Discrete Mathematics and Computational Complexity

1. **[Compulsory]**

Let $\mathscr{R}$ be the set of all relations on a set A.

a) Express the predicate $P(x)$, meaning that $x \in \mathscr{R}$ is a transitive relation, in terms of appropriate symbolic logic.

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b) Prove that the proposition p given by $\forall R\,(P(R) \leftrightarrow (\forall n \in Z^+ R^n \subseteq R))$ is true, where the universe of discourse is $\mathscr{R}$.

[12]

c) Prove that the proposition q given by $\forall R\,(P(R) \rightarrow (\forall n \in Z^+\,((n \geq 2) \rightarrow P(R^n))))$ is true, where the universe of discourse is $\mathscr{R}$.

[10]

d) Replacing the implication in the definition of q by its converse yields another proposition r. Prove that r is false.

[8]

e) A relation R' is said to be the transitive closure of R when R' is the smallest transitive relation containing R. Define the connectivity relation R^* and prove that $R^* = R'$.

[8]

SPECIMEN EE3 QUESTION - MODEL ANSWER

i. $P(x) \equiv \forall a \forall b \forall c \left((a,b) \in x \wedge (b,c) \in x \rightarrow (a,c) \in x \right)$
where the universe of discourse is A.

ii. $\forall R \left[P(R) \Leftrightarrow (\forall n \in \mathbb{Z}^+ \; R^n \subseteq R) \right]$

First prove $\forall R \left[P(R) \rightarrow (\forall n \in \mathbb{Z}^+ \; R^n \subseteq R) \right]$

For $n=1$, we have $\forall R \left[P(R) \rightarrow R \subseteq R \right]$
which is true as the RHS is always true.

Assume true for n & we induct for $n>1$.

Let $(a,b) \in R^{n+1} = R \cdot R^n$.

$\Rightarrow \exists x \left[(a,x) \in R \wedge (x,b) \in R^n \right]$

Since $R^n \subseteq R$, $(x,b) \in R$

$\Rightarrow (a,b) \in R$, i.e. $R^{n+1} \subseteq R$ □

Second prove $\forall R \left[(\forall n \in \mathbb{Z}^+ \; R^n \subseteq R) \rightarrow P(R) \right]$

To be false, we would need R s.t.

$(\forall n \in \mathbb{Z}^+ \; R^n \subseteq R) \wedge \neg P(R)$.

$\forall n \in \mathbb{Z}^+ \; R^n \subseteq R \Rightarrow R^2 \subseteq R$

Let $(a,b) \in R \wedge (b,c) \in R$. Then $(a,c) \in R^2 \subseteq R$

$\Rightarrow (a,c) \in R \Rightarrow P(R)$ □

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$\forall R\,[P(R) \to (\forall n \in Z^+ ((n>2) \to P(R^n)))]$

We will prove $\forall R\,[P(R) \to \forall n \in Z^+\, P(R^n)]$, which is a stronger statement.

For $n=1$, implication is $P(R) \to P(R)$, which is true.
Assume for n & use induction for $n>1$.
Separately show that $P(R^{2n})$ and $P(R^{2n+1})$ are true.

First, $P(R^{2n})$.

Let $(a,b) \in R^{2n} = R^n \cdot R^n$ $\quad (b,c) \in R^{2n} = R^n \cdot R^n$
Then $\exists x\, \exists y\, [(a,x) \in R^n \wedge (x,b) \in R^n \wedge (b,y) \in R^n \wedge (y,c) \in R^n]$

Since R^n is transitive, $(a,b) \in R^n \wedge (b,c) \in R^n$

$\Rightarrow (a,c) \in R^n \cdot R^n = R^{2n}$.

Second, $P(R^{2n+1})$

Let $(a,b) \in R^{2n+1} = R \cdot R^n \cdot R^n$ and $(b,c) \in R^{2n+1} = R^n \cdot R^n \cdot R$
Then $\exists p\, \exists q\, \exists x\, \exists y\, [(a,x) \in R^n \wedge (x,p) \in R^n \wedge (p,b) \in R^n \wedge (b,y) \in R^n \wedge (y,q) \in R^n \wedge (q,c) \in R^n]$

Since R^n is transitive, $(a,p) \in R^n \wedge (y,c) \in R^n$.
Since R is transitive, $(p,y) \in R^n$

$\Rightarrow (a,c) \in R^n \cdot R \cdot R^n = R^{2n+1}$ $\square$.

$\forall R\,(\cancel{R}\,(\forall n \in \mathbb{Z}^+((n \geq 2) \rightarrow P(R^n))) \rightarrow P(R))$

consider $R = \{(a,b),\ (b,c)\}$.

$R^2 = \{(a,c)\}$

$R^n = \emptyset$ for $n > 2$.

so $P(R)$ is false but $P(R^2)$, $P(R^3)$, etc. are true.

$\forall n \in \mathbb{Z}^+((n \geq 2) \rightarrow P(R^n))$ is TRUE.

so $\exists R\,(\forall n \in \mathbb{Z}^+((n \geq 2) \rightarrow P(R^n)) \wedge \neg P(R))$

TRUE $\Rightarrow$ original proposition is false.

$R^* = R \cup R^2 \cup R^3 \cup \ldots$ (†)

Need to prove $R' = R^*$

(i) $R \subseteq R^*$ directly

(ii) Need to show $R^* \subseteq S$ whenever $R \subseteq S$, S transitive.

If $(a,b) \in R^* \wedge (b,c) \in R^*$ it follows from (†)

that $(a,c) \in R^*$.

Let $R \subseteq S$. Then $P(S^n) \wedge S^n \subseteq S$.

$S^* = S \cup S^2 \ldots$ and $S^n \subseteq S$. So $S^* \subseteq S$.

$R \subseteq S \Rightarrow R^* \subseteq S^*$.

$\therefore R^* \subseteq S^* \subseteq S$ $\square$.

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