

SOLUTIONS: Feedback Systems

1. a) i) Let $y(t)$ be the position of the mass M . The force equations are
- $$f(t) = K_1 z(t) + D \dot{z}(t) + K_2 (z(t) - y(t)), \quad M \ddot{y}(t) + K_2 (y(t) - z(t)) = 0.$$

Taking Laplace transforms, substituting and eliminating $y(t)$ gives

$$G(s) = \frac{s^2 + 1}{s^3 + (1 + K_1)s^2 + s + K_1},$$

so, $n(s) = s^2 + 1$. [5]

- ii) The Routh array is:

$$\begin{array}{c|cc} s^3 & 1 & 1 \\ s^2 & 1 + K_1 & K_1 \\ s & 1 & 1 + K_1 \\ 1 & K_1 & \end{array}$$

So $K_1 > 0$ for stability. [5]

- iii) When $K_1 = 0$ the closed-loop is marginally stable. Substituting $K_1 = 0$ in $G(s)$ gives the poles as the roots of $s(s^2 + s + 1)$ which are $0, \frac{-1 \pm j\sqrt{3}}{2}$. [5]

- iv) Using the final value theorem and the fact that $f(s) = 1/s$,

$$z_{ss} := \lim_{t \rightarrow \infty} z(t) = \lim_{s \rightarrow 0} s z(s) = \lim_{s \rightarrow 0} s G(s) f(s) = \lim_{s \rightarrow 0} \frac{s G(s)}{s} = G(0) = \frac{1}{K_1}.$$

So for $y_{ss} = 2$, we need $K_1 = 0.5$. [5]

- b) i) The characteristic equation (CE) is $1 + K(s)G(s) = 0$.

- I. When $K(s) = K_P$, a proportional controller, the CE becomes $s^2 + 2K_P - 1 = 0$. The closed-loop cannot be stabilised since the coefficient of s is zero. [5]

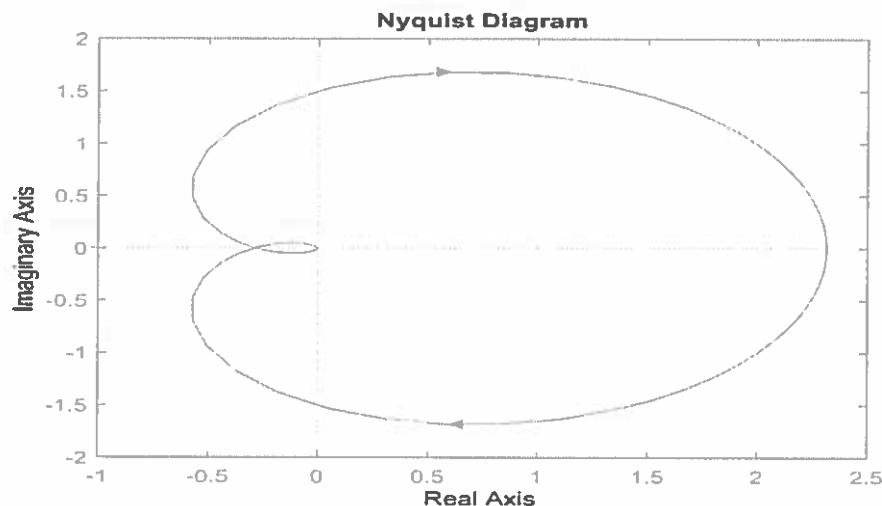
- II. When $K(s) = K_P + K_I s^{-1}$, a PI controller, the CE becomes $s^3 + (2K_P - 1)s + sK_I = 0$. The closed-loop cannot be stabilised since the coefficient of s^2 is zero. [5]

- III. When $K(s) = K_P + K_D s$, a PD controller, the CE becomes $s^2 + 2K_D s + (2K_P - 1) = 0$. The Routh array:
- $$\begin{array}{c|cc} s^2 & 1 & 2K_P - 1 \\ s & 2K_D & \\ 1 & 2K_P - 1 & \end{array}$$
- So, the closed-loop can be stabilised by any $K_D > 0, K_P > 0.5$. [5]

- ii) A PD compensator has the form $K(s) = K_P + K_D s$. For critical damping with a pole at $s = -1$, the closed-loop poles must be placed at $-1, -1$, and so the CE must be $s^2 + 2s + 1$. So we need $s^2 + 2K_D s + (2K_P - 1) = s^2 + 2s + 1$, and so $K_D = K_P = 1$. [5]

2. The transfer function used in fact was $G(s) = \frac{|j+a|^3}{(s+a)^3}$, where $a = 2/\sqrt{3}$, although this is not required.

- a) The real axis intercepts can be obtained from the frequency response (when the phase is 0° , -180° and -270° and are approximately given by 2.3, -0.3 and 0. The Nyquist plot is given below. [5]



- b) From the intercepts above, the gain margin is ~ 3.5 . The phase margin can be obtained from the frequency response (by inspecting the phase when the gain is 1) and is approximately 57° . Thus, the stability margins are adequate. [5]
- c) i) Let $K(s) = k$. The Nyquist criterion states that $N = Z - P$, where N is the number of clockwise encirclements by the Nyquist diagram of $-k^{-1}$, P is the number of unstable open-loop poles and Z is the number of unstable closed-loop poles. [5]
- ii) Since $G(s)$ is stable, $P = 0$. An inspection of the Nyquist diagram shows that

I. When $k = 1$, $N = 0$ so $Z = 0$. [5]

II. When $k = 10$, $N = 2$ so $Z = 2$. [5]

- d) An inspection of the frequency response reveals this is a proportional-plus-integral (PI) compensator. This can be written as

$$K(s) = K_P + \frac{K_I}{s} = \frac{K_I}{s} \left(1 + \frac{s}{K_I/K_P} \right)$$

It has high gain at frequencies below $\omega_0 = K_I/K_P$ and gain close to K_P beyond ω_0 . The phase is negative and large below ω_0 but insignificant above. It follows that by varying K_I and K_P we can use PI compensation to increase low frequency gain (hence improving tracking properties) without introducing phase-lag at high frequency (which would reduce the phase margin) by placing ω_0 in the 'middle' frequency range. Since the cross-over frequency for $G(s)$ is approximately 1 and ω_0 for $K(s)$ is approximately 0.2, this condition is satisfied. [5]

3. a) The CE is given by

$$1 + K_P G(s) = 0 \Rightarrow 1 + \frac{K_P}{s-1} = 0 \Rightarrow s - (1 - K_P) = 0.$$

It follows that the closed-loop poles are given by $1 - K_P$. [6]

- b) i) The CE is given by

$$1 + K_P \frac{s+z}{s(s-1)} = 0 \Rightarrow s^2 + (K_P - 1)s + K_P z = 0.$$

[6]

- ii) The specifications are satisfied by locating two closed-loop poles at -2. Thus the required CE is

$$(s+2)^2 = 0 \Rightarrow s^2 + 4s + 4 = 0$$

It follows that there exist values of K_P and z such that the two characteristic equations are equal. [6]

- iii) Equating the characteristic equations:

$$s^2 + (K_P - 1)s + K_P z = s^2 + 4s + 4.$$

It follows that $K_P = 5$. [6]

- iv) Using $K_P = 5$, we then get that $z = 4/5$. It therefore follows that $K_I = zK_P = 4$. [6]