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IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE
UNIVERSITY OF LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2001

MSc and EEE/ISE PART IV: M.Eng. and ACGI

DISCRETE-TIME SYSTEMS AND COMPUTER CONTROL

Friday, 11 May 10:00 am

There are SIX questions on this paper.

Answer FOUR questions.

Time allowed: 3:00 hours

Corrected Copy

Examiners: Allwright,J.C. and Vinter,R.B.

Special Information for Invigilators: **None**

Information for Candidates

Some notation

T is the sample period

q is the forward shift operator

$f^Z(z)$, $f^D(\gamma)$, $f^F(j\omega)$, $f^W(w)$ denote the Z-, Delta-, discrete-time Fourier and W-transforms, respectively, of $\{f_k\}$

$g^L(s)$ denotes the Laplace transform of $g(t)$

$'$ denotes transposition of a vector or matrix

$t_k = kT$

Some useful transforms

f_k	$f^Z(z)$	$f^D(\gamma)$
$i_k = 0^k$	1	T
1^k	$\frac{z}{z-1}$	$\frac{1+\gamma T}{\gamma}$
t_k	$\frac{Tz}{(z-1)^2}$	$\frac{1+\gamma T}{\gamma^2}$
α^k	$\frac{z}{z-\alpha}$	$\frac{1+\gamma T}{\gamma-\bar{\alpha}}$
$k\alpha^k$	$\frac{z\alpha}{(z-\alpha)^2}$	$\frac{(1+\gamma T)(1+\bar{\alpha}T)}{T(\gamma-\bar{\alpha})^2}$

where $\bar{\alpha} = \frac{\alpha-1}{T}$

$f^W(w) = f^Z(\frac{\mu+w}{\mu-w})$ where $\mu = \frac{2}{T}$.

The Routh Test

Every root of $a_0 w^n + a_1 w^{n-1} + \ldots. + a_n = 0$ has strictly negative real part iff all $n+1$ entries in the first column of the following Routh-table are non-zero and have the same sign:

1:	a_0	a_2	a_4	.....
2:	a_1	a_3	a_5	.....
3:	$\frac{a_1a_2-a_0a_3}{a_1}$	$\frac{a_1a_4-a_0a_5}{a_1}$	$\frac{a_1a_6-a_0a_7}{a_1}$	.....
.. :	....			
n+1:	....			

The Jury Test

Every root of $d(z) \triangleq \alpha_n z^n + \alpha_{n-1} z^{n-1} + \ldots. + \alpha_0 = 0$ has modulus strictly less than one iff

$$d(1) > 0,$$

and

$$d(-1) \quad \begin{cases} > 0 & \text{if } n \text{ is even} \\ < 0 & \text{if } n \text{ is odd} \end{cases}$$

and

$$|a_0| < a_n, |b_0| > |b_{n-1}|, |c_0| > |c_{n-2}|, \ldots,$$

where the b_i, c_i etc., are determined from the following Jury-table

1:	a_0	a_1	a_2	.....	a_n
2:	a_n	a_{n-1}	a_{n-2}	.....	a_0
3:	b_0	b_1	b_2	.....	b_{n-1}
	where $b_i = a_0 a_i - a_n a_{n-i}$				
4:	b_{n-1}	b_{n-2}	.....	b_0	
.. :	....				
2n−3:	....				

Here, for all i,

$$a_i = \begin{cases} \alpha_i & \text{if } \alpha_n > 0 \\ -\alpha_i & \text{if } \alpha_n < 0. \end{cases}$$

The Questions

1. (a) By considering the Z-transform of the sequence $\{x_k\}$ generated by the scalar system

$$x_{k+1} = \alpha x_k : x_0 = 1$$

for appropriate α, determine $Z\{(-1)^k\}$. **[5]**

(b) Find $x^Z(z)$ for the system

$$x_{k+2} = -\beta^2 x_k : x_0 = 0, x_1 = 2\beta.$$

Use a partial fraction expansion to determine from $x^Z(z)$ a formula for x_k which involves a trigonometric function. **[7]**

(c) Consider the discrete-time system S_d of Figure 1 below, with sample period T, input u_k and output y_k.

(i) Determine a state-space model for S_c and suppose that the eigenvalues associated with it, denoted λ_i, are distinct. State a first-order vector difference equation that relates x_{k+1} to x_k, where $x_k \triangleq x(kT)$. **[2]**

(ii) By considering spectral forms, determine the eigenvalues associated with the difference equation of part (i) in terms of the λ_i. Denote those eigenvalues by $\overline{\lambda}_i$. **[2]**

(iii) State, without proof, the relationship between BIBO-stability of the complete system S_d of Figure 1, its poles and the eigenvalues $\overline{\lambda}_i$. Use the relationship to show that the discrete-time system S_d is BIBO-stable if the eigenvalues λ_i associated with S_c all belong to the set $\{s \in \mathbb{C}\text{: Re}(s) < 0\}$. **[4]**

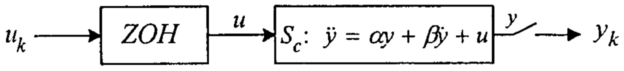


Figure 1

2. (a) Design the pole-zero pattern for a notch filter $G^Z(z)$ with 3 poles such that contributions at 0 Hz and 50 Hz in the continuous-time input $u(t)$ of Figure 2 do not appear (in discretized form) in the output signal y_k. The sample period is $T = (300)^{-1}$ second. The distance in the complex plane between any pole and any zero should be at least 0.1. [4]

(b) Determine a canonical direct realization of your $G^Z(z)$ from part (a). [5]

(c) Consider a system with zero initial conditions, transfer function

$$G^Z(z) = \frac{z-1}{(z-0.5)(z+0.5)},$$

input $u_k = \cos(\omega t_k)$ and output y_k. Note that a formula for $Z\{\cos(\omega t_k)\}$ is not needed here.

(i) State a formula that provides information about the output y_k in terms of the value of $G^Z(z)$ at a specific z. [2]

(ii) Use the integral inversion method to find a formula that predicts the output y_k when $\omega = 0$, exploiting the fact that $u_k = 1^k$ when $\omega = 0$.
What are the numerical values of y_0, y_1, y_2, y_3?
Check your values for y_0, y_1, y_2, y_3 by long division.
Discuss very briefly the consistency, or otherwise, of these values with your information about y_k from part (i). [9]

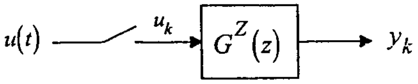


Figure 2

3. (a) Suppose $G^Z(z)$ is the pulse Z-transfer function from $u^Z(z)$ to $y^Z(z)$ of the system of Figure 3, and $G^D(\gamma)$ is the corresponding pulse Delta-transfer function.
In Figure 3, S_c denotes a continuous-time linear system and the sample period is T.

(i) Suppose S_c has the Laplace transfer function $\frac{1}{s(s+1)}$.
Find $G^Z(z)$ from the step response of S_c.
Determine $G^D(\gamma)$ from $G^Z(z)$. [6]

(ii) Suppose S_c has the model $\dot{x} = Ax + Bu,\ y = Cx$ where

$$x \in \mathbb{R}^2,\ A = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = [1 \quad 0].$$

Use $I + \frac{1}{2}AT$ to approximate

$$\psi(AT) \triangleq I + \frac{1}{2!}AT + \frac{1}{3!}A^2T^2 + \ldots$$

and hence approximate $G^D(\gamma)$ when $T = 0.1$ second. [6]

(b) Consider the system of Figure 4 below, where $G^Z(z) = \frac{z-0.5}{z-1}$, $r_k = 2 \times 1^k$, $d_k = 1^k$ and f is a scalar gain. Use the Final Value Theorem to determine whether y_k converges to a constant for some f, and determine the constant if such convergence takes place. [8]

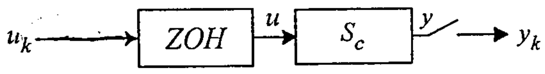


Figure 3

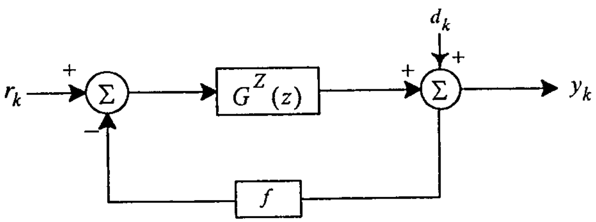


Figure 4

4. Consider the system of Figure 5 below, where $K > 0$.

(a) Suppose $G^Z(z) = \frac{z(z+1.5)}{(z-1.2)^2}$.

Draw the root-locus and determine from it the range of values of the gain K for which the closed-loop system is BIBO-stable,
perhaps using the fact that $G^Z(0.604+0.797j) \approx -2.272$. **[6]**
Confirm your results using the Jury test. **[4]**

(b) Suppose $G^Z(z) = \frac{(z+1)^4}{z^4}$ and $T = 2$ seconds.
Apply the W-transform followed by continuous-time Nyquist analysis to determine the range of values of K for which the closed-loop system is BIBO-stable, perhaps making use of the fact that $(1+j)^4 = -4$. **[6]**
Confirm your results using the Routh test. **[4]**

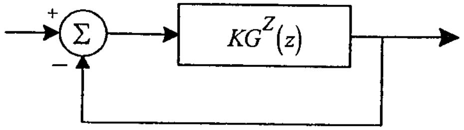


Figure 5

5. (a) Consider the system and observer defined below, where ′ denotes transposition:

System: $x_{k+1} = Ax_k + bu_k,\ y_k = c'x_k$
Observer: $\hat{x}_{k+1} = (A - \ell c')\hat{x}_k + \ell y_k + bu_k$

$$A = \begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix},\ b = \begin{bmatrix} 1 \\ 2 \end{bmatrix},\ c' = [1 \quad 1],\ \ell \in \mathbb{R}^2.$$

(i) Transform A' to companion form, using a transformation derived from the last row of the inverse of the relevant controllability matrix. **[9]**

(ii) Hence determine ℓ such that the eigenvalues associated with the observer are both zero. **[4]**

(b) Consider the system of Figure 6. Suppose $G^Z(z) = \frac{z-0.5}{(z-1)(z+0.5)}$ and $K > 0$.
A plot of $G^Z(e^{j\theta})$ as θ varies from 0.1 to 6.183 radians is shown in Figure 7, where the arrows indicate the movement of $G^Z(e^{j\theta})$ as θ increases from 0.1 radians.

Scale the real axis of the plot by evaluating $G^Z(-1)$. Use discrete-time Nyquist analysis and the plot to determine the range of values of $K > 0$ for which the closed-loop system is BIBO-stable. Give sufficient explanation to make your method clear. **[7]**

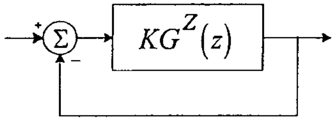


Figure 6

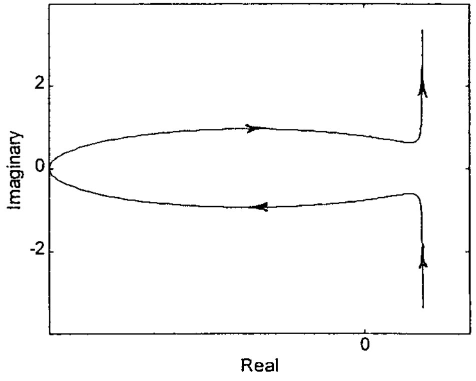


Figure 7

6. (a) Suppose it is required that $x_k \to 0$ for the control system consisting of

Plant: $x_{k+1} = Ax_k + bu_k,\ y_k = c'x_k$

Observer: $\widehat{x}_{k+1} = (A-\ell c')\widehat{x}_k + \ell y_k + bu_k$

Controller: $u_k = f'\widehat{x}_k$

where $x_k,\ \widehat{x}_k,\ b,\ c,\ f, \ell \in \mathbb{R}^n$ and $'$denotes transposition.

Suppose the eigenvalues of $A+bf'$, and of $A-\ell c'$, are all zero.

(i) By obtaining a difference equation for $e_k = x_k - \widehat{x}_k$ and using a companion form, show that $\widehat{x}_k = x_k$ for $k \geq n$ **[8]**

(ii) Use another companion form and the result of part (i) to show that $x_k = 0$ for all $k \geq 2n$. **[4]**

(b) Consider the second-order system with output y_k,

$$\begin{bmatrix} y_{k+1} \\ w_{k+1} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_k \\ w_k \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u_k$$

with the following reduced-order observer of ω_k:

$$\widehat{\omega}_k = v_k + hy_k$$

where

$$v_{k+1} = \ell v_k + my_k + nu_k.$$

Here $h,\ \ell,\ m,\ n \in \mathbb{R}$ and

$$\ell = \mathrm{a}_{22} - ha_{12} \text{ with } |\ell| < 1$$

$$m = a_{21} - ha_{11} + \ell h$$

$$n = b_2 - hb_1.$$

Let $e_k = w_k - \widehat{\omega}_k \in \mathbb{R},\ x_k = [y_k \ \ w_k]'$ and $\widehat{x}_k = [y_k \ \ \widehat{w}_k]'$.

(i) Show that $e_k = \ell^k e_0$ and that, as $k \to \infty$, $e_k \to 0$. Show also that $\widehat{x}_k - x_k \to 0$ as $k \to \infty$. **[5]**

(ii) By considering w_k in terms of e_k and $\widehat{w}_k$, show that the pulse Z-transfer function from $u^Z(z)$ to $\widehat{w}^Z(z)$ is equal to that from $u^Z(z)$ to $w^Z(z)$. **[3]**

1 (a) $\{x_{k+1}\} = \alpha\{x_k\}$ so $z x^z(z) - z x_0 = \alpha x^z(z)$ so $x^z(z) = (z-\alpha)^{-1} z x_0$.

Taking Z^{-1}: $x_k = \alpha^k x_0 = \alpha^k$.

Taking $\alpha = -1$ gives $x_k = (-1)^k$.

Hence $Z\{(-1)^k\} = \frac{z x_0}{(z-(-1))} = \frac{z}{z+1}$.

(b)
$$z^2(x^z(z) - \underset{\underset{0}{\|}}{x_0} - z^{-1}\underset{\underset{2\beta}{\|}}{x_1}) = -\beta^2 x^z(z)$$

$$\Rightarrow (z^2+\beta^2)x^z(z) = 2\beta z \quad \Rightarrow x^z(z) = \frac{2\beta z^2}{z^2+\beta^2} = \frac{2\beta z}{(z+j\beta)(z-j\beta)}$$

$$\Rightarrow \frac{x^z(z)}{z} = \frac{2\beta}{(z+j\beta)(z-j\beta)} = \frac{2\beta}{-2j\beta}\cdot\frac{1}{z+j\beta} + \frac{2\beta}{2j\beta}\cdot\frac{1}{z-j\beta}$$

$$= \frac{1}{j}\left(\frac{1}{z-j\beta} - \frac{1}{z+j\beta}\right)$$

$$\Rightarrow x^z(z) = \frac{1}{j}\left(\frac{z}{z-j\beta} - \frac{z}{z+j\beta}\right) \Rightarrow x_k = \frac{1}{j}[(j\beta)^k - (-j\beta)^k]$$

$$= \frac{1}{j}\beta^k[e^{j\frac{\pi}{2}k} - e^{-j\frac{\pi}{2}k}]$$

$$= 2\beta^k \sin(k\tfrac{\pi}{2}).$$

(c) (i) For $x = \begin{pmatrix} y \\ \dot{y}\end{pmatrix}$:
$$\dot{x} = \underbrace{\begin{pmatrix} 0 & 1 \\ \alpha & \beta \end{pmatrix}}_{A \underset{\boxed{\text{Spectral form}}}{=} V\Lambda V^{-1}} x + \begin{pmatrix} 0 \\ 1\end{pmatrix} u$$

(ii) Then $x_{k+1} = e^{AT} x_k + \int_0^T e^{A\theta} d\theta \, B u_k$ ($)

$\| \; V\begin{pmatrix} e^{\lambda_1 T} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n T}\end{pmatrix}V^{-1}$ so the eigenvalues associated with ($) are $e^{\lambda_i T} = \bar{\lambda}_i$.

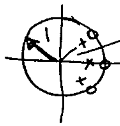
(iii) The poles of S_d are a subset of the eigenvalues of e^{AT} so S_d is BIBO-stable if $|e^{\lambda_i T}| < 1, \forall i$.

Now $\lambda_i \in \{\lambda \in \mathbb{C}: Re(\lambda) < 0\} \Rightarrow \lambda_i = \sigma_i + j\omega_i$ for some $\sigma_i < 0$

$$\Rightarrow |e^{\lambda_i T}| = |e^{\sigma_i T} e^{j\omega_i T}| = \underbrace{|e^{\sigma_i T}|}_{\boxed{<1}}\underbrace{|e^{j\omega_i T}|}_{\underset{1}{\|}} < 1, \forall i, \forall T > 0.$$

$\Rightarrow S_d$ = BIBO-stable

2 (a)



angle $\theta = \omega T = \frac{(2\pi 50)}{300}\frac{\cancel{0.0105}}{\pi} = 1.05$ rad.

For this filter, $G^z(z) = \frac{(z-1)(z-e^{j\theta})(z-e^{-j\theta})}{(z-0.9)(z-0.9e^{j\theta})(z-0.9e^{-j\theta})}$

$$= \frac{(z-1)(z^2 - z(e^{j\theta}+e^{-j\theta})+1)}{(z-0.9)(z^2-0.9(e^{j\theta}+e^{-j\theta})z+0.81)} = \frac{(z-1)(z^2-z+1)}{(z-0.9)(z^2-0.9z+0.81)}$$

$$= \frac{z^3-z^2+z-z^2+z-1}{z^3-0.9z^2+0.81z-0.9z^2+0.81z-0.729)} = \frac{z^3-2z^2+2z-1}{z^3-1.8z^2+1.62z-0.729}.$$

(b)
$$y^z(z) = (1-2z^{-1}+2z^{-2}-z^{-3})\underbrace{\left(\frac{u^z(z)}{1-1.8z^{-1}+1.62z^{-2}-0.729z^{-3}}\right)}_{w^z(z)}$$

so

$y_k = w_k - 2w_{k-1} + 2w_{k-2} - w_{k-3}$ ($)

where

$(1-1.8z^{-1}+1.62z^{-2}-0.729z^{-3})w^z(z) = u^z(z)$

i.e.

$w_k = 1.8w_{k-1} - 1.62w_{k-2} + 0.729w_{k-3} + u_k$ (£).

the canonical direct realization

(c) (i) One would expect that after the transients have died away
$$y_k = |G^z(e^{j\omega T})|\cos[\omega t_k + \measuredangle G^z(e^{j\omega T})]$$

When $\omega = 0$: $G^z(e^{j\omega T}) = G^z(1) = 0$ so $y_k = 0$

(ii) Also $\cos(\omega t_k) = 1, \forall k \geq 0$, so $u^z(z) = \frac{z}{z-1}$.

Hence $y^z(z) = \frac{\cancel{z-1}}{z^2-0.25}\cdot\frac{z}{\cancel{z-1}} = \frac{z}{(z-0.5)(z+0.5)}$

so $y_0 = \underset{|z|\to\infty}{L t}\, y^z(z) = 0$. For $k > 0$: $y_k = \frac{1}{2\pi j}\oint \underbrace{G^z(z)z^{k-1}}_{\frac{z^k}{(z-0.5)(z+0.5)}} dz$

so
$$y_k = \text{residue}\left[\frac{z^k}{(z-.5)(z+.5)} @ z = -0.5\right]$$
$$+ \text{residue}\left[\frac{z^k}{(z-.5)(z+.5)} @ z = 0.5\right]$$
$$= (0.5)^k - (-0.5)^k$$

Hence $\{y_k\} = \{0, 0.5+0.5, 0, 0.125+0.125, 0, \ldots\}$
$= \{0, 1, 0, 0.25, \ldots\}$

agrees

By long division:
$$z^2 - 0.25 \,\overline{\smash{)}\, z} \quad \text{quotient: } z^{-1} + 0.25z^{-3} + \cdots$$
$$z - 0.25z^{-1}$$
$$0.25z^{-1}$$

$\Rightarrow \{y_k\} = \{0, 1, 0, .25, ..\}$

Consistency: here we see the ~~transient~~ which dies to zero as $k \to \infty$.

3 (a) (i) Step response is $\mathcal{L}^{-1}\left[\frac{1}{s^2(1+s)}\right](t) = \mathcal{L}^{-1}\left[-\frac{1}{s}+\frac{1}{s^2}+\frac{1}{s+1}\right](t)$

$= -1 + t + e^{-t}$

The sampled step response is $-1 + kT + e^{kT}$

which has the z-transform $\frac{-z}{z-1} + \frac{Tz}{(z-1)^2} + \frac{z}{z-e^{-T}}$

Hence the pulse z-transfer fn is $\left(\frac{z-1}{z}\right)\left[\frac{-z}{z-1}+\frac{Tz}{(z-1)^2}+\frac{z}{z-e^{-T}}\right]$

$= -1 + \frac{T}{z-1} + \frac{(z-1)}{(z-e^{-T})} \triangleq G^z(z)$

Then

$G^D(\gamma) = G^z(1+\gamma T) = -1 + \frac{T}{1+\gamma T-1} + \frac{1+\gamma T-1}{1+\gamma T-e^{-T}}$

$= -1 + \frac{T}{\gamma} + \frac{\gamma T}{1+\gamma T-e^{-T}}$.

(ii) $\Psi(AT) \simeq I + \frac{1}{2}AT = \begin{bmatrix}1&0\\0&1\end{bmatrix} + 0.05\begin{bmatrix}0&1\\0&-2\end{bmatrix} = \begin{bmatrix}1&0.05\\0&0.9\end{bmatrix}$

$G^D(\gamma) = [1\ \ 0]\left[\gamma I - \underbrace{\Psi(AT)A}_{\parallel}\right]^{-1}\underbrace{\Psi(AT)B}$

$\begin{pmatrix}1&0.05\\0&0.9\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0.05\\0.9\end{pmatrix} = \tilde{B}$

$\begin{pmatrix}1&0.05\\0&0.9\end{pmatrix}\begin{pmatrix}0&1\\0&-2\end{pmatrix} = \begin{pmatrix}0&0.9\\0&-1.8\end{pmatrix} = \tilde{A}$

$= [1\ \ 0]\begin{bmatrix}\gamma&-0.9\\0&\gamma+1.8\end{bmatrix}^{-1}\begin{bmatrix}0.05\\0.9\end{bmatrix} = [1\ \ 0]\frac{\begin{bmatrix}\gamma+1.8&0.9\\0&\gamma\end{bmatrix}}{\gamma(\gamma+1.8)}\begin{bmatrix}0.05\\0.9\end{bmatrix}$

$= \frac{[\gamma+1.8\ \ \ 0.9]}{\gamma(\gamma+1.8)}\begin{bmatrix}0.05\\0.9\end{bmatrix} = \frac{0.05\gamma+0.9}{\gamma(\gamma+1.8)}$

(b) $y^z(z) = \frac{z}{z-1} + \left(\frac{z-0.5}{z-1}\right)\left[\frac{2z}{z-1} - fy^z(z)\right]$

$\Rightarrow \left(1+f\left(\frac{z-0.5}{z-1}\right)\right)y^z(z) = \frac{z}{z-1} + \frac{2z(z-0.5)}{(z-1)^2}$

$\Rightarrow$

$(z-1+f(z-0.5))y^z(z) = z + 2z\frac{(z-0.5)}{(z-1)}$

$\Rightarrow y^z(z) = \frac{z+2z\frac{(z-0.5)}{z-1}}{z-1+f(z-0.5)}$

$\Rightarrow (z-1)y^z(z) = \frac{z(z-1)+2z(z-0.5)}{(1+f)z-(1+0.5f)} = \left(\frac{1}{1+f}\right)\frac{z(z-1)+2z(z-0.5)}{z-\underbrace{\frac{(1+0.5f)}{(1+f)}}_{p_1(f)}}$

$= \frac{3z^2-2z}{(1+f)z-(1+0.5f)}$.

Here $|p_1(f)|<1$ if $f>0$.

Hence $y_k \to y_\infty = \left(\frac{1}{1+f}\right)\cdot\frac{1}{1-p_1(f)} = \frac{2}{f}$.

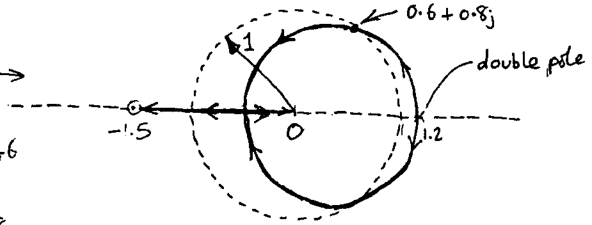
4 (a) Breakpoints: $\frac{1}{\sigma} + \frac{1}{\sigma+1.5} = \frac{2}{\sigma-1.2}$ i.e. $(\sigma+1.5)(\sigma-1.2)+\sigma(\sigma-1.2) = 2\sigma(\sigma+1.5)$

i.e. $(\sigma-1.2)(2\sigma+1.5) = 2\sigma(\sigma+1.5)$ i.e. $\cancel{2\sigma^2} - 2.4\sigma + 1.5\sigma - 1.8 = \cancel{2\sigma^2} + 3\sigma$

i.e. $-1.8 = 3.9\sigma$

i.e. $\sigma = -0.4615$.

Hence the root-locus is →



Therefore

$K_{min} = \frac{-1}{G^z(0.6+0.8j)} = 0.446$

$K_{max} = \frac{-1}{G^z(-1)} = \frac{-1}{\frac{-0.5}{(2.2)^2}} = 9.68$

Hence BIBO-stable for $\underbrace{0.446}_{\text{approx}} < K < 9.68$.

Jury: CL denominator is $d(z) = (z-1.2)^2 + Kz(z+1.5)$

$d(1) = 0.04 + 2.5K > 0$ for all $K > 0$

$d(-1) = (-2.2)^2 - K\times 0.5 = 4.84 - K/2 > 0$ if $K < 9.68$

Expanding $d(z)$: $d(z) = (1+K)z^2 + z(-2.4+1.5K) + 1.44$

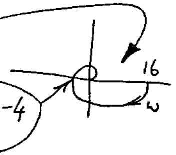
Hence row 1 of Jury table is: $1.44 \quad -2.4+1.5K \quad 1+K$.

Hence BIBO-stable iff also $1.44 < 1+K$, i.e. if $K>0.44$, i.e (finally) if $0.44 < K < 9.68$

consistent

(b) $G^w(w) = \left(\frac{z+1}{z}\right)^4\Big|_{z=\frac{\mu+w}{\mu-w}}$ $\boxed{\mu = 2/T = 1}$ $= \frac{\left(\frac{1+w}{1-w}+1\right)^4}{\left(\frac{1+w}{1-w}\right)^4} = \frac{\left(\frac{2}{1-w}\right)^4}{\left(\frac{1+w}{1-w}\right)^4} = \frac{16}{(1+w)^4}$

$G^w(jw) = \frac{16}{(1+jw)^4}$



$\frac{16}{(1-j)^4} = \frac{16}{-4} = -4$

$\Longrightarrow$ BIBO-stable for $0<K<0.25$

Routh Test: CL denom is $z^4 + K(z+1)^4$. Hence denom. for CL W-transform is $(1-w)^4\left[\left(\frac{1+w}{1-w}\right)^4 + K\left(\frac{2}{1-w}\right)^4\right] = (1+w)^4 + 16K$
$= w^4 + 4w^3 + 6w^2 + 4w + (16K+1)$.

Hence Routh Table is:

1.	1	6	16K+1
2.	4	4	0
3	5	16K+1	0
4	$\frac{20-4(16K+1)}{5}$	0	
5		(16K+1)	

$\Rightarrow$ all entries in col 1 are >0 if

$20 - 4(16K+1) > 0$

i.e. if $16 > 4\times 16K$

i.e. if $K < \frac{1}{4} = 0.25$

Hence system is BIBO-stable if $0<K<0.25$, consistent with the above Nyquist analysis

5 (a) Since the eigenvalues of $A-lc'$ are those of $(A-lc')'$, they can be assigned to zero by assigning those of $A'-cl'$ to zero - which can be done using the following procedure for choosing l!

$A' = \begin{bmatrix} 2 & -1 \\ 1 & -2 \end{bmatrix}$, $c = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The controllability matrix for (A', c)

is $M = [c \;\; A'c] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Therefore $M^{-1} = \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}/(-2) = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

Let p' be the last row of M^{-1}, so $p' = [\frac{1}{2} \;\; -\frac{1}{2}]$.

Then $p'A' = [\frac{1}{2} \;\; -\frac{1}{2}]\begin{bmatrix} 2 & -1 \\ 1 & -2 \end{bmatrix} = [\frac{1}{2} \;\; \frac{1}{2}]$ so $V = \begin{bmatrix} p' \\ p'A' \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

and $V^{-1} = \underbrace{\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}}_{\frac{1}{4}+\frac{1}{4}} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$.

Then A' is similar to the companion matrix

$$VA'V^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}\underbrace{\begin{bmatrix} 2 & -1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}}_{\begin{bmatrix} 3 & 1 \\ 3 & -1 \end{bmatrix}} = \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix}.$$

Then $V(A'-cl')V^{-1} = \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} l'V^{-1}$ so choose $l'V^{-1} = [3 \;\; 0]$

so $l' = [3 \;\; 0]V = [3 \;\; 0]\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = [\frac{3}{2} \;\; -\frac{3}{2}]$

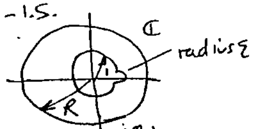
Therefore the l required is $l = \frac{1}{2}\begin{bmatrix} 3 \\ -3 \end{bmatrix}$.

<u>Check</u> $A - lc' = \begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix} - \begin{bmatrix} \frac{3}{2} \\ -\frac{3}{2} \end{bmatrix}[1 \;\; 1] = \begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix} + \begin{bmatrix} -\frac{3}{2} & -\frac{3}{2} \\ \frac{3}{2} & \frac{3}{2} \end{bmatrix}$

$= \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$ so $|\lambda - (A-lc')| = \left|\begin{pmatrix} \lambda-\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \lambda+\frac{1}{2} \end{pmatrix}\right| = (\lambda-\frac{1}{2})(\lambda+\frac{1}{2}) + \frac{1}{2}$
$= (\lambda-0)(\lambda-0)^2$. OK

(b) $G^z(-1) = \frac{-1.5}{(-2).(-0.5)} = -1.5.$

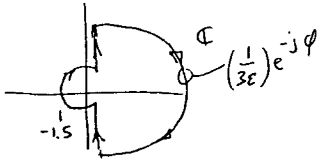
Nyquist path :



$R = \infty$

For $z = 1 + \varepsilon e^{j\varphi}$: $G^z(1+\varepsilon e^{j\varphi}) = \frac{1+\varepsilon e^{j\varphi} - 0.5}{\varepsilon e^{j\varphi}\,(1.5+\varepsilon e^{j\varphi})} \simeq \frac{0.5}{1.5\varepsilon} e^{-j\varphi}$
$= \frac{1}{(3\varepsilon)} e^{-j\varphi}.$

Hence Nyquist locus is



Since #(Q poles in unit disc) = 0, system is BIBO-stable for $-\frac{1}{K} < -1.5$, i.e. for $K < 0.666$.

6. (a) Let $e_k = x_k - \hat{x}_k$.

Then $e_{k+1} = x_{k+1} - \hat{x}_{k+1} = Ax_k + \cancel{bv_k} - (A-lc')\hat{x}_k - l\overset{c'x_k}{\overset{\|}{y_k}} - \cancel{bv_k}$
$= (A-lc')x_k - (A-lc')\hat{x}_k = (A-lc')(x_k - \hat{x}_k)$
$= (A-lc')e_k,$

so $e_k = (A-lc')^k e_0.$

Since the eigenvalues of $A-lc'$ are all 0, $A-lc'$ is similar to C_0, which is the companion matrix that has every entry in its last row equal to 0. For $n=3$: $C_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $C_0^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$,

$C_0^3 = \underset{C_0}{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}\underset{C_0^2}{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

for $C_0 \in R^{n \times n}$: $C_0^k = 0,\ \forall k \geq n.$

Hence $A-lc' = VC_0V^{-1}$ and $(A-lc')^k = VC_0^kV^{-1}$.

Therefore $(A-lc')^k = 0,\ \forall k \geq n.$

Hence $e_k = (A-lc')^k e_0 = 0,\ \forall k \geq n.$

Consequently $x_k - \hat{x}_k = e_k = 0,\ \forall k \geq n.$

Therefore the controlled system $x_{k+1} = Ax_k + bf'\hat{x}_k$ acts like $x_{k+1} = Ax_k + bf'x_k = (A+bf')x_k$ for $k \geq n$. Hence $x_{n+m} = (A+bf')^m x_n$.

Since $A+bf'$ has all its eigenvalues equal to zero, much as above we have $(A+bf') = VC_0V^{-1}$. Hence $(A+bf')^k = 0,\ \forall k \geq n$

Therefore $x_{n+m} = (A+bf')^m x_k = 0,\ \forall m \geq n,$
i.e. $x_k = 0,\ \forall k \geq 2n.$

$e_{k+1} = w_{k+1} - \hat{w}_{k+1} = a_{21}y_k + a_{22}w_k + b_2v_k - v_{k+1} - hy_{k+1}$
$= \underline{a_{21}y_k} + \utilde{a_{22}w_k} + \underline{\underline{b_2v_k}} - lv_k \underline{-my_k} \underline{\underline{-nv_k}} \underline{-ha_{11}y_k} \utilde{-ha_{12}w_k} \underline{\underline{-hb_1v_k}}$
$= y_k(a_{21} - \underset{a_{21}-ha_{11}+lh}{\overset{\|}{m}} - ha_{11}) + w_k\underbrace{(a_{22}-ha_{12})}_{l} + v_k(b_2 - \underset{b_2-hb_1}{\overset{\|}{n}} - hb_1) - lv_k$
$= -lv_k + l(w_k - hy_k) = l(w_k - [v_k + hy_k]) = l(w_k - \hat{w}_k) = le_k$

Hence $e_k = l^k e_0 \to 0$ as $k \to \infty$ since $|l| < 1$, so $w_k - \hat{w}_k \to 0$.

Therefore $x_k - \hat{x}_k \to 0.$

Now $w_k = \hat{w}_k + e_k$ ← indep. of v_k so the transfer fn from $v^z(z)$ to $w^z(z)$ is the same as that from $v^z(z)$ to $\hat{w}^z(z)$.