

(a) Let $x(t) = \begin{pmatrix} \beta(t) \\ -u(t)\beta(t) \end{pmatrix} \Rightarrow y(t) = \alpha(t) \Rightarrow \dot{\alpha}(t) = \dot{y}(t)$ [1]
 $\dot{x}(t) = \begin{bmatrix} \dot{\beta}(t) \\ -u(t)\dot{\beta}(t) \end{bmatrix} = \begin{bmatrix} \dot{\alpha}(t) \\ \dot{\beta}(t) \end{bmatrix} \Rightarrow \dot{\alpha}(t) = \dot{y}(t) \Rightarrow \dot{\beta}(t) = \dot{y}(t)$ [1]
 Check: $\dot{\beta}(t) = \dot{y}(t) = -u(t)\dot{y}(t)$, same as ODE.

(b) $\exp(M) := I + M + \frac{M^2}{2!} + \frac{M^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{M^n}{n!}$ [2]

(c) $M := \begin{bmatrix} 0 & h \\ 0 & 0 \end{bmatrix} \Rightarrow M^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow M^n = 0 \forall n \geq 2$ [1]
 $\Rightarrow \exp \begin{bmatrix} 0 & h \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & h \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix}$ [1]

(d) Let $X := \begin{bmatrix} 0 & h \\ 0 & -\alpha h \end{bmatrix} \Rightarrow X^2 = \begin{bmatrix} 0 & -h^2\alpha \\ 0 & h^2\alpha^2 \end{bmatrix} \Rightarrow X^3 = \begin{bmatrix} 0 & h^3\alpha^2 \\ 0 & -h^3\alpha^3 \end{bmatrix} \dots \text{etc.}$
 $\Rightarrow X^n = \begin{bmatrix} 0 & (-1)^{n-1} h^n \alpha^{n-1} \\ 0 & (-1)^n h^n \alpha^n \end{bmatrix} \forall n \geq 1$ [2]

Now $e^{-\alpha h} = \sum_{n=0}^{\infty} \frac{1}{n!} (-\alpha h)^n = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} (-1)^n \alpha^n h^n$ [1]

$\Rightarrow \frac{1 - e^{-\alpha h}}{\alpha} = \sum_{n=1}^{\infty} \frac{1}{n!} (-1)^{n-1} \alpha^{n-1} h^n$ [1]

$\exp(X) = I + \sum_{n=1}^{\infty} \frac{1}{n!} \begin{bmatrix} 0 & (-1)^{n-1} \alpha^{n-1} h^n \\ 0 & (-1)^n \alpha^n h^n \end{bmatrix}$
 $\Rightarrow \exp(X) = \begin{bmatrix} 1 & \sum_{n=1}^{\infty} \frac{1}{n!} (-1)^{n-1} \alpha^{n-1} h^n \\ 0 & 1 + \sum_{n=1}^{\infty} \frac{1}{n!} (-1)^n \alpha^n h^n \end{bmatrix} = \begin{bmatrix} 1 & (1 - e^{-\alpha h})/\alpha \\ 0 & e^{-\alpha h} \end{bmatrix}$ [1]

(e) $\forall t \in [kh, (k+1)h)$ we have
 $\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -u_k \end{bmatrix} x(t) \Rightarrow x((k+1)h) = \exp \left(\begin{bmatrix} 0 & 1 \\ 0 & -u_k \end{bmatrix} h \right) x(k)$ [2]
 $\Rightarrow x_{k+1} = \exp \left(\begin{bmatrix} 0 & h \\ 0 & -u_k h \end{bmatrix} \right) x(k)$ [2]

(f) $y_k = [1 \ 0] x_k = \dots \Rightarrow y_0 = [1 \ 0] \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1$ [1]
 $x_1 = A(u_0) x_0 = A(1) x_0 = \begin{pmatrix} 2 - e^{-1} \\ e^{-1} \end{pmatrix} \Rightarrow y_1 = 2 - e^{-1} = 1.6321$ [2]
 $\Rightarrow x_2 = A(u_1) x_1 = A(0) x_1 = \begin{pmatrix} 2 \\ e^{-1} \end{pmatrix} \Rightarrow y_2 = 2$ [2]
 where $A(0) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$
 $\Rightarrow x_3 = A(u_2) x_2 = A(1) x_2 = \begin{pmatrix} 2 - e^{-2} + e^{-1} \\ e^{-2} \end{pmatrix} \Rightarrow y_3 = 2 - e^{-2} + e^{-1} = 2.325$ [1]
 In above, $A(1) = \begin{bmatrix} 1 & 1 - e^{-1} \\ 0 & e^{-1} \end{bmatrix}$

$$2(a) \Theta := (u_0' \ u_1' \dots u_{N-1}' \ s_0' \ s_1' \dots s_{N-1}') \quad [2]$$

$$\text{where each } u_k \in \mathbb{R}^m \text{ and slack variable } s_k \in \mathbb{R}^p \quad [1]$$

$$(b) \ H \text{ is square, so has as many rows as columns } \left. \begin{array}{l} mN + Np \\ H \text{ has as many columns as } \Theta \text{ and } \Theta \in \mathbb{R}^{(mN+Np) \times (mN+Np)} \end{array} \right\} [1]$$

$$(c) \Rightarrow H \text{ as } (p+m)N \text{ rows} \rightarrow \text{Cost } \sum_{k=0}^{N-1} \|d(Fx_{k+1} - g)\|_1 = \min_{\substack{s_k \\ k=0, \dots, N-1}} \sum_{k=0}^{N-1} s_k \text{ s.t. } \begin{cases} Fx_{k+1} - g \leq s_k \\ s_k \geq 0 \end{cases} \quad [1]$$

$$\Rightarrow \text{there are } 2pN \text{ constraints to do with } l_1\text{-norm stage cost} [1]$$

$$\text{and } 2mN \text{ constraints due to input constraints} [1]$$

$$\Rightarrow M \text{ has } 2(p+m)N \text{ rows} [1]$$

(d) Many ways to answer. The following is probably the most likely way that students will attempt.

$$\bar{x} := \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} = \underline{\Theta} \hat{x} + \Gamma \underbrace{\begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{pmatrix}}_{\bar{u}}, \text{ where } \underline{\Theta} := \begin{pmatrix} A \\ A^2 \\ \vdots \\ A^N \end{pmatrix}, \Gamma := \begin{pmatrix} B & 0 & \dots & 0 \\ AB & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ A^{N-1}B & A^{N-2}B & \dots & AB & B \end{pmatrix}$$

no need to derive \Rightarrow can just quote as recalled from lecture

Input constraints can be written as $\mathbb{1}_N \otimes u_L \leq \bar{u} \leq \mathbb{1}_N \otimes u_H$, where $\mathbb{1}_N$ is a column vector of ones and \otimes denotes Kronecker product

$$\Leftrightarrow \begin{bmatrix} \mathbb{I}_{N_m} \\ -\mathbb{I}_{N_m} \end{bmatrix} \bar{u} \leq \begin{bmatrix} \mathbb{1}_N \otimes u_H \\ -\mathbb{1}_N \otimes u_L \end{bmatrix}$$

From part (c), the constraints due to the slack variables can be written as: $(\mathbb{I}_N \otimes F) \bar{x} - \mathbb{I}_{Np} \bar{s} \leq \mathbb{1}_N \otimes g$ (*)

$$-\mathbb{I}_{Np} \bar{s} \leq 0$$

$$\text{where } \bar{s} := (s_0' \ s_1' \ s_2' \dots s_{N-1}')'$$

$$\Theta = \begin{pmatrix} \bar{u} \\ \bar{s} \end{pmatrix}$$

$$(*) \Leftrightarrow (\mathbb{I}_N \otimes F)(\underline{\Theta} \hat{x} + \Gamma \bar{u}) - \mathbb{I}_{Np} \bar{s} \leq \mathbb{1}_N \otimes g$$

$$\Rightarrow M = \begin{pmatrix} \mathbb{I}_{N_m} & 0_{N_m \times Np} \\ -\mathbb{I}_{N_m} & 0_{N_m \times Np} \\ (\mathbb{I}_N \otimes F)\Gamma & -\mathbb{I}_{Np} \\ 0_{Np \times N_m} & -\mathbb{I}_{Np} \end{pmatrix} \quad \begin{array}{l} [1] \\ [1] \\ [1] \\ [1] \end{array}$$

$$e) \quad f = \begin{pmatrix} \mathbb{I}_N \otimes u_h \\ -\mathbb{I}_N \otimes u_l \\ \mathbb{I}_N \otimes g - (\mathbb{I}_N \otimes F) \Phi \hat{x} \\ \mathbf{0}_{N_p \times 1} \end{pmatrix} \quad \left. \begin{matrix} \} [1] \\ \} [1] \end{matrix} \right\}$$

$$\begin{aligned} (f) \quad Cost &= \underbrace{\|(\mathbb{I}_N \otimes Q) \bar{x}\|_2^2}_{\bar{Q}} + \underbrace{\|(\mathbb{I}_N \otimes R) \bar{u}\|_2^2}_{\bar{R}} + \rho \sum_{k=0}^{N-1} s_k \\ &= \left\| \begin{pmatrix} \bar{Q} \Phi \hat{x} + \bar{Q} \Gamma \bar{u} \\ \bar{R} \bar{u} \end{pmatrix} \right\|_2^2 + \rho \mathbb{1}_{N_p}^T \bar{s} \\ &= \left\| \underbrace{\begin{pmatrix} \bar{Q} \Gamma \\ \bar{R} \end{pmatrix}}_{W} \bar{u} + \underbrace{\begin{pmatrix} \bar{Q} \Phi \hat{x} \\ \mathbf{0} \end{pmatrix}}_g \right\|_2^2 + \rho \mathbb{1}_{N_p}^T \bar{s} \\ &= \bar{u}^T W^T W \bar{u} + 2g^T W \bar{u} + g^T g + \rho \mathbb{1}_{N_p}^T \bar{s} \\ &= \underbrace{\begin{pmatrix} \bar{u} \\ \bar{s} \\ \mathbf{0} \end{pmatrix}^T}_{\mathbf{0}} \begin{pmatrix} W^T W & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \underbrace{\begin{pmatrix} \bar{u} \\ \bar{s} \\ \mathbf{0} \end{pmatrix}}_{\mathbf{0}} + \underbrace{\begin{pmatrix} 2g^T W & \rho \mathbb{1}_{N_p}^T \end{pmatrix}}_{\mathbf{0}} \underbrace{\begin{pmatrix} \bar{u} \\ \bar{s} \\ \mathbf{0} \end{pmatrix}}_{\mathbf{0}} + g^T g \end{aligned}$$

↑ not function of $\mathbf{0}$, so can ignore.

$$\Rightarrow H = \begin{bmatrix} 2W^T W & \mathbf{0}_{n \times pN} \\ \mathbf{0}_{pN \times nN} & \mathbf{0}_{pN \times pN} \end{bmatrix} \quad \begin{matrix} [1] \\ [1] \end{matrix}$$

(Check defn of W by student is correct)

$$(g) \Rightarrow C = \begin{pmatrix} 2W^T g \\ \rho \mathbb{1}_{N_p} \end{pmatrix} \quad \begin{matrix} [1] \\ [2] \end{matrix}$$

(Check defn of g by student is correct. Size of $\mathbb{1}$ also has to be correct)

3 (a) The system is open-loop asymptotically stable \Rightarrow all e/values of A inside unit disk $\Rightarrow I-A$ has no e/values at zero $\Rightarrow (I-A)^{-1}$ exists [1]

(b) At equilibrium $x_e = Ax_e + BF r$

$$\Leftrightarrow x_e = (I-A)^{-1}BF r$$

$$\text{Want } Hx_e = r \Leftrightarrow H(I-A)^{-1}BF r = r \quad \forall r$$

$$\Leftrightarrow H(I-A)^{-1}BF = I$$

$$\Leftrightarrow F = 1/(H(I-A)^{-1}B) \text{ and } H(I-A)^{-1}B \neq 0$$

(c) We can eliminate s_j, w_j from constraints, so only inequality constraints remain. There are 2 inequality constraints from $0 \leq p \leq 1 \Leftrightarrow \begin{cases} p \leq 1 \\ -p \leq 0 \end{cases}$.

Since $C \in \mathbb{R}^{2 \times n}$, it follows there are qN remaining inequality constraints from $C_s \leq d, j=1, \dots, N$

$$\Rightarrow M = qN + 2$$

[2]

(d) Define the augmented state $\tilde{x}_j := \begin{pmatrix} s_j \\ w_{j-1} \end{pmatrix} \Rightarrow \tilde{x}_0 = \begin{pmatrix} x_k \\ v_{k-1} \end{pmatrix}$
 $\Rightarrow \tilde{x}_{j+1} = \begin{pmatrix} s_{j+1} \\ w_j \end{pmatrix} = \underbrace{\begin{pmatrix} A & BF \\ 0 & I \end{pmatrix}}_{\tilde{A}} \begin{pmatrix} s_j \\ w_{j-1} \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ r - v_{k-1} \end{pmatrix}}_{\tilde{B}} p$

Note $s_j = [I_n \ 0] \tilde{x}_j \Rightarrow C s_j = C [I_n \ 0] \tilde{x}_j$

$$\tilde{x}_1 = \tilde{A} \tilde{x}_0 + \tilde{B} p$$

$$\tilde{x}_2 = \tilde{A} \tilde{x}_1 + \tilde{B} p = \tilde{A}^2 \tilde{x}_0 + \tilde{A} \tilde{B} p + \tilde{B} p$$

$$\Rightarrow \tilde{x}_j = \tilde{A}^j \tilde{x}_0 + \sum_{i=0}^{j-1} \tilde{A}^i \tilde{B} p$$

$$\begin{pmatrix} C s_1 \\ C s_2 \\ \vdots \\ C s_N \end{pmatrix} = \begin{pmatrix} C [I_n \ 0] \tilde{x}_1 \\ C [I_n \ 0] \tilde{x}_2 \\ \vdots \\ C [I_n \ 0] \tilde{x}_N \end{pmatrix} = I_N \otimes [C [I_n \ 0]] \begin{pmatrix} \tilde{A} \\ \tilde{A}^2 \\ \vdots \\ \tilde{A}^N \end{pmatrix} \tilde{x}_0 + \begin{pmatrix} \tilde{A} \tilde{B} \\ \tilde{A}^2 \tilde{B} \\ \vdots \\ \sum_{i=0}^{N-1} \tilde{A}^i \tilde{B} \end{pmatrix} p$$

3 d) continued.

therefore the inequality constraints $C_{ij} \leq d$ $G = 1, \dots, N$ becomes

$$\underbrace{I_N \otimes [C[I_N \ 0]]}_{\bar{C}} \left[\underbrace{\begin{pmatrix} \tilde{A}^0 \\ \tilde{A}^2 \\ \vdots \\ \tilde{A}^N \end{pmatrix}}_{\Phi} \underbrace{\tilde{x}_0}_{\Gamma} + \underbrace{\begin{pmatrix} \tilde{B} \\ \tilde{A}\tilde{B} + \tilde{B} \\ \vdots \\ \sum_{i=0}^{N-1} \tilde{A}^i \tilde{B} \end{pmatrix}}_{\Gamma} \right] \leq \mathbb{1}_N \otimes d$$

Recall also that we have the constraints $0 \leq \rho \leq 1$

$$\Rightarrow G = \left[\begin{array}{c} \underbrace{I_N \otimes [C[I_N \ 0]]}_{\bar{C}} \underbrace{\begin{pmatrix} \tilde{B} \\ \tilde{A}\tilde{B} + \tilde{B} \\ \vdots \\ \sum_{i=0}^{N-1} \tilde{A}^i \tilde{B} \end{pmatrix}}_{\Gamma} \\ +1 \\ -1 \end{array} \right] \left. \begin{array}{l} [2] \text{ for } \Gamma \\ [2] \text{ for } \bar{C} \\ [1] \end{array} \right\}$$

and

$$h = \left[\begin{array}{c} \mathbb{1}_N \otimes d - \bar{C} \Phi \left(\frac{\partial \ell}{\partial x_{k-1}} \right) \\ +1 \\ 0 \end{array} \right] \left. \begin{array}{l} [1] \\ [1] \\ \text{[scribble]} \end{array} \right\}$$

G is a function of r , since \tilde{B} is a function of r , [1]
but h is not, since it is not a function of \tilde{B} .

There could be many variations on above derivation.

Important point to note is that G is a column vector, not a matrix.

- (a) At each sample instant, a new measurement is taken to produce a new state estimate. This is used to set up a new optimization problem, and only the first part of the solution is implemented as a new control input. At the next sample, a new measurement is used to recompute the control. [3]
- (b) To compensate for uncertainty. [1]
- (c) Feedback can destabilize an already open-loop stable system. Feedback can inject measurement noise into a system where there previously was no noise. [2]
- (d) In receding horizon control, an optimal control problem with a fixed horizon is solved at each sample instant. In variable horizon control, the horizon is also a decision/optimization variable. [2]
- (e) In a satellite docking / aircraft landing example. The time taken may need to be optimized and there is a chance in a receding horizon formulation that the target may never be reached in finite time. (Any other sensible example & motivation acceptable) [2]
- (f) When minimizing fuel in a satellite control example. Fuel is used is a linear and not quadratic function of thrust. (Any other is OK) [2]
- (g) An airplane coming into land. If you exceed the constraint, that the aircraft stay above the ground, you have a crash! (Any other OK) [2]
- (h) Temperature in a room. There are comfort ranges, but exceeding these for a short while is often acceptable. (Any other is OK) [2]
- (i) The value function is used as a Lyapunov function. The terminal weight of the cost function is a Control Lyapunov Function and the terminal constraint has to define a control invariant set associated with the same control law as the terminal weight. Using a shifting argument and applying the terminal control law, one can prove that ~~the~~ the control sequence is feasible and that the value function decreases. (Has to have at least 4 different keywords / ideas as highlighted) [4]