Imperial College London

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2015

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Mechanics

Date: Monday, 18 May 2015. Time: 10.00am – 12.00noon. Time allowed: 2 hours.

This paper has FOUR questions.

Candidates should start their solutions to each question in a new main answer book

Supplementary books may only be used after the relevant main book(s) are full.

Statistical tables will not be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers to each question as per the table below.

Raw mark	up to 12	13	14	15	16	17	18	19	20
Extra credit	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4

- Each question carries equal weight.
- Calculators may not be used.

1. (a) A particle of mass m moving along the positive x-axis is subject to a force with potential

$$V(x) = A\left[\frac{1}{3}\left(\frac{r_0}{x}\right)^6 - \frac{1}{2}\left(\frac{r_0}{x}\right)^4\right]$$

with constants $A > 0$ and $r_0 > 0$.

(i) Find the equilibrium point(s).

(ii) Find the speed needed to escape to infinity from the stable equilibrium point(s).

(b) Consider again a particle of mass m, but now moving under the influence of a central force with the same potential,

$$V(r) = A\left[\frac{1}{3}\left(\frac{r_0}{r}\right)^6 - \frac{1}{2}\left(\frac{r_0}{r}\right)^4\right].$$

(i) For the case where $mh^2 = 3Ar_0^2/8$, where h is the angular momentum per unit mass, show that there is a stable circular orbit, $r = r_0\sqrt{4/3}$, and an unstable circular orbit, $r = 2r_0$.

(ii) Find the radial speed needed to escape to infinity from $r = r_0\sqrt{4/3}$.

You may use without proof that the effective potential is

$$V_{EFF}(r) = V(r) + \frac{1}{2}\frac{mh^2}{r^2}.$$

2. (a) The position of a point on a curve is given by $r = x\hat{\imath} + y(x)\hat{\jmath}$. Derive the following expressions for the tangent, normal, and curvature:

$$\begin{aligned}
\hat{v} &= (1+y'^2)^{-1/2}(\hat{\imath} + y'\hat{\jmath}), \\
\hat{n} &= (y''/|y''|)(1+y'^2)^{-1/2}(-y'\hat{\imath} + \hat{\jmath}), \\
\kappa &= |y''|(1+y'^2)^{-3/2},
\end{aligned}$$

where $y' = dy/dx$.

(b) Suppose that a block of mass m in the xy-plane is placed on a surface with shape given by $y(x) = \cos x$. The coefficient of friction between the block and the surface is μ. The surface is accelerating to the right such that $\boldsymbol{a} = a\hat{\imath}$, where a is a positive constant. Gravity acts in the $-\hat{\jmath}$ direction.

Show that if the block is placed at $x = 3\pi/4$, then the maximum value of a for which the block will not move relative to the surface is

$$a = \frac{\mu + \sqrt{2}/2}{1 - \mu\sqrt{2}/2} g.$$

3. Three blocks with masses m_1, m_2, and m_3 are placed along a line on a frictionless table. Mass m_1 has position x_1, m_2 has position x_2, and m_3 has position x_3. They are arranged such that $x_1 < x_2 < x_3$. Masses m_1 and m_2 are connected by a spring with spring constant k, while m_2 and m_3 are connected by a spring with spring constant K. Both springs have equilibrium length L. Mass m_2 is also subject to the drag force $F_D = -C_D\dot{x}_2$ with $C_D > 0$.

Suppose that

$$\begin{aligned} x_1(t) &= A\cos(\omega t) - L \\ x_3(t) &= A\cos(\omega t) + L \end{aligned}$$

with $0 < A < L$.

(a) Show that the equation of motion for m_2 can be written in the form

$$\ddot{x}_2 + 2\mu\dot{x}_2 + \omega_0^2 x_2 = \frac{F_0}{m_2}\cos(\omega t),$$

and find the expressions for μ, ω_0, and F_0 in terms of k, K, C_D, and A.

(b) Find the amplitude and phase of the steady-state oscillation of m_2. You may leave your expressions for these quantities in terms of μ, ω_0, and F_0. (You may assume that L is sufficiently large so that the masses do not touch each other.)

4. A wheel of mass M and radius R rolls along a flat surface that makes an angle α with the horizontal. The coefficient of friction between the wheel and the surface is μ and the centre of the wheel is also its centre of mass. The moment of inertia of the wheel about its centre of mass is I. Gravity acts downwards. Take the coordinate system to be aligned such that $\hat{\imath}$ points down the surface and $\hat{\jmath}$ is normal to the surface.

(a) Show that the minimum value of μ for which the wheel rolls without slipping is

$$\mu = \left(1 + \frac{MR^2}{I}\right)^{-1} \tan\alpha.$$

(b) Suppose now there is also a spring with spring constant k and equilibrium length L that has its right end connected to the centre of the wheel and its left end connected to the point $(0, R)$. The force the spring exerts on the wheel is then $\boldsymbol{F}_{sp} = -k(x_{CM} - L)\hat{\imath}$, where x_{CM} is the centre of mass position along the surface.

(i) Assuming that the wheel rolls without slipping, show that the position of the centre of mass is of the form

$$x_{CM}(t) = A\sin(\omega_0 t) + B\cos(\omega_0 t) + x_0 \tag{1}$$

where the constants A and B are determined from initial conditions. Find expressions for x_0 and ω_0.

(ii) If at $t = 0$ we have $x_{CM} = x_0$, determine the maximum value of A for which the rolling without slipping assumption holds. You may use ω_0 in your expression.

Solutions for M1A1 2015 Summer Exam

1. (ALL UNSEEN)

(a) (i) Differentiating the potential with respect to x gives

$$V'(x) = A\left[-2r_0^6 x^{-7} + 2r_0^4 x^{-5}\right].$$

Setting this expression equal to zero, we find that

$$2Ar_0^4 x^{-5}\left[-r_0^2 x^{-2} + 1\right] = 0.$$

Thus, the derivative goes to zero as $x \to \infty$ and equals zero at the equilibrium point $x = r_0$. **(5 marks)**

(ii) The second derivative of the potential with respect to x is

$$V''(x) = A\left[14r_0^6 x^{-8} - 10r_0^4 x^{-6}\right]$$

which evaluated at $x = r_0$ is $V''(r_0) = 4Ar_0^{-2} > 0$. Therefore, the equilibrium point $x = r_0$ is a stable.

In order to escape from this point, we require that $E \geq V(\infty) = 0$. The total energy is given by

$$E = \frac{1}{2}mu^2 + V(r_0),$$

where $u > 0$ is the speed of the particle. Since $V(r_0) = -A/6$, the particle can escape to infinity if $mu^2/2 \geq A/6$, which occurs if $u \geq \sqrt{A/(3m)}$. **(5 marks)**

(b) (i) Taking $mh^2 = 3Ar_0^2/8$, the effective potential is

$$V_{EFF}(r) = \frac{3}{16}\frac{Ar_0^2}{r^2} + A\left[\frac{1}{3}\left(\frac{r_0}{r}\right)^6 - \frac{1}{2}\left(\frac{r_0}{r}\right)^4\right].$$

Differentiating this expression with respect to r we have

$$V'_{EFF}(r) = -\frac{3}{8}Ar_0^2 r^{-3} - 2Ar_0^6 r^{-7} + 2Ar_0^4 r^{-5}.$$

Factoring out terms and setting this equal to zero gives

$$Ar_0^2 r^{-3}\left[2r_0^4 r^{-4} - 2r_0^2 r^{-2} + \frac{3}{8}\right] = 0.$$

(3 marks)

The equation is satisfied as $r \to \infty$, but also when the expression in the brackets is zero. This gives a quadratic equation for $(r_0/r)^2$ whose solutions are

$$(r_0/r)^2 = 3/4\ ,\ (r_0/r)^2 = 1/4$$

which correspond to circular orbits with radii $r = r_0\sqrt{4/3}$ and $r = 2r_0$.

The second derivative of the effective potential is

$$V''_{EFF}(r) = Ar^{-2}\left[\frac{9}{8}\left(\frac{r_0}{r}\right)^2 + 14\left(\frac{r_0}{r}\right)^6 - 10\left(\frac{r_0}{r}\right)^4\right].$$

Since $Ar^{-2} > 0$, what remains in the brackets will determine if $V''_{EFF}(r)$ is positive or negative. For $(r_0/r)^2 = 3/4$, we have that

$$V''_{EFF}(r_0\sqrt{4/3}) = \frac{3}{4}Ar_0^{-2}\left[\frac{27}{32} + \frac{189}{32} - \frac{180}{32}\right] > 0,$$

while for $(r_0/r)^2 = 1/4$ we find,

$$V''_{EFF}(2r_0) = \frac{1}{4}Ar_0^{-2}\left[\frac{9}{32} + \frac{7}{32} - \frac{20}{32}\right] < 0.$$

Thus, we have that $r = r_0\sqrt{4/3}$ is a stable circular orbit. **(3 marks)**

(ii) To escape to infinity from the circular orbit with $r = r_0\sqrt{4/3}$, we must pass over the maximum at $r = 2r_0$. At these two radii, we have that $V_{EFF}(r_0\sqrt{4/3}) = 0$ and $V_{EFF}(2r_0) = A/48$. To escape to infinity, we require that

$$E = \frac{1}{2}m\dot{r}^2 + V(r_0\sqrt{4/3}) > V(2r_0)$$

which gives that $|\dot{r}| > (1/2)\sqrt{A/(6m)}$. **(4 marks)**

(Total 20 marks)

2. (a) (SEEN)The tangent is given by $\hat{\boldsymbol{v}} = d\boldsymbol{r}/ds$, where $\boldsymbol{r}$ is the position and s is the arc length. Using the chain rule, we may also express the tangent as $\hat{\boldsymbol{v}} = (d\boldsymbol{r}/dx)(dx/ds)$. Since the position is $\boldsymbol{r} = x\hat{\boldsymbol{\imath}} + y(x)\hat{\boldsymbol{\jmath}}$, we have that $d\boldsymbol{r}/dx = \hat{\boldsymbol{\imath}} + y'\hat{\boldsymbol{\jmath}}$. As $dx/ds = (1+y'^2)^{-1/2}$, the tangent is given by

$$\hat{\boldsymbol{v}} = (1+y'^2)^{-1/2}(\hat{\boldsymbol{\imath}} + y'\hat{\boldsymbol{\jmath}}).$$

(4 marks)

We also know that $d\hat{\boldsymbol{v}}/ds = (d\hat{\boldsymbol{v}}/dx)(dx/ds) = \kappa\hat{\boldsymbol{n}}$, where $\kappa = |d\hat{\boldsymbol{v}}/ds|$ is the curvature and $\hat{\boldsymbol{n}} = \kappa^{-1}(d\hat{\boldsymbol{v}}/ds)$ is the normal. Since

$$\frac{d\hat{\boldsymbol{v}}}{dx} = \frac{y''}{(1+y'^2)^{3/2}}(-y'\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}),$$

with our expression for dx/ds, we have that

$$\begin{aligned}\hat{\boldsymbol{n}} &= (y''/|y''|)(1+y'^2)^{-1/2}(-y'\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}),\\ \kappa &= |y''|(1+y'^2)^{-3/2}.\end{aligned}$$

(6 marks)

(b) (UNSEEN) There are three forces to consider in this problem: gravity, $\boldsymbol{F}_g = -mg\hat{\boldsymbol{\jmath}}$, friction, $\mathcal{F} = \mathcal{F}\hat{\boldsymbol{v}}$, and the normal force $\boldsymbol{N} = N\hat{\boldsymbol{n}}$, where we take $\hat{\boldsymbol{n}} = (1+y'^2)^{-1/2}(-y'\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}})$ since for $x > \pi/2$ we have that $y''/|y''| = 1$. Using $\hat{\boldsymbol{v}}$ and $\hat{\boldsymbol{n}}$, we express $\boldsymbol{a}$ and $\boldsymbol{F}_g$ as

$$\begin{aligned}\boldsymbol{a} &= a(1+y'^2)^{-1/2}(\hat{\boldsymbol{v}} - y'\hat{\boldsymbol{n}}),\\ \boldsymbol{F}_g &= -mg(1+y'^2)^{-1/2}(y'\hat{\boldsymbol{v}} + \hat{\boldsymbol{n}}).\end{aligned}$$

Since the block is not moving relative to the surface, the acceleration of the block must equal that of the surface. Thus, from Newton's second law, we know that

$$\begin{aligned}ma(1+y'^2)^{-1/2} &= \mathcal{F} - mgy'(1+y'^2)^{-1/2}\\ -may'(1+y'^2)^{-1/2} &= N - mg(1+y'^2)^{-1/2}.\end{aligned}$$

(6 marks)

Solving for N and $\mathcal{F}$, we see that

$$\begin{aligned}\mathcal{F} &= m(1+y'^2)^{-1/2}(a + gy')\\ N &= m(1+y'^2)^{-1/2}(g - ay')\end{aligned}$$

Since $|\mathcal{F}| \le \mu N$ and $y'(x) = -\sin x$, we have that

$$|a - g\sin x| \le \mu(g + a\sin x).$$

(2 marks)

The maximum value of a will be achieved when $\mathcal{F}$ is positive, which occurs when $a \ge g\sin x$. Thus, we have that

$$a \le \frac{\mu + \sin x}{1 - \mu\sin x}g.$$

Setting $x = 3\pi/4$, we have then that the maximum value of a is

$$a = \frac{\mu + \sqrt{2}/2}{1 - \mu\sqrt{2}/2}g.$$

Note to the marker: Students might set $x = 3\pi/4$ much earlier. This is perfectly fine and deserves full marks if done properly. Students might also solve this problem by introducing the angle that the tangent makes with $\hat{\imath}$. This is approach also deserves full marks if done properly.

(2 marks)
(Total 20 marks)

3. (a) (UNSEEN) Accounting for the spring forces and drag force on m_2, one finds that the equation of motion is

$$m_2\ddot{x}_2 = -k(x_2 - x_1 - L) + K(x_3 - x_2 - L) - C_D\dot{x}_2.$$

(4 marks)

Using the expressions for x_1 and x_3, we have that

$$m_2\ddot{x}_2 + C_D\dot{x}_2 + (k+K)x_2 = (k+K)A\cos\omega t \tag{1}$$

Dividing through by m_2, we have

$$\ddot{x}_2 + \frac{C_D}{m_2}\dot{x}_2 + \frac{k+K}{m_2}x_2 = \frac{(k+K)A}{m_2}\cos\omega t.$$

Thus, $\mu = C_D/(2m_2)$, $\omega_0^2 = (k+K)/m_2$, and $F_0 = (k+K)A$. **(4 marks)**

(b) (**SEEN**) We seek a solution for x_2 of the form $x_2 = B\cos(\omega t - \beta)$ where B is the amplitude and β is the phase. Substituting this into (1), we find

$$-B\omega^2\cos(\omega t-\beta) - 2\mu\omega B\sin(\omega t-\beta) + \omega_0^2 B\cos(\omega t-\beta) = (F_0/m_2)\cos\omega t$$

which in turn becomes

$$\begin{aligned}&\cos\omega t\left(-B\omega^2\cos\beta) + 2\mu\omega B\sin\beta + \omega_0^2 B\cos\beta - (F_0/m_2)\right)\\ &\quad + \sin\omega t\left(-B\omega^2\sin\beta) - 2\mu\omega B\cos\beta + \omega_0^2 B\sin\beta\right) = 0.\end{aligned}$$

(6 marks)

In order for this equation to be satisfied, both expressions in the brackets must be equal to zero. From these two expressions, one can obtain,

$$\begin{aligned}2\mu\omega B &= \frac{F_0}{m_2}\sin\beta\\ B(\omega_0^2-\omega^2) &= \frac{F_0}{m_2}\cos\beta.\end{aligned}$$

Thus,

$$\begin{aligned}\tan\beta &= \frac{2\mu\omega}{\omega_0^2-\omega^2}\\ B &= \frac{F_0}{m_2}\left[4\mu^2\omega^2 + (\omega^2-\omega_0^2)^2\right]^{-1/2}.\end{aligned}$$

(6 marks)

(Total 20 marks)

4. (a) (**SEEN**) We will have three forces acting on the wheel: gravity, $\boldsymbol{F}_g$, friction, $\mathcal{F}$, and the normal force $\boldsymbol{N}$. If we align our coordinate system such that $\hat{\imath}$ points down the incline and $\hat{\jmath}$ points normal to it, our forces are

$$\begin{aligned}\mathcal{F} &= -\mathcal{F}\hat{\imath}\\ \boldsymbol{N} &= N\hat{\jmath}\\ \boldsymbol{F}_g &= mg\sin\alpha\hat{\imath} - mg\cos\alpha\hat{\jmath}.\end{aligned}$$

We also know that friction will exert a torque, $\tau = -\mathcal{F}R$, on the wheel. With $\ddot{y} = 0$, the equations of motion are

$$M\ddot{x}_{CM} = mg\sin\alpha - \mathcal{F} \tag{2}$$
$$0 = N - mg\cos\alpha \tag{3}$$
$$I\dot{\omega} = -\mathcal{F}R. \tag{4}$$

(6 marks)

Thus, $N = mg\cos\alpha$. For rolling without slipping $\ddot{x}_{CM} = -R\dot{\omega}$, and using (4), we have that $\ddot{x} = \mathcal{F}R^2/I$. Substituting this expression into (2), we find that

$$\mathcal{F}\left(1 + \frac{MR^2}{I}\right) = mg\sin\alpha.$$

(2 marks)

Since $\mathcal{F} \leq \mu N = \mu mg\cos\alpha$, we have

$$\mu \geq \left(1 + \frac{MR^2}{I}\right)^{-1}\tan\alpha.$$

(2 marks)

(b) **(UNSEEN)**

(i) With the spring force, our equation of motion for x becomes

$$M\ddot{x}_{CM} = -k(x_{CM} - L) + mg\sin\alpha - \mathcal{F}. \quad (5)$$

We still have, however, that $N = mg\cos\alpha$ and $\dot{\omega} = -R\mathcal{F}/I$, as well as $\ddot{x}_{CM} = -R\dot{\omega}$. Thus, $\mathcal{F} = I\ddot{x}_{CM}/R^2$ and (5) becomes

$$M\left(1 + \frac{I}{MR^2}\right)\ddot{x}_{CM} + kx_{CM} = kL + mg\sin\alpha.$$

(3 marks)

The general solution for this equation is

$$x_{CM} = A\sin\omega_0 t + B\cos\omega_0 t + x_0$$

where

$$\begin{aligned} \omega_0^2 &= \frac{k}{M(1 + I/(MR^2))} \\ x_0 &= L + (mg/k)\sin\alpha \end{aligned}$$

(2 marks)

(ii) If at $t = 0$ we have $x_{CM} = x_0$ and we know then that $B = 0$. Thus,

$$x_{CM} = A\sin\omega_0 t + x_0.$$

To ensure that our rolling without slip assumption holds, we need that $\mu N \geq |\mathcal{F}| = I|\ddot{x}_{CM}|/R^2$. Since $\ddot{x}_{CM} = -A\omega_0^2\sin\omega_0 t$, we have that

$$\mu Mg\cos\alpha \geq A\frac{I}{R^2}\omega_0^2|\sin\omega_0 t|.$$

Since this inequality must hold for all times, we have that

$$\mu M g \cos\alpha \left(\frac{R^2}{I\omega_0^2}\right) \geq A$$

and the maximum value is therefore,

$$A = \mu M g \cos\alpha \left(\frac{R^2}{I\omega_0^2}\right)$$

or

$$A = \mu M g \cos\alpha \left(\frac{1 + MR^2/I}{k}\right)$$

Note to the marker: You may give full marks for a valid expression that is left in terms of ω_0.

(5 marks)

(Total 20 marks)

Imperial College
London

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2015

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Foundations of Analysis

Date: Tuesday, 12 May 2015. Time: 10.00am – 12.00noon. Time allowed: 2 hours.

This paper has FOUR questions.

Candidates should start their solutions to each question in a new main answer book

Supplementary books may only be used after the relevant main book(s) are full.

Statistical tables will not be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
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Extra credit	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4

- Each question carries equal weight.
- Calculators may not be used.

1. (i) Carefully state the principle of induction and the principle of *strong* induction.

(ii) Prove that every nonempty subset S of $\mathbb{N}$ has a smallest element.

(iii) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence. Suppose that $a_0 = 3$, $a_1 = 5$, and that for all $n \geq 1$

$$a_{n+1} = 4a_n - 3a_{n-1}\,.$$

Guess a formula for a_n.

(iv) Prove your formula by induction. Be careful to use induction and not strong induction.

2. It is often said that Earthlings use the decimal system because they have ten fingers. We see a Martian write down the equation:

$$x^2 - 19x + 76 = 0\,.$$

When asked to write down the difference between the larger and the smaller root, the Martian writes 9. How many fingers do Martians have?

Note: Martians write numbers between 0 and 9 exactly as Earthlings do.

3. In this question you may use any resuts proved in the lectures provided that you make it clear which you are using.

(i) Find all x, $y \in \mathbb{N}$ satisfying the equation:

$$7x + 11y = 85\,.$$

(ii) Find all solutions $x \in \mathbb{Z}/35\mathbb{Z}$ of the equation

$$x^{35} \equiv 5 \pmod{35}\,.$$

Make sure that you explain *why* the solutions you found are all the solutions.

4. (i) Let X be a set and R an equivalence relation on X. For an element $a \in X$, denote by $[a]$ the equivalence class of a, and by X/R the set of equivalence classes. By definition, the *projection to the quotient* is the function

$$\pi: X \to X/R,$$

such that $\pi(a) = [a]$. Prove from first principles the following statement. If Y is a set and $f: X \to Y$ a function, then there is a function $g: X/R \to Y$ such that $f = g \circ \pi$ if and only if for all $a, b \in X$, $(a, b) \in R \Rightarrow f(a) = f(b)$.

(ii) Let now $X = \mathbb{R}$ and

$$R = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x - y \in \mathbb{Z}\}.$$

Briefly argue that R is an equivalence relation. Next define a relation R^* on $[0, 1] \subset \mathbb{R}$ by saying that for all $x, y \in [0, 1]$, $(x, y) \in R^*$ if and only if $(x, y) \in R$. Describe the relation R^* and thus briefly argue that it is an equivalence relation. Using (i), construct a function $g: [0, 1]/R^* \to \mathbb{R}/R$. Prove that g is invertible.

M1F Examination Solutions 2015

Set by [redacted] Checked by [redacted]

Question 1

3+3=6 marks
seen

(i) INDUCTION Suppose given $\forall n \in \mathbb{N}$ proposition $P(n)$. Assume:

1. $P(0)$ is true;
2. $\forall n \in \mathbb{N}$, $(P(n) \Rightarrow P(n+1))$ is true.

Then $\forall n \in \mathbb{N}$ $P(n)$ is true.

STRONG INDUCTION Suppose given $\forall n \in \mathbb{N}$ proposition $P(n)$. Assume:

1. $P(0)$ is true;
2. $\forall n \in \mathbb{N}$, $\Big(\big(\forall k \leq n,\ P(k)\big) \Rightarrow P(n+1)\Big)$ is true.

Then $\forall n \in \mathbb{N}$ $P(n)$ is true.

4 marks
seen

(ii) Apply strong induction with $P(n)$ the proposition: If $n \in S$ then S has a smallest element.

(1) If $0 \in S$ then 0 is the smallest element, so $P(0)$ is true.

(2) Let $n \in \mathbb{N}$ and assume $\forall k \leq n,\ P(k)$. Suppose $n+1 \in S$. If $n+1$ is the smallest element of S, then S has a smallest element. Otherwise there exists $k < n+1$, $k \in S$. Necessarily $k \leq n$ and $k \in S$ so by $P(k)$ S has a smallest element. In all cases S has a smallest element, thus $P(n+1)$.

By strong induction for all n $P(n)$ is true. Suppose $\emptyset \neq S \subset \mathbb{N}$: then there exists $n \in S$ so by $P(n)$ S has a smallest element.[1]

4 marks
unseen

(iii) $a_n = 3^n + 2$.

6 marks
seen similar

(iv) The key thing is to declare what is the proposition $P(n)$. The examiner will be very tough here and award NO marks for this question unless the correct choice is clearly made. I made a big deal in class when I discussed the Fibonacci sequence, which presents a very similar issue.

Let $\forall n \in \mathbb{N}$ $Q(n)$ be the proposition: $a_n = 3^n + 2$. We apply induction with $P(n)$ the proposition: $\big(Q(n) \& Q(n+1)\big)$.

[1] This is the proof that I did in class. There are other proofs of this fact, some using induction not strong induction. Obviously any correct proof earns full credit.

[redacted]

It is clear that $P(0)$ is true. Next $P(n) \Rightarrow Q(n+2)$ for:

$$a_{n+2} = 4a_{n+1} - 3a_n = 4(3^{n+1} + 2) - 3(3^n + 2) = 3^{n+2} + 2$$

but clearly $P(n) \Rightarrow Q(n+1)$ so in fact $P(n) \Rightarrow \big(Q(n+1) \& Q(n+2)\big) = P(n+1)$. By induction $\forall n \in \mathbb{N}$, $P(n)$ is true and hence also $\forall n \in \mathbb{N}$, $Q(n)$ is true.

Question 2

20 marks
unseen

This is a pretty serious question. There is the risk that many students will not be able to get a sensible start at all.

If Martians have m fingers, then they write their numbers in base m. Let $r_1 \geq r_2$ be the roots. There are several ways to address the question but in the end it boils down to realise that the following three equations hold:

$$\begin{cases} r_1 + r_2 = m + 9 \\ r_1 - r_2 = 9 \\ r_1 r_2 = 7m + 6 \end{cases}$$

At this point there are several ways to proceed. For instance solving the first two equations we find that $r_1 = \frac{m}{2} + 9$ and $r_2 = \frac{m}{2}$, so

$$4r_1 r_2 = m(m + 18) = 28m + 24$$

and $m^2 - 10m - 24 = (m - 12)(m + 2) = 0$ giving $m = 12$ fingers.

Partial credit It may be necessary to be systematically generous in the award of partial credit.

- Award: (a) 5 marks for showing awareness of the meaning of expansions in base m, that is correctly interpreting $19 = m + 9$, $76 = 7m + 6$; and (b) 5 marks for showing awareness that $r_1 + r_2 = m + 9$, $r_1 r_2 = 7m + 6$.
- Some students may decide to work mod m from the start and conclude something like $m | 48$. Award 15 marks for this.
- SOME students may just guess $m = 12$. If they do this, award 10 marks for showing clearly that $m = 12$ is consistent.

Question 3

10 marks
seen similar

(i) The first taks is to find the general solution for x, $y \in \mathbb{Z}$ of the equation $7x + 11y = 85$. (Award 5 marks for this). A short calculation or guess gives $7 \times (-3) + 11 \times 2 = 1$.

In general, we know that if x_0, y_0 is an integer solution of $ax + by = c$. and $c = \mathrm{hcf}(a, b)$, $a = ca'$, $b = cb'$, then the general integer solution is given by $x = x_0 + b'm$, $y = y_0 - a'm$.

Thus, the general integer solution $x, y \in \mathbb{Z}$ of the given equation is:

$$\begin{cases} x = -255 + 11m \\ y = 170 - 7m \end{cases}$$

for $m \in \mathbb{Z}$.

Next we make sure that $x, y \geq 0$. From the first equation we see that $x \geq 0$ iff $m \geq \frac{255}{11}$, that is $m \geq 24$; and from the second equation $y \geq 0$ iff $m \leq \frac{170}{7}$, that is $m \leq 24$. So we get only one solution with $m = 24$: $x = 9$, $y = 2$.

Award full marks for guessing the answer and proving that the solution is unique. Award 3 marks for guessing the answer without a proof that the solution is unique.

10 marks
seen similar

(ii) The things to observe are: $35 = 5 \times 7$ is the product of two primes, and $35 \equiv 11 \pmod{4 \times 6}$ and $\mathrm{hcf}(11, 24) = 1$ so we are exactly in the RSA situation. General RSA theory gives that if k satisfies $11k \equiv 1 \pmod{24}$, then the function $x \mapsto x^{11}$ (from $\mathbb{Z}/35\mathbb{Z}$ to $\mathbb{Z}/35\mathbb{Z}$) has the inverse $y \mapsto y^k$. This shows uniqueness of the solution. (Award 3 marks for uniqueness.)

A small calculation with Euclid shows $1 = 11 \times 11 - 5 \times 24$ so $k = 11$ and $x \equiv 5^{11} \pmod{35}$.

There are several ways to finish the calculation and all will earn a student the mark. For me I would use the Chinese remainder theorem: $x \equiv 0 \pmod 5$ and $x \equiv 5^{11} \pmod 7$ and then little Fermat gives:

$$x \equiv 1/5 \equiv 3 \pmod 7$$

from which $x \equiv 10 \pmod{35}$.

Question 4

10 marks
seen similar

(i) It is clear that $\pi(a) = \pi(b)$ if and only if $(a, b) \in R$.

Suppose that for all $(a, b) \in X$, $(a, b) \in R$ implies $f(a) = f(b)$. For $Z \in X/R$, choose a such that $\pi(a) = Z$. There exists such an a, indeed any $a \in Z$ will do. Define $g(Z) = f(a)$. This is well-defined: if $\pi(a) = \pi(b)$ then $(a, b) \in R$ and by assumption $f(a) = f(b)$. By construction for all $a \in X$ $f(a) = g(\pi(a))$.

Viceversa suppose that there exists a function $f: X \to Y$ such that for all $a \in X$ $f(a) = g(\pi(a))$. By what we said: If $(a, b) \in R$, then $\pi(a) = \pi(b)$, then $f(a) = g(\pi(a)) = g(\pi(b)) = f(b)$.

10 marks
unseen

(ii) It is clear that: $x - x = 0 \in \mathbb{Z}$; if $x - y \in \mathbb{Z}$ then also $(y - x) = -(x - y) \in \mathbb{Z}$; and $x - y \in \mathbb{Z}$ and $y - z \in \mathbb{Z}$ implies $x - z = (x - y) + (y - z) \in \mathbb{Z}$, so R is an equivalence relation.

For $a, b \in [0, 1]$, then $(a, b) \in R^*$ if and only if either $a = b$ or $\{a, b\} \subset \{0, 1\}$. It is routine from here to see that R^* is an equivalence relation By (i) there is a function $g: [0, 1]/R^* \to \mathbb{R}/R$: indeed if $(a, b) \in R^*$ then also $(a, b) \in R$ and

[redacted]

(denoting by $\pi\colon \mathbb{R} \to \mathbb{R}/R$ the projection to the quotient), $\pi(a) = \pi(b)$. The function g is injective because for a, $b \in [0,1]$, $a - b \in \mathbb{Z}$, that is, if and only if $a = b$ or $\{a, b\} \in \{0, 1\}$ if and only if $(a, b) \in R^*$. The function g is surjective because for all $x \in \mathbb{R}$ there is $y \in [0,1]$ such that $x - y \in \mathbb{Z}$: for this, take the fractional part $y = \langle x \rangle \in [0, 1)$.

Imperial College
London

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2015

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Geometry & Linear Algebra

Date: Wednesday, 20 May 2015. Time: 10.00am – 12.00noon. Time allowed: 2 hours.

This paper has FOUR questions.

Candidates should start their solutions to each question in a new main answer book

Supplementary books may only be used after the relevant main book(s) are full.

Statistical tables will not be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers to each question as per the table below.

Raw mark	up to 12	13	14	15	16	17	18	19	20
Extra credit	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4

- Each question carries equal weight.
- Calculators may not be used.

1. (a) Let n be a positive integer. Prove that any vectors with real coordinates $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ satisfy the Cauchy–Schwarz inequality

$$(x_1y_1 + \ldots + x_ny_n)^2 \leq (x_1^2 + \ldots + x_n^2)(y_1^2 + \ldots + y_n^2).$$

(b) Find all complex numbers z such that the matrix

$$A = \begin{pmatrix} 1 & z & z^2 & z^3 \\ z & z^2 & z^3 & 1 \\ z^2 & z^3 & 1 & z \\ z^3 & 1 & z & z^2 \end{pmatrix}$$

is invertible. (Use any method you like. You need to justify your answer.)

(c) For each complex number z such that the matrix A from part (b) is not invertible, find the general solution of the linear system $Ax = 0$. (Use any method you like. You need to justify your answer.)

(d) Let B be a 3×3 matrix with real entries such that $B^t = -B$. Prove that B is not invertible. (You can use, without proof, any results from lectures if you state them correctly and clearly.)

2. *In this question all the matrices are 2×2 matrices with real entries.*

(a) Give a brief definition of a *rotation* matrix.

(b) Give a brief definition of a *reflection* matrix.

(c) Give a brief definition of an *orthogonal* matrix.

(d) Prove that an orthogonal matrix is either a rotation matrix or a reflection matrix.

(e) Are the following statements true or false? If you think that a statement is true, give a proof; if you think that it is false, give a counterexample.

- (i) Each reflection matrix has eigenvalues 1 and -1.
- (ii) If a rotation matrix A has an eigenvalue 1 or -1, then $A = \pm I$.
- (iii) Each matrix with eigenvalues 1 and -1 is a reflection matrix.
- (iv) Each matrix with complex eigenvalues i and $-i$ is a rotation matrix.
- (v) All rotation matrices other than I and $-I$ are equivalent.
- (vi) All reflection matrices are equivalent.

You can use, without proof, any results from lectures if you state them correctly and clearly.

3. (a) Let A be a symmetric 2×2 matrix with real entries such that $A \neq cI$, where $c \in \mathbb{R}$.

- (i) Prove that A has two *distinct real* eigenvalues.
- (ii) Prove that eigenvectors of A with different eigenvalues are perpendicular.

(b) Reduce the conic $x_1^2 - x_2^2 + 2\sqrt{3}x_1x_2 + 8x_1 = 0$ to standard form using a rotation and a translation. Determine the type of the conic.

(c) Using any method you like determine the type of the quadric surface

$$x_1x_2 - x_1 - 2x_2 + x_3 - 1 = 0.$$

(In parts (b) and (c) you need to justify your answers. You can use, without proof, any results from lectures if you state them correctly and clearly.)

4. (a) Let V be a vector space over $\mathbb{R}$. Suppose that R and S are finite bases of V. Sketch a proof that R and S have the same number of elements. (Explain the main idea of the proof in a few sentences; you are not asked to give a detailed proof.)

(b) Let M be the vector space of 3×3 matrices with real entries, with the usual addition and multiplication by real numbers. For each of the following subsets of M determine whether or not it is a *subspace* of M. (Give a brief justification of your answers.)

- (i) The set of matrices A such that $A^2 = 0$.
- (ii) The set of matrices of trace zero.
- (iii) The set of matrices A such that $Av = 0$, where v is the column vector $(1, 0, 0)^t$.
- (iv) The set of matrices A such that $A^t + A = 0$.

(c) Whenever a subset of M from part (~~a~~ b) is a subspace, find a basis of this subspace and hence determine its dimension. (No proof is required.)

M1GLA Geometry and Linear Algebra
Exam 2015 - Solutions

Question 1

(a) Write $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. For any $t \in \mathbb{R}$ we have $||x + ty||^2 \geq 0$, hence the quadratic equation $(x.x) + 2t(x.y) + t^2(y.y) = 0$ has at most one real root. This implies that $(x.y)^2 - (x.x)(y.y) \leq 0$. $\square$ seen

5 marks

(b) Subtracting the first row multiplied by z^{n-1} from the n-th row, where $n = 2, 3, 4$, we reduce A to

$$A' = \begin{pmatrix} 1 & z & z^2 & z^3 \\ 0 & 0 & 0 & 1-z^4 \\ 0 & 0 & 1-z^4 & z(1-z^4) \\ 0 & 1-z^4 & z(1-z^4) & z^2(1-z^4) \end{pmatrix}$$

Expanding in the first column we compute $det(A) = -(1-z^4)^3$. Thus A is invertible if and only if $z \notin \{\pm 1, \pm i\}$. unseen

7 marks

(c) When $z^4 = 1$ the equivalent system $A'x = 0$ has free variables x_2, x_3, x_4. So the general solution is $(-zx_2 - z^2x_3 - z^3x_4, x_2, x_3, x_4)$. unseen

3 marks

(d) From the explicit formula for a 3×3 determinant given in lectures it is clear that $det(B^t) = det(B)$. Multiplying a row of a matrix by -1 multiplies its determinant by -1. Since 3 is odd, we have $det(-B) = -det(B)$. Thus $det(B) = 0$, which, by lectures, is equivalent to B being non-invertible. unseen

5 marks

Question 2

(a) A rotation matrix is $\begin{pmatrix} cos(x) & -sin(x) \\ sin(x) & cos(x) \end{pmatrix}$, where $x \in \mathbb{R}$. seen

1 mark

(b) A reflection matrix is $\begin{pmatrix} cos(x) & sin(x) \\ sin(x) & -cos(x) \end{pmatrix}$, where $x \in \mathbb{R}$. seen

1 mark [In (a) and (b) it is OK to say that a rotation (respectively, reflection) matrix is an orthogonal matrix with determinant 1 (respectively, -1).]

(c) A matrix A is orthogonal if $A^tA = I$. seen

1 mark

(d) From the definition we see that A is orthogonal if and only if its columns are perpendicular unit vectors. Any unit vector can be written as $(cos(x), sin(x))$. There are exactly two unit vectors perpendicular to $(cos(x), sin(x))$, which differ by sign. The two resulting matrices are those from solutions to part (a) and part (b) above. seen

3 marks

(e) unseen

- (i) True, since the characteristic polynomial is $t^2 - 1$. **2 marks**
- (ii) True. The characteristic polynomial is $t^2 - 2cos(x)t + 1$. If ± 1 is a root, then $cos(x) = \pm 1$ and $sin(x) = 0$. **2 marks**
- (iii) False, e.g. for $\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$. **2 marks**
- (iv) False, e.g. for $\begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$. **3 marks**
- (v) False, because equivalent matrices have the same characteristic polynomial. **2 marks**
- (vi) True. In fact, any reflection matrix A can be written as $P\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}P^{-1}$, where the first column of P is an eigenvector of A with eigenvalue -1, and the second column of P is an eigenvector of A with eigenvalue 1. **3 marks**

Question 3

(a) (i) Write $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$. Then the characteristic polynomial of A is $t^2 - (a+c)t + (ac - b^2)$, so the eigenvalues are $\frac{1}{2}(a+c \pm \sqrt{(a-c)^2 + 4b^2})$. Since either $a - c \neq 0$ or $b \neq 0$, the eigenvalues are real and distinct. seen

3 marks

(ii) We have $Av = \lambda v$ and $Aw = \mu w$, where $\lambda \neq \mu$. Then $v^t A w = \mu(v.w)$ can also be written as $v^t A^t w = (Av)^t w = \lambda(v.w)$. Thus $(v.w) = 0$. seen

4 marks

(b) The associated symmetric 2×2 matrix is $A = \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$. The eigenvalues are 2 and -2. We can choose the respective eigenvectors as $(\sqrt{3}/2, 1/2)^t$ and $(-1/2, \sqrt{3}/2)^t$. The matrix with these columns is the rotation matrix $P = \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}$. Then $P^{-1}AP = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$. We have $x_1 = (\sqrt{3}/2)y_1 - (1/2)y_2$, $x_2 = (1/2)y_1 + (\sqrt{3}/2)y_2$. The conic reduces to $2y_1^2 - 2y_2^2 + 4\sqrt{3}y_1 - 4y_2 = 0$. After the translation $y_1 = z_1 - \sqrt{3}$, $y_2 = z_2 - 1$ we obtain $z_1^2 - z_2^2 = 2$, a hyperbola. unseen

8 marks [4 marks for showing it's a hyperbola, if correctly justified]

(c) We rewrite the equation as $(x_1 - 2)(x_2 - 1) + (x_3 - 3) = 0$. After the translation $x_1' = x_1 - 2$, $x_2' = x_2 - 1$, $x_3' = x_3 - 3$ we obtain $x_1'x_2' + x_3' = 0$, a hyperbolic paraboloid. [Any correct argument is OK.] unseen

5 marks

Question 4

(a) Suppose that $|R| = m$ and $|S| = n$. Write $R = \{r_1, \ldots, r_m\}$ and $S = \{s_1, \ldots, s_n\}$. Since S is a spanning set of V we have seen

$$(r_1, \ldots, r_m)^t = A(s_1, \ldots, s_n)^t$$

for some $m \times n$ matrix A. Similarly,

$$(s_1, \ldots, s_n)^t = B(r_1, \ldots, r_m)^t$$

for some $n \times m$ matrix B. Since R and S are bases we must have $AB = I_m$ and $BA = I_n$ (the identity matrices of sizes $m \times m$ and $n \times n$, respectively). However, it is known that the traces of AB and BA are equal, so $m = n$.
5 marks

(b) seen similar

(i) The set of matrices A such that $A^2 = 0$ is not a subspace, for example because the square of the sum of $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ is non-zero. **3 marks**

The set of matrices satisfying the condition in (ii) is a subspace because it is closed under the addition of matrices and multiplication of matrices by scalars. (**1 mark**) The same for (iii) (**1 mark**) and (iv) (**1 mark**). seen similar

(c)

(ii) A possible basis consists of the matrices with a non-diagonal entry 1 and 0 everywhere else, together with the diagonal matrices with $1, -1, 0$ and $0, 1, -1$ on the diagonal. The dimension is 8. **3 marks** seen similar

(iii) A possible basis consists of the matrices with an entry outside of the first column equal to 1, and 0 everywhere else. Hence the dimension is 6. **3 marks**

(iv) A possible basis consists of the matrices

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

The dimension is 3. **3 marks**

$$x_i^2 y_j^2 + x_j^2 y_i^2 - 2x_i y_i x_j y_j =$$

$$= x_i y_j$$

Imperial College
London

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2015

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Mathematical Methods I

Date: Monday, 11 May 2015. Time: 10.00am – 12.00 noon. Time allowed: 2 hours.

This paper has FOUR questions.

Candidates should start their solutions to each question in a new main answer book

Supplementary books may only be used after the relevant main book(s) are full.

Statistical tables will not be provided.

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Extra credit	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4

- Each question carries equal weight.
- Calculators may not be used.

1. (a) Plot on the same diagram between $x = 0$ and $x = 1$ the two functions

$$f_1(x) = \sin \pi x \qquad f_2(x) = 4x(1-x)$$

indicating carefully which curve is which, and justifying the distinction.

(b) Plot on the same diagram for $x > 0$ the functions

$$f_3(x) = \tfrac{2}{\pi}\tan^{-1} x, \qquad f_4(x) = \tanh\frac{2x}{\pi},$$

indicating carefully which curve is which, and justifying the distinction.

(c) Plot on the same diagram for $x > 0$ the functions

$$f_5(x) = \log x, \qquad f_6(x) = \frac{x-1}{x},$$

indicating carefully which curve is which, and justifying the distinction.

(d) Evaluate the limit

$$\lim_{x\to 1}\left[\frac{f_1(x) - f_6(x)}{f_5(x) - f_2(x)}\right].$$

(e) Calculate

$$\int_0^5 \Big[f_3(x) - f_4(x)\Big]\,dx.$$

(f) Extending the definitions of the functions to complex x in a standard way, find the imaginary x-values (i) for which f_3 is infinite and (ii) those for which f_4 is infinite.

2. (a) For a given constant λ, the function $y(x)$ obeys the differential equation and boundary conditions

$$(1-x^2)y'' - 2xy' + \lambda y = 0, \qquad y(0) = 1, \quad y'(0) = 0.$$

Obtain an expression for the $(n+2)^{th}$ derivative, $y^{(n+2)}(x)$ in terms of lower derivatives. Hence derive a series expansion for $y(x)$ about $x = 0$ when $\lambda = 3$, giving terms up to and including x^6.

(b) Find the radius of convergence of the series in part (a).

(c) Show that for certain special values of λ the infinite series terminates as a polynomial.

(d) If $y_0(x)$ is the solution when $\lambda = 0$, and $y_2(x)$ the solution when $\lambda = 6$, then evaluate the integral

$$\int_{-1}^{1} y_0 y_2\, dx.$$

3. (a) The differentiable function $f(x)$ has a root at $x = \alpha$ and $f(0) \neq 0$. For a given constant k, a sequence of approximations to α is sought by means of the scheme

$$x_0 = 0, \qquad x_{n+1} = x_n + kf(x_n) \quad \text{for } n = 0, 1, 2 \ldots$$

Use the Mean Value Theorem to show that

$$|x_{n+1} - \alpha| = K_n |x_n - \alpha|,$$

for a value of K_n which depends on k and a suitable value of the derivative of f.

(b) What extra condition on $f(x)$ makes it possible to choose k to guarantee that $x_n \to \alpha$ as $n \to \infty$?

If it is known that, for all x, $0 < f'(x) < M$, what range of values of k will give convergence?

(c) Show that there always exists a value of k such that $x_n \to \alpha$ as $n \to \infty$, even if we don't know what it is.

(d) Newton's method is similar to part (a), except that it uses a different value of k each iteration,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad x_0 = 0.$$

If $f(x)$ is twice differentiable, use Taylor's series with a remainder to show that

$$x_{n+1} - \alpha = O(x_n - \alpha)^2.$$

Discuss whether we can expect $x_n \to \alpha$ in this case.

4. (a) For $n \geqslant 2$ express the integral

$$I_n = \int_0^{\pi/4} \tan^n x \, dx$$

in terms of I_{n-2}, and hence find I_n for odd or even integers $n \geqslant 0$ as a finite series.

(b) Determine the limiting function

$$F(x) = \lim_{n \to \infty} \tan^n x \qquad \text{for} \quad 0 \leqslant x \leqslant \tfrac{1}{4}\pi.$$

Infer the limit of I_n as $n \to \infty$.

(c) Deduce from the above that

$$\pi = 4 - \tfrac{4}{3} + \tfrac{4}{5} - \tfrac{4}{7} + \tfrac{4}{9} - \ldots$$

and obtain a similar series whose sum is $\log 2$.

(d) Obtain the series for π and $\log 2$ in part (c) directly, by considering the series for $\log(1+x)$ and $1/(1+x^2)$, which you may quote.

Solutions [ALL UNSEEN, except where explicitly stated]

1. (a) Both curves pass through $(0,0)$, $(1,0)$ with a maximum at $(1/2,1)$. To distinguish the two, one could note that $f_2'(0) = 4 > \pi = f_1'(0)$, or

$$\int_0^1 f_1\,dx = \tfrac{2}{\pi} < \tfrac{2}{3} = \int_0^1 f_2(x)\,dx \qquad [3]$$

(b) Each curve is odd and asymptotes to ± 1 and $f_3'(0) = f_4'(0) = 2/\pi$. However f_4 approaches 1 exponentially, whereas f_3 does it algebraically. For example, for large x consider $f_3' \sim 1/x^2$ whereas $f_4' \sim \operatorname{sech}^2(2x/\pi) \sim \exp(-4x/\pi)$. [3]

(c) Each is infinite at $x = 0$, passes through $(1,0)$ with gradient 1. But for large x, f_5 slowly increases without limit but $f_6 \to 1$. Alternatively, f_6 tends to $-\infty$ more rapidly as x decreases to zero. [3]

(d) Using de l'Hôpital's rule, as the numerator and denominator are both zero at $x = 1$, the required limit is

$$\lim_{x\to 1}\left[\frac{f_1(x) - f_6(x)}{f_5(x) - f_2(x)}\right] = \lim_{x\to 1}\left[\frac{f_1'(x) - f_6'(x)}{f_5'(x) - f_2'(x)}\right] = \frac{-\pi - 1}{1+4} = -\tfrac{1}{5}(\pi + 1) \qquad [3].$$

(e) The integrals are regular, so consider them separately.

$$\int_0^5 \frac{2}{\pi}\tan^{-1} x\,dx = \left[\frac{2x}{\pi}\tan^{-1} x\right]_0^5 - \frac{2}{\pi}\int_0^5 \frac{x}{1+x^2}\,dx = \frac{10}{\pi}\tan^{-1} 5 - \frac{1}{\pi}\log 26.$$

$$\int_0^5 \tanh\frac{2x}{\pi}\,dx = \tfrac{\pi}{2}\Big[\log\cosh(2x/\pi)\Big]_0^5 = \frac{\pi}{2}\log\cosh(10/\pi)$$

Thus

$$\int_0^5 (f_3 - f_4)\,dx = \frac{10}{\pi}\tan^{-1} 5 - \frac{1}{\pi}\log 26 - \frac{\pi}{2}\log\cosh(10/\pi). \qquad [4]$$

(f) Now $\tanh(ix) = i\tan(x)$ and $\tan(ix) = i\tanh(x)$. It follows that $f_4(x)$ is infinite whenever $2x/\pi = i(\frac{1}{2}\pi + n\pi)$ or at $x = i\frac{1}{4}\pi^2(1+2n)$ [2]
Furthermore there are no values of x such that $\tanh x = \pm 1$, so that $\tan^{-1}(\pm i)$ is also singular, so that $f_3(\pm i)$ is formally infinite. (Or consider the derivative $1/(1+x^2)$.) [2]

Total : 20

2. (a) Differentiating n times by Leibniz, we have

$$(1-x^2)y^{(n+2)} - 2nxy^{(n+1)} - 2n(n-1)/2y^{(n)} - 2xy^{(n+1)} - 2ny^{(n)} + \lambda y^{(n)} = 0.$$

or

$$(1-x^2)y^{(n+2)} - (2n+2)xy^{(n+1)} = [n(n+1) - \lambda]y^{(n)}. \qquad [4]$$

Substituting $x = 0$, we have

$$y^{(n+2)}(0) = [n(n+1) - \lambda]y^{(n)}(0). \qquad [1]$$

Now as $y'(0) = 0$, all odd derivatives vanish at 0, and the series only has even terms. By repeated use of the above result, we have when $\lambda = 3$

$$y(0) = 1, \quad y''(0) = -3, \quad y^{(4)}(0) = 3y''(0) = -9, \quad y^{(6)}(0) = 17y^{(4)}(0) = -9 * 17$$

so that

$$y(x) + \sum_{n=0}^{\infty} \frac{y^{(n)}(0)x^n}{n!} = 1 - \frac{3}{2}x^2 - \frac{3}{8}x^4 - \frac{17}{80}x^6 + O(x^8) \qquad [5]$$

(b) Looking at the ratio of adjacent terms, we have

$$\left| \frac{y^{(n+2)}(0)x^{n+2}/(n+2)!}{y^{(n)}(0)x^n/n!} \right| = \frac{|n(n+1) - \lambda|x^2}{(n+1)(n+2)} \to x^2 \qquad \text{as } n \to \infty.$$

By the ratio test, the series converges for $|x| < 1$ so the radius of convergence is 1. [3]

(c) Now if $\lambda = k(k+1)$ for some positive integer k, then $y^{(k+2)}(0) = 0$ as are all higher derivatives. It follows that the series terminates as a polynomial (of order k – not required). [3]

(d) When $\lambda = 0$, the series terminates after the first term, so that $y_0(x) = 1$. When $\lambda = 6$, this corresponds to $k = 2$. The solution is $y_2(x) = 1 - 3x^2$. So the reuired integral is

$$\int_{-1}^{1} 1(1 - 3x^2)\, dx = \left[x - x^3 \right]_{-1}^{1} = 0. \qquad [4]$$

Total : 20

3. (a) We have $f(\alpha) = 0$. The MVT states that there exists a value ξ_n between x_n and α such that

$$f'(\xi_n)(x_n - \alpha) = f(x_n) - f(\alpha) = f(x_n).$$

It follows that

$$x_{n+1} - \alpha = x_n - \alpha + kf'(\xi_n)[(x_n - \alpha)] = [1 + kf'(\xi_n)](x_n - \alpha).$$

so we may define

$$K_n = |1 + kf'(\xi_n)|, \qquad \Longrightarrow \qquad |x_{n+1} - \alpha| = K_n|x_n - \alpha|. \qquad [5]$$

(b) Clearly, $K_n < 1$ iff $-2 < kf'(\xi_n) < 0$. However, $f'(\xi_n)$ may vary in sign for different n, in which case, no single value of k suffices. If we require that f' is of single sign over the domain of interest, then we can choose k to be of opposite sign. We must then choose $|k|$ small enough such that $1 + kf' > -1$. Thus if $0 < f' < M$, we will choose $0 > k > -2/M$. [5]

(c) If we choose $k = \alpha/f(x_0)$, then $x_1 = \alpha$ and then $x_2 = \alpha$ and so on. Clearly then $x_n \to \alpha$. As this value of k depends on α, we don't know what it is, however. [4]

(d) The Taylor series with remainder states for some η and μ

$$f(x_n) = f(\alpha) + (x_n - \alpha)f'(\alpha) + \tfrac{1}{2}(x_n - \alpha)^2 f''(\eta) \qquad \text{and} \qquad f'(x_n) = f'(\alpha) + (x_n - \alpha)f''(\mu)$$

Substituting in, we have

$$x_{n+1} = x_n - \frac{(x_n - \alpha)f'(\alpha) + O(x_n - \alpha)^2}{f'(\alpha) + O(x_n - \alpha)} = \alpha + O(x_n - \alpha)^2$$

Or as required

$$(x_{n+1} - \alpha) = O(x_n - \alpha)^2 < A(x_n - \alpha)^2, \qquad [4]$$

for some A. Thus provided $|x_n - \alpha|$ is small enough, $|x_{n+1} - \alpha|$ will be smaller still. So we expect Newton's method to converge provided our starting point ($x = 0$) is sufficiently close to the actual root ($x = \alpha$). [2]

Total : 20

4. (a) Writing $\tan^2 x = \sec^2 x - 1$, we have

$$I_n = \int_0^{\pi/4} \sec^2 x \tan^{n-2} x \, dx - I_{n-2} = \frac{1}{n-1}\Big[\tan^{n-1} x\Big]_0^{\pi/4} - I_{n-2} = \frac{1}{n-1} - I_{n-2} \quad [3]$$

Thus if n is even

$$I_n = \frac{1}{n-1} - \frac{1}{n-3} + \ldots - (-1)^{n/2}\left(1 - \int_0^{\pi/4} 1\, dx\right)$$

Or

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} + \ldots \pm \frac{1}{n-1} \mp I_n \quad [2]$$

If n is odd, then

$$I_n = \frac{1}{n-1} - \frac{1}{n-3} + \ldots - (-1)^{(n-1)/2}\left(\frac{1}{2} - \int_0^{\pi/4} \tan x\, dx\right)$$

Or

$$\Big[-\log\cos x\Big]_0^{\pi/4} = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} + \ldots \pm \frac{1}{n-1} \mp I_n,$$

and so

$$\tfrac{1}{2}\log 2 = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} + \ldots \pm \frac{1}{n-1} \mp I_n \quad [3]$$

(b) Now $0 \leqslant \tan x \leqslant 1$ over this range. Furthermore, as $n \to \infty$, $r^n \to 0$ for $0 < r < 1$. It follows that

$$F(x) = 0 \qquad \text{for } x \neq \tfrac{1}{4}\pi, \quad F(\tfrac{1}{4}\pi) = 1. \quad [2]$$

We infer that $I_n \to 0$ as $n \to \infty$. [1]

(c) Rearranging the above series, we have therefore in the limit as $n \to \infty$,

$$\pi = 4 - \tfrac{4}{3} + \tfrac{4}{5} - \tfrac{4}{7} + \tfrac{4}{9} \ldots = \sum_{m=1}^{\infty} \frac{4(-1)^{m-1}}{2m-1} \quad [2]$$

and

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \quad [3]$$

(d) [SEEN] We have $\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \ldots$ and substituting $x = 1$ gives the correct formula. Now

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 + \ldots \qquad \Longrightarrow \qquad \int_0^1 \frac{dx}{1+x^2} = \Big[x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \ldots\Big]_0^1$$

giving

$$\tan^{-1} 1 = \tfrac{1}{4}\pi = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \ldots \quad [4]$$

Total : 20

Imperial College London

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2015

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Mathematical Methods II

Date: Friday, 15 May 2015. Time: 10.00am – 12.00noon. Time allowed: 2 hours.

This paper has FOUR questions.

Candidates should start their solutions to each question in a new main answer book

Supplementary books may only be used after the relevant main book(s) are full.

Statistical tables will not be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers to each question as per the table below.

Raw mark	up to 12	13	14	15	16	17	18	19	20
Extra credit	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4

- Each question carries equal weight.
- Calculators may not be used.

1. (i) Find the general solution of the differential equation:

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 2\cosh(3x).$$

(ii) Consider the following first order nonlinear ODE:

$$\frac{dy}{dt} = y(y-2) - (r-2) \tag{1}$$

where $r \in \mathbb{R}$ is a parameter.

(a) Find the fixed points of the system y^* and determine their stability for all values of r.

(b) Sketch the possible values of y^* against r to produce the bifurcation diagram, and classify any bifurcation found.

(c) For $r = 2$, indicate the set of initial conditions that lead to divergence of $y(t)$ as $t \to \infty$.

2. (i) Consider the following system of coupled linear differential equations:

$$\begin{aligned}\frac{dx}{dt} &= x - 2y \\ \frac{dy}{dt} &= 4x + 3y\end{aligned} \tag{2}$$

(a) Find the general solution of this system in terms of its eigenvectors and eigenvalues. You may leave your answer in complex form.

(b) Sketch the phase portrait for this system in the (x, y) plane, using particular instances of the vector field to give correct qualitative detail to the phase portrait. Describe carefully the asymptotic behaviour of the system as $t \to \infty$.

(ii) Consider now the associated system

$$\begin{aligned}\frac{dx}{dt} &= x - 2y \\ \frac{dy}{dt} &= \left(4 + \frac{\varepsilon}{2}\right) x + (3 + \varepsilon) y\end{aligned}$$

where $\varepsilon \in \mathbb{R}$ is a parameter.

Either by sketching the trace-determinant $(\tau - \Delta)$ diagram or otherwise, find the value(s) of ε at which the system changes stability explaining in each case the manner in which the change in stability appears.

3. Consider the function

$$u(x, y) = y^2 - y + (x^2 - 4x)(y - 1).$$

(i) Find the set of points (x_*, y_*) that form the zero contour, i.e., $u(x_*, y_*) = 0$.

(ii) Find the stationary points of $u(x, y)$ and classify them, stating clearly the conditions used.

(iii) Sketch the contour plot of $u(x, y)$ indicating the zero contour ($u = 0$) and a few other representative contours. Indicate in your sketch the positions of the stationary points and the regions of positive and negative u.

(iv) Consider the closed domain R bounded by the zero contour $u(x_*, y_*) = 0$. Calculate the double integral over the domain R:

$$\iint_R dx\, dy$$

Leave your answer in the simplest form.

4. (i) Consider the following first order differential equation:

$$\frac{dy}{dx} = \frac{-e^y - \dfrac{xy}{\sqrt{1+x^2}}}{xe^y + \sqrt{1+x^2}}$$

(a) Determine if this equation is exact.

(b) Solve the equation either directly or with the aid of a suitable integrating factor.

(ii) Consider the following function $u = u(x, y)$ given in implicit form:

$$xu \log y + e^{yu} \cos x = 0.$$

Obtain the gradient $\nabla u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)$. You may leave your answer in implicit form.

(iii) The production Q of a factory depends on two positive variables $x, y > 0$

$$Q(x, y) = A\, x\, y^2,$$

where $A \in \mathbb{R}^+$ is a constant. The production needs to be optimised under the constraint

$$x + y = B,$$

where $B \in \mathbb{R}^+$ is the budget.

(a) By introducing a Lagrange multiplier λ or otherwise, find the values x^* and y^* that maximise $Q(x, y)$ under this constraint, and obtain the maximum production $Q^* = Q(x^*, y^*)$.

(b) Show that

$$\frac{dQ^*}{dB} = \lambda,$$

where λ is the Lagrange multiplier obtained above. Explain your answer.

EXAMINATION SOLUTIONS [redacted]
2014-15

Course **MIM2**

Question 1

Marks & seen/unseen

Parts

(i)

$$\mathcal{L}[y] = y'' + 6y' + 9y = 2\cosh(3x)$$

Method seen not this example

$y_{CF} = e^{\lambda x}$ (Ansatz)

$$\mathcal{L}[y_{CF}] = e^{\lambda x}\left(\lambda^2 + 6\lambda + 9\right) = 0$$

$$(\lambda+3)^2 = 0 \quad \lambda_1 = \lambda_2 = -3$$

$$\boxed{y_{CF} = C_1 e^{-3x} + C_2\, x e^{-3x}}$$

3

since $\{e^{-3x}, x e^{-3x}\}$ form a basis of the linear operator.

(no need to use variation of parameters to get $x e^{-3x}$)

---

y_{PI} : we split into two parts:

$$\mathcal{L}[y_{PI}] = \mathcal{L}[y_{PI_1}] + \mathcal{L}[y_{PI_2}] = e^{3x} + e^{-3x}$$

- $\mathcal{L}[y_{PI_1}] = e^{3x}$

$y_{PI_1} = B e^{3x}$ (ansatz and find B through MUC)

Setter's initials

Checker's initials [redacted]

	EXAMINATION SOLUTIONS [redacted] 2014-15	Course MIM2
Question 1		Marks & seen/unseen
Parts (i)	$\mathcal{L}[Be^{3x}] = Be^{3x}(9+6\cdot 3+9) = \underline{e^{3x}}$ $\Rightarrow \quad B = \frac{1}{36}$ $\underline{y_{PI_1} = \frac{1}{36} e^{3x}}$	3
	$\mathcal{L}[y_{PI_2}] = e^{-3x}$ Ansatz: $y_{PI_2} = C(x)\, e^{-3x}$ $\mathcal{L}[C(x)e^{-3x}] = 9Ce^{-3x} + 6(C'e^{-3x} - 3Ce^{-3x}) + (C''e^{-3x} - 3C'e^{-3x} - 3C'e^{-3x} + 9Ce^{-3x})$ $= C''e^{-3x} = \underline{e^{-3x}}$ $\Rightarrow \quad C'' = 1 \qquad C = \frac{x^2}{2}$ $\underline{y_{PI_2} = \frac{1}{2}x^2 e^{-3x}}$ (OK also if they guess $y_{PI_2} = Cx^2e^{-3x}$)	3
	Setter's initials	Checker's initials [redacted]	Page number 2

EXAMINATION SOLUTIONS [redacted]
2014-15

Course
MIM2

Question
1

Marks & seen/unseen

Parts
(i)

Full solution:

$$\boxed{y = y_{CF} + y_{PI} = C_1 e^{-3x} + C_2 x e^{-3x} + \frac{1}{36} e^{3x} + \frac{1}{2} x^2 e^{-3x}}$$

1

10 for part (i)

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EXAMINATION SOLUTIONS [redacted]
2014-15

Course **MIM2**

Question 1

Marks & seen/unseen

Parts

(ii) (a)

$$\frac{dy}{dt} = y(y-2) - (r-2) \quad = f(y)$$

Method seen [illegible] this example

Fixed points: y^*, $\left.\frac{dy}{dt}\right|_{y^*} = 0 = f(y^*)$

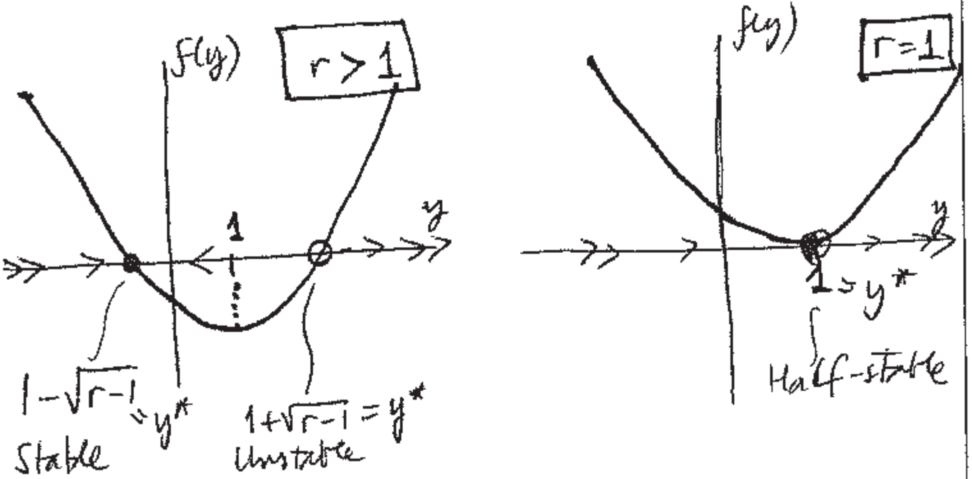
$$y^{*2} - 2y^* + 1 - r + 1 = 0$$

$$(y^* - 1)^2 - r + 1 = 0$$

$$\underline{y^* = 1 \pm \sqrt{r-1}}$$

2

Stability: Sketch $f(y)$ for relevant values of r:



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EXAMINATION SOLUTIONS [redacted]

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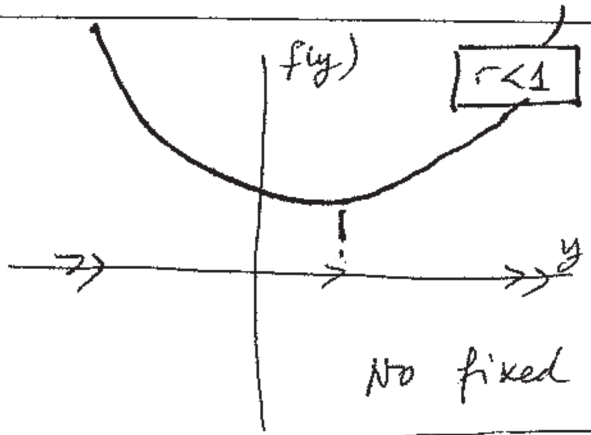
Course **MIM2**

Question

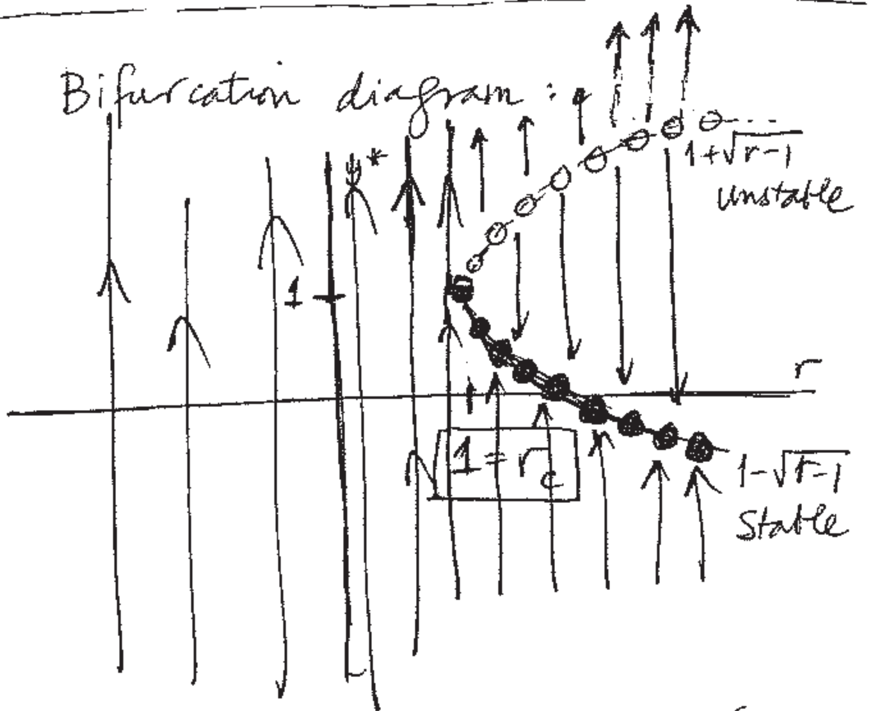
Marks & seen/unseen

Parts



No fixed points

(b) Bifurcation diagram:



3

There is a saddle-node bifurcation at $\underline{r_c = 1}$:

2

- For $r < r_c$, no F.P.
- For $r = r_c$, half-stable F.P.
- For $r > r_c$, $\left\{\begin{matrix}\text{stable} \\ \text{unstable}\end{matrix}\right\}$ F.P.

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EXAMINATION SOLUTIONS [redacted]

2014-15

Course: M1 M2

Question: 1

Marks & seen/unseen

Parts

(ii)

(c) For $r=2$, if $y(0)>2$ then

$$\lim_{t\to\infty} y(t) \to +\infty$$

1

10 for part (ii)

Setter's initials

Checker's initials [redacted]

EXAMINATION SOLUTIONS [redacted]

2014-15

Course

MIM2

Question

2

Marks & seen/unseen

Parts

(i)

(a)

Method seen (not this example)

$$A = \begin{pmatrix} 1 & -2 \\ 4 & 3 \end{pmatrix} \qquad \vec{y} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{vmatrix} 1-\lambda & -2 \\ 4 & 3-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(3-\lambda) + 8 = 0$$

$$\lambda^2 - 4\lambda + 11 = 0$$

$$\lambda = \frac{4 \pm \sqrt{16-44}}{2} = 2 \pm \sqrt{-7}$$

2

$$\underline{\lambda_1 = 2 + i\sqrt{7}} \qquad \underline{\lambda_2 = 2 - i\sqrt{7}}$$

$$\vec{V}_1 : \left[(1-2) - i\sqrt{7}\right] V_{1x} - 2\, V_{1y} = 0$$

$$V_{1y} = \left(-\frac{1}{2} - i\frac{\sqrt{7}}{2}\right) V_{1x}$$

$$\vec{V}_1 = \begin{pmatrix} 1 \\ -\frac{1}{2}(1+i\sqrt{7}) \end{pmatrix}$$

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2014-15

Course MIM2

Question 2

Marks & seen/unseen

Parts (i) (a)

$\vec{v}_2$: $\left[(1-2)+i\sqrt{7}\right] v_{2x} - 2 v_{2y} = 0$

$$v_{2y} = \left(-1 + i\sqrt{7}\right)\frac{1}{2}$$

$$\vec{v}_2 = \begin{pmatrix} 1 \\ -\frac{1}{2}(1-i\sqrt{7}) \end{pmatrix}$$ 2

$$\boxed{\begin{aligned} \vec{y} &= c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2 \\ &= e^{2t}\left[c_1 e^{i\sqrt{7}t}\begin{pmatrix} 1 \\ -\frac{1}{2}(1+i\sqrt{7}) \end{pmatrix} + \right. \\ &\qquad \left. + c_2 e^{-i\sqrt{7}t}\begin{pmatrix} 1 \\ -\frac{1}{2}(1-i\sqrt{7}) \end{pmatrix}\right] \end{aligned}}$$ 1

(OK to leave it in complex form.)

c_1 and c_2 are complex.

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EXAMINATION SOLUTIONS [redacted]
2014-15

Course MIM2

Question 2

Marks & seen/unseen

Parts (i) (b)

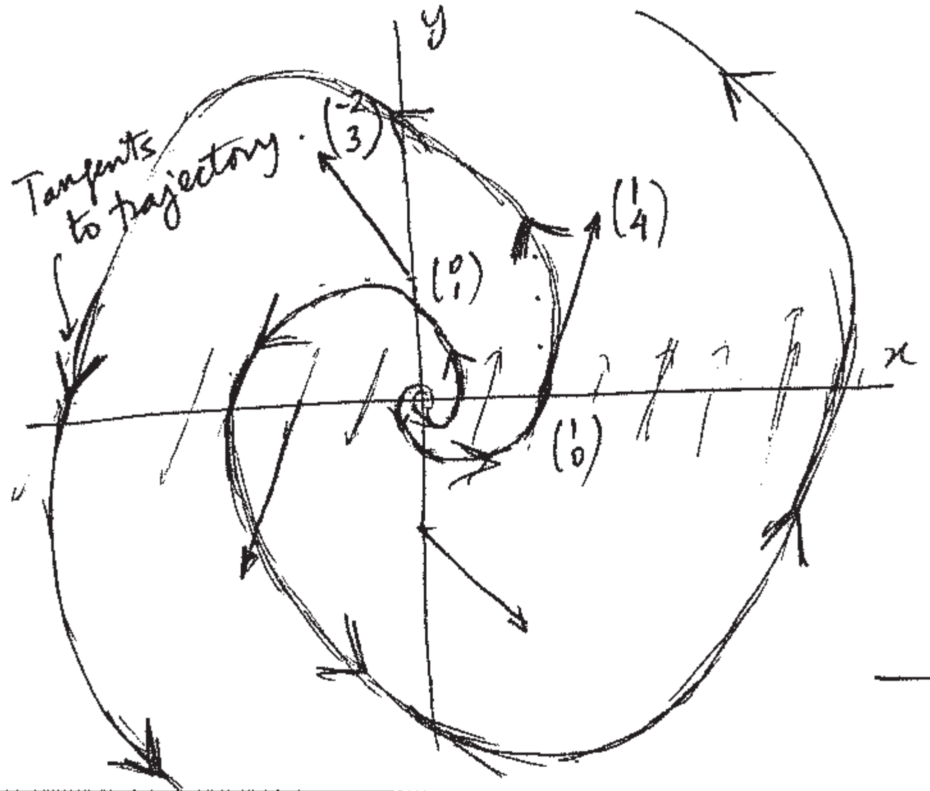
Phase portrait.

$\left.\begin{array}{l}\tau = 4 \\ \Delta = 11\end{array}\right\} \quad \tau^2 - 4\Delta = -28 < 0$

From eigenvalues also it is clearly an
<u>unstable or repelling spiral</u> — 1

To get the direction of the flow, use some instances of the vector field, e.g.

$A\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\4\end{pmatrix} \quad \& \quad A\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-2\\3\end{pmatrix}$ — 2



2 (careful with vector field)

10 for part (i)

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2014-15

Course **MIM2**

Question 2

Marks & seen/unseen

Parts (ii)

Associated system

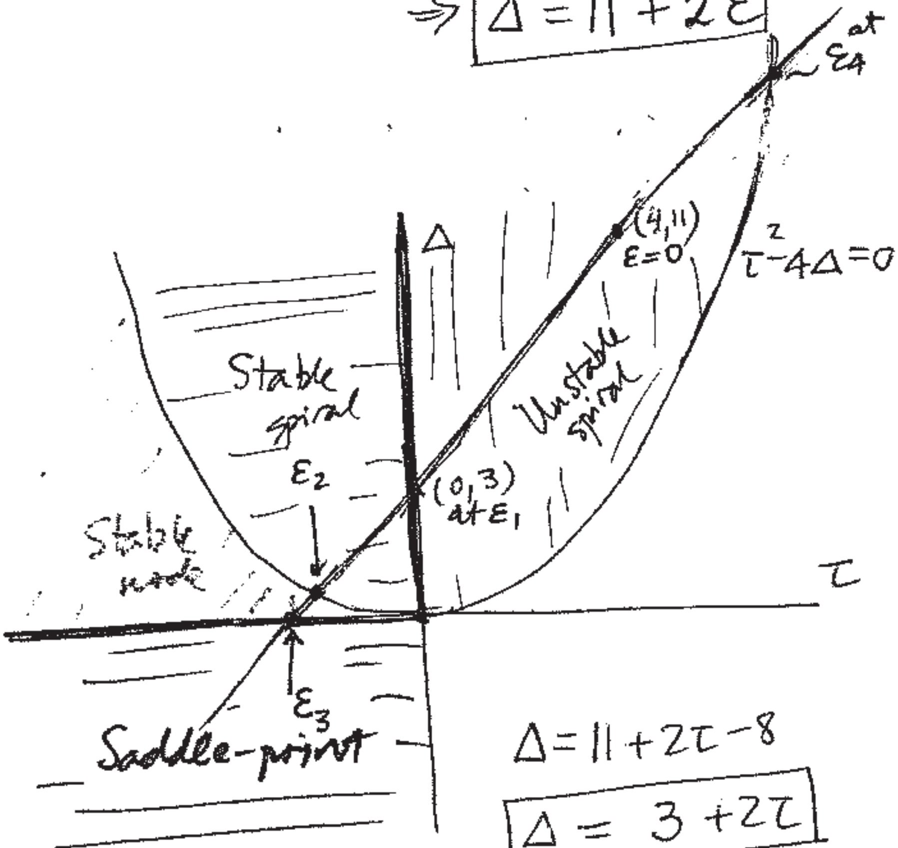
$$\begin{pmatrix} 1 & -2 \\ 4+\frac{\varepsilon}{2} & 3+\varepsilon \end{pmatrix}$$

$$\boxed{\tau = 4+\varepsilon}$$

$$\Delta = (3+\varepsilon)+2\left(4+\frac{\varepsilon}{2}\right)$$

$$\Rightarrow \boxed{\Delta = 11+2\varepsilon}$$

Method seen (not this example)



5

$$\Delta = 11+2\tau-8$$

$$\boxed{\Delta = 3+2\tau}$$

under changes in ε

Changes in stability of the system only occur at ε_1 and ε_3 indicated above

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EXAMINATION SOLUTIONS [redacted] 2014-15

Course MIM2

Question 2

Marks & seen/unseen

Parts

(ii)

At ε_1 corresponding to

$$\left.\begin{array}{l}\tau = 0\\ \Delta = 3\end{array}\right\} \quad \varepsilon_1 = -4$$

the system goes from an unstable spiral to a stable spiral through a centre. 2

At ε_3 corresponding to:

$$\left.\begin{array}{l}\Delta = 0\\ \tau = \frac{-3}{2}\end{array}\right\} \quad \varepsilon_3 = \frac{-11}{2}$$

the system goes from a stable node to a saddle point. 2

The other values ε_2 and ε_4 do not change the stability of the system. 1

10 total for (ii)

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EXAMINATION SOLUTIONS [redacted]
2014-15

Course **MIM2**

Question 3

Marks & seen/unseen

Parts

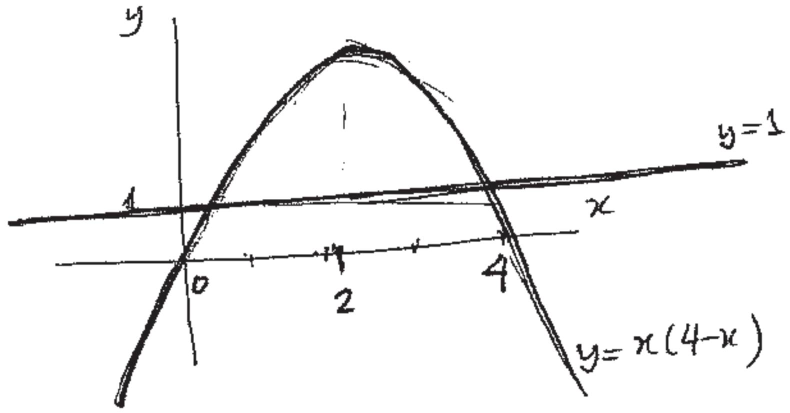
(i)

$$u(x,y) = y^2 - y + (x^2-4x)(y-1)$$
$$= (y-1)(y+x^2-4x)$$

Method seen not this example

Contour $u=0$ enclosed by

$$\begin{cases} y = 1 \\ y = 4x - x^2 = x(4-x) \end{cases}$$



2 for (i)

(ii)

Stationary points $\vec{x}^*$: $\nabla u(\vec{x}^*) = 0$

$$\left.\frac{\partial u}{\partial x}\right|_{\vec{x}^*} = (y^*-1)(2x^*-4) = 0 \qquad \begin{cases} y^* = 1 \\ \text{or} \\ x^* = 2 \end{cases}$$

$$\left.\frac{\partial u}{\partial y}\right|_{\vec{x}^*} = 2y^*-1 + x^{*2} - 4x^* = 0$$

• For $y^*=1$ $\qquad x^{*2} - 4x^* + 1 = 0$

$$x^* = \frac{4 \pm \sqrt{16-4}}{2}$$

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Page number

	EXAMINATION SOLUTIONS [redacted] 2014-15	Course MIM2
Question 3		Marks & seen/unseen

Parts

$$x^* = 2 \pm \sqrt{3}$$

- For $x^* = 2$

$$2y^* - 1 + 4 - 8 = 0$$

$$y^* = \frac{5}{2}$$

So the extrema/stationary points are:

$$P_3 = (2+\sqrt{3}, 1) \qquad P_1 = (2-\sqrt{3}, 1)$$

$$P_2 = \left(2, \frac{5}{2}\right)$$

3

Character of the points:

$$\left.\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= 2(y-1) \\ \frac{\partial^2 u}{\partial x \partial y} &= 2(x-2) \\ \frac{\partial^2 u}{\partial y^2} &= 2 \end{aligned}\right\} H(\vec{x}) = 2\begin{pmatrix} y-1 & x-2 \\ x-2 & 1 \end{pmatrix}$$

$$\frac{1}{2}\tau = y$$

$$\frac{1}{4}\Delta = y - 1 - (x-2)^2$$

To evaluate the character plug P_1, P_2, P_3 into $H(\vec{x})$ and check eigenvalues or (τ, Δ)

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EXAMINATION SOLUTIONS [redacted]

2014-15

Course **MIM2**

Question 3

Marks & seen/unseen

Parts

$P_1 = (2-\sqrt{3}, 1)$ $\quad H(P_1) = \begin{pmatrix} 0 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix} \cdot 2$

Saddle point $\quad \tau = 2 \quad \Delta = (-3)\cdot 4$

$P_2 = (2, \frac{5}{2})$ $\quad H(P_2) = \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & 1 \end{pmatrix} \cdot 2$

Minimum $\quad \tau = 5 \quad \Delta = 6$

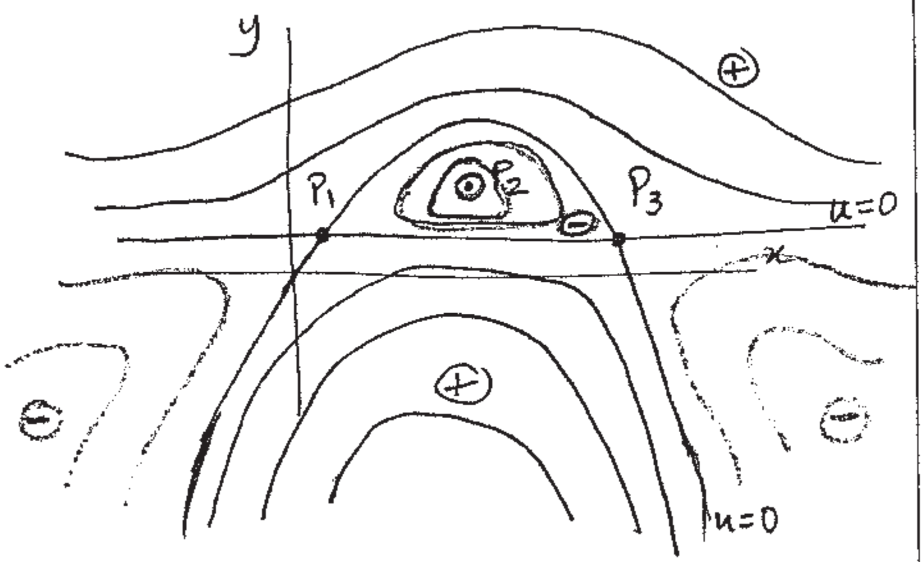
$P_3 = (2+\sqrt{3}, 1)$ $\quad H(P_3) = \begin{pmatrix} 0 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \cdot 2$

Saddle point $\quad \tau = 2 \quad \Delta = -12$

3

Tot: 6 for (ii)

(iii) Sketching the function with typical contours:



4

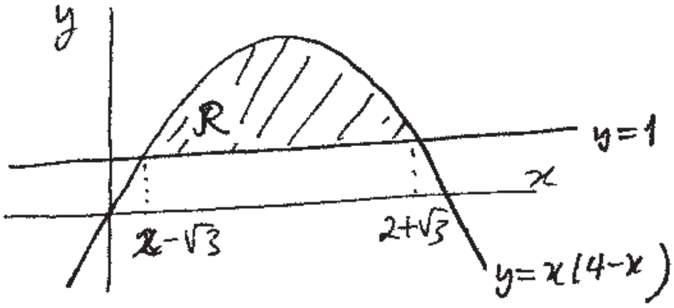
Tot: 4 for (iii)

	EXAMINATION SOLUTIONS [redacted] 2014-15	Course MIM2
Question 3		Marks & seen/unseen

Parts

(iv) Region contained within $u(x,y)=0$

Method seen not this example



$$\iint_R dx\,dy = \int_{x=2-\sqrt{3}}^{2+\sqrt{3}} \int_{y=1}^{x(4-x)} dx\,dy =$$ 2

$$= \int_{x=2-\sqrt{3}}^{2+\sqrt{3}} \Big[y\Big]_1^{x(4-x)} dx = \int_{x=2-\sqrt{3}}^{2+\sqrt{3}} \left[x(4-x)-1\right] dx$$

$$= \left[2x^2 - \frac{x^3}{3} - x\right]_{2-\sqrt{3}}^{2+\sqrt{3}} =$$

$$= \left[x\cdot\left(2x - \frac{x^2}{3} - 1\right)\right]_{2-\sqrt{3}}^{2+\sqrt{3}} \overset{*}{=} \text{ over}$$ 3

From above $x^2-4x+1=0$ for $x=2+\sqrt{3}$ and $x=2-\sqrt{3}$

So $2x - \frac{1}{3}x^2 - 1 = 2x - 1 - \frac{1}{3}(4x-1)$

$= \frac{2}{3}x - \frac{2}{3} = \frac{2}{3}(x-1)$ at those points

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2014-15

Course: MIM2

Question: 3

Parts: (iv)

Marks & seen/unseen

From ⊗ in previous page:

$$\iint_R dx\,dy = \left[x\cdot\frac{2}{3}(x-1)\right]_{2-\sqrt{3}}^{2+\sqrt{3}} =$$

$$= \frac{2}{3}\left[x^2-x\right]_{2-\sqrt{3}}^{2+\sqrt{3}} = \frac{2}{3}\left[4x-1-x\right]_{2-\sqrt{3}}^{2+\sqrt{3}}$$

$$= \frac{2}{3}\left[3x-1\right]_{2-\sqrt{3}}^{2+\sqrt{3}} = \frac{2}{\cancel{3}}\cdot\cancel{3}\left[\cancel{2}+\sqrt{3}-\cancel{2}+\sqrt{3}\right]$$

$$= 4\sqrt{3}$$

3

Tot: 8 for (iv)

Also OK if they obtain this result by slicing first along x and then y:

$$\int_1^4 2\sqrt{4-y}\,dy = 2\cdot\frac{2}{3}(4-y)^{3/2}\Big]_4^1 = \frac{4}{3}3^{3/2} = 4\sqrt{3}$$ ✓

Total for Q2:

2+6+4+8 = 20

Setter's initials

Checker's initials [redacted]

EXAMINATION SOLUTIONS [redacted]

2014-15

Course

MIM2

Question

4

Marks & seen/unseen

Parts

(i)

$$\frac{dy}{dx} = \frac{-\left(e^y + \frac{xy}{\sqrt{1+x^2}}\right)}{xe^y + \sqrt{1+x^2}}$$

Seen method not this example

$$\underbrace{\left(e^y + \frac{xy}{\sqrt{1+x^2}}\right)}_{F(x,y)} dx + \underbrace{\left(xe^y + \sqrt{1+x^2}\right)}_{G(x,y)} dy = 0$$

$$\left.\begin{aligned} \frac{\partial F}{\partial y} &= e^y + \frac{x}{\sqrt{1+x^2}} \\ \frac{\partial G}{\partial x} &= e^y + \frac{2x}{2\sqrt{1+x^2}} \end{aligned}\right\} \frac{\partial F}{\partial y} = \frac{\partial G}{\partial x}$$

Exact ✓

3

Then, $\exists \; u(x,y) = 0$ such that

$$\frac{\partial u}{\partial x} = F \quad \text{and} \quad \frac{\partial u}{\partial y} = G$$

$$\frac{\partial u}{\partial x} = e^y + \frac{xy}{\sqrt{1+x^2}}$$

$$u = xe^y + y\sqrt{1+x^2} + f(y) + C_1$$

$$\frac{\partial u}{\partial y} = xe^y + \sqrt{1+x^2} + \frac{df}{dy} = G \Rightarrow \frac{df}{dy} = 0$$

Setter's initials

Checker's initials [redacted]

Page number

	EXAMINATION SOLUTIONS [redacted] 2014-15	Course MIM2
Question 4		Marks & seen/unseen
Parts (i)	$$u(\cancel{x},y) = xe^{y} + y\sqrt{1+x^2} + C = 0$$ This is the solution in implicit form.	5 8 for (i)
	Setter's initials Checker's initials [redacted]	Page number 18

EXAMINATION SOLUTIONS [redacted]

2014-15

Course

MIM2

Question

4

Marks & seen/unseen

Parts

(ii)

$u = u(x,y)$ is given in implicit form as

$$F(x,y,u) = xu \ln y + e^{yu} \cos x = 0$$

Method seen (not this example)

$$dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial u} du = 0$$

$$\frac{\partial F}{\partial x} = u \ln y - \sin x \, e^{yu}$$

3

$$\frac{\partial F}{\partial y} = \frac{xu}{y} + u e^{yu} \cos x$$

$$\frac{\partial F}{\partial u} = x \ln y + y e^{yu} \cos x$$

$$du = \frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial u}} dx + \frac{-\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial u}} dy$$

$$= \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

3

$$\nabla u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right)$$

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EXAMINATION SOLUTIONS [redacted] 2014-15

Course MIM2

Question 4

Marks & seen/unseen

Parts

(ii)

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{-u\ln y + \sin x\, e^{yu}}{x\ln y + y e^{yu}\cos x} \\[2ex] \dfrac{\partial u}{\partial y} = \dfrac{-\frac{xu}{y} - u e^{yu}\cos x}{x\ln y + y e^{yu}\cos x} \end{cases}$$

This form is already ok. = Full marks.

$$\frac{\partial u}{\partial x} = \frac{-u\ln y - \frac{xu\ln y}{\cos x}\sin x}{x\ln y - xyu\ln y}$$

$$= \frac{-u\cancel{\ln y}\,(1 + x\tan x)}{x\cancel{\ln y}\,(1-uy)}$$

$$= \frac{\tan x + \frac{1}{x}}{y - \frac{1}{u}}$$

Other forms after simplification All ok.

Total 6 for (ii)

$$\frac{\partial u}{\partial y} = \frac{-\frac{xu}{y} + xu^2\ln y}{x\ln y\,(1-uy)} = \frac{\cancel{x}u\left(u\ln y - \frac{1}{y}\right)}{\cancel{x}\ln y\,(1-uy)}$$

$$= \frac{\frac{1}{y\ln y} - u}{y - \frac{1}{u}}$$

Setter's initials

Checker's initials [redacted]

EXAMINATION SOLUTIONS [redacted]
2014-15

Course **MIM2**

Question 4

Marks & seen/unseen

Parts

(iii)

Production

$$Q(x,y) = A\, x y^2$$

with budget $x + y = B$

Method seen.

Maximize production under constraints:

$$Q = A(B-y)y^2$$

$$\frac{dQ}{dy} = A\left[-y^2 + (B-y)2y\right]$$

$$\left.\frac{dQ}{dy}\right|_{y^*} = 0 \Rightarrow y^*\left[2(B-y^*) - y^*\right] = 0$$

$y^* = 0$ trivial minimum

$$y^* = \frac{2B}{3}$$

$$x^* = \frac{1}{3}B$$

3 (for solution either with or without Lagrange multipli

Maximum production:

$$Q^* = Q(x^*, y^*) = A\,\frac{B}{3}\left(\frac{2B}{3}\right)^2 = AB^3\,\frac{4}{27}$$

OK to solve like this or with Lagrange multipliers (over) →

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Checker's initials [redacted]

EXAMINATION SOLUTIONS [redacted]
2014-15

Course: MIM2

Question: 4

Marks & seen/unseen

Parts: (iii)

Consider now:

$$\frac{dQ^*}{dB} = A\, 3B^2 \frac{4}{27} = AB^2\frac{4}{9}$$

Lagrange multipliers:

$$\mathcal{L} = Axy^2 - \lambda(x+y-B)$$

$$\left.\begin{aligned} \frac{\partial \mathcal{L}}{\partial x} &= Ay^2 - \lambda \\ \frac{\partial \mathcal{L}}{\partial y} &= 2Axy - \lambda \\ \frac{\partial \mathcal{L}}{\partial \lambda} &= x+y-B \end{aligned}\right\} (x^*, y^*, \lambda^*) \text{ is the extremum}$$

$\Downarrow$

$$\left.\begin{aligned} Ay^{*2} &= \lambda^* \\ 2Ax^*y^* &= \lambda^* \end{aligned}\right\} y^* = 2x^*$$

$$x^* + y^* = B$$

$$y^* = \frac{2B}{3} \quad x^* = \frac{B}{3}$$

$$\lambda^* = A\left(\frac{2B}{3}\right)^2 = AB^2\frac{4}{9} = \frac{dQ^*}{dB}$$

3 (for showing that $\frac{dQ^*}{dB} = \lambda$ and explaining)

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Checker's initials [redacted]

EXAMINATION SOLUTIONS [redacted]

2014-15

Course

M1 M2

Question

4

Marks & seen/unseen

Parts

(iii)

The Lagrange multiplier gives the variation induced in Q^* by a change in the value of the constraint.

i.e. when we are at the optimum production, how much will our production increase if we relax the constraint.

6 for (iii)

8+6+6
(i) (ii) (iii)
= Total 20 for Q4

Setter's initials

Checker's initials [redacted]

Page number
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Imperial College
London

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2015

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Analysis I

Date: Wednesday, 13 May 2015. Time: 10.00am – 12.00noon. Time allowed: 2 hours.

This paper has FOUR questions.

Candidates should start their solutions to each question in a new main answer book

Supplementary books may only be used after the relevant main book(s) are full.

Statistical tables will not be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers to each question as per the table below.

Raw mark	up to 12	13	14	15	16	17	18	19	20
Extra credit	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4

- Each question carries equal weight.
- Calculators may not be used.

1. *In this question you should work from first principles.*

Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers.

(a) If $a_{n+1} \geq a_n \ \forall n$, prove that (a_n) is convergent if and only if it is bounded.

(b) Suppose instead that $a_n \geq 0 \ \forall n$. Prove that $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the sequence $s_n := \sum_{i=1}^{n} a_i$ is bounded.

(c) Suppose now that $a_n \geq a_{n+1} \geq 0 \ \forall n$. Show that

$$2^n a_{2^{n+1}} \leq \sum_{k=2^n}^{2^{n+1}-1} a_k \leq 2^n a_{2^n}.$$

(d) Deduce that $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} 2^n a_{2^n}$ converges.

(e) Deduce that $\sum_{n=1}^{\infty} \frac{1}{n}$ does not converge.

2. For each of the following statements, *either* give a **brief proof** *or* state (without proof) a **counterexample**. *You may use standard results from the course if you state them correctly.*

(a) A sequence of real numbers (a_n) is convergent if and only if the subsequences (a_{2n}) and (a_{3n}) are convergent.

(b) A continuous function $f\colon (0,1) \to \mathbb{R}$ can be extended to a continuous function $f\colon [0,1] \to \mathbb{R}$ if and only if it is bounded.

(c) Given $f\colon \mathbb{R} \to \mathbb{R}$, define $g\colon \mathbb{R} \to \mathbb{R}$ by $g(x) := xf(x)$. If f is continuous at $0 \in \mathbb{R}$ then g is differentiable at $0 \in \mathbb{R}$ with $g'(0) = f(0)$.

(d) A sequence (a_n) of rational numbers $a_n = p_n/q_n$ with bounded numerators $p_n \in \mathbb{Z}$ and bounded denominators $q_n \in \mathbb{N}$ has a subsequence convergent to a rational number $p/q \in \mathbb{Q}$.

(e) The power series $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=1}^{\infty} n a_n z^{n-1}$ have the same radius of convergence.

3. *Throughout this question you may use standard results from the course if you state them correctly.*

For $k, n \in \mathbb{N}$ let $a_k(n)$ be the coefficient of z^k in $\left(1+\frac{z}{n}\right)^n$, i.e.

$$\left(1+\frac{z}{n}\right)^n = \sum_{k\geq 0} a_k(n)\, z^k \quad \text{for } z \in \mathbb{C}.$$

(a) Fix $k \in \mathbb{N}$. Show that $a_k(n) \leq \frac{1}{k!}$ and converges to $\frac{1}{k!}$ as $n \to \infty$.

(b) Define the *radius of convergence* R of a complex power series. Determine R for

$$E(z) := \sum_{k=0}^{\infty} \frac{z^k}{k!}.$$

(c) For fixed z with $|z| < R$ prove that

$$\left(1+\frac{z}{n}\right)^n \to E(z) \quad \text{as } n \to \infty.$$

Hint: you could consider splitting $\sum_k \left|\frac{z^k}{k!} - a_k(n)z^k\right|$ *into a sum over* $k < N$ *and a sum over* $k \geq N$ *for an appropriately chosen* $N \in \mathbb{N}$.

4. Define what it means for a function $f\colon \mathbb{R} \to \mathbb{R}$ to be *continuous*.

We say that a subset $S \subset \mathbb{R}$ is *dense* in $\mathbb{R}$ if it satisfies the condition

$$\forall x \in \mathbb{R}\ \forall \delta > 0\ \exists s \in S \text{ such that } |x - s| < \delta.$$

(a) Give an example (without proof) of a dense subset $S \subset \mathbb{R}$ with infinite complement $\mathbb{R} \setminus S$.

(b) Fix a dense subset $S \subset \mathbb{R}$ and a point $x \in \mathbb{R}$. Show there exists a sequence $(s_n)_{n=1}^{\infty}$ with all $s_n \in S$ such that $s_n \to x$ as $n \to \infty$.

(c) Fix a dense subset $S \subset \mathbb{R}$. Suppose that $f, g : \mathbb{R} \to \mathbb{R}$ are both continuous and satisfy

$$f(s) = g(s) \quad \forall s \in S.$$

Prove from first principles that $f = g$.

[redacted]

Solutions: M1P1

1. (a), (b), (e) seen, (c), (d) unseen.

(a) If $a_{n+1} \geq a_n \; \forall n$ and $a_n \to a$ then $a_1 \leq a_n \leq a \; \forall n$ so a_n is bounded. (Proof: if $a_n > a$ then let $\epsilon = a_n - a > 0$ so $\exists N \in \mathbb{N}$ such that $n \geq N \Rightarrow |a_n - a| < \epsilon \Rightarrow a_n < a + \epsilon = a_n$; contradiction.)

Conversely, if $\{a_n : n \in \mathbb{N}\}$ is bounded above then since it's also nonempty we can let $a \in \mathbb{R}$ be its supremum. Fix $\epsilon > 0$, then $a - \epsilon$ is not an upper bound so $\exists N \in \mathbb{N}$ such that $a_N > a - \epsilon$. So then

$$n \geq N \Rightarrow a \geq a_n \geq a_N > a - \epsilon \Rightarrow |a_n - a| < \epsilon \Rightarrow a_n \to a.$$

4 marks

(b) Notice that $a_n \geq 0 \; \forall n \Rightarrow s_n := \sum_{i=1}^{n} a_i$ is monotonically increasing. So then by (a) this is convergent if and only if (s_n) is bounded.

2 marks

(c)
$$2^n a_{2^{n+1}} = \sum_{k=2^n}^{2^{n+1}-1} a_{2^{n+1}} \leq \sum_{k=2^n}^{2^{n+1}-1} a_k \leq \sum_{k=2^n}^{2^{n+1}-1} a_{2^n} = 2^n a_{2^n}. \quad (*)$$

3 marks

(d) It is enough to show that $s_n := \sum_{k=1}^{n} a_k$ is bounded $\iff \sigma_m := \sum_{n=1}^{m} 2^n a_{2^n}$ is bounded (by part (b)). Both bounded below by 0, so just need upper bounds.

Summing the RHS of $(*)$ over $n = 1, \ldots, m$ shows that if (σ_m) bounded above by σ then

$$\sum_{k=2}^{2^{m+1}-1} a_k \leq \sigma \qquad \forall m.$$

Therefore

$$s_n = \sum_{k=1}^{n} a_k \leq a_1 + \sum_{k=2}^{2^{n+1}-1} a_k \leq a_1 + \sigma$$

is bounded above, as required.

4 marks

Conversely suppose (s_n) is bounded above by $s \in \mathbb{R}$. Then summing the LHS of $(*)$ over $n = 0, \ldots, m-1$ gives

$$\sum_{n=0}^{m-1} 2^n a_{2^{n+1}} \leq \sum_{k=1}^{2^m - 1} a_k \leq s \qquad \forall m.$$

Thus

$$\sigma_m := \sum_{i=1}^{m} 2^i a_{2^i} = 2 \sum_{n=0}^{m-1} 2^n a_{2^{n+1}} \leq 2s$$

is bounded, as required.

4 marks

(e) By (d), $\sum_{n=1}^{\infty} \frac{1}{n}$ converges if and only if $\sum_{n=1}^{\infty} 2^n \frac{1}{2^n} = \sum_{n=1}^{\infty} 1$ converges.

2 marks

Thus by (b) $\sum_{n=1}^{\infty} \frac{1}{n}$ converges if and only if $\sigma_m = \sum_{n=1}^{m} 1$ is bounded. But $\sigma_m = m$ is not bounded, by the Archimedean axiom.

1 mark

2. All unseen. Quite short, but Q1 was quite long.

(a) False: eg if $a_n = 0$ for $n \equiv 1 \pmod 6$ and $a_n = 1$ otherwise. **4 marks**

(b) False: eg $f(x) = \sin\frac{1}{x}$ is continuous and bounded but no extension is continuous at 0. (We showed this in lectures. For completeness here's a proof: if continuous at 0 then $\exists\delta > 0$ such that $|f(x) - f(0)| < 1$ $\forall x \in (0, \delta)$; thus $|f(x) - f(y)| < 2$ $\forall x, y \in (0, \delta)$ by the triangle inequality. Taking $x = \left(2n\pi + \frac{\pi}{2}\right)^{-1}$ and $y = \left(2n\pi + \frac{3\pi}{2}\right)^{-1}$ for $n > \frac{1}{\delta}$ gives $|1 - (-1)| < 2$; contradiction.) **4 marks**

(c) True: $\lim_{h\to 0}\frac{g(h)-g(0)}{h} = \lim_{h\to 0}\frac{hf(h)-0}{h} = \lim_{h\to 0} f(h)$ exists and equals $f(0)$ by the continuity of f at 0. **4 marks**

(No marks for trying to use product rule, obviously: that would require f to be differentiable.)

(d) True. The p_ns take only finitely many values, so they take at least one value p infinitely often, so take a subsequence which is constantly p. Similarly a subsequence of this subsequence has constant numerator q. Therefore this sub-subsequence is a constant p/q, convergent to $p/q \in \mathbb{Q}$. (Could use Bolzano-Weierstrass on (p_n) and then on the resulting subsequence of (q_n), but that's overkill. Using it on p_n/q_n will only give a *real* limit, and so 1 mark.) **4 marks**

(e) True. Eg can use ratio test, since $|z|$ is $\leq$ the radius of convergence of $\sum b_n z^n$ if and only if $\frac{b_{n+1}z^{n+1}}{b_n z^n} \to 0$ as $n \to \infty$. But

$$\frac{(n+1)a_{n+1}z^n}{na_n z^{n-1}} = \left(1 + \frac{1}{n}\right)\frac{a_{n+1}z^{n+1}}{a_n z^n},$$

so, for fixed z, this $\to 0$ if and only if $\frac{a_{n+1}z^{n+1}}{a_n z^n} \to 0$. **4 marks**

3. (a), (c) unseen, (b) seen.

(a) By the binomial theorem,

$$a_k(n) = \binom{n}{k}\Big/n^k = 1.(1-1/n).(1-2/n)\cdots(1-(k-1)/n).\frac{1}{k!} \leq \frac{1}{k!}.$$

Each of the brackets $(1 - i/n) \to 1 - 0 = 1$ as $n \to \infty$ so by the algebra of limits their product (with a fixed number k of terms, notice) also $\to 1$. **4 marks**

(b)
$$R = \sup\{|z| : z \in \mathbb{C},\ a_n z_n \to 0 \text{ as } n \to \infty\} \in [0, \infty].$$ **3 marks**

Or can replace "$a_n z_n \to 0$" with "$a_n z_n$ is bounded" or "$\sum a_n z_n$ converges" or "converges absolutely", or .... Or can say it is the unique $R \in [0, \infty]$ such that $\sum a_n z_n$ is (absolutely) convergent for $|z| < R$ and divergent for $|z| > R$.

Applying the ratio test to $E(z)$ for fixed $z \in \mathbb{C}$:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{z^{n+1}/(n+1)!}{z^n/n!}\right| = \frac{|z|}{n+1} \to 0 \text{ as } n \to \infty.$$

Therefore the power series is absolutely convergent for all $z \in \mathbb{C}$, so $R = \infty$. **3 marks**

[redacted]

(c) By the algebra of limits and the triangle inequality,

$$\left|E(z) - \left(1+\frac{z}{n}\right)^n\right| = \left|\sum_{k=0}^{\infty} \frac{z^k}{k!} - a_k(n)z^k\right| \leq \sum_{k=0}^{N-1} \left|\frac{z^k}{k!} - a_k(n)z^k\right| + \sum_{k=N}^{\infty} \left|\frac{z^k}{k!}\right| + \sum_{k=N}^{\infty} |a_k(n)z^k|.$$

Fix $\epsilon > 0$ then by the absolute convergence of $\sum \frac{z^k}{k!}$ there exists $N \in \mathbb{N}$ such that $\sum_{k=N}^{\infty} \left|\frac{z^k}{k!}\right| < \epsilon$. Therefore by (a) and comparison, $\sum_{k=N}^{\infty} |a_k(n)z^k|$ is also $< \epsilon$.

Also by (a) each of the terms $\left|\frac{z^k}{k!} - a_k(n)z^k\right|$, $k = 0, 1, \ldots, N-1$, converges to 0 as $n \to \infty$.
So $\exists\, M \in \mathbb{N}$ such that

$$n \geq M \;\Rightarrow\; \left|\frac{z^k}{k!} - a_k(n)z^k\right| < \frac{\epsilon}{N} \quad \text{for } k = 0, 1, \ldots, N-1.$$

Putting it all together gives

$$n \geq M \;\Rightarrow\; \left|E(z) - \left(1+\frac{z}{n}\right)^n\right| < N\frac{\epsilon}{N} + \epsilon + \epsilon = 3\epsilon.$$

10 marks

4. (a) unseen, (b) seen similar, (c) unseen. Quite short, but this is new stuff, and Q3 was long.

$f\colon \mathbb{R} \to \mathbb{R}$ is continuous if and only if $\forall a \in \mathbb{R}\ \forall \epsilon > 0\ \exists \delta > 0$ such that

$$|x - a| < \delta \;\Rightarrow\; |f(x) - f(a)| < \epsilon.$$

2 marks

3 marks

(a) Any example like $S = \mathbb{Q}$, $\mathbb{R} \setminus \mathbb{Q}$, $\mathbb{R} \setminus \mathbb{N}$, $\mathbb{R} \setminus \mathbb{Z}, \ldots$ will do.

(b) Let $\delta = 1/n$ so $\exists s \in S$ with $|s - x| < 1/n$. Pick one of them and call it s_n.
We claim $s_n \to x$: for any $\epsilon > 0$ choose $N \in \mathbb{N}$ such that $N > 1/\epsilon$. Thus

$$n \geq N \;\Rightarrow\; |s_n - x| < 1/n \leq 1/N < \epsilon.$$

5 marks

(c) Suppose for a contradiction that $f(x) \neq g(x)$ for some $x \in \mathbb{R}$. Let $\epsilon := |f(x) - g(x)|/2 > 0$.
Then by continuity $\exists \delta_1 > 0$ and $\exists \delta_2 > 0$ such that

$$|s - x| < \delta_1 \;\Rightarrow\; |f(s) - f(x)| < \epsilon \qquad \text{and} \qquad |s - x| < \delta_2 \;\Rightarrow\; |g(s) - g(x)| < \epsilon.$$

Let $\delta := \min(\delta_1, \delta_2)$. Then by density $\exists s \in S$ such that $|s - x| < \delta$ so $f(s) = g(s)$ and the above gives

$$|f(s) - f(x)| < \epsilon \quad \text{and} \quad |f(s) - g(x)| < \epsilon.$$

Therefore by the triangle inequality, $|f(x) - g(x)| < 2\epsilon = |f(x) - g(x)|$; contradiction.

10 marks

Thus $f(x) = g(x)\ \forall x \in \mathbb{R}$ so $f = g$.

Imperial College
London

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2015

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Algebra I

Date: Tuesday, 19 May 2015. Time: 10.00am – 12.00noon. Time allowed: 2 hours.

This paper has FOUR questions.

Candidates should start their solutions to each question in a new main answer book

Supplementary books may only be used after the relevant main book(s) are full.

Statistical tables will not be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers to each question as per the table below.

Raw mark	up to 12	13	14	15	16	17	18	19	20
Extra credit	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4

- Each question carries equal weight.
- Calculators may not be used.

1. (a) Let V and W be vector spaces over a field F, and let $S : V \longrightarrow W$ be a function. What does it mean to say that S is a *linear transformation*?

(b) Let $B = \{v_1, \dots, v_n\}$ be a basis for V. Let $T : V \longrightarrow V$ be a linear transformation. What is meant by $[T]_B$, the matrix of T with respect to B?

(c) Suppose that V is the vector space of polynomials of degree at most 2, in a single variable x, with coefficients from $\mathbb{R}$. Let B be the basis $\{1, x, x^2\}$ for V. Let $T : V \longrightarrow V$ be the unique linear transformation such that

$$T(1) = 1 + x + x^2, \quad T(x) = 2x + x^2, \quad T(x^2) = x + 2x^2.$$

(i) Write down the matrix $[T]_B$.

(ii) Find a basis C for V such that $[T]_C$ is diagonal.

(iii) Write down the change of basis matrix from B to C.

(iv) Find the change of basis matrix from C to B.

2. (a) State and prove Lagrange's Theorem.

(b) Let G be a finite group, and H a subgroup of G.

(i) Let $x \in G$. What is meant by the *order* of x?

(ii) Show that if $x \notin H$, and if $Hx = Hx^{-1}$, then the order of x is even.

(iii) Suppose that $y^{-1} \in Hx$ for every $y \in Hx$. Show that $Hx = xH$.

3. (a) Let $n \in \mathbb{N}$. Let g be an element of the symmetric group S_n.

   (i) What is meant by the *cycle shape* of g?

   (ii) State and prove a formula for the order of g in terms of its cycle shape.

(b) Calculate the number of elements of order 6 in S_6.

(c) Prove that $\operatorname{ord} g \leq 3^{n/3}$ for all $g \in S_n$. (You may assume without proof that $3^t \geq t^3$ for all $t \in \mathbb{N}$.)

4. (a) Let R be a commutative ring.

   (i) Explain what is meant by a *unit* in R.

   (ii) Prove that the set of units in R form a group under multiplication.

   (iii) Explain what it means for an element of R to be *irreducible*.

(b) Show that $1 - \sqrt{-11}$ is irreducible in the ring $\mathbb{Z}[\sqrt{-11}]$.

(c) (i) Let a and b be elements of a ring R. Explain what is meant by a *highest common factor* for a and b.

   (ii) Find a highest common factor for $x^3 - 6x + 4$ and $2x^4 - 11x^2 + 5x + 2$ in the polynomial ring $\mathbb{R}[x]$.

M1P2 2015 Solutions

1. (a) **[Bookwork]** S is a linear transformation if for all $u, v \in V$ and $\lambda \in F$ we have $S(u+v) = S(u) + S(v)$, and $S(\lambda v) = \lambda S(v)$.
(2 marks)

(b) **[Bookwork]** Let $B = \{v_1, \dots, v_n\}$. Then $[T]_B$ is the matrix (λ_{ij}), where the scalars λ_{ij} are defined by $Tv_j = \sum_i \lambda_{ij} v_i$ for all j. **(3 marks)**

(c) **[Routine example]**

(i)

$$[T]_B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

(2 marks)

(ii) The characteristic polynomial of $[T]_B$ is $(x-1)^2(x-3)$. The vectors $(1,-1,0)^t$ and $(1,0,-1)^t$ are linearly independent eigenvectors with eigenvalue 1 [any other linearly independent pair of zero-sum vectors will do]. And $(0,1,1)^t$ is an eigenvector with eigenvalue 3. So a basis of V consisting of eigenvectors of T is $C = \{x^2 - x,\ x^2 - 1,\ x + 1\}$. **(5 marks)**

(iii) The change of basis matrix from B to C is

$$\begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}.$$

(This will vary with the choice and ordering of eigenvectors.)
(4 marks)

(iv) The change of basis matrix from C to B is the inverse of the matrix from (iii)—for the answer given here, this is

$$\begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}.$$

(4 marks)

2. (a) **[Bookwork]** *Lagrange's Theorem:* Let G be a finite group, and H a subgroup of G. Then $|H|$ divides $|G|$. **(2 marks)**

Proof: For $g \in G$ define the coset $Hg = \{hg \mid h \in H\}$. Every element $g \in G$ is in a coset, since $g = eg \in Hg$. Suppose that $x, y \in G$, and that $y \in Hx$. Then $y = h_0 x$ for some $h_0 \in H$. Now if $g \in Hy$ then $g = hy = hh_0 x$ for some $h \in H$, and so $g \in Hx$. And if $g \in Hx$ then $g = hx = hh_0^{-1} y$ for some $h \in H$, and so $g \in Hy$. Hence $Hx = Hy$. It follows that the cosets partition G. Since $h_1 x = h_2 x$ if and only if $h_1 = h_2$, we have $|Hg| = |H|$ for every $g \in G$. Hence $|G| = k|H|$, where k is the number of distinct cosets. **(8 marks)**

(b) (i) **[Bookwork]** The order of x is the smallest positive integer k such that $x^k = e$. **(1 mark)**

(ii) **[Unseen]** If $Hx = Hx^{-1}$ then $x^2 \in H$. Now it follows that

$$Hx^k = \begin{cases} Hx & \text{if } k \text{ is odd,} \\ H & \text{if } k \text{ is even.} \end{cases}$$

Now if $x^k = e$ then $x^k \in H$, and so k must be even; hence x has even order. **(4 marks)**

(iii) **[Unseen]** Since $x \in Hx$, we must have $x^{-1} \in Hx$, and so $Hx = Hx^{-1}$. Now

$$y \in Hx \Rightarrow y^{-1} \in Hx \Rightarrow y^{-1} \in Hx^{-1} \Rightarrow y^{-1} = hx^{-1}$$

for some $h \in H$. But $y^{-1} = hx^{-1} \Rightarrow y = xh^{-1}$, and so $y \in xH$. So $Hx \subseteq xH$, and now since $|xH| = |Hx|$, it follows that $Hx = xH$. **(5 marks)**

3. (a) (i) **[Bookwork]** The cycle shape of g is the sequence of lengths of cycles when g is written in disjoint cycle notation, in weakly decreasing order. **(2 marks)**

(ii) **[Bookwork]** The order of g is the least common multiple of the entries in its cycle shape. **(2 marks)**

Let $(r_1, \ldots, r_k)$ be the cycle shape. For $i = 1, \ldots, k$, let $c_1, \ldots, c_k$ be the disjoint cycles of g, so that c_i has length r_i. Then the cycles c_i and c_j commute. It follows that for $t \in \mathbb{Z}$ we have

$$g^t = c_1^t \cdots c_k^t,$$

and so $g^t = e$ if and only if $c_i^t = e$ for all i. But this occurs if and only if r_i divides t for all i, and so we see that $g^t = e$ if and only if t is a common multiple for the cycle lengths. Hence the order of g is the least common multiple, as required.

(5 marks)

(b) **[Similar to seen examples]** The possible cycle shapes are (6) and $(3, 2, 1)$. In constructing a 6-cycle we may take 1 as a starting point; then there are 5! ways of completing the cycle. So the number of 6-cycles is 120. For the cycle shape $(3, 2, 1)$, there are 6 ways of selecting the fixed point. Then there are $\binom{5}{2}$ ways of choosing the 2-cycle. Finally, there are 2 possible cycles on the remaining 3 points, and so there are $6 \cdot 10 \cdot 2 = 120$ permutations of this shape. So the total number of elements of order 6 is 240.

(4 marks)

(c) **[Unseen]** Let $r_1, \ldots, r_k$ be the cycle lengths of g. Then

$$\text{ord } g = \text{lcm}(r_1, \ldots, r_k) \leq r_1 \cdots r_k.$$

But $r_i < 3^{r_i/3}$ for all i by the fact provided, and so

$$\text{ord } g \leq 3^{(r_1 + \cdots + r_k)/3}.$$

But $r_1 + \cdots + r_k = n$, and the result follows. **(7 marks)**

4. (a) **[Bookwork]**

(i) An element x of R is a unit if there exists $z \in R$ such that $xz = zx = 1$. **(2 marks)**

(ii) Let w and x be units. Then there exist y and z such that $wy = yw = xz = zx = 1$. Now $(wx)(zy) = (zy)(wx) = 1$, and so wx is a unit in R. Hence multiplication gives a binary operation on the set of units of R. It is associative by the multiplicative axioms for a ring. Clearly 1 is a unit, since $1 \cdot 1 = 1$. And if $xz = zx = 1$ then z is a unit too, by definition, and an inverse for x. So the units form a group. **(5 marks)**

(iii) An non-zero non-unit element $x \in R$ is irreducible if it cannot be expressed as $x = ab$, for $a, b \in R$, in such a way that neither a nor b is a unit. **(2 marks)**

(b) **[Similar to seen examples]** Use the norm function, $N(a + b\sqrt{-11}) = a^2 + 11b^2 \in \mathbb{Z}$. This function has the property that $N(xy) = N(x)N(y)$ for elements $x, y \in \mathbb{Z}[\sqrt{-11}]$. Note also that $N(x) \geq 11$ unless $x \in \{0, \pm 1, \pm 2, \pm 3\}$. Now $N(1 - 1\sqrt{-11}) = 12$. Since $\pm 2, \pm 3$ clearly do not divide $1 - \sqrt{-11}$ in $\mathbb{Z}[\sqrt{-11}]$, we see that if $1 - \sqrt{-11} = xy$, then we must have $N(x) = 12$ and $y = \pm 1$ (or vice versa). But then y is a unit. **(5 marks)**

(c) (i) **[Bookwork]** x divides a in R if there exists $c \in R$ such that $xc = a$. We say x is a common divisor of a and b if x divides a and x divides b. And x is a greatest common divisor for a and b if it is a common divisor of a and b with the property that every common divisor of a and b divides x. **(3 marks)**

(ii) **[Routine example]** Use Euclid's algorithm: a greatest common divisor is $x - 2$. **(3 marks)**

Imperial College
London

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2015

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Probability & Statistics I

Date: Thursday, 14 May 2015. Time: 10.00am – 12.00noon. Time allowed: 2 hours.

This paper has FOUR questions.

Candidates should start their solutions to each question in a new main answer book

Supplementary books may only be used after the relevant main book(s) are full.

Statistical tables are provided on pages 5 & 6.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers to each question as per the table below.

Raw mark	up to 12	13	14	15	16	17	18	19	20
Extra credit	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4

- Each question carries equal weight.
- Calculators may not be used.

1. (a) State the three axioms of probability for events defined on a sample space Ω.

   (b) Prove from the axioms that, for any event $E \subseteq \Omega$, $\mathsf{P}(E) \leq 1$.

   (c) If E and F are independent events, prove that E and F^C are independent.

   (d) Professor Nasty challenges you to the following game, which involves three, three-sided dice: one red, one green and one blue. Each of the dice have equal probability of landing on each of the three sides independently of the other dice .

      The red die has the numbers $\{3,3,3\}$;
      The green die has the numbers $\{4,4,1\}$;
      The blue die has the numbers $\{2,2,5\}$.

   Professor Nasty lets you choose one of the dice and then she chooses one of the remaining two dice. You both roll your chosen dice once and the winner is the person whose die shows the bigger number.

   (i) What is the probability that Blue beats Green?

   (ii) What is the probability that Green beats Red?

   (iii) Given your calculations, you choose Blue. What colour should Professor Nasty choose to maximise her probability of winning?

   (iv) You play again, but this time Professor Nasty chooses first. She rolls each of the dice and chooses the colour with the highest score. Once she has chosen, you choose one of the two remaining dice at random. If you chose Blue and won, what is the probability that Professor Nasty chose Green?

2. The random variables X and Y are independently distributed with $X \sim Poisson(\lambda)$ and $Y \sim Gamma(2, 1)$.

(a) Find an expression for $\alpha = \mathsf{P}(X \geq 2)$.

(b) Prove that

$$\mathsf{P}(Y \leq \lambda) = \alpha.$$

(c) (i) Show that the moment generating function of X is given by

$$M_X(t) = \exp\{\lambda(\mathrm{e}^t - 1)\}.$$

(ii) Hence show that $\mathsf{E}_{f_X}(X) = \mathsf{var}_{f_X}(X) = \lambda$.

(d) Suppose X_1 and X_2 are independently distributed with the same distribution as X.

(i) Determine the probability mass function of $Z = X_1 + X_2$.

(ii) For what range of values of λ will $\mathsf{P}(Z > 0) > 0.5$?

3. Consider the continuous random variables $U \sim \mathit{Uniform}(0, 1)$ and $X = \sqrt{-2\log(1 - U)}$.

(a) Determine $F_U(u)$, the cumulative distribution function of U.

(b) Find an expression for $F_X(x)$ in terms of $F_U(\cdot)$.

(c) Hence prove that $f_X(x)$, the probability density function of X, is given by

$$f_X(x) = x\mathrm{e}^{-x^2/2}, \quad x > 0.$$

(d) If $Y \sim \mathit{Uniform}(-a, a), a > 0$, independent of X (where X is defined as above) and $Z = X + Y$, use the fact that

$$f_Z(z) = \int f_X(z - y) f_Y(y) \mathrm{d}y$$

to determine the form of $f_Z(z)$ when $z > a$.

(e) Show that

$$\mathsf{P}(Z > a) = \frac{1}{2a} \int_0^{2a} e^{-u^2/2} \mathrm{d}u.$$

4. Continuous random variables X and Y have joint probability density function (pdf), $f_{X,Y}(x, y)$.

(a) Prove that,

$$\text{var}_{f_{X,Y}}(aX + bY) = a^2\text{var}_{f_X}(X) + 2ab\text{cov}_{f_{X,Y}}(X, Y) + b^2\text{var}_{f_Y}(Y),$$

for constants a, b.

(b) Suppose the joint pdf of X and Y is given by,

$$f_{X,Y}(x, y) = \frac{2}{3}(x + 2y), \quad 0 \leq x \leq 1; 0 \leq y \leq 1.$$

(i) Determine $f_X(x)$, the marginal pdf of X.

(ii) Determine $f_Y(y)$, the marginal pdf of Y.

(iii) Determine $\text{E}_{f_X}(X)$ and $\text{E}_{f_Y}(Y)$ and $\text{E}_{f_{X,Y}}(XY)$.

(iv) Determine $\text{E}_{f_{X,Y}}(X - Y)$.

(v) Find $P(X > Y)$. Comment on this result in light of your answer to part (iv).

DISCRETE DISTRIBUTIONS

	RANGE $\mathbb{X}$	PARAMETERS	MASS FUNCTION f_X	CDF F_X	$\mathrm{E}_{f_X}[X]$	$\mathrm{Var}_{f_X}[X]$	MGF M_X
$Bernoulli(\theta)$	$\{0,1\}$	$\theta \in (0,1)$	$\theta^x(1-\theta)^{1-x}$		θ	$\theta(1-\theta)$	$1-\theta+\theta e^t$
$Binomial(n,\theta)$	$\{0,1,\ldots,n\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{n}{x}\theta^x(1-\theta)^{n-x}$		$n\theta$	$n\theta(1-\theta)$	$(1-\theta+\theta e^t)^n$
$Poisson(\lambda)$	$\{0,1,2,\ldots\}$	$\lambda \in \mathbb{R}^+$	$\dfrac{e^{-\lambda}\lambda^x}{x!}$		λ	λ	$\exp\{\lambda(e^t-1)\}$
$Geometric(\theta)$	$\{1,2,\ldots\}$	$\theta \in (0,1)$	$(1-\theta)^{x-1}\theta$	$1-(1-\theta)^x$	$\dfrac{1}{\theta}$	$\dfrac{(1-\theta)}{\theta^2}$	$\dfrac{\theta e^t}{1-e^t(1-\theta)}$
$NegBinomial(n,\theta)$	$\{n,n+1,\ldots\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{x-1}{n-1}\theta^n(1-\theta)^{x-n}$		$\dfrac{n}{\theta}$	$\dfrac{n(1-\theta)}{\theta^2}$	$\left(\dfrac{\theta e^t}{1-e^t(1-\theta)}\right)^n$
or	$\{0,1,2,\ldots\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{n+x-1}{x}\theta^n(1-\theta)^x$		$\dfrac{n(1-\theta)}{\theta}$	$\dfrac{n(1-\theta)}{\theta^2}$	$\left(\dfrac{\theta}{1-e^t(1-\theta)}\right)^n$

For **CONTINUOUS** distributions (see over), define the **GAMMA FUNCTION**

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx$$

and the LOCATION/SCALE transformation $Y = \mu + \sigma X$ gives

$$f_Y(y) = f_X\left(\frac{y-\mu}{\sigma}\right)\frac{1}{\sigma} \qquad F_Y(y) = F_X\left(\frac{y-\mu}{\sigma}\right) \qquad M_Y(t) = e^{\mu t}M_X(\sigma t) \qquad \mathrm{E}_{f_Y}[Y] = \mu + \sigma \mathrm{E}_{f_X}[X] \qquad \mathrm{Var}_{f_Y}[Y] = \sigma^2 \mathrm{Var}_{f_X}[X]$$

CONTINUOUS DISTRIBUTIONS							
	$\mathbb{X}$	PARAMS.	PDF f_X	CDF F_X	$\mathrm{E}_{f_X}[X]$	$\mathrm{Var}_{f_X}[X]$	MGF M_X
$Uniform(\alpha,\beta)$ (stand. model $\alpha=0,\beta=1$)	(α,β)	$\alpha<\beta\in\mathbb{R}$	$\dfrac{1}{\beta-\alpha}$	$\dfrac{x-\alpha}{\beta-\alpha}$	$\dfrac{(\alpha+\beta)}{2}$	$\dfrac{(\beta-\alpha)^2}{12}$	$\dfrac{e^{\beta t}-e^{\alpha t}}{t(\beta-\alpha)}$
$Exponential(\lambda)$ (stand. model $\lambda=1$)	$\mathbb{R}^+$	$\lambda\in\mathbb{R}^+$	$\lambda e^{-\lambda x}$	$1-e^{-\lambda x}$	$\dfrac{1}{\lambda}$	$\dfrac{1}{\lambda^2}$	$\left(\dfrac{\lambda}{\lambda-t}\right)$
$Gamma(\alpha,\beta)$ (stand. model $\beta=1$)	$\mathbb{R}^+$	$\alpha,\beta\in\mathbb{R}^+$	$\dfrac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}$		$\dfrac{\alpha}{\beta}$	$\dfrac{\alpha}{\beta^2}$	$\left(\dfrac{\beta}{\beta-t}\right)^\alpha$
$Weibull(\alpha,\beta)$ (stand. model $\beta=1$)	$\mathbb{R}^+$	$\alpha,\beta\in\mathbb{R}^+$	$\alpha\beta x^{\alpha-1}e^{-\beta x^\alpha}$	$1-e^{-\beta x^\alpha}$	$\dfrac{\Gamma(1+1/\alpha)}{\beta^{1/\alpha}}$	$\dfrac{\Gamma\left(1+\frac{2}{\alpha}\right)-\Gamma\left(1+\frac{1}{\alpha}\right)^2}{\beta^{2/\alpha}}$	
$Normal(\mu,\sigma^2)$ (stand. model $\mu=0,\sigma=1$)	$\mathbb{R}$	$\mu\in\mathbb{R},\sigma\in\mathbb{R}^+$	$\dfrac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\dfrac{(x-\mu)^2}{2\sigma^2}\right\}$		μ	σ^2	$e^{\{\mu t+\sigma^2t^2/2\}}$
$Student(\nu)$	$\mathbb{R}$	$\nu\in\mathbb{R}^+$	$\dfrac{(\pi\nu)^{-\frac{1}{2}}\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\left\{1+\frac{x^2}{\nu}\right\}^{(\nu+1)/2}}$		0 (if $\nu>1$)	$\dfrac{\nu}{\nu-2}$ (if $\nu>2$)	
$Pareto(\theta,\alpha)$	$\mathbb{R}^+$	$\theta,\alpha\in\mathbb{R}^+$	$\dfrac{\alpha\theta^\alpha}{(\theta+x)^{\alpha+1}}$	$1-\left(\dfrac{\theta}{\theta+x}\right)^\alpha$	$\dfrac{\theta}{\alpha-1}$ (if $\alpha>1$)	$\dfrac{\alpha\theta^2}{(\alpha-1)(\alpha-2)}$ (if $\alpha>2$)	
$Beta(\alpha,\beta)$	$(0,1)$	$\alpha,\beta\in\mathbb{R}^+$	$\dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$		$\dfrac{\alpha}{\alpha+\beta}$	$\dfrac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	

Imperial College
London

Course: M1S
Setter: [redacted]
Checker:
Editor:
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BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2015

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science .

M1S

Probability & Statistics I (Solutions)

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1\. (a) **Axioms of Probability**
Given a σ−field, $\mathcal{F}$ (a set of subsets of the sample space Ω.) For events $E, E_1, E_2, \ldots \in \mathcal{F}$, then the probability function, $\mathsf{P}(\cdot)$, must satisfy:

(I) $\mathsf{P}(E) \geq 0$.
(II) $\mathsf{P}(\Omega) = 1$.
(III) If $E_1, E_2, \ldots$, are pairwise disjoint then $\mathsf{P}(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mathsf{P}(E_i)$ (Countable additivity). $\boxed{3}$

(Do not need to specify σ−field, could instead say: for events $E, E_1, \ldots \subseteq \Omega$. Lose 1 mark if finite rather than countable additivity specified)

unseen $\Downarrow$

(b)

$$\begin{aligned}
E \cup E^C = \Omega &\Rightarrow \mathsf{P}(E) + \mathsf{P}(E^C) = \mathsf{P}(\Omega), \text{ axiom III as } E \cap E^C = \phi \\
&\Rightarrow \mathsf{P}(E) + \mathsf{P}(E^C) = 1, \text{ axiom II} \\
&\Rightarrow \mathsf{P}(E) \leq 1, \text{ axiom I as } \mathsf{P}(E^C) \geq 0 \text{ and } \mathsf{P}(E) \geq 0.
\end{aligned}$$

$\boxed{2}$

seen $\Downarrow$

(c)

$$\begin{aligned}
\mathsf{P}(E \cap F^C) &= \mathsf{P}(F^C \cap E) = \mathsf{P}(F^C \mid E)\mathsf{P}(E) = (1 - \mathsf{P}(F \mid E))\mathsf{P}(E) \\
&= (1 - \mathsf{P}(F))\mathsf{P}(E) \text{ (from } \mathsf{P}(F \mid E) = \mathsf{P}(F) \text{ as } E \text{ and } F \text{ indep)} \\
&= \mathsf{P}(F^C)\mathsf{P}(E),
\end{aligned}$$

therefore E and F^C are independent. $\boxed{2}$

(d) Let B, R and G reprensent the number on the Blue, Red and Green die respectively. Let B_i, R_i, G_i = event Blue, Red or Green chosen on the ith choice, $i = 1, 2$. Let W = event you win.

sim. seen $\Downarrow$

(i)

$$\begin{aligned}
\mathsf{P}(\text{Blue beats Green}) &= \mathsf{P}((B = 2 \cap G = 1) \cup (B = 5)) = \mathsf{P}(B = 2)\mathsf{P}(G = 1) + \mathsf{P}(B = 5) \\
&= \frac{2}{3} \cdot \frac{1}{3} + \frac{1}{3} = \frac{5}{9}.
\end{aligned}$$

$\boxed{2}$

(ii)

$$\mathsf{P}(\text{Green beats Red}) = \mathsf{P}(G = 4) = \frac{2}{3}.$$

$\boxed{1}$

(iii) We know that Blue beats Green with probability 5/9, need to calculate:

$$\mathsf{P}(\text{Blue beats Red}) = \mathsf{P}(B = 5) = \frac{1}{3}.$$

Professor Nasty should should Red to maximise her probability of winning. $\boxed{2}$

(iv)

$$\begin{aligned}
\mathsf{P}(R_1) &= \mathsf{P}(\text{Red beats both Blue and Green}) = \mathsf{P}(G = 1 \cap B = 2) \\
&= \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9}. \\
\mathsf{P}(B_1) &= \mathsf{P}(\text{Blue beats both Red and Green}) = \mathsf{P}(B = 5) = \frac{1}{3}. \\
\mathsf{P}(G_1) &= \mathsf{P}(\text{Green beats both Red and Blue}) = \mathsf{P}(G = 4 \cap B = 2) \\
&= \frac{2}{3} \cdot \frac{2}{3} = \frac{4}{9}.
\end{aligned}$$

$\boxed{3}$

Want to calculate:

$$\begin{aligned}
\mathsf{P}(G_1 \mid W \cap B_2) &= \frac{\mathsf{P}(W \cap B_2 \mid G_1)\mathsf{P}(G_1)}{\mathsf{P}(W \cap B_2)} \\
&= \frac{\mathsf{P}(W \mid G_1 \cap B_2)\mathsf{P}(B_2 \mid G_1)\mathsf{P}(G_1)}{\mathsf{P}(W \cap B_2)}.
\end{aligned}$$

$\boxed{2}$

R_1 and G_1 partition B_2, so, from the theorem of total probability:

$$\begin{aligned}
\mathsf{P}(W \cap B_2) &= \mathsf{P}(W \cap B_2 \mid G_1)\mathsf{P}(G_1) + \mathsf{P}(W \cap B_2 \mid R_1)\mathsf{P}(R_1) \\
&= \mathsf{P}(W \mid G_1 \cap B_2)\mathsf{P}(B_2 \mid G_1)\mathsf{P}(G_1) + \mathsf{P}(W \mid R_1 \cap B_2)\mathsf{P}(B_2 \mid R_1)\mathsf{P}(R_1) \\
&= \frac{5}{9} \cdot \frac{1}{2} \cdot \frac{4}{9} + \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{2}{9} = \frac{10}{81} + \frac{3}{81} = \frac{13}{81}.
\end{aligned}$$

$\boxed{2}$

So,

$$\mathsf{P}(G_1 \mid W \cap B_2) = \frac{\frac{10}{81}}{\frac{13}{81}} = \frac{10}{13}.$$

$\boxed{1}$

unseen ⇓

2. (a)

$$X \sim Poisson(\lambda) \Rightarrow f_X(x) = \frac{e^{-\lambda}\lambda^x}{x!}, x = 0, 1, 2, \dots$$

$$\begin{aligned}\alpha &= \mathsf{P}(X \geq 2) = 1 - \mathsf{P}(X \leq 1)\\ &= 1 - (\mathsf{P}(X = 0) + \mathsf{P}(X = 1)) = 1 - e^{-\lambda} - \lambda e^{-\lambda}.\end{aligned}$$

3

(b)

$$Y \sim \mathsf{Gamma}(2, 1) \Rightarrow f_Y(y) = xe^{-x},\ x \geq 0.$$

$$\begin{aligned}\mathsf{P}(Y \leq \lambda) &= \int_0^\lambda xe^{-x}\,\mathrm{d}x \qquad \begin{matrix} u = x & \frac{\mathrm{d}v}{\mathrm{d}x} = e^{-x} \\ \frac{\mathrm{d}u}{\mathrm{d}x} = 1 & v = -e^{-x}\end{matrix}\\ &= \left[-xe^{-x}\right]_0^\lambda + \int_0^\lambda e^{-x} = -\lambda e^{-\lambda} + \left[-e^{-x}\right]_0^\lambda\\ &= -\lambda e^{-\lambda} + (-e^{-\lambda} + 1) = 1 - e^{-\lambda} - \lambda e^{-\lambda} = \alpha,\end{aligned}$$

as required.

3

(c) (i)

seen ⇓

$$\begin{aligned}M_X(t) = \mathsf{E}_{f_X}(e^{tX}) &= \sum_{x=0}^{\infty} e^{tx}\frac{e^{-\lambda}\lambda^x}{x!}\\ &= e^{-\lambda}\sum_{x=0}^{\infty}\frac{(e^t\lambda)^x}{x!} = e^{-\lambda}\exp(e^t\lambda) = \exp(\lambda(e^t - 1)).\end{aligned}$$

as required.

3

(ii)

$$\begin{aligned}\mathsf{E}_{f_X}(X) &= M^{(1)}(0) = \exp(\lambda(e^t - 1))\lambda e^t|_{t=0}\\ &= \lambda\exp(\lambda(e^t - 1) + t)|_{t=0} = \lambda.\end{aligned}$$

2

$$\begin{aligned}\mathsf{var}_{f_X}(X) &= \mathsf{E}_{f_X}(X^2) - \mathsf{E}^2_{f_X}(X) = M^{(2)}(0) - \lambda^2\\ &= \lambda(\exp(\lambda(e^t - 1) + t))(\lambda e^t - 1)|_{t=0} - \lambda^2\\ &= \lambda(1 + \lambda) - \lambda^2 = \lambda.\end{aligned}$$

3

sim. seen ⇓

(d) $Z = X_1 + X_2$ where X_1 and X_2 are both independent *Poisson*(λ).

(i)

$$\begin{aligned} M_Z(t) &= M_{X_1}(t)M_{X_2}(t) = \exp(\lambda(e^t - 1))\exp(\lambda(e^t - 1)) \\ &= \exp(2\lambda(e^t - 1)) \end{aligned}$$

So $Z \sim$ *Poisson*(2λ) and

$$f_Z(z) = \frac{e^{-2\lambda}(2\lambda)^z}{z!}, z = 0, 1, 2, \dots$$

4

unseen ⇓

(ii)

$$P(Z > 0) = 1 - P(Z = 0) = 1 - e^{-2\lambda}.$$

So,

$$\begin{aligned} &P(Z > 0) > 0.5 \Rightarrow 1 - e^{-2\lambda} > 0.5 \Rightarrow e^{-2\lambda} < 0.5 \\ &\Rightarrow -2\lambda < \log(0.5) \Rightarrow \lambda > \frac{-\log(0.5)}{2} \\ &\Rightarrow \lambda > \frac{\log(2)}{2}. \end{aligned}$$

2

meth seen ⇓

3. (a) $U \sim Uniform(0,1)$, the range of U is $(0,1)$, so the range of $X = \sqrt{-2\log(1-U)}$ is $(0,\infty)$.

$F_U(u) = 0, u \le 0, F_U(u) = 1, u \ge 1$,

$$F_U(u) = \mathsf{P}(U \le u) = \int_0^u f_U(t)\mathrm{d}t = \int_0^u 1\mathrm{d}t = u$$
$$\Rightarrow F_U(u) = u, \quad u \in (0,1).$$

3

(b)

$$\begin{aligned} F_X(x) &= \mathsf{P}(X \le x) = \mathsf{P}(\sqrt{-2\log(1-U)} \le x) \\ &= \mathsf{P}(-2\log(1-U) \le x^2) = \mathsf{P}\left(\log(1-U) \ge -\frac{x^2}{2}\right) \\ &= \mathsf{P}\left(1-U \ge \mathrm{e}^{-x^2/2}\right) = \mathsf{P}(U \le 1-\mathrm{e}^{-x^2/2}) \\ &= F_U(1-\mathrm{e}^{-x^2/2}),\ x > 0. \end{aligned}$$

4

(c)

$$f_X(x) = \frac{\mathrm{d}}{\mathrm{d}x}F_X(x) = x\mathrm{e}^{-x^2/2},\ x > 0.$$

2

unseen ⇓

(d)

$$f_Y(y) = \frac{1}{2a}, y \in (-a,a), \quad f_X(x) = x\mathrm{e}^{-x^2/2}, x > 0.$$

$Z = X + Y$ so the range of Z is $(-a,\infty)$. 1

If $z > a$, then for $y \in (-a,a)$, we have that $z - y > 0$, so using convolution we have:

$$\begin{aligned} f_Z(z) &= \int f_X(z-y)f_Y(y)\,\mathrm{d}y = \int_{-a}^{a} (z-y)\mathrm{e}^{-(z-y)^2/2}\frac{1}{2a}\,\mathrm{d}y \\ &= \frac{1}{2a}\left[\mathrm{e}^{-(z-y)^2/2}\right]_{-a}^{a} \\ &= \frac{1}{2a}\left(\mathrm{e}^{-(z-a)^2/2} - \mathrm{e}^{-(z-a)^2/2}\right), z > a. \end{aligned}$$

4

(e)

$$\mathsf{P}(Z > a) = \int_a^\infty f_Z(z)\,\mathrm{d}z$$

$\boxed{2}$

$$\begin{aligned}
\mathsf{P}(Z > a) &= \frac{1}{2a}\int_a^\infty \mathrm{e}^{-(z-a)^2/2}\,\mathrm{d}z - \frac{1}{2a}\int_a^\infty \mathrm{e}^{-(z+a)^2/2}\,\mathrm{d}z \quad \text{let } u = z-a \text{ and } u = z+a \text{ resp.}\\
&= \frac{1}{2a}\int_0^\infty \mathrm{e}^{-u^2/2}\mathrm{d}u - \frac{1}{2a}\int_{2a}^\infty \mathrm{e}^{-u^2/2}\mathrm{d}u\\
&= \frac{\sqrt{2\pi}}{2a}\int_0^\infty \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-u^2/2}\mathrm{d}u - \frac{\sqrt{2\pi}}{2a}\int_{2a}^\infty \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-u^2/2}\mathrm{d}u\\
&= \frac{\sqrt{2\pi}}{2a}\left\{\left(\frac{1}{2} - \left(\frac{1}{2} - \int_0^{2a} \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-u^2/2}\mathrm{d}u\right)\right)\right\}\\
&= \frac{1}{2a}\int_0^{2a} \mathrm{e}^{-u^2/2}\mathrm{d}u.
\end{aligned}$$

using properties of the standard normal pdf. $\boxed{4}$

unseen $\Downarrow$

4. (a)

$$
\begin{aligned}
\mathrm{var}_{f_{X,Y}}(aX + bY) &= \mathsf{E}_{f_{X,Y}}((aX + bY)^2) - \mathsf{E}^2_{f_{X,Y}}(aX + bY) \\
&= \mathsf{E}_{f_{X,Y}}(a^2X^2 + 2abXY + b^2Y^2) - (a\mathsf{E}_{f_X}(X) + b\mathsf{E}_{f_Y}(Y))^2 \\
&= a^2\mathsf{E}_{f_X}(X^2) + 2ab\mathsf{E}_{f_{X,Y}}(XY) + b^2\mathsf{E}_{f_Y}(Y^2) - (a^2\mathsf{E}^2_{f_X}(X) + 2ab\mathsf{E}_{f_X}(X)\mathsf{E}_{f_Y}(Y) + b^2\mathsf{E}^2_{f_Y}(Y)) \\
&= a^2(\mathsf{E}_{f_X}(X^2) - \mathsf{E}^2_{f_X}(X)) + b^2(\mathsf{E}_{f_Y}(Y^2) - \mathsf{E}^2_{f_Y}(Y)) + 2ab(\mathsf{E}_{f_{X,Y}}(XY) - \mathsf{E}_{f_X}(X)\mathsf{E}_{f_Y}(Y)) \\
&= a^2\mathrm{var}_{f_X}(X) + b^2\mathrm{var}_{f_Y}(Y) + 2ab\mathrm{cov}_{f_{X,Y}}(X,Y).
\end{aligned}
$$

3

sim. seen $\Downarrow$

(b) (i)

$$
\begin{aligned}
f_X(x) &= \int_0^1 f_{X,Y}(x,y)\,\mathrm{d}y = \int_0^1 \frac{2}{3}(x+2y)\,\mathrm{d}y \\
&= \left[\frac{2xy}{3} + \frac{2y^2}{3}\right]_0^1 = \frac{2x}{3} + \frac{2}{3} \\
&= \frac{2}{3}(x+1), \quad 0 \le x \le 1.
\end{aligned}
$$

2

(ii)

$$
\begin{aligned}
f_Y(y) &= \int_0^1 f_{X,Y}(x,y)\,\mathrm{d}x = \int_0^1 \frac{2}{3}(x+2y)\,\mathrm{d}x \\
&= \left[\frac{x^2}{3} + \frac{4xy}{3}\right]_0^1 = \frac{1}{3} + \frac{4y}{3} \\
&= \frac{1}{3}(4y+1), \quad 0 \le y \le 1.
\end{aligned}
$$

2

(iii)

$$
\begin{aligned}
\mathsf{E}_{f_X}(X) &= \int_0^1 \frac{2}{3}(x^2+x)\,\mathrm{d}x = \frac{2}{3}\left[\frac{x^3}{3} + \frac{x^2}{2}\right]_0^1 \\
&= \frac{2}{3}\left(\frac{1}{3} + \frac{1}{2}\right) = \frac{2}{3}\cdot\frac{5}{6} = \frac{5}{9}.
\end{aligned}
$$

2

$$
\begin{aligned}
\mathsf{E}_{f_Y}(y) &= \int_0^1 \frac{1}{3}(4y^2+y)\,\mathrm{d}x = \frac{1}{3}\left[\frac{4y^3}{3} + \frac{y^2}{2}\right]_0^1 \\
&= \frac{1}{3}\left(\frac{4}{3} + \frac{1}{2}\right) = \frac{1}{3}\cdot\frac{11}{6} = \frac{11}{18}.
\end{aligned}
$$

2

$$
\begin{aligned}
\mathsf{E}_{f_{X,Y}}(XY) &= \int_0^1 \int_0^1 \frac{2}{3} xy(x+2y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \left[\frac{2}{3}\left(\frac{x^3 y}{3} + x^2 y^2\right)\right]_0^1 \mathrm{d}y \\
&= \int_0^1 \frac{2}{3}\left(\frac{y}{3} + y^2\right)\,\mathrm{d}y = \left[\frac{2}{3}\left(\frac{y^2}{6} + \frac{y^3}{3}\right)\right]_0^1 \\
&= \frac{2}{3}\left(\frac{1}{6} + \frac{1}{3}\right) = \frac{2}{3}\cdot\frac{3}{6} = \frac{1}{3}.
\end{aligned}
$$

$\boxed{3}$

$\boxed{\text{unseen} \Downarrow}$

(iv)

$$
\mathsf{E}_{f_{X,Y}}(X-Y) = \mathsf{E}_{f_X}(X) - \mathsf{E}_{f_Y}(Y) = \frac{5}{9} - \frac{11}{18} = -\frac{1}{18}.
$$

$\boxed{2}$

(v)

$$
\begin{aligned}
\mathsf{P}(X > Y) &= \int_0^1 \int_0^x f_{X,Y}(x,y)\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 \int_0^x \frac{2}{3}(x+2y)\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_0^1 \left[\frac{2}{3}(xy + y^2)\right]_0^x \mathrm{d}x = \int_0^1 \frac{2}{3}(x^2 + x^2)\,\mathrm{d}x \\
&= \left[\frac{2}{3}\cdot\frac{2x^3}{3}\right]_0^1 = \frac{4}{9}.
\end{aligned}
$$

$\boxed{3}$

Could also formulate as $\int_0^1 \int_y^1 f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y$.
$\mathsf{P}(X > Y) < 0.5$ which is consistent with $\mathsf{E}_{f_{X,Y}}(X - Y) < 0$. $\boxed{1}$

Imperial College
London

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2015

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Differential Equations

Date: Friday, 15 May 2015. Time: 2.00pm – 4.00pm. Time allowed: 2 hours.

This paper has FOUR questions.

Candidates should start their solutions to each question in a new main answer book

Supplementary books may only be used after the relevant main book(s) are full.

Statistical tables will not be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers to each question as per the table below.

Raw mark	up to 12	13	14	15	16	17	18	19	20
Extra credit	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4

- Each question carries equal weight.
- Calculators may not be used.

1.

(a) State the local Picard Theorem for the IVP
$$\frac{dx}{dt} = f(x), \quad x(0) = x_0,$$
where $f : U \to \mathbb{R}^n$ and U is an open subset of $\mathbb{R}^n$.

(b) Show that the Lipschitz constant of the function $[-2, 2] \ni x \mapsto \cos(x^2)$ is at most 4.

(c) Using (b) and the local Picard theorem (where we take $U = (-2, 2)$) give an explicit choice for $h > 0$ so that there exists a solution $x : (-h, h) \to \mathbb{R}$ of the IVP
$$\frac{dx}{dt} = \cos(x^2), x(0) = 1.$$

(d) Using results derived in the course, show that - in actual fact - this solution $x(t)$ exists and is unique for all $t \in \mathbb{R}$.

(e) Draw the phase diagram for the differential equation $\frac{dx}{dt} = \cos(x^2)$.

(f) Taking $x(t)$ to be the solution when $x(0) = 1$, compute $\lim_{t\to\infty} x(t)$ and prove your answer.

(g) Now also consider
$$\frac{dy}{dt} = \cos(y^2), y(0) = 0.5.$$
Show that $1 < y(1) < x(1)$. (Hint: Use that $\cos(1) > 0.5$.)

2. This question consists of several parts which are totally unrelated.

(a) Let $A = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$. It is given that this matrix has eigenvalue 1 with multiplicity 3, and only one eigenvector. Find a matrix T so that $T^{-1}AT = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

(b) Let $B = \begin{pmatrix} -2 & -1 & -1 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$ and determine e^{Bt} and $W^s(0)$.

(c) Consider
$$\begin{aligned} x' &= -4y - xy^2 \\ y' &= 2x - x^4 y \end{aligned}.$$
Is $(0, 0)$ a hyperbolic singularity of the differential equation? Find a Lyapounov function for $(0, 0)$. Can you conclude that $(0, 0)$ asymptotically stable? If so, give a careful argument.

(d) Let $g(x, y, z) = x^2 + y^2 + \sin(z) = 0$ and $M = \{(x, y, z); g(x, y, z) = 0\}$.

(i) Use the Implicit Function Theorem to show that near $(0, 0, 0)$ there exists $G : \mathbb{R}^2 \to \mathbb{R}$ so that $g(x, y, z) = 0$ is equivalent to $g(x, y, G(x, y)) = 0$.

(ii) Show that M is a manifold.

3. (a) Consider the differential equation

$$x^2 y''(x) - (x + x^2) y'(x) = 0$$

and assume that $y(x) = \sum_{n=0}^{\infty} a_n x^{n+m}$ is a solution and assume $a_0 \neq 0$. Determine m and a_1, a_2 in terms of a_0.

(b) Consider the differential equation

$$y''(x) + q(x) y(x) = 0$$

where

$$q(x) = 2 + \sin(e^x).$$

(i) Write this second order differential equation in the form $\frac{dz}{dx} = A(x)z(x)$ where $z(x)$ is a vector depending on x. Give a global Lipschitz constant K for this differential equation. Use the global Picard Theorem to derive that the solution $z(x)$ exists for all x.

(ii) Show that there exists no Lipschitz constant for q, i.e., there exists no L so that $|q(x) - q(y)| \leq L|x - y|$ for all x, y. Why does this not contradict the answer you found in question (i)? (Hint: use $x_n = \log(n\pi)$ and $y_n = \log(n\pi + \pi/2)$ and show that $|q(x_n) - q(y_n)|/|x_n - y_n| \to \infty$.)

(iii) Consider $y'' + q(x)y = 0$ and $v'' + v = 0$. Assume $a < b$ are two consecutive zeros of v. Show that y has a zero for some $x \in (a, b)$. (Hint: Use that $q(x) \geq 1$ for all x, consider $W = yv' - y'v$ and use methods you have seen before.)

(iv) Hence, or otherwise, show that y has infinitely many zeros.

4. (a) State the Poincaré-Bendixson Theorem.

(b) In this question we will consider the van der Pol equation

$$\begin{aligned} \dot{x} &= y - x^3 + x \\ \dot{y} &= -x. \end{aligned}$$

and let $\phi_t(x, y)$ be its flow.

(i) Determine the nullclines of the differential equation and sketch them. Show that $(0, 0)$ is the only singular point. Draw arrows indicating the direction of the flow in different regions of the (x, y)-plane.

(ii) Draw the region $A := \{(x, y); |y| \leq 10, |x| \leq 10, |x - y| \leq 10\}$.

(iii) Show that $\dot{x} - \dot{y} < 0$ when both $x - y = 10$ and $x \in [0, 10]$.

(iv) Using (i) and (iii) draw the arrows (indicating the direction of the flow) along the boundary of A. Show that when $(x, y) \in A$ then $\phi_t(x, y) \in A$ when $t \geq 0$.

(v) Take $V = x^2 + y^2$ and show that $\dot{V} \geq 0$ when $|x| \leq 1$. Conclude that if $V(x, y) \geq 1$ then $V(\phi_t(x, y)) \geq 1$ for $t \geq 0$.

(vi) Conclude that the van der Pol equation has a periodic solution. (In the lecture notes a proof was given that there exists precisely one periodic solution. You do NOT have to show this here.)

M2AA1 (2015) Solutions

Solution 1: [20 marks]

Solution 1a: [2 marks] Let K be a Lipschitz constant of f on U, i.e., so that $|f(x) - f(y)| \le K|x - y|$ for all $x, y \in U$. Then there exists a solution $(-h, h) \ni x(t) \to \mathbb{R}^n$ provided $h > 0$ is chosen so that $hK < 1$ and $\{x; |x - x_0| < hM\} \subset U$. **[Seen, in Lecture Notes.]**

Solution 1b: [2 marks] Write $f(x) = \cos(x^2)$. Then $f'(x) = -\sin(x^2)2x$ which is in norm at most $2|x| \le 4$ for $x \in [-2, 2]$. **[Seen, in assignments.]**

Solution 1c: [4 marks] $M = \sup_{x\in[-2,2]} |f(x)| \le 1$. According to the local Picard theorem we need that to choose $h > 0$ so that the interval $[x_0 - hM, x_0 + hM] \subset (-2, 2)$ and also so that $hK < 1$. Since $K = 4$, $x_0 = 1$ and $M = 1$, we can take $h = 1/5$ (or any choice with $h \in (0, 1/4)$. To marker: if the student uses a different theorem, then give full marks provided this is argued correctly. **[Only implicitly seen in assignments.]**

Solution 1d: [2 marks] According to the notes, if the maximal interval of existence is equal to (α, β) and $\beta < \infty$ then $|x(t)| \to \infty$ as $t \uparrow \beta$. However, $|x'(t)| \le 1$ so that $|x(t)| \le |t|$ and therefore $\beta < \infty$ is impossible. Similarly for $\alpha > -\infty$. Another way of seeing this, is to say that since $1 \in (-\sqrt{\pi/2}, \sqrt{\pi/2})$, $\cos(\pm\pi/2) = 0$, it follows that $|x(t)| < \sqrt{\pi/2}$ for all $t \in \mathbb{R}$. So again the solution exists for t. **[Seen, in lecture notes.]**

Solution 1e: [2 marks] Singularies at $\pm\sqrt{k\pi + \pi/2}$ and alternating arrow to the right and left. **[Seen in lecture notes.]**

Solution 1f: [4 marks] As $\dot{x} = \cos(x^2) > 0$ for $x \in (-\sqrt{\pi/2}, \sqrt{\pi/2})$ we have, $t \mapsto x(t)$ is increasing. For every $\epsilon > 0$ if $x \in (-\sqrt{\pi/2} + \epsilon, \sqrt{\pi/2} - \epsilon)$ there exists $\delta > 0$ so that $\dot{x} > \delta$. It follows that for $t > \sqrt{\pi/2}/\delta$ one has $x(t) \in (\sqrt{\pi/2} - \epsilon, \sqrt{\pi/2})$. Hence $x(t) \to \sqrt{\pi/2}$ for $t \to \infty$. **[Seen, in lecture notes.]**

Solution 1g: [4 marks] $y(t) < x(t)$ holds because $y(0) < x(0)$ and solutions do not intersect. Since $\dot{y} > 0$ we get $y(t) \ge 1/2$ for $t \ge 0$. While $y(t) \le 1$ one has $\dot{y} = \cos(y^2(t)) \ge \min_{y\in[1/2,1]} \cos(y^2(t)) \ge \cos(1) \ge 1/2$ (and a strict inequality holds for $t \approx 0$). Hence $y(t) - y(0) > t \cdot 0.5$ and so $y(1) > y(0) + 0.5 = 1$. **[Unseen.]**

Solution 2: [20 marks]
Solution 2a: [5 marks] Eigenvector is $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. The equation $Av_2 = v_1 + v_2$ has solution $v_2 = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$ and the eigenvector $Av_3 = v_3 + v_2$ has solution $v_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$. So if we take $T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$ then $T^{-1}AT$ has the desired form. **[Similar to problem classes / lecture notes]**

Solution 2b: [5 marks] Probably the easiest way is to write $B = -2I + N$ where $N = \begin{pmatrix} 0 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Note that $N^2 = 0$. So $\exp(tN) = \begin{pmatrix} 1 & -t & -t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and since N and I commute, we get $\exp(tB) = \begin{pmatrix} e^{-2t} & -te^{-2t} & -te^{-2t} \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{-2t} \end{pmatrix}$. Hence $\forall\, x_0 \in \mathbb{R}^3$, $\exp(tB)x_0 \to 0$ as $y \to \infty$ and therefore $W^s(0) = \mathbb{R}^3$. **[As in lecture notes]**

Solution 2c: [5 marks] The linear system is $x' = -4y, y' = 2x$ which has eigenvalues $\pm\sqrt{8}i$, and so $(0,0)$ is not hyperbolic. Take $V(x,y) = ax^2 + cy^2$. Then $\dot{V} = 2ax[-4y - xy^2] + 2cy[2x - x^4y]$. Taking $a = 1$ and $c = 2$ gives $\dot{V} = -2x^2y^2 - 4x^4y^2 \le 0$, and < 0 holds except when $x = 0$ or $y = 0$. So $V(x(t), y(t))$ is decreasing. If $\lim_{t\to\infty} V(x(t), y(t)) > 0$ then $\dot{V} \to 0$ and so $(x(t), y(t))$ tend to set where $\dot{V} = 0$, but $x(t), y(t)$ do not both tend to zero. However, $\dot{x} \neq 0$ when $x = 0$ and $y \neq 0$ and $\dot{y} \neq 0$ when $y = 0$ and $x \neq 0$, so a solution cannot remain near the set where $\dot{V} = 0$ (unless $x(t), y(t)$ tends to $(0,0)$). This contradiction show that $(0,0)$ is asymptotically stable. **[Similar to question discussed in class]**

Solution 2d: [5 marks] The linear part of g at $(0,0,0)$ is equal to

$$Dg_{(0,0,0)} = \begin{pmatrix} 2x & 2y & \cos(z) \end{pmatrix}\big|_{(x,y,z)=(0,0,0)}.$$

Since the last entry in this vector is $\neq 0$ one can apply the Implicit Function Theorem. More generally $Dg_{(x,y,x)} \neq 0$ for each $(x,y,z) \in M$, since if $\cos(z) = 0$ then $\sin(z) = \pm 1$ and since $(x,y,z) \in M$ this gives $x \neq 0$ or $y \neq 0$. Hence M is a manifold. **[Similar to questions discussed in class]**

Solution 3: [20 marks]
Solution 3a: [6 marks] Differentiating gives

$$0 = x^2[m(m-1)a_0x^{m-2} + (m+1)ma_1x^{m-1} + (m+2)(m+1)a_2x^m + \dots]$$
$$-(x+x^2)[ma_0x^{m-1} + (m+1)a_1x^m + (m+2)a_2x^{m+1} + \dots].$$

This gives that $m(m-1) - m = 0$ so $m = 0, 2$ (and a_0 is arbitrary), and

$$(m+1)ma_1 - ma_0 - (m+1)a_1 = 0$$

$$(m+2)(m+1)a_2 - (m+1)a_1 - (m+2)a_2 = 0.$$

When $m = 0$ this corresponds to $a_1 = 0$ and $a_2 = 0$, and when $m = 2$ this corresponds to $6a_1 - 2a_0 - 3a_1 = 0$ and $8a_2 - 3a_1 = 0$, i.e., $a_1 = (2/3)a_0$ and $a_2 = (3/8)a_1 = (2/8)a_0 = a_0/4$. **[Similar to other questions in assignments.]**

Solution 3b(i): [4 marks] Write $z = \begin{pmatrix} y \\ y' \end{pmatrix}$ then $z' = A(x)z$ where $A(x) = \begin{pmatrix} 0 & 1 \\ -q(x) & 0 \end{pmatrix}$. The matrix $A(x)$ has matrix norm which is bounded by, for example, 4 because for every vector $u \in \mathbb{R}^2$,

$$|A(x)u| \le \sqrt{u_2^2 + (q(x)u_1)^2} \le |u| + |q(x)||u| \le 4|u|.$$

Hence by the global Picard Theorem, and its corollaries, $z(x)$ exists for all x. **[New.]**

Solution 3b(ii): [4 marks] With the given choices $q(x_n) = 0$ and $q(y_n) = 1$, whereas $|x_n - y_n| = |\log(k\pi) - \log(k\pi + \pi/2)| = \log(1 + 1/2k) \to 0$. Hence $|q(x_n) - q(y_n)|/|x_n - y_n| \to \infty$. The reason this does not contradict the Picard theorem, because that theorem only requires in $\frac{dy}{dx} = f(x, y)$ that $|f(x, y_1) - f(x, y_2)| \le K|y_1 - y_2|$. No Lipschitz constant estimate is required in the first variable. **[Seen, but not in this context.]**

Solution 3b(iii): [4 marks] Assume that $v > 0$ on (a, b) (otherwise consider $-v$ instead). Assume by contradiction that $y(x) > 0$ on (a, b) (otherwise consider $-y$ instead). Note $v'(a) \ge 0$ and $v'(b) \le 0$. Moreover, $W' = yv'' - y''v = -yv + q(z)yv \ge 0$ because $q(x) \ge 1$ and $W'(x) > 0$ except when $\sin(e^x) = -1$. So $W(b) > W(a)$. But this contradicts that $W(a) = y(a)v'(a) \ge 0$ and $W(b) = y(b)v'(b) \le 0$. **[Similar to previous questions.]**

Solution 3b(iv): [2 marks] $v(x) = c_1\cos(x) + c_2\sin(x)$ so this has infinitely many zeros. Hence by the previous question y also has infinitely many zeros. **[Similar to previous questions.]**

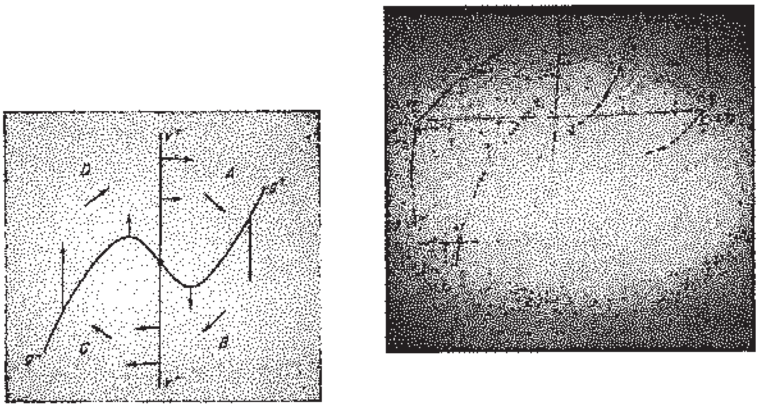


Figure 1: The nulclines of the van der Pol equation on the left and the region A on the right.

Solution (4): [20 marks]

Solution (4a): [2 marks] Consider an autonomous differential equation in the plane for which the assumptions of the local Picard theorem hold, and assume that the omega-limit of x, i.e., $\omega(x)$, is non-empty and does not contain a singular point. Then ω is a periodic orbit. **[In lecture note]**

Solution (4bi): [3 marks] The directions of the flow are shown in Figure 1 on the left. **[In lecture notes]**

Solution (4bii): [1 marks] See Figure 1 on the right. **[New]**

Solution (4biii): [4 marks] $\dot{x} - \dot{y} = y - x^3 + x + x = (x - 10) - x^3 + 2x = 3x - x^3 - 10$. This has extrema at $x = \pm 1$ and since $\dot{x} - \dot{y} < 0$ at $x = 0, 1, 10$ one has $\dot{x} - \dot{y} < 0$ for all $x \in [0, 10]$. **[Similar to seen.]**

Solution (4biv): [2 marks] This immediately follows from (i) and (iii). **[Similar to seen.]**

Solution (4bv): [4 marks] $\dot{V} = 2x\dot{x} + 2y\dot{y} = 2x(y - x^3 + x) + 2y(-x) = -2x^4 + 2x^2 \geq 0$ when $|x| \leq 1$. It follows that $\dot{V}$ is non-decreasing, and so if $V(x, y) \geq 1$ then $V(\phi_t(x, y)) \geq 1$ when $t \geq 0$. **[New.]**

Solution (4bvi): [4 marks] By (4biv) and (4bv) the set $A \setminus \{(x, y); |(x, y)| \leq 1\}$ is forward invariant. Since this annulus does not contain singularities and is forward invariant, the assumptions of the Poincaré-Bendixson theorem are satisfied and therefore there exists a periodic solution in this annulus. **[Similar to seen.]**

Imperial College
London

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2015

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Multivariable Calculus

Date: Wednesday, 13 May 2015. Time: 2.00pm – 4.00pm. Time allowed: 2 hours.

This paper has FOUR questions.

Candidates should start their solutions to each question in a new main answer book

Supplementary books may only be used after the relevant main book(s) are full.

Statistical tables will not be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers to each question as per the table below.

Raw mark	up to 12	13	14	15	16	17	18	19	20
Extra credit	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4

- Each question carries equal weight.
- Calculators may not be used.

1. (i) Use tensor notation to simplify the following expression:

$$\nabla^2(\psi \mathbf{A}) - \mathbf{A}\,\nabla^2\psi - 2(\nabla\psi\cdot\nabla)\,\mathbf{A},$$

where $\mathbf{A}, \psi$ are any sufficiently smooth vector and scalar fields respectively.

(ii) Consider the vector field

$$\mathbf{F} = (2xyz^3 + y\,\mathrm{e}^{xy} + \sin x)\,\mathbf{i} + (x^2z^3 + x\,\mathrm{e}^{xy})\,\mathbf{j} + (3x^2yz^2 + \cos z)\,\mathbf{k}.$$

Show that $\mathbf{F}$ is irrotational and find a corresponding potential function. Hence evaluate

$$\int_C \mathbf{F}\cdot d\mathbf{r},$$

where the curve C starts at the point $(\pi/2, 0, 0)$ and ends at $(0, 1, \pi/2)$.

(iii) Use the transformation

$$u = x + 2y, \quad v = x - 2y$$

to evaluate the integral

$$\int_R x \sin\left[\pi(x-2y)\right]\, dx\, dy,$$

where R is the finite region in the $x-y$ plane bounded by the straight lines

$$x - 2y = \pm 2, \quad x + 2y = \pm 2.$$

2. Let S be the **open** surface

$$z^2 = x^2 + y^2; \quad (0 \le z \le H),$$

and consider the vector field

$$\mathbf{A} = \alpha y\,\mathbf{i} - \beta x\,\mathbf{j} + \gamma xyz\,\mathbf{k},$$

where α, β, γ are constants.

Evaluate

$$\int_S (\text{curl}\,\mathbf{A})\cdot\widehat{\mathbf{n}}\,dS,$$

where $\widehat{\mathbf{n}}$ is the unit outward normal to S, by each of the following methods:

(i) use of Stokes theorem;

(ii) projection onto $z = 0$;

(iii) use of the divergence theorem (by considering an appropriate closed surface).

3. (i) Use the method of separation of variables to find the solution $u(x,t)$ to the heat equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\kappa}\frac{\partial u}{\partial t}, \quad (-L < x < L,\, t > 0),$$

that satisfies the boundary conditions

$$\frac{\partial u}{\partial x}(-L,t) = \frac{\partial u}{\partial x}(L,t) = 0,$$

and the initial condition

$$u(x,0) = \mathrm{e}^{-|x|}, \quad (-L < x < L).$$

(ii) Show that, in terms of the notation $\omega_n = n\pi/L, \delta\omega = \omega_{n+1} - \omega_n$, your solution can be written in the form

$$u(x,t) = \frac{1}{L}(1 - \mathrm{e}^{-L}) + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n\mathrm{e}^{-L}}{1+\omega_n^2}\cos(\omega_n x)\mathrm{e}^{-\omega_n^2\kappa t}\delta\omega. \tag{1}$$

(iii) Suppose that the problem described in (i) is now posed over $(-\infty,\infty)$, rather than $(-L,L)$. By considering expression (1), deduce the corresponding solution over the infinite domain. What procedure could we use to demonstrate the validity of this solution? (No detailed calculation is required here).

4. Green's theorem in vector form is given by

$$\oint_C \mathbf{A}\cdot\widehat{\mathbf{n}}\; ds = \int_R \operatorname{div}\mathbf{A}\;\, dx\, dy.$$

Here, $\mathbf{A}$ is any two-dimensional differentiable vector field, C is a simple closed curve bounding the region R, $\widehat{\mathbf{n}}$ is the unit outward normal to C and s represents arclength.

(i) In what direction must the curve C be traversed in order that the theorem as stated is correct?

(ii) Show that in Cartesian coordinates:

$$\widehat{\mathbf{n}}\, ds = dy\,\mathbf{i} - dx\,\mathbf{j}.$$

(iii) Obtain an expression for the area of the region R in terms of an integral around C.

Now consider the function

$$G = \frac{1}{4\pi}\ln(x^2+y^2).$$

(iv) Show that, except at the origin $(0,0)$:

$$\nabla^2 G = 0.$$

(v) Establish the result

$$\oint_C (\nabla G\cdot\widehat{\mathbf{n}})\; ds = \begin{cases} 0, & \text{if the origin is outside } C,\\ 1, & \text{if the origin is inside } C.\end{cases}$$

(vi) Explain briefly how results (iv) and (v) imply that

$$\nabla^2 G = \delta(\mathbf{r})$$

where δ is the Dirac delta function and $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}$.

EXAMINATION SOLUTIONS 2014-15

Course: M2AA2

Question 1

(i) unseen, but similar done in lectures & on Q. sheets
(ii) as above
(iii) as above

Marks & seen/unseen

Parts

(i)

$$\left[\nabla^2(\psi\underline{A}) - \underline{A}\nabla^2\psi - 2\left[(\nabla\psi\cdot\nabla)\underline{A}\right]\right]_i$$

$$= \frac{\partial^2}{\partial x_j^2}(\psi A_i) - A_i\frac{\partial^2\psi}{\partial x_j^2} - 2\frac{\partial\psi}{\partial x_j}\frac{\partial}{\partial x_j}A_i \qquad 2$$

$$= \psi\frac{\partial^2 A_i}{\partial x_j^2} + \cancel{2\frac{\partial\psi}{\partial x_j}\frac{\partial A_i}{\partial x_j}} + \cancel{A_i\frac{\partial^2\psi}{\partial x_j^2}} - \cancel{A_i\frac{\partial^2\psi}{\partial x_j^2}} - \cancel{2\frac{\partial\psi}{\partial x_j}\frac{\partial A_i}{\partial x_j}} \qquad 1$$

& hence the expression simplifies to $\underline{\psi\nabla^2\underline{A}}$. 1

(ii)

$$\text{curl }\underline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ 2xyz^3 + ye^{xy} + \sin x & x^2z^3 + xe^{xy} & 3x^2yz^2 + \cos z \end{vmatrix}$$

$$= \hat{i}(3x^2z^2 - 3x^2z^2) - \hat{j}(6xyz^2 - 6xyz^2) + \hat{k}\left((2xz^3 + e^{xy} + xye^{xy}) - (2xz^3 + e^{xy} + xye^{xy})\right)$$

$$= \underline{0}. \qquad 3$$

Set $\underline{F} = \nabla\varphi$.

Then: $\varphi_x = 2xyz^3 + ye^{xy} + \sin x \Rightarrow \varphi = x^2yz^3 + e^{xy} - \cos x + f_1(y,z)$

$\varphi_y = x^2z^3 + xe^{xy} \Rightarrow \varphi = x^2yz^3 + e^{xy} + f_2(x,z)$

$\varphi_z = 3x^2yz^2 + \cos z \Rightarrow \varphi = x^2yz^3 + \sin z + f_3(x,y)$.

Putting all the information together:

$$\underline{\varphi = x^2yz^3 + e^{xy} + \sin z - \cos x + C} \qquad 3$$

Then: $\int_C \underline{F}\cdot d\underline{r} = [\varphi]_{(\pi/2,0,0)}^{(0,1,\pi/2)} = (1+1-1+C) - (1+C)$

$= \underline{\underline{0}}$ 2

(iii) In terms of u, v the limits are $u = \pm 2$, $v = \pm 2$. 2

The integrand: $x\sin(\pi(x-2y)) = \frac{1}{2}(u+v)\sin(\pi v)$ 2

The Jacobian: $J = 1 \Big/ \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix}$

$= 1 \Big/ \begin{vmatrix} 1 & 2 \\ 1 & -2 \end{vmatrix} = -1/4$ 2

Then $I = \int_{-2}^{2}\int_{-2}^{2} \frac{1}{2}(u+v)\sin(\pi v)\,|J|\,du\,dv$

Setter's initials [redacted] Checker's initials [redacted]

Question 1 ctd

Marks & seen/unseen

Parts

$$\Rightarrow I = \frac{1}{8}\int_{-2}^{2}\int_{-2}^{2}\left(u\sin(\pi v) + v\sin(\pi v)\right)du\,dv$$

$$= \frac{1}{8}\int_{-2}^{2} \underbrace{\left[\frac{u^2}{2}\sin(\pi v)\right]_{-2}^{2}}_{\text{zero}} + \left[uv\sin(\pi v)\right]_{-2}^{2} dv$$

$$= \frac{1}{2}\int_{-2}^{2} v\sin(\pi v)\,dv$$

$$= \int_{0}^{2} v\sin(\pi v)\,dv = \left[-\frac{v}{\pi}\cos(\pi v)\right]_{0}^{2} + \underbrace{\frac{1}{\pi}\int_{0}^{2}\cos(\pi v)\,dv}_{\text{zero.}}$$

$$= \underline{\underline{-2/\pi}}$$

2

[Total 20]

EXAMINATION SOLUTIONS 2014-15

Course: M2AA2

Question 2

(i), (ii), (iii) all unseen, but similar examples done in lectures & on Q.sheets.

Marks & seen/unseen

Parts

(i) Stokes theorem: $\int_S (\mathrm{Curl}\,\underline{A})\cdot\hat{\underline{n}}\, dS = \oint_C \underline{A}\cdot d\underline{r}$ — 1

where C is the circular boundary of radius H at $z=H$, traversed CLOCKWISE according to right hand rule. — 1

on C, $x=H\cos\theta$, $y=H\sin\theta$ — 1

$\underline{A}\cdot d\underline{r} = (\alpha y, -\beta x, \gamma xyz)\cdot(dx\,\hat{\imath} + dy\,\hat{\jmath})$

$= \alpha y\,dx - \beta x\,dy$ — 1

$\therefore \oint_C \underline{A}\cdot d\underline{r} = \int_{2\pi}^{0} \alpha H\sin\theta(-H\sin\theta) - \beta H\cos\theta(H\cos\theta)\,d\theta$

$= \int_0^{2\pi} \alpha H^2\sin^2\theta + \beta H^2\cos^2\theta\, d\theta = \underline{\pi(\alpha+\beta)H^2}$ — 1

(ii) $\mathrm{Curl}\,\underline{A} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ \alpha y & -\beta x & \gamma xyz \end{vmatrix} = \gamma xz\,\hat{\imath} - \gamma yz\,\hat{\jmath} - (\alpha+\beta)\hat{k}$ — 2

$\hat{\underline{n}} = \pm\, \underline{\nabla}(z^2-x^2-y^2)/|\underline{\nabla}(z^2-x^2-y^2)|$

$= \pm(2z\hat{k} - 2x\hat{\imath} - 2y\hat{\jmath})/\sqrt{(4z^2+4x^2+4y^2)}$

$= \frac{1}{\sqrt{2}}\left(\frac{x\hat{\imath}+y\hat{\jmath}-z\hat{k}}{z}\right)$ for outward normal — 2 } this part seen.

Projecting onto $z=0$:

$\int_S \mathrm{Curl}\,\underline{A}\cdot\hat{n}\,dS = \int_\Sigma (\mathrm{Curl}\,\underline{A}\cdot\hat{\underline{n}})_S \frac{dx\,dy}{|\hat{\underline{n}}\cdot\hat{k}|}$

where Σ is a circle (disc) of radius H. — 2

Now, $|\hat{\underline{n}}\cdot\hat{k}| = 1/\sqrt{2}$ — 1

Integral becomes

$\int_\Sigma \frac{1}{\sqrt{2}}(\gamma x^2 - \gamma y^2 + (\alpha+\beta))\sqrt{2}\,dx\,dy$ | Set $x = r\cos\theta$, $y=r\sin\theta$, $dx\,dy \to r\,dr\,d\theta$ — 1

$= \int_0^{2\pi}\int_0^H (\gamma r^3\cos^2\theta - \gamma r^3\sin^2\theta + (\alpha+\beta)r)\,dr\,d\theta$

$= \int_0^H \cancel{\gamma r^3\pi} - \cancel{\gamma r^3\pi} + 2\pi(\alpha+\beta)r\,dr = \underline{\pi(\alpha+\beta)H^2}$ — 2

Setter's initials [redacted] Checker's initials [redacted]

Question 2 ctd

Marks & seen/unseen

Parts

(iii) Divergence theorem. Form a closed surface by including the circular disc at $z = H$. Let V be the enclosed volume. — 1

Then: $\int_V \text{div}(\text{curl}\,\underline{A})\,dV = \int_S (\text{curl}\,\underline{A}).\hat{n}\,dS + \int_{\text{disc}} (\text{curl}\,\underline{A}).\hat{\underline{n}}\,dS$ — 1

LHS $= 0$ since $\text{div}(\text{curl}\,\underline{A}) \equiv 0$. — 1

On disc $\hat{\underline{n}} = \hat{\underline{k}}$ (outward normal) — 1

Thus: $\int_S (\text{curl}\,\underline{A}).\hat{\underline{n}}\,dS = -\int_{\text{disc}} (\text{curl}\,\underline{A}).\hat{\underline{k}}\,dx\,dy$

$= -\int_0^{2\pi}\int_0^H -(\alpha+\beta)\,r\,dr\,d\theta$ (using curl $\underline{A}$ from earlier)

$= \underline{\pi(\alpha+\beta)H^2}$ — 1

[If part (ii) not done, give marks in (iii) for correct calculation of curl $\underline{A}$ and the switch to polars over the disc.]

Total 20

Setter's initials [redacted] Checker's initials [redacted]

EXAMINATION SOLUTIONS 2014-15

Course M2AA2

Question 3

(i) done on problem sheet for different initial condition and different domain.
(ii) unseen, but similar ideas discussed. [The FT problem has been seen].

Marks & seen/unseen

Parts

(i) Seek solution $u = X(x)T(t)$: $\dfrac{X''}{X} = \dfrac{T'}{\kappa T} = -\lambda^2$ for periodic soln in x. (& bounded soln in t). **2**

$\Rightarrow X = A\cos\lambda x + B\sin\lambda x$

with $X' = 0$ on $x = \pm L$

$\Rightarrow \lambda(B\cos\lambda L \pm A\sin\lambda L) = 0$ **2**

$[\Rightarrow \lambda = 0$ (leading to $X = \text{const}$)$]$ – incorporated below

or $B\cos\lambda L = 0$ and $A\sin\lambda L = 0$

Now, since initial condition is even in x we must have $A \neq 0$ and $B = 0$ & hence $\sin\lambda L = 0$. **2**

Thus $\lambda = n\pi/L$; $n = 0, 1, 2, \ldots$

Also: $T'/T = -\lambda^2\kappa \Rightarrow T \propto e^{-n^2\pi^2\kappa t/L^2}$ **2**

Then the full solution is

$$u(x,t) = \tfrac{1}{2}a_0 + \sum_1^\infty a_n\cos\left(\tfrac{n\pi x}{L}\right)e^{-n^2\pi^2\kappa t/L^2} \quad (*)$$ **1**

Applying the initial condition

$$e^{-|x|} = \tfrac{1}{2}a_0 + \sum_1^\infty a_n\cos(n\pi x/L) \quad (-L < x < L)$$

Fourier series:

$$a_n = \frac{1}{L}\int_{-L}^{L} e^{-|x|}\cos\left(\frac{n\pi x}{L}\right)dx \quad n = 0, 1, 2, \ldots$$

$$= \frac{2}{L}\,\mathrm{Re}\left\{\int_0^L e^{(-1+in\pi/L)x}dx\right\}$$ [or could integrate by parts twice]

$$= \frac{2}{L}\,\mathrm{Re}\left\{\left(e^{(-1+in\pi/L)L} - 1\right)\Big/\left(-1 + \frac{in\pi}{L}\right)\right\}$$

$$= \frac{2}{L}\,\mathrm{Re}\left\{\left(e^{-L}\cos(n\pi) - 1\right)\left(-1 - in\pi/L\right)\Big/\left(1 + \frac{n^2\pi^2}{L^2}\right)\right\}$$

$$= \frac{2}{L}\left(1 - (-1)^n e^{-L}\right)\Big/\left(1 + n^2\pi^2/L^2\right) \quad (+) \quad (n = 0, 1, 2, \ldots)$$ **4**

and so soln is given by $(*)$ & $(+)$.

(ii) Setting $\omega_n = n\pi/L$. We have $\delta\omega = \omega_{n+1} - \omega_n = \pi/L$.

Then: $2/L = (2/\pi)\delta\omega$

Also, note that $\tfrac{1}{2}a_0 = \tfrac{1}{L}(1 - e^{-L})$.

Setter's initials [redacted] Checker's initials [redacted]

Question 3 ctd

Marks & seen/unseen

Parts

Solution can therefore be written as

$$u = \frac{1}{L}(1-e^{-L}) + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(1-(-1)^n e^{-L})}{1+\omega_n^2}\cos(\omega_n x)e^{-\omega_n^2 \kappa t}\,\delta\omega$$

as required. 3

(iii) Letting $L \to \infty$ we have $\sum^{\infty} g(\omega_n)\delta\omega \to \int_0^\infty g(\omega)\,d\omega$ and hence it would seem that on $(-\infty,\infty)$ the solution is

$$u = \frac{2}{\pi}\int_0^\infty \frac{\cos(\omega x)}{1+\omega^2} e^{-\omega^2 \kappa t}\,d\omega.$$

(Since the 1st term $\to 0$ as $L \to \infty$) 2

This solution can be confirmed by taking an exponential Fourier transform in x. 2

[and indeed the solution has been obtained by this method on a problem sheet.]

Total 20.

EXAMINATION SOLUTIONS 2014-15

Course: M2AA2

Question 4

(i) seen; (ii) given as exercise; (iii) seen in scalar form (iv) seen in a different way; (v) unseen, but similar ideas seen. (vi) unseen in this form.

Marks & seen/unseen

Parts

(i)

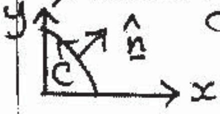
The curve C must be traversed in such a way that R is kept to the left at all times. (1 mark for anti-clockwise) — 2

(ii) The tangent vector $= dx\hat{\imath} + dy\hat{\jmath}$

vector orthogonal to this is $dy\hat{\imath} - dx\hat{\jmath}$

and a unit vector is therefore $\pm(dy\hat{\imath} - dx\hat{\jmath})/\sqrt{((dy)^2 + (dx)^2)}$

$\Rightarrow \hat{n}\,ds = \pm(dy\hat{\imath} - dx\hat{\jmath})$.



Then for outward normal we need to take + sign. [Alternatively use $\hat{n} = \underline{\nabla}(y-f)/|\underline{\nabla}(y-f)|$] — 3

(iii) If we take $\underline{A} = x\hat{\imath} + y\hat{\jmath}$, then G-T gives

$$\oint_C (x\hat{\imath} + y\hat{\jmath})\cdot(dy\hat{\imath} - dx\hat{\jmath}) = \int_R 2\,dx\,dy$$

$$\Rightarrow \text{area of } R = \frac{1}{2}\oint_C (x\,dy - y\,dx)$$ — 3

(iv) $G_x = \dfrac{x/2\pi}{x^2+y^2}$; $G_y = \dfrac{y/2\pi}{x^2+y^2}$; $G_{xx} = \dfrac{1/2\pi}{x^2+y^2} - \dfrac{x^2/\pi}{(x^2+y^2)^2}$

$G_{yy} = \dfrac{1/2\pi}{x^2+y^2} - \dfrac{y^2/\pi}{(x^2+y^2)^2} \Rightarrow G_{xx} + G_{yy} = \dfrac{1/\pi}{x^2+y^2} - \dfrac{1}{\pi}\dfrac{(x^2+y^2)}{(x^2+y^2)^2}$

$= 0$, as required. — 3

(v) (a) Suppose that O lies outside C.

Set $\underline{A} = \underline{\nabla}G$

Then $\operatorname{div}\underline{A} = 0$ throughout R by (iv) and hence $\int_R \operatorname{div}\underline{A}\,dx\,dy = 0$.

It follows by G-T that $\oint_C (\underline{\nabla}G\cdot\hat{n})\,ds = 0$. — 2

(b)

Now suppose O lies inside C. Exclude the origin as shown by surrounding by a small circle γ of radius ε.

Form the closed curve $C' = C + \overrightarrow{BA} + \overrightarrow{AB} +$ clockwise around γ.

This has the region R_1 as its interior and O is outside this region.

Hence, as in (a): $\oint_{C'} (\underline{\nabla}G\cdot\hat{n})\,ds = 0$

Setter's initials [redacted]

Checker's initials [redacted]

Parts

Since the contributions along AB are equal and opposite, we have

$$\oint_C (\underline{\nabla} G \cdot \hat{\underline{n}})\, ds = \underset{\text{(anti-clockwise)}}{\oint_\gamma (\underline{\nabla} G \cdot \hat{\underline{n}})\, ds}$$ using (ii)

Now, $\underline{\nabla} G \cdot \hat{\underline{n}}\, ds = \frac{1}{2\pi} \frac{(x\hat{\imath} + y\hat{\jmath})}{(x^2+y^2)} \cdot (dy\,\hat{\imath} - dx\,\hat{\jmath})$

$$= \frac{1}{2\pi\varepsilon^2}(x\,dy - y\,dx) \text{ on } \gamma$$

$$\therefore \oint_\gamma (\underline{\nabla} G \cdot \hat{\underline{n}})\, ds = \frac{1}{2\pi\varepsilon^2} \oint_\gamma (x\,dy - y\,dx)$$

$$= \frac{1}{\pi\varepsilon^2} \underbrace{(\text{area of } \gamma)}_{\pi\varepsilon^2} \text{, using (iii)}$$

$$= 1 \text{, as required.}$$ 4

(vi) Suppose $\nabla^2 G = f(\underline{r})$.

Then (iv) $\Rightarrow f(\underline{r}) \equiv 0$ for all $\underline{r} \neq 0$ (*)

and (v) implies, using G-T:

$\int_R \nabla^2 G\, dx\,dy = 1$ for any region R containing O.

It follows that $\int_R f(\underline{r})\, dx\,dy = 1$ (†).

Together, (*) & (†) imply that $f(\underline{r}) = \delta(\underline{r})$. 3

Total 20.

Imperial College
London

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2015

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Introduction to Numerical Analysis

Date: Monday, 11 May 2015. Time: 2.00pm – 4.00pm. Time allowed: 2 hours.

This paper has FOUR questions.

Candidates should start their solutions to each question in a new main answer book

Supplementary books may only be used after the relevant main book(s) are full.

Statistical tables will not be provided.

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- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers to each question as per the table below.

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Extra credit	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4

- Each question carries equal weight.
- Calculators may not be used.

1. (a) State the definition of a matrix $A \in \mathbb{R}^{n\times n}$ being nonnegative definite.

(b) Let A be a skew-symmetric matrix, i.e. $A^T = -A$. Prove that A is nonnegative definite.

(c) Let $B = A + \underline{u}\,\underline{v}^T$ for $A \in \mathbb{R}^{n\times n}$ invertible and $\underline{u}, \underline{v} \in \mathbb{R}^n$. Assuming that $\underline{v}^T A^{-1}\underline{u} = -1$, prove that B is singular.

(d) Define the Hessian $D^2 f(\underline{x})$ for a function $f : \mathbb{R}^n \to \mathbb{R}$.

(e) Let

$$A = \begin{pmatrix} -1 & 0 & 5 \\ 0 & 2 & 1 \\ 5 & 1 & 0 \end{pmatrix}.$$

Find a function $f : \mathbb{R}^3 \to \mathbb{R}$ such that

(i) $D^2 f(\underline{x}) = A$ for all $\underline{x} \in \mathbb{R}^3$.

(ii) $f(\underline{0}) = 0$.

(iii) $f(\underline{e}_1) = 1$, $f(\underline{e}_2) = 2$, $f(\underline{e}_3) = 3$.

Here $\{\underline{e}_i\}_{i=1}^3$ denotes the standard basis in $\mathbb{R}^3$.

(f) Find the Newton form of the cubic interpolating polynomial for the data $\{(x_i, f_i)\}_{i=0}^3 \equiv \{(-1,-5), (0,-1), (2,1), (3,11)\}$.

2. Let $A = (\underline{a}_1\ \underline{a}_2 \dots \underline{a}_n) \in \mathbb{R}^{m\times n}$ have n linearly independent columns $\underline{a}_i \in \mathbb{R}^m$, $i = 1, \dots, n$. Let $A = Q\,R$, where $R \in \mathbb{R}^{m\times n}$ is upper triangular, i.e. $R_{ij} = 0$ if $i > j$, and $Q = (\underline{q}_1\ \underline{q}_2 \cdots \underline{q}_m) \in \mathbb{R}^{m\times m}$.

(a) Prove that $\underline{a}_k \in \operatorname{span}\{\underline{q}_i\}_{i=1}^k$ for $k = 1, \dots, n$.

(b) Prove that $R_{ii} \neq 0$ for $i = 1, \dots, n$.

(c) State a necessary and sufficient condition on $\{\underline{q}_i\}_{i=1}^m$ for Q to be orthogonal.

(d) Assuming that Q is orthogonal and that $R_{ii} < 0$ for $i \in I \subset \{1, \dots, n\}$, find a QR factorization of A so that $A = \widetilde{Q}\,\widetilde{R}$, where $\widetilde{Q}$ is orthogonal and $\widetilde{R}$ is an upper triangular matrix with positive entries on the diagonal.

(e) Using Givens rotations, find such $\widetilde{Q}$ and $\widetilde{R}$ for the example matrix

$$A = \begin{pmatrix} 1 & 2 & -1 & -4 \\ 0 & 15 & 18 & 21 \\ -2 & -4 & -4 & -13 \\ -2 & -4 & -10 & -7 \end{pmatrix}.$$

Here you may state $\widetilde{Q}$ as a product of orthogonal matrices.

3. For a continuous function $f \in C(\mathbb{R})$ the Lagrange interpolant $L_n f \in C(\mathbb{R})$ for the distinct data points $\{x_i\}_{i=0}^n \subset \mathbb{R}$ is defined by

$$(L_n f)(x) = \sum_{i=0}^{n} f(x_i)\, \phi_i(x) \qquad \forall\, x \in \mathbb{R}\,,$$

where

$$\phi_i(x) = \prod_{j=0, j\neq i}^{n} \frac{x - x_j}{x_i - x_j}\,, \quad i = 0, \ldots, n\,.$$

(a) Prove that $L_n f \in \mathbb{P}_n$ is such that $(L_n f)(x_i) = f(x_i)$, $i = 0, \ldots, n$.

(b) Prove that if $q_n \in \mathbb{P}_n$ is such that $q_n(x_i) = f(x_i)$, $i = 0, \ldots, n$, then $q_n = L_n f$.

(c) Prove that $L_n : C(\mathbb{R}) \to \mathbb{P}_n$ is a projection, i.e. $L_n\, p = p$ for all $p \in \mathbb{P}_n$.

We define $f[x_0, \ldots, x_n]$ to be the coefficient of x^n in $L_n f$, and let $w_n = \prod_{i=0}^{n}(x - x_i)$.

(d) Derive an explicit formula for $f[x_0, \ldots, x_n]$ involving only $\{x_i\}_{i=0}^n$ and $\{f(x_i)\}_{i=0}^n$.

(e) Prove that, for $x \neq x_i$,

$$\phi_i(x) = c_i\, \frac{w_n(x)}{x - x_i}\,,$$

and state the value of $c_i \in \mathbb{R}$.

(f) Prove that

$$(L_n\, g)(x) - x\,(L_n f)(x) = -f[x_0, \ldots, x_n]\, w_n(x) \qquad \forall\, x \in \mathbb{R}\,,$$

where $g \in C(\mathbb{R})$ with $g(x) = x\, f(x)$ for all $x \in \mathbb{R}$.

4. For all $f,\, g \in C[a,b]$ let

$$\langle f,\, g\rangle = \int_a^b w(x)\, f(x)\, g(x)\, \mathrm{d}x\,,$$

where w is a positive weight function. Let $\{\varphi_k\}_{k=0}^{\infty} \subset C[a,b]$ be a sequence of orthogonal polynomials, i.e. $\varphi_k \in \mathbb{P}_k$ and $\langle \varphi_k, \varphi_l\rangle = 0$ if $k \neq l$, for $k, l \geq 0$.

(a) Prove that φ_{n+1} is orthogonal to $\mathbb{P}_n$, i.e.

$$\langle p, \varphi_{n+1}\rangle = 0 \qquad \forall\, p \in \mathbb{P}_n\,.$$

(b) Prove that φ_{n+1} is a polynomial of degree $n+1$, i.e. $\varphi_{n+1} \in \mathbb{P}_{n+1} \setminus \mathbb{P}_n$.

(c) Prove that φ_{n+1} has $n+1$ distinct zeros in (a,b).

(d) State the weights $\{\omega_i^\star\}_{i=0}^n$ and sampling points $\{x_i^\star\}_{i=0}^n$ for the Gaussian quadrature formula

$$I_n^\star(f) = \sum_{i=0}^{n} \omega_i^\star\, f(x_i^\star) \qquad \text{approximating} \qquad I(f) = \int_a^b w(x)\, f(x)\, \mathrm{d}x\,.$$

Hence prove that

$$I_n^\star(f) = I(f) \qquad \forall\, f \in \mathbb{P}_{2n+1}\,.$$

(e) For the case $[a,b] \equiv [-1,1]$ and $w(x) = 1 + 4\,|x|$ construct $I_0^\star(f)$.

1. (a) $A \in \mathbb{R}^{n\times n}$ is nonnegative definite iff $\underline{x}^T A\, \underline{x} \geq 0$ for all $\underline{x} \in \mathbb{R}^n$. SEEN 1 mark

(b) $\underline{x}^T A\, \underline{x} = (\underline{x}^T A\, \underline{x})^T = \underline{x}^T A^T \underline{x} = -\underline{x}^T A\, \underline{x}$, and so $\underline{x}^T A\, \underline{x} = 0$ for all $\underline{x} \in \mathbb{R}^n$. Hence the matrix A is nonnegative definite. SEEN SIMILAR 2 marks

(c) If $\underline{v}^T A^{-1} \underline{u} = -1$, then $\underline{z} = A^{-1} \underline{u} \neq \underline{0}$ is such that SEEN

$$B\, \underline{z} = (A + \underline{u}\, \underline{v}^T)\, \underline{z} = (A + \underline{u}\, \underline{v}^T)\, A^{-1} \underline{u} = (1 + \underline{v}^T A^{-1} \underline{u})\, \underline{u} = \underline{0}\,.$$

Therefore B is singular if $\underline{v}^T A^{-1} \underline{u} = -1$ as it has a zero eigenvalue. (Or as $B\, \underline{z} = \underline{0}$ does not imply $\underline{z} = \underline{0}$.) 4 marks

(d) The Hessian $D^2 f(\underline{x}) \in \mathbb{R}^{n\times n}$ of $f : \mathbb{R}^n \to \mathbb{R}$ is defined by its entries $[D^2 f(\underline{x})]_{ij} = \frac{\partial^2}{\partial x_i\, \partial x_j} f(\underline{x})$, $i, j = 1, \ldots, n$. SEEN 1 mark

(e) We know that $f(\underline{x}) = \frac{1}{2}\, \underline{x}^T A\, \underline{x} + \underline{b}^T \underline{x} + c$ is such that $D^2 f(\underline{x}) = A$ for all $\underline{x} \in \mathbb{R}^3$. Hence $c = f(\underline{0}) = 0$, i.e. $f(\underline{x}) = \frac{1}{2}\, \underline{x}^T A\, \underline{x} + \underline{b}^T \underline{x}$. Moreover, UNSEEN

$$\begin{aligned} f(\underline{e}_1) &= -\tfrac{1}{2} + b_1 = 1\,, \\ f(\underline{e}_2) &= 1 + b_2 = 2\,, \\ f(\underline{e}_3) &= 0 + b_3 = 3\,, \end{aligned}$$

which implies that $\underline{b} = (\frac{3}{2}, 1, 3)^T$. 4 marks

(f) Newton form of interpolating polynomial SEEN SIMILAR

$$\begin{aligned} p_3(x) = f[x_0] + f[x_0, x_1]\,(x - x_0) + f[x_0, x_1, x_2]\,(x - x_0)\,(x - x_1) \\ + f[x_0, x_1, x_2, x_3]\,(x - x_0)\,(x - x_1)\,(x - x_2), \end{aligned}$$

where

$$f[x_0, x_1, \cdots, x_j] = \frac{f[x_1, \cdots, x_j] - f[x_0, \cdots, x_{j-1}]}{x_j - x_0}\,, \qquad j = 1, \ldots, 3,$$

and $f[x_i] = f_i$, $i = 0, \ldots, 3$.

$$\begin{array}{lllll} x_0 = -1 & f[-1] = -5 & & & \\ x_1 = 0 & f[0] = -1 & f[-1, 0] = 4 & & \\ x_2 = 2 & f[2] = 1 & f[0, 2] = 1 & f[-1, 0, 2] = -1 & \\ x_3 = 3 & f[3] = 11 & f[2, 3] = 10 & f[0, 2, 3] = 3 & f[-1, 0, 2, 3] = 1 \end{array}$$

$$\Rightarrow \qquad p_3(x) = -5 + 4\,(x + 1) - (x + 1)\, x + (x + 1)\, x\, (x - 2)\,.$$

8 marks

TOTAL: 20 marks

2\. (a) UNSEEN

$$\underline{a}_k = A\,\underline{e}_k = (Q\,R)\,\underline{e}_k = Q\,R\,\underline{e}_k = \sum_{i=1}^{k} R_{ik}\,\underline{q}_i \in \operatorname{span}\{\underline{q}_i\}_{i=1}^{k}\,,$$ 3 marks

since $R\,\underline{e}_k = (R_{1k}, \ldots, R_{kk}, 0, \ldots, 0)^T$.

(b) Assume that $R_{kk} = 0$. Then it follows from (a) that $\operatorname{span}\{\underline{a}_i\}_{i=1}^{k} \subset \operatorname{span}\{\underline{q}_i\}_{i=1}^{k-1}$. However, the linear independence of $\{\underline{a}_i\}_{i=1}^{n}$ implies that $\dim \operatorname{span}\{\underline{a}_i\}_{i=1}^{k} = k$, while $\dim\{\underline{q}_i\}_{i=1}^{k-1} \leq k-1$. A contradiction. Hence $R_{kk} \neq 0$ for $k = 1, \ldots, n$. UNSEEN 3 marks

(c) Q is orthogonal $\iff$ $\{\underline{q}_i\}_{i=1}^{m}$ is orthonormal, i.e. $\underline{q}_i^T\,\underline{q}_j = \delta_{ij}$, $i, j = 1, \ldots, m$. SEEN 1 mark

(d) We know that $A = Q\,R$ with Q orthogonal and R upper triangular with $R_{ii} \neq 0$, $i = 1, \ldots, n$. Assume that $R_{ii} < 0$ for $i \in I \subset \{1, \ldots, n\}$. Then let $E_i \in \mathbb{R}^{m\times m}$ be a diagonal matrix with diagonal entries SEEN SIMILAR

$$[E_i]_{jj} = \begin{cases} 1 & j \neq i\,, \\ -1 & j = i\,. \end{cases}$$

Clearly, $E_i^2 = I$, and hence E_i is orthogonal. Hence $E := \prod_{i\in I} E_i \in \mathbb{R}^{m\times m}$ is orthogonal, and so is $\widetilde{Q} = Q\,E$. Finally, $\widetilde{R} = E\,R$ is clearly upper triangular with positive entries on the diagonal. On noting that

$$\widetilde{Q}\,\widetilde{R} = Q\,E\,E\,R = Q\,R = A\,,$$

3 marks

we have found the desired QR factorization of A.

SEEN SIMILAR

(e) We have $A = \begin{pmatrix} 1 & 2 & -1 & -4 \\ 0 & 15 & 18 & 21 \\ -2 & -4 & -4 & -13 \\ -2 & -4 & -10 & -7 \end{pmatrix}$. Hence

$$G_{13} = \begin{pmatrix} \frac{1}{\sqrt{5}} & 0 & -\frac{2}{\sqrt{5}} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Rightarrow A^{(1)} = G_{13}\,A = \begin{pmatrix} \sqrt{5} & 2\sqrt{5} & \frac{7}{\sqrt{5}} & \frac{22}{\sqrt{5}} \\ 0 & 15 & 18 & 21 \\ 0 & 0 & -\frac{6}{\sqrt{5}} & -\frac{21}{\sqrt{5}} \\ -2 & -4 & -10 & -7 \end{pmatrix}.$$

$$G_{14} = \begin{pmatrix} \frac{\sqrt{5}}{3} & 0 & 0 & -\frac{2}{3} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{2}{3} & 0 & 0 & \frac{\sqrt{5}}{3} \end{pmatrix} \Rightarrow A^{(2)} = G_{14}\, A^{(1)} = \begin{pmatrix} 3 & 6 & 9 & 12 \\ 0 & 15 & 18 & 21 \\ 0 & 0 & -\frac{6}{\sqrt{5}} & -\frac{21}{\sqrt{5}} \\ 0 & 0 & -\frac{12}{\sqrt{5}} & \frac{3}{\sqrt{5}} \end{pmatrix},$$

$$G_{34} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ 0 & 0 & \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{pmatrix} \Rightarrow R = A^{(3)} = G_{34}\, A^{(2)} = \begin{pmatrix} 3 & 6 & 9 & 12 \\ 0 & 15 & 18 & 21 \\ 0 & 0 & 6 & 3 \\ 0 & 0 & 0 & -9 \end{pmatrix}.$$

Hence $\widetilde{Q} = G_{13}^T G_{14}^T G_{34}^T E$ and $\widetilde{R} = \begin{pmatrix} 3 & 6 & 9 & 12 \\ 0 & 15 & 18 & 21 \\ 0 & 0 & 6 & 3 \\ 0 & 0 & 0 & 9 \end{pmatrix}$, where $E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$.

10 marks

TOTAL: 20 marks

3. (a) ϕ_i is the product of n linear functions, and so $\phi_i \in \mathbb{P}_n$. $L_n f \in \text{span}\{\phi_i\}_{i=0}^n$, and so $L_n f \in \mathbb{P}_n$. In addition $\phi_i(x_j) = \delta_{ij}$ for $i, j = 0, \ldots, n$, which implies that $(L_n f)(x_j) = \sum_{i=0}^n f(x_i)\,\phi_i(x_j) = \sum_{i=0}^n f(x_i)\,\delta_{ij} = f(x_j)$. SEEN 3 marks

(b) Clearly, $L_n f - q_n \in \mathbb{P}_n$ with (at least) $n+1$ distinct zeros. Hence the FTA implies that $L_n f - q_n = 0 \in \mathbb{P}_n$, i.e. $q_n = L_n f$. SEEN SIMILAR 2 marks

(c) This immediately follows from (b), since $p \in \mathbb{P}_n$ trivially interpolates the data $\{(x_i, p(x_i))\}_{i=0}^n$. 1 mark

(d) Clearly $\phi_i(x) = c_i\,x^n + q_{n-1}$, where $q_{n-1} \in \mathbb{P}_{n-1}$ and $c_i = [\prod_{j=0,j\neq i}^n (x_i - x_j)]^{-1}$. Hence

$$f[x_0, \ldots, x_n] = \sum_{i=0}^n c_i\,f(x_i) = \sum_{i=0}^n \frac{f(x_i)}{\prod_{j=0,j\neq i}^n (x_i - x_j)}\,.$$

3 marks

(e)

$$\begin{aligned}\phi_i(x) &= \prod_{j=0,j\neq i}^n \frac{x - x_j}{x_i - x_j} = \frac{x - x_i}{x - x_i}\prod_{j=0,j\neq i}^n \frac{x - x_j}{x_i - x_j} = \frac{1}{x - x_i}\,\frac{\prod_{j=0}^n (x - x_j)}{\prod_{j=0,j\neq i}^n (x_i - x_j)} \\ &= c_i\,\frac{1}{x - x_i}\,w_n(x)\,,\end{aligned}$$

where $c_i = [\prod_{j=0,j\neq i}^n (x_i - x_j)]^{-1}$ as above. 2 marks

(f) It follows from (d) and (e) that, for $x \notin \{x_i\}_{i=0}^n$, UNSEEN

$$\begin{aligned}f[x_0, \ldots, x_n] &= \sum_{i=0}^n c_i\,f(x_i) = \sum_{i=0}^n f(x_i)\,\frac{\phi_i(x)\,(x - x_i)}{w_n(x)} \\ &= [w_n(x)]^{-1}\sum_{i=0}^n f(x_i)\,\phi_i(x)\,(x - x_i)\,,\end{aligned}$$

and so

$$\begin{aligned}f[x_0, \ldots, x_n]\,w_n(x) &= \sum_{i=0}^n f(x_i)\,\phi_i(x)\,(x - x_i) = x\sum_{i=0}^n f(x_i)\,\phi_i(x) - \sum_{i=0}^n x_i\,f(x_i)\,\phi_i(x) \\ &= x\,(L_n f)(x) - (L_n(x\,f))(x)\,.\end{aligned}$$

For $x \in \{x_i\}_{i=0}^n$ the claim holds trivially, since $w(x_i) = 0$ and

$$x_i\,(L_n f)(x_i) = x_i\,f(x_i) = (L_n(x\,f))(x_i)\,.$$

Overall we obtain $L_n\,g - x\,L_n\,f = -f[x_0, \ldots, x_n]\,w_n$. 9 marks

TOTAL: 20 marks

4. (a) As φ_{n+1} is orthogonal to φ_k, $k = 0, \ldots, n$, it is sufficient to prove that $\{\varphi_k\}_{k=0}^n$ is a basis of $\mathbb{P}_n$. To this end, we first of all note that $\operatorname{span}\{\varphi_k\}_{k=0}^n \subset \mathbb{P}_n$. Moreover, the orthogonality of $\{\varphi_k\}_{k=0}^n$ implies that they are linearly independent. Hence $\dim \operatorname{span}\{\varphi_k\}_{k=0}^n = n+1 = \dim \mathbb{P}_n$ and so $\operatorname{span}\{\varphi_k\}_{k=0}^n = \mathbb{P}_n$. UNSEEN 4 marks

(b) This immediately follows from (a), as $\varphi_{n+1} \in \mathbb{P}_n$ would contradict the linear independence of $\{\varphi_k\}_{k=0}^{n+1}$. 1 mark

(c) Let $a < x_0^\star < x_1^\star < \cdots < x_{r-2}^\star < x_{r-1}^\star < b$ be the $r \leq (n+1)$ ordered points, where φ_{n+1} changes sign in (a, b). Let $q_r(x) = \prod_{i=0}^{r-1}(x - x_i^\star) \in \mathbb{P}_r$. SEEN

Then either $q_r(x)\,\varphi_{n+1}(x) > 0$ (or < 0) for all $x \in (a,b)$ with $x \neq x_i^\star$, $i = 0 \to (r-1)$, and so $\langle q_r, \varphi_{n+1}\rangle > 0$ or < 0, respectively.

If $r \leq n$, this contradicts φ_{n+1} being orthogonal to $\mathbb{P}_n$, and hence $r = n+1$. 4 marks

(d) Let $\{x_i^\star\}_{i=0}^n$ be the $n+1$ distinct zeros of φ_{n+1} in (a,b) and let

$$\omega_i^\star = \int_a^b w(x) \prod_{j=0,\, j\neq i}^n \frac{x - x_j^\star}{x_i^\star - x_j^\star}\, \mathrm{d}x\,, \qquad i = 0 \to n\,.$$

3 marks

For $f \in \mathbb{P}_{2n+1}$, let $p_n \in \mathbb{P}_n$ interpolate f at the distinct $\{x_i^\star\}_{i=0}^n$. Then the Lagrange form of p_n yields that

$$p_n(x) = \sum_{i=0}^n f(x_i^\star)\,\phi_i(x), \quad \text{where} \quad \phi_i(x) = \prod_{j=0,\, j\neq i}^n \frac{x - x_j^\star}{x_i^\star - x_j^\star},$$

and so

$$I_n^\star(f) = \sum_{i=0}^n \omega_i^\star\, f(x_i^\star) = \sum_{i=0}^n f(x_i^\star) \int_a^b w(x)\,\phi_i(x)\,\mathrm{d}x = I(p_n)\,.$$

[Note: In reference to Q3 students may simply write $p_n = L_n\, f$ or $p_n = L_n^\star\, f$.] 2 marks

It holds that $f - p_n \in \mathbb{P}_{2n+1}$ vanishes at $\{x_i^\star\}_{i=0}^n$, the distinct zeros of φ_{n+1}, and so $f - p_n = q_n\,\varphi_{n+1}$ for some $q_n \in \mathbb{P}_n$. Therefore it follows that

$$I(f) - I_n^\star(f) = I(f) - I(p_n) = \int_a^b w(x)\,q_n(x)\,\phi_{n+1}(x)\,\mathrm{d}x = \langle \phi_{n+1}, q_n\rangle = 0.$$

2 marks

(e) We need to construct φ_1. To this end, let $\varphi_0 = 1$ and $\varphi_1(x) = x - a$ such that $\langle \varphi_1, \varphi_0\rangle = 0$. Hence $a = \frac{\langle x,1\rangle}{\langle 1,1\rangle} = 0$, where we have noted that the integrand $x\,w(x)$ is odd. We obtain that $\varphi_1(x) = x$, and so $x_0^\star = 0$. The weight $\omega_0^\star$ needs to satisfy SEEN SIMILAR

$$\omega_0^\star = I_0^\star(1) = I_0(1) = \int_{-1}^1 (1 + 4\,|x|)\,\mathrm{d}x = 2\int_0^1 (1 + 4\,x)\,\mathrm{d}x = 6\,.$$

Hence $I_0^\star(f) = 6\,f(0)$. 4 marks

TOTAL: 20 marks

Imperial College
London

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2015

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Non-Linear Waves

Date: Wednesday, 20 May 2015. Time: 2.00pm – 4.00pm. Time allowed: 2 hours.

This paper has FOUR questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the main book is full.

Statistical tables will not be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers to each question as per the table below.

Raw mark	up to 12	13	14	15	16	17	18	19	20
Extra credit	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4

- Each question carries equal weight.
- Calculators may not be used.

1. (i) Given the conservation law

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \qquad x \in \mathbb{R}, \qquad t > 0, \tag{1}$$

write down (no need to derive them) the Rankine-Hugoniot conditions that govern the motion of any shocks that may be present.

(ii) Consider solutions of (1) subject to the initial condition

$$u(x,0) = \begin{cases} 2, & x \le -1, \\ 1 - x, & -1 < x \le 0, \\ 1, & x > 0. \end{cases}$$

(a) Sketch the initial condition and find and sketch the characteristics emanating from each of the intervals $x \le -1$, $-1 < x \le 0$ and $x > 0$.

(b) Find the solution in the time interval $0 < t \le 1$, and sketch it at $t = 1/2$ and $t = 1$. What happens at $t = 1$?

(c) Find the solution beyond $t = 1$ and sketch the characteristics pointing out any special features. Sketch the solution at $t = 2$.

2. The equation for traffic flow on an infinite single-lane road is modeled by the partial differential equation

$$\rho_t + q_x = 0,$$

where $\rho(x,t)$ is the density of cars at a given point x and time t, and $q(\rho)$ is the traffic flow flux assumed to be a function of ρ alone.

(i) In this model, what are the units of ρ and q? Show also that the units of q/ρ are the same as those for velocity, and hence provide a relationship between car density, traffic flux and car velocity.

(ii) A traffic engineer has suggested that the following model can be used

$$q(\rho) = u_{max}\, \rho \left(1 - \frac{\rho}{\rho_{max}}\right), \tag{2}$$

where u_{max} and ρ_{max} are positive constants. Clearly $q(0) = q(\rho_{max}) = 0$. Why is this reasonable and what do the constants u_{max} and ρ_{max} represent physically.

We want to achieve maximum traffic flow according to the model (2). What is the resulting car density and what should the speed limit be set to in this case?

(iii) Suppose traffic is flowing uniformly at density $\rho = \rho_{max}/2$, when a green light at $x = 0$ turns red at $t = 0$.

(a) Write down the partial differential equation that needs to be solved to determine the flow in $x < 0$, $t > 0$, ensuring that you specify clearly initial and boundary conditions.

(b) Use the method of characteristics to find the solution at any $t > 0$. Draw the characteristics in the $x - t$ plane and identify and find the position of any shocks that may form.

(c) Now suppose that the red light stays on for T seconds before turning green (and remaining green from then on). According to the model and your solution, will there be any drivers who will not have known about the light even turning red? Explain, and in particular calculate which segment of the road at $t = 0$ will not feel the effect of the red light.

3. Consider the viscous Burgers equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \epsilon\frac{\partial^2 u}{\partial x^2}, \qquad x \in \mathbb{R}, \qquad t > 0, \tag{3}$$

where $\epsilon > 0$ is a constant, subject to the initial condition

$$u(x, 0) = u_0(x).$$

(a) Suppose that the initial condition satisfies

$$\lim_{x\to-\infty} u_0(x) = U_1, \qquad \lim_{x\to\infty} u_0(x) = U_2,$$

where $U_1 > U_2$ are non-zero constants. Show that if at large times a travelling wave of permanent form $u(x,t) = U(\zeta = x - ct)$ emerges, then $c = \frac{1}{2}(U_1 + U_2)$.

(b) Now take

$$u_0(x) = \begin{cases} 1, & x \leq 0, \\ -1, & x > 0. \end{cases} \tag{4}$$

What is the speed of a large time travelling wave according to the formula above. What does this mean physically?

Show that the large time solution in this case is given by (you need to construct the solution, not simply verify it)

$$u = -\tanh\left(\frac{x}{2\epsilon}\right). \tag{5}$$

Sketch three solutions for decreasing values of ϵ. What happens to the solution (5) as $\epsilon \to 0$?

Now take $\epsilon = 0$ in (3) and solve it subject to the initial condition (4). Compare the two approaches.

(c) Next take the initial condition to be

$$u_0(x) = \begin{cases} 0, & x \leq 0, \\ 1, & x > 0. \end{cases} \tag{6}$$

If a travelling wave solution to (3) exists, what is its speed? (Do not attempt to find the solution.)

Now take $\epsilon = 0$ in (3) and find the solution starting from the initial condition (6) - use the method of characteristics to do this and provide a sketch of the characteristics in the $x - t$ plane. Does a travelling wave of permanent form emerge?

Use your results to guess what the solution for $\epsilon \neq 0$ would look like? In particular take $t = 1$ and sketch the $\epsilon = 0$ solution you just found, along with what you expect the solution to be for a small value of ϵ.

4. (i) The linearized equations of two-dimensional water waves in deep water are

$$\nabla^2\phi = 0 \qquad -\infty < x < \infty, \quad y < 0,$$
$$\mathbf{u} = \nabla\phi \to 0 \qquad \text{as} \qquad y \to -\infty,$$
$$\eta_t = \phi_y \qquad \text{on} \qquad y = 0,$$
$$\phi_t + g\eta = 0 \qquad \text{on} \qquad y = 0,$$

where $\phi(x, y, t)$ is the fluid potential, $\eta(x, t)$ denotes the wave surface and g is the gravitational acceleration. We want to construct standing wave solutions with ϕ having the form $\phi(x, y, t) = \cos(kx)\sin(\omega t)\,\hat{\phi}(y)$ where $\hat{\phi}(y)$ is a function to be found that satisfies $\hat{\phi}(0) = \phi_0$ with ϕ_0 a constant.

(a) What do the parameters k and ω represent physically and how are they related to the wave's wavelength and its period of oscillation? Given the form of ϕ explain why the wave amplitude must take the form $\eta(x, t) = \eta_0\cos(kx)\cos(\omega t)$, where η_0 is a constant.

(b) Show that for a given value of k, standing wave solutions are only possible if the dispersion relation $\omega^2 = gk$ holds. What is the function $\hat{\phi}(y)$?

(c) Show that the magnitude of the fluid velocity $|\mathbf{u}| = |\nabla\phi|$ anywhere in the domain satisfies

$$|\mathbf{u}| = |\phi_0|\,|\sin(\omega t)|\,k\,e^{ky},$$

and hence deduce that this velocity takes its maximum possible value at the free surface.

(d) If the wavelength of the standing wave is λ, show that at a depth of $\lambda/2$ the velocity in the fluid due to the standing wave motion is less than 5% of its maximum value at the free surface.
[You may need the result $e^{-\pi} \approx 0.0432$.]

EXAMINATION SOLUTIONS 20~~13~~-~~14~~ 14-15

Course: M2AM

Question 1

Marks & seen/unseen

Parts

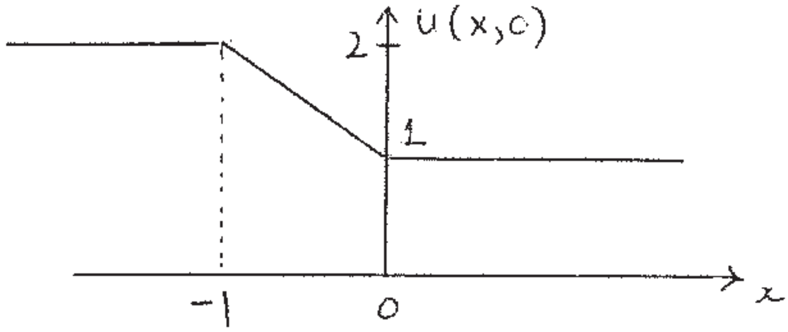
(i) Write as $u_t + \left(\frac{1}{2}u^2\right)_x = 0$

Rankine-Hugoniot for shock motion

$$\frac{ds}{dt} = \frac{\left[\frac{1}{2}u^2\right]}{[u]}$$

where $[\phi] = \phi_{behind} - \phi_{ahead}$ is the usual jump condition notation. — 5 marks (seen)

(ii)(a) Initial condition sketch — unseen



2

Characteristic forms:

$\underline{\xi < -1}$ $\quad u = 2$ on $\frac{dx}{dt} = 2$, i.e.

$u = 2$ on $x = 2t + \xi$

$\underline{\xi > 0}$ $\quad u = 1$ on $\frac{dx}{dt} = 1$, i.e.

$u = 1$ on $x = t + \xi$

Setter's initials [redacted] Checker's initials [redacted]

(ii)(a) CONT

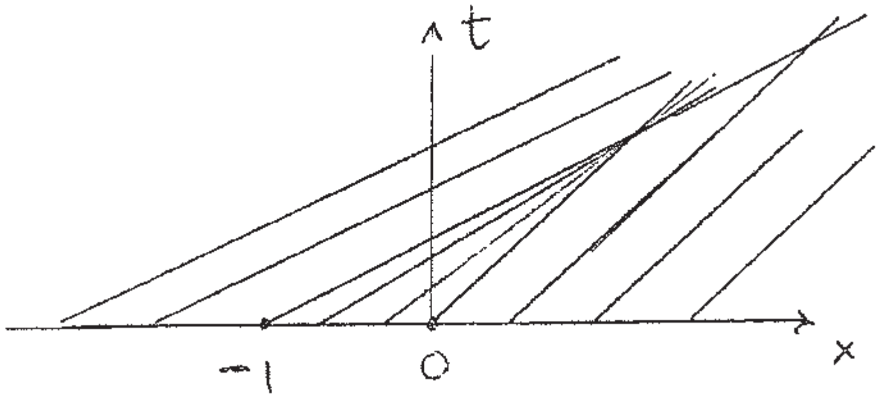
$\underline{-1<\xi<0}$ The initial condition is

$$u(x,0) = 1-x \quad \text{for} \quad -1<x<0$$

ie $\quad u = \text{const} = 1-\xi \quad \text{on} \quad \frac{dx}{dt} = 1-\xi$

$$u = 1-\xi \quad \text{on} \quad x = (1-\xi)t + \xi$$

Characteristics



3

All characteristics with $-1<\xi<0$ cross at $x=1, t=1$ as can be seen from the equation $x = (1-\xi)t+\xi$

(ii)(b) Solution for $0<t\leq 1$. Already found what happens for $\xi<-1$, $\xi>0$ — unseen

$$u = 2 \quad \text{for} \quad x-2t<-1$$

$$u = 1 \quad \text{for} \quad x-t>0$$

EXAMINATION SOLUTIONS 201~~3~~-1~~4~~ 4 5

Course M2AM

Question (1)

Marks & seen/unseen

Parts (ii)(b) CONT

$-1 < \xi < 0$

$u = 1 - \xi$ on $x = (1-\xi)t + \xi$,

ie. $\xi = \dfrac{x-t}{1-t}$

$\Rightarrow$

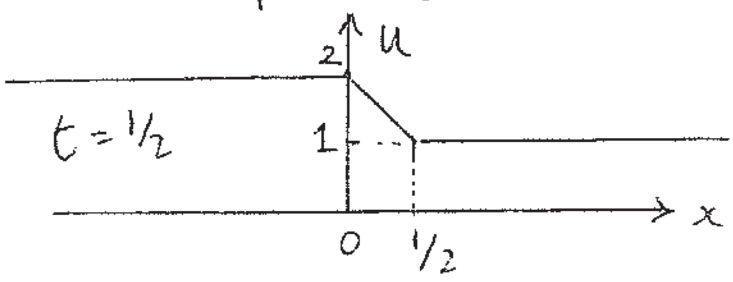
$$u = 1 - \frac{x-t}{1-t} = \frac{1-x}{1-t} \text{ for } -1 < \frac{x-t}{1-t} < 0$$

rearrange the inequality to find

$$2t - 1 < x < t$$

Solution is

$$u = \begin{cases} 2 & \text{for } x < 2t-1 \\ \dfrac{1-x}{1-t} & \text{for } 2t-1 < x < t \\ 1 & \text{for } x > t \end{cases}$$ 3



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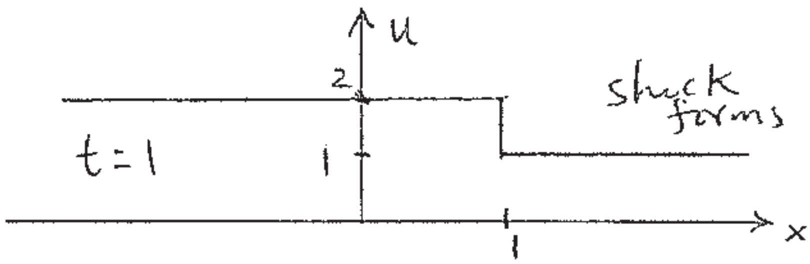

1

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EXAMINATION SOLUTIONS ~~2013-14~~ 2014-15

Course: M2AM

Question: 1

Marks & seen/unseen

Parts: (iii)(c)

A shock forms at $t=1$, $x=1$

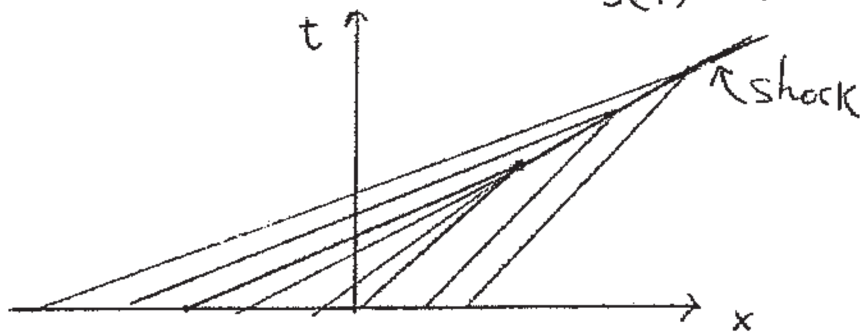
$u=2$ behind the shock

$u=1$ ahead of the shock

From R-H conditions

$$\frac{ds}{dt} = \frac{[\frac{1}{2}u^2]}{[u]} = \frac{1}{2}\frac{(4-1)}{(2-1)} = \frac{3}{2}$$

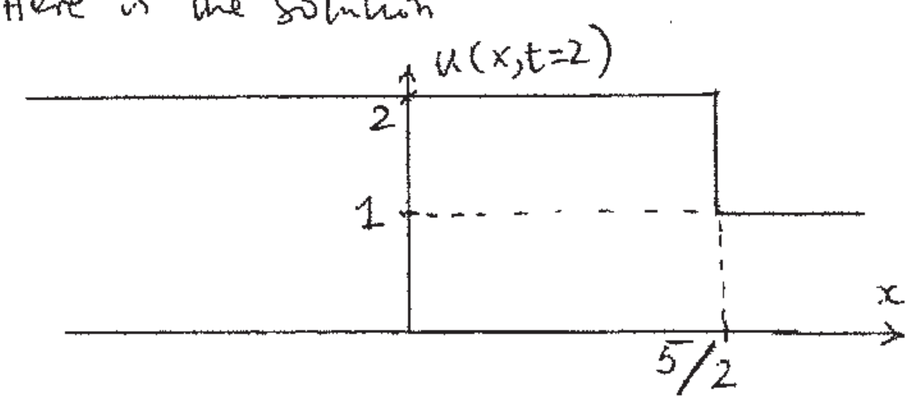
Solve $S = \frac{3}{2}t - \frac{1}{2}$ since $S(1)=1$ 2



2

At $t=2$ the shock is at $x=5/2$

Here is the solution



1

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	EXAMINATION SOLUTIONS ~~2013-14~~ 2014-15	Course M2AM
Question (2)		Marks & seen/unseen
Parts		Seen
(i)	Units of ρ - cars per unit length	1
	Units of q - cars per unit time	1
	But velocity is $\frac{\text{length}}{\text{time}}$ $\Rightarrow$	
	$\frac{q}{\rho}$ has units $\frac{\text{cars/time}}{\text{cars/length}} = \frac{\text{length}}{\text{time}}$ $= \text{velocity}$	1
	Define $q = \rho u$, u velocity of the traffic	1
(ii)	$q(0) = q(\rho_{max}) = 0$.	
	Reasonable because:	Seen
	(i) $\rho = 0$ should give zero flux - there are no cars present.	
	(ii) $\rho = \rho_{max}$, maximum packing ie traffic jam $\rightarrow$ no flux	
	ρ_{max} - maximum density	2
	$u = \frac{q}{\rho} = u_{max}\left(1 - \frac{\rho}{\rho_{max}}\right) \Rightarrow$ u_{max} is the max possible velocity.	
Setter's initials [redacted]	Checker's initials [redacted]	Page number 5

Parts

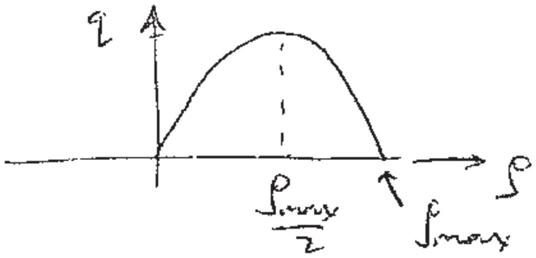
(iii) CONT

For maximum flux need to find ρ such that q is maximum — unseen

$$\frac{dq}{d\rho} = u_{max}\left(1 - \frac{2\rho}{\rho_{max}}\right) = 0$$

when $\rho = \frac{\rho_{max}}{2}$ — 1

This is a local maximum



$$q_{max} = q\left(\frac{\rho_{max}}{2}\right) = \frac{1}{4} u_{max} \rho_{max}$$

We need the u corresponding to this

Since $q = \rho u$ we have

$$\frac{1}{4} u_{max} \rho_{max} = \left(\frac{1}{2}\rho_{max}\right) u$$

ie $u = \frac{1}{2} u_{max}$ should be the speed limit. — 1

	EXAMINATION SOLUTIONS 20~~13~~-1~~4~~ 4 5	Course M2AM
Question (2)		Marks & seen/unseen

Parts

(iii)(a)

$$\rho_t + q'(\rho)\rho_x = 0 \quad , \quad \text{ie}$$

Seen

$$\rho_t + u_{max}\left(1 - \frac{2\rho}{\rho_{max}}\right)\rho_x = 0$$

Need to solve in the region $x < 0$

$$\rho(x,0) = \frac{\rho_{max}}{2} \quad \text{for } x \leq 0$$

$$\rho(0,t) = \rho_{max}$$

The condition at $x=0$ follows from the fact that traffic will reach maximum density once the light turns red. — 3

(iii)(b) Characteristic form

seen simular

$$\rho = \text{const. on } \frac{dx}{dt} = u_{max}\left(1 - \frac{2\rho}{\rho_{max}}\right)$$

For $\xi < 0$

$$\rho = \frac{\rho_{max}}{2} \quad \text{on} \quad \frac{dx}{dt} = u_{max}\left(1 - \frac{2\rho_{max}/2}{\rho_{max}}\right) = 0$$

Hence, characteristics are vertical

Setter's initials [redacted] Checker's initials [redacted] Page number 7

EXAMINATION SOLUTIONS 20~~13~~-1~~4~~ 4 5

Course: M2AM

Question: 2

Marks & seen/unseen

Parts: (iii)(b) CON'T

For the characteristics from the $x=0$ axis we have

$$\rho=\rho_{max} \quad \text{on} \quad \frac{dx}{dt} = -u_{max}$$

$$\text{ie} \quad x = -u_{max}t + u_{max}\tau$$

where τ is a label on the t-axis.

Clearly these cross with the $\xi<0$ characteristics $\Rightarrow$ a shock forms at $t=0$.

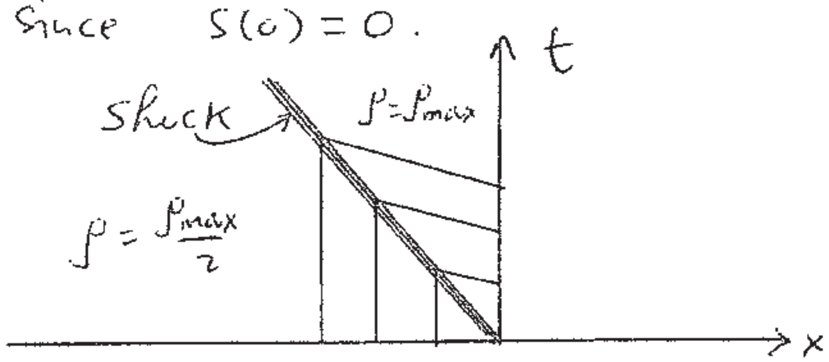
Shock position

$$\frac{ds}{dt} = \frac{[q]}{[\rho]} = \frac{\frac{1}{4}u_{max}\rho_{max} - 0}{\frac{\rho_{max}}{2} - \rho_{max}}$$

$$\frac{ds}{dt} = -\frac{1}{2}u_{max}$$

$$\text{ie} \quad s(t) = -\frac{1}{2}u_{max}t$$

since $s(0)=0$.



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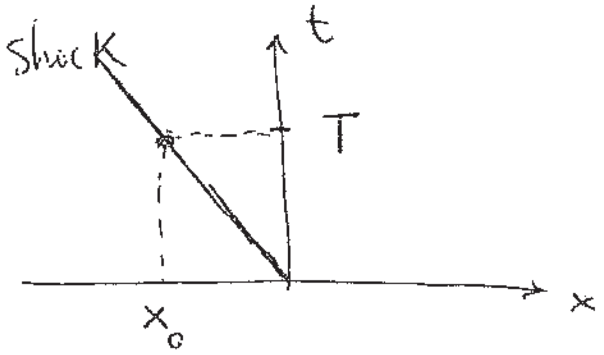
(iii)(c)



unseen

All traffic in $x < x_0$ will not come to a stop since the light will turn green again at $t = T$

From the equation of the shock

$$x = -\frac{1}{2} u_{max} t$$

we see that

$$x_0 = -\frac{1}{2} u_{max} T$$

2

Parts

(a) $u = U(x - ct)$, the equation becomes — seen

$$-cU' + UU' = \varepsilon U'' \quad \text{where } ' = \frac{d}{d\xi}$$

integrate

$$-c(U_1 - U_2) + \frac{1}{2}(U_1^2 - U_2^2) = \varepsilon [U']_{-\infty}^{\infty} = 0$$

ie $c = \frac{1}{2}(U_1 + U_2)$ — 3

(b) Here $U_1 = 1$, $U_2 = -1 \Rightarrow c = 0$

Physically we have a steady state since the speed is zero. (Also $\xi = x$) — 1

Equation becomes — UNSEEN

$$UU' = \varepsilon U''$$

Integrate once $\frac{1}{2}U^2 = \varepsilon U' + \text{const.}$

Send $|x| \to \infty$ to find const. $= \frac{1}{2}$

ie $\varepsilon U' = -\frac{1}{2}(1 - U^2)$

	EXAMINATION SOLUTIONS 20$\cancel{13}^{14}$-$\cancel{14}^{15}$	Course M2AM
Question 3		Marks & seen/unseen

Parts

(b) CONT

Separate variables and use partial fractions to find

$$\left(\frac{1}{1+U} + \frac{1}{1-U}\right) dU = -\frac{dx}{\varepsilon}$$

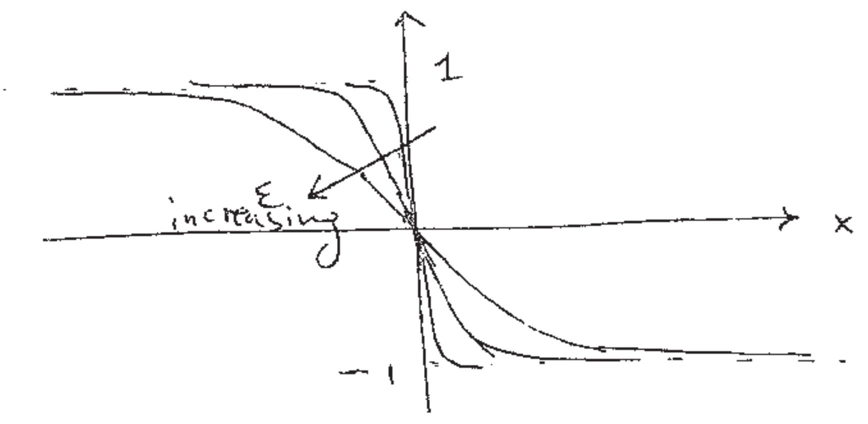
$$\ln\left(\frac{1+U}{1-U}\right) = -\frac{x}{\varepsilon} + K$$

But $U = 0$ at $x = 0 \Rightarrow K = 0$

$$\frac{1+U}{1-U} = e^{-x/\varepsilon} \Rightarrow U = -\frac{1-e^{-x/\varepsilon}}{1+e^{-x/\varepsilon}}$$

$$U = -\frac{e^{\frac{x}{2\varepsilon}} - e^{-\frac{x}{2\varepsilon}}}{e^{\frac{x}{2\varepsilon}} + e^{-\frac{x}{2\varepsilon}}} = -\tanh\left(\frac{x}{2\varepsilon}\right)$$ 4

As $\varepsilon \to 0$ $\quad U \to u_0(x)$



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(b) CONT

Need to solve

$$u_t + uu_x = 0$$

$$u(x,0) = \begin{cases} 1 & x \leq 0 \\ -1 & x > 0 \end{cases}$$

There is a shock at $t=0$

Shock position $\frac{ds}{dt} = \frac{[\frac{1}{2}u^2]}{[u]} = \frac{(1-1)\frac{1}{2}}{2} = 0$

$\Rightarrow$ solution is

$$u(x,t) = \begin{cases} 1 & x \leq 0 \\ -1 & x > 0 \end{cases}$$ 2

ie does not change.

The two approaches give the same solution in the limit $\varepsilon \to 0$. 1

EXAMINATION SOLUTIONS 201~~3~~4-1~~4~~5

Course: M2AM

Question 3

Marks & seen/unseen

Parts

(c) According to $c = \frac{1}{2}(U_1 + U_2)$, the speed is now $c = \frac{1}{2}$. 1

Need to solve

$$u_t + u u_x = 0$$

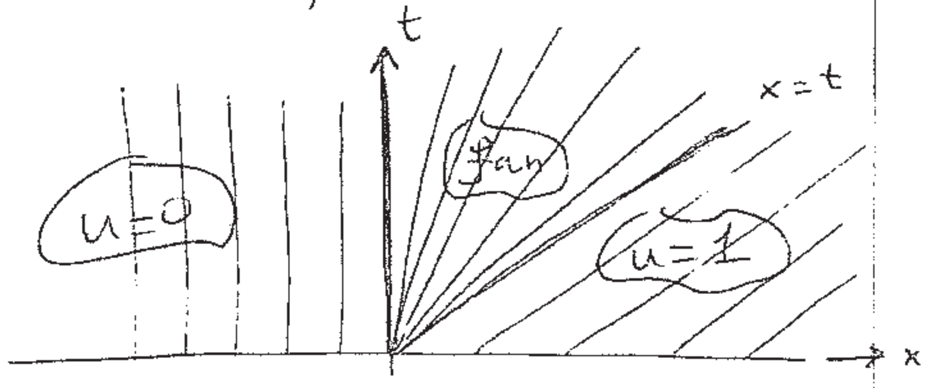
$$u(x,0) = \begin{cases} 0 & x \le 0 \\ 1 & x > 0 \end{cases}$$

Using characteristics,

$\xi < 0 \quad u = 0 \quad \text{on} \quad \frac{dx}{dt} = 0$

$\xi > 0 \quad u = 1 \quad \text{on} \quad \frac{dx}{dt} = 1$

$\Rightarrow$ A rarefaction forms



2

EXAMINATION SOLUTIONS ~~2013-14~~ 2014-15

Course M2AM

Question 3

Marks & seen/unseen

Parts

(c) CONT

In the fan region

$u = \text{const.}$ on $\frac{dx}{dt} = u$, ie on

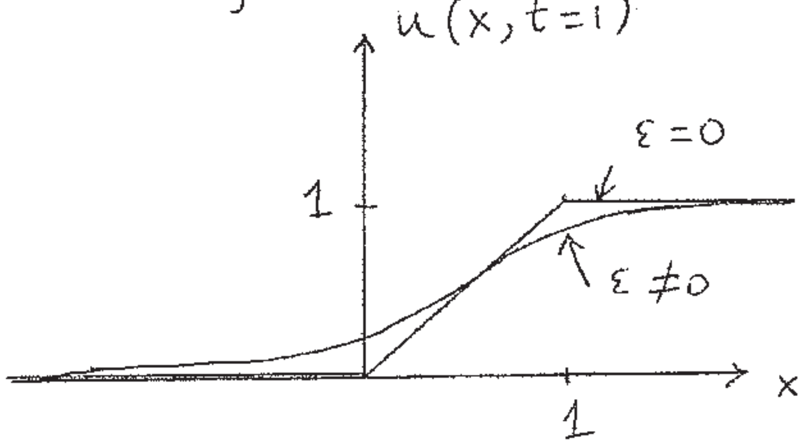
$x = ut \Rightarrow u = \frac{x}{t}$

Solution is

$$u = \begin{cases} 0 & x \leq 0 \\ \frac{x}{t} & 0 < x \leq t \\ 1 & x > t \end{cases}$$

2

This is not a traveling wave of permanent form.



2

Small ε solution will smooth out the corners as shown.

Setter's initials [redacted] Checker's initials [redacted]

Question 4

Marks & seen/unseen

Parts

(a)

k is the wavenumber of the wave (ie wavelength $\lambda = 2\pi/k$) — seen

ω is the frequency of oscillation ($\frac{2\pi}{\omega}$ is the period of time oscillations) — 2

$$\phi = \cos kx \sin \omega t \, \hat{\phi}(y)$$

From the conditions at $y=0$

$$\eta_t = \phi_y \Rightarrow \eta_t \propto \cos kx \sin \omega t$$

$$\Rightarrow \eta \propto \cos kx \cos \omega t$$

Check with last condition

$$\phi_t + g\eta = 0 \quad \text{on } y=0$$

ϕ_t: $\cos kx \cos \omega t$; $g\eta$: $\cos kx \cos \omega t$, ie consistent. — 3

(b)

$\nabla^2 \phi$ becomes — seen

$$\hat{\phi}'' - k^2 \hat{\phi} = 0$$

$$\hat{\phi} = A e^{ky} + B e^{-ky}$$

(b) CONT

For boundedness as $y \to -\infty$ we need to set $B = 0$

$\Rightarrow \quad \hat{\phi}(y) = A e^{ky}$. Let $\underline{A = \phi_0}$

Boundary conditions become.

$(\eta_t = \phi_y \quad y = 0)$

$-\eta_0 \omega \cos kx \sin \omega t = \cos kx \sin \omega t \times \phi_0 k$

ie $\quad -\eta_0 \omega = \phi_0 k \quad (*)$

$(\phi_t + g\eta = 0 \quad y = 0)$

$\omega \phi_0 = -g\eta_0 \Rightarrow \phi_0 = -\frac{g\eta_0}{\omega}$ 8

Substitute into $(*)$

$-\eta_0 \omega = -\frac{g\eta_0}{\omega} k$

$\Rightarrow \quad \omega^2 = gk$

$\hat{\phi}(y) = \phi_0 e^{ky}$

	EXAMINATION SOLUTIONS 201~~3~~4-1~~4~~5	Course M2AM
Question 4		Marks & seen/unseen

Parts

(c) Now — unseen

$$\phi(x,y,t) = \phi_0 \cos kx \sin\omega t \, e^{ky}$$

$$\phi_x = -k\phi_0 \sin kx \sin\omega t \, e^{ky}$$

$$\phi_y = k\phi_0 \cos kx \sin\omega t \, e^{ky}$$

$\Rightarrow$

$$\left(\phi_x^2 + \phi_y^2\right)^{1/2} = |\underline{u}| = k|\phi_0| e^{ky} |\sin\omega t| \times \left(\sin^2 kx + \cos^2 kx\right)^{1/2}$$

$$= |\phi_0| k e^{ky} |\sin\omega t|$$

Since $y \leq 0$, e^{ky} is a decreasing function $\Rightarrow$ $|\underline{u}|$ is max at $y=0$. — 5

(d) Need to take $y = -\frac{\lambda}{2}$. Now $\lambda = \frac{2\pi}{k} \Rightarrow \frac{\lambda}{2} = \frac{\pi}{k}$, and the velocity is smaller by a factor $e^{-k\frac{\lambda}{2}} = e^{-\pi} < 0.05$ — unseen, 2

ie less than 5% its value at $y=0$.

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Imperial College London

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2015

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Real Analysis

Date: Thursday, 14 May 2015. Time: 2.00pm – 4.00pm. Time allowed: 2 hours.

This paper has FOUR questions.

Candidates should use TWO main answer books (A & B) for their solutions as follows: book A - solutions to questions 1 & 2; book B - solutions to questions 3 & 4.

Supplementary books may only be used after the relevant main book(s) are full.

Statistical tables will not be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers to each question as per the table below.

Raw mark	up to 12	13	14	15	16	17	18	19	20
Extra credit	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4

- Each question carries equal weight.
- Calculators may not be used.

1. i. State the Mean Value Theorem (both the standard and Cauchy's version).

   ii. Suppose that $f : [0, 2] \to [0, \infty)$ is continuous on $[0, 2]$, differentiable on $(0, 2)$, that $f(0) = 0$, $f(2) = \pi$, and that $g(x) = \frac{f'(x)}{x}$ is a strictly increasing function on $(0, 2)$.

   a. Prove that for all $x \in (0, 2]$, there exists $\beta \in (0, 1)$ such that

$$\frac{f(x)}{x} = \frac{f'(x\beta)}{2\beta}.$$

   b. Prove that f is strictly increasing on $[0, 2]$.

   c. Prove that there exists a unique $\alpha \in (0, 2)$ such that $f(\alpha) = \pi/2$.

2. i. Define what it means to say that a bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$.

   ii. State and prove the ε-criterion for Riemann integrability.

   iii. Let $f : [0, 4] \to \mathbb{R}$ be given by $f(x) = 2$ if $x \in [0, 1)$, $f(x) = 1$ if $x \in (1, 4]$, and $f(1) = 0$. Prove using the ε-criterion of integrability that f is Riemann integrable and calculate its integral.

   iv. Is it possible to find an example of a Riemann integrable function $f : [0, 1] \to \mathbb{R}$ such that the function $A : [0, 1] \to \mathbb{R}$ given by $A(x) = \int_0^x f(s)\mathrm{d}s$ is differentiable at all $x \in (0, 1)$, but $A'(x_0) \neq f(x_0)$ for some $x_0 \in (0, 1)$? Justify your answer by giving a proof or a counterexample.

3. i. Define what it means for a collection Σ of subsets of $\mathbb{R}$ to be a σ-algebra. What is the Borel σ-algebra on $\mathbb{R}$?

   ii. Prove in full detail that the set of irrational numbers belongs to the Borel σ-algebra on $\mathbb{R}$.

   iii. Define what it means for a function $f : \mathbb{R}^2 \to \mathbb{R}$ to have directional derivative $\frac{\partial f}{\partial v}(a)$ at a point $a \in \mathbb{R}^2$ along a direction $v \in \mathbb{R}^2 \setminus \{0\}$. Prove that if f is differentiable at a, then $\frac{\partial f}{\partial v}(a) = \nabla f(a) \cdot v$.

   iv. Prove that the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$f(x, y) = \begin{cases} \frac{xy^2}{\sqrt{x^2+y^2}} & \text{for } (x, y) \neq (0, 0) \\ 0 & \text{for } (x, y) = (0, 0) \end{cases}$$

   is differentiable everywhere and compute its gradient $\nabla f(x, y)$ at every $(x, y) \in \mathbb{R}^2$.

4. Which of the following statements are true and which are false? No reasoning needs to be given: just answer T (true) or F (false) for each of them. For each correct answer you will be awarded 1 mark. For each incorrect answer you will lose 1 mark. If the number of incorrect answers is higher than the number of correct answers, then the total mark awarded for this question will be 0.

i. If $f : \mathbb{R} \to \mathbb{R}$ is differentiable at 15 and has a local maximum at 15, then $f'(15) = 0$.

ii. Any smooth function on $\mathbb{R}$ is analytic.

iii. There exist functions $f, g : [a, b] \to \mathbb{R}$ for which $U(f + g) < U(f) + U(g)$ (where $U(f)$ denotes the upper integral of f over $[a, b]$).

iv. If $f : \mathbb{R}^2 \to \mathbb{R}$ has partial derivatives at all points, then it is differentiable on $\mathbb{R}^2$.

v. The empty set is not a compact set.

vi. If $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable at 0 and $f''(0) > 0$, then f has a local maximum at 0.

vii. Any uniformly continuous function is also continuous.

viii. Any monotonic bounded function defined on $[0, 2015]$ is Riemann integrable on $[2014, 2015]$.

ix. There exists a function $f : \mathbb{R}^2 \to \mathbb{R}$ that is differentiable at $(2, 2)$ but is not continuous at $(2, 2)$.

x. The set $\bigcap_{n\geq 1}(-1 - \frac{1}{n}, n]$ is a compact set.

xi. If f is differentiable on (a, b), continuous on $[a, b]$, and such that $f(a) + b = f(b) + a$, then there exists some $c \in (a, b)$ such that $f'(c) = 1$.

xii. A set is closed if it is not an open set.

xiii. Let $f : [0, 1] \to \mathbb{R}$ be a Riemann integrable function. Then the function $A : [0, 1] \to \mathbb{R}$ defined as $A(x) = \int_0^x f(x)\mathrm{d}x$ is continuous on $[0, 1]$.

xiv. If $f(x) = 1/(1 + x^2)$ then, for any closed set K, $f^{-1}(K)$ is a closed set.

xv. The set $\cup_{n\geq 1}(n - \frac{1}{4}, n + \frac{1}{4})$ is an open set.

xvi. There are no Borel sets that are open and closed simultaneously.

xvii. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function. Suppose that for all $v \in \mathbb{R}^2 \setminus \{(0, 0)\}$, $\lim_{t\to 0} f(tv) = f((0, 0))$. Then f is continuous at $(0, 0)$.

xviii. For all $A \subset \mathbb{R}$, let $\chi_A : \mathbb{R} \to \mathbb{R}$ be given by $\chi_A(x) = 1$ if $x \in A$, and $\chi_A(x) = 0$ if $x \notin A$. The function χ_A is Borel measurable.

xix. The set $\{\frac{n}{n+1} : n \in \mathbb{N}\} \cup \{1\}$ is a compact set.

xx. Let $f, g : [5, 2015] \to \mathbb{R}$ be functions. If f and $f + g$ are differentiable at 2014, then g is also differentiable at 2014.

	EXAMINATION SOLUTIONS 2014-15	Course M2PM1
Question 1		Marks & seen/unseen

Parts

i

The Mean Value Theorem as stated in the lectures has two parts: **(4 Marks) Seen**

a. Let $f : [a, b] \to \mathbb{R}$ be a function continuous on $[a, b]$ and differentiable on (a, b). Then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

b. (Cauchy's version) Let $f, g : [a, b] \to \mathbb{R}$ be two functions continuous on $[a, b]$ and differentiable on (a, b). Then there exists $c \in (a, b)$ such that

$$(g(b) - g(a))f'(c) = (f(b) - f(a))g'(c).$$

ii.a

Fix $x \in (0, 2]$. Let $F, G : [0, 1] \to \mathbb{R}$ be the functions given by $F(t) = f(tx)$ and $G(t) = t^2$. Observe that F and G are continuous on $[0, 1]$ and differentiable on $(0, 1)$. Then, *the Cauchy's version of the Mean value theorem* yields that there exists $\beta \in (0, 1)$ such that **(6 Marks) Seen similar**

$$2\beta f(x) = G'(\beta)\,(F(1) - F(0)) = F'(\beta)\,(G(1) - G(0)) = f'(\beta x)x.$$

where the last equality follows from the facts that $G(1) = 1$, $G(0) = 0$ and from *The chain rule* applied to the function $F(t) = f(h(t))$ where $h(t) = tx$. Therefore,

$$\frac{f(x)}{x} = \frac{f'(\beta x)}{2\beta}.$$

ii.b

From the previous problem, we have in particular that for any $x \in (0, 2)$, there exists $\beta \in (0, 1)$ such that **(6 Marks) Unseen**

$$\frac{f(x)}{x} = \frac{f'(\beta x)}{2\beta} = \frac{x}{2}\frac{f'(\beta x)}{\beta x}.$$

Applying the hypothesis on the strict monotonicity of the function $f'(x)/x$ on $x \in (0, 2)$ on the previous equality, we obtain that for any $x \in (0, 2)$

$$\frac{f(x)}{x} = \frac{x}{2}\frac{f'(\beta x)}{\beta x} < \frac{x}{2}\frac{f'(x)}{x} = \frac{f'(x)}{2}.$$

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	EXAMINATION SOLUTIONS 2014-15	Course M2PM1
Question 1 (cont.)		Marks & seen/unseen

Parts

In particular, since $f(x) \geq 0$, for any $x \in (0, 2)$,

$$0 \leq \frac{2f(x)}{x} < f'(x).$$

Then, by one of the theorems proved in the lectures, it follows that f is a strictly increasing function.

ii.c

Since f is continuous on $[0, 2]$, $f(0) = 0$, $f(2) = \pi$ and $\pi/2 \in (0, \pi)$, by the *Intermediate Value Theorem*, there exists $\alpha \in (0, 2)$ such that $f(\alpha) = \pi/2$. Moreover, since we have proved that f is strictly increasing, f is in particular injective and so, this α is unique.

(4 Marks)
Seen similar

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	EXAMINATION SOLUTIONS 2014-15	Course M2PM1
Question 2		Marks & seen/unseen

Parts

i

A function f is integrable over the interval $[a,b]$ if its upper Riemann integral is equal to its lower Riemann integral. In that case, that value is called the integral of f on $[a,b]$ and we denote it by $\int_a^b f ds$. (3 Marks) Seen

ii

The ε-criterion for Riemann integration that we have presented in the lectures is the following one: (3 Marks)

Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Suppose that for any $\varepsilon > 0$ there is a partition Δ_ε of $[a,b]$ such that

$$S(f,\Delta_\varepsilon) - s(f,\Delta_\varepsilon) < \varepsilon. \tag{1}$$

Then f is Riemann integrable over $[a,b]$.

We have shown in the lectures that for any partition Δ of $[a,b]$, (4 Marks)

$$s(f,\Delta) \leq L(f) \leq U(f) \leq S(f,\Delta),$$

where $U(f)$ and $L(f)$ denote the upper and the lower integral of f on $[a,b]$ respectively. Thus, given $\varepsilon > 0$ and a partition Δ_ε satisfying (1), one has

$$0 \leq U(f) - L(f) \leq S(f,\Delta_\varepsilon) - s(f,\Delta_\varepsilon) < \varepsilon.$$

Since the inequality holds for any $\varepsilon > 0$, by the sandwich principle, it follows that $U(f) = L(f)$, and so that f is integrable on $[a,b]$.

iii

For any $N \geq 2$, take the partition $\Delta_N = \{x_0, x_1, x_2, x_3, x_4\}$ where $x_0 = 0$, $x_1 = 1 - \frac{1}{N}$, $x_2 = 1$, $x_3 = 1 + \frac{1}{N}$, $x_4 = 4$. (5 Marks) Seen similar

Hence,

$$\begin{aligned} S(f,\Delta_N) &= 2(x_1 - x_0) + 2(x_2 - x_1) + 1(x_3 - x_2) + 1(x_4 - x_3) \\ &= 2(x_2 - x_0) + 1(x_4 - x_2) = 5. \end{aligned}$$

Similarly

$$s(f,\Delta_N) = 2(x_1 - x_0) + 1(x_4 - x_3) = 2 - \frac{2}{N} + 3 - \frac{1}{N} = 5 - \frac{3}{N}.$$

Therefore $S(f,\Delta_N) - s(f,\Delta_N) = \frac{3}{N}$. Hence, for any $\varepsilon > 0$, taking $N > \frac{3}{\varepsilon}$, one has $S(f,\Delta_N) - s(f,\Delta_N) < \varepsilon$. Thus the ε-criterion yields the integrability of f.

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	EXAMINATION SOLUTIONS 2014-15	Course M2PM1
Question 2 (cont.)		Marks & seen/unseen

Parts

Moreover, since for any $N \geq 2$

$$s(f, \Delta_N) \leq L(f) \leq U(f) \leq S(f, \Delta_N)$$

taking limit in N, as N tends to infinity yields that $L(f) = U(f) = 5$. So, $\int_0^4 f \mathrm{d}s = 5$.

(2 Marks) Seen similar

iv The answer is yes, it is possible.

As an example, consider the function f on $[0, 1]$ given by $f(x) = 1$ if $x = 1/2$ and zero elsewhere. This function satisfies that it is bounded and Riemann integrable on any subinterval $[0, x]$ with $x \in [0, 1]$, and $A(x) = \int_0^x f \mathrm{d}s = 0$ (we have done this calculation during the course). Thus, since A is a constant function, it is differentiable on $(0, 1)$ and $A'(x) = 0$ for any $x \in (0, 1)$. In particular $A'(1/2) = 0 \neq f(1/2) = 1$.

(3 Marks) Unseen

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EXAMINATION SOLUTIONS 2014-15

Course M2PM1

Question 3

Marks & seen/unseen

Parts

i

A collection Σ of subsets of $\mathbb{R}$ is a σ-algebra if it satisfies the following three axioms: (3 Marks) Seen

a. $\emptyset \in \Sigma$.

b. If $A \in \Sigma$, then $\mathbb{R} \setminus A \in \Sigma$.

c. If $A_n \in \Sigma$ with $n \geq 1$ is a countable collection of subsets of $\mathbb{R}$, then $\cup_n A_n \in \Sigma$.

The Borel σ-algebra on $\mathbb{R}$ is the smallest σ-algebra that contains all the open intervals (a, b) with $a, b \in \mathbb{R}$ and $a < b$. That is, it is the σ-algebra generated by the open intervals of the form (a, b). (2 Marks)

ii

We will show first that the set of rational numbers $\mathbb{Q}$ is a Borel set. If we prove this, since the set of irrational numbers is the complementary of $\mathbb{Q}$, that is $\mathbb{R} \setminus \mathbb{Q}$, by the axiom b above, it follows that $\mathbb{R} \setminus \mathbb{Q}$ is a Borel set. (4 Marks) Seen similar

Since $\mathbb{Q}$ is countable, we can write $\mathbb{Q}$ as a countable union of singletons $\{x_n\}_{n \in \mathbb{N}}$. That is $\mathbb{Q} = \cup_{n \in \mathbb{N}} \{x_n\}$. So, if we show that each singleton is a Borel set, by the axiom c above, it will follow that $\mathbb{Q}$ is a Borel set. To this end, we observe that for any n, $\{x_n\} = \mathbb{R} \setminus (\cup_{m \geq 1} (-m, x_n) \cup (x_n, m))$. So by the axioms b and c above, $\{x_n\}$ is a Borel set.

iii

We say that f has directional derivative at a along the direction $v \in \mathbb{R}^n \setminus \{0\}$ if the following limit exists (2 Marks) Seen

$$\lim_{t \to 0} \frac{f(a + tv) - f(a)}{t}.$$

In that case, we denote that limit as $\frac{\partial f}{\partial v}(a)$, and we call it the directional derivative of f at a along the direction v.

If f is differentiable at a, $\lim_{x \to a} \frac{|f(x) - f(a) - \nabla f(a) \cdot (x-a)|}{\|x-a\|} = 0$. This implies in particular that (3 Marks) Seen

$$0 = \lim_{t \to 0} \frac{|f(a+tv) - f(a) - t\nabla f(a) \cdot v|}{|t|\|v\|} = \frac{1}{\|v\|} \lim_{t \to 0} \left| \frac{f(a+tv) - f(a)}{t} - \nabla f(a) \cdot v \right|,$$

which is equivalent to say that

$$\lim_{t \to 0} \frac{f(a + tv) - f(a)}{t} = \nabla f(a) \cdot v.$$

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	EXAMINATION SOLUTIONS 2014-15	Course M2PM1
Question 3 (Cont.)		Marks & seen/unseen

Parts

iv

Observe that for $(x,y) \neq (0,0)$ $f(x,y) = g(x,y)/h(x,y)$, where $g(x,y) = xy^2$ and $h(x,y) = h_1(h_2(x,y))$ with $h_1(t) = \sqrt{t}$ and $h_2(x,y) = x^2 + y^2$.

(6 Marks) Seen similar

Since h_1 is differentiable on $(0,\infty)$, $h_2(\mathbb{R}^2 \setminus \{(0,0)\}) \subset (0,\infty)$, and h_2 is differentiable on $\mathbb{R}^2 \setminus \{(0,0)\}$ (because it is a polynomial expression on x, y), *The Chain Rule Theorem* yields that h is differentiable on $\mathbb{R}^2 \setminus \{(0,0)\}$, and

$$\nabla h(x,y) = \frac{1}{\sqrt{x^2+y^2}}(x,y), \quad \forall (x,y) \neq (0,0).$$

Moreover, since g is a polynomial, it is differentiable on $\mathbb{R}^2$ and, in particular, on $\mathbb{R}^2 \setminus \{(0,0)\}$ and also

$$\nabla g(x,y) = (y^2, 2xy).$$

Observe also that $h(x,y) \neq 0$ for all $(x,y) \neq (0,0)$. Then, by the properties of differentiable functions, f is differentiable on $\mathbb{R}^2 \setminus \{(0,0)\}$ and for any $(x,y) \neq (0,0)$

$$\begin{aligned} \nabla f(x,y) &= \frac{\nabla g(x,y)h(x,y) - g(x,y)\nabla h(x,y)}{h^2(x,y)} \\ &= \frac{1}{(x^2+y^2)^{3/2}}\left(y^4, xy(2x^2+y^2)\right). \end{aligned}$$

Now, for $(x,y) = (0,0)$, $\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0} \frac{f(t,0)-f(0,0)}{t} = 0$ and similarly $\frac{\partial f}{\partial y}(0,0) = 0$. That is $\nabla f(0,0) = (0,0)$.

So, to check whether f is differentiable at $(0,0)$, we need to see if the following limit exists and it is zero

$$\lim_{(x,y)\to(0,0)} \frac{|f(x,y) - f(0,0) - \nabla f(0,0)\cdot(x,y)|}{\|(x,y)\|} = \lim_{(x,y)\to(0,0)} \frac{|x|y^2}{x^2+y^2}.$$

But observe that $0 \leq \frac{|x|y^2}{x^2+y^2} \leq |x| \leq \|(x,y)\|$, so it follows that the previous limit exists is zero, and so that f is also differentiable at $(0,0)$.

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	EXAMINATION SOLUTIONS 2014-15	Course M2PM1
Question 4		Marks & seen/unseen
Parts	Each correct answer is awarded 1 mark. Each incorrect answer is awarded -1 mark. If the number of incorrect answers is higher than the number of correct answers, then the total mark awarded for this question will be 0. i. T ii. F iii. T iv. F v. F vi. F vii. T viii. T ix. F x. T xi. T xii. F xiii. T xiv. T xv. T xvi. F xvii. F xviii. F xix. T xx. T	
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Imperial College
London

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2015

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Algebra II

Date: Tuesday, 19 May 2015. Time: 2.00pm – 4.00pm. Time allowed: 2 hours.

This paper has FOUR questions.

Candidates should start their solutions to each question in a new main answer book

Supplementary books may only be used after the relevant main book(s) are full.

Statistical tables will not be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers to each question as per the table below.

Raw mark	up to 12	13	14	15	16	17	18	19	20
Extra credit	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4

- Each question carries equal weight.
- Calculators may not be used.

Throughout this exam, standard facts and results from the course may be assumed, unless you are explicitly asked to prove them.

1. Let G be a group. We say elements x and y in G are *conjugate* if there exists $g \in G$ such that $g^{-1}xg = y$.

   (i) Write $x \sim y$ if x and y are conjugate. Prove that $\sim$ is an equivalence relation.

   The equivalence classes in (i) are called *conjugacy classes*. In the rest of this question we will examine conjugacy classes in a dihedral group.

   Say $n \geq 3$ and $G = D_{2n}$ is the dihedral group of size $2n$, generated by a rotation ρ of order n and a reflection σ of order 2, subject to the usual relation $\sigma\rho = \rho^{-1}\sigma$. You may assume, if it helps, that $\sigma\rho^k = \rho^{-k}\sigma$ for all $k \in \mathbb{Z}$.

   (ii) Prove that ρ and ρ^{-1} are conjugate.

   (iii) Say $j \in \mathbb{Z}$. Prove that ρ^j is conjugate to $x \in G$ if and only if $x = \rho^j$ or $x = \rho^{-j}$.

   (iv) Prove that σ is conjugate to $x \in G$ if and only if $x = \rho^{2m}\sigma$ for some $m \in \mathbb{Z}$.

   (v) Prove that if $n = 2d + 1$ is odd then G has exactly $2 + d$ conjugacy classes.

2. Recall that if G is a group and H is a subgroup of G, then a *right coset* of H in G is a subset of the form $Hx = \{hx : h \in H\}$ for some $x \in G$. Recall also that distinct right cosets are disjoint.

   Let G be a finite group and let $H \subseteq G$ be a subgroup.

   (i) What is the *index* of H in G? What does it mean for H to be a *normal subgroup* of G?

   Now assume that H is a subgroup of G of index 2.

   (ii) Prove that H is a normal subgroup of G.

   (iii) Prove that there is a surjective group homomorphism $\phi : G \to C_2$.

   (iv) Prove that if $g \in G$ with $g \notin H$ then the order of g is even.

   Now suppose A is a finite group, and B is a subgroup of A, of index 3.

   (v) Must B always be a normal subgroup of A? Proof or counterexample required.

   (vi) Must there always be a surjective group homomorphism $A \to C_3$? Proof or counterexample required.

   (vii) If $a \in A$ with $a \notin B$, must the order of a always be a multiple of 3? Proof or counterexample required.

3. (i) Define the *determinant* $\det(A)$ of an $n \times n$ matrix A.

(ii) Prove, directly from the definitions, that if A is an $n \times n$ matrix and B is a matrix obtained from A by swapping two (different) columns, then $\det(B) = -\det(A)$.

(iii) Let A_n be the following $n \times n$ matrix:

$$\begin{pmatrix} 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 1 & \cdots & 1 & 1 & 0 \\ 1 & 1 & \cdots & 1 & 0 & 0 \\ & & \vdots & & & \\ 1 & 1 & \cdots & 0 & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}.$$

[algebraically, $A_n = (a_{ij})_{1 \leq i,j \leq n}$ with $a_{ij} = 1$ if $i + j \leq n + 1$ and $a_{ij} = 0$ otherwise]. Prove that $\det(A_n) = +1$ if $n = 4d$ or $n = 4d + 1$, and $\det(A_n) = -1$ if $n = 4d + 2$ or $n = 4d + 3$ (here d is an integer).

(iv) What are the determinants of

(a)

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

(b)

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 7 \end{pmatrix}$$

(c)

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix}?$$

4. Let V be a finite-dimensional vector space and let $T : V \to V$ be a linear map.

(i) What is an *eigenvector* for T? What is an *eigenvalue* for T? What is the *eigenspace* for a given eigenvalue?

(ii) Prove that if $v_1, v_2, \ldots, v_k$ are eigenvectors for T with distinct eigenvalues, then the v_i are linearly independent.

(iii) Give, with proof, an example of a finite-dimensional vector space V over the complex numbers, and a linear map from this space to itself, such that the linear map only has one eigenvalue $\lambda = 7$, but the eigenspace for this eigenvalue is not all of V.

(iv) Suppose again V is finite-dimensional over the complex numbers, and $T : V \to V$ is a linear map. Suppose that the characteristic polynomial of T has at least three distinct eigenvalues, and the kernels of T and T^2 are not equal. Prove that the dimension of V is at least 4.

M2PM2 2015 SOLUTIONS

(1) (i)

Reflexivity: if $g = e$, the identity, then $g^{-1}xg = exe = x$.

Symmetry: if $x \sim y$ then there exists $g \in G$ such that $y = g^{-1}xg$. Now set $h = g^{-1}$; then $h^{-1}yh = gyg^{-1} = g(g^{-1}xg)g^{-1} = exe = x$, so $y \sim x$.

Transitivity: if $x \sim y$ and $y \sim z$ then there exists g, h such that $y = g^{-1}xg$ and $z = h^{-1}yh$. Then $z = h^{-1}(g^{-1}xg)h = (gh)^{-1}x(gh)$, so $x \sim z$. **Seen,3**

(ii) Set $g = \sigma$; then $g^2 = e$ so $g = g^{-1}$, and $g^{-1}\rho g = \sigma\rho\sigma = \rho^{-1}\sigma\sigma = \rho^{-1}$, and $\rho \sim \rho^{-1}$. **Seen sim,1**

(iii) It's a standard fact from the course that every element of G is of the form ρ^k or $\rho^k\sigma$. To figure out a complete list of the elements of G which are conjugate to ρ^j, we hence just have to compute $g^{-1}\rho^j g$ for $g = \rho^k$ and $g = \rho^k\sigma$. For $g = \rho^k$ we get

$$\begin{aligned} g^{-1}\rho^j g &= \rho^{-k}\rho^j\rho^k \\ &= \rho^{-k+j+k} \\ &= \rho^j. \end{aligned}$$

For $g = \rho^k\sigma$ we get

$$\begin{aligned} g^{-1}\rho^j g &= \sigma\rho^{-k}\rho^j\rho^k\sigma \\ &= \sigma\rho^j\sigma \\ &= \rho^{-j}\sigma\sigma \\ &= \rho^{-j}. \end{aligned}$$

Unseen,4

(iv) We use the same technique as in the previous part. If $g = \rho^k$ then $g^{-1}\sigma g = \rho^{-k}\sigma\rho^k = \rho^{-k}\rho^{-k}\sigma = \rho^{-2k}\sigma$, and if $g = \rho^k\sigma$ then $g^{-1}\sigma g = \sigma\rho^{-k}\sigma\rho^k\sigma = \rho^k\sigma\sigma\rho^k\sigma = \rho^{2k}\sigma$. The result follows. **Unseen, 4**

(v) If n is odd then *every* reflection is in the same conjugacy class. For we have seen that $\rho^{2m}\sigma$ is conjugate to σ, and if j is odd then $\rho^j\sigma = \rho^{j+n}\sigma$ and $j + n$ is even. Hence all the reflections form one conjugacy class.

The rotations: we know that conjugacy classes look like $\{\rho^j, \rho^{-j}\}$; every rotation is conjugate to itself and to its inverse. In the group $\langle \rho \rangle \cong C_n$, the only element equal to its own inverse is the identity, becaue if $x = x^{-1}$ then $x^2 = e$ and a group of odd order has no element of order 2. Hence the identity gives us one conjugacy class, and the remaining $n - 1 = 2d$ elements give us d conjugacy classes. So we get $d + 1 + 1 = d + 2$ conjugacy classes in total. **Unseen,8**

(2) (i) The *index* of H in G is the number of distinct right cosets for H in G. A *normal subgroup* of G is a subgroup H such that $g^{-1}Hg = H$ for all $g \in G$. **Seen,2**

(ii) We know that G is the disjoint union of H and Hx for some $x \in G$. By a result in lectures, to check normality it suffices to check that for all $g \in G$ we have $g^{-1}Hg \subseteq H$. In other words, what we need to do is to check that if $g \in G$ and $h \in H$ then $g^{-1}hg \in H$.

If $g \in H$ then $g^{-1} \in H$, so $g^{-1}hg$ is a product of elements of H, so in H.

If however $g \notin H$ then $g \in Hx$ so $g = jx$ for some $j \in H$. Then $g^{-1}hg = x^{-1}j^{-1}hjx = x^{-1}h'x$ for some $h' \in H$. We then have to show that $x^{-1}h'x \in H$. Let's prove this by contradiction. If $x^{-1}h'x \notin H$ then we must have $x^{-1}h'x \in Hx$, so $x^{-1}h'x = kx$ for some $k \in H$. Cancelling the x's we get $x^{-1}h' = k$ and hence $x = h'k^{-1}$ and in particular $x \in H$. But this is nonsense because $x \in Hx$ and Hx is disjoint from H. **Unseen,4**

(iii) We know H has index 2 and is normal, so the quotient group G/H has size 2 and hence must be isomorphic to the cyclic group of order 2. The natural map $G \to G/H$, composed with the isomorphism $G/H \cong C_2$, is a map $\phi : G \to C_2$, and if $G = H \cup Hx$ as before, then the image of x under this map is the non-trivial element of C_2, so $G \to C_2$ is surjective. **Seen sim,3**

(iv) If $C_2 = \{+1, -1\}$ then $\phi^{-1}(+1) = H$. Hence if $g \notin H$ then $\phi(g) = -1$, which has order 2 in C_2. By a result from lectures, the order of $\phi(g)$ divides the order of g, which is hence even (note that G is finite and hence g has finite order). **Seen,2**

(v) B might not be normal. For example if $A = S_3$ of order 6, and $B = \langle (1\ 2) \rangle$ of order 2, then B has index 3, but if $g = (1\ 3)$ then $g^{-1}(1\ 2)g = (2\ 3) \notin B$, so B is not normal in this case. **Seen sim,3**

(vi) Again there might not be. Again set $A = S_3$. I explicitly proved in lectures that there was no surjection $A \to C_3$, and arguably this is then a "standard result from the course". But here's a proof anyway: any group homomorphism $\psi : S_3 \to C_3$ must send an element of order 2 to an element of order dividing 2, but the only element of order dividing 2 in C_3 is the identity. Hence all three transpositions in S_3 are in the kernel of ψ. But this means that the kernel of ψ has size at least 4, and if ψ were surjective then by the first isomorphism theorem its kernel would have to have size exactly 2, a contradiction. $\boxed{\text{Seen, 3}}$

(vii) Again $A = S_3$ and $B = \langle(1\ 2)\rangle$ gives a counterexample; for $a = (1\ 3) \notin B = \{e, (1\ 2)\}$ and yet a has order 2, which is not a multiple of 3. $\boxed{\text{Unseen,3}}$

(3) (i) The determinant of $A = (a_{ij})$ is

$$\det(A) = \sum_{\pi \in S_n} \mathrm{sgn}\,(\pi) a_{1\,\pi(1)} a_{2\,\pi(2)} \cdots a_{n\,\pi(n)}.$$

$\boxed{\text{Seen,1}}$

(ii) Say we swap columns s and t of matrix $A = (a_{ij})$, to get matrix $B = (b_{ij})$. Then $b_{ij} = a_{i\,\tau(j)}$ for $\tau = (s\ t)$ the transposition. Then

$$\begin{aligned}\det(B) &= \sum_{\pi \in S_n} \mathrm{sgn}\,(\pi) b_{1\,\pi(1)} b_{2\,\pi(2)} \cdots \\ &= \sum_{\pi} \mathrm{sgn}\,(\pi) a_{1\,\tau\pi(1)} a_{2\,\tau\pi(2)} \cdots\end{aligned}$$

and as π runs through all of S_n, so does $\sigma = \tau\pi$. Hence changing the sum to one over σ and writing $\pi = \tau^{-1}\sigma = \tau\sigma$, we have

$$\begin{aligned}\det(B) &= \sum_{\sigma \in S_n} \mathrm{sgn}\,(\tau\sigma) a_{1\,\sigma(1)} a_{2\,\sigma(2)} \cdots \\ &= \sum_{\sigma \in S_n} \mathrm{sgn}\,(\tau) \mathrm{sgn}\,(\sigma) a_{1\,\sigma(1)} a_{2\,\sigma(2)} \cdots \\ &= -\sum_{\sigma \in S_n} \mathrm{sgn}\,(\sigma) a_{1\,\sigma(1)} a_{2\,\sigma(2)} \cdots \\ &= -\det(A)\end{aligned}$$

because $sgn(\tau) = -1$. $\boxed{\text{Seen,5}}$ (iii) We prove the result by induction. Expanding down the nth column we see immediately that $\det(A_n) = (-1)^{n-1}\det(A_{n-1})$, and hence $\det(A_n) = \pm\det(A_{n-1})$, with a $+$ sign if n is odd and a minus sign if n is even. Now $A_1 = (1)$ has determinant 1, and so the determinant of A_n for $n = 1, 2, 3, 4, 5, 6, \ldots$ is $+1, -1, -1, +1, +1, -1 \ldots$. Formally, one could now prove the result by induction on d, but basically it is obvious that the pattern continues. $\boxed{\text{Seen sim,3}}$

(iv) (a) The only non-zero term in the sum corresponds to the permutation (2 4 3), which has signature +1, so the determinant is +1. seensim,2

(b) Expand down the first column and then note that the resulting 3×3 matrix has a column of zeros, so the determinant is zero. seensim,3

(c) The determinant does not change if we subtract the second row from the third row (turning the third row into (4 4 4 4), and it still does not change if we subtract the first row from the second (turning the second row into (4 4 4 4). But the determinant of a matrix with two rows equal is zero, by a result from lectures. Unseen,6

(4) (i) An *eigenvector* is $0 \neq v \in V$ satisfying $T(v) = \lambda v$ for some $\lambda \in E$ (the ground field). The *eigenvalue* of this eigenvector is λ. The *eigenspace* for λ is the kernel of $T - \lambda I$, or equivalently the space $\{v \in V : T(v) = \lambda v\}$. Seen,3

(ii) Induction on k. The case $k = 1$ is clear because $v_1 \neq 0$ by definition. Now assume $k \geq 2$. Say v_i has eigenvalue λ_i, with the λ_i all distinct. If we have a linear relation

$$r_1 v_1 + r_2 v_2 + \cdots + r_k v_k = 0$$

then hitting this relation with T gives

$$r_1 T(v_1) + r_2 T(v_2) + \cdots + r_k T(v_k) = 0$$

and hence

$$r_1 \lambda_1 v_1 + r_2 \lambda_2 v_2 + \cdots + r_k \lambda_k v_k = 0.$$

Subtracting λ_k times the first equation gives

$$r_1(\lambda_1 - \lambda_k)v_1 + \cdots + r_{k-1}(\lambda_{k-1} - \lambda_k)v_{k-1} = 0$$

and by the inductive hypothesis this implies $r_i(\lambda_i - \lambda_k) = 0$ for all $1 \leq i \leq k-1$. But $\lambda_i - \lambda_k \neq 0$ by assumption, so $r_i = 0$ for all $i \leq k-1$, and substituting back into the first equation gives $r_k = 0$ as well. Hence all the r_j are zero, and the v_j are thus linearly independent. Seen,6

(iii) $V = \mathbb{C}^2$ and $T(v) = Av$ with $A = \begin{pmatrix} 7 & 1 \\ 0 & 7 \end{pmatrix}$. This is a linear map, and its eigenvalues are the roots of the characteristic polynomial of A, which is $(x-7)^2$, so the only eigenvalue is 7. However the eigenspace is the kernel of $A - 7I$ and this is not the zero matrix, hence the eigenspace is not all of V. Seen sim,3

(iv) Say the dimension of V is n. The characteristic polynomial of V is then a polynomial of degree n, and hence it has at most n distinct roots. Because V has at least three distinct eigenvalues, this means $n \geq 3$, and so we must rule out the case $n = 3$. If $n = 3$ then V must have three distinct eigenvalues each with algebraic multiplicity 1 (and hence geometric multiplicity 1). If none of these eigenvalues are zero then T is invertible and hence T^2 is too, which would mean that both T and T^2 had zero kernel and in particular the kernels of T and T^2 would coincide. We know the eigenvalues are distinct, so the only other case to consider is when the eigenvalues are $0, \lambda, \mu$ with λ, μ both non-zero. In this case the zero eigenspace for T is clearly contained within the zero eigenspace for T^2, but the eigenvalues of T^2 are $0, \lambda^2, \mu^2$ and hence the algebraic multiplicity of zero as an eigenvalue of T^2 is 1, so again the geometric multiplicity is 1; thus $\ker(T)$ and $\ker(T^2)$ are both one-dimensional, and one is contained within the other, so again they must be equal. This contradiction shows that in fact we must have $n \geq 4$. $\boxed{\textbf{Unseen,8}}$

Imperial College
London

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2015

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Complex Analysis

Date: Tuesday, 12 May 2015. Time: 2.00pm – 4.00pm. Time allowed: 2 hours.

This paper has FOUR questions.

Candidates should start their solutions to each question in a new main answer book

Supplementary books may only be used after the relevant main book(s) are full.

Statistical tables will not be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers to each question as per the table below.

Raw mark	up to 12	13	14	15	16	17	18	19	20
Extra credit	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4

- Each question carries equal weight.
- Calculators may not be used.

Question 1.

a. Give the definition of a function holomorphic at a point $z \in \mathbb{C}$ and in an open set $\Omega \subset \mathbb{C}$. What are Cauchy-Riemann equations?

b. Find all $(x, y) \in \mathbb{R}^2$ where

$$u(x,y) = e^{x^2-y^2} \sin 2xy$$

is harmonic. Find all harmonic conjugates.

c. If $f(z) = u + iv$, where u is defined in 1.b and v is its harmonic conjugate, find $f(z)$ in terms of z.

d. For the mapping $w = z + \frac{1}{z}$ find the images of the circles $|z| = R$, $R > 0$.

Question 2.

a. Find the maximum value of

$$\left|\frac{e^{-z}}{z+1}\right|, \qquad z \in \Omega_R = \{\operatorname{Re} z \geq 0\} \cap \{z : |z| \leq R\},$$

where $R > 0$.

b. Formulate Liouville's theorem (no proof is required).

c. Let $f(z)$ be a bounded entire function. Compute for R large and $z_0, z_1 \in \mathbb{C}$ ($|z_0|, |z_1| < R$) the value of the integral

$$\frac{1}{2\pi i} \oint_{|z|=R} \frac{f(z)}{(z-z_0)(z-z_1)}\, dz.$$

By taking $R \to \infty$, obtain a new proof of Liouville's theorem.

d. Let f be an entire function such that $|f(z)| \leq C(1+|z|)^n$, for some $n \in \mathbb{N}$ and where $C \geq 0$. Show that f is a polynomial of degree smaller or equal than n.

Question 3.

a. Give the definition of a zero of order m at z_0 for a holomorphic function f.

b. Prove that f has a zero of order m at z_0 if and only if it can be written in the form $f(z) = (z - z_0)^m g(z)$, where g is holomorphic in a neighbourhood of z_0 and $g(z_0) \neq 0$.

c. Find the Laurent series for

$$f(z) = \frac{\sin z}{(z-3)^2}, \qquad \forall\, z \neq 3.$$

d. Define residue of a function f at z_0.

e. Compute the integral

$$\int_0^{2\pi} \frac{1}{5 + 3\cos\theta}\, d\theta$$

by using the substitution $z = e^{i\theta}$.

Question 4.

a. State Rouche's theorem (no proof is required).

b. Prove that for every $\lambda > 1$, the equation $ze^{\lambda - z} = 1$ has exactly one root in the disk $\{z : |z| < 1\}$ and that this root is real.

c. Show that

$$\int_0^{\infty} \frac{x^{1/3}}{(x+1)^2}\, dx = \frac{2\pi}{3\sqrt{3}}.$$

[*Hint: Use for the branch cut of* $z^{1/3}$ *the semiaxis* $[0, \infty)$.]

d. Define Fourier coefficients and Fourier series for an integrable function on the interval $[-\pi, \pi]$ (no proof is required).

e. Let f be defined by $f(x) = |x|$, $x \in [-\pi, \pi]$. Use Parseval's identity for the function f in order to show that

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^4} = \frac{\pi^4}{96}.$$

M2PM3

Complex Analysis

Friday, May 12, 2015

14:00-16:00

Solutions

Question 1.

1a. Let $\Omega_1, \Omega_2 \subset \mathbb{C}$ be open sets and let $f : \Omega_1 \to \Omega_2$. We say that f is differentiable (holomorphic) at $z_0 \in \Omega_1$ if the quotient

$$\frac{f(z_0 + h) - f(z_0)}{h}$$

converges to a limit when $h \to 0$. Here $h \in \mathbb{C}$, $h \neq 0$ and $z_0 + h \in \Omega_1$. The limit of this quotient, when it exists, is denoted by $f'(z_0)$, and is called the derivative of f at z_0:

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}.$$

The function f is said to be holomorphic on open set Ω if f is holomorphic at every point of Ω .

If $f(z) = u(x, y) + iv(x, y)$ is holomorphic, then the functions u and v satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

(4 seen)

1b. For any $(x, y) \in \mathbb{R}^2$

$$u'_x = (2x \sin 2xy + 2y \cos 2xy)e^{x^2 - y^2}; \quad u'_y = (-2y \sin 2xy + 2x \cos 2xy)e^{x^2 - y^2};$$

$$\begin{aligned} u''_{xx} &= \Big((2 \sin 2xy + 4xy \cos 2xy - 4y^2 \sin 2xy) \\ &\qquad + (2x \sin 2xy + 2y \cos 2xy)2x\Big)e^{x^2 - y^2} \\ &= \Big(2 \sin 2xy + 8xy \cos 2xy + 4(x^2 - y^2) \sin 2xy\Big)e^{x^2 - y^2}, \end{aligned}$$

$$u''_{yy} = \Big((-2\sin 2xy - 4xy\cos 2xy - 4x^2\sin 2xy)$$
$$- (-2y\sin 2xy + 2x\cos 2xy)2y\Big)e^{x^2-y^2}$$
$$= \Big(-2\sin 2xy - 8xy\cos 2xy - 4(x^2-y^2)\sin 2xy\Big)e^{x^2-y^2}.$$

Therefore u is harmonic $\Delta u = 0$.

(4 unseen)

Using the C-R equation $u'_x = v'_y$ and integrating by parts we derive

$$v = \int u'_x\,dy = \int (2x\sin 2xy + 2y\cos 2xy)e^{x^2-y^2}\,dy$$
$$= \int (2x\sin 2xy)e^{x^2-y^2}\,dy - \int \cos 2xy(-2y)\,e^{x^2-y^2}\,dy$$
$$= \int (2x\sin 2xy)e^{x^2-y^2}\,dy - (\cos 2xy)\,e^{x^2-y^2} - \int 2x(\sin 2xy)e^{x^2-y^2}\,dy$$
$$= -(\cos 2xy)e^{x^2-y^2} + C(x).$$

(2 unseen)

The second C-R equation $v'_x = -u'_y$ gives

$$v'_x = -\Big((\cos 2xy)\,e^{x^2-y^2} - C(x)\Big)'_x = (2y\sin 2xy - (\cos 2xy)2x)e^{x^2-y^2} + C'(x)$$
$$= -u'_y = -(-2y\sin 2xy + 2x\cos 2xy)e^{x^2-y^2}.$$

This implies $C'(x) = 0$ and thus $C(x) = c = \mathrm{const} \in \mathbb{R}$.
Finally we obtain

$$v(x,y) = -(\cos 2xy)e^{x^2-y^2} + c.$$

(2 unseen)

1c.

$$f(z) = u + iv = e^{x^2-y^2}\sin 2xy - ie^{x^2-y^2}\cos 2xy + ic$$
$$= -i\left(e^{x^2-y^2}(\cos 2xy + i\sin 2xy)\right) + ic$$
$$= -ie^{x^2-y^2+i2xy} + ic = -ie^{z^2} + ic.$$

where $c \in \mathbb{R}$.

(2 unseen)

1d.
Note that if $|z| = R$, then

$$w = z + \frac{1}{z} = x + iy + \frac{x - iy}{x^2 + y^2} = \left(1 + \frac{1}{R^2}\right) x + \left(1 - \frac{1}{R^2}\right) iy.$$

Therefore, if we denote

$$u = \left(1 + \frac{1}{R^2}\right) x \qquad \text{and} \qquad v = \left(1 - \frac{1}{R^2}\right) y,$$

then for $R \neq 1$ we have

$$\frac{u^2}{(1 + 1/R^2)} + \frac{v^2}{(1 - 1/R^2)} = x^2 + y^2 = R^2,$$

and thus the mapping $z \to w$ maps the circles $|z| = R \neq 1$ onto the ellipses.

(4 unseen)

If $R = 1$, then $v \equiv 0$ and $u = 2x$. The equation $x^2 + y^2 = 1$ implies $-1 \leq x \leq 1$. Therefore the circle $|z| = 1$ maps onto the segment

$$\{v = 0, \ -2 \leq u \leq 2\}.$$

(2 unseen)

Question 2.

2a. Let

$$f(z) = \frac{e^{-z}}{z + 1}, \qquad z \in \Omega_R = \{\operatorname{Re} z \geq 0\} \cap \{z : \ |z| \leq R\}.$$

In order to find $\max_{z \in \Omega} |f(z)|$, by the maximum modulus principle we only need to check points on the boundary $\partial\Omega$.
If $z = iy$, then

$$\max_{z:\, z = iy,\, |y| \leq R} |f(z)| = \max_{y:\, |y| \leq R} \left|\frac{e^{-iy}}{iy + 1}\right| = \max_{y:\, |y| \leq R} \frac{1}{\sqrt{y^2 + 1}} = 1.$$

Moreover,

$$\max_{z:\, z = Re^{i\theta},\, \theta \in [-\pi/2, \pi/2]} |f(z)| = \max_{z:\, z = Re^{i\theta},\, \theta \in [-\pi/2, \pi/2]} \left|\frac{e^{-Re^{i\theta}}}{Re^{i\theta} + 1}\right|$$

$$= \max_{z:\, z = Re^{i\theta},\, \theta \in [-\pi/2, \pi/2]} \left|\frac{e^{-R(\cos\theta + i\sin\theta)}}{Re^{i\theta} + 1}\right| = \max_{z:\, z = Re^{i\theta},\, \theta \in [-\pi/2, \pi/2]} \left|\frac{e^{-R\cos\theta}}{Re^{i\theta} + 1}\right|.$$

Since $\min_{\theta:\,\theta\in[-\pi/2,\pi/2]}\cos\theta = 0$ we have

$$\max_{z:\,z=Re^{i\theta},\,\theta\in[-\pi/2,\pi/2]}\left|\frac{e^{-R\cos\theta}}{Re^{i\theta}+1}\right| \le \max_{\theta:\,\theta\in[-\pi/2,\pi/2]}\left|\frac{1}{Re^{i\theta}+1}\right|$$
$$= \max_{\theta:\,\theta\in[-\pi/2,\pi/2]}\left|\frac{1}{\sqrt{(R\cos\theta+1)^2+R^2\sin^2\theta}}\right|.$$

Note that

$$\min_{\theta:\,\theta\in[-\pi/2,\pi/2]}\left((R\cos\theta+1)^2+R^2\sin^2\theta\right) = R^2+1+2R\cos\theta = R^2+1.$$

Thus

$$\max_{\theta:\,\theta\in[-\pi/2,\pi/2]}\left|\frac{1}{\sqrt{(R^2\cos\theta+1)^2+r^2\sin^2\theta}}\right| = \frac{1}{\sqrt{R^2+1}} < 1, \qquad \forall R>0.$$

Answer: $\max_{\{z:\,z=iy,\,|y|\le R\}}|f(z)| = 1$.

(5 unseen)

2b. [Liouville's theorem] If an entire function is bounded, then it is constant.

(3 seen)

2c. Indeed, for any $z_0, z_1 \in \mathbb{C}$, $z_0 \neq z_1$ and R sufficiently large, we have

$$\frac{1}{2\pi i}\oint_{|z|=R}\frac{f(z)}{(z-z_0)(z-z_1)}\,dz$$
$$= \frac{1}{z_0-z_1}\left(\frac{1}{2\pi i}\oint_{|z|=R}\frac{f(z)}{(z-z_0)}\,dz - \frac{1}{2\pi i}\oint_{|z|=R}\frac{f(z)}{(z-z_1)}\,dz\right)$$
$$= \frac{1}{z_0-z_1}\left(f(z_0)-f(z_1)\right).$$

(3 unseen)

Since f is bounded, there is a constant M, such that $|f(z)| \le M$, $z \in \mathbb{C}$. Therefore using the ML-inequality we find

$$\left|\frac{1}{2\pi i}\oint_{|z|=R} \frac{f(z)}{(z-z_0)(z-z_1)}\,dz\right| \le M\,R \max_{z:|z|=R} \frac{1}{|(z-z_0)(z-z_1)|}$$
$$\le M\,R\,\frac{1}{(R-|z_0|)(R-|z_1|)}$$
$$= M\,R^{-1}\,\frac{1}{(1-|z_0|/R)(1-|z_1|/R)} \to 0,$$
$$\text{as} \quad R \to \infty$$

This implies

$$\frac{1}{z_1-z_0}\,(f(z_0)-f(z_1)) = 0$$

and thus $f(z_0) = f(z_1)$. Since z_0 and z_1 are arbitrary, we finally obtain that f is a constant function.

(4 unseen)

2d.

Note that if $n = 0$, then we simply apply Liouville's theorem.

Assume that $|f(z)| \le C(1+|z|)^n$ with some $C > 0$. Then for any $z_0 \in \mathbb{C}$ we have

$$|f^{(n+1)}(z_0)| = \left|\frac{1}{2i\pi}\oint_{|z-z_0|=R} \frac{f(z)}{(z-z_0)^{n+2}}\,dz\right|$$
$$\le \frac{C}{2\pi}\max_{z:|z-z_0|=R}(1+|z|)^n\,\frac{2\pi R}{R^{n+2}} \to 0,$$
$$\text{as} \quad R \to \infty.$$

Therefore $f^{(n+1)} \equiv 0$ and thus $f^{(n)}$ is a constant function. We conclude that $f(z)$ is a polynomial of degree at most n by integrating $f^{(n)}(z)$ n-times.

(5 unseen)

Question 3.

3a. We say that f has a zero of order m at $z_0 \in \mathbb{C}$ if

$$f^{(k)}(z_0) = 0, \qquad k = 0, 1, \dots m-1,$$

and $f^{(m)}(z_0) \neq 0$.

(2 seen)

3b. A holomorphic function f has a zero of order m at z_0 if and only if it can be written in the form

$$f(z) = (z - z_0)^m\, g(z),$$

where g is holomorphic at z_0 and $g(z_0) \neq 0$.

Proof.

$$\begin{aligned} f(z) &= \frac{f^{(m)}(z_0)}{m!}(z - z_0)^m + \frac{f^{(m+1)}(z_0)}{(m+1)!}(z - z_0)^{m+1} + \dots \\ &= (z - z_0)^m \left(\frac{f^{(m)}(z_0)}{m!} + \frac{f^{(m+1)}(z_0)}{(m+1)!}(z - z_0) + \dots \right). \end{aligned}$$

Then $f(z) = (z - z_0)^m\, g(z)$ where g is defined by

$$g(z) = \frac{f^{(m)}(z_0)}{m!} + \frac{f^{(m+1)}(z_0)}{(m+1)!}(z - z_0) + \dots.$$

The above series converges and thus g is holomorphic at z_0 and $g(z_0) \neq 0$.

(3 seen)

Conversely, if $f(z) = (z - z_0)^m\, g(z)$, where $g(z_0) \neq 0$, then $f^{(k)}(z_0) = 0$, $k = 0, 1 \dots, m - 1$ and $f^{(m)}(z_0) = m!\, g(z_0) \neq 0$. □

(2 seen)

3c.

$$\frac{\sin z}{(z-3)^2} = \frac{\sin(z - 3 + 3)}{(z-3)^2} = \frac{\sin(z-3)\cos 3 + \sin 3 \cos(z-3)}{(z-3)^2}$$

$$= \frac{1}{(z-3)^2}\left(\cos 3 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(z-3)^{2n+1} + \sin 3 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(z-3)^{2n} \right).$$

(5 unseen)

3d. Let

$$f(z) = \sum_{-\infty}^{\infty} a_n (z - z_0)^n, \qquad 0 < |z - z_0| < R.$$

be the Laurent series for f at z_0. The residue of f at z_0 is

$$\operatorname{Res}[f, z_0] = a_{-1}.$$

(2 seen)

3e. If $z = e^{i\theta}$, then

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + \frac{1}{z}}{2} = \frac{z^2+1}{2z}.$$

Since $d\theta = dz/iz$ we therefore obtain

$$\int_0^{2\pi} \frac{1}{5+3\cos\theta}\, d\theta = \oint_{|z|=1} \frac{1}{5+3\left(\frac{z^2+1}{2z}\right)} \frac{1}{iz}\, dz$$
$$= \oint_{|z|=1} \frac{-2i}{(3z+1)(z+3)}\, dz.$$

Let f be the function defined by

$$f(z) = -\frac{2i}{(3z+1)(z+3)}.$$

Then f has two poles, one at $-1/3$ and the other at -3, both of order 1. Only one pole $-1/3$ lies inside the curve $\gamma = \{z : |z| = 1\}$. Thus,

$$\int_0^{2\pi} \frac{1}{5+3\cos\theta}\, d\theta = \oint_{|z|=1} f(z)\, dz$$
$$= 2\pi i \operatorname{Res}[f, -1/3] = 2\pi i \lim_{z\to -1/3} \frac{-2i}{3(z+3)} = 2\pi i \frac{-2i}{8} = \frac{\pi}{2}.$$

(6 unseen)

Question 4.

4a. Rouche's Theorem:
Let f and g be holomorphic in an open set Ω and let $\gamma \subset \Omega$ be a simple, closed, piecewise-smooth curve that contains in its interior only points of Ω. If $|g(z)| < |f(z)|$, $z \in \gamma$, then the sums of the orders of the zeros of $f + g$ and f inside γ are the same.

(2 seen)

4b. The equation $ze^{\lambda - z} = 1$ is equivalent to the equation

$$e^{z-\lambda} - z = 0.$$

Let now $f(z) = -z$ and $g(z) = e^{z-\lambda}$.

For any $z : z = e^{i\theta} = \cos\theta + i\sin\theta$, $0 < \theta \leq 2\pi$, we have

$$|g(e^{i\theta})| = |e^{\cos\theta + i\sin\theta - \lambda}| = |e^{\cos\theta - \lambda}| < 1 = |e^{i\theta}| = |f(e^{i\theta})|,$$

where we have used that $\lambda > 1$. By applying Rouche's theorem we find that the number of solutions of the equation $e^{z-\lambda} - z = 0$ inside $\{z : |z| < 1\}$ is the same as the number of solutions of the equation $f(z) = -z = 0$, namely only one solution.

(4 unseen)

The solution z_0 of the equation $ze^{\lambda-z} = 1$ is real. Indeed, assume that $z_0 : |z_0| < 1$ is not real. Then $\bar{z}_0$ is also a solution and obviously $|z_0| < 1$. This contradicts to the fact that there is only one solution of the equation $ze^{\lambda-z} = 1$ inside the unit disc.

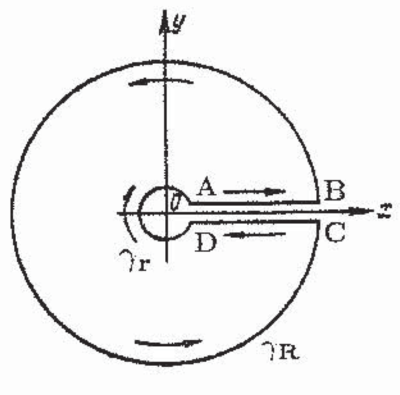
(2 unseen)

4c. Consider the integral

$$\oint_\gamma \frac{z^{1/3}}{(z+1)^2}\, dz,$$

where the curve γ consists of four components:

$$\gamma = [A, B] \cup \gamma_R \cup [C, D] \cup \gamma_r.$$



where $0 < r < 1 < R$.
The function $f(z) = z^{1/3}(z+1)^{-2}$ is single valued and holomorphic on and within γ except for the pole of order 2 at $z = -1 = e^{i\pi}$. Hence we can write

$$\oint_\gamma \frac{z^{1/3}}{(z+1)^2}\, dz = 2\pi i \operatorname{Res}(f(z), -1) = 2\pi i \frac{d}{dz} z^{1/3}\Big|_{z=e^{i\pi}}$$

$$= 2\pi i \frac{1}{3} z^{-2/3}\Big|_{z=e^{i\pi}} = 2\pi i \frac{1}{3} e^{-i2\pi/3} = 2\pi i \frac{1}{3}\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$$

$$= -\frac{\pi}{3} i + \frac{\pi\sqrt{3}}{3}.$$

On the other hand

$$\oint_{\gamma} \frac{z^{1/3}}{(z+1)^2}\,dz = \int_{[A,B]} \cdots + \int_{\gamma_R} \cdots + \int_{[C,D]} \cdots + \int_{\gamma_r} \cdots .$$

On $[A,B]$, $z = x\,e^{0\,i}$ and on $[C,D]$, $z = x\,e^{(0+2\pi)\,i} = x\,e^{2\pi i}$. Therefore

$$\int_{[A,B]} \cdots = \int_r^R \frac{(x\,e^{0\,i})^{1/3}}{(x\,e^{0\,i}+1)^2}\, e^{0\,i}\,dx = \int_r^R \frac{x^{1/3}}{(x+1)^2}\,dx$$

and

$$\int_{[C,D]} \cdots = \int_R^r \frac{(x\,e^{2\pi i})^{1/3}}{(x\,e^{2\pi i}+1)^2}\, e^{2\pi i}\,dx = \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \int_R^r \frac{x^{1/3}}{(x+1)^2}\,dx$$
$$= \left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) \int_r^R \frac{x^{1/3}}{(x+1)^2}\,dx.$$

Moreover, if now $z \in \gamma_R$, then $z = Re^{i\theta}$, $\theta \in (0, 2\pi)$ and by using the ML-inequality we obtain

$$\left|\int_{\gamma_R} \frac{z^{1/3}}{(z+1)^2}\,dz\right| \le 2\pi R \max_{z\in\gamma_R} \left|\frac{z^{1/3}}{(z+1)^2}\right|$$
$$= 2\pi R\, \frac{R^{1/3}}{(R-1)^2} \to 0, \quad \text{as } R \to \infty.$$

If $z \in \gamma_r$, then $z = re^{i\theta}$, $\theta \in (2\pi, 0)$ and therefore

$$\left|\int_{\gamma_r} \frac{z^{1/3}}{(z+1)^2}\,dz\right| \le 2\pi r \max_{z\in\gamma_r} \left|\frac{z^{1/3}}{(z+1)^2}\right|$$
$$= 2\pi r\, \frac{r^{1/3}}{(1-r)^2} \to 0, \quad \text{as } r \to 0.$$

as $r \to 0$.
Finally we arrive at

$$-\frac{\pi}{3}i + \frac{\pi\sqrt{3}}{3} = \oint_{\gamma} \frac{z^{1/3}}{(z+1)^2}\,dz \to \left(1 + \frac{1}{2} - i\frac{\sqrt{3}}{2}\right) \int_0^\infty \frac{x^{1/3}}{(x+1)^2}\,dx,$$
$$\text{as } r \to 0 \text{ and } R \to \infty.$$

Both real and imaginary parts of the latter imply

$$\int_0^\infty \frac{x^{1/3}}{(x+1)^2}\,dx = \frac{2\pi}{3\sqrt{3}}.$$

(6 unseen)

4d.
If f is an integrable function given on an interval $[-\pi, \pi]$, then the n-th Fourier coefficient of f is defined by

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx, \qquad n \in \mathbb{Z}.$$

The Fourier series of f is given by

$$\sum_{n=-\infty}^{\infty} \hat{f}(n) e^{inx}.$$

(2 seen)

4e. We compute the Fourier coefficients of f

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx = \frac{1}{2\pi} \left(-\int_{-\pi}^{0} x e^{-inx}\, dx + \int_{0}^{\pi} x e^{-inx}\, dx \right).$$

Note that if $n = 0$ we have

$$\hat{f}(0) = \frac{1}{2\pi} 2 \int_0^{\pi} x\, dx = \frac{\pi}{2}.$$

Moreover,

$$\int_{-\pi}^{0} x e^{-inx}\, dx = \frac{1}{-in} x e^{-inx}\Big|_{-\pi}^{0} - \frac{1}{-in} \int_{-\pi}^{0} e^{-inx}\, dx$$
$$= \frac{\pi}{-in}(-1)^n + \frac{1}{in}\frac{1}{(-in)} e^{-inx}\Big|_{-\pi}^{0} = \frac{\pi}{-in}(-1)^n + \frac{1}{n^2}\left(1 - (-1)^n\right),$$

and similarly

$$\int_{0}^{\pi} x e^{-inx}\, dx = \frac{1}{-in} x e^{-inx}\Big|_{0}^{\pi} - \frac{1}{-in} \int_{0}^{\pi} e^{-inx}\, dx$$
$$= \frac{\pi}{-in}(-1)^n + \frac{1}{in}\frac{1}{(-in)} e^{-inx}\Big|_{0}^{\pi} = \frac{\pi}{-in}(-1)^n + \frac{1}{n^2}\left((-1)^n - 1\right).$$

Therefore for $n \neq 0$

$$\hat{f}(n) = \frac{1}{2\pi}\frac{2}{n^2}\left((-1)^n - 1\right) = \frac{1}{\pi}\frac{1}{n^2}\left((-1)^n - 1\right).$$

Clearly

$$\int_{-\pi}^{\pi} f^2(x)\, dx = \int_{-\pi}^{\pi} x^2\, dx = \frac{1}{3} x^3\Big|_{-\pi}^{\pi} = \frac{2}{3}\pi^3.$$

Thus by using Parseval's identity we obtain

$$\frac{2}{3}\pi^3 = \int_{-\pi}^{\pi} f^2(x)\,dx = 2\pi\left(\hat{f}(0)^2 + \sum_{n\neq 0}|\hat{f}(n)|^2\right)$$

$$= 2\pi\left(\frac{\pi^2}{4} + \sum_{n\neq 0}\left(\frac{1}{\pi}\frac{1}{n^2}\left((-1)^n - 1\right)\right)^2\right)$$

$$= \frac{1}{2}\pi^3 + \frac{2}{\pi}\sum_{n\neq 0}\frac{1}{n^4}\left(1-(-1)^n\right)^2$$

$$= \frac{1}{2}\pi^3 + \frac{2}{\pi}2\sum_{k=0}^{\infty}\frac{4}{(2k+1)^4} = \frac{1}{2}\pi^3 + \frac{16}{\pi}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^4}.$$

This implies

$$\frac{16}{\pi}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^4} = \frac{\pi^3}{6},$$

and finally we have

$$\sum_{k=0}^{\infty}\frac{1}{(2k+1)^4} = \frac{\pi^4}{96}.$$

(4 unseen)

Imperial College
London

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2015

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Metric Spaces & Topology

Date: Wednesday, 20 May 2015. Time: 2.00pm – 4.00pm. Time allowed: 2 hours.

This paper has FOUR questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the main book is full.

Statistical tables will not be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers to each question as per the table below.

Raw mark	up to 12	13	14	15	16	17	18	19	20
Extra credit	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4

- Each question carries equal weight.
- Calculators may not be used.

1. (i) Let X, Y be topological spaces.

   (a) Give a definition of a continuous function $f : X \to Y$.

   (b) Prove that a constant map $f : X \to Y$, $f(x) = a$ for all $x \in X$, is a continuous function.

   (ii) Let d be a metric on a nonempty set A. Prove or disprove that its square $\widehat{d}(x,y) = d(x,y)^2 \ \forall x, y \in A$, is a metric on A.

2. (i) Show that the family of open balls forms a basis of the metric topology.

   (ii) Let $f : M \to M$ be a continuous mapping of a metric space M into itself. Prove that the set $\{x \in M : x = f(x)\}$ is closed.

   (iii) Are the following topological spaces homeomorphic? Give a short (one or two lines) argument for each case.

   (a) The interval $(-1, 1)$ and the real line $\mathbb{R}$ (Euclidean topology in both cases).

   (b) $\mathbb{R}^2$ and $\mathbb{R}^3$ (Euclidean topology in both cases).

   (c) Any 2 topological spaces on a set of 3 points.

3. (i) Give a definition of compactness of a topological space.

(ii) Prove or disprove that the following topological spaces are compact:

(a) The set of natural numbers $\mathbb{N} = \{1, 2, \ldots\}$ with the topology where the open sets are $\{\emptyset\}$, $\mathbb{N}$, and $\{1, 2, \ldots, n\}$ for all $n \in \mathbb{N}$.

(b) An infinite set X with the topology $\tau = \{\emptyset\} \cup \{B \subset X : X \setminus B \text{ is a finite set}\}$.

(c) A closed set K of points $x = (x_1, x_2, \ldots)$ in $\ell_2 = \{z = (z_1, z_2, \ldots) : z_j \subset \mathbb{R}, j = 1, 2 \ldots; \sum_{j=1}^{\infty} z_j^2 < \infty; d_2(z, y) = (\sum_{j=1}^{\infty} |z_j - y_j|^2)^{1/2}\}$ such that (1) $\sum_{j=1}^{\infty} x_j^2 \leq C$ for all $x \in K$, where $C > 0$ is a constant independent of x, and (2) for any $\epsilon > 0$ there exists N such that $\sum_{j=N}^{\infty} x_j^2 \leq \epsilon$ for all $x \in K$.

4. (i) Formulate and prove Baire's theorem.

(ii) Prove that the intersection of 2 complete subspaces of a metric space is complete (provided it is not empty).

EXAMINATION SOLUTIONS 2014-15

Course: M2PM5

Question: 1

Marks & seen/unseen

Parts		Marks & seen/unseen
i (a)	A function $f: X \to Y$ is called continuous if the preimage $f^{-1}(A)$ of any open set $A \subset Y$ is an open set in X.	5 seen
i (b)	Let $f(x) = a$ $\forall x \in X$. Let $A \subset Y$ be open. Then $f^{-1}(A) = \begin{cases} \emptyset & \text{if } a \notin A \\ X & \text{if } a \in A \end{cases}$, but both $\emptyset$ and X are open. Therefore f is continuous.	5 seen

Setter's initials [redacted] Checker's initials [redacted]

	EXAMINATION SOLUTIONS 2014-15	Course M2PM5
Question 1		Marks & seen/unseen
Parts ii	Let $d(x,y)$ be the Euclidean metric on $\mathbb{R}$. Then if d^2 is a metric, $1 = d^2(0,1) \leq$ $\leq d^2(0,1/2) + d^2(1/2,1)$ $= 1/4 + 1/4 = 1/2$. Contradiction. Thus d^2 is not a metric.	10 unseen
	Setter's initials [redacted] Checker's initials [redacted]	Page number 2

EXAMINATION SOLUTIONS 2014-15

Course: M2PM5

Question: 2

Marks & seen/unseen

Parts

i

We have to show that any open set G in a metric space can be represented as a union of open balls.

Since G is open, for any $a \in G$ there exists an open ball $B(a) \subset G$. Clearly,

$$G = \bigcup_{a \in G} B(a).$$

4 seen

ii

Let $A = \{x \in M : x = f(x)\}$, let $x \in \overline{A}$. Since M is a metric space, there exists a sequence $x_n \to x$, $x_n \in A$, $n = 1, 2..$

Thus $f(x_n) = x_n \to x$. On the other hand, since f is continuous $f(x_n) \to f(x)$. By unique-

4 unseen

Setter's initials [redacted] Checker's initials [redacted]

	EXAMINATION SOLUTIONS 2014-15	Course M2PM5
Question 2		Marks & seen/unseen
Parts	ness of the limit in metric spaces, $x = f(x)$, so $x \in A$. Thus $A = \overline{A}$, i.e. A is a closed set.	
iii	(a) yes, a homeomorphism is given by $f(x) = \tan(x \frac{\pi}{2})$, $f : (-1,1) \to (-\infty, \infty)$.	4 ~~unseen~~ seen
	(b) no, since the fundamental group of $\mathbb{R}^3 \setminus \{0\}$ is trivial, while that of $\mathbb{R}^2 \setminus \{0\}$ is not.	4 unseen
	(c) no, since only the discrete topology on this set gives a Hausdorff space and Hausdorffness is a topological property.	4 unseen
	Setter's initials [redacted]	Checker's initials [redacted]	Page number 4

EXAMINATION SOLUTIONS 2014-15

Course: M2PM5

Question: 3

Parts | Marks & seen/unseen

i

A topological space is called compact if any open cover of it contains a finite subcover

3 seen

ii (a)

Let $A_n = \{1, 2, \ldots, n\}$, $n = 1, 2, \ldots$

Then the cover $\mathbb{N} = \bigcup_{n=1}^{\infty} A_n$ does not have a finite subcover. So the space is not compact

3 unseen

(b)

Let $\{B_\alpha\}$ be a cover for X, $X = \bigcup_\alpha B_\alpha$, B_α - open sets. Take one of them, B_{α_0}.

$X \setminus B_{\alpha_0} = \{x_1, \ldots, x_n\}$ for some $n \in \mathbb{N}$, by the definition of B_{α_0}

So for some B_α's we have

6 unseen

Setter's initials [redacted] Checker's initials [redacted]

	EXAMINATION SOLUTIONS 2014-15	Course M2PM5
Question 3		Marks & seen/unseen

Parts

$x_1 \in B_{\alpha_1}, \ldots, x_n \in B_{\alpha_n}$.

Therefore $X = \bigcup_{j=0}^{n} B_{\alpha_j}$, so $\{B_{\alpha_j}\}_{j=1}^{n}$ form a finite sub-cover. So X is compact.

(c) Since ℓ_2 is complete, its closed and totally bounded subspace is compact. We will show that K is totally bounded. — 8 unseen / seen similar

Fix $\varepsilon > 0$ Then $\exists N$ s.t.

$\sum_{j=N}^{\infty} x_j^2 \leq \varepsilon \quad \forall x \in K$. For each $x = (x_1, x_2 \ldots) \in K$ let

$x^* = (x_1, x_2, \ldots, x_{N-1}, 0, 0 \ldots)$.

These points form a bounded (since $\sum_{j=1}^{N-1} x_j^2 \leq C \quad \forall x \in K$) subspace K^* of $\mathbb{R}^{N-1}$.

Therefore, K^* is totally bounded. Let S be its finite ε-net, $\forall x^* \in K^*$ $\exists z_k \in S$ s.t. $d_2(x^*, z_k) \leq \varepsilon$.

But then $\forall x \in K$ $\exists z_k \in S$ s.t. $d_2(x, z_k) \leq d_2(x, x^*) + d_2(x^*, z_k) \leq \varepsilon + \varepsilon = 2\varepsilon$

So S is a 2ε-net for K, hence K is totally bounded.

	EXAMINATION SOLUTIONS 2014-15	Course M2PM5
Question 4		Marks & seen/unseen
Parts		
i	<u>Thm</u> (Baire) A complete metric space cannot be represented as a union of a countable number of nowhere dense sets.	5 seen
	<u>Proof</u>. Assume the opposite: M - complete, but $M = \bigcup_{n=1}^{\infty} M_n$, M_n - nowhere dense $\forall n$. Let S_0 be a closed ball of radius 1. Since M_1 is nowhere dense, there exists a closed ball of radius less than $1/2$, S_1 s.t. $S_1 \subset S_0$, $S_1 \cap M_1 = \emptyset$. Since M_2 is nowhere dense, there exists a closed ball S_2 of	10 seen
	Setter's initials [redacted]	Checker's initials [redacted]	Page number 8

radius less than $1/3$ s.t.

$S_2 \subset S_1$, $S_2 \cap M_2 = \emptyset$, and

so on.

We obtain a decreasing sequence of closed balls of radii $r_n < 1/n \to 0$, $n \to \infty$; $S_n \cap M_n = \emptyset$ $\forall n$. By nested balls thm, $\exists\, x \in \bigcap_{n=1}^{\infty} S_n$. By construction, $x \notin M_n$ $\forall n$, therefore $x \notin \bigcup_{n=1}^{\infty} M_n = M$.

Contradiction.

	EXAMINATION SOLUTIONS 2014-15	Course M2PM5
Question 4		Marks & seen/unseen
Parts ii	Let $M, N \subset X$, where X is a metric space and M, N are complete. Therefore both M, N are closed, hence $M \cap N$ is closed. $M \cap N$ is a closed subspace of the complete space M, therefore complete	5 unseen
	Setter's initials [redacted] Checker's initials [redacted]	Page number 10

Imperial College London

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2015

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Probability & Statistics II

Date: Monday, 18 May 2015. Time: 2.00pm – 4.00pm. Time allowed: 2 hours.

This paper has FOUR questions.

Candidates should start their solutions to each question in a new main answer book

Supplementary books may only be used after the relevant main book(s) are full.

Statistical tables are provided on pages 5 & 6.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers to each question as per the table below.

Raw mark	up to 12	13	14	15	16	17	18	19	20
Extra credit	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4

- Each question carries equal weight.
- Calculators may not be used.

1. (a) Suppose $(X_1, \ldots, X_n)$ are independent and identically distributed normal random variables, that is $X_i \overset{\text{iid}}{\sim} \text{Norm}(\mu, \sigma^2)$ for $i = 1, \ldots, n$. Assume that both μ and σ^2 are unknown.

Give a pivot for μ and state the distribution of the pivot.
Derive a $100(1-\alpha)\%$ confidence interval for μ.

Identify the following distributions:

(i) The distribution of

$$V_1 = \left(\frac{X_1 - \mu}{\sigma}\right)^2$$

(ii) The distribution of

$$V_2 = \sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma}\right)^2$$

(iii) The distribution of

$$V_3 = \sum_{i=1}^{n} (X_i - \mu)^2$$

(b) Suppose X_1 and X_2 are independent random variables, such that $X_i \overset{\text{indep}}{\sim} \text{Norm}(0, \sigma_i^2)$.

Identify the distribution of $V_4 = \dfrac{X_1/\sigma_1}{|X_2/\sigma_2|}$.
Give an expression for the probability density function of $V_5 = X_1/|X_2|$.

(c) Suppose each $\boldsymbol{X}_i = (X_{1i}, X_{2i})^{\mathrm{T}}$ is a (2×1) random vector with

$$\boldsymbol{X}_i \overset{\text{indep}}{\sim} \mathrm{N}_2(\boldsymbol{\mu}, \Sigma), \quad \text{for} \quad i = 1, \ldots n,$$

where $\boldsymbol{\mu} = (\mu_1, \mu_2)^{\mathrm{T}}$ is a (2×1) mean vector and $\Sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$ is a (2×2) variance-covariance matrix. Let Φ be the cumulative distribution function of the standard normal distribution and M be the number of the $\boldsymbol{X}_i$ among $(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n)$ for which X_{1i} is greater than X_{2i}.

Give an expression for the joint probability density function of $(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n)$.
Using a theorem from the notes, derive the distribution of $U_i = X_{1i} - X_{2i}$.
Derive $\pi = \Pr(X_{1i} > X_{2i})$. You may express your answer in terms of Φ.
What is the distribution of M? Justify your answer.

2. This question compares frequency and Bayesian estimators in a particular Poisson model.

   (a) Suppose $X|\lambda \sim \textsc{Poisson}(\lambda)$ and $Y|\lambda, \xi \sim \textsc{Poisson}(\lambda\xi)$, with X and Y independent. Derive the maximum likelihood estimates of λ and ξ.
   For simplicity you may ignore the possibility of $x = 0$ and/or $y = 0$.

For the remainder of this problem, suppose $X_i \overset{\text{iid}}{\sim} \textsc{Poisson}(\lambda)$, for $i = 1, \ldots, n$.

   (b) For a Bayesian analysis, assign the prior distribution $\Lambda \sim \textsc{Gamma}(\alpha, \beta)$, where the gamma distribution is parameterized as in the formula sheet.

   Derive the posterior distribution for Λ given $X_1, \ldots, X_n$.
   Give an expression for the posterior mean of Λ. Denote this estimate of λ by $\widehat{\lambda}_1$ and the corresponding estimator of λ by $\widehat{\Lambda}_1$.
   Derive the marginal mean and variance of X_i. (Do not condition on $\Lambda = \lambda$.)
   Derive the correlation between X_i and X_j, for $i \neq j$. (Do not condition on $\Lambda = \lambda$.)

   (c) In this part you will compare the frequency properties of two estimators of λ, specifically $\widehat{\Lambda}_1$ derived in part (b) and $\widehat{\Lambda}_2 = \frac{1}{n}\sum_{i=1}^{n} X_i$. You should treat $X_1, \ldots, X_n$ as random variables and treat λ as a constant. (This does not preclude the use of the estimator, $\widehat{\Lambda}_1$, derived in part (b).)

   Derive the mean square error of $\widehat{\Lambda}_2$.
   Derive the mean square error of $\widehat{\Lambda}_1$. (For simplicity, set $\alpha = 0$ and $\beta = 1$.)
   For what values of λ does $\widehat{\Lambda}_1$ have a smaller mean square error than $\widehat{\Lambda}_2$?

3. Suppose $U_1, U_2, \ldots$ is a sequence of independent random variables that are uniformly distributed on $(0, 1)$. That is, $U_i \overset{\text{iid}}{\sim} \textsc{Uniform}(0, 1)$ for $i = 1, 2, \ldots$. For $n = 1, 2, \ldots$ and for some $\lambda > 0$, define $X_n = \prod_{i=1}^{n} U_i^{\lambda}$ and $Y_n = -\sum_{i=1}^{n} \log(U_i^{\lambda}) = -\log(X_n)$.

   (a) Derive the probability density function of Y_1.
   Use a result from the notes to obtain the distribution of Y_n.
   Construct a Normal Approximation for the distribution of Y_n.

   (b) Derive the probability density function of X_n.
   Derive $\mathrm{E}(X_n)$.

   (c) Prove that $\Pr(X_n \leq x) \geq 1 - (1 - \sqrt[\lambda]{x})^n$ for $0 \leq x \leq 1$.
   Prove that $X_n \xrightarrow{\mathcal{D}} 0$.

4. Suppose that I have a 10p and a 20p coin and that the coins are weighted so that when flipped the 10p coin has probability θ of coming up heads and the 20p coin has probability π of coming up heads. In each of n trials I flip both coins. Let X be the number among the n trials in which the 10p coin comes up heads and let Y be the number among the n trials in which *both* coins come up heads. You may assume (i) the trials are mutually independent, (ii) the outcomes of the two flips in each trial are independent, (iii) n is fixed in advance, and (iv) θ and π do not vary among the n trials.

(a) State the marginal distribution of X.
State the marginal distribution of Y.
State the conditional distribution of Y given X.
Derive the joint probability mass function of X and Y.
Are X and Y independent? Justify your answer.

(b) Derive the conditional probability mass function of X given Y.
Derive the conditional expectation of X given Y.

(c) Derive method of moments estimators of θ and π by setting up a system of two equations, one involving the first (marginal) moment of X and the other involving the first (marginal) moment of Y. Denote your estimators by $\widehat{\Theta}_{\text{MoM}}$ and $\widehat{\Pi}_{\text{MoM}}$.
For what values of X and Y are $\widehat{\Theta}_{\text{MoM}}$ and $\widehat{\Pi}_{\text{MoM}}$ defined?
Is the methods of moments estimator of θ biased? Prove your answer.
Consider the estimator of π given by

$$\widehat{\Pi} = \begin{cases} 0 & \text{if } X = 0 \\ \widehat{\Pi}_{\text{MoM}} & \text{otherwise} \end{cases}.$$

Is $\widehat{\Pi}$ an unbiased estimator or π? Justify your answer.

DISCRETE DISTRIBUTIONS

	range $\mathbb{X}$	parameters	pmf f_X	cdf F_X	$\mathsf{E}[X]$	$\mathsf{Var}[X]$	mgf M_X
$Bernoulli(\theta)$	$\{0,1\}$	$\theta \in (0,1)$	$\theta^x(1-\theta)^{1-x}$		θ	$\theta(1-\theta)$	$1-\theta+\theta e^t$
$Binomial(n,\theta)$	$\{0,1,...,n\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{n}{x}\theta^x(1-\theta)^{n-x}$		$n\theta$	$n\theta(1-\theta)$	$(1-\theta+\theta e^t)^n$
$Poisson(\lambda)$	$\{0,1,2,...\}$	$\lambda \in \mathbb{R}^+$	$\dfrac{e^{-\lambda}\lambda^x}{x!}$		λ	λ	$\exp\{\lambda(e^t-1)\}$
$Geometric(\theta)$	$\{1,2,...\}$	$\theta \in (0,1)$	$(1-\theta)^{x-1}\theta$	$1-(1-\theta)^x$	$\dfrac{1}{\theta}$	$\dfrac{(1-\theta)}{\theta^2}$	$\dfrac{\theta e^t}{1-e^t(1-\theta)}$
$NegBinomial(n,\theta)$	$\{n, n+1, ...\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{x-1}{n-1}\theta^n(1-\theta)^{x-n}$		$\dfrac{n}{\theta}$	$\dfrac{n(1-\theta)}{\theta^2}$	$\left(\dfrac{\theta e^t}{1-e^t(1-\theta)}\right)^n$
or	$\{0,1,2,...\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{n+x-1}{x}\theta^n(1-\theta)^x$		$\dfrac{n(1-\theta)}{\theta}$	$\dfrac{n(1-\theta)}{\theta^2}$	$\left(\dfrac{\theta}{1-e^t(1-\theta)}\right)^n$

The *gamma function* is given by $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx$.

The location/scale transformation $Y = \mu + \sigma X$ gives

$$f_Y(y) = f_X\left(\frac{y-\mu}{\sigma}\right)\frac{1}{\sigma} \qquad F_Y(y) = F_X\left(\frac{y-\mu}{\sigma}\right) \qquad M_Y(t) = e^{\mu t}M_X(\sigma t) \qquad \mathsf{E}[Y] = \mu + \sigma\mathsf{E}[X] \qquad \mathsf{Var}[Y] = \sigma^2\mathsf{Var}[X]$$

CONTINUOUS DISTRIBUTIONS

		parameters	*pdf*	*cdf*	E[X]	Var[X]	*mgf*
$Uniform(\alpha,\beta)$ (stand. model $\alpha=0,\beta=1$)	(α,β)	$\alpha<\beta\in\mathbb{R}$	$\dfrac{1}{\beta-\alpha}$	$\dfrac{x-\alpha}{\beta-\alpha}$	$\dfrac{(\alpha+\beta)}{2}$	$\dfrac{(\beta-\alpha)^2}{12}$	$\dfrac{e^{\beta t}-e^{\alpha t}}{t(\beta-\alpha)}$
$Exponential(\lambda)$ (stand. model $\lambda=1$)	$\mathbb{R}^+$	$\lambda\in\mathbb{R}^+$	$\lambda e^{-\lambda x}$	$1-e^{-\lambda x}$	$\dfrac{1}{\lambda}$	$\dfrac{1}{\lambda^2}$	$\left(\dfrac{\lambda}{\lambda-t}\right)$
$Gamma(\alpha,\beta)$ (stand. model $\beta=1$)	$\mathbb{R}^+$	$\alpha,\beta\in\mathbb{R}^+$	$\dfrac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}$		$\dfrac{\alpha}{\beta}$	$\dfrac{\alpha}{\beta^2}$	$\left(\dfrac{\beta}{\beta-t}\right)^\alpha$
$Weibull(\alpha,\beta)$ (stand. model $\beta=1$)	$\mathbb{R}^+$	$\alpha,\beta\in\mathbb{R}^+$	$\alpha\beta x^{\alpha-1}e^{-\beta x^\alpha}$	$1-e^{-\beta x^\alpha}$	$\dfrac{\Gamma(1+1/\alpha)}{\beta^{1/\alpha}}$	$\dfrac{\Gamma\left(1+\frac{2}{\alpha}\right)-\Gamma\left(1+\frac{1}{\alpha}\right)^2}{\beta^{2/\alpha}}$	
$Normal(\mu,\sigma^2)$ (stand. model $\mu=0,\sigma=1$)	$\mathbb{R}$	$\mu\in\mathbb{R},\sigma\in\mathbb{R}^+$	$\dfrac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\dfrac{(x-\mu)^2}{2\sigma^2}\right\}$		μ	σ^2	$e^{\{\mu t+\sigma^2t^2/2\}}$
$Student(\nu)$	$\mathbb{R}$	$\nu\in\mathbb{R}^+$	$\dfrac{(\pi\nu)^{-\frac{1}{2}}\Gamma\left(\dfrac{\nu+1}{2}\right)}{\Gamma\left(\dfrac{\nu}{2}\right)\left\{1+\dfrac{x^2}{\nu}\right\}^{(\nu+1)/2}}$		0 (if $\nu>1$)	$\dfrac{\nu}{\nu-2}$ (if $\nu>2$)	
$Pareto(\theta,\alpha)$	$\mathbb{R}^+$	$\theta,\alpha\in\mathbb{R}^+$	$\dfrac{\alpha\theta^\alpha}{(\theta+x)^{\alpha+1}}$	$1-\left(\dfrac{\theta}{\theta+x}\right)^\alpha$	$\dfrac{\theta}{\alpha-1}$ (if $\alpha>1$)	$\dfrac{\alpha\theta^2}{(\alpha-1)^2(\alpha-2)}$ (if $\alpha>2$)	
$Beta(\alpha,\beta)$	$(0,1)$	$\alpha,\beta\in\mathbb{R}^+$	$\dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$		$\dfrac{\alpha}{\alpha+\beta}$	$\dfrac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	

M2S1 — May 2015 Exam — Solution

1. (a) *[Seen]* Let $T = \dfrac{\bar{X} - \mu}{S/\sqrt{n}}$, where $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $S = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2}$.
$T \sim t_{n-1}$ is a pivot for μ.

Let F_T be the cumulative distribution function of the t_{n-1} distribution and $F_T(t_{\alpha/2}) = 1 - \dfrac{\alpha}{2}$, then

$$\Pr\left(-t_{\alpha/2} \ \le\ \frac{\bar{X} - \mu}{S/\sqrt{n}} \ \le\ t_{\alpha/2}\right) = 100(1-\alpha)\%,$$

so that

$$\Pr\left(-\frac{t_{\alpha/2}S}{\sqrt{n}} - \bar{X} \ \le\ -\mu \ \le\ \frac{t_{\alpha/2}S}{\sqrt{n}} - \bar{X}\right) = 100(1-\alpha)\%,$$

and

$$\Pr\left(\bar{X} - \frac{t_{\alpha/2}S}{\sqrt{n}} \ \le\ \mu \ \le\ \bar{X} + \frac{t_{\alpha/2}S}{\sqrt{n}}\right) = 100(1-\alpha)\%,$$

i.e., $\bar{X} \pm \dfrac{S}{\sqrt{n}} t_{\alpha/2}$ is a $100(1-\alpha)\%$ CI for μ.

The distributions are:

(i) *[Seen]* V_1 is the square of a standard normal, thus $V_1 \sim \chi^2_1$, or $V_1 \sim \text{GAMMA}\left(\frac{1}{2}, \frac{1}{2}\right)$.

(ii) *[Seen]* V_2 is the sum of n independent χ^2_1 random variables, thus $V_2 \sim \chi^2_n$, or $V_2 \sim \text{GAMMA}\left(\frac{n}{2}, \frac{1}{2}\right)$.

(iii) *[Seen Similar]* $V_3 = \sigma^2 V_2$ and identifying the scale parameter of the gamma distribution, $V_3 \sim \text{GAMMA}\left(\frac{n}{2}, \frac{1}{2\sigma^2}\right)$.

[8 marks: One mark each for stating (i) the pivot, (ii) its distribution, and the distributions of (iii) V_1, (iv) V_2, and (v) V_3. Three marks for deriving the confidence interval.]

(b) *[Seen]* V_4 is a standard normal random variable divided by an independent $\sqrt{\chi^2_\nu/\nu}$, with $\nu = 1$, thus $V_4 \sim t_1$, or $V_4 \sim \text{CAUCHY}$.

[Seen Similar] Because $V_5 = \sigma_1 V_4/\sigma_2$ we can express its density function by identifying σ_1/σ_2 as a scale parameter,

$$f_{V_5}(v) = \frac{\sigma_2}{\sigma_1} f_{V_4}\left(\frac{\sigma_2 v}{\sigma_1}\right) = \frac{\sigma_2}{\sigma_1 \pi\left(1 + \left(\frac{\sigma_2}{\sigma_1}v\right)^2\right)} \quad \text{for} \quad -\infty < v < +\infty$$

[3 marks: One mark each for (i) stating the distribution of V_4, (ii) implicitly using the Cauchy density, and (iii) giving the final pdf of V_5.]

(c) *[Seen]* The joint density function can be written in either of two ways: (i) in scalar notation,

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = \left(\frac{1}{2\pi\sqrt{\sigma_1^2\sigma_2^2(1-\rho^2)}}\right)^n \exp\left\{-\frac{1}{2(1-\rho^2)}\sum_{i=1}^{n}\left[\left(\frac{x_{1i}-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_{1i}-\mu_1}{\sigma_1}\right)\left(\frac{x_{2i}-\mu_2}{\sigma_2}\right) + \left(\frac{x_{2i}-\mu_2}{\sigma_2}\right)^2\right]\right\},$$

where $\boldsymbol{X} = (\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n)$ and $\boldsymbol{x} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$ with $\boldsymbol{x}_i = (x_{1i}, x_{2i})^{\mathrm{T}}$
or (ii) more succinctly in matrix notation

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = \left(\frac{1}{2\pi|\Sigma|^{1/2}}\right)^n \exp\left\{-\frac{1}{2}\sum_{i=1}^{n}(\boldsymbol{x}_i - \boldsymbol{\mu})^T\Sigma^{-1}(\boldsymbol{x}_i - \boldsymbol{\mu})\right\},$$

for (in both cases) $-\infty < x_{1i}, x_{2i} < +\infty$.

[Seen Similar] We know that $B\boldsymbol{X}_i \sim \mathrm{N}(B\boldsymbol{\mu}, B\Sigma B^{\mathrm{T}})$. Setting the first row of B to $(1, -1)$, this implies that $U_i = X_{1i} - X_{2i} \sim \text{Norm}(\mu_1 - \mu_2,\ \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2)$.

[Seen Similar] Letting Z be a standard normal random variable,

$$\begin{aligned}\pi &= \Pr(X_{1i} > X_{2i}) = \Pr(U_i > 0) = \Pr\left(Z > \frac{0 - (\mu_1 - \mu_2)}{\sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}}\right)\\ &= 1 - \Phi\left(\frac{\mu_2 - \mu_1}{\sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}}\right) = \Phi\left(\frac{\mu_1 - \mu_2}{\sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}}\right).\end{aligned}$$

(Either of the last two expressions is acceptable.)

[Seen Similar] Because

(i) there is a fixed number of X_i, i.e., n;

(ii) for each i, either X_{i1} is greater than X_{i2} or it is not;

(iii) the probability that X_{i1} is greater than X_{i2} is the same for each i; and

(iv) the occurrences of X_{i1} being greater than X_{i2} are independent among the X_i

we know $M \sim \text{BINOMIAL}(n, \pi)$.

[9 marks: Two marks each for (i) the joint pdf of X and (ii) deriving π. One mark each for (i) justifying that the distribution of U is normal, (ii) computing its mean, (iii) computing its variance, (iv) stating the distribution of M, and (v) justifying the distribution of M.]

2. (a) *[Seen]* The likelihood function is $L(\lambda, \xi \mid x, y) = \dfrac{e^{-\lambda}\lambda^x}{x!}\dfrac{e^{-\lambda\xi}(\lambda\xi)^y}{y!}$ and the loglikelihood function is $\ell(\lambda, \xi \mid x, y) = -\lambda(1+\xi) + (x+y)\log(\lambda) + y\log(\xi)$.

To find a candidate value for the maximum likelihood estimate, set

$$\frac{\partial \ell}{\partial \lambda} = -(1+\xi) + \frac{x+y}{\lambda} = 0 \quad \text{and} \quad \frac{\partial \ell}{\partial \xi} = -\lambda + \frac{y}{\xi} = 0.$$

Solving these equations yields candidate estimates $\hat{\lambda}_{\text{MLE}} = x$ and $\hat{\xi}_{\text{MLE}} = y/x$ (since $x > 0$).

To verify that the candidate estimates indeed maximize the log likelihood function, we note

$$\frac{\partial^2 \ell}{\partial \lambda^2} = -\frac{x+y}{\lambda^2}, \quad \frac{\partial^2 \ell}{\partial \xi^2} = -\frac{y}{\xi^2}, \quad \text{and} \quad \frac{\partial^2 \ell}{\partial \lambda \cdot \partial \xi} = -1.$$

Evaluating at the candidate estimates,

$$\left.\frac{\partial^2 \ell}{\partial \lambda^2}\right|_{\hat{\lambda},\hat{\xi}} = -\frac{x+y}{x^2} \le 0, \quad \left.\frac{\partial^2 \ell}{\partial \xi^2}\right|_{\hat{\lambda},\hat{\xi}} = -\frac{x^2}{y} \le 0, \quad \text{and} \quad \left.\frac{\partial^2 \ell}{\partial \lambda \cdot \partial \xi}\right|_{\hat{\lambda},\hat{\xi}} = -1.$$

Because the inequalities are strict when $x, y > 0$, this verifies two of the three conditions for the two-dimensional second derivative test. The third is

$$\left[\frac{\partial^2 \ell}{\partial \lambda^2} \cdot \frac{\partial^2 \ell}{\partial \xi^2} - \left(\frac{\partial^2 \ell}{\partial \lambda \cdot \partial \xi}\right)^2\right]_{\lambda=\hat{\lambda}, \xi=\hat{\xi}} = \frac{x+y}{y} - 1 = \frac{x}{y} \ge 0.$$

(If $y = 0$, $\partial\ell/\partial\xi$ is everywhere negative and the estimate of ξ is zero.) Thus, the maximum likelihood estimates are $\hat{\lambda}_{\text{MLE}} = x$ and $\hat{\xi}_{\text{MLE}} = x/y$ if $x > 0$. (The estimate of ξ is not defined if $x = 0$.)

[7 marks: One mark each for (i) the likelihood function, (ii) the loglikelihood function, (iii)-(iv) each of the derivatives of the loglikelihood function, (iv) the candidate estimates, (vi), the matrix of partial derivatives (vii) checking the signs of the diagonal terms and the determinant of the matrix of partial derivatives.]

(b) *[Seen]* Setting $\boldsymbol{X} = (X_1, \ldots, X_n)$, the posterior distribution is

$$f_{\Lambda|\boldsymbol{X}}(\lambda) \propto \prod_{i=1}^{n} f_{X_i|\Lambda}(x_i|\lambda) f_\Lambda(\lambda) = \frac{e^{-n\lambda}\lambda^{\sum_{i=1}^n x_i}}{\prod_{i=1}^n x_i!}\frac{\beta^\alpha}{\Gamma(\alpha)}\lambda^{\alpha-1}e^{-\beta\lambda} \propto \lambda^{\sum_{i=1}^n x_i+\alpha-1}e^{-(\beta+n)\lambda},$$

i.e., $\Lambda|X_1, \ldots, X_n \sim \text{GAMMA}(\alpha + \sum_{i=1}^n X_i, \beta + n)$.

The posterior expectation is $\hat{\lambda}_1 = (\alpha + \sum_{i=1}^n x_i)/(\beta+n)$. The corresponding estimator is $\widehat{\Lambda}_1 = (\alpha + \sum_{i=1}^n X_i)/(\beta+n)$.

The marginal moments of X_i can be derived as *[Seen]*

$$\text{E}(X_i) = \text{E}\Big[\text{E}(X_i|\Lambda)\Big] = E(\Lambda) = \alpha/\beta,$$

$$\text{Var}(X_i) = \text{E}\Big[\text{Var}(X_i|\Lambda)\Big] + \text{Var}\Big[\text{E}(X_i|\Lambda)\Big] = E(\Lambda) + \text{Var}(\Lambda) = \alpha(\beta+1)/\beta^2,$$

[Unseen]

$$\mathrm{E}(X_iX_j) = \mathrm{E}\Big[\mathrm{E}(X_iX_j|\Lambda)\Big] = \mathrm{E}(\Lambda^2) = \mathrm{Var}(\Lambda) + [\mathrm{E}(\Lambda)]^2 = \alpha(\alpha+1)/\beta^2.$$

and

$$\mathrm{Corr}(X_i, X_j) = \frac{\mathrm{Cov}(X_iX_j)}{\sqrt{\mathrm{Var}(X_i)\mathrm{Var}(X_j)}} = \frac{\mathrm{E}(X_iX_j) - \mathrm{E}(X_i)\mathrm{E}(X_j)}{\sqrt{\mathrm{Var}(X_i)\mathrm{Var}(X_j)}} = \frac{\alpha(\alpha+1)-\alpha^2}{\alpha(\beta+1)} = \frac{1}{\beta+1}.$$

[7 marks: One mark each for (i) the general formula for a posterior distribution, (ii) the correct gamma posterior distribution, (iii) the posterior expectation, (iv) $\mathrm{E}(X_i)$, *(v)* $\mathrm{Var}(X_i)$, *(vi)* $\mathrm{E}(X_iX_j)$, *and (vii)* $\mathrm{Corr}(X_i, X_j)$.*]*

(c) *[Seen Similar]* The mean square error of $\widehat{\Lambda}_2$ can be computed as

$$\mathrm{mse}(\widehat{\Lambda}_2) = \Big(\mathrm{bias}(\widehat{\Lambda}_2)\Big)^2 + \mathrm{Var}(\widehat{\Lambda}_2) = \Big(\mathrm{E}(\widehat{\Lambda}_2) - \lambda\Big)^2 + \mathrm{Var}(\widehat{\Lambda}_2) = \mathrm{Var}(\widehat{\Lambda}_2) = \lambda/n.$$

[Unseen] Setting $\alpha = 0$ and $\beta = 1$, the mean square error of $\widehat{\Lambda}_1$ can be computed as

$$\mathrm{mse}(\widehat{\Lambda}_1) = \Big(\mathrm{E}(\widehat{\Lambda}_1) - \lambda\Big)^2 + \mathrm{Var}(\widehat{\Lambda}_1) = \left(\frac{n\lambda}{n+1} - \lambda\right)^2 + \mathrm{Var}\left(\frac{\sum_{i=1}^n X_i}{n+1}\right) = \frac{(\lambda^2+n\lambda)}{(n+1)^2} = \lambda\frac{(n+\lambda)}{(n+1)^2}.$$

[Unseen] For $\mathrm{mse}(\widehat{\Lambda}_1) < \mathrm{mse}(\widehat{\Lambda}_2)$, we need $(n+1)^2/(n+\lambda) > n$, i.e., $\lambda < 2 + \frac{1}{n}$.

[6 marks: 2 marks each for (i) mse of $\widehat{\Lambda}_2$, *(ii)* $\widehat{\Lambda}_1$, *and (iii) range of* λ *with* $\mathrm{mse}(\widehat{\Lambda}_1) < \mathrm{mse}(\widehat{\Lambda}_2)$.*]*

3. (a) *[Seen Similar]* Inverting the transformation, $Y_1 = -\log(U_1^\lambda)$, yields $U_1 = \exp(-Y_1/\lambda)$ and the Jacobian, $J = -\frac{1}{\lambda}\exp(-y/\lambda)$. Thus the density of Y_1 is

$$f_{Y_1}(y) = f_{U_1}(e^{-y/\lambda})|J| = \frac{1}{\lambda}\exp(-y/\lambda) \quad \text{for} \quad 0 < y < +\infty.$$

[Seen] Recognizing this probability density function, $Y_1 \sim \textsc{Exponential}(1/\lambda)$. Since the U_i are independent and identically distributed, we have $-\log(U_i^\lambda) \overset{\text{iid}}{\sim} \textsc{Exponential}(1/\lambda)$ and thus $Y_n = -\sum_{i=1}^n \log(U_i^\lambda) \sim \textsc{Gamma}(n, 1/\lambda)$.

[Seen Similar] For the normal approximation we use the fact that $Y_n = \sum_{i=1}^n V_i$ where $V_i \sim \textsc{Exponential}(1/\lambda)$ and apply the Central Limit Theorem with $\mu = \mathrm{E}(V_i) = \lambda$ and $\sigma^2 = \mathrm{Var}(V_i) = \lambda^2$. This yields,

$$Z_n = \frac{\sum_{i=1}^n V_i - n\lambda}{\lambda\sqrt{n}} \xrightarrow{\mathcal{D}} \mathrm{Norm}(0,1) \text{ and hence for large } n,\ Y_n \overset{\text{approx}}{\sim} \mathrm{Norm}\left(n\lambda, n\lambda^2\right).$$

[7 marks: Two marks for the change of variable calculations and one mark each for (i) the final pdf of Y_1, *(ii) stating the distribution of* Y_n, *(iii) justification for stated distribution of* Y_n, *(iv) correctly applying the central limit theorem, and (v) the final normal approximation for* Y_n.*]*

(b) *[Seen Similar]* We know $X_n = \exp(-Y_n)$ and that $Y_n \sim \textsc{Gamma}(n, 1/\lambda)$. Inverting this transformation, we have $Y_n = -\log(X_n)$ and Jacobian, $J = -1/x$ so that

$$f_{X_n}(x) = f_{Y_n}(-\log x)|J| = \frac{1}{\lambda^n\Gamma(n)}(-\log x)^{n-1}x^{\frac{1}{\lambda}-1} \quad \text{for} \quad 0 < x < 1,$$

and *[Seen Similar]*

$$\begin{aligned}\mathrm{E}(X_n) = \int_0^1 v f_V(v)\mathrm{d}v &= -\int_0^1 \frac{1}{\lambda^n\Gamma(n)}(-\log x)^{n-1}x^{(\frac{1}{\lambda}+1)-1}\mathrm{d}x \\ &= \frac{(\frac{1}{\lambda}+1)^{-n}}{\lambda^n}\int_0^1 \frac{(\frac{1}{\lambda}+1)^n}{\Gamma(n)}(-\log x)^{n-1}x^{(\frac{1}{\lambda}+1)-1}\mathrm{d}x = \frac{1}{(\lambda+1)^n},\end{aligned}$$

where the second integral is one because it is a density (of the form as $f_{X_n}(x)$) integrated over its support.

[6 marks: Two marks for the change of variable calculations. One mark each for (i) the final pdf of X_n, *(ii) setting up the integral for* $\mathrm{E}(X_n)$ *including the range of integration, (iii) recognizing the pdf in the integrand, and (iv) the final expression for* $\mathrm{E}(X_n)$.*]*

(c) *[Unseen]* By construction $X_n \leq \min_{i=1}^n U_i^\lambda$. Thus, $U_i^\lambda \leq x$ for any i in $(1, 2, \ldots, n)$ implies that $X_n \leq x$ and

$$\Pr(X_n \leq x) \geq \Pr\left(\bigcup_{i=1}^n \{U_i^\lambda \leq x\}\right) = 1 - \Pr\left(\bigcap_{i=1}^n \{U_i \geq \sqrt[\lambda]{x}\}\right) = 1 - \left(1 - \sqrt[\lambda]{x}\right)^n,$$

where the inequality follows from a basic property of probability (i.e., if A implies B, then $\Pr(A) \leq \Pr(B)$), the first equality follows from De Morgan's Law, and the second equality holds for any $0 \leq x \leq 1$.

[Unseen] For any n and $0 \leq x \leq 1$, we have $1 - \left(1 - \sqrt[\lambda]{x}\right)^n \leq \Pr(X_n \leq x) \leq 1$. Taking the limit as $n \to \infty$ yields $\lim_{n\to\infty} \Pr(X_n \leq x) = 1$ for $0 \leq x \leq 1$. Whereas $\Pr(X_n \leq x) = 0$ for $x < 0$ and $\Pr(X_n \leq x) = 1$ for any $x > 1$ we have that $X_n \xrightarrow{\mathcal{D}} 0$ (i.e., X_n is degenerate at zero).

[7 marks: Two marks for the general outline of the first proof; one mark for the detailed justification for the first proof; two marks for convergence of $\Pr(X_n \leq x)$ *for* $0 \leq x \leq 1$*; one mark for convergence for other* x*; and one mark for establishing convergence in distribution.]*

4. (a) *[Seen]* $X \sim \textsc{Binomial}(n, \theta)$; *[Seen Similar]* $Y \sim \textsc{Binomial}(n, \theta\pi)$; and *[Unseen]* $Y \mid X = x \sim \textsc{Binomial}(x, \pi)$.

[Seen Similar] The joint probability mass function is given by

$$f_{XY}(x, y) = \binom{n}{x}\theta^x(1-\theta)^{n-x}\binom{x}{y}\pi^y(1-\pi)^{x-y} = \frac{n!}{(n-x)!(x-y)!y!}(1-\theta)^{n-x}(\theta-\theta\pi)^{x-y}(\theta\pi)^y$$

for integer x and y such that $0 \leq y \leq x \leq n$.

[Seen Similar] X and Y are not independent because their joint support is not the cross product of their marginal supports and their joint mass function does not factor.

[7 marks: One mark each for (i) the marginal distribution of X*, (ii) the marginal distribution of* Y*, (iii) the conditional distribution of* Y *given* X*, (iv) the joint mass function, (v) its support, (vi) noting that* X *and* Y *are not independent, and (vii) justification this.]*

(b) *[Seen Similar]* Since $f_{X|Y}(x|y) = f_{XY}(x,y)/f_Y(y)$, the conditional probability mass function can be written

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{n!}{(n-x)!(x-y)!y!}\,\frac{y!(n-y)!}{n!}\,\frac{(1-\theta)^{n-x}(\theta-\theta\pi)^{x-y}(\theta\pi)^y}{(\theta\pi)^y(1-\theta\pi)^{n-y}} \\ &= \frac{(n-y)!}{(n-x)!(x-y)!}\left(\frac{1-\theta}{1-\theta\pi}\right)^{n-x}\left(\frac{\theta-\theta\pi}{1-\theta\pi}\right)^{x-y} \quad \text{for } y \leq x \leq n, \end{aligned}$$

[Seen Similar] Alternatively, since $f_{X|Y}(x|y) \propto f_{XY}(x,y)$, the conditional mass function can be written

$$f_{X|Y}(x|y) = \frac{k}{(n-x)!(x-y)!}(1-\theta)^{n-x}(\theta-\theta\pi)^{x-y} \quad \text{for } y \leq x \leq n,$$

for some positive k. Letting $z = x - y$,

$$\begin{aligned} \frac{1}{k} &= \sum_{z=0}^{n-y}\frac{1}{(n-y-z)!z!}(1-\theta)^{n-y-z}(\theta-\theta\pi)^z \\ &= \frac{(1-\theta\pi)^{n-y}}{(n-y)!}\sum_{z=0}^{n-y}\frac{(n-y)!}{(n-y-z)!z!}\left(\frac{1-\theta}{1-\theta\pi}\right)^{n-y-z}\left(\frac{\theta-\theta\pi}{1-\theta\pi}\right)^z = \frac{(1-\theta\pi)^{n-y}}{(n-y)!} \end{aligned}$$

and

$$f_{X|Y}(x|y) = \frac{(n-y)!}{(n-x)!(x-y)!}\left(\frac{1-\theta}{1-\theta\pi}\right)^{n-x}\left(\frac{\theta-\theta\pi}{1-\theta\pi}\right)^{x-y} \quad \text{for } y \leq x \leq n.$$

[Unseen] Noting that $X|Y \sim Y + \textsc{Binomial}\left(n - Y, \dfrac{\theta-\theta\pi}{1-\theta\pi}\right)$, we have $\mathrm{E}(X|Y) = Y + \dfrac{(n-Y)(\theta-\theta\pi)}{1-\theta\pi}$.

[Seen Similar] Alternatively, the conditional expectation may be computed directly, again letting $z = x - y$.

$$\begin{aligned}
\mathrm{E}(X|Y=y) &= \sum_{z=0}^{n-y} \frac{(n-y)!(z+y)}{(n-y-z)!z!}\left(\frac{1-\theta}{1-\theta\pi}\right)^{n-y-z}\left(\frac{\theta-\theta\pi}{1-\theta\pi}\right)^{z} \\
&= y\sum_{z=0}^{n-y} \frac{(n-y)!}{(n-y-z)!z!}\left(\frac{1-\theta}{1-\theta\pi}\right)^{n-y-z}\left(\frac{\theta-\theta\pi}{1-\theta\pi}\right)^{z} + \sum_{z=0}^{n-y} \frac{(n-y)!z}{(n-y-z)!z!}\left(\frac{1-\theta}{1-\theta\pi}\right)^{n-y-z}\left(\frac{\theta-\theta\pi}{1-\theta\pi}\right)^{z} \\
&= y + \frac{(n-y)(\theta-\theta\pi)}{1-\theta\pi}.
\end{aligned}$$

[6 marks: One mark each for (i) knowing a method for computing a conditional distribution, (ii) the correctly normalized probability mass function, (iii) the support of the mass function, and (iv) correct formulae for the conditional expectation. Two marks for deriving the conditional expectation.]

(c) *[Unseen]* From the marginal distributions of X and Y, we know $\mathrm{E}(X) = n\theta$ and $\mathrm{E}(Y) = n\theta\pi$, so we can set up a system of two equations: (i) $x = n\theta$ and (ii) $y = n\theta\pi$. Solving yields the method of moments estimates, $\hat{\theta} = x/n$ and $\hat{\pi} = y/x$; although $\hat{\theta}$ is always defined, $\hat{\pi}$ only exists if $x > 0$. Hence the method of moments estimators are $\widehat{\Theta}_{\mathrm{MoM}} = X/n$ and $\widehat{\Pi}_{\mathrm{MoM}} = Y/X$, if $X > 0$.

[Seen Similar] The estimator, $\widehat{\Theta}_{\mathrm{MoM}}$ is unbiased for θ because $\mathrm{E}(\widehat{\Theta}_{\mathrm{MoM}}) = \mathrm{E}(X/n) = n\theta/n = \theta$.

[Unseen] The expectation of $\widehat{\Pi}$ is

$$\begin{aligned}
\mathrm{E}(\widehat{\Pi}) = \mathrm{E}\Big[\mathrm{E}(\widehat{\Pi}|X)\Big] &= \mathrm{E}\left[\mathrm{E}\left(0I\{X=0\} + \frac{Y}{X}I\{X>0\} \;\Big|\; X\right)\right] = \mathrm{E}\left(\frac{X\pi}{X}I\{X>0\}\right) \\
&= \pi\Big[\Pr(X>0)\Big] < \pi.
\end{aligned}$$

Thus $\widehat{\Pi}$ is a biased estimator of π.

[7 marks: One mark each for (i) the system of equations, (ii) the method of moments estimators, (iii) noting that the estimate (or estimator) of π does not exist if $x = 0$ (or $X = 0$), (iv) arguing that $\widehat{\Theta}_{\mathrm{MoM}}$ is unbiased, and (v) stating that $\widehat{\Pi}$ is biased. Two marks for the expectation of $\widehat{\Pi}$.]

Imperial College
London

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2015

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Statistical Modelling I

Date: Wednesday, 20 May 2015. Time: 2.00pm – 4.00pm. Time allowed: 2 hours.

This paper has FOUR questions.

Candidates should start their solutions to each question in a new main answer book

Supplementary books may only be used after the relevant main

Statistical tables will not be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers to each question as per the table below.

Raw mark	up to 12	13	14	15	16	17	18	19	20
Extra credit	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4

- Each question carries equal weight.
- Calculators may not be used.

1. Consider the model $X \sim$ Binomial(n,p), where n is a known positive integer and $p \in [0,1]$ is an unknown parameter.
Recall: The probability mass function (pmf) of a random variable $Z \sim Binomial(n,p)$ is $f(z) = \binom{n}{z}p^z(1-p)^{n-z}$, where $z \in \{0,1,...,n\}$. $E(Z) = np$ and $Var(Z) = np(1-p)$.

(a) Consider $S = \frac{X}{n}$ as an estimator for the unknown parameter p.

(i) Define the bias of an estimator.

(ii) Show that S is an unbiased estimator for p.

(iii) Calculate the variance of the estimator S, and hence write down the mean squared error of S.

(b) Now consider an alternative estimator, $T = \frac{X+1}{n+2}$.

(i) Calculate the bias of T.

(ii) Calculate the variance of T.

(c) Compare the mean squared error of these two estimators, S and T. Comment on their relative performance for different values of p.

2. Let $Y_1, ..., Y_n \sim$ Exp(λ) independently for some unknown parameter $\lambda > 0$.
Recall: The probability density function (pdf) of a random variable $Z \sim Exp(\lambda)$ is $f(z) = \lambda \exp(-\lambda z)$ for $z > 0$, $\lambda > 0$.

(a) Derive the maximum likelihood estimator for λ.

(b) Calculate the large sample properties of this maximum likelihood estimator.

(c) Write down an asymptotic 95% confidence interval for λ.

(d) We could also employ a Bayesian approach for estimating λ, in which case we must define a prior distribution for the unknown parameter. It is often convenient to choose one that is conjugate to the likelihood.

(i) Define the term *conjugate prior*.

(ii) Show that the gamma distribution is a conjugate prior for an exponential likelihood.
Recall: The probability density function (pdf) of a random variable $Z \sim Gamma(\alpha, \beta)$ is $f(z) = \frac{\beta^\alpha}{\Gamma(\alpha)} z^{\alpha-1} \exp(-\beta z)$ for $z > 0$, $\alpha > 0$, $\beta > 0$.

3. (a) Write down the general form of a linear model (using matrix notation) and fully describe each term in the equation.

   (b) The least squares estimator for a linear model is given by $\widehat{\beta} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{Y}$.

      (i) Write down an expression for the vector of fitted values, $\widehat{\mathbf{Y}}$.

      (ii) Define the term *projection matrix*.

      (iii) Show how the vector of fitted values may be written as a projection of the original data vector, $\mathbf{Y}$, and prove that this matrix does indeed satisfy the required properties of a projection matrix.

   (c) (i) State the Gauss Markov Theorem.

      (ii) Give a specific and concrete example of a modelling problem that could be tackled with a linear model. State clearly what you are trying to estimate and describe whether or not the Gauss Markov Theorem would influence your choice of estimator.

4. (a) Define the non-central t-distribution.

   (b) In this part we consider the linear model you defined in question 3 part (a) and assume the Normal Theory Assumptions.

      (i) Calculate the expected value and variance of the maximum likelihood estimator of some linear combination of the parameters, i.e. $\mathrm{E}(\mathbf{c}^T\widehat{\beta})$ and $\mathrm{Var}(\mathbf{c}^T\widehat{\beta})$, for some deterministic column vector $\mathbf{c}$.

      (ii) Show that $\mathrm{RSS} = \mathbf{Y}^T\mathbf{Q}\mathbf{Y}$.
      Recall: RSS is the residual sum of squares, $\mathbf{Y}$ is the measurement vector and $\mathbf{Q}$ is the projection onto the complement of the space spanned by the columns of the design matrix.

      (iii) State what distribution the following test statistic has, and sketch out how you would prove this.

$$\frac{\mathbf{c}^T\widehat{\beta} - \mathbf{c}^T\beta}{\sqrt{\mathbf{c}^T(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{c}\frac{\mathrm{RSS}}{n-p}}}$$

      Recall: $\frac{RSS}{\sigma^2} \sim \chi^2_{n-p}$, where p is the rank of the design matrix.

   (c) Define the non-central F-distribution.

   (d) Explain how the Fisher-Cochran Theorem can be used to prove that a test statistic has a non-central F distribution.

Imperial College London

IMPERIAL COLLEGE LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

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This paper is also taken for the relevant examination for the Associateship.

M2S2

Statistical Modelling (Solutions)

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1. (a) $\boxed{\text{seen} \Downarrow}$

(i) $\text{bias}_\theta(S) = \mathrm{E}_\theta(S) - \theta$. $\boxed{2}$

(ii) $\forall p, \text{bias}_p(S) = \mathsf{E}_p(S - p) = \frac{1}{n}\mathsf{E}(X) - p = 0$. Thus, S is unbiased for p. $\boxed{2}$

(iii) $\text{Var}_p(S) = \frac{1}{n^2}\text{Var}_p(X) = \frac{p(1-p)}{n}$

$\text{MSE}_p(S) = \text{Var}(S) + \text{bias}(S)^2 = \frac{p(1-p)}{n}$ $\boxed{4}$

(b) *This is an example in the lecture notes.*

(i) $\text{bias}_p(T) = \mathsf{E}_p(T - p) = \frac{\mathsf{E}_p(X)+1}{n+2} - p = \frac{np+1}{n+2} - p = \frac{1-2p}{n+2} \neq 0$ $\boxed{3}$

(ii) $\text{Var}_p(T) = \frac{1}{(n+2)^2}\text{Var}_p(X) = \frac{np(1-p)}{(n+2)^2}$ $\boxed{3}$

(c) *This is an example in the lecture notes.*

$\text{MSE}_p(T) = \text{Var}_p(T) + \text{bias}_p(T)^2 = \frac{np(1-p)}{(n+2)^2} + \frac{(1-2p)^2}{(n+2)^2}$

For $p = 0$ and $p = 1$, $\text{MSE}_p(T) = \frac{1}{(n+2)^2} > 0 = \text{MSE}_p(S)$.

But for $p = 0.5$, $\text{MSE}_p(T) = \frac{n}{4(n+2)^2} < \frac{n}{4n^2} = \frac{1}{4n} = \text{MSE}_p(S)$.

The performance of the estimator therefore depends on the true value of p.

Sometimes a biased estimator gives a better estimate than an unbiased estimator, according to the MSE criterion. $\boxed{6}$

2. (a) *This part appeared in the lecture notes.* $\boxed{\text{sim. seen} \Downarrow}$

The likelihood is $p(\mathbf{Y}|\lambda) = \prod_{i=1}^{n} \lambda \exp(-\lambda \mathbf{y}_i)$.
The log-likelihood is $\log p(\mathbf{Y}|\lambda) = n \log \lambda - \lambda \sum_{i=1}^{n} \mathbf{y}_i$.
The derivative of the log-likelihood follows as $\frac{d}{d\lambda} \log p(\mathbf{Y}|\lambda) = \frac{n}{\lambda} - \sum_{i=1}^{n} \mathbf{y}_i$.
Setting this expression for the derivative equal to zero implies that $\lambda_{MLE} = \frac{n}{\sum_{i=1}^{n} y_i}$.
We then confirm it is a maximum by showing that the Hessian is always negative, $\frac{d^2}{d\lambda^2} \log p(\mathbf{Y}|\lambda) = -\frac{n}{\lambda^2} \leq 0$. $\boxed{4}$

(b) *This part is similar to an example in the lecture notes.*
The Fisher Information is $-\mathsf{E}\left(\frac{d^2}{d\lambda^2} \log p(\mathbf{Y}|\lambda)\right) = -\mathsf{E}(-\frac{n}{\lambda^2}) = \frac{n}{\lambda^2}$.
If λ_0 is the "true" parameter, then $\sqrt{n}(\lambda_{MLE} - \lambda_0) \to^d N(0, \lambda_0^2)$. $\boxed{4}$

(c) *This part is similar to an example in the lecture notes.*
From the previous part we can conclude that, $Pr\left(c_1 < \frac{\sqrt{n}(\lambda_{MLE} - \lambda_0)}{\lambda_0} < c_2\right) = 1 - \alpha$.
Choosing $\alpha = 0.05$, c_1 such that $\Phi(c_1) = \frac{\alpha}{2}$, and c_2 such that $\Phi(c_2) = 1 - \frac{\alpha}{2}$, then rearranging the inequality results in $\frac{c_1/\sqrt{n}+1}{\lambda_{MLE}} < \frac{1}{\lambda_0} < \frac{c_2/\sqrt{n}+1}{\lambda_{MLE}}$, and so the random interval is given by $\left(\frac{\lambda_{MLE}}{c_2/\sqrt{n}+1}, \frac{\lambda_{MLE}}{c_1/\sqrt{n}+1}\right)$ $\boxed{6}$

(d) (i) A family of prior probability distributions P is said to be conjugate to a family of observational distributions L, if for every prior $p \in P$ and every observational distribution $l \in L$, the resulting posterior distribution also belongs to P. $\boxed{2}$

(ii) $p(y_1|\lambda)p(\lambda) = \lambda \exp(-\lambda y_1) \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} \exp(-\beta\lambda) \propto \lambda^{(\alpha+1)-1} \exp(-\lambda(\beta + y_1))$ which is also gamma distributed with parameters $\alpha_{new} = \alpha + 1$ and $\beta_{new} = \beta + y_1$. $\boxed{4}$

3. (a) $\mathbf{Y} = \mathbf{X}\beta + \epsilon$ $\boxed{\text{seen} \Downarrow}$

$\mathbf{Y}$ is an $n \times 1$ vector of observations.

$\mathbf{X}$ is an $n \times p$ design matrix.

β is an $p \times 1$ vector of parameters.

ϵ is an $n \times 1$ vector of random variables describing the error. $\boxed{4}$

(b) *(Seen before in class)*

(i) $\widehat{\mathbf{Y}} = \mathbf{X}\widehat{\beta} = \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{Y}$ $\boxed{2}$

(ii) Let L be a linear subspace of $\mathbb{R}^n$, $\dim L = r \leq n$. $P \in \mathbb{R}^{n\times n}$ is a projection matrix onto L, if

1. $P\boldsymbol{x} = \boldsymbol{x} \quad \forall \boldsymbol{x} \in L$
2. $P\boldsymbol{x} = \mathbf{0} \quad \forall \boldsymbol{x} \in L^\perp = \{\boldsymbol{z} \in \mathbb{R}^n : \boldsymbol{z}^T\boldsymbol{y} = \mathbf{0}\,\forall \boldsymbol{y} \in L\}$

Alternatively, candidates can define a projection matrix as follows.

Let $A \in \mathbb{R}^{n\times n}$. A is called a projection matrix if it is symmetric ($A^T = A$) and idempotent ($AA = A$). $\boxed{3}$

(iii) $\widehat{\mathbf{Y}} = \mathbf{X}\widehat{\beta} = \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{Y} = P\mathbf{Y}$, where $P = \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T$ is a projection matrix, since $P^T = P$ and $P^2 = P$, both of which should be shown algebraically. $\boxed{3}$

(c) (i) Let $c \in \mathbb{R}^p$ and let $\widehat{\beta}$ be a least squares estimator of β in a linear model, where we assume full rank and second order assumptions. Then the estimator $c^T\widehat{\beta}$ has the smallest variance among all linear unbiased estimators for $c^T\beta$. $\boxed{3}$

(ii) This is an open question and any reasonable description of a linear model, where we are interested in estimating some linear combination of parameters, i.e. $c^T\beta$, is acceptable. $\boxed{\text{unseen} \Downarrow}$

If we want an unbiased estimator for $c^T\beta$, then the Gauss Markov theorem says that we should use $c^T\widehat{\beta}$, as per part (i).

However, we may be able to find a biased estimator with lower variance, and hence lower MSE, in which case we might choose to ignore the Gauss Markov theorem. $\boxed{5}$

4. (a) If $X \sim N(\delta, 1)$, and $U \sim \chi^2_n$ independently then $\boxed{\text{seen} \Downarrow}$

$$Y = \frac{X}{\sqrt{U/n}}$$

is said to have a non-central t-distribution with n d.f. and n.c.p.$=\delta$.

$\boxed{3}$

(b) (i) Since $\mathbf{Y} \sim N(\mathbf{X}\boldsymbol{\beta}, \sigma^2 I)$, we have that

$$\mathsf{E}(c^T\boldsymbol{\beta}) = \mathsf{E}(c^T(X^TX)^{-1}X^TY) = c^T(X^TX)^{-1}X^TX\boldsymbol{\beta} = c^T\boldsymbol{\beta}$$

$$\begin{aligned}\mathsf{Var}(c^T\boldsymbol{\beta}) &= \mathsf{Var}(c^T(X^TX)^{-1}X^TY) \\ &= c^T(X^TX)^{-1}X^T\mathsf{Cov}(Y)X(X^TX)^{-1}c \\ &= c^T(X^TX)^{-1}c\sigma^2\end{aligned}$$

$\boxed{3}$

(ii)

$$\begin{aligned}\mathsf{RSS} &= e^Te \\ &= ((I-P)Y)^T((I-P)Y) \\ &= Y^TQ^TQY \\ &= Y^TQY\end{aligned}$$

$\boxed{2}$

(iii) This statistic is t_{n-p}-distributed. From part (i) we know $c^T\widehat{\beta} \sim N(c^T\beta, c^T(X^TX)^{-1}c\sigma^2)$, and so $\boxed{\text{sim. seen} \Downarrow}$

$$A = \frac{c^T\widehat{\beta} - c^T\beta}{\sqrt{c^T(X^TX)^{-1}c\sigma^2}} \sim N(0,1)$$

Let $B = \frac{\mathsf{RSS}}{\sigma^2} \sim \chi^2_{n-p}$. We can first prove that A and B are independent, then use the fact from part (a) that $\frac{A}{\sqrt{B/n}} \sim t_n$. $\boxed{4}$

(c) If $W_1 \sim \chi^2_{n_1}(\delta)$, $W_2 \sim \chi^2_{n_2}$ independently then

$$F = \frac{W_1/n_1}{W_2/n_2}$$

is said to have a non-central F distribution with (n_1, n_2) d.f. and n.c.p.$=\delta$. $\boxed{3}$

(d) The Fisher-Cochran theorem states that if $A_1, \ldots, A_k$ are $n \times n$ projection matrices such that $\sum_{i=1}^{n} A_i = I_n$, and if $Z \sim N(\boldsymbol{\mu}, I_n)$, then $Z^T A_1 Z, \ldots, Z^T A_k Z$ are independent and

$$Z^T A_i Z \sim \chi^2_{r_i}(\delta_i), \quad \text{where } r_i = \operatorname{rank} A_i \text{ and } \delta_i^2 = \boldsymbol{\mu}^T A_i \boldsymbol{\mu}.$$

seen ⇓

If a test statistic can be written in the form defined in part (c), then once we have shown independence of the two chi squared distributions using the Fisher-Cochran theorem, we can conclude that the test statistic is F distributed. 5