UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 2004

MEng Honours Degree in Information Systems Engineering Part IV
MSci Honours Degree in Mathematics and Computer Science Part IV
MEng Honours Degrees in Computing Part IV
MSc in Advanced Computing
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the Associateship of the City and Guilds of London Institute
This paper is also taken for the relevant examinations for the Associateship of the Royal College of Science

PAPER C478=I4.37

ADVANCED OPERATIONS RESEARCH

Wednesday 5 May 2004, 10:00
Duration: 120 minutes

Answer THREE questions

Paper contains 4 questions
Calculators required

1 a Simplify and convert the following linear programming constraints into standard LP equalities. Indicate the type of the associated logical variable. Try to combine constraints if possible.

$$x_4 + x_2 \geq x_1 + 2x_3 \quad (1)$$
$$-x_2 + 2x_1 + 3x_3 - 4x_5 \geq -5 - x_1 - x_2 + x_4 \quad (2)$$
$$3x_1 - x_2 - 4 \leq -2x_3 + x_4 \quad (3)$$
$$6 \geq 3x_1 - 2x_2 + x_3 - 4x_4 \geq -6 \quad (4)$$
$$2x_1 - 3x_4 = x_1 + x_2 - x_3 \quad (5)$$
$$x_2 - 2x_3 \leq 3 + 3x_1 - x_4 \quad (6)$$

b Total Hotels is enjoying a boom in business and decides to extend capacities. There are three types of rooms that can be added to the existing ones.

Room	Size (m^2)	Cost (£1000)
Small	12	18
Medium	20	26
Executive	32	50

Management thinks it would be desirable to add 25 small, 15 medium rooms and 5 executive suites. The expansion should be around $700m^2$ and the total cost is limited to £1,000,000.

Write a goal programming model for the above problem if (i) the financial constraint cannot be exceeded, (ii) it is undesirable to overachieve or underachieve the $700m^2$ goal, underachievement being twice as bad as overachievement and (iii) it is equally undesirable to underachieve the number of different rooms. [Hint: note the three separate requirement in (iii).]

Explain your work.

c Assume a linear programming problem is provided in a general form: $\min\{\mathbf{c}^T\mathbf{x} \mid \mathbf{A}\mathbf{x} = \mathbf{b}\}$ and variables are subject to type specifications. A basic feasible solution (BFS) with $z = -6$ and a type-1 incoming variable with reduced cost $d_q = 4$ and individual upper bound of $u_q = 5$ are given. Also, $\boldsymbol{\alpha}_q = \mathbf{B}^{-1}\mathbf{a}_q$ is available. The relevant part of the problem is the following:

i	x_{Bi}	type(x_{Bi})	u_{Bi}	α_{iq}
1	3	1	9	-1
2	2	2	$+\infty$	-5
3	0	3	$+\infty$	5
4	2	1	4	1
5	6	2	$+\infty$	2
6	1	1	5	-2

Answer the following questions in full.

What is the original (starting nonbasic) value of the incoming variable? Is the BFS degenerate? Determine the ratios, the value of the incoming variable, the variable leaving the basis (if any), the new BFS and the new value of the objective function. What role does the individual upper bound of the improving variable play in this case? Is the new BFS degenerate?

The three parts carry, respectively, 30%, 40%, and 30% of the marks.

2a Determine the type of each variable in the following problem, min $\mathbf{c}^T\mathbf{x}$, $\mathbf{A}\mathbf{x} = \mathbf{b}$. Is the given solution feasible? Does it satisfy the optimality conditions? Is the solution degenerate?

i)

	x_1	x_2	x_3	x_4	x_5	x_6
ℓ_j	0	0	$-\infty$	0	0	0
u_j	1	$+\infty$	$+\infty$	0	10	$+\infty$
In the solution:						
B/N	B	B	B	N	N	N
Value	1	1	-1	0	10	0
d_j	0	0	0	-10	10	0

b A company investigates whether it is worth expanding the financial constraints of their optimal production plan obtained from a linear programming (LP) model in such a way that the structure of the production remains unchanged. The current optimal value is 12,500 (in units of thousands of pounds).

Here is a fragment from the solution of their standard form LP, $\min\{\mathbf{c}^T\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq 0\}$. The inverse of the optimal basis and $\mathbf{c}_B$ are given:

$$\mathbf{B}^{-1} = \begin{bmatrix} 4 & 0 & -\frac{1}{2} \\ -2 & 2 & \frac{1}{2} \\ \frac{1}{2} & -1 & \frac{3}{2} \end{bmatrix} \quad \text{and} \quad \mathbf{c}_B = \begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix}.$$

The financial constraint is the first one in the model.

Determine the rate of change of the optimal objective value as a function of the expansion. Write the functional relationship of the change. What subject area of LP is applicable to this situation?

c You are given the following linear programming problem.

$$\begin{array}{lrcrcrcrcr}
\min & -4x_1 & + & 3x_2 & + & 4x_3 & + & 2x_4 & & \\
\text{s. t.} & -x_1 & + & 5x_2 & - & 2x_3 & + & x_4 & \leq & 1 \\
 & x_1 & & & + & 2x_3 & - & 2x_4 & \leq & -3 \\
 & 2x_1 & + & 4x_2 & + & x_3 & & & \leq & -1 \\
\end{array}$$
$$x_1 \leq 0,\ x_j \geq 0,\ j = 2, \ldots, 4$$

Solve it by the dual simplex method.

At the end, determine the dual solution from its defining equation including the recomputation of the value of the dual objective function. How can one recover $\mathbf{B}^{-1}$?

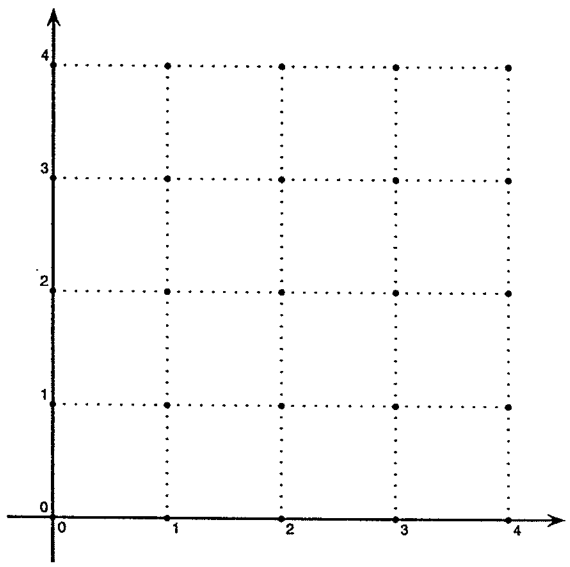
The three parts carry, respectively, 15%, 35%, and 50% of the marks.

3 a Give a sufficient condition that is easy to check and which ensures that the optimal solution of the LP relaxation of an integer programming problem is all integer, therefore it is an optimal solution to the original problem, as well.

b Explain the types of variables and constraints in a general linear programming problem. Show why the standard form is a special case of this general form.

c Solve the following MILP problem graphically using the graph printed here or your own drawing. It need not be very accurate. If in doubt, rely on the numerical data given below.

The objective is to maximize $z = -2x_1 + x_2$, where $x_1 \geq 0$ and x_2 is a general nonnegative integer. The feasible region of the LP relaxation of the problem is determined by the polygon with vertices:
$P_1(0,0)$, $P_2(0,1)$, $P_3(1,3)$, $P_4(3,4)$, $P_5(4,3)$ and $P_6(2,0)$. Where are the feasible solutions of the problem located? Determine an optimal solution. Is it unique? If not, can you find them all? How many are there? Compare the situation with continuous LP.



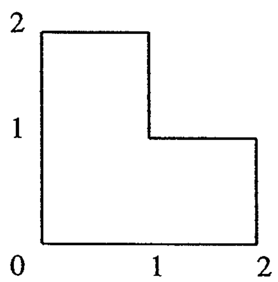
The three parts carry, respectively, 20%, 30%, and 50% of the marks.

4 a Define the simplex multiplier and explain its role in the revised simplex method.

b Bring the following set of constraints to a computational form of equalities. Perform conversion, introducing new variables if necessary. Carefully explain your work.

$$\begin{array}{rcrcrcr} 2x_1 & + & 3x_2 & - & 4x_4 & \leq & 5 \\ x_1 & - & x_2 & + & 2x_3 & \leq & -1 \end{array}$$
$$-2 \leq x_1 \leq -1,\ x_2 \geq 2,\ x_3 \geq 0.$$

c You are given the following nonconvex region in the xy plane with x and y coordinates indicated. Write the appropriate linear inequalities and logic expressions to describe the points in the closed area. Introduce integer variable(s) if needed and give a MIP formulation of the region. Verify your solution by showing that point $(1.5, 1.5)$ does not satisfy your constraints while $(0.5, 1.5)$ does.



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