E4.45
CS7.21
SO22
ISE4.47

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2009

MSc and EEE/ISE PART IV: MEng and ACGI

Corrected Copy

WAVELETS AND APPLICATIONS

Monday, 11 May 10:00 am

Time allowed: 3:00 hours

There are FOUR questions on this paper.

Answer THREE questions.

All questions carry equal marks.

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s) :	P.L. Dragotti
	Second Marker(s) :	M. Petrou

Special Information for the Invigilators: NONE

Information for Candidates:

Sub-sampling by an integer N

$$x_{\downarrow N}[n] \longleftrightarrow \frac{1}{N}\sum_{k=0}^{N-1} X(e^{j(\omega-2\pi k)/N}) = \frac{1}{N}\sum_{k=0}^{N-1} X(W_N^k z^{1/N}),$$

where

$$W_N^k = e^{-j2\pi k/N}.$$

Optimal bit allocation:

Given N zero-mean Gaussian components with variances $\sigma_1^2, \sigma_2^2, ..., \sigma_N^2$ and a total bit budget R, the optimal bit allocation is given by:

$$R_i = \frac{R}{N} + \frac{1}{2}\log_2 \frac{\sigma_i^2}{(\prod_{j=1}^{N}\sigma_j^2)^{1/N}}, \quad i = 1, 2, ..., N.$$

The Questions

1. *Multi-Rate Signal Processing.*

    (a) Consider the multi-rate system shown in Figure 1a, where $G_0(z)$ is an ideal low-pass filter with cutoff frequency $\pi/2$ and $G_1(z)$ is an ideal high-pass filter with cutoff frequency $\pi/2$. The two filters are shown in Figure 1b.
    Sketch and dimension the four spectra $Y_1(e^{j\omega})$, $Y_2(e^{j\omega})$, $Y_3(e^{j\omega})$, $Y_4(e^{j\omega})$ assuming that $x[n]$ has the spectrum shown in Figure 1c.

    [5]

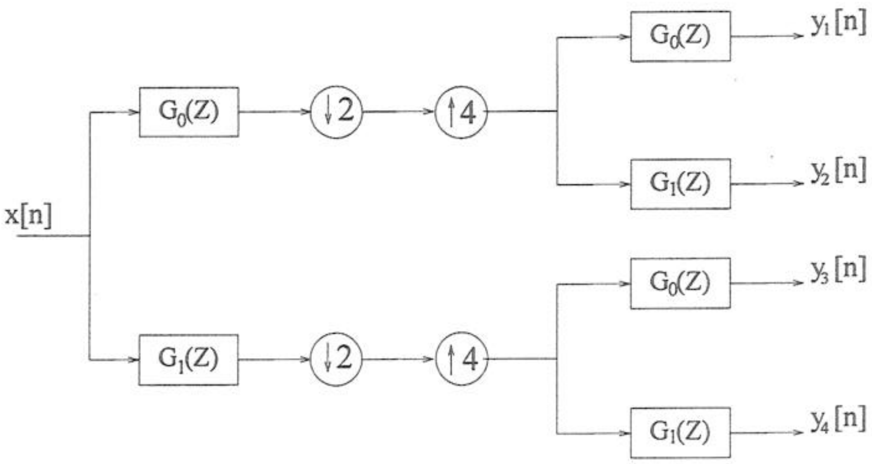


Figure 1a: The multi-rate system.

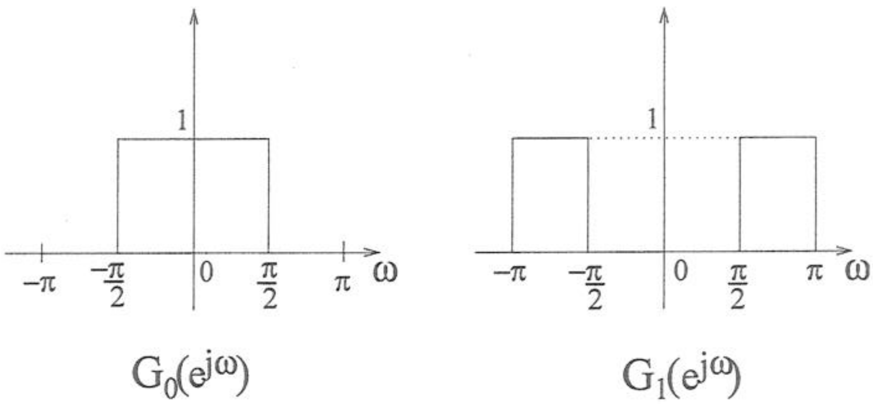


Figure 1b: The ideal low-pass and high-pass filters $G_0(z)$, $G_1(z)$.

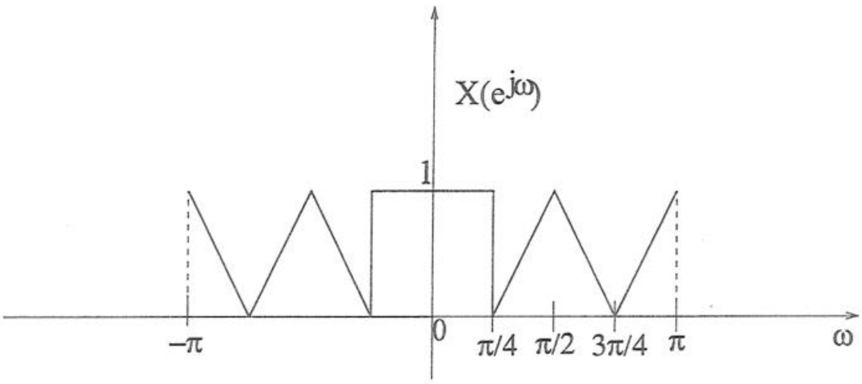
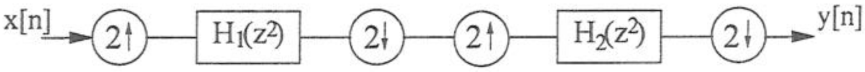


Figure 1c: Spectrum of $x[n]$.

(b) Technically, one cannot talk of transfer function in the case of multi-rate systems since changes in the sampling rates are not time invariant. However, there are cases where by carefully designing the processing chain, the input/output relationship can indeed by modeled with an equivalent transfer function.

i. Find the equivalent transfer function $H(z) = Y(z)/X(z)$ of the following system:



[5]

ii. Consider the system described by the block diagram of Figure 1d where $H(z)$ is an ideal low-pass filter with cutoff frequency $\pi/4$. Compute the transfer function of the system for $M = 2$ and for $M = 4$.

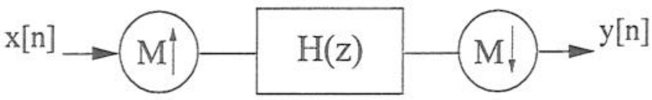


Figure 1d: Multirate filtering.

[5]

iii. Consider the system in Figure 1e where $H(z)$ is an ideal low-pass filter with cutoff frequency π/M. Show that this system implements a fractional delay (i.e., show that the equivalent transfer function of the system is that of a pure delay, where the delay is not necessarily an integer).

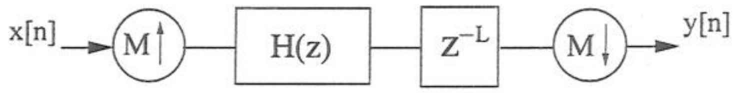


Figure 1e: Multirate system implementing a fractional delay.

[5]

2. Consider the two-channel filter bank of Figure 2.

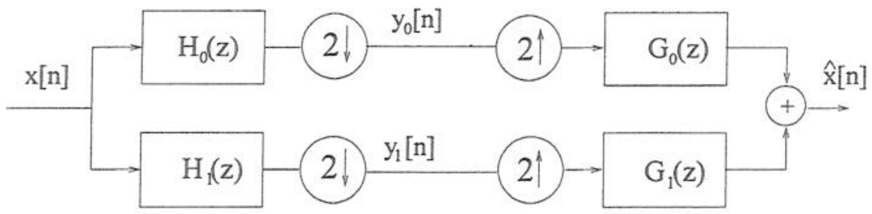


Figure 2: Two-channel filter bank.

(a) Express $\hat{X}(z)$ as a function of $X(z)$ and the filters. Then, derive the two perfect reconstruction conditions the filters have to satisfy.

[5]

(b) In 1984, Smith and Barnwell suggested that the product filter $P(z) = H_0(z)G_0(z)$ should have the following form:

$$p[n] = \begin{cases} 1 & \text{for } n = 0 \\ \frac{\sin(\pi n/2)}{\pi n} w[n] & \text{otherwise} \end{cases}$$

where $w[n]$ is a window function.

i. Assume that $w[n]$ is the rectangular window:

$$w[n] = \begin{cases} 1 & \text{for } n = -M, -M+1, .., 0, 1, .., M \\ 0 & \text{otherwise} \end{cases}$$

Show that the proposed $P(z)$ satisfies the half-band condition for any choice of M: $P(z) + P(-z) = 2$ for any M.

[5]

ii. Assume that $M = 1$. Factorize the resulting $P(z)$. Assign one zero of $P(z)$ to $G_0(z)$ and choose $G_0(z)$ to be minimum phase. Design the other filters in order to have perfect reconstruction biorthogonal filter banks.

[5]

iii. Now, consider the limit function

$$\hat{\varphi}(\omega) = \lim_{J\to\infty} \prod_{k=1}^{J} M_0\left(\frac{\omega}{2^k}\right),$$

where $M_0(\omega) = \frac{G_0(e^{j\omega})}{\sqrt{2}}$ and $G_0(z)$ is the filter you found in part (ii). What can you say about convergence of the above limit?

[5]

3. *Multiresolution analysis.* Consider a scaling function $\varphi(t) \in V_0$ satisfying the axioms of the multiresolution analysis. That is,

$$\{\varphi(t-n)\}_{n\in\mathbb{Z}}$$

is an orthonormal basis for the subspace V_0 and the sequence of embedded subspaces

$$...V_2 \subset V_1 \subset V_0 \subset V_{-1}...$$

is such that

(a) Upward Completeness

$$\lim_{m\to-\infty} V_m = L_2(\mathbb{R})$$

(b) Downward Completeness

$$\lim_{m\to\infty} V_m = \{0\}$$

(c) Scale Invariance

$$f(t) \in V_m \leftrightarrow f(2^m t) \in V_0$$

(d) Shift Invariance

$$f(t) \in V_0 \rightarrow f(t-n) \in V_0 \quad \text{for all} \quad n \in \mathbb{Z}.$$

Demonstrate the following:

(a) $\varphi(t)$ satisfies the two-scale equation

$$\varphi(t) = \sqrt{2}\sum_{n=-\infty}^{\infty} g_0[n]\varphi(2t-n),$$

where $G_0(e^{j\omega}) = \sum_{n=-\infty}^{\infty} g_0[n]e^{-j\omega n}$ satisfies the following equation:

$$|G_0(e^{j\omega})|^2 + |G_0(e^{j\omega+\pi})|^2 = 2.$$

[Hint: use the fact that $\varphi(t)$ satisfies the Riesz criterion

$$\sum_{k=-\infty}^{\infty} |\hat{\varphi}(\omega + 2\pi k)|^2 = 1.$$

where $\hat{\varphi}(\omega)$ is the Fourier transform of $\varphi(t)$.]

[5]

(b) The wavelet function $\psi(t) = \sqrt{2}\sum_{n=-\infty}^{\infty}(-1)^n g_0[1-n]\varphi(2t-n)$ is an orthonormal basis. That is, show that

$$\sum_{k=-\infty}^{\infty} |\hat{\psi}(\omega + 2\pi k)|^2 = 1,$$

where $\hat{\psi}(\omega)$ is the Fourier transform of $\psi(t)$.

[5]

(c) Denote with W_0 the space spanned by $\{\psi(t-n)\}_{n\in\mathbb{Z}}$. Show that $W_0 \perp V_0$ and that $V_{-1} = V_0 \oplus W_0$

[5]

(d) Finally, show that

$$L_2(\mathbb{R}) = \bigoplus_{j=-\infty}^{\infty} W_j.$$

where W_j is the space spanned by

$$\{\frac{1}{\sqrt{2^j}}\psi(2^{-j}t-n)\}_{n\in\mathbb{Z}}.$$

[5]

4. *Transform Coding.* Consider a jointly Gaussian zero-mean vector $\mathbf{x} = (x_1, x_2)^T$ with covariance matrix

$$R_x = \begin{pmatrix} 17/2 & 15/2 \\ 15/2 & 17/2 \end{pmatrix}.$$

You are given R bits to encode this vector. You first apply an orthonormal transform T to $\mathbf{x}$ leading to the transformed vector $\mathbf{y} = T\mathbf{x}$ with $\mathbf{y} = (y_1, y_2)^T$. Then the two transformed components y_1 and y_2 are encoded independently. We assume that the encoder involved can achieve the rate-distortion bound for a Gaussian source given by:

$$D(R) = \sigma^2 2^{-2R},$$

where σ^2 is the variance of the source.

(a) If we assume that T is the identity matrix, how would you allocate the bits between y_1 and y_2? Compute the average distortion you achieve in this case when the total number of bits available is $R = 8$ bits.

[5]

(b) Now, assume that T is the the Karhunen-Loève transform (KLT) of R_x. That is, T is the transform that diagonalizes the covariance matrix R_x. Assuming $R = 8$, find the optimal number of bits that should be allocated to y_1 and y_2. Then evaluate the average distortion achieved in this case.

[5]

(c) Now assume that T is the Haar transform:

$$T = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix}.$$

If $R = 8$, what is the average distortion that can be achieved in this last scenario?

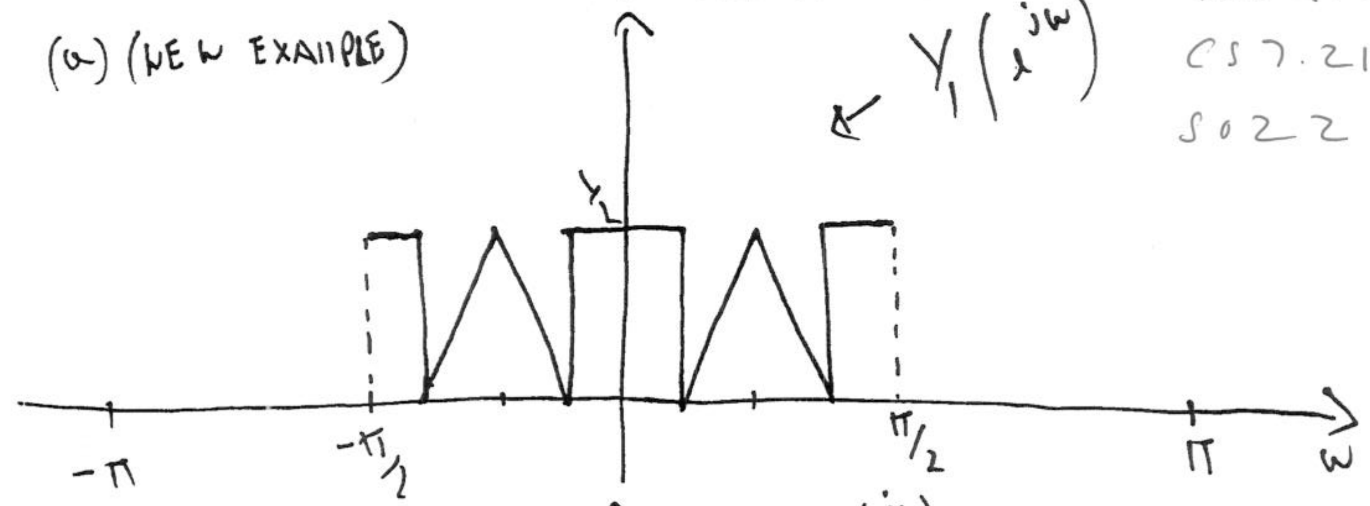
[5]

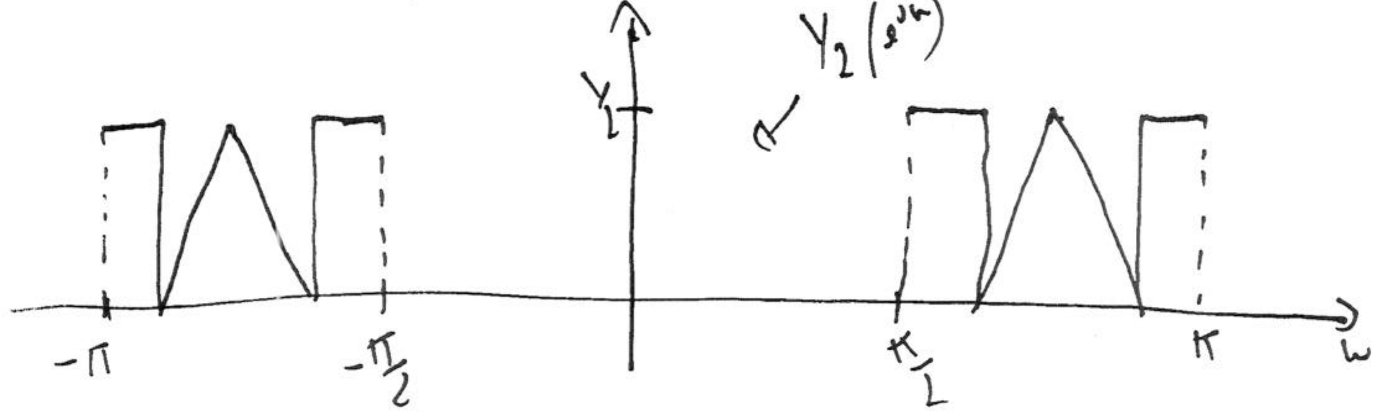
(d) The KLT is the optimal transform when dealing with jointly Gaussian vectors. However, this is not the case when the vectors involved are not Gaussian. Consider a source that produces 4-D vectors $\mathbf{x} = (x_1, x_2, x_3, x_4)^T$ with the following properties: a) only one component of $\mathbf{x}$ is non-zero, b) the location of the non-zero component is uniformly distributed, c) the amplitude of the non-zero component follows a Gaussian distribution with zero mean and variance σ^2. Demonstrate that under these hypotheses an encoder based on the KLT achieves $D(R) = c_1\sigma^2 2^{-R/2}$, whereas an encoder that transmits first the location of the non-zero component and then encodes its amplitude achieves $D(R) = c_2\sigma^2 2^{-2(R-2)}$, where c_1 and c_2 are two constants.

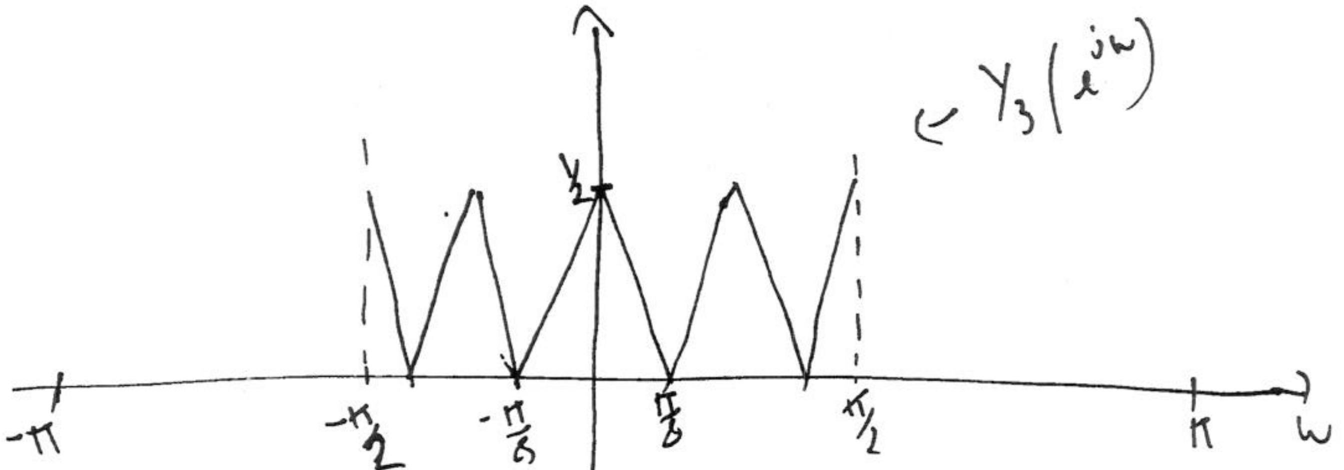
[5]

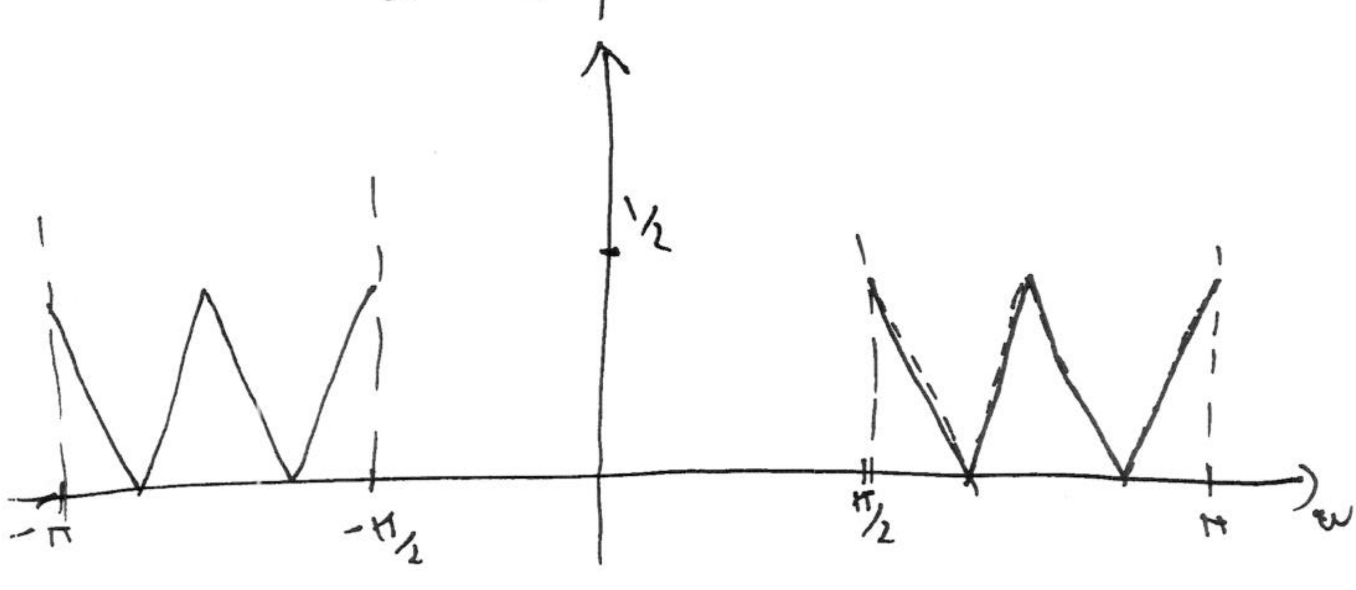
QUESTION 1

(a) (NEW EXAMPLE)



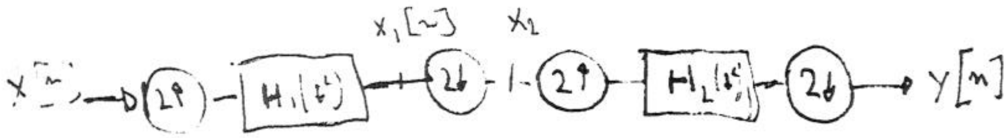






b) (NEW EXAMPLE)

i.



$$X_1(z) = H_1(z^2)\, X(z^2)$$

$$X_2(z) = H_1(z)\, X(z)$$

$$Y(z) = H_2(z)\, H_1(z)\, X(z) \Rightarrow$$

$$H_{EQ}(z) = H_2(z)\, H_1(z)$$

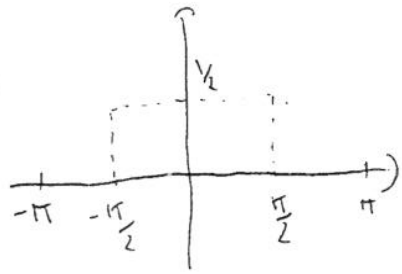
ii. (NEW EXAMPLE)

WHEN $n = 2$

$$Y(z) = X(z)\, \frac{1}{2}\left(H(z^{1/2}) + H(-z^{1/2})\right)$$

SINCE $H(z)$ IS A LOW-PASS FILTER WITH CUT-OFF AT $\pi/4$

$$\frac{1}{2}\left(H(z^{1/2}) + H(-z^{1/2})\right) \quad (=)$$

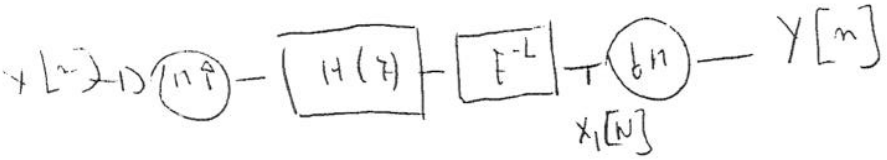


IS EQUIVALENT TO A LOW-PASS FILTER WITH CUT-OFF AT $\frac{\pi}{2}$ AND AMPLITUDE $1/2$

WHEN $n = 4$

WE GET AN ALL PASS-FILTER WITH AMPLITUDE $1/4$

iii (NEW EXAMPLE)



$$X_1(z) = H(z)\, z^{-L}\, X(z^M)$$

$$Y(z) = X(z) \cdot \frac{1}{M} \sum_{k=0}^{M-1} H\left(W_M^k z^{1/M}\right) z^{-L/M}\, W_M^{k}$$

SINCE $H(z)$ IS IDEAL-LOW PASS WITH CUT-OFF AT π/M, ALL THE ALIASED COMPONENT DON'T CONTRIBUTE AND $H(z^{1/M})$ IS AN ALL-PASS FILTER ⇒

$$\frac{1}{M} \sum_{k=0}^{M-1} H\left(W_M^k z^{1/M}\right) z^{-L/M}\, W_M^k = \frac{1}{M} z^{-L/M}$$

THUS

$$H_{EQ}(z) = \frac{1}{M} z^{-L/M}$$

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QUESTION 2

(a) (TEXT BOOK)

$$\hat{X}(z) = \frac{1}{2} G_0(z)\left(H_0(z)X(z) + H_0(-z)X(-z)\right) + \frac{1}{2} G_1(z)\left(H_1(z)X(z) + H_1(-z)X(-z)\right).$$

PR CONDITIONS:

$$\begin{cases} G_0(z)H_0(z) + G_1(z)H_1(z) = 2 \\ G_0(z)H_0(-z) + G_1(z)H_1(-z) = 0 \end{cases}$$

(b) (NOVEL EXAMPLE)

(i) THE CHOSEN $P(z)$ IS SYMMETRIC.

$P(z) + P(-z) = 2$ IS SATISFIED WHEN $p[0] = 1$, AND $p[2n] = 0$.

SINCE $p[0] = 1$ BY CONSTRUCTION

AND $p[2n] = \dfrac{\sin(\pi n)}{2\pi n} = 0$

CONDITIONS ARE SATISFIED.

1b) (ii) (NEW EXAMPLE)

$M=1$

$$P(z) = \frac{1}{\pi} z + 1 + \frac{1}{\pi} z^{-1} = \frac{1}{\pi} z^{-1}\left(z^{+2} + \pi z^{+1} + 1\right)$$

ROOTS OF $P(z)$

$$\alpha_{1,2} = \frac{-\pi \pm \sqrt{\pi^2 - 4}}{2} \cong \begin{cases} -2.78 \\ -0.36 \end{cases}$$

$\alpha_1 = -0.36$, $\alpha_2 = -2.78$. NOTICE $\alpha_1 = 1/\alpha_2$

$$P(z) = \frac{z^{-1}}{\pi}(z-\alpha_1)(z-\alpha_2)$$

$$G_0(z) = \frac{1}{\sqrt{\pi}}\left(1 - \alpha_1 z^{-1}\right)$$

$$H_0(z) = \frac{1}{\sqrt{\pi}}\left(z - \alpha_2\right) = \frac{1}{\sqrt{\pi}}\left(1 - \frac{1}{\alpha_1}\right)$$

$$G_1(z) = z^{-1} H_0(-z) = -\frac{1}{\sqrt{\pi}}\left(1 + \frac{z^{-1}}{\alpha_1}\right)$$

$$H_1(z) = z\, G_0(-z) = \frac{1}{\sqrt{\pi}}\left(z + \alpha_1\right)$$

(iii) (NEW EXAMPLE)

THE LIMIT DOES NOT CONVERGE

SINCE $\quad G_0(e^{j\omega}) \neq 0 \quad$ FOR $\omega = \pi$

AND $\quad G_0(e^{j\omega}) \neq \sqrt{2} \quad$ FOR $\omega = 0$

QUESTION 3

(a) FROM MY LECTURE NOTES

The multiresolution analysis leads directly to the *two-scale equation.* Since V_1 is included in V_0 if $\varphi(t/2)$ belongs to V_1, it belongs to V_0 as well. Moreover, since $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$ is a basis for V_0, we can express $\varphi(t/2)$ as a linear combination of $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$, thus, we have that $\varphi(x/2) = \sqrt{2}\sum_{n=-\infty}^{\infty} g_0[n]\varphi(x-n)$. Replace x with $2t$ to obtain the two-scale relation

$$\varphi(t) = \sqrt{2}\sum_{n=-\infty}^{\infty} g_0[n]\varphi(2t-n).$$

Now, take the Fourier transform of both sides, we obtain

$$\hat{\varphi}(\omega) = \frac{1}{\sqrt{2}}G_0(e^{j\omega/2})\hat{\varphi}(\omega/2)$$

where

$$G_0(e^{j\omega}) = \sum_n g_0[n]e^{-j\omega n}.$$

Because of the orthogonality of $\varphi(t)$, $G(e^{j\omega})$ satisfies the following property

$$|G_0(e^{j\omega})|^2 + |G_0(e^{j(\omega+\pi)})|^2 = 2.$$

Proof: We know that

$$\sum_{k=-\infty}^{\infty} |\hat{\varphi}(2\omega + 2k\pi)|^2 = 1.$$

Thus we have that

$$\begin{aligned}
1 &= \tfrac{1}{2}\textstyle\sum_k |G_0(e^{j(\omega+k\pi)})|^2|\hat{\varphi}(\omega+k\pi)|^2 \\
&= \tfrac{1}{2}\textstyle\sum_k |G_0(e^{j(\omega+2k\pi)})|^2|\hat{\varphi}(\omega+2k\pi)|^2 \\
&+ \tfrac{1}{2}\textstyle\sum_k |G_0(e^{j(\omega+(2k+1)\pi)})|^2|\hat{\varphi}(\omega+(2k+1)\pi)|^2 \\
&= \tfrac{1}{2}|G_0(e^{j\omega})|^2\textstyle\sum_k |\hat{\varphi}(\omega+2k\pi)|^2 \\
&+ \tfrac{1}{2}|G_0(e^{j(\omega+\pi)})|^2\textstyle\sum_k |\hat{\varphi}(\omega+(2k+1)\pi)|^2 \\
&= \tfrac{1}{2}(|G_0(e^{j\omega})|^2 + |G_0(e^{j(\omega+\pi)})|^2)
\end{aligned}$$

which completes the proof.

(b) THE WAVELET IS [TEXT BOOK]

$$\psi(t) = \sqrt{2}\sum_n g_1[n]\varphi(2t-n)$$

WITH $g_1[n] = (-1)^n g_0[n]$.

NOW $\langle\psi(t), \psi(t-n)\rangle = \delta_n \iff \sum_k |\hat{\psi}(\omega+2\pi k)|^2 = 1$

AND THIS LAST CONDITION IS SATISFIED WHEN

$$\left|G_1\left(e^{jw}\right)\right|^2 + \left|G_1\left(e^{jw+\pi}\right)\right|^2 = 2 \qquad (1)$$

BUT $$G_1\left(e^{jw}\right) = -e^{-jw} G_0^*\left(e^{j(w+\pi)}\right) \qquad (2)$$

AND WE KNOW THAT

$$\left|G_0\left(e^{jw}\right)\right|^2 + \left|G_0\left(e^{jw+\pi}\right)\right|^2 = 1 \qquad (3)$$

THUS ~~EQUATION (1)~~ BY COMPARING EQ. (2) WITH EQ. (3) WE SEE THAT (1) IS SATISFIED.

[TEXT BOOK]

(c) WE NEED TO SHOW THAT

$$\langle \varphi(t), \psi(t-n) \rangle = 0$$

THIS IS EQUIVALENT TO

$$\sum_l \hat{\varphi}(w+2\pi l)\,\hat{\psi}^*(w+2\pi l) = 0$$

WHICH IN TURN IMPLIES THAT

$$G_1(e^{jw})\,G_0^*(e^{jw}) + G_1(e^{jw+\pi})\,G_0^*(e^{jw+\pi}) = 0$$

~~WHICH~~ AND THIS LAST IDENTITY IS SATISFIED SINCE

$$G_1(e^{jw}) = -e^{-jw}\,G_0^*(e^{j(w+\pi)})$$

$\square$

$V_{-1} = V_0 \oplus W_0$ (4) ~~EXAMPLE~~ [TEXT BOOK]

$\{\varphi(2t-n)\}_{n\in\mathbb{Z}}$ IS A BASIS OF V_{-1}

THUS ANY SIGNAL $f(t) \in V_{-1}$ CAN BE WRITTEN AS

$$f(t) = \sqrt{2}\sum_n a[n]\,\varphi(2t-n)$$

CONDITION (4) IS SATISFIED IF ANY $f(t) \in V_{-1}$ CAN BE EXPRESSED IN TERMS OF $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$ AN $\{\psi(t-n)\}_{n\in\mathbb{Z}}$

OR

$$f(t) = \sqrt{2}\sum_n a[n]\,\varphi(2t-n) = \sum_n b[n]\,\varphi(t-n) + \sum_n c[n]\,\psi(t-n) \qquad (5)$$

FOR A PROPER CHOICE OF THE COEFFICIENTS $b[n]$ AND $c[n]$.

BY WRITING (5) IN THE FOURIER DOMAIN WE OBTAIN

$$\frac{1}{\sqrt{2}} A\left(e^{j\frac{w}{2}}\right)\hat{\varphi}\left(\frac{w}{2}\right) = B\left(e^{jw}\right)\hat{\varphi}(w) + C\left(e^{jw}\right)\hat{\psi}(w)$$

THE TWO SCALE RELATIONS ARE:

$$\hat{\varphi}(w) = \frac{1}{\sqrt{2}} G_0\left(e^{j\frac{w}{2}}\right)\hat{\varphi}\left(\frac{w}{2}\right)$$

$$\hat{\psi}(w) = \frac{1}{\sqrt{2}} G_1\left(e^{j\frac{w}{2}}\right)\hat{\varphi}\left(\frac{w}{2}\right)$$

THUS, USING THE ABOVE RELATIONS WE OBTAIN:

$$A\left(e^{j\frac{w}{2}}\right) = B\left(e^{jw}\right) G_0\left(e^{j\frac{w}{2}}\right) + C\left(e^{jw}\right) G_1\left(e^{j\frac{w}{2}}\right)$$

THIS EQUALITY IS ALWAYS SATISFIED WHEN CHOOSING

$$B\left(e^{j2w}\right) = \frac{1}{2}\left[A\left(e^{jw}\right) G_0^*\left(e^{jw}\right) + A\left(e^{j(w+\pi)}\right) G_0^*\left(e^{j(w+\pi)}\right)\right]$$

$$C\left(e^{j2w}\right) = \frac{1}{2}\left[A\left(e^{jw}\right) G_1^*\left(e^{jw}\right) + A\left(e^{j(w+\pi)}\right) G_1^*\left(e^{j(w+\pi)}\right)\right]$$

$\square$

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(d) FROM MY LECTURE NOTES:

CONSTRUCTION

Proof: We start the proof by noticing that the detail spaces $\{W_j\}_{n\in\mathbb{Z}}$ are orthogonal. Indeed, by [redacted] $W_j \perp V_j$, moreover $W_l \subset V_{l-1} \subset V_j$ for $j < l$. Therefore W_j and W_l are orthogonal. We can also decompose $L_2(\mathbb{R})$ into the mutually orthogonal subspace W_j or

$$L_2(\mathbb{R}) = \bigoplus_{j=-\infty}^{\infty} W_j. \tag{3.2}$$

Indeed, since $V_{j-1} = V_j \oplus W_j$, we can write

$$V_L = \bigoplus_{j=L+1}^{J} W_j \oplus V_J.$$

Because of the upward/downward completeness properties, V_L and V_J tend respectively to $L_2(\mathbb{R})$ and $\{0\}$ when L and J go respectively to $-\infty$ and ∞ which leads to (3.2).

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QUESTION 4

a) SINCE $\sigma_1^2 = \sigma_2^2 = 17/2$. THE SAME RATE SHOULD BE ALLOCATED ~~THE a~~ TO Y_1 AND Y_2.

THE AVERAGE DISTORTION IS:

$$D(R) = \frac{1}{2}\left[\sigma_1^2\, 2^{-\frac{2R}{2}} + \sigma_2^2\, 2^{-\frac{2R}{2}}\right] = \sigma_1^2\, 2^{-\frac{2R}{2}}$$

$$= \frac{17}{2}\, 2^{-8} \cong 0.033$$

b) THE EIGENVALUES OF

$$R_x = \begin{pmatrix} \frac{17}{2} & \frac{15}{2} \\ \frac{15}{2} & \frac{17}{2} \end{pmatrix}$$

ARE:

$$\lambda_1 = \frac{17}{2} + \frac{15}{2} = \frac{32}{2} = 16$$

$$\lambda_2 = \frac{17}{2} - \frac{15}{2} = 1$$

THUS

$$R_1 = \frac{R}{2} + \frac{1}{2}\log_2 \frac{\lambda_1^2}{\sqrt{\lambda_1^2 \lambda_2^2}} = 4 + \frac{1}{2}\log_2 \frac{16}{4} = 4 + 1 = 5$$

$$R_2 = \frac{R}{2} + \frac{1}{2}\log_2 \frac{\lambda_2^2}{\sqrt{\lambda_1^2 \lambda_2^2}} = 4 - \frac{1}{2}\log_2 \frac{16}{4} = 4 - 1 = 3$$

THE AVERAGE DISTORTION IS

$$D(R) = \frac{1}{2}\left[\cancel{\tfrac{1}{2}}\lambda_1^2 2^{-2R_1} + \cancel{\tfrac{1}{2}}\lambda_2^2 2^{-2R_2}\right] = \frac{1}{2}\left(8\,2^{-10} + \frac{1}{2}\,2^{-6}\right) =$$

$$= \cancel{0.0078} = 0.0156$$

c) IN THIS CASE THE HAAR COINCIDES WITH THE KLT. THEREFORE IT ACHIEVES THE SAME DISTORTION AS IN PART (B): $D(R) = 0.0078$

d) THE ~~KLT IS IN THIS CASE~~ CORRELATION MATRIX IS IN THIS CASE

$$R_x = \frac{\sigma^2}{N}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \frac{\sigma^2}{4} I$$

AND KLT $= I$. THUS THE SAME NUMBER OF BITS IS ALLOCATED TO THE FOUR COMPONENTS AND THE AVERAGE DISTORTION IS

$$D(R) = \cancel{4}c_1\sigma^2\, 2^{-2R/4} = \cancel{4}c_1\sigma^2\, 2^{-R/2}$$

IN THE SECOND SCENARIO $\log_2 4 = 2$ BITS ARE USED TO ENCODE THE LOCATION ON THE NON-ZERO COMPONENT AND THE REMAINING BITS $(R-2)$ ARE USED TO QUANTIZE IT SINCE THE COMPONENT IS GAUSSIAN WITH VARIANCE σ^2, THE FINAL $D(R)$ IS:

$$D(R) = \sigma^2 \, 2^{-2(R-2)}.$$

THIS DISTORTION IS MUCH BETTER THAN THE OTHER FOR LARGE VALUES OF R.