

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2018

MSc and EEE PART IV: MEng and ACGI

Corrected copy

MODELLING AND CONTROL OF MULTI-BODY MECHANICAL SYSTEMS

Friday, 4 May 10:00 am

Time allowed: 3:00 hours

There are FOUR questions on this paper.

Answer ALL questions

All questions carry equal marks.

This is an OPEN BOOK examination.

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : S.A. Evangelou
Second Marker(s) : A. Astolfi

MODELLING AND CONTROL OF MULTIBODY MECHANICAL SYSTEMS

1. A compound 'ball-jointed' pendulum of mass m consists of a symmetric rigid body fixed at one of its end points along its axis of symmetry (point O in Figure 1.1), and which is free to rotate about the fixed point in any direction under the influence of gravity. The centre of mass of the pendulum is on the axis of symmetry at distance f from the fixed point. A body-fixed Cartesian coordinate system has its origin at the fixed point and its z -axis (Z') along the axis of symmetry of the body. The body-fixed x -, y -, and z -axes ($X'Y'Z'$) correspond to principal axes with principal moments of inertia I_{xx} , I_{yy} , and I_{zz} respectively. In the nominal configuration the body-fixed x -, y -, and z -axes are respectively aligned with the x -, y -, and z -axes (XYZ) of an earth-fixed Cartesian coordinate system, in which the x -axis is horizontal and the z -axis points downwards.

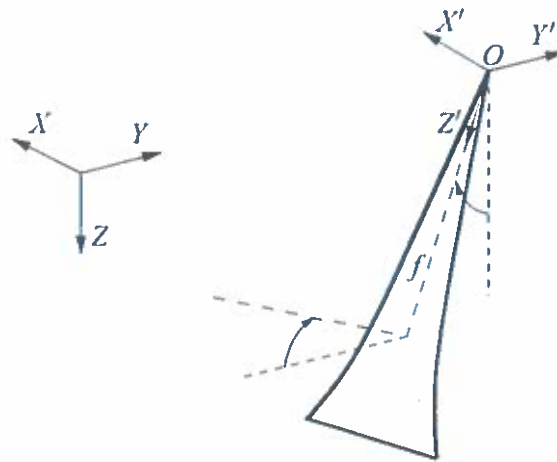


Figure 1.1 Compound 'ball-jointed' pendulum.

The angular configuration of the body is given by three Euler angles in the 3-2-3 convention. In this convention the body is first rotated from its nominal configuration by an angle ψ about the z -axis, then by an angle θ about the intermediate y -axis of the body and finally by an angle ϕ about the new z -axis of the body.

- a) By making use of the standard single-axis-rotation transformation matrices, calculate
 - i) the complete transformation matrix A from earth-fixed to body-fixed coordinates, and [4]
 - ii) the body angular velocity vector in the body-fixed coordinate system, [4]
 in terms of the generalised coordinates.
- b) By expressing the unit vector of the earth-fixed z -axis in body-fixed coordinates, compute the total external moment N acting on the rigid body due to gravity in the body-fixed coordinate system. [5]
- c) Make use of the Euler equations of motion for a rigid body with one point fixed to derive the equations of motion of the compound pendulum in terms of the generalised coordinates. [7]

2. A 2-wheeled self-balancing robot comprises a small uniform wheel of radius r_2 that rolls by an angle ϕ_2 without slipping on a large uniform wheel of radius r_1 under the influence of gravity, as shown in Figure 2.1; the small wheel is constrained to always touch the large wheel by a massless link (not shown) joining the centres of the two wheels. The large wheel rolls by an angle ϕ_1 without slipping on a horizontal surface, and the large wheel centre moves horizontally by a distance x . The mass and spin inertia of the large wheel are m_1 and I_1 respectively, and the corresponding quantities for the small wheel are m_2 and I_2 respectively. A control torque T_c is applied on the large wheel and reacts on earth.

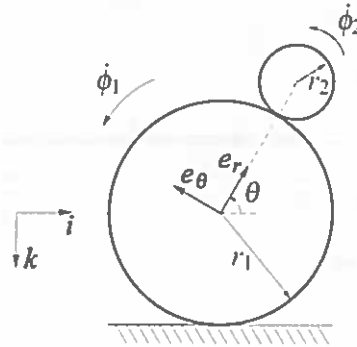


Figure 2.1 Self-balancing robot with two wheels.

A fixed Cartesian coordinate system with unit vectors i and k , and a moving Cartesian coordinate system with unit vectors e_r and e_θ (see Figure 2.1) are used to analyse the motion of the two wheels. The moving coordinate system has its origin at the centre of the large wheel and it rotates with the line joining the centres of the two wheels by an angle θ .

- Write the position vectors of the centre of mass of the large wheel and of the centre of mass of the small wheel. [2]
- Determine the velocity vectors of the centre of mass of the large wheel and of the centre of mass of the small wheel. [3]
- Compute the total kinetic energy and potential energy of the system, and hence determine the Lagrangian function. [5]
- Derive the equation of the rolling constraint between the large wheel and the horizontal surface. [2]
- The equation of the rolling constraint between the two wheels is given by

$$A_1 \dot{\theta} + A_2 \dot{\phi}_1 + A_3 \dot{\phi}_2 = 0.$$

By considering the velocity of the instantaneous contact point between the two wheels, determine A_1 , A_2 , and A_3 . [3]

- Use the two rolling constraint equations to eliminate the x and ϕ_2 generalised coordinates in the Lagrangian function. Use the Lagrangian approach to derive the equations of motion of the wheel in terms of the generalised coordinates ϕ_1 and θ . [5]

3. Two identical uniform wheels of mass m and radius R each are held at their centre of mass respectively at the left and right ends of a uniform axle of length l , as shown in Figure 3.1. The axle has only one rotational degree of freedom about a vertical axis through its middle point, A , represented by an angle ψ . The axle moment of inertia about its axis of rotation is I_{axe} . The left and right wheels spin relative to the axle by angles θ_l and θ_r respectively. Each of the wheels is axisymmetric with spin moment of inertia I_{yy} and radial moment of inertia, passing through the wheel centre of mass, I_{xx} .

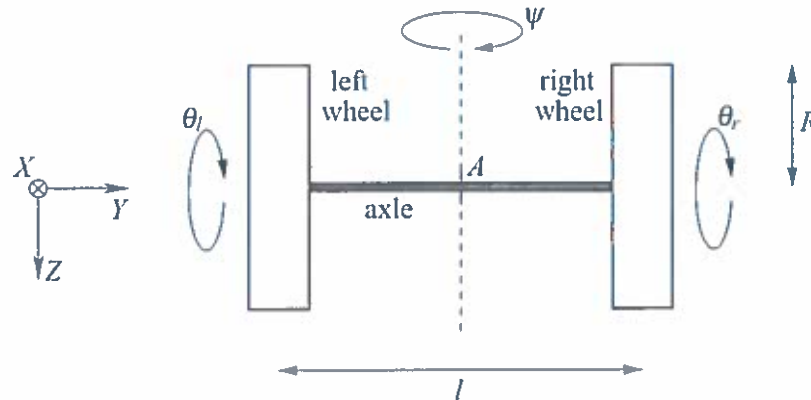


Figure 3.1 Two wheels on an axle.

Three body-fixed Cartesian coordinate systems, fixed in the left wheel, the axle and the right wheel respectively, are used to analyse the motion of the system. These coordinate systems have their origin at the left wheel centre of mass, point A , and the right wheel centre of mass respectively, and their unit vectors are initially respectively aligned with the earth-fixed axes X , Y and Z , in which X is into the page, as shown in Figure 3.1.

- Write the generalised coordinates of the system. [2]
- The rotation of each of the left and right wheels is represented respectively by three Euler angles in the yaw-roll-pitch convention. By making use of the standard single-axis-rotation transformation matrices, calculate the body angular velocity vector of each wheel in its respective body-fixed coordinate system, in terms of the generalised coordinates. [4]
- Compute the kinetic energy of the left wheel. [4]
- Calculate the total kinetic energy of the system. [3]
- Assume that the wheels rest on a rough horizontal road such that their contact points with the road are instantaneously at rest, as the axle rotates.
 - Write the constraint equation for each wheel due to their road contact point being at rest. [2]
 - Use the Lagrangian approach to derive the equations of motion of the system when a moment M is acting on the axle about the vertical axis. [5]

4. A 2-wheeled self-balancing robot comprises a small uniform wheel of radius r_2 that rolls by an angle ϕ_2 without slipping on a large uniform wheel of radius r_1 under the influence of gravity, as shown in Figure 4.1; the small wheel is constrained to always touch the large wheel by a massless link (not shown) joining the centres of the two wheels. The large wheel rolls by an angle ϕ_1 without slipping on a horizontal surface, and the large wheel centre moves horizontally by a distance x . The mass and spin inertia of the large wheel are m_1 and I_1 respectively, and the corresponding quantities for the small wheel are m_2 and I_2 respectively. A control torque T_c is applied on the large wheel and reacts on earth.

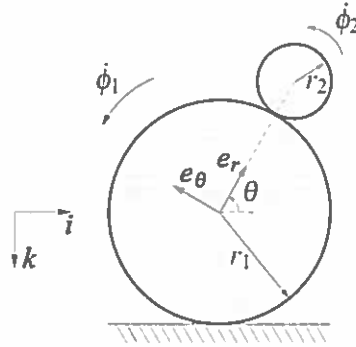


Figure 4.1 Self-balancing robot with two wheels.

A fixed Cartesian coordinate system with unit vectors i and k , and a moving Cartesian coordinate system with unit vectors e_r and e_θ (see Figure 4.1) are used to analyse the motion of the two wheels. The moving coordinate system has its origin at the centre of the large wheel and it rotates with the line joining the centres of the two wheels by an angle θ .

- Write the generalised coordinates of the system. [2]
- Determine the acceleration vectors of the centre of mass of the large wheel and of the centre of mass of the small wheel. [3]
- F_{roll1} and R_1 are the unknown constraint forces acting on the large wheel and reacting on earth that respectively maintain the rolling constraint between the large wheel and the horizontal surface, and prevent the large wheel from sinking into the horizontal surface. F_{roll2} and R_2 are the unknown constraint forces acting on the small wheel and reacting on the large wheel that respectively maintain the rolling constraint between the two wheels and the radial contact between the two wheels. Write the total force that acts on each of the wheels in vector form in terms of the unknown constraint forces. [3]
- By considering the translational motion of the centre of mass of each of the wheels, use the vectorial approach to derive F_{roll2} , R_2 , F_{roll1} , and R_1 in terms of the generalised coordinates. [6]
- By considering the rotational motion about the centre of mass of each of the wheels, use the vectorial approach to derive the equations of motion of the system with respect to the generalised coordinates. [6]

