EE4-26
EE9-CS5-3
EE9-FPN2-07

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2017

MSc and EEE PART IV: MEng and ACGI

ESTIMATION AND FAULT DETECTION

Monday, 22 May 10:00 am

Time allowed: 3:00 hours

There are FOUR questions on this paper.

Answer ALL questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : T. Parisini
Second Marker(s) : D. Angeli

ESTIMATION AND FAULT DETECTION

Information for candidates:

- **One-step ahead Kalman predictor:**

$$\hat{x}(t+1|t) = F\hat{x}(t|t-1) + K(t)[y(t) - H\hat{x}(t|t-1)]$$

- **Kalman predictor gain**

$$K(t) = FP(t)H^\top \left(V_2 + HP(t)H^\top\right)^{-1}, \quad t = 1,2,\ldots$$

- **Riccati equation**

$$P(t+1) = F\left[P(t) - P(t)H^\top \left(V_2 + HP(t)H^\top\right)^{-1} HP(t)\right]F^\top + V_1, \quad t = 1,2,\ldots$$

- **Realization: observer canonical form**

Given

$$Y(s)/U(s) = \frac{b_m s^m + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots a_1 s + a_0} \quad \text{with} \quad m < n$$

then:

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & -a_{n-2} \\ 0 & \cdots & 0 & 1 & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_m \\ 0 \\ \vdots \\ 0 \end{bmatrix} u \\[2em] y = [0 \cdots 0\, 1] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} \end{cases}$$

1. Consider the model of an active vehicle suspension system, as illustrated in Fig. 1.1.

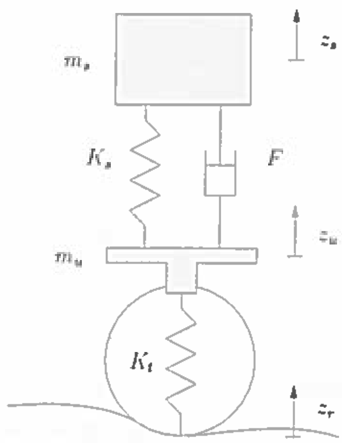

Figure 1.1 Scheme of the active suspension system of Question 1.

By neglecting the effect of the gravity forces, the system dynamics is described by

$$m_s\ddot{z}_s + k_s(z_s - z_u) = F \tag{1.1}$$

$$m_u\ddot{z}_u - k_s(z_s - z_u) + k_t(z_u - z_r) = -F$$

where m_s and m_u represent the sprung and the unsprung masses respectively, k_s and k_t are the spring coefficients of suspension and tire respectively, z_s and z_u are the displacements of the sprung and of the unsprung masses, respectively, and z_r is the terrain input disturbance. F is the electromagnetic force, which is the control input of the system. All the parameters m_u, m_s, k_s, k_t are positive constants.

a) Consider Equations (1.1). Determine a state-space representation of the system supposing that the output is given by $y(t) = z_s(t)$. **[4 marks]**

b) Assume that $k_t z_r = 0$. Set $m_u = 0.4$, $m_s = 2.5$, $k_s = 500$, $k_t = 80$. Discuss the observability properties from the output $y(t)$. In case the input is generated as $u(t) = hy(t)$, discuss the observability properties from the output $y(t)$ as a function of the value of the scalar h. **[6 marks]**

c) Denote by $\hat{x}(t)$ the estimate of the state $x(t)$ associated with the state-space description derived in your answer to Question 1-a), let $e(t) = x(t) - \hat{x}(t)$ denote the state estimation error and suppose that its dynamics obey

$$\dot{e}(t) = Fe(t),$$

and $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the eigenvalues of F. Design (if possible) a full-order state observer such that

$$\lambda_1 = -5,\ \lambda_2 = -5,\ \lambda_3 = -10,\ \lambda_4 = -10.$$

[7 marks]

d) Determine the order of a reduced-order state observer for the system described by the state-space description obtained in your answer to Question 1-a). *Hint: do not attempt to design such a reduced-order observer* **[3 marks]**

2. Consider the hydraulic system in Fig. 2.1 consisting of two interconnected tanks

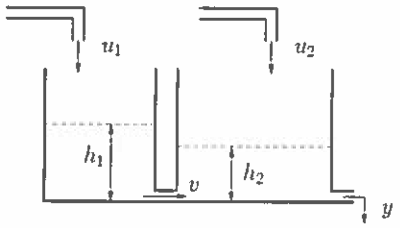

Figure 2.1 Scheme of the hydraulic system considered in Question 2

where the time-behaviours of the levels of water h_1, h_2 and of the outflow rate y are supposed to be described by

$$A_1\dot{h}_1 = u_1 - v,\ A_2\dot{h}_2 = u_2 + v - y,\ v = \beta(h_1 - h_2),\ u_1 = \gamma_1\bar{u}_1,\ u_2 = \gamma_2\bar{u}_2,\ y = \alpha h_2 \tag{2.1}$$

where A_1, A_2, α, β, γ_1, and γ_2 are given positive scalars. $\bar{u}_1$ and $\bar{u}_2$ are known nominal values of the input flow rates.

a) Determine a state-space representation of the system illustrated in Fig. 2.1 using Equations (2.1). Moreover, determine the vector transfer function from the inputs u_1, u_2 to the output y. **[3 marks]**

b) Analyse the observability properties of the system from the output y as a function of the values of α, β, A_1, A_2. **[3 marks]**

c) Denote by $\hat{x}(t)$ the estimate of the state $x(t)$ using the state equations determined in the answer to Question 2-a) and set $A_1 = A_2 = 1$, $\alpha = 1/2$, $\beta = 2$, $\gamma_1 = 1$, $\gamma_2 = 1$. Let $e(t) = x(t) - \hat{x}(t)$ denote the state estimation error. Design a full-order state observer such that

$$\dot{e}(t) = Fe(t)$$

where $\lambda_1 = -3, \lambda_2 = -5$ are the eigenvalues of F. Determine the time behaviour of the output residual

$$\varepsilon(t) = Ce(t),$$

for a given value $\tilde{e}$ of the initial estimation error $e(0)$, where C is the output matrix determined in the answer to Question 2-a). **[6 marks]**

d) Suppose that an abrupt actuator fault occurs at some unknown time $T_0 > 0$ "blocking" the input flow rate u_1, that is

$$\gamma_1 = 1,\ \forall t \in [0, T_0) \text{ and } \gamma_1 = 0,\ \forall t \geq T_0$$

Design a fault detection scheme based on the observer determined in the answer to Question 2-c). Specifically, assuming that $|x(0)| \leq 4$, design a detection threshold $\bar{\varepsilon}(t)$ such that the above actuator fault is detected at some finite time $T_d > T_0$ if the output residual $\varepsilon(T_d)$ satisfies

$$\varepsilon(T_d) > \bar{\varepsilon}(T_d).$$

[8 marks]

3. Consider the discrete-time dynamic system described in Fig. 3.1

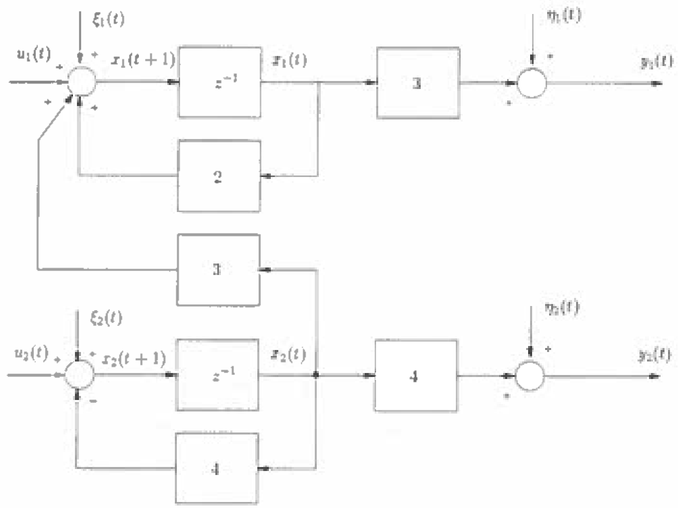

Figure 3.1 Block diagram for Question 3.

where $\xi_1(\cdot) \sim WGN(0,4)$, $\xi_2(\cdot) \sim WGN(0,1)$, $\eta_1(\cdot) \sim WGN(0,1)$, $\eta_2(\cdot) \sim WGN(0,9)$ (Gaussian zero-mean stochastic processes) and the stochastic processes $\xi_1(\cdot)$, $\xi_2(\cdot)$, $\eta_1(\cdot)$, $\eta_2(\cdot)$ are supposed to be independent to each other.

a) Referring to the system sketched in Fig. 3.1, write the state and output equations where the state, input, and output vectors are defined as $x := [x_1, x_2]^\top \in \Re^2$, $u := [u_1, u_2]^\top \in \Re^2$, and $y := [y_1, y_2]^\top \in \Re^2$, respectively.

[3 Marks]

b) Suppose that the inputs u_1 and u_2 are given by:

$$u_1(t) = -\frac{5}{2}x_1(t) - 3x_2(t); \quad u_2(t) = \frac{13}{3}x_2(t)$$

Consider the one-step ahead optimal steady-state Kalman predictor of the state x. Show that the Algebraic Riccati Equation admits a feasible matrix solution $\bar{P}$ and compute it. Compute the corresponding gain vector $\bar{K}$ and write the difference equation yielding the steady-state Kalman prediction $\hat{x}(t|t-1)$.

[6 Marks]

c) Compute the covariance matrix of the prediction error, $\mathrm{Cov}[x(t) - \hat{x}(t|t-1)]$, and the covariance matrix of the process, $\mathrm{Cov}[x(t)]$. Compare $\mathrm{Cov}[x(t) - \hat{x}(t|t-1)]$ with $\mathrm{Cov}[x(t)]$ and comment on your findings.

[5 Marks]

d) Consider the optimal steady-state Kalman filter yielding $\hat{x}(t|t)$. From your answer to Question 3-b), compute the constant gain vector $\bar{K}_0$ of the optimal steady-state Kalman filter. Compute the covariance matrix of the filtering error, $\mathrm{Cov}[x(t) - \hat{x}(t|t)]$ and compare it with $\mathrm{Cov}[x(t) - \hat{x}(t|t-1)]$ and $\mathrm{Cov}[x(t)]$ computed in your answer to Question 3-c). Comment on your findings.

[6 Marks]

4. Consider the continuous-time dynamic system depicted in Fig. 4.1.

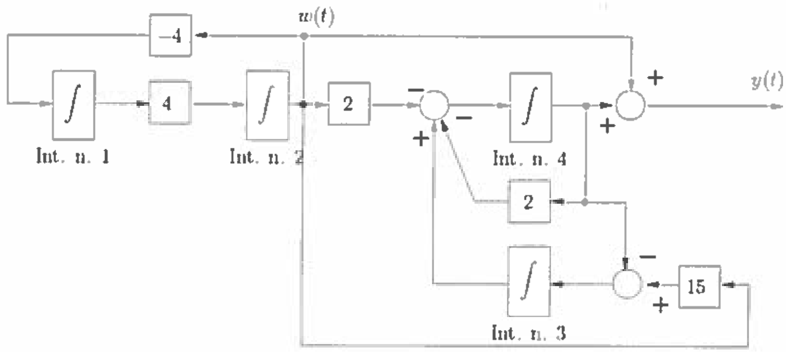

Figure 4.1 Block diagram for Question 4.

The system depicted in Fig. 4.1 has no inputs and comprises four integrators "Int. n. 1", "Int. n. 2", "Int. n. 3", and "Int. n. 4", and several "gain" blocks (their output is given their input multiplied by the scalar shown inside the block).

a) Define a state variable for each of the four integrators in Fig. 4.1 and construct a state-space description of the whole interconnected system where $y(t)$ is the output. **[3 marks]**

b) i) Analyse the observability of the whole system in Fig. 4.1 from the output $y(t)$.

[3 marks]

ii) Determine a state-space description of the dynamic system depicted in Fig. 4.1, which is equivalent to that determined in your answer to Question 4-a) in which the observable and the non-observable sub-systems (if any) are clearly identified.

[5 marks]

iii) If the non-observable vector subspace is non trivial, that is, if $X_{no} \neq \{0\}$, determine a basis for the non-observable vector subspace X_{no}.

[3 marks]

c) Denoting by $Y(s) = \mathscr{L}[y(t)]$ the Laplace transform of the output $y(t)$ and by $x(0^-)$ the initial state in the state space realisation determined in your answer to Question 4-a), determine the analytical expression of $Y(s)$ as a function of $x(0^-)$. Compare the degree of the denominator in the expression of $Y(s)$ with the dimension of the state vector. Comment on your findings.

[3 marks]

d) If $x(0^-) \neq 0$, show that the variable $w(t)$ in the scheme of Fig. 4.1 is characterised by a non-vanishing oscillatory time-behaviour, whereas such an oscillatory time-behaviour does not show up in the variable $y(t)$. Comment on your findings.

[3 marks]