Imperial College London

BSc/MSci EXAMINATION May 2012

This paper is also taken for the relevant Examination for the Associateship

PHYSICS COMPREHENSIVE II

For Third and Fourth Year Physics Students

11 May 2012: 10:00 to 13:10

You may attempt as many questions as you wish. Only the answers to the best EIGHT questions over the two papers will contribute to your mark.

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

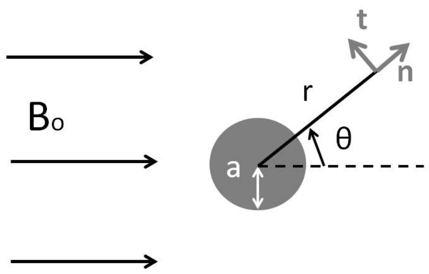
USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in all answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

1. This question examines the distortion of an external magnetic field by a superconducting solid, under steady-state conditions.



The figure displays an infinitely long grey cylinder with its axis orthogonal to the page, of radius a subject to an external magnetic field $\mathbf{B}_{ext}$ in free space. Far from the object $\mathbf{B}_{ext}$ is of constant magnitude B_o and direction given in the figure. Close enough to the object, however, both magnitude and direction differ from those shown in the figure. The geometry is polar, with radius r and angle θ. The grey vectors **n** and **t** indicate the associated unit vectors (radial and tangential, respectively).

(i) (a) Write down Maxwell's equations for magnetostatics given a magnetic field **B** and current density **J**. [2 marks]

(b) In a region of space where there is no net current, a scalar magnetic potential ψ can be defined using $\mathbf{B} \equiv -\nabla\psi$. What is the partial differential equation satisfied by ψ irrespective of the field **B**? [2 marks]

(c) At sufficiently low temperature the solid can become superconducting with the surprising property that magnetic field lines do not penetrate it. Check that in this case, ψ given below provides a valid description of $\mathbf{B} = \mathbf{B}_{ext}$:

$$\psi = -B_o\left(r + \frac{a^2}{r}\right)\cos\theta \quad \text{for } r \geq a \,. \tag{1}$$

n.b. you are expected to show that ψ in Eq. (1) satisfies the equation found in part (i)(b) and are expected to pay particular attention to the boundary conditions satisfied by **B** far from the cylinder and at its surface ($r = a$). [6 marks]

(d) Sketch the magnetic field lines associated with Eq. (1). [2 marks]

(ii) We now consider a slightly different situation in which the superconductor in the figure, still immersed in $\mathbf{B}_{ext}$, carries an electric current along its axis, flowing into the page and of magnitude I.

(a) Does $\mathbf{B}_{ext}$ exert a force on the superconductor? If yes, what is its direction? [1 mark]

[This question continues on the next page . . .]

(b) Obtain an expression for the total magnetic field created by the electric current for $r \geq a$. Explain qualitatively why it is still valid to use a magnetic scalar potential outside the superconductor. [3 marks]

(c) Show that ψ is then given by:

$$\psi = -B_o \cos\theta \left(r + \frac{a^2}{r} \right) + \frac{\mu_o I}{2\pi}\theta \quad \text{for } r \geq a \tag{2}$$

where μ_o is the permeability of free space. [2 marks]

(d) Plot the direction and magnitude of the tangential component of **B** at the surface of the superconductor as a function of θ when $I = 4\pi a B_o/\mu_o$. [2 marks]

[Total 20 marks]

Gradient in cylindrical polar coordinates:

$$\nabla = \left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \theta} \right)$$

Laplacian in cylindrical polar coordinates:

$$\nabla^2 \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

2. (i) Using dimensional analysis (or otherwise), derive an expression for the gravitational binding energy of a spherically symmetric star of mass M and radius R, ignoring numerical factors. On what do those numerical factors (the constant of proportionality) depend? [4 marks]

(ii) Just before it goes supernova, a seven solar mass star has a radius of four solar radii. Just after the collapse, about 1.5 solar masses are concentrated in a core of radius $r \sim 10$ km. What is the density of the core? What is the number density of nucleons in the core? What is the ratio of the gravitational binding energy of the core to the original binding energy of the whole star? [2 marks]

When the pre-supernova star collapses, a significant fraction of this binding energy is converted into neutrinos.

(iii) The neutrinos interact with the nucleons with a cross-section σ. If the density of the matter in the supernova is ρ, what is the typical length that a neutrino will travel between interactions? How high must the cross section be in order for a typical neutrino to interact in the core of the supernova? [3 marks]

(iv) The final binding energy of the core is about $E = 6 \times 10^{46}$ J. The neutrinos are created in an approximately thermal distribution of temperature $T_\nu \simeq 10^{11}$ K. If 10% of the binding energy is released in the form of these neutrinos, argue that the total number of neutrinos produced is approximately 10^{57}. [3 marks]

A supernova was observed in the Large Magellanic Cloud, at a distance of $L \simeq 51$ kpc in 1987. A total of about 20 neutrinos was observed in various experiments operating on Earth at the time.

(v) We now know that neutrinos have nonzero mass. If we assume $m_\nu < 1\ \mathrm{eV}/c^2$, what is the maximum expected time difference for a typical thermal neutrino from the supernova between the arrival of neutrinos and photons at the earth? (Hint: your calculator may not be able to do the required calculation.)

[5 marks]

(vi) A recent experiment performed between CERN and a neutrino detector 730 km away in Italy seemed to show that neutrinos arrived 60 ns earlier than the light-travel time. If this neutrino velocity were universal, when would we have expected to see the neutrino burst from SN1987a? In fact, the neutrino burst did arrive at roughly the same time as the photons. Comment. [3 marks]

[Total 20 marks]

You may find the following useful for this question:

$$\text{Mass of Sun,} \quad M_\odot = 2 \times 10^{30}\ \mathrm{kg}$$

$$\text{Radius of Sun,} \quad R_\odot = 7 \times 10^{8}\ \mathrm{m}$$

$$1\mathrm{pc} = 3.1 \times 10^{16}\mathrm{m}$$

3. A thermionic valve contains a piece of pure solid silver heated in a sealed vacuum to a temperature T. Electrons are released from the metal into the vacuum. Silver has a work function of ϕ = 4.5 eV. This is the minimum energy needed to excite an electron from the metal into the vacuum.

Assume that we can ignore electron-electron interactions. The density of states $g(E)$ of non-interacting electrons of mass m in free space is

$$g(E) = \frac{V}{\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}(E - E_0)^{1/2}$$

where V is the volume, E is the energy of the electron state and E_0 is the lowest possible energy in free space.

The Fermi-Dirac distribution $f(E)$ for the occupation of an energy level at energy E is

$$f(E) = \frac{1}{e^{(E-E_F)/k_B T} + 1}$$

where E_F is the Fermi level and k_B is the Boltzmann constant.

(i) (a) Sketch the occupation probability $f(E)$, of an electron state of energy E, in the regime $k_B T \ll E_F$. Show how the two energy scales, $k_B T$ and E_F, are related to features in your sketch. [3 marks]

(b) Using the definition of the work function above, write down a relation between E_F, E_0 and ϕ. Hence, sketch on the *same* diagram the density of states for the free electron outside the metal. [3 marks]

(ii) The metal is heated to close to its melting point of 1508 K. Compare the work function with the thermal energy at this temperature. Hence, explain why the emitted electrons can be treated as a classical gas. [3 marks]

(iii) Using the given density of states, show that, at thermal equilibrium at temperature $T \lesssim 1508$ K, the number of electrons emitted out of the metal into the vacuum is well approximated by

$$N(T) \simeq \frac{V}{\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} e^{-\phi/k_B T}\int_0^\infty \epsilon^{1/2} e^{-\epsilon/k_B T}\, d\epsilon\,.$$

[3 marks]

(iv) Hence, assuming results for a classical ideal gas, show that the pressure of the electron gas outside the metal is proportional to $T^{5/2}e^{-\phi/k_B T}$. [3 marks]

(v) Write down the charge current density of an electron gas in terms of its density and velocity. Hence show that in equilibrium the current density of electrons outside the metal hitting the surface of the metal is proportional to $T^2 e^{-\phi/k_B T}$. You may assume that this electron gas obeys the classical equipartition theorem. [3 marks]

(vi) An electric field is applied to sweep away the thermally emitted electrons to an electrode. Assume that the field does not alter the electron density significantly and the silver sample is earthed. The maximum possible current collected at the electrode is also found to be proportional to $T^2 e^{-\phi/k_B T}$. Briefly explain. [2 marks]

[Total 20 marks]

4. A thin rod A, with moment of inertia I, rotates at its midpoint about a fixed perpendicular axis. The coordinate ϕ_A represents the rotation of the rod, and we will treat this as a quantum mechanical system, with a wavefunction $\Psi(\phi_A, t)$ representing its state. The angular momentum operator can be written in terms of the angular coordinate in the form $\hat{L}_A = -i\hbar\frac{d}{d\phi_A}$.

(i) Show that the eigenfunctions for $\hat{L}_A$ can be written as $\Psi_A \propto \exp im_A\phi_A$, and explain why m_A must be an integer. [2 marks]

(ii) The Hamiltonian of this system is given by $\hat{H}_A = \hat{L}_A^2/2I$. What are its energy levels? [3 marks]

We now add another rod B, which has the same moment of inertia as A, but is distinguishable from it, and rotates about the same axis, and is described by the coordinate ϕ_B.

(iii) What are the first five energy levels of this system, and their degrees of degeneracy? [3 marks]

(iv) What is the lowest energy level with greater degeneracy than the lowest five levels? [2 marks]

An interaction between the two rods, given by the potential $V(\phi_A, \phi_B) = k\,[1 - \cos(\phi_A - \phi_B)]$, is now introduced, where k is positive.

(v) Write down the time-independent Schrödinger equation for the wavefunction $\Psi(\phi_A, \phi_B)$ for the system. [2 marks]

If k is much greater than $\hbar^2/I$, the potential energy of the system tends to dominate over the kinetic energy.

(vi) It is observed that in this case, the low-kinetic energy states are found to have energy levels which belong to equally-spaced bands. Explain why this is the case, discussing qualitatively how the parameters of the system affect the spacing of the energy levels.
Hint: Consider a new variable $\Delta_- = \phi_A - \phi_B$ and expand the potential around zero. [3 marks]

When the value of k is small compared to $\hbar^2/I$, the attractive potential can be considered using perturbation theory.

(vii) Give the first-order perturbation to the ground-state energy of the system. [3 marks]

(viii) Sketch an example of a classical system with rods A and B that would be described by the above rotational kinetic energies and attractive potential. [2 marks]

[Total 20 marks]

5. This question concerns statistical physics and partition functions. Consider an electric dipole, situated at the origin $x = 0$, that is always aligned along the x-axis. The dipole has the same energy E, irrespective of whether it is pointing in the $+x$ or $-x$ direction. We will call these dipole orientations the + and − states. The temperature is T.

(i) (a) What is the partition function Z of the dipole in terms of the degeneracy g of its states? [2 marks]

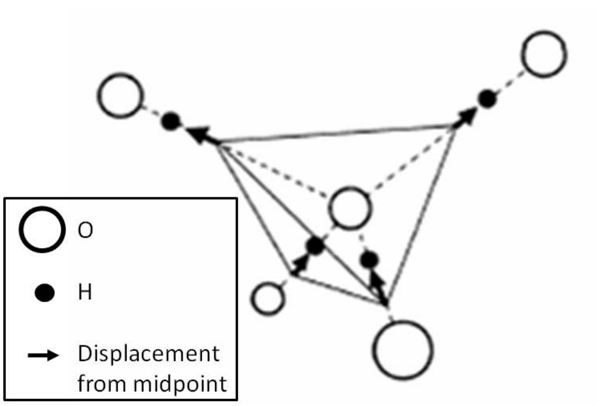
(b) Thus calculate the probabilities P_+ and P_- of the dipole being in the + and − states. [1 mark]

(ii) Now consider a second dipole at a different position $x = d$. It is also always aligned along the x-axis, with identical energy E, irrespective of whether it is pointing in the $+x$ or $-x$ direction. In addition, there is now an interaction between the two dipoles with energy $-E_I$ for parallel alignment and $+E_I$ for antiparallel alignment.

(a) What is the partition function Z of the two dipole system in terms of the degeneracy g of its states? [2 marks]

(b) Express the probability of the dipoles being aligned antiparallel to each other as a function of E_I and thus show that for $E_I = \frac{k_B T}{2}$ the probability of the dipoles being in the antiparallel state is $P_a = \frac{1}{1+e}$. [4 marks]

(iii) In ice crystals each oxygen atom is surrounded by four other oxygen atoms in a tetrahedral arrangement. One hydrogen atom sits along each $O - O$ bond, displaced from the midpoint of the $O - O$ bond to give one short $O - H$ bond (covalent bond). The interaction between the hydrogen atom and the oxygen atom at the other end of the bond is weak. Thus each hydrogen atom sits randomly in one of two equivalent positions along the $O - O$ bond, as shown in the figure.



(a) In the tetrahedron around a single oxygen atom, containing four randomly positioned hydrogen atoms, calculate the number of possible hydrogen con-

[This question continues on the next page . . .]

figurations. Treat the possible configurations as distinguishable microstates. [1 mark]

(b) In an ice crystal containing N oxygen atoms, where N is very large, calculate the number of possible hydrogen configurations. Treat the possible configurations as distinguishable microstates. [2 marks]

(c) However, as shown in the figure, each tetrahedron contains a water molecule (i.e. exactly two short (covalent) bonds). What fraction of the possible hydrogen configurations within a single tetrahedron are consistent with molecular bonding in water? [3 marks]

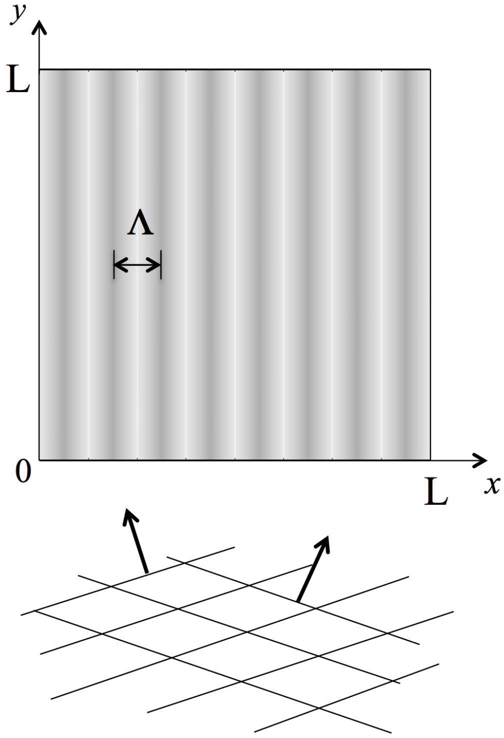
(d) How will the bonding requirement of having exactly two short bonds per water molecule reduce the total number of microstates for N water molecules? [1 mark]

(e) By considering the fraction of the total number of microstates in the system that are consistent with the bonding in water, show that the residual configurational entropy at 0 Kelvin (*zero – point* entropy) in an ice crystal containing N oxygen atoms is $Nk_B \ln\left(\frac{3}{2}\right)$. [3 marks]

(f) The third law of thermodynamics states that, for a system in thermodynamic equilibrium, 'The entropy of a perfect crystal approaches zero as temperature approaches absolute zero.' Briefly comment on the finite entropy of ice at zero temperature. [1 mark]

[Total 20 marks]

6. This question concerns an experiment to measure the thermal conductivity D of a gas. The idea is to infer D from the subsequent evolution of temperature in the gas after being subjected to rapid heating by a short-duration laser pulse. The homogeneous gas is held in a transparent box, which thermally insulates it from the outside. The laser is set up to make interference fringes in the gas, such that the intensity is modulated as $I(x) = I_o \cos^2(\pi x/\Lambda)$ in the x direction but is constant in the other two directions. I_o is the peak intensity. The box is aligned with the coordinate axes, as shown. Assume that the fringe spacing Λ is small compared to the length L of the cubic box. **Give all numerical answers to 2 significant figures.**



A heat flux $\mathbf{q}$ (units Wm^{-2}) is driven by temperature gradients in the gas according to,

$$\mathbf{q} = -D\nabla T \quad . \tag{1}$$

In the absence of heating, the temperature evolution is governed by the diffusion equation

$$c_v \frac{\partial T}{\partial t} = D\nabla^2 T \tag{2}$$

where c_v is the heat capacity at constant volume, per unit volume of the gas.

(i) Explain the difference between intensive and extensive physical quantities. Identify whether each quantity in equations (1) and (2) is intensive or extensive. [3 marks]

(ii) Heating by the laser instantaneously imparts a temperature perturbation $\delta T = T_1 \cos(k_x x)$ at $t = 0$, where T_1 is the amplitude and k_x is the wavenumber of the spatial perturbation. Find an expression for the time τ for the temperature perturbation to decay by a factor $1/e$. [4 marks]

[This question continues on the next page . . .]

(iii) The decay time is measured to be 0.10 s when $\Lambda = 2.0$ mm. Given that the gas has number density $n = 1.0 \times 10^{26}\,\mathrm{m}^{-3}$ and behaves like an ideal monatomic gas, and $T_1 = 30$ K, calculate the thermal conductivity and the peak heat flow. [3 marks]

(iv) Starting from the Fundamental Equation of Thermodynamics, show that the entropy per unit volume for an ideal gas when the volume is fixed is

$$s = s_o + \frac{3}{2} n k_B \ln(T)$$

where n is the number density, and s_o is an integration constant. [3 marks]

(v) Decay of this perturbation eventually returns the gas to a uniform temperature T_0. By integrating the entropy over the volume of the gas, and comparing it before and after the decay, show that the average entropy change of the gas (per unit volume) caused by this decay can be expressed as

$$\Delta s = \frac{3 n k_B}{2L} \int_o^L \left[a \cos^2(k_x x) - b \cos(k_x x) \right] dx$$

when the initial perturbation amplitude T_1 is small compared to T_0, i.e., $T_1 \ll T_0$ and find expressions for a and b. (*Note: 2nd order terms should be retained.*) [4 marks]

(vi) Calculate the entropy change given that $T_0 = 300$ K. Comment on the thermodynamic significance of the sign of Δs. [3 marks]

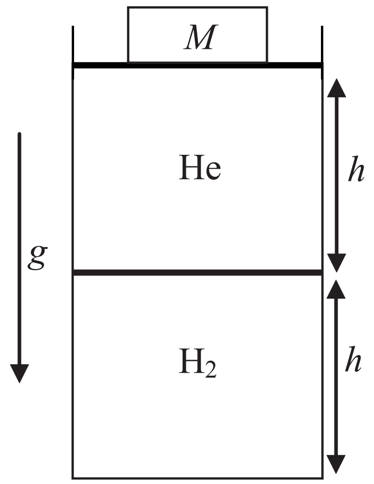
Useful formula:

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

[Total 20 marks]

7. (i) What is an ideal gas? Write down a formula for the equation of state of an ideal gas and clearly define the quantities appearing in it. [2 marks]

(ii) He and H_2 at room temperature, T_0, can be treated as ideal gases. They are contained in a cylinder with two frictionless and massless pistons as shown in the figure. A mass M is placed on the top piston. The system is in a gravitational field and is in equilibrium when the two gas volumes are equal.



(a) Find expressions for the number of molecules and hence the mass of each gas required for such equilibrium. You may assume that the external pressure can be neglected and the masses of both gases are much smaller than M. [4 marks]

(b) An additional mass, nine times larger than M, is slowly added to the initial mass such that the system reaches a new equilibrium state. Assuming no heat loss to the walls and the pistons, and no heat exchange between the two gases, find expressions for the temperatures of the two gases and the new positions of the two pistons. The expression for each gas should include γ, the ratio of the specific heat at constant pressure to constant volume. [5 marks]

(c) Which gas is more compressed and which has a higher temperature? Explain why. [3 marks]

(iii) The same experiment is repeated but with He gas used in both volumes.

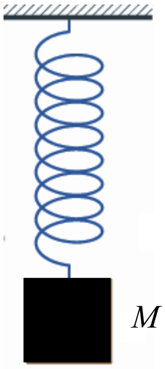
(a) Find an expression for the change of the total internal energy of the system after the addition of the larger mass. [4 marks]

(b) Compare your answer to part (iii)(a) with the change of the potential energy and comment on the result. [2 marks]

[Total 20 marks]

8. For small oscillations the period of both a spring driven clock and a pendulum clock driven by gravity does not depend on the amplitude of the oscillation. In this question you may assume that all oscillations are small.

(i) A mass M hangs under gravity from a spring that is fixed as shown in the figure. When displaced vertically from its equilibrium position by an amount X, the



mass oscillates under the action of a restoring force given by $F = -kX$. Write down the equation of motion of the mass and derive expressions for the oscillation period of its displacement, and for the oscillation period of both its potential and kinetic energy. [5 marks]

(ii) The oscillation period for the angular displacement of a pendulum of length ℓ is given by $T = 2\pi(\ell/g)^{1/2}$. A rocket is launched vertically upwards from the surface of the Earth with a constant acceleration a. The rocket engine fires for a time t, as measured by a spring clock. Derive an expression for the engine firing time t_1, as measured by a pendulum clock located inside the rocket. Relativistic effects and the variation of gravity with height can be ignored. [5 marks]

(iii) An aircraft moves with a constant speed $v = 250\ \mathrm{ms}^{-1}$ in a circular trajectory of radius $R = 25$ km at constant height. A pendulum clock of length ℓ and a spring clock are located on the plane. Both clocks are calibrated before the plane takes off. Calculate the time t_1 measured by the pendulum clock during the flight if the time measured by the spring clock is $t = 1$ hour. The variation of gravity with height and the Coriolis effect can be neglected. [5 marks]

(iv) A pendulum of length ℓ is located at the surface of the Earth and calibrated against a spring clock. A spherical cavity of radius $r = 8$ m is excavated under the pendulum at a vertical distance $h = 20$ m between the pendulum and the centre of the cavity (n.b. $h >> \ell$). The average density of the Earth is $\rho_0 = 5.5 \times 10^3\ \mathrm{kgm}^{-3}$, whilst the density of the Earth in the vicinity of the cavity is $\rho = 2.75 \times 10^3 \mathrm{kgm}^{-3}$. The radius of the Earth is 6400 km. Calculate the change in the period of the pendulum compared with the period of the spring clock. *(Hint: consider the cavity as a centre of attraction with negative mass.)*

[5 marks]

[Total 20 marks]

9. This question examines the properties of highly-excited hydrogen atoms such as their size and typical energy scales.

(i) What are the assumptions made in the Bohr model regarding physical properties such as energy and angular momentum of the orbits? [2 marks]

(ii) By equating the centripetal and electrostatic forces, and making use of the quantisation condition for angular momentum, show that the energies and the orbital radii are

$$r_n = \frac{4\pi\varepsilon_0\hbar^2}{m_e e^2} n^2\,, \quad E_n = -\frac{m_e e^4}{2(4\pi\varepsilon_0)^2\hbar^2 n^2}\,.$$

[5 marks]

(iii) Atoms in quantum states with large principal quantum numbers are known as Rydberg atoms. They have some remarkable properties.

(a) How large is the ionization energy (in SI units and in eV) of a Bohr-model hydrogen atom with $n = 50$? [2 marks]

(b) What is the diameter of a Bohr-model hydrogen atom with $n = 100$? [1 mark]

(c) Show that the transition frequency between two neighboring energy levels with principal quantum numbers n and $n - 1$ scales approximately as n^{-3} for large n. [2 marks]

(iv) A Fabry-Perot resonator consists of two parallel perfectly conducting mirrors. It is possible to build such a resonator with a width of $L = 1.7$mm.

(a) What are the boundary conditions for the electric field on the mirrors? Explain the physical reason for them. [2 marks]

(b) If the electric field $\mathbf{E}(\mathbf{r})$ is expanded into plane-wave modes of the form $\sin(\mathbf{k}\cdot\mathbf{r})$, which of these modes obeys these boundary conditions in a cavity of plate separation L? [3 marks]

(c) What is the largest value of n for which a transition between neighbouring levels is resonant with such a cavity? [3 marks]

[Total 20 marks]

10. Write short notes on FIVE of the following topics.

(i) Rocket Motion.

(ii) The use of complex numbers in mechanics, optics and electronics

(iii) Stern-Gerlach experiments

(iv) Electromagnetics of plasmas

(v) Central limit theorem

(vi) Phonons

(vii) Fermat's principle

(viii) Lamb shift

(ix) Bragg diffraction

(x) Neutrino oscillations

[Total 20 marks]