E4.27
CS5.2
ISE4.41

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2009

MSc and EEE/ISE PART IV: MEng and ACGI

IDENTIFICATION AND ADAPTIVE CONTROL

Corrected Copy

Wednesday, 29 April 10:00 am

Time allowed: 3:00 hours

There are FIVE questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

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Information for candidates:

The probability density $N(m, Q)$ of an n-vector, normal random variable with mean m and covariance matrix Q ($Q > 0$) is

$$N(m, Q)(x) = \frac{1}{(\sqrt{2\pi})^{\frac{n}{2}} (\det Q)^{\frac{1}{2}}} \exp -\frac{1}{2} \left((x - m)^T Q^{-1} (x - m) \right) .$$

In the case that $n = 1$, m is a scalar and $Q = \sigma^2$ ($\sigma^2 > 0$),

$$N(m, \sigma^2)(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left(-\frac{(x - m)^2}{2\sigma^2} \right)$$

and, if X is a scalar random variable with probability density $N(m, \sigma^2)$,

$$\text{Prob}\{m - 2\sigma \leq X \leq m + 2\sigma\} \approx 0.95 .$$

1. Two coupled, stationary, chemical processes y_t and z_t are governed by the equations

$$\begin{aligned} y_t &= \alpha y_{t-1} + e_t \\ z_t &= \alpha z_{t-1} + \gamma y_{t-1} , \end{aligned}$$

in which e_t is a white noise process with variance $\sigma^2 > 0$. The constant α ($|\alpha| < 1$) is the reaction rate parameter for both processes. The constant γ is the coupling coefficient.

Let $x_t = (y_t,\ z_t)^T$. Develop a state space model for x_t, of the form

$$x_t = F x_{t-1} + g e_t .$$

[2]

Show that $R = E[x_t x_t^T]$ satisfies the equation

$$R = FRF^T + bb^T \sigma^2$$

and derive formulae for the entries of R in terms of σ^2, α and γ. [4]

[6]

Now assume that the value of the reaction rate parameter is

$$\alpha = 0.5 .$$

Assume also that, by means of an identification experiment, it has been possible to establish the following relation between the variances $r_z = E[z_t^2]$ and $r_y = E[y_t^2]$:

$$\frac{r_z}{r_y} = 1.25 .$$

Determine the value of the coupling coefficient γ. [8]

2. An N-dimensional measurement vector is assumed to be modelled by the equation

$$y = x\theta + e$$

in which x is a known, non-zero, deterministic N-vector and e is a normal N-vector random variable zero mean and covariance matrix Q ($Q > 0$). θ is an unknown scalar parameter.

Consider the linear estimate $\hat{\theta}$ of θ given y:

$$\hat{\theta} = \left(x^T Q^{-1} x\right)^{-1} x^T Q^{-1} y \, .$$

Show that the estimate $\hat{\theta}$ is unbiased. [3]

Determine the variance of the estimate $\hat{\theta}$. [3]

Show that the estimate $\hat{\theta}$ minimizes the mean square error

$$E\left[|\hat{\hat{\theta}} - \theta|^2\right]$$

over all unbiased linear estimates $\hat{\hat{\theta}}$ of θ given y. [10]

Hint: Use the fact that an arbitrary linear, unbiased estimate $\hat{\hat{\theta}}$ can be expressed as

$$\hat{\hat{\theta}} = \left(x^T Q^{-1} x\right)^{-1} x^T Q^{-1} y \, + \, b^T y \, ,$$

where b is an N-vector satisfying the condition:

$$b^T x = 0 \, .$$

Now assume that, for some integer $N \geq 2$, x is the N-vector $[1, \ldots, 1]^T$ and

$$Q^{-1} = \begin{bmatrix} 1 & 0.5 & 0 & 0 & . & 0 \\ 0.5 & 1 & 0.5 & 0 & . & 0 \\ . & . & . & . & . & \\ 0 & . & 0 & 0.5 & 1 & 0.5 \\ 0 & . & 0 & 0 & 0.5 & 1 \end{bmatrix} .$$

Determine a 0.95 confidence interval for θ, given the estimate $\hat{\theta}$. [4]

3. Let y_t be a stationary, ergodic, scalar process satisfying

$$y_t + a y_{t-1} = e_t ,$$

in which e_t is a Gaussian, white noise process, with variance $\sigma^2 > 0$. The number a is an unknown, scalar parameter satisfying $|a| < 1$. Write

$$\hat{R}^N = \frac{1}{N} \sum_{t=0}^{N-1} y_t^2 \quad \text{(the sample covariance function of } y_t) .$$

What is the linear least squares estimate $\hat{a}_N$ of the parameter a, based on observations y_t, $t = 0, 1, \ldots, N$? [2]

Show that, as $N \to \infty$,

$$\hat{a}_N \to a .$$

[8]

Show further that

$$\hat{R}^N (\hat{a}_N - a) = -\sum_{i=1}^{N} y_{i-1} e_i .$$

[5]

Hence, or otherwise, show that

$$\text{var}\{\hat{R}^N (\hat{a}_N - a)\} \leq \frac{1}{N} \times \frac{\sigma^2}{(1 - a^2)} ,$$

for all $N > 1$. [5]

Hint: To evaluate var$\{\hat{R}^N (\hat{a}_N - a)\}$*, use the fact that*

$$E[y_{t-1} e_t y_{t'-1} e_{t'}] = 0 \quad \textit{for } t' \neq t.$$

4 A control system relating the scalar control signal u_t and the scalar output signal y_t is modelled by the equations

$$y_t - ay_{t-1} = bu_t + e_t$$

in which e_t is a white noise sequence with variance σ^2. Here, a and b are scalar parameters. The true value of the parameter a is $a = 0$. b is non-zero.

In an identification experiment, the input signal is chosen to be samples of a process modelled as

$$u_t = v_t + 0.5v_{t-1}$$

in which v_t is a white noise sequence with unit variance, uncorrelated with e_t. It can be assumed that the joint process (y_t, u_t) is stationary and ergodic.

Let $(\hat{a}, \hat{b})$ be linear least squares estimate of (a, b), given $\{y_0, \ldots, y_N\}$ and $\{u_1, \ldots, u_N\}$, based on the assumption that both a and b are unknown parameters.

Let $\hat{\hat{b}}$ be the linear least squares estimate of b, based on the assumption that $a = 0$.

(a): Calculate $R_u(0) = E[u_t^2]$, $R_y(0) = E[y_t^2]$ and $R_{uy}(1) = E[u_t y_{t-1}]$. (In performing this calculation you should assume that $a = 0$ and b is an arbitrary non-zero number). [5]

(b): Obtain formulae for $\hat{b}$ and $\hat{\hat{b}}$, expressed in terms of sample convariances and cross-covariances of y_y and u_t, and the constant b. [7]

(c): Show that the conditional covariances $\hat{\gamma}$ and $\hat{\hat{\gamma}}$ of $\hat{b}$ and $\hat{\hat{b}}$ respectively, given $\{y_0, \ldots, y_N\}$ and $\{u_1, \ldots, u_N\}$, are, for N large, approximately

$$\hat{\gamma} = \frac{1}{N}(b^2 + \frac{4}{5})\sigma^2 \quad \text{and} \quad \hat{\hat{\gamma}} = \frac{1}{N} \times \frac{4}{5}\sigma^2 \, .$$

Comment on the relative magnitudes of the $\hat{\gamma}$ and $\hat{\hat{\gamma}}$. [7]

In (c) you should assume that sample covariances/cross-covariances can be replaced by covariances/cross-covariances. [1]

5. **(a):** Measurements y_1 and y_2 are taken at times $t = 1$ and $t = 2$ of a process governed by the ARMA model equations

$$y_t = e_t + c\,e_{t-1} \,,$$

in which $e_0 = 0$, and e_1 and e_2 are independent, zero mean, normal random variables, each with variance σ^2. c and σ^2 are unknown parameters.

Show that the 2-vector random variables $y = (y_1, y_2)^T$ and $e = (e_1, e_2)^T$ are related by

$$y = \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix} e \,.$$

Calculate the log likelihood function $L(c, \sigma^2; y)$ of c and σ^2:

$$L(c, \sigma^2; y) = \log_e p(y|c, \sigma^2)$$

in which $p(y|c, \sigma^2)$ denotes the probability density of y, given c and σ^2. [3]

Calculate the maximum likelihood estimates $\hat{c}$ and $\hat{\sigma^2}$ of c and σ^2; that is, the values of c and σ^2 maximizing $L(c, \sigma^2)$. [7]

Show that $\hat{\sigma^2}$ is an biased estimate of σ^2. [2]

Hint: Maximize the likelihood function first over c (for fixed σ^2) and then over σ^2.

(b): Describe the Generalized Least Squares Algorithm for estimating the parameters $a_1, \ldots, a_n$ and $d_1, \ldots, d_n$ in the model

$$\begin{cases} A(z)y_t = B(z)u_t + n_t \\ D(z)n_t = e_t \end{cases}$$

given measurements $y_1, \ldots, y_N$, $u_1, \ldots, u_N$ (and appropriate starting values). Here

$A(z) = 1 + a_1 z^{-1} + \ldots + a_n z^{-n}$, $B(z) = b_0 + b_1 z^{-1} + \ldots + b_n z^{-n}$, $D(z) = 1 + d_1 z^{-1} + \ldots + d_n z^{-n}$,

and $\{e_t\}$ is a white noise sequence with random variables, each with variance σ^2. [8]

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Model Answers. Identification + Adaptive Control, 2009

1. Set $\underline{x}_t = \begin{pmatrix} y_t \\ z_t \end{pmatrix}$. Then $x_t^{(1)} = \alpha x_{t-1}^{(1)} + e_t$, $x_t^{(2)} = \gamma x_{t-1}^{(1)} + \alpha x_{t-1}^{(2)}$.

It follows
$$x_t = \begin{bmatrix} \alpha & 0 \\ \gamma & \alpha \end{bmatrix} x_{t-1} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e_t \qquad \left(E\{e_t\} = 0,\ \text{var}\{e_t\} = \sigma^2 \right)$$

Write $x_t = F x_{t-1} + b e_t$, with
$$F = \begin{bmatrix} \alpha & 0 \\ \gamma & \alpha \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Write $R = E[x_t x_t^T]$. We have
$$E[x_t x_t^T] = E[F x_t x_t^T F^T] + b b^T \sigma^2$$
or
$$R = F R F^T + b b^T \sigma^2$$

Expand as
$$\begin{bmatrix} r_{11} & r_{12} \\ r_{12} & r_{22} \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ \gamma & \alpha \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} \\ r_{12} & r_{22} \end{bmatrix} \begin{bmatrix} \alpha & \gamma \\ 0 & \alpha \end{bmatrix} + \sigma^2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} \alpha & 0 \\ \gamma & \alpha \end{bmatrix} \begin{bmatrix} \alpha r_{11} & \gamma r_{11} + \alpha r_{12} \\ \alpha r_{12} & \gamma r_{12} + \alpha r_{22} \end{bmatrix} + \sigma^2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} \alpha^2 r_{11} + \sigma^2 & \alpha\gamma r_{11} + \alpha^2 r_{12} \\ \alpha\gamma r_{11} + \alpha^2 r_{12} & \gamma^2 r_{11} + \alpha\gamma r_{12} + \alpha\gamma r_{12} + \alpha^2 r_{22} \end{bmatrix}$$

Equating matrix entries gives
$$r_{11} = \alpha^2 r_{11} + \sigma^2, \quad r_{12} = \alpha\gamma r_{11} + \alpha^2 r_{12}$$
and
$$r_{22} = \gamma^2 r_{11} + 2\alpha\gamma r_{12} + \alpha^2 r_{22}$$

Hence
$$\underline{r_{11} = \frac{\sigma^2}{1-\alpha^2}}, \quad r_{12} = \frac{\alpha\gamma}{(1-\alpha^2)} \times \frac{\sigma^2}{1-\alpha^2} = \frac{\alpha\gamma\sigma^2}{(1-\alpha^2)^2}$$
$$r_{22} = \frac{1}{(1-\alpha^2)} \left(\frac{\gamma^2\sigma^2}{1-\alpha^2} + \frac{2\alpha^2\gamma^2}{1-\alpha^2} \times \frac{\sigma^2}{1-\alpha^2} \right)$$
$$= \frac{1}{(1-\alpha^2)^3} \left(\gamma^2\sigma^2(1-\alpha^2) + 2\alpha^2\gamma^2\sigma^2 \right) = \underline{\frac{\sigma^2(1+\alpha^2)\gamma^2}{(1-\alpha^2)^3}}$$

If $\alpha = 1/2$
$$r_{11} = 4/3\,\sigma^2, \quad r_{22} = \left(\frac{4^3}{3^3}\right) \times 1.25\,\gamma^2\sigma^2$$
and
$$r_{22}/r_{11} = \frac{4^2}{3^2} \times 1.25\,\gamma^2 = 1.25$$

So $\underline{\gamma = 3/4}$

2. Take $\hat{\theta} = (X^T Q^{-1} X)^{-1} X^T Q^{-1} y$, $y = X\theta + e$, $e \sim N(0, Q)$

Then $E[\hat{\theta}] = E[(X^T Q^{-1} X)^{-1} X^T Q^{-1}(X\theta + e)] = \theta + E[(X^T Q^{-1} X)^{-1} X^T Q^{-1} \underbrace{e]}_{=0}$

We have shown $\hat{\theta}$ is <u>unbiased</u>.

$\hat{\theta} - \theta = (X^T Q^{-1} X)^{-1} X^T Q^{-1} e$ so

$E|\hat{\theta} - \theta|^2 = E[(X^T Q^{-1} X)^{-2} X^T Q^{-1} e e^T Q^{-1} X] = (X^T Q^{-1} X)^{-2} (X^T Q^{-1} X)$

$= \underline{(X^T Q^{-1} X)^{-1}}$

An arbitrary linear, unbiased estimator can be written

$$\hat{\hat{\theta}} = (X^T Q^{-1} X)^{-1} X^T Q^{-1} y + b^T y$$

for some N-vector b. Because $\hat{\hat{\theta}}$ is unbiased

$E[\hat{\hat{\theta}}] = \theta + (X^T Q^{-1} X)^{-1} X^T Q^{-1} \overset{\to 0}{E[e]} + b^T X\theta + \overset{\to 0}{b^T E[e]}$

Since $E[\hat{\hat{\theta}}] = \theta$, for all θ, we must have

$b^T X = 0$ —— (1)

Now examine mean square error of estimate

$E[|\hat{\hat{\theta}} - \theta|^2] = E|(X^T Q^{-1} X)^{-1} X^T Q^{-1} e + \overset{\to 0 \text{ (by (1))}}{b^T X\theta} + b^T e|^2$

$= E[[(X^T Q^{-1} X)^{-1} X^T Q^{-1} + b^T] e e^T [Q^{-1} X (X^T Q^{-1} X)^{-1} + b]$

$= [\quad \ldots \quad] Q [\quad \ldots \quad]$

$= (X^T Q^{-1} X)^{-1} + 0 + 0 + b^T Q b$

(we have used (1))

Since $b^T Q b \geq 0$, we have shown

$$E[|\hat{\hat{\theta}} - \theta|^2] \geq (X^T Q^{-1} X)^{-1} = E[|\hat{\theta} - \theta|^2].$$

This establishes that $\hat{\theta}$ is BLUE ("best linear, unbiased estimator.)

We know that $\hat{\theta}$ has mean θ and variance $(X^T Q^{-1} X)^{-1}$. Furthermore, it is an affine function of e, so it is <u>normal</u>: $\hat{\theta} \sim N(\theta, (X^T Q^{-1} X))$

It is now supposed that

$X = (\underbrace{1, 1, \ldots, 1, 1}_{N \text{ terms}})^T$ and $Q^{-1} = \begin{bmatrix} 1 & 0.5 & 0 & \\ 0.5 & 1 & 0.5 & \\ 0 & 0.5 & 1 & \\ & & & \ddots & 1 \end{bmatrix}$

We see $X^T Q^{-1} X = [1, \ldots, 1] \begin{bmatrix} 1.5 \\ 2 \\ \vdots \\ 2 \\ 1.5 \end{bmatrix} = 2 \times 1.5 + (N-2) \times 2$

$= N - 1 \quad (N > 2)$

Hence $\hat{\theta} \sim N(\theta, \frac{1}{N-1})$. So $|\hat{\theta} - \theta| \leq 2/\sqrt{N-1}$ w.p. 0.95

We conclude

$$\hat{\theta} - \frac{2}{\sqrt{N-1}} \leq \theta \leq \hat{\theta} + \frac{2}{\sqrt{N-1}} \quad \text{w.p. } 0.95$$

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3. $y_t = -a y_{t-1} + e_t \quad — (1)$

implies $\begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix} = \begin{pmatrix} -y_0 \\ \vdots \\ -y_{N-1} \end{pmatrix} a + \begin{pmatrix} e_1 \\ \vdots \\ e_N \end{pmatrix}$; write as $\underline{y} = X\underline{a} + \underline{e}$

The linear l.s. estimate $\hat{a} = (X^T X)^{-1} X^T y = -\sum_{t=1}^{N} y_i y_0 \Big/ \sum_{t=1}^{N} y_{t-1}^2$

But $N^{-1} \sum_i^N y_i y_0 \to R_y(1)$, $N^{-1} \sum_i^N y_i^2 \to R_y(0)$ as $N \to \infty$.

It follows that $\hat{a} \to -R_y(1) / R_y(0)$

But, from (1), $E\{y_t y_{t-1}\} = -a E[y_{t-1}^2] + \underbrace{E[e_t y_{t-1}]}_{0}$,

whence $R_y(1) = -a R_y(0)$

We deduce that

$$\hat{a} \to a \quad \text{as } N \to \infty.$$

We have

$$\left(\frac{1}{N} \sum_{t=1}^N y_{t-1}^2\right) \hat{a} = -\frac{1}{N} \sum_{t=1}^N y_t y_{t-1}$$

Multiply across 1 by y_{t-1} and summing gives

$$\frac{1}{N} \sum_{i=1}^N y_i y_{i-1} = -a \frac{1}{N} \sum_{i=1}^N y_{t-1}^2 + \frac{1}{N} \sum_{t=0}^N y_{t-1} e_t$$

It follows

$$\left(\frac{1}{N} \sum_{t=0}^{N-1} y_t^2\right)(\hat{a} - a) = \cancel{-\frac{1}{N} \sum y_t y_{t-1}} + \cancel{\frac{1}{N} \sum y_t y_{t-1}} - \frac{1}{N} \sum_{t=0}^N y_{t-1} e_t.$$

Hence

$$\hat{R}_y(0) \times (\hat{a} - a) = -\frac{1}{N} \sum_{t=0}^N y_{t-1} e_t$$

We see

$$E[\hat{R}_y(0) \times (\hat{a} - a)] = -E\left[\frac{1}{N} \ldots\right] = 0$$

Hence,

$$\mathrm{var}\{\hat{R}_y(0) \times (\hat{a} - a)\} = \frac{1}{N^2} E\left[\left(\sum_{t=1}^N y_{t-1} e_t\right) \times \left(\sum_{t=1}^N y_{t-1} e_t\right)\right]$$

$$= \frac{1}{N^2} E\left[\sum_{t=1}^N y_{t-1}^2\right] \sigma^2 + 0 + \ldots + 0.$$

$$= \frac{\sigma^2}{N} R_y(0).$$

But, from (1) $R_y(0) = E[y_t^2] = E[(-a y_{t-1} + e_t)^2] = a^2 R_y(0) + \sigma^2$

Hence $R_y(0) = \sigma^2 / (1 - a^2)$

But then

$$\mathrm{var}\{\hat{R}_y(0) \times (\hat{a} - a)\} = \underbrace{\frac{1}{N} \times \frac{\sigma^4}{(1 - a^2)}}$$

4. (i) $R_u(0) = E[(v_t + \frac{1}{2}v_{t-1})(v_t + \frac{1}{2}v_{t-1})] = E[v_t^2] + 0 + 0 + \frac{1}{4}E[v_t^2] = \frac{5}{4}$

$R_y(0) = E[(b(v_t + \frac{1}{2}v_{t-1}) + e_t)^2] = b^2(1 + \frac{1}{4}) + \sigma^2 = \frac{5}{4}b^2 + \sigma^2$

$R_{uy}(L) = E[(v_t + \frac{1}{2}v_{t-1})(b(v_t + \frac{1}{2}v_{t-1}) + e_t)] = \frac{5}{4}b$

(ii) We write
$$\begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix} = \begin{bmatrix} y_0 & u_1 \\ \vdots & \vdots \\ y_{N-1} & u_N \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix} \equiv y = X\theta + e$$

Then
$$\begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix} = (X^TX)^{-1}X^Ty = \begin{bmatrix} \frac{1}{N}\sum_{i=0}^{N-1} y_i^2 & \frac{1}{N}\sum_{i=1}^{N} u_i y_{i-1} \\ \frac{1}{N}\sum_{i=1}^{N} u_i y_{i-1} & \frac{1}{N}\sum_{i=1}^{N} u_i^2 \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{N}\sum_{i=1}^{N} y_i y_{i-1} \\ \frac{1}{N}\sum_{i=1}^{N} u_i y_i \end{bmatrix}$$

$$= \begin{bmatrix} \hat{R}_y(0) & \hat{R}_{uy}(1) \\ R_{uy}(L) & \hat{R}_u(0) \end{bmatrix}^{-1} \begin{bmatrix} R_y(1) \\ R_{uy}(L) \end{bmatrix}. \text{ So } \hat{b} = [0\ 1] \begin{bmatrix} \hat{R}_y(0) & \hat{R}_{uy}(1) \\ \hat{R}_{uy}(1) & R_u(0) \end{bmatrix}^{-1} \begin{bmatrix} R_y(1) \\ R_{uy}(0) \end{bmatrix}$$

If however we assume $a = 0$, $\begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix} = \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix} b + \begin{bmatrix} e_1 \\ \vdots \\ e_r \end{bmatrix} \equiv y = \bar{X}\bar{\theta} + e$

In this case,
$$\hat{\hat{b}} = (\bar{X}^T\bar{X})^{-1}\bar{X}^Ty = \left(\frac{1}{N}\sum_{i=1}^{N} u_i^2\right)^{-1}\left(\frac{1}{N}\sum_{i=1}^{N} u_i y_i\right)$$
giving $\hat{\hat{b}} = \hat{R}_u(0)^{-1} R_{uy}(0)$.

(iii) The conditional variance of $\hat{b}$ (given $\{y_0, \dots, y_N\}$ and $\{u_1, \dots, u_N\}$):
$$\hat{\gamma} = \sigma^2 [0\ 1](X^TX)^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{\sigma^2}{N} \times [0\ 1] \begin{bmatrix} \frac{1}{N}\sum_i y_i^2 & \frac{1}{N}\sum_i u_i y_{i-1} \\ \frac{1}{N}\sum_i u_i y_{i-1} & \frac{1}{N}\sum_i u_i^2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$= \frac{\sigma^2}{N} \times \frac{1}{\left(\hat{R}_y(0)\hat{R}_u(0) - \frac{1}{N}\hat{R}_{uy}^2(0)\right)} \times \hat{R}_y(0)$$

Also, the conditional variance of $\hat{\hat{b}}$ is
$$\hat{\hat{\gamma}} = \sigma^2(\bar{X}^TX)^{-1} = \sigma^2/N \left(\frac{1}{N}\sum_i u_i^2\right)^{-1} = \frac{\sigma^2}{N}\hat{R}_u(0)^{-1}$$

Using the results from (i), and assuming $\hat{R}_y(0) = R_y(0)$, etc, we have
$$\hat{\gamma} = \frac{\sigma^2}{N}\frac{(5/4\, b^2 + \sigma^2)}{5/4(5/4\, b^2 + \sigma^2) - (\frac{5}{4})^2 b^2} = \frac{1}{N}(b^2 + 4/5\, \sigma^2)$$
$$\hat{\hat{\gamma}} = 1/N \times 4/5\, \sigma^2$$

We observe that $\hat{\gamma} > \hat{\hat{\gamma}}$, reflecting the fact that the model (with a unknown) is over-parameterized and therefore gives rise to estimates of non-zero parameters with increased variance.

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5.(a) $y_1 = e_1 + 0$ and $y_2 = e_2 + c e_1$, so $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$

The prob. density of y is:

$$p(y \mid c, \sigma^2) = \frac{1}{2\pi\, \sigma^2 \det Q} \exp\left\{ -\frac{\sigma^{-2}}{2} y^T Q^{-1} y \right\}$$

in which $Q = \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & c \\ c & 1+c^2 \end{bmatrix}$. Note, $\det Q = (1 + c^2 - c^2) = 1$

So $Q^{-1} = \begin{bmatrix} 1+c^2 & -c \\ -c & 1 \end{bmatrix}$.

$$L(c, \sigma^2) = \log_e p(y \mid c, \sigma^2) = -\log(2\pi) - \log(\sigma^2) - \frac{1}{2\sigma^2}(y_1, y_2) \begin{bmatrix} 1+c^2 & -c \\ -c & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= -\log(2\pi) - \log(\sigma^2) - \frac{1}{2\sigma^2}\left((1+c^2) y_1^2 - 2c\, y_1 y_2 + y_2^2 \right)$$

For fixed σ^2, the maximizing $c = \hat{c}$ satisfies

$$\partial/\partial c \left[\hat{c}^2 y_1^2 - 2\hat{c}\, y_1 y_2 \right] = 0 \implies \hat{c} = y_1 y_2 / y_1^2$$

Then

$$L(\hat{c}, \sigma^2) = -\log(2\pi) - \log(\sigma^2) - \frac{1}{2\sigma^2}\left(y_1^2 + y_2^2 - \frac{(y_1 y_2)^2}{y_1^2} \right)$$

$$= \text{const.} - \log(\sigma^2) - \frac{1}{2\sigma^2} y_1^2$$

The maximizing $\sigma^2 = \hat{\sigma}^2$ satisfies

$$\partial/\partial \sigma^2\, L(\hat{c}, \hat{\sigma}^2) = -\frac{1}{\hat{\sigma}^2} + \frac{1}{2\hat{\sigma}^4} y_1^2 = 0$$

Hence $\hat{\sigma}^2 = \frac{1}{2} y_1^2$.

We have shown that the maximum likelihood estimates are

$\hat{c} = y_1 / y_2$ and $\hat{\sigma}^2 = \frac{1}{2} y_1^2$.

We see $E[\hat{\sigma}^2] = \frac{1}{2} E[y_1^2] = \frac{1}{2} E[e_1^2] = \frac{1}{2}\sigma^2$. bias $= \frac{1}{2}\sigma^2$

(b) Generalized Least Squares Algorithm:

Choose $D^1(z)$ ($=1$, say). Compute $y_t' = D^0(z) y_t$, $u_t' = D^0(z) u_t$

Obtain LS estimate $A'(z), B'(z)$ of $A(z), B(z)$ for model

$$A(z)\, y_t' = B(z)\, u_t' + \text{"noise"}$$

given y_t', u_t'.

Calculate residuals $n_t' = A(z) y_t' - B(z) u_t'$.

Obtain LS estimate $D^2(z)$ of $D(z)$ for model

$$D(z)\, n_t' = 0 + \text{"noise"}$$

Repeat to obtain $(A^2(z), B^2(z))$, (D^3, A^3, B^3), ...