UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 1998

MEng Honours Degrees in Computing Part IV
MSci Honours Degree in Mathematics and Computer Science Part IV
MSc Degree in Advanced Computing
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the
Diploma of Membership of Imperial College
Associateship of the Royal College of Science
Associateship of the City and Guilds of London Institute

PAPER 4.73

DOMAIN THEORY AND FRACTALS

Friday, May 8th 1998, 2.00 - 4.00

Answer THREE questions

For admin. only: paper contains 4 questions

1a Let $\mathbb{B}_\perp$ be the flat domain of $\mathbb{B} = \{\mathsf{tt}, \mathsf{ff}\}$ and $\mathbf{2} = \{\perp, \top\}$ be the two-element domain with $\perp \sqsubseteq \top$. Draw a picture of the function space $[\mathbb{B}_\perp \to \mathbf{2}]$.

b Consider $f\colon \mathbb{N} \to \mathbb{N}$ defined by

$$f(n) = \mathsf{if}\ n = 0\ \mathsf{then}\ 0\ \mathsf{else}\ n + f(n-1).$$

Obtain f using $\mathsf{fix}(H)$ for suitable $H\colon [\mathbb{N}_\perp \to_s \mathbb{N}_\perp] \to [\mathbb{N}_\perp \to_s \mathbb{N}_\perp]$. What does f compute?

c Suppose D is a cpo and $f, g \in [D \to D]$ with $f(\perp) = g(\perp)$ and $f \circ g = g \circ f$. Show that $\mathsf{fix}(f) = \mathsf{fix}(g)$.

The three parts carry, respectively, 25%, 40%, 35% of the marks.

2a The real numbers $a = 1 + \sqrt{3}$ and $b = \frac{1}{2}(1 + \sqrt{3})$ satisfy the equations

$$a = 2 + \frac{1}{b} \qquad \text{and} \qquad b = 1 + \frac{1}{a}.$$

i) Using these equations, or otherwise, give a continued fraction expansion for a.

ii) Express a as a normal product. Calculate the first three intervals approximating a.

iii) Suppose $f\colon \mathbb{R}^* \to \mathbb{R}^*$, where $\mathbb{R}^* = \mathbb{R} \cup \{\infty\}$, is defined by

$$f(x) = \frac{x+2}{x+1}.$$

Express $f(a)$ as a normal product.

b Define an ideal in a poset. What is the ideal completion of a poset? Give a precise definition.

c The cpo's D and E, depicted below, are defined as

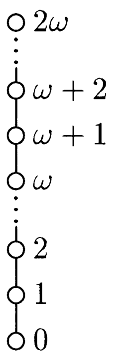
$D = \{0, 1, 2, \cdots\} \cup \{\omega, \omega+1, \omega+2, \omega+3, \ldots\} \cup \{2\omega\}$

with the order $0 \sqsubseteq 1 \sqsubseteq 2 \sqsubseteq \cdots \sqsubseteq \omega \sqsubseteq \omega+1 \sqsubseteq \omega+2 \cdots \sqsubseteq 2\omega$ and

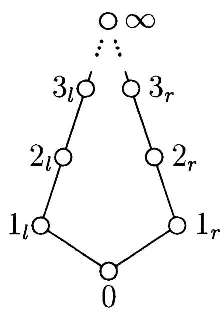
$E = \{0\} \cup \{1_l, 2_l, 3_l, \ldots\} \cup \{1_r, 2_r, 3_r, \ldots\} \cup \{\infty\}$

with the order $0 \sqsubseteq 1_l \sqsubseteq 2_l \sqsubseteq \cdots \sqsubseteq \infty$ and $0 \sqsubseteq 1_r \sqsubseteq 2_r \sqsubseteq \cdots \sqsubseteq \infty$.

D



E



i) Without giving a proof, determine the finite elements of D. Deduce that D is ω-algebraic.

ii) Show that $1_l \in E$ is not finite. Is E ω-algebraic? Justify your answer.

iii) Describe explicitly the isomorphism $\mathsf{Idl}(\mathsf{K}(D)) \to D$.

The three parts carry, respectively, 40%, 20%, 40% of the marks.

Turn over...

3a An iterated function system is given by the two maps $f_1, f_2 \colon \mathbb{R}^2 \to \mathbb{R}^2$ with

$$f_1((x,y)) = \left(\frac{x}{3}, \frac{y+2}{3}\right) \qquad f_2((x,y)) = \left(\frac{x+2}{3}, \frac{y}{3}\right).$$

i) Find the fixed points and the contractivity factors of these maps.

ii) Find the smallest square with sides parallel to the coordinate axes which the functions f_1 and f_2 map into itself.

iii) Define a mapping $f \colon D \to D$ on a cpo D of subsets of $\mathbb{R}^2$, ordered by reverse inclusion, whose fixed point gives the attractor of the IFS.

iv) Determine the similarity dimension of the attractor.

b Consider a monotone function $f \colon D \to E$ between cpo's D, E. For each of the following statements, either give a proof or show that it is false.

i) If D is finite then f is continuous.

ii) If E is finite then f is continuous.

iii) If all chains in D and E are eventually constant then f is a finite element of the function space $[D \to E]$.

The two parts carry, respectively, 50%, 50% of the marks.

4a Given cpo's D and E define their coalesced sum $D \oplus E$ and their disjoint sum $D + E$.

b Consider the domain equation

$$D \cong D + D$$

and the corresponding functor F with $F(D) = D + D$.

i) Find the iterates $D_n = F^n(\{\perp\})$ for $n = 0, 1, 2$ and the corresponding embedding-projection pairs

$$(e_n, p_n) \colon D_n \lhd D_{n+1}$$

for $n = 0, 1$.

ii) Obtain the solution of the domain equation. What is the corresponding isomorphism in the equation?

c Let $\mathbb{B}_\perp$ be the flat domain of $\mathbb{B} = \{\mathsf{tt}, \mathsf{ff}\}$. Draw the three pictures of the upper, lower, and Plotkin power domain of $\mathbb{B}_\perp$.

The three parts carry, respectively, 20%, 45%, 35% of the marks.

END OF PAPER