Master

IMPERIAL COLLEGE LONDON

E4.10
C2.1
SC4

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2004

MSc and EEE PART IV: M.Eng. and ACGI

PROBABILITY AND STOCHASTIC PROCESSES

Time allowed: 3:00 hours

There are SIX questions on this paper.
Answer FOUR questions.

Any special instructions for invigilators and information for candidates are on page 1.

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Information for Candidates

The Kalman Filter

Consider signal and observation processes $\{x_k\}$ and $\{y_k\}$, respectively, that satisfy:

$$\begin{aligned} x_k &= Ax_{k-1} + e_{k-1} \\ y_k &= Cx_k + v_k, \quad k = 1, 2, \ldots \end{aligned}$$

Here, $\{e_k\}$ and $\{v_k\}$ are zero mean Gaussian processes with covariances

$$\mathrm{cov}\{e_k\} = Q^{(s)} \quad \text{and} \quad \mathrm{cov}\{v_k\} = Q^{(0)} \quad \text{for all } k\,.$$

The initial state x_0 is a Gaussian random variable with specified mean and covariance:

$$E[x_0] = \hat{x}_0 \quad \text{and} \quad \mathrm{cov}\{x_0\} = P_0.$$

Assume that

x_0, $\{e_k\}$ and $\{v_k\}$ are independent random variables and $Q^{(0)}$ is invertible.

The conditional mean $\hat{x}_k$ and conditional variance P_k of x_k given $y_1, \ldots, y_k$ are related to the conditional mean $\hat{x}_{k-1}$ and conditional variance P_{k-1} of x_{k-1} given $y_1, \ldots, y_{k-1}$, via the intermediate variable $P_{k|k-1}$, by the following equations:

$$\begin{aligned} P_{k|k-1} &= AP_{k-1}A^T + Q^{(s)} \\ P_k &= P_{k|k-1} - P_{k|k-1}C^T(CP_{k|k-1}C^T + Q^{(o)})^{-1}CP_{k|k-1} \\ K(k) &= P_{k|k-1}C^T(CP_{k|k-1}C^T + Q^{(o)})^{-1}. \\ \hat{x}_k &= A\hat{x}_{k-1} + K(k)(y_k - CA\hat{x}_{k-1})\,. \end{aligned}$$

1. (a) Possible failures in a communication link joining points a and b are represented by the state ('open' or 'closed') of switches $S_1, \ldots, S_5$ in Figure 1(a). Assume that the switches fail (are open) independently and

$$P[S_i \text{ is closed}] = p \quad \text{for } i = 1, 2, \ldots, 5$$

for some constant p, $0 < p < 1$. What is the probability that the path between a and b will be closed? [10]

Hint: Consider separately the cases 'S_3 is closed' and S_3 is open, i.e. use the formula

$$P[E] = P[E|\mathcal{S}_3]P[\mathcal{S}_3] + P[E|\bar{\mathcal{S}}_3]P[\bar{\mathcal{S}}_3]$$

where $E = \{$'there is a closed path from a to b'$\}$ and, for $i = 1, \ldots, 5$, $\mathcal{S}_i = \{$'S_i is closed'$\}$.

(b) A discrete random signal S, that takes values $S = 1$ or $S = 2$, is transmitted at point A in Fig. 1(b). The signal is received at point B, after passage through a channel that is modelled as an amplifier with gain K. The amplifier fails randomly:

$$K = \begin{cases} 2 & \text{if amplifier is functioning} \\ 1 & \text{if amplifier fails (no amplification).} \end{cases}$$

Assume that amplifier failure is independent of the value of the signal and

$$P[S = 1] = \alpha, \quad P[S = 2] = (1 - \alpha)$$
$$P[K = 2] = \beta, \quad P[K = 1] = (1 - \beta)$$

for some constants α, $0 < \alpha < 1$, and β, $0 < \beta < 1$.

The received signal at B is $R = 2$. What is the probability that the amplifier has failed? [10]

Hint: Calculate $P[R = 2|K = 1]$ and $P[R = 2|K = 2]$ and use Bayes Rule.

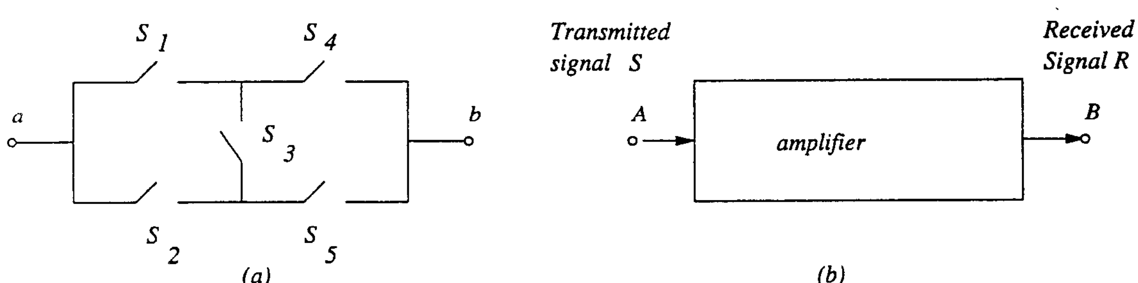

Figure 1

2. (a) A random variable $T(\omega)$ has the exponential probability density

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t}, & \text{for } t \geq 0 \\ 0, & \text{for } t < 0\,. \end{cases}$$

Here $\lambda\,(>0)$ is a parameter.

Calculate the characteristic function of $T(\omega)$. Hence determine the mean m_T and the variance σ_T^2 of $T(\omega)$. [8]

2. (b) The lifetime $T(\omega)$ of an electronic device is modelled as a random variable with exponential distribution. For fixed $S > 0$ and $t > S$ calculate the conditional probability

$$P[T(\omega) \leq t \,|\, T(\omega) > S]\,.$$

Hence calculate the conditional density of $T(\omega)$, given the event '$T(\omega) \geq S$'. [4]

Show that the conditional mean and variance of $T(\omega)$ given '$T(\omega) \geq S$' (i.e. the expected life time and its variance, given the the device has not failed before time S) are

$$m_{T|T\geq S} = S + m_T \quad \text{and} \quad \sigma^2_{T|T\geq S} = \sigma_T^2\,,$$

where m_T and σ_T^2 are the unconditional mean and variance of $T(\omega)$, calculated in (a). [6]

Comment on the suitability of modelling the lifetime of a device using the exponential distribution, in the light of your calculation. [2]

Hint: When evaluating integrals of the form

$$I = \int_S^\infty g(t)dt$$

use a change of variables '$t' = t - S$', i.e. express the integral

$$I = \int_0^\infty g(t' + S)dt'\,.$$

Note also that you have formulae for the moments $\int_0^\infty t^k f_T(t)dt$, $k = 1, 2$, from part (a).

3. A noisy measurement $Y(\omega)$ is made of the position $X(\omega)$ of an object along a line. Assume that the measurement noise is additive, i.e.

$$Y(\omega) \;=\; X(\omega) + N(\omega)\,.$$

Assume also that the noise $N(\omega)$ and the signal $X(\omega)$ are independent, that $X(\omega)$ is uniformly distributed on $[-a, a]$:

$$f_X(x) \;=\; \begin{cases} (1/2)a & \text{for } -a \le x \le a \\ 0 & \text{otherwise} \end{cases}$$

for some parameter $a > 0$, and that $N(\omega)$ is normally distributed, with zero mean and unit variance:

$$f_N(n) \;=\; \frac{1}{(2\pi)^{\frac{1}{2}}} e^{-\frac{1}{2}n^2}\,.$$

Derive expressions for the conditional density of $X(\omega)$ given $Y(\omega)$, $f_{X|Y}(x|y)$. [10]

Hence derive expressions for the (nonlinear) least squares estimate $\hat{X}(y)$ of $X(\omega)$ given $Y(\omega) = y$. [6]

(In these expressions, you do not have to evaluate the integrals involved).

Show that, as $a \to \infty$,

$$\hat{X}(y) \to \hat{x}_{ML}$$

where, for fixed y, $\hat{x}_{ML}$ maximizes the 'likelihood function'

$$x \to f_{Y|X}(y|x)$$

[4]

Hint: Obtain a formula for the joint probability density $f_{XY}(x, y)$ by using the formula $f_{XY}(x, y) = f_{Y|X}(y|x)f_X(x)$.

4. The position $X(\omega)$ of an object in one dimensional space needs to be estimated from noisy measurements. For this purpose, K cheap, identical sensors are to be used.

Assume that the i^{th} sensor measurement $Y_i(\omega)$ is related to $X(\omega)$ according to

$$Y_i(\omega) \;=\; X(\omega) + N_i(\omega) \quad \text{for } i = 1, 2, \ldots, K\,,$$

for random variables $N_1(\omega), \ldots N_K(\omega)$. Assume further that $X(\omega)$, $N_1(\omega), \ldots, N_K(\omega)$ are zero mean, independent random variables and

$$var\{X\} \;=\; \sigma_X^2 \quad \text{and} \quad var\{N_1\} = \ldots = var\{N_K\} = \sigma_N^2\,.$$

Determine

(i): the linear least squares estimate $\hat{X}$ of X given $Y_1, \ldots, Y_n$ [12]

(ii): the mean square estimation error of $\hat{X}$. [6]

Now assume that $\sigma_N^2 = 1\ m^2$ and $\sigma_X^2 = 0.5\ m^2$. Suppose that we require

$$E|\hat{X} - X|^2 \;\leq\; 0.1\, m^2\,.$$

Determine the minimum number of sensors K required to achieve this specification. [2]

Hint: Determine the linear least squares estimate $\hat{X}$ from first principles, *not* by using the general formula for multi-dimensional linear least squares estimation. Use the fact that, by symmetry, all the weights in the linear least squares estimator are the same.

5. (a) Consider a stationary scalar output process $\{y_k\}$ and vector state process $\{x_k\}$ governed by the equations

$$\begin{cases} x_{k+1} &= Ax_k + be_k \\ y_k &= c^T x_k \,. \end{cases} \tag{1}$$

Here, A is a given $n \times n$ matrix and b and c are given n-vectors. $\{e_k\}$ is a sequence of zero mean, uncorrelated random variables, each with unit variance. Develop formulae for the covariance matrix of x_k and the variance of y_k:

$$R_x(0) = E[x_k x_k^T] \quad \text{and} \quad R_y(0) = E[y_k^2]\,.$$

[8]

(b) Now suppose the matrices in (5.1) are as follows:

$$A = \begin{bmatrix} -k & -\alpha \\ 1 & 0 \end{bmatrix} \text{ and } b = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } c^T = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Here, α is a system constant. k is a design parameter (the gain in some inner feedback loop) that is to be chosen to reduce the variance of $\{y_k\}$.

(i): For fixed α and k, obtain a formula for the variance of y_k. [8]

(ii): Assume $\alpha = 0.25$. Determine the value of k that minimizes $E[y_k^2]$. [4]

You should assume that all the values of α and k considered are such that (1) is a stable dynamical system.

6. The position of a stationary object along a line is modelled as the scalar Gaussian random variable x_0. Measurements y_k of the position are taken at times $k = 1, 2, \ldots$ It is assumed that

$$y_k = x_0 + v_k$$

where $\{v_k\}$ is a sequence of zero mean, independent Gaussian random variables, each with variance σ_0^2. Assume also that x_0 has zero mean and variance P_0.

Let $\hat{x}_k$ and P_k be the conditional mean and variance of x_0, given $y_1, \ldots, y_k$, for $k = 1, 2, \ldots$ Use the Kalman filter equations to derive the following recursive equations for P_k^{-1}

$$P_{k+1}^{-1} = \sigma_0^{-2} + P_k^{-1} \tag{2}$$

and

$$\hat{x}_{k+1} = (1 - \sigma_0^{-2} P_{k+1})\hat{x}_k + \sigma_0^{-2} P_{k+1} y_{k+1} \, .$$

[16]

By using (2), or otherwise, show that

$$P_k \to 0 \quad \text{as } k \to \infty \, .$$

Hint: Introduce the 'state equation' $x_{k+1} = Ax_k + Be_k$ with $A = 1$ and $B = 0$. [4]

Master 16/2/04

E4.10, C21, SC4 Prob. + Stochastic Processes Exam 2004. Answers

1(a) Write S_i = "switch S_i is closed", for $i = 1, \dots, 5$. Note that, for any i, j, $i \neq j$,

$$P(S_i \cup S_j) = 1 - P(\bar{S}_i \cap \bar{S}_j) \quad \text{(by de Morgan's rule)}$$
$$= 1 - P(\bar{S}_i)P(\bar{S}_j) \quad \text{(by independence)}$$
$$= 1 - (1-p)^2$$

The event E = "there is a closed path from a to b" has probability

$$P[E \mid S_3]\, P[S_3] + P[E \mid \bar{S}_3]\, P[\bar{S}_3] \qquad (\bar{S}_3 = \text{complement}\{S_3\})$$
$$= P[(S_1 \cup S_2) \cap (S_4 \cup S_5)]\, p + P[(S_1 \cap S_4) \cup (S_2 \cap S_5)]\, (1-p)$$
$$= P[S_1 \cup S_2] \cdot P[S_4 \cup S_5]\, p + (1 - P[\overline{S_1 \cap S_4}])(1 - P[\overline{S_2 \cap S_5}]))(1-p)$$
$$= (1-(1-p)^2)^2 p + (1-(1-p^2)^2)(1-p)$$
$$= (2p-p^2)^2 p + (2p^2 - p^4)(1-p) = (2-p)^2 p^3 + (2-p^2)(1-p)p^2$$

[10] $$= 4p^3 - 4p^4 + p^5 + 2p^2 - 2p^3 - p^4 + p^5 = 2p^2 + 2p^3 - 5p^4 + 2p^5$$
$$= p^2(2 + 2p - 5p^2 + 2p^3)$$

(b) $P[R=2 \mid K=1] = P[S=2] = (1-\alpha)$

Also, $P[R=2 \mid K=2] = P[S=1] = \alpha$, $P[K=1] = (1-\beta)$, $P[K=2] = \beta$

By Bayes Rule

$$P[\text{'amplifier fails'} \mid R=2] = P[K=1 \mid R=2]$$
$$= \frac{P[R=2 \mid K=1]\, P[K=1]}{P[R=2 \mid K=1]\, P[K=1] + P[R=2 \mid K=2]\, P[K=2]}$$

[10] $$= \frac{(1-\alpha)(1-\beta)}{(1-\alpha)(1-\beta) + \alpha\beta}$$

Prob. + Stoch. Processes Exam 2004 Answers

2 (a) $\Phi_T(\theta) = E\{e^{j\theta T(\omega)}\} = \int_0^\infty e^{j\theta t}\lambda e^{-\lambda t}dt = \lambda\int_0^\infty e^{-(\lambda - j\theta)t}dt$

$= -\lambda(\lambda-j\theta)^{-1}\exp\{(\lambda-j\theta)t\}\Big|_{t=0}^{\infty} = \underline{\lambda(\lambda-j\theta)^{-1}}$

$\frac{d}{d\theta}\Phi_T(\theta)\Big|_{\theta=0} = \frac{+j\lambda}{(\lambda-j\theta)^2}\Big|_{\theta=0} = \frac{j}{\lambda} = j\, m_T \Rightarrow \underline{m_T = \frac{1}{\lambda}}$

$\frac{d^2}{d\theta^2}\Phi_T\Big|_{\theta=0} = \frac{2\lambda j^2}{(\lambda-j\theta)^3}\Big|_{\theta=0} = \frac{2}{\lambda^2}j^2 = j^2E\{T^2\} \Rightarrow E\{T^2\} = \frac{2}{\lambda^2}$

So $\sigma_T^2 = E\{T^2\} - m_T^2 = \underline{1/\lambda^2}$.

(b) $P[T(\omega)\le t \mid T(\omega) > s] = \dfrac{P[s < T(\omega)\le t]}{1 - P[T(\omega)\le s]} = \dfrac{\int_s^t \lambda e^{-\lambda t}dt}{1-\int_0^s \lambda e^{-\lambda t}dt}$

$= -e^{-\lambda t}\Big|_s^t \Big/ e^{-\lambda s} = (e^{-\lambda s} - e^{-\lambda t})/e^{-\lambda s} = \underline{1 - e^{-\lambda(t-s)}}$ (for $t > s$)

Hence

conditional density of $T(\omega)$, given $T(\omega)\ge s$ is

$$f_{T/T>s}(t \mid T>s) = \frac{d}{dt}\{\cdots\} = \begin{cases}\lambda e^{-\lambda(t-s)} & t\ge s\\ 0 & \text{otherwise}\end{cases}$$

[4]

We have

$m_{T|T>s} = \int_s^\infty \lambda t e^{-\lambda(t-s)}dt = \int_0^\infty \lambda(t+s)e^{-\lambda t}ds$

$= \int_0^\infty \lambda t e^{-\lambda t}ds + s\lambda\int_0^\infty e^{-\lambda t}ds$

$= m_T + s\, e^{-\lambda t}\Big|_0^\infty = s + 1/\lambda$ (by part (a))

$= \underline{s + m_T}$

Also,

$\sigma^2_{T|T>s} = \int_s^\infty \lambda t^2 e^{-\lambda(t-s)}dt - m^2_{T|T>s}$

$= \int_0^\infty \lambda(t+s)^2e^{-\lambda t}dt - m^2_{T|T>s} = \int_0^\infty \lambda t^2 e^{-\lambda t}dt - 2s\int \lambda t e^{-\lambda t}dt + s^2\int_0^\infty \lambda e^{-\lambda t}dt - m^2_{T|T>s}$

$= \frac{2}{\lambda^2} + 2s\times\frac{1}{\lambda} + s^2 - \left(s + \frac{1}{\lambda}\right)^2$

[6] $= \frac{1}{\lambda^2} + \left(\frac{1}{\lambda^2} + 2s\times 1/\lambda + s^2\right) - \left(\frac{1}{\lambda^2} + 2s\times\frac{1}{\lambda} + s^2\right) = 1/\lambda^2$

$= \sigma^2_{T}$

Notice that, if the device has not failed up to time s, then its remaining expected lifetime $m_{T|T>s} - s$ is the same as when it was new. This is a reasonable model, over the medium term, for many electronic devices. However it can be expected to

[2] be a poor model in the long term.

Prob. + Stoch. Processes Exam 2004. Answers

3. $Y = X + N$. Since X and N are independent, it follows that

$$f_{Y/X}(y|x) = \frac{1}{(2\pi)^{1/2}} \exp\{-\tfrac{1}{2}(y-x)^2\}$$

Since $f_X(x) = \begin{cases} (2a)^{-1} & \text{if } -a \le x \le +a \\ 0 & \text{otherwise} \end{cases}$

It follows $f_{XY}(x,y) = (2a)^{-1}(2\pi)^{-1/2} \exp\{-\tfrac{1}{2}(y-x)^2\}$

But then the conditional density of X given Y is (for $-a \le x \le b$)

$$f_{X/Y}(x|y) = \frac{f_{XY}}{f_Y} = \frac{\cancel{(2a)^{-1}}(2\pi)^{-1/2}\exp\{-\tfrac{1}{2}(y-x)^2\}}{\cancel{(2a)^{-1}}(2\pi)^{-\frac{1}{2}}\int_{-a}^{+a}\exp\{-\tfrac{1}{2}(y-x)^2\}dx}$$

We have

$$f_{X/Y}(x|y) = \begin{cases} \dfrac{\frac{1}{(2\pi)^{1/2}}\exp\{-\frac{1}{2}(y-x)^2\}}{\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-a}^{+a}\exp\{-\frac{1}{2}(y-x)^2\}dx} & -a \le x \le +a \\ 0 & \text{otherwise} \end{cases}$$

[10]

The conditional mean of X given Y is therefore

$$\hat{x} \left(= \int x f_{X/Y}\, dx\right) = \frac{ⓐ}{ⓑ}$$

where $ⓐ = \frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-a}^{+a} x \exp\{-\tfrac{1}{2}(y-x)^2\}dx$

and

[6] $ⓑ = \frac{1}{(2\pi)^{1/2}}\int_{-a}^{+a}\exp\{-\tfrac{1}{2}(y-x)^2\}dx$

As $a \to \infty$

$$ⓐ \to \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty} x\exp\{-\tfrac{1}{2}(y-x)^2\}dx = y$$

$$ⓑ \to \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}\exp\{-\tfrac{1}{2}(y-x)^2\}dx = 1$$

(We have used here the facts that $\frac{1}{(2\pi)^{\frac{1}{2}}}\exp\{-\frac{1}{2}(y-x)^2\}$ is a probability density with mean y.) We see that

$$\hat{x} \to y.$$

But $\hat{x}_{ml}$ maximizes $x \to f_{Y/X}(y|x) = \frac{1}{(2\pi)^{\frac{1}{2}}}\exp\{-\frac{1}{2}(y-x)^2\}$.

Clearly $\hat{x}_{ml} = y$, so we have shown

[4] $$\hat{x} \to \hat{x}_{ml} \quad \text{as} \quad a \to \infty$$

Prob + Stochastic Processes Exam 2004, Answers

4 Since all random variables involved are zero mean, the 'constant' component in the linear least squares estimator is zero. By symmetry

$$\hat{x} = \alpha \sum_{i=1}^{n} Y_i$$

The mean square error is

$$J(\alpha) = E[|X - \alpha \sum_{i=1}^{n} Y_i|^2] = E[|X - \alpha \sum_{i=1}^{n} (X + N_i)|^2]$$

$$= E[|(1-\alpha N)X - \alpha \sum_{i=1}^{n} N_i|^2]$$

$$= (\alpha n - 1)^2 E[X^2] + \alpha^2 n E[N_i^2]$$

$$= (\alpha n - 1)^2 \sigma_x^2 + \alpha^2 n \sigma_N^2$$

Minimizing parameter, α^*, satisfies

$\frac{d}{d\alpha} J(\alpha^*) = 0$, i.e.

$$2(\alpha^* n - 1) n \sigma_x^2 + 2\alpha^* n \sigma_N^2 = 0$$

whence $\alpha^* = \sigma_x^2 / (n\sigma_x^2 + \sigma_N^2)$.

[12] Linear least squares estimate is $\hat{x} = \dfrac{\sigma_x^2}{n\sigma_x^2 + \sigma_N^2} \sum_{i=1}^{n} Y_i$

The mean square error is

$$J(\alpha^*) = \left(\frac{n\sigma_x^2}{n\sigma_x^2 + \sigma_N^2} - 1\right)^2 \sigma_x^2 + \frac{n\sigma_x^4 \sigma_N^2}{(n\sigma_x^2 + \sigma_N^2)^2}$$

[6] $$= \frac{\sigma_N^4 \sigma_x^2 + n\sigma_x^4 \sigma_N^2}{(n\sigma_x^2 + \sigma_N^2)^2} = \frac{\sigma_x^2 \sigma_N^2 \cancel{(\sigma_N^2 + n\sigma_x^2)}}{(\sigma_N^2 + n\sigma_x^2)^{\cancel{2}}} = \frac{\sigma_x^2 \sigma_N^2}{\sigma_N^2 + n\sigma_x^2}$$

For $\sigma_N^2 = 1\,m^2$, $\sigma_x = 0.5\,m^2$, $J(\alpha^*) = \dfrac{0.5}{1 + 0.5n}$

We require $J(\alpha^*) \leq 0.1\,m^2$, i.e.

$$0.5 \leq 0.1 \times (1 + 0.5n)$$

or $\quad 5 \leq 1 + 0.5n \quad$ or $\quad 8 \leq n$

[2] The least number of sensors required then is 8.

(a) Prob. + Stochastic Processes. Exam 2004. Answers

5 ~~(i)~~ We have $E\{x_{k+1}x_{k+1}^T\} = E\{(Ax_k + be_k)(Ax_k + be_k)^T\}$ (from state equations)

But x_k is zero mean and a linear function of $e_{k-1}, e_{k-2},\ldots$ Since the e_k's are uncorrelated and zero mean, $E\{e_k x_k^T\} = 0$. Expanding, we have

$R_x(0) = E\{x_{k+1}x_{k+1}^T\} = A\,E\{x_k x_k^T\}A^T + 0 + b\,E\{e_k e_k^T\}b^T = \underbrace{A R_x(0) A + bb^T}$.

[6] But then $R_y(0) = E\{c^T x_k x_k^T c\} = c^T R_x(0) c$.

Lyapunov equation

(b) ~~(ii)~~ ~~(i)~~ For $A = \begin{bmatrix} -k & -\alpha \\ 1 & 0 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, the Lyapunov equation is

(i) $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} = \begin{bmatrix} -k & -\alpha \\ 1 & 0 \end{bmatrix}\begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{12} \end{bmatrix}\begin{bmatrix} -k & 1 \\ -\alpha & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix}$

$$= \begin{bmatrix} k^2P_{11} + 2\alpha k P_{12} + \alpha^2 P_{22} & -kP_{11} - \alpha P_{12} \\ -kP_{11} - \alpha P_{12} & P_{11} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Equating entries in this matrix equation gives.

$$P_{11} = +k^2 P_{11} + 2\alpha k P_{12} + \alpha^2 P_{22} + 1$$

$$P_{12} = -kP_{11} - \alpha P_{12}$$

$$P_{22} = P_{11}$$

We have $P_{12} = -\dfrac{kP_{11}}{1+\alpha}$

Hence $P_{11} = \left(k^2 - \dfrac{2\alpha k^2}{1+\alpha} + \alpha^2\right) P_{11} + 1$

and

$$P_{11} = \frac{1}{1 - (1 - \frac{2\alpha}{1+\alpha})k^2 - \alpha^2}$$

[8] Then $R_y(0) = c^T R_x(0) c = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = P_{11}$

$$= \frac{1}{1 - (1 - \frac{2\alpha}{1+\alpha})k^2 - \alpha^2}$$

for $\alpha = 0.25$

[4] ~~(ii)~~ (ii). Since, $1 - \frac{2\alpha}{1+\alpha} > 0$, $R_Y(0)$ is minimized by $k = 0$.

Prob. + Stochastic Processes. Exam 2004. Answers

6 Let $\{x_k\}$ be the sequence $\{x_0, x_0, \ldots\}$ We model $\{x_k\}$ and $\{y_k\}$ as

$$\begin{cases} x_{k+1} = a x_k + b e_k \\ y_k = c x_k + v_k \end{cases}$$

with $a=1$, $b=0$ and $c=1$. Here $\{v_k\}$ is white noise with $\operatorname{cov}\{v_k\} = \sigma_0^2$.

The Kalman filter equations give update equations for the conditional mean and covariance of x_k, $\hat{x}_k$ and P_k respectively:

$$P_{k+1|k} = A P_k A^T + 0 = P_k$$

$$P_{k+1} = P_k - P_k C^T (C P C^T + \sigma_0^2)^{-1} C P_k = P_k - \frac{P_k^2}{P_k + \sigma_0^2} = \frac{\sigma_0^2 P_k}{P_k + \sigma_0^2} \quad \text{— (1)}$$

$$K_{k+1} = P_k C (C P_k C^T + \sigma_0^2)^{-1} = \frac{P_k}{P_k + \sigma_0^2}$$

and

$$\hat{x}_{k+1} = \hat{x}_k + \left(P_k / P_k + \sigma_0^2\right)\left(y_{k+1} - \hat{x}_k\right). \quad \text{— (2)}$$

We can write (2) as

$$\hat{x}_{k+1} = \frac{\sigma_0^2}{P_k + \sigma_0^2} \hat{x}_k + \frac{P_k}{P_k + \sigma_0^2} y_{k+1} = \left(1 - \sigma_0^{-2} P_{k+1}\right) \hat{x}_k + \sigma_0^{-2} P_{k+1} y_{k+1}$$

(we have used (1)).

Also

$$P_{k+1}^{-1} = \sigma_0^{-2} + P_k^{-1}, \quad k = 0, 1, \ldots \quad \text{— (3)}$$

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(3) implies

$$P_k^{-1} = \sigma_0^{-2} + k P_0^{-1}$$

It follows from this equation that

$$P_k^{-1} \to \infty \quad \text{as} \quad k \to \infty$$

Hence

$$P_k \to 0 \quad \text{as} \quad k \to \infty.$$

[4]