EE2-08A MATHEMATICS

The below comments apply to common errors. Where there is no comment, the question was done well by most students.

1. Given the complex mapping from $z = x + iy$ to $w = u + iv$:

$$w = \frac{1}{z+i}$$

a) Show that circles $x^2 + (y+1)^2 = a^2$ in the $z-$plane map to circles in the $w-$plane, and give the equation of the circles in terms of u, v. [4]

SOLUTION

Begin with

$$w = \frac{1}{z+i} = \frac{1}{[x+i(y+1)]} \frac{x-i(y+1)}{[x-i(y+1)]} = \frac{x}{x^2+(y+1)^2} - i\frac{y+1}{x^2+(y+1)^2},$$

so that

$$u = \frac{x}{x^2+(y+1)^2}, \quad \text{and} \quad v = -\frac{y+1}{x^2+(y+1)^2}.$$

Hence

$$u^2+v^2 = \frac{x^2}{[x^2+(y+1)^2]^2} + \frac{(y+1)^2}{[x^2+(y+1)^2]^2} = \frac{1}{x^2+(y+1)^2} = \frac{1}{a^2}$$

So circles $x^2 + (y+1)^2 = a^2$ in the $z-$plane map to circles $u^2 + v^2 = 1/a^2$, with center at origin, radius $1/a$ in the $w-$plane.

b) Show that the axes in the $z-$plane map to an axis and a circle in the $w-$plane. Obtain the axes and circle. [3]

SOLUTION

For the $y-$axis, $x = 0$ so that $u = 0$ and $v = -1/(y+1)$ giving the $v-$, or vertical axis in the $w-$plane.

For the $x-$axis, $y = 0$ and

$$u = \frac{x}{x^2+1}, \quad \text{and} \quad v = -\frac{1}{x^2+1},$$

so that

$$u^2+v^2 = \frac{1}{x^2+1} = -v \Rightarrow u^2 + \left(v+\frac{1}{2}\right)^2 = \frac{1}{4},$$

so the $x-$axis maps to the circle with centre $(0,-1/2)$ and radius $1/2$ in the $w-$plane.

For the y-axis, many people got to $u=0$ and $v=-1/(y+1)$ and left this without explaining the significance. The second equation does not determine the line in the $u,v-$plane, only which point on the line. Only the first equation determines the line, namely the $v-$axis. Similarly, for the $x-$axis, many people got to $u^2+v^2=1/(x^2+1)$ and left is as that, stating this is a circle with centre at the origin, and in some cases claiming that the circle has radius $1/(x^2+1)$ or similar.

c) Obtain the images in w of the lines $y=x-1$ and $y=-1$. [3]

SOLUTION

If $y=x-1$ then $x=y+1$ and we substitute to get

$$u=\frac{1}{2(y+1)}, \quad \text{and} \quad v=-\frac{1}{2(y+1)}=-u$$

so the line $y=x-1$ maps to the line $v=-u$.

If $y=-1$ we have

$$u=\frac{1}{x} \quad \text{and} \quad v=0$$

so the line $y=-1$ maps to the horizontal $u-$axis.

Same problem as in (b). Lots of people got to $v=-1/2(y+1)$ and left it, not seeing the line $v=-u$.

2. Given the real integral

$$I=\int_0^{2\pi}\frac{d\theta}{(5+3\cos\theta)^2},$$

a) Use the substitution $z=e^{i\theta}$ to show that

$$I=-i\oint_C\frac{4z\,dz}{(3z+1)^2(z+3)^2},$$

where C is the unit circle in the complex plane. [6]

SOLUTION

The substitution $z=e^{i\theta}$ describes the unit circle for $\theta=0\ldots2\pi$ and gives $d\theta=\dfrac{dz}{iz}$ and we use $\cos\theta=\frac{1}{2}\left(z+\frac{1}{z}\right)$ to get

$$I=\oint_C\frac{1}{\left[5+\frac{3}{2}\left(z+\frac{1}{z}\right)\right]^2}\frac{dz}{iz}=-i\oint_C\frac{4z\,dz}{z^2\left[10+3\left(z+\frac{1}{z}\right)\right]^2}$$

$$= -i \oint_C \frac{4z\, dz}{(10z+3z^2+3)^2}$$

Solve

$$3z^2+10z+3=0 \Rightarrow z=-\frac{5}{3}\pm\frac{4}{3}=-\frac{1}{3},-3$$

so that $3z^2+10z+3=3(z+1/3)(z+3)=(3z+1)(z+3)$, and the integral becomes

$$I=-i\oint_C \frac{4z\, dz}{(3z+1)^2(z+3)^2},$$

as required.

Lots of people expanded the hard way around, starting with

$$4z/(3z+1)^2(z+3)^2,$$

substituting $z=e^{i\theta}$ and trying to get to $5+3\cos\theta$ in the denominator, much harder than the other way, as in the solution, many got stuck. Many people did the substitution as above, but expanded at the end of the first line, getting $9z^4+60z^3+118z^2+60z+9$ in the denominator, or something close to this, with arithmetic error. Even those that got the correct degree-4 polynomial then found it difficult to factorize. Incorrect or correct polynomial, many then just wrote down the required answer at this point, leaving it unclear whether they actually factorized anything.

b) Using Cauchy's residue theorem, or otherwise, calculate I. [4]

Recall that the residue of a complex function $F(z)$ at a pole $z=a$ of multiplicity m is given by the expression

$$\lim_{z\to a}\left\{\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[(z-a)^m F(z)\right]\right\}.$$

SOLUTION

To use the residue theorem, we need to calculate the residues due to each pole inside C, and there is only the double pole at $z=-1/3$, as $z=-3$ is outside the unit circle. Using the residue formula we obtain:

$$\lim_{z\to-1/3}\frac{d}{dz}\left((z-1/3)^2\frac{4z}{(3z+1)^2(z+3)^2}\right)=\lim_{z\to-1/3}\frac{d}{dz}\left(\frac{4z}{9(z+3)^2}\right)$$

$$=\frac{4}{9}\lim_{z\to-1/3}\frac{3-z}{(z+3)^3}=\frac{4}{9}\times\frac{3+\frac{1}{3}}{\left(-\frac{1}{3}+3\right)^3}=\frac{5}{64},$$

after some simplification. Cauchy's residue theorem gives that

$$I=-i\times 2\pi i\times(\sum \text{Residues inside } C)=\frac{5\pi}{32}.$$

Many multiplied by $(3z+1)^2$ before differentiating. Many, many differentiated the correct fraction, but did not simplify sufficiently before taking the limit $z \to -1/3$, complicating the expression and leading to arithmetic errors. Many forgot to multiply by $-i$ in the last step, leaving the answer complex.

3. a) The complex function

$$F(z) = \frac{e^{imz}}{(z^2+4)^2}$$

has two double poles. Find the residue at the pole lying in the upper half of the complex plane. [5]

SOLUTION

The poles are at $z = \pm 2i$, with $z = 2i$ in the upper half-plane. The residue is obtained as

$$\lim_{z\to 2i} \frac{d}{dz}\left((z-2i)^2 \frac{e^{imz}}{(z^2+4)^2} \right) = \lim_{z\to 2i} \frac{d}{dz}\left(\frac{e^{imz}}{(z+2i)^2} \right)$$

$$= \lim_{z\to 2i} e^{imz}\left(\frac{im}{(z+2i)^2} - \frac{2}{(z+2i)^3} \right) = e^{im(2i)}\left(\frac{im}{(4i)^2} - \frac{2}{(4i)^3} \right) = -\frac{ie^{-2m}(2m+1)}{32},$$

after some simplification.

b) Consider the contour integral $I = \oint_C \frac{e^{imz}}{(z^2+4)^2}\, dz$,

where the closed contour C consists of a semi-circle in the complex upper half-plane, taken in the anti-clockwise sense, and $m > 0$.

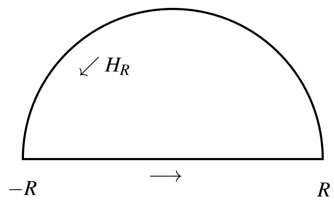
Using the result from (a), Cauchy's Residue Theorem and Jordan's lemma, show that

$$\int_{-\infty}^{\infty} \frac{\cos mx}{(x^2+4)^2}\, dx = \frac{(2m+1)\pi}{16} e^{-2m}.$$

[10]

SOLUTION

We use the contour C as the union of the semi-circular arc H_R^+ in the upper-half plane and the real interval $[-R, R]$:



The contour integral can be written in two parts:

$$I = \oint_C \frac{e^{imz}}{(z^2+4)^2}\,dz = \int_{H_R} \frac{e^{imz}}{(z^2+4)^2}\,dz + \int_{-R}^{R} \frac{e^{imx}}{(x^2+4)^2}\,dx$$

Using Jordan's lemma

$$\lim_{R\to\infty} \int_{H_R} \frac{e^{imz}}{(z^2+4)^2}\,dz = 0\,,$$

because:
(i) $m > 0$,
(ii) $\left|\frac{1}{(z^2+4)^2}\right| \to 0$ as $R \to \infty$,
(iii) all singularities are poles.

Using Cauchy's residue theorem,

$$I = \oint_C \frac{e^{imz}}{(z^2+4)^2}\,dz = 2\pi i \times \text{ sum of residues in the upper half-plane}$$

$$= \frac{(2m+1)\pi}{16} e^{-2m}$$

and taking the limit as $R \to \infty$ we have

$$\int_{-\infty}^{\infty} \frac{e^{imx}}{(x^2+4)^2}\,dx = \int_{-\infty}^{\infty} \frac{\cos mx + i \sin mx}{(x^2+4)^2}\,dx = \frac{\pi}{16}(2m+1)e^{-2m}\,,$$

and the sine part vanishes as it's the symmetric integral of an odd function, giving the required result.

Lots of small errors, leaving out details such as stating the correct conditions for Jordan's lemma, taking limit $R \to \infty$ at wrong step, forgetting last part regarding complex exponential and cosine/sine terms, rewriting integral in terms of contour parts without a picture and then giving no explanation as to what the parts were, leaving an integral on the real line in terms of z, and similar. More serious errors lay in having arithmetic errors in (a) and trying to force the correct answer out of this.

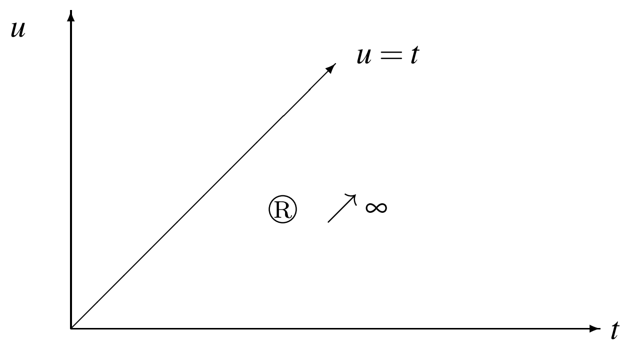
4. a) Two functions $f(t)$ and $g(t)$ have Laplace transforms $\overline{f}(s) = \mathscr{L}[f(t)]$ and
$\overline{g}(s) = \mathscr{L}[g(t)]$, respectively. If the convolution of $f(t)$ with $g(t)$ is defined as

$$f \star g = \int_0^t f(u)g(t-u)\,du\,,$$

prove that $\mathscr{L}[f \star g] = \overline{f}(s)\overline{g}(s)$. [5]

SOLUTION

Take the Laplace transform of the convolution product and exchange the order of the integrals, as in the below figure,



The region of integration is shown as Ⓡ.

$$\mathscr{L}\big(f \star g\big) = \int_0^\infty e^{-st}\left(\int_0^t f(u)g(t-u)\,du\right)dt = \int_0^\infty \left(\int_{t=u}^{t=\infty} e^{-st}g(t-u)\,dt\right)f(u)\,du$$

now substitute $\tau = t - u$

$$= \int_0^\infty \left(\int_{\tau=0}^\infty e^{-s(\tau+u)}g(\tau)\,d\tau\right)f(u)\,du = \int_0^\infty e^{-su}f(u)\,du \int_0^\infty e^{-s\tau}g(\tau)\,d\tau$$

$$= \overline{f}(s)\,\overline{g}(s)\,,$$

as required.

Most common error: sloppiness implementing change of order of integration. At end of first line, can't assume that $t = u \ldots \infty$ magically becomes $t = 0 \ldots \infty$ and similar.

b) Use the Laplace convolution theorem to solve the second order ordinary differential equation

$$\frac{d^2x}{dt^2} + 9x = \sin 3t\,,$$

with initial conditions $x(0) = x'(0) = 0$. [10]

[Recall the identity $2\sin A \sin B = \cos(A-B) - \cos(A+B)$.]

SOLUTION

Taking transforms we have

$$s^2\bar{x} + 9\bar{x} = \frac{3}{s^2+9} \Rightarrow \bar{x}(s) = \frac{1}{3}\left(\frac{3}{s^2+9}\right)\left(\frac{3}{s^2+9}\right),$$

and using the convolution theorem we have $\overline{f}(s) = \overline{g}(s) = 3/(s^2+9)$ so that $f(t) = g(t) = \sin 3t$ and

$$x(t) = \frac{1}{3} f \star g = \frac{1}{3}\int_0^t \sin 3u \, \sin 3(t-u)\, du,$$

and using the trigonometric identity with $A = 3u$ and $B = 3(t-u)$ we have

$$\begin{aligned} x(t) &= \frac{1}{6}\int_0^t \cos(6u-3t) - \cos 3t \; du \\ &= \frac{1}{6}\left[\frac{1}{6}\sin(6u-3t) - u\cos 3t\right]_0^t \\ &= \frac{1}{6}\left(\frac{1}{6}\sin 3t - \frac{1}{6}\sin(-3t) - t\cos 3t\right) \\ &= \frac{1}{18}\sin 3t - \frac{1}{6} t\cos 3t \end{aligned}$$

Lots of errors after incorrect use of trigonometric identity. Lots of people integrated $\cos 3t$ *to* $\frac{1}{3}\sin(3t)$. *Others had trouble with t as the upper limit of integration. A few ignored the instruction to use the convolution theorem and solved directly using the usual inverse transforms.*