**E2.5 Signals and Linear Systems**
**Solutions 2008**

**All questions are unseen.**

**Question 1 is compulsory.**

**Answer to Question 1**

a)

If $x_1 \to y_1$ and $x_2 \to y_2$, for a linear system,

$k_1x_1 + k_2x_2 \to k_1y_1 + k_2y_2$ where $k_1$ and $k_2$ are constants.

[2]

b)

$$x(t) = e^{j\theta} = \cos\theta + j\sin\theta$$

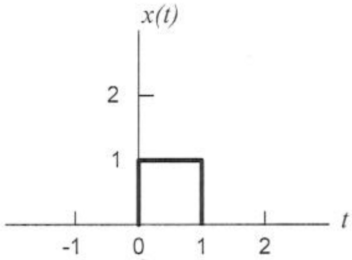
Therefore,

Even: $\cos\theta$

Odd: $j\sin\theta$.

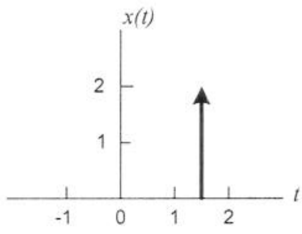
[2]

c) *i)*




[3]

*ii)*




[3]

d) *i*)

$$v_s(t) = Ri(t) + v_c(t)$$
$$v_c(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau$$
$$x(t) = v_s(t),\ \ y(t) = i(t).$$
$$\therefore\ \ Ry(t) + \frac{1}{C}\int_{-\infty}^{t} y(\tau)d\tau = x(t).$$

Differentiate both sides wrt $t$:

$$R\frac{dy}{dt} + \frac{1}{C}y(t) = \frac{dx}{dt}$$
$$\Rightarrow \frac{dy}{dt} + \frac{1}{RC}y(t) = \frac{1}{R}\frac{dx}{dt}$$

[3]

*ii*) Take Laplace transform on both sides:

$$(s + \frac{1}{RC})\,Y(s) = \frac{1}{R}sX(s).$$

$$\therefore H(s) = \frac{Y(s)}{X(s)} = C \times \frac{s}{RCs+1}.$$

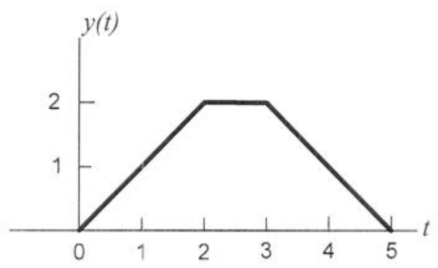
[3]

e)

$$\begin{aligned} y(t) &= h(t) * x(t) \\ &= (2e^{-3t} - e^{-2t})u(t) * e^{-t}u(t) \\ &= 2e^{-3t}u(t) * e^{-t}u(t) - e^{-2t}u(t) * e^{-t}u(t) \\ &= \left[\frac{2(e^{-t} - e^{-3t})}{2} - \frac{(e^{-t} - e^{-2t})}{1}\right]u(t) \\ &= \left(e^{-2t} - e^{-3t}\right)u(t) \end{aligned}$$

[4]

f)




[4]

g) The complex zeros are given by:

$$z^2 - z + \frac{5}{2} = 0\,.$$

Therefore the zeros are at:

$$z = \frac{1 \pm \sqrt{1\text{-}10}}{2} = \frac{1}{2} \pm j\frac{3}{2}.$$

The poles are given by:

$$p^2 + 5p + 5 = (p+4)(p+1) = 0.$$

Therefore the poles are at:

$$p = -1 \text{ and } p = -4.$$

[4]

h) By definition of Fourier transform,

$$\text{FT of } x(t-t_0) = \int_{-\infty}^{\infty} x(t-t_0)e^{-j\omega t}\,dt\,.$$

Let $\tau = t - t_0$,

$$\int_{-\infty}^{\infty} x(t-t_0)e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} x(\tau)e^{-j\omega(\tau+t_0)}\,d\tau$$

$$= e^{-j\omega t_0}\int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}\,d\tau$$

$$= e^{-j\omega t_0}X(\omega)$$

[4]

i) Divide F[z] by z, and perform partial fraction:

$$\frac{F[z]}{z} = \frac{z-7}{z^2-5z+4} = \frac{z-7}{(z-1)(z-4)} = \frac{2}{z-1} - \frac{1}{z-4}\,.$$

$$F[z] = 2\frac{z}{z-1} - \frac{z}{z-4}$$

$$\therefore\ f[k] = [2 - 4^k]u[k].$$

[4]

j)

i) Nyquist rate is $2 \times 4.5 \times 10^6 = 9$ MHz.
Therefore the actual sampling rate = 9 MHz$\times$1.2=10.8 MHz.

[2]

ii) 1024 levels require 10 bits per sample. Therefore bit-rate is:

$10.8 \times 10^6 \times 10 = 108$ Mbits/sec.

[2]

**Answer to Question 2**

a) Express the differential equation in terms of D operators:

$$(D^2+6D+9)y(t)=(2D+9)x(t)\Rightarrow Q(D)y(t)=P(D)x(t)$$
$$Q(D)=(D^2+6D+9),\qquad P(D)=(2D+9)$$

The characteristic equation is therefore:

$$(\lambda^2+6\lambda+9)=0\Rightarrow(\lambda+3)^2=0.$$

$$\therefore\ y_0(t)=(c_1+c_2t)e^{-3t}\quad\text{and}\quad \dot{y}_0(t)=[-3(c_1+c_2t)+c_2]e^{-3t}$$

Setting t = 0, and substituting $e^{-3t}$ $y_0(0)=0$ and $\dot{y}_0(0)=1$, gives

$$\left.\begin{aligned}0&=c_1\\1&=-3c_1+c_2\end{aligned}\right\}\Rightarrow\begin{aligned}c_1&=0\\c_2&=1\end{aligned}$$

$$\therefore\ y_0(t)=te^{-3t}\quad\text{and}\quad \dot{y}_0(t)=(-3t+1)e^{-3t}$$

Now the impulse response can be calculated:

$$\begin{aligned}h(t)&=[P(D)y_0(t)]u(t)\\&=[2y_0(t)+9\dot{y}_0(t)]u(t)\\&=(-6te^{-3t}+2e^{-3t}+9te^{-3t})\,u(t)\\&=(2+3t)e^{-3t}\,u(t)\end{aligned}$$

[15]

b) The system response to $u(t)$ is $g(t)$, and the response to the step $u(t-\tau)$ is $g(t-\tau)$ (time-invariant property).

It is given that $\Delta f=\dfrac{\Delta f}{\Delta\tau}\Delta\tau=\dot{f}(\tau)\Delta\tau$. The step component at $t=n\Delta\tau$ therefore has a height of $\dot{f}(n\Delta\tau)\Delta\tau$, and can be expressed as $\left[\dot{f}(n\Delta\tau)\Delta\tau\right]u(t-n\Delta\tau)$. This gives a response $\Delta y(t)$ at the output, where

$$\Delta y(t)=\left[\dot{f}(n\Delta\tau)\Delta\tau\right]g(t-n\Delta\tau).$$

Therefore, the total response due to ALL step components is:

$$\begin{aligned}y(t)&=\lim_{\Delta\tau\to0}\sum_{n=-\infty}^{\infty}\dot{f}(n\Delta\tau)g(t-n\Delta\tau)\Delta\tau\\&=\int_{-\infty}^{\infty}\dot{f}(\tau)g(t-\tau)\,d\tau\\&=\dot{f}(\tau)*g(\tau)\\&=\dot{f}(t)*g(t).\end{aligned}$$

[15]

**Answer to Question 3**

a) i) From definition of Fourier transform,

$$
\begin{aligned}
F(\omega) &= \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt \\
&= \int_{-\tau}^{0} e^{-j\omega t}\,dt - \int_{0}^{\tau} e^{-j\omega t}\,dt \\
&= -\frac{1}{j\omega}e^{-j\omega t}\Big|_{-\tau}^{0} - \frac{1}{j\omega}e^{-j\omega t}\Big|_{0}^{\tau} \\
&= -\frac{1}{j\omega} + \frac{1}{j\omega}e^{j\omega t} + \frac{1}{j\omega}e^{-j\omega t} - \frac{1}{j\omega} \\
&= -\frac{2}{j\omega} + \frac{2}{j\omega}\cos\omega\tau \\
&= j\frac{4}{\omega}\sin^2\left(\frac{\omega\tau}{2}\right)
\end{aligned}
$$

[10]

ii) Express *f(t)* as sum of two rectangular functions:

$$
f(t) = rect\left(\frac{t+\tau/2}{\tau}\right) - rect\left(\frac{t-\tau/2}{\tau}\right)
$$

Given that

$$
rect\left(\frac{t}{\tau}\right) \Leftrightarrow \tau\,\mathrm{sinc}\left(\frac{\omega\tau}{2}\right),
$$

apply time-shifting property gives

$$
rect\left(\frac{t\pm\tau/2}{\tau}\right) \Leftrightarrow \tau\,\mathrm{sinc}\left(\frac{\omega\tau}{2}\right)e^{\pm j\omega\tau/2}.
$$

Therefore

$$
\begin{aligned}
F(\omega) &= \tau\,\mathrm{sinc}\left(\frac{\omega\tau}{2}\right)e^{+j\omega\tau/2} - \tau\,\mathrm{sinc}\left(\frac{\omega\tau}{2}\right)e^{-j\omega\tau/2} \\
&= 2j\,\tau\,\mathrm{sinc}\left(\frac{\omega\tau}{2}\right)\sin\left(\frac{\omega\tau}{2}\right) \\
&= j\frac{4}{\omega}\sin^2\left(\frac{\omega\tau}{2}\right)
\end{aligned}
$$

[10]

b)

$$f(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma^2}}$$

and $$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma^2}} \Leftrightarrow e^{-\sigma^2\omega^2/2}.$$

Parseval's Theorem states:

$$E_f = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2 \, d\omega .$$

Given

$$F(\omega) = e^{-\sigma^2\omega^2/2}$$

we obtain:

$$E_f = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\sigma^2\omega^2} \, d\omega .$$

Let $\sigma\omega = \frac{x}{\sqrt{2}}$, then $\sigma^2\omega^2 = \frac{x^2}{2}$ and $d\omega = \frac{1}{\sigma\sqrt{2}}dx$.

Therefore

$$E_f = \frac{1}{2\pi}\frac{1}{\sigma\sqrt{2}}\int_{-\infty}^{\infty} e^{-x^2/2}dx = \frac{1}{2\sigma\sqrt{\pi}} .$$

[10]

**Answer to Question 4**

a) Taking $z$-transform of both sides:

$$zY[z]-0.5Y[z]=zF[z]+0.8F[z].$$

Therefore

$$H[z]=\frac{Y[z]}{F[z]}=\frac{z+0.8}{z-0.5}.$$

b) The frequency response is given by:

$$H[e^{j\Omega}]=\frac{e^{j\Omega}+0.8}{e^{j\Omega}-0.5}=\frac{(\cos\Omega+0.8)+j\sin\Omega}{(\cos\Omega-0.5)+j\sin\Omega}.$$

Therefore, the amplitude response is

$$\left|H[e^{j\Omega}]\right|^2=H[e^{j\Omega}]H[e^{-j\Omega}].$$

$$=\frac{(e^{j\Omega}+0.8)(e^{-j\Omega}+0.8)}{(e^{j\Omega}-0.5)(e^{-j\Omega}-0.5)}.$$

$$=\frac{1.64+1.6\cos\Omega}{1.25-\cos\Omega}$$

The phase response is

$$\angle H[e^{j\Omega}]=\tan^{-1}\left(\frac{\sin\Omega}{\cos\Omega+0.8}\right)-\tan^{-1}\left(\frac{\sin\Omega}{\cos\Omega-0.5}\right).$$

c) Since $f[k]=\cos(0.5k-\frac{\pi}{3})$, $\Omega=0.5$.

Therefore

$$\left|H[e^{j\Omega}]\right|^2=\frac{1.64+1.6\cos 0.5}{1.25-\cos 0.5}=8.174$$

$$\left|H[e^{j\Omega}]\right|=2.86$$

$$\angle H[e^{j\Omega}]=\tan^{-1}\left(\frac{\sin 0.5}{\cos 0.5+0.8}\right)-\tan^{-1}\left(\frac{\sin 0.5}{\cos 0.5-0.5}\right).$$

$$=0.2784-0.9037$$

$$=-0.6253 \text{ radian or } 35.83^\circ.$$

Therefore, the system response is

$$y[k]=2.86\cos(0.5k-\frac{\pi}{3}-0.6253)=2.86\cos(0.5k-1.6725).$$