

E1.10 MATHS 1
(EE - 1st year)

UNIVERSITY OF LONDON

[I(1) 2005]

B.ENG. AND M.ENG. EXAMINATIONS 2005

For Internal Students of Imperial College London

This paper is also taken for the relevant examination for the Associateship of the City & Guilds of London Institute

PART I : MATHEMATICS 1

Wednesday 1st June 2005 10.00 am - 1.00 pm

Answer *EIGHT* questions.

Formulae sheet provided.

[Before starting, please make sure that the paper is complete; there should be 6 pages, with a total of 10 questions. Ask the invigilator for a replacement if your copy is faulty.]

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1. (i) Classify the following functions as odd, even or neither :

(a) $(x^3 + 2x) \cos x$,

(b) $(\cos x + \sin x)^2 - 1$,

(c) xe^{x^3} .

(ii) Compute $f(f(x))$ when $f(x) = \frac{5x+2}{3x-5}$.

(iii) Find the inverse functions of $\sqrt{1-x}$ and $\ln\left(\frac{x+1}{x-1}\right)$.

2. Consider the function

$$f(x) = (x^2 - 4)e^{-x} .$$

- (i) Find the points where $f(x) = 0$.
- (ii) Find the vertical and horizontal asymptotes of f , if any.
- (iii) Use (i) and (ii) to determine the sign of $f(x)$, for all x .
- (iv) Find the points where $f'(x) = 0$.
- (v) Determine the local minima and maxima of f .
- (vi) Sketch the graph of f .

PLEASE TURN OVER

3. Find $\frac{dy}{dx}$ in terms of x in the following four cases, simplifying your answer where necessary :

(i) $y = \frac{\sin 2x}{x^2 + 2} ;$

(ii) $y = \sinh^{-1} \left[\frac{3x}{4} \right] ;$

(iii) $y = \ln \left[x + (1 + x^2)^{1/2} \right] ;$

(iv) $y = (\sin x)^x .$

4. (i) If $x = t + \sin t$, $y = t + \cos t$, show that

$$(1 + \cos t)^3 \frac{d^2y}{dx^2} = \sin t - \cos t - 1 .$$

- (ii) Given that

$$x^2 + y^2 + xy = 1 ,$$

find dy/dx .

Given that $y = 1$ when $x = 0$, use the formula

$$\frac{dy}{dx} = \lim_{\delta x \rightarrow 0} \frac{y(x + \delta x) - y(x)}{\delta x}$$

to find the approximate value of y at $x = 0.1$.

5. (i) You are given the limit

$$\lim_{\theta \rightarrow 0} \frac{\sin(\theta)}{\theta} = 1 .$$

Use this to evaluate the following limits:

$$\lim_{\theta \rightarrow 0} \frac{\sin(\theta^2)}{(\sin \theta)^2} \quad \text{and} \quad \lim_{\theta \rightarrow \infty} \theta \sin(2/\theta) .$$

- (ii) State l'Hôpital's Rule and use it to evaluate the limit :

$$\lim_{x \rightarrow 2} \frac{x^3 - 8}{x^2 + x - 6} .$$

6. (i) Use appropriate substitutions to evaluate the indefinite integrals

$$\int \frac{1}{x \ln x} dx \quad \text{and} \quad \int \sqrt{2 - x^2} dx .$$

- (ii) Evaluate the integrals

$$\int_0^\infty (\cos nx) e^{-x} dx \quad \text{and} \quad \int_0^\infty (\sin nx) e^{-x} dx ,$$

in terms of n , where n is a given positive integer.

Hence show that

$$\lim_{n \rightarrow \infty} \int_0^\infty (\cos nx) e^{-x} dx = 0$$

and

$$\lim_{n \rightarrow \infty} \int_0^\infty (\sin nx) e^{-x} dx = 0 .$$

PLEASE TURN OVER

7. Let $I_n = \int_0^\infty x^{2n+1} e^{-x} dx$ where $n \geq 0$ is an integer.

- (i) Find I_0 .
- (ii) Show that $I_n = (2n+1)! I_0$.
- (iii) Evaluate I_5 in terms of a factorial.

8. (i) Explain what it means for a series

$$\sum_{n=1}^{\infty} a_n$$

to be convergent.

(ii) Show that the series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

is divergent.

Hence, or otherwise, show that the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \quad \text{is also divergent.}$$

(iii) Find the radius of convergence R of

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}.$$

9. (i) Using De Moivre's theorem, show that

$$\cos^6 \theta = \frac{1}{32} [\cos 6\theta + 6 \cos 4\theta + 15 \cos 2\theta + 10] .$$

- (ii) If

$$z_1 = -2 + 4i ,$$

$$z_2 = 2 + 2i ,$$

find $z_1 z_2$ and $\frac{z_1}{z_2}$ and show that $|z_1 z_2| = |z_1| |z_2|$.

10. (i) Prove that

$$\cosh(x + iy) = \cosh x \cos y + i \sinh x \sin y$$

and show that

$$|\cosh(x + iy)|^2 = \frac{\cosh 2x + \cos 2y}{2} .$$

- (ii) If

$$y = \tanh^{-1} x ,$$

show that

$$\operatorname{sech} y = \sqrt{1 - x^2}$$

and hence that

$$\tanh^{-1} x = \frac{1}{2} \log_e \left(\frac{1+x}{1-x} \right) .$$

END OF PAPER

MATHEMATICAL FORMULAE

1. VECTOR ALGEBRA

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1, a_2, a_3)$$

Scalar (dot) product: $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

Vector (cross) product:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Scalar triple product:

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = \mathbf{b} \cdot \mathbf{c} \times \mathbf{a} = \mathbf{c} \cdot \mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product: $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c} \cdot \mathbf{a})\mathbf{b} - (\mathbf{b} \cdot \mathbf{a})\mathbf{c}$

2. SERIES

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \dots \quad (\alpha \text{ arbitrary, } |x| < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^n \frac{x^{2n}}{(2n)!} + \dots,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \dots,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^n \frac{x^{n+1}}{(n+1)} + \dots \quad (-1 < x \leq 1)$$

$$\begin{aligned} \sin(a+b) &= \sin a \cos b + \cos a \sin b; \\ \cos(a+b) &= \cos a \cos b - \sin a \sin b. \\ \cos iz &= \cosh z; \quad \cosh iz = \cos z; \quad \sin iz = i \sinh z; \quad \sinh iz = i \sin z. \end{aligned}$$

4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f D^n g + \binom{n}{1} Df D^{n-1} g + \dots + \binom{n}{n} D^n f D^{n-n} g + \dots + D^n f g.$$

(b) Taylor's expansion of $f(x)$ about $x = a$:

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \dots + \frac{h^n}{n!}f^{(n)}(a) + \epsilon_n(h),$$

where $\epsilon_n(h) = h^{n+1}f^{(n+1)}(a + \theta h)/(n+1)!$, $0 < \theta < 1$.

(c) Taylor's expansion of $f(x, y)$ about (a, b) :

$$f(a+h, b+k) = f(a, b) + [hf_x + kf_y]_{a,b} + \frac{1}{2!} [h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}]_{a,b} + \dots$$

(d) Partial differentiation of $f(x, y)$:

- If $y = y(x)$, then $f = F(x)$, and $\frac{dF}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx}$.
- If $x = x(t)$, $y = y(t)$, then $f = F(t)$, and $\frac{dF}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$.
- If $x = x(u, v)$, $y = y(u, v)$, then $f = F(u, v)$, and

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u}, \quad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial v}.$$

- Stationary points of $f(x, y)$ occur where $f_x = 0$, $f_y = 0$ simultaneously. Let (a, b) be a stationary point: examine $D = [f_{xx}f_{yy} - (f_{xy})^2]_{a,b}$. If $D > 0$ and $f_{xx}(a, b) < 0$, then (a, b) is a maximum; If $D > 0$ and $f_{xx}(a, b) > 0$, then (a, b) is a minimum; If $D < 0$ then (a, b) is a saddle-point.

(f) Differential equations:

- The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating factor $I(x) = \exp[\int P(x)dx]$, so that $\frac{d}{dx}(Iy) = IQ$.
ii. $P(x, y)dx + Q(x, y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y$.

5 INTEGRAL CALCULUS

- (a) An important substitution: $\tan(\theta/2) = t$;
 $\sin \theta = 2t/(1+t^2)$, $\cos \theta = (1-t^2)/(1+t^2)$, $d\theta = 2dt/(1+t^2)$.

- (b) Some indefinite integrals:

$$\int (a^2 - x^2)^{-1/2} dx = \sin^{-1} \left(\frac{x}{a} \right), \quad |x| < a.$$

$$\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1} \left(\frac{x}{a} \right) = \ln \left\{ \frac{x}{a} + \left(1 + \frac{x^2}{a^2} \right)^{1/2} \right\}.$$

$$\int (x^2 - a^2)^{-1/2} dx = \cosh^{-1} \left(\frac{x}{a} \right) = \ln \left| \frac{x}{a} + \left(\frac{x^2}{a^2} - 1 \right)^{1/2} \right|.$$

$$\int (a^2 + x^2)^{-1} dx = \left(\frac{1}{a} \right) \tan^{-1} \left(\frac{x}{a} \right).$$

6. NUMERICAL METHODS

- (a) Approximate solution of an algebraic equation:

If a root of $f(x) = 0$ occurs near $x = a$, take $x_0 = a$ and $x_{n+1} = x_n - [f(x_n)/f'(x_n)]$, $n = 0, 1, 2 \dots$

(Newton Raphson method).

- (b) Formulae for numerical integration: Write $x_n = x_0 + nh$, $y_n = y(x_n)$.

- i. Trapezium rule (1-strip): $\int_{x_0}^{x_1} y(x) dx \approx (h/2)[y_0 + y_1]$.
 ii. Simpson's rule (2-strip): $\int_{x_0}^{x_2} y(x) dx \approx (h/3)[y_0 + 4y_1 + y_2]$.

- (c) Richardson's extrapolation method: Let $I = \int_a^b f(x) dx$ and let I_1, I_2 be two estimates of I obtained by using Simpson's rule with intervals h and $h/2$.

Then, provided h is small enough,

$$I_2 + (I_2 - I_1)/15,$$

is a better estimate of I .

7. LAPLACE TRANSFORMS

Function	Transform	Function	Transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t) dt$	$af(t) + bg(t)$	$aF(s) + bG(s)$
df/dt	$sF(s) - f(0)$	$d^2 f/dt^2$	$s^2 F(s) - sf(0) - f'(0)$
$e^{at} f(t)$	$F(s-a)$	$tf(t)$	$-dF(s)/ds$
$(\partial/\partial \alpha) f(t, \alpha)$	$(\partial/\partial \alpha) F(s, \alpha)$	$\int_0^t f(t) dt$	$F(s)/s$
$\int_0^t f(u) g(t-u) du$	$F(s)G(s)$		
1	$1/s$	$t^n (n = 1, 2 \dots)$	$n!/s^{n+1}, (s > 0)$
e^{at}	$1/(s-a), (s > a)$	$\sin \omega t$	$\omega/(s^2 + \omega^2), (s > 0)$
$\cos \omega t$	$s/(s^2 + \omega^2), (s > 0)$	$H(t-T) = \begin{cases} 0, & t < T \\ 1, & t > T \end{cases}$	$e^{-sT}/s, (s, T > 0)$

8. FOURIER SERIES

If $f(x)$ is periodic of period $2L$, then $f(x+2L) = f(x)$, and

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \text{ where}$$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx, \quad n = 0, 1, 2, \dots, \text{ and}$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx, \quad n = 1, 2, 3, \dots$$

Parseval's theorem

$$\frac{1}{L} \int_{-L}^L [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

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a) i) odd ;

1

$$ii) (\cos x + \sin x)^2 - 1 = 2 \cos x \sin x ,$$

so this function is also odd ;

2

iii) neither.

2

$$b) f(f(x)) = \frac{5(5x+2) + 2(3x-5)}{3(5x+2) - 5(3x-5)} = \frac{31x}{31} = x .$$

4

c) Write $y = \sqrt{1+x}$, then $x = y^2 - 1$ is the inverse function. If $y = \ln\left(\frac{x+1}{x-1}\right)$, then

2

$$e^y = \frac{x+1}{x-1} = 1 + \frac{2}{x-1} . \text{ It follows that}$$

$$x-1 = \frac{2}{e^y-1} \text{ which implies } x = \frac{e^y+1}{e^y-1} .$$

4

Setter : A. Skoro bogata

Setter's signature :

Checker : WZKATO

Checker's signature :

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(1) $f(x) = 0 \Leftrightarrow x^2 - 4 = 0$ or $e^{-x} = 0$
 $\Leftrightarrow x = \pm 2$ \times

(2) f is defined everywhere \Rightarrow no vertical asymptotes

$x \rightarrow +\infty \Rightarrow f(x) \rightarrow 0$, because the

~~exponential wins from the polynomial~~ $\rightarrow 0$

\Rightarrow horizontal asymptote $y = 0$ for $x \rightarrow +\infty$

$x \rightarrow -\infty \Rightarrow f(x) \rightarrow +\infty$, so no horizontal asymptote for $x \rightarrow -\infty$.

(3) $\begin{array}{c} +++ \quad - - - \quad +++ \\ -2 \quad \quad 2 \quad x \end{array} \text{ sign}(f)$

(4) $f'(x) = 2x e^{-x} - (x^2 - 4) e^{-x}$
 $= e^{-x} (2x - x^2 + 4)$

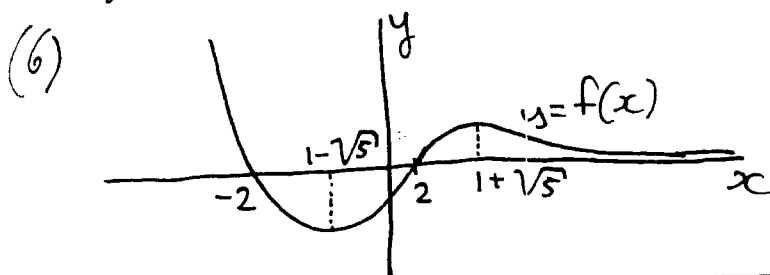
$f'(x) = 0 \Leftrightarrow x^2 - 2x - 4 = 0$

$\Leftrightarrow x = \frac{2 \pm \sqrt{20}}{2} = 1 \pm \sqrt{5} \quad (\approx -1, 3)$

(5) $\begin{array}{c} \text{loc. min} \quad \text{loc. max} \quad \text{sign}(f') \\ - - - \quad + + + \quad + \quad - - - \\ 1 - \sqrt{5} \quad \quad 1 + \sqrt{5} \quad x \end{array}$

loc min $x = 1 - \sqrt{5} : f(x) = (6 - 2\sqrt{5}) e^{-1 + \sqrt{5}}$

loc max $x = 1 + \sqrt{5} : f(x) = (6 + 2\sqrt{5}) e^{-1 - \sqrt{5}}$



Setter : M. Noort

Checker : J.R. CASU

Setter's signature :

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$$i) y = \frac{\sin 2x}{x^2+2} \quad \frac{dy}{dx} = \frac{2(x^2+2)\cos 2x - 2x\sin 2x}{(x^2+2)^2}$$

by quotient rule //

$$ii) y = \sinh^{-1} \left[\frac{3x}{4} \right]$$

$$\sinh y = \frac{3x}{4} \quad \cosh y \frac{dy}{dx} = \frac{3}{4}$$

$$\frac{dy}{dx} = \frac{3}{4} \left[\frac{1}{1 + \sinh^2 y} \right]^{1/2}$$

$$= \frac{3}{4} \left[\frac{1}{1 + \frac{9x^2}{16}} \right]^{1/2} = \frac{3}{(16+9x^2)^{1/2}} //$$

$$iii) y = \ln \left[x + (1+x^2)^{1/2} \right] \quad \frac{dy}{dx} = \frac{1 + x(1+x^2)^{-1/2}}{x + (1+x^2)^{1/2}}$$

$$\text{So } \frac{dy}{dx} = (1+x^2)^{-1/2} \left(\frac{(1+x^2)^{1/2} + x}{x + (1+x^2)^{1/2}} \right) = (1+x^2)^{-1/2} //$$

$$iv) y = (\sin x)^x \Rightarrow \ln y = x \ln(\sin x)$$

$$\frac{1}{y} \frac{dy}{dx} = \ln(\sin x) + x \frac{\cos x}{\sin x}$$

$$\text{So } \frac{dy}{dx} = (\sin x)^x \left[\ln(\sin x) + x \cot x \right] //$$

Setter : J. R. CASH

Checker : M. v. Noort

Setter's signature : J. R. Cash

Checker's signature :

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$$x = t + \sin t, \quad y = t + \cos t \quad \therefore \frac{dx}{dt} = 1 + \cos t, \quad \frac{dy}{dt} = 1 - \sin t$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1 - \sin t}{1 + \cos t}$$

$$\text{So } (1 + \cos t) \left(\frac{dy}{dx} \right) = 1 - \sin t$$

Differentiate again

$$-\sin t \frac{dy}{dx} + (1 + \cos t) \frac{d^2y}{dx^2} \frac{dx}{dt} = -\cos t$$

$$\begin{aligned} \therefore (1 + \cos t)^2 \frac{d^2y}{dx^2} &= \sin t \left(\frac{1 - \sin t}{1 + \cos t} \right) - \cos t \\ &= \frac{\sin t - \sin^2 t - \cos t - \cos^2 t}{1 + \cos t} \end{aligned}$$

$$\therefore (1 + \cos t)^3 \frac{d^2y}{dx^2} = \sin t - \cos t - 1$$

$$\text{ii) } x^2 + y^2 + xy = 1$$

$$2x + 2y \frac{dy}{dx} + y + x \frac{dy}{dx} = 0$$

$$\text{So } \frac{dy}{dx} = \frac{-(2x + y)}{x + 2y}$$

$$\text{Formula is } y(x + \delta x) - y(x) \approx \delta x y'(x)$$

$$\text{So put } x = 0, \delta x = 0.1$$

$$\left. \frac{dy}{dx} \right|_{x=0} = -\left(\frac{1}{2} \right) = -\frac{1}{2}$$

$$\text{So } y(0.1) \approx 1 - \frac{1}{2} \times 0.1 = 1 - 0.05 = 0.95$$

Setter : J.R. CASH

Checker : Mv Noort

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Mv Noort

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Q5 (a) i) Using the fact that

$$\frac{\sin(\theta^2)}{(\sin \theta)^2} = \frac{\sin(\theta^2)}{\theta^2} \cdot \frac{\theta^2}{(\sin \theta)^2} \text{ we find}$$

$$\lim_{\theta \rightarrow 0} \frac{\sin(\theta^2)}{(\sin \theta)^2} = \lim_{\theta \rightarrow 0} \frac{\sin(\theta^2)}{\theta^2} \cdot \left(\lim_{\theta \rightarrow 0} \frac{\theta}{\sin \theta} \right)^2$$

$$= 1 \times 1^2 = 1,$$

using algebra of limits.

ii) Since $\theta \cdot \sin(2/\theta) = \frac{\sin(2/\theta)}{2/\theta} \cdot 2$

we obtain

$$\lim_{\theta \rightarrow \infty} \theta \sin(2/\theta) = \lim_{\theta \rightarrow \infty} \frac{\sin(2/\theta)}{2/\theta} \cdot 2$$

$$= \left(\lim_{x \rightarrow 0} \frac{\sin x}{x} \right) 2 = 2.$$

(b) L'Hôpital's rule states that if f and g are differentiable functions such that $f(a) = g(a) = 0$, then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)},$$

provided $g'(a) \neq 0$.

Setter : R. BARNMORE

Checker : R. CRATER

Setter's signature : R. B. L.

Checker's signature : R. C.

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Q5 (b) (cont'd)

Hence, in the case $f(x) = x^2 - 8$ and
 $g(x) = x^2 + x - 6$, we find $f(2) = 0$

and $g(2) = 0$. However,

$$f'(x) = 2x \quad \text{and} \quad g'(x) = 2x + 1,$$

so that $g'(2) = 5 \neq 0$.

As a result,

$$\lim_{x \rightarrow 2} \frac{f(x)}{g(x)} = \frac{3 \cdot 2^2}{5} = 12/5.$$

[3]

Setter : R. BOARDMAN

Setter's signature : *R. Boardman*

Checker : R. CRATER

Checker's signature : *R. Crater*

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Q6 (a) To find $\int \frac{1}{x \ln x} dx$, put $x = e^u$

$$\text{so that } \int \frac{1}{x \ln x} dx = \int \frac{e^u du}{e^u \cdot u} = \int \frac{du}{u}$$

$$= \ln |u| + c,$$

$$= \ln |\ln x| + c \quad \text{for } x > 0.$$

To find $\int \sqrt{2-x^2} dx$, put $x = \frac{\sqrt{2}}{1} \sin \theta$, then

$$\int \sqrt{2-x^2} dx = \int \sqrt{2-2\sin^2 \theta} |\sqrt{2} \cos \theta| d\theta$$

$$= 2 \int \cos^2 \theta d\theta = \int 1 + \cos 2\theta d\theta$$

$$= \theta + \frac{\sin 2\theta}{2} = \theta + \sin \theta \cos \theta$$

$$= \sin^{-1} \left(\frac{x}{\sqrt{2}} \right) + \frac{x}{\sqrt{2}} \sqrt{1 - \frac{x^2}{2}}.$$

Setter : R. BOARDMAN

Checker : R. CLASTER

Setter's signature :

R. Boardman

Checker's signature :

R. Claster

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Q6(b) If, $C_n = \int_0^{\infty} e^{-nx} \cos nx \, dx$ and

$S_n = \int_0^{\infty} e^{-nx} \sin nx \, dx$ then

$$\begin{aligned} C_n &= \left[-e^{-nx} \cos nx \right]_0^{\infty} - \int_0^{\infty} e^{-nx} \sin nx \, dx \cdot n \\ &= 1 - nS_n \end{aligned}$$

and

$$\begin{aligned} S_n &= \left[-e^{-nx} \sin nx \right]_0^{\infty} + \int_0^{\infty} e^{-nx} \cos nx \, dx \cdot n \\ &= nC_n \end{aligned}$$

$$\text{Hence } C_n = 1 - n^2 C_n \Rightarrow C_n = \frac{1}{1+n^2}$$

$$\therefore S_n = \frac{n}{1+n^2}$$

It is immediate that $\lim_{n \rightarrow \infty} C_n = \lim_{n \rightarrow \infty} S_n = 0$.

Setter : R. BEARDMORE

Setter's signature :

Checker : R. CASTER

Checker's signature :

R. Beardmore

RVC

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$$(i) I_0 = \int_0^{\infty} x e^{-x} dx$$

$$= [-x e^{-x}]_0^{\infty} + \int_0^{\infty} e^{-x} dx = 1$$

$$(ii) I_n = [-x^{2n+1} e^{-x}]_0^{\infty} + (2n+1) \int_0^{\infty} x^{2n} e^{-x} dx$$

$$= (2n+1) [-x^{2n} e^{-x}]_0^{\infty} + (2n+1)(2n) \int_0^{\infty} x^{2n-1} e^{-x} dx$$

$$= (2n+1)(2n) I_{n-1}$$

Applying this recursively gives

$$I_n = (2n+1)! I_0$$

$$(iii) I_3 = 11! \text{ in terms of factorial (514)}$$

Setter : LUZENTAO

Setter's signature : *Stella Luente*

Checker : *A. Shombogator*

Checker's signature : *Aliu*

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i) A series is convergent if the limit of partial sums $S_n = a_1 + \dots + a_n$ exists:

$$\lim_{n \rightarrow \infty} S_n = S < \infty.$$

ii) Use the integral test:

$$\sum_{n=1}^{\infty} a_n \geq \int_1^{\infty} \frac{1}{n} dn = [\ln n]_1^{\infty} = \infty$$

By the comparison test $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is

divergent since $\frac{1}{\sqrt{n}} > \frac{1}{n}$ for all $n > 2$.

iii) Applying the ratio test

$$\lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+2} x^{n+1}}{n+1} \cdot \frac{n}{(-1)^{n+1} x^n} \right| = |x|$$

So $R = 1$.

Setter : *Luzzatto*

Setter's signature : *Stefano Luzzatto*

Checker : *A. Skovboegh*

Checker's signature : *A. Skovboegh*

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(i)

$$\text{let } z = \cos \theta + i \sin \theta$$

$$\frac{1}{z} = \cos \theta - i \sin \theta$$

$$z + \frac{1}{z} = 2 \cos \theta$$

$$2^6 \cos^6 \theta = \left(z + \frac{1}{z} \right)^6$$

$$= z^6 + \frac{1}{z^6} + 6 \left(z^4 + \frac{1}{z^4} \right) + 15 \left(z^2 + \frac{1}{z^2} \right)$$

$$+ 20 (+)$$

De Moivre.

$$(\cos \theta + i \sin \theta)^n = z^n = \cos n\theta + i \sin n\theta$$

$$(\cos \theta - i \sin \theta)^n = \frac{1}{z^n} = \cos n\theta - i \sin n\theta$$

$$\therefore (+) = 2 \cos 6\theta + 2.6 \cos 4\theta + 15.2 \cos 2\theta + 20$$

$$\therefore \cos^6 \theta = \frac{1}{2^5} [\cos 6\theta + 6 \cos 4\theta + 15 \cos 2\theta + 10]$$

Setter : R. CRASTER

Setter's signature : R. Craster

Checker : R. BEARDMORE

Checker's signature : R. Beardmore

EXAMINATION QUESTION / SOLUTION

2004 -- 2005

QUESTION

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SOLUTION

(ii) If $z_1 = -2 + 4i$ $z_2 = 2 + 2i$

9

$$z_1 z_2 = 2(-1 + 2i)(1 + i) = 4(-3 + i)$$

1

$$\frac{z_1}{z_2} = \frac{2(-1 + 2i)(1 - i)}{2(1 + i)(1 - i)} = \frac{1 + 3i}{2}$$

1

$$|z_1| = 2\sqrt{1 + 2^2} = 2\sqrt{5}$$

2

$$|z_2| = 2\sqrt{1 + 1} = 2\sqrt{2}$$

$$|z_1||z_2| = 4\sqrt{10}$$

$$|z_1 z_2| = 4\sqrt{1 + 3^2} = 4\sqrt{10}$$

1

Setter : R CASTER

Setter's signature :

Checker : R. BEARDMORE

Checker's signature :

15

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$$\begin{aligned} \text{(i) } \cosh(x+iy) &= \cos(i(x+iy)) = \cos[ix-y] \\ &= \cos ix \cos(y) + \sin ix \sin y \\ &= \cosh x \cos y + i \sinh x \sin y \end{aligned}$$

2

$$\begin{aligned} |\cosh(x+iy)|^2 &= \cosh^2 x \cos^2 y + \sinh^2 x \sin^2 y \\ &= \cosh^2 x \cos^2 y + (\cosh^2 x - 1) \sin^2 y \\ &= \cosh^2 x - \sin^2 y \\ &= \frac{1}{2}(1 + \cosh 2x) - \frac{1}{2}[1 - \cos 2y] \\ &= \cosh 2x + \cos 2y \end{aligned}$$

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$$\text{(ii) } \tanh y = x$$

$$\cosh^2 y - \sinh^2 y = 1$$

$$1 - \tanh^2 y = \operatorname{sech}^2 y$$

$$\sqrt{1-x^2} = \operatorname{sech} y = \frac{1}{\cosh y}$$

$$\cosh y = \frac{1}{\sqrt{1-x^2}}$$

$$\sinh y = \sqrt{1 + \frac{1}{1-x^2}}$$

$$= \frac{\sqrt{1-1+x^2}}{\sqrt{1-x^2}} = \frac{x}{\sqrt{1-x^2}}$$

$$e^y = \cosh y + \sinh y = \frac{x+1}{\sqrt{1-x^2}} = \sqrt{\frac{x+1}{1-x}}$$

$$y = \frac{1}{2} \log \sqrt{\frac{x+1}{1-x}}$$

2

2

3

1

Setter : R. CRASTER

Setter's signature : Rvc

Checker : L. BEARDMORE

Checker's signature : R. D. D.

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