EE1-10B MATHEMATICS II

1. a) Obtain the Fourier Transform of the function [7]

$$f(t) = \begin{cases} \cos(\pi t) & -1 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$$

SOLUTION

$$\begin{aligned}
F(\omega) &= \int_{-\infty}^{+\infty} f(t)e^{-i\omega t}\,dt = \int_{-1}^{1} \cos(\pi t)e^{-i\omega t}\,dt = \frac{1}{2}\int_{-1}^{1}\left[e^{-i(\omega+\pi)t} + e^{-i(\omega-\pi)t}\right]dt \\
&= \frac{1}{2}\left[\frac{e^{-i(\omega+\pi)t}}{-i(\omega+\pi)} + \frac{e^{-i(\omega-\pi)t}}{-i(\omega-\pi)}\right]_{-1}^{1} \\
&= \frac{i}{2}\left\{\frac{1}{\omega+\pi}\left[e^{-i(\omega+\pi)} - e^{i(\omega+\pi)}\right] + \frac{1}{\omega-\pi}\left[e^{-i(\omega-\pi)} - e^{i(\omega-\pi)}\right]\right\} \qquad (**) \\
&= \frac{1}{2i}\left[\frac{1}{\omega+\pi}\left(e^{i\omega} - e^{-i\omega}\right) + \frac{1}{\omega-\pi}\left(e^{i\omega} - e^{-i\omega}\right)\right] \\
&= -\left[\left(\frac{1}{\omega+\pi} + \frac{1}{\omega-\pi}\right)\frac{e^{i\omega} - e^{-i\omega}}{2i}\right] = -\frac{2\omega}{(\omega^2-\pi^2)}\sin\omega,
\end{aligned}$$

where we have used $e^{\pm i\pi} = -1$ in $(**)$

b) (i) Let $\mathscr{F}[f(t)]$ be the Fourier transform of $f(t)$ and $\mathscr{F}^{-1}[F(\omega)]$ the inverse Fourier transform of $F(\omega)$. Use the inverse Fourier Transform to show that

$$\mathscr{F}[\cos(at)] = \pi\left[\ \delta(\omega+a)+\delta(\omega-a)\right]. \qquad [\,3\,]$$

SOLUTION

$$\begin{aligned} f(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}\,d\omega \\ &= \frac{1}{2}\int_{-\infty}^{\infty}\delta(\omega+a)e^{i\omega t}\,d\omega + \frac{1}{2}\int_{-\infty}^{\infty}\delta(\omega-a)e^{i\omega t}\,d\omega \\ &= \frac{1}{2}\left(e^{-iat}+e^{iat}\right) = \cos(at) \end{aligned}$$

where we have used the property $\int_{-\infty}^{\infty} f(t)\delta(t-t_0)\,dt = f(t_0)$, of the delta function.

(ii) Similarly, obtain $\mathscr{F}[\sin(at)]$. [4]

SOLUTION

Following the experience with (i), we need terms $e^{iat}-e^{-iat}$ after integration, so we begin with $\delta(\omega-a)-\delta(\omega+a)$ and worry about the constants later:

$$\begin{aligned} f(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}\,d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\delta(\omega-a)e^{i\omega t}\,d\omega - \frac{1}{2\pi}\int_{-\infty}^{\infty}\delta(\omega+a)e^{i\omega t}\,d\omega \\ &= \frac{1}{2\pi}\left(e^{iat}-e^{iat}\right) = \frac{1}{2\pi}2i\sin(at) \end{aligned}$$

As this is the inverse Fourier transform of $\delta(\omega-a)-\delta(\omega+a)$ we multiply by πi and conclude that

$$\mathscr{F}[\sin(at)] = \pi i[\delta(\omega+a)-\delta(\omega-a)].$$

c) A cube has edges of length 2, with one corner at the origin and another corner at $P(2,2,2)$. A line L_1 has equation $\underline{\mathbf{r}} = (2,1,4) + \lambda(1,2,1)$. Find:

(i) the vector equation of the line L_2 through the origin and P; [2]

SOLUTION

As the cube has edges of length 2, the position of the two given corners constrains the cube to lie entirely in the positive octant ($x,y,z \geq 0$), with centre at $(1,1,1)$ and $L_2 = \lambda(1,1,1)$ for any real λ is also a main diagonal of the cube.

(ii) the cartesian equation of the plane Π through the centre of the cube and equidistant from the origin and P; [3]

SOLUTION

Distance of point to plane is perpendicular distance, so this is along L_2 which is perpendicular to the plane, intersecting at the centre of the cube. Hence the plane has equation

$$(x,y,z)\cdot(1,1,1) = (1,1,1)\cdot(1,1,1) \Rightarrow x+y+z=3.$$

(iii) the point Q, the intersection of Π and L_1; [3]

SOLUTION

Substituting $(x,y,z) = (2,1,4) + \lambda(1,2,1)$ into Π we have

$$(2+\lambda)+(1+2\lambda)+(4+\lambda) = 3 \Rightarrow \lambda = -1$$

and the intersection is $Q(1,-1,3)$.

(iv) the distance between line L_2 and point Q. [3]

SOLUTION

Can calculate this using standard technique for distance of point/line. Or observe that Q is on the plane Π which is perpendicular to L_2, so the nearest point to Q on the line L_2 is the intersection of Π and L_2: the centre of the cube at $(1,1,1)$. So the distance is

$$|(1,1,1)-(1,-1,3)| = |(0,2,-2)| = 2\sqrt{2}.$$

2. a) The vector $\underline{\mathbf{a}}$ satisfies

$$\underline{\mathbf{a}} \times \underline{\mathbf{v}} = \underline{\mathbf{w}}, \quad \text{and} \quad \underline{\mathbf{a}} \cdot \underline{\mathbf{v}} = 2,$$

where $\underline{\mathbf{v}} = (-1, 2, 1)$ and $\underline{\mathbf{w}} = (3, -10, 23)$. By taking the vector product of the first equation with one of the given vectors, or otherwise, find $\underline{\mathbf{a}}$.
[5]

SOLUTION

To use the second equation we need to obtain the scalar product $\underline{\mathbf{a}} \cdot \underline{\mathbf{v}}$ when taking the vector product of a given vector with the first equation. This entails the triple vector product and we need to choose $\underline{\mathbf{v}}$. Hence

$$\underline{\mathbf{v}} \times (\underline{\mathbf{a}} \times \underline{\mathbf{v}}) = (\underline{\mathbf{v}} \cdot \underline{\mathbf{v}})\underline{\mathbf{a}} - (\underline{\mathbf{v}} \cdot \underline{\mathbf{a}})\underline{\mathbf{v}} = \underline{\mathbf{v}} \times \underline{\mathbf{w}}$$

Calculating the scalar product $\underline{\mathbf{v}} \cdot \underline{\mathbf{v}} = 6$ and vector product $\underline{\mathbf{v}} \times \underline{\mathbf{w}} = (56, 26, 4)$ we get

$$6\underline{\mathbf{a}} - 2(-1, 2, 1) = (56, 26, 4) \Rightarrow \underline{\mathbf{a}} = (9, 5, 1).$$

b) Consider the matrices

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -2 & 3 \\ -1 & 1 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & -1 & 3 \\ -2 & 1 & 0 \\ 3 & -1 & 1 \end{pmatrix}.$$

Obtain (i) $\det(A)$, (ii) $\det(B)$, (iii) $\det(AB)$, and (iv) $\det(BA)$. [4]

SOLUTION

Expanding by the first row

(i) $\det(A) = 1(-4-3) - 1(2+3) - 1(1-2) = -11$

(ii) $\det(B) = 2(1-0) + 1(-2-0) + 3(2-3) = -3$

(iii) and (iv): given $\det(AB) = \det(A)\det(B)$, both are 33.

c) You are given a system of linear equations

$$\begin{array}{rcrcrcl} -x & + & 2y & - & z & = & 4 \\ 2x & - & 3y & + & z & = & \alpha \\ -3x & + & 2y & + & \beta z & = & 2 \end{array}$$

where α and β are constants.

(i) Write the system in the form $A\underline{\mathbf{x}} = \underline{\mathbf{b}}$ and find $\det(A)$. Hence find the conditions on α, β required so that the sytem has (I) no solution, (II) a unique solution, and (III) infinitely many solutions. In case (III) find these solutions. [6]

SOLUTION

(i) In matrix form, we have $(A : \underline{\mathbf{b}}) = \begin{pmatrix} -1 & 2 & -1 & 4 \\ 2 & -3 & 1 & \alpha \\ -3 & 2 & \beta & 2 \end{pmatrix}$

and expanding by the first row,

$\det(A) = -(-3\beta - 2) - 2(2\beta + 3) - (4 - 9) = 1 - \beta$. For (II), a unique solution, we require the matrix to be invertible, so $\beta \neq 1$ with no restrictions on α. For (I) and (III) we need a singular matrix, so let $\beta = 1$ and begin by obtaining echelon form, using row operations, immediately multiplying the first row by -1

$$(A : \underline{\mathbf{b}}) = \begin{pmatrix} 1 & -2 & 1 & -4 \\ 2 & -3 & 1 & \alpha \\ -3 & 2 & 1 & 2 \end{pmatrix} \underset{\sim}{\overset{R_2 - 2R_1}{R_3 + 3R1}} \begin{pmatrix} 1 & -2 & 1 & -4 \\ 0 & 1 & -1 & \alpha + 8 \\ 0 & -4 & 4 & -10 \end{pmatrix}$$

$$\overset{R_3 + 4R_2}{\sim} \begin{pmatrix} 1 & -2 & 1 & -4 \\ 0 & 1 & -1 & \alpha + 8 \\ 0 & 0 & 0 & 4\alpha + 22 \end{pmatrix}$$

Hence (III) infinitely many solutions exist if $\beta = 1$ and $4\alpha + 22 = 0 \Rightarrow \alpha = -11/2$. Similarly, (I) no solutions exist, the system is inconsistent, if $\beta = 1$ and $4\alpha + 22 \neq 0 \Rightarrow \alpha \neq -11/2$.

In case (III), letting $\alpha = -11/2$, the last row is all zeroes and the second row gives $y - z = -11/2 + 8 = 5/2$. Choose the free variable $z = \lambda$, so that $y = 5/2 + \lambda$ and substituting into the top row we get $x = 2y - z - 4 = 1 + \lambda$ and in vector form the solution is the equation of a line:

$$(x, y, z) = (1, 5/2, 0) + \lambda(1, 1, 1)$$

(ii) For $\beta = 0$, use Gaussian elimination to find the inverse matrix A^{-1}. [4]

SOLUTION Begin by augmenting A with the identity matrix I and use

row operations to obtain I in the left half $(A : I) =$

$$\left(\begin{array}{rrr|rrr} -1 & 2 & -1 & 1 & 0 & 0 \\ 2 & -3 & 1 & 0 & 1 & 0 \\ -3 & 2 & 0 & 0 & 0 & 1 \end{array}\right) \underset{\substack{-R_1 \\ R_2-2R_1 \\ R_3+3R_1}}{\sim} \left(\begin{array}{rrr|rrr} 1 & -2 & 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 2 & 1 & 0 \\ 0 & -4 & 3 & -3 & 0 & 1 \end{array}\right)$$

$$\underset{\substack{R_1+2R_2 \\ R_3+4R_2}}{\sim} \left(\begin{array}{rrr|rrr} 1 & 0 & -1 & 3 & 2 & 0 \\ 0 & 1 & -1 & 2 & 1 & 0 \\ 0 & 0 & -1 & 5 & 4 & 1 \end{array}\right) \underset{\substack{-R_3 \\ R_2-R_3 \\ R_1-R_3}}{\sim} \left(\begin{array}{rrr|rrr} 1 & 0 & 0 & -2 & -2 & -1 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -5 & -4 & -1 \end{array}\right)$$

so the inverse is

$$A^{-1} = \begin{pmatrix} -2 & -2 & -1 \\ -3 & -3 & -1 \\ -5 & -4 & -1 \end{pmatrix}$$

d) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2 matrix, with distinct eigenvalues λ_1 and λ_2.

Considering $\det(A - \lambda I)$ for both eigenvalues, or otherwise, prove that

(i) $\text{Trace}(A) = \lambda_1 + \lambda_2$, and (ii) $\det(A) = \lambda_1 \lambda_2$. [6]

[Recall that the trace of a matrix is the sum of its diagonal elements.]

SOLUTION

(i) Following the hint, $\lambda_{1,2}$ satisfy: $\det(A - \lambda) = \det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = 0$

$$\Rightarrow (a-\lambda)(d-\lambda) - bc = 0 \Rightarrow ad - bc - \lambda(a+d) + \lambda^2 = 0$$

which involves both trace and determinant. Substituting the eigenvalues gives two equations:

$$ad - bc - \lambda_1(a+d) + \lambda_1^2 = 0 \quad \text{and} \quad ad - bc - \lambda_2(a+d) + \lambda_2^2 = 0.$$

Subtracting the two equations we obtain

$$(\lambda_1 - \lambda_2)(a+d) = \lambda_1^2 - \lambda_2^2 = (\lambda_1 - \lambda_2)(\lambda_1 + \lambda_2)$$

and since the eigenvalues are distinct, we can cancel $(\lambda_1 - \lambda_2)$ and obtain

$$\lambda_1 + \lambda_2 = a + d = \text{Trace}(A),$$

as required.

(ii) Substitute $\lambda_1 + \lambda_2 = a + d$ into $ad - bc - \lambda_1(a+d) + \lambda_1^2 = 0$ (the other equation works as well) and obtain

$$\det(A) = ad - bc = \lambda_1(\lambda_1 + \lambda_2) - \lambda_1^2 = \lambda_1 \lambda_2,$$

as required.

3. a) Given the matrix

$$A = \begin{pmatrix} 4 & -2 & -1 \\ 0 & 2 & -1 \\ 3 & -3 & 1 \end{pmatrix},$$

show that $\lambda = 2$ is one of the eigenvalues of A, find the other two eigenvalues, and obtain an eigenvector for $\lambda = 2$. [5]

SOLUTION

Expand the determinant of

$$A - \lambda I = \begin{pmatrix} 4-\lambda & -2 & -1 \\ 0 & 2-\lambda & -1 \\ 3 & -3 & 1-\lambda \end{pmatrix},$$

by the first column, as it includes a zero:

$(4-\lambda)[(2-\lambda)(1-\lambda)-3] + 3[(2+2-\lambda)$

now spot the factorization, rather than multiply out:

$= (4-\lambda)[(2-\lambda)(1-\lambda)-3] + 3(4-\lambda) = (4-\lambda)[(2-\lambda)(1-\lambda)$

and the eigenvalues are $1, 2, 4$.

To find the eigenvector for $\lambda = 2$ we solve $(A - 2I)\underline{\mathbf{x}} = \mathbf{0}$:

$$\begin{pmatrix} 2 & -2 & -1 \\ 0 & 0 & -1 \\ 3 & -3 & -1 \end{pmatrix} \underline{\mathbf{x}} = \mathbf{0}$$

and there's no need for Gaussian elimination. The second row gives $z=0$ which makes both first and third row into $x-y=0 \Rightarrow x=y$ and the eigenvector is $\mu(1,1,0)$ for any non-zero μ.

b) Solve the first order ODE

$$(x^2+1)\frac{dy}{dx}+xy=\sqrt{x^2+1},$$

satisfying the initial condition $y(\sqrt{3})=\sqrt{3}$. [6]

SOLUTION This is a linear first order ODE, divide by x^2+1 to get it into standard form and find the integrating factor:

$$\frac{dy}{dx}+\frac{x}{x^2+1}y=\frac{1}{\sqrt{x^2+1}}$$

To find the IF $\mu(x)$, identify $P(x)=\dfrac{x}{x^2+1}$ and calculate

$$\mu(x)=\exp\left(\int\frac{x}{x^2+1}\,dx\right)=\exp\left[\frac{1}{2}\ln(x^2+1)\right]=\sqrt{x^2+1}$$

Now multiply by $\mu(x)$:

$$\sqrt{x^2+1}\frac{dy}{dx}+\frac{x}{\sqrt{x^2+1}}\,y=\frac{d}{dx}\left(\sqrt{x^2+1}\,y\right)=1$$

and integrate:

$$\sqrt{x^2+1}\,y=\int 1\,dx=x+C \Rightarrow y=\frac{x+C}{\sqrt{x^2+1}}$$

substituting the initial condition $y(\sqrt{3})=\sqrt{3}$ we have

$$\sqrt{3}=\frac{\sqrt{3}+C}{\sqrt{3+1}} \Rightarrow C=\sqrt{3},$$

and the solution is

$$y=\frac{x+\sqrt{3}}{\sqrt{x^2+1}}.$$

c) A solution of the second order differential equation

$$(1-x^2)\,\frac{d^2y}{dx^2}+2x\frac{dy}{dx}=0,$$

can be found in the form of a series, using the Leibnitz-Maclaurin method.

(i) Differentiate the ODE n times and evaluate at $x=0$ to obtain the recurrence relation

$$y^{(n+2)}(0)=n(n-3)y^{(n)}(0), \quad (n\geq 0),$$

where $y^{(k)}(0)$ is the k^{th} derivative of y, evaluated at zero. [4]

SOLUTION Begin by differentiating the ODE n times, using Leibniz' theorem:

$$(1-x^2)y^{(n+2)}-2nxy^{(n+1)}-2\frac{n(n-1)}{2}y^{(n)}+2\left(xy^{(n+1)}+ny^{(n)}\right)=0.$$

Now substitute $x=0$ and two terms vanish, collect like tems, leaving

$$y^{(n+2)}(0)+[-n(n-1)+2n]y^{(n)}(0)=0\Rightarrow y^{(n+2)}(0)=n(n-3)y^{(n)}(0),$$

as required.

(ii) If initial conditions are $y(0)=1$ and $y'(0)=1$, show that $y^{(n)}(0)=0$ for all $n>3$ and hence find the solution $y(x)$. . [4]

SOLUTION

We have initial conditions
$y^{(0)}(0)=y(0)=1$
$y^{(1)}(0)=y'(0)=1$
now use the recurrence relation:
$n=0: y^{(2)}(0)=0(0-3)y^{(0)}(0)=0$
$n=1: y^{(3)}(0)=1(1-3)y^{(1)}(0)=-2$
$n=2: y^{(4)}(0)=2(2-3)y^{(2)}(0)=0$
$n=3: y^{(5)}(0)=3(3-3)y^{(3)}(0)=0$
and all other terms are zero as they are multiples of either $y^{(4)}(0)=0$ or $y^{(5)}(0)=0$.

Using the Maclaurin series we now have

$$y=y(0)+y'(0)x+\frac{y''(0)}{2!}x^2+\frac{y'''(0)}{3!}x^3+\cdots=1+x-\frac{1}{3}x^3.$$

(iii) The substitution $z = y'$ reduces the second-order ODE to a separable first-order ODE for $z(x)$. Solve this for z, and hence obtain $y(x)$, confirming your result from (ii). [6]

SOLUTION Rewriting in terms of z we get

$$(1-x^2)\,\frac{dz}{dx} + 2x\,z = 0 \Rightarrow \frac{dz}{dx} = -\frac{2x}{1-x^2}z$$

a separable first-order ODE in z, as expected, and we integrate

$$\int \frac{1}{z}\,dz = \int -\frac{2x}{1-x^2}\,dx \Rightarrow \ln z = \ln(1-x^2) + K \Rightarrow z = C(1-x^2).$$

The initial condition $y'(0) = (0) = 1$ gives $C = 1$ and we can integrate again

$$z = \frac{dy}{dx} = 1 - x^2 \Rightarrow y = \int 1 - x^2\,dx = x - \frac{1}{3}x^3 + C',$$

and the initial condition $y(0) = 1$ gives $C' = 1$, confirming the solution obtained in (ii).

4. a) Find the solution of the differential equation

$$t^2\frac{dx}{dt} = t^2 + x^2 + tx.$$

satisfying the condition $x(1) = 1$. [6]

SOLUTION

Divide by t^2 to see that this is a homogenous first-order ODE in $x(t)$:

$$\frac{dx}{dt} = 1 + \left(\frac{x}{t}\right)^2 + \frac{x}{t},$$

and we substitute

$$v = x/t \Rightarrow x = vt \Rightarrow \frac{dx}{dt} = v + t\frac{dv}{dt}$$

so the ODE becomes

$$v + t\frac{dv}{dt} = 1 + v^2 + v \Rightarrow t\frac{dv}{dt} = v^2 + 1,$$

which is separable and can be integrated easily:

$$\int \frac{1}{v^2+1}\,dv = \int \frac{1}{t}\,dt \Rightarrow \tan^{-1} v = \ln t + C \Rightarrow \tan^{-1}\left(\frac{x}{t}\right) = \ln t + C$$

and substituting the I.C. $x(1) = 1$, we get $\tan^{-1}(1) = C \Rightarrow C = \pi/4$ and we rearrange to obtain the explicit solution

$$x = t\tan\left(\ln t + \frac{\pi}{4}\right).$$

b) We would like to estimate the speed of light using Einstein's celebrated formula

$$E = mc^2.$$

To do so we carry out a controlled and very small (i.e. safe) nuclear detonation, where we measure the amount of energy (E) generated, and the amount of mass (m) consumed in the process. If the energy is measured to within 0.01% accuracy and the mass to within 0.03%, use the total differential to estimate the maximum percentage error in the calculated speed of light, c. [6]

SOLUTION

First rearrange $c = \sqrt{E/m} = E^{1/2}m^{-1/2}$. The total differential for $c(m,E)$ gives

$$dc = \frac{\partial c}{\partial m}dm + \frac{\partial c}{\partial E}dE \Rightarrow \Delta c \approx \frac{\partial c}{\partial m}\Delta m + \frac{\partial c}{\partial E}\Delta E$$

where Δx represents a small absolute error in x. Differentiating c we have

$$\Delta c \approx -\frac{1}{2}E^{1/2}m^{-3/2}\Delta m + \frac{1}{2}E^{-1/2}m^{-1/2}\Delta E$$

and we need to divide by c to get relative (percentage) error:

$$\frac{\Delta c}{c} \approx \frac{1}{E^{1/2}m^{-1/2}}\left(-\frac{1}{2}E^{1/2}m^{-3/2}\Delta m + \frac{1}{2}E^{-1/2}m^{-1/2}\Delta E\right) = -\frac{1}{2}\frac{\Delta m}{m} + \frac{1}{2}\frac{\Delta E}{E}$$

Now apply the triangle inequality:

$$\left|\frac{\Delta c}{c}\right| \approx \left|-\frac{1}{2}\frac{\Delta m}{m} + \frac{1}{2}\frac{\Delta E}{E}\right| \leq \frac{1}{2}\left|\frac{\Delta m}{m}\right| + \frac{1}{2}\left|\frac{\Delta E}{E}\right| = \frac{1}{2}(0.0003) + \frac{1}{2}(0.0001) = 0.0002$$

so the maximum percentage error in c is 0.02%.

c) A function of two variables is given as

$$f(x,y) = x^3 + xy^2 - x.$$

(i) Find the stationary points of $f(x,y)$ and determine their nature using the Hessian determinant. [7]

SOLUTION Begin by obtaining the first-order partial derivatives and setting them equal to zero:

$$f_x = 3x^2 + y^2 - 1 = 0 \quad \text{and} \quad f_y = 2xy = 0.$$

The second equation gives $x = 0$ or $y = 0$. Substituting into the first equation:

$$x = 0 \Rightarrow y^2 - 1 = 0 \Rightarrow y = \pm 1,$$

and we have stationary points $P_1(0,1)$ and $P_2(0,-1)$.

$$y = 0 \Rightarrow 3x^2 - 1 = 0 \Rightarrow x = \pm 1/\sqrt{3}.$$

and we have stationary points $P_3(1/\sqrt{3},0)$ and $P_4(-1/\sqrt{3},0)$.

To classify the stationary points we calculate the second-order partial derivatives:

$$f_{xx} = 6x\,, f_{yy} = 2x\,, f_{xy} = f_{yx} = 2y$$

so the Hessian determinant is $\det\begin{pmatrix} 6x & 2y \\ 2y & 2x \end{pmatrix} = 12x^2 - 4y^2 = D.$

Evaluating D at the stationary points, we get $D(P_{1,2}) = -4 < 0 \Rightarrow P_{1,2}$ are saddle points;

$D(P_{3,4}) = 4 > 0 \Rightarrow P_{3,4}$ are extrema;

evaluate $f_{xx}(P_3) = 6/\sqrt{3} > 0$ so P_3 is a minimum;

evaluate $f_{xx}(P_4) = -6/\sqrt{3} < 0$ so P_4 is a maximum.

(ii) Sketch the contours of the surface $z = f(x, y)$. [6]

SOLUTION

Begin by factorizing $f(x, y) = x(x^2 + y^2 - 1)$ so we see that $f = 0$ on the contours $x = 0$, the $y-$ axis, and $x^2 + y^2 - 1 = 0$, the unit circle. Axis and circle intersect at $(0, \pm 1)$, confirming that saddle points are at intersections of contour lines. The minimum point P_3 is inside the right-hand half of the unit circle and the maximum point, P_4, symmetrically inside the left-hand half. This is enough information to draw the sketch, which should look approximately as: