UNIVERSITY OF LONDON [C145 2003]

B.ENG. AND M.ENG. EXAMINATIONS 2003

For Internal Students of Imperial College London

This paper is also taken for the relevant examination for the Associateship of the City & Guilds of London Institute

COMPUTING C145

MATHEMATICAL METHODS AND GRAPHICS

Date Thursday 8th May 2003 3.30 - 5.30 pm

Answer FOUR questions

[*Before starting, please make sure that the paper is complete. There should be a total of SIX questions. Ask the invigilator for a replacement if this copy is faulty.*]

1. (i) What is the purpose of a projection and how is it achieved? Give two examples of commonly used projections and mention the differences between them.

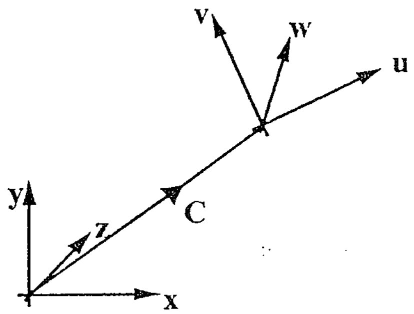
   (ii) How would you define 3D homogeneous coordinates? Explain why they are used and compare the two types.

   (iii) A simple definition of a convex object is one where the line joining any two points on the surface is completely contained within the volume. An object produced by the intersection of infinite planes is always convex. Write a simple algorithm in pseudocode to test the convexity of an object bounded by planar facets.

   (iv) One of the faces of a convex object has vertices $(-1, 0, 0)$ $(0, -1, 0)$ $(0, 0, -1)$. Obtain the cartesian equation of the plane in which the face lies. If an internal point of the object is $(6, -2, 6)$, determine if the face is visible when the viewpoint is at the origin.

   (*The four parts carry equal marks.*)

2. A graphics scene is made up of points defined in a world coordinate system denoted by $[x, y, z]$. As part of an animation sequence, the graphics scene is to be viewed from a point $\mathbf{C} = [C\mathrm{x}, C\mathrm{y}, C\mathrm{z}]$ (in the world coordinate system). The viewing coordinate system is defined by three unit vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ defined in the world coordinate system as indicated in the diagram below.



(i) Given a point $\mathbf{P}$ in the world coordinate system, calculate the corresponding point in the $[u, v, w]$ space using the dot product.

(ii) Using the result from part (i), derive the transformation matrix that will transform the points in the scene from world coordinates to $[u, v, w]$ coordinates.

(iii) Each row of the transformation matrix obtained in part (ii) can be treated as a vector. Explain the meaning of each of these four vectors in terms of the two coordinate systems.

(iv) After transformation, the scene is to be drawn using perspective projection with the view plane at a distance of 2 units from the origin along the $\mathbf{w}$ direction. Find the projection matrix for perspective projection and the matrix that will first transform the points and then project them.

(*The four parts carry equal marks.*)

3. (i) Find $\underline{a} \wedge \underline{b}$ where $\underline{a} = (1, 2, 2)$ and $\underline{b} = (2, 1, 1)$. Hence or otherwise find the equation of the plane that passes through the point with coordinates $(4, 2, 1)$ and is parallel to the vectors $\underline{a}$ and $\underline{b}$.

(ii) Find the determinant of the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 8 \\ 0 & 1 & 2 \end{pmatrix}.$$

(iii) Consider the simultaneous equations

$$\begin{array}{rcrcrcl} x & + & 2y & + & 3z & = & 1 \\ 2x & + & 5y & + & 8z & = & 3 \\ & & y & + & 2z & = & b \end{array}$$

where b is a constant. By using Gaussian elimination, or otherwise, find the value of b for which these equations have solutions, and obtain the general solution in this case. Explain how this is related to the result of part (ii).

(Parts (i), (ii) and (iii) carry 30%, 20% and 50% of the marks respectively.)

4. (i) Express the following complex numbers in the form $x + i\,y$, where x and y are real numbers, expressing numbers using powers, e.g. in the form 2^n, if appropriate.

(a) $\dfrac{5}{4+3i}$. (b) $(1+i)^{100}$.

(ii) Find the solutions of the following equation in the polar form $r\,e^{i\theta}$.

$$z^4 - z^2 + 1 = 0.$$

Sketch the positions of the solutions in the complex plane.

(iii) Find all the complex numbers which satisfy the relation

$$|z - 1| = |z - i|.$$

(Parts (i), (ii) and (iii) carry 35%, 45% and 20% of the marks respectively.)

5. Find the general solution of the difference equation

$$u_n - 5u_{n-1} + 6u_{n-2} = F(n)$$

in the following cases.

(i) $F(n) = 0$.

(ii) $F(n) = 2n + 3$.

(iii) $F(n) = 2^n$.

Find the solution in case (ii) when $u_0 = 5$ and $u_1 = 7$.

(The cases (i), (ii), (iii) and the last part carry 20%, 30%, 30% and 20% of the marks respectively.)

6. (i) Show that the function $u(x,y) = x^2 e^{x/y}$ satisfies the equation

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u.$$

(ii) Find the stationary points of the function

$$f(x,y) = y^2 + x^2 y - 2y,$$

but do not give their nature.

iii) The surface area of a closed circular cylinder of radius x and length y is given by

$$S(x,y) = 2\pi x^2 + 2\pi xy.$$

Find the total derivative of $S(x,y)$.

Using the above, find the *approximate* change in $S(x,y)$ if the radius x increases by $0.01\,\text{cm}$ from $1\,\text{cm}$, and the length y increases by $0.02\,\text{cm}$ from $3\,\text{cm}$.

(Parts (i), (ii) and (iii) carry 30%, 40% and 30% of the marks respectively.)

END OF PAPER

MATHEMATICS DEPARTMENT

MATHEMATICAL FORMULAE

1. VECTOR ALGEBRA

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1,\, a_2,\, a_3)$$

Scalar (dot) product: $\mathbf{a}\,.\,\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

Vector (cross) product:

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Scalar triple product:

$$[\mathbf{a},\,\mathbf{b},\,\mathbf{c}] = \mathbf{a}\,.\,\mathbf{b}\times\mathbf{c} = \mathbf{b}\,.\,\mathbf{c}\times\mathbf{a} = \mathbf{c}\,.\,\mathbf{a}\times\mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product: $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{c}\,.\,\mathbf{a})\mathbf{b} - (\mathbf{b}\,.\,\mathbf{a})\mathbf{c}$

2. SERIES

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \dots \quad (\alpha \text{ arbitrary, } |x| < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots ,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^n\frac{x^{2n}}{(2n)!} + \dots ,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + \dots ,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^n\frac{x^{n+1}}{(n+1)} + \dots (-1 < x \le 1)$$

3. TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin(a+b) = \sin a\,\cos b + \cos a\,\sin b\ ;$$

$$\cos(a+b) = \cos a\,\cos b - \sin a\,\sin b\,.$$

$$\cos iz = \cosh z\,;\quad \cosh iz = \cos z\,;\quad \sin iz = i\sinh z\,;\quad \sinh iz = i\sin z\,.$$

4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f\,D^n g + \tbinom{n}{1}\,Df\,D^{n-1}g + \dots + \tbinom{n}{r}\,D^r f\,D^{n-r}g + \dots + D^n f\,g\,.$$

(b) Taylor's expansion of $f(x)$ about $x = a$:

$$f(a+h) = f(a) + hf'(a) + h^2f''(a)/2! + \dots + h^nf^{(n)}(a)/n! + \epsilon_n(h)\,,$$

where $\epsilon_n(h) = h^{n+1}f^{(n+1)}(a+\theta h)/(n+1)!,\quad 0 < \theta < 1\,.$

(c) Taylor's expansion of $f(x,\,y)$ about $(a,\,b)$:

$$f(a+h,\,b+k) = f(a,\,b) + [hf_x + kf_y]_{a.b} + 1/2!\left[h^2f_{xx} + 2hkf_{xy} + k^2f_{yy}\right]_{a.b} + \dots$$

(d) Partial differentiation of $f(x,\,y)$:

i. If $y = y(x)$, then $f = F(x)$, and $\dfrac{dF}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dx}$.

ii. If $x = x(t)$, $y = y(t)$, then $f = F(t)$, and $\dfrac{dF}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$.

iii. If $x = x(u,\,v)$, $y = y(u,\,v)$, then $f = F(u,\,v)$, and

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}\,,\qquad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}\,.$$

(e) Stationary points of $f(x,\,y)$ occur where $f_x = 0$, $f_y = 0$ simultaneously.
Let $(a,\,b)$ be a stationary point: examine $D = [f_{xx}f_{yy} - (f_{xy})^2]_{a.b}$.
If $D > 0$ and $f_{xx}(a,\,b) < 0$, then $(a,\,b)$ is a maximum;
If $D > 0$ and $f_{xx}(a,\,b) > 0$, then $(a,\,b)$ is a minimum;
If $D < 0$ then $(a,\,b)$ is a saddle-point.

(f) Differential equations:

i. The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating factor $I(x) = \exp[\int P(x)(dx]$, so that $\dfrac{d}{dx}(Iy) = IQ$.

ii. $P(x,\,y)dx + Q(x,\,y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y$.

5. INTEGRAL CALCULUS

(a) An important substitution: $\tan(\theta/2) = t$:
$\sin\theta = 2t/(1+t^2)$, $\cos\theta = (1-t^2)/(1+t^2)$, $d\theta = 2\,dt/(1+t^2)$.

(b) Some indefinite integrals:

$$\int (a^2 - x^2)^{-1/2} dx = \sin^{-1}\left(\frac{x}{a}\right), \quad |x| < a .$$

$$\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1}\left(\frac{x}{a}\right) = \ln\left\{\frac{x}{a} + \left(1 + \frac{x^2}{a^2}\right)^{1/2}\right\} .$$

$$\int (x^2 - a^2)^{-1/2} dx = \cosh^{-1}\left(\frac{x}{a}\right) = \ln\left|\frac{x}{a} + \left(\frac{x^2}{a^2} - 1\right)^{1/2}\right| .$$

$$\int (a^2 + x^2)^{-1} dx = \left(\frac{1}{a}\right)\tan^{-1}\left(\frac{x}{a}\right) .$$

6. NUMERICAL METHODS

(a) Approximate solution of an algebraic equation:

If a root of $f(x) = 0$ occurs near $x = a$, take $x_0 = a$ and
$x_{n+1} = x_n - [f(x_n)/f'(x_n)]$, $n = 0, 1, 2 \ldots$

(Newton Raphson method).

(b) Formulae for numerical integration: Write $x_n = x_0 + nh$, $y_n = y(x_n)$.

i. Trapezium rule (1-strip): $\int_{x_0}^{x_1} y(x)dx \approx (h/2)[y_0 + y_1]$.

ii. Simpson's rule (2-strip): $\int_{x_0}^{x_2} y(x)dx \approx (h/3)[y_0 + 4y_1 + y_2]$.

(c) Richardson's extrapolation method: Let $I = \int_a^b f(x)dx$ and let I_1, I_2 be two estimates of I obtained by using Simpson's rule with intervals h and $h/2$.

Then, provided h is small enough,

$$I_2 + (I_2 - I_1)/15 ,$$

is a better estimate of I .

7. LAPLACE TRANSFORMS

Function	Transform	Function	Transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)dt$	$af(t) + bg(t)$	$aF(s) + bG(s)$
df/dt	$sF(s) - f(0)$	d^2f/dt^2	$s^2F(s) - sf(0) - f'(0)$
$e^{at}f(t)$	$F(s-a)$	$tf(t)$	$-dF(s)/ds$
$(\partial/\partial\alpha)f(t,\alpha)$	$(\partial/\partial\alpha)F(s,\alpha)$	$\int_0^t f(t)dt$	$F(s)/s$
$\int_0^t f(u)g(t-u)du$	$F(s)G(s)$		
1	$1/s$	$t^n (n = 1, 2 \ldots)$	$n!/s^{n+1}$, $(s > 0)$
e^{at}	$1/(s-a)$, $(s > a)$	$\sin\omega t$	$\omega/(s^2+\omega^2)$, $(s > 0)$
$\cos\omega t$	$s/(s^2+\omega^2)$, $(s > 0)$	$H(t-T) = \begin{cases} 0, & t < T \\ 1, & t > T \end{cases}$	e^{-sT}/s , $(s, T > 0)$

8. FOURIER SERIES

If $f(x)$ is periodic of period $2L$, then $f(x + 2L) = f(x)$, and

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} , \text{ where}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad n = 0, 1, 2, \ldots , \text{ and}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1, 2, 3, \ldots$$

Parseval's theorem

$$\frac{1}{L}\int_{-L}^{L} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right) .$$

September 2000