EE4-25
EE9-CS1-2
EE9-FPN2-06

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2016

MSc and EEE/EIE PART IV: MEng and ACGI

Corrected copy

**DESIGN OF LINEAR MULTIVARIABLE CONTROL SYSTEMS**

Monday, 16 May 10:00 am

Time allowed: 3:00 hours

**There are FOUR questions on this paper.**

**Answer ALL questions**

*All questions carry equal marks*

**Any special instructions for invigilators and information for candidates are on page 1.**


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| Examiners responsible | First Marker(s) : | I.M. Jaimoukha |
| | Second Marker(s) : | E.C. Kerrigan |

1. Let the $n$-th order transfer matrix $G(s)$ have a state space realisation

$$G(s) \stackrel{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$$

and let

$$A^T Q + QA + C^T C = 0,$$

and

$$AP + PA^T + BB^T = 0$$

for some $Q = Q^T$ and $P = P^T$.

Suppose that

$$Q = \begin{bmatrix} 0 & 0 \\ 0 & Q_2 \end{bmatrix},$$

and

$$P = \begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix}$$

where $\mathscr{R}^{n_1 \times n_1} \ni P_1 \succ 0$ and $\mathscr{R}^{n_2 \times n_2} \ni Q_2 \succ 0$ and where $n_1 + n_2 = n$. Assume that $A$ has no eigenvalues on the imaginary axis.

a) By partitioning the realisation for $G(s)$ compatibly with $P$ and $Q$, prove that the realisation can be decomposed into two subsystems:

i) A subsystem with $n_1$ modes that are stable, controllable and unobservable. [ 7 ]

ii) A subsystem with $n_2$ modes that are stable, uncontrollable and observable. [ 7 ]

b) Draw a diagram illustrating the two subsystems of $G(s)$. [ 6 ]

  

2. a) Consider a state–variable model described by the dynamics

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t) + Du(t),$$

and let $H(s) = D + C(sI - A)^{-1}B$ denote the corresponding transfer matrix. Suppose there exists a $P = P^T \succ 0$ such that

$$\begin{bmatrix} A^T P + PA + C^T C & PB + C^T D \\ B^T P + D^T C & D^T D - \gamma^2 I \end{bmatrix} \prec 0.$$

i) Prove that $A$ is stable. [ 4 ]

ii) By defining suitable Lyapunov and cost functions and completing a square, prove that

$$\|H\|_\infty < \gamma.$$

[ 6 ]

b) Consider the output injection problem shown in Figure 2. Let $w = [\ w_1^T \quad w_2^T\ ]^T$ and let $T_{yw}(s)$ denote the transfer matrix from $w$ to $y$. An internally stabilizing output injection gain matrix $L$ is to be designed such that, for a given $\gamma > 0$, $\|T_{yw}\|_\infty < \gamma$.

i) Derive a state space realization for $T_{yw}(s)$. [ 4 ]

ii) By using the answer to Part (a) above, or otherwise, derive sufficient conditions for the existence of a feasible $L$. Your conditions should be in the form of the existence of certain solutions to linear matrix inequalities. [ 6 ]

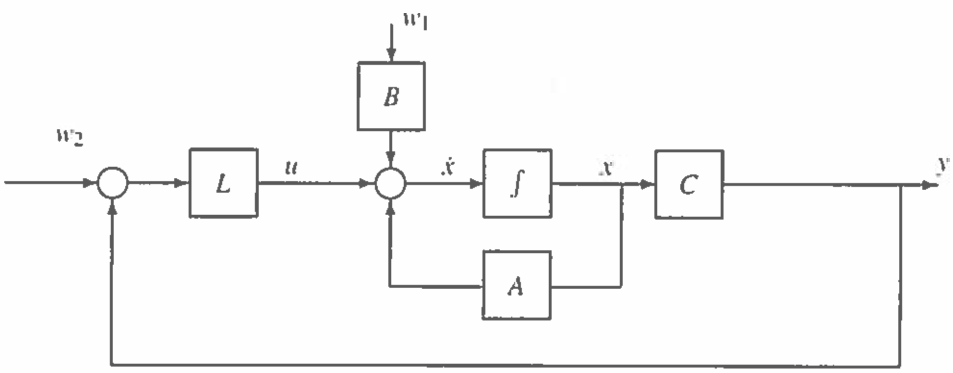


*Figure 2*

  

3. Consider the $\mathcal{H}_\infty$ filter for estimating $C_z x$ shown in Figure 3.

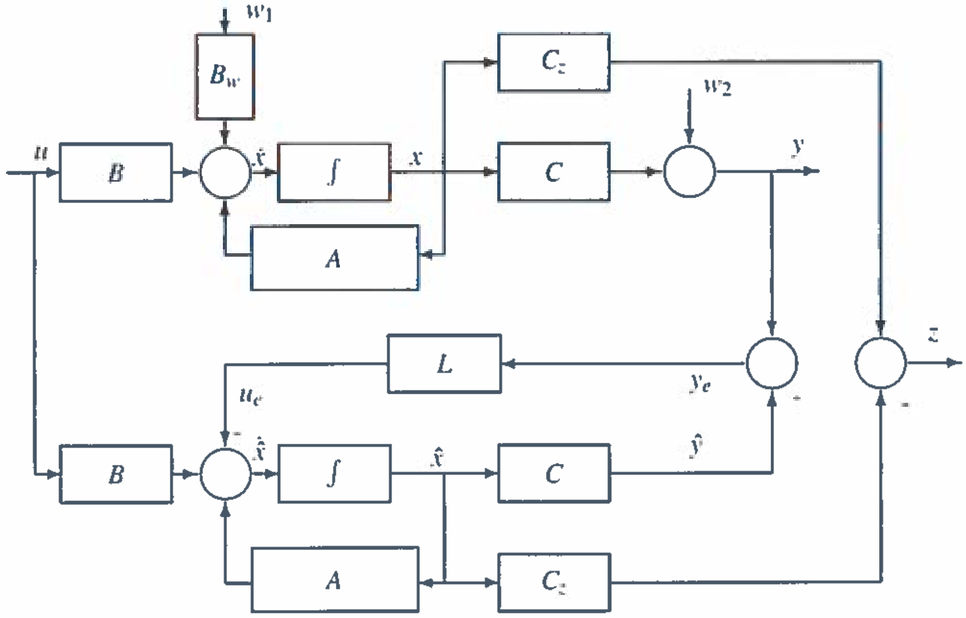


*Figure 3*

Let $T_{zw}(s)$ denote the transfer matrix from $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ to $z$. A stabilizing filter gain matrix $L$ is to be designed such that, for $\gamma > 0$, $\|T_{zw}\|_\infty < \gamma$.

a) Derive a state-space realisation for the transfer matrix $T_{zw}(s)$. [ 5 ]

b) Use the Bounded Real Lemma, stated in Question 2a above, to derive necessary and sufficient conditions for $\|T_{zw}\|_\infty < \gamma$. These conditions should be in the form of matrix inequality constraints. [ 5 ]

c) Suggest a transformation of variables that turns the matrix inequality derived in Part (b) above into a linear matrix inequality. [ 5 ]

d) Suppose that $A = -1, B_w = C = 1, C_z = \sqrt{2}$. Find the optimal $\gamma$ and the corresponding filter gain $L$. [ 5 ]

4. Consider the regulator shown in Figure 4 for which it is assumed that the pair $(A,B)$ is controllable and $x(0)=0$.

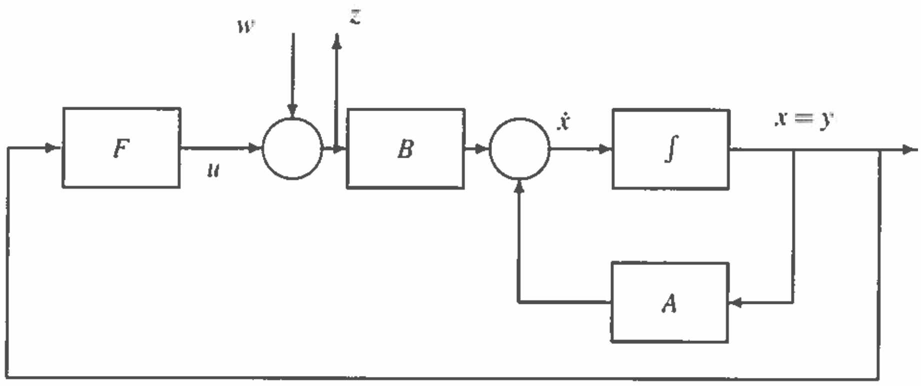


*Figure 4*

Let $H(s)$ denote the transfer matrix from $w$ to $z$. A stabilizing state-feedback gain matrix $F$ is to be designed such that, for $\gamma>0$, $\|H\|_\infty<\gamma$. Assume that $\gamma>1$.

a) Write down the generalized regulator system for this design problem. [ 4 ]

b) By defining suitable Lyapunov and cost functions as well as two completion of squares procedures, derive sufficient conditions for the solution of the design problem. Your conditions should be in the form of the existence of a certain solution to an algebraic Riccati equation. It should also include an expression for $F$ and an expression for the worst-case disturbance $w$. [ 8 ]

c) What is the smallest $\gamma$ for which your sufficient conditions guarantee the existence of $F$ satisfying the design specifications. Justify your answer. [ 4 ]

d) Suppose that $A$ is stable. Show that $F=0$ is a solution. [ 2 ]

e) Suppose that $-A$ is stable. Show that the solution of the Riccati equation reduces to the solution of a Lyapunov equation. [ 2 ]

  