Imperial College
London

M4P19/M5P19

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May-June 2016

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Integration Theory and Applications

Date: **Friday 6th May 2016**

Time: **14.00 – 16.30**

Time Allowed: **2 Hours 30 Mins**

This paper has Five Questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the relevant main book(s) are full.

Statistical tables will not be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers to each question as per the table below.

Raw Mark	Up to 12	13	14	15	16	17	18	19	20
Extra Credit	0	½	1	1 ½	2	2 ½	3	3 ½	4

- Each question carries equal weight.
- Calculators may not be used.

1.

(a) Let Ω be a set and $\mathcal{F}$ a collection of its subsets.

(i) What properties does $\mathcal{F}$ need to statisfy to be an algebra of sets?

(ii) When is $\mathcal{F}$ a σ-algebra?

(iii) Let $\mathcal{F}$ be a σ-algebra. What properties does the map $\mu : \mathcal{F} \to [0, \infty]$ need to satisfy to be a measure on $(\Omega, \mathcal{F})$?

(b) Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, i.e. μ is a measure on the measurable space $(\Omega, \mathcal{F})$. Define the map $\nu : \mathcal{F} \to [0, \infty]$ by

$$\nu(A) := \sup\{\mu(B) : B \subseteq A,\ B \in \mathcal{F} \text{ and } \mu(B) < \infty\}.$$

(i) Prove that ν is a measure on $(\Omega, \mathcal{F})$.

(ii) Show that if μ is σ-finite, then $\nu = \mu$.

(iii) Assume further that $\mu(A) = \infty$ for any $A \in \mathcal{F}$, such that $A \neq \emptyset$, and $\mu(A) = 0$ if $A = \emptyset$. Find the corresponding measure ν in this case.

2. Let $(\Omega, \mathcal{F})$ be a measurable space.

(a) Let μ_n, $n \in \mathbb{N}$, be an increasing family of measures on $(\Omega, \mathcal{F})$:

$$\mu_n(F) \leq \mu_{n+1}(F) \text{ for all } n \in \mathbb{N} \text{ and } F \in \mathcal{F}.$$

Prove that the formula $\mu(F) := \lim_{n\to\infty} \mu_n(F)$, for any $F \in \mathcal{F}$, defines a measure on $(\Omega, \mathcal{F})$.

(b) Let μ_n, $n \in \mathbb{N}$, be an arbitrary family of measures on $(\Omega, \mathcal{F})$. Show that $\mu(F) := \sum_{n\in\mathbb{N}} \mu_n(F)$ defines a measure on $(\Omega, \mathcal{F})$.

(c) Assume you are given the Lebesgue measure λ on the measurable space $([0,1), \mathcal{B}([0,1)))$. Apply part (b) to construct a measure λ' on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, such that

$$\lambda'([a,b]) = b - a \text{ for any interval } [a,b] \text{ with } -\infty < a \leq b < \infty \tag{1}$$

(here $\mathcal{B}(\mathbb{R})$, resp. $\mathcal{B}([0,1))$, is the Borel σ-algebra on the real line, resp. on $[0,1)$). Why is this measure a unique measure satisfying property (1)?

3.

(a) Let Ω be a set. Write short notes to define the following concepts:

(i) a Π-system of subsets of Ω;

(ii) the independence of an infinite sequence of σ-algebras $\mathcal{F}_i \subseteq \mathcal{F}$, $i \in \mathbb{N}$, on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

(b) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Suppose that Π_1, Π_2 are Π-systems of events from $\mathcal{F}$, such that $\mathbb{P}[A \cap B] = \mathbb{P}[A]\,\mathbb{P}[B]$ for all $A \in \Pi_1$, $B \in \Pi_2$. If $\mathcal{G}_1$, $\mathcal{G}_2$ are the σ-algebras generated by Π_1, Π_2 respectively, then show that $\mathcal{G}_1$, $\mathcal{G}_2$ are independent σ-algebras. (In your proof you may use basic results from lectures about Π-systems, so long as you state them clearly.)

(c) Suppose that X, Y are two random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and are such that

$$\mathbb{P}[X \leq x, Y \leq y] \quad = \quad \mathbb{P}[X \leq x]\,\mathbb{P}[Y \leq y] \qquad \text{for } x, y \in \mathbb{R}.$$

Two random variables X and Y are said to be independent if and only if the σ-algebras $\sigma(X)$ and $\sigma(Y)$, generated by X and Y respectively, are independent. Prove, using part (b) of this question, that the above equality implies the independence of X and Y. Show that if X, Y are integer-valued then

$$\mathbb{P}[X + Y \text{ is odd}] \quad = \quad \mathbb{P}[X \text{ is even}]\,\mathbb{P}[Y \text{ is odd}] + \mathbb{P}[X \text{ is odd}]\,\mathbb{P}[Y \text{ is even}]\,.$$

4.

(a) State (without proof) Carathéodory's extension theorem for σ-additive set functions on an algebra of sets.

(b) Construct the Lebesgue measure λ on the Borel σ-algebra $\mathcal{B}([0,1))$ on the interval $[0,1)$ by applying Carathéodory's extension theorem. Carefully describe the key steps in the construction of the Lebesgue measure using Carathéodorys extension theorem. (You do not need to prove any assertions you make in detail, and you may refer to results proved in the lectures provided they are carefully stated.)

(c) Prove that for any Borel set $A \in \mathcal{B}([0,1))$ it holds that

$$\lambda(A) = \inf\{\lambda(U) \, : \, U \text{ open in } [0,1) \text{ and } A \subseteq U\},$$

where λ is the Lebesgue measure constructed in part (b) of this question.

(d) Prove that any countable set $A \subseteq [0,1)$ is in $\mathcal{B}([0,1))$ and find $\lambda(A)$.

(e) Let A denote the set $\mathbb{Q} \cap [0,1)$ of all non-negative rational numbers smaller than one. Is $[0,1)$ the only open subset of $[0,1)$ that contains A? Either provide a proof or find an open set U in $[0,1)$ such that $A \subseteq U$ and $U \neq [0,1)$.

5. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

(a) Write short notes to define the following notions of convergence of random variables, and state the implications between them.

   (i) convergence almost surely;

   (ii) convergence in 1-norm;

   (iii) convergence in probability.

(b) Consider a sequence of random variables Z_1, Z_2, ... defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and such that each Z_n takes only the values 0 and $\sqrt{n}$, and suppose that $\mathbb{P}[Z_n = \sqrt{n}] = 1/n^2$. Explain with reasons whether the sequence converges almost surely, in 1-norm, or in probability.

(c) Suppose that a sequence of random variables X_1, X_2, ... is defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and satisfies the bound $|X_n| \leq W$, where W is a non-negative integrable random variable. Define the notion of uniform integrability for a countable family of random variables and prove that the sequence X_1, X_2, ... is uniformly integrable.

(d) Suppose that $X_n \to 1$ almost surely and that $|X_n| \leq 2$ for all n. Let U_n, for any $n \in \mathbb{N}$, be a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ that satisfies $\mathbb{P}[a \leq U_n \leq b] = b - a$ for any $0 \leq a \leq b \leq 1$. Consider the sequence

$$Y_n = (1 + \mathbb{I}[U_n \leq 1/n])X_n,$$

where $\mathbb{I}[U_n \leq 1/n] = 1$ exactly when $U_n \leq 1/n$, and otherwise is zero. Prove the following statements:

   (i) The sequence Y_1, Y_2, ... is uniformly integrable.

   (ii) Assume that X_n and U_n are independent for all $n \in \mathbb{N}$ and prove that $Y_n \to 1$ in 1-norm. Hint: You may use without proof that for any independent random variables X and Y the equality $\mathbb{E}[f(X)g(Y)] = \mathbb{E}[f(X)]\mathbb{E}[g(Y)]$ holds (for any bounded measurable functions f and g), where $\mathbb{E}[\cdot]$ denotes the Lebesgue integral with respect to the measure $\mathbb{P}$.

Measure and Integration M3P19 – Solutions

Setter:

February 13, 2016

1: (a) (i) **(BOOKWORK)**

A algebra is a family of subsets of a fixed set Ω such that:

1. the family contains $\emptyset$;
2. it is closed under complements;
3. it is closed under finite union.

(or any equivalent variation on these axioms!)
Marks: 2

(ii) **(BOOKWORK)**

σ-algebra is an algebra that is also closed under countable union.
Marks: 1

(iii) **(BOOKWORK)**

μ is a measure if it is σ-additive: $\mu(\cup_{n\in\mathbb{N}}A_n) = \sum_{n\in\mathbb{N}}\mu(A_n)$ for any pairwise disjoint collection of sets $A_n \in \mathcal{F}$, $n \in \mathbb{N}$.
Marks: 2

(b) (i) **(UNSEEN)**

First note that

$$\nu(A) = \mu(A) \qquad \forall A \in \mathcal{F} \text{ such that } \mu(A) < \infty. \tag{1}$$

Hence by the definition of ν we have for any disjoint family $A_n \in \mathcal{F}$:

$$\nu(\cup_{n\in\mathbb{N}}A_n) = \sup\{\sum_{n\in\mathbb{N}}\mu(B\cap A_n) : B \subseteq \cup_{n\in\mathbb{N}}A_n,\ B \in \mathcal{F} \text{ and } \mu(B) < \infty\} \leq \sum_{n\in\mathbb{N}}\nu(A_n).$$

If $\nu(\cup_{n\in\mathbb{N}}A_n) = \infty$, then by this inequality we have $\nu(\cup_{n\in\mathbb{N}}A_n) = \sum_{n\in\mathbb{N}}\nu(A_n)$. If $\nu(\cup_{n\in\mathbb{N}}A_n) < \infty$, then by (1) and countable additivity of μ it holds that

(c) **(SIMILAR TO EXAMPLE IN LECTURES)**
Define $A_n := [n, n+1)$ for any $n \in \mathbb{Z}$ and note that in the lectures we have constructed the Lebesgue measure on λ on the Borel σ-algebra of A_0. We can define the family of measures λ_n on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ by

$$\lambda_n(B) := \lambda(\{x \in A_0 : x + n \in B \cap A_n\}) \qquad \forall B \in \mathcal{B}(\mathbb{R}).$$

The set $\{x \in A_0 : x + n \in B \cap A_n\}$ is Borel measurable as it is a preimage under a measurable function of the measurable set $B \cap A_n$. Then by part (b) we have that the set function $\lambda' : \mathcal{B}(\mathbb{R}) \to [0, \infty]$, defined by

$$\lambda'(B) := \sum_{n \in \mathbb{Z}} \lambda_n(B) \qquad \forall B \in \mathcal{B}(\mathbb{R}),$$

is in fact a measure on $\mathcal{B}(\mathbb{R})$. It is clear from the properties of the Lebesgue measure λ on $\mathcal{B}(A_0)$ and this definition that λ' maps each bounded interval to its length. To prove that λ' is unique, note that for any $n \in \mathbb{N}$ its restriction on $[-n, n)$ is a finite measure, determined uniquely by its values on the Π-system $\{[a, b) : -n \leq a \leq b \leq n\}$ which generates $\mathcal{B}([-n, n)) = [-n, n) \cap \mathcal{B}(\mathbb{R})$ (this equality proved in the lectures). Hence uniqueness follows by the uniqueness-of-extension lemma from the lectures and part (a) of this question.
Marks: 8

Total mark: 20

3: (a) (i) **(BOOKWORK)**
A collection of subsets of Ω is a Π-system if it is closed under intersection.
Marks: 2

(ii) **(BOOKWORK)**
Sub-sigma algebras $\mathcal{F}_i$, $i \in \mathbb{N}$, are independent if and only if for any finite collection of indices $i_1 < \cdots < i_n$ and any sets $A_{i_j} \in \mathcal{F}_{i_j}$ it holds that

$$\mathbb{P}[A_{i_1} \cap \cdots \cap A_{i_n}] = \mathbb{P}[A_{i_1}] \cdots \mathbb{P}[A_{i_n}].$$

Marks: 4

4: (a) **(BOOKWORK)**

Theorem. Let $\mathcal{F}_0$ be an algebra on a set Ω and $\mu_0 : \mathcal{F}_0 \to [0, \infty]$ a countably additive map. Then there exists a measure μ on the measurable space $(\Omega, \sigma(\mathcal{F}_0))$ such that $\mu = \mu_0$ on $\mathcal{F}_0$. Furthermore, if $\mu_0(\Omega) < \infty$, then the measure μ is unique.

Marks: 4

(b) **(BOOKWORK)**

We apply the theorem from part (a) to the following collection of subsets of $[0, 1)$:

$$\mathcal{F}_0 := \{F \subseteq [0,1) : \exists 0 \leq a_1 \leq b_1 \leq a_2 \leq b_2 \leq \ldots \leq a_r \leq b_r \leq 1, \text{ where } F = \cup_{i=1}^r [a_i, b_i)\}.$$

$\mathcal{F}_0$ is an algebra since $[0,1) \in \mathcal{F}_0$, $F \in \mathcal{F}_0$ implies $F^c \in \mathcal{F}_0$, $F, G \in \mathcal{F}_0$ implies $F \cap G \in \mathcal{F}_0$. Define $\mu_0 : \mathcal{F}_0 \to [0, 1]$ by

$$\mu_0(F) := \sum_{i=1}^r (b_i - a_i).$$

μ_0 is clearly well defined and finitely additive ($\mu_0(F \cup G) = \mu_0(F) + \mu_0(G)$ for disjoint $F, G \in \mathcal{F}_0$). We proved in the lectures that μ_0 is σ-additive. Since $\mu_0([0,1)) = 1 < \infty$, by theorem in (a) there exists a unique extension of μ_0, called the Lebesgue measure λ on $([0,1), \mathcal{B}([0,1)))$.

Marks: 4

(c) **(UNSEEN)**

There are several ways of proving this statement. The easiest is to recall from the lectures that in the proof of theorem in (a) we show that the measure λ from (b) above satisfies

$$\lambda(G) = \inf\{\sum_{n \in \mathbb{N}} \lambda(F_n) : F_n \in \mathcal{F}_0 \forall n \in \mathbb{N} \text{ and } G \subseteq \cup_{n \in \mathbb{N}} F_n\}.$$

Since each F_n is contained in an open set U_n, such that $\lambda(U_n \setminus F_n)$ is arbitrarily small, and the union of open sets is open, the statement follows.

Marks 5

(d) **(SIMILAR TO EXAMPLE IN LECTURES)**

Since the set A is of the form $\{x_n : n \in \mathbb{N}\}$ and σ-algebra is closed under countable union, it suffices to prove that a singelton $\{x_n\}$ is in $\mathcal{B}([0,1))$. This is clear as it can be expressed as a countable intersection of open intervals. This also shows that $\lambda(\{x_n\}) = 0$ and hence $\lambda(A) = 0$ by countable additivity of measures.

Marks: 3

(c) **(BOOKWORK)**
We need to show, for every $\varepsilon > 0$ we can find K such that $\mathbb{E}[|X_n|; |X_n| > K] < \varepsilon$ for all n.
Now $W\mathbb{I}[W > K] \to 0$ almost surely as $K \to \infty$ so, by dominated convergence theorem, $\mathbb{E}[W; W > K] \to 0$. Thus given ε we can find K such that $\mathbb{E}[W; W > K] \leq \varepsilon$.
Hence for every n

$$[\mathbb{E}[|X_n|; |X_n| > K] \quad \leq \quad \mathbb{E}[W; |X_n| > K] \quad \leq \quad \mathbb{E}[W; W > K] \quad \leq \quad \varepsilon.$$

Hence uniform integrability holds.
Mark 5

(d) (i) **(UNSEEN)**

$$|Y_n| \quad = \quad |(1 + \mathbb{I}[U_n \leq 1/n])X_n| \quad \leq \quad 2|X_n| \quad \leq \quad 4,$$

so uniform integrability holds by work in previous parts.
Mark 2

(ii) **(UNSEEN)**
Note that $|Y_n - 1| \leq |X_n - 1| + |X_n|\,\mathbb{I}[U_n \leq 1/n]$. Hence, by independence of U_n and X_n

$$\mathbb{E}[|Y_n - 1|] \quad \leq \quad \mathbb{E}[|X_n - 1|] + \tfrac{1}{n}\mathbb{E}[|X_n|] \leq \quad \mathbb{E}[|X_n - 1|] + \tfrac{2}{n}.$$

Now by dominated convergence $\mathbb{E}[|X_n - 1|] \to 0$, since $|X_n - 1| \leq 3$, and so $\mathbb{E}[|Y_n - 1|] \to 0$.
Mark 3

Total mark: 20

1. a) *State Poincaré's lemma for the compactly supported cohomology group $H_c^n(\mathbb{R}^n)$ and give a sketch of how it is proven*[**seen; 8 marks**]

The integration map defines an isomorphism

$$\int : H_c^n(\mathbb{R}^n) \to \mathbb{R}$$

The proof in the $n = 1$ case is by elementary calculus so we assume that $n \geq 2$. It suffices to show that the map is well-defined and injective (surjectivity is trivial). It is not difficult to see that the condition

$$\int_{\mathbb{R}^n} \omega = 0$$

is necessary for there to exist a compactly supported form α such that $d\alpha = \omega$. Namely because the form α is supposed compactly supported, we have that it is zero when restricted to a large enough sphere S_R^{n-1} of radius $R >> 0$. Then we have that

$$\int_{\mathbb{R}^n} \omega = \int_{B_R} \omega = \int_{S_R^{n-1}} \alpha = 0$$

To see the converse, assume that we are given two balls $B_R \subset B_{R'}$ which both contain the support of ω. Next, consider an arbitrary primitive α_0. This is not necessarily compactly supported. However, by Stokes' theorem, this form has the property that the integral

$$\int_{S_R^{n-1}} \alpha_0 = 0$$

(and similarly for R'). This implies that α_0 is itself *exact* when restricted to $U = \mathbb{R}^n \setminus B_R$ (notice that $d\alpha_0|U = 0$). That is to say that there exists η such that $d\eta = \alpha_0|U$. We consider a bump function ϕ such that

- $\phi = 0$ inside of B_R
- $\phi = 1$ in $U' = \mathbb{R}^n \setminus B'_R$.

Notice that the differential form $\phi\eta$ now can be extended to all of $\mathbb{R}^n$ by setting it to equal zero inside of B_R. Now consider the form

$$\alpha = \alpha_0 - d(\phi\eta)$$

Then α is compactly supported because in U', we have that

$$d(\phi\eta) = d\eta = \alpha_0$$

We also have

$$d\alpha = d\alpha_0 - d \circ d(\phi\eta) = d\alpha_0 = \omega$$

b) *Fix a star-shaped domain U in $\mathbb{R}^3$. Prove that for any vector field $\vec{F}$ on U with $curl(\vec{F}) = 0$, $\vec{F} = \nabla(S)$ for some function $S : U \to \mathbb{R}$ (∇ denotes the gradient of the function).*[**seen; 6 marks**]

Write $\vec{F} = (F_1, F_2, F_3)$ and set $\omega_F = F_1dx_1 + F_2dx_2 + F_3dx_3$. The condition $curl(\vec{F}) = 0$ translates to the condition that the one-form $d(F_1dx_1 + F_2dx_2 + F_3dx_3) = 0$. This implies that ω is exact by Poincaré's lemma and hence that $\omega_F = dS$. This in turn implies that $\vec{F} = \nabla(S)$.

c) *Calculate $H_c^p(\mathbb{R}^2 \setminus \{0\})$. Note: A suitable picture will suffice to show why the hypotheses of Poincaré duality hold.* [**unseen;6 marks**]

The manifold $\mathbb{R}^2 \setminus \{0\}$ is orientable because $\mathbb{R}^2$ is and has a good cover(for example, it can be divided into sectors whose pairwise intersections are either empty or overlap in sectors and all higher intersections are empty). Therefore Poincaré duality applies. We have an isomorphism of ordinary cohomologies $H^p(\mathbb{R}^2 \setminus \{0\}) \cong H^p(S^1)$. So the duality theorem implies that

$$H^p_c(\mathbb{R}^2 \setminus \{0\}) \cong (H^{2-p}(S^1))^\vee$$

Hence we have that $H^p(\mathbb{R}^2 \setminus \{0\}) =$

- 0 if $p \neq 1, 2$
- $\mathbb{R}$ if $p = 1, 2$

2. a) *Briefly describe the construction of the Lie algebra* $\mathfrak{g}$ *associated to any smooth Lie group* G. [Seen, **4 marks**]

There is an isomorphism between vector space $\mathfrak{g} = T_{id}G$ and the vector space of left-invariant vector fields on G. Left invariant vector fields are closed under Lie bracket and hence define an operation on $\mathfrak{g}$ which verifies all necessary identities.

b) *Explain why the Lie group* $SU(2)$ *is diffeomorphic to the three sphere* S^3. (Seen, [**4 marks**])

$SU(2)$ is by definition the group of complex 2×2 matrices A such that $det(A) = 1$ and [**2 marks**]

$$A^* A = Id$$

By comparing the formula for the inverse of a 2×2 matrix and the formula for the Hermitian transpose, we see that $SU(2)$ is precisely matrices of the form.

$$\begin{bmatrix} z_1 & -\overline{z_2} \\ z_2 & \overline{z_1} \end{bmatrix}$$

such that $|z_1|^2 + |z_2|^2 = 1$ [2 marks]. This is in turn the definition of the three-sphere. The identification preserves the smooth structure and so the result is proven.

c) *Fix a basis* v_i *for the Lie algebra of* $SU(2)$ *and write formulae for the brackets between all pairs* $[v_i, v_j]$ *in terms of your fixed basis.* (Seen, [**6 marks**])

We have calculated in class that the Lie algebra consists of matrices such that $tr(A) = 0$ and $A^* = -A$ [**2 marks**] . A basis for this is given by (for instance the matrices) [**2 marks**]

$$v_1 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$v_3 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

We have the relations

$$[v_1, v_2] = 2v_3$$

$$[v_3, v_1] = 2v_2$$

$$[v_2, v_3] = 2v_1 \quad \textbf{[2 marks]}$$

d) *Use the theorem of Chevalley-Eilenberg to give an alternative computation of* $H^1(S^3)$ *to that presented in class.* (Seen, [**6 marks**])

Let us denote the Lie algebra cohomology complex by C^*. We have that the map $C^0 \to C^1$ is zero since this corresponds to the de Rham differential of constant functions [**1 mark**]. In degree one, we have that C^1 is generated by vectors of the form v_1^*, v_2^*, v_3^* [**2 marks**]. We must prove that the kernel of d is zero. In this case, Cartan's formula simplifies:

$$d\omega(v_i, v_j) = \sum_{i<j}(-1)^{i+j}\omega([v_i, v_j])$$

So we have

$$dv_1^* = \pm 2v_2^* \wedge v_3^*$$
$$dv_2^* = \pm 2v_3^* \wedge v_1^*$$
$$dv_3^* = \pm 2v_1^* \wedge v_2^* \quad \textbf{[2 marks]}$$

The vectors $v_2^* \wedge v_3^*, v_3^* \wedge v_1^*, v_1^* \wedge v_2^*$ are linearly independent **[1 mark]** and hence the kernel of $d = 0$. Therefore $H^1(S^3) = 0$.

3. a) *Identify the cohomology ring of* $\mathbb{C}P^n$.**[Seen: 3 marks]**

$\mathbb{R}[u]/u^{n+1}$ where $deg(u) = 2$.

b) *In the case of* $\mathbb{C}P^2$, *describe explicit differential forms which represent generators of* $H^2(\mathbb{C}P^2)$ *and* $H^4(\mathbb{C}P^2)$. **[Unseen; 6 marks]**

Let us consider the chart $[1 : u : v]$ on $\mathbb{C}P^2$. Write $u = r_1 e^{i\theta_1}$, $v = r_2 e^{i\theta_2}$. Let $\eta = \frac{r_1^2 d\theta_1 + r_2^2 d\theta_2}{(1+r_1^2+r_2^2)}$ be the one-form on this chart. One checks in local coordinates that this the two-form $\omega = d\eta$ extends to all of $\mathbb{C}P^2$ **(2 marks)**. Notice that the restriction of the two-form ω to the $\mathbb{C}P^1$ defined as the locus of points with coordinates $[1 : u : 0]$ is a positive multiple of the volume form for every point of $\mathbb{C} \subset \mathbb{C}P^1$ and hence has a non-trivial integral over $\mathbb{C}P^1$ **(2 marks)**. As calculated in class, the restriction map

$$H^2(\mathbb{C}P^2) \to H^2(\mathbb{C}P^1)$$

is an isomorphism, and hence we have that $[\omega]$ is non-trivial in $H^2(\mathbb{C}P^1)$. Hence $[\omega^2]$ generates $H^4(\mathbb{C}P^2)$ **(2 marks)**.

c) *Let* M *and* N *be smooth, compact, orientable manifolds. Prove that if* $M \times N$ *is diffeomorphic to* $\mathbb{C}P^n$ *then either* M *or* N *is a point.* **[Unseen ; 6 marks]**

We have that $\mathbb{C}P^n$ is connected so both M and N are connected **(1 mark)**. Because $H^2(\mathbb{C}P^n)$ is one dimensional, we have that by the Kunneth formula that either M nor N has a non-trivial cohomology class $\bar{u}$ in degree 2 such that $u = \bar{u} \otimes 1$ (or $1 \otimes \bar{u}$) **(2 marks)**. In view of the ring structure we have that $\bar{u}^n \neq 0$ **(2 marks)** and hence that M has dimension at least $2n$. It follows that N must be a point **(1 mark)**.

d) *Prove that there is no orientation reversing diffeomorphism of* $\mathbb{C}P^2$.

Consider the generator u in $H^2(\mathbb{C}P^2)$. Any diffeomorphism ϕ would satisfy $\phi^*(u) = cu$ with $c \in \mathbb{R}$ non-zero. Thus $\phi^*(u^2) = c^2u^2$. Since $c^2 > 0$, ϕ cannot be orientation reversing. **[Seen; 5 marks]**

4. *Let* n *be an integer* ≥ 2 *and* $f : S^{2n-1} \to S^n$ *be a smooth map. Fix an* n*-form* ω *on* S^n *such that* $\int_{S^n} \omega = 1$.

a) For any f, *show that* $f^*(\omega)$ *is exact i.e. there exists* $\theta \in \Omega^{n-1}(S^{2n-1})$ *such that*

$$f^*(\omega) = d\theta.$$

[Unseen; 3 marks]

Observe that $d(f^*(\omega)) = f^*(d\omega) = 0$. Since $H^n(S^{2n-1}) = 0$, we have that $[f^*(\omega)] = 0$ and hence that $\omega = d\theta$.

Let θ be such a form. Define the Hopf invariant of f to be given by the formula

$$h(f) = \int_{S^{2n-1}} \theta \wedge f^*(\omega)$$

b) Prove the Hopf invariant is independent of the choice of θ. [Unseen; **6 marks**]

Suppose that θ' is a different form such that $d\theta' = \omega$. Then $d(\theta' - \theta) = 0$ and hence $\theta' - \theta = d\beta$ for some $\beta \in H^{n-2}(S^{2n-1})$ (**2 marks**). Then we have that the form

$$d(\beta \wedge f^*(\omega)) = d\beta \wedge f^*(\omega) = (\theta' - \theta) \wedge f^*(\omega)$$

(**2 marks**). Hence by Stokes' theorem, we have that the integral

$$\int (\theta' - \theta) \wedge f^*(\omega)$$

vanishes (**2 marks**).

c) Prove that the Hopf invariant is independent of the choice of ω.[Unseen; **5 marks**]
Suppose that ω' is a different choice. Then $\omega = \omega' + d\beta$ (**1 marks**). So we can choose $\theta' = \theta + f^*(\beta)$. We have that

$$\theta' \wedge f^*(\omega') = \theta \wedge f^*(\omega) + \theta \wedge f^* d\beta + f^*(\beta) \wedge f^*(\omega) + f^*(\beta) \wedge f^*(d\beta)$$

We have that $f^*(\beta) \wedge f^*(\omega) = f^*(\beta \wedge \omega) = 0$ and similarly for the fourth term(**2 marks**). Finally, we have that $\theta \wedge f^*(d\beta) = \pm d(\theta \wedge f^*(\beta))$ (**2 marks**).

d) Prove that given two smoothly homotopic maps f_1 and f_2, $h(f_1) = h(f_2)$. [Unseen; **6 marks**

We have a smooth map $H : S^{2n-1} \times I \to S^n$ which restricts to f_1 and f_2 along the two boundaries. Choose some primitive $\overline{\theta}$ on $S^{2n-1} \times I$ for $H^*(\omega)$. Consider $\overline{\theta} \wedge H^*(\omega)$. This form is closed because

$$d(\overline{\theta} \wedge H^*(\omega)) = H^*(\omega) \wedge H^*(\omega) = H^*(\omega \wedge \omega) = 0$$

Hence we have that the integral

$$\int_{\partial(S^{2n-1} \times I)} \overline{\theta} \wedge H^*(\omega) = 0$$

by Stokes' theorem and therefore that $h(f_1) = h(f_2)$.

5. *a) Prove that if n is odd, $h(f) = 0$.* [Unseen ; **4 marks**]

If n is odd, $\theta \wedge d\theta = 1/2 d(\theta \wedge \theta)$, since θ is an even degree form. The then result follows immediately from Stokes' theorem.

b) *Give a formula for a volume form on ω on S^2 such that $\int_{S^2} \omega = 1$.* [Seen; **4 marks**]
Viewing $S^2 \subset \mathbb{R}^3$ with coordinates (u_1, u_2, u_3) we have seen that the standard volume form is given by the restriction of

$$1/4\pi(u_1 du_2 du_3 - u_2 du_1 du_2 + u_3 du_1 du_2)$$

to S^2.

c) Let $f : S^3 \to S^2 \cong \mathbb{C}P^1$ be the standard Hopf map. Find an explicit differential form $\theta \in \Omega^1(S^3)$ such that $d\theta = f^*\omega$. [Unseen; **6 marks**]

View $\mathbb{C}P^1$ as $\mathbb{C} \cup \infty$. Under the Hopf map, and setting $dz = dx + idy$, $d\bar{z} = dx - idy$ a straightforward calculation shows that ω pulls back to

$$-i/2\pi(dz \wedge d\bar{z})/(1+|z|^2)^2$$

Since $z = z_0/z_1$, this extends over all of $\mathbb{C}P^1$ with projective coordinates $[z_0 : z_1]$ to the form

$$-i/2\pi\frac{(z_1 dz_0 - z_0 dz_1) \wedge (\bar{z}_1 d\bar{z}_0 - \bar{z}_0 d\bar{z}_0)}{(|z_0|^2 + |z_1|^2)^2}$$

Now consider coordinates x_1, x_2, x_3, x_4 on $\mathbb{R}^4$. We have that in these coordinates, this form is

$$-\frac{1}{\pi}(dx_1 dx_2 + dx_3 dx_4)$$

which is in turn

$$d(-\frac{1}{\pi}x_1 dx_2 + x_3 dx_4)$$

d) *Calculate $h(f)$ for the standard Hopf map.* [Unseen; **6 marks**]

We have that this is equal to the integral

$$-1/\pi^2(\int_{S^3} x_1 dx_2 dx_3 dx_4 + x_3 dx_1 dx_2 dx_4)$$

by symmetry this is the same as the integral

$$-2/\pi^2 \int_{S^3} x_1 dx_2 dx_3 dx_4$$

By Stokes' theorem this is the integral

$$-2/\pi^2 \int_{S^3} x_1 dx_2 dx_3 dx_4 = \int_{D^4} dx_1 dx_2 dx_3 dx_4$$

Since the volume of the standard 4-ball is $\pi^2/2$, we have that this integral equals to -1.