

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING  
EXAMINATIONS 2012

MSc and EEE PART IV: MEng and ACGI

**MODELLING AND CONTROL OF MULTI-BODY MECHANICAL SYSTEMS**

Friday, 4 May 2:30 pm

Time allowed: 3:00 hours

**There are SIX questions on this paper.**

**Answer FOUR questions.**

*All questions carry equal marks.*

*This is an OPEN BOOK examination.*

**Any special instructions for invigilators and information for candidates are on page 1.**

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## MODELLING AND CONTROL OF MULTIBODY MECHANICAL SYSTEMS

1. The body-fixed axes of a rigid body are initially aligned with an earth-fixed set of axes. The rotation of this body is represented by three Euler angles  $\psi$ ,  $\phi$  and  $\theta$  in the yaw-roll-pitch convention. In this convention the body is first rotated from its nominal configuration by an angle  $\psi$  about the  $z$ -axis, then by an angle  $\phi$  about the intermediate  $x$ -axis of the body and finally by an angle  $\theta$  about the new  $y$ -axis of the body.

- a) By making use of the standard single-axis-rotation transformation matrices, show that the complete transformation from earth-fixed coordinates to body-fixed coordinates is given by the matrix

$$\begin{bmatrix} \cos \theta \cos \psi - \sin \phi \sin \theta \sin \psi & \cos \theta \sin \psi + \sin \phi \sin \theta \cos \psi & -\cos \phi \sin \theta \\ -\cos \phi \sin \psi & \cos \phi \cos \psi & \sin \phi \\ \sin \theta \cos \psi + \sin \phi \cos \theta \sin \psi & \sin \theta \sin \psi - \sin \phi \cos \theta \cos \psi & \cos \phi \cos \theta \end{bmatrix}$$

[ 10 marks ]

- b) Write the body angular velocity vector,  $\boldsymbol{\Omega}$ , in terms of the Euler angles, in the body-fixed coordinate system.

[ 10 marks ]

2. Consider a mass  $m$  suspended from a spring of negligible mass as shown in Figure 2.1. Assume that  $m$  is free to move in a vertical plane under the action of gravity, the spring and two controlling forces  $F_r$  and  $F_\theta$ . The unstretched length of the spring is  $r_0$  and its stiffness is  $c$ .

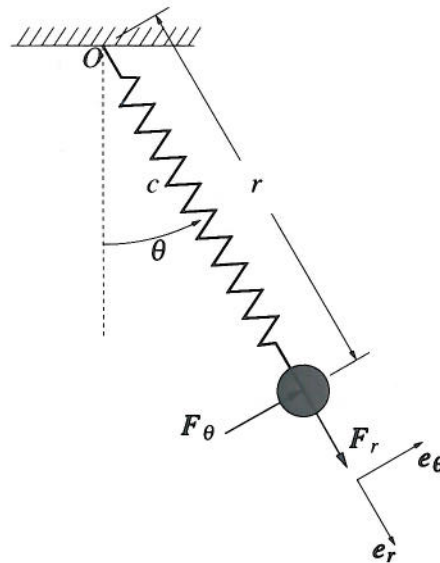


Figure 2.1 Springy pendulum.

A moving Cartesian coordinate system with unit vectors  $e_r$  and  $e_\theta$  is used to analyse the motion of the mass. This coordinate system has a fixed origin  $O$  but it rotates by an angle  $\theta$ .

- State the number of degrees of freedom of the system, and the associated generalised coordinates. [ 1 ]
- Write the position vector,  $r$ , of the mass in the moving coordinate system. [ 1 ]
- Determine the velocity vector of the mass. [ 1 ]
- Compute the total kinetic energy and potential energy of the mass, and hence determine the Lagrangian function. [ 3 ]
- Use the Lagrangian approach to derive the equations of motion of the mass. [ 6 ]
- Assume that the control force  $F_r$  is given by

$$F_r = -mr\ddot{\theta}^2 - mg\cos\theta,$$

and that at time  $t = 0$  the radial distance and radial velocity of the mass are given by  $r = r_0 + \varepsilon$  and  $\dot{r} = 0$ , in which  $\varepsilon$  is a small quantity.

- Determine the motion of the mass,  $r(t)$ , in the radial direction  $e_r$ . [ 4 ]
- An extra term is added to the control force  $F_r$  to provide damping for the oscillations in the radial motion of the mass. Calculate this term. Hence determine an expression for the control force  $F_\theta$  which will force the springy pendulum to behave like a simple pendulum. [ 4 ]

3. A uniform wheel of radius  $r_2$  rolls without slipping on a circular surface (fixed to earth) of radius  $r_1$  under the influence of gravity, as shown in Figure 3.1; the wheel is constrained to always touch the circular surface by a massless link (not shown) joining the centres of the wheel and the circular surface. The mass and spin inertia of the wheel are  $m$  and  $I$  respectively. A driving torque  $T_d$  is applied on the wheel and reacts on earth.

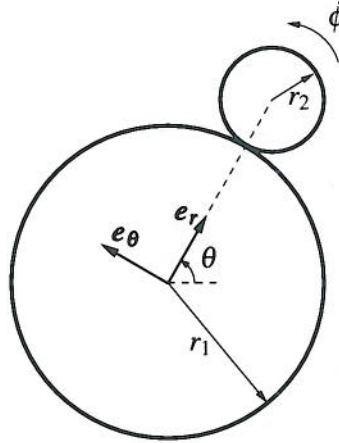


Figure 3.1 Wheel rolling on circular surface.

Polar unit vectors  $e_r$  and  $e_\theta$  are used to analyse the motion of the wheel. This coordinate system has a fixed origin at the centre of the circular surface but it rotates with the line joining the centres of the surface and the wheel, by an angle  $\theta$ . The wheel rotates by an angle  $\phi$ .

- Write the position vector,  $r$ , of the centre of mass of the wheel in the moving coordinate system. [ 1 ]
- Determine the velocity vector of the centre of mass of the wheel. [ 1 ]
- Compute the total kinetic energy and potential energy of the wheel, and hence determine the Lagrangian function. [ 3 ]
- By considering the velocity of the instantaneous contact point between the wheel and the surface derive the equation of the rolling constraint. [ 3 ]
- Use the Lagrangian approach to derive the equation of motion of the wheel in terms of the generalised coordinate  $\theta$ . [ 8 ]
- If the driving torque  $T_d$  reacts on the massless link holding the wheel on the circular surface (e.g. a motor is attached on the link and drives the wheel) then show that the equation of motion of the wheel is

$$(mr_2^2 + I) \ddot{\theta} + \frac{mgr_2^2}{r_1 + r_2} \cos \theta = \frac{r_2(r_1 + 2r_2)}{(r_1 + r_2)^2} T_d.$$

[ 4 ]

4. A uniform wheel of radius  $r_2$  rolls without slipping on a circular surface (fixed to earth) of radius  $r_1$  under the influence of gravity, as shown in Figure 4.1; the wheel is constrained to always touch the circular surface by a massless link (not shown) joining the centres of the wheel and the circular surface. The mass and spin inertia of the wheel are  $m$  and  $I$  respectively. A driving torque  $T_d$  is applied on the wheel and reacts on earth.

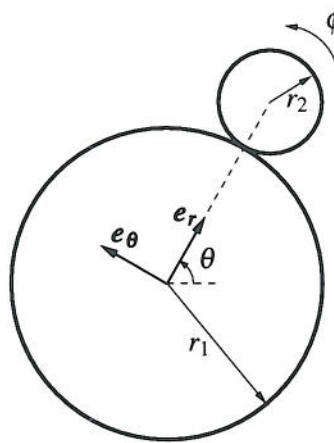


Figure 4.1 Wheel rolling on circular surface.

Polar unit vectors  $e_r$  and  $e_\theta$  are used to analyse the motion of the wheel. This coordinate system has a fixed origin at the centre of the circular surface but it rotates with the line joining the centres of the surface and the wheel, by an angle  $\theta$ . The wheel rotates by an angle  $\phi$ .

- a) By considering the velocity of the instantaneous contact point between the wheel and the circular surface derive the equation of the rolling constraint. [ 3 ]
- b) Determine the acceleration vector of the centre of mass of the wheel. [ 2 ]
- c) Use the Newtonian approach to derive for the wheel:
  - i) The equation of motion with respect to the generalised coordinate  $\theta$ . [ 6 ]
  - ii) The force of constraint which prevents the wheel from moving away from the circular surface. [ 3 ]
  - iii) The force of constraint which maintains the rolling constraint. [ 3 ]
- d) Specify the type of motion the wheel will perform for small perturbations from the  $\theta = -90^\circ$  position when  $T_d = 0$ . [ 3 ]



5. Two particles of mass  $m$  each are attached at the two ends of a rigid rod of length  $l$  and of negligible mass that is free to rotate by an angle  $\psi$  about the vertical axis and by an angle  $\theta$  about a horizontal axis which is perpendicular to the rod, as shown in Figure 5.1. Both axes of rotation pass through the centre of the rod. A moving Cartesian coordinate system attached to the rod with fixed origin  $O$  and with unit vectors  $\mathbf{i}$ ,  $\mathbf{j}$  and  $\mathbf{k}$  is used to analyse the motion of the system. The  $\mathbf{i}$  vector has a direction into the page at the instant shown and it is along the axis of the  $\theta$  rotation. A moment  $N$  is applied on the rod in the  $\mathbf{k}$  direction. The effect of gravity is neglected.

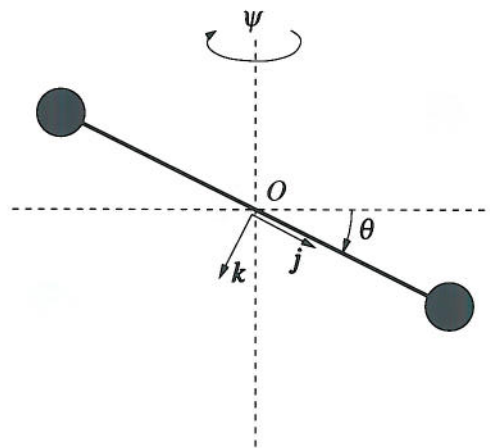


Figure 5.1 Two masses on a massless link.

Consider the two masses and the rod as one rigid system.

- Write an expression for the angular velocity vector of the system in terms of  $\mathbf{i}$ ,  $\mathbf{j}$  and  $\mathbf{k}$ . [ 3 ]
- Calculate the inertia matrix of the system. [ 3 ]
- Compute the angular momentum vector of the system. [ 2 ]
- Use the vectorial approach to derive the equations of motion. [ 12 ]

6. Two particles of mass  $m$  each are attached at the two ends of a rigid rod of length  $l$  and of negligible mass that is free to rotate by an angle  $\psi$  about the vertical axis and by an angle  $\theta$  about a horizontal axis which is perpendicular to the rod, as shown in Figure 6.1. Both axes of rotation pass through the centre of the rod. A moving Cartesian coordinate system attached to the rod with fixed origin  $O$  and with unit vectors  $i$ ,  $j$  and  $k$  is used to analyse the motion of the system. The  $i$  vector has a direction into the page at the instant shown and it is along the axis of the  $\theta$  rotation. A moment  $N$  is applied on the rod in the  $k$  direction. The effect of gravity is neglected.

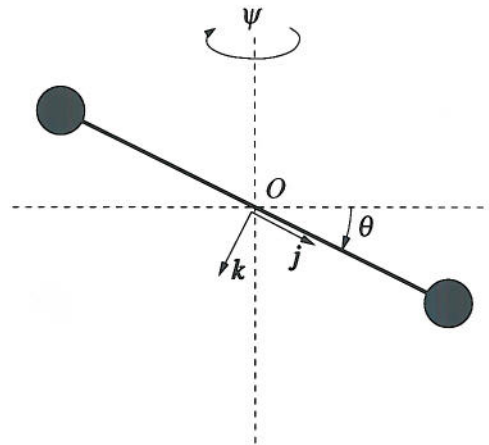


Figure 6.1 Two masses on a massless link.

- a) Compute the total kinetic energy of the system. [ 4 ]
- b) Use the Lagrangian approach to derive the equations of motion. [ 12 ]
- c) Assume the angular velocity  $\dot{\psi}$  about the vertical axis is constant.
  - i) Calculate the moment  $N$  which is required to impose this constraint. [ 2 ]
  - ii) Determine the equation of motion of the system when  $\theta$  is a small angle and hence write the angular frequency of the oscillations of the motion. [ 2 ]



## Modelling and control of multibody mechanical systems

Model answers 2012

## Question 1

a) Three single-axis-rotation transformation matrices are needed.

$$D_\psi = \begin{bmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which is the rotation matrix by angle  $\psi$  about a  $z$  axis.

$$C_\theta = \begin{bmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{bmatrix},$$

which is the rotation matrix by angle  $\theta$  about a  $y$  axis.

$$B_\phi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & \sin \phi \\ 0 & -\sin \phi & \cos \phi \end{bmatrix},$$

which is the rotation matrix by angle  $\phi$  about an  $x$  axis.

The complete transformation from earth-fixed coordinates to body-fixed coordinates is  $A = C_\theta B_\phi D_\psi$  and it amounts to

$$\begin{bmatrix} \cos \theta \cos \psi - \sin \phi \sin \theta \sin \psi & \cos \theta \sin \psi + \sin \phi \sin \theta \cos \psi & -\cos \phi \sin \theta \\ -\cos \phi \sin \psi & \cos \phi \cos \psi & \sin \phi \\ \sin \theta \cos \psi + \sin \phi \cos \theta \sin \psi & \sin \theta \sin \psi - \sin \phi \cos \theta \cos \psi & \cos \phi \cos \theta \end{bmatrix}$$

b)

$$\Omega = C_\theta B_\phi \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} + C_\theta \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\dot{\psi} \cos \phi \sin \theta + \dot{\phi} \cos \theta \\ \dot{\psi} \sin \phi + \dot{\theta} \\ \dot{\psi} \cos \phi \cos \theta + \dot{\phi} \sin \theta \end{bmatrix}$$

## Question 2

a) 2 degrees of freedom. Generalised coordinates:  $r, \theta$ .

b)  $\mathbf{r} = r\mathbf{e}_r$ .

c)  $\dot{\mathbf{r}} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta$ .

d) The kinetic energy is  $T = \frac{1}{2}m\dot{\mathbf{r}} \cdot \dot{\mathbf{r}} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$ .

The potential energy is  $V = -mgr \cos \theta + \frac{1}{2}c(r - r_0)^2$ , with the level of point  $O$  corresponding to zero gravitational potential energy.

The Lagrangian is  $L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + mgr \cos \theta - \frac{1}{2}c(r - r_0)^2$ .

e) The Lagrangian equation with respect to the generalised coordinate  $r$  is

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{r}} \right) - \frac{\partial L}{\partial r} = F_r,$$

or

$$\frac{d}{dt} (m\dot{r}) - mr\dot{\theta}^2 - mg \cos \theta + c(r - r_0) = F_r,$$

or

$$m\ddot{r} - mr\dot{\theta}^2 - mg \cos \theta + c(r - r_0) = F_r,$$

or

$$\ddot{r} - r\dot{\theta}^2 - g \cos \theta + \frac{c}{m}(r - r_0) = \frac{F_r}{m}. \quad (1)$$

The Lagrangian equation with respect to the generalised coordinate  $\theta$  is

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = rF_\theta,$$

or

$$\frac{d}{dt} (mr^2\dot{\theta}) + mgr \sin \theta = rF_\theta,$$

or

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} + g \sin \theta = \frac{F_\theta}{m}.$$

f) i) Substitute in Equation (1)  $F_r = -mr\dot{\theta}^2 - mg \cos \theta$ . Therefore

$$\ddot{r} + \frac{c}{m}(r - r_0) = 0.$$

Therefore

$$r - r_0 = A \cos \left( \sqrt{\frac{c}{m}} t \right) + B \sin \left( \sqrt{\frac{c}{m}} t \right).$$

But at  $t = 0$ ,  $r = r_0 + \epsilon$  and therefore  $A = \epsilon$ . Also,  $\dot{r} = 0$  at  $t = 0$  therefore  $B = 0$ . Then

$$r(t) = r_0 + \epsilon \cos \left( \sqrt{\frac{c}{m}} t \right).$$

ii) The extra term is  $-\mu(r - r_0)$  in which  $\mu > 0$  is the damping constant. Once the radial oscillations die  $r$  will be constant at  $r_0$  and  $\dot{r} = 0$ . Therefore if  $F_\theta = 0$  it can be seen from the second equation of motion that this equation reduces to the same equation as that of a simple pendulum of length  $r$ .

### Question 3

- a) The position vector is  $\mathbf{r} = (r_1 + r_2)\mathbf{e}_r$ .
- b) The velocity vector is  $\dot{\mathbf{r}} = (r_1 + r_2)\dot{\theta}\mathbf{e}_\theta$ .
- c) The kinetic energy is  $T = \frac{1}{2}m(r_1 + r_2)^2\dot{\theta}^2 + \frac{1}{2}I\dot{\phi}^2$ .  
 The potential energy is  $V = mg(r_1 + r_2)\sin\theta$  with the centre of the circular surface the zero potential energy level.  
 Hence the Lagrangian is  $L = T - V = T = \frac{1}{2}m(r_1 + r_2)^2\dot{\theta}^2 + \frac{1}{2}I\dot{\phi}^2 - mg(r_1 + r_2)\sin\theta$ .
- d) The velocity of the centre of mass of the wheel is  $\dot{\mathbf{r}} = (r_1 + r_2)\dot{\theta}\mathbf{e}_\theta$  and the velocity of the contact point caused by the rotation of the wheel is  $-r_2\dot{\phi}\mathbf{e}_\theta$ . The total velocity of the material contact point is zero, therefore the constraint equation is

$$(r_1 + r_2)\dot{\theta} - r_2\dot{\phi} = 0.$$

- e) The Lagrangian equation with respect to the generalised coordinate  $\theta$  is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} + \lambda(r_1 + r_2) = 0,$$

in which  $\lambda$  is the Lagrange multiplier corresponding to the rolling constraint. The RHS of the equation is zero because the torque  $T_d$  reacts on earth. Then

$$\frac{d}{dt}\left(m(r_1 + r_2)^2\dot{\theta}\right) + mg(r_1 + r_2)\cos\theta + \lambda(r_1 + r_2) = 0,$$

or

$$m(r_1 + r_2)^2\ddot{\theta} + mg(r_1 + r_2)\cos\theta + \lambda(r_1 + r_2) = 0.$$

The Lagrangian equation with respect to the generalised coordinate  $\phi$  is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} - \lambda r_2 = T_d,$$

or

$$\frac{d}{dt}(I\dot{\phi}) - \lambda r_2 = T_d,$$

or

$$I\ddot{\phi} - \lambda r_2 = T_d,$$

which implies that

$$\lambda = \frac{I}{r_2}\ddot{\phi} - \frac{T_d}{r_2},$$

and after making use of the rolling constraint equation

$$\lambda = \frac{I(r_1 + r_2)}{r_2^2}\ddot{\theta} - \frac{T_d}{r_2}.$$

After substitution of  $\lambda$  into the first Lagrangian equation we obtain the equation of motion

$$(mr_2^2 + I)\ddot{\theta} + \frac{mgr_2^2}{r_1 + r_2}\cos\theta = \frac{r_2}{r_1 + r_2}T_d.$$

f) The RHS of the first Lagrangian equation is  $T_d$  so that

$$m(r_1 + r_2)^2 \ddot{\theta} + mg(r_1 + r_2) \cos \theta + \lambda(r_1 + r_2) = T_d.$$

This is because the torque reacting on the massless link causes this link to apply a force on the centre of the wheel of magnitude  $T_d/(r_1 + r_2)$  (consider a free body diagram of the link). The next steps are the same as previously and give

$$\left( mr_2^2 + I \right) \ddot{\theta} + \frac{mgr_2^2}{r_1 + r_2} \cos \theta = \frac{r_2(r_1 + 2r_2)}{(r_1 + r_2)^2} T_d.$$

## Question 4

- a) The velocity of the centre of mass of the wheel is  $\dot{\mathbf{r}} = (r_1 + r_2)\dot{\theta}\mathbf{e}_\theta$  and the velocity of the contact point caused by the rotation of the wheel is  $-r_2\dot{\phi}\mathbf{e}_\theta$ . The total velocity of the material contact point is zero, therefore the constraint equation is

$$(r_1 + r_2)\dot{\theta} - r_2\dot{\phi} = 0.$$

- b) By differentiating twice the position vector  $\mathbf{r} = (r_1 + r_2)\mathbf{e}_r$  the acceleration vector is given by  $\ddot{\mathbf{r}} = -(r_1 + r_2)\dot{\theta}^2\mathbf{e}_r + (r_1 + r_2)\ddot{\theta}\mathbf{e}_\theta$ .
- c) Newton's second law of motion gives  $\mathbf{F} = m\ddot{\mathbf{r}}$  and therefore

$$F_r\mathbf{e}_r + (F_\theta + F_{\theta roll})\mathbf{e}_\theta + mg\mathbf{k} = m\left(-(r_1 + r_2)\dot{\theta}^2\mathbf{e}_r + (r_1 + r_2)\ddot{\theta}\mathbf{e}_\theta\right),$$

in which  $F_r$  is the radial force of constraint applied by the massless link to hold the wheel on the surface,  $F_\theta$  is the force applied by the massless link on the wheel in the  $\mathbf{e}_\theta$  direction, and  $F_{\theta roll}$  is the force from the surface on the wheel which maintains the rolling constraint.  $\mathbf{k}$  is a unit vector in the vertical downwards direction. When  $T_d$  reacts on earth  $F_\theta = 0$  and therefore

$$F_r = mg \sin \theta - m(r_1 + r_2)\dot{\theta}^2,$$

$$F_{\theta roll} = mg \cos \theta + m(r_1 + r_2)\ddot{\theta}.$$

By considering the motion of the wheel about its centre of mass,  $d\mathbf{H}/dt = \mathbf{N}$  in which the angular momentum is  $\mathbf{H} = I\dot{\phi}$ . Therefore

$$I\ddot{\phi} = -F_{\theta roll}r_2 + T_d,$$

or after substitution of  $F_{\theta roll}$

$$I\ddot{\phi} = -mgr_2 \cos \theta - m(r_1 + r_2)r_2\ddot{\theta} + T_d.$$

By making use of the rolling constraint equation and substituting  $\ddot{\phi}$  we obtain the equation of motion

$$\left(mr_2^2 + I\right)\ddot{\theta} + \frac{mgr_2^2}{r_1 + r_2} \cos \theta = \frac{r_2}{r_1 + r_2}T_d.$$

- d) For  $T_d = 0$  and  $\theta = -\pi/2 + \epsilon$  in which  $\epsilon$  is a small perturbation, the equation of motion becomes

$$\left(mr_2^2 + I\right)\ddot{\epsilon} + \frac{mgr_2^2}{r_1 + r_2}\epsilon = 0,$$

which describes simple harmonic motion with angular frequency of oscillation

$$\omega = \sqrt{\frac{\frac{mgr_2^2}{r_1 + r_2}}{mr_2^2 + I}}.$$

## Question 5

- a) The angular velocity of the system about the vertical axis is  $\dot{\psi}$  and in the  $\mathbf{i}$  direction it is  $\dot{\theta}$ . All together it is

$$\boldsymbol{\Omega} = \dot{\theta}\mathbf{i} + \dot{\psi}\sin\theta\mathbf{j} + \dot{\psi}\cos\theta\mathbf{k}.$$

- b) The moment of inertia about the axis in the  $\mathbf{i}$  direction is

$$I_{xx} = m\left(\frac{l}{2}\right)^2 + m\left(\frac{l}{2}\right)^2 = \frac{1}{2}ml^2.$$

The moment of inertia about the axis in the  $\mathbf{j}$  direction is zero and the moment of inertia about the axis in the  $\mathbf{k}$  direction is the same as  $I_{xx}$ . These three axes are principal and therefore the inertia matrix is

$$I = \begin{bmatrix} \frac{1}{2}ml^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}ml^2 \end{bmatrix}.$$

- c) The angular momentum is given by  $\mathbf{H} = I\boldsymbol{\Omega}$  which in vector form is

$$\mathbf{H} = \frac{1}{2}ml^2\dot{\theta}\mathbf{i} + \frac{1}{2}ml^2\dot{\psi}\cos\theta\mathbf{k}.$$

- d) The motion about the centre of mass is given by

$$\frac{d'\mathbf{H}}{dt} + \boldsymbol{\Omega} \times \mathbf{H} = \mathbf{N},$$

or

$$\frac{1}{2}ml^2\ddot{\theta}\mathbf{i} + \frac{1}{2}ml^2(\ddot{\psi}\cos\theta - \dot{\psi}\dot{\theta}\sin\theta)\mathbf{k} + \frac{1}{2}ml^2(\dot{\theta}\mathbf{i} + \dot{\psi}\sin\theta\mathbf{j} + \dot{\psi}\cos\theta\mathbf{k}) \times (\dot{\theta}\mathbf{i} + \dot{\psi}\cos\theta\mathbf{k}) = N\mathbf{k},$$

or

$$\frac{1}{2}ml^2\left((\dot{\psi}^2\sin\theta\cos\theta + \ddot{\theta})\mathbf{i} + (-\dot{\theta}\dot{\psi}\cos\theta + \dot{\theta}\dot{\psi}\cos\theta)\mathbf{j} + (-2\dot{\psi}\dot{\theta}\sin\theta + \ddot{\psi}\cos\theta)\mathbf{k}\right) = N\mathbf{k}.$$

Therefore the two equations of motion are

$$\ddot{\theta} + \dot{\psi}^2\sin\theta\cos\theta = 0,$$

and

$$\frac{1}{2}ml^2(\ddot{\psi}\cos\theta - 2\dot{\psi}\dot{\theta}\sin\theta) = N.$$



## Question 6

- a) The velocity vector of each mass is given by

$$-\frac{l}{2}\dot{\psi}\cos\theta\mathbf{i} + \frac{l}{2}\dot{\theta}\mathbf{k}.$$

Therefore the total kinetic energy of the system is

$$T = \frac{1}{2}m \left( \left( -\frac{l}{2}\dot{\psi}\cos\theta \right)^2 + \left( \frac{l}{2}\dot{\theta} \right)^2 \right) \times 2,$$

or

$$T = \frac{1}{4}ml^2\dot{\theta}^2 + \frac{1}{4}ml^2\dot{\psi}^2\cos^2\theta.$$

- b) Since gravity is neglected the potential energy is zero and  $L = T$ . The Lagrangian equation with respect to the generalised coordinate  $\psi$  is

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\psi}} \right) - \frac{\partial L}{\partial \psi} = N \cos \theta,$$

or

$$\frac{d}{dt} \left( \frac{1}{2}ml^2\dot{\psi}\cos^2\theta \right) = N \cos \theta,$$

or

$$\frac{1}{2}ml^2 \left( \ddot{\psi}\cos^2\theta - 2\dot{\psi}\dot{\theta}\cos\theta\sin\theta \right) = N \cos \theta.$$

Therefore the first equation of motion is

$$\frac{1}{2}ml^2 \left( \ddot{\psi}\cos\theta - 2\dot{\psi}\dot{\theta}\sin\theta \right) = N.$$

The Lagrangian equation with respect to the generalised coordinate  $\theta$  is

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0,$$

or

$$\frac{d}{dt} \left( \frac{1}{2}ml^2\dot{\theta} \right) + \frac{1}{2}ml^2\dot{\psi}^2\cos\theta\sin\theta = 0.$$

Therefore the second equation of motion is

$$\ddot{\theta} + \dot{\psi}^2\sin\theta\cos\theta = 0.$$

- c) i) If  $\dot{\psi}$  is constant then  $\ddot{\psi} = 0$  and from the first equation of motion

$$N = -ml^2\dot{\psi}\dot{\theta}\sin\theta.$$

- ii) When  $\theta$  is small, from the second equation of motion

$$\ddot{\theta} + \dot{\psi}^2\theta = 0,$$

which is simple harmonic motion in  $\theta$  with angular frequency of the oscillations equal to  $\dot{\psi}$ .

