Imperial College
London

M4/5P19

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May-June 2018

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Measure and Integration

Date: Tuesday, 08 May 2018

Time: 2:00 PM - 4:30 PM

Time Allowed: 2.5 hours

This paper has 5 questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the relevant main book(s) are full.

All required additional material will be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Each question carries equal weight.
- Calculators may not be used.

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1. (a) Let X be a nonempty set. Suppose a measure m is initially given on a semiring of sets from X. Give the definition of the outer measure of a subset of X, a measurable set and its measure μ.

Consider a measure space $(X, \mathcal{M}, \mu)$.

(b) Give the definition of a measurable function $f : X \to \mathbb{R}$.

(c) Let $\mu(X) < \infty$. Show that convergence in $L^2(X, \mu)$ implies convergence in $L^1(X, \mu)$.

2. (a) For any $\epsilon \in (0,1)$ construct an open set $A \subset [0,1]$ such that it is dense in $[0,1]$ (i.e. its closure contains $[0,1]$) and the Lebesgue measure $\mu(A) \leq \epsilon$.

(b) Construct a set satisfying the same conditions as in Part a of this question, but with $\mu(A) = \epsilon$ instead of $\mu(A) \leq \epsilon$. Hint: Assume that the set in Part a is already constructed and consider its union with the interval $(1/2 - x, 1/2 + x)$.

3. (a) State without proof the Levi (monotone convergence) theorem.

(b) Show for a integrable function $f : X \to \mathbb{R}$ on a measure space $(X, \mathcal{M}, \mu)$ that if $\int_A |f| d\mu = 0$ then $f(x) = 0$ a.e. on the set A.

4. (a) Describe without proof the type of discontinuity a nondecreasing function $f : [a, b] \to \mathbb{R}$ can have at a point $x \in (a, b)$.

(b) Let $f : [a, b] \to \mathbb{R}$ be a nondecreasing absolutely continuous function. State what it means for f to be absolutely continuous. Show that for a measurable $A \subset (a, b)$, if $\mu(A) = 0$ then $\mu(f(A)) = 0$, where μ is the Lebesgue measure on $\mathbb{R}$.

5. Consider the representation of a number $x \in (0,1)$ by its continued fraction

$$x = [0; a_1, a_2, ...] = \frac{1}{a_1 + \frac{1}{a_2 + \cdots}},$$

(which is finite if x is rational), where all a_j's are positive integers.

Let the transformation $T : (0,1) \to (0,1)$ be given by

$$Tx = \frac{1}{x} - \left[\frac{1}{x}\right],$$

where $[\cdot]$ stands for the integer part, i.e., $T[0; a_1, a_2, a_3, ...] = [0; a_2, a_3, ...]$.

(a) Show that T preserves the following measure

$$\mu(A) = \frac{1}{\log 2} \int_A \frac{dx}{1+x},$$

where A is a measurable set with respect to the Lebesgue measure. Hint: For any $0 < a < 1$, compute first $T((\frac{1}{n+a}, \frac{1}{n}))$ and show that $\mu(T^{-1}(0,a)) = \mu((0,a))$.

(b) The transformation T is ergodic. Define what this statement means.

	EXAMINATION SOLUTIONS ~~2017~~-18	Course P19
Question 1		Marks & seen/unseen
Parts a	The outer measure of a set $A \in X$ is $\mu^*(A) = \inf_{A \subset \bigcup P_k} \sum_k m(P_k)$ over all coverings of A by finite or countable number of elements P_k of the semiring $\mathcal{L}$, where $m(P_k)$ is the measure of P_k. Let $\mathcal{R}(\mathcal{L})$ be the minimal ring generated by $\mathcal{L}$. A set $A \in X$ is called measurable if $\forall \varepsilon > 0 \ \exists B \in \mathcal{R}(\mathcal{L})$ such that $\mu^*(A \triangle B) < \varepsilon$. μ^* restricted to measurable sets is called measure (μ).	7 / seen
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	EXAMINATION SOLUTIONS 2017-18	Course P19
Question 1		Marks & seen/unseen
Parts b	A function $f: X \to \mathbb{R}$ is called measurable if $f^{-1}(B)$ is a measurable set for any Borel $B \subset \mathbb{R}$.	3 / seen
c	Let $f_n \to f$ in $L^2(X, \mu)$, i.e. $\int_X \lvert f_n - f\rvert^2 d\mu \to 0$ as $n \to \infty$. By Cauchy-Schwarz inequality $\int_X \lvert f_n - f \rvert d\mu = \int_X \lvert f_n - f\rvert \cdot 1 \, d\mu$ $\leq \left(\int_X \lvert f_n - f\rvert^2 d\mu \right)^{1/2} \cdot \mu(X)^{1/2}$ $\to 0$ as $n \to \infty$. Therefore $f_n \to f$ in $L^1(X, \mu)$	10 / ~~unseen~~ seen
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	EXAMINATION SOLUTIONS 2017-18	Course P19
Question 2		Marks & seen/unseen
Parts		
a	Let q_n, $n=1,2\ldots$ be rational numbers in $[0,1]$. Consider $A' = \bigcup_{n=1}^{\infty} \left(q_n - \frac{\varepsilon}{2^{n+1}}, q_n + \frac{\varepsilon}{2^{n+1}}\right)$. It is measurable and open as countable union of open intervals; q_n are dense in $[0,1]$, so A' is dense in $[0,1]$, $\mu(A') \leqslant \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon$ by subadditivity. Now $A = A' \cap (0,1)$.	10 / unseen
b	Take the set A constructed in question 2a. We have $\varepsilon' = \mu(A) \leqslant \varepsilon < 1$.	10 / unseen
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Consider now

$B_x = (\frac{1}{2}-x, \frac{1}{2}+x) \cup A, \quad x \in (0, \frac{1}{2})$.

It is open and dense in $[0,1]$

Its measure $f(x) = \mu(B_x)$

satisfies : $f(0) = \varepsilon'$, $f(\frac{1}{2}) = 1$

and since $B_x \subset B_y$ for $x < y$,

$f(y) - f(x) = \mu(B_y \setminus B_x) =$

$\mu\left((\frac{1}{2}-y, \frac{1}{2}-x] \cup [\frac{1}{2}+x, \frac{1}{2}+y) \cup A\right)$

$\leq 2(y-x)$. Thus $f(x)$ is

continuous and therefore takes

any value between ε' and 1.

So $\exists\, x_0 \in (0, \frac{1}{2})$ s.t. $f(x_0) = \varepsilon$,

i.e. B_{x_0} is the set

with $\mu(B_{x_0}) = \varepsilon$.

	EXAMINATION SOLUTIONS 2017-18	Course P19
Question 3		Marks & seen/unseen
Parts		
a	Let $f_1(x) \le f_2(x) \le \cdots$ on A, $f_n(x)$, $n=1,2\ldots$ are integrable and $\int_A f_n \, d\mu \le K$ $\forall n$, where $K>0$ is a constant. Then there exists a.e. a finite limit $f(x) = \lim_{n\to\infty} f_n(x)$, $f(x)$ is integrable and $\int_A f_n \, d\mu \to \int f \, d\mu$.	10/seen

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	EXAMINATION SOLUTIONS 2017-18	Course P19
Question 3		Marks & seen/unseen
Parts b	For $\varepsilon > 0$, let $A_\varepsilon = \{x \in A : \|f\| \geq \varepsilon\}$. By Chebyshev inequality, $\mu(A_\varepsilon) \leq \frac{1}{\varepsilon} \int_A \|f\| d\mu = 0$, so $\mu(A_\varepsilon) = 0$.	~~seen~~/unseen 5
	The set $B = \{x \in A : \|f\| > 0\} = \bigcup_{j=1}^{\infty} A_{1/j}$, but $A_1 \subset A_{1/2} \subset A_{1/3} \subset \cdots$ By continuity of measure, $\mu(B) = \lim_{n \to \infty} \mu(A_{1/n}) = 0$. So $\|f\| = 0$ a.e. on A $\Rightarrow f = 0$ a.e on A.	5
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	EXAMINATION SOLUTIONS 2017-18	Course P19
Question 4		Marks & seen/unseen
Parts a	The only discontinuity a nondecreasing function $f:[a,b]\to\mathbb{R}$ can have at a point $x\in(a,b)$ is the one where both limits $f(x+0)$, $f(x-0)$ exist but are not equal.	5/seen
b	Let $f:[a,b]\to\mathbb{R}$ be non-decreasing and a.c., $A\in[a,b]$, $\mu(A)=0$. Fix $\varepsilon>0$. As f is a.c. $\exists\,\delta$ s.t. for any nonintersecting (a_j,b_j), $j=1,\dots,n$ with $\sum_1^n(b_j-a_j)<\delta$ we have	5/seen
	Setter's initials Checker's initials	Page number 7

Question 4

Marks & seen/unseen

Parts

$$\sum_1^n (f(b_j) - f(a_j)) < \varepsilon .$$

Choose an open set U s.t. 10/unseen

$A \subset U \subset (a,b)$ and $\mu(U \setminus A) < \delta$

Hence $\mu(U) = \mu(A) + \mu(U \setminus A) < \delta$

As an open set in $\mathbb{R}$, U has a representation

$U = \bigcup_1^\infty (a_j, b_j)$ – union of non-intersecting intervals. Since

$\sum_1^\infty (b_j - a_j) < \delta \Rightarrow$

$\sum_1^n (f(b_j) - f(a_j)) < \varepsilon \quad \forall n$

$\Rightarrow \sum_1^\infty (f(b_j) - f(a_j)) \leq \varepsilon$

But $f(A) \subset \bigcup_1^\infty (f(a_j), f(b_j))$

as f is nondecreasing.

Setter's initials Checker's initials

Parts

So $\mu^*(f(A)) \leq \sum_1^\infty (f(b_j) - f(a_j))$

$\leq \varepsilon$.

Since $\varepsilon > 0$ is arbitrary,

$\mu^*(f(A)) = 0$

$\Rightarrow \mu(f(A)) = 0$.

	EXAMINATION SOLUTIONS 2017-18	Course P19
Question 5		Marks & seen/unseen
Parts a	Let $0<a<1$.	10

Note that $T\left(\left(\frac{1}{n+a},\frac{1}{n}\right)\right) = (0,a) \quad \forall n=1,2,\dots$

$\Rightarrow$

$$T^{-1}((0,a)) = \bigcup_{n=1}^{\infty}\left(\frac{1}{n+a},\frac{1}{n}\right);$$

$$\mu(T^{-1}((0,a))) = \sum_{n=1}^{\infty}\mu\left(\left(\frac{1}{n+a},\frac{1}{n}\right)\right) =$$

$$= \sum_{n=1}^{\infty}\frac{1}{\log 2}\int_{\frac{1}{n+a}}^{1/n}\frac{dx}{1+x} =$$

$$= \sum_{n=1}^{\infty}\frac{1}{\log 2}\left(\log\frac{1/n+1}{\frac{1}{n+a}+1}\right) =$$

$$= \sum_{n=1}^{\infty}\frac{1}{\log 2}\left(\log\frac{n+1}{n+1+a} - \log\frac{n}{n+a}\right)$$

$$= \frac{1}{\log 2}\log\frac{1+a}{1} = \frac{1}{\log 2}\int_0^a\frac{dx}{1+x} =$$

$$= \mu((0,a))$$

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	EXAMINATION SOLUTIONS 2017-18	Course P19
Question 5		Marks & seen/unseen
Parts	Therefore, for $0<a<b<1$, $\mu((a,b)) = \mu((0,b)) - \mu((0,a))$ $= \mu(T^{-1}(0,b)) - \mu(T^{-1}(0,a)) =$ $= \mu(T^{-1}(a,b))$ since $T^{-1}(a,b) \cup T^{-1}(0,a] = T^{-1}(0,b)$. Hence, the measures $\mu(A)$, $\eta(A) = \mu(T^{-1}A)$ coincide on the ring. Since μ, η are σ-additive, they also coincide on any Lebesgue-measurable set by uniqueness of the σ-additive extension.	5
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	EXAMINATION SOLUTIONS 2017-18	Course P19
Question 5		Marks & seen/unseen
Parts b	The measure-preserving transformation T is ergodic if for any measurable A s.t. $A = T^{-1}A$, either $\mu(A) = 1$ or $\mu(A) = 0$.	5
	Setter's initials	Checker's initials	Page number 12