EE4-45
EE9-CS7-21
EE9-SO22
EE9-FPN2-09

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2018

MSc and EEE/EIE PART IV: MEng and ACGI

WAVELETS AND APPLICATIONS

Corrected copy

Tuesday, 15th May 10:00 am

Time allowed: 3:00 hours

There are FOUR questions on this paper.

Answer ALL questions.

All questions carry equal marks.

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : P.L. Dragotti
Second Marker(s) : A. Manikas

Special Information for the Invigilators: NONE

Information for Candidates:

Sub-sampling by an integer N

$$x_{\downarrow N}[n] \longleftrightarrow \frac{1}{N}\sum_{k=0}^{N-1} X(e^{j(\omega-2\pi k)/N}) = \frac{1}{N}\sum_{k=0}^{N-1} X(W_N^k z^{1/N}),$$

where

$$W_N^k = e^{-j2\pi k/N}.$$

Dual Basis:
Given a basis $\{\varphi_i(t)\}_{i\in\mathbb{Z}}$, the dual basis is given by the set of elements $\{\tilde{\varphi}_i(t)\}_{i\in\mathbb{Z}}$ satisfying:

$$\langle \varphi_i(t), \tilde{\varphi}_j(t)\rangle = \delta_{i,j}.$$

A useful trigonometric identity

$$\sin x \cos y = \frac{1}{2}\left[\sin(x-y) + \sin(x+y)\right]$$

Poisson Summation Formula

$$\sum_{n=-\infty}^{\infty} \varphi(t-nT) = \frac{1}{T}\sum_{k=-\infty}^{\infty} \hat{\varphi}\left(\frac{2\pi k}{T}\right) e^{-j2\pi k t/T}$$

The Questions

1. (a) Consider the system shown in Fig. 1a. Express $y[n]$ in terms of $x[n]$.

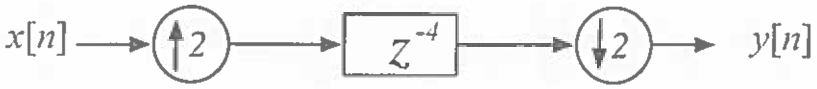

Figure 1a: A multi-rate system with a delay.

[5]

(b) Consider now the system shown in Fig. 1b.

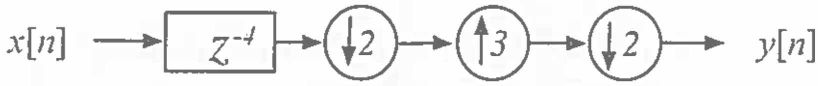

Figure 1b: A second multi-rate system with a delay.

i. Express $Y(z)$ in terms of $X(z)$. [7]

ii. Find $y[n]$ for the following inputs:

A. $x[n] = \delta[n]$, [4]

B. $x[n] = (-1)^n$. [4]

Question continues on next page.

(c) Consider the system shown in Fig. 1c, where $G_0(z)$ is an ideal low-pass filter with cut-off frequency $\omega = \pi/2$. Sketch and dimension the three spectra $Y_1(e^{j\omega})$, $Y_2(e^{j\omega})$ and $Y_3(e^{j\omega})$ assuming that $x[n]$ has the spectrum shown in Fig. 1d

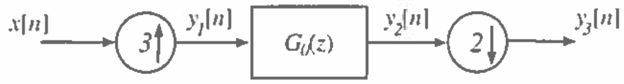

Figure 1c: Multi-rate system with low-pass filter $G_0(z)$.

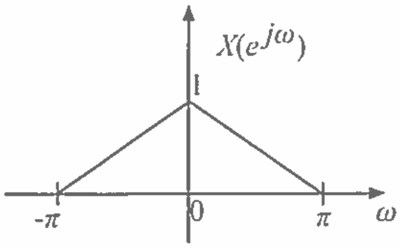

Figure 1d: Discrete-time Fourier transform of $x[n]$.

[5]

2. Consider the two-channel filter bank of Figure 2a.

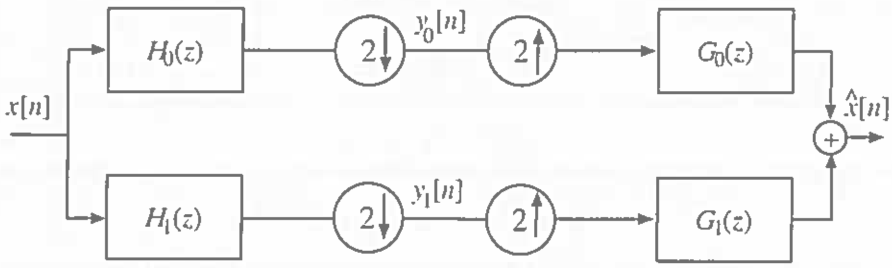

Figure 2a: Two-channel filter bank.

(a) Express $\hat{X}(z)$ as a function of $X(z)$ and the filters. Then derive the two perfect reconstruction (PR) conditions the filters have to satisfy. [5]

(b) Assume that $G_0(z) = (z+2+z^{-1})/(2\sqrt{2})$ and $G_1(z) = \sqrt{2}(z+2+3z^{-1}+2z^{-2}+z^{-3})$, find two analysis filters $H_0(z)$ and $H_1(z)$ that would lead to a perfect reconstruction filter-bank. [5]

(c) Consider $P(z) = H_0(z)G_0(z) = (z+2+z^{-1})^3 Q(z)$ with

$$Q(z) = \frac{1}{256}(3z^2 - 18z + 38 - 18z^{-1} + 3z^{-2})$$

and assume that $P(z)$ satisfies the half-band condition: $P(z)+P(-z) = 2$. Find the roots of $P(z)$. [Hint: Note that $P(z)$ is symmetric and has real-valued coefficients]. [5]

(d) Given $P(z) = H_0(z)G_0(z)$ of part (c), design the filters $H_0(z), H_1(z), G_0(z), G_1(z)$ in order to have a perfect reconstruction orthogonal filter-bank. [5]

(e) Using again $P(z)$ of part (c), design a biorthogonal perfect-reconstruction filter bank that would lead to an analysis wavelet function with six vanishing moments. Justify your answer. [Hint: Remember that $\tilde{\psi}(t) = \sqrt{2}\sum_n h_1[n]\tilde{\varphi}(2t-n)$.] [5]

3. Consider the two functions $\varphi_1(t)$ and $\varphi_2(t)$ defined as follows:

$$\varphi_1(t) = \begin{cases} 1, & \text{for } t \in [0,1] \\ 0, & \text{otherwise} \end{cases}$$

and

$$\varphi_2(t) = \begin{cases} \sin(\pi t), & \text{for } t \in [0,1] \\ 0, & \text{otherwise} \end{cases}$$

and denote with $V = \text{span}\{\varphi_1(t), \varphi_2(t)\}$ the sub-space generated by $\varphi_i(t)$ with $i = 1, 2$ over the interval $t \in [0, 1]$.

(a) Determine the two dual-basis functions $\tilde{\varphi}_i(t)$, $i = 1, 2$. [Hint: Remember that since $\tilde{\varphi}_i(t) \in V$ we can write $\tilde{\varphi}_i(t) = \sum_{k=1}^{2} \alpha_{i,k}\varphi_k(t)$. Using this fact, you just need to find the coefficients $\alpha_{i,k}$, $i = 1, 2$ and $k = 1, 2$.] [7]

(b) Given the dual basis and the signal

$$x(t) = \begin{cases} \cos(2\pi t), & \text{for } t \in [0,1] \\ 0 & \text{otherwise.} \end{cases}$$

i. Compute the inner products $\langle x(t), \tilde{\varphi}_i(t)\rangle$, $i = 1, 2$. [6]

ii. Write the exact expression for $x_v(t)$, the orthogonal projection of $x(t)$ onto V, which is given by $x_v(t) = \sum_{i=1}^{2} \langle x(t), \tilde{\varphi}_i(t)\rangle \varphi_i(t)$. [6]

iii. Verify that the error $e(t) = x(t) - x_v(t)$ is orthogonal to V. [6]

4. Suppose you are given a two-channel FIR filter bank with real coefficients and synthesis lowpass filter

$$g_0[n] = \frac{1}{2\sqrt{2}}(\delta_n + 2\delta_{n-1} + \delta_{n-2}).$$

Consider the equivalent filter

$$G_0^{(i)}(z) = \prod_{k=0}^{i-1} G_0(z^{2^k})$$

obtained by iterating the filter bank decomposition i times. Consider the function

$$\varphi^{(i)}(t) = 2^{i/2} g_0^{(i)}[n], \qquad n/2^i \leq t < (n+1)/2^i.$$

(a) Can you say anything about the convergence of $\lim_{i\to\infty} \varphi^{(i)}(t)$?

[5]

(b) Assume that $\varphi(t) = \lim_{i\to\infty} \varphi^{(i)}(t)$ exists.

i. Show that $\varphi(t)$ satisfies partition of unity, that is, show that

$$\sum_{n=-\infty}^{\infty} \varphi(t-n) = 1.$$

[Hint: Use Poisson summation formula].

[5]

ii. Show that $\varphi(t)$ satisfies the two-scale equation, that is, show that

$$\varphi(t) = \sqrt{2} \sum_n g_0[n] \varphi(2t - n).$$

[5]

(c) We know that, in the case of convergence, $\varphi(t)$ is a valid scaling function. Can you say anything about continuity of this function?

[5]

(d) State the number of vanishing moments of the analysis wavelet function obtained from $\varphi(t)$.

[5]