Final

E4.45
SO22

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2010

MSc and EEE/ISE PART IV: MEng and ACGI

WAVELETS AND APPLICATIONS

Thursday, 20 May 10:00 am

Time allowed: 3:00 hours

There are FOUR questions on this paper.

Answer THREE questions.

All questions carry equal marks.

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s) :	P.L. Dragotti
	Second Marker(s) :	K.D. Harris

Special Information for the Invigilators: NONE

Information for Candidates:

Sub-sampling by an integer N:

$$x_{\downarrow N}[n] \longleftrightarrow \frac{1}{N}\sum_{k=0}^{N-1} X(e^{j(\omega-2\pi k)/N}) = \frac{1}{N}\sum_{k=0}^{N-1} X(W_N^k z^{1/N}),$$

where

$$W_N^k = e^{-j2\pi k/N}.$$

Parseval's identity:

$$\langle g(t), f^*(t)\rangle = \frac{1}{2\pi}\langle \hat{g}(\omega), \hat{f}^*(\omega)\rangle,$$

where $\hat{g}(\omega)$ and $\hat{f}(\omega)$ are the Fourier transforms of $g(t)$ and $f(t)$ respectively.

Poisson summation formula:

$$\sum_{n=-\infty}^{\infty} f(t-n) = \sum_{k=-\infty}^{\infty} \hat{f}(2\pi k)e^{j2\pi kt}.$$

Fourier transform pair:

$$\frac{\sin \pi t}{\pi t} \Longleftrightarrow \mathrm{rect}\left(\frac{\omega}{2\pi}\right),$$

where

$$\mathrm{rect}(x) = \begin{cases} 1 & |x| \leq 1/2 \\ 0 & \text{otherwise} \end{cases}$$

The Questions

1. Consider the oversampled three-channel filter bank shown in Figure 1a. Note that the down-sampling as well as the up-sampling factor is two.

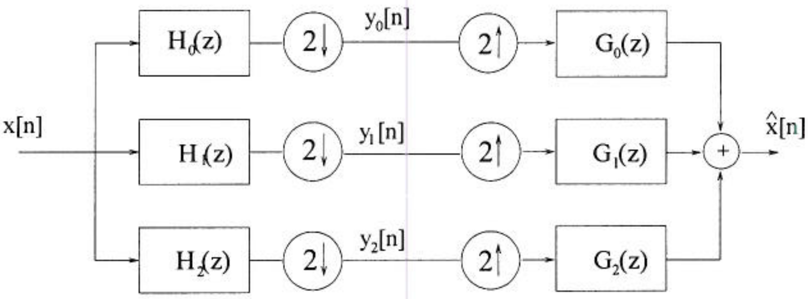

Figure 1a: Three-channel filter bank with down-sampling by 2.

(a) Express $\hat{X}(z)$ as a function of $X(z)$ and the filters. Then, derive the two perfect reconstruction conditions the filters have to satisfy.

[7]

(b) Assume that $G_0(z)$, $G_1(z)$ and $G_2(z)$ are the ideal filters shown in Figure 1b, (check carefully the cut-off frequencies). Assume that $H_i(z) = \alpha G_i(z^{-1})$, for $i = 0, 1, 2$. Here α is a constant. Sketch and dimension the Fourier transform of $y_0[n], y_1[n], y_2[n]$ and $\hat{x}[n]$ assuming that $x[n]$ has the spectrum shown in Figure 1c.

[5]

(c) Choose the constant α, so that $\hat{X}(z) = X(z)$.

[2]

Question 1 continues on the next page

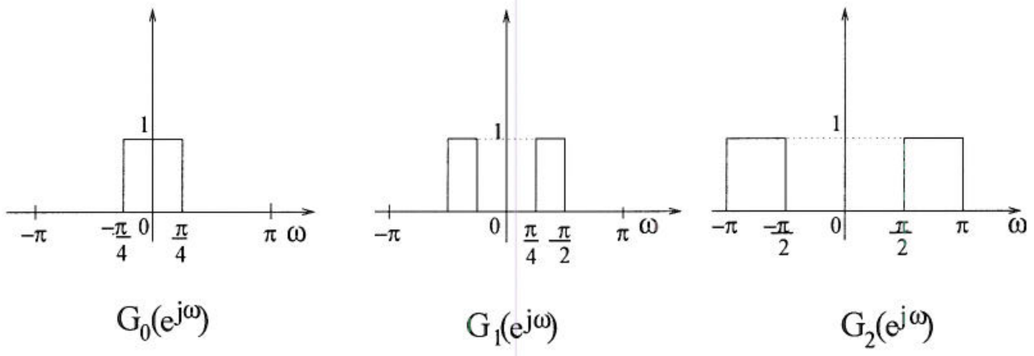

Figure 1b: Fourier transforms of the synthesis filters of Figure 1a.

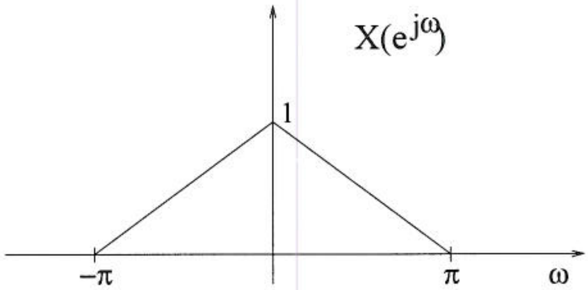

Figure 1c: Spectrum of $x[n]$.

(d) Now, the filter bank is iterated on the H_0 branch to form a 2-level decomposition.

i. Draw either the synthesis or the analysis filter bank of the equivalent 5-channel filter bank clearly specifying the transfer functions and downsampling factors.

[3]

ii. If the filters are those shown in Figure 1b, draw the Fourier transform of the equivalent filters of each branch before downsampling.

[3]

2. Consider the two-channel filter bank shown in Figure 2.

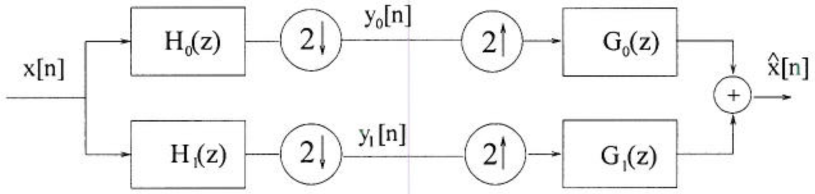

Figure 2: Two-channel filter bank.

(a) You are first asked to design an orthogonal filter-bank.

i. Start by designing the shortest possible filter $G_0(z)$ with a zero at $\omega = \pi$ and at least one zero at $\pi/3$. You want a filter with real-valued coefficients. [Hint: Recall that if the coefficients of a filter $G(z)$ are real then if z_k is a complex root of $G(z)$ so is z_k^*, where * denotes the complex conjugate].

[4]

ii. Choose $H_0(z) = G_0(z^{-1})$ and make sure that $P(z) = H_0(z)G_0(z)$ satisfies the half-band condition: $P(z) + P(-z) = 2$. [Hint: you might have to multiply both filters $H_0(z)$ and $G_0(z)$ by a proper coefficient α].

[4]

iii. Design the filters $H_1(z)$ and $G_1(z)$ in order to have a perfect reconstruction orthogonal filter-bank.

[4]

(b) You are now willing to design a symmetric/antisymmetric biorthogonal filter bank. Take

$$P(z) = (z + 1 + z^{-1})\left(bz^2 + \frac{1}{4}z + \frac{1}{2} + \frac{1}{4}z^{-1} + bz^{-2}\right).$$

i. Find b so that $P(z) + P(-z) = 2$.

[4]

ii. Choose $G_0(z) = (z + 1 + z^{-1})$. Design the filters $H_0(z), H_1(z)$ and $G_1(z)$ in order to have a perfect reconstruction bio-orthogonal filter-bank.

[4]

3. *Shannon Multiresolution Analysis.* For $j \in \mathbb{Z}$, let V_j be the space of all finite energy signals f for which the Fourier transform $\hat{f}(\omega) = 0$ outside of the interval $[-2^j\pi, 2^j\pi]$. We want to show that the function $\varphi(t) = \frac{\sin \pi t}{\pi t}$ with Fourier transform $\hat{\varphi}(\omega) = \mathrm{rect}(\omega/2\pi)$ is the scaling function of this multiresolution analysis. We need to show that $\varphi(t)$ satisfies the three criteria of a valid scaling function. More specifically:

   (a) Show that $\langle \varphi(t-n), \varphi(t-m) \rangle = \delta_{n,m}$. This is equivalent to showing that $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$ is an orthonormal basis of the space $V_0 = \mathrm{span}\{\varphi(t-n)\}_{n\in\mathbb{Z}}$. [Hint: use Parseval's identity].

   [5]

   (b) Show that $\varphi(t)$ satisfies partition of unity:

   $$\sum_{n=-\infty}^{\infty} \varphi(t-n) = 1.$$

   [5]

   (c) Finally, derive the coefficients $g_0[n]$ that lead to the two-scale equation

   $$\varphi(t) = \sqrt{2} \sum_{n=-\infty}^{\infty} g_0[n]\varphi(2t-n).$$

   [5]

   (d) Given $\varphi(t)$ and the two-scale equation, derive the corresponding wavelet $\psi(t)$.

   [5]

4. Let $\varphi(t)$ and $\psi(t)$ be the Haar scaling and wavelet functions, respectively. Let V_j and W_j be the spaces generated by $\varphi_{j,n}(t) = \sqrt{2^{-j}}\varphi(2^{-j}t-n)$, $n \in \mathbb{Z}$ and $\psi_{j,n}(t) = \sqrt{2^{-j}}\psi(2^{-j}t-n)$, $n \in \mathbb{Z}$, respectively. Consider the function defined on $0 \leq t < 2$ given by

$$f(t) = \begin{cases} 0 & 0 \leq t < 1/4 \\ 1 & 1/4 \leq t < 1/2 \\ 0 & 1/2 \leq t < 2. \end{cases}$$

(a) Decompose $f(t)$ into its component parts W_{-1}, W_0, and V_0. In other words, find the coefficients $c_{0,n}$, $d_{-1,n}$ and $d_{0,n}$, $n \in \mathbb{Z}$ that leads to the following decomposition

$$f(t) = \sum_{n=0}^{1} c_{0,n}\varphi_{0,n}(t) + \sum_{j=-1}^{0}\sum_{n=0}^{2^{-j+1}} d_{j,n}\psi_{j,n}(t).$$

[6]

(b) Verify the Parseval equality. That is, verify that:

$$\|f(t)\|^2 = \sum_{n} |c_{0,n}|^2 + \sum_{j=-1}^{0}\sum_{n} |d_{j,n}|^2.$$

[6]

(c) You now want to compress $f(t)$ using R bits in total. Assume that R is large.

i. Compute an approximated operational distortion-rate curve $D(R)$ that you obtain by scalar quantizing each of the 8 transform coefficient of the above decomposition and by allocating the same amount of bits to each coefficient. Justify your answer.

[4]

ii. Devise a non-linear strategy where only the non-zero coefficients are scalar quantized. Derive the new $D(R)$. Justify your answer.

[4]

QUESTIONS

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QUESTION 1

(a)

$$Y_i(z) = \frac{1}{2} H_i(z^{1/2}) X(z^{1/2}) + \frac{1}{2} H_i(-z^{1/2}) X(-z^{1/2}) \qquad i=0,1,2$$

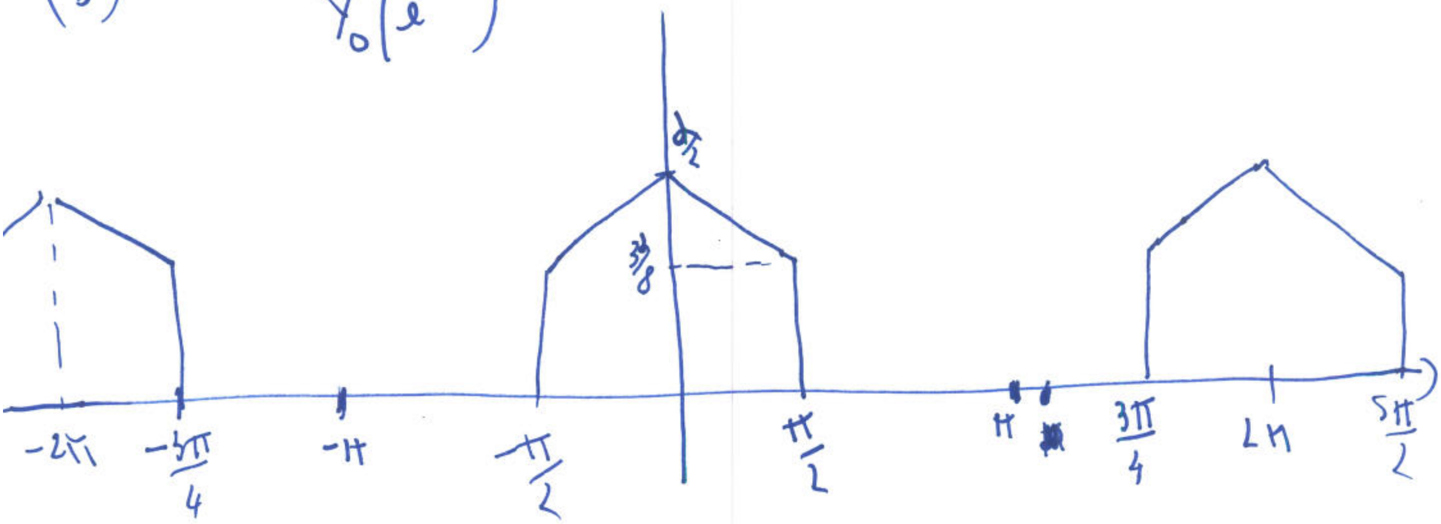
$$\hat{X}(z) = \frac{1}{2} X(z)\left(G_0(z)H_0(z) + G_1(z)H_1(z) + G_2(z)H_2(z)\right)$$
$$+ \frac{1}{2} X(-z)\left(G_0(z)H_0(-z) + G_1(z)H_1(-z) + G_2(z)H_2(-z)\right)$$

PR CONDITIONS

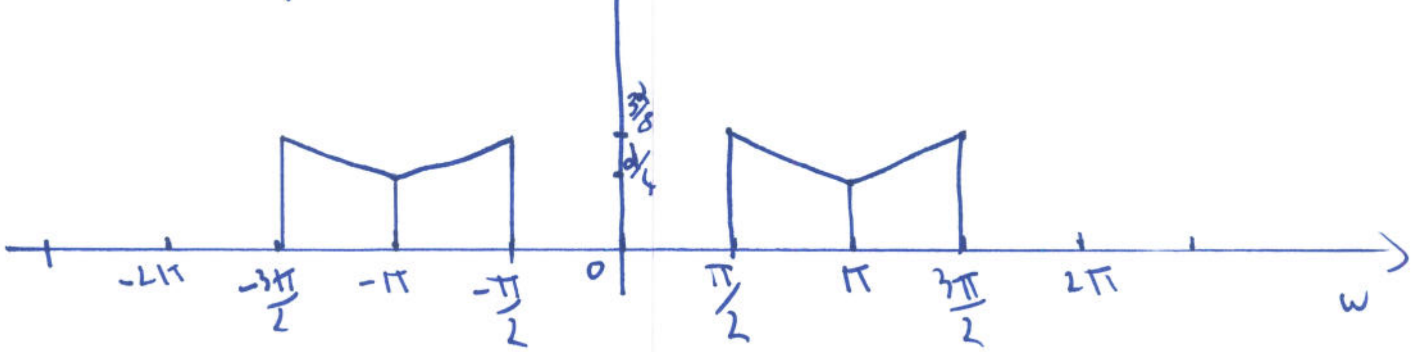
$$\begin{cases} G_0(z)H_0(z) + G_1(z)H_1(z) + G_2(z)H_2(z) = 2 \\ G_0(z)H_0(-z) + G_1(z)H_1(-z) + G_2(z)H_2(-z) = 0 \end{cases}$$

(b)

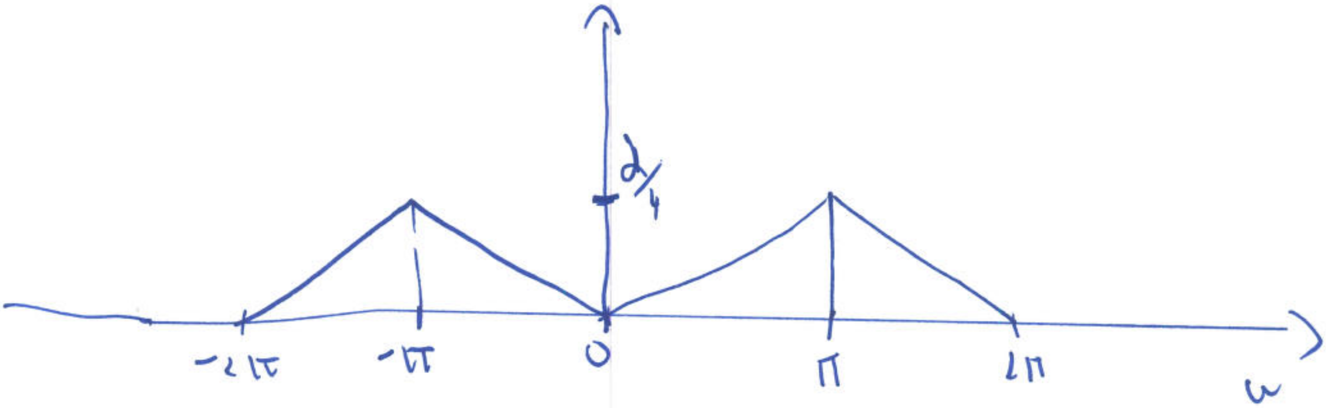
$Y_0(e^{j\omega})$



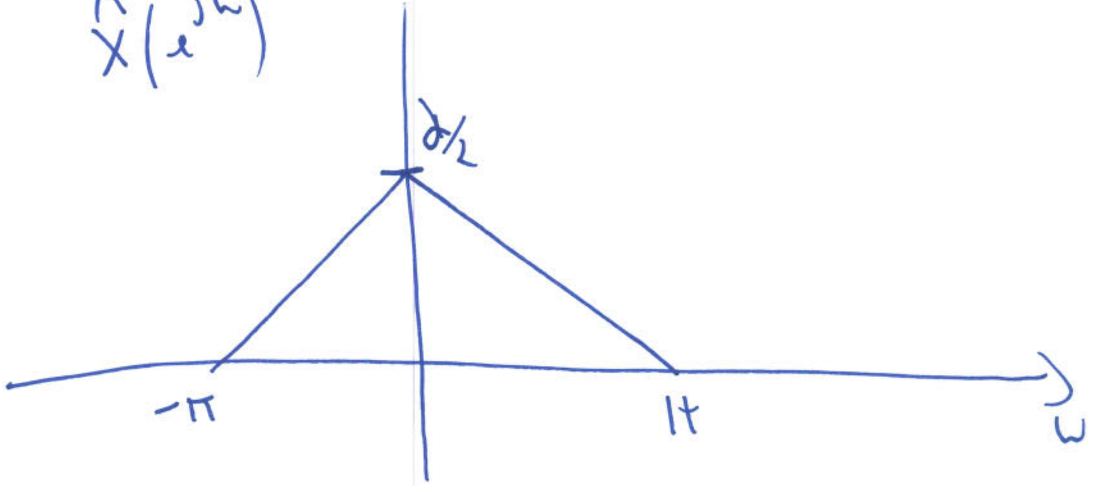
$Y_1(e^{jw})$



$Y_2(e^{jw})$



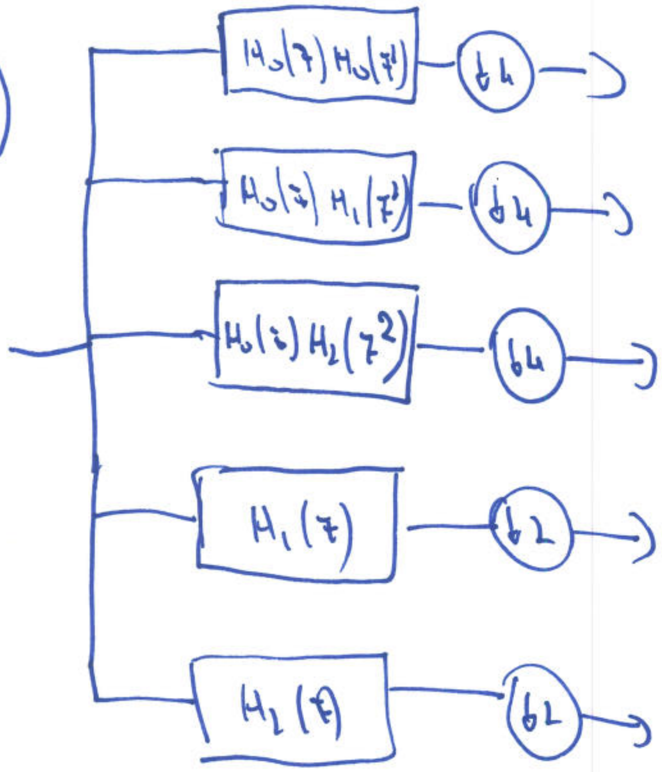
$\hat{X}(e^{jw})$



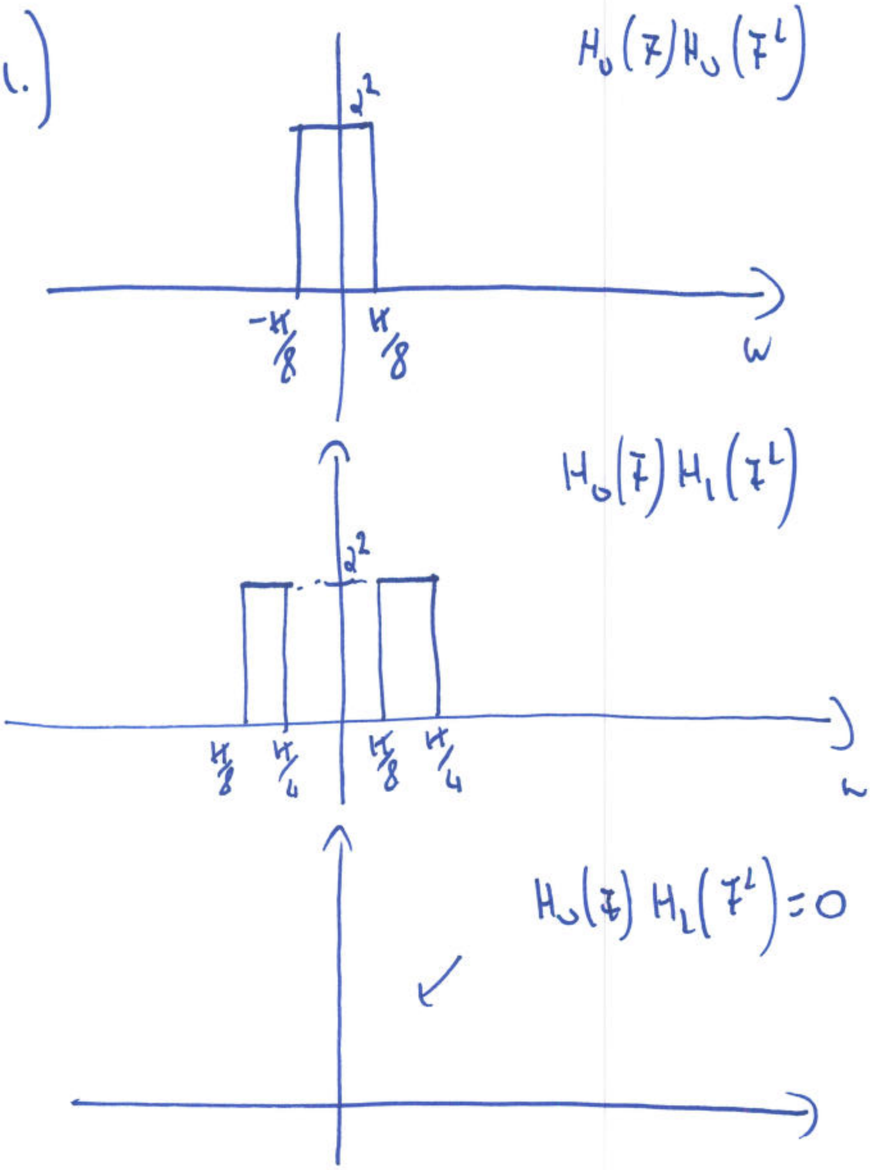
c) $d = 2$ IN ORDER TO HAVE $X(z) = \hat{X}(z)$

d)

i.)



ii.)



$H_1(z)$ and $H_2(z)$ as before

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QUESTION 2

a)

i.) ~~S~~HORTEST FILTER WITH REAL COEFFICIENTS:

$$G_0(z) = \alpha(z^{-1}+1)\left(z^{-1} - e^{j\frac{\pi}{3}}\right)\left(z^{-1} - e^{-j\frac{\pi}{3}}\right)$$

$$= \alpha(z^{-1}+1)\left(z^{-2} + 2\cos\frac{\pi}{3}\cdot z^{-1} + 1\right)$$

$$= \alpha(z^{-1}+1)(z^{-2} - z^{-1} + 1)$$

$$= \alpha(1 + z^{-3})$$

$$H_0(z) = G_0(z^{-1}) = \alpha(1+z^3)$$

ii.)

$$P(z) = (1+z^{-3})(1+z^3) = \alpha^2(1 + z^3 + z^{-3} + 1)$$

$$P(z) = P(-z) = 4\alpha^2 = 1 \Rightarrow \alpha^2 = \frac{1}{2}. \Rightarrow \alpha = \frac{1}{\sqrt{2}}$$

iii.)

$$G_1(z) = -z^{-1}G_0(z^{-1}) = -\frac{z^{-1}}{\sqrt{2}}(1 - z^3) = \frac{z^{2}}{\sqrt{2}} - \frac{z^{-1}}{\sqrt{2}}$$

$$H_1(z) = G_1(z^{-1}) = \frac{z^{-2}}{\sqrt{2}} - \frac{z}{\sqrt{2}}$$

QUESTION 2

b)

i.) CLEARLY $P(z)+P(-z)=2$ WHEN

$b=-1/4$

i.i.)

$$H_0(z) = -\frac{1}{4}z^2 + \frac{1}{4}z + \frac{1}{2} + \frac{1}{4}z^{-1} - \frac{1}{4}z^2$$

$$H_1(z) = z\,G_0(-z) = z\left(-z+1-z^{-1}\right) = -z^2+z-1$$

$$G_1(z) = z^{-1}H_0(-z) = z^{-1}\left(-\frac{1}{4}z^2 - \frac{1}{4}z + \frac{1}{2} - \frac{1}{4}z^{-1} - \frac{1}{4}z^2\right)$$

$$= \left(-\frac{1}{4}z - \frac{1}{4} + \frac{1}{2}z^{-1} - \frac{1}{4}z^{-2} - \frac{1}{4}z^{-3}\right)$$

QUESTION 3 $\quad\quad$ 6

(a) PARSEVAL EQUALITY: $\langle g(t), f^*(t)\rangle = \frac{1}{2\pi}\langle \hat{g}(\omega), \hat{f}^*(\omega)\rangle$.

THEREFORE, WE HAVE THAT

$$\langle \varphi(t-n), \varphi(t-m)\rangle = \frac{1}{2\pi}\left\langle \mathrm{RECT}\left(\frac{\omega}{2\pi}\right)e^{-j\omega n}, \mathrm{RECT}\left(\frac{\omega}{2\pi}\right)e^{j\omega m}\right\rangle$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j\omega(n-m)}\, d\omega = \cancel{\frac{\sin \pi(n-m)}{\pi k}}$$

$$= \frac{\sin \pi(n-m)}{\pi(n-m)} = \begin{cases} 1 & \text{IF } n=m \\ 0 & \text{OTHERWISE} \end{cases}$$

(b)

USING POISSON SUM FORMULA, WE HAVE THAT

$$\sum_{n=-\infty}^{\infty} \varphi(t-n) = \sum_{k=-\infty}^{\infty} \hat{\varphi}(2\pi k)e^{j2\pi kt} = \sum_{k=-\infty}^{\infty} \mathrm{RECT}\left(\frac{2\pi k}{2\pi}\right)e^{j2\pi kt}$$

AND CLEARLY

$$\sum_{k=-\infty}^{\infty} \text{RECT}\left(\frac{2\pi k}{2\pi}\right) e^{j2\pi k t} = 1$$

(c) WE FIRST SHOW THAT

$$g_0[k] = \sqrt{2} \langle \varphi(t), \varphi(2t-k) \rangle .$$

SINCE $\varphi(t)$ SATISFIES THE TWO-SCALE RELATION $\varphi(t) = \sqrt{2} \sum_{n=-\infty}^{\infty} g_0[n] \varphi(2t-n)$, (1)

WE HAVE THAT

$$\langle \varphi(t), \varphi(2t-k) \rangle = \sqrt{2} \sum_{n=-\infty}^{\infty} g_0[n] \langle \varphi(2t-n), \varphi(2t-k) \rangle$$

WHERE I HAVE USED (1) AND THE LINEARITY OF INNER PRODUCT.

BY REPLACING $2t = x$ WE HAVE THAT

$$\langle \varphi(t), \varphi(2t-k) \rangle = \frac{\sqrt{2}}{2} \sum_{n=-\infty}^{\infty} g_0[n] \underbrace{\langle \varphi(x-n), \varphi(x-k) \rangle}_{\delta_{n,k}} =$$

$$= \frac{1}{\sqrt{2}} g_0[k],$$

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WHERE I USED THE FACT THAT $\varphi(t)$ IS ORTHOGONAL TO ITS SHIFTS.

THEREFORE

$$g_0[k] = \sqrt{2}\langle \varphi(t), \varphi(2t-k)\rangle$$

USING PARSEVAL WE GET:

$$g_0[k] = \sqrt{2}\langle \varphi(t), \varphi(2t-k)\rangle = \frac{\sqrt{2}}{2\pi}\langle \hat{\varphi}(w), \frac{1}{2}\hat{\varphi}^*\left(\frac{w}{2}\right)e^{j\frac{kw}{2}}\rangle$$

$$= \frac{\sqrt{2}}{2\pi}\int_{-\pi}^{\pi}\frac{1}{2}e^{j\frac{kw}{2}}dw = \frac{\sqrt{2}}{2}\frac{\sin\frac{k\pi}{2}}{\frac{k\pi}{2}}$$

THEREFORE WE CAN WRITE

$$\varphi(t) = \sum_{m=-\infty}^{\infty}\frac{\sin\frac{m\pi}{2}}{\frac{m\pi}{2}}\varphi(2t-m).$$

(d)

$$\psi(t) = \sqrt{2}\sum_{n=-\infty}^{\infty}(-1)^n g_0[1-n]\varphi(2t-n)$$

$$= \sum_{m=-\infty}^{\infty}(-1)^m\frac{\sin(m-1)\frac{\pi}{2}}{\frac{(m-1)\pi}{2}}\varphi(2t-m).$$

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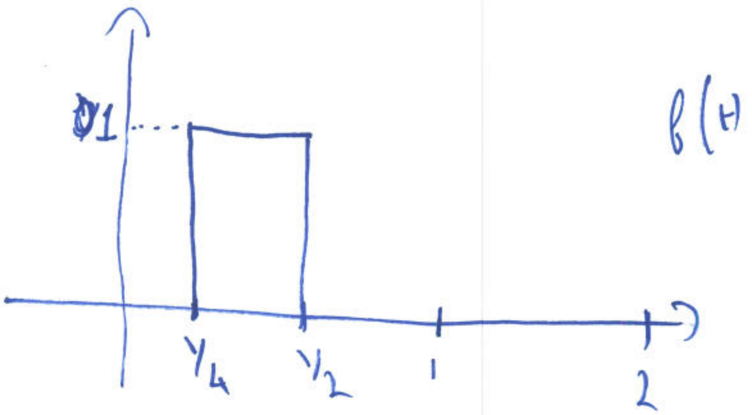
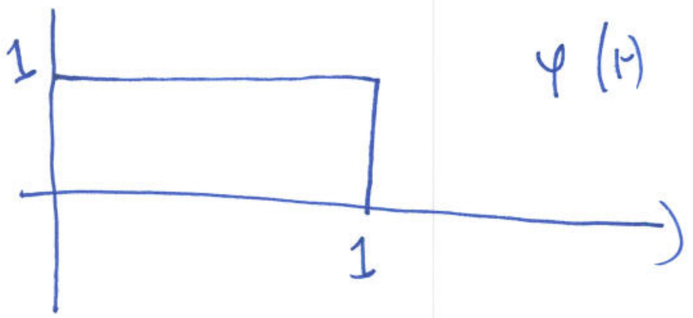
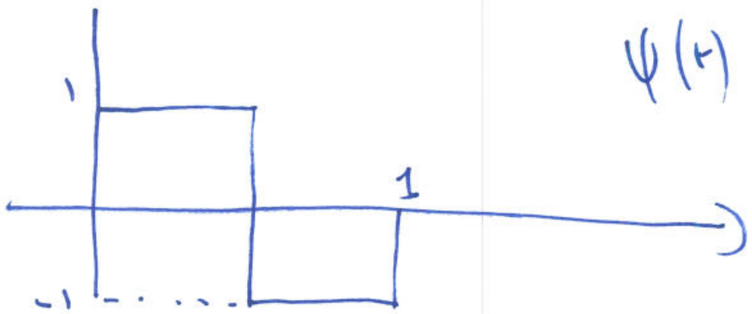
QUESTION 4

(a)







CLEARLY

$$c_{0,m} = \langle f(t), \varphi(t-m) \rangle = \begin{cases} 1/4 & \text{FOR } m=0 \\ 0 & \text{OTHERWISE} \end{cases}$$

$$d_{0,m} = \langle f(t), \psi(t-m) \rangle = \begin{cases} 1/4 & \text{FOR } m=0 \\ 0 & \text{OTHERWISE} \end{cases}$$

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$$d_{-1,n} = \sqrt{2}\, \langle f(t), \psi(2t-n) \rangle = \begin{cases} -\frac{\sqrt{2}}{4} & n=0 \\ 0 & \text{OTHERWISE} \end{cases}$$

SO

$$f(t) = \frac{1}{4}\varphi(t) + \frac{1}{4}\psi(t) - \frac{\sqrt{2}}{4}\psi_{-1,0}(t).$$

(b)

$$\|f\|^2 = \int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{1/4}^{1/2} dt = 1/4$$

$$\|f\|^2 = |c_{0,0}|^2 + |d_{0,0}|^2 + |d_{-1,0}|^2 = \frac{1}{16} + \frac{1}{16} + \frac{2}{16} = 1/4.$$

PARSEVAL VERIFIED

(c) i) AT HIGH RATE THE MSE DISTORTION OF A SCALAR QUANTIZER IS

$$D(R) = c\, 2^{-2R}$$

WHERE c DEPENDS ON THE STATISTICS OF THE SOURCE AND THE QUANTIZER USED.

~~BECAU~~ THE TOTAL BIT RATE IS R AND WE HAVE 8 COEFFICIENTS TO QUANTIZE. THEREFORE THE DISTORTION RELATED TO EACH OF THEM IS $D(R) = C\,2^{-2R/8}$

~~SINCE~~ BECAUSE OF PARSEVAL IDENTITY WE HAVE THAT:

$$D(R) = E\left[\|f - \hat{f}\|^2\right] = E\left[|c_{0,0} - \hat{c}_{0,0}|^2 + |d_{0,0} - \hat{d}_{0,0}|^2 + |d_{-1,0} - \hat{d}_{-1,0}|^2\right].$$

WHERE $\hat{c}_{0,0}, \hat{d}_{0,0}, \hat{d}_{-1,0}$ ARE THE QUANTIZED COEFFICIENTS.

WE THEREFORE HAVE THAT

$$D(R) \cong (c_1 + c_2 + c_3)\,2^{-2R/8} = C\,2^{-2R/8}$$

i.i. WE CAN ALTERNATIVELY QUANTIZE ONLY THE NON-ZERO COEFFICIENTS. WE NEED 3 BITS TO LOCATE EACH NON-ZERO COEFFICIENTS. THIS MEANS THAT WE NEED 9 BITS TO LOCATE THEM. WE CAN THEN USE THE REMAINING $R-9$ BITS TO QUANTIZE THEM. THIS LEADS TO THE FOLLOWING $D(R)$ CURVE : $D(R) \cong C\,2^{-2(R-9)/3}$