

MATHEMATICS FOR SIGNALS AND SYSTEMS

1. Let $A \in \mathbb{R}^{n \times m}$ be a matrix with n rows and m columns whose entries are real numbers. We assume that $n \geq m$ and that A has rank m .

We assume that we know the QR -decomposition of the matrix A , i.e. there exist two matrices $Q \in \mathbb{R}^{n \times n}$ and $R \in \mathbb{R}^{n \times m}$ such that $A = QR$, Q is orthogonal ($Q^T Q = I$ where I is the identity matrix), and R is an upper triangular matrix (the upper $m \times m$ block of R is upper triangular and the rest of the entries of R are equal to zero). More precisely

$$R = \begin{pmatrix} r_{11} & r_{12} & r_{13} & \dots & r_{1m} \\ 0 & r_{22} & r_{23} & \ddots & \vdots \\ \vdots & \ddots & r_{33} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & r_{mm} \\ 0 & \dots & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & \dots & 0 \end{pmatrix}$$

In this problem the goal is to devise an algorithm to derive the QR -decomposition of the matrix $\tilde{A} \in \mathbb{R}^{(n+1) \times m}$ obtained by adding a row to the matrix A . More precisely,

$$\tilde{A} = \begin{pmatrix} A_1 \\ z^T \\ A_2 \end{pmatrix} \quad \text{where} \quad A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

with $A_1 \in \mathbb{R}^{n_1 \times m}$, $A_2 \in \mathbb{R}^{n_2 \times m}$, $n_1 + n_2 = n$, and $z \in \mathbb{R}^m$.

- a) Let us also assume that

$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} R,$$

where Q_1 and Q_2 have the same number of rows as A_1 and A_2 respectively.

- i) Show that

$$\tilde{A} = \begin{pmatrix} \mathbf{0} & Q_1 \\ 1 & \mathbf{0}^T \\ \mathbf{0} & Q_2 \end{pmatrix} \begin{pmatrix} z^T \\ R \end{pmatrix}, \quad (1.1)$$

where $\mathbf{0}$ represent zero vectors of the appropriate dimensions. [2]

- ii) Show that $\hat{Q} = \begin{pmatrix} \mathbf{0} & Q_1 \\ 1 & \mathbf{0}^T \\ \mathbf{0} & Q_2 \end{pmatrix}$ is an orthogonal matrix. [3]

- iii) Is (1.1) a QR decomposition? Justify your answer. [2]

b) We define the matrix U_1 as follows, for $\theta \in \mathbb{R}$

$$U_1 = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 & \dots & 0 \\ \sin(\theta) & \cos(\theta) & 0 & \ddots & \vdots \\ 0 & 0 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

- i) Show that U_1 is orthogonal. [1]
- ii) Find a value of θ such that $U_1^T \begin{pmatrix} z^T \\ R \end{pmatrix}$ has 0 in the entry on its second row and first column. [2]
- iii) More generally show that there is a sequence of orthogonal matrices U_1, \dots, U_m such that $\tilde{R} = U_m^T \dots U_1^T \begin{pmatrix} z^T \\ R \end{pmatrix} \in \mathbb{R}^{(n+1) \times m}$ is upper triangular. [2]
- iv) Show that $\tilde{Q} = \hat{Q}U_1 \dots U_m$ is an orthogonal matrix. [2]
- v) Describe an algorithm that updates the QR of a matrix if a row is added to it. [2]
- vi) Is the procedure described in b) v) more efficient than performing the QR decomposition of \tilde{A} from scratch, i.e. without relying on an update of the QR decomposition of A ? Justify your answer carefully. [4]

2. Define $\|x\| = \sqrt{x^T x}$.

a) Given a vector x and an orthogonal projection P , i.e. $P^2 = P$ and $P^T = P$.

i) Show that Px and $x - Px$ are orthogonal. [2]

ii) Show that $\|x\|^2 = \|Px\|^2 + \|x - Px\|^2$. [2]

b) Let U a subspace of \mathbb{R}^d of dimension k and let u_1, \dots, u_k an orthonormal basis of U , i.e. $u_i^T u_i = 1$, and for all $i \neq j$ $u_i^T u_j = 0$. Consider the matrix

$$P_U = \sum_{i=1}^k u_i u_i^T = u_1 u_1^T + u_2 u_2^T + \dots + u_k u_k^T$$

i) Show that P_U is an orthogonal projection and derive its range and null-space. [3]

ii) Let a_1, \dots, a_n be n vectors in \mathbb{R}^d . Show that

$$\sum_{j=1}^n \|P_U a_j\|^2 = \sum_{j=1}^n \sum_{i=1}^k (u_i^T a_j)^2.$$

Hint: start by showing that for a given vector $x \in \mathbb{R}^d$ we have that $\|P_U x\|^2 = \sum_{i=1}^k (u_i^T x)^2$. [2]

c) We say that a subspace U of dimension k is the *best-fit k -dimensional subspace* if it maximizes the sum of the squared lengths of the orthogonal projections onto it, i.e. the subspace U such

$$\sum_{j=1}^n \|P_U a_j\|^2 = \max_{\substack{V \text{ subspace of } \mathbb{R}^d \\ \dim(V)=k}} \sum_{i=1}^n \|P_V a_i\|^2.$$

Using questions a.ii show that the subset U that maximize the sum of the squared lengths of the projections onto the subspace does also minimize the sum of squared distances to the subspace. [2]

d) Let $A \in \mathbb{R}^{n \times n}$. In what follows we will describe a procedure for deriving the singular vectors of A . Let a_1^T, \dots, a_n^T the vectors representing the rows of A (a_i^T being the i th row of A).

i) Show that for any vector $v \in \mathbb{R}^n$ we have

$$\sum_{i=1}^n (a_i^T v)^2 = \|Av\|^2. \quad [2]$$

ii) Let $v_1 \in \mathbb{R}^n$ be such that $Av_1 = \max_{\|v\|=1} \|Av\|$. Using c and d.i, show that for all $v \in \mathbb{R}^n$

$$\sum_{i=1}^n \|a_i - (v_1^T a_i)v\| \geq \sum_{i=1}^n \|a_i - (v_1^T a_i)v_1\|. \quad [2]$$

e) We now define a greedy procedure for deriving the singular vectors of A . Recall that

$$Av_1 = \max_{\|v\|=1} \|Av\|.$$

Let $v_2 \in \mathbb{R}^n$ such that

$$Av_2 = \max_{\|v\|=1, v^T v_1=0} \|Av\|.$$

Similarly let $v_3 \in \mathbb{R}^n$ such that

$$Av_3 = \max_{\|v\|=1, v^T v_1=0, v^T v_2=0} \|Av\|.$$

We stop the process when we found vectors v_1, v_2, \dots, v_r as singular vectors and

$$\max_{\|v\|=1, v^T v_1=0, \dots, v^T v_r=0} \|Av\| = 0.$$

Using an induction show that the subspace $\text{Span}\{v_1, \dots, v_r\}$ spanned by $\{v_1, \dots, v_r\}$ is the best-fit k -dimensional subspace for the vectors a_1, \dots, a_n . [5]

3. Let $\mathbb{R}[X]$ be the vector space of polynomials with real coefficients, and $\mathbb{R}_n[X]$ be the subspace of polynomials with degree smaller or equal to n . For P and Q in $\mathbb{R}[X]$, we define

$$\langle P, Q \rangle = \int_{-1}^1 P(x)Q(x) \frac{1}{\sqrt{1-x^2}} dx.$$

- a) Define $T_k(x)$ the polynomials such that, for $k \geq 1$ and $\theta \in (0, \pi)$

$$T_k(\cos(\theta)) = \cos(k\theta), \quad T_0 = 1,$$

known as Chebyshev's polynomials.

- i) Give the expressions of T_1 , T_2 and T_3 . [3]
 ii) Show that, for $k \geq 1$, we have

$$T_{k+1} = 2XT_k - T_{k-1}.$$

[3]

- iii) Using the change of variable $\theta = \arccos(x)$, compute $\langle T_n, T_m \rangle$, when $n = m$ and $n \neq m$. [3]
 iv) Derive an orthonormal basis for $\mathbb{R}_3[X]$. Justify your answer. [2]

- b) Consider the application on $\mathbb{R}_n[X]$ defined by

$$D(P) = (1 - X^2)P'' - XP'$$

where $P \in \mathbb{R}_n[X]$, P' and P'' are its first and second derivatives respectively.

- i) Using that the fact that $T_k(\cos(\theta)) = \cos(k\theta)$, show that

$$-\cos(\theta)T'_k(\cos(\theta)) + \sin(\theta)^2 T''_k(\cos(\theta)) = -k^2 T_k(\cos(\theta)).$$

[3]

- ii) Show that $D(T_k) = -k^2 T_k$. [3]
 iii) Derive the eigenvalues and eigenvectors of the transformation D on $\mathbb{R}_n[X]$. [3]