E4.05
C5.22
SO7

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2006

MSc and EEE PART IV: MEng and ACGI

TRAFFIC THEORY & QUEUEING SYSTEMS

Monday, 15 May 10:00 am

Time allowed: 3:00 hours

Corrected Copy

There are FIVE questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : J.A. Barria

Second Marker(s) : J.C. Allwright

Especial Information for Invigilators: NIL

Information for Candidates:

1. Erlang Loss formula recursive evaluation:

$$E_N(\rho) = \frac{\rho E_{N-1}(\rho)}{N + \rho E_{N-1}(\rho)}$$
$$E_0(\rho) = 1.$$

2. Engset Loss formula recursive evaluation (for a fixed M and $p = \alpha/1+\alpha$):

$$e_N = \frac{(M-N+1)\alpha e_{N-1}}{N + (M-N+1)\alpha e_{N-1}}$$
$$e_0 = 1.$$
$$\alpha = \lambda/\mu.$$

3. Traffic capacity on basis of Erlang B formula (next page).

1.

a) For the system in Figure 1.1 assume:

- The M sources act independently.
- The call arrival rate from a free source is Poisson with parameter λ.
- The link is composed of N channels.
- The channel holding time is a negative exponential with parameter $1/\mu$.
- There is full availability access.

i) Derive the Global balance equations. [6]

ii) Derive the Local balance equations. [6]

b) Consider the system of part a) and assume the following parameters:

- N = 4 (channels link).
- M = 6 (sources of traffic).
- The call rate of a free source = 0.5 [calls/minute].
- The mean holding time = 2 [minutes].

Determine the total traffic carried by the link. [8]

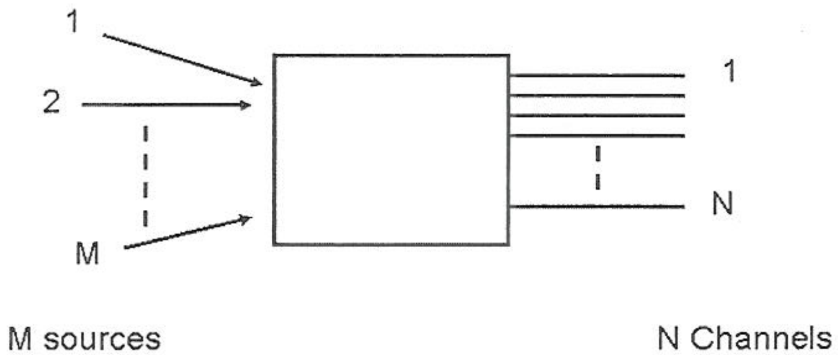

Figure 1.1

2.

a) A 15-channel communication link is offered pure chance traffic with an arrival rate of 3 [demands/minute] and a mean holding time of 3 [minutes].

If blocked calls are allowed to overflow onto a single-channel overflow link:

i) Determine the mean of the traffic offered to the overflow link.

i) Determine the mean traffic carried by the overflow link.

[10]

b) In practice a local access switch may not give full-availability access to the link. That is, an arriving demand may be blocked even when the link is not saturated.

i) Using a known definition of "loss factor", describe and discuss a restricted availability model.

ii) For the model described in b) i) derive the birth and death coefficients.

[10]

3.

a) For an M/M/K/N system

i) Show that the probability that the system is empty can be expressed by:

$$\pi_0 = \frac{1}{\left(A^K / K!\right)} \left[\frac{(1-\rho)E_K(A)}{(1-\rho)+\rho(1-\rho^B)E_K(A)} \right]$$

ii) Define A, B, K, ρ and $E_K(A)$.

iii) Derive the probability that new arrivals are delayed.

iv) Derive the probability that new arrivals are blocked.

State clearly all the assumptions made.
Explain all the steps of your derivations.

[10]

b) For the following priority schemes:

i) For the pre-emptive priority scheme: determine the expected value of residual time seen by class k traffic.

ii) For the non-pre-emptive priority scheme: determine the mean waiting time for class k traffic.

State clearly all the assumptions made.
Explain all the steps of your derivation.

[10]

4.

a) Consider a fluid flow model. The stationary probability *F(x)* that the buffer occupancy is less than or equal to *x*, given *i* sources in talkspurt, can be obtained from the following equation:

$$(i-C)\alpha\frac{\partial F_i(x)}{\partial x}=[N-(i-1)]\lambda F_{i-1}(x)-[(N-i)\lambda+i\alpha]F_i(x)+(i+1)\alpha F_{i+1}(x)$$

This can be written in matrix form as

$$\frac{dF(x)}{dx}\mathrm{D}=F(x)\mathrm{M}$$

The dimension of D and M is NxN, where N is the number of traffic sources, and hence the general solution is given by:

$$F(x)=\sum_{i:\mathrm{Re}[Z_i<0]}^{N}a_i\phi_i e^{z_i x}$$

where z_i is the i-th eigenvalue and ϕ_i is the corresponding i-th eigenvector of the eigenvector equation $z_j\ \phi_j\ \mathrm{D}=\phi_j\mathrm{M}$.

i) Define the traffic source model and explain the meaning of the parameter α.

ii) Define and construct M and D for the system when N = 1.

iii) Obtain the eigenvalues for the system when N = 1.

iv) Discuss your results.

[10]

b) In ATM admission control mechanisms, the measure of equivalent capacity is frequently used.

i) Describe the motivations for using a stationary approximation of equivalent capacity

ii) Derive the stationary approximation of equivalent capacity.

iii) Describe the motivation for using a fluid-flow approximation of equivalent capacity.

iv) Derive the fluid-flow approximation of equivalent capacity.

State clearly all the assumptions made.
Explain all the steps of your derivations.

[10]

5.

a)

i) Describe a way in which a composite Performance/Reliability assessment could be realised using the Performability model.

ii) Describe and discuss assumptions made on a Markov Reward Model (MRM).

iii) For the system in Figure 5.1.

assume:
- The Link 1 failure rate is α .
- The Link 2 failure rate is β .
- The system can only be fully repaired at a rate μ if both links are simultaneously in failure condition.

a) Define the state space of the system.

b) Derive the Markov chain representation of the fault tolerant system.

[10]

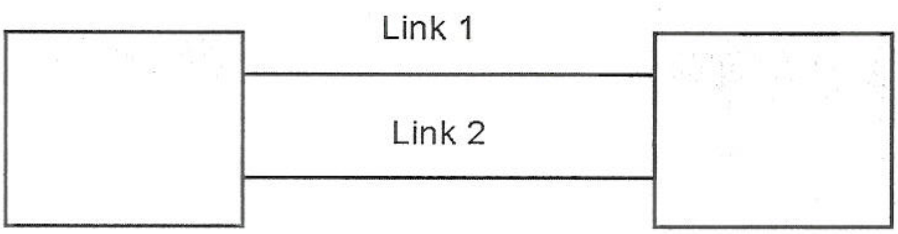

Figure 5.1

b) For a stochastic knapsack problem:

i) Derive the blocking probability of an arriving class-k object.

ii) Discuss the relationship between the knapsack problem and the Erlang loss system.

[10]

SOLUTIONS – 05/06 Final 1/10

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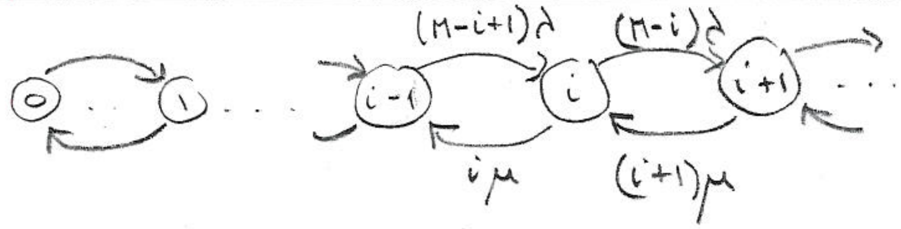
Model Answers and Mark Schemes First Examiner:

Paper Code: E4.05 – SC7 – CS22 Second Examiner: Traffic Theory and Queueing Systems

Question Number etc. in left margin Mark allocation in right margin

Q1

a) 

2

i) Global

$$[(M-i)\lambda + i\mu]\pi_i = (M-i+1)\lambda\,\pi_{i-1} + (i+1)\mu\,\pi_{i+1}$$

M = no. sources

π_i = equilibrium state probability in state i

λ = calling rate / free source

μ = release rate / channel 4

ii) Local

$$(M-i+1)\lambda\,\pi_{i-1} = i\mu\,\pi_i$$

4

+ discussion (bookwork extension)

2/

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Q1 a)

$M = 6, \; N = 4$

For $N < M$

$$\rho_c = E(N_+) = Mp - [\Pi_N(M,p) \times (M-N)p]$$ 3

$$\Pi_N(M,p) = e_N = \frac{(M-N+1)\alpha \, e_{N-1}}{N + (M-N+1)\alpha \, e_{N-1}}, \quad e_0 = 1$$ 3

(Recursion)

$$p = \frac{\lambda}{\lambda+\mu} = 0.5 \qquad \alpha = \frac{\lambda}{\mu} = 1 = 0.5 \times 2$$

$N=1 \quad e_1 = \frac{(6-1+1)}{1+6-1+1} = \frac{6}{7}$

$N=2 \quad e_2 = \frac{(6-2+1)\,6/7}{2+(6-2+1)\,6/7} = \frac{30/7}{(14+30)/7} = \frac{30}{44}$ 2

$N=3 \quad e_3 = 30/63$

$N=4 \quad e_4 = \frac{(6-4+1)\;30/63}{4+(6-4+1)\,30/63} = \frac{90}{342}$

$$\rho_c = 6 \cdot 0.5 - \frac{90}{342} \cdot (6-4)\,0.5$$ 2

$= 2.7368$

(calculation of new example)

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Q2

a) offered traffic $\rho = \lambda \bar{h} = 9$ Erlangs

Using Kosten model

Mean of overflow traffic

$m = E(N_t) = \rho E_M(\rho)$

from table

$m = 9[0.02] = 0.18$ 5

Using Brockmeyer model

Mean of the carried traffic

$m = E(N_t) = \rho [E_M(\rho) - E_{M+N}(\rho)]$

from table

$m = 9[0.02 - 0.011] = 0.081$ (calculated, of new example) 5

b) Restricted availability

Loss factor $\beta_i = P[\text{arrival is blocked} \mid N_t = i]$ $i = 0, 1, \ldots$

For restricted availability access

$0 < \beta_i < 1$ for at least one $i < N$

The new birth coefficients $\lambda_i' = (1 - \beta_i)\lambda_i$

Since $P[\text{arrival in } (t, t+\Delta t) \text{ and not blocked} \mid N_t = i]$

$= (\lambda_i \Delta t)(1 - \beta_i)$

The death coefficient $\mu_i = i\mu$ 10

(bookwork)

4)

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Q3 M|M|K|N

a)

local balance equation

$$\pi_i = \left(\frac{A^i}{i!}\right)\pi_0 \qquad 0 \le i \le K$$

$$= \left(\frac{A^K}{K!}\right)\rho^{i-K}\pi_0 \qquad K \le i \le K+B$$

$$\sum \pi_i = 1 = \left(\sum_{i=1}^{K}\left(\frac{A^i}{i!}\right) + \sum_{j=1}^{B}\left(\frac{A^K}{K!}\right)\rho^j\right)\pi_0$$

$$\Rightarrow S = \left(\frac{A^K}{K!}\right)\left[E_K^{-1}(A) + \frac{\rho(1-\rho^B)}{1-\rho}\right] \qquad \left(\pi_0 = \frac{1}{S}\right)$$ 3

note
$$\sum_{j=1}^{B}\rho^j = \rho\sum_{j=0}^{B-1}\rho^j = \rho\left(\sum_{j=0}^{\infty}\rho^j - \sum_{k=B}^{\infty}\rho^k\right)$$

$$= \rho\left(\frac{1}{1-\rho} - \frac{\rho^B}{1-\rho}\right)$$

$$= \rho\left(\frac{1-\rho^B}{1-\rho}\right)$$

$N = K+B$ (size system)

K = nr servers $\qquad B$ = size Buffer 3

$\rho = A/K \qquad A = \lambda/\mu$

$$E_K(A) = \frac{A^K/K!}{\sum_{j=0}^{N}(A^j/j!)}$$

(bookwork extension)

Q3 a) (cont) iii)

$$P[\text{Delay}] = P[\text{all } K \text{ servers busy} \mid \text{buffer not full}]$$
$$= P[K \le N_t < K+B]$$
$$= \pi_K \left[\frac{1-\rho^B}{1-\rho} \right]$$ 2

$$P[\text{loss}] = P[\text{Buffer full}]$$
$$= P[N_t = K+B]$$
$$= \pi_K \rho^B$$ 2

b) i)

$$E(R) = \frac{1}{2} \sum_{i=1}^{K} \lambda_i E(S_i^2) \sim E(R_K)$$

class K items can interrupt any lower-class service in progress. So the only contribute to E(R) will be from class K and above 3

ii)

$$E(W_K) = \frac{E(R)}{(1-\sigma_{K-1})(1-\sigma_K)}$$

$$\sigma_{K-1} = \sum_{i=1}^{K-1} \rho_i$$

derived starting from

$$E(W_1) = E(R) + E(Q_1) E(S_1)$$
$$= E(R) + \lambda_1 E(W_1) E(S_1)$$

$$E(W_1) = \frac{E(R)}{1-\rho_1}$$

(bookwork + discussion) 5

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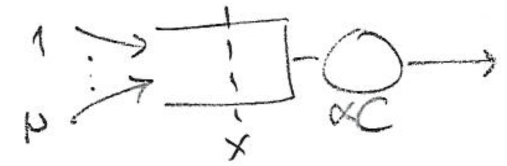
Q4 a)



1

$1/\alpha$ = average length of talk spurt

$$\frac{dF(x)}{dx} D = F(x) M \;\rightarrow\; \frac{dF(x)}{dx} = F(x) M'$$

$$M' = MD'$$

$N=1$ (one voice source)

$$M = \begin{bmatrix} -\lambda & \lambda \\ \alpha & -\alpha \end{bmatrix} = \begin{bmatrix} -\gamma & \gamma \\ 1 & -1 \end{bmatrix} \qquad \gamma \equiv \frac{\lambda}{\alpha}$$ 3

$$D = \begin{bmatrix} -c\alpha & 0 \\ 0 & (1-c)\alpha \end{bmatrix} \qquad D' = \begin{bmatrix} \frac{1}{-c\alpha} & 0 \\ 0 & \frac{1}{(1-c)\alpha} \end{bmatrix}$$

$$M' = \begin{bmatrix} \gamma/c & \gamma/(1-c) \\ -1/c & -1/(1-c) \end{bmatrix} \Bigg|_{\alpha=1}$$ (normalised version) 3

Eigenvalues given by:

$$zI - M' = 0$$

$$z = 0$$

$$z = \frac{\gamma}{c} - \frac{1}{1-c}$$

Stable system $\Rightarrow \rho \equiv \left(\frac{\gamma}{1-\gamma}\right)\frac{1}{c} < 1$

$$z = -\frac{(1-\rho)(1+\gamma)}{1-c} \qquad (z<0)$$ (calculate of new example) 3

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Q4 b) Equivalent capacity

$C_L = (m + K\sigma) R_p$ $\quad$ $m R_p$ = mean, σR_p = standard deviation

$K = K(QoS)$

$m = Np$

$\sigma^2 = Np(1-p) = m(1-p)$

$C = \frac{C_L}{R_p} \rightarrow C = m + K\sigma = Np + K\sqrt{Np(1-p)}$

$K(QoS) \sim P_L$

i) $P_L = \sum_{i=J_0}^{N} \frac{(i-c)\pi_i}{m}$ $\quad$ ii) $\varepsilon = \sum_{i=J_0}^{N} \pi_i$ 2

i) large number of sources multiplexed

$N \gg 1$, $p \ll 1$

$\pi_i = \binom{N}{i} p^i (1-p)^{N-i}$

Approximated by the normal distribution

$m = Np$, $\sigma^2 = Np(1-p)$

$$P_L = \frac{1}{m}\int_{J_0}^{\infty} \frac{e^{-(x-m)^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}(x-c)\, dx$$

$$\varepsilon = \int_{J_0}^{\infty} \frac{e^{-(x-m)^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}\, dx$$ 4

if $(c-m) > 3\sqrt{2}\sigma$

$$\varepsilon = \frac{\sigma e^{-(c-m)^2/2\sigma^2}}{\sqrt{2\pi}(c-m)} \quad ; \quad P_L = \frac{1-p}{c-m}\varepsilon$$

$$\ln(\sqrt{2\pi}\,\varepsilon) = \ln(\sigma/(c-m)) - (c-m)^2/2\sigma^2$$

$$C_{LS} = m R_p + \sigma \underbrace{\sqrt{-\ln(2\pi) - 2\ln\varepsilon}}_{K} R_p$$

(derivation + good work explanation)

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Q4 ii) Effect of the access buffer

$$G(x) \sim A_N \rho^N e^{-\beta_R x / R_p}$$

(probability buffer occupancy > x)

$$R = (1-\rho)\left(1+\frac{\alpha}{\beta}\right) \Big/ \left(1 - \frac{C_L}{N R_p}\right)$$

$$\rho = \frac{N_p R_p}{C_L}$$

if $\rho \sim 1$ $A_N \rho^N \sim 1$

$$P_L = e^{-\beta_R x / R_p}$$

$$-\ln P_L = \beta_R x / R_p$$

$$\frac{C_L}{R_p N} = \frac{1-k}{2} + \sqrt{\left(\frac{1-k}{2}\right)^2 + k\rho}$$

$$C_{LF} = R_p N \left(\frac{1-k}{2}\right) + R_p N \sqrt{\left(\frac{1-k}{2}\right)^2 + k\rho}$$ 4

(bookwork + explanation)

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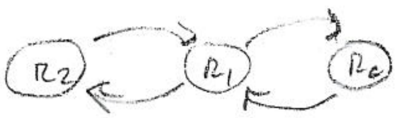
5a i) Performability is a measure that quantifies a system's ability to perform in the presence of faults.

A Performability model is obtained by:
- Modelling the process X that represents the behaviour of the system and
- Defining the performance variable Y for each state of the system

ii) MRM 2
- If X(t) can be represented by a continuous-time finite-state Markov chain (CTMC), and
- Extend this process X(t) assigning reward rates to its states 2



$$Q \qquad Y(t) = \sum_{i=0}^{n} r_i z_i$$

$$P_i(t, y) = P[\, Y(t) \le y \mid X(0) = s_i \,]$$

iii)

a) 00, 01, 10, 11 (link 1 failure, link 2 failure) 3

b)

$$Q = \begin{array}{c|cccc} & 00 & 01 & 10 & 11 \\ \hline 00 & -(\alpha+\mu) & \mu & \alpha & 0 \\ 01 & 0 & -\alpha & 0 & \alpha \\ 10 & 0 & 0 & -\mu & \mu \\ 11 & \mu & 0 & 0 & -\mu \end{array}$$

3

(Calculation of new example)

Q5 b) i)

- C resource units
- Poisson arrival λ_K
- Exponential holding time $1/\mu_K$
- hold b_K resource units
- pure loss system: $n = (n_1, \dots, n_K)$; $b = (b_1, \dots, b_K)$

Admit class-k arrival if $b_K < C - b \cdot n$

Dynamic Knapsack Problem

$S \equiv \{n \in I^K : b \cdot n \leq C\}$

$X(t) = (X_1(t), \dots, X_K(t))$ state at t

$\{X(t)\}$ = aperiodic and irreducible Markov process over S

Blocking probability of class-k

$S_K = \{n \in S : b \cdot n \leq C - b_K\}$

Since arrivals are Poisson $B_K = 1 - \sum_{n \in S_K} \pi(n)$ 5

ii) The stochastic knapsack generalises the Erlang loss system. If there is only one class and all objects have size of unit, then the stochastic knapsack reduces to the Erlang loss system. The stochastic knapsack result is a multidimensional generalisation of the Erlang

5

(bookwork)