IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2017

EEE/EIE PART II: MEng, BEng and ACGI

Corrected Copy

COMMUNICATION SYSTEMS

Monday, 5 June 10:00 am

11.04 am

Time allowed: 2:00 hours

Q 3a (iv) correction

There are THREE questions on this paper.

Answer ALL questions.
Q1 carries 40% of the marks. Questions 2 and 3 carry equal marks (30% each).

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : D. Gunduz

Second Marker(s) : J.A. Barria

EXAM QUESTIONS

Information for Students

Fourier Transform Pairs

Pair Number	$x(t)$	$X(f)$
1.	$\Pi\left(\frac{t}{\tau}\right)$	$\tau \operatorname{sinc} \tau f$
2.	$2W \operatorname{sinc} 2Wt$	$\Pi\left(\frac{f}{2W}\right)$
3.	$\Lambda\left(\frac{t}{\tau}\right)$	$\tau \operatorname{sinc}^2 \tau f$
4.	$\exp(-\alpha t)u(t),\ \alpha > 0$	$\frac{1}{\alpha + j2\pi f}$
5.	$t\exp(-\alpha t)u(t),\ \alpha > 0$	$\frac{1}{(\alpha + j2\pi f)^2}$
6.	$\exp(-\alpha\lvert t\rvert),\ \alpha > 0$	$\frac{2\alpha}{\alpha^2 + (2\pi f)^2}$
7.	$e^{-\pi(t/\tau)^2}$	$\tau e^{-\pi(f\tau)^2}$
8.	$\delta(t)$	1
9.	1	$\delta(f)$
10.	$\delta(t - t_0)$	$\exp(-j2\pi f t_0)$
11.	$\exp(j2\pi f_0 t)$	$\delta(f - f_0)$
12.	$\cos 2\pi f_0 t$	$\frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)$
13.	$\sin 2\pi f_0 t$	$\frac{1}{2j}\delta(f - f_0) - \frac{1}{2j}\delta(f + f_0)$
14.	$u(t)$	$(j2\pi f)^{-1} + \frac{1}{2}\delta(f)$
15.	$\operatorname{sgn} t$	$(j\pi f)^{-1}$
16.	$\frac{1}{\pi t}$	$-j\operatorname{sgn}(f)$
17.	$\hat{x}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(\lambda)}{t - \lambda}\, d\lambda$	$-j\operatorname{sgn}(f)X(f)$
18.	$\sum_{m=-\infty}^{\infty} \delta(t - mT_s)$	$f_s \sum_{m=-\infty}^{\infty} \delta(f - mf_s)$, $f_s = T_s^{-1}$

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

$$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

$$\tan(\frac{\pi}{2} - x) = \cot x$$

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x} = \frac{2\cot x}{\cot^2 x - 1} = \frac{2}{\cot x - \tan x}$$

$$\tan\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}} = \frac{\sin x}{1+\cos x} = \frac{1-\cos x}{\sin x}$$

$$\sin x \cos y = \frac{1}{2}[\sin(x+y) + \sin(x-y)]$$

$$\cos x \sin y = \frac{1}{2}[\sin(x+y) - \sin(x-y)]$$

$$\sin x \sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$$

$$\cos x \cos y = \frac{1}{2}[\cos(x+y) + \cos(x-y)]$$

$$\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$$

$$\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$$

$$\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$$

$$\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$$

Fourier Transform Theorems[a]

Name of Theorem		
1. Superposition (a_1 and a_2 arbitrary constants)	$a_1x_1(t) + a_2x_2(t)$	$a_1X_1(f) + a_2X_2(f)$
2. Time delay	$x(t - t_0)$	$X(f)e^{-j2\pi f t_0}$
3a. Scale change	$x(at)$	$\lvert a\rvert^{-1}X\left(\frac{f}{a}\right)$
b. Time reversal	$x(-t)$	$X(-f) = X^*(f)$
4. Duality	$X(t)$	$x(-f)$
5a. Frequency translation	$x(t)e^{j\omega_0 t}$	$X(f - f_0)$
b. Modulation	$x(t)\cos\omega_0 t$	$\frac{1}{2}X(f - f_0) + \frac{1}{2}X(f + f_0)$
6. Differentiation	$\frac{d^n x(t)}{dt^n}$	$(j2\pi f)^n X(f)$
7. Integration	$\int_{-\infty}^{t} x(t')\,dt'$	$(j2\pi f)^{-1}X(f) + \frac{1}{2}X(0)\delta(f)$
8. Convolution	$\int_{-\infty}^{\infty} x_1(t - t')x_2(t')\,dt'$ $= \int_{-\infty}^{\infty} x_1(t')x_2(t - t')\,dt'$	$X_1(f)X_2(f)$
9. Multiplication	$x_1(t)x_2(t)$	$\int_{-\infty}^{\infty} X_1(f - f')X_2(f')\,df'$ $= \int_{-\infty}^{\infty} X_1(f')X_2(f - f')\,df'$

Differentiation Rule of Leibnitz

Let $F(z) = \int_{a(z)}^{b(z)} f(x,z)dx$. Then we have

$$\frac{dF(z)}{dz} = \frac{db(z)}{dz}f(b(z),z) - \frac{da(z)}{dz}f(a(z),z) + \int_{a(z)}^{b(z)} \frac{\partial f(x,z)}{\partial z}dx$$

Joint Gaussian density

The joint probability density function (pdf) of two correlated Gaussian random variables X and Y is given by

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}e^{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_X)^2}{\sigma_X^2} - \frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} + \frac{(y-\mu_Y)^2}{\sigma_Y^2}\right]}.$$

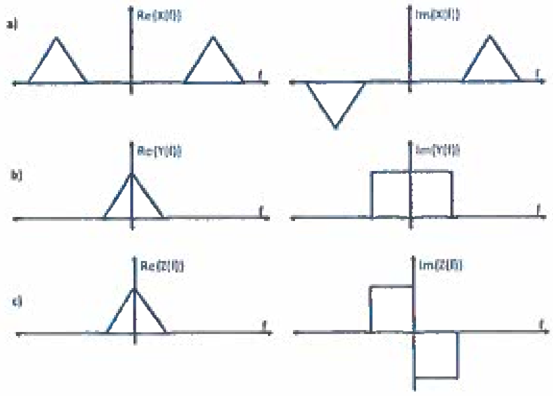
where $\mu_X = E[X]$, $\mu_Y = E[Y]$ are the mean values, σ_X and σ_Y are the standard deviation of X and Y, respectively, and ρ is the correlation coefficient defined as

$$\rho = \frac{\mathrm{Cov}(X,Y)}{\sigma_X\sigma_Y}.$$

1. a) i) Explain how you can convert a continuous signal into sequences of bits that can be transmitted over a digital communication system. [2]

ii) Write down the definitions of baseband and passband signals. [2]

iii) The spectrum of three signals $x(t)$, $y(t)$, and $z(t)$ are depicted below. Which of these (there may be multiple) may represent a real baseband signal? Explain your answer.



[3]

iv) Consider a binary frequency shift keying (FSK) communication system, in which bit 0 is transmitted with signal $A\cos(2\pi f_0 t)$ and bit 1 is transmitted with signal $A\cos(2\pi f_1 t)$.

Draw the diagram of a coherent FSK receiver for this system, and explain the function of each component of the receiver. [5]

b) State whether each of the following statements are true or false, and discuss your answer:

i) $R_X(t,s) = \sin(t+s)$ cannot be the autocorrelation function of a random process. [2]

ii) If the input to a linear time-invariant (LTI) system is wide sense stationary, so is the output. [2]

iii) Let $\Phi(x)$ denote the cumulative distribution function of a standard Gaussian random variable, and $Q(x) \triangleq \frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-t^2/2}dt$. The following relation between these two functions hold for any real x:

$$\Phi(x) + Q(x) = 1.$$

[2]

c) Consider two independent Gaussian random variables X and Y. Suppose X has a mean value of -2 and variance 2, i.e., $X \sim \mathcal{N}(-2,2)$, while Y has a mean value of 1 and variance 3, i.e., $Y \sim \mathcal{N}(1,3)$.

Express the following probabilities in terms of the Q function.

i) $\mathrm{P}\{-5 < X < 2\}$. [2]

ii) $\mathrm{P}\{X^2 - 2X > 3\}$. [3]

iii) $\mathrm{P}\{X \cdot Y - Y + 3X < 3\}$. [4]

iv) $\mathrm{P}\left\{\frac{X+2}{\sqrt{2}} > \frac{Y-1}{\sqrt{3}}\right\}$. [3]

d) Assume that $X(t)$ is a real wide sense stationary (WSS) random process whose power spectral density (PSD) is given as follows:

$$S_X(f) = \frac{2\alpha}{\alpha^2 + (2\pi f)^2} + \frac{1}{2}\left[\delta(f-\frac{1}{4}) + \delta(f+\frac{1}{4})\right] + 3\delta(f)$$

for some $\alpha > 0$.

i) What is the second moment of the random variable $X(1) - X(-1)$? *(Hint: Second moment of a random variable Y is given by $E[Y^2]$.)* [5]

ii) What is the second moment of the following random variable?

$$X(2) + X(0) - X(-2)$$

[5]

2. a) Consider the random process

$$X(t) = Y(t)\cos(2\pi f_c t) - Z(t)\sin(2\pi f_c t),$$

where $Y(t)$ and $Z(t)$ are two independent random processes.

i) Find the conditions on $Y(t)$ and $Z(t)$ under which the mean of $X(t)$ is shift-invariant, i.e., $E[X(t)]$ does not depend on t. [4]

ii) Assume that both $Y(t)$ and $Z(t)$ are zero-mean processes, i.e., $E[Y(t)] = E[Z(t)] = 0$, $\forall t$.

Find the conditions on $Y(t)$ and $Z(t)$ under which $X(t)$ is a wide sense stationary (WSS) process. [6]

iii) Assume that both $Y(t)$ and $Z(t)$ are zero-mean WSS processes, and their autocorrelation functions are identical, i.e., $E[Y(t)] = E[Z(t)] = 0$, $\forall t$, and $R_Y(\tau) = R_Z(\tau)$, $\forall \tau$.

If the power spectral density (PSD) of $Y(t)$ is $S_Y(f)$, find the PSD of $X(t)$ in terms of $S_Y(f)$. [3]

iv) Assume that both $Y(t)$ and $Z(t)$ are zero-mean white Gaussian noise processes with PSD $N_o/2$.

What is the PSD of $X(t)$? Is $X(t)$ strict sense stationary? [3]

b) Consider a binary communication system. When a 0 is transmitted, probability of error is p_0; while when a 1 is transmitted, probability of error is p_1.

i) Assuming that bit 0 is transmitted with probability q_0, if the decoder outputs 1, what is the probability of the input being 0? Express this probability in terms of q_0, p_0 and p_1. [3]

ii) Consider using (7,4) Hamming code to communicate over this channel. The generator matrix of this code is given by

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Remember that the codeword for any 4-bit message $\mathbf{u} = [u_1\ u_2\ u_3\ u_4]$ is generated by $\mathbf{x} = \mathbf{u} \cdot \mathbf{G}$.

If we transmit message $\mathbf{u} = [0\,0\,0\,0]$ using this code, what is the probability of an error at the receiver? If $p_0 = 10^{-3}$ and $p_1 = 10^{-1}$, find the approximate value of this block error probability. [6]

iii) Assume that $p_0 = p_1 = 10^{-3}$. What is the average error probability if the 4-bit message sequences are generated as independent outcomes of a Bernoulli distribution, such that $\Pr(u_i = 1) = 0.3$, for $i = 1, \ldots, 4$. For example, $\Pr\{\mathbf{u} = [0111]\} = 0.7 \times 0.3^3$. [5]

3. a) Consider a binary communication system, where bit "0" is transmitted with a pulse of amplitude 0, and bit "1" is transmitted with a pulse of amplitude A. The channel is an additive Laplacian noise channel: For an input signal X, where $X \in \{0, A\}$, the output Y is given by

$$Y = X + W,$$

where W is the zero-mean additive noise component, which is independent of X and has the following probability density function (pdf):

$$f_W(w) = \frac{1}{2b} e^{-\frac{|w|}{b}}.$$

Assume that the detection threshold at the receiver is $T \in (0, A)$; that is, if $Y \geq T$, the transmitted bit is estimated as 1, while if $Y < T$, it is estimated as 0.

i) Given that a bit 0 was sent, derive the error probability P_{e0} in terms of T and b. [3]

ii) Given that a bit 1 was sent, derive the error probability P_{e1} in terms of A, T and b. [3]

iii) If a bit 0 is sent with probability p_0 and a bit 1 is sent with probability p_1, write down the total error probability P_e in terms of p_1, P_{e0} and P_{e1}. [2]

iv) Assume $p_1 = 2/3$, $A = 2$ and $b = 1/\ln 2$. Find the detection threshold T that minimizes P_e, ~~in terms of A~~.

What is the corresponding error probability? [10]

b) Consider a language which has only 3 letters in its alphabet: $\{x, y, z\}$. This language has 6 words in total: $\{xxx, xyz, yyy, yzx, zzz, zxy\}$. We take a sufficiently long book written in this language, and choose a random word. The probabilities of different words are given as follows:

Word	xxx	xyz	yyy	yzx	zzz	zxy
Probability	0.3	0.25	0.2	0.15	0.05	0.05

i) What is the word entropy of this language; that is, if W is the random variable denoting the randomly chosen word, what is $H(W)$? [2]

ii) If you choose a random letter from the book, what is the probability of encountering each letter? What is the entropy of the randomly chosen letter? [3]

iii) Among i) and ii) above, which one has a higher entropy per letter? Explain why the two numbers are different? [3]

iv) Consider removing the last letter of each word. What is the word entropy of this new language? What would be the advantages/disadvantages of this new language compared to the original one? [4]