Imperial College
London

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2013

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Probability & Statistics I

Date: Wednesday, 22 May 2013. Time: 10.00am. Time allowed: 2 hours.

This paper has FOUR questions.

Candidates should start their solutions to each question in a new main answer book

Supplementary books may only be used after the relevant main book(s) are full.

Statistical tables will not be provided.

Formula sheets are provided on pages 4 & 5

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Answer all the questions. Each question carries equal weight.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Calculators may not be used.

1. (a) State the axioms of probability for events defined on a sample space Ω.

(b) For events $E \subseteq \Omega$ and $F \subseteq \Omega$, prove from the axioms that $\mathsf{P}(E) \geq \mathsf{P}(E \cap F)$.

(c) Events $E_1, E_2, E_3 \subseteq \Omega$ are such that, E_1 and E_3 are disjoint, E_1 and E_2 are independent and E_2 and E_3 are independent. Given $\mathsf{P}(E_i) = 1/2^i$, $i = 1, 2, 3$, determine the probability that out of E_1, E_2 and E_3, only E_2 occurs.

(d) The discrete random variable X has range $\{0, 1, 2, \ldots\}$. Prove that if $\mathsf{E}_{f_X}(X) = 0$ then $\mathsf{P}(X = 0) = 1$.

(e) The discrete random variable X has the following probability mass function (pmf):

$$f_X(-2) = \frac{1}{8}; \;\; f_X(-1) = \frac{1}{8}; \;\; f_X(0) = \frac{1}{8}; \;\; f_X(1) = \frac{1}{4}, \;\; f_X(2) = \frac{3}{8}.$$

(i) Determine the pmf of $Y = X + 2$.

(ii) Determine the pmf of $Z = X^2$.

(f) The continuous random variables α and β are both identically and independently distributed with a uniform distribution on the interval $[-1, 1]$. Consider the quadratic equation given by

$$x^2 + \alpha x + \beta = 0.$$

Determine the probability that the roots of this equation are real.

2. The continuous random variable X has Moment Generating Function (MGF), $M_X(t)$. Consider the random variable $Y = \mu + \sigma X$.

(a) Prove that the mean and variance of Y (μ_Y and σ_Y^2 respectively) may be written in terms of the mean and variance of X (μ_X and σ_X^2 respectively) as follows:

$$\mu_Y = \mu + \sigma \mu_X; \quad \sigma_Y^2 = \sigma^2 \sigma_X^2.$$

(b) Prove that the MGF of Y is given by

$$M_Y(t) = \mathrm{e}^{\mu t} M_X(\sigma t).$$

(c) Express the mean and variance of Y in terms of $M_X(t)$ and its derivatives.

(d) Suppose that $M_X(t) = \mathrm{e}^{t+6t^2}$.

(i) Determine μ_X and σ_X using the MGF.

(ii) Find the mean and variance of $Y = \frac{1}{3}(X - 1)$.

3. (a) For discrete variables Y and N prove that

$$\mathsf{E}_{f_N}(N) = \mathsf{E}_{f_Y}(E_{f_{N|Y}}(N|Y))$$

(b) Consider a sequence of independently, identically distributed random variables $Y_i, i = 1, 2, \ldots$ such that

$$Y_i = \begin{cases} 1 & \text{with probability } \theta,\ 0 \leq \theta \leq 1; \\ 0 & \text{otherwise,} \end{cases}$$

and let N be the discrete random variable representing the number of trials until a one is observed, i.e. $N = n$ if: $Y_n = 1$ and $Y_i = 0, i = 1, \ldots, n-1$.

(i) Determine $\mathsf{E}_{f_{Y_1}}(Y_1)$.

(ii) Derive the probability mass function of N and state the name of the distribution of N.

(iii) Using part (a) and letting $Y = Y_1$, prove that

$$\mathsf{E}_{f_N}(N) = (1 + \mathsf{E}_{f_N}(N))(1-\theta) + \theta,$$

hence find $\mathsf{E}_{f_N}(N)$.

4. The continuous random variables X and Y have joint probability density function given by

$$f_{X,Y}(x,y) = \begin{cases} \frac{x}{3} + cy & 0 < x < 1,\ 0 < y < 3; \\ 0 & \text{otherwise,} \end{cases}$$

for some constant c.

(a) Determine c.

(b) Find the marginal density functions, $f_X(x)$ of X and $f_Y(y)$ of Y.

(c) Find the marginal distribution functions, $F_X(x)$ of X and $F_Y(y)$ of Y.

(d) Find the joint distribution function:

$$F_{X,Y}(x,y) = \int_0^y \int_0^x f_{X,Y}(u,v)\, \mathrm{d}u\mathrm{d}v, \quad 0 < x < 1,\ 0 < y < 3.$$

(e) Show that X and Y are not independent.

(f) Determine $\mathsf{P}(X > 1/2 | Y > 1)$. What would this probability be if X and Y were independent with marginals determined in part (c)?

DISCRETE DISTRIBUTIONS

	RANGE $\mathbb{X}$	PARAMETERS	MASS FUNCTION f_X	CDF F_X	$\mathsf{E}_{f_X}[X]$	$\mathsf{Var}_{f_X}[X]$	MGF M_X
$Bernoulli(\theta)$	$\{0,1\}$	$\theta \in (0,1)$	$\theta^x(1-\theta)^{1-x}$		θ	$\theta(1-\theta)$	$1-\theta+\theta e^t$
$Binomial(n,\theta)$	$\{0,1,...,n\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{n}{x}\theta^x(1-\theta)^{n-x}$		$n\theta$	$n\theta(1-\theta)$	$(1-\theta+\theta e^t)^n$
$Poisson(\lambda)$	$\{0,1,2,...\}$	$\lambda \in \mathbb{R}^+$	$\dfrac{e^{-\lambda}\lambda^x}{x!}$		λ	λ	$\exp\left\{\lambda\left(e^t-1\right)\right\}$
$Geometric(\theta)$	$\{1,2,...\}$	$\theta \in (0,1)$	$(1-\theta)^{x-1}\theta$	$1-(1-\theta)^x$	$\dfrac{1}{\theta}$	$\dfrac{(1-\theta)}{\theta^2}$	$\dfrac{\theta e^t}{1-e^t(1-\theta)}$
$NegBinomial(n,\theta)$	$\{n,n+1,...\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{x-1}{n-1}\theta^n(1-\theta)^{x-n}$		$\dfrac{n}{\theta}$	$\dfrac{n(1-\theta)}{\theta^2}$	$\left(\dfrac{\theta e^t}{1-e^t(1-\theta)}\right)^n$
or	$\{0,1,2,...\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{n+x-1}{x}\theta^n(1-\theta)^x$		$\dfrac{n(1-\theta)}{\theta}$	$\dfrac{n(1-\theta)}{\theta^2}$	$\left(\dfrac{\theta}{1-e^t(1-\theta)}\right)^n$

For **CONTINUOUS** distributions (see over), define the **GAMMA FUNCTION**

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx$$

and the LOCATION/SCALE transformation $Y = \mu + \sigma X$ gives

$$f_Y(y) = f_X\left(\frac{y-\mu}{\sigma}\right)\frac{1}{\sigma} \qquad F_Y(y) = F_X\left(\frac{y-\mu}{\sigma}\right) \qquad M_Y(t) = e^{\mu t}M_X(\sigma t) \qquad \mathsf{E}_{f_Y}[Y] = \mu + \sigma\mathsf{E}_{f_X}[X] \qquad \mathsf{Var}_{f_Y}[Y] = \sigma^2\mathsf{Var}_{f_X}[X]$$

CONTINUOUS DISTRIBUTIONS

	$\mathbb{X}$	PARAMS.	PDF f_X	CDF F_X	$\mathsf{E}_{f_X}[X]$	$\mathsf{Var}_{f_X}[X]$	MGF M_X
$Uniform(\alpha,\beta)$ (stand. model $\alpha=0,\beta=1$)	(α,β)	$\alpha<\beta\in\mathbb{R}$	$\frac{1}{\beta-\alpha}$	$\frac{x-\alpha}{\beta-\alpha}$	$\frac{(\alpha+\beta)}{2}$	$\frac{(\beta-\alpha)^2}{12}$	$\frac{e^{\beta t}-e^{\alpha t}}{t(\beta-\alpha)}$
$Exponential(\lambda)$ (stand. model $\lambda=1$)	$\mathbb{R}^+$	$\lambda\in\mathbb{R}^+$	$\lambda e^{-\lambda x}$	$1-e^{-\lambda x}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\left(\frac{\lambda}{\lambda-t}\right)$
$Gamma(\alpha,\beta)$ (stand. model $\beta=1$)	$\mathbb{R}^+$	$\alpha,\beta\in\mathbb{R}^+$	$\frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}$		$\frac{\alpha}{\beta}$	$\frac{\alpha}{\beta^2}$	$\left(\frac{\beta}{\beta-t}\right)^\alpha$
$Weibull(\alpha,\beta)$ (stand. model $\beta=1$)	$\mathbb{R}^+$	$\alpha,\beta\in\mathbb{R}^+$	$\alpha\beta x^{\alpha-1}e^{-\beta x^\alpha}$	$1-e^{-\beta x^\alpha}$	$\frac{\Gamma(1+1/\alpha)}{\beta^{1/\alpha}}$	$\frac{\Gamma\left(1+\frac{2}{\alpha}\right)-\Gamma\left(1+\frac{1}{\alpha}\right)^2}{\beta^{2/\alpha}}$	
$Normal(\mu,\sigma^2)$ (stand. model $\mu=0,\sigma=1$)	$\mathbb{R}$	$\mu\in\mathbb{R},\sigma\in\mathbb{R}^+$	$\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$		μ	σ^2	$e^{\{\mu t+\sigma^2t^2/2\}}$
$Student(\nu)$	$\mathbb{R}$	$\nu\in\mathbb{R}^+$	$\frac{(\pi\nu)^{-\frac{1}{2}}\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\left\{1+\frac{x^2}{\nu}\right\}^{(\nu+1)/2}}$		0 (if $\nu>1$)	$\frac{\nu}{\nu-2}$ (if $\nu>2$)	
$Pareto(\theta,\alpha)$	$\mathbb{R}^+$	$\theta,\alpha\in\mathbb{R}^+$	$\frac{\alpha\theta^\alpha}{(\theta+x)^{\alpha+1}}$	$1-\left(\frac{\theta}{\theta+x}\right)^\alpha$	$\frac{\theta}{\alpha-1}$ (if $\alpha>1$)	$\frac{\alpha\theta^2}{(\alpha-1)(\alpha-2)}$ (if $\alpha>2$)	
$Beta(\alpha,\beta)$	$(0,1)$	$\alpha,\beta\in\mathbb{R}^+$	$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$		$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	

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M1S

Probability & Statistics I (Solutions)

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1. (a) **Axioms of Probability**

For events $E, F \subseteq \Omega$

(I) $0 \leq \mathsf{P}(E) \leq 1$

(II) $\mathsf{P}(\Omega) = 1$

(III) If $E \cap F = \phi$, then $\mathsf{P}(E \cup F) = \mathsf{P}(E) + \mathsf{P}(F)$ (Addition rule)

$\boxed{3}$

sim. seen $\Downarrow$

(b)

$$
\begin{aligned}
E &= (E \cap F) \cup (E \cap F') \\
\Rightarrow \mathsf{P}(E) &= \mathsf{P}(E \cap F) + \mathsf{P}(E \cap F') \text{ axiom } \mathbf{III} \text{ as } (E \cap f) \cap (E \cap F') = \phi \\
\Rightarrow \mathsf{P}(E) &\geq \mathsf{P}(E \cap F) \text{ axiom } \mathbf{I} \text{ (} \mathsf{P}(E \cap F') \geq 0)
\end{aligned}
$$

$\boxed{3}$

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(c)

$$
\begin{aligned}
\mathsf{P}(\text{only } E_2) &= \mathsf{P}(E_2 \cap E_1' \cap E_3') = \mathsf{P}(E_2)\mathsf{P}(E_1' \mid E_2)\mathsf{P}(E_3' \mid E_1' \cap E_2) \\
&= \mathsf{P}(E_2)\mathsf{P}(E_1')\mathsf{P}(E_3' \mid E_1') \quad \text{independence of } E_1, E_2 \text{ and } E_2, E_3 \\
&= \frac{1}{4} \times \frac{1}{2}\left(\frac{\mathsf{P}(E_3' \cap E_1')}{\mathsf{P}(E_1')}\right) = \frac{1}{8}\left(\frac{\mathsf{P}((E_3 \cup E_1)')}{1/2}\right) \\
&= \frac{1}{4}(1 - \mathsf{P}(E_3 \cup E_1)) = \frac{1}{4}(1 - \mathsf{P}(E_3) - \mathsf{P}(E_1)) \quad \text{as } E_3 \cap E_1 = \phi \\
&= \frac{1}{4}\left(1 - \frac{1}{8} - \frac{1}{2}\right) = \frac{3}{32}.
\end{aligned}
$$

Alternatively,

$$
\begin{aligned}
\mathsf{P}(\text{only } E_2) &= \mathsf{P}(E_2) - \mathsf{P}(E_2 \cap E_1) - (E_2 \cap E_3) \\
&= \mathsf{P}(E_2) - \mathsf{P}(E_2 \mid E_1)\mathsf{P}(E_1) + \mathsf{P}(E_2 \mid E_3)\mathsf{P}(E_3) \\
&= \mathsf{P}(E_2) - \mathsf{P}(E_2)\mathsf{P}(E_1) + \mathsf{P}(E_2)\mathsf{P}(E_3) \quad \text{independence} \\
&= \frac{1}{4} - \frac{1}{4} \times \frac{1}{2} + \frac{1}{4} \times \frac{1}{8} = \frac{3}{32}.
\end{aligned}
$$

$\boxed{4}$

meth seen $\Downarrow$

(d)

$$
\mathsf{E}_{f_X}(X) = \sum_{x=0}^{\infty} x f_X(x) = f_X(1) + 2f_X(2) + 3f_X(3) + \ldots
$$

$$
f_X(x) \geq 0 \,\forall x \Rightarrow \mathsf{E}_{f_X}(X) = 0 \Leftrightarrow f_X(x) = 0 \,\forall x \geq 1.
$$

$$
\sum_{i=0}^{\infty} f_X(x) = 1 \Rightarrow f_X(0) = \mathsf{P}(X = 0) = 1.
$$

$\boxed{2}$

$\boxed{\text{sim. seen} \Downarrow}$

(e) (i) $Y = X + 2$, so range of Y is $\{0, 1, 2, 3, 4\}$

$$f_Y(y) = \mathsf{P}(Y = y) = \mathsf{P}(X = y - 2) = f_X(y - 2), y \in \{0, 1, 2, 3, 4\}$$

Giving

$$f_Y(0) = f_Y(1) = f_Y(2) = \frac{1}{8}, f_Y(3) = \frac{1}{4}, f_Y(4) = \frac{3}{8}.$$

$\boxed{2}$

(ii) $Z = X^2$, so range of Z is $\{0, 1, 4\}$

$$f_Z(z) = \mathsf{P}(Z = x) = \mathsf{P}(X = \pm\sqrt{x}), \mathsf{P}((X = -\sqrt{x}) \cup (X = \sqrt{x})), x \in \{0, 1, 4\}.$$

Giving

$$f_Z(0) = \frac{1}{8}, f_Z(1) = \frac{3}{8}, f_Z(4) = \frac{1}{2}.$$

$\boxed{2}$

$\boxed{\text{unseen} \Downarrow}$

(f) Roots are real if $\alpha^2 - 4\beta \geq 0$.

$$\mathsf{P}(\alpha^2 - 4\beta \geq 0) = \mathsf{P}(\beta \leq \alpha^2/4)$$

α and β are both $Uniform[-1, 1]$ independently so

$$f_{\alpha,\beta}(x, y) = f_\alpha(x) f_\beta(y) = \frac{1}{4}, -1 \leq x, y \leq 1$$

$$\begin{aligned}
\mathsf{P}(\beta \leq \alpha^2/4) &= \int_{-1}^{1} \int_{-1}^{\alpha^2/4} \frac{1}{4} \, \mathsf{d}\beta \mathsf{d}\alpha \\
&= \frac{1}{4} \int_{-1}^{1} [\beta]_{-1}^{\alpha^2/4} \, \mathsf{d}\alpha = \frac{1}{4} \int_{-1}^{1} \left(\frac{\alpha^2}{4} + 1 \right) \mathsf{d}\alpha \\
&= \frac{1}{4} \left[\frac{\alpha^3}{12} + \alpha \right]_{-1}^{1} = \frac{13}{24}.
\end{aligned}$$

$\boxed{4}$

$\boxed{\text{meth seen} \Downarrow}$

2. (a)

$$\mu_Y = \mathsf{E}_{f_Y}(Y) = \mathsf{E}_{f_X}(\mu + \sigma X) = \mu + \sigma \mathsf{E}_{f_X}(X) = \mu + \sigma\mu_X,$$

$\boxed{2}$

$$\begin{aligned}
\sigma_Y^2 &= \mathsf{var}_{f_Y}(Y) = \mathsf{var}_{f_X}(\mu + \sigma X) \\
&= \mathsf{E}_{f_X}((\mu + \sigma X)^2) - \mathsf{E}_{f_X}^2(\mu + \sigma X) \\
&= \mu^2 + 2\mu\sigma \mathsf{E}_{f_X}(X) + \sigma^2 \mathsf{E}_{f_X}(X^2) - (\mu + \sigma \mathsf{E}_{f_X}(X))^2 \\
&= \mu^2 + 2\mu\sigma \mathsf{E}_{f_X}(X) + \sigma^2 \mathsf{E}_{f_X}(X^2) - (\mu^2 + 2\mu\sigma \mathsf{E}_{f_X}(X) + \sigma^2 \mathsf{E}_{f_X}^2(X)) \\
&= \sigma^2(\mathsf{E}_{f_X}(X^2) - E_{f_X}^2(X)) = \sigma^2\sigma_X^2.
\end{aligned}$$

Note: Can also assume that $\mathsf{var}(aX + b) = a^2\mathsf{var}(X)$ which gives much shorter solution. $\boxed{3}$

(b)

$$\begin{aligned}
M_Y(t) &= \mathsf{E}_{f_Y}(\mathsf{e}^{tY}) = \mathsf{E}_{f_X}(\mathsf{e}^{t(\mu+\sigma X)}) \\
&= \mathsf{e}^{\mu t} E_{f_X}(\mathsf{e}^{t\sigma X}) = \mathsf{e}^{\mu t} E_{f_X}(\mathsf{e}^{t\sigma X}) = \mathsf{e}^{\mu t} M_X(\sigma t)
\end{aligned}$$

$\boxed{3}$

(c) We have $\mu_X = M_X'(0)$ and $\sigma_X^2 = \mathsf{E}_{f_X}(X^2) - \mathsf{E}_{f_X}^2(X) = M_X''(0) - (M_X'(0))^2$, so, from (a)

$$\begin{aligned}
\mu_Y &= \mu + \sigma M_X'(0) \\
\sigma_Y^2 &= \sigma^2(M_X''(0) - (M_X'(0))^2).
\end{aligned}$$

Alternatively,

$$\begin{aligned}
&M_Y'(t) = \mu \mathsf{e}^{\mu t} M_X(\sigma t) + \sigma \mathsf{e}^{\mu t} M_X'(\sigma t) \\
&\Rightarrow \mu_Y = M_Y'(0) = \mu M_X(0) + \sigma M_X'(0) = \mu + \sigma M_X'(0),
\end{aligned}$$

and,

$$\begin{aligned}
M_Y''(t) &= \mu^2 \mathsf{e}^{\mu t} M_X(\sigma t) + \mu\sigma \mathsf{e}^{\mu t} M_X'(\sigma t) + \sigma\mu \mathsf{e}^{\mu t} M_X'(\sigma t) + \sigma^2 \mathsf{e}^{\mu t} M_X''(\sigma t) \\
\Rightarrow M_Y''(0) &= \mu^2 M_X(0) + 2\mu\sigma M_X'(0) + \sigma^2 M_X''(0) \\
&= \mu^2 + 2\mu\sigma M_X'(0) + \sigma^2 M_X''(0) \\
\Rightarrow \sigma_Y^2 &= \mathsf{E}_{f_Y}(Y^2) - \mathsf{E}_{f_Y}^2(Y) \\
&= M_Y''(0) - (M_Y'(0))^2 \\
&= \mu^2 + 2\mu\sigma M_X'(0) + \sigma^2 M_X''(0) - (\mu + \sigma M_X'(0))^2 \\
&= \sigma^2(M_X''(0) - [M_X'(0)]^2).
\end{aligned}$$

$\boxed{4}$

$\boxed{\text{unseen} \Downarrow}$

(d) $M_X(t) = \mathrm{e}^{t+6t^2}$, so $X \sim N(1, 12)$.

(i)

$$M'_X(t) = (1 + 12t)\mathrm{e}^{t+6t^2} \Rightarrow \mu_X = M'_X(0) = 1.$$ $\boxed{2}$

$$\begin{aligned} M''_X(t) &= (1 + 12t)^2\mathrm{e}^{t+6t^2} + 12\mathrm{e}^{t+6t^2} \\ \sigma_X^2 &= M''_X(0) - (M'_X(0))^2 \\ &= 1 + 12 - 1 = 12. \\ \Rightarrow \sigma_X &= \sqrt{12} = 2\sqrt{3}. \end{aligned}$$

$\boxed{3}$

as expected.

(ii) $Y = (X - 1)/3 \Rightarrow \mu = 1/3, \sigma = -1/3$,

$$\mu_Y = \mu + \sigma\mu_X = \frac{1}{3} - \frac{1}{3} = 0.$$

$$\sigma_Y^2 = \sigma^2\sigma_X^2 = \frac{1}{9} \times 12 = \frac{4}{3}.$$ $\boxed{3}$

$\boxed{\text{seen} \Downarrow}$

3. (a)

$$
\begin{aligned}
\mathsf{E}_{f_Y}(\mathsf{E}_{f_{N|Y}}(N \mid Y = y)) &= \sum_y \mathsf{E}_{f_{N|Y}}(N \mid Y = y) f_Y(y) \\
&= \sum_y \sum_n n f_{N|Y}(n \mid Y = y) f_Y(y) \\
&= \sum_n n \sum_y f_{N,Y}(n, y) = \sum_n n f_N(n) \\
&= \mathsf{E}_{f_N}(N).
\end{aligned}
$$

$\boxed{4}$

(b) (i)

$$
\begin{aligned}
\mathsf{E}_{f_{Y_1}}(Y_1) &= \sum_{y=0}^{1} y f_{Y_1}(y) = 0 \times f_{Y_1}(0) + 1 \times f_{Y_1}(1) \\
&= f_{Y_1}(1) = \theta.
\end{aligned}
$$

$\boxed{2}$

(ii)

$$
\begin{aligned}
f_N(n) = \mathsf{P}(N = n) &= \mathsf{P}(Y_1 = 0 \cap \ldots \cap Y_{n-1} = 0 \cap Y_n = 1) \\
&= (1-\theta)^{n-1}\theta \quad n = 1, 2, \ldots \quad \text{as } Y_i\text{s independent},
\end{aligned}
$$

$\boxed{3}$

i.e. $N \sim Geometric(\theta)$ $\boxed{1}$

$\boxed{\text{unseen} \Downarrow}$

(iii) from (a)

$$\mathsf{E}_{f_N}(N) = \mathsf{E}_{f_{Y_1}}(\mathsf{E}_{f_{N|Y_1}}(N \mid Y_1 = y))$$

$\boxed{2}$

$$\begin{aligned}\mathsf{E}_{f_N}(N) &= \sum_{y=0}^{1} \mathsf{E}_{f_{N|Y_1}}(N \mid Y_1 = y) f_{Y_1}(y) \\ &= \mathsf{E}_{f_{N|Y_1}}(N \mid Y_1 = 0)(1-\theta) + \mathsf{E}_{f_{N|Y_1}}(N \mid Y_1 = 1)(\theta).\end{aligned}$$

$$\begin{aligned}\mathsf{E}_{f_{N|Y_1}}(N \mid Y_1 = 0) &= \sum_{n=1}^{\infty} n f_{N|Y_1}(n \mid Y_1 = 0) \\ &= \sum_{n=2}^{\infty} n \frac{\mathsf{P}(N = n \cap Y_1 = 0)}{\mathsf{P}(Y_1 = 0)} = \sum_{n=2}^{\infty} n \frac{(1-\theta)^{n-1}\theta}{(1-\theta)} \\ &= \sum_{n=2}^{\infty} n(1-\theta)^{n-2}\theta.\end{aligned}$$

Let $M = N + 1$, then $\mathsf{E}_{f_M}(M) = \mathsf{E}_{f_N}(N) + 1$ and

$$f_M(m) = (1-\theta)^{m-2}\theta, \quad m = 2, 3, \ldots$$

So,

$$\mathsf{E}_{f_{N|Y_1}}(N \mid Y_1 = 0) = \mathsf{E}_{f_M}(M) = \mathsf{E}_{f_N}(N) + 1.$$

Also. $f_{N|Y_1}(n \mid Y_1 = 1) = 1, n = 1$, and $f_{N|Y_1}(n \mid Y_1 = 1) = 0, n > 1$, so $\mathsf{E}_{f_{N|Y_1}}(N \mid Y_1 = 1)) = 1$.

Giving,

$$\mathsf{E}_{f_N}(N) = (1 + E_{f_N}(N))(1-\theta) + \theta,$$

as required.

Could also use argument (if well explained) that if $Y_1 = 0$, the game starts over.

Hence, $\boxed{5}$

$$\begin{aligned}\mathsf{E}_{f_N}(N) &= (1 + E_{f_N}(N))(1-\theta) + \theta \\ \Rightarrow \mathsf{E}_{f_N}(N)\,(1 - (1-\theta)) &= (1-\theta) + \theta \\ \Rightarrow \mathsf{E}_{f_N}(N) &= \frac{1}{\theta},\end{aligned}$$

as expected given $N \sim Geometric(\theta)$. $\boxed{3}$

sim. seen ⇓

4. (a)

$$\int_0^1 \int_0^3 \left(\frac{x}{3} + cy\right) \, \mathrm{d}y \, \mathrm{d}x = \int_0^1 \left[\frac{xy}{3} + \frac{cy^2}{2}\right]_0^3 \, \mathrm{d}x$$
$$= \int_0^2 \left(x + \frac{9c}{2}\right) \, \mathrm{d}x = \left[\frac{x^2}{2} + \frac{9cx}{2}\right]_0^1$$
$$= \frac{1}{2} + \frac{9c}{2},$$
$$\Rightarrow \frac{1}{2} + \frac{9c}{2} = 1 \Rightarrow c = \frac{1}{9}.$$

$\boxed{2}$

(b)

$$f_X(x) = \frac{1}{9}\int_0^3 3x + y \, \mathrm{d}y = \frac{1}{9}\left[3xy + \frac{y^2}{2}\right]_0^3$$
$$= x + \frac{1}{2}, \quad 0 < x < 1.$$

$\boxed{2}$

$$f_Y(y) = \frac{1}{9}\int_0^1 3x + y \, \mathrm{d}x = \frac{1}{9}\left[\frac{3x^2}{2} + yx\right]_0^1$$
$$= \frac{1}{9}\left(y + \frac{3}{2}\right), \quad 0 < y < 3.$$

$\boxed{2}$

(c) $F_X(x) = 0, x \leq 0$; $F_X(x) = 1, x \geq 1$;

$$F_X(x) = \int_0^x f_X(v) \, \mathrm{d}v = \int_0^x v + \frac{1}{2} \, \mathrm{d}v$$
$$= \left[\frac{v^2}{2} + \frac{v}{2}\right]_0^x = \frac{x^2}{2} + \frac{x}{2} = \frac{x}{2}(x+1), \ 0 < x < 1.$$

$\boxed{2}$

$F_Y(y) = 0, y \leq 0$; $F_Y(y) = 1, y \geq 3$;

$$F_Y(y) = \int_0^y f_Y(v) \, \mathrm{d}v = \frac{1}{9}\int_0^y \left(v + \frac{3}{2}\right) \, \mathrm{d}v$$
$$= \frac{1}{9}\left[\frac{v^2}{2} + \frac{3v}{2}\right]_0^y = \frac{1}{9}\left(\frac{y^2}{2} + \frac{3y}{2}\right) = \frac{y}{18}(y+3), \ 0 < y < 3.$$

$\boxed{2}$

(d)

$$
\begin{aligned}
F_{X,Y}(x,y) &= \frac{1}{9}\int_0^x \int_0^y 3u+v \,\mathrm{d}v\mathrm{d}u \\
&= \frac{1}{9}\int_0^x \left[3uv+\frac{v^2}{2}\right]_0^y \mathrm{d}u \\
&= \frac{1}{9}\int_0^x 3uy+\frac{y^2}{2}\,\mathrm{d}u = \frac{1}{9}\left[\frac{3u^2y}{2}+\frac{y^2u}{2}\right]_0^x \\
&= \frac{1}{9}\left(\frac{3x^2y}{2}+\frac{xy^2}{2}\right) = \frac{xy}{18}(3x+y)\,,\ 0<x<1; 0<y<1.
\end{aligned}
$$

$\boxed{3}$

(e)

$$
f_X(x)f_Y(y) = \frac{1}{9}\left(x+\frac{1}{2}\right)\left(y+\frac{3}{2}\right) \neq f_{X,Y}(x,y).
$$

NOT independent.
Alternatively,

$$
F_X(x)F_Y(y) = \frac{x}{2}(x+1)\frac{y}{18}(y+3) \neq F_{X,Y}(x,y).
$$

$\boxed{1}$

$\boxed{\text{unseen} \Downarrow}$

(f)

$$
\begin{aligned}
\mathsf{P}(X>0.5 \mid Y>1) &= \frac{\mathsf{P}(X>0.5 \cap Y>1)}{\mathsf{P}(Y>1)} \\
\mathsf{P}(X>0.5 \cap Y>1) &= 1-(F_X(0.5)+F_Y(1)-F_{X,Y}(0.5,1)) \\
&= 1-\left(\frac{1}{4}\left(\frac{1}{2}+1\right)+\frac{1}{18}(1+3)-\frac{1}{36}\left(\frac{3}{2}+1\right)\right) \\
&= 1-\left(\frac{3}{8}+\frac{2}{9}-\frac{5}{72}\right) = 1-\left(\frac{27+16-5}{72}\right) \\
&= 1-\frac{19}{36} = \frac{17}{36}. \\
\Rightarrow \mathsf{P}(X>0.5 \mid Y>1) &= \frac{17/36}{1-2/9} = \frac{17/36}{7/9} = \frac{17}{28}.
\end{aligned}
$$

Alternatively,

$$
\begin{aligned}
\mathsf{P}(X>0.5 \cap Y>1) &= \frac{1}{9}\int_{0.5}^1 \int_1^3 3x+y \,\mathrm{d}y\,\mathrm{d}x \\
&= \frac{1}{9}\int_{0/5}^1 \left[3xy+\frac{y^2}{2}\right]_1^3 \mathrm{d}x = \frac{1}{9}\int_{0/5}^1 \left(9x+\frac{9}{2}\right)-\left(3x+\frac{1}{2}\right)\,\mathrm{d}x \\
&= \frac{1}{9}\int_{0/5}^1 6x+4\,\mathrm{d}x = \frac{1}{9}\left[3x^2+4x\right]_{0.5}^1 \\
&= \frac{1}{9}\left((3+4)-\left(\frac{3}{4}+2\right)\right) = \frac{1}{9}\left(7-\frac{11}{4}\right) = \frac{17}{36}.
\end{aligned}
$$

$\boxed{5}$

If independent, then

$$
\mathsf{P}(X>0.5 \mid Y>1) = \mathsf{P}(X>0.5) = 1-F_X(0.5) = 1-\frac{3}{8} = \frac{5}{8}.
$$

$\boxed{1}$