EE4-29
EE9-CS3-2

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2015

MSc and EEE/EIE PART IV: MEng and ACGI

Corrected Copy

OPTIMIZATION

Thursday, 30 April 10:00 am

Time allowed: 3:00 hours

There are FOUR questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s) :	A. Astolfi
	Second Marker(s) :	P.L. Dragotti

OPTIMISATION

1. Consider the function

$$f(x) = (x_1 - 2)^4 + (x_1 - 2)^2 x_2^2 + (x_2 + 1)^2.$$

a) Compute the unique stationary point $x_\star$ of the function f. [2 marks]

b) Using second order sufficient conditions of optimality show that the stationary point determined in part a) is a local minimizer. Hence, show that f is radially unbounded and that the stationary point determined in part a) is the global minimizer of f. [4 marks]

c) Write the modified Newton's iteration for the minimization of the function f given by

$$x_{k+1} = x_k - \left[\nabla^2 f(x_\star)\right]^{-1} \nabla f(x_k).$$

[4 marks]

d) Run five steps of the modified Newton's iteration in part c) from the starting point $(1.5, 0)$. [4 marks]

e) Run four steps of the modified Newton's iteration in part c) from the starting point $(1, 0)$. [2 marks]

f) Show that the research directions generated by the modified Newton's iteration in part c) are descent directions satisfying the condition of angle. Explain why the iteration is not globally convergent. [4 marks]

2. Consider the function

$$f(x) = \frac{1}{2}x_1^2 + \frac{m}{2}x_2^2,$$

with $m > 0$. The function has a global minimizer at $x_\star = 0$.

a) Show that the gradient algorithm with exact line search for the function f can be written as

$$x_{k+1} = x_k - \frac{x_{1,k}^2 + m^2 x_{2,k}^2}{x_{1,k}^2 + m^3 x_{2,k}^2} \begin{bmatrix} x_{1,k} \\ m\, x_{2,k} \end{bmatrix}$$

[6 marks]

b) Let $m = 9$ and $x_0 = [9, 1]^T$. Show that the sequence of points generated by the gradient algorithm is given by

$$x_k = \begin{bmatrix} 9 \\ (-1)^k \end{bmatrix} (0.8)^k.$$

(Hint: assume that for the given values of m and x_0 the quantity

$$\frac{x_1^2 + m^2 x_2^2}{x_1^2 + m^3 x_2^2}$$

remains constant for all iterations of the algorithm.)

[8 marks]

c) Compute the speed of convergence of the sequence generated by the algorithm and in particular show that

$$\frac{\|x_{k+1} - x_\star\|}{\|x_k - x_\star\|} = \text{constant}$$

for every k, where $\|v\| = \sqrt{v^T v}$. [6 marks]

3. Consider the optimization problem

$$\min_{x_1,x_2} 2x_1^2 + 9x_2,$$

$$x_1 + x_2 \geq 4.$$

a) State first order necessary conditions of optimality for such a constrained optimization problem. [4 marks]

b) Using the conditions derived in part a) compute candidate optimal solutions. [4 marks]

c) This constrained optimization problem can be transformed into an unconstrained optimization problem by defining the so-called barrier function

$$B_r(x) = 2x_1^2 + 9x_2 + r\frac{-1}{4-x_1-x_2},$$

with $r > 0$. Determine the stationary points of the function B_r. Show that there is only one stationary point in the admissible set, for all $r > 0$, and that this stationary point converges to the candidate optimal solution determined in part b). [6 marks]

d) By comparing the necessary conditions of optimality for the constrained optimization problem and the necessary conditions of optimality for the minimization of the function B_r, compute the optimal multiplier of the problem and show that this coincides with the value determined in part b). [6 marks]

4. Consider the optimization problem

$$\min_{x_1,x_2} 2(x_1^2+x_2^2-1)-x_1,$$
$$x_1^2+x_2^2-1=0.$$

This is a well-known problem which is used to illustrate the so-called Maratos effect: the slow convergence of algorithms exploiting the linearization of the constraints.

a) State first order necessary conditions of optimality for this constrained optimisation problem. [2 marks]

b) Using the conditions derived in part a) compute candidate optimal solutions. [4 marks]

c) Using second order sufficient conditions of optimality determine the solution $x_\star$ of the optimization problem and the corresponding optimal multiplier $\lambda_\star$. [4 marks]

d) Consider a point $x_k = (x_{1,k}, x_{2,k}) = (\cos\theta_k, \sin\theta_k)$. This point is a feasible point. To determine an update for the point x_k one has to update the value of θ_k. This can be done using the following procedure.

i) Show that the linearization of the constraint around the point x_k is the constraint

$$2\cos\theta_k x_1 + 2\sin\theta_k x_2 - 2 = 0.$$

[2 marks]

ii) Consider the constrained minimization problem given by

$$\min_{x_1,x_2} 2(x_1^2+x_2^2-1)-x_1,$$
$$2\cos\theta_k x_1 + 2\sin\theta_k x_2 - 2 = 0.$$

Using first order necessary conditions of optimality determine a candidate optimal solution

$$x_\star(\theta_k) = \begin{bmatrix} x_{1,\star}(\theta_k) \\ x_{2,\star}(\theta_k) \end{bmatrix}.$$

[4 marks]

iii) Consider the update law

$$\theta_{k+1} = \left(\frac{x_{2,\star}}{x_{1,\star}}\right),$$

with $x_{1,\star}(\theta_k)$ and $x_{2,\star}(\theta_k)$ as determined in part d.ii). Let $\theta_0 = 0.1$ and run five iterations of the algorithm. Argue that the sequence converges to $\theta = 0$ with linear speed of convergence. [4 marks]