Imperial College London

[E1.10 (Maths 1) 2011]

B.ENG. and M.ENG. EXAMINATIONS 2011

PART I : MATHEMATICS 1 (ELECTRICAL ENGINEERING)

Date Wednesday 8th June 2011 10.00 - 12.00

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.

Answer Question 1 and THREE of the remaining five

Question 1 carries twice the marks of each of the other questions.

CALCULATORS MAY **NOT** BE USED.

A mathematical formulae sheet is provided.

[Before starting, please make sure that the paper is complete; there should be 5 pages, with a total of SIX questions. Ask the invigilator for a replacement if your copy is faulty.]

1. (i) Express in polar form:

$$i\,,\quad 1-i\,,\quad (2+2i)^2\,,\quad (2+i)\,(2-i)\,,\quad 1/2-\sqrt{3/4}\,i\,.$$

(ii) Determine real and imaginary parts in the form
$x+iy$ where $x,\,y\in\mathbb{R}$ and $z\in\mathcal{C}$:

$$(1+i)^3,\ \exp(z),\ \exp(iz),\ \sinh(iz)\,,\cos(iz)\ .$$

Note: $\sinh(x) \;=\; \frac{1}{2}(e^{x}-e^{-x})$.

(iii) Find the limits :

$$\lim_{x\to\infty}\ \frac{(x+1)^3-x(x+1)^2}{x^2}\ ; \qquad [\textit{do not use L'Hôpital's rule}]$$

$$\lim_{x\to\infty}\ y^{1/x} \qquad \text{for} \qquad y>0\ ;$$

$$\lim_{x\to 1}\ \frac{x^{1/3}-x^{1/2}}{x-1}\ ; \qquad [\textit{do not use L'Hôpital's rule}]$$

$$\lim_{x\to 1}\ \frac{x^{1/3}-x^{1/2}}{x-1}\ . \qquad [\textit{use L'Hôpital's rule}]$$

(iv) Differentiate the following functions with respect to x :

$$f(x) \;=\; (1+\cos(x))^4\ ;$$

$$f(x) \;=\; \ln(\ln(x))\ ;$$

$$f(x) \;=\; x\,\ln(x/a)\ ;$$

$$f(x) \;=\; x^{\exp(x)}\ .$$

PLEASE TURN OVER

(v) Determine the following definite integrals :

$$\int_0^{2\pi} \frac{dx}{\sin^2 x + \cos^2 x} ;$$

$$\int_1^2 \frac{dx}{\sqrt{x-1}} ;$$

$$\int_1^2 \frac{x + x^5}{2x^2 + x^6}\, dx .$$

(vi) Determine the indefinite integral

$$\int \frac{dx}{\cos^2 x - \sin^2 x} .$$

(vii) Find the Taylor expansion of $\tan(x)$ about $x = \pi/4$ to first order (up to and including the term linear in x) and state the remainder term $R_2(x)$.

(viii) Determine the radius of convergence of the following two series :

$$\sum_{n=1}^{\infty} \frac{x^n}{n} ;$$

$$\sum_{n=1}^{\infty} \frac{x^{2n}}{n!} .$$

(ix) Find the general solutions of the following ODEs :

$$y''(x) = 0 ;$$

$$y'(x) = \frac{y^2(x)}{x^2} ;$$

$$x\,y'(x) = \frac{y(x)}{x} + \frac{1}{x} .$$

(x) Find the general solutions of the following second order ODEs :

$$y''(x) + 2y'(x) + y(x) = 0 ;$$

$$3y''(x) - 2y'(x) + 2y(x) = 0 .$$

2. Find $\dfrac{dy}{dx}$ as a function of x in each of the following cases :

(i) $y = e^{x+x^2}$;

(ii) $y = \dfrac{x \sin x}{x+1}$;

(iii) $y = \sin(\ln x)$;

(iv) $y = x^{\sin x}$;

(v) $e^x + e^y = e^{x+y}$;

(vi) $e^x = \ln(x+y)$.

(vii) Show by induction that

$$\frac{d^n}{dx^n}\left(\frac{1}{1-x}\right) = \frac{n\,!}{(1-x)^{n+1}} \qquad \forall n \geq 1 .$$

3. Evaluate the following limits :

(i) $\displaystyle\lim_{x\to 1} \frac{(x-2)\,(x+2)}{(x-3)\,(x+1)}$;

(ii) $\displaystyle\lim_{x\to\infty} \frac{\sin(\sinh x)}{x}$;

(iii) $\displaystyle\lim_{x\to 0} x^x$;

(iv) $\displaystyle\lim_{x\to -2} \frac{\sqrt{-2x}-2}{x+2}$;

(v) $\displaystyle\lim_{x\to\infty} \left(\frac{x}{2}\right)^{1/2} \left[(2x+1)^{1/2} - (2x-3)^{1/2}\right]$.

Note: You can assume $\displaystyle\lim_{x\to 0} x \ln x = 0$.

PLEASE TURN OVER

4. (i) State whether the improper integral $\displaystyle\int_0^1 \frac{\ln x}{x}\, dx$ is finite and calculate its value if so.

   (ii) Integrate $\displaystyle\int 5^{6x+7}\, dx\,.$

   (iii) Integrate $\displaystyle\int \frac{dx}{\sqrt{3+2x-x^2}}\,.$

5. (i) Find all complex number solutions of the equation

$$z^4 \;-\; z^2 \;+\; 1 \;=\; 0\;.$$

   (ii) Sketch the subset of the complex plane described by the equation

$$|\,z-i-1\,| \;=\; \mathrm{Im}\,(\mathrm{z}+2\mathrm{i})\;,$$

   where, for $z = x+iy$ a complex number, $\mathrm{Im}\,\mathrm{z} = \mathrm{y}$ is the imaginary part.

6. (i) Find the solution $y(x)$ of the differential equation

$$\frac{d^2y}{dx^2} \;+\; \frac{dy}{dx} \;-\; 2y \;=\; 2e^x\;,$$

   that satisfies $y(0)=0,\ \dfrac{dy}{dx}(0)=1\,.$

   (ii) Find the general solution for the following ODE using the $x = e^t$ substitution (or any other method).

$$x^2\,\frac{d^2y}{dx^2} \;+\; 3x\,\frac{dy}{dx} \;+\; y \;=\; 0\;.$$

END OF PAPER

M A T H E M A T I C S   D E P A R T M E N T

MATHEMATICAL FORMULAE

1. VECTOR ALGEBRA

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1,\ a_2,\ a_3)$$

Scalar (dot) product: $\mathbf{a}\,.\,\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

Vector (cross) product:

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Scalar triple product:

$$[\mathbf{a},\ \mathbf{b},\ \mathbf{c}] = \mathbf{a}\,.\,\mathbf{b}\times\mathbf{c} = \mathbf{b}\,.\,\mathbf{c}\times\mathbf{a} = \mathbf{c}\,.\,\mathbf{a}\times\mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product: $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{c}\,.\,\mathbf{a})\mathbf{b} - (\mathbf{b}\,.\,\mathbf{a})\mathbf{c}$

2. SERIES

$$(1+x)^\alpha = 1 + \alpha\,x + \frac{\alpha(\alpha-1)}{2!}\,x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}\,x^3 + \dots \quad (\alpha \text{ arbitrary, } |x|<1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots\ ,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^n\frac{x^{2n}}{(2n)!} + \dots\ ,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + \dots\ ,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^n\frac{x^{n+1}}{(n+1)} + \dots (-1 < x \le 1)$$

3. TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin(a+b) = \sin a \, \cos b + \cos a \, \sin b \; ;$$

$$\cos(a+b) = \cos a \, \cos b - \sin a \, \sin b \, .$$

$$\cos iz = \cosh z \, ; \quad \cosh iz = \cos z \, ; \quad \sin iz = i \sinh z \, ; \quad \sinh iz = i \sin z \, .$$

4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f\,D^n g + \tbinom{n}{1} Df\,D^{n-1}\,g + \ldots + \tbinom{n}{r} D^r f\,D^{n-r} g + \ldots + D^n f\; g \, .$$

(b) Taylor's expansion of $f(x)$ about $x = a$:

$$f(a+h) = f(a) + hf'(a) + h^2 f''(a)/2! + \ldots + h^n f^{(n)}(a)/n! + \epsilon_n(h)\,,$$

where $\quad \epsilon_n(h) = h^{n+1} f^{(n+1)}(a+\theta h)/(n+1)!, \quad 0 < \theta < 1\,.$

(c) Taylor's expansion of $f(x, y)$ about (a, b):

$$f(a+h,\, b+k) = f(a,\, b) + [hf_x + kf_y]_{a.b} + 1/2! \left[h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}\right]_{a.b} + \cdots$$

(d) Partial differentiation of $f(x, y)$:

i. If $y = y(x)$, then $f = F(x)$, and $\dfrac{dF}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dx}\,.$

ii. If $x = x(t)$, $y = y(t)$, then $f = F(t)$, and $\dfrac{dF}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}\,.$

iii. If $x = x(u, v)$, $y = y(u, v)$, then $f = F(u, v)$, and

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}\,, \quad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}\,.$$

(e) Stationary points of $f(x, y)$ occur where $f_x = 0$, $f_y = 0$ simultaneously.
Let (a, b) be a stationary point: examine $D = [f_{xx} f_{yy} - (f_{xy})^2]_{a.b}\,.$
If $D > 0$ and $f_{xx}(a, b) < 0$, then (a, b) is a maximum;
If $D > 0$ and $f_{xx}(a, b) > 0$, then (a, b) is a minimum;
If $D < 0$ then (a, b) is a saddle-point.

(f) Differential equations:

i. The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating factor $\quad I(x) = \exp[\int P(x)(dx]$, so that $\dfrac{d}{dx}(Iy) = IQ$.

ii. $P(x, y)dx + Q(x, y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y\,.$

5. INTEGRAL CALCULUS

(a) An important substitution: $\tan(\theta/2) = t$:
$\sin\theta = 2\,t/(1+t^2)\,, \quad \cos\theta = (1-t^2)/(1+t^2)\,, \quad d\theta = 2\,dt/(1+t^2)\,.$

(b) Some indefinite integrals:

$$\int (a^2 - x^2)^{-1/2} dx = \sin^{-1}\left(\frac{x}{a}\right), \quad |x| < a\,.$$

$$\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1}\left(\frac{x}{a}\right) = \ln\left\{\frac{x}{a} + \left(1 + \frac{x^2}{a^2}\right)^{1/2}\right\}.$$

$$\int (x^2 - a^2)^{-1/2} dx = \cosh^{-1}\left(\frac{x}{a}\right) = \ln\left|\frac{x}{a} + \left(\frac{x^2}{a^2} - 1\right)^{1/2}\right|.$$

$$\int (a^2 + x^2)^{-1} dx = \left(\frac{1}{a}\right)\tan^{-1}\left(\frac{x}{a}\right).$$

6. NUMERICAL METHODS

(a) Approximate solution of an algebraic equation:

If a root of $f(x) = 0$ occurs near $x = a$, take $x_0 = a$ and
$x_{n+1} = x_n - [f(x_n)/f'(x_n)]\,, \quad n = 0, 1, 2 \ldots$

(Newton Raphson method).

(b) Formulae for numerical integration: Write $x_n = x_0 + nh\,, \quad y_n = y(x_n)$.

i. Trapezium rule (1-strip): $\int_{x_0}^{x_1} y(x)dx \approx (h/2)\,[y_0 + y_1]$.

ii. Simpson's rule (2-strip): $\int_{x_0}^{x_2} y(x)dx \approx (h/3)\,[y_0 + 4y_1 + y_2]$.

(c) Richardson's extrapolation method: Let $I = \int_a^b f(x)dx$ and let $I_1,\ I_2$ be two estimates of I obtained by using Simpson's rule with intervals h and $h/2$. Then, provided h is small enough,

$$I_2 + (I_2 - I_1)/15\,,$$

is a better estimate of I.

7. LAPLACE TRANSFORMS

Function	Transform	Function	Transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)dt$	$af(t) + bg(t)$	$aF(s) + bG(s)$
df/dt	$sF(s) - f(0)$	d^2f/dt^2	$s^2F(s) - sf(0) - f'(0)$
$e^{at}f(t)$	$F(s-a)$	$tf(t)$	$-dF(s)/ds$
$(\partial/\partial\alpha)f(t, \alpha)$	$(\partial/\partial\alpha)F(s, \alpha)$	$\int_0^t f(t)dt$	$F(s)/s$
$\int_0^t f(u)g(t-u)du$	$F(s)G(s)$		
1	$1/s$	$t^n (n = 1, 2 \ldots)$	$n!/s^{n+1}$, $(s > 0)$
e^{at}	$1/(s-a)$, $(s > a)$	$\sin\omega t$	$\omega/(s^2+\omega^2)$, $(s > 0)$
$\cos\omega t$	$s/(s^2+\omega^2)$, $(s > 0)$	$H(t-T) = \begin{cases} 0, & t < T \\ 1, & t > T \end{cases}$	e^{-sT}/s, $(s, T > 0)$

8. FOURIER SERIES

If $f(x)$ is periodic of period $2L$, then $f(x+2L) = f(x)$, and

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \text{ where}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad n = 0, 1, 2, \ldots, \text{ and}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1, 2, 3, \ldots$$

Parseval's theorem

$$\frac{1}{L}\int_{-L}^{L} [f(x)]^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right).$$

EXAMINATION QUESTIONS/SOLUTIONS 2010-2011

Course: EE1(1) (1)

Question: Q1

Marks & seen/unseen

Parts

i)

$$i = \underline{\underline{e^{i\frac{\pi}{2}}}}$$

$$1-i = \sqrt{2}\,e^{-i\frac{\pi}{4}} = \underline{\underline{\sqrt{2}\,e^{i\frac{7\pi}{4}}}}$$

$$(2+2i)^2 = 8i = \underline{\underline{8\,e^{i\frac{\pi}{2}}}}$$

$$(2+i)(2-i) = \underline{\underline{5}}$$

$$\frac{1}{2} - \sqrt{\tfrac{3}{4}}\,i = e^{-i\frac{\pi}{3}} = \underline{\underline{e^{i\frac{5\pi}{3}}}}$$

4 seen similar

ii)

$$(1+i)^3 = \underline{\underline{-2+i2}}$$

$$z = x+iy$$

$$e^{z} = \underline{e^{x}\cos y + i e^{x}\sin y}$$

$$e^{iz} = \underline{\underline{e^{-y}\cos x + i e^{-y}\sin x}}$$

$$\sinh(iz) = \frac{1}{2}\left\{e^{-y}\cos x + i e^{-y}\sin x - e^{y}\cos x + i e^{y}\sin x\right\}$$

$$= \underline{\underline{\frac{1}{2}\left\{(e^{-y}-e^{y})\cos x + i(e^{-y}+e^{y})\sin x\right\}}}$$

$$\text{or} \quad = -\sinh y\cos x + i\cosh y\sin x$$

4 seen similar

$$\cos iz = \frac{1}{2}\left(e^{-z}+e^{z}\right) = \underline{\frac{1}{2}\left(e^{x}+e^{-x}\right)\cos y +}$$
$$\underline{i\,\frac{1}{2}\left(e^{x}-e^{-x}\right)\sin y}$$

$$\text{or} = \cosh x\cos y + i\sinh x\sin y$$

Setter's initials: GP

Checker's initials: RW

	EXAMINATION QUESTIONS/SOLUTIONS 2010-2011	Course EE1(1.1) 1
Question Q1		Marks & seen/unseen

Parts

iii)

$$\lim_{x\to\infty} \frac{(x+1)^3 - x(x+1)^2}{x^2} = \lim_{x\to\infty} \frac{3x^2-2x^2+\dots}{x^2} = \underline{\underline{1}}$$

4

seen similar

$$\lim_{x\to\infty} y^{\frac{1}{x}} = y^0 = \underline{\underline{1}}$$

$$\lim_{x\to 1} \frac{x^{1/3}-x^{1/2}}{x-1} = \underline{\underline{-\frac{1}{6}}}$$

using

$$\frac{x^{1/3}-x^{1/2}}{x-1} = -x^{\frac{1}{3}} \frac{1-x^{\frac{1}{6}}}{1-x}$$

$$= -x^{1/3}\frac{1-y}{1-y^6} = -x^{1/3}\frac{1-y}{(1+y+y^2+y^3+y^4+y^5)(1-y)}$$

$$= -x^{1/3}\frac{1}{1+y+y^2+y^3+y^4+y^5}$$

where $y = x^{1/6}$ and $\lim_{x\to 1} y = 1$

Using L'Hopital gives

$$\lim_{x\to 1} \frac{\frac{1}{3}x^{-\frac{2}{3}} - \frac{1}{2}x^{-\frac{1}{2}}}{1} = \underline{\underline{-\frac{1}{6}}}$$

Setter's initials GP | Checker's initials RW | Page number S2

	EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2010-2011	Course EE1(1)
Question		Marks & seen/unseen

Parts

iv)

$$\frac{d}{dx}\left(1+\cos(x)\right)^4 = \underline{\underline{-4\sin(x)\left(1+\cos(x)\right)^3}}$$

$$\frac{d}{dx}\ln(\ln(x)) = \underline{\underline{\frac{1}{x}\frac{1}{\ln(x)}}}$$

$$\frac{d}{dx}\ln\left(\frac{x}{a}\right)x = \underline{\underline{\ln\left(\frac{x}{a}\right)+1}}$$

$$\frac{d}{dx}x^{\exp(x)} = \frac{d}{dx}\exp(\ln(x)\exp(x))$$

$$= \left(\frac{1}{x}\exp(x)+\ln(x)\exp(x)\right)x^{\exp(x)}$$

$$= \underline{\underline{x^{\exp(x)}e^x\left(\frac{1}{x}+\ln x\right)}}$$

4 seen similar

v)

$$\int_0^{2\pi}\frac{dx}{\sin^2x+\cos^2x} = \underline{\underline{2\pi}}$$

$$\int_1^2\frac{dx}{\sqrt{x-1}} = \lim_{\varepsilon\to 0}\left[2\sqrt{x-1}\right]_{1+\varepsilon}^{2} = \underline{\underline{2}}$$

$$\int_1^2\frac{x+x^5}{2x^2+x^6}dx = \frac{1}{8}\int_1^2\frac{8x^3+8x^7}{2x^4+x^8}dx = \frac{1}{8}\left[\ln(2x^4+x^8)\right]_1^2$$

$$= \frac{1}{8}\ln\left(\frac{288}{3}\right) = \underline{\underline{\frac{1}{8}\ln(96)}}$$

or $\displaystyle\int_1^2\frac{x+x^5}{2x^2+x^6}dx = \int_1^2\frac{1+x^4}{2x+x^5}dx = \frac{1}{2}\int_1^2\frac{2+5x^4-3x^4}{2x+x^5}$

4 seen similar

Setter's initials GP	Checker's initials RLJ	Page number S3

$$= \frac{1}{2}\left[\ln(2x+x^5)\right]_1^2 - \frac{3}{2}\int_1^2 \frac{x^3}{2+x^4}$$

$$= \frac{1}{2}\ln(12) - \frac{3}{8}\left[\ln(2+x^4)\right]_1^2$$

$$= \frac{1}{2}\ln(12) - \frac{3}{8}\ln(6) = \frac{1}{8}\ln(12^4/6^3)$$

$$= \frac{1}{8}\ln 96$$

vi)

$$\int \frac{dx}{\cos^2 x - \sin^2 x} = \int \frac{dt}{1+t^2}\,\frac{1+t^2}{1-t^2}$$

$$= \frac{1}{2}\int \frac{1}{1+t} + \frac{1}{1-t}\,dt = \frac{1}{2}\ln\left(\frac{t+1}{t-1}\right) + C_1$$

$$= \frac{1}{2}\ln\left(\frac{\sin x + \cos(x)}{\sin x - \cos(x)}\right) + C_1$$

4

seen similar

using $t = \tan x$ $\quad \frac{dt}{dx} = \frac{1}{\cos^2 x} = 1+t^2$

$$\sin^2 x = \frac{t^2}{1+t^2} \qquad \cos^2 x = \frac{1}{1+t^2}$$

Ans can also be written:

$$\frac{1}{2}\ln(\sec 2x + \tan 2x) + c$$

	EXAMINATION QUESTIONS/SOLUTIONS 2010-2011	Course EE1(1)
Question		Marks & seen/unseen

Parts

vii)

$$\tan\left(\frac{\pi}{4}\right) = 1$$

$$\left.\frac{d}{dx}\right|_{x=\frac{\pi}{4}} \tan(x) = \frac{1}{\cos^2(\frac{\pi}{4})} = 2$$

$$\frac{d^2}{dx^2}\tan x = \frac{2\sin(x)}{\cos^3(x)}$$

4 seen similar

$$\Rightarrow \quad \underline{\underline{\tan(x) = 1 + 2\left(x - \frac{\pi}{4}\right) + R_2(x)}}$$

$$R_2(x) = \underline{\underline{\frac{1}{2}\left(x-\frac{\pi}{4}\right)^2 \frac{2\sin(\xi)}{\cos^3(\xi)} \quad \xi \in [\tfrac{\pi}{4}, x]}}$$

viii)

$$a_n = \frac{x^n}{n} \Rightarrow \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x| \lim_{n\to\infty}\frac{n}{n+1} = |x|$$

$\Rightarrow$ radius of convergence $\underline{\underline{1}}$

$$a_n = \frac{x^{2n}}{n!} \Rightarrow \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x|^2 \lim_{n\to\infty}\frac{1}{n+1} = 0$$

$\Rightarrow$ radius of convergence $\underline{\underline{\infty}}$

4 seen similar

Setter's initials GP	Checker's initials RLJ	Page number 55

	EXAMINATION QUESTIONS/SOLUTIONS 2010-2011	Course EE1(I) ①
Question		Marks & seen/unseen
Parts ix)	$y''=0 \Rightarrow \underline{y(x) = Ax + B}$ $y'(x) = \frac{y^2}{x^2}$ separation of variables: $-\frac{1}{y} = -\frac{1}{x} + A \Rightarrow \underline{y = \frac{x}{1-Ax}}$ or via homogeneous ODE: $\ln x = \int \frac{dv}{v^2 - v}$ with $v = \frac{y}{x}$ $= \int -\frac{1}{v} + \frac{1}{1-v}\, dv = \ln\left(\frac{1-v}{v}\right) + C'$ $\Rightarrow x = \tilde{A}\frac{x-y}{y} \Rightarrow \underline{y = \frac{\tilde{A}x}{\tilde{A}+x}} = \frac{x}{1+\underbrace{\frac{1}{\tilde{A}}}_{=-A \text{ above}} x}$ $y'x = \frac{y}{x} + x^{-1}$ $y' - \frac{y}{x^2} = x^{-2}$ Integrating factor $-R\frac{1}{x^2} = R' \Rightarrow R = e^{1/x}$ $\Rightarrow Ry = \int dx\, x^{-2} e^{1/x} = -e^{1/x} + C'$ $\Rightarrow \underline{y = -1 + C' e^{-1/x}}$	4 seen similar
	Setter's initials GP Checker's initials RW	Page number S6

	EXAMINATION QUESTIONS/SOLUTIONS 2010-2011	Course EE 1(II) ①
Question Q1		Marks & seen/unseen
Parts x)	$$y'' + 2y' + y = 0$$ $$ay'' + by' + cy = 0$$ $a=1,\ b=2,\ c=1 \Rightarrow 4ac = b^2$ Critical case, $-b/2a = -1$ $$y(x) = \underline{\underline{Ae^{-x} + Bxe^{-x}}}$$ $3y'' - 2y' + 2y = 0$ $\quad 4ac = 24$ $a=3 \quad b=-2 \quad c=2 \quad b^2 = 4$ oscillatory case $\Omega = \frac{\sqrt{4ac-b^2}}{2a} = \frac{\sqrt{5}}{3}$ $$-\frac{b}{2a} = \frac{1}{3}$$ $$y(x) = \underline{\underline{e^{\frac{x}{3}}\left(A\sin\left(\frac{\sqrt{5}}{3}x\right) + B\cos\left(\frac{\sqrt{5}}{3}x\right)\right)}}$$	4 Seen Similar
	Setter's initials GP Checker's initials RLJ	Page number 57

CORE 1

EXAMINATION QUESTIONS/SOLUTIONS 2010-2011

Course: EE I(1)

2

Question

Marks & seen/unseen

Parts

(i) $\frac{dy}{dx} = (1+2x)e^{x+x^2}$ Chain Rule (2)

ii) $\frac{dy}{dx} = \frac{\sin x}{x+1} + \frac{x\cos x}{x+1} - \frac{x\sin x}{(x+1)^2}$ Product Rule (3)

$$= \frac{\sin x + (x+1)x\cos x}{(x+1)^2}$$

iii) $\frac{dy}{dx} = \frac{\cos(\ln x)}{x}$ (2)

iv) $y = x^{\sin x} = e^{(\ln x)\sin x}$

$$\frac{dy}{dx} = e^{(\ln x)\sin x}\left(\frac{\sin x}{x} + \cos x \ln x\right)$$

$$= x^{\sin x}\left[\frac{\sin x}{x} + (\ln x)\cos x\right]$$ (3)

v) $e^x + e^y = e^{x+y} \Rightarrow e^y = \frac{e^x}{e^x-1} = \frac{1}{1-e^{-x}}$

$$\therefore \frac{dy}{dx}e^y = \frac{-e^{-x}}{(1-e^{-x})^2}$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{1}{1-e^{-x}}\right) = \frac{-e^{-x}}{(1-e^{-x})^2} \Rightarrow \frac{dy}{dx} = \frac{-e^{-x}}{1-e^{-x}} = \frac{1}{1-e^{+x}}$$ (4)

vi) $e^x = \ln(x+y) \Rightarrow x+y = e^{e^x}$

$$\therefore 1 + \frac{dy}{dx} = e^x e^{e^x} \Rightarrow \frac{dy}{dx} = e^x e^{e^x} - 1$$ (3)

vii) True for $n=1$. Assume $\frac{d^n}{dx^n}\left(\frac{1}{1-x}\right) = \frac{n!}{(1-x)^{n+1}}$ (3)

Then $\frac{d^{n+1}}{dx^{n+1}}\frac{1}{1-x} = \frac{d}{dx}\left(\frac{n!}{(1-x)^{n+1}}\right) = \frac{(n+1)!}{(1-x)^{n+2}}$

Setter's initials: JRC

Checker's initials: MWL

Page number

Core 2

	EXAMINATION QUESTIONS/SOLUTIONS 2010-2011	Course EE I(1) 3
Question		Marks & seen/unseen
Parts		

i) $\lim_{x\to 1} \frac{(x-2)(x+2)}{(x-3)(x+1)} = \frac{-1\cdot 3}{-2\cdot 2} = 3/4$ (2)

ii) $L = \lim_{x\to\infty} \frac{\sin(\sinh x)}{x}$. $|L| < \lim_{x\to\infty} \frac{1}{x} = 0$ (3)

iii) Let $y = x^x$ then $\ln y = x\ln x$

as $x\to 0 \Rightarrow \ln y \to 1$ $\therefore \lim_{x\to 0} x^x = 1$ (5)

iv) $\lim_{x\to -2} \frac{\sqrt{-2x}-2}{x+2} = \lim_{x\to -2} \frac{(\sqrt{-2x}-2)(\sqrt{-2x}+2)}{(x+2)(\sqrt{-2x}+2)}$

$= \lim_{x\to -2} \frac{-2x-4}{(x+2)(\sqrt{-2x}+2)}$

$= \lim_{x\to -2} \frac{-2}{\sqrt{-2x}+2} = -\frac{1}{2}.$ (5)

v) $\lim_{x\to\infty} \left(\frac{x}{2}\right)^{1/2}\left[(2x+1)^{1/2} - (2x-3)^{1/2}\right]$

$= \lim_{x\to\infty} \left(\frac{x}{2}\right)^{1/2}(2x)^{1/2}\left[\left(1+\frac{1}{2x}\right)^{1/2} - \left(1-\frac{3}{2x}\right)^{1/2}\right]$

$= \lim_{x\to\infty} x\left[\left(1+\frac{1}{4x}+\dots\right) - \left(1 - 3/4x + \dots\right)\right]$

$= \lim_{x\to\infty} x\left[1/x + \dots\right] = 1$ (5)

or $\lim_{x\to\infty} \left(\frac{x}{2}\right)^{1/2} = \left[(2x+1)^{1/2} - (2x-3)^{1/2}\right]\left[\frac{(2x+1)^{1/2} + (2x-3)^{1/2}}{(2x+1)^{1/2} + (2x-3)^{1/2}}\right]$

etc

Setter's initials JRC	Checker's initials MWL	Page numbe

Core 3

EE I (1)

4

	INTEGRAL CORE EXAM SOLUTION 2010-2011	Course
Core 3		Marks & seen/unseen
Part A	Consider $\int_c^1 \frac{\ln x}{x}\, dx = \frac{1}{2}(\ln x)^2\vert_c^1 = -\frac{1}{2}(\ln c)^2$. Taking the limit as $c \to 0^+$ yields $-\infty$. Therefore the integral diverges.	4 3
B	Let $u = 6x+7$, so $du = 6dx$. Then the integral becomes $\frac{1}{6}\int 5^u\, du = \frac{1}{6} * \frac{5^u}{\ln 5} + c = \frac{5^{6x+7}}{6\ln 5} + c$	2 4
C	$I(x) = \int \frac{dx}{\sqrt{3+2x-x^2}} = \int \frac{dx}{\sqrt{4-(x-1)^2}}$. Now let $u = x-1$, giving $du = dx$ and hence $I(x) = \int du/\sqrt{4-u^2}$. Let $u = 2\sin\theta \Rightarrow du = 2\cos\theta\, d\theta$. $I(x) = \int \frac{2\cos\theta\, d\theta}{\sqrt{4-4\sin^2\theta}}$ $= \int \frac{2\cos\theta\, d\theta}{2\cos\theta} = \theta + c$ $= \sin^{-1}\left(\frac{u}{2}\right) + c = \sin^{-1}\left(\frac{x-1}{2}\right) + c$	1 1 2 1 1 1
	Setter's initials: DB Checker's initials AC	Page number

Core 4

EXAMINATION QUESTIONS/SOLUTIONS 2010-2011

Course: EE1 (1) (5)

Question: C4

Marks & seen/unseen

Parts

(a) I first solve

$$z^2 - z + 1 = 0$$

$$z_{1,2} = \frac{1 \pm i\sqrt{3}}{2} = e^{\frac{\pi}{3} i}, \quad e^{-\frac{\pi}{3} i}$$

4 marks

next I solve for x:

$$x^2 = z_1 \qquad x_{1,2} = \begin{cases} e^{\frac{\pi}{6} i} = \frac{\sqrt{3}}{2} + \frac{1}{2} i \\ -e^{+\frac{\pi}{6} i} = -\frac{\sqrt{3}}{2} - \frac{1}{2} i \end{cases}$$

3 marks

$$x^2 = z_2 \qquad x_{3,4} = \begin{cases} e^{-\frac{\pi}{6} i} = \frac{\sqrt{3}}{2} - \frac{1}{2} i \\ -e^{-\pi/6\, i} = -\frac{\sqrt{3}}{2} + \frac{1}{2} i \end{cases}$$

3 marks

(In summary $x = \pm \frac{\sqrt{3}}{2} \pm \frac{1}{2} i$)

Setter's initials: AC

Checker's initials: RB

Page number: 2/3

EXAMINATION QUESTIONS/SOLUTIONS 2010-2011

Course: Core 4

Question: C4

Marks & seen/unseen

Parts

unseen (I hope)

10 mark

(b) $|z-i-1| = \mathrm{Im}(z+2i)$

$z = x+iy$:

$|x-1+i(y-1)| = y+2$ (note: $y \geq -2$)

square both sides:

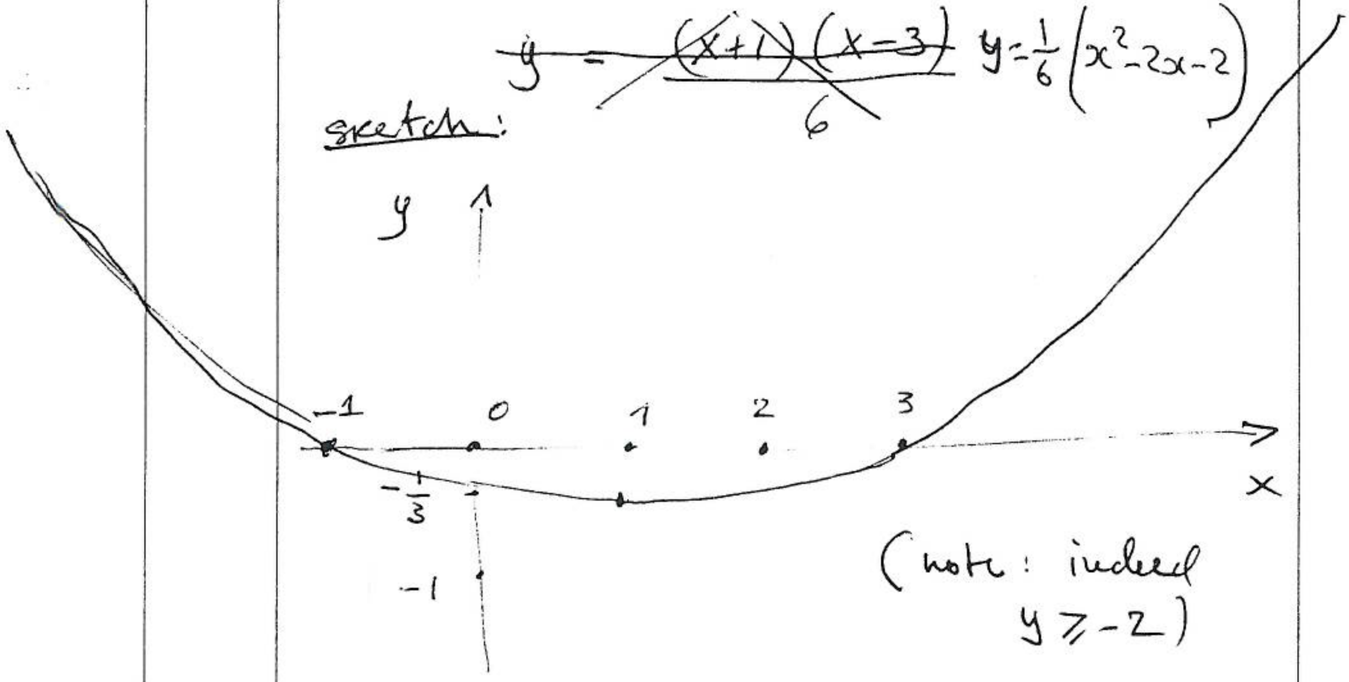
$(x-1)^2+(y-1)^2 = (y+2)^2$

$x^2-2x+1+\cancel{y^2}-2y+1 = \cancel{y^2}+4y+4$

$6y = x^2-2x-2$

$\cancel{y = \frac{(x+1)(x-3)}{6}}$ $y=\frac{1}{6}\left(x^2-2x-2\right)$

sketch:



(note: indeed $y \geq -2$)

Setter's initials: AC

Checker's initials: JMC

Page number

	EXAMINATION QUESTIONS/SOLUTIONS 2010-2011	Course EE1 (1) 6
Question G10	Second order ODE (solution)	Marks & seen/unseen

Parts

(i) Finding the complementary function of the ODE by solving the auxiliary equation:

$$\lambda^2 + \lambda - 2 = 0 \Rightarrow$$ 2

$$\lambda = -2, +1$$

CF: $Y(x) = A_1 e^{-2x} + A_2 e^{x}$ 2

This is a degenerate case, therefore for the particular integral (PI) we try

$$y = C x e^{x}$$ 2

$$\frac{dy}{dx} = Cxe^{x} + Ce^{x}$$

$$\frac{d^2y}{dx^2} = Cxe^{x} + 2Ce^{x}$$

$$Cxe^{x} + 2Ce^{x} + Cxe^{x} + Ce^{x} - 2Cxe^{x} = 2e^{x}$$

$$3C = 2 \Rightarrow C = \frac{2}{3}$$

General solution: $y(x) = A_1 e^{-2x} + A_2 e^{x} + \frac{2}{3}xe^{x}$ 2

$y(0) = A_1 + A_2 = 0$

$y'(0) = -2A_1 + A_2 + \frac{2}{3} = 1$ $\Rightarrow -3A_1 = \frac{1}{3} \Rightarrow A_1 = -\frac{1}{9}$ 1

$A_2 = \frac{1}{9}$ 1

$$y(x) = -\frac{1}{9}e^{-2x} + \frac{1}{9}e^{x} + \frac{2}{3}xe^{x}$$

Setter's initials V.S.	Checker's initials	Page number 2

	EXAMINATION QUESTIONS/SOLUTIONS 2010-2011	Course EE I(1) 6
Question C1a	Second order ODE (solution)	Marks & seen/unseen

Parts

(ii)

$$x = e^{t} \Rightarrow \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}e^{-t}$$ 2

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dt}e^{-t}\right)e^{-t} = e^{-2t}\frac{d^2y}{dt^2} - \frac{dy}{dt}e^{-2t}$$ 2

Substitution gives:

$$\cancel{e^{2t}}\left(e^{-2t}\frac{d^2y}{dt^2} - \frac{dy}{dt}\cancel{e^{-2t}}\right) + 3\cancel{e^{-t}}\cancel{e^{t}}\frac{dy}{dt} + y = 0$$

$$\frac{d^2y}{dt} + 2\frac{dy}{dt} + y = 0$$ 2

Now an equation with constant coefficients, we solve the auxiliary equation:

$$\lambda^2 + 2\lambda + 1 = 0 \Rightarrow \lambda = -1.$$ 1

Repeated root! therefore CF is

$$y(t) = (a + bt)e^{-t}$$ 2

$$t = \log x \Rightarrow y(x) = \frac{(a + b\log x)}{x}$$ 1

Note: If students used $y = Ax^b$, substitution and guessed the $\log x$ solution should get full mark. If they only find $\frac{a}{x}$ solution 5 mark only.

Setter's initials V.S.	Checker's initials [signature]	Page number 3