ANALYSIS OF CIRCUITS

**** Solutions 2016 ****

Information for Candidates:

- Numerical answers must be given as fully evaluated decimal values and not as unevaluated arithmetic expressions.

Notation

The following notation is used in this paper:

1. The voltage waveform at node X in a circuit is denoted by $x(t)$, the phasor voltage by X and the root-mean-square (or RMS) phasor voltage by $\widetilde{X} = \frac{X}{\sqrt{2}}$. The complex conjugate of X is X^*.
2. Component and source values in a circuit are normally given in Ohms, Farads, Henrys, Volts or Amps with the unit symbol omitted. Where an imaginary number is specified, it represents the complex impedance or phasor value.
3. Times are given in seconds unless otherwise stated.
4. Unless otherwise indicated, frequency response graphs should use a linear axis for phase and logarithmic axes for frequency and magnitude.
5. The real and imaginary parts of a complex number, X, are written $\Re(X)$ and $\Im(X)$ respectively.

Key: B=bookwork, U=unseen example

1. a) Using nodal analysis, calculate the voltages at nodes X and Y of Figure 1.1. [4]

[U] KCL at node X gives

$$\begin{aligned}\frac{X-18}{4}+\frac{X}{3}+\frac{X-Y}{2} &= 0\\ \Rightarrow 3X-54+4X+6X-6Y &= 0\\ \Rightarrow 13X-6Y &= 54\end{aligned}$$

KCL at node Y gives

$$\begin{aligned}\frac{Y-X}{2}+\frac{Y}{1}-3 &= 0\\ \Rightarrow -X+3Y &= 6\end{aligned}$$

Solving these simultaneous equations gives

$$X=6, \qquad Y=4.$$

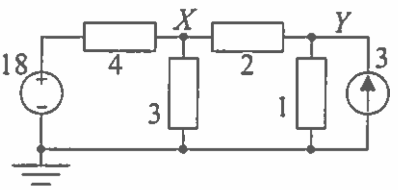


Figure 1.1

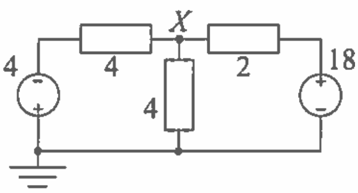


Figure 1.2

b) Use the principle of superposition to find the voltage X in Figure 1.2. [4]

[U] If we short circuit the 18 V *source, the* $2\,\Omega$ *and* $4\,\Omega$ *resistors are inparallel and are equivalent to a* $\frac{2\times 4}{2+4}=\frac{8}{6}=1.333\,\Omega$ *resistor. The circuit is now a potential divider and the voltage at X is given by* $X_1=\frac{1.333}{4+1.333}\times -4=\frac{1.333}{5.333}\times -4=-1\,\text{V}$.

If we now short circuit the 4 V *voltage source, the two* $4\,\Omega$ *resistors are in parallel and equal* $2\,\Omega$*. The voltage at X is then* $X_2=\frac{2}{2+2}\times 18=\frac{1}{2}\times 18=9\,\text{V}$.

By superposition, the total voltage is therefore $X=X_1+X_2=-1+9=8\,\text{V}$.

c) Draw the Thévenin equivalent circuit of the two-terminal network in Figure 1.3 and find the values of its components. [4]

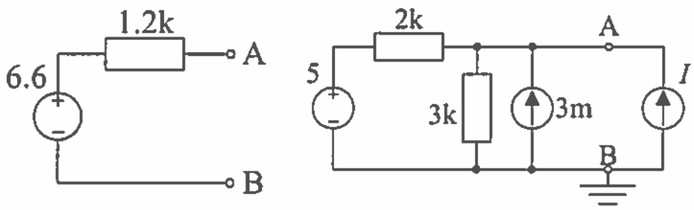
[U] We can find the Thévenin resistance by short-circuiting the voltage source and open-circuiting the current source. This leaves two resistors in parallel with an equivalent resistance of $R_{Thev} = \frac{2\times 3}{2+3} = 1.2\,\text{k}\Omega$.

We can find the open circuit voltage by nodal analysis or by superposition.

(i) Using nodal analysis (and grounding node B): $\frac{A-5}{2} + \frac{A}{3} - 3 = 0$ *from which* $V_{Thev} = A = \frac{33}{5} = 6.6\,\text{V}$.

(ii) By superposition: $V_{5\text{V}} = \frac{3}{3+2} \times 5 = 3\,\text{V}$ *and* $V_{3\text{m}} = \frac{2\times 3}{2+3} \times 3 = 3.6\,\text{V}$ *from which* $V_{Thev} = 3 + 3.6 = 6.6\,\text{V}$.

Either way, we get the diagram on the left below. Alternatively we can ground node B and append a current source, I, as shown in the rightmost diagram below. Now doing KCL at node A gives $\frac{A-5}{2} + \frac{A}{3} - 3 - I = 0$ *from which* $A = 6.6 + 1.2I$ *which gives* V_{Thev} *and* R_{Thev} *directly.*



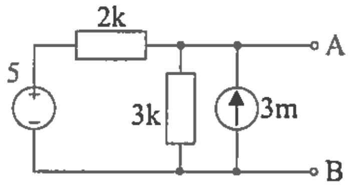


Figure 1.3

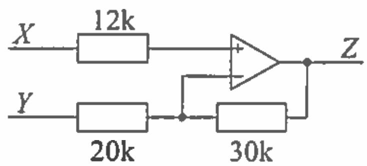


Figure 1.4

d) Assuming the opamp in the circuit of Figure 1.4 is ideal, give an expression for Z in terms of X and Y. [4]

[U] There is no current flowing through the $12\,\text{k}\Omega$ *resistor, so* $V_+ = X$. *The circuit has negative feedback and so we also have* $V_- = X$. *Now, doing KCL at* V_- *gives*

$$\frac{X-Y}{20} + \frac{X-Z}{30} = 0$$
$$\Rightarrow \quad 5X - 3Y - 2Z = 0$$
$$\Rightarrow \quad Z = 2.5X - 1.5Y$$

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e) The diode in the circuit of Figure 1.5 has a forward voltage of 0.7V when conducting but is otherwise ideal. Determine the output voltage, Y, when
(i) $X = 1\,\text{V}$,

(ii) $X = 5\,\mathrm{V}$
(iii) $X = -5\,\mathrm{V}$. [5]

[U]If the diode is not conducting, then the circuit is a potential divider and $X = 0.75Y$ and the voltage across the diode is $0.25X$. Thus, the diode will be off when $0.25X < 0.7 \Rightarrow X < 2.8$. If the diode is conducting, then $Y = X - 0.7$.

(i) when $X = 1\,\mathrm{V}$, the diode is off and $Y = 0.25X = 0.25\,\mathrm{V}$. (ii) when $X = 5\,\mathrm{V}$, the diode is conducting and $Y = X - 0.7 = 4.3\,\mathrm{V}$. (iii) when $X = -5\,\mathrm{V}$, the diode is off and $Y = 0.25X = -1.25\,\mathrm{V}$.

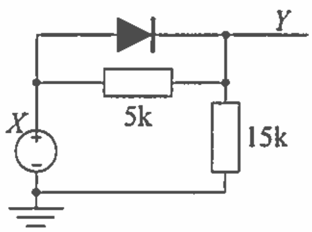

Figure 1.5

f) i) The diagram of Figure 1.6 shows an AC source with r.m.s. voltage $\widetilde{V} = 230\,\text{V}$ driving a load with impedance $50+25j\,\Omega$ through a line with impedance $2\,\Omega$. Determine the complex powers, given by $S = \widetilde{V}\times\widetilde{I}^*$, absorbed both by the load and by the 2Ω resistor. [4]

[U] The current phasor is $\widetilde{I} = \frac{\widetilde{V}}{52+25j} = 3.593 - 1.727j$. *The complex power absorbed by an impedance is* $S = \widetilde{V}\times\widetilde{I}^* = \left|\widetilde{I}\right|^2 Z = 15.891Z$. *So the power absorbed by the resistor is* $S_R = 15.891 \times 2 = 31.781\,\text{W}$. *The power absorbed by the load is* $S_L = 15.891 \times (50+25j) = 794.5+397.3j\,\text{VA}$.

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ii) A capacitor with impedance $-200j$ is now connected across the load, as indicated in Figure 1.7. Determine the complex powers absorbed both by the load and by the 2Ω resistor. [4]

[U] The combined load+capacitor impedance is now $Z_{LC} = \frac{-200j(50+25j)}{50+25j-200j} = 60.38+11.32j\,\Omega$. *So the voltage across the load+capacitor is* $\frac{Z_{LC}}{2+Z_{LC}} \times \widetilde{V} = \frac{(60.38+11.32j)230}{62.38+11.32j} = 222.86+129.57j$. *The source current is now* $\frac{V}{2+Z_{LC}} = \frac{230}{62.38+11.32j} = 3.570-0.648j$.

So the power absorbed by the resistor is $S_R = |3.570-0.648j|^2 \times 2 = 13.162\times 2 = 26.32\,\text{W}$, *a decrease of 17%. The power absorbed by the load is* $S_L = \frac{|V_L|^2}{Z_L^*} = \frac{|222.86+129.57j|^2}{50-25j} = \frac{49669}{50-25j} = 794.7+397.3j\,\text{VA}$ *which is almost exactly the same as before.*

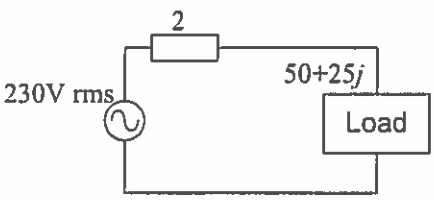

Figure 1.6

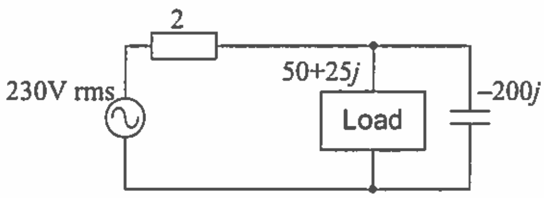

Figure 1.7

g) Determine the gain, $\frac{Y}{X}$, for the block diagram shown in Figure 1.8. The rectangular blocks are drawn with inputs at the left and outputs at the right and have gains of F, G and H respectively. The open circle represents an adder/subtractor; its three inputs have the signs indicated on the diagram and its output is V. [4]

[U] We can write down the following equations from the block diagram:

$$\begin{aligned} V &= X - Y - FHV \\ Y &= FGV \end{aligned}$$

We need to eliminate V from these equations:

$$
\begin{aligned}
V &= \frac{Y}{FG}\\
\Rightarrow \frac{1}{FG}Y &= X - Y - \frac{H}{G}Y\\
\Rightarrow \left(\frac{1}{FG}+1+\frac{H}{G}\right)Y &= X\\
\frac{1+FG+FH}{FG}Y &= X\\
\frac{Y}{X} &= \frac{FG}{1+F(G+H)}
\end{aligned}
$$

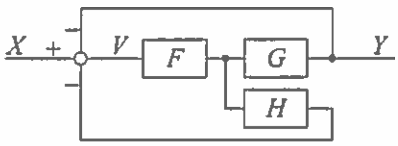

Figure 1.8

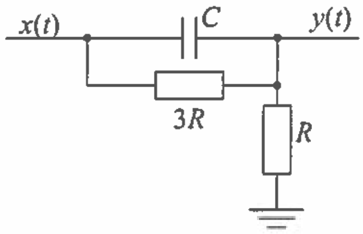

Figure 1.9

h) The input voltage in Figure 1.9 is given by

$$x(t) = \begin{cases} 0 & t < 0 \\ 8\,\mathrm{V} & t \geq 0. \end{cases}$$

i) Determine the time constant of the circuit. [2]

[U] The time constant is given by $\tau = R_{Thev}C$ where R_{Thev} is the Thévenin resistance across the terminals of the capacitor. If we short circuit the source, $x(t)$, we find $R_{Thev} = \frac{3R\times R}{3R+R} = 0.75R$ so the time constant is $\tau = 0.75RC$.

An alternative method is to calculate the transfor function of the circuit as

$$\frac{Y}{X} = \frac{R}{R+\frac{1}{j\omega C+\frac{1}{3R}}} = \frac{R}{R+\frac{3R}{j\omega 3RC+1}} = \frac{1}{j\omega 3RC+1+3} = \frac{1}{j\omega 3RC+4}$$

from which the time constant is the reciprocal of the denominator corner frequency and therefore equals $\tau = 0.75RC$.

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ii) Determine an expression for $y(t)$ for $t > 0$. [5]

[U] Since the DC gain of the circuit is 0.25 (obtained by treating the capacitor as an open circuit), the steady state output for $t \geq 0$ is $y_{SS}(t) = 0.25x(t) = 2$.

At time $t = 0$, the capacitor voltage, $y - x$, cannot change instantaneously. Therefore, $y(0+) - x(0+) = y(0-) - x(0-) = 0$ and hence $y(0+) = x(0+) = 8$. The transient amplitude is therefore $y(0+) - y_{SS}(0+) = 8 - 2 = 6$. The complete output is therefore $y(t) = 2 + 6e^{-\frac{t}{\tau}}$.

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2. The frequency response of a circuit is given by

$$H(j\omega) = \frac{aj\omega}{(j\omega)^2 + 2\zeta\omega_0 j\omega + \omega_0^2}$$

where a, ζ and ω_0 are real numbers.

a) i) By dividing the numerator and denominator of $H(j\omega)$ by $j\omega$ and then multiplying the resultant expression by its complex conjugate, show that $|H(j\omega)|^2 = \dfrac{a^2}{4\zeta^2\omega_0^2 + \left(\omega - \frac{\omega_0^2}{\omega}\right)^2}$. [3]

[U] Dividing numerator and denominator by $j\omega$ gives

$$\begin{aligned} H(j\omega) &= \frac{a}{2\zeta\omega_0 + j\omega + \frac{\omega_0^2}{j\omega}} \\ &= \frac{a}{2\zeta\omega_0 + j\left(\omega - \frac{\omega_0^2}{\omega}\right)}. \end{aligned}$$

To multiply by its complex conjugate we take the sum of the real and imaginary parts in both numerator and denominator to obtain

$$|H(j\omega)|^2 = \frac{a^2}{4\zeta^2\omega_0^2 + \left(\omega - \frac{\omega_0^2}{\omega}\right)^2}.$$

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ii) Explain why the maximum value of $|H(j\omega)|^2$ occurs when the quantity $\left(\omega - \frac{\omega_0^2}{\omega}\right)$ equals zero. Hence show that the maximum occurs at $\omega = \omega_0$ and determine $|H(j\omega_0)|^2$. [2]

[U] The denominator of $|H(j\omega)|^2$ is the sum of two squares of which only one involves ω. Therefore the denominator in minimized (and $|H(j\omega)|^2$ is maximized) when this term is zero:

$$\left(\omega - \frac{\omega_0^2}{\omega}\right)^2 = 0 \quad \Rightarrow \quad \omega = \frac{\omega_0^2}{\omega} \quad \Rightarrow \quad \omega = \pm\omega_0.$$

Substituting this into the expression for $|H(j\omega)|^2$ gives

$$\max\left\{|H(j\omega)|^2\right\} = \frac{a^2}{4\zeta^2\omega_0^2}.$$

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iii) Find expressions for the two positive values of ω for which $|H(j\omega)|^2 = \dfrac{a^2}{8\zeta^2\omega_0^2}$ and determine a simplified expression for the difference between them. [4]

[U] We have

$$|H(j\omega)|^2 = \frac{a^2}{4\zeta^2\omega_0^2 + \left(\omega - \frac{\omega_0^2}{\omega}\right)^2} = \frac{a^2}{8\zeta^2\omega_0^2}$$

$$\Rightarrow \quad \left(\omega - \frac{\omega_0^2}{\omega}\right)^2 = 4\zeta^2\omega_0^2$$

$$\omega - \frac{\omega_0^2}{\omega} = \pm 2\zeta\omega_0$$

$$\omega^2 \pm 2\zeta\omega_0\omega - \omega_0^2 = 0$$

$$\omega = \frac{\pm 2\zeta\omega_0 \pm \sqrt{4\zeta^2\omega_0^2 + 4\omega_0^2}}{2}$$

$$= \pm\zeta\omega_0 \pm \sqrt{\zeta^2\omega_0^2 + \omega_0^2}.$$

Since the square-root term is larger in magnitude than the first term, the two positive roots will be when the square root term is positive:

$$\omega_{1,2} = \pm\zeta\omega_0 + \sqrt{\zeta^2\omega_0^2 + \omega_0^2} = \left(\pm\zeta + \sqrt{\zeta^2 + 1}\right)\omega_0$$

Thus the difference between these two roots will be $\omega_2 - \omega_1 = 2\zeta\omega_0$ *(since the square root term cancels out in the subtraction). At these values of* ω*, the response has fallen* 3 dB *from its peak.*

b) Suppose now that $a = 5000\,\mathrm{s}^{-1}$, $\zeta = 0.1$ and $\omega_0 = 5000\,\mathrm{rad/s}$.

i) Determine the low and high frequency asymptotes of $H(j\omega)$. [2]

[U] The LF asymptote is found by taking the terms with the lowest power of $j\omega$ *in numerator and denomiator and is*

$$H_L(j\omega) = ja\omega_0^{-2}\omega = j2\times 10^{-4}\omega$$

. Similarly, the HF asymptote is

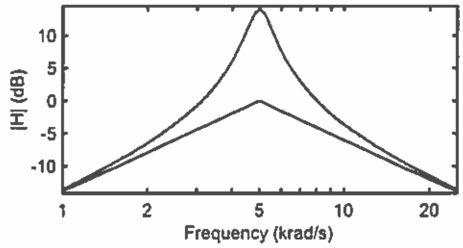
$$H_H(j\omega) = -ja\omega^{-1} = -j5000\omega^{-1}$$

.

ii) Draw a dimensioned sketch showing the high and low frequency asymptotes as well as the true magnitude response, $|H(j\omega)|$. Indicate on your graph in dB the peak value of $|H(j\omega)|$ and the value of the asymptotes at their point of intersection. [5]

[U] The magnitude asymptotes cross when $a\omega_0^{-2}\omega = a\omega^{-1} \Rightarrow \omega = \omega_0 = 5000$*. At this point, their value is* $a\omega_0^{-1} = 1 = 0\,\mathrm{dB}$*. From*

part ii), the peak magnitude gain is $\sqrt{\frac{a^2}{4\zeta^2\omega_0^2}} = \frac{a}{2\zeta\omega_0} = 5 = 14\,\text{dB}$ *at* $\omega = 5000$. *We also know from part iii) that the* $3\,dB$ *bandwidth is* $2\zeta\omega_0 = 1000\,\text{rad/s}$. *Thus we can draw the graph as shown.*



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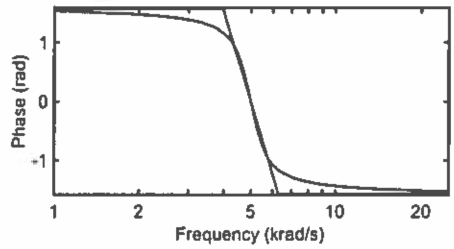
iii) Draw a dimensioned sketch of the straight-line approximation to the phase response, $\angle H(j\omega)$. You may assume without proof that the gradient of the approximation at ω_0 is equal to $-0.5\pi\zeta^{-1}$ radians per decade where "decade" means a factor of 10 in frequency. [4]

[U] From part i), the LF and HF phase shifts are $+\frac{\pi}{2}$ *and* $-\frac{\pi}{2}$. *Also, at* $\omega = \omega_0$, *the outer terms of the quadratic cancel and the phase shift is 0. At* ω_0, *the gradient is* $-0.5\pi\zeta^{-1}$, *so the central line of the approximation will hit* $\pm\frac{\pi}{2}$ *at* $\omega = \omega_0 \pm \zeta$ decades. $\zeta = 0.1$ *decades is a factor of* $10^{0.1} = 1.259$. *So the sloping segement goes between* $\omega = [3972, 6295]$



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c) i) Show that the frequency response, $\frac{Y(j\omega)}{X(j\omega)}$ of the circuit shown in Figure 2.1 is given by [5]

$$\frac{Y(j\omega)}{X(j\omega)} = \frac{-j\omega R_2 C}{(j\omega)^2 R_1 R_2 C^2 + 2j\omega R_1 C + 1}.$$

[U] KCL at the –ve opamp input (which is a virtual ground) gives

$$j\omega C(0-V) + \frac{0-Y}{R_2} = 0$$

$$\Rightarrow \quad V = \frac{-Y}{j\omega R_2 C}.$$

KCL at V gives

$$\frac{V-X}{R_1} + j\omega C(V-Y) + j\omega C(V-0) = 0$$

$$V(1+2j\omega R_1 C) - X - j\omega R_1 C Y = 0.$$

Substituting from the first equation gives

$$\frac{-Y}{j\omega R_2C}(1+2j\omega R_1C)-X-j\omega R_1CY=0.$$

$$\left(1+2j\omega R_1C+(j\omega)^2R_1R_2C^2\right)Y=-j\omega R_2CX$$

$$\Rightarrow \quad \frac{Y(j\omega)}{X(j\omega)}=\frac{-j\omega R_2C}{(j\omega)^2R_1R_2C^2+2j\omega R_1C+1}.$$

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ii) Determine simplified expressions for a, ζ and ω_0 so that the expression given in part c)i) equals that given above for $H(j\omega)$. [3]

[U] We can rewrite the equation as

$$\frac{Y(j\omega)}{X(j\omega)}=\frac{-(R_1C)^{-1}j\omega}{(j\omega)^2+2(R_2C)^{-1}j\omega+(R_1R_2C^2)^{-1}}.$$

Matching coefficients gives

$$a=-(R_1C)^{-1}$$

$$\omega_0=\left(R_1R_2C^2\right)^{-0.5}=\frac{1}{C\sqrt{R_1R_2}}$$

$$\zeta=\frac{1}{R_2C\omega_0}=\frac{C\sqrt{R_1R_2}}{R_2C}=\sqrt{\frac{R_1}{R_2}}.$$

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iii) Given that $C=10\,\text{nF}$, determine the values of R_1 and R_2 so that $\omega_0=5000\,\text{rad/s}$ and $\zeta=0.1$. [2]

[U] From part ii),

$$\zeta=\frac{1}{R_2C\omega_0} \quad\Rightarrow\quad R_2=\frac{1}{\zeta C\omega_0}=200\,\text{k}\Omega.$$

Now we can write

$$\zeta=\sqrt{\frac{R_1}{R_2}} \quad\Rightarrow\quad R_1=R_2\zeta^2=0.012=2\,\text{k}\Omega.$$

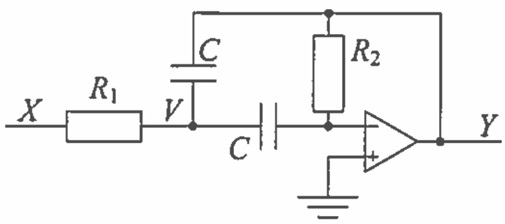

Figure 2.1

3. Figure 3.1 shows a shows a transmission line of length $L = 10\,\text{m}$ whose characteristic impedance is $Z_0 = 120\,\Omega$ and whose propagation velocity is $u = 2 \times 10^8\,\text{m/s}$. Distance along the line is denoted by x and the two points $x = 0$ and $x = L$ are marked in the figure.

At a point x on the line, the line voltage and current are given by $v_x(t) = f_x(t) + g_x(t)$ and $i_x(t) = Z_0^{-1}(f_x(t) - g_x(t))$ where $f_x(t) = f_0(t - u^{-1}x)$ and $g_x(t) = g_0(t + u^{-1}x)$ are the forward and backward waves respectively.

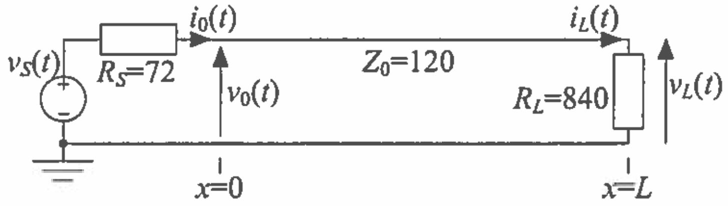

Figure 3.1

a) i) At the position $x = L$, the backward wave is given by $g_L(t) = \rho_L f_L(t)$ where $\rho_L = 0.75$ is the reflection coefficient at $x = L$.

Show that $g_0(t) = \rho_L f_0(t - 2u^{-1}L)$. [3]

[B] We substitute the given expressions, $f_x(t) = f_0(t - u^{-1}x)$ and $g_x(t) = g_0(t + u^{-1}x)$ into $g_L(t) = \rho_L f_L(t)$ to obtain

$$\begin{aligned} g_L(t) &= \rho_L f_L(t) \\ g_0(t + u^{-1}L) &= \rho_L f_0(t - u^{-1}L) \\ g_0(t') &= \rho_L f_0(t' - 2u^{-1}L) \end{aligned}$$

where in the final line we make the substitution $t' = t + u^{-1}L$.

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ii) At $x = 0$, show that $v_s(t) = v_0(t) + R_S i_0(t)$. Hence show that $f_0(t)$ can be written in the form $f_0(t) = \tau_0 v_s(t) + \rho_0 g_0(t)$ and determine the numerical values of τ_0 and ρ_0. [6]

[U] Applying Kirchoff's Current law at the rightmost end of R_S gives $\frac{v_0 - v_s}{R_S} + i_0 = 0$ from which $v_s = v_0 + R_S i_0$. [2]

Substituting for $v_0$and i_0 (and omitting the t argument) results in [2]

$$\begin{aligned} v_s &= (f_0 + g_0) + R_S Z_0^{-1}(f_0 - g_0) \\ &= \left(1 + R_S Z_0^{-1}\right) f_0 + \left(1 - R_S Z_0^{-1}\right) g_0 \\ \Rightarrow \quad f_0 &= \frac{1}{1 + R_S Z_0^{-1}} v_s - \frac{1 - R_S Z_0^{-1}}{1 + R_S Z_0^{-1}} g_0 \\ &= \frac{Z_0}{R_S + Z_0} v_s + \frac{R_S - Z_0}{R_S + Z_0} g_0 \end{aligned}$$

from which $\tau_0 = \frac{Z_0}{R_S + Z_0} = \frac{120}{192} = 0.625$ and $\rho_0 = \frac{R_S - Z_0}{R_S + Z_0} = \frac{-48}{192} = -0.25$.[2]

iii) By combining the results of parts i) and ii) show that

$$f_0(t) = \tau_0 v_s(t) + \rho_0\rho_L f_0(t - 2u^{-1}L).$$

Hence prove, by using induction or otherwise, that

$$f_0(t) = \textstyle\sum_{n=0}^{\infty} \tau_0\rho_0^n\rho_L^n v_s\left(t - 2nu^{-1}L\right). \qquad [6]$$

[U] Substituting part i) into part ii) gives $f_0(t) = \tau_0 v_s(t) + \rho_0\rho_L f_0(t - 2u^{-1}L)$ directly.

We now prove by induction that $f_0(t) = \rho_0^N\rho_L^N f_0\left(t - 2Nu^{-1}L\right) + \sum_{n=0}^{N-1} \tau_0\rho_0^n\rho_L^n v_s\left(t - 2nu^{-1}L\right)$.

When $N = 1$, this is true because the summation has only one term and it becomes the result in the first line.

We now assume it is true for $N = N_0$ and prove it for $N = N_0 + 1$ by substituting the result from the first line into the initial term:

$$\begin{aligned}
f_0(t) &= \rho_0^{N_0}\rho_L^{N_0} f_0\left(t - 2N_0u^{-1}L\right) + \sum_{n=0}^{N_0-1} \tau_0\rho_0^n\rho_L^n v_s\left(t - 2nu^{-1}L\right)\\
&= \rho_0^{N_0}\rho_L^{N_0}\left(\tau_0 v_s(t - 2N_0u^{-1}L) + \rho_0\rho_L f_0(t - 2N_0u^{-1}L - 2u^{-1}L)\right) +\\
&\quad \sum_{n=0}^{N_0-1} \tau_0\rho_0^n\rho_L^n v_s\left(t - 2nu^{-1}L\right)\\
&= \rho_0\rho_L\rho_0^{N_0}\rho_L^{N_0} f_0(t - 2N_0u^{-1}L - 2u^{-1}L) + \tau_0\rho_0^{N_0}\rho_L^{N_0} v_s(t - 2N_0u^{-1}L) +\\
&\quad \sum_{n=0}^{N_0-1} \tau_0\rho_0^n\rho_L^n v_s\left(t - 2nu^{-1}L\right)\\
&= \rho_0^{N_0+1}\rho_L^{N_0+1} f_0(t - 2\left(N_0+1\right)u^{-1}L) + \sum_{n=0}^{N_0} \tau_0\rho_0^n\rho_L^n v_s\left(t - 2nu^{-1}L\right)
\end{aligned}$$

As, $N_0 \to \infty$, the initial term tends to zero because $|\rho_0|, |\rho_L| < 1$ from which

$$f_0(t) = \sum_{n=0}^{\infty} \tau_0\rho_0^n\rho_L^n v_s\left(t - 2nu^{-1}L\right).$$

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b) If the source is a 30 ns pulse given by

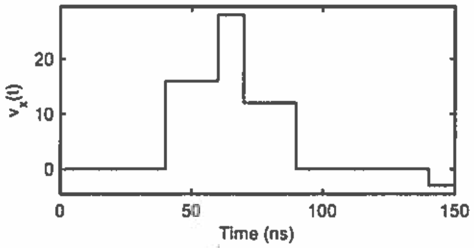
$$v_s(t) = \begin{cases} 25.6\,\text{V} & \text{for } 0 \le t \le 30\,\text{ns} \\ 0 & \text{otherwise} \end{cases},$$

draw a dimensioned sketch of the waveform $v_x(t)$ on the line at the point $x = 8$ m for the time interval $0 \le t \le 150$ ns. Give the times of all discontinuities and the values of all horizontal portions of the waveform. [6]

[U] The propagatin velocity is $u = 2 \times 10^8$ which equals 5 ns per metre. So the pulse arrives at x at $8 \times 5 = 40$ ns, reflects off the load and returns at $12 \times 5 = 60$ ns. Subsequent arrivals are at these times pulse multiples of the round trip time, $20 \times 5 = 100$ ns so only the transition at 140 ns lies within the

plotted range. The initial forward wave amplitude is $8 \times \tau_0 = 6.4$ and subsequent amplitudes are $5 \times \rho_L = 4.8$, $3.75 \times \rho_0 = -1.2$, $-0.9375 \times \rho_L = -0.9$.

putting all this together, we get transitions at $t = \{40, 60, 70, 90, 140\}$ of voltages $\delta v = \{16, 12, -16, -12, -3\}$. The voltage after each transition is therefore $v_x = \{16, 28, 12, 0, -3\}$.



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c) Now assume that all voltages and currents are sinusoidal with angular frequency ω. The uppercase letter, V_x, denotes the phasor corresponding to $v_x(t)$.

i) The waveform $f_0(t) = A\cos(\omega t + \theta)$ is represented by the phasor $F_0 = Ae^{j\theta}$. Show that $F_x = F_0 e^{-jkx}$ where $k = u^{-1}\omega$. [3]

[B] We know that $f_x(t) = f_0(t - u^{-1}x) = f_0(t) = A\cos\left(\omega(t - u^{-1}x) + \theta\right) = A\cos\left(\omega t + \theta - \omega u^{-1}x\right)$. The corresponding phasor is therefore $F_x = Ae^{j(\theta - \omega u^{-1}x)} = Ae^{j\theta}e^{-j\omega u^{-1}x} = F_0 e^{-jkx}$.

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ii) By converting the first equation given in part a)iii) into phasor form, determine an expression for F_0 in terms of V_s. [3]

[U] Converting $f_0(t) = \tau_0 v_s(t) + \rho_0\rho_L f_0(t - 2u^{-1}L)$ into phasor form gives

$$F_0 = \tau_0 V_S + \rho_0\rho_L F_0 e^{-j2kL}$$

$$\Rightarrow \quad F_0\left(1 - \rho_0\rho_L e^{-j2kL}\right) = \tau_0 V_S$$

$$\Rightarrow \quad F_0 = \frac{\tau_0}{1 - \rho_0\rho_L e^{-j2kL}} V_S$$

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iii) Determine an expression for V_x in terms of V_s. [3]

[U] We know that

$$\begin{aligned} V_x &= F_x + G_x \\ &= F_0 e^{-jkx} + G_0 e^{jkx} \\ &= F_0 e^{-jkx} + \rho_L F_0 e^{-j2kL} e^{jkx} \\ &= F_0\left(e^{-jkx} + \rho_L e^{-jk(2L-x)}\right) \\ &= \frac{\tau_0\left(e^{-jkx} + \rho_L e^{-jk(2L-x)}\right)}{1 - \rho_0\rho_L e^{-j2kL}} V_S \end{aligned}$$