EE4-29

OPTIMISATION - MODEL ANSWERS 2014

Question 1

a) The revenue for sales is given by

$$\text{revenue} = p(2000 + 4\sqrt{a} - 20p).$$

[2 marks]

b) The costs are

$$\begin{aligned} \text{production cost} &= 2(2000 + 4\sqrt{a} - 20p), \\ \text{development cost} &= 20000, \\ \text{advertising cost} &= a. \end{aligned}$$

Hence

$$\text{total cost} = 24000 + 8\sqrt{a} - 40p + a.$$

[2 marks]

c) The profit is given by

$$\text{profit} = p(2000 + 4\sqrt{a} - 20p) - (24000 + 8\sqrt{a} - 40p + a).$$

[2 marks]

d) The optimization problem is

$$\max_{a,p} = p(2000 + 4\sqrt{a} - 20p) - (24000 + 8\sqrt{a} - 40p + a).$$

[2 marks]

e) The statiory points of the profit are the solutions of the equations

$$0 = \frac{\partial \text{profit}}{\partial a} = 2\frac{p}{\sqrt{a}} - \frac{4}{\sqrt{a}} - 1, \qquad 0 = \frac{\partial \text{profit}}{\partial p} = 2\frac{p}{\sqrt{a}} - \frac{4}{\sqrt{a}} - 1.$$

The only solution is

$$a^\star = \frac{60025}{4} = 15006.25, \qquad p^\star = \frac{253}{4} = 63.25.$$

The Hessian of the profit at the stationary point is

$$H(a^\star, p^\star) = -\begin{bmatrix} \frac{2}{60025} & -\frac{4}{245} \\ -\frac{4}{245} & 40 \end{bmatrix}$$

which is negative definite, hence the point $(a^\star, p^\star)$ is a local maximizer.

[4 marks]

f) The profit for fixed price is

$$\text{profit fix price} = \bar{p}(2000 + 4\sqrt{a} - 20\bar{p}) - (24000 + 8\sqrt{a} - 40\bar{p} + a).$$

i) The optimal advertising cost $\tilde{a}^*$ is given by the solution of the equation

$$0 = \frac{\partial \text{profit fix price}}{\partial a},$$

which gives

$$\tilde{a}^* = 4(\bar{p} - 2)^2.$$

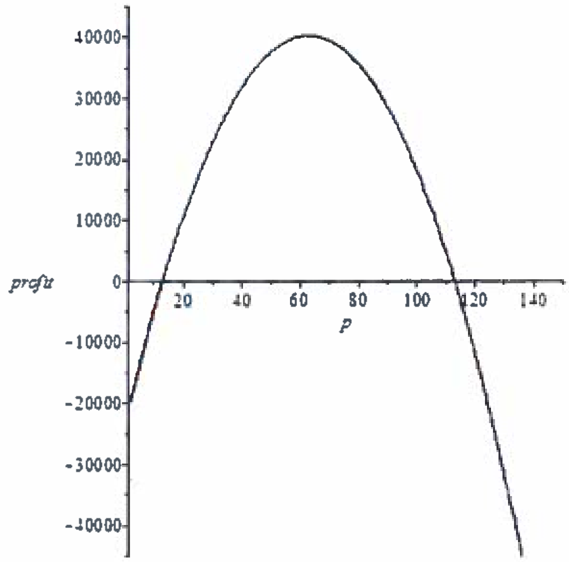
[4 marks]

ii) The resulting optimal profit is

$$\text{profit fix price}^* = 2024\bar{p} - 16\bar{p}^2 - 23984.$$

[2 marks]

iii) The optimal profit as a function of the fixed price $\bar{p}$ is displayed in the graph below. Note that, as $\bar{p}$ increases the optimal profit becomes negative (because of the term $-16\bar{p}^2$).



[2 marks]

Question 2

a) Note that

$$\text{running distance} = a - x$$

and, by Pythagoras' theorem,

$$\text{rowing distance} = \sqrt{x^2+9}.$$

[2 marks]

b) Note that

$$\text{running time} = \frac{a-x}{8}$$

and

$$\text{rowing time} = \frac{\sqrt{x^2+9}}{6},$$

thus

$$T(x,a) = \frac{a-x}{8} + \frac{\sqrt{x^2+9}}{6}.$$

[2 marks]

c) The selection $a = 8$ yields

$$T(x,8) = 1 - \frac{1}{8}x + \frac{\sqrt{x^2+9}}{6}.$$

The minimizers of $T(x,8)$ are either at $x = 0$, or $x = 8$, or at a stationary point in the interval $x \in [0,8]$. The stationary points of $T(x,8)$ are the solutions of

$$0 = \frac{\partial T(x,8)}{\partial x} = -\frac{1}{8} + \frac{1}{6}\frac{x}{\sqrt{x^2+9}}.$$

This equation has a unique solution

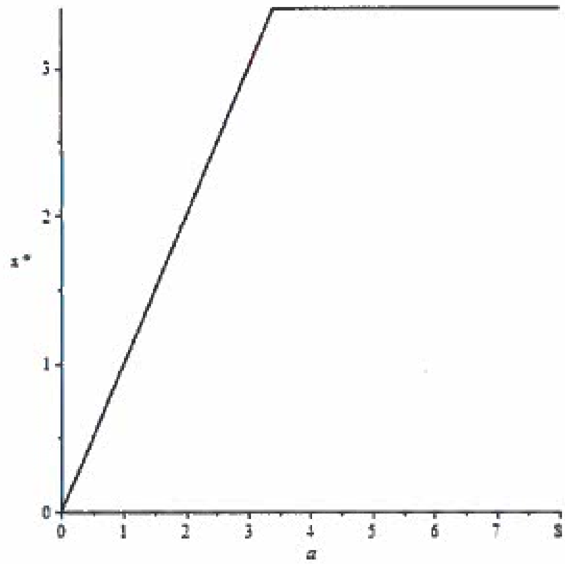
$$x^\star = 9\frac{\sqrt{7}}{7}.$$

Note now that

$$T(0,8) = 1.5, \qquad T(x^\star,8) = 1.3307, \qquad T(8,8) = 1.49.$$

As a result, the function $T(x,8)$ attains its minimum at $x = x^\star$. [4 marks]

d) The optimal solution of the problem, as a function of a, is obtained as in the previous point, considering the stationary points of the function $T(x,a)$ in the interval $x = [0,a]$ and the values of the function at the boundary, that is $x = 0$ and $x = a$. Note that the stationary point of the function $T(x,a)$ is, for any a located at $x^\star$ (that is the value computed in the previous point). Note now that the function $T(x,a)$ is decreasing for all $a \in [0,x^\star)$, and increasing for $a > x^\star$. As a result, if $a \in [0,x^\star]$ then the optimal x is $x = a$, whereas if $a > x^\star$ then the optimal x is $x^\star$. The graph of the optimal x as a function of a is given below.



[6 marks]

e) The constraints $T(x,a) = 1$ can be solved to give

$$a = 8 + x - \frac{4}{3}\sqrt{x^2+9}.$$

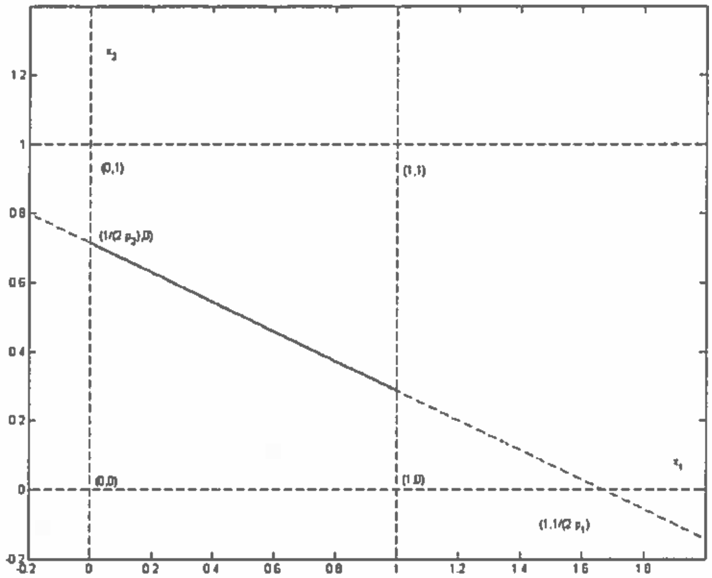
Replacing the value of a in the total distance gives

$$D_{total} = 8 - \frac{1}{3}\sqrt{x^2+9}.$$

This function is maximized at $x = 0$. These results imply that if the man wants to maximize the distance travelled in a fixed time, that is one hour, he has to row straight to C and then start running. This is consistent with the fact that to maximize the distance travelled one has to minimize the time spent rowing, since rowing is performed at a slower speed than running. [6 marks]

Question 3

a) The lines $x_1 = 0$, $x_1 = 1$, $x_2 = 0$, $x_2 = 1$ and $x_1 p_1 + x_2 p_2 - 1/2 = 0$ are represented by the dashed lines in the figure below, and the admissible set by the solid line. Note that the graph exploits the conditions $0 < p_1 < 1/2 < p_2 < 1$ and the condition $p_1 + p_2 = 1$.



[2 marks]

b) Clearly, admissible points are either on the boundary of the admissible set, that is with $x_1 = 0$ or $x_1 = 1$, or are in the *middle* of the admissible set which gives the three types

$$Type_1 = (0,\star), \qquad Type_2 = (\star,1), \qquad Type_3 = (\star,\star).$$

[4 marks]

c) The Lagrangian of the problem is

$$\begin{aligned} L(x_1,x_2,\lambda,\rho_1,\rho_2,\rho_3,\rho_4) &= 1-(x_1q_1+x_2q_2)+\lambda(x_1p_1+x_2p_2-1/2)+ \\ &\quad \rho_1(-x_1)+\rho_2(x_1-1)+\rho_3(-x_2)+\rho_4(x_2-1). \end{aligned}$$

The necessary conditions of optimality are

$$0 = \frac{\partial L}{\partial x_1} = -q_1 + \lambda p_1 - \rho_1 + \rho_2, \quad 0 = \frac{\partial L}{\partial x_2} = -q_2 + \lambda p_2 - \rho_3 + \rho_4,$$

$$0 = x_1p_1 + x_2p_2 - 1/2, \quad -x_1 \leq 0, \quad x_1 - 1 \leq 0, \quad -x_2 \leq 0, \quad x_2 - 1 \leq 0,$$

$$\rho_1 \geq 0, \quad \rho_2 \geq 0, \quad \rho_3 \geq 0, \quad \rho_4 \geq 0,$$

$$\rho_1(-x_1) = 0, \qquad \rho_2(x_1-1) = 0, \qquad \rho_3(-x_2) = 0, \qquad \rho_4(x_2-1) = 0.$$

[4 marks]

d) Note that the constrains $x_2 \geq 0$ and $x_2 \leq 1$ are never active, hence $\rho_3 = 0$ and $\rho_4 = 0$. As a result one has the following results.

- At the point of Type 1 the necessary conditions give

$$x_1 = 0, \qquad \rho_2 = 0, \qquad \lambda = \frac{q_2}{p_2}, \qquad x_2 = \frac{1}{2p_2}, \qquad \rho_1 = -\frac{p_2 q_1 - p_1 q_2}{p_2}.$$

- At the point of Type 2 the necessary conditions give

$$x_1 = 1, \qquad \rho_1 = 0, \qquad \lambda = \frac{q_2}{p_2}, \qquad x_2 = \frac{1 - 2p_1}{2p_2}, \quad \rho_2 = \frac{p_2 q_1 - p_1 q_2}{p_2}.$$

- At the points of Type 3 the necessary conditions give

$$\rho_1 = 0, \rho_2 = 0, \qquad \lambda = \frac{q_1}{p_1} = \frac{q_2}{p_2}, \qquad x_1 p_1 + x_2 p_2 - 1/2 = 0.$$

As a result of the above discussion the following holds.

i) If $q_1 p_2 - q_2 p_1 < 0$ then the point of Type 1 is the only candidate point, and the optimal solution is the point $(0, \frac{1}{2p_2})$. [4 marks]

ii) If $q_1 p_2 - q_2 p_1 > 0$ then the point of Type 2 is the only candidate point, and the optimal solution is the point $(1, \frac{1 - 2p_1}{2p_2})$. [2 marks]

iii) If $q_1 p_2 - q_2 p_1 = 0$, then $\frac{q_1}{p_1} = \frac{q_2}{p_2}$, hence all points (that is Type 1, Type 2 and Type 3) are candidate optimal solutions. However, in this case the objective function can be rewritten as

$$\begin{aligned} 1 - (x_1 q_1 + x_2 q_2) &= 1 - \left(x_1 q_1 + x_2 \frac{p_2}{p_1} q_1 \right) \\ &= 1 - \frac{q_1}{p_1}(p_1 x_1 + p_2 x_2) \\ &= 1 - \frac{q_1}{p_1}\frac{1}{2}, \end{aligned}$$

that is the objective function is constant on the admissible set, and all admissible points are optimal. [4 marks]

Question 4

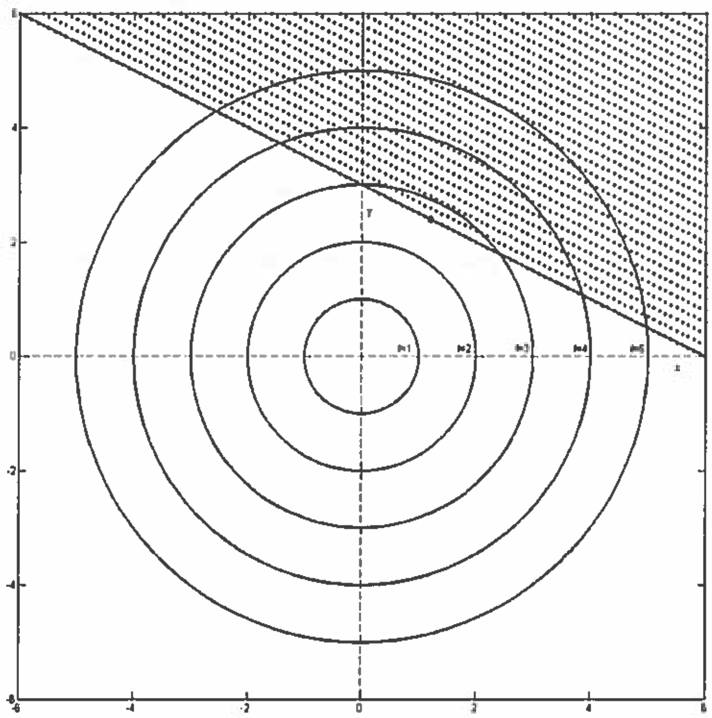
a) The admissible set is the dashed area in the figure below, and the level lines of the objective functions are circles, centered at the origin, with the *level* indicated in the figure. The solution of the optimization problem is given by the point in which the boundary of the admissible set is tangent to a level line, indicated with a black dot in the figure. This point can be calculated, replacing the equation

$$0 = 6 - x - 2y$$

in the objective function, yielding

$$(6-2y)^2 + y^2,$$

and minimizing with respect to y, thus yielding $y = \frac{12}{5}$, hence $x = \frac{6}{5}$.



[4 marks]

b) i) The stationary points of the function $B_\varepsilon(x,y)$ are the solutions of the equations

$$0 = \frac{\partial B_\varepsilon}{\partial x} = 2x + \frac{\varepsilon}{6-x-2y}, \qquad 0 = \frac{\partial B_\varepsilon}{\partial x} = 2y + \frac{2\varepsilon}{6-x-2y}.$$

Note that solutions of these equations are such that

$$x = \frac{1}{2}y.$$

Using this condition yields the equation

$$0 = \frac{5y^2 - 12y - 2\varepsilon}{5y - 12},$$

hence the two stationary points

$$P_1 = \left(\frac{3}{5} + \frac{\sqrt{36+10\varepsilon}}{10}, \frac{6}{5} + \frac{\sqrt{36+10\varepsilon}}{5}\right),$$

$$P_2 = \left(\frac{3}{5} - \frac{\sqrt{36+10\varepsilon}}{10}, \frac{6}{5} - \frac{\sqrt{36+10\varepsilon}}{5}\right).$$

Note that P_1 is admissible for all $\varepsilon \geq 0$, whereas P_2 is not admissible for all $\varepsilon \geq 0$. [8 marks]

ii) The Hessian of the function $B_\varepsilon(x,y)$ at P_1 is

$$\nabla^2 B_\varepsilon(P_1) = 2I + \frac{\varepsilon}{(6-\sqrt{36+10\varepsilon})^2}\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix},$$

which is positive definite for all $\varepsilon \geq 0$, hence P_1 is a local minimizer of B_ε. [6 marks]

iii) Note that

$$\lim_{\varepsilon \to 0} P_1 = \left(\frac{6}{5}, \frac{12}{5}\right)$$

which coincides with the optimal solution determined in part a). [2 marks]