DTS AND COMPUTER CONTROL

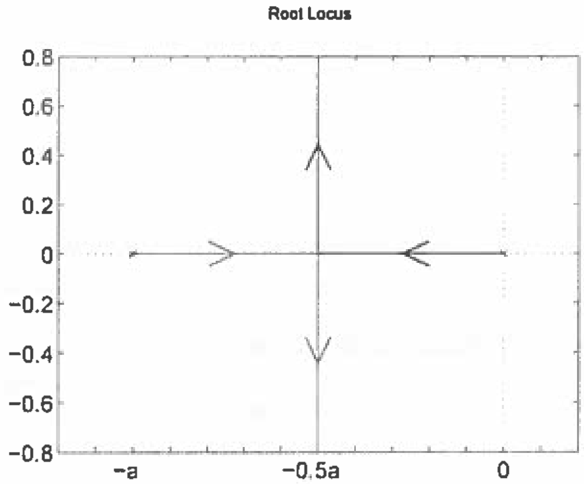


Figure 1.1 Root locus of $C(s)P(s)$.

1. a) Figure 1.1 shows the root locus of $C(s)P(s)$. The closed-loop transfer function is

$$\frac{C(s)P(s)}{1+C(s)P(s)} = \frac{K}{s^2+as+K}.$$

The characteristic polynomial of the closed-loop continuous-time system is

$$s^2+as+K.$$

By Routh test, all roots of the above polynomial are in the left half of the complex plane if $a>0$ and $K>0$. [4 marks]

b) The equivalent discrete-time model of the plant is

$$HP(z) = Z\left(\frac{1-e^{-sT}}{s}P(s)\right) = (1-z^{-1})Z\left(\frac{P(s)}{s}\right) = (1-z^{-1})Z\left(\frac{K}{s^2}\right)$$
$$= \frac{TKz^{-1}}{1-z^{-1}} = \frac{TK}{z-1}.$$

[3 marks]

c) To determine $C(z)$ with the impulse response invariance method we compute the z-transform of $C(s)$. As a result

$$C_I(z) = Z(C(s)) = \frac{z}{z-e^{-aT}}.$$

Using the pole-zero correspondence method yields

$$C_{PZ}(z) = k\,\frac{z+1}{z-e^{-aT}},$$

with $k = \dfrac{1-e^{-aT}}{2a}$ to match the DC gain ($s=0$, $z=1$) of the continuous-time controller. [3 marks]

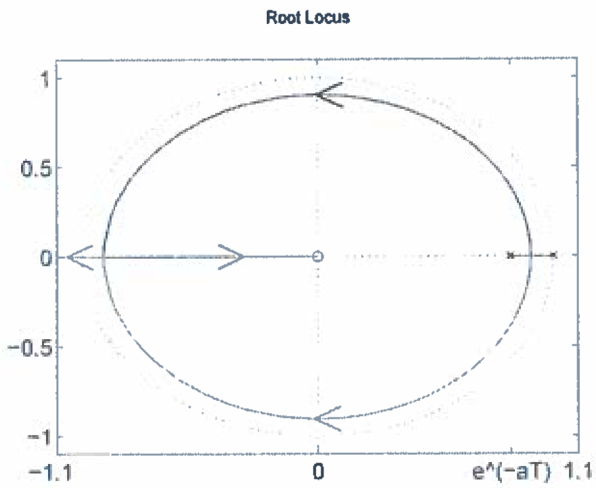


Figure 1.2 Root locus of $C_I(z)P(z)$.

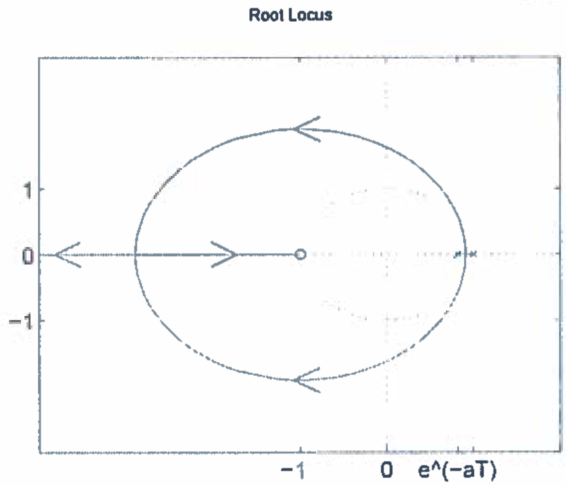


Figure 1.3 Root locus of $C_{PZ}(z)P(z)$.

d) Figure 1.2 shows the root locus of $C_I(z)HP(z)$. Figure 1.3 shows the root locus of $C_{PZ}(z)HP(z)$. [4 marks]

e) The characteristic polynomial when we use $C_I(z)$ is

$$z^2-(1+e^{-aT}-TK)z+e^{-aT}.$$

To determine the location of the roots of this polynomial we can use the bilinear transformation $z=\dfrac{w-1}{w+1}$ and then use the Routh test, or we can recall that the polynomial

$$z^2+\alpha z+\beta$$

has all roots in the unit circle if

$$\begin{aligned} &1+\alpha+\beta>0,\\ &1-\beta>0,\\ &1-\alpha+\beta>0. \end{aligned} \tag{1.1}$$

Applying these conditions yields that the closed-loop discrete-time system is asymptotically stable, when we use the controller $C_I(z)$, for any value of K such that

$$0 < K < \frac{2}{T}\left(1+e^{-aT}\right).$$

The characteristic polynomial when we use $C_{ZP}(z)$ is

$$z^2 - \left(1+e^{-aT} - \left(\frac{T\left(1-e^{-aT}\right)}{2a}\right)K\right)z + e^{-aT} + \left(\frac{T\left(1-e^{-aT}\right)}{2a}\right)K.$$

Applying again the conditions (1.1) yields

$$0 < K < \frac{2}{T}a.$$

Comparing the three ranges of K we observe that the discretization introduces a maximum value of the gain for which the closed-loop system is asymptotically stable. Moreover, in the case of $C_I(z)$, the maximum gain is influenced by a only marginally. If $T \to 0$ than we recover the continuous-time case. In the case of $C_{ZP}(z)$, both T and a greatly influence the maximum gain. If $T \to 0$ or/and $a \to +\infty$ than we recover the continuous-time case. [6 marks]

2. a) Since the sampling time is $T = 3.92699$, the primary strip is enclosed between $-\frac{\omega_s}{2} = -\frac{\pi}{T} = -0.8$ and $\frac{\omega_s}{2} = \frac{\pi}{T} = 0.8$. The s-plane region is mapped into the shaded z-plane region shown in Figure 2.1. [4 marks]

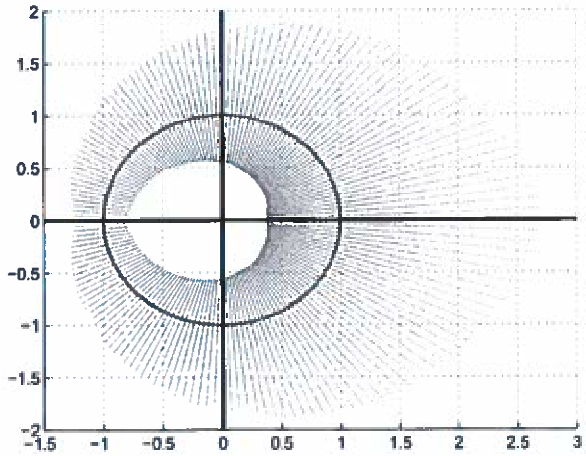


Figure 2.1 Shaded z-plane region.

b) We introduce the auxiliary variables $e_1(t)$ and $e_2(t)$ at the entrance of the first two samplers. At the exit of these two samplers we introduce the variables $e_1^*(t)$ and $e_2^*(t)$. We can now write down the relations between these variables in the Laplace domain, namely

$$
\begin{aligned}
E_1(s) &= R(s) - H_1(s)Y^*(s), \\
E_2(s) &= C_1(s)E_1^*(s) - H_2(s)Y^*(s), \\
Y(s) &= G(s)C_2(s)E_2^*(s).
\end{aligned}
$$

We determine the *starred* version of the first and of the second equation, and we replace the result in the third equation

$$
\begin{aligned}
E_1^*(s) &= R^*(s) - H_1^*(s)Y^*(s), \\
E_2^*(s) &= C_1^*(s)E_1^*(s) - H_2^*(s)Y^*(s), \\
Y(s) &= G(s)C_2(s)\left[C_1^*(s)R^*(s) - C_1^*(s)H_1^*(s)Y^*(s) - H_2^*(s)Y^*(s)\right].
\end{aligned}
$$

We now determine the *starred* version of the third equation

$$Y^*(s) = [G(s)C_2(s)]^*\left[C_1^*(s)R^*(s) - C_1^*(s)H_1^*(s)Y^*(s) - H_2^*(s)Y^*(s)\right].$$

From this equation, we solve with respect to $Y^*(s)$ and divide by $R^*(s)$, yielding

$$\frac{Y^*(s)}{R^*(s)} = \frac{C_1^*(s)\,[G(s)C_2(s)]^*}{1 + C_1^*(s)H_1^*(s)\,[G(s)C_2(s)]^* + H_2^*(s)\,[G(s)C_2(s)]^*}.$$

From this last equation we can write directly the pulse transfer function

$$\frac{Y(z)}{R(z)} = \frac{C_1(z)GC_2(z)}{1 + C_1(z)H_1(z)GC_2(z) + H_2(z)GC_2(z)}.$$

where $GC_2(z) = Z[G(s)C_2(s)]$. [4 marks]

c) We introduce the auxiliary variables $E(z)$ and $E_{D_1}(z)$ before the blocks $C_1(z)$ and $C_2(z)$ respectively. We can now write down the relations between $D_1(z)$, $E(z)$, $E_{D_1}(z)$ and $Y(z)$, namely

$$\begin{aligned} Y(z) &= G(z)C_2(z)E_{D_1}(z), \\ E_{D_1}(z) &= D_1(z) + C_1(z)E(z) - H_2(z)Y(z), \\ E(z) &= -H_1(z)Y(z). \end{aligned}$$

Substituting the third equation in the second and the resulting second equation in the first yields

$$Y(z) = G(z)C_2(z)D_1(z) - G(z)C_2(z)C_1(z)H_1(z)Y(z) - G(z)C_2(z)H_2(z)Y(z).$$

Solving with respect to $Y(z)$ and dividing by $D_1(z)$ yields

$$\frac{Y(z)}{D_1(z)} = \frac{G(z)C_2(z)}{1 + G(z)C_2(z)C_1(z)H_1(z) + G(z)C_2(z)H_2(z)}.$$

To minimize the effect of the disturbance on the output the gain of $C_1(z)$ should be selected as large as possible, whereas the gain of $C_2(z)$ should be selected as small as possible. [4 marks]

d) We introduce the auxiliary variables $E_1(z)$, $E_2(z)$ and $E_{D_2}(z)$ before the blocks $C_1(z)$, $C_2(z)$ and $G(z)$, respectively. We can now write down the relations between $D_2(z)$, $E_1(z)$, $E_2(z)$, $E_{D_2}(z)$ and $Y(z)$, namely

$$\begin{aligned} Y(z) &= G(z)E_{D_2}(z), \\ E_{D_2}(z) &= D_2(z) + C_2(z)E_2(z), \\ E_2(z) &= C_1(z)E_1(z) - H_2(z)Y(z), \\ E_1(z) &= -H_1(z)Y(z). \end{aligned}$$

Substituting the forth equation in the third, the resulting third in the second and the resulting second in the first yields

$$Y(z) = G(z)D_2(z) - G(z)C_2(z)C_1(z)H_1(z)Y(z) - G(z)C_2(z)H_2(z)Y(z).$$

Solving with respect to $Y(z)$ and dividing by $D_2(z)$ yields

$$\frac{Y(z)}{D_2(z)} = \frac{G(z)}{1 + G(z)C_2(z)C_1(z)H_1(z) + G(z)C_2(z)H_2(z)}.$$

To minimize the effect of the disturbance on the output the gains of $C_1(z)$ and $C_2(z)$ should be selected as large as possible. [4 marks]

e) We introduce the auxiliary variables $E_1(z)$ and $E_2(z)$ before the blocks $C_1(z)$ and $C_2(z)$, respectively. We can now write down the relations between $D_3(z)$, $E_1(z)$, $E_2(z)$ and $Y(z)$, namely

$$\begin{aligned} Y(z) &= G(z)C_2(z)E_2(z), \\ E_2(z) &= C_1(z)E_1(z) - H_2(z)Y(z), \\ E_1(z) &= -H_1(z)(D_3(z) + Y(z)). \end{aligned}$$

Substituting the third equation in the second and the resulting second in the first yields

$$Y(z) = -G(z)C_2(z)\left[C_1(z)H_1(z)D_3(z) + C_1(z)H_1(z)Y(z) + H_2(z)Y(z)\right].$$

Solving with respect to $Y(z)$ and dividing by $D_3(z)$ yields

$$\frac{Y(z)}{D_3(z)} = \frac{-G(z)C_2(z)C_1(z)H_1(z)}{1 + G(z)C_2(z)C_1(z)H_1(z) + G(z)C_2(z)H_2(z)}.$$

To minimize the effect of the disturbance on the output the gains of $C_1(z)$ and $C_2(z)$ should be selected as small as possible. [4 marks]

3. a) The equivalent discrete-time model is

$$\begin{aligned} HP(z) &= \frac{z-1}{z}Z\left(\frac{P(s)}{s}\right) = \frac{z-1}{z}Z\left(\frac{2.5}{s} - \frac{5}{s+1} + \frac{2.5}{s+2}\right) \\ &= \frac{0.0205z+0.0226}{(z-0.9048)(z-0.8187)}. \end{aligned}$$

[4 marks]

b) The transfer function in the w-plane is (recall that $T = 0.1$)

$$HP(w) = HP(z)|_{z=\left(\frac{1+0.2w}{1-0.2w}\right)} = -0.000622\frac{(w+400.33)(w-20)}{(w+1.9934)(w+0.9992)}.$$

[3 marks]

c) The velocity constant in the w-plane is defined as

$$K_v = \lim_{w\to 0} w\, C_1(w)\, HP(w) = 1.$$

Selecting $r = 0$ yields $K_v = 0$, whereas selecting $r \geq 2$ yields $K_v = \infty$. Selecting $r = 1$ yields

$$K_v = -0.000622\frac{(0+400.33)(0-20)}{(0+1.9934)(0+0.9992)}k = 1$$

that is

$$K_v = 2.5k = 1.$$

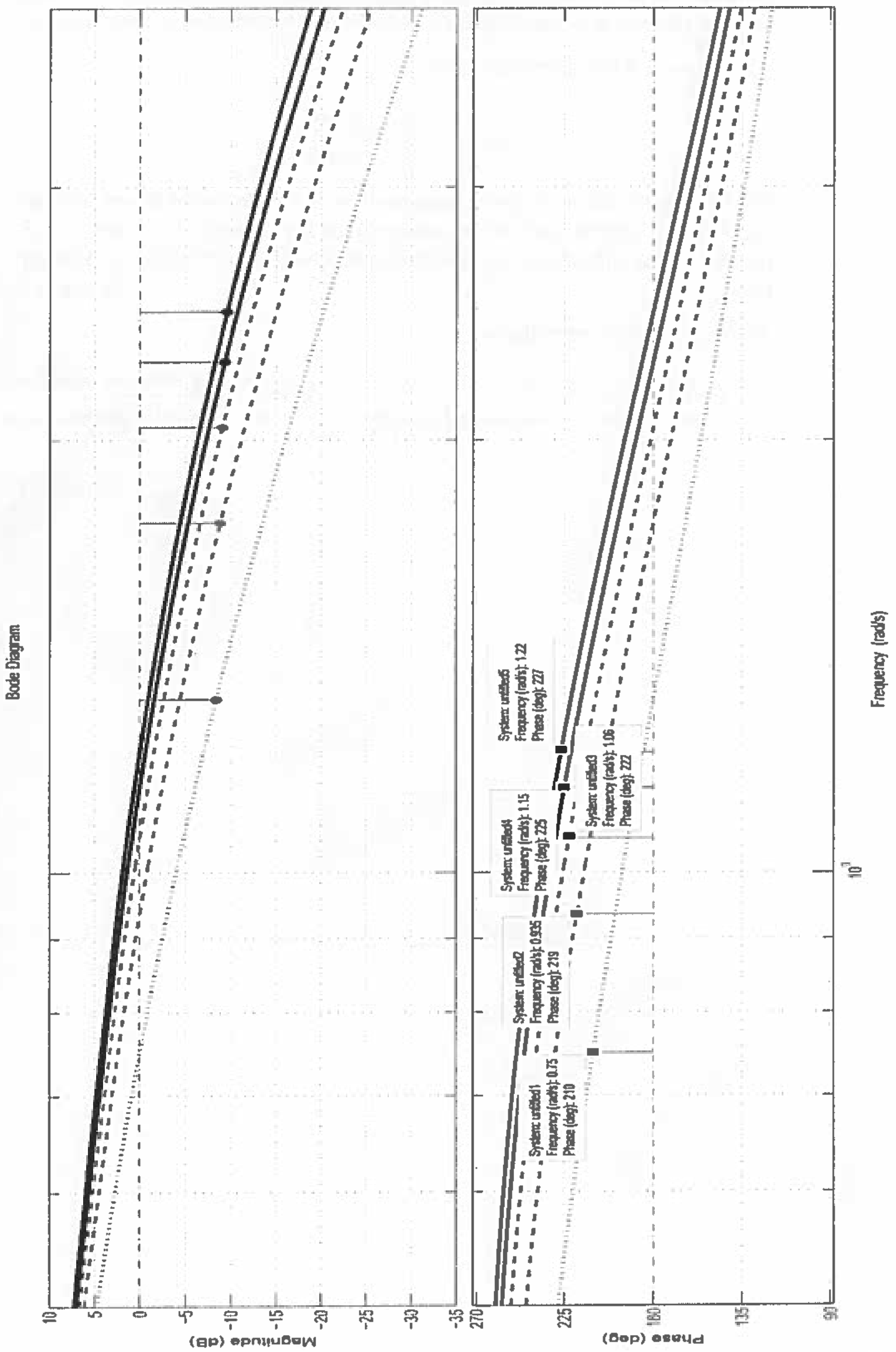
Thus $k = 0.4$ and $C_1(w) = \frac{0.4}{w}$. The open-loop transfer function is given by

$$C_1(w)HP(w) = -0.000249\frac{(w+400.33)(w-20)}{w(w+1.9934)(w+0.9992)}.$$

[4 marks]

d) The phase margin of the open-loop transfer function is approximatively 30°. We need at least 15° more.

- We start selecting $m = 2$. The phase increase is approximatively 20°. The magnitude decrease is approximatively 3 dB. From the Bode plot, we read that -3 dB is approximatively at $\omega = 0.935$ for which the phase margin is 19°. Since $20 + 19 = 39 < 45°$, this compensator does not satisfy the phase margin requirement.
- We select $m = 3$. The phase increase is approximatively 30°. The magnitude decrease is approximatively 4.8 dB. From the Bode plot, we read that -4.8 dB is approximatively at $\omega = 1.05$ for which the phase margin is 13°. Since $30 + 13 = 43 < 45°$, this compensator does not satisfy the phase margin requirement.
- We select $m = 4$. The phase increase is approximatively 37°. The magnitude decrease is approximatively 6 dB. From the Bode plot, we read that -6 dB is approximatively at $\omega = 1.14$ for which the phase margin is 8°. Since $37 + 8 = 45 = 45°$, this is an acceptable solution.
- We select $m = 5$. The phase increase is approximatively 43°. The magnitude decrease is approximatively 7 dB. From the Bode plot, we read that -7 dB is approximatively at $\omega = 1.22$ for which the phase margin is 4°. Since $43 + 4 < 47°$, this is an acceptable solution.


Bode Diagram
Magnitude (dB)
Phase (deg)
Frequency (rad/s)
System: untitled5
Frequency (rad/s): 1.22
Phase (deg): 227
System: untitled4
Frequency (rad/s): 1.15
Phase (deg): 225
System: untitled3
Frequency (rad/s): 1.06
Phase (deg): 222
System: untitled2
Frequency (rad/s): 0.935
Phase (deg): 219
System: untitled1
Frequency (rad/s): 0.75
Phase (deg): 210

Note that for any value of $m \geq 4$ the phase margin becomes at least $45°$. Any of these value of m is acceptable. In the following we select $m = 5$ which gives $\tau = \frac{\sqrt{5}}{1.22} = 1.8328$ and the controller

$$C_2(w) = \frac{1 + 1.8328w}{1 + 0.3666w}.$$

The full-page figure in the previous page shows the Bode plot of the uncompensated system (dotted line), of the compensated system with $m = 2$ and $m = 3$ (dashed lines) and of the compensated system with $m = 4$ and $m = 5$ (solid lines). [6 marks]

e) The discrete-time controller is

$$C(z) = [C_1(w)C_2(w)]_{w = 20\left(\frac{z-1}{z+1}\right)} = 0.0904\frac{(z-0.9469)(z+1)}{(z-1)(z-0.75994)}.$$

[3 marks]

4. a) The equivalent discrete-time model is

$$\begin{aligned}
HP(z) &= (1-z^{-1})Z\left(\frac{P(s)}{s}\right) = (1-z^{-1})Z\left(\frac{e^{-s}}{s(2s+1)}\right) \\
&= (1-z^{-1})z^{-1}Z\left(\frac{1}{s(2s+1)}\right) = (z^{-1}-z^{-2})Z\left(\frac{1}{s}-\frac{1}{s+0.5}\right) \\
&= (z^{-1}-z^{-2})\left(\frac{1}{1-z^{-1}}-\frac{1}{1-0.6065z^{-1}}\right) = \frac{0.3935z^{-2}}{1-0.6065z^{-1}}.
\end{aligned}$$

[4 marks]

b) Since $\alpha(1-e^{-0.5(3-1)}) = 1$, we have

$$\alpha = \frac{e}{e-1} = 1.5820.$$

The sequence $\tilde{y}(kT)$ is

$$\begin{aligned}
&\tilde{y}(0) = 0 \\
&\tilde{y}(T) = 0 \\
&\tilde{y}(2T) = \frac{e}{e-1}(1-e^{-0.5(2-1)}) = 0.6225 \\
&\tilde{y}(kT) = 1, \qquad k = 3,4,5,\ldots
\end{aligned}$$

Hence,

$$\begin{aligned}
C(z) &= 0.6225z^{-2}+z^{-3}+z^{-4}+z^{-5}+\cdots = 0.6225z^{-2}+z^{-3}\frac{1}{1-z^{-1}} \\
&= \frac{0.6225z^{-2}+0.3775z^{-3}}{1-z^{-1}}.
\end{aligned}$$

[6 marks]

c) To determine the controller $C(z)$ we compute the closed-loop transfer function between the reference and the output, namely

$$\frac{\tilde{Y}(z)}{R(z)} = \frac{C(z)HP(z)}{1+C(z)HP(z)}.$$

Since $R(z)$, $\tilde{Y}(z)$ and $HP(z)$ are given, it is sufficient to solve this equation with respect to $C(z)$ yielding

$$C(z) = \frac{-\tilde{Y}(z)}{HP(z)(\tilde{Y}(z)-R(z))}.$$

After some computation we obtain

$$C(z) = \frac{1.5820(1+0.6065z^{-1})(1-0.6065z^{-1})}{(1-0.6225z^{-2}-0.3775z^{-3})} = \frac{1.5820(1-0.3678z^{-2})}{(1-0.6225z^{-2}-0.3775z^{-3})}.$$

We observe that the denominator of $C(z)$ can be factorized as

$$1-0.6225z^{-2}-0.3775z^{-3} = (1-z^{-1})(1+z^{-1}+0.3775z^{-2}),$$

which shows that $C(z)$ has a pole at $z=1$. [6 marks]

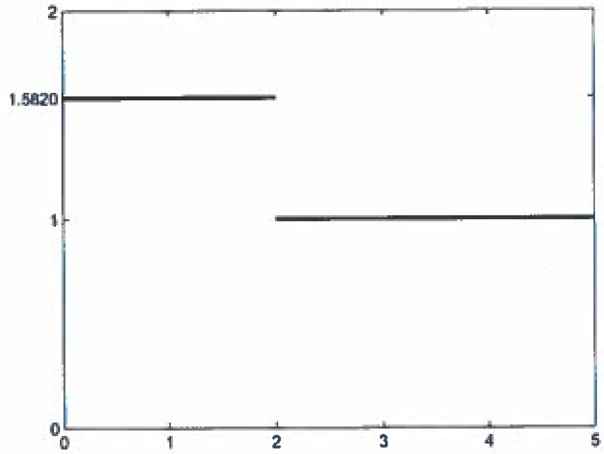


Figure 4.1 Plot of $u(t)$.

d) Note that $\tilde{Y}(z) = HP(z)U(z)$. Hence,

$$U(z) = \frac{\tilde{Y}(z)}{HP(z)} = \frac{1.5820(1-0.3678z^{-2})}{1-z^{-1}}.$$

By dividing the numerator by the denominator yields

$$U(z) = 1.5820 + 1.5820z^{-1} + z^{-2} + z^{-3} + z^{-4} + \dots$$

which corresponds to the signal $u(0) = 1.5820$, $u(T) = 1.5820$, $u(kT) = 1$ for all $k \geq 2$. The signal $u(t)$ is shown in Figure 4.1.

[4 marks]