Imperial College London

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2012

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

M3P6/M4P6

Probability Theory

Date: examdate Time: examtime

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

Notation: Σ - the σ-algebra of Lebesgue measurable sets on real line $\mathbb{R}$, λ - the Lebesgue measure on $(\mathbb{R}, \Sigma)$.

1. (1.a) Give a definition of a probability space and of a random variable.
   (1.b) Explain giving reasons, which of the following random variables are independent and which are not :

   (1.b.i) $X_1 \equiv -\chi_{[-1,0]}$ and $X_2 \equiv \chi_{[0,+1]}$ in $([-1,+1], \Sigma \cap [-1,+1], \mu \equiv \frac{1}{2}\lambda$ with Σ denoting the σ-algebra of Lebesgue measurable sets in $\mathbb{R}$ and λ the Lebesgue measure;

   (1.b.ii) $\pi_n(\omega) \equiv \omega_n$, $n \in \mathbb{N}$, on a product space $(\Omega_0, \Sigma_0, \mu_0)^{\mathbb{N}}$

   (1.b.iii) $X_n \equiv \prod_{k=0}^{9} \pi_{n+k}$, $n \in \mathbb{N}$, on a product space $(\Omega_0, \Sigma_0, \mu_0)^{\mathbb{N}}$.

2. (2.a) Give a definition of convergence in probability and almost everywhere.
   Which of the sequences below converge or do not converge in the following sense
   (*) pointwise;
   (**) in probability;
   (***) almost everywhere;
   (****) in $\mathcal{L}_1$:

   (2.a.i) $k^2 \chi_{[0,\frac{1}{k}]}$ in $([0,1], \Sigma \cap [0,1], \lambda)$;
   (2.a.ii) $e^{k^2} \chi_{(k,\infty)}$ in $(\mathbb{R}^+, \Sigma \cap \mathbb{R}^+, e^{-x}\lambda)$;
   (2.a.iii) $\chi_{(|\sin(\frac{\pi}{2}\sqrt{k})|,|\sin(\frac{\pi}{2}\sqrt{k+1})|]}$ in $([0,1], \Sigma \cap [0,1], \lambda)$

   (2.b) State and prove the Weak Law of Large Numbers.

3. (3.a) State and prove Borel-Cantelli lemma.

(3.b) Suppose one chooses randomly 14 signs from a set including all letters of the English alphabet and additional typographical sign "-". Prove or disprove that with probability one in infinite sequence of trials one can find the exact expression "Borel-Cantelli" infinitely many times.

4. (4.a) Define the characteristic function ϕ_X of a random variable X on a probability space $(\Omega, \mathcal{A}, \mu)$ and prove or disprove the following properties :

(4.a.i) $\phi(0) = 1$;

(4.a.ii) $\sum_{j,k=1}^{N} \phi(t_j - t_k)\bar{z}_k z_j \geq 0$ for any $N \in \mathbb{N}$ and any $z_j \in \mathbb{C}$, $j = 1, .., N$.

(4.a.iii) If random variables X and Y are independent, then

$$\phi_{X+Y}(t) = \phi_X(t)\phi_Y(t).$$

(4.b) State the Central Limit Theorem.

(4.c) Let X_j, $j \in \mathbb{N}$, be real valued independent random variables with mean zero and finite variance σ^2 defined on a probability space $(\mathbb{R}^+, \Sigma, e^{-x}d\lambda)^{\mathbb{N}}$. Suppose additionally that $\exists c, L \in (0, \infty)$ such that $|X_j(x)|\chi(|X_j| \geq L) \leq cx$. Prove or disprove that

$$\lim_{n\to\infty} \int \exp\{t\frac{1}{\sqrt{n}} \sum_{i=1,..,n} X_i\} d\mu = e^{\frac{1}{2}\sigma^2 t^2}$$

(1)

Solution 1.

(1.a) (1.a) A probability space is by definition a triple (Ω, Σ, μ) consisting of a non-empty set Ω, a σ-algebra Σ, (that is a family of subsets in Ω including this set and closed with respect to operations of taking complement and countable unions), and a probability measure μ, (that is a non-negative function on Σ, such that $\mu(\Omega) = 1$ and for any disjoint sets $A_n \in \Sigma$, $n \in \mathbb{N}$, $\mu(\bigcup_n A_n) = \sum_n \mu(A_n)$). A measurable function $X : \Omega \to \mathbb{R}$, i.e. a function such that $\{\omega \in \Omega : X(\omega) < x\} \in \Sigma$, for any $x \in \mathbb{R}$, is called a random variable. [seen] [5pts]

(1.b.i) (1.b.i) No. By direct calculation one has [unseen] [5pts]

$$0 = \mu(X_1 X_2) \neq \mu(X_1)\mu(X_2) = \frac{1}{4}$$

(1.b.ii) (1.b.ii) Yes. This follows from the fact that in the product probability space by definition all σ-algebras $\pi_n^{-1}(\Sigma)$ are independent. [unseen] [5pts]

(1.b.iii) (1.b.iii) No. For example, for a product probability measure $\mu \equiv \mu_0^{\otimes \mathbb{N}}$, we have [unseen] [5pts]

$$\mu X_n X_{n+1} = \mu_0 \pi_n \prod_{k=1}^{9} \mu_0 \pi_{n+k}^2 \, \mu_0 \pi_{n+10} \neq \mu X_n \cdot \mu X_{n+1}$$

since in general $\mu_0 \pi_{n+k}^2 \neq (\mu_0 \pi_{n+k})^2$.

(2) Solution 2.

(2.a) (2.a) A sequence of random variables X_n, $n \in \mathbb{N}$, on a probability space (Ω, Σ, μ) converges in probability to a random variable X iff [seen] [3pts]

$$\forall \varepsilon > 0 \ \lim_{n\to\infty} \mu\{|X_n - X| > \varepsilon\} = 0$$

A sequence of random variables X_n, $n \in \mathbb{N}$, on a probability space (Ω, Σ, μ) converges almost everywhere to a random variable X iff

$$\mu\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\} = 1$$

(2.a.i) (2.a.i) The sequence $k^2\chi_{[0,\frac{1}{k}]}$ converges pointwise to zero everywhere except the point 0 where it diverges. Since the set $\{0\}$ has measure zero, we have convergence to zero a.s. and so in probability; but it does not converge in $\mathcal{L}_1$ because [unseen] [4pts]

$$\int k^2\chi_{[0,\frac{1}{k}]}d\lambda = k \to_{k\to\infty} \infty.$$

(2.a.ii) (2.a.ii) The sequence $e^{k^2}\chi_{(k,\infty)}$ converges to zero pointwise everywhere, so also a.s. and in probability, but [unseen] [4pts]

$$\int_{\mathbb{R}^+} e^{k^2}\chi_{(k,\infty)}e^{-x}d\lambda = e^{-k}e^{k^2} \to_{k\to\infty} \infty$$

so not in $\mathcal{L}_1$

(2.a.iii) (2.a.iii) To consider $\chi_{(|\sin(\frac{\pi}{2}\sqrt{k})|,|\sin(\frac{\pi}{2}\sqrt{k+1})|]}$, observe that [unseen] [4pts]

$$|\sin(\frac{\pi}{2}\sqrt{k}) - \sin(\frac{\pi}{2}\sqrt{k+1})| \le \frac{\pi}{2}|\sqrt{k} - \sqrt{k+1}| = \frac{\pi}{2(\sqrt{k} + \sqrt{k+1})}$$

Hence the sequence converges to zero in probability and, since it is bounded, also in $\mathcal{L}_1$. The sequence does not converge to zero pointwise and a.s..

(2.b) (2.b) [seen] [5pts]

Weak Law of Large Numbers : *Suppose X_j, $j \in \mathbb{N}$, are square integrable random variables on a probability space (Ω, Σ, μ), such that $\forall j \in \mathbb{N} \mu(X_j) = 0$, $\sup_j \mu X_j^2 < \infty$ and $\mu(X_i X_j) = 0$ for $i \neq j$. Then*

$$s_n \equiv \frac{1}{n} \sum_{j=1}^{n} X_j \to_{n \to \infty} 0$$

in probability.

Proof: Since by Tchebyshev inequality

$$\mu\{|s_n| > \varepsilon\} \leq \frac{1}{\varepsilon^2} \mu |s_n|^2$$

and, since by our assumption $\mu(X_i X_j) = 0$, we have

$$\mu |s_n|^2 = \frac{1}{n^2} \sum_{i,j=1,..,n} \mu X_j X_i \leq \frac{1}{n} \sup_j \mu X_j^2$$

so

$$\mu\{|s_n| > \varepsilon\} \leq \frac{1}{n} \frac{1}{\varepsilon^2} \sup_j \mu X_j^2$$

Thus

$$\lim_{n \to \infty} \mu\{|s_n| > \varepsilon\} = 0.$$

for any $\varepsilon > 0$.

(3)

Solution 3.

(3.a)

(3.a) Borel-Cantelli Lemma : [seen]

(B-C.1) Suppose a sequence of events A_n, $n \in \mathbb{N}$, satisfies [8pts]

$$\sum_{n\in\mathbb{N}} \mu(A_n) < \infty.$$

Then

$$\mu(\bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} A_k) = 0$$

(B-C.2) Suppose a sequence of independent events A_n, $n \in \mathbb{N}$, satisfies

$$\sum_{n\in\mathbb{N}} \mu(A_n) = \infty.$$

Then

$$\mu(\bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} A_k) = 1$$

Proof: (B-C.1) Note that

$$\mu(\bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} A_n) = \lim_{n\to\infty} \mu(\bigcup_{k\geq n} A_n) \leq \lim_{n\to\infty} \sum_{k\geq n} \mu(A_k) = 0$$

(B-C.2) It is sufficient to show that

$$\Omega \setminus \bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} A_n = \bigcup_{n\in\mathbb{N}} \bigcap_{k\geq n} \Omega \setminus A_n$$

has probability zero. Since

$$\mu(\bigcup_{n\in\mathbb{N}} \bigcap_{k\geq n} \Omega \setminus A_n) \leq \sum_{n\in\mathbb{N}} \mu(\bigcap_{k\geq n} \Omega \setminus A_n),$$

so using independence of the events, we get

$$\mu(\bigcup_{n\in\mathbb{N}} \bigcap_{k\geq n} \Omega \setminus A_n) \leq \sum_{n\in\mathbb{N}} \prod_{k\geq n} (1 - \mu A_n)$$

To conclude, we observe that

$$\prod_{N\geq k\geq n}(1-\mu A_n)=\exp\{\sum_{N\geq k\geq n}\log(1-\mu A_n)\}\leq\exp\{-\sum_{N\geq k\geq n}\mu A_n\}\to_{N\to\infty}0$$

(3.b)

(3.b) Let $(\Omega,\Sigma,\mu)\equiv(\Omega_0,\Sigma_0,\mu_0)^{\mathbb{N}}$, where Ω_0 is a set containing all letters of English alphabet and the sign "–", $\Sigma_0\equiv 2^{\Omega_0}$ and μ_0 a probability measure on (Ω_0,Σ_0) such that each one point set has nonzero probability. Define [unseen] [12pts]

$$A_n\equiv\{\omega\in\Omega:\omega_{14n+1}...\omega_{14(n+1)=Borel-Cantelli}\}$$

Then all A_n's are jointly independent and by our assumption $\mu(A_n)\equiv q>0$. Hence by second Borell-Cantelli lema the set $\bigcap_{n\in\mathbb{N}}\bigcup_{k\geq n}A_n$ consisting of sequences containing the pattern "Borel-Cantelli" infinitely many times has probability one.

(4)

Solution 4.

(4.a)

(4.a) The characteristic function ϕ_X of a random variable X defiend on a probability space $(\Omega, \mathcal{A}, \mu)$ isdefined by [seen] [5pts]

$$\phi_X(t) := \mu(e^{itX})$$

for any $t \in \mathbb{R}$.

(4.a.i)

The condition (4.a.i) follows from normalisation condition of a probability measure.

(4.a.ii)

To show (4.a.ii) note that for any complex numbers z_j, $j = 1, .., N$, $N \in \mathbb{N}$, we have

$$\sum_{j,k=1,..,N} \phi(t_j - t_k)\bar{z}_k z_j = \mu|\sum_{j=1,..,N} e^{it_j} z_j|^2 \geq 0$$

(4.a.iii)

The property (4.a.iii) follows from the fact that for independent random variables X and Y, with for any boundend measurable functions f and g, one has

$$\mu(f(X)g(Y)) = \mu(f(X))\mu(g(Y)).$$

(4.b)

(4.b) Central Limit Theorem: Suppose X_j, $j \in \mathbb{N}$, are square integrable independent random variables with mean 0 and variance σ^2. Then a sequence [seen] [3pts]

$$\xi_n \equiv \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i$$

converges in distribution to the Gaussian random variable with mean 0 and variance σ^2.

$\boxed{\texttt{(4.c)}}$ (4.c) First we note that by the assumption that $\exists c, L \in (0,\infty)$ such that $|X_j(x)|\chi(|X_j| \geq L) \leq cx$, for any $n \geq 4(ct)^2$, we have [unseen] [12pts]

$$\int \exp\{t\frac{1}{\sqrt{n}}X_i\}d\mu \leq L + 2 < \infty$$

Moreover, by independence of all X_j's, we have

$$\int \exp\{t\frac{1}{\sqrt{n}}\sum_{i=1,..,n} X_i\}d\mu = \prod_{j=1,..,n}\int \exp\{t\frac{1}{\sqrt{n}}X_i\}d\mu$$

Next we note that

$$\int \exp\{t\frac{1}{\sqrt{n}}X_i\}d\mu = 1 + \frac{t^2\sigma^2}{2n}$$

$$+\frac{1}{n^{3/2}}\int_0^1 ds_1\int_0^{s_1} ds_2\int_0^{s_2} ds_3\int t^3X_j^3\exp\{s_3t\frac{1}{\sqrt{n}}X_i\}d\mu$$

By our assumption we have that for $n \geq 4(ct)^2$

$$\int t^3X_j^3\exp\{s_3t\frac{1}{\sqrt{n}}X_i\}d\mu \leq L^3 + 16$$

Hence we obtain that

$$(1+\frac{t^2\sigma^2}{2n})^n \leq \prod_{j=1,..,n}\int \exp\{t\frac{1}{\sqrt{n}}X_i\}d\mu \leq \left(1 + \frac{t^2\sigma^2}{2n} + \frac{t^3}{6n^{3/2}}(L^3+16)\right)^n$$

From this, we conclude that

$$\lim_{n\to\infty}\prod_{j=1,..,n}\int \exp\{t\frac{1}{\sqrt{n}}X_i\}d\mu = e^{\frac{t^2\sigma^2}{2}}$$