DIGITAL SIGNAL PROCESSING AND DIGITAL FILTERS

*********** Solutions ***********

Information for Candidates:

Where a question requires a numerical answer, it must be given as a fully evaluated decimal number and not as an unevaluated arithmetic expression.

Notation

- All signals and filter coefficients are real-valued unless explicitly noted otherwise.
- Unless otherwise specified, upper and lower case letters are used for sequences and their z-transforms respectively. The signal at a block diagram node V is $v[n]$ and its z-transform is $V(z)$.
- $x[n] = [a, b, c, d, e, f]$ means that $x[0] = a, \ldots x[5] = f$ and that $x[n] = 0$ outside this range.
- $\Re(z)$, $\Im(z)$, z^*, $|z|$ and $\angle z$ denote respectively the real part, imaginary part, complex conjugate, magnitude and argument of a complex number z.
- The expected value of x is denoted $E\{x\}$.
- In block diagrams: solid arrows denote the direction of signal flow; an open triangle denotes a gain element with the gain indicated adjacently; a "+" in a circle denotes an adder/subtractor whose inputs may be labelled "+" or "−" according to their sign; the sample rate of a signal may be indicated in the form "@f".

Abbreviations

BIBO	Bounded Input, Bounded Output
CTFT	Continuous-Time Fourier Transform
DCT	Discrete Cosine Transform
DFT	Discrete Fourier Transform
DTFT	Discrete-Time Fourier Transform
FIR	Finite Impulse Response

IIR	Infinite Impulse Response
LTI	Linear Time-Invariant
MDCT	Modified Discrete Cosine Transform
PSD	Power Spectral Density
SNR	Signal-to-Noise Ratio

A datasheet is included at the end of the examination paper.

********* Questions and Solutions *********

1. a) A finite-length complex exponential signal is given by $x[n] = e^{j\omega n}$ for $n \in [0, N-1]$. The DFT of $x[n]$ satisfies

$$|X[k]| = \frac{\left|\sin \frac{2\pi k - N\omega}{2}\right|}{\left|\sin \frac{2\pi k - N\omega}{2N}\right|}.$$

i) By using the approximation $\sin\theta \approx \theta$ for $|\theta| < 0.2$ rad, show that $|X[k]|$ is approximately bounded by $2\left(\frac{2\pi k}{N} - \omega\right)^{-1}$ for a suitable range of k. Give the range of k for which this bound applies and explain the significance of the term: $\left(\frac{2\pi k}{N} - \omega\right)$. [4]

The argument of sin *in the denominator is "small" if* $\frac{2\pi k - N\omega}{2N} < 0.2 \Leftrightarrow \left|k - \frac{\omega N}{2\pi}\right| < \frac{0.4N}{2\pi} = \frac{N}{15.7}$. *Within this range,* $|X[n]| \approx \left|\sin \frac{2\pi k - N\omega}{2}\right| \times \frac{2N}{2\pi k - N\omega} = \left|\sin \frac{2\pi k - N\omega}{2}\right| \times 2\left(\frac{2\pi k}{N} - \omega\right)^{-1}$. *This proves the required result since the* sin *term is bounded by* 1. *Since* $X[k]$ *corresponds to a frequency of* $\frac{2\pi k}{N}$, *the term in parentheses gives the distance that a spectral component, k, is away from the frequency* ω *in rad/sample. Thus, we have shown that, when using a rectangular window, the spectral leakage falls as* $\frac{1}{k-k_0}$ *where* $k_0 = \frac{\omega N}{2\pi}$.

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ii) Explain why it is customary to multiply a signal, $x[n]$, by a window before performing a DFT and explain the tradeoffs that affect the choice of window function. [3]

By multiplying $x[n]$ *by a window before taking the DFT, we can reduce the spectral leakage either by making it decay faster or by reducing its amplitude. The principal tradeoffs are between, (a) the width of the main lobe (which determines how much spectral components are broadened), (b) the amplitude of the maximum sidelobe and (c) the rate at which the sidelobe peaks decay with* $|k - k_0|$.

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b) i) Explain what is meant by saying that a linear time invariant system is "BIBO stable". [2]

An LTI system is BIBO stable if a bounded input sequence, $x[n]$, *always gives a bounded output sequence,* $y[n]$. *That is,*

$$|x[n]| \leq B \,\forall n \Rightarrow |y[m]| \leq f(B) < \infty \,\forall m$$

for some function $f(B)$.

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ii) Prove that if a linear time invariant system is BIBO stable, then its impulse response, $h[n]$, satisfies $\sum_{n=-\infty}^{+\infty} |h[n]| < \infty$. [3]

Define

$$x[n] = \begin{cases} +1 & h[-n] \geq 0 \\ -1 & h[-n] < 0 \end{cases}.$$

This clearly satisfies $|x[n]| \leq 1\ \forall n$, so it follows from the BIBO condition that $|y[m]| \leq K < \infty\ \forall m$ for some fixed K. In particular, $y[0] = \sum_{r=-\infty}^{+\infty} h[r]x[0-r] = \sum_{r=-\infty}^{+\infty} |h[r]| \leq K < \infty$.

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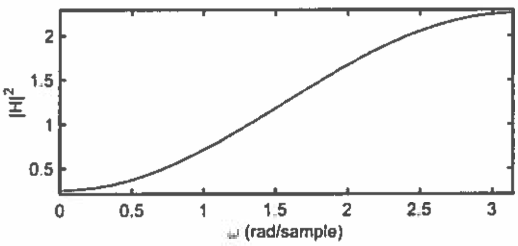
c) A first-order FIR filter is given by $H(z) = 1 - 0.5z^{-1}$.

i) Determine a simplified expression for the squared magnitude response, $|H(e^{j\omega})|^2$, and sketch its graph for $\omega \in [0, \pi]$. [4]

Since $H(z) = 1 - 0.5z^{-1}$, $H(e^{j\omega}) = 1 - 0.5e^{-j\omega}$ and

$$\begin{aligned}|H(e^{j\omega})|^2 &= H(e^{j\omega})H(e^{-j\omega}) = \left(1 - 0.5e^{-j\omega}\right)\left(1 - 0.5e^{j\omega}\right) \\ &= 1 - \cos\omega + 0.25 = 1.25 - \cos\omega.\end{aligned}$$

Its graph is



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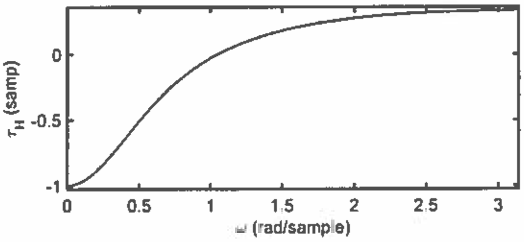
ii) Using the formula in the datasheet, or otherwise, determine the group delay of the filter, $\tau_H(e^{j\omega})$, and sketch its graph for $\omega \in [0, \pi]$. [4]

From the datasheet,

$$\begin{aligned}\tau_H(e^{j\omega}) &= \Re\left(\frac{\mathscr{F}(nh[n])}{\mathscr{F}(h[n])}\right) \\ &= \Re\left(\frac{0 - 0.5e^{-j\omega}}{1 - 0.5e^{-j\omega}}\right) = \Re\left(\frac{-e^{-j\omega}}{2 - e^{-j\omega}}\right) \\ &= \Re\left(\frac{-e^{-j\omega}\left(2 - e^{j\omega}\right)}{\left(2 - e^{-j\omega}\right)\left(2 - e^{j\omega}\right)}\right) \\ &= \Re\left(\frac{1 - 2e^{-j\omega}}{4 - 4\cos\omega + 1}\right) \\ &= \frac{1 - 2\cos\omega}{5 - 4\cos\omega}\end{aligned}$$

The graph is



with particular points of interest at $\tau_H(e^{j0}) = -1$, $\tau_H(e^{j\frac{\pi}{3}}) = 0$, $\tau_H(e^{j\frac{\pi}{2}}) = 0.2$ *and* $\tau_H(e^{j\pi}) = 0.333$.

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d) In the block diagram of Figure 1.1, all elements are drawn with their outputs on the right. The input and output signals are $x[n]$ and $y[n]$ respectively.

i) Determine the transfer function of the system, $H(z) = \frac{Y(z)}{X(z)}$. [3]

We can write

$$W = X - z^{-1}aW \Rightarrow \left(1 + az^{-1}\right)W = X \Rightarrow W = \frac{1}{1 + az^{-1}}X$$

$$Y = z^{-1}W + aW = \left(a + z^{-1}\right)W = \frac{a + z^{-1}}{1 + az^{-1}}X$$

Although not requested, this is a first order allpass filter.

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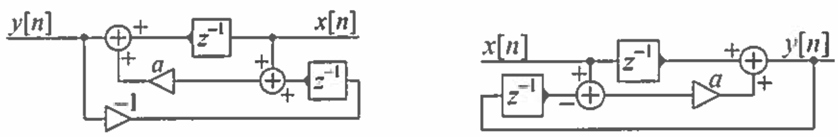
ii) Draw the transposed form of the block diagram. [4]

The transposed block diagram is obtained by reversing the direction of all elements and interchanging junctions and adders. This gives the left diagram which can be re-drawn to give the right diagram (in which the gain of -1 has been absorbed into the adder that follows).



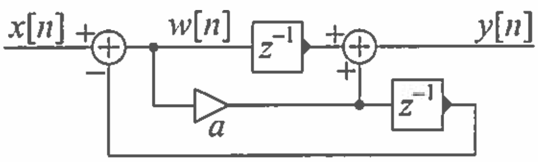


Figure 1.1

c) i) If a bounded discrete-time signal, $x[n]$, is stationary ergodic then its average power (i.e. the average energy per sample) is equal to $E\{x^2[n]\}$ for any n. Explain why the average power of such a signal is unchanged by downsampling. [3]

Downsampling by K retains only every K^{th} sample. If the signal is stationary ergodic, then all samples have the same average power,. $E\{x^2[n]\}$, and so retaining only every K^{th} sample leaves the average power unchanged.

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ii) Figure 1.2 shows the power spectral density (PSD) of a real-valued stationary ergodic signal, $x[n]$; the horizontal portions of the PSD have values 1 or 4.

The signal $y[m] = x[3m]$ is obtained by downsampling $x[n]$ by a factor of 3. Draw a dimensioned sketch of the PSD of $y[m]$ giving the values of all horizontal portions of the graph and the values of all frequencies at which there is a discontinuity in the PSD. [4]

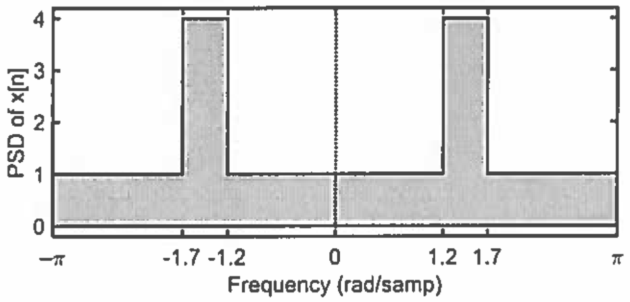
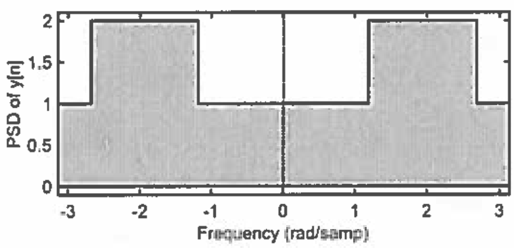


Figure 1.2

Downsampling by 3 multiplies all the frequencies by 3 and so $\{-1.7, -1.2, 1.2, 1.7\}$ becomes $\{-5.1, -3.6, 3.6, 5.1\}$. Since these values are outside the range $\pm\pi$, we add/subtract 2π to alias them into the correct range. Thus they become s $\{1.18, 2.68, -2.68, -1.18\}$.

We can regard the original signal as the sum of a broadband signal with a PSD of 1 and a band-limited signal with a PSD of 3. The PSD of the broadband component will remain unchanged while that of the bandlimited component will be divided by 3 (so that its total power remains unchanged). Thus the resultant PSD is



The horizontal portions of the graph have values of 1 or 2 and the discontinuities are at $\omega = \{\pm 1.18, \pm 2.68\}$.

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f) i) In the block diagram of Figure 1.3 the input is $x[m]$ and the output is $y[n]$. Determine a simplified expression for $Y(z)$ in terms of $X(z)$ and the filters $H_p(z)$ for $p \in [0, 2]$. [3]

From the datasheet, $V_p(z) = H_p(z^3)X(z^3)$. So we can write

$$\begin{aligned}
Y(z) &= V_0(z) + z^{-1}V_1(z) + z^{-2}V_2(z) \\
&= H_0(z^3)X(z^3) + z^{-1}H_1(z^3)X(z^3) + z^{-2}H_2(z^3)X(z^3) \\
&= \left(H_0(z^3) + z^{-1}H_1(z^3) + z^{-2}H_2(z^3)\right)X(z^3)
\end{aligned}$$

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ii) If $H_p(z) = \sum_{m=0}^{M} h_p[m]z^{-m}$, derive an expression for $g[n]$ in terms of the $h_p[m]$ so that the block diagram of Figure 1.4 is equivalent to that of Figure 1.3. [3]

From the diagram (and using the datasheet) $Y(z) = G(z)X(z^3)$. Thus we need to have $G(z) = H_0(z^3) + z^{-1}H_1(z^3) + z^{-2}H_2(z^3)$.

By writing $n = 3m + p$ where $m = \lfloor \frac{n}{3} \rfloor$ and $p = n - 3m \in [0, 2]$, we can write

$$\begin{aligned}
G(z) &= \sum_{n=0}^{3M+2} g[n]z^{-n} \\
&= \sum_{p=0}^{2}\sum_{m=0}^{M} g[3m+p]z^{-3m-p} \\
&= \sum_{p=0}^{2} z^{-p} \sum_{m=0}^{M} g[3m+p]z^{-3m} \\
&= \sum_{p=0}^{2} z^{-p} \sum_{m=0}^{M} h_p[m]z^{-3m} \\
&= \sum_{p=0}^{2} z^{-p} H_p(z^3)
\end{aligned}$$

This is of the required form with $g[3m+p] = h_p[m]$.

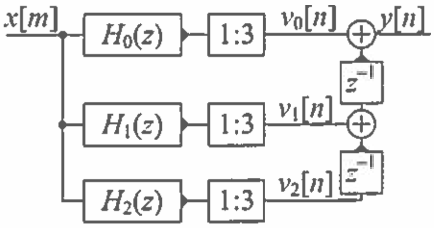

Figure 1.3

Figure 1.4

2. a) Outline the relative advantages of the bilinear and impulse-invariant transformations for converting a continuous-time filter into a discrete-time filter. [2]

The bilinear mapping preserves both the magnitude and phase of the frequency response exactly but at the expense of a non-linear transformation of the frequency axis. In contrast, the impulse-invariant transformation preserves an undistorted frequency axis but introduces aliasing into the frequency response.

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b) If p is a complex-valued constant, show that the z-transform of the causal sequence $v[n] = e^{pn}$ is given by $V(z) = \left(1 - e^{p}z^{-1}\right)^{-1}$ and give its region of convergence. [3]

From the datasheet we have (summing from $n = 0$ since $v[n]$ is causal),

$$\begin{aligned} V(z) &= \sum_{n=0}^{\infty} v[n]z^{-n} \\ &= \sum_{n=0}^{\infty} e^{pn}z^{-n} \\ &= \frac{1}{1 - e^{p}z^{-1}} \end{aligned}$$

where, from the datasheet, the last line is true provided that $\left|e^{p}z^{-1}\right| < 1$ which is equivalent to $|z| > |e^{p}|$ for the ROC.

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c) For $t \geq 0$, the impulse response of the causal continuous-time filter $H(s) = \frac{\Omega_0^2}{(s+\alpha)^2+\Omega_0^2}$ is given by

$$\begin{aligned} h(t) &= \Omega_0 e^{-\alpha t}\sin(\Omega_0 t) \\ &= -0.5j\Omega_0 e^{-\alpha t}\left(e^{j\Omega_0 t} - e^{-j\Omega_0 t}\right). \end{aligned}$$

i) Use the result of part b) to find a simplified expression for the z-transform, $G(z)$, of the causal sequence given by $g[n] = T \times h(nT)$ where T is the sample period. Express $G(z)$ as a ratio of polynomials in z^{-1}. [6]

We have

$$\begin{aligned} g[n] &= T \times h(nT) \\ &= -0.5jT\Omega_0 e^{-\alpha Tn}\left(e^{j\Omega_0 Tn} - e^{-j\Omega_0 Tn}\right) \\ &= -0.5jT\Omega_0\left(e^{(-\alpha T+j\Omega_0 T)n} - e^{(-\alpha T-j\Omega_0 T)n}\right). \end{aligned}$$

From the result of part b), we can write

$$
\begin{aligned}
G(z) &= -0.5jT\Omega_0\left(\frac{1}{1-e^{-\alpha T+j\Omega_0 T}z^{-1}} - \frac{1}{1-e^{-\alpha T-j\Omega_0 T}z^{-1}}\right)\\
&= -0.5jT\Omega_0\left(\frac{e^{-\alpha T+j\Omega_0 T}z^{-1}-e^{-\alpha T-j\Omega_0 T}z^{-1}}{(1-e^{-\alpha T+j\Omega_0 T}z^{-1})(1-e^{-\alpha T-j\Omega_0 T}z^{-1})}\right)\\
&= -0.5jT\Omega_0 z^{-1}e^{-\alpha T}\left(\frac{e^{j\Omega_0 T}-e^{-j\Omega_0 T}}{1-e^{-\alpha T}\left(e^{j\Omega_0 T}+e^{-j\Omega_0 T}\right)z^{-1}+e^{-2\alpha T}z^{-2}}\right)\\
&= \frac{\Omega_0 T e^{-\alpha T}\sin(\Omega_0 T)z^{-1}}{1-2e^{-\alpha T}\cos(\Omega_0 T)z^{-1}+e^{-2\alpha T}z^{-2}}
\end{aligned}
$$

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ii) If $T = 10^{-4}$ s, $\Omega_0 = 5000$ rad/s and $\alpha = 800\,\mathrm{s}^{-1}$, give the numerical values of the coefficients of $G(z)$ to 3 decimal places after normalizing to make the leading denominator coefficient unity. [3]

Substituting the given values into the above formula gives

$$
\begin{aligned}
\Omega_0 T &= 0.5\\
e^{-\alpha T} &= 0.923\\
G(z) &= \frac{0.221z^{-1}}{1-1.620z^{-1}+0.852z^{-2}}.
\end{aligned}
$$

So the numerator and denominator coefficients are $[0, 0.221]$ *and* $[1, -1.620, 0.852]$.

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d) i) Show that, under the mapping $s = \kappa\frac{z-1}{z+1}$, the value $s = j\Omega_0$ corresponds to $z = e^{j\omega_0}$ where $\Omega_0 = \kappa\tan(0.5\omega_0)$. Determine the numerical value of κ such that $\omega_0 = \Omega_0 T$ when T and Ω_0 have the values given in part c)ii). [4]

Substituting $\Omega_0 = \kappa\tan(0.5\omega_0)$ *into the mapping equation gives*

$$
\begin{aligned}
j\Omega_0 &= j\kappa\frac{\sin 0.5\omega_0}{\cos 0.5\omega_0}\\
&= \kappa\frac{e^{j0.5\omega_0}-e^{-j0.5\omega_0}}{e^{j0.5\omega_0}+e^{-j0.5\omega_0}}\\
&= \kappa\frac{e^{j\omega_0}-1}{e^{j\omega_0}+1} = \kappa\frac{z-1}{z+1}.
\end{aligned}
$$

Substituting the given values to determine κ *gives*

$$
\begin{aligned}
\kappa &= \frac{\Omega_0}{\tan(0.5\Omega_0 T)}\\
&= \frac{5000}{\tan 0.25}\\
&= \frac{5000}{0.255} = 19582.
\end{aligned}
$$

ii) Use the bilinear mapping from part d)i) to transform the filter $H(s)$ from part c) into a discrete time filter, $F(z)$, and give the numerical values of its coefficients to 3 decimal places after normalizing to make the leading denominator coefficient unity. [6]

Writing $\bar{\kappa} = \frac{\kappa}{\Omega_0}$ and $\alpha = \frac{\alpha}{\Omega_0}$, we have

$$\begin{aligned}
F(z) &= \frac{\Omega_0^2}{(\bar{\kappa}\Omega_0\frac{z-1}{z+1} + \bar{\alpha}\Omega_0)^2 + \Omega_0^2} \\
&= \frac{(z+1)^2}{(\bar{\kappa}(z-1) + \bar{\alpha}(z+1))^2 + (z+1)^2} \\
&= \frac{z^2+2z+1}{((\bar{\alpha}+\bar{\kappa})z + (\bar{\alpha}-\bar{\kappa}))^2 + (z+1)^2} \\
&= \frac{z^2+2z+1}{((\bar{\alpha}+\bar{\kappa})^2+1)z^2 + 2(\bar{\alpha}^2-\bar{\kappa}^2+1)z + ((\bar{\alpha}-\bar{\kappa})^2+1)}.
\end{aligned}$$

Substituting in the values gives

$$\begin{aligned}
\bar{\kappa} &= \frac{19582}{5000} = 3.916 \\
\bar{\alpha} &= \frac{800}{5000} = 0.16 \\
F(z) &= \frac{z^2+2z+1}{((0.16+3.916)^2+1)z^2 + 2(0.16^2-3.916^2+1)z + ((0.16-3.916)^2+1)} \\
&= \frac{z^2+2z+1}{(16.616+1)z^2 + 2(0.026-15.338+1)z + (14.110+1)} \\
&= \frac{z^2+2z+1}{17.616z^2 - 28.624z + 15.110}.
\end{aligned}$$

Thus the numerator and denominator filter coefficients are [1, 2, 1] *and* [17.616, −28.624, 15.110] *or, after normalizing,* [0.057, 0.1135, 0.057] *and* [1, −1.625, 0.858] .

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e) Using the values given in part c)ii), determine the pole and zero positions of $H(s)$, $G(z)$ and $F(z)$ and comment on their relationship to the properties of the three filters. [6]

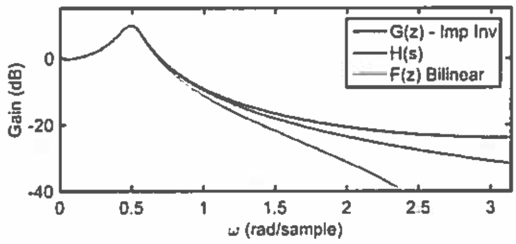
For $H(s) = \frac{\Omega_0^2}{(s+\alpha)^2+\Omega_0^2}$, *we have a complex conjugate pair of poles at* $-\alpha \pm j\Omega_0 = -800 \pm j5000$.

For $G(z) = \frac{0.221z^{-1}}{1-1.620z^{-1}+0.852z^{-2}} = \frac{0.221z}{z^2-1.620z+0.852}$, *we have a zero at* $z=0$ *and poles at* $z = \frac{1.62\pm\sqrt{2.624-3.408}}{2} = 0.810 \pm j0.443 = 0.923\angle\pm 0.5$.

For $F(z) = \frac{z^2+2z+1}{17.616(z^2-1.625z+0.858)}$ *we have a double zero at* $z=-1$ *and poles at* $z = \frac{1.625\pm\sqrt{2.640-3.431}}{2} = 0.812 \pm j0.445 = 0.926\angle\pm 0.5$.

Thus we see that $G(z)$ and $H(s)$ both have a single complex pole pair (excluding the pole at $z = 0$ which affects only the phase response) which, by construction, gives rise to identical impulse responses consisting of an exponentially decaying sine wave. $F(z)$ also has a complex pole pair at almost the same place but, in addition, has a double zero at $z = 0$ which means that $F(e^{j\pi}) = 0$ unlike $G(e^{j\pi}) = \frac{-0.221}{1+1.62+0.852} = -0.0637$.

From the magnitude response plot shown below (not requested) we see that the responses are very similar at low frequencies but differ substantially at high frequencies.



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3. The FM radio baseband spectrum shown in Figure 3.1 comprises (i) a mono signal (L+R) with a bandwidth of 15 kHz, (ii) a 19 kHz pilot tone and (iii) stereo information (L–R) modulated on a suppressed 38 kHz subcarrier. To demodulate the stereo component it is necessary to regenerate the 38 kHz subcarrier by isolating the 19 kHz pilot tone and multiplying its frequency by 2. The baseband signal is sampled at $f_s = 200\,\text{kHz}$.

All filters in this question are lowpass FIR filters with a stopband attenuation of 60 dB whose order may be estimated using the datatsheet formula $M = \frac{60}{3.5\Delta\omega}$ where $\Delta\omega$ is the transition bandwidth in rad/sample.

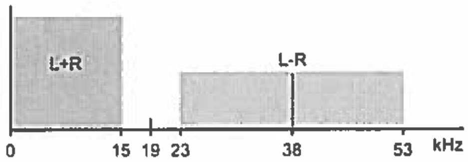

Figure 3.1

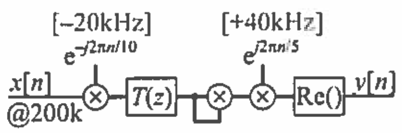

Figure 3.2

a) A block diagram for obtaining the 38 kHz subcarrier, $y[n]$, is shown in Figure 3.2 in which complex-valued signal paths are shown as bold lines. The baseband FM signal, $x[n]$, is translated down in frequency by 20 kHz and lowpass filtered by $T(z)$ to isolated the pilot tone component. The output of $T(z)$ is squared and translated up in frequency by 40 kHz and then the subcarrier, $y[n]$, is obtained by taking the real part of the signal.

i) The pilot tone component of $x[n]$ is given by $x_p[n] = \cos\omega_p n$ and has a frequency of $\omega_p = 2\pi \times \frac{19}{200} = 0.597\,\text{rad/sample}$.

Give the signed frequencies, in rad/sample, of the complex exponential components of the pilot tone signal at each stage of the processing, i.e. for each horizontal line segment in Figure 3.2. [3]

At successive nodes along the signal path, the true frequency is

$$\{\pm 19, -39\,\& -1, -1, -2, 38, \pm 38\}\ \text{kHz}.$$

To obtain the normalized frequencies, we multiply these values by $\frac{2\pi}{f_s} = 3.14 \times 10^{-5}$ *to obtain*

$\{\pm 0.597, -1.225\,\& -0.031, -0.031, -0.063, 1.194, \pm 1.194\}$ rad/sample.

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ii) Determine the passband edge frequency and the width of the transition band, $\Delta\omega$, for the lowpass filter, $T(z)$. Hence determine the required FIR filter order using the formula given at the start of the question. [3]

The lowpass filter must pass the pilot tone at $-1\,\text{kHz}$ *but must block the nearby signal components at* $\{15, 23\} - 20 = \{-5, 3\}$. *Hence the transition band is* $[1, 3]\,\text{kHz}$ *and its width is* $2\,\text{kHz} = 0.063\,\text{rad/sample}$. *Substituting this into the given formula (and rounding up) gives* $M = 273$.

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iii) Explain why squaring the output of $T(z)$ doubles the frequency of the pilot tone component. [3]

Following, $T(z)$, the pilot tone components is $e^{j\omega_p n}$ (assuming that the passband gain of $T(z)$ is $1\angle 0$). Squaring this gives $\left(e^{j\omega_p n}\right)^2 = e^{j2\omega_p n}$ which is a complex exponential with twice the frequency.

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iv) Estimate the number of real multiplications per second needed to implement the block diagram of Figure 3.2. [3]

Multiplications per input sample are: 2 for the frequency downshift, $2 \times 274 = 548$ for $T(z)$, 3 for squaring a complex number and 2 for the frequency upshift (since only the real part is required). The total is therefore 555 per input sample or 1.11×10^8 per second.

[Question continues on next page]

b) An alternative block diagram for generating the 38 kHz subcarrier is shown in Figure 3.3 in which $T(z)$ has been replaced by a lowpass filter, $G(z)$, operating at a sample frequency of 10 kHz.

i) Explain the reason that the lowpass filters $F(z)$ and $H(z)$ are needed. [2]

The antialiasing filter, $F(z)$, eliminates any signal components above the new Nyquist frequency of $0.5K^{-1}f_s$ to prevent aliasing. The reconstruction filter, $H(z)$, with the same cutoff frequency eliminates the image components that are introduced by the upsampling.

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ii) Determine the passband edge, transition band width and filter order for each of the lowpass filters $F(z)$, $G(z)$ and $H(z)$. [6]

The intermediate sample frequency is 10 kHz. *The antialiasing filter, $F(z)$, must pass the pilot tone at* -1 kHz $= -0.597$ rad/s *but must block anything that might alias onto this frequency, e.g.* $-1 \pm \frac{f_s}{K} = -11$ *or* $+9$ kHz. *Thus, the passband edge is* 1 kHz $= 0.031$ rad/s *and the transition band width is* $9-1 = 8$ kHz $= 0.251$ rad/s *giving* $M_F = 68.3 \approx 69$. *This has increased by a factor of slightly more than 2 because of the larger decimation factor.*

For $G(z)$, the passband edge is 1 kHz $= 0.628$ rad/s *and the transition band width is, as before,* 2 kHz $= 1.257$ rad/s *giving* $M_H = 13.6 \approx 14$. *This has reduced by a factor of 2 because of the lower intermediate sample frequency.*

For $H(z)$, the pilot tone is now at 2 kHz *so the passband edge is now* 2 kHz $= 0.063$ rad/s *and the nearest image frequency is at* 8 kHz *giving a transition band width of* $8-2 = 6$ kHz $= 0.188$ rad/s *and a filter order of* $M_G = 91.2 \approx 92$.

---

iii) Estimate the number of real multiplications per second needed to implement the block diagram assuming that $F(z)$ and $H(z)$ both use a polyphase implementation that incorporates the associated upsampler/downsampler. You may assume without proof that a polyphase filter of order M acting on a complex-valued signal requires $(2M+2)$ multiplications per sample at the lower of the two sample rates. [3]

The frequency downshift requires $2f_s$, the filters F, H, G require

$$(2M_F+2)K^{-1}f_s + (2M_H+2)K^{-1}f_s + (2M_G+2)K^{-1}f_s = 354K^{-1}f_s,$$

the squaring requires $3K^{-1}f_s$ and the frequency upshift requires $2f_s$ for a total of $(2+17.7+0.3+2)f_s = 21.85f_s = 4.37\times10^6$. *This is very much less than in part a)iv).*

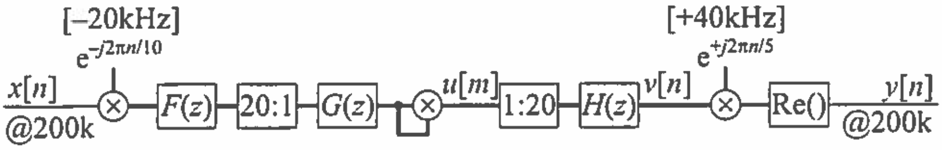


Figure 3.3

c) Suppose now that the upsampling is performed in two stages as illustrated in Figure 3.4 which replaces the blocks "1 : 20" and "$H(z)$" in Figure 3.3.

i) Determine the cutoff frequency, transition bandwidth and filter order for each of the lowpass filters $P(z)$ and $Q(z)$. [4]

Following $P(z)$ the sample rate is $20\,\text{kHz}$ *and the transition band of the filter needs to be* $[2, 8]\,\text{kHz}$ *for a transition band width of* $6\,\text{kHz} = 1.885\,\text{rad/s}$ *and* $M_P = 9.1 \approx 10$.

Following $Q(z)$ the sample rate is $200\,\text{kHz}$ *so the transition band of the filter needs to be* $[2, 18]\,\text{kHz}$ *for a transition band width of* $16\,\text{kHz} = 0.503\,\text{rad/s}$ *and* $M_Q = 34.1 \approx 35$.

---

ii) Estimate the number of real multiplications per second needed to implement the block diagram of Figure 3.4 assuming that a polyphase implementation is used for $P(z)$ and $Q(z)$. Compare this with the number of multiplications needed for the corresponding part of Figure 3.3. [3]

The number of multiplications needed is

$$\frac{2M_P+2}{20}f_s + \frac{2M_Q+2}{10}f_s = \left(\frac{22}{20}+\frac{72}{10}\right)f_s = (1.1+7.2)\,f_s = 8.3f_s = 1.66\times10^6$$

. *We can compare this to the multiplication rate required for $H(z)$ which is*

$$\frac{(2M_H+2)}{20}f_s = \frac{186}{20}f_s = 9.3f_s = 1.86\times10^6$$

. *So performing the decimation in stages reduces the computation a little.*

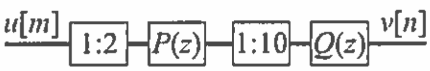


Figure 3.4

4. a) Explain briefly the advantages of processing signals in subbands. [2]

After splitting a signal into subbands, the subband signals are bandlimited and so it is possible to reduce the sample rate within each subband. This generally results in lower computational requirements than for full-band processing. A second advantage is that adaptive filtering applications converge more rapidly because the signal power spectrum is flatter in any subband than in the full band. Finally, processing the signal independently in subbands allows parallelism.

---

b) Figure 4.1 shows the analysis and synthesis stages of a 2-subband system. Show that $Y(z) = T(z)X(z)$ where $T(z) = \frac{1}{2}(H(z) - H(-z))(H(z) + H(-z))$. [4]

For $p \in [0, 1]$ you may assume without proof that $W_p(z) = U_p(z^2)$ and that $U_p(z) = \frac{1}{2}\left\{V_p\left(z^{\frac{1}{2}}\right) + V_p\left(-z^{\frac{1}{2}}\right)\right\}$.

Working backwards from the output to the input, we can write

$$\begin{aligned}
Y(z) &= H(z)W_0(z) - H(-z)W_1(z) \\
&= H(z)U_0\left(z^2\right) - H(-z)U_1\left(z^2\right) \\
&= \frac{1}{2}\left(H(z)\left(V_0(z) + V_0(-z)\right) - H(-z)\left(V_1(z) + V_1(-z)\right)\right) \\
&= \frac{1}{2}\left(H(z)\left(H(z)X(z) + H(-z)X(-z)\right) - H(-z)\left(H(-z)X(z) + H(z)X(-z)\right)\right) \\
&= \frac{1}{2}\left(H^2(z) - H^2(-z)\right)X(z) \\
&= \frac{1}{2}\left(H(z) - H(-z)\right)\left(H(z) + H(-z)\right)X(z)
\end{aligned}$$

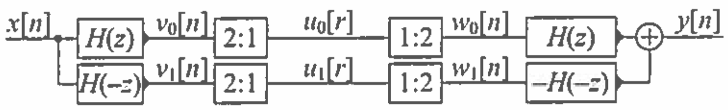


Figure 4.1

c) Given that the impulse response, $h[n]$, is causal and of odd order M, we define

$$t[n] = \frac{1}{2}\left(h[n] + (-1)^n h[n]\right) * \left(h[n] - (-1)^n h[n]\right)$$

where * denotes convolution.

i) Show that the z-transform of $t[n]$ is [3]

$$T(z) = \frac{1}{2}\left(H(z) - H(-z)\right)\left(H(z) + H(-z)\right).$$

If we define $s[n] = (-1)^n h[n]$, we can write

$$S(z) = \sum s[n] z^{-n} = \sum h[n] (-1)^n z^{-n}$$
$$= \sum h[n] \left(\frac{z}{-1}\right)^{-n} = \sum h[n] (-z)^{-n} = H(-z).$$

Hence

$$T(z) = \frac{1}{2}(H(z) - G(z))(H(z) + G(z)) = \frac{1}{2}(H(z) - H(-z))(H(z) + H(-z))$$

where convolution in the time domain corresponds to multiplication in the z-transform domain.

---

ii) Show that, if $h[n]$ satisfies the symmetry condition $h[M-n] = h[n]$, then $t[n]$ satisfies the condition $t[2M-n] = t[n]$. [3]

Writing out the convolution explicitly, we have

$$2t[n] = \sum_r (h[r] - (-1)^r h[r]) \left(h[n-r] + (-1)^{n-r} h[n-r]\right).$$

Now we can write

$$\begin{aligned}
2t[2M-n] &= \sum_r (h[r] + (-1)^r h[r]) \left(h[2M-n-r] - (-1)^{2M-n-r} h[2M-n-r]\right) \\
&= \sum_r (h[M-r] + (-1)^r h[M-r]) \left(h[n+r-M] - (-1)^{2M-n-r} h[n+r-M]\right) \\
&= \sum_s \left(h[s] + (-1)^{M-s} h[s]\right) \left(h[n-s] - (-1)^{M-n+s} h[n-s]\right) \\
&= \sum_s \left(h[s] + (-1)^M (-1)^s h[s]\right) \left(h[n-s] - (-1)^M (-1)^{n-s} h[n-s]\right) \\
&= \sum_s (h[s] - (-1)^s h[s]) \left(h[n-s] + (-1)^{n-s} h[n-s]\right) = 2t[n]
\end{aligned}$$

where the second line follows from $h[M-n] = h[n]$, the third line uses the substitution $s = M - r$ and the fifth line uses $(-1)^M = -1$since M is odd.

If M is even, the result is still true but the proof requires the factors in the summand to be interchanged.

---

iii) Deduce the group delay function, $\tau_T\left(e^{j\omega}\right)$, of the filter $T(z)$ from the symmetry condition of part ii). [2]

The group delay of a symmetric filter satisfying $t[2M-n] = t[n]$ is independent of $\omega$and equals M samples.

[Question continues on next page]

d) i) By using the inverse DTFT, show that the impulse response of an ideal lowpass filter whose frequency response is

$$G\left(e^{j\omega}\right) = \begin{cases} e^{-j0.5M\omega} & |\omega| \le \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < |\omega| \le \pi \end{cases}$$

is given by [3]

$$g[n] = \frac{\sin\left(0.5\pi\left(n - 0.5M\right)\right)}{\pi\left(n - 0.5M\right)}.$$

From the inverse DTFT given in the datasheet,

$$\begin{aligned} g[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} G\left(e^{j\omega}\right)e^{j\omega n}d\omega \\ &= \frac{1}{2\pi}\int_{-0.5\pi}^{0.5\pi} e^{-j0.5M\omega}e^{j\omega n}d\omega \\ &= \frac{1}{2\pi}\left[\frac{e^{j\omega(n-0.5M)}}{j(n-0.5M)}\right]_{-0.5\pi}^{0.5\pi} \\ &= \frac{2j\sin\left(0.5\pi(n-0.5M)\right)}{2\pi j(n-0.5M)} \\ &= \frac{\sin\left(0.5\pi(n-0.5M)\right)}{\pi(n-0.5M)} \end{aligned}$$

---

ii) A causal Hamming window of length $M+1$ is given by

$$w[n] = 0.54 - 0.46\cos\left(\frac{2n\pi}{M}\right)$$

for $n \in [0, M]$. Using the window design method with $g[n]$ and $w[n]$, design a causal FIR filter, $H(z)$, of order $M = 7$ with a cutoff frequency of $\omega = \frac{\pi}{2}$. Determine the numerical values of the filter coefficients, $h[n]$, to 3 decimal places. [4]

Substituting $n \in [0, M]$ into the given formula gives

$$w[n] = [0.080, 0.253, 0.642, 0.954, 0.954, 0.642, 0.253, 0.080].$$

Likewise, from part i), we obtain

$$g[n] = [-0.064, -0.090, 0.150, 0.450, 0.450, 0.150, -0.090, -0.064].$$

Multiplying these together gives

$$h[n] = [-0.005, -0.023, 0.096, 0.429, 0.429, 0.096, -0.023, -0.005].$$

---

iii) The filter, $H(z)$, from part ii) is used in the block diagram shown in Figure 4.1. If $T(z) = \frac{Y(z)}{X(z)}$, determine the magnitude gain, $|T(e^{j\omega})|$ for $\omega = 0$, $\frac{\pi}{2}$ and π. [4]

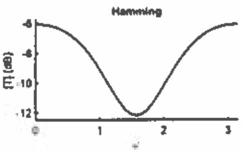
The gain is given by $|T(e^{j\omega})| = H^2(e^{j\omega}) - H^2(-e^{j\omega})$. *Taking* **h** *to be the column vector of filter coefficients,*

$$
\begin{aligned}
H(e^{j0}) &= H(1) = [1\,1\,1\,1\,1\,1\,1\,1]\mathbf{h} = 0.994 \\
H(e^{j\frac{\pi}{2}}) &= H(j) = [1\; j\; -1\; -j\; 1\; j\; -1\; -j]\mathbf{h} = 0.351 - 0.351j \\
&\Rightarrow H(-j) = 0.351 + 0.351j \\
H(e^{j\pi}) &= H(-1) = [1\; -1\; 1\; -1\; 1\; -1\; 1\; -1]\mathbf{h} = 0
\end{aligned}
$$

from which

$$
\begin{aligned}
|T(e^{j0})| &= 0.5\left|H^2(1) - H^2(-1)\right| = 0.5 \times 0.994^2 = 0.5 \times 0.988 = 0.494 \\
\left|T(e^{j\frac{\pi}{2}})\right| &= 0.5\left|H^2(j) - H^2(-j)\right| = 0.5 \times |-0.246j - 0.246j| = 0.5 \times 0.492 = 0.246 \\
|T(e^{j\pi})| &= 0.5\left|H^2(-1) - H^2(1)\right| = 0.5 \times 0.994^2 = 0.5 \times 0.988 = 0.494.
\end{aligned}
$$

The response, $T(e^{j\omega})$ *has a 6 dB dip at* $\omega = \frac{\pi}{2}$. *The complete response (although not requested) is*



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e) A "Johnston half-band filter" selects the coefficients, $h[n]$, to minimize the cost function

$$\alpha \int_{\frac{\pi}{2}+\Delta}^{\pi} \left|H(e^{j\omega})\right|^2 d\omega + (1-\alpha) \int_0^{\pi} \left(\left|H^2(e^{j\omega}) - H^2(-e^{j\omega})\right| - 1\right)^2 d\omega$$

for suitable choices of α and Δ.

i) Explain the significance of the two integrals in the cost function and hence explain the effect of reducing the value of α. [2]

The filter $H(z)$ *has a pass band of* $(0, \frac{\pi}{2})$ *and a stop band* $(\frac{\pi}{2}, \pi)$. *The first term of the cost function integrates the squared response over the stopband but allows a transition region* $(\frac{\pi}{2}, \frac{\pi}{2} + \Delta)$; *it is therefore a measure of the stop band attenuation. The overall magnitude gain* $|T(e^{j\omega})|$ *should ideally equal 1 at all frequencies. The second term in the cost function integrates the squared error in the overall magnitude gain over the entire band and is therefore a measure of the flatness of the overall response. Decreasing* α *will therefore make the overall response,* $|T(e^{j\omega})|$, *flatter but at the expense of reducing the stopband attenuation of* $H(z)$.

---

ii) For $M = 7$, $\alpha = 0.5$ and $\Delta = 0.07$, the $h[n]$ are given by

$$h[0] = h[7] = 0.009, \qquad h[1] = h[6] = -0.071$$
$$h[2] = h[5] = 0.069, \qquad h[3] = h[4] = 0.490$$

Determine the magnitude gain, $|T(e^{j\omega})|$ for $\omega = 0$, $\frac{\pi}{2}$ and π. [3]

The gain is given by $|T(e^{j\omega})| = H^2(e^{j\omega}) - H^2(-e^{j\omega})$. *Taking* **h** *to be the column vector of filter coefficients,*

$$H(e^{j0}) = H(1) = [1\,1\,1\,1\,1\,1\,1\,1]\mathbf{h} = 0.994$$
$$H(e^{j\frac{\pi}{2}}) = H(j) = [1\; j\; -1\; -j\; 1\; j\; -1\; -j]\mathbf{h} = 0.501 - 0.501j$$
$$\Rightarrow H(-j) = 0.501 + 0.501j$$
$$H(e^{j\pi}) = H(-1) = [1\; -1\; 1\; -1\; 1\; -1\; 1\; -1]\mathbf{h} = 0$$

from which

$$|T(e^{j0})| = 0.5\,|H^2(1) - H^2(-1)| = 0.5 \times 0.994^2 = 0.5 \times 0.988 = 0.494$$
$$|T(e^{j\frac{\pi}{2}})| = 0.5\,|H^2(j) - H^2(-j)| = 0.5 \times |-0.502j - 0.502j| = 0.5 \times 1.004 = 0.502$$
$$|T(e^{j\pi})| = 0.5\,|H^2(-1) - H^2(1)| = 0.5 \times 0.994^2 = 0.5 \times 0.988 = 0.494.$$

The complete response (not requested) is shown below and varies by only a small fraction of a dB.