IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2005

EEE/ISE PART II: MEng, BEng and ACGI

Corrected Copy

SIGNALS AND LINEAR SYSTEMS

Monday, 23 May 2:00 pm

Time allowed: 2:00 hours

There are FOUR questions on this paper.

Q1 is compulsory.
Answer Q1 and any two of questions 2-4.
Q1 carries 40% of the marks. Questions 2 to 4 carry equal marks (30% each).

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s) :	R.U. Nabar, R.U. Nabar
	Second Marker(s) :	P.T. Stathaki, P.T. Stathaki

Formulae

Convolution: $x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$

Continuous Time Fourier Transform: $x(t) \xleftrightarrow{F} X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$

Useful Properties & Relations

1. $\delta(t-a) \xleftrightarrow{F} e^{-j\omega a}$
2. $x(t-a) \xleftrightarrow{F} e^{-j\omega a}X(\omega)$
3. $\mathrm{sinc}\left(\frac{t-a}{b}\right) \xleftrightarrow{F} |b|\,e^{-j\omega a}\Pi\left(\frac{b\omega}{2\pi}\right)$
4. $e^{-at}u(t) \xleftrightarrow{F} \frac{1}{a+j\omega}, a>0$
5. $x(-t) \xleftrightarrow{F} X(-\omega)$
6. $e^{jat} \xleftrightarrow{F} 2\pi\delta(\omega-a)$

Parseval's Relation: $\int_{-\infty}^{\infty} |x(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2\,d\omega$

Discrete Time Fourier Transform: $x[n] \xleftrightarrow{F} X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$

Useful Properties & Relations

1. $\delta[n-a] \xleftrightarrow{F} e^{-j\omega a}$
2. $e^{jna}x[n] \xleftrightarrow{F} X(e^{j(\omega-a)})$

Laplace Transform: $x(t) \xleftrightarrow{L} X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$, ROC

Useful Properties & Relations

1. $\delta(t-a) \xleftrightarrow{L} e^{-sa}$ ROC: entire s plane
2. $x'(t) \xleftrightarrow{L} sX(s)$, same ROC
3. $te^{-at}u(t) \xleftrightarrow{L} \frac{1}{(s+a)^2}$, ROC: Re{s}> $-a$
4. $u(t) \xleftrightarrow{L} \frac{1}{s}$, ROC: Re{s}> 0

Useful Identities:

1. $\cos(t) = \frac{e^{jt}+e^{-jt}}{2}$
2. $\sin(t) = \frac{e^{jt}-e^{-jt}}{2j}$
3. $(-1)^n = e^{jn\pi}$

The Questions

[Compulsory]

Consider the wireless communication system shown in Figure 1.1 below. The signal $x(t)$ is transmitted and $y(t)$ is the received signal. There is no direct path between transmitter and receiver. The receiver receives two copies of the transmitted signal, through two reflections that we name Reflection 1 and Reflection 2. The first copy (Reflection 1) arrives at the receiver with a delay of 1 second and is attenuated in amplitude by a factor of 0.5. The second copy (Reflection 2) arrives with a delay of 2 seconds, also attenuated in amplitude by a factor of 0.5.

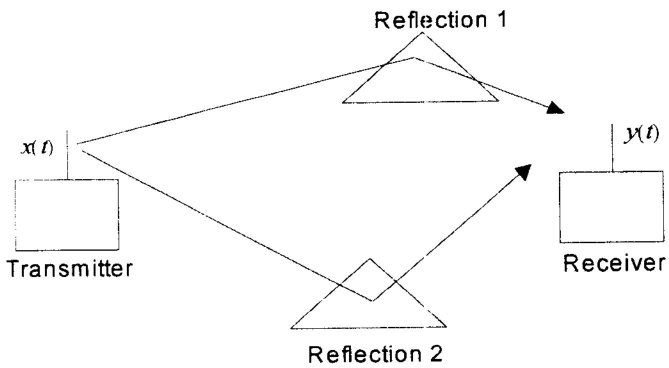


Figure 1.1

(a) Is the system LTI? Is the system causal? [6]

(b) Determine the impulse response, $h(t)$, of the system. [6]

(c) Show that the output of the system to the input $x(t) = \sin(t)$ is

$$y(t) = \cos(0.5)\sin(t-1.5)$$ [6]

(d) Sketch the frequency response $H(\omega)$ (magnitude and phase) of the system. Which transmission frequencies, if any, should the transmitter avoid using? [8]

(e) Evaluate the magnitude of the frequency response of the system if Reflection 1 is delayed by 3 seconds (instead of 1 second) and Reflection 2 is unchanged. [4]

(f) Determine the Laplace tranform of the impulse response of the system and the associated region of convergence. Reflections 1 and 2 are as in the original description. [4]

(g) Sketch the step response of the system, i.e., the output of the system when the input is the unit-step function $u(t)$? Reflections 1 and 2 are as in the original description. [6]

2 Evaluate the following

(a) $\int_{-\infty}^{\infty} \text{sinc}\left(\frac{\tau - t}{2}\right) \text{sinc}\left(\frac{\tau}{3}\right) d\tau$ [8]

(b) $\int_{-\infty}^{\infty} \text{sinc}(t)\,\text{sinc}(t)\,dt$ [7]

(c) $\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{j\omega t}}{6 + j\omega + \omega^2} d\omega$ [8]

(d) $u(t) * e^{t}\, u(-t)$, where $u(t)$ is the unit-step function and * denotes convolution [7]

3 Consider a continuous time bandwidth limited signal $x(t)$ that is uniformly sampled with period T to produce the discrete time signal $x[n]$. The sampling period **satisfies** the Nyquist criterion. The Discrete Time Fourier Transform (DTFT) of $x[n]$, denoted by $X(e^{j\omega})$, is shown below (Assume that the phase response is 0 for all frequencies).

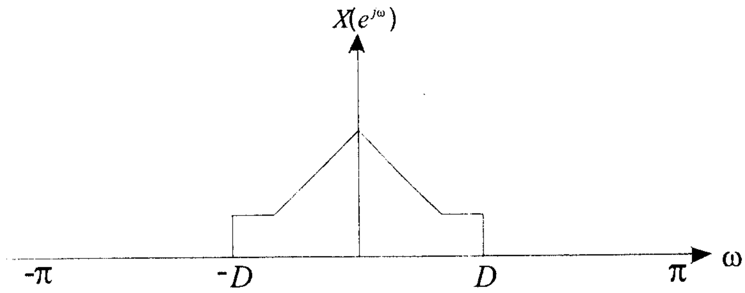


Figure 3.1

(a) What is the bandwidth of the signal $x(t)$? Express your answer in terms of D, T and constants. [3]

(b) Now, suppose we form the new discrete time signal $y[n]$ such that

$$y[n] = x[n], \text{ if n is even, and 0 otherwise.}$$

Express the DTFT of $y[n]$ in terms of $X(e^{j\omega})$. [6]

(c) Derive a condition on D that will allow a digital-to-analog converter to perfectly reconstruct $x(t)$ from the samples $y[n]$. [6]

(d) We now linearly interpolate the zeroed out samples of x[n] to produce z[n] such that

$$z[n] = x[n] \text{ if } n \text{ is even, and } (x[n+1] + x[n-1])/2 \text{ otherwise}$$

Express the DTFT of $z[n]$ as a function of $X(e^{j\omega})$. [15]

4 Consider the RLC circuit shown below

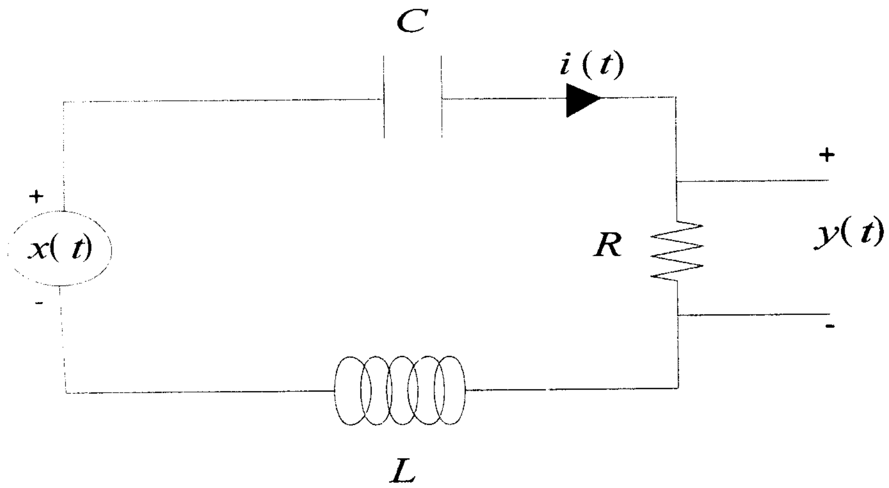


Figure 4.1

The source voltage is $x(t)$ and is the input to the system. The output of the system is the voltage $y(t)$ across the resistance. The circuit equation is

$$C\,x'(t) = i(t) + RC\,i'(t) + LC\,i''(t),$$

where $i(t)$ is the current in the circuit and $'$ and $''$ denote the derivative and double derivative respectively. In the following assume $C = 2$ F, $R = 1\ \Omega$ and L = 0.5 H.

(a) Find the Laplace transform of the system impulse response, H(s). Sketch its region of convergence (ROC). [6]

(b) Find the step response, $w(t)$, of the system. [12]

(c) Now assume $y(t)$ is the input to an LTI system with impulse response $g(t)$. Determine $g(t)$ in order to recover $x(t)$ perfectly at the output of the system. (Assume $x(t)$ has no DC component). [12]

E2.5

Solutions - Signals & Linear Systems

2005

1. **[Compulsory]**

(a) $y(t) = \frac{x(t-1)}{2} + \frac{x(t-2)}{2}$ (Bookwork)

The system is LTI. The system is causal. [6]

(b) $h(t) = \frac{\delta(t-1)}{2} + \frac{\delta(t-2)}{2}$ (Bookwork)

[6]

(c) $\sin(t) = \frac{e^{jt} - e^{-jt}}{2j}$ (New Application of theory)

Frequency response of system is [6]

$$H(\omega) = \frac{1}{2}\left(e^{-j\omega} + e^{-2j\omega}\right)$$

$$= \frac{e^{-j3\omega/2}}{2}\left(e^{j\omega/2} + e^{-j\omega/2}\right)$$

$$= e^{-j3\omega/2}\cos(\omega/2)$$

Complex exponentials are eigenfunctions of LTI systems.

$$e^{j\omega_0 t} \longrightarrow \boxed{H(\omega)} \longrightarrow H(\omega_0)e^{j\omega_0 t}$$

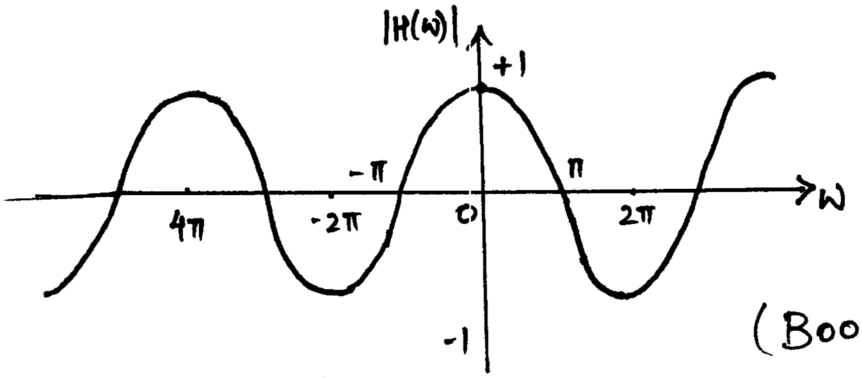
Hence, $$y(t) = \frac{e^{-j3/2}\cos(1/2)\,e^{jt} - e^{j3/2}\cos(1/2)\,e^{-jt}}{2j}$$

$$= \cos(1/2)\,\frac{e^{j(t-3/2)} - e^{-j(t-3/2)}}{2j}$$

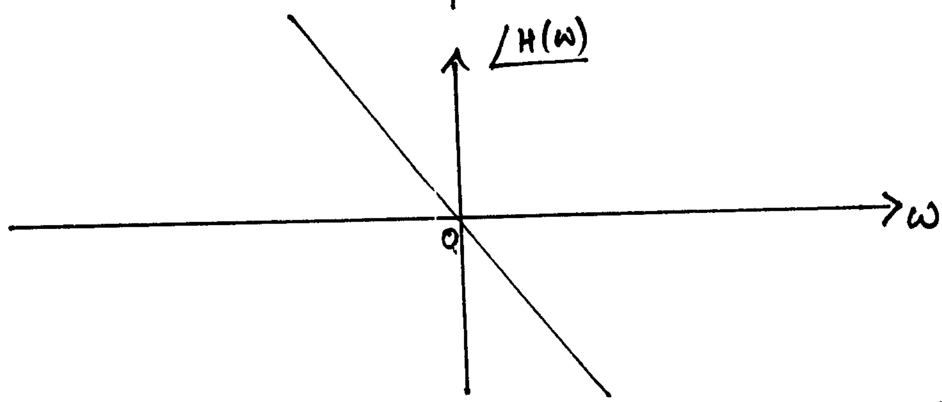
$$= \cos(1/2)\sin(t-3/2)$$

(d) $|H(\omega)| = \cos(\omega/2)$ $\quad \angle H(\omega) = \dfrac{-3\omega}{2}$

[8]



(Bookwork)



The transmitter should avoid transmitting at frequencies where $|H(\omega)| = 0$

(New Application) i.e. $\frac{\omega}{2} = (2n+1)\frac{\pi}{2} \Rightarrow \omega = (2n+1)\pi$

(e) If Reflection 1 is delayed by 3 seconds

$$h_{new}(t) = \frac{\delta(t-2)}{2} + \frac{\delta(t-3)}{2},$$

which is the new impulse response.

Hence, $h_{new}(t) = h(t-1)$ [4]

(New Application)

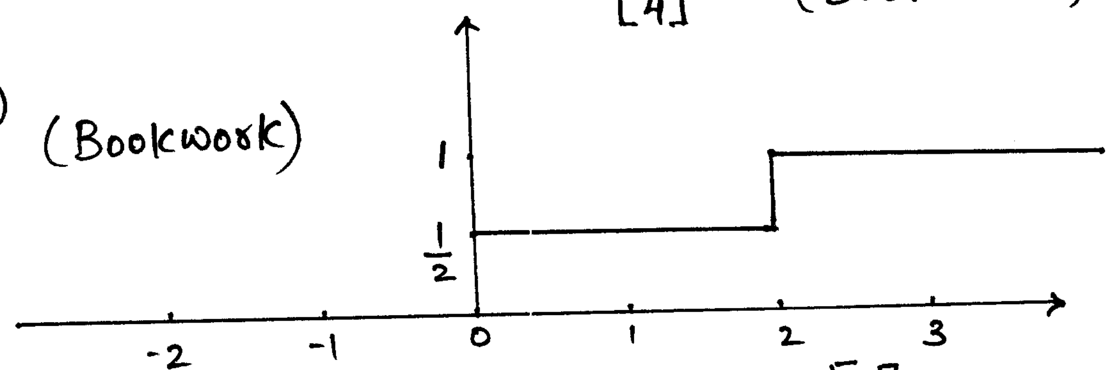
$$\Rightarrow H_{new}(\omega) = H(\omega)\, e^{-j\omega}$$

$$\Rightarrow |H_{new}(\omega)| = \cos(\omega/2)$$

(f) $H(s) = e^{-s} + e^{-2s}$ ROC: entire s plane

[4] (Bookwork)

(g) (Bookwork)



[6]

2.

(a) $$\int_{-\infty}^{\infty} \text{sinc}\left(\frac{\tau - t}{2}\right) \text{sinc}(\tau/3)\, d\tau$$ (New Computed Example)

$$= \text{sinc}\left(\frac{t}{2}\right) * \text{sinc}(t/3)$$ [8]

$$= F^{-1}\left[2\Pi\left(\frac{\omega}{\pi}\right) \cdot 3\,\Pi\left(\frac{3\omega}{2\pi}\right)\right]$$

$$= F^{-1}\left[6\,\Pi\left(\frac{3\omega}{2\pi}\right)\right]$$

$$= 2\ \mathrm{sinc}\left(\frac{t}{3}\right)$$

(b) $$\int_{-\infty}^{\infty} \mathrm{sinc}(t)\,\mathrm{sinc}(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Pi^2\left(\frac{\omega}{2\pi}\right)d\omega$$ [7]

(New Computed Example) $$= 1$$ (Parseval's relation)

(c) $$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{j\omega t}}{6+j\omega+\omega^2}\,d\omega = F^{-1}\left[\frac{1}{6+j\omega+\omega^2}\right]$$

(Bookwork)

$$\frac{1}{6+j\omega+\omega^2} = \frac{1}{6+j\omega-(j\omega)^2} = \frac{-1}{(j\omega+2)(j\omega-3)}$$

Using partial fraction expansion

$$\frac{-1}{(j\omega+2)(j\omega-3)} = \frac{1/5}{j\omega+2} + \frac{-1/5}{j\omega-3}$$ [8]

$$= \frac{1/5}{j\omega+2} + \frac{1/5}{-j\omega+3}$$

$$\Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{j\omega t}}{6+j\omega+\omega^2}\,d\omega = \frac{1}{5}e^{-2t}u(t) + \frac{1}{5}e^{3t}u(-t)$$

(d) For $t < 0$ (New computed example)

$$u(t) * e^{t}u(-t) = \int_{-\infty}^{t} e^{\tau}\, d\tau = e^{t}$$

[7]

For $t \geq 0$

$$u(t) * e^{t}u(-t) = \int_{-\infty}^{0} e^{\tau}\, d\tau = 1$$

Hence, $u(t) * e^{t}u(-t) = \begin{cases} e^{t}, & t < 0 \\ 1, & t \geq 0 \end{cases}$

3.

(a) $B = \dfrac{D}{2\pi T}$ (Bookwork) [3]

(b) $y[n] = \dfrac{x[n] + (-1)^{n} x[n]}{2} = \dfrac{x[n] + e^{jn\pi} x[n]}{2}$

$$\Rightarrow Y(e^{j\omega}) = \frac{X(e^{j\omega}) + X\left(e^{j(\omega - \pi)}\right)}{2}$$ [6]

(New Application of Theory)

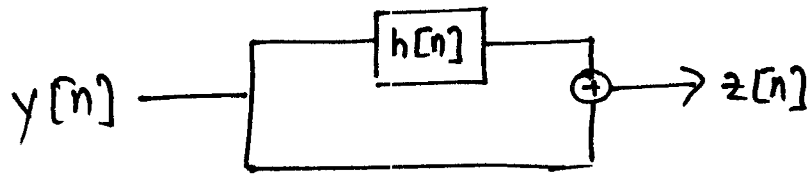
(c) Retaining even samples is equivalent to increasing sampling period by 2. Hence, to satisfy Nyquist criterion.-

(New Application of Theory) $2T < \dfrac{1}{2B} \Rightarrow T < \dfrac{\pi T}{2D}$ [6]

$$\Rightarrow D < \pi/2$$

(d) Note that $z[n]$ can be obtained by filtering $y[n]$ through a filter with impulse response $h[n] = \frac{\delta[n+1] + \delta[n-1]}{2}$

and summing the result with $y[n]$.



$$\Rightarrow Z(e^{j\omega}) = Y(e^{j\omega}) H(e^{j\omega}) + Y(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{e^{j\omega} + e^{-j\omega}}{2} = \cos(\omega)$$

Hence $Z(e^{j\omega}) = Y(e^{j\omega}) \left(1 + \cos(\omega)\right)$

$$= \frac{X(e^{j\omega}) + X(e^{j(\omega-\pi)})}{2} (1+\cos(\omega))$$

(New Application of Theory) [15]

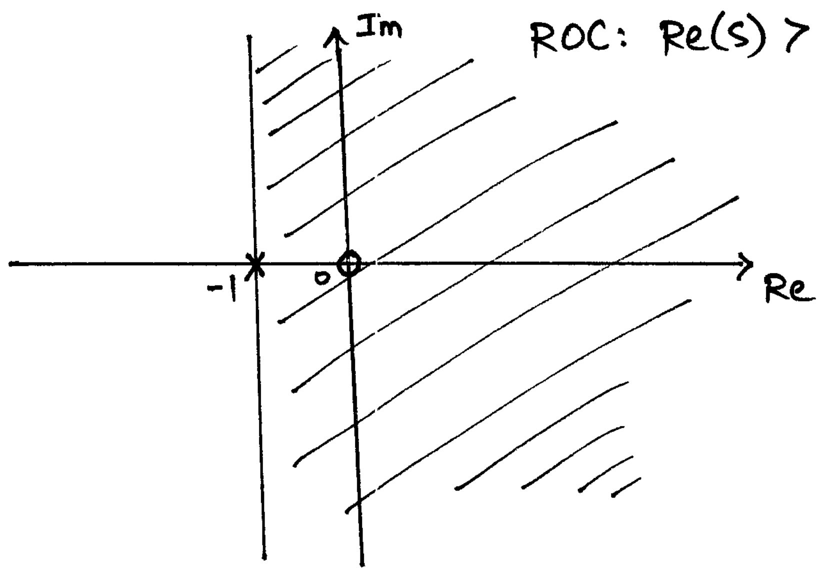
4.

Since $R = 1\,\Omega$, $y(t) = i(t)$

(a) $Cs\,X(s) = Y(s) + RCs\,Y(s) + LCs^2\,Y(s)$

$$2s\,X(s) = Y(s) + 2s\,Y(s) + s^2 Y(s)$$

$$\Rightarrow H(s) = \frac{2s}{(s+1)^2}$$ (Bookwork)



ROC: $Re(s) > -1$

[6]

(b) $\frac{1}{(s+1)^2} \overset{L^{-1}}{\longleftrightarrow} t e^{-t} u(t)$ (Bookwork)

$w(t) = u(t) * h(t)$ [12]

$\Rightarrow W(s) = U(s)\,H(s)$

$= 2/(s+1)^2$

$\Rightarrow w(t) = 2te^{-t}u(t)$

(c): $G(s) = \frac{1}{H(s)}$ (New application of theory)

$$= \frac{(s+1)^2}{2s}$$

[12]

$$= \frac{s^2 + 1 + 2s}{2s}$$

$$= \frac{s}{2} + \frac{1}{2s} + 1$$

$$\Rightarrow g(t) = \frac{1}{2}\delta'(t) + \frac{1}{2}u(t) + \delta(t)$$