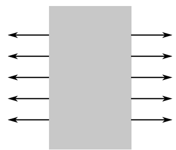
Energy Conversion: Answers

Question 1. calculated problem [a) = 5], [b) = 15], [c) = 20]

a) Symmetry considerations give the following three properties of the fields:

1. The magnitude of the field is constant in any plane parallel to the slab [1]. 2. If we choose two planes at the opposite sides of the slab but at the same distance from it, the fields will point in the opposite directions [1] but will have equal magnitudes [1]. It follows from the mirror symmetry. 3. The fields are directed perpendicular to the slab everywhere. It follows from the rotation symmetry. [1]

The field lines outside the slab are uniformly spaced horizontal lines pointing either towards or away from the slab. [1]



b) This is a symmetrical charge distribution, and we can apply Gauss's law to find the field. An appropriate choice of the Gauss surface is a cylinder or parallelepiped whose axis is perpendicular to the slab. We will take a cylinder here. The cylinder should be place symmetrically, so that the middle of its axis is in the middle of the slab. [3]

According to the field properties found above, the flux through the side of the cylinder is zero. The flux will be non-zero only through the cylinder bases. Assuming that the area of a base is S, the total flux is $\Phi = 2S\varepsilon_0 E$.

The flux must be equal to the charge enclosed by the surface. If the cylinder bases lie outside the slab, the charge enclosed is $q = \rho a S$. Then the field outside the slab is

$$E_{\text{out}} = \frac{\rho a}{2\varepsilon_0} \qquad [4]$$

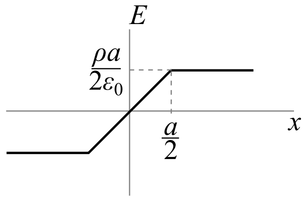
If the cylinder lies inside the slab, and its axis has the length $2x$, the charge enclosed is $q = \rho 2xS$. The field is then

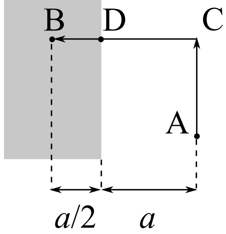
$$E_{\text{in}} = \frac{\rho x}{\varepsilon_0} \qquad [5]$$

The field variation on the distance is shown below assuming positive ρ [3]. Note the negative values for negative x. Also note that the field is continuous and is zero in the middle of the slab.

c) We can calculate the voltage by interating the field, $U_{\text{AB}} = \int_{\text{A}}^{\text{B}} (\mathbf{E} \cdot \mathrm{d}\mathbf{l})$. The path between points A and B can be arbitrary, but it is most convenient to go along the vertical and horizontal paths AC and CB as shown below [5].

The field is horizontal, so the integral along AC is zero and we have $U_{\text{AB}} = \int_{\text{C}}^{\text{B}} (\mathbf{E} \cdot \mathrm{d}\mathbf{l})$. We can split the path CB into two segments, CD and DB, along which the field behaves in a different manner. The field





is constant along CD, so that we have (note the negative sign)

$$U_{\mathrm{CD}} = -\frac{\rho a^2}{2\varepsilon_0} \qquad [6]$$

The field is varying along DB, so that

$$U_{\mathrm{DB}} = \frac{\rho}{\varepsilon_0}\int_{a/2}^{0} x\mathrm{d}x = -\frac{\rho a^2}{8\varepsilon_0} \qquad [6]$$

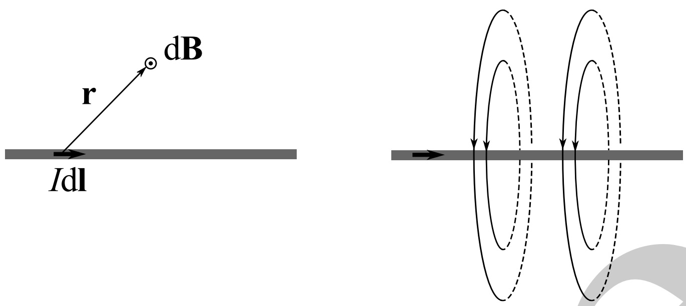
The answer is then

$$U_{\mathrm{AB}} = U_{\mathrm{CB}} = U_{\mathrm{CD}} + U_{\mathrm{DB}} = -\frac{\rho a^2}{2\varepsilon_0}(1+\frac{1}{4}) = -\frac{5\rho a^2}{8\varepsilon_0}$$

The sign of the voltage is negative for positive ρ, because the work to move a unit positive charge from A to B (which is voltage by definition) is done against the field [3].

Question 2. bookwork [a) = 5], [b) = 10], [c) = 15]

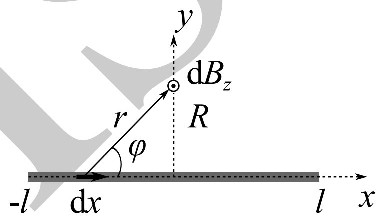
a) From the symmetry, the field must be the same on any circle whose centre coincides with the wire and whose plane is perpendicular to it [1]. According to the Biot-Savart law, a small straight element $\mathbf{dl}$ carrying a current I creates a magnetic field at a point $\mathbf{r}$ equal to $\mathrm{d}\mathbf{B} = \mu_0 I[\mathrm{d}\mathbf{l} \times \mathbf{r}]/(4\pi r^3)$ [1]. Assume that this point lies together with the wire in the plane of the paper, as shown below. The current flows from the left to the right. Using the right-hand rule, we find that $\mathrm{d}\mathbf{B}$ will be pointing out of the plane of the paper, perpendicular to it. The field created by any other element will have the same direction, so the total field will also point out of the plane of the paper. Therefore, the magnetic field lines will be circles as shown below. Their direction will be linked to the direction of the current by the right-hand rule [3].



b) Ampere's law says that the circulation of H along any path, $\oint_l (\mathbf{H}\cdot \mathrm{d}\mathbf{l})$, equals to the current enclosed [1]. Hence, the circulation along a field line shown above must be equal to I [3]. Because the field is constant along a field line and is parallel to $\mathrm{d}\mathbf{l}$, we have $\oint_l (\mathbf{H}\cdot \mathrm{d}\mathbf{l}) = H 2\pi R$, where R is the radius of the field line. Hence the field at a distance r from the wire is

$$H = \frac{I}{2\pi R} \qquad \left(B = \frac{\mu_0 I}{2\pi R}\right) \qquad [6]$$

c) We now have to derive the same result using the Biot-Savart law. We first assume that the wire has a finite length $2l$ and find the field above the centre of the wire. Place the x-axis along the wire, the y-axis perpendicular to it, and the origin at the centre of the wire (see below). [3]



Then for an small element we have $\mathrm{d}\mathbf{l} = \mathrm{d}x\mathbf{e}_x$, so that it creates the field equal to

$$\mathrm{d}B_z = \frac{\mu_0 I}{4\pi}\frac{\sin\varphi}{r^2}\mathrm{d}x \qquad [2]$$

where the angle φ is shown in the figure, and $\sin\varphi = R/r$, so that

$$\mathrm{d}B_z = \frac{\mu_0 I R}{4\pi}\frac{\mathrm{d}x}{r^3}$$

and (use the formula sheet)

$$B_z = \frac{\mu_0 I R}{4\pi}\int_{-l}^{l}\frac{\mathrm{d}x}{(x^2+R^2)^{3/2}} = \frac{\mu_0 I R}{4\pi}\frac{2l}{R^2\sqrt{l^2+R^2}} = \frac{\mu_0 I}{2\pi R}\frac{l}{\sqrt{l^2+R^2}} \qquad [5]$$

Assuming now that $l \to \infty$, we have $l/\sqrt{l^2+R^2} \to 1$, and

$$B_z = \frac{\mu_0 I}{2\pi R} \qquad [5]$$

Question 3. bookwork [a) = 5], [b) = 10], [c) = 15]

a) Two assumptions are made. First, that the flux flowing through the primary and secondary coils is the same (there is no flux leakage) [2]. Second, that H in the core is zero [2]. It is zero because μ_r is very high, but B is finite, so that $B = \mu_0\mu_r H$ requires negligible H [1].

b) Use the assumption of the total flux linkage, and denote the flux by Φ. The voltages in the primary and secondary coils are then, due to Faraday's law,

$$U_{1,2} = N_{1,2}\frac{\mathrm{d}\Phi}{\mathrm{d}t} \qquad [2]$$

Dividing one by the other, we get

$$\frac{U_1}{U_2} = \frac{N_1}{N_2} \qquad [3]$$

To find the relationship between the currents, use Ampere's law. Because $H = 0$, we have $0 = N_1 I_1 + N_2 I_2$ and

$$\frac{I_1}{I_2} = -\frac{N_2}{N_1} \qquad [5]$$

c) All the components are referred to the primary coil [1]. R_t characterises the Ohmic loss in the coils. [1]

X_t is due to the flux leakage [2]. Not all flux generated by the primary coil will reach the secondary one. It means that if we apply a voltage U_1 across the primary coil, not all of it will be able to generate a voltage across the secondary one. The reduction of the voltage can be modelled by X_t in series with the coil. [2]

X_m is due to H being non-zero in real cores [2]. Non-zero H leads, due to Ampere's law, to a reduced current I_2, which can be modelled by placing X_m in parallel with the coil. [2]

R_i characterises the loss in the core. It has two sources [1]. One is eddy currents in the conducting core excited by the ac magnetic field [2]. The other loss during the magnetisation/de-magnetisation cycles of the ferromagnetic material [2].