E2.4

OK

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2007

EEE/ISE PART II: MEng, BEng and ACGI

COMMUNICATIONS 2

Monday, 11 June 2:00 pm

Corrected Copy

Time allowed: 2:00 hours

There are FOUR questions on this paper.

Q1 is compulsory.
Answer Q1 and any two of questions 2-4.
Q1 carries 40% of the marks. Questions 2 to 4 carry equal marks (30% each).

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : M. Petrou, M. Petrou
Second Marker(s) : J.A. Barria, J.A. Barria

[Compulsory]

1. a) i) Starting from the representation of the noise as $n(t) = \sum_k A_k \cos(2\pi f_k t + \theta_k)$ where A_k, f_k and θ_k are random variables, show that $n(t)$ may be written as

$$n(t) = n_c(t)\cos(\omega_c t) - n_s(t)\sin(\omega_c t) \tag{1.1}$$

where $n_c(t)$ and $n_s(t)$ are suitably defined expressions (which you should derive) and $\omega_c \equiv 2\pi f_c$ with f_c being the carrier frequency of the modulation system used.

[8]

ii) Explain what the term "Gaussian noise" means.

[2]

iii) Explain what the term "uncorrelated noise" means.

[2]

b) i) Explain what is meant by the term "ergodicity".

[2]

ii) When is a signal considered to be ergodic with respect to the mean?

[3]

iii) When is a signal considered to be ergodic with respect to the autocorrelation function?

[4]

iv) What use do we make of the assumption of ergodicity in signal processing?

[3]

[Question continuous on the next page]

c) i) Define the term "channel capacity".

[3]

ii) State the Hartley Shannon theorem and state the assumption under which it is exactly valid. Explain the meaning of each symbol you use in its mathematical expression.

[3]

iii) According to information theory which property would make a modulation scheme ideal?

[3]

iv) Use the Hartley Shannon theorem to work out the relationship between the input and the output signal to noise ratio for the ideal modulation scheme.

[5]

v) What is the meaning of the term "source entropy"?

[2]

2. a) Consider the signal $x(t) = m(t)\cos(\omega_c t + \theta)$ where $m(t)$ is some function of time, ω_c is a constant and θ is a random variable taking values from a uniform distribution in the range $[0, \pi/4]$. (You are reminded that $\cos(\pi/4) = \sin(\pi/4) = \sqrt{2}/2$.)

i) Compute the ensemble mean value of $x(t)$.

[6]

ii) Compute the ensemble mean value of $x(t)^2$.

[6]

b) You know that a true signal is flat with constant value 5. However, when you receive it, you receive the following string of numbers due to channel noise: $4,5,6,3,5,5,3,6,7,4,5,5,4,4,6,6,6,5,5,5$.

i) How will you check whether the channel noise is Gaussian?

[5]

ii) How will you check whether the channel noise is white?

[5]

c) You use noise signal $n(t)$ as an input to the system of figure 2.1.

i) Show that the two branches of the system will output the two components $n_c(t)$ and $n_s(t)$.

[6]

ii) Explain how you would choose the low pass filter in each branch so that the above is true.

[2]

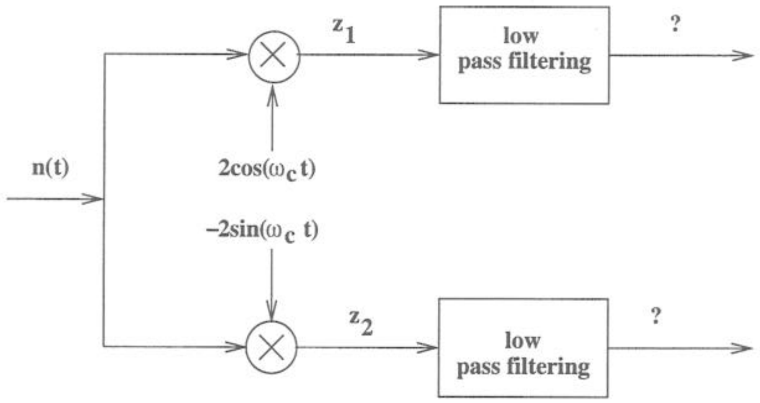

Figure 2.1 The system used in part (c) of question 2

3. a) A 5-sample long digital signal is transmitted down a noisy communication line five times. Five versions of that signal are received, and they constitute an ensemble of signals. These versions are:

$$(5,4,3,7,6)$$
$$(4,3,7,5,6)$$
$$(4,4,7,3,7)$$
$$(6,7,5,5,2)$$
$$(6,7,3,5,4)$$

i) Calculate the ensemble mean signal.

[3]

ii) Is the signal stationary with respect to the mean? Justify your answer.

[3]

iii) Is this signal ergodic with respect to the mean? Justify your answer.

[3]

iv) Calculate the ensemble autocorrelation function of this signal $R(t_2, t_3)$, where t_2 and t_3 refer to the times the second and the third sample of the signal are received.

[5]

v) Is the signal stationary with respect to the autocorrelation function? Justify your answer.

[5]

vi) Calculate the temporal autocorrelation function of the fifth instantiation of this signal for shift $\tau = 3$.

[5]

vii) Is this signal ergodic with respect to the autocorrelation function? Justify your answer.

[1]

b) A source produces symbols from a four-symbol alphabet $\{E, V, O, L\}$, with corresponding frequencies $0.5, 0.15, 0.25, 0.1$, respectively.

i) Use Huffman coding to construct a coding scheme for this source.

[4]

ii) Decipher the sequence 111101100111010.

[1]

4. Consider a quantiser with uniform separation of Δ volts between the quantisation levels.

a) Write down an expression for the probability density function of the quantisation error q.

[1]

b) Work out the mean square error of the quantiser.

[3]

c) If the quantiser is used to quantise symbols with n bits each, what is the maximum peak-to-peak dynamic range of the quantiser?

[1]

d) Assuming that a message signal fully loads the quantiser, what is its maximum absolute value m_p of the message signal?

[1]

e) Write down an expression for the signal to noise ratio of the quantiser output in terms of Δ, if the power of the message signal is P.

[1]

f) Work out the signal to noise ratio of the quantiser output in terms of m_p.

[3]

g) Consider a full load signal: $m(t) = A_m \sin(2\pi ft)\cos(2\pi ft)$.

i) Compute the average power P of this signal.

[10]

ii) Derive an expression of the signal to noise ratio at the output of the quantiser described in parts (a)–(f) when this signal is represented by symbols with n bits each.

[6]

iii) Express the result in the above question in dB.

[3]

iv) Comment on the effect on the quantisation noise as the number of quantisation levels increases.

[1]

E2.4

SOLUTIONS TO EXAM QUESTIONS 2007

1. a) i) **Starting from the representation of the noise as $n(t) = \sum_k A_k \cos(2\pi f_k t + \theta_k)$ where A_k, f_k and θ_k are random variables, show that $n(t)$ may be written as**

$$n(t) = n_c(t)\cos(\omega_c t) - n_s(t)\sin(\omega_c t) \qquad (1.1)$$

where $n_c(t)$ and $n_s(t)$ are suitably defined expressions (which you should derive) and $\omega_c \equiv 2\pi f_c$ with f_c being the carrier frequency of the modulation system used.

We start by writing first $f_k = f_k - f_c + f_c$ and substituting into the given expression for $n(t)$:

$$\begin{aligned}
n(t) &= \sum_k A_k \cos(2\pi f_k t + \theta_k) \\
&= \sum_k A_k \cos\left(2\pi(f_k - f_c + f_c)t + \theta_k\right) \\
&= \sum_k A_k \cos\left(2\pi(f_k - f_c) + \theta_k + 2\pi f_c t\right) \\
&= \sum_k A_k \cos\left([2\pi(f_k - f_c) + \theta_k] + [2\pi f_c t]\right) \\
&= \sum_k A_k \left(\cos[2\pi(f_k - f_c) + \theta_k]\cos[2\pi f_c t] - \sin[2\pi(f_k - f_c) + \theta_k]\sin[2\pi f_c t]\right) \\
&= \cos[2\pi f_c t]\sum_k A_k \cos[2\pi(f_k - f_c) + \theta_k] - \sin[2\pi f_c t]\sum_k A_k \sin[2\pi(f_k - f_c) + \theta_k] \\
&= n_c(t)\cos(\omega_c t) - n_s(t)\sin(\omega_c t)
\end{aligned}$$

where

$$n_c(t) \equiv \sum_k A_k \cos[2\pi(f_k - f_c) + \theta_k] \qquad n_s(t) \equiv \sum_k A_k \sin[2\pi(f_k - f_c) + \theta_k] \qquad (1.2)$$

[Bookwork] [8]

ii) **Explain what the term “Gaussian noise” means.**

It means that the random number that affects the value of a signal afflicted by this noise is drawn from a Gaussian probability density function.

[Bookwork] [2]

iii) **Explain what the term “uncorrelated noise” means.**

It means that the power spectral density of the signal is flat, ie it has the same value for all frequencies. OR: It means that the correlation function of the noise sequence is a delta function. If they simply say “it means that the noise is uncorrelated”, no marks.

[Bookwork] [2]

b) i) **Explain what is meant by the term “ergodicity”.**

The ensemble statistics of a random process are equal to its temporal statistics.

[*Bookwork*] [2]

ii) **When is a signal considered to be ergodic with respect to the mean?**

When it is stationary with respect to the mean (ie it has the same ensemble mean at any instant in time) and its ensemble average is the same as the temporal average of any instantiation of it.

[*Bookwork*] [3]

iii) **When is a signal considered to be ergodic with respect to the autocorrelation function?**

When it is stationary with respect to its autocorrelation function (ie the ensemble autocorrelation function has the same value for any pair of instances at the same relative time shift from each other) and this value is the same as the value of its temporal autocorrelation function for the same time shift, and this is valid for all time shifts.

[*Bookwork*] [4]

iv) **What use do we make of the assumption of ergodicity in signal processing?**

We replace the calculation of ensemble statistics with the calculation of temporal statistics over the single instantiation we have at our disposal.

[*Bookwork*] [3]

c) i) **Define the term "channel capacity".**

It is the maximum rate at which error free information may be transmitted through the channel, even when the channel is noisy.

[*Bookwork*] [3]

ii) **State the Hartley Shannon theorem and state the assumption under which it is exactly valid. Explain the meaning of each symbol you use in the mathematical expression.**

For additive white Gaussian noise, the capacity C of a channel is given by

$$C = B\log_2\left(1+\frac{S}{N}\right) \tag{1.3}$$

where B is the bandwidth of the channel, S is the average signal power at the receiver and N is the average noise power at the receiver.

[*Bookwork*] [3]

iii) **According to information theory which property would make a modulation scheme ideal?**

The channel capacity should be the same before and after demodulation.

[*Bookwork*] [3]

iv) **Use the Hartley Shannon theorem to work out the relationship between the input and the output signal to noise ratio for the ideal modulation scheme.**

The maximum rate at which information may arrive at the receiver is

$$C_{in} = B\log_2(1+SNR_{in}) \tag{1.4}$$

where SNR_{in} is the predetection signal to noise ratio at the input of the demodulator.

After demodulation, the signal is low pass filtered with bandwidth W, that of the message. The maximum rate at which information can leave the receiver is:

$$C_o = W \log_2 (1 + SNR_o) \tag{1.5}$$

where SNR_o is the SNR at the output of the post-detection filter.

For an ideal demodulation scheme, $C_{in} = C_o$. Therefore:

$$B \log_2 (1 + SNR_{in}) = W \log_2 (1 + SNR_o) \Rightarrow SNR_o = [1 + SNR_{in}]^{B/W} - 1 \tag{1.6}$$

[*Bookwork*] [5]

v) **What is the meaning of the term "source entropy"?**

It gives the average amount of information per source symbol. [*Bookwork*] [2]

2. a) **Consider the signal $x(t) = m(t)\cos(\omega_c t + \theta)$ where $m(t)$ is some function of time, ω_c is a constant and θ is a random variable taking values from a uniform distribution in the range $[0, \pi/4]$. (You are reminded that $\cos(\pi/4) = \sin(\pi/4) = \sqrt{2}/2$.)**

i) **Compute the ensemble mean value of $x(t)$.**

The only random variable is θ. It has prob. density function:

$$p(\theta) = \begin{cases} \frac{4}{\pi} & \text{for } 0 \le \theta < \frac{\pi}{4} \\ 0 & \text{elsewhere} \end{cases} \tag{2.1}$$

The ensemble mean then is

$$\begin{aligned}
< x(t) > &= \int_{-\infty}^{\infty} x(t) p(\theta) d\theta \\
&= \frac{4}{\pi} \int_0^{\pi/4} m(t)\cos(\omega_c t + \theta) d\theta \\
&= \frac{4m(t)}{\pi} [\sin(\omega_c t + \theta)|_0^{\pi/4} \\
&= \frac{4m(t)}{\pi} \left[\sin(\omega_c t + \frac{\pi}{4}) - \sin(\omega_c t)\right] \\
&= \frac{4m(t)}{\pi} \left[\sin(\omega_c t)\cos\frac{\pi}{4} + \cos(\omega_c t)\sin\frac{\pi}{4} - \sin(\omega_c t)\right] \\
&= \frac{2m(t)}{\pi} \left[\sin(\omega_c t)\sqrt{2} + \cos(\omega_c t)\sqrt{2} - 2\sin(\omega_c t)\right] \\
&= \frac{2m(t)}{\pi} \left[(\sqrt{2} - 2)\sin(\omega_c t) + \cos(\omega_c t)\sqrt{2}\right]
\end{aligned}$$

[*Calculation of new example*] [6]

ii) **Compute the ensemble mean value of $x(t)^2$.**

$$\begin{aligned}
< x(t)^2 > &= \int_{-\infty}^{\infty} x(t)^2 p(\theta) d\theta \\
&= \frac{4}{\pi} \int_0^{\pi/4} m(t)^2 \cos^2(\omega_c t + \theta) d\theta \\
&= \frac{4m(t)^2}{\pi} \int_0^{\pi/4} \frac{1 + \cos(2\omega_c t + 2\theta)}{2} d\theta \\
&= \frac{2m(t)^2}{\pi} \left[\theta + \frac{\sin(2\omega_c t + 2\theta)}{2}\right|_0^{\pi/4} \\
&= \frac{2m(t)^2}{\pi} \left[\frac{\pi}{4} + \frac{\cos(2\omega_c t)}{2} - \frac{\sin(2\omega_c t)}{2}\right] \\
&= \frac{m(t)^2}{\pi} \left[\frac{\pi}{2} + \cos(2\omega_c t) - \sin(2\omega_c t)\right]
\end{aligned}$$

[*Calculation of new example*] [6]

b) **You know that a true signal is flat with constant value 5. However, when you receive it, you receive the following string of numbers due to channel noise:**
$4, 5, 6, 3, 5, 5, 3, 6, 7, 4, 5, 5, 4, 4, 6, 6, 6, 5, 5, 5$.

i) **How will you check whether the channel noise is Gaussian?**

I will compute the histogram of these numbers, normalise it so that it corresponds to a probability density function (pdf) and then perform statistical tests to see whether this pdf can be modelled as as Gaussian. (This is the ideal answer, but given that I did not give them any such example in the course, any answer that makes good sense will get good marks.)

[New example] [5]

ii) **How will you check whether the channel noise is white?**

I will compute the Fourier spectrum of the string of numbers I receive and check whether it is flat or not. (This is the ideal answer, but given that I did not give them any such example in the course, any answer that makes good sense will get good marks.)

[New example] [5]

c) **You use noise signal $n(t)$ as an input to the system of figure 2.1.**

i) **Show that the two branches of the system will output the two components $n_c(t)$ and $n_s(t)$.**

Write down expressions for the two signals z_1 and z_2:

$$\begin{aligned} z_1 &= n(t)2\cos(\omega_c t) \\ &= (n_c(t)\cos(\omega_c t) - n_s(t)\sin(\omega_c t))2\cos(\omega_c t) \\ &= n_c(t)2\cos^2(\omega_c t) - n_s(t)2\sin(\omega_c t)\cos(\omega_c t) \\ &= n_c(t)(1+\cos(2\omega_c t)) - n_s(t)\sin(2\omega_c t) \end{aligned}$$

LPF can be used to allow only $n_c(t)$ to pass.

$$\begin{aligned} z_2 &= -n(t)2\sin(\omega_c t) \\ &= -(n_c(t)\cos(\omega_c t) - n_s(t)\sin(\omega_c t))2\sin(\omega_c t) \\ &= -n_c(t)2\cos(\omega_c t)\sin(\omega_c t) + n_s(t)2\sin^2(\omega_c t) \\ &= -n_c(t)\sin(2\omega_c t) + n_s(t)(1-\cos(2\omega_c t)) \end{aligned}$$

LPF can be used to allow only $n_s(t)$ to pass.

[Material related to a tutorial problem] [6]

ii) **Explain how you would choose the low pass filter in each branch so that the above is true.**

The LPF should be chosen so that its bandwidth is less than $2f_c$ (where $\omega_c = 2\pi f_c$) to cut off the trigonometric terms in the output signals. Also, it should have flat response for frequencies below $2f_c$ so that it does not distort the component that passes.

[Material related to a tutorial problem] [2]

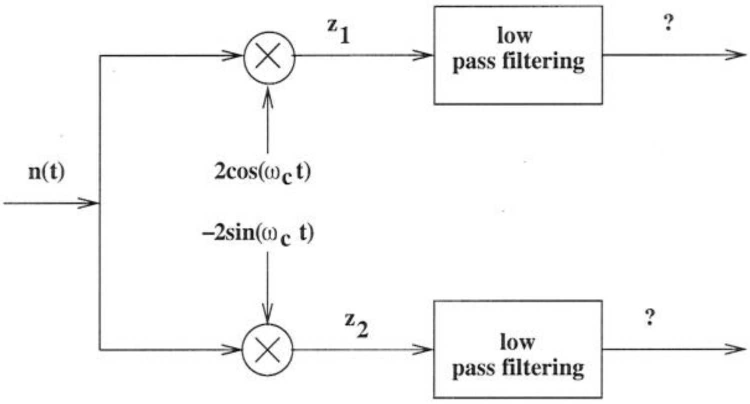

Figure 2.1 The system used in part (c) of question 2

3. a) **A 5-sample long digital signal is transmitted down a noisy communication line five times. Five versions of that signal are received, and they constitute an ensemble of signals. These versions are:**

$$(5,4,3,7,6)$$
$$(4,3,7,5,6)$$
$$(4,4,7,3,7)$$
$$(6,7,5,5,2)$$
$$(6,7,3,5,4)$$

i) **Calculate the ensemble mean signal.**

Average the columns to get: $(5,5,5,5,5)$

[*New example*] [3]

ii) **Is the signal stationary with respect to the mean? Justify your answer.**

Yes, because all samples have the same mean.

[*New example*] [3]

iii) **Is this signal ergodic with respect to the mean? Justify your answer.**

We must compute the average of each row. All rows give 5. So, the temporal average of any instantiation of the signal is the same as its ensemble average at any instant in time. So, it is ergodic with respect to the mean.

[*New example*] [3]

iv) **Calculate the ensemble autocorrelation function of this signal $R(t_2,t_3)$, where t_2 and t_3 refer to the times the second and the third sample of the signal are received.**

$$R(t_2,t_3) = \frac{4\times3+3\times7+4\times7+7\times5+7\times3}{5} = \frac{117}{5} = 23.4 \quad (3.1)$$

[*New example*] [5]

v) **Is the signal stationary with respect to the autocorrelation function? Justify your answer.**

We compute the value of the autocorrelation function for another pair of positions, at the same relative shift from each other, say $R(t_3,t_4)$:

$$R(t_3,t_4) = \frac{3\times7+7\times5+7\times3+5\times5+3\times5}{5} = \frac{117}{5} = 23.4 \quad (3.2)$$

This value is the same as $R(t_2,t_3)$, so this signal might be stationary with respect to the autocorrelation function. I will have to check another pair of samples:

$$R(t_4,t_5) = \frac{7\times6+5\times6+3\times7+5\times2+5\times4}{5} = \frac{123}{5} \neq 23.4 \quad (3.3)$$

This value is different from the previous two, so the signal is not stationary with respect to the autocorrelation function.

vi) **Calculate the temporal autocorrelation function of the fifth instantiation of this signal for shift $\tau = 3$.**

$$R(3) = \frac{6 \times 5 + 7 \times 4}{2} = \frac{58}{2} = 29 \tag{3.4}$$

[New example] [5]

vii) **Is this signal ergodic with respect to the autocorrelation function? Justify your answer.**

Since it is not stationary with respect to the autocorrelation function, it cannot be ergodic.

[New example] [1]

b) **A source produces symbols from a four-symbol alphabet $\{E, V, O, L\}$, with corresponding frequencies $0.5, 0.15, 0.25, 0.1$, respectively.**

i) **Use Huffman coding to construct a coding scheme for this source.**

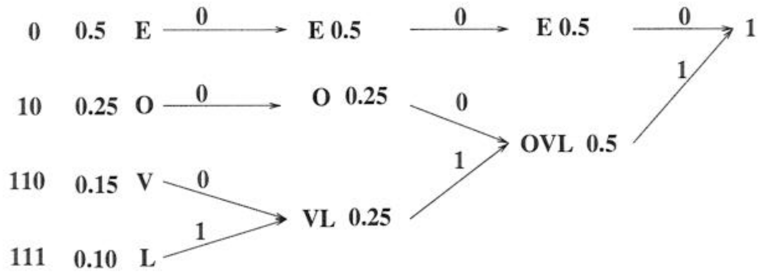

Figure 3.1 Huffman coding of the 4 symbol source

[New example] [4]

ii) **Decipher the string:** 111101100111010.

LOVELEO

[New example] [1]

4. **Consider a quantiser with uniform separation of Δ volts between the quantisation levels.**

a) **Write down an expression for the probability density function of the quantisation error q.**

It will be uniformly distributed in the range $[-\frac{\Delta}{2}, \frac{\Delta}{2}]$. So,

$$p(q) = \begin{cases} \frac{1}{\Delta} & \text{for } -\frac{\Delta}{2} \leq q \leq \frac{\Delta}{2} \\ 0 & \text{elsewhere} \end{cases} \tag{4.1}$$

[*Bookwork*] [1]

b) **Work out the mean square error of the quantiser.**

$$\begin{aligned} <e^2> &= \int_{-\infty}^{\infty} q^2 p(q) dq \\ &= \frac{1}{\Delta} \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} q^2 dq \\ &= \frac{1}{\Delta} \left[\frac{q^3}{3} \right|_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} \\ &= \frac{1}{3\Delta} \left[\left(\frac{\Delta}{2}\right)^3 - \left(-\frac{\Delta}{2}\right)^3 \right] \\ &= \frac{1}{3\Delta} 2 \frac{\Delta^3}{8} \\ &= \frac{\Delta^2}{12} \end{aligned}$$

[*Bookwork*] [3]

c) **If the quantiser is used to quantise symbols with n bits each, what is the maximum peak-to-peak dynamic range of the quantiser?**

The maximum number of quantisation levels will be 2^n. So, the total dynamic range of the quantiser will be $2^n\Delta$.

[*Bookwork*] [1]

d) **Assuming that a message signal fully loads the quantiser, what is its maximum absolute value m_p of the message signal?**

$m_p = \frac{2^n\Delta}{2} = 2^{n-1}\Delta$

[*Bookwork*] [1]

e) **Write down an expression for the signal to noise ratio of the quantiser output in terms of Δ, if the power of the message signal is P.**

$$SNR = \frac{P}{P_N} = \frac{P}{\frac{\Delta^2}{12}} = \frac{12P}{\Delta^2} \tag{4.2}$$

[*Bookwork*] [1]

f) **Work out the signal to noise ratio of the quantiser output in terms of m_p.**

$$m_p = 2^{n-1}\Delta \Rightarrow \Delta = \frac{m_p}{2^{n-1}} \tag{4.3}$$

Then

$$SNR = \frac{12P}{\Delta^2} = \frac{12P}{\frac{m_p^2}{2^{2(n-1)}}} = \frac{12P2^{2(n-1)}}{m_p^2} = \frac{3P2^{2n}}{m_p^2} \tag{4.4}$$

[*Bookwork*] [3]

g) **Consider a full load signal:** $m(t) = A_m \sin(2\pi ft)\cos(2\pi ft)$.

i) **Compute the average power P of this signal.**

We note that $m(t) = \frac{A_m}{2}\sin(4\pi ft)$.

This is a periodic signal with period $T = \frac{1}{2f}$. Then

$$\begin{aligned} P &= \frac{1}{T}\int_{-T/2}^{T/2} \frac{A_m^2}{4}\sin^2(4\pi ft)dt \\ &= \frac{A_m^2}{4T}\int_{-T/2}^{T/2} \frac{1-\cos(8\pi ft)}{2}dt \\ &= \frac{A_m^2}{8T}(t - \sin(8\pi ft)|_{-T/2}^{T/2} \\ &= \frac{A_m^2}{8T}T \\ &= \frac{A_m^2}{8} \end{aligned}$$

[*New example*] [10]

ii) **Derive an expression of the signal to noise ratio at the output of the quantiser described in parts (a)–(f) when this signal is represented by symbols with n bits each.**

Here $m_p = A_m/2$, so:

$$SNR = \frac{3\frac{A_m^2}{8}2^{2n}}{\frac{A_m^2}{4}} = \frac{3}{2}2^{2n} \tag{4.5}$$

[*New example*] [6]

iii) **Express the result in the above question in dB.**

$$SNR = 10log_{10}1.5 + 2n10\log_{10}2 = 1.76 + 6.02n \tag{4.6}$$

[*New example*] [3]

iv) **Comment on the effect on the quantisation noise as the number of quantisation levels increases.**

We note that increasing the bits used to represent each symbol by 1, adds about 6db to the SNR.

[*Bookwork*] [1]