EE4-65

THE ANSWERS

Notations:

(a) B - Bookwork

(b) E - New example

(c) A - New application

1. a) User channels are orthogonal. We can simply apply

$$\mathbf{p}_1 = \begin{bmatrix} 1 & 1-j & 1 & 1+j \end{bmatrix}^T / \sqrt{6},$$

to match with user 1 channel and null out the multi-user interference.

[3 - E]

This is a (normalized) transmit matched filter w.r.t. the co-scheduled user channel. Given that the user channel are orthogonal the matched filter maximizes the SNR, nulls out the interference and acts as a ZFBF.

[3 - E]

b) i) (3) goes with (a) since (3) leads to the lowest diversity gain among the three schemes.

[1 - E]

(2) goes with (b) since (2) leads to the same diversity gain as (1) but benefit from an array gain originating from the CSIT.

[1 - E]

(1) goes with (c) since (1) leads to the same diversity gain as (2) but does not benefit from any array gain.

[1 - E]

ii) The average symbol error rate of (c) at high SNR can be upper bounded as $\bar{P} \leq \bar{N}_e \left(\frac{\rho d_{min}^2}{8}\right)^{-2}$ where $\bar{N}_e$ is teh number of nearest neighbors, ρ is teh average SNR and d_{min} is the minimum distance of the constellation.

[2 - B]

The average symbol error rate of (b) at high SNR can be upper bounded as $\bar{P} \leq \bar{N}_e \left(\frac{\rho d_{min}^2}{4}\right)^{-2}$.

[2 - B]

Comparing the two expressions, we see a factor of 1/2 difference that explains why (c) incurs a 3dB loss in terms of SNR compared to (b).

[1 - B]

c) i) The multiplexing gain is the pre-log factor of the ergodic capacity at high SNR, i.e. $g_s = \lim_{\rho\to\infty} \frac{C_{CDIT}}{\log_2(\rho)}$. Hence by increasing the SNR by 3dB (e.g. from 17dB to 20dB), the ergodic capacity increases by g_s bits/s/Hz.

(a) $g_s = 1$.

[1 - E]

(b) $g_s = 3$.

[1 - E]

(c) $g_s = 3$.

[1 - E]

(d) $g_s = 2$.

[1 - E]

(e) $g_s = 2$.

[1 - E]

ii) There are several possible configurations that satisfy to $n_r + n_t = 7$, namely 4×3, 3×4, 5×2, 2×5, 6×1 and 1×6. The matching between curves and antenna configurations is easily identified by using the following two arguments: 1) The multiplexing gain with CDIT at high SNR is given by $\min\{n_t, n_r\}$. 2) With CDIT only, the input covariance matrix in i.i.d. channel is $\mathbf{Q} = 1/n_t \mathbf{I}_{n_t}$. This implies that 4×3, 5×2 and 6×1 outperform 3×4, 2×5 and 1×6, respectively.

(a) $n_r \times n_t = 6 \times 1$ or 1×6

[1 - E]

(b) $n_r \times n_t = 4 \times 3$

[1 - E]

(c) $n_r \times n_t = 3 \times 4$

[1 - E]

(d) $n_r \times n_t = 5 \times 2$

[1 - E]

(e) $n_r \times n_t = 2 \times 5$

[1 - E]

d) i) The transmit correlation depends on the angle spread and inter-element spacing, the larger the angle spread and the larger the inter-element spacing, the lower the spatial correlation. Hence scenario 1 would lead to a higher spatial correlation than scenario 2.

[2 - A]

ii) At high SNR, the error rate performance relates to the diversity gain. An increase in the spatial correlation is harmful from a diversity gain persepctive. Hence, the larger the spatial correlation, the higher the error rate of O-STBC. Scenario 2 would therefore leads to a lower error rate.

[3 - A]

iii) The average output SNR is given by $\bar{\rho} = \rho \mathcal{E}\left\{\|\mathbf{h}\|_F^2\right\}$. We can write

$$\begin{aligned}
\mathcal{E}\left\{\|\mathbf{h}\|_F^2\right\} &= \mathcal{E}\left\{\mathbf{h}\mathbf{h}^H\right\} \\
&= \mathcal{E}\left\{\mathbf{h}_w \mathbf{R}_t^{1/2} \mathbf{R}_t^{H/2} \mathbf{h}_w^H\right\} \\
&= \mathcal{E}\left\{\mathrm{Tr}\left\{\mathbf{R}_t^{H/2} \mathbf{h}_w^H \mathbf{h}_w \mathbf{R}_t^{1/2}\right\}\right\} \\
&= \mathrm{Tr}\left\{\mathbf{R}_t^{H/2} \mathcal{E}\left\{\mathbf{h}_w^H \mathbf{h}_w\right\} \mathbf{R}_t^{1/2}\right\} \\
&= \mathrm{Tr}\left\{\mathbf{R}_t^{H/2} \mathbf{R}_t^{1/2}\right\} \\
&= \mathrm{Tr}\left\{\mathbf{R}_t\right\} = n_t
\end{aligned}$$

Hence both scenarios will lead to the same array gain. [3 - A]

e) i) The conditional PEP writes as

$$P(\mathbf{C} \to \mathbf{E} \mid \mathbf{H}) = Q\left(\sqrt{\frac{\rho}{2}\|\mathbf{H}(\mathbf{C}-\mathbf{E})\|_F^2}\right),$$

which can be upper bounded using the Chernoff bound as

$$P(\mathbf{C} \to \mathbf{E} \mid \mathbf{H}) \leq e^{-\frac{\rho}{4}\|\mathbf{H}(\mathbf{C}-\mathbf{E})\|_F^2}.$$

[2 - B]

The average PEP over Rayleigh slow fading channels is

$$P(\mathbf{C} \to \mathbf{E}) = \mathcal{E}_{\mathbf{H}}\left\{P(\mathbf{C} \to \mathbf{E}|\mathbf{H})\right\} \leq M_\Gamma(-1)\, d\beta$$

where $M_\Gamma(\gamma)$ moment generating function (MGF) of

$$\Gamma = \frac{\rho}{4}\|\mathbf{H}(\mathbf{C}-\mathbf{E})\|_F^2.$$

[2 - B]

Since

$$\|\mathbf{H}(\mathbf{C}-\mathbf{E})\|_F^2 = \mathrm{Tr}\left\{\mathbf{H}\tilde{\mathbf{E}}\mathbf{H}^H\right\} = \mathrm{vec}\left(\mathbf{H}^H\right)^H \left(\mathbf{I}_{n_r} \otimes \tilde{\mathbf{E}}\right) \mathrm{vec}\left(\mathbf{H}^H\right)$$

where $\tilde{\mathbf{E}} = (\mathbf{C}-\mathbf{E})(\mathbf{C}-\mathbf{E})^H$, we can find from hermitian quadratic form of complex Gaussian random variables

$$\begin{aligned}
P(\mathbf{C} \to \mathbf{E}) &\leq \left[\det\left(\mathbf{I}_{n_t} + \frac{\eta}{4}\tilde{\mathbf{E}}\right)\right]^{-n_r} \\
&= \left(1 + \frac{\rho}{4n_t}\sum_{q=1}^{n_t}\left|c_q - e_q\right|^2\right)^{-n_r},
\end{aligned}$$

where c_q, e_q are symbols chosen in the QAM constellation.

[2 - B]

ii) At high SNR,

$$P(\mathbf{C} \to \mathbf{E}) \leq \left(\frac{\rho}{4n_t}\right)^{-n_r}\left(\sum_{q=1}^{n_t}\left|c_q - e_q\right|^2\right)^{-n_r}.$$

The exponent of the SNR relates to the diversity gain. This highlights a diversity gain of n_r. [2 - B]

2. a) Define $\mathsf{h}_q = \Lambda_q^{-1/2} h_q / \sigma_{n,q}$, $q = 1,2$. Assume $|\mathsf{h}_1|^2 \geq |\mathsf{h}_2|^2$. The achievable rates are

$$R_1 \leq \log_2\left(1 + |\mathsf{h}_1|^2 s_1\right)$$

$$R_2 \leq \log_2\left(1 + \frac{|\mathsf{h}_2|^2 s_2}{1 + |\mathsf{h}_2|^2 s_1}\right)$$

subject to $s_1 + s_2 = E_s$.

[3 - B]

The sum-rate writes as

$$\begin{aligned} R_1 + R_2 &\leq \log_2\left(1 + |\mathsf{h}_1|^2 s_1\right) + \log_2\left(1 + \frac{|\mathsf{h}_2|^2 s_2}{1 + |\mathsf{h}_2|^2 s_1}\right) \\ &= \log_2\left(1 + |\mathsf{h}_1|^2 s_1\right) - \log_2\left(1 + |\mathsf{h}_2|^2 s_1\right) + \log_2\left(1 + |\mathsf{h}_2|^2 [s_2 + s_1]\right) \\ &= \log_2\left(1 + |\mathsf{h}_1|^2 s_1\right) - \log_2\left(1 + |\mathsf{h}_2|^2 s_1\right) + \log_2\left(1 + |\mathsf{h}_2|^2 E_s\right). \end{aligned}$$

It should be clear from the last expression that the power s_1 that maximizes the difference $\log_2\left(1 + |\mathsf{h}_1|^2 s_1\right) - \log_2\left(1 + |\mathsf{h}_2|^2 s_1\right)$ is given by $s_1 = E_s$ since $|\mathsf{h}_1|^2 > |\mathsf{h}_2|^2$. Hence, the sum-rate capacity of the SISO BC is achieved by allocating the transmit power to the strongest user and

$$C_{BC} = \log_2\left(1 + E_s |\mathsf{h}_1|^2\right).$$

[3 - B]

b) The channel is diagonal. The capacity over the deterministic channel writes as

$$C(\mathbf{H}) = \max_{P_1, P_2}\left(\log_2\left(1 + \frac{P_1}{\sigma_{n,1}^2}|a|^2\right) + \log_2\left(1 + \frac{P_2}{\sigma_{n,2}^2}|b|^2\right)\right)$$

with $P_1 + P_2 = P$. The optimal power allocation is given by the water-filling solution

$$P_1^\star = \left(\mu - \frac{\sigma_{n,1}^2}{|a|^2}\right)^+, \quad P_2^\star = \left(\mu - \frac{\sigma_{n,2}^2}{|b|^2}\right)^+$$

with μ computed such that $P_1^\star + P_2^\star = P$.

[2 - A]

Assuming $P_1^\star$ and $P_2^\star$ are positive, $\mu = \frac{P}{2} + \frac{1}{2}\left(\frac{\sigma_{n,1}^2}{|a|^2} + \frac{\sigma_{n,2}^2}{|b|^2}\right)$. If $\mu - \frac{\sigma_{n,2}^2}{|b|^2} \leq 0$, $P_2^\star = 0$ and $P_1^\star = P$. The capacity writes as

$$C(\mathbf{H}) = \log_2\left(1 + \frac{P}{\sigma_{n,1}^2}|a|^2\right).$$

[2 - A]

If $\mu - \frac{\sigma_{n,2}^2}{|b|^2} > 0$, $P_1^\star = \frac{P}{2} - \frac{\sigma_{n,1}^2}{2|a|^2} + \frac{\sigma_{n,2}^2}{2|b|^2}$ and $P_2^\star = \frac{P}{2} + \frac{\sigma_{n,1}^2}{2|a|^2} - \frac{\sigma_{n,2}^2}{2|b|^2}$. The capacity writes as

$$C(\mathbf{H}) = \log_2\left(1 + \frac{P_1^\star}{\sigma_{n,1}^2}|a|^2\right) + \log_2\left(1 + \frac{P_2^\star}{\sigma_{n,2}^2}|b|^2\right).$$

[2 - A]

c) By the distance-product criterion, the diversity gain is given by $n_r L_{min} = n_r \min l_{\mathbf{C},\mathbf{E}}$ where $\min l_{\mathbf{C},\mathbf{E}}$ refers to the minimum effective length over all possible non-zero error matrices.

[3 - A]

Given the four codewords, given the circulant behavior of the four codewords, the error matrices will have no zero entries, therefore leading to effective lengths of 4. Hence, $l_{\mathbf{x},\mathbf{y}} = 4$, for $\mathbf{x},\mathbf{y} = \mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}$ with $\mathbf{x} \neq \mathbf{y}$, leading to $L_{min} = 4$ and a total diversity gain of 4 over fast fading channels.

[3 - A]

d) i) The capacity region of the two-user MAC with a single transmit antenna at the two transmitters is a pentagon.

[2 - A]

Hence on the edge where the single-user rate constraint of user 2 is the tighter constraint, an increase of user 1 rate does not lead to a decrease of user 2 rate, i.e. user 2 can still transmit at its single user rate and user 1 enjoys a non-zero rate.

[2 - A]

However in the region where the sum-rate constraint is the tighter constraint, any increase in user 1 rate leads to a decrease of user 2 rate.

[2 - A]

ii) We have by Jensen's inequality

$$\mathscr{E}\{f(X)\} \geq f(\mathscr{E}\{X\})$$

for a convex function and therefore

$$\mathscr{E}\{f(X)\} \leq f(\mathscr{E}\{X\})$$

for a concave function.

[3 - A]

The log function is a concave function. Hence, we have

$$\mathscr{E}\left\{\log_2\left(1+\rho\,|h|^2\right)\right\} \leq \log_2\left(1+\rho\mathscr{E}\left\{|h|^2\right\}\right) = \log_2(1+\rho).$$

The sign should therefore be written in the reversed order.

[3 - A]

3. a) The received signal of terminal 1 in cell 1 writes as

$$\mathbf{y} = \mathbf{h}_{1,1}c_1 + \mathbf{h}_{1,2}c_2 + \mathbf{h}_{1,3}c_3 + \mathbf{h}_{1,4}c_4 + \mathbf{n}$$

where $\mathbf{y} =$ is the $[n_r \times 1]$ received signal at receiver 1, $\mathbf{h}_{1,i}$ is the channel between transmitter i and receiver 1, c_i are independent symbols with power P, i.e. $\mathscr{E}\left\{|c_i|^2\right\} = P$. For stream 1, the first term is the intended signal, the second term refers to the intra-cell interference and the third/fourth term refer to the inter-cell interference.

[5-A]

b) i) $n_{r,min} = 2$ corresponding to a 2-dimensional space because one dimension will be used to detect stream 1 and one dimension to zero-force stream 2.

[3-A]

ii) Assuming $n_r = 2$, the choice is unique and consists designing the combiner $\mathbf{g}_1$ such that

$$\mathbf{g}_1 \mathbf{h}_{1,2} = 0$$

so that

$$\mathbf{g}_1 \mathbf{y} = \mathbf{g}_1 \mathbf{h}_{1,1}c_1 + \mathbf{g}_1 \mathbf{h}_{1,3}c_3 + \mathbf{g}_1 \mathbf{h}_{1,4}c_4 + \mathbf{g}_1 \mathbf{n}.$$

Since $\mathbf{h}_{1,2}$ is 2×1 vector, we can find a single vector such that $\mathbf{g}_1 \mathbf{h}_{1,2} = 0$. Writing $\mathbf{h}_{1,2} = \left[\begin{array}{cc} a & b \end{array}\right]^T$, we choose $\mathbf{g}_1 = \left[\begin{array}{cc} -b & a \end{array}\right] / \sqrt{a^2 + b^2}$.

[4-A]

iii) Assuming $n_r > n_{r,min}$, we have multiple choices to null the interference and the best precoder would be such that it also maximizes the SNR under the constraint it nulls the interference. We can write

$$\mathbf{h}_{1,2} = \left[\begin{array}{cc} \mathbf{U}' & \tilde{\mathbf{U}} \end{array}\right] \mathbf{S} \mathbf{V}^H$$

where $\tilde{\mathbf{U}}$ is the matrix containing the left singular vectors corresponding to the null singular values. There are $n_r - 1$ of those. By choosing $\mathbf{g}_1$ as any linear combination of the rows of $\tilde{\mathbf{U}}^H$ would null the interference, since

$$\tilde{\mathbf{U}}^H \mathbf{y} = \tilde{\mathbf{U}}^H \mathbf{h}_{1,1}c_1 + \tilde{\mathbf{U}}^H \mathbf{h}_{1,3}c_3 + \tilde{\mathbf{U}}^H \mathbf{h}_{1,4}c_4 + \tilde{\mathbf{U}}^H \mathbf{n}.$$

In order to maximize the SNR, we should further matched to $\tilde{\mathbf{U}}^H \mathbf{h}_{1,1}$. This gives $\mathbf{g}_1 = \mathbf{h}_{1,1}^H \tilde{\mathbf{U}} \tilde{\mathbf{U}}^H$. This can be further normalized to guarantee $\|\mathbf{g}_1\|^2 = 1$.

[2-A]

The rate achieved by stream 1 is given as

$$R_1 = \log_2\left(1 + \frac{|\mathbf{g}_1 \mathbf{h}_{1,1}|^2 P}{|\mathbf{g}_1 \mathbf{h}_{1,3}|^2 P + |\mathbf{g}_1 \mathbf{h}_{1,4}|^2 P + \|\mathbf{g}_1\|^2 \sigma_n^2}\right).$$

[2-A]

iv) As P increase, the SINR saturates to $\frac{|\mathbf{g}_1 \mathbf{h}_{1,1}|^2}{|\mathbf{g}_1 \mathbf{h}_{1,3}|^2 + |\mathbf{g}_1 \mathbf{h}_{1,4}|^2}$, leading to a saturation of the rate and therefore a multiplexing gain of 0.

[4-A]

c) i) To null both intra and inter-cell interference, $n'_{r,min} = 4$ as there are 3 interfering streams.

[3-A]

ii) Assuming $n_r = 4$, there is a single choice for the combiner $\mathbf{g}_1'$. We can write

$$\begin{bmatrix} \mathbf{h}_{1,2} & \mathbf{h}_{1,3} & \mathbf{h}_{1,4} \end{bmatrix} = \begin{bmatrix} \mathbf{U}' & \tilde{\mathbf{u}} \end{bmatrix} \mathbf{S}\mathbf{V}^H$$

where $\tilde{\mathbf{u}}$ is a $n_r \times 1$ vector. We can simply choose $\mathbf{g}_1' = \tilde{\mathbf{u}}^H$.

[2-A]

Stream 1 does ot experience any interference and its rate is

$$R_1 = \log_2\left(1 + \frac{|\mathbf{g}_1\mathbf{h}_{1,1}|^2 P}{\|\mathbf{g}_1\|^2 \sigma_n^2}\right).$$

[2-A]

iii) In the high SNR regime (i.e. P large), since the SINR of stream 1 is not limited by interference, the rate will scale with P such that the multiplexing gain is equal to 1.

[3-A]