Imperial College London

[E2.11 (Maths) ISE 2009]

B.ENG. and M.ENG. EXAMINATIONS 2009

MATHEMATICS (INFORMATION SYSTEMS ENGINEERING E2.11)

Date Thursday 4th June 2009 2.00 - 4.00 pm

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

Answer FOUR questions, to include at least one from Section B.

Answers to questions from Section A and Section B should be written in different answer books.

Mathematical and statistics formulae sheets are provided.

[Before starting, please make sure that the paper is complete; there should be 6 pages, with a total of 6 questions. Ask the invigilator for a replacement if your copy is faulty.]

Section A

1. The Fourier transform (FT) of $f(t)$ is given by

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt \,.$$

Establish the following results:

(i) $FT\,\{f(-t)\} = \widehat{f}\,(-\omega)\,,$

(ii) $FT\,\{t\,f(t)\} = i\dfrac{d}{d\omega}\,\widehat{f}\,(\omega)\,,$

(iii) $FT\,\left\{\widehat{f}\,(t)\right\} = 2\pi f\,(-\omega)\,.$

The functions $f\,(t)$, $h\,(t)$ and $k\,(t)$ are defined for $-\infty < t < \infty$ by

$$g\,(t) = \begin{cases} e^{-t}\,, & t \geq 0\,, \\ 0\,, & t < 0\,, \end{cases} \qquad h\,(t) = tg\,(t),\; k\,(t) = |t|e^{-|t|}\,.$$

Find $\widehat{g}\,(\omega)$ and hence, or otherwise, find $\widehat{h}\,(\omega)$ and show that

$$\widehat{k}\,(\omega) = \frac{2(1-\omega^2)}{(1+\omega^2)^2}\,.$$

Use the above results to find the Fourier transforms of the functions

$$\frac{1}{(1+it)^2} \qquad \text{and} \qquad \frac{1-t^2}{(1+t^2)^2}\,.$$

Hence deduce the value of the integral

$$\int_0^{\infty} \frac{(1-t^2)}{(1+t^2)^2}\,\cos t\, dt\,.$$

You may assume that the inverse Fourier transform is given by

$$f\,(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}\,(\omega)\, e^{i\omega t}\, d\omega\,.$$

PLEASE TURN OVER

2. (i) Evaluate the path integral

$$I = \int_C (k\,x\,y + \pi\,y\sin(\pi\,x\,y))\; dx + \left(x^2 + \pi\,x\sin(\pi\,x\,y)\right)\; dy$$

where k is a constant, in the cases:

(a) C consists of the straight line segment from (0, 0) to (1, 0), followed by the straight line segment from (1, 0) to (1, 1).

(b) C consists of the straight line segment from (0, 0) to (1, 1).

Find the value of k that ensures that I has the same value for all paths C that start at (0, 0) and end at (1, 1). Construct the corresponding potential function.

(ii) Sketch the region of the $x - y$ plane over which the double integral

$$\int_{x=0}^{x=1} \left\{ \int_{y=x}^{y=\sqrt{(2x-x^2)}} \frac{y}{x^2+y^2}\, dy \right\} dx$$

is taken. Use polar coordinates to evaluate this integral.

3. (i) Classify the singularities of the function

$$f(z) = \frac{e^{i\alpha z}}{2z^2+1}\,, \qquad \alpha > 0\,,$$

in the complex z-plane, and determine the residues at the poles.

(ii) Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\cos(\alpha x)}{2x^2+1}\, dx$$

in the following cases:

(a) $\alpha = 0$, using a standard substitution,

(b) $\alpha > 0$, using contour integration.

Demonstrate that, upon setting $\alpha = 0$, the result for (b) reduces to that obtained in (a).

4. (i) By using the residue theorem, show that the inverse Laplace transform of

$$\frac{1}{s\,(2s^2+11s+5)}$$

is given by

$$\frac{1}{5} - \frac{2}{9}\,e^{-\frac{1}{2}t} + \frac{1}{45}\,e^{-5t}\,.$$

(ii) The function $u\,(x,\,t)$ satisfies

$$2\,\frac{\partial^2 u}{\partial t^2} + 11\,\frac{\partial u}{\partial t} + x\,\frac{\partial u}{\partial x} = x^5\,, \quad x>0, \quad t>0\,,$$

and is subject to the initial conditions

$$u\,(x,\,0) = \frac{\partial u}{\partial t}\,(x,\,0) = 0$$

and the boundary condition

$$u\,(0,\,t) = 0\,.$$

By taking the Laplace transform with respect to t, and using the result of part (i), determine the solution for $u\,(x,\,t)$.

PLEASE TURN OVER

5. (i) In a binary symmetric channel, with all transmissions independent, the following probabilities hold,

P(1 received | 1 transmitted) $= 0.8$ P(1 received | 0 transmitted) $= 0.2$
P(0 received | 0 transmitted) $= 0.8$ P(0 received | 1 transmitted) $= 0.2$

The probability of a 1 being transmitted is 0.4.

(a) What is the probability that a 0 is received?

(b) If a 0 is received, what is the probability that a 1 was transmitted?

To reduce transmission errors, a repetition code of length 5 is employed as follows: to transmit a 0, we transmit the 5 bit string: 00000 and if the received 5 bit string contains 3 or more 0s, we decode it as 0. To transmit a 1, we transmit 11111 and if the received 5 bit string contains 3 or more 1s, we decode it as 1.

Using this repetition scheme:

(c) What is the probability that a 0 is decoded correctly?

(d) What is the probability that the message 000 is decoded correctly, where each of the 0s is transmitted and decoded as in part (c)?

(ii) A helpdesk responds to reported errors for two application systems, A and B. The number of errors reported per week have independent Poisson distributions with parameter $\lambda = 0.5$ for system A and $\lambda = 1$ for system B.

(a) What is the probability that, in any particular week, no errors will be reported for system A?

(b) What is the probability that, in any particular week, no errors will be reported for system B?

(c) The helpdesk staff all take their fortnight holiday at the same time. What is the probability that there are no reported errors while they are on holiday?

PLEASE TURN OVER

6. A high-availability cluster has two nodes and functions if either of the nodes is functioning. The lifetime, T, in weeks, of both nodes are independently exponentially distributed with probability density function

$$f(t) = \begin{cases} \dfrac{1}{10}\, e^{-t/10} & t > 0, \\ 0 & \text{otherwise.} \end{cases}$$

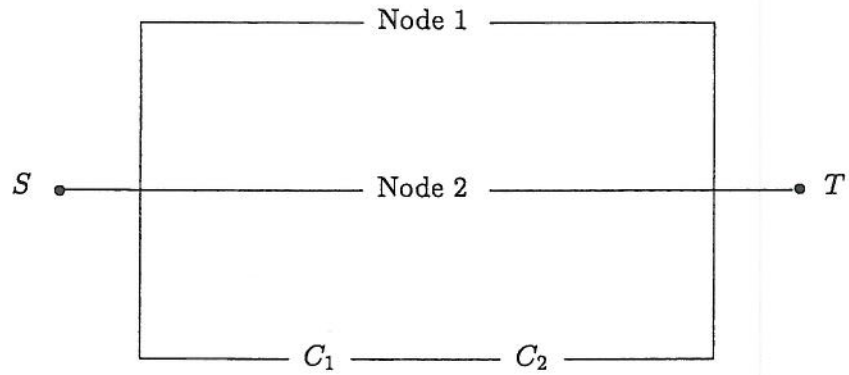
Find the corresponding,

(i) distribution function, $F(t)$;

(ii) reliability $R(t)$ and hazard function $h(t)$;
what is the reliability and hazard at 8 weeks?

(iii) if one of the nodes has not failed at 2 weeks, determine the probability that it will still not have failed at 10 weeks? How does this compare to the reliability at 8 weeks? What property of the exponential distribution does this illustrate?

An additional independent node is added which has two independent components, C_1 and C_2, which have the same lifetime distribution and will fail if either of these components fail. The reliability of C_1 and C_2 at 8 weeks is 0.8.
Such a cluster can be represented as follows, where the cluster operates if there is a path of functioning components between S and T.



(iv) Determine the reliability of the cluster at 8 weeks.

(v) What is the minimum reliability of C_1 and C_2 for the overall reliability at 8 weeks to be 0.95?

END OF PAPER

1. **Probabilities for events**

For events A, B, and C $\quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$

More generally $\quad P(\bigcup A_i) = \sum P(A_i) - \sum P(A_i \cap A_j) + \sum P(A_i \cap A_j \cap A_k) - \cdots$

The odds in favour of A $\quad P(A)\,/\,P(\overline{A})$

Conditional probability $\quad P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$ provided that $P(B) > 0$

Chain rule $\quad P(A \cap B \cap C) = P(A)\,P(B \mid A)\,P(C \mid A \cap B)$

Bayes' rule $\quad P(A \mid B) = \dfrac{P(A)\,P(B \mid A)}{P(A)\,P(B \mid A) + P(\overline{A})\,P(B \mid \overline{A})}$

A and B are independent if $\quad P(B \mid A) = P(B)$

A, B, and C are independent if $\quad P(A \cap B \cap C) = P(A)P(B)P(C)$, and

$$P(A \cap B) = P(A)P(B), \quad P(B \cap C) = P(B)P(C), \quad P(C \cap A) = P(C)P(A)$$

2. **Probability distribution, expectation and variance**

The probability distribution for a discrete random variable X is called the probability mass function (pmf) and is the complete set of probabilities $\{p_x\} = \{P(X = x)\}$

Expectation $\quad E(X) = \mu = \sum_x x p_x$

For function $g(x)$ of x, $\quad E\{g(X)\} = \sum_x g(x) p_x$, $\quad$ so $\quad E(X^2) = \sum_x x^2 p_x$

Sample mean $\quad \overline{x} = \dfrac{1}{n} \sum_k x_k$ $\quad$ estimates μ from random sample $x_1, x_2, \ldots, x_n$

Variance $\quad \mathrm{var}(X) = \sigma^2 = E\{(X-\mu)^2\} = E(X^2) - \mu^2$

Sample variance $\quad s^2 = \dfrac{1}{n-1}\left\{\sum_k x_k^2 - \dfrac{1}{n}\left(\sum_j x_j\right)^2\right\}$ $\quad$ estimates σ^2

Standard deviation $\quad \mathrm{sd}(X) = \sigma$

If value y is observed with frequency n_y

$$n = \sum_y n_y, \quad \sum_k x_k = \sum_y y n_y, \quad \sum_k x_k^2 = \sum_y y^2 n_y$$

Skewness $\quad \beta_1 = E\left(\dfrac{X-\mu}{\sigma}\right)^3$ $\quad$ is estimated by $\quad \dfrac{1}{n-1} \sum \left(\dfrac{x_i - \overline{x}}{s}\right)^3$

Kurtosis $\quad \beta_2 = E\left(\dfrac{X-\mu}{\sigma}\right)^4 - 3$ $\quad$ is estimated by $\quad \dfrac{1}{n-1} \sum \left(\dfrac{x_i - \overline{x}}{s}\right)^4 - 3$

Sample median $\quad \tilde{x}$ or x_{med}. Half the sample values are smaller and half larger

If the sample values $x_1, \ldots, x_n$ are ordered as $x_{(1)} \le x_{(2)} \le \cdots \le x_{(n)}$,
then $\tilde{x} = x_{(\frac{n+1}{2})}$ if n is odd, and $\tilde{x} = \frac{1}{2}\left(x_{(\frac{n}{2})} + x_{(\frac{n+2}{2})}\right)$ if n is even

α-quantile $Q(\alpha)$ is such that $P(X \leq Q(\alpha)) = \alpha$

Sample α-quantile $\widehat{Q}(\alpha)$ Proportion α of the data values are smaller

Lower quartile Q1 = $\widehat{Q}(0.25)$ one quarter are smaller

Upper quartile Q3 = $\widehat{Q}(0.75)$ three quarters are smaller

Sample median $\tilde{x} = \widehat{Q}(0.5)$ estimates the population median $Q(0.5)$

3. Probability distribution for a continuous random variable

The cumulative distribution function (cdf) $$F(x) = P(X \leq x) = \int_{x_0=-\infty}^{x} f(x_0)\mathrm{d}x_0$$

The probability density function (pdf) $$f(x) = \frac{\mathrm{d}F(x)}{\mathrm{d}x}$$

$$E(X) = \mu = \int_{-\infty}^{\infty} x\, f(x)\mathrm{d}x\,, \quad \mathrm{var}\,(X) = \sigma^2 = E(X^2) - \mu^2, \text{ where } E(X^2) = \int_{-\infty}^{\infty} x^2\, f(x)\mathrm{d}x$$

4. Discrete probability distributions

Discrete Uniform *Uniform* (n)

$$p_x = \frac{1}{n} \quad (x = 1, 2, \ldots, n) \qquad \mu = (n+1)/2,\ \sigma^2 = (n^2-1)/12$$

Binomial distribution *Binomial* (n, θ)

$$p_x = \binom{n}{x} \theta^x (1-\theta)^{n-x} \quad (x = 0, 1, 2, \ldots, n) \qquad \mu = n\theta\,,\ \sigma^2 = n\theta(1-\theta)$$

Poisson distribution *Poisson* (λ)

$$p_x = \frac{\lambda^x e^{-\lambda}}{x!} \quad (x = 0, 1, 2, \ldots) \quad (\text{with } \lambda > 0) \qquad \mu = \lambda\,,\ \sigma^2 = \lambda$$

Geometric distribution *Geometric* (θ)

$$p_x = (1-\theta)^{x-1}\theta \quad (x = 1, 2, 3, \ldots) \qquad \mu = \frac{1}{\theta}\,,\ \sigma^2 = \frac{1-\theta}{\theta^2}$$

5. Continuous probability distributions

Uniform distribution *Uniform* (α, β)

$$f(x) = \begin{cases} \dfrac{1}{\beta - \alpha} & (\alpha < x < \beta), \\ 0 & (\text{otherwise}). \end{cases} \qquad \mu = (\alpha+\beta)/2\,,\ \sigma^2 = (\beta-\alpha)^2/12$$

Exponential distribution *Exponential* (λ)

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & (0 < x < \infty), \\ 0 & (-\infty < x \leq 0). \end{cases} \qquad \mu = 1/\lambda\,,\ \sigma^2 = 1/\lambda^2$$

Normal distribution $N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\} \quad (-\infty < x < \infty), \quad E(X) = \mu, \quad \mathrm{var}(X) = \sigma^2$$

Standard normal distribution $N(0,1)$

If X is $N(\mu, \sigma^2)$, then $Y = \dfrac{X-\mu}{\sigma}$ is $N(0,1)$

6. Reliability

For a device in continuous operation with failure time random variable T having pdf $f(t)$ $(t > 0)$

The reliability function at time t	$R(t) = P(T > t)$
The failure rate or hazard function	$h(t) = f(t)/R(t)$
The cumulative hazard function	$H(t) = \int_0^t h(t_0)\,\mathrm{d}t_0 = -\ln\{R(t)\}$
The Weibull(α, β) distribution has	$H(t) = \beta t^\alpha$

7. System reliability

For a system of k devices, which operate independently, let

$R_i = P(D_i) = P(\text{"device } i \text{ operates"})$

The system reliability, R, is the probability of a path of operating devices

A system of devices in series operates only if every device operates

$R = P(D_1 \cap D_2 \cap \cdots \cap D_k) = R_1 R_2 \cdots R_k$

A system of devices in parallel operates if any device operates

$R = P(D_1 \cup D_2 \cup \cdots \cup D_k) = 1 - (1 - R_1)(1 - R_2) \cdots (1 - R_k)$

8. Covariance and correlation

The covariance of X and Y $\quad \mathrm{cov}(X, Y) = E(XY) - \{E(X)\}\{E(Y)\}$

From pairs of observations $(x_1, y_1), \ldots, (x_n, y_n)$ $\quad S_{xy} = \sum_k x_k y_k - \frac{1}{n}(\sum_i x_i)(\sum_j y_j)$

$$S_{xx} = \sum_k x_k^2 - \frac{1}{n}(\sum_i x_i)^2, \qquad S_{yy} = \sum_k y_k^2 - \frac{1}{n}(\sum_j y_j)^2$$

Sample covariance	$s_{xy} = \dfrac{1}{n-1} S_{xy}$ estimates $\mathrm{cov}(X, Y)$
Correlation coefficient	$\rho = \mathrm{corr}(X, Y) = \dfrac{\mathrm{cov}(X, Y)}{\mathrm{sd}(X) \cdot \mathrm{sd}(Y)}$
Sample correlation coefficient	$r = \dfrac{S_{xy}}{\sqrt{S_{xx} S_{yy}}}$ estimates ρ

9. Sums of random variables

$$E(X+Y) = E(X)+E(Y)$$
$$\mathrm{var}\,(X+Y) = \mathrm{var}\,(X)+\mathrm{var}\,(Y)+2\,\mathrm{cov}\,(X,Y)$$
$$\mathrm{cov}\,(aX+bY,\ cX+dY) = (ac)\,\mathrm{var}\,(X)+(bd)\,\mathrm{var}\,(Y)+(ad+bc)\,\mathrm{cov}\,(X,Y)$$

If X is $N(\mu_1, \sigma_1^2)$, Y is $N(\mu_2, \sigma_2^2)$, and $\mathrm{cov}\,(X,Y)=c$, then $X+Y$ is $N(\mu_1+\mu_2,\ \sigma_1^2+\sigma_2^2+2c)$

10. Bias, standard error, mean square error

If t estimates θ (with random variable T giving t)

Bias of t $\quad \mathrm{bias}\,(t) = E(T)-\theta$

Standard error of t $\quad \mathrm{se}\,(t) = \mathrm{sd}\,(T)$

Mean square error of t $\quad \mathrm{MSE}(t) = E\{(T-\theta)^2\} = \{\mathrm{se}\,(t)\}^2+\{\mathrm{bias}\,(t)\}^2$

If $\overline{x}$ estimates μ, then $\mathrm{bias}\,(\overline{x})=0$, $\mathrm{se}\,(\overline{x})=\sigma/\sqrt{n}$, $\mathrm{MSE}(\overline{x})=\sigma^2/n$, $\widehat{\mathrm{se}}\,(\overline{x})=s/\sqrt{n}$

Central limit property $\quad$ If n is fairly large, $\overline{x}$ is from $N(\mu,\ \sigma^2/n)$ approximately

11. Likelihood

The likelihood is the joint probability as a function of the unknown parameter θ.

For a random sample $x_1, x_2, \ldots, x_n$

$$\ell(\theta;\ x_1, x_2, \ldots, x_n) = P(X_1=x_1 \mid \theta) \cdots P(X_n=x_n \mid \theta) \qquad \text{(discrete distribution)}$$
$$\ell(\theta;\ x_1, x_2, \ldots, x_n) = f(x_1 \mid \theta)\, f(x_2 \mid \theta) \cdots f(x_n \mid \theta) \qquad \text{(continuous distribution)}$$

The maximum likelihood estimator (MLE) is $\widehat{\theta}$ for which the likelihood is a maximum

12. Confidence intervals

If $x_1, x_2, \ldots, x_n$ are a random sample from $N(\mu, \sigma^2)$ and σ^2 is known, then the 95% confidence interval for μ is $(\overline{x}-1.96\frac{\sigma}{\sqrt{n}},\ \overline{x}+1.96\frac{\sigma}{\sqrt{n}})$

If σ^2 is estimated, then from the Student t table for t_{n-1} we find $t_0 = t_{n-1,0.05}$

The 95% confidence interval for μ is $(\overline{x}-t_0\frac{s}{\sqrt{n}},\ \overline{x}+t_0\frac{s}{\sqrt{n}})$

13. Standard normal table — Values of pdf $\phi(y) = f(y)$ and cdf $\Phi(y) = F(y)$

y	$\phi(y)$	$\Phi(y)$	y	$\phi(y)$	$\Phi(y)$	y	$\phi(y)$	$\Phi(y)$	y	$\Phi(y)$
0	.399	.5	.9	.266	.816	1.8	.079	.964	2.8	.997
.1	.397	.540	1.0	.242	.841	1.9	.066	.971	3.0	.999
.2	.391	.579	1.1	.218	.864	2.0	.054	.977	0.841	.8
.3	.381	.618	1.2	.194	.885	2.1	.044	.982	1.282	.9
.4	.368	.655	1.3	.171	.903	2.2	.035	.986	1.645	.95
.5	.352	.691	1.4	.150	.919	2.3	.028	.989	1.96	.975
.6	.333	.726	1.5	.130	.933	2.4	.022	.992	2.326	.99
.7	.312	.758	1.6	.111	.945	2.5	.018	.994	2.576	.995
.8	.290	.788	1.7	.094	.955	2.6	.014	.995	3.09	.999

14. Student t table — Values $t_{m,p}$ of x for which $P(|X| > x) = p$, when X is t_m

m	p= 0.10	0.05	0.02	0.01	m	p= 0.10	0.05	0.02	0.01
1	6.31	12.71	31.82	63.66	9	1.83	2.26	2.82	3.25
2	2.92	4.30	6.96	9.92	10	1.81	2.23	2.76	3.17
3	2.35	3.18	4.54	5.84	12	1.78	2.18	2.68	3.05
4	2.13	2.78	3.75	4.60	15	1.75	2.13	2.60	2.95
5	2.02	2.57	3.36	4.03	20	1.72	2.09	2.53	2.85
6	1.94	2.45	3.14	3.71	25	1.71	2.06	2.48	2.78
7	1.89	2.36	3.00	3.50	40	1.68	2.02	2.42	2.70
8	1.86	2.31	2.90	3.36	∞	1.645	1.96	2.326	2.576

15. Chi-squared table — Values $\chi^2_{k,p}$ of x for which $P(X > x) = p$, when X is χ^2_k and p = .995, .975, *etc*

k	.995	.975	.05	.025	.01	.005	k	.995	.975	.05	.025	.01	.005
1	.000	.001	3.84	5.02	6.63	7.88	18	6.26	8.23	28.87	31.53	34.81	37.16
2	.010	.051	5.99	7.38	9.21	10.60	20	7.43	9.59	31.42	34.17	37.57	40.00
3	.072	.216	7.81	9.35	11.34	12.84	22	8.64	10.98	33.92	36.78	40.29	42.80
4	.207	.484	9.49	11.14	13.28	14.86	24	9.89	12.40	36.42	39.36	42.98	45.56
5	.412	.831	11.07	12.83	15.09	16.75	26	11.16	13.84	38.89	41.92	45.64	48.29
6	.676	1.24	12.59	14.45	16.81	18.55	28	12.46	15.31	41.34	44.46	48.28	50.99
7	.990	1.69	14.07	16.01	18.48	20.28	30	13.79	16.79	43.77	46.98	50.89	53.67
8	1.34	2.18	15.51	17.53	20.09	21.95	40	20.71	24.43	55.76	59.34	63.69	66.77
9	1.73	2.70	16.92	19.02	21.67	23.59	50	27.99	32.36	67.50	71.41	76.15	79.49
10	2.16	3.25	13.31	20.48	23.21	25.19	60	35.53	40.48	79.08	83.30	88.38	91.95
12	3.07	4.40	21.03	23.34	26.22	28.30	70	43.28	48.76	90.53	95.02	100.4	104.2
14	4.07	5.63	23.68	26.12	29.14	31.32	80	51.17	57.15	101.9	106.6	112.3	116.3
16	5.14	6.91	26.30	28.85	32.00	34.27	100	67.33	74.22	124.3	129.6	135.8	140.2

16. The chi-squared goodness-of-fit test

The frequencies n_y are grouped so that the fitted frequency $\widehat{n}_y$ for every group exceeds about 5.

$X^2 = \sum_y \frac{(n_y - \widehat{n}_y)^2}{\widehat{n}_y}$ is referred to the table of χ^2_k with significance point p,

where k is the number of terms summed, less one for each constraint, *eg* matching total frequency, and matching $\overline{x}$ with μ

17. Joint probability distributions

Discrete distribution $\{p_{xy}\}$, where $p_{xy} = P(\{X = x\} \cap \{Y = y\})$.

Let $p_{x\bullet} = P(X = x)$, and $p_{\bullet y} = P(Y = y)$, then

$$p_{x\bullet} = \sum_y p_{xy} \quad \text{and} \quad P(X = x \,|\, Y = y) = \frac{p_{xy}}{p_{\bullet y}}$$

Continuous distribution

Joint cdf $\quad F(x, y) = P(\{X \le x\} \cap \{Y \le y\}) = \int_{x_0=-\infty}^{x} \int_{y_0=-\infty}^{y} f(x_0, y_0)\, \mathrm{d}x_0\, \mathrm{d}y_0$

Joint pdf $\quad f(x, y) = \frac{\mathrm{d}^2 F(x, y)}{\mathrm{d}x\, \mathrm{d}y}$

Marginal pdf of X $\quad f_X(x) = \int_{-\infty}^{\infty} f(x, y_0)\, \mathrm{d}y_0$

Conditional pdf of X given $Y = y$ $\quad f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(y)}$ (provided $f_Y(y) > 0$)

18. Linear regression

To fit the linear regression model $y = \alpha + \beta x$ by $\widehat{y}_x = \widehat{\alpha} + \widehat{\beta} x$ from observations $(x_1, y_1), \dots, (x_n, y_n)$, the least squares fit is $\quad \widehat{\alpha} = \overline{y} - \overline{x}\widehat{\beta}, \quad \widehat{\beta} = \frac{S_{xy}}{S_{xx}}$

The residual sum of squares $\quad \text{RSS} = S_{yy} - \frac{S_{xy}^2}{S_{xx}}$

$\widehat{\sigma^2} = \frac{\text{RSS}}{n-2} \qquad \frac{n-2}{\sigma^2}\widehat{\sigma^2}$ is from χ^2_{n-2}

$$E(\widehat{\alpha}) = \alpha, \quad E(\widehat{\beta}) = \beta,$$

$$\text{var}(\widehat{\alpha}) = \frac{\sum x_i^2}{n S_{xx}}\sigma^2, \quad \text{var}(\widehat{\beta}) = \frac{\sigma^2}{S_{xx}}, \quad \text{cov}(\widehat{\alpha}, \widehat{\beta}) = -\frac{\overline{x}}{S_{xx}}\sigma^2$$

$$\widehat{y}_x = \widehat{\alpha} + \widehat{\beta} x, \qquad E(\widehat{y}_x) = \alpha + \beta x, \qquad \text{var}(\widehat{y}_x) = \left\{\frac{1}{n} + \frac{(x - \overline{x})^2}{S_{xx}}\right\}\sigma^2$$

$\frac{\widehat{\alpha} - \alpha}{\widehat{\text{se}}(\widehat{\alpha})}, \quad \frac{\widehat{\beta} - \beta}{\widehat{\text{se}}(\widehat{\beta})}, \quad \frac{\widehat{y}_x - \alpha - \beta x}{\widehat{\text{se}}(\widehat{y}_x)}$ are each from t_{n-2}

M A T H E M A T I C S   D E P A R T M E N T

MATHEMATICAL FORMULAE

1. VECTOR ALGEBRA

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1, a_2, a_3)$$

Scalar (dot) product: $\mathbf{a}\,.\,\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

Vector (cross) product:

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Scalar triple product:

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a}\,.\,\mathbf{b}\times\mathbf{c} = \mathbf{b}\,.\,\mathbf{c}\times\mathbf{a} = \mathbf{c}\,.\,\mathbf{a}\times\mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product: $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{c}\,.\,\mathbf{a})\mathbf{b} - (\mathbf{b}\,.\,\mathbf{a})\mathbf{c}$

2. SERIES

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \ldots \quad (\alpha \text{ arbitrary}, |x| < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n\frac{x^{2n}}{(2n)!} + \ldots,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + \ldots,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^n\frac{x^{n+1}}{(n+1)} + \ldots (-1 < x \le 1)$$

3. TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin(a+b) = \sin a\cos b + \cos a\sin b\,;$$

$$\cos(a+b) = \cos a\cos b - \sin a\sin b\,.$$

$$\cos iz = \cosh z\,;\quad \cosh iz = \cos z\,;\quad \sin iz = i\sinh z\,;\quad \sinh iz = i\sin z\,.$$

4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f\,D^ng + \binom{n}{1}Df\,D^{n-1}g + \ldots + \binom{n}{r}D^rf\,D^{n-r}g + \ldots + D^nf\,g\,.$$

(b) Taylor's expansion of $f(x)$ about $x = a$:

$$f(a+h) = f(a) + hf'(a) + h^2f''(a)/2! + \ldots + h^nf^{(n)}(a)/n! + \epsilon_n(h)\,,$$

where $\epsilon_n(h) = h^{n+1}f^{(n+1)}(a+\theta h)/(n+1)!,\quad 0 < \theta < 1\,.$

(c) Taylor's expansion of $f(x, y)$ about (a, b):

$$f(a+h, b+k) = f(a,b) + [hf_x + kf_y]_{a,b} + 1/2!\left[h^2f_{xx} + 2hkf_{xy} + k^2f_{yy}\right]_{a,b} + \ldots$$

(d) Partial differentiation of $f(x, y)$:

i. If $y = y(x)$, then $f = F(x)$, and $\dfrac{dF}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dx}$.

ii. If $x = x(t)$, $y = y(t)$, then $f = F(t)$, and $\dfrac{dF}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$.

iii. If $x = x(u, v)$, $y = y(u, v)$, then $f = F(u, v)$, and

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}, \quad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$

(e) Stationary points of $f(x, y)$ occur where $f_x = 0$, $f_y = 0$ simultaneously.
Let (a, b) be a stationary point: examine $D = [f_{xx}f_{yy} - (f_{xy})^2]_{a,b}$.
If $D > 0$ and $f_{xx}(a, b) < 0$, then (a, b) is a maximum;
If $D > 0$ and $f_{xx}(a, b) > 0$, then (a, b) is a minimum;
If $D < 0$ then (a, b) is a saddle-point.

(f) Differential equations:

i. The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating factor $I(x) = \exp[\int P(x)(dx]$, so that $\dfrac{d}{dx}(Iy) = IQ$.

ii. $P(x, y)dx + Q(x, y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y$.

5. INTEGRAL CALCULUS

(a) An important substitution: $\tan(\theta/2) = t$:
$\sin\theta = 2t/(1+t^2)$, $\cos\theta = (1-t^2)/(1+t^2)$, $d\theta = 2\,dt/(1+t^2)$.

(b) Some indefinite integrals:

$$\int (a^2 - x^2)^{-1/2} dx = \sin^{-1}\left(\frac{x}{a}\right), \quad |x| < a .$$

$$\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1}\left(\frac{x}{a}\right) = \ln\left\{\frac{x}{a} + \left(1 + \frac{x^2}{a^2}\right)^{1/2}\right\} .$$

$$\int (x^2 - a^2)^{-1/2} dx = \cosh^{-1}\left(\frac{x}{a}\right) = \ln\left|\frac{x}{a} + \left(\frac{x^2}{a^2} - 1\right)^{1/2}\right| .$$

$$\int (a^2 + x^2)^{-1} dx = \left(\frac{1}{a}\right)\tan^{-1}\left(\frac{x}{a}\right) .$$

6. NUMERICAL METHODS

(a) Approximate solution of an algebraic equation:

If a root of $f(x) = 0$ occurs near $x = a$, take $x_0 = a$ and
$x_{n+1} = x_n - [f(x_n)/f'(x_n)]$, $n = 0, 1, 2 \ldots$

(Newton Raphson method).

(b) Formulae for numerical integration: Write $x_n = x_0 + nh$, $y_n = y(x_n)$.

i. Trapezium rule (1-strip): $\int_{x_0}^{x_1} y(x)dx \approx (h/2)\,[y_0 + y_1]$.

ii. Simpson's rule (2-strip): $\int_{x_0}^{x_2} y(x)dx \approx (h/3)\,[y_0 + 4y_1 + y_2]$.

(c) Richardson's extrapolation method: Let $I = \int_a^b f(x)dx$ and let I_1, I_2 be two estimates of I obtained by using Simpson's rule with intervals h and $h/2$.

Then, provided h is small enough,

$$I_2 + (I_2 - I_1)/15 ,$$

is a better estimate of I .

7. LAPLACE TRANSFORMS

Function	Transform	Function	Transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)dt$	$af(t) + bg(t)$	$aF(s) + bG(s)$
df/dt	$sF(s) - f(0)$	d^2f/dt^2	$s^2F(s) - sf(0) - f'(0)$
$e^{at}f(t)$	$F(s-a)$	$tf(t)$	$-dF(s)/ds$
$(\partial/\partial\alpha)f(t,\alpha)$	$(\partial/\partial\alpha)F(s,\alpha)$	$\int_0^t f(t)dt$	$F(s)/s$
$\int_0^t f(u)g(t-u)du$	$F(s)G(s)$		
1	$1/s$	$t^n (n = 1, 2 \ldots)$	$n!/s^{n+1}$, $(s > 0)$
e^{at}	$1/(s-a)$, $(s > a)$	$\sin\omega t$	$\omega/(s^2+\omega^2)$, $(s > 0)$
$\cos\omega t$	$s/(s^2+\omega^2)$, $(s > 0)$	$H(t-T) = \begin{cases} 0, & t < T \\ 1, & t > T \end{cases}$	e^{-sT}/s , $(s, T > 0)$

8. FOURIER SERIES

If $f(x)$ is periodic of period $2L$, then $f(x + 2L) = f(x)$, and

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} , \text{ where}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad n = 0, 1, 2, \ldots, \text{ and}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1, 2, 3, \ldots$$

Parseval's theorem

$$\frac{1}{L}\int_{-L}^{L} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right) .$$

Question

Solution 1

Marks & seen/unseen

Parts

(i) $FT\{f(-t)\} = \int_{-\infty}^{\infty} f(-t)e^{-i\omega t}\,dt \overset{(s=-t)}{=} -\int_{\infty}^{-\infty} f(s)e^{i\omega s}\,ds$

$= \int_{-\infty}^{\infty} f(s)e^{-i(-\omega)s}\,ds = \underline{\underline{\hat{f}(-\omega)}}$ ← 2

seen

(ii) $FT\{tf(t)\} = \int_{-\infty}^{\infty} tf(t)e^{-i\omega t}\,dt = \frac{1}{-i}\frac{d}{d\omega}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt$

$= \underline{\underline{i\frac{d}{d\omega}\hat{f}(\omega)}}$ ← 2

(iii) We know $f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{i\omega t}\,d\omega \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(s)e^{ist}\,ds$ [Alternatively, work in terms of delta function]

Then put $t=-\omega$: $f(-\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(s)e^{-i\omega s}\,ds$

Hence $\underline{\underline{FT\{\hat{f}(s)\} = 2\pi f(-\omega)}}$ (Put $s=t$ to obtain result.) 2

$\hat{g}(\omega) = \int_0^{\infty} e^{-t}e^{-i\omega t}\,dt = \int_0^{\infty} e^{-(1+i\omega)t}\,dt = \left[-\frac{e^{-(1+i\omega)t}}{1+i\omega}\right]_0^{\infty} = \underline{\underline{1/(1+i\omega)}}$ 2

unseen but similar done

Using result (ii) above: $\hat{h}(\omega) = i\frac{d}{d\omega}\hat{g}(\omega)$

$= i(-i)/(1+i\omega)^2 = \underline{\underline{1/(1+i\omega)^2}}$ ← 2

Observe that $k(t) = h(t) + h(-t)$

$\Rightarrow \hat{k}(\omega) = \hat{h}(\omega) + \hat{h}(-\omega)$ using property (i)

$= \frac{1}{(1+i\omega)^2} + \frac{1}{(1-i\omega)^2} = \underline{\underline{\frac{2(1-\omega^2)}{(1+\omega^2)^2}}}$ as req'd. 3

Using symmetry formula (iii):

$\frac{1}{(1+it)^2}$ has FT $2\pi h(-\omega) = \underline{\underline{\begin{cases} 0 & \text{for } \omega>0 \\ -2\pi\omega e^{\omega} & \text{for } \omega\leqslant 0\end{cases}}}$ 2

and

$\frac{1-t^2}{(1+t^2)^2}$ has FT $\pi k(-\omega) = \underline{\underline{\pi|\omega|e^{-|\omega|}}}$ 2

From this last result we see that $\int_{-\infty}^{\infty}\frac{1-t^2}{(1+t^2)^2}e^{-i\omega t}\,dt = \pi|\omega|e^{-|\omega|}$

	EXAMINATION QUESTIONS/SOLUTIONS 2008-09	Course ISE 2.6
Question Solution 1 (ctd)		Marks & seen/unseen
Parts	Taking the real part we have $$\underbrace{\int_{-\infty}^{\infty} \frac{1-t^2}{(1+t^2)^2} \cos \omega t \, dt}_{= 2\int_0^\infty \text{ Since integrand is even.}} = \pi \lvert\omega\rvert e^{-\lvert\omega\rvert}$$	1
	Hence, setting $\omega = 1$: $$\int_0^{\infty} \frac{1-t^2}{(1+t^2)^2} \cos t \, dt = \frac{\pi}{2e}$$	1
		Total 20
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EXAMINATION QUESTIONS/SOLUTIONS 2008-09

Course: ISE 2.6

Question: Solution 2

Marks & seen/unseen

Parts

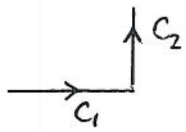
(i) (a) On the line C_1 say, from $(0,0)$ to $(1,0)$ put $x=t, y=0$ for $0 \leq t \leq 1$.

Then $\int_{C_1}$ becomes $\int_{t=0}^{t=1} (0+0)\,dt + (\ \ )\overset{dy}{(0)} = 0$. — 1

On C_2 say, from $(1,0)$ to $(1,1)$ we put $x=1$ $(dx=0)$, $y=t$ $(dy=1)$ for $0 \leq t \leq 1$).

Then $\int_{C_2}$ is $\int_0^1 (\ \ )\overset{dx}{(0)} + (1+\pi \sin(\pi t))\,dt$

$= [t - \cos(\pi t)]_0^1 = 1 - \cos\pi + \cos 0 = 3.$ — 2



$\therefore \int_C = \int_{C_1} + \int_{C_2} = \underline{\underline{3}}$

(b) This time $x=t, y=t$ (for $0 \leq t \leq 1$) so that $dx = dy = dt$

Then $\int_C = \int_0^1 kt^2 + \pi t \sin(\pi t^2)\,dt + (t^2 + \pi t \sin(\pi t^2))\,dt$

$= \int_0^1 (k+1)t^2 + \underbrace{2\pi t \sin(\pi t^2)}_{-\frac{d}{dt}\cos(\pi t^2)}\,dt = \left[\frac{(k+1)t^3}{3} - \cos(\pi t^2)\right]_0^1$

$= \underline{\underline{\frac{k+1}{3} + 2}}$ — 2

all unseen but students have seen similar problems

Thus answers to (a) & (b) are only the same if $k=2$ — 1

To show that integral is path-independent in this case we take $P = kxy + \pi y \sin(\pi xy)$ & $Q = x^2 + \pi x \sin(\pi xy)$

Then $\frac{\partial P}{\partial y} = kx + \pi \sin(\pi xy) + \pi^2 xy \cos(\pi xy)$

& $\frac{\partial Q}{\partial x} = 2x + \pi \sin(\pi xy) + \pi^2 xy \cos(\pi xy)$

$\therefore P_y = Q_x$ iff $k=2$ & integral is path-independent. — 2

Potential function V satisfies $\frac{\partial V}{\partial x} = P$ & $\frac{\partial V}{\partial y} = Q$

Integrating: $V = x^2 y - \cos(\pi xy) + f(y)$ (with $k=2$)

$\Rightarrow x^2 + \pi x \sin(\pi xy) + f'(y) = Q \Rightarrow f'(y) = 0 \Rightarrow f = \text{const}.$

$\therefore \underline{\underline{V = x^2 y - \cos(\pi xy) + \text{const}}}$. is the corresponding potential function. — 2

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Question

Solution 2 (ctd)

Marks & seen/unseen

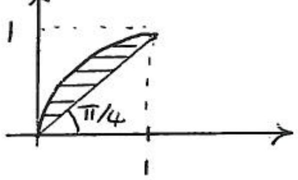
Parts

(ii) $y=\sqrt{(2x-x^2)} \Rightarrow x^2-2x+y^2=0$ with $y>0$

i.e. $(x-1)^2+y^2=1$ — " — CIRCLE Centre $(1,0)$ radius 1 — 3 for Sketch

$y=x$ is st. line

So region of integration is shaded below



Set $x=r\cos\theta,\ y=r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

Then r goes from 0 to its value on the circular arc $x^2+y^2=2x$

i.e. $r^2=2r\cos\theta \Rightarrow r=2\cos\theta$

and θ goes from $\pi/4$ to $\pi/2$ to cover region. — 4

$$\therefore I = \int_{\theta=\pi/4}^{\theta=\pi/2}\int_{r=0}^{r=2\cos\theta} \frac{\cancel{r}\sin\theta}{\cancel{r^2}}\cancel{r}\,dr\,d\theta$$

$$= \int_{\pi/4}^{\pi/2} 2\cos\theta\sin\theta\,d\theta = \left[\sin^2\theta\right]_{\pi/4}^{\pi/2} = \underline{\underline{\tfrac{1}{2}}}$$

— 3

Total 20

	EXAMINATION QUESTIONS/SOLUTIONS 2008-09	Course ISE 2.6
Question Solution 3		Marks & seen/unseen

Parts

(i) $f(z) = \dfrac{e^{i\alpha z}}{2z^2+1}$ has poles when $2z^2+1=0$ i.e. when $z = \pm i/\sqrt{2}$ — 2

These are simple because the roots of the quadratic are distinct. — 1

$$\operatorname*{Res}_{z=\pm i/\sqrt{2}} f(z) = \lim_{z\to \pm \frac{i}{\sqrt{2}}} \left(z \mp \frac{i}{\sqrt{2}}\right) \frac{e^{i\alpha z}}{2\left(z+\frac{i}{\sqrt{2}}\right)\left(z-\frac{i}{\sqrt{2}}\right)}$$

$$= \frac{e^{\mp\alpha/\sqrt{2}}}{\pm 2\left(\frac{i}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)} = \frac{e^{\mp\alpha/\sqrt{2}}}{\pm 2i\sqrt{2}}$$ — 4

all unseen but students have seen similar problems

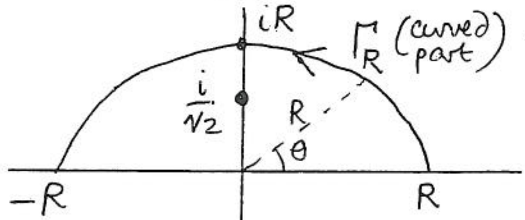
(ii) (a) $\alpha = 0$

We have $\displaystyle\int_{-\infty}^{\infty} \frac{1}{2x^2+1}\,dx \equiv \int_{-\infty}^{\infty} \frac{1/2}{x^2+(1/\sqrt{2})^2}\,dx$

using formula sheet $= \dfrac{1}{2}\left[\dfrac{1}{1/\sqrt{2}} \tan^{-1}\left(\dfrac{x}{1/\sqrt{2}}\right)\right]_{-\infty}^{\infty} = \dfrac{\sqrt{2}}{2}\left(\dfrac{\pi}{2}+\dfrac{\pi}{2}\right)$

$= \pi\sqrt{2}/2 = \pi/\sqrt{2}$ — 3

(b) Take semi-circular contour C as shown with $R > 1/\sqrt{2}$



Note that only 1 pole of $f(z)$ lies inside C.

Then
$$\int_C f(z)\,dz = \int_{-R}^{R} \frac{e^{i\alpha x}}{2x^2+1}\,dx + \underbrace{\int_{\Gamma_R} \frac{e^{i\alpha z}}{2z^2+1}\,dz}_{\text{with } z=Re^{i\theta}} \qquad (*)$$ — 3

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	EXAMINATION QUESTIONS/SOLUTIONS 2008-09	Course ISE 2.6
Question Solution 3 (ctd)		Marks & seen/unseen
Parts	By residue theorem LHS of $(*) = 2\pi i \cdot \dfrac{e^{-\alpha/\sqrt{2}}}{2i\sqrt{2}}$ $= \pi e^{-\alpha/\sqrt{2}} \cdot \dfrac{\sqrt{2}}{2}$	2
	Now, $\displaystyle\int_{\Gamma_R} = \int_0^{\pi} \frac{e^{i\alpha(R\cos\theta + iR\sin\theta)}}{2R^2e^{2i\theta}+1} Rie^{i\theta}d\theta$ $\longrightarrow 0$ as $R \to \infty$ since $\alpha > 0$ & $\sin\theta \geq 0$ on Γ_R.	2
	Thus, letting $R \to \infty$ in $(*)$: $\displaystyle\frac{\pi\sqrt{2}}{2} e^{-\alpha/\sqrt{2}} = \int_{-\infty}^{\infty} \frac{e^{i\alpha x}}{2x^2+1} dx$ and taking the real part: $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(\alpha x)}{2x^2+1} dx = \frac{\pi\sqrt{2}}{2} e^{-\alpha/\sqrt{2}}$	2
	and this $= \dfrac{\pi\sqrt{2}}{2}$ when $\alpha = 0$, in agreement with (ii)(a).	1
		Total 20.
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	EXAMINATION QUESTIONS/SOLUTIONS 2008-09	Course ISE 2.6
Question ISE	Solution 4	Marks & seen/unseen

Parts

(i) We know that $(LT)^{-1}(F(s))$ = Sum of residues of $F(s)e^{st}$ at poles of $F(s)$. — 1

In this case $F(s) = \dfrac{1}{s(2s^2+11s+5)}$ and so we have poles when $s=0$ & when $2s^2+11s+5=0$ i.e. $(2s+1)(s+5)=0$ — 3

So the poles are at $\underline{s=0,\ s=-\frac{1}{2},\ s=-5}$

(all simple, since distinct roots). — 1

all unseen but students have seen similar problems.

$$\operatorname*{Res}_{s=0} F(s)e^{st} = \lim_{s\to 0} \frac{e^{st}}{(2s+1)(s+5)} = \frac{1}{5}$$ — 2

$$\operatorname*{Res}_{s=-\frac{1}{2}} F(s)e^{st} = \lim_{s\to -\frac{1}{2}} \frac{(s+\frac{1}{2})e^{st}}{s(2s+1)(s+5)} = -\frac{2}{9}e^{-\frac{1}{2}t}$$ — 2

$$\operatorname*{Res}_{s=-5} F(s)e^{st} = \lim_{s\to -5} \frac{e^{st}}{s(2s+1)} = \frac{e^{-5t}}{45}$$ — 2

Hence the inverse transform is $f(t) = \frac{1}{5} - \frac{2}{9}e^{-\frac{1}{2}t} + \frac{e^{-5t}}{45}$

(ii) Take LT of PDE and use results on formula sheet: (with $U = LT\{u\}$)

$$2\left(s^2 U - \frac{\partial u}{\partial t}(x,0) - su(x,0)\right) + 11\left(sU - u(x,0)\right) + x\frac{\partial U}{\partial x} = LT\{x^5\} = \frac{x^5}{s} \text{ using tables}$$ — 4

Applying zero initial conditions we get

$$x\frac{\partial U}{\partial x} + (2s^2+11s)U = \frac{x^5}{s}.$$

Use integrating factor $e^{\int \frac{2s^2+11s}{x}dx} = x^{2s^2+11s}$

Then: $\dfrac{\partial}{\partial x}\left(x^{2s^2+11s}U\right) = \dfrac{x^{2s^2+11s+4}}{s}$ — 2

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	EXAMINATION QUESTIONS/SOLUTIONS 2008-09	Course ISE 2.6
Question Solution 4 (ctd)		Marks & seen/unseen

Parts

Integrating & applying $U=0$ on $x=0$ (from $u=0$ on $x=0$) we get:

$$U = \frac{x^{-(2s^2+11s)}}{s}\int_0^x x^{2s^2+11s+4}\,dx = \frac{x^5}{s(2s^2+11s+5)} \equiv x^5 F(s)$$ 2

Inverting, using result from part (i)

$$u = x^5 f(t)$$

$$= x^5\left(\frac{1}{5} - \frac{2}{9}e^{-\frac{1}{2}t} + \frac{e^{-5t}}{45}\right)$$ 1

Total 20

Setter's initials	Checker's initials	Page number
AGW	XWU	

Parts

(i) Let X – digit received; Y – digit transmitted.

$$\begin{array}{ll} P(X=0 \mid Y=0)=0.8, & P(X=1 \mid Y=1)=0.8, \\ P(X=0 \mid Y=1)=0.2, & P(X=1 \mid Y=0)=0.2 \\ P(Y=1)=0.4 & P(Y=0)=0.6 \end{array}$$

(a)

$$\begin{aligned} P(X=0) &= P(X=0|Y=0)P(Y=0)+P(X=0|Y=1)P(Y=1) \\ &= 0.8\times 0.6+0.2\times 0.4 = \boxed{0.56} \end{aligned}$$

3

(b)

$$\begin{aligned} P(Y=1 \mid X=0) &= \frac{P(X=0 \mid Y=1)P(Y=1)}{P(X=0)} \\ &= \frac{0.2\times 0.4}{0.56} = \boxed{0.1429} \end{aligned}$$

2

(c) 0 is sent as 00000 - 5 0s are transmitted.
Let N – number of 0s in 5 digit received signal,
then $N \sim \mathit{Binomial}(p,5)$ where $p = P(X=0 \mid Y=0)$. We have

$$P(N=x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0,1,2,3,4,5$$

2

$$\begin{aligned} P(\text{o decoded correctly}) &= P(N\geq 3) \\ &= P(N=3)+P(N=4)+P(N=5) \\ &= \binom{5}{3}p^3(1-p)^2+\binom{5}{4}p^4(1-p)+\binom{5}{5}p^5(1-p)^0 \\ &= 10(0.8)^3(0.2)^2+5(0.8)^4(0.2)+(0.8)^5 \\ &= \boxed{0.9421} \end{aligned}$$

4

(d)

$$P(000\to 000) = (0.9421)^3 = \boxed{0.8361}$$

1

Question 5.

Marks & seen/unseen

Parts

(ii) Let X = number of errors for A, Y = number of errors for B in a week. We have

$$P(X = x) = \frac{e^{-0.5}(0.5)^x}{x!}, x = 0, 1, 2, \ldots; \; P(Y = y) = \frac{e^{-1}}{y!}, y = 0, 1, 2, \ldots.$$

(a)

$$P(X = 0) = e^{-0.5} = \boxed{0.6065}$$

2

(b)

$$P(Y = 0) = e^{-1} = \boxed{0.3679}$$

2

(c) Let W = number of errors in A in a fortnight, then $W \sim Poisson(1)$.
Let Z = number of errors in B in a fortnight, then $Z \sim Poisson(2)$

$$\begin{aligned} P(\text{no errors reported in a fortnight}) &= P(W = 0 \cap Z = 0) \\ &= e^{-1}e^{-2} = e^{-3} = \boxed{0.0498} \end{aligned}$$

4

Parts

(i)

$$F(t) = \int_0^t \frac{1}{10} e^{-x/10}\, \mathrm{d}x = \left[-\mathrm{e}^{-x/10}\right]_0^t = \boxed{1 - \mathrm{e}^{-t/10}}$$

2

(ii)

$$R(t) = 1 - F(t) = \boxed{\mathrm{e}^{-t/10}}$$

1

$$h(t) = \frac{f(t)}{R(t)} = \frac{\mathrm{e}^{-t/10}/10}{\mathrm{e}^{-t/10}} = \boxed{\tfrac{1}{10}}$$

2

$$R(8) = \mathrm{e}^{-8/10} = \boxed{\mathrm{e}^{-0.8} = 0.4493}$$

1

$$h(8) = \boxed{\tfrac{1}{10}}$$

1

(iii)

$$\begin{aligned} \mathrm{P}(T > 10 \mid T > 2) &= \frac{\mathrm{P}((T > 10) \cap (T > 2))}{\mathrm{P}(T > 2)} \\ &= \frac{\mathrm{P}(T > 10)}{\mathrm{P}(T > 2)} = \frac{R(10)}{R(2)} \\ &= \frac{\mathrm{e}^{-10/10}}{\mathrm{e}^{-2/10}} = \mathrm{e}^{-(1-1/5)} = \boxed{\mathrm{e}^{-0.8} = 0.4493} \end{aligned}$$

3

$\mathrm{P}(T > 10 \mid T > 2)$ is the same as $\mathrm{P}(T > 8)$. This demonstrates the lack-of-memory property of the exponential distribution.

2

Question 6.

Marks & seen/unseen

Parts

(iv) Let T be the lifetime of the system and N_1, N_2, C_1, C_2 be the events that each component is operating at 8 weeks.

$$\begin{aligned}
\mathrm{P}(T > 8) &= \mathrm{P}(N_1 \cup N_2 \cup (C_1 \cap C_2)) \\
&= 1 - \mathrm{P}((N_1 \cup N_2 \cup (C_1 \cap C_2))') \\
&= 1 - \mathrm{P}(N_1' \cap N_2' \cap (C_1 \cap C_2)') \\
&= 1 - \mathrm{P}(N_1')\mathrm{P}(N_2')\mathrm{P}((C_1 \cap C_2)') \\
&= 1 - (1 - \mathrm{P}(N_1))(1 - \mathrm{P}(N_2))(1 - \mathrm{P}(C_1)\mathrm{P}(C_2)) \\
&= 1 - (1 - e^{-0.8})^2(1 - 0.8^2) = \boxed{0.8908}
\end{aligned}$$

5

(v) Let p be the reliability of C_1 and C_2 at 8 weeks.

$$\begin{aligned}
&\mathrm{P}(T > 8) > 0.95 \\
\Rightarrow\ & 1 - (1 - e^{-0.8})^2(1 - p^2) > 0.95 \\
\Rightarrow\ & (1 - e^{-0.8})^2(1 - p^2) < 0.05 \\
\Rightarrow\ & p^2 > 1 - \frac{0.05}{(1 - e^{-0.8})^2} \\
\Rightarrow\ & p > 0.9138
\end{aligned}$$

3