Imperial College London

MATH97056 MATH97167

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2020

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Probability Theory

Date: 2nd June 2020

Time: 09.00am - 11.30am (BST)

Time Allowed: 2 Hours 30 Minutes

Upload Time Allowed: 30 Minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS AS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD INCLUDING A COMPLETED COVERSHEET WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

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1. (1.a) **5 pts** *seen*

Give the definition of a random variable on a probability space.

(1.b) Explain giving reasons which of the following is a random variable and which is not. **2 pts** *unseen*

(i) Let (Ω, Σ, μ) be a probability space defined with $\Omega \equiv \mathbb{N}$, $\Sigma \equiv$ *smallest σ-algebra containing subsets consisting of even numbers*, and $\mu(\{2n\}) \equiv \frac{1}{2^n}, n \in \mathbb{N}$. Let

$$X \equiv -\frac{1}{2}\chi_{\{4,6\}} + \frac{1}{2}\chi_{\{1,2,3\}}$$

where χ_A denotes the characteristic function of a set A.

(ii) Let $(\mathbb{R}, \Sigma_L, \gamma)$ be a probability space, where $\Sigma_L \equiv$ *Lebesgue measurable sets in* $\mathbb{R}$, and γ is the Gaussian measure with variance one and mean zero. Let **3 pts** *unseen*

$$X_n(x) \equiv x^n.$$

(iii) Let $([0,1], \Sigma_L \cap [0,1], \lambda)$, where $\Sigma_L \equiv$ is as in (ii), and λ is the *Lebesgue measure*. Let $A \subset [0,1]$ and $A \notin \Sigma_L \cap [0,1]$. Let **4 pts** *unseen*

$$X(x) \equiv \chi_{A\cap\mathbb{Q}}.$$

(1.c) Prove or disprove that there are uncountably many distribution functions on the real line which have jumps at the rational points. **6 pts** *unseen*

2. (2.a) **4 pts** *seen*

Give the definition of mutually independent σ-algebras, explaining carefully all notions involved.

(2.b) Let $[0, 2\pi]^{\times\mathbb{N}} \ni \omega \equiv (\omega_j)_{j\in\mathbb{N}} \mapsto \pi_j(\omega) := \omega_j$. Are the random variables **6 pts** *useen*

$$\{Y_j \equiv \cos(j\pi_j)\cos((j+1)\pi_{j+1})\}_{j\in\mathbb{N}}$$

on a probability space $([0,2\pi], \Sigma_L \cap [0,2\pi], \frac{1}{2\pi}\lambda)^{\otimes\mathbb{N}}$ mutually independent.

(2.c) Prove the Weak Law of Large Numbers for the family $Z_k \equiv X_k X_{k+1}$, $k \in \mathbb{N}$, where (X_j) are mutually independent Gaussian random variables with mean zero and variance $Var(X_j) = \log(1+j)$, $j \in \mathbb{N}$, on the probability space $([0,1], \Sigma_L \cap [0,1], \lambda)$. **10 pts** *unseen*

3. (3.a) $\boxed{\textbf{5 pts}}$ $\boxed{\textit{seen}}$

Let A_n , $n \in \mathbb{N}$, be a sequence of events in a probability space. Define the corresponding $\limsup_n A_n$ event. Prove that $\omega \in \limsup_n A_n$ if and only if ω belongs to infinitely many sets A_n.

$\boxed{\textbf{8 pts}}$ $\boxed{\textit{unseen}}$

(3.b) Let $\Omega \equiv \{0,1\}^{\mathbb{N}}$ and, for a finite sequence $\alpha_k \in \{0,1\}$, $k = 1,..,6$, let

$$A_n \equiv \{\omega : \omega_{12n+k} = \alpha_k, k = 1,..,6\}.$$

Suppose on Ω we are given a product probability measure

$$\mu \equiv \otimes_{j\in\mathbb{N}}\nu_j$$

where $\nu_j(\{0\}) = j^{-\frac{1}{3}}$. Explain giving reasons whether or not the pattern $\alpha_k = 0$, for all $k = 1,..,6$, will appear infinitely many times with probability one ? State the necessary theorems without giving their proofs.

(3.c) $\boxed{\textbf{7 pts}}$ $\boxed{\textit{unseen}}$

Let r_j, $j \in N$, be the Rademacher random variables on unit interval with the Lebesgue measure. Prove the Strong Law of Large Numbers for the random variables $R_j \equiv r_j r_{j+1} r_{j+2}$, $j \in \mathbb{N}$.

$\boxed{\textbf{5 pts}}$ $\boxed{\textit{seen}}$

4. (4.a)

(i) Give the definition of the characteristic function of a real valued random variable.

(ii) Prove that the product of n characteristic functions satisfies all properties of a characteristic function.

$\boxed{\textbf{6 pts}}$ $\boxed{\textit{unseen}}$

(4.b) Let random variables r_j , $j \in \mathbb{N}$, be as in Problem (3.c).
Find a characteristic function of the following random variables

$$Y_j \equiv r_j r_{j+1}, \qquad j \in \mathbb{N}$$

on the probability space $([0,1], \Sigma_L \cap [0,1], \lambda)$.

$\boxed{\textbf{9 pts}}$ $\boxed{\textit{unseen}}$

(4.c) Prove the Central Limit Theorem for the random variables $\{Y_{2j}\}_{j\in\mathbb{N}}$ on a probability space $([0,1], \Sigma_L \cap [0,1], \lambda)$.

5. (5.a) State the Birkhoff Ergodic Theorem explaining carefully all notions involved. **5 pts** *seen*

7 pts *unseen*

(5.b) Let r_k, $k \in \mathbb{N}$, be the Rademacher functions on $([0,1], \Sigma_L \cap [0,1], \lambda)$. Using the Birkhoff Ergodic Theorem prove the Strong Law of Large Numbers for r_k, $k \in \mathbb{N}$.

8 pts *unseen*

(5.c) Let $(\Omega, \Sigma, \mu) \equiv ([0, 2\pi], \Sigma_L \cap [0, 2\pi], \frac{1}{2\pi}\lambda)^{\mathbb{N}}$, where λ denotes the Lebesgue measure on $\Sigma_L \cap [0, 2\pi]$.

Let

$$f_k(\omega) = \sum_{n \in \mathbb{N}} \frac{1}{n^2} \prod_{j=1}^{n} \omega_{j+k}^{\frac{1}{n}} \cos(n \sum_{j=1}^{n} \omega_{j+k}).$$

Using the Birkhoff Ergodic Theorem prove that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} f_k = 0$$

in $\mathbb{L}_1$ and almost everywhere.

Module: MATH
Setter:
Checker:
Editor:
External: external
Date: March 10, 2020
Version: Draft version for checking

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2020

MATH Probability Theory

The following information must be completed:

Is the paper suitable for resitting students from previous years: Yes

Category A marks: available for basic, routine material (excluding any mastery question) (40 percent = 32/80 for 4 questions):
eg 1(a) 6 marks; 1(b.i) 7 marks; 2(a) 5 marks; 3(a) 9 marks; 4(a) 6 marks.

For example : 1a 5 pts, 1b.i& ii 5 pts , 2a 4 pts, 3a 5 pts, 3c 7 pts, 4a 5 pts

Category B marks: Further 25 percent of marks (20/ 80 for 4 questions) for demonstration of a sound knowledge of a good part of the material and the solution of straightforward problems and examples with reasonable accuracy (excluding mastery question):
eg 1(b.ii) 8 marks; 3(b.i) 5 marks; 3(b.ii) 6 marks; 2(c) 1 mark computation of convolution of kernels.

For example : 1b.iii 4 pts, 2b 6 pts, 3a pts, 4b 6 pts, 4c (statement of Thm) 4 pts

Category C marks: the next 15 percent of the marks (= 12/80 for 4 questions) for parts of questions at the high 2:1 or 1st class level (excluding mastery question):
eg 2(b) 6 marks; 4(c.i) 6 marks.

For example : 1c 6 pts, 4c (proof of Thm) 5pts

Category D marks: Most challenging 20 percent (16/80 marks for 4 questions) of the paper (excluding mastery question):
eg 2(c) 8 marks; 4(c.ii) 8 marks.

For example : 2c 10 pts, 3b 8pts

Signatures are required for the final version:

Setter's signature	Checker's signature	Editor's signature
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BSc, MSc and MSci EXAMINATIONS (MATHEMATICS)

May – June 2020

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Probability Theory

Date: ??
Time: ??
Time Allowed: 2 Hours for MATH96 paper; 2.5 Hours for MATH97 papers
This paper has *4 Questions (MATH96 version); 5 Questions (MATH97 versions)*.
Candidates should start their solutions to each question in a new main answer book.
Supplementary books may only be used after the relevant main book(s) are full.
Statistical tables will not be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted.
- Each question carries equal weight.
- Calculators may not be used.

Solutions

1. Solution

5 pts *seen*

(1.a)

Let (Ω, Σ, μ) be a probability space. A function $X : \Omega \to \mathbb{R}$ is called a random variable iff for any Borel set $A \subset \mathbb{R}$ we have

$$X^{-1}(A) \equiv \{\omega \in \Omega : X(\omega) \in A\} \in \Sigma.$$

2 pts *unseen*

(1.b) (i) Let $\Omega \equiv \mathbb{N}$, $\Sigma \equiv$ *smallest σ-algebra containing subsets consisting of even numbers*, and $\mu(\{2n\}) \equiv \frac{1}{2^n}, n \in \mathbb{N}$. If

$$X \equiv -\frac{1}{2}\chi_{\{4,6\}} + \frac{1}{2}\chi_{\{1,2,3\}}$$

where χ_A denotes the characteristic function of a set A, then we have

$$X^{-1}(\left\{\frac{1}{2}\right\}) = \{1, 2, 3, 4, 6\}$$

which does not belong to Σ. Hence X is not a random variable.

3 pts *unseen*

(ii) Let $(\mathbb{R}, \Sigma_L, \gamma)$, where $\Sigma_L \equiv$ *Lebesgue measurable sets in* $\mathbb{R}$, and γ is the Gaussian measure with variance one and mean zero. Let

$$X_n(x) \equiv x^n.$$

Since preimage by X_n of an interval is an interval and intervals generate the Borel σ-algebra which is contained in Σ_L,X_n is a random variable.

4 pts *unseen*

(iii) Let $([0, 1], \Sigma_L \cap [0, 1], \lambda)$, where $\Sigma_L \equiv$ *Lebesgue measurable sets in* $\mathbb{R}$, and λ is the *Lebesgue measure*. Let $A \subset [0, 1]$ and $A \notin \Sigma_L \cap [0, 1]$ Let

$$X(x) \equiv \chi_{A\cap\mathbb{Q}}.$$

We note that rational numbers have Lebesgue measure zero and hence $A \cap \mathbb{Q}$ is of measure zero too. Since sets of measure zero belong to Σ_L by definition of the σ-algebra of Lebesgue measurable sets, this implies that X is a random variable.

6 pts *unseen*

(1.c) Since rationals form a countable set $\mathbb{Q} \equiv (q_n)_{n\in\mathbb{N}}$ we can assign a probability measure $\nu(\cdot)$ on the σ-algebra of Borel or Lebesgue measurable sets by setting

$$\nu(\{q_n\}) = \alpha_n, \qquad with \qquad \alpha_n \geq 0, \qquad \sum_n \alpha_n = 1$$

and consequently zero for any set not containing rational points. It is clear that there is uncountable many choices of such measures. Then the distribution function of the function id on $\mathbb{R}$ has jumps on the set of rational points.

2. Solution

$\boxed{\textbf{4 pts}}$ $\boxed{seen}$

(2.a)

Let (Ω, Σ, μ) be a probability space and let Σ_i, $i = 1, .., n$, $n \in \mathbb{N}$, be a collection of sub σ-algebras in Σ. We say that the σ-algebras in this collection are mutually independent iff

$$\forall A_i \in \Sigma_i, \quad i = 1, .., n, \qquad \mu\left(\cap_{i=1,..,n} A_i\right) = \prod_{i=1,..,n} \mu\left(A_i\right).$$

$\boxed{\textbf{6 pts}}$ $\boxed{useen}$

(2.b) Let $[0, 2\pi]^{\times \mathbb{N}} \ni \omega \equiv (\omega_j)_{j\in\mathbb{N}} \mapsto \pi_j(\omega) := \omega_j$. By definition of the product measure the σ-algebra $\pi_j^{-1}(\mathcal{B}_\mathbb{R})$ are mutually independent. Hence all random variables Y_j and Y_k for $|j-k| \geq 2$ are mutually independent. Next we note that

$$E\left(Y_j^2\right) = E\left(\cos^2(j\pi_j)\cos^2((j+1)\pi_{j+1})\right) = E\left(\frac{1}{2}(\cos(2j\pi_j)+1)\frac{1}{2}(\cos(2(j+1)\pi_{j+1})+1)\right) = \frac{1}{4}$$

and so

$$E\left(Y_{j-1}^2\right)E\left(Y_j^2\right) = \frac{1}{16}$$

On the other hand

$$E\left(Y_{j-1}^2 Y_j^2\right) = E\left(\cos^2((j-1)\pi_{j-1})\cos^4(j\pi_j)\cos^2((j+1)\pi_{j+1})\right)$$

$$= E\left(\frac{1}{2}(\cos(2(j-1)\pi_{j-1})+1)\left(\frac{1}{2}(\cos(2j\pi_j)+1)\right)^2 \frac{1}{2}(\cos(2(j+1)\pi_{j+1})+1)\right)$$

$$= E\left(\frac{1}{2}(\cos(2(j-1)\pi_{j-1})+1)\right)E\left(\frac{1}{2}(\cos(2j\pi_j)+1)\right)^2 E\left(\frac{1}{2}(\cos(2(j+1)\pi_{j+1})+1)\right) = \frac{3}{16}$$

Thus

$$E\left(Y_{j-1}^2 Y_j^2\right) \neq E\left(Y_{j-1}^2\right)E\left(Y_j^2\right)$$

and hence the random variables Y_{j-1} and Y_j as well as Y_{j+1} and Y_j are not mutually independent;

$\boxed{\textbf{10 pts}}$ $\boxed{unseen}$

(2.c) Using mutual independence of the Gaussian random variables X_j and $EX_j = 0$, we have

$$E\left(\frac{1}{n}\sum_{j=1}^n Y_j\right)^2 = \frac{1}{n^2}\sum_{j=1}^n E\left(Y_j^2\right).$$

We also note that

$$E\left(Y_j^2\right) = EX_j^2 X_{j+1}^2 = \log(1+j)\log(2+j).$$

Hence

$$E\left(\frac{1}{n}\sum_{j=1}^n Y_j\right)^2 = \frac{1}{n^2}\sum_{j=1}^n \log(1+j)\log(2+j) \leq \frac{\log(1+n)\log(2+n)}{n}$$

which goes to zero as $n \to \infty$. This implies that

$$\frac{1}{n}\sum_{j=1}^n Y_j \to_{n\to\infty} 0$$

in probability.

3. (3.a) $\boxed{\textbf{5 pts}}$ $\boxed{\textit{seen}}$

For be a sequence of events A_n , $n \in \mathbb{N}$, in some probability space (Ω, Σ, μ), the corresponding $\limsup_n A_n$ event is defined as follows

$$\limsup_n A_n := \bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} A_k$$

By definition of the intersection of sets in the set theory, we have

$$\bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} A_k = \{\omega \in \Omega : \forall n \in \mathbb{N} \quad \omega \in \bigcup_{k\geq n} A_k\}$$

so, using the definition of the union of the sets, we get the following equivalence

$$\omega \in \limsup_n A_n \Longleftrightarrow \forall n \in \mathbb{N} \quad \exists k_n \geq n \qquad \omega \in A_{k_n}$$

i.e. exists an infinite sequence A_{k_n}, $k_n \in \mathbb{N}$, such that for $\omega \in \limsup_n A_n$, we have $\omega \in A_{k_n}$ for infinitely many sets A_{k_n}.

(3.b) Let $\Omega \equiv \{0,1\}^{\mathbb{N}}$ and, for a finite sequence $\alpha_k \in \{0,1\}$, $k = 1,..,6$, let $\boxed{\textbf{8 pts}}$ $\boxed{\textit{unseen}}$

$$A_n \equiv \{\omega : \omega_{12n+k} = \alpha_k, k = 1,..,6\}.$$

Suppose we are given a product probability measure

$$\mu \equiv \otimes_{j\in\mathbb{N}}\nu_j$$

where $\nu_j(\{0\}) = j^{-\frac{1}{3}}$. Then we have

$$\mu(A_n) = \mu\left(\bigcap_{k=1,..,6} \{\omega_{12n+k} = 0\}\right) = \prod_{k=1,..,6} \nu_{12n+k}(\{0\}) = \prod_{k=1,..,6} (12n+k)^{-\frac{1}{3}} \leq (12n)^{-2}$$

Since $\sum_n (12n)^{-2} < \infty$, we get

$$\sum_n \mu(A_n) < \infty.$$

Next we use the following theorem.
Borel-Cantelli Lemma: For a sequence of events A_n, $n \in \mathbb{N}$, in a probability space (Ω, Σ, μ), if

$$\sum_n \mu(A_n) < \infty,$$

then

$$\mu(\limsup_n A_n) = 0.$$

This implies that in our case the pattern of six zeros appears infinitely many times with probability zero.

(3.c) **7 pts** *unseen*

Let r_j, $j \in N$, be the Rademacher random variables on unit interval with the Lebesgue measure and let $R_j \equiv r_j r_{j+1} r_{j+2}$, $j \in \mathbb{N}$. Define

$$s_n \equiv \frac{1}{n} \sum_{j=1,..,n} R_j.$$

We have

$$E\left(s_n^4\right) = \frac{1}{n^4} \sum_{i,j,k,l=1,..,n} E(R_i R_j R_k R_l)$$

Since the Rademacher random variables are mutually independent, have mean zero and their square equals to one, we have

$$\begin{aligned} n^4 \cdot E\left(s_n^4\right) = & \sum_{i=j=k=l=1,..,n} E(R_i R_j R_k R_l) + \sum_{i=j,k=l \in \{1,..,n\}} E(R_i R_j R_k R_l) \\ & + \sum_{i=k,j=l \in \{1,..,n\}} E(R_i R_j R_k R_l) + \sum_{i=l,j=k \in \{1,..,n\}} E(R_i R_j R_k R_l) \\ & = n + 3n^2. \end{aligned}$$

This implies that

$$E\left(s_n^4\right) \leq \frac{1}{n^4}(n + 3n^2) \leq 4\frac{1}{n^2}$$

and so we have

$$\sum_n E\left(s_n^4\right) < \infty.$$

From this, using the monotone convergence theorem and necessary condition for convergence of series, we conclude that

$$s_n \to_{n \to \infty} 0$$

with probability one.

4. (4.a) $\boxed{\textbf{5 pts}}$ $\boxed{\textit{seen}}$

(i) Let X be a real valued random variable on a probability space (Ω, Σ, μ). A characteristic function of X is defined by

$$\varphi_X(t) := \int e^{itX} d\mu.$$

(ii) Let $\varphi_i(t)$, $i = 1, .., n$, be characteristic functions of some random variables X_i on some probability spaces $(\Omega_i, \Sigma_i, \mu_i)$. Let

$$\varphi(t) \equiv \prod_{j=1,..,n} \varphi_j(t).$$

Since all $\varphi_i(t)$ are uniformly continuous and have value one at $t = 0$, so does the function $\varphi(t)$. The uniform continuity follows from the following bound

$$\begin{aligned} |\varphi(t+h) - \varphi(t)| &\leq |\varphi_1(t+h) - \varphi_1(t)| \prod_{k>1} |\varphi_k(t)| \\ + \sum_{j=1,..,n-1} \prod_{k<j} |\varphi_k(t+h)| &\prod_{k>j} |\varphi_k(t)| \cdot |\varphi_j(t+h) - \varphi_j(t)| + \\ &\prod_{k<n} |\varphi_k(t+h)| \cdot |\varphi_n(t+h) - \varphi_n(t)| \\ &\leq \sum_{j=1,..,n} |\varphi_j(t+h) - \varphi_j(t)| \end{aligned}$$

where in the last step one uses the fact that any characteristic function satisfies $|\varphi_k(t)| \leq 1$. To show positive definitness, it is sufficient to show that for $n = 2$ and use the mathematical induction for the general case. We note that any positive definite matrix C can be represented as follows

$$C = A^*A$$

with some matrix A and with A^* denoting the adjoint of A. Hence if

$$\varphi_1(t_i - t_j) = (A^*A)_{ij} \quad and \quad \varphi_2(t_i - t_j) = (B^*B)_{ij},$$

we have

$$\begin{aligned} \sum_{i,j} \varphi_1(t_i - t_j)\varphi_2(t_i - t_j)\bar{z}_i z_j &= \sum_{i,j} (A^*A)_{ij}(B^*B)_{ij}\bar{z}_i z_j \\ = \sum_{k,l} \sum_{i,j} A^*_{ik} A_{kj} B^*_{il} B_{lj} \bar{z}_i z_j &= \sum_{k,l} \left| \sum_j A_{kj} B_{lj} z_j \right|^2 \geq 2 \end{aligned}$$

That is the product of characteristic functions satisfy the positive definitness condition.

(4.b) Let **6 pts** *unseen*

$$Y_j \equiv r_j r_{j+1}, \qquad j \in \mathbb{N}$$

on the probability space $([0,1], \Sigma_L \cap [0,1], \lambda)$. Since r_j are mutually independent Bernoulli random variable taking on values ± 1 with equal probability, we have

$$\begin{aligned} \int e^{itY_j} d\lambda = e^{it}\lambda(\{Y_j = 1\}) + e^{-it}\lambda(\{Y_j = -1\}) \\ = e^{it}\lambda(\{r_j = 1, r_{j+1} = 1\} \cup \{r_j = -1, r_{j+1} = -1\}) \\ +e^{-it}\lambda(\{\{r_j = 1, r_{j+1} = -1\} \cup \{r_j = -1, r_{j+1} = 1\}\}) \\ = \cos(t) \end{aligned}$$

where we have used

$$\begin{aligned} &\lambda(\{r_j = 1, r_{j+1} = 1\} \cup \{r_j = -1, r_{j+1} = -1\}) \\ &\qquad = \lambda(\{r_j = 1, r_{j+1} = 1\}) + \lambda(\{r_j = -1, r_{j+1} = -1\}) \\ &\qquad = \lambda(\{r_j = 1\})\lambda(\{r_{j+1} = 1\}) + \lambda(\{r_j = -1\})\lambda(\{r_{j+1} = -1\}) \\ &\qquad = \frac{1}{2} \end{aligned}$$

and similarly for the probability of $\{Y_j = -1\}$.

(4.c) Central Limit Theorem : If $\{Y_{2j} \equiv r_{2j} r_{2j+1}\}_{j\in\mathbb{N}}$ are the random variables on a probability space $([0,1], \Sigma_L \cap [0,1], \lambda)$, then **9 pts** *unseen*

$$\frac{1}{\sqrt{n}} \sum_{j=1,..,n} Y_{2j}$$

converges in distribution to the Gaussian random variable with mean zero and covariance one.

Proof : By definition and mutual independence of the Rademacher functions r_k, $k \in \mathbb{N}$, the random variables $Y_{2j} \equiv r_{2j} r_{2j+1}$, $j \in \mathbb{N}$, are i.i.d. with mean zero and variance one. Using (4.b) we have

$$\int \exp\left(it \frac{1}{\sqrt{n}} \sum_{j=1,..,n} Y_{2j} \right) d\lambda = \left(\cos(\frac{t}{\sqrt{n}}) \right)^n = \left(1 - \frac{t^2}{2n} + \mathcal{O}(\frac{1}{n^{3/2}}) \right)^n \to_{n\to\infty} = e^{-\frac{1}{2}t^2}.$$

and hence by Levy's continuity theorem $\frac{1}{\sqrt{n}} \sum_{j=1,..,n} Y_{2j}$ converges in distribution to the Gaussian random variable with mean zero and covariance one.

5. (5.a) **5 pts** *seen*

Let (Ω, Σ, μ) be a probability space. A map $T : \Omega \to \Omega$ is called measure preserving if for any $A \in \Sigma$

$$\mu(T^{-1}A) = \mu(A).$$

A set $A \in \Sigma$ is called T-invariant iff the symmetric difference $A \setminus T^{-1}A \cup T^{-1}A \setminus A$ has probability zero. Let $\mathcal{I}$ denote σ-algebra of T-invariant sets and let $E_\mu(\cdot|\mathcal{I})$ be the corresponding conditional expectation associated to the probability measure μ.

Birkhoff's Ergodic Theorem : Suppose $X \in \mathbb{L}_1(\Omega, \Sigma, \mu)$. Then

$$\frac{1}{n}\sum_{j=0}^{n} X(T^j\omega) \to_{n\to\infty} E_\mu(X|\mathcal{I})$$

in $\mathbb{L}_1(\Omega, \Sigma, \mu)$ and almost surely.

7 pts *unseen*

(5.b) Let r_k be the Rademacher functions on $([0,1], \Sigma_L \cap [0,1], \lambda)$.

If

$$T(t) = \begin{cases} 2t & for\ 0 \le t \le \frac{1}{2} \\ 2(t - \frac{1}{2}) & for\ \frac{1}{2} < t \le 1 \end{cases}$$

Let

$$\epsilon(t) = \begin{cases} 1 & for\ 0 \le t \le \frac{1}{2} \\ 0 & for\ \frac{1}{2} < t \le 1 \end{cases}$$

and define

$$\epsilon_k(t) \equiv \epsilon(T^{k-1}(t)).$$

Then the Rademacher functions are related to ϵ_k,s as follows

$$r_k(t) = 1 - 2\epsilon_k(t)$$

and so

$$r_k(t) = r_1(T^{k-1}(t)).$$

We note that the corresponding σ-algebra at infinity is given by

$$\bigcap_n \sigma(\{r_k, k \ge n\}),$$

where $\sigma(\{r_k, k \ge n\})$ denotes the smallest σ-algebra s.t. all $\{r_k, k \ge n\}$ are measurable. By the Kolmogorov 0-1 Law this σ-algebra is trivial. On the other hand we note that

$$T^{-1}\sigma(\{r_k, k \ge n+1\}) = \sigma(\{r_k, k \ge n\}),$$

which implies that the σ-algebra at infinity contains the T invariant sets. Hence, using the triviality of σ-algebra at infinity, the Birkhoff's Ergodic Theorem implies the Strong Law of Large numbers for Rademacher functions.

$\boxed{\textbf{8 pts}}$
$\boxed{unseen}$

(5.c) Let $(\Omega, \Sigma, \mu) \equiv ([0, 2\pi], \Sigma_L \cap [0, 2\pi], \frac{1}{2\pi}\lambda)^{\mathbb{N}}$, where λ denotes the Lebesgue measure on $\Sigma_L \cap [0, 2\pi]$.

Let

$$f_k(\omega) = \sum_{n\in\mathbb{N}} \frac{1}{n^{n+2}} \prod_{j=1}^{n} \omega_{j+k}^{\frac{1}{n}} \cos(n \sum_{j=1}^{n} \omega_{j+k})$$

We note first that

$$\int \prod_{j=1}^{n} \omega_j^{\frac{1}{n}} d\mu = \prod_{j=1}^{n} \frac{1}{2\pi} \int_0^{2\pi} \omega_j^{\frac{1}{n}} d\omega_j = n^n 2\pi$$

and so

$$\int |f_k| d\mu = \int |f_0| d\mu \leq \sum_{n\in\mathbb{N}} \frac{1}{n^{2+\varepsilon}} \int \prod_{j=1}^{n} \omega_j^{-1+\frac{1}{n}} d\mu \leq 2\pi \sum_{n\in\mathbb{N}} \frac{1}{n^2} < \infty$$

This means that $f_k \in \mathbb{L}_1(\mu)$. A map $T : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ defined by

$$(T\omega)_j = \omega_{j+1}$$

is preserving the product measure. Since we have $\pi_{j+1} = \pi_j \circ T$, we get

$$f_k = f_1 \circ T^{k-1}$$

Using the Birkhoff ergodic theorem we conclude that

$$\frac{1}{n} \sum_{k=1}^{n} f_k = \frac{1}{n} \sum_{k=1}^{n} f_1 \circ T^{k-1} \to_{n\to\infty} 0$$

almost everywhere and in $\mathbb{L}_1$. We use here the fact that by Kolmogorov 0-1 the σ-algebra at infinity is trivial.