E4.45
C5.21
SO22
ISE4.47

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2007

MSc and EEE/ISE PART IV: MEng and ACGI

WAVELETS AND APPLICATIONS

Corrected Copy

Thursday, 3 May 10:00 am

Time allowed: 3:00 hours

There are FOUR questions on this paper.

Answer THREE questions.

All questions carry equal marks.

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : P.L. Dragotti
Second Marker(s) : P.A. Naylor

Special Information for the Invigilators: NONE

Information for Candidates:

1. *Sub-sampling by an integer N:*

$$x_{\downarrow N}[n] \longleftrightarrow \frac{1}{N}\sum_{k=0}^{N-1} X(e^{j(\omega-2\pi k)/N}) = \frac{1}{N}\sum_{k=0}^{N-1} X(W_N^k z^{1/N}),$$

where

$$W_N^k = e^{-j2\pi k/N}.$$

2. *Poisson Summation formula:*

$$\sum_{n=-\infty}^{\infty} \varphi(t-n) = \sum_{k=-\infty}^{\infty} \hat{\varphi}(2\pi k)e^{j2\pi kt}.$$

3. *Geometric Series:*

$$\sum_{n=0}^{\infty} \rho^n = \frac{1}{1-\rho} \qquad |\rho| < 1.$$

4. *Admissible Scaling Function:*
A function $\varphi(t)$ is an admissible scaling function of $L_2(\mathbb{R})$ if and only if it satisfies the three following conditions:

   (a) Riesz basis criterion: there exists two constants $A > 0$ and $B < +\infty$ such that

$$A \leq \sum_{n\in\mathbb{Z}} |\hat{\varphi}(\omega + 2\pi n)|^2 \leq B$$

   (b) Two scale relation

$$\varphi(t) = \sqrt{2}\sum_{k\in\mathbb{Z}} g_0[k]\varphi(2t-k)$$

   (c) Partition of unity

$$\sum_{k\in\mathbb{Z}} \varphi(t-k) = 1.$$

The Questions

1. Consider the four-channel filter bank shown in Figure 1

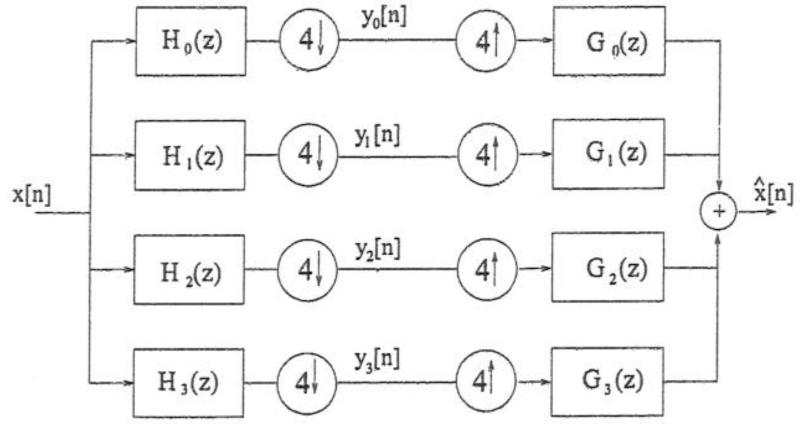

Figure 1: Four-channel filter bank.

(a) Express $\hat{X}(z)$ as a function of $X(z)$ and the filters. Then, derive the four perfect reconstruction conditions the filters have to satisfy.

[8]

(b) Assume that $G_0(z) = \frac{1}{2}(1 + z^{-1} + z^{-2} + z^{-3})$, and $G_1(z) = \frac{1}{2}(1 + z^{-1} - z^{-2} - z^{-3})$, design two four-taps filters $G_2(z)$ and $G_3(z)$ such that the following conditions are satisfied:

$$\langle g_i[n], g_j[n-4k]\rangle = \delta_{i,j} \cdot \delta_k \quad i,j = 0,1,2,3 \text{ and } k \in \mathbb{Z}.$$

[6]

(c) Given the synthesis filters $g_i[n]$ of part (b), choose $H_i(z) = G_i(z^{-1})$, for $i = 0,1,2,3$. Now, the filter bank is iterated on the H_0 branch to form a 2-level decomposition. Draw either the synthesis or the analysis filter bank of the equivalent 7-channel filter bank clearly specifying all the transfer functions and downsampling factors.

[6]

2. Consider the two-channel filter bank of Figure 2.

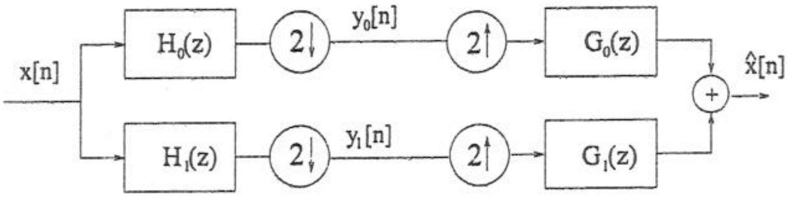

Figure 2: Two-channel filter bank.

(a) Assume that $G_0(z) = \frac{1}{2\sqrt{2}}(1+z^{-1})(1+z)$ and assume that $H_0(z) = (1+z)(1+z^{-1})B(z)$. Determine the shortest symmetric polynomial $B(z)$ such that $P(z) + P(-z) = 2$, where $P(z) = H_0(z)G_0(z)$.

[8]

(b) Given the filters $G_0(z)$ and $H_0(z)$ of part (a), design the filters $H_1(z)$ and $G_1(z)$ in order to have a perfect reconstruction biorthogonal filter bank.

[6]

(c) Based on the polynomial $P(z)$ of part (a), construct an orthogonal filter bank.

[6]

3. Assume that two functions $\varphi_0(t)$ and $\varphi_1(t)$ are valid scaling functions. Show that the function $\varphi_2(t) = \varphi_0(t) * \varphi_1(t)$ given by the convolution of $\varphi_0(t)$ with $\varphi_1(t)$ satisfies:

   (a) The partition of unity:

   $$\sum_{n=-\infty}^{\infty} \varphi_2(t-n) = 1$$

   (Hint: Use Poisson sum formula).

   [7]

   (b) The two-scale equation:

   $$\varphi_2(t) = \sqrt{2}\sum_n g_2[n]\varphi_2(2t-n).$$

   [7]

   (c) Now assume that $\varphi_0(t) = \beta_0(t)$ and $\varphi_1(t) = \beta_1(t)$, where $\beta_0(t)$ is the box function with Fourier transform $\hat{\beta}_0(\omega) = \frac{1-e^{j\omega}}{j\omega}$ and $\beta_1(t) = \beta_0(t) * \beta_0(t)$. Thus, $\varphi_2(t) = \beta_0(t) * \beta_1(t)$. Find the exact expression of the filter $g_2[n]$ that leads to the two-scale equation:

   $$\varphi_2(t) = \sqrt{2}\sum_n g_2[n]\varphi_2(2t-n).$$

   [6]

4. Consider the system shown in Figure 3

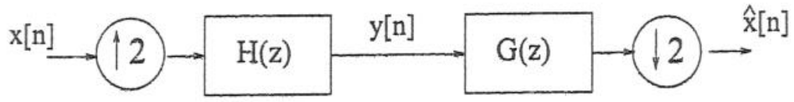

Figure 3: An interpolator.

(a) What does the product filter $P(z) = H(z)G(z)$ have to satisfy in order to have perfect reconstruction such that $\hat{x}[n] = x[n]$?

[4]

(b) Assume that $H(z) = (z^{-2} + z^{-1} + 1 + z + z^2)$. Find the shortest symmetric filter $G(z)$ such that perfect reconstruction is achieved.

[4]

(c) Now assume that $H(z) = (1 + z + z^2 + z^3)$. Design $G(z)$ so that the output $\hat{X}(z) = 0$.

[4]

(d) *Infinite products and Haar scaling function*

i. Consider the following product:

$$p_i = \prod_{k=0}^{i} a^{b^k} \quad |b| < 1,$$

show that $\lim_{i\to\infty} p_i = a^{1/(1-b)}$.

[4]

ii. Assume that $M_0(\omega) = G_0(e^{j\omega})/\sqrt{2}$ where $G_0(e^{j\omega}) = (1+e^{-j\omega})/\sqrt{2}$ is the Haar low-pass filter. Show that

$$\lim_{i\to\infty} \prod_{k=1}^{i} M_0(\omega/2^k) = e^{-j\omega/2}\frac{\sin(\omega/2)}{\omega/2}.$$

Hint: Use the identity $\cos(\omega) = \sin(2\omega)/2\sin(\omega)$.

[4]

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QUESTION 1

(a)
$$\hat{X}(z) = \frac{1}{4}\Big[G_0(z)\Big(X(z)H_0(z) + X(W_4^1 z)H_0(W_4^1 z) + X(W_4^2 z)H_0(W_4^2 z) + X(W_4^3 z)H_0(W_4^3 z)\Big)$$
$$+ G_1(z)\Big(X(z)H_1(z) + X(W_4^1 z)H_1(W_4^1 z) + X(W_4^2 z)H_1(W_4^2 z) + X(W_4^3 z)H_1(W_4^3 z)\Big)$$
$$+ G_2(z)\Big(X(z)H_2(z) + X(W_4^1 z)H_2(W_4^1 z) + X(W_4^2 z)H_2(W_4^2 z) + X(W_4^3 z)H_2(W_4^3 z)\Big)$$
$$+ G_3(z)\Big(X(z)H_3(z) + X(W_4^1 z)H_3(W_4^1 z) + X(W_4^2 z)H_3(W_4^2 z) + X(W_4^3 z)H_3(W_4^3 z)\Big)\Big].$$

PR CONDITIONS

$$G_0(z)H_0(z) + G_1(z)H_1(z) + G_2(z)H_2(z) + G_3(z)H_3(z) = 4$$

$$G_0(z)H_0(W_3^i z) + G_1(z)H_1(W_3^i z) + G_2(z)H_2(W_3^i z) + G_3(z)H_3(W_3^i z) = 0 \quad \text{FOR } i=1,2,3.$$

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Patrick Naylor

(b)

$$G_2(z) = a + bz^{-1} + cz^{-2} + dz^{-3}$$

CONDITIONS:

$$\left.\begin{array}{l} g_2[n] \perp g_0[n] \\ g_2[n] \perp g_1[n] \end{array}\right. \Rightarrow \begin{cases} a+b = -(c+d) \\ a+b = c+d \end{cases}$$

THUS $a=-b$ AND $c=-d$

WE CHOOSE $a=c$ AND $b=d=-a$

CONDITION $\|g_2[n]\|^2 = 1$ LEADS TO $a = \frac{1}{2}$

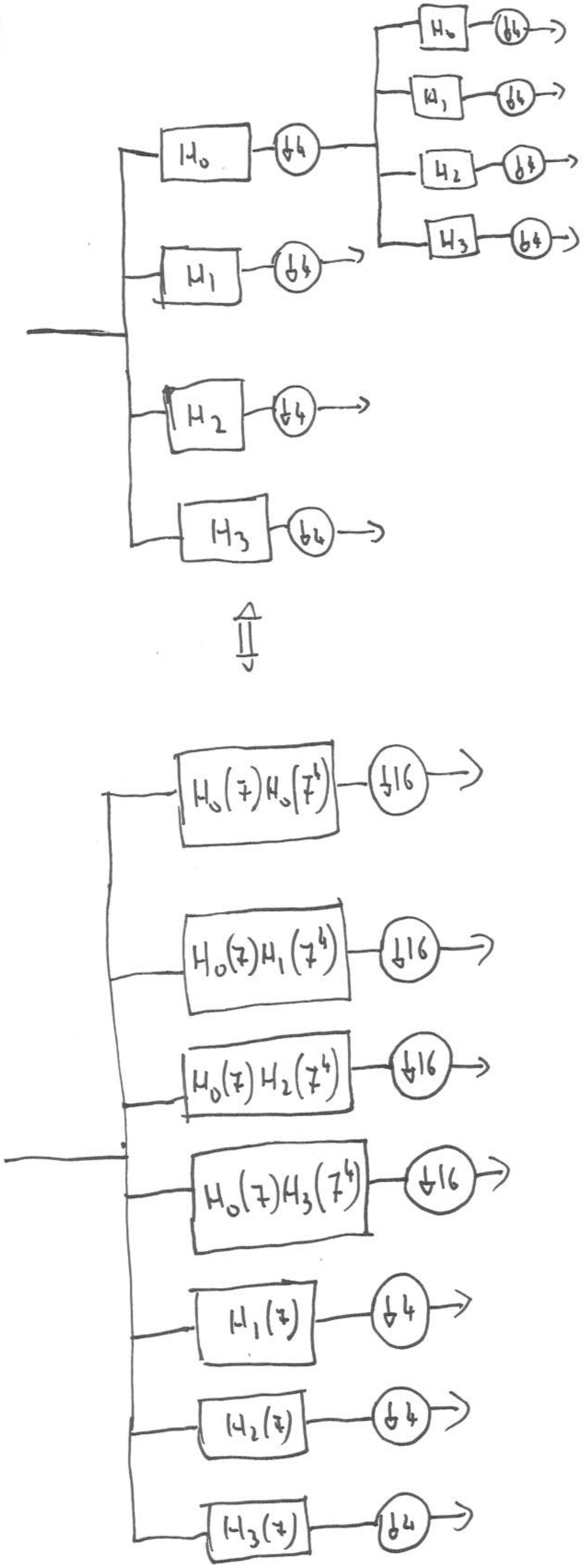
THUS

$$G_2(z) = \frac{1}{2}\left(1 - z^{-1} + z^{-2} - z^{-3}\right)$$

THE ~~THREE~~ FOUR CONDITIONS

$$\left.\begin{array}{l} g_3[n] \perp g_0[n] \\ g_3[n] \perp g_1[n] \\ g_3[n] \perp g_2[n] \\ \|g_3[n]\|^2 = 1 \end{array}\right. \Rightarrow G_3(z) = \frac{1}{2}\left(-1 + z^{-1} + z^{-2} - z^{-3}\right)$$

(c)



QUESTION 2

(a)

$$P(z) = \frac{1}{2\sqrt{2}} (1+z)^2 (1+z^{-1})^2 B(z)$$

IF $B(z) = a$ $\quad P(z) + P(-z) \neq 2$

SO LET'S TRY $B(z) = (a z^{-1} + b + a z)$

$$P(z) = \frac{1}{2\sqrt{2}} (1 + 2z + z^2)(1 + 2z^{-1} + z^{-2})(a z^{-1} + b + a z) =$$

$$= \Big(a z^{-3} + (b+4a) z^{-2} + (7a+4b) z^{-1} + (8a+6b) + (7a+4b) z + (b+4a) z^{-2} + a z^3\Big) \Big/ 2\sqrt{2}$$

THE HALF-BAND CONDITION $P(z) + P(-z) = 2$ IMPLIES THAT:

$$\left.\begin{aligned} (b+4a) &= 0 \\ \frac{1}{2\sqrt{2}} (8a+6b) &= 1 \end{aligned}\right\} \Rightarrow \quad a = -\frac{2\sqrt{2}}{16} \qquad b = \frac{2\sqrt{2}}{4}$$

$$P(z) = \frac{1}{16} (1+z)^2 (1+z^{-1})^2 (-z^{-1} + 4 - z)$$

AND

$$H_0(z) = \frac{1}{4\sqrt{2}} (1+z)(1+z^{-1})(-z + 4 - z^{-1})$$

#

(b) $H_1(z) = z\, G_0(-z) = \frac{1}{2\sqrt{2}} z(1-z)(1-z^{-1})$

$G_1(z) = z^{-1} H_0(-z) = \frac{z^{-1}}{4\sqrt{2}} (1-z)(1-z^{-1})(z+4+z^{-1})$

(c) ROOTS OF z^2+4z+1 ARE

$z_0 = -2-\sqrt{3}$

$z_1 = -2+\sqrt{3}$

NOTICE THAT $z_0 = 1/z_1$.

THUS

$$P(z) = \frac{1}{16a}(1+z^2)(1+z^{-2})(1-az^{-1})(1-az)$$

WHERE $a = (-2+\sqrt{3})$

THUS

$$G_0(z) = \frac{1}{4\sqrt{a}}(1+z^{-2})(1-az^{-1})$$

$H_0(z) = G_0(z^{-1})$

$G_1(z) = -z^{-1} G_0(-z^{-1})$

$H_1(z) = G_1(z^{-1})$

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QUESTION 3

(a)

PARTITION OF UNITY

$$\sum_{m=-\infty}^{\infty} \varphi(t-m) = \sum_{k=-\infty}^{\infty} \hat{\varphi}(2\pi k) e^{j2\pi k t} = 1$$

IMPLIES THAT

$$\begin{cases} \hat{\varphi}(2\pi k) = 1 & \text{FOR } k=0 \\ \hat{\varphi}(2\pi k) = 0 & \text{FOR } k \neq 0 \text{ AND } k \in \mathbb{Z} \end{cases}$$

NOW, $\varphi_0(t)$ AND $\varphi_1(t)$ SATISFY PARTITION OF UNITY, MOREOVER $\varphi_2(t) = \varphi_0(t) * \varphi_1(t)$

$$\Updownarrow$$

$$\hat{\varphi}_2(w) = \hat{\varphi}_0(w)\,\hat{\varphi}_1(w)$$

THUS $\hat{\varphi}_2(2\pi k) = 1$ FOR $k=0$ AND $\hat{\varphi}_2(2\pi k) = 0$ $k \neq 0$

AND

$$\sum_m \varphi_2(t-m) = \sum_k \hat{\varphi}_2(2\pi k) e^{j2\pi k t} = 1$$

\#

(b) $\varphi_0(t)$ AND $\varphi_1(t)$ SATISFY THE TWO-SCALE RELATION THUS IN FOURIER DOMAIN WE HAVE THAT

$$\hat{\varphi}_0(w) = \frac{1}{\sqrt{2}} G_0\left(e^{\frac{jw}{2}}\right) \hat{\varphi}_0\left(\frac{w}{2}\right)$$

$$\hat{\varphi}_1(w) = \frac{1}{\sqrt{2}} G_0\left(e^{\frac{jw}{2}}\right) \hat{\varphi}_1\left(\frac{w}{2}\right)$$

AND

$$\hat{\varphi}_2(w) = \hat{\varphi}_0(w)\,\hat{\varphi}_1(w) = \frac{1}{\sqrt{2}}\left[\cdot\frac{1}{\sqrt{2}}\,G_0\left(e^{\frac{jw}{2}}\right)G_1\left(e^{\frac{jw}{2}}\right)\right]\hat{\varphi}_0\left(\frac{w}{2}\right)\hat{\varphi}_1\left(\frac{w}{2}\right) =$$

$$= \frac{1}{\sqrt{2}}\,G_2\left(e^{\frac{jw}{2}}\right)\hat{\varphi}_2\left(\frac{w}{2}\right) \quad \text{WITH} \quad g_2[n] = \frac{1}{\sqrt{2}}\,g_0[n] * g_1[n]$$

#

(c) IN THIS CASE

$$g_0[n] = \frac{1}{\sqrt{2}}\left(\delta[n] + \delta[n-1]\right) \iff G_0(z) = \frac{(1+z^{-1})}{\sqrt{2}}$$

$$g_1[n] = \frac{1}{\sqrt{2}}\left(\frac{1}{2}\delta[n+1] + \delta[n] + \frac{1}{2}\delta[n-1]\right) \iff G_1(z) = \frac{(1+z^{-1})^2}{2\sqrt{2}}$$

THUS

$$g_2[n] = \frac{1}{\sqrt{2}}\,g_0[n] * g_1[n] \iff G_2(z)\,\frac{1}{\sqrt{2}}\,\frac{(1+z^{-1})^3}{4}$$

#

QUESTION 4

(a)

$$\hat{X}(z) = \frac{1}{2}\left[G(z^{1/2})H(z^{1/2}) + G(-z^{1/2})H(-z^{1/2})\right]X(z)$$

PR CONDITION:

$$G(z)H(z) + G(-z)H(-z) = 2 \quad \text{OR}$$

$$P(z) + P(-z) = 2$$

(b) $G(z) = a$ DOES NOT WORK

TRY $G(z) = az^{-1} + b + az$

$$P(z) = (z^{-2} + z^{-1} + 1 + z + z^2)(az^{-1} + b + az) =$$

$$= az^{-3} + (a+b)z^{-2} + (2a+b)z^{-1} + (2a+b) + (2a+b)z + (a+b)z^2 + az^3.$$

$$P(z) + P(-z) = 2 \Rightarrow \begin{cases} a+b=0 \\ 2a+b=1 \end{cases} \Rightarrow \begin{matrix} a=1 \\ b=-1 \end{matrix}$$

$$G(z) = (z^{-1} - 1 + z)$$

(c) $\hat{X}(z) = 0 \iff P(z) + P(-z) = 0$

THIS IS ACHIEVED WHEN $G(z) = (1 - z^{-1})$

IN THIS CASE WE HAVE

$$P(z) = (1 + z + z^2 + z^3)(1 - z^{-1}) = z^3 - z^{-1}$$

THUS

$$P(z) + P(-z) = 0$$

#

(d)

(i)

$$p_i = \prod_{k=0}^{i} a^{b^k} = a^{\sum_{k=0}^{i} b^k}$$

USING $\sum_{k=0}^{\infty} \rho^k = \frac{1}{1-\rho}$ $|\rho|<1$

WE OBTAIN

$$p = \lim_{i \to \infty} p_i = \lim_{i \to \infty} a^{\sum_{k=0}^{i} b^k} = a^{\sum_{k=0}^{\infty} b^k} = a^{\frac{1}{1-b}} \quad (1)$$

#

(ii)

$$\prod_{k=1}^{i} m_0\left(\frac{\omega}{2^k}\right) = \prod_{k=1}^{i} e^{-\frac{j\omega}{2^{k+1}}} \left(\frac{e^{\frac{j\omega}{2^{k+1}}} + e^{-\frac{j\omega}{2^{k+1}}}}{2} \right) =$$

$$= \prod_{k=1}^{i} e^{-\frac{j\omega}{2^{k+1}}} \prod_{k=1}^{i} \cos\left(\frac{\omega}{2^{k+1}}\right)$$

USING THE RESULT OF PART (i),

(a) $\lim_{i \to \infty} \prod_{k=1}^{i} e^{-j\frac{\omega}{2^{k+1}}} = e^{-\frac{j\omega}{2}}$

$$\prod_{k=1}^{i} \cos \frac{\omega}{2^{k+1}} = \prod_{k=1}^{i} \frac{\sin\left(\frac{\omega}{2^k}\right)}{2\sin\left(\frac{\omega}{2^{k+1}}\right)} = \frac{1}{2^i} \frac{\sin \frac{\omega}{2}}{\sin\left(\frac{\omega}{2^{i+1}}\right)}$$

(b) $\lim_{i \to \infty} \frac{1}{2^i} \frac{\sin \frac{\omega}{2}}{\sin\left(\frac{\omega}{2^{i+1}}\right)} = \frac{\sin \omega/2}{\omega/2}$

BY COMBINING (a) WITH (b) WE OBTAIN THE RESULT #