EXAM SOLUTIONS

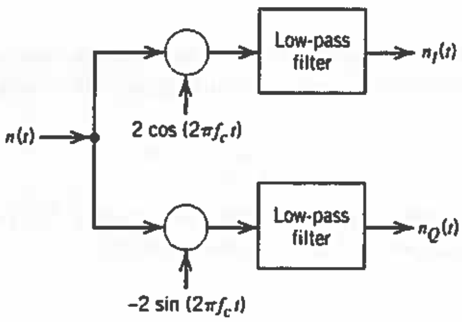
1. a) **This question is straightforward bookwork.**

i) [2]

A coherent receiver is phase-locked to the transmitter, while in the case of a non-coherent receiver there is no synchronization between the transmitter and the receiver.

ii) [3]

$$n(t) = n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t).$$



iii) [2]

The SNR performance of DSB-SC is the same as that of baseband transmission.

iv) [3]

The SNR performance of FM is significantly higher than DSB-SC, but it also requires a higher transmission bandwidth.

b) **This question requires an understanding of the properties of the autocorrelation function of WSS processes, which were given in the class.**

i) [2]

No. $R_X(t_1,t_2)$ should depend only on $t_1 - t_2$ for a WSS process.

ii) [3]

No. We should have $R_X(t_1,t_2) = R_X(t_2,t_1)$.

iii) [4]

No. We should have $R_X(\tau) \leq R_X(0)$ for all τ.

c) **This question is straightforward bookwork.**

i) [2]

From the Nyquist theorem the minimum sampling rate is 8 KHz.

ii) [3]

Maximum quantization error for a uniform quantizer is equal to $\Delta/2$, i.e., half the size of the quantization region. Then we have

$$\frac{\Delta/2}{L \cdot \Delta} \leq 0.004,$$

where L is the number of quantization regions. We get $L \geq 125$. We need a minimum of 7 bits per source sample.

iii) [2]

Bit rate is $7 * 8 = 56$ KHz.

iv) [3]

We should satisfy

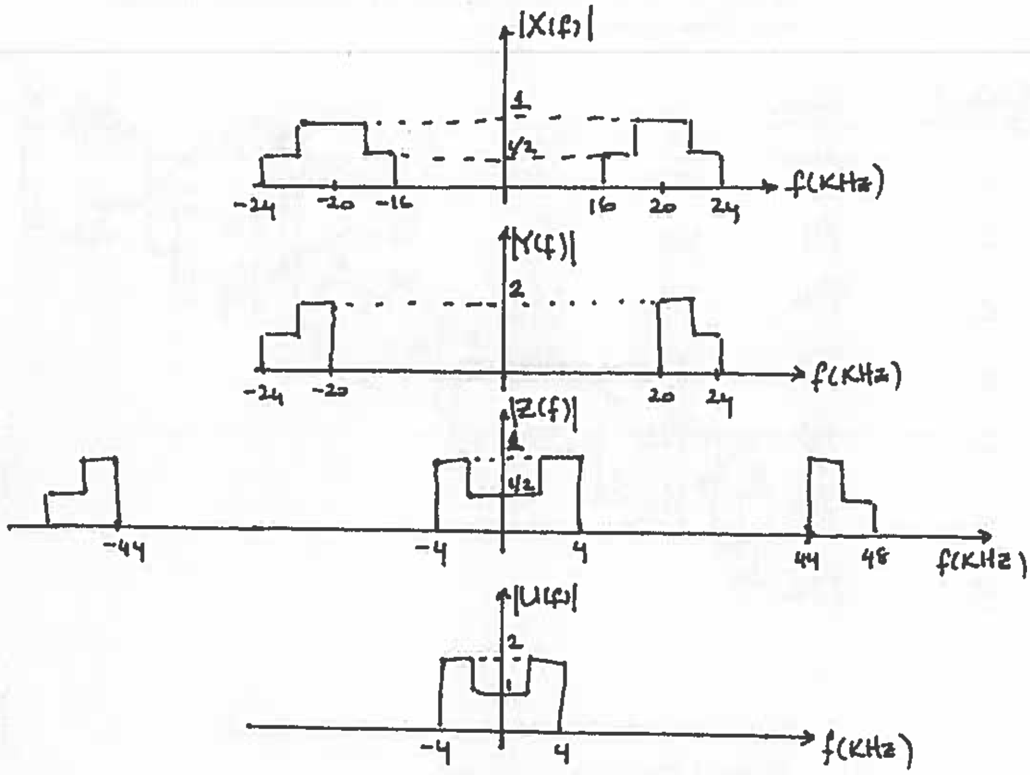
$$56K \text{ bits/sec} = \frac{4 \text{ bits/symbol}}{T \text{ sec/symbol}} = \frac{4}{T} \text{ bits/sec}$$

From here we get $T \approx 71.4$ microseconds.

d) **This question requires an understanding of modulation, and its effect on the spectrum.**

i) [5]

The magnitude spectrum of signals $x(t)$, $y(t)$, $z(t)$ and $u(t)$ are as follows:



ii) [3]

The system simply reversed the spectrum of $m(t)$. If we apply the same on $u(t)$, we obtain $m(t)$.

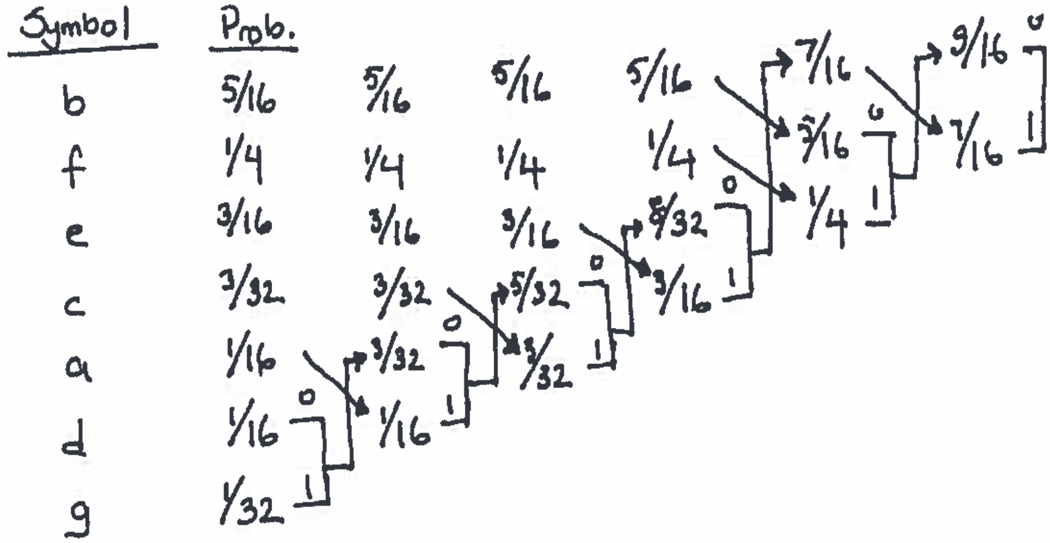
iii) [2]

$$w(t) = m(t).$$

2. a) **This question requires understanding the algorithm that generates the Huffman codewords. The students should also know how to compute the entropy of a memoryless source, and its relation to the average codeword length of any given source code.**

i) [8]

We order the symbols from the most likely to the least likely, and apply the Huffman algorithm as below.



Then we obtain the following codewords for each symbol:

Symbol	Probability	Codeword
a	1/16	1001
b	5/16	00
c	3/32	101
d	1/16	10000
e	3/16	11
f	1/4	01
g	1/32	10001

ii) [2]

The average codeword length is given by

$$4\times\frac{1}{16}+2\times\frac{5}{16}+3\times\frac{3}{32}+5\times\frac{1}{16}+2\times\frac{3}{16}+2\times\frac{1}{4}+5\times\frac{1}{32}=2.5 \text{ bits per sample.}$$

iii) [4]

The entropy of the source is found as

$$H(S) = \frac{1}{16}\log_2 16 + \frac{5}{16}\log_2\frac{16}{5} + \frac{3}{32}\log_2\frac{32}{3} + \frac{1}{16}\log_2 16$$
$$+ \frac{3}{16}\log_2\frac{16}{3} + \frac{1}{4}\log_2 4 + \frac{1}{32}\log_2 32$$
$$= 2.45 \text{ bits per sample.}$$

iv) [2]

The code is not optimal since its rate is slightly below the entropy bound.

b) **Similar problems have been solved in the class. It is slightly harder than the ones done in the class, which typically consisted of only one sinusoidal.**

i) [5]

We have

$$\begin{aligned} E[X(t)] &= E[N_1\cos(2\pi f_c t) - N_2\sin(2\pi f_c t)] \\ &= E[N_1]\cos(2\pi f_c t) - E[N_2]\sin(2\pi f_c t) \\ &= 0 \end{aligned}$$

$$\begin{aligned} R_X(t_1,t_2) &= E[X(t_1)X(t_2)] \\ &= E\left[(N_1\cos(2\pi f_c t_1) - N_2\sin(2\pi f_c t_1))(N_1\cos(2\pi f_c t_2) - N_2\sin(2\pi f_c t_2))\right] \\ &= E[N_1^2]\cos(2\pi f_c t_1)\cos(2\pi f_c t_2) + E[N_2^2]\sin(2\pi f_c t_1)\sin(2\pi f_c t_2) \\ &\quad - E[N_1N_2](\cos(2\pi f_c t_1)\sin(2\pi f_c t_2) + \sin(2\pi f_c t_1)\cos(2\pi f_c t_2)) \\ &= \sigma^2[\cos(2\pi f_c t_1)\cos(2\pi f_c t_2) + \sin(2\pi f_c t_1)\sin(2\pi f_c t_2)] \\ &= \sigma^2\cos(2\pi f_c(t_1 - t_2)), \end{aligned}$$

where we have used the linearity of the expectation operation, and the independence of N_1 and N_2, i.e., $E[N_1N_2] = E[N_1]E[N_2] = 0$. We can write the autocorrelation function as follows:

$$R_X(\tau) = \sigma^2\cos(2\pi f_c\tau).$$

ii) [2]

Since the mean is independent of time, and the autocorrelation function depends only on the time difference, $X(t)$ is WSS.

iii) [3]

We have

$$\begin{aligned} S_X(f) &= \mathbb{F}(R_X(\tau)) \\ &= \mathbb{F}\big(\sigma^2\cos(2\pi f_c\tau)\big) \\ &= \frac{\sigma^2}{2}[\delta(f-f_c) + \delta(f+f_c)]. \end{aligned}$$

iv) [4]

We can consider $Y(t)$ as the output of an LTI filter with frequency response $H(f) = (j2\pi f)^n$. Then the PSD of $Y(t)$ is found as:

$$\begin{aligned} S_Y(f) &= |H(f)|^2 S_X(f) \\ &= |j2\pi f|^{2n} S_X(f) \\ &= \frac{\sigma^2}{2}(2\pi f_c)^{2n}[\delta(f-f_c)+\delta(f+f_c)]. \end{aligned}$$

3. a) **This problem requires a good understanding of probability of error in digital communication systems.**

i) [2]

Due to the symmetry in the system, the optimal threshold in this case is trivial, and is given by 0.

ii) [6]

Probability of error when a "0" is transmitted is given by

$$P_{e0} = \int_{1+T}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2},$$

where T is the threshold of the detector.

Probability of error when a "1" is transmitted is given by

$$P_{e1} = \int_{1-T}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2}.$$

The overall probability of error can be written as

$$P_e = p_0 \int_{1+T}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2} + p_1 \int_{1-T}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2}.$$

The optimal threshold is found by solving for $\frac{dP_e}{dT} = 0$. We use the Leibnitz rule to take the derivative, and obtain the following:

$$\frac{dP_e}{dT} = -p_0 \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(1+T)^2/2\sigma^2} + p_1 \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(1-T)^2/2\sigma^2} = 0.$$

From here we obtain

$$\ln\frac{p_0}{p_1} = \frac{(1+T)^2-(1-T)^2}{2\sigma^2} = \frac{2T}{\sigma^2} = 4T.$$

We find $T = \frac{\ln 4}{4} \approx 0.35$.

iii) [6]

The error probability can be written as

$$P_e = p_0 \cdot Q(1+T) + p1 \cdot Q(1-T).$$

The receiver uses the T from above, given by $T = 0.35$. Then we have

$$\begin{aligned} P_e &= 0.3 \cdot Q(1.35) + 0.7 \cdot Q(0.65) \\ &\approx 0.3 \times 9 \times 10^{-2} + 0.7 \times 3 \times 10^{-1} \qquad \approx 0.237. \end{aligned}$$

This is almost double the probability of error the receiver was expecting to achieve, which is found as

$$\begin{aligned} P_e &= 0.8 \cdot Q(1.35) + 0.2 \cdot Q(0.65) \\ &\approx 0.8 \times 9 \times 10^{-2} + 0.2 \times 3 \times 10^{-1} \qquad \approx 0.132. \end{aligned}$$

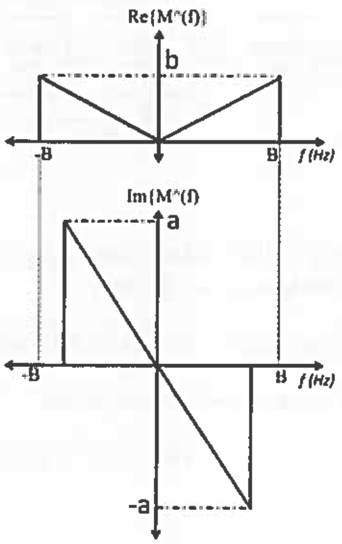
b) **This problem requires a good understanding of the Fourier transform, and the impact of modulation on the spectrum. They also need to know/ derive the modulator/demodulator characteristics of an SSB system, and be able analyse its SNR behaviour in the presence of noise.**

i) [2]

$\hat{m}(t) = m(t) * \frac{1}{\pi t}$.

ii) [4]

$\hat{M}(f) = -j \cdot sgn(f) \cdot M(f)$



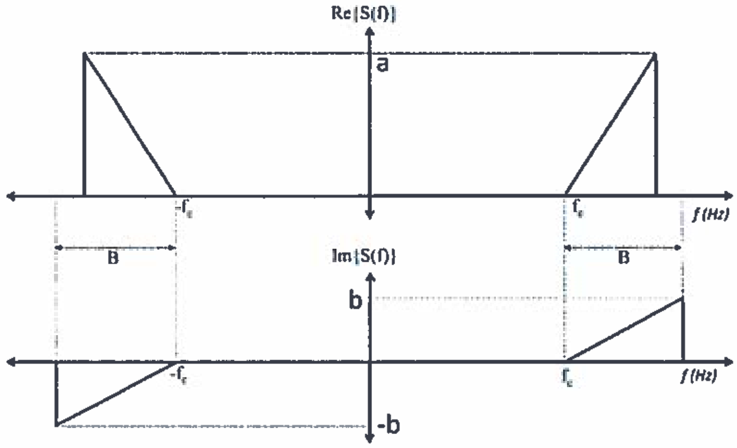
iii) [4]

The students may remember from class that the USB-SSB transmits only the upper halves of the spectrum of the signal, centered around carrier frequency f_c.

They should also be able to derive this directly from the basic principles as follows:

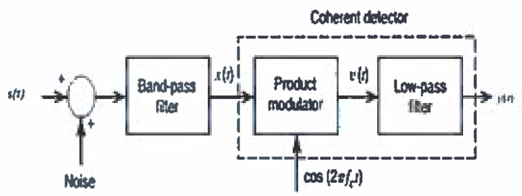
$$S(f) = \frac{1}{2}\left[M(f-f_c) + M(f+f_c)\right] - \frac{1}{2j}\left[\hat{M}(f-f_c) - \hat{M}(f+f_c)\right]$$

By simply shifting the spectrum of $M(f)$ and $\hat{M}(f)$ and applying the appropriate sign and conjugate adjustments they get the spectrum as below.



iv) [4]

The coherent receiver diagram is as follows:



At the output of the bandpass filter we get $x(t) = s(t) + n(t)$, where $n(t)$ is the bandpass noise. We have

$$x(t) = m(t)\cos(2\pi f_c t) - \hat{m}(t)\sin(2\pi f_c t) + n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t).$$

At the end of the demodulator we obtain

$$y(t) = m(t) + n_I(t).$$

v) [2]

SSB-SC has the same SNR performance as DSB-SC; however, it uses only half the bandwidth of DSB-SC.