Imperial College
London

M4/5A6

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May-June 2017

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Special Relativity and Electromagnetism

Date: Tuesday 09 May 2017

Time: 14:00 - 16:00

Time Allowed: 2.5 Hours

This paper has 5 Questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the relevant main book(s) are full.

All required additional material will be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers to each question as per the table below.

Raw Mark	Up to 12	13	14	15	16	17	18	19	20
Extra Credit	0	½	1	1 ½	2	2 ½	3	3 ½	4

- Each question carries equal weight.
- Calculators may not be used.

1. (a) State the Lorentz transformation rule, by expressing the contravariant four-vector A^i measured in coordinate system K in terms of the contravariant four-vector A'^i, measured in coordinate system K' which is observed from K to move away with constant relative velocity V along the x-axis. Their origins coincide at time $t = 0 = t'$ and their axes are parallel.

(b) Inertial frame K' is observed from K to move away with constant, positive relative velocity V along the x-axis. Their origins coincide at time $t = 0 = t'$ and their axes are parallel.

Two events take place in K simultaneously at time $t_1 = t_2 = 0$, one at its origin, $\mathbf{r}_1 = (0, 0, 0)$, the other one at position $\mathbf{r}_2 = (L, 0, 0)$, where L is a positive, but otherwise arbitrary constant (length).

The same two events are observed in coordinate system K', the first one at time $t'_1 = 0$ at its origin, $\mathbf{r}'_1 = (0, 0, 0)$, and the second at time t'_2 at $\mathbf{r}'_2 = (\frac{5}{3}L, 0, 0)$.

(i) Determine the velocity V.

(ii) Determine the time t'_2.

(c) Inertial frame K' is observed from K to move away with constant relative velocity V along the x-axis. Their origins coincide at time $t = 0 = t'$ and their axes are parallel.

Two events take place in K, with contravariant (world, *i.e.* four-vector) coordinates $(a, a.0, 0)$ and $(a/2, 2a, 0, 0)$ respectively, where $a \neq 0$ is a non-vanishing, but otherwise arbitrary constant (length).

(i) Determine V such that the two events occur simultaneously in K'.

(ii) Determine the time t' when the events are observed in K'.

2. (a) Starting from the contravariant four-vector (world-point) x^i and the infinitesimal interval $\mathrm{d}s$,

(i) state the four-velocity

$$u^i = \frac{\mathrm{d}}{\mathrm{d}s} x^i$$

in terms of derivatives with respect to time t. Further

(ii) calculate $u^i u_i$ and

(iii) calculate

$$u_i \frac{\mathrm{d}u^i}{\mathrm{d}s}.$$

(b) Derive an expression for $p^i u_i$, where p^i denotes the four-momentum.

(c) Given the electromagnetic field tensor F^{ij}, determine

$$\mathcal{F} = \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} F^{ij}$$

and state how it Lorentz-transforms.

(d) Inertial frame K' is observed from K to move away with constant relative velocity $V > 0$ along the x-axis. Their axes are parallel. A particle moves in K' along the y-axis with velocity $v'_y > 0$.

(i) Determine the magnitude of the particle's velocity w as observed from K.

(ii) Demonstrate that if $V < c$ and $v'_y < c$ then the velocity w obtained in (i) obeys $w < c$.

3. A particle with charge e and mass m moves in a constant electric field $\mathbf{E} = (0, E, 0)$. At time $t = 0$ the particle has position $\mathbf{r} = (0, 0, z_0)$ and velocity $\mathbf{v} = (v_0, 0, 0)$.

(a) Determine the particle's kinetic energy $\mathcal{E}$ as a function of time t.

(b) Determine

(i) the particle's velocity $\mathbf{v}(t)$ as a function of time (and initial conditions) and

(ii) integrate it to determine the particle's position as a function of time.
Hint: $\int \mathrm{d}x\, 1/\sqrt{1+x^2} = \operatorname{arcsinh}(x) + \text{const.}$.

(c) Determine the electric $\mathbf{E}'$ and magnetic $\mathbf{H}'$ fields at time $t = 0$ in the rest frame of the particle. State $\mathbf{E} \cdot \mathbf{H}$ and $\mathbf{E}' \cdot \mathbf{H}'$.

4. (a) The electric field of a circular polarised, monochromatic wave in vacuum (in the absence of any charges and currents) may be given as

$$\mathbf{E}(\mathbf{r}, t) = E\Big(0, \cos(kx - \omega t), \sin(kx - \omega t)\Big) ,$$

with $\mathbf{r} = (x, y, z)$. State the corresponding magnetic field.

(b) A particle with mass m, charge e and velocity $\mathbf{v}(t)$ is placed in the electric and magnetic fields of the wave in (a). State the force $\mathrm{d}\mathbf{p}/\mathrm{d}t$ acting on it (explicitly, component by component).

(c) Assuming that the particle is located at the origin at $t = 0$,

(i) determine its initial velocity so that the particle's energy $\mathcal{E}$ remains constant in time.

(ii) Qualitatively describe the motion of the particle.

It can be assumed that the particle is massive, *i.e.* its speed must be strictly less than the speed of light. Hint: It will turn out that the x-component of the velocity, $v_x(t)$, is constant in time and therefore determined solely by the initial condition.

5. (a) State the principle of least action in terms of a generalised Lagrangian L.

(b) State the Lagrangian L for a free point particle (no external potential or force) in

(i) a classical system (*i.e.* in classical analytical mechanics) and

(ii) a Lorentz-invariant system (*i.e.* in special relativity).

(c) Demonstrate that the Lorentz-invariant free particle reduces to a classical one as $c \to \infty$.

(d) Energy and momentum in a classical system are given by

$$\mathcal{E} = \dot{\mathbf{r}} \cdot \nabla_{\dot{\mathbf{r}}} L - L \qquad \text{and} \qquad \mathbf{p} = \nabla_{\dot{\mathbf{r}}} L$$

respectively. Here, $\nabla_{\dot{\mathbf{r}}} L$ denotes the gradient of L with respect to $\dot{\mathbf{r}}$.
State the conditions of and derive from the symmetries of a classical Lagrangian:

(i) energy conservation and

(ii) momentum conservation.

[M345A6 (2017) Solutions. Setter: Pruessner; Checker: Barnett]

I should say that some of the questions below are paraphased or slightly adopted from Nolting, "Grundkurs: Theoretische Physik", Vol. 2 and Vol. 3, Verlag Zimmermann-Neufang, Ulmen 1993.

1a *(5 marks, seen many times, most basic bookwork)*
Lorentz transform:

$$\boxed{A^0 = \frac{A'^0 + (V/c)A'^1}{\sqrt{1 - V^2/c^2}} \qquad A^1 = \frac{A'^1 + (V/c)A'^0}{\sqrt{1 - V^2/c^2}} \qquad A^2 = A'^2 \qquad A^3 = A'^3}$$

1b *(7 marks, application of the above, but inverse here)*
Using the notation $\mathbf{r}_2 = (x, 0, 0)$ and $\mathbf{r}'_2 = (x', 0, 0)$, the inverse Lorentz transform gives $x' = \gamma(x - Vt_2)$ with $\gamma = 1/\sqrt{1 - V^2/c^2}$ and therefore $\gamma = 5/3$, because $t_2 = 0$. It follows that (i) $\boxed{V = (4/5)c}$. Using the inverse transform again, $ct'_2 = \gamma(ct_2 - (V/c)x)$, and therefore (ii) $\boxed{t'_2 = -(5/3)(4/5)L/c = -(4/3)L/c}$.

1c *(8 marks, application of Lorentz transform, but inverse and additional application to simultaneity)*
Using the notation $\mathbf{e}_1^i = (a, a, 0, 0)$ and $\mathbf{e}_2^i = (a/2, 2a, 0, 0)$ their (inverse) transforms are

$$\mathbf{e}_1'^i = ((1 - \beta)\gamma a, (1 - \beta)\gamma a, 0, 0) \qquad \text{and} \qquad \mathbf{e}_2'^i = ((1/2 - 2\beta)\gamma a, (2 - \beta/2)\gamma a, 0, 0) \quad ,$$

where $\gamma = 1/\sqrt{1 - V^2/c^2}$ and $\beta = V/c$. To have both events occur simultaneously, we need $1 - \beta = 1/2 - 2\beta$ and therefore $\beta = -1/2$, *i.e.* (i) $\boxed{V = -c/2}$. It follows that $\gamma = 2/\sqrt{3}$ and therefore $(1 - \beta)\gamma = \sqrt{3}$, so that (ii) $\boxed{t' = \sqrt{3}a/c}$, using $\mathbf{e}_1'^1 = t'c = (1 - \beta)\gamma a$.

2a *(5 marks, seen similar)*
This could be quoted straight from the book (LL), alternatively $\mathrm{d}s = c\sqrt{1 - V^2/c^2}\mathrm{d}t$ and $x^i = (ct, \mathbf{r})$, so (i)

$$\boxed{u^i = \left(\frac{1}{\sqrt{1 - V^2/c^2}}, \frac{\dot{\mathbf{r}}}{c\sqrt{1 - V^2/c^2}}\right)}$$

where $V = |\dot{\mathbf{r}}|$. By explicit calculation (ii)

$$\boxed{u^i u_i = \frac{1}{1 - V^2/c^2} - \frac{\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}/c^2}{1 - V^2/c^2} = 1 \ .}$$

Since $u^i \mathrm{d}u_i/\mathrm{d}s = u_i \mathrm{d}u^i/\mathrm{d}s$, we have $\mathrm{d}/\mathrm{d}s\left(u^i u_i\right) = 2u_i \mathrm{d}u^i/\mathrm{d}s$ and thus (iii) $\boxed{u_i \frac{\mathrm{d}u^i}{\mathrm{d}s} = 0}$.

2b *(5 marks, unseen)*
The four-momentum is $p^i = (\mathcal{E}/c, m\dot{\mathbf{r}}/\sqrt{1 - V^2/c^2})$ with $\mathcal{E} = mc^2/\sqrt{1 - V^2/c^2}$ and so

$$\boxed{p^i u_i = \frac{\mathcal{E}}{c\sqrt{1 - V^2/c^2}} - \frac{mV^2}{c(1 - V^2/c^2)} = mc\left(\frac{1}{1 - V^2/c^2} - \frac{V^2/c^2}{1 - V^2/c^2}\right) = mc \ .}$$

2c *(5 marks, unseen)*
From $\partial F^{ik}/\partial x^k = -\frac{4\pi}{c} j^i$ and $\partial j^i/\partial x^i = 0$ it follows that $\boxed{\mathcal{F} = \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} F^{ij} = 0}$, which is a Lorentz-scalar and thus remains constant under Lorentz transformations.

2d *(5 marks, seen similar)*
The transformed velocity in the K frame is

$$v_y = \frac{v'_y\sqrt{1 - V^2/c^2}}{1 + v'_x V/c^2}$$

with $v'_x = 0$, so the magnitude w is given by (i)

$$\boxed{w^2 = V^2 + v_y^2 = V^2 + v_y'^2(1 - V^2/c^2) = V^2 + v_y'^2 - v_y'^2 V^2/c^2}\ .$$

To show that $w < c$ for $v'_y, V < c$, we introduce $\alpha = v'_y/c < 1$, $\beta = V/c < 1$, $\omega = w/c$, so $\omega^2 = \alpha^2 + \beta^2 - \alpha^2\beta^2$, and we need to show $\omega^2 < 1$. Since $\alpha^2 < 1$ and $\beta^2 < 1$ we have $(1 - \alpha^2)(1 - \beta^2) > 0$, and therefore (ii) $\boxed{1 > \beta^2 + \alpha^2 - \alpha^2\beta^2 = \omega^2}$.

3a *(5 marks, seen similar)*
From $\dot{\mathbf{p}} = e\mathbf{E}$ we have immediately $\mathbf{p} = (p_x, eEt, 0)$ where p_x is determined solely by the initial condition, $p_x = mv_0/\sqrt{1 - v_0^2/c^2}$. From $\mathcal{E}^2 = m^2c^4 + \mathbf{p}^2c^2$ the kinetic energy is therefore

$$\begin{aligned}\mathcal{E}(t) &= \sqrt{m^2c^4 + (eEtc)^2 + (mv_0c)^2/(1 - v_0^2/c^2)} \\ &= \sqrt{(eEtc)^2 + m^2c^4/(1 - v_0^2/c^2)} = \boxed{\sqrt{(eEtc)^2 + \mathcal{E}(0)^2} = \mathcal{E}(t)}\ ,\end{aligned}$$

where $\mathcal{E}(0) = mc^2/\sqrt{1 - v_0^2/c^2}$.

3b *(10 marks, seen similar)*
From $\mathbf{p} = m\mathbf{v}/\sqrt{1 - V^2/c^2}$ and $\mathcal{E} = mc^2/\sqrt{1 - V^2/c^2}$ it follows that $\mathbf{v} = c^2\mathbf{p}/\mathcal{E}$ and thus (i)

$$\boxed{v_x(t) = \frac{c^2 p_x}{\sqrt{(eEtc)^2 + \mathcal{E}(0)^2}}}$$

$$\boxed{v_y(t) = \frac{c^2 eEt}{\sqrt{(eEtc)^2 + \mathcal{E}(0)^2}}}$$

$$\boxed{v_z(t) = 0\ .}$$

Integrating $v_x(t)$ is based on the hint, (ii)

$$x(t) = \int_0^t \mathrm{d}t' \frac{c^2 p_x}{\sqrt{(eEt'c)^2 + \mathcal{E}(0)^2}} = \frac{cp_x}{eE}\int_0^{cEtc/\mathcal{E}(0)} \mathrm{d}x \frac{1}{\sqrt{x^2 + 1}} = \boxed{\frac{cp_x}{eE}\,\mathrm{arcsinh}\left(\frac{eEtc}{\mathcal{E}(0)}\right) = x(t)}\,.$$

Integrating $v_y(t)$ is a matter of inspection,

$$v_y(t) = \dot{y}(t) = \frac{1}{eE}\frac{\mathrm{d}}{\mathrm{d}t}\sqrt{(eEtc)^2 + \mathcal{E}(0)^2}$$

so that

$$\boxed{y(t) = \frac{1}{eE}\left(\sqrt{(eEtc)^2 + \mathcal{E}(0)^2} - \mathcal{E}(0)\right)}\ ,$$

as $y(0) = 0$. Finally, $\boxed{z(t) = z_0}$ remains unchanged.

3c *(5 marks, bookwork)*
Standard transformations (see §24 in LL) simplify significantly as the magnetic field $\mathbf{H}$ vanishes in the rest frame of the electric field and so $\boxed{\mathbf{E}' = (0, E/\sqrt{1 - v_0^2/c^2}, 0)}$ and $\boxed{\mathbf{H}' = (0, 0, -v_0 E/(c\sqrt{1 - v_0^2/c^2}))}$, as v_0 is the initial velocity of the particle frame moving away from the field frame. The dot-product $\mathbf{E}\cdot\mathbf{H}$ is of course an invariant, $\mathbf{E}\cdot\mathbf{H} = 0 = \mathbf{E}'\cdot\mathbf{H}'$ as is $\mathbf{E}^2 - \mathbf{H}^2 = E^2 = \mathbf{E}'^2 - \mathbf{H}'^2$. To determine $\mathbf{H}'$ it therefore suffices to know how $\mathbf{E}$ transforms at $\mathbf{H} = 0$.

4a *(5 marks, bookwork)*
The easy way is to notice that the wave is propagating in the x-direction, so the normal along the direction of propagation is $\mathbf{n} = (1, 0, 0)$ and $\mathbf{H} = \mathbf{n} \times \mathbf{E}$, so

$$\boxed{\mathbf{H} = E\big(0, -\sin(kx - \omega t), \cos(kx - \omega t)\big)\ .}$$

Alternatively, $\nabla \times \mathbf{E} = -\frac{1}{c}\dot{\mathbf{H}}$, so

$$-\frac{1}{c}\dot{\mathbf{H}} = \nabla \times \mathbf{E} = E\big(0, -k\cos(kx - \omega t), -k\sin(kx - \omega t)\big)\ .$$

In the absence of charges and currents the integration constant vanishes, so

$$-\frac{1}{c}\mathbf{H} = E\frac{k}{\omega}(0, \sin(kx - \omega t), -\cos(kx - \omega t))\ ,$$

which reproduces the result above, as $kc = \omega$.

4b *(5 marks, unseen)*
This is a matter of evaluating

$$\dot{\mathbf{p}} = e\mathbf{E} + \frac{e}{c}\mathbf{v} \times \mathbf{H}\ ,$$

and the only mild complication is $\mathbf{v} \times \mathbf{H}$, resulting in

$$\boxed{\dot{p}_x = E\frac{e}{c}\big(v_y \cos(kx - \omega t) + v_z \sin(kx - \omega t)\big)}$$

$$\boxed{\dot{p}_y = eE\cos(kx - \omega t) + E\frac{e}{c}\big(-v_x \cos(kx - \omega t)\big)}$$

$$\boxed{\dot{p}_z = eE\sin(kx - \omega t) + E\frac{e}{c}\big(-v_x \sin(kx - \omega t)\big)}$$

4c *(10 marks, unseen)*
(i) If $\mathcal{E} = mc^2/\sqrt{1 - V^2/c^2}$ does not change in time, then $V = |\mathbf{v}|$ remains constant and so $\dot{\mathbf{p}} = \dot{\mathbf{v}}m/\sqrt{1 - V^2/c^2} = \dot{\mathbf{v}}\tilde{m}$ with $\tilde{m} = m/\sqrt{1 - V^2/c^2}$, *i.e.* $\dot{\mathbf{p}} \propto \dot{\mathbf{v}}$. The equation of motion therefore is

$$\begin{aligned}
\dot{v}_x &= \frac{eE}{\tilde{m}}\Big(\frac{v_y}{c}\cos(kx - \omega t) + \frac{v_z}{c}\sin(kx - \omega t)\Big) \\
\dot{v}_y &= \frac{eE}{\tilde{m}}\cos(kx - \omega t)\Big(1 - \frac{v_x}{c}\Big) \\
\dot{v}_z &= \frac{eE}{\tilde{m}}\sin(kx - \omega t)\Big(1 - \frac{v_x}{c}\Big)
\end{aligned}$$

where x and all velocities on the RHS are functions of time. Since $\mathrm{d}/\mathrm{d}t\mathcal{E} = e\mathbf{E}\cdot\mathbf{v} = 0$ we necessarily have

$$0 = v_y \cos(kx - \omega t) + v_z \sin(kx - \omega t)$$

and it follows from the equation of motion that $\dot{v}_x = 0$, *i.e.* v_x is constant in time as well, $v_x(t) = v_x(0)$ and $x(t) = v_x(0)t$ from $x(0) = 0$, so that the equation of motion can be summarised as

$$\dot{\mathbf{v}} = A\Big(0, \cos(kv_x(0)t - \omega t), \sin(kv_x(0)t - \omega t)\Big)$$

with $A = eE(1 - v_x(0)/c)/\tilde{m}$. By integration it follows that

$$\mathbf{v}(t) = \mathbf{v}(0) + \frac{A}{kv_x(0) - \omega}\Big(0, \sin(kv_x(0)t - \omega t), -\cos(kv_x(0)t - \omega t)\Big) ,$$

which is singular only if $kv_x(0) = \omega$, *i.e.* $v_x(0) = c$, which is not allowed by assumption.

All of the above follows necessarily from $\mathrm{d}/\mathrm{d}t\mathcal{E} = 0$ and the equation of motion effectively determined in part (b). We still need to determine a sufficient $\mathbf{v}(0)$ such that $\mathrm{d}/\mathrm{d}t\mathcal{E} = 0$ for all times. Using the result for $\mathbf{v}(t)$ obtained above in $\mathrm{d}/\mathrm{d}t\mathcal{E} = e\mathbf{E}\cdot\mathbf{v} = 0$ we need

$$\begin{aligned} 0 = \big(v_y(0) - \frac{A}{\omega}\sin(kx - \omega t)\big)\cos(kx - \omega t) + \big(v_z(0) + \frac{A}{\omega}\cos(kx - \omega t)\big)\sin(kx - \omega t) \\ = v_y(0)\cos(kx - \omega t) + v_z(0)\sin(kx - \omega t) \end{aligned}$$

for all t, which means $v_y(0) = 0$ and $v_z(0) = 0$, unless $kx - \omega t = kv_x(0)t - \omega t$ is constant in time, which is possible only if $v_x(0) = c$ (not allowed by assumption).

The particle's velocity therefore is

$$\mathbf{v}(t) = (v_x(0), 0, 0) + \frac{eE}{\tilde{m}\omega}(1 - \frac{v_x(0)}{c})\Big(0, -\sin(kv_x(0)t - \omega t), \cos(kv_x(0)t - \omega t)\Big)$$

(which confirms $|\mathbf{v}(t)|^2$ constant in time) and thus initially

$$\boxed{\mathbf{v}(0) = \Big(v_x(0), 0, \frac{eE}{\tilde{m}\omega}(1 - \frac{v_x(0)}{c})\Big)}$$

with arbitrary $|v_x(0)| < c$.

(ii) Qualitatively, the particle $\boxed{\text{moves along a circular path}}$, which is expected as $\mathbf{E}\cdot\mathbf{v}$ vanishes (*i.e.* the particle's velocity is always perpendicular to the electric field of circular polarised electromagnetic wave). In fact $V = |\mathbf{v}|$ being constant at constant $v_x(t)$ and non-constant $v_y(t)$ and $v_z(t)$ allows only circular motion.

5a *(4 marks, bookwork)*
The principle of least action is $\delta S = 0$ with

$$\boxed{S = \int_{t_1}^{t_2} \mathrm{d}t\, L(\mathbf{q}, \dot{\mathbf{q}}, t)}$$

5b *(4 marks, bookwork)*
The Lagrangian of a free particle is (i) classically

$$\boxed{L_{\mathrm{cl.}}(\mathbf{r}, \dot{\mathbf{r}}, t) = \frac{1}{2}m\dot{r}^2 = L_{\mathrm{cl.}}(\dot{r})}$$

and (ii) relativistically

$$\boxed{L_{\mathrm{rel.}}(\mathbf{r}, \dot{\mathbf{r}}, t) = -mc^2\sqrt{1 - \dot{r}^2/c^2} = L_{\mathrm{rel.}}(\dot{r})\ .}$$

5c *(4 marks, seen similar)*

The key-step is to show that $L_{\text{rel.}}$ becomes $L_{\text{cl.}}$ plus a constant in large c, or more precisely, that there is a constant C (independent of particle coordinates and time, so that differentiation does not produce extra terms) such that

$$\lim_{c\to\infty} L_{\text{rel.}} - C = L_{\text{cl.}}$$

Choosing $C = -mc^2$ the left hand side is the limit of $-mc^2(\sqrt{1-\dot{r}^2/c^2}-1) = -mc^2(-(1/2)\dot{r}^2/c^2 +$ h.o.t.), so that in fact

$$\boxed{\lim_{c\to\infty} L_{\text{rel.}} + mc^2 = \frac{1}{2}m\dot{r}^2 = L_{\text{cl.}}}$$

as required.

5d *(8 marks, unseen)*

(i) The Lagrangian of a $\boxed{\text{closed system does not depend explicitly on time}}$ and so the classical Lagrangian is of the form $L_{\text{cl.}}(\mathbf{r}, \dot{\mathbf{r}})$, so that

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \dot{\mathbf{r}} \cdot \nabla_{\mathbf{r}} L + \ddot{\mathbf{r}} \cdot \nabla_{\dot{\mathbf{r}}} L$$

From Euler-Lagrange $\nabla_{\mathbf{r}} L = \mathrm{d}/\mathrm{d}t\,(\nabla_{\dot{\mathbf{r}}} L)$, so

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \dot{\mathbf{r}} \cdot \left(\frac{\mathrm{d}}{\mathrm{d}t}\nabla_{\dot{\mathbf{r}}} L\right) + \left(\frac{\mathrm{d}}{\mathrm{d}t}\dot{\mathbf{r}}\right) \cdot \nabla_{\dot{\mathbf{r}}} L = \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{\mathbf{r}} \cdot \nabla_{\dot{\mathbf{r}}} L\right)$$

and therefore

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{\mathbf{r}} \cdot \nabla_{\dot{\mathbf{r}}} L - L\right) = \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} = 0}$$

which means that $\mathcal{E} = (\dot{\mathbf{r}} \cdot \nabla_{\dot{\mathbf{r}}} L - L)$, also known as the energy, is a conserved quantity.

(ii) The mechanical properties of a $\boxed{\text{spatially homogeneous, closed system}}$ do not change under any parallel displacement of the entire system in space, so that

$$\nabla_{\mathbf{r}} L = 0\ .$$

Invoking again the Euler-Lagrange equation, it follows that

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\nabla_{\dot{\mathbf{r}}} L = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{p} = 0}$$

which implies that $\mathbf{p} = \nabla_{\dot{\mathbf{r}}} L$, also known as the momentum, is a conserved quantity.

Imperial College London
Department of Mathematics

Examiner's Comments

Exam: M3A6 **Session: 2016-2107**

Question 1

Please use the space below to comment on the candidates' overall performance in the exam. A brief paragraph highlighting common mistakes and parts of questions done badly (or well) is sufficient. Do not refer to individual candidates. The purpose of this exercise is to provide guidance to the external examiners, and to the candidates themselves, on how you feel the cohort faired. Your comments will be available to students online.

Most candidates scored highly in this question, which asks for the basics of Lorentz-transformations. I have marked this question generously, ~~[illegible]~~ discounting occasional sign mistakes and confusions of units (superfluous factors of c etc.).
The only ~~pattern~~ pattern I could spot in the rare errors that were made is the failure to solve for v/c in equations that contain combinations of powers of γ and β.

Marker: Gunnar Pruessner

Signature: [illegible] **Date:** 13 May 2017

Please return with exam marks (one report per marker)

Imperial College London
Department of Mathematics

Examiner's Comments

Exam: M3A6 **Session: 2016-2107**

Question 2

Please use the space below to comment on the candidates' overall performance in the exam. A brief paragraph highlighting common mistakes and parts of questions done badly (or well) is sufficient. Do not refer to individual candidates. The purpose of this exercise is to provide guidance to the external examiners, and to the candidates themselves, on how you feel the cohort faired. Your comments will be available to students online.

This question did contain a good part of bookwork and students did generally well. Some have made extensive use of more ~~d~~ sophisticated identities. Not all realized that Lorentz-scalars don't transform (2c) and most took rather brutal short-cuts to derive the inequality in 2d, which I allowed.
Surprisingly many candidates failed to state v_y^* in terms of v_y' and V in 2d.

Marker: Gunnar Pruessner

Signature: [signature] **Date:** 13 May 2017

Please return with exam marks (one report per marker)

Imperial College London
Department of Mathematics

Examiner's Comments

Exam: M3A6 **Session: 2016-2107**

Question 3

Please use the space below to comment on the candidates' overall performance in the exam. A brief paragraph highlighting common mistakes and parts of questions done badly (or well) is sufficient. Do not refer to individual candidates. The purpose of this exercise is to provide guidance to the external examiners, and to the candidates themselves, on how you feel the cohort faired. Your comments will be available to students online.

Most candidates did well in ~~the~~ part a, but many lost marks deriving the velocities in part b and subsequently integrating them.
Most candidates used some of the standard recipes, but some failed to notice basic features, such as $p_y = eEt$.
Most who managed to derive the velocities also managed to integrate them.

Marker: Gunnar Pruessner

Signature: [signature] **Date:** 13 May 2017

Please return with exam marks (one report per marker)

Imperial College London
Department of Mathematics

Examiner's Comments

Exam: M3A6 **Session: 2016-2107**

Question 4

Please use the space below to comment on the candidates' overall performance in the exam. A brief paragraph highlighting common mistakes and parts of questions done badly (or well) is sufficient. Do not refer to individual candidates. The purpose of this exercise is to provide guidance to the external examiners, and to the candidates themselves, on how you feel the cohort faired. Your comments will be available to students online.

Most candidates had little or no problems with part a and b (but some struggled with the details). Part c was generally not answered well. Some managed to get the basic properties out by some guessing, but in general a systematic path was rarely taken.

Some students struggled with basic vector cross products.

Marker: Gunnar Pruessner

Signature:

Date: 14 May 2017

Please return with exam marks (one report per marker)

Imperial College London
Department of Mathematics

Examiner's Comments

Exam: TBA6 **Session: 2016-2107**

Question ~~1~~ 5

Please use the space below to comment on the candidates' overall performance in the exam. A brief paragraph highlighting common mistakes and parts of questions done badly (or well) is sufficient. Do not refer to individual candidates. The purpose of this exercise is to provide guidance to the external examiners, and to the candidates themselves, on how you feel the cohort faired. Your comments will be available to students online.

This was mostly book work. Some candidates struggled with expanding $\sqrt{1-v^2/c^2}$. I was strict with signs and details, but most candidates did rather well in this question

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