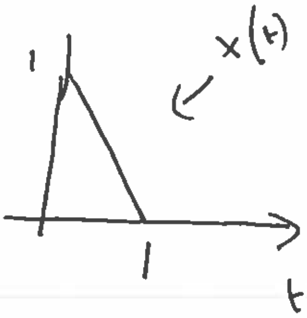
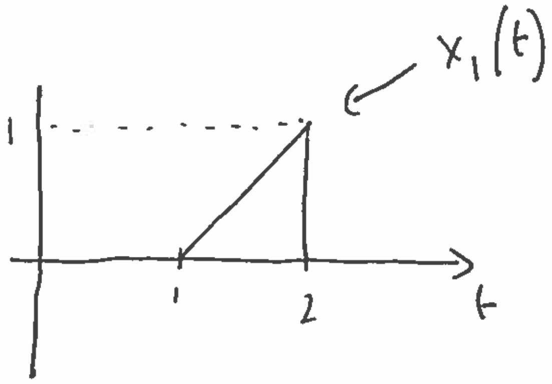
SOLUTIONS EE2-05

QUESTION 1

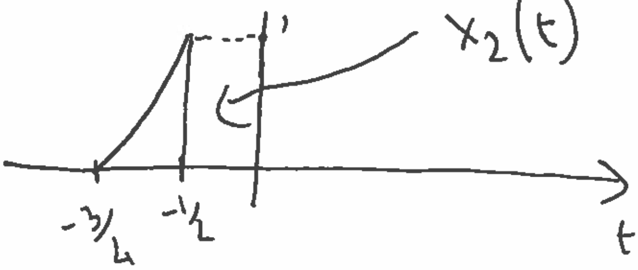
(a)



i.



ii.



(b) A CAUSAL SYSTEM WITH RATIONAL LAPLACE TRANSFORM $H(s)$ IS STABLE IF AND ONLY IF ALL THE POLES OF $H(s)$ HAVE A NEGATIVE REAL PART.

i. THUS, SYSTEM $H_1(s)$ IS STABLE SINCE THE ROOTS OF $s^2+4s+13$ ARE $s_1=-2+j3$ AND $s_2=-2-j3$

ii. SINCE $H_2(s)$ IS UNSTABLE SINCE THE ROOTS OF s^2+s-2 ARE $s_1=-2$ AND $s_2=1$. (SYSTEM UNSTABLE BECAUSE OF $s_2=1$).

SOLUTIONS

Q.1

(c) A SYSTEM IS BIBO STABLE IF AND ONLY IF

$$\int_{-\infty}^{\infty} |h(t)|\,dt < \infty \qquad (1)$$

THE CONDITION $|h(t)| \leq K$ DOES NOT GUARANTEE (1). ASSUME FOR EXAMPLE THAT $|h(t)| = K$ FOR ANY t THEN

$$\int_{-\infty}^{\infty} |h(t)|\,dt = K \int_{-\infty}^{\infty} dt = \infty$$

(d)

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

$$= \int_{0}^{1} e^{-\tau} x_2(t-\tau)\, d\tau$$

THEREFORE

$$x_1(t) * x_2(t) = \begin{cases} 0 & t < 0 \\ \int_0^t e^{-\tau}\, d\tau = 1 - e^{-t} & 0 \leq t < 1 \\ \int_{-1+t}^{1} e^{-\tau}\, d\tau = e\, e^{-t} - \frac{1}{e} & 1 \leq t < 2 \\ 0 & t \geq 2 \end{cases}$$

SOLUTIONS

Q. 1

(a)

i. CHARACTERISTIC POLYNOMIAL : $x^2 + 7x + 10$

CHARACTERISTIC ROOTS : $x_{1,2} = \frac{-7 \pm \sqrt{49-40}}{2} = \begin{cases} -5 \\ -2 \end{cases}$

CHARACTERISTIC MODES : e^{-5t}, e^{-2t}

ii. $y(t) = c_1 e^{-5t} + c_2 e^{-2t}$

$$\left.\begin{aligned} y(0) &= c_1 + c_2 = 1 \\ \dot{y}(0) &= -5c_1 - 2c_2 = 0 \end{aligned}\right\} \Rightarrow \begin{aligned} c_1 &= -2/3 \\ c_2 &= 5/3 \end{aligned}$$

$$y(t) = \left(\frac{5}{3} e^{-2t} - \frac{2}{3} e^{-5t}\right) u(t)$$

iii IN THE LAPLACE DOMAIN WE HAVE :

$$s^2 Y(s) + 7s Y(s) + 10 Y(s) = X(s) \Rightarrow$$

$$\therefore Y(s) = \frac{X(s)}{s^2 + 7s + 10}$$

SINCE $x(t) = e^{-t} u(t) \Rightarrow X(s) = \frac{1}{s+1}$

THEREFORE

$$Y(s) = \frac{1}{(s+5)(s+2)(s+1)}$$

SOLUTIONS

Q.1

(a) iii

USING PARTIAL FRACTIONS

$$Y(s) = \frac{A}{s+5} + \frac{B}{s+2} + \frac{C}{s+1}$$

$$= \frac{1}{12}\frac{1}{s+5} - \frac{1}{3(s+2)} + \frac{1}{4(s+1)}$$

THEREFORE

$$y(t) = \left(\frac{1}{12}e^{-5t} - \frac{1}{3}e^{-2t} + \frac{1}{4}e^{-t}\right)u(t)$$

iv

THE TOTAL RESPONSE IS THE SUM OF THE RESPONSES OF PART ii AND iii

$$y(t) = \left(\frac{4}{3}e^{-2t} - \frac{7}{12}e^{-5t} + \frac{1}{4}e^{-t}\right)u(t)$$

(b) IN THE LAPLACE DOMAIN

$$Y(s) = H(s)X(s) \Rightarrow H(s) = \frac{Y(s)}{X(s)}$$

SINCE $x(t) = e^{-2t}u(t) \Rightarrow X(s) = \dfrac{1}{s+2}$

MOREOVER, SINCE $y(t) = \left(e^{-t} - e^{-2t}\right)u(t) \Rightarrow$

$$Y(s) = \frac{1}{s+1} - \frac{1}{s+2} = \frac{1}{(s+1)(s+2)}$$

SOLUTIONS

Q.1 (f)

CONSEQUENTLY

$$H(s) = \frac{Y(s)}{X(s)} = \frac{(s+2)}{(s+1)(s+2)} \Rightarrow h(t) = e^{-t}u(t)$$

(g)

$$X[z] = \frac{z}{z^2 - 5z + 6} = \frac{z}{z-3} - \frac{z}{z-2}$$

~~CONSE~~ HENCE

$$x[n] = \left(3^n - 2^n\right)u[n]$$

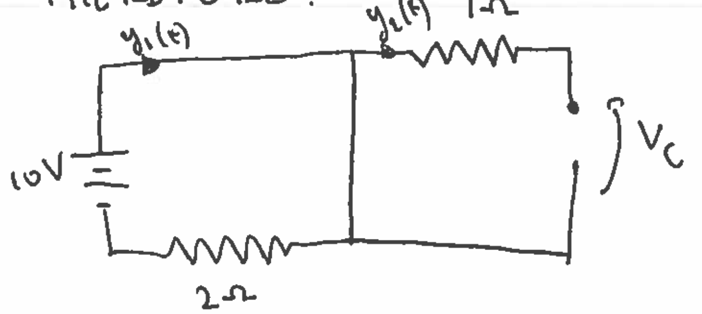
NOTE THAT EVENTHOUGH $x[n]$ DIVERGES, THE Z-TRANSFORM ~~IS~~ ~~WELL D~~ EXISTS FOR $|z| > 3$

SOLUTIONS

QUESTION 2

(a) IN STADY STATE INDUCTORS BEHAVE LIKE SHORT CIRCUIT AND CAPACITORS AS OPEN CIRCUITS.

THEREFORE:



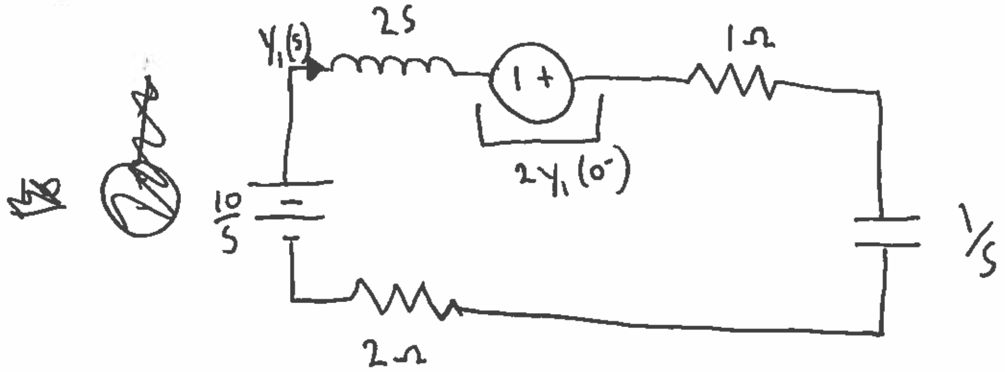
[2/5]

CONSEQUENTLY THE INITIAL CONDITIONS ARE:

$$y_2(0^-) = 0 \,, \quad V_c(0^-) = 0 \quad \text{AND} \quad y_1(0^-) = \frac{10}{2} = 5\text{ A}$$

[5/5]

(b) FOR $t > 0$ WE HAVE IN THE LAPLACE DOMAIN:



[3/5]

THEREFORE

$$\frac{10}{s} = 3Y_1(s) + 2s\,Y_1(s) - 2y_1(0^-) + \frac{1}{s}Y_1(s);$$

$$\left(2s + 3 + \frac{1}{s}\right)Y_1(s) = \frac{10}{s} + 10$$

[5/5]

SOLUTIONS

Q.2

(c) $\left(2S^2+3S+1\right)Y_1(S) = 10 + 10S$

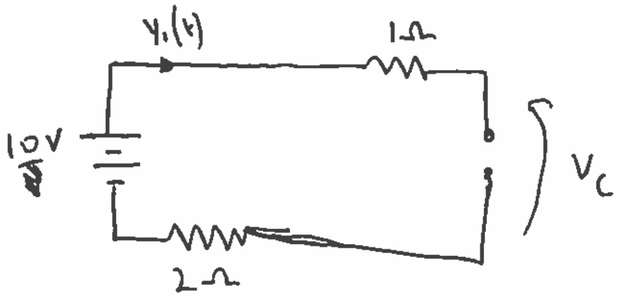
$$Y_1(S) = \frac{10+10S}{\left(2S^2+3S+1\right)} = \frac{5(S+1)}{(S+1)\left(S+\frac{1}{2}\right)} = \frac{5}{S+\frac{1}{2}}$$

THEREFORE

$$y_1(t) = 5e^{-t/2}u(t)$$

(d)

i. STEADY STATE CIRCUIT

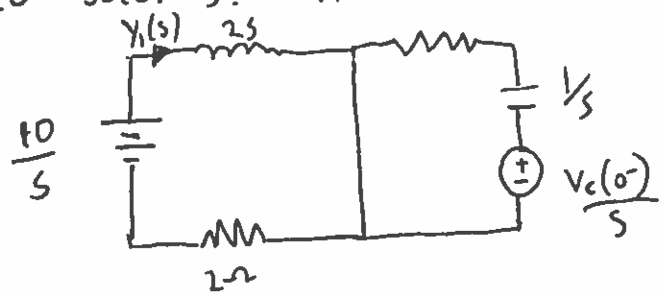


THIS IMPLIES

$$y_1(10^-) = 0 \quad , \quad y_2(10^-) = 0 \quad \text{AND} \quad V_C(10^-) = 10$$

ii.

SINCE THE CIRCUIT IS TIME-INVARIANT I CAN ASSUME THAT $t=0$ RATHER THEN $t=10$ AND THEN SHIFT THE FINAL SOLUTION BY 10 SECONDS. THEREFORE, IN THE LAPLACE DOMAIN WE HAVE:



$\left[\frac{2}{5}\right]$

Q.2 SOLUTIONS

~~THE~~

THE LOOP EQUATION IS

$$\frac{10}{s} = 2s\,Y_1(s) + 2Y_1(s) \Rightarrow Y_1(s) = \frac{10}{2s(s+1)}$$

USING PARTIAL FRACTIONS

$$Y_1(s) = \frac{5}{s} - \frac{5}{s+1} \Rightarrow y_1(t) = \left(5 - 5e^{-t}\right)u(t)$$

SHIFTING BY 10 SECONDS WE ARRIVE AT THE CORRECT ANSWER:

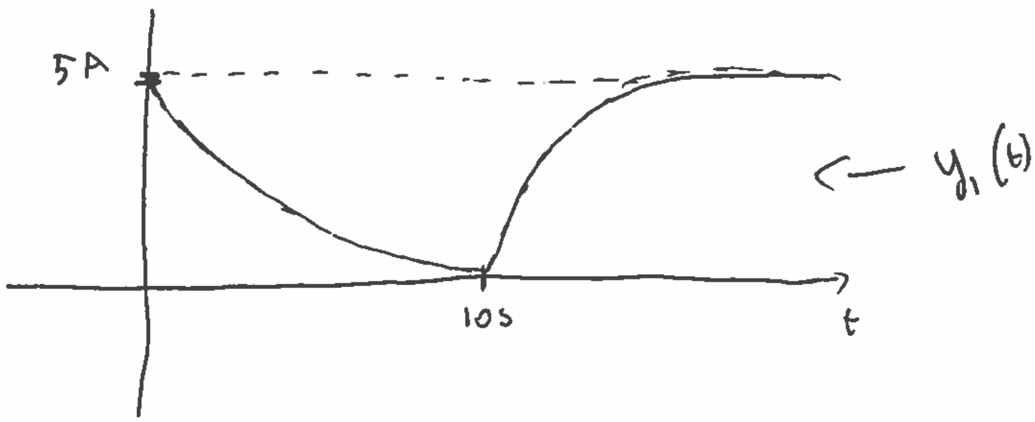
$$y_1(t) = \left(5 - 5e^{-(t-10)}\right)u(t-10).$$

[5/5]

(e) COMPLETE $y_1(t)$ FOR $t>0$:

$$y_1(t) \approx 5e^{-t/2}\left(u(t) - u(t-10)\right) + \left(5 - 5e^{-(t-10)}\right)u(t-10)$$

THIS SOLUTION IS APPROXIMATELY CORRECT SINCE WE HAVE ASSUMED STEADY-STATE AT t=10 SECONDS.



SOLUTIONS

QUESTION 3

(a)

THE RESPONSE OF THE CASCADE OF TWO LTI SYSTEM IS THE PRODUCT OF THE TWO TRANSFER FUNCTIONS :

$G(s) \cdot H(s)$ AND WE WANT THIS TO BE ONE $\Rightarrow$ $G(s) = s^2 + 2s + 1$.

(b)

ASSUME THE INPUT WITH TRANSFER FUNCTION $H(s)$ TO BE $x(t) = e^{s_0 t}$ FOR SOME CONSTANT s_0.

THE OUTPUT $y(t)$ IS GIVEN BY

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau) e^{s_0 t} e^{-s_0 \tau} d\tau$$

$$= H(s_0) e^{s_0 t}$$

2/4

WE WANT TO FIND s_0 SUCH THAT $y(t) = 0$

WE THUS NEED TO FIND THE s_0 SUCH THAT $H(s_0) = 0$

Q.3

ROOTS OF S^2+4 ARE $S_1 = 2j$, $S_2 = -2j$

THUS AN INPUT $x(t) = e^{\pm 2j}$ LEADS TO $y(t) = 0$

4/7

WE WANT A REAL-VALUED INPUT, USING EULER IDENTITIES WE HAVE THAT

$$x_*(t) = 2\cos 2t = e^{2j} + e^{-2j} \Rightarrow \text{THAT}$$

$$y(t) = h(t) * 2\cos 2t = H(2j)e^{2j} + H(-2j)e^{-2j} = 0$$

SO GIVEN $x_1(t)$, IF WE SET $x_2(t) = x_1(t) + \cos 2t$

THE TWO REAL-VALUED INPUTS PRODUCE THE SAME OUTPUT

7/7

(c) i. THE INPUT TO $g(t)$ IS

$$y(t) = h_0 x(t) + h_1 x(t-T) + h_2 x(t-2T) + \ldots + \ldots \quad \text{EQ. 1.}$$

THE OUTPUT OF $g(t)$ IS

$$f(t) = g_0 y(t) + g_1 y(t-T) + g_2 y(t-2T) + \ldots \quad \text{EQ. 2.}$$

AND WE WANT $f(t) = x(t)$

3/6

SOLUTIONS

Q.3

BY REPLACING $y(t)$ OF EQ. 1 INTO EQ. 2
~~[illegible]~~ WE OBTAIN

$$x(t) = g_0 h_0 x(t) + g_0 h_1 x(t-T) + g_0 h_2 x(t-2T) + \dots$$
$$+ g_1 h_0 x(t-T) + g_1 h_1 x(t-2T) + \dots$$
$$+ g_2 h_0 x(t-2T) + \dots$$

WHICH LEADS TO THE FOLLOWING EQUATIONS:

$g_0 h_0 = 1$ EQ. 1

$g_0 h_1 + g_1 h_0 = 0$ EQ. 2

$g_0 h_2 + g_1 h_1 + g_2 h_0 = 0$ EQ. 3

$\vdots$

$\sum_{k=0}^{L-1} g_k h_{L-1-k} = 0$ EQ. L

6/6

(c) ii.

$h_0 = 1$, $h_1 = \frac{1}{2}$, $h_2 = \frac{1}{4}$

$\Rightarrow$ $g_0 = 1$

$g_1 = -g_0 h_1 = -\frac{1}{2}$

$g_2 = -\frac{1}{4} + \frac{1}{4} = 0$

IN FACT $g_k = 0$ $k \geq 2$

AN ALTERNATIVE WAY TO PROOVE THIS FACT IS BY NOTING THE FOLLOWING

$$\delta(t-kT) \Longleftrightarrow e^{-kTs}$$

FOR THIS REASON, WE HAVE:

$$H(s) = \sum_{k=0}^{\infty} h_k e^{-kTs} = \sum_{k=0}^{\infty} \left(\frac{1}{2e^{Ts}}\right)^k$$

$$\overset{(a)}{=} \frac{1}{1 - \frac{1}{2e^{Ts}}}$$

WHERE IN (a) WE USED $\sum_{k=0}^{\infty} \rho^k = \frac{1}{1-\rho}$ FOR $|\rho|<1$

NOW $G(s) = \frac{1}{H(s)} = 1 - \frac{1}{2} e^{-Ts} \Rightarrow g(t) = 1 - \frac{1}{2}\delta(t-T)$

$\Rightarrow g_0 = 1 \quad g_1 = -\frac{1}{2} \quad g_k = 0$ FOR $k \geq 2$.

Q. 3 BY OPERATING IN THE LAPLACE DOMAIN WE HAVE THAT

$$F(s) = H_0(s)\left[Y(s) - H_1(s)\,F(s)\right] \Rightarrow$$

$$F(s)\left[1 + H_0(s)H_1(s)\right] = H_0(s)\,Y(s)$$

THUS

$$G(s) = \frac{F(s)}{Y(s)} = \frac{H_0(s)}{1 + H_0(s)H_1(s)}$$

3/2

WE WANT $G(s) \simeq \frac{1}{H(s)} \Rightarrow \frac{H_0(s)}{1 + H_0(s)H_1(s)} \simeq \frac{1}{H(s)}$

WE SET $H_0(s) = K$ FOR SOME CONSTANT K

$\Rightarrow \frac{K}{1 + KH_1(s)} \simeq \frac{1}{H(s)}$ THEN IF $K >> 1$ AND

$H_1(s) = H(s)$, WE HAVE THAT $G(s) \simeq \frac{1}{H(s)}$.

7/7