master

UNIVERSITY OF LONDON

[E1.10 (Maths 1) 2007]

B.ENG. AND M.ENG. EXAMINATIONS 2007

For Internal Students of Imperial College London

This paper is also taken for the relevant examination for the Associateship of the City & Guilds of London Institute

PART I : MATHEMATICS 1 (ELECTRICAL ENGINEERING)

Wednesday 30th May 2007 10.00 am - 1.00 pm

Answer EIGHT questions.

[Before starting, please make sure that the paper is complete; there should be 6 pages, with a total of 10 questions. Ask the invigilator for a replacement if your copy is faulty.]

1. (i) Find $\dfrac{dy}{dx}$ as a function of x in cases (a) and (b) and as a function of x and y in case (c).

   (a) $\quad y = \ln(\cos x)\ ;$

   (b) $\quad y = (\ln x)^x\ ;$

   (c) $\quad y^2 = \sin(xy)\ .$

   (ii) If $x(t) = 1 - \cos t$ and $y(t) = t - \sin t$, show that

$$\frac{dy}{dx} = \tan\left(\frac{t}{2}\right) .$$

2. Evaluate the following limits:

(i)
$$\lim_{x \to 1} \frac{(x-2)\,(x+2)}{(x-3)\,(x+1)}\ ;$$

(ii)
$$\lim_{x \to 0} \frac{1 - \cos x}{\tan^2 x}\ ;$$

(iii)
$$\lim_{x \to 0} x^x\ ;$$

You may assume $\lim_{x \to 0} x \ln x = 0$.

(iv)
$$\lim_{x \to -2} \frac{\sqrt{-2x} - 2}{x + 2}\ .$$

PLEASE TURN OVER

3. (i) Integrate the following rational functions of x :

$$\text{(a)}\ \frac{x+1}{x}\ ; \qquad \text{(b)}\ \frac{x}{x+1}\ : \qquad \text{(c)}\ \frac{x+1}{x-1}\ ; \qquad \text{(d)}\ \frac{2x^2-x+2}{x^3-x}\ .$$

(ii) Evaluate the following :

$$\int_0^\infty x^5\, e^{-x^2}\, dx\ .$$

4. (i) Put the following complex numbers into standard form i.e. in the form $x+iy$ for some real x and y :

$$\text{(a)}\ \frac{1+i}{1-i}\ ; \qquad \text{(b)}\ \frac{1}{1+\sqrt{3}\,i}\ .$$

(ii) Find all complex solutions to the following equations :

$$\text{(a)}\ z^7 \ = \ -1\ ; \qquad \text{(b)}\ e^z \ = \ -2\ .$$

(iii) If $z = e^{i\theta}$,

(a) find a formula for $\cos n\theta$ in terms of powers of z ;

(b) find a formula for $\cos^6\theta$ in terms of $\cos 2\theta$, $\cos 4\theta$ and $\cos 6\theta$.

5. Consider the function

$$f(x) = (x^2 + x - 2)\, e^{-2x} .$$

(i) Find the points where $f(x) = 0$.

(ii) Find any vertical and horizontal asymptotes.

(iii) Use (i) and (ii) to determine the sign of $f(x)$, for all x.

(iv) Find the points where $f'(x) = 0$.

(v) Determine any local minima and maxima of f.

(vi) Sketch the graph of f.

6. (i) Given any three non-coplanar vectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$, explain why $(\boldsymbol{u}\times\boldsymbol{v})\times(\boldsymbol{v}\times\boldsymbol{w})$ is given by $k\boldsymbol{v}$, where k is a scalar, and find k in terms of $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$.
Hence find an expression for

$$(\boldsymbol{w} \times \boldsymbol{u}) \times [(\boldsymbol{u} \times \boldsymbol{v}) \times (\boldsymbol{v} \times \boldsymbol{w})]$$

in the form $\alpha\boldsymbol{u} + \beta\boldsymbol{w}$, where α, β are scalars.

(ii) Consider the planes

$$x + y - 2z = 3 \quad \text{and}$$

$$2x + 2y + z = 1 .$$

(a) Find a vector parallel to the line of intersection of the planes.

(b) Find the equation of the plane through the origin which is perpendicular to the line of intersection of the planes.

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7. Factorise the matrix

$$A = \begin{pmatrix} 1 & -2 & 3 \\ 2 & -1 & 3 \\ 3 & -3 & 7 \end{pmatrix}$$

into a product LU, where L and U are lower and upper triangular matrices, respectively, with ones down the main diagonal of L.

Find L^{-1} and U^{-1}, and hence A^{-1}.

8. (i) Find the general solution $y(x)$ of the differential equation

$$\frac{dy}{dx} + 2\,\frac{y}{x} = \ln x\,.$$

(ii) Find the solution $y(x)$ of the differential equation

$$\frac{dy}{dx} = \frac{2x - y}{2y - 2x} + \frac{3y}{2x}$$

that satisfies $y\,(2) = 4$.

9. (i) Find the solution $y(x)$ of the differential equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$$

that satisfies $y(1) = 1$ and $y(2) = 0$.

(ii) Find the general solution $y(x)$ of the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = \sin 2x .$$

10. The function $f(x)$ is defined as

$$f(x) = (1-x^2)^{1/4}$$

Compute the derivative $f'(x)$ and show that $f'(0) = 0$.

Compute the second derivative $f''(x)$ and show that f satisfies the differential equation

$$(1-x^2)\, f'' - \frac{3}{2}xf' + \frac{1}{2}f = 0 .$$

Use the Leibnitz formula to differentiate this equation n times and show that at $x = 0$

$$f^{(n+2)}(0) = \left(n^2 + \frac{1}{2}n - \frac{1}{2}\right) f^{(n)}(0) \quad \text{for} \quad n \geq 0 .$$

Here $f^{(n)}$ denotes the nth derivative of f and $f^{(0)}(0) \equiv f(0)$.

Hence find the first three non-zero terms in the Maclaurin expansion for $f(x)$.

Use the binomial expansion to check your result.

END OF PAPER

MATHEMATICS DEPARTMENT

MATHEMATICAL FORMULAE

1. VECTOR ALGEBRA

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1,\, a_2,\, a_3)$$

Scalar (dot) product: $\mathbf{a}\,.\,\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

Vector (cross) product:

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Scalar triple product:

$$[\mathbf{a},\,\mathbf{b},\,\mathbf{c}] = \mathbf{a}\,.\,\mathbf{b}\times\mathbf{c} = \mathbf{b}\,.\,\mathbf{c}\times\mathbf{a} = \mathbf{c}\,.\,\mathbf{a}\times\mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product: $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{c}\,.\,\mathbf{a})\mathbf{b} - (\mathbf{b}\,.\,\mathbf{a})\mathbf{c}$

2. SERIES

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \dots \quad (\alpha \text{ arbitrary, } |x| < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots\,,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^n\frac{x^{2n}}{(2n)!} + \dots\,,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + \dots\,,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^n\frac{x^{n+1}}{(n+1)} + \dots (-1 < x \le 1)$$

3. TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin(a+b) = \sin a\,\cos b + \cos a\,\sin b\,;$$

$$\cos(a+b) = \cos a\,\cos b - \sin a\,\sin b\,.$$

$$\cos iz = \cosh z\,;\quad \cosh iz = \cos z\,;\quad \sin iz = i\sinh z\,;\quad \sinh iz = i\sin z\,.$$

4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f\,D^n g + \binom{n}{1} Df\,D^{n-1}g + \dots + \binom{n}{r} D^r f\,D^{n-r}g + \dots + D^n f\;g\,.$$

(b) Taylor's expansion of $f(x)$ about $x = a$:

$$f(a+h) = f(a) + hf'(a) + h^2 f''(a)/2! + \dots + h^n f^{(n)}(a)/n! + \epsilon_n(h)\,,$$

where $\epsilon_n(h) = h^{n+1}f^{(n+1)}(a+\theta h)/(n+1)!, \quad 0 < \theta < 1\,.$

(c) Taylor's expansion of $f(x,\,y)$ about $(a,\,b)$:

$$f(a+h,\,b+k) = f(a,\,b) + [hf_x + kf_y]_{a.b} + 1/2!\left[h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}\right]_{a.b} + \dots$$

(d) Partial differentiation of $f(x,\,y)$:

i. If $y = y(x)$, then $f = F(x)$, and $\dfrac{dF}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dx}$.

ii. If $x = x(t)$, $y = y(t)$, then $f = F(t)$, and $\dfrac{dF}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$.

iii. If $x = x(u,\,v)$, $y = y(u,\,v)$, then $f = F(u,\,v)$, and

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}\,,\qquad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}\,.$$

(e) Stationary points of $f(x,\,y)$ occur where $f_x = 0$, $f_y = 0$ simultaneously.
Let $(a,\,b)$ be a stationary point: examine $D = [f_{xx}f_{yy} - (f_{xy})^2]_{a.b}$.
If $D > 0$ and $f_{xx}(a,\,b) < 0$, then $(a,\,b)$ is a maximum;
If $D > 0$ and $f_{xx}(a,\,b) > 0$, then $(a,\,b)$ is a minimum;
If $D < 0$ then $(a,\,b)$ is a saddle-point.

(f) Differential equations:

i. The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating factor $I(x) = \exp[\int P(x)(dx)]$, so that $\dfrac{d}{dx}(Iy) = IQ$.

ii. $P(x,\,y)dx + Q(x,\,y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y$.

5. INTEGRAL CALCULUS

(a) An important substitution: $\tan(\theta/2) = t$:
$\sin\theta = 2t/(1+t^2)$, $\cos\theta = (1-t^2)/(1+t^2)$, $d\theta = 2\,dt/(1+t^2)$.

(b) Some indefinite integrals:

$$\int (a^2 - x^2)^{-1/2} dx = \sin^{-1}\left(\frac{x}{a}\right), \quad |x| < a.$$

$$\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1}\left(\frac{x}{a}\right) = \ln\left\{\frac{x}{a} + \left(1 + \frac{x^2}{a^2}\right)^{1/2}\right\}.$$

$$\int (x^2 - a^2)^{-1/2} dx = \cosh^{-1}\left(\frac{x}{a}\right) = \ln\left|\frac{x}{a} + \left(\frac{x^2}{a^2} - 1\right)^{1/2}\right|.$$

$$\int (a^2 + x^2)^{-1} dx = \left(\frac{1}{a}\right)\tan^{-1}\left(\frac{x}{a}\right).$$

6. NUMERICAL METHODS

(a) Approximate solution of an algebraic equation:

If a root of $f(x) = 0$ occurs near $x = a$, take $x_0 = a$ and
$x_{n+1} = x_n - [f(x_n)/f'(x_n)]$, $n = 0, 1, 2 \ldots$

(Newton Raphson method).

(b) Formulae for numerical integration: Write $x_n = x_0 + nh$, $y_n = y(x_n)$.

i. Trapezium rule (1-strip): $\int_{x_0}^{x_1} y(x)dx \approx (h/2)\,[y_0 + y_1]$.

ii. Simpson's rule (2-strip): $\int_{x_0}^{x_2} y(x)dx \approx (h/3)\,[y_0 + 4y_1 + y_2]$.

(c) Richardson's extrapolation method: Let $I = \int_a^b f(x)dx$ and let I_1, I_2 be two estimates of I obtained by using Simpson's rule with intervals h and $h/2$.

Then, provided h is small enough,

$$I_2 + (I_2 - I_1)/15,$$

is a better estimate of I.

7. LAPLACE TRANSFORMS

Function	Transform	Function	Transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)dt$	$af(t) + bg(t)$	$aF(s) + bG(s)$
df/dt	$sF(s) - f(0)$	d^2f/dt^2	$s^2F(s) - sf(0) - f'(0)$
$e^{at}f(t)$	$F(s-a)$	$tf(t)$	$-dF(s)/ds$
$(\partial/\partial\alpha)f(t,\alpha)$	$(\partial/\partial\alpha)F(s,\alpha)$	$\int_0^t f(t)dt$	$F(s)/s$
$\int_0^t f(u)g(t-u)du$	$F(s)G(s)$		
1	$1/s$	$t^n (n = 1, 2 \ldots)$	$n!/s^{n+1}$, $(s > 0)$
e^{at}	$1/(s-a)$, $(s > a)$	$\sin\omega t$	$\omega/(s^2+\omega^2)$, $(s > 0)$
$\cos\omega t$	$s/(s^2+\omega^2)$, $(s > 0)$	$H(t-T) = \begin{cases} 0, & t < T \\ 1, & t > T \end{cases}$	e^{-sT}/s, $(s, T > 0)$

8. FOURIER SERIES

If $f(x)$ is periodic of period $2L$, then $f(x+2L) = f(x)$, and

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \text{ where}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad n = 0, 1, 2, \ldots, \text{ and}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1, 2, 3, \ldots$$

Parseval's theorem

$$\frac{1}{L}\int_{-L}^{L} [f(x)]^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right).$$

EXAMINATION QUESTIONS/SOLUTIONS 2006-07

Course: CORE

~~Question~~ Solution C1

Marks & seen/unseen

Parts

(a) $\dfrac{dy}{dx} = -\dfrac{\sin x}{\cos x} = -\tan x$ — 3

(b) Take logs — 1

$\ln y = x \ln(\ln x)$

$\dfrac{d}{dx}\ln y = \dfrac{d}{dx}\left[x\ln(\ln x)\right]$ — 2

$\dfrac{1}{y}\dfrac{dy}{dx} = \ln(\ln x) + x\,\dfrac{1}{\ln x}\cdot\dfrac{1}{x}$ — 2

$\dfrac{dy}{dx} = (\ln x)^x \ln(\ln x) + (\ln x)^{x-1}$ — 1

(c) $y^2 = \sin(xy)$

$\dfrac{d}{dx}y^2 = \dfrac{d}{dx}\sin(xy)$ — 1

$\dfrac{d}{dy}y^2\,\dfrac{dy}{dx} = \dfrac{d}{dx}\sin(xy)$ — 2

$2y\dfrac{dy}{dx} = \cos(xy)\left[y + x\dfrac{dy}{dx}\right]$ — 2

$\dfrac{dy}{dx} = \dfrac{y\cos(xy)}{2y - x\cos(xy)}$ — 2

(ii) $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{1-\cos t}{\sin t} = \dfrac{2\sin^2(t/2)}{2\sin(t/2)\cos(t/2)}$ — 3

$= \tan(t/2)$ — 1

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	EXAMINATION QUESTIONS/SOLUTIONS 2006-07	Course CORE
Question C2		Marks & seen/unseen

Parts

i) $\lim_{x\to 1} \dfrac{(x-2)(x+2)}{(x-3)(x+1)} = \dfrac{(-1)(3)}{-2(2)} = \dfrac{3}{4}$ (3)

ii) $\lim_{x\to 0} \dfrac{1-\cos x}{\tan^2 x} = \lim_{x\to 0} \dfrac{\sin x}{2\tan x \sec^2 x}$ by L'Hopital

$= \lim_{x\to 0} \dfrac{\cos x}{2\sec^4 x + 2\tan x \frac{d}{dx}\sec^2 x}$ (7)

$= \dfrac{1}{2+0} = \frac{1}{2}$ (note $\frac{d}{dx}\sec^2 x = 2\sec x\left(\frac{\sin x}{\cos^2 x}\right)$)

(Quick way $\frac{1-\cos x}{\tan^2 x} = \frac{x^2/2}{x^2}$ $\lim = \frac{1}{2}$)

iii) Let $y = x^x$

Then $\ln y = x \ln x$

as $x \to 0$ ~~then~~ $\ln y \to 0$ Hence $y \to 1$ (5)

~~$\therefore \lim x \to 0$~~ $\therefore \lim_{x\to 0} x^x = 1$

iv) $\lim_{x\to -2} \dfrac{\sqrt{-2x}-2}{x+2} = \lim_{x\to -2} \dfrac{(\sqrt{-2x}-2)(\sqrt{-2x}+2)}{(x+2)(\sqrt{-2x}+2)}$

$= \lim_{x\to -2} \dfrac{-2x-4}{(x+2)(\sqrt{-2x}+2)}$ ~~L'Hopital~~ (5)

$= \lim_{x\to -2} \dfrac{-2}{\sqrt{-2x}+2} = -\dfrac{1}{2}$

(Or use L'hopital's rule.)

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	EXAMINATION QUESTIONS/SOLUTIONS 2006-07	Course
Question C3	Solutions	Marks & seen/unseen
Parts		
	a) $\int \frac{x+1}{x}\,dx = \int \left(1+\frac{1}{x}\right)dx = x + \ln\lvert x\rvert + C$	2
	b) $\int \frac{x}{x+1}\,dx = \int \left(1-\frac{1}{x+1}\right)dx = x - \ln\lvert x+1\rvert + C$	2
	c) $\int \frac{x+1}{x-1}\,dx = \int\left(1+\frac{2}{x-1}\right)dx = x + 2\ln\lvert x-1\rvert + C$	2
	d) $\frac{2x^2-x+2}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$	2
	$\Rightarrow 2x^2-x+2 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1)$	
	$x=0 \Rightarrow A=-2,\quad x=1 \Rightarrow 3=2B,\quad x=-1 \Rightarrow 5=2C$	3
	So $I = \int -\frac{2}{x} + \frac{3/2}{x-1} + \frac{5/2}{x+1}$	
	$= -2\ln\lvert x\rvert + \frac{3}{2}\ln\lvert x-1\rvert + \frac{5}{2}\ln\lvert x+1\rvert + C$	3
	ii) $I_5 = \int_0^\infty x^5 e^{-x^2}\,dx = \cancel{\left[-\frac{x^4}{2}e^{-x^2}\right]_0^\infty}$	3
	$+ \frac{4}{2}\int_0^\infty x^3 e^{-x^2}\,dx$	
	$= 2I_3$	
	$I_3 = \int_0^\infty x^3 e^{-x^2}\,dx = \cancel{\left[-\frac{x^2}{2}e^{-x^2}\right]_0^\infty} + \int_0^\infty x e^{-x^2}\,dx$	3
	So $I_5 = 2I_1 = 2\int_0^\infty x e^{-x^2}\,dx = 1$	
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I EE(1) (4)

	EXAMINATION QUESTIONS/SOLUTIONS 2006-07	Course
Question C4	SOLUTION	Marks & seen/unseen
Parts		

i) a) $\frac{1+i}{1-i} = \frac{(1+i)}{(1-i)}\frac{(1-i)}{(1-i)} = \frac{1+1}{1-2i-1} = -\frac{1}{i}$ 1

$= -\frac{1}{i}\cdot\frac{i}{i} = i$

b) $\frac{1}{1+i\sqrt{3}} = \frac{1}{1+i\sqrt{3}}\cdot\frac{1-i\sqrt{3}}{1-i\sqrt{3}} = \frac{1-i\sqrt{3}}{1+3}$ 1

$= \frac{1-i\sqrt{3}}{4}$

ii) a) $z^7 = -1 = e^{i\pi + 2k\pi i}$ so 2

$z = e^{\frac{i\pi}{7} + \frac{2k\pi i}{7}}$ $k = 0, \dots, 6$ 1

b) $e^z = -2 = 2e^{i\pi + 2\pi k i}$ 2

$= e^{\log 2 + i\pi + 2\pi k i}$

so $z = \log 2 + i\pi + 2\pi k i$ $k \in \mathbb{Z}$. 1

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IEE (1) (4)

	EXAMINATION QUESTIONS/SOLUTIONS 2006-07	Course
Question C4	SOLUTION	Marks & seen/unseen
Parts	i(i) a) $z = e^{i\theta}$ then	
	$z^n = e^{in\theta} = \cos n\theta + i\sin n\theta$	1
	$z^{-n} = e^{-in\theta} = \cos n\theta - i\sin n\theta$	1
	So $2\cos n\theta = z^n + z^{-n}$	2
	b) $2^6 \cos^6\theta = (z + z^{-1})^6$	2
	$= z^6 + 6z^4 + 15z^2 + 20 + \frac{15}{z^2} + \frac{6}{z^4} + \frac{1}{z^6}$	2
	$= \left(z^6 + \frac{1}{z^6}\right) + 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) + 20$	2
	$= 2\cos 6\theta + \cancel{6}\,12 \cos 4\theta + \cancel{18}\,30 \cos 2\theta + 20$	2
	Setter's initials: JL / Checker's initials: JKC.	Page number

EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2006-07 | Course

Question C5 | Marks & seen/unseen

Parts

(i) $f(x) = 0 \Rightarrow (x^2+x-2) \equiv (x+2)(x-1) = 0$
$\Rightarrow (-2,0), (+1,0)$. 1+2

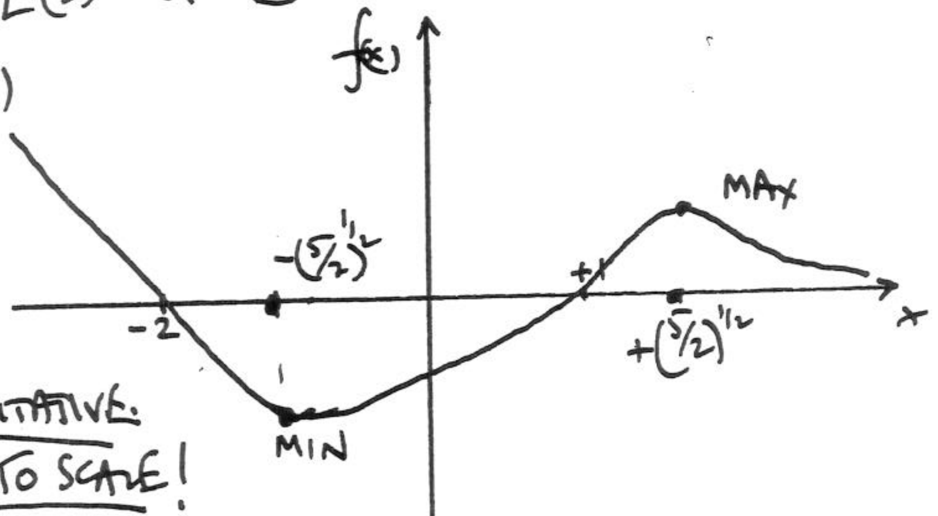
(ii) No vertical asymptote.
Horizontal asymptote $f \to y = 0^+$ as $x \to +\infty$. 1+2

(iii) Since $f(x)$ changes sign only at $x=-2$, and $x=+1$.
we must have $f > 0$ $(1 < x < \infty)$
$f < 0$ $(-2 < x < +1)$, $f > 0$ $(x < -2)$.
[$f = 0$ at $x = -2$, $x = +1$ ONLY]. 4

(iv) $f'(x) = [(2x+1) - 2(x^2+x-2)]e^{-2x}$.
$= (5 - 2x^2)\, e^{-2x} = 0$ when $x = \pm\left(\frac{5}{2}\right)^{1/2}$. 2

(v) USING (iii) [or f''] we must have
$\left[-\left(\frac{5}{2}\right)^{1/2}, f\left(-\left(\frac{5}{2}\right)^{1/2}\right)\right]$ MINIMUM, $\left[+\left(\frac{5}{2}\right)^{1/2}, f\left(+\left(\frac{5}{2}\right)^{1/2}\right)\right]$ MAXIMUM 2

(vi)



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EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2006-07

Course

Question C6

Marks & seen/unseen

Parts

(i) $\underline{u}\times\underline{v}$ is $\perp$r to $\underline{u}$ and to $\underline{v}$

$\underline{v}\times\underline{w}$ is $\perp$r to $\underline{v}$ and to $\underline{w}$

$(\ )\times(\ )$ is $\perp$r to each of $(\ )$ and $(\ )$

so is parallel to each of the planes defined by ($\underline{u}$ and $\underline{v}$), ($\underline{v}$ and $\underline{w}$) respectively

hence it is parallel to $\underline{v}$ and $= k\underline{v}$ — 3

with k a scalar.

$$(\underline{u}\times\underline{v})\times(\underline{v}\times\underline{w}) = \left[(\underline{u}\times\underline{v})\cdot\underline{w}\right]\underline{v} - \underbrace{\left[\cancel{(\underline{u}\times\underline{v})\cdot\underline{v}}\right]}_{=0}\underline{w}.$$

Hence $k = (\underline{u}\times\underline{v})\cdot\underline{w}$. — 3

$$(\underline{w}\times\underline{u})\times[\ .\ .\ ] \equiv k\left[(\underline{w}\times\underline{u})\times\underline{v}\right] = k\left[(\underline{v}\cdot\underline{w})\underline{u} - (\underline{v}\cdot\underline{u})\underline{w}\right].$$

$$\equiv \alpha\,\underline{u} + \beta\,\underline{w}$$

$\alpha \equiv \left[(\underline{u}\times\underline{v})\cdot\underline{w}\right](\underline{v}\cdot\underline{w})$, $\beta = -\left[(\underline{u}\times\underline{v})\cdot\underline{w}\right](\underline{v}\cdot\underline{u})$. — 2+2

(ii) Planes are $\underline{r}\cdot(1,1,-2) = 3$

and $\underline{r}\cdot(2,2,1) = 1$. — 2

(a) line of intersection is perpendicular to the normals to both planes $\therefore$ take $(1,1,-2)\times(2,2,1)$

$\equiv (5,-5,0)$. — 2+2

(b) We need to take $\underline{r}\cdot\underline{n} = p$ with p the perp distance $=0$ and $\underline{n} \parallel (5,-5,0)$ — 2+2

$\therefore \underline{r}\cdot(5,-5,0) = 0$ ie. $5x - 5y = 0$ or equivalent

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IEE (1) 7

	EXAMINATION QUESTIONS/SOLUTIONS 2006-07	Course: CORE
Question: 7		Marks & seen/unseen

Parts

$$\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix} \begin{pmatrix} e & f & g \\ 0 & h & i \\ 0 & 0 & j \end{pmatrix} = \begin{pmatrix} e & f & g \\ ae & af+h & ag+i \\ be & bf+ch & bg+ci+j \end{pmatrix}$$

$$\quad L \qquad\qquad U \qquad\qquad\qquad A$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 & 3 \\ 0 & 3 & -3 \\ 0 & 0 & 1 \end{pmatrix} \Leftarrow \begin{pmatrix} 1 & -2 & 3 \\ 2 & -1 & 3 \\ 3 & -3 & 7 \end{pmatrix}$$ 9

$\det L = 1$, $\det U = 3$ 2

$$L^{-1} = \frac{\text{Adj } L}{\det L} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}$$ 3

$$U^{-1} = \frac{\text{Adj } U}{\det U} = \frac{1}{3}\begin{pmatrix} 3 & 2 & -3 \\ 0 & 1 & 3 \\ 0 & 0 & 3 \end{pmatrix}$$ 4

$$A = LU \Rightarrow A^{-1} = U^{-1}L^{-1}$$

$$= \frac{1}{3}\begin{pmatrix} 3 & 2 & -3 \\ 0 & 1 & 3 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 2 & 5 & -3 \\ -5 & -2 & 3 \\ -3 & -3 & 3 \end{pmatrix}$$ 2

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IEE(1) 8

	EXAMINATION SOLUTIONS 2006-07	Course
Question C9		Marks & ~~seen~~/unseen
Part		
a)	Use the integrating factor $$\exp\left(\int \frac{2}{x}\,dx\right) = \exp\left(2\ln x\right) = x^2,$$	2
	to obtain $$\frac{d}{dx}\left(x^2 y\right) = x^2 \ln x.$$	1
	Now integrate by parts, $$\begin{aligned} x^2 y &= \int x^2 \ln x \, dx, \\ &= \frac{1}{3}x^3 \ln x - \int \frac{1}{3}x^3 \frac{1}{x}\,dx, \\ &= \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + c, \end{aligned}$$	3
	and rearrange into $$y = \frac{1}{3}x \ln x - \frac{x}{9} + \frac{c}{x^2}.$$	2
		Part a out of 8
	Setter's initials PJD / Checker's initials	Page number 1/2

IEE (1) 8

	EXAMINATION SOLUTIONS 2006-07	Course
Question C9		Marks & ~~seen~~/unseen

Part

b) Put $y(x) = xu(x)$ to get

$$x\frac{du}{dx} + u = \frac{1}{2}\frac{2-u}{u-1} + \frac{3}{2}u,$$

$$x\frac{du}{dx} = \frac{1}{2}\frac{2-u}{u-1} + \frac{1}{2}u = \frac{\frac{1}{2}u^2 - u + 1}{u-1}.$$

3

Now separate and integrate,

$$\begin{aligned} \int \frac{u-1}{\frac{1}{2}u^2 - u + 1}du &= \int \frac{dx}{x}, \\ \ln\left(\frac{1}{2}u^2 - u + 1\right) &= \ln x + \ln c, \end{aligned}$$

4

where c is an arbitrary constant.

Exponentiate both sides,

$$\tfrac{1}{2}u^2 - u + 1 = cx,$$

and solve the quadratic for u,

$$u = 1 \pm \sqrt{cx - 1}.$$

3

Returning to $y(x) = xu(x)$,

$$y(x) = x\left(1 \pm \sqrt{cx-1}\right).$$

To satisfy $y(2) = 4$ we need the positive root and $c = 1$, giving

$$y(x) = x\left(1 + \sqrt{x-1}\right).$$

2

Part b out of 12

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	EXAMINATION SOLUTIONS 2006-07	Course
Question C10		Marks & ~~seen~~/unseen
Part		

a) Try $y = e^{\lambda x}$. The ODE

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$$

then implies

$$\lambda^2 + 4\lambda + 4 = (\lambda + 2)^2 = 0.$$ 2

This has a repeated root at $\lambda = -2$, so the general solution is

$$y = (A + Bx)\, e^{-2x}.$$ 3

Putting $y(2) = (A + 2B)\, e^{-4} = 0$ gives $A = -2B$.

Then $y(1) = (A + B)\, e^{-2} = -B\, e^{-2} = 1$, so $B = -e^2$ and $A = 2e^2$.

The solution is $y(x) = (2 - x)\, e^{2(1-x)}$. 3

Part a out of 8.

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PJD	JRC.	1/2

IEE (1) 9

	EXAMINATION SOLUTIONS 2006-07	Course
Question C10		Marks & ~~seen~~/unseen

Part

b) For the complementary function try $y = e^{\lambda x}$. The ODE

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 0$$

then implies

$$\lambda^2 + 2\lambda + 5 = (\lambda + 1 + 2i)(\lambda + 1 - 2i) = 0.$$ 3

The two roots are the complex conjugate pair $\lambda = -1 \pm 2i$, so the general solution is

$$y = (A \sin 2x + B \cos 2x)\, e^{-x}.$$ 2

For the particular integral try

$$y = C \sin 2x + D \cos 2x.$$ 3

Differentiate,

$$\frac{dy}{dx} = 2C \cos 2x - 2D \sin 2x, \quad \frac{d^2y}{dx^2} = -4C \sin 2x - 4D \cos 2x,$$

and substitute,

$$\begin{aligned}[-4C \sin 2x - 4D \cos 2x] \quad &+ \quad 2[2C \cos 2x - 2D \sin 2x] \\ &+ \quad 5[C \sin 2x + D \cos 2x] \;=\; \sin 2x.\end{aligned}$$

Coefficient of $\sin 2x$: $-4D + C = 1$.
Coefficient of $\cos 2x$: $4C + D = 0$, so $D = -4C$.

Previous equation now gives $17C = 1$, so $C = 1/17$ and $D = -4/17$.

Solution is

$$y = (A \sin 2x + B \cos 2x)\, e^{-x} + \frac{1}{17} \sin 2x - \frac{4}{17} \cos 2x.$$ 4

Part b out of 12

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IEE (1) 10

	EXAMINATION QUESTIONS/SOLUTIONS 2006-07	Course Core
~~Question~~ Solution C11		Marks & seen/unseen

Parts

$$f'(x) = \frac{1}{4}(1-x^2)^{-3/4}\cdot(-2x) = -\frac{x}{2}(1-x^2)^{-3/4}$$

so $f'(0)=0$. — 2

$$f''(x) = -\frac{(1-x^2)^{-3/4}}{2} - \frac{2x^2\cdot 3}{2\cdot 4}(1-x^2)^{-7/4}$$ — 2

So $(1-x^2)f'' - \tfrac{3}{2}xf' + \tfrac{1}{2}f$

$$= -\frac{(1-x^2)^{1/4}}{2} - \frac{3}{4}x^2(1-x^2)^{-3/4} + \tfrac{3}{4}x^2(1-x^2)^{-3/4}$$

$$+ \tfrac{1}{2}(1-x^2)^{1/4} = 0$$ — 3

Differentiate this n times by Leibnitz

$$(1-x^2)f^{n+2} - n\,2xf^{n+1} - \frac{n(n-1)}{2}\cdot 2f^{(n)}$$

$$-\tfrac{3}{2}xf^{n+1} - \frac{3}{2}n\,f^{(n)} + \tfrac{1}{2}f^n = 0$$ — 3

Put $x=0$ $\quad f^{(n+2)}(0) - n(n-1)f^{(n)}(0) + \tfrac{1}{2}f^n(0) - \tfrac{3}{2}n\,f^{(n)}(0) = 0$

So $f^{(n+2)}(0) = (n^2 - n + \tfrac{3}{2}n - \tfrac{1}{2})f^{(n)}(0)$

$$= (n^2 + \tfrac{1}{2}n - \tfrac{1}{2})f^{(n)}(0)$$ — 4

So McClaurin is $f(x) = \underset{1}{f(0)} + x\underset{0}{f'(0)} + \frac{x^2}{2}\underset{-1/2}{f''(0)} + \frac{x^3}{6}\underset{0}{f'''(0)}$ — 2

$$+ \frac{x^4}{4!}\cdot\frac{9}{2}f^{IV}(0)$$

$\qquad\qquad 9/2\cdot(-1/2)$

So $f(x) = 1 - \frac{x^2}{4} + \frac{x^4}{24}\cdot\frac{9}{2} = 1 - x^2/4 - \tfrac{3}{32}x^4$ — 2

Binomial $f(x) = 1 - \tfrac{1}{4}x^2 - \tfrac{1}{4}\cdot\tfrac{3}{4}x^4/2!$

$= 1 - \tfrac{1}{4}x^2 - \tfrac{3}{32}x^4 \ldots$ agrees — 2

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