Master

E3.07
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ISE3.11

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2004

EEE/ISE PART III/IV: MEng, BEng and ACGI

DIGITAL SIGNAL PROCESSING

Wednesday, 12 May 10:00 am

Time allowed: 3:00 hours

There are SIX questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : P.A. Naylor
Second Marker(s) : D.P. Mandic

Special Instructions for Invigilators: None

Information for Candidates:

Sequence	z-transform
$\delta(n)$	1
$u(n)$	$\dfrac{1}{1-z^{-1}}$
$a^n u(n)$	$\dfrac{1}{1-az^{-1}}$
$\left(r^n \cos\omega_0 n\right)u(n)$	$\dfrac{1-(r\cos\omega_0)z^{-1}}{1-(2r\cos\omega_0)z^{-1}+r^2z^{-2}}$

Table 1 : z-transform pairs

$\delta(n)$ is defined to be the unit impulse function.
$u(n)$ is defined to be the unit step function.

Numbers in square brackets against the right margin of the following pages are a guide to the marking scheme.

1.

(a) Give a summary description of the processing steps of the overlap-add block filtering method for convolution of a time series $x(n)$ with a system impulse response $h(n)$. Be precise in your description and include diagrams where appropriate. [8]

(b) Use the overlap-add block filtering method to calculate the output of a filter with impulse response [12]

$$h(n) = [1, 0, 1]$$

and input signal

$$x(n) = [1, 2, 3, -3, -2, -1, 0, -1, -2, -3, -2, -1, 0, 3, 2, 1, \ldots].$$

Use only the first 2 data blocks to calculate the output. Show your method clearly and include any relevant diagrams.

[Hint: your solution must employ, and be illustrative of, the overlap-add block filter approach but consider carefully how the internal calculations can be performed most efficiently for this particular case of $h(n)$.]

2. Let $X(k)$ be the DFT of the real discrete-time sequence $x(n)$ of length N.

(a) Describe what is meant by the term basis function in the context of the DFT. Write down the basis functions corresponding to a 4-point DFT and compute the samples of each basis function. [5]

(b) Show that $X(k)$ can be real under certain conditions on $x(n)$. State these conditions and give an illustrative example. [5]

(c) If $x(n)$ satisfies $x(n) = -x(M+n)$ with $N = 2M$, show that $X(2l) = 0$ for $l = 0, 1, \ldots, M-1$. [5]

(d) State any conditions on the magnitude and phase of $X(k)$ that must be satisfied for $x(n)$ to be real and draw a labelled sketch of the magnitude and phase of a non-trivial example of such a function $X(k)$. [5]

3.

(a) Figure 1 shows multirate processing blocks for decimation and expansion. Derive expressions for $Y_D(e^{j\omega})$ and $Y_E(e^{j\omega})$ in terms of $X(e^{j\omega})$, M and L. Describe the effect of each multirate processing block in the frequency domain using one or two sentences and an illustrative diagram for each. [6]

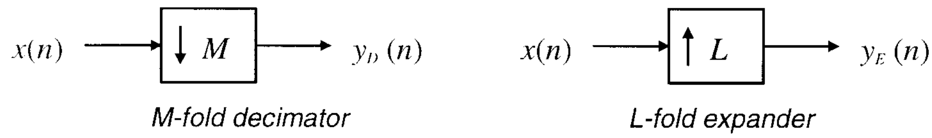

Figure 1

(b) Write down an expression for $Y(z)$ in terms of $X(z)$ for the discrete-time system of Figure 2. [7]

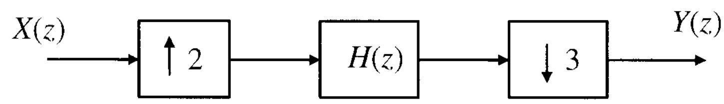

Figure 2

Part of the magnitude spectrum corresponding to $X(z)$ is shown in Figure 3. Draw a labelled sketch of the magnitude spectrum corresponding to $Y(z)$ over the range of frequencies $-2\pi < \omega < 6\pi$. Briefly explain the main features of your sketch.

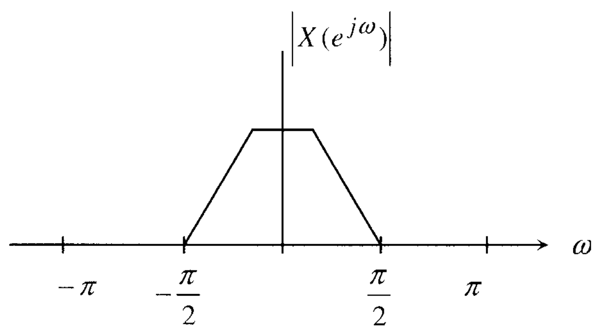

Figure 3

c) Draw and label the implementation of the structure of Figure 2 using Type 1 polyphase decomposition. State any advantages or disadvantages of your polyphase implementation compared to the structure of Figure 2. [7]

4.

(a) Draw a labelled sketch covering the range $-2\pi < \omega < 2\pi$ of the magnitude response of a discrete-time ideal highpass filter with a cut-off frequency of $P/4$. [4]

Write down an expression for the impulse response of this ideal filter and state why it is not suitable in practical implementation. [6]

(b) Let $H(z)$ denote the transfer function of a digital filter. Formulate a general expression in terms of a and b for $H(z)$ having two zeros [10]

$$z_1 = a + jb \text{ and } z_2 = a - jb .$$

Write a modified expression to satisfy the condition that the two zeros lie on the unit circle and for this case draw labelled sketch-graphs of $\left|H(e^{j\omega})\right|$ as a function of parameters a and b for $\omega = \pi/2$.

Hence, deduce and sketch the magnitude response of $H(z)$ when

$$a = 0.5 \text{ and } b = \sin(P/3)$$

and compute the output sequence, $y(n)$, of this filter for the input sequence

$$x(n) = \sin\left(\frac{n\pi}{3}\right), \quad n = 0,\ 1,\ 2,\ \ldots,\ 9 .$$

5.

(a) State the principal characteristics of IIR filters and the advantages and disadvantages of IIR filters compared to FIR filters. [5]

(b) Describe in detail the bilinear transform method for IIR filter design for a lowpass filter. [7]

(c) Consider the following transfer function $H(s)$ for a continuous time notch filter for which the notch frequency is 1 rad/s. [8]

$$H(s) = \frac{1+s^2}{1+s+s^2}.$$

Using $H(s)$ and the bilinear transform, determine the coefficients of the corresponding discrete time IIR notch filter with notch frequency 1600 Hz. Assume that the sampling frequency is 8 kHz.

6.

(a) Give the formulae for the two-sided z-transform and the inverse z-transform and state what is meant by the Region of Convergence of a z-transform. [6]

Explain the relationship between the Region of Convergence and system stability.

(b) Find the inverse z-transform of the function [7]

$$X(z) = 1 + \frac{7z + 20.6}{z^2 + 5.2z + 1}.$$

and state whether $X(z)$ describes a stable causal system. Explain your answer.

(c) Figure 4 shows the block diagram for a discrete time system for which the z-transforms of the input signal and output signal are $X(z)$ and $Y(z)$ respectively. [7]

Work out the transfer function $H(z) = \dfrac{Y(z)}{X(z)}$ for this system and state its main characteristics.

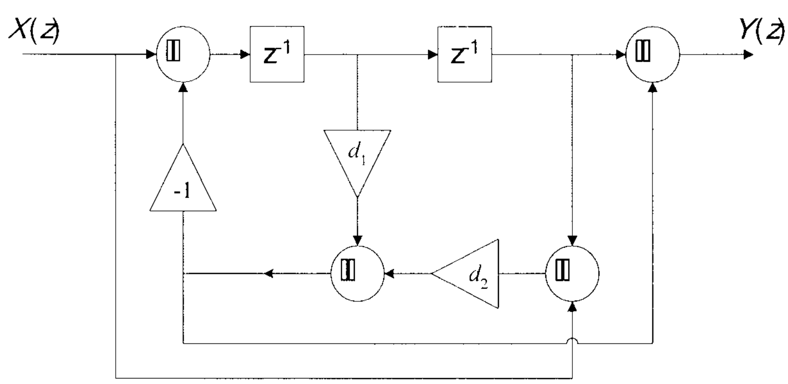

Figure 4

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DSP Solutions

2003/2004

1.
a)

1. Determine the FFT size, N, the data block size, N_1, and the impulse response block size, N_2 such that N is an integer power of 2 and greater than $N_1 + N_2 - 1$.
2. Zero-pad the data block with $N_2 - 1$ zeros and zero-pad the impulse response block with $N_1 - 1$ zeros.
3. Compute and store the FFT of the zero-padded impulse response.
4. For each data block, calculate the IFFT of the product of the FFT of the zero-padded impulse response with the FFT of the zero-padded data block.
5. Determine the final output from the IFFTs overlapped by $N_2 - 1$ samples and forming the sum in the overlapping regions.

b)
Let $x(n)$ be divided up into data blocks of length N1=6 so that N = N1 + N2 -1 = 8. Now we have the zero padded impulse response as
h[n] = [1, 0, 1, 0, 0, 0, 0, 0]
The first 2 data blocks (including zero padding) are:
x1 = [1, 2, 3, -3, -2, -1, 0, 0] and x2 = [0, -1, -2, -3, -2, -1, 0, 0]
The convolutions of h*x1 and h*x2 are now performed as the IFFT of the product of their FFTs. This result is easier to obtain in the time domain than through the FFT for this particular example because of the simple nature of h, as follows:
y1 = h*x1 = [1, 2, 4, -1, 1, -4, -2, -1]
y2 = h*x2 = [0, -1, -2, -4, -4, -4, -2, -1]
These outputs are then overlapped as follows:

```
1, 2, 4, -1, 1, -4, -2, -1
                  0, -1, -2, -4, -4, -4, -2, -1
```

to give
y = [1, 2, 4, -1, 1, -4, -2, -2, -2, -4, -4, -4, -2, -1]

2.

(a)
The basis functions are an orthogonal set of functions from which $x(n)$ can be can be built as a linear combination.

For N=4, the basis functions are $f_k(n) = e^{-j2pkn/N} = e^{-jpkn/2}, \quad n = 0, 1, 2, 3$

k=0: [1 1 1 1]
k=1: [1 –j –1 j]
k=2: [1 -1 1 -1]
k=3: [1 j -1 -j]

(b) [5]
$x(n)$ must have even symmetry such that $x(n) = x(N-n)$.
Example: x(n) = [1 1 1 0 0 0 1 1];

(c) [5]

$$X(k) = \frac{1}{N} \mathring{a}_{n=0}^{N-1} x(n) e^{-j2pnk/N}$$

$$= \frac{1}{N} \mathring{a}_{n=0}^{M-1} x(n) e^{-j2pnl/M} + \frac{1}{N} \mathring{a}_{p=0}^{M-1} x(p) e^{-j2ppl/M} e^{-j2pl}$$

using $p = n + M$. Since $x(n) = -x(p)$ this sums to zero when $e^{-j2pl} = 1$.

(d) [5]
For $x(n)$ to be real, $|X(k)|$ must be even symmetric and $\angle X(k)$ must be odd-symmetric.
A simple sketch graph demonstrating that the student has understood even and odd symmetric functions of k is sufficient. The sketch must be labelled appropriately.

3.
(a)

$$Y_D(e^{j\omega}) = \frac{1}{M}\sum_{k=0}^{M-1} X\left(e^{j(\omega-2\pi k)/M}\right)$$

YD is a stretched version of X on the frequency axis. M-1 copies of the stretched version of X are created and each shifted by successive multiples of 2pi and superimposed (added). The resultant spectrum is then divided by M.

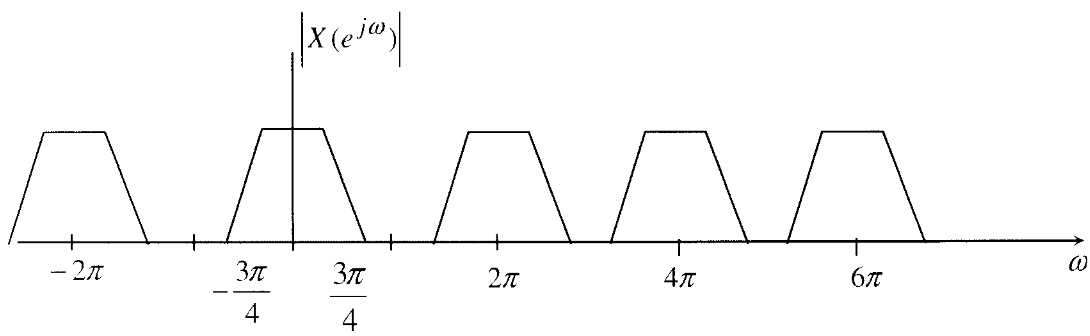
$$Y_E(e^{j\omega}) = X(e^{j\omega L})$$

YE is a compressed version of X on the frequency axis. L-1 images of X are created in YE between w=0 and 2pi.

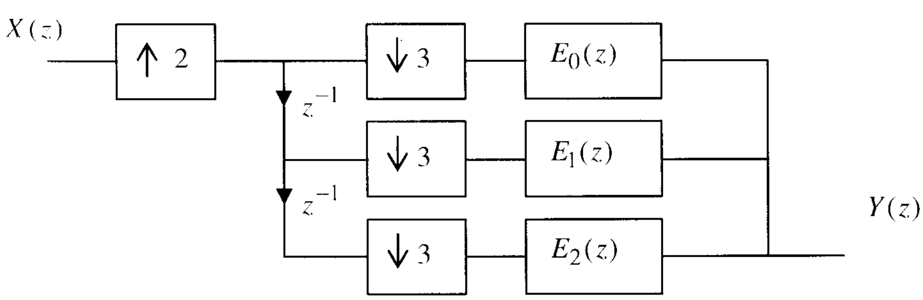
(b)

$$Y(z) = \frac{1}{3}\sum_{k=0}^{2} X(z^{2/3}W^{2k})H(z^{1/3}W^{k}) \qquad W = e^{-j2\pi/3}$$

Main features are stretched original spectrum with 2 images each shifted by 2π .



c)

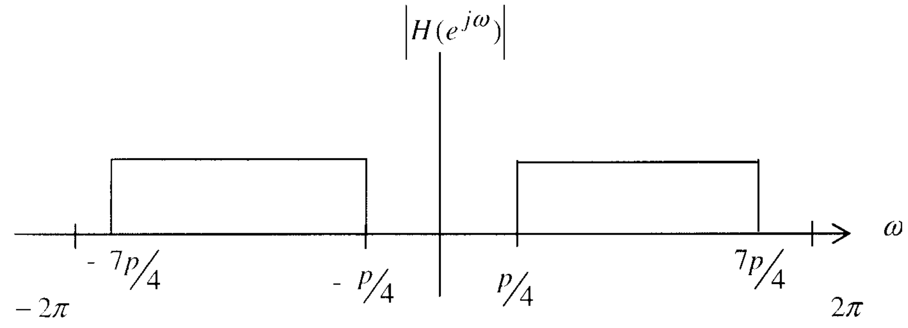


Advantage of polyphase is that filtering is done at lower sampling rate (2 fs / 3) than in Figure 2 (2 fs)

4.

(a)

[4]



The impulse response of the ideal filter can be found using the inverse DTFT. A simple approach is to determine the impulse response for a lowpass filter of the same bandwidth and then modulate this impulse response by e^{jp} giving

$$h(n) = \frac{3T}{4}\frac{\sin(3pnT\ /\ 4)}{3pnT\ /\ 4} \times (-1)^n$$

It is not possible to implement this filter in a real-time system because:
(i) its impulse response is of infinite duration
(ii) its impulse response is non-causal.

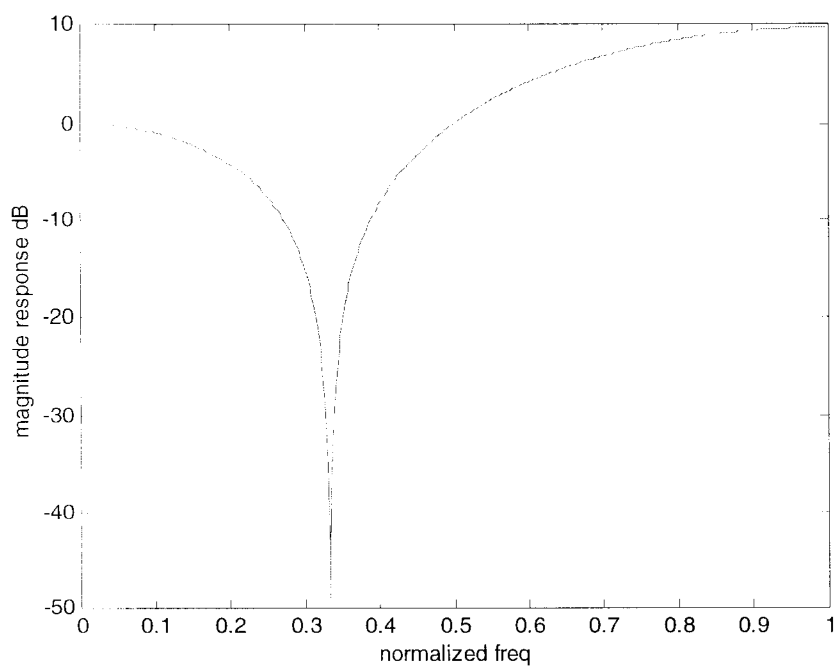
(b)

[10]

$H(z) = z^2 - 2az + a^2 + b^2$

For zeros on the unit circle: $a^2 + b^2 = 1$, $H(z) = z^2 - 2az + 1$

The magnitude response will have a single null corresponding to the zeros on the unit circle at frequency $p\ /\ 3$.



The output sequence is found by convolution with the impulse respone h(n)=[1 -1 1].

n	**x(n)**	**y(n)**
0	0	0
1	0.866	0.866
2	0.866	0
3	0	0
4	-0.866	0
5	-0.866	0
etc.		

5.

a)

- z-transform of the impulse response contains both zeros and poles
- recursive structure (output depends on previous outputs and also current and previous inputs)
- can be unstable. This happens if any poles outside |z|=1.
- can achieve high frequency selectivity with fewer coefficients compared to FIR
- non-linear phase typically (dispersive)
- implementation in fixed point requires careful treatment for rounding errors (limit cycles, precision)

b)

Starting from a prototype analogue filter $H(s)$ normalized for a corner frequency of unity, pre-warping is applied to set the effective corner frequency $\omega_c^{'}$ to the specified value:

Pre-warping: $\omega_c^{'} = \tan\left(\frac{\omega_c T}{2}\right)$

Frequency scaling is then applied to convert the filter from a corner frequency of unity to the desired corner frequency $\omega_c^{'}$: s is replaced by $\frac{s}{\omega_c^{'}}$

The bilinear transform is then applied to $H(s)$ to obtain $H(z)$: $s = \frac{z-1}{z+1}$

c)

Pre-warp: $\omega_c^{'} = \tan\left(\frac{\omega_c T}{2}\right) = \tan\left(\frac{2\pi.1600}{2\times 8000}\right) = 0.72654$

Scaling: s replaced by $\frac{s}{\omega_c^{'}}$ gives

$$H^{'}(s) = \frac{1+\left(s/\omega_c^{'}\right)^2}{1+\left(s/\omega_c^{'}\right)+\left(s/\omega_c^{'}\right)^2} = \frac{\omega_c^{'2}+s^2}{\omega_c^{'2}+s\omega_c^{'}+s^2} = \frac{0.52786+s^2}{0.52786+0.72654s+s^2}$$

$$= \frac{(1+0.52786)z^2+2(0.52786\text{-}1)z+(0.52786+1)}{(1+0.52786+0.72654)z^2+2(0.52786\text{-}1)z+0.52786}$$

$$= \frac{1.52786\text{-}0.94428z^{-1}+1.52786z^{-2}}{2.2544\text{-}0.94428z^{-1}+0.8015z^{-2}}$$

6.
a)

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

$$x(n) = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz$$

Region of Convergence is that part of the z-plane for which $X(z)$ is a convergent series.

ROC must include the unit circle in z for stability.

b)

$$X(z) = 1 + \frac{4}{z+0.2} + \frac{3}{z+5}$$

$$x(n) = \delta(n) + 4.(-0.2)^{n-1}u(n-1) + 3.(-5)^{n-1}u(n-1)$$

This is an unstable system because in order to be stable causal the ROC must included the unit circle. In this case the ROC is $|z| > 5$.

c)

1) $p = x - q$
2) $q = pz^{-1}d_1 + d_2 r$
3) $r = x + pz^{-2}$
4) $y = pz^{-2} + q$

$$\begin{aligned} q &= xd_1z^{-1} + xd_2z^{-2} - qz^{-1}d_1 - qz^{-2}d_2 \\ &= x\frac{d_1z^{-1} + d_2z^{-2} + d_2}{1 + d_1z^{-1} + d_2z^{-2}} \\ y &= (x-q)z^{-2} + q \\ &= xz^{-2} + x\frac{d_1z^{-1} + d_2z^{-2} + d_2}{1 + d_1z^{-1} + d_2z^{-2}}(1 - z^{-2}) \end{aligned}$$

$$H(z) = \frac{d_2 + d_1z^{-1} + z^{-2}}{1 + d_1z^{-1} + d_2z^{-2}}$$

This is an allpass filter