Imperial College London

[E2.8 (Maths 3) 2009]

B.ENG. AND M.ENG. EXAMINATIONS 2009

PART II Paper 3 : MATHEMATICS (ELECTRICAL ENGINEERING)

Date Wednesday 3rd June 2009 2.00 - 5.00 pm

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.

Answer EIGHT questions.

Please answer questions from Section A and Section B in separate answer-books.

A mathematical formulae sheet is provided.

Statistical data sheets are provided.

[Before starting, please make sure that the paper is complete; there should be EIGHT pages, with a total of TWELVE questions. Ask the invigilator for a replacement if your copy is faulty.]

1. The complex variable w is related to the complex variable $z = x + iy$ by

$$w = \frac{e^z - i}{e^z + i} .$$

Find $|w|$ as a function of x and y and deduce that $|w| < 1$ if $0 < y < \pi$.

Illustrate in a diagram in the w-plane the curves corresponding to the lines $y = 0$, $y = \frac{1}{2}\pi$ and $y = \pi$.

Find z as a function of w.

The curves $\mathcal{I}m(z) =$ constant are known as streamlines. Sketch these in the z-plane, and infer the streamlines in the w-plane.

2. Use the Residue Theorem to show that :

(i)

$$\oint_{C_1} \frac{z^3\,dz}{(z-i)^4} = 2\pi i ,$$

(ii)

$$\oint_{C_1} \frac{z\,dz}{(z-1)(z-i)^2} = 0 ,$$

where C_1 is the circle centred at $z = 0$ of radius 2.

(iii) If the circle C_1 is changed into a new circle C_2 centred at $z = 1$ of radius $1/2$, what is the value of the contour integral

$$\oint_{C_2} \frac{z\,dz}{(z-1)(z-i)^2} \ ?$$

The residue of a complex function $f(z)$ at a pole at $z = a$ of multiplicity m is given by

$$\lim_{z\to a} \frac{1}{(m-1)!} \left[\frac{d^{m-1}}{dz^{m-1}} \left\{ (z-a)^m f(z) \right\} \right] .$$

PLEASE TURN OVER

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3. The complex function

$$\frac{e^{iz}}{z(z^2+1)^2}$$

has a single pole at $z = 0$ and a double pole at $z = i$ in the upper half-plane.

(i) Show that the residue at $z = 0$ is 1.

(ii) Calculate the residue at $z = i$.

Now consider the contour integral

$$\oint_C \frac{e^{iz}dz}{z(z^2+1)^2},$$

where C is taken to be a closed semi-circle of radius R lying in the upper half-plane centred at $z = 0$, but with a small semi-circle of radius r ($r < R$), also centred at $z = 0$, indented into the lower half-plane.

(iii) Show that the contribution to the above integral from this indentation, in the limit $r \to 0$, is $i\pi$.

(iv) In the limit $R \to \infty$, show why the contribution from the arc of the larger semi-circle is zero.

(v) Hence deduce the value of

$$\int_{-\infty}^{\infty} \frac{\sin x \, dx}{x(x^2+1)^2}.$$

The residue of a complex function $f(z)$ *at a pole at* $z = a$ *of multiplicity* m *is given by*

$$\lim_{z\to a} \frac{1}{(m-1)!}\left[\frac{d^{m-1}}{dz^{m-1}}\left\{(z-a)^m f(z)\right\}\right].$$

4. Two functions of time $f(t)$ and $g(t)$ have Laplace Transforms $\mathcal{L}\{f(t)\} = \overline{f}(s)$ and $\mathcal{L}\{g(t)\} = \overline{g}(s)$ respectively. The convolution product between the two is defined by

$$f(t) * g(t) = \int_0^t f(u)g(t-u)du .$$

(i) Prove the Laplace convolution theorem

$$\mathcal{L}(f * g) = \overline{f}(s)\overline{g}(s) .$$

(ii) Find the inverse transforms of the following in terms of convolution integrals:

(a) $$\mathcal{L}^{-1}\left\{ \frac{\overline{f}(s)}{s^2 - 3s + 2} \right\} ,$$

(b) $$\mathcal{L}^{-1}\left\{ \frac{\overline{f}(s)}{s^2 + 2s + 5} \right\} .$$

5. Consider the ordinary differential equation

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 5x = F(t)$$

where $F(t)$ has a finite Laplace Transform $\overline{F}(s)$.

The initial condition for $\dfrac{dx}{dt}$ is

$$\frac{dx}{dt}(0) = -2x(0) .$$

Show that

(i) $$\overline{x}(s) = \frac{(s+2)}{s^2 + 4s + 5}\, x(0) + \frac{\overline{F}(s)}{s^2 + 4s + 5} ,$$

(ii) $$x(t) = x(0)\, p(t) + \int_0^t F(t-u)\, q(u)\, du ,$$

where p and q are functions to be determined.

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6. You are given a function $f(t)$ that is periodic in time with a period T such that $f(t) = f(t \pm nT)$ for $n = 1, 2, 3, \ldots$.

Show that the Laplace transform $\overline{f}(s)$, is given by

$$\overline{f}(s) = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t) dt \,, \quad (s > 0) \,.$$

If $f(t)$ is the periodic function of period T

$$f(t) = \begin{cases} e^{at} & 0 \le t \le \frac{1}{2}T \,, \\ 0 & \frac{1}{2}T \le t \le T \,, \end{cases}$$

find its Laplace transform for $s > 0$.

7. Consider a two-dimensional region R bounded by a closed piecewise smooth curve C. Using Green's Theorem in a plane (given below), choose the components of a vector field $\boldsymbol{u}(x, y)$ in terms of two arbitrary differentiable functions $P(x, y)$ and $Q(x, y)$ to prove the two-dimensional form of the Divergence Theorem

$$\oint_C \boldsymbol{u} \cdot \widehat{\boldsymbol{n}} \, ds = \int\!\!\int_R \operatorname{div} \boldsymbol{u} \, dx \, dy \tag{1}$$

where $\widehat{\boldsymbol{n}}$ is a unit vector normal to C.

Suppose R is the wedge-shaped region lying between the lines $y = ax$ and $y = bx$ $(0 < a < b)$ and bounded by the vertical line $x = 1$.

Sketch the region R.

If

$$\boldsymbol{u} = \boldsymbol{i}\, x^2 \sin y + \boldsymbol{j}\, x \cos y$$

show that the double integral on the right hand side of (1) is

$$\int\!\!\int_R \operatorname{div} \boldsymbol{u} \, dx \, dy = \frac{\sin a}{a} - \frac{\sin b}{b} + \frac{\cos a - 1}{a^2} - \frac{\cos b - 1}{b^2} \,.$$

Green's Theorem in a plane states that for a two-dimensional region R bounded by a closed, piecewise smooth curve C, then

$$\oint_C \{P(x, y)\, dx + Q(x, y)\, dy\} = \int\!\!\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \, dy \,.$$

8. (i) The region S is the sector of a disc lying between the lines $y = x$, $y = -x$ and the circle $x^2 + y^2 = a^2$ with $y \geq 0$.
Find the coordinates $(\overline{x}, \overline{y})$ of the centroid of S and the value I_{oz} of the moment of inertia about the z-axis of S.

Note: The coordinates of the centroid are given by

$$\overline{x} = \frac{1}{A} \iint_S x \, dx \, dy \, ,$$

$$\overline{y} = \frac{1}{A} \iint_S y \, dx \, dy \, ,$$

where A is the area of S.

The moment of inertia is given by

$$I_{0z} = \iint_S (x^2 + y^2) \, dx \, dy \, .$$

(ii) The integral $\iint_R (x^4 - y^4) \, dx \, dy$ is to be evaluated over the region R bounded by the four hyperbolae $xy = 1$, $xy = 2$, $x^2 - y^2 = 1$ and $x^2 - y^2 = -1$ with $x > 0$ and $y > 0$.

Sketch the region R and find the values of the new variables $u = xy$ and $v = x^2 - y^2$ on each part of the boundary of R.

Evaluate the integral by first changing to the new variables, and using the Jacobian for the transformation.

9. Show for any twice differentiable scalar field $\phi(x, y, z)$ and any twice differentiable vector field $\boldsymbol{E}(x, y, z)$ that

$$\text{curl grad } \phi = 0 \quad \text{and} \quad \text{div curl } \boldsymbol{E} = 0 \, .$$

Suppose that the vector field $\boldsymbol{E}(x, y, z)$ is given by

$$\boldsymbol{E} = (x^2 - y^2)\boldsymbol{i} + (axy + z^2)\boldsymbol{j} + (byz + z)\boldsymbol{k} \, .$$

Find the values of the constants a and b that make $\text{curl}\, \boldsymbol{E} = 0$.

For these values of a and b find the potential $\phi(x, y, z)$ that satisfies

$$\boldsymbol{E} = \text{grad}\, \phi \, .$$

PLEASE TURN OVER

10. Suppose R is a bounded simply-connected region in the x-y plane. Green's theorem in the plane states that

$$\oint_C (f\,dx + g\,dy) = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx\,dy$$

where C is the counter-clockwise boundary of R.

(i) Suppose there is a bounded simply-connected region R^* where the condition $\partial g/\partial x = \partial f/\partial y$ holds, and let C_1 be a path joining the points A and B. Use Green's theorem to show that the line integral

$$I = \int_{C_1} (f\,dx + g\,dy)$$

is independent of the path C_1, provided that the points A and B and the path lie in the region R^*.

Suppose now that the functions f and g are given by

$$f = \frac{x-y}{x^2+y^2}\,, \quad g = \frac{x+y}{x^2+y^2}$$

and the points A and B are $(1, 0)$ and $(0, 1)$ respectively.

(ii) Show that $\partial g/\partial x = \partial f/\partial y$ everywhere except at the origin $(0, 0)$.

(iii) Calculate I in the following cases:

(a) C_1 is the counter-clockwise quarter circle joining A to B along the curve $x^2 + y^2 = 1$;

(b) C_1 is the clockwise three quarter circle joining A to B along the curve $x^2 + y^2 = 1$.

(iv) Why are the answers to (a) and (b) different?

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11. A fair six-sided die is thrown twice.

    (i) Write down a suitable sample space Ω for this experiment that takes the order of the throws into account.

    (ii) Let A be the event that the first number thrown is even.
    Let B be the event that the sum of the numbers thrown is odd.

      (a) Write down A and B in set notation.

      (b) Are A and B independent? Justify your answer.

    (iii) Let C be the event that the sum of the numbers thrown equals 4.

      (a) Write down C in set notation.

      (b) Find the conditional probability of A given C.

    (iv) Using only the definition of conditional probability, prove Bayes' theorem

$$P(D|E) = \frac{P(E|D)\,P(D)}{P(E)}$$

    for events D, E with $P(E) > 0$ and $P(D) > 0$.

12. Consider the time series $\{y_t, t = 0, 1, \ldots\}$ given by

$$y_t = 1 + \sum_{s=0}^{t} e_s, \quad t = 0, 1, 2, \ldots$$

    where $\{e_t\}$ is white noise with $E(e_t) = 0$ and $\mathrm{Var}(e_t) = 1$.

    (i) Find $E(y_t)$ for $t = 0, 1, 2, \ldots$.

    (ii) Find $\mathrm{Var}(y_t)$ for $t = 0, 1, 2, \ldots$.

    (iii) Find the covariance $\mathrm{cov}(y_t, y_{t+s})$ of $\{y_t\}$ for $s = 1, 2, 3, \ldots$

    (iv) Define when a time series is called weakly stationary.

    (v) Is $\{y_t\}$ weakly stationary? Justify your answer.

    (vi) Define when a time series is called strictly stationary.

    (vii) Is $\{y_t\}$ strictly stationary? Justify your answer.

END OF PAPER

MATHEMATICS DEPARTMENT

MATHEMATICAL FORMULAE

1. VECTOR ALGEBRA

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1, a_2, a_3)$$

Scalar (dot) product: $\mathbf{a}\,.\,\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

Vector (cross) product:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Scalar triple product:

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a}\,.\,\mathbf{b} \times \mathbf{c} = \mathbf{b}\,.\,\mathbf{c} \times \mathbf{a} = \mathbf{c}\,.\,\mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product: $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c}\,.\,\mathbf{a})\mathbf{b} - (\mathbf{b}\,.\,\mathbf{a})\mathbf{c}$

2. SERIES

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \ldots \quad (\alpha \text{ arbitrary, } |x| < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \ldots,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^n \frac{x^{n+1}}{(n+1)} + \ldots (-1 < x \le 1)$$

3. TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin(a+b) = \sin a \cos b + \cos a \sin b\,;$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b\,.$$

$$\cos iz = \cosh z\,; \quad \cosh iz = \cos z\,; \quad \sin iz = i \sinh z\,; \quad \sinh iz = i \sin z\,.$$

4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f\,D^n g + \binom{n}{1} Df\,D^{n-1}g + \ldots + \binom{n}{r} D^r f\,D^{n-r}g + \ldots + D^n f\,g\,.$$

(b) Taylor's expansion of $f(x)$ about $x = a$:

$$f(a+h) = f(a) + hf'(a) + h^2 f''(a)/2! + \ldots + h^n f^{(n)}(a)/n! + \epsilon_n(h)\,,$$

where $\epsilon_n(h) = h^{n+1} f^{(n+1)}(a + \theta h)/(n+1)!, \quad 0 < \theta < 1\,.$

(c) Taylor's expansion of $f(x, y)$ about (a, b):

$$f(a+h, b+k) = f(a, b) + [hf_x + kf_y]_{a,b} + 1/2!\left[h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}\right]_{a,b} + \ldots$$

(d) Partial differentiation of $f(x, y)$:

i. If $y = y(x)$, then $f = F(x)$, and $\dfrac{dF}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dx}$.

ii. If $x = x(t)$, $y = y(t)$, then $f = F(t)$, and $\dfrac{dF}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$.

iii. If $x = x(u, v)$, $y = y(u, v)$, then $f = F(u, v)$, and

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}, \quad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$

(e) Stationary points of $f(x, y)$ occur where $f_x = 0$, $f_y = 0$ simultaneously.
Let (a, b) be a stationary point: examine $D = [f_{xx}f_{yy} - (f_{xy})^2]_{a,b}$.
If $D > 0$ and $f_{xx}(a, b) < 0$, then (a, b) is a maximum;
If $D > 0$ and $f_{xx}(a, b) > 0$, then (a, b) is a minimum;
If $D < 0$ then (a, b) is a saddle-point.

(f) Differential equations:

i. The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating factor $I(x) = \exp[\int P(x)(dx]$, so that $\dfrac{d}{dx}(Iy) = IQ$.

ii. $P(x, y)dx + Q(x, y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y$.

5. INTEGRAL CALCULUS

(a) An important substitution: $\tan(\theta/2) = t$:
$\sin\theta = 2t/(1+t^2)$, $\cos\theta = (1-t^2)/(1+t^2)$, $d\theta = 2\,dt/(1+t^2)$.

(b) Some indefinite integrals:

$$\int (a^2 - x^2)^{-1/2} dx = \sin^{-1}\left(\frac{x}{a}\right), \quad |x| < a .$$

$$\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1}\left(\frac{x}{a}\right) = \ln\left\{\frac{x}{a} + \left(1 + \frac{x^2}{a^2}\right)^{1/2}\right\} .$$

$$\int (x^2 - a^2)^{-1/2} dx = \cosh^{-1}\left(\frac{x}{a}\right) = \ln\left|\frac{x}{a} + \left(\frac{x^2}{a^2} - 1\right)^{1/2}\right| .$$

$$\int (a^2 + x^2)^{-1} dx = \left(\frac{1}{a}\right)\tan^{-1}\left(\frac{x}{a}\right) .$$

6. NUMERICAL METHODS

(a) Approximate solution of an algebraic equation:

If a root of $f(x) = 0$ occurs near $x = a$, take $x_0 = a$ and
$x_{n+1} = x_n - [f(x_n)/f'(x_n)]$, $n = 0, 1, 2 \ldots$

(Newton Raphson method).

(b) Formulae for numerical integration: Write $x_n = x_0 + nh$, $y_n = y(x_n)$.

i. Trapezium rule (1-strip): $\int_{x_0}^{x_1} y(x)dx \approx (h/2)\,[y_0 + y_1]$.

ii. Simpson's rule (2-strip): $\int_{x_0}^{x_2} y(x)dx \approx (h/3)\,[y_0 + 4y_1 + y_2]$.

(c) Richardson's extrapolation method: Let $I = \int_a^b f(x)dx$ and let I_1, I_2 be two estimates of I obtained by using Simpson's rule with intervals h and $h/2$. Then, provided h is small enough,

$$I_2 + (I_2 - I_1)/15 ,$$

is a better estimate of I .

7. LAPLACE TRANSFORMS

Function	Transform	Function	Transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)dt$	$af(t) + bg(t)$	$aF(s) + bG(s)$
df/dt	$sF(s) - f(0)$	d^2f/dt^2	$s^2F(s) - sf(0) - f'(0)$
$e^{at}f(t)$	$F(s-a)$	$tf(t)$	$-dF(s)/ds$
$(\partial/\partial\alpha)f(t,\alpha)$	$(\partial/\partial\alpha)F(s,\alpha)$	$\int_0^t f(t)dt$	$F(s)/s$
$\int_0^t f(u)g(t-u)du$	$F(s)G(s)$		
1	$1/s$	$t^n (n = 1, 2 \ldots)$	$n!/s^{n+1}$, $(s > 0)$
e^{at}	$1/(s-a)$, $(s > a)$	$\sin\omega t$	$\omega/(s^2+\omega^2)$, $(s > 0)$
$\cos\omega t$	$s/(s^2+\omega^2)$, $(s > 0)$	$H(t-T) = \begin{cases} 0, & t < T \\ 1, & t > T \end{cases}$	e^{-sT}/s , $(s, T > 0)$

8. FOURIER SERIES

If $f(x)$ is periodic of period $2L$, then $f(x+2L) = f(x)$, and

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} , \text{ where}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad n = 0, 1, 2, \ldots , \text{ and}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1, 2, 3, \ldots$$

Parseval's theorem

$$\frac{1}{L}\int_{-L}^{L} [f(x)]^2\, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right) .$$

1. **Probabilities for events**

For events A, B, and C $\quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$

More generally $\quad P(\bigcup A_i) = \sum P(A_i) - \sum P(A_i \cap A_j) + \sum P(A_i \cap A_j \cap A_k) - \cdots$

The odds in favour of A $\quad P(A) \,/\, P(\overline{A})$

Conditional probability $\quad P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$ provided that $P(B) > 0$

Chain rule $\quad P(A \cap B \cap C) = P(A)\,P(B \mid A)\,P(C \mid A \cap B)$

Bayes' rule $\quad P(A \mid B) = \dfrac{P(A)\,P(B \mid A)}{P(A)\,P(B \mid A) + P(\overline{A})\,P(B \mid \overline{A})}$

A and B are independent if $\quad P(B \mid A) = P(B)$

A, B, and C are independent if $\quad P(A \cap B \cap C) = P(A)P(B)P(C)$, and

$$P(A \cap B) = P(A)P(B), \quad P(B \cap C) = P(B)P(C), \quad P(C \cap A) = P(C)P(A)$$

2. **Probability distribution, expectation and variance**

The probability distribution for a discrete random variable X is called the probability mass function (pmf) and is the complete set of probabilities $\{p_x\} = \{P(X = x)\}$

Expectation $\quad E(X) = \mu = \sum_x x p_x$

For function $g(x)$ of x, $\quad E\{g(X)\} = \sum_x g(x) p_x$, $\quad$ so $\quad E(X^2) = \sum_x x^2 p_x$

Sample mean $\quad \overline{x} = \dfrac{1}{n} \sum_k x_k$ estimates μ from random sample $x_1, x_2, \ldots, x_n$

Variance $\quad \operatorname{var}(X) = \sigma^2 = E\{(X-\mu)^2\} = E(X^2) - \mu^2$

Sample variance $\quad s^2 = \dfrac{1}{n-1}\left\{\sum_k x_k^2 - \dfrac{1}{n}\left(\sum_j x_j\right)^2\right\}$ estimates σ^2

Standard deviation $\quad \operatorname{sd}(X) = \sigma$

If value y is observed with frequency n_y

$$n = \sum_y n_y, \quad \sum_k x_k = \sum_y y n_y, \quad \sum_k x_k^2 = \sum_y y^2 n_y$$

Skewness $\quad \beta_1 = E\left(\dfrac{X-\mu}{\sigma}\right)^3$ $\quad$ is estimated by $\quad \dfrac{1}{n-1} \sum \left(\dfrac{x_i - \overline{x}}{s}\right)^3$

Kurtosis $\quad \beta_2 = E\left(\dfrac{X-\mu}{\sigma}\right)^4 - 3$ $\quad$ is estimated by $\quad \dfrac{1}{n-1} \sum \left(\dfrac{x_i - \overline{x}}{s}\right)^4 - 3$

Sample median $\quad \widetilde{x}$ or x_{med}. Half the sample values are smaller and half larger

If the sample values $x_1, \ldots, x_n$ are ordered as $x_{(1)} \le x_{(2)} \le \cdots \le x_{(n)}$,
then $\widetilde{x} = x_{(\frac{n+1}{2})}$ if n is odd, and $\widetilde{x} = \frac{1}{2}\left(x_{(\frac{n}{2})} + x_{(\frac{n+2}{2})}\right)$ if n is even

α-quantile $Q(\alpha)$ is such that $P(X \le Q(\alpha)) = \alpha$

Sample α-quantile $\widehat{Q}(\alpha)$ Proportion α of the data values are smaller

Lower quartile $\mathrm{Q1} = \widehat{Q}(0.25)$ one quarter are smaller

Upper quartile $\mathrm{Q3} = \widehat{Q}(0.75)$ three quarters are smaller

Sample median $\tilde{x} = \widehat{Q}(0.5)$ estimates the population median $Q(0.5)$

3. Probability distribution for a continuous random variable

The cumulative distribution function (cdf) $F(x) = P(X \le x) = \int_{x_0=-\infty}^{x} f(x_0)\mathrm{d}x_0$

The probability density function (pdf) $f(x) = \dfrac{\mathrm{d}F(x)}{\mathrm{d}x}$

$E(X) = \mu = \int_{-\infty}^{\infty} x\, f(x)\mathrm{d}x$, $\mathrm{var}\,(X) = \sigma^2 = E(X^2) - \mu^2$, where $E(X^2) = \int_{-\infty}^{\infty} x^2\, f(x)\mathrm{d}x$

4. Discrete probability distributions

Discrete Uniform *Uniform* (n)

$$p_x = \frac{1}{n} \quad (x = 1, 2, \ldots, n) \qquad \mu = (n+1)/2,\ \sigma^2 = (n^2-1)/12$$

Binomial distribution *Binomial* (n, θ)

$$p_x = \binom{n}{x} \theta^x (1-\theta)^{n-x} \quad (x = 0, 1, 2, \ldots, n) \qquad \mu = n\theta\,,\ \sigma^2 = n\theta(1-\theta)$$

Poisson distribution *Poisson* (λ)

$$p_x = \frac{\lambda^x e^{-\lambda}}{x!} \quad (x = 0, 1, 2, \ldots) \quad (\text{with } \lambda > 0) \qquad \mu = \lambda\,,\ \sigma^2 = \lambda$$

Geometric distribution *Geometric* (θ)

$$p_x = (1-\theta)^{x-1}\theta \quad (x = 1, 2, 3, \ldots) \qquad \mu = \frac{1}{\theta}\,,\ \sigma^2 = \frac{1-\theta}{\theta^2}$$

5. Continuous probability distributions

Uniform distribution *Uniform* (α, β)

$$f(x) = \begin{cases} \dfrac{1}{\beta - \alpha} & (\alpha < x < \beta), \\ 0 & (\text{otherwise}). \end{cases} \qquad \mu = (\alpha+\beta)/2\,,\ \sigma^2 = (\beta-\alpha)^2/12$$

Exponential distribution *Exponential* (λ)

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & (0 < x < \infty), \\ 0 & (-\infty < x \le 0). \end{cases} \qquad \mu = 1/\lambda\,,\ \sigma^2 = 1/\lambda^2$$

Normal distribution $N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\} \quad (-\infty < x < \infty), \quad E(X) = \mu, \quad \mathrm{var}(X) = \sigma^2$$

Standard normal distribution $N(0,1)$

If X is $N(\mu, \sigma^2)$, then $Y = \dfrac{X-\mu}{\sigma}$ is $N(0,1)$

6. Reliability

For a device in continuous operation with failure time random variable T having pdf $f(t)$ $(t > 0)$

The reliability function at time t	$R(t)$	$= P(T > t)$
The failure rate or hazard function	$h(t)$	$= f(t)/R(t)$
The cumulative hazard function	$H(t)$	$= \int_0^t h(t_0)\,dt_0 = -\ln\{R(t)\}$
The Weibull(α, β) distribution has	$H(t)$	$= \beta t^\alpha$

7. System reliability

For a system of k devices, which operate independently, let

$R_i = P(D_i) = P($"device i operates"$)$

The system reliability, R, is the probability of a path of operating devices

A system of devices in series operates only if every device operates

$$R = P(D_1 \cap D_2 \cap \cdots \cap D_k) = R_1 R_2 \cdots R_k$$

A system of devices in parallel operates if any device operates

$$R = P(D_1 \cup D_2 \cup \cdots \cup D_k) = 1 - (1-R_1)(1-R_2)\cdots(1-R_k)$$

8. Covariance and correlation

The covariance of X and Y $\quad \mathrm{cov}(X,Y) = E(XY) - \{E(X)\}\{E(Y)\}$

From pairs of observations $(x_1, y_1), \ldots, (x_n, y_n)$ $\quad S_{xy} = \sum_k x_k y_k - \frac{1}{n}(\sum_i x_i)(\sum_j y_j)$

$$S_{xx} = \sum_k x_k^2 - \frac{1}{n}(\sum_i x_i)^2, \qquad S_{yy} = \sum_k y_k^2 - \frac{1}{n}(\sum_j y_j)^2$$

Sample covariance	s_{xy}	$= \dfrac{1}{n-1} S_{xy}$	estimates $\mathrm{cov}(X,Y)$
Correlation coefficient	ρ	$= \mathrm{corr}(X,Y) = \dfrac{\mathrm{cov}(X,Y)}{\mathrm{sd}(X)\cdot\mathrm{sd}(Y)}$	
Sample correlation coefficient	r	$= \dfrac{S_{xy}}{\sqrt{S_{xx}S_{yy}}}$	estimates ρ

9. Sums of random variables

$$E(X+Y) = E(X)+E(Y)$$

$$\text{var}(X+Y) = \text{var}(X)+\text{var}(Y)+2\,\text{cov}(X,Y)$$

$$\text{cov}(aX+bY,\ cX+dY) = (ac)\,\text{var}(X)+(bd)\,\text{var}(Y)+(ad+bc)\,\text{cov}(X,Y)$$

If X is $N(\mu_1,\sigma_1^2)$, Y is $N(\mu_2,\sigma_2^2)$, and $\text{cov}(X,Y)=c$, then $X+Y$ is $N(\mu_1+\mu_2,\ \sigma_1^2+\sigma_2^2+2c)$

10. Bias, standard error, mean square error

If t estimates θ (with random variable T giving t)

Bias of t	$\text{bias}(t)$	$= E(T)-\theta$
Standard error of t	$\text{se}(t)$	$= \text{sd}(T)$
Mean square error of t	$\text{MSE}(t)$	$= E\{(T-\theta)^2\} = \{\text{se}(t)\}^2+\{\text{bias}(t)\}^2$

If $\overline{x}$ estimates μ, then $\text{bias}(\overline{x})=0$, $\text{se}(\overline{x})=\sigma/\sqrt{n}$, $\text{MSE}(\overline{x})=\sigma^2/n$, $\widehat{\text{se}}(\overline{x})=s/\sqrt{n}$

Central limit property   If n is fairly large, $\overline{x}$ is from $N(\mu,\ \sigma^2/n)$ approximately

11. Likelihood

The likelihood is the joint probability as a function of the unknown parameter θ.

For a random sample $x_1,x_2,\ldots,x_n$

$$\ell(\theta;\,x_1,x_2,\ldots,x_n) = P(X_1=x_1\,|\,\theta)\cdots P(X_n=x_n\,|\,\theta) \qquad \text{(discrete distribution)}$$

$$\ell(\theta;\,x_1,x_2,\ldots,x_n) = f(x_1\,|\,\theta)\,f(x_2\,|\,\theta)\cdots f(x_n\,|\,\theta) \qquad \text{(continuous distribution)}$$

The maximum likelihood estimator (MLE) is $\widehat{\theta}$ for which the likelihood is a maximum

12. Confidence intervals

If $x_1,x_2,\ldots,x_n$ are a random sample from $N(\mu,\sigma^2)$ and σ^2 is known, then the 95% confidence interval for μ is $(\overline{x}-1.96\frac{\sigma}{\sqrt{n}},\ \overline{x}+1.96\frac{\sigma}{\sqrt{n}})$

If σ^2 is estimated, then from the Student t table for t_{n-1} we find $t_0=t_{n-1,0.05}$

The 95% confidence interval for μ is $(\overline{x}-t_0\frac{s}{\sqrt{n}},\ \overline{x}+t_0\frac{s}{\sqrt{n}})$

13. Standard normal table — Values of pdf $\phi(y) = f(y)$ and cdf $\Phi(y) = F(y)$

y	$\phi(y)$	$\Phi(y)$	y	$\phi(y)$	$\Phi(y)$	y	$\phi(y)$	$\Phi(y)$	y	$\Phi(y)$
0	.399	.5	.9	.266	.816	1.8	.079	.964	2.8	.997
.1	.397	.540	1.0	.242	.841	1.9	.066	.971	3.0	.999
.2	.391	.579	1.1	.218	.864	2.0	.054	.977	0.841	.8
.3	.381	.618	1.2	.194	.885	2.1	.044	.982	1.282	.9
.4	.368	.655	1.3	.171	.903	2.2	.035	.986	1.645	.95
.5	.352	.691	1.4	.150	.919	2.3	.028	.989	1.96	.975
.6	.333	.726	1.5	.130	.933	2.4	.022	.992	2.326	.99
.7	.312	.758	1.6	.111	.945	2.5	.018	.994	2.576	.995
.8	.290	.788	1.7	.094	.955	2.6	.014	.995	3.09	.999

14. Student t table — Values $t_{m,p}$ of x for which $P(|X| > x) = p$, when X is t_m

m	p= 0.10	0.05	0.02	0.01	m	p= 0.10	0.05	0.02	0.01
1	6.31	12.71	31.82	63.66	9	1.83	2.26	2.82	3.25
2	2.92	4.30	6.96	9.92	10	1.81	2.23	2.76	3.17
3	2.35	3.18	4.54	5.84	12	1.78	2.18	2.68	3.05
4	2.13	2.78	3.75	4.60	15	1.75	2.13	2.60	2.95
5	2.02	2.57	3.36	4.03	20	1.72	2.09	2.53	2.85
6	1.94	2.45	3.14	3.71	25	1.71	2.06	2.48	2.78
7	1.89	2.36	3.00	3.50	40	1.68	2.02	2.42	2.70
8	1.86	2.31	2.90	3.36	∞	1.645	1.96	2.326	2.576

15. Chi-squared table — Values $\chi^2_{k,p}$ of x for which $P(X > x) = p$, when X is χ^2_k and $p = .995, .975$, *etc*

k	.995	.975	.05	.025	.01	.005	k	.995	.975	.05	.025	.01	.005
1	.000	.001	3.84	5.02	6.63	7.88	18	6.26	8.23	28.87	31.53	34.81	37.16
2	.010	.051	5.99	7.38	9.21	10.60	20	7.43	9.59	31.42	34.17	37.57	40.00
3	.072	.216	7.81	9.35	11.34	12.84	22	8.64	10.98	33.92	36.78	40.29	42.80
4	.207	.484	9.49	11.14	13.28	14.86	24	9.89	12.40	36.42	39.36	42.98	45.56
5	.412	.831	11.07	12.83	15.09	16.75	26	11.16	13.84	38.89	41.92	45.64	48.29
6	.676	1.24	12.59	14.45	16.81	18.55	28	12.46	15.31	41.34	44.46	48.28	50.99
7	.990	1.69	14.07	16.01	18.48	20.28	30	13.79	16.79	43.77	46.98	50.89	53.67
8	1.34	2.18	15.51	17.53	20.09	21.95	40	20.71	24.43	55.76	59.34	63.69	66.77
9	1.73	2.70	16.92	19.02	21.67	23.59	50	27.99	32.36	67.50	71.41	76.15	79.49
10	2.16	3.25	13.31	20.48	23.21	25.19	60	35.53	40.48	79.08	83.30	88.38	91.95
12	3.07	4.40	21.03	23.34	26.22	28.30	70	43.28	48.76	90.53	95.02	100.4	104.2
14	4.07	5.63	23.68	26.12	29.14	31.32	80	51.17	57.15	101.9	106.6	112.3	116.3
16	5.14	6.91	26.30	28.85	32.00	34.27	100	67.33	74.22	124.3	129.6	135.8	140.2

16. The chi-squared goodness-of-fit test

The frequencies n_y are grouped so that the fitted frequency $\widehat{n}_y$ for every group exceeds about 5.

$X^2 = \sum_y \frac{(n_y - \widehat{n}_y)^2}{\widehat{n}_y}$ is referred to the table of χ^2_k with significance point p,

where k is the number of terms summed, less one for each constraint, *eg* matching total frequency, and matching $\overline{x}$ with μ

17. Joint probability distributions

Discrete distribution $\{p_{xy}\}$, where $p_{xy} = P(\{X = x\} \cap \{Y = y\})$.

Let $p_{x\bullet} = P(X = x)$, and $p_{\bullet y} = P(Y = y)$, then

$$p_{x\bullet} = \sum_y p_{xy} \quad \text{and} \quad P(X = x \,|\, Y = y) = \frac{p_{xy}}{p_{\bullet y}}$$

Continuous distribution

Joint cdf $\quad F(x,y) = P(\{X \le x\} \cap \{Y \le y\}) = \int_{x_0=-\infty}^{x} \int_{y_0=-\infty}^{y} f(x_0, y_0)\, dx_0\, dy_0$

Joint pdf $\quad f(x,y) = \frac{d^2 F(x,y)}{dx\, dy}$

Marginal pdf of X $\quad f_X(x) = \int_{-\infty}^{\infty} f(x, y_0)\, dy_0$

Conditional pdf of X given $Y = y$ $\quad f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$ (provided $f_Y(y) > 0$)

18. Linear regression

To fit the linear regression model $y = \alpha + \beta x$ by $\widehat{y}_x = \widehat{\alpha} + \widehat{\beta} x$ from observations $(x_1, y_1), \ldots, (x_n, y_n)$, the least squares fit is $\quad \widehat{\alpha} = \overline{y} - \overline{x}\widehat{\beta}, \quad \widehat{\beta} = \frac{S_{xy}}{S_{xx}}$

The residual sum of squares $\quad \text{RSS} = S_{yy} - \frac{S_{xy}^2}{S_{xx}}$

$$\widehat{\sigma^2} = \frac{\text{RSS}}{n-2} \qquad \frac{n-2}{\sigma^2}\,\widehat{\sigma^2} \text{ is from } \chi^2_{n-2}$$

$$E(\widehat{\alpha}) = \alpha, \quad E(\widehat{\beta}) = \beta,$$

$$\text{var}(\widehat{\alpha}) = \frac{\sum x_i^2}{n\, S_{xx}}\sigma^2, \quad \text{var}(\widehat{\beta}) = \frac{\sigma^2}{S_{xx}}, \quad \text{cov}(\widehat{\alpha}, \widehat{\beta}) = -\frac{\overline{x}}{S_{xx}}\sigma^2$$

$$\widehat{y}_x = \widehat{\alpha} + \widehat{\beta} x, \qquad E(\widehat{y}_x) = \alpha + \beta x, \qquad \text{var}(\widehat{y}_x) = \left\{\frac{1}{n} + \frac{(x - \overline{x})^2}{S_{xx}}\right\}\sigma^2$$

$$\frac{\widehat{\alpha} - \alpha}{\widehat{\text{se}}(\widehat{\alpha})}, \quad \frac{\widehat{\beta} - \beta}{\widehat{\text{se}}(\widehat{\beta})}, \quad \frac{\widehat{y}_x - \alpha - \beta x}{\widehat{\text{se}}(\widehat{y}_x)} \quad \text{are each from } t_{n-2}$$

Engineering 2 Core, Solutions:

C8. We have $e^z = e^x(\cos y + i\sin y)$ and hence

$$|W|^2 = \frac{e^{2x}\cos^2 y + (e^x \sin y - 1)^2}{e^{2x}\cos^2 y + (e^x \sin y + 1)^2} = \frac{e^{2x} + 1 - 2e^x \sin y}{e^{2x} + 1 + 2e^x \sin y}.$$

Now if $0 < y < \pi$ then $0 < \sin y < 1$. The denominator is thus the sum of two positive numbers and the numerator is their (smaller) difference. So we have $|W| < 1$. 5

If $y = \frac{1}{2}\pi$ then $e^z = ie^x$ and $W = (e^x - 1)/(e^x + 1) = \tanh(x/2)$ which is real. As $x \to \pm\infty, W \to \pm 1$. So we have the line segment with $|W| \leqslant 1$ (W real). 3

If $y = 0$, i.e. $= (e^x - i)/(e^x + i) = (e^x - i)^2/(e^{2x} + 1)$. Thus $|W| = 1$ with $\Im m(W) < 0$, (lower) half of the unit circle. If $y = \pi$ then we have the other (upper) half of the unit circle. See figure. 4

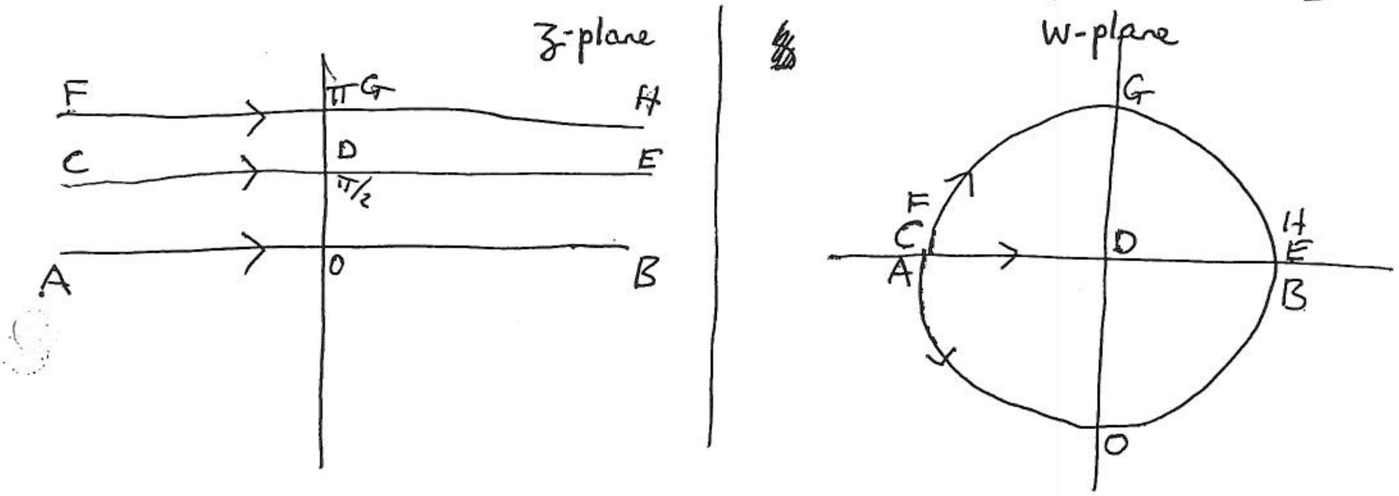
Now

$$W = \frac{e^z - i}{e^z + i} \quad \Longrightarrow \quad e^z = i\frac{1+W}{1-W}$$

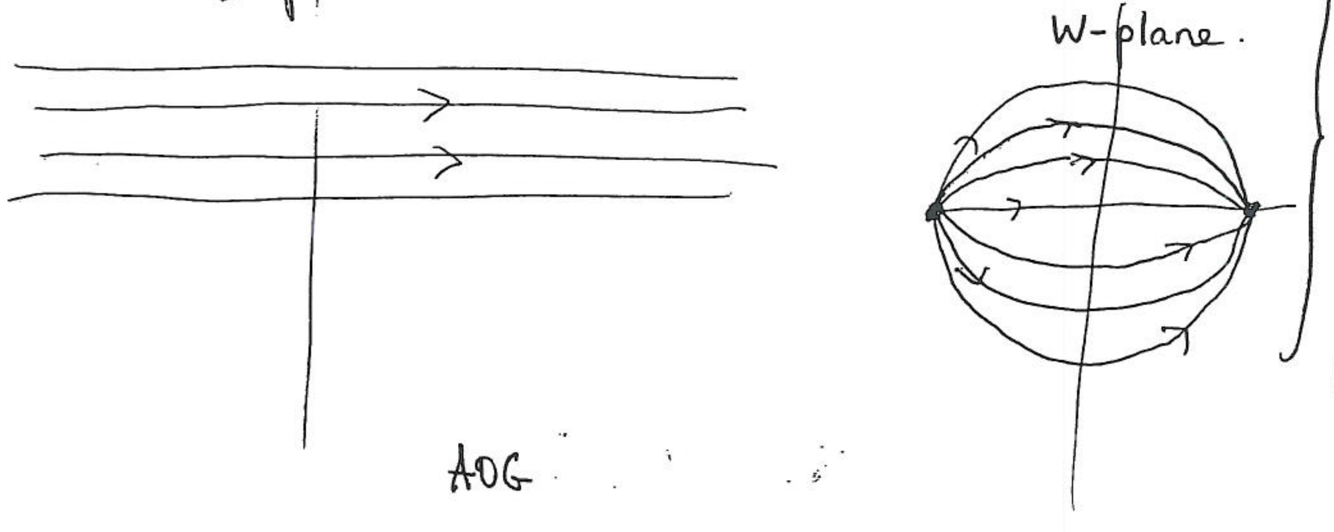
Thus

$$z = \log i + \log(1+W) - \log(1-W) \quad \Longrightarrow \quad Z = \tfrac{1}{2}i\pi + \log\frac{1+W}{1-W}$$ 4

Now in the z-plane [redacted] $Im(Z) = const$ is uniform flow in the x-direction. In the W-plane, all the streamlines emerge from the point $W = -1$, which corresponds to $x = -\infty$ and head into the point $W = 1$ ($x = +\infty$). [Thus the flow is of a source/sink pair, as illustrated in the diagram]



Z-plane Streamlines for final part



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AOG

	EXAMINATION QUESTIONS/SOLUTIONS 2008-09	Course (3) EE2
Soln		
Question 2		Marks & seen/unseen

Parts

(i) The residue at the pole of multiplicity 4 at $z=i$ is $\lim_{z\to i}\left\{\frac{1}{3!}\frac{d^3}{dz^3}\left[(z-i)^4\frac{z^3}{(z-i)^4}\right]\right\}$

$$= \lim_{z\to i}\frac{1}{3!}\frac{d^3}{dz^3}z^3 = 1$$ 4

$\therefore$ By R.T. $\oint_{c_1} = 2\pi i\times 1 = 2\pi i$ 2

(ii) Residue at the simple pole at $z=1$ is

$$\lim_{z\to 1}\left\{(z-1)\frac{z}{(z-1)(z-i)^2}\right\} = \frac{1}{(1-i)^2} = \frac{1}{-2i} = i/2$$ 3

Residue at the double pole at $z=i$ is

$$\lim_{z\to i}\frac{d}{dz}\left\{(z-i)^2\frac{z}{(z-1)(z-i)^2}\right\} = \lim_{z\to i}\frac{d}{dz}\left\{\frac{z}{z-1}\right\}$$

$$= \lim_{z\to i}\left\{\frac{1.(z-1)-z}{(z-1)^2}\right\} = -\frac{1}{(i-1)^2}$$ 5

$$= \frac{1}{2i} = -i/2$$

$\therefore$ $\oint_{c_1} = 2\pi i\{i/2 - i/2\} = 0$ 2

by the Residue Theorem.

(iii) For the new contour, the pole at $z=1$ is included by the double pole at $z=i$ is excluded so 4

$$\oint_{c_1} = 2\pi i\times i/2 = -\pi$$

20 ok

Setter's initials	Checker's initials	Page number
JDG	AOG	1

EXAMINATION QUESTIONS/SOLUTIONS 2008-09

Course (3) EE2

Soln.

Question 3

Marks & seen/unseen

Parts

(i) Residue at $z=0$ is $\lim_{z\to 0}\left\{\dfrac{z.e^{iz}}{z(z^2+1)^2}\right\} = 1$ — 3

(ii) " " $z=i$ is $\lim_{z\to i}\dfrac{d}{dz}\left\{\dfrac{(z-i)^2e^{iz}}{z(z+i)^2(z-i)^2}\right\}$

$= \lim_{z\to i}\dfrac{d}{dz}\left\{\dfrac{e^{iz}}{z(z+i)^2}\right\} = e^{-1}\left\{\dfrac{4+4+4}{(4i)^2}\right\} = -\dfrac{3}{4e}$ — 5

(iii) $(z=re^{i\theta})$ $(dz = rie^{i\theta}d\theta)$

$\lim_{r\to 0}\displaystyle\int_{\pi}^{2\pi}\dfrac{e^{ire^{i\theta}}\,i\cancel{r}e^{i\theta}d\theta}{\cancel{r}e^{i\theta}(r^2e^{2i\theta}+1)} = i\int_{\pi}^{2\pi}d\theta = \pi i$ — 3

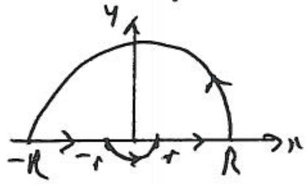
(iv) Use Jordan's Lemma: conditions for its use are

i) Only singularities are poles: yes ✓

ii) $m=1$ ✓

iii) $\left|\dfrac{e^{iz}}{z(z^2+1)^2}\right| \to 0$ as $R\to\infty$ fast enough. Yes. ✓

Then $\lim_{R\to\infty}\displaystyle\int\dfrac{e^{iz}}{z(z^2+1)^2}dz = 0$ — 4



(v) By the Residue Thm

$$\oint_C \frac{e^{iz}dz}{z(z^2+1)^2} = 2\pi i\left(1-\frac{3}{4e}\right)$$ — 2

$$= \int_{-\infty}^{\infty}\frac{e^{ix}dx}{x(x^2+1)^2} + 0 + \pi i$$

$$\therefore \pi i\left(1-\frac{3}{2e}\right) = \int_{-\infty}^{\infty}\frac{(\cos x + i\sin x)dx}{x(x^2+1)^2}$$ — 2

The cos-integral is zero as integrand is odd.

$$\therefore \int_{-\infty}^{\infty}\frac{\sin x\,dx}{x(x^2+1)^2} = \frac{\pi}{2e}(2e-3)$$ — 1

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OK

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Checker's initials: AOG

EXAMINATION QUESTIONS/SOLUTIONS 2008-09

Course (3) EE2

Soln

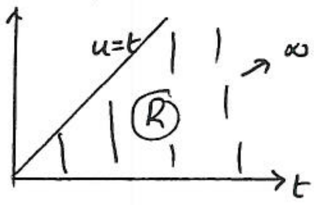
Question 4

Marks & seen/unseen

Parts

a) $\mathcal{L}(f*g) = \int_0^\infty e^{-st}\left\{\int_0^t f(u)g(t-u)\,du\right\}dt$ — 2

re-arranging order of integn.



$= \int_0^\infty f(u)\left(\int_u^\infty e^{-st} g(t-u)\,dt\right)du$ — 2 (pic)

Substitute $t-u=\tau$

$\mathcal{L}(f*g) = \int_0^\infty f(u)\left(\int_0^\infty e^{-s(u+\tau)} g(\tau)\,d\tau\right)du$ — 4

$= \left(\int_0^\infty f(u)e^{-su}\,du\right)\left(\int_0^\infty e^{-s\tau} g(\tau)\,d\tau\right) = \bar{f}(s)\bar{g}(s)$

b) (i) $\bar{g}(s) = \dfrac{1}{s^2-3s+2} = \dfrac{1}{(s-1)(s-2)} = \dfrac{1}{s-2} - \dfrac{1}{s-1}$

$\therefore g(t) = e^{2t} - e^t$ — 6

$\therefore \mathcal{L}^{-1}\{\bar{f}(s)\bar{g}(s)\} = f(t)*g(t)$

$= \int_0^t f(t-u)\left(e^{2u}-e^u\right)du.$

(ii) $\bar{g}(s) = \dfrac{1}{s^2+2s+5} = \dfrac{1}{(s+1)^2+2^2} = \dfrac{1}{2}\left\{\dfrac{2}{(s+1)^2+2^2}\right\}.$

$\therefore g(t) = \frac{1}{2}e^{-t}\sin 2t$ by Shift Thm. — 6

$\therefore \mathcal{L}^{-1}\{\cdot\} = \frac{1}{2}\int_0^t f(t-u)e^{-u}\sin 2u\,du.$

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EXAMINATION QUESTIONS/SOLUTIONS 2008-09

Course: (3) EE 2

Question: 5

Marks & seen/unseen

Parts

L.T. $\ddot{x} + 4\dot{x} + 5x = F(t)$ to get

$$[s^2 \bar{x}(s) - s\,x(0) - \dot{x}(0)] + 4[s\bar{x}(s) - x(0)] + 5\bar{x} = \bar{F}(s) \qquad 3$$

$$\therefore \bar{x}(s)(s^2 + 4s + 5) = (s+4)x(0) + \dot{x}(0) + \bar{F}(s)$$

$$= (s+2)\,x(0) + \bar{F}(s) \qquad 3$$

Use ICs to get:

$$\therefore \bar{x}(s) = \frac{s+2}{s^2+4s+5}\,x(0) + \frac{\bar{F}(s)}{s^2+4s+5} \quad \text{as requested.} \qquad 4$$

Re-write this as

$$\bar{x}(s) = \frac{s+2}{(s+2)^2+1}\,x(0) + \frac{\bar{F}(s)}{(s+2)^2+1} \qquad 3$$

(which is the answer to (i))

Now $\mathcal{L}(\sin t) = \frac{1}{s^2+1}$; $\mathcal{L}(\cos t) = \frac{s}{s^2+1}$

$\therefore$ Using the shift + convolution thms, we have $\qquad 3 + 3$

$$x(t) = x(0)e^{-2t}\cos t + e^{-2t}\sin t * F(t)$$

or

$$x(t) = x(0)e^{-2t}\cos t + \int_0^t F(t-u)\,e^{-2u}\sin u\,du \qquad 1$$

i.e. $p(t) = e^{-2t}\cos t$

& $q(u) = e^{-2u}\sin u$

answer to (ii)

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Checker's initials: AOG

Page number

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	EXAMINATION QUESTIONS/SOLUTIONS 2008-09	Course (3) EE2
Question 6		Marks & seen/unseen
Parts	$\int_0^\infty = \int_0^T + \sum_{n=1}^{\infty} \int_{nT}^{(n+1)T}$ and, with $\tau_n = t - nT$ $\int_{nT}^{(n+1)T} e^{-st} f(t)\,dt = \int_0^T e^{-s(\tau_n + nT)} f(\tau_n + nT)\,d\tau_n$ periodic $= e^{-snT} \int_0^T e^{-s\tau_n} f(\tau_n)\,d\tau_n$	4 4
	$\therefore \int_0^\infty = \left(\sum_{n=0}^{\infty} e^{-snT}\right) \int_0^T$ $= \dfrac{1}{1-e^{-sT}} \int_0^T$ Only for $s > 0$	4
	If $f(t) = \begin{cases} e^{at} & 0 \le t \le \frac{1}{2}T \\ 0 & \frac{1}{2}T \le t \le T \end{cases}$ then $\int_0^T f(t) e^{-st}\,dt = \int_0^{\frac{1}{2}T} e^{-(s-a)t}\,dt$ $= -\frac{1}{s-a}\left[e^{-(s-a)t}\right]_0^{\frac{1}{2}T}$ $= \dfrac{1 - e^{-\frac{1}{2}(s-a)T}}{s-a}$	6
	$\therefore \int_0^\infty f(t) e^{-st}\,dt = \dfrac{1}{s-a}\left(\dfrac{1 - e^{-\frac{1}{2}(s-a)T}}{1 - e^{-sT}}\right)$	2
		20
	Setter's initials JDG Checker's initials AOG	Page number 1

EXAMINATION QUESTIONS/SOLUTIONS 2008-09

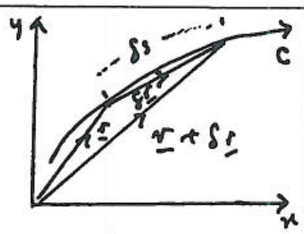
Course (3) EE2

Question 7

Marks & seen/unseen

Parts

Arc length δs; chord vector $\delta \underline{r}$



Unit tangent vector $\underline{\hat{t}} = d\underline{r}/ds$

$\therefore \underline{\hat{t}} = \underline{\hat{i}}\frac{dx}{ds} + \underline{\hat{j}}\frac{dy}{ds}$ $\therefore$ construct $\underline{\hat{n}}$ — 4 (fig)

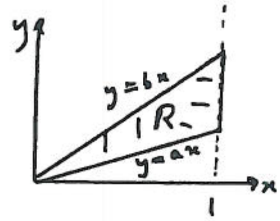
from $\underline{\hat{n}}\cdot\underline{\hat{t}} = 0 \Rightarrow \underline{\hat{n}} = \pm\left(\underline{\hat{i}}\frac{dy}{ds} - \underline{\hat{j}}\frac{dx}{ds}\right)$ + outer normal — 2

Now try $\underline{u} = \underline{\hat{i}}Q - \underline{\hat{j}}P \Rightarrow \operatorname{div}\underline{u} = Q_x - P_y$ — 2

and $\underline{u}\cdot\underline{\hat{n}} = Q\frac{dy}{ds} + P\frac{dx}{ds}$. Thus, from G.T. — 2

$$\oint_C P\,dx + Q\,dy = \oint_C \underline{u}\cdot\underline{\hat{n}}\,ds = \iint_R (Q_x - P_y)\,dx\,dy = \iint_R \operatorname{div}\underline{u}\,dx\,dy$$ — 2

If $\underline{u} = \underline{\hat{i}}\,x^2\sin y + x\cos y\,\underline{\hat{j}}$ — 2 (fig)



$\therefore \operatorname{div}\underline{u} = 2x\sin y - x\sin y = x\sin y$

$\therefore \iint_R \operatorname{div}\underline{u}\,dx\,dy = \int_0^1 x\left(\int_{ax}^{bx}\sin y\,dy\right)dx$ — 2

$= \int_0^1 x(\cos ax - \cos bx)\,dx$ — 1

Now $\int_0^1 x\cos ax\,dx = \frac{1}{a}\int x\,d(\sin ax)$

$= \frac{1}{a}\left\{[x\sin ax]_0^1 + \frac{1}{a}[\cos ax]_0^1\right\} = \frac{1}{a}\left\{\sin a + \frac{1}{a}(\cos a - 1)\right\}$ — 3

$$\therefore \iint_R \operatorname{div}\underline{u}\,dx\,dy = \frac{\sin a}{a} - \frac{\sin b}{b} + \frac{\cos a - 1}{a^2} - \frac{\cos b - 1}{b^2}$$

20

Setter's initials: JDG

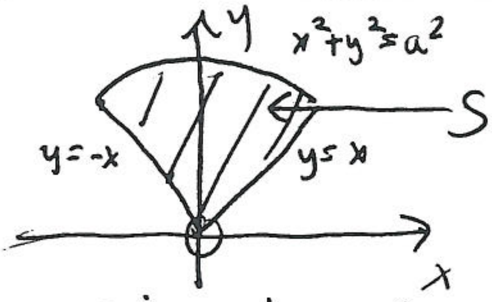
Checker's initials: ADG

EE II (3) (8)

	EXAMINATION QUESTIONS/SOLUTIONS 2008-09	Course
SOLN		C3
~~Question~~ C3		Marks & seen/unseen

Parts

(i)



S is a $\frac{1}{4}$ circle $\therefore$ $A = \frac{1}{4}\pi a^2$ — 2

$\bar{x} = \frac{1}{A}\iint_S x \, dx dy = 0$ — 2

by symmetry or by (longer) calculation

$\bar{y} = \frac{1}{A}\iint_S y \, dx dy$

$= \frac{1}{A}\int_{\theta=\frac{\pi}{4}}^{\frac{3\pi}{4}}\int_{r=0}^{a} r\sin\theta \; r \, dr \, d\theta$ (polars)

$= \frac{1}{A}\frac{a^3}{3}\left[-\cos\theta\right]_{\pi/4}^{3\pi/4} = \frac{4\sqrt{2}}{\pi a^2}\frac{a^3}{3}$ ✓ — 3

$= \frac{4\sqrt{2}}{3\pi}a.$

$I_{0z} = \iint_S (x^2+y^2)\, dx dy$

$= \int_{\theta=\frac{\pi}{4}}^{3\pi/4}\int_{r=0}^{a} r^2 \, r \, dr d\theta$ (polars)

$= \frac{\pi}{2}\frac{a^4}{4} = \frac{\pi a^4}{8}$ ✓ — 3

Setter's initials	Checker's initials	Page number
RLS	AOG	1

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EXAMINATION QUESTIONS/SOLUTIONS 2008-09

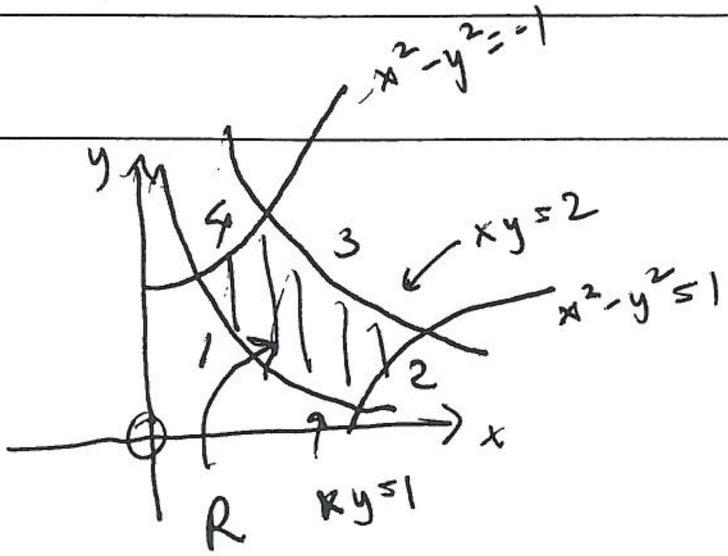
Course: CORE

SOLN

Question: C3

Marks & seen/unseen

Parts: (ii)



2

On 1 $u=1$

On 3 $u=2$

On 2 $v=1$

On 4 $v=-1$

2

$$J' = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} y & x \\ 2x & -2y \end{vmatrix} = -2(y^2+x^2)$$ 2

$$J = 1/J' = -\frac{1}{2(x^2+y^2)}$$ 1

$$\text{Int} = \iint_{R^*} (x^4-y^4)\,|J|\,du\,dv$$ 1

$$= \iint_{R^*} \frac{1}{2}(x^2-y^2)\,du\,dv$$ 1

$$= \int_{v=-1}^{1}\int_{u=1}^{2} \frac{1}{2} v\,du\,dv = \underline{\underline{0}}$$ 1

Total: 20

Setter's initials: RW

Checker's initials

	EXAMINATION QUESTIONS/SOLUTIONS 2008-09	Course CORE
SOLN		
Question C4		Marks & seen/unseen

Parts

$$\text{grad}\,\varphi = \underset{\sim}{i}\frac{\partial\varphi}{\partial x} + \underset{\sim}{j}\frac{\partial\varphi}{\partial y} + \underset{\sim}{k}\frac{\partial\varphi}{\partial z}$$

$$\text{curl grad}\,\varphi = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial_x & \partial_y & \partial_z \\ \frac{\partial\varphi}{\partial x} & \frac{\partial\varphi}{\partial y} & \frac{\partial\varphi}{\partial z} \end{vmatrix}$$

$$= \underset{\sim}{i}\left[\partial_y\left(\frac{\partial\varphi}{\partial z}\right) - \partial_z\left(\frac{\partial\varphi}{\partial y}\right)\right] - \underset{\sim}{j}\left[\partial_x\left(\frac{\partial\varphi}{\partial z}\right) - \partial_z\left(\frac{\partial\varphi}{\partial x}\right)\right]$$
$$+ \underset{\sim}{k}\left[\partial_x\left(\frac{\partial\varphi}{\partial y}\right) - \partial_y\left(\frac{\partial\varphi}{\partial x}\right)\right]$$ 5

Each square bracket vanishes because

$\frac{\partial}{\partial y}\left(\frac{\partial\varphi}{\partial z}\right) = \frac{\partial}{\partial z}\left(\frac{\partial\varphi}{\partial y}\right)$ and two other similar relations.

Hence result

$$\text{div curl}\,\underset{\sim}{E} = \frac{\partial}{\partial x}(\text{curl}\,\underset{\sim}{E})_x + \frac{\partial}{\partial y}(\text{curl}\,\underset{\sim}{E})_y + \frac{\partial}{\partial z}(\text{curl}\,\underset{\sim}{E})_z$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial E_3}{\partial y} - \frac{\partial E_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial E_1}{\partial z} - \frac{\partial E_3}{\partial x}\right)$$
$$+ \frac{\partial}{\partial z}\left(\frac{\partial E_2}{\partial x} - \frac{\partial E_x}{\partial y}\right)$$ 5

Terms cancel in pairs such as $\frac{\partial}{\partial x}\frac{\partial E_3}{\partial y} - \frac{\partial}{\partial y}\frac{\partial E_3}{\partial x} = 0$

Hence result.

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RLJ	AOG	1

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SOLN	EXAMINATION QUESTIONS/SOLUTIONS 2008-09	Course CORE
Question C4		Marks & seen/unseen

Parts

$$\text{curl}\, \underset{\sim}{E} = \underset{\sim}{i}\,(bz - 2z) + \underset{\sim}{j}\,(0 - 0) + \underset{\sim}{k}\,(ay - (-2y))$$ 2

$= 0$

if $b = 2$, $a = -2$ 2

$\underset{\sim}{E} = \text{grad}\,\varphi$

$\Rightarrow \quad x^2 - y^2 = \dfrac{\partial\varphi}{\partial x}$ — (1)

$-2xy + z^2 = \dfrac{\partial\varphi}{\partial y}$ — (2)

$2yz + z = \dfrac{\partial\varphi}{\partial z}$ — (3) 2

arbitrary fn.

(1) $\Rightarrow \quad \varphi = \dfrac{x^3}{3} - xy^2 + h(y,z)$ — (4)

(2)+(4) $\Rightarrow \quad -2xy + z^2 = -2xy + \dfrac{\partial h}{\partial y}$ ← partial der

$\Rightarrow \quad h(y,z) = yz^2 + g(z)$ — (5) 2

ord. der.

(3)+(4)+(5) $\Rightarrow \quad 2yz + z = 2yz + \dfrac{dg}{dz}$ ✓

$\Rightarrow \quad g = \dfrac{z^2}{2} + c$ ← arb. const

$\Rightarrow \quad \varphi(x,y,z) = \dfrac{x^3}{3} - xy^2 + yz^2 + \dfrac{z^2}{2} + c$ 2

Total 20

Setter's initials RLJ	Checker's initials ADG	Page number 2

EE II (3) (10)

	EXAMINATION QUESTIONS/SOLUTIONS 2008-09	Course CORE
SOLN Question C5		Marks & seen/unseen

Parts

(i) If $\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y}$ everywhere inside R in GT

then $\oint_C = 0$. If C is made up of two bits C_1 and C_2 this implies

$$\int_{C_1, A}^{B} = \int_{C_2, A}^{B}$$

but note that R is the whole region enclosed by C so it is simply connected.

R^* is a larger region surrounding C_1, C_2, A and B

Thus if $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial y}$ everywhere inside R^*

and R^* is simply connected

then $\int_{C_1, A}^{B} = \int_{C_2, A}^{B}$ for any pair of curves joining A to B.

5 for part (i)

[Students must state $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$

and R^* is simply connected

and present some fairly logical argument.]

Setter's initials: R.L.J

Checker's initials: AOG

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(ii) If $f = \frac{x-y}{x^2+y^2}$, $g = \frac{x+y}{x^2+y^2}$

then $\frac{\partial f}{\partial y} = \frac{-1}{x^2+y^2} - \frac{(x-y)}{(x^2+y^2)^2} 2y$

$$= \frac{y^2-x^2-2xy}{(x^2+y^2)^2} \quad (1)$$

$$\frac{\partial g}{\partial x} = \frac{1}{x^2+y^2} - \frac{x+y}{(x^2+y^2)^2} 2x$$

$$= \frac{y^2-x^2-2xy}{(x^2+y^2)^2} \quad (2) \qquad 4$$

(1) + (2) are the same everywhere except possibly at $x=0, y=0$. ← 2

[Thus we have ~~true~~ path independence for $\int_c (f\,dx + g\,dy)$ in any simply connected region excluding 0.]

(iii) (a)

$x = \cos\theta$, $y = \sin\theta$

$dx = -\sin\theta\, d\theta$, $dy = \cos\theta\, d\theta$

$$\int_{C_1} = \int_{C_1} \left(\frac{x-y}{x^2+y^2} dx + \frac{x+y}{x^2+y^2} dy \right) \quad \text{use polars}$$

$$= \int_{\theta=0}^{\pi/2} \left[\frac{\cos\theta - \sin\theta}{\cos^2\theta + \sin^2\theta} (-\sin\theta) + \frac{\cos\theta + \sin\theta}{\cos^2\theta + \sin^2\theta} \cos\theta \right] d\theta$$

EXAMINATION QUESTIONS/SOLUTIONS 2008-09

Course

SolN

~~Question~~ C5

Marks & seen/unseen

Parts

$$= \int_0^{\pi/2} \left(\cancel{-\sin\theta\cos\theta} + \sin^2\theta + \cos^2\theta + \cancel{\sin\theta\cos\theta} \right) d\theta$$

$$= \int_0^{\pi/2} d\theta$$ ✓

$$= \pi/2$$

7 marks for part (iii) [5 for showing that integrand equals unity +1 each for the two sets of limits]

(b)

$$\int_{C_1} = \int_{\theta=0}^{-3\pi/2} (\cdots\cdots) \, d\theta$$

$$= -\frac{3\pi}{2}$$

Answers differ because C_1 and C_2 are <u>not</u> in a simply connected region in which $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$ because O is excluded

2

20

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Checker's initials: AoB

EXAMINATION SOLUTIONS 2008-09

Course EE2(3)

Question 11

Marks & seen/unseen

Parts

sim. seen ⇓

(i) $\Omega = \{(i,j) : i = 1, \dots, 6, j = 1, \dots, 6\}$ 2

(ii) (a) $A = \{(i,j) : i = 2,4,6, j = 1, \dots, 6\}$
$B = \{(i,j) : i + j \text{ is odd}\}$ 4

(b) $\mathrm{P}(A) = \frac{1}{2}$, $\mathrm{P}(B) = \frac{1}{2}$,
$\mathrm{P}(A \cap B) = \mathrm{P}(\{(i,j) : i = 2,4,6, j = 1,3,5\}) = 9/36 = 1/4$.
Thus $\mathrm{P}(A)\,\mathrm{P}(B) = \mathrm{P}(A \cap B)$, showing that A and B are independent. 4

(iii) (a) $C = \{(1,3),(2,2),(3,1)\}$ 2

(b)

$$\mathrm{P}(A|C) = \frac{\mathrm{P}(A \cap C)}{\mathrm{P}(C)} = \frac{\mathrm{P}(\{(2,2)\})}{3/36} = \frac{1}{3}$$

3

seen ⇓

(iv) By the definition of conditional probability,

$$\mathrm{P}(D|E) = \frac{\mathrm{P}(D \cap E)}{\mathrm{P}(E)} \quad (*)$$

Also by the definition of conditional probability, $\mathrm{P}(E|D) = \frac{\mathrm{P}(D \cap E)}{\mathrm{P}(D)}$ and thus $\mathrm{P}(D \cap E) = \mathrm{P}(E|D)\,\mathrm{P}(D)$.

Plugging this into $(*)$ yields

$$\mathrm{P}(D|E) = \frac{\mathrm{P}(E|D)\,\mathrm{P}(D)}{\mathrm{P}(E)}$$

5

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EXAMINATION SOLUTIONS 2008-09

Course EE2(3)

Question 12

Marks & seen/unseen

Parts

sim. seen ⇓

(i) $\mathrm{E}(y_t) = \mathrm{E}(1 + \sum_{s=0}^{t} e_s) = 1 + \sum_{s=0}^{t} \mathrm{E}(e_s) = 1 + 0 = 1$ — 3

(ii) $\mathrm{Var}(y_t) = \mathrm{Var}(1 + \sum_{s=0}^{t} e_s) = 0 + \sum_{s=0}^{t} \mathrm{Var}(e_s) = t + 1$ — 4

(iii) For $s > 0$,

$$\begin{aligned}\mathrm{cov}(y_t, y_{t+s}) &= \mathrm{cov}(y_t, (y_{t+s} - y_t) + y_t) = \mathrm{cov}(y_t, y_{t+s} - y_t) + \mathrm{cov}(y_t, y_t)\\ &= \mathrm{cov}(\sum_{u=0}^{t} e_u, \sum_{v=t+1}^{t+s} e_v) + \mathrm{Var}(y_t)\\ &= \sum_{u=0}^{t} \sum_{v=t+1}^{t+s} \mathrm{cov}(e_u, e_v) + t + 1 = t + 1.\end{aligned}$$

5

(iv) A time series y_t is called weakly stationary if $\mathrm{E}(y_t)$ and $\mathrm{cov}(y_t, y_{t+s})$, $s = 0, 1, 2, \dots$ are independent of t. — 3

(v) No, since $\mathrm{cov}(y_t, y_{t+s})$ depends on t. — 2

(vi) A time series $\{y_t\}$ is called strictly stationary if for all $m = 1, 2, \dots$ and all indices $t_1, \dots, t_m$, the joint distribution $(y_{t_1+s}, \dots, y_{t_m+s})$ is the same as that of $(y_{t_1}, \dots, y_{t_m})$ for any s. — 3

(vii) No. Strict stationarity would imply weak stationarity. Since y_t is not weakly stationary it cannot be strictly stationary. — 2

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