Imperial College London

[MP1 2012]

B.Sc. and M.Sci. EXAMINATIONS 2012

FIRST YEAR STUDENTS OF PHYSICS

MATHEMATICS - M.PHYS 1

Date Tuesday 1st May 2012 10.00 am - 1.00 pm

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

Do not attempt more than SIX questions.

Please use a separate answerbook for each question.

A mathematical formulae sheet is provided.

[*Before starting, please make sure that the paper is complete; there should be EIGHT pages, with a total of TEN questions. Ask the invigilator for a replacement if your copy is faulty.*]

1. (i) Using L'Hôpital's rule, or otherwise, evaluate the limit

$$\lim_{x \to 0} \frac{(1 - \cos x)^{3/2}}{x - \sin x} \ .$$

(ii) Determine the limit

$$\lim_{x \to \infty} \ x^{3/2} \ \left\{ (x+1)^{1/2} \ + \ (x-1)^{1/2} \ - \ 2x^{1/2} \ + \ x^{1/2} \ \sin^2 \ \frac{1}{x} \right\} \ .$$

(iii) Evaluate the definite integral

$$I \ = \ \int_0^1 \ \frac{x^5}{x^2+1} \ dx \ .$$

(iv) Evaluate the definite integral

$$J \ = \ \int_0^\infty \ e^{-x} \ \cos \, x \, dx \ .$$

2. (i) What is the area of the largest rectangle, with vertical sides, that can be inscribed in the ellipse

$$x^2 \ + \ \frac{y^2}{4} \ = \ 1 \ .$$

(ii) Determine the volume, (total) surface area and centre of mass of a solid cone of height L and base radius R.

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3. (i) If $u = x \ln(x^2 + y^2) - 2y \tan^{-1}(y/x)$, for what value of α does the function satisfy

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = u + \alpha x \, .$$

(ii) If $u = \sin x \, \sin y \, \sin(x + y)$, show that

$$\frac{\partial u}{\partial x} = \sin y \, \sin(2x + y) \, ,$$

and similarly determine $\partial u / \partial y$.

Consider the function $u(x,\, y)$ in the square region defined by the lines

$$x = \frac{\pi}{4} \, , \quad x = \frac{\pi}{2} \quad \text{and} \quad y = \frac{\pi}{4} \, , \quad y = \frac{\pi}{2} \, .$$

Determine the location of the stationary points in this square and the associated value of u.

What is the nature of the stationary point ?

4. Consider the complex number $z = -1 + i$:

(i) Write z in polar form.

(ii) Plot z in the complex plane.

(iii) Find z^9 and express the result in real and imaginary form.

(iv) Find $z^{1/3} \equiv \sqrt[3]{z}$ and express the results in polar form.

(v) Find the value of $\cos(z)$ expressed in real and imaginary form.

For the following complex functions determine if they are differentiable for all values $z \in \mathbb{C}$. If yes, find $f'(z)$.

(vi) $f(z) = \mathrm{Im}\,(z)$.

(vii) $f(z) = e^{-z}$.

5. Consider the three vectors $\mathbf{a} = \mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$, $\mathbf{b} = \mathbf{i} - 4\mathbf{j} - 13\mathbf{k}$ and $\mathbf{c} = -3\mathbf{i} + 5\mathbf{j} + 4\mathbf{k}$ where $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ denote the natural basis vectors for $\mathbb{R}^3$.

   (i) Calculate the lengths $|\mathbf{a}|$, $|\mathbf{b}|$ of the vectors $\mathbf{a}$ and $\mathbf{b}$.

   (ii) Calculate the dot (scalar) product $\mathbf{a} \cdot \mathbf{b}$.

   (iii) From the sign of the dot product $\mathbf{a} \cdot \mathbf{b}$, what can you conclude about the angle θ $(0 \le \theta \le \pi)$ between the two vectors $\mathbf{a}$ and $\mathbf{b}$?

   (iv) Find the angle θ $(0 \le \theta \le \pi)$ between the two vectors $\mathbf{a}$ and $\mathbf{b}$.

   (v) Calculate the area of the parallelogram spanned by the two vectors $\mathbf{a}$ and $\mathbf{c}$.

   Consider three planes in $\mathbb{R}^3$ specified by the equations

$$x - 2y - 3z = 2, \tag{1a}$$
$$x - 4y - 13z = 14, \tag{1b}$$
$$-3x + 5y + 4z = 0. \tag{1c}$$

   (vi) Let $\mathbf{A}$ denote the matrix of coefficients for this system of three linear equations in three unknowns. Calculate $\det \mathbf{A}$, the determinant of $\mathbf{A}$.

   (vii) Does the system of linear equations (1a), (1b) and (1c) have a unique solution?

   (viii) Using Gaussian elimination or otherwise, show that the solution $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ to equations (1a), (1b) and (1c) can be written in the form

$$\mathbf{r} = \mathbf{r}_0 + \mu\,\mathbf{d}, \quad \mu \in \mathbb{R}$$

   and identify possible values for $\mathbf{r}_0$ and $\mathbf{d}$.

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6. Consider the 2×2 matrix
$$\mathbf{B} = \begin{pmatrix} 2 & -3 \\ 2 & -5 \end{pmatrix}.$$

   (i) Calculate $\det \mathbf{B}$, the determinant of the matrix $\mathbf{B}$.

   (ii) Find the two eigenvalues $\lambda_1 > \lambda_2$ of the matrix $\mathbf{B}$.

   (iii) Show that $\mathbf{x}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ is an eigenvector of the matrix $\mathbf{B}$ with eigenvalue λ_1.

   (iv) Find the eigenvector $\mathbf{x}_2$ associated with the smallest eigenvalue λ_2.

   (v) Let $\mathbf{S} = (\mathbf{x}_1\, \mathbf{x}_2)$ denote the 2×2 matrix of eigenvectors. Calculate $\mathbf{S}^{-1}\mathbf{B}\mathbf{S}$ explicitly.

   (vi) Why does the inverse $\mathbf{B}^{-1}$ exist?

   (vii) What are the two eigenvalues of $\mathbf{B}^{-1}$?

   (viii) What are the two eigenvectors of $\mathbf{B}^{-1}$?

7. Consider an accelerator that is producing isotopes of type A at a constant rate k in a sample. The isotope of type A is radioactive with a decay rate α and decays into an isotope B which is stable.

   (i) Write down the differential equations governing the time-dependent populations $A(t)$ and $B(t)$.

   (ii) Write down the full name for this set of equations according to the classification of differential equations.

   (iii) Find the steady-state solution to $A(t)$, meaning the solution where $A(t)$ is constant.

   (iv) Find the general solution to $A(t)$ to show that you end up with
$$A(t) \;=\; A_0 e^{-\alpha t} \;+\; \frac{k}{\alpha}\;.$$

   (v) Draw a sketch of $A(t)$ for the two special cases where $A(t = 0) = 0$, and where $A(t = 0) > k/\alpha$.

   (vi) Confirm through simple insertion in the differential equation, or otherwise, that
$$B(t) \;=\; c_1 t \;+\; c_2\, e^{-\alpha t} \;+\; c_3$$
   is a solution to $B(t)$ and find the solution satisfying the initial state conditions $A(t) \;=\; B(t) \;=\; 0$ at $t = 0$.

8. (i) A function $f(x, y, z)$ is given by

$$f(x, y, z) = x^2 e^y + 3 \sin z \; .$$

(a) At the point $(1, 0, \pi/4)$, find the direction in which f is increasing most rapidly.

(b) Find two distinct directions that correspond to a minimum change in $|f|$ at this point.

(c) Find the equation of the plane that is tangent to the surface of constant f at the point $(1, 0, \pi/4)$.

(ii) Consider a variable transformation given by

$$\begin{aligned} u &= x + y \; , \\ v &= x - y \; , \\ w &= z^2 \; . \end{aligned}$$

(a) Write down the position vector $\mathbf{r} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$ in terms of u, v and w. Show that the line element $\mathrm{d}\mathbf{r}$ in (u, v, w) is given by

$$\mathrm{d}\mathbf{r} = \frac{\sqrt{2}}{2}\, du \left[\frac{\sqrt{2}}{2}\,(\hat{\mathbf{i}} + \hat{\mathbf{j}})\right] + \frac{\sqrt{2}}{2}\, dv \left[\frac{\sqrt{2}}{2}\,(\hat{\mathbf{i}} - \hat{\mathbf{j}})\right] + \frac{1}{2\sqrt{w}}\, dw \,[\hat{\mathbf{k}}]$$

and verify that the corresponding unit vectors $\hat{\mathbf{u}}$, $\hat{\mathbf{v}}$ and $\hat{\mathbf{w}}$ (shown within the square braces above) are mutually orthogonal.

(b) A function g is expressed in (u, v, w) coordinates. By considering the expression $\nabla g \cdot \mathrm{d}\mathbf{r}$ or otherwise, show that the $\hat{\mathbf{u}}$ component of ∇g is given by

$$(\nabla g)_u = \sqrt{2}\, \frac{\partial g}{\partial u} \; .$$

(c) Find the corresponding $\hat{\mathbf{v}}$ and $\hat{\mathbf{w}}$ component of ∇g.

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9. (i) A vector field $\mathbf{F} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$ is found within a closed volume V that is bounded below by the $x-y$ plane and above by the surface S satisfying $x^2 + y^2 + \beta^2\, z^2 = a^2$ where β and a are constants.

(a) Show that

$$\iiint_V \nabla \cdot \boldsymbol{F}\, dV \;=\; 2\pi\,\frac{a^3}{\beta}\,.$$

where the lefthand side is the integral of $\nabla \cdot F$ over V

You might find it convenient to transform to new variables $(u,\, v,\, w) = (x,\, y,\, \beta z)$.

(b) Calculate the flux of $\boldsymbol{F}$ entering through the bottom face ($z = 0$) of V.

(c) Hence write down, with suitable explanation, the flux of $\mathbf{F}$ leaving V through the top surface S.

(ii) (a) Show that $\nabla \cdot (\mathbf{A} \times \mathbf{B}) \;=\; \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B}$.

(b) Hence show that

$$\iiint_V \nabla \times \mathbf{F}\, dV \;=\; -\oiint_S \mathbf{F} \times \mathrm{d}\mathbf{S}\;,$$

where S is a closed surface bounding the volume V.

Hint: Consider the vector field $\mathbf{G} \equiv \mathbf{c} \times \mathbf{F}$ *where* $\mathbf{c}$ *is a constant vector and apply the Divergence Theorem.*

10. (i) State Stokes' Theorem, indicating clearly the meaning of all the symbols. Illustrate your answer with a sketch showing the direction and orientation of all surfaces and integrals.

(ii) A vector field $\mathbf{B}$ is given in cylindrical polar coordinates by $\mathbf{B} = \rho z\hat{\boldsymbol{\phi}}$. Define the surface S by that portion of a cylinder, centred on the z-axis, that lies in the first quadrant $(0 < \phi < \pi/2)$ between the planes $z = 0$ and $z = b$. Let the normal to this surface be defined to point away from the z-axis.

(a) Calculate $\displaystyle\iint_S \nabla \times \mathbf{B} \cdot \mathrm{d}\mathbf{S}$ over this surface.

(b) Verify Stokes' Theorem by calculating a suitable line integral.

You may assume the form of the curl operator in cylindrical polar coordinates :

$$\nabla\times\mathbf{A} = \left(\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{\boldsymbol{\rho}} + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right)\hat{\boldsymbol{\phi}} + \left(\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\phi) - \frac{1}{\rho}\frac{\partial A_\rho}{\partial \phi}\right)\hat{\mathbf{k}}\,.$$

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