UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 2002

MEng Honours Degree in Electrical Engineering Part IV
MEng Honours Degree in Information Systems Engineering Part IV
MSci Honours Degree in Mathematics and Computer Science Part IV
MEng Honours Degrees in Computing Part IV
MSc in Advanced Computing
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the Associateship of the City and Guilds of London Institute
This paper is also taken for the relevant examinations for the Associateship of the Royal College of Science

PAPER C478=I4.37

ADVANCED OPERATIONS RESEARCH

Friday 10 May 2002, 10:00
Duration: 120 minutes

Answer THREE questions

Paper contains 4 questions
Calculators required

1a You are given the following set of linear programming (LP) constraints. Simplify and convert them into standard LP equalities. Indicate the type of the associated logical variable for each constraint. Try to combine constraints if possible.

$$x_1 + x_2 + x_3 + x_4 <> 0 \tag{1}$$
$$2x_1 - x_2 + 3x_3 - 4x_5 \geq -5 - x_1 + x_4 \tag{2}$$
$$-x_1 + x_2 \geq 2x_3 - x_4 \tag{3}$$
$$3x_1 - x_2 - 4 \leq -2x_3 + x_4 \tag{4}$$
$$6 \geq 3x_1 - 2x_2 + x_3 - 4x_4 \geq -6 \tag{5}$$
$$2x_1 - 3x_4 = x_1 + x_2 - x_3 \tag{6}$$
$$x_2 - 2x_3 \leq 3 + 3x_1 - x_4 \tag{7}$$

b Solve the following linear programming problem using the dual simplex method.

$$\begin{array}{lrcrcrcrcr}
\min & 4x_1 & + & 3x_2 & + & 4x_3 & + & 2x_4 & & \\
\text{s. t.} & -x_1 & & & + & 2x_3 & - & 2x_4 & \leq & -3 \\
 & -2x_1 & + & 4x_2 & + & x_3 & & & \leq & -1 \\
 & x_1 & + & 5x_2 & - & 2x_3 & + & x_4 & \leq & 1 \\
 & \multicolumn{9}{c}{x_j \geq 0,\ j = 1, \ldots, 4}
\end{array}$$

First verify that the conditions of applying it are satisfied.
At the end, determine the dual solution from its defining equation including the recomputation of the value of the dual objective function. Where do you find $\boldsymbol{B}^{-1}$?

c Assume there is an empty column in an LP model, i.e., all coefficients of the column are zero. This can be viewed as an 'activity' that does not consume resources. Quite likely it is a modelling error that went unnoticed, however, the solution algorithm should be able to handle it. What can be the consequences of such a situation? Discuss the possibilities if the corresponding variable is (i) nonnegative (type-2), or (ii) free (type-3). Hint: Investigate the role of the objective coefficient of the variable.

(The three parts carry, respectively, 30%, 50% and 20% of the marks).

2a A food processing company can produce tinned vegetables in four different versions. As part of the company's production planning system the following problem has to be solved.

$$\begin{array}{llcccccccccc}
\max & \text{Revenue } z = & 5x_1 & + & 6x_2 & + & 9x_3 & + & 4x_4 & & \\
\text{s.t.} & \text{Preprocessing} & 2x_1 & + & 3x_2 & + & 4x_3 & + & 2x_4 & \leq & 320 \\
 & \text{Cooking} & 4x_1 & + & x_2 & + & 4x_3 & + & 2x_4 & \leq & 240 \\
 & \text{Filling} & 3x_1 & + & 2x_2 & + & 3x_3 & + & 5x_4 & \leq & 240 \\
 & & & & x_1, & x_2, & x_3, & x_4 \geq 0. & & &
\end{array}$$

An optimal solution is achieved with $x_2 = 40$, $x_3 = 50$ and $y_3 = 10$ giving an objective value of $z = 690$, where y_3 is the logical variable of constraint 3. y_3 can also referred to as x_7.

The optimal basis is $B = \{7, 3, 2\}$. The basis matrix is

$$\boldsymbol{B} = \begin{bmatrix} 0 & 4 & 3 \\ 0 & 4 & 1 \\ 1 & 3 & 2 \end{bmatrix} \qquad \text{its inverse} \qquad \boldsymbol{B}^{-1} = \begin{bmatrix} -\frac{5}{8} & -\frac{1}{8} & 1 \\ -\frac{1}{8} & \frac{3}{8} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix}$$

(i) Determine the simplex multiplier associated with $\boldsymbol{B}$ and the reduced costs of nonbasic variables.

(ii) Determine the ranges of the original right-hand-side coefficients within which the solution remains optimal.

b You are given an LP problem in the form of $\min\{\boldsymbol{c}^T\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}\}$ and variables are subject to type specifications. A basic feasible solution (BFS) with $z = 4$ and a type-2 incoming variable with reduced cost $d_q = 3$ are given. Also, $\boldsymbol{\alpha}_q = \boldsymbol{B}^{-1}\boldsymbol{a}_q$ is available. The table below shows the relevant part of the problem:

i	x_{Bi}	type(x_{Bi})	u_{Bi}	α_{iq}	t_i
1	2	2	$+\infty$	-1	
2	0	3	$+\infty$	1	
3	2	1	4	1	
4	6	2	$+\infty$	2	
5	1	1	5	-2	

Is the BFS degenerate? Determine the ratios, the value of the incoming variable, the variable leaving the basis (if any), the new BFS and the new value of the objective function. Is the new BFS degenerate?

c (i) An integer variable x is restricted to take a value in the interval $[-6, 1]$. How can you express this condition using as few 0/1 variables as possible?

(ii) A variable has to be included in a model that can take only the following values $(1.1, 0, -2.3, 9.44, -5)$. How can you formulate this requirement using integer variables?

(The three parts carry, respectively, 40%, 40% and 20% of the marks).

3a The *transportation problem* can be described as follows. A certain product, say computers, is available at m supply points (depots) in quantities of s_i $(i = 1, \ldots, m)$. They fulfill the demand d_j $(j = 1, \ldots, n)$ at n delivery points (destinations). The unit cost of a shipment from depot i to destination j is c_{ij}. We assume that the total supply is equal to the total demand. In a complete shipment allocation no computers remain at any of the depots and the demand at every destination is satisfied.

An allocation is sought that minimizes total transportation costs. Set up an integer programming model for the problem and show that its matrix satisfies Property K. Explain the importance of this property.

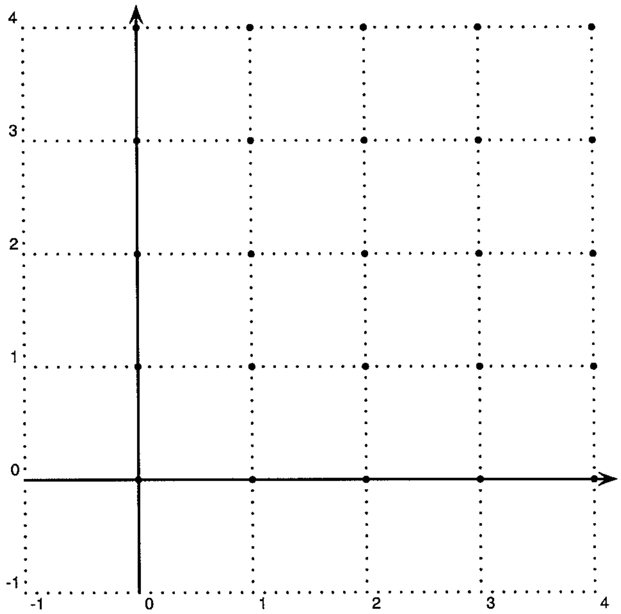
What is the size of the problem in terms of constraints and variables?

Hint: define the decision variables first.

b Graphical solution of mixed integer linear programming problems (MILP) is possible if there are not more than two variables.

Solve the following MILP problem using the graph printed here or your own drawing. It need not be very accurate. If in doubt, rely on the numerical data given below.

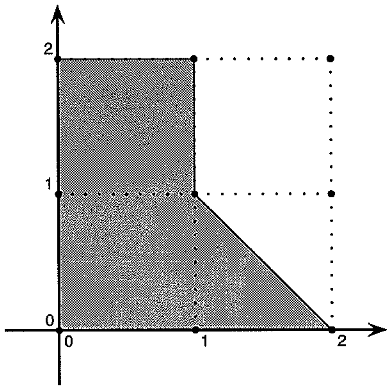
The objective is to maximize $z = 2x_1 + x_2$, where x_2 is a general nonnegative integer, x_1 is nonnegative. The feasible region of the LP relaxation of the problem is determined by the polygon with vertices: $P_1(0, 1)$, $P_2(0, 3.5)$ and $P_3(2.95, 0)$. Where are the feasible solutions located? Determine an optimal solution. Is it unique?



c When do we call a mixed integer programming (MIP) formulation tight? What is its advantage?

(The three parts carry, respectively, 40%, 40% and 20% of the marks).

4a The figure below shows a nonconvex region R. Write the appropriate linear inequalities and logic expressions to describe the points in the shaded area. Introduce indicator variable(s) if needed and give a MIP formulation of the region. Verify your solution by showing that point (0.5,0.5) satisfies your constraints and point (1.5,1.5) does not.



b Discuss the product form of the inverse and summarize its computational advantages in the simplex method. Explain the operations with it.

c You are given a general LP problem with a feasible basis $\boldsymbol{B}$. Show that the reduced cost of every basic variable is zero.

(The three parts carry, respectively, 50%, 30% and 20% of the marks).