Master

E4.29
CS3.2
ISE4.55

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2009

MSc and EEE/ISE PART IV: MEng and ACGI

OPTIMIZATION

Tuesday, 12 May 10:00 am

Time allowed: 3:00 hours

There are SIX questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : A. Astolfi
Second Marker(s) : M.M. Draief

Optimisation

1. Consider the problem of minimizing the function

$$f(x_1,x_2) = 4x_1^2 - 2x_1^4 + \frac{1}{3}x_1^6 + x_1x_2 + \frac{1}{4}x_2^2.$$

a) Compute all stationary points of the function. [4 marks]

b) Using second order sufficient conditions *classify* the stationary points determined in part a), *i.e.* say which is a local minimum, or a local maximum, or a saddle point. [6 marks]

c) Show that the function f is radially unbounded and hence compute the global minimum of f. Is the global minimizer unique? [4 marks]

d) Using the results of parts a), b) and c) sketch the level lines of the function f. [6 marks]

2. Consider the problem of minimizing the function

$$f(x_1,x_2) = \frac{1}{2}x_1^2\left(\frac{1}{6}x_1^2 + 1\right) + x_2 \arctan x_2 - \frac{1}{2}\ln(x_2^2 + 1).$$

a) Compute the unique stationary point of the function.
(Hint: $\dfrac{d \arctan x}{dx} = \dfrac{1}{1+x^2}$.) [2 marks]

b) Using second order sufficient conditions show that the stationary point determined in part a) is a local minimizer. [4 marks]

c)
i) Write Newton's iteration for the considered problem. [2 marks]

ii) Show that Newton's direction is a descent direction for f at any point which is not a stationary point. [2 marks]

iii) Compute four steps of Newton's algorithm from the initial point $(1, 0.5)$. Compute four steps of Newton's algorithm from the initial point $(1, 2)$. [4 marks]

iv) Discuss why the second sequence computed in part c.iii) does not converge to the global minimizer, despite the fact that Newton's direction is always a descent direction. Propose a simple modification of Newton's iteration that would guarantee global convergence to the minimizer. [6 marks]

3. A chain with three links, each of length one, hangs between two points at the same height, a distance $L > 1$ apart (see Figure 3). To find the form in which the chain hangs we minimize the potential energy.

Let (x_i, y_i) be the displacement of the right end of the ith link , from the right end of the $(i-1)$th link.

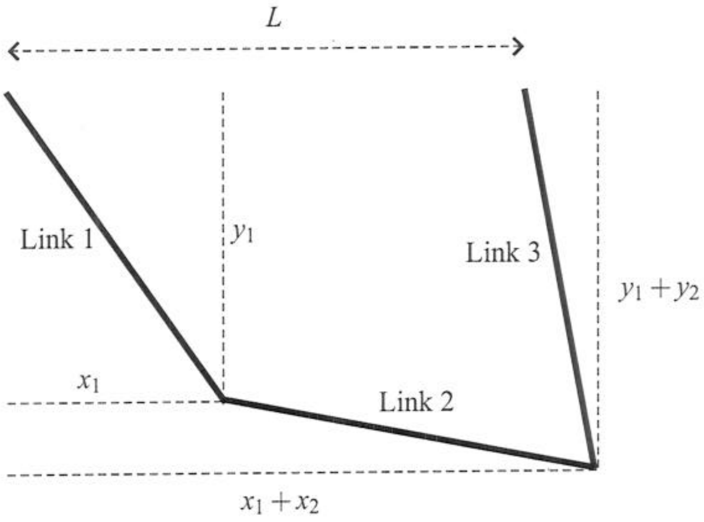


Figure 3: Sketch of the configuration of the chain.

The potential energy is therefore

$$V(y_1, y_2, y_3) = \frac{1}{2}y_1 + (y_1 + \frac{1}{2}y_2) + (y_1 + y_2 + \frac{1}{2}y_3).$$

a) The condition that the hanging points are a distance L apart can be translated in the constraint

$$x_1 + x_2 + x_3 = L.$$

Express this constraint in terms of the variables y_i.
(Hint: use Pythagoras' Theorem!) [2 marks]

b) Show that the condition that the height of the hanging points is the same can be expressed with the constraint

$$y_1 + y_2 + y_3 = 0.$$

[4 marks]

c) Consider the problem of minimizing the potential energy $V(y_1, y_2, y_3)$ subject to the constraints determined in parts b) and c).

i) Write necessary conditions of optimality for the considered optimization problem. [4 marks]

ii) Using physical considerations, it may be noted that candidate optimal solutions should be such that the chain has a _/ shape or a /‾\\ shape. Show that these two shapes yield values for y_1, y_2, y_3 and for the Lagrangian multipliers such that the necessary conditions of optimality are met.
(Hint: note that for both shapes $y_2 = 0$.) [8 marks]

iii) By evaluating the potential energy at the candidate optimal solutions determined in part c.ii) determine the shape that minimizes the potential energy. [2 marks]

4. The economy class luggage policy of an airline, on a transatlantic flight, reads:

Each passenger is allowed one piece of luggage. The three linear dimensions, when added together, must not exceed 150 *cm.*

The problem of maximizing the volume of the luggage can be posed and solved with the following steps.

a) Let $x_1 > 0$, $x_2 > 0$ and $x_3 > 0$ be the three linear dimensions (in cm) of a piece of luggage. Write the considered optimization problem as a minimization problem subject to one inequality constraint. (Do not include the constraints $x_1 > 0$, $x_2 > 0$ and $x_3 > 0$ in the formulation of the problem.) [2 marks]

b) State first order necessary conditions of optimality for this constrained optimization problem. [2 marks]

c) Using the conditions derived in part b) compute candidate optimal solutions. [8 marks]

d) Using second order sufficient conditions of optimality determine which of the candidate optimal solutions determined in part c) is a local maximizer. [6 marks]

e) Which is the geometric shape of the 'optimal luggage'? [2 marks]

5. Consider the optimization problem

$$\begin{cases} \min\limits_{x_1,x_2} x_1 x_2^2, \\ x_1^2 + x_2^2 \leq 2. \end{cases}$$

a) State first order necessary conditions of optimality for this constrained optimization problem. [2 marks]

b) Using the conditions derived in part a) compute candidate optimal solutions. [6 marks]

c) Evaluating the objective function at the candidate optimal solutions determined in part b) derive the solution of the considered optimization problem. [2 marks]

d) The considered constrained optimization problem can be solved minimizing the so-called logarithmic penalty function given by

$$P_l(x_1,x_2) = x_1 x_2^2 - \varepsilon \log(2 - x_1^2 - x_2^2),$$

with $\varepsilon > 0$.

i) State first order necessary condition of optimality for P_l. [2 marks]

ii) Show that the stationary points of P_l are such that

$$x_2^2 = 2x_1^2.$$

[4 marks]

iii) Using the results in part d.ii) show that the stationary points of P_l are such that

$$x_1(3x_1^3 - 2x_1 - \varepsilon) = 0.$$

Hence argue that, as ε approaches zero the stationary points of P_l approach candidate optimal solutions for the considered problem. [4 marks]

6. Consider the optimization problem

$$\begin{cases} \max\limits_{x_1,x_2,x_3} x_1 + 2x_2 + x_3, \\ x_1^2 + x_2^2 + x_3^2 \leq 1. \end{cases}$$

a) State first order necessary condition of optimality for this constrained optimization problem. [2 marks]

b) Using the conditions derived in part a) compute candidate optimal solutions. [4 marks]

c) Using second order sufficient conditions of optimality determine the solution of the optimization problem. [4 marks]

d) Consider the change of variables

$$x_1 = r\cos\theta\sin\phi \qquad x_2 = r\sin\theta\sin\phi \qquad x_3 = r\cos\phi$$

with $r \geq 0$, $\theta \in [0, 2\pi)$, and $\phi = [0, 2\pi)$.

i) Rewrite the considered optimization problem in the new variables and show that the resulting problem can be written in the form

$$\begin{cases} \max\limits_{r,\theta,\phi} r\Psi(\theta,\phi), \\ r \leq 1 \\ \theta \in [0, 2\pi) \\ \phi \in [0, 2\pi) \end{cases}$$

Determine the function $\Psi(\theta,\phi)$. [4 marks]

ii) Argue that the problem is equivalent to the unconstrained optimization problem

$$\max_{\theta,\phi} \Psi(\theta,\phi).$$

[2 marks]

iii) Find candidate solutions of the uncontrained optimization problem in part d.ii), and show that one of the candidate solutions coincides with the optimal solution determined in part c). [4 marks]

E4.29
CS 3.2
ISE 4.55

Optimisation - Model answers 2009

(Note to external examiners: all questions involve mostly applications of standard methods and concepts to unseen examples.)

Question 1

a) The stationary points of the function f are computed by solving the equations

$$0 = \nabla f = \left[\begin{array}{c} 8x_1 - 8x_1^3 + 2x_1^5 + x_2 \\ x_1 + \frac{1}{2}x_2 \end{array} \right].$$

The second equation yields $x_2 = -2x_1$, which replaced in the first equation yields

$$0 = 2x_1(x_1 - 1)(x_1 + 1)(x_1^2 - 3).$$

As a result, the function f has five stationary points

$$P_1 = (0, 0) \qquad P_2 = (-1, 2) \qquad P_3 = (1, -2) \qquad P_4 = (\sqrt{3}, -2\sqrt{3}) \qquad P_5 = (-\sqrt{3}, 2\sqrt{3})$$

b) Note that

$$\nabla^2 f = \left[\begin{array}{cc} 8 - 24x_1^2 + 10x_1^4 & 1 \\ 1 & \dfrac{1}{2} \end{array} \right].$$

As a result:

$$\nabla^2 f(P_1) = \left[\begin{array}{cc} 8 & 1 \\ 1 & \frac{1}{2} \end{array} \right],$$

which is a positive definite matrix, hence P_1 is a local minimizer;

$$\nabla^2 f(P_2) = \nabla^2 f(P_3) \left[\begin{array}{cc} -6 & 1 \\ 1 & \frac{1}{2} \end{array} \right],$$

which is an indefinite matrix, hence P_2 and P_3 are saddle points;

$$\nabla^2 f(P_4) = \nabla^2 f(P_5) \left[\begin{array}{cc} 26 & 1 \\ 1 & \frac{1}{2} \end{array} \right],$$

which is a positive definite matrix, hence P_4 and P_5 are local minimizers.

c) The function f can be written as

$$f = \left(x_1 + \frac{1}{2}x_2 \right)^2 + \frac{x_1^2}{3} \left(x_1^2 - 3 \right)^2.$$

Hence f is a *sum of squares*, and all variables x_1 and x_2 are present in one of the squares. As a result the function is radially unbounded and the local minimum of f is also a global minimum. Note that

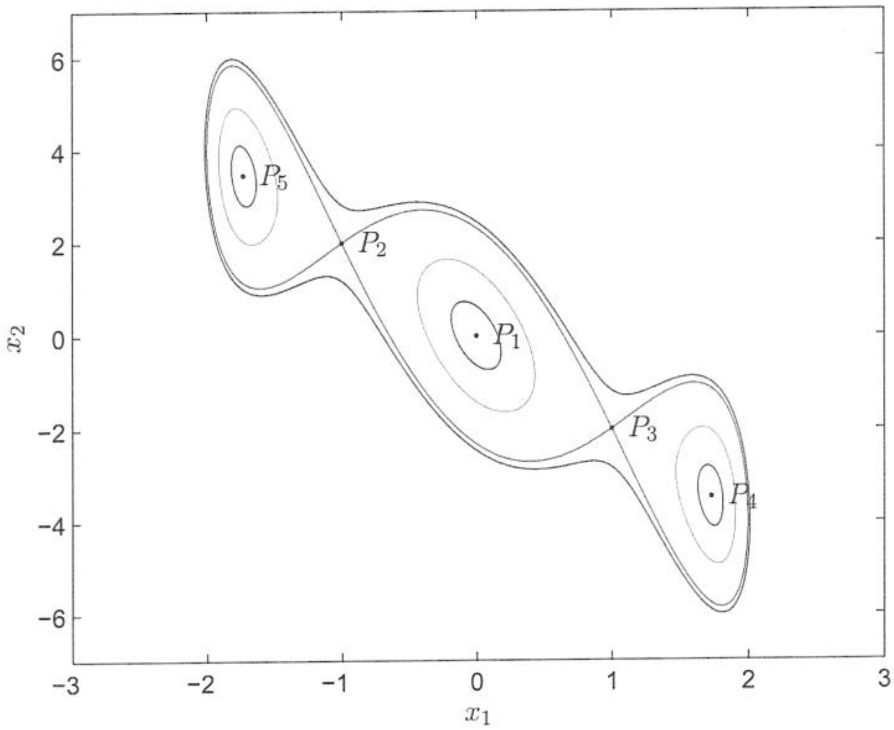
$$f(P_1) = f(P_4) = f(P_5) = 0$$

hence P_1, P_4 and P_5 are all global minimizers.

d) The level lines of f can be sketched using the following considerations.

- Around the minimizers the level lines are closed.
- The value of f at the saddle points P_2 and P_3 is $4/3$. There is a level lines that *connects* the saddle points. Close to the saddle points this level lines is composed of two curves.

A sketch of the level lines is in the figure below.



Question 2

a) The stationary points of the function f are computed by solving the equations

$$0 = \nabla f = \begin{bmatrix} \frac{1}{3}x_1^3 + x_1 \\ \arctan x_2 \end{bmatrix}.$$

These equations have the unique solution $x_1 = x_2 = 0$, which is therefore the unique stationary point of f.

b) Note that

$$\nabla^2 f = \begin{bmatrix} x_1^2 + 1 & 0 \\ 0 & \dfrac{1}{1 + x_2^2} \end{bmatrix}.$$

Hence $\nabla^2 f(0,0) = \mathrm{diag}(1,1)$, which is a positive definite matrix. The stationary point is a local minimizer.

c) i) Newton's iteration is

$$x_{k+1} = x_k - [\nabla^2 f(x_k)]^{-1} \nabla f(x_k)$$

hence

$$x_{1,k+1} = \frac{2x_{1,k}^3}{3(x_{1,k}^2 + 1)} \qquad x_{2,k+1} = x_{2,k} - (1 + x_{2,k}^2) \arctan x_{2,k}.$$

ii) Newton's direction is

$$d = -[\nabla^2 f(x)]^{-1} \nabla f(x).$$

Note that

$$\nabla f' \, d = -\nabla f' [\nabla^2 f(x)]^{-1} \nabla f(x) < 0,$$

for all points such that $\nabla f(x) \neq 0$, since $\nabla^2 f$ is positive definite. As a result, d is a descent direction for f for all $x \neq 0$.

iii) A direct computation yields

$$x_0 = (1, 1/2) \qquad x_1 = (1/3, -0.079) \qquad x_2 = (0.022, 0.00033)$$
$$x_3 = (0.000007, -2.5 \ 10^{-11}) \qquad x_4 = (2.6 \ 10^{-16}, 0)$$

and

$$x_0 = (1, 2) \qquad x_1 = (1/3, -3.53) \qquad x_2 = (0.022, 13.95)$$
$$x_3 = (0.000007, -279.34) \qquad x_4 = (2.6 \ 10^{-16}, 1.2 \ 10^5).$$

iv) The second sequence does not converge since Newton's method guarantee only local convergence properties. To achieve global convergence, since Newton's direction is a descent direction for f at any $x \neq 0$, it is enough to introduce a line search parameter, *i.e.* to consider the iteration

$$x_{k+1} = x_k - \alpha [\nabla^2 f(x_k)]^{-1} \nabla f(x_k),$$

with $\alpha > 0$, and determined using a line search algorithm.

Question 3

a) From Figure 3 in the Exam Paper we obtain, for $i = 1,2,3$, $x_i^2 + y_i^2 = 1$, hence $x_i = \sqrt{1-y_i^2}$, yielding the constrain

$$\sqrt{1-y_1^2} + \sqrt{1-y_2^2} + \sqrt{1-y_3^2} = L.$$

b) Since the height of the left hanging point is at zero, and the y-coordinate of the last link is $y_1 + y_2 + y_3$ then the condition that both hanging points are at the same height is given by $y_1 + y_2 + y_3 = 0$.

c) The optimization problem to solve is thus

$$\begin{cases} \displaystyle\min_{y_1,y_1,y_3} \frac{5}{2}y_1 + \frac{3}{2}y_2 + \frac{1}{2}y_3, \\ \sqrt{1-y_1^2} + \sqrt{1-y_2^2} + \sqrt{1-y_3^2} - L = 0, \\ y_1 + y_2 + y_3 = 0. \end{cases}$$

i) Define the Lagrangian

$$L(y_1,y_2,y_3,\lambda_1,\lambda_2) = \frac{5}{2}y_1 + \frac{3}{2}y_2 + \frac{1}{2}y_3 + \lambda_1(\sqrt{1-y_1^2} + \sqrt{1-y_2^2} + \sqrt{1-y_3^2} - L) + \lambda_2(y_1+y_2+y_3).$$

The first order necessary conditions of optimality are

$$0 = \frac{\partial L}{\partial y_1} = \frac{5}{2} - \lambda_1 \frac{y_1}{\sqrt{1-y_1^2}} + \lambda_2 \qquad 0 = \frac{\partial L}{\partial y_2} = \frac{3}{2} - \lambda_1 \frac{y_2}{\sqrt{1-y_2^2}} + \lambda_2$$

$$0 = \frac{\partial L}{\partial y_3} = \frac{1}{2} - \lambda_1 \frac{y_3}{\sqrt{1-y_3^2}} + \lambda_2$$

$$\sqrt{1-y_1^2} + \sqrt{1-y_2^2} + \sqrt{1-y_3^2} - L = 0 \qquad y_1 + y_2 + y_3 = 0.$$

ii) The indicated shapes are such that

$$y_2 = 0 \qquad y_1 = -y_3.$$

Replacing these conditions in the necessary conditions of optimality yields

$$0 = \frac{\partial L}{\partial y_1} = \frac{5}{2} - \lambda_1 \frac{y_1}{\sqrt{1-y_1^2}} + \lambda_2 \qquad 0 = \frac{\partial L}{\partial y_2} = \frac{3}{2} - \lambda_2$$

$$0 = \frac{\partial L}{\partial y_3} = \frac{1}{2} - \lambda_1 \frac{y_1}{\sqrt{1-y_1^2}} + \lambda_2$$

$$2\sqrt{1-y_1^2} + 1 - L = 0 \qquad 0 = 0.$$

These equations have the two solutions

$$y_1 = \pm\frac{1}{2}\sqrt{3+2L-L^2} \qquad \lambda_1 = \pm\frac{L-1}{\sqrt{3+2L-L^2}} \qquad \lambda_2 = \frac{3}{2}.$$

The one with positive y_1 corresponds to the $\diagup\!\overline{\quad}\!\diagdown$ shape, the one with negative y_1 corresponds to the $\diagdown\!\underline{\quad}\!\diagup$ shape. Note that all square roots are well-defined since $L > 1$.

iii) The potential energy for the above shapes is $V(y_1,y_2,y_3) = 2y_1$. Hence the candidate optimal solution with negative y_1 yields a local minimizer.

Question 4

a) The considered optimization problem can be written as

$$\begin{cases} \max\limits_{x_1,x_2,x_3} \; x_1x_2x_3 \\ x_1 + x_2 + x_3 \le 150. \end{cases}$$

b) Define the Lagrangian (note the change in sign of the objective function)

$$L(x_1, x_2, x_3, \rho) = -x_1x_2x_3 + \rho(x_1 + x_2 + x_3 - 150).$$

The first order necessary conditions of optimality are

$$\begin{array}{ccc} 0 = \dfrac{\partial L}{\partial x_1} = -x_2x_3 + \rho & 0 = \dfrac{\partial L}{\partial x_2} = -x_1x_3 + \rho & 0 = \dfrac{\partial L}{\partial x_3} = -x_2x_1 + \rho \\ \rho \ge 0 & x_1 + x_2 + x_3 - 150 \le 0 & \rho(x_1 + x_2 + x_3 - 150) = 0. \end{array}$$

c) Using the complementarity condition, *i.e.* the last condition, we have two cases.
Case 1: $\rho = 0$. This implies $x_1x_2 = x_2x_3 = x_1x_3 = 0$, yielding the sets of candidate solutions

$$x_1 = x_2 = \rho = 0, x_3 \le 150, \qquad x_1 = x_3 = \rho = 0, x_2 \le 150, \qquad x_2 = x_3 = \rho = 0, x_1 \le 150.$$

Case 2: $x_1 + x_2 + x_3 = 150$. This yields the candidate solutions

$$x_1 = x_2 = \rho = 0, x_3 = 150, \qquad x_1 = x_3 = \rho = 0, x_2 = 150, \qquad x_2 = x_3 = \rho = 0, x_1 = 150,$$

and

$$x_1 = x_2 = x_3 = 50, \rho = 50^2.$$

d) Note that

$$\nabla^2 L(x_1, x_2, x_3) = -\begin{bmatrix} 0 & x_3 & x_2 \\ x_3 & 0 & x_1 \\ x_2 & x_1 & 0 \end{bmatrix}.$$

All candidate solutions obtained in Case 1, for which no constrain is active, are such that $\nabla^2 L$ has a positive, a negative and a zero eigenvalue. As a result, all solutions obtained in Case 1, are saddle points.
Consider now the candidate solutions obtained in Case 2, and such that $\rho = 0$. For such solutions the condition of strict complementarity does not hold, hence it is not possible to use second order sufficient conditions to classify these points.
Finally, consider the candidate optimal solution

$$x_1 = x_2 = x_3 = 50, \rho = 50^2.$$

The second order sufficient condition require

$$s' \nabla^2 L(50, 50, 50) s > 0$$

for all non-zero s such that

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} s = 0.$$

Such s can be parameterized as

$$s = \begin{bmatrix} s_1 & s_2 & -s_1 - s_2 \end{bmatrix},$$

yielding

$$s' \nabla^2 L(50, 50, 50) s = 100(s_1^2 + s_2^2 + s_1 s_2),$$

which is positive for all non-zero s_1 and s_2. As a result, this candidate optimal solution is a local minimizer. (It is a local maximizer for the original problem).

e) The optimal luggage is a cube!

Question 5

a) Define the Lagrangian

$$L(x_1, x_2, \rho) = x_1 x_2^2 + \rho(x_1^2 + x_2^2 - 2).$$

The first order necessary conditions of optimality are

$$0 = \frac{\partial L}{\partial x_1} = x_2^2 + 2\rho x_1 \qquad 0 = \frac{\partial L}{\partial x_2} = 2x_1 x_2 + 2\rho x_2$$

$$x_1^2 + x_2^2 - 2 \le 0 \qquad \rho > 0$$

$$\rho(x_1^2 + x_2^2 - 2) = 0.$$

b) Using the complementarity conditions, *i.e.* the last condition, we have two possibilities.

- $\rho = 0$. This yields the candidate optimal solutions

$$P_1 : \ (x_1, x_2) = (\alpha, 0)$$

with $|\alpha| \le \sqrt{2}$. Note that at $(x_1, x_2) = (\pm\sqrt{2}, 0)$ the strict condition of complementarity does not hold.

- $x_1^2 + x_2^2 - 2 = 0$. This yields the candidate optimal solutions

$$P_2 : \ (x_1, x_2) = (\pm\sqrt{2}, 0),$$

with $\rho \ge 0$, and

$$P_3 : \ (x_1, x_2) = (-\frac{\sqrt{6}}{3}, \pm\frac{\sqrt{12}}{3}),$$

with $\rho = \frac{\sqrt{6}}{3}$.

In summary there are infinitely many candidate optimal solutions, some of which such that second order sufficient conditions cannot be used.

c) The values of the objective function at candidate optimal points are

$$f(P_1) = 0 \qquad f(P_2) = 0 \qquad f(P_3) = -\frac{4}{9}\sqrt{6}.$$

Hence P_3 is the solution of the considered problem.

d) i) The first order necessary condition of optimality for P_l are

$$0 = \frac{\partial P_l}{\partial x_1} = x_2^2 + 2\epsilon\frac{x_1}{2 - x_1^2 - x_2^2} \qquad 0 = \frac{\partial P_l}{\partial x_2} = 2x_1 x_2 + 2\epsilon\frac{x_2}{2 - x_1^2 - x_2^2}.$$

ii) The equations defining the stationary points of P_l yield, for nonzero x_1 and x_2,

$$-2\epsilon\frac{1}{2 - x_1^2 - x_2^2} = \frac{x_2^2}{x_1} = 2x_1,$$

hence stationary points are such that

$$x_2^2 = 2x_1^2.$$

If $x_1 = 0$ then the necessary conditions yield $x_2 = 0$, and similarly for $x_2 = 0$. Hence, the above relation holds for any x_1 and x_2.

iii) Replacing the above relation in the equation

$$0 = \frac{\partial P_l}{\partial x_1}$$

yields

$$2x_1 \frac{3x_1^3 - 2x_1 - \epsilon}{3x_1^2 - 2} = 0$$

from which we infer that, as $\epsilon \to 0$, $x_1 \to 0$ or $x_1 \to \pm\frac{\sqrt{6}}{3}$. As a result, as ϵ goes to zero the stationary points of P_l *approach* the candidate optimal solutions of the problem.

Question 6

a) Define the Lagrangian (note the sign change due to the transformation of the maximization problem into a minimization problem)

$$L(x_1, x_2, x_3, \rho) = -x_1 - 2x_2 - x_3 + \rho(x_1^2 + x_2^2 + x_3^2 - 1).$$

The first order necessary conditions of optimality are

$$0 = \frac{\partial L}{\partial x_1} = -1 + 2\rho x_1 \qquad 0 = \frac{\partial L}{\partial x_2} = -1 + 2\rho x_2 \qquad 0 = \frac{\partial L}{\partial x_2} = -1 + 2\rho x_3$$

$$\rho \geq 0 \qquad x_1^2 + x_2^2 + x_3^2 - 1 \leq 0$$

$$\rho(x_1^2 + x_2^2 + x_3^2 - 1) = 0.$$

b) Using the complementarity conditions, *i.e.* the last condition, we have two possibilities.

- $\rho = 0$. This does not yield any candidate optimal solution.
- $x_1^2 + x_2^2 + x_3^2 - 1 = 0$, $\rho > 0$. This yields the candidate optimal solution

$$P: \ (x_1, x_2, x_3) = \frac{1}{2\rho}(1, 2, 1)$$

with $\rho \geq 0$ such that

$$\frac{3}{2\rho^2} = 1.$$

In summary there is only one candidate optimal solution given by

$$P: \ (x_1, x_2, x3) = \frac{\sqrt{6}}{6}(1, 2, 1).$$

c) The Hessian of the Lagrangian is

$$\nabla^2 L = 2\rho I,$$

with I the identity matrix. Hence, the Hessian is positive definite at the candidate optimal solution which is therefore a (local) minimizer for the problem (note that we have changed the sign of the objective function to transform the maximization problem into a minimization one).

d) i) Applying the change of variable to the objective function yields the transformed objective function

$$r(\cos\theta \sin\phi + 2\sin\theta \sin\phi + \cos\phi),$$

whereas the constraint is transformed into $r^2 \leq 1$, which is equivalent to $r \leq 1$ since $r \geq 0$. As a result, the function Ψ is given by

$$\Psi(\theta, \phi) = (\cos\theta \sin\phi + 2\sin\theta \sin\phi + \cos\phi).$$

ii) The objective function in the transformed variables is separable, *i.e.* it is the product of two functions or different variables, namely r and Ψ. As a result, the maximization is achieved maximizing Ψ and r. The latter is maximized for $r = 1$. The former has to be maximized for $\theta \in [0, 2\pi)$ and $\phi \in [0, 2\pi)$. However, since Ψ is periodic in θ and ϕ it can be maximized disregarding the constraints.

iii) The stationary points of Ψ are the solutions of

$$\sin\phi(2\cos\theta - \sin\theta) = 0 \qquad\qquad \cos\theta\cos\phi + 2\sin\theta\cos\phi + \sin\phi = 0.$$

The first equation yields

- $\phi = 0$ or $\phi = \pi$, which replaced in the second equation yield $\theta = -\arctan 2$;
- $\theta = \arctan 2$, yielding $\phi = \arctan\sqrt{5}$.

In the original coordinates the first candidate solutions yield $(x_1, x_2, x_3) = (0, 0, \pm 1)$, whereas the second candidate solution give the optimal solution determined in part c).