PROBLEM 1

(a) Let the signal $y[n] = x[n] * h[n]$ where $h[n] = d[n - n_0], n_0 > 0$. Determine $y[n]$.

(b) Let the signal $y[n] = x[n] * h[n]$ where $x[n] = h[n] = u[n-1]$. Determine $y[n]$.

(c) Consider the Linear Time Invariant (LTI) system that is described by the following input-output relationship

$$y[n] + 2y[n-1] = x[n] + 2x[n-1]$$

where $x[n]$ is the input and $y[n]$ is the output of the system. Find the output of the system to the following input:

$$x[n] = \begin{cases} 1, & n = -2 \\ 2, & n = -1 \\ 3, & n = 0 \\ 2, & n = 1 \\ 2, & n = 2 \\ 1, & n = 3 \\ 0, & \text{otherwise} \end{cases}$$

Assume that $y[n] = 0, n < -2$.

PROBLEM 2

(a) Let $x(t)$ be a periodic signal with period 4 whose Fourier series coefficients are

$$a_k = \begin{cases} jk, & |k| < 4 \\ 0, & \text{otherwise} \end{cases}$$

Determine $x(t)$.

(b) Find the Fourier series coefficients of:

(i) The signal $y_1(t) = x^*(t)$.

(ii) The signal $y_2(t) = x(-t)$.

(c) Consider an LTI system whose response to the input $x(t) = e^{-at}u(t)$ is $y(t) = e^{-bt}u(t)$. Assume that the real part of a and b is positive and that $u(t)$ is the continuous unit step function defined as

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

(i) Find the frequency response of this system.

(ii) Determine the system's impulse response.

(iii) Find the differential equation relating the input and the output of this system.

PROBLEM 3

The output $y(t)$ of a causal LTI system is related to the input $x(t)$ by the differential equation

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 4y(t) = x(t)$$

Determine the frequency response of the system and sketch its Bode plots.

PROBLEM 4

(a) (i) Find the analytical expression and the region of convergence (ROC) of the Laplace transform of the continuous causal signal $x(t) = e^{-at}u(t)$, with a real and positive and $u(t)$ the discrete unit step function.

(ii) Find the analytical expression and the region of convergence (ROC) of the Laplace transform of the continuous anti-causal signal $x(t) = -e^{-at}u(-t)$, with a real and positive and $u(t)$ the discrete unit step function.

(iii) Is the analytical expression $X(s)$ of the Laplace transform of a signal sufficient in order to determine the analytical expression $x(t)$ of the signal in time?

(b) The output $y(t)$ of a causal LTI system is related to the input $x(t)$ by the differential equation

$$\frac{d^2y(t)}{dt^2} - \frac{dy(t)}{dt} - 2y(t) = x(t)$$

Let $X(s)$ and $Y(s)$ denote Laplace transforms of $x(t)$ and $y(t)$, respectively, and let $H(s)$ denote the Laplace transform of $h(t)$, the system's impulse response.

(i) Determine $H(s)$ as a ration of two polynomials.

(ii) Determine $h(t)$ for each of the following cases:

1. The system is stable.
2. The system is causal.
3. The system is neither stable nor causal.

PROBLEM 5

Consider an LTI system for which the input $x[n]$ and output $y[n]$ satisfy the linear constant coefficient difference equation

$$y[n] - \frac{1}{2}y[n-1] = x[n] + \frac{1}{3}x[n-1]$$

Find the two distinct impulse responses that are consistent with the above difference equation.

Use the fact that the z-transform $\frac{1}{1-az^{-1}}$ corresponds to the function $a^n u[n]$ in discrete time if $|z| > |a|$ and the function $-a^n u[-n-1]$ if $|z| < |a|$.

Answer 1

(a) $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]d[n-n_0-k] = x[n-n_0]$

(b) $y[n] = x[n] * h[n] = (n-1)u[n-2]$

(c) $y[n] = -2y[n-1] + x[n] + 2x[n-2]$.

$y[-2] = x[-2] = 1$

$y[-1] = -2y[-2] + x[-1] + 2x[-2] = -2 + 2 + 2 = 2$

$y[0] = -2y[-1] + x[0] + 2x[-1] = -4 + 3 + 4 = 3$

$y[1] = -2y[0] + x[1] + 2x[0] = -6 + 2 + 6 = 2$

$y[2] = -2y[1] + x[2] + 2x[1] = -4 + 2 + 4 = 2$

$y[3] = -2y[2] + x[3] + 2x[2] = -4 + 1 + 4 = 1$

$y[4] = -2y[3] + x[4] + 2x[3] = -2 + 2 = 0$

$y[n] = 0, n > 5$

We observe that $y[n] = x[n]$.

If we use z-transforms in both sides we see that

$(1+2z^{-1})Y(z) = (1+2z^{-1})X(z) \Rightarrow Y(z) = X(z) \Rightarrow y[n] = x[n]$ as already have shown.

Answer 2

(a)

$$x(t) = \sum_{k=-3}^{3} jke^{jk(2p/T)t} = \sum_{1}^{3} jk(e^{jk(2p/T)t} - e^{-jk(2p/T)t}) = \sum_{1}^{3} jk2j\sin[k(2p/T)t] = \sum_{1}^{3} -2k\sin[k(p/2)t]$$

(b)

(i) The signal $y_1(t) = x^*(t)$ has Fourier series coefficients a_{-k}^* with a_k the FS coefficients of $x(t)$. You are not supposed to remember something like this but you are supposed to be able to prove it in the exam. In that case

$$a_{-k}^* = \begin{cases} jk, & |k| < 4 \\ 0, & \text{otherwise} \end{cases} = a_k$$

(ii) The signal $y_2(t) = x(-t)$ has Fourier series coefficients a_{-k} with a_k the FS coefficients of $x(t)$. You are not supposed to remember something like this but you are supposed to be able to prove it in the exam. In that case

$$a_{-k} = \begin{cases} -jk, & |k| < 4 \\ 0, & \text{otherwise} \end{cases} = -a_k$$

(c) Consider an LTI system whose response to the input $x(t) = e^{-at}u(t)$ is $y(t) = e^{-bt}u(t)$. We have the Fourier transform of $x(t)$ being $X(jw) = \frac{1}{a+jw}$ and the Fourier transform of $y(t)$ being $Y(jw) = \frac{1}{b+jw}$.

(i) Find the frequency response of this system. We call this $H(jw) = \frac{Y(jw)}{X(jw)} = \frac{a+jw}{b+jw}$

(ii) Determine the system's impulse response.

$$H(jw) = \frac{Y(jw)}{X(jw)} = \frac{a+jw}{b+jw} = \frac{a}{b+jw} + jw\frac{1}{b+jw} \Rightarrow h(t) = ae^{-bt}u(t) + \frac{d}{dt}e^{-bt}u(t).$$

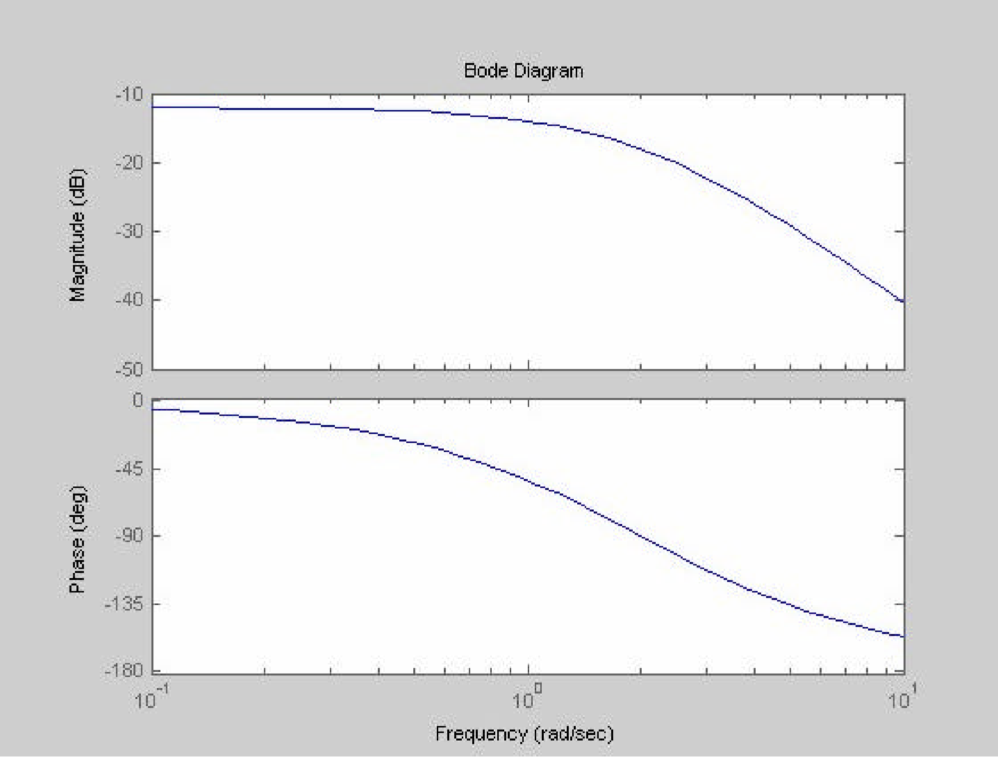
(iii) Find the differential equation relating the input and the output of this system.

$$H(jw) = \frac{Y(jw)}{X(jw)} = \frac{a + jw}{b + jw} \Rightarrow Y(jw)(b + jw) = X(jw)(a + jw) \Rightarrow$$

$$by(t) + \frac{d}{dt}y(t) = ax(t) + \frac{d}{dt}x(t)$$

Answer 3

From $\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 4y(t) = x(t)$ if we take the Fourier transform in both sides we get $Y(jw)[(jw)^2 + 4jw + 4] = X(jw) \Rightarrow H(jw) = \frac{Y(jw)}{X(jw)} = \frac{1}{(jw+2)^2}$. You can treat this function easily since for the Bode plots of $H(jw)$ you need to find the Bode plots of the function $\frac{1}{(jw+2)}$ and multiply it by 2.



Answer 4

(a) (i) Consider the signal $x(t) = e^{-at}u(t)$. The Fourier transform $X(jw)$ converges only for $a > 0$ as shown in the following.

$$X(jw) = \int_{-\infty}^{+\infty} e^{-at}u(t)e^{-jwt}dt = \int_{0}^{+\infty} e^{-at}e^{-jwt}dt = \frac{1}{-(jw+a)}e^{-(jw+a)t}\Big|_0^{+\infty} = \frac{1}{jw+a}, a > 0$$

The Laplace transform is

$$X(s) = \int_{-\infty}^{+\infty} e^{-at}u(t)e^{-st}dt = \int_{0}^{+\infty} e^{-(s+a)t}dt$$

With $s = s + jw$ we have

$$X(s + jw) = \int_{0}^{+\infty} e^{-(s+a)t} e^{-jwt} dt$$

The above is the Fourier transform of $e^{-(s+a)t}u(t)$, and as shown above

$$X(s + jw) = \frac{1}{(s + a) + jw}, s + a > 0$$

or since $s = s + jw$ and $s = \mathrm{Re}\{s\}$, we have

$$X(s) = \frac{1}{s + a}, \text{ ROC: } \mathrm{Re}\{s\} > -a$$

(ii) Consider the signal $x(t) = -e^{-at}u(-t)$. The Fourier transform $X(jw)$ converges only for $a < 0$ as shown in the following.

$$X(jw) = \int_{-\infty}^{+\infty} -e^{-at}u(-t)e^{-jwt} dt = \int_{-\infty}^{0} -e^{-at}e^{-jwt} dt = \frac{1}{jw + a} e^{-(jw+a)t}\Big|_{-\infty}^{0} = \frac{1}{jw + a}, a < 0$$

The Laplace transform is

$$X(s) = \int_{-\infty}^{+\infty} -e^{-at}u(-t)e^{-st} dt = -\int_{-\infty}^{0} e^{-(s+a)t} dt$$

With $s = s + jw$ we have

$$X(s + jw) = -\int_{-\infty}^{0} e^{-(s+a)t} e^{-jwt} dt$$

The above is the Fourier transform of $-e^{-(s+a)t}u(-t)$, and thus,

$$X(s + jw) = \frac{1}{(s + a) + jw}, s + a < 0$$

or since $s = s + jw$ and $s = \mathrm{Re}\{s\}$, we have

$$X(s) = \frac{1}{s + a}, \text{ ROC: } \mathrm{Re}\{s\} < -a$$

(iii) From (i) and (ii) is obvious that the analytical expression $X(s)$ of the Laplace transform of a signal is NOT sufficient in order to determine the analytical expression $x(t)$ of the signal in time. The ROC is also necessary.

(b) The output $y(t)$ of a causal LTI system is related to the input $x(t)$ by the differential equation

$$\frac{d^2y(t)}{dt^2} - \frac{dy(t)}{dt} - 2y(t) = x(t)$$

(i) Determine $H(s)$ as a ration of two polynomials. By taking the Laplace transform in both sides we get:

$$s^2Y(s) - sY(s) - 2Y(s) = X(s) \Rightarrow \frac{Y(s)}{X(s)} = H(s) = \frac{1}{(s-2)(s+1)} \text{ or } H(s) = \frac{1}{3}\frac{1}{s-2} - \frac{1}{3}\frac{1}{s+1}$$

(ii) Determine $h(t)$.

Since we have no information about the ROC's, the factor $\frac{1}{3}\frac{1}{s-2}$ in time is either the function $\frac{1}{3}e^{2t}u(t)$ or the function $-\frac{1}{3}e^{2t}u(-t)$. Also, the factor $\frac{1}{3}\frac{1}{s+1}$ in time is either the function $\frac{1}{3}e^{-t}u(t)$ or the function $-\frac{1}{3}e^{-t}u(-t)$.

1. The system is stable. In that case $h(t) = -\frac{1}{3}e^{2t}u(-t) + \frac{1}{3}e^{-t}u(t)$
2. The system is causal. In that case $h(t) = \frac{1}{3}e^{2t}u(t) + \frac{1}{3}e^{-t}u(t)$

3. The system is neither stable nor causal. In that case $h(t) = -\frac{1}{3}e^{2t}u(-t) - \frac{1}{3}e^{-t}u(-t)$ or $h(t) = \frac{1}{3}e^{2t}u(t) - \frac{1}{3}e^{-t}u(-t)$.

Answer 5

By taking the z-transform in both sides of the input-output relationship we end up with the following expression for the z-transform of the system.

$$Y(z) - \frac{1}{2}z^{-1}Y(z) = X(z) + \frac{1}{3}z^{-1}X(z) \Rightarrow \frac{Y(z)}{X(z)} = H(z) = \frac{1+\frac{1}{3}z^{-1}}{1-\frac{1}{2}z^{-1}} \Rightarrow$$

$$H(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{3}\frac{z^{-1}}{1-\frac{1}{2}z^{-1}}$$

Since we do not have any information about the ROC of the system's transfer function we have two different possible functions for the system's impulse response as follows:

1. The system is causal so the transform $\frac{1}{1-\frac{1}{2}z^{-1}}$ corresponds to the function $(\frac{1}{2})^n u[n]$. We also need to use the property that if the function $x[n]$ has z-transform $X(z)$, the function $x[n-1]$ has z-transform $z^{-1}X(z)$. In that case we have

$$H(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{3}\frac{z^{-1}}{1-\frac{1}{2}z^{-1}} \Rightarrow h[n] = (\frac{1}{2})^n u[n] + \frac{1}{3}(\frac{1}{2})^{n-1}u[n-1]$$

2. The system is anti-causal so the transform $\frac{1}{1-\frac{1}{2}z^{-1}}$ corresponds to the function $-(\frac{1}{2})^n u[-n-1]$.

   In that case we have

$$H(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{3}\frac{z^{-1}}{1-\frac{1}{2}z^{-1}} \Rightarrow$$

$$h[n] = -(\frac{1}{2})^n u[-n-1] - \frac{1}{3}(\frac{1}{2})^{n-1}u[-(n-1)-1] = -(\frac{1}{2})^n u[-n-1] - \frac{1}{3}(\frac{1}{2})^{n-1}u[-n]$$

   The system is anti-causal because $h[n] = 0$ for $n \geq 0$ and unstable because of the term $(\frac{1}{2})^n$ that becomes infinite when $n \to +\infty$.
