Imperial College
London

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2015

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Probability Theory

Date: Wednesday, 13 May 2015. Time: 2.00pm – 4.00pm. Time allowed: 2 hours.

This paper has FOUR questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the main book is full.

Statistical tables will not be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers to each question as per the table below.

Raw mark	up to 12	13	14	15	16	17	18	19	20
Extra credit	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4

- Each question carries equal weight.
- Calculators may not be used.

1.

(a) Write short notes to define the following concepts:

  (i) a σ-algebra of events;

  (ii) a Π-system of events;

  (iii) the independence of an infinite sequence of σ-algebras $\mathcal{F}_1$, $\mathcal{F}_2$, ...

(b) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Suppose that Π_1, Π_2 are Π-systems of events from $\mathcal{F}$, such that $\mathbb{P}[A \cap B] = \mathbb{P}[A]\,\mathbb{P}[B]$ for all $A \in \Pi_1$, $B \in \Pi_2$. If $\mathcal{G}_1$, $\mathcal{G}_2$ are the σ-algebras generated by Π_1, Π_2 respectively, then show that $\mathcal{G}_1$, $\mathcal{G}_2$ are independent σ-algebras. (In your proof you may use basic results from lectures about Π-systems, so long as you state them clearly.)

(c) Suppose that X, Y are two random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and such that

$$\mathbb{P}[X \leq x, Y \leq y] \quad = \quad \mathbb{P}[X \leq x]\,\mathbb{P}[Y \leq y] \qquad \text{for } x, y \in \mathbb{R}.$$

Explain how the theory of Π-systems can be used to produce a rigorous proof of the independence of X and Y. Show that if X, Y are integer-valued then

$$\mathbb{P}[X + Y \text{ is odd}] \quad = \quad \mathbb{P}[X \text{ is even}]\,\mathbb{P}[Y \text{ is odd}] + \mathbb{P}[X \text{ is odd}]\,\mathbb{P}[Y \text{ is even}]\ .$$

(d) Suppose that X, Y are independent absolutely continuous random variables with the same standard normal density

$$f(z) \quad = \quad \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) \qquad \text{for all } z\,.$$

By using polar coordinates or otherwise, show that $H = \sqrt{X^2 + Y^2}$ and $T = |Y/X|$ are independent random variables and compute the density of H. .

2.

(a) Write short notes to define the following notions of convergence, and state the implications between them.

(i) convergence almost surely;

(ii) convergence in 1-norm;

(iii) convergence in probability.

(b) Consider a sequence of independent random variables Z_1, Z_2, ... defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and such that each Z_n takes only the values 0 and $\sqrt{n}$, and suppose that $\mathbb{P}[Z_n = \sqrt{n}] = p_n$. In each of the following cases, explain with reasons whether the sequence converges almost surely, in 1-norm, or in probability.

(i) $p_n = 1/n^2$;

(ii) $p_n = 1/n$;

(iii) $p_n = 1/\sqrt{n}$.

(c) Suppose that a sequence of random variables X_1, X_2, ... is defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and satisfies the bound $|X_n| \leq W$, where W is a non-negative random variable of finite expectation. Prove that the sequence X_1, X_2, ... is uniformly integrable.

(d) Suppose that $X_n \to 1$ almost surely, that $|X_n| \leq 2$ for all n, and that U_1, U_2, ... are Uniform([0,1]) random variables, independent of each other and of the X_n. Consider the sequence $Y_n = (1 + \mathbb{I}[U_n \leq 1/n])X_n$ (where $\mathbb{I}[U_n \leq 1/n] = 1$ exactly when $U_n \leq 1/n$, and otherwise is zero) and show that:

(i) the sequence Y_1, Y_2, ... is uniformly integrable;

(ii) $Y_n \to 1$ in probability but not almost surely;

(iii) $Y_n \to 1$ in 1-norm.

3.

(a) (i) Define the notion of weak convergence;

(ii) Define the notion of convergence in distribution;

(iii) Carefully describe the circumstances and the extent to which these two notions can be related to the notion of convergence in probability.

(b) Prove that if $X_n \to X$ almost surely then $X_n \to X$ weakly.

(c) Suppose that U_1, U_2, ... are independent random variables, each uniformly distributed over $[0, 1]$. Set $M_n = \min\{U_1, \ldots, U_n\}$ and show that nM_n converges weakly and identify its limiting distribution.

(d) Suppose that U_1, U_2, ..., V_1, V_2, ... are independent random variables, each uniformly distributed over $[0, 1]$, so that the random points (U_1, V_1), (U_2, V_2), ... are independently uniformly distributed over the unit square. Let R_n be the radius of the largest circle centred on $(0, 0)$ which contains none of the points (U_i, V_i) for $i = 1, \ldots, n$. By considering $\mathbb{P}[\sqrt{n}R_n > r]$ for $r > 0$, show that $\sqrt{n}R_n$ converges weakly to a distribution with a probability density, and calculate what is the probability density function of the limit.

4.

(a) Write down definitions of the following concepts:

(i) a filtration of σ-algebras;

(ii) an adapted process;

(iii) a martingale;

(iv) a supermartingale;

(v) a submartingale.

(b) Suppose that X is a random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and suppose that $\mathcal{G} \subseteq \mathcal{H} \subseteq \mathcal{F}$ are sub-σ-algebras. Use the general definition of the conditional expectation to prove the tower property, that if $\mathbb{E}[|X|] < \infty$ then $\mathbb{E}[\mathbb{E}[X|\mathcal{H}]|\mathcal{G}] = \mathbb{E}[X|\mathcal{G}]$.

(c) Suppose that $X_0 = 0$, X_1, X_2, ... is a simple symmetric random walk begun at 0, so that the jumps $X_1 - X_0$, $X_2 - X_1$, ... are independent identically distributed random variables, each equally likely to equal $+1$ or -1. State the Optional Stopping Theorem for martingales, and show how to use it to calculate the probability that the random walk hits level a before level $-b$, where a and b are positive integers.

(d) Now suppose that $X_0 = 0$, X_1, X_2, ... actually forms an asymmetric random walk, so that $X_1 - X_0$, $X_2 - X_1$, ... are still independent identically distributed random variables, but now $\mathbb{P}[X_n - X_{n-1} = +1] = p = 1 - \mathbb{P}[X_n - X_{n-1} = -1]$ for some fixed $p \in (0, 1)$, with $p \neq \frac{1}{2}$. Find a $\gamma > 0$ such that

$$Y_n \quad = \quad \gamma^{X_n}$$

defines a martingale, and use this martingale to calculate the new value for the hitting probability described in (c) above.

Probability Theory M3P6 – Solutions

Setter:

February 20, 2015

1: (a) (i) **(BOOKWORK)**

A σ-algebra of events is a family of subsets of a fixed sample space Ω such that:

1. the family contains $\emptyset$;
2. it is closed under complements;
3. it is closed under countable union.

(or any equivalent variation on these axioms!)
Marks: 2

(ii) **(BOOKWORK)**

A Π-system of events is a subset of a σ-algebra which is closed under finite intersections.
Marks: 1

(iii) **(BOOKWORK)**

The σ-algebras $\mathcal{F}_1$, $\mathcal{F}_2$, ... are independent if, for any finite strictly increasing sequence $i_1 < i_2 < \ldots < i_k$ of indices,

$$\mathbb{P}[A_{i_1} \cap \ldots \cap A_{i_k}] \quad = \quad \mathbb{P}[A_{i_1}] \times \ldots \times \mathbb{P}[A_{i_k}]$$

whenever $A_{i_j} \in \mathcal{F}_{i_j}$ for $j = 1, \ldots, k$..
Marks: 2

(b) **(BOOKWORK)**

Here are the crucial elements of the proof.

We will need the Uniqueness Lemma for Π-systems: if two measures agree on a Π-system Π then they must agree on the generated σ-algebra $\sigma(\Pi)$.

Fix $B \in \Pi_2$ and consider the two measures

$$\begin{aligned} A &\mapsto \mathbb{P}[A \cap B] \,, \\ A &\mapsto \mathbb{P}[A]\,\mathbb{P}[B] \,. \end{aligned}$$

These agree on Π_1 hence must agree on $\sigma(\Pi_1) = \mathcal{G}_1$.
Fix $A \in \mathcal{G}_1$ and consider the two measures

$$\begin{aligned} B &\mapsto \mathbb{P}[A \cap B] \,, \\ B &\mapsto \mathbb{P}[A]\,\mathbb{P}[B] \,. \end{aligned}$$

These agree on Π_2 hence must agree on $\sigma(\Pi_2) = \mathcal{G}_2$.
The independence of $\mathcal{G}_1$ and $\mathcal{G}_2$ now follows.
Marks: 5

(c) **(SIMILAR TO EXERCISES)**
The families of events $\{[X \leq x] : x \in \mathbb{R}\}$, $\{[Y \leq x] : x \in \mathbb{R}\}$ are both Π-systems. Hence we can apply the previous argument to show that $\mathbb{P}[A \cap B] = \mathbb{P}[A]\,\mathbb{P}[B]$ if $A \in \sigma(X)$, $B \in \sigma(Y)$. So X, Y are independent. Now argue that, since the corresponding events belong to $\sigma(X)$, $\sigma(Y)$ as appropriate,

$$\begin{aligned} \mathbb{P}[X + Y \text{ is odd}] &= \mathbb{P}[X \text{ is even and } Y \text{ is odd}] + \mathbb{P}[X \text{ is odd and } Y \text{ is even}] \\ &= \mathbb{P}[X \text{ is even}]\,\mathbb{P}[Y \text{ is odd}] + \mathbb{P}[X \text{ is odd}]\,\mathbb{P}[Y \text{ is even}] \,. \end{aligned}$$

(Half-marks if student omits Π-system material.)
Marks: 4

(d) **(UNSEEN)**
Joint density of X, Y is $\frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)}$. Hence

$$\mathbb{P}[H \leq \rho, T \leq c] \quad = \quad \frac{1}{2\pi} \iint_{x^2+y^2 \leq \rho^2, |y/x| \leq c} e^{-\frac{1}{2}(x^2-y^2)} \, \mathrm{d}x \, \mathrm{d}y \,.$$

Converting to polar coordinates,

$$\begin{aligned} \mathbb{P}[H \leq \rho, T \leq c] &= \frac{1}{2\pi} \int_{|\tan\theta| \leq c} \int_0^\rho e^{-r^2/2} r \, \mathrm{d}r \, \mathrm{d}\theta \\ &= (1 - e^{-\rho^2/2}) \times \frac{1}{2\pi} \text{length}\{\theta \in [0, 2\pi) : |\tan\theta| \leq c\} \,. \end{aligned}$$

Both factors are distribution functions, as can be seen by suitable choice of ρ, c. Indeed for $\rho \geq 0$ and $\theta \in [0, 2\pi)$

$$\begin{aligned} \mathbb{P}[H \leq \rho] &= 1 - e^{-\rho^2/2}, \\ \mathbb{P}[T \leq c] &= \tfrac{1}{2\pi}\text{length}\{\theta \in [0, 2\pi) : |\tan\theta| \leq c\}. \end{aligned}$$

So the above arguments apply, and H and T are independent and the density of H is $re^{-r^2/2}$ for $r \geq 0$.
Marks: 6

Total mark: 20

2: (BOOKWORK)

Implications: (ii) implies (iii), (i) implies (iii), no other implications hold in general.
Mark 2

(a) (i) (BOOKWORK)
$X_n \to X$ almost surely if $\mathbb{P}\{\omega : X_n(\omega) \to X(\omega)\} = 1$;
Marks: 1

(ii) (BOOKWORK)
$X_n \to X$ in 1-norm if $\mathbb{E}[|X_n - X|] \to 0$;
Mark 1

(iii) (BOOKWORK)
$X_n \to X$ in probability if for any $\varepsilon > 0$ we have $\mathbb{P}[|X_n - X| \geq \varepsilon] \to 0$.
Mark 1

(b) (i) (SIMILAR TO EXERCISES)
Set $Z = 0$. For $\varepsilon > 0$, eventually

$$\mathbb{P}[|Z_n - Z| > \varepsilon] = \mathbb{P}[Z_n = \sqrt{n}] = \tfrac{1}{n^2} \to 0.$$

So $Z_n \to Z$ in probability.
Moreover $\sum \mathbb{P}[|Z_n - Z| > \varepsilon] = \sum \frac{1}{n^2} < \infty$ so, by Borel-Cantelli lemmas, with probability one $|Z_n - Z| \geq \varepsilon$ for only finitely many n. Hence $Z_n \to Z$ almost surely.
Finally $\mathbb{E}[|Z_n - Z|] = \frac{1}{n^{3/2}} \to 0$ so convergence in 1-norm holds.
Mark 2

(ii) (SIMILAR TO EXERCISES)

By similar arguments we obtain convergence in probability and in 1-norm. However $\sum \mathbb{P}[|Z_n - Z| > \varepsilon] = \sum \frac{1}{n} = \infty$, and since the Z_n are independent we can deduce by the Borel-Cantelli lemmas that with probability one we obtain $|Z_n - Z| \geq \varepsilon$ for infinitely many n. So convergence almost surely does not occur.

Mark 2

(iii) (SIMILAR TO EXERCISES)

By similar arguments we obtain convergence in probability and do not obtain convergence almost surely. However $\mathbb{E}[|Z_n - Z|] = 1 \not\to 0$, so 1-norm convergence does not occur.

Mark 1

(c) (BOOKWORK)

We need to show, for every $\varepsilon > 0$ we can find K such that $\mathbb{E}[|X_n|; |X_n| > K] < \varepsilon$ for all n.

Now $W\,\mathbb{I}[W > K] \to 0$ almost surely as $K \to \infty$ so, by dominated convergence theorem, $\mathbb{E}[W; W > K] \to 0$. Thus given ε we can find K such that $\mathbb{E}[W; W > K] \leq \varepsilon$.

Hence for every n

$$\mathbb{E}[|X_n|; |X_n| > K] \quad \leq \quad \mathbb{E}[W; |X_n| > K] \quad \leq \quad \mathbb{E}[W; W > K] \quad \leq \quad \varepsilon\,.$$

Hence uniform integrability holds.

Mark 5

(d) (i) (UNSEEN)

$$|Y_n| \quad = \quad |(1 + \mathbb{I}[U_n \leq 1/n])X_n| \quad \leq \quad 2|X_n| \quad \leq \quad 4\,,$$

so uniform integrability holds by work in previous parts.

Mark 1

(ii) (UNSEEN)

Consider

$$\mathbb{P}[|Y_n - 1| \geq \varepsilon] \quad \leq \quad \mathbb{P}\left[U_n \leq \tfrac{1}{n}\right] + \mathbb{P}[|X_n - 1| \geq \varepsilon] \quad = \quad \frac{1}{n} + \mathbb{P}[|X_n - 1| \geq \varepsilon]$$

which tends to 0 as $n \to \infty$ (note that almost sure convergence implies convergence in probability).

However, since $\sum \frac{1}{n} = \infty$ and the U_n are independent, it follows that with probability one $U_n \leq \frac{1}{n}$ infinitely often, so $Y_n = 2X_n$ for infinitely many n, so if $X_n \to 1$ almost surely then Y_n gets arbitrarily close to 2 for infinitely many n. Hence Y_n cannot converge almost surely.
Mark 2

(iii) (UNSEEN)
Consider

$$\mathbb{E}[|Y_n - 1|] = (1-\tfrac{1}{n})\,\mathbb{E}[|X_n - 1|] + \tfrac{1}{n}\,\mathbb{E}[|2X_n - 1|] \leq (1-\tfrac{1}{n})\,\mathbb{E}[|X_n - 1|] + \tfrac{5}{n}.$$

Now by dominated convergence $\mathbb{E}[|X_n - 1|] \to 0$. since $|X_n - 1| \leq 3$, and so $\mathbb{E}[|Y_n - 1|] \to 0$.
Mark 2

Total mark: 20

3: (a) (BOOKWORK)
$X_n \to X$ weakly if $\mathbb{E}[f(X_n)] \to \mathbb{E}[f(X)]$ for all bounded continuous f.

$X_n \to X$ in distribution (for real-valued X_n, X) if $F_n(x) \to F(x)$ at all continuity points x of F, where F_n, F are distribution functions for X_n, X,

These relate to convergence in probability as follows. Convergence in probability implies both (which are equivalent concepts for real-valued random variables). For random variables defined on the same probability space, weak convergence to a constant implies convergence in probability. Alternatively, it is possible to realize a weakly convergent sequence of random variables on the same probability space so as to achive convergence almost surely and hence convergence in probability.
Marks 6

(b) (BOOKWORK)
Suppose $X_n \to X$ almost surely. If f is a bounded continuous function then (a) by continuity $f(X_n) \to f(X)$ almost surely, (b) $|f(X_n)|$ is uniformly bounded for all n. Hence by the dominated convergence theorem $\mathbb{E}[f(X_n)] \to \mathbb{E}[f(X)]$, and so weak convergence applies.
Marks 4

(c) (SIMILAR TO EXERCISES)

$$\mathbb{P}[nM_n > x] \quad = \quad \mathbb{P}\left[U_m > \tfrac{x}{n} \text{ for } m \leq n\right] \quad = \quad \left(1 - \tfrac{x}{n}\right)^n \quad \rightarrow \quad e^{-x}$$

which shows that nM_n converges in distribution (and hence weakly) to a unit Exponential random variable.
Marks 5

(d) (SIMILAR TO EXERCISES)
Here (U_1, V_1) is uniformly distributed over the unit square, so $\mathbb{P}[R_n > r] = (1 - \frac{\pi}{4}r^2)^n$ at least if $r \leq 1$. So

$$\mathbb{P}\left[\sqrt{n}R_n > r\right] \quad = \quad \left(1 - \frac{\pi}{4}\frac{r^2}{n}\right)^n$$

at least if $r < \sqrt{\frac{4n}{\pi}}$, and this converges to the limit $e^{-\pi r^2/4}$. Hence

$$\mathbb{P}\left[\sqrt{n}R_n \leq r\right] \quad \rightarrow \quad 1 - e^{-\pi r^2/4}$$

So the limit probability density function is $\frac{\pi}{2} r e^{-\pi r^2/4}$ for $r \geq 0$.
Marks 5

Total mark: 20

4: (a) (i) (BOOKWORK)
This is an increasing sequence of σ-algebras $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \ldots$.
Mark 1

(ii) (BOOKWORK)
$\{X_0, X_1, \ldots\}$ such that X_n is $\mathcal{F}_n$-measurable for all $n \geq 0$.
Mark 1

(iii) (BOOKWORK)
$\{X_0, X_1, \ldots\}$ (adapted, though this is implied by the following equation), such that $\mathbb{E}[X_{n+1}|\mathcal{F}_n] = X_n$ almost surely for all $n \geq 0$.
Mark 1

(iv) (BOOKWORK)
$\{X_0, X_1, \ldots\}$ *adapted* such that $\mathbb{E}[X_{n+1}|\mathcal{F}_n] \leq X_n$ almost surely for all $n \geq 0$.
Mark 1

(v) (BOOKWORK)
$\{X_0, X_1, \ldots\}$ *adapted* such that $\mathbb{E}[X_{n+1}|\mathcal{F}_n] \geq X_n$ almost surely for all $n \geq 0$.
Mark 1

(b) (BOOKWORK)
General definition of $\mathbb{E}[X|\mathcal{G}]$: suppose that X is in L^1. Then $\mathbb{E}[X|\mathcal{G}]$ exists and is unique up to almost-sure equivalence, defined by

* $\mathbb{E}[X|\mathcal{G}]$ is in L^1;
* $\mathbb{E}[X|\mathcal{G}]$ is $\mathcal{G}$-measurable;
* $\mathbb{E}[\mathbb{E}[X|\mathcal{G}]; A] = \mathbb{E}[X; A]$ for all $A \in \mathcal{G}$.

So consider $\mathbb{E}[\mathbb{E}[X|\mathcal{H}]|\mathcal{G}]$ when $\mathcal{G} \subseteq \mathcal{H}$. Certainly the first two requirements hold. Suppose $A \in \mathcal{G}$. Then by third requirement applied to $\mathbb{E}[\ldots|\mathcal{G}]$, since $A \in \mathcal{G}$,

$$\mathbb{E}[\mathbb{E}[\mathbb{E}[X|\mathcal{H}]|\mathcal{G}]; A] \quad = \quad \mathbb{E}[\mathbb{E}[X|\mathcal{H}]; A] .$$

But now $A \in \mathcal{G} \subseteq \mathcal{H}$, so by third requirement again but now applied to $\mathbb{E}[\ldots|\mathcal{H}]$

$$\mathbb{E}[\mathbb{E}[X|\mathcal{H}]; A] \quad = \quad \mathbb{E}[X; A] .$$

Hence the third requirment holds for $\mathbb{E}[\mathbb{E}[X|\mathcal{H}]|\mathcal{G}]$.
Marks 5

(c) (BOOKWORK)
Optional Stopping Theorem: suppose X is a martingale. Consider stopping times $0 \leq S \leq T \leq K$ for a non-random finite bound K. Then

$$\mathbb{E}[X_T|\mathcal{F}_S] \quad = \quad X_S .$$

Moreover this holds without a finite upper bound K so long as $\{X_{n\wedge T} : n \geq 0\}$ is uniformly integrable.
Here we take $S = 0$ and $T = \inf\{n : X_n = a \text{ or } X_n = -b\}$. Since $X_{n\wedge T}$ lies between $-b$ and a we know the desired uniform integrability obtains. Hence

$$\mathbb{E}[X_T|\mathcal{F}_0] \quad = \quad \mathbb{E}[X_T] \quad = \quad X_0 \quad = \quad 0 .$$

But $\mathbb{E}[X_T] = a\,\mathbb{P}[X_T = a] - b\,\mathbb{P}[X_T = -b]$ and thus

$$\mathbb{P}[X \text{ hits } a \text{ before } -b] \quad = \quad \mathbb{P}[X_T = a] \quad = \quad \frac{b}{a+b} .$$

Marks 5

(d) **(SIMILAR TO EXAMPLE IN LECTURES)**

Now we have

$$\mathbb{E}\left[Y_{n+1}|\mathcal{F}_n\right] \quad = \quad \left(p\gamma + \frac{1-p}{\gamma}\right) Y_n$$

and for the martingale property to hold we need $p\gamma^2 - \gamma + (1-p) = 0$. Hence $\gamma = \frac{1-p}{p}$ will suffice. (The other root $\gamma = 1$ conveys no information!)

So argue as before:

$$\mathbb{E}\left[Y_T\right] \quad = \quad 1 \quad = \quad \gamma^a \,\mathbb{P}\left[X_T = a\right] + \gamma^{-b}\,\mathbb{P}\left[X_T = -b\right]$$

and so

$$\mathbb{P}\left[X_T = a\right] \quad = \quad \frac{1-\gamma^{-b}}{\gamma^a - \gamma^{-b}}\,.$$

Marks 5

Total mark: 20