Paper Number(s): **E1.1**

IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE
UNIVERSITY OF LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2010

EEE/ISE PART I: MEng, BEng and ACGI

ANALYSIS OF CIRCUITS

Wednesday, 9 June 10:00 am

There are FOUR questions on this paper.

Q1 is compulsory.
Answer Q1 and any two of questions 2-4.
Q1 carries 40% of the marks. Questions 2 to 4 carry equal marks (30% each).

Time allowed: 2:00 hours

Examiners responsible:

First Marker(s): D.M. Brookes D.M. Brookes

Second Marker(s): P.D. Mitcheson P.D. Mitcheson

Information for Candidates:

The following notation is used in this paper:

1. The voltage waveform at node X in a circuit is denoted by $x(t)$, the phasor voltage by X and the root-mean-square phasor voltage by $\tilde{X} = \frac{X}{\sqrt{2}}$.
2. Component and source values in a circuit are normally given in Ohms, Farads, Henrys, Volts or Amps with the unit symbol omitted. Where an imaginary number is specified, it represents the complex impedance.
3. Times are given in seconds unless otherwise stated.

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1. (a) Using nodal analysis calculate the voltages at nodes X and Y in *Figure 1.1*. [5]

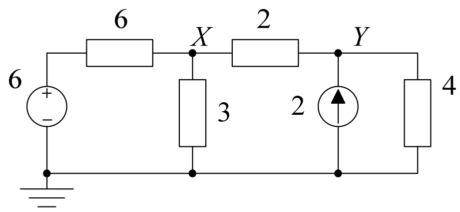

Figure 1.1

(b) Use the principle of superposition to find the current I in *Figure 1.2*. [5]

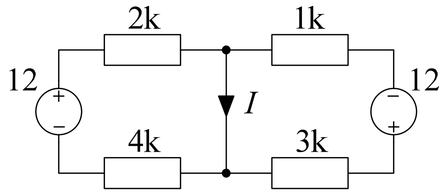

Figure 1.2

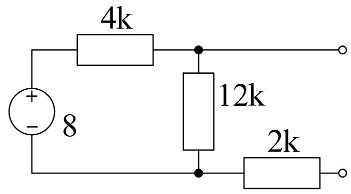

Figure 1.3

(c) Draw the Thévenin equivalent circuit of the network in *Figure 1.3* and find the values of its components. [5]

(d) Assuming the opamp in the circuit of *Figure 1.4* is ideal, give an expression for Z in terms of X and Y. [5]

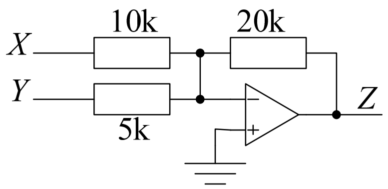

Figure 1.4

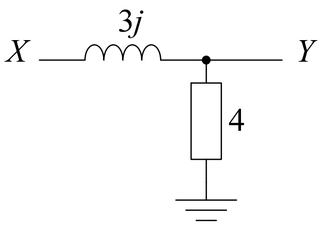

Figure 1.5

(e) (i) The phasor representing the voltage at X in *Figure 1.5* has the value $50j$. Components are labelled with their complex impedances and $x(t)$ has a frequency of ω. Calculate the phasor representing the voltage at Y. [3]

(ii) Express the waveform at Y in the form $y(t) = A\cos(\omega t) + B\sin(\omega t)$. [1]

(iii) If $\omega = 300$ rad/s, calculate the value of the inductance in Henries. [1]

(f) Calculate the frequency response, $\frac{Y}{X}(j\omega)$, of the circuit shown in *Figure 1.6*. Draw a dimensioned graph of the magnitude response in dB versus frequency using a logarithmic frequency scale. Indicate on your graph the corner frequency values and the gain of any horizontal portions of the magnitude response. [5]

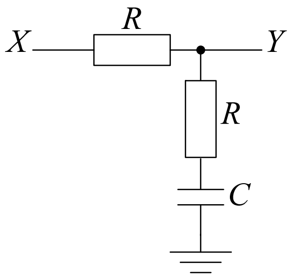

Figure 1.6

(g) In the circuit of *Figure 1.7*, the r.m.s. phasor $\tilde{X}$ has the value 100. Determine the value of the phasor current $\tilde{J}$ and the complex power $\tilde{V} \times \tilde{I}^*$ absorbed by each of the four components. Component values represent complex impedances. [5]

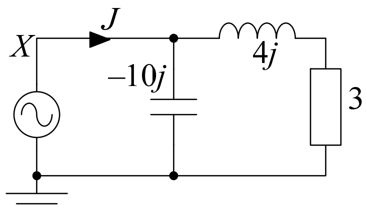

Figure 1.7

(h) In *Figure 1.8*, the input voltage, $x(t) = \begin{cases} -1 & \text{for } t < 0 \\ +1 & \text{for } t \geq 0 \end{cases}$.

(i) Give the steady state capacitor voltage, V_{YX}, for $t < 0$. [1]

(ii) Determine an expression for $y(t)$ for $t \geq 0$ and sketch a graph showing $y(t)$ for $-1\text{ ms} < t < 5\text{ ms}$. [4]

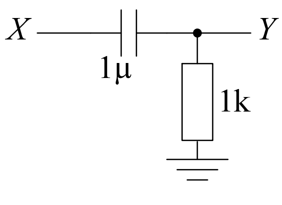

Figure 1.8

2. (a) Assuming that the op-amp in the circuit of *Figure 2.1* has a finite gain such that $Y = A(X - W)$ but is otherwise ideal. Show that [12]

$$\frac{Y}{X} = \frac{R_1 + R_2}{R_1 + A^{-1}(R_1 + R_2)}$$

(b) Determine the transfer function, $\frac{Y}{X}(j\omega)$ if $R_1 = 1$ k, $R_2 = 99$ k and the op-amp gain is given by

$$A(j\omega) = \frac{A_0}{1 + \frac{j\omega}{\omega_0}}$$

where $A_0 = 10^5$ and $\omega_0 = 50$ rad/s. Find the DC gain and corner frequency of the response $\frac{Y}{X}(j\omega)$ and sketch a dimensioned graph of the magnitude response, $\left|\frac{Y}{X}(j\omega)\right|$. [12]

(c) Determine the transfer function of the circuit shown in *Figure 2.2* if both op-amps have the gain, $A(j\omega)$ given in part (b) above. Find the DC gain and the corner frequencies of the response and sketch a dimensioned graph of the magnitude response, $\left|\frac{Y}{X}(j\omega)\right|$. [6]

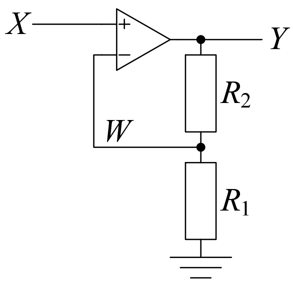

Figure 2.1

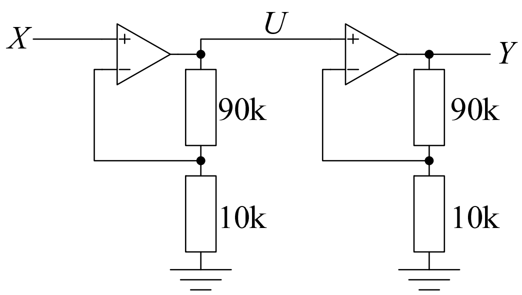

Figure 2.2

3. The circuit of *Figure 3.1* has the transfer function

$$\frac{Y}{X}(j\omega) = \frac{1}{mn(j\omega RC)^2 + (m+1)j\omega RC + 1}.$$

(a) Show that this may be written in the form

$$\frac{1}{\left(\frac{j\omega}{\omega_0}\right)^2 + \frac{2\zeta(j\omega)}{\omega_0} + 1}$$

and determine expressions for ω_0 and ζ.

Give expressions for the high and low frequency asymptotes. Determine the exact value of $\frac{Y}{X}$ when $\omega = \omega_0$. [12]

(b) Using $n = 2$ and $C = 10$ nF, choose values for m and R such that $\omega_0 = 2000\pi$ and $\zeta = \sqrt{0.5} = 0.707$.

Sketch a graph of $\left|\frac{Y}{X}(j\omega)\right|$ using a logarithmic frequency axis and a decibel scale for gain. [12]

(c) In the circuit of *Figure 3.2*, the positions of the capacitors and resistors have been interchanged relative to *Figure 3.1*. For each figure, $Z_{①}(j\omega)$ denotes the complex impedance of the component in position ①; e.g. . $Z_{①} = \frac{1}{j\omega nC}$ in *Figure 3.1* but $Z_{①} = R$ in *Figure 3.2*.

Show that the impedance ratio $\frac{Z_{①}}{Z_{②}}(j\omega_2)$ in *Figure 3.2* is equal to the complex conjugate of $\frac{Z_{①}}{Z_{②}}(j\omega_1)$ in *Figure 3.1* where $\omega_1 = \frac{\omega_0^2}{\omega_2}$.

Explain why this relationship also holds for all other impedance ratios in the two figures.

Hence, assuming that the frequency response depends only on the impedance ratios, sketch a graph of $\left|\frac{V}{U}(j\omega)\right|$. [6]

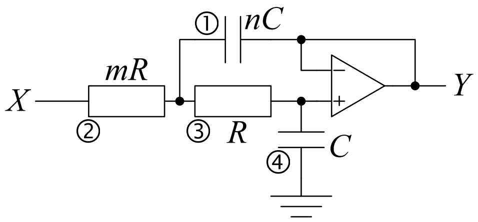

Figure 3.1

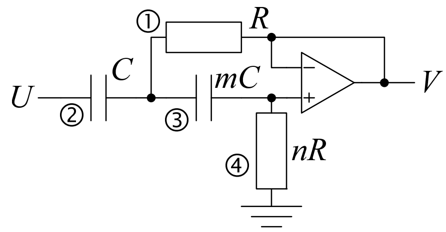

Figure 3.2

4. In the circuit of *Figure 4.1*, the input voltage phasor is X and the output voltage phasor is Y. In this question t denotes the time in seconds.

(a) Determine an expression for the transfer function $\frac{Y}{X}(j\omega)$ and calculate its value at the frequency $\omega = 500\ \text{rad/s}$. [8]

(b) If the input voltage is $x(t) = 10\sin(500t)$ for $t > 0$, use phasor analysis to determine an expression for the steady-state output $y_{SS}(t)$ in the form $y_{SS}(t) = A\cos(500t) + B\sin(500t)$ where t is in seconds. [8]

(c) If the output voltage at $t = 0$ is $y(0) = U$, give a complete expression for $y(t)$ for $t > 0$. The input voltage $x(t)$ is the same as in part (b). [8]

(d) Determine the value of U for which it is also true that $y\left(\frac{\pi}{500}\right) = U$. Hence give an expression for $y(t)$ for $0 \le t < \frac{\pi}{500}$ when $x(t) = |10\sin(500t)|$ for all t. [6]

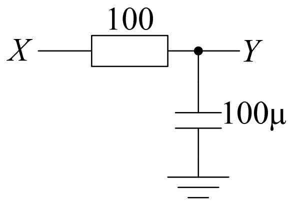

Figure 4.1

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2010 E1.1: Analysis of Circuits - Solutions

Key to letters on mark scheme: B=Bookwork, C=New computed example, A=Analysis of new circuit, D=design of new circuit

1. (a) Nodal equation at X gives $\frac{X-6}{6}+\frac{X}{3}+\frac{X-Y}{2}=0$ from which $6X-3Y=6$. [2A]

Nodal equation at Y gives $\frac{Y-X}{2}-2+\frac{Y}{4}=0$ from which $3Y-2X=8$. [2A]

Adding these together gives $4X=14$ so $X=3.5$. Substituting for X then gives $Y=5$. [1A]

Mostly OK. Many made algebraic errors in solving the simultaneous equations. These can be avoided by using the calculator simultaneous equation function and/or by verifying the result with the original equations.

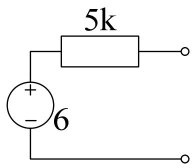
(b) Setting the rightmost source to zero gives $I_1=\frac{12}{2+4}=2$ mA (note that the 1k and 2k resistors are short circuited and so have no effect). Likewise, setting the leftmost source to zero gives $I_2=\frac{-12}{1+3}=-3$ mA. Combining these gives $I=2-3=-1$ mA. [5A]

Some people ignored the sign of the current when adding them together. Many did not realize that when the rightmost source is set to zero, there is no current through the 1k and 3k resistors because they are shorted out by the link that carries current I. Several people ignored the k's and got an answer of –1 A.

(c) The Thévenin resistance (obtained by setting the voltage source to zero) is $2+4||12=2+\frac{4\times 12}{4+12}=2+3=5$ kΩ. [2A]

The open circuit voltage is just the voltage across the 12k resistor (since the 2k resistor has no current flowing through it). Viewing the 4k and 12k resistors as a potential divider gives $V_{Th}=8\times\frac{12}{4+12}=6$ V. [2A]

The Thévenin equivalent is therefore:



[1A]

(d) Inverting summing amplifier: $Z = -2X - 4Y$. [5A]

(e) (i) $Y = 50j \times \frac{4}{4+3j} = 24 + 32j$ [3A]

Surprisingly many people solved this using nodal analysis rather than just writing down the potential divider formula; this gets the right answer (usually) but is a lot more effort.

(ii) $y(t) = 24\cos(\omega t) - 32\sin(\omega t)$ [1A]

Many people omitted the minus sign.

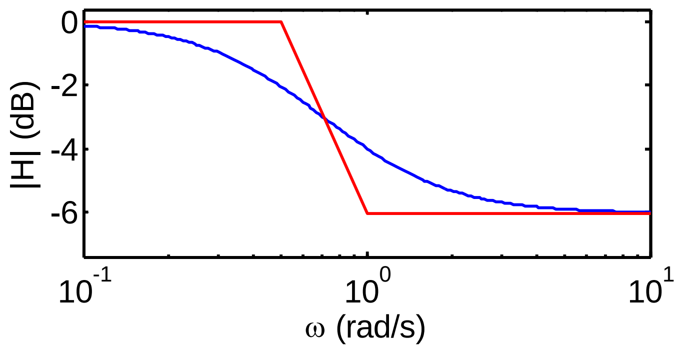
(iii) $j\omega L = 3j$ so $L = \frac{3}{\omega} = 10$ mH [1A]

(f) This circuit is a potential divider, so its transfer function is

$$\frac{Y}{X} = \frac{R+\frac{1}{j\omega C}}{2R+\frac{1}{j\omega C}} = \frac{j\omega RC+1}{2j\omega RC+1}$$ [3A]

Several tried to use the correct formula $\frac{Y}{X} = \frac{1}{1+Z_1Y_2}$ where Y_2 is the admittance of the RC combination. However, everyone who did it this way got the wrong expression for Y_2; in most cases they assumed $Y_2 = Y_R + Y_C$ but there were several other incorrect expressions as well.

Corner frequencies at $\omega = \frac{1}{2RC}, \frac{1}{RC}$. LF asymptote = 1, HF asymptote = 0.5 = −6 dB. [2A]



(g) Capacitor current: $\tilde{I}_C = \frac{\tilde{X}}{-10j} = 10j$.

L-R current: $\tilde{I}_L = \frac{\tilde{X}}{3+4j} = 12 - 16j$ ($|\tilde{I}_L| = 20$).

A surprising number of people treated R and L as in parallel or else R and C as in parallel. Also quite a lot think $\frac{100}{-10j} = -10j$.

Total Current: $\tilde{J} = \tilde{I}_C + \tilde{I}_L = 12 - 6j$ ($|\tilde{J}| = \sqrt{180} = 13.4$). [2A]

A messier approach: $Z = \frac{-10j(3+4j)}{-10j+3+4j} = \frac{40-30j}{3-6j} = \frac{20+10j}{3}$

$$\Rightarrow J = \frac{V}{Z} = \frac{100(3-6j)}{40-30j} = 12 - 6j$$

Complex power absorbed by a component is $\tilde{V} \times \tilde{I}^* = |\tilde{I}|^2 Z = \frac{|\tilde{V}|^2}{Z^*}$.

Several people used the last of these three expressions but omitted the "". Other people ignored the modulus signs in the second and third expressions. Since you already know the current through the components, it is easiest to use the first expression for the source and the second for the other components. Several people did complex calculations manually instead of with the calculator and often got the wrong answer. Quite a few people used the same current, $\tilde{J}$, for all components instead of using the current that actually flowed through the component in question. Many people worked out J first (via the total impedance) and then worked out the current through C and $L + R$ using the current divider formula: this was much more effort than working them out directly.*

Complex power absorbed by: [3A]

- C is $10^2 \times -10j = -1000j$ VA (i.e. it is generating VARs)
- R is $20^2 \times 3$ =1200 W. ($\tilde{V}_R = 36 - 48j$)
- L is $20^2 \times 4j = 1600j$ VA. ($\tilde{V}_L = 64 + 48j$)
- Source is $-100 \times (12 + 6j) = -1200 - 600j$. This also equals $-(1200 + 1600j - 1000j)$ from power conservation (Tellegen).

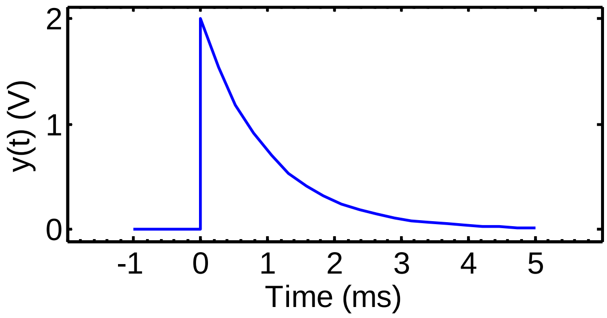
(h) (i) There is no DC current through the capacitor so the steady state output voltage is $Y_{SS} = 0$. Hence the steady state capacitor voltage is $V_{YX} = 0 - (-1) = 1$. [1A]

Surprisingly, many people got this wrong and gave -1 or 0 as the answer. Many thought that V_{YX} was the same as Y rather than equalling $Y - X$.

(ii) The steady state output voltage is $y_{SS} = 0$. So the total output voltage is $y(t) = Ae^{-\frac{t}{\tau}}$ where $\tau = RC = 1$ ms.

At time $t = 0$ the capacitor voltage must be unchanged from its previous value of 1 V so $y(0+) = x(0+) + 1 = 2$ V. So $A = 2$ and $y(t) = 2e^{-\frac{t}{\tau}}$ for $t > 0$. $y(0.005) = 2e^{-5} = 0.0135$. [4A]

Although most people correctly said the capacitor voltage cannot change instantly, quite a lot assumed wrongly that this meant the output voltage cannot change instantly. In fact it means that if X jumps by 2V then Y must jump by the same amount.



2. (a) We eliminate W from $Y = A(X - W)$ and $W = \frac{R_1}{R_1+R_2} \times Y$. To get

$Y = A\left(X - \frac{YR_1}{R_1+R_2}\right) \Rightarrow \frac{Y}{X} = \frac{R_1+R_2}{R_1+A^{-1}(R_1+R_2)}$ [12A]

An easy question which most people got right. A few people made the assumption that $X = W$ but this is only true under the ideal op-amp assumption that $A = \infty$ which is explicitly not the case in this question.

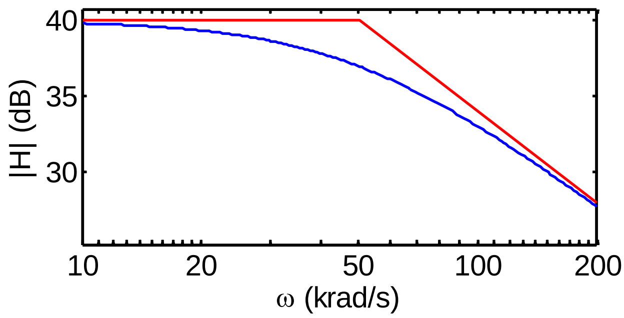
(b) Substituting the expression for A into the formula from part (b) gives

$$\frac{Y}{X} = \frac{A_0\omega_0(R_1 + R_2)}{j\omega(R_1 + R_2) + \omega_0(R_1 + R_2 + A_0R_1)}.$$

Before substituting for A, several people manipulated the expression from part (a) so that A appeared twice; this is a bad idea since it complicates the expression. Many many people made algebraic errors and/or omitted factors of 10^3.

The DC gain is $\frac{A_0(R_1+R_2)}{R_1+R_2+A_0R_1} = 99.9001 = 39.991$ dB.

The corner frequency is $\omega = \frac{\omega_0(R_1+R_2+A_0R_1)}{R_1+R_2} = 50.05\ \text{krad/s}$ [12A]



The HF asymptote has a slope of -1 and hits unity gain at $\omega = 5$ Mrad/s.

Most people did not give any indication of the slope of the second graph segment. A few people plotted the op-amp frequency respones, A, rather than $\frac{Y}{X}$.

(c) Since we have two identical stages, the response is just

$$\frac{Y}{X} = \left(\frac{A_0\omega_0(R_1 + R_2)}{j\omega(R_1 + R_2) + \omega_0(R_1 + R_2 + A_0R_1)}\right)^2.$$

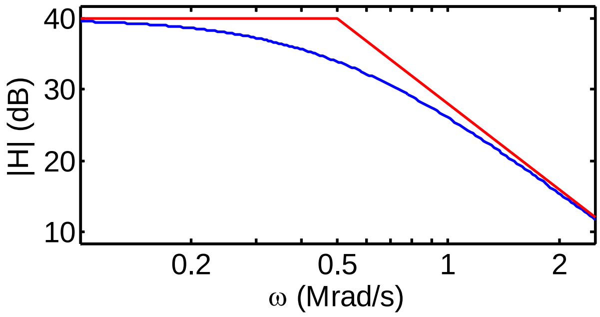
but with $R_2 = 90$ k.

Many people pointlessly multiplied out the denominator and then tried to factorize the resulting quadratic (often with wrong or inaccurate factors). It is almost always best to keep it factorized.

The DC gain is $\left(\frac{A_0(R_1+R_2)}{R_1+R_2+A_0R_1}\right)^2 = 9.999^2 = 99.98 = 39.998$ dB..

The corner frequencies are both at $\omega = \frac{\omega_0(R_1+R_2+A_0R_1)}{R_1+R_2} = 500.05\ \text{krad/s}$ [6A]

The gradient of the high frequency asymptote is -2 (i.e. -12 dB/octave).



The HF asymptote has a slope of -2 and still hits unity gain at $\omega = 5$ Mrad/s.

3. (a) Identifying coefficients gives: $mnR^2C^2 = \omega_0^{-2}$ and $(m+1)RC = 2\zeta\omega_0^{-1}$.

Hence $\omega_0 = \frac{1}{RC\sqrt{mn}}$ and $\zeta = \frac{m+1}{2\sqrt{mn}}$.

The LF asymptote is 1 and the HF asymptote is $-\left(\frac{\omega_0}{\omega}\right)^2$.

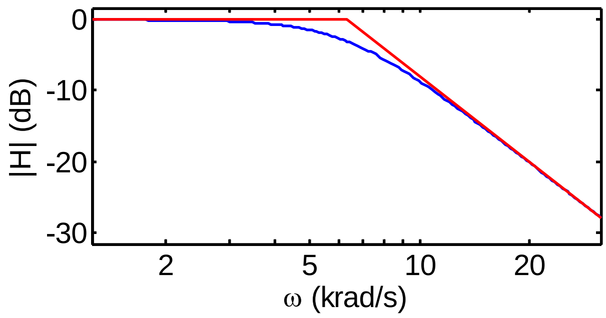
Many people omitted the ω^{-2} factor from the HF asymptote.

At $\omega = \omega_0$ the outer terms of the denominator cancel and so $\frac{Y}{X} = \frac{1}{2j\zeta} = -0.5\zeta^{-1}j = \frac{-j\sqrt{mn}}{m+1}$. [12A]

(b) We have $4\zeta^2 mn = (m+1)^2$ from which $m^2 - 2m + 1 = 0$. Hence $m = 1$.

Finally $R = \frac{1}{\omega_0 C\sqrt{mn}} = 11.25\ \text{k}\Omega$. [12D]

The corner frequency is $\omega = 2000\pi = 6285\ \text{rad/s}$ and the gain at the corner frequency is $\frac{1}{2\zeta} = 0.707 = -3$ dB.



Several people gave the response a positive slope at low frequencies despite having worked out the correct LF asymptote in part (a).

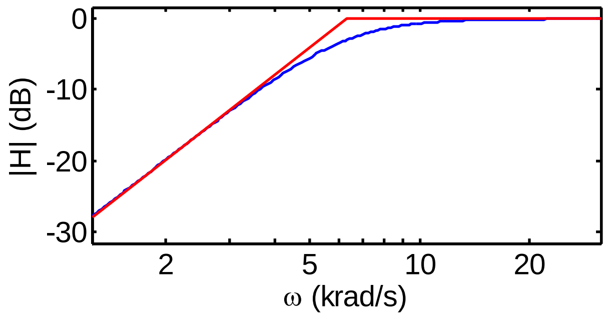
(c) Since the impedance of a capacitor is inversely proportional to its capacitance $\frac{Z_C}{Z_{mC}} = m = \frac{Z_{mR}}{Z_R} = \left(\frac{Z_{mR}}{Z_R}\right)^*$ as required. (and similarly for $\frac{Z_R}{Z_{nR}}$). [6B]

In the second circuit $\frac{Z_R}{Z_C} = j\omega_2 RC$. In the first circuit $\frac{Z_{nC}}{Z_{mR}} = \frac{1}{j\omega_1 mnRC}$. Substituting $\omega_1 = \frac{\omega_0^2}{\omega_2}$ we get $\frac{Z_{nC}}{Z_{mR}} = \frac{\omega_2}{j\omega_0^2 mnRC} = \frac{\omega_2 RC}{j} = (j\omega_2 RC)^*$ as required.

If you exchange one of the components for another of the same type, you divide both numerators or both denominators by m or n so the relationship remains valid. If both impedances are capacitors or resistors then the ratios are real and both equal m or n.

Clear proofs were very rare.

The magnitude response graph is reflected around the frequency ω_0.



4. (a) The transfer function is $\frac{Y}{X} = \frac{1}{j\omega RC+1} = \frac{1}{1+0.01j\omega}$.

At $\omega = 500$, $\frac{Y}{X} = \frac{1}{1+5j} = \frac{1-5j}{26} = 0.0385 - 0.1923j = 0.1961\angle -78.69°$. [8A]

Some people calculated the magnitude of the transfer function rather than its value (which is complex).

(b) The input phasor is $X = -10j$, so the output phasor is $Y = X \times \frac{Y}{X} = -1.923 - 0.385j$. So the steady state output is therefore $y_{SS}(t) = -1.923\cos(500t) + 0.385\sin(500t)$. [8A]

Some people thought that "steady state" meant that they should use the DC gain rather than the gain at $\omega = 500$. Several forgot that the phasor for $\sin\omega t$ is $-j$ and not $+j$.

(c) The complete output is $y(t) = y_{SS}(t) + Ae^{\frac{-t}{\tau}}$ where $\tau = RC = 0.01$.

So at $t = 0$, $y(0) = U = -1.923 + A \Rightarrow A = U + 1.923$.

Or equivalently, you can use the formula: $A = y(0+) - y_{SS}(0+)$.

Therefore $y(t) = y_{SS}(t) + \left(U - y_{SS}(0+)\right)e^{-100t} =$

$-1.923\cos(500t) + 0.385\sin(500t) + (U + 1.923)e^{-100t}$. [8A]

Several people used $U - y_{SS}(t)$ as the transient amplitude instead of $U - y_{SS}(0+)$. Transient amplitudes in the circuits in this course are <u>never</u> complex and <u>never</u> depend on t.

(d) $U = y\left(\frac{\pi}{500}\right) = 1.923 + (U + 1.923)e^{-0.628} = 1.923 + (U + 1.923) \times 0.5335$.

Equating this to U gives $U = \frac{1.923\times1.5335}{1-0.5335} = 6.321$.

Mostly fine.

When $x(t) = |10\sin(500t)|$ for all t, $y(t)$ must be periodic with period $T = \frac{\pi}{500}$ which implies that $y\left(\frac{\pi}{500}\right) = y(0)$ which is the premise of this part. Therefore, one complete cycle of $y(t)$ is given by $y(t) = -1.923\cos(500t) + 0.385\sin(500t) + 8.245e^{-100t}$ for $0 \le t < \frac{\pi}{500}$. [6A]

Almost no-one gave a plausible argument for this last bit.