E2.9 - MATHS 4
(EE - 2nd yr)

UNIVERSITY OF LONDON

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B.ENG. AND M.ENG. EXAMINATIONS 2005

For Internal Students of Imperial College

This paper is also taken for the relevant examination for the Associateship of the City & Guilds of London Institute

PART II : MATHEMATICS 4 (ELECTRICAL ENGINEERING)

Thursday 2nd June 2005   2.00 - 4.00 pm

Answer FOUR questions.

Answer questions from Section A and Section B in separate answerbooks.

A statistics data sheet is provided.

[Before starting, please make sure that the paper is complete; there should be 4 pages, with a total of 6 questions. Ask the invigilator for a replacement if your copy is faulty.]

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SECTION A

1. Find the eigenvalues and normalized eigenvectors of the matrix

$$A = \begin{pmatrix} 2 & \sqrt{6} & 0 \\ \sqrt{6} & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Using these, or otherwise, show that the matrix

$$P = \begin{pmatrix} \sqrt{\frac{3}{5}} & \sqrt{\frac{2}{5}} & 0 \\ \sqrt{\frac{2}{5}} & -\sqrt{\frac{3}{5}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

diagonalizes A such that

$$P^{-1}AP = \begin{pmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

2. Show that the quadratic form

$$Q = x_1^2 + 4x_1x_2 + 4x_2^2 + 6x_3^2$$

can be written as

$$Q = \boldsymbol{x}^T A \boldsymbol{x},$$

where $\boldsymbol{x} = (x_1, x_2, x_3)^T$ and A is a real symmetric matrix, which is to be found. Hence show that Q can be re-expressed in the diagonal form

$$Q = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2,$$

where the λ_i are to be determined, by finding a matrix P that satisfies $\boldsymbol{x} = P\boldsymbol{y}$ where $\boldsymbol{y} = (y_1, y_2, y_3)^T$. Find y_1, y_2 and y_3 in terms of x_1, x_2 and x_3 from the matrix P.

PLEASE TURN OVER

SECTION B

3 (i) Suppose that

$$P(A) = \frac{1}{2}, \quad P(B) = \frac{7}{12}, \quad P(C) = \frac{3}{4},$$

$$P(A \cap B) = P(A \cap C) = P(B \cap C) = \frac{1}{3} \text{ and } P(A \cap B \cap C) = \frac{1}{6}\,.$$

Calculate $P(B \cup C)$, $P(A \cup B)$, $P(A \,|\, B)$ and $P(A \cup B \,|\, B \cap C)$.

(ii) In a sequence of success/failure trials the probability of a success is p if the preceding trial was a success, and q if it was a failure. The initial state, preceding trial number 1, is taken to be a success.

What are the probabilities that

(a) trials 2 and 3 are both successes ?

(b) no two consecutive trials have the same outcome in trials 1 to 6?

4. A certain discrete distribution, in which the random variable N takes integer values $(0, 1, 2, \ldots)$, is defined by the system of equations

$$P(N = r + 1) \;=\; \frac{\lambda}{r+1}\, P(N = r) \quad \text{for} \quad r = 0, 1, 2, \ldots\,.$$

Why does this system of equations imply that $\lambda \geq 0$? What does $\lambda = 0$ imply about $P(N = r)$?

Derive the relationship

$$P(N = r + 1) \;=\; \frac{\lambda^2}{(r+1)r}\; P(N = r - 1)\,,$$

express $P(N = r)$ in terms of $P(N = 0)$, and derive an explicit expression for $P(N = r)$ in terms of r and λ.

Calculate $P(N = r \,|\, N > 0)$ and $E(N \,|\, N > 0)$.

5. The random variables Y_1 and Y_2 are independent, each generated from the distribution with density $f(y) = ye^{-y}$ on $(0, \infty)$. Write down the joint density of Y_1 and Y_2 and the conditional density of Y_2 given Y_1.

   Calculate $E(Y_1)$, $E(3Y_1 - 2Y_2)$ and $E(Y_1 Y_2)$.

   Derive the identity

$$P(Y_2 > 2Y_1) = \int_0^\infty y(2y+1)e^{-3y}dy .$$

   Given that this integral has value 7/27, calculate the probability that the larger of Y_1 and Y_2 exceeds twice the smaller of Y_1 and Y_2.

6. A random sample, $(y_1, \ldots, y_n)$, is available from the uniform distribution on $(0, \theta)$. Let $y_{min} = \min(y_1, \ldots, y_n)$ and $y_{max} = \max(y_1, \ldots, y_n)$ denote the sample minimum and maximum, respectively.

   Derive

$$P(y_{max} \le u) = (u/\theta)^n \qquad (0 \le u \le \theta) .$$

   Hence, write down the density function of y_{max} and use it to calculate $E(y_{max})$ and $\mathrm{var}(y_{max})$.

   Given that $E(y_{min}) = \theta/(n+1)$, compare the estimators y_{max} and ny_{min} of θ in terms of bias.

   Given also that $\mathrm{var}(y_{min}) = n\theta^2/\{(n+1)^2(n+2)\}$, calculate the ratio of variances, $\mathrm{var}(ny_{min})/\mathrm{var}(y_{max})$.

END OF PAPER

MATHEMATICS DEPARTMENT

MATHEMATICAL FORMULAE

1. VECTOR ALGEBRA

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1,\, a_2,\, a_3)$$

Scalar (dot) product: $\mathbf{a}\,.\,\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

Vector (cross) product:

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Scalar triple product:

$$[\mathbf{a},\, \mathbf{b},\, \mathbf{c}] = \mathbf{a}\,.\,\mathbf{b}\times\mathbf{c} = \mathbf{b}\,.\,\mathbf{c}\times\mathbf{a} = \mathbf{c}\,.\,\mathbf{a}\times\mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product: $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{c}\,.\,\mathbf{a})\mathbf{b} - (\mathbf{b}\,.\,\mathbf{a})\mathbf{c}$

2. SERIES

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \ldots \quad (\alpha \text{ arbitrary},\ |x|<1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots\,,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n\frac{x^{2n}}{(2n)!} + \ldots\,,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + \ldots\,,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^n\frac{x^{n+1}}{(n+1)} + \ldots (-1 < x \le 1)$$

3. TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin(a+b) = \sin a\, \cos b + \cos a\, \sin b\,;$$

$$\cos(a+b) = \cos a\, \cos b - \sin a\, \sin b$$

$$\cos iz = \cosh z\,;\quad \cosh iz = \cos z\,;\quad \sin iz = i\sinh z\,;\quad \sinh iz = i\sin z\,.$$

4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f\,D^n g + \binom{n}{1} Df\,D^{n-1}g + \ldots + \binom{n}{r} D^r f\,D^{n-r}g + \ldots + D^n f\,g\,.$$

(b) Taylor's expansion of $f(x)$ about $x=a$:

$$f(a+h) = f(a) + hf'(a) + h^2f''(a)/2! + \ldots + h^nf^{(n)}(a)/n! + \epsilon_n(h)\,,$$

where $\epsilon_n(h) = h^{n+1}f^{(n+1)}(a+\theta h)/(n+1)!, \quad 0<\theta<1\,.$

(c) Taylor's expansion of $f(x,\,y)$ about $(a,\,b)$:

$$f(a+h,\, b+k) = f(a,\,b) + [hf_x + kf_y]_{a.b} + 1/2!\left[h^2f_{xx} + 2hkf_{xy} + k^2f_{yy}\right]_{a.b} + \ldots$$

(d) Partial differentiation of $f(x,\,y)$:

i. If $y = y(x)$, then $f = F(x)$, and $\dfrac{dF}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dx}$.

ii. If $x = x(t)$, $y = y(t)$, then $f = F(t)$, and $\dfrac{dF}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$.

iii. If $x = x(u,\,v)$, $y = y(u,\,v)$, then $f = F(u,\,v)$, and

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}\,,\qquad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}\,.$$

(e) Stationary points of $f(x,\,y)$ occur where $f_x = 0$, $f_y = 0$ simultaneously.
Let $(a,\,b)$ be a stationary point: examine $D = [f_{xx}f_{yy} - (f_{xy})^2]_{a.b}$.
If $D>0$ and $f_{xx}(a,\,b)<0$, then $(a,\,b)$ is a maximum;
If $D>0$ and $f_{xx}(a,\,b)>0$, then $(a,\,b)$ is a minimum;
If $D<0$ then $(a,\,b)$ is a saddle-point.

(f) Differential equations:

i. The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating factor $I(x) = \exp[\int P(x)(dx)]$, so that $\dfrac{d}{dx}(Iy) = IQ$.

ii. $P(x,\,y)dx + Q(x,\,y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y$.

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5. INTEGRAL CALCULUS

(a) An important substitution: $\tan(\theta/2) = t$:
$\sin\theta = 2t/(1+t^2)$, $\cos\theta = (1-t^2)/(1+t^2)$, $d\theta = 2\,dt/(1+t^2)$.

(b) Some indefinite integrals:

$$\int (a^2 - x^2)^{-1/2} dx = \sin^{-1}\left(\frac{x}{a}\right), \quad |x| < a.$$

$$\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1}\left(\frac{x}{a}\right) = \ln\left\{\frac{x}{a} + \left(1 + \frac{x^2}{a^2}\right)^{1/2}\right\}.$$

$$\int (x^2 - a^2)^{-1/2} dx = \cosh^{-1}\left(\frac{x}{a}\right) = \ln\left|\frac{x}{a} + \left(\frac{x^2}{a^2} - 1\right)^{1/2}\right|.$$

$$\int (a^2 + x^2)^{-1} dx = \left(\frac{1}{a}\right)\tan^{-1}\left(\frac{x}{a}\right).$$

6. NUMERICAL METHODS

(a) Approximate solution of an algebraic equation:

If a root of $f(x) = 0$ occurs near $x = a$, take $x_0 = a$ and
$x_{n+1} = x_n - [f(x_n)/f'(x_n)]$, $n = 0, 1, 2 \ldots$

(Newton Raphson method).

(b) Formulae for numerical integration: Write $x_n = x_0 + nh$, $y_n = y(x_n)$.

i. Trapezium rule (1-strip): $\int_{x_0}^{x_1} y(x)dx \approx (h/2)[y_0 + y_1]$.

ii. Simpson's rule (2-strip): $\int_{x_0}^{x_2} y(x)dx \approx (h/3)[y_0 + 4y_1 + y_2]$.

(c) Richardson's extrapolation method: Let $I = \int_a^b f(x)dx$ and let I_1, I_2 be two estimates of I obtained by using Simpson's rule with intervals h and $h/2$.

Then, provided h is small enough,

$$I_2 + (I_2 - I_1)/15,$$

is a better estimate of I.

7. LAPLACE TRANSFORMS

Function	Transform	Function	Transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)dt$	$af(t) + bg(t)$	$aF(s) + bG(s)$
df/dt	$sF(s) - f(0)$	d^2f/dt^2	$s^2F(s) - sf(0) - f'(0)$
$e^{at}f(t)$	$F(s-a)$	$tf(t)$	$-dF(s)/ds$
$(\partial/\partial\alpha)f(t,\alpha)$	$(\partial/\partial\alpha)F(s,\alpha)$	$\int_0^t f(t)dt$	$F(s)/s$
$\int_0^t f(u)g(t-u)du$	$F(s)G(s)$		
1	$1/s$	$t^n (n = 1, 2 \ldots)$	$n!/s^{n+1}$, $(s > 0)$
e^{at}	$1/(s-a)$, $(s > a)$	$\sin\omega t$	$\omega/(s^2+\omega^2)$, $(s > 0)$
$\cos\omega t$	$s/(s^2+\omega^2)$, $(s > 0)$	$H(t-T) = \begin{cases} 0, & t < T \\ 1, & t > T \end{cases}$	e^{-sT}/s, $(s, T > 0)$

8. FOURIER SERIES

If $f(x)$ is periodic of period $2L$, then $f(x+2L) = f(x)$, and

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \text{ where}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad n = 0, 1, 2, \ldots, \text{ and}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1, 2, 3, \ldots$$

Parseval's theorem

$$\frac{1}{L}\int_{-L}^{L} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right).$$

1. Probabilities for events

For events A, B, and C $\quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$

More generally $\quad P(\bigcup A_i) = \sum P(A_i) - \sum P(A_i \cap A_j) + \sum P(A_i \cap A_j \cap A_k) - \cdots$

The odds in favour of A $\quad P(A) \,/\, P(\overline{A})$

Conditional probability $\quad P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$ provided that $P(B) > 0$

Chain rule $\quad P(A \cap B \cap C) = P(A)\,P(B \mid A)\,P(C \mid A \cap B)$

Bayes' rule $\quad P(A \mid B) = \dfrac{P(A)\,P(B \mid A)}{P(A)\,P(B \mid A) \;+\; P(\overline{A})\,P(B \mid \overline{A})}$

A and B are independent if $\quad P(B \mid A) = P(B)$

A, B, and C are independent if $\quad P(A \cap B \cap C) = P(A)P(B)P(C)$, and

$$P(A \cap B) = P(A)P(B), \quad P(B \cap C) = P(B)P(C), \quad P(C \cap A) = P(C)P(A)$$

2. Probability distribution, expectation and variance

The probability distribution for a discrete random variable X is the complete set of probabilities $\{p_x\} = \{P(X = x)\}$

Expectation $\quad E(X) = \mu = \sum_x x p_x$

Sample mean $\quad \overline{x} = \dfrac{1}{n} \sum_k x_k$ estimates μ from random sample $x_1, x_2, \ldots, x_n$

Variance $\quad \operatorname{var}(X) = \sigma^2 = E\{(X - \mu)^2\} = E(X^2) - \mu^2$, where $E(X^2) = \sum_x x^2 p_x$

Sample variance $\quad s^2 = \dfrac{1}{n-1} \left\{ \sum_k x_k^2 - \dfrac{1}{n} \left(\sum_j x_j \right)^2 \right\}$ estimates σ^2

Standard deviation $\quad \operatorname{sd}(X) = \sigma$

If value y is observed with frequency n_y

$$n = \sum_y n_y, \quad \sum_k x_k = \sum_y y n_y, \quad \sum_k x_k^2 = \sum_y y^2 n_y$$

For function $g(x)$ of x, $\quad E\{g(X)\} = \sum_x g(x) p_x$

Skewness $\quad \beta_1 = E\left(\dfrac{X - \mu}{\sigma}\right)^3$ is estimated by $\dfrac{1}{n-1} \sum \left(\dfrac{x_i - \overline{x}}{s}\right)^3$

Kurtosis $\quad \beta_2 = E\left(\dfrac{X - \mu}{\sigma}\right)^4 - 3$ is estimated by $\dfrac{1}{n-1} \sum \left(\dfrac{x_i - \overline{x}}{s}\right)^4 - 3$

Sample median $\tilde{x}$. If the sample values $x_1, \ldots, x_n$ are ordered $x_{(1)} \le x_{(2)} \le \cdots \le x_{(n)}$

$$\tilde{x} = x_{(\frac{n+1}{2})} \text{ if } n \text{ is odd, and } \tilde{x} = \tfrac{1}{2}\left(x_{(\frac{n}{2})} + x_{(\frac{n+2}{2})}\right) \text{ if } n \text{ is even.}$$

α-quantile $Q(\alpha)$ is such that $P(X \le Q(\alpha)) = \alpha$

Sample α-quantile $\widehat{Q}(\alpha)$ is the sample value for which the proportion of values $\le \widehat{Q}(\alpha)$ is α (using linear interpolation between values on either side)

The sample median $\tilde{x}$ estimates the population median $Q(0.5)$.

3. Probability distribution for a continuous random variable

The cumulative distribution function (cdf) $\quad F(x) = P(X \le x) = \int_{x_0=-\infty}^{x} f(x_0)\mathrm{d}x_0$

The probability density function (pdf) $\quad f(x) = \dfrac{\mathrm{d}F(x)}{\mathrm{d}x}$

$$E(X) = \mu = \int_{-\infty}^{\infty} x\, f(x)\mathrm{d}x, \quad \mathrm{var}(X) = \sigma^2 = E(X^2) - \mu^2,$$

where $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\mathrm{d}x$

4. Discrete probability distributions

Discrete Uniform *Uniform* (n)

$$p_x = \frac{1}{n} \quad (x = 1, 2, \ldots, n) \qquad \mu = \tfrac{1}{2}(n+1), \quad \sigma^2 = \tfrac{1}{12}(n^2 - 1)$$

Binomial distribution *Binomial* (n, θ)

$$p_x = \binom{n}{x} \theta^x (1-\theta)^{n-x} \quad (x = 0, 1, 2, \ldots, n) \qquad \mu = n\theta, \quad \sigma^2 = n\theta(1-\theta)$$

Poisson distribution *Poisson* (λ)

$$p_x = \frac{\lambda^x e^{-\lambda}}{x!} \quad (x = 0, 1, 2, \ldots) \quad (\text{with } \lambda > 0) \qquad \mu = \lambda, \quad \sigma^2 = \lambda$$

Geometric distribution *Geometric* (θ)

$$p_x = (1-\theta)^{x-1}\theta \quad (x = 1, 2, 3, \ldots) \qquad \mu = \frac{1}{\theta}, \quad \sigma^2 = \frac{1-\theta}{\theta^2}$$

5. Continuous probability distributions

Uniform distribution *Uniform* (α, β)

$$f(x) = \begin{cases} \dfrac{1}{\beta - \alpha} & (\alpha < x < \beta), \\ 0 & (\text{otherwise}). \end{cases} \qquad \mu = (\alpha + \beta)/2, \quad \sigma^2 = (\beta - \alpha)^2/12.$$

Exponential distribution $\mathit{Exponential}\,(\lambda)$

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & (0 < x < \infty), \\ 0 & (-\infty < x \le 0). \end{cases} \qquad \mu = 1/\lambda, \quad \sigma^2 = 1/\lambda^2.$$

Normal distribution $N\,(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2 \right\} \quad (-\infty < x < \infty)$$

$$E(X) = \mu, \quad \mathrm{var}\,(X) = \sigma^2$$

Standard normal distribution $N\,(0,1)$

If X is $N(\mu, \sigma^2)$, then $Y = \dfrac{X-\mu}{\sigma}$ is $N(0,1)$

6. $\boxed{\text{Reliability}}$

For a device in continuous operation with failure time random variable T having pdf $f(t)$ $(t > 0)$

The reliability function at time t	$R(t)$	$=$	$P(T > t)$		
The failure rate or hazard function	$h(t)$	$=$	$f(t)/R(t)$		
The cumulative hazard	$H(t)$	$=$	$\int_0^t h(t_0)\,\mathrm{d}t_0$	$=$	$-\ln\{R(t)\}$
The Weibull(α, β) distribution has	$H(t)$	$=$	βt^α		

7. $\boxed{\text{System reliability}}$

For a system of k devices, which operate independently, let

$R_i = P(D_i) = P(\text{"device } i \text{ operates"})$

The system reliability, R, is the probability of a path of operating devices

A system of devices in series operates only if every device operates

$R = P(D_1 \cap D_2 \cap \cdots \cap D_k) = R_1 R_2 \cdots R_k$

A system of devices in parallel operates if any device operates

$R = P(D_1 \cup D_2 \cup \cdots \cup D_k) = 1 - (1 - R_1)(1 - R_2)\cdots(1 - R_k)$

8. $\boxed{\text{Covariance and correlation}}$

The covariance of X and Y $\quad \mathrm{cov}\,(X,Y) = E(XY) - \{E(X)\}\{E(Y)\}$

From pairs of observations $(x_1, y_1), \ldots, (x_n, y_n)$ $\quad S_{xy} = \sum_k x_k y_k - \frac{1}{n}(\sum_i x_i)(\sum_j y_j)$

$$S_{xx} = \sum_k x_k^2 - \frac{1}{n}(\sum_i x_i)^2, \qquad S_{yy} = \sum_k y_k^2 - \frac{1}{n}(\sum_j y_j)^2$$

Sample covariance $\quad s_{xy} = \frac{1}{n-1} S_{xy}$ estimates $\operatorname{cov}(X,Y)$

Correlation coefficient $\quad \rho = \operatorname{corr}(X,Y) = \frac{\operatorname{cov}(X,Y)}{\operatorname{sd}(X) \cdot \operatorname{sd}(Y)}$

Sample correlation coefficient $\quad r = \frac{S_{xy}}{\sqrt{S_{xx} S_{yy}}}$ estimates ρ

9. Sums of random variables

$$E(X+Y) = E(X) + E(Y)$$

$$\operatorname{var}(X+Y) = \operatorname{var}(X) + \operatorname{var}(Y) + 2\operatorname{cov}(X,Y)$$

$$\operatorname{cov}(aX+bY,\ cX+dY) = (ac)\operatorname{var}(X) + (bd)\operatorname{var}(Y) + (ad+bc)\operatorname{cov}(X,Y)$$

If X is $N(\mu_1, \sigma_1^2)$, Y is $N(\mu_2, \sigma_2^2)$, and $\operatorname{cov}(X,Y) = c$,
then $X+Y$ is $N(\mu_1+\mu_2,\ \sigma_1^2+\sigma_2^2+2c)$

10 Bias, standard error, mean square error

If t estimates θ (with random variable T giving t)

Bias of t $\quad \operatorname{bias}(t) = E(T) - \theta$

Standard error of t $\quad \operatorname{se}(t) = \operatorname{sd}(T)$

Mean square error of t $\quad \mathrm{MSE}(t) = E\{(T-\theta)^2\} = \{\operatorname{se}(t)\}^2 + \{\operatorname{bias}(t)\}^2$

If $\bar{x}$ estimates μ, then $\operatorname{bias}(\bar{x}) = 0$, $\operatorname{se}(\bar{x}) = \sigma/\sqrt{n}$, $\mathrm{MSE}(\bar{x}) = \sigma^2/n$, $\widehat{\operatorname{se}}(\bar{x}) = s/\sqrt{n}$

Central limit property $\quad$ if n is fairly large, $\bar{x}$ is from $N(\mu,\ \sigma^2/n)$ approximately

11 Likelihood

The likelihood is the joint probability as a function of the unknown parameter θ.
For a random sample $x_1, x_2, \ldots, x_n$

$$\ell(\theta;\ x_1, x_2, \ldots, x_n) = P(X_1 = x_1 \mid \theta) \cdots P(X_n = x_n \mid \theta) \quad \text{(discrete distribution)}$$

$$\ell(\theta;\ x_1, x_2, \ldots, x_n) = f(x_1 \mid \theta) f(x_2 \mid \theta) \cdots f(x_n \mid \theta) \quad \text{(continuous distribution)}$$

The maximum likelihood estimator (MLE) is $\widehat{\theta}$ for which the likelihood is a maximum.

12. **Confidence intervals**

If $x_1, x_2, \ldots, x_n$ are a random sample from $N(\mu, \sigma^2)$ and σ^2 is known, then the 95% confidence interval for μ is $(\overline{x} - 1.96\frac{\sigma}{\sqrt{n}},\ \overline{x} + 1.96\frac{\sigma}{\sqrt{n}})$

If σ^2 is estimated, then from the Student t table for t_{n-1} we find $t_0 = t_{n-1,0.05}$

The 95% confidence interval for μ is $(\overline{x} - t_0\frac{s}{\sqrt{n}},\ \overline{x} + t_0\frac{s}{\sqrt{n}})$

13. **Standard normal table** Values of pdf $\phi(y) = f(y)$ and cdf $\Phi(y) = F(y)$

y	$\phi(y)$	$\Phi(y)$	y	$\phi(y)$	$\Phi(y)$	y	$\phi(y)$	$\Phi(y)$	y	$\Phi(y)$
0	.399	.5	.9	.266	.816	1.8	.079	.964	2.8	.997
.1	.397	.540	1.0	.242	.841	1.9	.066	.971	3.0	.998
.2	.391	.579	1.1	.218	.864	2.0	.054	.977	0.841	.8
.3	.381	.618	1.2	.194	.885	2.1	.044	.982	1.282	.9
.4	.368	.655	1.3	.171	.903	2.2	.035	.986	1.645	.95
.5	.352	.691	1.4	.150	.919	2.3	.028	.989	1.96	.975
.6	.333	.726	1.5	.130	.933	2.4	.022	.992	2.326	.99
.7	.312	.758	1.6	.111	.945	2.5	.018	.994	2.576	.995
.8	.290	.788	1.7	.094	.955	2.6	.014	.995	3.09	.999

14. **Student t table** Values $t_{m,p}$ of x for which $P(|X| > x) = p$, when X is t_m

	p	.10	.05	.02	0.01		p	.10	.05	.02	0.01
m	1	6.31	12.71	31.82	63.66	m	9	1.83	2.26	2.82	3.25
	2	2.92	4.30	6.96	9.92		10	1.81	2.23	2.76	3.17
	3	2.35	3.18	4.54	5.84		12	1.78	2.18	2.68	3.05
	4	2.13	2.78	3.75	4.60		15	1.75	2.13	2.60	2.95
	5	2.02	2.57	3.36	4.03		20	1.72	2.09	2.53	2.85
	6	1.94	2.45	3.14	3.71		25	1.71	2.06	2.48	2.78
	7	1.89	2.36	3.00	3.50		40	1.68	2.02	2.42	2.70
	8	1.86	2.31	2.90	3.36		∞	1.645	1.96	2.326	2.576

15. Chi-squared table

Values $\chi^2_{k,p}$ of x for which $P(X > x) = p$, when X is χ^2_k and $p = .995,\ .975,$ *etc*

k	.995	.975	.05	.025	.01	.005	k	.995	.975	.05	.025	.01	.005
1	.000	.001	3.84	5.02	6.63	7.88	18	6.26	8.23	28.87	31.53	34.81	37.16
2	.010	.051	5.99	7.38	9.21	10.60	20	7.43	9.59	31.42	34.17	37.57	40.00
3	.072	.216	7.81	9.35	11.34	12.84	22	8.64	10.98	33.92	36.78	40.29	42.80
4	.207	.484	9.49	11.14	13.28	14.86	24	9.89	12.40	36.42	39.36	42.98	45.56
5	.412	.831	11.07	12.83	15.09	16.75	26	11.16	13.84	38.89	41.92	45.64	48.29
6	.676	1.24	12.59	14.45	16.81	18.55	28	12.46	15.31	41.34	44.46	48.28	50.99
7	.990	1.69	14.07	16.01	18.48	20.28	30	13.79	16.79	43.77	46.98	50.89	53.67
8	1.34	2.18	15.51	17.53	20.09	21.95	40	20.71	24.43	55.76	59.34	63.69	66.77
9	1.73	2.70	16.92	19.02	21.67	23.59	50	27.99	32.36	67.50	71.41	76.15	79.49
10	2.16	3.25	13.31	20.48	23.21	25.19	60	35.53	40.48	79.08	83.30	88.38	91.95
12	3.07	4.40	21.03	23.34	26.22	28.30	70	43.28	48.76	90.53	95.02	100.4	104.2
14	4.07	5.63	23.68	26.12	29.14	31.32	80	51.17	57.15	101.9	106.6	112.3	116.3
16	5.14	6.91	26.30	28.85	32.00	34.27	100	67.33	74.22	124.3	129.6	135.8	140.2

16. The chi-squared goodness-of-fit test

The frequencies n_y are grouped so that the fitted frequency $\widehat{n}_y$ for every group exceeds about 5.

$X^2 = \sum_y \frac{(n_y - \widehat{n}_y)^2}{\widehat{n}_y}$ is referred to the table of χ^2_k with significance point p,

where k is the number of terms summed, less one for each constraint, *eg* matching total frequency, and matching $\overline{x}$ with μ.

17 Joint probability distributions

Discrete distribution $\{p_{xy}\}$, where $p_{xy} = P(\{X = x\} \cap \{Y = y\})$.

Let $p_{x\bullet} = P(X = x)$, and $p_{\bullet y} = P(Y = y)$, then

$$p_{x\bullet} = \sum_y p_{xy}, \quad \text{and} \quad P(X = x \,|\, Y = y) = \frac{p_{xy}}{p_{\bullet y}}.$$

Continuous distribution

Joint cdf $\quad F(x,y) = P(\{X \le x\} \cap \{Y \le y\}) = \int_{x_0=-\infty}^{x} \int_{y_0=-\infty}^{y} f(x_0, y_0)\, \mathrm{d}x_0\, \mathrm{d}y_0$

Joint pdf $\quad f(x,y) = \dfrac{\mathrm{d}^2 F(x,y)}{\mathrm{d}x\, \mathrm{d}y}$

Marginal pdf of X $\quad f_X(x) = \int_{-\infty}^{\infty} f(x, y_0)\, \mathrm{d}y_0$

Conditional pdf of X given $Y = y$ $\quad f_{X|Y}(x|y) = \dfrac{f(x,y)}{f_Y(y)}$ (provided $f_Y(y) > 0$)

18. Linear regression

To fit the linear regression model $y = \alpha + \beta x$ by $\widehat{y}_x = \widehat{\alpha} + \widehat{\beta} x$ from observations $(x_1, y_1), \ldots, (x_n, y_n)$, the least squares fit is

$$\widehat{\alpha} = \overline{y} - \overline{x}\widehat{\beta}, \quad \widehat{\beta} = S_{xy}/S_{xx}$$

The residual sum of squares $\quad \text{RSS} = S_{yy} - \dfrac{S_{xy}^2}{S_{xx}}$

$$\widehat{\sigma^2} = \frac{\text{RSS}}{n-2}, \qquad \frac{n-2}{\sigma^2}\widehat{\sigma^2} \text{ is from } \chi^2_{n-2}$$

$$E(\widehat{\alpha}) = \alpha, \quad E(\widehat{\beta}) = \beta,$$

$$\operatorname{var}(\widehat{\alpha}) = \frac{\sum x_i^2}{n\, S_{xx}}\sigma^2, \quad \operatorname{var}(\widehat{\beta}) = \frac{\sigma^2}{S_{xx}}, \quad \operatorname{cov}(\widehat{\alpha}, \widehat{\beta}) = -\frac{\overline{x}}{S_{xx}}\sigma^2$$

$$\widehat{y}_x = \widehat{\alpha} + \widehat{\beta} x, \qquad E(\widehat{y}_x) = \alpha + \beta x, \qquad \operatorname{var}(\widehat{y}_x) = \left\{ \frac{1}{n} + \frac{(x-\overline{x})^2}{S_{xx}} \right\} \sigma^2$$

$$\frac{\widehat{\alpha} - \alpha}{\widehat{\text{se}}\,(\widehat{\alpha})}, \quad \frac{\widehat{\beta} - \beta}{\widehat{\text{se}}\,(\widehat{\beta})}, \quad \frac{\widehat{y}_x - \alpha - \beta x}{\widehat{\text{se}}\,(\widehat{y}_x)} \quad \text{are each from } t_{n-2}$$

19. Design matrix for factorial experiments $\quad$ With 3 factors each at 2 levels

$$\mathbf{X} = \begin{pmatrix} 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & \sqrt{6} & 0 \\ \sqrt{6} & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$\lambda_3 = 3$ $\quad \begin{vmatrix} 2-\lambda & \sqrt{6} \\ \sqrt{6} & 1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 3\lambda - 4 = 0$ 3

$\lambda_1 = 4 \;;\quad \lambda_2 = -1 \quad \lambda_3 = 3$

$\lambda_1 = 4: \begin{pmatrix} -2 & \sqrt{6} & 0 \\ \sqrt{6} & -3 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix} = 0 \quad p = q\sqrt{3/2},\ \underline{a}_1 = \sqrt{\tfrac{2}{5}}\begin{pmatrix} \sqrt{3/2} \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{3/5} \\ \sqrt{2/5} \\ 0 \end{pmatrix}$ 4

$\lambda_2 = -1 \begin{pmatrix} 3 & \sqrt{6} & 0 \\ \sqrt{6} & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix} = 0 \quad q = -p\sqrt{\tfrac{3}{2}},\ \underline{a}_2 = \sqrt{\tfrac{2}{5}}\begin{pmatrix} 1 \\ -\sqrt{\tfrac{3}{2}} \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{2/5} \\ -\sqrt{3/5} \\ 0 \end{pmatrix}$ 4

$\lambda_3 = 3 \qquad \underline{a}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ 1

Choose P such that

$$P = (\underline{a}_1\ \underline{a}_2\ \underline{a}_3)$$

$$\therefore\ AP = (\lambda_1 \underline{a}_1\ \lambda_2 \underline{a}_2\ \lambda_3 \underline{a}_3)$$

$= P\Lambda \qquad \Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$ 6

$$\therefore\ P^{-1}AP = \underline{\Lambda} = \begin{pmatrix} 4 & & 0 \\ & -1 & \\ 0 & & 3 \end{pmatrix}$$

$P = \begin{pmatrix} \sqrt{3/5} & \sqrt{2/5} & 0 \\ \sqrt{2/5} & -\sqrt{3/5} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ 2

Acceptable to write out $P^{-1}AP$ longhand but time-consuming

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$$Q = x_1^2 + 4x_1x_2 + 4x_2^2 + 6x_3^2$$

$$= \underline{x}^T A \underline{x} \qquad A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

$\therefore (\lambda-1)(\lambda-4)-4=0 \quad \lambda(\lambda-5)=0 \Rightarrow \lambda_1 = 5, \ \lambda_2 = 0, \ \lambda_3 = 6$

$\lambda_3 = 6$ 3

$\lambda_1 = 5 \quad \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = 0 \quad q = 2p \quad \underline{a}_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}\frac{1}{\sqrt{5}}$ 3

$\lambda_2 = 0 \quad \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = 0 \quad p = -2q \quad \underline{a}_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}\frac{1}{\sqrt{5}}$ 3

$\lambda_3 = 6 \quad \underline{a}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ 1

$\therefore$ Write $P = (\underline{a}_1\ \underline{a}_2\ \underline{a}_3) = \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} & 0 \\ 2/\sqrt{5} & 1/\sqrt{5} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ 3

Choose $\underline{x} = P\underline{y}$

$\therefore \quad Q = \underline{x}^T A \underline{x} = \underline{y}^T (P^T A P)\underline{y}$

$AP = P\Lambda \Rightarrow P^{-1}AP = \Lambda$

$P^T A P = \Lambda$ 5

$\therefore \quad Q = \underline{y}^T \Lambda \underline{y} = 5y_1^2 + 6y_3^2$

$\underline{x} = P\underline{y} \Rightarrow \underline{y} = P^{-1}\underline{x} = P^T\underline{x} = \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} & 0 \\ -2/\sqrt{5} & 1/\sqrt{5} & 0 \\ 0 & 0 & 1 \end{pmatrix}\underline{x}$

$\therefore \sqrt{5}\, y_1 = x_1 + x_2$

$\sqrt{5}\, y_2 = -2x_1 + x_2$ 2

$y_3 = x_3$

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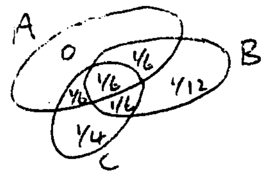
PAPER II(4) E

QUESTION 3

SOLUTION

3.



(i) $P(B \cup C) = 1$, 2

$P(A \cup B) =$ 3/4, 2

$P(A \mid B) = \frac{1}{3} \div \frac{7}{12} = \frac{4}{7}$, 3

$P(A \cup B \mid B \cap C) = P\{(A \cup B) \cap B \cap C) \div P(B \cap C) = \frac{1}{3} \div \frac{1}{3} = 1.$ 4

(ii) (a) $P(sss) + P(fss) = p^3 + (1-p)qp.$ 4

(b) $P(sfsfsf) + P(fsfsfs)$

$= p(1-p)q(1-p)q(1-p) + (1-p)q(1-p)q(1-p)q$

$= (p+q)(1-p)^3q^2.$ 5

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QUESTION

4

SOLUTION

4.

$\lambda \geq 0$ because $P(N = r+1) \geq 0$ and $P(N = r)/(r+1) \geq 0$. 2

$\lambda = 0$ implies $P(N = r+1) = 0$ for $r = 0, 1, 2, \ldots$, so $P(N = 0) = 1$. 2

$P(N = r+1) = \frac{\lambda}{r+1} P(N = r) = \frac{\lambda}{r+1} \frac{\lambda}{r} P(N = r-1)$. 2

$P(N = r) = \frac{\lambda}{r} P(N = r-1) = \ldots = \frac{\lambda^r}{r!} P(N = 0)$ 2

$1 = \sum_{r=0}^{\infty} P(N = r) = \sum_{r=0}^{\infty} \frac{\lambda^r}{r!} P(N = 0) = e^{\lambda} P(N = 0)$,

so $P(N = r) = e^{-\lambda} \lambda^r / r!$. 4

$P(N = r \mid N > 0) = P(N = r \cap N > 0)/P(N = 0)$

$= (e^{-\lambda} \lambda^r / r!) \div (1 - e^{-\lambda})$ $(r = 1, 2, \ldots)$ 4

$E(N \mid N > 0) = \sum_{r=1}^{\infty} r (e^{-\lambda} \lambda^r / r!)/(1 - e^{-\lambda})$

$= (1 - e^{-\lambda})^{-1} e^{-\lambda} \lambda \sum_{r=1}^{\infty} \lambda^{r-1}/(r-1)! = \lambda/(1 - e^{-\lambda})$. 4

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QUESTION 5

SOLUTION

5.

joint density $f(y_1, y_2) = (y_1 e^{-y_1}) \times (y_2 e^{-y_2})$ (since independent) 2

conditional density $f(y_2 \mid y_1) = y_2 e^{-y_2}$ (since independent) 2

$\mathrm{E}(Y_1) = \int_0^\infty y_1^2 e^{-y_1} dy_1 = 2$ 2

$\mathrm{E}(3Y_1 - 2Y_2) = (3 \times 2) - (2 \times 2) = 2$ 2

$\mathrm{E}(Y_1 Y_2) = 2 \times 2 = 4$ (since independent) 2

$$\mathrm{P}(Y_2 > 2Y_1) = \int_0^\infty dy_1 \int_{2y_1}^\infty dy_2 \{y_1 e^{-y_1} y_2 e^{-y_2}\}$$

$$= \int_0^\infty \{[-y e^{-y}]_{2y_1}^\infty + \int_{2y_1}^\infty e^{-y} dy\} dy_1$$

$$= \int_0^\infty y_1 e^{-y_1} (2y_1 e^{-2y_1} + e^{-2y_1}) dy_1 = \textit{result given.}$$

6

$\mathrm{P}\{\max(Y_1, Y_2) > 2\min(Y_1, Y_2)\} = \mathrm{P}(Y_2 > 2Y_1) + \mathrm{P}(Y_1 > 2Y_2) = 14/27.$ 4

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QUESTION 6

SOLUTION

6

$\mathrm{P}(y_{max} \le u) = \mathrm{P}(y_1 \le u) \times \mathrm{P}(y_2 \le u) \times ... = (u/\theta)^n.$ 2

density fn $f(y_{max}) = \frac{d}{du}\mathrm{P}(y_{max} \le u) = n\theta^{-n}u^{n-1}.$ 2

$\mathrm{E}(y_{max}) = \int_0^\theta u \times n\theta^{-n}u^{n-1}du = n\theta/(n+1)$ 3

$\mathrm{E}(y_{max}^2) = \int_0^\theta u^2 \times n\theta^{-n}u^{n-1}du = n\theta^2/(n+2)$ 3

$\Rightarrow \quad \mathrm{var}(y_{max}) = \theta^2(\frac{n}{n+2} - \frac{n^2}{(n+1)^2}) = n\theta^2/\{(n+1)^2(n+2)\}.$ 2

estimators

bias $\mathrm{E}(y_{max}) - \theta = \theta(\frac{n}{n+1} - 1) = -\theta/(n+1)$ 2

bias $\mathrm{E}(ny_{min}) - \theta = \theta(\frac{n}{n+1} - 1) = -\theta/(n+1)$ 2

variance $\mathrm{var}(y_{max}) = n\theta^2/\{(n+1)^2(n+2)\}$ 2

variance $\mathrm{var}(ny_{min}) = n^2 \times n\theta^2/\{(n+1)^2(n+2)\}$ 2

so $ratio = n^2$

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