EE4-27
EE9-CS5-2
EE9-FPN2-08

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2016

MSc and EEE/EIE PART IV: MEng and ACGI

**Corrected copy**

**SYSTEMS IDENTIFICATION**

Tuesday, 3 May 10:00 am

Time allowed: 3:00 hours

**There are FOUR questions on this paper.**

**Answer FOUR questions.**

*All questions carry equal marks*

**Any special instructions for invigilators and information for candidates are on page 1.**

Examiners responsible First Marker(s) : T. Parisini
Second Marker(s) : S.A. Evangelou

1. Consider a discrete-time stochastic system described by the block-scheme shown in Fig. 1.1.

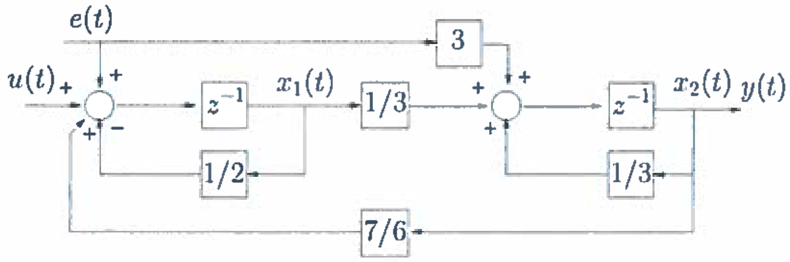


Figure 1.1 Block-scheme of the stochastic dynamic system.

where $u(t)$ is a *known deterministic* input and $e(\cdot) \sim WN(0,1)$ is a white stochastic process.

a) Show that the stochastic system depicted in Fig. 1.1 can be described by a model belonging to the family of $ARMAX(n,n)$ stochastic models in canonical form, that is

$$A(z)y(t) = B(z)u(t) + C(z)e(t)$$

where $A(z) = 1 + a_1 z^{-1} + \cdots + a_n z^{-n}$, $B(z) = b_1 z^{-1} + \cdots + b_n z^{-n}$, and $C(z) = 1 + c_1 z^{-1} + \cdots + c_n z^{-n}$.

[ 6 Marks ]

b) Determine the difference equation by which the optimal one-step ahead prediction $\hat{y}(t+1|t)$ can be computed using the knowledge of the sequences $y(t)$ and $u(t)$ till time-instant $t$.

[ 3 Marks ]

c) Set $u(t) = 0$, $\forall t \geq 0$.

i) Sketch approximately the spectrum $\Gamma_y(\omega)$ of the process $y(\cdot)$ described by the model obtained in your answer to Question 1a) in the interval $\omega \in [-\pi, \pi]$.

[ 4 Marks ]

ii) Determine the difference equation by which the optimal two-step ahead prediction $\hat{y}(t+2|t)$ can be computed using the knowledge of the sequence $y(t)$ till time-instant $t$.

[ 4 Marks ]

iii) Compute the variance of the prediction error $\varepsilon_1(t) = y(t+1) - \hat{y}(t+1|t)$ associated with the optimal one-step ahead predictor determined in your answer to Question 1b) and the variance of the prediction error $\varepsilon_2(t) = y(t+2) - \hat{y}(t+2|t)$ associated with the optimal two-step ahead predictor determined in your answer to Question 1c)ii). Compare the two variances and discuss your findings.

[ 3 Marks ]

2. Consider a stochastic process $v(\cdot)$ and an arbitrarily large number $N$ of measurements $\{v(1), v(2), \dots, v(N)\}$. Moreover, consider the following two families of AR models:

$$\mathcal{M}_1(\theta_1): \quad v(t) = av(t-1) + \xi(t), \quad \theta_1 = a$$
$$\mathcal{M}_2(\theta_2): \quad v(t) = a_1 v(t-1) + a_2 v(t-2) + \xi(t), \quad \theta_2 = [a_1\,, a_2]^\top .$$

where $\xi(\cdot)$ is a generic white process uncorrelated with $v(\cdot)$. Denote with $\widehat{\theta}_1(N) = \widehat{a}(N)$ the least squares estimate of $\theta_1$ based on $N$ measurements $\{v(1), v(2), \dots, v(N)\}$. Likewise denote with $\widehat{\theta}_2(N) = [\widehat{a}_1(N), \widehat{a}_2(N)]^\top$ the least squares estimate of $\theta_2$ based on $N$ measurements $\{v(1), v(2), \dots, v(N)\}$.

a) **Case 1.** Suppose that the process is generated as

$$v(t) = e(t) + \frac{1}{2}e(t-1), \quad e(\cdot) \sim WN(0,1)$$

i) Determine the value $\overline{\theta}_1^{(1)}$ the estimate $\widehat{\theta}_1(N)$ approaches for large values of $N$ (that is, $\overline{\theta}_1^{(1)} = \lim\limits_{N\to\infty} \widehat{\theta}_1(N)$, a.s.) when the family $\mathcal{M}_1$ of models is used.

[ 3 Marks ]

ii) Determine the value $\overline{\theta}_2^{(1)}$ the estimate $\widehat{\theta}_2(N)$ approaches for large values of $N$ (that is, $\overline{\theta}_2^{(1)} = \lim\limits_{N\to\infty} \widehat{\theta}_2(N)$, a.s.) when the family $\mathcal{M}_2$ of models is used.

[ 3 Marks ]

iii) Compute the variances of the prediction errors when using the models $\mathcal{M}_1(\overline{\theta}_1^{(1)})$ and $\mathcal{M}_2(\overline{\theta}_2^{(1)})$ and compare them. Discuss your findings.

[ 3 Marks ]

iv) Determine the analytical expressions of the correlation function $\gamma_\varepsilon(\tau)$, $\forall \tau \geq 0$ and of the spectrum $\Gamma_\varepsilon(\omega)$, $\omega \in [-\pi, \pi]$ of the prediction error $\varepsilon(\cdot)$ when the model $\mathcal{M}_1(\overline{\theta}_1^{(1)})$ is used.

[ 3 Marks ]

b) **Case 2.** Suppose that the process is generated as

$$v(t) = e(t) + \frac{1}{2}e(t-1) + \frac{1}{4}e(t-2), \quad e(\cdot) \sim WN(0,1)$$

i) Determine the value $\overline{\theta}_1^{(2)}$ the estimate $\widehat{\theta}_1(N)$ approaches for large values of $N$ (that is, $\overline{\theta}_1^{(2)} = \lim\limits_{N\to\infty} \widehat{\theta}_1(N)$, a.s.) when the family $\mathcal{M}_1$ of models is used.

[ 4 Marks ]

ii) Determine the value $\overline{\theta}_2^{(2)}$ the estimate $\widehat{\theta}_2(N)$ approaches for large values of $N$ (that is, $\overline{\theta}_2^{(2)} = \lim\limits_{N\to\infty} \widehat{\theta}_2(N)$, a.s.) when the family $\mathcal{M}_2$ of models is used.

[ 4 Marks ]

3. Consider the discrete-time stochastic dynamic system described by the block-scheme shown in Fig. 3.1.

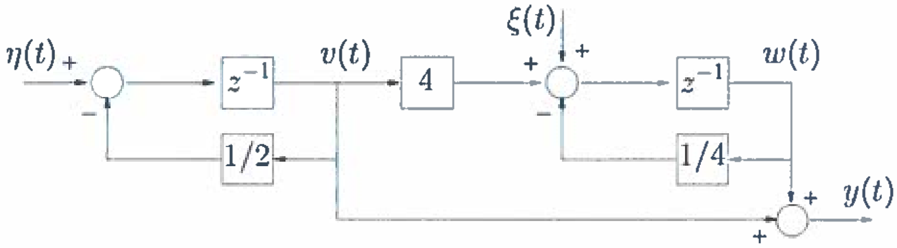


Figure 3.1 Block-scheme of the stochastic dynamic system.

where $\eta(\cdot) \sim WN(1,1)$ and $\xi(\cdot) \sim WN(3,1)$ are white non-zero mean uncorrelated stochastic processes.

a) Show that the steady-state stochastic process $y(\cdot)$ is stationary.

[ 3 Marks ]

b) Compute the expected value $m_v = \mathbb{E}(v)$ of the steady-state stochastic process $v(\cdot)$.

[ 3 Marks ]

c) Compute the expected value $m_w = \mathbb{E}(w)$ of the steady-state stochastic process $w(\cdot)$.

[ 3 Marks ]

d) Compute the covariance matrix of the stochastic processes $v(\cdot)$ and $w(\cdot)$, given by

$$\Lambda = \begin{bmatrix} \lambda_{vv} & \lambda_{vw} \\ \lambda_{vw} & \lambda_{ww} \end{bmatrix}$$

where

$$\lambda_{vv} = \text{var}[v(t)] = \mathbb{E}[(v(t) - m_v)^2]$$
$$\lambda_{ww} = \text{var}[w(t)] = \mathbb{E}[(w(t) - m_w)^2]$$
$$\lambda_{vw} = \text{cov}[v(t), w(t)] = \mathbb{E}[(v(t) - m_v)(w(t) - m_w)]$$

[ 7 Marks ]

e) Determine the transfer function $G_{\eta y}(z)$ from the input $\eta(t)$ to the output $y(t)$ and the transfer function $G_{\xi y}(z)$ from the input $\xi(t)$ to the output $y(t)$.

[ 4 Marks ]

4. Consider three visual sensors $VS_1$, $VS_2$, and $VS_3$ respectively providing three samples $x_i$, $i = 1,2,3$ of the position $x$ of a given object $O$ as depicted in Fig. 4.1.

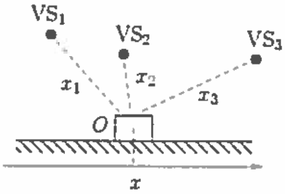


Figure 4.1 Scheme of the position measurements provided by three visual sensors $VS_1$, $VS_2$, and $VS_3$.

$x_i$, $i = 1,2,3$ are realisations of *Gaussian* and *independent* random variables:

$$x_1 \sim \mathscr{G}(\bar{x}, 1/3), \quad x_2 \sim \mathscr{G}(0.1 + \bar{x}, 2), \quad x_3 \sim \mathscr{G}(\bar{x}, 1).$$

where the second measurement is affected by a bias equal to 0.1. The objective is to design an estimator of the mean position $\bar{x}$ as a function of the samples $x_i$, $i = 1,2,3$.

a) Consider the empirical mean estimator $\tilde{x}$ of $\bar{x}$ given by:

$$\tilde{x} = \frac{1}{3}(x_1 + x_2 + x_3)$$

Compute $\mathbb{E}(\tilde{x})$ and $\mathrm{var}(\tilde{x})$.

[ 3 Marks ]

b) Consider an estimator of the form

$$\hat{x}(a,b,c,d) = ax_1 + bx_2 + cx_3 + d,$$

where $a,b,c,d \in \mathbb{R}$ are four parameters to be determined. Determine, a set of constraints on $a,b,c,d$ guaranteeing that the estimator $\hat{x}(a,b,c,d)$ is unbiased.

[ 4 Marks ]

c) Consider an estimator of the form

$$\check{x}(\alpha,\beta,\gamma) = \alpha x_1 + \beta x_2 + \gamma x_3,$$

where $\alpha,\beta,\gamma \in \mathbb{R}$ are three parameters to be determined. Determine, a set of constraints on $\alpha,\beta,\gamma$ guaranteeing that the estimator $\check{x}(\alpha,\beta,\gamma)$ is unbiased.

[ 4 Marks ]

d) Among the values of $a,b,c,d$ determined in your answer to Question 4b) and among the values of $\alpha,\beta,\gamma$ determined in your answer to Question 4c), find:

$$(a^\circ,b^\circ,c^\circ,d^\circ) = \arg\min_{a,b,c,d} \mathrm{var}[\hat{x}(a,b,c,d)]; \; (\alpha^\circ,\beta^\circ,\gamma^\circ) = \arg\min_{\alpha,\beta,\gamma} \mathrm{var}[\check{x}(\alpha,\beta,\gamma)]$$

that is, find $(a^\circ,b^\circ,c^\circ,d^\circ)$ and $(\alpha^\circ,\beta^\circ,\gamma^\circ)$ such that $\mathrm{var}(\hat{x})$ and $\mathrm{var}(\check{x})$, respectively, take on the smallest value possible.

[ 6 Marks ]

e) Compute $\mathrm{var}[\hat{x}(a^\circ,b^\circ,c^\circ,d^\circ)]$ and $\mathrm{var}[\check{x}(\alpha^\circ,\beta^\circ,\gamma^\circ)]$. Compare these variances among themselves and also with $\mathrm{var}(\tilde{x})$ obtained in the answer to Question 4a). Discuss your findings.

[ 3 Marks ]