EEE - stream (E1.10)

# UNIVERSITY OF LONDON

[I(1) 2002]

## B.ENG. AND M.ENG. EXAMINATIONS 2002

For Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examination for the Associateship.

**PART I : MATHEMATICS 1**

**Wednesday 29th May 2002 10.00 am - 1.00 pm**

*Answer EIGHT questions.*

[*Before starting, please make sure that the paper is complete; there should be 6 pages, with a total of 10 questions. Ask the invigilator for a replacement if your copy is faulty.*]

1. (i) Classify each of the following functions as even, odd or neither:

$$\cos(x^3)\,; \qquad e^x\,; \qquad e^{|x|}\,; \qquad x^2 \sin x\,; \qquad \cos^2 x - \sin^2 x\,.$$

(ii) For each of the five functions in (i), state whether or not it is periodic. For any case in which the function is periodic, sketch its graph and find its (smallest positive) period.

(iii) If $f(x) = \dfrac{x+2}{x-3}$, find the inverse function $f^{-1}$.

Find also $f(f(x))$.

What is $f^{-1}(f(x))$?

2. Express the function

$$f(x) = \frac{x^2 + 3x + 1}{x+3}$$

in the form

$$Ax + \frac{B}{x+3},$$

where $A$ and $B$ are constants.

Find the stationary points of $f(x)$ and, by evaluating the second derivative $f''(x)$ or otherwise, determine which of these are maxima or minima.

Sketch the graph of $f(x)$, indicating clearly all the asymptotes and stationary points.

**PLEASE TURN OVER**

3. (i) Find $\dfrac{dy}{dx}$ as a function of $x$ in cases (a)-(c), and as a function of $x$ and $y$ in case (d).

(a) $$y = \ln(\cos x)\,.$$

(b) $$y = \frac{xe^{2x}}{1+x^2}\,.$$

(c) $$y = (\ln x)^x\,.$$

(d) $$y^2 = \sin(xy)\,.$$

(ii) If $x(t) = 1 - \cos t$ and $y(t) = t - \sin t$, show that

$$\frac{dy}{dx} = \tan\left(\frac{t}{2}\right).$$

4. (i) Find a general formula for $\dfrac{d^n y}{dx^n}$ in each of the following cases:

(a) $y = e^{2x}$; (b) $y = \ln(1-x)$; (c) $y = x^2 e^{2x}$ .

(ii) A cylindrical container with closed ends is designed to have height $h = 3\,\text{cm}$. and radius $r = 1\,\text{cm}$. Due to a manufacturing error the radius becomes 1.04 cm., though the height is correct. By considering the total external surface area, including ends, as a function $A(r)$ of $r$, and using the formula

$$\frac{dA}{dr} = \lim_{\delta r \to 0} \frac{A(r+\delta r) - A(r)}{\delta r}\,,$$

find the approximate increase in surface area as a proportion of the designed surface area.

5. Evaluate the following limits:

(i) $$\lim_{x \to \pi/4} \frac{\cos 2x}{\tan x - 1} \, ;$$

(ii) $$\lim_{x \to \pi/4} \frac{\cos 2x}{\tan(\sqrt{x}) - 1} \, ;$$

(iii) $$\lim_{x \to 2} \frac{\sqrt{(x+2)} - 2}{\sqrt{(x^3 - 4)} - 2} \, ;$$

(iv) $$\lim_{x \to \infty} \left( \frac{x+3}{x} \right)^x .$$

6. (i) Evaluate the following integrals :

(a) $$\int_1^2 \frac{x^2}{1+x^3} \, dx \, ;$$

(b) $$\int_0^1 x \tan^{-1} x \, dx.$$

(ii) Use partial fractions to evaluate

$$\int \frac{dx}{(x-1)^2 (x+3)} .$$

7. (i) Use partial fractions to find the following integral:

$$\int \frac{2x^2 + 6x + 10}{x^2 + 2x + 2}\, dx \; .$$

(ii) Let

$$I_m \;=\; \int_0^{\pi} \cos^m x \, dx, \qquad m = 0,\, 1,\, 2,\, \ldots$$

Derive a formula relating $I_m$ and $I_{m-2}$ (for $m \geq 2$).

Use this formula to calculate $I_4$.

8. (i) Obtain the radius of convergence for each of the following series:

(a)
$$\sum_{n=0}^{\infty} 3^n \, x^n \; ;$$

(b)
$$\sum_{n=0}^{\infty} \frac{1}{n!} \, x^{2n} \; .$$

(ii) Expand the function

$$f(x) \;=\; \frac{1}{1-x}$$

about $x = 0$ in a series up to the first four non-zero terms.

Use this expansion to find an approximation for $f(x)$ at the point $x = 0.1$. Using the remainder term, what error bound can you give for the difference between the exact value of $f(x)$ at $x = 0.1$ and the approximation computed from the series expansion?

9. (i) Given the two complex numbers $z_1 = 1+i$, $z_2 = 2-3i$, express the following complex numbers in the standard form $x+iy$:

(a) $z_1 z_2$, (b) $\dfrac{z_1}{z_2}$, (c) $z_1^{10}$.

(ii) Find the six roots of the equation

$$z^6 + 2z^3 + 4 = 0$$

and indicate their positions in the complex plane.

10. (i) State the definition of $\sinh x$ and $\cosh x$ (in terms of the exponential function) and show that $\cosh^2 x - \sinh^2 x = 1$ for all $x$.

(ii) Find all real solutions $x$ of the equation

$$3\sinh x - \cosh x = 1.$$

(iii) Find all complex solutions $z$ of the equation

$$3\sinh z - \cosh z = 1.$$

(iv) Hence, or otherwise, find all real solutions $x$ of the equation

$$3i\sin x - \cos x = 1.$$

**END OF PAPER**

M A T H E M A T I C S   D E P A R T M E N T

# MATHEMATICAL FORMULAE

## 1. VECTOR ALGEBRA

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1, a_2, a_3)$$

Scalar (dot) product: $\mathbf{a}\,.\,\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

Vector (cross) product:

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Scalar triple product:

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a}\,.\,\mathbf{b}\times\mathbf{c} = \mathbf{b}\,.\,\mathbf{c}\times\mathbf{a} = \mathbf{c}\,.\,\mathbf{a}\times\mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product: $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{c}\,.\,\mathbf{a})\mathbf{b} - (\mathbf{b}\,.\,\mathbf{a})\mathbf{c}$

## 2. SERIES

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \ldots \quad (\alpha \text{ arbitrary, } |x| < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n\frac{x^{2n}}{(2n)!} + \ldots,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + \ldots,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^n\frac{x^{n+1}}{(n+1)} + \ldots (-1 < x \le 1)$$



## 3. TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin(a+b) = \sin a\cos b + \cos a \sin b\,;$$

$$\cos(a+b) = \cos a\cos b - \sin a\sin b\,.$$

$$\cos iz = \cosh z\,;\quad \cosh iz = \cos z\,;\quad \sin iz = i\sinh z\,;\quad \sinh iz = i\sin z\,.$$

## 4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f\,D^n g + \binom{n}{1} Df\,D^{n-1}g + \ldots + \binom{n}{r} D^r f\,D^{n-r}g + \ldots + D^n f\,g\,.$$

(b) Taylor's expansion of $f(x)$ about $x = a$:

$$f(a+h) = f(a) + hf'(a) + h^2 f''(a)/2! + \ldots + h^n f^{(n)}(a)/n! + \epsilon_n(h)\,,$$

where $\epsilon_n(h) = h^{n+1}f^{(n+1)}(a+\theta h)/(n+1)!$, $0 < \theta < 1$.

(c) Taylor's expansion of $f(x, y)$ about $(a, b)$:

$$f(a+h, b+k) = f(a,b) + [hf_x + kf_y]_{a,b} + 1/2!\left[h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}\right]_{a,b} + \ldots$$

(d) Partial differentiation of $f(x, y)$:

i. If $y = y(x)$, then $f = F(x)$, and $\dfrac{dF}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dx}$.

ii. If $x = x(t)$, $y = y(t)$, then $f = F(t)$, and $\dfrac{dF}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$.

iii. If $x = x(u, v)$, $y = y(u, v)$, then $f = F(u, v)$, and

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}, \quad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$

(e) Stationary points of $f(x, y)$ occur where $f_x = 0$, $f_y = 0$ simultaneously.
Let $(a, b)$ be a stationary point: examine $D = [f_{xx}f_{yy} - (f_{xy})^2]_{a,b}$.
If $D > 0$ and $f_{xx}(a, b) < 0$, then $(a, b)$ is a maximum;
If $D > 0$ and $f_{xx}(a, b) > 0$, then $(a, b)$ is a minimum;
If $D < 0$ then $(a, b)$ is a saddle-point.

(f) Differential equations:

i. The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating factor $I(x) = \exp[\int P(x)(dx]$, so that $\dfrac{d}{dx}(Iy) = IQ$.

ii. $P(x, y)dx + Q(x, y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y$.

## 5. INTEGRAL CALCULUS

(a) An important substitution: $\tan(\theta/2) = t$ :
$\sin\theta = 2t/(1+t^2)$, $\cos\theta = (1-t^2)/(1+t^2)$, $d\theta = 2\,dt/(1+t^2)$.

(b) Some indefinite integrals:

$$\int (a^2 - x^2)^{-1/2} dx = \sin^{-1}\left(\frac{x}{a}\right), \quad |x| < a.$$

$$\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1}\left(\frac{x}{a}\right) = \ln\left\{\frac{x}{a} + \left(1 + \frac{x^2}{a^2}\right)^{1/2}\right\}.$$

$$\int (x^2 - a^2)^{-1/2} dx = \cosh^{-1}\left(\frac{x}{a}\right) = \ln\left|\frac{x}{a} + \left(\frac{x^2}{a^2} - 1\right)^{1/2}\right|.$$

$$\int (a^2 + x^2)^{-1} dx = \left(\frac{1}{a}\right)\tan^{-1}\left(\frac{x}{a}\right).$$

## 6. NUMERICAL METHODS

(a) Approximate solution of an algebraic equation:

If a root of $f(x) = 0$ occurs near $x = a$, take $x_0 = a$ and
$x_{n+1} = x_n - [f(x_n)/f'(x_n)]$, $n = 0, 1, 2 \ldots$

(Newton Raphson method).

(b) Formulae for numerical integration: Write $x_n = x_0 + nh$, $y_n = y(x_n)$.

i. Trapezium rule (1-strip): $\int_{x_0}^{x_1} y(x)dx \approx (h/2)[y_0 + y_1]$.

ii. Simpson's rule (2-strip): $\int_{x_0}^{x_2} y(x)dx \approx (h/3)[y_0 + 4y_1 + y_2]$.

(c) Richardson's extrapolation method: Let $I = \int_a^b f(x)dx$ and let $I_1$, $I_2$ be two estimates of $I$ obtained by using Simpson's rule with intervals $h$ and $h/2$.

Then, provided $h$ is small enough,

$$I_2 + (I_2 - I_1)/15,$$

is a better estimate of $I$.



## 7. LAPLACE TRANSFORMS

| Function | Transform | Function | Transform |
|---|---|---|---|
| $f(t)$ | $F(s) = \int_0^\infty e^{-st} f(t)dt$ | $af(t) + bg(t)$ | $aF(s) + bG(s)$ |
| $df/dt$ | $sF(s) - f(0)$ | $d^2f/dt^2$ | $s^2F(s) - sf(0) - f'(0)$ |
| $e^{at}f(t)$ | $F(s-a)$ | $tf(t)$ | $-dF(s)/ds$ |
| $(\partial/\partial\alpha)f(t,\alpha)$ | $(\partial/\partial\alpha)F(s,\alpha)$ | $\int_0^t f(t)dt$ | $F(s)/s$ |
| $\int_0^t f(u)g(t-u)du$ | $F(s)G(s)$ | | |
| $1$ | $1/s$ | $t^n (n = 1, 2 \ldots)$ | $n!/s^{n+1}$, $(s > 0)$ |
| $e^{at}$ | $1/(s-a)$, $(s > a)$ | $\sin\omega t$ | $\omega/(s^2+\omega^2)$, $(s > 0)$ |
| $\cos\omega t$ | $s/(s^2+\omega^2)$, $(s > 0)$ | $H(t-T) = \begin{cases} 0, & t < T \\ 1, & t > T \end{cases}$ | $e^{-sT}/s$, $(s, T > 0)$ |

## 8. FOURIER SERIES

If $f(x)$ is periodic of period $2L$, then $f(x+2L) = f(x)$, and

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \text{ where}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad n = 0, 1, 2, \ldots, \text{ and}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1, 2, 3, \ldots$$

Parseval's theorem

$$\frac{1}{L}\int_{-L}^{L} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right).$$

September 2000

Solutions - Part I - Maths I - 2002

MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

SESSION : 2001-2002

(EE Stream - paper E1.10)

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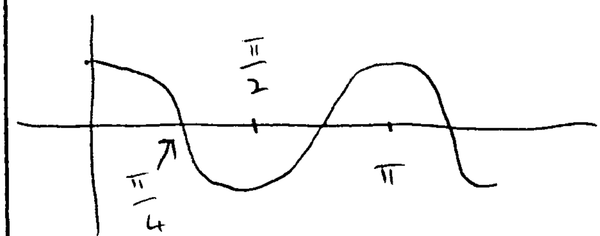
PAPER I

QUESTION

SOLUTION 1

(i) $\cos(x^3)$ is even; $e^x$ is neither even nor odd; $e^{|x|}$ is even; $x^2 \sin x$ is odd; $\cos^2 x - \sin^2 x$ is even. — 1 each ($\Sigma = 5$)

(ii) Of these 5 functions only the last one $\cos^2 x - \sin^2 x = \cos 2x$ is periodic. — 2




The period is $\pi$. — 2

(iii) $y = \frac{x+2}{x-3} \iff xy - 3y = x + 2$
$\iff x = \frac{3y+2}{y-1}$. Thus $f^{-1}(x) = \frac{3x+2}{x-1}$, — 3

$$f(f(x)) = \frac{\frac{x+2}{x-3} + 2}{\frac{x+2}{x-3} - 3} = \frac{3x-4}{-2x+11};$$ — 2

and $f^{-1}(f(x)) = x$. — 1

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PAPER I(1)

QUESTION

SOLUTION 2

- long division

$$\frac{x^2+3x+1}{x+3} = x + \frac{1}{x+3} \Rightarrow A = B = 1$$ 2

- $f'(x) = 1 - \frac{1}{(x+3)^2}$, stationary points. 2

$$0 = 1 - \frac{1}{(x+3)^2} = (x+3)^2 - 1 = x^2+6x+8 = (x+2)(x+4)$$ 3

↳ stationary points at $x_1 = -2$, $x_2 = -4$

- $f''(x) = \frac{2}{(x+3)^3}$, $f''(-2) > 0 \Rightarrow$ minima

$f''(-4) < 0 \Rightarrow$ maxima 3

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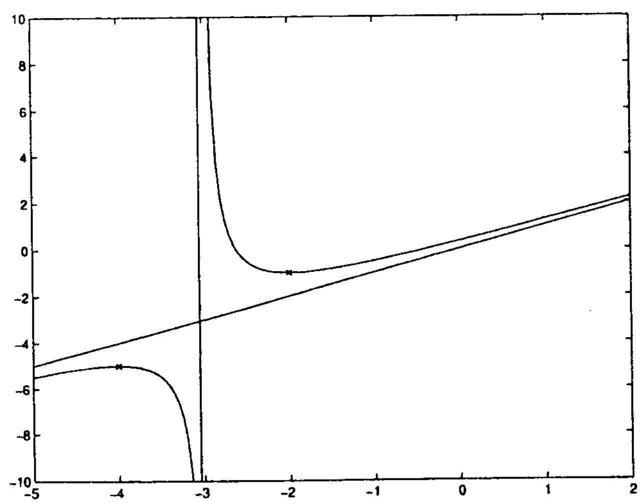


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(i)

(a) $$y' = -\frac{\sin x}{\cos x} = -\tan x$$ 2

(b) $$y' = \frac{\left(e^{2x} + 2x e^{2x}\right)\left(1+x^2\right) - 2x\, x\, e^{2x}}{\left(1+x^2\right)^2}$$

$$= \frac{\left(1 + 2x - x^2 + 2x^3\right) e^{2x}}{\left(1+x^2\right)^2}$$ 3

(c) $$\ln y = x \ln(\ln x)$$ 1

$$y' y^{-1} = \ln(\ln x) + x\, x^{-1} (\ln x)^{-1}$$ 1

$$y' = (\ln x)^x \ln(\ln x) + (\ln x)^{x-1}$$ 1

(d) $$2 y y' = (y + x y') \cos(xy)$$ 2

$$y' = \frac{y \cos(xy)}{2y - x\cos(xy)}$$ 1

(ii) $$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{1-\cos t}{\sin t}$$

$$= \frac{2\sin^2\left(\frac{t}{2}\right)}{2\sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right)} = \tan\left(\frac{t}{2}\right).$$ 4

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PAPER I.1

QUESTION

SOLUTION 4

(i) (a) $y' = 2e^{2x}$, $y'' = 2^2 e^{2x}$, ..., $y^{(n)} = 2^n e^{2x}$. 2

(b) $y' = -(1-x)^{-1}$, $y'' = -(1-x)^{-2}$, $y''' = -2(1-x)^{-3}$,

..., $y^{(n)} = -(n-1)!\,(1-x)^{-n}$. 3

(c) Using Leibniz's formula (Formula Sheet) with $f = x^2$, $g = e^{2x}$,

$$\frac{d^n}{dx^n}\left(x^2 e^{2x}\right) = f g^{(n)} + n f' g^{(n-1)} + \frac{n(n-1)}{2!} f'' g^{(n-2)} + \dots$$

$$= x^2 2^n e^{2x} + n\,2x\,2^{n-1} e^{2x} + \frac{n(n-1)}{2} 2.2^{n-2} e^{2x}$$

$$= \left(2^n x^2 + n 2^n x + n(n-1) 2^{n-2}\right) e^{2x}.$$ 3

(ii) $A = 2\pi r^2 + 2\pi r.3 = 2\pi(r^2+3r)$ 3

$\delta A = A(r+\delta r) - A(r) \simeq \frac{dA}{dr}\delta r$

$= 2\pi(2r+3)\delta r$ 1

$\therefore \frac{\delta A}{A} \simeq \frac{2r+3}{r^2+3r}\delta r$ 2

$= \frac{5}{4} \times 0.04 = 0.05.$ 1

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MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

SESSION : 2001-2002

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PAPER I

QUESTION

SOLUTION 5

(i) Denominator and numerator are 0 when $x = \frac{\pi}{4}$,

so it $= \lim\limits_{x \to \pi/4} \dfrac{-2\sin 2x}{\sec^2 x} = \dfrac{-2}{(2/\sqrt{2})^2} = -1$ 4

(ii) When $x = \pi/4$, numerator is 0 and denominator is not, so the limit $= 0$. 3

(iii) As in (i), l'Hôpital's rule applies,

so the limit $= \lim\limits_{x \to 2} \dfrac{\frac{1}{2}(x+2)^{-\frac{1}{2}}}{\frac{1}{2}(x^3-4)^{-\frac{1}{2}} \cdot 3x^2}$ 4

$= \dfrac{1}{12}$

(iv) Let $y = \left(\dfrac{x+3}{x}\right)^x$, so that

$\ln y = x \ln\left(1 + \frac{3}{x}\right)$. For $x$ large,

this $= x\left(\frac{3}{x} - \frac{9}{2x^2} + \dots\right) = 3 - \frac{9}{2x} + \dots$ 4

So $\lim\limits_{x\to\infty} \ln y = 3$, hence $\lim\limits_{x \to \infty} y = e^3$

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MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

SESSION : 2001-2002

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PAPER I

QUESTION

SOLUTION 6

(i)(a) $\int_1^2 \frac{x^2}{1+x^3}dx = \frac{1}{3}\int_1^2 \frac{3x^2}{1+x^3}dx = \frac{1}{3}\ln(1+x^3)\Big|_1^2 =$ 3

$= \frac{1}{3}[\ln 9 - \ln 2]$

(b) $\int_0^1 x\tan^{-1}x\,dx = \frac{x^2}{2}\tan^{-1}x\Big|_0^1 - \frac{1}{2}\int_0^1 \frac{x^2}{1+x^2}dx$

$= \frac{1}{2}\tan^{-1}1 - \frac{1}{2}\int_0^1\left(1-\frac{1}{1+x^2}\right)dx$ 5

$= \frac{1}{2}\tan^{-1}x - \frac{1}{2}\left[x - \tan^{-1}x\right]_0^1 = \tan^{-1}1 - \frac{1}{2} = \frac{\pi}{4} - \frac{1}{2}.$

(ii) $\frac{1}{(x-1)^2(x+3)} = \frac{A}{x+3} + \frac{B}{(x-1)^2} + \frac{C}{x-1}$

$A = \frac{1}{(x-1)^2}\Big|_{x=-3} = \frac{1}{16}, \quad B = \frac{1}{x+3}\Big|_{x=1} = \frac{1}{4},$

$C = \frac{d}{dx}\frac{1}{(x+3)}\Big|_{x=1} = -\frac{1}{(x+3)^2}\Big|_{x=1} = -\frac{1}{16}$ 4

Thus

$\int \frac{dx}{(x-1)^2(x+3)} = \frac{1}{16}\ln|x+3| - \frac{1}{16}\ln|x-1| - \frac{1}{4(x-1)} + \text{const}$ 3

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(i)

$$\frac{2x^2+6x+10}{x^2+2x+2} = \frac{2x^2+4x+4}{x^2+2x+2} + \frac{2x+2}{x^2+2x+2} + \frac{4}{x^2+2x+2}$$

$$= 2 + \frac{2x+2}{x^2+2x+2} + \frac{4}{1+(x+1)^2}$$ 5

Thus

$$\int \frac{2x^2+6x+0}{x^2+2x+2}dx = 2x + \ln(x^2+2x+2) + 4\tan^{-1}(x+1) + c.$$ 3

(ii) $$\int (\cos x)^n dx = \int \cos x\,(\cos x)^{n-1} dx$$

$$= \sin x\,(\cos x)^{n-1} + \cancel{(n-1)\int (\sin x)^2 (\cos x)^{n-2} dx}$$

$$(n-1)\int (\cos x)^{n-2} dx - (n-1)\int (\cos x)^n dx$$

Thus

$$n\int (\cos x)^n dx = \sin x\,(\cos x)^{n-1} + (n-1)\int (\cos x)^{n-2} dx,$$

or $$I_n = \frac{n-1}{n} I_{n-2}.$$ 5

$$I_4 = \frac{4-1}{4} I_2 = \frac{3}{4}\,\frac{2-1}{2} I_0 = \frac{3}{4}\,\frac{1}{2}\int_0^{\pi} dx = \frac{3\pi}{8}.$$ 2

15

PAPER I(1)

QUESTION

SOLUTION 8

(i) ratio test

$$\sum_{n=0}^{\infty} b_n \quad , \quad \lim_{n\to\infty} \left| \frac{b_{n+1}}{b_n} \right| = \ell$$ 2

a) $\ell = \frac{|x|}{R}$ , $b_n = 3^n x^n$

$$\lim_{n\to\infty} \left| \frac{3^{n+1} x^{n+1}}{3^n x^n} \right| = 3|x| = \frac{|x|}{R} \Rightarrow R = \frac{1}{3}$$ 4

converges for $|x| < \frac{1}{3}$

b) $\ell = \frac{|x^2|}{R}$ , $b_n = \frac{1}{n!} x^{2n}$

$$\lim_{n\to\infty} \left| \frac{1/(n+1)!\; x^{2n+2}}{1/n!\; x^{2n}} \right| = \lim_{n\to\infty} \frac{1}{n+1} |x^2| = \frac{|x^2|}{R} \Rightarrow$$ 4

$$R = \infty$$

converges for all $|x| < \infty$

(ii) • $f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}} \Rightarrow$ 3

$f(x) \approx 1 + x + x^2 + x^3$ , $f(0.1) \approx 1.111$

$$E_4(0.1) = \frac{1}{(1-\bar{x})^5} 0.1^4 \le 0.0001 \cdot 1.7$$

$$\le 0.00017$$ 2

$$\uparrow$$

$$0 \le \bar{x} \le 0.1$$

Setter : S. REICH

Setter's signature : S. Reich

Checker : M CHARALAMBIDES

Checker's signature :

PAPER I

QUESTION

SOLUTION 9

(i) (a) $z_1 z_2 = (1+i)(2-3i) = 2-3i+2i-3i^2 = \underline{\underline{5-i}}$ 1

(b) $\dfrac{z_1}{z_2} = \dfrac{1+i}{2-3i} = \dfrac{(1+i)(2+3i)}{2^2+3^2} = \dfrac{2+3i+2i+3i^2}{13}$

$= \underline{\underline{-\dfrac{1}{13}+\dfrac{5i}{13}}}$ 2

(c) $z_1 = \sqrt{2}\,e^{i\frac{\pi}{4}} \rightarrow z_1^{10} = (\sqrt{2})^{10} e^{i\frac{10\pi}{4}}$

$= 2^5 e^{i\frac{\pi}{2}} = \underline{\underline{32i}}$ 4

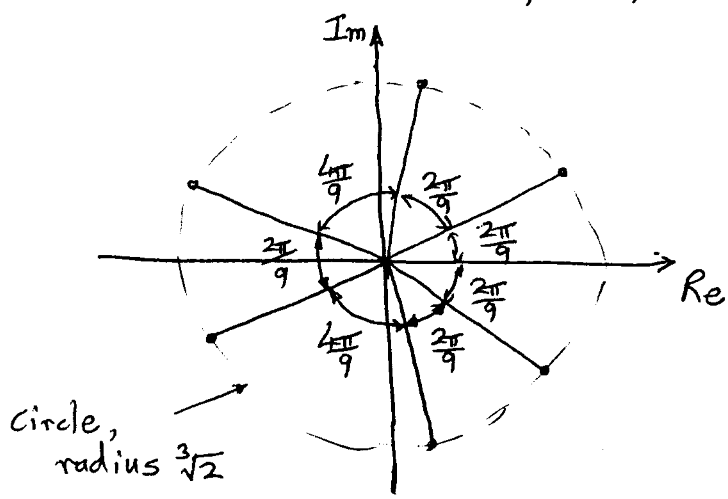
(ii) $z^3 = -1 \pm \sqrt{1^2-4} = -1 \pm \sqrt{3}\,i$ 1

$= 2e^{i\theta}$ where $\theta = \dfrac{2\pi}{3}, \dfrac{4\pi}{3}$ 3

$z = \sqrt[3]{2}\,e^{i\phi}$, where $\phi = \dfrac{2\pi}{9}, \dfrac{2\pi}{9}+\dfrac{2\pi}{3}, \dfrac{2\pi}{9}+\dfrac{4\pi}{3}, \dfrac{4\pi}{9},$

$\dfrac{4\pi}{9}+\dfrac{2\pi}{3}, \dfrac{4\pi}{9}+\dfrac{4\pi}{3}$

$\phi = \dfrac{2\pi}{9}, \dfrac{8\pi}{9}, \dfrac{14\pi}{9}, \dfrac{4\pi}{9}, \dfrac{10\pi}{9}, \dfrac{16\pi}{9}$ 2

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MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

SESSION : 2001-2002

Please write on this side only, legibly and neatly, between the margins

PAPER I

QUESTION

SOLUTION 10

(i) $\sinh x = \frac{1}{2}(e^x - e^{-x})$, $\cosh x = \frac{1}{2}(e^x + e^{-x})$ 1

So $\cosh^2 x - \sinh^2 x = \frac{1}{4}(e^{2x} + e^{-2x} + 2)$

$-\frac{1}{4}(e^{2x} + e^{-2x} - 2) = \frac{1}{2} + \frac{1}{2} = 1$ 3

(ii) Equation is $\frac{3}{2}(e^x - e^{-x}) - \frac{1}{2}(e^x + e^{-x}) = 1$

$\iff e^x - 2e^{-x} = 1 \iff (e^x)^2 - e^x - 2 = 0$

$\iff (e^x - 2)(e^x + 1) = 0 \iff e^x = 2$ or $-1$ 5

So the only real solution is $x = \ln 2$.

(iii) As in (ii) the equation is equivalent to $e^z = 2$ or $-1$, with solutions

$z = \ln 2 + 2n\pi i$, $z = (2n+1)\pi i$ 4

(where $n$ can be any integer).

(iv) If $x$ is real so are $\sin x$ and $\cos x$, so the equation is equivalent to $\sin x = 0$, $\cos x = -1$, with solutions 2

$x = (2n+1)\pi$ ($n$ any integer)

Aliter: use (iii).

Setter : WILSON Setter's signature : J. Wilson

Checker : Skorobogatov Checker's signature : [signature]