Imperial College London

MSci/MSc EXAMINATION May 2012

This paper is also taken for the relevant Examination for the Associateship

UNIFICATION

For 4th-Year MSci and MSc Physics Students

Monday, May 21st 2012: 10:00 to 12:00

Answer TWO out of the following FOUR questions.
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the 2 answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in 2 answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

1. Take a theory describing three particle types, represented by three real scalar fields ϕ_1, ϕ_2 and ϕ_3 with a Lagrangian

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi_i\partial^\mu\phi_i + \frac{1}{2}\mu^2\phi_i\phi_i - \lambda(\phi_i\phi_i)^2 \qquad i = 1, 2, 3 \; . \tag{1.1}$$

(i) The Lagrangian $\mathcal{L}$ in (1.1) is invariant under continuous transformations given by a 3×3 constant matrix $\mathbf{M}$,

$$\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix} \to \begin{pmatrix}\phi_1'\\ \phi_2'\\ \phi_3'\end{pmatrix} = \mathbf{M}\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix} \; . \tag{1.2}$$

Find the weakest conditions that $\mathbf{M}$ must satisfy in order for this transformation to be a symmetry of the Lagrangian. What is the symmetry group $\hat{G}$ that these matrices $\mathbf{M}$ belong to? Is this group connected? What nontrivial continuous symmetry subgroups of $\hat{G}$ are there? (Trivial subgroups are either the whole of $\hat{G}$ or the trivial group consisting just of the identity.) [4 marks]

(ii) Consider transformations infinitesimally close to the identity matrix $\mathbb{1}$, *i.e.* let $\mathbf{M} = \mathbb{1} + \mathrm{i}\delta\theta^a\mathbf{T}^a$ and find explicitly a basis for the Lie algebra generators $\mathbf{T}^a$ for such infinitesimal transformations. How many independent generators $\mathbf{T}^a$ are there? What is the group G corresponding to transformations that can be continuously connected to the identity? [3 marks]

(iii) Find the equations of motion for this theory. [3 marks]

(iv) Show that this theory has a set of conserved currents of the form

$$J_i^\mu = \epsilon_{ijk}\phi_j(\partial^\mu\phi_k) \; . \tag{1.3}$$

Explain how this may be derived by considering transformations of the form (1.2) but with $\mathbf{M}$ spacetime *dependent*. Show that the integrated charges $Q = \int d^3x J_i^0$ are conserved in time. [4 marks]

(v) Which unitary group is related to the connected group G by a homomorphism? Explain the precise relation between G and that group. [3 marks]

(vi) State Goldstone's theorem and explain how this relates to the mass spectrum of this model theory when $\mu^2 > 0$. Give the masses of all modes in terms of μ and λ. [3 marks]

[Total 20 marks]

2. (i) Consider a theory of a single complex field $\Phi(x) \in \mathbb{C}$ with a Lagrangian

$$\mathcal{L}_\Phi = -(\partial_\mu \Phi)^\dagger (\partial^\mu \Phi) - m^2 |\Phi|^2 - \lambda |\Phi|^4 \ .$$

Suppose this theory were considered in D spacetime dimensions, so the above Lagrangian would have to be integrated over $\int d^D x$ to give the theory's action. If a quantity has units of $(\text{mass})^k$, one says that it has dimension k. Given that the action should be dimensionless overall, find the dimension of the complex scalar field in this D-dimensional theory, *i.e.* let the dimension of Φ be $(\text{mass})^k$ and find k.

What are the dimensions of the parameters m and λ in this case? Given that renormalizable theories cannot have any parameters with negative powers of mass dimensions, what is the largest power of $|\Phi|$ that can appear in a renormalizable theory's potential term if $D = 4$? What is the largest power in a potential term if $D = 3$?

Show that the above Lagrangian is invariant under rigid U(1) transformations

$$\Phi(x) \to \Phi'(x) = e^{\mathrm{i}\theta} \Phi(x) \quad \text{where } \partial_\mu \theta = 0 \ , \theta \in \mathbb{R} \ .$$

[4 marks]

(ii) Now consider a theory of a single Dirac spinor field $\psi(x)$, with Lagrangian

$$\mathcal{L}_\psi = -\mathrm{i}\, \bar{\psi} \gamma^\mu \partial_\mu \psi - \mathrm{i}\, m \bar{\psi} \psi \ .$$

(a) Find the dimension of ψ in D spacetime dimensions. What is the dimension of the parameter m in this case?

(b) Show that this fermionic Lagrangian is invariant under a rigid U(1) transformation

$$\psi \to \psi' = e^{\mathrm{i}\rho} \psi \quad \text{where } \partial_\mu \rho = 0 \ , \theta \in \mathbb{R} \ .$$

(c) Show that if $m = 0$, but not for $m \neq 0$, this fermionic Lagrangian has an additional "γ_5-symmetry" $\psi \to e^{i\alpha\gamma_5}\psi$, where $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ and α is a real constant parameter. For $m = 0$, find two quartic forms of self-interaction terms for the fermionic field $\psi(x)$ that are Lorentz-invariant and U(1)-invariant and γ_5-invariant. In which integral spacetime dimension D might such interactions be renormalizable?

[6 marks]

(iii) Now put the above elements together and consider a theory of a complex scalar field $\Phi(x) \in \mathbb{C}$ together with two Dirac fermions $\psi_1(x)$ and $\psi_2(x)$

$$\begin{aligned} \mathcal{L} = \ & - \ (\partial_\mu \Phi)^\dagger (\partial^\mu \Phi) - m^2 |\Phi|^2 - \lambda |\Phi|^4 \\ & - \ \mathrm{i}\, \bar{\psi}_1 \gamma^\mu \partial_\mu \psi_1 - \mathrm{i}\, m_1 \bar{\psi}_1 \psi_1 - \mathrm{i}\, \bar{\psi}_2 \gamma^\mu \partial_\mu \psi_2 - \mathrm{i}\, m_2 \bar{\psi}_2 \psi_2 - g(\Phi \bar{\psi}_1 \psi_2 + \Phi^\dagger \bar{\psi}_2 \psi_1) \ , \end{aligned}$$

where m^2, m_1, m_2, λ and g are all real positive constants.

(a) Find the symmetry group of this theory in the case $g = 0$, but m, m_1 and $m_2 \neq 0$.

(b) Find the symmetry group of this theory when $g \neq 0$. [4 marks]

(iv) (a) What is the unbroken symmetry of the vacuum in the above theory when m^2, m_1, m_2, λ and g are all positive?

(b) What is the unbroken symmetry if $m^2 < 0$ but the other parameters remain positive?

(c) What does Goldstone's theorem say about the boson masses in the $m^2 < 0$ case? Find the masses of the non-zero-mass bosonic states. [6 marks]

[Total 20 marks]

3. Consider a Lagrangian $\mathcal{L}$ for n complex scalar fields Φ_a $(a = 1, \ldots, n)$ and d real Yang-Mills gauge fields A^i_μ $(i = 1, \ldots, d)$. Let this be invariant under local symmetry transformations corresponding to a unitary Lie group G, with $\dim(G) = d$ independent generators $\mathbf{T}^i$ (matrix-valued quantities are in bold; their components are ${T^i}_a{}^b$) and structure constants f^{ijk}:

$$\begin{aligned}
\mathcal{L} &= -\tfrac{1}{4}F^i_{\mu\nu}F^{i\,\mu\nu} - (D_\mu\Phi)^{\dagger\,a}(D^\mu\Phi)_a - V(\Phi^{\dagger\,a}\Phi_a) \\
(D_\mu\Phi)_a &= \partial_\mu\Phi_a - \mathrm{i}\,gA_\mu\cdot T_a{}^b\Phi_b\ , \quad A_\mu\cdot\mathbf{T} = A^i_\mu\mathbf{T}^i \\
F_{\mu\nu}\cdot\mathbf{T} &= F^i_{\mu\nu}\mathbf{T}^i = \tfrac{\mathrm{i}}{g}[\mathbf{D}_\mu.\mathbf{D}_\nu]\ , \quad \mathbf{D}_\mu = \mathbb{1}\partial_\mu - \mathrm{i}\,gA_\mu\cdot\mathbf{T} \\
[\mathbf{T}^i, \mathbf{T}^j] &= \mathrm{i}\,f^{ijk}\mathbf{T}^k \\
\mathrm{Tr}\{\mathbf{T}^i\mathbf{T}^j\} &= \tfrac{1}{2}\delta^{ij}\ .
\end{aligned} \tag{3.1}$$

The unitary symmetry transformation for the scalar fields is of the form

$$\mathbf{\Phi}(x) \to \mathbf{\Phi}'(x) = \mathbf{U}(x)\cdot\mathbf{\Phi}(x)\ , \quad \mathbf{U}(x) \in G\ . \tag{3.2}$$

(i) Work out explicitly the form of $F^i_{\mu\nu}$ in terms of A^i_μ, g and f^{ijk}. [3 marks]

(ii) (a) The covariant derivative transforms as

$$\mathbf{D}_\mu \to \mathbf{D}'_\mu = \mathbf{U}\cdot\mathbf{D}_\mu\cdot\mathbf{U}^{-1}\ . \tag{3.3}$$

Give the transformation of the Yang-Mills field strength $F_{\mu\nu}\cdot\mathbf{T}$. What representation of the group does this correspond to?

(b) Show that $\mathcal{L}$ is invariant under the local transformations corresponding to the unitary group G.

(c) Then find how the Lie algebra valued gauge field $\mathbf{A}_\mu = A_\mu\cdot\mathbf{T} = A^i_\mu\mathbf{T}^i$ transforms. Explain the relation between this gauge field transformation and the linear adjoint realisation of the group symmetry, the generators of which are $(T^i_{\mathrm{adj}})^{jk} := -\mathrm{i}f^{ijk}$. [6 marks]

(iii) Derive the transformation of the gauge fields A^i_μ under *infinitesimal* gauge transformations by considering a gauge transformation of the form $\mathbf{U} = \exp(\mathrm{i}\epsilon^i\mathbf{T}^i)$ with $|\epsilon^i| << 1$. Express your result in terms of A^i_μ, ϵ^i and the adjoint representation generators $(T^i_{\mathrm{adj}})^{jk}$. [5 marks]

(iv) (a) Derive the classical equations of motion for the gauge fields A^i_μ and the scalar fields Φ_a. Show that these can be written in the form

$$(D^\mu_{\mathrm{adj}}F_{\mu\nu})^i = gJ^i_\nu \tag{3.4}$$

(b) Take a covariant divergence of the left-hand side of (3.4) and use the expression for the Yang-Mills field strength $F^i_{\mu\nu}$ given in (3.1) together with the properties of the structure constants f^{ijk} to show that J^i_μ is covariantly conserved, *i.e.* that $(D^\mu_{\mathrm{adj}}J_\mu)^i = 0$. [6 marks]

[Total 20 marks]

4. (i) Work out the electromagnetic charge and weak hypercharge assignments of the u_L, u_R, d_L and d_R quarks. In doing this, you may use the following information. The unbroken electromagnetic $U(1)_{em}$ generator in the Standard Model is conventionally taken to be $Q_{\text{em}} = T_L^3 + Y$, where $T_L^3 = \frac{1}{2}\sigma^3$ and Y is weak hypercharge, *i.e.* the generator of the $U(1)_Y$ factor in the $SU(2)_L \times U(1)_Y$ gauge group. The left-handed spinorial (u_L, d_L) quarks transform as a doublet under $SU(2)_L$, while the right-handed spinorial versions of these quarks are $SU(2)_L$ singlets. As for electromagnetic charges, note that the left-handed component of the proton $(uud)_L$ has electromagnetic charge $+1$ while the left-handed component of the neutron $(udd)_L$ has electromagnetic charge zero. The right-handed components have the same electromagnetic charges as the left-handed components.

[6 marks]

(ii) The Standard Model gauge group $SU(3) \times SU(2)_L \times U(1)_Y$ fits nicely as a subgroup into the Georgi-Glashow candidate Grand Unification gauge group $SU(5)$.

(a) How does the fundamental $\mathbf{5}$ representation ψ_A $(A = 1, 2, 3, 4, 5)$ of $SU(5)$ decompose into (R_3, R_2) representations R_3 of $SU(3)$ and R_2 of $SU(2)$?

(b) Let the $\mathbf{5}^T$ representation of $SU(5)$ be written as the row $(f_1, f_2, f_3, h_1, h_2)$. Given that the Standard Model gauge group is the direct product $SU(3) \times SU(2)_L \times U(1)_Y$, explain why the $f_{\tilde{a}}$ $(\tilde{a} = 1, 2, 3)$ must correspond to the same $U(1)_Y$ hypercharge y_f and the h_b $(b = 1, 2)$ correspond to the same hypercharge y_h. From the requirement that the hypercharge Y generator must be chosen from among the $SU(5)$ generators, derive a linear relation involving y_f and y_h.

(c) Hence, given your value for the hypercharge assignment for the d_R quarks found in part (i) above, find the corresponding values of y_f and y_h for the $SU(5)$ representation that contains the d_R quarks.

(d) Show then that the h_b part of the $\mathbf{5}$ representation containing the d_R quarks can only be $(\tilde{L}_C)_b = \epsilon_{bd}(L_C)^d$, where $(L_C)^b$ is the charge conjugate of the Standard Model $L_b = (\nu_L, e_L)$ doublet (suppressing Lorentz spinor indices) containing the left-handed neutrino and electron components.

(e) The conventional normalisation of the non-abelian $SU(5)$ generators T^i is such that $\text{Tr}(T^i T^j) = \frac{1}{2}\delta^{ij}$. Show that in order to agree with this normalisation, the Y generator found in part (c) above needs to be rescaled to $\tilde{Y} = \sqrt{\frac{3}{5}}\, Y$.

[10 marks]

(iii) Explain how the charge-conjugated right-handed electron singlet $(e_R)_C$, the charge-conjugated $SU(3)$ triplet of right-handed $(u_R)_C$ quarks and the ($SU(3)$ triplet, $SU(2)$ doublet) of left-handed $(Q_L)_{\tilde{a}b}$ quarks (again suppressing Lorentz spinor indices) fit into the $SU(5)$ antisymmetric tensor representation $\chi_{[AB]}$. Hence, show that the left-handed fermions of the Standard Model fit precisely into the $SU(5)$ representation $\bar{5} \oplus 10$.

[4 marks]

[Total 20 marks]