Imperial College London

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2012

This paper is also taken for the relevant examination for the Associateship.

M2S1

PROBABILITY AND STATISTICS II

Date: Tuesday, 22nd May 2012 Time: 2 pm – 4 pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

Formula sheets are included as pages 5 and 6.

1. (a) Let X_1, X_2, X_3, X_4 be independent, identically distributed $N(0,1)$ random variables.

Identify the following distributions:

(i) The distribution of

$$Y_1 = 3X_1 - 5X_2 + X_3;$$

(ii) The distribution of

$$Y_2 = X_1/|X_2|;$$

(iii) The distribution of

$$Y_3 = \sum_{i=1}^{4} X_i^2;$$

(iv) The distribution of

$$Y_4 = \frac{\sqrt{2}X_1}{\sqrt{(X_2^2 + X_3^2)}}.$$

(b) Let $X_1, \ldots, X_n$ be independent, identically distributed $N(\mu, \sigma^2)$, where μ and σ^2 are both unknown.

Obtain the maximum likelihood estimators of μ and σ^2. Show that the maximum likelihood estimator of σ^2 is biased, but that some multiple of it is not.

State, without proof, the joint distribution of the random variables $\bar{X} = n^{-1}\sum_{i=1}^{n} X_i$ and $S^2 = (n-1)^{-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$.

Explain clearly how the joint distribution allows construction of an appropriate test statistic for testing the null hypothesis $H_0 : \mu = \mu_0$ against the alternative hypothesis $H_1 : \mu \neq \mu_0$. Describe in detail how you would carry out the test.

Explain how to construct a $100(1-\alpha)\%$ confidence interval for μ.

How would you construct a $100(1-\alpha)\%$ confidence interval for μ in the situation where the value of σ^2 is *known*?

2. (a) Let $X_1, X_2, \ldots$ be a sequence of independent, identically distributed random variables, having common moment generating function $M_X(t)$, and let $N(\geq 0)$ be a random variable which is independent of the X_i, with probability generating function $G_N(s)$.

What is the moment generating function of $Z = X_1 + \ldots + X_N$?

(b) A hen lays X eggs, where X is Poisson with parameter λ. Each egg hatches, independently of the other eggs, with probability p.

Let Y be the number of eggs which do hatch and Z the number which do not.

Find the joint moment generating function of Y and Z and deduce that they are independent and that Y is Poisson with parameter λp. What is the correlation between X and Y?

3. Let U and X be independent random variables, such that U is uniformly distributed on $(0, 1)$ and X has density $f_X(x)$. Suppose that there exists a constant $a > 0$, such that for all x the function $f_S(x)$ satisfies

$$0 \leq f_S(x) \leq a f_X(x)$$

and $\int_{-\infty}^{\infty} f_S(x)dx = 1$.

(i) What is $P(aUf_X(X) \leq f_S(X))$?

(ii) Show that

$$P(X \leq x | aUf_X(X) \leq f_S(X)) = \int_{-\infty}^{x} f_S(y)dy.$$

(iii) Explain how the result in (ii) may be used to produce realisations of a random variable S with density $f_S(s)$.

(iv) A 'half normal' random variable S is defined as $S = |Z|$, where Z is standard normal.

What is the probability density function of S?

Let X have exponential density $f_X(x) = \lambda e^{-\lambda x},\ x > 0$.

Show that the ratio $f_S(x)/f_X(x)$ is maximised at $x = \lambda$ and find the value of $a = \sup_x \{f_S(x)/f_X(x)\}$.

In using the procedure developed in (iii) to produce realisations of S for this choice of X, how would you choose λ?

4. (a) Explain what is meant by a $\sigma-$field $\mathcal{A}$ of subsets of a sample space Ω.

Let A and B belong to some $\sigma-$field $\mathcal{A}$. Show that $\mathcal{A}$ contains the set $A \cap B$.

What is meant by a *probability function*? What is a *probability space*?

(b) What does it mean to say that a sequence of random variables $X_1, X_2, \ldots,$ (i) *converges in probability*, (ii) *converges in quadratic mean*, (iii) *converges in distribution*, to a random variable X?

Show that convergence in quadratic mean implies convergence in probability. Is the converse true?

(c) Suppose that $X_1, \ldots, X_n$ are independently uniformly distributed on $(0, \theta)$.

Show that $\max\{X_1, \ldots, X_n\}$ converges in probability to θ.

DISCRETE DISTRIBUTIONS

	RANGE $\mathbb{X}$	PARAMETERS	MASS FUNCTION f_X	CDF F_X	$\mathsf{E}_{f_X}[X]$	$\mathsf{Var}_{f_X}[X]$	MGF M_X
$Bernoulli(\theta)$	$\{0,1\}$	$\theta \in (0,1)$	$\theta^x(1-\theta)^{1-x}$		θ	$\theta(1-\theta)$	$1-\theta+\theta e^t$
$Binomial(n,\theta)$	$\{0,1,...,n\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{n}{x}\theta^x(1-\theta)^{n-x}$		$n\theta$	$n\theta(1-\theta)$	$\left(1-\theta+\theta e^t\right)^n$
$Poisson(\lambda)$	$\{0,1,2,...\}$	$\lambda \in \mathbb{R}^+$	$\dfrac{e^{-\lambda}\lambda^x}{x!}$		λ	λ	$\exp\left\{\lambda\left(e^t-1\right)\right\}$
$Geometric(\theta)$	$\{1,2,...\}$	$\theta \in (0,1)$	$(1-\theta)^{x-1}\theta$	$1-(1-\theta)^x$	$\dfrac{1}{\theta}$	$\dfrac{(1-\theta)}{\theta^2}$	$\dfrac{\theta e^t}{1-e^t(1-\theta)}$
$NegBinomial(n,\theta)$	$\{n,n+1,...\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{x-1}{n-1}\theta^n(1-\theta)^{x-n}$		$\dfrac{n}{\theta}$	$\dfrac{n(1-\theta)}{\theta^2}$	$\left(\dfrac{\theta e^t}{1-e^t(1-\theta)}\right)^n$
or	$\{0,1,2,...\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{n+x-1}{x}\theta^n(1-\theta)^x$		$\dfrac{n(1-\theta)}{\theta}$	$\dfrac{n(1-\theta)}{\theta^2}$	$\left(\dfrac{\theta}{1-e^t(1-\theta)}\right)^n$

For **CONTINUOUS** distributions (see over), define the **GAMMA FUNCTION**

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx$$

and the LOCATION/SCALE transformation $Y = \mu + \sigma X$ gives

$$f_Y(y) = f_X\left(\frac{y-\mu}{\sigma}\right)\frac{1}{\sigma} \qquad F_Y(y) = F_X\left(\frac{y-\mu}{\sigma}\right) \qquad M_Y(t) = e^{\mu t}M_X(\sigma t) \qquad \mathsf{E}_{f_Y}[Y] = \mu + \sigma\mathsf{E}_{f_X}[X] \qquad \mathsf{Var}_{f_Y}[Y] = \sigma^2\mathsf{Var}_{f_X}[X]$$

CONTINUOUS DISTRIBUTIONS

	$\mathbb{X}$	PARAMS.	PDF f_X	CDF F_X	$\mathsf{E}_{f_X}[X]$	$\mathsf{Var}_{f_X}[X]$	MGF M_X
$Uniform(\alpha,\beta)$ (stand. model $\alpha=0,\beta=1$)	(α,β)	$\alpha<\beta\in\mathbb{R}$	$\dfrac{1}{\beta-\alpha}$	$\dfrac{x-\alpha}{\beta-\alpha}$	$\dfrac{(\alpha+\beta)}{2}$	$\dfrac{(\beta-\alpha)^2}{12}$	$\dfrac{e^{\beta t}-e^{\alpha t}}{t(\beta-\alpha)}$
$Exponential(\lambda)$ (stand. model $\lambda=1$)	$\mathbb{R}^+$	$\lambda\in\mathbb{R}^+$	$\lambda e^{-\lambda x}$	$1-e^{-\lambda x}$	$\dfrac{1}{\lambda}$	$\dfrac{1}{\lambda^2}$	$\left(\dfrac{\lambda}{\lambda-t}\right)$
$Gamma(\alpha,\beta)$ (stand. model $\beta=1$)	$\mathbb{R}^+$	$\alpha,\beta\in\mathbb{R}^+$	$\dfrac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}$		$\dfrac{\alpha}{\beta}$	$\dfrac{\alpha}{\beta^2}$	$\left(\dfrac{\beta}{\beta-t}\right)^\alpha$
$Weibull(\alpha,\beta)$ (stand. model $\beta=1$)	$\mathbb{R}^+$	$\alpha,\beta\in\mathbb{R}^+$	$\alpha\beta x^{\alpha-1}e^{-\beta x^\alpha}$	$1-e^{-\beta x^\alpha}$	$\dfrac{\Gamma(1+1/\alpha)}{\beta^{1/\alpha}}$	$\dfrac{\Gamma\left(1+\frac{2}{\alpha}\right)-\Gamma\left(1+\frac{1}{\alpha}\right)^2}{\beta^{2/\alpha}}$	
$Normal(\mu,\sigma^2)$ (stand. model $\mu=0,\sigma=1$)	$\mathbb{R}$	$\mu\in\mathbb{R},\sigma\in\mathbb{R}^+$	$\dfrac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\dfrac{(x-\mu)^2}{2\sigma^2}\right\}$		μ	σ^2	$e^{\{\mu t+\sigma^2t^2/2\}}$
$Student(\nu)$	$\mathbb{R}$	$\nu\in\mathbb{R}^+$	$\dfrac{(\pi\nu)^{-\frac{1}{2}}\Gamma\left(\dfrac{\nu+1}{2}\right)}{\Gamma\left(\dfrac{\nu}{2}\right)\left\{1+\dfrac{x^2}{\nu}\right\}^{(\nu+1)/2}}$		0 (if $\nu>1$)	$\dfrac{\nu}{\nu-2}$ (if $\nu>2$)	
$Pareto(\theta,\alpha)$	$\mathbb{R}^+$	$\theta,\alpha\in\mathbb{R}^+$	$\dfrac{\alpha\theta^\alpha}{(\theta+x)^{\alpha+1}}$	$1-\left(\dfrac{\theta}{\theta+x}\right)^\alpha$	$\dfrac{\theta}{\alpha-1}$ (if $\alpha>1$)	$\dfrac{\alpha\theta^2}{(\alpha-1)(\alpha-2)}$ (if $\alpha>2$)	
$Beta(\alpha,\beta)$	$(0,1)$	$\alpha,\beta\in\mathbb{R}^+$	$\dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$		$\dfrac{\alpha}{\alpha+\beta}$	$\dfrac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	

	EXAMINATION SOLUTIONS 2011-12	Course M2S1
Question 1		Marks & seen/unseen
Parts		
(a)	i. $N(0, 3^2+5^2+1^2) = N(0, 35)$	1
	ii $Y_2 \equiv \dfrac{X_1}{\sqrt{X_2^2/1}} \sim t_1$,	
	Student's t-distribution on 1 degree of freedom [or Cauchy]	1
	iii χ^2_4	1
	iv $Y_4 \equiv \dfrac{X_1}{\sqrt{(X_2^2+X_3^2)/2}} \sim t_2$	1
(b).	Likelihood function is	
	$L(\mu, \sigma^2) = \prod_{i=1}^{n} f(x_i; \mu, \sigma^2)$	
	$= (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left\{-\frac{1}{2\sigma^2}\sum(x_i-\mu)^2\right\}$.	
	Log-likelihood is [apart from a constant]	

Very standard, seen lots of examples.

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	EXAMINATION SOLUTIONS 2011-12	Course M2S1
Question 1		Marks & seen/unseen
Parts	$$l(\mu,\sigma^2) = -\frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\sum(x_i-\mu)^2$$ $$\frac{\partial l}{\partial\mu} = \frac{1}{\sigma^2}\sum(x_i-\mu)$$ $$\frac{\partial l}{\partial\sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2(\sigma^2)^2}\sum(x_i-\mu)^2.$$ Solving $\left.\begin{array}{l}\frac{\partial l}{\partial\mu}=0\\ \frac{\partial l}{\partial\sigma^2}=0\end{array}\right\}$ at $(\hat\mu,\hat\sigma^2)$ gives $\hat\mu = \bar{x} = n^{-1}\sum_i x_i$, $\hat\sigma^2 = \frac{1}{n}\sum_i(x_i-\bar{x})^2$. So, MLEs are $\hat\mu = \bar{X}$, $\hat\sigma^2 = \frac{1}{n}\sum_i(X_i-\bar{X})^2$.	Bookwork. 3
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	EXAMINATION SOLUTIONS 2011-12	Course M2S1
Question 1		Marks & seen/unseen

Parts

Directly, $E(\hat{\sigma}^2)$

$$= \frac{1}{n} E\left\{ \sum_i X_i^2 - n\bar{X}^2 \right\}$$

$$= \frac{1}{n}\left\{ n(\sigma^2+\mu^2) - n\left(\frac{\sigma^2}{n}+\mu^2\right)\right\}$$

$$= \frac{n-1}{n}\sigma^2 \neq \sigma^2,$$ so $\hat{\sigma}^2$

is biased, but $\frac{n}{n-1}\hat{\sigma}^2$ is <u>not</u>. 2

Done in lectures.

$\bar{X}$ and S^2 are <u>independent</u>, 2

$\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$, $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$ Standard, seen.

So, $T = \dfrac{\bar{X}-\mu}{\sqrt{S^2/n}} = \dfrac{\bar{X}-\mu}{S/\sqrt{n}}$ 2

$\sim t_{n-1}$, by def^n of t-dist^n. Standard, bookwork

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Parts

So, to test $H_0: \mu = \mu_0$ against $H_1: \mu \neq \mu_0$, use test statistic $T_0 = \dfrac{\bar{X} - \mu_0}{S/\sqrt{n}}$. $T_0 \sim t_{n-1}$ iff H_0 is true. Let $t_{\alpha/2}$ be the $1-\alpha/2$ quantile of t_{n-1} [so that $P(t_{n-1} < t_{\alpha/2}) = 1-\alpha/2$]. Then, in a test of significance level α, reject H_0 iff $|t| > t_{\alpha/2}$, where t is value of the statistic T_0 calculated from data.

Bookwork, seen.

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A $100(1-\alpha)\%$ CI for μ $\equiv \{\mu_0 : H_0 : \mu = \mu_0$ is <u>accepted</u>$\}$

<u>OR</u>,

$$P(-t_{\alpha/2} < T < t_{\alpha/2}) = 1-\alpha,$$

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Question: 1

Marks & seen/unseen

Parts

so

$$P\left(\bar{X} - t_{\alpha/2}\frac{S}{\sqrt{n}} < \mu < \bar{X} + t_{\alpha/2}\frac{S}{\sqrt{n}}\right) = 1-\alpha,$$

2. Derivation seen.

$\left(\bar{X} - t_{\alpha/2}\frac{S}{\sqrt{n}}, \bar{X} + t_{\alpha/2}\frac{S}{\sqrt{n}}\right)$ is a $1-\alpha$ confidence interval.

If σ^2 is <u>known</u>, use

2. Standard, seen, done in lectures.

$$\frac{\bar{X}-\mu}{(\sigma^2/n)^{1/2}} \sim N(0,1)$$ to obtain

$1-\alpha$ confidence interval of form

$$\left(\bar{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right),$$

in terms of $z_{\alpha/2}$, $1-\alpha/2$ quantile of $N(0,1)$.

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	EXAMINATION SOLUTIONS 2011-12	Course M2S1
Question 2		Marks & seen/unseen

Parts

(a) The moment generating function of Z is given by

$$M_Z(t) = E(e^{tZ})$$

$$= \sum_n E(e^{tZ} \mid N=n) P(N=n)$$

$$= \sum_n E(e^{t(X_1 + \dots + X_n)}) P(N=n)$$

$$= \sum_n E\left(\prod_{i=1}^{n} e^{tX_i}\right) P(N=n)$$

$$= \sum_n M_X(t)^n P(N=n),$$

since X_i indep

$$= \underline{G_N(M_X(t))}$$

Very standard calculation. Seen lots of examples.

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	EXAMINATION SOLUTIONS 2011-12	Course M2S1
Question 2		Marks & seen/unseen
Parts (b)	Conditional on $X = x$, Y is Binomial (x, p), so $E(e^{tY} \mid X = x)$ $= (pe^t + 1 - p)^x$. The joint mgf of Y, Z is $M_{Y,Z}(s,t) = E(e^{sY + tZ})$ $= E[E[e^{sY + t(X-Y)} \mid X]]$ $= E[e^{tX} E[e^{(s-t)Y} \mid X]]$ $= E[e^{tX}(pe^{s-t} + 1 - p)^X]$ $= E[(pe^s + (1-p)e^t)^X]$ $= G_X(pe^s + (1-p)e^t)$ $= \exp\{\lambda(pe^s + (1-p)e^t - 1)\}$.	Standard calculation of joint mgf using iterated expectation but this particular case unseen. Tricky
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	EXAMINATION SOLUTIONS 2011-12	Course M2S1
Question 2		Marks & seen/unseen
Parts	Since X is Poisson(λ).	
	Then, $M_{Y,Z}(s,t)$	6
	$= \exp\{\lambda p(e^s - 1)\} \times \exp\{\lambda(1-p)(e^t - 1)\}$	mgf
	$\lambda p + \lambda(1-p) = \lambda$	Easy.
	$= M_Y(s) M_Z(t)$.	3
	So Y and Z are independent, and $Y \sim$ Poisson(λp).	deduction.
	Directly, noting that $E(Y \mid X = x) = xp$, we have	
	$E(XY) = E[E(XY \mid X)]$ $= E(X^2 p) = pE(X^2)$ $= p(\lambda + \lambda^2)$, and	
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	EXAMINATION SOLUTIONS 2011-12	Course M2S1
Question 2		Marks & seen/unseen
Parts	$$\mathrm{corr}(X,Y) = \frac{E(XY) - EX.EY}{\sqrt{\mathrm{var}(X).\mathrm{var}(Y)}}$$ $$= \frac{p\lambda^2 + p\lambda - \lambda.\lambda p}{\sqrt{\lambda.\lambda p}} = \underline{\sqrt{p}}.$$	Unseen 5 — 20
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	EXAMINATION SOLUTIONS 2011-12	Course M2S1
Question 3		Marks & seen/unseen
Parts		

(i) Conditional on $X = x$, the probability is $P\left(U \leq \frac{f_S(x)}{a f_X(x)}\right) = \frac{f_S(x)}{a f_X(x)}$.

Then, iterated expectation gives

$$P(aUf_X(X) \leq f_S(X))$$
$$= E\left[\frac{f_S(X)}{a f_X(X)}\right]$$
$$= \int_{-\infty}^{\infty} \frac{f_S(x)}{a f_X(x)} \cdot f_X(x)\,dx$$
$$= \frac{1}{a}\int_{-\infty}^{\infty} f_S(x)\,dx = \frac{1}{a}.$$

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Unseen, but standard. techniques iterated expectation.

(ii) Similarly,

$$P(X \leq x \mid aUf_X(X) \leq f_S(X))$$

$$=$$

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Parts

$$\frac{P(X \le x,\ aUf_X(X) \le f_S(X))}{P(aUf_X(X) \le f_S(X))}$$

$$= \int_{-\infty}^{x} \frac{f_S(y)}{a f_X(y)} f_X(y)\,dy \Big/ \frac{1}{a}$$ 3

$$= \int_{-\infty}^{x} f_S(y)\,dy$$, as required.

(iii) Suppose we have a sequence $(U_K, X_K, K \geq 1)$ of rv's which have the same distribution as (U, X).
For every pair for which $aU_K f_X(X_K) \le f_S(X_K)$ the rv X_K has pdf f_S: 'accept' X_K for which the condition holds. 3
Accepted X_K have pdf f_S.

EXAMINATION SOLUTIONS 2011-12

Course M2S1

Question 3

Marks & seen/unseen

Parts

(iv) $P(S \le s) = P(|Z| \le s)$

$= P(-s \le Z \le s) = 2\Phi(s) - 1$,

in terms of $N(0,1)$ cdf Φ.

So, S has pdf

$$2\phi(s) = \sqrt{\frac{2}{\pi}}\, e^{-\frac{1}{2}s^2}, \quad s > 0.$$

2 ⌜pdf⌟

$$\frac{f_S(x)}{f_X(x)} \propto \exp\left\{-\frac{1}{2}x^2 + \lambda x\right\},$$

maximised where $g(x) = \lambda x - \frac{1}{2}x^2$ is maximised. $g'(x) = -x + \lambda$ $= 0 \iff x = \lambda$ ⌜$g''(\lambda) < 0$⌟

Then $a = \dfrac{\sqrt{2/\pi}\, \exp(\frac{1}{2}\lambda^2)}{\lambda}$.

4 ⌜ratio and value of a⌟

Since the probability an X is accepted as a realisation of S is $1/a$, choose λ to minimise $a \equiv a(\lambda)$. So, minimise

Setter's initials

Checker's initials

	EXAMINATION SOLUTIONS 2011-12	Course M2S1
Question 3		Marks & seen/unseen
Parts	$h(\lambda) = \frac{1}{2}\lambda^2 - \log\lambda$ $h'(\lambda) = 0 \iff \lambda = 1 \quad [h''(1) > 0]$ So, take $\underline{\lambda = 1}$.	4 $\overline{20}$
	Setter's initials	Checker's initials

	EXAMINATION SOLUTIONS 2011-12	Course M2S1
Question 4		Marks & seen/unseen
Parts		
(a)	$\mathcal{A}$ is a σ-field if:	Seen. Basic definition.
	I. $\emptyset \in \mathcal{A}$	
	II. $A_1, A_2, \ldots \in \mathcal{A} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$	2
	III $A \in \mathcal{A} \Rightarrow A' \in \mathcal{A}$.	
	$A \cap B = (A' \cup B')'$ —— $*$	2
	If $A, B \in \mathcal{A}$, $A', B' \in \mathcal{A}$, by III. Then $A' \cup B' \in \mathcal{A}$, by II, and finally $A \cap B \in \mathcal{A}$, using $*$ and III	Unseen, but seen similar
	A probability function P on $(\Omega, \mathcal{A})$ is a function $P: \mathcal{A} \rightarrow [0,1]$ satisfying	Seen. Basic definition
	(a) $P(\emptyset) = 0$, $P(\Omega) = 1$;	
	(b) If $A_1, A_2, \ldots$ are disjoint members of $\mathcal{A}$ ($A_i \cap A_j = \emptyset$,	2
	Setter's initials	Checker's initials

	EXAMINATION SOLUTIONS 2011-12	Course M2S1
Question 4		Marks & seen/unseen
Parts	$i \neq j$. $$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i).$$ A <u>probability space</u> is a triple $(\Omega, \mathcal{A}, P(\cdot))$ where Ω is sample space, $\mathcal{A}$ is σ-field of subsets of Ω, P is a probability function on $(\Omega, \mathcal{A})$.	Seen. Basic Definition 1.
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EXAMINATION SOLUTIONS 2011-12

Course M2S1

Question 4

Marks & seen/unseen

Parts

(b) Let X have cdf F, X_n have cdf F_n.

i. Convergence in probability:

$X_n \xrightarrow{p} X$ if $\forall \epsilon > 0$, $P(|X_n - X| > \epsilon) \to 0$ as $n \to \infty$. 2

ii Convergence in quadratic mean:

$X_n \xrightarrow{qm} X$ if $E\{(X_n - X)^2\} \to 0$ as $n \to \infty$. 2

iii Convergence in distribution:

$X_n \xrightarrow{d} X$ if $F_n(x) \to F(x)$, at all x at which F is continuous. 2

Basic definitions

Let $\epsilon > 0$. Use Markov's Inequality Seen

$P(|X_n - X| > \epsilon) = P((X_n - X)^2 > \epsilon^2)$

$\leq E\{(X_n - X)^2\}/\epsilon^2 \to 0$ as $n \to \infty$, since $X_n \xrightarrow{qm} X$. 2

Setter's initials

Checker's initials

EXAMINATION SOLUTIONS 2011-12

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Question: 4

Marks & seen/unseen

Parts

The converse is false. — Seen 1

(c) $Y_n = \max\{X_1, \ldots, X_n\}$ has cdf $F(y) = P(Y_n \le y)$ — Standard.

$= P(\underline{\text{all}}\ X_i \le y) = \left(\frac{y}{\theta}\right)^n$.

Then, $P(|Y_n - \theta| > \epsilon)$ — Seen Similar

$= P(Y_n \le \theta - \epsilon)$

$= \left(1 - \frac{\epsilon}{\theta}\right)^n \longrightarrow 0$ as $n \to \infty$

(since $\epsilon > 0$).

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