Final Copy - May 2011

EE4-24
EE9CS1-3

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2011

MSc and EEE/ISE PART IV: MEng and ACGI

DISCRETE-TIME SYSTEMS AND COMPUTER CONTROL

Wednesday, 4 May 2:30 pm

Time allowed: 3:00 hours

There are SIX questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s) :	A. Astolfi
	Second Marker(s) :	E.C. Kerrigan

DTS AND COMPUTER CONTROL

Information for candidates:

- $Z\left(\frac{1}{s}\right) = \frac{z}{z-1} = \frac{1}{1-z^{-1}}$

- $Z\left(\frac{1}{s+a}\right) = \frac{z}{z-e^{-aT}} = \frac{1}{1-z^{-1}e^{-aT}}$

- $Z\left(\frac{1}{s^2}\right) = T\frac{z}{(z-1)^2} = T\frac{z^{-1}}{(1-z^{-1})^2}$

- $Z\left(\frac{1}{s^3}\right) = \frac{T^2}{2}\frac{z(z+1)}{(z-1)^3} = \frac{T^2}{2}\frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3}$

- Transfer function of the ZOH: $H_0(s) = \frac{1-e^{-sT}}{s}$

- Definition of the w-plane: $z = \frac{1+\frac{wT}{2}}{1-\frac{wT}{2}}$, $w = \frac{2}{T}\frac{z-1}{z+1}$

- Note that, for a given signal r, or $r(t)$, $R(z)$ denotes its Z-transform.

1. Consider the digital control system in Figure 1 with K constant.

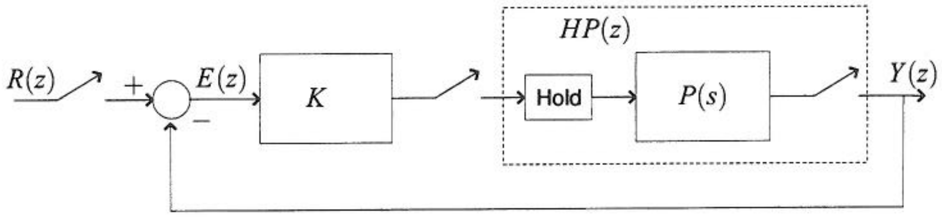

Figure 1: Block diagram for question 1.

Let

$$P(s) = \frac{1}{s(s+1)},$$

and $K > 0$. Assume the hold is a ZOH and let the sampling period be $T > 0$.

a) Compute the equivalent discrete-time model $HP(z)$ for the plant interconnected to the hold and the sampler. [4 marks]

b) Write the closed-loop discrete-time transfer function from the input r to the output y. [4 marks]

c) Show that there exists a function $\kappa(T)$ such that the closed-loop system is asymptotically stable for all $K \in (0, \kappa(T))$. Show that

$$\lim_{T \to 0} \kappa(T) = \infty.$$

[6 marks]

d) Assume that r is a constant. Determine the steady-state values of $e(kT)$. Explain why this value does not depend upon K. [2 marks]

e) Assume that r is a unity ramp. Determine the steady-state values of $e(kT)$. Compare this result with the result of part d). [4 marks]

2. Consider the digital control system in Figure 2.

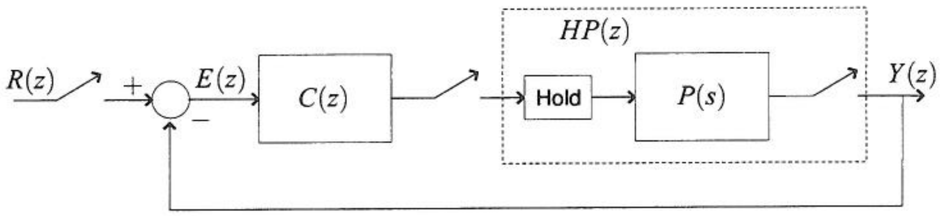

Figure 2: Block diagram for question 2.

Let

$$P(s) = \frac{K}{s}\,\frac{1}{\tau s+1},$$

with $\tau \geq 0$. The term $\frac{1}{\tau s+1}$ describes unmodelled dynamics. Assume the hold is a ZOH and let the sampling period be $T = 1$ sec.

a) Compute the equivalent discrete-time model $HP(z)$ for the plant interconnected to the hold and the sampler. [4 marks]

b) Assume $\tau = 0$.

i) Study the stability properties of the closed-loop system as a function of K. [2 marks]

ii) Determine the value of K such that the closed-loop transfer function has all poles at $z = 0$. [2 marks]

c) Assume $\tau > 0$ and let K be as in part b.ii).

i) Study the stability properties of the closed-loop system as a function of τ. [4 marks]

ii) Design a controller

$$C(z) = \alpha\,\frac{z - e^{-1/\tau}}{z+\beta}$$

such that the closed-loop system has all poles at $z = 0$. [4 marks]

iii) Consider the controller designed in part c.ii) in closed-loop with the discrete-time model of the system for $\tau = 0$. Study the stability properties of the closed-loop system as a function of $\tau \geq 0$. [2 marks]

d) Discuss the robustness, to variations in τ, of the controllers designed in part b.ii) and in part c.ii). [2 marks]

3. The transfer function of a simple mechanical system is given by

$$P(s) = \frac{1}{s^2}.$$

Assume the system is interconnected to a ZOH and a sampler. Let $T = 1$ sec be the sampling time.

a) Compute the equivalent discrete-time model $HP(z)$ for the plant interconnected to the hold and the sampler. [4 marks]

b) Using the definition of the w-plane, determine the transfer function $HP(w)$. [2 marks]

c) Design, in the w-plane, a controller $C(w)$ such that the closed-loop system is asymptotically stable. [8 marks]

d) Compute the transfer function $C(z)$ of the discrete-time controller. Verify if the discrete-time closed-loop system resulting from the use of $C(z)$ is asymptotically stable. [6 marks]

4. Consider a system with transfer function

$$P(s) = \frac{1}{s+1}$$

and the problem of designing a feedback controller such that the closed-loop system is asymptotically stable and the system is of type 1.

a) Show that the PI control law

$$C(s) = 1 + \frac{2}{s}$$

is such that the closed-loop system is asymptotically stable. Compute the poles of the closed-loop system and, using the transformation $z = e^{sT}$, with $T = 1$, determine the location of these poles in the complex z-plane. [4 marks]

b) Discretize the controller in part a) using the backward difference and the Tustin transformations with sampling time $T = 1$. Let $C_B(z)$ and $C_T(z)$ be the resulting controllers. [2 marks]

c) Compute the equivalent discrete-time model $HP(z)$ for the plant interconnected to a ZOH and a sampler. Let $T = 1$ sec be the sampling time. [2 marks]

d) Consider the discrete-time model in part c) in closed-loop with the controller $C_B(z)$ determined in part b). Compute the poles of the closed-loop system. [4 marks]

e) Repeat part d) using the controller $C_T(z)$. [4 marks]

f) Using the results in parts a), d) and e) compare the three designs in terms of locations of the closed-loop poles and performance of the resulting closed-loop systems. [4 marks]

5. Consider a feedback system with open-loop transfer function

$$HP(z) = k\frac{\frac{z}{10} + \frac{11}{125}}{z^2 - \frac{143}{100}z + \frac{3}{5}}$$

with $k > 0$.

a) Let $k = 1$. Design a controller $C(z)$ such that the closed-loop system transfer function

$$\frac{C(z)HP(z)}{1+C(z)HP(z)}$$

has only two poles at $z = 0$. [10 marks]

b) To assess the robustness of the design in the presence of variations in the gain k, consider the feedback interconnection of the discrete-time model determined in part a) with the controller determined in part b). Study the stability of the resulting closed-loop system as a function of $k > 0$. [6 marks]

c) Discuss briefly the results in part b). [4 marks]

6. Consider a continuous-time system described by the transfer function

$$P(s) = e^{-s}\frac{1}{s-1}.$$

a) Assume the system is connected to a ZOH and a sampler. Let $T = 1$ sec be the sampling period. Determine the discrete-time equivalent transfer function $HP(z)$. [4 marks]

b) Design a digital controller $C(z)$ such that the closed-loop system is asymptotically stable. [6 marks]

c) Suppose now that the continuous-time system is connected to a ZOH and a sampler with $T = 1/2$ sec. Let the controller be as in part b).

i) Determine the discrete-time equivalent transfer function $HP(z)$. [2 marks]

ii) Compute the characteristic polynomial of the resulting closed-loop system. [4 marks]

iii) Study the stability of the polynomial determined in part c.ii). [4 marks]

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DTS and Computer Control

Model answers 2011

Question 1

a) Note that

$$\begin{aligned} HP(z) &= (1-z^{-1})Z\left(\frac{1}{s^2(s+1)}\right) \\ &= (1-z^{-1})Z\left(\frac{1}{s+1}-\frac{1}{s}+\frac{1}{s^2}\right) \\ &= \frac{z(T-1+e^{-T})+(1-e^{-T}-Te^{-T})}{(z-1)(z-e^{-T})} \end{aligned}$$

b) The closed-loop transfer function from r to y is

$$\begin{aligned} W_{ry}(z) &= \frac{KHP(z)}{1+KHP(z)} \\ &= K\frac{z(T-1+e^{-T})+(1-e^{-T}-Te^{-T})}{z^2+z(-K-1+KT+e^{-T}K-e^{-T})+(K+e^{-T}-e^{-T}KT-e^{-T}K)}. \end{aligned}$$

c) The closed-loop system is asymptotically stable if the roots of the denominator polynomial of $W_{ry}(z)$ are inside the unity disk. To test this condition let

$$a=-K-1+KT+e^{-T}K-e^{-T} \qquad b=K+e^{-T}-e^{-T}KT-e^{-T}K$$

and recall that the roots of the polynomial z^2+az+b are inside the unity disk if and only if

$$1+a+b>0 \qquad 1-a+b>0 \qquad |b|<1.$$

These conditions, for the considered polynomial, are

$$KT\overbrace{(1-e^{-T})}^{>0}>0, \qquad 2+2e^{-T}-\overbrace{(2e^{-T}-2+T+Te^{-T})}^{>0}K>0,$$

$$-1<e^{-T}+\overbrace{(1-e^{-T}-Te^{-T})}^{>0}K<1.$$

As a result K should be such that

$$0<K<\min\left\{\frac{1-e^{-T}}{1-e^{-T}-Te^{-T}},\frac{2+2e^{-T}}{2e^{-T}-2+T+Te^{-T}}\right\}=\kappa(T).$$

Note that both functions inside the min go to infinity as T goes to zero.

d) The steady-state value of $e(kT)$ is zero, since the system is of type 1, that is the open loop transfer function has a pole at $z=1$.

e) The velocity constant is $k_v=\lim_{z\to 1}\frac{(1-z^{-1})KHP(z)}{T}=K$, hence the steady state error is

$$e(kT)=\frac{1}{K}<\frac{1}{\kappa(T)}.$$

Note that the velocity error cannot be reduced arbitrarily, since K is upperbounded by $\kappa(T)$ for stability.

Question 2

a) Note that

$$\begin{aligned} HP(z) &= K(1-z^{-1})Z\left(\frac{1}{s^2(\tau s+1)}\right) \\ &= K(1-z^{-1})Z\left(\frac{1}{s^2}-\frac{\tau}{s}+\frac{\tau}{s+1/\tau}\right) \\ &= -K\frac{z(\tau-1-\tau e^{-1/\tau})+(\tau e^{-1/\tau}+e^{-1/\tau}-\tau)}{(z-1)(z-e^{-1/\tau})}. \end{aligned}$$

b) For $\tau = 0$ the discrete-time model is

$$HP_0(z) = \frac{K}{z-1}.$$

i) The characteristic polynomial of the closed-loop system is

$$z - 1 + K,$$

hence the close-loop system is asymptotically stable for

$$0 < K < 2.$$

ii) The closed-loop poles are all at $z = 0$ for $K = 1$.

c) The characteristic polynomial of the closed-loop system, with $\tau \neq 0$ and $K = 1$, is

$$z^2 + z(e^{-1/\tau} - \tau - e^{-1/\tau}) + \tau(1 - e^{-1/\tau}).$$

i) The closed-loop characteristic polynomial has all roots inside the unity disk for all $\tau > 0$.

ii) Let

$$C(z) = \alpha\,\frac{z - e^{-1/\tau}}{z + \beta}.$$

The closed-loop characteristic polynomial is

$$z^2 + z(\alpha\tau e^{-1/\tau} - \alpha\tau + \alpha + \beta - 1) + (-\alpha\tau e^{-1/\tau} - \alpha e^{-1/\tau} + \alpha\tau - \beta).$$

Setting

$$\alpha = \frac{1}{1 - e^{-1/\tau}} \qquad \beta = \frac{\tau e^{-1/\tau} + e^{-1/\tau} - \tau}{e^{-1/\tau} - 1}$$

yields the closed-loop characteristic polynomial z^2.

iii) Consider the controller designed in part c.ii) in closed-loop with the discrete-time model of the system with $\tau = 0$, that is $HP_0(z)$. The closed-loop characteristic polynomial is

$$z^2 + \tau z - \tau,$$

and this has all roots inside the unity circle if, and only if, $\tau \in [0, 1/2)$.

d) The controller designed on the model with $\tau = 0$ stabilizes, for any $\tau \geq 0$, the model that includes the unmodelled dynamics. The controller designed for $\tau \neq 0$ does not stabilize, for all τ, the model with $\tau = 0$. As a result, the controller designed on the basis of the model for $\tau = 0$ is more robust.

Question 3

a) Note that

$$\begin{aligned} HP(z) &= (1-z^{-1})Z\left(\frac{1}{s^3}\right) \\ &= \frac{1}{2}\frac{z+1}{(z-1)^2} \end{aligned}$$

b) The transfer function in the w-plane is given by

$$HP(w) = HP(z)\Big|_{z=\frac{1+w/2}{1-w/2}} = \frac{1}{2}\,\frac{2-w}{w^2}.$$

c) Let, for example,

$$C(w) = \frac{\alpha w + \beta}{w + \delta}$$

and select α, β and δ such that all closed-loop poles are at $w = -1$. (Other designs are possible.) The closed-loop characteristic polynomial is

$$2w^3 + w^2(2\delta - \alpha) + w(2\alpha - \beta) + 2\beta.$$

Selecting

$$\alpha = \frac{7}{2} \qquad \beta = 1 \qquad \delta = \frac{19}{4}$$

yields the polynomial $2(w+1)^3$. The resulting controller is

$$C(w) = \frac{\frac{7}{2}w + 1}{w + \frac{19}{4}}.$$

d) The discrete-time controller is

$$C(z) = C(w)\Big|_{w=2\frac{z-1}{z+1}} = 8\,\frac{4z-3}{27z+11}.$$

The characteristic polynomial of the discrete-time closed-loop system is

$$27z^3 - 27z^2 + 9z - 1 = (3z-1)^3,$$

hence the discrete-time closed-loop system is asymptotically stable.

Question 4

a) The closed-loop characteristic polynomial is

$$s^2 + 2s + 2$$

with roots $s = -1 \pm I$, hence the closed-loop system is stable. These roots are mapped in the z-plane to

$$e^{-1} e^{\pm I} \approx 0.198 \pm 0.31I$$

Finally, the system is of type 1 because of the integrator in $C(s)$.

b) The controllers are given by

$$C_B(z) = C(s)\Big|_{s=1-\frac{1}{z}} = \frac{3z-1}{z-1}$$

and

$$C_T(z) = C(s)\Big|_{s=2\frac{z-1}{z+1}} = 2\frac{z}{z-1}.$$

Both controllers have a pole at $z = 1$, hence the resulting closed-loop systems are of type 1.

c) The equivalent discrete-time model is

$$\begin{aligned} HP(z) &= (1 - z^{-1}) Z\left(\frac{1}{s(s+1)}\right) \\ &= \frac{1 - e^{-1}}{z - e^{-1}}. \end{aligned}$$

d) The poles of the closed-loop system are the roots of the characteristic polynomial

$$z^2 + z(2 - 4e^{-1}) + 2e^{-1} - 1,$$

namely

$$z_{B1} \approx -0.842 \qquad z_{B2} \approx 0.313.$$

e) The poles of the closed-loop system are the roots of the characteristic polynomial

$$z^2 + z(1 - 3e^{-1}) + e^{-1},$$

namely

$$z_T \approx 0.05 \pm 0.604I.$$

f) All three designs yields stabilizing controllers. The discrete-time equivalent poles of the continuous-time design are complex conjugate with imaginary part 1.5 times the real part, giving a good damping coefficient. The backward difference based controller yields two real poles, one positive and one negative. The negative pole is *slower* than the equivalent poles of the continuous-time design and introduces *spurious* oscillations. The Tustin based controller yields two complex conjugate poles with imaginary part twelve times bigger than the real part. These poles are slower than the equivalent continuous-time poles and have a worse damping coefficient.

Question 5

a) Let

$$C(z) = \frac{\alpha z + \beta}{z + \delta}$$

and note that since the desired closed-loop system has only two poles, the controller has to cancel the zero of $HP(z)$. This is achieved setting $\delta = 22/25$. The closed-loop system is then given by

$$\frac{C(z)HP(z)}{1 + C(z)HP(z)} = 10\frac{\alpha z + \beta}{100z^2 + z(10\alpha - 143) + (60 + 10\beta)}.$$

Selecting $\alpha = 143/10$ and $\beta = -6$ yields

$$\frac{C(z)HP(z)}{1 + C(z)HP(z)} = \frac{143z - 60}{100z^2}$$

as requested.

b) The characteristic polynomial of the closed-loop system is

$$100z^2 + 143z(k-1) + 60(k-1).$$

and this has all roots inside the unity disk provided $k \in [0, 303/203)$.

c) The results in part c) shows that the proposed design yields some robustness in terms of *gain margin*, but it is not *very* robust: if the gain increases by 50% the closed-loop system is unstable.

Question 6

a) The discrete-time equivalent transfer function is

$$HP(z) = Z\left(\frac{1-e^{-s}}{s}e^{-s}\frac{1}{s-1}\right) = (1-z^{-1})\frac{1}{z}Z\left(\frac{1}{s(s-1)}\right) = \frac{e-1}{z(z-e)}.$$

b) Let (other selections are possible)

$$C(z) = k\frac{z}{z+e}.$$

The closed-loop characteristic polynomial is

$$z^2 - e^2 + k(e-1),$$

hence selecting

$$k = \frac{e^2}{e-1}$$

yields a closed-loop system with two poles at $z = 0$.

c) If $T = 1/2$ sec, then the delay e^{-s} yields to poles at $z = 0$.

i) The discrete-time equivalent model is

$$HP(z) = Z\left(\frac{1-e^{-s/2}}{s}e^{-s}\frac{1}{s-1}\right) = (1-z^{-1})\frac{1}{z^2}Z\left(\frac{1}{s(s-1)}\right) = \frac{e^{1/2}-1}{z^2(z-e^{1/2})}.$$

ii) The characteristic polynomial of the resulting closed-loop system is

$$p(z) \approx 1.718z^3 + 1.83z^2 - 7.7z + 4.79.$$

iii) Applying the bilinear transformation to $p(z)$ yields the polynomial

$$p(s) \approx 12.61w^3 - 25.39w^2 - 0.31w - 0.648.$$

Since all coefficients of the polynomial $p(s)$ do not have the same sign, this polynomial is not stable. As a result, the closed-loop system is unstable.