UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 2004

BSc Honours Degree in Mathematics and Computer Science Part I
MSci Honours Degree in Mathematics and Computer Science Part I
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the Associateship of the Royal College of Science

PAPER MC140

LOGIC (JMC)

Thursday 29 April 2004, 14:30
Duration: 90 minutes
(Reading time 5 minutes)

Answer THREE questions

Paper contains 4 questions
Calculators not required

1a Let A be the formula $((p \to \bot) \to \bot) \to p$.

i) Draw the formation tree of the formula A.

ii) Use truth tables to show that A is valid.

iii) Use propositional equivalences to show that A and $\top$ are logically equivalent.

iv) Using natural deduction, show that $\vdash A$. [You may assume the lemma $B \vee \neg B$, for any suitable B. Do not rewrite any formula using equivalences.]

b i) Define the relations $\models$ and $\vdash$ on propositional formulae.

ii) Show that $\neg(p \to q) \models p$ holds.

iii) State clearly the relationship which holds between $\models$ and $\vdash$ which allows us to deduce $\neg(p \to q) \vdash p$ from the result demonstrated in (b)ii). What is this relationship/property called?

c Recall that the natural deduction rule for $\to$-elimination is 'from A and $A \to B$ we deduce B'. Recall also that this rule is sound in the sense that in any situation in which A and $A \to B$ are both true, B is true too.

i) Write the truth tables for all possible binary connectives $*$ which make the $*$-elimination rule 'from A and $A * B$ we deduce B' sound. [Hint: There are 8 of them.]

ii) Using natural deduction with $*$-elimination added to the rules, show that $p * q \vdash p \to q$. [Do not rewrite any formula using equivalences.]

The three parts carry, respectively, 50%, 25%, and 25% of the marks.

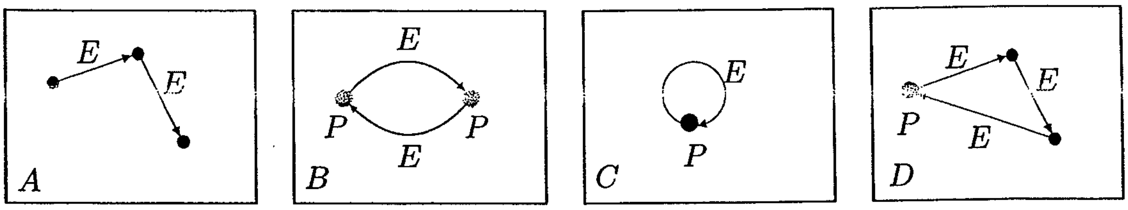
2 a Explain the following italicised terms:

i) *domain* of a structure,

ii) *free variable* (of a formula),

iii) *sentence.*

b Let L be the first-order signature consisting of a unary relation symbol P and a binary relation symbol E. Let A, B, C, D be L-structures as shown below:



An E-labelled arrow from a black circle a to a black circle b means that $E(a, b)$ is true. The objects satisfying P are the large black circles, labelled 'P'.

i) Which of the following sentences are true in which of the structures A, B, C, D?

1. $\forall x \forall y \forall z (E(x, y) \wedge E(y, z) \rightarrow E(x, z))$

2. $\forall x \exists y E(x, y)$

3. $\forall x (P(x) \rightarrow \exists y (y \neq x \wedge P(y))$

ii) For each of the structures A, B, C, D in turn, write down an L-sentence that is true in that structure and false in the other three.

c Now let L' be the signature consisting of a unary function symbol f and a constant c. Let S be the L'-sentence $\forall x(x \neq f(x) \wedge f(f(x)) = x)$.

i) Draw a suitably-labelled diagram of an L'-structure M with four objects in its domain and such that $M \models S$.

ii) Suppose that M is an L'-structure with finite domain D, and $M \models S$. What can you say about the number of objects in D? Explain your answer.

iii) Write an L'-sentence T such that for any finite domain D, the following is true: there is some L'-structure N with domain D and with $N \models T$ *if and only if* there are an odd number of objects in D. Justify your answer.

The three parts carry, respectively, 20%, 40%, and 40% of the marks.

3 a In what sense are preconditions and postconditions a contract between user and programmer? Who benefits by a weakening of the precondition and who benefits by a weakening of the postcondition?

b Complete in logic the definitions i) to iv) which are informally described in English. You can use the order relations $<$ and $>$ on natural numbers and the predicate $in(x, ys)$ which holds iff the natural number x is in the list ys.

Example:

$$\forall xs, ys : [Nat](lessthan0(xs, ys) \leftrightarrow \textit{all elements of xs are smaller than all elements of ys})$$

Answer:

$$\forall xs, ys : [Nat](lessthan0(xs, ys) \leftrightarrow \forall x, y : Nat(in(x, xs) \wedge in(y, ys) \rightarrow x < y))$$

i)

$$\forall xs, ys : [Nat](lessthan1(xs, ys) \leftrightarrow \textit{for each element of xs there is a larger element in ys})$$

ii)

$$\forall xs, ys : [Nat](lessthan2(xs, ys) \leftrightarrow \textit{there is an element in ys which is larger than all elements in xs})$$

iii)

$$\forall xs, ys : [Nat](lessthan3(xs, ys) \leftrightarrow \textit{for each element of xs there is a larger element in ys and for each element of ys there is a smaller element in xs})$$

iv) Let $bs : [Nat]$ be a given list of natural numbers.

$$\forall xs, ys : [Nat](lessthan4(xs, ys) \leftrightarrow \textit{for each element of } xs, \textit{ and any element of the given list } bs \textit{ which is less than that element of } xs, \textit{ there is an element of } ys \textit{ larger than that element of } bs)$$

c With $lessthan0$, $lessthan1$, $lessthan2$, $lessthan3$ and $lessthan4$ defined as in part (b):

i) Give an example of lists xs and ys such that $lessthan1(xs, ys)$ is true and $lessthan0(xs, ys)$ is false.

ii) Give an example of lists xs and ys such that $lessthan1(xs, ys)$ is false and $lessthan0(xs, ys)$ is true.

iii) Give an example of lists xs and ys such that $lessthan1(xs, ys)$ is true and $lessthan2(xs, ys)$ is false.

iv) Let the given list bs be $[7, 12]$. Give an example of lists xs and ys such that $lessthan0(xs, ys)$ and $lessthan2(xs, ys)$ and $lessthan4(ys, xs)$ are all true. (Notice the order of the arguments in $lessthan4$.)

The three parts carry, respectively, 20%, 40%, and 40% of the marks.

4a Explain what '$A \models B$' means, where A, B are first-order formulas of some signature L. If you use the term 'situation', you should explain what it means.

b Show by a direct argument that $\exists y \forall x R(x,y) \models \forall x \exists y R(x,y)$.

c
i) Give an example of a structure M with $M \models \forall x \exists y R(x,y)$ and $M \not\models \exists y \forall x R(x,y)$.

ii) Here is an incorrect natural deduction proof that $\forall x \exists y R(x,y) \vdash \exists y \forall x R(x,y)$:

1	$\forall x \exists y R(x,y)$	given
2	c	$\forall I$ const
3	$\exists y R(c,y)$	$\forall E(1)$
4	$R(c,d)$	ass
5	$R(c,d)$	$\checkmark(4)$
6	$R(c,d)$	$\exists E(3,4,5)$
7	$\forall x R(x,d)$	$\forall I(2,6)$
8	$\exists y \forall x R(x,y)$	$\exists I(7)$

Which line(s) in the proof are incorrect? Explain precisely how they violate the natural deduction rules.

d Show by natural deduction that

$$\left.\begin{array}{l} \forall x (R(x,a) \vee R(x,b)) \\ \forall x \forall y \forall z (R(x,y) \wedge R(y,z) \rightarrow R(x,z)) \\ R(a,b) \end{array}\right\} \vdash \exists y \forall x R(x,y).$$

The four parts carry, respectively, 15%, 20%, 30%, and 35% of the marks.