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EE9-SO22

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2013

MSc and EEE/EIE PART IV: MEng and ACGI

WAVELETS AND APPLICATIONS

Thursday, 16 May 10:00 am

Time allowed: 3:00 hours

There are FOUR questions on this paper.

Answer ALL questions.

All questions carry equal marks.

Any special instructions for invigilators and information for candidates are on page 1.

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Special Information for the Invigilators: NONE

Information for Candidates:

Causal spline $\beta_n^+(t)$

The causal spline $\beta_n^+(t)$ of order n is obtained from the $(n+1)$-fold convolution of the causal box function $\beta_0^+(t)$. Specifically,

$$\beta_n^+(t) = \underbrace{\beta_0^+(t) * \beta_0^+(t) \ldots * \beta_0^+(t)}_{n+1 \text{ times}}$$

with

$$\beta_0^+(t) = \begin{cases} 1 & t \in [0,1) \\ 0 & \text{otherwise.} \end{cases}$$

Poisson summation formula:

$$\sum_{n=-\infty}^{\infty} f(t-n) = \sum_{k=-\infty}^{\infty} \hat{f}(2\pi k) e^{j2\pi k t}.$$

The Questions

1. (a) Find the overall transfer function $Y(z)/X(z)$ of the system in Figure 1a.

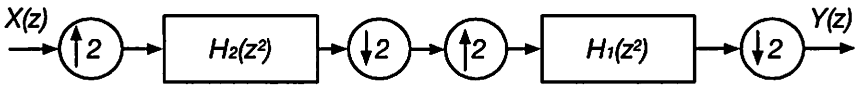


Figure 1a: Multirate system one.

[6]

(b) In the system in Figure 1b, if $H(z) = H_0(z^2) + z^{-1}H_1(z^2)$, prove that $Y(z) = X(z)H_0(z)$.

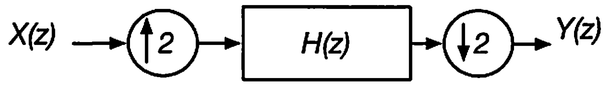


Figure 1b: Multirate system two.

[6]

(c) Let $H(z)$, $F(z)$ and $G(z)$ be filters satisfying

$$H(z)G(z) + H(-z)G(-z) = 2,$$

$$H(z)F(z) + H(-z)F(-z) = 0.$$

Prove that for one of the systems in Figure 1c $Y(z)/X(z) = 1$, while for the other $Y(z)/X(z) = 0$.

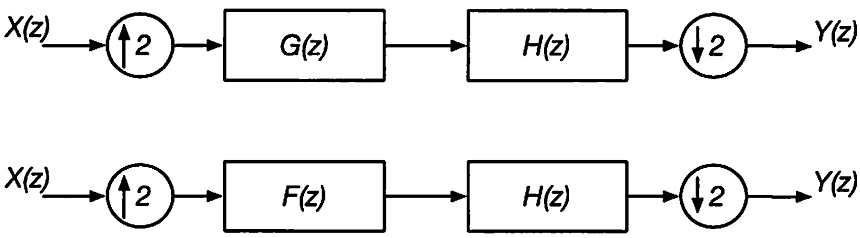


Figure 1c: Two examples of multirate systems.

[6]

(d) Show that the filter $H(z) = (1 - z)^2$ annihilates discrete-time polynomial of maximum degree 1. Specifically, show that the convolution $h[n] * x[n] = 0$ when $x[n]$ is a discrete-time polynomial of maximum degree 1.

[7]

2. *Rudin-Shapiro Polynomial*

Rudin-Shapiro polynomials are defined by the following recursive equations

$$P_0(z) = Q_0(z) = 1$$
$$P_{n+1}(z) = P_n(z) + z^k Q_n(z)$$
$$Q_{n+1}(z) = P_n(z) - z^k Q_n(z)$$

where $k = 2^n$.

(a) Derive the Rudin-Shapiro polynomial pair (P, Q) of degree 3.

[6]

(b) Prove that for $n > 0$, P_n and Q_n lead to a two-channel perfect reconstruction (PR) orthogonal filter bank, that is, show the following:

$$P_n(z)P_n(z^{-1}) + Q_n(z)Q_n(z^{-1}) = k_n,$$
$$P_n(z)P_n(-z^{-1}) + Q_n(z)Q_n(-z^{-1}) = 0.$$

Determine the constant k_n.

[9]

(c) Consider now the 2-channel filter bank of Fig. 2 with $G_0(z) = P_2(z)$ and $G_1(z) = Q_2(z)$.

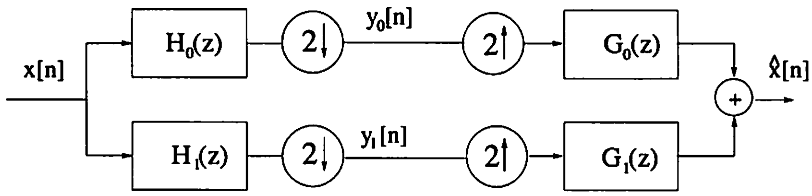


Figure 2: Two-channel filter bank.

i. Assume $x[n] = 1$ and ignore any boundary effect. Which of the signals $y_0[n]$, $y_1[n]$ is nonzero? (Justify your answer)

[5]

ii. Can you design a 2-channel perfect reconstruction orthogonal filter bank with filters of the same length as $P_2(z)$ and $Q_2(z)$ such that $y_1[n] = 0$ when $x[n] = n$? (Justify your answer)

[5]

3. Denote with $V_n = span(\{\beta_n^+(t-k)\}_{n\in\mathbf{Z}})$ the shift-invariant space generated by the causal spline $\beta_n^+(t)$ and its integer shifts. Given the signal $x(t) \in V_n$ with

$$x(t) = \sum_k \alpha_k \beta_n^+(t-k),$$

we want to show that

$$\frac{dx(t)}{dt} = \sum_k \tilde{\alpha}_k \beta_{n-1}^+(t-k)$$

with $\tilde{\alpha}_k = \alpha_k - \alpha_{k-1}$. Thus the computation of continuous-time derivative of $x(t)$ only requires computing the finite difference of the sequence α_k.

(a) Begin by showing that

$$\frac{d\beta_n^+(t)}{dt} = \beta_{n-1}^+(t) - \beta_{n-1}^+(t-1).$$

[Hint: use the fact that $\frac{d\beta_0^+(t)}{dt} = \delta(t) - \delta(t-1)$.]

[5]

(b) Now show that

$$\frac{dx(t)}{dt} = \sum_k \tilde{\alpha}_k \beta_{n-1}^+(t-k)$$

with $\tilde{\alpha}_k = \alpha_k - \alpha_{k-1}$.

[5]

(c) Using the above fact compute the derivative of $x(t)$ with

$$x(t) = \sum_{k=-1}^{1} \alpha_k \beta_1^+(t-k)$$

and $\alpha_{-1} = \alpha_0 = \alpha_1 = 2$.

[5]

(d) Sketch and dimension $dx(t)/dt$ of part (c).

[5]

(e) Compute the orthogonal projection of $x(t)$ of part (c) onto $V_0 = \text{span}(\{\beta_0^+(t-k)\}_{n\in\mathbf{Z}})$.

[5]

4. Consider the symmetric linear spline given by $\varphi(t) = \beta_0(t) * \beta_0(t)$, with

$$\beta_0(t) = \begin{cases} 1, & |t| < 1/2 \\ 1/2 & |t| = 1/2 \\ 0, & \text{otherwise.} \end{cases}$$

We know that $\varphi(t)$ is a valid scaling function. However, the linear spline is not orthogonal. It is our aim to orthogonalize it.

(a) Compute the deterministic autocorrelation function

$$a[n] = \langle \varphi(t), \varphi(t-n) \rangle.$$

Denote with $\hat{\varphi}(\omega)$ the Fourier transform of $\varphi(t)$ and with $A(e^{j\omega})$ the discrete-time Fourier transform of $a[n]$. Show that the new function $\phi(t)$ with Fourier transform

$$\hat{\phi}(\omega) = \frac{\hat{\varphi}(\omega)}{\sqrt{A(e^{j\omega})}}$$

is an orthogonal basis of the subspace $V_0 =$ span $\{\phi(t-n)\}_{n\in\mathbb{Z}}$. (Hint: Show that the Riesz basis criterion $A \leq \sum_{n\in\mathbb{Z}} |\hat{\phi}(\omega + 2\pi n)|^2 \leq B$ is satisfied with $A = B = 1$).

[9]

(b) Find the z-domain expression of the filter $H_0(z)$ that leads to the two-scale equation:

$$\phi(t) = \sqrt{2} \sum_n h_0[n] \phi(2t-n).$$

(Hint: Use the fact that $\hat{\varphi}(\omega) = \frac{G_0(e^{j\omega/2})}{\sqrt{2}} \hat{\varphi}(\omega/2)$ and the fact that $\hat{\phi}(\omega) = \frac{\hat{\varphi}(\omega)}{\sqrt{A(e^{j\omega})}}$.)

[6]

(c) Now consider the function

$$\varphi(t) = \beta_1(t) + \frac{1}{2}\frac{d\beta_1(t)}{dt},$$

where $\beta_1(t) = \beta_0(t) * \beta_0(t)$. Is $\varphi(t)$ a valid scaling function? (Justify your answer).

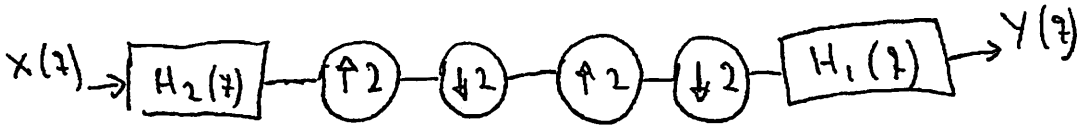
[10]

SOLUTIONS

1.

(a) WE USE THE FACT THAT

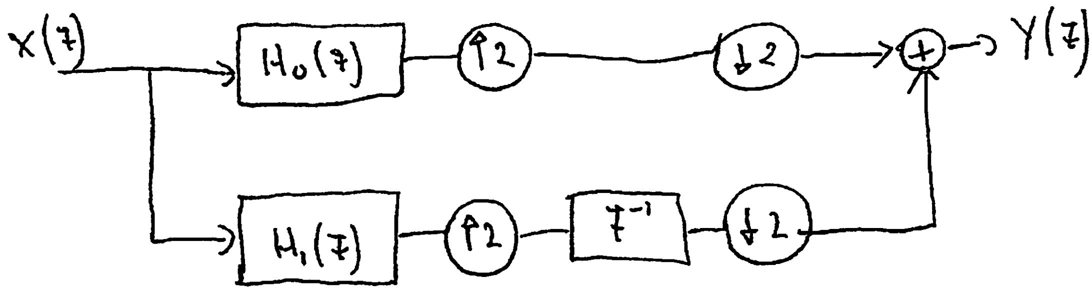
TO OBTAIN



CONSEQUENTLY

$$\frac{Y(z)}{X(z)} = H_1(z)H_2(z)$$

(b) THE EASIEST WAY TO SHOW THIS IS BY NOTING THAT WE CAN RE-DRAW THE SYSTEM AS FOLLOWS



THE LOWER BRANCH CONTAINS AN UPSAMPLER FOLLOWED BY A DELAY AND A DOWNSAMPLER. THE OUTPUT OF SUCH A SYSTEM IS ZERO.

THEREFORE $\quad \dfrac{Y(z)}{X(z)} = H_0(z)$

(c) FOR THE FIRST SYSTEM, THE INPUT/OUTPUT RELATIONSHIP IS

$$\frac{Y(z)}{X(z)} = \frac{1}{2}\left[H\left(z^{1/2}\right)G\left(z^{1/2}\right) + H\left(-z^{1/2}\right)G\left(-z^{1/2}\right)\right] \overset{(i)}{=} 1$$

WHERE (i) FOLLOWS FROM:

$$H(z)\,G(z) + H(-z)\,G(-z) = 2$$

FOR THE SECOND SYSTEM, THE INPUT/OUTPUT RELATIONSHIP IS

$$\frac{Y(z)}{X(z)} = \frac{1}{2}\left[H\left(z^{1/2}\right)F\left(z^{1/2}\right) + H\left(-z^{1/2}\right)F\left(-z^{1/2}\right)\right] \overset{(i)}{=} 0$$

WHERE (i) FOLLOWS FROM:

$$H(z)\,F(z) + H(-z)\,F(-z) = 0$$

(d) USING THE DEFINITION OF THE Z-TRANSFORM WE HAVE THAT

$$H(z) = (1 - z)^2 = \sum_{k=-\infty}^{\infty} h[k]\, z^{-k}$$

WE NOTE THE FOLLOWING

$$H(z)\Big|_{z=1} = \sum_{k=-\infty}^{\infty} h[k] = 0 \qquad (1)$$

ALSO IF WE EVALUATE THE DERIVATIVE OF $H(z)$ AT $z=1$ WE OBTAIN

$$\frac{dH(z)}{dz}\Big|_{z=1} = -\sum_{k=-\infty}^{\infty} k\,h[k]\,z^{-k-1}\Big|_{z=1} = -\sum_{k} k\,h[k] = 2(1-z)\Big|_{z=1} = 0 \qquad (2)$$

~~FINALLY~~

$$\cancel{\frac{d^2 H(z)}{dz^2}}$$

CONSEQUENTLY, IF $x[n] = a_0$

THEN $$h[n] * x[n] = \sum_k h[k]\,x[n-k] = a_0 \underbrace{\sum_k h[k]}_{=0} = 0$$

BECAUSE OF (1)

IF $x[n] = a_0 + a_1 n$

THEN
$$h[n] * x[n] = \sum_k h[k]\,x[n-k]$$
$$= a_0 \underbrace{\sum_k h[k]}_{=0} + a_1 \sum_k (n-k)\,h[k]$$
$$= a_1 n \underbrace{\sum_k h[k]}_{=0 \text{ BECAUSE OF } (1)} - a_1 \underbrace{\sum_k k\,h[k]}_{=0 \text{ BECAUSE OF } (2)} = 0$$

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QUESTION 2

(a)

$$P_1(z) = P_0(z) + z^{2^0} Q_0(z) = 1 + z$$

$$Q_1(z) = P_0(z) - z^{2^0} Q_0(z) = 1 - z$$

$$P_2(z) = P_1(z) + z^2 Q_1(z) = 1 + z + z^2 - z^3$$

$$Q_2(z) = P_1(z) - z^2 Q_1(z) = 1 + z - z^2 + z^3$$

//

(b) WE PROVE THIS BY INDUCTION:

$$P_1(z) P_1(z^{-1}) + Q_1(z) Q_1(z^{-1}) = (1+z)(1+z^{-1}) + (1-z)(1-z^{-1}) = 4$$

$$P_1(z) P_1(-z^{-1}) + Q_1(z) Q_1(-z^{-1}) = (1+z)(1-z^{-1}) + (1-z)(1+z^{-1}) = 0$$

WE NOW ASSUME THAT PR CONDITIONS ARE SATISFIED BY $P_n(z)$, $Q_n(z)$ AND SHOW THAT THIS IMPLIES THAT THE PR CONDITIONS ARE ALSO SATISFIED BY $P_{n+1}(q)$, $Q_{n+1}(z)$:

$$P_{n+1}(z) P_{n+1}(z^{-1}) + Q_{n+1}(z) Q_{n+1}(z^{-1}) =$$

$$= \left(P_n(z) + z^{2^n} Q_n(z)\right)\left(P_n(z^{-1}) + z^{-2^n} Q_n(z^{-1})\right) +$$

$$+ \left(P_n(z) - z^{2^n} Q_n(z)\right)\left(P_n(z^{-1}) - z^{-2^n} Q_n(z^{-1})\right) =$$

$$= P_m(z)P_m(z^{-1}) + Q_m(z)Q_m(z^{-1}) + z^{-2^m}P_m(z)Q_m(z)+$$

$$+ z^{2^m}Q_m(z)P_m(z^{-1}) - z^{-2^m}P_m(z)Q_m(z^{-1}) - z^{2^m}Q_m(z)P_m(z^{-1})+$$

$$+ P_m(z)P_m(z^{-1}) + Q_m(z)Q_m(z^{-1}) =$$

$$= 2\Big[\underbrace{P_m(z)P_m(z^{-1}) + Q_m(z)Q_m(z^{-1})}_{=K_m \text{ (BY HYPOTHESIS)}}\Big] = 2K_m .$$

SIMILARLY BY INDUCTION WE CAN SHOW THAT

$$P_{m+1}(z)\,P_{m+1}(-z^{-1}) + Q_{m+1}(z)\,Q_{m+1}(-z^{-1}) = 0$$

FINALLY, SINCE $K_{m+1} = 2K_m$ AND $K_1 = 4$

WE CONCLUDE THAT

$$K_m = 2^{(m+1)}$$

//

(c) i. $y_0[m]$ OR $y_1[m]$ ARE ZERO ONLY IF $P_2(z)$ OR $Q_2(z)$ HAVE A ZERO AT $\omega = 0$ $(z=1)$.

HOWEVER

$$P_2(z)\Big|_{z=1} = 4 \quad \text{AND} \quad Q_2(z)\Big|_{z=1} = 2$$

THEREFORE BOTH $y_0[n]$ AND $y_1[n]$ WILL BE NON-ZERO

ii.

DAUBECHIES FILTERS ARE THE SHORTEST FILTERS WITH THE MAXIMUM NUMBER OF ZEROS AT $w=0$ LEADING TO PR ORTHOGONAL FILTER BANKS.

WE ALSO KNOW FROM THE COURSE THAT IF WE WANT p ZEROS AT $w=0$ THEN THE 4 FILTERS HAVE LENGTH $2 \cdot p$. CONSEQUENTLY, SINCE OUR FILTERS HAVE LENGTH 4 WE CAN HAVE AT MOST $p=2$ ZEROS AT $w=0$. THIS IS ENOUGH TO ANNIHILATE POLYNOMIAL OF MAXIMUM DEGREE ONE.

THEREFORE THE ANSWER IS YES.

THE FILTER WILL BE DAUBECHIES 2.

7

QUESTION 3

(a)

$$\beta_m^+(t) = \beta_{m-1}^+(t) * \beta_0^+(t) \; ;$$

$$\frac{d\,\beta_m^+(t)}{dt} = \frac{d}{dt}\left[\beta_{m-1}^+(t) * \beta_0^+(t)\right]$$

$$= \beta_{m-1}^+(t) * \frac{d\beta_0^+(t)}{dt}$$

$$= \beta_{m-1}^+(t) * \left[\delta(t) - \delta(t-1)\right]$$

$$= \beta_{m-1}(t) - \beta_{m-1}(t-1) .$$

//

(b) $$x(t) = \sum_{k=-\infty}^{\infty} \alpha_k \beta_m^+(t-k) \; ;$$

$$\frac{dx(t)}{dt} = \frac{d}{dt}\left[\sum_k \alpha_k \beta_m^+(t-k)\right]$$

$$= \sum_k \alpha_k \frac{d\beta_m^+(t-k)}{dt}$$

$$= \sum_k \alpha_k\left[\beta_{m-1}^+(t-k) - \beta_{m-1}^+(t-k-1)\right]$$

$$= \sum_k \alpha_k \beta_{m-1}^+(t-k) - \sum_\ell \alpha_\ell \beta_{m-1}^+(t-\ell-1) ,$$

REPLACING $l+1=k$ IN THE SECOND SUM YIELDS :

$$\frac{d\,x(t)}{dt} = \sum_k d_k \beta^+_{m-1}(t-k) - \sum_k d_{k-1} \beta^+_{m-1}(t-k)$$

$$= \sum_k (d_k - d_{k-1}) \beta^+_{m-1}(t-k)$$

$$= \sum_k \tilde{d}_k \beta^+_{m-1}(t-k)$$

$\square$

(c)

$$\frac{d\,x(t)}{dt} = \sum_{k=-\infty}^{\infty} \tilde{d}_k \beta^+_0(t-k)$$

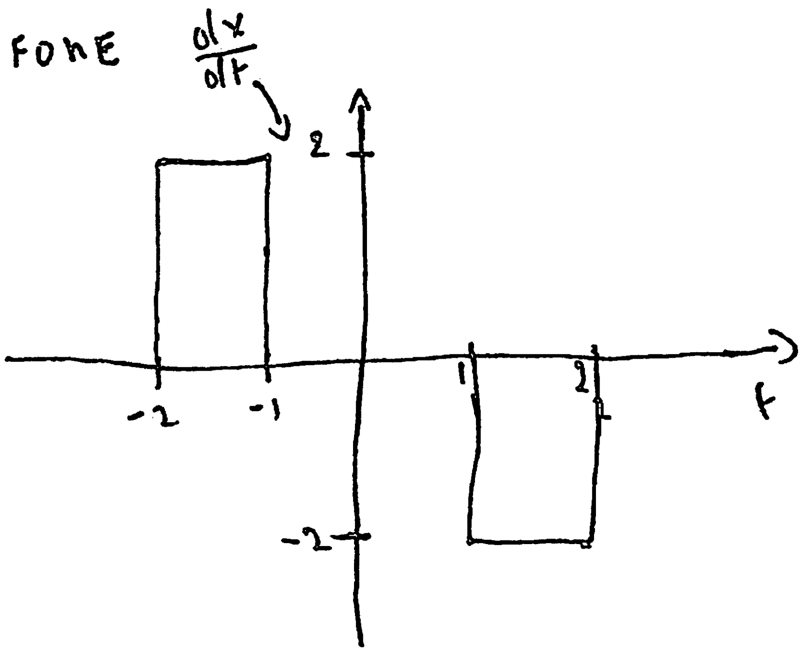
WITH $\tilde{d}_{-1} = 2$, $\tilde{d}_1 = -2$ AND

$\tilde{d}_k = 0$ OTHERWISE

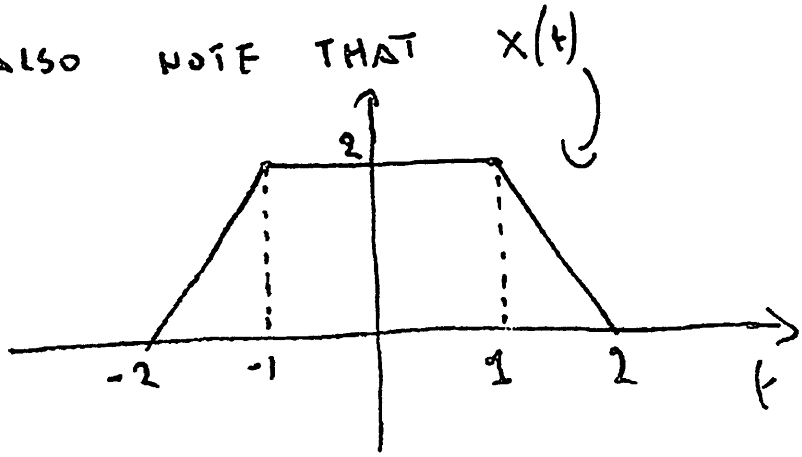
(d) $$\frac{dx}{dt} = \sum_k \tilde{d}_k \, \beta_0^+(t-k) \qquad (1)$$

WITH $$\tilde{d}_k = \begin{cases} 2 & \text{FOR } k=-1 \\ -2 & \text{FOR } k=1 \\ 0 & \text{OTHERWISE} \end{cases}$$

THEREFORE



WE ALSO NOTE THAT



SO $\frac{dx}{dt}$ GIVEN BY (1) IS THE CORRECT DERIVATIVE OF $x(t)$

(2) CONTRARY TO $\{\beta_1^+(t-k)\}_{k\in Z}$, THE SET $\{\beta_0^+(t-k)\}_{k\in Z}$ LEADS TO AN ORTHONORMAL BASIS.

THEREFORE, THE ORTHOGONAL PROJECTION OF $x(t)$ ONTO V_0 IS GIVEN BY

$$x_\perp(t) = \sum_k c_k \beta_0^+(t-k)$$

WITH

$$c_k = \langle x(t), \beta_0^+(t-k) \rangle$$

SINCE

$$x(t) = \sum_{l=-1}^{1} d_l \beta_1^+(t-l) = 2 \sum_{l=-1}^{1} \beta_1^+(t-l)$$

WE HAVE THAT:

$$c_k = \langle 2\sum_{l=-1}^{1} \beta_1^+(t-l), \beta_0^+(t-k) \rangle$$

$$= 2 \sum_{l=-1}^{1} \langle \beta_1^+(t-l), \beta_0^+(t-k) \rangle .$$

WE NOTE THAT:

$$\langle \beta_1^+(t), \beta_0^+(t-k) \rangle = \begin{cases} 1/2 & \text{FOR } k=0,1 \\ 0 & \text{OTHERWISE} \end{cases}$$

CONSEQUENTLY

$$c_k = \begin{cases} 1 & k=-2 \\ 2 & k=-1,0, \\ 1 & k=2 \\ 0 & \text{OTHERWISE} \end{cases}$$

//

QUESTION 4

(a)
$$a[n] = \begin{cases} 2/3 & \text{FOR } n = 0 \\ 1/6 & \text{FOR } n = \pm 1 \\ 0 & \text{OTHERWISE} \end{cases}$$

$$A(e^{jw}) = \sum_{k=-\infty}^{\infty} |\hat{\varphi}(w + 2\pi k)|^2 \quad (1)$$

THUS, IF

$$\hat{\phi}(w) = \frac{\hat{\varphi}(w)}{\sqrt{A(e^{jw})}} \qquad (2)$$

WE HAVE THAT

$$\sum_{k=-\infty}^{\infty} |\hat{\phi}(w + 2\pi k)|^2 \overset{(a)}{=} \sum_{k=-\infty}^{\infty} \left| \frac{\hat{\varphi}(w+2\pi k)}{\sqrt{A(e^{jw})}} \right|^2$$

$$= \frac{1}{A(e^{jw})} \sum_{k} |\hat{\varphi}(w + 2\pi k)|^2 \overset{(b)}{=} 1$$

WHERE (a) FOLLOWS FROM EQ. (2) AND FROM THE FACT THAT $A(e^{jw})$ IS PERIODIC OF PERIOD 2π, AND (b) FROM (1).

(b)

$$\hat{\varphi}(\omega) = \frac{G_0(e^{j\omega/2})}{\sqrt{2}} \hat{\varphi}(\omega/2) \Rightarrow$$

$$\hat{\phi}(\omega) = \frac{\hat{\varphi}(\omega)}{\sqrt{A(e^{j\omega})}} = \frac{G_0(e^{j\omega/2})}{\sqrt{2A(e^{j\omega})}} \hat{\varphi}\left(\frac{\omega}{2}\right) \Rightarrow$$

$$\hat{\phi}(\omega) = \frac{1}{\sqrt{2}} G_0(e^{j\omega/2}) \sqrt{\frac{A(e^{j\omega/2})}{A(e^{j\omega})}} \hat{\phi}\left(\frac{\omega}{2}\right)$$

THUS

$$H_0(e^{j\omega/2}) = G_0(e^{j\omega/2}) \sqrt{\frac{A(e^{j\omega/2})}{A(e^{j\omega})}}$$

AND

$$H_0(z) = G_0(z) \sqrt{\frac{A(z)}{A(z^2)}}$$

#

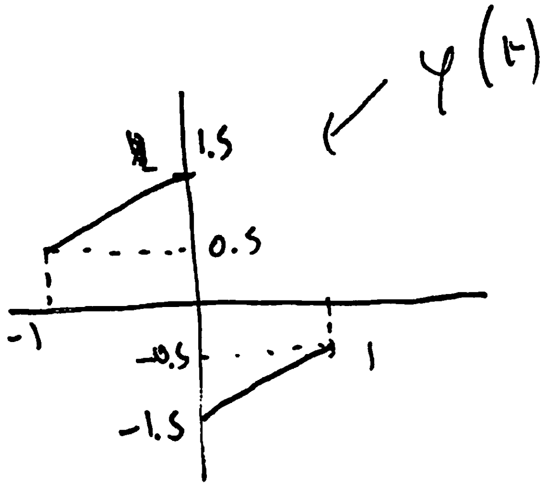
(c) NO, BECAUSE ONE OF THE THREE CRITERIA OF A VALID SCALING FUNCTION ~~...~~ IS NOT SATISFIED

GRAPHICALLY WE SEE THAT

(i) PARTITION OF UNITY IS SATISFIED:

$$\sum_k \varphi(t-k) = 1$$

AS



(ii) BUT THE TWO SCALE EQUATION

$$\varphi(t) = \sqrt{2} \sum_m g_0[m] \varphi(2t-m)$$

~~...~~ IS NOT SATISFIED

(iii)

MESTH BOUND CRITERION IS SATISFIED WITH

$$A = 1 \quad \text{AND} \quad B = 4/3$$