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IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2006

MSc and EEE PART IV: MEng and ACGI

LINEAR OPTIMAL CONTROL

Wednesday, 26 April 10:00 am

Time allowed: 3:00 hours

There are SIX questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible   First Marker(s) : A. Astolfi
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LINEAR OPTIMAL CONTROL

1. Consider the system

$$\dot{x} = Ax + Bu = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

a) Show that the system is controllable. [4 marks]

b) Assume u_2=0. Show that the system with input u_1 is not controllable and not stabilizable. [4 marks]

c) Determine a state feedback control law

$$u = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \end{bmatrix} x$$

with $K_{22} = K_{23} = K_{24} = 0$ such that the closed-loop system has all eigenvalues equal to -1. Show that there are infinitely many selections of the gains K_{11}, K_{12}, K_{13}, K_{14}, K_{21} which achieve this objective. [8 marks]

d) Consider the feedback law

$$u_1 = v_1 \qquad u_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} x.$$

Show that the system with input v_1 is controllable. [4 marks]

2. Consider the system

$$\dot{x} = Ax, \qquad y = Cx,$$

with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -\alpha \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

a) Show that the system is observable for any real α. [2 marks]

b) Design an asymptotic observer for the system. Select the output injection gain L such that the matrix $A - LC$ has two eigenvalues equal to -3. [4 marks]

c) Suppose that one can measure $y(t)$ and a delayed copy of $y(t)$ given by $y(t-\tau)$, with $\tau > 0$. Assume (for simplicity) that $\alpha \neq 0$.

i) For $t \geq \tau$, express the vector

$$Y(t) = \begin{bmatrix} y(t) \\ y(t-\tau) \end{bmatrix}$$

from $x(0)$. [6 marks]

ii) Show that the relation determined in part c)i) can be used, for any $\tau > 0$, to compute $x(0)$ as a function of $Y(t)$, where $t \geq \tau$. [4 marks]

iii) Argue that the relation determined in part c)i) can be used to determine $x(t)$ from $Y(t)$, for $t \geq \tau$, exactly. [4 marks]

3. Consider the system

$$J\ddot{\theta} + \beta\dot{\theta} = Ju.$$

a) Let $x = (x_1, x_2)'$ with $x_1 = \theta$ and $x_2 = \dot{\theta}$. Write the equations of the system in the standard state space form

$$\dot{x} = Ax + Bu.$$

[2 marks]

b) The goal of the control is to drive the state x_1 to a reference value $\bar{x}_1$ and x_2 to zero, while minimizing the cost

$$J = \int_0^\infty ((x_1(\tau) - \bar{x}_1)^2 + ru^2(\tau))d\tau.$$

Show that this tracking problem can be transformed into a standard LQR problem. Write explicitly this LQR problem. [6 marks]

c) Write the algebraic Riccati equation associated with the LQR problem in part b). [2 marks]

d) For $\beta = 1$ and $J = 1$, find the positive definite solution of the algebraic Riccati equation determined in part c). [8 marks]

e) Let $\beta = 1$ and $J = 1$. Write the optimal control law, and the optimal closed-loop system, for the problem in part b). Show that the optimal closed-loop system is stable for any $r > 0$. [2 marks]

4. Consider a cart of unity mass moving along a straight line without friction. Suppose that at $t = 0$ its position is $s(0)$ and its velocity is $\dot{s}(0)$. In the time interval $[0, T]$, with T known, we want to apply a force u to minimize the cost

$$J = cs^2(T) + \int_0^T u^2(\tau)d\tau,$$

with $c \geq 0$.

a) Let $x = (x_1, x_2)$ with $x_1 = s$ and $x_2 = \dot{s}$ and determine matrices Q, R, M, A and B such that

$$J = \int_0^T [x(\tau)'Qx(\tau) + Ru^2(\tau)]d\tau + x(T)'Mx(T)$$

and

$$\dot{x} = Ax + Bu.$$

[2 marks]

b) Write the Hamiltonian matrix H and the differential Riccati equation associated with the considered optimal control problem. [4 marks]

c) The solution of the differential Riccati equation associated with the problem can be computed by integrating the system

$$\begin{bmatrix} \dot{X} \\ \dot{Y} \end{bmatrix} = H \begin{bmatrix} X \\ Y \end{bmatrix},$$

with appropriate final conditions $X(T)$ and $Y(T)$.

i) Choose $X(T)$ and $Y(T)$. [2 marks]

ii) Determine $X(t)$ and $Y(t)$. (Hint: use the fact that $H^4 = 0$.) [6 marks]

iii) Determine the solution $P(t)$ of the differential Riccati equation. [4 marks]

d) Determine the optimal control law. [2 marks]

5. Consider the problem of determining the optimal investment plan for a production unit. Denoting the rate of investment by u, the production level is described by

$$\dot{x} = -\alpha x + u$$

with $\alpha > 0$ and $x(0) > 0$, and the index to maximize is

$$J = \beta x(T) + \int_0^T (x(\tau) - u(\tau))d\tau$$

with $\beta > 0$ and $T > 0$ known.

Suppose $0 \leq u \leq \bar{u}$.

a) Write the necessary conditions of optimality for normal extremals. [4 marks]

b) Write the optimal control as a function of the optimal costate. [2 marks]

c) Integrate the differential equations of the costate with $\lambda^\star(T) = -\beta$. [2 marks]

d) Determine the optimal control as a function of t.

i) Show that if α and β are such that

$$D(t) = e^{\alpha(t-T)}(1/\alpha - \beta) - 1/\alpha + 1 \neq 0 \tag{5.1}$$

for all $t \in [0, T]$ then the optimal control law is constant. [4 marks]

ii) Show that $D(t)$ in equation (5.1) can change sign only once. If $D(t)$ changes sign at $t = t_s$ the optimal control has a jump at t_s. Compute t_s as a function of α and β. Show that if $D(t)$ changes sign and $\beta > 1$ then the optimal control is

$$u^\star = \begin{cases} 0 & \text{for } t \in [0, t_s) \\ \bar{u} & \text{for } t \in [t_s, T] \end{cases}$$

[8 marks]

6. Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u \end{aligned}$$

and the cost to minimize

$$J = -x_1(T) + \frac{1}{2}\int_0^T u^2(\tau)d\tau.$$

The cost represents a tradeoff between the maximization of $x_1(T)$ and the minimization of the control effort.

a) Write the necessary conditions of optimality for normal extremals. (Hint: use the condition $\frac{\partial H}{\partial u} = 0$ for minimizing H with respect to u.) [4 marks]

b) Write the optimal control as a function of the optimal costate. [2 marks]

c) Integrate the differential equations of the costate. Note that the optimal costate should be such that $\lambda_1^\star(T) = -1$ and $\lambda_2^\star(T) = 0$. [4 marks]

d) Determine the optimal control as a function of t. [2 marks]

e) Integrate the optimal state equations with initial conditions $x_1^\star(0) = 0$ and $x_2^\star(0) = 0$. Determine $x_1^\star(T)$. [4 marks]

f) Compute the optimal cost $J^\star$ corresponding to the initial conditions $x_1^\star(0) = 0$ and $x_2^\star(0) = 0$. [4 marks]

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Linear Optimal Control - Model answers 2006

Question 1

a) Consider the following submatrix of the controllability matrix

$$\tilde{\mathcal{C}} = [B, AB, A^2B] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and note that the first, second, third and fifth columns are linearly independent. Hence the system is controllable.

b) If $u_2 = 0$ we have

$$\dot{x} = Ax + b_1u_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_1.$$

The controllability pencil is

$$[sI - A|b_1] = \left[\begin{array}{cccc|c} s & 0 & 0 & 0 & 0 \\ 0 & s & -1 & 0 & 0 \\ 0 & 0 & s & -1 & 0 \\ 0 & 0 & -1 & s & 1 \end{array}\right]$$

and this loses rank for $s = 0$. Hence the system is neither controllable nor stabilizable.

c) Let

$$A_{cl} = A + BK = \begin{bmatrix} K_{21} & K_{22} & K_{23} & K_{24} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ K_{11} & K_{12} & K_{13}+1 & K_{14} \end{bmatrix} = \left[\begin{array}{c|ccc} K_{21} & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ K_{11} & K_{12} & K_{13}+1 & K_{14} \end{array}\right].$$

We have partitioned the last matrix to show that K_{21} is an eigenvalue of A_{cl} and the eigenvalues of the right lower 3×3 block are also eigenvalues of A_{cl}. Hence, setting $K_{21} = -1$, $K_{12} = -1$, $K_{13} = -4$, $K_{14} = -3$ yields

$$A_{cl} = \left[\begin{array}{c|ccc} -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ K_{11} & -1 & -3 & -3 \end{array}\right]$$

which has all eigenvalues equal to -1 for any K_{11}.

d) Setting $u_1 = v_1$ and $u_2 = [0\ 1\ 0\ 0]x$, we obtain

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} v_1.$$

The controllability matrix of this system is

$$\mathcal{C} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

This is full rank, hence the system is controllable.

Question 2

a) The observability matrix is

$$\mathcal{O} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

which does not depend upon α and has rank two. Hence the system is observable for all α.

b) An asymptotic observer is described by

$$\dot{\xi} = A\xi + L(y - C\xi) = (A - LC)\xi + Ly$$

for some L, where ξ is the asymptotic estimate of x provided the matrix $A - LC$ has all eigenvalues with negative real part. Note that

$$A - LC = \begin{bmatrix} -L_1 & 1 \\ -L_2 & -\alpha \end{bmatrix}$$

and its characteristic polynomial is

$$s^2 + s(\alpha + L_1) + \alpha L_1 + L_2.$$

This should be equal to $(s+3)^2 = s^2 + 6s + 9$, yielding

$$L_1 = -\alpha + 6 \qquad L_2 = 9 - (6 - \alpha)\alpha.$$

c) Note that

$$y(t) = Ce^{At}x(0)$$

and replacing t with $t - \tau$ one has

$$y(t - \tau) = Ce^{A(t-\tau)}x(0)$$

Then, for $t \geq \tau$,

$$Y(t) = \begin{bmatrix} y(t) \\ y(t-\tau) \end{bmatrix} = \begin{bmatrix} C \\ Ce^{-A\tau} \end{bmatrix} e^{At}x(0).$$

For the given A and C (using $\alpha \neq 0$) we have

$$\begin{bmatrix} C \\ Ce^{-A\tau} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -\frac{e^{\alpha\tau}-1}{\alpha} \end{bmatrix}$$

which is invertible for all $\alpha \neq 0$ and all $\tau > 0$. Hence

$$x(t) = e^{At}x(0) = \begin{bmatrix} 1 & 0 \\ 1 & -\frac{e^{\alpha\tau}-1}{\alpha} \end{bmatrix}^{-1} Y(t).$$

The above relation implies that, for all $t \geq \tau$ it is possible to obtain exactly $x(t)$.

Question 3

a) The equations of the system in standard state-space form are

$$\dot{x} = Ax + Bu = \begin{bmatrix} 0 & 1 \\ 0 & -\beta/J \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u.$$

b) The reference signal to be tracked is $w = [\bar{x}_1\ 0]'$. Note that

$$Aw = 0,$$

hence the optimal tracking problem can be transformed into a standard LQR problem. To this end, set $\xi = x - w$ and note that the optimal tracking problem can be written as

$$\min_u \int_0^\infty (\xi_1^2(\tau) + ru^2(\tau))d\tau$$

with

$$\dot{\xi} = A\xi + Bu,$$

i.e. as a standard LQR regulator problem.

c) Set

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}$$

The ARE associated with the problem is

$$0 = A'P + PA + Q - \frac{1}{r}PBB'P = \begin{bmatrix} 1 - \frac{P_{12}^2}{r} & P_{11} - \frac{P_{12}\beta}{J} - \frac{P_{12}P_{22}}{r} \\ P_{11} - \frac{P_{12}\beta}{J} - \frac{P_{12}P_{22}}{r} & 2P_{12} - \frac{2\beta P_{22}}{J} - \frac{P_{22}^2}{r} \end{bmatrix}.$$

d) From the $(1,1)$ block we have $P_{12} = \pm\sqrt{r}$. Then, from the $(2,2)$ block and keeping in mind that P_{22} should be positive one has (one has to select $P_{12} = \sqrt{r}$)

$$P_{22} = -r + \sqrt{r^2 + 2r\sqrt{r}}).$$

Then, from the $(1,2)$ block one has (recall $P_{11} > 0$)

$$P_{11} = \sqrt{r + 2\sqrt{r}}.$$

Hence the matrix

$$P = \begin{bmatrix} \sqrt{r + 2\sqrt{r}} & \sqrt{r} \\ \sqrt{r} & -r + \sqrt{r^2 + 2r\sqrt{r}}) \end{bmatrix}$$

is the positive definite solution of the ARE.

e) The optimal control law is $u = -K\xi$, with

$$K = \begin{bmatrix} \frac{1}{\sqrt{r}} & -1 + \frac{\sqrt{r+2\sqrt{r}}}{\sqrt{r}} \end{bmatrix}.$$

The optimal closed loop system is

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1/\sqrt{r} & -\frac{\sqrt{r+2\sqrt{r}}}{\sqrt{r}} \end{bmatrix} x$$

The characteristic polynomial of the optimal closed-loop system is

$$s^2 + s\frac{\sqrt{r+2\sqrt{r}}}{\sqrt{r}} + 1/\sqrt{r},$$

hence the optimal closed loop system is stable for any $r > 0$.

Question 4

a) The matrices are

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$Q = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad R = 1 \qquad M = \begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix}.$$

b) The Hamiltonian matrix is

$$H = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right].$$

The DRE is

$$-\dot{P} = A'P + PA + Q - PBR^{-1}B'P.$$

c) The matrices are

$$X(T) = I, \qquad Y(T) = M.$$

To determine $X(t)$ and $Y(t)$ note that

$$\begin{bmatrix} X(t) \\ Y(t) \end{bmatrix} = e^{H(t-T)} \begin{bmatrix} X(T) \\ Y(T) \end{bmatrix} = e^{H(t-T)} \begin{bmatrix} I \\ M \end{bmatrix}$$

and that

$$e^{H(t-T)} = I + H(t-T) + H^2 \frac{(t-T)^2}{2} + H^3 \frac{(t-T)^3}{3!}$$

$$= \left[\begin{array}{cc|cc} 1 & t-T & -\frac{1}{6}(-t+T)^3 & -\frac{1}{2}(-t+T)^2 \\ 0 & 1 & \frac{1}{2}(-t+T)^2 & T-t \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & T-t & 1 \end{array}\right]$$

As a result

$$X(t) = \begin{bmatrix} 1 + \frac{1}{6}c(t-T)^3 & t-T \\ \frac{1}{2}c(t-T)^2 & 1 \end{bmatrix} \qquad Y(t) = \begin{bmatrix} c & 0 \\ -c(t-T)^2 & 0 \end{bmatrix}.$$

Finally

$$P(t) = Y(t)X^{-1}(t) = \frac{1}{\det(X(t))} c \begin{bmatrix} 1 & -(t-T) \\ -(t-T) & (t-T)^2 \end{bmatrix}$$

with

$$\det(X(t)) = 1 + \frac{1}{3}c(T-t)^3.$$

d) The optimal control law is

$$u = -K(t)x = -R^{-1}B'P(t)x$$

with

$$K(t) = \frac{c}{\det(X(t))} \begin{bmatrix} (T-t) & (T-t)^2 \end{bmatrix}.$$

Question 5

a) Note that the cost function should be changed to $-J$ to have a minimization problem. Let

$$H = -x + u + \lambda(-\alpha x + u).$$

The necessary conditions of optimality for normal extremals are

$$\dot{x} = -\alpha x + u \qquad \dot{\lambda} = 1 + \alpha\lambda,$$

$$(1+\lambda)u \leq (1+\lambda)\omega \quad \forall \omega \in [0, \bar{u}].$$

b) The optimal control as a function of the costate is

$$u^\star(t) = \begin{cases} 0 \text{ if } 1 + \lambda^\star(t) > 0 \\ \bar{u} \text{ if } 1 + \lambda^\star(t) < 0 \end{cases}$$

If $1 + \lambda^\star(t) = 0$ we do not have information on the optimal control.

c) The optimal costate is

$$\lambda^\star(t) = e^{\alpha(t-T)}(\frac{1}{\alpha} - \beta) - \frac{1}{\alpha}.$$

d) The optimal control is (recall equation (5.1))

$$u^\star(t) = \begin{cases} 0 \text{ if } D(t) > 0 \\ \bar{u} \text{ if } D(t) < 0 \end{cases}$$

If $D(t) = 0$ we do not have information on the optimal control.
If $D(t)$ does not change sign then the optimal control is constant.
$D(t)$ changes sign if

$$e^{\alpha(t-T)} = \frac{1 - \frac{1}{\alpha}}{\beta - \frac{1}{\alpha}}.$$

This equation may have at most one solution

$$t_s = T + \frac{1}{\alpha} \log \frac{1 - \frac{1}{\alpha}}{\beta - \frac{1}{\alpha}}.$$

If $D(t)$ changes sign and $\beta > 1$ then $D(T) = 1 - \beta < 0$, hence the optimal control is such that $u^\star(T) = \bar{u}$. As a result the optimal control is equal to zero for $t \in [0, t_s)$ and equal to $\bar{u}$ otherwise.

Question 6

a) Let

$$H = \frac{u^2}{2} + \lambda_1 x_2 + \lambda_2 u.$$

The necessary conditions of optimality for normal extremals are

$$\dot{x}_1 = x_2 \qquad \dot{x}_2 = u$$

$$\dot{\lambda}_1 = 0 \qquad \dot{\lambda}_2 = -\lambda_1$$

$$\tfrac{\partial H}{\partial u} = u + \lambda_2 = 0.$$

b) The optimal control as a function of the optimal costate is

$$u^\star(t) = -\lambda_2^\star(t).$$

c) From the differential equations of the costate we obtain

$$\lambda_1^\star(t) = -1 \qquad \lambda_2^\star(t) = (t - T).$$

d) The optimal control as a function of time is

$$u^\star(t) = -(t - T).$$

e) From the differential equations of the state with the optimal control we obtain

$$x_1^\star(t) = \frac{T}{2}t^2 - \frac{t^3}{6} \qquad x_2^\star(t) = Tt - \frac{t^2}{2}.$$

Hence $x_1^\star(T) = \frac{T^3}{3}$.

f) The optimal cost is

$$J^\star = -\frac{T^3}{6}.$$