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ISE3.7

IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE
UNIVERSITY OF LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
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EEE/ISE PART III/IV: M.Eng., B.Eng. and ACGI

MATHEMATICS FOR SIGNALS AND SYSTEMS

Friday, 3 May 10:00 am

There are FIVE questions on this paper.

Answer THREE questions.

Corrected Copy

Time allowed: 3:00 hours

Examiners responsible:

First Marker(s): Weiss,G.

Second Marker(s): Allwright,J.C.

Special instructions to invigilators: None

Information for candidates: None

1. On the real vector space $\mathbb{R}^{3\times 3}$ (which contains all the real 3×3 matrices), we define an inner product by

$$< a, b > \;=\; \frac{1}{2}\,\text{trace}\; a^T b,$$

where a^T is the transpose of a. We define the subspaces

$$S = \{a \in \mathbb{R}^{3\times 3} \mid a^T = a\}$$

(these are the so-called symmetric, or self-adjoint real matrices), and

$$A = \{a \in \mathbb{R}^{3\times 3} \mid a^T = -a\}$$

(these are the so-called anti-symmetric, or skew-adjoint real matrices).

(a) What are the dimensions of $\mathbb{R}^{3\times 3}$, S and A? [2]

(b) Find an orthonormal basis in A. [4]

(c) Show that A is orthogonal to S. From here, using your answer to part (a), conclude that in fact, A is the orthogonal complement of S. [4]

(d) Show that if $a \in A$, then the eigenvalues of a are imaginary. (Hint: the complex matrix ia is self-adjoint.) [3]

(e) Show that if $a \in A$, then $\det a = 0$. (Hint: use part (d) and a certain symmetry of the eigenvalues.) [3]

(f) We denote the orthogonal projectors from $\mathbb{R}^{3\times 3}$ onto S and A by $\mathbf{P}_S$ and $\mathbf{P}_A$ (thus, $\mathbf{P}_S + \mathbf{P}_A = I$, the identity operator acting on $\mathbb{R}^{3\times 3}$). Check that these projectors are given by

$$\mathbf{P}_S x = \frac{1}{2}(x + x^T), \qquad \mathbf{P}_A x = \frac{1}{2}(x - x^T).$$

(Hint: use the conclusion from part (c).) [4]

2. We denote by $\mathbf{c}_0$ is the space of sequences convergent to zero, and by $\mathbf{c}$ the space of convergent sequences. We consider the indices (i.e., the discrete time) to run from 0 to ∞.

(a) Give an example of a sequence $a \in l^1$ that has infinitely many nonzero terms, and also infinitely many zero terms. [3]

(b) Which of the inclusions $\mathbf{c}_0 \subset l^2$ or $l^2 \subset \mathbf{c}_0$ is true? Give a very brief explanation of your answer, and show that $\mathbf{c}_0 \neq l^2$. [3]

(c) Give an example of a sequence $b \in l^\infty$ such that $b \notin \mathbf{c}$, and compute its norm in l^∞. [2]

(d) Compute the $\mathcal{Z}$ transforms of the sequences u and y given by

$$u_k = k\,, \qquad y_k = (-1)^k\,.$$

For each of these $\mathcal{Z}$ transforms, indicate a domain (the largest domain that you can determine) where the series defining the $\mathcal{Z}$ transform is convergent. [4]

(e) If possible, find a linear system which, starting from initial state zero, if it receives the input u, it produces the output y. Here, u and y are the signals from part (d). If you think that this is impossible, then explain why you think so. [4]

(f) Give an example of a sequence $q = (q_k)$ such that the series defining its $\mathcal{Z}$ transform does not converge for any value of the variable z. Hint: think of the $\mathcal{Z}$ transform as a Taylor series in the variable $\zeta = z^{-1}$. How do you compute the radius of convergence of this series? Make this radius zero. [4]

3. In this question, S_τ denotes the right shift operator by τ on $L^2[0,\infty)$ and $*$ denotes the convolution product.

(a) Define the natural inner product and the corresponding norm on the space $L^2[0,\infty)$. For $s \in \mathbb{C}_+$ and $\varphi \in L^2[0,\infty)$ defined by $\varphi(t) = e^{-st}$, compute $\|\varphi\|_2$. [3]

(b) Let $g \in L^2[0,\infty)$ and let $\mathcal{L}g$ denote its Laplace transform. Show that

$$|(\mathcal{L}g)(s)| \le \frac{\|g\|_2}{\sqrt{2\,\mathrm{Re}\, s}} \quad \text{for all} \quad s \in \mathbb{C}_+ .$$

Hint: use the result about $\|\varphi\|_2$ from part (a) and the Cauchy-Schwarz inequality. [3]

(c) In the sequel, consider f to be the characteristic function of the interval $[0,2]$ and $g(t) = e^{-5t}$, $t \ge 0$. (Thus, $f(t) = 1$ for $t \in [0,2]$ and $f(t) = 0$ for $t > 2$.) Compute the Laplace transforms $F = \mathcal{L}f$ and $G = \mathcal{L}g$. [3]

(d) Compute $\|f\|_2$, $< f, g >$ and $\|g\|_2$ and check that the Cauchy-Schwarz inequality holds for them. [4]

(e) Define $h = S_3 g$, i.e., h is obtained by delaying g by 3 time units. Compute

$$H = \mathcal{L}h, \qquad \|h\|_2 \qquad \text{and} \quad P = \mathcal{L}(h * g). \qquad [4]$$

(f) Compute

$$\|G\|_2, \qquad \|H\|_2 \quad \text{and} \quad < F, G > ,$$

where the norms and the scalar products correspond to the Hardy space $H^2(\mathbb{C}_+)$ and F, G, H are as defined above. [3]

4. Consider the system described by

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \alpha \\ \beta \end{bmatrix} u,$$

$$y = [0 \;\; 1]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

where u is the input signal, x is the state (with two components), y is the output signal and α, β are real constants.

(a) For which values of α, β is the system stable? [2]

(b) Compute the transfer function $\mathbf{G}$ of this system. [3]

(c) For $\alpha = \beta = 1$, compute the impulse response and the step response of this system, as functions of $t \geq 0$. [2]

(d) Still considering $\alpha = \beta = 1$, compute $\|\mathbf{G}\|_\infty$ and $\|\mathbf{G}\|_2$ (i.e., the norms of $\mathbf{G}$ in $H^\infty(\mathbb{C}_+)$ and in $H^2(\mathbb{C}_+)$). [3]

(e) Still considering $\alpha = \beta = 1$, if $u(t) = te^{-3t}$ and $x(0) = 0$, compute the output signal y as a function of t. [2]

(f) For $\alpha = 1$ and $\beta = 0$ (be careful, β has changed), consider the cascade connection of the system with a delay line of 2 time units. Thus, if z is the output signal of the delay line, then $z(t) = y(t-2)$. Compute the transfer function $\mathbf{H}$ from u to z. [2]

(g) Compute $\|\mathbf{H}\|_\infty$, where $\mathbf{H}$ is the transfer function from part (f). [3]

(h) Suppose now that α and β are functions of t: $\alpha(t) = \cos t$ and $\beta(t) = \sin t$. Is the the system with input u and output y still linear? Does this system have a transfer function? Explain very briefly your answer. [3]

5. (a) Explain briefly what is meant by a time-invariant operator on l^2. [3]

(b) State the discrete-time version of the Fourés-Segal theorem and discuss briefly its connections with systems theory. [7]

(c) Define the space $BL(\omega_b)$ of band-limited functions with angular frequencies not higher than ω_b. Give two examples of functions in this space which are linearly independent. [4]

(d) State the sampling theorem and discuss briefly its significance for the transmission and storage of signals. [6]

[END]

Mathematics for Signals and Systems

Exam of May 2002

SOLUTIONS

Question 1 (a) 9, 6 and 3.

(b) Matrices in A are of the form

$$m = \begin{bmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0 \end{bmatrix}.$$

The scalar product of two matrices $a, b \in \mathbb{R}^{3\times3}$ can also be written in the form

$$\langle a, b\rangle = \frac{1}{2}\sum_{k=1}^{3}\sum_{j=1}^{3} a_{jk}\, b_{jk}.$$

Hence, the following is an orthonormal basis in A:

$$e_1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

(The matrix m above is $m = \alpha e_1 + \beta e_2 + \gamma e_3$.)

(c) If $a \in S$, then $a_{jk} = a_{kj}$ $(k, j = 1,2,3)$. If $b \in A$, then $b_{jk} = -b_{kj}$, in particular, $b_{jj} = 0$. Hence,

$$\langle a, b\rangle = \sum_{k>j} a_{jk}\, b_{jk} + \sum_{k<j} a_{jk}\, b_{jk}$$

$$= \sum_{k>j} \left(a_{jk}\, b_{jk} + a_{kj}\, b_{kj} \right) = 0$$

(because $a_{kj}\, b_{kj} = - a_{jk}\, b_{jk}$). (This argument has been written such that it remains valid for square matrices of arbitrary dimensions, in the 3×3 case there are only three terms in the last sum: $(k,j) = (3,1), (3,2), (2,1)$.) Since $\dim S + \dim A = \dim \mathbb{R}^{3\times 3}$ and S is orthogonal to A, it follows that A is the orthogonal complement of S (i.e., the space of all matrices orthogonal to S).

(d) If $a \in A$, then $(ia)^* = (\bar{i})a^T = (-i)(-a) = ia$, so that ia (being self-adjoint) has only real eigenvalues. Hence, the eigenvalues of a are on $i\mathbb{R}$.

(e) Since $a \in A$ is real, its eigenvalues are located symmetrically with respect to the real axis. Since a has 3 eigenvalues on $i\mathbb{R}$, one of them must be zero. Since $\det a = \lambda_1 \lambda_2 \lambda_3$, where λ_j are the eigenvalues of a, we get $\det a = 0$.

An entirely different way to see that $\det a = 0$ is the following: using the structure from the answer to part (b), we have $ax = 0$, where $x = \begin{bmatrix} \gamma \\ -\beta \\ \alpha \end{bmatrix}$.

(f) If we denote $a = \frac{1}{2}(x + x^T)$, $b = \frac{1}{2}(x - x^T)$, then it is easy to see that $a^T = a$ and $b^T = -b$, i.e., $a \in S$ and $b \in A$. Moreover, we have $x = a + b$. Since such a decomposition is unique, we must have $a = P_S\, x$ and $b = P_A\, x$.

Question 2

(a) $a_k = \begin{cases} \frac{1}{k^2} & \text{if } k \text{ is odd}, \\ 0 & \text{if } k \text{ is even}. \end{cases}$

(b) We have $l^2 \subset c_0$. Indeed, if $a \in l^2$, i.e., $\sum_{k=0}^{\infty} |a_k|^2 < \infty$, then $\lim a_k = 0$. We have $l^2 \neq c_0$ because the sequence $b_k = \frac{1}{\sqrt{k+1}}$ is in c_0, but not in l^2.

(c) $d_k = (-1)^k$, $d \notin c$, $\|d\|_\infty = 1$.

(d) If $u_k = k$, $y_k = (-1)^k$, $k = 0, 1, 2, \ldots$, then

$$\hat{u}(z) = \frac{z}{(z-1)^2}, \qquad \hat{y}(z) = \frac{z}{z+1}.$$

$\hat{u}$ has a singularity (a pole) at $z=1$, hence the series is convergent for $|z|>1$. $\hat{y}$ has a singularity at $z=-1$, hence its series is also convergent for $|z|>1$.

(e) This is impossible, because any linear system is causal, i.e., at any time k, the response is caused by the past input only (up to k). In our case, for $k=0$, we have $u_0 = 0$, which together with $x_0 = 0$ (initial state zero) implies $y_0 = 0$, but we have $y_0 = 1$.

(f) Take $q_k = 2^{2^k}$. Denoting $\xi = \frac{1}{z}$, we have

$$\hat{q}(z) = q_0 + q_1 \xi + q_2 \xi^2 + q_3 \xi^3 + \ldots,$$

for all $\xi \in \mathbb{C}$ with $|\xi| < R$. The radius of convergence R of this Taylor series is given by $\frac{1}{R} = \limsup_{n\to\infty} |q_n|^{\frac{1}{n}}$. We get $\frac{1}{R} = \infty$, hence $R = 0$.

$\boxed{\text{Question 3}}$ (a) On the space $L^2[0,\infty)$,

$$<f,g> = \int_0^\infty f(t)\overline{g(t)}\,dt, \quad \|f\|_2^2 = \int_0^\infty |f(t)|^2 dt.$$

If $\varphi(t) = e^{-st}$, then $\|\varphi\|_2^2 = \int_0^\infty e^{-2(\mathrm{Re}\,s)t}\,dt = \frac{1}{2\,\mathrm{Re}\,s}$.

Hence, $\|\varphi\|_2 = \frac{1}{\sqrt{2\,\mathrm{Re}\,s}}$.

(b) We have, for any $g \in L^2[0,\infty)$,

$$|(\mathcal{L}g)(s)| = \left|\int_0^\infty g(t)e^{-st}dt\right| = <g,\overline{\varphi}>,$$

where $\overline{\varphi}$ is the complex conjugate of φ introduced in part (a). By the Cauchy-Schwarz inequality, we get

$$|(\mathcal{L}g)(s)| \leq \|g\|_2 \cdot \|\overline{\varphi}\|_2 = \|g\|_2 \cdot \|\varphi\|_2$$
$$= \|g\|_2 \cdot \frac{1}{\sqrt{2\,\mathrm{Re}\,s}}.$$

(c) $F(s) = \frac{1}{s}(1-e^{-2s})$, $G(s) = \frac{1}{s+5}$

(d) $\|f\|_2^2 = \int_0^2 dt = 2$, hence $\|f\|_2 = \sqrt{2}$.

$\|g\|_2^2 = \int_0^\infty e^{-10t}\,dt = \frac{1}{10}$, so $\|g\|_2 = \frac{1}{\sqrt{10}}$.

$<f,g> = \int_0^2 e^{-5t}\,dt = \frac{1}{5}(1-e^{-10}) \approx \frac{1}{5}$

Cauchy - Schwarz: $\frac{1}{5} < \frac{1}{\sqrt{10}} \cdot \sqrt{2} \left(= \frac{1}{\sqrt{5}}\right)$.

(e) $H(s) = e^{-3s}\dfrac{1}{s+5}$, $\|h\|_2 = \|g\|_2 = \dfrac{1}{\sqrt{10}}$

$$P(s) = H(s)\,G(s) = e^{-3s}\frac{1}{(s+5)^2} .$$

(f) $\|G\|_2 = \|g\|_2 = \dfrac{1}{\sqrt{10}}$, by Paley-Wiener.

Similarly,
$$\|H\|_2 = \|h\|_2 = \frac{1}{\sqrt{10}} .$$

Using the Paley-Wiener theorem a third time, we have

$$\langle F, G\rangle = \langle f, g\rangle = \frac{1}{5}(1-e^{-10}) .$$

Question 4

(a) $\dot{x} = Ax + Bu$, $B = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$,

$A = \begin{bmatrix} 0 & -3 \\ 1 & -4 \end{bmatrix}$, $\det(sI - A) = s^2 + 4s + 3$ (this is the characteristic polynomial), $\sigma(A) = \{-1, -3\}$. The system is stable regardless of α, β.

(b) $C = [0 \ 1]$, $y = Cx$, $G(s) = C(sI-A)^{-1}B$.

$$C(sI-A)^{-1} = [0 \ 1] \frac{1}{s^2+4s+3} \begin{bmatrix} s+4 & -3 \\ 1 & s \end{bmatrix} = \frac{1}{s^2+4s+3}[1 \ s],$$

hence $$G(s) = \frac{\beta s + \alpha}{s^2+4s+3} = \frac{\beta s + \alpha}{(s+1)(s+3)}.$$

(c) For $\alpha = \beta = 1$, $G(s) = \dfrac{1}{s+3}$, hence the impulse response is $g = \mathcal{L}^{-1}G$, $g(t) = e^{-3t}$. The step response is $y_{step}(t) = \int_0^t g(\sigma)\,d\sigma = \frac{1}{3}(1 - e^{-3t})$.

(d) $\|G\|_\infty = \left\| \frac{1}{s+3} \right\|_\infty = \frac{1}{3}$, $\|G\|_2 = \|g\|_2 = \frac{1}{\sqrt{6}}$.

(e) $u(t) = te^{-3t}$, $\hat{u}(s) = \dfrac{1}{(s+3)^2}$, $\hat{y}(s) = \dfrac{1}{(s+3)^3}$, $y(t) = \dfrac{t^2}{2}e^{-3t}$.

(f) For $\alpha = 1, \beta = 0$, $G(s) = \dfrac{1}{(s+1)(s+3)}$, the delay transfer function is e^{-2s}, hence $H(s) = e^{-2s}/(s+1)(s+3)$.

(g) $\|H\|_\infty = \|G\|_\infty$, because $|e^{-2i\omega}| = 1$ for $\omega \in \mathbb{R}$. G attains its sup at $s = 0$, hence $\|H\|_\infty = \frac{1}{3}$.

(h) With α, β functions of t, the system is still linear but it is not time-invariant. Hence, the system has no transfer function.

$\boxed{\text{Question 5}}$ (a) We denote by S the operator of right shift (or delay) by one step on ℓ^2 (the indices are from 0 to ∞). Thus,

$$S(u_0\ u_1\ u_2\ \dots) = (0\ u_0\ u_1\ \dots).$$

A bounded operator T from ℓ^2 to ℓ^2 is called time invariant if $TS = ST$.

(b) **Theorem (Fourès-Segal)** Let T be a bounded linear operator from ℓ^2 to ℓ^2. T is time-invariant if and only if there exists $G \in H^\infty(\mathcal{E})$ such that

$$T = \mathcal{Z}^{-1} G \mathcal{Z} \qquad (\mathcal{Z} = \mathcal{Z}\text{-transform}).$$

If this is the case, then $\|T\| = \|G\|_\infty$.

Consider a linear system described by

$$\begin{cases} x_{k+1} = Ax_k + Bu_k \\ y_k = Cx_k + Du_k \end{cases} \quad \begin{array}{|l} u = \text{input signal} \\ x = \text{state} \\ y = \text{output signal} \end{array}$$

where A, B, C, D are constant matrices and the eigenvalues of A are in $\mathcal{D}$ (i.e., A is stable). If $x(0) = 0$, then $y = Tu$, where T is the input-output operator of the system. This is bounded on ℓ^2 and time-invariant. According to the Fourès-Segal theorem, $T = \mathcal{Z}^{-1} G \mathcal{Z}$, with $G \in H^\infty(\mathcal{E})$. It can be checked that $G(z) = C(zI - A)^{-1}B + D$. The norm $\|G\|_\infty$ can be seen from the magnitude Bode plot of G.

(c) $BL(\omega_b)$ is the subspace of $L^2(-\infty,\infty)$ consisting of those functions whose Fourier transform is in $L^2[-i\omega_b, i\omega_b]$ (in other words, $u \in BL(\omega_b)$ if $(\mathcal{F}u)(i\omega) = 0$ for $|\omega| > \omega_b$). Such functions are analytic on all $\mathbb{C}$, in particular, they are infinitely differentiable. In practice, signals will usually not belong to such a space, but they can be approximated very well by band-limited functions. The following functions form an orthonormal basis in $BL(\omega_b)$:

$$e_k(t) = \frac{\sin \omega_b (t-k\tau)}{\sqrt{\pi \omega_b}\,(t-k\tau)}, \qquad k \in \mathbb{Z}, \quad \tau = \pi/\omega_b .$$

Notice that e_k is obtained by shifting e_0 to the right by the amount $k\tau$ (if $k<0$ then we are actually shifting to the left). Thus, for example, e_0 and e_7 are linearly independent functions in $BL(\omega_b)$.

(d) **Theorem (Whittaker – Kotelnikov – Shannon).**

If $u \in BL(\omega_b)$ and $\tau \in \left(0, \frac{\pi}{\omega_b}\right]$, then for all $t \in \mathbb{R}$,

$$u(t) = \sum_{k\in\mathbb{Z}} u(k\tau) \frac{\sin \omega_b (t-k\tau)}{\omega_b (t-k\tau)} .$$

This shows that if we sample the signal at the time instants $k\tau$, $k \in \mathbb{Z}$, where τ is the sampling period, then u can be completely reconstructed from these samples. It is easier to store and/or transmit samples of a signal than the whole signal.

In practice, signals are not exactly bandlimited, just "almost" bandlimited. This means that $u = v + e$, where $v \in BL(\omega_b)$ and e is a small error (deviation). Also, the samples $u(k\tau)$ cannot be taken for all $k \in \mathbb{Z}$, only for a finite (but possibly very large) set of integers. Then, the formula will hold approximately, for values of t which are not close to the end of the time interval in which samples were taken. The condition $\tau \leq \frac{\pi}{\omega_b}$ means that the sampling frequency $\frac{1}{\tau} \geqslant 2$ times the highest frequency components of u.

[END]