

SOLUTIONS: COMPLEX CALCULUS EE2L

1. Exercise

a) Differentiating u with respect to x and y yields:

$$\frac{\partial u}{\partial x} = 2y - \frac{e^y \sin(x)}{2} - \frac{e^{-y} \sin(x)}{2}$$

$$\frac{\partial u}{\partial y} = 2x + \frac{e^y \cos(x)}{2} - \frac{e^{-y} \cos(x)}{2}.$$

Taking second derivatives with respect to x and y , yields:

$$\frac{\partial^2 u}{\partial x^2} = -\frac{e^y \cos(x)}{2} - \frac{\cos(x)}{2e^y},$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{e^y \cos(x)}{2} + \frac{\cos(x)}{2e^y}.$$

Notice that:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Hence, u is harmonic.

b) From Cauchy-Riemann equations we have that $u_x = v_y$, hence:

$$\frac{\partial v}{\partial y} = 2y - \frac{e^y \sin(x)}{2} - \frac{e^{-y} \sin(x)}{2}.$$

Taking indefinite integrals of the above equation with respect to y yields:

$$v = y^2 - \sin(x) \frac{e^y - e^{-y}}{2} + c_1(x).$$

Moreover, $v_x = -u_y$, hence:

$$\frac{\partial v}{\partial x} = -2x - \frac{e^y \cos(x)}{2} + \frac{e^{-y} \cos(x)}{2}.$$

Taking indefinite integrals of the above equation with respect to x yields:

$$v = -x^2 - \sin(x) \frac{e^y - e^{-y}}{2} + c_2(y).$$

Equating the two expressions we see that:

$$c_1(x) + y^2 = c_2(y) - x^2$$

and therefore, for some complex $c \in \mathbb{C}$:

$$c_1(x) = c - x^2$$

$$c_2(y) = c + y^2.$$

Therefore, v is determined up to an additive constant as follows:

$$v = y^2 - x^2 - \sin(x) \frac{e^y - e^{-y}}{2} + c.$$

- c) Notice that, for $z = x + i0$, it holds:

$$f(z) = u(x, 0) + iv(x, 0) = \cos(x) + i(c - x^2)$$

Hence, we may take:

$$f(z) = \cos(z) + i(c - z^2).$$

- a) The function is a ratio of polynomials. The denominator vanishes for $z \in \{\pm i, \pm 2i, \pm 3i\}$. The numerator only vanishes for $z = 0$. Hence, the function is holomorphic in $\mathbb{C} \setminus \{\pm i, \pm 2i, \pm 3i\}$.
- b) The function has poles in $\pm i, \pm 2i$ and $\pm 3i$. Computation of the Residues yields:

$$\text{Res}(i) = \lim_{z \rightarrow i} (z - i)f(z) = \lim_{z \rightarrow i} \frac{z}{(z + i)(z^2 + 4)(z^2 + 9)} = \frac{1}{48}$$

$$\text{Res}(2i) = \lim_{z \rightarrow 2i} (z - 2i)f(z) = \lim_{z \rightarrow 2i} \frac{z}{(z^2 + 1)(z + 2i)(z^2 + 9)} = -\frac{1}{30}$$

$$\text{Res}(3i) = \lim_{z \rightarrow 3i} (z - 3i)f(z) = \lim_{z \rightarrow 3i} \frac{z}{(z^2 + 1)(z^2 + 4)(z + 3i)} = \frac{1}{80}$$

- c) In order to compute the improper integral, it is enough to realize that:

$$\int_{-\infty}^{+\infty} f(x)dx = \lim_{R \rightarrow +\infty} \int_{-R}^R f(x)dx = \lim_{R \rightarrow +\infty} \left[\int_{\Gamma_R} f(z)dz - \int_{\gamma_R} f(z)dz \right],$$

where $\gamma_R : [0, \pi] \rightarrow \mathbb{C}$, is the curve $\gamma_R(t) = Re^{it}$, while Γ_R is the concatenation of the segment $[-R, R]$ with the curve γ_R . Notice that Γ_R is a simple closed curve. By Cauchy's formula, for all $R > 3$ we have:

$$\int_{\Gamma_R} f(z)dz = 2\pi i(1/48 + 1/80 - 1/30) = 0$$

Notice that the function $|f(z)|$ decreases as $|z|^{-6}$ when $|z| \rightarrow +\infty$. Therefore, $|z||f(z)|$ also approaches zero as $|z| \rightarrow +\infty$. As a consequence, $\lim_{R \rightarrow +\infty} \int_{\gamma_R} f(z)dz = 0$. Combining the previous considerations we see:

$$\int_{-\infty}^{+\infty} f(x)dx = 0.$$

- d) Notice that $f(x) = -f(-x)$. Hence, since the integral converges absolutely, we have:

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^0 f(x)dx + \int_0^{+\infty} f(x)dx = -\int_0^{+\infty} f(x)dx + \int_0^{+\infty} f(x)dx = 0.$$

2. Exercise

- a) Let $X(s)$ denote $\mathcal{L}[x]$. Then $\mathcal{L}[\dot{x}] = sX(s) - 1$ and $\mathcal{L}[\ddot{x}] = s^2X(s) - s$. Finally $\mathcal{L}[x^{(3)}] = s^3X(s) - s^2$. Substituting in the differential equations we see that:

$$s^3X(s) - s^2 = -s^2X(s) + s - sX(s) + 1 - X(s).$$

Solving with respect to $X(s)$ yields:

$$X(s) = \frac{s^2 + s + 1}{s^3 + s^2 + s + 1}$$

Notice that: $(s^3 + s^2 + s + 1) = (s + 1)(s^2 + 1)$, therefore:

$$X(s) = \frac{A}{s+1} + \frac{Bs+C}{s^2+1}$$

with:

$$A = \lim_{s \rightarrow -1} \frac{s^2 + s + 1}{s^2 + 1} = 1/2$$

and $B = C = 1/2$. Taking inverse Laplace's transforms yields:

$$x(t) = \frac{1}{2}e^{-t} + \frac{1}{2}\cos(t) + \frac{1}{2}\sin(t).$$

