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EE4-05
EE9CS7-2
EE9SO7

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2011

MSc and EEE PART IV: MEng and ACGI

TRAFFIC THEORY & QUEUEING SYSTEMS

Friday, 13 May 10:00 am

Time allowed: 3:00 hours

There are FOUR questions on this paper.

Answer ALL questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : J.A. Barria

Second Marker(s) : D.P. Mandic

Special instructions for students

1. Erlang Loss formula recursive evaluation:

$$E_N(\rho) = \frac{\rho E_{N-1}(\rho)}{N + \rho E_{N-1}(\rho)}$$
$$E_0(\rho) = 1$$

2. Engset Loss formula recursive evaluation (for a fixed M and $p = \alpha/1+\alpha$):

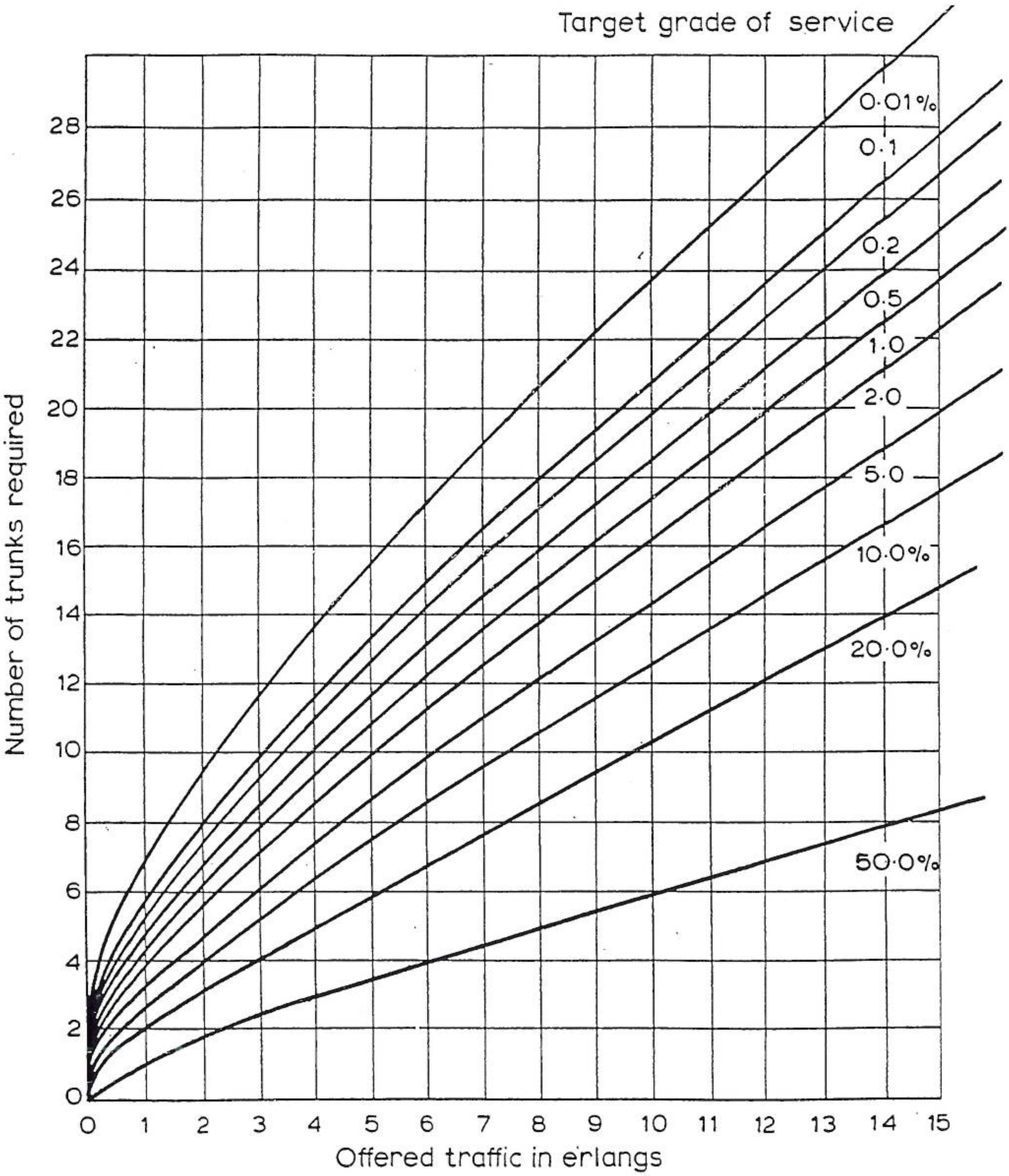
$$e_N = \frac{(M-N+1)\alpha e_{N-1}}{N + (M-N+1)\alpha e_{N-1}}$$
$$e_0 = 1$$
$$\alpha = \lambda/\mu$$

3. Traffic capacity on basis of Erlang B formula (next page).

Note: for large ρ, N is approximately linear: $N \approx 1.33\rho + 5$

4. Expected residual time

$$E[R] = \frac{1}{2}\lambda E[S^2]$$



Traffic capacity on basis of Erlang B. formula.

The Questions

1.

a) Pure chance traffic refers to traffic for which variance of the number of busy channels is equal to the mean of the number of busy channels.

Let N_t denote the number of busy channels at time t.

i) Derive the Markov chain $\{N_t\}$ for the pure chance traffic system described above. [4]

ii) Derive the probability distribution of the Markov chain $\{N_t\}$. [4]

iii) Show that variance of $\{N_t\}$ is equal to the mean of $\{N_t\}$. [4]

b) For a single ON-OFF source model.

i) Derive the Markov chain model of one ON-OFF source. [2]

ii) Derive the Markov Modulated Poisson Process (MMPP) model of N multiplexed ON-OFF sources. [2]

iii) Assuming that the mean length of the arriving packets is negative exponentially distributed $1/\gamma$,

Define the state space of an N ON-OFF sources multiplexer.

Derive the N ON-OFF sources multiplexer Markov chain. Define and identify all transition rates. [2]

iv) State the condition for the multiplexer to be stable. [2]

2.

a) In a network with automatic alternative routing, two exchanges are connected by a first choice link of size M, and a second-choice link of size N. Calls are offered to the second-choice link only if the first-choice link is saturated.

i) Assuming that the total offered traffic is pure chance traffic with parameters (λ, μ) derive an interrupted Poisson process (IPP) model for the traffic on the second-choice link and draw a state transition diagram for your model.

[5]

ii) Determine the mean ON and OFF times for the true overflow traffic process when $M = 12,\ N = 6,\ \lambda = 0.75s^{-1}$ and $\mu = 0.08s^{-1}$.

[5]

iii) Choose the parameters of the IPP model to give the same ON and OFF times as those of the true overflow process.

Discuss one alternative way to choose the IPP parameters.

[5]

b) For an M/G/1 System, in equilibrium, the instantaneous residual service time R_t seen by a virtual arrival at time t is shown in Fig. 2.1.

Using Fig. 2.1., derive the expected value of $R_t = E[R_t]$.

Show your derivations step by step.

[5]

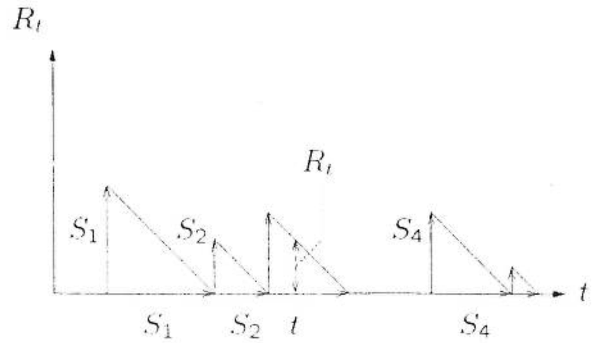


Figure. 2.1.

3.

a) Departing packets from an $M / M / K$ queuing system are either with probability p, fed-back immediately for re-processing, or, with probability $(1-p)$, leave the system forever.

Obtain an expression for the mean number of packets in the buffer, in terms of the external arrival rate, λ, the service rate per channel, μ, and the feedback probability, p.

[8]

b) Traffic from M independently acting sources is offered to an N - channel communication link.

i) If each source acts as a Poisson source and the channel holding times are exponential, show that the resulting equilibrium traffic distribution is truncated binomial when $M > N$.

[6]

i) Obtain an expression for the mean traffic carried by the link in terms of the offered traffic per free source, α.

[6]

4.

a) A Poisson stream of messages with a rate of 3000 [message /s] is fed to a single-channel data link via a large buffer. The packet stream consists of a random mixture of 10-packets and 30-packets messages, all packets being of length 40 bits.

The channel operates at 2 [Mbit/s] and 75% of the messages are 10-packet messages.

Determine the overall mean message waiting time when the queue discipline for the link is first-in first-out.

[10]

b) Figure 4.1 shows the reliability block diagram of a system composed of two types of units. All type 1units are connected in parallel while all type 2 units are arranged in a series configuration.

Assume that the failure rate of each type 1 unit is λ and the failure rate of each type 2 unit is γ.

If the system type 1 units represent processors, and the type 2 units represent buffers to hold jobs not being serviced by the processors then, Figure 4.1 can be thought as an M/M/a/a+b queueing system.

The repair rate for each type 1 unit be μ and the repair rate of type 2 units be τ.

Assume:

- A single repair person is devoted to the repair of each type of unit,

- Components do not fail in system failure state,

- The system is in failure state if any one of type 2 units fails and/or all type 1 units fail.

i) Define the state space of the system.

Clearly identify the operational states.

[5]

ii) Derive all transition rates.

[5]

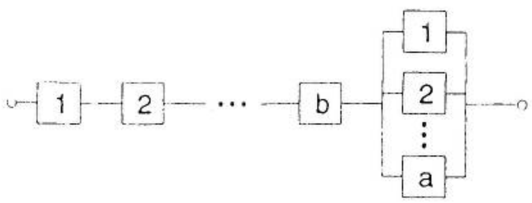


Figure 4.1

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Model Answers and Mark Schemes 2011 — First Examiner: J. Barria — EE4-05

Paper Code: Traffic Theory & Queuing Systems — Second Examiner: D. Mandic — EE9507, EE9C57-2

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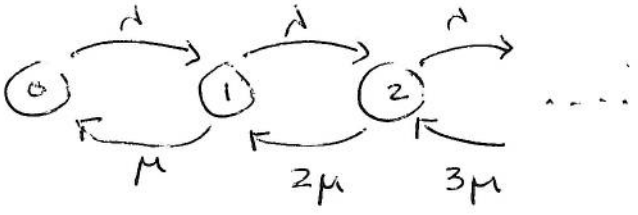
Q1

a)



$$\pi_i = \left(\frac{\lambda_{i-1}}{\mu_i}\right)\pi_{i-1} = \left(\frac{\lambda}{i\mu}\right)\pi_{i-1} = \left(\frac{\rho}{i}\right)\pi_{i-1}$$

Recursive solution

$$\pi_i = \left(\frac{\rho^i}{i!}\right)\pi_0 \qquad i = 1, 2, 3, \dots$$

where $\pi_0 = e^{-\rho}$

$$\pi_i = \frac{\rho^i}{i!} e^{-\rho}, \quad i = 0, 1, 2 \dots$$

$$E(N_t) = \sum_{i=0}^{\infty} i\pi_i$$

$$= \sum i\frac{\rho^i}{i!}\pi_0 = \frac{1}{e^\rho}\sum i\frac{\rho^i}{i!} = \frac{1}{e^\rho}\left(\frac{1\rho^1}{1!} + \frac{2\rho^2}{2!} + \dots\right.$$

$$= \frac{1}{e^\rho}\rho\left(1 + \frac{\rho}{1!} + \frac{\rho^2}{2!} + \dots\right)$$

$$= \frac{1}{e^\rho}\rho e^\rho = \rho$$

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Q1 a

$$\text{Variance} = E[(x-\mu)^2]$$

$$E(x^2 - 2x\mu + \mu^2) = E(x^2) - E(2x\mu) + E(\mu^2)$$

$$= E(x^2) - (E(x))^2$$

$$E(N_t^2) = \sum_{i=0}^{\infty} i^2 \pi_i = \frac{1}{e^\rho} \sum i^2 \frac{\rho^i}{i!}$$

$$= \frac{1}{e^\rho}\left(1\frac{\rho^1}{1!} + 4\frac{\rho^2}{2!} + 9\frac{\rho^3}{3!} + 16\frac{\rho^4}{4!} + \dots\right.$$

$$= \frac{1}{e^\rho}\left(\rho + \frac{2\rho^2}{2!} + \frac{3\rho^3}{3!} + \frac{4\rho^4}{4!} + \dots\right.$$

$$+ \frac{1}{e^\rho}\left(\frac{2\rho^2}{2!} + \frac{6\rho^3}{3!} + \frac{14\rho^4}{4!} + \dots\right.$$

$$= \frac{1}{e^\rho}\rho e^\rho + \frac{1}{e^\rho}\rho^2\left(\frac{2}{2!} + \rho + \frac{\rho^2}{2!} + \dots\right.$$

$$= \rho + \frac{1}{e^\rho}\rho^2 e^\rho$$

$$= \rho + \rho^2$$

$$\text{Variance} = \rho + \rho^2 - \rho^2 = \rho$$

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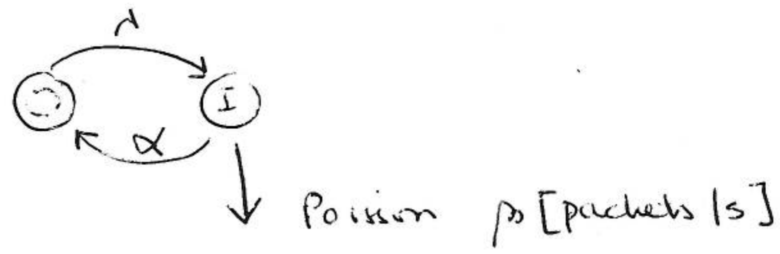
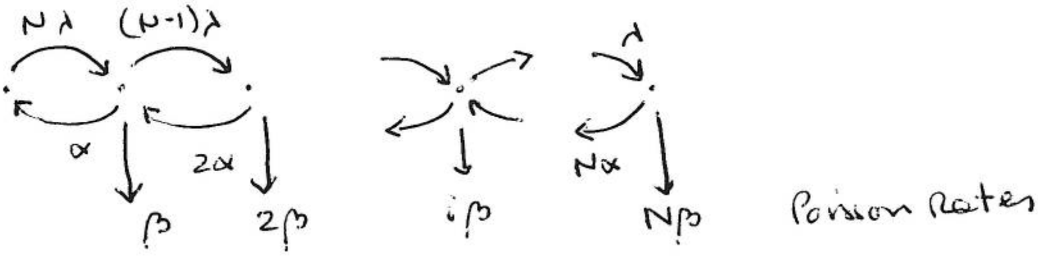
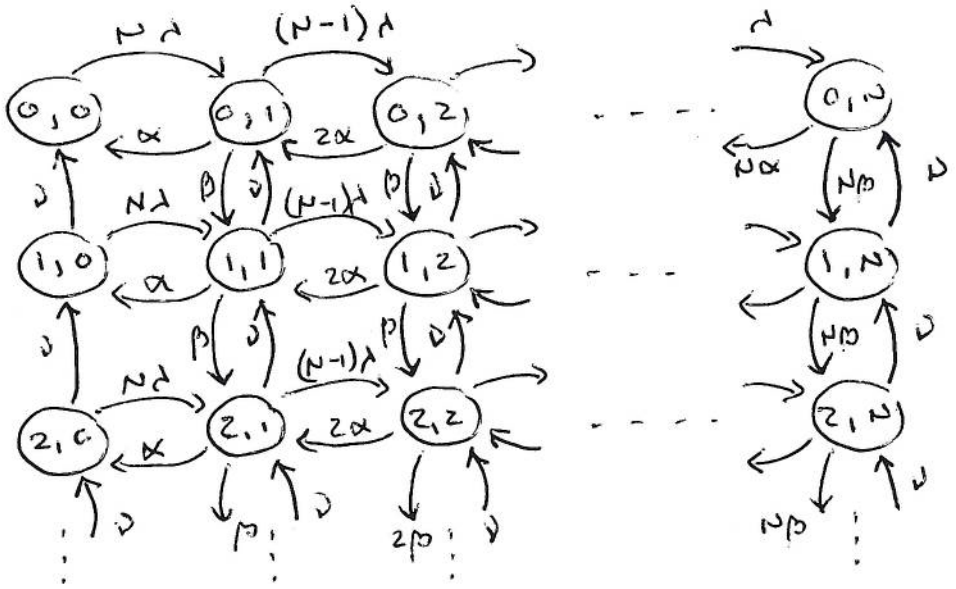
Q1
b







Each source β packets/s

N active sources Average Arrival Rate = $N\beta \frac{\lambda}{\alpha+\lambda}$ $\left[\frac{\text{packets}}{s}\right]$

System stable iff: $N\beta \frac{\lambda}{\alpha+\lambda} < \nu$

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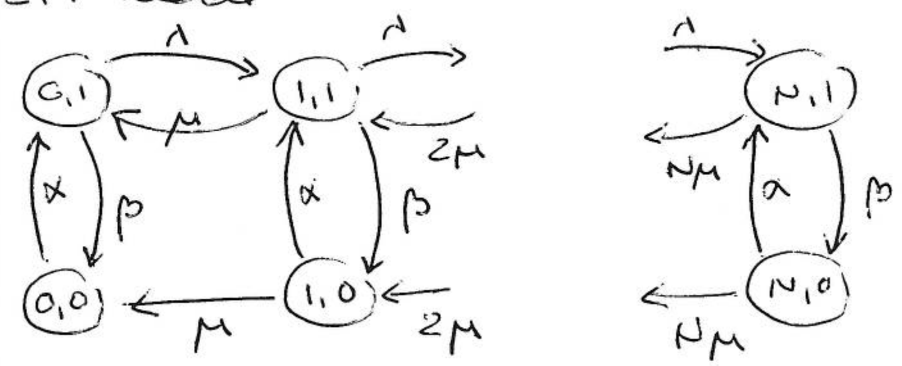
Q2 a) IPP model



where

(i,j) is the state $(N_t = i,\ y_t = j)$

$y_t = 0$ arrival stream OFF

$y_t = 1$ arrival stream ON

with $\lambda = 0.75$ and $\mu = 0.08$

offered traffic $\rho = \dfrac{\lambda}{\mu} = 9.4$ Erlangs

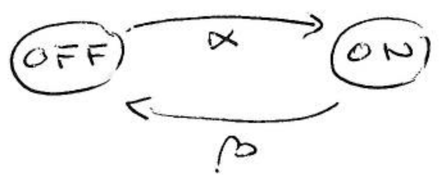
For link of size 12

$P[\text{link saturation}] = E_{12}(9.4) = 0.1$

overflow process has (by ergodicity)

$$\frac{T_{ON}}{T_{ON} + T_{OFF}} = P(\text{link saturation}) = 0.1 \Rightarrow T_{ON}/T_{OFF} = 1/9$$

For IPP we have



and so $\dfrac{\pi_{ON}}{\pi_{OFF}} = \dfrac{\alpha}{\beta} \Rightarrow \beta = 9\alpha$

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Q2 a

But $E[T_{on}] = E[\text{Sojourn time in saturation}]$

$$= \frac{1}{N\mu} = 2.08\ s$$

$$\beta = \frac{1}{E[T_{on}]} = 0.48\ s^{-1}$$

$$\alpha = \frac{1}{9}\beta = 0.0533\ s^{-1}$$

An alternative way is to choose α and β such that the mean and variance of the overflow traffic is matched

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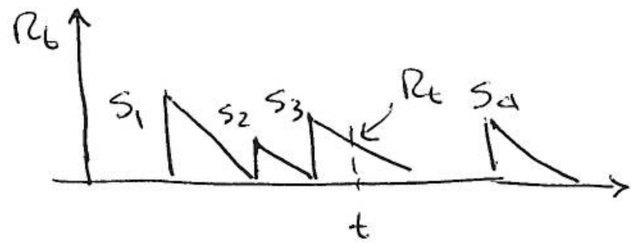
Q2 b)

R_t = residual service time seen by a virtual arrival at time t.

At equilibrium $\{R_t\}$ is a continuous-time stochastic process which looks like



Assuming that $\{R_t\}$ is ergodic (in mean)

$$E[R_t] = \lim_{T\to\infty} \frac{1}{T}\int_0^T R_t\,dt = \lim_{T\to\infty} \frac{1}{T}\sum_{i=1}^{M_T}\left(\frac{1}{2}S_i^2\right)$$

where

$\frac{1}{2}S_i^2$ = area of the ith triangle

M_t = number of completed service in $[0,T]$

Rewriting this equation gives

$$E[R_t] = \lim_{T\to\infty} \frac{1}{2}\left(\frac{M_T}{T}\right)\left[\frac{1}{M_T}\sum_{i=1}^{M_T} S_i^2\right]$$

where

M_T/T = service completion rate = mean arrival rate $=\lambda$

$\frac{1}{M_T}\sum_{i=1}^{M_T} S_i^2$ = mean squared service time $= E[S^2]$

M/G/1 $\quad E[R_t] = \frac{1}{2}\lambda E[S^2]$

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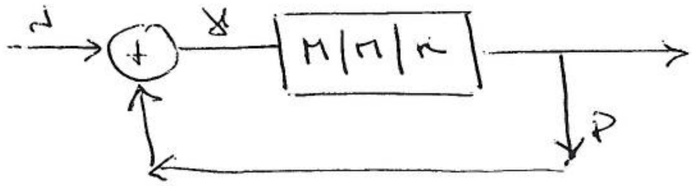
Q3

a)



The M|M|K box retains its M|M|K properties under feedback conditions, but with total arrival, $\gamma = \dfrac{\lambda}{1-p}$

Mean number in buffer is given by

$$P(Q_t = i \mid N_t \geq K) = \frac{P(N_t = K+i)}{\sum_{j=0}^{\infty} P(N_t = K+j)} = (1-\rho)\rho^i \qquad i = 0,1,2\ldots$$

$$E[Q_t \mid \text{Delay}] = \frac{\rho}{1-\rho}$$

$$P(Q_t = i) = P(\text{delay})\,P(Q_t = i \mid \text{delay}) + P(\text{no delay})\,P(Q_t = i \mid \text{no delay})$$

$$= \begin{cases} 1, & \text{if } i = 0 \\ 0, & \text{if } i > 0 \end{cases}$$

$$E[Q_t] = D_K(A)\left(\frac{\rho}{1-\rho}\right)$$

$$A = \frac{\gamma}{\mu}$$

$$D_K(A) = \frac{E_K(A)}{(1-\rho) + \rho E_K(A)}$$

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Q3 b) With Poisson sources and exponential holding times, the number of busy channels N_t is a birth/death process equilibrium equations for N_t are

$$(i\mu)\pi_i = (M-i+1)\lambda\,\pi_{i-1} \quad \text{for} \quad 0 < i \le i_{max}$$

$$\therefore \pi_i = \binom{M}{i}\alpha^i \pi_0 \qquad \text{where} \quad \alpha = \frac{\lambda}{\mu}$$

$$\text{and} \quad \pi_0 = \sum_{j=0}^{i_{max}} \binom{M}{j}\alpha^j$$

$$\alpha = \frac{p}{1-p} \quad \text{and multiply by } (1-p)^M$$

$$\pi_i = \frac{\binom{M}{i} p^i (1-p)^{M-i}}{\sum_{j=0}^{i_{max}} \binom{M}{j} p^j (1-p)^{M-j}}$$

Truncated binomial since $i_{max} = N$ $(M > N)$

Mean carried traffic ρ_c = [mean offered traffic × (1 − B)]

= [Mean nr of free sources × α] × (1 − B)

$$\rho_c = (M-\rho_c) \times \alpha \times (1-B)$$

$$\rho_c = \frac{(1-B)\alpha M}{1+\alpha(1-B}$$

Note: Mean nr of free sources $= M - \rho_c$ As the mean nr of busy sources is equal to mean nr of busy channels

Q4

a) Residual time, R, is the time to the next service completion

For an M|G|1 system

$$E(R) = \frac{1}{2}\lambda E(S^2)$$

λ = arrival rate

S = service time

FIFO / no priority

Mean message/packet waiting time $= \dfrac{\lambda E(S^2)}{2(1-\rho)}$

where $\lambda = 3000$ messages/packets / s

$$E(S) = \frac{3/4\ 400 + 1/4\ 1200}{2\times10^6} = 3\times10^{-4}\ s$$

so that

$$\rho = \lambda E(S) = 0.9$$

Also

$$E(S^2) = \frac{3/4\,(400)^2 + 1/4\,(1200)^2}{4\times10^{12}} = 0.12\times10^{-6}\ s$$

$\Rightarrow$ Mean waiting time $= \dfrac{0.36\times10^{-3}}{0.2} = 1.8$ ms

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Q4 b)