Final

Imperial College
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[E2.8 (Maths 3) 2010]

B.ENG. AND M.ENG. EXAMINATIONS 2010

PART II Paper 3 : MATHEMATICS (ELECTRICAL ENGINEERING)

Date Wednesday 2nd June 2010 2.00 - 5.00 pm

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.

Answer EIGHT questions.

Please answer questions from Section A and Section B in separate answer-books.

A mathematical formulae sheet is provided.

Statistical data sheets are provided.

[Before starting, please make sure that the paper is complete; there should be SEVEN pages, with a total of TWELVE questions. Ask the invigilator for a replacement if your copy is faulty.]

1. Consider the mapping
$$w = \frac{1}{(z-i)^2}$$
from the z-plane to the w-plane, where $z = x + iy$ and $w = u + iv$.

(i) What corresponds in the w-plane to the family of circles $x^2 + (y-1)^2 = c^2$ in the z-plane?
Sketch both families.

(ii) Sketch the lines $x = 0$ and $y = 1$ in the z-plane. To what does this correspond in the w-plane?
Sketch the result.

(iii) $y = ax + 1$ represents a family of straight lines of gradient a. What is the corresponding family of lines in the w-plane?
Sketch both families.

(iv) In the two special cases when $a = 1$ & $a = -1$, what corresponds to this in the w-plane?
Sketch the result.

2. The complex function
$$\frac{e^{iz}}{z(z^2+1)(z^2+4)}$$
has simple poles in the upper half-plane at $z = i$ and $z = 2i$, with another at $z = 0$ and two simple poles in the lower half-plane at $z = -i$ and $z = -2i$. Show that the residues at the poles lying at $z = 0$, $z = i$ and $z = 2i$ are respectively $1/4$, $-e^{-1}/6$ and $e^{-2}/24$.

Now consider the contour integral
$$\oint_C \frac{e^{iz}\,dz}{z(z^2+1)(z^2+4)}$$
where C is taken to be a semi-circle in the upper half of the complex plane, with an additional small semi-circular indentation below the pole at $z = 0$. Show that the contribution to the above integral from this indentation, in the limit when its radius goes to zero, is $\pi i/4$.

Using all this information, show that
$$\int_{-\infty}^{\infty} \frac{\sin x\,dx}{x(x^2+1)(x^2+4)} = \frac{\pi(3e-1)(e-1)}{12e^2}.$$

PLEASE TURN OVER

3. Using the unit circle $z = e^{i\theta}$ as your contour C, convert the integral

$$I = \int_0^{2\pi} \frac{d\theta}{4 + \sin\theta}$$

to a complex integral over C, and hence show that

$$I = \frac{2\pi}{\sqrt{15}}.$$

4. If $\overline{f}(\omega)$ is the Fourier transform of $f(t)$, prove Parseval's equality

$$\int_{-\infty}^{\infty} |f(t)|^2\, dt \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} |\overline{f}(\omega)|^2\, d\omega\,.$$

The sinc-function sinc (t) and the tent function $\Lambda(t)$ are defined respectively by

$$\text{sinc}\,(t) \;=\; \frac{\sin(t/2)}{(t/2)}\,,$$

$$\Lambda(t) \;=\; \begin{cases} 1+t, & -1 \le t \le 0\,, \\ 1-t, & 0 \le t \le 1\,. \end{cases}$$

Show that

(i) $$\overline{\Lambda}(\omega) \;=\; \text{sinc}^2\,(\omega)\,,$$

(ii) $$\int_{-\infty}^{\infty} \text{sinc}^4\,(\omega)\, d\omega \;=\; \frac{4\pi}{3}\,.$$

5. A second order ordinary differential equation, with initial values, takes the form

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = f(t)\,. \tag{1}$$

$x(t)$ has an initial value $x = x_0$ at $t = 0$, where x_0 is an unspecified non-zero constant, while $dx/dt = -x_0$ at $t = 0$.

By taking a Laplace transform of (1)and using the convolution theorem, show that

$$x(t) = x_0 e^{-t}\cos 2t + \frac{1}{2}\int_0^t e^{-t'}\sin(2t')\,f(t-t')\,dt'$$

is a solution of (1)and its initial conditions.

6. $\overline{f}(s) = \mathcal{L}\{f(t)\}$ and $\overline{g}(s) = \mathcal{L}\{g(t)\}$ are the Laplace transforms of two functions $f(t)$ and $g(t)$ respectively. The convolution of $f(t)$ with $g(t)$ is defined as

$$f * g = \int_0^t f(t')g(t-t')\,dt'\,.$$

Use double integration to prove the Laplace convolution theorem

$$\mathcal{L}\{f * g\} = \overline{f}(s)\overline{g}(s)$$

providing a sketch of the region over which the integration takes place.

Hence, or otherwise, show that

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \frac{1}{2}t\sin t\,.$$

PLEASE TURN OVER

7. A closed boundary curve C is formed from two parabolae which enclose a region R. The lower of the two is $y = x^2$ while the upper is $y^2 = x$. Sketch these curves, marking where they intersect. Evaluate the double integral

$$\int\int_R dxdy\,,$$

to show the area of the region R between the two parabolae is $1/3$.

Use Green's theorem in a plane to show that

$$\oint_C \frac{2y^4\,dx - xy^3\,dy}{2x^2} \;=\; -\,\frac{3}{4}\,.$$

Evaluate the line integral directly, showing that the contribution from the lower parabola is zero.

Green's Theorem in a plane states that for a two-dimensional region R bounded by a closed, piecewise smooth curve C:

$$\oint_C \{P(x,y)dx \;+\; Q(x,y)dy\} \;=\; \int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dxdy\,.$$

8. Consider the transformation of variables

$$u \;=\; \frac{x}{x^2+y^2}\,, \qquad v \;=\; \frac{y}{x^2+y^2}\,.$$

Show that

$$u^2 \;+\; v^2 \;=\; \frac{1}{x^2+y^2}$$

and hence show that the circle $x^2 + y^2 = a^2$ in the $x - y$ plane transforms to the circles $u^2 + v^2 = \frac{1}{a^2}$ in the $u - v$ plane.

Show that the line $y = mx$ in the $x - y$ plane transforms to the line $v = mu$ in the $u - v$ plane.

Show also that the Jacobian J' for the transformation from $(u,\,v)$ to $(x,\,y)$ satisfies

$$|J'| \;=\; \frac{1}{(x^2+y^2)^2}\,.$$

Hence deduce $|J|$ where J is the Jacobian for the transformation from $(x,\,y)$ to $(u,\,v)$.

Finally evaluate the integral

$$\int\int_R \frac{1}{(x^2+y^2)^2}\,\exp\left(\frac{1}{x^2+y^2}\right)\,dx\,dy$$

where R is the region in the first quadrant between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ and the lines $y = \frac{1}{\sqrt{3}}x$ and $y = \sqrt{3}x$.

9. Let $\boldsymbol{F} = (F_1,\, F_2,\, F_3)$ and $\boldsymbol{A} = (A_1,\, A_2,\, A_3)$ be differentiable vector fields and ϕ and ψ be differentiable scalar fields in three dimensions. Define grad ϕ, div $\boldsymbol{F}$ and curl $\boldsymbol{F}$.

   Show that

   (i) div curl $\boldsymbol{A} = 0$ and curl grad $\psi = 0$,

   (ii) div $(\phi \boldsymbol{F}) = \boldsymbol{F} \,.\, \mathrm{grad}\, \phi + \phi\, \mathrm{div}\, \boldsymbol{F}$,

   (iii) curl $(\phi \boldsymbol{F}) = \phi\, \mathrm{curl}\, \boldsymbol{F} + \mathrm{grad}\, \phi \times \boldsymbol{F}$.

   In case (iii) *it is only necessary to prove the result for one component.*

   When $\boldsymbol{F} = \mathrm{curl}\, \boldsymbol{A}$, write down div $(\phi \boldsymbol{F})$ in terms of ϕ and the components of $\boldsymbol{F}$.

   When $\boldsymbol{F} = \mathrm{grad}\, \psi$ write down curl $(\phi \boldsymbol{F})$ in terms of ϕ and ψ.

   If $\boldsymbol{F} = (x{+}yz,\; y{+}zx,\; z{+}xy)$, show that curl $\boldsymbol{F} = 0$ and find ψ such that grad $\psi = \boldsymbol{F}$.

   (*Note alternative notation for* $\mathrm{grad}\phi = \nabla\phi$, $\mathrm{curl}\boldsymbol{F} = \nabla \times \boldsymbol{F}$ *and* $\mathrm{div}\boldsymbol{F} = \nabla \,.\, \boldsymbol{F}$.)

10. Write down the condition for the line integral

$$\int_{P,A}^{B} \left[\, f(x,\, y)\, dx + g(x,\, y)\, dy \,\right]$$

   to be independent of the path P joining the initial point A to the final point B.

   Find the value of α that makes the line integral

$$\int_{P,A}^{B} (x^3\, y^2\, dx + \alpha x^4\, y\, dy)$$

   independent of the path.

   Evaluate the line integral (which depends on β)

$$\int_{P,A}^{B} (x^3\, y^2\, dx + \beta\, x^4\, y\, dy)$$

   for two paths *of your own choice* joining $A(0,\, 0)$ to $B(1,\, 1)$ and verify that the answers are the same if the value of β is equal to the value of α obtained above.

   Find the potential function $\psi(x,\, y)$ defined by the line integral for the value of α obtained above.

PLEASE TURN OVER

11. The random variable T has a truncated exponential distribution with probability density function

$$f(t) = \begin{cases} Ce^{-\lambda t}, & t > 1, \\ 0, & t \leq 1, \end{cases}$$

where $\lambda > 0$.

(i) Show that $C = \lambda e^{\lambda}$, derive the cumulative distribution function of T and hence find the median of T in terms of λ.

(ii) Find $P(T > t + s | T > s)$, where $s > 1$ and $t > 0$.

(iii) Suppose we have a random sample of size n, $t_1, \ldots, t_n$, from this truncated exponential distribution. Show that the maximum likelihood estimator of λ is

$$\widehat{\lambda} = \frac{1}{\bar{t} - 1},$$

where $\bar{t}$ is the sample mean.

12. Consider the time series

$$y_t = t + 0.4e_t + 0.3e_{t-1} + 0.3e_{t-2},$$

where $\{e_t\}$ is white noise with $\text{Var}(e_t) = 1$.

(i) Define "white noise".

(ii) Find $\text{cov}(y_t, y_{t+s})$ for all t and $s = 0, 1, 2, 3, \ldots$.

(iii) Find the autocorrelation ρ_k, $k = 1, 2, \ldots$ of $\{y_t\}$.

(iv) Is $\{y_t\}$ (weakly) stationary? Justify your answer.

(v) Consider the time series

$$x_t = y_t - t.$$

Is $\{x_t\}$ (weakly) stationary? Justify your answer.

(vi) Find the spectrum $f(w)$ of $\{x_t\}$.

END OF PAPER

M A T H E M A T I C S   D E P A R T M E N T

MATHEMATICAL FORMULAE

1. VECTOR ALGEBRA

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1, a_2, a_3)$$

Scalar (dot) product: $\mathbf{a}\,.\,\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

Vector (cross) product:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix};$$

Scalar triple product:

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a}\,.\,\mathbf{b} \times \mathbf{c} = \mathbf{b}\,.\,\mathbf{c} \times \mathbf{a} = \mathbf{c}\,.\,\mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product: $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c}\,.\,\mathbf{a})\mathbf{b} - (\mathbf{b}\,.\,\mathbf{a})\mathbf{c}$

2. SERIES

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \ldots \quad (\alpha \text{ arbitrary, } |x| < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n\frac{x^{2n}}{(2n)!} + \ldots,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + \ldots,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^n\frac{x^{n+1}}{(n+1)} + \ldots (-1 < x \le 1)$$

3. TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin(a+b) = \sin a \cos b + \cos a \sin b\,;$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b\,.$$

$$\cos iz = \cosh z\,; \quad \cosh iz = \cos z\,; \quad \sin iz = i\sinh z\,; \quad \sinh iz = i \sin z\,.$$

4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f\,D^ng + \binom{n}{1} Df\,D^{n-1}g + \ldots + \binom{n}{r} D^rf\,D^{n-r}g + \ldots + D^nf\,g\,.$$

(b) Taylor's expansion of $f(x)$ about $x = a$:

$$f(a+h) = f(a) + hf'(a) + h^2f''(a)/2! + \ldots + h^nf^{(n)}(a)/n! + \epsilon_n(h)\,,$$

where $\epsilon_n(h) = h^{n+1}f^{(n+1)}(a+\theta h)/(n+1)!, \quad 0 < \theta < 1\,.$

(c) Taylor's expansion of $f(x, y)$ about (a, b):

$$f(a+h,\, b+k) = f(a, b) + [hf_x + kf_y]_{a,b} + 1/2!\left[h^2f_{xx} + 2hkf_{xy} + k^2f_{yy}\right]_{a,b} + \ldots$$

(d) Partial differentiation of $f(x, y)$:

i. If $y = y(x)$, then $f = F(x)$, and $\dfrac{dF}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dx}$.

ii. If $x = x(t)$, $y = y(t)$, then $f = F(t)$, and $\dfrac{dF}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$.

iii. If $x = x(u, v)$, $y = y(u, v)$, then $f = F(u, v)$, and

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}, \quad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$

(e) Stationary points of $f(x, y)$ occur where $f_x = 0$, $f_y = 0$ simultaneously.
Let (a, b) be a stationary point: examine $D = [f_{xx}f_{yy} - (f_{xy})^2]_{a,b}$.
If $D > 0$ and $f_{xx}(a, b) < 0$, then (a, b) is a maximum;
If $D > 0$ and $f_{xx}(a, b) > 0$, then (a, b) is a minimum;
If $D < 0$ then (a, b) is a saddle-point.

(f) Differential equations:

i. The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating factor $I(x) = \exp[\int P(x)(dx)]$, so that $\dfrac{d}{dx}(Iy) = IQ$.

ii. $P(x, y)dx + Q(x, y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y$.

5. INTEGRAL CALCULUS

(a) An important substitution: $\tan(\theta/2) = t$:
$\sin\theta = 2t/(1+t^2)$, $\cos\theta = (1-t^2)/(1+t^2)$, $d\theta = 2\,dt/(1+t^2)$.

(b) Some indefinite integrals:

$$\int (a^2 - x^2)^{-1/2} dx = \sin^{-1}\left(\frac{x}{a}\right), \quad |x| < a.$$

$$\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1}\left(\frac{x}{a}\right) = \ln\left\{\frac{x}{a} + \left(1 + \frac{x^2}{a^2}\right)^{1/2}\right\}.$$

$$\int (x^2 - a^2)^{-1/2} dx = \cosh^{-1}\left(\frac{x}{a}\right) = \ln\left|\frac{x}{a} + \left(\frac{x^2}{a^2} - 1\right)^{1/2}\right|.$$

$$\int (a^2 + x^2)^{-1} dx = \left(\frac{1}{a}\right)\tan^{-1}\left(\frac{x}{a}\right).$$

6. NUMERICAL METHODS

(a) Approximate solution of an algebraic equation:

If a root of $f(x) = 0$ occurs near $x = a$, take $x_0 = a$ and
$x_{n+1} = x_n - [f(x_n)/f'(x_n)]$, $n = 0, 1, 2 \ldots$

(Newton Raphson method).

(b) Formulae for numerical integration: Write $x_n = x_0 + nh$, $y_n = y(x_n)$.

i. Trapezium rule (1-strip): $\int_{x_0}^{x_1} y(x)dx \approx (h/2)[y_0 + y_1]$.

ii. Simpson's rule (2-strip): $\int_{x_0}^{x_2} y(x)dx \approx (h/3)[y_0 + 4y_1 + y_2]$.

(c) Richardson's extrapolation method: Let $I = \int_a^b f(x)dx$ and let I_1, I_2 be two estimates of I obtained by using Simpson's rule with intervals h and $h/2$.

Then, provided h is small enough,

$$I_2 + (I_2 - I_1)/15,$$

is a better estimate of I.

7. LAPLACE TRANSFORMS

Function	Transform	Function	Transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)dt$	$af(t) + bg(t)$	$aF(s) + bG(s)$
df/dt	$sF(s) - f(0)$	d^2f/dt^2	$s^2F(s) - sf(0) - f'(0)$
$e^{at}f(t)$	$F(s-a)$	$tf(t)$	$-dF(s)/ds$
$(\partial/\partial\alpha)f(t,\alpha)$	$(\partial/\partial\alpha)F(s,\alpha)$	$\int_0^t f(t)dt$	$F(s)/s$
$\int_0^t f(u)g(t-u)du$	$F(s)G(s)$		
1	$1/s$	$t^n (n = 1, 2 \ldots)$	$n!/s^{n+1}$, $(s > 0)$
e^{at}	$1/(s-a)$, $(s > a)$	$\sin\omega t$	$\omega/(s^2+\omega^2)$, $(s > 0)$
$\cos\omega t$	$s/(s^2+\omega^2)$, $(s > 0)$	$H(t-T) = \begin{cases} 0, & t < T \\ 1, & t > T \end{cases}$	e^{-sT}/s, $(s, T > 0)$

8. FOURIER SERIES

If $f(x)$ is periodic of period $2L$, then $f(x+2L) = f(x)$, and

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \text{ where}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad n = 0, 1, 2, \ldots, \text{ and}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1, 2, 3, \ldots$$

Parseval's theorem

$$\frac{1}{L}\int_{-L}^{L} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right).$$

1. Probabilities for events

For events A, B, and C $\qquad P(A \cup B) = P(A) + P(B) - P(A \cap B)$

More generally $\quad P(\bigcup A_i) = \sum P(A_i) - \sum P(A_i \cap A_j) + \sum P(A_i \cap A_j \cap A_k) - \cdots$

The odds in favour of A $\qquad P(A)\,/\,P(\overline{A})$

Conditional probability $\qquad P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$ provided that $P(B) > 0$

Chain rule $\qquad P(A \cap B \cap C) = P(A)\,P(B \mid A)\,P(C \mid A \cap B)$

Bayes' rule $\qquad P(A \mid B) = \dfrac{P(A)\,P(B \mid A)}{P(A)\,P(B \mid A) + P(\overline{A})\,P(B \mid \overline{A})}$

A and B are independent if $\qquad P(B \mid A) = P(B)$

A, B, and C are independent if $\qquad P(A \cap B \cap C) = P(A)P(B)P(C)$, and

$$P(A \cap B) = P(A)P(B), \quad P(B \cap C) = P(B)P(C), \quad P(C \cap A) = P(C)P(A)$$

2. Probability distribution, expectation and variance

The probability distribution for a discrete random variable X is called the probability mass function (pmf) and is the complete set of probabilities $\{p_x\} = \{P(X = x)\}$

Expectation $\quad E(X) = \mu = \sum_x x p_x$

For function $g(x)$ of x, $\quad E\{g(X)\} = \sum_x g(x) p_x$, so $E(X^2) = \sum_x x^2 p_x$

Sample mean $\quad \overline{x} = \dfrac{1}{n} \sum_k x_k$ estimates μ from random sample $x_1, x_2, \ldots, x_n$

Variance $\quad \operatorname{var}(X) = \sigma^2 = E\{(X - \mu)^2\} = E(X^2) - \mu^2$

Sample variance $\quad s^2 = \dfrac{1}{n-1} \left\{ \sum_k x_k^2 - \dfrac{1}{n} \left(\sum_j x_j \right)^2 \right\}$ estimates σ^2

Standard deviation $\quad \operatorname{sd}(X) = \sigma$

If value y is observed with frequency n_y

$$n = \sum_y n_y, \quad \sum_k x_k = \sum_y y n_y, \quad \sum_k x_k^2 = \sum_y y^2 n_y$$

Skewness $\quad \beta_1 = E\left(\dfrac{X - \mu}{\sigma}\right)^3$ is estimated by $\dfrac{1}{n-1} \sum \left(\dfrac{x_i - \overline{x}}{s}\right)^3$

Kurtosis $\quad \beta_2 = E\left(\dfrac{X - \mu}{\sigma}\right)^4 - 3$ is estimated by $\dfrac{1}{n-1} \sum \left(\dfrac{x_i - \overline{x}}{s}\right)^4 - 3$

Sample median $\quad \tilde{x}$ or x_{med}. Half the sample values are smaller and half larger

If the sample values $x_1, \ldots, x_n$ are ordered as $x_{(1)} \le x_{(2)} \le \cdots \le x_{(n)}$, then $\tilde{x} = x_{(\frac{n+1}{2})}$ if n is odd, and $\tilde{x} = \frac{1}{2}\left(x_{(\frac{n}{2})} + x_{(\frac{n+2}{2})}\right)$ if n is even

α-quantile $Q(\alpha)$ is such that $P(X \le Q(\alpha)) = \alpha$

Sample α-quantile $\widehat{Q}(\alpha)$ Proportion α of the data values are smaller

Lower quartile Q1 $= \widehat{Q}(0.25)$ one quarter are smaller

Upper quartile Q3 $= \widehat{Q}(0.75)$ three quarters are smaller

Sample median $\widetilde{x} = \widehat{Q}(0.5)$ estimates the population median $Q(0.5)$

3. Probability distribution for a continuous random variable

The cumulative distribution function (cdf) $F(x) = P(X \le x) = \int_{x_0=-\infty}^{x} f(x_0)\mathrm{d}x_0$

The probability density function (pdf) $f(x) = \dfrac{\mathrm{d}F(x)}{\mathrm{d}x}$

$E(X) = \mu = \int_{-\infty}^{\infty} x\, f(x)\mathrm{d}x$, $\mathrm{var}\,(X) = \sigma^2 = E(X^2) - \mu^2$, where $E(X^2) = \int_{-\infty}^{\infty} x^2\, f(x)\mathrm{d}x$

4. Discrete probability distributions

Discrete Uniform *Uniform* (n)

$p_x = \dfrac{1}{n} \quad (x = 1, 2, \ldots, n)$ $\qquad \mu = (n+1)/2,\ \sigma^2 = (n^2-1)/12$

Binomial distribution *Binomial* (n, θ)

$p_x = \binom{n}{x} \theta^x (1-\theta)^{n-x} \quad (x = 0, 1, 2, \ldots, n)$ $\qquad \mu = n\theta,\ \sigma^2 = n\theta(1-\theta)$

Poisson distribution *Poisson* (λ)

$p_x = \dfrac{\lambda^x e^{-\lambda}}{x!} \quad (x = 0, 1, 2, \ldots)$ (with $\lambda > 0$) $\qquad \mu = \lambda,\ \sigma^2 = \lambda$

Geometric distribution *Geometric* (θ)

$p_x = (1-\theta)^{x-1}\theta \quad (x = 1, 2, 3, \ldots)$ $\qquad \mu = \dfrac{1}{\theta},\ \sigma^2 = \dfrac{1-\theta}{\theta^2}$

5. Continuous probability distributions

Uniform distribution *Uniform* (α, β)

$$f(x) = \begin{cases} \dfrac{1}{\beta - \alpha} & (\alpha < x < \beta), \\ 0 & (\text{otherwise}). \end{cases}$$

$\mu = (\alpha+\beta)/2,\ \sigma^2 = (\beta-\alpha)^2/12$

Exponential distribution *Exponential* (λ)

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & (0 < x < \infty), \\ 0 & (-\infty < x \le 0). \end{cases}$$

$\mu = 1/\lambda,\ \sigma^2 = 1/\lambda^2$

Normal distribution $N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\} \quad (-\infty < x < \infty), \quad E(X) = \mu, \quad \mathrm{var}\,(X) = \sigma^2$$

Standard normal distribution $N(0,1)$

If X is $N(\mu, \sigma^2)$, then $Y = \dfrac{X-\mu}{\sigma}$ is $N(0,1)$

6. Reliability

For a device in continuous operation with failure time random variable T having pdf $f(t)$ $(t > 0)$

The reliability function at time t	$R(t)$	$= P(T > t)$
The failure rate or hazard function	$h(t)$	$= f(t)/R(t)$
The cumulative hazard function	$H(t)$	$= \int_0^t h(t_0)\,\mathrm{d}t_0 \;=\; -\ln\{R(t)\}$
The Weibull(α, β) distribution has	$H(t)$	$= \beta t^\alpha$

7. System reliability

For a system of k devices, which operate independently, let

$R_i = P(D_i) = P($"device i operates"$)$

The system reliability, R, is the probability of a path of operating devices

A system of devices in series operates only if every device operates

$$R = P(D_1 \cap D_2 \cap \cdots \cap D_k) = R_1 R_2 \cdots R_k$$

A system of devices in parallel operates if any device operates

$$R = P(D_1 \cup D_2 \cup \cdots \cup D_k) = 1 - (1-R_1)(1-R_2)\cdots(1-R_k)$$

8. Covariance and correlation

The covariance of X and Y $\quad \mathrm{cov}\,(X,Y) = E(XY) - \{E(X)\}\{E(Y)\}$

From pairs of observations $(x_1, y_1), \ldots, (x_n, y_n)$ $\quad S_{xy} = \sum_k x_k y_k - \frac{1}{n}(\sum_i x_i)(\sum_j y_j)$

$$S_{xx} = \sum_k x_k^2 - \frac{1}{n}(\sum_i x_i)^2, \qquad S_{yy} = \sum_k y_k^2 - \frac{1}{n}(\sum_j y_j)^2$$

Sample covariance	$s_{xy} = \dfrac{1}{n-1} S_{xy}$	estimates $\mathrm{cov}\,(X,Y)$
Correlation coefficient	$\rho = \mathrm{corr}\,(X,Y) = \dfrac{\mathrm{cov}\,(X,Y)}{\mathrm{sd}\,(X)\cdot\mathrm{sd}\,(Y)}$	
Sample correlation coefficient	$r = \dfrac{S_{xy}}{\sqrt{S_{xx}S_{yy}}}$	estimates ρ

9. Sums of random variables

$$E(X+Y) = E(X)+E(Y)$$
$$\text{var}(X+Y) = \text{var}(X)+\text{var}(Y)+2\,\text{cov}(X,Y)$$
$$\text{cov}(aX+bY,\ cX+dY) = (ac)\,\text{var}(X)+(bd)\,\text{var}(Y)+(ad+bc)\,\text{cov}(X,Y)$$

If X is $N(\mu_1,\sigma_1^2)$, Y is $N(\mu_2,\sigma_2^2)$, and $\text{cov}(X,Y)=c$, then $X+Y$ is $N(\mu_1+\mu_2,\ \sigma_1^2+\sigma_2^2+2c)$

10. Bias, standard error, mean square error

If t estimates θ (with random variable T giving t)

Bias of t	$\text{bias}(t)$	$= E(T)-\theta$
Standard error of t	$\text{se}(t)$	$= \text{sd}(T)$
Mean square error of t	$\text{MSE}(t)$	$= E\{(T-\theta)^2\} = \{\text{se}(t)\}^2+\{\text{bias}(t)\}^2$

If $\bar{x}$ estimates μ, then $\text{bias}(\bar{x})=0$, $\text{se}(\bar{x})=\sigma/\sqrt{n}$, $\text{MSE}(\bar{x})=\sigma^2/n$, $\widehat{\text{se}}(\bar{x})=s/\sqrt{n}$

Central limit property   If n is fairly large, $\bar{x}$ is from $N(\mu,\ \sigma^2/n)$ approximately

11. Likelihood

The likelihood is the joint probability as a function of the unknown parameter θ.

For a random sample $x_1,x_2,\ldots,x_n$

$$\ell(\theta;\ x_1,x_2,\ldots,x_n) = P(X_1=x_1\,|\,\theta)\cdots P(X_n=x_n\,|\,\theta) \quad \text{(discrete distribution)}$$
$$\ell(\theta;\ x_1,x_2,\ldots,x_n) = f(x_1\,|\,\theta)\,f(x_2\,|\,\theta)\cdots f(x_n\,|\,\theta) \quad \text{(continuous distribution)}$$

The maximum likelihood estimator (MLE) is $\hat{\theta}$ for which the likelihood is a maximum

12. Confidence intervals

If $x_1,x_2,\ldots,x_n$ are a random sample from $N(\mu,\sigma^2)$ and σ^2 is known, then the 95% confidence interval for μ is $(\bar{x}-1.96\frac{\sigma}{\sqrt{n}},\ \bar{x}+1.96\frac{\sigma}{\sqrt{n}})$

If σ^2 is estimated, then from the Student t table for t_{n-1} we find $t_0=t_{n-1,0.05}$

The 95% confidence interval for μ is $(\bar{x}-t_0\frac{s}{\sqrt{n}},\ \bar{x}+t_0\frac{s}{\sqrt{n}})$

13. Standard normal table — Values of pdf $\phi(y) = f(y)$ and cdf $\Phi(y) = F(y)$

y	$\phi(y)$	$\Phi(y)$	y	$\phi(y)$	$\Phi(y)$	y	$\phi(y)$	$\Phi(y)$	y	$\Phi(y)$
0	.399	.5	.9	.266	.816	1.8	.079	.964	2.8	.997
.1	.397	.540	1.0	.242	.841	1.9	.066	.971	3.0	.999
.2	.391	.579	1.1	.218	.864	2.0	.054	.977	0.841	.8
.3	.381	.618	1.2	.194	.885	2.1	.044	.982	1.282	.9
.4	.368	.655	1.3	.171	.903	2.2	.035	.986	1.645	.95
.5	.352	.691	1.4	.150	.919	2.3	.028	.989	1.96	.975
.6	.333	.726	1.5	.130	.933	2.4	.022	.992	2.326	.99
.7	.312	.758	1.6	.111	.945	2.5	.018	.994	2.576	.995
.8	.290	.788	1.7	.094	.955	2.6	.014	.995	3.09	.999

14. Student t table — Values $t_{m,p}$ of x for which $P(|X| > x) = p$, when X is t_m

m	$p=$ 0.10	0.05	0.02	0.01	m	$p=$ 0.10	0.05	0.02	0.01
1	6.31	12.71	31.82	63.66	9	1.83	2.26	2.82	3.25
2	2.92	4.30	6.96	9.92	10	1.81	2.23	2.76	3.17
3	2.35	3.18	4.54	5.84	12	1.78	2.18	2.68	3.05
4	2.13	2.78	3.75	4.60	15	1.75	2.13	2.60	2.95
5	2.02	2.57	3.36	4.03	20	1.72	2.09	2.53	2.85
6	1.94	2.45	3.14	3.71	25	1.71	2.06	2.48	2.78
7	1.89	2.36	3.00	3.50	40	1.68	2.02	2.42	2.70
8	1.86	2.31	2.90	3.36	∞	1.645	1.96	2.326	2.576

15. Chi-squared table — Values $\chi^2_{k,p}$ of x for which $P(X > x) = p$, when X is χ^2_k and $p = .995$, .975, *etc*

k	.995	.975	.05	.025	.01	.005	k	.995	.975	.05	.025	.01	.005
1	.000	.001	3.84	5.02	6.63	7.88	18	6.26	8.23	28.87	31.53	34.81	37.16
2	.010	.051	5.99	7.38	9.21	10.60	20	7.43	9.59	31.42	34.17	37.57	40.00
3	.072	.216	7.81	9.35	11.34	12.84	22	8.64	10.98	33.92	36.78	40.29	42.80
4	.207	.484	9.49	11.14	13.28	14.86	24	9.89	12.40	36.42	39.36	42.98	45.56
5	.412	.831	11.07	12.83	15.09	16.75	26	11.16	13.84	38.89	41.92	45.64	48.29
6	.676	1.24	12.59	14.45	16.81	18.55	28	12.46	15.31	41.34	44.46	48.28	50.99
7	.990	1.69	14.07	16.01	18.48	20.28	30	13.79	16.79	43.77	46.98	50.89	53.67
8	1.34	2.18	15.51	17.53	20.09	21.95	40	20.71	24.43	55.76	59.34	63.69	66.77
9	1.73	2.70	16.92	19.02	21.67	23.59	50	27.99	32.36	67.50	71.41	76.15	79.49
10	2.16	3.25	13.31	20.48	23.21	25.19	60	35.53	40.48	79.08	83.30	88.38	91.95
12	3.07	4.40	21.03	23.34	26.22	28.30	70	43.28	48.76	90.53	95.02	100.4	104.2
14	4.07	5.63	23.68	26.12	29.14	31.32	80	51.17	57.15	101.9	106.6	112.3	116.3
16	5.14	6.91	26.30	28.85	32.00	34.27	100	67.33	74.22	124.3	129.6	135.8	140.2

16. The chi-squared goodness-of-fit test

The frequencies n_y are grouped so that the fitted frequency $\widehat{n}_y$ for every group exceeds about 5.

$X^2 = \sum_y \frac{(n_y - \widehat{n}_y)^2}{\widehat{n}_y}$ is referred to the table of χ^2_k with significance point p,

where k is the number of terms summed, less one for each constraint, *eg* matching total frequency, and matching $\overline{x}$ with μ

17. Joint probability distributions

Discrete distribution $\{p_{xy}\}$, where $p_{xy} = P(\{X = x\} \cap \{Y = y\})$.

Let $p_{x\bullet} = P(X = x)$, and $p_{\bullet y} = P(Y = y)$, then

$$p_{x\bullet} = \sum_y p_{xy} \quad \text{and} \quad P(X = x \,|\, Y = y) = \frac{p_{xy}}{p_{\bullet y}}$$

Continuous distribution

Joint cdf $\quad F(x,y) = P(\{X \le x\} \cap \{Y \le y\}) = \int_{x_0=-\infty}^{x} \int_{y_0=-\infty}^{y} f(x_0, y_0)\, \mathrm{d}x_0\, \mathrm{d}y_0$

Joint pdf $\quad f(x,y) = \frac{\mathrm{d}^2 F(x,y)}{\mathrm{d}x\, \mathrm{d}y}$

Marginal pdf of X $\quad f_X(x) = \int_{-\infty}^{\infty} f(x, y_0)\, \mathrm{d}y_0$

Conditional pdf of X given $Y = y$ $\quad f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$ (provided $f_Y(y) > 0$)

18. Linear regression

To fit the linear regression model $y = \alpha + \beta x$ by $\widehat{y}_x = \widehat{\alpha} + \widehat{\beta} x$ from observations $(x_1, y_1), \ldots, (x_n, y_n)$, the least squares fit is $\quad \widehat{\alpha} = \overline{y} - \overline{x}\widehat{\beta}, \quad \widehat{\beta} = \frac{S_{xy}}{S_{xx}}$

The residual sum of squares $\quad \text{RSS} = S_{yy} - \frac{S_{xy}^2}{S_{xx}}$

$\widehat{\sigma^2} = \frac{\text{RSS}}{n-2} \qquad \frac{n-2}{\sigma^2}\widehat{\sigma^2}$ is from χ^2_{n-2}

$$E(\widehat{\alpha}) = \alpha, \quad E(\widehat{\beta}) = \beta,$$

$$\text{var}(\widehat{\alpha}) = \frac{\sum x_i^2}{n S_{xx}} \sigma^2, \quad \text{var}(\widehat{\beta}) = \frac{\sigma^2}{S_{xx}}, \quad \text{cov}(\widehat{\alpha}, \widehat{\beta}) = -\frac{\overline{x}}{S_{xx}} \sigma^2$$

$$\widehat{y}_x = \widehat{\alpha} + \widehat{\beta} x, \qquad E(\widehat{y}_x) = \alpha + \beta x, \qquad \text{var}(\widehat{y}_x) = \left\{ \frac{1}{n} + \frac{(x - \overline{x})^2}{S_{xx}} \right\} \sigma^2$$

$\frac{\widehat{\alpha} - \alpha}{\widehat{\text{se}}(\widehat{\alpha})}, \quad \frac{\widehat{\beta} - \beta}{\widehat{\text{se}}(\widehat{\beta})}, \quad \frac{\widehat{y}_x - \alpha - \beta x}{\widehat{\text{se}}(\widehat{y}_x)}$ are each from t_{n-2}

EE2 - Maths Paper 3 - (E2.8) EE 1

EXAMINATION QUESTIONS/SOLUTIONS 2009-2010

Course: EE2 + AE2 paper 3

Question

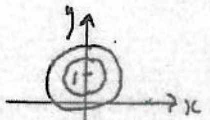
Marks & seen/unseen

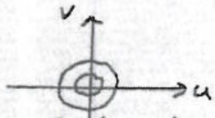
Parts

$$w = \frac{1}{(z-i)^2} = \frac{1}{[x+i(y-1)]^2} = \frac{[x-i(y-1)]^2}{[x^2+(y-1)^2]^2} = u+iv$$

$$u = \frac{x^2-(y-1)^2}{[x^2+(y-1)^2]^2} \;;\quad v = \frac{-2x(y-1)}{[x^2+(y-1)^2]^2} \quad (*)$$ 4

(i) $u^2+v^2 = \dfrac{1}{[x^2+(y-1)^2]^2}$





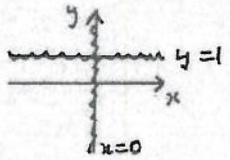
$\therefore\ x^2+(y-1)^2 = c^2$ corresponds to $u^2+v^2 = 1/c^4$ 6

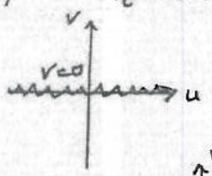
Circles radius c centred $(0,1)$

Circles radius $1/c^2$ centred $(0,0)$

(ii) $x=0 \rightarrow v=0$

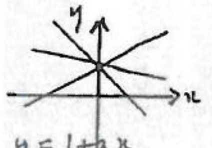
$y=1 \nearrow$ from $(*)$





4

(iii)

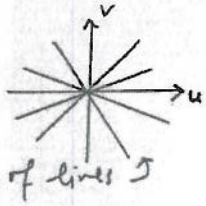


$y = 1+ax$

Family of lines.

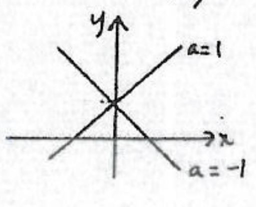
$$u = \frac{(1-a^2)}{x^2(1+a^2)^2}, \quad v = \frac{-2a}{x^2(1+a^2)^2}$$

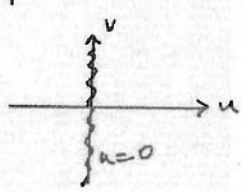
$(1-a^2)v = -2au$ Family of lines



4

(iv) When $a=\pm1$, $a^2=1$ so $(1-a^2)v = -2au \Rightarrow u=0$





2

Note to markers:
It is possible to do this question without explicitly writing down $(*)$ [e.g. for part (i) we have $|z-i|=c$] - in this case allocate the 4 marks for $(*)$ equally to parts (i)-(iv).

Setter's initials: JDG.

Checker's initials: AGW

Page number

EE 2

	EXAMINATION QUESTIONS/SOLUTIONS 2009-2010	Course EE2 paper 3
Question EE2		Marks & seen/unseen

Parts

$\frac{e^{iz}}{z(z^2+1)(z^2+4)}$ has poles in the upper-z-plane at $0, i, 2i$

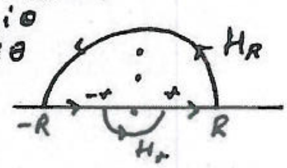
Residue at $z=0$ is $\left[\frac{e^{iz}}{(z^2+1)(z^2+4)}\right]_{z=0} = \frac{1}{4}$

Residue at $z=i$ is $[e^{iz}z^{-1}(z^2+4)^{-1}(z+i)^{-1}]_{z=i} = -e^{-1}/6$ — 3

Residue at $z=2i$ is $[e^{iz}z^{-1}(z^2+1)^{-1}(z+2i)^{-1}]_{z=2i} = e^{-2}/24$

Residue Theorem:

$H_R = Re^{i\theta}$
$H_r = re^{i\theta}$



2 (pic)

$$(*)\begin{cases}\oint_C F(z)\,dz = 2\pi i\left\{\frac{1}{4} - \frac{1}{6e} + \frac{1}{24e^2}\right\}\\ \oint_C F(z)\,dz = \left(\int_{-R}^{-r} + \int_r^R\right)\frac{e^{ix}dx}{x(x^2+1)(x^2+4)} + \int_{H_r} F(z)\,dz + \int_{H_R} F(z)\,dz\end{cases}$$ — 4

$$\therefore \lim_{r\to 0}\int_{H_r} F(z)\,dz = \lim_{r\to 0}\int_{\pi}^{2\pi}\frac{\exp(ire^{i\theta})\,ire^{i\theta}d\theta}{re^{i\theta}(1+r^2e^{2i\theta})(4+r^2e^{2i\theta})}$$

$$= i\int_{\pi}^{2\pi}\frac{1}{4}\,d\theta = \pi i/4$$ — 4

$\lim_{R\to\infty}\int_{H_R} F(z)\,dz = 0$ by Jordan's Lemma — i) $m=1>0$ ii) Only poles iii) $|F(z)|\to 0$ as $R\to\infty$. — 4

Thus in the limits $r\to 0$ & $R\to\infty$

$(*)$ becomes

$$2\pi i\left(\frac{1}{4} - \frac{1}{6e} + \frac{1}{24e^2}\right) = \frac{\pi i}{4} + \int_{-\infty}^{\infty}\frac{e^{ix}dx}{x(x^2+1)(x^2+4)}$$

Now $\int_{-\infty}^{\infty}\frac{\cos x\,dx}{x(x^2+1)(x^2+4)} = 0$ as the integrand is odd.

3

$$\therefore \int_{-\infty}^{\infty}\frac{\sin x\,dx}{x(x^2+1)(x^2+4)} = 2\pi\left(\frac{1}{8} - \frac{1}{6e} + \frac{1}{24e^2}\right)$$

$$= \frac{\pi}{12e^2}(3e^2 - 4e + 1)$$

$$= \frac{\pi}{12e^2}(3e-1)(e-1)$$

Unseen.

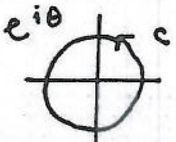
Setter's initials	Checker's initials	Page number
JDG	ADG	

EE 3

	EXAMINATION QUESTIONS/SOLUTIONS 2009-2010	Course EE 2 paper 3
Question EE3		Marks & seen/unseen

Parts

$\sin\theta = \frac{1}{2i}(z - z^{-1})$ $\qquad z = e^{i\theta}$



$I = \int_0^{2\pi} \frac{d\theta}{4+\sin\theta} = \frac{1}{i}\oint_C \frac{dz}{z[4+(z-z^{-1})/2i]}$ because $dz = iz\,d\theta$ 6

$= 2\oint_C \frac{dz}{z^2+8iz-1}$

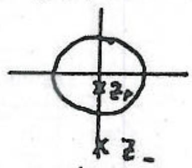
Now $z^2+8iz-1=0$ has roots at $z=z_+$ & $z=z_-$

where $2z_\pm = -8i \pm [-64+4]^{1/2}$ 4

or $z_\pm = -4i \pm i\sqrt{15}$ $\qquad z_+$ inside C

z_- outside C

$\therefore \frac{1}{2} I = \oint_C \frac{dz}{(z-z_+)(z-z_-)}$



R.T. $= 2\pi i \times \left(\text{Res. of integrand at } z_+\right)$ 6

Residue of $(z-z_+)^{-1}(z-z_-)^{-1}$ at $z=z_+$

$= (z_+-z_-)^{-1}$

$= (2i\sqrt{15})^{-1}$ 4

$\therefore \quad I = 2\pi/\sqrt{15}$.

Seen similar

Setter's initials JDG $\qquad$ Checker's initials AOG $\qquad$ Page number

EE 4

EXAMINATION QUESTIONS/SOLUTIONS 2009-2010

Course: EE2 paper 3

Question: EE4

Marks & seen/unseen

Parts

a) $$\int_{-\infty}^{\infty}|f(t)|^2dt=\int_{-\infty}^{\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\omega t}\bar{f}(\omega)d\omega\right)\left(\int_{-\infty}^{\infty}\frac{1}{2\pi}e^{i\omega' t}\bar{f}(\omega')d\omega'\right)^{*}dt$$

$$=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}e^{i\omega t}\bar{f}(\omega)d\omega\right)\left(\int_{-\infty}^{\infty}e^{-i\omega' t}\bar{f}^{*}(\omega')d\omega'\right)dt$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\bar{f}(\omega)\left(\int_{-\infty}^{\infty}\bar{f}^{*}(\omega')\underbrace{\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i(\omega-\omega')t}dt\right\}}_{\delta(\omega-\omega')}d\omega'\right)d\omega$$ 8

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\bar{f}(\omega)\left(\int_{-\infty}^{\infty}\bar{f}^{*}(\omega')\delta(\omega-\omega')d\omega'\right)d\omega$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\bar{f}(\omega)\bar{f}^{*}(\omega)d\omega=\frac{1}{2\pi}\int_{-\infty}^{\infty}|\bar{f}(\omega)|^2d\omega \quad \text{Q.E.D.}$$ seen

b) i) $$\bar{\Lambda}(\omega)=\int_{-1}^{0}e^{-i\omega t}(1+t)dt+\int_{0}^{1}e^{-i\omega t}(1-t)dt$$

$$=-\frac{1}{i\omega}\left\{\int_{-1}^{0}(1+t)d(e^{-i\omega t})+\int_{0}^{1}(1-t)d(e^{-i\omega t})\right\}$$

$$\overset{\text{parts}}{=}-\frac{1}{i\omega}\int_{0}^{1}(e^{-i\omega t}-e^{i\omega t})dt=\frac{2}{\omega}\int_{0}^{1}\sin\omega t\,dt$$ 6

$$=\frac{2}{\omega^2}(1-\cos\omega)=\frac{4\sin^2\omega}{\omega^2}=\operatorname{sinc}^2\omega$$ seen

ii) $$\int_{-\infty}^{\infty}\operatorname{sinc}^4\omega\,d\omega=\int_{-\infty}^{\infty}|\bar{\Lambda}(\omega)|^2d\omega\overset{\text{Parseval}}{=}2\pi\int_{-\infty}^{\infty}|\Lambda(t)|^2dt$$

$$=2\pi\left\{\int_{-1}^{0}(1+t)^2dt+\int_{0}^{1}(1-t)^2dt\right\}$$

$$=2\pi\left\{\left[t+t^2+t^3/3\right]_{-1}^{0}+\left[t-t^2+t^3/3\right]_{0}^{1}\right\}$$ 6

$$=4\pi/3.$$ unseen

Setter's initials

Checker's initials ADG

Page number

EE5

	EXAMINATION QUESTIONS/SOLUTIONS 2009-2010	Course EE2 paper 3
Question EE5		Marks & seen/unseen

Parts

$$\ddot{x} + 2\dot{x} + 5x = f(t) \;;\; x(0) = x_0 ;\; \dot{x}(0) = -x_0$$

$$\therefore \left(s^2\bar{x}(s) - s x_0 + x_0\right) + 2\left(s\bar{x}(s) - x_0\right) + 5\bar{x}(s) = \bar{f}(s)$$ 4

$$\therefore \left(s^2 + 2s + 5\right)\bar{x}(s) = (s+2)x_0 - x_0 + \bar{f}(s)$$

$$\therefore \quad \bar{x}(s) = \underbrace{\frac{(s+1)x_0}{(s+1)^2 + 4}}_{\text{C.F.}} + \underbrace{\frac{\bar{f}(s)}{(s+1)^2+4}}_{\text{P.I.}}$$ Completing the square 3

$$\therefore \quad \bar{x}(s) = x_0\,\bar{g}_1(s) + \bar{f}(s)\,\bar{g}_2(s)$$

$$\bar{g}_1(s) = \frac{s+1}{(s+1)^2+2^2} \Rightarrow g_1(t) = e^{-t}\mathcal{L}^{-1}\left(\frac{s}{s^2+2^2}\right)$$ 4

$$= e^{-t}\cos 2t$$ Shift Thm

$$\bar{g}_2(s) = \frac{1}{2}\cdot\frac{2}{(s+1)^2+2^2} = \frac{1}{2}e^{-t}\mathcal{L}^{-1}\left(\frac{2}{s^2+2^2}\right)$$ 4

$$= \frac{1}{2}e^{-t}\sin 2t$$

$$\therefore x(t) = x_0\,g_1(t) + \mathcal{L}^{-1}\{\bar{f}(s)\,\bar{g}_2(s)\}$$

$$= x_0\,g_1(t) + f(t) * g_2(t)$$ Convolution Thm 3

$$= x_0 e^{-t}\cos 2t + \frac{1}{2}\int_0^t e^{-t'}\sin(2t')f(t-t')\,dt'$$ 2

Unseen but similar

Setter's initials	Checker's initials	Page number
J.D.G.	AOG	

EE 6

EXAMINATION QUESTIONS/SOLUTIONS 2009-2010

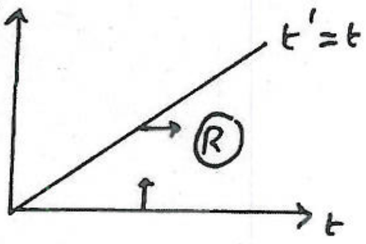
Course: EE2 paper 3

Question: EE6

Parts | Marks & seen/unseen

a) $\mathcal{L}(f*g) = \int_0^\infty e^{-st}\left(\int_0^t f(t')g(t-t')\,dt'\right)dt$

$= \int_0^\infty f(t')\left\{\int_{t'}^\infty e^{-st} g(t-t')\,dt\right\}dt'$ — 4 picture, 2 change of limits



The change of limits occurs in order to cover the wedge-like region R after a change of order of integration.

Now let $\tau = t - t'$ to get

$\mathcal{L}(f*g) = \int_0^\infty f(t')\left(e^{-st'}\int_0^\infty e^{-s\tau} g(\tau)\,d\tau\right)dt'$ — 4

$= \underline{\int_0^\infty f(t')e^{-st'}\,dt'}\int_0^\infty e^{-s\tau} g(\tau)\,d\tau = \bar{f}(s)\bar{g}(s)$ — (seen)

b) Let $\dfrac{s}{(s+1)^2} = \dfrac{s}{s^2+1}\cdot\dfrac{1}{s^2+1} \equiv \bar{f}(s)\bar{g}(s)$ — 2

$\therefore \bar{f}(s) = \dfrac{s}{s^2+1} \rightarrow f(t) = \cos t$ — 2

$\bar{g}(s) = \dfrac{1}{s^2+1} \rightarrow g(t) = \sin t$ — 2

$\therefore \mathcal{L}^{-1}\left(\dfrac{s}{(s^2+1)^2}\right) = \cos t * \sin t$ Convolution Thm

$= \int_0^t \sin(t-t')\cos t'\,dt'$

$= \frac{1}{2}\int_0^t \{\sin t + \sin(t-2t')\}\,dt'$

$= \frac{1}{2} t\sin t + \frac{1}{4}\left[\cos(t-2t')\right]_0^t$ — 4

$= \frac{1}{2} t\sin t$

(unseen) but similar

Setter's initials: JDG

Checker's initials: AOG

Page number

EE 7

EXAMINATION QUESTIONS/SOLUTIONS 2009-2010

Course: EE 2 paper 3

Question: EE7

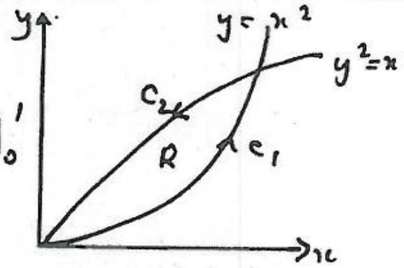
Marks & seen/unseen

Parts

a) $\iint dx\,dy = \int_0^1 \left(\int_{x^2}^{\sqrt{x}} dy \right) dx$

$= \int_0^1 (x^{1/2} - x^2)\,dx = \left[\frac{2}{3} x^{3/2} - \frac{1}{3} x^3 \right]_0^1$

$= 1/3$ — 2 + 2



b) $\oint_C P\,dx + Q\,dy \rightarrow P = y^4/x^2$ & $Q = -y^3/2x$

G.T. $\quad Q_x - P_y = -7/2 \cdot y^3/x^2$ — 2

$= \iint_R (Q_x - P_y)\,dx\,dy = -7/2 \iint (y^3/x^2)\,dx\,dy$ — 4

$= -7/2 \int_0^1 x^{-2} \left(\int_{x^2}^{x^{1/2}} y^3\,dy \right) dx = -\frac{7}{8} \int_0^1 x^{-2} (x^2 - x^8)\,dx$

$= -7/8 \int_0^1 (1 - x^6)\,dx = -7/8 \cdot 6/7 = -3/4$ — 4

c) $\int_{C_1} \frac{2y^4\,dx - xy^3\,dy}{2x^2}$ on $y = x^2 \Rightarrow dy = 2x\,dx$.

$= \int_0^1 \frac{1}{2x^2} (2x^8 - 2x^8)\,dx = 0$ — 3

$\int_{C_2} \left(\frac{\cdot}{\cdot} \right) = \int_1^0 \frac{1}{2y^4} (4y^5 - y^5)\,dy$ on $x = y^2$

$= -\frac{3}{2} \int_0^1 y\,dy = -3/4.$ — 3

Unseen

Setter's initials: J.D.G.

Checker's initials: ADG

Page number

EE8

EXAMINATION QUESTIONS/SOLUTIONS 2009-2010

Course: CORE

Question: C3 p2

Marks & ~~seen~~/unseen

Parts

To evaluate the integral we first note that $R \to R^*$ where R^* is the region between the two circles $u^2+v^2 = \frac{1}{4}$ and $u^2+v^2=1$ and the two lines $v = \frac{1}{\sqrt{3}}u$ and $v=\sqrt{3}u$. — 3 for specification of R^* including diagrams below

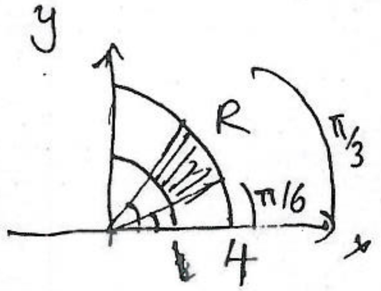
Then $\iint_R \frac{1}{(x^2+y^2)^2} \exp\left(\frac{1}{x^2+y^2}\right) dx\, dy$

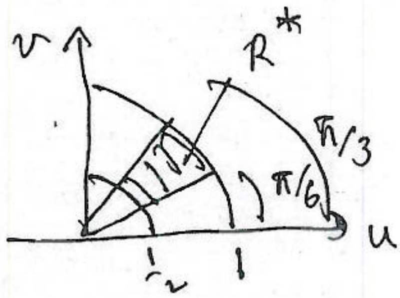
$= \iint_{R^*} \exp(u^2+v^2)\, du\, dv$ — 3

Now change to polar coordinates ρ and θ where $u = \rho\cos\theta$, $v = \rho\sin\theta$, $|J| = \rho$

Thus integral $= \int_{\theta=\pi/6}^{\pi/3} \int_{\rho=\frac{1}{2}}^{1} \rho e^{\rho^2}\, d\rho\, d\theta$ — 4

$= \frac{\pi}{6}\left[\frac{1}{2}e^{\rho^2}\right]_{1/2}^{1} = \frac{\pi}{12}\left[e - e^{1/4}\right]$





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Setter's initials: RCJ.

Checker's initials: AOG

Page number: 2/2

EE 8

EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2009-2010

Course

CORE

Question

C3 p1

Marks & ~~seen~~/unseen

Parts

$$u^2+v^2 = \frac{x^2}{(x^2+y^2)^2} + \frac{y^2}{(x^2+y^2)^2} = \frac{1}{x^2+y^2}$$

Thus if $x^2+y^2=a^2$, $u^2+v^2 = \frac{1}{a^2}$ 2

If $y=mx$ then $u = \frac{x}{x^2+y^2}$ and $v = \frac{mx}{x^2+y^2}$ 2

and $\therefore$ $v = mu$.

$$\frac{\partial u}{\partial x} = \frac{1}{x^2+y^2} - \frac{x\,2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = -\frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial x} = -\frac{2xy}{(x^2+y^2)^2}, \quad \frac{\partial v}{\partial y} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

Thus $J' = \begin{vmatrix} \frac{y^2-x^2}{(x^2+y^2)^2} & -\frac{2xy}{(x^2+y^2)^2} \\ \frac{-2xy}{(x^2+y^2)^2} & \frac{x^2-y^2}{(x^2+y^2)^2} \end{vmatrix} = +\frac{1}{(x^2+y^2)^2}$ 4

Hence $|J'| = \frac{1}{(x^2+y^2)^2}$ 2

J and J' satisfy $|J| = \frac{1}{|J'|} = (x^2+y^2)^2$

Setter's initials RLJ

Checker's initials AOG

EE9

EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2009-2010

Course

COR5

Question

C4 p1

Marks & seen/unseen

Parts

$$\text{grad}\,\varphi = \underline{i}\frac{\partial\varphi}{\partial x} + \underline{j}\frac{\partial\varphi}{\partial y} + \underline{k}\frac{\partial\varphi}{\partial z}$$

$$\text{div}\,\underline{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\text{curl}\,\underline{F} = \begin{pmatrix} \hat{i} & \hat{j} & \underline{k} \\ \partial_x & \partial_y & \partial_z \\ F_1 & F_2 & F_3 \end{pmatrix} = \underline{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \hat{j}(\ldots) + \underline{k}(\ldots)$$ 3

(i) $$\text{div curl}\,\underline{F} = \frac{\partial}{\partial x}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) = 0$$ 2

$$[\text{curl grad}\,\psi]_1 = \frac{\partial}{\partial y}\left(\frac{\partial\psi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial\psi}{\partial y}\right) = 0$$

similarly for other components

(ii) $$\text{div}(\varphi\underline{F}) = \frac{\partial}{\partial x}(\varphi F_1) + \frac{\partial}{\partial y}(\varphi F_2) + \frac{\partial}{\partial z}(\varphi F_3)$$

$$= F_1\frac{\partial\varphi}{\partial x} + F_2\frac{\partial\varphi}{\partial y} + F_3\frac{\partial\varphi}{\partial z} + \varphi\frac{\partial F_1}{\partial x} + \varphi\frac{\partial F_2}{\partial y} + \varphi\frac{\partial F_3}{\partial z}$$ 2

$$= \underline{F}\cdot\text{grad}\,\varphi + \varphi\,\text{div}\,\underline{F}$$

(iii) $$[\text{curl}(\varphi\underline{F})]_1 = \frac{\partial}{\partial y}(\varphi F_3) - \frac{\partial}{\partial z}(\varphi F_2)$$

$$= \varphi\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \frac{\partial\varphi}{\partial y}F_3 - \frac{\partial\varphi}{\partial z}F_2$$ 2

$$= \varphi[\text{curl}\,\underline{F}]_1 + [\text{grad}\,\varphi\times\underline{F}]_1$$

similarly for other components

ALL SEEN

Setter's initials RLi

Checker's initials AOG

Page number 1/2

EE9

EXAMINATION QUESTIONS/SOLUTIONS 2009-2010

Course: CORE

Question: C4 p2

Marks & seen/unseen

Parts

When $\underset{\sim}{F} = \text{curl}\,\underset{\sim}{A}$, $\text{div}\,\underset{\sim}{F} = \text{div}\,\text{curl}\,\underset{\sim}{A} = 0$ by (i). Hence by (ii)

$$\text{div}(\varphi \underset{\sim}{F}) = \underset{\sim}{F} \cdot \text{grad}\,\varphi = F_1 \frac{\partial \varphi}{\partial x} + F_2 \frac{\partial \varphi}{\partial y} + F_3 \frac{\partial \varphi}{\partial z}$$ 3

When $\underset{\sim}{F} = \text{grad}\,\psi$, $\text{curl}\,\underset{\sim}{F} = \text{curl}\,\text{grad}\,\psi = 0$ by (i). Hence by (iii)

$$\text{curl}(\varphi \underset{\sim}{F}) = \text{grad}\,\varphi \times \underset{\sim}{F} = \text{grad}\,\varphi \times \text{grad}\,\psi$$ 3

$$\text{curl}\,\underset{\sim}{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial_x & \partial_y & \partial_z \\ x+yz & y+zx & z+xy \end{vmatrix} = \underline{i}\,(x-x) - \underline{j}\,(y-y) + \underline{k}\,(z-z) = 0$$

$$\frac{\partial \psi}{\partial x} = x + yz, \quad \frac{\partial \psi}{\partial y} = y + zx, \quad \frac{\partial \psi}{\partial z} = z + xy$$

$$\frac{\partial \psi}{\partial x} = x + yz \Rightarrow \psi(x,y,z) = \tfrac{1}{2}x^2 + xyz + f(y,z)$$

Substitute into eqn. for $\frac{\partial \psi}{\partial y}$ to get

$$xz + \frac{\partial f}{\partial y} = y + zx \Rightarrow f(y,z) = \tfrac{1}{2}y^2 + g(z)$$

Substitute ψ into eqn. for $\frac{\partial \psi}{\partial z}$ to get

$$xy + \frac{dg}{dz} = z + xy \Rightarrow g = \tfrac{1}{2}z^2 + C$$

$$\Rightarrow \psi(x,y,z) = \tfrac{1}{2}(x^2 + y^2 + z^2) + xyz + C.$$ 5

20

ALL UNSEEN

Setter's initials: RLJ

Checker's initials: AOG

Core Q5 Solution to C5 EE10

Line integral is independent of path
if $\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y}$. 2

In this case $f = x^3y^2$, $g = \alpha x^4 y$
so l.i. is independent of path if
$$4\alpha x^3 y = 2x^3 y$$ 2
i.e. $\alpha = 1/2$.

Many paths are possible but two simple paths are P_1, a straight line of equation $y=x$ joining A to B, and P_2, a path of two parts with the first being a horizontal straight line $y=0$ running from A to $C(1,0)$ and the second a vertical straight line $x=1$ running from C to B.

Then $\int_{P_1 A}^{B} = \int_0^1 (x^5\,dx + \beta x^5)\,dx$ —— (1)
$= \frac{1}{6}(1+\beta)$

and $\int_{P_2, A}^{B} = \int_A^C + \int_C^B$
$= 0 + \beta\int_0^1 y\,dy$

The first term is zero because y and dy are both zero on P_1. The second term uses the fact that $x=1$ and $dx=0$ on P_2 8

thus $\int_{P_2, A}^{B} = \frac{1}{2}\beta$ —— (2)

RLJ. ADG (1)+(2) $\Rightarrow \beta = \frac{1}{2}$ same as α

CONTINUED

Core Q 5 EE 10

Solution to C5 (Continued)

$$\frac{\partial\varphi}{\partial x} = f = x^3 y^2 , \quad \frac{\partial\varphi}{\partial y} = g = \frac{1}{2}x^4 y$$

$$\frac{\partial\varphi}{\partial x} = x^3 y^2 \Rightarrow \varphi(x,y) = \frac{x^4 y^2}{4} + \cancel{g}\, h(y)$$

Substitute into $\frac{\partial\varphi}{\partial y}$ to get

$$\frac{2x^4 y}{2} + \frac{dh}{dy} = \frac{1}{2}x^4 y$$

$$\therefore \frac{dh}{dy} = 0 \Rightarrow h = c$$

$$\varphi(x,y) = \frac{x^4 y^2}{4} + c .$$ 8

Total 20

RLJ.

AOG

EE II (3) 11

i. We require

$$\int_1^\infty Ce^{-\lambda t}\,\mathrm{d}t = 1$$
$$C\left[-\frac{e^{-\lambda t}}{\lambda}\right]_1^\infty = 1$$
$$\frac{C}{\lambda}(0 + (e^{-\lambda})) = 1$$
$$C = \lambda e^{\lambda}$$

[4]

For $x \geq 1$, the cdf is

$$\begin{aligned} F(x) &= \lambda e^{\lambda} \int_1^x e^{-\lambda t}\,\mathrm{d}t \\ &= \lambda e^{\lambda}\left[-\frac{e^{-\lambda t}}{\lambda}\right]_1^x \\ &= \lambda e^{\lambda}\left(-\frac{e^{-\lambda x}}{\lambda} + \frac{e^{-\lambda}}{\lambda}\right) \\ &= 1 - e^{\lambda(1-x)} \end{aligned}$$

[2]

So $F(x) = 0$ for $x \leq 1$, $F(x) = 1 - e^{\lambda(1-x)}$ for $x > 1$.

[1]

The median is the value x such that $F(x) = 0.5$

$$\begin{aligned} 1 - e^{\lambda(1-x)} &= 0.5 \\ e^{\lambda(1-x)} &= 0.5 \\ \lambda(1-x) &= \log(0.5) \\ x &= 1 - \frac{\log(0.5)}{\lambda} \end{aligned}$$

[3]

NA

ii.

$$\begin{aligned} P(T > t+s|T > s) &= \frac{P(\{T > t+s\} \cap \{T > s\})}{P(T > s)} \\ &= \frac{P(T > t+s)}{P(T > s)} = \frac{1 - F(t+s)}{1 - F(s)} \\ &= \frac{e^{\lambda(1-(t+s))}}{e^{\lambda(1-s)}} = \underline{e^{-\lambda t}} \end{aligned}$$

[3]

iii.

$$\begin{aligned} L(\lambda) &= \prod_{i=1}^{n} \lambda e^{\lambda} e^{-\lambda t_i} \\ &= \lambda^n e^{n\lambda} e^{-\lambda \sum_{i=1}^{n} t_i} \end{aligned}$$

taking logs

$$\log(L(\lambda)) = n\log(\lambda) + n\lambda - \lambda \sum_{i=1}^{n} t_i$$

we seek a maximum

$$\frac{d\log(L(\lambda))}{d\lambda} = \frac{n}{\lambda} + n - \sum_{i=1}^{n} t_i$$

turning points occur when the derivative is 0, so

$$0 = \frac{n}{\hat{\lambda}} + n - \sum_{i=1}^{n} t_i = n + n\hat{\lambda} - \hat{\lambda} \sum_{i=1}^{n} t_i$$

9 agreed

so the maximum likelihood estimator of λ is

$$\hat{\lambda} = \frac{-n}{n - \sum_{i=1}^{n} t_i} = \frac{1}{\frac{1}{n}\sum_{i=1}^{n} t_i - 1} = \frac{1}{\bar{t} - 1}$$

and examine the second derivative

$$\frac{d^2\log(L(\lambda))}{d\lambda^2} = -\frac{n}{\lambda^2}, < 0 \ \forall \ \lambda \text{ including } \hat{\lambda}$$

to verify that this solution is a maximum.

[7]

TOTAL [20]

EXAMINATION SOLUTIONS 2009-10

Question		Course
12		EE2(3)
Parts		Marks & seen/unseen
(i)	A time series $\{e_t\}$ is called white noise if $\mathrm{E}(e_t) = 0$ for all t, $\mathrm{cov}(e_t, e_s) = 0$ for all $t \neq s$ $\mathrm{Var}(e_t)$ does not depend on t.	3
(ii)	$\mathrm{cov}(y_t, y_t) = \mathrm{Var}(y_t) = 0.4^2\,\mathrm{Var}(e_t) + 0.3^2\,\mathrm{Var}(e_{t-1}) + 0.3^2\,\mathrm{Var}(e_{t-2})$ $= 0.4^2 + 0.3^2 + 0.3^2 = 0.34$ $\mathrm{cov}(y_t, y_{t+1}) = 0.4 \cdot 0.3\,\mathrm{cov}(e_t, e_t) + 0.3 \cdot 0.3\,\mathrm{cov}(e_{t-1}, e_{t-1})$ $= 0.4 \cdot 0.3 + 0.3 \cdot 0.3 = 0.21$ $\mathrm{cov}(y_t, y_{t+2}) = 0.4 \cdot 0.3 = 0.12$ $\mathrm{cov}(y_t, y_{t+k}) = 0$ for $k = 3, 4, \ldots$	6
(iii)	$\rho_1 = 0.21/0.34 \approx 0.618$ $\rho_2 = 0.12/0.34 \approx 0.353$ $\rho_k = 0$ for $k = 3, 4, \ldots$.	3
(iv)	$\mathrm{E}(y_t) = t + 0.3\,\mathrm{E}(e_t) + 0.5\,\mathrm{E}(e_{t-1}) + 0.2\,\mathrm{E}(e_{t-2}) = t$ $\{y_t\}$ is not stationary because $\mathrm{E}(y_t)$ depends on t.	2
(v)	$\mathrm{E}(x_t) = \mathrm{E}(y_t) - t = 0$, $\mathrm{cov}(x_t, x_{t+s}) = \mathrm{cov}(y_t, y_{t+s})$. Since both $\mathrm{E}(x_t)$ and $\mathrm{cov}(x_t, x_{t+s})$ do not depend on t, the time series $\{x_t\}$ is stationary.	3
(vi)	The spectrum is given by $f(\omega) = \mathrm{cov}(x_t, x_t) + 2\sum_{k=1}^{\infty} \mathrm{cov}(x_t, x_{t+k}) \cos(k\omega)$ $= 0.34 + 0.42\cos(\omega) + 0.24\cos(2\omega)$.	3

Setter's initials	Checker's initials	Page number

i. We require

$$\int_1^\infty Ce^{-\lambda t}\,\mathrm{d}t = 1$$
$$C\left[-\frac{e^{-\lambda t}}{\lambda}\right]_1^\infty = 1$$
$$\frac{C}{\lambda}(0 + (e^{-\lambda})) = 1$$
$$C = \lambda e^{\lambda}$$

[4]

For $x \geq 1$, the cdf is

$$\begin{aligned} F(x) &= \lambda e^{\lambda}\int_1^x e^{-\lambda t}\,\mathrm{d}t \\ &= \lambda e^{\lambda}\left[-\frac{e^{-\lambda t}}{\lambda}\right]_1^x \\ &= \lambda e^{\lambda}\left(-\frac{e^{-\lambda x}}{\lambda} + \frac{e^{-\lambda}}{\lambda}\right) \\ &= 1 - e^{\lambda(1-x)} \end{aligned}$$

[2]

So $F(x) = 0$ for $x \leq 1$, $F(x) = 1 - e^{\lambda(1-x)}$ for $x > 1$.

[1]

The median is the value x such that $F(x) = 0.5$

$$\begin{aligned} 1 - e^{\lambda(1-x)} &= 0.5 \\ e^{\lambda(1-x)} &= 0.5 \\ \lambda(1-x) &= \log(0.5) \\ x &= 1 - \frac{\log(0.5)}{\lambda} \end{aligned}$$

[3]

NA

ii.

$$
\begin{aligned}
P(T > t + s | T > s) &= \frac{P(\{T > t + s\} \cap \{T > s\})}{P(T > s)} \\
&= \frac{P(T > t + s)}{P(T > s)} = \frac{1 - F(t + s)}{1 - F(s)} \\
&= \frac{e^{\lambda(1-(t+s))}}{e^{\lambda(1-s)}} = \underline{e^{-\lambda t}}
\end{aligned}
$$

[3]

iii.

$$
\begin{aligned}
L(\lambda) &= \prod_{i=1}^{n} \lambda e^{\lambda} e^{-\lambda t_i} \\
&= \lambda^n e^{n\lambda} e^{-\lambda \sum_{i=1}^{n} t_i}
\end{aligned}
$$

taking logs

$$
\log(L(\lambda)) = n\log(\lambda) + n\lambda - \lambda \sum_{i=1}^{n} t_i
$$

we seek a maximum

$$
\frac{d\log(L(\lambda))}{d\lambda} = \frac{n}{\lambda} + n - \sum_{i=1}^{n} t_i
$$

turning points occur when the derivative is 0, so

$$
0 = \frac{n}{\hat{\lambda}} + n - \sum_{i=1}^{n} t_i = n + n\hat{\lambda} - \hat{\lambda} \sum_{i=1}^{n} t_i
$$

9 agreed

so the maximum likelihood estimator of λ is

$$
\hat{\lambda} = \frac{-n}{n - \sum_{i=1}^{n} t_i} = \frac{1}{\frac{1}{n}\sum_{i=1}^{n} t_i - 1} = \frac{1}{\bar{t} - 1}
$$

and examine the second derivative

$$
\frac{d^2\log(L(\lambda))}{d\lambda^2} = -\frac{n}{\lambda^2}, < 0 \ \forall\ \lambda \text{ including } \hat{\lambda}
$$

to verify that this solution is a maximum.

[7]

TOTAL [20]

EXAMINATION SOLUTIONS 2009-10

Course EE2(3)

Question 12

Marks & seen/unseen

Parts

(i) A time series $\{e_t\}$ is called white noise if
$E(e_t) = 0$ for all t,
$\text{cov}(e_t, e_s) = 0$ for all $t \neq s$
$\text{Var}(e_t)$ does not depend on t. — 3

(ii)
$$\begin{aligned}\text{cov}(y_t, y_t) &= \text{Var}(y_t) = 0.4^2\,\text{Var}(e_t) + 0.3^2\,\text{Var}(e_{t-1}) + 0.3^2\,\text{Var}(e_{t-2})\\ &= 0.4^2 + 0.3^2 + 0.3^2 = 0.34\\ \text{cov}(y_t, y_{t+1}) &= 0.4 \cdot 0.3\,\text{cov}(e_t, e_t) + 0.3 \cdot 0.3\,\text{cov}(e_{t-1}, e_{t-1})\\ &= 0.4 \cdot 0.3 + 0.3 \cdot 0.3 = 0.21\\ \text{cov}(y_t, y_{t+2}) &= 0.4 \cdot 0.3 = 0.12\\ \text{cov}(y_t, y_{t+k}) &= 0 \text{ for } k = 3, 4, \ldots\end{aligned}$$
— 6

(iii)
$\rho_1 = 0.21/0.34 \approx 0.618$
$\rho_2 = 0.12/0.34 \approx 0.353$
$\rho_k = 0$ for $k = 3, 4, \ldots$. — 3

(iv) $E(y_t) = t + 0.3\,E(e_t) + 0.5\,E(e_{t-1}) + 0.2\,E(e_{t-2}) = t$
$\{y_t\}$ is not stationary because $E(y_t)$ depends on t. — 2

(v) $E(x_t) = E(y_t) - t = 0$, $\text{cov}(x_t, x_{t+s}) = \text{cov}(y_t, y_{t+s})$.
Since both $E(x_t)$ and $\text{cov}(x_t, x_{t+s})$ do not depend on t, the time series $\{x_t\}$ is stationary. — 3

(vi) The spectrum is given by

$$\begin{aligned}f(\omega) &= \text{cov}(x_t, x_t) + 2\sum_{k=1}^{\infty} \text{cov}(x_t, x_{t+k}) \cos(k\omega)\\ &= 0.34 + 0.42\cos(\omega) + 0.24\cos(2\omega).\end{aligned}$$

— 3

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