# Imperial College London

# BSc/MSci EXAMINATION June 2012

*This paper is also taken for the relevant Examination for the Associateship*

# PLASMA PHYSICS

## For 3rd and 4th Year Physics Students

Friday, 1st June 2012: 10:00 to 12:00

*This paper consists of two sections: A & B. Section A contains one question. Section B contains four questions.*
*Answer ALL parts of Section A and TWO questions from Section B.*
*Marks shown on this paper are indicative of those the Examiners anticipate assigning.*

**General Instructions**

Complete the front cover of each of the THREE answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

**You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.**

## SECTION A

**1.** (i) Three of the fundamental characteristic scales for plasma are; $\omega_p$, $r_L$ and $v_{th}$.

(a) For each of the three quantities above, give its name and a brief example of a phenomenon that it relates to.

(b) State how each of the three quantities above scale with temperature, density and magnetic field strength.

(c) State two additional fundamental characteristic scales for plasma. Give an example of a phenomenon that each relates to.

[5 marks]

(ii) (a) Sketch the density–temperature parameter space, indicating approximately the conditions of tokamak plasmas, the sun's core and interstellar plasmas.

(b) Calculate the electron number density necessary for $N_D = 1$ (where $N_D$ is the number of electrons in the Debye sphere) at a temperature of $T = 100$ eV.

(c) Indicate on your answer to (ii)(a) where plasma is considered to be "non-ideal".

[5 marks]

(iii) (a) Sketch the trajectory of an ion moving perpendicular to a uniform magnetic field that points in the $x$-direction ($\mathbf{B} = B_x\hat{\mathbf{x}}$), when there is also a uniform electric field present that points in the $z$-direction ($\mathbf{E} = E_z\hat{\mathbf{z}}$). Your sketch should be in the $y$–$z$ plane, and you should clearly indicate the direction that the ion moves along the trajectory.

(b) Explain qualitatively the physical origin of the drift.

(c) Calculate the magnitude and direction of the drift velocity for the uniform fields, $\mathbf{B} = 2\hat{\mathbf{x}}$ T and $\mathbf{E} = (3\hat{\mathbf{x}} + 10\hat{\mathbf{z}})$ V/m.

[5 marks]

(iv) (a) For the following reaction, calculate the energies of each product (in MeV);

$$ {}^{2}_{1}\mathrm{D} + {}^{3}_{2}\mathrm{He} \rightarrow {}^{4}_{2}\mathrm{He} + {}^{1}_{1}\mathrm{H} $$

(b) Derive Lawson's criterion for breakeven

$$ n\tau_E = \left(\frac{1-\eta}{\eta}\right)\frac{12k_BT}{\langle\sigma v\rangle E_{DT}} $$

where $\eta$ is efficiency, $T$ temperature, $\sigma$ thermonuclear reaction cross section, $v$ ion velocity and $E_{DT}$ is the thermonuclear energy released per deuterium–tritium reaction.

[5 marks]

[Total 20 marks]

Useful formulae & data:

$$ \lambda_D = \sqrt{\frac{\epsilon_o k_B T_e}{n_e e^2}} $$

Rest masses in a.m.u. ($1.66 \times 10^{-27}$ kg):

${}^{4}_{2}He \rightarrow 4.0026$ , ${}^{3}_{2}He \rightarrow 3.016$ , ${}^{2}_{1}D \rightarrow 2.014$ , ${}^{1}_{1}H \rightarrow 1.0078$

## SECTION B

**2.** (i) Draw a cross-section sketch of plasma in a tokamak, indicating the poloidal and toroidal directions, the major & minor radii, a flux surface and a banana orbit. [3 marks]

(ii) Explain qualitatively how banana orbits arise. [2 marks]

(iii) Starting from the definition for the magnetic moment of a current loop, show that the magnitude of the magnetic moment of a particle on a Larmor orbit is

$$\mu = \frac{mv_\perp^2}{2B}$$

[2 marks]

(iv) A charged particle moving along a field line into a region of changing B-field strength experiences a force $\mathbf{F}_\parallel = -\mu\nabla_\parallel B$ . Using this expression for $\mathbf{F}_\parallel$, show that the magnetic moment is conserved. [2 marks]

(v) Consider a tokamak plasma with a circular cross-section, major and minor radii of $R_o = 10$ m and $a = 2.5$ m, respectively, and B-field strength of $B = 5$ T in the centre of the plasma (i.e. $r = 0$). A deuteron at 20 keV starts off on the outboard side of the plasma (i.e. $\theta = 0$) on a flux surface at $r = 2$ m. The deuteron initially has $v_{\perp o} = 6v_{\parallel o}$. (*You may use the large aspect-ratio approximation throughout this question.*)

(a) Show that the smallest major radius $R_m$ that the deuteron can reach is

$$R_m = (R_o + r)\frac{v_{\perp o}^2}{v_{\parallel o}^2 + v_{\perp o}^2} \quad .$$

[2 marks]

(b) Assuming that the deuteron stays on its original flux surface, and that this surface has a safety factor of $q = rB_\phi/R_oB_\theta = 2$, show that the distance travelled along the field line to reach $R_m$ is approximately 12 m, and calculate the value of $\theta$ at $R_m$. [2 marks]

(c) $v_\parallel$ varies sinusoidally in time for this trapped deuteron. Estimate the distance that the deuteron's guiding centre drifts when travelling to $R_m$. Also state the direction of the drift.
(*HINT: The projection of the Larmor orbital velocity onto one axis also varies sinusoidally in time.* )

[2 marks]

<u>Useful formulae:</u>

$$\mathbf{V}_g = -\frac{1}{2}mv_\perp^2\frac{\nabla B\times\mathbf{B}}{qB^3} \quad , \quad \mathbf{V}_c = -\frac{mv_\parallel^2}{R_c^2}\frac{\mathbf{R}_c\times\mathbf{B}}{qB^2}$$

$m_p = 1.67\times10^{-27}$ kg , $e = 1.6\times10^{-19}$ C

[Total 15 marks]

**3.** Langmuir waves have the following dispersion relation:

$$\omega^2 = \omega_p^2 \left(1 + 3k^2\lambda_D^2\right) \tag{1}$$

where $\omega$ is angular frequency, $k$ is wavenumber, $\omega_p$ is the plasma frequency and $\lambda_D$ is the Debye length.

(i) State whether Langmuir waves are transverse or longitudinal, and fast or slow. Briefly explain how Langmuir waves work. [2 marks]

(ii) Sketch an example of the dispersion relation of a Langmuir wave, an EM wave in plasma and an EM wave in vacuum, on the same plot, clearly labelling each of your three curves. [2 marks]

(iii) (a) Show that the group velocity of light in plasma is

$$\left(v_g\right)_{EM} = c\sqrt{1 - \frac{n_e}{n_{cr}}}$$

where $n_{cr}$ is the critical density and $n_e$ the actual electron number density.

(b) Find an expression for the wavelength $\lambda_p$ of a Langmuir wave that is kicked up in the wake of a laser pulse propagating through plasma. Express your answer in terms of $n_e$, $n_{cr}$, the electron thermal speed $v_{th}$, and the limiting plasma wavelength $\lambda_o = 2\pi c/\omega_p$, where $c$ is the speed of light in vacuum.

[4 marks]

(iv) The following equations describe Langmuir waves propagating in the $x$-direction

$$\frac{d}{dt}\left(P_e n_e^{-\gamma}\right) = 0 \tag{2}$$

$$m_e n_e \frac{dv_x}{dt} = -\frac{\partial P_e}{\partial x} - e n_e E_x \tag{3}$$

$$\frac{\partial E_x}{\partial x} = \frac{(Zn_i - n_e)e}{\epsilon_o} \tag{4}$$

$$\frac{\partial n_e}{\partial t} + \frac{\partial}{\partial x}(v_x n_e) = 0 \tag{5}$$

$m_e$ is the electron mass, $v_x$ the electron fluid velocity ($x$-component), $P_e$ the electron pressure, $E_x$ the electric field, $\gamma$ the ratio of specific heats, $n_i$ the ion number density and $Z$ the ion charge state.

(a) State the name of each of the 4 equations above. [2 marks]

(b) Show that equation (2) becomes the following after linearization

$$P_1 = \gamma \frac{n_1}{n_o} P_o$$

where $P_1$ and $P_o$ are the perturbed and unperturbed pressure, respectively. Similarly, $n_1$ and $n_o$ are the perturbed and unperturbed electron number density. [2 marks]

(c) The Langmuir dispersion relation, equation (1), results from a 2nd order PDE for $n_1$. Derive this PDE from the expression given in (iv)(b) and the other equations given in question (iv). [3 marks]

[Total 15 marks]

**4.** (i) The induction equation for the magnetic field $\mathbf{B}$ is

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \frac{\eta}{\mu_o} \nabla^2 \mathbf{B}$$

where $\mathbf{u}$ is the plasma velocity, $\eta$ the electrical resistivity and $\mu_o$ the permeability of free space.

(a) State the physical phenomena caused by each term on the right hand side of the above equation.

(b) Show how this equation is obtained from Maxwell's equations.

[3 marks]

An imploding Z-pinch comprises a thin-walled, hollow cylinder of perfectly conducting, dense plasma with initial radius $r_o$ = 2 cm. This is the 'liner'. Inside the liner is a low density, deuterium plasma (the 'fuel') with an axial B-field. This axial field only exists in the fuel, is uniform and is initially $B_{zo}$ = 2 T . The fuel resistivity is very low. A constant axial current of $I$ = 3 MA is passed through the liner, compressing it to an equilibrium radius $r_1$.

(ii) Assuming that the fuel thermal pressure is negligible, show that the compression ratio is given by

$$\frac{r_o}{r_1} = \frac{\mu_o I}{2\pi r_o B_{zo}}$$

[3 marks]

(iii) (a) The fuel is compressed in an adiabatic fashion, with $\gamma = \frac{5}{3}$ (ratio of specific heats). Given that the plasma beta of the fuel is initially $\beta_o$ = 0.01, find $\beta$ after compression.

(b) Determine the magnetization parameter of the deuterons before and after compression, if initially $T_o$ = 10 eV and the density of deuterons is $n_{Do} = 10^{20}$ m$^{-3}$.

[5 marks]

Someone has the idea to use this device to generate fusion power. An external microwave heating source is to be added to heat up and maintain the deuterium fuel at a temperature of $T_f$ = 20 keV. In the uncompressed state the product $n\tau_E$ is 10% of that needed to satisfy Lawson's criterion. However, the device is still able to compress the deuterium fuel, though to a different radius $r_2$ since the plasma thermal pressure is now significant.

(iv) Determine how $n\tau_E$ scales with the compression ratio $r_o/r_2$, assuming classical diffusion. Therefore explain the effect that compression has on meeting Lawson's criterion.

[2 marks]

(v) If anomalous (i.e. Bohm) diffusion were to occur instead of classical diffusion, what is the effect of compression on satisfying Lawson's criterion? (Make sure that you justify your answer.)

[2 marks]

*[This question continues on the next page . . . ]*

Useful formulae:

$$\nabla \times (\nabla \times \mathbf{a}) = \nabla(\nabla \cdot \mathbf{a}) - \nabla^2 \mathbf{a}$$

$$\tau_{ci} = 2.9 \times 10^{12} \frac{T_i^{\frac{3}{2}}}{Z^4 n_i} \text{ s} \qquad \text{(for } T_i \text{ in eV)}$$

$\mu_o = 4\pi \times 10^{-7}$ H/m , $m_p = 1.67 \times 10^{-27}$ kg , $e = 1.6 \times 10^{-19}$ C

[Total 15 marks]

**5.** (i) The MHD momentum equation can be written as

$$\rho \frac{d\mathbf{u}}{dt} = -\nabla P + \mathbf{j} \times \mathbf{B}$$

where $\rho$ is the mass density, $\mathbf{u}$ the flow speed, $P$ the thermal pressure, $\mathbf{j}$ the current density and $\mathbf{B}$ the magnetic field.

(a) State 3 requirements for ideal MHD to be valid.

(b) Show how the above equation can also be written in terms of magnetic pressure and tension forces.

[3 marks]

(ii) (a) Show how the force balance condition for a cylindrically symmetric plasma can be expressed as

$$r^2 \frac{\partial P}{\partial r} = -\frac{1}{2\mu_o} \frac{\partial}{\partial r} \left( r^2 B_\theta^2 \right)$$

(b) Show how the MHD momentum equation can be used to derive the Bennett relation

$$k_B T = \frac{\mu_o I^2}{8\pi (Z+1) N}$$

for an equilibrium Z-pinch with uniform temperature and surrounded by vacuum, stating any assumptions used. $N$ is the ion line-density, $I$ the electrical current, $T$ the temperature and $Z$ the ion charge state.

[4 marks]

(iii) Consider a fully ionized carbon Z-pinch carrying an axial current of $I = 4$ MA. Its length, temperature and ion density are $L = 1$ cm, $T = 2$ keV and $n_i = 10^{28}$ m$^{-3}$, respectively. The density is uniform.

(a) Calculate the radius of the pinch that is necessary for equilibrium.

(b) Calculate the maximum azimuthal magnetic field.

[2 marks]

(iv) Describe two different modes of MHD instability in a Z-pinch plasma and explain how each works. Also explain how these instabilities can be stabilized.

[4 marks]

(v) Estimate the growth time of MHD instabilities for the Z-pinch in question (iii).

[2 marks]

<u>Useful formulae:</u>

$$\nabla(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \times (\nabla \times \mathbf{b}) + \mathbf{b} \times (\nabla \times \mathbf{a}) + (\mathbf{a} \cdot \nabla)\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{a}$$

$$\int u\,dv = uv - \int v\,du$$

$$v_A = \sqrt{P_{mag}/\rho}$$

In cyl. polar coords: $$(\nabla \times \mathbf{v})_z = \frac{1}{r}\left[ \frac{\partial (r v_\theta)}{\partial r} - \frac{\partial v_r}{\partial \theta} \right]$$

$\mu_o = 4\pi \times 10^{-7}$ H/m , $e = 1.6 \times 10^{-19}$ C , $m_p = 1.67 \times 10^{-27}$ kg

[Total 15 marks]