# SOLUTIONS: STABILITY AND CONTROL OF NONLINEAR SYSTEMS<br>MASTER IN CONTROL

1. Exercise

a) The function $\max\{|x_1|,1\}\text{sign}(x_1)$ is discontinuous for $x_1 = 0$, since:

$$\lim_{x_1\to 0^+} \max\{|x_1|,1\}\text{sign}(x_1) = +1,$$

and

$$\lim_{x_1\to 0^-} \max\{|x_1|,1\}\text{sign}(x_1) = -1.$$

Hence, the function $f(x)$ is not Lipschitz continuous (the remaining terms are all continuous). Notice that in each subinterval $(-\infty,0)$ and $(0,+\infty)$ the function $\max\{|x_1|,1\}\text{sign}(x_1)$ is locally Lipschitz continuous with Lipschitz constant $L$ equal to 1. [ 2 ]

b) The vector-field is Lipschitz continuous ouside the set $\{x : x_1 = 0\}$ as sum of Lipschitz continuous functions (as from previous answer) and polynomials (which are smooth). Overall solutions exist and are uniquely defined over their interval of definition within the set $\{x : x_1 \neq 0\}$ . [ 2 ]

c) The first nullcline, $\mathcal{N}_1$ is given by:

$$\mathcal{N}_1 = \{(x_1,x_2) : x_2 = x_1^3 - \frac{7}{2}x_1\}.$$

It is therefore a cubic function which intersects the $x_1$ axis in 0 and $\pm\frac{\sqrt{7}}{\sqrt{2}}$. The second nullcline is the discontinuous function described in the answer to the first item:

$$\mathcal{N}_2 = \{(x_1,x_2) : x_2 = \max\{|x_1|,1\}\text{sign}(x_1)\}.$$

[ 3 ]

d) A graphical sketch of the nullclines is shown in the Figure 1.1, with the different regions labeled as R1,R2,R3,R4,R5,R6. In particular, the vector-field's orientations are given as: North-East, in region R1, South-West in region R2, South-West in region R3, North-East in region R4, South-East in region R5 and North-West in region R6. [ 3 ]

e) Notice that regions R1,R2, R3 and R4 are forward invariant. Indeed, the vector field at their boundary is either tangent to the boundary or pointing towards its interior. [ 3 ]

f) Equilibria are found at the intersection of the Nullclines. As it can be seen graphically, there are 3 intersection points between $\mathcal{N}_1$ and $\mathcal{N}_2$. These occur in $\{[0,0], \pm[3/\sqrt{2}, 3/\sqrt{2}]\}$. [ 2 ]

g) Linearization around $[0,0]$ is not possible because of discontinuity of the vector field.

For $x = \pm[3/\sqrt{2}, 3/\sqrt{2}]'$ we see that:

$$\frac{\partial f}{\partial x} = \left[\begin{array}{cc} -3x_1^2 + \frac{7}{2} & 1 \\ 1 & -1 \end{array}\right]_{x=\pm[3/\sqrt{2},3/\sqrt{2}]} = \left[\begin{array}{cc} -10 & 1 \\ 1 & -1 \end{array}\right].$$

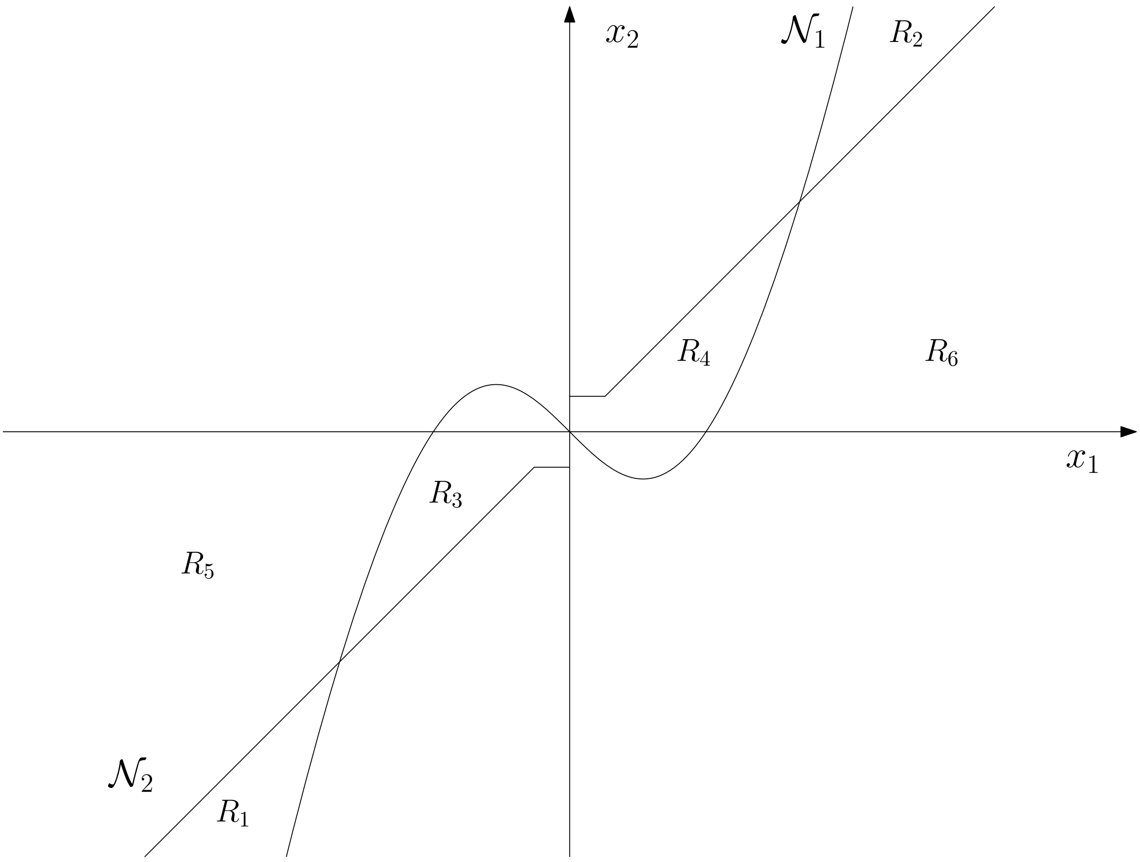


Figure 1.1 Nullclines and regions in state-space

The eigenvalues are both real and negative. Hence, the equilibria in $x = \pm[3/\sqrt{2}, 3/\sqrt{2}]$ are both stable nodes. [ 3 ]

h) A sketch of the global phase-portrait is shown in Fig. 1.2.

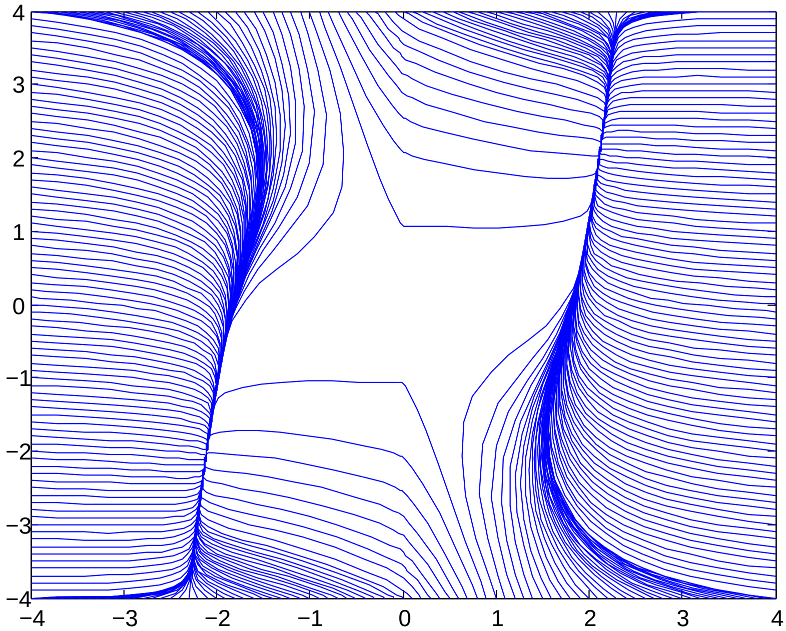




Figure 1.2 Phase portrait

2. Exercise

a) We regard the system as the feedback interconnections of two scalar systems:

$$\dot{x}_1 = -\alpha x_1^3 + \beta d_1^3, \qquad \dot{x}_2 = -\delta x_2^3 + \gamma d_2^3,$$

under the identifications $d_1 = x_2$ and $d_2 = x_1$.

b) To show ISS of the subsystems we use 2 candidate ISS Lyapunov functions, $V_1(x_1) = x_1^2/2$ and $V_2(x_2) = x_2^2/2$. Taking derivatives along solutions yields:

$$\begin{aligned}\dot{V}_1 = x_1(-\alpha x_1^3 + \beta d_1^3) &= -\varepsilon\alpha x_1^4 - (1-\varepsilon)\alpha x_1^4 + \beta x_1 d_1^3 \\ &\leq -\varepsilon\alpha|x_1|^4 - (1-\varepsilon)\alpha|x_1|^4 + \beta|x_1||d_1|^3.\end{aligned}$$

Hence:

$$|x_1| \geq |d_1|\sqrt[3]{\frac{\beta}{\alpha(1-\varepsilon)}} \Rightarrow \dot{V}_1 \leq -\varepsilon\alpha|x_1|^4.$$

This shows that the first subsystem is ISS. Similarly for the second subsystem:

$$|x_2| \geq |d_2|\sqrt[3]{\frac{\gamma}{\delta(1-\varepsilon)}} \Rightarrow \dot{V}_2 \leq -\varepsilon\delta|x_2|^4.$$

Therefore the second subsystem is also ISS.

c) The tightest gains of subsystem 1 and 2, can be expressed as:

$$\gamma_1(r) = \sqrt[3]{\frac{\beta}{\alpha(1-\varepsilon)}}r$$

$$\gamma_2(r) = \sqrt[3]{\frac{\gamma}{\delta(1-\varepsilon)}}r$$

d) The small gain theorem can be applied to conclude GAS of the closed-loop system provided the composition of gains is less than the identity. In this case this is true provided for some $\varepsilon > 0$

$$\gamma_1(\gamma_2(r)) < r \Leftrightarrow \sqrt[3]{\frac{\beta}{\alpha(1-\varepsilon)}}\sqrt[3]{\frac{\gamma}{\delta(1-\varepsilon)}} < 1$$

Equivalently:

$$\sqrt[3]{\frac{\beta}{\alpha}}\sqrt[3]{\frac{\gamma}{\delta}} < 1 \Leftrightarrow \frac{\beta}{\alpha}\frac{\gamma}{\delta} < 1$$

e) For positive $\alpha, \beta, \gamma$ and $\delta$, the matrix $A$ is Hurwitz iff its determinant is positive:

$$\det(A) = \alpha\delta - \beta\gamma > 0.$$

This is exactly the same region as the ISS region obtained thanks to the small-gain theorem.

f) Let us take the derivative of $V_1(x)$ along solutions of the nonlinear system:

$$\dot{V}_1(x) = 2(x_1 - x_2)(\dot{x}_1 - \dot{x}_2) = 2(x_1 - x_2)3(x_2^3 - x_1^3) = -6(x_1 - x_2)^2(x_1^2 + x_1x_2 + x_2^2) \leq 0$$

The last inequality holds since $x_1^2 + x_1x_2 + x_2^2$ is positive definite and $(x_1 - x_2)^2$ is positive semidefinite.Similarly we see that:

$$\dot{V}_2(x) = 2(x_1 + x_2)(\dot{x}_1 + \dot{x}_2) = 2(x_1 + x_2)(x_1^3 + x_2^3) = 2(x_1 + x_2)^2(x_1^2 - x_1x_2 + x_2^2) \geq 0.$$

The last inequality holds since $x_1^2 - x_1x_2 + x_2^2$ is positive definite and $(x_1 + x_2)^2$ is positive semidefinite.

g) We define $W(x) = V_2(x) - V_1(x)$. Notice that

$$\dot{W}(x) = \dot{V}_2(x) - \dot{V}_1(x) \geq 0.$$

Moreover, $\dot{W} = 0$ iff $\dot{V}_2 = 0$ and $\dot{V}_1 = 0$. This is true iff $(x_1 + x_2) = 0$ and $(x_1 - x_2) = 0$, viz. iff $x = 0$. Hence $\dot{W}$ is positive definite. Notice that 0 belongs to the closure of the set of points where $W$ is positive. To see this notice that, $W(1/n, 1/n) > 0$ for all $n \in \mathbb{N}$. As a result we may apply Lyapunov's instability criterion to conclude that the origin is unstable.

3. Exercise

a) In order to compute the relative degree we start differentiating the output variable. This yields:

$$\dot{y} = \dot{x}_1 + \dot{x}_2 = x_1 + x_2 + 2x_2x_3 + [2 + \cos(x_2)]u$$

Notice that the coefficient of $u$ in the expression for $\dot{y}$ equals $2 + \cos(x_2) > 0$ for all $x \in \mathbb{R}^3$. Hence the relative degree is 1 and is globally defined. [ 4 ]

b) We may define the Input-Output linearizing feedback as:

$$u = \frac{-x_1 - x_2 - 2x_2x_3 + v}{2 + \cos(x_2)}$$

This yields:

$$\dot{y} = v$$

[ 4 ]

c) In order to write the system in normal form we pick $\xi = [\xi_1, \xi_2]' = [x_2, x_3]'$. This yields the following set of equations:

$$\begin{aligned}
\dot{z} &= v \\
\dot{\xi}_1 &= -\xi_1 + \sin(y) \\
\dot{\xi}_2 &= -\xi_2^3 + \xi_1\xi_2 \\
\\
y &= z.
\end{aligned}$$

[ 4 ]

d) The internal dynamics are two-dimensional.

$$\begin{aligned}
\dot{\xi}_1 &= -\xi_1 + \sin(y) \\
\dot{\xi}_2 &= -\xi_2^3 + \xi_1\xi_2.
\end{aligned}$$

The variable $y$ is the input of the system. Notice that the $\xi_1$ equation is trivially an ISS system, when regarded as a scalar system of input $y$. Moreover, the $\xi_2$ equation also defines an ISS scalar system with respect to the input $\xi_1$ (because the negative term $-\xi_2^3$ is of higher degree than the coefficient of $\xi_1$ (degree 1). Overall, then, the Internal Dynamics are a cascade of ISS systems and are therefore ISS with respect to the input $y$. [ 4 ]

e) A globally stabilizing feedback is simply achieved by letting $v = -y$. [ 2 ]
This results in the closed-loop system:

$$\begin{aligned}
\dot{z} &= -z \\
\dot{\xi}_1 &= -\xi_1 + \sin(y) \\
\dot{\xi}_2 &= -\xi_2^3 + \xi_1\xi_2,
\end{aligned}$$

which is a cascade of a GAS (exponentially stable) and an ISS system. Hence this yields global asymptotic stability of the origin. [ 2 ]

4. Exercise

a) For an affine control system to be passive and loss-less the following equations need to be fulfilled:

$$\frac{\partial S}{\partial x}(x)\begin{bmatrix} g(x_2) \\ -g(x_1) \end{bmatrix} = 0$$

$$\frac{\partial S}{\partial x}(x)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = h(x).$$

The first equation yields:

$$\frac{\partial S}{\partial x_1}(x)g(x_2) - \frac{\partial S}{\partial x_2}(x)g(x_1) = 0$$

which can be solved by letting

$$S(x_1,x_2) = \int_0^{x_1} g(r)dr + \int_0^{x_2} g(r)dr.$$

Taking into account the previous expression for $S(x)$, the second equation yields:

$$h(x) = \frac{\partial S}{\partial x_2}(x) = g(x_2).$$

b) For $g(r) = e^r - e^{-r}$, we have:

$$S(x) = e^{x_1} + e^{-x_1} + e^{x_2} + e^{-x_2} - 4.$$

Notice that $S(x)$ is smooth, positive definite and radially unbounded. In fact

$$e^r \geq 1 + r + r^2/2 + r^3/6.$$

Similarly:

$$e^{-r} \geq 1 - r + r^2/2 - r^3/6.$$

Then, $e^r + e^{-r} \geq 2 + r^2$, and

$$S(x) \geq x_1^2 + x_2^2.$$

On the other hand, taking derivative along solutions of the closed-loop system we have:

$$\dot{S} = uy = -x_2 h(x) = -x_2(e^{x_2} - e^{-x_2}) \leq 0.$$

Hence, $\dot{S}$ is negative semi-definite. The set $\{x : \dot{S}(x) = 0\} = \{x : x_2 = 0\}$. Hence, the largest invariant set contained for which $\dot{S}$ vanishes is also contained in:

$$\{x : x_2 = 0 \,\&\, \dot{x}_2 = 0\} = \{x : x_2 = 0 \,\&\, -g(x_1) = 0\} = \{0\}.$$

Therefore we may apply the Lasalle's stability criterion to claim that the origin is globally asymptotically stable.

c) A similar result could be achieved by letting $u = -\mathrm{atan}(x_2)/2$. Indeed, the derivative of $S(x)$ reads:

$$\dot{S}(x) = yu = -\mathrm{atan}(x_2)g(x_2) \leq 0.$$

since both atan and $g$ are increasing odd functions. Moreover, the Kernel of $\dot{S}$ is unchanged and the largest invariant set therein contained is still the origin.

d) Pick as a candidate function for the Lyapunov criterion:

$$V(x) = x_1 - x_2.$$

Taking derivatives along solutions, for $u = 0$ yields:

$$\dot{V} = \dot{x}_1 - \dot{x}_2 = x_2^2 - (-x_1^2) = x_2^2 + x_1^2.$$

Thus $\dot{V}$ is positive definite. Moreover $V(0) = 0$ and

$$0 \in \mathrm{cl}\{x : V(x) > 0\}$$

as it follows by choosing the sequence $x_n = [1/n, 0]'$. Therefore the origin is unstable.