

EXAM SOLUTIONS

1. a) This question is straightforward bookwork.

i) [2]

White noise is a random process which has a flat power spectral density spanning the entire spectrum.

ii) [2]

White noise cannot be realized since it has infinite average power. It is a reasonable modelling assumption as long as the bandwidth of the noise process is larger than the bandwidth of the communication system itself.

iii) [2]

$$S_N(f) = \frac{N_o}{2}, \forall f.$$

$$R_N(\tau) = \frac{N_o}{2} \delta(\tau).$$

iv) [3]

A process is Gaussian when the probability density function of its samples follow a Gaussian (Normal) distribution. Gaussian distribution for noise is motivated based on the *central limit theorem*, which states that the arithmetic mean of a sufficiently large number of realizations of independent random variables is approximately Gaussian distributed.

v) [2]

Since it is white, its samples are uncorrelated:

$$E[X(t_1)X(t_2)] = R_N(t_2 - t_1) = 0, \text{ for } t_1 \neq t_2.$$

Since it is also Gaussian, samples are independent.

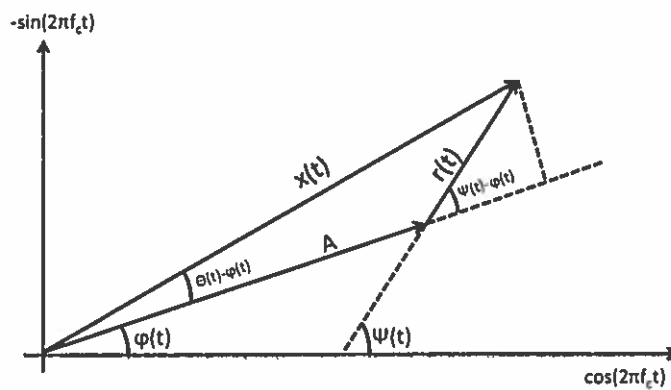
b) This question is straightforward bookwork.

i) [2]

Bandpass filter at the input of the receiver removes the noise components out of the pass-band of the transmitted FM signal $f_c \pm \frac{B_T}{2}$ and $-f_c \pm \frac{B_T}{2}$

ii) [3]

$$\begin{aligned} x(t) &= s(t) + n(t), \\ &= A \cos[2\pi f_c t + \phi(t)] + r(t) \cos[2\pi f_c t + \psi(t)]. \end{aligned}$$



iii) [2]

Since the FM modulated signal has a constant envelope, limiter removes the amplitude variations.

iv) [3]

$$v(t) = \frac{1}{2\pi} \frac{d\theta(t)}{dt} \simeq k_f m(t) + n_d(t).$$

v) [2]

$$n_d(t) = \frac{1}{2\pi A} \frac{dn_Q(t)}{dt}.$$

From this, we can find

$$S_{N_d}(f) = \frac{f^2}{A^2} S_{N_Q}(f).$$

Using the PSD of the quadrature component of a narrowband white noise signal, we get

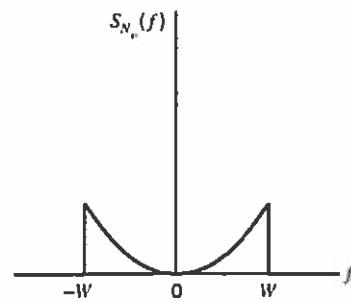
$$S_{N_d}(f) = \begin{cases} \frac{N_0 f^2}{A^2}, & \text{if } |f| \leq \frac{B_T}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

vi)

[2]

We have

$$S_{N_o}(f) = \begin{cases} \frac{N_o f^2}{A^2}, & \text{if } |f| \leq W, \\ 0, & \text{otherwise.} \end{cases}$$



- c) This question requires simply understanding the definition of entropy (provided here) and applying it to a joint distribution. The second part requires the use of the first part, otherwise they will end up spending a lot of time.

i)

[3]

If S and T are independent, then

$$P\{(S, T) = (s_k, t_m)\} = P\{S = s_k\}P\{T = t_m\},$$

that is, $r_{k,m} = p_k q_m$ for any $k = 1, \dots, K$ and $m = 1, \dots, M$. Then we can write:

$$\begin{aligned} H(S, T) &= \sum_{k=1}^K \sum_{m=1}^M r_{k,m} \log_2 \left(\frac{1}{r_{k,m}} \right) \\ &= \sum_{k=1}^K \sum_{m=1}^M p_k q_m \log_2 \left(\frac{1}{p_k q_m} \right) \\ &= \sum_{k=1}^K \sum_{m=1}^M p_k q_m \left[\log_2 \left(\frac{1}{p_k} \right) + \log_2 \left(\frac{1}{q_m} \right) \right] \\ &= \sum_{k=1}^K \sum_{m=1}^M p_k q_m \log_2 \left(\frac{1}{p_k} \right) + \sum_{k=1}^K \sum_{m=1}^M p_k q_m \log_2 \left(\frac{1}{q_m} \right) \\ &= \sum_{k=1}^K p_k \log_2 \left(\frac{1}{p_k} \right) + \sum_{m=1}^M q_m \log_2 \left(\frac{1}{q_m} \right) \\ &= H(S) + H(T). \end{aligned}$$

ii)

[4]

Let X_i denote the random variable corresponding to the outcome of the i th coin toss. We have

$$\begin{aligned} H(X_i) &= 0.2 \times \log_2 \left(\frac{1}{0.2} \right) + 0.8 \times \log_2 \left(\frac{1}{0.8} \right) \\ &\approx 0.72. \end{aligned}$$

Note that the entropy of X is nothing but the joint entropy of the outcome of the 8 coin tosses. Since the coin tosses are independent of each other, it follows from the previous question that the joint entropy is simply the sum of the individual entropies. Hence, we have

$$H(X) = \sum_{i=1}^8 H(X_i) = 8 \times 0.72 = 5.76$$

d) A very similar problem to this one was solved in the class.

i) [2]

For $n = 10$ we have 10^5 bits per second. This corresponds to

$$2 \times 60 \times 60 \times 10^5 = 72 \times 10^7$$

bits in 2 hours. This is equivalent to 90 Mbytes of memory space.

ii) [3]

Since the signal is Gaussian with standard deviation, its average power is $P_S = \sigma^2$. The quantization step is chosen such that a range of 6σ can be covered with 2^n quantization levels. This corresponds to $\Delta = \frac{6\sigma}{2^n}$.

The mean square error of a uniform quantizer is given by

$$P_N = \frac{\Delta^2}{12} = \frac{36\sigma^2}{12 \times 2^{2n}}.$$

The signal to noise ratio is found to be

$$SNR = \frac{P_S}{P_N} = \frac{\sigma^2}{3\sigma^2 2^{-2n}} = \frac{2^{2n}}{3}.$$

Changing to the dB scale, we obtain

$$\begin{aligned} SNR_{dB} &= 10 \log_{10} \frac{1}{3} + 20n \log_{10} 2 \\ &= -4.77 + 6.02n. \end{aligned}$$

iii) [3]

Let X denote a zero-mean Gaussian random variable with standard deviation σ . The overload probability is found as follows:

$$\begin{aligned} P_{\text{overload}} &= P\{X < -3\sigma\} + P\{X > 3\sigma\} \\ &= 2 \int_{3\sigma}^{\infty} \frac{1}{\sqrt{2\pi}\sigma^2} e^{-\frac{x^2}{2\sigma^2}} dx \\ &= 2 \cdot Q(3) \approx 2 \times 10^{-3}. \end{aligned}$$

2. a) This problem requires basic knowledge of probability and stochastic processes; in particular, definitions of variance, uncorrelated, independent, wide-sense stationary, power spectral density).

i) [2]

We have

$$\begin{aligned} E[ZT] &= E[3XY + 15X] \\ &= 3E[XY] + 15E[X] \\ &= 3E[X]E[Y] + 15E[X] \\ &= 3E[X](E[Y] + 5) \\ &= E[3X]E[Y + 5] = E[Z]E[T]. \end{aligned}$$

ii) [4]

No. To show that U and V are not always independent, we just need one counter-example. For example, consider $Y = 0$ independent of X . In this case, we have $U = V = X$, which are not independent.

iii) [4]

$$\begin{aligned} E[UV] &= E[(X+Y)(X-Y)] = E[X^2 - Y^2] \\ &= E[X^2] - E[Y^2] \\ &= \text{Var}(X) + E[X]^2 - \text{Var}(Y) - E[Y]^2 \\ &= E[X]^2 - E[Y]^2 \\ &= (E[X] + E[Y])(E[X] - E[Y]) = E[U]E[V]. \end{aligned}$$

b) i) [6]

$$\begin{aligned} E[Y(t)] &= \int_{a=-\infty}^{\infty} a \cos(2\pi f_c t) \cdot f_A(a) da \\ &= \frac{1}{4} \cos(2\pi f_c t) \int_{a=-2}^2 a da \\ &= 0. \end{aligned}$$

$$\begin{aligned} E[Y(t_1)Y(t_2)] &= \int_{a=-2}^2 a^2 \cos(2\pi f_c t_1) \cos(2\pi f_c t_2) f_A(a) da \\ &= \frac{1}{4} \cdot \frac{1}{2} [\cos(2\pi f_c(t_2 - t_1)) + \cos(2\pi f_c(t_1 + t_2))] \int_{a=-2}^2 a^2 da \\ &= \frac{2}{3} \cdot \frac{1}{2} [\cos(2\pi f_c(t_2 - t_1)) + \cos(2\pi f_c(t_1 + t_2))] \end{aligned}$$

Since the autocorrelation function is not only a function of $(t_2 - t_1)$, $Y(t)$ is not a wide-sense stationary process.

ii)

[6]

$$\begin{aligned}
E[Z(t)] &= E[A \cos(2\pi f_c t + \Theta)] \\
&= E[A] \cdot E[\cos(2\pi f_c t + \Theta)] \\
&= 0
\end{aligned}$$

where the first line follows from the independence of A and Θ , and the second line follows since both expectations are equal to 0.

$$\begin{aligned}
E[Z(t_1)Z(t_2)] &= E[A^2 \cos(2\pi f_c t_1 + \Theta) \cos(2\pi f_c t_2 + \Theta)] \\
&= E[A^2] E[\cos(2\pi f_c t_1 + \Theta) \cos(2\pi f_c t_2 + \Theta)]
\end{aligned}$$

due to independence. We have

$$\begin{aligned}
E[A^2] &= \int_{a=-2}^2 a^2 f_A(a) da \\
&= \frac{1}{4} \frac{a^3}{3} \Big|_{a=-2}^2 = \frac{4}{3}
\end{aligned}$$

and

$$\begin{aligned}
&E[\cos(2\pi f_c t_1 + \Theta) \cos(2\pi f_c t_2 + \Theta)] \\
&= \frac{1}{2\pi} \int_{\theta=-\pi}^{\pi} \cos(2\pi f_c t_1 + \theta) \cos(2\pi f_c t_2 + \theta) d\theta \\
&= \frac{1}{4\pi} \int_{\theta=-\pi}^{\pi} [\cos(2\pi f_c(t_2 - t_1)) + \cos(2\pi f_c(t_1 + t_2) + 2\theta)] d\theta \\
&= \frac{1}{4\pi} \cos(2\pi f_c(t_2 - t_1))
\end{aligned}$$

Since $E[Z(t)]$ is independent of time t , and the autocorrelation function is a function of $(t_2 - t_1)$, $Z(t)$ is a wide-sense stationary process.

c)

i)

[4]

We have

$$E[Y(t)] = E[X(t+a)] - E[X(t-a)] = 0,$$

since $E[X(t)]$ is independent of time t . We can also write

$$\begin{aligned}
E[Y(t+\tau)Y(t)] &= E[(X(t+\tau+a) - X(t+\tau-a))(X(t+a) - X(t-a))] \\
&= E[X(t+\tau+a)X(t+a)] - E[X(t+\tau+a)X(t-a)] \\
&\quad - E[X(t+\tau-a)X(t-a)] + E[X(t+\tau-a)X(t+a)] \\
&= R_X(\tau) - R_X(\tau+2a) - R_X(\tau-2a) + R_X(\tau) \\
&= 2R_X(\tau) - R_X(\tau+2a) - R_X(\tau-2a).
\end{aligned}$$

Since the expectation of $Y(t)$ is independent of time t , and its autocorrelation function is a function of τ , $Y(t)$ is a wide-sense stationary process.

ii)

[6]

Remember that the power spectral density for a wide-sense stationary process is obtained by taking the Fourier transform of its autocorrelation function. We have

$$\begin{aligned}
 S_Y(f) &= 2S_X(f) - S_X(f)e^{j2\pi f(2a)} - S_X(f)e^{-j2\pi f(2a)} \\
 &= S_X(f) [2 - e^{j4\pi fa} - e^{-j4\pi fa}] \\
 &= S_X(f) [2 - \cos(4\pi fa) - j\sin(4\pi fa) - \cos(-4\pi fa) - j\sin(-4\pi fa)] \\
 &= S_X(f) [2 - 2\cos(4\pi fa)] \\
 &= S_X(f) [4 - 4\cos^2(2\pi fa)] \\
 &= 4S_X(f) \sin^2(2\pi fa).
 \end{aligned}$$

Alternative Solution:

We can consider $Y(t)$ as $X(t)$ going through a linear time-invariant (LTI) system with impulse response $h(t) = \delta(t+a) - \delta(t-a)$. We know that when a WSS process is applied to an LTI system, the output is also a WSS process.

Moreover, it is known that $S_Y(f) = |H(f)|^2 S_X(f)$. We have

$$H(f) = e^{j2\pi fa} - e^{-j2\pi fa} = j \times 2 \sin(2\pi fa).$$

Hence,

$$S_Y(f) = 4 \sin^2(2\pi fa) S_X(f).$$

3. a) First three parts are standard steps that have been done in the lecture. They must have understood the main ideas so they can apply it to this slightly modified setup (message-dependent noise). The last part requires the use of the Leibnitz rule, which has been done in the lecture and class multiple times.

i) [4]

When a 0 is transmitted, the receiver makes an error if the received signal is above the threshold T . Then we can write the error probability as

$$\begin{aligned} P_{e0} &= \int_T^{\infty} \frac{1}{\sqrt{2\pi\sigma_0^2}} e^{-\frac{x^2}{2\sigma_0^2}} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{T/\sigma_0}^{\infty} e^{-\frac{x^2}{2}} dx \\ &= Q\left(\frac{T}{\sigma_0}\right). \end{aligned}$$

ii) [2]

When a 1 is transmitted, the receiver makes an error if the received signal is below the threshold T . Then the corresponding error probability is

$$\begin{aligned} P_{e1} &= P\{A + N_1 < T\} = P\{N_1 < T - A\} = P\{N_1 > A - T\} \\ &= \int_{A-T}^{\infty} \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{x^2}{2\sigma_1^2}} dx \\ &= Q\left(\frac{A-T}{\sigma_1}\right). \end{aligned}$$

iii) [2]

$$P_e = p_0 P_{e0} + p_1 P_{e1}.$$

iv) [10]

$$P_e = p_0 \frac{1}{\sqrt{2\pi\sigma_0^2}} \int_T^{\infty} e^{-\frac{x^2}{2\sigma_0^2}} dx + p_1 \int_{A-T}^{\infty} \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{x^2}{2\sigma_1^2}} dx.$$

To find the optimal threshold, we find T which satisfies $\frac{dP_e}{dT} = 0$. We first need to take the derivative of P_e with respect to T . We use the Leibnitz rule for this. We have

$$\frac{dP_e}{dT} = -p_0 \frac{1}{\sqrt{2\pi\sigma_0^2}} e^{-\frac{T^2}{2\sigma_0^2}} + p_1 \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{(A-T)^2}{2\sigma_1^2}} = 0.$$

This is equivalent to

$$\frac{1}{3} \frac{1}{\sqrt{2\pi\sigma_0^2}} e^{-\frac{T^2}{2\sigma_0^2}} = \frac{2}{3} \frac{1}{\sqrt{2\pi 4\sigma_0^2}} e^{-\frac{(A-T)^2}{8\sigma_0^2}},$$

which reduces to

$$4T^2 = (A - T)^2,$$

and, finally, we get $T = \frac{A}{3}$.

- b) **This problem requires the direct application of the noise analysis for baseband and AM systems, that have been done in the lecture and class. A very similar problem (with non-Gaussian noise) was done in class.**

i) [6]

The message signal is a sinusoidal wave with amplitude $A_m = 10$. Its average power is found as $P_S = \frac{1}{2}A_m^2 = 50$ W.

Due to the 20 dB attenuation in the channel, the received signal power is $P_R = 10^{-2}$ times the transmitted signal power.

Note that the message bandwidth is $W = 10$ KHz. This is also the minimum required bandwidth for baseband transmission.

The PSD of noise within the message bandwidth is $N_0/2$. Hence, the noise power is $2W \times \frac{N_0}{2} = 1 \times 10^{-10}$.

Hence, the SNR at the receiver output is

$$SNR_{baseband} = \frac{P_R}{P_N} = \frac{10^{-2} \times 50}{1 \times 10^{-10}} = 0.5 \times 10^{-10}.$$

In the dB scale, we have $SNR_{baseband,dB} = 10\log_{10}(0.5) + 100 = 97$ dB.

ii) [6]

Remember that, when there is no attenuation, the recovered signal at the output of a coherent DSB-SC receiver is $A_c m(t) + n_I(t)$, where $n_I(t)$ is the in-phase component of the band-pass noise.

Then the signal power at the receiver output is $A_c^2 P_S = 50$ W. With attenuation, we have $P_R = 50 \times 10^{-2}$ W.

Required bandwidth for DSB-SC is double the bandwidth of the message: 20 KHz.

The noise PSD in the transmission bandwidth around the carrier frequency of 1 MHz is $N_0/4$, and the PSD of the in-phase noise component is $N_0/2$. Then the average noise power at the receiver output is $2W \times \frac{N_0}{2} = 1 \times 10^{-10}$.

Hence, the SNR at the receiver output is

$$SNR_{DSB-SC} = \frac{P_R}{P_N} = \frac{10^{-2} \times 50}{1 \times 10^{-10}} = 0.5 \times 10^{-10}.$$

In the dB scale, we have $SNR_{DSB-SC,dB} = 10\log_{10}(0.5) + 100 = 97$ dB.

Note that we get the same SNR despite the fact that we have used half the transmit power $P_T = \frac{1}{2}A_c^2 P_S$, compared to baseband modulation. This is due to the reduced noise PSD within the transmitted bandwidth.