\rangle$ states can be rewritten as (see diagram):



$$|2p_z\rangle = \psi_{210} \sim r\cos\theta\, e^{-Zr/2a_0}$$

$$= \underbrace{z}_{\text{axis}}\, e^{-Zr/2a_0}$$ [2]

and

$$|2p_{\pm1}\rangle = \psi_{21,\pm1} = e^{-Zr/2a_0} e^{\pm i\phi} r\sin\theta$$

$$= e^{-Zr/2a_0}(r\sin\theta)(\cos\phi \pm i\sin\phi)$$

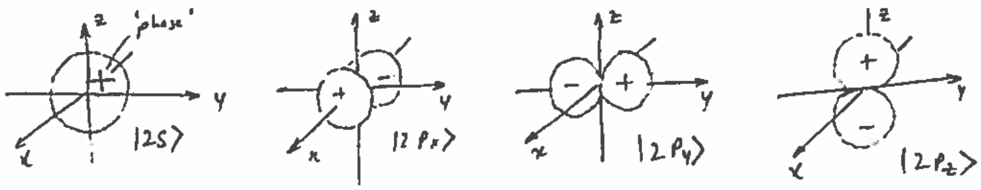
$$= e^{-Zr/2a_0}(x \pm iy)$$

We may then construct the linear combinations:

$$|2p_x\rangle = \frac{\psi_{21,1} + \psi_{21,-1}}{2} = x e^{-Zr/2a_0}$$ [2]

$$|2p_y\rangle = \frac{\psi_{21,1} - \psi_{21,-1}}{2i} = y e^{-Zr/2a_0}$$ [2]

(ii) The $|2s\rangle$, $|2p_x\rangle$, $|2p_y\rangle$ and $|2p_z\rangle$ states are sketched below:



[1 mark each]

Solution 4:

a) The bending equation for the deflection $v(x)$ of a buckled beam that is pinned at both ends and subject to an axial end load is:

$$EI\frac{d^2v}{dx^2} = -Fv$$

where E is Young's modulus, I is the second moment of area of the beam, and F is the axial load. The general solution is of the form:

$$v = A\cos\kappa x + B\sin\kappa x \quad , \quad \kappa = \sqrt{\frac{F}{EI}}$$

The boundary conditions are $v = 0$ at $x = 0$ and $x = L$, from which it follows that $A = 0$ and $\kappa = n\pi/L$ where n is an integer.

The lowest order solution, where n = 1, is the only one that occurs in practice, and for this solution we have $k = \pi/L$. The end load on the buckled beam, which is also the critical load for buckling, is therefore:

$$F = F_0 = EI\kappa^2 = \frac{\pi^2 EI}{L^2} \qquad [6]$$

b) From the information given in the question we can write the transverse stiffness k in the form:

$$k = k_0(1 - F/F_0) = \frac{k_0}{F_0}(F_0 - F)$$

But the stiffness of a flexure at zero axial load is $k_0 = 12EI/L^3$ so, using the result from part a), we have $k_0/F_0 = 12/(\pi^2 L)$, and the required result follows. [6]

c) The Joule heating will lead to a thermal strain $\varepsilon = \alpha\Delta T$ where ΔT is the average temperature rise and α is the CTE. This will produce an axial load of $F = whE\alpha\Delta T$ where E is Young's modulus and w & h are the width & depth of the beams. This will reduce the suspension stiffness, causing the resonant frequency to drop. [3]

For a 5% drop in resonant frequency (proportional to $\sqrt{k}$), the stiffness needs to fall by $\{1 - (0.95)^2\}$ = 9.75%. So, we require $F = 0.0975F_0$, or:

$$whE\alpha\Delta T = 0.0975\frac{\pi^2 Ew^3h}{12L^2} \quad \Rightarrow \quad \Delta T = 0.0975\frac{\pi^2 w^2}{12L^2\alpha} \qquad [3]$$

Putting w = 10 μm, L = 500 μm, $\alpha = 2.5 \times 10^{-6}$ /K gives **ΔT = 12.83 °C**. (13.16 °C corresponding to 10% assumed shift in k also acceptable.) [2]

If the device were operated in vacuum then the cooling of the proof mass would be much less effective and the proof mass would warm up. This would increase the thermal strain for a given temperature rise, so the temperature rise required for a given frequency shift would be smaller. [2]

c) The 1-D heat flow equation (see *Info for Candidates*) is:

$$\frac{d^2T}{dx^2} = -c \quad \text{where} \quad c = \frac{I^2R}{\kappa A} = \frac{I^2\rho}{\kappa A^2}$$

In a uniform beam, c is constant and we can integrate twice to obtain $T(x) = a + bx - cx^2/2$. In the present case the boundary conditions are assumed to be $T = 0$ at $x = 0$ and $x = L$. It follows that $a = 0$ and $b = cL/2$. The temperature profile is therefore $T(x) = c/2 \cdot (Lx - x^2)$ and the average temperature rise is:

$$\Delta T = \frac{cL^2}{12} = \frac{I^2\rho L^2}{12\kappa A^2}$$ [6]

With $\rho = 10\ \Omega\text{cm} = 0.1\ \Omega\text{m}$, $L = 500\ \mu\text{m} = 5 \times 10^{-4}$ m, $A = 10\ \mu\text{m} \times 20\ \mu\text{m} = 2 \times 10^{-10}\ \text{m}^2$, $\kappa = 150\ \text{Wm}^{-1}\text{K}^{-1}$, and $\Delta T = 12.83$ °C, this gives $\boldsymbol{I} = \mathbf{192\ \mu A}$. [2]

Solution 5:

a) In a piezoelectric material, applied stress will generate dielectric polarization and hence surface charge. This is the direct piezoelectric effect. A given stress component may generate polarisation in more than one axis, and the general relationship between the stress and the polarisation is of the form:

$$P_i = d_{ij}\sigma_j$$

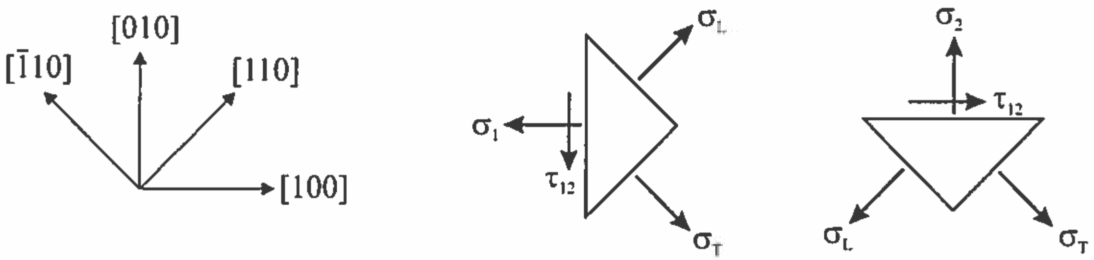
where d_{ij} is a 3 x 6 matrix of piezoelectric coefficients. (In the presence of electric field the above equation will also include a normal dielectric response term.) The effect is reversible, so an applied electric field will generate mechanical strain. The direct effect is used for transduction, while the inverse effect is used for actuation. The piezoelectric effect is observed only in non-centrosymmetric crystals, so silicon is not piezoelectric.

In a piezoresistive material, applied stress leads to a change in electrical resistivity. In general both the resistivity and the piezoresistive response will be anisotropic, and the generalised relation between electric field and current density is of the form:

$$\boldsymbol{E} = (\rho_0 + \boldsymbol{\Pi} \cdot \boldsymbol{\sigma}) \cdot \boldsymbol{J}$$

where ρ_0, Π and σ are all tensors. Usually symmetries allow these equations to be vastly simplified. [6]

b) Suggested approach is to consider the equilibrium of small prismatic elements (alternative would be to use coordinate rotation of stress tensor):



Resolving forces along [100] in the middle diagram (bearing in mind the relative lengths of the different sides) gives:

$$\sigma_1 \cdot \sqrt{2}a = \sigma_L a \cdot \frac{1}{\sqrt{2}} + \sigma_T a \cdot \frac{1}{\sqrt{2}} \quad \Rightarrow \quad \sigma_1 = \frac{\sigma_L + \sigma_T}{2}$$

Similarly:

$$\sigma_2 = \frac{\sigma_L + \sigma_T}{2} \quad \text{and} \quad \tau_{12} = \frac{\sigma_L - \sigma_T}{2}$$ [6]

To obtain the next result, we need to apply the standard piezoresistive equations (see *Info for Candidates*) to the case where the electric field and current both lie along the [110] direction. In this case the $\boldsymbol{E}$ and $\boldsymbol{J}$ vectors referred to the usual <100> axes are $\boldsymbol{E} = (E/\sqrt{2}, E\sqrt{2}, 0)$

and $J = (J/\sqrt{2}, J\sqrt{2}, 0)$, and we can write either the first or second piezoelectric equation as:

$$E/\rho_e = J[1 + \pi_{11}\sigma_1 + \pi_{12}\sigma_2] + J\pi_{44}\tau_{12}$$

Substituting for σ_1, σ_2 and τ_{12} from above, the resistivity becomes:

$$\frac{E}{J} = \rho_e\left\{1 + \pi_{11}\frac{\sigma_L + \sigma_T}{2} + \pi_{12}\frac{\sigma_L + \sigma_T}{2} + \pi_{44}\frac{\sigma_L - \sigma_T}{2}\right\} = \rho_e\{1 + \sigma_L\pi_L + \sigma_T\pi_T\}$$

The fractional change in resistivity, and hence in the overall resistance of the piezoresistor, is then:

$$\frac{\Delta R}{R} = \frac{\Delta\rho}{\rho_e} = \sigma_L\pi_L + \sigma_T\pi_T \qquad \text{as required}$$ [6]

c) Using the given equation, the maximum stress for the membrane in question at a unit applied pressure of 1 Pa will be $\sigma_{max} \approx 0.3 \times (20000/20)^2 = 3$ kPa. Also, from the given values for π_{ij}, the longitudinal and transverse piezoresistive coefficients are $\pi_L = 71.8 \times 10^{-11}$ Pa^{-1}, and $\pi_T = -66.3 \times 10^{-11}$ Pa^{-1}. [2]

One pair of piezoresistors (R_1, R_3 say) will experience predominantly longitudinal stress and will have a fractional resistance change of:

$$\Delta R_1/R_1 = \Delta R_3/R_3 = \sigma_{max}(\pi_L + \nu\pi_T) = +2.027 \times 10^{-6} = \alpha$$

The other pair will experience mainly transverse stress and will have a fractional resistance change of:

$$\Delta R_2/R_2 = \Delta R_4/R_4 = \sigma_{max}(\pi_T + \nu\pi_L) = -1.851 \times 10^{-6} = \beta$$ [4]

The bridge output voltage will then be:

$$V_{out} = \frac{(R_1R_3 - R_2R_4)}{(R_1 + R_2)(R_3 + R_4)}V_s = \frac{(1+\alpha)^2 - (1+\beta)^2}{(2+\alpha+\beta)^2}V_s$$

where we have used the fact that all four resistors have the same nominal value. With the above values for α and β, and with a bridge supply of $V_S = 5$ V, this gives **V_{out} = 9.7 μV/Pa.** [6]