UNIVERSITY OF LONDON [C145 2004]

B.ENG. AND M.ENG. EXAMINATIONS 2004

For Internal Students of Imperial College London

This paper is also taken for the relevant examination for the Associateship of the City & Guilds of London Institute

COMPUTING C145

MATHEMATICAL METHODS AND GRAPHICS

Wednesday 5th May 2004 2.30 - 4.30 pm

Answer FOUR questions.
Please use separate answerbooks for questions 1 and 2, and questions 3 - 6.

[*Before starting, please make sure that the paper is complete. There should be a total of SIX questions. Ask the invigilator for a replacement if this copy is faulty.*]

1. The vector equation of a plane is $\boldsymbol{n} \bullet \boldsymbol{P} = k$ where $\boldsymbol{n}$ is a normal vector to the plane, $\boldsymbol{P}$ is the position vector of a point and k is a scalar constant.

   The vector equation of a line segment is

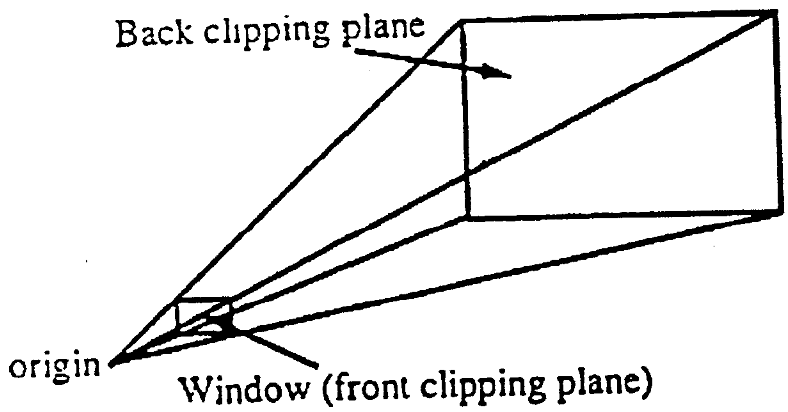
$$\boldsymbol{P} = \mu \boldsymbol{P1} + (1-\mu)\boldsymbol{P2}$$

   where $\boldsymbol{P1}$ and $\boldsymbol{P2}$ are the position vectors of the end points and μ is a scalar in the interval $[0 \ldots 1]$.

   (i) Find an equation for the intersection of a plane and a line segment.

   (ii) Use this equation to determine the point of intersection of the line joining the points [8, 2, −3] to [−6, 4, 5] with the plane $z = 3$.

   (iii) In a flight simulator, the clipping is carried out in 3D (object) space. The objects to be drawn are planar polygons with uniform colour. Perspective projection is being used and the three dimensional viewing volume is a convex polyhedron bounded by the four planes through the viewpoint and the corners of the viewing window, and a front and back clipping plane.



   Explain how, using the equations of the six planes that bound the viewing volume, it is possible to determine whether any vertex of a polygon is visible or not.

   (iv) Describe what further extensions would be required to determine the part of the polygon that is visible.

(*The four parts carry equal marks.*)

2. Matrix Transformations

(i) In computer games and flight simulators, the scene data are normally expressed in an absolute $[x, y, z]$ co-ordinate system. Explain briefly why it is desirable to be able to transform the data to a new co-ordinate system.

(ii) Given a point $\mathbf{P} = [P_x, P_y]$ defined in the normal two dimensional Cartesian co-ordinate system $[x, y]$, and a new system which is defined by a point $\mathbf{C} = [C_x, C_y]$ and two orthogonal unit vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, show, with the aid of a suitable diagram, that the co-ordinates of $\mathbf{P}$ expressed in the $[u, v]$ system are : $\boldsymbol{u} \bullet (\mathbf{P} - \mathbf{C})$ and $\boldsymbol{v} \bullet (\mathbf{P} - \mathbf{C})$.

(iii) Extend your result to three dimensions and derive a transformation matrix which will take co-ordinates with values expressed in an $[x, y, z]$ system to values expressed in a co-ordinate system whose origin is at point $\mathbf{C} = [C_x, C_y, C_z]$ and whose axis directions are expressed by the orthogonal unit vectors $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$.

(iv) A scene in a computer game is to be transformed into a new co-ordinate system with the following specification:

(a) The origin is at (10, 15, 5).

(b) The view direction (equivalent to the w axis direction) is defined by the vector $(-6, 0, 8)$.

(c) The new u axis is to remain perpendicular to the old y axis, so a vector in the u direction may be written $(p_x, 0, p_z)$.

(d) The new v axis must have a positive y component, so a vector in the v direction may be written $(q_x, 1, q_z)$.

Find the three unit vectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ describing the new co-ordinate system and hence derive the transformation matrix.

(v) This transformation is called affine. What does the word affine mean in this context? Give an example of a non-affine transformation matrix.

(The five parts carry equal marks.)

3. (i) Find the plane through the points $(0, 0, 0)$, $(1, 3, 4)$ and $(1, 1, 8)$.

   For what value of the parameter a do the points $(0, 1, a)$ and $(-1, a, -6)$ also lie on this plane?

   (ii) Use Gaussian elimination to reduce to upper triangular form the system of equations

$$\begin{array}{ccccccc} x & & & - & z & = & 1\,, \\ 3x & + & y & + & bz & = & 1\,, \\ 4x & + & by & - & 6z & = & 8\,, \end{array}$$

   where b is a parameter.

   Hence find all solutions to these equations, paying particular attention to any special values of b which may arise.

(Parts (i) and (ii) carry 30% and 70% of the marks respectively.)

4. (i) Show that the only finite stationary point of the function

$$f(x, y, z) \;=\; x^2 \;+\; 4xy \;+\; y^2 \;+\; e^{-3z^2}$$

   is $x = y = z = 0$.

   (ii) Calculate the matrix of second partial derivatives of f at the stationary point.

   (iii) Find the eigenvalues and normalised eigenvectors of this matrix.

   (iv) What kind of stationary point is $(0, 0, 0)$?

(Parts (i), (ii), (iii) and (iv) carry 20%, 20%, 50% and 10% of the marks respectively.)

5. (i) Show for every integer n that

$$(1+i)^n - (1-i)^n = 2^{(n+2)/2}\, i \sin\left(\frac{n\pi}{4}\right) .$$

(ii) The sequence $\{u_n\}$ is defined by the recurrence relation

$$u_{n+1} - 2u_n + 2u_{n-1} = n \qquad \text{with } u_0 = u_1 = 1 .$$

Obtain an explicit formula for u_n, and describe qualitatively the behaviour of u_n as $n \to \infty$.

(Parts (i) and (ii) carry 25% and 75% of the marks respectively.)

6. (i) An iterative scheme is defined for a given smooth $f(x)$ by

$$x_{n+1} = (1-k)\,x_n + k\,f(x_n)$$

where $k > 0$ is constant.

If the sequence $\{x_n\}$ converges to a limit X, infer an equation satisfied by X.

Defining $\epsilon_n = x_n - X$ for every n, show that

$$\epsilon_{n+1} = (1 + k\,[\,f'(X) - 1\,])\,\epsilon_n + O(\epsilon_n^2) .$$

(ii) Write down a necessary bound on $f'(X)$ in terms of k for the scheme to converge.

What value of k gives fastest convergence?

Compare the scheme for this value with Newton's method, and explain why Newton's method is more practical.

(iii) If $\epsilon_n \to 0$ as $n \to \infty$, discuss whether $\sum_{n=1}^{\infty} x_n$ is convergent, stating which convergence test you use.

(Parts (i), (ii) and (iii) carry 25%, 60% and 15% of the marks respectively.)

END OF PAPER