EE4-25

SOLUTIONS: DESIGN OF LINEAR MULTIVARIABLE CONTROL SYSTEMS

1. a) Let the realisation be partitioned compatibly with $P$ and $Q$ as

$$G(s) \stackrel{s}{=} \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & 0 \end{array}\right].$$

Then

$$A^TQ+QA+C^TC = \begin{bmatrix} C_1^TC_1 & A_{21}^TQ_2+C_1^TC_2 \\ \star & A_{22}^TQ_2+Q_2A_{22}+C_2^TC_2 \end{bmatrix} = 0 \qquad (1.1)$$

$$AP+PA^T+BB^T = \begin{bmatrix} A_{11}P_1+P_1A_{11}^T+B_1B_1^T & P_1A_{21}+B_1B_2^T \\ \star & B_2B_2^T \end{bmatrix} = 0 \qquad (1.2)$$

It follows from the (1,1) entry of (1.1) and the (2,2) entry of (1.2) that $C_1 = 0$ and $B_2 = 0$. Also, it follows from the (1,2) entry of (1.1) and the assumption that $Q_2 \succ 0$ that $A_{21} = 0$. So, the realisation for $G(s)$ has the form

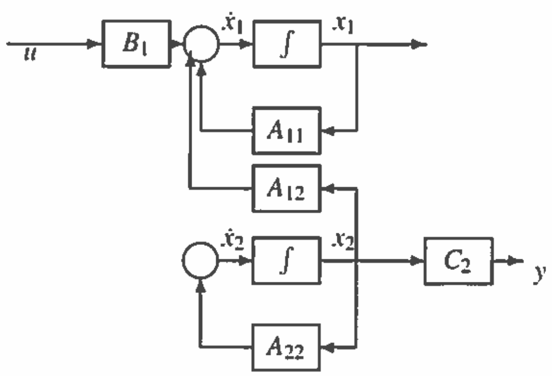
$$G(s) \stackrel{s}{=} \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ 0 & A_{22} & 0 \\ \hline 0 & C_2 & 0 \end{array}\right]. \qquad (1.3)$$

Thus we can decompose the realisation into two subsystems $G_1 : \dot{x}_1 = A_{11}x_1 + B_1u + A_{12}x_2$ with $n_1$ modes and $G_2 : \dot{x}_2 = A_{22}x_2$, $y = C_2x_2$ with $n_2$ modes.

i) For $G_1$ it is clear all the modes are unobservable since the $C$-matrix is zero. We prove controllability and stability. Let $\lambda$ be an eigenvalue of $A_{11}$ and $z \neq 0$ the corresponding left eigenvector. Then $z'A_{11} = \lambda z'$. Pre- and post-multiplying the the (1,1) entry of (1.2) by $z'$ and $z$, respectively, we get $(\lambda + \bar{\lambda})z'P_1z + z'B_1B_1'z = 0$. If $z'B_1 = 0$ then $\lambda + \bar{\lambda} = 0$ which contradicts the assumption that $A$ has no eigenvalues on the imaginary axis. Thus the realisation is controllable. Since $z \neq 0$ and $P_1 \succ 0$, $z'P_1z > 0$ and $z'B_1B_1'z > 0$ then $\lambda + \bar{\lambda} < 0$ and $A_{11}$ is stable. [7]

ii) For the subsystem $G_2$ it is clear that all the modes are uncontrollable since the $B$-matrix is zero. We prove observability and stability. Using a duality argument, $G_2$ is stable and observable if and only if the pair $A_{22}^T$ is stable and $(A_{22}^T, C_2^T)$ is controllable. But this follows from an argument dual to that used above. [7]

b)




[6]

2. a) i) The $(1,1)$ block of the inequality gives the inequality $A^TP+PA \prec 0$. Let $z \neq 0$ be a right eigenvector of $A$ and let $\lambda$ be the corresponding eigenvalue. Then multiplying the inequality from the left by $z^T$ and from the right by $z$ gives $(\lambda+\bar{\lambda})z^TPz < 0$. Since $P \succ 0$ it follows that $z^TPz > 0$ and it follows that $\lambda+\bar{\lambda} < 0$ so that $A$ is stable. [4]

ii) Since $A$ is stable, $\|H\|_\infty < \gamma$ if and only if, with $x(0)=0$, $J := \int_0^\infty [y^Ty - \gamma^2 u^Tu]dt < 0$, for all $u(t)$ such that $\|u\|_2 < \infty$. If $\|u\|_2$ is bounded, then $\lim_{t\to\infty} x(t) = 0$. Now, $\int_0^\infty \frac{d}{dt}[x^TPx]dt = x(\infty)^TPx(\infty) - x(0)^TPx(0) = 0$. So, [6]

$$0 = \int_0^\infty (\dot{x}^TPx + x^TP\dot{x})\,dt = \int_0^\infty [x^T(A^TP+PA)x + x^TPBu + u^TB^TPx]dt.$$

Use $y = Cx+Du$ and add the last expression to $J$

$$J = \int_0^\infty [x^T(A^TP+PA+C^TC)x + 2x^T(PB+C^TD)u + u^T(D^TD-\gamma^2I)u]dt$$

$$= \int_0^\infty \begin{bmatrix} x^T & u^T \end{bmatrix} \overbrace{\begin{bmatrix} A^TP+PA+C^TC & PB+C^TD \\ B^TP+D^TC & D^TD-\gamma^2I \end{bmatrix}}^{M} \begin{bmatrix} x \\ u \end{bmatrix} dt.$$

It follows that $J < 0$, and so $\|H\|_\infty < \gamma$, if $M \prec 0$. This proves the result.

b) i) Substituting $u = Lw_2 + Cx$, $y = Cx$ into the state equation gives [4]

$$\dot{x} = \underbrace{(A+LC)}_{A_c}x + \underbrace{\begin{bmatrix} B & L \end{bmatrix}}_{B_c}w, \qquad y = \underbrace{C}_{C_c}x + \underbrace{0}_{D_c}w.$$

It follows that $T_{yw}(s) = D_c + C_c(sI-A_c)^{-1}B_c$.

ii) Using the results of part (a), by replacing $A$, $B$, $C$ and $D$ by $A_c$, $B_c$, $C_c$ and $D_c$, we have that there exists a feasible $L$ if there exists $P = P^T \succ 0$ such that [6]

$$\begin{bmatrix} (A+LC)^TP + P(A+LC) + C^TC & PB & PL \\ B^TP & -\gamma^2I & 0 \\ L^TP & 0 & -\gamma^2I \end{bmatrix} \prec 0.$$

Noting that the only nonlinearity is due to the product $PL$, we define $Z = PL$ and so there exists a feasible $L$ if there exists $P = P^T \succ 0$ and $Z$ such that

$$\begin{bmatrix} A^TP + PA + ZC + C^TZ^T + C^TC & PB & Z \\ B^TP & -\gamma^2I & 0 \\ Z^T & 0 & -\gamma^2I \end{bmatrix} \prec 0.$$

3. a) An inspection of Figure 3 shows that

$$\begin{aligned} \dot{x}-\dot{\hat{x}} &= (A+LC)(x-\hat{x}) + \begin{bmatrix} B_w & L \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \\ z &= C_z(x-\hat{x}) \end{aligned}$$

[5]

It follows that

$$T_{zw}(s) \stackrel{s}{=} \left[\begin{array}{c|c} A+LC & \begin{bmatrix} B_w & L \end{bmatrix} \\ \hline C_z & \begin{bmatrix} 0 & 0 \end{bmatrix} \end{array}\right] \stackrel{s}{=}: \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right]$$

b) The Bounded Real Lemma states that $A_c$ is stable $\|T_{zw}\|_\infty < \gamma$ if there exists a $P = P^T$ such that

$$\begin{aligned} \begin{bmatrix} A_c^TP + PA_c + C_c^TC_c & PB_c + C_c^TD_c \\ B_c^TP + D_c^TC_c & D_c^TD_c - \gamma^2 I \end{bmatrix} &\prec 0 \\ P = P^T &\succ 0 \end{aligned}$$

[5]

By substituting the expressions for $A_c, B_c, C_c$ and $D_c$, this becomes

$$\begin{aligned} \begin{bmatrix} (A+LC)^TP + P(A+LC) + C_z^TC_z & PB_w & PL \\ \star & -\gamma^2 I & 0 \\ \star & \star & -\gamma^2 I \end{bmatrix} &\prec 0 \\ P = P^T &\succ 0 \end{aligned}$$

where $\star$ denotes terms easily inferred from symmetry.

c) By defining $Y = PL$, the matrix inequalities become

$$\begin{aligned} \begin{bmatrix} PA + A^TP + YC + C^TY^T + C_z^TC_z & PB_w & Y \\ \star & -\gamma^2 I & 0 \\ \star & \star & -\gamma^2 I \end{bmatrix} &\prec 0 \\ P = P^T &\succ 0 \end{aligned}$$

[5]

which are linear.

d) Putting the numbers into the LMI:

$$\begin{aligned} \begin{bmatrix} -2P + 2Y + 2 & P & Y \\ \star & -\gamma^2 I & 0 \\ \star & \star & -\gamma^2 I \end{bmatrix} &\prec 0 \\ P = P^T &\succ 0 \end{aligned}$$

[5]

effecting a Schur complement, this is equivalent to

$$-2P + 2Y + 2 + \gamma^{-2}Y^2 + \gamma^{-2}P^2 \prec 0, \qquad P \succ 0$$

which when completing two squares become

$$(\gamma^{-1}P - \gamma)^2 + (\gamma^{-1}Y + \gamma)^2 + 2 - 2\gamma^2 \prec 0, \qquad P \succ 0$$

and so $2\gamma^2 > 2$ or $\gamma > 1$. In the limit when $\gamma \to 1$, $P \to 1$, $Y \to -1$ and so $L \to -1$.

4. a) The generalized regulator formulation is given by

$$\begin{bmatrix} z(s) \\ y(s) \end{bmatrix} = P(s)\begin{bmatrix} w(s) \\ u(s) \end{bmatrix},\ u(s)=Fy(s), P(s)=\begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{c||c|c} A & B & B \\ \hline 0 & I & I \\ \hline I & 0 & 0 \end{array}\right].$$ [4]

b) The requirement $\|H\|_\infty < \gamma$ is equivalent to $J := \|z\|_2^2 - \gamma^2\|w\|_2^2 < 0$. Let $V = x^T X x$ and set $u = Fx$. Provided that $X = X^T \succ 0$ and $\dot{V} < 0$ along the closed-loop trajectory, we can assume $\lim_{t\to\infty} x(t) = 0$. Then [8]

$$\dot{V} = \dot{x}^T X x + x^T X \dot{x} = x^T\left(A^T X + XA + F^T B^T X + XBF\right)x + x^T XBw + w^T B^T X x.$$

Integrating from 0 to $\infty$ and using $x(0) = x(\infty) = 0$,

$$0 = \int_0^\infty [x^T\left(A^T X + XA + F^T B^T X + XBF\right)x + x^T XBw + w^T B^T X x]dt.$$

Using the definition of $J$ and adding the last equation, $J =$

$$J = \int_0^\infty \{x^T[A^T X + XA + F^T F + F^T B^T X + XBF]x - [\beta^2 w^T w - x^T Z^T w - w^T Zx]\}dt$$

where $Z = F + B^T X$ and $\beta^2 = \gamma^2 - 1 > 0$ since $\gamma > 1$ by assumption. Now

$$\begin{aligned} Z^T Z &= F^T F + F^T B^T X + XBF + XBB^T X \\ \|(\beta w - \beta^{-1} Zx)\|^2 &= \beta^2 w^T w - w^T Zx - x^T Z^T w + \beta^{-2} x^T Z^T Zx, \end{aligned}$$

$$J = \int_0^\infty \{x^T[A^T X + XA - XBB^T X]x + (1+\beta^{-2})\|Zx\|^2 - \|\beta w - \beta^{-1} Zx\|^2\}dt.$$

Thus two sufficient conditions for $J < 0$ are the existence of $X$ such that

$$A^T X + XA - XBB^T X = 0, \qquad X = X^T \succ 0.$$

Setting $Z = 0$, $F = -B^T X$. The worst case disturbance is $w^* = \beta^{-2} Zx = 0$. The closed-loop with $u = Fx$ and $w = w^*$ is $\dot{x} = [A - BB^T X]x$ and a third condition is $Re\ \lambda_i[A - BB^T X] < 0$, $\forall\ i$. To prove $\dot{V} < 0$ for $u = Fx$ and $w = 0$,

$$\dot{V} = x^T\left(A^T X + XA + F^T B^T X + XBF\right)x = -x^T(XBB^T X)x < 0$$

for all $x \neq 0$ (since $(A, B)$ is assumed controllable) proving closed-loop stability.

c) It is clear that our procedure breaks down if $\gamma \leq 1$ since in that case $\beta^2 \leq 0$. Thus the smallest value of $\gamma$ is 1. [4]

d) If $A$ is stable, in the limit $X = 0$ and hence $F = 0$ is a solution. [2]

e) If $-A$ is stable, pre- and post-multiplying the Riccati equation by $X^{-1}$,

$$AX^{-1} + X^{-1}A^T - BB^T = 0 \Leftrightarrow (A - BB^T X)X^{-1} + X^{-1}(A - BB^T X) + BB^T = 0$$ [2]

which has a unique solution $X^{-1} \succ 0$ if $-A$ is stable and so $A - BB^T X$ is stable.