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IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2009

MSc and EEE PART III/IV: MEng, BEng.and ACGI

Corrected Copy

**OPTOELECTRONICS**

Tuesday, 5 May 10:00 am

Time allowed: 3:00 hours

**There are SIX questions on this paper.**

**Answer FOUR questions.**

*All questions carry equal marks*

**Any special instructions for invigilators and information for candidates are on page 1.**

Examiners responsible First Marker(s) : R.R.A. Syms
Second Marker(s) : W.T. Pike

Fundamental constants

$e = 1.6 \times 10^{-19}$ C

$m_e = 9.1 \times 10^{-31}$ kg

$\varepsilon_0 = 8.85 \times 10^{-12}$ F/m

$\mu_0 = 4\pi \times 10^{-7}$ m kg/C$^2$

$c = 3 \times 10^8$ m/s

$h = 6.62 \times 10^{-34}$ Js

Maxwell's equations – integral form

$\iint_A \underline{\mathbf{D}} \cdot d\underline{a} = \iiint_V \rho \, dv$

$\iint_A \underline{\mathbf{B}} \cdot d\underline{a} = 0$

$\int_L \underline{\mathbf{E}} \cdot d\underline{L} = -\iint_A \partial\underline{\mathbf{B}}/\partial t \cdot d\underline{a}$

$\int_L \underline{\mathbf{H}} \cdot d\underline{L} = \iint_A [\underline{\mathbf{J}} + \partial\underline{\mathbf{D}}/\partial t] \cdot d\underline{a}$

Maxwell's equations – differential form

$\mathrm{div}(\underline{\mathbf{D}}) = \rho$

$\mathrm{div}(\underline{\mathbf{B}}) = 0$

$\mathrm{curl}(\underline{\mathbf{E}}) = -\partial\underline{\mathbf{B}}/\partial t$

$\mathrm{curl}(\underline{\mathbf{H}}) = \underline{\mathbf{J}} + \partial\underline{\mathbf{D}}/\partial t$

Material equations

$\underline{\mathbf{J}} = \sigma \, \underline{\mathbf{E}}$

$\underline{\mathbf{D}} = \varepsilon \, \underline{\mathbf{E}}$

$\underline{\mathbf{B}} = \mu \, \underline{\mathbf{H}}$

Vector calculus (Cartesian co-ordinates)

$\mathrm{grad}(\phi) = \partial\phi/\partial x \, \underline{i} + \partial\phi/\partial y \, \underline{j} + \partial\phi/\partial z \, \underline{k}$

$\mathrm{div}(\underline{\mathbf{F}}) = \partial\mathbf{F}_x/\partial x + \partial\mathbf{F}_y/\partial y + \partial\mathbf{F}_z/\partial z$

$\mathrm{curl}(\underline{\mathbf{F}}) = \underline{i} \, \{\partial\mathbf{F}_z/\partial y - \partial\mathbf{F}_y/\partial z\} + \underline{j} \, \{\partial\mathbf{F}_x/\partial z - \partial\mathbf{F}_z/\partial x\} + \underline{k} \, \{\partial\mathbf{F}_y/\partial x - \partial\mathbf{F}_x/\partial y\}$

$\mathrm{curl}\,\{\mathrm{curl}(\underline{\mathbf{F}})\} = \mathrm{grad}\,\{\mathrm{div}(\underline{\mathbf{F}})\} - \nabla^2\underline{\mathbf{F}}$

$\iint_A \underline{\mathbf{F}} \cdot d\underline{a} = \iiint_V \mathrm{div}(\underline{\mathbf{F}}) \, dv$

$\int_L \underline{\mathbf{F}} \cdot d\underline{L} = \iint_A \mathrm{curl}(\underline{\mathbf{F}}) \cdot d\underline{a}$

1. Using the information given on the previous page, explain:

a) Which of the integral forms of Maxwell's equations represent Gauss' Law and Faraday's law, and describe their meaning using words and diagrams.

**[10]**

b) How the differential forms of these two equations may be derived from the integral forms.

**[4]**

c) Maxwell's modification to Ampere's Law, and his justification for the change.

**[6]**

2. Figure 1 below shows a plane wave incident on an interface between dielectric media.

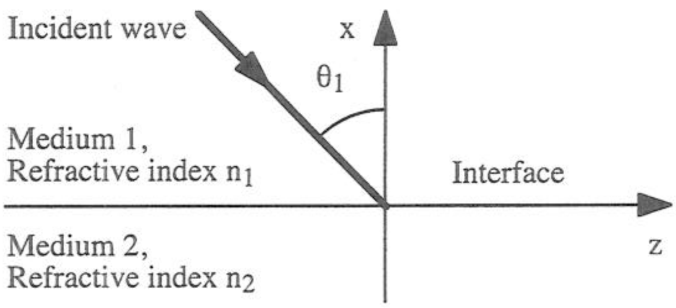



Figure 1.

a) Sketch the ray directions of any additional waves arising, and explain how they are calculated. Explain the phenomenon of total internal reflection, and state the conditions under which it occurs.

**[8]**

b) The critical angle for an interface between two dielectric media is 41.81°. If one medium is air, what is the refractive index of the other medium? Find the power reflection coefficient, for normal incidence from the high-index side of the interface.

**[4]**

c) Estimate the external efficiency of a LED formed in a semiconductor with refractive index $n = 3.5$.

**[8]**

3. Figure 2a shows a symmetric slab waveguide formed from three layers of dielectric. Figure 2b shows a Y-junction formed by tracks of similar guide. Explain:

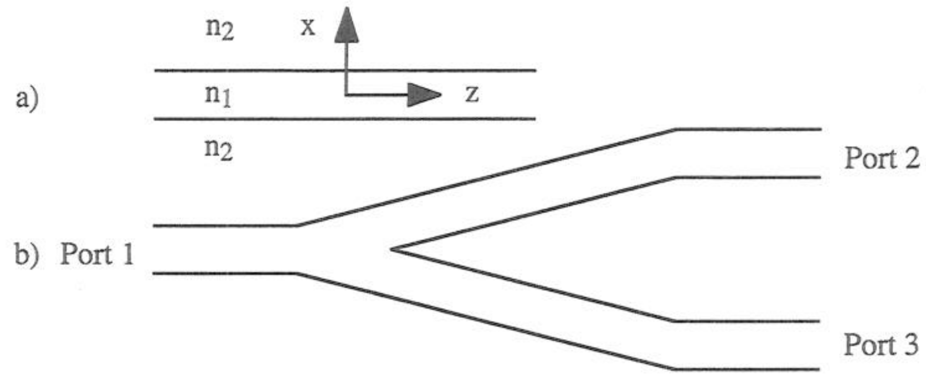



Figure 2.

a) The range of modes supported by the slab waveguide.

**[6]**

b) Operation of the junction for inputs to i) Port 1 and ii) Port 2 in terms of these modes.

**[14]**

4. Figure 3 shows a radiative star, which consists of a symmetric array of 2N+1 input and 2N+1 output channel guides separated by a planar guide with propagation constant $\beta$ and curved boundaries of radius R. Adjacent ports are separated by an angle $\Delta\theta$.

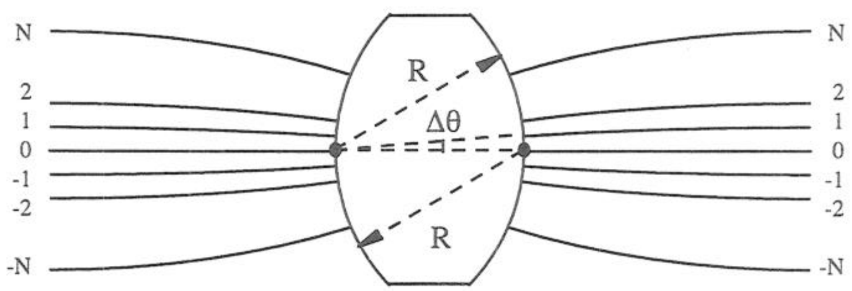



Figure 3.

a) Find the transfer function relating an input to the $p^{th}$ port to an output from the $q^{th}$ port. Show that the ratio of the amplitudes of adjacent output ports is $\exp\{jp\beta R\Delta\theta^2\}$.

**[10]**

b) Sketch an arrayed waveguide grating multiplexer, and find the outputs for an input to Port 0. Show that the wavelengths for peak transmission vary linearly across the array.

**[10]**

5. A photodiode is to be constructed from silicon, whose energy gap is 1.14 eV.

a) Explain the operation of the photodiode in terms of a band diagram.

[6]

b) Sketch the ideal and actual spectral variations of responsivity, explaining any differences.

[7]

c) Estimate the peak responsivity, and the wavelength at which it occurs.

[7]

6. Figure 4 shows a lumped element model of a Fabry-Perot semiconductor laser.

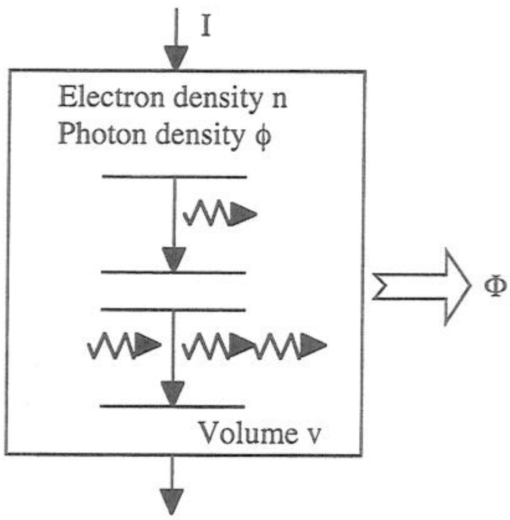



Figure 5.

a) What processes are being modelled? Write down the separate rate equations for electrons and photons, identifying each term with one of the processes shown.

[8]

b) What determines the photon lifetime, in i) an LED and ii) a laser? An InGaAlAs laser with a refractive index of 3.5 has a photon lifetime of 2.5 ps. How long is the cavity?

[6]

c) The laser above has a threshold current of 16.8 mA. Assuming that the stripe cross-section measures 3 μm x 0.1 μm, calculate the electron density during lasing. Assuming also that InGaAlAs has an electron lifetime of 1 ns and a gain constant of $10^{-12}$ $m^3/s$, estimate the electron density needed to achieve transparency.

[6]

Marker - April 09

EJ.12

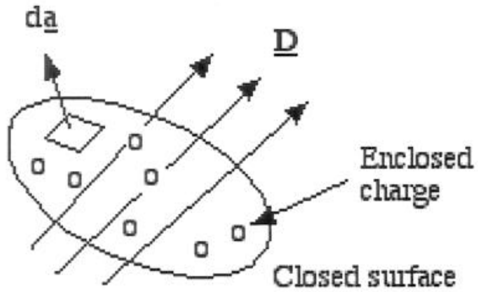
SolJ.

Optoelectronics 2009 - Solutions

1. a) Integral forms of Guass' Law and Faraday's Law are:

Gauss' Law: $\iint_A \underline{\mathbf{D}} \cdot d\underline{a} = \iiint_V \rho \, dv$ **[1]**

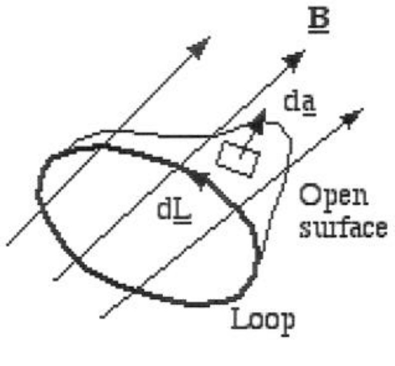
Electric flux flowing out of a closed surface (LHS) is equal to the electric charge enclosed inside the volume (RHS). **[2]**




**[2]**

Faraday's Law: $\int_L \underline{\mathbf{E}} \cdot d\underline{L} = -\iint_A \partial\underline{\mathbf{B}}/\partial t \cdot d\underline{a}$ **[1]**

EMF induced around a loop (LHS) is equal and opposite to the time derivative of the magnetic flux passing through the loop (RHS). **[2]**




**[2]**

b) Transformation to differential forms:

Gauss' Law $\iint_A \underline{\mathbf{D}} \cdot d\underline{a} = \iiint_V \rho \, dv$

Gauss' Theorem $\iint_A \underline{\mathbf{F}} \cdot d\underline{a} = \iiint_V \mathrm{div}(\underline{\mathbf{F}}) \, dv$

Applying to Gauss' Law $\iint_A \underline{\mathbf{D}} \cdot d\underline{a} = \iiint_V \mathrm{div}(\underline{\mathbf{D}}) \, dv = \iiint_V \rho \, dv$

Comparing integrands $\mathrm{div}(\underline{\mathbf{D}}) = \rho$

**[2]**

Faraday's Law $\int_L \underline{\mathbf{E}} \cdot d\underline{L} = -\iint_A \partial\underline{\mathbf{B}}/\partial t \cdot d\underline{a}$

Stokes' Theorem $\int_L \underline{\mathbf{F}} \cdot d\underline{L} = \iint_A \mathrm{curl}(\underline{\mathbf{F}}) \cdot d\underline{a}$

Applying to Faraday's Law $\int_L \underline{\mathbf{E}} \cdot d\underline{L} = \iint_A \text{curl}(\underline{\mathbf{E}}) \cdot d\underline{a} = -\iint_A \partial\underline{\mathbf{B}}/\partial t \cdot d\underline{a}$

Comparing integrands $\text{curl}(\underline{\mathbf{E}}) = -\partial\underline{\mathbf{B}}/\partial t$

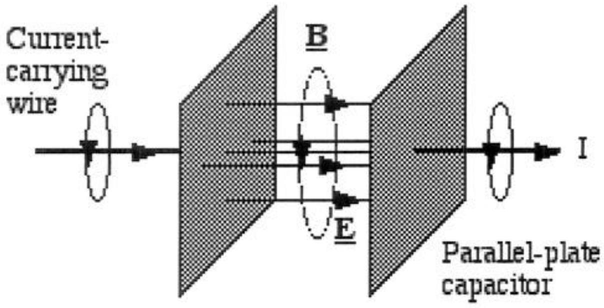
**[2]**

c) Ampere's Law $\int_L \underline{\mathbf{H}} \cdot d\underline{L} = \iint_A \underline{\mathbf{J}} \cdot d\underline{a}$

Maxwell's modification $\int_L \underline{\mathbf{H}} \cdot d\underline{L} = \iint_A [\underline{\mathbf{J}} + \partial\underline{\mathbf{D}}/\partial t] \cdot d\underline{a}$ **[2]**

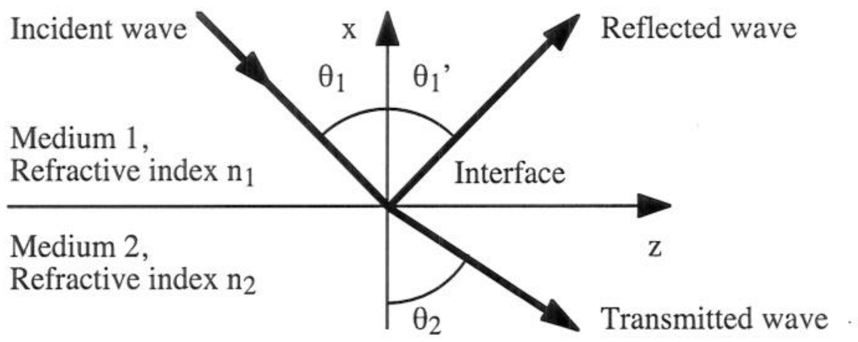
Maxwell's justification was based on current continuity in a circuit involving a parallel plate capacitor. If A is the area of each plate, and Q is the charge on it, then the electric field E between the plates is $E = Q/\varepsilon A$. As the charge varies, the electric field changes, so that $\varepsilon\, dE/dt = I/A$ is effectively a current density. Maxwell therefore defined a vector displacement current density $\underline{\mathbf{J}}_D$ as $\underline{\mathbf{J}}_D = \varepsilon\, \partial\underline{\mathbf{E}}/\partial t = \partial\underline{\mathbf{D}}/\partial t$

This displacement current must be added into any calculation involving the 'normal' conduction current, so the total current density is $\underline{\mathbf{J}}_T = \underline{\mathbf{J}} + \partial\underline{\mathbf{D}}/\partial t$.




**[4]**

2. a) Generally a plane wave incident on a boundary between two dielectric media will give rise to reflected and refracted waves as shown below.




**[2]**

The reflected wave direction is found from Alhazen's Law, $\theta_1 = \theta_1$'

The transmitted wave direction is found from Snell's Law, $n_1 \sin(\theta_1) = n_2 \sin(\theta_2)$ **[2]**

Rearranging, we get $\theta_2 = \sin^{-1}\{(n_1/n_2) \sin(\theta_1)\}$

When the argument of the inverse sine above is greater than unity there is no real solution for $\theta_2$, so a propagating transmitted wave cannot arise and total internal reflection occurs. The onset of TIR requires $\theta_1 \geq \sin^{-1}(n_2/n_1)$ and hence that $n_1 > n_2$. Incidence must therefore be from the high index side of the interface. **[4]**

b) If the critical angle is $\theta_{1c} = 41.81°$, then $n_2/n_1 = \sin(41.81°) = 0.666 = 1/1.5$. If the low index medium is air ($n_2 = 1$) the refractive index of medium 1 must be $n_1 = 1.5$. **[2]**

The amplitude reflection coefficient for normal incidence at an interface between two media is $R = (n_1 - n_2) / (n_1 + n_2) = (1 - n_2/n_1) / (1 + n_2/n_1)$.

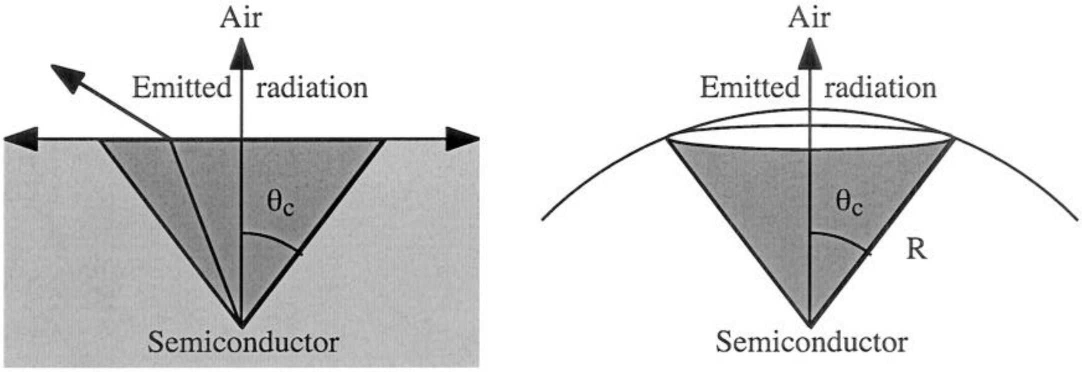
In this case we obtain $R = (1 - 0.666) / (1 + 0.666) = 0.2$.

The power reflection coefficient is then $P_R = 0.2^2 = 0.04$ **[4]**

c) Only internally generated light striking the interface between the emitting semiconductor and air (the usual external medium) at less than the critical angle can escape.The fraction of isotropically generated emission lying inside the shaded cone is:

$F = \pi(R\theta_c)^2/4\pi R^2 = \theta_c^2/4$

The critical angle is $\theta_C = \sin^{-1}(n_2/n_1) = \sin^{-1}(1/n)$ where n is the refractive index of the semiconductor. If n is large enough, we obtain $\theta_C \approx 1/n$. Hence $F = 1/4n^2$




**[4]**

The total emission escaping from the semiconductor is this fraction, multiplied by a factor 1 - $P_R$ that describes the transmitted power. If n is large enough, the value of $P_R$ at normal incidence may be used. Hence:

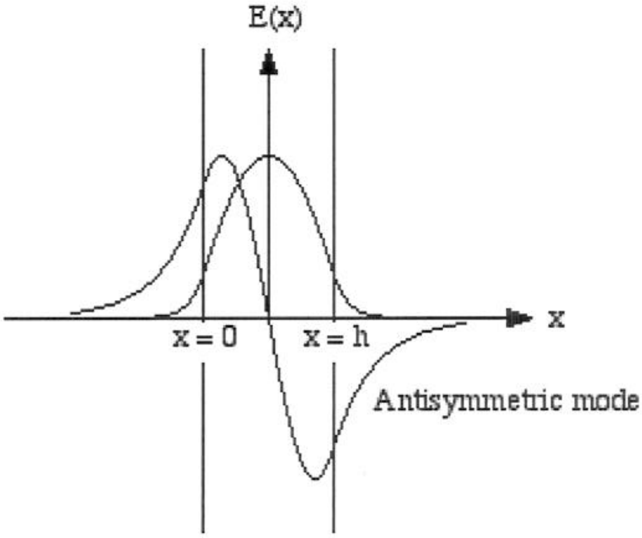
$1 - P_R = 1 - \{(n_1 - n_2) / (n_1 + n_2)\}^2 = 4n_1n_2/(n_1 + n_2)^2$

If air is the external medium, we then obtain $1 - P_R = 4n/(1 + n)^2$

Hence, the external efficiency is $\eta_E = F(1 - P_R) = 1/n(1 + n)^2$ **[3]**

If $n = 3.5$, we then obtain $\eta_E = 0.141$ or 1.4% **[1]**

3. a) The symmetric slab supports guided and radiation modes. Each is a field structure that is unaltered apart from a phase change as it propagates along the guide. For TE modes, the electric field may be written as $\underline{E}(x, z) = E_T(x) \exp(-j\beta z)\, \underline{j}$. The transverse variation of the field is described by the function $E_T(x)$. Modes are guided by total internal reflection at the two interfaces. For guided modes, $E_T$ varies sinusoidally inside the guide and decays exponentially outside, with a smooth junction at the interfaces. Guided modes are either symmetric or antisymmetric. The lowest-order mode is symmetric, and has no cutoff. The second mode is antisymmetric, and does have a cutoff, when total internal reflection breaks down and the guide radiates. The third is symmetric, the fourth is symmetric, etc.



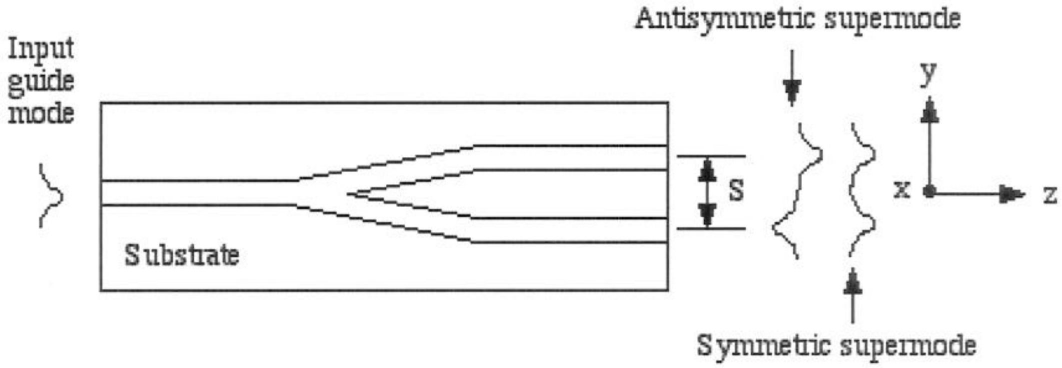

**[6]**

b) The Y-junction consists of a single-moded input guide, which is gradually separated by a forked transition region into two similar output guides. At the left-hand end, the structure supports a characteristic mode with a transverse field distribution $E_T(x)$. At the right-hand end, the device consists of two single-moded guides, separated by a distance S. If the transverse field patterns of the upper and lower guides are $E_{RU}$ and $E_{RL}$, we may write:

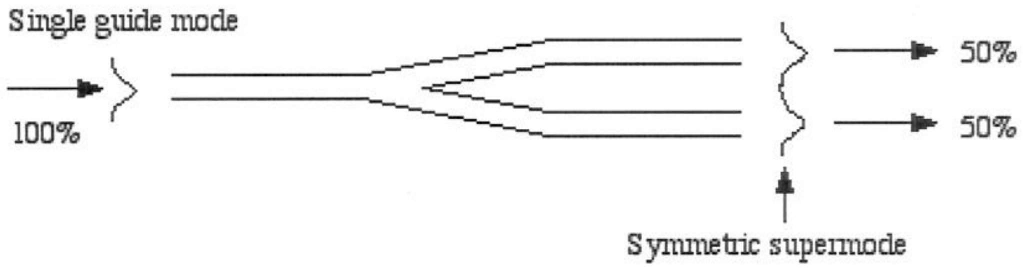
$E_{TRU}$ (x, y) $= E_T(x - S/2)$ and $E_{TRL}(x) = E_T(x + S/2)$

The structure at the right-hand end can also be thought of as a composite guide, which supports its own set of characteristic modes. There are only two such modes, since the whole structure has twice the guiding cross-section of a single guide. Each can be written as a linear combination of the modes of each individual guide. Finally, since the structure is inherently symmetric, the modes must be symmetric and anti-symmetric fields, of the form:

$E_{TRS} = E_{RU} + E_{RL}$ and $E_{TRA} = E_{RU} - E_{RL}$

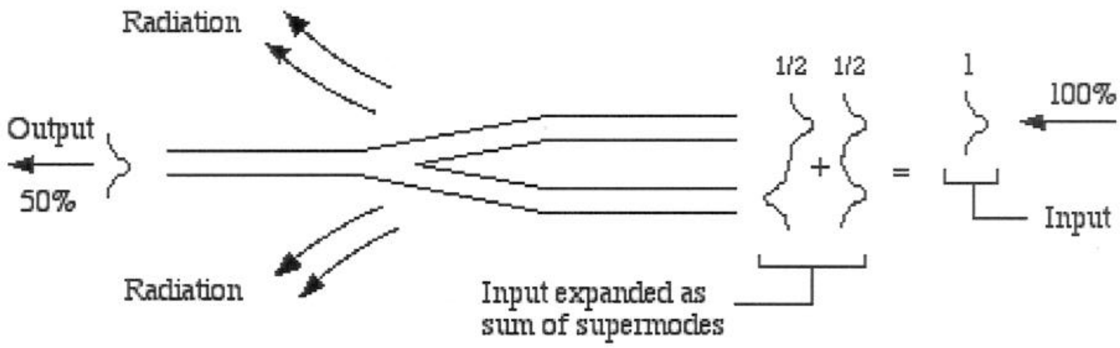



For an input to Port 1, as shown below, the input mode is gradually converted into the symmetric mode in the tapered transition region. No power is radiated, and the conversion is ideally 100% efficient. By symmetry, each output carries 50% of the input power.




[7]

An input to Port 2 may be considered as a sum of the symmetric and antisymmetric modes, excited in equal proportions so that the amplitudes add in Port 2 and exactly cancel in Port 3 to match the boundary conditions. The two fields propagate into the tapered transition region together, where at some point the antisymmetric mode becomes cut off. At this point, 50% of the power must be lost to radiation. The symmetric mode is not cut off, but its two lobes gradually merge together to form the transverse field of the single output guide. This mode emerges as a guided output, but with only 50% efficiency.

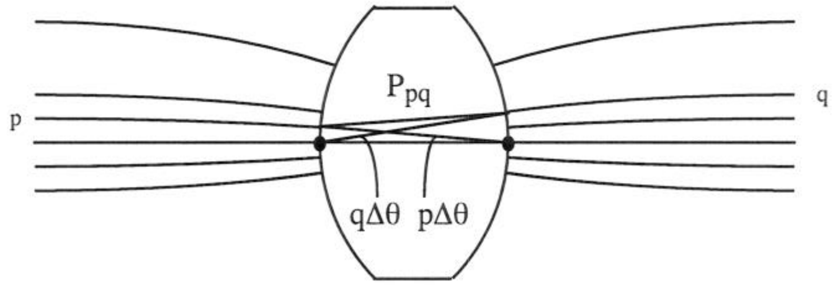



[7]

4.a) To calculate the transfer function, we first assume that each input distributes its power uniformly over the outputs, so that output powers are a reduced by a factor 1/N where N = 2M + 1. Amplitudes are therefore reduced by a factor $1/\sqrt{N}$.

**[2]**

Amplitudes are also affected by the phase delay between input and output. For the path $P_{pq}$ between the $p^{th}$ input and the $q^{th}$ output, the lengths measured in the x- and y-directions are:




$L_x = R - R\{1 - (1 - \cos(p\Delta\theta))\} - R\{1 - (1 - \cos(q\Delta\theta))\}$

$L_y = R \sin(p\Delta\theta) - R \sin(q\Delta\theta)$

Assuming small angles, these expressions can be approximated as:

$L_x \approx R - R\, q^2\Delta\theta^2/2 - R\, p^2\Delta\theta^2/2$

$L_y \approx R\, (q\Delta\theta) - R\, (p\Delta\theta)$

So the total distance is $P_{pq}^{\ 2} = L_x^{\ 2} + L_y^{\ 2}$

Or: $P_{pq}^{\ 2} \approx R^2\{1 - p^2\Delta\theta^2/2 - \ q^2\Delta\theta^2/2\}^2 + R^2\{q\Delta\theta - p\Delta\theta\}^2$

Or: $P_{pq}^{\ 2} \approx R^2\{1 - p^2\Delta\theta^2 - p^2\Delta\theta^2\} + R^2\{q^2\Delta\theta^2 - 2pq\Delta\theta^2 + p^2\Delta\theta^2\}$

Or: $P_{pq}^{\ 2} \approx R^2\{1 - 2pq\Delta\theta^2\}$

So that: $P_{pq} \approx R\{1 - pq\Delta\theta^2\}$

**[4]**

Assuming that the propagation constant of the planar guide is $\beta$, the relationship between an input amplitude at guide p and an output at guide q is:

$A_q = (A_p/\sqrt{N}) \exp\{-j\beta P_{pq}\}$

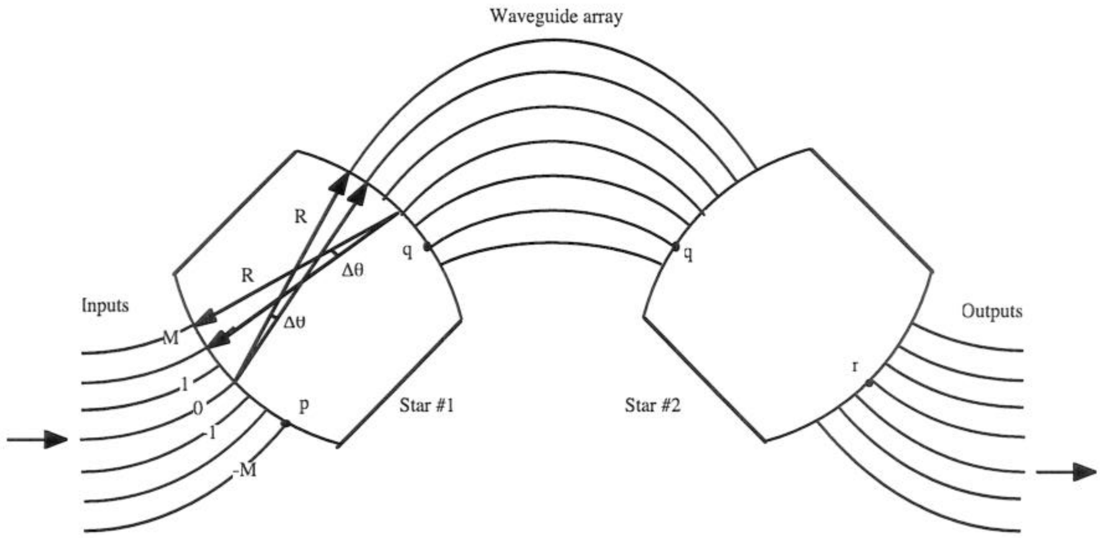
$\quad = (A_p/\sqrt{N}) \exp\{-j\beta R\} \exp\{+jpq\beta R\Delta\theta^2\}$

**[2]**

Hence, the ratio between adjacent outputs is $A_{q+1}/A_q = \exp\{jp\beta R\Delta\theta^2\}$

**[2]**

b) The AWG MUX is constructed from two back-to-back radiative stars, which are linked by an array of waveguides with linearly varying length as shown below.




**[2]**

For an input to Port 0, the outputs are:

$A_q = (A_0/\sqrt{N}) \exp\{-j\beta R\}$

Assuming that the length of the $q^{th}$ waveguide in the array is $L_0 + q\Delta L$, the input to the $q^{th}$ port of the second star is:

$A_q' = (A_0/\sqrt{N}) \exp\{-j\beta R\} \exp\{-j\beta(L_0 + q\Delta L)\}$

The output from the $r^{th}$ port of the second star due to the contribution travelling via q is:

$A_r = (A_q'/\sqrt{N}) \exp\{-j\beta P_{qr}\}$

$\quad = (A_0/N) \exp\{-\beta(L_0 + 2R)\} \exp\{-jq\beta(\Delta L - rR\Delta\theta^2)\}$

The output from the $r^{th}$ port of the second star due to all contributions is:

$A_r = (A_0/N) \exp\{-\beta(L_0 + 2R)\} \sum_{q=-M}^{M} \exp\{-jq\beta(\Delta L - rR\Delta\theta^2)\}$

**[6]**

The terms in the summation will add constructively when

$\beta(\Delta L - rR\Delta\theta^2) = 2\nu\pi$

Since $\beta = 2\pi n_{eff}/\lambda$, where $n_{eff}$ is the effective index, we get:

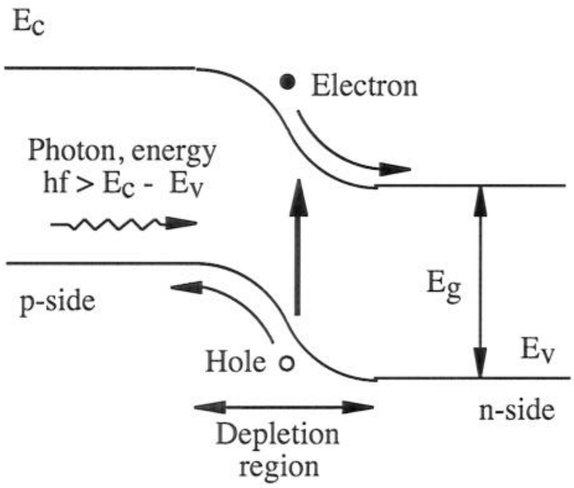
$\lambda = n_{eff}(\Delta L - rR\Delta\theta^2)/\nu$

The wavelengths for peak transmission therefore vary linearly with output port number r.

**[2]**

5. a) A reverse-biased diode may be used as a photodetector as shown below. An electron may be promoted from the valence band into the conduction band by absorption of a photon of energy $hf > E_g$, where $E_g = E_c - E_v$ is the energy gap. If the absorption occurs in a region with a high in-built electric field, such as the depletion region of a p-n junction, the photo-generated carriers will be swept apart before they can recombine, allowing detection with high efficiency. The electrons will be swept towards the n-side, and the corresponding hole towards the p-side. Once out of the depletion layer, the carriers diffuse to the contacts. The effect is equivalent to a single electron transiting the entire device.

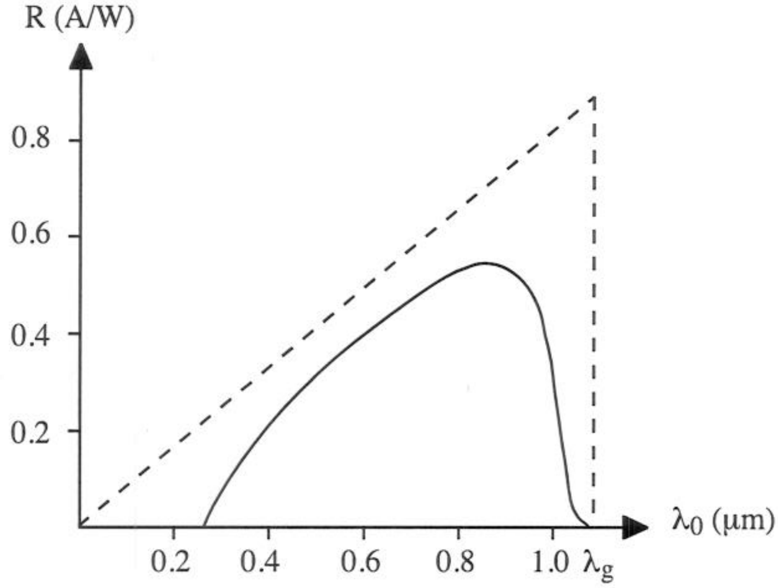
**[3]**




**[3]**

c) Assuming uniform quantum efficiency, the responsivity should be linearly proportional to wavelength up to $\lambda_g$ and then fall to zero, as shown by the dashed line below. However, not all the photons will be absorbed, and not all the carriers generated will reach the contacts. We may model this by defining a quantum efficiency $\eta$, representing the number of useful carrier-pairs per photon. In practice, the quantum efficiency of a surface-entry device is variable, and peaks at about 80%. At short wavelengths, $\eta$ is reduced, because the attenuation is so strong that the light is absorbed before reaching the depletion layer, and at long wavelengths it falls because the light passes right through. The actual variation is then as shown by the full line.

**[3]**




**[4]**

c) Interband transitions will only occur for photons with sufficient energy. Less energetic photons will not contribute, so the response must fall to zero for $\lambda > hc/E_g$.

For Si ($E_g$ = 1.14 eV), $\lambda_{max} = 6.62 \times 10^{-34} \times 3 \times 10^8 / (1.14 \times 1.6 \times 10^{-19})$ m = 1.09 μm

**[2]**

Assuming that the optical power falling on the photodiode is P, the number of photons arriving per second is

$P/hf = P\lambda/hc$.

Assuming that each photon generates an electron-hole pair, the ideal photocurrent is

$I_P = Pe\lambda/hc$

The responsivity $R = I_P/P$ is then

$R = e\lambda/hc$

The peak responsivity is

$R_{max} = (e/hc)(hc/E_g) = e/E_g$, or $R_{max} = 1/E_g$ with $E_g$ in eV

For Si, $R_{max} = 1/1.14 = 0.88$ A/W

**[5]**

6. a) The processes modelled are current injection, spontaneous emission, stimulated emission and optical output.

**[2]**

Rate equations are:

$dn/dt = I/ev - n/\tau_e - G\phi(n - n_0)$

$d\phi/dt = \beta n/\tau_{rr} + G\phi(n - n_0) - \phi/\tau_p$

**[2]**

Where the individual terms model:

| | |
|---|---|
| $dn/dt$ | Rate of change of electron density |
| $I/ev$ | Rate of injection of electrons |
| $n/\tau_e$ | Rate of recombination of electrons, by all processes |
| $G\phi(n - n_0)$ | Rate of stimulated emission (and absorption, if $n < n_0$) |
| $d\phi/dt$ | Rate of change of photon density |
| $\beta n/\tau_{rr}$ | Rate of radiative recombination |
| $\phi/\tau_p$ | Rate of escape of photons |

**[4]**

b) The photon lifetime $\tau_p$ in an LED is determined by the transit time of a photon across the active volume, so that $\tau_p = L/v_g$, where L is the length of the active volume and $v_g \approx c/n$ is the group velocity and n is here the refractive index.

**[1]**

The photon lifetime in a laser of length L is increased by multiple internal reflections from the cavity end mirrors to $\tau_p = L/\{v_g \log_e(1/R_1R_2)\}$, where $R_1$ and $R_2$ are the mirror reflectivities. For emission at normal incidence into air, $R_1 = R_2 = (n - 1) / (n + 1)$.

**[2]**

For a refractive index of 3.5, we obtain $R_1 = R_2 = 2.5/4.5 = 0.556$.

Re-arranging the expression for photon lifetime we get $L = \tau_p v_g \log_e(1/R_1R_2)$

For the data given, $L = 2.5 \times 10^{-12} \times (3 \times 10^8/3.5) \log_e(1/0.556^2) = 250 \times 10^{-6}$, or 250 μm

**[3]**

c) In the steady state, the rate equations reduce to:

$0 = I/ev - n/\tau_e - G\phi(n - n_0)$

$0 = \beta n/\tau_{rr} + G\phi(n - n_0) - \phi/\tau_p$

During lasing, we may neglect spontaneous emission, so from the photon rate equation:

$G\phi(n - n_0) = \phi/\tau_p$

The electron rate equation then reduces to:

$I/ev - n/\tau_e = \phi/\tau_p$

The RHS here is the rate of escape of photons per unit volume and is a linear function of the current. Regrouping the terms we obtain:

$(I - I_t)/ev = \phi/\tau_p$ where $I_t = nev/\tau_e$ is the threshold current.

Re-arranging, we can obtain the electron density during lasing as

$n = I_t\tau_e/ev$

The active volume is $v = 250 \times 3 \times 0.1 \times 10^{-18} = 75 \times 10^{-18}\,m^3$

Hence, $n = 16.8 \times 10^{-3} \times 10^{-9}/(1.6 \times 10^{-19} \times 75 \times 10^{-18}) = 1.4 \times 10^{24}\ m^{-3}$

**[3]**

Returning to the photon rate equation, the electron density during lasing can also be written:

$n = n_0 + 1/G\tau_p$

Hence, the electron density at transparency is:

$n_0 = n - 1/G\tau_p$

For the data given,

$n_0 = 1.4 \times 10^{24} - 1/(10^{-12} \times 2.5 \times 10^{-12}) = 10^{24}\ m^{-3}$

**[3]**