DTS AND COMPUTER CONTROL

1. a) Applying the forward difference property of the z-transform we obtain

$$z^2X(z) - z^2x(0) - zx(1) + \alpha(zX(z) - zx(0)) + \beta X(z) = U(z).$$

Solving this equation with respect to $X(z)$ yields

$$X(z) = \frac{z^2+\alpha z}{z^2+\alpha z+\beta}x(0) + \frac{z}{z^2+\alpha z+\beta}x(1) + \frac{1}{z^2+\alpha z+\beta}U(z).$$

[3 marks]

b) For $\alpha = -2$, $\beta = 1$, $x(1) = 0$ and the input

$$U(z) = \frac{z(z-1)}{(z-\frac{1}{2})^2},$$

the function $X(z)$ becomes

$$X(z) = \frac{z^2-2z}{z^2-2z+1}x(0) + \frac{1}{z^2-2z+1}\frac{z(z-1)}{(z-\frac{1}{2})^2}.$$

i) Note that $X(z)$ can be rewritten as

$$X(z) = \frac{z^2-2z}{(z-1)^2}x(0) + \frac{z}{(z-1)(z-\frac{1}{2})^2}.$$

Computing the partial fraction expansion of $\dfrac{X(z)}{z}$ we obtain

$$X(z) = \frac{z}{z-1}x(0) - \frac{z}{(z-1)^2}x(0) + 4\frac{z}{z-1} - 4\frac{z}{z-\frac{1}{2}} - 2\frac{z}{(z-\frac{1}{2})^2}.$$

The inverse z-transform of this last expression is

$$x(k) = (1-k)x(0) + 4 - 4(1+k)\left(\frac{1}{2}\right)^k.$$

[3 marks]

ii) Note that

$$\lim_{k\to\infty}(1+k)\left(\frac{1}{2}\right)^k \to 0,$$

Hence, $x(k)$ behaves as $(1-k)x(0)+4$. If $x(0) = 0$ the solution $x(k)$ converges asymptotically to 4. If $x(0) \neq 0$, the solution $x(k)$ diverges to ∞.

[3 marks]

c) For $\alpha = -\dfrac{3}{2}$, $\beta = \dfrac{1}{2}$, $x(0) = 0$, $x(1) = 0$ and the input

$$u(k) = \sin\left(\frac{\pi}{2}k\right),$$

the function $X(z)$ becomes

$$X(z) = \frac{1}{z^2-\frac{3}{2}z+\frac{1}{2}}\frac{z}{z^2+1} = \frac{z}{(z-1)(z-\frac{1}{2})(z^2+1)}.$$

i) Using the partial fraction expansion we obtain

$$X(z) = \frac{1}{5}\frac{3z^2 - z}{z^2+1} + \frac{z}{z-1} - \frac{8}{5}\frac{z}{z-\frac{1}{2}}.$$

The inverse z-transform of this last expression gives

$$x(k) = -\frac{1}{5}\sin\left(\frac{\pi}{2}k\right) + \frac{3}{5}\cos\left(\frac{\pi}{2}k\right) + 1 - \frac{8}{5}\left(\frac{1}{2}\right)^k.$$

Hence, the transient response is

$$x_{tr}(k) = -\frac{8}{5}\left(\frac{1}{2}\right)^k,$$

whereas the steady-state response is

$$x_{ss}(k) = 1 - \frac{1}{5}\sin\left(\frac{\pi}{2}k\right) + \frac{3}{5}\cos\left(\frac{\pi}{2}k\right).$$

[3 marks]

ii) In this case the function $X(z)$ is

$$X(z) = \frac{1}{(z-1)(z-\frac{1}{2})}\left(\frac{z}{z-1} + \frac{z}{z^2+1}\right).$$

To compute the response to the new input we just need to compute the contribution of 1 because using linearity the contribution due to $\sin\left(\frac{\pi}{2}k\right)$ remains unchanged. Note that

$$\frac{z}{(z-1)^2(z-\frac{1}{2})} = -4\frac{z}{z-1} + 2\frac{z}{(z-1)^2} + 4\frac{z}{z-\frac{1}{2}}.$$

So the new contribution is

$$-4 + 2k + 4\left(\frac{1}{2}\right)^k.$$

Adding this contribution to the previous result yields

$$x(k) = -\frac{1}{5}\sin\left(\frac{\pi}{2}k\right) + \frac{3}{5}\cos\left(\frac{\pi}{2}k\right) - 3 + \frac{12}{5}\left(\frac{1}{2}\right)^k + 2k.$$

The steady-state response is not properly defined as $x(k) - x_{tr}(k) \to \infty$. The reason of this change is that the input 1 "resonates" with the pole in -1 already in the system. The interconnected system has two poles in -1. This produces terms in k and, similarly to part b), the solution diverges to ∞.

[3 marks]

d) i) For $a = b$,

$$X(z) = \frac{z}{(z-a)^2}.$$

From the formula sheet it follows that

$$x(k) = ka^{k-1}.$$

If $a = b = 1$, then $x(k) = k$ and the solution diverges.

[2 marks]

ii) For $a \neq b$

$$X(z) = \frac{z}{(z-a)(z-b)}.$$

It follows that

$$x(k) = \frac{1}{a-b}a^k + \frac{1}{b-a}b^k.$$

When $a = 1$, this last expression becomes

$$x(k) = \frac{1}{1-b} + \frac{1}{b-1}b^k.$$

If $|b| > 1$, then $x(k)$ diverges. If $|b| < 1$, then $x(k)$ converges to $\frac{1}{1-b}$.
If $b = -1$, then

$$x(k) = \begin{cases} 0, & k \text{ even}, \\ \frac{2}{1-b}, & k \text{ odd}. \end{cases}$$

[3 marks]

2. a) The closed-loop characteristic polynomial is $s+k+2$, hence the closed-loop system is asymptotically stable for all $k>0$.

[1 marks]

b) i) The equivalent discrete-time model is

$$\begin{aligned} H_0P(z) &= \frac{z-1}{z}Z\left(\frac{P(s)}{s}\right) = k\frac{z-1}{z}Z\left(\frac{1}{2}\frac{1}{s}-\frac{1}{2}\frac{1}{s+2}\right) \\ &= k\frac{z-1}{z}\left(\frac{1}{2}\frac{1}{1-z^{-1}}-\frac{1}{2}\frac{1}{1-e^{-2T}z^{-1}}\right) \\ &= \frac{k}{2}\frac{1-e^{-2T}}{z-e^{-2T}}. \end{aligned}$$

[2 marks]

ii) The characteristic polynomial of the closed-loop system is

$$z+\frac{k}{2}(1-e^{-2T})-e^{-2T}.$$

The roots of this polynomial are all inside the unity circle if

$$k<2\frac{1+e^{-2T}}{1-e^{-2T}}=\overline{K}.$$

[2 marks]

iii) Note that $\lim_{T\to 0}\overline{K}=\infty$ and $\lim_{T\to\infty}\overline{K}=2$.

[2 marks]

c) i) The equivalent discrete-time model is

$$\begin{aligned} H_0P(z) &= \frac{(z-1)^2}{z^2}Z\left(\frac{1+Ts}{T}\frac{P(s)}{s^2}\right) \\ &= \frac{k}{T}\frac{(z-1)^2}{z^2}Z\left(\frac{1}{2}\frac{1}{s^2}-\frac{1}{4}\frac{2T-1}{s+2}+\frac{1}{4}\frac{2T-1}{s}\right) \\ &= \frac{k}{T}\frac{(z-1)^2}{z^2}\left(\frac{1}{2}\frac{Tz}{(z-1)^2}+\frac{1}{4}\frac{(2T-1)z}{z-1}-\frac{1}{4}\frac{(2T-1)z}{z-e^{-2T}}\right) \\ &= \frac{k}{4T}\frac{(-1+4T-2e^{-2T}T+e^{-2T})z+1-2T-e^{-2T}}{z(z-e^{-2T})}. \end{aligned}$$

[4 marks]

ii) The characteristic polynomial of the closed-loop system is

$$z^2+\left(\frac{k}{4T}(-1+4T-2e^{-2T}T+e^{-2T})-e^{-2T}\right)z+\frac{k}{4T}(1-2T-e^{-2T}).$$

To determine the location of the roots of this polynomial we can use the bilinear transformation $z=\frac{w+1}{w-1}$. A quicker and safer route is to recall that the polynomial

$$z^2+\alpha z+\beta$$

has all roots in the unit circle if

$$\begin{aligned} &\beta>-1+\alpha, \\ &\beta<1, \qquad\qquad (2.1) \\ &\beta>-1-\alpha. \end{aligned}$$

These conditions corresponds, respectively, to

$$
\begin{aligned}
&k < \frac{2(1+e^{-2T})}{(3-e^{-2T}-\frac{1}{T}(1-e^{-2T}))}, \\
&k > \frac{4}{(\frac{1}{T}-2-\frac{1}{T}e^{-2T})}, \\
&k > -2.
\end{aligned}
$$

Note that the last two conditions are verified for $k > 0$. Hence, the roots of the characteristic polynomial are all inside the unity circle if

$$k < \frac{2(1+e^{-2T})}{(3-e^{-2T}-\frac{1}{T}(1-e^{-2T}))} = \overline{\overline{K}}.$$

[5 marks]

iii) For $T \to 0$

$$\overline{\overline{K}} \approx \frac{2(1+1-2T)}{(3-1+2T-\frac{1}{T}(1-1+2T)} \approx \frac{2}{T}.$$

Hence, $\lim_{T\to 0} \overline{\overline{K}} = \infty$ and $\lim_{T\to\infty} \overline{\overline{K}} = \frac{2}{3}$.

[3 marks]

d) For sufficiently small values of T, the behavior of the system interconnected to a ZOH or to a FOH approaches the behavior of the continuous-time system. As the sampling time increases, the discretized systems becomes increasingly unstable for larger values of k. The maximum k achievable using the FOH is $\frac{1}{3}$ of the maximum value achievable using the ZOH.

[1 marks]

3. a) The relation between the matrices (A,B,C) and (F,G,H) are the following

$$A = e^{FT}, \qquad B = \int_0^T e^{F\lambda} G d\lambda, \qquad C = H.$$

To compute these matrices we first compute $(sI-F)^{-1}$, namely

$$(sI-F)^{-1} = \begin{bmatrix} \frac{1}{s} & \frac{1}{s(s+1)} \\ 0 & \frac{1}{s+1} \end{bmatrix}.$$

It follows that

$$A = \begin{bmatrix} 1 & 1-e^{-1} \\ 0 & e^{-1} \end{bmatrix}, \qquad B = \begin{bmatrix} e^{-1} \\ 1-e^{-1} \end{bmatrix}.$$

[4 marks]

b) To compute the transfer function we first determine the term

$$(zI-A)^{-1} = \begin{bmatrix} \frac{1}{z-1} & \frac{1-e^{-1}}{(z-1)(z-e^{-1})} \\ 0 & \frac{1}{z-e^{-1}} \end{bmatrix}.$$

The input-output transfer function is given by

$$\frac{Y(z)}{U(z)} = C(zI-A)^{-1}B = \frac{e^{-1}z+1-2e^{-1}}{(z-1)(z-e^{-1})}.$$

[3 marks]

c) Exploiting the result of part a), the computation of the transfer function is straightforward

$$\frac{Y(s)}{U(s)} = P(s) = H(sI-F)^{-1}G = \frac{1}{s(s+1)}.$$

[2 marks]

d) The equivalent discrete-time model is

$$\begin{aligned} HP(z) &= \frac{z-1}{z} Z\left(\frac{P(s)}{s}\right) = \frac{z-1}{z} Z\left(\frac{1}{s+1} - \frac{1}{s} + \frac{1}{s^2}\right) \\ &= \frac{z-1}{z}\left(\frac{1}{1-e^{-1}z^{-1}} - \frac{1}{1-z^{-1}} + \frac{Tz^{-1}}{(1-z^{-1})^2}\right) \\ &= \frac{e^{-1}z+1-2e^{-1}}{(z-1)(z-e^{-1})}. \end{aligned}$$

as in b).

[3 marks]

e) Selecting $u(k) = -Kx(k)$ we obtain the closed-loop state equation

$$x(k+1) = (A-BK)x(k).$$

To achieve a deadbeat response we place the eigenvalues of $(A-BK)$ in zero. Let $K = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$. Then,

$$\det(\lambda I - (A-BK)) = \lambda^2 - (1-K_2+e^{-1})\lambda - ((e^{-1}-1)K_1 + K_2 - e^{-1}).$$

To place the eigenvalues in zero we solve the equations

$$1 - K_2 + e^{-1} = 0,$$
$$(e^{-1} - 1)K_1 + K_2 - e^{-1} = 0,$$

which gives

$$K_1 = \frac{1}{1 - e^{-1}}, \qquad K_2 = 1 + e^{-1}.$$

[4 marks]

f) The closed-loop transfer function of the block diagram is given by

$$\frac{Y(z)}{R(z)} = \frac{C(z)HP(z)}{1 + C(z)HP(z)}.$$

Thus, the characteristic polynomial is

$$\mathrm{num}(1 + C(z) + HP(z)).$$

The problem has a unique solution if we select the control $C(z)$ as

$$C(z) = \frac{s_0 z + s_1}{z + v_1},$$

because $m = \deg(\mathrm{den}(HP(z))) - 1$. Thus,

$$\mathrm{num}(1 + C(z)HP(z)) = z^3 + (v_1 - (1 + e^{-1}) + s_0 e^{-1})z^2 +$$
$$+ (-v_1(1 + e^{-1}) + e^{-1} + e^{-1}s_1 + (1 - 2e^{-1})s_0)z + (e^{-1}v_1 + (1 - 2e^{-1})s_1).$$

To achieve a deadbeat response, we require that $\mathrm{num}(1 + C(z)HP(z)) = z^3$, which yields

$$\begin{bmatrix} e^{-1} & 0 & 1 \\ 1 - 2e^{-1} & e^{-1} & -1 - e^{-1} \\ 0 & 1 - 2e^{-1} & e^{-1} \end{bmatrix} \begin{bmatrix} s_0 \\ s_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} 1 + e^{-1} \\ -e^{-1} \\ 0 \end{bmatrix}.$$

(For the sake of completeness the solution of this linear system is $s_0 = 2.3055$, $s_1 = -0.7236$, $v_1 = 0.5197$. However, it was not required to determine these values.) [4 marks]

4. a) i) The reachability matrix is

$$P = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} d & ad \\ 0 & bd \end{bmatrix}.$$

We easily see that full rank is achieved if and only if $b \neq 0$ and $d \neq 0$. If either b or d is zero, the system is not reachable.

[2 marks]

ii) The observability matrix is

$$Q = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & e \\ be & ce \end{bmatrix}.$$

We easily see that full rank is achieved if and only if $b \neq 0$ and $e \neq 0$. If either b or e is zero, the system is not observable.

[2 marks]

b) The system is controllable if and only if

$$\text{rank}\left(\begin{bmatrix} P & A^2 \end{bmatrix}\right) = \text{rank}\,(P),$$

which yields

$$\text{rank}\left(\begin{bmatrix} d & ad & a^2 & 0 \\ 0 & bd & ab+cb & c^2 \end{bmatrix}\right) = \text{rank}\left(\begin{bmatrix} d & ad \\ 0 & bd \end{bmatrix}\right).$$

We distinguish three cases.

- If $b \neq 0$ and $d \neq 0$, the system is reachable. This implies that it is also controllable.
- If $b = 0$ and $d \neq 0$, $\text{rank}\,(P) = 1$. If $c = 0$, then $\text{rank}\left(\begin{bmatrix} P & A^2 \end{bmatrix}\right) = 1$, which implies that the system is controllable. If $c \neq 0$, then $\text{rank}\left(\begin{bmatrix} P & A^2 \end{bmatrix}\right) = 2$, which implies that the system is not controllable.
- If $b = d = 0$, $\text{rank}\,(P) = 0$. If $a = c = 0$, then $\text{rank}\left(\begin{bmatrix} P & A^2 \end{bmatrix}\right) = 0$, which implies that the system is controllable. If $a \neq 0$ or $c \neq 0$, then $\text{rank}\left(\begin{bmatrix} P & A^2 \end{bmatrix}\right)$ is either 1 or 2, which implies that the system is not controllable.

[8 marks]

c) i) Since $b = d = 1$, the system is reachable. Let $x(0) = \begin{bmatrix} x_1(0) & x_2(0) \end{bmatrix}^{\top}$. Then

$$\begin{bmatrix} x_1(1) \\ x_2(1) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(0) = \begin{bmatrix} u(0) \\ x_1(0) + x_2(0) \end{bmatrix}$$

and

$$\begin{bmatrix} x_1(2) \\ x_2(2) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(1) \\ x_2(1) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(1) = \begin{bmatrix} u(1) \\ u(0) + x_1(0) + x_2(0) \end{bmatrix}.$$

Hence, the problem is solved selecting $u(0) = 1 - x_1(0) - x_2(0)$ and $u(1) = 1$.

[2 marks]

ii) The equations

$$y(0) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}, \qquad y(1) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(1) \\ x_2(1) \end{bmatrix}$$

imply $x_2(0) = 1$ and $x_2(1) = 2$. Substituting these values in the equation

$$\begin{bmatrix} x_1(1) \\ x_2(1) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(0)$$

yields $x_1(1) = 2$ and $x_1(0) = 1$. Finally, $x(2)$ can be computed directly from the state equation, yielding $x_1(2) = 1$ and $x_2(2) = 4$.

[2 marks]

d) Recall from part b) that since $b = 0$, the controllability of the system depends on the value of c.

i) If $c = 0$, then the system is controllable. In fact,

$$\begin{bmatrix} x_1(1) \\ x_2(1) \end{bmatrix} = \begin{bmatrix} x_{10} + u(0) \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} x_1(2) \\ x_2(2) \end{bmatrix} = \begin{bmatrix} x_{10} + u(0) + u(1) \\ 0 \end{bmatrix}.$$

The system can be controlled to zero in one step selecting $u(0) = -x_{10}$ and $u(k) = 0$ for all $k > 0$.

[2 marks]

ii) If $c \neq 0$, then the system is not controllable. However, note that there is still a set of initial states that can be controlled to zero. In fact,

$$\begin{bmatrix} x_1(1) \\ x_2(1) \end{bmatrix} = \begin{bmatrix} x_{10} + u(0) \\ x_{20} \end{bmatrix} \qquad \begin{bmatrix} x_1(2) \\ x_2(2) \end{bmatrix} = \begin{bmatrix} x_{10} + u(0) + u(1) \\ x_{20} \end{bmatrix}$$

implies that any initial state such that $x_{20} = 0$ can be controlled to zero in one step selecting $u(0) = -x_{10}$ and $u(k) = 0$ for all $k > 0$.

[2 marks]