Imperial College London

MEng (Engineering) Examination 2016

Year 1

AE1-107 Mathematics Term II

Tuesday 24th May 2016: 14.00 to 16.00
[2 hours]

The paper is divided into Section A and Section B

Both sections carry the same weight

Candidates may obtain full marks for complete answers to ***ALL*** questions.

You must answer each section in a separate answer booklet

The use of lecture notes is NOT allowed.

Section A

1. Consider the ODE:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{xe^{x^2}}{y}.$$

(a) Find the general solution to the ODE. [15%]

Now consider the ODE:

$$y\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{6xy^2 - y^3}{3xy - 6x^2}.$$

(b) Show that the ODE is inexact. [10%]

(c) Find an integrating factor that makes the ODE exact.
Hint: the required integrating factor is a function of y alone. [30%]

(d) Find the general solution to the ODE. [25%]

Finally consider the ODE:

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y = y^3.$$

(e) Find the general solution to the ODE. [20%]

2. The Laplace transform $F(s)$ of a function $f(t)$ is defined as:

$$F(s) = \int_0^\infty f(t)e^{-st}\,\mathrm{d}t$$

for some appropriate range of s.

(a) Derive expressions for the Laplace transform of $\dfrac{\mathrm{d}f}{\mathrm{d}t}$ and the Laplace transform of $\dfrac{\mathrm{d}^2 f}{\mathrm{d}t^2}$ in terms of the Laplace transform of $f(t)$. Show all your workings. [25%]

Consider the ODE:

$$a\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + y = f(x).$$

(b) If $a = 1$, $b = 2$, $f(x) = 0$, find the general solution to the ODE. [15%]

(c) If $a = 1/4$, $b = 0$, $f(x) = \sin(2x)$, find the general solution to the ODE. [35%]

(d) If $a = 2$, $b = 1$, $f(x) = x^2$, use Laplace transforms to find the Laplace transform of a particular solution to the ODE that satisfies $\dfrac{\mathrm{d}y}{\mathrm{d}x}(0) = 1$ and $y(0) = 0$. [25%]

Section B

3. Let

$$A(p) = \begin{bmatrix} 1 & -1 & 8 \\ 2 & 0 & 6 \\ -1 & 5 & p \\ 3 & 1 & 4 \end{bmatrix}, b(q) = \begin{bmatrix} 2 \\ -4 \\ q \\ -10 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(a) Show that if $p \neq -28$, the set of equations $A(p)x = b(q)$ has a unique solution for all q. Find this solution x in terms of p and q. [35%]

(b) Show that $A(-28)x = b(q)$ has no solution if $q \neq -18$. Find all possible solutions x in this case when $q = -18$. [35%]

Let

$$B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 2 & 1 & 4 \end{bmatrix}$$

(c) Find the eigenvalues and eigenvectors of B. [30%]

4. Consider the system

$$\frac{\mathrm{d}X(t)}{\mathrm{d}t} = AX(t)$$

where

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

and

$$X(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

(a) Find the eigenvalues and eigenvectors of A. [30%]

(b) Solve the system to find $x_1(t)$ and $x_2(t)$. [40%]

(c) Evaluate A^{50}. [30%]

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Marks

1.) a.)

$$\frac{dy}{dx} = \frac{xe^{x^2}}{y}$$

$$\Rightarrow \int y\,dy = \int xe^{x^2}\,dx$$

$$\Rightarrow \frac{1}{2}y^2 = \frac{1}{2}e^{x^2} + C$$

$$\Rightarrow y^2 = e^{x^2} + D$$ Ok

$$\Rightarrow y = \pm\sqrt{e^{x^2} + D}$$ 3

1.) b.)

$$y\frac{dy}{dx} = \frac{6xy^2 - y^3}{3xy - 6x^2}$$

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$$\Rightarrow \frac{dy}{dx} = -\frac{f(x,y)}{g(x,y)} \quad \text{where}$$

$$f(x,y) = y^2 - 6xy$$

$$g(x,y) = 3xy - 6x^2$$

$$\Rightarrow \frac{\partial f}{\partial y} = 2y - 6x$$

$$\frac{\partial g}{\partial x} = 3y - 12x$$

Not equal! $\Rightarrow$ Inexact — 2

1.)c.) Require $I(y)$ s.t.

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$$\frac{\partial}{\partial y}\left[I(y)\left(y^2-6xy\right)\right] =$$

$$\frac{\partial}{\partial x}\left[I(y)\left(3xy-6x^2\right)\right]$$

$$\Rightarrow 2y\,I(y) + \frac{dI(y)}{dy}y^2 - 6xy\frac{dI(y)}{dy}$$

$$- I(y)6x =$$

$$3I(y)y - 12I(y)x$$

$$\Rightarrow \frac{dI(y)}{dy}\left(y^2-6xy\right) =$$

$$I(y)\left(y-6x\right)$$

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Marks

$$\Rightarrow \frac{1}{I(y)}\frac{dI(y)}{dy} = \frac{1}{y}\left(\frac{y-6x}{y-6x}\right) = \frac{1}{y}$$

$\Rightarrow I(y) = y$ is an integrating factor. 6

1.)d.) $f = y(y^2 - 6xy)$

$g = y(3xy - 6x^2)$

Need $u(x,y)$ s.t.

$$\frac{\partial u}{\partial x} = f \quad , \quad \frac{\partial u}{\partial y} = g$$

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	Marks
$\Rightarrow u = y^3x - 3x^2y^2 + C(y)$	
$u = y^3x - 3x^2y^2 + D(x)$	
$\Rightarrow C(y) = D(x) = 0$	
$\Rightarrow u = y^3x - 3x^2y^2$	
$\Rightarrow y^3x - 3x^2y^2 = F$	5
1)e.) Let $y = u^{-\frac{1}{2}}$	
$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = -\frac{1}{2}u^{-\frac{3}{2}}\frac{du}{dx}$	

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$$\Rightarrow -\frac{1}{2}u^{-\frac{3}{2}}\frac{du}{dx} + u^{-\frac{1}{2}} = u^{-\frac{3}{2}}$$

$$\Rightarrow \frac{du}{dx} - 2u = -2$$

Linear

$$\text{I.F.} \quad I = \exp\left[\int -2\,dx\right] = e^{-2x}$$

$$u = -2e^{+2x}\int e^{-2x}\,dx$$

$$= -2e^{2x}\left[-\frac{1}{2}e^{-2x} + C\right]$$

$$= 1 - 2Ce^{2x}$$

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$\Rightarrow y = \dfrac{1}{\sqrt{1-2Ce^{2x}}}$	4

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2.) a.) $$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

$$L\left\{\frac{df}{dt}\right\} = \int_0^\infty \frac{df}{dt} e^{-st}\, dt$$

$$= -f(0) + s\int_0^\infty f(t)\, e^{-st}\, dt$$

$$= -f(0) + sL\{f\}$$

$$\Rightarrow L\left\{\frac{df}{dt}\right\} = sL\{f\} - f(0)$$

$$\Rightarrow L\left\{\frac{d^2f}{dt^2}\right\} = sL\left\{\frac{df}{dt}\right\} - \frac{df}{dt}(0)$$

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$$\Rightarrow L\left\{\frac{d^2 f}{dt^2}\right\} = s\left(s\,L\{f\} - f(0)\right) - \frac{df}{dt}(0)$$

$$= s^2 L\{f\} - sf(0) - \frac{df}{dt}(0)$$ 5

2) b) $\alpha^2 + 2\alpha + 1 = 0$

$$\Rightarrow \alpha = \frac{-2 \pm \sqrt{4-4}}{2}$$

$$= -1$$

$$\Rightarrow y = (A + Bx)e^{-x}$$ 3

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Marks

2(f.)

$$\frac{\alpha^2}{4} + 1 = 0$$

$$\Rightarrow \quad \alpha = \pm 2i$$

$$\Rightarrow y_h = A\sin(2x) + B\cos(2x)$$

For y_p, try $y_p = xC\sin(2x) + xD\cos(2x)$

(since $C\sin(2x) + D\cos(2x) \equiv y_h$)

$$y_p' = 2xC\cos(2x) + C\sin(2x) - 2xD\sin(2x) + D\cos(2x)$$

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$$y_p'' = -4xC\sin(2x) + 4C\cos(2x) - 4xD\cos(2x) - 4D\sin(2x)$$

Sub in to ODE $\Rightarrow$

$$-xC\sin(2x) + C\cos(2x) - xD\cos(2x) - D\sin(2x) + xC\sin(2x) + xD\cos(2x) = \sin(2x)$$

$\Rightarrow C=0, \quad D=-1 \Rightarrow$

$$y_p = -x\cos(2x) \Rightarrow$$

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$$y = y_c + y_p =$$

$$A\sin(2x) + B\cos(2x)$$

$$- x\cos(2x)$$

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2.) d.)

$$2L\left\{\frac{d^2y}{dx^2}\right\} + L\left\{\frac{dy}{dx}\right\} + L\{y\} = L\{x^2\}$$

$$\Rightarrow 2\left[s^2L\{y\} - sy(0) - y'(0)\right]$$

$$+ sL\{y\} - y(0) + L\{y\}$$

$$= \frac{2}{s^3}$$

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$$\Rightarrow 2s^2 L\{y\} - 2 + sL\{y\} + L\{y\} = \frac{2}{s^3}$$

$$\Rightarrow L\{y\}\left(2s^2 + s + 1\right) = \frac{2}{s^3} + 2$$

$$\Rightarrow L\{y\} = \frac{2 + 2s^3}{s^3\left(2s^2 + s + 1\right)}$$

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Section B

1. Let

$$A(p) = \begin{bmatrix} 1 & -1 & 8 \\ 2 & 0 & 6 \\ -1 & 5 & p \\ 3 & 1 & 4 \end{bmatrix}, b(q) = \begin{bmatrix} 2 \\ -4 \\ q \\ -10 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(a) Show that if $p \neq -28$, the set of equations $A(p)x = b(q)$ has a unique solution for all q. Find this solution x in terms of p and q [35%]

$$\begin{bmatrix} 1 & -1 & 8 \\ 2 & 0 & 6 \\ -1 & 5 & p \\ 3 & 1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -4 \\ q \\ -10 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 8 & 2 \\ 0 & 2 & -10 & -8 \\ 0 & 4 & p+8 & q+2 \\ 0 & 4 & -20 & -16 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 8 & 2 \\ 0 & 2 & -10 & -8 \\ 0 & 0 & p+28 & q+18 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$p \neq -28$ a unique solution

$$x_3 = \frac{q+18}{p+28}$$

$$x_2 = -4 + 5\frac{q+18}{p+28} = \frac{-4p+5q-22}{p+28}$$

$$x_3 = 2 - 8\frac{q+18}{p+28} + \frac{-4p+5q-22}{p+28} = \frac{-2p+3q-110}{p+28}$$

(b) Show that $A(-28)x = b(q)$ has no solution if $q \neq -18$. Find all possible solutions x in this case when $q = -18$. [35%]

$p = -28, q \neq -18$
$\rightarrow 0 = q + 18 \neq 0 \rightarrow$ no solution
$p = -28, q = -18$
$x_3 = t$ arbitrary
$x_2 = -4 + 5t$
$x_1 = -2 - 3t$

(c) Let

$$B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 2 & 1 & 4 \end{bmatrix}$$

find eigenvalues and eigenvectors [30%]

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The characteristic equation is $(3-\lambda)^2(4-\lambda)=0$ so there are two coincident eigenvalues $\lambda_1=\lambda_2=3$ and an independent one $\lambda_3=4$. For $\lambda_3=4$ the system

$$B=\begin{bmatrix}3-4 & 0 & 0\\ 0 & 3-4 & 0\\ 2 & 1 & 4-4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

becomes a system of 2 equations in 3 unknowns $x=y=0$; so the eigenvectors of B for $\lambda_3=4$ are the eigenvectors

$$v_3=\begin{bmatrix}0\\ 0\\ x_3\end{bmatrix}$$

For $\lambda_1=\lambda_2=3$ the system becomes $2x+y+z=0$; so the eigenvectors are

$$v=\begin{bmatrix}x_1\\ x_2\\ -2x_1-x_2\end{bmatrix}=\begin{bmatrix}1\\ 0\\ -2\end{bmatrix}x_1+\begin{bmatrix}0\\ 1\\ -1\end{bmatrix}x_2$$

and this allows more than one solution linearly independent. For example for x=1 and y=0 or x=0 and y=1.

$$v_1=\begin{bmatrix}1\\ 0\\ -2\end{bmatrix}v_2=\begin{bmatrix}0\\ 1\\ -1\end{bmatrix}$$

or for x=1 and y=-2 or x=0 and y=1.

$$v_1=\begin{bmatrix}1\\ -2\\ 0\end{bmatrix}v_2=\begin{bmatrix}0\\ 1\\ -1\end{bmatrix}$$

so the eigenvectors matrix

$$V=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -2 & -1 & 1\end{bmatrix}$$

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2. Consider the system

$$\frac{dX(t)}{dt} = AX(t)$$

where

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

and

$$X(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

(a) Find the eigenvalues and eigenvectors of A. [30%]
Eigenvalues of A

$$(2-\lambda)(2-\lambda) - 1 = 0$$

$$\lambda_1 = 1, \lambda_2 = 3$$

Eigenvector of A for $\lambda_1 = 1$

$$A = \begin{bmatrix} 2-1 & 1 \\ 1 & 2-1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$u_1 + u_2 = 0$$

$$U_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Eigenvector of A for $\lambda_2 = 3$

$$A = \begin{bmatrix} 2-3 & 1 \\ 1 & 2-3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$u_1 - u_2 = 0$$

$$U_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

(b) Solve the system to find $x_1(t)$ and $x_2(t)$. [40%]
So we can write

$$A = U\Lambda U^{-1}$$

where

$$U = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

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so

$$\frac{dX}{dt} = AX$$

$$\frac{dX}{dt} = U\Lambda U^{-1}X$$

$$\frac{dU^{-1}X}{dt} = \Lambda U^{-1}X$$

$$\frac{dZ}{dt} = \Lambda Z, Z = U^{-1}X$$

so

$$\frac{dz_1}{dt} = \lambda_1 z_1(t) = 1z_1(t),$$
$$\frac{dz_2}{dt} = \lambda_2 z_1(t) = 3z_1(t)$$

and

$$z_1(t) = \alpha e^{\lambda_1 t} = \alpha e^{1t},$$
$$z_2(t) = \beta e^{\lambda_2 t} = \beta e^{3t}$$

$$X = UZ = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \alpha e^{1t} \\ \beta e^{3t} \end{bmatrix}$$

$$x_1(t) = \alpha e^{1t} + \beta e^{3t},$$
$$x_2(t) = \alpha e^{1t} - \beta e^{3t},$$

since $x_1(0) = 1$ and $x_2(0) = 3$

$$1 = \alpha + \beta,$$
$$3 = \alpha - \beta,$$
$$\alpha = 1, \beta = -2$$

$$x_1(t) = 1e^{1t} - 2e^{3t},$$
$$x_2(t) = 1e^{1t} + 2e^{3t},$$

(c) Evaluate A^{50}. [30%]

$$A = U\Lambda U^{-1}$$
$$A^{50} = (U\Lambda U^{-1})^{50}$$
$$A^{50} = U\Lambda^{50}U^{-1}$$

$$U = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

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$$U^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\Lambda^{50} = \begin{bmatrix} 1 & 0 \\ 0 & 3^{50} \end{bmatrix}$$

$$A^{50} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3^{50} \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} = 10^{23} \begin{bmatrix} 3.5895 & -3.5895 \\ -3.5895 & 3.5895 \end{bmatrix}$$