IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING

MOCK EXAM 2017

MATHEMATICS FOR SIGNALS AND SYSTEMS SOLUTIONS

SOLUTIONS

1. (a) We put the 4 vectors along the rows of the matrix $\mathbf{A}$ and row reduce:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & -1 & 4 & 1 \\ 2 & 2 & -6 & 0 \\ -5 & 0 & -5 & -5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & -1 & 4 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore S has dimension 2 and a possible basis is: $[1,0,1,1]^T, [0,-1,4,1]^T$. The orthogonal basis is $\mathbf{u}_1 = [1,0,1,1]^T$, the second vector must be orthogonal to $\mathbf{u}_1$. We denote with $\mathbf{b} = [0,-1,4,1]^T$, therefore, $\mathbf{u_2} = \mathbf{b} - \mathbf{u_1}(\mathbf{u_1^T u_1})^{-\mathbf{1}}\mathbf{u_1^T b} = [-5/3, -1, 7/3, -2/3]^T$. The final answer is $\mathbf{q}_1 = \mathbf{u}_1/\|\mathbf{u}_1\|$ and $\mathbf{q}_2 = \mathbf{u}_2/\|\mathbf{u}_2\|$.

(b) We first row reduce the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & -1 & -5/3 \\ 1/2 & 1 & 1/2 & -1/6 \\ -1 & -1 & 0 & 1 \end{bmatrix}$$

to find the dimension of its null space and range space. We have

$$\begin{bmatrix} 1 & 0 & -1 & -5/3 \\ 1/2 & 1 & 1/2 & -1/6 \\ -1 & -1 & -1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 & -5/3 \\ 0 & -1 & -1 & -2/3 \\ 0 & -1 & -1 & -2/3 \end{bmatrix}.$$

Column one and two are linearly independent so range space has dimension 2 and the null space has dimension $n - rank(A) = 4 - 2 = 2$. We then find a basis for the null space by solving $\mathbf{Ax} = \mathbf{0}$ with $\mathbf{A}$ in echelon form. We have

$$\begin{bmatrix} x_1 \\ x_2 \\ -2/3x_1 + 5/3x_2 \\ x_1 + x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ -2/3 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ -5/3 \\ 1 \end{bmatrix}.$$

So a basis for the null space is $(1,0,-2/3,1)^T, (0,1,-5/3,1)^T$.

(c) The matrix $\mathbf{A}$ is orthogonal, therefore $\mathbf{A}^{-1} = \mathbf{A}^T$.

(d) $\det(5\mathbf{A}) = 5^3{*}10 = 1250$, $\det(3\mathbf{A}^{-1}) = 3^3/10 = 2.7$ and $\det(3\mathbf{A}^3) = 3^3{*}10^3 = 27000$

2. (a) The answer is yes. We show this by either computing the pseudo-inverse of $\mathbf{A}$ and by showing that it equals $\mathbf{B}$. Alternatively we can simply verify that

   i. $\mathbf{ABA} = \mathbf{A}$
   ii. $\mathbf{BAB} = \mathbf{B}$
   iii. $\mathbf{AB} = (\mathbf{AB})^T$
   iv. $\mathbf{BA} = (\mathbf{BA})^T$

   (b) The inverse exists if and only if the determinant is non-zero. We have

$$|\mathbf{A}| = \begin{vmatrix} 1 & 1 & 1 \\ 0 & a & b \\ 1 & 0 & 1 \end{vmatrix} = a \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} - b \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = b.$$

   Therefore, the inverse exists if and only if $b \neq 0$, it exists for any $a \in \mathbb{C}$.

   (c) The equations are satisfied when $x + y = 3 \Leftrightarrow x = 3 - y$ and $y + z = 3 \Leftrightarrow z = 3 - y$. We can define the set of solutions as $\begin{bmatrix} 3-\alpha \\ \alpha \\ 3-\alpha \end{bmatrix}$, where $\alpha \in \mathbb{R}$ is a free parameter. The minimum norm solution can be found from the derivate of the norm with respect to alpha:

$$\frac{d}{d\alpha}\left(2(3-\alpha)^2 + \alpha^2\right) = -4(3-\alpha) + 2\alpha.$$

   The derivative is zero at $\alpha = 2$. By verifying the second derivative is positive at this value of α, we know that this the minimum norm solution occurs when $\alpha = 2$ and is $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

   (d) Consider $\mathbf{Ax} = \lambda\mathbf{x}$ with $\mathbf{A}$ possibly complex. If we take complex conjugate on both sides we get $\mathbf{A}^*\mathbf{x}^* = \lambda^*\mathbf{x}^*$. If we take transpose on both sides we get $\mathbf{x^H A^H} = \lambda^*\mathbf{x^H}$. We now multiply both sides from the right with $\mathbf{x}$ we get $\mathbf{x^H A^H x} = \lambda^*\mathbf{x^H x}$. We now take $\mathbf{Ax} = \lambda\mathbf{x}$. We now multiply both sides from the left with $\mathbf{x}^H$ and we get $\mathbf{x^H A x} = \lambda\mathbf{x^H x}$. From the above we see that if $\mathbf{A}^H = \mathbf{A}$ then $\lambda^*\mathbf{x^H x} = \lambda\mathbf{x^H x}$ and since $\mathbf{x^H x} = \|\mathbf{x}\|^2 \neq 0$ then $\lambda = \lambda^*$ and so λ is real.

3. (a) Denote with

$$\mathbf{A} = [\mathbf{p}_1, \mathbf{p}_2] = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 1 \end{bmatrix}.$$

We know that the nearest vector (in the least-squares sense) is given by the projection of $\hat{\mathbf{x}}$ onto S. The projection matrix is:

$$\mathbf{P} = \mathbf{A}(\mathbf{A^H A})^{-1}\mathbf{A}^H = \begin{bmatrix} 2/3 & 1/3 & 1/3 \\ 1/3 & 2/3 & -1/3 \\ 1/3 & -1/3 & 2/3 \end{bmatrix}$$

and $\hat{\mathbf{x}} = \mathbf{P}\mathbf{x} = [11/3, -2/3, 13/3]^T$.

(b) We denote with

$$\mathbf{u}_n = \begin{bmatrix} x_n \\ y_n \end{bmatrix}.$$

So that we can write the difference equation as follows:

$$\mathbf{u}_{n+1} = \mathbf{A}\mathbf{u}_n,$$

with

$$\mathbf{A} = \begin{bmatrix} -7 & 10 \\ -5 & 8 \end{bmatrix}.$$

Consequently,

$$\mathbf{u}_k = \mathbf{A}^k \mathbf{u}_0,$$

We compute the eigenvalues and eigenvectors of $\mathbf{A}$ in order to diagonalize $\mathbf{A}$ and we obtain:

$$\mathbf{A} = \mathbf{S\Lambda S}^{-1} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}.$$

Consequently:

$$\mathbf{A}^4 = \mathbf{S\Lambda^4 S}^{-1} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3^4 & 0 \\ 0 & (-2)^4 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -49 & 130 \\ -65 & 146 \end{bmatrix}$$

and

$$\mathbf{u}_4 = \mathbf{A}^4 \mathbf{u}_0 = \begin{bmatrix} -49 & 130 \\ -65 & 146 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 16 \\ 8 \end{bmatrix}.$$

(c) The singular values of $\mathbf{A}$ are the square roots of the eigenvalues of $\mathbf{AA}^\top$ and $\mathbf{A}^\top\mathbf{A}$. We compute the eigenvalues of the smaller matrix, $\mathbf{AA}^\top$, which is quicker. We have

$$\mathbf{AA}^\top = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 8 \\ 8 & 9 \end{bmatrix},$$

and

$$\begin{aligned} |\mathbf{AA}^\top - \lambda\mathbf{I}| &= \begin{vmatrix} 9-\lambda & 8 \\ 8 & 9-\lambda \end{vmatrix} = (9-\lambda)^2 - 64 \\ &= \lambda^2 - 18\lambda + 17 = (\lambda - 1)(\lambda - 17). \end{aligned}$$

The eigenvalues of $\mathbf{AA}^\top$ are therefore 1 and 17 and the singular values of $\mathbf{A}$ are 1 and $\sqrt{17}$.

(d) The singular value decomposition is $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top$, where the columns of $\mathbf{U}$ and $\mathbf{V}$ are the (unit-norm) eigenvectors of $\mathbf{A}\mathbf{A}^\top$ and $\mathbf{A}^\top\mathbf{A}$, respectively, and $\mathbf{\Sigma}$ is a diagonal matrix whose non-zero entries are the singular values of $\mathbf{A}$. From the previous question, we have

$$\mathbf{\Sigma} = \begin{bmatrix} \sqrt{17} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

The eigenvector corresponding to the eigenvalue $\lambda = 1$ of $\mathbf{A}\mathbf{A}^\top$ satisfies

$$\begin{aligned} &\left(\mathbf{A}\mathbf{A}^\top - \mathbf{I}\right) \begin{bmatrix} x \\ y \end{bmatrix} = 0 \\ \Rightarrow &\begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \\ \Rightarrow &\, x = -y. \end{aligned}$$

The unit-norm eigenvector is therefore $\frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. The eigenvector corresponding to the eigenvalue $\lambda = 17$ of $\mathbf{A}\mathbf{A}^\top$ satisfies

$$\begin{aligned} &\left(\mathbf{A}\mathbf{A}^\top - 17\,\mathbf{I}\right) \begin{bmatrix} x \\ y \end{bmatrix} = 0 \\ \Rightarrow &\begin{bmatrix} -8 & 8 \\ 8 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \\ \Rightarrow &\, x = y. \end{aligned}$$

The unit-norm eigenvector is therefore $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. From the two eigenvectors, we have

$$\mathbf{U} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

We now calculate the eigenvectors of $\mathbf{A}^\top\mathbf{A}$, which has the same eigenvalues as $\mathbf{A}\mathbf{A}^\top$ plus one zero eigenvalue. We have

$$\mathbf{A}^\top\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 6 & 4 \\ 6 & 8 & 6 \\ 4 & 6 & 5 \end{bmatrix}.$$

The eigenvector corresponding to $\lambda = 1$, satisfies

$$\begin{aligned} &\left(\mathbf{A}^\top\mathbf{A} - \mathbf{I}\right) \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \\ \Rightarrow &\begin{bmatrix} 4 & 6 & 4 \\ 6 & 7 & 6 \\ 4 & 6 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \\ \Rightarrow &\, 4x + 6y + 4z = 0 \quad \text{and} \quad 6x + 7y + 6z = 0 \\ \Rightarrow &\, x + z = -\frac{6}{4}y \quad \text{and} \quad x + z = -\frac{7}{6}y \\ \Rightarrow &\, y = 0 \quad \text{and} \quad x = -z. \end{aligned}$$

The unit-norm eigenvector is therefore $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. The eigenvector corresponding to $\lambda = 17$, satisfies

$$
\begin{aligned}
&\left(\mathbf{A}^\top \mathbf{A} - 17\,\mathbf{I}\right)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \\
&\Rightarrow \begin{bmatrix} -12 & 6 & 4 \\ 6 & -9 & 6 \\ 4 & 6 & -12 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \\
&\Rightarrow -12x + 6y + 4z = 0 \quad \text{and} \quad 4x + 6y - 12z = 0 \\
&\Rightarrow 6y = 12x - 4z \quad \text{and} \quad 6y = 12z - 4x \\
&\Rightarrow 12x - 4z = 12z - 4x \\
&\Rightarrow x = z \quad \text{and} \quad y = \frac{4}{3}z.
\end{aligned}
$$

The unit-norm eigenvector is therefore $\frac{1}{\sqrt{34}}\begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}$. Finally, we must find an eigenvector corresponding to $\lambda = 0$, that vector satisfies

$$
\begin{aligned}
&\begin{bmatrix} 5 & 6 & 4 \\ 6 & 8 & 6 \\ 4 & 6 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \\
&\Rightarrow 6x + 8y + 6z = 0 \quad \text{and} \quad 5x + 6y + 4z = 0 \\
&\Rightarrow y = -\frac{6}{8}(x+z) \quad \text{and} \quad 40x - 36(x+z) + 32z = 0 \\
&\Rightarrow x = z \quad \text{and} \quad y = -\frac{3}{2}z
\end{aligned}
$$

The unit-norm eigenvector is therefore $\frac{1}{\sqrt{17}}\begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}$. From the three eigenvectors, we have

$$
\mathbf{V} = \begin{bmatrix} \frac{3}{\sqrt{34}} & \frac{1}{\sqrt{2}} & \frac{2}{\sqrt{17}} \\ \frac{4}{\sqrt{34}} & 0 & -\frac{3}{\sqrt{17}} \\ \frac{3}{\sqrt{34}} & -\frac{1}{\sqrt{2}} & \frac{2}{\sqrt{17}} \end{bmatrix}.
$$

Finally

$$
\mathbf{A} = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 2 & 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \sqrt{17} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} \frac{3}{\sqrt{34}} & \frac{1}{\sqrt{2}} & \frac{2}{\sqrt{17}} \\ \frac{4}{\sqrt{34}} & 0 & -\frac{3}{\sqrt{17}} \\ \frac{3}{\sqrt{34}} & -\frac{1}{\sqrt{2}} & \frac{2}{\sqrt{17}} \end{bmatrix}^\top.
$$