IMPERIAL COLLEGE LONDON

**E4.10**
**C2.1**
**SC4**

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2005

MSc and EEE PART IV: MEng and ACGI

Corrected Copy

**PROBABILITY AND STOCHASTIC PROCESSES**

Monday, 16 May 10:00 am

Time allowed: 3:00 hours

**There are SIX questions on this paper.**

**Answer FOUR questions.**

*All questions carry equal marks*

**Any special instructions for invigilators and information for candidates are on page 1.**

| Examiners responsible | First Marker(s) : | K.K. Leung |
|---|---|---|
| | Second Marker(s) : | R.B. Vinter |

Special Instructions for Invigilator: **None**

Information for Students: **None**

a. Assume that scalar random variables, $X_1$, $X_2$,..., $X_n$, are independent and identically distributed with a common probability distribution function (PDF) $F_X(x)$. Define two new random variables $U$ and $V$ as

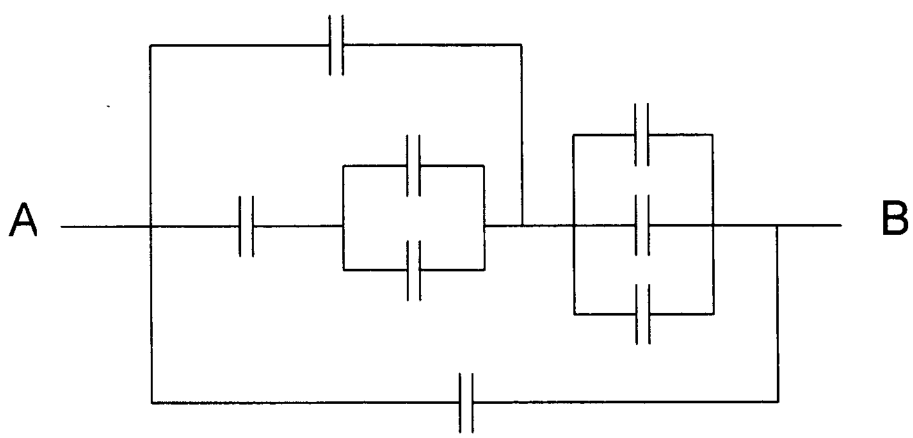
$$U = \min\{X_1, X_2, ..., X_n\}$$

$$\text{and } V = \max\{X_1, X_2, ..., X_n\}.$$

Determine the PDFs for $U$ and $V$. [14]

b. In the following diagram, each --||-- represents one communication link. Link failures are independent and each link has a probability of 0.5 of being out of service. Towns A and B can communicate as long as they are connected in the communication network by at least one path which contains only in-service links. Determine the probability that A and B can communicate.

[11]



2. For any given scalar random variables $X$, $Y$ and $Z$, prove the following properties of covariance.

a. Show that $\operatorname{cov}(X, Y+Z) = \operatorname{cov}(X, Y) + \operatorname{cov}(X, Z)$. [3]

b. Show that $\operatorname{cov}(X, Y) = \operatorname{cov}(X, E[Y \mid X])$. [5]

c. Suppose that $E[Y \mid X] = \alpha + \beta X$ for some constants $\alpha$ and $\beta$. Using the result in part b above, show that

$$\beta = \frac{\operatorname{cov}(X, Y)}{\operatorname{var}(X)}.$$ [5]

d. If $E[Y \mid X] = 1$, show that

$$\operatorname{var}(XY) \geq \operatorname{var}(X).$$ [12]

(Hint: Part d does not make use of results in parts a to c. Use the fact that for any random variable $U$, $E[U^2] \geq \{E[U]\}^2$, and consider the conditional expectation $E[X^2Y^2 \mid X]$ to first prove that $E[X^2Y^2] \geq E[X^2]$.)

3. Adam and Brian are the only participants in a race, and their elapsed times are characterized by two independent random variables X and Y, respectively. Their respective probability density functions (pdf's) are given by

$$f_X(x) = \begin{cases} 0.0 & x < 1 \\ 1.0 & 1 \le x \le 2 \\ 0.0 & x > 2 \end{cases} \quad \text{and} \quad f_Y(\overset{y}{\cancel{x}}) = \begin{cases} 0.0 & y < 1 \\ 0.5 & 1 \le y \le 3 \\ 0.0 & y > 3 \end{cases}$$

11-25

A participant wins a race if his elapsed time is shorter than the other. Let $A$ denote the event "Adam won the race."

a. Find the joint probability density function for X and Y. [3]

b. Determine the probability of event A, $P(A)$. [3]

c. Find the conditional probability density function $f_{X \mid A}(x \mid A)$. [6]

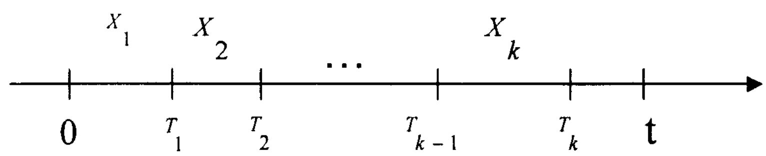
d. Let $W = Y - X$. Determine the mean $E[W]$ and the conditional mean $E[W \mid A]$. (Hint: To find the conditional mean, consider a new random variable $Z = -X$ so that $W = Y + Z$ and use the convolution integral to first determine the pdf for $W$.) [13]

4. a. A random variable X has an exponential probability density function (pdf) with parameter $\mu > 0$. That is, $f_X(x) = \mu e^{-\mu x}$ if $x \geq 0$, and 0 otherwise. Determine its characteristic function $\Phi_X(\omega)$. Hence, or otherwise, evaluate the mean and standard deviation of X. [12]

b. The number of calls arriving at a telephone switch in the time interval $[0, t]$ is modelled by a random variable $N(t)$ which has a Poisson probability mass function with parameter $\mu t$. That is,

$$P[N(t) = k] = \frac{(\mu t)^k}{k!} e^{-\mu t} \quad \text{for } k = 0, 1, 2, \ldots$$

Let $T_n$ be the arrival time of the n-th call and $X_n$ be the time interval between the (n-1)st and n-th call arrivals as shown in the following diagram. For simplicity, it is assumed that a call arrives immediately before time 0, which is not shown in the diagram.




Define $S_k = \sum_{n=1}^{k} X_n$ for all $k \geq 1$.

i. Show that $S_k$ has an Erlangian distribution. That is, the pdf for $S_k$ is

$$f_{S_k}(t) = \mu e^{-\mu t} \frac{(\mu t)^{k-1}}{(k-1)!} \quad \text{for } t \geq 0.$$ [9]

(Hint: For each $k$, the event $\{N(t) \geq k\}$ is identical to the event $\{S_k \leq t\}$.)

ii. Find the pdf for the interarrival times $X_n$'s. [4]

(Hint: Use result in part i and the fact that all $X_n$'s are independent and identically distributed.)

5. a. Let $X$, $Y$ and $Z$ be zero-mean random variables. Determine the linear least squares estimate $\hat{Z} = \alpha X + \beta Y$ of $Z$ given $X$ and $Y$, i.e., find $\alpha$ and $\beta$ to minimize the mean square error. You should express the optimal values of $\alpha$ and $\beta$ in terms of variances and covariances of the random variables. [14]

b. A stationary, second order, stochastic process $\{y_k\}$ is given by the following difference equation:

$$y_k + a y_{k-1} + b y_{k-2} = e_k$$

where $a$ and $b$ are constants and $\{e_k\}$ is a sequence of zero mean, uncorrelated random variables with unit variance.

i. Find the first three values of the covariance function $r_y(k)$, $k = 0$, 1 and 2 for the sequence $\{y_k\}$. [7]

ii. Using results in part a, determine the linear least squares estimate of $y_k$ given $y_{k-1}$ and $y_{k-2}$. [4]

6. Consider an urn with 3 balls. Each ball is colored either white (W) or black (B). At the end of each day, exactly one ball is withdrawn at random from the urn and it is examined, possibly repainted, and then returned to the urn as follows:

i. If the drawn ball is B, it is then repainted W and returned.

ii. If the drawn ball is W, then with probability $\alpha$, the ball is repainted B and returned. With probability $1-\alpha$, the ball is returned as is.

a. Let $M(n)$ be the number of B balls in the urn at the beginning of the n-th day for n=1, 2, 3,... Define a random process with $M(n)$ as its state and argue why the random process is a Markov chain. What are the possible states of the Markov chain? [4]

b. Find the matrix of state transition probabilities P for the Markov chain. [9]

c. Let the state probability vector at the beginning of the n-th day be denoted by $\underline{\pi}(n) = [\pi_0(n), \pi_1(n), \pi_2(n), \ldots]$ where $\pi_k(n) = P[M(n) = k]$ for $k = 0, 1, 2, \ldots$ Assume that all balls are B in the urn at the beginning of the first day. Derive a general formula for the state probability vector, $\underline{\pi}(4)$, at the beginning of the 4-th day. (You are not required to carry out detailed calculations.) [4]

d. Assume that $\alpha = 0.5$. Find the limiting state probabilities for $n \to \infty$ and the mean recurrence time for each state. [8]

Model Answer for 2005 exam - Prob. & Stochastic Processes

KKL

Q.1:

a

$$F_U(y) = P[U \le y] = 1 - P[U > y]$$

$$\Rightarrow \quad F_U(y) = 1 - P[\min\{X_1, \cdots, X_n\} > y]$$

$$= 1 - P[X_1 > y, X_2 > y, \cdots, X_n > y]$$

$$= 1 - P[X_1 > y] P[X_2 > y] \cdots P[X_n > y] \quad \because \text{independence}$$

$$= 1 - \prod_{i=1}^{n} P[X_i > y]$$

$$\Rightarrow \quad F_U(y) = 1 - \prod_{i=1}^{n} [1 - F_X(y)]$$

$$\Rightarrow \quad F_U(y) = 1 - [1 - F_X(y)]^n$$

For random variable V,

$$F_V(y) = P[V \le y]$$

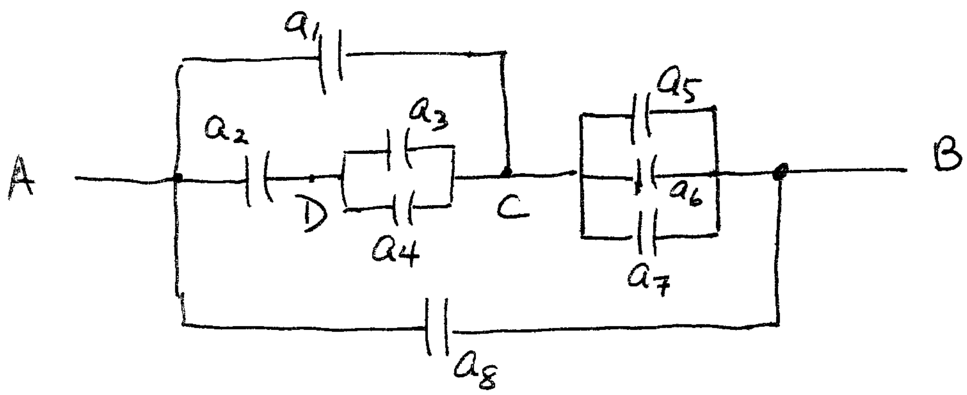
$$= P[\max\{X_1, \cdots, X_n\} \le y]$$

$$= P[X_1 \le y, X_2 \le y, \cdots, X_n \le y]$$

$$= \prod_{i=1}^{n} P[X_i \le y] \quad \because \text{independence}$$

$$\Rightarrow \quad F_V(y) = [F_X(y)]^n$$

b




Let $X$ and $\bar{X}$ denote a path or link in working order and out of order, respectively.

To find $P(AB)$, let us consider $P(\overline{AB})$ i.e., prob that A and B cannot communicate.

$$P(\overline{AB}) = P(\bar{a}_8)\, P(\overline{AC} \text{ or } \overline{CB})$$

$$= \frac{1}{2} P(\overline{AC} \text{ or } \overline{CB}) \quad \text{—— ①}$$

Now let's consider

$$P(AC \text{ and } CB) = P(AC) \cdot P(CB)$$

$$= P(AC) \cdot \left[1 - \left(\tfrac{1}{2}\right)^3\right]$$

$$\Rightarrow P(AC \text{ and } CB) = \frac{7}{8} P(AC)$$

$$= \frac{7}{8}\left[1 - P(\overline{AC})\right] \quad \text{—— ②}$$

We have $\quad P(\overline{AC}) = P(\bar{a}_1) \cdot P(\overline{ADC})$

$$= \frac{1}{2}\left[1 - P(ADC)\right]$$

Since $\quad P(ADC) = P(a_2)\, P(a_3 \text{ or } a_4)$

$$= \frac{1}{2}\left[1 - \left(\tfrac{1}{2}\right)^2\right] = \frac{1}{2}\left(\frac{3}{4}\right) = \frac{3}{8},$$

we have

$$P(\overline{AC}) = \frac{1}{2}\left[1 - \frac{3}{8}\right] = \frac{1}{2} \cdot \frac{5}{8} = \frac{5}{16}$$

Subs. the above into ②, we get

$$P(AC \text{ and } CB) = \frac{7}{8}\left[1 - \frac{5}{16}\right] = \frac{7}{8} \cdot \frac{11}{16} = \frac{77}{128}$$

From ①,

$$P(\overline{AB}) = \frac{1}{2}\left[1 - P(AC \text{ and } CB)\right]$$

$$= \frac{1}{2}\left[1 - \frac{77}{128}\right] = \frac{51}{256}$$

Thus, $\quad P(AB) = 1 - P(\overline{AB}) = 1 - \frac{51}{256} = \frac{205}{256}.$

Q2:



a) $\text{cov}(X, Y+Z) = E[X(Y+Z)] - E(X)E[Y+Z]$

$= E[XY] + E[XZ] - E(X)E(Y) - E(X)E(Z)$

$\Rightarrow \text{cov}(X, Y+Z) = \text{cov}(X,Y) + \text{cov}(X,Z)$

b) $\text{cov}(X, E(Y/X)) = E(X\,E(Y/X)) - E(X)\,E(E(Y/X))$

$\Rightarrow \text{cov}(X, E(Y/X)) = E(XY) - E(X)E(Y)$

$= \text{cov}(X,Y).$

c) given $E(Y/X) = a + bX$,

$\text{cov}(X,Y) = \text{cov}(X, E(Y/X))$ from part b.

$= \text{cov}(X, a+bX)$ as $E(Y/X) = a+bX$

$= E[X(a+bX)] - E(X)E(a+bX)$

$\Rightarrow \text{cov}(X,Y) = \cancel{aE(X)} + bE(X^2) - \cancel{aE(X)} - bE(X)E(X)$

$= b[E(X^2) - \{E(X)\}^2]$

$\Rightarrow \text{cov}(X,Y) = b\,\text{var}(X) \Rightarrow b = \dfrac{\text{cov}(X,Y)}{\text{var}(X)}.$

d) $E[X^2Y^2/X] = X^2E[Y^2/X]$

$\geq X^2\{E[Y/X]\}^2$ $\because$ for any R.V. $U$, $E[U^2] \geq E^2[U]$

$= X^2$ $\because E[Y/X] = 1$, given.

$\Rightarrow E\{E[X^2Y^2/X]\} \geq E[X^2]$

$\Rightarrow E[X^2Y^2] \geq E[X^2]$

As $E[XY] = E[E[XY|X]] = E[X \underbrace{E[Y|X]}_{=1}]$

$$= E[X]$$

Therefore, $E[X^2Y^2] - \{E[XY]\}^2 \geq E[X^2] - \{E[X]\}^2$

$\Rightarrow \quad var(XY) \geq var(X).$

Q3 :

a) To express the pdf's for X and Y in a compact form, we have

$$f_X(x) = u(x-1) - u(x-2)$$

and

$$f_Y(y) = \frac{1}{2}\left[u(y-1) - u(y-3)\right]$$

where

$$u(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Since X and Y are independent, the joint pdf for X and Y is

$$f_{XY}(x,y) = \frac{1}{2}\left[u(x-1) - u(x-2)\right] \cdot \left[u(y-1) - u(y-2)\right]$$

That is,

$$f_{XY}(x,y) = \begin{cases} \frac{1}{2} & \text{if } 1 \leq x \leq 2 \text{ \& } 1 \leq y \leq 3 \\ 0 & \text{otherwise.} \end{cases}$$

b

$$P(A) = P_r(X < Y)$$

$$= \int_{x=1}^{2} \int_{y=x}^{3} f_{XY}(x,y)\,dy\,dx$$

$$= \int_{x=1}^{2} \int_{y=x}^{3} \frac{1}{2}\,dy\,dx$$

$$= \int_{x=1}^{2} \frac{1}{2}(3-x)\,dx$$

$$\Rightarrow P(A) = \frac{3}{4}$$

c. To find the conditional pdf $f_{X/A}(x/A)$, let us consider the conditional distribution function

$$F_{X/A}(x/A) = P(X \le x / A) = P(X \le x / X \le Y)$$

$$\Rightarrow F_{X/A}(x/A) = \frac{P(X \le x \text{ and } X \le Y)}{P(X \le Y)} \quad -\ ①$$

To find $P(X \le x \text{ and } X \le Y)$,

$$P(X \le x \text{ and } X \le Y) = \int_{u=-\infty}^{x} \int_{v=u}^{\infty} f_{XY}(u,v)\, dv\, du$$

For $1 \le x \le 2$,

$$P(X \le x \text{ and } X \le Y) = \int_{u=1}^{x} \int_{v=u}^{3} \frac{1}{2}\, dv\, du$$

$$= \int_{u=1}^{x} \frac{1}{2}(3-u)\, du$$

$$= \frac{1}{2}\left[3u - \frac{u^2}{2}\right]_{u=1}^{x}$$

$$\Rightarrow P(X \le x \text{ and } X \le Y) = \frac{3}{2}x - \frac{x^2}{4} - \frac{5}{4}$$

For $x > 2$,

$$P(X \le x \text{ and } X \le Y) = \int_{u=1}^{2} \int_{v=u}^{3} \frac{1}{2}\, dv\, du$$

$$= \frac{1}{2}\left[3u - \frac{u^2}{2}\right]_{u=1}^{2}$$

$$= \frac{3}{4}. \quad \text{i.e. } P(X \le Y)$$

For $x < 1$,

$$P(X \le x \text{ and } X \le 4) = 0.$$

Therefore,

$$F_{X/A}(x/A) = \frac{P(X \le x \text{ and } X \le 4)}{P(X \le 4)}$$

$$= \frac{P(X \le x \text{ and } X \le 4)}{3/4}$$

$\Rightarrow$

$$F_{X/A}(x/A) = \begin{cases} 0 & \text{if } x < 1 \\ 2x - \frac{x^2}{3} - \frac{5}{3} & \text{if } 1 \le x \le 2 \\ 1 & \text{if } x \ge 2 \end{cases}$$

Differentiate the above w.r.t. $x$, we get

$$f_{X/A}(x/A) = \begin{cases} 2 - \frac{2}{3}x & \text{if } 1 \le x \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

d. $W = Y - X \quad \Rightarrow \quad E[W] = E[Y] - E[X]$

$\Rightarrow \quad E[W] = 2 - 1\frac{1}{2} = \frac{1}{2}.$

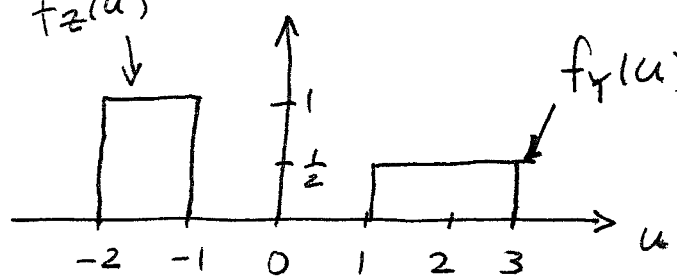
To find $E[W|A]$, we need to obtain $f_{W|A}(w|A)$.

Towards this goal, let us first find $f_W(w)$.

Given $W = Y - X$. We define $Z = -X$, Then $W = Y + Z$.

The pdf's for $Y$ and $Z$ are given in the following diagram:

Fig. 1.

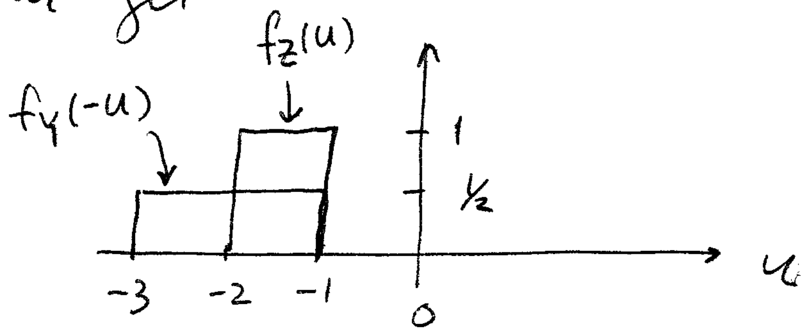



Since $Y$ and $Z$ are independent, we have by convolution,

$$f_W(w) = \int_{-\infty}^{\infty} f_Y(w-u)\, f_Z(u)\, du \quad — \text{②}$$

From Fig 1, we get

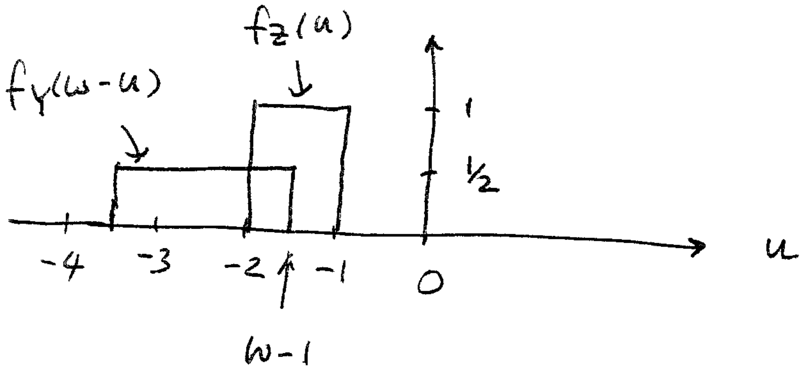
Fig. 2




From Fig 2, we can see that $f_Y(w-u) \cdot f_Z(u) = 0$ if $w \leq -1$ or $w \geq 2$ because $f_Y(w-u)$ is shifted (located) too far to the left or to the right, respectively.

We can divide $\omega$ (given $-1 \le \omega \le 2$) into 3 regions to carry out the integration in ②.

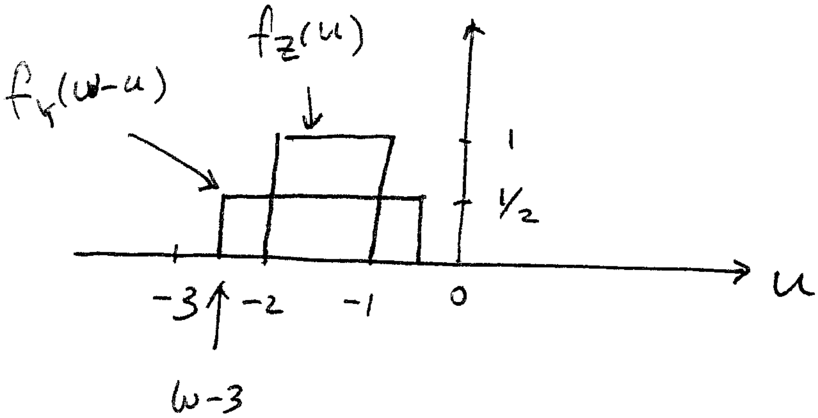
$1^{st}$ region: For $-1 \le \omega \le 0$




$$f_W(\omega) = \int_{-2}^{\omega-1} f_Y(\omega-u)\, f_Z(u)\, du = \int_{-2}^{\omega-1} \frac{1}{2} \cdot 1\, du$$

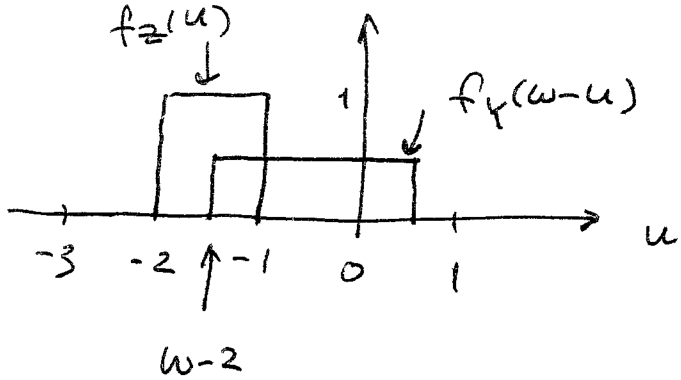
$$\Rightarrow \quad f_W(\omega) = \frac{\omega+1}{2}.$$

$2^{nd}$ region: For $0 \le \omega \le 1$




$$f_W(\omega) = \int_{-2}^{-1} f_Y(\omega-u)\, f_Z(u)\, du = \int_{-2}^{-1} \frac{1}{2} \cdot 1 \cdot du = \frac{1}{2}.$$

3rd region: For $1 \le w \le 2$ :




$$f_W(w) = \int_{w-2}^{-1} f_Y(w-u) f_Z(u)\, du = \int_{w-2}^{-1} \frac{1}{2} \cdot 1 \cdot du$$

$$\Rightarrow \quad f_W(w) = \frac{1-w}{2}.$$

By conditional pdf,

$$f_{W|A}(w|A)\, dw = \frac{P(w < W \le w+dw \text{ and } w>0)}{P(W>0)}$$

(Note that event $\{W>0\}$ is equivalent to the event A — Adam wins.)

Since $P(W>0) = P(A) = 3/4$ from part a,

$$f_{W|A}(w|A) = \begin{cases} 0 & w \le 0 \\ \frac{1}{2} \cdot \frac{4}{3} = \frac{2}{3} & 0 \le w \le 1 \\ \frac{1-w}{2} \cdot \frac{4}{3} = (1-w) \cdot \frac{2}{3} & 1 \le w \le 2 \end{cases}$$

$$E[W|A] = \int_{-\infty}^{\infty} w \, f_{W/A}(w/A) \, dw$$

$$= \int_0^1 w \cdot \frac{2}{3} \, dw + \int_1^2 \frac{2}{3}(1-w) w \, dw$$

$$\Rightarrow E[W/A] = 7/9$$

Q4 

a. $f_X(x) = \mu e^{-\mu x}$ for $x \geq 0$.

$$\phi_X(\theta) = \int^{\infty} f_X(x) e^{j\omega x} dx = \int^{\infty} \mu e^{-\mu x} e^{j\omega x} dx$$

$$\Rightarrow \phi_X(\theta) = \int^{\infty} \mu e^{-(\mu - j\omega)x} dx$$

$$= \frac{\mu}{\mu - j\omega}$$

$$E(x) = \int_0^{\infty} \mu e^{-\mu x} x\, dx = -\int_0^{\infty} x\, d e^{-\mu x}$$

$$\Rightarrow E(x) = -x e^{-\mu x}\Big|_0^{\infty} + \int_0^{\infty} e^{-\mu x} dx$$

$$= \frac{-1}{\mu} e^{-\mu x}\Big|_0^{\infty} = \frac{1}{\mu}$$

$$E(x^2) = \int_0^{\infty} \mu e^{-\mu x} x^2 dx = -\int_0^{\infty} x^2 d e^{-\mu x}$$

$$\Rightarrow E(x^2) = -x^2 e^{-\mu x}\Big|_0^{\infty} + \int_0^{\infty} e^{-\mu x} d x^2$$

$$= \int_0^{\infty} e^{-\mu x} 2x\, dx = 2\int_0^{\infty} x e^{-\mu x} dx$$

$$\Rightarrow E(x^2) = -\frac{2}{\mu} \int_0^{\infty} x\, d e^{-\mu x}$$

$$= -\frac{2}{\mu}\left[ x e^{-\mu x}\Big|_0^{\infty} - \int_0^{\infty} e^{-\mu x} dx \right]$$

$$= -\frac{2}{\mu}\left[ \frac{1}{\mu} \int_0^{\infty} d e^{-\mu x} \right] = \frac{-2}{\mu^2}\left[ e^{-\mu x}\Big|_0^{\infty} \right]$$

$$\Rightarrow E(x^2) = 2/\mu^2$$

$$\mathrm{var}(X) = E(X^2) - \{E(X)\}^2$$

$$= \frac{2}{\mu^2} - \left(\frac{1}{\mu}\right)^2 = \frac{1}{\mu^2}$$

$$\text{s.d. of } X = \frac{1}{\mu}$$

b. i) Note that the two events are equivalent:

$$\{N(t) = k\} \iff \{S_k \le t\}$$

Thus,
$$P(S_k \le t) = P(N(t) \ge k)$$
$$= \sum_{j=k}^{\infty} e^{-\mu t} \frac{(\mu t)^j}{j!}$$

Differentiate the above. We get

$$f_{S_k}(t) = -\sum_{j=k}^{\infty} \mu e^{-\mu t} \frac{(\mu t)^j}{j!} + \sum_{j=k}^{\infty} \mu e^{-\mu t} \frac{(\mu t)^{j-1}}{(j-1)!}$$
$$= \mu e^{-\mu t} \cdot \frac{(\mu t)^{k-1}}{(k-1)!} \quad \text{for } t \ge 0.$$

ii) Since $S_k = \sum_{i=1}^{k} X_i$, we can obtain from

$f_{S_k}(t)$ that the pdf for $X_i$'s

$$f_X(t) = \mu e^{-\mu t}.$$

This is so because the sum of k independent, exponentially distributed random variables has an Erlangian pdf.

Q5:

a) Let $f = E[(Z-\alpha X-\beta Y)^2]$ find $\alpha, \beta$ to minimize $f$:

$$\frac{\partial f}{\partial \alpha} = E[2(Z-\alpha X-\beta Y)(-X)] = 0$$

$$\Rightarrow \quad E[XZ-\alpha X^2+\beta XY] = 0 \qquad \text{—} ①$$

$$\frac{\partial f}{\partial \beta} = E[2(Z-\alpha X-\beta Y)(-Y)] = 0$$

$$\Rightarrow \quad E[YZ-\alpha XY+\beta Y^2] = 0 \qquad \text{—} ②$$

From ① and ②, we have

$$E(X^2)\alpha - E(XY)\beta = E(XZ)$$

$$E(XY)\alpha - E(Y^2)\beta = E(YZ)$$

Solving the two equations and using the facts that $E(X)=E(Y)=E(Z)=0$ yield

$$\alpha = \frac{var(Y)\,cov(X,Z) - cov(X,Y)\,cov(Y,Z)}{var(X)\,var(Y) - \{cov(X,Y)\}^2}$$

$$\beta = \frac{var(X)\,cov(Y,Z) - cov(X,Y)\,cov(X,Z)}{var(X)\,var(Y) - \{cov(X,Y)\}^2}$$

b. Given $y_k + a y_{k-1} + b y_{k-2} = e_k$

multiply $y_k$ and take expectation:

$$E(y_k^2) + aE(y_k y_{k-1}) + bE(y_k y_{k-2}) = E(y_k e_k)$$

$$\Rightarrow \quad r_y(0) + a r_y(1) + b r_y(2) = E(y_k e_k) \quad — (3)$$

multiply $y_{k-1}$ and take expectation:

$$E(y_k y_{k-1}) + aE(y_{k-1} y_{k-1}) + bE(y_{k-2} y_{k-1}) = \underbrace{E(y_{k-1} e_k)}_{0}$$

$$\Rightarrow \quad r_y(1) + a r_y(0) + b r_y(1) = 0 \quad — (4)$$

Multiply $y_{k-2}$ and take expectation:

$$E(y_k y_{k-2}) + aE(y_{k-1} y_{k-2}) + bE(y_{k-2} y_{k-2}) = \underbrace{E(y_{k-2} e_k)}_{0}$$

$$\Rightarrow \quad r_y(2) + a r_y(1) + b r_y(0) = 0 \quad — (5)$$

Multiple $e_k$ and take expectation:

$$E(y_k e_k) + a\overbrace{\cancel{E(y_{k-1} e_k)}}^{0} + b\overbrace{\cancel{E(y_{k-2} e_k)}}^{0} = E(e_k e_k)$$

$$\Rightarrow \quad E(y_k e_k) = 1 \quad —$$

Put the above in (3). We have 3 linear equations:

$$r_y(0) + a r_y(1) + b r_y(2) = 1 \quad — (6)$$

$$a r_y(0) + (1+b) r_y(1) = 0 \quad — (7)$$

$$b r_y(0) + a r_y(1) + r_y(2) = 0 \quad — (8)$$

From (8), $$r_y(1) = \frac{-a}{1+b}\, r_y(0) \qquad \text{—— (9)}$$

Subs. this into (6) and (8):

$$\begin{cases} r_y(0) \quad \dfrac{a^2}{1+b} r_y(0) + b\, r_y(2) = 1 \\ b\, r_y(0) - \dfrac{a^2}{1+b} r_y(0) + r_y(2) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \left(1 - \dfrac{a^2}{1+b}\right) r_y(0) + b\, r_y(2) = 1 \\ \left(b - \dfrac{a^2}{1+b}\right) r_y(0) + r_y(2) = 0 \end{cases}$$

From the last eqn.

$$r_y(2) = \left(-b + \frac{a^2}{1+b}\right) r_y(0)$$

$$\Rightarrow \left(1 - \frac{a^2}{1+b}\right) r_y(0) + b\left(-b + \frac{a^2}{1+b}\right) r_y(0) = 1$$

$$\Rightarrow \qquad r_y(0) = \frac{1}{1-b^2}$$

$$\Rightarrow \qquad r_y(1) = \frac{-a}{1+b} \cdot \frac{1}{1-b^2}$$

$$\Rightarrow \qquad r_y(2) = \left(-b + \frac{a^2}{1+b}\right) \cdot \frac{1}{1-b^2}$$

Let the least square estimate

$$\hat{y}_k = \alpha y_{k-1} + \beta y_{k-2}$$

i.e., $y_k \Leftrightarrow Z$, $y_{k-1} \Leftrightarrow X$ and $y_{k-2} \Leftrightarrow Y$

Using results in Part a,

$$\alpha = \frac{r_y(0)\, r_y(1) - r_y(1)\, r_y(2)}{\{r_y(0)\}^2 - \{r_y(1)\}^2}$$

and $$\beta = \frac{r_y(0)\, r_y(2) - \{r_y(1)\}^2}{\{r_y(0)\}^2 - \{r_y(1)\}^2}.$$

where $r_y(0)$, $r_y(1)$ and $r_y(2)$ are given above

Q6



a. The random value of $M(n)$ is a random process as a function of $n$. The process is a Markov chain because only the exact value of $M(n)$ is relevant in predicting $M(k)$ for $k>n$ (in the future). That is, as of how the process evolves up to day $n$ is irrelevant in predicting the future.

The possible states $\mathcal{S} = \{0, 1, 2, 3\}$

b.

$$
\begin{array}{cc|cccc|l}
\text{\# of black balls} & \text{\# of white balls} & 3 & 2 & 1 & 0 & \text{\# of white balls} \\
 & & 0 & 1 & 2 & 3 & \text{\# of black balls} \leftarrow \text{future state} \\
0 & 3 & 1-\alpha & \alpha & 0 & 0 & \\
1 & 2 & 1/3 & \frac{2}{3}(1-\alpha) & \frac{2}{3}\alpha & 0 & \\
2 & 1 & 0 & 2/3 & \frac{1}{3}(1-\alpha) & \frac{1}{3}\alpha & \\
3 & 0 & 0 & 0 & 1 & 0 &
\end{array}
$$

$\uparrow$ current state

$\underbrace{\qquad\qquad}_{\text{state transition matrix } P}$

c.

$$\underline{\Pi}(1) = (0, 0, 0, 1)$$

$$\underline{\Pi}(n) = \underline{\Pi}(n-1) P$$

$$\Rightarrow \quad \underline{\Pi}(4) = \underline{\Pi}(1) P^3$$

$$\underline{\Pi}(4) = (0,0,0,1) \begin{pmatrix} 1-\alpha & \alpha & 0 & 0 \\ 1/3 & \frac{2}{3}(1-\alpha) & \frac{2}{3}\alpha & 0 \\ 0 & 2/3 & \frac{1}{3}(1-\alpha) & \frac{1}{3}\alpha \\ 0 & 0 & 1 & 0 \end{pmatrix}^3$$

d, When $\alpha = 0.5$,

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 \\ 0 & 2/3 & 1/6 & 1/6 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Let $\underline{\pi}$ be the limiting state probs:

$$\underline{\pi} = (\pi_0, \pi_1, \pi_2, \pi_3)$$

$$\underline{\pi} = \underline{\pi} P \quad \text{and} \quad \sum_{i=0}^{3} \pi_i = 1$$

$$\Rightarrow \quad (\pi_0, \pi_1, \pi_2, \pi_3) \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 \\ 0 & 2/3 & 1/6 & 1/6 \\ 0 & 0 & 1 & 0 \end{pmatrix} = (\pi_0 \ \pi_1 \ \pi_2 \ \pi_3)$$

$$\frac{1}{2}\pi_0 + \frac{1}{3}\pi_1 = \pi_0 \quad \text{—} \ ①$$

$$\frac{1}{2}\pi_0 + \frac{1}{3}\pi_1 + \frac{2}{3}\pi_2 = \pi_1 \quad \text{—} \ ②$$

$$\frac{1}{3}\pi_1 + \frac{1}{6}\pi_2 + \pi_3 = \pi_2 \quad \text{—} \ ③$$

$$\frac{1}{6}\pi_2 = \pi_3 \quad \text{—} \ ④$$

$$\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1 \quad \text{—} \ ⑤$$

Subs. ④ into ③ : $\Rightarrow \ \pi_2 = \frac{1}{2}\pi_1$ Using ④, $\pi_3 = \frac{1}{12}\pi_1$

Subs. the above into ② : $\frac{1}{2}\pi_0 + \frac{1}{3}\pi_1 + \frac{1}{3}\pi_1 = \pi_1$

$$\Rightarrow \quad \frac{1}{2}\pi_0 = \frac{1}{3}\pi_1$$

$$\pi_0 = \frac{2}{3}\pi_1$$

Put $\pi_3 = \frac{1}{12}\pi_1$, $\pi_2 = \frac{1}{2}\pi_1$, $\pi_0 = \frac{2}{3}\pi_1$ into (5):

$$\frac{2}{3}\pi_1 + \pi_1 + \frac{1}{2}\pi_1 + \frac{1}{12}\pi_1 = 1$$

$$\Rightarrow \left(\frac{2}{3} + 1 + \frac{1}{2} + \frac{1}{12}\right)\pi_1 = 1$$

$$\Rightarrow \pi_1 = 4/9$$

$$\pi_2 = \frac{1}{2}\pi_1 = \frac{2}{9}$$

$$\pi_3 = \frac{1}{12}\pi_1 = \frac{1}{12}\cdot\frac{4}{9} = \frac{1}{27}$$

$$\pi_0 = \frac{2}{3}\pi_1 = \frac{2}{3}\cdot\frac{4}{9} = \frac{8}{27}$$

$$\Rightarrow \underline{\pi} = \left(8/27,\ 4/9,\ 2/9,\ 1/27\right)$$

Let the mean recurrence time for state $i$ be $L_i$

$$L_i = \frac{1}{\pi_i} \quad \text{for } i = 0, 1, 2, 3.$$

Thus, the corresponding mean recurrence times are

$$L_0 = \frac{27}{8}, \quad L_1 = \frac{9}{4} \quad L_2 = \frac{9}{2}, \ L_3 = 27$$