UNIVERSITY OF LONDON [ISE 1.6 2002]

B.ENG. AND M.ENG. EXAMINATIONS 2002

For Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examination for the Associateship of the City and Guilds of London Institute.

INFORMATION SYSTEMS ENGINEERING 1.6

MATHEMATICS

Date Wednesday 29th May 2002 10.00 am - 1.00 pm

Answer SEVEN questions

Answers to Section A *questions must be written in a different answer book from answers to* Section B *questions.*

[Before starting, please make sure that the paper is complete. There should be SIX pages, with a total of NINE questions. Ask the invigilator for a replacement if this copy is faulty.]

SECTION A **[ISE 1.6 2002]**

1. (i) Use the Binomial theorem and de Moivre's theorem and the fact that

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$$

to show that

$$\cos^{10}\theta = \frac{1}{2^9}\left[\cos 10\theta + 10\cos 8\theta + 45\cos 6\theta + 120\cos 4\theta + 210\cos 2\theta + 126\right].$$

Verify that your result is correct when $\theta = 0$ and when $\theta = \pi/4$.

(ii) Find all the roots of the equation

$$z^6 + z^3 + 1 = 0$$

in the form $z = R(\cos\theta + i\sin\theta)$ where R and θ are to be obtained.

(iii) Sketch the graph in the Cartesian plane corresponding to the equation

$$\mathrm{Re}\left(z^2\right) = 1.$$

The three parts carry, respectively, 40% , 30% and 30% of the marks.

2. (i) Find the general solution of the equation

$$\frac{dy}{dx} = \frac{1+y^2}{1+x^2},$$

expressing the solution in the form $y = f(x)$.

You may use the following formula without proof. $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\,\tan B}$.

(ii) Solve the equation

$$\frac{dy}{dx} = \frac{1}{2(x+y)} - 1$$

subject to the condition $y = 0$ when $x = 0$.

(iii) Find the general solution of the equation

$$\frac{dy}{dx} + 3x^2 y = e^{-x^3}.$$

The three parts carry, respectively, 40% , 40% and 20% of the marks.

PLEASE TURN OVER

3. (i) Find the solution of the differential equation

$$y'' + 4y = e^{2x} + \sin x$$

which satisfies the conditions $y = 1$ and $y' = 0$ at $x = 0$.

(ii) Find the general solution of the differential equation

$$y'' + 5y' + 4y = e^{-x} + e^{x}.$$

The two parts carry, respectively, 50% and 50% of the marks.

4. (i) Given

$$A = \begin{pmatrix} 2 & 1 & 1 & 3 \\ 4 & 3 & 2 & 1 \\ 2 & 2 & 2 & 2 \\ 2 & 3 & 4 & \alpha \end{pmatrix},$$

find an upper triangular matrix U, and a lower triangular matrix L, (with 1's down the main diagonal), such that $A = LU$.
Hence, or otherwise, evaluate the determinant of A and show that A^{-1} does not exist if $\alpha = 5$.

(ii) Show that the equations

$$\begin{aligned} 2x_1 + x_2 + x_3 + 3x_4 &= -1, \\ 4x_1 + 3x_2 + 2x_3 + x_4 &= 2, \\ 2x_1 + 2x_2 + 2x_3 + 2x_4 &= 0, \\ 2x_1 + 3x_2 + 4x_3 + 5x_4 &= \beta, \end{aligned}$$

do not have a solution unless $\beta = -2$.

Find x_1, x_2 and x_3 in terms of x_4, if $\beta = -2$.

The two parts carry, respectively, 60% and 40% of the marks.

PLEASE TURN OVER

5. Find the eigenvalues and the corresponding eigenvectors, normalized to one, of the matrix

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 3 & 1 & 4 \\ 0 & 4 & 1 \end{pmatrix} .$$

Verify that the eigenvectors are orthogonal.

Hence write down an orthogonal matrix U, such that $U^T AU = Q$ is diagonal and write down Q.

Evaluate the following limit

$$\lim_{n\to\infty} \frac{1}{6^n} A^n .$$

PLEASE TURN OVER

SECTION B

6. (i) If $u = x + y$, $v = xy$ and f is a function of x and y, express $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ in terms of $\dfrac{\partial f}{\partial u}$ and $\dfrac{\partial f}{\partial v}$, and show that

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial u^2} + u\frac{\partial^2 f}{\partial u \partial v} + v\frac{\partial^2 f}{\partial v^2} + \frac{\partial f}{\partial v}.$$

(ii) Find the six stationary points of the function

$$f(x, y) = x^3 y + xy^2 - xy$$

and determine their nature.

The two parts carry, respectively, 40% and 60% of the marks.

7. (i) Use standard tests to determine whether or not the following series converge:

$$\text{(a)} \sum_{n=1}^{\infty} \frac{3}{(1.1)^n}, \qquad \text{(b)} \sum_{n=1}^{\infty} \frac{n^3 + 3n^2 - 2}{2n^4 - 1},$$

$$\text{(c)} \sum_{n=1}^{\infty} \frac{n}{2^n}, \qquad \text{(d)} \sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2 - n - 1}}.$$

(ii) Find the range of values of x for which the following series converge :

$$\text{(a)} \sum_{n=1}^{\infty} \frac{(-1)^n n^3 x^n}{2^n}, \qquad \text{(b)} \sum_{n=1}^{\infty} \frac{(x+1)^{2n}}{(x^2 + x + 7)^n}.$$

The two parts each carry, respectively, 40% and 60% of the marks.

PLEASE TURN OVER

8. A periodic function $f(x)$ of period 2π is defined for $0 \leq x \leq \pi$ by

$$f(x) = \begin{cases} x , & \left(0 \leq x \leq \frac{\pi}{2}\right) , \\ \pi - x , & \left(\frac{\pi}{2} \leq x \leq \pi\right) . \end{cases}$$

Sketch the graph of $f(x)$ for $-2\pi \leq x \leq 2\pi$ in the cases where

(i) f is an even function; (ii) f is an odd function.

Show that the Fourier series expansion that represents the even function is

$$\frac{\pi}{4} - \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{\cos(4k-2)x}{(2k-1)^2} .$$

Deduce that

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8} .$$

Using Parseval's formula, deduce also that

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^4} = \frac{\pi^4}{96} .$$

9. The Laplace transform of a function $f(x)$ is defined as

$$\mathcal{L}(f(x)) = F(t) = \int_0^{\infty} e^{-tx} f(x) dx .$$

Assuming that $e^{-tx} f(x) \to 0$ and $e^{-tx} f'(x) \to 0$ as $x \to \infty$, show that

$$\begin{aligned} \mathcal{L}(f'(x)) &= t\mathcal{L}(f(x)) - f(0) , \\ \mathcal{L}(f''(x)) &= t^2 \mathcal{L}(f(x)) - t f(0) - f'(0) . \end{aligned}$$

Use Laplace transforms to solve the simultaneous differential equations

$$\begin{aligned} \frac{d^2 y}{dx^2} &= z - y , \\ \frac{d^2 z}{dx^2} &= y - z , \end{aligned}$$

where y, z are functions of x satisfying the conditions

$$y(0) = 3, \quad y'(0) = 0, \quad z(0) = 1, \quad z'(0) = 0.$$

END OF PAPER

MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

SESSION : 2001-2002

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PAPER JSE1.6

QUESTION

SOLUTION 1

(i) $\cos^{10}\theta = \frac{1}{2^{10}}\left(e^{i\theta}+e^{-i\theta}\right)^{10}$

$= \frac{1}{2^{10}}\Big(e^{i10\theta} + 10e^{i8\theta} + \frac{10.9}{1.2}e^{i6\theta} + \frac{10.9.8}{1.2.3}e^{i4\theta} + \frac{10.9.8.7}{1.2.3.4}e^{i2\theta} + \frac{10.9.8.7.6}{1.2.3.4.5}$

$+ \ldots\ldots + e^{-i10\theta}\Big)$ from binomial th.

$= \frac{1}{2^9}\big(\cos 10\theta + 10\cos 8\theta + 45\cos 6\theta + 120\cos 4\theta + 210\cos 2\theta + 126\big)$ 6

$\theta = 0 \Rightarrow 1 = \frac{1}{2^9}(1+10+45+120+210+126)$ 2

$\theta = \pi/2 \Rightarrow \frac{1}{2^5} = \frac{1}{2^9}(0+10+0-120+0+126)$

(ii) $z^6+z^3+1=0 \Rightarrow z^3 = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} = e^{\pm i 2\pi/3}$

$\Rightarrow z = \cos\left(\pm\frac{2\pi}{9}+\frac{2n\pi}{3}\right) + i\sin\left(\pm\frac{2\pi}{9}+\frac{2n\pi}{3}\right)$ 6

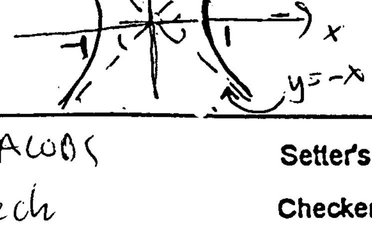
$n = 0, 1, 2$

$R = 1,\ \theta = \pm\frac{2\pi}{9}+\frac{2n\pi}{3}$

(iii) $\mathrm{Re}\, z^2 = 1 \Rightarrow x^2 - y^2 = 1$ ← Rectangular hyperbola 6

Asymptotes are $x = \pm y$

Hence



No graphs in 2 sectors because of + sign in front of 1.

Setter : R-L·JACOBS Setter's signature : RLJ

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EXAMINATION QUESTION / SOLUTION

SESSION : 2001-2002

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QUESTION

SOLUTION 2

(i) Separate $\int \frac{dy}{1+y^2} = \int \frac{dx}{1+x^2} \Rightarrow \tan^{-1} y = \tan^{-1} x + c$ 4

$$\Rightarrow y = \tan(\tan^{-1} x + c)$$

$$= \frac{\tan(\tan^{-1} x) + \tan c}{1 - \tan(\tan^{-1} x) . \tan c}$$ [Using $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$] 4

$$= \frac{x + A}{1 - Ax}$$ A arbitrary

(ii) Put $u = x + y \Rightarrow \frac{dy}{dx} = \frac{du}{dx} - 1$

$\therefore \frac{du}{dx} = \frac{1}{2u} \Rightarrow u^2 = x + c$

$\Rightarrow y = -x \pm \sqrt{x+c}$ 4

If $y = 0$ when $x = 0$ then $c = 0$ so $y = -x \pm \sqrt{x}$ 2

(iii) $IF = \exp\left(\int 3x^2 dx\right) = e^{x^3}$ 3

$\therefore \frac{d}{dx}(e^{x^3} y) = 1 \Rightarrow y = (x + c) e^{-x^3}$ 3

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QUESTION

SOLUTION 3

(i) $y_{CF} = A\cos 2x + B\sin 2x$

$y_{PI} = a e^{2x} + b\sin x + c\cos x$ where A, B are arbitrary and a, b, c are to be determined — 2

$y_{PI}' = 2ae^{2x} + b\cos x - c\sin x$

$y_{PI}'' = 4ae^{2x} - b\sin x - c\cos x$

$\Rightarrow 4ae^{2x} - b\sin x - c\cos x + 4ae^{2x} + 4b\sin x$

$+ 4c\cos x = e^{2x} + \sin x$ — 4

$\Rightarrow 8a = 1,\ 3b = 1,\ 3c = 0$

$\Rightarrow$ G.S. $y = A\cos 2x + B\sin 2x + \frac{1}{8}e^{2x} + \frac{1}{3}\sin x.$ — 2

$y' = -2A\sin 2x + 2B\cos 2x + \frac{2}{8}e^{2x} + \frac{1}{3}\cos x$ — 2

Conds, $\Rightarrow 1 = A + \frac{1}{8}$ $A = 7/8$

$0 = 2B + \frac{1}{4} + \frac{1}{3},\ B = -7/24.$

(ii) $y_{CF} = Ae^{-4x} + Be^{-x}$ — 4

e^{-x} appears alone in inhomogeneous term $\therefore$ use as trial

$y_{PI} = axe^{-x} + be^{+x}$ where a and b are to be determined

$y_{PI}' = ae^{-x} - axe^{-x} + be^{x}$

$y_{PI}'' = -2ae^{-x} + axe^{-x} + be^{x}$ — 6

$\therefore -2ae^{-x} \cancel{+axe^{-x}} + be^{x} + 5ae^{-x} \cancel{-5axe^{-x}} + 5be^{x} + \cancel{4axe^{-x}} + 4be^{x}$

$= e^{-x} + e^{x}$

$\Rightarrow 3a = 1 \quad 10b = 1 \Rightarrow$ G.S. $y = Ae^{-4x} + Be^{x} + \frac{1}{3}xe^{-x} + \frac{1}{10}e^{x}$

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QUESTION

SOLUTION 4

(i) Row operations

$$\begin{pmatrix} 2 & 1 & 1 & 3 \\ 4 & 3 & 2 & 1 \\ 2 & 2 & 2 & 2 \\ 2 & 3 & 4 & \alpha \end{pmatrix} \to \begin{pmatrix} 2 & 1 & 1 & 3 \\ 0 & 1 & 0 & -5 \\ 0 & 1 & 1 & -1 \\ 0 & 2 & 3 & \alpha-3 \end{pmatrix} \begin{matrix} r_2 - 2r_1 \\ r_3 - 1r_1 \\ r_4 - 1r_1 \end{matrix}$$

$$\to \begin{pmatrix} 2 & 1 & 1 & 3 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 3 & \alpha+7 \end{pmatrix} \begin{matrix} r_3 - 1.r_2 \\ r_4 - 2r_2 \end{matrix}$$

$$\to \begin{pmatrix} 2 & 1 & 1 & 3 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & \alpha-5 \end{pmatrix} r_4 - 3r_3$$ 4

$$\Rightarrow L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 1 \end{pmatrix} \quad U = \begin{pmatrix} 2 & 1 & 1 & 3 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & \alpha-5 \end{pmatrix}$$ 4

from multipliers above

$A = LU \Rightarrow |A| = |L|\,|U|$ since L and U are square

$= 1 \,.\, 2(\alpha-5)$ products of diag. els. 2

If $\alpha = 5$ $|A| = 0$ $\therefore A^{-1}$ does not exist 2

(ii) Carry out same row operations on augmented matrix to find

$$\left(\begin{array}{cccc|c} 2 & 1 & 1 & 3 & -1 \\ 4 & 3 & 2 & 1 & 2 \\ 2 & 2 & 2 & 2 & 0 \\ 2 & 3 & 4 & 5 & \beta \end{array}\right) \to \left(\begin{array}{cccc|c} 2 & 1 & 1 & 3 & -1 \\ 0 & 1 & 0 & -5 & 4 \\ 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & \beta+2 \end{array}\right)$$ 4

Last equation is now $0 = \beta + 2$ which $\Rightarrow \beta = -2$ 2

$x_3 = -3 - 4x_4$ $\quad x_2 = 4 + 5x_4$, $2x_1 = -1 - 3x_4 - 4 - 5x_4 + 3 + 4x_4 = -2 - 4x_4$ 2

$x_1 = -1 - 2x_4$

Setter : R.L. JACOBS Setter's signature : R.L.J

Checker : Checker's signature :

Eigenvalues from $|A - \lambda I| = 0$

$$\lambda^3 - 3\lambda^2 - 22\lambda + 24 = 0$$ 2

Trial and error shows that $\lambda_1 = 1$ satisfies equ.
Other roots are $\lambda_2 = 6$, $\lambda_3 = -4$.

Eigenvector corr. to $\lambda_1 = 1$ $\quad \underline{x} = \frac{1}{5}\begin{pmatrix} -4 \\ 0 \\ 3 \end{pmatrix}$ 2

Eigenvector corr. to $\lambda_2 = 6$ $\quad \underline{y} = \frac{1}{\sqrt{50}}\begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix}$ 2

Eigenvector corr. to $\lambda_3 = -4$ $\quad \underline{z} = \frac{1}{\sqrt{50}}\begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix}$ 2

Verify $\underline{x}^T \underline{y} = \underline{y}^T \underline{z} = \underline{z}^T \underline{x} = 0$ 2

$$U = \begin{pmatrix} -\frac{4}{5} & \frac{3}{\sqrt{50}} & \frac{3}{\sqrt{50}} \\ 0 & \frac{5}{\sqrt{50}} & -\frac{5}{\sqrt{50}} \\ \frac{3}{5} & \frac{4}{\sqrt{50}} & 4/\sqrt{50} \end{pmatrix} \qquad Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -4 \end{pmatrix}$$ 2+2

$$\frac{1}{6^n} A^n = \frac{1}{6^n} U Q^n U^T = U \begin{pmatrix} \frac{1}{6^n} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{(-4)^n}{6^n} \end{pmatrix} U^T$$

$$\to U \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} U^T \quad \text{as } n \to \infty$$ 6

$$= \frac{1}{50}\begin{pmatrix} 9 & 15 & 12 \\ 15 & 25 & 20 \\ 12 & 20 & 16 \end{pmatrix}$$

$\boxed{20}$

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2 (i) $f_x = f_u + y f_v, \quad f_y = f_u + x f_v.$

Hence

$$f_{xy} = f_{uu} + x f_{uv} + f_v + y(f_{vu} + x f_{vv})$$

6 $$= f_{uu} + (x+y) f_{uv} + xy f_{vv} + f_v$$

$$= f_{uu} + u f_{uv} + v f_{vv} + f_v.$$

(ii) $f_x = 3x^2 y + y^2 - y = y(3x^2 + y - 1)$

2 $f_y = x^3 + 2xy - x = x(x^2 + 2y - 1).$

Set both equal to zero: possibilities are

$y = 0, \quad x = 0$ or ± 1

$x = 0, \quad y = 0$ or 1

or $x, y \neq 0, \quad 3x^2 + y - 1 = x^2 + 2y - 1 = 0$

$\Rightarrow y = \frac{2}{5}, \quad x = \pm \frac{1}{\sqrt{5}}.$

5 So stationary pts are

$(0,0), (\pm 1, 0), (0,1), (\pm \frac{1}{\sqrt{5}}, \frac{2}{5}).$

Now $f_{xx} = 6xy, \ f_{xy} = 3x^2 + 2y - 1, \ f_{yy} = 2x.$

So setting $\Delta = f_{xx} f_{yy} - f_{xy}^2$, have

pt.	(0,0)	(1,0)	(-1,0)	(0,1)	$(\frac{1}{\sqrt{5}}, \frac{2}{5})$	$(-\frac{1}{\sqrt{5}}, \frac{2}{5})$
Δ	-1	-4	-4	-1	$\frac{4}{5}$	$\frac{4}{5}$
f_{xx}					>0	<0
NATURE	SADDLE	SADDLE	SADDLE	SADDLE	MIN	MAX

5

Setter: M. LIEBECK Signature: MLiebeck

Checker Signature: [illegible]

MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

SESSION : 2001-2002

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QUESTION

SOLUTION 7

2 (i) a) Geometric series with common ratio $\frac{1}{1.1} < 1$ $\therefore$ CONVERGENT

2 b) $a_n = \dfrac{n^3 + 3n^2 - 2}{2n^4 - 1} > \dfrac{1}{2n}$, so DIVERGENT by comparison test with $\sum \frac{1}{n}$.

2 c) $\dfrac{a_{n+1}}{a_n} = \dfrac{n+1}{2^{n+1}} \cdot \dfrac{2^n}{n} = \dfrac{1}{2}\,\dfrac{n+1}{n} \to \dfrac{1}{2} < 1$.

$\therefore$ CONVERGENT by Ratio Test

2 (d) $\dfrac{1}{\sqrt{n^2 - n - 1}} > \dfrac{1}{n}$, so DIVERGENT by comparison with $\sum \frac{1}{n}$.

(ii) (a) $\left|\dfrac{a_{n+1}}{a_n}\right| = \dfrac{(n+1)^3\,|x|^{n+1}}{2^{n+1}} \cdot \dfrac{2^n}{n^3\,|x|^n} = \dfrac{|x|}{2}\cdot\dfrac{(n+1)^3}{n^3} \to \dfrac{|x|}{2}$.

So by Ratio test, converges for $-2 < x < 2$,

4 diverges for $|x| > 2$.

1 For $x = \pm 2$, series is $\sum (-1)^n n^3$ or $\sum n^3$, divergent.

(b) $\left|\dfrac{a_{n+1}}{a_n}\right| = \left| \dfrac{(x+1)^{2n+2}}{(x^2+x+7)^{n+1}} \cdot \dfrac{(x^2+x+7)^n}{(x+1)^{2n}} \right| = \left| \dfrac{(x+1)^2}{x^2+x+7} \right|$.

By Ratio test, converges when

$$\left| \frac{x^2+2x+1}{x^2+x+7} \right| < 1 .$$

All terms are +ve, so this holds $\Leftrightarrow$ $x^2+2x+1 < x^2+x+7$

6 i.e. $x < 6$

1 Diverges when $\dfrac{x^2+2x+1}{x^2+x+7} > 1$; and when $=$, $x = 6$ and series is $\sum 1$, divgt.

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MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

SESSION : 2001-2002

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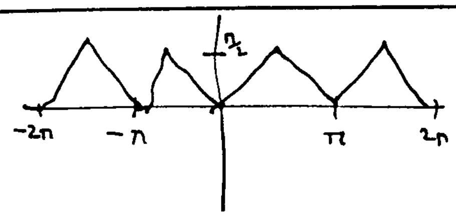
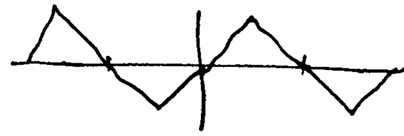
QUESTION

SOLUTION 8.

2+2

Sketch:

(a) 

(b) 

Even fn has Fourier series $\frac{a_0}{2} + \sum_1^\infty a_n \cos nx$

where $a_0 = \frac{2}{\pi}\int_0^\pi f(x)\,dx, \quad a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx.$

Here
$$a_0 = \frac{2}{\pi}\left(\left[\frac{x^2}{2}\right]_0^{\frac{\pi}{2}} - \left[\frac{(\pi-x)^2}{2}\right]_{\frac{\pi}{2}}^{\pi}\right) = \frac{\pi}{2}$$
$$a_n = \frac{2}{\pi}\left(\int_0^{\frac{\pi}{2}} x\cos nx\,dx + \int_{\frac{\pi}{2}}^{\pi}(\pi-x)\cos nx\,dx\right)$$

Putting $u = \pi - x$, have
$$\int_{\frac{\pi}{2}}^{\pi}(\pi-x)\cos nx\,dx = \int_{\frac{\pi}{2}}^{0} u\cos n(\pi-u)\,(-du)$$
$$= \begin{cases}\int_0^{\frac{\pi}{2}} u\cos nu\,du & \text{if } n \text{ even}\\ -\int_0^{\frac{\pi}{2}} u\cos nu\,du & \text{if } n \text{ odd.}\end{cases}$$

Hence
$$a_n = \begin{cases}\frac{4}{\pi}\int_0^{\frac{\pi}{2}} x\cos nx\,dx & \text{if } n \text{ even}\\ 0 & \text{if } n \text{ odd}\end{cases}$$

Setter : M. LIEBECK Setter's signature : M Liebeck

Checker : Checker's signature : R.L.J

MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

SESSION : 2001-2002

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QUESTION

SOLUTION 8, cd

Now I for n even, $\displaystyle\int_0^{\frac{\pi}{2}} x\cos nx\,dx = \left[\frac{x\sin nx}{n}\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \frac{\sin nx}{n}\,dx$

$$= \underset{(n\text{ even})}{0} + \frac{1}{n^2}\left[\cos nx\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{n^2}\left(\cos\frac{n\pi}{2} - 1\right) = \begin{cases} \frac{-2}{n^2} & \text{if } n = 4k-2 \\ 0 & \text{if } n = 4k \end{cases}$$

Hence $a_n = \dfrac{-8}{\pi(4k-2)^2} = \dfrac{-2}{\pi(2k-1)^2}$ if $n = 4k-2$

and 0 otherwise.

So Fourier Series is

12 $$\frac{\pi}{4} - \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{\cos(4k-2)x}{(2k-1)^2}$$

Putting $\underline{x=0}$:

1 $$f(0) = 0 = \frac{\pi}{4} - \frac{2}{\pi}\sum \frac{1}{(2k-1)^2}$$

$$\Rightarrow \sum_1^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}.$$

<u>Parseval</u>: $\displaystyle\frac{2}{\pi}\int_0^{\pi} f(x)^2\,dx = \frac{a_0^2}{2} + \sum a_n^2$

3 $$\text{LHS} = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} x^2\,dx + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi}(\pi-x)^2\,dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\frac{\pi}{2}} + \frac{2}{\pi}\left[\frac{(\pi-x)^3}{3}\right]_{\pi}^{\frac{\pi}{2}}$$

$$= \frac{\pi^2}{6}.$$

So $\displaystyle\frac{\pi^2}{6} = \frac{\pi^2}{8} + \frac{4}{\pi^2}\sum_1^{\infty}\frac{1}{(2k-1)^4} \Rightarrow \sum\frac{1}{(2k-1)^4} = \frac{\pi^4}{96}.$

Setter : M.LIEBECK Setter's signature : MLiebeck

Checker : Checker's signature : R.L.V

PAPER ISE L6

QUESTION

SOLUTION 9.

$$\mathcal{L}(f'(x)) = \int_0^\infty f'(x)e^{-tx}\,dx = \left[f(x)e^{-tx}\right]_0^\infty + t\int_0^\infty f(x)e^{-tx}\,dx$$

2 $$= -f(0) + t\,\mathcal{L}(f).$$

$$\mathcal{L}(f''(x)) = \int_0^\infty f''(x)e^{-tx}\,dx = \left[f'(x)e^{-tx}\right]_0^\infty + t\int_0^\infty f'(x)e^{-tx}$$

$$= -f'(0) + t\,\mathcal{L}(f')$$

3 $$= -f'(0) - tf(0) + t^2\mathcal{L}(f).$$

Take Laplace transforms of both diff eqns:

4
$$① \quad -3t + t^2 L(y) = L(z) - L(y)$$
$$② \quad -t + t^2 L(z) = L(y) - L(z)$$

$$\Rightarrow ① \quad (t^2+1)L(y) - L(z) - 3t = 0$$
$$② \quad -L(y) + (t^2+1)L(z) - t = 0$$

Eliminate $L(z)$:

$$L(y)\left((t^2+1)^2 - 1\right) - t\left(3(t^2+1)+1\right) = 0$$

4 $$\Rightarrow L(y)\left(t^2(t^2+2)\right) = t(3t^2+4)$$

$$\Rightarrow L(y) = \frac{3t^2+4}{t(t^2+2)} = \frac{a}{t} + \frac{bt+c}{t^2+2}$$

Then $a(t^2+2) + t(bt+c) \equiv 3t^2+4 \Rightarrow a=2,\ c=0,\ b=1$

4 Hence $L(y) = \dfrac{2}{t} + \dfrac{t}{t^2+2} \Rightarrow y = 2 + \cos\sqrt{2}\,x$

Then $y'' = -2\cos\sqrt{2}\,x = z - y \Rightarrow z = y - 2\cos\sqrt{2}\,x$

3 $$\Rightarrow z = 2 - \cos\sqrt{2}\,x$$

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Checker :

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