EE2-08A MATHEMATICS

1. Given the complex mapping from $z = x + iy$ to $w = u + iv$:

$$w = \frac{1}{z+i}$$

a) Show that circles $x^2 + (y+1)^2 = a^2$ in the $z-$plane map to circles in the $w-$plane, and give the equation of the circles in terms of u, v. [4]

SOLUTION

Begin with

$$w = \frac{1}{z+i} = \frac{1}{[x+i(y+1)]}\frac{x-i(y+1)}{[x-i(y+1)]} = \frac{x}{x^2+(y+1)^2} - i\frac{y+1}{x^2+(y+1)^2},$$

so that

$$u = \frac{x}{x^2+(y+1)^2}, \quad \text{and} \quad v = -\frac{y+1}{x^2+(y+1)^2}.$$

Hence

$$u^2+v^2 = \frac{x^2}{[x^2+(y+1)^2]^2} + \frac{(y+1)^2}{[x^2+(y+1)^2]^2} = \frac{1}{x^2+(y+1)^2} = \frac{1}{a^2}$$

So circles $x^2 + (y+1)^2 = a^2$ in the $z-$plane map to circles $u^2 + v^2 = 1/a^2$, with center at origin, radius $1/a$ in the $w-$plane.

b) Show that the axes in the $z-$plane map to an axis and a circle in the $w-$plane. Obtain the axes and circle. [3]

SOLUTION

For the $y-$axis, $x = 0$ so that $u = 0$ and $v = -1/(y+1)$ giving the $v-$, or vertical axis in the $w-$plane.

For the $x-$axis, $y = 0$ and

$$u = \frac{x}{x^2+1}, \quad \text{and} \quad v = -\frac{1}{x^2+1},$$

so that

$$u^2+v^2 = \frac{1}{x^2+1} = -v \Rightarrow u^2 + \left(v+\frac{1}{2}\right)^2 = \frac{1}{4},$$

so the $x-$axis maps to the circle with centre $(0, -1/2)$ and radius $1/2$ in the $w-$plane.

c) Obtain the images in w of the lines $y = x - 1$ and $y = -1$. [3]

SOLUTION

If $y = x - 1$ then $x = y + 1$ and we substitute to get

$$u = \frac{1}{2(y+1)}, \quad \text{and} \quad v = -\frac{1}{2(y+1)} = -u$$

so the line $y = x - 1$ maps to the line $v = -u$.

If $y = -1$ we have

$$u = \frac{1}{x} \quad \text{and} \quad v = 0$$

so the line $y = -1$ maps to the horizontal $u-$axis.

2. Given the real integral

$$I = \int_0^{2\pi} \frac{d\theta}{(5+3\cos\theta)^2},$$

a) Use the substitution $z = e^{i\theta}$ to show that

$$I = -i \oint_C \frac{4z\,dz}{(3z+1)^2(z+3)^2},$$

where C is the unit circle in the complex plane. [6]

SOLUTION

The substitution $z = e^{i\theta}$ describes the unit circle for $\theta = 0 \ldots 2\pi$ and gives $d\theta = \frac{dz}{iz}$ and we use $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$ to get

$$I = \oint_C \frac{1}{\left[5 + \frac{3}{2}\left(z+\frac{1}{z}\right)\right]^2} \frac{dz}{iz} = -i \oint_C \frac{4z\,dz}{z^2\left[10 + 3\left(z+\frac{1}{z}\right)\right]^2}$$

$$= -i \oint_C \frac{4z\,dz}{(10z+3z^2+3)^2}$$

Solve

$$3z^2 + 10z + 3 = 0 \Rightarrow z = -\frac{5}{3} \pm \frac{4}{3} = -\frac{1}{3}, -3$$

so that $3z^2 + 10z + 3 = 3(z+1/3)(z+3) = (3z+1)(z+3)$, and the integral becomes

$$I = -i \oint_C \frac{4z\,dz}{(3z+1)^2(z+3)^2},$$

as required.

b) Using Cauchy's residue theorem, or otherwise, calculate I. [4]

Recall that the residue of a complex function $F(z)$ at a pole $z = a$ of multiplicity m is given by the expression

$$\lim_{z\to a} \left\{ \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m F(z)\right] \right\}.$$

SOLUTION

To use the residue theorem, we need to calculate the residues due to each pole inside C, and there is only the double pole at $z = -1/3$, as $z = -3$ is outside the unit circle. Using the residue formula we obtain:

$$\lim_{z\to -1/3} \frac{d}{dz}\left((z-1/3)^2 \frac{4z}{(3z+1)^2(z+3)^2}\right) = \lim_{z\to -1/3} \frac{d}{dz}\left(\frac{4z}{9(z+3)^2}\right)$$

$$= \frac{4}{9} \lim_{z\to -1/3} \frac{3-z}{(z+3)^3} = \frac{4}{9} \times \frac{3+\frac{1}{3}}{\left(-\frac{1}{3}+3\right)^3} = \frac{5}{64},$$

after some simplification. Cauchy's residue theorem gives that

$$I = -i \times 2\pi i \times \left(\sum \text{Residues inside } C\right) = \frac{5\pi}{32}.$$

3. a) The complex function

$$F(z) = \frac{e^{imz}}{(z^2+4)^2}$$

has two double poles. Find the residue at the pole lying in the upper half of the complex plane. [5]

SOLUTION

The poles are at $z = \pm 2i$, with $z = 2i$ in the upper half-plane. The residue is obtained as

$$\lim_{z\to 2i} \frac{d}{dz}\left((z-2i)^2 \frac{e^{imz}}{(z^2+4)^2}\right) = \lim_{z\to 2i} \frac{d}{dz}\left(\frac{e^{imz}}{(z+2i)^2}\right)$$

$$= \lim_{z\to 2i} e^{imz}\left(\frac{im}{(z+2i)^2} - \frac{2}{(z+2i)^3}\right) = e^{im(2i)}\left(\frac{im}{(4i)^2} - \frac{2}{(4i)^3}\right) = -\frac{ie^{-2m}(2m+1)}{32},$$

after some simplification.

b) Consider the contour integral $I = \oint_C \frac{e^{imz}}{(z^2+4)^2}\, dz$,

where the closed contour C consists of a semi-circle in the complex upper half-plane, taken in the anti-clockwise sense, and $m > 0$.

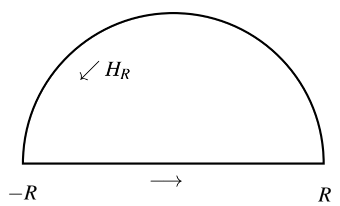
Using the result from (a), Cauchy's Residue Theorem and Jordan's lemma, show that

$$\int_{-\infty}^{\infty} \frac{\cos mx}{(x^2+4)^2}\, dx = \frac{(2m+1)\pi}{16} e^{-2m}.$$

[10]

SOLUTION

We use the contour C as the union of the semi-circular arc H_R^+ in the upper-half plane and the real interval $[-R, R]$:



The contour integral can be written in two parts:

$$I = \oint_C \frac{e^{imz}}{(z^2+4)^2}\,dz = \int_{H_R} \frac{e^{imz}}{(z^2+4)^2}\,dz + \int_{-R}^{R} \frac{e^{imx}}{(x^2+4)^2}\,dx$$

Using Jordan's lemma

$$\lim_{R\to\infty} \int_{H_R} \frac{e^{imz}}{(z^2+4)^2}\,dz = 0\,,$$

because:
(i) $m > 0\,,$
(ii) $\left|\frac{1}{(z^2+4)^2}\right| \to 0$ as $R \to \infty\,,$
(iii) all singularities are poles.

Using Cauchy's residue theorem,

$$I = \oint_C \frac{e^{imz}}{(z^2+4)^2}\,dz = 2\pi i \times \text{ sum of residues in the upper half-plane}$$

$$= \frac{(2m+1)\pi}{16} e^{-2m}$$

and taking the limit as $R \to \infty$ we have

$$\int_{-\infty}^{\infty} \frac{e^{imx}}{(x^2+4)^2}\,dx = \int_{-\infty}^{\infty} \frac{\cos mx + i \sin mx}{(x^2+4)^2}\,dx = \frac{\pi}{16}(2m+1)e^{-2m}\,,$$

and the sine part vanishes as it's the symmetric integral of an odd function, giving the required result.

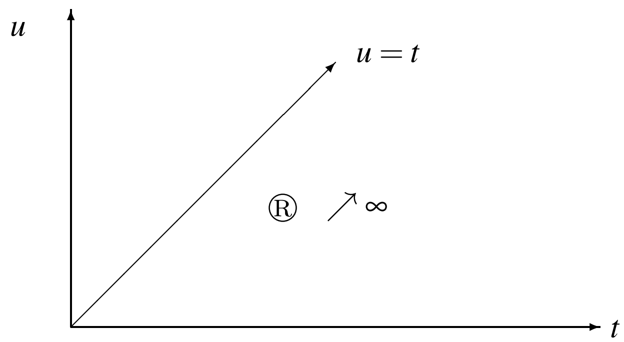
4. a) Two functions $f(t)$ and $g(t)$ have Laplace transforms $\overline{f}(s) = \mathscr{L}[f(t)]$ and
$\overline{g}(s) = \mathscr{L}[g(t)]$, respectively. If the convolution of $f(t)$ with $g(t)$ is defined as

$$f \star g = \int_0^t f(u)g(t-u)\,du\,,$$

prove that $\mathscr{L}[f \star g] = \overline{f}(s)\overline{g}(s)$. [5]

SOLUTION

Take the Laplace transform of the convolution product and exchange the order of the integrals, as in the below figure,



The region of integration is shown as Ⓡ.

$$\mathscr{L}\big(f \star g\big) = \int_0^\infty e^{-st}\left(\int_0^t f(u)g(t-u)\,du\right)dt = \int_0^\infty \left(\int_{t=u}^{t=\infty} e^{-st}g(t-u)\,dt\right) f(u)\,du$$

now substitute $\tau = t-u$

$$= \int_0^\infty \left(\int_{\tau=0}^\infty e^{-s(\tau+u)}g(\tau)\,d\tau\right) f(u)\,du = \int_0^\infty e^{-su}f(u)\,du \int_0^\infty e^{-s\tau}g(\tau)\,d\tau$$

$$= \overline{f}(s)\,\overline{g}(s)\,,$$

as required.

b) Use the Laplace convolution theorem to solve the second order ordinary differential equation

$$\frac{d^2x}{dt^2} + 9x = \sin 3t\,,$$

with initial conditions $x(0) = x'(0) = 0$. [10]

[Recall the identity $2\sin A \sin B = \cos(A-B) - \cos(A+B)$.]

SOLUTION

Taking transforms we have

$$s^2\bar{x} + 9\bar{x} = \frac{3}{s^2+9} \Rightarrow \bar{x}(s) = \frac{1}{3}\left(\frac{3}{s^2+9}\right)\left(\frac{3}{s^2+9}\right),$$

and using the convolution theorem we have $\overline{f}(s) = \overline{g}(s) = 3/(s^2+9)$ so that $f(t) = g(t) = \sin 3t$ and

$$x(t) = \frac{1}{3} f \star g = \frac{1}{3}\int_0^t \sin 3u \,\sin 3(t-u)\; du\,,$$

and using the trigonometric identity with $A = 3u$ and $B = 3(t-u)$ we have

$$\begin{aligned}
x(t) &= \frac{1}{6}\int_0^t \cos(6u-3t) - \cos 3t\; du \\
&= \frac{1}{6}\left[\frac{1}{6}\sin(6u-3t) - u\cos 3t\right]_0^t \\
&= \frac{1}{6}\left(\frac{1}{6}\sin 3t - \frac{1}{6}\sin(-3t) - t\cos 3t\right) \\
&= \frac{1}{18}\sin 3t - \frac{1}{6}t\cos 3t
\end{aligned}$$