Imperial College London

BSc/MSci EXAMINATION June 2012

This paper is also taken for the relevant Examination for the Associateship

COSMOLOGY

For 4th-Year Physics Students

Friday, 1st June 2012: 14:00 to 16:00

Answer ALL of Section A and any TWO questions from Section B.

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the THREE answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

SECTION A

1. (i) (a) Calculate the approximate redshift at which the Dark Energy become the dominating factor in the expansion of the Universe in the standard "Concordance" or "ΛCDM" cosmology. [3 marks]

(b) Argue on dimensional grounds that the energy density of blackbody radiation at temperature T should scale as T^4. Use this to find the redshift of matter-radiation equality to an order of magnitude. [5 marks]

(ii) (a) Explain why the nearly uniform temperature of the CMB is a problem for the standard big bang cosmology, and how inflation may solve it. Describe how inflation also provides the seeds for the departures from uniformity we see in the CMB. [4 marks]

(b) What are the flatness problems and the cosmological constant problem? Does inflation solve each of these, and, if so, how? You should use the expression for Ω_k given below as appropriate. [5 marks]

(iii) From Big-Bang Nucleosynthesis calculations and appropriate observations, the contribution of the present-day baryon density to the critical density is measured to be $\Omega_{b0} \simeq 0.045$, assuming $H_0 = 72\text{km s}^{-1}\text{ Mpc}^{=1}$. If the measurement of the Hubble constant were revised to $H_0 = 65\text{km s}^{-1}\text{ Mpc}^{=1}$, how would the value of Ω_{b0} be changed as a result? [3 marks]

[Total 20 marks]

You may find the following useful:

- The curvature "density" parameter is given by $\Omega_k = -k/(aH)^2$ where a is the scale factor and $H = \dot{a}/a$.
- The Planck Energy and Planck Length are given by

$$E_{\rm Pl} = \sqrt{\frac{\hbar c^5}{G}} \approx 1.96 \times 10^9\ \text{J} \qquad \ell_{\rm Pl} = \sqrt{\frac{\hbar G}{c^3}} \approx 1.62\ \text{m}$$

- The combination

$$\hbar c = 3.16 \times 10^{-26}\ \text{m}^3\ \text{kg s}^{-2}.$$

- The critical *energy* density is

$$\rho_c c^2 = \frac{3H_0^2 c^2}{8\pi G} = 8.8h^2 \times 10^{-10}\ \text{J m}^{-3}$$

SECTION B

2. (i) Define the meaning of the "particle horizon" in an expanding Universe. You may find a diagram useful. [3 marks]

(ii) Show that it is given by the expression

$$d_H(z) = \frac{cH_0^{-1}}{1+z}\int_z^\infty \frac{dz'}{E(z')}$$

where the function $E(z)$ is given at the end of the exam. [4 marks]

(iii) What is the behaviour of $d_H(z)$ with z in a Universe with only a cosmological constant? How does it differ from the behaviour in a Unverse with only matter? Give a physical interpretation. [4 marks]

(iv) Consider the expression

$$d_+(z) = \frac{cH_0^{-1}}{1+z}\int_{-1}^{z} \frac{dz'}{E(z')}$$

Note that the limits of integration differ from d_H. What do values $-1 < z < 0$ represent? Give a physical interpretation of this quantity. Show that this expression converges for a flat Universe with only matter and diverges for a flat Universe with only Λ. [4 marks]

[Total 15 marks]

3. The FRW metric is given by

$$ds^2 = c^2 dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right]$$

where $d\Omega$ is a differential element of 2-dimensional angular polar coordinates.

(i) Show that the comoving distance χ_e to a point on our light cone at redshift z_e is given by the expression

$$a_0 H_0 c^{-1} \chi_e = \int_0^{z_e} \frac{dz}{E(z)} \tag{1}$$

[4 marks]

In class we found that the comoving coordinate r_e of a point on our light cone at comoving distance χ_e is given by

$$r_e = |k|^{-1/2} \sin_k\left(\sqrt{|k|}\,\chi_e\right)$$

where k gives the curvature of the Universe and $\sin_k(x) = \{\sin(x), x, \sinh(x)\}$ depending on the sign of k.

(ii) Using the previous expressions, or otherwise, show that the angular diameter distance to an object at redshift z is given by

$$d_A(z) = cH_0^{-1}(1+z)^{-1}|\Omega_k|^{-1/2} \sin_k\left(\sqrt{|\Omega_k|}\int_0^z \frac{dz'}{E(z')}\right) \ .$$

[4 marks]

(iii) Calculate $d_A(z)$ for a flat, matter-dominated Universe. [4 marks]

(iv) Use this to show how the angular size of a standard ruler behaves as a function of redshift. [3 marks]

[Total 15 marks]

4. In a matter-dominated expanding FRW Universe with average density $\bar{\rho}(t)$, the density contrast $\delta = (\rho - \bar{\rho})/\bar{\rho}$ of the dark matter obeys

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G\bar{\rho}\delta = 0 \tag{1}$$

where a is the scale factor and overdots refer to time derivatives.

(i) In a flat, matter-dominated Universe, show that this differential equation can be written as

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - \frac{3}{2}\left(\frac{\dot{a}}{a}\right)^2 \delta = 0\ .$$

By using the Ansatz $\delta = At^n$ or otherwise, solve for the growing and decaying modes in a flat $\Omega_m = 1$ universe. [5 marks]

(ii) Starting again from Eq. 1, derive the solution for $\Omega_m \to 0$ for late times. Use the results in these two cases to describe how galaxy redshift surveys can be used to constrain the cosmological parameters. [4 marks]

(iii) State whether matter fluctuations on scales smaller than the horizon grow in a Universe that is

- matter-dominated;
- radiation-dominated;
- curvature-dominated; or
- cosmological-constant-dominated.

[2 marks]

(iv) By considering scales that "enter the horizon" in matter- and radiation-dominated periods, discuss the effect of the differing growth rates at different times on the matter power spectrum, $P(k)$, and sketch it assuming an initial Harrison-Zeldovich spectrum as expected from inflation, being sure to quantitatively describe the resulting shape. [4 marks]

[Total 15 marks]

5. (i) How does the energy density of *matter* vary with redshift in a matter-dominated Universe? In a radiation-dominated Universe? How does temperature of the Universe vary in both of those cases? [3 marks]

(ii) (a) What is the approximate number of baryons in the Universe for every photon? What are the important measurements and theoretical ideas that have gone into the determination of these two numbers? [3 marks]

(b) The present magnitude of the baryon-to-photon ratio is affected by the rate of interactions in the early universe. What are two such interactions and what quantitative effect does the baryon-to-photon ratio have? [3 marks]

(iii) The freeze-out of interactions between neutrons and protons occurs at $kT_F \simeq 0.8$ MeV. Determine the neutron-proton ratio at this time. [3 marks]

(iv) If there is no subsequent conversion between neutrons and protons, what is the final mass fraction of Helium in the Universe when it is formed at approximately $t \approx 3$ minutes? What physical effect is being ignored? [3 marks]

[Total 15 marks]

The neutron mass-energy is $m_n c^2 = 939.565$ MeV and the proton mass-energy is $m_p c^2 = 938.272$ MeV

The expansion rate is given by

$$\frac{\dot{a}}{a} = H_0 E(z) = H_0 \sqrt{(1+z)^3\Omega_m + (1+z)^4\Omega_r + \Omega_\Lambda + (1+z)^2\Omega_k}$$

and the Hubble Constant is

$$H_0 = 100h \text{ km s}^{-1} \text{ Mpc}^{-1}$$

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