Imperial College London

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2012

This paper is also taken for the relevant examination for the Associateship.

# M34PM19

# MeasureTheory & Integration

Date: examdate Time: examtime

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. i. Consider sets of the plane in $E = \{0 \leq x \leq 1, 0 \leq y \leq 1\}$. Give the definition of the outer measure, a measurable set (in Lebesgue sense) and its measure.
ii. Let $f : X \to \mathbb{R}$, $g : X \to \mathbb{R}$ be measurable functions on a measure space. Show that $f + g$ is measurable.

2. i. Show that convergence in $L^1(X, \mu)$ implies convergence in measure.
ii. For any $\epsilon \in (0, 1)$ construct an open set $A \in [0, 1]$ such that it is dense in $[0, 1]$ (i.e. its closure contains $[0, 1]$) and the Lebesgue measure $\mu(A) \leq \epsilon$.

3. Let $(X, M, \mu)$ be a complete measure space. Let $f : X \to [0, +\infty)$ be measurable and $\int_X f d\mu = c$, where $0 < c < \infty$. Prove the following:

$$\lim_{n\to\infty} \int_X n \log\left(1 + \frac{f}{n}\right) d\mu = c,$$

where "log" is the natural logarithm.

4. i. State the Poincare recurrence theorem.
ii. Show that the product of two absolutely continuous functions on a bounded interval $[a, b]$ is absolutely continuous.

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| Question 1 | | Marks & seen/unseen |
| Parts i. | The outer measure of a set $A$ is $\mu^*(A) = \inf\limits_{A \subset \cup P_k} \sum_k m(P_k)$ over all coverings of $A$ by finite or countable number of rectangles $P_k$; $m(P_k)$ is the area (measure) of rectangle $P_k$. Let $\mathcal{R}(\mathcal{L})$ be the minimal ring generated by the semiring $\mathcal{L}$ of rectangles. A set $A$ is called measurable if $\forall \varepsilon > 0 \ \exists B \in \mathcal{R}(\mathcal{L})$ such that $\mu^*(A \triangle B) < \varepsilon$. $\mu^*$ restricted to measurable sets is called measure ($\mu$). | 10 seen |
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| Question 1 | | Marks & seen/unseen |
| Parts *ii* | $f: X \to \mathbb{R}$, $g: X \to \mathbb{R}$ measurable means $\forall c \in \mathbb{R}$ the sets $\{x : f(x) < c\}$, $\{x : g(x) < c\}$ are measurable.<br>Therefore, $a \cdot f$, $a + f$, $a = \text{const}$ are measurable functions.<br>We have $\{x : f(x) > g(x)\} =$<br>$= \bigcup_{k=1}^{\infty} \left( \{x : f(x) > r_k\} \cap \{x : g(x) < r_k\} \right)$, $r_k \in \mathbb{Q}$. This set is measurable because measurable sets form a $\sigma$-algebra. Therefore also<br>$\{x : f(x) > a + (-1) g(x)\} =$<br>$= \{x : f(x) + g(x) > a\}$ is measurable $\forall a$. Thus,<br>$f + g$ is a measurable function. | 10 seen |
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| Question 2 | | Marks & seen/unseen |
| Parts i | Consider a measure space $(X, \mathcal{M}, \mu)$ | 5 seen |

Let a sequence of measurable functions $f_n$ converge to $f$ in $\mathcal{L}^1(X,\mu)$, i.e.,

$$\int_X |f - f_n| \, d\mu \to 0, \quad n \to \infty,$$

$$f_n, f \in \mathcal{L}^1.$$

By Chebyshev inequality, for $\delta > 0$,

$$\mu\{x : |f_n - f| \geq \delta\} \leq \frac{1}{\delta} \int_X |f - f_n| \, d\mu$$

and therefore

$$\lim_{n \to \infty} \mu\{x : |f_n - f| \geq \delta\} = 0$$

$\forall \delta > 0$, i.e.,

$f_n \to f$ in measure.

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| Question 2 | | Marks & seen/unseen |
| Parts | | |
| ii | Let $q_n$, $n=1,2,\dots$ be rational numbers in $[0,1]$<br>Consider $A=\bigcup_{n=1}^{\infty}\left(q_n-\frac{\varepsilon}{2^{n+1}}, q_n+\frac{\varepsilon}{2^{n+1}}\right)$<br>– measurable and open as countable union of open intervals<br>$q_n$ are dense in $[0,1]$ so $A$ is dense in $[0,1]$<br>$\mu(A)\le\sum_{n=1}^{\infty}\frac{\varepsilon}{2^n}=\varepsilon$<br>By subadditivity. | 15<br>unseen |
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| Question 3 | | Marks & seen/unseen |
| Parts | $f : X \to [0,+\infty)$, $\int_X f\,d\mu = c$ | 20 unseen |

First, note that

$$\log(1+z) \le z\ , \quad z \ge 0$$

Indeed, let $g = \log(1+z) - z$

$$g' = \frac{1}{1+z} - 1 = \frac{-z}{1+z} \le 0\ , \ z \ge 0$$

$g(0) = 0$ and $g(z)$ is decreasing so $g(z) \le 0$, $z \ge 0$.

Thus, $0 \le n\log(1+\frac{f}{n}) \le f$,

$\int f\,d\mu < \infty$ ; $n\log(1+\frac{f}{n}) \xrightarrow[n\to\infty]{} f$.

Therefore, by Lebesgue's thm

$$\lim_{n\to\infty} \int_X n\log(1+\frac{f}{n})\,d\mu =$$

$$= \int_X \lim_{n\to\infty} n\log(1+\frac{f}{n})\,d\mu =$$

$$= \int_X f\,d\mu = c\ .$$

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| Question 4 | | Marks & seen/unseen |
| Parts i | Let $(X, \mathcal{L}_X, \mu)$ - probability space, let $E \in \mathcal{L}_X$, $\mu(E) > 0$, $T: X \to X$ - measure-preserving. Then almost all points of $E$ return infinitely often to $E$ under positive iteration by $T$ (i.e. there exists $F \subset E$, $\mu(F) = \mu(E)$ such that $\forall x \in F$ there is a sequence of positive integers $n_1, n_2, \dots$ with $T^{n_j}(x) \in F$, $j = 1, 2, \dots$) | 5 seen |
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| Question 4 | | Marks & seen/unseen |
| Parts ii | Let $f, g$ be absolutely continuous on $[a,b]$, i.e, $\forall \varepsilon_1 > 0 \ \exists \delta_1 > 0$ s.t. $$\sum_{j=1}^{n} \lvert f(b_j) - f(a_j) \rvert < \varepsilon_1$$ for any finite family of disjoint subintervals $(a_j, b_j)$ such that $\sum_{j=1}^{n} \lvert b_j - a_j \rvert < \delta_1$. Similarly, for $g$ (with $\varepsilon_2, \delta_2$) In particular, $f, g$ are continuous on $[a,b]$ and therefore bounded, as $[a,b]$ is compact. So $$\sum_{j=1}^{n} \lvert f(b_j) g(b_j) - f(a_j) g(a_j) \rvert =$$ | 15 unseen |
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| Question 4 | | Marks & seen/unseen |
| Parts | | |

$$= \sum_{j=1}^{n} \left| \left[f(b_j) - f(a_j)\right] g(b_j) + f(a_j)\cdot\left(g(b_j) - g(a_j)\right) \right|$$

$$\leq C_1 \sum_{j=1}^{n} |f(b_j) - f(a_j)| + C_2 \sum_{j=1}^{n} |g(b_j) - g(a_j)|$$

$$< \varepsilon_1 C_1 + \varepsilon_2 C_2 < \varepsilon$$

Taking $\delta = \min(\delta_1, \delta_2)$, we obtain that $f+g$ is a.c. on $[a,b]$.

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