E3.09 | ISE 3.9

Control engineering exam paper - Model answers

2008

Question 1

a) To show that system (1.2) matches system (1.1) at s_1 and s_2 we need to verify that

$$H(s_iI - F)^{-1}G = C(s_iI - A)^{-1}B,$$

for $i = 1, 2$. Note that

$$F = \begin{bmatrix} s_1 - \Delta_1 & -\Delta_1 \\ -\Delta_2 & s_2 - \Delta_2 \end{bmatrix}$$

and that

$$(sI - F)^{-1} = \frac{1}{\Delta(s)} \begin{bmatrix} s - s_2 + \Delta_2 & -\Delta_1 \\ -\Delta_2 & s - s_1 + \Delta_1. \end{bmatrix},$$

where

$$\Delta(s) = s^2 + s((\Delta_1 - s_1) + (\Delta_2 - s_2)) + (s_1 s_2 - s_1\Delta_2 - s_2\Delta_1).$$

A direct calculation shows that

$$(s_1I - F)^{-1}G = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad (s_2I - F)^{-1}G = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

which proves the claim.

b) The reachability matrix is

$$R = \begin{bmatrix} \Delta_1 & \Delta_1(s_1 - \Delta_1) - \Delta_1\Delta_2 \\ \Delta_2 & \Delta_2(s_2 - \Delta_2) - \Delta_1\Delta_2 \end{bmatrix},$$

and $\det(R) = (s_2 - s_1)\Delta_1\Delta_2$. Hence, since $s_1 \neq s_2$, the reachability matrix is full rank if and only if $\Delta_1\Delta_2 \neq 0$.

c) If $C(s_1I - A)^{-1}B = C(s_2I - A)^{-1}B = \kappa$ then the observability matrix is

$$O = \begin{bmatrix} \kappa & \kappa \\ \kappa(s_1 - \Delta_1 - \Delta_2) & \kappa(s_2 - \Delta_2 - \Delta_1) \end{bmatrix},$$

and $\det(O) = \kappa^2(s_2 - s_1)$. Hence the system is observable if and only if $\kappa \neq 0$.

d) Since $s_1 = 0$ and $s_2 = 1$ we have

$$F = \begin{bmatrix} -\Delta_1 & -\Delta_1 \\ -\Delta_2 & 1 - \Delta_2 \end{bmatrix}.$$

The characteristic polynomial of F is

$$s^2 + s(\Delta_1 + \Delta_2 - 1) - \Delta_1$$

and this should be equal to $(s + 1)^2 = s^2 + 2s + 1$. As a result

$$\Delta_1 = -1 \qquad \Delta_2 = 4.$$

Question 2

a) Note that the system can be written as

$$x(k+1) = \begin{bmatrix} 0 & I \\ G & 0 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ B \end{bmatrix} u(k).$$

The reachability matrix of this system is

$$R = \begin{bmatrix} 0 & B & 0 & GB & 0 & G^2B & \cdots \\ B & 0 & GB & 0 & G^2B & 0 & \cdots \end{bmatrix},$$

hence the rank of R is twice the rank of the matrix

$$\begin{bmatrix} B & GB & G^2B & \cdots \end{bmatrix},$$

which is the reachability matrix of the system

$$\xi(k+1) = G\xi(k) + Bv(k).$$

Hence, system (2.1) is reachable if and only if the above system is reachable.

b) If $m = n$ and $B = I$ then the system

$$\xi(k+1) = G\xi(k) + Bv(k) = G\xi(k) + v(k)$$

is reachable, hence by part a), the system (2.1) is reachable.

c) i) A family of matrices $K = K'$ which achieves the stabilization goal is $K = \alpha I$, with $\alpha \in (-1, 1)$.

ii) Note that

$$e(k) = x_2(k) - \alpha x_1(k),$$

hence

$$e(k+1) = x_2(k+1) - \alpha x_1(k+1) = Gx_1(k) + u(k) - \alpha x_2(k).$$

iii) Selecting

$$u(k) = -Gx_1(k) + \alpha x_2(k)$$

yields $e(k+1) = 0$, which implies that $e(k) = 0$ for all $k \geq 1$ (and all $e(0)$).

iv) The control law determined in part c.iii) stabilizes the discrete-time system because, for $k \geq 1$ we have $e(k) = 0$, which implies $x_2(k) = \alpha x_1(k)$, for all $k \geq 1$. Therefore, for $k \geq 1$

$$x_1(k+1) = \alpha x_1(k),$$

hence $x_1(k)$ converges to zero as $k \to \infty$. This, together with the fact that $e(k) = 0$, for all $k \geq 1$, implies that $x_2(k)$ also converges to zero as $k \to \infty$. As a result, the zero equilibrium of the system is attractive and, since this is a linear, time-invariant system, attractivity implies stability, *i.e.* the system is asymptotically stable.

Question 3

a) The approximate discrete-time Euler model is

$$x(k+1) = (I + TA)x(k) + TBu(k) = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ T \end{bmatrix} u(k).$$

b) The reachability matrix of the Euler model is

$$R = \begin{bmatrix} 0 & T^2 \\ T & T, \end{bmatrix}$$

and R has full rank for all $T > 0$, *i.e.* the approximate model is reachable for all $T > 0$.

c) Consider the matrix

$$I + TA + TBK = \begin{bmatrix} 1 & T \\ TK_1 & 1 + TK_2 \end{bmatrix}.$$

Its characteristic polynomial is

$$s^2 + s(-2 - TK_2) + (1 + TK_2 - T^2 K_1),$$

and this should be equal to s^2. As a result

$$K_1 = -\frac{1}{T^2} \qquad K_2 = -\frac{2}{T}.$$

d) i) By definition of the matrix exponential we have

$$A_d = e^{AT} = I + AT + A^2 \frac{T^2}{2} + \cdots.$$

Since $A^2 = A^3 = \cdots = 0$, then

$$A_d = I + AT = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}$$

and

$$B_d = \int_0^T \begin{bmatrix} 1 & T - \tau \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} d\tau = \int_0^T \begin{bmatrix} T - \tau \\ 1 \end{bmatrix} d\tau = \begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix}.$$

Note that the matrix A_d coincides with the A matrix of the Euler model, whereas the matrix B_d is different.

ii) The matrix $A_d + B_d K$, with K as in part c) is

$$\begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{T} & -1 \end{bmatrix}.$$

This matrix has eigenvalues equal to -1 and $1/2$. Hence, the resulting system is stable, not asymptotically, *i.e.* the design based on the Euler approximate model is not adequate to stabilize the sampled-data system. (A more careful analysis, note requested, would reveal that the main reason why the design is not effective is the selection of the closed-loop eigenvalues.)

Question 4

a) i) The linearized system at $x = 0$ is described by

$$\dot{\delta}_x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \delta_x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \delta_u.$$

ii) The reachability matrix of the linearized system is

$$R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Note that $\text{rank} R = 2$, hence the system is reachable.

iii) A direct application of Ackerman formula yields

$$K_a = \begin{bmatrix} -1 & -2 \end{bmatrix}.$$

iv) The linearized closed-loop system is asymptotically stable hence, by the principle of stability in the first approximation, the zero equilibrium of the controlled nonlinear system is locally asymptotically (exponentially) stable.

b) i) Note that

$$\dot{y} = \dot{x}_1 = x_1^2 + x_2.$$

Hence

$$\ddot{y} = 2x_1\dot{x}_1 + \dot{x}_2 = 2x_1(x_1^2 + x_2) + x_1x_2 + u.$$

As a result

$$\ddot{y} + \left(-2x_1^3 - 3x_1x_2\right) = u.$$

ii) Setting

$$K_b(x) = \left(-2x_1^3 - 3x_1x_2\right) - 2\dot{y} - y = \left(-2x_1^3 - 3x_1x_2\right) - 2(x_1^2 + x_2) - x_1,$$

yields

$$\ddot{y} + 2\dot{y} + y = 0,$$

as requested.

iii) The controlled nonlinear system can be written using the coordinates $(y, \dot{y})$. In these coordinates the system is linear and the zero equilibrium is globally asymptotically stable.

c) Clearly the control law $K_b(x)$ is more complex, *e.g.* requires more computation, than the control law $K_a(x)$. However, while $K_b(x)$ is such that the zero equilibrium of the closed-loop system is globally asymptotically stable, $K_a(x)$ only guaranteees a local property. (Note, in addition, that the system in closed loop with the controller $u = K_a x$ has two equilibria: $(0, 0)$ and $(1, -1)$. This proves that the zero equilibrium cannot be globally asymptotically stable.)

Question 5

a) The observability matrix of the system is

$$O = \begin{bmatrix} 1 & 0 & 1 \\ 0 & \alpha & -\frac{1}{2} \\ -\alpha^2 & 0 & \alpha + \frac{1}{4} \end{bmatrix}.$$

The determinant of the observability matrix is

$$\det O = \alpha^3 + \alpha^2 + \frac{1}{4}\alpha.$$

Hence the observability matrix loses rank for $\alpha = 0$ and $\alpha = -1/2$. The system is therefore observable for all $\alpha \neq 0$ and $\alpha \neq -1/2$.
For $\alpha = 0$ the observability pencil is

$$\begin{bmatrix} sI - A \\ C \end{bmatrix} = \begin{bmatrix} s & 0 & 0 \\ 0 & s & -1 \\ 0 & 0 & s + 1/2 \\ 1 & 0 & 1 \end{bmatrix}$$

and it loses rank only for $s = 0$. Therefore the unobservable mode is for $s = 0$, hence the system is detectable and reconstructable for $\alpha = 0$.
For $\alpha = -1/2$ the observability pencil is

$$\begin{bmatrix} sI - A \\ C \end{bmatrix} = \begin{bmatrix} s & 1/2 & 0 \\ -1/2 & s & -1 \\ 0 & 0 & s + 1/2 \\ 1 & 0 & 1 \end{bmatrix}$$

and it loses rank only for $s = -1/2$. Therefore the unobservable mode is for $s = -1/2$, hence the system is detectable, but not reconstructable, for $\alpha = -1/2$.

b) To design an observer with the requested property it is necessary to find a matrix L such that the eigenvalues of $A + LC$ are all equal to zero. Such an L exists if (and only if) the system is observable or reconstructable. As a result, it is possible to design the observer with the requested property for all $\alpha \neq -1/2$.

c) For $\alpha = 0$ the observability matrix is

$$O = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{4} \end{bmatrix}.$$

The matrix has rank equal to two. A basis for the unobservable subspace is

$$\ker O = \text{span} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

To write the system in observability canonical form consider the change of coordinates $x = L\hat{x}$, with

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Note that the last column of L spans the kernel of O. In the new coordinates $\hat{x}$ the system is described by

$$\hat{x}(k+1) = L^{-1}AL\hat{x} + L^{-1}Bu = \hat{A}\hat{x} + \hat{B}u \qquad y(k) = CL\hat{x}, = \hat{C}\hat{x},$$

where

$$\hat{A} = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & -1/2 & 0 \\ \hline 0 & 1 & 0 \end{array}\right] \qquad \hat{B} = \left[\begin{array}{c} 0 \\ 1 \\ \hline 1 \end{array}\right] \qquad \hat{C} = \left[\begin{array}{cc|c} 1 & 1 & 0 \end{array}\right],$$

which clearly shows the decomposition into observable and unobservable subsystems.

Question 6

a) Note that

$$CB = 0 \qquad CAB = 1 \qquad\qquad CP = 0 \qquad CAP = 0$$

and

$$\text{rank}\begin{bmatrix} C \\ CA \end{bmatrix} = \text{rank}\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = 2.$$

Hence $\kappa = 2$ is such that condition (C) holds.

b) The equations of the system can be rewritten as

$$\begin{aligned} \dot{x}_1 &= \hat{A}_{11}x_1 + \hat{A}_{12}x_2 + B_1u + P_1d, \\ \dot{x}_2 &= \hat{A}_{21}x_1 + \hat{A}_{22}x_2 + B_2u, \\ y &= C_2x_2, \end{aligned}$$

hence the output equation is already in the desired form. Note now that

$$\hat{A}_{21} = B_2L_1 \qquad\qquad \hat{A}_{22} = S + B_2L_2$$

for some matrices L_1 and L_2. As a result

$$\dot{x}_2 = Sx_2 + B_2(u + L_1x_1 + L_2x_2),$$

which shows that the $\dot{x}_2$ equation is in the desired form, with $L = [L_1\ L_2]$. Finally

$$\begin{aligned} \dot{x}_1 &= \hat{A}_{11}x_1 + \hat{A}_{12}x_2 + B_1u + P_1d, \\ &= \hat{A}_{11}x_1 + \hat{A}_{12}x_2 + B_1(u + L_1x_1 + L_2x_2) - B_1L_1x_1 - B_1L_2x_2 + P_1d, \\ &= (\hat{A}_{11} - B_1L_1)x_1 + (\hat{A}_{12} - B_2L_2)x_2 + B_1(u + L_1x_1 + L_2x_2) + P_1d, \end{aligned}$$

which shows that the $\dot{x}_1$ equation can be written as requested, with $A_{11} = \hat{A}_{11} - B_1L_1$, $A_{12} = \hat{A}_{12} - B_1L_2$.

c) Setting $u = -Lx + K_2x_2$ yields the equations

$$\begin{aligned} \dot{x}_1 &= A_{11}x_1 + (A_{12} + B_1K_2)x_2 + P_1d, \\ \dot{x}_2 &= (S + B_2K_2)x_2, \\ y &= C_2x_2. \end{aligned}$$

The x_2 subsystem is not directly affected by the disturbance d and by the x_1 subsystem, which is not observable. This implies that y is not affected by d. In addition, the second of the above equations yields

$$x_2(t) = e^{(S+B_2K_2)t}x_2(0),$$

hence

$$y(t) = c_2e^{(S+B_2K_2)t}x_2(0),$$

which shows again that d is not acting on y, *i.e.* the control law has decoupled the effect of the disturbance from the output.