UNIVERSITY OF LONDON [I(2)E 2004]

B.ENG. AND M.ENG. EXAMINATIONS 2004

For Internal Students of Imperial College London

This paper is also taken for the relevant examination for the Associateship of the City & Guilds of London Institute

PART I : MATHEMATICS 2 (ELECTRICAL ENGINEERING)

Thursday 3rd June 2004 10.00 am - 1.00 pm

Answer EIGHT questions.

Corrected Copy

[Before starting, please make sure that the paper is complete; there should be 6 pages, with a total of 10 questions. Ask the invigilator for a replacement if your copy is faulty.]

1. Find the stationary points of

$$f(x,\, y) \;=\; y(x-2)^2 + y^2 - y$$

and determine their nature.

Sketch the contours of the surface $z = f(x,\, y)$.

2. (i) If $z = f(y/x)$, show that

$$x\,\frac{\partial z}{\partial x} \;+\; y\,\frac{\partial z}{\partial y} \;=\; 0\,.$$

(ii) The volume V of a cylinder with radius r and height h is given by $V = \pi r^2 h$.

Find the approximate change in volume when the radius increases from 5cm to 5.02cm and the height decreases from 10cm to 9.9cm.

PLEASE TURN OVER

3. Given the semi-circle

$$x^2 + y^2 = a^2 \qquad (y > 0)$$

and the curve

$$y = \tan x\ ,$$

show graphically that the equation

$$\tan x = +\sqrt{a^2 - x^2}$$

has exactly one root if $0 < a < \pi/2$ but if $\pi < a \leq 3\pi/2$ has three roots, of which two are positive.

If $a = 4$, use the Newton-Raphson method to find both positive roots correct to four decimal places, taking as a first estimate 1.5 for the smaller root and 3.7 for the larger root.

4. Let $\boldsymbol{v}_1 = (1, -2, -1)$, $\boldsymbol{v}_2 = (4, 5, 4)$, $\boldsymbol{v}_3 = (0, 8, 5)$.

(i) Compute $\boldsymbol{v}_2 \times \boldsymbol{v}_3$, and verify that $\boldsymbol{v}_1 \cdot (\boldsymbol{v}_2 \times \boldsymbol{v}_3) = 1$.

(ii) Let $\boldsymbol{w}_1 = \boldsymbol{v}_2 \times \boldsymbol{v}_3$, $\boldsymbol{w}_2 = \boldsymbol{v}_3 \times \boldsymbol{v}_1$, $\boldsymbol{w}_3 = \boldsymbol{v}_1 \times \boldsymbol{v}_2$.

Show that

$$\boldsymbol{v}_i \cdot \boldsymbol{w}_j = \begin{cases} 1 & \text{if } i = j\ , \\ 0 & \text{if } i \neq j\ . \end{cases}$$

(iii) Express each of $\boldsymbol{w}_2 \times \boldsymbol{w}_3$, $\boldsymbol{w}_3 \times \boldsymbol{w}_1$, $\boldsymbol{w}_1 \times \boldsymbol{w}_2$ in terms of $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$.

5. Consider a plane P given by the equation

$$x + y + z = 10$$

and a line L given by

$$(x, y, z) = (-1, -3, 4) + s(1, 0, 0).$$

(i) Find the point of intersection between L and P.

(ii) Find the minimum distance from the point $(1, 0, 0)$ to the plane P.

(iii) Find an equation for the plane Q which contains the line L and is perpendicular to the plane P.

6. Let

$$A = \begin{pmatrix} 0 & -2 & 0 \\ -1 & 1 & 0 \\ 2 & 2 & 2 \end{pmatrix}.$$

(i) Compute A^2 and A^3.

Verify that

$$A^3 - 3A^2 + 4I = 0$$

where

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(ii) Determine scalars b, c, for which

$$A^2 + bA + cI = 0.$$

(iii) Using (ii), or otherwise, determine scalars p, q, for which

$$A^{-1} = pI + qA.$$

PLEASE TURN OVER

7. (i) Show that the substitution $u = x + y$ reduces the differential equation

$$e^y \left(\frac{dy}{dx} + 1\right) = e^{-x}$$

to the form

$$e^u \frac{du}{dx} = 1 .$$

Hence find the solution $y = y(x)$ for which $y(1) = 0$.

(ii) Find the solution of the differential equation

$$\frac{dy}{dx} + y = y^2$$

satisfying $y\,(0) = \frac{1}{2}$.

Hint: *put* $y = \frac{1}{v}$.

8. (i) Solve the differential equations

$$\frac{dy}{dx} = \frac{y}{x} \quad \text{and} \quad \frac{dy}{dx} = -\frac{x}{y} .$$

Show that the two families of solutions are *orthogonal* because whenever two curves from each family intersect, they do so at right angles.

Draw curves in the xy plane to represent these families.

(ii) Solve the differential equation

$$x\frac{dy}{dx} - y = x^2 , \qquad y\,(1) = 2 ,$$

using the integrating factor method, or otherwise.

9. (i) Find the general solution to the differential equation

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = e^x .$$

(ii) Find the general solution to the differential equation

$$\frac{d^2y}{dx^2} + 4y = 5\cos x + \sin 2x .$$

10. Show that the Fourier expansion of the function

$$f(x) = \begin{cases} 1 + (x/\pi) , & -\pi \le x \le 0 , \\ 1 - (x/\pi) , & 0 \le x \le \pi , \end{cases}$$

in the range $-\pi \le x \le \pi$ is

$$f(x) = \frac{1}{2} + \frac{4}{\pi^2}\left(\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \ldots\right) .$$

Sketch $f(x)$ over the range $-3\pi \le x \le 3\pi$.

Use the above result to deduce that

$$\frac{\pi^2}{8} = \sum_{r=1}^{\infty} \frac{1}{(2r-1)^2} .$$

END OF PAPER

M A T H E M A T I C S   D E P A R T M E N T

MATHEMATICAL FORMULAE

1. VECTOR ALGEBRA

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1,\, a_2,\, a_3)$$

Scalar (dot) product: $\mathbf{a}\,.\,\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

Vector (cross) product:

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Scalar triple product:

$$[\mathbf{a},\,\mathbf{b},\,\mathbf{c}] = \mathbf{a}\,.\,\mathbf{b}\times\mathbf{c} = \mathbf{b}\,.\,\mathbf{c}\times\mathbf{a} = \mathbf{c}\,.\,\mathbf{a}\times\mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product: $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{c}\,.\,\mathbf{a})\mathbf{b} - (\mathbf{b}\,.\,\mathbf{a})\mathbf{c}$

2. SERIES

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \ldots \quad (\alpha \text{ arbitrary, } |x|<1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots\,,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n\frac{x^{2n}}{(2n)!} + \ldots\,,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + \ldots\,,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^n\frac{x^{n+1}}{(n+1)} + \ldots (-1 < x \le 1)$$

3. TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin(a+b) = \sin a\,\cos b + \cos a\,\sin b\,;$$

$$\cos(a+b) = \cos a\,\cos b - \sin a\,\sin b\,.$$

$$\cos iz = \cosh z\,; \quad \cosh iz = \cos z\,; \quad \sin iz = i\sinh z\,; \quad \sinh iz = i\sin z\,.$$

4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f\,D^n g + \tbinom{n}{1} Df\,D^{n-1}g + \ldots + \tbinom{n}{r} D^r f\,D^{n-r}g + \ldots + D^n f\,g\,.$$

(b) Taylor's expansion of $f(x)$ about $x = a$:

$$f(a+h) = f(a) + hf'(a) + h^2f''(a)/2! + \ldots + h^nf^{(n)}(a)/n! + \epsilon_n(h)\,,$$

where $\epsilon_n(h) = h^{n+1}f^{(n+1)}(a+\theta h)/(n+1)!, \quad 0<\theta<1\,.$

(c) Taylor's expansion of $f(x,\,y)$ about $(a,\,b)$:

$$f(a+h,\,b+k) = f(a,\,b) + [hf_x + kf_y]_{a.b} + 1/2!\,\left[h^2f_{xx} + 2hkf_{xy} + k^2f_{yy}\right]_{a.b} + \ldots$$

(d) Partial differentiation of $f(x,\,y)$:

i. If $y = y(x)$, then $f = F(x)$, and $\dfrac{dF}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dx}$.

ii. If $x = x(t)$, $y = y(t)$, then $f = F(t)$, and $\dfrac{dF}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$.

iii. If $x = x(u,\,v)$, $y = y(u,\,v)$, then $f = F(u,\,v)$, and

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}\,, \qquad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}\,.$$

(e) Stationary points of $f(x,\,y)$ occur where $f_x = 0$, $f_y = 0$ simultaneously.
Let $(a,\,b)$ be a stationary point: examine $D = [f_{xx}f_{yy} - (f_{xy})^2]_{a.b}$.
If $D>0$ and $f_{xx}(a,\,b)<0$, then $(a,\,b)$ is a maximum;
If $D>0$ and $f_{xx}(a,\,b)>0$, then $(a,\,b)$ is a minimum;
If $D<0$ then $(a,\,b)$ is a saddle-point.

(f) Differential equations:

i. The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating factor $I(x) = \exp[\int P(x)(dx]$, so that $\dfrac{d}{dx}(Iy) = IQ$.

ii. $P(x,\,y)dx + Q(x,\,y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y$.

5. INTEGRAL CALCULUS

(a) An important substitution: $\tan(\theta/2) = t$:
$\sin\theta = 2t/(1+t^2)$, $\cos\theta = (1-t^2)/(1+t^2)$, $d\theta = 2\,dt/(1+t^2)$.

(b) Some indefinite integrals:

$$\int (a^2 - x^2)^{-1/2} dx = \sin^{-1}\left(\frac{x}{a}\right), \quad |x| < a.$$

$$\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1}\left(\frac{x}{a}\right) = \ln\left\{\frac{x}{a} + \left(1 + \frac{x^2}{a^2}\right)^{1/2}\right\}.$$

$$\int (x^2 - a^2)^{-1/2} dx = \cosh^{-1}\left(\frac{x}{a}\right) = \ln\left|\frac{x}{a} + \left(\frac{x^2}{a^2} - 1\right)^{1/2}\right|.$$

$$\int (a^2 + x^2)^{-1} dx = \left(\frac{1}{a}\right)\tan^{-1}\left(\frac{x}{a}\right).$$

6. NUMERICAL METHODS

(a) Approximate solution of an algebraic equation:

If a root of $f(x) = 0$ occurs near $x = a$, take $x_0 = a$ and
$x_{n+1} = x_n - [f(x_n)/f'(x_n)]$, $n = 0, 1, 2 \ldots$

(Newton Raphson method).

(b) Formulae for numerical integration: Write $x_n = x_0 + nh$, $y_n = y(x_n)$.

i. Trapezium rule (1-strip): $\int_{x_0}^{x_1} y(x)dx \approx (h/2)[y_0 + y_1]$.

ii. Simpson's rule (2-strip): $\int_{x_0}^{x_2} y(x)dx \approx (h/3)[y_0 + 4y_1 + y_2]$.

(c) Richardson's extrapolation method: Let $I = \int_a^b f(x)dx$ and let I_1, I_2 be two estimates of I obtained by using Simpson's rule with intervals h and $h/2$.

Then, provided h is small enough,

$$I_2 + (I_2 - I_1)/15,$$

is a better estimate of I.

7. LAPLACE TRANSFORMS

Function	Transform	Function	Transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)dt$	$af(t) + bg(t)$	$aF(s) + bG(s)$
df/dt	$sF(s) - f(0)$	d^2f/dt^2	$s^2F(s) - sf(0) - f'(0)$
$e^{at}f(t)$	$F(s-a)$	$tf(t)$	$-dF(s)/ds$
$(\partial/\partial\alpha)f(t,\alpha)$	$(\partial/\partial\alpha)F(s,\alpha)$	$\int_0^t f(t)dt$	$F(s)/s$
$\int_0^t f(u)g(t-u)du$	$F(s)G(s)$		
1	$1/s$	$t^n (n = 1, 2 \ldots)$	$n!/s^{n+1}$, $(s > 0)$
e^{at}	$1/(s-a)$, $(s > a)$	$\sin\omega t$	$\omega/(s^2+\omega^2)$, $(s > 0)$
$\cos\omega t$	$s/(s^2+\omega^2)$, $(s > 0)$	$H(t-T) = \begin{cases} 0, & t < T \\ 1, & t > T \end{cases}$	e^{-sT}/s, $(s, T > 0)$

8. FOURIER SERIES

If $f(x)$ is periodic of period $2L$, then $f(x+2L) = f(x)$, and

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \text{ where}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad n = 0, 1, 2, \ldots, \text{ and}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1, 2, 3, \ldots$$

Parseval's theorem

$$\frac{1}{L}\int_{-L}^{L} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right).$$

EE
1st yr
Paper 2
2004

MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

2003 – 2004

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E1

PAPER I(2) 2004

QUESTION

SOLUTION 1

$$\frac{\partial f}{\partial x} = 2y(x-2)$$ 1

$$\frac{\partial f}{\partial y} = (x-2)^2 + 2y - 1$$ 1

$$\frac{\partial f}{\partial x} = 0 \iff y = 0 \text{ and/or } x = 2$$ 1

If $x = 2$ $\quad \frac{\partial f}{\partial y} = 0 \iff 2y - 1 = 0 \iff y = 1/2$ 1

If $y = 0$ $\quad \frac{\partial f}{\partial y} = 0 \iff (x-2)^2 - 1 = 0$

$$\iff (x-2)^2 = 1$$

$$\iff x = 1 \text{ or } 3$$ 1

Stationary points $(2, 1/2), (1, 0), (3, 0)$. 1

$$\frac{\partial^2 f}{\partial x^2} = 2y$$ 1

$$\frac{\partial^2 f}{\partial y^2} = 2$$ 1

$$\frac{\partial^2 f}{\partial x \partial y} = 2(x-2)$$ 1

Setter: LUZZATTO Setter's signature: Stefano Luzzatto

Checker: Skorobogatov Checker's signature:

$$D = f_{xx}f_{yy} - (f_{xy})^2 = 4y - 4(x-2)^2$$

$D(2, 1/2) = 2 - 0 = 2 > 0$

$f_{xx}(2, 1/2) = 1 > 0$

$\therefore$ $(2, 1/2)$ is a minimum 1

$D(1,0) = -4 < 0$ 1

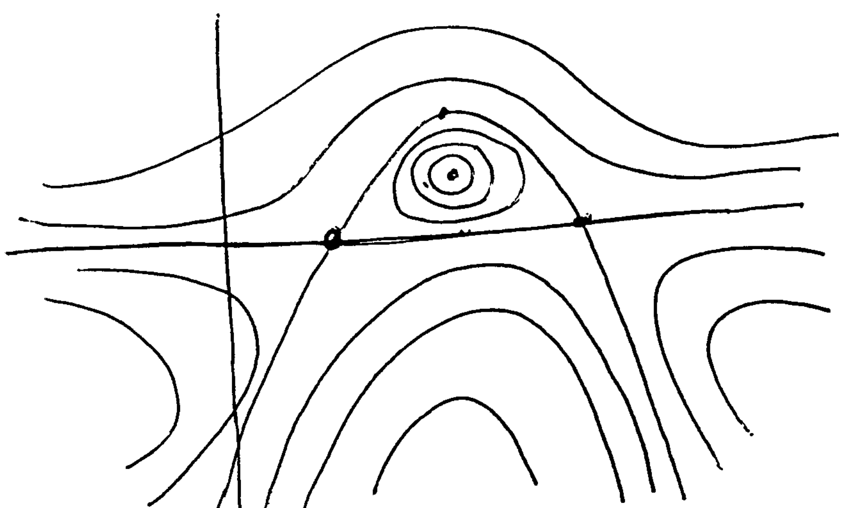
$\therefore$ $(1,0)$ saddle

$D(3,0) = -4 < 0$ 1

$\therefore$ $(3,0)$ saddle

$f(x,y) = 0 \iff y(y + (x-2)^2 - 1) = 0$ 1

$\iff y = 0$ or $y = -(x-2)^2 + 1$



2

i)

$$\frac{\partial z}{\partial x} = f'(y/x) \cdot \frac{\partial}{\partial x}(y/x)$$

$$= f'(y/x)(-y/x^2) = -\frac{y}{x^2} f'(y/x) \qquad 3$$

$$\frac{\partial z}{\partial y} = f'(y/x) \cdot \frac{\partial}{\partial y}(y/x)$$

$$= \frac{1}{x} f'(y/x) \qquad 3$$

$$\therefore\ x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = x\left(-\frac{y}{x^2}\right) f'(y/x) + y(1/x)f'(y/x) \qquad 2$$

$$= -y/x\, f'(y/x) + y/x\, f'(y/x) = 0$$

ii)

$$\frac{\partial V}{\partial r} = 2\pi r h = 100\pi \qquad 1$$

$$\frac{\partial V}{\partial h} = \pi r^2 = 25\pi \qquad 1$$

$$\delta r = 0.2, \quad \delta h = -0.1 \qquad 1$$

$$\delta V \simeq \frac{\partial V}{\partial r}\delta r + \frac{\partial V}{\partial h}\delta h \qquad 2$$

$$= 100\pi(0.2) + 25\pi(-0.1)$$

$$= 17.5\pi \approx 54.98\ \text{cm}^3 \qquad 2$$

Setter : LUTHSTONE

Setter's signature : [signature]

Checker : Skorobogatov

Checker's signature : [signature]

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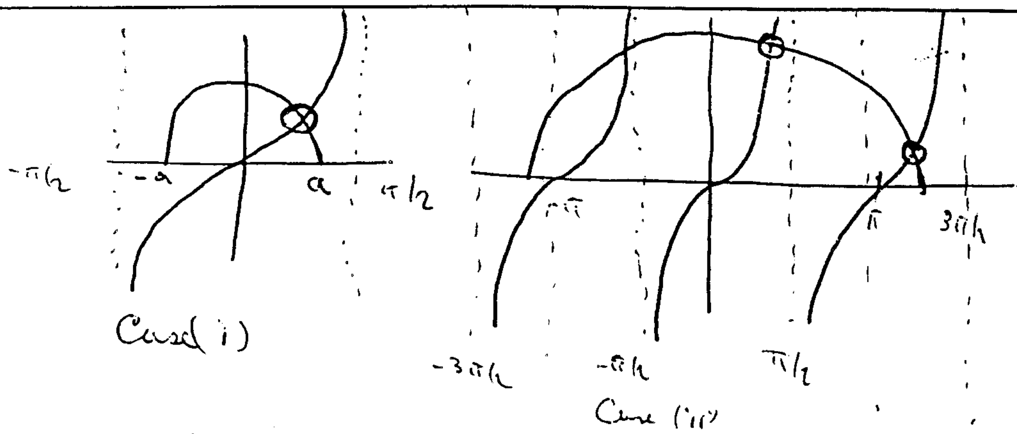

Can see in Case (i) $0<a<\pi/2$ so only one root between 0 and $\pi/2$ (2)

in Case (ii) $\pi < a < 3\pi/2$ so there are 2 positive roots (3)

So $0 = f(x) = \tan x - \sqrt{a^2-x^2}$

$$f'(x) = \sec^2 x + \frac{x}{\sqrt{a^2-x^2}}$$

So Newton Raphson is

$$x_{n+1} = x_n - \frac{\left[\tan x_n - \sqrt{a^2-x_n^2}\right]}{\sec^2 x_n + x_n/\sqrt{a^2-x_n^2}}$$ (4)

With $x_0 = 1.5$ get

$x_1 = 1.44810$ $\quad x_4 = 1.31342$

$x_2 = 1.38286$ $\quad x_5 = 1.31209$

$x_3 = 1.33100$ $\quad x_6 = 1.31208$ (3)

$\therefore x = 1.3121$ to 4 dp.

With $x_0 = 3.7$ $\quad x_3 = 3.89054$

$x_1 = 3.93404$ $\quad x_4 = 3.89049$

$x_2 = 3.89510$ $\quad x_5 = 3.89049$

So $x = 3.8905$ to 4 dp (3)

Setter : J.R. CASH

Checker : C.J. RIDLER-ROWE

Setter's signature : J R Cash

Checker's signature : C J Ridler-Rowe

(i) $\underline{v}_2 \times \underline{v}_3 = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 4 & 5 & 4 \\ 0 & 8 & 5 \end{vmatrix} = (-7, -20, 32)$ 3

$\underline{v}_1 \cdot (\underline{v}_2 \times \underline{v}_3) = -7 + 40 - 32 = 1.$ 2

(ii) From above, $\underline{v}_1 \cdot \underline{w}_1 = 1$.

$\underline{v}_2 \cdot \underline{w}_2 = \underline{v}_2 \cdot (\underline{v}_3 \times \underline{v}_1) = \underline{v}_1 \cdot (\underline{v}_2 \times \underline{v}_3) = 1.$

$\underline{v}_3 \cdot \underline{w}_3 = \underline{v}_3 \cdot (\underline{v}_1 \times \underline{v}_2) = 1$ similarly. 3

If $i \neq j$ then $\underline{w}_j$ is the cross product of $\underline{v}_i$ with something, so the triple product $\underline{v}_i \cdot \underline{w}_j$ is zero. 3

(iii) $\underline{w}_2 \times \underline{w}_3 = (\underline{v}_3 \times \underline{v}_1) \times (\underline{v}_1 \times \underline{v}_2)$

$= ((\underline{v}_3 \times \underline{v}_1) \cdot \underline{v}_2)\underline{v}_1 - \cancel{((\underline{v}_3 \times \underline{v}_1) \cdot \underline{v}_1)\underline{v}_2}$

$= 1 . \underline{v}_1$

Similarly,

$\underline{w}_3 \times \underline{w}_1 = (\underline{v}_1 \times \underline{v}_2) \times (\underline{v}_2 \times \underline{v}_3)$

$= ((\underline{v}_1 \times \underline{v}_2) \cdot \underline{v}_3)\, \underline{v}_2 = \underline{v}_2$ 4

$\underline{w}_1 \times \underline{w}_2 = (\underline{v}_2 \times \underline{v}_3) \times (\underline{v}_3 \times \underline{v}_1)$

$= ((\underline{v}_2 \times \underline{v}_3) \cdot \underline{v}_1)\, \underline{v}_3 = \underline{v}_3 .$

15

Setter : S. Salamon

Checker: S Reid

Setter's signature :

Checker's signature :

i) the point of intersection satisfies

$-1-3+4+s=10$. Hence $s=10$ 3

and the point of intersection is

$(9,-3,4)$.

ii) The line perpendicular to P through

$(1,0,0)$ is given by

$(x,y,z) = (1,0,0) + s(1,1,1)$. 5

It intersects the plane for $s=3$.

Hence the distance is $3\cdot\sqrt{3}$.

iii) The plane Q is characterized by

$ax + by + cy = d$. We have $a\cdot 1 = 0$

and $a+b+c=0$ (orthogonality). Hence 7

we have $a=0$ and we may set

$b=1$, $c=-1$. Since $(-1,-3,4)$

is in the plane, $d=-7$ and

Q is given by

$$y - z = -7.$$ 15

E6

PAPER 2

QUESTION

SOLUTION 7

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(i) $A^2 = \begin{pmatrix} 2 & -2 & 0 \\ -1 & 3 & 0 \\ 2 & 2 & 4 \end{pmatrix}$. $A^3 = \begin{pmatrix} 2 & -6 & 0 \\ -3 & 5 & 0 \\ 6 & 6 & 8 \end{pmatrix}$. 2

$A^3 - 3A^2 = \begin{pmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{pmatrix}$, so 2

relation is valid.

(ii) To eliminate off-diagonal entries we must take $b = -1$, giving

$$A^2 - A = 2I.$$ 4

Thus $c = -2$.

(iii) Multiplying (ii) by A^{-1},

$$A + bI + cA^{-1} = 0$$

So $A^{-1} = -\frac{1}{c}(A + bI)$, 5

and $p = -b/c = -1/2$, $q = -1/c = 1/2$.

15

Setter: S. Salaman
Checker: S Reid

Setter's signature:
Checker's signature:

(i) We have $\frac{du}{dx} = 1 + \frac{dy}{dx}$, so

$$e^{u-x}\frac{du}{dx} = e^{-x}$$

$$\Rightarrow e^{u}\frac{du}{dx} = 1.$$ 2

Thus $\int e^{u}du = \int dx + c$

$$\Rightarrow e^{u} = x + c$$

$$\Rightarrow u = \ln(x+c)$$

$$\Rightarrow y = \ln(x+c) - x$$ 3

Also $0 = \ln(1+c) - 1 \Rightarrow 1+c = e$

So solution is

$$y = \ln(x+e-1) - x.$$ 2

(ii) We have $\frac{dy}{dx} = -\frac{1}{v^2}\frac{dv}{dx}$, so

$$-\frac{1}{v^2}\frac{dv}{dx} + \frac{1}{v} = \frac{1}{v^2}$$

$$\Rightarrow \frac{dv}{dx} - v = -1,$$ 3

a linear equation with integrating factor $e^{-\int dx} = e^{-x}$

$$\Rightarrow \frac{d}{dx}(e^{-x}v) = -e^{-x}$$

$$\Rightarrow e^{-x} v = e^{-x} + c$$

$$\Rightarrow v = 1 + ce^{x}$$

$$\Rightarrow y = \frac{1}{1+ce^{x}}$$ 3

Also

$$\frac{1}{2} = \frac{1}{1+c} \Rightarrow c = 1.$$

So solution is

$$y = \frac{1}{1+e^{x}}.$$ 2

15

A10 a) From $\frac{dy}{dx} = y/x$ we find that

$$\int \frac{1}{y} dy = \int \frac{1}{x} dx + c,$$

for some constant c and therefore

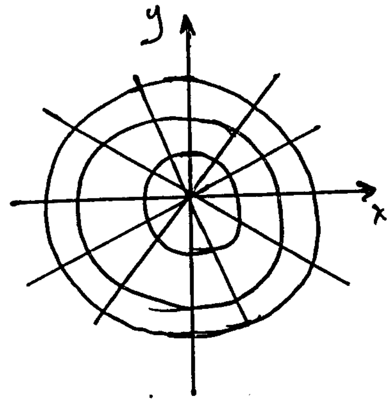
$$\ln|y| = \ln|x| + c.$$

Hence there is a constant $A' \in \mathbb{R}$ such that $|y| = A'|x|$, in other words

$$y = Ax$$

satisfies the given ODE for any $A \in \mathbb{R}$.
From $\frac{dy}{dx} = -x/y$ we find that

$$\int y dy = -\int x dx + c,$$



so that

$$x^2 + y^2 = B$$

for some constant B.
The two families of curves are straight lines through the origin and circles centred on the origin, and therefore they intersect orthogonally. 8

b) Given

$$\frac{dy}{dx} - \frac{y}{x} = x,$$

we find that an integrating factor I is given by

$$I = \exp - \int \frac{1}{x} dx = 1/x, \qquad x > 0.$$

Using

$$x = \frac{dy}{dx} - \frac{y}{x} = x \frac{d}{dx}(y/x)$$

we obtain

$$1 = \frac{d}{dx}(y/x) \Longrightarrow y(x) = x^2 + cx,$$

for some $c \in \mathbb{R}$. Using the given boundary condition we obtain $c = 1$ so that

$$y(x) = x^2 + x.$$

7

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1) $y'' + 6y' + 9y = e^x$

C.F satisfies $y'' + 6y' + 9y = 0$ (2)

Try $y = Ae^{mx} \Rightarrow m^2 + 6m + 9 = 0$

$\Rightarrow (m+3)^2 = 0$ (2)

$\therefore y = (A + Bx)e^{-3x}$ (1)

For the P.I. try $y = \bar{A}e^x \Rightarrow (\bar{A} + 6\bar{A} + 9\bar{A})e^x = e^x$

$\Rightarrow \bar{A} = \frac{1}{16}$ (1)

$\therefore$ General Solution is

$y = (A + Bx)e^{-3x} + \frac{1}{16}e^x$.

2) $\frac{d^2y}{dx^2} + 4y = 5\cos x + \sin 2x$

C.F. is $y = A\sin 2x + B\cos 2x$ (2)

First find P.I. for the cos term $\Rightarrow y'' + 4y = 5\cos x$

Try $y = C\cos x + D\sin x \Rightarrow y' = -C\sin x + D\cos x \Rightarrow y'' = -C\cos x - D\sin x$ (1)

$\therefore -C\cos x - D\sin x + 4C\cos x + 4D\sin x = 5\cos x$

$\Rightarrow C = 5/3,\ D = 0 \quad \therefore y = \frac{5}{3}\cos x$ is P.I for "cos term" (2)

For sin term try $y = Ax\sin 2x + Bx\cos 2x$ (2)

$y' = A\sin 2x + 2Ax\cos 2x + B\cos 2x - 2Bx\sin 2x$

$y'' = 2A\cos 2x + 2A\cos 2x - 4Ax\sin 2x - 2B\sin 2x$
$- 2B\sin 2x - 4Bx\cos 2x$ (1)

$\Rightarrow B = -1/4,\ A = 0$

$\therefore y = A\sin 2x + B\cos 2x + \frac{5}{3}\cos x - \frac{1}{4}x\cos 2x$, (1) E10

PAPER 2

QUESTION

SOLUTION Q4/

Given $f(x) = \begin{cases} 1+(x/\pi), & -\pi \le x \le 0 \\ 1-(x/\pi), & 0 \le x \le \pi \end{cases}$

Observe that $f(x)$ is <u>even</u> $\Rightarrow b_n = 0 \;\forall n$. 2

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{x}{\pi}\right)dx = \frac{2}{\pi}\left(\pi - \frac{\pi}{2}\right)$

$= 1$ 2

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi}(1-x/\pi)\cos nx\,dx$

$= -\frac{2}{\pi^2}\int_0^{\pi} x\cos nx\,dx$

$= -\frac{2}{\pi^2}\left(x\frac{\sin nx}{n}\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin nx\right)$

$= \frac{2}{n\pi^2}\left(-\frac{\cos nx}{n}\right)\Big|_0^{\pi} = \frac{2}{n^2\pi^2}\left(1-(-1)^n\right)$

ie, $a_n = 0$, n even

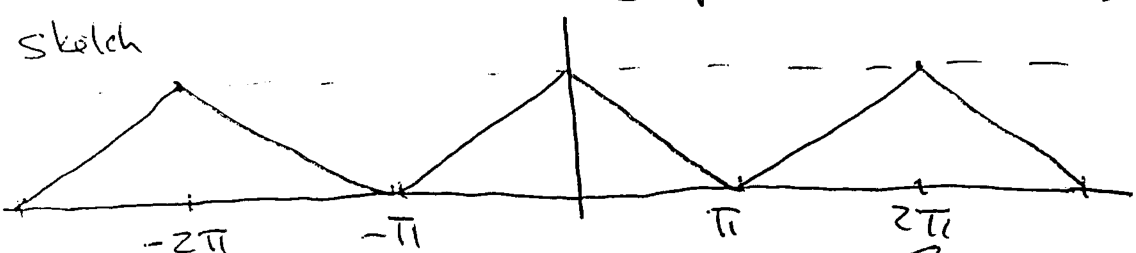
$= \frac{4}{n^2\pi^2}$, n odd 4

Hence $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + b_n\sin nx$

$= \frac{a_0}{2} + \frac{4}{\pi^2}\sum_{\substack{n=1\\ n\text{ odd}}}^{\infty}\frac{\cos nx}{n^2}$

$= \frac{1}{2} + \frac{4}{\pi^2}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \cdots\right)$ 1

Sketch



2

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Since $f(x)$ is continuous, series converges to $f(x)$ $\forall x$. Put $x = 0$ 1

$$\Rightarrow f(0) = 1 = \frac{1}{2} + \frac{4}{\pi^2} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{\cos(n.0)}{n^2}$$

$$\text{or} \quad \frac{\pi^2}{8} = \sum_{n \text{ odd}} \frac{1}{n^2}, \quad \cos(n0) = 1$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \pi^2/8 \quad \text{QED}$$ 3

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