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IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2005

MSc and EEE PART IV: MEng and ACGI

LINEAR OPTIMAL CONTROL

Friday, 29 April 10:00 am

Time allowed: 3:00 hours

There are SIX questions on this paper.

Answer FOUR questions.

Corrected Copy

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s) :	A. Astolfi
	Second Marker(s) :	G. Weiss

Special instructions for invigilators: None

Information for candidates:

Hamilton Jacobi theory:

$$\dot{x}(t) = f(t,x,u),\ x(0) = x_0$$

$$J(x_0,u) = \int_\tau^T L(t,x,u)dt + m(x(T)),$$

$$-\frac{\partial V}{\partial t} = \min_u \left[L(t,x,u) + \frac{\partial V}{\partial x} f(t,x,u)\right],\ V(x,T) = m(x)$$

Linear Quadratic Regulator:

$$\dot{x}(t) = Ax(t) + Bu(t),\ x(0) = x_0$$

$$J(x_0,u) = \int_\tau^T \left[x(t)'Qx(t) + u(t)'Ru(t)\right] dt + x(T)'Mx(T)$$

$$Q = Q' \geq 0,\ R = R' > 0,\ M = M' \geq 0$$

$$-\dot{P} = A'P + PA + Q - PBR^{-1}B'P,\ P(T) = M$$

$$u(t) = -R^{-1}B'Px(t) = -Kx(t).$$

The matrices A, B, Q, R, P and K may depend upon t.

Minimum principle:

$$\dot{x} = f(x,u),\ u \in \mathcal{U}$$

$$J(x_0,u) = \int_0^{t_f} L(x(t),u(t))dt,$$

$$H(x,u,\lambda_0,\lambda) = \lambda_0 L(x,u) + \lambda' f(x,u),$$

$$\dot{\lambda}^\star = -\frac{\partial H}{\partial x}\bigg|'_{(x^\star,u^\star,\lambda_0^\star,\lambda^\star)},$$

$$H(x^\star,\omega,\lambda_0^\star,\lambda^\star) \geq H(x^\star,u^\star,\lambda_0^\star,\lambda^\star),\ \forall \omega \in \mathcal{U},$$

$$H(x^\star,u^\star,\lambda_0^\star,\lambda^\star) = k \quad \text{for fixed } t_f$$
$$H(x^\star,u^\star,\lambda_0^\star,\lambda^\star) = 0 \quad \text{for free } t_f$$

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1. Consider the system with one-dimensional state

$$\dot{x} = f(x) + u$$

with initial state $x_0 > 0$ and with the cost to be minimised

$$J(x_0, u) = \int_0^\infty L(x, u)dt$$

where

$$L(x, u) = (q(x))^2 + u^2.$$

(a) Write the Hamilton-Jacobi equation associated with this optimal control problem.
(Hint: use the fact that the value function $V(x, t)$ is independent of t). [2]

(b) Solve the Hamilton-Jacobi equation derived in part (a).
(Hint: note that the Hamilton-Jacobi equation is quadratic in $\frac{\partial V}{\partial x}$. Hence you can solve for $\frac{\partial V}{\partial x}$, and then integrate formally in x.) [8]

(c) Assume $f(x) = -q(x)$ and $q(x) = x + x^3$. Compute a positive definite solution of the Hamilton-Jacobi equation which is also such that $V(0) = 0$. [4]

(d) For $f(x)$ and $q(x)$ as in part (c), compute the optimal control and the optimal closed-loop system. [2]

(e) Show that $x = 0$ is the only equilibrium of the optimal closed-loop system. Discuss the stability of this equilibrium. (Hint: study the signum of $\dot{x}$ as a function of x.) [4]

2. Consider the simplified model of a ship described by the equation

$$M\ddot{\theta} + d\dot{\theta} + c\alpha = w$$
$$\dot{\alpha} + \alpha = u$$

where θ denotes the heading angle error (the angle between the ship's heading and the desired heading), α denotes the rudder angle, w denotes a disturbance due to wind, and u is the control input. M and c are positive parameters, and d is a non-negative parameter.

(a) Write the equation of the system, with state $(\theta, \dot{\theta}, \alpha)$, input (w, u) and output θ in standard state space form. [4]

(b) Consider the system determined in part (a) with $w = 0$. Verify that the system is controllable. [4]

(c) Consider the system determined in part (a). Verify that the system is observable. [4]

(d) Consider the system determined in part (a) with $w = 0$. Assume $M = 1$, $c = 1$ and $d = 0$. Design an output feedback controller applying the separation principle. In particular, select the state feedback gain K such that the matrix $A - BK$ has three eigenvalues equal to -1 and the output injection gain L such that the matrix $A - LC$ has three eigenvalues equal to -3. [8]

3. Consider the system

$$\dot{x} = Ax + Bu \quad = \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad + \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

with initial state x_0, with the quadratic cost to be minimised

$$J(x_0, u) = \int_0^\infty ((x_1(t) + \beta x_2(t))^2 + u^2(t))dt$$

with $\beta \in I\!R$ and $\beta \neq 0$.

(a) Verify that the (sufficient) conditions for the existence and uniqueness of an optimal feedback control law are met. [2]

(b) Write the Riccati equation associated with this optimal control problem and find all its solutions. [4]

(c) Find the positive definite solution of the Riccati equation determined in part (b). [8]

(d) Compute the optimal control law and the optimal closed-loop system. [2]

(e) Compute the eigenvalues of the optimal closed loop system determined in part (d). Show that for $0 < |\beta| < \sqrt{2}$ the eigenvalues are complex conjugate, for $|\beta| = \sqrt{2}$ the eigenvalues are real and coincide, and for $|\beta| > \sqrt{2}$ the eigenvalues are real. Show, moreover, that as $|\beta| \to \infty$ one eigenvalues approaches zero and the other approaches $-\infty$. [4]

4. Consider the system

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_1 + u\\ y &= \alpha x_1 + x_2.\end{aligned}$$

Consider a reference signal $w(t) = [1, 0]'$ and consider the problem of designing a linear static error feedback control law such that the state of the closed-loop system asymptotically tracks the signal w.

(a) Show that there is no linear static error feedback control law solving the considered problem. [2]

(b) Consider the system with input y

$$\dot{\xi} = \lambda\xi + y,$$

and set $u = \xi + v$, where v is a new input signal. Write state space equations for the extended system with state $x_e = [x_1, x_2, \xi]'$ and input v. [4]

(c) Show that it is possible to select the parameter λ of the extended system, determined in part (b), in a way that makes the following problem solvable: design a linear static error feedback control law such that the state of the extended closed-loop system asymptotically tracks a signal of the form $w_e = [1, 0, \bar{\xi}]'$, with $\bar{\xi}$ constant. [6]

(d) Let λ be as determined in part (d). Design a control law $u = -Kx_e + Kw_e$ which solves the asymptotic tracking problem and which is such that the eigenvalues of the closed-loop extended system are all equal to -1. Show that there is such a K only if $\alpha = 1$. [4]

(e) Let λ be as determined in part (c). By computing the rank of the controllability matrix of the extended system explain the results obtained in part (d). [4]

5. Consider the system

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + u\end{aligned}$$

with $u(t) \in [0,1]$, initial state $x(0) = [x_{10}, x_{20}]'$, free final state $x(T)$, and the problem of maximizing $x_1(T)$, at some fixed time $T > 0$.

(a) Show that the considered maximization problem can be recast as the problem of minimizing

$$J = \int_0^T -x_2(t)dt.$$

[4]

(b) Write the necessary conditions of optimality for normal extremals. [4]

(c) Write the optimal control as a function of the optimal costate. [2]

(d) Consider the differential equations of the costate. Assume $\lambda_1^\star(T) = \lambda_2^\star(T) = 0$ and $\lambda_1^\star(t) = A\sin t + B\cos t + C$. Determine $\lambda_1^\star(t)$ and $\lambda_2^\star(t)$. [6]

(e) Write the optimal control as a function of time. Assume $T = 10$. Compute how many times the optimal control switches. [4]

6. Consider the system

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= u\end{aligned}$$

with initial state $x(0) = [x_{10}, x_{20}]'$, final state $x(T) = [0, 0]'$, and the problem of minimizing the cost

$$J = \int_0^T u^2(t)dt$$

with fixed final time $T > 0$.

(a) Write the necessary conditions of optimality for normal extremals. (Hint: use the condition $\frac{\partial H}{\partial u} = 0$ to compute the optimal control.) [6]

(b) Write the optimal control as a function of the optimal costate. [2]

(c) Integrate the differential equations of the costate with initial conditions $\lambda^\star(0) = [\lambda^\star_{10}, \lambda^\star_{20}]$. [2]

(d) Determine the optimal control as a function of time and of $\lambda^\star(0)$. [2]

(e) Integrate the state equations with the optimal control, and use the boundary condition $x(T) = 0$ to determine $\lambda^\star_{10}$ and $\lambda^\star_{20}$ as a function of T, x_{10} and x_{20}. Write the optimal control as a function of T, x_{10} and x_{20}. Determine the initial condition x_{10} and x_{20} for which the corresponding optimal control is constant for all t. [8]

MA 1TE2

Linear Optimal Control - Model answers 2005

Question 1

(a) The Hamilton-Jacobi equation is (note that $\frac{\partial V}{\partial t} = 0$)

$$0 = \min_u \left[(q(x))^2 + u^2 + \frac{\partial V}{\partial x}(f(x) + u) \right].$$

Performing the minimization yields the optimal control (as a function of x and $\frac{\partial V}{\partial x}$), namely

$$u^\star = -\frac{1}{2}\frac{\partial V}{\partial x}$$

and the Hamilton-Jacobi equation

$$0 = (q(x))^2 - \frac{1}{4}\left(\frac{\partial V}{\partial x}\right)^2 + \frac{\partial V}{\partial x} f(x).$$

(b) Note that the Hamilton-Jacobi equation is quadratic in $\frac{\partial V}{\partial x}$. Hence,

$$\frac{\partial V}{\partial x} = 2\left(f(x) \pm \sqrt{(f(x))^2 + (q(x))^2} \right)$$

yielding

$$V(x) = \int_0^x 2\left(f(\xi) \pm \sqrt{(f(\xi))^2 + (q(\xi))^2} \right) d\xi + c,$$

where c is a constant.

(c) Setting $f(x) = -q(x)$, and $q(x) = x + x^3$, and taking the $-$ sign in the integral yields

$$V(x) = 2(\sqrt{2} - 1)\left(\frac{x^2}{2} + \frac{x^4}{4} \right) + c.$$

Setting $c = 0$ gives the desired positive definite solution.

(d) The optimal control is

$$u^\star(x) = -(\sqrt{2} - 1)\left(x + x^3 \right)$$

and the optimal closed-loop system is

$$\dot{x} = f(x) + u^\star(x) = -\sqrt{2}x(1 + x^2).$$

(e) The equation $\dot{x} = 0$ has only one solution, namely $x = 0$. Note now that $\dot{x} < 0$ for $x > 0$ and $\dot{x} > 0$ for $x < 0$, hence all trajectories approach the zero equilibrium.

Question 2

(a) The description of the system in standard state space form is (set $x = (\theta, \dot{\theta}, \alpha)'$)

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -d/M & -c/M \\ 0 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 1/M & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x.$$

(b) The controllability matrix is

$$\mathcal{C} = \begin{bmatrix} 0 & 0 & -c/M \\ 0 & -c/M & c/M(d/M+1) \\ 1 & -1 & 1 \end{bmatrix}$$

and this has full rank for all positive c and M. The system is controllable.

(c) The observability matrix is

$$\mathcal{O} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -d/M & -c/M \end{bmatrix}$$

and this has full rank for all positive c and M. The system is observable.

(d) Let $K = [k_1\ k_2\ k_3]$ and note that

$$A - BK = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -k_1 & -k_2 & -1-k_3 \end{bmatrix},$$

and that the characteristic polynomial of this matrix is $s^3 + (1+k_3)s^2 + (-k_2)s + (-k_1)$. Hence the selection

$$k_1 = -1 \qquad k_2 = -3 \qquad k_3 = 2$$

is such that the eigenvalues of $A - BK$ are equal to -1. Let $L = [l_1\ l_2\ l_3]'$ and note that

$$A - LC = \begin{bmatrix} -l_1 & 1 & 0 \\ -l_2 & 0 & -1 \\ -l_3 & 0 & -1 \end{bmatrix},$$

and that the characteristic polynomial of this matrix is $s^3 + (1+l_1)s^2 + (l_1+l_2)s + (l_2-l_3)$. Hence the selection

$$l_1 = 8 \qquad l_2 = 19 \qquad l_3 = -8$$

is such that the eigenvalues of $A - LC$ are equal to -3. Finally, the controller is $\dot{\xi} = (A - BK - LC)\xi + Ly$, $u = -K\xi$.

Question 3

(a) The pair (A, B) is controllable, $R = 1 > 0$, $Q = [1\ \beta]'[1\ \beta] \geq 0$, and the pair $(A, [1\ \beta])$ is observable.

(b) Set

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}.$$

The ARE is

$$\begin{bmatrix} 1 - p_{12}^2 & p_{11} - p_{12}p_{22} + \beta \\ p_{11} - p_{12}p_{22} + \beta & 2p_{12} - p_{22}^2 + \beta^2 \end{bmatrix} = 0.$$

We obtain $p_{12} = \pm 1$. Selecting $p_{12} = 1$ yields $p_{22} = \pm\sqrt{2 + \beta^2}$ and $p_{11} = -\beta \pm \sqrt{2 + \beta^2}$. Selecting $p_{12} = -1$ yields $p_{22} = \pm\sqrt{-2 + \beta^2}$ and $p_{11} = -\beta \mp \sqrt{-2 + \beta^2}$.

(c) For any β, the only positive definite solution is

$$P = \begin{bmatrix} \sqrt{2 + \beta^2} - \beta & 1 \\ 1 & \sqrt{2 + \beta^2} \end{bmatrix}.$$

(d) The optimal control is

$$u = -[1,\ \sqrt{2 + \beta^2}]x$$

and the optimal closed-loop systems is

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -\sqrt{2 + \beta^2} \end{bmatrix} x.$$

(e) The eigenvalues of the optimal closed-loop system are

$$\lambda_1 = -\frac{1}{2}\sqrt{\beta^2 + 2} + \frac{1}{2}\sqrt{\beta^2 - 2} \qquad \lambda_2 = -\frac{1}{2}\sqrt{\beta^2 + 2} - \frac{1}{2}\sqrt{\beta^2 - 2}.$$

Clearly, for $|\beta| < \sqrt{2}$ these eigenvalues are complex conjugate, for $|\beta| = \sqrt{2}$ they are real and coincide (equal to -1), and for $|\beta| > \sqrt{2}$ they are real. Finally, as $|\beta| \to \infty$, we have $\lambda_1 \to 0$ and $\lambda_2 \to -\infty$.

Question 4

(a) To achieve asymptotic tracking with the stated class of feedback, it is necessary that $\dot{w} - Aw = 0$. In particular this is not the case for the given signal. In fact, if $w(t) = [1, 0]'$ then $\dot{w}(t) = [0, 0]'$ and

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \neq 0.$$

(b) The state space equations for the extended system are

$$\dot{x}_e = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ \alpha & 1 & \lambda \end{bmatrix} x_e + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} v = A_e x_e + B_e v$$

(c) Note now that $w_e = [1, 0, \bar{\xi}]'$, $\dot{w}_e = [0, 0, 0]'$ and

$$\dot{w}_e - A_e w_e = \begin{bmatrix} 0 \\ 1 - \bar{\xi} \\ -\alpha - \lambda\bar{\xi} \end{bmatrix}.$$

Hence, selecting $\bar{\xi} = 1$ and $\lambda = -\alpha$ we satisfy the condition for the existence of a static error feedback achieving asymptotic tracking.

(d) Let $K = [k_1, k_2, k_3]$ and note that

$$A_e - B_e K = \begin{bmatrix} 0 & 1 & 0 \\ -1 - k_1 & -k_2 & 1 - k_3 \\ \alpha & 1 & -\alpha \end{bmatrix}.$$

The characteristic polynomial of the above matrix is

$$s^3 + (\alpha + k_2)s^2 + (k_1 + k_3 + \alpha k_2)s + \alpha(k_1 + k_3),$$

and this should be equal to $(s+1)^3$, *i.e.* we have to solve the equations

$$\alpha + k_2 = 3 \qquad k_1 + k_3 + \alpha k_2 = 3 \qquad \alpha(k_1 + k_3) = 1.$$

Note that these equations are not independent. From the first one we obtain $k_2 = 3 - \alpha$ Substituting in the second one we have $k_1 + k_3 = 3 - \alpha(3 - \alpha)$ however, from the third equation we obtain $k_1 + k_3 = \frac{1}{\alpha}$. Therefore

$$\frac{1}{\alpha} = 3 - \alpha(3 - \alpha).$$

Plotting the left-hand side and the right-hand side we see that this equation has only one solution: $\alpha = 1$. Hence, there is a K solving the stated problem problem only if $\alpha = 1$, namely

$$K = [1 - k_3, 2, k_3].$$

(e) The controllability matrix, for $\lambda = -\alpha$, is

$$\mathcal{C} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and this has rank 2. It is therefore not possible to assign arbitrarily the eigenvalues of $A_e - B_e K$. Note however, that for $\alpha = 1$ the uncontrollable mode is $s = -1$, hence there is a feedback gain assigning the desired eigenvalues.

Question 5

(a) From the $\dot{x}_1$ equation we have

$$x_1(T) = x_1(0) + \int_0^T x_2(t)dt,$$

hence, maximizing $x_1(T)$ is equivalent to maximizing

$$\int_0^T x_2(t)dt,$$

which is equivalent to minimizing

$$\int_0^T -x_2(t)dt.$$

(b) Let

$$H = -x_2 + \lambda_1 x_2 + \lambda_2(-x_1 + u).$$

The necessary conditions of optimality, for normal extremals, are

$$\dot{x}_1 = x_2 \qquad \dot{x}_2 = -x_1 + u$$

$$\dot{\lambda}_1 = \lambda_2 \qquad \dot{\lambda}_2 = 1 - \lambda_1$$

$$\lambda_2 u \leq \lambda_2 \omega, \ \forall \omega \in [0, 1].$$

(c) The optimal control as a function of the costate is

$$u^\star(t) = \begin{cases} 0 \text{ if} \lambda_2^\star(t) > 0 \\ 1 \text{ if} \lambda_2^\star(t) < 0 \end{cases}$$

If $\lambda_2^\star(t) = 0$ we do not have information on the optimal control.

(d) From $\lambda_1^\star = A\sin t + B\cos t + C$, we have $\lambda_2^\star = A\cos t - B\sin t$. Setting $\lambda_1(T) = \lambda_2(T) = 0$ and solving for A, B and C yields

$$\lambda_1^\star = 1 - \cos(t - T) \qquad \lambda_2^\star = \sin(t - T).$$

(e) The optimal control as a function of time is

$$u^\star(t) = \begin{cases} 0 \text{ if} \sin(t - T) > 0 \\ 1 \text{ if} \sin(t - T) < 0 \end{cases}$$

Hence, for $T = 10$ the optimal control switches 3 times.

Question 6

(a) Let

$$H = u^2 + \lambda_1 x_2 + \lambda_2 u.$$

The necessary conditions of optimality, for normal extremals, are

$$\dot{x}_1 = x_2 \quad \dot{x}_2 = u$$

$$\dot{\lambda}_1 = 0 \quad \dot{\lambda}_2 = -\lambda_1$$

$$2u + \lambda_2 = 0.$$

(b) The optimal control as a function of the costate is

$$u^\star = -\frac{1}{2}(\lambda_2^\star(t)).$$

(c) From the necessary conditions in part (a) we obtain

$$\begin{aligned} \lambda_1^\star(t) &= \lambda_{10}^\star \\ \lambda_2^\star(t) &= \lambda_{20}^\star - \lambda_{10}^\star t. \end{aligned}$$

(d) The optimal control is

$$u^\star = -\frac{1}{2}(\lambda_{20}^\star - \lambda_{10}^\star t).$$

(e) Integrating the state equations with the optimal control we obtain

$$x_1^\star(t) = x_{10} + x_{20}t - \frac{1}{4}\lambda_{20}^\star t^2 + \frac{1}{12}\lambda_{10}^\star t^3$$

$$x_2^\star(t) = x_{20} - \frac{1}{2}\lambda_{20}^\star t + \frac{1}{4}\lambda_{10}^\star t^2.$$

Setting $x_1(T) = x_2(T) = 0$ and solving for $\lambda_{10}^\star$ and $\lambda_{20}^\star$ yields

$$\lambda_{10}^\star = 12\frac{x_{20}T + 2x_{10}}{T^3} \qquad \lambda_{20}^\star = -4\frac{2x_{20}T + 3x_{10}}{T^2}.$$

As a result, the optimal control is

$$u^\star(t) = -2\frac{3x_{10} + 2x_{20}T}{T^2} + 6\frac{x_{20}T + 2x_{10}}{T^3}t.$$

This is constant for all initial states such that

$$x_{20}T + 2x_{10} = 0.$$

Finally, for such initial conditions the optimal control is $u^\star(t) = -\frac{x_{20}}{T}$.