SOLUTIONS: Control Engineering

1. a) i) Applying Kirchhoff's law on the loop,

$$\boxed{v_i(t) = L\ddot{q}(t) + R\dot{q}(t) + C^{-1}q(t).}$$

ii) Taking Laplace transform gives the transfer function

$$\boxed{\frac{q(s)}{v_i(s)} = \frac{1}{Ls^2 + Rs + C^{-1}}}$$

iii) Comparing the transfer function with the standard second order form

$$G(s) = C\frac{(LC)^{-1}}{s^2 + sRL^{-1} + (LC)^{-1}} = K\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

gives $K = C$, $\omega_n = \frac{1}{\sqrt{LC}}$ and $\zeta = 0.5R\sqrt{\frac{C}{L}}$. The second specification demands $\zeta = \frac{1}{\sqrt{2}}$ for 5% maximum overshoot while the first demands $\frac{4}{\zeta\omega_n} = 10^{-3}$.

A. It follows that $\boxed{R = 8 \times 10^3\ \Omega}$ and $\boxed{C = 31.25 \times 10^{-9}\ F}$.

B. The steady state output is simply $G(0) = C$ and so $\boxed{q_{ss} = 31.25 \times 10^{-9}.}$

b) i) A computation gives

$$\boxed{\frac{e(s)}{v_i(s)} = \frac{s(s^2 + K_2 s + K_2)}{s^3 + K_2 s^2 + K_2 s + K_1}}$$

ii) The Routh array is given by

$$\begin{array}{c|cc} s^3 & 1 & K_2 \\ s^2 & K_2 & K_1 \\ s & \frac{K_2^2 - K_1}{K_2} & \\ 1 & K_1 & \end{array}$$

It follows that $\boxed{K_2 > 0,\ K_1 > 0 \text{ and } K_1 < K_2^2}$ for closed-loop stability.

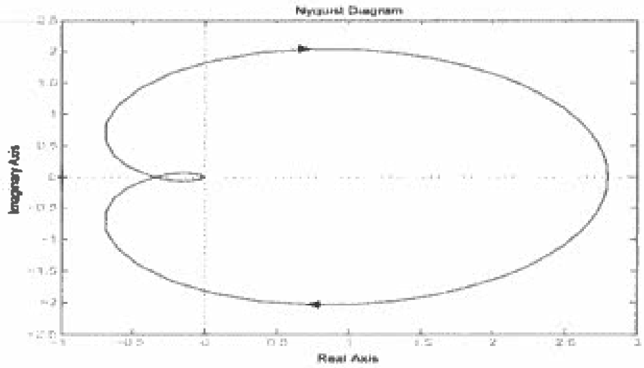
iii) For a ramp, $v_i(s) = 1/s^2$. Using the final value theorem:

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} se(s) = \lim_{s\to 0} s\frac{1}{s^2}\frac{s(s^2 + K_2 s + K_2)}{s^3 + K_2 s^2 + K_2 s + K_1} = \boxed{\frac{K_2}{K_1}}$$

iv) Since $K_2 = 1$, $K_1 < 1$ for stability and the steady-state error is $1/K_1$. It follows that the minimum value of the steady-state error is $\boxed{1}$

2. The transfer function used in fact was $G(s) = 0.35/(s+0.5)^3$, although this is not required.

a) The real axis intercepts can be obtained from the frequency response (when the phase is 0, -180° and -270° and are approximately given by $\boxed{2.8, -0.35 \text{ and } 0.}$ The Nyquist plot is given below.



b) From the intercepts above, the gain margin is approximately $\boxed{2.9}$. The phase margin can be obtained from the frequency response (by inspecting the phase when the gain is 1) and is approximately $\boxed{45^\circ}$.

c) Let $K(s) = k$. The Nyquist criterion states that $N = Z - P$, where N is the number of clockwise encirclements by the Nyquist diagram of the point $-k^{-1}$, P is the number of unstable open–loop poles and Z is the number of unstable closed–loop poles. Since $G(s)$ is stable, $P = 0$. An inspection of the Nyquist diagram shows that

i) When $k = 1$, $N = 0$ so $\boxed{Z = 0}$

ii) When $k = 10$, $N = 2$ so $\boxed{Z = 2}$

d) An inspection of the frequency response reveals this is a proportional–plus–integral (PI) compensator. This can be written as

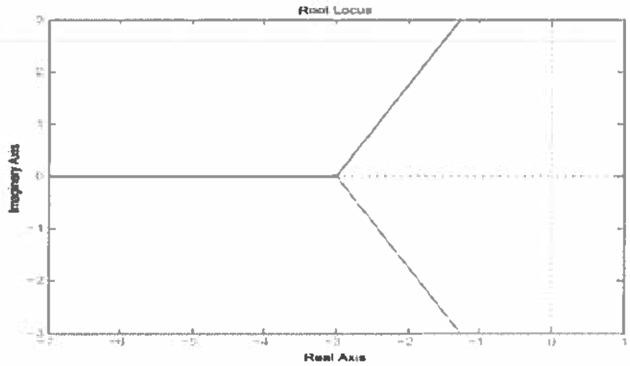
$$K(s) = K_P + \frac{K_I}{s} = K_I \frac{1 + \dfrac{s}{K_I/K_P}}{s}$$

It has high gain at frequencies below $\omega_0 = K_I/K_P$ and gain close to K_P beyond ω_0. The phase is negative and large below ω_0 but insignificant above. It follows that by varying K_I and K_P we can use PI compensation to increase low frequency gain (hence improving tracking properties) without introducing phase-lag at high frequency (which would reduce the phase margin) by placing w_0 in the 'middle' frequency range. Since the cross–over frequency for $G(s)$ is approximately 0.8 and ω_0 for $K(s)$ is approximately 0.1, this condition is satisfied.

3. a) For a maximum overshoot of 5% and a settling time of 2 seconds the closed-loop poles must be placed at $s_1, \bar{s}_1 = -2 \pm j2$.

b) We set $K(s) = k$ where $k > 0$.

i) The root-locus plot is shown below.



ii) The closed-loop poles are the roots of $1 + kG(s) = 0$ or

$$s^3 + 9s^2 + 27s + 27 + k = 0.$$

The Routh array is given by

$$\begin{array}{c|cc} s^3 & 1 & 27 \\ s^2 & 9 & 27+k \\ s & \frac{216-k}{9} & \\ 1 & 27+k & \end{array}$$

Thus for closed-loop stability, $-27 < k < 216$ and since $k > 0$ by assumption, $\boxed{k < 216.}$

iii) To achieve the design specifications, $s_1 = -2 + j2$ must lie on the root-locus. Equivalently, $1/G(s_1) = (s_1 + 3)^3$ must be negative. However

$$(s_1 + 3)^3 = (1 + j2)^3 = -11 - j2$$

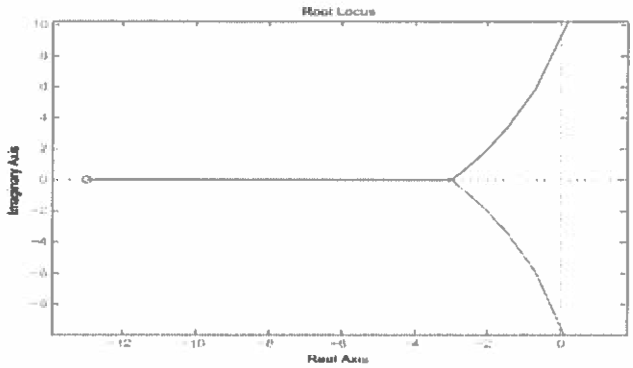
from the hint and so the design specifications cannot be satisfied.

c) A PD compensator has the form

$$K(s) = K_P + sK_D = K_D(s + K_P/K_D) = K_D(s - z), \qquad z = -K_P/K_D.$$

i) We use the angle criterion to find z. The sum of the angles from s_1 to the three open–loop poles at -3 is $3\tan^{-1}(2) \approx 3 \times 63.4° \approx 190.3°$ from the question hint. So, the angle that the zero makes with s_1 is $190.3° - 180° \approx 10.3°$. It follows that the zero must be at $\boxed{-13}$ from the second hint.

ii) The root–locus is shown below.



iii) Using the gain criterion, $K_D = -(s_1+3)^3/(s_1+13) = -(1+j2)^3/(11+j2) = 1$. It follows that $K_P = 13$. So,

$$\boxed{K(s) = 13 + s}$$