EE3-07

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2012

EEE/ISE PART III/IV: MEng, BEng and ACGI

DIGITAL SIGNAL PROCESSING

Monday, 21 May 2:30 pm

Time allowed: 3:00 hours

There are FOUR questions on this paper.

Answer ALL questions.

All questions carry equal marks.

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s) :	P.A. Naylor
	Second Marker(s) :	W. Dai

DIGITAL SIGNAL PROCESSING

1. A 2-channel maximally decimated filter bank structure is shown in Fig. 1.1 in which the analysis filter bank is directly connected to the synthesis filter bank. The input and output signals are $x(n)$ and $\hat{x}(n)$ with corresponding z-transforms $X(z)$ and $\hat{X}(z)$ respectively.

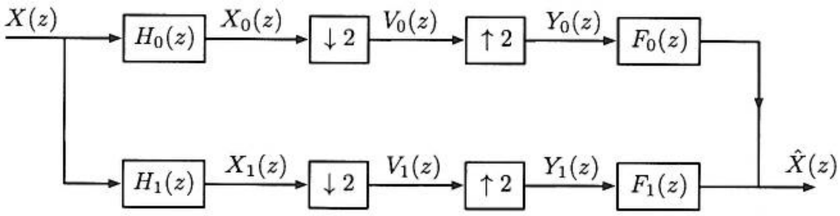

Figure 1.1 Maximally decimated filter bank.

a) Deduce expressions for $X_0(z)$, $X_1(z)$, $V_0(z)$, $V_1(z)$, $Y_0(z)$ and $Y_1(z)$ in terms of $X(z)$. [6]

b) Hence show that the z-transform of the output signal $\hat{X}(z)$ can be written

$$\hat{X}(z) = \frac{1}{2}\begin{bmatrix} X(z) & X(-z) \end{bmatrix} \tilde{\mathbf{H}}(z) \begin{bmatrix} F_0(z) \\ F_1(z) \end{bmatrix}.$$

The matrix $\tilde{\mathbf{H}}(z)$ is known as the aliasing component matrix. Write out $\tilde{\mathbf{H}}(z)$ in terms of $H_0(z)$ and $H_1(z)$. [4]

c) i) Write a brief description of quadrature mirror filters. [2]

ii) Draw a labelled illustrative sketch of the magnitude frequency response of $H_0(z)$ and $H_1(z)$ in the case that they are quadrature mirror filters. [1]

iii) Write down the relationship between $H_0(z)$ and $H_1(z)$ when they are quadrature mirror filters. [1]

iv) Also write down the corresponding relationship between the impulse responses of these filters. [1]

v) Deduce the condition on the synthesis filter bank such that $\hat{X}(z)$ is free from aliasing. [2]

vi) Make an estimate of the order of $H_0(z)$ necessary to achieve 50 dB stopband attenuation. [3]

2. a) State the definition of the DFT $X(k)$ of an N-point sequence $x(n)$. If $x(n)$ is a 2-point sequence, write out expressions for its DFT $X(0)$ and $X(1)$. [4]

b) Describe in about one paragraph the key concepts involved in the radix-2 decimation-in-time fast Fourier transform algorithm. Hence, show that a 4-point DFT can be computed as two 2-point DFTs and give the recombination equations. Illustrate your solution using a signal flow graph and include equations for all nodes of the flow graph as well as the outputs. [6]

c) i) Derive the complexity of the DFT of N samples in terms of the number of complex multiplications. State any constraints on N. [2]

ii) Derive the complexity of the radix-2 decimation-in-time FFT of N samples in terms of the number of butterflies. State any constraints on N. [3]

iii) Comment on the complexity of the radix-2 decimation-in-time FFT in comparison to the complexity of the DFT. Determine the ratio of the complexity of the radix-2 decimation-in-time FFT to the complexity of the DFT for $N = 2048$. [3]

iv) When computing the radix-2 decimation-in-time FFT using a microprocessor, state two desirable features of the microprocessor architecture which significantly impact the efficiency of the computations. [2]

3. a) Derive the transfer function $Y(z)/X(z)$ of the filter which is shown in Fig. 3.1. [7]

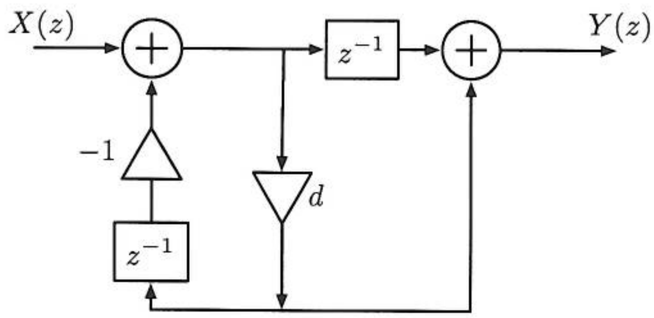

Figure 3.1 Signal flow diagram.

b) Show that the frequency response of the filter has an allpass characteristic.

[6]

c) Assume that the sampling frequency is $f_s = 48$ kHz. Find the value of d for which the phase response of the filter at a frequency of 16 kHz is -30 degrees. [7]

4. a) Let $u(n)$ denote the unit step function.

A discrete-time system is described by

$$y(n) = x(n) + y(n-1) - 0.5y(n-2).$$

i) Consider an input signal $x_a(n) = \sin(\pi n/3 + \pi/6)$. Compute the sample values of $y(n)$ for $0 \leq n < 6$. [3]

ii) Next consider an input signal $x_b(n) = \sin(\pi n/3 + \pi/6)u(n)$. Compute the sample values of $y(n)$ for this case for $0 \leq n < 6$. [3]

iii) Hence determine and sketch the sample values of the associated start-up transient. [3]

b) i) Give the definition of the z-transform of a discrete-time signal $x(n)$. What is meant by the *region of convergence* in the context of the z-transform? [2]

ii) Find the z-transform of

$$x(n) = \{1, 3, -3, -1\}$$
$$\qquad\qquad\qquad\uparrow$$

and state the region of convergence. [2]

iii) Find the z-transform of $y(n) = \cos(\omega_0 n) \cdot u(n)$. [2]

iv) Prove that if

$$x_1(n) \stackrel{z}{\longleftrightarrow} X_1(z) \quad \text{and} \quad x_2(n) \stackrel{z}{\longleftrightarrow} X_2(z)$$

then

$$x(n) = x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} x_1(k)x_2(n-k) \stackrel{z}{\longleftrightarrow} X(z) = X_1(z)X_2(z).$$

Hence find the z-transform $Z\{x(n) = x_1(n) * x_2(n)\}$ for the case when

$$x_1(n) = \{1, -1, 2\} \text{ and } x_2(n) = \begin{cases} 1, & 0 \leq n \leq 5 \\ 0, & \text{elsewhere} \end{cases}$$

and sketch $x(n)$.

[5]

EE3-07

DIGITAL SIGNAL PROCESSING - Solutions 2012

1. A 2-channel maximally decimated filter bank structure is shown in Fig. 1.1 in which the analysis filter bank is directly connected to the synthesis filter bank. The input and output signals are $x(n)$ and $\hat{x}(n)$ with corresponding z-transforms $X(z)$ and $\hat{X}(z)$ respectively.

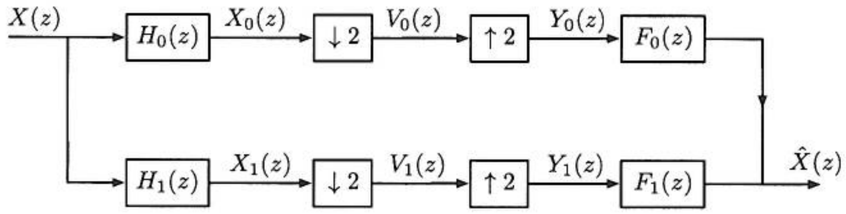

Figure 1.1 Maximally decimated filter bank.

a) Deduce expressions for $X_0(z)$, $X_1(z)$, $V_0(z)$, $V_1(z)$, $Y_0(z)$ and $Y_1(z)$ in terms of $X(z)$. [6]

Solution:

$$X_k(z) = H_k(z)X(z), \quad k = 0, 1$$

$$V_k(z) = \frac{1}{2}\left(X_k(z^{1/2}) + X_k(-z^{1/2})\right) = \frac{1}{2}\left(H_k(z^{1/2})X(z^{1/2}) + H_k(-z^{1/2})X(-z^{1/2})\right), \quad k = 0, 1$$

$$Y_k(z) = \frac{1}{2}\left(X_k(z) + X_k(-z)\right) = \frac{1}{2}\left(H_k(z)X(z) + H_k(-z)X(-z)\right), \quad k = 0, 1$$

b) Hence show that the z-transform of the output signal $\hat{X}(z)$ can be written

$$\hat{X}(z) = \frac{1}{2}\begin{bmatrix} X(z) & X(-z) \end{bmatrix}\tilde{\mathbf{H}}(z)\begin{bmatrix} F_0(z) \\ F_1(z) \end{bmatrix}.$$

Solution: This is shown by first writing

$$\begin{aligned}\hat{X}(z) &= F_0(z)Y_0(z) + F_1(z)Y_1(z) \\ &= \frac{1}{2}\left(H_0(z)F_0(z) + H_1(z)F_1(z)\right)X(z) \\ &+ \frac{1}{2}\left(H_0(-z)F_0(z) + H_1(-z)F_1(z)\right)X(-z)\end{aligned}$$

and then collecting terms.

The matrix $\tilde{\mathbf{H}}(z)$ is known as the aliasing component matrix. Write out $\tilde{\mathbf{H}}(z)$ in terms of $H_0(z)$ and $H_1(z)$. [4]

Solution:

$$\tilde{\mathbf{H}}(z) = \begin{bmatrix} H_0(z) & H_1(z) \\ H_0(-z) & H_1(-z) \end{bmatrix}$$

c) i) Write a brief description of quadrature mirror filters. [2]

Solution: QMF filters are lowpass and highpass filters with mirror symmetry at $2\pi/4$.

ii) Draw a labelled illustrative sketch of the magnitude frequency response of $H_0(z)$ and $H_1(z)$ in the case that they are quadrature mirror filters. [1]

Solution:

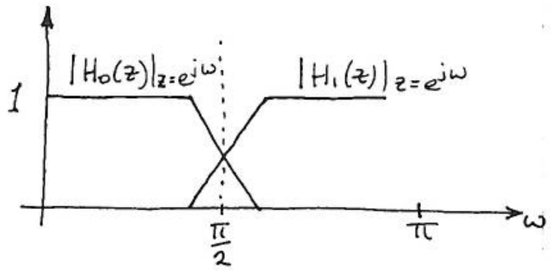

Figure 1.2

iii) Write down the relationship between $H_0(z)$ and $H_1(z)$ when they are quadrature mirror filters. [1]

Solution: From a lowpass prototype $H_o(z)$, the highpass mirror filter can be obtained as

$$H_1(z) = H_0(-z).$$

iv) Also write down the corresponding relationship between the impulse responses of these filters. [1]

Solution: For the corresponding impulse responses, $h_1(n) = h_0(n)(-1)^n$.

v) Deduce the condition on the synthesis filter bank such that $\hat{X}(z)$ is free from aliasing. [2]

Solution: We require

$$H_0(-z)F_0(z) + H_1(-z)F_1(z) = 0$$

leading to $F_0(z) = H_1(-z)$ *and* $F_1(z) = -H_0(-z)$.

vi) Make an estimate of the order of $H_0(z)$ necessary to achieve 50 dB stopband attenuation. [3]

Solution: Any answer in the range 32 to 256 would be reasonable. A bonus mark is available for a systematic approach including reference to the 'rule of thumb' formula

$$N = \frac{Attenuation(dB)}{22(Fstop - Fpass)}.$$

2. a) State the definition of the DFT $X(k)$ of an N-point sequence $x(n)$. If $x(n)$ is a 2-point sequence, write out expressions for its DFT $X(0)$ and $X(1)$. [4]

Solution:

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-jkn2\pi/N} \quad k = 0, 1, 2, \ldots, N-1.$$

For $N = 2, X(0) = x(0) + x(1)$ and $X(1) = x(0) - x(1)$.

b) Describe in about one paragraph the key concepts involved in the radix-2 decimation-in-time fast Fourier transform algorithm. Hence, show that a 4-point DFT can be computed as 2 2-point DFTs and give the recombination equations. Illustrate your solution using a signal flow graph and include equations for all nodes of the flow graph as well as the outputs. [6]

Solution: The radix-2 DIT FFT decomposes a transform into a combination of 2-point transforms. In order to be able to do this, the number of points must be an integer power of 2. The advantage comes from the fact that the 2-point transforms require only sum and difference operations - no multiplies. After the 2-point transforms are computed, the final result is obtained through a process known as 'recombination' in which the recombination equations are applied to the output of the 2-point transforms.

The 4-point DFT can be computed using 2 2-point transforms as:

$$\begin{aligned}
X_1(0) &= x(0) + x(2) \\
X_1(1) &= x(0) - x(2) \\
X_2(0) &= x(1) + x(3) \\
X_2(1) &= x(1) - x(3)
\end{aligned}$$

$$\begin{aligned}
X(0) &= X_1(0) + X_2(0) \\
X(1) &= X_1(1) + W_4^1 X_2(1) \\
X(2) &= X_1(0) - X_2(0) \\
X(3) &= X_1(1) - W_4^1 X_2(1)
\end{aligned}$$

where $W_N = e^{-j2\pi/N}$.

The figure is bookwork.

c) i) Derive the complexity of the DFT of N samples in terms of the number of complex multiplications. State any constraints on N. [2]

Solution:

The complexity of the DFT is N^2 *for any* N.

ii) Derive the complexity of the radix-2 decimation-in-time FFT of N samples in terms of the number of butterflies. State any constraints on N. [3]

Solution:

The number of 2-point DFTs (= number of butterflies) is $N/2$.

The number of subsequent recombination stages satisfies $\frac{N}{2^R} = 2$ so that $R = \log_2 N$.

Total number of stages (DFTs plus recombination) is $\frac{N}{2}\log_2 N$.

The constraint is that Nmust be an integer power of 2.

iii) Comment on the complexity of the radix-2 decimation-in-time FFT in comparison to the complexity of the DFT. Determine the ratio of the complexity of the radix-2 decimation-in-time FFT to the complexity of the DFT for $N = 2048$. [3]

Solution:

The FFT is substantially more efficient for $N > 128$. In general, the ratio is given by

$$\frac{1}{2N}\log_2 N.$$

For $N = 2048$, this gives 0.0027.

iv) When computing the radix-2 decimation-in-time FFT using a microprocessor, state two desirable features of the microprocessor architecture which significantly impact the efficiency of the computations. [2]

Solution:

Bit-reverse addressing mode and fast single-cycle MAC.

3. a) Derive the transfer function $Y(z)/X(z)$ of the filter which is shown in Fig. 3.1. [6]

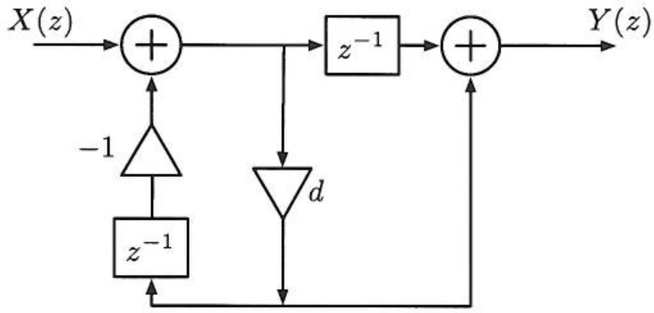

Figure 3.1 Signal flow diagram.

Solution:

$$
\begin{aligned}
Y &= Xz^{-1} + X_2(1 - z^{-2}) \\
X_2 &= d_1(X - X_2 z^{-1}) \\
&= \frac{dX}{1 + dz^{-1}} \\
Y &= Xz^{-1} + \frac{dX}{1 + dz^{-1}}(1 - z^{-2}) \\
\frac{Y}{X} &= \frac{d + z^{-1}}{1 + dz^{-1}}.
\end{aligned}
$$

b) Show that the frequency response of the filter has an allpass characteristic. [7]

Solution:

To show that the frequency response is allpass, we need to show that the magnitude is constant with ω.

$$
\begin{aligned}
\left|\frac{Y(e^{j\omega})}{X(e^{j\omega})}\right|^2 &= \frac{d^2 + \cos^2\omega + 2d\cos\omega + \sin^2\omega}{1 + d^2\cos^2\omega + 2d\cos\omega + d^2\sin^2\omega} \\
&= \frac{d^2 + 2d\cos\omega + 1}{1 + 2d\cos\omega + d^2}.
\end{aligned}
$$

c) Assume that the sampling frequency is $f_s = 48$ kHz. Find the value of d for which the phase response of the filter at a frequency of 16 kHz is -30 degrees. [7]

Solution:

The frequency of interest is $\omega = 2\pi\frac{16}{48} = 2\pi/3$.

The value of d then comes from the solution for the phase angle as

$$\angle \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \tan^{-1}\left(\frac{-\sin\omega}{d+\cos\omega}\right) - \tan^{-1}\left(\frac{-d\sin\omega}{1+d\cos\omega}\right) = -30\frac{\pi}{180}.$$

Taking tan of both sides and using the identity

$$\tan^{-1}(\alpha) \pm \tan^{-1}(\beta) = \tan^{-1}\left(\frac{\alpha \pm \beta}{1 \mp \alpha\beta}\right)$$

we can write

$$\frac{\frac{-0.866}{d-0.5} + \frac{0.866d}{1-0.5d}}{1 + \frac{0.866d^2}{(d-0.5)(1-0,5d)}} = 0.5774.$$

Solving next for d:

$$\begin{aligned} -0.866 + 0.5 \times d \times 0.866 + 0.866d^2 - 0.5 \times d \times 0.866 &= -0.5774\,(d - 0.5d^2 - 0.5 + 0.25d + 0.866^2 d \\ \Rightarrow 0.5773d^2 + 1.1548d - 1.1548 &= 0 \\ \Rightarrow d^2 + 2d - 2 &= 0 \\ d &= 0.7321, -2.7321. \end{aligned}$$

You may use the identity $\tan^{-1}(\alpha) \pm \tan^{-1}(\beta) = \tan^{-1}\left(\frac{\alpha\pm\beta}{1\mp\alpha\beta}\right)$.

4. a) Let $u(n)$ denote the unit step function.

A discrete-time system is described by

$$y(n) = x(n) + y(n-1) - 0.5y(n-2).$$

i) Consider an input signal $x_a(n) = \sin(\pi n/3 + \pi/6)$. Compute the sample values of $y(n)$ for $0 \leq n < 6$. [3]

ii) Next consider an input signal $x_b(n) = \sin(\pi n/3 + \pi/6)u(n)$. Compute the sample values of $y(n)$ for this case for $0 \leq n < 6$. [3]

iii) Hence determine and sketch the sample values of the associated start-up transient. [3]

Solution:

n	xa	xb	ya	yb	ya-yb
-2	-1.0	0	0	0	0.0
-1	-0.5	0	0	0	0.0
0	0.5	0.5	0	0	-0.500
1	1.0	1.0	1.500	1.500	0.0
2	0.5	0.5	1.500	1.250	0.250
3	-0.5	-0.5	-0.750	-0.750	0.0
4	-1.0	-1.0	-2.250	-2.125	-0.125
5	-0.5	-0.5	-1.125	-1.125	0.0

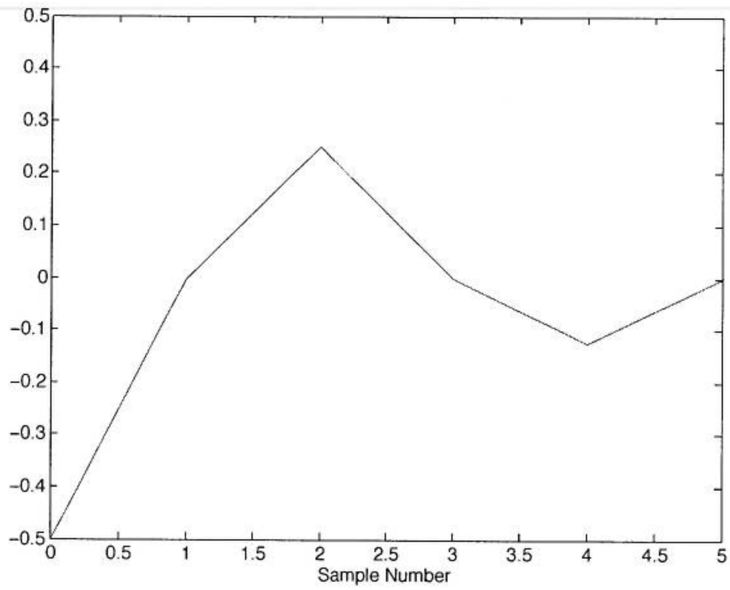

Figure 4.1 Startup transient

b) i) Give the definition of the z-transform of a discrete-time signal $x(n)$. What is meant by the *region of convergence* in the context of the z-transform? [2]

Solution:

$$X(z) = \sum_{-\infty}^{\infty} x(n) z^{-n}.$$

The region of convergence is the region of the z plane for which the summation in the z-transform converges.

ii) Find the z-transform of

$$x(n) = \{1, 3, -3, -1\}$$
$$\uparrow$$

and state the region of convergence. [2]

Solution:

$X(z) = z^2 + 3z - 3 - z^{-1}$.

ROC: $z \neq 0, \infty$.

iii) Find the z-transform of $y(n) = \cos(\omega_0 n) \cdot u(n)$. [2]

Solution:

$$\begin{aligned} Y(z) &= \frac{1}{2} Z\left\{e^{j\omega_0 n} u(n)\right\} + \frac{1}{2} Z\left\{e^{-j\omega_0 n} u(n)\right\} \\ &= \frac{1}{2}\left\{\frac{1}{1-e^{j\omega_0} z^{-1}} + \frac{1}{1-e^{-j\omega_0} z^{-1}}\right\} \\ &= \frac{1-z^{-1}\cos\omega_0}{1-2z^{-1}\cos\omega_0 + z^{-2}} \end{aligned}$$

iv) Prove that if

$$x_1(n) \overset{z}{\longleftrightarrow} X_1(z) \quad \text{and} \quad x_2(n) \overset{z}{\longleftrightarrow} X_2(z)$$

then

$$x(n) = x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} x_1(k) x_2(n-k) \overset{z}{\longleftrightarrow} X(z) = X_1(z) X_2(z).$$

Hence find the z-transform $Z\{x(n) = x_1(n) * x_2(n)\}$ for the case when

$x_1(n) = [1, -1, 2]$ and $x_2(n) = \begin{cases} 1, & 0 \leq n \leq 5 \\ 0, & \text{elsewhere} \end{cases}$

and sketch $x(n)$.

[5]

Solution:

Starting from the definition of the z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} = \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x_1(k)x_2(n-k) \right] z^{-n}.$$

Then changing the order of summation and exploit the z-transform property of shifting in time gives

$$X(z) = \sum_{k=-\infty}^{\infty} x_1(k) \left[\sum_{n=-\infty}^{\infty} x_2(n-k)z^{-n} \right] = X_2(z) \sum_{k=-\infty}^{\infty} x_1(k)z^{-k} = X_1(z)X_2(z).$$

To obtain the convolution, first write down the z-transforms

$$X_1(z) = 1 - z^{-1} + 2z^{-2}$$
$$X_2(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5}$$

and then multiply to give

$$X(z) = X_1(z)X_2(z) = 1 + 2z^2\left(1 + z^{-1} + z^{-2} + z^{-3} + z^{-5}\right) + z^{-6}.$$

By taking the inverse z-transform (by inspection) we obtain (and sketch)

$$x(n) = [1, 0, 2, 2, 2, 2, 1, 2].$$