UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 2000

MEng Honours Degree in Mathematics and Computer Science Part IV
MEng Honours Degrees in Computing Part IV
MSc in Advanced Computing
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the Associateship of the City and Guilds of London Institute
This paper is also taken for the relevant examinations for the Associateship of the Royal College of Science

PAPER C473

DOMAIN THEORY AND FRACTALS

Wednesday 3 May 2000, 10:00
Duration: 120 minutes

Answer THREE questions

Paper contains 4 questions

1a Suppose that $f : D \to D$ is a monotone function on the cpo D.

i) Prove that

$$\bigsqcup_{n\geq 0} f^n(\perp) \sqsubseteq f(\bigsqcup_{n\geq 0} f^n(\perp)).$$

ii) Show by an example that

$$\bigsqcup_{n\geq 0} f^n(\perp) = f(\bigsqcup_{n\geq 0} f^n(\perp))$$

does not necessarily hold.

b The function $f : \mathbb{N} \to \mathbb{N}$ is defined by

$$f(x) := \text{if } x = 0 \text{ then } 0 \text{ else } 2 * x + f(x-1) - 1.$$

Obtain f as the least fixed point of a continuous function

$$F : [\mathbb{N}_\perp \to_s \mathbb{N}_\perp] \to [\mathbb{N}_\perp \to_s \mathbb{N}_\perp],$$

where $[\mathbb{N}_\perp \to_s \mathbb{N}_\perp]$ is the strict function space of $\mathbb{N}_\perp$. (You need not prove that F is continuous, or that it maps continuous functions to continuous functions.)
Compute $F^n(\perp_{[\mathbb{N}_\perp \to_s \mathbb{N}_\perp]})$ for $n \in \mathbb{N}$. What does f compute?

c Let $\Sigma = \{0, 1\}$, and let $\mathsf{Str}(\Sigma)$ be the cpo of finite and infinite strings over Σ, ordered by the prefix ordering.

i) Let $f : \mathsf{Str}(\Sigma) \times \mathsf{Str}(\Sigma) \to \mathsf{Str}(\Sigma)$ be defined by concatenation:

$$f(x, y) = \begin{cases} xy & \text{if } x \text{ is finite} \\ x & \text{if } x \text{ is infinite.} \end{cases}$$

Check whether f is continuous, and prove your answer.

ii) Let $g : \mathsf{Str}(\Sigma) \to \mathsf{Str}(\Sigma)$ be recursively defined by

$g(\varepsilon) = \varepsilon$, $g(0x) = g(x)$, and $g(1x) = 1g(x)$.

α) Prove that $g(xy) = g(x)g(y)$ for all *finite* x in $\mathsf{Str}(\Sigma)$ and all y in $\mathsf{Str}(\Sigma)$.

β) Hence or otherwise prove that g is continuous.

The three parts carry, respectively, 30%, 40%, and 30% of the marks.

2a i) Let D and E be two cpos. Define

α) an *embedding-projection pair* $(e, p) : D \lhd E$, and

β) the *smash product* $D \otimes E$.

ii) Prove that a projection preserves all existing greatest lower bounds, but not necessarily least upper bounds.

b Consider the domain equation

$$D \cong F(D) = \{\perp\} + D$$

in **CPO**.

i) Find the iterates $D_n = F^n(\{\perp\})$ for $n = 0, 1, 2$ and the corresponding embedding-projection pairs $(e_n, p_n) : D_n \lhd D_{n+1}$ for $n = 0, 1$.

ii) Obtain the solution of the domain equation and check that it is a cpo.

iii) Prove directly that your solution satisfies the domain equation by obtaining the corresponding isomorphism in the equation.

The two parts carry, respectively, 35% and 65% of the marks.

3a i) Define

α) a *finite* element of a cpo,

β) an ω-*algebraic* cpo, and

γ) a *Scott domain.*

ii) For each of the following functions, check whether they are finite elements of the function space $[\mathbb{N}_\perp \to \mathbb{N}_\perp]$, and prove your answer.

α) $f : \mathbb{N}_\perp \to \mathbb{N}_\perp$ which maps all arguments (including $\perp$) to the number 0.

β) $g : \mathbb{N}_\perp \to \mathbb{N}_\perp$ which maps $\perp$ to $\perp$ and all other arguments to the number 0.

b Prove that equality of a real number with 0 is undecidable.

c Consider the normal product

$$x = \begin{pmatrix} 1 & 4 \\ 2 & 1 \end{pmatrix}^\omega = \begin{pmatrix} 1 & 4 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ 2 & 1 \end{pmatrix}\cdots .$$

i) Compute the first two intervals approximating x.

ii) From the normal product, obtain a fixed point equation for x, and hence determine x.

iii) Express $\frac{x+1}{x-1}$ as a normal product, and compute its first two approximating intervals.

The three parts carry, respectively, 35%, 25%, and 40% of the marks.

4a i) Define the *convex power domain* of an ω-algebraic cpo in terms of its poset of finite elements.

ii) Let $D = \{0, 1, 2\}$ be the cpo with order $0 \sqsubseteq 1 \sqsubseteq 2$. Construct explicitly the pre-order representing the convex power domain of D.

b Consider the iterated function system $\{f_1, f_2, f_3\}$ in $\mathbb{R}$ given by the maps $f_1, f_2, f_3 : \mathbb{R} \to \mathbb{R}$ defined by

$$\begin{aligned} f_1(x) &= \frac{x}{5} \\ f_2(x) &= \frac{x+4}{5} \\ f_3(x) &= \frac{x+8}{5}. \end{aligned}$$

i) Find the fixed points and the contractivity factors of f_1, f_2, and f_3.

ii) Find the smallest closed interval $I = [l, u]$ which is mapped by all three functions into itself.

iii) Define a function $f : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ which can be used to obtain the attractor, and compute the first iterate $f(I)$.

iv) Devise a labelling scheme to model the generation of the attractor of the IFS by the cpo $(\mathrm{Str}\{1, 2, 3\}, \sqsubseteq)$, i.e. the set of finite and infinite strings of $\{1, 2, 3\}$ with the prefix ordering.

v) Draw I and $f(I)$, and indicate how $f^2(I)$ would look like (in your drawing, the lengths and positions of the subintervals need not be accurate). Label all parts according to the labeling scheme devised in part (iv).

vi) Are there points of the attractor which have more than one address?

vii) What is the similarity dimension of the attractor?

The two parts carry, respectively, 30% and 70% of the marks.