SOLUTIONS: STABILITY AND CONTROL OF NONLINEAR SYSTEMS MASTER IN CONTROL

1. Exercise

a) The vector field $f(x)$ is of class $\mathscr{C}^\infty$. It is therefore differentiable an infinite number of times, and in particular locally Lipschitz continuous. Therefore solutions exist and are unique on some open maximal interval of definition. [2]

b) The first nullcline, $\mathscr{N}_1$ is given by:

$$\mathscr{N}_1 = \{(x_1,x_2) : -\sin(x_1 - x_2) = 0\} = \{(x_1,x_2) : \exists k \in \mathbb{Z} : x_1 - x_2 = k\pi\}.$$

It is therefore a family of parallel equally spaced lines with angular coefficient $+1$. The second nullcline is the parabola described below:

$$\mathscr{N}_2 = \{(x_1,x_2) : x_1 = x_2^2 - x_2\}.$$

[3]

c) A graphical sketch of the nullclines is shown in the Figure 1.1, with the different regions labeled with the corresponding orientations of the vector-field (North-East, North-West, South-East and South-West). [3]

d) The NE region between the parabula and the line $x_1 = x_2$ is the only forward invariant region. [3]

e) In order to compute the equilibria we need to find the solution to the following system of equations:

$$\begin{cases} -\sin(x_1 - x_2) & = & 0 \\ x_2 - x_2^2 + x_1 & = & 0 \end{cases}$$

Solving the first equation yields: $x_1 = x_2 + k\pi$ for some integer k. Substituting in the second we see that the x_2 component of equilibria needs to fulfill:

$$x_2^2 - 2x_2 - k\pi = 0$$

This yields:

$$x_2 = 1 \pm \sqrt{1 + k\pi} \qquad \forall k \in \mathbb{N}.$$

Correspondingly:

$$x_1 = 1 \pm \sqrt{1 + k\pi} + k\pi \qquad \forall k \in \mathbb{N}.$$

Hence there are infinitely may equilibria of coordinates:

$$(1 \pm \sqrt{1 + k\pi} + k\pi, 1 \pm \sqrt{1 + k\pi}) \qquad \forall k \in \mathbb{N}.$$

[2]

f) Taking the Jacobian of $f(x)$ yields:

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -\cos(x_1 - x_2) & \cos(x_1 - x_2) \\ 1 & 1 - 2x_2 \end{bmatrix}.$$

Evaluating it at equilibria yields:

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -\cos(k\pi) & \cos(k\pi) \\ 1 & -1 \mp 2\sqrt{1+k\pi} \end{bmatrix}.$$

[1]

For k even we have:

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \mp 2\sqrt{1+k\pi} \end{bmatrix}.$$

The characteristic polynomial is:

$$\chi(s) = s^2 + 2(1 \pm \sqrt{1+k\pi})s \pm 2\sqrt{1+k\pi}.$$

Hence, the equilibria for positive x_2 (k even) are locally asymptotically stable. The local phase portrait is a node (focus is not compatible with nullclines). The equilibrium for negative (or 0) x_2 is unstable and a saddle.
For k odd we have:

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \mp 2\sqrt{1+k\pi} \end{bmatrix}.$$

The characteristic polynomial is:

$$\chi(s) = s^2 \pm 2\sqrt{1+k\pi}s \mp 2\sqrt{1+k\pi}.$$

Both eigenvalues are real. For positive x_2 we have one permanence and one variation in the sequence of signs of coefficients of the characteristic polynomial. Hence, the local phase portrait is a saddle. For negative x_2 we have two variations. Hence, the local phase-portrait is an unstable node. [3]

g) The phase-portrait of the system is sketched in Fig. 1.2. [3]

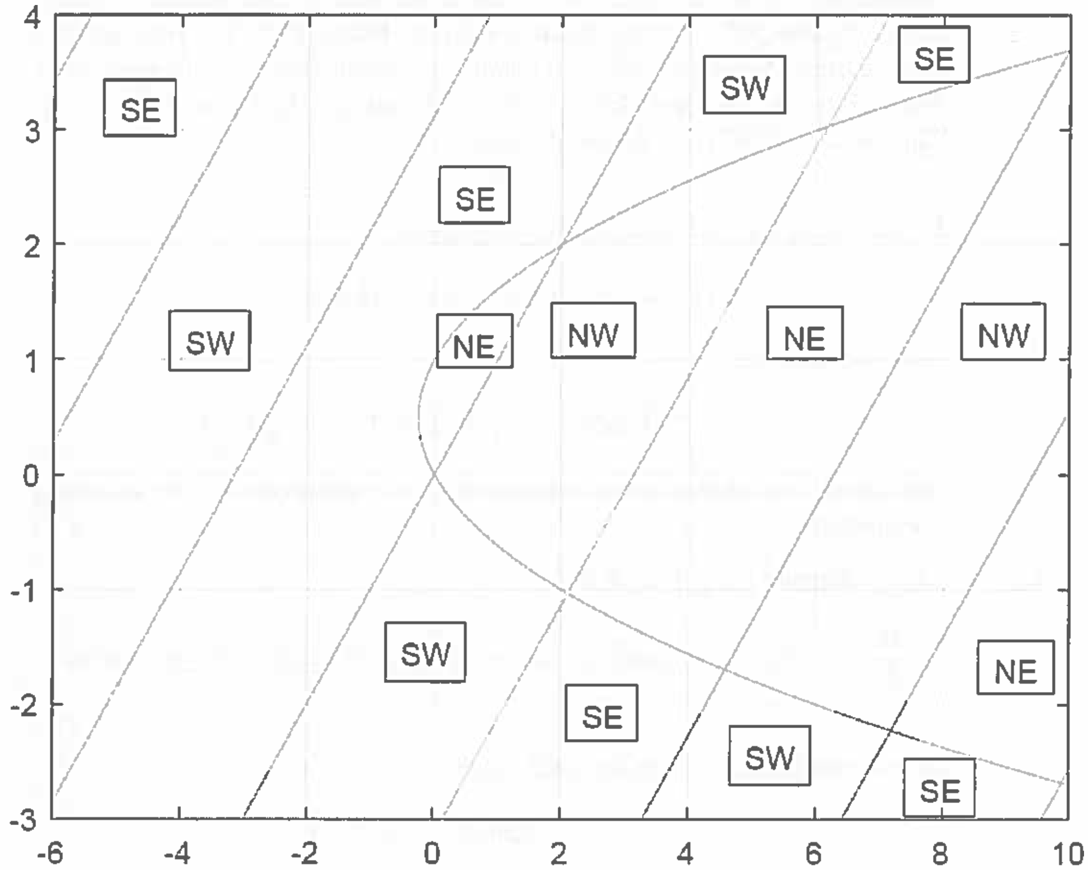

Figure 1.1 Nullclines and regions in state-space

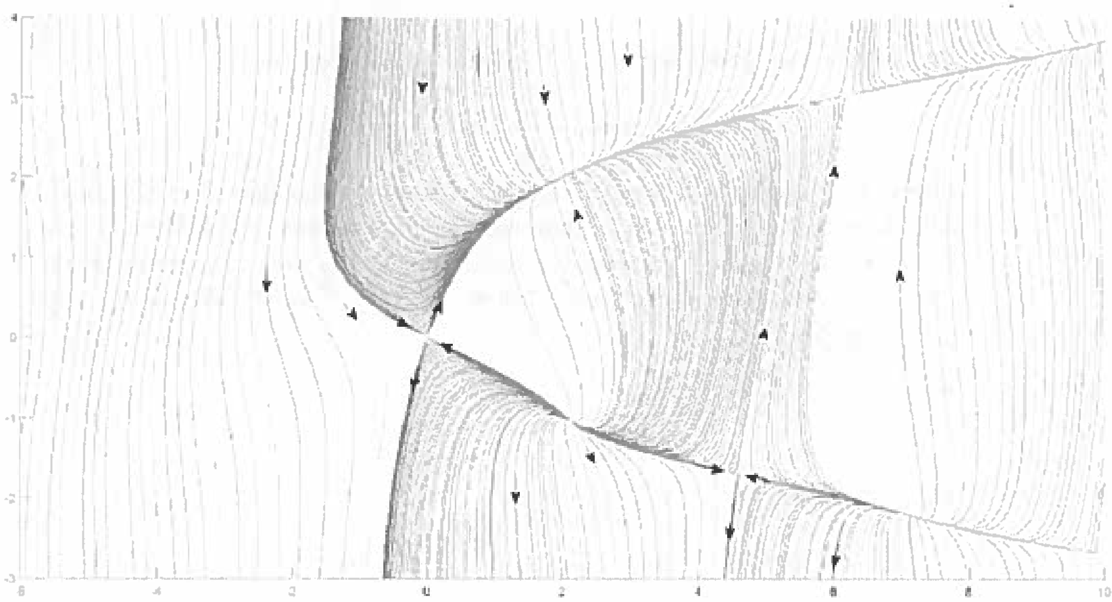

Figure 1.2 Global Phase-Portrait

2. Exercise

a) Notice that, for α and β positive, $V(x)$ is the sum of 3 non-negative terms: $x_1^2/2$, αx_2^2 and βx_3^4 (each one being a square). Hence $V(x)$ is at least positive semi-definite. Moreover, $V(x) = 0$ if and only if each of the 3 terms equal zezo. This is only the case provided $x_1 = 0$, $x_2 = 0$ and $x_3 = 0$, that is only if $x = 0$. This proves that $V(x)$ is positive definite. [3]

In addition, $V(x) \leq M$ for some $M > 0$ implies:

$$x_1^2 \leq 2M \text{ and } x_2^2 \leq M/\alpha \text{ and } x_3^4 \leq M/\beta.$$

As a consequence:

$$|x_1| \leq \sqrt{2M} \text{ and } |x_2| \leq \sqrt{M/\alpha} \text{ and } |x_3| \leq \sqrt[4]{M/\beta}.$$

This shows that sublevel sets of $V(x)$ are bounded, and therefore $V(x)$ is radially unbounded. [2]

b) Let us compute $\dot{V}(x)$. We see that:

$$\frac{\partial V}{\partial x}(x) \cdot f(x) = -x_1 \operatorname{atan}(x_1) + x_1 x_2 - 4\alpha x_1 x_2 + 2\alpha x_2 x_3^3 - 4\beta x_2 x_3^3 - 4\beta x_3^4.$$

[2]

Hence, taking $4\alpha = 1$ and $2\alpha = 4\beta$ yields:

$$\dot{V}(x) = -x_1 \operatorname{atan}(x_1) - 4\beta x_3^4 \leq 0.$$

This is achieved for $\alpha = 1/4$ and $\beta = 1/8$. [3]

c) Since $\dot{V}$ is only negative semi-definite we wish to apply Lasalle's invariance principle. To this end notice that

$$\text{Ker}[\dot{V}(x)] := \{x : \dot{V}(x) = 0\} = \{x : x_1 = 0 \text{ and } x_3 = 0\}.$$

[2]

Any invariant set contained in $\text{Ker}[\dot{V}(x)]$ must also belong to:

$$\{x : x_1 = 0 \text{ and } x_3 = 0 \text{ and } \dot{x}_3 = 0\} = \{0\}.$$

Hence, $V(x)$ is a Lyapunov function which, for $\alpha = 1/4$ and $\beta = 1/8$ is positive definite, radially unbounded, with a negative semi-definite derivative and $\{0\}$ is the only invariant set contained in $\text{Ker}[\dot{V}(x)]$. Hence the assumptions of the Lasalle's invariance principle are fulfilled and 0 is a Globally Asymptotically Stable equilibrium. [3]

d) We choose:

$$W(x) = -\frac{x_1^2}{2} - \frac{x_2^2}{4} + \frac{x_3^4}{8}.$$

For this choice of W we see that:

$$\dot{W}(x) = x_1 \operatorname{atan}(x_1) + x_3^4/2 \geq 0.$$

Hence W is a function with a positive semi-definite derivative. Moreover, 0 is an accumulation point of the set $\{x : W(x) > 0\}$. [2]
Unfortunately $\dot{W}$ is only positive semi-definite. To conclude instability we need

to have $\dot{W}$ positive definite in a neighborhood of the origin. To this end, pick an extra term $\varepsilon x_1 x_2$. Its derivative along solutions reads:

$$\frac{\partial \varepsilon x_1, x_2}{\partial x} \cdot \dot{x} = \varepsilon(-\text{atan}(x_1)x_2 + x_2^2 - 2x_1^2 + x_2 x_3^3).$$

Overall, letting $\bar{W} = W + \varepsilon x_1 x_2$ yields:

$$\dot{\bar{W}}(x) = x_1 \text{atan}(x_1) - \varepsilon \text{atan}(x_1)x_2 + \varepsilon x_2^2 + \varepsilon x_2 x_3^3 + x_3^4/8.$$

In order to dominate the terms which are not sign-definite is useful to split εx_2^2 in two parts:

$$\dot{\bar{W}}(x) = (x_1 \text{atan}(x_1) - \varepsilon \text{atan}(x_1)x_2 + \varepsilon x_2^2/2) + (\varepsilon x_2^2/2 + \varepsilon x_2 x_3^3 + x_3^4/8).$$

Both terms are positive semidefinite in a neighborhood of 0. Moreover they only vanish if $x_1 = x_2 = 0$ and $x_2 = x_3 = 0$ respectively. Hence $\dot{\bar{W}}$ is locally positive definite. We remark that 0 is an accumulation point of the set $\{x : \bar{W}(x) > 0\}$ as it follows by considering the sequence $x_n = (0, 0, 1/n)^T$ which indeed fulfills $\bar{W}(x_n) > 0$ for all $n \in \mathbb{N}$. [3]

3. Exercise

a) We choose the 3 subsystems as:

$$\dot{x}_1 = \operatorname{atan}(d_1 - x_1)$$

with one dimensional state x_1 and scalar input d_1;

$$\dot{x}_2 = -x_2^5 + d_2$$

with state x_2 and scalar input d_2 and

$$\dot{x}_3 = -x_3 + d_3^3.$$

[3]

The overall system can be obtained as the cascade interconnection resulting from assigning, $d_3 = x_1$, and $d_2 = x_1^2 x_3^2$. For the first system we let of course $d_1 = d$. Notice that there are no directed loops in the block-diagram of the interconnected system. This shows that indeed we have decomposed the system as a cascade. [1]

b) Let us consider $V_1(x_1) = x_1^2$. This is clearly positive definite and radially unbounded. Moreover:

$$|x_1| \geq 2|d_1| \Rightarrow \dot{V}_1 = 2x_1 \operatorname{atan}(d - x_1) \leq -2x_1 \operatorname{atan}(x_1/2).$$

This shows ISS of the first subsystem, with gain $\gamma_1(r) = 2r$ [2]

For the third subsystem we choose:

$$V_3(x_3) = x_3^2, \qquad \dot{V}_3 = 2x_3(d_3^3 - x_3).$$

In particular:

$$|x_3| \geq 2|d_3^3| \Rightarrow \dot{V}_3 \leq -x_3^2.$$

This shows ISS of subsystem 3 with gain $\gamma_3(r) = 2r^3$. [2]

For the second subsystem we pick:

$$V_2(x_2) = x_2^2, \qquad \dot{V}_2 = 2x_2(-x_2^5 + d_2).$$

In particular then:

$$|x_2| \geq \sqrt[5]{2|d_2|} \Rightarrow \dot{V}_2 \leq -x_2^6.$$

This shows ISS of the second subsystem with ISS gain $\gamma_2(r) = \sqrt[5]{2|r|}$. [2]

c) Notice that:

$$\limsup_{t \to +\infty} |x_1(t)| \leq \gamma_1(\|d\|_\infty).$$

[1]

As a consequence:

$$\limsup_{t \to +\infty} |x_3(t)| \leq \gamma_3\left(\limsup_{t \to +\infty} |x_1(t)|\right) \leq \gamma_3(\gamma_1(\|d\|_\infty)).$$

[2]

Finally:

$$\limsup_{t \to +\infty} |x_2(t)| \leq \gamma_2\left(\limsup_{t \to +\infty} d_2(t)\right) \leq \gamma_1^2 \cdot (\gamma_3 \circ \gamma_1)^2(\|d\|_\infty).$$

[2]

Overall then:

$$\limsup_{t \to +\infty} |x(t)| \leq \limsup_{t \to +\infty} |x_1(t)| + |x_2(t)| + |x_3(t)| \leq [\gamma_1 + \gamma_3 \circ \gamma_1 + \gamma_1^2 \cdot (\gamma_3 \circ \gamma_1)^2](\|d\|_\infty).$$

[1]

d) The caracterization of dissipativity by means of dissipation inequalities is as follows:

$$\dot{S} \leq s(y,u)$$

Letting $V(x) = S(x)$ and $y = |x|$, $s(y,u) = -\alpha(|x|) + \gamma(|u|)$ for some class $\mathcal{K}_\infty$ functions α and γ, existence of an ISS Lyapunov function amounts to:

$$\dot{V} \leq -\alpha(|x|) + \gamma(|u|)$$

which is therefore a special case of the previous condition. [2]
Integrating both sides of the previous inequality along solutions of a nonlinear system we get:

$$V(x(t)) - V(x(0)) \leq -\int_0^t \alpha(|x(s)|)\,ds + \int_0^t \gamma(|u(s)|)\,ds.$$

[2]

4. Exercise

a) Taking derivatives along solutions of the system of the output equation $y = x_1$ yields:

$$\begin{aligned}
\dot{y} &= x_1 + x_3 \\
\ddot{y} &= x_1 + x_3 + x_2 + \sin(x_1) - x_3 = x_1 + \sin(x_1) + x_2 \\
\dddot{y} &= (x_1 + x_3)(1 + \cos(x_1)) - x_2^2 + u.
\end{aligned}$$

[3]
Notice that u first appears only after three differentiations. Moreover its coefficient is 1 (and therefore always different from 0). This means that the global relative degree is 3. [1]

b) An input-output linearizing feedback is given as:

$$u = -(x_1 + x_3)(1 + \cos(x_1)) + x_2^2 + v,$$

where $v \in \mathbb{R}$ is the new input variable. After this preliminary feedback loop the system reads $\dddot{y} = v$ and is therefore linear. [3]

c) Taking the stable polynomial $P(s) = (s+1)^3 = s^3 + 3s^2 + 3s + 1$ one may achieve stability and tracking by means of the following pole-placement feedback controller:

$$v = -3\ddot{y} - 3\dot{y} - (y - r),$$

where r is the set-point variable. [3]

d) Choosing $y = x_3$ yields,

$$\begin{aligned}
\dot{y} &= -x_3 + x_2 + \sin(x_1) \\
\ddot{y} &= -(-x_3 + x_2 + \sin(x_1)) + \cos(x_1)(x_3 + x_1) - x_2^2 + u.
\end{aligned}$$

Hence the global relative degree is 2 and the normal form can be achieved by letting, $z = [y, \dot{y}]'$, $\xi = x_1$. [3]
Accordingly, appying the preliminary feedback

$$(-x_3 + x_2 + \sin(x_1)) - \cos(x_1)(x_3 + x_1) + x_2^2 + v = u$$

the equations become:

$$\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= v \\
\dot{\xi} &= x_1 + x_3 = \xi + y.
\end{aligned}$$

This is the normal form of the system. [3]

e) The zero-dynamics are obtained for $y \equiv 0$ and read as:

$$\dot{\xi} = \xi.$$

[2]

and are clearly exponentially unstable. As a consequence the system cannot be stabilized straightforwardly by using feedback linearization and this particular selection of output signal. [2]