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IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE
UNIVERSITY OF LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2001

MSc and EEE/ISE PART IV: M.Eng. and ACGI

SPECTRAL ESTIMATION AND ADAPTIVE SIGNAL PROCESSING

Monday, 21 May 10:00 am

There are FIVE questions on this paper.

Answer THREE questions.

Time allowed: 3:00 hours

Corrected Copy
@5c @ 11.35 am.

Examiners: Clark, J.M.C. and Allwright, J.C.

Special instructions for invigilators:

None

Information for candidates:

None

1.

The mean square error performance function for the N coefficient Finite Impulse Response (FIR) filter represented in Figure 1 is given by

$$J(\underline{w}) = \sigma_d^2 - 2\underline{p}^T \underline{w} + \underline{w}^T \underline{R} \underline{w}$$

where σ_d^2 is the variance of the desired response $\{d[n]\}$, \underline{R} is the autocorrelation matrix, $E\{\underline{x}[n]\underline{x}^T[n]\}$, with $\underline{x}[n] = [x[n], x[n-1], \dots, x[n-N+1]]^T$, \underline{p} is the cross-correlation vector $E\{d[n]\underline{x}[n]\}$ and $\underline{w} = [w_1, w_2, \dots, w_N]^T$ is the vector of coefficients.

(a) State the assumption on the nature of the autocorrelation matrix \underline{R} so that $J(\underline{w})$ has a unique minimum, and give an example input signal, $\{x[n]\}$, which would satisfy this assumption. **[2 marks]**

(b) Sketch the contours of constant $J(\underline{w})$, as a function of \underline{w} , when

(i) $\underline{p} = \begin{bmatrix} 0.9 \\ 0.85 \end{bmatrix}$, $\underline{R} = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}$ and (ii) $\underline{p} = \begin{bmatrix} 0.9 \\ 0.85 \end{bmatrix}$, $\underline{R} = \begin{bmatrix} 1.0 & 0.9 \\ 0.9 & 1.0 \end{bmatrix}$ **[6 marks]**

(c) Describe how the method of steepest descent, as described by the recursion

$$\underline{w}[k+1] = \underline{w}[k] - \frac{\alpha}{2} \nabla_{\underline{w}} J(\underline{w}[k])$$

may be used to converge in the mean to the minimum of $J(\underline{w})$ **[4 marks]**

(d) For $J(\underline{w})$ in (b) (ii) and given that the autocorrelation matrix \underline{R} can be replaced by the similarity transform

$$\underline{R} = \underline{Q} \underline{\Lambda} \underline{Q}^T$$

where \underline{Q} is the matrix of normalised eigenvectors of \underline{R} and $\underline{\Lambda}$ is the diagonal matrix of corresponding eigenvalues; hence, or otherwise, show that the steepest descent solution may be written as

$$\begin{bmatrix} w_1[k] \\ w_2[k] \end{bmatrix} = \begin{bmatrix} 0.71 - 0.46(1 - 1.9\alpha)^k - 0.25(1 - 0.1\alpha)^k \\ 0.21 - 0.46(1 - 1.9\alpha)^k + 0.25(1 - 0.1\alpha)^k \end{bmatrix}$$

[13 marks]

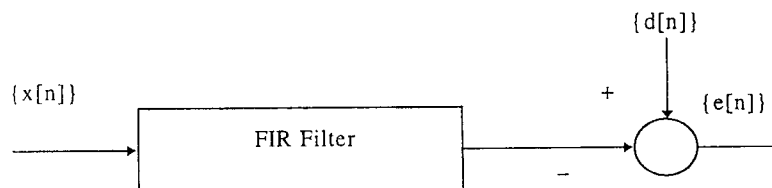


Figure 1

2.

(a) Discuss the relationship between linear prediction and autoregressive modelling. **[5 marks]**

(b) For an autoregressive process generated by the difference equation

$$x[n] = \frac{14}{24}x[n-1] - \frac{9}{24}x[n-2] - \frac{1}{24}x[n-3] + w[n]$$

where $w[n]$ is a zero mean statistically stationary white noise discrete time signal with variance σ_w^2

(i) Calculate the coefficients of the optimum linear predictor. **[1 mark]**

(ii) Using the step down algorithm, as described by

$$a_{k-1}[i] = \frac{a_k[i] - a_k[k]a_k[k-i]}{1 - a_k^2[k]} \quad i = 1, 2, \dots, k-1; k = p, p-1, \dots, 2$$

evaluate the reflection coefficients which correspond to the optimum linear predictor and show how these can be used in a lattice structure realisation. **[6 marks]**

(iii) Determine the autocorrelation sequence $r_{xx}[\tau]$ for $|\tau| \leq 2$. **[6 marks]**

(c) Describe the steps involved in the autocorrelation method for power spectrum estimation from an observation $\{x[0], x[1], \dots, x[N-1]\}$, commenting upon the computational complexity of each step. **[7 marks]**

3.

(a) Describe where moving average models are used to represent elements of mobile communication systems.

[3 marks]

(b) A zero mean, white noise process, $\{w[n]\}$, with variance σ_w^2 is input to a moving average MA(q) filter with impulse response sequence $\{b[0], b[1], \dots, b[q]\}$, calculate the mean value of the output of such a filter and verify that its autocorrelation function is given by

$$r_{MA}[\tau] = \sigma_w^2 \sum_{m=0}^{q-|\tau|} b[m]b[m+\tau]$$

[7 marks]

(c) The autocorrelation sequence of an MA(2) process is found to be $r_{MA}[0] = 6\sigma_w^2$, $r_{MA}[\pm 1] = -4\sigma_w^2$ and $r_{MA}[\pm 2] = 2\sigma_w^2$, and is otherwise zero.

- (i) Evaluate the impulse response sequence of the MA(2) filter.
- (ii) State whether the solution in (i) is unique and, if there is more than one solution, describe the different solutions and why they exist.

[10 marks]

(d) If the output of a MA model is corrupted by additive coloured Gaussian measurement noise, suggest a method to estimate the impulse response of the model which is immune to such noise, and state any assumptions that are necessary.

[5 marks]

4.

A set of linear equations is represented in matrix form by

$$A\mathbf{x} = \mathbf{b}$$

where A is an $n \times m$ matrix with known complex elements, \mathbf{x} is an m -dimensional vector, the elements of which are the unknowns, and \mathbf{b} is an n -dimension vector with known complex elements.

(a) Show, for the three cases $n = m$, $n > m$, and $n < m$, the form of the solution for \mathbf{x} , and the corresponding value of the cost function $J = \mathbf{e}^H \mathbf{e}$, where $\mathbf{e} = \mathbf{b} - A\mathbf{x}$, and $(.)^H$ denotes Hermitian transpose.

[8 marks]

(b) A communications array measurement signal is modelled in the form

$$m[k] = \mu + \alpha \exp(j2\pi f_o k) ; k = 0, 1, \dots, N-1$$

where μ is a complex d.c. level and α is the amplitude of the complex sinusoid. Formulate the solution for μ and α as a set of overdetermined equations and show that the least squares solution for μ and α is given by

$$\begin{bmatrix} \mu \\ \alpha \end{bmatrix} = \begin{bmatrix} \frac{N \text{DFT}(0) - \exp(j\pi f_o [N-1]) S(f_o) \text{DFT}(f_o)}{N^2 - S^2(f_o)} \\ \frac{N \text{DFT}(f_o) - \exp(-j\pi f_o [N-1]) S(f_o) \text{DFT}(0)}{N^2 - S^2(f_o)} \end{bmatrix}$$

$$\text{where } S(f_o) = \frac{\sin \pi f_o N}{\sin \pi f_o} \text{ and } \text{DFT}(f) = \sum_{k=0}^{N-1} m[k] \exp(-j2\pi f k).$$

[10 marks]

(c) Show how the least squares solution in (b) simplifies when N is even and

$$f_o = \frac{p}{N} \text{ where } p \text{ is a nonzero integer in the range } \left[-\frac{N}{2}, \frac{N}{2} - 1 \right].$$

Comment upon the result.

[4 marks]

(d) As the frequency f_o of the model is generally unknown suggest methods by which this may be estimated.

[3 marks]

5.

A family of stochastic gradient algorithms is based upon approximately minimising cost functions of the form

$$J = E\{e^{2p}[n]\} \quad p = 1, 2, 3, \dots$$

where $e[n] = d[n] - \hat{d}[n]$, namely the difference between the desired response $d[n]$ and the output of the adaptive filter $\hat{d}[n] = \underline{w}^T[n]\underline{x}[n]$, where $\underline{w}[n] = [w_1[n], w_2[n], \dots, w_N[n]]^T$ is the coefficient vector of an N -tap, finite impulse response, adaptive filter with input vector $\underline{x}[n] = [x[n], x[n-1], \dots, x[n-N+1]]^T$.

(a) Explain when it would be advantageous to use an adaptive algorithm based on $p \geq 2$ and give an example application.

[3 marks]

(b) Verify that a least mean square (LMS) type coefficient update for $\underline{w}[n]$, based upon J , is given by

$$\underline{w}[n+1] = \underline{w}[n] + 2p\mu e^{2p-1}[n]\underline{x}[n]$$

[3 marks]

(c) Given that $d[n] = \underline{w}^T \underline{x}[n] + v[n]$ where $\underline{w} = [w_1, w_2, \dots, w_N]^T$ is a vector of fixed, but unknown parameters, and $v[n]$ is zero mean independent identically distributed white noise with symmetric probability density function, which is statistically independent of $\underline{x}[n]$, and that the weight error vector, $\underline{c}[n] = \underline{w}[n] - \underline{w}$, is close to zero, show that

$$E\{\underline{c}[n+1]\} = \left[I - \mu p(2p-1)E\{v^{2p-2}[n]\}R \right] E\{\underline{c}[n]\}$$

[12 marks]

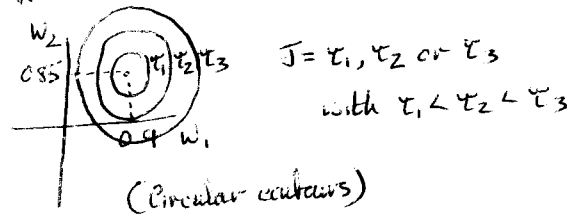
(d) Establish the conditions on the adaptation gain, μ , that assures that the mean $E\{\underline{w}[n]\}$ of the coefficient vector of the adaptive filter converges to the desired vector \underline{w} .

[7 marks]

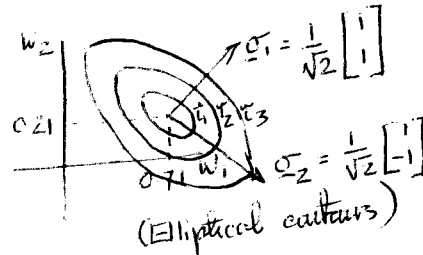
- (1) (a) Positive definite, $x^T R x > 0 \quad \forall x \neq 0$
white noise, w. N sinusoids with different frequencies

(2)

(b) (i) $w_{opt} = R^{-1} p = \begin{bmatrix} 0.9 \\ 0.55 \end{bmatrix}$



(ii) $w_{opt} = R^{-1} p = \begin{bmatrix} 0.71 \\ 0.21 \end{bmatrix}$



(6)

- (c) Begin with $w(0) = 0$, iterate for $k=1,2$,
Select α to satisfy $0 < \alpha < \frac{1}{\lambda_{max}}$

$\nabla_w J(w) = -2p + 2Rw$, $w[k+1] = w[k] + \alpha(p - R w[k])$

$\lim_{k \rightarrow \infty} w[k+1] \rightarrow w_{opt} = R^{-1} p$

(4)

(d) $w[k+1] = w[k] + \alpha(R w_{opt} - R w[k])$
 $w[k+1] - w_{opt} = (w[k] - w_{opt}) - \alpha R (w[k] - w_{opt})$, $v[k] \triangleq (w[k] - w_{opt})$
 $v[k+1] = v[k] - \alpha R v[k] = (I - \alpha R)v[k] - \text{①}$

$R = Q \Lambda Q^T$ - eigenvalues, found from $|R - \lambda I| = 0$
So, from characteristic equation $\lambda_1 = 1.9$, with $\sigma_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as in (b) (ii),
and $Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ $\lambda_2 = 0.1$, with $\sigma_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\Lambda = \text{Diag}(1.9, 0.1)$

From ①, and using $Q^T Q = I$

$Q^T v[k+1] = (I - \alpha \Lambda) Q^T v[k]$, $v'[k] = Q^T v[k]$,

thus $v'[k+1] = (I - \alpha \Lambda) v'[k]$, by induction $v'[k] = (I - \alpha \Lambda)^k v'[0]$
with $v'[0] = Q^T v[0] = \frac{1}{\sqrt{2}} \begin{bmatrix} -0.92 \\ -0.55 \end{bmatrix}$

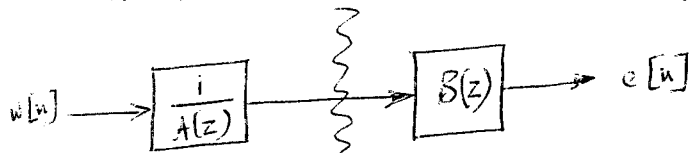
$v[k] = Q v'[k] = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -0.92(1 - \alpha 1.9)^k \\ -0.55(1 - \alpha 0.1)^k \end{bmatrix}$

and $w[k] = v[k] + w_{opt} = \begin{bmatrix} 0.71 - 0.46(1 - \alpha 1.9)^k - 0.25(1 - \alpha 0.1)^k \\ 0.21 - 0.46(1 - \alpha 1.9)^k + 0.25(1 - \alpha 0.1)^k \end{bmatrix}$

(13)

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(2) (a) AR Model Predictor (Linear forward prediction error filter)



$A(z) = 1 + a_1 z^{-1} + \dots + a_p z^{-p}$ $B(z) = 1 + b_1 z^{-1} + \dots + b_p z^{-p}$
 $b_i, i=1, 2, \dots, p$ designed to minimize $E\{e^2[n]\}$,
 then $b_i = a_i \forall i$ and hence $e[n] = w[n]$.

(5)

(b) (i) $B(z) = 1 + a_3[1]z^{-1} + a_3[2]z^{-2} + a_3[3]z^{-3}$
 $a_3[1] = \frac{-14}{24}, a_3[2] = \frac{9}{24}, a_3[3] = \frac{1}{24}$

(1)

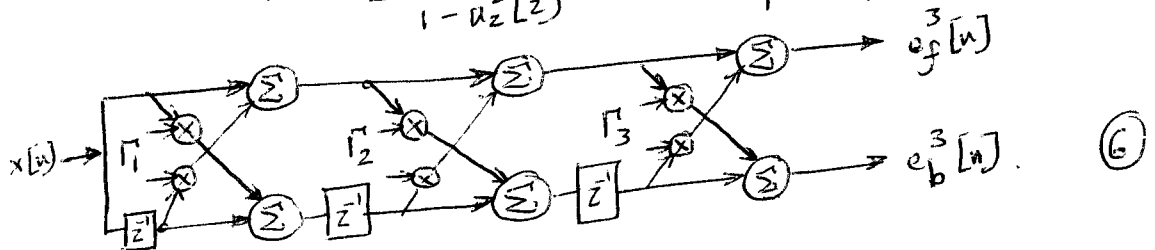
(ii) $a_3[3] = \frac{1}{24} = \Gamma_3$

Using the step down algorithm

$k=3, i=1 \quad a_2[1] = \frac{a_3[1] - a_3[3]a_3[2]}{1 - a_3^2[3]} = \frac{-\frac{14}{24} - \frac{1}{24} \times \frac{9}{24}}{1 - \frac{1}{24 \times 24}} = -0.6$

$k=3, i=2 \quad a_2[2] = \frac{a_3[2] - a_3[3]a_3[1]}{1 - a_3^2[3]} = \frac{\frac{9}{24} + \frac{1}{24} \times \frac{14}{24}}{1 - \frac{1}{24 \times 24}} = 0.4 = \Gamma_2$

$k=2, i=1 \quad a_1[1] = \frac{a_2[1] - a_2[2]a_2[1]}{1 - a_2^2[2]} = \frac{-0.6 + 0.4 \times 0.6}{1 - 0.4^2} = -\frac{3}{7} = \Gamma_1$



(iii) $r_{xx}(0) = \frac{\sigma_w^2}{\prod_{i=1}^3 (1 - \Gamma_i^2)} = 1456 \sigma_w^2$

$r_{xx}(-1) = r_{xx}(1) = -a_1(1) r_{xx}(0) = \frac{3}{7} \times 1456 \sigma_w^2 = 0.625 \sigma_w^2$

$\rho_1 = r_{xx}(0) (1 - \Gamma_1^2) = 1.193 \sigma_w^2$

$r_{xx}(-2) = r_{xx}(2) = -a_1(1) r_{xx}(1) - a_2(2) \rho_1 = -0.208 \sigma_w^2$

(6)

(c) Step 1

+0 Estimation

$\hat{r}_{xx}(k) = \frac{1}{N} \sum_{n=0}^{N-1-k} x[n]x[n+k], k=0, 1, \dots, p$
 $= \hat{r}_{xx}(k) \sim O(N(p+1))$

Step 2

Solve normal eqns with Lev algo

$\begin{bmatrix} \hat{r}_{xx}(0) & \dots & \hat{r}_{xx}(p-1) \\ \hat{r}_{xx}(1) & \dots & \hat{r}_{xx}(p) \end{bmatrix} \begin{bmatrix} \hat{a}[1] \\ \vdots \\ \hat{a}[p] \end{bmatrix} = - \begin{bmatrix} \hat{r}_{xx}(1) \\ \vdots \\ \hat{r}_{xx}(p) \end{bmatrix} \sim O(p^2)$

$\hat{\sigma}^2 = \hat{r}_{xx}(0) + \sum_{k=1}^p \hat{a}[k] \hat{r}_{xx}(-k) \sim O(p)$

Step 3

$\hat{P}_{xx}(f) = \frac{\hat{\sigma}^2}{|1 + \hat{a}[1]e^{-j2\pi f} + \dots + \hat{a}[p]e^{-j2\pi fp}|^2}$

Use FFT
 $O(N \log_2 N)$

(25/25)

- 3) a) Multipath channel modelling as in MLSE equalizer, as in GSM Inter loudspeaker, microphone, impulse response, as in acoustic echo cancellation

(3)

$$b) \quad y[k] = \sum_{m=0}^q b(m)w[k-m]$$

$$E\{w[k]\} = 0, \quad E\{w[k]w[k+\tau]\} = \sigma_w^2 \delta[\tau]$$

$$E\{y[k]\} = \sum_{m=0}^q b(m)E\{w[k-m]\} = 0$$

$$r_{yy}(\tau) = E\{y[k]y[k+\tau]\} = \sum_{m=0}^q \sum_{s=0}^q b(m)b(s) \underbrace{E\{w[k-m]w[k+\tau-s]\}}_{\sigma_w^2 \delta(s-\tau+m)}$$

$$= \sigma_w^2 \sum_{m=0}^q b(m)b(m+\tau)$$

Since $b(m) = 0 \forall |m| > q$ (7)

$$= \sigma_w^2 \sum_{m=0}^{q-|\tau|} b(m)b(m+\tau)$$

c) From $r_{yy}(\tau)$

$$\sigma_w^2 (b^2(0) + b^2(1) + b^2(2)) = 6\sigma_w^2 \quad \text{--- (I)}$$

$$\sigma_w^2 (b(1)(b(0) + b(2))) = -4\sigma_w^2 \quad \text{--- (II)}$$

$$\sigma_w^2 (b(0)b(2)) = 2\sigma_w^2 \quad \text{--- (III)}$$

Using II + III

$$b(1) = \frac{-4b(0)}{b^2(0) + 2} \quad b(0) = \frac{2}{b(2)}$$

With $s \triangleq b^2(0)$, from (I) $(s-2)(s^3-s) = s^4 - 2s^3 - 8s + 16 = (s-2)^2(s^2+2s+4) = 0$

From the real root, $b^2(0) = 2 \Rightarrow b(0) = \pm\sqrt{2}$

(i) $b(0) = \sqrt{2}, \quad b(1) = -\sqrt{2}, \quad b(2) = \sqrt{2}$

(ii) Not unique, $b(0) = -\sqrt{2}, \quad b(1) = \sqrt{2}, \quad b(2) = -\sqrt{2}$ another solution;
ACF is symmetric, no phase information

(10)

- d) Employ higher order statistics, assume input to AR model is third order white

$$y(n) = x_{AR}(n) + w_{AR}(n)$$

$$\begin{aligned} r_y(\tau_1, \tau_2) &= r_{x_{AR}}(\tau_1, \tau_2) \quad \text{since } r_{w_{AR}}(\tau_1, \tau_2) = 0 \quad \forall \tau_1, \tau_2 \\ &= \delta_3 \sum_{m=0}^{q-\max(|\tau_1|, |\tau_2|)} b(m)b(m+\tau_1)b(m+\tau_2) \end{aligned}$$

Use cumulant matching to solve for $b(i)$

$$c \triangleq \min_{b(i)} \left| \hat{r}_y(\tau_1, \tau_2) - \delta_3 \sum_{m=0}^{q-\max(|\tau_1|, |\tau_2|)} b(m)b(m+\tau_1)b(m+\tau_2) \right|^2$$

$\tau_1, \tau_2 \in \mathcal{L}_{E\&}$

(5)

(25/25)

- 4) a) $n = m$ exactly determined
 $\underline{x} = A^{-1} \underline{b}$ - provided $\text{rank}(A) = n$, $\underline{J} = 0$.
 $n > m$ - overdetermined
 $\underline{z} = (A^H A)^{-1} A^H \underline{b}$, $\underline{J} = (\underline{b} - A \underline{z})^H (\underline{b} - A \underline{z}) \big|_{\underline{z}} = (A^H A)^{-1} A^H \underline{b}$
 $= \underline{b}^H \underline{b} - \underline{b}^H A (A^H A)^{-1} A^H \underline{b}$ since $A^H \underline{e} = 0$
 $= \underline{b}^H P^\perp \underline{b}$ where $P^\perp = (I - A(A^H A)^{-1} A^H)$

$n < m$ - underdetermined

One approach, $\min \|\underline{x}\|_2$ s.t. $A \underline{x} = \underline{b}$

$\underline{x} = A^H (A A^H)^{-1} \underline{b}$ where $A^H (A A^H)^{-1}$ - pseudo inverse, $\underline{J} = 0$. (8)

b)
$$\begin{bmatrix} 1 & \exp(j2\pi f_0) \\ 1 & \exp(j2\pi f_0(N-1)) \end{bmatrix} \begin{bmatrix} \mu \\ \alpha \end{bmatrix} = \begin{bmatrix} m[0] \\ m[N-1] \end{bmatrix}$$

c)
$$A \underline{x} = \underline{b}$$

$N > 2$, $\begin{bmatrix} \mu \\ \alpha \end{bmatrix}_{LS} = (A^H A)^{-1} A^H \underline{b}$, $A^H A = \begin{bmatrix} N & \sum_{k=0}^{N-1} \exp(j2\pi f_0 k) \\ \sum_{k=0}^{N-1} \exp(-j2\pi f_0 k) & N \end{bmatrix}$.

$\sum_{k=0}^{N-1} \exp(j2\pi f_0 k) = \exp(j2\pi f_0 (N-1)) \frac{\sin \pi f_0 N}{\sin \pi f_0}$, thus $(A^H A)^{-1} = \frac{1}{N^2 - S^2(f_0)} \begin{bmatrix} N & -\exp(j\pi f_0 (N-1)) S(f_0) \\ -\exp(-j\pi f_0 (N-1)) S(f_0) & N \end{bmatrix}$

$A^H \underline{b} = \begin{bmatrix} \sum_{k=0}^{N-1} m[k] \\ \sum_{k=0}^{N-1} m[k] \exp(-j2\pi f_0 k) \end{bmatrix} = \begin{bmatrix} \text{DFT}(0) \\ \text{DFT}(f_0) \end{bmatrix}$, $\begin{bmatrix} \mu \\ \alpha \end{bmatrix}_{LS} = \frac{1}{N^2 - S^2(f_0)} \begin{bmatrix} N \text{DFT}(0) - \exp(j\pi f_0 (N-1)) S(f_0) \text{DFT}(f_0) \\ N \text{DFT}(f_0) - \exp(-j\pi f_0 (N-1)) S(f_0) \text{DFT}(0) \end{bmatrix}$

d) $f_0 = \frac{p}{N}$, $S(f_0) = \frac{\sin \pi p}{\sin \pi f_0} = 0$, p non zero integer

$\Rightarrow \begin{bmatrix} \mu \\ \alpha \end{bmatrix}_{LS} = \begin{bmatrix} \frac{1}{N} \sum_{k=0}^{N-1} m[k] \\ \frac{1}{N} \sum_{k=0}^{N-1} m[k] \exp(-j2\pi p k / N) \end{bmatrix} = \begin{bmatrix} \text{Sample mean} \\ \text{Sample mean of input shifted to } f = p/N \end{bmatrix}$ (4)

e) many possibilities - non linear least squares, peak detection of FFT following mean removal (3)

- 5 a) In system identification if the measurement noise has a sub-Gaussian probability density function, $p=2$ would yield lower misadjustment than $p=1$.
 b) LMS-based minimization of instantaneous error squared with an equation of the form - (3)

$$\underline{w}[n+1] = \underline{w}[n] - \mu \nabla_{\underline{w}} \hat{J} \Big|_{\underline{w}=\underline{w}[n]}$$

$$\hat{J} = e^{2p}[n] \quad \nabla_{\underline{w}} \hat{J} = 2pe^{2p-1}[n] \nabla_{\underline{w}} e[n]$$

$$e[n] = d[n] - \underline{w}^T[n] \underline{x}[n], \quad \nabla_{\underline{w}} \hat{J} = -2pe^{2p-1}[n] \underline{x}[n], \quad \underline{w}[n+1] = \underline{w}[n] + 2\mu pe^{2p-1}[n] \underline{x}[n] \quad (3)$$

$$\begin{aligned} e[n] &= d[n] - \underline{w}^T[n] \underline{x}[n] \\ &= (\underline{w} - \underline{w}[n])^T \underline{x}[n] + v[n] = -\underline{e}^T[n] \underline{x}[n] + v[n] \end{aligned}$$

$$e^{2p-1}[n] = (-\underline{e}^T[n] \underline{x}[n] + v[n])^{2p-1}$$

$$\text{Since } \underline{e}[n] \approx 0$$

$$e^{2p-1}[n] \approx -(2p-1) \underline{x}^T[n] \underline{e}[n] v^{2p-2}[n] + v^{2p-1}[n]$$

Thus, from update equation in b)

$$\underline{w}[n+1] - \underline{w} = \underline{w}[n] - \underline{w} + 2\mu pe^{2p-1}[n] \underline{x}[n]$$

$$\Rightarrow \underline{e}[n+1] \approx \underline{e}[n] - 2\mu p(2p-1) v^{2p-2}[n] \underline{x}[n] \underline{x}^T[n] \underline{e}[n] + 2\mu p v^{2p-1}[n] \underline{x}[n]$$

Taking $E\{\}$

$$E\{\underline{e}[n+1]\} \approx (I - 2\mu p(2p-1) E\{v^{2p-2}[n]\} R_{xx}) E\{\underline{e}[n]\} \quad (12)$$

1) $\underline{e} = \underline{Q} \underline{\Lambda} \underline{Q}^T$, thus

$$E\{\underline{e}[n+1]\} \approx (I - 2\mu p(2p-1) E\{v^{2p-2}[n]\} \underline{Q} \underline{\Lambda} \underline{Q}^T) E\{\underline{e}[n]\}$$

$$\Rightarrow E\{\underline{Q}^T \underline{e}[n+1]\} \approx (I - 2\mu p(2p-1) E\{v^{2p-2}[n]\} \underline{\Lambda}) E\{\underline{Q}^T \underline{e}[n]\}, \text{ where } \underline{\Lambda} = \text{Diag}(\lambda_1, \dots, \lambda_N)$$

For convergence of all modes

$$|1 - 2\mu p(2p-1) E\{v^{2p-2}[n]\} \lambda_i| < 1 \quad \forall i$$

$$\Rightarrow -1 < 1 - 2\mu p(2p-1) E\{v^{2(p-1)}[n]\} \lambda_i < 1$$

$$\Rightarrow 0 < \mu < \frac{1}{p(2p-1) E\{v^{2(p-1)}[n]\} \lambda_i} \quad (7)$$

$$\text{Worst case } \lambda_i = \lambda_1 = \lambda_{\max}$$

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