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UNIVERSITY OF LONDON

[II(4)E 2004]

B.ENG. AND M.ENG. EXAMINATIONS 2004

For Internal Students of Imperial College

This paper is also taken for the relevant examination for the Associateship of the City & Guilds of London Institute

PART II : MATHEMATICS 4 (ELECTRICAL ENGINEERING)

Thursday 3rd June 2004 2.00 - 4.00 pm

Answer FOUR questions.

Corrected Copy

[Before starting, please make sure that the paper is complete; there should be 4 pages, with a total of 6 questions. Ask the invigilator for a replacement if your copy is faulty.]

1. Find the eigenvalues λ_1, λ_2, λ_3 ($\lambda_1 > \lambda_2 > \lambda_3$) and the normalised eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 0 & \sqrt{3}/2 \\ 0 & 2 & 0 \\ \sqrt{3}/2 & 0 & 2 \end{bmatrix} .$$

Using these, or otherwise, find the diagonalising matrix P that satisfies

$$P^T A P \;=\; \Lambda \;=\; \mathrm{diag}(\lambda_1,\, \lambda_2,\, \lambda_3) \;.$$

Find the expression for A^n, n being a positive integer, in terms of Λ and P.

Hence calculate the limiting form

$$B \;=\; \lim_{n\to\infty} \frac{1}{\lambda_1^n} A^n \;,$$

where λ_1 is the largest eigenvalue.

2. Find the eigenvalues and normalised eigenvectors of the matrix

$$A = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 4 & 1 \\ 1 & 1 & 2 \end{bmatrix} .$$

The quadratic form Q is defined as $Q = \mathbf{x}^T A\mathbf{x}$, where $\mathbf{x}^T = (x_1,\, x_2,\, x_3)$.

Find a matrix P such that the substitution $\mathbf{x} = P\mathbf{y}$ brings Q into a diagonal form in terms of $\mathbf{y}$.

Write down the resulting diagonal form and prove that $Q \geq 0$ for all $\mathbf{y}$.

PLEASE TURN OVER

3. (i) The events A, B and C have probabilities given by $\mathrm{P}(A) = 0.3$, $\mathrm{P}(B) = 0.2$, $\mathrm{P}(C) = 0.2$, and $\mathrm{P}(B \cap C) = 0.05$. Also, A and B are independent, and A and C are exclusive.

Calculate $\mathrm{P}(A \cap B)$, $\mathrm{P}(A \cup C)$, $\mathrm{P}(B \mid C)$, and the probability that exactly one of A, B and C occurs.

(ii) Manufacturer A supplies 80% of the electrical components used by a firm, and a proportion p_A of them are defective. Manufacturer B supplies the other 20%, of which a proportion p_B is defective.

Give an expression for the probability that a randomly-selected component is defective.

Two components selected at random are both defective.

Obtain an expression for the probability that both are from Manufacturer A.

4. (i) The discrete random variable X has probability function $p(x) = k/x$ for $x = 1, 2, 3, 4$; $p(x)$ is zero for any other value of x.

Show that $k = 12/25$. Calculate the mean and variance of X, and compute $\mathrm{cov}(X, X^2)$.

(ii) The continuous random variable Y has density function $f(y) = 6y(1-y)$ on (0,1); $f(y)$ is zero outside the range (0,1).

Calculate the mean and variance of Y, and compute $\mathrm{E}[\{Y(1-Y)\}^{-1}]$.

5. The distribution of lifetimes to failure of certain electronic units is well described by the survivor function $\bar{F}(t) = \exp(-0.2t - 0.1t^{3/2})$ for $0 < t < \infty$.

   Compute the probability that a randomly-selected unit will fail before time $t = 1$.

   Calculate the hazard function and the time by which the hazard function has doubled from its value at time 0.

   Two such independently-functioning units are arranged in series, so that the system fails as soon as one of the units fails.

   Calculate the survivor and hazard functions for the system lifetime.

   Write down the form of these functions for a system comprising m such units in series.

   You are reminded that the survivor and hazard functions are related by

$$\bar{F}(t) = \exp\{-\int_0^t h(s)ds\}.$$

6. The random variable Y has distribution function $F(y) = 1 - \frac{1}{3}(e^{-y} + 2e^{-y/\theta})$ on $(0, \infty)$, with $\theta > 0$.

   Write down its density function, and compute $E(Y)$ and $\text{var}(Y)$. From a random sample of n such Y-values the sample sum, $S = Y_1 + ... + Y_n$, is formed.

   Show that $\frac{1}{2}(\frac{3S}{n} - 1)$ is an unbiased estimator of θ and calculate its variance. Is it consistent?

   Now suppose that the null hypothesis $H_0 : \theta = 1$ is to be tested against the alternative $H_1 : \theta > 1$.

   Write down the form of the appropriate p-value based on a single observation, $Y = y$.

END OF PAPER

MATHEMATICS DEPARTMENT

MATHEMATICAL FORMULAE

1. VECTOR ALGEBRA

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1, a_2, a_3)$$

Scalar (dot) product: $\mathbf{a}\,.\,\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

Vector (cross) product:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Scalar triple product:

$$[\mathbf{a},\,\mathbf{b},\,\mathbf{c}] = \mathbf{a}\,.\,\mathbf{b}\times\mathbf{c} = \mathbf{b}\,.\,\mathbf{c}\times\mathbf{a} = \mathbf{c}\,.\,\mathbf{a}\times\mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product: $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c}\,.\,\mathbf{a})\mathbf{b} - (\mathbf{b}\,.\,\mathbf{a})\mathbf{c}$

2. SERIES

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \ldots \quad (\alpha \text{ arbitrary, } |x| < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \cdots,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n\frac{x^{2n}}{(2n)!} + \cdots,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + \cdots,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^n\frac{x^{n+1}}{(n+1)} + \ldots (-1 < x \leq 1)$$

3. TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin(a+b) = \sin a\,\cos b + \cos a\,\sin b\,;$$

$$\cos(a+b) = \cos a\,\cos b - \sin a\,\sin b\,.$$

$$\cos iz = \cosh z\,;\quad \cosh iz = \cos z\,;\quad \sin iz = i\sinh z\,;\quad \sinh iz = i\sin z\,.$$

4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f\,D^n g + \tbinom{n}{1}\,Df\,D^{n-1}g + \ldots + \tbinom{n}{r}\,D^r f\,D^{n-r}g + \ldots + D^n f\,g\,.$$

(b) Taylor's expansion of $f(x)$ about $x = a$:

$$f(a+h) = f(a) + hf'(a) + h^2 f''(a)/2! + \ldots + h^n f^{(n)}(a)/n! + \epsilon_n(h)\,,$$

where $\epsilon_n(h) = h^{n+1}f^{(n+1)}(a+\theta h)/(n+1)!, \quad 0 < \theta < 1\,.$

(c) Taylor's expansion of $f(x, y)$ about (a, b):

$$f(a+h,\,b+k) = f(a,\,b) + [hf_x + kf_y]_{a.b} + 1/2!\left[h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}\right]_{a.b} + \cdots$$

(d) Partial differentiation of $f(x, y)$:

i. If $y = y(x)$, then $f = F(x)$, and $\dfrac{dF}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dx}$.

ii. If $x = x(t)$, $y = y(t)$, then $f = F(t)$, and $\dfrac{dF}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$.

iii. If $x = x(u, v)$, $y = y(u, v)$, then $f = F(u, v)$, and

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}, \quad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$

(e) Stationary points of $f(x, y)$ occur where $f_x = 0$, $f_y = 0$ simultaneously.
Let (a, b) be a stationary point: examine $D = [f_{xx}f_{yy} - (f_{xy})^2]_{a.b}$.
If $D > 0$ and $f_{xx}(a, b) < 0$, then (a, b) is a maximum;
If $D > 0$ and $f_{xx}(a, b) > 0$, then (a, b) is a minimum;
If $D < 0$ then (a, b) is a saddle-point.

(f) Differential equations:

i. The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating factor $I(x) = \exp[\int P(x)(dx]$, so that $\dfrac{d}{dx}(Iy) = IQ$.

ii. $P(x, y)dx + Q(x, y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y$.

5. INTEGRAL CALCULUS

(a) An important substitution: $\tan(\theta/2) = t$:
$\sin\theta = 2t/(1+t^2)$, $\cos\theta = (1-t^2)/(1+t^2)$, $d\theta = 2\,dt/(1+t^2)$.

(b) Some indefinite integrals:

$$\int (a^2 - x^2)^{-1/2} dx = \sin^{-1}\left(\frac{x}{a}\right), \quad |x| < a.$$

$$\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1}\left(\frac{x}{a}\right) = \ln\left\{\frac{x}{a} + \left(1 + \frac{x^2}{a^2}\right)^{1/2}\right\}.$$

$$\int (x^2 - a^2)^{-1/2} dx = \cosh^{-1}\left(\frac{x}{a}\right) = \ln\left|\frac{x}{a} + \left(\frac{x^2}{a^2} - 1\right)^{1/2}\right|.$$

$$\int (a^2 + x^2)^{-1} dx = \left(\frac{1}{a}\right)\tan^{-1}\left(\frac{x}{a}\right).$$

6. NUMERICAL METHODS

(a) Approximate solution of an algebraic equation:

If a root of $f(x) = 0$ occurs near $x = a$, take $x_0 = a$ and
$x_{n+1} = x_n - [f(x_n)/f'(x_n)]$, $n = 0, 1, 2 \ldots$

(Newton Raphson method).

(b) Formulae for numerical integration: Write $x_n = x_0 + nh$, $y_n = y(x_n)$.

i. Trapezium rule (1-strip): $\int_{x_0}^{x_1} y(x)dx \approx (h/2)[y_0 + y_1]$.

ii. Simpson's rule (2-strip): $\int_{x_0}^{x_2} y(x)dx \approx (h/3)[y_0 + 4y_1 + y_2]$.

(c) Richardson's extrapolation method: Let $I = \int_a^b f(x)dx$ and let I_1, I_2 be two estimates of I obtained by using Simpson's rule with intervals h and $h/2$.

Then, provided h is small enough,

$$I_2 + (I_2 - I_1)/15,$$

is a better estimate of I.

7. LAPLACE TRANSFORMS

Function	Transform	Function	Transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)dt$	$af(t) + bg(t)$	$aF(s) + bG(s)$
df/dt	$sF(s) - f(0)$	d^2f/dt^2	$s^2F(s) - sf(0) - f'(0)$
$e^{at}f(t)$	$F(s-a)$	$tf(t)$	$-dF(s)/ds$
$(\partial/\partial\alpha)f(t,\alpha)$	$(\partial/\partial\alpha)F(s,\alpha)$	$\int_0^t f(t)dt$	$F(s)/s$
$\int_0^t f(u)g(t-u)du$	$F(s)G(s)$		
1	$1/s$	$t^n (n = 1, 2 \ldots)$	$n!/s^{n+1}$, $(s > 0)$
e^{at}	$1/(s-a)$, $(s > a)$	$\sin\omega t$	$\omega/(s^2+\omega^2)$, $(s > 0)$
$\cos\omega t$	$s/(s^2+\omega^2)$, $(s > 0)$	$H(t-T) = \begin{cases} 0, & t < T \\ 1, & t > T \end{cases}$	e^{-sT}/s, $(s, T > 0)$

8. FOURIER SERIES

If $f(x)$ is periodic of period $2L$, then $f(x+2L) = f(x)$, and

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \text{ where}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad n = 0, 1, 2, \ldots, \text{ and}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1, 2, 3, \ldots$$

Parseval's theorem

$$\frac{1}{L}\int_{-L}^{L} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right).$$

Eigenvalues: $\begin{vmatrix} 1-\lambda & 0 & \sqrt{3}/2 \\ 0 & 2-\lambda & 0 \\ \sqrt{3}/2 & 0 & 2-\lambda \end{vmatrix} = 0 \Rightarrow \underbrace{(\lambda-2)}_{\lambda=2}\underbrace{\left[(\lambda-1)(\lambda-2)-\frac{3}{4}\right]}_{\lambda^2-3\lambda+5/4=(\lambda-1/2)(\lambda-5/2)} = 0$ 6

Hence $\lambda_1 = 5/2,\ \lambda_2 = 2,\ \lambda_3 = 1/2\quad (\lambda_1 > \lambda_2 > \lambda_3)$.

Eigenvectors:

① $\begin{bmatrix} -3/2 & 0 & \sqrt{3}/2 \\ 0 & -1/2 & 0 \\ \sqrt{3}/2 & 0 & -1/2 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = 0 \Rightarrow \begin{matrix} a_2 = 0, \\ a_3 = \sqrt{3}a_1 \end{matrix} \Rightarrow \begin{bmatrix} 1 \\ 0 \\ \sqrt{3} \end{bmatrix} \xmapsto[\text{normalised}]{} \underset{\sim}{e}_1 = \begin{bmatrix} 1/2 \\ 0 \\ \sqrt{3}/2 \end{bmatrix}$ 2

② $\underset{\sim}{e}_2^T = (0, 1, 0)$ 2

③ $\begin{bmatrix} 1/2 & 0 & \sqrt{3}/2 \\ 0 & 3/2 & 0 \\ \sqrt{3}/2 & & 3/2 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = 0 \Rightarrow \begin{matrix} a_2 = 0, \\ a_1 = -\sqrt{3}a_3 \end{matrix} \Rightarrow \begin{bmatrix} \sqrt{3} \\ 0 \\ -1 \end{bmatrix} \xrightarrow[\text{normalised}]{} \underset{\sim}{e}_3 = \begin{bmatrix} \mp\sqrt{3}/2 \\ 0 \\ -1/2 \end{bmatrix}$ 2

Diag. matrix: $P = (\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3) = \begin{bmatrix} 1/2 & 0 & \sqrt{3}/2 \\ 0 & 1 & 0 \\ \sqrt{3}/2 & 0 & -1/2 \end{bmatrix}$ 1

Powers: $P^{-1}AP = \Lambda \Rightarrow A = P\Lambda P^{-1}$

$A^n = \underbrace{P\Lambda P^{-1}P\Lambda P^{-1} \times \ldots \times P\Lambda P^{-1}}_{n \text{ times}} = P\Lambda^n P^{-1}$. 2

$B = \lim_{n\to\infty} P\,\frac{\Lambda^n}{\lambda_1^n}\,P^{-1} = P\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}P^{-1}$ as $\lim_{n\to\infty}(\lambda_{2,3}/\lambda_1)^n = 0$. 2

Substitute P, then

$B = \begin{bmatrix} 1/2 & 0 & \sqrt{3}/2 \\ 0 & 1 & 0 \\ \sqrt{3}/2 & 0 & -1/2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1/2 & 0 & \sqrt{3}/2 \\ 0 & 1 & 0 \\ \sqrt{3}/2 & 0 & -1/2 \end{bmatrix} = \begin{bmatrix} 1/4 & 0 & \sqrt{3}/4 \\ 0 & 0 & 0 \\ \sqrt{3}/4 & 0 & 3/4 \end{bmatrix}$. 3

Total $\boxed{20}$

Setter : Gogolin — Setter's signature : A. Gogolin

Checker : J.D. GIBBON — Checker's signature : J.D. Gibbon

Eigenvalues: $\begin{vmatrix} 4-\lambda & -1 & 1 \\ -1 & 4-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix} = -(\lambda-4)\left[(\lambda-4)(\lambda-2)-1\right] + \left(-(2-\lambda)-1\right) + \left(-1-(4-\lambda)\right) = \left\{\lambda-3+\lambda-5 = 2(\lambda-4)\right\}$ 6

$= -(\lambda-4)(\lambda^2-6\lambda+5) = -(\lambda-4)(\lambda-1)(\lambda-5)$

$\Rightarrow \lambda_1 = 5,\ \lambda_2 = 4,\ \lambda_3 = 1$ ✓

Eigenvectors:

① $\begin{bmatrix} -1 & -1 & 1 \\ -1 & -1 & 1 \\ 1 & 1 & -3 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = 0 \Rightarrow \underset{\sim}{e}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ 2

② $\begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = 0 \Rightarrow \underset{\sim}{e}_2 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ 2

③ $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = 0 \Rightarrow \underset{\sim}{e}_3 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ 2

Quadratic form: realise that P must diagonalise A:

$Q = \underset{\sim}{x}^T A \underset{\sim}{x} = \underset{\sim}{y}^T P^{-1} A P \underset{\sim}{y} = \underset{\sim}{y}^T \Lambda \underset{\sim}{y}$ – diagonal, so that 4

$P = (\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3) = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \end{bmatrix}$.

The result is 4

$Q = \underset{\sim}{y}^T \Lambda \underset{\sim}{y} = 5y_1^2 + 4y_2^2 + y_3^2 \geq 0$ because all the eigenvalues are positive.

Total 20

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Checker : J.D. GIBBON   Checker's signature : J.D. Gibbon

3.

(a) $P(A \cap B) = 0.3 \times 0.2 = 0.06$ 3

$P(A \cup C) = 0.3 + 0.2 = 0.5$ 3

$P(B \mid C) = P(B \cap C)/P(C) = 0.05/0.2 = 0.25$ 3

$P(exactly\ one) = P(A) + P(B) + P(C) - 2P(A \cap B) - 2P(B \cap C)$

$= 0.3 + 0.2 + 0.2 - 0.12 - 0.10 = 0.48$ 5

(b) $P(compt\ def) = P(def \mid A)P(A) + P(def \mid B)P(B) = 0.8p_A + 0.2p_B$.

$P(both\ A \mid both\ def) = P(both\ def \cap both\ A)/P(both\ def)$

$= (0.8p_A)^2/(0.8p_A + 0.2p_B)^2$ 6

20

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Checker: R COLEMAN Checker's signature: R Coleman

MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

2003-2004

Please write on this side only, legibly and neatly, between the margins

PAPER 4 EE2

QUESTION 4

SOLUTION

4.

(a) $1 = \sum p(x) = k(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}) = k(\frac{25}{12})$, so $k = \frac{12}{25} = 0.48$ 2

mean $\mu = \mathrm{E}(X) = \sum xp(x) = 4k = 48/25 = 1.92$ 3

variance $\sigma^2 = \mathrm{var}(X) = \sum x^2 p(x) - \mu^2 = 10k - 1.92^2 = 4.8 - 3.6864 = 1.1136$ 3

$\mathrm{cov}(X, X^2) = \mathrm{E}(X^3) - \mathrm{E}(X)\mathrm{E}(X^2) = \sum(kx^2) - (1.92 \times 4.8)$

$= 30k - 9.216 = 5.184$ 4

(b) mean $\mu = \int yf(y)dy = \int_0^1 6y^2(1-y)dy$

$= 6[y^3/3 - y^4/4]_0^1 = \frac{1}{2} = 0.5$ 3

variance $\sigma^2 = \int y^2 f(y)dy - \mu^2 = 6[y^4/4 - y^5/5]_0^1 - \frac{1}{4}$

$= \frac{3}{10} - \frac{1}{4} = \frac{1}{20} = 0.05$ 3

$\mathrm{E}[\{Y(1-Y)\}^{-1}] = \int \{y(1-y)\}^{-1} f(y)dy = \int_0^1 6dy = 6$ 2

20

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PAPER 4 EE2

QUESTION 5

SOLUTION

5.

$$P(T < 1) = 1 - \bar{F}(1) = 1 - \exp(-0.2 - 0.1)$$

$$= 1 - e^{-0.3} = 1 - 0.7408 = 0.2592$$ 3

hazard function $h(t) = -\frac{d}{dt} \log \bar{F}(t) = 0.2 + 0.15t^{1/2}$ 3

time at which doubled: solve $0.2 + 0.15t^{1/2} = 2 \times 0.2$

$$\Rightarrow t = (\tfrac{0.4-0.2}{0.15})^2 = 1.778$$ 3

Series system

survivor function: $P(\textit{system failure time} > t) = P(T_1 > t \cap T_2 > t)$

$$= \bar{F}(t)^2 = \exp(-0.4t - 0.2t^{3/2})$$ 4

hazard function $0.4 + 0.3t^{1/2}$ 3

m units

survivor function $\bar{F}(t)^m = \exp(-0.2mt - 0.1mt^{3/2})$ 2

hazard function $0.2m + 0.15mt^{1/2}$ 2

20

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PAPER 4 EE2

QUESTION 6

SOLUTION

6.

density function $f(y) = \frac{d}{dy}F(y) = \frac{1}{3}(e^{-y} + 2\theta^{-1}e^{-y/\theta})$ 2

$E(Y) = \int yf(y)dy = \frac{1}{3}\int_0^\infty ye^{-y}dy + \frac{2}{3}\theta^{-1}\int_0^\infty ye^{-y/\theta}dy = \frac{1}{3} + \frac{2\theta}{3}$ 3

$\mathrm{var}(Y) = E(Y^2) - E(Y)^2$

$= \frac{1}{3}\int_0^\infty y^2e^{-y}dy + \frac{2}{3}\theta^{-1}\int_0^\infty y^2e^{-y/\theta}dy - (\frac{1}{3} + \frac{2\theta}{3})^2$

$= (\frac{2}{3} + \frac{4\theta^2}{3}) - (\frac{1}{3} + \frac{2\theta}{3})^2 = \frac{1}{9}(5 - 4\theta + 8\theta^2) = \frac{1}{2} + \frac{8}{9}(\theta - \frac{1}{4})^2$ 4

sample sum $E(S) = nE(Y) = \frac{n}{3}(1 + 2\theta)$

$\Rightarrow E\{\frac{1}{2}(\frac{3S}{n} - 1)\} = \theta$ (unbiased) 3

$\mathrm{var}\{\frac{1}{2}(\frac{3S}{n} - 1)\} = \frac{1}{4}\mathrm{var}(\frac{3S}{n}) = \frac{9}{4n}\mathrm{var}(Y) = n^{-1}\{\frac{9}{8} + 2(\theta - \frac{1}{4})^2\}$ 3

consistent: yes, because bias=0 and variance $\rightarrow 0$ 2

p-value: $\theta > 1 \Rightarrow$ larger values of Y are more probable,

so appropriate probability is

$P_0(Y > y) = 1 - F_0(y) = \frac{1}{3}(e^{-y} + 2e^{-y/1}) = e^{-y}$ 3

(20)

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Setter's signature: MJ Crowder

Checker: R COLEMAN

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