Imperial College
London

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2014

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

M1S

Probability and Statistics

Date: Monday, 12 May 2014 Time: 10:00-12:00

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

Statistical tables will not be available.

Formula sheets are provided on pages 4 & 5.

1. (a) State the three axioms of probability for events defined on a sample space Ω.

(b) Prove from the axioms that if $E \subset F$ then $\mathsf{P}(E) \leq \mathsf{P}(F)$, where $E, F \subseteq \Omega$.

(c) Let A and B be events in Ω. Given that events $E_1, E_2, \ldots, E_n$ partition Ω and $\mathsf{P}(B \cap E_i) > 0$, $\forall\ 1 \leq i \leq n$, prove that

$$\mathsf{P}(A \mid B) = \sum_{i=1}^{n} \mathsf{P}(A \mid B \cap E_i)\mathsf{P}(E_i \mid B).$$

(d) You are playing a game in which your opponent has two cards in a bag, one card is red on both sides, whilst the other is red on one side and black on the other. Your opponent draws a card at random from the bag, discards the other card, and then repeatedly, at random, shows you one side of the selected card (without revealing the other side of the card on each showing).

(i) What is the probability that he reveals a red side on the first showing?

(ii) What is the probability that he reveals a red side on the second showing given that he showed you a red side on the first showing?

2. (a) For events $E_1, E_2, \ldots, E_n$ in Ω with $\mathsf{P}(E_i) > 0\ \forall\ 1 \leq i \leq n$, prove the chain rule:

$$\mathsf{P}(E_1 \cap E_2 \cap E_3 \cap \ldots \cap E_n) = \mathsf{P}(E_1)\mathsf{P}(E_2 \mid E_1)\mathsf{P}(E_3 \mid E_1 \cap E_2) \ldots \mathsf{P}(E_n \mid E_1 \cap E_2 \cap \ldots \cap E_{n-1})$$

by induction.

(b) Prove that, if $X \sim Binomial(n, \theta)$ then $\mathsf{E}(X) = n\theta$.

(c) In order to estimate the probability that students cheat in exams, we construct the following scheme which uses randomisation so that students can answer truthfully and their answers remain anonymous. We ask the student to roll a fair, six sided die without showing us the outcome, if they roll a 1 or a 2 (event S) they answer "Yes" (event Y) or "No" (event N) to the question: "Is it true that you have cheated in an exam?" If they roll a number bigger than 2 (event S') then they answer "Yes" (event Y) or "No" (event N) to the converse question: "Is it true that you have never cheated in an exam?" Assume that students answer truthfully and let $p = \mathsf{P}(Y \mid S)$.

(i) Determine $\mathsf{P}(Y)$ in terms of p.

(ii) If we carry out the above procedure for a sample of n students, assuming independence, what is the distribution of R, the number of students answering "Yes"?

(iii) Prove that

$$\mathsf{E}(R) = n\left(\frac{2-p}{3}\right).$$

(iv) Determine, in terms of n, the maximum value of the variance of R, and find the associated value of p.

3. The continuous random variable Y has probability density function (pdf) given by

$$f_Y(y) = \begin{cases} a\exp(-\lambda(y+1)) & y > 0; \\ 0 & \text{otherwise}, \end{cases}$$

with $\lambda > 1$.

(a) Find the value of a in terms of λ.

(b) Find $\mathsf{P}\left(Y > \frac{1}{\lambda} \mid Y > \frac{1}{2\lambda}\right)$.

(c) Determine $M_Y(t)$, the moment generating function (mgf) of Y.

(d) Determine $\mu = \mathsf{E}_{f_Y}(Y)$.

(e) Let $X = Y - \mu$. Find the mgf of X and show that $M_X^{(1)}(0) = 0$.

4. Suppose X and Y are continuous random variables with joint probability density function (pdf) given by,

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{12}\exp\left(-\frac{x}{3} - \frac{y}{4}\right) & x > 0, y > 0; \\ 0 & \text{otherwise}. \end{cases}$$

(a) (i) Determine, using integration, $f_X(x)$, the marginal pdf of X.

(ii) Determine, using integration, $f_Y(y)$, the marginal pdf of Y.

(b) Prove that X and Y are independent.

The joint moment generating function of X and Y is defined as

$$M_{X,Y}(s,t) = \mathsf{E}_{f_{X,Y}}(\exp(sX + tY)).$$

(c) Determine the joint moment generating function of X and Y defined above.

(d) Show that $M_{X,Y}(t,t) = M_Z(t)$ where $Z = X + Y$.

(e) Show that you obtain the same result for determining $\mathsf{E}_{f_Z}(Z)$ using $\mathsf{E}_{f_Y}(Y)$ and $\mathsf{E}_{f_X}(X)$ derived from the pdfs calculated in part (a) and the mfg derived in part (d).

(f) Justifying your answer, would the result in part (d) hold for random variables X and Y which are not independent?

DISCRETE DISTRIBUTIONS

	RANGE $\mathbb{X}$	PARAMETERS	MASS FUNCTION f_X	CDF F_X	$\mathsf{E}_{f_X}[X]$	$\mathsf{Var}_{f_X}[X]$	MGF M_X
$Bernoulli(\theta)$	$\{0,1\}$	$\theta \in (0,1)$	$\theta^x(1-\theta)^{1-x}$		θ	$\theta(1-\theta)$	$1-\theta+\theta e^t$
$Binomial(n,\theta)$	$\{0,1,\ldots,n\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{n}{x}\theta^x(1-\theta)^{n-x}$		$n\theta$	$n\theta(1-\theta)$	$(1-\theta+\theta e^t)^n$
$Poisson(\lambda)$	$\{0,1,2,\ldots\}$	$\lambda \in \mathbb{R}^+$	$\dfrac{e^{-\lambda}\lambda^x}{x!}$		λ	λ	$\exp\{\lambda(e^t-1)\}$
$Geometric(\theta)$	$\{1,2,\ldots\}$	$\theta \in (0,1)$	$(1-\theta)^{x-1}\theta$	$1-(1-\theta)^x$	$\dfrac{1}{\theta}$	$\dfrac{(1-\theta)}{\theta^2}$	$\dfrac{\theta e^t}{1-e^t(1-\theta)}$
$NegBinomial(n,\theta)$	$\{n,n+1,\ldots\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{x-1}{n-1}\theta^n(1-\theta)^{x-n}$		$\dfrac{n}{\theta}$	$\dfrac{n(1-\theta)}{\theta^2}$	$\left(\dfrac{\theta e^t}{1-e^t(1-\theta)}\right)^n$
or	$\{0,1,2,\ldots\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{n+x-1}{x}\theta^n(1-\theta)^x$		$\dfrac{n(1-\theta)}{\theta}$	$\dfrac{n(1-\theta)}{\theta^2}$	$\left(\dfrac{\theta}{1-e^t(1-\theta)}\right)^n$

For **CONTINUOUS** distributions (see over), define the **GAMMA FUNCTION**

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx$$

and the LOCATION/SCALE transformation $Y = \mu + \sigma X$ gives

$$f_Y(y) = f_X\left(\frac{y-\mu}{\sigma}\right)\frac{1}{\sigma} \qquad F_Y(y) = F_X\left(\frac{y-\mu}{\sigma}\right) \qquad M_Y(t) = e^{\mu t}M_X(\sigma t) \qquad \mathsf{E}_{f_Y}[Y] = \mu + \sigma\mathsf{E}_{f_X}[X] \qquad \mathsf{Var}_{f_Y}[Y] = \sigma^2\mathsf{Var}_{f_X}[X]$$

CONTINUOUS DISTRIBUTIONS

	$\mathbb{X}$	PARAMS.	PDF f_X	CDF F_X	$\mathrm{E}_{f_X}[X]$	$\mathrm{Var}_{f_X}[X]$	MGF M_X
$Uniform(\alpha,\beta)$ (stand. model $\alpha=0, \beta=1$)	(α,β)	$\alpha<\beta\in\mathbb{R}$	$\dfrac{1}{\beta-\alpha}$	$\dfrac{x-\alpha}{\beta-\alpha}$	$\dfrac{(\alpha+\beta)}{2}$	$\dfrac{(\beta-\alpha)^2}{12}$	$\dfrac{e^{\beta t}-e^{\alpha t}}{t(\beta-\alpha)}$
$Exponential(\lambda)$ (stand. model $\lambda=1$)	$\mathbb{R}^+$	$\lambda\in\mathbb{R}^+$	$\lambda e^{-\lambda x}$	$1-e^{-\lambda x}$	$\dfrac{1}{\lambda}$	$\dfrac{1}{\lambda^2}$	$\left(\dfrac{\lambda}{\lambda-t}\right)$
$Gamma(\alpha,\beta)$ (stand. model $\beta=1$)	$\mathbb{R}^+$	$\alpha,\beta\in\mathbb{R}^+$	$\dfrac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}$		$\dfrac{\alpha}{\beta}$	$\dfrac{\alpha}{\beta^2}$	$\left(\dfrac{\beta}{\beta-t}\right)^\alpha$
$Weibull(\alpha,\beta)$ (stand. model $\beta=1$)	$\mathbb{R}^+$	$\alpha,\beta\in\mathbb{R}^+$	$\alpha\beta x^{\alpha-1}e^{-\beta x^\alpha}$	$1-e^{-\beta x^\alpha}$	$\dfrac{\Gamma(1+1/\alpha)}{\beta^{1/\alpha}}$	$\dfrac{\Gamma\left(1+\frac{2}{\alpha}\right)-\Gamma\left(1+\frac{1}{\alpha}\right)^2}{\beta^{2/\alpha}}$	
$Normal(\mu,\sigma^2)$ (stand. model $\mu=0, \sigma=1$)	$\mathbb{R}$	$\mu\in\mathbb{R}, \sigma\in\mathbb{R}^+$	$\dfrac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\dfrac{(x-\mu)^2}{2\sigma^2}\right\}$		μ	σ^2	$e^{\{\mu t+\sigma^2t^2/2\}}$
$Student(\nu)$	$\mathbb{R}$	$\nu\in\mathbb{R}^+$	$\dfrac{(\pi\nu)^{-\frac{1}{2}}\Gamma\left(\dfrac{\nu+1}{2}\right)}{\Gamma\left(\dfrac{\nu}{2}\right)\left\{1+\dfrac{x^2}{\nu}\right\}^{(\nu+1)/2}}$		0 (if $\nu>1$)	$\dfrac{\nu}{\nu-2}$ (if $\nu>2$)	
$Pareto(\theta,\alpha)$	$\mathbb{R}^+$	$\theta,\alpha\in\mathbb{R}^+$	$\dfrac{\alpha\theta^\alpha}{(\theta+x)^{\alpha+1}}$	$1-\left(\dfrac{\theta}{\theta+x}\right)^\alpha$	$\dfrac{\theta}{\alpha-1}$ (if $\alpha>1$)	$\dfrac{\alpha\theta^2}{(\alpha-1)(\alpha-2)}$ (if $\alpha>2$)	
$Beta(\alpha,\beta)$	$(0,1)$	$\alpha,\beta\in\mathbb{R}^+$	$\dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$		$\dfrac{\alpha}{\alpha+\beta}$	$\dfrac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	

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M1S

Probability & Statistics I (Solutions)

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seen ⇓

1. (a) **Axioms of Probability**

For events $E, F \subseteq \Omega$

(I) $0 \leq \mathsf{P}(E) \leq 1$.

(II) $\mathsf{P}(\Omega) = 1$.

(III) If $E \cap F = \phi$, then $\mathsf{P}(E \cup F) = \mathsf{P}(E) + \mathsf{P}(F)$ (Addition rule).

3

seen ⇓

(b)

$$\begin{aligned} E \subset F &\Rightarrow F = E \cup (F \cap E') \\ &\Rightarrow \mathsf{P}(F) = \mathsf{P}(E) + \mathsf{P}(F \cap E') \text{ axiom III} \\ &\Rightarrow \mathsf{P}(F) > \mathsf{P}(E) \text{ axiom I as } \mathsf{P}(F \cap E') > 0 \\ &\Rightarrow \mathsf{P}(E) < \mathsf{P}(F). \end{aligned}$$

3

unseen ⇓

(c) Note that

$$A \cap B = \bigcup_{i=1}^{n} (A \cap B \cap E_i) \text{ as } E_i \text{ partition } \Omega$$

$$\Rightarrow \mathsf{P}(A \cap B) = \sum_{i=1}^{n} \mathsf{P}(A \cap B \cap E_1) \text{ axiom III as } (A \cap B \cap E_i) \cap (A \cap B \cap E_j) = \phi \,\forall i \neq j$$

So,

$$\begin{aligned} \mathsf{P}(A \mid B) = \frac{\mathsf{P}(A \cap B)}{\mathsf{P}(B)} &= \sum_{i=1}^{n} \frac{\mathsf{P}(A \cap B \cap E_i)}{\mathsf{P}(B)} \text{ as } E_i \text{ partition } \Omega \\ &= \sum_{i=1}^{n} \frac{\mathsf{P}(A \cap B \cap E_i)}{\mathsf{P}(B \cap E_i)} \times \frac{\mathsf{P}(B \cap E_i)}{\mathsf{P}(B)} \\ &= \sum_{i=1}^{n} \mathsf{P}(A \mid B \cap E_i)\mathsf{P}(E_i \mid B), \text{ as required.} \end{aligned}$$

4

sim. seen ⇓

(d) (i) Let R_i = event a red side is revealed on showing i.
Let C_1 = event red/red card is chosen and C_2 = event red/black card is chosen.

$$\begin{aligned} \mathsf{P}(R_1) &= \mathsf{P}(R_1 \mid C_1)\mathsf{P}(C_1) + \mathsf{P}(R_1 \mid C_2)\mathsf{P}(C_2) \\ &= 1 \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{3}{4}. \end{aligned}$$

4

unseen ⇓

(ii) As C_1 and C_2 partition Ω, we have from (c) that

$$\begin{aligned} \mathsf{P}(R_2 \mid R_1) &= \mathsf{P}(R_2 \mid R_1 \cap C_1)\mathsf{P}(C_1 \mid R_1) + \mathsf{P}(R_2 \mid R_1 \cap C_2)\mathsf{P}(C_2 \mid R_1) \\ &= 1 \times \mathsf{P}(C_1 \mid R_1) + \frac{1}{2} \times \mathsf{P}(C_2 \mid R_1). \end{aligned}$$

Now,

$$\mathsf{P}(C_1 \mid R_1) = \frac{\mathsf{P}(R_1 \mid C_1)\mathsf{P}(C_1)}{\mathsf{P}(R_1)} = \frac{1 \times \frac{1}{2}}{\frac{3}{4}} = \frac{2}{3},$$
$$\mathsf{P}(C_2 \mid R_1) = \frac{\mathsf{P}(R_1 \mid C_2)\mathsf{P}(C_2)}{\mathsf{P}(R_1)} = \frac{\frac{1}{2} \times \frac{1}{2}}{\frac{3}{4}} = \frac{1}{3}.$$

So,

$$\mathsf{P}(R_2 \mid R_1) = \frac{2}{3} + \frac{1}{2} \times \frac{1}{3} = \frac{5}{6}.$$ $\boxed{6}$

$\boxed{\text{sim. seen} \Downarrow}$

2. (a) From the definition of conditional probability we have
$\mathsf{P}(E_1 \cap E_2) = \mathsf{P}(E_1 \mid E_2)\mathsf{P}(E_2)$, so true for $n = 2$. Assume true for $n = k$:

$$\mathsf{P}(E_1 \cap E_2 \cap E_3 \cap \ldots \cap E_k) = \mathsf{P}(E_1)\mathsf{P}(E_2 \mid E_1)\mathsf{P}(E_3 \mid E_1 \cap E_2)\ldots\mathsf{P}(E_k \mid E_1 \cap E_2 \cap \ldots \cap E_{k-1})$$

Let $n = k + 1$

$$\begin{aligned}
\mathsf{P}(E_1 \cap \ldots \cap E_{k+1}) &= \mathsf{P}((E_1 \cap E_2 \cap \ldots \cap E_k) \cap E_{k+1}) \\
&= \mathsf{P}(E_1 \cap E_2 \cap \ldots \cap E_k)\mathsf{P}(E_{k+1} \mid E_1 \cap E_2 \cap \ldots \cap E_k) \\
&= \mathsf{P}(E_1)\mathsf{P}(E_2 \mid E_1)\ldots\mathsf{P}(E_k \mid E_1 \cap E_2 \cap \ldots \cap E_{k-1})\mathsf{P}(E_{k+1} \mid E_1 \cap \ldots \cap E_k) \\
&\quad \text{from assumption.}
\end{aligned}$$

So, true for $n = k + 1$ if true for $n = k$, true for $n = 2$ and result follows by induction. $\boxed{3}$

$\boxed{\text{seen} \Downarrow}$

(b) If $N \sim Binomial(n, \theta)$ then

$$f_N(x) = \binom{n}{x}\theta^x(1-\theta)^{n-x}, \quad x = 0, \ldots, n.$$

and

$$\begin{aligned}
\mathsf{E}_{f_N}(N) &= \sum_{x=0}^{n} x f_N(x) = \sum_{x=0}^{n} x \binom{n}{x}\theta^x(1-\theta)^{n-x} \\
&= \sum_{x=1}^{n} \frac{xn!}{x!(n-x)!}\theta^x(1-\theta)^{n-x} \\
&= \sum_{x=1}^{n} n\frac{(n-1)!}{(x-1)!(n-x)!}\theta\theta^{x-1}(1-\theta)^{n-x} \\
&= n\theta\sum_{y=0}^{n-1} \frac{(n-1)!}{y!(n-1-y)!}\theta^y(1-\theta)^{n-1-y} \quad \text{(letting } y = x-1\text{)} \\
&= n\theta\sum_{y=0}^{n-1} \binom{n-1}{y}\theta^y(1-\theta)^{(n-1)-y} \\
&= n\theta,
\end{aligned}$$

as required (could also use binomial expansion or sum of Bernoulli rvs). $\boxed{4}$

$\boxed{\text{sim. seen} \Downarrow}$

(c) We are given

$$P(S) = \frac{2}{6} = \frac{1}{3};\ \mathsf{P}(S') = \frac{4}{6} = \frac{2}{3};\ p = \mathsf{P}(Y \mid S).$$

Note that the probability that the statement: "student cheated" is True is the same as the probability that the converse statement: "student never cheated" is False, i.e. $\mathsf{P}(N \mid S') = \mathsf{P}(Y \mid S) = p$.

(i)

$$\begin{aligned}\mathsf{P}(Y) = \mathsf{P}(Y \mid S)\mathsf{P}(S) + \mathsf{P}(Y \mid S')\mathsf{P}(S') &= p \times \frac{1}{3} + (1-p) \times \frac{2}{3} \\ &= \frac{2-p}{3}.\end{aligned}$$

$\boxed{3}$

(ii) R = number of students answering Y, and we have that $R \sim Binomial(n, \theta)$ where $\theta = \mathsf{P}(Y) = \frac{2-p}{3}$. $\boxed{2}$

(iii) From (b) we have that $\mathsf{E}_{f_R}(R) = n\theta = n\left(\frac{2-p}{3}\right)$. $\boxed{2}$

(iv) $\boxed{\text{unseen} \Downarrow}$

$$\begin{aligned}\mathsf{var}_{f_R}(R) &= n\theta(1-\theta) = n(\theta - \theta^2) \\ &= n\left(-\left(\theta - \frac{1}{2}\right)^2 + \frac{1}{4}\right) = \frac{n}{4} - n\left(\theta - \frac{1}{2}\right)^2,\end{aligned}$$

so, variance is maximal when

$$\left(\theta - \frac{1}{2}\right) = 0 \Rightarrow \theta = \frac{1}{2}.$$

Giving

$$\mathsf{var}_{f_R}(R) = \frac{n}{4}.$$

$\boxed{4}$

The associated value of $p = 2 - 3\theta = \frac{1}{2}$.
Alternatively determine maximum via derivatives. $\boxed{2}$

unseen $\Downarrow$

3. (a) (i)

$$\int_0^\infty a\exp(-\lambda(y+1))\ \mathrm{d}y = \left[-a\exp(-\lambda)\frac{\exp(-\lambda y)}{\lambda}\right]_0^\infty = \frac{a\exp(-\lambda)}{\lambda}$$
$$\Rightarrow \frac{a\exp(-\lambda)}{\lambda} = 1 \Rightarrow a = \lambda\exp(\lambda).$$

Note that $Y \sim Exponential(\lambda)$ as $f_Y(y) = \lambda e^{\lambda} e^{-\lambda(y+1)} = \lambda e^{-\lambda y}$. $\boxed{3}$

(ii)

meth seen $\Downarrow$

$$\mathrm{P}\left(Y > \frac{1}{\lambda} \mid Y > \frac{1}{2\lambda}\right) = \frac{\mathrm{P}\left(\left(Y > \frac{1}{\lambda}\right) \cap \left(Y > \frac{1}{2\lambda}\right)\right)}{\mathrm{P}\left(Y > \frac{1}{2\lambda}\right)} = \frac{P\left(Y > \frac{1}{\lambda}\right)}{\mathrm{P}\left(Y > \frac{1}{2\lambda}\right)}.$$

Now

$$\mathrm{P}(Y > y) = \int_y^\infty \lambda\exp(-\lambda x)\ \mathrm{d}x = [-\exp(-\lambda x)]_y^\infty = \exp(-\lambda y).$$

So,

$$\mathrm{P}\left(Y > \frac{1}{\lambda} \mid Y > \frac{1}{2\lambda}\right) = \frac{e^{-1}}{e^{-1/2}} = e^{-1/2}.$$

$\boxed{5}$

(c)

$$\begin{aligned} M_Y(t) &= \int_0^\infty f_Y(y)e^{ty}\ \mathrm{d}y \\ &= \int_0^\infty \lambda\exp(\lambda)\exp(-\lambda(y+1))e^{ty}\ \mathrm{d}y \\ &= \int_0^\infty \lambda\exp(-y(\lambda-t))\ \mathrm{d}y = \left[-\frac{\lambda\exp(-y(\lambda-t))}{\lambda-t}\right]_0^\infty \\ &= \frac{\lambda}{\lambda-t}, \end{aligned}$$

as expected (fine to just state the form of the mgf if noted that Y is exponential.) $\boxed{4}$

(d)

seen $\Downarrow$

$$\mu = M_Y^{(1)}(0) = \left.\frac{\lambda}{(\lambda-t)^2}\right|_{t=0} = \frac{1}{\lambda}.$$

$\boxed{2}$

$\boxed{\text{unseen} \Downarrow}$

(e) As $X = Y - \mu$ we have

$$\begin{aligned}\Rightarrow M_X(t) &= \mathsf{E}_{f_X}(\mathrm{e}^{tX}) = \mathsf{E}_{f_X}(\mathrm{e}^{tY-t\mu}) = \mathrm{e}^{-t\mu}\mathsf{E}_{f_Y}(\mathrm{e}^{tY}) \\ &= \mathrm{e}^{-t\mu}\frac{\lambda}{\lambda - t} = \frac{\lambda \mathrm{e}^{-t/\lambda}}{\lambda - t}.\end{aligned}$$

Note that we have used the result that for $Y = g(X)$:

$$\begin{aligned}\mathsf{E}_{f_Y}(Y) &= \int_y y f_Y(y)\,\mathrm{d}y = \int_y y \int_x f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_x g(x) \int_y f_{X,Y}(x,y)\,\mathrm{d}y\,\mathrm{d}x = \int_x g(x) f_X(x)\,\mathrm{d}x \\ &= \mathsf{E}_{f_X}(g(X))\end{aligned}$$

$\boxed{3}$

So,

$$\begin{aligned}M_X^{(1)}(t) &= \frac{\lambda \mathrm{e}^{-t/\lambda}}{(\lambda - t)^2} - \frac{\lambda \mathrm{e}^{-t/\lambda}}{\lambda(\lambda - t)} \\ \Rightarrow M_X^{(1)}(0) &= \frac{1}{\lambda} - \frac{1}{\lambda} = 0,\end{aligned}$$

as required. $\boxed{3}$

sim. seen ⇓

4. (a) (i)

$$
\begin{aligned}
f_X(x) &= \int_0^\infty \frac{1}{12} \mathrm{e}^{-(x/3+y/4)}\ \mathrm{d}y \\
&= \frac{1}{12}\mathrm{e}^{-x/3} \int_0^\infty \mathrm{e}^{-y/4}\ \mathrm{d}y \\
&= \frac{1}{12}\mathrm{e}^{-x/3} \left[-4\mathrm{e}^{-y/4}\right]_0^\infty \\
&= \frac{1}{12}\mathrm{e}^{-x/3}4 = \frac{1}{3}\mathrm{e}^{-x/3},\ x > 0.
\end{aligned}
$$

So $X \sim \mathit{Exponential}(1/3)$. 3

(ii)

$$
\begin{aligned}
f_X(x) &= \int_0^\infty \frac{1}{12} \mathrm{e}^{-(x/3+y/4)}\ \mathrm{d}y \\
&= \frac{1}{12}\mathrm{e}^{-y/4} \int_0^\infty \mathrm{e}^{-x/3}\ \mathrm{d}y \\
&= \frac{1}{12}\mathrm{e}^{-y/4} \left[-3\mathrm{e}^{-y/3}\right]_0^\infty \\
&= \frac{1}{12}\mathrm{e}^{-y/4}3 = \frac{1}{4}\mathrm{e}^{-y/4},\ y > 0.
\end{aligned}
$$

So $Y \sim \mathit{Exponential}(1/4)$. 2

(b)

$$
f_{X,Y}(x,y) = \frac{1}{12}\mathrm{e}^{-(x/3+y/4)} = \frac{1}{3}\mathrm{e}^{-x/3} \times \frac{1}{4}\mathrm{e}^{-y/4} = f_X(x) f_Y(y), x > 0, y > 0.
$$

So, X and Y are independent. 2

(c) unseen ⇓

$$
\begin{aligned}
M_{X,Y}(s,t) = \mathrm{E}_{f_{X,Y}}(\exp(sX+tY)) &= \int_0^\infty \int_0^\infty \mathrm{e}^{sx+ty} \frac{1}{12} \mathrm{e}^{-(x/3+y/4)}\ \mathrm{d}y\,\mathrm{d}x \\
&= \frac{1}{12} \int_0^\infty \int_0^\infty \mathrm{e}^{-(x/3+y/4-sx-ty)}\ \mathrm{d}y\,\mathrm{d}x \\
&= \frac{1}{12} \int_0^\infty \int_0^\infty \mathrm{e}^{-x(1/3-s)} \mathrm{e}^{-y(1/4-t)}\ \mathrm{d}y\,\mathrm{d}x \\
&= \frac{1}{12} \int_0^\infty \left[-\frac{\mathrm{e}^{-x(1/3-s)}}{1/3-s}\right]_0^\infty \mathrm{e}^{-y(1/4-t)}\ \mathrm{d}y = \frac{1}{12}\int_0^\infty \frac{3}{1-3s}\mathrm{e}^{-y(1/4-t)}\ \mathrm{d}y \\
&= \frac{1}{12}\frac{3}{1-3s}\left[-\frac{\mathrm{e}^{-y(1/4-t)}}{1/4-t}\right]_0^\infty = \frac{1}{12}\left(\frac{3}{1-3s}\right)\left(\frac{4}{1-4t}\right) \\
&= \frac{1}{(1-3s)(1-4t)}.
\end{aligned}
$$

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(d) We have,

$$M_{X,Y}(t,t) = \frac{1}{(1-3t)(1-4t)}$$

as X and Y are independent $M_Z(t) = M_X(t)M_Y(t)$. Can either calculate directly or note: $M_X(t) = \mathsf{E}_{f_X}(\mathrm{e}^{tX}) = M_{X,Y}(t,0)$ and $M_Y(t) = \mathsf{E}_{f_Y}(\mathrm{e}^{tY}) = M_{X,Y}(0,t)$, giving

$$M_X(t) = \frac{1}{1-3t}, \quad M_Y(t) = \frac{1}{1-4t}, \quad M_Z(t) = \frac{1}{(1-3t)(1-4t)}.$$

So, $M_Z(t) = M_{X,Y}(t,t)$ as required. $\boxed{3}$

(e) From the mgf, we have $\boxed{\text{sim. seen} \Downarrow}$

$$E_{f_Z}(Z) = M_Z^{(1)}(0) = \left.\frac{3(1-3t)+4(1-4t)}{((1-3t)(1-4t))^2}\right|_{t=0} = 3+4 = 7.$$

$X \sim Exponential(1/3)$ so $\mathsf{E}_{f_X}(X) = 3$ and $Y \sim Exponential(1/4)$ so $\mathsf{E}_{f_X}(X) = 4$

$$E_{f_Z}(Z) = \mathsf{E}_{f_X}(X) + \mathsf{E}_{f_Y}(Y) = 3+4 = 7,$$

as required. $\boxed{3}$

(f) If X and Y are not independent then it is not necessarily true that for functions g and h, that $\mathsf{E}_{f_{X,Y}}(g(X)h(Y)) = \mathsf{E}_{f_X}(g(X))\mathsf{E}_{f_Y}(h(Y))$. Hence, we could have the case that:

$$M_{X,Y}(t,t) = \mathsf{E}_{f_{X,Y}}(\mathrm{e}^{tX+tY}) \neq \mathsf{E}_{f_X}(\mathrm{e}^{tX})\mathsf{E}_{f_Y}(\mathrm{e}^{tY}) = M_X(t)M_Y(t) = M_Z(t).$$

In which case $M_{X,Y}(t) \neq M_Z(t)$. So part (d) will not in general hold. $\boxed{2}$