E3.10
C1.5
ISE3.7

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2006

MSc and EEE/ISE PART III/IV: MEng, BEng and ACGI

MATHEMATICS FOR SIGNALS AND SYSTEMS

Tuesday, 25 April 10:00 am

Time allowed: 3:00 hours

Corrected Copy

There are FIVE questions on this paper.

Answer THREE questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : G. Weiss
Second Marker(s) : J.C. Allwright

1. Consider the space $\mathcal{H} = \mathbb{C}^{3\times 3}$ of matrices with three rows and three columns. We define an inner product on $\mathcal{H}$ by $< A, B > \; = \text{trace} B^* A$, where B^* is the complex conjugate of the transpose of B, and we define the corresponding norm on $\mathcal{H}$ by $\|A\|_{\mathcal{H}}^2 = \; < A, A >$.

   (a) What is the dimension of $\mathcal{H}$? [1]

   (b) In the sequel we denote

   $$S = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

   Compute $\|S\|_{\mathcal{H}}$. Compute also the norm $\|S\|$ when S is regarded as an operator from $\mathbb{C}^3$ to $\mathbb{C}^3$. (Hint: be careful, the norm of S as an operator is not the same as $\|S\|_{\mathcal{H}}$.) [3]

   (c) We say that a matrix $A \in \mathcal{H}$ is *S-invariant* if $AS = SA$. In the sequel we denote by $\mathcal{F}$ the set of all the S-invariant matrices in $\mathcal{H}$. Show that $\mathcal{F}$ is actually a subspace of $\mathcal{H}$. [2]

   (d) Determine the dimension of $\mathcal{F}$. [2]

   (e) Show that if $A, B \in \mathcal{F}$, then also $AB \in \mathcal{F}$ and $AB = BA$. [3]

   (f) Find an orthonormal basis in $\mathcal{F}$. [3]

   (g) Show that if $A \in \mathcal{F}$, then A has only one eigenvalue. [2]

   (h) Find a non-zero vector $x \in \mathbb{C}^3$ such that for every S-invariant matrix A, x is an eigenvector of A. [3]

   (i) Explicitly describe all the matrices $A \in \mathcal{F}$ for which $A^* \in \mathcal{F}$. [1]

2. For $1 \leq p < \infty$, we denote by l^p the space of all sequences u indexed by $k \in \{0, 1, 2, 3, ...\}$ for which $\Sigma_{k=0}^{\infty}|u_k|^p < \infty$. For such sequences u, we use the notation $\|u\|_p = \left(\Sigma_{k=0}^{\infty}|u_k|^p\right)^{\frac{1}{p}}$. We denote by l^∞ the space of all bounded sequences, and let $\|u\|_\infty = \sup |u_k|$.

A linear discrete-time system with input u and output y is defined by the formula

$$y_k = u_k + 2u_{k-1}, \qquad k = 0, 1, 2, 3, \ldots.$$

The signals u and y are defined for integer times $k \geq 0$ and we consider $u_{-1} = 0$ (this occurs for $k = 0$ in the above formula).

(a) In the sequel, we denote by T the input-output operator of the above system. Is T time-invariant? Determine its impulse response g and compute its transfer function $\mathbf{G}$. [2]

(b) With the notation from part (a), is $\mathbf{G}$ stable? Is it strictly proper? Is this a finite impulse response (FIR) system? What is the DC gain of this system? [2]

(c) Show that for every p $(1 \leq p \leq \infty)$, if $u \in l^p$ and $y = Tu$, then also $y \in l^p$ and

$$\|y\|_p \leq 3\|u\|_p.$$ [3]

(d) Show that if $u \in l^2$ and $y = Tu$, then

$$\|y\|_2 \geq \|u\|_2.$$ [5]

(e) Let $\hat{y}$ denote the $\mathcal{Z}$-transform of y. Show that if $u \in l^2$ and $y = Tu$, then $\hat{y}(-2) = 0$. Show that the operator $T \in \mathcal{L}(l^2, l^2)$ is not onto. [4]

(f) Show that there exist operators $L \in \mathcal{L}(l^2, l^2)$ such that $LT = I$ (the identity on l^2). Show that L can be chosen such that $\|L\| \leq 1$. Show that L cannot be chosen such that it is time-invariant. Hint: Denote $V = \{Tu \,|\, u \in l^2\}$ (this is the range space of T). Define L on V using $\mathcal{Z}$-transforms, while L on $V^\perp$ can be chosen in an arbitrary way. [4]

3. In this question, $\mathbf{S}_\tau$ denotes the right shift operator by τ on $L^2[0,\infty)$.

(a) Define the natural inner product and the corresponding norm on the space $L^2[0,\infty)$. For $s \in \mathbb{C}_+$ and $\varphi \in L^2[0,\infty)$ defined by $\varphi(t) = e^{-st}$, compute $\|\varphi\|_2$. [3]

(b) Let $u \in L^2[0,\infty)$ and let $\hat{u}$ denote its Laplace transform. Show that

$$|\hat{u}(s)| \leq \frac{\|u\|_2}{\sqrt{2\operatorname{Re} s}} \quad \text{for all} \quad s \in \mathbb{C}_+ .$$

Hint: use the result about $\|\varphi\|_2$ from part (a) and the Cauchy-Schwarz inequality. [3]

(c) Let $y \in L^1[0,\infty)$, let $\hat{y}$ denote its Laplace transform and, as usual, denote $\|y\|_1 = \int_0^\infty |y(t)|\,dt$. Show that

$$|\hat{y}(s)| \leq \|y\|_1 \quad \text{for all} \quad s \in \mathbb{C}_+ .$$ [3]

(d) In the sequel, consider f to be the characteristic function of the interval $[0,4]$ and $g(t) = e^{5t}$, $t \geq 0$. (Thus, $f(t) = 1$ for $t \in [0,4]$ and $f(t) = 0$ for $t > 4$.) We also define $m = fg$. Compute the Laplace transforms $\hat{f}$, $\hat{g}$ and $\hat{m}$. [2]

(e) Which of $\hat{f}, \hat{g}$ and $\hat{m}$ is rational? Which of these functions belongs to $H^\infty(\mathbb{C}_+)$? Determine the poles of $\hat{f}, \hat{g}$ and $\hat{m}$. Hint: for the question concerning $H^\infty(\mathbb{C}_+)$, you may use the result from part (c). [3]

(f) Define $h = \mathbf{S}_4 m$, i.e., h is obtained by delaying m by 4 time units. Compute its Laplace transform $\hat{h}$, its norm $\|h\|_2$ and the inner product $<m, h>$. [3]

(g) Compute

$$\|\hat{m}\|_2, \qquad \|\hat{h}\|_2 \quad \text{and} \quad <\hat{m}, \hat{h}> ,$$

where the norms and the scalar products correspond to the Hardy space $H^2(\mathbb{C}_+)$. [3]

4. Let $L \in \mathbb{C}^{2\times 2}$, $H \in \mathbb{C}^{2\times 1}$ and consider the system described by

$$\begin{aligned} \dot{p}(t) &= Lq(t)\,, \\ \dot{q}(t) &= -L^* p(t) + Hu(t)\,, \\ y(t) &= H^* q(t)\,, \end{aligned}$$

where u is the scalar input signal, $p(t), q(t) \in \mathbb{C}^2$ and y is the scalar output signal. (As usual, L^* denotes the complex conjugate of the transpose of L and a dot denotes differentiation with respect to the time.) We define the "energy in the system" by

$$E(t) = \frac{1}{2}\left(\|p(t)\|^2 + \|q(t)\|^2\right).$$

(a) Write the equations of the system in the form

$$\dot{x}(t) = Ax(t) + Bu(t)\,, \qquad y(t) = Cx(t)\,,$$

where x is the state of the system and $A^* = -A$, $C = B^*$. [2]
The notation A, B, C and $x(t)$ will be used also in the sequel.

(b) Prove that all the eigenvalues of A are on the imaginary axis. Is this system stable? Hint: iA is self-adjoint. [3]

(c) Show that $\dot{E}(t) = \operatorname{Re} u(t)\overline{y(t)}$. [3]

(d) Express the transfer function $\mathbf{G}$ of this system, in terms of the matrices L, H, and also in terms of A, B, C. [3]

(e) Recall that if $u = 0$, then $x(t) = e^{tA}x(0)$. Show that e^{tA} is a unitary operator on $\mathbb{C}^4$ (for every $t \geq 0$). Hint: use part (c). [3]

(f) For $\mathbf{G}$ as in part (d), find a function $k : \mathbb{C}_+ \to \mathbb{R}$ such that

$$\mathbf{G}(s) + \mathbf{G}(s)^* = k(s)C(sI - A)^{-1}(\bar{s}I - A^*)^{-1}B\,,$$

for all $s \in \mathbb{C}_+$. Hint: for every s, β that are not eigenvalues of A,
$(sI - A)^{-1} - (\beta I - A)^{-1} = (\beta - s)(sI - A)^{-1}(\beta I - A)^{-1}$. [3]

(g) Assume that $H \neq 0$. Using the result from part (f), show that the transfer function $\mathbf{G}$ is "strictly positive", which means that

$$\operatorname{Re}\mathbf{G}(s) > 0 \quad \text{for all} \quad s \in \mathbb{C}_+\,.$$ [3]

5. Consider the system with input u and output y described by the differential equation

$$\ddot{y} + 0.2\dot{y} + 100y = 2\dot{u} - u.$$

We denote its transfer function by $\mathbf{G}$.

(a) Compute $\mathbf{G}$ and determine if it is stable. [2]

(b) Sketch the Bode amplitude plot of $\mathbf{G}$ and estimate $\|\mathbf{G}\|_\infty$ with a precision of $\pm 20\%$. [3]

(c) Define the space $BL(\omega_b)$ of band-limited functions with angular frequencies not higher than ω_b. [3]

(d) Find an orthonormal basis in $BL(\omega_b)$. [3]

(e) Suppose that $u \in BL(3)$ and $v(t) = u(t)\cos 50t$ for all $t \in \mathbb{R}$. Determine if v is band-limited and, if yes, what is its band-limit (i.e., the smallest $\omega_b > 0$ such that $v \in BL(\omega_b)$). [3]

(f) Show that u and v from part (e) are orthogonal to each other. [3]

(g) Suppose that u from part (e) is the input signal of the system considered earlier, and y is the corresponding output function (defined for all $t \in \mathbb{R}$). Show that $y \in BL(3)$ and $\|y\| \leq \|u\|$ (these norms are computed in $L^2(\mathbb{R})$). Hint: you will need the Bode plot from part (b) to answer this part. [3]

[END]

Mathematics for Signals & Systems

Exam of May 2006

SOLUTIONS

Question 1 Note that if $A = [A_{ij}]$, $i,j = 1,2,3$, then $\|A\|_{\mathcal{H}}^2 = \sum_{i,j \in \{1,2,3\}} |A_{ij}|^2$.

(a) $\dim \mathcal{H} = 9$.

(b) $\|S\|_{\mathcal{H}} = \sqrt{2}$. $S^*S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$,

$\sigma(S^*S) = \{0,1\}$, hence $\|S\| = 1$.

(c) If $AS = SA$ and $BS = SB$, then clearly $(A+B)S = S(A+B)$ and $(\lambda A)S = S(\lambda A)$ for every $\lambda \in \mathbb{C}$. Thus, $\mathcal{F}$ is a subspace of $\mathcal{H}$.

(d)
$$AS = \begin{bmatrix} A_{12} & A_{13} & 0 \\ A_{22} & A_{23} & 0 \\ A_{32} & A_{33} & 0 \end{bmatrix}, \qquad SA = \begin{bmatrix} 0 & 0 & 0 \\ A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix},$$

if the above are equal (i.e., $A \in \mathcal{F}$) then A must have the structure

$$A = \begin{bmatrix} \alpha & 0 & 0 \\ \beta & \alpha & 0 \\ \gamma & \beta & \alpha \end{bmatrix}, \quad \text{with } \alpha, \beta, \gamma \in \mathbb{C}.$$

From here it is clear that $\dim \mathcal{F} = 3$.

(e) If $A = \begin{bmatrix} \alpha & 0 & 0 \\ \beta & \alpha & 0 \\ \gamma & \beta & \alpha \end{bmatrix}$ and $B = \begin{bmatrix} \delta & 0 & 0 \\ \varepsilon & \delta & 0 \\ \eta & \varepsilon & \delta \end{bmatrix}$, then

$$AB = BA = \begin{bmatrix} \alpha\delta & 0 & 0 \\ \beta\delta + \alpha\varepsilon & \alpha\delta & 0 \\ \gamma\delta + \beta\varepsilon + \alpha\eta & \beta\delta + \alpha\varepsilon & \alpha\delta \end{bmatrix}.$$

(f) An orthonormal basis in $\mathcal{F}$ is

$$E_1 = \begin{bmatrix} 1/\sqrt{3} & 0 & 0 \\ 0 & 1/\sqrt{3} & 0 \\ 0 & 0 & 1/\sqrt{3} \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1/\sqrt{2} & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 \end{bmatrix},$$

$$E_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Recall that for any triangular matrix, the eigenvalues are the numbers on the diagonal.

(g) If A is as described in the answer to (d), then it has only one eigenvalue, α.

(h) $x = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is an eigenvector for every $A \in \mathcal{F}$.

We remark that if $\beta \neq 0$ then $A = \begin{bmatrix} \alpha & 0 & 0 \\ \beta & \alpha & 0 \\ \gamma & \beta & \alpha \end{bmatrix}$ has no other independent eigenvectors.

(i) If A is as described in the answer to (d), then $A^* = \begin{bmatrix} \bar{\alpha} & \bar{\beta} & \bar{\gamma} \\ 0 & \bar{\alpha} & \bar{\beta} \\ 0 & 0 & \bar{\alpha} \end{bmatrix}$. For $A^* \in \mathcal{F}$ we must have $\beta = \gamma = 0$, so that $A = \alpha I$.

$\boxed{\text{Question 2}}$ (a) T is time-invariant. Its impulse response is $g = (1, 2, 0, 0, 0, \ldots)$ and its transfer function is $G(z) = 1 + 2z^{-1} = \dfrac{z+2}{z}$.

(b) The only pole of G is at $z = 0$, and it is proper, hence it is stable. It is not strictly proper, since $G(\infty) = 1$. The system is FIR and its DC gain is $G(1) = 3$.

(c) Denote the right shift operator (delay by one step) by S. Then $y = u + 2Su$, hence (by the triangle inequality in ℓ^p) $\|y\|_p \leq \|u\|_p + 2\|Su\|_p = 3\|u\|_p$.

(d) We have $\hat{y}(z) = (1 + 2z^{-1})\hat{u}(z)$ and (by the Paley Wiener theorem) $\|y\|_2 = \|\hat{y}\|_2$ (the last norm is in $H^2(\mathcal{E})$). Thus

$$\|y\|_2^2 = \|\hat{y}\|_2^2 = \frac{1}{2\pi}\int_{\mathcal{C}_1} |1 + 2z^{-1}|^2 \cdot |\hat{u}(z)|^2 \cdot |dz|$$

where $\mathcal{C}_1$ is the unit circle in $\mathbb{C}$.

Notice that $1 + 2z^{-1}$ is on a circle centered at 1 and with radius 2, so that $|1 + 2z^{-1}| \geq 1$ for all $z \in \mathcal{C}_1$.

Hence

$$\|y\|_2^2 \geq \frac{1}{2\pi}\int_{\mathcal{C}_1} |\hat{u}(z)|^2 \cdot |dz| = \|\hat{u}\|_2^2 = \|u\|_2^2.$$

(e) If $y = Tu$ then $\hat{y} = G\hat{u}$. Since $G(-2) = 0$, we obtain $\hat{y}(-2) = 0$. Thus, we cannot obtain as Tu those signals $w \in \ell^2$ for which $\hat{w}(-2) \neq 0$.

(f) On $V = \{Tu \mid u \in \ell^2\}$ we define L by

$$u = Ly \iff \hat{u} = G^{-1}\hat{y} \iff y = Tu,$$

so that $LT = I$. According to the result of part (d), $\|Ly\|_2 = \|u\|_2 \leq \|y\|_2$, so that L is bounded from V to ℓ^2 and $\|L\| \leq 1$ (as an operator in $\mathcal{L}(V, \ell^2)$).

Now note that <u>V is closed</u>. Indeed, if (y_n) is a sequence in V and $y_n \to y_0 \in \ell^2$, then define $u_n = Ly_n$. It is clear that (u_n) is a Cauchy sequence in ℓ^2, hence $u_n \to u_0$ for some $u_0 \in \ell^2$. Since $y_n = Tu_n$ and T is continuous, it follows that $y_n \to Tu_0$, so that $y_0 = Tu_0 \in V$. Thus, V contains its limit points.

We decompose $\ell^2 = V + V^\perp$ and we define L on $V^\perp$ in an arbitrary linear way, for example $Lw = 0$ for all $w \in V^\perp$. Now L is defined on all of ℓ^2 and we still have that $\|L\| \leq 1$. (In fact, we have $\|L\| = 1$, but this is not important.)

If L were time-invariant (for some choice of its restriction to $V^\perp$) then, according to the Fourés – Segal theorem, we would have $u = Ly$ iff $\hat{u} = F\hat{y}$, where $F \in H^\infty(\mathcal{E})$. Taking $y \in V$ we see that we must have $F = G^{-1}$, but G^{-1} is unstable.

Question 3 (a) $\langle f, g \rangle = \int_0^\infty f(t)\overline{g(t)}\,dt$,

$\|f\|_2 = \left(\int_0^\infty |f(t)|^2 dt \right)^{\frac{1}{2}}$. If $\varphi(t) = e^{-\bar{s}t}$, where $\operatorname{Re} s > 0$, then $\varphi \in L^2[0,\infty)$ and $\|\varphi\|_2 = \frac{1}{\sqrt{2 \operatorname{Re} s}}$.

(b) $u \in L^2[0,\infty)$, $\hat{u}(s) = \int_0^\infty e^{-st} u(t)\,dt = \langle \varphi, u \rangle$, where φ is as in part (a). By the Cauchy-Schwarz inequality, $|\hat{u}(s)| \leq \|\varphi\|_2 \cdot \|u\|_2$.

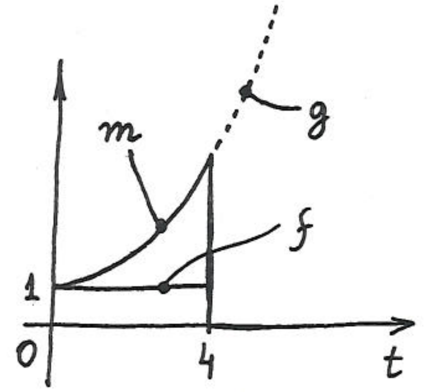
(c) If $y \in L^1[0,\infty)$, then

$$|\hat{y}(s)| = \left| \int_0^\infty e^{-st} y(t)\,dt \right| \leq \int_0^\infty |e^{-st}| \cdot |y(t)|\,dt.$$

If $\operatorname{Re} s > 0$, then $|e^{-st}| < 1$ for all $t > 0$, so that

$$|\hat{y}(s)| \leq \int_0^\infty |y(t)|\,dt = \|y\|_1.$$

(d)



$\hat{f}(s) = \frac{1}{s}\left(1 - e^{-4s}\right)$,

$\hat{g}(s) = \frac{1}{s-5}$,

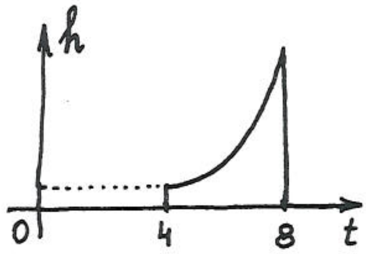
$\hat{m}(s) = \hat{f}(s-5)$

$= \frac{1}{s-5}\left(1 - e^{20} e^{-4s}\right)$.

(e) $\hat{g}$ is rational, $\hat{f}$ and $\hat{m}$ are not rational.
We have $f, m \in L^1[0,\infty)$,
hence (by part (c)) $\hat{f}, \hat{m} \in H^\infty(\mathbb{C}_+)$. $\hat{f}$ and $\hat{m}$ have no poles. $\hat{g}$ has a pole at 5, hence it is not in $H^\infty(\mathbb{C}_+)$.

(f) $h = S_4 m$, hence $\hat{h}(s) = e^{-4s}\,\hat{m}(s)$, so that

$$\hat{h}(s) = \frac{e^{-4s}}{s-5}\left(1 - e^{20}e^{-4s}\right).$$



$$\|h\|_2 = \|m\|_2 = \left(\int_0^4 e^{10t}\,dt\right)^{\frac{1}{2}}$$

$$= \left(\frac{1}{10}\, e^{10t}\Big|_0^4\right)^{\frac{1}{2}} = \frac{1}{\sqrt{10}}\left(e^{40}-1\right)^{\frac{1}{2}}.$$

$\langle m, h\rangle = 0$ because $m(t)\,h(t) = 0$ for almost every $t \geq 0$.

(g) According to the Paley–Wiener theorem (the continuous-time version), we have

$$\|\hat{m}\|_2 = \|m\|_2 = \frac{1}{\sqrt{10}}\left(e^{40}-1\right)^{\frac{1}{2}},$$

$$\|\hat{h}\|_2 = \|h\|_2 = \frac{1}{\sqrt{10}}\left(e^{40}-1\right)^{\frac{1}{2}},$$

$$\langle \hat{m}, \hat{h}\rangle = \langle m, h\rangle = 0.$$

Question 4

(a) The system can be described by $\dot{x}(t) = Ax(t) + Bu(t)$, $y(t) = Cx(t)$, where

$$x(t) = \begin{bmatrix} p(t) \\ q(t) \end{bmatrix}, \quad A = \begin{bmatrix} 0 & L \\ -L^* & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ H \end{bmatrix}$$

and $C = \begin{bmatrix} 0 & H^* \end{bmatrix}$. Note that $A^* = -A$, $C = B^*$.

(b) $A^* = -A$ implies that iA is self-adjoint, so that iA has only real eigenvalues. Hence, all the eigenvalues of A are on $i\mathbb{R}$, so that A is not stable.

(c) We have $\frac{d}{dt}\left(\frac{1}{2}\|p\|^2\right) = \frac{1}{2}\frac{d}{dt} \langle p, p \rangle$

$$= \frac{1}{2}\left(\langle \dot{p}, p \rangle + \langle p, \dot{p} \rangle\right) = \frac{1}{2}\left(\langle \dot{p}, p \rangle + \overline{\langle \dot{p}, p \rangle}\right)$$

$= \operatorname{Re}\langle \dot{p}, p \rangle$, similarly for q in place of p,

hence

$$\dot{E} = \operatorname{Re}\left(\langle \dot{p}, p \rangle + \langle \dot{q}, q \rangle\right)$$

$$= \operatorname{Re}\left(\langle Lq, p \rangle - \langle L^* p, q \rangle + \langle Hu, q \rangle\right)$$

$$= \underbrace{\operatorname{Re}\left(\langle Lq, p \rangle - \overline{\langle Lq, p \rangle}\right)}_{0} + \operatorname{Re}\langle u, H^* q \rangle$$

$$= \operatorname{Re}(u\bar{y}).$$

(d) $G(s) = C(sI - A)^{-1}B$. Applying the Laplace transformation to the system equations, with zero initial conditions, we get $s\hat{p} = L\hat{q}$, $s\hat{q} = -L^*\hat{p} + H\hat{u}$ hence $s^2\hat{q} = -L^*L\hat{q} + sH\hat{u}$.

$$(s^2 I + L^* L)\hat{q} = sH\hat{u}, \quad \hat{y} = H^*\hat{q}, \quad \text{hence}$$

$$\boxed{G(s) = sH^*(s^2 I + L^* L)^{-1} H.}$$

(e) We have $E(t) = \frac{1}{2}\|x(t)\|^2$. If $u = 0$ then, according to the result from part (c), $\dot{E}(t) = 0$, so that $\|x(t)\| = \|x(0)\|$. Thus, e^{At} is isometric, hence $\operatorname{Ker} e^{At} = \{0\}$, hence $\det e^{At} \neq 0$, hence e^{At} is invertible, hence e^{At} is unitary. (We remark that e^{At} is actually invertible for every square matrix A.)

(f)
$$\begin{aligned}
G(s) + G(s)^* &= C(sI-A)^{-1}B + B^*(\bar{s}I-A^*)^{-1}C^* \\
&= C(sI-A)^{-1}B + C(\bar{s}I+A)^{-1}B \\
&= C\left[(sI-A)^{-1} - (-\bar{s}I-A)^{-1}\right]B \\
&= (-\bar{s}-s)\, C(sI-A)^{-1}(-\bar{s}I-A)^{-1}B \\
&= (2\operatorname{Re} s)\, C(sI-A)^{-1}(\bar{s}I-A^*)^{-1}B,
\end{aligned}$$

this is the same as $\overline{G(s)}$

so that we have $k(s) = 2\operatorname{Re} s$.

(g) Denote $z(s) = (\bar{s}I - A^*)^{-1}B$, so that $z(s) \in \mathbb{C}^{4\times 1}$. We have, for $\operatorname{Re} s > 0$,

$$\begin{aligned}
\operatorname{Re} G(s) &= \frac{1}{2}\left[G(s) + G(s)^*\right] \\
&= (\operatorname{Re} s)\, z(s)^* z(s) \\
&= (\operatorname{Re} s)\, \|z(s)\|^2 > 0.
\end{aligned}$$

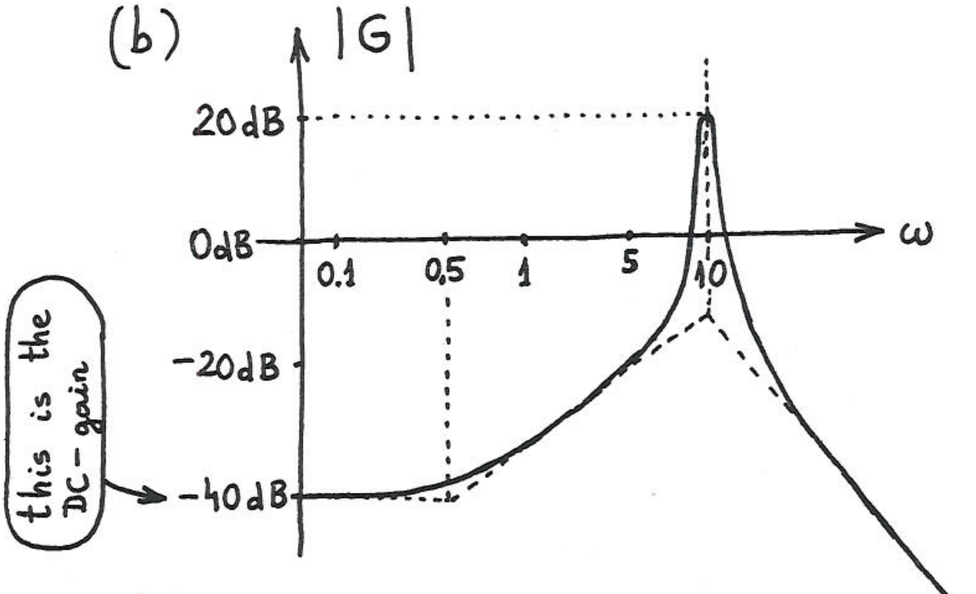
according to part (f)

In the last step we have used that $H \neq 0 \Rightarrow B \neq 0 \Rightarrow z(s) \neq 0$.

Question 5 (a) $G(s) = \dfrac{2s-1}{s^2+0.2s+100}$.

G is stable, because the denominator is a polynomial of degree two with positive coefficients.

(b)



Comments: The zero at 0.5 causes the plot to rise at a slope of 20dB/dec. The pair of poles with absolute value $\omega_n = \sqrt{100} = 10$ causes the linear approximation to the plot to bend down with -20dB/dec. The peak value is approximately at $\omega_n = 10$.

To estimate the peak value, we compute

$$G(10i) = \frac{20i-1}{2i} = 10 + 0.5i,$$

so that $|G(10i)| \approx 10$ (with an error less than 1%). Thus, $\|G\|_\infty \approx 10$.

(c) $u \in L^2(\mathbb{R})$ belongs to $BL(\omega_b)$ if

$$(\mathcal{F}u)(i\omega) = 0 \quad \text{for} \quad |\omega| > \omega_b.$$

Here, $\omega_b > 0$ and $\mathcal{F}$ denotes the Fourier transform.

(d) $$e_k(t) = \frac{1}{\sqrt{\pi\omega_b}} \cdot \frac{\sin \omega_b(t-k\tau)}{\omega_b(t-k\tau)},$$

where $k \in \mathbb{Z}$ and $\tau = \dfrac{\pi}{\omega_b}$. This basis is obtained as the inverse Fourier transform of the usual Fourier orthonormal basis in $L^2[-i\omega_b, i\omega_b]$.

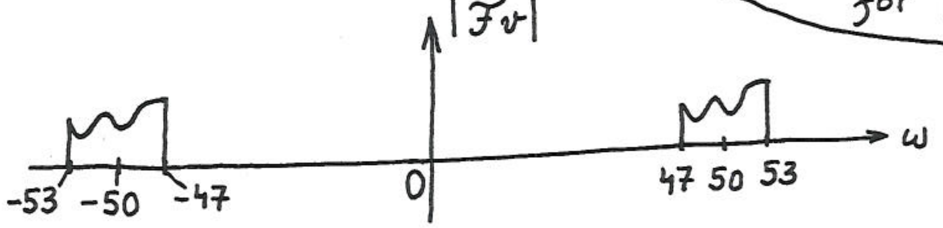
(e) If $u \in BL(3)$ and $v(t) = u(t)\cos 50t$ then,

using that $\cos 50t = \frac{1}{2}\left(e^{i50t} + e^{-i50t}\right)$,

we obtain (as in amplitude modulation)

$$(\mathcal{F}v)(i\omega) = \frac{1}{2}\Big[(\mathcal{F}u)(i\omega - i50) + (\mathcal{F}u)(i\omega + i50)\Big]$$

so that $(\mathcal{F}v)(i\omega) = 0$ for $|\omega| > 53$ (and also for $|\omega| < 47$).



Thus, $v \in BL(53)$.

(f) The Fourier transformation from $L^2(\mathbb{R})$ to $L^2(i\mathbb{R})$ is isometric (this is the Parseval equ lity), hence $\langle u, v\rangle = \langle \mathcal{F}u, \mathcal{F}v\rangle$.
Since $(\mathcal{F}v)(i\omega) = 0$ for $|\omega| < 47$, in particular for $|\omega| < 3$, we have $\langle \mathcal{F}u, \mathcal{F}v\rangle = 0$ (because $(\mathcal{F}u)(i\omega)\overline{(\mathcal{F}v)(i\omega)}$ is zero for all $\omega \in \mathbb{R}$).

(g) If $u \in BL(3)$ is the input signal and y is the output signal, then $(\mathcal{F}y)(i\omega) = G(i\omega)(\mathcal{F}u)(i\omega)$.
Hence, $(\mathcal{F}y)(i\omega) = 0$ for $|\omega| > 3$, so that $y \in BL(3)$.

Using again that $\mathcal{F}$ is isometric, we have

$$\|y\|_2^2 = \|\mathcal{F}y\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |(\mathcal{F}y)(i\omega)|^2\, d\omega$$

$$= \frac{1}{2\pi}\int_{-3}^{3} |G(i\omega)|^2 \cdot |(\mathcal{F}u)(i\omega)|^2\, d\omega.$$ From the Bode plot in part (b) we see that $|G(i\omega)| \leq 1$ for $|\omega| < 3$.
Hence $\|y\|_2^2 \leq \frac{1}{2\pi}\int_{-3}^{3} |(\mathcal{F}u)(i\omega)|^2\, d\omega = \|\mathcal{F}u\|_2^2 = \|u\|_2^2$.