M3/M4/M5
P19

Imperial College
London

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2014

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Measure & Integration

Date: Wednesday, 07 May 2014. Time: 2.00pm – 4.00pm. Time allowed: 2 hours.

This paper has FOUR questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the main book is full.

Statistical tables will not be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers to each question as per the table below.

Raw mark	up to 12	13	14	15	16	17	18	19	20
Extra credit	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4

- Each question carries equal weight.
- Calculators may not be used.

1. i. Let μ be a complete measure, A be a set such that $\mu(A) = 0$, f be a function on A, $f : A \to \mathbb{R}$. Prove or disprove that f is measurable.
ii. Formulate Egorov's theorem (without proof).

2. i. Given a measure on a ring and the outer measure, define what it means for a set to be measurable (in Lebesgue sense).
ii. Let μ be a complete σ-additive measure on A, and let a function f be integrable on A. Assuming other properties of the integral, show that given $\epsilon > 0$ there is a $\delta > 0$ such that for any measurable set $E \subset A$ with $\mu(E) < \delta$, we have

$$\left| \int_E f d\mu \right| < \epsilon.$$

3. i. State the property of σ-additivity of a measure.
ii. Compute the limit (if it exists):

$$\lim_{n\to\infty} \int_0^n \left(1 + \frac{x}{n}\right)^n e^{-2x} dx,$$

where the integral is with respect to the Lebesgue measure on $[0, \infty)$. Justify your reasoning (you can use elementary results about the exponential function without proof).

4. Let $f(x)$ be an absolutely continuous function on $[0, 1-\delta]$ for any $1 > \delta > 0$. Moreover, let this $f(x)$ be continuous at $x = 1$ and of bounded variation on $[0, 1]$. Show that f is absolutely continuous on $[0, 1]$.

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	EXAMINATION SOLUTIONS 2013-14	Course
Question 1		Marks & seen/unseen
Parts		
i	μ-complete, $\mu(A)=0$, $f: A \to \mathbb{R}$. Since μ is complete, any subset of A is measurable. In particular, the preimage $f^{-1}(B)$ of any Borel set is measurable (as $f^{-1}(B) \subset A$). Hence, f is measurable by definition.	10
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	EXAMINATION SOLUTIONS 2013 - M...	Course
Question 1		Marks & seen/unseen
Parts ii	Egorov's theorem: Let E be a set of finite measure, $f_n(x)$ – a sequence of measurable functions converging to a function f a.e. on E Then for any $\delta > 0$ there exists a measurable subset $E_\delta \subset E$ s.t. 1) $\mu(E_\delta) > \mu(E) - \delta$ 2) $f_n \to f$ uniformly on E_δ.	10 seen
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	EXAMINATION SOLUTIONS 2013-14	Course
Question 2		Marks & seen/unseen
Parts i	A set A is called measurable if $\forall \varepsilon > 0$ there exists a set in the ring, ~~set~~ call it B, s.t. the outer measure $\mu^*(A \triangle B) < \varepsilon$.	5 seen
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Question		Marks & seen/unseen
2		

Parts

ii

Let μ – complete measure on A, f - integrable on A. We now show that $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. for any measurable $B \subset A$ with $\mu(B) < \delta$ we have $\left| \int_B f \, d\mu \right| < \varepsilon$ (absolute continuity of the integral).

seen

1) If f is bounded, i.e. if $|f| \leq M$ on A, then by a property of the integral
$$\left| \int_B f \, d\mu \right| \leq M \mu(B) < M\delta$$
and so the result follows by choosing $\delta < \varepsilon / M$.

		Course
Question		Marks & seen/unseen
Parts		

2) In general, let

$$A_n = \{x \in A : n \leq |f(x)| < n+1\},$$

$$B_N = \bigcup_{n=0}^{N} A_n \quad , \quad C_N = A \setminus B_N$$

By σ-additivity of the integral,

$$\int_A |f| \, d\mu = \sum_{n=0}^{\infty} \int_{A_n} |f| \, d\mu$$

Using the fact that the series converge, for an $\varepsilon > 0$ choose N s.t.

$$\sum_{n=N+1}^{\infty} \int_{A_n} |f| \, d\mu = \int_{C_N} |f| \, d\mu < \varepsilon/2 .$$

Let $0 < \delta < \frac{\varepsilon}{2(N+1)}$

If $\mu(B) < \delta$, we have by the properties of the integral,

	EXAMINATION SOLUTIONS 2013 - M...	Course
Question		Marks & seen/unseen
Parts		

$$\left|\int_B f\,d\mu\right| \leq \int_B |f|\,d\mu =$$

$$= \int_{B\cap B_N} + \int_{B\cap C_N} .$$

By construction and properties of the integral,

$$\int_{B\cap B_N} |f|\,d\mu \leq (N+1)\,\mu(B) <$$

$$< (N+1)\,\delta < \varepsilon/2 ;$$

$$\int_{B\cap C_N} |f|\,d\mu < \int_{C_N} |f|\,d\mu < \varepsilon/2 .$$

Therefore,

$$\left|\int_B f\,d\mu\right| < \varepsilon .$$

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	EXAMINATION SOLUTIONS 2013 - ...	Course
Question 3		Marks & seen/unseen
Parts i	Let $\{A_k,\ k=1,2,\dots\}$ be a countable collection of pairwise disjoint measurable sets. A measure μ is σ-additive if $\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k)$.	3
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EXAMINATION SOLUTIONS 2013-14

Course

Question 3

Marks & seen/unseen

Parts ii

unseen

We use the elementary result

$$\left(1+\frac{x}{n}\right)^n \leq e^x \text{ for } \frac{x}{n} < 1,$$

if $x \geq 0$.

Let $f_n = \begin{cases} (1+\frac{x}{n})^n e^{-2x}, & 0 < x < n \\ 0, & x \geq n \end{cases}$

For a fixed n, this function is bounded and measurable (as a product of a continuous function and the characteristic function of the set $(0,n)$ — both measurable), and therefore integrable over any finite interval $[0, x_0]$ and, clearly, over $[0,\infty)$.

Moreover, $|f_n(x)| \leq e^x \cdot e^{-2x} = e^{-x}$

Setter's initials Checker's initials

The bound is an integrable function, and

$$\int_0^\infty e^{-x}\,dx = 1 .$$

For any $x \geq 0$,

$$\lim_{n\to\infty} f_n(x) = e^{x}\cdot e^{-2x} = e^{-x} .$$

By Lebesgue's theorem,

$$\lim_{n\to\infty} \int_0^n \left(1+\frac{x}{n}\right)^n e^{-2x}\,dx =$$

$$= \lim_{n\to\infty} \int_0^\infty f_n(x)\,dx =$$

$$= \int_0^\infty e^{-x}\,dx = 1 .$$

	EXAMINATION SOLUTIONS 2013-14	Course
Question 4		Marks & seen/unseen
Parts	Let $f(x)$ be a.c. on $[0,1-\delta]$, $\delta>0$; f - continuous at $x=1$; f of B.V. on $[0,1]$. Let $V_0^x(f)$ be the variation of f on $[0,x]$. Since f is continuous at 1, we have that $V_0^x(f)$ is continuous at 1. Therefore, for any fixed $\varepsilon>0$ there is $\delta>0$ s.t. $V_x^1 = V_0^1 - V_0^x < \varepsilon/2$ for all $x\in[1-\delta,1]$. (1) However, $\sum_{k=0}^{N} \lvert f(b_k)-f(a_k)\rvert \le V_{1-\delta}^1$ for any finite division of $[1-\delta,1]$ (2)	20 unseen
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Question 4 | Marks & seen/unseen

Parts

Since f is a.c. on $[0,1-\delta]$, there is $\delta'>0$ s.t.

$$\sum_k |f(b_k')-f(a_k')| < \varepsilon/2 \qquad (3)$$

for any family of disjoint intervals $(a_k', b_k') \subset [0,1-\delta]$ satisfying

$$\sum_{k=0}^{L} (b_k'-a_k') < \delta' \qquad (4)$$

Take any finite family F of disjoint intervals $(a_k'', b_k'') \subset$ $\subset [0,1]$ satisfying

$$\sum_k (b_k''-a_k'') < \delta'$$

Relable points $a_k'', b_k'' \subset [0,1-\delta]$ by a_k', b_k'. If $1-\delta \in (a_{k_0}'', b_{k_0}'')$ for some k_0, let $b_L' = 1-\delta$, $a_0 = 1-\delta$.

Considering the part of F in $[0, 1-\delta]$ and the part of F in $[1-\delta, 1]$ separately, we obtain by (4), (3), (2), and (1), that

$$\sum_k |f(b_k'') - f(a_k'')| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

which establishes the result.