B.ENG. and M.ENG. EXAMINATIONS 2012

PART I : MATHEMATICS 1 (ELECTRICAL AND INFORMATION SYSTEMS ENGINEERING)

Date Thursday 7th June 2012 10.00 - 12.00

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.

Answer Question 1 and THREE of the remaining five

Answer Section A and Section B in different answerbooks.

Question 1 carries twice the marks of each of the other questions.

CALCULATORS MAY **NOT** BE USED.

A mathematical formulae sheet is provided.

[*Before starting, please make sure that the paper is complete; there should be 6 pages, with a total of SIX questions. Ask the invigilator for a replacement if your copy is faulty.*]

SECTION A

[E1.10 (Maths 1) 2012]

1. (i) Write

$$z^2 + z^{*2} = -8$$

in Cartesian form.

Given that $z = x + iy$, *solve the above equation for all possible values of* y *in terms of* x.

(ii) Find all possible values of

$$(2 - 2i)^{2/3}$$

in polar form.

(iii) Find q so that the limit

$$\lim_{x\to\infty} e^{qx} \left(e^{2x/3} - (2 + e^x)^{2/3} \right)$$

is finite and non-zero.

Do not use L'Hôpital's rule.

(iv) Find the limit

$$\lim_{x\to 0} \frac{\cos(x) - 1}{x^2}$$

Do not use L'Hôpital's rule.

You can use $\lim_{x\to 0} \sin(x)/x = 1$.

(v) Differentiate

$$(\sin(x))^{\sin(x)}$$

(vi) Integrate

$$\int \frac{\sin(x) + 1}{\sin(x) - 1} \, \mathrm{d}x$$

(vii) Integrate

$$\int_0^1 \frac{x}{(1 - x^2)^{1/3}} \, \mathrm{d}x$$

Q1 CONTINUES ON THE NEXT PAGE

(viii) Find the Taylor expansion of

$$\frac{\ln(x-1)}{x}$$

about $x = 2$ to first order (up to and including the term linear in x) and state the remainder term $R_2\,(x)$.

(ix) Find the general solution of the following first order ODE:

$$y'(x) = \frac{y(x)}{x} + 1$$

(x) Find the general solutions of the following second order ODE:

$$y''(x) + 4y'(x) + 4y(x) = \exp(x)$$

SECTION B

[E1.10 (Maths 1) 2012]

2. Find $\dfrac{dy}{dx}$ as a function of x in each of the following cases :

   (i) $y \;=\; \dfrac{\exp(x)}{\exp(x^2)}$;

   (ii) $y = \exp(\sin^{-1}(x))$;

   Note: $\sin^{-1}(x)$ *denotes the inverse* $\sin$ *function*

   (iii) $y \;=\; \ln\left(\cos\left(x^{\sin(x)}\right)\right)$;

   (iv) $\cos(y) = \sin(x)$;

   Find the following nth derivative

   (v) $\dfrac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^2 \exp(x/2)\right)$.

3. Evaluate the following limits *without using L'Hôpital's rule unless specified*:

   (i) $$\lim_{x\to -1} \frac{(x-2)\,(x+2)}{(x-3)\,(x-1)} \;;$$

   Do not use L'Hôpital's rule.

   (ii) $$\lim_{x\to -1} \frac{(x-2)\,(2x^2-2)}{(x-3)\,(x+1)} \;;$$

   Do not use L'Hôpital's rule.

   (iii) $$\lim_{x\to \infty} \sqrt{x}\left(\sqrt{1+x}-\sqrt{x}\right) \;;$$

   Do not use L'Hôpital's rule.

   (iv) $$\lim_{x\to 1} \frac{x^n-1}{x-1} \quad \text{for integer } n>0;$$

   Do not use L'Hôpital's rule.

   (v) $$\lim_{x\to \pi} \frac{\sin(x)+(x-\pi)}{(x-\pi)^3} \;;$$

   Use L'Hôpital's rule here.

PLEASE TURN OVER

4. (i) State whether the improper integral $\int_0^1 x \ln(x)\, \mathrm{d}x$ is finite and calculate its value if it is.

   Note: $\lim_{x \to 0} x \ln(x) = 0$.

   (ii) Show that $\int_0^\infty x^n \exp(-x)\, \mathrm{d}x = n \int_0^\infty x^{n-1} \exp(-x)\, \mathrm{d}x$ for any positive integer n, i.e. $0 < n \in \mathbb{N}$

   (iii) Integrate $\int \cos^3(x) dx$.

   (iv) Integrate $\int \frac{x^2 - 1}{x^3 + 2x^2 - 3x} dx$.

5. (i) Write the following equation in Cartesian form and then solve it for all possible values of y in terms of x :

$$|z + 1| = z^* + z$$

   (ii) Express in polar form

$$1 + 2i \qquad \text{and} \qquad (\sqrt{3} + i)^{1/3}$$

   (iii) Express $\sin(4\theta)$ in terms of $\sin(\theta)$ and $\cos(\theta)$.

   (iv) Integrate $\int \cos(x) \exp(2x) \mathrm{d}x$.

   Note: You can use $\int \exp(ax)\, \mathrm{d}x = \frac{1}{a} \exp(ax) + C$ *for complex* $a \in \mathbb{C}$.

6. (i) Find the solution $y(x)$ of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2 + x^2 + 2(x+y) + 2}{2x^2 + 4x + 2}$$

Note: It might help to solve the differential equation for Y=y+1 as a function of X=x+1.

(ii) Find the solution $y(x)$ of the differential equation

$$\frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{x^2} + \exp(x) = 0$$

(iii) Find the solution $y(x)$ of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} \;+\; 2\frac{\mathrm{d}y}{\mathrm{d}x} \;-\; 3y \;=\; 2\sin(3x)\ ,$$

that satisfies $y(0) = 1,\ \dfrac{\mathrm{d}y}{\mathrm{d}x}(0) = 0\,.$

END OF PAPER

EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2011-2012

Course: EE1.1

Question: Q1

TOPIC: General (long question)

Parts		Marks & seen/unseen
(i)	$z^2 + z^{*2} = -8$ find all roots $2x^2 - 2y^2 = -8 \Rightarrow y = \pm\sqrt{x^2+4}$	4 seen similar
(ii)	$(2-2i)^{2/3}$ in polar form. $(2-2i)^{2/3} = 8^{\frac{1}{3}} e^{-i\left(\frac{\pi}{4}+2\pi k\right)\frac{2}{3}}$ $= 2\, e^{-i\left(\frac{\pi}{6}+\frac{4}{3}\pi k\right)}$ $\quad k=0,1,2$ so: $2e^{-i\frac{\pi}{6}}$, $2e^{-i\frac{3\pi}{2}} = 2e^{i\frac{\pi}{2}}$ and $2e^{-i\frac{17}{6}\pi} = 2e^{-i\frac{5}{6}\pi} = 2e^{i\frac{7}{6}\pi}$	4 seen similar

$(2+2i)^{\frac{1}{3}}$ in polar form:

$(2+2i)^{\frac{1}{3}} = 8^{\frac{1}{6}} e^{i\left(\frac{\pi}{4}+2\pi k\right)\frac{1}{3}}$

$= \sqrt{2}\, e^{i\left(\frac{\pi}{12}+\frac{2k}{3}\pi\right)}$ $\quad k=0,1,2$

so: $\sqrt{2}e^{i\frac{\pi}{12}}$, $\sqrt{2}e^{i\frac{3}{4}\pi} = \sqrt{2}e^{-i\frac{5}{4}\pi}$

and $\sqrt{2}e^{i\frac{17}{12}\pi} = \sqrt{2}e^{-i\frac{7}{12}\pi}$

EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2011-2012

Course: EE1.1

Question: Q1

TOPIC

Marks & seen/unseen

Parts

(iii) Find q so that $\lim_{x\to\infty} e^{qx}\left(e^{\frac{2x}{3}} - (2+e^{x})^{\frac{2}{3}}\right)$ is finite.

$A = e^{\frac{2x}{3}}$, $B = (2+e^{x})^{\frac{2}{3}}$ and

$$A - B = \frac{A^3 - B^3}{A^2 + AB + B^2} = \frac{e^{2x} - (2+e^{x})^2}{A^2 + AB + B^2}$$

$$= -\frac{4e^{x} + 4}{A^2 + AB + B^2} = -\frac{e^{x}}{e^{\frac{4x}{3}}}\;\frac{4 + 4e^{-x}}{1 + (2e^{-x}+1) + (2e^{-x}+1)^2}$$

$$\Rightarrow e^{qx}(A-B) = -e^{(q-\frac{1}{3})x}\;\frac{4+4e^{-x}}{1 + (2e^{-x}+1) + (2e^{-x}+1)}$$

For $q = \frac{1}{3}$ this converges to $-\frac{4}{3}$ as $x \to \infty$. — 4 unseen

(iv)
$$\lim_{x\to 0} \frac{\cos x - 1}{x^2} = \lim_{x \to 0} \frac{-\sin^2 x}{x^2(1+\cos x)} = -\frac{1}{2}$$
seen 4

(v)
$$(\sin x)^{\sin x} = \exp(\sin x \ln \sin x)$$

$$\frac{d}{dx}(\sin x)^{\sin x} = (\cos x \ln \sin x + \cos x)(\sin x)^{\sin x}$$

$$= \cos x\,(1 + \ln(\sin x))\,(\sin x)^{\sin x}$$

unseen 4

Setter's initials: GP

Checker's initials: M.

EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2011-2012

Course: EE1.1

Question: Q1

TOPIC

Marks & seen/unseen

Parts

(vi)

$$\int \frac{\sin x + 1}{\sin x - 1}\,dx = \int \frac{(\sin x + 1)^2}{-\cos^2 x}\,dx$$

$$= \int -\frac{1}{\cos^2 x} - 2\frac{\sin x}{\cos^2 x} - \underbrace{\tan^2 x}_{\frac{1}{\cos^2 x} - 1}\,dx$$

$$= x - 2\tan x - 2\frac{1}{\cos x} + C' \quad \triangle$$

4 seen similar

Alternatively

$$t = \tan\left(\frac{x}{2}\right), \quad \sin(x) = \frac{2t}{1+t^2}, \quad dx = \frac{2}{1+t^2}\,dt$$

$$\int \frac{\sin x + 1}{\sin x - 1}\,dx = x + 2\int \frac{1}{\sin x - 1}\,dx$$

$$= x + 2\int \frac{1}{\frac{2t}{1+t^2} - 1}\,\frac{2}{1+t^2}\,dt = x - 4\int \frac{dt}{(1-t)^2}$$

$$= x - 4\frac{1}{1 - \tan\frac{x}{2}} + C' \quad \square$$

(differs by a const from above)

$$\triangle - \square = 2$$

(vii)

$$\int_0^{1} \frac{x}{(1-x^2)^{1/3}}\,dx = \frac{1}{2}\int_0^1 \frac{du}{(1-u)^{1/3}} = \frac{1}{2}\lim_{\varepsilon \to 0^-}\left[-\frac{3}{2}(1-u)^{2/3}\right]_0^{1+\varepsilon}$$

$$= \frac{1}{2}\left(-\frac{3}{2}(-1)\right) = \frac{3}{4}$$

4 unseen

Setter's initials: GP

Checker's initials: MH

EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2011-2012

Course: EE1.1

Question: Q1

TOPIC

Marks & seen/unseen

Parts

(viii)

$$f(x) = \frac{\ln(x-1)}{x} \quad \text{Taylor expansion:}$$

4 unseen

$$f'(x) = -\frac{\ln(x-1)}{x^2} + \frac{1}{x(x-1)}$$

$$f''(x) = 2\frac{\ln(x-1)}{x^3} - \frac{1}{x^2(x-1)} - \frac{2x-1}{x^2(x-1)^2}$$

$$f(2) = 0$$

$$f'(2) = \frac{1}{2}$$

$$\Rightarrow f(x) = \frac{1}{2}(x-2) + R_2(x)$$

$$\text{with } R_2(x) = \frac{(x-2)^2}{2} f''(\xi), \quad \xi \in [2, x]$$

(ix)

$$y' = \underbrace{\frac{y}{x} + 1}_{f(v)} \quad \text{homogeneous}, \quad v = \frac{y}{x}, \ f(v) = v+1$$

4 seen similar

$$\ln x = \int \frac{1}{f(v) - v}\, dv = v + C' \Rightarrow y = x(\ln x - C')$$

(x)

$$y'' + 4y' + 4y = \exp(x)$$

4 seen similar

$$a = 1, \ b = 4, \ c = 4 \qquad b^2 - 4ac = 0 \quad \text{critical case}$$

$$\lambda = -\frac{b}{2a} = -2$$

$$\left.\begin{array}{l} \Rightarrow y_{CF} = Ae^{-2x} + Bxe^{-2x} \\ y_{PI} = \frac{1}{9}\exp(x) \end{array}\right\} \Rightarrow y(x) = Ae^{-2x} + Bxe^{-2x} + \frac{1}{9}e^{x}$$

Setter's initials: GP

Checker's initials: MH

EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2011-2012

Course EE1.1

Question Q2

TOPIC Differentiation

Marks & seen/unseen

Parts

(i)

$$y = \frac{e^x}{e^{x^2}}$$

$$y' = \frac{e^x}{e^{x^2}} - 2x e^{x^2} \frac{e^x}{e^{2x^2}} = (1-2x) e^{x-x^2}$$

3 seen similar

(ii)

$$y = \exp(\sin^{-1}(x))$$

$$y' = \frac{1}{\sqrt{1-x^2}} \exp(\sin^{-1}(x))$$

3 unseen

(iii)

$$y = \ln(\cos(x^{\sin x}))$$

$$= \ln(\cos(\exp(\sin(x)\ln(x))))$$

$$y' = -\left(\frac{\sin x}{x} + \cos(x)\ln(x)\right) \exp(\sin(x)\ln(x)) \tan(\exp(\sin(x)\ln(x)))$$

4 unseen

$$= -\left(\frac{\sin x}{x} + \cos(x)\ln(x)\right) x^{\sin x} \tan(x^{\sin x})$$

Setter's initials GP

Checker's initials HM

EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2011-2012

Course EE1.1

Question 2 | TOPIC | Marks & seen/unseen

Parts

(iv)

$$\cos(y) = \sin(x)$$

$$\Rightarrow \quad -y'\sin(y) = \cos x$$

$$\Rightarrow \quad y' = \frac{-\cos x}{\pm\sqrt{1-\sin^2(x)}} = \pm 1$$

4 unseen

Award all marks for either sign

However if you use principle values it is -1,

(v)

$$\frac{d^n}{dx^n}\left(x^2 \exp(\tfrac{1}{2}x)\right) = \sum_{i=0}^{n} \binom{n}{i} \left(x^2\right)^{(i)} \left(\exp(\tfrac{1}{2}x)\right)^{(n-i)}$$

3 unseen

i	$(x^2)^{(i)}$	$\binom{n}{i}$
0	x^2	1
1	$2x$	n
2	2	$\frac{n(n-1)}{2}$
3	0	⋮
⋮	⋮	

$$\left(\exp(\tfrac{1}{2}x)\right)^{(n-i)} = \left(\tfrac{1}{2}\right)^{(n-i)} \exp(\tfrac{1}{2}x)$$

$$\Rightarrow \frac{d^n}{dx^n}\left(x^2 \exp(\tfrac{1}{2}x)\right) = e^{\frac{1}{2}x}\left\{\frac{x^2}{2^n} + \frac{nx}{2^{n-2}} + \frac{n(n-1)}{2^{n-2}}\right\}$$

3

Setter's initials GP

Checker's initials MM

EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2011-2012

Course: EE1.1

Question 3

TOPIC: Limits

Marks & seen/unseen

Parts

(i) $\lim_{x\to -1} \frac{(x-2)(x+2)}{(x-3)(x-1)} = \frac{-3}{8}$ — 3, seen similar

(ii) $\lim_{x\to -1} \frac{(x-2)(2x^2-2)}{(x-3)(x+1)} = \lim_{x\to -1} 2\,\frac{(x-2)(x-1)}{(x-3)}$ — 3, seen similar

$= 2\,\frac{6}{-4} = -3$

(iii) $\sqrt{1+x} - \sqrt{x} = \frac{1}{\sqrt{1+x}+\sqrt{x}}$ — 4, seen similar

$\Rightarrow \lim_{x\to\infty} \sqrt{x}\left(\sqrt{1+x}-\sqrt{x}\right) = \lim_{x\to\infty} \frac{\sqrt{x}}{\sqrt{1+x}+\sqrt{x}} = \frac{1}{2}$

(iv) $\lim_{x\to 1} \frac{x^n-1}{x-1} = \lim_{x\to 1} \underbrace{x^{n-1}+x^{n-2}+\dots+1}_{n \text{ terms}} = n$ — 5, seen similar

(v) $\lim_{x\to\pi} \frac{\sin(x)+(x-\pi)}{(x-\pi)^3}$ — 5, unseen

To use L'Hôpital, show that π is a root in numerator and denominator. Latter obvious for 0th, 1st, 2nd but not third derivative. Former obvious for 0th derivative, and then

1st	$\cos(x)+1$	$=0$	at $x=\pi$
2nd	$-\sin(x)$	$=0$	at $x=\pi$
3rd	$-\cos(x)$	$=1$	at $x=\pi$

Setter's initials: GP

Checker's initials: MK

Page number: 5

	EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2011-2012	Course EE1.1
Question 3	TOPIC	Marks & seen/unseen

Parts

Third derivative of denominator: 6

$$\Rightarrow \lim_{x\to\pi} \frac{\sin(x)+(x-\pi)}{(x-\pi)^3} = \frac{1}{6}$$

Setter's initials GP

Checker's initials MI

EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2011-2012

Course: EE1.1

Question 4

TOPIC: Integration

Marks & seen/unseen

Parts

(i) $\int_0^1 x \ln x \, dx$ integrand singular at $x=0$. — 2 unseen

$$\int x \ln x \, dx = \tfrac{1}{2}x^2 \ln x - \int \tfrac{1}{2} x \, dx = \tfrac{1}{2}x^2 \ln x - \tfrac{1}{4}x^2 + C'$$

$$\Rightarrow \int_0^1 x \ln x \, dx = \lim_{\varepsilon \to 0^+} \left[\tfrac{1}{2}x^2 \ln x - \tfrac{1}{4}x^2\right]_\varepsilon^1$$

$$= -\tfrac{1}{4}, \text{ using } \lim_{x\to 0} x^2 \ln x = 0$$

3

since $\lim_{x\to 0} x \ln x = 0$.

(ii)

$$\int_0^\infty x^n e^{-x} dx = \underbrace{\left[-x^n e^{-x}\right]_0^\infty}_{=0} + \int_0^\infty n x^{n-1} e^{-x} dx$$

5 unseen

$$= n \int_0^\infty x^{n-1} e^{-x} dx$$

(iii)

$$\int \cos^3(x)\, dx = \int (1-\sin^2(x)) \cos(x)\, dx$$

4 seen similar

$$= \sin(x) - \tfrac{1}{3}\sin^3(x) + C'$$

Setter's initials: GP

Checker's initials: [illegible]

EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2011-2012

Course: EE1.1

Question: 4

TOPIC

Marks & seen/unseen

Parts

(iv) $\displaystyle\int \frac{x^2-1}{x^3+2x^2-3x}\,dx = I$

Denominator: $x^3+2x^2-3x = x(x^2+2x-3) = x(x+3)(x-1)$

Numerator: $(x+1)(x-1)$

$\Rightarrow \quad I = \displaystyle\int \frac{x+1}{x(x+3)}\,dx$

$\dfrac{x+1}{x(x+3)} = \dfrac{A}{x} + \dfrac{B}{x+3}. \quad \Rightarrow A = \frac{1}{3} \quad B = \frac{2}{3}$ — 3 seen similar

$\Rightarrow \quad I = \frac{1}{3}\displaystyle\int \frac{1}{x} + \frac{2}{x+3}\,dx = \frac{1}{3}\ln|x| + \frac{2}{3}\ln|x+3| + C'$ — 3

(v) $\displaystyle\int \frac{dx}{\sqrt{8+2x-x^2}} = I$ — seen similar

$8+2x-x^2 = -(x-4)(x+2)$

$y = x-1$

$\Rightarrow 8+2x-x^2 = -(y-3)(y+3) = 9-y^2$

$I = \displaystyle\int \frac{1}{3}\frac{dy}{\sqrt{1-y^2/9}} = \sin^{-1}\left(\frac{y}{3}\right) + C'$

$= \sin^{-1}\left(\frac{x-1}{3}\right) + C'$

Setter's initials: GP

Checker's initials: LM

Page number: S10

	EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2011-2012	Course EE1.1
Question 5	TOPIC Complex numbers + Taylor series	Marks & seen/unseen

Parts

(i) $|z+1| = z^* + z \underset{z=x+iy}{\Rightarrow} (x+1)^2 + y^2 = 4x^2$

$\Rightarrow y^2 = 3x^2 - 2x - 1$

$y = \pm\sqrt{3x^2-2x-1}$ for $3x^2-2x-1 \geq 0$, $x \in \mathbb{R}$

Roots of $3x^2-2x-1$ by inspection: $x=1$

$\Rightarrow 3x^2-2x-1 = 3(x-1)(x+\frac{1}{3})$

other root $x = -\frac{1}{3}$

$\Rightarrow x \geq 1$ and $x \leq -\frac{1}{3}$, $x \in (-\infty, -\frac{1}{3}] \cup [1, \infty)$ — 5 seen similar

~~$\arg(z(z-2)) = \pi \Rightarrow$~~

~~$z(z-2) = -r$ with $0 \leq r \in \mathbb{R}$~~

~~Where is $z(z-2)$ negative?~~

~~$z \in (0,2) \subset \mathbb{R}$, $z = 1 \pm i\sqrt{\frac{r}{2}-1}$ for $r > 2$~~ — seen similar

(ii) $1+2i = \sqrt{5}\, e^{i \tan^{-1}(2)}$ — 2 seen similar

$(\sqrt{3}+i)^{\frac{1}{3}} = 2^{\frac{1}{3}} e^{i \underbrace{\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)}_{\frac{\pi}{6}} \cdot \frac{1}{3} + i 2\pi \frac{k}{3}}$ $\quad k = 0,1,2$

$2^{\frac{1}{3}} e^{i\frac{\pi}{18}}, \quad 2^{\frac{1}{3}} e^{i\frac{13\pi}{18}}, \quad 2^{\frac{1}{3}} e^{i\frac{25\pi}{18}} = 2^{\frac{1}{3}} e^{-i\frac{11\pi}{18}}$ — 3 seen similar

	EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2011-2012	Course EE1.1
Question 5	TOPIC	Marks & seen/unseen

Parts

(iii)

$$\sin(4\theta) = \mathrm{Im}\left(e^{i4\theta}\right) = \mathrm{Im}\left((\cos\theta + i\sin\theta)^4\right)$$

$$= 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$$

4 seen similar

(iv)

$$\int \cos(x)e^{2x}\,dx = \int \tfrac{1}{2}\left(e^{ix}+e^{-ix}\right)e^{2x}\,dx$$

$$= \frac{1}{2}\frac{1}{2+i}e^{(2+i)x} + \frac{1}{2}\frac{1}{2-i}e^{(2-i)x} + C'$$

3 unseen

$$= \frac{1}{2}e^{2x}\frac{1}{5}\left\{(2-i)e^{ix} + (2+i)e^{-ix}\right\} + C'$$

$$= \frac{1}{5}e^{2x}\left\{2\cos(x) + \sin(x)\right\} + C'$$

3

Alternatively integration by parts twice:

$$\int \cos(x)e^{2x}\,dx = \sin x\, e^{2x} - 2\int \sin(x)e^{2x}\,dx$$

$$= \sin x\, e^{2x} + 2\cos x\, e^{2x} - 4\int \cos(x)e^{2x}\,dx$$

$$\Rightarrow \int \cos(x)e^{2x}\,dx = \frac{1}{5}e^{2x}\left\{2\cos(x) + \sin(x)\right\} + C'$$

(v)

$$\cancel{f(x) = x\ln(x)}$$

$$\cancel{f'(x) = \ln(x) + 1}$$

$$\cancel{f''(x) = \frac{1}{x}}$$

$$\cancel{f'''(x) = -\frac{1}{x^2}}$$

$$\cancel{f(x) = (x-1) + \tfrac{1}{2}(x-1)^2 + R_3(x)}$$

$$\cancel{R_3(x) = \frac{(x-1)^3}{6}\left(-\frac{1}{\xi^2}\right)}$$

$$\cancel{\xi \in [1,x]}$$

unseen

Setter's initials	Checker's initials	Page number
GP	[illegible]	S12

EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2011-2012

Course: EE1.1

Question 6

TOPIC: ODEs

Marks & seen/unseen

Parts

(i)

$$y' = \frac{y^2+x^2+2(x+y)+2}{2x^2+4x+2} = \frac{(y+1)^2+(x+1)^2}{2(x+1)^2}$$

$$X = x+1 \quad Y = y+1 \quad Y' = \frac{Y^2+X^2}{2X^2} = \underbrace{\tfrac{1}{2}(v^2+1)}_{f(v)}$$

with $v = \frac{Y}{X}$

3 seen similar

$$\Rightarrow \ln X = \int \frac{dv}{f(v)-v} = \int \frac{2}{(1-v)^2}\,dv = \frac{2}{1-v} + C_1'$$

$$\frac{1}{2}(\ln X - C_1') = \frac{1}{1-v}$$

$$v = 1 - \frac{2}{\ln X - C_1'}$$

$$\Rightarrow y = (x+1)\left\{1 - \frac{2}{\ln(x+1) - C_1'}\right\} - 1$$

4

(ii)

$$\frac{1}{x}y' + \frac{y}{x^2} + \exp(x) = 0 \Rightarrow$$

$$y' + \frac{y}{x} = -xe^x \quad \text{linear} \quad R = e^{\int \frac{1}{x}dx} = x$$

$$\Rightarrow y(x) = \frac{-1}{x}\int x^2e^x\,dx$$

6 seen similar

$$\int x^2e^x\,dx = x^2e^x - 2\int xe^x\,dx = x^2e^x - 2xe^x + 2e^x + C$$

$$\Rightarrow y(x) = -e^x\,\frac{x^2-2x+2}{x} + \frac{C_1'}{x}$$

Setter's initials: GP

Checker's initials: MC

Page number: S13

	EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2011-2012	Course EE1.1
Question 6	TOPIC	Marks & seen/unseen

Parts

(iii)

$$y'' + 2y' - 3y = 2\sin(3x)$$

$a=1,\ b=2,\ c=-3$ $\qquad b^2-4ac = 16 > 0$

$\lambda_1 = \dfrac{-2+\sqrt{16}}{2} = 1$ $\qquad \lambda_2 = \dfrac{-2-\sqrt{16}}{2} = -3$

$y_{CF} = Ae^{x} + Be^{-3x}$ — 2 seen similar

$y_{P1} = \alpha\sin(3x) + \beta\cos(3x)$

$y_{P1}' = 3\alpha\cos(3x) - 3\beta\sin(3x)$

$y_{P1}'' = -9\alpha\sin(3x) - 9\beta\cos(3x)$

$$\begin{aligned} y_{P1}'' + 2y_{P1}' - 3y_{P1} = &-9\alpha\sin(3x) - 9\beta\cos(3x) \\ &-6\beta\sin(3x) + 6\alpha\cos(3x) \\ &-3\alpha\sin(3x) - 3\beta\cos(3x) \end{aligned}$$

$-9\alpha - 6\beta - 3\alpha = 2 \quad \Rightarrow \quad -3\beta - 1 = 6\alpha$

$-9\beta + 6\alpha - 3\beta = 0 \quad \Rightarrow \quad -9\beta - 3\beta - 1 - 3\beta = 0$

$\beta = -\frac{1}{15}$

$\alpha = -\frac{2}{15}$

$\Rightarrow y_{P1}(x) = -\frac{2}{15}\sin(3x) - \frac{1}{15}\cos(3x)$ — 2

$y(x) = Ae^{x} + Be^{-3x} - \frac{2}{15}\sin(3x) - \frac{1}{15}\cos(3x)$

$y(0) = A + B - \frac{1}{15} = 1 \quad \Rightarrow \quad A + B = \frac{16}{15}$

$y'(x) = Ae^{x} - 3Be^{-3x} - \frac{2}{5}\cos(3x) + \frac{1}{5}\sin(3x)$

$y'(0) = A - 3B - \frac{2}{5} = 0 \quad \Rightarrow \quad \frac{16}{15} - 4B - \frac{2}{5} = 0 \Rightarrow B = \frac{1}{6}$ — 3

$A = \frac{16}{15} - \frac{1}{6} = \frac{27}{30} = \frac{9}{10}$

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