EE 2-04 Communication Systems

EXAM SOLUTIONS

1. a) i) [3]

Modulation is the modification of the source signals at the transmitter into a form that is suitable to be transmitted over the channel.

We have studied analog and digital modulation schemes. Analog modulation maps a continuous signal to a parameter of the carrier waveform continuously; whereas in digital communication, we first discretize the message signal, and map the corresponding values to discrete values of a paramater of the carrier waveform.

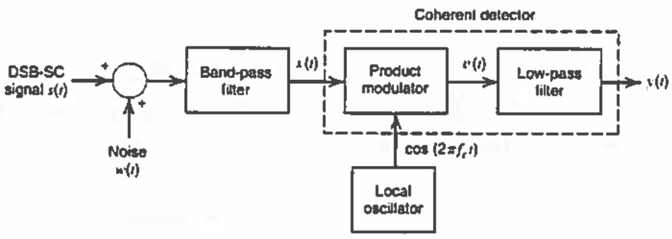
We use signal-to-noise ratio (SNR) to measure the performance of analog modulation, and error probability for digital communications.

ii) [3]

The complex envelope of $x(t)$ is given by $\tilde{x}(t) = x_I(t) + jx_Q(t)$, and we have

$$x(t) = \mathrm{Re}\{\tilde{x}(t)e^{j2\pi f_c t}\}.$$

iii) [4]



Bandpass filter at the front end removes the out-of-band noise. We then multiply the filtered received signal with $2\cos(2\pi f_c t)$ to obtain a baseband signal corresponding to the in-phase component. Low-pass filter removes the high frequency components that are located around $\pm 4f_c$.

iv) [2]

SSB uses half the bandwidth of DSB-SC while achieving the same SNR performance.

b) i) True. [1]

ii) True. [1]

iii) False. $S_X(f) = S_X(-f), \ \forall f$. [1]

iv) False. $R_X(0) = \int_{-\infty}^{+\infty} S_X(f)df$. [1]

v) True. $E[Y(t)] = H(0)E[X(t)] = 0$, where $H(\cdot)$ is the Fourier transform corresponding to Hilbert transform, which is given by $-j\mathrm{sgn}(f)$. [2]

c) i) [3]

We have $Y \sim \mathcal{N}(-4,4)$. We can write

$$\Pr(Y > 3) = \Pr((Y+4)/2 > 3.5) = \Pr(\bar{Y} > 3.5),$$

where $\bar{Y}$ is a standard normal distribution. Then we have $\Pr(Y > 3) = Q(3.5)$.

ii) [4]

We have $Y \sim \mathcal{N}(0,4)$. We can write

$$\Pr(3 \leq Y \leq 10) = \Pr(1.5 \leq Y/2 \leq 5) = \Pr(1.5 \leq \bar{Y} \leq 5),$$

where $\bar{Y}$ is a standard normal distribution. We have $\Pr(3 \leq Y \leq 10) = Q(1.5) - Q(5)$.

iii) [5]

We have $Y \sim \mathcal{N}(1,25)$. We can write

$$\Pr(Y \leq 4) = \Pr((Y-1)/5 \leq 4) = \Pr(\bar{Y} \leq 0.6),$$

where $\bar{Y}$ is a standard normal distribution. Then we have $\Pr(Y \leq 4) = \Pr(\bar{Y} \geq -0.6) = Q(-0.6)$.

d) i) [2]

$P_S = A^2 + \frac{B^2}{2}$.

ii) [2]

Quantization noise is assumed to have a uniform distribution with pdf $f_Q(q) = \frac{1}{\Delta}$ for $-\frac{\Delta}{2} < f \leq \frac{\Delta}{2}$, where Δ is the length of the quantization interval. It follows that the quantization noise variance is given by $P_N = \Delta^2/12$.

The signal range is $[A-B, A+B]$. Hence, we have

$$\Delta = \frac{2B}{2^n} = \frac{B}{2^{n-1}}.$$

iii) [3]

$$\begin{aligned} SNR &= \frac{P_S}{P_N} \\ &= \frac{A^2 + \frac{B^2}{2}}{\frac{B^2}{12 \cdot 2^{2n-2}}} \\ &= 3\left(\frac{A^2}{B^2} + \frac{1}{2}\right) 2^{2n}. \end{aligned}$$

Then

$$SNR_{dB} = 6.02n + 4.77 + 10 \cdot \log_{10}\left(\frac{A^2}{B^2} + \frac{1}{2}\right).$$

iv) [3]

We need $n \geq 15$ to have $SNR \geq 100\,dB$.

2. a) i) [4]

We obtain the following codewords for each symbol pair

Symbol	Probability	Codeword
cz	7/32	10
dz	6/32	11
dy	5/32	001
dx	4/32	010
bt	4/32	011
by	3/32	0000
cx	2/32	00010
ax	1/32	00011

The average codeword length is $92/32 = 2.875$.

ii) [5]

If we consider each source separately, we obtain the following codes:

	Source S_1	
Symbol	Probability	Codeword
d	15/32	1
c	9/32	00
b	7/32	010
a	1/32	011

The average codeword length per source symbol for source S_1 is $57/32 = 1.7813$ bits.

	Source S_2	
Symbol	Probability	Codeword
z	13/32	1
y	8/32	01
x	7/32	000
t	4/32	001

The average codeword length per source symbol for source S_2 is $62/32 = 1.9375$ bits.

We find the average code length for the proposed code by averaging the sum code length for each symbol pair. We have

$$\sum_{s_1 \in \mathscr{S}_1, s_2 \in \mathscr{S}_2} p(s_1, s_2)(\text{length}(s_1) + \text{length}(s_2)),$$

where length(s_i) is the length of the corresponding codeword for each symbol obtained from the two separate codes. We find the average length per symbol pair as $135/32 = 4.2188$.

iii) [4]

Joint encoding achieves a lower average code length. The answer would still be the same if the sources were independent; this is the idea behind Shannon's source coding theorem.

iv) [5]

$H(S_1, S_2) = 2.8273$. This is smaller than the average length of both codes. The average code length reduces as we code over more and more source symbol pairs.

No, entropy is a lower bound on the minimum average code length for any prefix code.

b) For a real WSS process, we have

$$E\left[\left(X(t+\tau)-X(t)\right)^2\right] = E[X^2(t+\tau) - 2X(t+\tau)X(t) + X^2(t)]$$
$$= 2R_X(0) - 2R_X(\tau)$$
$$= 2[R_X(0) - R_X(\tau)].$$

i) [4]

For $\tau = 3$, we have

$$2[R_X(0) - R_X(\tau)] = 2[A - Ae^{-3\alpha} + 1].$$

ii) [3]

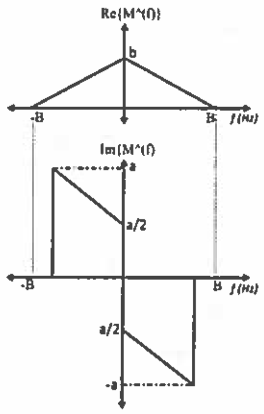
For $\tau = -3$, we get the same result as above since $R_X(\tau)$ is an even function.

iii) [5]

$$E\left[\left(X(t+\tau)+X(t)-X(t-\tau)\right)^2\right] = E[X^2(t+\tau) + X^2(t) + X^2(t-\tau) + 2X(t+\tau)X(t)$$
$$- 2X(t+\tau)X(t-\tau) - 2X(t)X(t-\tau)]$$
$$= 3R_X(0) - 2R_X(2\tau)$$
$$= 3A - 2(Ae^{-6\alpha} - 2)$$

3. a) i) [6]

We have $\hat{M}(f) = -j \cdot sgn(f) \cdot M(f)$



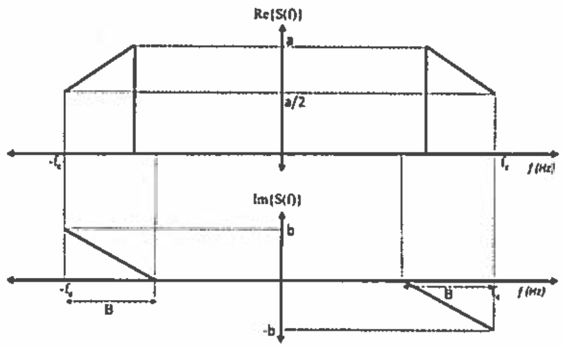
ii) [8]

The students may remember from class that the USB-LSB transmits only the lower halves of the spectrum of the signal, centered around carrier frequency f_c.

They should also be able to derive this directly from the basic principles as follows:

$$S(f) = \frac{1}{2}\left[M(f-f_c)+M(f+f_c)\right]+\frac{1}{2j}\left[\hat{M}(f-f_c)-\hat{M}(f+f_c)\right]$$

By simply shifting the spectrum of $M(f)$ and $\hat{M}(f)$ and applying the appropriate sign and conjugate adjustments they get the spectrum as below.



iii) [2]

We need a filter with bandwidth B, which passes only frequencies between $[-f_c, -f_c+B]$ and $[f_c-B, f_c]$.

iv) [4]

Multiplication with $\cos(2\pi f_c t + 3\pi/2) = \sin(2\pi f_c t)$ will decode the quadrature component of the modulated signal. Therefore, after low pass filtering, we will obtain $\hat{m}(t)$.

Since Hilbert transform is one to one, we can recover the message signal, by taking the Hilbert transform of $\hat{m}(t)$, we get $-m(t)$.

b) i) [2]

Let X denote the input bit, and Y denote the decoded bit.

$$P(X=0|Y=1) = \frac{P(X=0, Y=1)}{P(Y=1)} = \frac{0.5(1-p_0)}{0.5(1-p_0-p_1)} = \frac{1-p_0}{1-p_0-p_1}.$$

ii) [3]

Probability of error is minimized by mapping the more likely outcome to the more reliable bit. We transmit a 0 for tails, and a 1 for heads. The probability of error is then given by $0.7(1-p_0)+0.3(1-p_1) = 1-0.7p_0-0.3p_1$.

iii) [5]

We transmit 000 for tails, and 111 for heads. For an error, at least two bits should be flipped. Probability of error if 0 is transmitted is given by

$$Pe_0 = 3(1-p_0)^2 p_0 + (1-p_0)^3$$

Probability of error if 1 is transmitted is given by

$$Pe_1 = 3(1-p_1)^2 p_1 + (1-p_1)^3$$

Note that $Pe_1 > Pe_0$. Probability of error is then given by $0.7Pe_0 + 0.3Pe_1$.