Mathematics for signal & systems 2007/2008. ①

SOLUTIONS - 2008.

①

a) i) $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

(ii) $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(iii) We will show that $\mathcal{M}$ is a subspace of $\mathcal{M}_3(\mathbb{C})$

First, $\mathcal{M} \subset \mathcal{M}_3(\mathbb{C})$ and $\mathcal{M} \neq \emptyset$. (as $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{M}$).

Let $a \in \mathbb{C}$ and $A, B \in \mathcal{M}$ i.e.

$$l_i(A) = c_j(A) \quad \text{and} \quad l_i(B) = c_j(B) \quad (*)$$

for $i,j = 1..3$

It is not difficult to see that l_i and c_j are linear

so that $l_i(aA+B) = a\,l_i(A) + l_i(B)$

similarly for c_j. Hence $l_i(\alpha A+B) = c_j(\alpha A+B)$

for $i,j = 1,2,3$ by $(*)$ above.

If $A, B \in \mathcal{M}$ and $a \in \mathbb{C}$, then

$$\alpha(aA+B) = l_1(aA+B) = a\,l_1(A) + l_1(B)$$

$$= a\,\alpha(A) + \alpha(B)$$

Hence α is linear.

(ii) It is straightforward to check that $J \in \mathcal{M}$

as $\quad l_i(J) = c_j(J) = 3$.

Let $A = (a_{ij})_{i,j=1,2,3}$ then

$$AJ = \begin{bmatrix} a_{11}+a_{12}+a_{13} & a_{11}+a_{12}+a_{13} & a_{11}+a_{12}+a_{13} \\ a_{21}+a_{22}+a_{23} & a_{21}+a_{22}+a_{23} & a_{21}+a_{22}+a_{23} \\ a_{31}+a_{32}+a_{33} & a_{31}+a_{32}+a_{33} & a_{31}+a_{32}+a_{33} \end{bmatrix}$$

$$JA = \begin{bmatrix} a_{11}+a_{21}+a_{31} & a_{12}+a_{22}+a_{32} & a_{13}+a_{23}+a_{33} \\ a_{11}+a_{21}+a_{31} & a_{12}+a_{22}+a_{32} & a_{13}+a_{23}+a_{33} \\ a_{11}+a_{21}+a_{31} & a_{12}+a_{22}+a_{32} & a_{13}+a_{23}+a_{33} \end{bmatrix}$$

so that $\quad AJ = \begin{pmatrix} l_1(a) & l_1(a) & l_1(a) \\ l_2(a) & l_2(a) & l_2(a) \\ l_3(a) & l_3(a) & l_3(a) \end{pmatrix}$

$$JA = \begin{pmatrix} c_1(a) & c_2(a) & c_3(a) \\ c_1(a) & c_2(a) & c_3(a) \\ c_1(a) & c_2(a) & c_3(a) \end{pmatrix}.$$

- If $\quad AJ = JA = \lambda A$

  then from the previous computation $l_i(A) = c_j(A) = \lambda$

  and $\lambda = \alpha(A)$.

- If $\quad A \in \mathcal{M}$ then $\quad AJ = JA = \alpha(A)\, J$.

b) (i) Let $a \in \mathbb{C}$, $A, B \in \mathcal{M}^0$.

We show that $\mathcal{M}^0$ is a subspace of $\mathcal{M}$.

. $\mathcal{M}^0 \subset \mathcal{M}$ and $\mathcal{M}^0 \neq \emptyset$ as $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{M}^0$.

as α, tr and anti are linear operators on $\mathcal{M}$, it is not difficult to check that

if A and B are such $\alpha(A) = \cancel{\mathrm{tr}}(A) = \mathrm{anti}(A)$
$\alpha(B) = \mathrm{tr}(B) = \mathrm{anti}(B)$

then the same holds for $aA + B$.

(ii) (G, G^T, J) independent.

$$a G + b G^T + c J = 0$$

$$\Rightarrow \quad a + b + c = 0 \;;\; -2a + c = 0 \;,\; a - b + c = 0$$
$$c = 0 \;,\; 2b + c = 0 \;,\; -a - b + c = 0.$$

$\Rightarrow \quad a = b = c = 0 \quad \Rightarrow \quad (G, G^T, J)$ independent.

$\alpha(J) = 3$; $\mathrm{tr}\, J = 3$ & $\mathrm{anti}\; J = 3 \Rightarrow J \in \mathcal{M}^0$.

$\alpha(G) = 0$, $\mathrm{anti}\; G = 0$ and $\mathrm{tr}\, G = 0 \Rightarrow G \in \mathcal{M}^0$

Similarly for G^T.

If $A \in \mathcal{M}^{\circ} \Rightarrow$ $a_{k1} + a_{k2} + a_{k3} =$
$a_{1k} + a_{2k} + a_{3k} =$
$a_{11} + a_{22} + a_{33} =$
$a_{13} + a_{22} + a_{31}$

$$a_{11} + a_{12} + a_{13} = a_{21} + a_{22} + a_{23} = a_{31} + a_{33} + a_{32} =$$
$$a_{11} + a_{21} + a_{31} = a_{12} + a_{22} + a_{32} = a_{13} + a_{23} + a_{33} =$$
$$a_{11} + a_{22} + a_{33} = a_{13} + a_{22} + a_{32}.$$

Solving the above system, it is not difficult to show that.

$$A = \frac{a_{22} - a_{12}}{2} G + \frac{a_{23} - a_{22}}{2} G^T + a_{22} J.$$

Hence $\mathcal{M}^{\circ} = \operatorname{Span}(G, G^T, J)$ and $\dim \mathcal{M}^{\circ} = 3$.

(2) Bookwork (treated in pb sheet 3).

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2 (a) ?

(i) The only difficult point is to show that if

$\langle P, P \rangle = 0$ then $P = 0$

$$\int_{-1}^{1} \frac{P^2(t)}{\sqrt{1-t^2}}\, dt = 0 \quad \Rightarrow \quad P(t) = 0 \text{ on } (-1, 1)$$

$P(t)$ is a polynomial with an infinite number of roots (or zeros) then it is the zero polynomial.

(b) (c)

(ii) By induction.

$T_0(\cos\theta) = 1$ for $T_0(x) = 1$
$T_1(\cos\theta) = \cos\theta$ for $T_1(x) = x$.

Suppose that for any $k \leq n$ there exists a T_k such that $\cos k\theta = T_k(\cos\theta)$.

$$\cos(n+1)\theta = \cos(n\theta)\cos\theta - \sin\theta \sin((n\cancel{+1})\theta)$$
$$= T_n(\cos\theta)\cos\theta - \frac{1}{2}\left(-\cos(n+1)\theta + \cos(n-1)\theta\right)$$

$$\Rightarrow \cos(n+1)\theta = 2T_n(\cos\theta)\cos\theta - T_{n-1}(\cos\theta).$$

if we let $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$

then $\cos(n+1)\theta = T_{n+1}(\cos\theta)$.

The uniqueness stems from the fact that if P_n is such that $P_n(\cos\theta) = \cos n\theta$ then the polynomial $T_n - P_n$ is 0 on $[-1, 1]$; Hence $T_n - P_n = 0$.

where is solution for c)?

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~~(iii)~~ (d) for $m, n \in \mathbb{N}$.

$$\int_{-1}^{1} \frac{T_n(x)\, T_m(x)}{\sqrt{1-x^2}}\, dx = \int_{\pi}^{0} \frac{T_n(\cos\theta)\, T_m(\cos\theta)}{\sin\theta} \, -\sin\theta \, d\theta$$

with the change of variable $t = \cos\theta$

Therefore, $\langle T_n, T_m \rangle = \int_0^{\pi} \cos(n\theta)\cos(m\theta)\, d\theta$.

$$= \int_0^{\pi} \frac{\cos(n+m)\theta}{2}\, d\theta + \int_0^{\pi} \frac{\cos(n-m)\theta}{2}\, d\theta$$

$= 0$ if $m \neq n$.

- $m = n > 0$ then $\langle T_n, T_n \rangle = \int_0^{\pi} \frac{1 - \cancel{1}\cos^2 2n\theta}{2}\, d\theta$

$= \pi/2$

- $m = n = 0$ then $\langle T_0, T_0 \rangle = \pi$.

e) We can in addition show that T_n satisfies a differential equation.

On one hand

$$\frac{\partial}{\partial\theta} T_n(\cos\theta) = -\sin\theta\, T_n'(\cos\theta)$$

$$\frac{\partial^2}{\partial\theta^2} T_n(\cos\theta) = -\cos\theta\, T_n'(\cos\theta) + \sin^2\theta\, T_n''(\cos\theta)$$

$$= (1-\cos^2\theta)\, T_n''(\cos\theta) - \cos\theta\, T_n'(\cancel{\theta}\cos\theta).$$

On the other hand, $\frac{\partial^2}{\partial^2\theta} T_n(\cos\theta) = \frac{\partial^2}{\partial\theta^2}\cos(n\theta) = -n^2\cos n\theta$

$$\Rightarrow -n^2\, T_n(\cos\theta) = (1-\cos^2\theta)\, T_n''(\cos\theta) - \cos\theta\, T_n'(\cos\theta)$$

T_n is the solution of $\boxed{(1-x^2)y'' - x\,y' = -n^2 y}$

(3)

a) Bookwork.

b)

(i)

We will show that S_n, and A_n are subspaces of $M_n(\mathbb{R})$

$* \, S_n \subset M_n(\mathbb{R})$ $\quad S_n \neq \emptyset$ as $\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \in S_n$.

α, A, B ; $\alpha \in \mathbb{R}$ and $A, B \in S_n$.

$$(\alpha A + B)^T = \alpha A^T + B^T = \alpha A + B$$

Hence S_n is a vector space.

$* \, A_n \subset M_n(\mathbb{R})$ $\quad A_n \neq \emptyset$ as $\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \in A_n$.

$\alpha \in \mathbb{R}$ and $A, B \in A_n$.

$$(\alpha A + B)^T = \alpha A^T + B^T = -\alpha A - B$$
$$= -(\alpha A + B)$$

Hence A_n is a vector space.

If $A \in S_n$ then. $\quad A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & \ddots & & \vdots \\ \vdots & & \ddots & \\ a_{1n} & \cdots & & a_{nn} \end{pmatrix}$

The only "degrees of freedom" are the upper triangular coefficients

Hence $\boxed{\dim(S_n) = \frac{n(n+1)}{2}}$

If $A \in \mathcal{A}_n$ then $\begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ -a_{12} & \ddots & & \\ \vdots & & \ddots & a_{n-1\,n} \\ -a_{1n} & \cdots & & 0 \end{pmatrix}$

Hence $\boxed{\dim(\mathcal{A}_n) = \frac{n(n-1)}{2}}$

(ii)

- $(M + M^T)^T = M^T + M$. so that $M + M^T \in \mathcal{S}_n$.
- $(M - M^T)^T = M^T - M = -(M - M^T) \Rightarrow M - M^T \in \mathcal{A}_n$.

(iii) Let $A \in \mathcal{S}_n$ and $B \in \mathcal{A}_n$.

$(A, B) = \operatorname{tr}(A^T B) = \operatorname{tr}(AB)$

$(B, A) = \operatorname{tr}(B^T A) = -\operatorname{tr}(BA)$

as $(A, B) = (B, A)$ since $(\cdot, \cdot)$ is an inner product

and $\operatorname{tr}(AB) = \operatorname{tr}(BA)$, this implies that

$\operatorname{tr}(AB) = (A, B) = 0$ so that $A \perp B$

This being true for any $A \in \mathcal{S}_n$ and $B \in \mathcal{A}_n$, we have $\mathcal{S}_n$ orthogonal to $\mathcal{A}_n$.

Let $M \in \mathcal{M}_n(\mathbb{R})$ we can see that

$$M = \left(\frac{M + M^T}{2}\right) + \left(\frac{M - M^T}{2}\right).$$

by (ii) $\frac{M + M^T}{2} \in \mathcal{S}_n$ and $\frac{M - M^T}{2} \in \mathcal{A}_n$.

This shows the existence of such decomposition.

Let us prove the uniqueness.

$M = S + A$ $\quad S \in \mathcal{S}_n$ and $A \in \mathcal{A}_n$.

then $M^T = S - A$ $\Rightarrow$ $\begin{cases} S = \frac{M + M^T}{2} \\ A = \frac{M - M^T}{2}. \end{cases}$

which concludes the proof.

(iv) by the previous question it is easily seen that the orthogonal projection on $\mathcal{S}_n$ is given by

$$P_{\mathcal{S}_n}(M) = \frac{M + M^T}{2}$$

and $P_{\mathcal{A}_n}(M) = \frac{M - M^T}{2}$.

(iv)

(iii) Let $M \in M_n(\mathbb{R})$ ~~we~~ we can see that

$$M = \left(\frac{M + M^T}{2}\right) + \left(\frac{M - M^T}{2}\right)$$

and by the first part of this question.

$\frac{M + M^T}{2} \in S_n$ and

4 Bookwork.

5 Bookwork.

Solution Pb 4

The pathology of Banach spaces cannot occur in Hilbert spaces.

Theorem :

Let X be a Hilbert space and let M be a closed linear subspace of X.

For $x_0 \notin M$, consider

$$\delta = \inf \{ \|x_0 - y\| \ ; y \in M \}.$$

Then, (i) There exist a unique $y_0 \in M$ such that $\|x_0 - y_0\| = \delta$

(ii) $x_0 - y_0$ is orthogonal to M ($\forall y \in M$; $\langle x_0 - y_0, y \rangle = 0$

Remark : This theorem says that the unique point in M closest to x_0 is found by "dropping a perpendicular from x_0 to M".
It is important to note that the theorem is not true for inner product spaces that are not complete.

Proof :

By the definition of the infimum of a set, there exists a sequence $(y_n)_n$ in M such that

$$\delta_n = \|x - y_n\| \quad \text{and} \quad \delta_n \longrightarrow \delta \text{ when } n \to +\infty.$$

We first show that $(y_n)_n$ is a Cauchy sequence.

$$\| y_n - y_m \|^2 = \| (y_n - x) + (x - y_m) \|^2 = \| (y_n - x) - (y_m - x) \|^2.$$

By the parallelogramme equality : $\left(\|a+b\|^2 + \|a-b\|^2 = 2 (\|a\|^2 + \|b\|^2 \right)$

we have,

$$\|y_n - y_m\|^2 \leqslant 2\left(\|y_n - x\|^2 + \|y_m - x\|^2\right) - \|y_n + y_m - 2x\|^2$$
$$= 2(\delta_n^2 + \delta_m^2) - 2\left\|\frac{y_n + y_m}{2} - x\right\|^2.$$

As M is a subspace of X and $y_n, y_m \in X$; $\frac{y_n + y_m}{2} \in M$. as well.

By the definition of δ as the smallest $\|x_0 - y\|$ for $y \in M$.
$\left\|\left(\frac{y_n + y_m}{2}\right) - x\right\| \geqslant \delta.$

Hence, $\|y_n - y_m\|^2 \leqslant 2(\delta_n^2 + \delta_m^2) - 2\delta^2.$

as $\delta_n \to \delta$ when $n \to \infty$, taking $n \to +\infty$ and $m \to +\infty$ in the previous inequality yields:

$$\lim_{n, m \to +\infty} \|y_n - y_m\| = 0.$$

So $(y_n)_n$ is indeed a Cauchy sequence.

X is a Hilbert space, in particular a complete space, thus there exists a $y_0 \in X$ such that $\lim_{n \to +\infty} y_n = y_0$. Moreover $(y_n)_n$ is a sequence in M which is closed, this implies that $y_0 \in M$.

This shows that, there is a $y_0 \in M$ such that

$$\delta = \inf\{\|x_0 - y\|;\ y \in M\}$$
$$= \|x_0 - y_0\|$$

To prove (i) of the theorem, we need to prove the uniqueness of y_0.

Assume that $y_0 \in M$ and $y_* \in M$ both satisfy:

$$\|x - y_0\| = \delta \quad \text{and} \quad \|x - y_*\| = \delta.$$

We want to show that $y_0 = y_*$.

By the parallelogramme equality,

$$\|y_* - y_0\|^2 = \|(y_* - x) - (y_0 - x)\|^2$$

$$= 2\|y_0 - x\|^2 + 2\|y_* - x\|^2 - \|(y_* - x) + (y_0 - x)\|^2$$

$$= 2\delta^2 + 2\delta^2 - 4\left\|\left(\frac{y_* + y_0}{2}\right) - x\right\|^2$$

$\frac{y_* + y_0}{2} \in M$; $\quad \leq 4\delta^2 - 4\delta^2 = 0. \quad \Rightarrow y_* = y_0.$

To conclude the proof of the Theorem, it remains to prove (ii) $x_0 - y_0$ is orthogonal to M.

Let $y \in M$, as M is a vector space $y_0 + \alpha y \in M$ for any $\alpha \in \mathbb{C}$;

From the definition of δ $\|x_0 - (y_0 + \alpha y)\|^2 \geq \delta^2$, so that

$$\delta^2 \leq \langle x_0 - y_0 - \alpha y, x_0 - y_0 - \alpha y\rangle = \|x_0 - y_0\|^2 + |\alpha|^2\|y\|^2 - 2\,\mathrm{Re}(\alpha\langle y, x_0 - y_0\rangle).$$

since $\|x_0 - y_0\| = \delta$ $\quad = \delta^2 + |\alpha|^2\|y\|^2 - 2\,\mathrm{Re}(\alpha\langle y, x_0 - y_0\rangle)$

Which implies that $\quad |\alpha|^2\|y\|^2 - 2\,\mathrm{Re}(\alpha\langle y, x_0 - y_0\rangle) \geq 0$

Let $\alpha = \beta\langle x_0 - y_0, y\rangle$; $\beta \in \mathbb{R}$, we have.

$$\beta^2 |\langle x_0 - y_0, y\rangle|^2 \|y\|^2 - 2\beta |\langle x_0 - y_0, y\rangle|^2 \geq 0.$$

This inequality holds for all $\beta \in \mathbb{R}$. This cannot occur unless the coefficient of β is equal to 0, otherwise the left hand side of the inequality will change sign.

Hence, $\quad < x_0 - y_0 , y > = 0$

$\blacksquare$

<u>Example 5</u>: SOLUTION PBS

(i) Find the minimum over $(a,b) \in \mathbb{R}^2$ of

$$I(a,b) = \int_0^{\pi} \left(\sin t - (at^2 + bt) \right)^2 dt.$$

Let $L^2[0,\pi] = \{ \text{function } f \text{ such that } \int_0^{\pi} |f(t)|^2 dt < +\infty \}$

with the inner product $\langle f,g \rangle = \int_0^{\pi} f(t)\, \bar{g}(t)\, dt$.

$$I(a,b) = \left(\begin{array}{c} \text{the distance between } \sin(t) \text{ and the} \\ \text{vector space spanned by } t^2 \text{ and } t \end{array} \right)^2$$

We can use the previous theorem to minimise this distance. To this end we need the orthogonal projection of $\sin t$ on $\mathrm{Span}(t, t^2) = \{ at^2 + bt,\ a,b \in \mathbb{R} \}$.

$$\begin{cases} \langle \sin t - (\alpha t^2 + \beta t), t \rangle = 0 & (1) \\ \langle \sin t - (\alpha t^2 + \beta t), t^2 \rangle = 0 & (2) \end{cases}$$

$$(1) \Rightarrow \int_0^{\pi} t \sin t \, dt - \alpha \int_0^{\pi} t^3 \, dt - \beta \int_0^{\pi} t^2 \, dt = 0$$

$$\Rightarrow \left[t(-\cos t) \right]_0^{\pi} + \int_0^{\pi} \cos t \, dt - \alpha \frac{\pi^4}{4} - \beta \frac{\pi^3}{3} = 0$$

$$\Rightarrow \boxed{\alpha \frac{\pi^4}{4} + \beta \frac{\pi^3}{3} = \pi.} \quad (1')$$

$$(2) \Rightarrow \int_0^{\pi} t^2 \sin t \, dt - \alpha \int_0^{\pi} t^4 \, dt - \beta \int_0^{\pi} t^3 \, dt = 0$$

$$\Rightarrow \left[t^2(-\cos t)\right]_0^{\pi} + \int_0^{\pi} 2t \cos t \, dt - \alpha \frac{\pi^5}{5} - \beta \frac{\pi^4}{4} = 0.$$

$$\Rightarrow \quad \pi^2 + 2\left[t(\sin t)\right]_0^{\pi} - 2\int_0^{\pi} \sin t \, dt - \alpha \frac{\pi^5}{5} - \beta \frac{\pi^4}{4} = 0.$$

$$\boxed{\alpha \frac{\pi^5}{5} + \beta \frac{\pi^4}{4} = \pi^2 + 2\left[\cos t\right]_0^{\pi} = \pi^2 - 4} \quad (2').$$

$$(1') \Rightarrow \qquad 3\pi\alpha + 4\beta = \frac{12}{\pi^2}$$

$$(2') \Rightarrow \qquad 4\pi\alpha + 5\beta = \frac{20}{\pi^2} - \frac{80}{\pi}$$

$$5 \times (1') - 4(2') \Rightarrow \qquad -\pi\alpha = \frac{60}{\pi^2} - \frac{80}{\pi^2} + \frac{320}{\pi}$$

$$\boxed{\alpha = \frac{20}{\pi^3} - \frac{320}{\pi^2}}$$

$$4(1') - 3(2') \Rightarrow \qquad \beta = \frac{48}{\pi^2} - \frac{60}{\pi^2} + \frac{240}{\pi}$$

$$\boxed{\beta = \frac{240}{\pi} - \frac{12}{\pi^2}}$$

The orthogonal projection of $\sin t$ on $\operatorname{Span}\{t^2, t\}$ is given by

$$\left(\frac{20}{\pi^3} - \frac{320}{\pi^2}\right) t^2 + \left(\frac{240}{\pi} - \frac{12}{\pi^2}\right) t.$$

To compute the minimum of $I(a,b)$ above, we only need to compute the following integral.

$$\int_0^{\pi} \left(\sin t - \alpha t^2 - \beta t\right)^2 dt \qquad \alpha, \beta \text{ defined above.}$$

$$= \int_0^{\pi} \sin^2 t \, dt + \int_0^{\pi} \alpha^2 t^4 dt + \int_0^{\pi} \beta^2 t^2 dt$$
$$- 2\alpha \int_0^{\pi} t^2 \sin t \, dt - 2\beta \int_0^{\pi} t \sin t \, dt + 2\alpha\beta \int_0^{\pi} t^3 dt$$

$$= \int_0^{\pi} \left(\frac{1 - \cos 2t}{2}\right) dt + \alpha^2 \frac{\pi^5}{5} + \beta^2 \frac{\pi^3}{3} - 2\alpha\left(\pi^2 - 4\right) - 2\beta\pi + \frac{2\alpha\beta}{4}\pi^4$$

$$= \frac{\pi}{2} + \alpha^2 \frac{\pi^5}{5} + \beta^2 \frac{\pi^3}{3} - 2\alpha\pi^4 + 8\alpha - 2\beta\pi + \frac{\alpha\beta\pi^4}{2}$$

Replacing α and β by their computed values give

the $\min\limits_{a,b \in \mathbb{R}} I(a,b) = I(\alpha, \beta)$.

(ii) Find the minimum of $\int_{-1}^{1} \left(x^3 - ax^2 - bx - c\right)^2 dx := I(a,b,c)$

$a, b, c \in \mathbb{R}$.

To minimise this integral that can be interpreted as the distance between x^3 and the vector space $\mathrm{Span}\{x^2, x, 1\}$.

We need to find the orthogonal projection of x^3 on $\mathrm{Span}\{x^2, x, 1$

for the inner product $\langle f, g \rangle = \int_{-1}^{1} f(t)\, \overline{g}(t)\, dt$.