QUESTION 1

SOLUTIONS – E1.6 Communications I 2008

a) i. $E_{x_1} = \int_{-\infty}^{\infty} |x_1(t)|^2 dt = \int_0^1 \sin^2(2\pi t)\, dt =$

$$= \frac{1}{2} - \frac{1}{2}\int_0^1 \cos(4\pi t)\, dt = \frac{1}{2}$$

ii. $E_{x_2} = E_{x_1} = \frac{1}{2}$ (ENERGY IS NOT INFLUENCED BY A CHANGE OF SIGN OR A TIME-SHIFT)

iii. $E_{(x_1+x_2)} = \int_{-\infty}^{\infty} (x_1(t) + x_2(t))^2 dt = \int_{-\infty}^{\infty} x_1^2(t)\, dt +$

$$+ \int_{-\infty}^{\infty} x_2^2(t)\, dt + 2\int_{-\infty}^{\infty} x_1(t)\, x_2(t)\, dt =$$

$$= \frac{1}{2} + \frac{1}{2} + 2\int_{-\infty}^{\infty} x_1(t)\, x_2(t)\, dt = 1 + 2\int_{0.5}^{1} \sin^2(2\pi t)\, dt =$$

$$= 1 + 2\left(\frac{1}{4} - \frac{1}{2}\int_{0.5}^{1} \sin(4\pi t)\, dt\right) = 1 + \frac{1}{2} = \frac{3}{2}$$

b) $\omega_0 = \frac{2\pi}{T_0} = \frac{\pi}{2}$

i. $x(t)$ IS EVEN $\Rightarrow$ $b_n = 0$

$$a_0 = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\,dt = \frac{1}{4}\int_{-2}^{2} x(t)\,dt = \frac{1}{2}$$

$$a_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\cos n\omega_0 t\,dt = \frac{1}{2}\int_{\cancel{0},2}^{2} x(t)\cos n\omega_0 t\,dt =$$

$$= \int_0^2 \cos\frac{n\pi}{2}t\,dt = \frac{2}{n\pi}\sin\left(n\pi/2\right)$$

$$x(t) = \frac{1}{2} + \sum_{n=1}^{\infty}\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)\cos\frac{n\pi}{2}t$$

ii. THE FILTER REMOVES ALL THE HARMONIC WITH THE EXCEPTION OF THE FIRST ONE:

$$y(t) = a_0 + a_1\cos\pi t/2 = \frac{1}{2} + \frac{2}{\pi}\cos\frac{\pi t}{2}$$

c) $X(\omega) = \frac{1}{\omega^2}\left(e^{j\omega} - j\omega e^{j\omega} - 1\right)$

$$y(t) = x(t-1) + x(-t-1)$$

$$x(t-1) \iff X(\omega)\, e^{-j\omega}$$

$$x(-t) \iff X(-\omega)$$

THEREFORE

$$Y(\omega) = \left[\left(X(\omega) + X(-\omega)\right) e^{-j\omega}\right]$$

$$= \frac{2e^{-j\omega}}{\omega^2}\left(\cos\omega + \omega\sin\omega - 1\right)$$

d) i.

$$R_x(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} \cos\omega_0 t \cos(\omega_0 t + \omega_0 \tau)\, dt =$$

$$= \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} \cos^2\omega_0 t \cos\omega_0\tau\, dt -$$

$$\underbrace{\lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} \sin\omega_0\tau \cos\omega_0 t \sin\omega_0 t\, dt}_{=0} =$$

$$= \frac{\cos\omega_0\tau}{2}$$

$$R_x(\tau) = \frac{\cos \omega_0 \tau}{2} = \frac{\cos 100\tau}{2}$$

d) i.i.

$$R_x(\tau) \iff S_x(\omega)$$

WHERE $S_x(\omega)$ IS THE POWER SPECTRAL DENSITY OF $x(t)$.

THUS

$$S_x(\omega) = \frac{\pi}{2}\left[\delta(\omega - 100) + \delta(\omega + 100)\right]$$

e)

$$\varphi_{FM}(t) = A\cos\left(\omega_c t + k_f \int_{-\infty}^{t} m_1(\alpha)\, d\alpha\right)$$

$$= A\cos\left(\omega_c t + \int_{0}^{t} (a_1 \alpha + a_0)\, d\alpha\right)$$

$$= A\cos\left(\omega_c t + \frac{a_1 t^2}{2} + a_0 t\right).$$

$$\varphi_{PM} = A\cos\left(\omega_c t + k_p m_2(t)\right) = A\cos\left(\omega_c t + b_2 t^2 + b_1 t + b_0\right).$$

THE TWO OUTPUTS ARE THE SAME WHEN

$$b_2 = \frac{a_1}{2}, \quad b_1 = a_0, \quad b_0 = 0.$$

f) $Z_{IN} = j50\,\tan(KL)$

Z_{IN} IS ZERO WHEN ~~TAN~~ $KL = \pi$

BUT $K = \frac{\omega}{u}$ $\Rightarrow$ $\frac{\omega}{u} \cdot L = \pi$

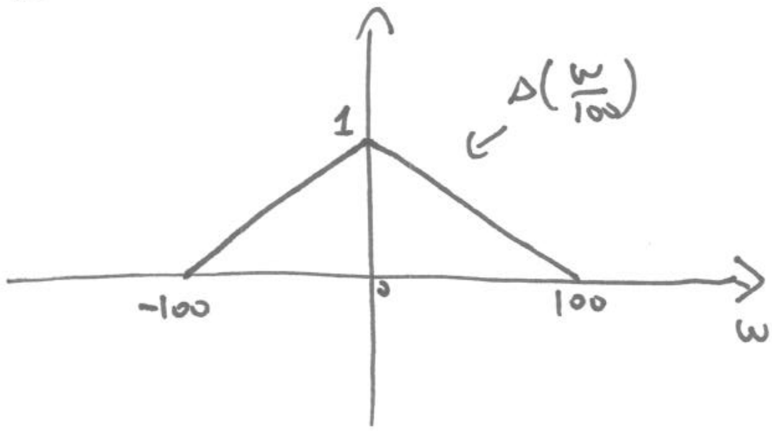
$\frac{2\pi f_0}{u} \cdot L = \cancel{\pi}$ $\quad f_0 = \frac{u}{200} = 1\,MHz$

QUESTION 2

a) i. $\frac{\alpha^2}{2\pi} \operatorname{SINC}^2\left(\frac{\alpha t}{2}\right) \Leftrightarrow \Delta\left(\frac{\omega}{\alpha}\right)$

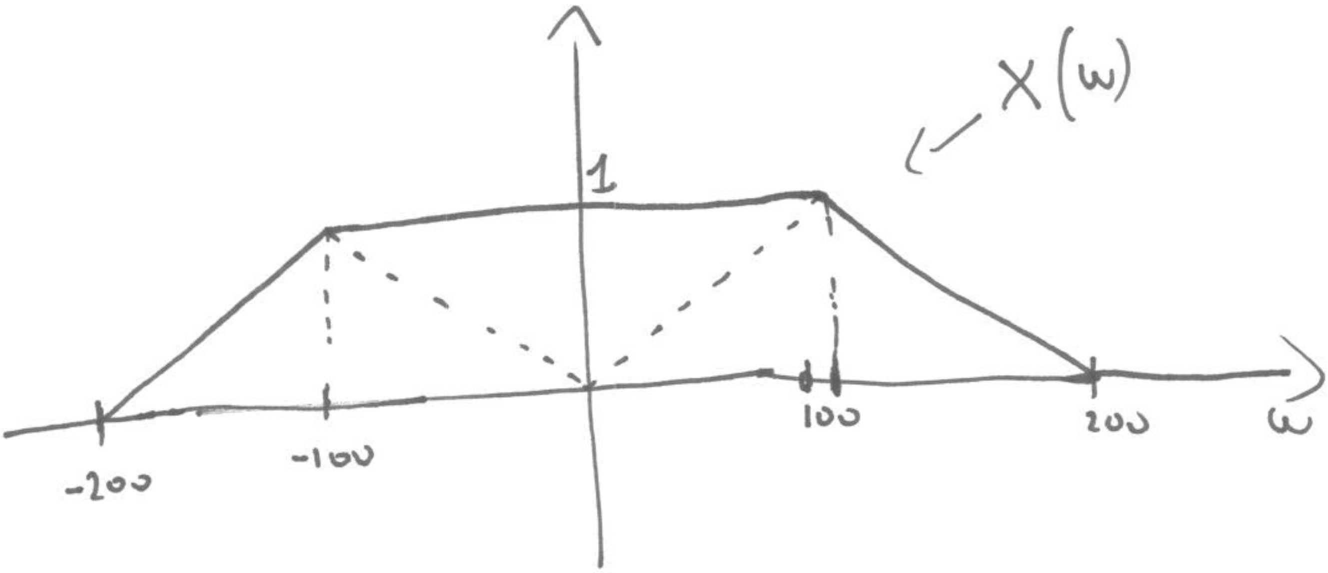
IN OUR CASE $\alpha = 100$

$$\frac{5000}{\pi} \operatorname{SINC}^2(50t) \Leftrightarrow \Delta\left(\frac{\omega}{100}\right)$$



ii. $x(t) = \frac{5000}{\pi} \operatorname{SINC}^2(50t) + \frac{10000}{\pi} \operatorname{SINC}^2(50t) \cos 100t$

$$X(\omega) = \Delta\left(\frac{\omega}{100}\right) + \Delta\left(\frac{\omega - 100}{100}\right) + \Delta\left(\frac{\omega + 100}{100}\right)$$



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$$B_{FM} = 2\left(\Delta f + B\right) = 2\left(\frac{k_f \cancel{A} p}{2\pi} + B\right)$$

$$B = \frac{200}{2\pi}\ Hz$$

$$\cancel{A}p = x(0) = \frac{5000}{\pi} + \frac{10000}{\pi} = \frac{15000}{\pi}$$

$$B_{FM} = 2\left(\frac{\cancel{\pi} \cdot \frac{15000}{\cancel{\pi}}}{2\pi} + \frac{200}{2\pi}\right) = \frac{34000}{2\pi}\ Hz$$

b) i. $\beta = \frac{\Delta f}{B} = \frac{k_f \cdot A_m}{2\pi \cdot f_m} = \frac{k_f \cdot 2}{2\pi \cdot 1000}$

FIRST ROOT AT $\beta = 2.405 \Rightarrow$

$$\frac{k_f \cdot \cancel{2}}{\cancel{2}\pi \cdot 1000} = 2.405 \qquad k_f = 2405\pi$$

b) ii. $5.5 = \frac{k_f A_m}{2\pi \cdot 1000} \Rightarrow$

$$A_m = \frac{11\cancel{\pi} \cdot 1000}{2405\cancel{\pi}} = 4.57$$

QUESTION 3

(a)

$$i(t) = I_0\left(e^{-\frac{V(t)}{V_T}} - 1\right).$$

$$e^x = 1 + x + \frac{x^2}{2} + \dots$$

$$i(t) = I_0\left(1 - \frac{V(t)}{V_T} + \frac{V^2(t)}{2V_T^2} - 1\right) = I_0\left(\frac{V^2(t)}{2V_T^2} - \frac{V(t)}{V_T}\right).$$

$$v(t) = m(t) - c(t)$$

THEREFORE

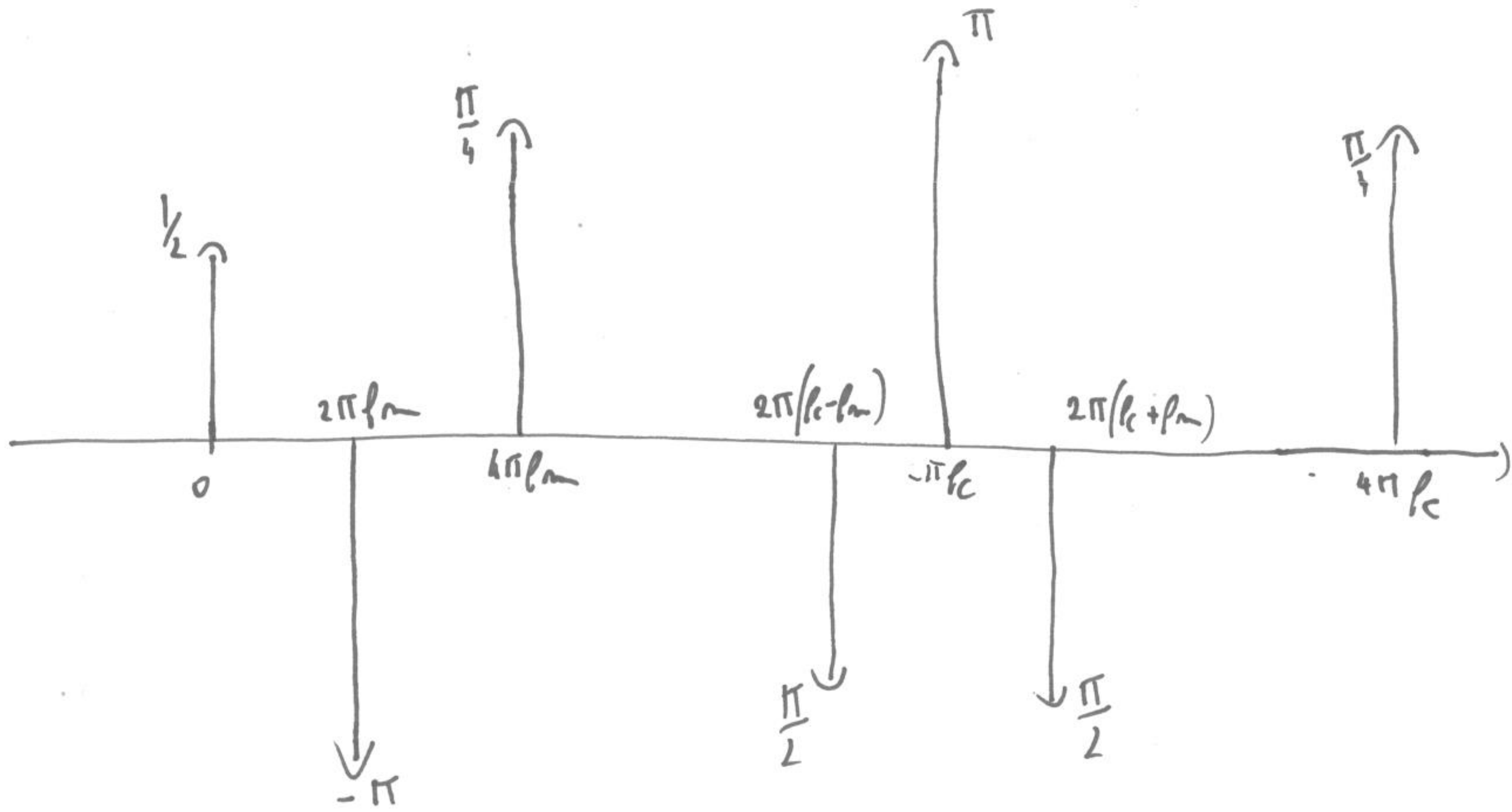
$$i(t) = I_0\left(\frac{m^2(t)}{2V_T} + \frac{c^2(t)}{2V_T^2} - \frac{m(t)\,c(t)}{V_T^2} - \frac{m(t)}{V_T} + \frac{c(t)}{V_T}\right)$$

$$= I_0\left(\frac{\cos^2 2\pi f_m t}{2} + \frac{\cos^2 2\pi f_c t}{2} - \cos 2\pi f_m t \cos 2\pi f_c t + \right.$$
$$\left. - \cos 2\pi f_m t + \cos 2\pi f_c t\right)$$

$$= I_0\left(\frac{1}{4} + \frac{1}{4}\cos 4\pi f_m t + \frac{1}{4} + \frac{1}{4}\cos 4\pi f_c t + \right.$$
$$- \frac{1}{2}\cos 2\pi(f_c - f_m)t - \frac{1}{2}\cos 2\pi(f_c + f_m)t +$$
$$\left. - \cos 2\pi f_m t + \cos 2\pi f_c t\right)$$

$$= I_0\left(\frac{1}{2} + \frac{1}{4}\cos 4\pi f_m t + \frac{1}{4}\cos 4\pi f_c t - \frac{1}{2}\cos 2\pi(f_c - f_m)t \right.$$
$$\left. - \frac{1}{2}\cos 2\pi(f_c + f_m)t - \cos 2\pi f_m t + \cos 2\pi f_c t\right).$$

SPECTRUM FOR $\omega \geq 0$ (THE SPECTRUM IS EVEN)



c) $2\pi(f_c+f_m) < W < 4\pi f_c$

THE OUTPUT IS

$$y(t) = I_0\left(1-\cos 2\pi f_m t\right)\cos 2\pi f_c t =$$

$$= \frac{I_0}{V_T}\left(V_T - m(t)\right)\cos 2\pi f_c t$$

d) SINCE $\varphi_{AM}(t) = \frac{I_0}{V_T}\left(V_T - m(t)\right)\cos 2\pi f_c t$

THEN

$$\mu = \frac{m_p}{A} = \frac{V_T}{V_T} = 1$$

e) $$\eta = \frac{P_s}{P_c+P_s} = \frac{1/4}{1/2+1/4} = \frac{1}{3}$$

QUESTION 4

a) $K_p = |K_v|^2$ AND WE WANT $K_p = 0.01$

THE LOAD AT THE JUNCTION IS $Z_0 // Z_1$

SO K_V IS GIVEN BY

$$K_V = \frac{Z_0//Z_1 - Z_0}{Z_0//Z_1 + Z_0} = \frac{\frac{Z_0 \cdot Z_1}{Z_0 + Z_1} - Z_0}{\frac{Z_0 Z_1}{Z_0+Z_1} + Z_0} = - \frac{Z_0}{2Z_1 + Z_0} = -\frac{1}{1+2\alpha}$$

WHERE $\alpha = \frac{Z_1}{Z_0}$

$$K_p = |K_V|^2 = \frac{1}{(1+2\alpha)^2} = \frac{1}{100} \Rightarrow 1 + 2\alpha = 10$$

$$\alpha = 4.5 \Rightarrow Z_1 = 4.5\, Z_0 = 225\,\Omega$$

b) $V_2 = K_V V_+ = -\frac{1}{1+2\alpha} \cdot 1 \qquad \alpha = 2 \Rightarrow V_2 = -0.2$ VOLTS

$I_2 = K_I \cdot I_+ = -K_V \frac{V_+}{Z_0} = +\frac{1}{5} \cdot \frac{1}{50} = 4$ mA

$V_T = V_2 + V_+ = -0.2 + 1 = 0.8$ VOLTS

$I_T = I_2 + I_+ = 16$ mA

c) $P_2 = \frac{1}{2} |V_2 \cdot I_2| = \frac{1}{2} (0.2 \cdot 4 \cdot 10^{-3}) W = 0.4 \, mW$

$P_T = \frac{1}{2} |V_T \cdot I_T| = \frac{1}{2} (0.8 \cdot 16 \cdot 10^{-3}) W = 6.4 \, mW$