

**Imperial College
London**

[E1.11 (Maths) ISE 2010]

B.ENG. and M.ENG. EXAMINATIONS 2010

MATHEMATICS (INFORMATION SYSTEMS ENGINEERING E1.11)

Date Wednesday 2nd June 2010 10.00 am - 1.00 pm

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

Answer ANY SEVEN questions.

Answers to questions from Section A and Section B should be written in different answer books.

CALCULATORS MAY NOT BE USED.

A mathematical formulae sheet is provided.

[Before starting, please make sure that the paper is complete; there should be 7 pages, with a total of 9 questions. Ask the invigilator for a replacement if your copy is faulty.]

SECTION A

1. (i) If $z_1 = 1 + 7i$, and $z_2 = 4 + 3i$, find $|z_1|$ and $|z_2|$.

Find the real and imaginary parts of $\frac{z_1}{z_2}$.

What are the modulus and argument of this number?

Verify directly in this case that $|z_1/z_2| = |z_1|/|z_2|$.

- (ii) If x and y are real, find the real and imaginary parts of

$$\cos(x + iy).$$

Hence show that

$$|\cos(x + iy)|^2 = \cos^2(x) + \sinh^2(y).$$

PLEASE TURN OVER

2. (i) Evaluate the partial sum

$$S_N = \sum_{n=1}^N \left(\frac{1}{n(n+1)} \right).$$

Evaluate the limit, if it exists,

$$\lim_{N \rightarrow \infty} S_N.$$

State whether the infinite series

$$\sum_{n=1}^{\infty} \left(\frac{1}{n(n+1)} \right)$$

is convergent or not.

Hence, using the comparison test, state whether the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

is convergent or divergent.

- (ii) Explain what is meant by the *radius of convergence* of a power series.

Calculate the radius of convergence of the following power series:

$$\sum_{n=0}^{\infty} \frac{1}{2n+1} z^{2n+1}.$$

Investigate its convergence at both of the endpoints of the interval of convergence.

- (iii) Calculate the Maclaurin series for the function $f(x) = \ln(1+x)$. Split this function into its odd and even parts f_{odd} and f_{even} , and write down both f_{odd} and its Maclaurin series.

PLEASE TURN OVER

3. (i) Evaluate the following limits, or else show that they do not exist:

(a)
$$\lim_{n \rightarrow 1} \frac{n^3 - 1}{n^4 - 4n + 3} ,$$

(b)
$$\lim_{n \rightarrow 1} \frac{n^3 - 1}{n^4 - 4n^2 + 3} ,$$

(c)
$$\lim_{x \rightarrow \pi/2} \frac{\sin x - \operatorname{cosec} x}{\cos^2(x)} ,$$

(d)
$$\lim_{n \rightarrow \infty} \frac{n^2 + 1}{(n - 2)(n - 3)} .$$

(ii) Using L'Hôpital's rule, evaluate

$$\lim_{x \rightarrow 0} \frac{\sin(x^2)}{1 - \cos(x)} .$$

4. Evaluate the definite integrals

(i)
$$\int_0^{\infty} x^3 \exp(-x) \, dx ;$$

(ii)
$$\int_e^{e^2} \frac{1}{x \ln x} \, dx ;$$

(iii)
$$\int_3^4 \frac{(x + 1)}{(x^2 - 3x + 2)} \, dx .$$

PLEASE TURN OVER

5. Solve the ordinary differential equations

(i)

$$\frac{dy}{dx} = -\frac{2x+3y}{x+y} ;$$

(ii)

$$\frac{dy}{dx} - xy = \exp\left(\frac{x^2}{2}\right) , \text{ with } y(0) = 1 ;$$

(iii)

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = \exp(x), \text{ with } y(0) = 0, \text{ and } y'(0) = 0 .$$

PLEASE TURN OVER

SECTION B

6. (i) Let $u = u(x, y)$ where $x = s^2 + t^2$ and $y = 2st$.

Prove that

$$s \frac{\partial u}{\partial s} + t \frac{\partial u}{\partial t} = 2 \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right).$$

- (ii) Find the stationary points of the function $f(x, y) = x^3 - 3xy^2 + 12y$, and determine their nature.

7. Define the function $f(x)$ on the interval $-\pi < x \leq \pi$ by

$$f(x) = \begin{cases} x, & \text{if } 0 \leq x \leq \pi \\ 0, & \text{if } -\pi < x < 0. \end{cases}$$

Show that the Fourier series of $f(x)$ is

$$\frac{\pi}{4} - \frac{2}{\pi} \sum_{m=0}^{\infty} \frac{\cos[(2m+1)x]}{(2m+1)^2} + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin(nx)}{n}.$$

Hence evaluate the infinite sums

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \quad \text{and} \quad \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}.$$

PLEASE TURN OVER

8. (i) Use Laplace transforms to find a function $y = y(t)$ satisfying the differential equation

$$\frac{d^2y}{dt^2} - y + t^2 = 0$$

with $y(0) = 2$, $y'(0) = 0$.

(No credit will be given if you use another method.)

- (ii) Use Laplace transforms to find functions x, y of t satisfying the simultaneous differential equations

$$\frac{dx}{dt} + \frac{dy}{dt} - y = 2,$$

$$\frac{dx}{dt} - \frac{dy}{dt} + x = t^2 + 2t,$$

with $x(0) = 2$, $y(0) = 0$.

9. (i) Consider the three planes given by the equations

$$\begin{aligned}\mathbf{v} \cdot (1, 1, 1) &= -1, \\ \mathbf{v} \cdot (1, -1, a) &= -3, \\ \mathbf{v} \cdot (2, 0, 1) &= b,\end{aligned}$$

where $\mathbf{v} = (x, y, z)$. For which values of a and b do these three planes

- (a) meet in exactly one point ?
- (b) meet in a line ?
- (c) not meet at all ?

- (ii) Let

$$A = \begin{pmatrix} -1 & -10 \\ 5 & 14 \end{pmatrix}.$$

- (a) Find an invertible 2×2 matrix P such that $P^{-1}AP$ is diagonal.
- (b) Find a 2×2 matrix B such that $B^2 = A$.

END OF PAPER

MATHEMATICAL FORMULAE

1. VECTOR ALGEBRA

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1, a_2, a_3)$$

Scalar (dot) product:

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

Vector (cross) product:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Scalar triple product:

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = \mathbf{b} \cdot \mathbf{c} \times \mathbf{a} = \mathbf{c} \cdot \mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c} \cdot \mathbf{a})\mathbf{b} - (\mathbf{b} \cdot \mathbf{a})\mathbf{c}$$

2. SERIES

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \dots \quad (\alpha \text{ arbitrary, } |x| < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^n \frac{x^{2n}}{(2n)!} + \dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \dots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^n \frac{x^{n+1}}{(n+1)!} + \dots \quad (-1 < x \leq 1)$$

3. TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin(a+b) = \sin a \cos b + \cos a \sin b;$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b.$$

$$\cos iz = \cosh z; \quad \cosh iz = \cos z; \quad \sin iz = i \sinh z; \quad \sinh iz = i \sin z.$$

4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f D^n g + \binom{n}{1} Df D^{n-1} g + \dots + \binom{n}{n-1} D^{n-1} f Dg + \dots + D^n f g.$$

(b) Taylor's expansion of $f(x)$ about $x = a$:

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \dots + \frac{h^n}{n!}f^{(n)}(a) + \epsilon_n(h),$$

where $\epsilon_n(h) = h^{n+1}f^{(n+1)}(a + \theta h)/(n+1)!$, $0 < \theta < 1$.

(c) Taylor's expansion of $f(x, y)$ about (a, b) :

$$f(a+h, b+k) = f(a, b) + [hf_x + kf_y]_{a,b} + \frac{1}{2!} [h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}]_{a,b} + \dots$$

(d) Partial differentiation of $f(x, y)$:

$$\text{i. If } y = y(x), \text{ then } f = F(x), \text{ and } \frac{dF}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx}.$$

$$\text{ii. If } z = x(t), y = y(t), \text{ then } f = F(t), \text{ and } \frac{dF}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}.$$

$$\text{iii. If } z = x(u, v), y = y(u, v), \text{ then } f = F(u, v), \text{ and}$$

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u}, \quad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial v}.$$

(e) Stationary points of $f(x, y)$ occur where $f_x = 0$, $f_y = 0$ simultaneously.

Let (a, b) be a stationary point: examine $D = [f_{xx}f_{yy} - (f_{xy})^2]_{a,b}$.

If $D > 0$ and $f_{xx}(a, b) < 0$, then (a, b) is a maximum;

If $D > 0$ and $f_{xx}(a, b) > 0$, then (a, b) is a minimum;

If $D < 0$ then (a, b) is a saddle-point.

(f) Differential equations:

i. The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating factor $I(x) = \exp[\int P(x)dx]$, so that $\frac{d}{dx}(Iy) = IQ$.

ii. $P(x, y)dx + Q(x, y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y$.

5. INTEGRAL CALCULUS

(a) An important substitution: $\tan(\theta/2) = t$;
 $\sin \theta = 2t/(1+t^2)$, $\cos \theta = (1-t^2)/(1+t^2)$, $d\theta = 2 dt/(1+t^2)$.

(b) Some indefinite integrals:

$$\int (a^2 - x^2)^{-1/2} dx = \sin^{-1} \left(\frac{x}{a} \right), \quad |x| < a.$$

$$\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1} \left(\frac{x}{a} \right) = \ln \left\{ \frac{x}{a} + \left(1 + \frac{x^2}{a^2} \right)^{1/2} \right\}.$$

$$\int (x^2 - a^2)^{-1/2} dx = \cosh^{-1} \left(\frac{x}{a} \right) = \ln \left[\frac{x}{a} + \left(\frac{x^2}{a^2} - 1 \right)^{1/2} \right].$$

$$\int (a^2 + x^2)^{-1} dx = \left(\frac{1}{a} \right) \tan^{-1} \left(\frac{x}{a} \right).$$

6. NUMERICAL METHODS

(a) Approximate solution of an algebraic equation:

If a root of $f(x) = 0$ occurs near $x = a$, take $x_0 = a$ and
 $x_{n+1} = x_n - [f(x_n)/f'(x_n)]$, $n = 0, 1, 2, \dots$

(Newton Raphson method).

(b) Formulae for numerical integration: Write $x_n = x_0 + nh$, $y_n = y(x_n)$.

i. Trapezium rule (1-strip): $\int_{x_0}^{x_1} y(x) dx \approx (h/2) [y_0 + y_1]$.

ii. Simpson's rule (2-strip): $\int_{x_0}^{x_2} y(x) dx \approx (h/3) [y_0 + 4y_1 + y_2]$.

(c) Richardson's extrapolation method: Let $I = \int_a^b f(x) dx$ and let I_1, I_2 be two

estimates of I obtained by using Simpson's rule with intervals h and $h/2$.

Then, provided h is small enough,

$$I_2 + (I_2 - I_1)/15,$$

is a better estimate of I .

7. LAPLACE TRANSFORMS

Function	Transform	Function	Transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t) dt$	$af(t) + bg(t)$	$aF(s) + bG(s)$
d/dt	$sF(s) - f(0)$	$d^2 f/dt^2$	$s^2 F(s) - sf(0) - f'(0)$
$e^{at} f(t)$	$F(s-a)$	$tf(t)$	$-dF(s)/ds$
$(\partial/\partial \alpha) f(t, \alpha)$	$(\partial/\partial \alpha) F(s, \alpha)$	$\int_0^t f(t) dt$	$F(s)/s$
$\int_0^t f(u)g(t-u) du$	$F(s)G(s)$		
1	$1/s$	$t^n (n = 1, 2, \dots)$	$n!/s^{n+1}, (n > 0)$
e^{at}	$1/(s-a), (s > a)$	$\sin \omega t$	$\omega/(s^2 + \omega^2), (s > 0)$
$\cos \omega t$	$s/(s^2 + \omega^2), (s > 0)$	$I f(t-T) = \begin{cases} 0, & t < T \\ 1, & t > T \end{cases}$	$e^{-sT}/s, (s, T > 0)$

8. FOURIER SERIES

If $f(x)$ is periodic of period $2L$, then $f(x+2L) = f(x)$, and

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \text{ where}$$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx, \quad n = 0, 1, 2, \dots, \text{ and}$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx, \quad n = 1, 2, 3, \dots$$

Parseval's Theorem

$$\frac{1}{L} \int_{-L}^L [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

E1.11 - ISE1 - Maths - 2010
Solutions -

ISE 1 ①
E1.11
1/14

A1. (i) If $z_1 = 1 + 7i$, and $z_2 = 4 + 3i$, then

$$|z_1| = \sqrt{1^2 + 7^2} = \sqrt{50} = 5\sqrt{2}$$

and

$$|z_2| = \sqrt{4^2 + 3^2} = 5.$$

We have

$$\frac{z_1}{z_2} = \frac{1 + 7i}{4 + 3i} = \frac{(1 + 7i)(4 - 3i)}{4^2 + 3^2}.$$

Expanding the numerator and denominator, we find

$$\frac{z_1}{z_2} = \frac{(4 + 21) + (28 - 3)i}{25} = 1 + i.$$

Thus $\Re(\frac{z_1}{z_2}) = \Im(\frac{z_1}{z_2}) = 1$. The modulus

$$|\frac{z_1}{z_2}| = \sqrt{2},$$

its argument is

$$\arg(\frac{z_1}{z_2}) = \arg(1 + i) = \tan^{-1}(1) = \pi/4.$$

We see easily that $|z_1/z_2| = |z_1|/|z_2|$; that is

$$\sqrt{2} = \frac{5\sqrt{2}}{5}.$$

(ii) If x and y are real,

$$\begin{aligned} \cos(x + iy) &= \cos(x) \cos(iy) - \sin(x) \sin(iy) = \\ &= \cos(x) \cosh(y) - i \sin(x) \sinh(y). \end{aligned}$$

Hence

$$\begin{aligned} |\cos(x + iy)|^2 &= \cos^2(x) \cosh^2(y) + \sin^2(x) \sinh^2(y) = \\ &= \cos^2(x)(1 + \sinh^2(y)) + (1 - \cos^2(x)) \sinh^2(y) = \cos^2(x) + \sinh^2(y), \end{aligned}$$

as required.

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A2. (i) The partial sum

$$\begin{aligned} S_N &= \sum_{n=1}^N \left(\frac{1}{n(n+1)} \right) \\ &= \sum_{n=1}^N \left[\frac{1}{n} - \frac{1}{n+1} \right] \\ &= 1 - \frac{1}{N+1}. \end{aligned}$$

The limit exists, and is

$$\lim_{n \rightarrow \infty} S_N = 1.$$

Hence the infinite series

$$\sum_{n=1}^{\infty} \left(\frac{1}{n(n+1)} \right)$$

is convergent; the limit is 1.

Using the comparison test, noting that for $n > 0$,

$$\frac{1}{n^2} > \frac{1}{n(n+1)} > \frac{1}{(n+1)^2},$$

we see that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

is convergent.

- (ii) If a power series $\sum_{n=0}^{\infty} a_n z^n$ converges for some z_0 , then by the comparison test it converges for $|z| < |z_0|$. The series thus converges in some interval $|z| < R$, and diverges for $|z| > R$. R is called the *radius of convergence* of the power series.

To calculate the radius of convergence of the power series:

$$\sum_{n=0}^{\infty} \frac{1}{2n+1} z^{2n+1},$$

consider the ratio of successive terms, applying the ratio test.

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{z^{2n+1}}{2n+1} \frac{2n-1}{z^{2n-1}} \\ = \lim_{n \rightarrow \infty} \frac{2n-1}{2n+1} z^2 = z^2. \end{aligned}$$

The series converges if this limit $z^2 < 1$.

For $|z| = 1$, the series diverges, by comparison with $\sum_{n=0}^{\infty} \frac{1}{n}$.

- (iii) The Maclaurin series for the function $f(x) = \ln(1+x)$ is found by successively differentiating, and evaluating these derivatives at $x = 0$:

$$f(x) = \ln(1+x), \quad f(0) = 0,$$

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ISE 1 (2)
E1.11

$$f'(x) = \frac{1}{1+x}, \quad f'(0) = 1,$$

$$f''(x) = -\frac{1}{(1+x)^2}, \quad f''(0) = -1,$$

$$f^{(n)}(x) = (-1)^{(n+1)}(n-1)! \frac{1}{(1+x)^n}, \quad f^{(n)}(0) = (-1)^{(n+1)}(n-1)! \dots$$

The series is given by:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

Here this gives

$$f(x) = \sum_{n=1}^{\infty} (-1)^{(n+1)} \frac{x^n}{n}.$$

Note the zeroth term vanishes.

Splitting $f(x)$ into its odd and even parts we get

$$f_{\text{odd}} = \frac{1}{2}(\ln(1+x) - \ln(1-x)) = \ln\left(\sqrt{\frac{1+x}{1-x}}\right)$$

and

$$f_{\text{even}} = \frac{1}{2}(\ln(1+x) + \ln(1-x)) = \ln(\sqrt{1-x^2}).$$

The Maclaurin series for f_{odd} consists of the odd power terms in the series for f :

$$f_{\text{odd}}(x) = \sum_{n=0}^{\infty} \frac{1}{2n+1} x^{2n+1}.$$

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A3. (a) (i) To evaluate $\lim_{n \rightarrow 1} \frac{n^3 - 1}{n^4 - 4n + 3}$, divide top and bottom by $n - 1$; we get:

$$\lim_{n \rightarrow 1} \frac{n^2 + n + 1}{n^3 + n^2 + n - 3}$$

Here the numerator has the limit 3, but the denominator has the limit zero - the limit of the quotient does not exist.

(ii) Similarly

$$\begin{aligned} \lim_{n \rightarrow 1} \frac{n^3 - 1}{n^4 - 4n^2 + 3} &= \\ \lim_{n \rightarrow 1} \frac{n^2 + n + 1}{n^3 + n^2 - 3n - 3} &= \\ \lim_{n \rightarrow 1} \frac{n^2 + n + 1}{n^3 + n^2 - 3n - 3} &= \\ \frac{3}{-4} &= -\frac{3}{4}. \end{aligned}$$

(iii) Multiply top and bottom by $\sin(x)$:

$$\begin{aligned} \lim_{x \rightarrow \pi/2} \frac{\sin x - \operatorname{cosec} x}{\cos^2(x)} &= \\ \lim_{x \rightarrow \pi/2} \frac{\sin^2 x - 1}{\cos^2(x) \sin x} &= \\ \lim_{x \rightarrow \pi/2} \frac{-1}{\sin x} &= -1. \end{aligned}$$

(iv) Divide top and bottom by n^2 :

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{n^2 + 1}{(n - 2)(n - 3)} &= \\ \lim_{n \rightarrow \infty} \frac{1 + 1/n^2}{(1 - 2/n)(1 - 3/n)} &= 1. \end{aligned}$$

(b) The numerator and denominator both vanish as $x \rightarrow 0$. Hence, using L'Hôpital's rule, we find:

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sin(x^2)}{1 - \cos(x)} &= \\ \lim_{x \rightarrow 0} \frac{2x \cos(x^2)}{\sin(x)} &= \\ \lim_{x \rightarrow 0} 2 \frac{x}{\sin(x)} \lim_{x \rightarrow 0} \cos(x^2) &= 2. \end{aligned}$$

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ISE 1
E1.11 (4)

A4. (i) To evaluate

$$\int_0^{\infty} x^3 \exp(-x) dx,$$

integrate by parts repeatedly:

$$\begin{aligned} & \int_0^{\infty} x^3 \exp(-x) dx \\ &= \int_0^{\infty} 3x^2 \exp(-x) dx \\ &= \int_0^{\infty} 6x \exp(-x) dx \end{aligned}$$

$$\int_0^{\infty} 6 \exp(-x) dx = 6[-\exp(-x)]_0^{\infty} = 6.$$

(ii) To evaluate $\int_e^{e^2} \frac{1}{x \ln x} dx$, substitute $u = \ln(x)$, $dx/x = du$:

$$\int_{x=e}^{e^2} \frac{1}{x \ln x} dx =$$

$$\int_{u=1}^2 \frac{1}{u} du =$$

$$[\ln(u)]_1^2 = \ln(2).$$

(iii) Put the integrand into partial fractions:

$$\int_3^4 \frac{(x+1)}{(x^2-3x+2)} dx =$$

$$\int_3^4 \frac{a}{x-1} + \frac{b}{x-2}$$

We find $a(x-2) + b(x-1) = x+1$, so $a = -2$, $b = 3$.

$$\int_3^4 \frac{-2}{x-1} + \frac{3}{x-2} =$$

$$\begin{aligned} & [-2 \ln(|x-1|) + 3 \ln(|x-2|)]_3^4 = -2 \ln(3/2) + 3 \ln(2/1) \\ & = 5 \ln(2) - 2 \ln(3). \end{aligned}$$

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A5. (i) To solve the homogeneous ode

$$\frac{dy}{dx} = -\frac{2x+3y}{x+y},$$

set $y = xu(x)$:

$$u + x \frac{du}{dx} = -\frac{2+3u}{1+u}$$

so

$$\begin{aligned} x \frac{du}{dx} &= -\left(\frac{2+3u}{1+u} + \frac{u+u^2}{1+u}\right) \\ &= -\frac{2+2u+u^2}{1+u} \end{aligned}$$

This is separable

$$\begin{aligned} \int^x \frac{dx'}{x'} &= - \int^{u(x)} \frac{(1+u')du'}{2+2u'+u'^2} \\ &= -\frac{1}{2} \ln(2+2u+u^2) + k. \end{aligned}$$

Thus, integrating and exponentiating,

$$x \sqrt{2 + 2\frac{y}{x} + \frac{y^2}{x^2}} = K,$$

that is

$$(2x^2 + 2xy + y^2) = K^2.$$

(ii)

$$\begin{aligned} \frac{dy}{dx} - xy &= \exp\left(\frac{x^2}{2}\right), \\ \text{with } y(0) &= 1. \end{aligned}$$

Multiply by the integrating factor $\exp(-x^2/2)$:

$$\frac{d}{dx}(y \exp(-x^2/2)) = 1,$$

so that

$$y \exp(-x^2/2) = x - x_0,$$

or

$$y = (x - x_0) \exp(x^2/2).$$

Set $x = 0$, $y = 1$, giving $1 = -x_0$, so that

$$y = (x + 1) \exp(x^2/2).$$

(iii)

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = \exp(x), \quad \text{with } y(0) = 0, \text{ and } y'(0) = 0.$$

ISEI ⑤
E1.11

The auxiliary equation is $m^2 + 3m + 2 = 0$; its roots are $m = -1$, $m = -2$. Hence the CF is

$$y_{CF} = A \exp(-x) + B \exp(-2x).$$

The PI must be a multiple of the exponential on the rhs - try

$$y_{PI} = \alpha \exp(x);$$

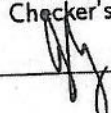
we see that this is a solution if $\alpha = 1/6$. Hence the general solution is

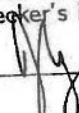
$$y = A \exp(-x) + B \exp(-2x) + \frac{1}{6} \exp(x).$$


This satisfies $y(0) = A + B + 1/6$, $y'(0) = -A - 2B + 1/6$, so $A = -1/2$, $B = 1/3$, giving the required solution:

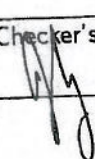
$$y = -\frac{1}{2} \exp(-x) + \frac{1}{3} \exp(-2x) + \frac{1}{6} \exp(x).$$

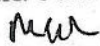

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	EXAMINATION SOLUTIONS 2006-07 9-10	Course ISC I
Question 6		Marks & seen/unseen
Parts	<p>(a) $\frac{\partial u}{\partial s} = u_x \frac{\partial x}{\partial s} + u_y \frac{\partial y}{\partial s}$ $= 2s u_x + 2t u_y$ $\frac{\partial u}{\partial t} = 2t u_x + 2s u_y$</p> <p>$\therefore s u_s + t u_t = s(2s u_x + 2t u_y) + t(2t u_x + 2s u_y)$ $= (2s^2 + 2t^2) u_x + 4st u_y$ $= 2x u_x + 2y u_y.$</p> <p>(b) $f(x, y) = x^3 - 3xy^2 + 12y$ $\therefore f_x = 3x^2 - 3y^2, \quad f_y = -6xy + 12.$ At stationary pts, $x^2 - y^2 = 0 \quad (1)$ $xy = 2 \quad (2)$ By (1), $y = \pm x$. If $y = x$, (2) $\Rightarrow x^2 = 2$ $\therefore y = x = \pm \sqrt{2}.$ If $y = -x$, (2) $\Rightarrow x^2 = -2 \times$ \therefore stationary pts are $(\sqrt{2}, \sqrt{2}), (-\sqrt{2}, -\sqrt{2}).$ Now $f_{xx} = 6x, f_{xy} = -6y, f_{yy} = -6x.$ At $(\sqrt{2}, \sqrt{2}), f_{xx} > 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 < 0 \therefore \underline{\text{MIN}}$ At $(-\sqrt{2}, -\sqrt{2}), f_{xx} < 0 \& f_{xx}f_{yy} - f_{xy}^2 < 0 \therefore \underline{\text{MAX}}$</p>	<p>2</p> <p>2</p> <p>4</p> <p>2</p> <p>6</p> <p>2</p> <p>2</p>
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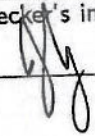
	EXAMINATION SOLUTIONS 2006-07	Course
Question 7, ch		Marks & seen/unseen
Parts	$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ $= \frac{\pi}{4} - \frac{2}{\pi} \sum_{n \text{ odd}} \frac{\cos nx}{n^2} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin nx}{n}$ $= \frac{\pi}{4} - \frac{2}{\pi} \sum_{m=0}^{\infty} \frac{\cos(2m+1)x}{(2m+1)^2} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin nx}{n}$ <hr/> <p>Put $x=0$: As $f(x)$ is continuous at $x=0$, Fourier series equals $f(0)$, so</p> $0 = \frac{\pi}{4} - \frac{2}{\pi} \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2}$ <p>Hence $\sum \frac{1}{(2m+1)^2} = \frac{\pi^2}{8}$.</p> <p>Put $x = \frac{\pi}{2}$: as $\cos(2m+1)\frac{\pi}{2} = 0$ and $\sin n\frac{\pi}{2} = \begin{cases} 0 & \text{if } n \text{ even} \\ (-1)^k & \text{if } n = 2k+1 \end{cases}$</p> <p>Fourier series gives</p> $f(\frac{\pi}{2}) = \frac{\pi}{2} = \frac{\pi}{4} + \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$ <p>So $\sum \frac{(-1)^k}{2k+1} = \frac{\pi}{4}$</p>	<p>2</p> <p>3</p> <p>4</p>
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	EXAMINATION SOLUTIONS 2006-07	Course
Question 8		Marks & seen/unseen
Parts	<p>(a) $y'' - y + t^2 = 0, \quad y(0) = 2, y'(0) = 0.$</p> <p>Take Laplace transforms:</p> $s^2 L(y) - 2s - L(y) + L(t^2) = 0$ <p>As $L(t^2) = \frac{2}{s^3}$ this gives</p> $(s^2 - 1) L(y) = 2s - \frac{2}{s^3} = \frac{2(s^4 - 1)}{s^3}$ <p>Hence</p> $L(y) = \frac{2(s^2 + 1)}{s^3} = \frac{2}{s} + \frac{2}{s^3}$ <p>Thus</p> $\underline{y = 2 + t^2}$ <p>(b) $x' + y' - y = 2$ $x' - y' + x = t^2 + 2t$</p> <p>Take Laplace transforms:</p> <p>(1) $sL(x) - 2 + sL(y) - L(y) = \frac{2}{s}$</p> <p>(2) $sL(x) - 2 - sL(y) + L(x) = \frac{2}{s^3} + \frac{2}{s^2}$</p> <p>So</p> <p>(1) $sL(x) + (s-1)L(y) = 2 + \frac{2}{s} = \frac{2(s+1)}{s}$</p> <p>(2) $(s+1)L(x) - sL(y) = \frac{2}{s^3} + \frac{2}{s^2} + 2$</p> $= \frac{2(s^3 + s + 1)}{s^3}$	<p>3</p> <p>3</p> <p>2</p> <p>4</p>
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	EXAMINATION SOLUTIONS 2006-07	Course ISE 1
Question 8, ch		Marks & seen/unseen
Parts	<p>So (1) $\times (s+1) - (2) \times s$ gives</p> $(2s^2-1)L(y) = \frac{2(s+1)^2}{s} - \frac{2(s^3+s+1)}{s^2}$ $= \frac{2(s^3+2s^2+s-s^3-s-1)}{s^2}$ $= \frac{2(2s^2-1)}{s^2}$ <p>So $L(y) = \frac{2}{s^2}$, hence <u>$y = 2t$</u></p> <p>From (1),</p> $sL(x) = 2 + \frac{2}{s} - (s-1)L(y)$ $= 2 + \frac{2}{s} - \frac{(2s-2)}{s^2}$ $= 2 + \frac{2}{s^2}$ <p>So $L(x) = \frac{2}{s} + \frac{2}{s^3}$, so <u>$x = 2 + t^2$</u></p>	<p>4</p> <p>2</p> <p>2</p>
	Setter's initials MLR <div style="display: inline-block; vertical-align: middle; margin-left: 100px;"> Checker's initials  </div>	Page number

	EXAMINATION SOLUTIONS 2006-07	Course ISE 1
Question 9		Marks & seen/unseen
Parts	<p>10) Solving the three eqns simultaneously gives system with augmented matrix</p> $\begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & -1 & a & -3 \\ 2 & 0 & 1 & b \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 & -1 \\ 0 & -2 & a-1 & -2 \\ 0 & -2 & -1 & b+2 \end{pmatrix}$ $\longrightarrow \begin{pmatrix} 1 & 1 & 1 & -1 \\ 0 & -2 & a-1 & -2 \\ 0 & 0 & -a & b+4 \end{pmatrix}$ <p>So last eqn is $-az = b+4$.</p> <p>Therefore</p> <p>(i) system has 1 soln (ie. planes meet in 1 pt) if <u>$a \neq 0$</u></p> <p>(ii) system has a line of solns (ie. planes meet in a line) if <u>$a = 0, b = -4$</u></p> <p>(iii) system has no solns (ie. planes don't meet) if <u>$a = 0, b \neq -4$</u></p>	<p>4</p> <p>2</p> <p>2</p> <p>2</p>
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	EXAMINATION SOLUTIONS 2006-07	Course ISE 1
Question 9, d)		Marks & seen/unseen
Parts	<p>(b) Eigenvalues of A :</p> $\begin{vmatrix} -1-\lambda & -10 \\ 5 & 14-\lambda \end{vmatrix} = \lambda^2 - 13\lambda + 36$ $= (\lambda - 9)(\lambda - 4).$ <p>So eigenvalues are 4, 9.</p> <p><u>$\lambda = 4$</u> Eigenvectors are solutions of</p> $\begin{pmatrix} -5 & -10 \\ 5 & 10 \end{pmatrix} \rightarrow \text{an eigenvector } \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ <p><u>$\lambda = 9$</u> $\begin{pmatrix} -10 & -10 \\ 5 & 5 \end{pmatrix} \rightarrow \text{an eigenvector } \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$</p> <p>So <u>$P = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$</u> will work</p> <p>(other answers consist of course).</p> <p>Take So $P^{-1}AP = \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}.$</p> <p>Take $B = P \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} P^{-1}$</p> $= \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ $= \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}.$ <p>Then $B^2 = (PDP^{-1})^2 \quad (D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix})$</p> $= PD^2P^{-1} = A.$	<p>3</p> <p>3</p> <p>4</p>
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