# Digital Signal Processing and Digital Filters

## ********* Solutions *********

**Information for Candidates:**

Where a question requires a numerical answer, it must be given as a fully evaluated decimal number and not as an unevaluated arithmetic expression.

**Notation**

- All signals and filter coefficients are real-valued unless explicitly noted otherwise.
- Unless otherwise specified, upper and lower case letters are used for sequences and their $z$-transforms respectively. The signal at a block diagram node $V$ is $v[n]$ and its $z$-transform is $V(z)$.
- $x[n] = [a, b, c, d, e, f]$ means that $x[0] = a, \ldots x[5] = f$ and that $x[n] = 0$ outside this range.
- $\Re(z)$, $\Im(z)$, $z^*$, $|z|$ and $\angle z$ denote respectively the real part, imaginary part, complex conjugate, magnitude and argument of a complex number $z$.
- The expected value of $x$ is denoted $E\{x\}$.
- In block diagrams: solid arrows denote the direction of signal flow; an open triangle denotes a gain element with the gain indicated adjacently; a "+" in a circle denotes an adder/subtractor whose inputs may be labelled "+" or "−" according to their sign; the sample rate, $f$, of a signal in Hz may be indicated in the form "@$f$".

**Abbreviations**

| | | | |
|---|---|---|---|
| BIBO | Bounded Input, Bounded Output | IIR | Infinite Impulse Response |
| CTFT | Continuous-Time Fourier Transform | LTI | Linear Time-Invariant |
| DCT | Discrete Cosine Transform | MDCT | Modified Discrete Cosine Transform |
| DFT | Discrete Fourier Transform | PSD | Power Spectral Density |
| DTFT | Discrete-Time Fourier Transform | SNR | Signal-to-Noise Ratio |
| FIR | Finite Impulse Response | | |

**A datasheet is included at the end of the examination paper.**

## Key: B=bookwork, U=unseen example, T=Novel theory

  

## ********* Questions and Solutions *********

1. a) The signal $x[n]$ is zero outside the range $n \in [0, N-1]$. The impulse response $h[n]$ is zero outside the range $n \in [0, M]$.

i) The convolution of $x[n]$ and $h[n]$ is given by $y[n] = \sum_{r=0}^{M} h[r]x[n-r]$. Giving your reasons fully, determine the range of $n$ for which $y[n]$ may be non-zero. [ 3 ]

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*[B] The sum may be non-zero if any of the terms are non-zero. This will be true if any of the referenced values, $x[n-r]$ have indices in the range $[0, N-1]$ as $r$ ranges over 0 to $M$. This will be false if either $n < 0$ or if $n - M > N - 1$ and so it will be true if $n \geq 0$ and $n - M \leq N - 1$. Hence the range of $n$ for which $y[n]$ may be non-zero is $0 \leq n \leq N + M - 1$.*

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ii) The signal $w[n] = \sum_{r=0}^{M} h[r]x[(n-r)_{\text{mod}\,K}]$ is the length-$K$ circular convolution of $x[n]$ and $h[n]$. Giving your reasons fully, determine the smallest value of $K$ to ensure that $w[n] = y[n]$ for $n \in [0, K-1]$ where $y[n]$is defined in part i). Note that $m_{\text{mod}\,K}$ is the remainder when $m$ is divided by $K$ and satisfies $m_{\text{mod}\,K} \in [0, K-1]$. [ 3 ]

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*[T] To ensure $w[n] = y[n]$, we require that, for each referenced value, $x[(n-r)\ _{\text{mod}\,K}] = x[n-r]$. This means that either $(n-r)\ _{\text{mod}\,K} = n - r$ or else $x[(n-r)\ _{\text{mod}\,K}] = x[n-r] = 0$. When , $n - r$ is positivewe have $(n-r)\ _{\text{mod}\,K} = n - r$ and so problems occur only when $n - r$ is negative meaning that, $x[(n-r)\ _{\text{mod}\,K}] = x[n-r+K]$. To ensure that these values are 0 , we need to have $n - r + K > N - 1$ for all values of $n$ and $r$. The worst case is $n = 0$ and $r = M$ which gives $-M + K > N - 1$ or $K \geq N + M$.*

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b) i) Determine the $z$-transform of $x[n] = a^n u[n-1]$ and its region of convergence when $a$ is a positive real number and $u[n]$ is the unit step function. [ 3 ]

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*[U] Using the geometric progression formula given in the datasheet, we can write*

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n]z^{-n} = \sum_{n=1}^{\infty} a^n z^{-n}$$
$$= \sum_{n=1}^{\infty} (az^{-1})^n = \frac{az^{-1}}{1 - az^{-1}}$$

*provided that $|az^{-1}| < 1 \Leftrightarrow |z| > a$. Thus the z-transform is $\frac{az^{-1}}{1-az^{-1}}$ and the ROC is $|z| > a$.*

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ii) Show that $y[n] = -a^n u[-n]$ has the same $z$-transform but a different region of convergence. [ 2 ]

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*[U]We can write*

$$Y(z) = \sum_{n=-\infty}^{+\infty} y[n]z^{-n} = \sum_{n=-\infty}^{0} -\alpha^n z^{-n}$$

*We now make the substitution $n = -m$ to obtain*

$$Y(z) = -\sum_{m=0}^{\infty} \alpha^{-m} z^m = -\sum_{m=0}^{\infty} \left(\alpha^{-1}z\right)^m = \frac{-1}{1-\alpha^{-1}z}$$

*provided that $\left|\alpha^{-1}z\right| < 1 \Leftrightarrow |z| < \alpha$. Thus the z-transform is $\frac{-1}{1-\alpha^{-1}z} = \frac{\alpha z^{-1}}{1-\alpha z^{-1}}$ and the ROC is $|z| < \alpha$.*

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c) A first-order IIR filter is given by $H(z) = \frac{1-z^{-1}}{1-az^{-1}}$ where $a$ is real with $|a| < 1$.

i) If $a = 0.8$, sketch a diagram of the complex plane showing the unit circle, $|z| = 1$, and indicating any poles and zeros of $H(z)$ by crosses and circles respectively. [2]

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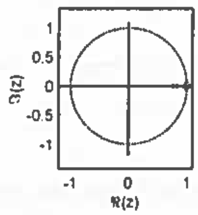
*[U] There is one zero at $z = 1$ and one pole at $z = a = 0.8$.*

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ii) Calculate $\left|H(e^{j\omega})\right|$ for $\omega = \left\{0, \frac{\pi}{2}, \pi\right\}$ and hence sketch a dimensioned graph of $\left|H(e^{j\omega})\right|$ for $\omega \in [0, \pi]$. [3]

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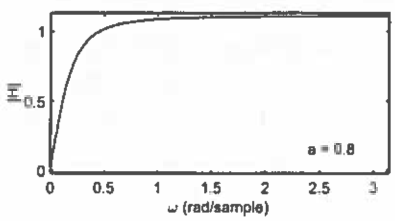
*[U] At $\omega = 0$, $z = e^{j\omega} = 1$ and $\left|H(e^{j\omega})\right| = \left|\frac{0}{0.2}\right| = 0$.*

*At $\omega = 0.5\pi$, $z = e^{j\omega} = j$ and $\left|H(e^{j\omega})\right| = \left|\frac{1+j}{1+0.8j}\right| = \left|\frac{1.4142}{1.2806}\right| = 1.1043$.*

*At $\omega = \pi$, $z = e^{j\omega} = -1$ and $\left|H(e^{j\omega})\right| = \left|\frac{1+1}{1+0.8}\right| = \left|\frac{2}{1.8}\right| = 1.1111$.*

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iii) Determine the value of $a$ such that $\left|H(e^{j0.1})\right|^2 = \frac{1}{2}\left|H(e^{j\pi})\right|^2$. [4]

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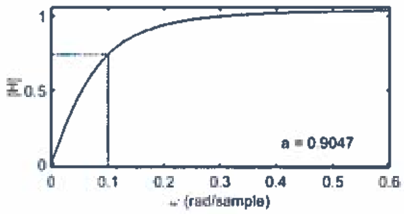
*[U] At* $\omega=\pi$, $0.5\left|H(e^{j\pi})\right|^2 = 0.5\times\left(\frac{2}{1+a}\right)^2 = \frac{2}{(1+a)^2}$. *So we need* $\frac{(1-e^{-j\omega_0})(1-e^{j\omega_0})}{(1-ae^{-j\omega_0})(1-ae^{j\omega_0})} = \frac{2-2\cos\omega_0}{1+a^2-2a\cos\omega_0} = \frac{2}{(1+a)^2}$. *Cross-multipying gives*

$$
\begin{aligned}
2(1-\cos\omega_0)(1+a)^2 &= 2\left(1+a^2-2a\cos\omega_0\right)\\
(1-\cos\omega_0)\left(1+2a+a^2\right) &= 1+a^2-2a\cos\omega_0\\
\Rightarrow (\cos\omega_0)a^2-2a+\cos\omega_0 &= 0\\
\Rightarrow a &= \frac{2\pm\sqrt{4-4\cos^2\omega_0}}{2\cos\omega_0}
\end{aligned}
$$

*Substituting in* $\omega_0=0.1 \Rightarrow \cos\omega_0 = 0.995$, *gives*

$$
a=\frac{2\pm\sqrt{4-4\times 0.99}}{2\times 0.995}=\frac{2\pm 0.1997}{1.99}=\frac{1.8003}{1.99}=0.9047
$$

*The magnitude response (not requested) is*

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d) i) The frequency response of an ideal lowpass filter is given by

$$
H(e^{j\omega}) = \begin{cases} 1 & |\omega|\le\omega_0 \\ 0 & |\omega|>\omega_0 \end{cases}.
$$

By taking the inverse DTFT of $H(e^{j\omega})$, show that the corresponding impluse response is $h[n]=\dfrac{\sin\omega_0 n}{\pi n}$. [ 3 ]

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*[B] From the datasheet,* $h[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi}H(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0}e^{j\omega n}d\omega = \frac{1}{j2n\pi}\left[e^{j\omega}\right]_{-\omega_0}^{\omega_0} = \frac{2j\sin\omega_0}{j2n\pi} = \frac{\sin\omega_0 n}{\pi n}$.

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ii) By multiplying an ideal filter impulse response by a Hamming window, determine an expression for the real-valued coefficients of a causal FIR highpass filter of even order, $M$, whose passband is $\omega\ge 1$.

For even $M$, a non-causal symmetric Hamming window is given by $w[n]=0.54+0.46\cos\frac{2\pi n}{M+1}$ for $-0.5M\le n\le 0.5M$. [ 3 ]

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*[U] We can obtain the ideal response of a highpass filter,* $g[n]$, *in one of two ways (they give the same answer). Either take* $G(e^{j\omega})=1-H(e^{j\omega})$ *from which* $g[n]=\delta[n]-h[n]$. *Alternatively take a lowpass filter, H(z), with a*

*cutoff frequency of $\pi - 1$ and then take $G(z) = H(-z)$ by negating all odd-number coefficients as $g[n] = (-1)^n h[n]$.*

*Using the first form, we obtain for the filter, $f[n+\frac{M}{2}] = w[n]\left(\delta[n] - \frac{\sin n}{\pi n}\right) \Rightarrow$ $f[n] = w[n-\frac{M}{2}]\left(\delta[n-\frac{M}{2}] - \frac{\sin(n-\frac{M}{2})}{\pi(n-\frac{M}{2})}\right)$.*

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iii) The lowest value of $\omega > 0$ for which $W(e^{j\omega}) = 0$ is $\omega \approx \frac{4\pi}{M+1}$ where $W(e^{j\omega})$ is the DTFT of $w[n]$. Estimate the filter order, $M$, that results in a transition width, $\Delta\omega$, of approximately 0.2.

[ 2 ]

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*[U] The transition width is conservatively equal to the main lobe width which is $\frac{8\pi}{M+1}$. Therefore we need $\frac{8\pi}{M+1} = 0.2 \Rightarrow M = 40\pi - 1 = 124.66 \approx 124$ where I have rounded to the nearest even number.*

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e) Figure 1.1 shows the power spectral density (PSD) of a real-valued signal $x[n]$. The horizontal portions of the PSD have values 2 and 1 respectively.

The signal $y[n]$ is obtained by downsampling $x[n]$ by a factor of 2. Draw a dimensioned sketch showing the PSD of $y[n]$ for $0 \le \omega \le \pi$. You should assume that components of $x[n]$ at different frequencies are uncorrelated and may assume without proof that $Y(z) = \frac{1}{2}\left\{X(z^{\frac{1}{2}}) + X(-z^{\frac{1}{2}})\right\}$.

Determine the value of each horizontal portion of the PSD of $y[n]$ and each of the angular frequencies at which its value changes. [ 3 ]

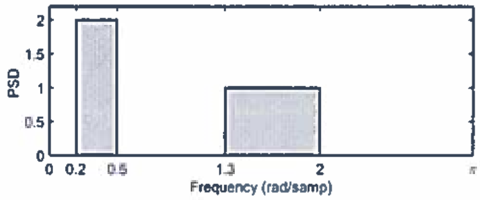


Figure 1.1

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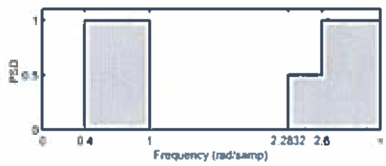
*[U]Because of the factor of $\frac{1}{2}$ in $Y(z) = \frac{1}{2}\left\{X(z^{\frac{1}{2}}) + X(-z^{\frac{1}{2}})\right\}$, the energy per second of each block will derease by a factor of 4 in the absesnce of aliasing. However, the number of samples has also decreased by a factor of 2 so the PSD only decreases by a factor of 2. The width of each block increases by a factor of 2 and so the total power remains the same. The transition frequencies will all multiply by 2 except that, since $2 > \pi$, the frequency $\omega = 2$ will map to $2\pi - 4 = 2.2832$ and the power of the aliased portion will add onto that of the unaliased portion.*

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f) Figure 1.2 shows the block diagram of a two-band analysis and synthesis processor. You may assume without proof that, for $m = 0$ or $1$, $U_m(z) = \frac{1}{2}\left\{V_m(z^{\frac{1}{2}}) + V_m(-z^{\frac{1}{2}})\right\}$ and $W_m(z) = U_m(z^2)$.

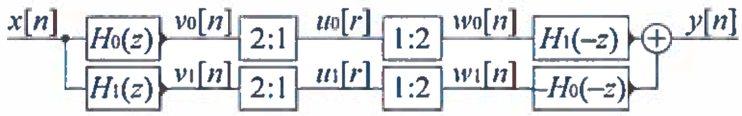


Figure 1.2

i) Derive a simplified expression for $Y(z)$ in terms of $X(z)$. [ 3 ]

*[B] Working backwards from the output*

$$
\begin{aligned}
Y(z) &= H_1(-z)W_0(z) - H_0(-z)W_1(z) \\
&= H_1(-z)U_0(z^2) - H_0(-z)U_1(z^2) \\
&= \frac{1}{2}H_1(-z)\left(V(z)+V_0(-z)\right) - \frac{1}{2}H_0(-z)\left(V_1(z)+V_1(-z)\right) \\
&= \frac{1}{2}\left(H_1(-z)\left(H_0(z)X(z)+H_0(-z)X(-z)\right) - H_0(-z)\left(H_1(z)X(z)+H_1(-z)X(-z)\right)\right) \\
&= \frac{1}{2}\left(H_0(z)H_1(-z) - H_0(-z)H_1(z)\right)X(z)
\end{aligned}
$$

*Note that the aliased terms in $X(-z)$ cancel out.*

ii) Show that the transfer function, $\frac{Y(z)}{X(z)}$, may be written in the form $\frac{1}{2}\{G(z)-G(-z)\}$ and describe how the coefficients of $\frac{Y(z)}{X(z)}$ are related to those of $G(z)$. [ 2 ]

*[U] By setting $G(z) = H_0(z)H_1(-z)$, we can write $\frac{Y(z)}{X(z)} = \frac{1}{2}\{G(z)-G(-z)\}$. To obtain $G(-z)$ from $G(z)$, we need to negate the coefficients of all odd powers of z. Thus, in $\frac{1}{2}\{G(z)-G(-z)\}$, the coefficients of even powers of z will be zero since they cancel out, while the coefficients of odd powers of z will be the same as in $G(z)$.*

iii) If $H_0(z) = 4\left(1+z^{-1}\right)^2$ and $H_1(z) = -\left(1-z^{-1}\right)^2\left(1+4z^{-1}+z^{-2}\right)$, determine the transfer function $\frac{Y(z)}{X(z)}$. [ 4 ]

*[U] We first calculate*

$$
\begin{aligned}
G(z) = H_0(z)H_1(-z) &= -4\left(1+z^{-1}\right)^2\left(1+z^{-1}\right)^2\left(1-4z^{-1}+z^{-2}\right) \\
&= -4\left(1+4z^{-1}+6z^{-2}+4z^{-3}+z^{-4}\right)\left(1-4z^{-1}+z^{-2}\right) \\
&= -4\left(\left(1+z^{-6}\right)+(6-16+1)\left(z^{-2}+z^{-4}\right)+(4-24+4)z^{-3}\right) \\
&= -4\left(1-9z^{-2}-16z^{-3}-9z^{-4}+z^{-6}\right)
\end{aligned}
$$

*So we can write*

$$
\begin{aligned}
\frac{Y(z)}{X(z)} &= \frac{1}{2}\left(G(z)-G(-z)\right)X(z) \\
&= -2\left(\left(1-9z^{-2}-16z^{-3}-9z^{-4}+z^{-6}\right)-\left(1-9z^{-2}+16z^{-3}-9z^{-4}+z^{-6}\right)\right) \\
&= -2\left(-32z^{-3}\right) \\
&= 64z^{-3}
\end{aligned}
$$

2. A causal symmetric high-pass FIR filter, $H(z)$, of even order, $M$, is to be designed such that its magnitude response lies within the unshaded region of the graph in Figure 2.1. The impulse response of $H(z)$ satisfies $h[M-n]=h[n]$. We define $\overline{H}(\omega)\triangleq H(e^{j\omega})e^{0.5jM\omega}$.

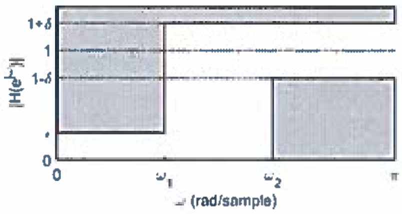


Figure 2.1

a) i) Show that $\overline{H}(\omega)$ (defined above) is given by the real-valued expression $\overline{H}(\omega)=h[0.5M]+2\sum_{n=1}^{0.5M}h[n+0.5M]\cos(n\omega)$. [ 4 ]

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*[B] We can write*

$$\begin{aligned}\overline{H}(\omega) &= H(e^{j\omega})e^{0.5jM\omega}\\ &= e^{0.5jM\omega}\sum_{m=0}^{M}h[m]e^{-jm\omega}\\ &= e^{0.5jM\omega}\left(h[0.5M]e^{-0.5jM\omega}+\sum_{m=0}^{0.5M-1}h[m]e^{-jm\omega}+\sum_{m=0.5M+1}^{M}h[m]e^{-jm\omega}\right)\end{aligned}$$

*We now make the substitution $m=0.5M-n$ in the first summation and $m=n+0.5M$ in the second to obtain*

$$\overline{H}(\omega)=h[0.5M]+e^{0.5jM\omega}\left(\sum_{n=1}^{0.5M}h[0.5M-n]e^{-j0.5M\omega}e^{jn\omega}+\sum_{n=1}^{0.5M}h[n+0.5M]e^{-j0.5M\omega}e^{-jn\omega}\right)$$

*We now note that, for the symmetry condition given at the start of the question, $h[0.5M-n]=h[M-(0.5M-n)]=h[n+0.5M]$ giving*

$$\begin{aligned}\overline{H}(\omega) &= h[0.5M]+e^{0.5jM\omega}\left(\sum_{n=1}^{0.5M}h[n+0.5M]e^{-j0.5M\omega}e^{jn\omega}+\sum_{n=1}^{0.5M}h[n+0.5M]e^{-j0.5M\omega}e^{-jn\omega}\right)\\ &= h[0.5M]+\sum_{n=1}^{0.5M}h[n+0.5M]\left(e^{jn\omega}+e^{-jn\omega}\right)\\ &= h[0.5M]+2\sum_{n=1}^{0.5M}h[n+0.5M]\cos(n\omega)\end{aligned}$$

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ii) Consider the hypothesis that $\cos(n\omega)=T_n(\cos\omega)$ where $T_n(\cdot)$ is a polynomial of order $n$. From the trigonometrical identity,

$$\cos((n+1)\omega)+\cos((n-1)\omega)=2\cos(\omega)\cos(n\omega),$$

show that if the hypothesis is true for both $n$ and $n-1$ then it is also true for $n+1$. Hence prove by induction that the hypothesis is true for all $n$. [ 3 ]

*[B] The hypothesis is true for $n = 0$ and $n = 1$ since $\cos(0\omega) = 1$ and $\cos(1\omega) = \cos\omega$ and both of these expressions are polynomials in $\cos\omega$ of order 0 and 1 respectively. We now assume that the hypothesis is true for $n$ and $n-1$ which, using the identity given in the question, allows us to write*

$$\begin{aligned}\cos((n+1)\omega) &= 2\cos(\omega)\cos(n\omega) - \cos((n-1)\omega)\\ &= 2\cos(\omega)T_n(\cos\omega) - T_{n-1}(\cos\omega)\end{aligned}$$

*The first term on the right hand side is $2\cos\omega$ multipied by an order-$n$ polynomial in $\cos\omega$ and is therefore an order-$(n+1)$ polynomial in $\cos\omega$. The second term is an order-$(n-1)$ polynomial in $\cos\omega$ and so their sum is an order-$(n+1)$ polynomial in $\cos\omega$. It follows that the hypothesis is true for $n+1$ and hence, by induction, for all $n$.*

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iii) Show that $\overline{H}(\omega)$ has at most $0.5M+1$ stationary values within the range $\omega \in [0, \pi]$. [ 3 ]

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*[T] Combining parts i) and ii), $\overline{H}(\omega)$ is an order-$0.5M$ polynomial in $\cos\omega$, say $\overline{H}(\omega) = P(\cos\omega) = P(x)$ where $x = \cos\omega$. For a stationary value, we want*

$$0 = \frac{d\overline{H}(\omega)}{d\omega} = \frac{dP}{dx} \times \frac{dx}{d\omega} = -P'(\cos\omega)\sin\omega$$

*where $P'(x)$ is the derivative of $P(x)$ and is therefore a polynomial of order $0.5M-1$. Since $P'(\cos\omega)$ is an order-$(0.5M-1)$ polynomial in $\cos\omega$, it has at most $0.5M-1$ zeros for $0 \le \omega \le \pi$ which corresponds 1-1 with $1 \ge \cos\omega \ge -1$. An additional two zeros at $\omega = \{0, \pi\}$ come from $\sin\omega$ making a total of $0.5M+1$ as required.*

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b) For a target response of $d(\omega) = \begin{cases} 0 & \text{for } \omega < \omega_2 \\ 1 & \text{for } \omega \ge \omega_2 \end{cases}$, the weighted error of $\overline{H}(\omega)$ is defined as $e(\omega) = s(\omega)\left(\overline{H}(\omega) - d(\omega)\right)$. Determine the positive real-valued weighting function, $s(\omega)$, in each of the three frequency bands bounded by $\omega_i = \{0, \omega_1, \omega_2, \pi\}$ so that $|e(\omega)| \le 1\ \forall\omega$ if and only if the filter satisfies its specification. [ 3 ]

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*[U] We require* $s(\omega) = \begin{cases} \frac{1}{\varepsilon} & 0 \le \omega \le \omega_1 \\ \frac{1}{1+\delta} & \omega_1 < \omega < \omega_2 \\ \frac{1}{\delta} & \omega_2 \le \omega \le \pi \end{cases}.$

*It is easily seen that, with these weights, $|e(\omega)| = 1$ on the specification bounds shown in Figure 2.1.*

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c) If $\delta = \varepsilon = 0.25$ in Figure 2.1, determine the values in dB of the the maximum stopband gain and of the maximum passband ripple. [ 2 ]

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*[U] The maximum stopband gain is $20\log_{10}\varepsilon = 20\log_{10}0.25 = -12.04\,\text{dB}$.*

*The maximum passband gain is* $20\log_{10}(1+\delta) = 20\log_{10} 1.25 = +1.938\,\text{dB}$. *The minimum passband gain is* $20\log_{10}(1-\delta) = 20\log_{10} 0.75 = -2.499\,\text{dB}$. *From these two values, the maximum passband ripple is* $2.499\,\text{dB}$.

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d) i) State the "Alternation Theorem" in the context of finding the minimax-optimal polynomial fit, $y = f(x)$, to a set of data pairs, $(x_i, y_i)$. "Minimax-optimal" means the worst-case absolute error is as small as possible for a given polynomial order. [ 2 ]

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*[B] If we define the error at point i to be* $e_i = f(x_i) - y_i$, *then the Alternation Theorem states that if* $f(x)$ *is an order-n polynomial then it is a minimax-optimal fit if and only the maximum value of* $|e_i|$ *occurs for* $n+2$ *values of i with alternating signs.*

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ii) By constructing equations of the form $y_i = mx_i + c \pm e$, determine the constants $m$ and $c$ such that $y = mx + c$ is the minimax-optimal linear fit to the data pairs (1, 3), (4, 5), (8, 17). [ 4 ]

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*[U] Since there are only three data points, they must all have the same error but with alternating signs. So we can write* $y_i = mx_i + c \pm e$. *This gives us three equations for three unknowns:* $m + c + e = 3$, $4m + c - e = 5$, $8m + c + e = 17$. *Subtracting the first equation form the last gives* $7m = 14 \Rightarrow m = 2$. *Now adding the first two equations gives* $5m + 2c = 8 \Rightarrow c = 4 - 2.5m = 4 - 5 = -1$. *Finally substituting these values into the first equation gives* $e = 3 - m - c = 2$. *Although not requested, the general symbolic solution is* $m = \frac{y_3 - y_1}{x_3 - x_1}$ *and* $c = 0.5\left(y_2 + y_1 - m\left(x_2 + x_1\right)\right)$.

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e) Explain why, if the frequency range $0 \le \omega \le \pi$ is partitioned into a finite number of bands and $d(\omega)$ and $s(\omega)$ are both constant within each band, then all maxima of $|e(\omega)|$ must occur either at the band edges or at stationary values of $\overline{H}(\omega)$. [ 2 ]

---

*[T] Within a band,* $d(\omega)$ *and* $s(\omega)$ *are both constant and so a maximum or minimum of* $e(\omega) = s\left(\overline{H}(\omega) - d\right)$ *will occur if and only if there is a maximum or minimum of* $\overline{H}(\omega)$. *At the band edges, both* $s(\omega)$ *and* $d(\omega)$ *may change abruptly and so it is possible for* $e(\omega)$ *to have a maximum or minimum without having a stationary value.*

---

f) If $M = 2$, determine a minimax-optimal solution, $\overline{H}(\cos^{-1} x) = mx + c$, corresponding to the filter design problem specified in Figure 2.1 using the values $\omega_1 = 1$, $\omega_2 = 2$, $\delta = \varepsilon = 0.25$ and $s(\omega) \equiv 4\ \forall\omega$. You may assume without proof that, in the minimax-optimal solution, the maximal values of $|e(\omega)|$ occur at $\omega = \{\omega_1, \omega_2, \pi\}$. [ 5 ]

---

*[U] We are told that the maximal values of* $|e(\omega)|$ *occur at* $\omega = \{\omega_1, \omega_2, \pi\}$ *in the minimax-optimal solution. Writing* $y(x) = \overline{H}(\cos^{-1} x) = mx + c$, *we can write the following equation or the form* $y(\cos\omega) = d(\omega) \pm \frac{1}{s(\omega)}\hat{e}$ *where* $\hat{e}$ *is the maximum value of*

$|e(\omega)|$:

$$\begin{aligned} y(0.5403) &= 0 + 0.25\hat{e} = 0.5403m + c \\ y(-0.4161) &= 1 - 0.25\hat{e} = -0.4161m + c \\ y(-1) &= 1 + 0.25\hat{e} = -m + c \end{aligned}$$

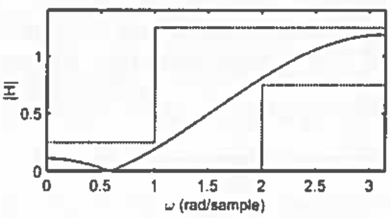
*Subtracting the first equation from the third gives* $1 = -1.5403m \Rightarrow m = -0.6492$. *Adding the second and third equations gives* $2 = -1.4161m + 2c$ *from which* $c = 0.5(2 + 1.4161m) = 0.5403$. *Substituting these values into the third equation gives* $\hat{e} = (-1 - m + c)/0.25 = 0.7581$*; since this is less than* 1*, the specification has been satisfied.*

---

g) Give the coefficients, $h[0]$, $h[1]$ and $h[2]$ of the filter found in part f). [ 2 ]

---

*[U] From the equation given in part a)i), we can write* $m = 2h[0] = 2h[2]$ *and* $c = h[1]$. *The minimax-optimal filter therefore has coefficients* $[-0.3246, 0.5403, -0.3246]$. *Although not requested, the filter magnitude response and the design constraints are plotted below.*




---

3. a) The transfer function $G(z)$ is given by $G(z) = 1 + g_1 z^{-1} + g_2 z^{-2}$ where $g_1$ and $g_2$ are real-valued.

i) By considering $G(z)G(z^{-1})$, or otherwise, show that [ 4 ]

$$|G(e^{j\omega})|^2 = 4g_2 \cos^2\omega + 2g_1(1+g_2)\cos\omega + g_1^2 + (1-g_2)^2.$$

---

*[U] We can write*

$$\begin{aligned} G(z)G(z^{-1}) &= \left(1 + g_1 z^{-1} + g_2 z^{-2}\right)\left(1 + g_1 z^1 + g_2 z^2\right) \\ &= g_2\left(z^2 + z^{-2}\right) + g_1(1+g_2)\left(z + z^{-1}\right) + g_2^2 + g_1^2 + 1 \end{aligned}$$

*Now we make the substitution $z = e^{j\omega}$ which gives $z + z^{-1} = 2\cos\omega$ and $z^2 + z^{-2} = 2\cos 2\omega = 4\cos^2\omega - 2$. This gives*

$$\begin{aligned} G(e^{j\omega})G(e^{-j\omega}) = \left|G(e^{j\omega})\right|^2 &= g_2\left(4\cos^2\omega - 2\right) + g_1(1+g_2)(2\cos\omega) + g_2^2 + g_1^2 + 1 \\ &= 4g_2\cos^2\omega + 2g_1(1+g_2)\cos\omega + g_1^2 + (1-g_2)^2. \end{aligned}$$

---

ii) Hence determine a condition for $|G(e^{j\omega})|$ to have a stationary value in the range $0 \le \omega \le \pi$ and show that, if it exists, its value is given by $|G(e^{j\omega})| = \sqrt{g_1^2 + (1-g_2)^2 - 0.25 g_1^2 g_2^{-1}(1+g_2)^2}$. [ 4 ]

---

*[T] A quadratic of the form $y = ax^2 + bx + c$ has a unique stationary point when $x_0 = \frac{-b}{2a}$ and its value at the stationary point is $y_0 = c - \frac{b^2}{4a}$. Therefore, a stationary value of $\left|G\left(e^{j\omega}\right)\right|$ will occur when $\omega_m = \cos^{-1}\left(\frac{-g_1(1+g_2)}{4g_2}\right) = \cos^{-1}\left(-0.25 g_1\left(1+g_2^{-1}\right)\right)$; this only happens if $\left|0.25 g_1\left(1+g_2^{-1}\right)\right| \le 1$. At this point*

$$\begin{aligned} \left|G(e^{j\omega_m})\right|^2 &= g_1^2 + (1-g_2)^2 - \frac{g_1^2(1+g_2)^2}{4g_2} \\ \Rightarrow \left|G(e^{j\omega_m})\right| &= \sqrt{g_1^2 + (1-g_2)^2 - 0.25 g_1^2 g_2^{-1}(1+g_2)^2} \end{aligned}$$

---

b) The block diagram in Figure 3.1 has two input signals, $x[n]$ and $u[n]$ and one output, $y[n]$. The five multipliers have gains $k_0, \cdots, k_4$ as shown. Determine the transfer function of the block diagram in the form $Y(z) = \frac{B(z)}{A(z)}X(z) + \frac{D(z)}{C(z)}U(z)$ where $A(z), \cdots, D(z)$ are polynomials in $z^{-1}$. Give the coefficients of the polynomials in terms of the $k_i$.[ 4 ]

---

*[U] From the diagram, we can write*

$$y[n] = u[n] + k_0 x[n] + k_1 x[n-1] + k_2 x[n-2] + k_3 y[n-1] + k_4 y[n-2]$$

*which can be rewritten as*

$$y[n] - k_3 y[n-1] - k_4 y[n-2] = u[n] + k_0 x[n] + k_1 x[n-1] + k_2 x[n-2].$$

*Taking z-transforms gives*

$$A(z)Y(z) = U(z) + B(z)X(z)$$

*where* $A(z) = 1 - k_3 z^{-1} - k_4 z^{-2}$ *and* $B(z) = k_0 + k_1 z^{-1} + k_2 z^{-2}$. *Thus we can write*

$$Y(z) = \frac{B(z)}{A(z)}X(z) + \frac{1}{A(z)}U(z)$$

*which is of the required form with* $C(z) = A(z)$ *and* $D(z) = 1$. *The coefficients of* $A(z)$ *and* $B(z)$ *are given above.*

---

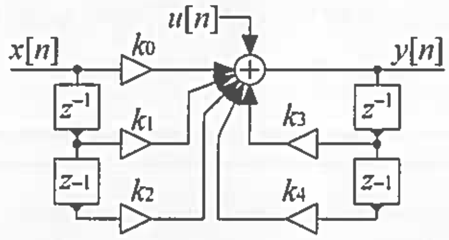



Figure 3.1

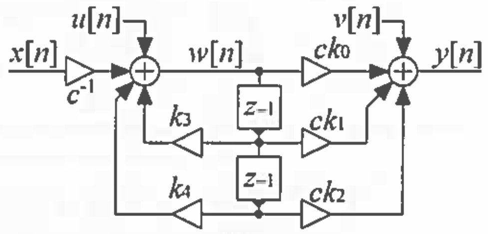



Figure 3.2

c) The transfer function of a second-order lowpass elliptic filter with cutoff frequency $\omega_0 = 1$ is $H(z) = \frac{B(z)}{A(z)}$ where $B(z) = 0.1696 + 0.082z^{-1} + 0.1696z^{-2}$, and $A(z) = 1 - 0.9887z^{-1} + 0.5837z^{-2}$.
You are given that $\frac{1}{2\pi}\int_0^{2\pi} |H(e^{j\omega})|^2 d\omega = 0.2508$ and $\frac{1}{2\pi}\int_0^{2\pi} |A(e^{j\omega})|^{-2} d\omega = 2.4854$.

i) Determine the poles and zeros of $H(z)$ and express them in polar form to 3 decimal places. Explain the relationship between the positions of the poles and zeros on the complex plane and the shapes of the magnitude responses $|B(e^{j\omega})|$, $|A(e^{j\omega})|^{-1}$ and $|H(e^{j\omega})|$. [ 4 ]

---

*[U] The poles are the roots of* $A(z)$ *and are at* $0.4943 \pm 0.5825j = 0.764\angle \pm 0.8671 = 0.764\angle \pm 49.68°$. *The zeros are the roots of* $B(z)$ *and are at* $-0.2417 \pm 0.9703j = 1\angle \pm 1.8149 = 1\angle \pm 103.99°$. *The zeros lie on the unit circle and so, as* $\omega = 0 \to \pi$, $|B(e^{j\omega})|$ *will fall to zero at* $\omega = \pm 1.8149$ *and then rise again. The poles have a radius of* $0.764$ *and so will give rise to a peak in the response of* $|A(e^{j\omega})|^{-1}$ *close to* $\omega = 0.8671$. *This peak will cancel the falling response of* $|B(e^{j\omega})|$ *to give an approximately flat pass-band response in* $|H(e^{j\omega})|$.

---

ii) When $H(z)$ is implemented as in Figure 3.1 on a fixed point processor, the effect of truncation error is to add zero-mean white noise of variance $\sigma^2$ at the input $u[n]$. Determine the variance of the resultant noise at $y[n]$. [ 4 ]

---

*[U] From part b),* $u[n]$ *is filtered by* $\frac{1}{A(z)}$. *Since* $u[n]$ *has a white spectrum, its variance will be multiplied by* $\frac{1}{2\pi}\int_0^{2\pi} |A(e^{j\omega})|^{-2} d\omega$ *whose value is given in part c), as* $2.4854$. *Therefore the noise variance at* $y[n]$ *will be* $2.4854\sigma^2$.

---

d) If the filter, $H(z)$, is instead implemented as in Figure 3.2, the truncation error introduces uncorrelated zero-mean white noise signals of variance $\sigma^2$ at both $u[n]$ and $v[n]$.

i) Determine the value of the smallest constant $c$ in Figure 3.2 that will ensure that the magnitude gain, $\left|\frac{W(e^{j\omega})}{X(e^{j\omega})}\right|$, is $\leq 1$ at all frequencies. [ 4 ]

---

*[U] The transfer function is* $\frac{W(z)}{X(z)} = \frac{1}{cA(z)}$. *From part 3.ii), the stationary value of* $A(e^{j\omega}) = 1 - 0.9887e^{-j\omega} + 0.5837e^{-2j\omega}$ *occurs when* $\cos\omega_1 = -0.25a_1\left(1+a_2^{-1}\right) = 0.6706$. *Since this lies in the range* $\pm 1$, *a stationary value exists in the magnitude response at* $\omega_1 = \cos^{-1}0.6706 = 0.8357$ *(not requested) and, since* $a_2 > 0$, *it is a minimum. From part 3.ii),* $\left|A(e^{j\omega_1})\right| = \sqrt{a_1^2 + (1-a_2)^2 - 0.25a_1^2a_2^{-1}(1+a_2)^2} = \sqrt{0.9775 + 0.1733 - 1.0501} = \sqrt{0.1008} = 0.3174$ *So the peak value of* $\left|\frac{W(e^{j\omega})}{X(e^{j\omega})}\right|$ *is* $\frac{1}{0.3174c}$ *from which we require* $c \geq \frac{1}{0.3174} = 3.1504$.

---

ii) Using the value of $c$ calculated in the previous part, determine the variance of the total noise at $y[n]$ due to $u[n]$ and $v[n]$. [ 3 ]

---

*[U] Since* $u[n]$ *and* $v[n]$ *are uncorrelated, the corresponding noise components at* $y[n]$ *add in power. The transfer functions are* $\frac{Y(z)}{U(z)} = cH(z)$ *and* $\frac{Y(z)}{V(z)} = 1$ *so the total noise variance at* $y[n]$ *is* $\left(1 + \frac{c^2}{2\pi}\int_0^{2\pi}\left|H(e^{j\omega})\right|^2 d\omega\right)\sigma^2 = \left(1+0.2508c^2\right)\sigma^2 = 3.4895\sigma^2$.

---

e) Using the appropriate $z$-plane transformation from the datasheet, determine the transformation needed to convert $H(z)$ into a highpass filter, $\widehat{H}(\hat{z})$, with a cutoff frequency of $\omega = 1.5$. Give the value of the transformation parameter $\lambda$ to 3 decimal places and give an expression for $\widehat{H}(\hat{z})$. It is not necessary to calculate the numerical values of the coefficients of $\widehat{H}(\hat{z})$. [ 3 ]

---

*[U] We apply the lowpass-to-highpass transformation with* $\omega_0 = 1$ *and* $\omega_1 = 1.5$. *From this, we get* $\lambda = \frac{\cos 1.25}{\cos -0.25} = 0.3254$. *Now we want to make the substitution* $z^{-1} = -\frac{\hat{z}^{-1}+\lambda}{1+\lambda\hat{z}^{-1}}$ *to obtain*

$$\widehat{H}(\hat{z}) = \frac{b_0\left(1+\lambda\hat{z}^{-1}\right)^2 - b_1\left(\lambda+\hat{z}^{-1}\right)\left(1+\lambda\hat{z}^{-1}\right) + b_2\left(\lambda+\hat{z}^{-1}\right)^2}{\left(1+\lambda\hat{z}^{-1}\right)^2 - a_1\left(\lambda+\hat{z}^{-1}\right)\left(1+\lambda\hat{z}^{-1}\right) + a_2\left(\lambda+\hat{z}^{-1}\right)^2}$$

---

4. Figure 4.1 shows the block diagram of a sample rate converter. The input signal, $x[n]$, has a bandwidth of 4 kHz with a sample rate of 21 kHz while the output signal, $y[m]$, has a sample rate of 12 kHz. The causal lowpass FIR filter, $H(z)$, has coefficients $h[0], \cdots, h[M]$ where $M = 59$.


Figure 4.1

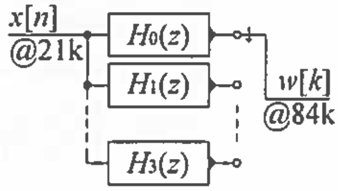



Figure 4.2

a) i) Explain the purpose of the lowpass filter, $H(z)$. Giving reasons for your choices, determine appropriate values in radians per sample for its passband edge and transition width. [ 4 ]

---

*[B] The filter removes images introduced by the upsampler and also removes high frequency components that would cause aliasing in the downsampler.*

*The normalized bandwidth af $x[n]$ is $\frac{8}{21}\pi = 1.197$. The normalized bandwidth of $v[k]$ is therefore $\frac{2}{21}\pi = 0.299$ and this is therefore the passband edge. The centre of the transition band corresponds to the lower of the input and output Nyquist frequencies (i.e. the output Nyquist frequency) and is therefore $\frac{1}{7}\pi = 0.449$. It follows that the transition width is $\Delta\omega = 2(0.449 - 0.299) = \frac{2}{21}\pi = 0.299$.*

---

ii) Assume that the order of $H(z)$ is given by $M = \frac{a}{3.5\Delta\omega}$ where $a$ is the stopband attenuation in dB and $\Delta\omega$ is the transition width in radians/sample. Given that $M = 59$, determine the stopband attenuation, $a$. [ 2 ]

---

*[U] We can write $a = 3.5M\Delta\omega = 3.5 \times 59 \times \frac{2}{21}\pi = 61.8$ dB.*

---

iii) Suppose that the signal to noise ratio of the input signal, $x[n]$, is 60 dB and that the noise spectrum is white. Stating any assumptions that you make, estimate the signal to noise ratio of the output signal, $y[n]$. [ 5 ]

---

*[U] We take the signal power in $x[n]$ to be 0 dB and the noise power to be $-55$ dB. The wanted signal power is divided by 16 due to 1 : 4 and left unchanged by $H(z)$ and 7 : 1 so the overall signal power is changed by $-10\log_{10} 16 = -12.04$ dB. The three unwanted signal images from 1 : 4 each have the same power as the wanted signal but are attenuated by $H(z)$ and so end up with a total power of $-12.04 + 10\log_{10} 3 - 61.8 = -69.1$ dB. The white noise power is divided by 4 due to 1 : 4, is attenuated by $H(z)$ and left unchanged by 7 : 1. The effect of $H(z)$ depends on the precise response within the transition band. If we assume that the equivalent rectangular bandwidth of $H(z)$ extends to the centre of the transition band, it will attenuate the noise power by a factor of 7, so the total power of this component in the output will be $-60 - 10\log_{10} 28 = -74.5$ dB. Combining these two noise terms gives a total noise $-68$ dB and hence the output SNR*

*is* $-12.04+68=55.9$dB.

---

iv) Estimate the number of mutiplications per second required for a direct implementation of Figure 4.1. [ 2 ]

---

*[U] The filter* $H(z)$ *has* 60 *coefficients and operates at a sample rate of* $4\times 21\,kHz$. *The number of multiplications per second is therefore* $5.04\times 10^6$.

---

b) i) Prove that, if $k$ is written as $k=4r+p$ with $0\le p\le 3$, then $w[k]=\sum_{s=0}^{(M-3)/4} h[4s+p]x[r-s]$. [ 5 ]

---

*[B] We can write* $w[k]=\sum_{l=0}^{M} h[l]v[k-l]$. *If we now decompose* $l=4s+q$ *with* $0\le q\le 3$ *and* $0\le s\le \frac{M-3}{4}$

*we get (note that* $M-3=56=4\times 14$ *is a multiple of 4)*

$$\begin{aligned} w[4r+p] &= \sum_{l=0}^{M} h[l]v[4r+p-l] \\ &= \sum_{q=0}^{3}\sum_{s=0}^{(M-3)/4} h[4s+q]v[4r+p-4s-q] \\ &= \sum_{q=0}^{3}\sum_{s=0}^{(M-3)/4} h[4s+q]v[4(r-s)+p-q]. \end{aligned}$$

*However, we know that* $v[k]=v[4r+p]=\begin{cases} x[r] & p=0 \\ 0 & p\ne 0\end{cases}$. *So the final factor is zero except when* $q=p$. *So we can replace* $q$ *by* $p$ *and omit the first summation to obtain*

$$w[4r+p]=\sum_{s=0}^{(M-3)/4} h[4s+q]x[r-s].$$

---

ii) If the first two blocks in Figure 4.1 are replaced by the polyphase implementation shown in Figure 4.2, determine the number of coefficients in each of the subfilters and give an expression for the coefficients, $h_3[n]$, of $H_3(z)$ in terms of the coefficients, $h[k]$, of $H(z)$. [ 3 ]

---

*[U] The number of coefficients in each sub-filter is* $\frac{M+1}{4}=15$, *so the order of each subfilter is* $15-1=14$. *The coefficients of* $H_3(z)$ *are* $h_3[n]=h[4n+3]$.

---

c) i) Explain how, by reordering the subfilters and changing the commutator frequency in Figure 4.2 it is possible to eliminate the downsampler and to generate the samples, $y[m]$, directly. Explaining your answer fully, give the revised order of the subfilters. [ 4 ]

---

*[U] The downsampler takes only every seventh value of $w[x]$ and so it is only necessary to evalute these samples. We have $y[m] = w[k]$ where $k = 7m = 4r + p$ using the notaion of part b)i). We can make the following table*

| $m$ | 0 | 1 | 2 | 3 | 4 | 5 | $\cdots$ |
|---|---|---|---|---|---|---|---|
| $k$ | 0 | 7 | 14 | 21 | 28 | 35 | $\cdots$ |
| $r$ | 0 | 1 | 3 | 5 | 7 | 8 | $\cdots$ |
| $p$ | 0 | 3 | 2 | 1 | 0 | 3 | $\cdots$ |

*We see that incrementing $m$ by one, adds 7 onto $k$ which corresponds to either (a) adding 1 to $r$ and 3 to $p$ or else (b) adding 2 to $r$ and subtractiong 1 from $p$. Thus we change the commmutator frequency to 12 kHz, the output sample rate, and we reorder the subfilters into the order 0, 3, 2, 1 according to the above table.*

ii) Estimate the number of multiplications per second required for the resultant system. [ 2 ]

*[U] The subfilters each have 15 coefficients and we need to evaluate one for each output sample. Hence the multiplication rate is $15 \times 12000 = 1.8 \times 10^5$.*

iii) Determine which of the input samples, $x[n]$, are used to calculate the output sample $y[90]$ and identify which subfilter, $H_p(z)$, is used. [ 3 ]

*[U] The ouptut sample $y[90] = w[630]$. Hence $k = 630 = 4 \times 157 + 2$ and so $r = 157$ and $p = 2$. In part b) i), $s$ ranges over $[0, 14]$, so $x[r - s]$ ranges over $x[r - 14] = x[143]$ to $x[r - 0] = x[157]$. So the range of input samples used is $x[143 : 157]$ and the subfilter used is $H_2(z)$.*

# Datasheet:

## Standard Sequences

- $\delta[n] = 1$ for $n = 0$ and 0 otherwise.
- $\delta_{\text{condition}}[n] = 1$ whenever "condition" is true and 0 otherwise.
- $u[n] = 1$ for $n \geq 0$ and 0 otherwise.

## Geometric Progression

- $\sum_{n=0}^{r} \alpha^n z^{-n} = \frac{1-\alpha^{r+1}z^{-r-1}}{1-\alpha z^{-1}}$ provided that $\alpha z^{-1} \neq 1$.
- $\sum_{n=0}^{\infty} \alpha^n z^{-n} = \frac{1}{1-\alpha z^{-1}}$ provided that $\left|\alpha z^{-1}\right| < 1$.

## Forward and Inverse Transforms

| | | |
|---|---|---|
| z: | $X(z) = \sum_{-\infty}^{\infty} x[n] z^{-n}$ | $x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$ |
| CTFT: | $X(j\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt$ | $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega) e^{j\Omega t} d\Omega$ |
| DTFT: | $X(e^{j\omega}) = \sum_{-\infty}^{\infty} x[n] e^{-j\omega n}$ | $x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$ |
| DFT: | $X[k] = \sum_{0}^{N-1} x[n] e^{-j2\pi \frac{kn}{N}}$ | $x[n] = \frac{1}{N} \sum_{0}^{N-1} X[k] e^{j2\pi \frac{kn}{N}}$ |
| DCT: | $X[k] = \sum_{n=0}^{N-1} x[n] \cos \frac{2\pi(2n+1)k}{4N}$ | $x[n] = \frac{X[0]}{N} + \frac{2}{N} \sum_{n=1}^{N-1} X[k] \cos \frac{2\pi(2n+1)k}{4N}$ |
| MDCT: | $X[k] = \sum_{n=0}^{2N-1} x[n] \cos \frac{2\pi(2n+1+N)(2k+1)}{8N}$ | $y[n] = \frac{1}{N} \sum_{0}^{N-1} X[k] \cos \frac{2\pi(2n+1+N)(2k+1)}{8N}$ |

## Convolution

DTFT:

$$v[n] = x[n] * y[n] \triangleq \sum_{r=-\infty}^{\infty} x[r] y[n-r] \quad \Leftrightarrow \quad V\left(e^{j\omega}\right) = X\left(e^{j\omega}\right) Y\left(e^{j\omega}\right)$$

$$v[n] = x[n] y[n] \quad \Leftrightarrow \quad V\left(e^{j\omega}\right) = \frac{1}{2\pi} X\left(e^{j\omega}\right) \circledast Y\left(e^{j\omega}\right) \triangleq \frac{1}{2\pi} \int_{-\pi}^{\pi} X\left(e^{j\theta}\right) Y\left(e^{j(\omega-\theta)}\right) d\theta$$

DFT:

$$v[n] = x[n] \circledast_N y[n] \triangleq \sum_{r=0}^{N-1} x[r] y[(n-r)_{\bmod N}] \quad \Leftrightarrow \quad V[k] = X[k] Y[k]$$

$$v[n] = x[n] y[n] \quad \Leftrightarrow \quad V[k] = \frac{1}{N} X[k] \circledast_N Y[k] \triangleq \frac{1}{N} \sum_{r=0}^{N-1} X[r] Y[(k-r)_{\bmod N}]$$

## Group Delay

The group delay of a filter, $H(z)$, is $\tau_H(e^{j\omega}) = -\frac{d\angle H(e^{j\omega})}{d\omega} = \Re\left(\frac{-z}{H(z)} \frac{dH(z)}{dz}\right)\Big|_{z=e^{j\omega}} = \Re\left(\frac{\mathscr{F}(nh[n])}{\mathscr{F}(h[n])}\right)$ where $\mathscr{F}()$ denotes the DTFT.

## Order Estimation for FIR Filters

Three increasingly sophisticated formulae for estimating the minimum order of an FIR filter with unity gain passbands:

1. $M \approx \frac{a}{3.5\Delta\omega}$
2. $M \approx \frac{a-8}{2.2\Delta\omega}$
3. $M \approx \frac{a-1.2-20\log_{10} b}{4.6\Delta\omega}$

where $a$ =stop band attenuation in dB, $b$ = peak-to-peak passband ripple in dB and $\Delta\omega$ = width of smallest transition band in radians per sample.

## z-plane Transformations

A lowpass filter, $H(z)$, with cutoff frequency $\omega_0$ may be transformed into the filter $H(\hat{z})$ as follows:

| Target $H(\hat{z})$ | Substitute | Parameters |
|---|---|---|
| Lowpass $\hat{\omega} < \hat{\omega}_1$ | $z^{-1} = \frac{\hat{z}^{-1}-\lambda}{1-\lambda\hat{z}^{-1}}$ | $\lambda = \frac{\sin\left(\frac{\omega_0-\hat{\omega}_1}{2}\right)}{\sin\left(\frac{\omega_0+\hat{\omega}_1}{2}\right)}$ |
| Highpass $\hat{\omega} > \hat{\omega}_1$ | $z^{-1} = -\frac{\hat{z}^{-1}+\lambda}{1+\lambda\hat{z}^{-1}}$ | $\lambda = \frac{\cos\left(\frac{\omega_0+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\omega_0-\hat{\omega}_1}{2}\right)}$ |
| Bandpass $\hat{\omega}_1 < \hat{\omega} < \hat{\omega}_2$ | $z^{-1} = -\frac{(\rho-1)-2\lambda\rho\hat{z}^{-1}+(\rho+1)\hat{z}^{-2}}{(\rho+1)-2\lambda\rho\hat{z}^{-1}+(\rho-1)\hat{z}^{-2}}$ | $\lambda = \frac{\cos\left(\frac{\hat{\omega}_2+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)}, \rho = \cot\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)\tan\left(\frac{\omega_0}{2}\right)$ |
| Bandstop $\hat{\omega}_1 \not< \hat{\omega} \not< \hat{\omega}_2$ | $z^{-1} = \frac{(1-\rho)-2\lambda\hat{z}^{-1}+(\rho+1)\hat{z}^{-2}}{(\rho+1)-2\lambda\hat{z}^{-1}+(1-\rho)\hat{z}^{-2}}$ | $\lambda = \frac{\cos\left(\frac{\hat{\omega}_2+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)}, \rho = \tan\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)\tan\left(\frac{\omega_0}{2}\right)$ |

## Noble Identities

$$-\boxed{Q{:}1}-\boxed{H(z)}- \;=\; -\boxed{H(z^Q)}-\boxed{Q{:}1}-$$

$$-\boxed{H(z)}-\boxed{1{:}Q}- \;=\; -\boxed{1{:}Q}-\boxed{H(z^Q)}-$$

## Multirate Spectra

Upsample: $\xrightarrow{v[n]}\boxed{1{:}Q}\xrightarrow{x[r]} \;\Rightarrow\; x[r] = \begin{cases} v\left[\frac{r}{Q}\right] & \text{if } Q \mid r \\ 0 & \text{if } Q \nmid r \end{cases} \;\Rightarrow\; X(z) = V(z^Q)$

Downsample: $\xrightarrow{v[n]}\boxed{Q{:}1}\xrightarrow{y[m]} \;\Rightarrow\; y[m] = v[Qm] \;\Rightarrow\; Y(z) = \frac{1}{Q}\sum_{k=0}^{Q-1} V\left(e^{\frac{-j2\pi k}{Q}} z^{\frac{1}{Q}}\right)$

**Multirate Commutators**