IMPERIAL COLLEGE LONDON

E4.04
SC6
ISE4.9

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING

EXAMINATIONS 2005

MSc and EEE/ISE PART IV: M.Eng. and ACGI

ADVANCED DATA COMMUNICATIONS

Time Allowed: 3:00 hours

There are FOUR questions on this paper.
Answer THREE questions.

Corrected Copy

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible

First Marker: Gurcan, M.K.
Second Marker: Stathaki, T.

Special Instructions for Invigilators: None

Information for candidates:

Notations

BSC	Binary Symmetric Channel
NNPE	Nearest Neighbour constellation point Probability of Error

Useful equations

Suppose $g(t)$ and $G(f)$ are Fourier transform pairs such that

$$g(t) \Leftrightarrow G(f) = \mathcal{F}(g(t))$$

where

$$G(f) = \mathcal{F}\{g(t)\} = \int_{-\infty}^{\infty} g(t)\exp(-j2\pi f t)\,dt \text{ and}$$

$$g(t) = \mathcal{F}^{-1}\{G(f)\} = \int_{-\infty}^{\infty} G(f)\exp(j2\pi f t)\,df .$$

Then the following Fourier transform relationships might be useful

$$g(t) = \mathrm{rect}\left(\frac{t}{T}\right) \Leftrightarrow G(f) = T\,\mathrm{sinc}(f\,T)$$

$$g(t) = \delta(t) \Leftrightarrow G(f) = 1$$

$$g(t) = \mathrm{rect}\left(\frac{t}{\alpha T}\right)\cos\left(\frac{\pi t}{\alpha T}\right) \Leftrightarrow G(f) = \frac{2\alpha T}{\pi}\frac{\cos(\pi f \alpha T)}{1-4f^2(\alpha T)^2}$$

Duality principle

$$Y(f) = \mathcal{F}\{x(t)\} \Rightarrow X(f) = \mathcal{F}\{y(-t)\}$$

1. a) Consider two possible signal constellations for 8-ary signalling in white Gaussian noise with double-sided noise power-spectral-density $N_0/2$. Both sets of signals are in two-dimensional signal space. Set A has all 8 signals on the circumference of a circle of radius $\sqrt{E}$ centred at the origin. The signals in Set B are at (±3, 0), (±1, 0), and (±1,±2). [2]

i) If the two signal sets have the same average energy, what is the value of $\sqrt{E}$? [3]

ii) Sketch the maximum-likelihood decision regions for each set of signals.

iii) Show that for each signal in Set A, the Nearest Neighbour constellation point Probability Error $P_{e,i}$ is $2 \cdot Q\left(d_{\min,A}/\sqrt{2N_0}\right)$. [3]

iv) Show that two of the signals in Set B have Nearest Neighbour constellation point Probability Error (NNPE) $Q\left(d_{\min,B}/\sqrt{2N_0}\right)$, four have NNPE $2 \cdot Q\left(d_{\min,B}/\sqrt{2N_0}\right)$ and two have NNPE $4 \cdot Q\left(d_{\min,B}/\sqrt{2N_0}\right)$. [3]

v) The average NNPE for Set A is $2 \cdot Q\left(d_{\min,A}/\sqrt{2N_0}\right)$ while for Set B it is $2.25 \cdot Q\left(d_{\min,B}/\sqrt{2N_0}\right)$. Ignoring the difference between 2.25 and 2, what should be the value of E such that both sets have the same average NNPE? Under these conditions (and still ignoring the difference between 2.25 and 2) show that Set A needs approximately 1.36 dB more signal energy to achieve the same average NNPE as Set B. [5]

Question continued over

b) Figure 1.1 shows one dimensional *8-ary* constellation points for pulse amplitude modulation scheme and a received value of $y = -5.5$.

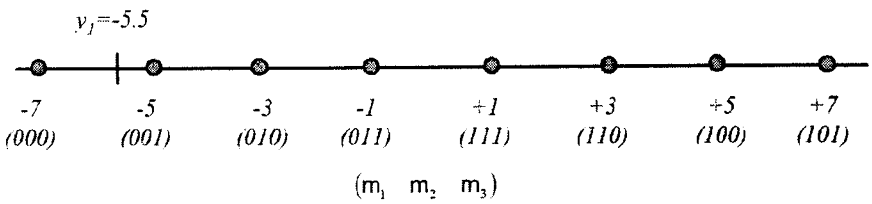


Figure 1.1

The received value of -5.5 corresponds to the 3 bits $(m_1, \quad m_2, \quad m_3)$.

i) Find the probability of having $p(m_i = 0)$ for $i = 1,2,3$ and $p(m_i = 1)$ for $i = 1,2,3$ for *a priori* probabilities of $p_i = 1/2$ for $i = 1,2,3$. [3]

ii) Calculate the log likelihood ratios $LLR(m_i)$ for $i = 1,2,3$ when the log likelihoods are approximated by using the dominant terms in each sum of event probabilities. [3]

iii) Show that the log likelihood ratio when using the dominant terms is [3]

$$LLR = \frac{y}{\sigma^2}(b-a) + \frac{a^2 - b^2}{2\sigma^2}$$

Where "b" is the closest 1-bit point in the constellation and "a" is the closest 0-bit point in the constellation, σ^2 is noise variance. .

2 Consider the random processes

$$\mathcal{X}(t) = \sum_{k=-\infty}^{\infty} (-1)^{b_{2k}} \operatorname{rect}\left((t-2kT)/2T\right) \cos(\pi t/2T)$$

$$\mathcal{Y}(t) = \sum_{k=-\infty}^{\infty} (-1)^{b_{2k+1}} \operatorname{rect}\left((t-(2k+1)T)/2T\right) \sin(\pi t/2T)$$

where the b_k's are i.i.d. random variables taking on values 0 and 1 with equal probability. These are the baseband inphase and quadrature signals in a Minimum Shift Keying (MSK) system.

a) Show that we can write $\mathcal{X}(t) = \sum_{m=-\infty}^{\infty} \mathcal{A}_m \, g(t-2mT)$ where the $\mathcal{A}_m$'s are i.i.d. random variables related to the b_{2k}'s and with finite variance σ^2. The term g(·) is a suitably defined finite energy pulse shaping function, *not necessarily of duration at most T.* [4]

b) If $\mathcal{T}$ is uniformly distributed on [0, 2T) and independent of the b_k's, show that [6]

11.35

$$E\left[\mathcal{X}(t-\mathcal{T})\mathcal{X}(t+\tau-\mathcal{T})\right] = \sigma^2 (2T)^{-1} \int_{-\infty}^{\infty} g(u)\, g(u-\tau)\, du$$

c) Show that the autocorrelation function, $R_{\mathcal{X}}(\tau)$, of the Wide Sense Stationary (WSS) process $\mathcal{X}(t-\mathcal{T})$ is [7]

$$R_{\mathcal{X}}(\tau) = \left[\left(\frac{1}{4T}\right)(2T-|\tau|)\cos\left(\frac{\pi\tau}{2T}\right) + \left(\frac{1}{2\pi}\right)\sin\left(\frac{\pi|\tau|}{2T}\right)\right]\operatorname{rect}\left(\frac{t}{4T}\right) \qquad \frac{\tau}{4T} \qquad 10:00$$

This also happens to be the autocorrelation function of the WSS random process $\mathcal{Y}(t-\mathcal{T})$, a fact that you may assume without proving it.

d) Assume that [8]

$$\mathcal{Z}(t) = \mathcal{X}(t-\mathcal{T})\cos(2\pi f_c t + \Theta) - \mathcal{Y}(t-\mathcal{T})\sin(2\pi f_c t + \Theta)$$

where Θ is a random variable uniformly distributed on $[0, 2\pi)$ and independent of the b_k's and $\mathcal{T}$, is a Wide Sense Stationary process. What is $\mathcal{Z}(t)$'s power spectral density?

3 a) Answer the following questions

i) Show that the Fourier transform of $s(t) = \text{rect}(t/T)\cos(\pi t/T)$ is [5]

$$S(f) = \left(\frac{2T}{\pi}\right)\frac{\cos(\pi f T)}{1-(2fT)^2}$$

ii) From the result of Problem 3 a) i) use duality principle (see useful equations) to deduce the Fourier transform of $x(t) = \cos(\pi\alpha t/T)/\left[1-(2\alpha t/T)^2\right]$. Assume henceforth that $|\alpha| \le 1$. [5]

Draw a neat sketch of $X(f)$ and $Y(f) = T\cdot\text{rect}(fT)$ on the same axes. Be sure to mark the points $f = \pm T^{-1}$ on the frequency axis.

iii) Find the Fourier transform of $x(t)y(t)$ via convolution and duality. Comment on the result from the Nyquist third criterion point of view. [5]

b) A rate $2/3$ convolutional code is described by the 3 dimensional vector sequence $v(D) = [v_3(D),\ v_2(D),\ v_1(D)]$ where $v(D)$ is any sequence generated by all the binary possibilities for an input bit sequence $u(D) = [u_2(D),\ u_1(D)]$ according to

$$v(D) = u(D)\cdot\begin{bmatrix} 1, & D, & 1+D \\ 0, & 1, & 1+D \end{bmatrix},$$

i) Write the equations for the output bits at any time k in terms of the input bits at that same time k and k-1. [4]

ii) Using the state label $[u_{2,k-1},\ u_{1,k-1}]$, draw the trellis for this code. [3]

iii) Use Viterbi decoding with a Hamming distance metric to estimate the encoder input if $y(D) = [010,\ 110,\ 111,\ 000,\ 000,\ 000,\ \cdots]$ is received at the output of a Binary Symmetric Channel (BSC). Assume that you have started in state 00. Note that the order of the triplets in the received output is: $v_3v_2v_1$. [3]

4. a) The data rate, $R = b/T$, for an OFDM system with a set of 8 sub channels requires maximization. Where $1/T$ is the symbol rate, and b is

$$b = \frac{1}{2}\sum_{n=1}^{8} b_n = \frac{1}{2}\sum_{n=1}^{8} \log_2\left(1+\varepsilon_n g_n\right)$$

the largest number of bits that can be transmitted over the parallel set of 8 channels. In this equation, $g_n = |H_n|^2/\sigma_n^2$ represents the subchannel signal-to-noise ratio when the transmitter applies unit energy to that subchannel. The terms ε_n, $|H_n|^2$ and σ_n^2 correspond to the energy, the channel gain and noise variance in the n^{th} subchannel respectively.

i) Using Lagrange multiplier method show that the following set of linear equations

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & -1 \\ 0 & 1 & 0 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & -1 \\ 1 & 1 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_8 \\ K \end{bmatrix} = \begin{bmatrix} -1/g_1 \\ -1/g_2 \\ \vdots \\ -1/g_8 \\ 8\bar{\varepsilon}_x \end{bmatrix}$$

provide solutions for energy distributions in each subchannel. Where K is a constant and $8\bar{\varepsilon}_x = \sum_{n=1}^{8} \varepsilon_n$. [6]

ii) Assume that the channel gain and noise variance in each subchannel are given by [6]

$$H_n = \begin{cases} 1 + 0.9 \bullet \exp\left(\dfrac{j(n-1)\pi}{4}\right) & \textit{for } n = 1, \cdots 8 \\ 0 & \text{otherwise} \end{cases}$$ and

$\sigma_n^2 = 0.181$ respectively. Find the energy distribution in each subchannel and constant K for $8\bar{\varepsilon}_x = 8$. Calculate the corresponding number of bits, b_n, per subchannel.

Ouestion continued over

b) The convolutional code in Figure 4.1 is based upon a modulo-2 linear combination of current and past input bits.

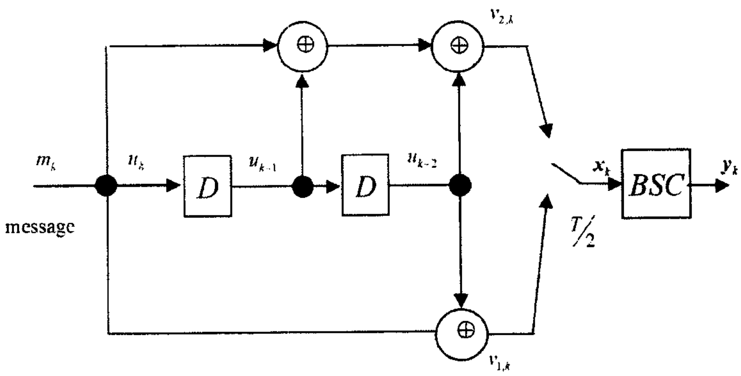


Figure 4.1

The encoder has four states represented by the four possible values of the ordered pair $(u_{k-2},\ u_{k-1})$. The trellis diagram for this code is shown in Figure 4.2.

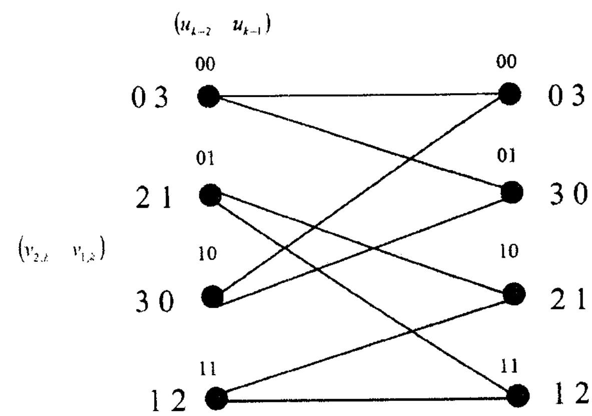


Figure 4.2

The upper branch emanating from each state corresponds to an input bit of zero while the lower branch corresponds to an input bit of one. The outputs transmitted from each state are listed in modulo-4 notation to the left of each starting state with the left most corresponding to the upper branch and rightmost corresponding to the lower branch. To the right of each ending state, the upper branch is listed as the leftmost, and the lower branch is given as the rightmost value. The data is transmitted over a BSC channel with error probability $p = 0.25$.

The trellis diagram of a sequential MAP detector for this encoder is given in Figure 4.3 for *a priori* probabilities of 0.5.

Question continued over

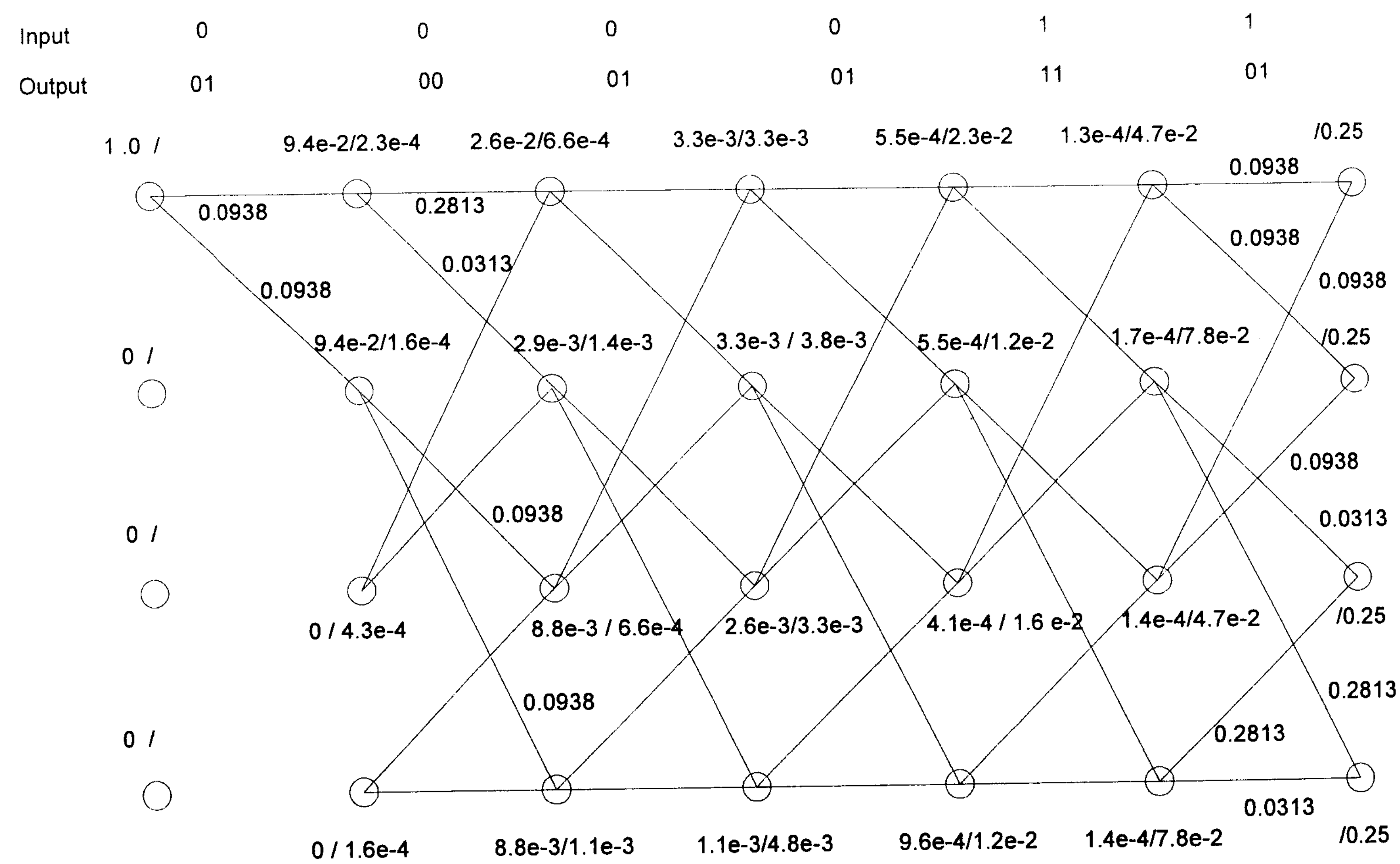


Figure 4.3 MAP detector trellis diagram

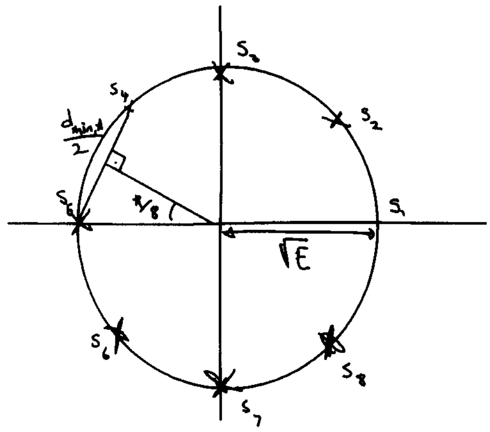
Question continued over

The data sequence at the output of the BSC is given above the trellis diagram. At each state of the trellis in Figure 4.3, the values of α_k/β_k appear while the values of γ_k appear~~s~~ on each branch for the first two time epochs. Where γ_k is the current branch probability, α_k and β_k are the state occupancy probabilities for beginning and ending states respectively, Each of the upper emanating branches has the value $\gamma_0 = 0.0938 = 1/2 \cdot (0.25) \cdot (0.75)$. Where the term 0.25 corresponds to the first-code output bit not matching the channel-output 0. While the 0.75 corresponds to second code-output bit matching the second channel output bit of 1. The factor 1/2 represents the *a priori* probability. The final transmission probability for each bit is computed by summing the probabilities of the branches corresponding to "1" and also a second sum for the "0" branches at each trellis stage and then normalising the two measures so that they add to 1.

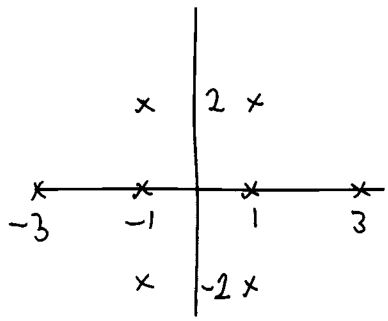
i) For the trellis diagram given in Figure 4.3, calculate the probabilities of transmitting a "0" and "1" at each time epoch. [10]

ii) Using the probabilities calculated in 4 b) i) identify the data sequence transmitted. [3]

1.a i



A



B

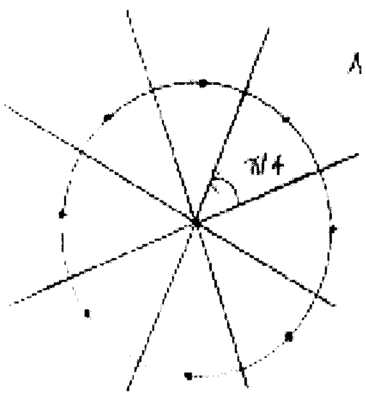
(2)

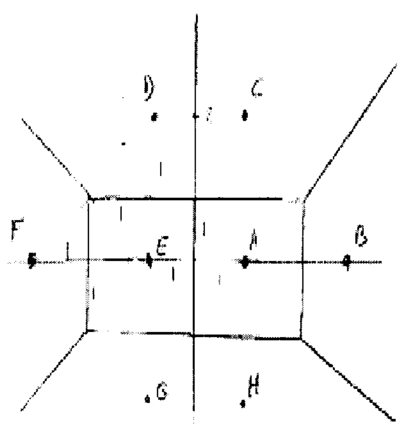
$$\bar{E}_A = \frac{1}{8}(\sqrt{E})^2\, 8 = E$$

$$\bar{E}_B = \frac{1}{8}\left(2\cdot 1^2 + 2\cdot 3^2 + 4(2^2+1^2)\right) = \frac{40}{8} = 5$$

$$E = 5 \Rightarrow \sqrt{E} = \sqrt{5}$$

1a ii





1a iii Each signal has 2 nearest members. The distance is $d_{min,A}$ Hence

$$P_e \leq 2Q\left(\frac{d_{min,A}/2}{\sqrt{\frac{N_0}{2}}}\right) = 2Q\left(\frac{d_{min,A}}{\sqrt{2N_0}}\right) = NNEP$$ (3)

From basic trigonometry $d_{min,A} = 2\sqrt{E}\sin\left(\frac{\pi}{8}\right)$

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1a IV Points F & B have 1 nearest neighbour with distance $2 = d_{min,B}$ so

$$P_e < Q\left(\frac{d_{min,B}/2}{\sqrt{N_0/2}}\right) = 2Q\left(\frac{d_{min,B}}{\sqrt{2N_0}}\right)$$

Points E & A have 4 neighbours with distance $d_{min,B} = 2$

$$P_e < 4Q\left(\frac{d_{min,B}}{\sqrt{2N_0}}\right)$$

Points D, C, G, H have 2 neighbours

$$P_e < 2Q\left(\frac{d_{min,B}}{\sqrt{2N_0}}\right)$$ (3)

For E & A they are NN bounds

——///——

1a V We need, $\frac{d_{min,A}}{\sqrt{2N_0}} = \frac{d_{min,B}}{\sqrt{2N_0}} \Rightarrow d_{min,A} = 2$

So

$$2E\sin\left(\frac{\pi}{8}\right) = 2 \Rightarrow E = 6.828$$

$$10\log_{10}\left(\frac{\overline{E}_A}{\overline{E}_B}\right) = 10\log_{10}\left(\frac{6.828}{5}\right) = 1.3536\,dB$$ (5)

1b i)

$$p(m_1=0) = c_1 \frac{1}{\sqrt{2\pi\sigma^2}} \left(e^{-\frac{(0.5)^2}{2\sigma^2}} + e^{-\frac{(1.5)^2}{2\sigma^2}} + e^{-\frac{(2.5)^2}{2\sigma^2}} + e^{-\frac{(4.5)^2}{2\sigma^2}} \right) \cdot (1-P_1)$$

$$p(m_1=1) = c_1 \frac{1}{\sqrt{2\pi\sigma^2}} \left(e^{-\frac{(6.5)^2}{2\sigma^2}} + e^{-\frac{(8.5)^2}{2\sigma^2}} + e^{-\frac{(10.5)}{2\sigma^2}} + e^{-\frac{(12.5)^2}{2\sigma^2}} \right) \cdot P_1$$

$$p(m_2=0) = c_2 \frac{1}{\sqrt{2\pi\sigma^2}} \left(e^{-\frac{(0.5)^2}{2\sigma^2}} + e^{-\frac{(1.5)^2}{2\sigma^2}} + e^{-\frac{(10.5)^2}{2\sigma^2}} + e^{-\frac{(12.5)^2}{2\sigma^2}} \right) \cdot (1-P_2)$$

$$p(m_2=1) = c_2 \frac{1}{\sqrt{2\pi\sigma^2}} \left(e^{-\frac{(2.5)^2}{2\sigma^2}} + e^{-\frac{(4.5)^2}{2\sigma^2}} + e^{-\frac{(6.5)^2}{2\sigma^2}} + e^{-\frac{(8.5)^2}{2\sigma^2}} \right) \cdot P_2$$

$$p(m_3=0) = c_3 \frac{1}{\sqrt{2\pi\sigma^2}} \left(e^{-\frac{(1.5)^2}{2\sigma^2}} + e^{-\frac{(2.5)^2}{2\sigma^2}} + e^{-\frac{(8.5)^2}{2\sigma^2}} + e^{-\frac{(10.5)^2}{2\sigma^2}} \right) \cdot (1-P_3)$$ (3)

$$p(m_3=1) = c_3 \frac{1}{\sqrt{2\pi\sigma^2}} \left(e^{-\frac{(0.5)^2}{2\sigma^2}} + e^{-\frac{(4.5)^2}{2\sigma^2}} + e^{-\frac{(6.5)^2}{2\sigma^2}} + e^{-\frac{(12.5)^2}{2\sigma^2}} \right) \cdot P_3$$

1b ii)

$$LLR(m_1) = \ln\left[e^{-\frac{[(6.5)^2-(0.5)^2]}{2\sigma^2}} \right] = -\frac{1}{2\sigma^2}[42] \quad \text{Favours } 0$$

$$LLR(m_2) = \ln\left[e^{-\frac{(2.5)^2-(0.5)^2}{2\sigma^2}} \right] = -\frac{1}{2\sigma^2}[6] \quad \text{Favours } 0$$ (3)

$$LLR(m_3) = \ln\left[e^{-\frac{(0.5)^2-(1.5)^2}{2\sigma^2}} \right] = \frac{1}{\sigma^2} \quad \text{Favours } 1$$

1b iii)

The log likelihood ratio when using the dominant terms is (3)

$$LLR = \frac{y}{\sigma^2}(b-a) + \frac{a^2-b^2}{2\sigma^2}$$

where b is the closest 1-bit point in the constellation a a is the closest 0-bit point in the constellation.

The LLR generalises to the form

$$LLR = \mathrm{Re}\left\{ \frac{y^*(b-a)}{\sigma^2} + \frac{\|a\|^2-\|b\|^2}{2\sigma^2} \right\}$$

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MODEL ANSWERs and MARKING SCHEME

First Examiner: Gurcan, M.K. **Paper Code :** E4.04, SC6, ISE4.9

Second Examiner: Stathaki, T **Question** 2 **Page** 4 *out of* 11

Question labels in left margin **Marks allocations in right margin**

2 a Define $g(t) = \cos\left(\frac{\pi t}{2T}\right) \operatorname{rect}\left(\frac{t}{2T}\right)$

$A_m = (-1)^{b_{2m}}$ ~ Bernoulli with $p = 1/2$ (4)

$E(A_m A_n) = \delta_{mn}$ $\quad G(f) = \frac{4T}{\pi} \frac{\cos(2\pi f T)}{1 - 16 f^2 T^2}$

2b

$$E\left[x(t-\Upsilon)\, x(t+\tau-\Upsilon)\right]$$

$$= E\left[\sum_m \sum_n A_m A_n\, g(t - mT - \Upsilon)\, g(t+\tau-mT-\Upsilon)\right]$$

$$= \sum_m \sigma^2 E\left[g(t-mT\,\Upsilon)\, g(t+\tau-mT-\Upsilon)\right]$$

$$= \frac{1}{2T} \sum_m \sigma^2 \int_{t+\tau-mT}^{t+\tau-(m+1)T} g(u-\tau)\, g(u)\, du$$ (6)

$$R_{x(t-\tau)}(\tau) = \frac{\sigma^2}{2T} \int_{-\infty}^{\infty} g(u-\tau)\, g(u)\, du$$

$$= \frac{\sigma^2}{2T} R_{g(t)}(\tau)$$

Auto correlation function of g

2 c

$$S_{x(t-\tau)}(f) = \mathcal{F}\left\{R_{x(t-\tau)}\right\}$$

$$S_g(g) = \mathcal{F}\left(R_g(\tau)\right)$$

$$S_x(f) = \frac{\sigma^2}{2T} S_g(f) = \frac{1}{2T} |G(f)|^2 = \frac{8T}{\pi^2} \left(\frac{\cos(2\pi f T)}{1 - 16 f^2 T^2}\right)^2$$

$$R_x(\tau) = \frac{\sigma^2}{2T} \int_{-\infty}^{\infty} g(u)\, g(u-\tau)\, du$$

$$= \frac{1}{2T} \int_{-\infty}^{\infty} \operatorname{rect}\left(\frac{u}{2T}\right) \cos\left(\frac{\pi u}{2T}\right) \operatorname{rect}\left(\frac{u-\tau}{2T}\right) \cos\left(\frac{\pi(u-\tau)}{2T}\right) du$$ (8)

which can be shown to be

$$R_x(\tau) = \left(\frac{1}{4T}(2T - |\tau|)\cos\left(\frac{\pi\tau}{2T}\right) + \frac{1}{2\pi}\sin\left(\frac{\pi|\tau|}{2T}\right)\right]\mathrm{rect}\left(\frac{\tau}{4T}\right)$$ 7

—///—

2d $R_z(t, t+\tau) = E[z(t)\, z(t-\tau)]$

$$= E\Big\{\big(x(t-\Upsilon)\cos(2\pi f_c t + \theta) - y(t-\Upsilon)\sin(2\pi f_c t + \theta)\big)$$

$$\big(x(t-\tau-\Upsilon)\cos(2\pi f_c(t-\tau)\theta) - y(t-\tau-\Upsilon)\sin(2\pi f_c(t-\tau)+\theta$$

$$= \frac{1}{2}R_x(\tau)\cos 2\pi f_c t + \frac{1}{2}R_y(\tau)\cos 2\pi f_c \tau$$

$$-E\{x(t-\tau)\,y(t-\tau-\Upsilon)\}\,(\cdots\ \cdots\cdots)$$

$$-E(x^2(t-\tau-\Upsilon)\,y(t-\tau)\}\cdots\ -\ \cdots$$

The expectations vanish since

$$E[x(t-\Upsilon)\,y(t-\tau-\Upsilon)] = E\{\sum_m\sum_n A_n B_m\, g(\cdot)h(\cdot)\}$$

$$= \sum_m\sum_n E(A_n)E(B_m)\,E[g(\cdot)h(\cdot)]$$

$$= 0 \qquad \forall\ t, \tau \quad \text{hence}$$

$$R_z(t, t+\tau) = R_z(\tau) = \frac{1}{2}\big(R_x(\tau) + R_y(\tau)\big)\cos 2\pi f_c\tau$$

$$= R_x(\tau)\cos 2\pi f_c \tau$$

$$S_z(f) = \frac{1}{2}\left(S_x(f+f_c) + S_x(f-f_c)\right)$$ 8

where $S_x(f) = \dfrac{8T}{\pi^2}\left(\dfrac{\cos 2\pi f T}{1-16f^2T^2}\right)^2$

3ai $s(t) = \mathrm{rect}\left(\frac{t}{T}\right) \cos\left(\frac{\pi t}{T}\right)$ (5)

$$S(f) = T\,\mathrm{sinc}(Tf) * \left(\delta\left(f-\frac{1}{2T}\right) + \delta\left(f+\frac{1}{2T}\right)\right)$$

$$= \frac{T}{2}\left[\mathrm{sinc}\left[\left(f-\frac{1}{2T}\right)/T\right] + \mathrm{sinc}\left[\left(f+\frac{1}{2T}\right)/T\right]\right]$$

$$= \frac{T}{2\pi}\left[-\frac{\cos(\pi f T)}{fT-1/2} + \frac{\cos(\pi f T)}{fT+1/2}\right]$$

$$= \frac{2T}{\pi}\,\frac{\cos(\pi f T)}{1-4f^2T^2}$$

———////———

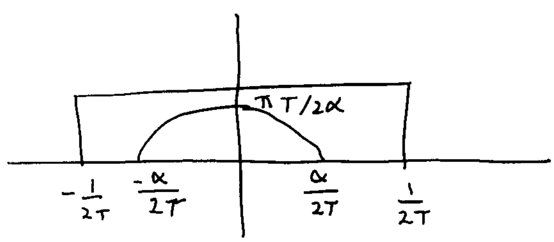
3aii $Y(f) = \mathcal{F}\{x(t)\} \Longrightarrow X(f) = \mathcal{F}\{y(-t)\}$ (5)

we know from previous section

$$\mathrm{rect}\left(\frac{f}{T}\right)\cos\left(\frac{\pi f}{T}\right) = \mathcal{F}\left\{\frac{2T}{\pi}\,\frac{\cos(-\pi t T)}{1-4T^2(-t)^2}\right\}$$

$$= \mathcal{F}\left\{\frac{2T}{\pi}\,\frac{\cos(\pi t T)}{1-4T^2t^2}\right\}$$

$$x(t) = \frac{\cos(\pi\alpha t/T)}{1-(2\alpha t/T)^2} \Longleftrightarrow \frac{\pi T}{2|\alpha|}\,\mathrm{rect}\left(\frac{fT}{\alpha}\right)\cos\left(\frac{\pi f T}{\alpha}\right)$$



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MODEL ANSWERs and MARKING SCHEME

First Examiner: Gurcan, M.K. **Paper Code : E4.04, SC6, ISE4.9**

Second Examiner: Stathaki, T **Question 3 Page 7 *out of 11***

Question labels in left margin **Marks allocations in right margin**

3a iii $x(t)\, y(t) \overset{F}{\Longleftrightarrow} X(f) * Y(f)$

$$X(f) * Y(f) = \begin{cases} \int_{-\frac{\alpha}{2T}}^{f+\frac{1}{2T}} \frac{\pi T^2}{2\alpha} \cos\left(\frac{\pi \tau T}{\alpha}\right) d\tau & \text{if } \left|f+\frac{1}{2T}\right| \leq \frac{\alpha}{2T} \\ \int_{f-\frac{1}{2T}}^{\frac{\alpha}{2T}} \frac{\pi T^2}{2\alpha} \cos\left(\frac{\pi \tau T}{\alpha}\right) d\tau & \text{if } \left|f-\frac{1}{2T}\right| \leq \frac{\alpha}{2T} \\ \int_{-\frac{\alpha}{2T}}^{\frac{\alpha}{2T}} \frac{\pi T^2}{2\alpha} \cos\left(\frac{\pi \tau T}{\alpha}\right) d\tau & \text{if } |f| \leq \frac{1-\alpha}{2T} \\ 0 & \text{otherwise} \end{cases}$$

(5)

—///—

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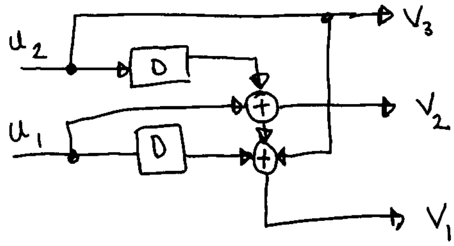
3bi By inspection, we have

$v_{3,k} = u_{2,k}$

$v_{2,k} = u_{2,k-1} + u_{1,k}$

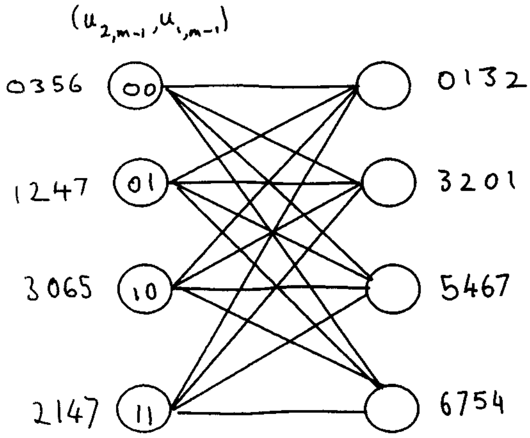
$v_{1,k} = u_{2,k} + u_{2,k-1} + u_{1,k} + u_{1,k-1}$

The convolutional encoder is as shown in figure



4

3bii

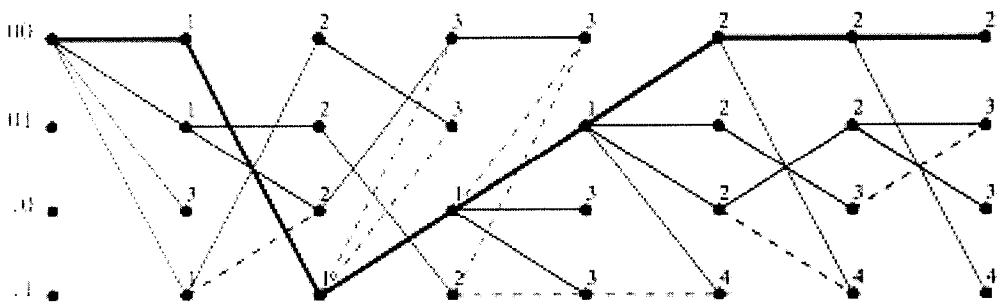


Input 00
01
10
11

Output = (v_3, v_2, v_1)
in octal notation

3

3biii



3

Decoded sequence is {00 11 10 01 00 00 00 ...}

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4a i To maximise the data rate, $R=b/T$, for a set of parallel subchannels when the symbol rate $1/T$ is fixed requires maximisation of achievable $b=\sum_n b_n$ and ε_n. The largest number of bits that can be transmitted over a parallel set of subchannels must maximize the sum.

$$b=\frac{1}{2}\sum_{n=1}^{8}\log_2(1+\varepsilon_n g_n)$$

where g_n represents the subchannel signal-to-noise ratio when the transmitter applies unit energy to that subchannel. (For multitone $g_n=|H_n|^2/\sigma_n^2$.) g_n is a fixed function of the channel, but ε_n can be varied to maximize b, subject to an energy constraint that

$$\sum_{n=1}^{8}\varepsilon_n = 8\bar{\varepsilon}_x$$

Using La Grange multipliers, the cost function to maximize

$$b=\frac{1}{2}\sum_{n=1}^{\infty}\log_2(1+\varepsilon_n g_n)$$

subject to the constraint in

$$\frac{1}{2\ln(2)}\sum_n \ln(1+\varepsilon_n g_n)+\lambda\left(\sum_n \varepsilon_n-8\bar{\varepsilon}_x\right)$$

Differentiating with respect to ε_n produces

$$\frac{1}{2\ln(2)}\frac{1}{\varepsilon_n+\frac{1}{g_n}}=-\frac{\lambda}{g_n}$$

Thus

$$b=\frac{1}{2}\sum_{n=1}^{\infty}\log_2(1+\varepsilon_n g_n)$$ is maximized when

$$\varepsilon_n+\frac{1}{g_n}=\text{Constant}$$

The set of linear equations that has the water-fill distribution as its solution is

$$\varepsilon_1+\frac{1}{g_1}=K$$

$$\varepsilon_2+\frac{1}{g_2}=K$$

$$\vdots \quad = \quad \vdots$$

$$\varepsilon_n+\frac{1}{g_8}=K$$

$$\varepsilon_1+\varepsilon_2+\cdots\ \varepsilon_8=8\bar{\varepsilon}_x$$

(3)

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4a In matrix form, the equations become

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & -1 \\ 0 & 1 & 0 & 0 & \cdots & -1 \\ \vdots & \vdots & \ddots & \ddots & \ddots & -1 \\ 1 & 1 & 1 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} \mathcal{E}_1 \\ \mathcal{E}_2 \\ \vdots \\ \mathcal{E}_8 \\ K \end{bmatrix} = \begin{bmatrix} -\frac{1}{g_1} \\ -1/g_2 \\ \vdots \\ -1/g_8 \\ 8\bar{\mathcal{E}}_x \end{bmatrix}$$

An alternative solution sums the first 8 equations to obtain (3)

$$K = \frac{1}{8}\left[8\bar{\mathcal{E}}_x + \sum_{n=1}^{8}\frac{1}{g_n}\right], \quad \mathcal{E}_n = K - \frac{1}{g_n} \quad \forall\, n = 1, \ldots, 8$$

4a ii First, the subchannels are characterized by

$$g_0 = \frac{1.9^2}{0.181} = 19.94$$

$$g_1 = \frac{1.7558^2}{0.181} = 17.03$$

$$g_2 = \frac{1.3454^2}{0.181} = 10.00$$

$$g_3 = \frac{0.7329^2}{0.181} = 2.968$$

$$g_4 = \frac{0.1^2}{0.181} = 0.0552$$

$$K = \frac{1}{8}\left[8 + \frac{1}{19.94} + \frac{2}{17.03} + \frac{2}{2.968} + \frac{1}{0.0552}\right] = 3.3947$$

(6)

$$\mathcal{E}_4 = 3.3447 - \frac{1}{0.05525} = -14.7 < 0$$

so the subchannel $n=4$ should be eliminated

$$K = \frac{1}{7}\left[8 + \frac{1}{20} + \frac{2}{17} + \frac{2}{10} + \frac{2}{3}\right] = 1.292$$

$\bar{\mathcal{E}}_n$ on the subchannels are 1.24, 1.23, 1.19 ...... 0.96

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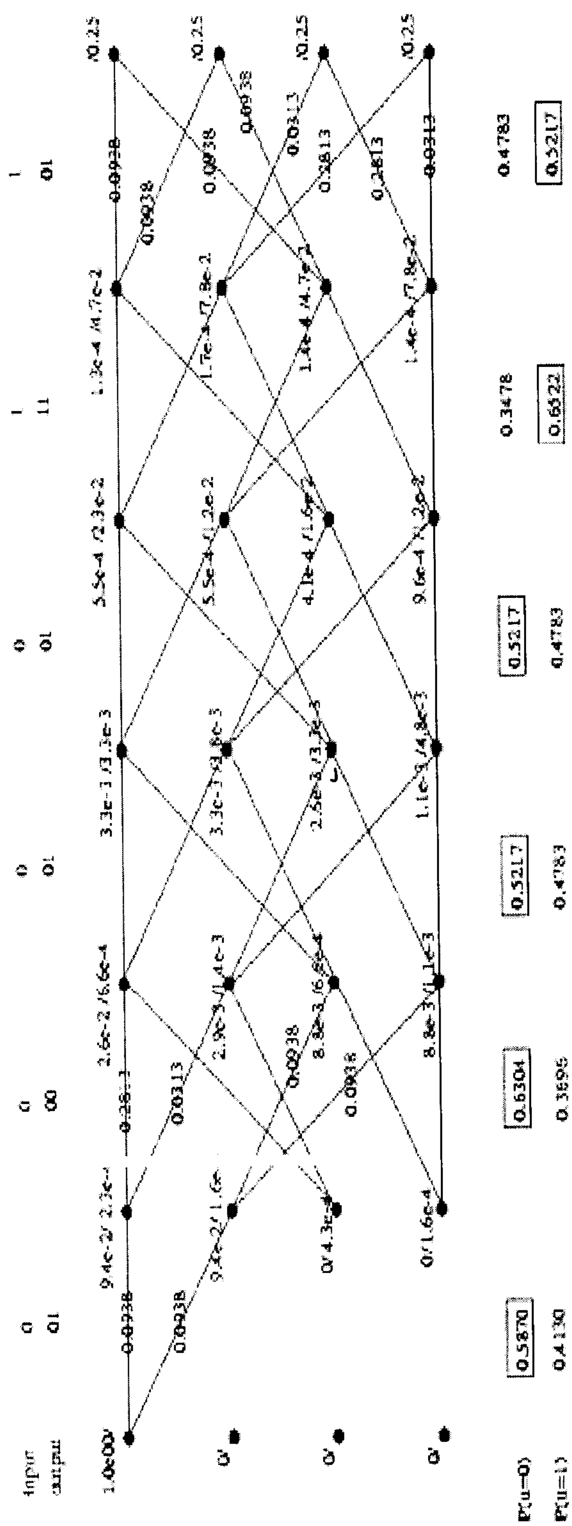
4b



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3