SOLUTIONS: ESTIMATION AND FAULT DETECTION

1. Solution

a) Consider the mechanical system depicted in Fig. 1.1 in the text of the exam paper. From standard Newton rotational mechanics, for $t < t_0$ one immediately gets the following differential equation:

$$T_m - \beta \frac{\mathrm{d}}{\mathrm{d}t}\theta = J\frac{\mathrm{d}^2}{\mathrm{d}t^2}\theta$$

Using the notation $\dot{\theta} := \frac{\mathrm{d}}{\mathrm{d}t}\theta$, $\ddot{\theta} := \frac{\mathrm{d}^2}{\mathrm{d}t^2}\theta$, and setting $x_1 := \theta$ and $x_2 := \dot{\theta}$, the following state equations can be devised:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\dfrac{\beta}{J}x_2 + \dfrac{T_m}{J} \\ y = x_1 \end{cases}$$

and in matrix form:

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\beta/J \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/J \end{bmatrix} T_m \\ y = [1\ 0]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{cases}$$

[4 marks]

b) Letting

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -\beta/J \end{bmatrix}, \qquad C = [1\ 0]$$

after some easy algebra, the observability matrix is given by

$$Q = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Since

$$\det Q \neq 0$$

we conclude that the the pair (A, C) is fully observable.

[3 marks]

c) The action of the braking torque T_b for $t \geq t_0$ can be represented as an additional constant input $b(t)$ in the mechanical differential equation considered in the answer to Question 1-a):

$$T_m - \beta\dot{\theta} - b(t) = J\ddot{\theta}, \quad \forall t \geq t_0.$$

where $b(t) := T_b \cdot 1(t - t_0)$ and $b(t)$ can be generated as follows:

$$\begin{cases} \dot{z}(t) = 0 \\ b(t) = z(t) \end{cases}$$

with $z(t_0^-) = T_b$. Therefore, introducing the augmented state vector

$$x_a := \begin{bmatrix} x_1 \\ x_2 \\ z \end{bmatrix}$$

the following augmented state equations can be written:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\dfrac{\beta}{J}x_2 - \dfrac{1}{J}z + \dfrac{T_m}{J} \\ \dot{z} = 0 \\ y = x_1 \end{cases}$$

and in matrix form:

$$\begin{cases} \dot{x}_a = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{z} \end{bmatrix} = Ax_a + Bu = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\beta/J & -1/J \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ 1/J \\ 0 \end{bmatrix} T_m \\ \\ y = Cx_a = [1\ 0\ 0] \begin{bmatrix} x_1 \\ x_2 \\ z \end{bmatrix} \end{cases}$$

Therefore, a third-order Luenberger observer architecture can be devised that, in case of full observability of the pair (A,C), provides an asymptotic estimate of the augmented state x_a, hence also providing an estimate of the unknown constant input $b(t) = T_b \cdot 1(t - t_0)$ (which coincides with the third component of x_a).

After some easy algebra, the observability matrix is given by

$$Q = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\beta/J & -1/J \end{bmatrix}$$

Since

$$\det Q \neq 0$$

we conclude that the the pair (A,C) is fully observable.

[6 marks]

d) Setting $J = 10$, $\beta = 1$, matrices A and C defined in the answer to Question 1-c) are given by:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1/10 & -1/10 \\ 0 & 0 & 1 \end{bmatrix}; \quad C = [1\ 0\ 0]$$

In the answer to Question 1-c) it has been established that a third order asymptotic observer can be designed. This design consists in determining a matrix L such that the eigenvalues of $F = A - LC$ are:

$$\lambda_1 = -1, \lambda_2 = -1, \lambda_3 = -1 .$$

A direct design is possible. Letting

$$L = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix}$$

some easy algebra gives

$$A-LC=\begin{bmatrix} 0 & 1 & 0 \\ 0 & -1/10 & -1/10 \\ 0 & 0 & 1 \end{bmatrix}-\begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix}\cdot[1\,0\,0]=\begin{bmatrix} -l_1 & 1 & 0 \\ -l_2 & -1/10 & -1/10 \\ -l_3 & 0 & 1 \end{bmatrix}$$

and the characteristic polynomial is:

$$p_F(\lambda)=\det(\lambda I-A+LC)=\lambda^3+(l_1-\frac{9}{10})\lambda^2+(l_2-\frac{9}{10}l_1-\frac{1}{10})\lambda+(-\frac{1}{10}l_1-l_2-\frac{1}{10}l_3)$$

The desired eigenvalues $\lambda_i=-1, i=1,\ldots,3$ are the roots of the polynomial $(\lambda+1)^3$ and hence, after some algebra, one gets:

$$L=\begin{bmatrix} \frac{39}{10} \\ \frac{661}{100} \\ -80 \end{bmatrix}$$

[7 marks]

2. Solution

a) By analysing the block diagram in Question 2, a possible choice of the state variables is:

$$x_1(t) := w(t), \quad x_2(t) := \rho(t)$$

Accordingly, one gets immediately:

$$\begin{cases} x_1(t+1) = -\frac{1}{3}x_1(t) + u(t) \\ w(t) = x_1(t) \end{cases}$$

and

$$\begin{cases} x_2(t+1) = -\frac{1}{2}x_2(t) + v(t) \\ \rho(t) = x_2(t) \end{cases}$$

A further inspection of the block diagram in Question 2 and some algebra gives:

$$\begin{cases} \begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} = \begin{bmatrix} -4/3 & 0 \\ 0 & -1/2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} r(t) + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix} \\ \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \eta_1(t) \\ \eta_2(t) \end{bmatrix} \end{cases}$$

[4 marks]

b) According to the answer to Question 2-a), the conditions on observability and reachability of the theorem on convergence of the recursive Riccati equation hold. Hence a positive-definite solution of the Algebraic Riccati equation does exist.

Letting

$$\tilde{\xi}_2 := \xi_1 + \xi_2$$

and owing to the mutual independence of $\xi_1(\cdot)$ and $\xi_2(\cdot)$, it follows that

$$\mathrm{var}(\tilde{\xi}_2) = \mathrm{var}(\xi_1) + \mathrm{var}(\xi_2) = 2$$

Then, the state equations given in the answer to Question 2-a) can be equivalently written as follows:

$$\begin{cases} \begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} = \begin{bmatrix} -4/3 & 0 \\ 0 & -1/2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} r(t) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix} \\ \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \eta_1(t) \\ \eta_2(t) \end{bmatrix} \end{cases}$$

It is now easy to see that the two state variables x_1 and x_2 are decoupled. It is then possible to separately compute the optimal steady-state Kalman predictor for each state component. Let us first consider variable x_1:

$$\begin{cases} x_1(t+1) = -\frac{4}{3}x_1(t) + \xi_1(t) \\ y_1(t) = x_1(t) + \eta_1(t) \end{cases}$$

where $\xi_1(\cdot) \sim WGN(0,1)$ and $\eta_1(\cdot) \sim WGN(0,9)$. The general algebraic Riccati equation is

$$P = F\left[P - PH^\top\left(V_2 + HPH^\top\right)^{-1}HP\right]F^\top + V_1$$

Letting $F_1 = -4/3$, $H_1 = 1$, $V_{1,1} = 1$, $V_{1,2} = 9$, we have

$$P_1 = \frac{16}{9}\left(P_1 - \frac{P_1^2}{9+P_1}\right) + 1 \Longrightarrow P_1^2 - 8P_1 - 9 = 0$$

thus obtaining the two solutions

$$\bar{P}_{1,1} = -1 \quad \text{and} \quad \bar{P}_{1,2} = 9$$

Clearly, the only admissible solution is the positive one. Thus $\bar{P}_1 = \bar{P}_{1,2} = 9$. Accordingly:

$$\bar{K}_1 = F_1\bar{P}_1H_1^\top\left(V_{1,2} + H_1\bar{P}_1H_1^\top\right)^{-1} = -\frac{2}{3} \simeq -0.67$$

Repeating the same procedure for the second state variable one gets:

$$\begin{cases} x_2(t+1) = -\frac{1}{2}x_2(t) + \tilde{\xi}_2(t) \\ y_2(t) = x_2(t) + \eta_2(t) \end{cases}$$

where $\tilde{\xi}_2(\cdot) \sim WGN(0,2)$ and $\eta_2(\cdot) \sim WGN(0,4)$. Letting $F_2 = -1/2$, $H_2 = 1$, $V_{2,1} = 2$, $V_{2,2} = 4$, the algebraic Riccati equation is

$$P_2 = \frac{1}{4}\left(P_2 - \frac{P_2^2}{4+P_2}\right) + 2 \Longrightarrow P_2^2 + P_2 - 8 = 0$$

thus obtaining the two solutions

$$\bar{P}_{2,1} = -\frac{1}{2}(1+\sqrt{33}) \simeq -3.37 \quad \text{and} \quad \bar{P}_{2,2} = -\frac{1}{2}(1-\sqrt{33}) \simeq 2.37$$

Again, the only admissible solution is the positive one. Thus $\bar{P}_2 = \bar{P}_{2,2} \simeq 2.37$. Accordingly:

$$\bar{K}_2 = F_2\bar{P}_2H_2^\top\left(V_{2,2} + H_2\bar{P}_2H_2^\top\right)^{-1} \simeq -0.19$$

The steady state Kalman predictor obeys to the following equations:

$$\begin{cases} \hat{x}(t+1|t) = \begin{bmatrix} -\frac{4}{3} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}\hat{x}(t|t-1) + \bar{K}e(t) \\ \hat{y}(t+1|t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\hat{x}(t+1|t) \\ e(t) = y(t) - \hat{y}(t|t-1) \end{cases}$$

where $\bar{K} \simeq \begin{bmatrix} -0.67 & 0 \\ 0 & -0.19 \end{bmatrix}$.

[5 marks]

c) We have

$$\mathrm{Cov}[x(t) - \hat{x}(t|t-1)] = \bar{P} = \begin{bmatrix} \bar{P}_1 & 0 \\ 0 & \bar{P}_2 \end{bmatrix} \simeq \begin{bmatrix} 9 & 0 \\ 0 & 2.37 \end{bmatrix}$$

Let us now compute $\mathrm{Cov}[x(t)]$. The stochastic process $x(\cdot)$ is generated by the system

$$\begin{cases} x_1(t+1) = -\frac{1}{3}x_1(t) + \xi_1(t) \\ \\ x_2(t+1) = -\frac{1}{2}x_2(t) + \tilde{\xi}_2(t) \end{cases}$$

and clearly it is not stationary because the first state equation is unstable. Therefore, $\mathrm{Cov}[x(t)]$ turns out to be time-dependent and should not be computed. However, it is worth noting that the second state variable $x_2(t)$ is a scalar stationary process and hence $\mathrm{var}[x_2(t)] = \mathrm{var}[x_2(t-1)]$. Since $\mathbb{E}[x_2(t)] = 0$, one gets

$$\mathrm{var}[x_2(t)] = \frac{1}{4}\mathrm{var}[x_2(t)] + 2 \Longrightarrow \mathrm{var}[x_2(t)] = \frac{8}{3}$$

and

$$\mathrm{var}[x_2(t) - \hat{x}_2(t|t-1)] \simeq 2.37 < \frac{8}{3} = \mathrm{var}[x_2(t)]$$

As expected, the Kalman estimator allows to predict the second state variable with a smaller variance with respect to the a-priori one thanks to the use of the measurements $y_2(t)$.

[5 marks]

d) The steady-state Kalman filter obeys to the following equation:

$$\hat{x}(t|t) = \hat{x}(t|t-1) + K_0 e(t)$$

The following relationship exists between the Kalman predictor gain $K(t)$ and the Kalman filter gain $K_0(t)$:

$$K(t) = FK_0(t).$$

Therefore, we compute the constant gain vector $\bar{K}_0$ of the optimal steady-state Kalman filter as

$$\bar{K}_0 = F^{-1}\bar{K} \simeq \begin{bmatrix} 0.5 & 0 \\ 0 & 0.38 \end{bmatrix}$$

Since $\hat{x}(t+1|t) = F\hat{x}(t|t)$, then we can write

$$\hat{x}(t|t) = F^{-1}\hat{x}(t+1|t).$$

We can then compute the covariance matrix of the filtering error $\mathrm{Cov}[x(t) - \hat{x}(t|t)]$. We can write the following

$$\nu(t) = x(t) - \hat{x}(t|t) = x(t) - \hat{x}(t|t-1) - K_0(t)e(t) =$$

$$= (I - K_0 H)(x(t) - \hat{x}(t|t-1)) - K_0(t)\eta(t)$$

Since both $x(t)$ and $\hat{x}(t|t-1)$ are not correlated with the measurement noise $\eta(t)$, then we compute

$$\mathrm{Cov}[x(t) - \hat{x}(t|t)] = (I - K_0 H)\mathrm{Cov}[x(t) - \hat{x}(t|t-1)](I - K_0 H)^\top + K_0 V_2 K_0^\top$$

$$\simeq \begin{bmatrix} 4.5 & 0 \\ 0 & 1.49 \end{bmatrix}$$

Now:

$$\mathrm{var}[x_1(t) - \hat{x}_1(t|t)] = 4.5 < \mathrm{var}[x_1(t) - \hat{x}_1(t|t-1)] = 9$$

$$\mathrm{var}[x_2(t)-\hat{x}_2(t|t)] \simeq 1.49 < \mathrm{var}[x_2(t)-\hat{x}_2(t|t-1)] \simeq 2.37 < \frac{8}{3} = \mathrm{var}[x_2(t)]$$

As can be seen, the Kalman filter allows a further reduction of the variance of the estimation error with respect to the variance obtained by the Kalman one-step-ahead predictor in the answer to Question 2-c), thanks to the use of an additional output measurement.

[6 marks]

3. Solution

a) The transfer functions $G_1(s)$, $G_2(s)$, and $G_3(s)$ can be obtained by selecting the outputs $y_1(t)$, $y_2(t)$, and $y_3(t)$, respectively, one at a time neglegting the other ones.

By inspection of the block-diagram shown in Fig. 3.1 of the text of the exam paper, we immediately obtain

$$Y_1(s) = \frac{1}{s+10}U(s)$$

$$Y_2(s) = \frac{1}{(s+3)(s+10)}U(s)$$

and

$$Y_3(s) = 2\left(\frac{1}{s+3} - \frac{1}{s+4+K}\right)\frac{1}{s+10}U(s)$$

Moreover, after a little algebra, one gets:

$$U(s) = \frac{(s+3)(s+10)(s+4+K)}{(s+3)(s+10)(s+4+K)+2(K+1)}R(s)$$

Then:

$$G_1(s) = \frac{(s+3)(s+4+K)}{(s+3)(s+10)(s+4+K)+2(K+1)}$$

$$G_2(s) = \frac{(s+4+K)}{(s+3)(s+10)(s+4+K)+2(K+1)}$$

$$G_3(s) = \frac{2(K+1)}{(s+3)(s+10)(s+4+K)+2(K+1)}$$

[5 marks]

b) A possible choice of the state variables is the following: x_1 set to the output of the block with transfer function $\frac{1}{s+10}$, x_2 set to the output of the integrator, and x_3 associated with the output of the block with transfer function $\frac{1}{s+4}$.

Then, one gets:

$$\begin{cases} \dot{x}_1 = -10x_1 + u \\ y_1 = x_1 \end{cases}, \quad \begin{cases} \dot{x}_2 = -3x_2 + x_1 \\ y_2 = x_2 \end{cases}, \quad \begin{cases} \dot{x}_3 = x_1 - (4+K)x_3 \\ y_3 = 2x_2 - 2x_3 \end{cases}$$

Using $u = r - 2(x_2 - x_3)$, one finally obtains

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = A \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + B \cdot r = \begin{bmatrix} -10 & -2 & 2 \\ 1 & -3 & 0 \\ 1 & 0 & -(4+K) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} r \\ \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = C \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \end{cases}$$

[3 marks]

c) With reference to the whole system depicted in Fig.3.1 of the text of the exam paper, in order to analyse its observability from each output $y_i(t)$, $i = 1,2,3$ taken separately, one has to select each row $C_i, i = 1,2,3$ of the output matrix C determined in the answer to Question 3-b) and analyse the observability of the pair (A, C_i) for $i = 1,2,3$.

One has:

$$Q_1 := \begin{bmatrix} C_1 \\ C_1A \\ C_1A^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -10 & -2 & 2 \\ 118 & 26 & -20-2(4+K) \end{bmatrix}$$

Since

$$\det Q_1 = 0 \quad \text{if} \quad K = -1$$

one concludes that for $K = -1$ the system is not observable from output y_1 without access to the other outputs.

Then

$$Q_2 := \begin{bmatrix} C_2 \\ C_2A \\ C_2A^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -3 & 0 \\ -13 & 7 & 2 \end{bmatrix}$$

In this case, Q_2 does not depend on the parameter K. Since

$$\det Q_2 = -2$$

one concludes the system is always observable from output y_2 without access to the other outputs.

Finally:

$$Q_3 := \begin{bmatrix} C_3 \\ C_3A \\ C_3A^2 \end{bmatrix} = \begin{bmatrix} 0 & 2 & -2 \\ 0 & -6 & 2(4+K) \\ -6+2(4+K) & 18 & -2(4+K)^2 \end{bmatrix}$$

Since

$$\det Q_3 = 0 \quad \text{if} \quad K = -1$$

one concludes that for $K = -1$ the system is not observable from output y_3 without access to the other outputs.

[5 marks]

d) Setting $K = -1$ and selecting the output y_3 (see the answer to Question 3-c)), the state equations become:

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = A \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + B \cdot r = \begin{bmatrix} -10 & -2 & 2 \\ 1 & -3 & 0 \\ 1 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} r \\ y_3 = C_3 \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \end{cases}$$

Now, to determine the observability canonical form a basis for $\ker(Q_3)$ has to be constructed. One has:

$$Q_3 = \begin{bmatrix} 0 & 2 & -2 \\ 0 & -6 & 6 \\ 0 & 18 & -18 \end{bmatrix} \Longrightarrow \text{rank}\,[\ker(Q_3)] = 3 - \text{rank}(Q_3) = 2$$

Hence

$$Q_3 v = 0 \Longrightarrow \begin{bmatrix} 0 & 2 & -2 \\ 0 & -6 & 6 \\ 0 & 18 & -18 \end{bmatrix} v = 0 \Longrightarrow v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

The orthogonal complement to $\ker(Q_3)$ is one-dimensional and a possible choice for the basis is the vector

$$v_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Selecting the matrix

$$T = [v_3 \,|\, v_1 \,|\, v_2] = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

and computing the inverse

$$T^{-1} = \begin{bmatrix} 0 & 1/2 & -1/2 \\ 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

By setting $x = Tz$, one gets the following equivalent observability canonical form:

$$\begin{cases} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = T^{-1}ATz + T^{-1}Br = \begin{bmatrix} -3 & 0 & 0 \\ -4 & 10 & 0 \\ 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} r \\ y_3 = CTz = [4\,0\,0] \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \end{cases}$$

Hence, the non-observable sub-system is the 2-dimensional system described via the transformed state variables z_2 and z_3.

[7 marks]

4. Solution

a) Setting $d(t)=0, \forall t$ and $f(t)=0, \forall t$, the state equations (4.1) in the text of the exam paper take on the simplified form

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = A \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + B \cdot u = \begin{bmatrix} -3 & 1 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = C \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{cases}$$

The pair (A,C) is clearly observable and hence a full-order observer can be designed:

$$\begin{cases} \dot{\widehat{x}} = A\widehat{x} + Bu + L(y - C\widehat{x}) \\ \widehat{y} = C\widehat{x} \end{cases}$$

where, in general, L has the structure

$$L = \begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{bmatrix}$$

Since there is not a unique choice of the four elements of L in order to assign the two eigenvalues of $F = A - LC$ to arbitrarily chosen values, a possible solution is to restrict the number of parameters of L such as

$$L = \begin{bmatrix} l_1 & 0 \\ l_2 & 0 \end{bmatrix}$$

Then, one gets

$$F = A - LC = \begin{bmatrix} -3 - l_1 & 1 - l_1 \\ 1 - l_2 & -4 - l_2 \end{bmatrix}$$

After some algebra, by selecting

$$L = \begin{bmatrix} 7 & 0 \\ -11 & 0 \end{bmatrix}$$

one obtains $\det(\lambda I - F) = \lambda^2 + 3\lambda + 2$ and hence $\lambda_1 = -1, \lambda_2 = -2$.

[5 marks]

b) The state equations of the dynamic system take now the form:

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = A \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + B \cdot u = \begin{bmatrix} -3 & 1 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 1 \\ 0 \end{bmatrix} d \\ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = C \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{cases}$$

where $d(t) = 0.1\sin(t) \cdot 1(t)$. Moreover, replacing into the matrix $F = A - LC$ the observer gain L obtained in the answer to Question 4-a), one gets

$$F = \begin{bmatrix} -10 & -6 \\ 12 & 7 \end{bmatrix}$$

Hence, the state error dynamics is described in terms of

$$\dot{e}(t) = \begin{bmatrix} -10 & -6 \\ 12 & 7 \end{bmatrix} e(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} d(t)$$

Applying the Laplace transform operator, one gets

$$\mathscr{L}[e(t)] = (sI-F)^{-1}\bar{e}_0 + (sI-F)^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\mathscr{L}[d(t)]$$

and thus

$$\mathscr{L}[\varepsilon(t)] = C(sI-F)^{-1}\bar{e}_0 + C(sI-F)^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\mathscr{L}[d(t)]$$

Some easy algebra gives

$$(sI-F)^{-1} = \begin{bmatrix} s+10 & 6 \\ -12 & s-7 \end{bmatrix}^{-1} = \frac{1}{s^2+3s+2}\begin{bmatrix} s-7 & -6 \\ 12 & s+10 \end{bmatrix}$$

and thus

$$\mathscr{L}[\varepsilon(t)] = \frac{1}{s^2+3s+2}\begin{bmatrix} s+5 & s+4 \\ 2s-2 & s-2 \end{bmatrix}\tilde{e} + \frac{0.1}{s^2+3s+2}\begin{bmatrix} s+5 \\ 2s-2 \end{bmatrix}\frac{1}{s^2+1}$$

[5 marks]

c) Under the action of both the disturbance d and the fault f, the state equations of the dynamic system take on the form:

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = A\cdot\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + B\cdot u + N\cdot d + M\cdot f \\ \qquad = \begin{bmatrix} -3 & 1 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}u + \begin{bmatrix} 1 \\ 0 \end{bmatrix}d + \begin{bmatrix} 0 \\ 1 \end{bmatrix}f \\ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = C\cdot\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{cases}$$

The UIO has the general form

$$\begin{cases} \dot{z} = \widetilde{F}z + TBu + Ky \\ \widehat{x} = z + Hy \end{cases}$$

where H has to be designed such that $(I-HC)N = 0$. Since N is full column-rank, such a matrix H does exist if and only if

$$\operatorname{rank}(CN) = \operatorname{rank}(N)$$

and this rank condition is clearly satisfied in the present case. A solution (not unique) for H is given using the left-inverse, that is, using the above state equations, one obtains

$$H^* = N\left[(CN)^{\top}(CN)\right]^{-1}(CN)^{\top} = \begin{bmatrix} 1/5 & 2/5 \\ 0 & 0 \end{bmatrix}$$

Accordingly, some algebra gives

$$A - H^*CA = \begin{bmatrix} -3/5 & 12/5 \\ 1 & -4 \end{bmatrix}$$

The pair $(A-H^*CA, C)$ is fully observable since

$$\operatorname{rank}\begin{bmatrix} C \\ C(A-H^*CA) \end{bmatrix} = \operatorname{rank}(C) = 2$$

Therefore, a matrix

$$\widetilde{K} = \begin{bmatrix} \widetilde{k}_{11} & \widetilde{k}_{12} \\ \widetilde{k}_{21} & \widetilde{k}_{22} \end{bmatrix}$$

exists such that the eigenvalues of $\widehat{F} - \widehat{K}C$ can be arbitrarily assigned where $\widehat{F} := A - H^*CA$. Analogously to the answer to Question 4-a), there is not a unique choice of the four elements of $\widehat{K}$ in order to assign the two eigenvalues of $\widehat{F} - \widehat{K}C$ to arbitrarily chosen values, a possible solution is to restrict the number of parameters of $\widehat{K}$ such as

$$\widehat{K} = \begin{bmatrix} \widehat{k}_{11} & 0 \\ \widehat{k}_{21} & 0 \end{bmatrix}$$

Then, one gets

$$\widehat{F} - \widehat{K}C = \begin{bmatrix} -3/5 - \widehat{k}_{11} & 12/5 - \widehat{k}_{11} \\ 1 - \widehat{k}_{21} & -4 - \widehat{k}_{21} \end{bmatrix}$$

After some algebra, it turns out that by selecting

$$\widehat{K} = \begin{bmatrix} 17/5 & 0 \\ -5 & 0 \end{bmatrix}$$

one obtains $\det(\lambda I - \widehat{F} + \widehat{K}C) = \lambda^2 + 3\lambda + 2$ and hence $\lambda_1 = -1, \lambda_2 = -2$.

Hence, matrices $\widetilde{F}$, K and T to be used in the UIO are given by

$$\widetilde{F} = \widehat{F} - \widehat{K}C = \begin{bmatrix} -4 & -1 \\ 6 & 1 \end{bmatrix}, \quad K = \widetilde{K} + \widetilde{F}H^* = \begin{bmatrix} 13/5 & -8/5 \\ -19/5 & 12/5 \end{bmatrix},$$

$$T = I - H^*C = \begin{bmatrix} 0 & -3/5 \\ 0 & 1 \end{bmatrix}$$

[6 marks]

d) It is easy to see that, before the occurrence of the fault, that is for $t < 20$, the state error dynamics $\widetilde{e}(t) := x(t) - \widetilde{x}(t)$ is described by

$$\dot{\widetilde{e}}(t) = \widetilde{F}\widetilde{e}(t) = \begin{bmatrix} -4 & -1 \\ 6 & 1 \end{bmatrix} e(t)$$

Applying the Laplace transform operator, one gets

$$\mathcal{L}[\widetilde{e}(t)] = (sI - \widetilde{F})^{-1}\widetilde{e}_0$$

and thus

$$\mathcal{L}[\widetilde{\varepsilon}(t)] = C(sI - \widetilde{F})^{-1}\widetilde{e}_0$$

Some easy algebra gives

$$(sI - \widetilde{F})^{-1} = \begin{bmatrix} s+4 & 1 \\ -6 & s+1 \end{bmatrix}^{-1} = \frac{1}{s^2+3s+2}\begin{bmatrix} s+1 & -1 \\ 6 & s+4 \end{bmatrix}$$

and thus

$$\mathcal{L}[\varepsilon(t)] = \frac{1}{s^2+3s+2}\begin{bmatrix} s+7 & s+3 \\ 2s+8 & s+2 \end{bmatrix}\widetilde{e}_0$$

which, as said above, is valid for $t < 20$. After the occurrence of the fault, that is for $t \geq 20$, the above-defined state error dynamics $\tilde{e}(t)$ is described by

$$\dot{\tilde{e}}(t) = \tilde{F}\tilde{e}(t) + (I - H^{*}C)\cdot M \cdot f(t) = \begin{bmatrix} -4 & -1 \\ 6 & 1 \end{bmatrix} e(t) + \begin{bmatrix} -3/5 \\ 1 \end{bmatrix} \cdot 2$$

Applying the Laplace transform operator and using the results in the answer to Question 4-c), one gets

$$\mathcal{L}[\tilde{e}(t)] = \frac{1}{s^2+3s+2}\begin{bmatrix} s+1 & -1 \\ 6 & s+4 \end{bmatrix}\tilde{e}_0 + \begin{bmatrix} -6/5 \\ 2 \end{bmatrix}\cdot\frac{1}{s}$$

and thus

$$\mathcal{L}[\tilde{\varepsilon}(t)] = \frac{1}{s^2+3s+2}\begin{bmatrix} s+7 & s+3 \\ 2s+8 & s+2 \end{bmatrix}\tilde{e}_0 + \begin{bmatrix} 4/5 \\ -2/5 \end{bmatrix}\cdot\frac{1}{s}$$

which, as said above, is valid for $t \geq 20$.

From the comparison between $\mathcal{L}[\varepsilon(t)]$ determined in the answer to Question 4-b) and $\mathcal{L}[\tilde{\varepsilon}(t)]$ computed above, it is immediate to see that before the occurrence of the fault, the residual $\varepsilon(t)$ is influenced by the action of the sinusoidal disturbance $d(t)$ whereas the residual $\tilde{\varepsilon}(t)$ is not influenced at all by the disturbance. Therefore, the design of a fault detection scheme using the UIO designed in the answer to Question 4-c) should be more effective in detecting faults when disturbances act on the first component of the state and fault possibly affect the second component of the state.

[4 marks]