IMPERIAL COLLEGE LONDON

Master - June 08

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ISE4.53

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2008

MSc and EEE/ISE PART IV: MEng and ACGI

LINEAR OPTIMAL CONTROL

Thursday, 15 May 10:00 am

Time allowed: 3:00 hours

There are SIX questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s) :	E.C. Kerrigan
	Second Marker(s) :	A. Astolfi

LINEAR OPTIMAL CONTROL

1. Consider the following finite-horizon discrete-time linear quadratic regulator problem:

$$\pi^*(x_0) := \arg\min_{\pi} x_N' Q x_N + \sum_{k=0}^{N-1} \left(x_k' Q x_k + u_k' R u_k \right)$$

where the system dynamics is given by

$$x_{k+1} = Ax_k + Bu_k,$$

and the policy $\pi := \{\mu_0, \mu_1, \ldots, \mu_{N-1}\}$ defines the state feedback control law

$$u_k = \mu_k(x_k).$$

The matrices Q and R are assumed to be symmetric.

a) Show that the optimal control policy is given by

$$\mu_k^*(x_k) = -(B'P_{k+1}B + R)^{-1} B'P_{k+1}Ax_k$$

where P_k is given by

$$P_k = A'(P_{k+1} - P_{k+1}B(B'P_{k+1}B + R)^{-1}B'P_{k+1})A + Q$$

with boundary condition $P_N = Q$.

State any additional assumptions you have made and the reasons for making these additional assumptions. [16]

b) How would you modify your assumptions and results if Q and R were not symmetric? Justify your answer. [4]

2. Consider the following scalar discrete-time system:

$$x_{k+1} = ax_k + u_k + w_k$$

where $a \in \mathbb{R}$, $x_k \in \mathbb{R}$ is the system state, $u_k \in \mathbb{R}$ is the control input and $w_k \in \mathbb{W}$ is an unmeasurable disturbance that satisfies

$$-1 \leq w_k \leq 1, \quad k = 0, 1, \ldots$$

Consider the design of a state feedback gain $L \in \mathbb{R}$ such that

$$u_k = Lx_k, \quad k = 0, 1, \ldots$$

In the following, assume that the initial state $x_0 = 0$.

a) Show that, if c is any given scalar, then

$$\max_{-1 \leq w_k \leq 1} cw_k = |c|$$

[4]

b) Show that, for the closed-loop system,

$$s_k := \max_{w_0, w_1, \ldots, w_{k-1}} |x_k| = \max_{w_0, w_1, \ldots, w_{k-1}} x_k, \quad k = 1, 2, \ldots,$$

where the constraints on the disturbance sequence $\{w_0, \ldots, w_{k-1}\}$ are as above. Hence, show that the sequence of the maximum magnitude of the state, i.e. $\{s_1, s_1, s_2, \ldots\}$, is non-decreasing and that the state trajectory of the closed-loop system is bounded if and only if $|a + L| < 1$. [8]

c) Compute the feedback gain L that minimizes the maximum deviation of the state from the origin over all time, i.e. compute the gain that solves the following optimal control problem:

$$L^* := \arg\min_L \left\{ \max_{k=0,1,\ldots} s_k \right\}.$$

Hint: You may wish to use the fact that $\sum_{n=0}^{\infty} r^n = 1/(1-r)$ if and only if $|r| < 1$. [8]

3. a) Consider the following continuous-time optimal control problem:

$$v^*(\cdot) = \arg\min_{v(\cdot)} \int_0^\infty \left(z(t)'\bar{Q}z(t) + v(t)'\bar{R}v(t)\right) dt,$$

where $\bar{Q}$ and $\bar{R}$ are constant (time-invariant) matrices and the continuous-time dynamics are given by

$$\dot{z} = \bar{A}z + \bar{B}v.$$

From standard LQR theory it can be shown that the optimal control law is given by

$$v^*(t) = \bar{L}z(t) = -\bar{R}^{-1}\bar{B}'\bar{P}z(t),$$

where $\bar{P}$ is the positive semidefinite solution of the continuous-time algebraic Riccati equation (ARE)

$$\bar{A}'\bar{P} + \bar{P}\bar{A} + \bar{Q} - \bar{P}\bar{B}\bar{R}^{-1}\bar{B}'\bar{P} = 0.$$

Give conditions on $\bar{Q}$, $\bar{R}$, $\bar{A}$ and $\bar{B}$ which guarantee that the ARE has a unique stabilizing solution. [4]

b) It is sometimes desired to have all the eigenvalues of a closed-loop system with real parts less than some negative number $-\alpha$, $\alpha > 0$. This is commonly referred to as "degree of stability $-\alpha$". It turns out that this is simple to design with LQR theory, given suitable time-varying choices of state penalty $Q(t)$ and input penalty $R(t)$.

In particular, consider now the continuous-time optimal control problem:

$$u^*(\cdot) := \arg\min_{u(\cdot)} \int_0^\infty \left(x(t)'Q(t)x(t) + u(t)'R(t)u(t)\right) dt,$$

where the continuous-time dynamics are given by

$$\dot{x} = Ax + Bu.$$

Show that if

$$Q(t) := e^{2\alpha t}\bar{Q} \text{ and } R(t) := e^{2\alpha t}\bar{R},$$

then the optimal LQR control law, which guarantees that the closed-loop system has "degree of stability $-\alpha$", is given by

$$u^*(t) = Lx(t) = -\bar{R}^{-1}B'\bar{P}x(t)$$

where $\bar{P}$ satisfies

$$(A + \alpha I)'\bar{P} + \bar{P}(A + \alpha I) + \bar{Q} - \bar{P}B\bar{R}^{-1}B'\bar{P} = 0.$$

Hint: If one can show that $x(t) = e^{-\alpha t}z(t)$, where $z(t)$ is the state of an asymptotically stable system, then the closed-loop system $\dot{x} = (A + BL)x$ has "degree of stability $-\alpha$". [16]

4. Consider the discrete-time optimal control problem:

$$\pi^*(x_0) := \arg\min_{\pi} \mathbb{E}\left\{ c'x_N + \sum_{k=0}^{N-1} \left[c'x_k + d\left(u_k\right)\right] \right\}$$

where the system dynamics are given by

$$x_{k+1} = A_k x_k + \beta(u_k) + w_k,$$

and the policy $\pi := \{\mu_0, \mu_1, \dots, \mu_{N-1}\}$ defines the state feedback control law

$$u_k = \mu_k(x_k).$$

In the above, c is a given vector, $d(\cdot)$ and $\beta(\cdot)$ are given nonlinear functions, A_k and w_k are random $n \times n$ matrices and n-dimensional vectors, respectively, with given probability distributions that do not depend on x_k, u_k or prior values of A_k and w_k. The expectation $\mathbb{E}\{\cdot\}$ is taken with respect to A_k and w_k, $k = 0, 1, \dots, N-1$.

a) State the "principle of optimality" in words. [4]

b) Show that the cost-to-go functions of the Dynamic Programming algorithm for the above optimal control problem are affine (linear plus a constant). [10]

c) What is meant with "certainty equivalence"? Determine whether or not "certainty equivalence" holds for the above optimal control problem. [6]

5. a) Consider the quadratic form

$$\ell(z,u) := z'Qz + u'Ru + 2u'Sz,$$

where R is positive definite. Show that $\ell(z,u) \geq 0$ for all (z,u) if and only if $Q - S'R^{-1}S$ is positive semidefinite.

Hint: You may wish to use the fact that $\ell(z,u) \geq 0$ for all (z,u) if and only if the function $L(z) := \min_u \ell(z,u) \geq 0$ for all z. [6]

b) A popular cost function, especially in predictive control applications, is the following:

$$J := y_N'My_N + \sum_{k=0}^{N-1} \left(y_k'My_k + u_k'Vu_k + (\Delta u_k)'W\Delta u_k \right),$$

where M, V and W are symmetric matrices, the discrete-time dynamics are given by

$$x_{k+1} = Ax_k + Bu_k, \qquad y_k = Cx_k,$$

and the change in control input at time k is defined as

$$\Delta u_k := u_k - u_{k-1}.$$

Show that, by defining the augmented state vector

$$z_k := [x_k' \; u_{k-1}']',$$

one can rewrite the cost function in the form

$$J = z_N'Qz_N + \sum_{k=0}^{N-1} \left(z_k'Qz_k + u_k'Ru_k + 2u_k'Sz_k \right)$$

where the augmented discrete-time dynamics are given by

$$z_{k+1} = \bar{A}z_k + \bar{B}u_k$$

with $\bar{A}$, $\bar{B}$, Q, R and S suitably defined. [10]

c) Give sufficient conditions on M, V and W such that R is positive definite and $Q - S'R^{-1}S$ is positive semi-definite, with Q, R and S as in part b). [4]

6. From standard LQR theory, it can be shown that the solution to the problem

$$u^*(\cdot) := \arg\min_{u(\cdot)} \int_0^\infty \left(x(t)'Qx(t) + u(t)'Ru(t)\right) dt,$$

where the continuous-time dynamics are given by

$$\dot{x} = Ax + Bu,$$

is given by

$$u^*(t) = -R^{-1}B'Px(t),$$

where P satisfies the continuous-time algebraic Riccati equation (ARE)

$$A'P + PA + Q - PBR^{-1}B'P = 0.$$

The linearized continuous-time state space model for an inverted pendulum is given by

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ \gamma & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u,$$

where $\gamma > 0$ is a given scalar. A control law is sought to minimize the performance index

$$\int_0^\infty \left(x_1(t)^2 + \frac{u(t)^2}{c} \right) dt,$$

where $c > 0$ is a given scalar.

a) Show that a stabilizing solution exists to the above LQR problem for the inverted pendulum. [6]

b) Show that the optimal LQR control law is given by

$$u(t) = -\left[\gamma + \sqrt{\gamma^2 + c} \quad \sqrt{2(\gamma + \sqrt{\gamma^2 + c})}\right] x(t).$$

[14]