M3S1/M4S1
Setter:
Checker:
Editor:
External:
March 27, 2014

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2014

M3S1/M4S1 Statistical Theory I

1. (a) (i) What is the *sufficiency principle*?

(ii) What is its importance?

(iii) Explain what is meant by a *minimal sufficient statistic* for a family of distributions parameterised by an unknown parameter θ.

(b) For cases (i) and (ii) below find, giving your reasoning, a minimal sufficient statistic for positive θ.

(i) From a random sample $\boldsymbol{x} = \{x_1, x_2, \ldots, x_n\}$ having the probability density function

$$f(x|\theta) = \begin{cases} \theta e^{\theta(\theta - x)} & (x > \theta), \\ 0 & (x \leq \theta). \end{cases}$$

(ii) In bio-assay, $P(\text{positive response at dosage } z) = P(X = 1 \,|\, z, \theta) = \dfrac{e^{\theta z}}{1 + e^{\theta z}}$

Here $X_1, X_2, \ldots, X_n$ are independent *Bernoulli* random variables with

$$P(X_k = x_k \,|\, z_k, \theta) = \frac{e^{\theta z_k x_k}}{1 + e^{\theta z_k x_k}} \quad (x_k \in \{0, 1\})$$

where $\boldsymbol{z} = \{z_1, z_2, \ldots, z_n\}$ are known constants.

2. (a) What is the *monotone likelihood ratio criterion*?
What is its importance?

(b) (i) From the combined independent random samples $\boldsymbol{x} = \{x_1, x_2, \ldots, x_n\}$ from *Poisson*(θ) and $\boldsymbol{y} = \{y_1, y_2, \ldots, y_n\}$ from *Poisson*($c\theta$), where $c > 0$ is a known constant, show that $t(\boldsymbol{x}, \boldsymbol{y}) = \sum_1^n (x_i + y_i)$ is sufficient for θ.

(ii) Find the most powerful size α test of $H_0 : \theta \leq 1$ against $H_1 : \theta > 1$, where α is small. *Give your reasoning.*

(iii) Consider the size α test of $H_0^* : \theta = 1$ against $H_1 : \theta > 1$.
Obtain a normal approximation for the distribution of $T = t(\boldsymbol{X}, \boldsymbol{Y})$ for given θ and large n.
If ξ is such that $\alpha = P(T > \xi | \theta = 1)$, find an approximation for ξ.
Obtain an approximation to the power function $\beta(\theta)$.

3. (a) What is meant by a *pivotal quantity*?

(b) Let $\boldsymbol{x} = \{x_1, x_2, \ldots, x_n\}$ be a random sample from a distribution having probability density function

$$f(x|\theta) = \begin{cases} \dfrac{x}{\theta} \exp\left(-\dfrac{1}{2}\dfrac{x^2}{\theta}\right) & (x > 0), \\ 0 & (x \le 0), \end{cases}$$

where θ is an unknown positive parameter.

(i) Obtain the efficient total score $U_\bullet(\theta)$.

(ii) From the form of $U_\bullet(\theta)$ write down

- the total Fisher information $I_\bullet(\theta)$,
- the maximum likelihood estimator $\widehat{\theta}$ of θ,
- $\mathrm{var}\,(\widehat{\theta})$.

(iii) Which theorem guarantees that $\mathrm{var}\,(\widehat{\theta})$ minimises the variance over all unbiased estimators of θ?

(iv) Show that $Z = \widehat{\theta}/\theta$ is a pivotal quantity having a *Gamma*$(n, 1)$ distribution.

(v) From (iv) construct a $100(1-\alpha)\%$ confidence interval for θ having equal tail probabilities for small α.

4. (a) State the Lehmann-Scheffé Theorem for finding a minimum variance unbiased estimator (MVUE).

(b) For a random sample $\boldsymbol{x} = \{x_1, x_2, \ldots, x_n\}$ from the Delayed Exponential distribution having probability density function

$$f(x|\theta) = \begin{cases} e^{-(x-\theta)} & (x > \theta), \\ 0 & (x \le \theta), \end{cases}$$

by considering the pivotal quantity $Z = X - \theta$, or otherwise, find the distribution of X_{min}.

Find an unbiased estimator of θ that is a function of X_{min} alone.

(c) (i) Find a non-zero function $h(t)$, for which $E\{h(T)\} = 0$, to show that the *Uniform*$(-\theta, \theta)$ family of distributions, where $\theta > 0$, is not complete.

(ii) Let $\boldsymbol{x} = \{x_1, x_2, \ldots, x_n\}$ be a random sample from *Uniform*$(-\theta, \theta)$ $(\theta > 0)$, and let the prior probability for θ be *Pareto* with probability density function

$$\pi(\theta\,|\alpha, \beta) = \frac{\beta\alpha^\beta}{\theta^{\beta+1}}\, H(\theta > \alpha)$$

where α and β are known positive constants, and $H(A) = 1$ if A is true, and 0 if A is false.

Show that the posterior distribution for θ is *Pareto*(α^*, β^*), where α^* and β^* are to be determined.

	M3S1/M4S1 EXAMINATION SOLUTIONS May-June 2014	M3S1 M4S1
Question 1		Marks & seen/unseen

Parts

a) i) The sufficiency principle is that: if statistic t is sufficient for the family of distributions parameterised by θ, then analysis of the data should be only through t. — Bookwork — 2

ii) Possible responses regarding its importance are:
- the reduction of a set of data to t,
- a simplified analysis for hypothesis testing etc eg criteria for most powerful tests of composite hypotheses, ~~using~~
- Identifying MVUE by Lehmann-Scheffé

2

iii) A minimal sufficient statistic is a function of every sufficient statistic, so any further reduction would not yield a sufficient statistic.
It is essentially unique through the equivalence relation $t(\underset{\sim}{x}) = t(\underset{\sim}{y})$ iff likelihood ratio $\ell(\theta;\underset{\sim}{x})/\ell(\theta;\underset{\sim}{y})$ does not depend on θ.

2

b) i) Unseen

$$\ell(\theta;\underset{\sim}{x}) = (\theta e^{\theta^2})^n e^{-n\theta\bar{x}} H(x_{min} > \theta)$$
$$= g(\theta, \underset{\sim}{x}) = g(\theta, \bar{x}, x_{min})$$

so $\bar{x}, x_{min}$ are jointly sufficient for θ by Neyman factorisation.

$$\frac{\ell(\theta;\underset{\sim}{x})}{\ell(\theta;\underset{\sim}{y})} = e^{-n\theta(\bar{x}-\bar{y})} \frac{H(x_{min}>\theta)}{H(y_{min}>\theta)}$$

This does not depend on θ when $\bar{x}=\bar{y}$ & $x_{min}=x_{max}$ — 7

ii)
$$\ell(\theta;\underset{\sim}{x},\underset{\sim}{z}) = e^{\theta\sum z_k x_k} / \prod(1+e^{\theta z_k})$$

so $t(\underset{\sim}{x}) = \sum z_k x_k$ is sufficient for θ.
Minimal sufficient because dimension 1 cannot be reduced further.

~~7~~

Alt loglik $L(\theta;\underset{\sim}{x},\underset{\sim}{z}) = \theta\sum z_k x_k - \sum \ln(1+e^{\theta z_k})$

$L(\theta;\underset{\sim}{x},\underset{\sim}{z}) - L(\theta;\underset{\sim}{y},\underset{\sim}{z}) = \theta\{t(\underset{\sim}{x}) - t(\underset{\sim}{y})\}$ does not depend on θ if $t(x)=t(y)$. — 7

	Setter's initials	Checker's initials	Page number
			1

Question 2 | Marks & seen/unseen

Parts

a) The monotone likelihood ratio criterion holds if the likelihood ratio $\lambda(\underset{\sim}{x})$ is a non-increasing (or non-decreasing) function of $t(\underset{\sim}{x})$, a sufficient statistic for θ.
Its importance is that if the criterion is satisfied, the test is UMP. — Bookwork, 3

b) i) $f_{\underset{\sim}{X},\underset{\sim}{Y}}(\underset{\sim}{x},\underset{\sim}{y}|\theta) = \ell(\theta;\underset{\sim}{x},\underset{\sim}{y},c) = \prod_i \left\{ \frac{\theta^{x_i} e^{-\theta}}{x_i!} \cdot \frac{(c\theta)^{y_i} e^{-c\theta}}{y_i!} \right\}$ — Unseen

$= \frac{c^{\Sigma y_i}}{\prod\{x_i! y_i!\}} \theta^{\Sigma(x_i+y_i)} e^{-(c+1)n\theta}$

Let $t(\underset{\sim}{x},\underset{\sim}{y}) = \Sigma(x_i+y_i)$ & $w=(c+1)n$
then t is sufficient for θ by Neyman factorisation. — 3

ii) Loglikelihood $L(\theta;\underset{\sim}{x},\underset{\sim}{y}) = t\ln\theta - w\theta$

$\frac{\partial L}{\partial\theta} = \frac{t}{\theta} - w$, $\frac{\partial^2 L}{\partial\theta^2} = -\frac{t}{\theta^2} < 0$ so $L\uparrow$ as $\theta\uparrow$

So criterion is satisfied. (Note: mle $\hat{\theta} = \frac{t}{w}$) — 5

Note: $H_0: \theta=\theta_0<1$ v. $H_1: \theta=\theta_1>1$ is MP by Neyman-Pearson
Holds for all θ_0, θ_1,
so the MP test is to reject $H_0: \theta\le 1$ if t is too large. — 2

iii) Under $H_0^*: \theta=1$ v. $H_1: \theta>1$, $T = \Sigma(X_i+Y_i)$

$E(T) = w\theta$, $var(T) = w\theta$ (by independence)

$Z = \frac{T-E(T)}{\sqrt{var(T)}} = \frac{T-w\theta}{\sqrt{w\theta}} \sim N(0,1)$ (for large n)

$P(T>\xi|\theta) = P\left(Z > \frac{\xi-w\theta}{\sqrt{w\theta}}\right) \sim 1-\Phi\left(\frac{\xi-w\theta}{\sqrt{w\theta}}\right)$

Although ~~Since~~ T is integer valued, for large n we can approximate ξ.

If $\theta=1$, for given size α, $\alpha \approx 1-\Phi\left(\frac{\xi-w}{\sqrt{w}}\right)$

i.e. $\xi \approx w+\sqrt{w}\,\Phi^{-1}(1-\alpha)$

so $\beta(\theta) \approx 1-\Phi\left(\frac{\xi-w\theta}{\sqrt{w\theta}}\right)$. — 7

Setter's initials | Checker's initials | Page number

M3S1/M4S1 EXAMINATION SOLUTIONS — M3S1 M4S1

May-June 2014

Question 3 — Marks & seen/unseen

Parts

a) A pivotal quantity $z(\underset{\sim}{x}, \theta)$ has a known sampling distribution that does not depend on θ — Seen, 2

b) i) — unseen

$$\ln f = \ln x - \ln\theta - \frac{1}{2}\frac{x^2}{\theta}$$

$$\frac{\partial \ln f}{\partial \theta} = -\frac{1}{\theta} + \frac{1}{2}x^2\left(\frac{1}{-\theta^2}\right) \quad \text{so } U(\theta) = \frac{1}{\theta^2}\left(\frac{1}{2}X^2 - \theta\right)$$

$$U_\cdot(\theta) = \frac{n}{\theta^2}\left(\frac{1}{2}\overline{X^2} - \theta\right)$$ — 4

ii) $I_\cdot(\theta) = \frac{n}{\theta^2}$ — 2

MLE $\hat{\theta} = \frac{1}{2}\overline{x^2} = \frac{1}{2n}\Sigma x_i^2$ — 2

$\mathrm{var}(\hat{\theta}) = 1/I_\cdot(\theta) = \frac{\theta^2}{n}$ — 2

iii) Cramér-Rao Theorem — 2

iv) Let $y = \frac{x^2}{2\theta}$ $\quad dy = \frac{x}{\theta}dx$

$$\int_0^x f(x_0|\theta)dx_0 = \int_0^x e^{-\frac{1}{2\theta}x_0^2}\cdot\frac{1}{\theta}x_0 dx_0$$

$$= \int_0^y e^{-y_0}dy_0 = 1 - e^{-y}$$

so $Y = \frac{X^2}{2\theta}$ is Exponential(1)

$$Z = \sum_1^n Y_i^2 = \frac{1}{2\theta}\Sigma X_i^2 = \frac{\hat{\theta}}{\theta} \quad \text{is Gamma}(n, 1)$$ — 3

v)

$$\tfrac{1}{2}\alpha = P\left(\frac{\hat{\theta}}{\theta} > c_u\right) = P\left(\theta < \frac{\hat{\theta}}{c_u}\right)$$

$$\tfrac{1}{2}\alpha = P\left(\frac{\hat{\theta}}{\theta} < c_L\right) = P\left(\theta > \frac{\hat{\theta}}{c_L}\right)$$

so the $100(1-\alpha)\%$ CI for θ is $\left(\frac{\hat{\theta}}{c_u}, \frac{\hat{\theta}}{c_L}\right)$ — 3

Setter's initials — Checker's initials — Page number

Question 4		Marks & seen/unseen
Parts a)	If S is a complete sufficient statistic, then any function of S is a MVUE of its expectation.	bookwork 2
b)	Pivot $Z = X - \theta$ is Exponential(1) $P(Z_{min} > z) = P(\text{each } Z_i > z) = (e^{-z})^n = e^{-nz} \quad (z>0)$ so Z_{min} is Exponential(n) $E(Z_{min}) = \frac{1}{n}$ so $E\big(n(X_{min}-\theta)\big) = 1$ i.e. $E\left(X_{min} - \frac{1}{n}\right) = \theta$ so $X_{min} - \cancel{\frac{1}{n}}$ is an unbiased estimator of θ and is a function of n alone.	unseen 6
c) i)	$E(X) = 0$, $E(\overline{X}) = 0$ for example.	3
ii)	$\pi(\theta \mid \alpha, \beta, \underset{\sim}{x}) = \frac{\beta\alpha^\beta}{\theta^{\beta+1}} H(\theta > \alpha) \cdot \frac{1}{(2\theta)^n} H(-\theta < \text{each } x_i < \theta)$ $\propto \frac{1}{\theta^{\beta+n+1}} H(\theta > \alpha)\, H(\theta > \lvert x_{min} \rvert)\, H(\theta > \lvert x_{max} \rvert)$ $\propto \frac{\beta^* \alpha^{*\beta^*}}{\theta^{\beta^*}} H(\theta > \alpha^*)$ where $\alpha^* = \max\{\alpha, \lvert x_{min} \rvert, \lvert x_{max} \rvert\}$, $\beta^* = \beta + n$.	9

Setter's initials	Checker's initials	Page number
		4