The Solutions for E3.18, 2008

Model answer to Q 1(a): Derivations and Computed Example

For a half-height waveguide (i.e. its height dimension 'b' is half that of the width dimension 'a':

i) Using (1.1), derive an expression for the length of the cavity in terms of 'a' and the various frequency terms.

An ideal air-filled rectangular waveguide has a guided-wavelength given by the following expression:

$$\lambda_g = \frac{\lambda_o}{\sqrt{1-\left(\frac{f_c}{f_o}\right)^2}} \qquad (1.1)$$

All variables have their usual meaning.

$$for \quad half-height: b = \frac{a}{2}$$

$$\lambda_c = 2a \quad \therefore f_c = \frac{c}{2a} \quad \therefore a = \frac{\lambda_c}{2} = \frac{c}{2f_c} \quad also \quad \lambda_o = \frac{c}{f_o} \quad \therefore a = \frac{\lambda_o}{2}\left(\frac{f_o}{f_c}\right)$$

$$for \quad TE_{101}\ \mathrm{mode}: l \equiv \frac{\lambda g}{2} = \frac{\lambda_o/2}{\sqrt{1-\left(\frac{f_c}{f_o}\right)^2}} = \frac{a}{\left(\frac{f_o}{f_c}\right)\sqrt{1-\left(\frac{f_c}{f_o}\right)^2}} = \frac{a}{\sqrt{\left(\frac{f_o}{f_c}\right)^2-1}}$$

[4]

ii) Using (i), derive an expression for the internal volume of the cavity.

$$Volume \equiv a\,b\,l = \frac{a^3}{2\sqrt{\left(\frac{f_o}{f_c}\right)^2-1}}$$

[2]

iii) Using (i), derive an expression for the internal area of the cavity.

$$Area \equiv 2(a\,l + a\,b + b\,l) = 2a^2\left[\frac{1}{2} + \frac{1}{\sqrt{\left(\frac{f_o}{f_c}\right)^2-1}} + \frac{1}{2\sqrt{\left(\frac{f_o}{f_c}\right)^2-1}}\right] = a^2\left[1 + \frac{3}{\sqrt{\left(\frac{f_o}{f_c}\right)^2-1}}\right]$$

[3]

iv) Using (1.2) and assuming that $f_o/f_c = \sqrt{2}$, derive an expression for the unloaded-Q-factor in terms of 'a' and classical skin depth.

For the TE_{101} mode, the unloaded Q-factor for an air-filled rectangular waveguide resonant cavity is given by the following expression:

$$Q_u\big|_{TE101} \cong \frac{2\,Volume}{\delta_o\;Area} \qquad (1.2)$$

All variables have their usual meaning.

$$Volume = \frac{a^3}{2\sqrt{\left(\frac{f_o}{f_c}\right)^2 - 1}} \to \frac{a^3}{2} \quad and \quad Area = a^2\left[1 + \frac{3}{\sqrt{\left(\frac{f_o}{f_c}\right)^2 - 1}}\right] \to 4a^2$$

$$\therefore Q_u\big|_{TE101} \cong \frac{2\,Volume}{\delta_o\;Area} \to \frac{a}{4\delta_o}$$

[4]

v) Using (iv), calculate the unloaded-Q-factor for a 15.5 GHz resonant cavity made with copper walls having a DC bulk conductivity of 5.8 x 10^7 S/m.

$$\lambda_o = \frac{c}{f_o} = 19.355mm \quad and \quad a = \frac{\lambda_o}{2}\left(\frac{f_o}{f_c}\right) = 13.686mm$$

$$\delta_o = \sqrt{\frac{2}{\omega_o \mu_o \sigma_o}} = \sqrt{\frac{2}{2\pi 15.5\times10^9 \times 4\pi\times10^{-7}\times5.8\times10^7}} = 0.531\mu m$$

$$\therefore Q_u\big|_{TE101} \cong \frac{a}{4\delta_o} = 6{,}446$$

[4]

Model answer to Q 1(b): Derivation

For a cubical cavity (i.e. all internal dimensions are equal) derive an expression for the unloaded-Q-factor in terms of 'a' and classical skin depth and show that this has a 33.333% higher unloaded Q-factor.

$$Volume = a^3 \quad and \quad Area = 6a^2$$

$$\therefore Q_u\big|_{TE101} \cong \frac{2\,Volume}{\delta_o\;Area} \to \frac{a}{3\delta_o} \quad which\ is\ 4/3 = 1.3333333\ higher\ than\ with\ half-height$$

[3]

Model answer to Q 2(a): Computed Example

$$\varepsilon_r' = 12.86 \quad and\ \tan\delta = \frac{\varepsilon_r''}{\varepsilon_r'} = 0.0006 \quad \therefore \varepsilon_r'' = 7.716\times10^{-3}$$

$$\sigma = \sigma' - j\sigma'' = j\omega\varepsilon_o(\varepsilon_r - 1) \quad and \quad \varepsilon_r = \varepsilon_r' - j\varepsilon_r''$$

$$\therefore \sigma' = \omega\varepsilon_o\varepsilon_r'' \quad and \quad -\sigma'' = \omega\varepsilon_o(\varepsilon_r' - 1)$$

$$\therefore \sigma = 0.129 - j(-197.932)\ S/m$$

[5]

Model answer to Q 2(b): Computed Example

$$\rho_o = 8\ k\Omega\cdot cm = 80\ \Omega\cdot m$$

$$\therefore \sigma_o = \frac{1}{\rho_o} = 0.0125\ S/m$$

Therefore, it can be seen that at 300 GHz the conductivity is 10.32 times greater than the originally quoted value suggests.

[3]

Model answer to Q 2(c): Computed Example

$$\rho = \frac{\eta - \eta_o}{\eta - \eta_o} \quad where \quad \eta_o = \sqrt{\frac{\mu_o}{\varepsilon_o}} \quad and \quad \eta = \sqrt{\frac{\mu_o\mu_r}{\varepsilon_o\varepsilon_r}} \to \sqrt{\frac{\mu_o}{\varepsilon_o\varepsilon_r'}}$$

$$\therefore \rho = \frac{1-\sqrt{\varepsilon_r'}}{1+\sqrt{\varepsilon_r'}} = -0.564$$

$$\Gamma = |\rho|^2 = 31.8\%$$

[5]

Model answer to Q 2(d): New Derivation

$$H(z) = H(0)e^{-\gamma z} + H(0)e^{+\gamma z}$$

$$P_{ABSORBED} = |H(z)|^2_{z=0} R_s = 4|H(0)|^2 R_s$$

$$P_{INCIDENCE} = |H(0)|^2 \eta_o$$

$$\Gamma = \frac{P_{REFLECTED}}{P_{INCIDENCE}} \quad where \quad P_{REFLECTED} = P_{INCIDENCE} - P_{ABSORBED}$$

$$\therefore \Gamma = 1 - \frac{P_{ABSORBED}}{P_{INCIDENCE}} = 1 - 4\frac{R_s}{\eta_o}$$

[5]

Model answer to Q 2(e): Computed Example

$$\Gamma = 1 - 4\frac{R_s}{\eta_o} = 1 - 4\frac{0.1}{120\pi} = 99.89\%$$

[2]

$$H(z) = H(0)e^{-\gamma z} + H(0)e^{+\gamma z}$$

$$P_{ABSORBED} = |H(z)|^2_{z=0} R_s = 4|H(0)|^2 R_s$$

$$P_{INCIDENCE} = |H(0)|^2 \eta_o$$

$$\Gamma = \frac{P_{REFLECTED}}{P_{INCIDENCE}} \quad where \quad P_{REFLECTED} = P_{INCIDENCE} - P_{ABSORBED}$$

$$\therefore \Gamma = 1 - \frac{P_{ABSORBED}}{P_{INCIDENCE}} = 1 - 4\frac{R_s}{\eta_o}$$

[7]

Model answer to Q 3(a): Bookwork and New Derivation

$$R_{DC} = \frac{1}{\sigma_o}\left(\frac{length, l}{area}\right) \rightarrow \frac{R_{DC}}{l} = \frac{1}{\sigma_o(\pi R^2)}$$

[2]

Model answer to Q 3(b): Bookwork and New Derivation

$$W_m = \frac{\mu_o}{2}\int_{volume} H^2 \cdot dv \quad where \quad v = (2\pi r)rl \quad \text{and} \quad H = \frac{I}{2\pi R}\left(\frac{r}{R}\right) \quad where \quad r < R$$

$$\therefore W_m = \frac{\mu_o}{2}\int_{volume}\left(\frac{I}{2\pi R}\left(\frac{r}{R}\right)\right)^2 \cdot dv \quad where \quad v = (2\pi r)rl$$

$$\therefore W_m = \frac{\mu_o}{2}\left(\frac{I}{2\pi R}\left(\frac{1}{R}\right)\right)^2 (2\pi r)l\int_0^R r^3 \cdot dr = \frac{\mu_o}{2}\left(\frac{I}{2\pi R}\left(\frac{1}{R}\right)\right)^2 (2\pi r)l\left[\frac{r^4}{4}\right]_0^R = \frac{\mu_o I^2 l}{16\pi}$$

$$W_m \equiv \frac{1}{2}L_{DC}I^2 \rightarrow \frac{L_{DC}}{l} = \frac{\mu_o}{8\pi} \neq f(R)$$

[4]

Model answer to Q 3(c): Bookwork and New Derivation

$$\text{Im}\,pedance, Z = Z_s\left(\frac{length, l}{width, 2\pi R}\right) \quad with \quad Z_s = R_S(1+j) \quad and \quad R_s = \frac{1}{\sigma_o\delta_o}$$

[2]

Model answer to Q 3(d): Bookwork and New Derivation

$$\frac{Z}{l} \equiv \frac{R_{HF}}{l} + j\omega\frac{L_{HF}}{l}$$

$$\therefore \frac{R_{HF}}{l} = \frac{1}{\sigma_o\delta_o 2\pi R} = \left(\frac{R}{2\delta_o}\right)\left(\frac{R_{DC}}{l}\right)$$

[2]

Model answer to Q 3(e): Bookwork and New Derivation

$$\therefore \frac{L_{HF}}{l} = \frac{1}{\sigma_o\delta_o 2\pi R\omega} = \left(\frac{2\delta_o}{R}\right)\left(\frac{L_{DC}}{l}\right)$$

[2]

Model answer to Q 3(f): Bookwork and New Derivation

No, since the resistance and inductive reactance are equal at all frequencies.

[2]

Model answer to Q 3(e): Bookwork and New Derivation

$$\frac{L_{HF}}{l} = \left(\frac{2\delta_o}{R}\right)\left(\frac{L_{DC}}{l}\right) = \left(\frac{2\delta_o}{R}\right)\left(\frac{\mu_o}{8\pi}\right) = \frac{\mu_o \delta_o}{2\times(2\pi R)}$$

$$\therefore L_{HF} = \frac{\mu_o \delta_o}{2}\left(\frac{length,\, l}{width,\, 2\pi R}\right)$$

[2]

Model answer to Q 3(f): Bookwork and Computed example

$$\delta_o = \sqrt{\frac{2}{\omega \mu_o \sigma_o}}$$

$$\sigma_o = \frac{1}{22.14\times 10^{-9}\,\Omega m}$$

$$\therefore \delta_o = 1.248\ \mu m$$

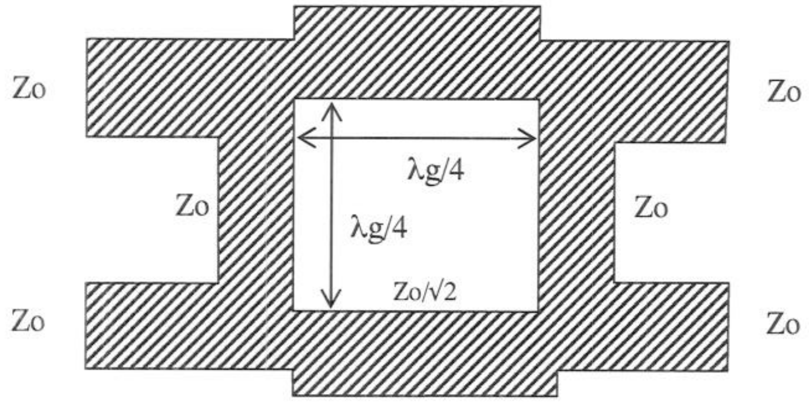
$$\therefore L_{HF} = \frac{\mu_o \delta_o}{2}\left(\frac{10^{-3}}{2\pi\times 25\times 10^{-6}/2}\right) = 0.01\, nH$$

$$\therefore R_{HF} = \omega L_{HF} = 0.226\ \Omega$$

[4]

Model answer to Q 4(a): Bookwork

90^o 3dB Directional Coupler (Branch-line Coupler)



- Works on the interference principle, therefore, narrow fractional bandwidth (15% maximum)
- No bond-wires or isolation resistors required
- Wider tracks make it easier to fabricate and is, therefore, good for lower loss and higher power applications
- Simple design but large
- Meandered lines are possible for lower frequency applications

[5]

Model answer to Q 4(b): Bookwork and Computed Example

The lumped-element equivalent of a $\lambda_g/4$ transmission line is shown below.

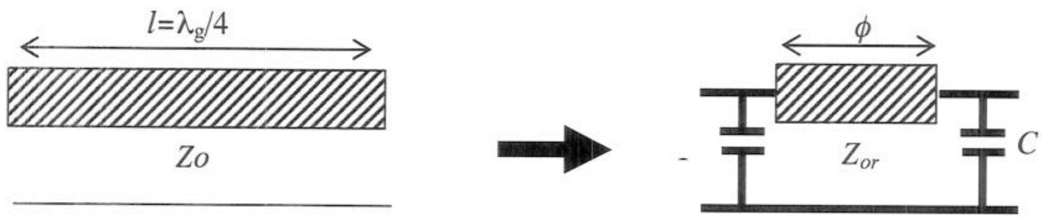
All the previous distributed-element couplers can be transformed into equivalent lumped-element couplers by simply replacing all the $\lambda_g/4$ lengths of transmission lines with the above π-network. Since lumped-element components have a lower Q-factor, when compared to distributed-element components, there is an insertion loss penalty. Also, because this π-network is clearly a low-pass filter, having a cut-off frequency, $f_c = \dfrac{1}{2\pi\sqrt{LC}}$, there is also a bandwidth penalty.

L = 4.3 nH and C = 1.7 pF for the Zo = 50 Ω sections of line
L = 3.0 nH and C = 2.4 pF for the Zo = 35 Ω sections of line

[5]

Model answer to Q 4(c): Bookwork and Computed Example

- Lumped-Distributed Couplers



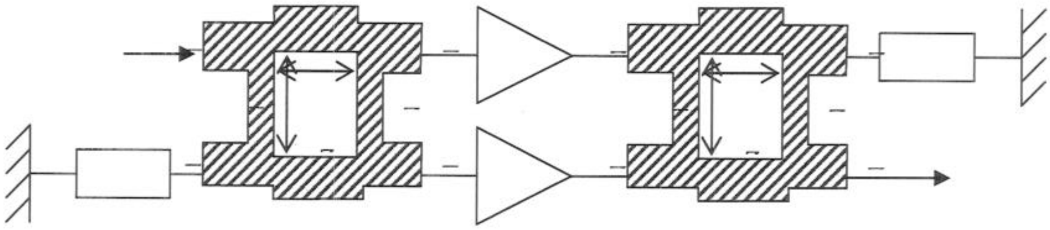
In this 'reduced-size' technique, each $\lambda_g/4$ line is replaced with the above π-network.

$$Z_{or} = \frac{Zo}{Sin\phi} \quad and \quad C = \frac{Cos\phi}{\omega Zo}$$

With $\phi = 45°$,
Z_{or} = 70.7 Ω and C = 1.22 pF for the Zo = 50 Ω sections of line
Z_{or} = 50.0 Ω and C = 1.72 pF for the Zo = 35 Ω sections of line

[5]

Model answer to Q 4(d): Solution given in class



[5]

Model answer to Q5(a): New Derivation

$$z_{IN} = \frac{z + jz_{TX} \tan\vartheta}{z_{TX} + jz \tan\theta} \equiv z_o$$

$$\therefore Z_{TX}\left(Z + jZ_{TX} \tan\vartheta\right) = Zo(Z_{TX} + jZ \tan\vartheta)$$

$$\mathrm{Re}\{LHS\} \equiv \mathrm{Re}\{RHS\}$$

$$\therefore \theta = \tan^{-1}\left\{\frac{Z_{TX}\left(Zo - R\right)}{XZo}\right\}$$

$$\mathrm{Im}\{LHS\} \equiv \mathrm{Im}\{RHS\}$$

$$\tan\theta = \frac{Z_{TX}X}{ZoR - Z_{TX}^2} \equiv \frac{Z_{TX}\left(Zo - R\right)}{XZo}$$

$$\therefore Z_{TX} = \sqrt{ZoR - \frac{X^2 Zo}{Zo - R}}$$

[7]

Model answer to Q5(b): New Derivation

From the last expression in 4(a), the limits are:

$$R \neq Zo \quad and \quad X < \sqrt{R(Zo - R)}$$

[3]

Model answer to Q5(c): Computed Example

For a 2 nH inductance in series with a 3 Ω resistance at 900 MHz, the termination load impedance is Z = 2 + j11.31 Ω.

Using the expressions from 5(b), R is not equal to 50 Ω and X < 11.87 Ω, so both values are within the acceptable mathematical limits.

Using the expressions from 5(a), Z_{TX} = 3.73 Ω and ϑ = 16.5°.

[7]

Model answer to Q5(d): Bookwork

The value of Z_{TX} calculated in 5(c) would be considered very low in general. In practice, a conventional microstrip line could not be used to implement such a low impedance because the width of the signal line would be too wide. However, thin-film microstrip technology may be suitable as the widths of the lines are much narrower.

[3]

Model answer to Q 6(a): Bookwork and Derivation Exercise

The voltage and current on the line can be represented as :

$$V(z) = V_+\left(e^{-\gamma z} + \rho(0)e^{+\gamma z}\right)$$
$$I(z) = I_+\left(e^{-\gamma z} - \rho(0)e^{+\gamma z}\right)$$

$$It\ can\ be\ found\ that: V_+ = 0.5\left(V(0) + ZoI(0)\right) \quad and \quad V_- = 0.5\left(V(0) - ZoI(0)\right)$$

$$\therefore incident\ wave\ power,\ P_+ = \frac{|V_+|^2}{Zo} \quad and \quad reflected\ wave\ power,\ P_- = \frac{|V_-|^2}{Zo}$$

If Zo is taken to be purely real, the time-average power flow along the line is:

$$\mathrm{P(z)} = \mathrm{Re}\{V(z)I(z)^*\} = \mathrm{Re}\left\{V_+\left(e^{-\gamma z} + \rho(0)e^{+\gamma z}\right)I_+{}^*\left(e^{-\gamma z} - \rho(0)e^{+\gamma z}\right)^*\right\}$$

$$where,\ \rho(z) = \rho(0)e^{+2\gamma z} \equiv \rho(0)e^{+j2\beta z}\ for\ a\ lossless\ line$$

$$\mathrm{P(z)} = \mathrm{Re}\left\{\frac{|V_+|^2}{Zo}(1+\rho(z))(1-\rho(z))^*\right\} = \mathrm{Re}\left\{\frac{|V_+|^2}{Zo}(1+\rho(z))(1-\rho(z)^*)\right\} = \frac{|V_+|^2}{Zo}\left(1-|\rho(z)|^2\right)$$

$$but,\ |\rho(z)| = |\rho(0)|\ for\ a\ lossless\ transmission\ line$$

$$\therefore \mathrm{P(z)} = \frac{|V_+|^2}{Zo}\left(1-|\rho(0)|^2\right) = P_+\left(1-\frac{P_-}{P_+}\right) = \frac{|V_+|^2}{Zo}\left(1-|\rho(0)|^2\right) = P_+\left(1-\frac{P_-}{P_+}\right) = \left(P_+ - P_-\right)$$

This shows that, for a lossless transmission line, time-average power flow is independent of the line length and is equal to the incident wave power minus the reflected wave power.

[5]

Model answer to Q 6(b): Bookwork

The guided wavelength, λg, is defined as the distance between two successive points of equal phase on the wave at a fixed instance in time. The phase velocity of a wave is defined as the speed at which a constant phase point travels down the line. Frequency dispersion is said to occur when $\beta \neq \omega$.constant. Dispersion can occur when $vp = f(\omega)$, i.e. when $Dk = f(\omega)$. It can be shown that zero dispersion in a lossy line can also occur, but only when RC = GL:

$$\gamma^2 = (R + j\omega L)(G + j\omega C) \quad and \quad RC = GL$$

$$\therefore \alpha(\omega) = \alpha(0) = \sqrt{RG} \neq f(\omega) \quad and \quad \beta = \omega\sqrt{LC}$$

$$also, Group\ Velocity,\ Vg = \frac{\partial\omega}{\partial\beta} = \frac{1}{\sqrt{LC}} \equiv vp \neq f(\omega)$$

[5]

Model answer to Q 6(c): Bookwork

$$Zin = j\omega L + \frac{Zo\frac{1}{j\omega C}}{Zo + \frac{1}{j\omega C}} \equiv Zo$$

$$\therefore Zo = \frac{j\omega L}{2}\left(1 \mp \sqrt{1 - \frac{4}{\omega^2 LC}}\right)$$

$\therefore$ *Cut-off frequency,* $fc = \dfrac{1}{\pi\sqrt{LC}}$ representing the bandwidth, i.e. when $\dfrac{4}{\omega^2 LC} = 1$

$$Zo = \begin{cases} \sqrt{\frac{L}{C}} & when \quad \omega << \omega_c \quad i.e.\ purely\ real \\ Complex & when \quad 0 << \omega < \omega_c \\ j\sqrt{\frac{L}{C}} & when \quad \omega = \omega_c \quad i.e.\ purely\ imajinary \\ Imajinary & when \quad \omega \geq \omega_c \end{cases}$$

[5]

Model answer to Q 6(d): Computed Example

$$Zin(\omega_c) = jZo(\omega << \omega_c) \quad and \quad |\rho(\omega_c)|^2 = 1 \Rightarrow 0\ dB \quad and \quad |\tau(\omega_c)|^2 = 1 - |\rho(\omega_c)|^2 = 0 \Rightarrow -\infty\ dB$$

[5]