EE4-25

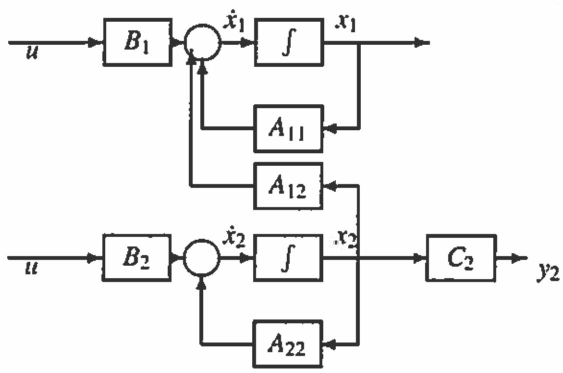
SOLUTIONS: DESIGN OF LINEAR MULTIVARIABLE CONTROL SYSTEMS

1. a) i) The PBH test states that the realisation is observable if and only if $[(A-\lambda I)^T \;\; C^T]^T$ has full rank for all complex λ. The matrix loses rank if λ is an eigenvalue of A_{11} so the realisation is unobservable.

ii) It follows that the unobservable modes that can be deduced from the structure are the eigenvalues of A_{11}.

iii) A realisation is detectable if and only if all the unobservable modes are stable. Since A_{22} is stable, and the modes of A_{11} are all unobservable, the realisation is detectable if and only if A_{11} is stable.

iv) The diagram is shown below. The subsystem with x_1 is unobservable.



b) i) Applying the suggested similarity transformation with $\star$ replaced by X and using the given relations gives

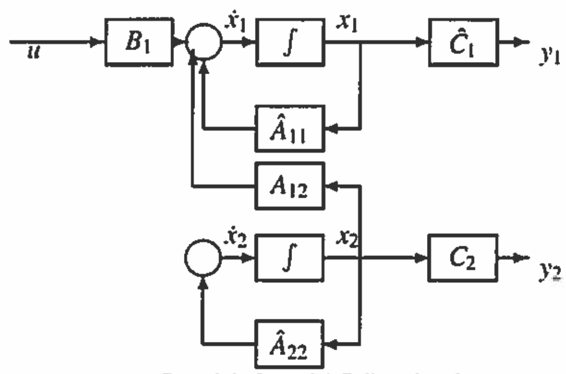
$$G(s) \stackrel{s}{=} \left[\begin{array}{c|c} T^{-1}AT & T^{-1}B \\ \hline CT & D \end{array}\right] \stackrel{s}{=} \left[\begin{array}{cc|c} A_{11}+A_{12}X & A_{12} & B_1 \\ 0 & A_{22}-XA_{12} & 0 \\ \hline C_1+C_2X & C_2 & 0 \end{array}\right].$$

The PBH test now shows that the realisation is uncontrollable.

ii) These are the modes of $A_{22}-XA_{12}$.

iii) Since $A_{12}X+A_{11}$ is stable, a necessary and sufficient condition is that $A_{22}-XA_{12}$ is stable

iv) The diagram is shown below with $\hat{A}_{11}=A_{11}+A_{12}X$, $\hat{A}_{22}=A_{22}-XA_{12}$ and $\hat{C}_1=C_1+C_2X$. The subsystem with x_2 is uncontrollable.



2. a) An inspection of Figure 2 shows that

$$\begin{aligned} \dot{x}-\dot{\hat{x}} &= (A+LC)(x-\hat{x})+\begin{bmatrix} B_w & -L \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \\ z &= C_z(x-\hat{x}) \end{aligned}$$

It follows that

$$T_{zw}(s) \stackrel{s}{=} \left[\begin{array}{c|c} A+LC & \begin{bmatrix} B_w & -L \end{bmatrix} \\ \hline C_z & \begin{bmatrix} 0 & 0 \end{bmatrix} \end{array}\right] \stackrel{s}{=}: \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right]$$

b) The Bounded Real Lemma states that A_c is stable $\|T_{zw}\|_\infty < \gamma$ if there exists a $P = P'$ such that

$$\begin{aligned} \begin{bmatrix} A_c'P+PA_c+C_c'C_c & PB_c+C_c'D_c \\ B_c'P+D_c'C_c & D_c'D_c-\gamma^2 I \end{bmatrix} &\prec 0 \\ P=P' &\succ 0 \end{aligned}$$

By substituting the expressions for A_c, B_c, C_c and D_c, this becomes

$$\begin{aligned} \begin{bmatrix} (A+LC)'P+P(A+LC)+C_z'C_z & PB_w & -PL \\ \star & -\gamma^2 I & 0 \\ \star & \star & -\gamma^2 I \end{bmatrix} &\prec 0 \\ P=P' &\succ 0 \end{aligned}$$

where $\star$ denotes terms easily inferred from symmetry.

c) By defining $Y = PL$, the matrix inequalities become

$$\begin{aligned} \begin{bmatrix} PA+A'P+YC+C'Y'+C_z'C_z & PB_w & -Y \\ \star & -\gamma^2 I & 0 \\ \star & \star & -\gamma^2 I \end{bmatrix} &\prec 0 \\ P=P' &\succ 0 \end{aligned}$$

which are linear.

d) Putting the numbers into the LMI:

$$\begin{aligned} \begin{bmatrix} -2P+2Y+2 & P & -Y \\ \star & -\gamma^2 I & 0 \\ \star & \star & -\gamma^2 I \end{bmatrix} &\prec 0 \\ P=P' &\succ 0 \end{aligned}$$

effecting a Schur complement, this is equivalent to

$$-2P+2Y+2+\gamma^{-2}Y^2+\gamma^{-2}P^2 \prec 0, \qquad P \succ 0$$

which when completing two squares become

$$(\gamma^{-1}P-\gamma)^2+(\gamma^{-1}Y+\gamma)^2+2-2\gamma^2 \prec 0, \qquad P \succ 0$$

and so $2\gamma^2 > 2$ or $\gamma > 1$. In the limit when $\gamma \to 1$, $P \to 1$, $Y \to -1$ and so $L \to -1$.

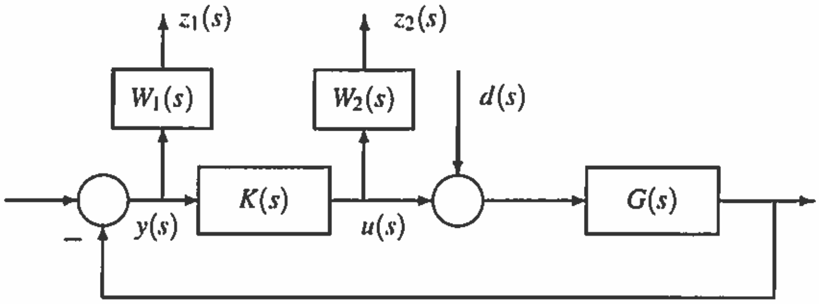
3. a) It is clear that we require $K(s)$ to be internally stabilising.

A calculation shows that $y(s) = T_{yd}(s)d(s)$ where $T_{yd}(s) = -(I + G(s)K(s))^{-1}G(s)$. It follows that a sufficient condition to achieve the first specification is $\|T_{yd}(j\omega)\| < |w_1(j\omega)^{-1}|\ \forall\omega$ or, equivalently, $\|W_1T_{yd}\|_\infty < 1$, where $W_1(s) = w_1(s)I$.

A similar calculation shows that $u(s) = T_{ud}(s)d(s)$ where $T_{ud}(s) = -K(s)\,(I + G(s)K(s))^{-1}G(s)$. It follows that a sufficient condition to achieve the second specification is $\|T_{ud}(j\omega)\| < |w_2(j\omega)^{-1}|\ \forall\omega$ or, equivalently, $\|W_2T_{ud}\|_\infty < 1$, where $W_2(s) = w_2(s)I$.

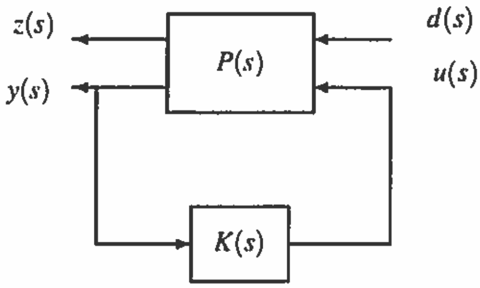
To satisfy both design requirements, it is sufficient that $\left\| \begin{bmatrix} W_1T_{yd} \\ W_2T_{ud} \end{bmatrix} \right\|_\infty < 1$.

b) The cost signals are given as $z_1(s) = W_1(s)y(s)$ and $z_2(s) = W_2(s)u(s)$. The block diagram incorporating $z_1(s)$ and $z_2(s)$ is shown below.



c) The corresponding generalised regulator formulation is to find an internally stabilising $K(s)$ such that $\|\mathcal{F}_l(P,K)\|_\infty < 1$ where

$$z(s) = \begin{bmatrix} z_1(s) \\ z_2(s) \end{bmatrix},\ P(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} = \left[\begin{array}{c|c} -W_1(s)G(s) & -W_1(s)G(s) \\ 0 & W_2(s) \\ \hline -G(s) & -G(s) \end{array}\right].$$



d) Suppose that Δ and S are stable. Then the feedback loop is stable if $\|\Delta S\|_\infty < 1$.

e) Let $K(s)$ be replaced by $K(s) + \Delta(s)$ in Figure 3 and let ε be the input and δ be the output of Δ. Then $\varepsilon = -(I + GK)^{-1}G\delta$. Using the small gain theorem the maximum stability radius is $|w_1^{-1}(j\omega)|$.

4. a) A suitable Lyapunov function for regulating x is $V = x'Px$ where $P = P'$.

b) Set $u = -Fx$. Provided that $P = P' \succ 0$ and $\dot{V} < 0$ along closed-loop trajectories, we can assume $\lim_{t\to\infty} x(t) = 0$. Then

$$\dot{V} = \dot{x}'Px + x'P\dot{x} = x'\left(A'P + PA - F'B'P - PBF\right)x.$$

Using $x(\infty) = 0$,

$$\int_0^\infty x'\left(A'P + PA - F'B'P - PBF\right)x\,dt = -x_0'Px_0.$$

c) Adding the last equation to the expression for J and completing a square:

$$J = x_0'Px_0 + \int_0^\infty \{x'[A'P + PA + C'C - PBB'P]x + \|(F - B'P)x\|^2\}dt.$$

Since the last term is always nonnegative, it follows that the minimizing value of F is given by $F = B'P$. We can set the term in square brackets to zero provided P satisfies the Riccati equation,

$$A'P + PA + C'C - PBB'P = 0.$$

It follows that the minimum value of J is $x_0'Px_0$.

d) We need to prove that $A_c := A - BB'P$ is stable. The Riccati equation can be written as $A_c'P + PA_c + C'C + PBB'P = 0$. Let $\lambda \in \mathscr{C}$ be an eigenvalue of A_c and $y \neq 0$ be the corresponding eigenvector. Pre– and post–multiplying the Riccati equation be y' and y respectively gives $(\lambda + \bar{\lambda})y'Py + y'C'Cy + y'PBB'Py = 0$. Since $P \succ 0$ and $y \neq 0$, $y'Py > 0$, $y'y > 0$ and $y'PBB'Py \geq 0$. It follows that $\lambda + \bar{\lambda} < 0$ and the closed loop is stable.

e) Since $\|w\|_2 \leq 1$, then an upper bound on $\|z\|_2$ is $\|T_{zw}\|_\infty$. Now,

$$\begin{aligned} \dot{x} &= Ax + Bu = Ax + B(w - Fx) \\ &= (A - BF)x + Bw \\ z &= Cx \end{aligned}$$

it follows that $T_{zw} \stackrel{s}{=} (A - BF, B, C, 0)$. It follows from the bounded real lemma that $\|T_{zw}\|_\infty < 1$ if there exists $P = P' \succ 0$ such that

$$\begin{bmatrix} P(A - BF) + (A - BF)'P + C'C & PB \\ B'P & -I \end{bmatrix} \prec 0$$

Using a Schur complement argument, this inequality is equivalent to

$$P(A - BF) + (A - BF)'P + C'C + PBB'P \prec 0.$$

However, it follows from the Riccati equation in Part b above that $P(A - BF) + (A - BF)'P + C'C + PBB'P = 0$. This proves that $\|T_{zw}\|_\infty < 1$ and so $\|z\|_2 < 1$.

5. a) i) The $(1,1)$ block of the inequality gives the inequality $A'P + PA \prec 0$. Let $z \neq 0$ be a right eigenvector of A and let λ be the corresponding eigenvalue. Then multiplying the inequality from the left by z' and from the right by z gives $(\lambda + \bar{\lambda})z'Pz < 0$. Since $P \succ 0$ it follows that $z'Pz > 0$ and it follows that $\lambda + \bar{\lambda} < 0$ so that A is stable.

ii) Let x, u and y denote the state, inout and output for $H(s)$. Since A is stable, $\|H\|_\infty < \gamma$ if and only if, with $x(0) = 0$, $J := \int_0^\infty [y'y - \gamma^2 u'u]dt < 0$, for all $u(t)$ such that $\|u\|_2 < \infty$. If $\|u\|_2$ is bounded, then $\lim_{t\to\infty} x(t) = 0$. Now, $\int_0^\infty \frac{d}{dt}[x'Px]dt = x(\infty)'Px(\infty) - x(0)'Px(0) = 0$. So,

$$0 = \int_0^\infty (\dot{x}'Px + x'P\dot{x})\, dt = \int_0^\infty [x'(A'P + PA)x + x'PBu + u'B'Px]dt.$$

Use $y = Cx + Du$ and add the last expression to J

$$\begin{aligned} J &= \int_0^\infty [x'(A'P+PA+C'C)x + 2x'(PB+C'D)u + u'(D'D-\gamma^2 I)u[dt \\ &= \int_0^\infty \begin{bmatrix} x' & u' \end{bmatrix} \overbrace{\begin{bmatrix} A'P+PA+C'C & PB+C'D \\ B'P+D'C & D'D-\gamma^2 I \end{bmatrix}}^{M} \begin{bmatrix} x \\ u \end{bmatrix} dt. \end{aligned}$$

It follows that $J < 0$, and so $\|H\|_\infty < \gamma$, if $M \prec 0$. This proves the result.

b) i) The state equations for Figure 5 give

$$\dot{x} = \underbrace{(A + BFC)}_{A_c}x + \underbrace{BF}_{B_c}r, \qquad z = \underbrace{C}_{C_c}x + \underbrace{I}_{D_c}r.$$

It follows that $T_{zr}(s) = D_c + C_c(sI - A_c)^{-1}B_c$.

ii) The transfer matrix T_{zr} is the sensitivity for the feedback-loop and limiting its infinity norm will improve the tracking properties of the loop.

iii) Using the results of part (a), by replacing A, B, C and D by A_c, B_c, C_c and D_c, we have that there exists a feasible F if there exists $P = P'$ such that

$$\begin{aligned} \begin{bmatrix} (A+BFC)'P + P(A+BFC) + C'C & PBF + C' \\ \star & I - \gamma^2 I \end{bmatrix} &\prec 0 \\ P &\succ 0 \end{aligned}$$

Noting that the only nonlinearity is due to the product PBF, and that B is square and nonsingular, we define $Z = PBF$ and so there exists a feasible F if there exists $P = P'$ and Z such that

$$\begin{aligned} \begin{bmatrix} PA + A'P + ZC + C'Z' + C'C & Z + C' \\ \star & I - \gamma^2 I \end{bmatrix} &\prec 0 \\ P &\succ 0 \end{aligned}$$

in which case $F = B^{-1}P^{-1}Z$.

6. a) The generalized regulator formulation is given by

$$\begin{bmatrix} z(s) \\ y(s) \end{bmatrix} = P(s)\begin{bmatrix} w(s) \\ u(s) \end{bmatrix}, u(s) = Fy(s), P(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{c||c|c} A & B & B \\ \hline C & 0 & 0 \\ 0 & 0 & I \\ \hline I & 0 & 0 \end{array}\right].$$

b) The requirement $\|H\|_\infty < \gamma$ is equivalent to $J := \|z\|_2^2 - \gamma^2\|w\|_2^2 < 0$. Let $V = x^T X x$ and set $u = Fx$. Provided that $X = X^T > 0$ and $\dot{V} < 0$ along the closed-loop trajectory, we can assume $\lim_{t\to\infty} x(t) = 0$. Then

$\dot{V} = \dot{x}^T Xx + x^T X\dot{x} = x^T\left(A^T X + XA + F^T B^T X + XBF\right)x + x^T XBw + w^T B^T Xx.$

Integrating from 0 to ∞ and using $x(0) = x(\infty) = 0$,

$$0 = \int_0^\infty [x^T\left(A^T X + XA + F^T B^T X + XBF\right)x + x^T XBw + w^T B^T Xx]dt. \qquad (6.1)$$

Using the definition of J and adding the last equation, $J =$

$$\int_0^\infty \{x^T[A^T X + XA + C^T C + F^T F + F^T B^T X + XBF]x - [\gamma^2 w^T w - x^T XBw - w^T B^T Xx]\}dt.$$

Let $Z = F + B^T X$. Completing the squares gives

$$J = \int_0^\infty \{x^T[A^T X + XA + C^T C - (1-\gamma^{-2})XBB^T X]x + \|Zx\|^2 - \left\|\gamma w - \gamma^{-1}B^T Xx\right\|^2\}dt.$$

Thus two sufficient conditions for $J < 0$ are the existence of X such that

$$A^T X + XA + C^T C - (1-\gamma^{-2})XBB^T X = 0, \qquad X = X^T > 0.$$

The state feedback gain is $F = -B^T X$ (ensuring $Z = 0$) and the worst case disturbance is $w^* = \gamma^{-2}B^T Xx$. The closed-loop with these feedback laws is $\dot{x} = [A - (1-\gamma^{-2})BB^T X]x$ and a third condition is therefore $Re\ \lambda_i[A - (1-\gamma^{-2})BB^T X[<$ 0, $\forall\, i$.

It remains to prove $\dot{V} < 0$ along state-trajectory with $u = Fx$ and $w = 0$. But

$\dot{V} = x^T\left(A^T X + XA + F^T B^T X + XBF\right)x = -x^T(C^T C + (1+\gamma^{-2})XBB^T X)x < 0$

for all $x \neq 0$ (since (A, B, C) is assumed minimal) proving closed-loop stability.

c) Setting $w = 0$ and $\gamma \to \infty$ and assuming $x(0) = x_0 \neq 0$ implies that (6.1) now becomes

$$-x_0' X x_0 = \int_0^\infty [x^T\left(A^T X + XA + F^T B^T X + XBF\right)x + x^T XBw + w^T B^T Xx]dt.$$

Adding this to the cost function and proceeding as before gives the Riccati equation as

$$A^T X + XA + C^T C - XBB^T X = 0, \qquad X = X^T > 0.$$

and the cost function as

$$J = x_0' X x_0$$

This may be recognised as the solution of the LQR problem of minimizing $\|z\|_2$.

