EE4-26
EE9-CS5-3

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2013

MSc and EEE PART IV: MEng and ACGI

ESTIMATION AND FAULT DETECTION

Tuesday, 30 April 10:00 am

Time allowed: 3:00 hours

There are FIVE questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : R.B. Vinter

Second Marker(s) : D. Angeli

Information for candidates:

Some formulae relevant to the questions.

The normal $\mathcal{N}(m, \sigma^2)$ density:

$$\mathcal{N}(m, \sigma^2)(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-m)^2}{2\sigma^2}\right)$$

System equations:

$$\begin{aligned} \mathbf{x}_t &= F\mathbf{x}_{t-1} + \mathbf{u}^s + \mathbf{w}_{t-1} \\ \mathbf{y}_t &= H\mathbf{x}_t + \mathbf{u}^o + \mathbf{v}_t \,. \end{aligned}$$

Here, $\{\mathbf{w}_t\}$ and $\{\mathbf{v}_t\}$ are white noise sequences with covariances Q^s and Q^o respectively.

The Kalman filter equations are

$$\begin{aligned} P_{t|t-1} &= FP_{t-1|t-1}F^T + Q^s \\ P_t &= P_{t|t-1} - P_{t|t-1}H^T(HP_{t|t-1}H^T + Q^o)^{-1}HP_{t|t-1} \,, \\ K_t &= P_{t|t-1}H^T(HP_{t|t-1}H^T + Q^o)^{-1} \,, \\ \hat{\mathbf{x}}_t &= \hat{\mathbf{x}}_{t|t-1} + K_t(\mathbf{y}_t - \hat{\mathbf{y}}_{t|t-1}) \,, \end{aligned}$$

in which $\hat{\mathbf{x}}_{t|t-1} = F\hat{\mathbf{x}}_{t-1} + \mathbf{u}^s$ and $\hat{\mathbf{y}}_{t|t-1} = H\hat{\mathbf{x}}_{t|t-1} + \mathbf{u}^o$

1. A scalar continous time signal $z(t)$ has the description

$$dz(t)/dt = n(t) ,$$

'a pure integrator system driven by coloured noise'. The noise process $n(t)$ is modelled as

$$dn(t)/dt = -2n(t) + v(t) , \qquad (1)$$

in which $\{v(t)\}$ is a continuous time white noise process with unit intensity, i.e.

$$E[e(t)e(s)] = 1 \times \delta(t-s) .$$

(i): Derive a state space model for the vector process $\mathbf{x}(t) = [z(t), n(t)]^T$. Show that $\mathbf{x}(t)$ satisfies an equation of the type

$$\mathbf{x}(t) = L(t-s)\mathbf{x}(s) + \int_s^t \mathbf{g}(t-\sigma)e(\sigma)d\sigma$$

and determine the matrix function $L(t)$ and vector function $\mathbf{g}(t)$. [10]

Hint: to obtain the transition matrix solve first eqn. (1) for $n(t)$ given $n(s)$ when $v(t) = 0$.

(ii): The signal is sampled at times $t = kh$ for $k = 0, 1, \ldots$ to yield the discrete time process $\mathbf{x}_k = \mathbf{x}(kh)$. Show that $\mathbf{x}_k$ satisfies an equation of the type

$$\mathbf{x}_{k+1} = \tilde{A}\mathbf{x}_k + \mathbf{v}_{k+1} ,$$

in which $\mathbf{v}_k$ is a white noise process. Determine $\tilde{A}$ and the covariance matrix $\tilde{Q}$ of $\mathbf{v}_k$. [7]

(iii): A Kalman filter is used to generate least squares estimates $\hat{\mathbf{x}}_k$ of $\mathbf{x}_k$ from noisy measurements $\mathbf{y}_i$ of $\mathbf{x}_i$, for $i = 1, \ldots, k$. Give an expression for the least squares estimate

$$E[\mathbf{x}((k+\frac{1}{2}))h) \,|\, \mathbf{y}_{1:k}]$$

of the inter-sample value of the state $\mathbf{x}(k(h+\frac{1}{2}))$ in terms of $\hat{\mathbf{x}}_k$. [3]

2. Consider a zero mean random 2-vector random variable $\mathbf{n} = \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$. Write

$$Q = E[nn^T].$$

(i): Show that, if Q is singular, then n_1 and n_2 are linearly dependent, i.e. there exist scalars c_1, c_2, not both zero, such that $c_1 n_1 + c_2 n_2 = 0$. [4]

(ii): Now assume that

$$Q = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix},$$

i.e. $E[n_1^2] = E[n_2^2] = 1$ and n_1, n_2 have correlation coefficient ρ, $-1 \leq \rho \leq 1$.

Take x to be a zero mean, scalar random variable which is uncorrelated with n_1 and n_2, and write $\sigma^2 = \text{var}\{x\}$.

Two measurements y_1 and y_2 are take of x:

$$y_1 = x + n_1 \quad \text{and} \quad y_2 = x + n_2 .$$

Show that the best linear least squares estimate $\hat{x}$ of x given y_1 and y_2, and the mean square estimation error are

$$\hat{x} = \frac{\sigma^2}{2\sigma^2 + 1 + \rho} \times (y_1 + y_2)$$

and

$$J^*(\rho) := E[|x - \hat{x}|^2] = \frac{\sigma^2(1+\rho)}{2\sigma^2 + 1 + \rho}.$$

[10]

You can use the formulae providing the solution to the standard vector linear least squares problem.

Plot $J^*(\rho)$ as a function of the correlation coefficient ρ. [3]

Comment on the values of $J^*(\rho)$ at $\rho = +1, 0, -1$. [3]

3. Signal and measurement processes $\{\mathbf{x}_k\}$ and $\{\mathbf{y}_k\}$ are modelled as

$$\begin{cases} \mathbf{x}_k = F\mathbf{x}_{k-1} \\ \mathbf{y}_k = H\mathbf{x}_k + \mathbf{v}_k \,, \end{cases} \qquad k = 0, 1, 2, \ldots \tag{2}$$

in which F, H are constant matrices, and $\{\mathbf{v}_k\}$ is a non-stationary sequence of independent, zero mean, Gaussian, vector random variables with *time-varying* covariances:

$$E[\mathbf{v}_k \mathbf{v}_k^T] = Q_k, \quad k = 1, 2, \ldots$$

It is assumed that $\mathbf{x}_0 \sim \mathcal{N}(\hat{\mathbf{x}}_0, P_0)$, and that $\mathbf{x}_0$ and $\{\mathbf{v}_k\}$ are independent. Write

$$P_k = \operatorname{cov}\{\mathbf{x}_k \,|\, \mathbf{y}_{1:k}\} \quad \text{and} \quad P_{k|k-1} = \operatorname{cov}\{\mathbf{x}_k \,|\, \mathbf{y}_{1:k-1}\}.$$

(i): Deduce from Bayes' Rule, and the fact that $p(\mathbf{y}_k|\mathbf{x}_k, \mathbf{y}_{1:k-1}) = p(\mathbf{y}_k|\mathbf{x}_k)$, that

$$p(\mathbf{x}_k|\mathbf{y}_{1:k}) = \frac{1}{c} \times p(\mathbf{y}_k|\mathbf{x}_k)p(\mathbf{x}_k|\mathbf{y}_{1:k-1}) \tag{3}$$

in which c is a constant (for a given set of measurements). [3]

Hence show that, for $k = 1, 2, \ldots$

$$P_k^{-1} = H^T Q_k^{-1} H + P_{k|k-1}^{-1} \,.$$

[10]

Hint: Assume $p(\mathbf{x}_k|\mathbf{y}_{1:k})$ and $p(\mathbf{x}_k|\mathbf{y}_{1:k-1})$ have the structure

$$p(\mathbf{x}_k|\mathbf{y}_{1:k}) = \mathcal{N}(\hat{\mathbf{x}}_k, P_k)(\mathbf{x}_k), \quad p(\mathbf{x}_k|\mathbf{y}_{1:k-1}) = \mathcal{N}(\hat{\mathbf{x}}_{k|k-1}, P_{k|k-1})(\mathbf{x}_k)\,,$$

substitute into (3), and equate quadratic exponents in $\mathbf{x}_k$.

(ii): Let $\{x_k\}$ be a sequence of increasingly noisy measurements of a scalar random variable $x \sim \mathcal{N}(\hat{x}_0, p_0)$. Assume

$$y_k = x + v_k, \quad k = 1, 2, \ldots$$

for some sequence of zero mean, independent random variables $\{v_k\}$, independent of x_0 such that

$$\operatorname{var}\{v_k\} = c^{k-1}, \quad k = 1, 2, \ldots$$

for some constant $c > 1$.

Using part (i) in which $F = 1$, show that the estimation error variance p_k after k measurements is

$$p_k^{-1} = p_0^{-1} + \left(1 + \left(\frac{1}{c}\right) + \left(\frac{1}{c}\right)^2 + \ldots + \left(\frac{1}{c}\right)^{k-1}\right).$$

[4]

Show that the asymptotic error covariance for an infinite number of measurements is

$$p_\infty = \frac{p_0}{1 + p_0 \times \left(\frac{c}{c-1}\right)}.$$

[3]

4. (i): A signal and measurement process are described by the equations

$$\begin{cases} \mathbf{x}_k &= F\mathbf{x}_{k-1} + +\mathbf{u}^s + \mathbf{e}_k \\ \mathbf{y}_k &= H\mathbf{x}_k + \mathbf{u}^o + \mathbf{v}_k \end{cases}$$

in which $\{\mathbf{e}_k\}$ and $\{\mathbf{v}_k\}$ are independent, Gaussian, white noise processes with covariancs Q^s and Q^o respectively. Assuming that the error covariance P_k, the predicted error covariance $P_{k|k-1}$ and the Kalman gain K_k converge as $k \to \infty$, derive equations for

$$P = \lim_{k\to\infty} P_k,\ S = \lim_{k\to\infty} P_{k|k-1} \text{ and } K = \lim_{k\to\infty} K_k\,.$$

[6]

Give conditions on the matrices F and H under which convergence occurs and these equations have a unique solution (for which P is a covariance matrix). [2]

Assuming these conditions hold, show that the error covariance converges to the zero matrix if $Q^s = 0$. (The 'no process noise' case.) [2]

(ii) A signal $\{y_k\}$ is described by the stochastic difference equation

$$y_k - ay_{k-1} - \frac{1}{2}y_{k-2} \;=\; e_k$$

in which $\{e_k\}$ is a Gaussian white noise process with unit variance. a is an unknown parameter in the model that we need to identify, taking account of some prior statistical information about the nature of a.

Treating a as a random variable with probability density $a \sim \mathcal{N}(x_0, p_0)$, independent of $\{e_k\}$, show that $\hat{a}_k$ satisfies a recursive equation of the form:

$$\hat{a}_k \;=\; \hat{a}_{k-1} + K_k\left(y_k - \hat{a}_{k-1}y_{k-1} - \frac{1}{2}y_{k-2}\right)\,,$$

for some matrix K_k. Derive recursive equations for K_k and also the error variance p_k:

$$p_k \;:=\; E[|a - \hat{a}_k|^2|y_{1:k}]\,.$$

[10]

Hint: write $x_k = a$ for all k. Notice that $E[x_k|y_{1:k}] = \hat{a}_k$, and y_k and x_k satisfy the standard equations for application of the Kalman filter:

$$\begin{cases} x_k = Fx_{k-1} \\ y_k = h(y_{1:k-1})x_k + u^o(y_{1:k-1}) + e_k\,, \end{cases}$$

in which F, h, and u^o are appropriate scalars. You should briefly explain why the Kalman filter yields least squares estimates, even though h and u^0 depend on past measurements $y_{1:k-1}$.

5. (i): Measurements y_k are taken at sample times kT, $k = 1, 2, \ldots$ of the position of a vehicle moving along a straight line in a viscous medium. The position z_k and velocity v_k of the vehicle, and the measurement y_k, are assumed to be the solutions of the equations:

$$\begin{cases} z_k = z_{k-1} + Tv_{k-1} \\ v_k = v_{k-1} - Td(v_{k-1}) + e_k \\ y_k = h(x_k) + w_k \; , \end{cases}$$

The nonlinear functions in the second and third equations, which take account of the viscous drag and of the 'soft saturation' of the sensor, are

$$d(v) = v^3 \quad \text{and} \quad h(z) = \begin{cases} (1 - e^{-|z|}) & \text{if } z \geq 0 \\ -(1 - e^{-|z|}) & \text{if } z < 0 \, . \end{cases}$$

Assume that $\{e_k\}$ and $\{w_k\}$ are scalar Gaussian white noise processes with variances σ_s^2 and σ_m^2 respectively, and that $\{e_k\}$, $\{w_k\}$, z_0 and v_0 are independent.

Develop an extended Kalman filter for the online estimation of (z_k, v_k) given $y_{1:k}$, briefly explaining the ideas which underly the construction. [10]

(ii): A sensor provides a noisy measurement y of the state x of a device. It is assumed that x and y are scalar random variables related by the equation

$$y = x + d + e \, .$$

Here, e is a random variable that is independent of x. It is further assumed that

$$E[x] = 1, \; \text{var}\{x\} = 0.07, \; E[e] = 0, \; \text{var}\{e\} = 0.02 \, .$$

d is a number taking values either 1 or 0, depending on whether or not a sensor failure has occurred, causing a measurement bias. Consider the two hypotheses:

$$\begin{aligned} &(H_0): \; d = 0 \quad \text{'the device has not failed'}, \\ &(H_1): \; d = 1 \quad \text{'the device has failed'}. \end{aligned}$$

Treating (H_0) as the null hypothesis, construct a Neyman Pearson decision function to test if the device has failed, at the significance level of $\alpha \times 100$ percent. [6]

Derive a formula (expressed in terms of the error function erfc(.)) for the power of the test, where [4]

$$\text{erfc}(y) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{y} e^{-s^2/2} ds \, .$$

Estimation and Fault Detection

2013 Estimation + Fault Detection Exam : Model Answers

1 (i) $\dot{z} = n$, $\dot{n} = -2n + v$. Write $x_1 = z$, $x_2 = n$. Then $\dot{x}_1 = x_2$ and $\dot{x}_2 = -2x_2 + e$. The state space equations are $\dot{x} = Ax + be$, in which $A = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}$ and $b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

The transition matrix e^{At} satisfies
$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = e^{At} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$ when $\dot{x}_1(t) = x_2(t)$, $\dot{x}_2 = -2x_2 + (e = 0)$
The last equation gives: $x_2(t) = e^{-2t} x_2(0)$. Then

$$x_1(t) = x_1(0) + \int_0^t x_2(t')\,dt' = x_1(0) + \tfrac{1}{2}\left(1 - e^{-2t}\right) x_2(0).$$

So $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{bmatrix} 1 & \frac{1}{2}(1 - e^{-2t}) \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$, So $e^{At} = \begin{bmatrix} 1 & \frac{1}{2}(1 - e^{-2t}) \\ 0 & e^{-2t} \end{bmatrix}$

By the variation of constants formula

$$x(t) = \begin{bmatrix} 1 & \frac{1}{2}(1 - e^{-2(t-s)}) \\ 0 & e^{-2(t-s)} \end{bmatrix} x(s) + \int_s^t \begin{bmatrix} 1 & \frac{1}{2}(1 - e^{-2(t-\sigma)}) \\ 0 & e^{-2(t-\sigma)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} e(\sigma)\,d\sigma$$

$$= L(t-s)\,x(s) + \int_s^t g(t-\sigma)\,e(\sigma)\,d\sigma, \text{ when}$$

$$L(t) = \begin{bmatrix} 1 & \frac{1}{2}(1 - e^{-2t}) \\ 0 & e^{-2t} \end{bmatrix},\quad g(t) = \begin{bmatrix} \frac{1}{2}(1 - e^{-2t}) \\ e^{-2t} \end{bmatrix}$$

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(ii) From (i)

$$x_{k+1} = x((k+1)h) = \tilde{A}\,x(kh) + v_{k+1}, \text{ where}$$

$\tilde{A} = \begin{bmatrix} 1 & \frac{1}{2}(1 - e^{-2h}) \\ 0 & e^{-2h} \end{bmatrix}$ and $\{v_k\}$ is "white noise" with covariance:

$$\tilde{Q} = \int_{kh}^{(k+1)h} \begin{bmatrix} \frac{1}{2}(1 - e^{-2((k+1)h - \sigma)}) \\ e^{-2((k+1)h - \sigma)} \end{bmatrix} \begin{bmatrix} \cdot & \cdot \end{bmatrix} d\sigma$$

$$(\sigma' = \sigma - kh) = \int_0^h \begin{bmatrix} \frac{1}{2}(1 - e^{-2(h-\sigma')}) \\ e^{-2(h-\sigma')} \end{bmatrix} \begin{bmatrix} \frac{1}{2}(1 - e^{-2(h-\sigma')}) & e^{-2(h-\sigma')} \end{bmatrix} d\sigma'$$

$$= \begin{bmatrix} \frac{1}{4}\left(h + (1 - e^{-2h}) + \frac{1}{4}(1 - e^{-4h})\right) & \frac{1}{2}\left(\frac{1}{2}(1 - e^{-2h}) - \frac{1}{4}(1 - e^{-4h})\right) \\ \frac{1}{2}\left(\frac{1}{2}(1 - e^{-2h}) - \frac{1}{4}(1 - e^{-4h})\right) & \frac{1}{4}\left(1 - e^{-4h}\right) \end{bmatrix}$$

[7]

(iii) $$x((k+\tfrac{1}{2})h) = L(\tfrac{1}{2}h)\,x(kh) + \int_{kh}^{(k+\frac{1}{2})h} g((k+1)h - \sigma)\,v(\sigma)\,d\sigma$$

Taking conditional expectations $E\{(\cdot\cdot) \mid y_{1:k}\}$ gives

$E\{x((k+\tfrac{1}{2})h) \mid y$ ... $\begin{bmatrix} 1 & \frac{1}{2}(1 - e^{-h}) \end{bmatrix}$ ...

2 (i) If Q is singular, the is a non-zero vector $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \neq 0$ such that $Q\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 0$

Then $0 = [c_1\ c_2]\, Q \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = [c_1\ c_2]\, E\left[\begin{pmatrix} n_1 \\ n_2 \end{pmatrix}(n_1\ n_2)\right]\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = E|c_1 n_1 + c_2 n_2|^2$

But then $c_1 n_1 + c_2 n_2 = 0$ (c_1, c_2 not both zero)

(ii) $y = \begin{bmatrix} 1 \\ 1 \end{bmatrix} x + n$ and x has zero mean, $E[x^2] = \sigma^2$. So

$\mathrm{cov}\{x,y\} = E[x(x[1\ 1] + n)] = \sigma^2 [1\ 1] + 0$

$\mathrm{cov}\{y\} = E\left[\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} x + n\right)(x[1\ 1] + n)\right] = \sigma^2\begin{bmatrix} 1 \\ 1 \end{bmatrix}[1\ 1] + Q = \begin{bmatrix} 1+\sigma^2 & \rho+\sigma^2 \\ \rho+\sigma^2 & 1+\sigma^2 \end{bmatrix}$

$\mathrm{cov}\{y\}^{-1} = \frac{1}{\alpha} \cdot \begin{bmatrix} 1+\sigma^2 & -(\rho+\sigma^2) \\ -(\rho+\sigma^2) & 1+\sigma^2 \end{bmatrix}$ in which

$\alpha = \det\{\mathrm{cov}\{y\}\} = 1 + 2\sigma^2 + \cancel{\sigma^4} - \rho^2 - 2\rho\sigma^2 - \cancel{\sigma^4} = (1-\rho^2) + 2(1-\rho)\sigma^2$
$= (1-\rho)(1+\rho+2\sigma^2)$

Then

$\hat{x} = \mathrm{cov}\{x,y\}\,\mathrm{cov}\{y\}^{-1} y = \frac{\sigma^2}{\alpha}[1\ 1]\begin{bmatrix} 1+\sigma^2 & -(\rho+\sigma^2) \\ -(\rho+\sigma^2) & 1+\sigma^2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

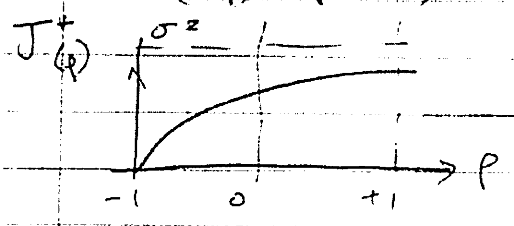
$= \frac{(1-\rho)\sigma^2}{(1-\rho)(1+\rho+2\sigma^2)}(y_1+y_2) = \frac{\sigma^2}{1+\rho+2\sigma^2}(y_1+y_2)$

$J(\rho) = E\{|x|^2\} - \mathrm{cov}\{x,y\}\,\mathrm{cov}\{y\}^{-1}\mathrm{cov}\{y,x\}$

$= \sigma^2 - \frac{\sigma^4}{(1-\rho)(1+\rho+2\sigma^2)}[1\ 1]\begin{bmatrix} (1+\sigma^2) & -(\rho+\sigma^2) \\ -(\rho+\sigma^2) & (1+\sigma^2) \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$= \sigma^2 - \frac{\sigma^4}{(1-\rho)(1+\rho+2\sigma^2)}\left(2(1+\sigma^2) - 2(\rho+\sigma^2)\right) = \sigma^2 - \sigma^4\, 2\cancel{(1-\rho)} / \cancel{(1-\rho)}(1+\rho+2\sigma^2)$

$= \sigma^2 - \frac{2\sigma^4}{1+\rho+2\sigma^2} = \frac{(1+\rho)\sigma^2}{1+\rho+2\sigma^2}$



$\rho = 1$. Error covariance $= \frac{\sigma^2}{1+\sigma^2}$, $\hat{x} = \frac{\sigma^2}{2+2\sigma^2}(y_1+y_2)$

In this case $\mathrm{cov}\, Q = 1$, so $n_1 = n_2$. Since the two measurements are just a single measurement we expect $J^*(\rho=1)$ to be high.

$\rho = 0$. Error covariance $= \frac{\sigma^2}{1+2\sigma^2}$, $\hat{x} = \frac{1}{2}(y_1+y_2)$

The two measurements are not the same so $J^*(\rho=0)$ is smaller.

$\rho = -1$. In this case $n_1 = -n_2$, so $y_1 + y_2 = x + n_1 + x - n_1 = 2x$

i.e. $x = \frac{1}{2}(y_1+y_2)$. Since we can exactly reconstruct

3 (i) From Bayes rule, $p(x_k \mid y_{1:k}) = p(x_k \mid y_k, y_{1:k-1})$

$$= \frac{p(y_k \mid x_k, y_{1:k-1}) \times p(x_k \mid y_{1:k-1})}{p(y_k \mid y_{1:k-1})} \qquad (\dagger)$$

But $p(y_k \mid x_k, y_{1:k-1}) = p(y_k \mid x_k)$ (by the Markov properties of (x_k))

Also $c = p(y_k \mid y_{1:k-1})$ is a number that depends only on $y_{1:k}$

So $p(x_k \mid y_{1:k}) = \frac{1}{c(y_{1:k})} \times p(y_k \mid x_k) \times p(x_k \mid y_{1:k-1})$

For $x_k = F x_{k-1}$ and $y_k = H x_k + (c^{\frac{1}{2}})^k v_k$,

$p(x_k \mid y_{1:k}) = N(\hat{x}_k, P_k)(x_k)$,

$p(y_k \mid x_k) = N(H x_k, Q_k)(y_k)$.

Inserting into $(\dagger)$ yields

$$\frac{1}{c'} \exp\left\{-\frac{1}{2}(x_k - \hat{x}_k)^T P_k^{-1}(x_k - \hat{x}_k)\right\} = \frac{1}{c''} \exp\left\{-\frac{1}{2}(y_k - Hx_k) Q_k^{-1} (y_k - Hx_k) - \frac{1}{2}(x_k - \hat{x}_{k-1}) P_{k|k-1}^{-1} (x_k - \hat{x}_{k-1})\right\}$$

Equating quadratic terms in x_k in the exponents of the two sides of the equation gives

$$-\frac{1}{2} x_k^T P_k^{-1} x_k + (\ldots) = -\frac{1}{2}\left[x_k^T H^T Q_k^{-1} H x_k + x_k^T P_{k|k-1}^{-1} x_k\right]$$

$$\Rightarrow \quad P_k^{-1} = H^T Q_k^{-1} H + P_{k|k-1}^{-1}$$

(ii) Write $x_k = x$, $k = 1, 2$. Then $x_k = x_{k-1}$ $(= x)$, $y_k = x_k + v_k$

We have $P_k = \mathrm{cov}\{x \mid y_{1:k}\} = \mathrm{cov}\{x_k \mid y_{1:k}\}$

and $P_{k|k-1} = \mathrm{cov}\{x \mid y_{1:k}\} = \mathrm{cov}\{x_k \mid y_{1:k-1}\} = P_{k-1}$

From (i)

$$P_k^{-1} = \left(\frac{1}{c}\right)^{k-1} + P_{k-1}^{-1}.$$

This implies $P_k^{-1} = P_0^{-1} + \left(1 + \left(\frac{1}{c}\right) + \ldots + \left(\frac{1}{c}\right)^{k-1}\right)$

In the limit as $k \to \infty$,

$$P_\infty^{-1} = \lim_{k\to\infty} P_k^{-1} = P_0^{-1} + \frac{1}{(1 - 1/c)}$$

Hence $$P_\infty = \frac{(1 - 1/c)\, P_0}{P_0 + (1 - \frac{1}{c})} = \frac{P_0}{1 + P_0 \times \left(\frac{c}{c-1}\right)}$$

4 (i) We know $P_k = P_{k|k-1} - P_{k|k-1} H^T [H P_{k|k-1} H^T + Q^o]^{-1} H P_{k|k-1}$

$K_k = P_{k|k-1} H^T [\ldots]$ and $P_{k|k-1} = F P_{k-1} F^T + Q^s$

To get asymptotic values, set $P_k = P_{k-1} = P$, $K_k = K$, $P_{k|k-1} = S$

Then $S = FPF^T + Q^s$ and $P = S - SH^T[HSH^T + Q^o]^{-1} HS$

$$\left.\begin{array}{l} \Rightarrow \quad S = FSF^T - FSH^T[HSH^T + Q^o]^{-1} HSF^T + Q^s \\ \text{also} \quad K = SH[HSH^T + Q^o]^{-1} \\ \text{and} \quad \hat{x}_k = F\hat{x}_{k-1} + u^s + K[y - H(F\hat{x}_{k-1} + u^s) + u^o] \end{array}\right\}$$

The equations have a unique solution (with P a covariance matrix) (P, S, K) if (F, H) is an observable pair

If $Q^s = 0$, then $(P, S, K) = (0,0,0)$ is a solution to the equations ($P=0$ is a cov. matrix). By uniqueness

$$P_k = \operatorname{cov}\{x_k - \hat{x}_k\} \to 0.$$

(ii) We treat $x = a$ as a state variable. Then

$$\begin{cases} x_{k+1} = x_k & (x_0 \sim N(\hat{x}_0, P_0)) \\ y_k = \underbrace{y_{k-1}}_{h} \cdot x_k + \overbrace{\tfrac{1}{2} y_{k-1}}^{u^o} + e_k \end{cases}$$

The Kalman filter, at time k, yields the conditional mean and covariance of x_k given y_k, taking as prior $p(x_k \mid y_{1:k-1})$. Because the measurement noise at time k, e_k, is independent of $\{x_0, e_1, \ldots, e_{k-1}\}$ and therefore $y_{1:k-1}$, $y_{1:k-1}$ can be treated as deterministic, in which case the Kalman filter still yields the conditional mean.

The Kalman filter equations give:

$$P_{k|k-1} = P_{k-1}, \text{ so } K_k = P_{k-1} y_{k-1} / (y_{k-1}^2 P_{k-1} + 1)$$

$$P_k = P_{k-1} - \frac{P_{k-1}^2 y_{k-1}^2}{P_{k-1} y_{k-1}^2 + 1}$$

and so $\hat{a}_k = \hat{a}_{k-1} + \underbrace{P_{k-1} y_{k-1} \cdot}_{K_k} (y_k - \hat{a}\, y_{k-1} - \tfrac{1}{2} y_{k-1})$

5 (i) Write $x_k = \begin{pmatrix} z_k \\ v_k \end{pmatrix}$. Assume $p(x_{k-1} \mid y_{1:k-1}) = N(\hat{x}_{k-1}, P_{k-1})$

The system and measurement processes are of the form

$$(S) \quad \begin{cases} x_k = f(x_{k-1}) + \tilde{e}_k, & \text{where } \operatorname{cov}\{\tilde{e}\} = \sigma_s^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\ y_k = h(x_k) + \omega_k, & \operatorname{var}\{\omega_k\} = \sigma_m^2. \end{cases}$$

The construction of the Extended Kalman filter is based on approximating $f(\cdot)$ and $h(\cdot)$ by linearizing them about $\hat{x}_{k-1}$ and $f(\hat{x}_{k-1})$, assumed to be good estimates of x_{k-1}, x_k:

$$f(x) \simeq f(\hat{x}_{k-1}) + \nabla f(\hat{x}_{k-1}) \cdot (x - \hat{x}_{k-1})$$

$$h(x) \simeq h(f(\hat{x}_{k-1})) + \nabla h(f(\hat{x}_{k-1}))(x_k - f(\hat{x}_{k-1}))$$

(S) then becomes a linear system. The standard KF eqns give

$$\hat{x}_k = f(\hat{x}_{k-1}) + K_k (y_k - h(f(\hat{x}_{k+1})))$$

in which K_k, $P_{k|k-1}$ (the predicted error covariance) and P_k (the error covariance) are given by

$$P_{k|k-1} = \nabla f P_{k-1} (\nabla f)^T + Q_s, \quad K_k = P_{k|k-1} \nabla h^T \left[\nabla h^T P_{k|k-1} \nabla h + \sigma_m^2 \right]^{-1}$$

and $P_k = P_{k-1} - K_k \nabla h P_{k|k-1}$. Here $Q = \sigma_s^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

$$\nabla f(\hat{x}_{k-1}) = \underbrace{\begin{bmatrix} 1 & T \\ 0 & 1 - 3\hat{v}_{k-1}^2 \end{bmatrix}}$$

and

$$\nabla h(f(\hat{x}_{k-1})) = \underbrace{\operatorname{sign}\{\hat{z}_{k-1} - T\hat{v}_{k-1}\} e^{-(\hat{z}_{k-1} - T\hat{v}_{k-1})}}$$

where $\begin{bmatrix} \hat{z}_{k-1} \\ \hat{v}_{k-1} \end{bmatrix} = \hat{x}_{k-1}$.

(ii) (H0) $y \sim N(1, \sigma^2)$ $\quad (\sigma = 0.3)$

(H1) $y \sim N(2, \sigma^2)$

likelihood ratio $= \exp\{-\frac{1}{2\sigma^2}|y-1|^2 + \frac{1}{2\sigma^2}|y-2|^2\}$

log likelihood $= \frac{1}{2\sigma^2}\left[|y-2|^2 - |y-1|^2\right] = \frac{1}{2\sigma^2}(-2y + 3)$

N-P decision function $\delta(y) = \begin{cases} 1 & y < c \\ 0 & y \geq c \end{cases}$

where c is chosen so that $P[y < c \mid y \sim N(1, \sigma^2)] = \alpha$

But $P[\ldots] = P\left[\frac{y-1}{\sigma} < \frac{c-1}{\sigma} \,\middle|\, \frac{y-1}{\sigma} \sim N(0,1)\right]$

so $\int_{-\infty}^{(c-1)/\sigma} N(0,1)(s)\, ds = \alpha$ or $\underbrace{\operatorname{erf}\left(\frac{c-1}{\sigma}\right) = \alpha}$

Power:

$$P[y \geq c \mid y \sim N(2, \sigma^2)] = P\left[\frac{y-2}{\sigma} \geq \frac{c-2}{\sigma} \,\middle|\, \frac{y-2}{\sigma} \sim N(0,1)\right]$$

$$= \int_{(c-2)/\sigma}^{\infty} N(0,1)(s)\, ds = \underbrace{1 - \operatorname{erf}\left(\frac{c-2}{\sigma}\right)}_{\text{power of test}} \quad (\sigma = 0.3)$$

Examination Paper Submission document for 2012-2013 academic year.

For this exam, please write the main course code and the course title below.

Code:

Title:

We, the exam setter and the second marker, confirm that the following points have been discussed and agreed between us.

1. There is no full or partial reuse of questions.
2. This examination yields an appropriate range of marks that is well balanced, reflecting the quality of student (with weak students failing, capable students getting at least 40% and bright industrious students obtaining more than 70%)
3. The model answers give a fair indication of the amount of work needed to answer the questions. Each part has a comment indicating to the external examiners the nature of the question; i.e. whether it is bookwork, new theory, a new theoretical application, a calculation for a new example, etc.
4. The exam paper does not contain any grammar and spelling mistakes.
5. The marking schedule is shown in the answers document and the resolution of each allocated mark is better than 3/20 for each question.
6. The examination paper can be completed by the students within time allowed.

Signed (Setter):

Date: 8-2-2013

Signed (Second Marker):

Date: 8-2-2013

Please submit this form with exam paper and model answers, and associated coursework to the Undergraduate Office on Level 6 by the required submission date.