THE ANSWERS

Notations:

(a) B - Bookwork

(b) E - New example

(c) A - New application

1. The questions are similar to past exam papers and problems solved during lectures, classes and courseworks. Most students got them right.

a) The two columns are orthogonal. We can simply apply

$$\mathbf{g}_1 = \begin{bmatrix} 1 & 1-j & 1 & 1+j \end{bmatrix} / \sqrt{6},$$

to match with stream 1 and null out the interference from the second stream.

[3 - E]

Similarly, for the second stream, we can apply

$$\mathbf{g}_2 = \begin{bmatrix} 1 & 0 & -1 & 0 \end{bmatrix} / \sqrt{2},$$

The combiner is given by

$$\mathbf{G} = \begin{bmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \end{bmatrix}$$

[3 - E]

This is a (normalized) received matched filter. Given that the columns are orthogonal the matched filter maximizes the SNR, nulls out the inter-stream interference.

[4 - E]

b) i) (3) goes with (a) since (3) leads to the lowest diversity gain among the three schemes.

[2 - E]

(2) goes with (b) since (2) leads to the same diversity gain as (1) but benefit from an array gain originating from the CSIT.

[2 - E]

(1) goes with (c) since (1) leads to the same diversity gain as (2) but does not benefit from any array gain.

[2 - E]

ii) The average symbol error rate of (c) at high SNR can be upper bounded as $\bar{P} \leq \bar{N}_e \left(\frac{\rho d_{min}^2}{8} \right)^{-2}$ where $\bar{N}_e$ is the number of nearest neighbors, ρ is the average SNR and d_{min} is the minimum distance of the constellation.

[1 - B]

The average symbol error rate of (b) at high SNR can be upper bounded as $\bar{P} \leq \bar{N}_e \left(\frac{\rho d_{min}^2}{4}\right)^{-2}$.

[1 - B]

Comparing the two expressions, we see a factor of 1/2 difference that explains why (c) incurs a 3dB loss in terms of SNR compared to (b).

[2 - B]

c) i) Assume that the flat fading channel remains constant over the two successive symbol periods, and is denoted by $\mathbf{h} = [h_1\ h_2]$.

Two symbols c_1 and c_2 are transmitted simultaneously from antennas 1 and 2 during the first symbol period, followed by symbols $-c_2^*$ and c_1^*, transmitted from antennas 1 and 2 during the next symbol period:

$$y_1 = \sqrt{E_s} h_1 \frac{c_1}{\sqrt{2}} + \sqrt{E_s} h_2 \frac{c_2}{\sqrt{2}} + n_1, \qquad \text{(first symbol period)}$$

$$y_2 = -\sqrt{E_s} h_1 \frac{c_2^*}{\sqrt{2}} + \sqrt{E_s} h_2 \frac{c_1^*}{\sqrt{2}} + n_2. \qquad \text{(second symbol period)}$$

The two symbols are spread over two antennas and over two symbol periods.

[3 - B]

ii) Equivalently

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2^* \end{bmatrix} = \sqrt{E_s} \underbrace{\begin{bmatrix} h_1 & h_2 \\ h_2^* & -h_1^* \end{bmatrix}}_{\mathbf{H}_{eff}} \underbrace{\begin{bmatrix} c_1/\sqrt{2} \\ c_2/\sqrt{2} \end{bmatrix}}_{\mathbf{c}} + \begin{bmatrix} n_1 \\ n_2^* \end{bmatrix}.$$

Applying the matched filter $\mathbf{H}_{eff}^H$ to the received vector $\mathbf{y}$ effectively decouples the transmitted symbols as shown below

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \mathbf{H}_{eff}^H \begin{bmatrix} y_1 \\ y_2^* \end{bmatrix} = \sqrt{E_s}\left[|h_1|^2 + |h_2|^2\right] \mathbf{I}_2 \begin{bmatrix} c_1/\sqrt{2} \\ c_2/\sqrt{2} \end{bmatrix} + \mathbf{H}_{eff}^H \begin{bmatrix} n_1 \\ n_2^* \end{bmatrix}$$

For both symbol c_i, $i = 1, 2$, we can apply a SISO ML decoder that will find the symbol in the constellation that minimizes $\left|z_i - \sqrt{E_s/2}\left[|h_1|^2 + |h_2|^2\right] c_i\right|$.

[4 - B]

iii) If the channel is not constant over two consecutive symbol durations, by applying the same procedure, we can see that the decoupling does not hold anymore and symbols are still subject to inter-stream interference. In order to benefit from full diversity, a MIMO ML decoder is necessary, which leads to higher complexity. [3 - A]

d) i) The multiplexing gain is the pre-log factor of the ergodic capacity at high SNR, i.e. $g_s = \lim_{\rho \to \infty} \frac{C_{CDIT}}{\log_2(\rho)}$. Hence by increasing the SNR by 3dB (e.g. from 27dB to 30dB), the ergodic capacity increases by g_s bits/s/Hz.

(a) $g_s = 1$.

[1 - E]

(b) $g_s = 2$.

[1 - E]

(c) $g_s = 2$.

[1 - E]

(d) $g_s = 3$.

[1 - E]

(e) $g_s = 4$.

[1 - E]

ii) There are several possible configurations that satisfy to $n_r + n_t = 9$, namely 5×4, 4×5, 6×3, 3×6, 7×2, 2×7, 1×8 and 8×1. The matching between curves and antenna configurations is easily identified by using the following two arguments: 1) The multiplexing gain with CDIT at high SNR is given by $\min\{n_t, n_r\}$. 2) With CDIT only, the input covariance matrix in i.i.d. channel is $\mathbf{Q} = 1/n_t \mathbf{I}_{n_t}$. This implies that 5×4, 6×3, 7×2 and 8×1 outperform 4×5, 3×6, 2×7 and 1×8, respectively.

(a) $n_r \times n_t = 8 \times 1$ or 1×8

[1 - E]

(b) $n_r \times n_t = 2 \times 7$

[1 - E]

(c) $n_r \times n_t = 7 \times 2$

[1 - E]

(d) $n_r \times n_t = 3 \times 6$ or 6×3

[1 - E]

(e) $n_r \times n_t = 4 \times 5$ or 5×4

[1 - E]

e) i) The capacity region is given by all the pairs (R_1, R_2) satisfying the following three inequalities

$$R_1 \leq \log_2\left(1 + \frac{P_1}{\sigma_n^2}|h_1|^2\right),$$

$$R_2 \leq \log_2\left(1 + \frac{P_2}{\sigma_n^2}|h_2|^2\right),$$

$$R_1 + R_2 \leq \log_2\left(1 + \frac{P_1}{\sigma_n^2}|h_1|^2 + \frac{P_2}{\sigma_n^2}|h_2|^2\right).$$

The first two inequalities come from the fact that the rates cannot be larger than the capacity of a point to point scenario. The last inequality tells us the sum of rate cannot be larger than the capacity of a point-to-point AWGN channel with the sum of the received powers of the two users. Given the presence of a single transmit antenna at each transmitter, transmission is performed at the maximum transmit power.

[5 - B]

ii) The three inequalities define a pentagon with two corner points. The corner points $(R1, R_2')$ where transmitter 1 transmits at full rate and transmitter 2 transmits at rate R_2' can be computed from

$$R_2' = \log_2\left(1 + \frac{P_1}{\sigma_n^2}|h_1|^2 + \frac{P_2}{\sigma_n^2}|h_2|^2\right) - \log_2\left(1 + \frac{P_1}{\sigma_n^1}|h_1|^2\right)$$

or equivalently

$$R_2' = \log_2\left(1 + \frac{P_2}{\sigma_n^2}|h_2|^2\left(1 + \frac{P_1}{\sigma_n^2}|h_1|^2\right)^{-1}\right).$$

The argument of the log in the last equation is the SINR where the interference from user 1 is treated as noise. (R_1, R_2') is achieved by decoding transmitter 2's signal first by treating transmitter 1's signal as noise. Decode transmitter 2's signal and cancel it from the received signal using SIC and then decode transmitter 1's signal. (R_1', R_2) can be obtained similarly by changing the decoding order.

[5 - B]

2. The questions are similar to past exam papers and problems solved during lectures, classes and courseworks. Most students got them right.

a) The channel is diagonal. The capacity over the deterministic channel writes as

$$C(\mathbf{H}) = \max_{P_1,P_2} \left(\log_2 \left(1 + \frac{P_1}{\sigma_{n,1}^2} |a|^2 \right) + \log_2 \left(1 + \frac{P_2}{\sigma_{n,2}^2} |b|^2 \right) \right)$$

with $P_1 + P_2 = P$. The optimal power allocation is given by the water-filling solution

$$P_1^\star = \left(\mu - \frac{\sigma_{n,1}^2}{|a|^2} \right)^+, \quad P_2^\star = \left(\mu - \frac{\sigma_{n,2}^2}{|b|^2} \right)^+$$

with μ computed such that $P_1^\star + P_2^\star = P$.

[4 - A]

Assuming $P_1^\star$ and $P_2^\star$ are positive, $\mu = \frac{P}{2} + \frac{1}{2}\left(\frac{\sigma_{n,1}^2}{|a|^2} + \frac{\sigma_{n,2}^2}{|b|^2} \right)$. If $\mu - \frac{\sigma_{n,2}^2}{|b|^2} \leq 0$, $P_2^\star = 0$ and $P_1^\star = P$. The capacity writes as

$$C(\mathbf{H}) = \log_2 \left(1 + \frac{P}{\sigma_{n,1}^2} |a|^2 \right).$$

[3 - A]

If $\mu - \frac{\sigma_{n,2}^2}{|b|^2} > 0$, $P_1^\star = \frac{P}{2} - \frac{\sigma_{n,1}^2}{2|a|^2} + \frac{\sigma_{n,2}^2}{2|b|^2}$ and $P_2^\star = \frac{P}{2} + \frac{\sigma_{n,1}^2}{2|a|^2} - \frac{\sigma_{n,2}^2}{2|b|^2}$. The capacity writes as

$$C(\mathbf{H}) = \log_2 \left(1 + \frac{P_1^\star}{\sigma_{n,1}^2} |a|^2 \right) + \log_2 \left(1 + \frac{P_2^\star}{\sigma_{n,2}^2} |b|^2 \right).$$

[3 - A]

b) The matrix is clearly rank 1. Hence we transmit a single data stream. [5 - A]

The capacity over this channel can be simply achieved by taking the SVD of the matrix and transmitting a single stream along the right dominant singular vector of the channel matrix, namely $[1,1,1]^T/\sqrt{3}$, and combining along the left dominant singular vector of the channel matrix, namely $[1,2,3]/\sqrt{14}$. All the transmit power is allocated to that single stream. [5 - A]

c) i) The diversity gain of 3 can be obtained with and without CSIT. Without CSIT, such a diversity gain can for instance be achieved with the use of an O-STBC or delay-diversity.

[10 - A]

ii) We transmit $c' = c'_1 + c'_2$ (with power of c'_q denoted as s_q). User 1 cancels user 2's signal c'_2 so as to be left with its own Gaussian noise. User 2 decodes its signal by treating user 1's signal c'_1 as Gaussian noise. This is called superposition coding with SIC. The achievable rates of such strategy (with sum-power constraint $s_1 + s_2 = E_s$)

$$R_1 = \log_2 \left(1 + \frac{\Lambda_1^{-1} s_1}{\sigma_{n,1}^2} |h_1|^2 \right)$$

$$R_2 = \log_2 \left(1 + \frac{\Lambda_2^{-1} |h_2|^2 s_2}{\sigma_{n,2}^2 + \Lambda_2^{-1} |h_2|^2 s_1} \right).$$

For user 1 to be able to correctly cancel user 2's signal, user 1's channel has to be good enough to support R_2, i.e.

$$R_2 \leq \log_2\left(1+\frac{\Lambda_1^{-1}\left|h_1\right|^2 s_2}{\sigma_{n,1}^2+\Lambda_1^{-1}\left|h_1\right|^2 s_1}\right).$$

The channel gains normalized w.r.t. their respective noise power should be ordered

$$\frac{\Lambda_2^{-1}\left|h_2\right|^2}{\sigma_{n,2}^2} \leq \frac{\Lambda_1^{-1}\left|h_1\right|^2}{\sigma_{n,1}^2}.$$

If the ordering condition is satisfied, the above strategy achieves the boundary of the capacity region of the two-user SISO BC for any power allocation s_1 and s_2 satisfying $s_1+s_2=E_s$.

[10 - B]

iii) In a MIMO point to point communication system based on Spatial Multiplexing, a Zero-Forcing receiver would outperform a Matched Filter only at high SNR when the inter-stream interference is dominating the noise. At low SNR, on the other hand, the inter-stream interference is negligible compared to the noise and the matched filter outperforms ZF as it maximizes the SNR. ZF would provide exactly the same performance as matched filter at low SNR if the columns of the channel matrix are orthogonal.

[10 - A]