Imperial College
London

[E2.8 (Maths 3) 2008]

B.ENG. AND M.ENG. EXAMINATIONS 2008

PART II Paper 3 : MATHEMATICS (ELECTRICAL ENGINEERING)

Date Wednesday 4th June 2008 2.00 - 5.00 pm

Answer EIGHT questions.

Please answer questions from Section A and Section B in separate answer-books.

A mathematical formulae sheet is provided.

Statistical data sheets are provided.

[*Before starting, please make sure that the paper is complete; there should be EIGHT pages, with a total of TWELVE questions. Ask the invigilator for a replacement if your copy is faulty.*]

1. Consider the function $w = \ln z$, for a complex number z.

   (i) Make this multi-valued function definite.

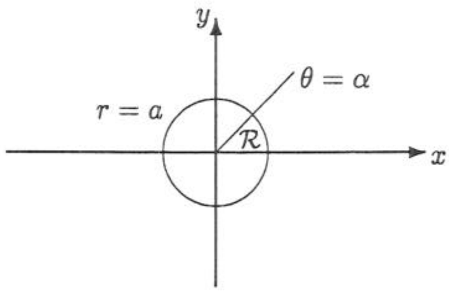
   (ii) Is this mapping conformal?

   (iii) Write w in terms of z in polar form, $re^{i\theta}$, and determine where the positive real axis is mapped to.

   (iv) Similarly, determine where the half-line $\theta = \alpha\,, \; -\pi < \alpha \leq \pi$ is mapped to.

   (v) Determine the image of the circle with center at the origin and radius a under this mapping.

   (vi) Thus plot the image of the region $\mathcal{R}$, as shown below.



2. Consider the complex function

$$f(z) = \frac{z}{(z-1)^2(z-i)^2}\,.$$

   (i) Show that the residue of $f(z)$ at the pole at $z = 1$ is $\frac{1}{2}$;

   (ii) Show that the residue of $f(z)$ at the pole at $z = i$ is $-\frac{1}{2}$.

   (iii) If C is the contour consisting of the circle of radius 2 centred at the origin, find the value of $\oint_C f(z)\,dz$.

The residue of a complex function $f(z)$ at a pole $z = a$ of multiplicity m is given by the expression

$$\lim_{z\to a} \frac{1}{(m-1)!} \left[\frac{d^{m-1}}{dz^{m-1}} \left\{ (z-a)^m f(z) \right\} \right].$$

PLEASE TURN OVER

3. Consider the complex function

$$f(z) = \frac{e^{iz}}{z(z^2+4)} .$$

(i) Show that $f(z)$ has three simple poles, one lying at the origin and the other two lying on the imaginary axis.

(ii) Find the residue of $f(z)$ at the pole at the origin and also the residue at the pole in the upper half-plane.

(iii) By considering a semi-circular contour in the upper half-plane of radius R, with a suitable semi-circular indentation into the lower half-plane ($\pi \leq \theta \leq 2\pi$) of radius r around $z = 0$, use Jordan's Lemma and the Residue Theorem to show that in the limits $R \to \infty$ and $r \to 0$:

$$\int_{-\infty}^{\infty} \frac{\sin x \, dx}{x(x^2+4)} = \frac{\pi}{4}\left(1 - e^{-2}\right).$$

4. Given that

$$\int_{-\infty}^{\infty} \frac{e^{it}}{t}\, dt = i\pi ,$$

and if q is an arbitrary real number, show that

$$\int_{-\infty}^{\infty} \frac{e^{iqt}}{t}\, dt = \begin{cases} +i\pi, & q > 0 , \\ -i\pi, & q < 0 . \end{cases}$$

Hence show that the Fourier transform $\overline{f}(\omega)$ of the function

$$f(t) = \frac{\sin \frac{1}{2}t}{\frac{1}{2}t}$$

is given by

$$\overline{f}(\omega) = \begin{cases} 2\pi & -\frac{1}{2} < \omega < \frac{1}{2} , \\ 0 & \omega < -\frac{1}{2} , \quad \frac{1}{2} < \omega . \end{cases}$$

5. The Dirac delta-function has an integral representation of the form

$$\int_{-\infty}^{\infty} e^{\pm i\Omega t}\, dt \;=\; 2\pi\delta(\Omega)\,.$$

Use this to prove Plancherel's integral relation between the two functions $f(t)$ and $g(t)$ and their Fourier transforms $\overline{f}(\omega)$ and $\overline{g}(\omega)$:

$$\int_{-\infty}^{\infty} f(t)g^*(t)\, dt \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{f}(\omega)\,\overline{g}^*\,(\omega)\, d\omega\,,$$

where $*$ represents the complex conjugate.

If $f(t) = e^{-|t|}$ and $g(t) = \cos(\omega_0 t)$, where ω_0 is a constant frequency, show that

$$\int_{-\infty}^{\infty} e^{-|t|}\,\cos\left(\omega_0 t\right) dt \;=\; \frac{2}{1+\omega_0^2}\,.$$

6. A third order ordinary differential equation takes the form

$$\frac{d^3x}{dt^3} \;+\; 3\,\frac{d^2x}{dt^2} \;+\; 3\,\frac{dx}{dt} \;+\; x \;=\; f(t)$$

where $f(t)$ is an arbitrary piecewise smooth function with Laplace transform $\overline{f}(s)$. The function $x(t)$ and its first two derivatives satisfy the initial conditions

$$x \;=\; \frac{dx}{dt} \;=\; \frac{d^2x}{dt^2} \;=\; 0 \;\text{ when }\; t=0.$$

Use the Laplace convolution and shift theorems to show that

$$x(t) \;=\; \frac{1}{2}\int_0^t e^{-u}\, u^2 f(t-u)\, du.$$

PLEASE TURN OVER

7. Consider a two-dimensional region R bounded by a closed piecewise smooth curve C. Using Green's Theorem in a plane (see below), choose the components of a vector field $\boldsymbol{v}(x,y)$ in terms of $P(x,y)$ and $Q(x,y)$ to prove the two-dimensional form of Stokes' Theorem

$$\int\!\!\int_R \boldsymbol{k}.(\operatorname{curl}\boldsymbol{v})\,dxdy \;=\; \oint_C \boldsymbol{v}.d\boldsymbol{r} \tag{1}$$

where $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j}$.

Now consider R to be the region bounded by the hyperbola $y = \frac{1}{x}$, the lines $y = x$ and $x = 2$ and the x-axis ($y = 0$), and suppose that $\boldsymbol{v} \;=\; \frac{1}{2}(y^2\boldsymbol{i} + x^2\boldsymbol{j})$.

Sketch the region R in the $x - y$ plane.

By evaluating the line integral on the right hand side of (1), or otherwise, show that

$$\int\!\!\int_R \boldsymbol{k}.\operatorname{curl}\boldsymbol{v}\,dxdy \;=\; \frac{11}{12}\,.$$

Green's Theorem in a plane states that for a two-dimensional region R bounded by a closed, piecewise smooth curve C, then

$$\oint_C \{P(x,\,y)\,dx \;+\; Q\,(x,\,y)\,dy\} \;=\; \int\!\!\int_R \left(\frac{\partial Q}{\partial x} \;-\; \frac{\partial P}{\partial y}\right)\,dx\,dy\,.$$

8. The integral

$$\int\int_{\mathcal{R}} \frac{y^2}{x}\, e^{xy}\, dx\, dy$$

is to be evaluated over the region $\mathcal{R}$ bounded by all four of the parabolas $y = x^2$, $y = 2x^2$, $x = y^2$ and $x = 2y^2$.

Sketch the region $\mathcal{R}$. Find the values of the new variables

$$u = x^2/y \qquad \text{and} \qquad v = y^2/x$$

on each part of the boundary of $\mathcal{R}$. Determine x and y in terms of u and v and hence or otherwise evaluate the Jacobian of the transformation.

Evaluate the integral.

9. Show for any twice differentiable scalar field $\phi(x, y, z)$ that

$$\text{curl grad}\, \phi = 0 .$$

Show also for any twice differentiable vector field $\boldsymbol{E}(x, y, z)$ that

$$\text{div curl}\, \boldsymbol{E} = 0 .$$

Now consider two vector fields, $\boldsymbol{E}(x, y, z)$ and $\boldsymbol{A}(x, y, z)$, and a scalar field $\phi(x, y, z)$ that satisfy

$$\boldsymbol{E} = \boldsymbol{A} + \text{grad}\, \phi . \qquad (*)$$

Show that $\text{curl}\, \boldsymbol{E} = \text{curl}\, \boldsymbol{A}$.

In the case where

$$\boldsymbol{E} = e^{ax} \cos y\, \boldsymbol{i} + e^{ax} \sin y\, \boldsymbol{j} + z^2\, \boldsymbol{k}$$

$$\boldsymbol{A} = (2\, e^{ax} \cos y + x)\boldsymbol{i} + z\, \boldsymbol{j} + y\, \boldsymbol{k}$$

Show that for a certain value of a (which is to be found)

$$\text{curl}\, \boldsymbol{E} = \text{curl}\, \boldsymbol{A} .$$

For this value of a find $\phi(x, y, z)$ from relation $(*)$ above.

PLEASE TURN OVER

10. Consider the two-dimensional vector field

$$\boldsymbol{E} = f(x, y)\,\boldsymbol{i} + g(x, y)\,\boldsymbol{j}$$

where

$$f(x, y) = e^{x}(x\cos y - y\sin y) \qquad \text{and} \qquad g(x, y) = -e^{x}(y\cos y + x\sin y)\,.$$

Show that the vector field is conservative i.e. that $\dfrac{\partial g}{\partial x} = \dfrac{\partial f}{\partial y}$.

Find a scalar potential $\phi(x, y)$ such that $\boldsymbol{E} = \text{grad}\,\phi$.

Hence or otherwise evaluate the line integral

$$\int_C (f(x, y)\,dx + g(x, y)\,dy)$$

where $f(x, y)$ and $g(x, y)$ are defined above and C is a path between the points $(0, 0)$ and $(1, \pi)$ consisting of two straight lines $y = 0$ and then $x = 1$.

Finally evaluate the integral

$$\int_C ([f(x, y) + x]\,dx + [g(x, y) + \sin(y/2)]\,dy)$$

along the same path between the same two points.

11. Suppose that the probability density function f of the random variable X is given by $f(x) = k\sin(x)$ for $0 \le x \le \pi$ and $f(x) = 0$ otherwise.

    (i) Show that $k = 1/2$.

    (ii) Find $P(1 \le X \le 4)$.

    (iii) Find $P(X > 2 \,|\, X > 1)$.

    (iv) Find $E(X)$ and $Var(X)$.

    You may use that

$$\frac{d}{dx}\left((2 - x^2)\cos(x) + 2x\sin(x) \right) = x^2\sin(x) .$$

12. Consider the time series

$$y_t = 0.3e_t + 0.5e_{t-1} + 0.2e_{t-2} ,$$

where $\{e_t\}$ is white noise with $Var(e_t) = 1$.

    (i) Define 'white noise'.

    (ii) Find the covariance $\gamma(t, t+s)$ of $\{y_t\}$ for $s = 0, 1, 2, 3, \ldots$

    (iii) Is $\{y_t\}$ stationary? Justify your answer.

    (iv) Find the autocorrelation γ_k of $\{y_t\}$.

    (v) Find the spectrum $f(\omega)$ of $\{y_t\}$.

END OF PAPER

MATHEMATICS DEPARTMENT

MATHEMATICAL FORMULAE

1. VECTOR ALGEBRA

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1, a_2, a_3)$$

Scalar (dot) product: $\mathbf{a}\,.\,\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

Vector (cross) product:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix};$$

Scalar triple product:

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a}\,.\,\mathbf{b} \times \mathbf{c} = \mathbf{b}\,.\,\mathbf{c} \times \mathbf{a} = \mathbf{c}\,.\,\mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product: $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c}\,.\,\mathbf{a})\mathbf{b} - (\mathbf{b}\,.\,\mathbf{a})\mathbf{c}$

2. SERIES

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \dots \quad (\alpha \text{ arbitrary, } |x| < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^n\frac{x^{2n}}{(2n)!} + \dots,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + \dots,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^n\frac{x^{n+1}}{(n+1)} + \dots (-1 < x \leq 1)$$

3. TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin(a+b) = \sin a \cos b + \cos a \sin b\,;$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b\,.$$

$$\cos iz = \cosh z\,; \quad \cosh iz = \cos z\,; \quad \sin iz = i\sinh z\,; \quad \sinh iz = i\sin z\,.$$

4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f\,D^n g + \tbinom{n}{1} Df\,D^{n-1}g + \dots + \tbinom{n}{r} D^r f\,D^{n-r}g + \dots + D^n f\,g\,.$$

(b) Taylor's expansion of $f(x)$ about $x = a$:

$$f(a+h) = f(a) + hf'(a) + h^2 f''(a)/2! + \dots + h^n f^{(n)}(a)/n! + \epsilon_n(h)\,,$$

where $\epsilon_n(h) = h^{n+1} f^{(n+1)}(a + \theta h)/(n+1)!, \quad 0 < \theta < 1\,.$

(c) Taylor's expansion of $f(x, y)$ about (a, b):

$$f(a+h, b+k) = f(a,b) + [hf_x + kf_y]_{a,b} + 1/2!\left[h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}\right]_{a,b} + \dots$$

(d) Partial differentiation of $f(x, y)$:

i. If $y = y(x)$, then $f = F(x)$, and $\dfrac{dF}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dx}$.

ii. If $x = x(t)$, $y = y(t)$, then $f = F(t)$, and $\dfrac{dF}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$.

iii. If $x = x(u, v)$, $y = y(u, v)$, then $f = F(u, v)$, and

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}, \quad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$

(e) Stationary points of $f(x, y)$ occur where $f_x = 0$, $f_y = 0$ simultaneously.
Let (a, b) be a stationary point: examine $D = [f_{xx}f_{yy} - (f_{xy})^2]_{a,b}$.
If $D > 0$ and $f_{xx}(a, b) < 0$, then (a, b) is a maximum;
If $D > 0$ and $f_{xx}(a, b) > 0$, then (a, b) is a minimum;
If $D < 0$ then (a, b) is a saddle-point.

(f) Differential equations:

i. The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating factor $I(x) = \exp[\int P(x)(dx)]$, so that $\dfrac{d}{dx}(Iy) = IQ$.

ii. $P(x, y)dx + Q(x, y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y$.

5. INTEGRAL CALCULUS

(a) An important substitution: $\tan(\theta/2) = t$:
$\sin\theta = 2t/(1+t^2)$, $\cos\theta = (1-t^2)/(1+t^2)$, $d\theta = 2\,dt/(1+t^2)$.

(b) Some indefinite integrals:

$$\int (a^2 - x^2)^{-1/2} dx = \sin^{-1}\left(\frac{x}{a}\right), \quad |x| < a.$$

$$\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1}\left(\frac{x}{a}\right) = \ln\left\{\frac{x}{a} + \left(1 + \frac{x^2}{a^2}\right)^{1/2}\right\}.$$

$$\int (x^2 - a^2)^{-1/2} dx = \cosh^{-1}\left(\frac{x}{a}\right) = \ln\left|\frac{x}{a} + \left(\frac{x^2}{a^2} - 1\right)^{1/2}\right|.$$

$$\int (a^2 + x^2)^{-1} dx = \left(\frac{1}{a}\right)\tan^{-1}\left(\frac{x}{a}\right).$$

6. NUMERICAL METHODS

(a) Approximate solution of an algebraic equation:

If a root of $f(x) = 0$ occurs near $x = a$, take $x_0 = a$ and
$x_{n+1} = x_n - [f(x_n)/f'(x_n)]$, $n = 0, 1, 2 \ldots$

(Newton Raphson method).

(b) Formulae for numerical integration: Write $x_n = x_0 + nh$, $y_n = y(x_n)$.

i. Trapezium rule (1-strip): $\int_{x_0}^{x_1} y(x)dx \approx (h/2)[y_0 + y_1]$.

ii. Simpson's rule (2-strip): $\int_{x_0}^{x_2} y(x)dx \approx (h/3)[y_0 + 4y_1 + y_2]$.

(c) Richardson's extrapolation method: Let $I = \int_a^b f(x)dx$ and let I_1, I_2 be two estimates of I obtained by using Simpson's rule with intervals h and $h/2$.

Then, provided h is small enough,

$$I_2 + (I_2 - I_1)/15,$$

is a better estimate of I.

7. LAPLACE TRANSFORMS

Function	Transform	Function	Transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)dt$	$af(t) + bg(t)$	$aF(s) + bG(s)$
df/dt	$sF(s) - f(0)$	d^2f/dt^2	$s^2F(s) - sf(0) - f'(0)$
$e^{at}f(t)$	$F(s-a)$	$tf(t)$	$-dF(s)/ds$
$(\partial/\partial\alpha)f(t,\alpha)$	$(\partial/\partial\alpha)F(s,\alpha)$	$\int_0^t f(t)dt$	$F(s)/s$
$\int_0^t f(u)g(t-u)du$	$F(s)G(s)$		
1	$1/s$	$t^n (n = 1, 2 \ldots)$	$n!/s^{n+1}$, $(s > 0)$
e^{at}	$1/(s-a)$, $(s > a)$	$\sin\omega t$	$\omega/(s^2+\omega^2)$, $(s > 0)$
$\cos\omega t$	$s/(s^2+\omega^2)$, $(s > 0)$	$H(t-T) = \begin{cases} 0, & t < T \\ 1, & t > T \end{cases}$	e^{-sT}/s, $(s, T > 0)$

8. FOURIER SERIES

If $f(x)$ is periodic of period $2L$, then $f(x+2L) = f(x)$, and

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \text{ where}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad n = 0, 1, 2, \ldots, \text{ and}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1, 2, 3, \ldots$$

Parseval's theorem

$$\frac{1}{L}\int_{-L}^{L} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right).$$

September 2000

1. Probabilities for events

For events A, B, and C $\qquad P(A \cup B) = P(A) + P(B) - P(A \cap B)$

More generally $\quad P(\bigcup A_i) = \sum P(A_i) - \sum P(A_i \cap A_j) + \sum P(A_i \cap A_j \cap A_k) - \cdots$

The <u>odds</u> in favour of A $\qquad P(A) \,/\, P(\overline{A})$

<u>Conditional probability</u> $\qquad P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$ provided that $P(B) > 0$

<u>Chain rule</u> $\qquad P(A \cap B \cap C) = P(A)\, P(B \mid A)\, P(C \mid A \cap B)$

<u>Bayes' rule</u> $\qquad P(A \mid B) = \dfrac{P(A)\, P(B \mid A)}{P(A)\, P(B \mid A) + P(\overline{A})\, P(B \mid \overline{A})}$

A and B are <u>independent</u> if $\qquad P(B \mid A) = P(B)$

A, B, and C are <u>independent</u> if $\qquad P(A \cap B \cap C) = P(A)P(B)P(C)$, and

$$P(A \cap B) = P(A)P(B), \quad P(B \cap C) = P(B)P(C), \quad P(C \cap A) = P(C)P(A)$$

2. Probability distribution, expectation and variance

The <u>probability distribution</u> for a <u>discrete</u> random variable X is called the <u>probability mass function</u> (pmf) and is the complete set of probabilities $\{p_x\} = \{P(X = x)\}$

<u>Expectation</u> $\quad E(X) = \mu = \sum_x x p_x$

For function $g(x)$ of x, $\quad E\{g(X)\} = \sum_x g(x) p_x$, $\quad$ so $\quad E(X^2) = \sum_x x^2 p_x$

<u>Sample mean</u> $\quad \overline{x} = \dfrac{1}{n} \sum_k x_k$ estimates μ from random sample $x_1, x_2, \ldots, x_n$

<u>Variance</u> $\quad \operatorname{var}(X) = \sigma^2 = E\{(X - \mu)^2\} = E(X^2) - \mu^2$

<u>Sample variance</u> $\quad s^2 = \dfrac{1}{n-1} \left\{ \sum_k x_k^2 - \dfrac{1}{n} \left(\sum_j x_j \right)^2 \right\}$ estimates σ^2

<u>Standard deviation</u> $\quad \operatorname{sd}(X) = \sigma$

If value y is observed with frequency n_y

$$n = \sum_y n_y, \quad \sum_k x_k = \sum_y y n_y, \quad \sum_k x_k^2 = \sum_y y^2 n_y$$

<u>Skewness</u> $\quad \beta_1 = E\left(\dfrac{X - \mu}{\sigma} \right)^3 \qquad$ is estimated by $\quad \dfrac{1}{n-1} \sum \left(\dfrac{x_i - \overline{x}}{s} \right)^3$

<u>Kurtosis</u> $\quad \beta_2 = E\left(\dfrac{X - \mu}{\sigma} \right)^4 - 3 \qquad$ is estimated by $\quad \dfrac{1}{n-1} \sum \left(\dfrac{x_i - \overline{x}}{s} \right)^4 - 3$

<u>Sample median</u> $\quad \widetilde{x}$ or x_{med}. Half the sample values are smaller and half larger

If the sample values $x_1, \ldots, x_n$ are ordered as $x_{(1)} \le x_{(2)} \le \cdots \le x_{(n)}$, then $\widetilde{x} = x_{(\frac{n+1}{2})}$ if n is odd, and $\widetilde{x} = \frac{1}{2}\left(x_{(\frac{n}{2})} + x_{(\frac{n+2}{2})}\right)$ if n is even

α-quantile $Q(\alpha)$ is such that $P(X \le Q(\alpha)) = \alpha$

Sample α-quantile $\widehat{Q}(\alpha)$ Proportion α of the data values are smaller

Lower quartile Q1 $= \widehat{Q}(0.25)$ one quarter are smaller

Upper quartile Q3 $= \widehat{Q}(0.75)$ three quarters are smaller

Sample median $\tilde{x} = \widehat{Q}(0.5)$ estimates the population median $Q(0.5)$

3. Probability distribution for a continuous random variable

The cumulative distribution function (cdf) $$F(x) = P(X \le x) = \int_{x_0=-\infty}^{x} f(x_0)\mathrm{d}x_0$$

The probability density function (pdf) $$f(x) = \frac{\mathrm{d}F(x)}{\mathrm{d}x}$$

$$E(X) = \mu = \int_{-\infty}^{\infty} x\, f(x)\mathrm{d}x\,, \quad \mathrm{var}\,(X) = \sigma^2 = E(X^2) - \mu^2, \text{ where } E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\mathrm{d}x$$

4. Discrete probability distributions

Discrete Uniform *Uniform* (n)

$$p_x = \frac{1}{n} \quad (x = 1, 2, \ldots, n) \qquad \mu = (n+1)/2, \;\; \sigma^2 = (n^2-1)/12$$

Binomial distribution *Binomial* (n, θ)

$$p_x = \binom{n}{x} \theta^x (1-\theta)^{n-x} \quad (x = 0, 1, 2, \ldots, n) \qquad \mu = n\theta\,, \;\; \sigma^2 = n\theta(1-\theta)$$

Poisson distribution *Poisson* (λ)

$$p_x = \frac{\lambda^x e^{-\lambda}}{x!} \quad (x = 0, 1, 2, \ldots) \quad (\text{with } \lambda > 0) \qquad \mu = \lambda\,, \;\; \sigma^2 = \lambda$$

Geometric distribution *Geometric* (θ)

$$p_x = (1-\theta)^{x-1}\theta \quad (x = 1, 2, 3, \ldots) \qquad \mu = \frac{1}{\theta}\,, \;\; \sigma^2 = \frac{1-\theta}{\theta^2}$$

5. Continuous probability distributions

Uniform distribution *Uniform* (α, β)

$$f(x) = \begin{cases} \dfrac{1}{\beta-\alpha} & (\alpha < x < \beta), \\ 0 & (\text{otherwise}). \end{cases} \qquad \mu = (\alpha+\beta)/2\,, \;\; \sigma^2 = (\beta-\alpha)^2/12$$

Exponential distribution *Exponential* (λ)

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & (0 < x < \infty), \\ 0 & (-\infty < x \le 0). \end{cases} \qquad \mu = 1/\lambda\,, \;\; \sigma^2 = 1/\lambda^2$$

Normal distribution $N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\} \quad (-\infty < x < \infty), \quad E(X) = \mu, \quad \operatorname{var}(X) = \sigma^2$$

Standard normal distribution $N(0,1)$

If X is $N(\mu, \sigma^2)$, then $Y = \dfrac{X-\mu}{\sigma}$ is $N(0,1)$

6. Reliability

For a device in continuous operation with failure time random variable T having pdf $f(t)$ $(t > 0)$

The reliability function at time t $\quad R(t) = P(T > t)$

The failure rate or hazard function $\quad h(t) = f(t)/R(t)$

The cumulative hazard function $\quad H(t) = \int_0^t h(t_0)\,\mathrm{d}t_0 = -\ln\{R(t)\}$

The Weibull(α, β) distribution has $\quad H(t) = \beta t^{\alpha}$

7. System reliability

For a system of k devices, which operate independently, let

$R_i = P(D_i) = P(\text{"device } i \text{ operates"})$

The system reliability, R, is the probability of a path of operating devices

A system of devices in series operates only if every device operates

$R = P(D_1 \cap D_2 \cap \cdots \cap D_k) = R_1 R_2 \cdots R_k$

A system of devices in parallel operates if any device operates

$R = P(D_1 \cup D_2 \cup \cdots \cup D_k) = 1 - (1 - R_1)(1 - R_2)\cdots(1 - R_k)$

8. Covariance and correlation

The covariance of X and Y $\quad \operatorname{cov}(X, Y) = E(XY) - \{E(X)\}\{E(Y)\}$

From pairs of observations $(x_1, y_1), \ldots, (x_n, y_n)$ $\quad S_{xy} = \sum_k x_k y_k - \frac{1}{n}(\sum_i x_i)(\sum_j y_j)$

$$S_{xx} = \sum_k x_k^2 - \frac{1}{n}(\sum_i x_i)^2, \qquad S_{yy} = \sum_k y_k^2 - \frac{1}{n}(\sum_j y_j)^2$$

Sample covariance $\quad s_{xy} = \dfrac{1}{n-1} S_{xy}$ estimates $\operatorname{cov}(X, Y)$

Correlation coefficient $\quad \rho = \operatorname{corr}(X, Y) = \dfrac{\operatorname{cov}(X, Y)}{\operatorname{sd}(X) \cdot \operatorname{sd}(Y)}$

Sample correlation coefficient $\quad r = \dfrac{S_{xy}}{\sqrt{S_{xx} S_{yy}}}$ estimates ρ

9. Sums of random variables

$$E(X+Y) = E(X)+E(Y)$$

$$\mathrm{var}\,(X+Y) = \mathrm{var}\,(X)+\mathrm{var}\,(Y)+2\,\mathrm{cov}\,(X,Y)$$

$$\mathrm{cov}\,(aX+bY,\ cX+dY) = (ac)\,\mathrm{var}\,(X)+(bd)\,\mathrm{var}\,(Y)+(ad+bc)\,\mathrm{cov}\,(X,Y)$$

If X is $N(\mu_1,\sigma_1^2)$, Y is $N(\mu_2,\sigma_2^2)$, and $\mathrm{cov}\,(X,Y)=c$, then $X+Y$ is $N(\mu_1+\mu_2,\ \sigma_1^2+\sigma_2^2+2c)$

10. Bias, standard error, mean square error

If t estimates θ (with random variable T giving t)

Bias of t — $\mathrm{bias}\,(t) = E(T)-\theta$

Standard error of t — $\mathrm{se}\,(t) = \mathrm{sd}\,(T)$

Mean square error of t — $\mathrm{MSE}(t) = E\{(T-\theta)^2\} = \{\mathrm{se}\,(t)\}^2+\{\mathrm{bias}\,(t)\}^2$

If $\overline{x}$ estimates μ, then $\mathrm{bias}\,(\overline{x})=0$, $\mathrm{se}\,(\overline{x})=\sigma/\sqrt{n}$, $\mathrm{MSE}(\overline{x})=\sigma^2/n$, $\widehat{\mathrm{se}}\,(\overline{x})=s/\sqrt{n}$

Central limit property — If n is fairly large, $\overline{x}$ is from $N(\mu,\ \sigma^2/n)$ approximately

11. Likelihood

The likelihood is the joint probability as a function of the unknown parameter θ.

For a random sample $x_1,x_2,\ldots,x_n$

$$\ell(\theta;\,x_1,x_2,\ldots,x_n) = P(X_1=x_1\mid\theta)\cdots P(X_n=x_n\mid\theta) \qquad \text{(discrete distribution)}$$

$$\ell(\theta;\,x_1,x_2,\ldots,x_n) = f(x_1\mid\theta)\,f(x_2\mid\theta)\cdots f(x_n\mid\theta) \qquad \text{(continuous distribution)}$$

The maximum likelihood estimator (MLE) is $\widehat{\theta}$ for which the likelihood is a maximum

12. Confidence intervals

If $x_1,x_2,\ldots,x_n$ are a random sample from $N(\mu,\sigma^2)$ and σ^2 is known, then the 95% confidence interval for μ is $(\overline{x}-1.96\frac{\sigma}{\sqrt{n}},\ \overline{x}+1.96\frac{\sigma}{\sqrt{n}})$

If σ^2 is estimated, then from the Student t table for t_{n-1} we find $t_0=t_{n-1,0.05}$

The 95% confidence interval for μ is $(\overline{x}-t_0\frac{s}{\sqrt{n}},\ \overline{x}+t_0\frac{s}{\sqrt{n}})$

13. Standard normal table — Values of pdf $\phi(y) = f(y)$ and cdf $\Phi(y) = F(y)$

y	$\phi(y)$	$\Phi(y)$	y	$\phi(y)$	$\Phi(y)$	y	$\phi(y)$	$\Phi(y)$	y	$\Phi(y)$
0	.399	.5	.9	.266	.816	1.8	.079	.964	2.8	.997
.1	.397	.540	1.0	.242	.841	1.9	.066	.971	3.0	.999
.2	.391	.579	1.1	.218	.864	2.0	.054	.977	0.841	.8
.3	.381	.618	1.2	.194	.885	2.1	.044	.982	1.282	.9
.4	.368	.655	1.3	.171	.903	2.2	.035	.986	1.645	.95
.5	.352	.691	1.4	.150	.919	2.3	.028	.989	1.96	.975
.6	.333	.726	1.5	.130	.933	2.4	.022	.992	2.326	.99
.7	.312	.758	1.6	.111	.945	2.5	.018	.994	2.576	.995
.8	.290	.788	1.7	.094	.955	2.6	.014	.995	3.09	.999

14. Student t table — Values $t_{m,p}$ of x for which $P(|X| > x) = p$, when X is t_m

m	$p=$ 0.10	0.05	0.02	0.01	m	$p=$ 0.10	0.05	0.02	0.01
1	6.31	12.71	31.82	63.66	9	1.83	2.26	2.82	3.25
2	2.92	4.30	6.96	9.92	10	1.81	2.23	2.76	3.17
3	2.35	3.18	4.54	5.84	12	1.78	2.18	2.68	3.05
4	2.13	2.78	3.75	4.60	15	1.75	2.13	2.60	2.95
5	2.02	2.57	3.36	4.03	20	1.72	2.09	2.53	2.85
6	1.94	2.45	3.14	3.71	25	1.71	2.06	2.48	2.78
7	1.89	2.36	3.00	3.50	40	1.68	2.02	2.42	2.70
8	1.86	2.31	2.90	3.36	∞	1.645	1.96	2.326	2.576

15. Chi-squared table — Values $\chi^2_{k,p}$ of x for which $P(X > x) = p$, when X is χ^2_k and $p = .995, .975,$ *etc*

k	.995	.975	.05	.025	.01	.005	k	.995	.975	.05	.025	.01	.005
1	.000	.001	3.84	5.02	6.63	7.88	18	6.26	8.23	28.87	31.53	34.81	37.16
2	.010	.051	5.99	7.38	9.21	10.60	20	7.43	9.59	31.42	34.17	37.57	40.00
3	.072	.216	7.81	9.35	11.34	12.84	22	8.64	10.98	33.92	36.78	40.29	42.80
4	.207	.484	9.49	11.14	13.28	14.86	24	9.89	12.40	36.42	39.36	42.98	45.56
5	.412	.831	11.07	12.83	15.09	16.75	26	11.16	13.84	38.89	41.92	45.64	48.29
6	.676	1.24	12.59	14.45	16.81	18.55	28	12.46	15.31	41.34	44.46	48.28	50.99
7	.990	1.69	14.07	16.01	18.48	20.28	30	13.79	16.79	43.77	46.98	50.89	53.67
8	1.34	2.18	15.51	17.53	20.09	21.95	40	20.71	24.43	55.76	59.34	63.69	66.77
9	1.73	2.70	16.92	19.02	21.67	23.59	50	27.99	32.36	67.50	71.41	76.15	79.49
10	2.16	3.25	13.31	20.48	23.21	25.19	60	35.53	40.48	79.08	83.30	88.38	91.95
12	3.07	4.40	21.03	23.34	26.22	28.30	70	43.28	48.76	90.53	95.02	100.4	104.2
14	4.07	5.63	23.68	26.12	29.14	31.32	80	51.17	57.15	101.9	106.6	112.3	116.3
16	5.14	6.91	26.30	28.85	32.00	34.27	100	67.33	74.22	124.3	129.6	135.8	140.2

16. The chi-squared goodness-of-fit test

The frequencies n_y are grouped so that the fitted frequency $\widehat{n}_y$ for every group exceeds about 5.

$X^2 = \sum_y \frac{(n_y - \widehat{n}_y)^2}{\widehat{n}_y}$ is referred to the table of χ^2_k with significance point p,

where k is the number of terms summed, less one for each constraint, *eg* matching total frequency, and matching $\overline{x}$ with μ

17. Joint probability distributions

Discrete distribution $\{p_{xy}\}$, where $p_{xy} = P(\{X = x\} \cap \{Y = y\})$.

Let $p_{x\bullet} = P(X = x)$, and $p_{\bullet y} = P(Y = y)$, then

$$p_{x\bullet} = \sum_y p_{xy} \quad \text{and} \quad P(X = x \,|\, Y = y) = \frac{p_{xy}}{p_{\bullet y}}$$

Continuous distribution

Joint cdf $F(x,y) = P(\{X \le x\} \cap \{Y \le y\}) = \int_{x_0=-\infty}^{x} \int_{y_0=-\infty}^{y} f(x_0\,, y_0)\, \mathrm{d}x_0\, \mathrm{d}y_0$

Joint pdf $f(x,y) = \frac{\mathrm{d}^2 F(x,y)}{\mathrm{d}x\, \mathrm{d}y}$

Marginal pdf of X $f_X(x) = \int_{-\infty}^{\infty} f(x, y_0)\, \mathrm{d}y_0$

Conditional pdf of X given $Y = y$ $f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$ (provided $f_Y(y) > 0$)

18. Linear regression

To fit the linear regression model $y = \alpha + \beta x$ by $\widehat{y}_x = \widehat{\alpha} + \widehat{\beta} x$ from observations $(x_1, y_1), \ldots, (x_n, y_n)$, the least squares fit is $\widehat{\alpha} = \overline{y} - \overline{x}\widehat{\beta}$, $\widehat{\beta} = \frac{S_{xy}}{S_{xx}}$

The residual sum of squares $\mathrm{RSS} = S_{yy} - \frac{S_{xy}^2}{S_{xx}}$

$\widehat{\sigma^2} = \frac{\mathrm{RSS}}{n-2}$ $\quad \frac{n-2}{\sigma^2}\widehat{\sigma^2}$ is from χ^2_{n-2}

$$E(\widehat{\alpha}) = \alpha, \quad E(\widehat{\beta}) = \beta,$$

$$\mathrm{var}\,(\widehat{\alpha}) = \frac{\sum x_i^2}{n\, S_{xx}}\sigma^2, \quad \mathrm{var}\,(\widehat{\beta}) = \frac{\sigma^2}{S_{xx}}, \quad \mathrm{cov}\,(\widehat{\alpha}, \widehat{\beta}) = -\frac{\overline{x}}{S_{xx}}\sigma^2$$

$$\widehat{y}_x = \widehat{\alpha} + \widehat{\beta} x\,, \qquad E(\widehat{y}_x) = \alpha + \beta x\,, \qquad \mathrm{var}\,(\widehat{y}_x) = \left\{\frac{1}{n} + \frac{(x - \overline{x})^2}{S_{xx}}\right\}\sigma^2$$

$\frac{\widehat{\alpha} - \alpha}{\widehat{\mathrm{se}}\,(\widehat{\alpha})}$, $\frac{\widehat{\beta} - \beta}{\widehat{\mathrm{se}}\,(\widehat{\beta})}$, $\frac{\widehat{y}_x - \alpha - \beta x}{\widehat{\mathrm{se}}\,(\widehat{y}_x)}$ are each from t_{n-2}