Modelling and control of multibody mechanical systems

Model answers

Question 1

a) i) The position vector of the centre of mass is given by

$$\boldsymbol{r} = x'\boldsymbol{i}' + y'\boldsymbol{j}',$$

and therefore the velocity vector by differentiation is

$$\dot{\boldsymbol{r}} = (\dot{x}' - y'\dot{\psi})\boldsymbol{i}' + (\dot{y}' + x'\dot{\psi})\boldsymbol{j}'.$$

[2 marks]

ii) The equations of the rolling constraint are

$$\dot{x}' - y'\dot{\psi} + a\dot{\theta} = 0, \qquad (1)$$

$$\dot{y}' + x'\dot{\psi} = 0. \qquad (2)$$

[4 marks]

iii) By substituting Equations 1 and 2 in the velocity vector expression

$$\dot{\boldsymbol{r}} = -a\dot{\theta}\boldsymbol{i}'.$$

Then by differentiation the acceleration vector is

$$\ddot{\boldsymbol{r}} = -a\ddot{\theta}\boldsymbol{i}' - a\dot{\theta}\dot{\psi}\boldsymbol{j}'.$$

[4 marks]

b) i) $\boldsymbol{F} = m\ddot{\boldsymbol{r}}$, or

$$F_{long}\boldsymbol{i}' + F_{lat}\boldsymbol{j}' = -ma\ddot{\theta}\boldsymbol{i}' - ma\dot{\theta}\dot{\psi}\boldsymbol{j}',$$

therefore

$$F_{long} = -ma\ddot{\theta},$$

and

$$F_{lat} = -ma\dot{\theta}\dot{\psi}.$$

[4 marks]

ii) The angular momentum vector of the body about C is

$$\boldsymbol{H} = I_{yy}\dot{\theta}\boldsymbol{j}' + I_{zz}\dot{\psi}\boldsymbol{k}',$$

therefore

$$\frac{d\boldsymbol{H}}{dt} = -I_{yy}\dot{\theta}\dot{\psi}\boldsymbol{i}' + I_{yy}\ddot{\theta}\boldsymbol{j}' + I_{zz}\ddot{\psi}\boldsymbol{k}'.$$

The total moment vector acting on the body is

$$\boldsymbol{N} = a\boldsymbol{k}' \times F_{long}\boldsymbol{i}' + a\boldsymbol{k}' \times F_{lat}\boldsymbol{j}',$$

or

$$\boldsymbol{N} = -ma^2\ddot{\theta}\boldsymbol{j}' + ma^2\dot{\theta}\dot{\psi}\boldsymbol{i}'.$$

Therefore

$$N = \frac{dH}{dt},$$

which gives the equations of motion

$$(I_{yy} + ma^2)\dot{\theta}\dot{\psi} = 0,$$

$$(I_{yy} + ma^2)\ddot{\theta} = 0,$$

$$I_{zz}\ddot{\psi} = 0.$$

This implies that either one of θ or ψ will be constant. [6 marks]

Question 2

a) i) We consider the moment of inertia about the axis of symmetry (z axis):

$$I_{zz} = \int r^2 \cos^2\phi \, \mathrm{d}m = \rho \int_V r^2 \cos^2\phi \, \mathrm{d}V = \rho \int\int\int r^2 \cos^2\phi \; r\cos\phi \mathrm{d}\theta \; r\mathrm{d}\phi \; \mathrm{d}r =$$

$$= \rho \int_0^a \int_0^{\frac{\pi}{2}} \int_0^{2\pi} r^4 \cos^3\phi \; \mathrm{d}\theta \; \mathrm{d}\phi \; \mathrm{d}r = 2\pi\rho \int_0^a \int_0^{\frac{\pi}{2}} r^4 \cos^3\phi \; \mathrm{d}\phi \; \mathrm{d}r =$$

$$= 2\pi\rho \int_0^a r^4 \left(\frac{1}{3}\cos^2\phi \sin\phi + \frac{2}{3}\sin\phi\right) \Big|_0^{\pi/2} \; \mathrm{d}r = \frac{4}{3}\pi\rho \int_0^a r^4 \; \mathrm{d}r = \frac{4}{15}\pi\rho a^5.$$

Note now that

$$m = \rho V = \rho \times \frac{1}{2} \times \frac{4}{3}\pi a^3,$$

and therefore

$$I_{zz} = \frac{2}{5} m a^2.$$

[6 marks]

ii) We consider the moment of inertia about the x axis which is perpendicular to z:

$$I_{xx} = \int r^2 \cos^2\phi \, \mathrm{d}m = \rho \int_V r^2 \cos^2\phi \, \mathrm{d}V = \rho \int\int\int r^2 \cos^2\phi \; r\cos\phi \mathrm{d}\theta \; r\mathrm{d}\phi \; \mathrm{d}r =$$

$$= \rho \int_0^a \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\pi} r^4 \cos^3\phi \; \mathrm{d}\theta \; \mathrm{d}\phi \; \mathrm{d}r = \pi\rho \int_0^a \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r^4 \cos^3\phi \; \mathrm{d}\phi \; \mathrm{d}r =$$

$$= \pi\rho \int_0^a r^4 \left(\frac{1}{3}\cos^2\phi \sin\phi + \frac{2}{3}\sin\phi\right) \Big|_{-\pi/2}^{\pi/2} \; \mathrm{d}r = \frac{4}{3}\pi\rho \int_0^a r^4 \; \mathrm{d}r = \frac{4}{15}\pi\rho a^5.$$

Note now that

$$m = \rho V = \rho \times \frac{1}{2} \times \frac{4}{3}\pi a^3,$$

and therefore

$$I_{xx} = \frac{2}{5} m a^2.$$

[6 marks]

b) The principal moments of inertia are all the same due to symmetry and have a value of $\frac{2}{5}ma^2$. For example,

$$I_{xx} = I_{yy} = I_{zz} = \frac{2}{5} m a^2,$$

where I_{yy} is the moment of inertia about the y axis which is perpendicular to both x and z axes. Any set of three mutually orthogonal axes passing through the centre of the flat side of the hemisphere are principal axes. [4 marks]

c) We consider the moment of inertia about the parallel s axis at perpendicular distance a from the axis of symmetry (passing through the centre of mass). By using the parallel axis theorem:

$$I_s = I_{zz} + ma^2 = \frac{2}{5}ma^2 + ma^2 = \frac{7}{5}ma^2.$$

[4 marks]

Question 3

a) $\boldsymbol{r_M} = x\boldsymbol{i}$ and $\boldsymbol{r_m} = x\boldsymbol{i} + l\boldsymbol{e_r}$. [1 mark]

b) By differentiating the position vector $\dot{\boldsymbol{r}}_M = \dot{x}\boldsymbol{i}$ and

$$\dot{\boldsymbol{r}}_m = \dot{x}\boldsymbol{i} + l\dot{\theta}\boldsymbol{e_\theta} = \dot{x}\cos\theta\boldsymbol{e_r} + (-\dot{x}\sin\theta + l\dot{\theta})\boldsymbol{e_\theta}.$$

[2 marks]

c)

$$T = \frac{1}{2}M\dot{\boldsymbol{r}}_M\cdot\dot{\boldsymbol{r}}_M + \frac{1}{2}I\dot{\phi}^2 + \frac{1}{2}m\dot{\boldsymbol{r}}_m\cdot\dot{\boldsymbol{r}}_m = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}I\dot{\phi}^2 + \frac{1}{2}m\left(\dot{x}^2 - 2\dot{x}\dot{\theta}l\sin\theta + l^2\dot{\theta}^2\right).$$

[2 marks]

d) The horizontal level at the wheel centre is taken as the zero potential energy level, therefore

$$V = -m\boldsymbol{r_m}\cdot\boldsymbol{g} = mgl\sin\theta.$$

[2 marks]

e)

$$L = T - V = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}I\dot{\phi}^2 + \frac{1}{2}m\left(\dot{x}^2 - 2\dot{x}\dot{\theta}l\sin\theta + l^2\dot{\theta}^2\right) - mgl\sin\theta.$$

[2 marks]

f) $x + a\phi = 0$ or $\dot{x} + a\dot{\phi} = 0$ [1 mark]

g) The Lagrangian equation with respect to the generalised coordinate x is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{x}}\right) - \frac{\partial L}{\partial x} + \lambda = 0,$$

yielding

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(M\dot{x} + m\dot{x} - m\dot{\theta}l\sin\theta\right) + \lambda = 0,$$

hence

$$(M+m)\ddot{x} - ml\sin\theta\ddot{\theta} - ml\dot{\theta}^2\cos\theta + \lambda = 0,$$

and finally it gives

$$-\lambda = (M+m)\ddot{x} - ml\sin\theta\ddot{\theta} - ml\dot{\theta}^2\cos\theta.$$

The Lagrangian equation with respect to the generalised coordinate θ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0,$$

yielding

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(-m\dot{x}l\sin\theta + ml^2\dot{\theta}\right) + m\dot{x}\dot{\theta}l\cos\theta + mgl\cos\theta = 0,$$

hence

$$-\sin\theta\ddot{x} + l\ddot{\theta} + g\cos\theta = 0.$$

The Lagrangian equation with respect to the generalised coordinate ϕ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} + a\lambda = T_d,$$

yielding

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(I\dot{\phi}\right) + a\lambda = T_d,$$

and by substituting λ from the equation above

$$I\ddot{\phi} - a\left((M+m)\ddot{x} - ml\sin\theta\ddot{\theta} - ml\dot{\theta}^2\cos\theta\right) = T_d.$$

Finally substituting $\ddot{x} = -a\ddot{\phi}$ from the constraint equation and $l\ddot{\theta} = \sin\theta\ddot{x} - g\cos\theta$ from the above Lagrangian equation yields

$$\left(I + Ma^2 + ma^2\cos^2\theta\right)\ddot{\phi} + mal\cos\theta\dot{\theta}^2 - mga\sin\theta\cos\theta = T_d.$$

[10 marks]

Question 4

a) Three single-axis-rotation transformation matrices are needed.

$$D_\psi = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which is the rotation matrix by angle ψ about a z axis.

$$C_\theta = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix},$$

which is the rotation matrix by angle θ about a y axis.

$$D_\phi = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which is the rotation matrix by angle ϕ about a z axis.

i) The complete transformation from Earth-fixed coordinates to body-fixed coordinates is $A = D_\phi C_\theta D_\psi$ and it amounts to

$$\begin{bmatrix} \cos\phi\cos\theta\cos\psi - \sin\phi\sin\psi & \cos\phi\cos\theta\sin\psi + \sin\phi\cos\psi & -\cos\phi\sin\theta \\ -\sin\phi\cos\theta\cos\psi - \cos\phi\sin\psi & -\sin\phi\cos\theta\sin\psi + \cos\phi\cos\psi & \sin\phi\sin\theta \\ \sin\theta\cos\psi & \sin\theta\sin\psi & \cos\theta \end{bmatrix}$$

[8 marks]

ii) The complete transformation from body-fixed coordinates to Earth-fixed coordinates is $A^{-1} = A^T$.

[4 marks]

b)

$$\Omega = \begin{bmatrix} 0 \\ 0 \\ \dot\psi \end{bmatrix} + D_\psi^T \begin{bmatrix} 0 \\ \dot\theta \\ 0 \end{bmatrix} + D_\psi^T C_\theta^T \begin{bmatrix} 0 \\ 0 \\ \dot\phi \end{bmatrix}$$

$$= \begin{bmatrix} \dot\phi\cos\psi\sin\theta - \dot\theta\sin\psi \\ \dot\phi\sin\psi\sin\theta + \dot\theta\cos\psi \\ \dot\psi + \dot\phi\cos\theta \end{bmatrix}$$

[8 marks]