E2.3A
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IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2007

EEE PART II: MEng, BEng and ACGI

Corrected Copy

A: DEVICES AND FIELDS

Tuesday, 29 May 2:00 pm

Time allowed: 2:00 hours

There are SIX questions on this paper.

Question ONE and Question FOUR are compuslory. Answer Question One, Question Four, plus one additional question from Section A and one additional question from Section B.

All questions carry equal marks and weighting.

Use a separate answer book for each section.

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s) :	K. Fobelets, R.R.A. Syms
	Second Marker(s) :	W.T. Pike, C.A. Hernandez-Aramburo

Special instructions for invigilators

This exam consists of **2 sections**. Section A: **Devices** and section B: **Fields**. Each section has to be solved in their respective answer books. Check that 2 different answer books are available for the students. Questions 1 and 4 are obligatory.

Special instructions for students

Use different answers books for each section:

Devices: answer book **A**
Fields: answer book **B**

Questions **1** and **4** are obligatory.

Constants and Formulae for section A: Devices

permittivity of free space:	$\varepsilon_o = 8.85 \times 10^{-12}$ F/m
permeability of free space:	$\mu_o = 4\pi \times 10^{-7}$ H/m
intrinsic carrier concentration in Si:	$n_i = 1.45 \times 10^{10}$ cm^{-3} at $T = 300$K
dielectric constant of Si:	$\varepsilon_{Si} = 11$
dielectric constant of SiO_2:	$\varepsilon_{ox} = 4$
thermal voltage:	$kT/e = 0.026$V at $T = 300$K
charge of an electron:	$e = 1.6 \times 10^{-19}$ C

$$\left.\begin{aligned} J_n(x) &= e\mu_n n(x)E(x) + eD_n\frac{dn(x)}{dx} \\ J_p(x) &= e\mu_p p(x)E(x) - eD_p\frac{dp(x)}{dx} \end{aligned}\right\}$$

$$I_{DS} = \frac{\mu C_{ox}W}{L}\left((V_{GS} - V_T)V_{DS} - \frac{V_{DS}^2}{2}\right)$$

$$\left.\begin{aligned} J_n &= \frac{eD_n n_p}{L_n}\left(e^{\frac{eV}{kT}} - 1\right) \\ J_p &= \frac{eD_p p_n}{L_p}\left(e^{\frac{eV}{kT}} - 1\right) \end{aligned}\right\}$$

$$V_0 = \frac{kT}{e}\ln\left(\frac{N_A N_D}{n_i^2}\right)$$ Built-in voltage

$$c = c_0 \exp\left(\frac{eV}{kT}\right) \text{ with } \begin{cases} c = p_n \text{ or } n_p \\ c_0 \text{ bulk minority carrier concentration} \end{cases}$$ Minority carrier injection under bias V

$$\delta c = \Delta c \exp\left(\frac{-x}{L}\right) \text{ with } \begin{cases} \delta c = \delta p_n \text{ or } \delta n_p \\ \Delta c \text{ the excess carrier concentration} \\ \text{at the edge of the depletion region} \end{cases}$$ Excess carrier concentration as a function of distance

$$L = \sqrt{D\tau}$$ Diffusion length

$$D = \frac{kT}{e}\mu$$ Einstein relation

Constants and Formulae for section B: Fields

Vector calculus (Cartesian co-ordinates)

$\nabla = \underline{i}\ \partial/\partial x + \underline{j}\ \partial/\partial y + \underline{k}\ \partial/\partial z$

$\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$

$\mathrm{grad}(\phi) = \nabla\phi = \underline{i}\ \partial\phi/\partial x + \underline{j}\ \partial\phi/\partial y + \underline{k}\ \partial\phi/\partial z$

$\mathrm{div}(\underline{\mathbf{F}}) = \nabla . \underline{\mathbf{F}} = \partial \mathbf{F}_x/\partial x + \partial \mathbf{F}_y/\partial y + \partial \mathbf{F}_z/\partial z$

$\mathrm{curl}(\underline{\mathbf{F}}) = \nabla \times \underline{\mathbf{F}} = \underline{i}\ \{\partial \mathbf{F}_z/\partial y - \partial \mathbf{F}_y/\partial z\} + \underline{j}\ \{\partial \mathbf{F}_x/\partial z - \partial \mathbf{F}_z/\partial x\} + \underline{k}\ \{\partial \mathbf{F}_y/\partial x - \partial \mathbf{F}_x/\partial y\}$

Where ϕ is a scalar field and $\underline{\mathbf{F}}$ is a vector field

Maxwell's equations – integral form

$\iint_A \underline{\mathbf{D}} . d\underline{a} = \iiint_V \rho\ dv$

$\iint_A \underline{\mathbf{B}} . d\underline{a} = 0$

$\int_L \underline{\mathbf{E}} . d\underline{L} = - \iint_A \partial\underline{\mathbf{B}}/\partial t . d\underline{a}$

$\int_L \underline{\mathbf{H}} . d\underline{L} = \iint_A [\underline{\mathbf{J}} + \partial\underline{\mathbf{D}}/\partial t] . d\underline{a}$

Where $\underline{\mathbf{D}}$, $\underline{\mathbf{B}}$, $\underline{\mathbf{E}}$, $\underline{\mathbf{H}}$, $\underline{\mathbf{J}}$ are time-varying vector fields

Maxwell's equations – differential form

$\mathrm{div}(\underline{\mathbf{D}}) = \rho$

$\mathrm{div}(\underline{\mathbf{B}}) = 0$

$\mathrm{curl}(\underline{\mathbf{E}}) = -\partial\underline{\mathbf{B}}/\partial t$

$\mathrm{curl}(\underline{\mathbf{H}}) = \underline{\mathbf{J}} + \partial\underline{\mathbf{D}}/\partial t$

Material equations

$\underline{\mathbf{J}} = \sigma\ \underline{\mathbf{E}}$

$\underline{\mathbf{D}} = \varepsilon\ \underline{\mathbf{E}}$

$\underline{\mathbf{B}} = \mu\ \underline{\mathbf{H}}$

Theorems

$\iint_A \underline{\mathbf{F}} . d\underline{a} = \iiint_V \mathrm{div}(\underline{\mathbf{F}})\ dv$ – Gauss' theorem

$\int_L \underline{\mathbf{F}} . d\underline{L} = \iint_A \mathrm{curl}(\underline{\mathbf{F}}) . d\underline{a}$ – Stokes' theorem

$\mathrm{curl}\ \{\mathrm{curl}(\underline{\mathbf{F}})\} = \mathrm{grad}\ \{\mathrm{div}(\underline{\mathbf{F}})\} - \nabla^2\underline{\mathbf{F}}$

SECTION A: SEMICONDUCTOR DEVICES

1. This question is **obligatory**.

a) Explain briefly the processes causing generation-recombination of carriers in semiconductors. [2]

b) What is the physical reason for the delay in minority carrier devices? Explain briefly. Note: stating "capacitance effects" is not sufficient, you have to explain the cause of the capacitive effect. [2]

c) Consider two identical enhancement mode p-channel MOSFETs (labelled 1 and 2) that only differ in the type of the gate contact. The gate contact workfunction relationship is: $\phi_1 > \phi_2$.
How do the threshold voltages, V_{th1} and V_{th2}, of these MOSFET relate (<,>,=)? Justify your answer. [5]

d) Many pn diodes do not have a perfect exponential current-voltage characteristic, but the current is better described by: $I = I_0 \exp\left(\frac{eV}{nkT}\right)$ with n the ideality factor (emission factor in SPICE). What are the causes for this deviation? [5]

e) Using formulae from the list on p.2, show that the current of an n^+pn BJT in active mode increases at large voltages across the base-collector junction. [6]

2. Given is a p^+n diode with the length of the p-region smaller than the minority carrier diffusion length and of the n-region larger than the minority carrier diffusion length. The doping density of the p region is $N_A=10^{18}$ cm^{-3} and of the n-region $N_D=10^{15}$ cm^{-3}. The area of the diode is 10^{-4}cm^2.

For this diode, answer the following questions.

a) Sketch the excess minority carrier concentration as a function of distance in both regions of the diode. Label your graphs, define all parameters you use and make sure that the relative magnitudes of the values of the parameters are correct. [5]

b) Sketch the variation of the electron I_n, hole I_p and total current I with distance in all regions of the pn diode under the assumption that there is no recombination in the depletion region. Label your graphs and make sure that the relative magnitudes of the graphs values of the parameters are correct. [5]

c) Derive the expression of the excess charge Q as a function of geometry and material parameters (doping density) in both regions of the diode from the equations given in the formulae sheet and with reference to your answer in a). [4]

d) Explain briefly why in order to reduce the switching time of this diode the length of the n-type region should be decreased. [2]

e) What is the maximum allowed length of the n-region in order to fulfil the requirement of d) given that the diode current at 0.7V forward bias is 10mA and the lifetime of the minority carriers is 1μs. Write down all assumptions you make. [4]

3. In the lab researchers measure the switching behaviour of a pnp bipolar transistor (BJT) at room temperature. They record the base and collector current and plot these characteristics together with the applied base voltage in Figure 3.1. They need your help to understand the shape of the collector current.

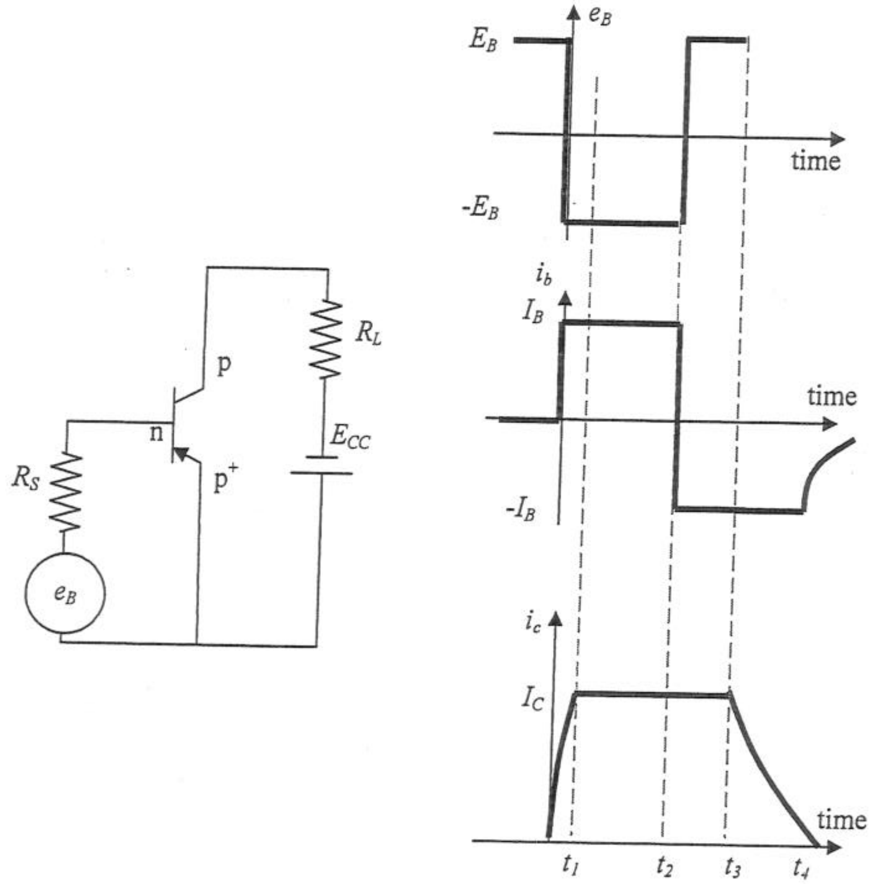


Figure 3.1 Left: The circuit for the switching experiment. Right: the measured graphs. From top to bottom: input signal on the base, base current variation and collector current variation as a function of time.

a) What happens at $t = t_1$? Give the theoretical expression of the approximate collector current I_C for $t_1 < t < t_3$. [3]

b) Give the relation between the base current and collector current for $t < t_1$. [2]

c) Sketch the variation of the charge in the base, Q_B as a function of time for $0 < t < t_4$. Indicate the times t_1, t_2, t_3 and t_4 on your plot. [4]

d) Give one plot with the minority carrier concentration as a function of distance in the small base region at both t_1 and t_2, and indicate on this plot the excess charge difference in 1D between t_1 and t_2. [4]

Question continues overleaf

e) In order to verify the experiment, the researcher also performs the switching experiment in SPICE.

i) Draw the large signal equivalent circuit transient model that SPICE uses to perform this simulation and state and define 4 DC SPICE parameters that would be needed for the input to the simulation to specify the model. [4]

ii) There are 3 types of AC parameters that need to be specified (in the BJT and diodes models) that SPICE uses to calculate the capacitances of the two junctions. What are these three parameters? [3]

SECTION B: ELECTROMAGNETIC FIELDS

4. This question is **obligatory**.

a) Explain how Marconi was able to communicate by low-frequency radio across the Atlantic ocean. How does modern long-distance radio communication operate? [4]

b) The main factors limiting the propagation of signals in the atmosphere are absorption, scattering and diffraction. Explain briefly the origin and frequency dependence of each effect. [4]

c) The governing equations for a transmission line are

$$dV/dz = -j\omega LI \text{ and } dI/dz = -j\omega CV$$

Where V is the voltage, I is the current and ω is the angular frequency, and C and L are the capacitance per unit length and the inductance per unit length, respectively. Show that the solutions $V = V_0 \exp(-jkz)$ and $I = I_0 \exp(-jkz)$ satisfy the equations, where V_0, I_0 and k and constants, and find the value of k. [4]

d) A light wave is incident on the interface between two dielectric media, as shown in fig. 4.1 below. Sketch the directions of the reflected and transmitted waves, assuming that $n_1 < n_2$. State Snell's law. Under what conditions may total internal reflection occur? [4]

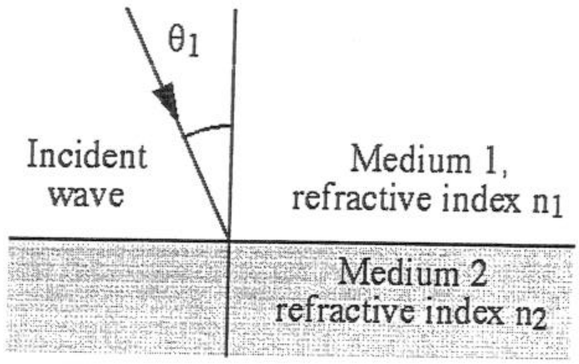


Figure 4.1 Light wave incident on the interface between two dielectric media.

Question continues overleaf

e) State the relation between the object distance u, the image distance v and the focal length f in an imaging operation by a lens. Show how a guiding structure may be formed from a set of such lenses. [4]

5. The figure 5.1 shows a coaxial transmission line. The inner conductor has a diameter of 0.9 mm. The dielectric between the inner and outer conductors has a diameter of 3.2 mm and a relative dielectric constant of $\varepsilon_r = 2.3$.

a) Assuming that the central conductor carries a current I, and has an instantaneous line charge q per unit length, obtain analytic expressions for the electric and magnetic field variations. Sketch the electric and magnetic field lines. [8]

b) Calculate the capacitance and inductance of the line, per unit length. A voltage of 1 V is applied between the inner and outer conductor. Where is the peak electric field located, and what is its value? [12]

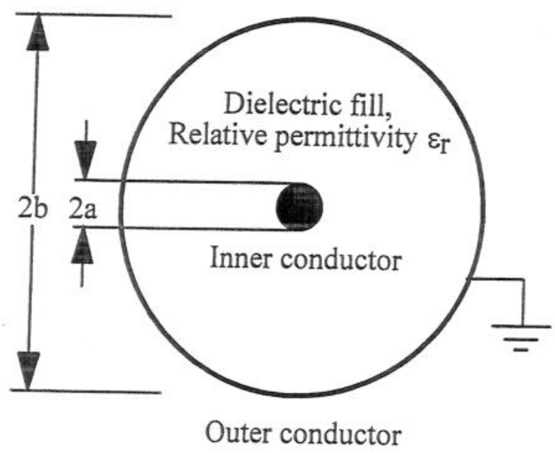


Figure 5.1: Schematic view of a coaxial transmission line

6.

a) In the differential form of Maxwell's equations, identify Gauss' law, Faraday's law and Ampere's law. Which term represents the displacement current density? What simplifications are made to describe propagation of electromagnetic waves in a dielectric medium? [6]

b) Derive a time-independent vector wave equation for single-frequency waves in a dielectric medium, in terms of the electric field alone. [10]

c) Assuming that the electric field is polarized in the x-direction, derive a time-independent scalar wave equation for electromagnetic waves travelling in the z-direction. Find the propagation constant and phase velocity of the wave. [4]

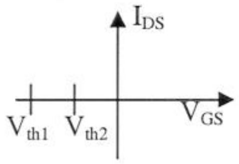
SECTION A: SEMICONDUCTOR DEVICES **ANSWERS**

1. This question is **obligatory**.

a) Generation of carriers is the creation of free carrier due to electron hole pair formation via the use of thermal energy. Recombination is the loss of free carriers via recombination of an electron and a hole or via recombination of a carrier and an ion (e.g. ionised doping atoms) [3]

b) Under forward bias there is an excess of minority carriers in each side of the pn junction with a magnitude proportional to the forward bias current. This excess has to be removed before the diode can switch off. [3]

c) $|V_{th1}|>|V_{th2}|$ or [4]



d) When the depletion region is large compared to the diffusion length of the carriers that are travelling through this region, recombination can occur that gives a loss of carriers and thus a loss of current. This happens at lower voltages. At higher voltages a large amount of majority carriers is injected which causes a deviation to the assumption that majority carriers remain constant. Both effects can be described by the introduction of n in the exponential. [5]

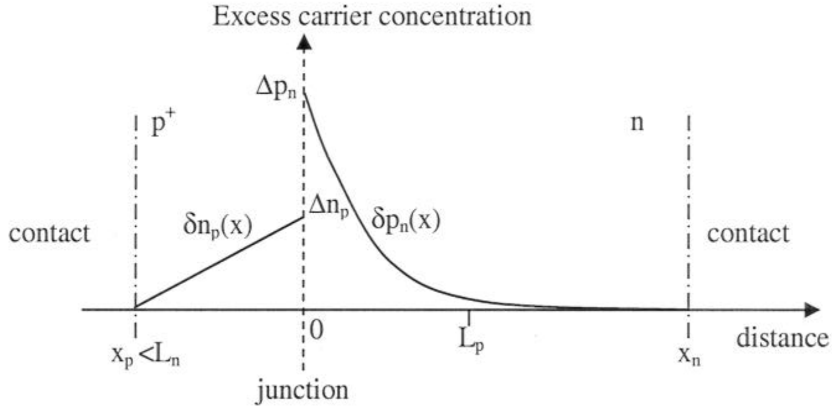
e) In an n^+pn BJT the main current component are electrons injected from the emitter into the base and then collected by the collector. This means that the collector current is mainly determined by diffusion of electrons in the base region and that is given by the electron current component in a pn junction in the formulae list:

$$I_n = \frac{eD_n n_p}{L_n}\left(e^{\frac{eV}{kT}}-1\right)A$$

(note the voltage V here is the base-emitter forward bias voltage).
Since the base is short in a BJT, the parameter L_n in the equation becomes W_b, the width of the base. In active mode the base collector junction is reverse biased and thus the depletion region extending into the base will increase with increasing reverse bias, thus making the effective base width smaller than the metallurgic base width. Since $I_c \approx I_n \propto \frac{1}{W_b}$ and W_b is decreasing thus I_c is increasing. [6]

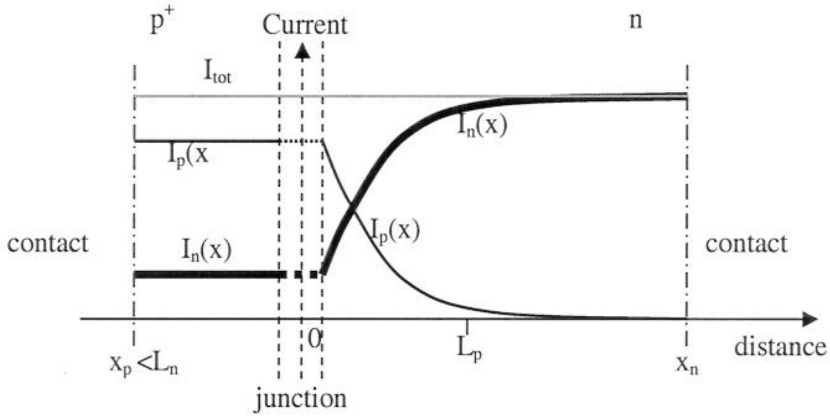
2. Given is a p^+n diode with the length of the p-region smaller than the minority carrier diffusion length and of the n-region larger than the minority carrier diffusion length. The doping density of the p region is $N_A=10^{18}$ cm^{-3} and of the n-region $N_D=10^{15}$ cm^{-3}. The area of the diode is 0.1cm^2.
For this diode, answer the following questions.

a) [5]



Definition of parameters: $x_{p,n}$ are respectively the lengths of the p and n region. $L_{p,n}$ are the diffusion lengths of the minority carriers in each material. δ indicates the excess carrier variation as a function of x and since it is excess the graphs should start from zero. Δ gives the excess carrier concentration at the edge of the depletion region. Note that the depletion region is ignored in the drawing.

b) [5]



The total current is $I_{tot}=I_p+I_n$ and is the grey line.
At x=0 the hole current (thin line) should be higher than the electron current (fat line). In the thin region, no recombination occurs and thus the current components are constant as a function of x. In the long region, recombination occurs and current components vary exponentially.

c) With reference to a) the excess charge Q in each region is given by the area underneath the graphs for the excess carriers. [4]
In the p-type region this is a triangle thus:

$$Q_n = -e\frac{\Delta n_p x_p}{2}A = -e\frac{n'_p x_p}{2}A = -e\frac{n_{p_0}\exp\left(\frac{eV}{kT}\right)x_p}{2}A = -e\frac{n_i^2\exp\left(\frac{eV}{kT}\right)x_p}{2N_A}A$$

In the n-type region the excess carrier concentration varies exponentially and thus the area under the curve has to be calculated via integrals:

$$Q_p = eA\int_0^{x_n}\delta p_n(x)dx = eA\int_0^{x_n}\Delta p_n \exp(\frac{-x}{L_p})dx = -eA\Delta p_n L_p \exp(\frac{-x}{L_p})\Big|_0^{x_n}$$

$$Q_p = -eA\Delta p_n L_p(0-1) = eA\Delta p_n L_p = e\frac{n_i^2\exp\left(\frac{eV}{kT}\right)L_p}{N_D}A$$

d) If the length of the n-region is reduced to below the minority carrier diffusion length (here minority carriers are holes, thus length x_n of n region has to be smaller than L_p) then the excess charge will reduce and thus the time to remove this excess charge too. Note that this can be easily deduced from the results in c). [2]

e) We have to determine L_p following the conclusion in d) as the requirement is that $x_n < L_p$. [5]

We have a p$^+$n diode with a doping density of the p-region much larger than the n-region. Thus the measured current will be approximately equal to the hole current ($I_p >> I_n$ in p$^+$n region), thus:

$I_{tot} \approx I_p \approx 1mA$

From the formulae sheet on page 1 we can find the hole current in a pn diode:

$$I_p = \frac{eD_p p_{n_0}}{L_p}\left(e^{\frac{eV}{kT}}-1\right)A \approx \frac{eD_p p_{n_0}e^{\frac{eV}{kT}}}{L_p}A \text{ (eq. 1) for large forward bias.}$$

From the excess charge derivation in c) we know that:

$$Q_p = eA\Delta p_n L_p = eAp_{n_0}e^{\frac{eV}{kT}}L_p \text{ (eq. 2)}$$

Combining (1) and (2) and $L = \sqrt{D\tau}$ (see formulae list) gives:

$$I_p = \frac{Q_p}{\tau_p} \text{ thus } Q_p = I_p\tau_p$$

From (2) we then find:

$$L_p = \frac{Q_p}{eAp_{n_0}e^{\frac{eV}{kT}}} = \frac{I_p\tau_p N_D}{eAn_i^2 e^{\frac{eV}{kT}}} = \frac{10^{-2}A\,10^{-6}s\,10^{15}cm^{-3}}{1.610^{-19}C\,10^{-4}cm^2\left(1.45\ 10^{10}\ \text{cm}^{-3}\right)^2\exp\left(\frac{0.7}{0.026}\right)}$$

$$L_p = 0.006cm$$

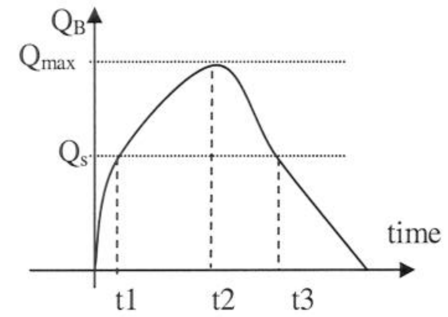
Thus x_n<0.006cm=60μm

3. BJT switching characteristics

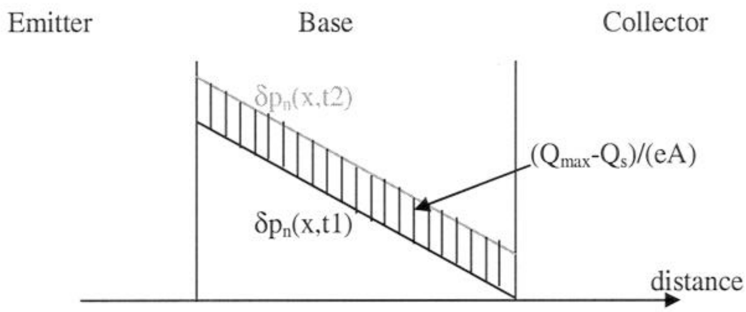
a) At t1 the BJT goes into saturation. This means that both base emitter and base collector junctions are forward biased. The charge builds up in the base to accommodate for the inflow of carriers. In saturation: $I_c \approx \dfrac{E_{CC}}{R_L}$. [3]

b) $I_c = \beta I_b$ [2]

c) [4]



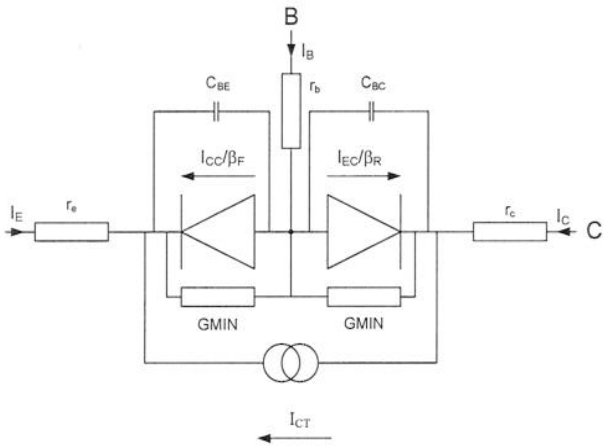
d) [4]



e)

i) [bookwork] [4]

The large signal transient model is the Ebers-Moll model with capacitances and is shown below:



The main DC parameters are:

IS (I_s)	The transistor saturation current
RE (r_e)	The Ohmic resistance of the contact and bond wire at the emitter
RB (r_b)	The Ohmic resistance of the contact and bond wire at the base
RC (r_c)	The Ohmic resistance of the contact and bond wire at the collector
NF (n)	The emission (or ideality) coefficient for the base-emitter junction
NR (n)	The emission (or ideality) coefficient for the base-collector junction
BF (β_F)	The forward current gain
BR (β_R)	The reverse current gain

The students should name 4 of them.

ii) [bookwork] [3]

For each junction, SPICE needs to know 3 parameters to specify the diffusion and junction capacitances:

Zero bias junction capacitance
Transit time
Built in voltage

2E Electromagnetic Fields 2007 - Answers

4. a) Marconi was able to communicate across the Atlantic because the low frequency radio waves he used were reflected from the ionosphere over the horizon limit.

Modern long distance radio communications systems operate at much higher frequencies, to which the ionosphere is transparent. Consequently, the signal can travel through the ionosphere to reach a geostationary satellite, which can re-broadcast the signal far beyond the terrestrial horizon.

[4]

b) Absorption – loss of energy caused by the excitation of molecular bond vibrations; generally limited to a narrow band near a reasonant absorption frequency.

Scattering – diversion of energy into random directions, caused by inhomogeneities or small particles in the atmosphere. Rises steadily at high frequencies or short wavelengths, increasingas $1/\lambda^4$.

Diffraction – the spread of a bounded beam, due to the wave nature of electromagnetic radiation. Rises as the frequency decreases, when the wavelength approaches the size of the emitting aperture.

[4]

c) Start with $dV/dz = -j\omega LI$ and $dI/dz = -j\omega CV$

Differentiating:

$d^2V/dz^2 = -j\omega L dI/dz = -\omega^2 LCV$

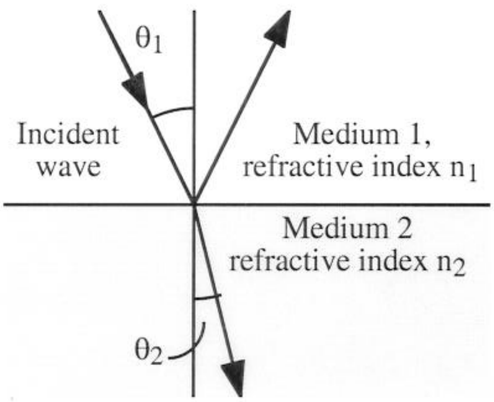
$d^2I/dz^2 = -j\omega C dV/dz = -\omega^2 LCI$

$I = I_0 \exp(-jkz)$ and $V = V_0 \exp(-jkz)$ are solutions if $-k^2 = -\omega^2 LC$

Hence $k = \omega(LC)^{1/2}$

[4]

d) Ray diagram is as shown below:



Snell's law is $n_1 \sin(\theta_1) = n_2 \sin(\theta_2)$

4

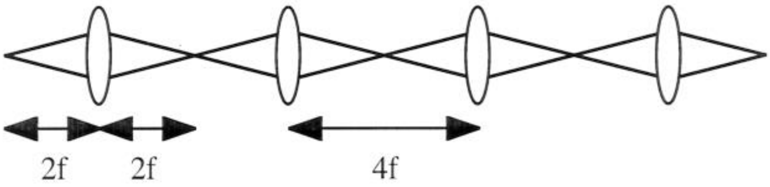
Hence, the direction of the refracted ray is found from $\sin(\theta_2) = \{(n_1/n_2) \sin(\theta_1)\}$

Total internal reflection can therefore only occur if $n_1 > n_2$, and if $\sin(\theta_1) > (n_2/n_1)$

[4]

e) Imaging relation is $1/u + 1/v = 1/f$.

From the above, we can see that having $u = 2f$ gives $v = 2f$. A chain of lenses on a periodic spacing of 4f therefore gives a form of waveguide operating by repetitive imaging.



[4]

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5 a) Analytic expression for electric field:

The flux $\underline{D}$ is radial by symmetry; since $\underline{D} = \varepsilon\underline{E}$, the electric field $\underline{E}$ is similarly radial

Assume a line charge q per unit length;

Assume a Gaussian surface is located at radius r

Gauss' law (flux out = charge enclosed) gives: $2\pi r D_r = q$

Since $D_r = \varepsilon E_r$, we must have $2\pi r \varepsilon E_r = q$ and $E_r = q/2\pi r\varepsilon$

[2]

The magnetic field $\underline{H}$ is circumferential by symmetry

Analytic expression for magnetic field:

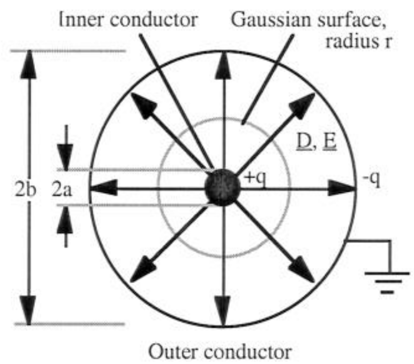
Assume central conductor carries current I into paper

Assume circular path, of radius r

Ampere's law gives $2\pi r H_\phi = I$, hence $H_\phi = I/2\pi r$

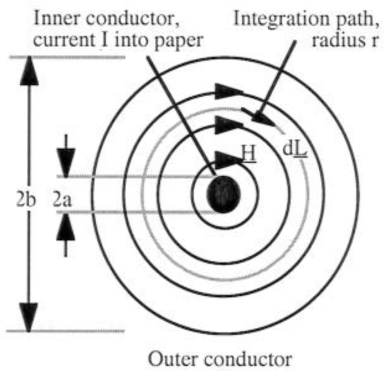
[2]

Electric field



[2]

Magnetic field:



[2]

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b) Capacitance per unit length

Potential V found from $E_r = -dV/dr$

Hence $V(b) - V(a) = -\int_a^b (q/2\pi r\varepsilon)\, dr = -(q/2\pi\varepsilon) \log_e(b/a)$

If outer conductor is grounded, $V(a) = (q/2\pi\varepsilon) \log_e(b/a)$

Capacitance per unit length is $C = q/V = (2\pi\varepsilon) \div \log_e(b/a)$ F/m

If a = 0.45 mm, b = 1.6 mm and $\varepsilon_r = 2.3$, then:

$C = (2\pi \times 8.85 \times 10^{-12} \times 2.3) \div \log_e(1.6/0.45) = 1 \times 10^{-10}$ F/m = 100 pF/m

[4]

Inductance per unit length

Flux linked between conductors per unit length is:

$\Phi = \int_a^b B_\phi\, dr = \int_a^b \mu_0 H_\phi\, dr = \int_a^b (\mu_0 I/2\pi r)\, dr = (\mu_0 I/2\pi) \log_e(b/a)$

Inductance per unit length is:

$L = \Phi/I = (\mu_0/2\pi) \log_e(b/a)$

If a = 0.45 mm, b =1.6 mm and $\varepsilon_r = 2.3$, then:

$L = (2 \times 10^{-7}) \log_e(1.6/0.45) = 2.54 \times 10^{-7}$ H/m = 254 nH/m

[4]

The peak electric field is found at the inner conductor, namely $E_{rmax} = q/2\pi a\varepsilon$,

If V = 1 V, the charge per unit length is $q = CV = 1 \times 10^{-10} \times 1 = 1 \times 10^{-10}$ C/m

Hence $E_{rmax} = 1 \times 10^{-10}/(2\pi \times 0.45 \times 10^{-3} \times 8.85 \times 10^{-12} \times 2.3) = 1752$ V/m

[4]

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6. a) Maxwell's equations:

$\text{div}(\underline{\mathbf{D}}) = \rho$	Gauss's law
$\text{div}(\underline{\mathbf{B}}) = 0$	Magnetic equivalent of Gauss's law
$\text{curl}(\underline{\mathbf{E}}) = -\partial\underline{\mathbf{B}}/\partial t$	Faraday's law
$\text{curl}(\underline{\mathbf{H}}) = \underline{\mathbf{J}} + \partial\underline{\mathbf{D}}/\partial t$	Ampere's law; $\partial\underline{\mathbf{D}}/\partial t$ is displacement current density

[3]

Approximations for dielectric media:

$\rho = 0$; $\sigma = 0$ (so $\underline{\mathbf{J}} = 0$); $\mu = \mu_0$

No variation in ε with position

Hence, Maxwell's equations simplify to:

$\text{div}(\underline{\mathbf{D}}) = 0$ (1)

$\text{div}(\underline{\mathbf{B}}) = 0$ (2)

$\text{curl}(\underline{\mathbf{E}}) = -\partial\underline{\mathbf{B}}/\partial t = -\mu_0\partial\underline{\mathbf{H}}/\partial t$ (3)

$\text{curl}(\underline{\mathbf{H}}) = \partial\underline{\mathbf{D}}/\partial t = \varepsilon\, \partial\underline{\mathbf{E}}/\partial t$ (4)

[3]

b) Taking curl of (3):

$\text{curl}\,[\text{curl}(\underline{\mathbf{E}})] = -\mu_0\, \text{curl}(\partial\underline{\mathbf{H}}/\partial t) = -\mu_0\, \partial\{\text{curl}(\underline{\mathbf{H}})\}/\partial t$

Substituting using (4):

$\text{curl}\,[\text{curl}(\underline{\mathbf{E}})] = -\mu_0\varepsilon\, \partial^2\underline{\mathbf{E}}/\partial t^2$

Using the identity $\text{curl}\,[\text{curl}(\underline{\mathbf{E}})] = \text{grad}\,[\text{div}(\underline{\mathbf{E}})] - \nabla^2\underline{\mathbf{E}}$

$\text{grad}\,[\text{div}(\underline{\mathbf{E}})] - \nabla^2\underline{\mathbf{E}} = -\mu_0\varepsilon\, \partial^2\underline{\mathbf{E}}/\partial t^2$

Equation (1) implies that $\text{div}(\underline{\mathbf{D}}) = \text{div}(\varepsilon\underline{\mathbf{D}}) = 0$, so that:

$\nabla^2\underline{\mathbf{E}} = \mu_0\varepsilon\, \partial^2\underline{\mathbf{E}}/\partial t^2$ – time dependent vector wave equation

If $\underline{\mathbf{E}}(x, y, z, t) = \underline{\mathrm{E}}(x, y, z)\exp(j\omega t)$ then:

$\nabla^2\underline{\mathbf{E}} = \nabla^2\underline{\mathrm{E}}\exp(j\omega t)$

$\partial^2\underline{\mathbf{E}}/\partial t^2 = -\omega^2\underline{\mathrm{E}}\exp(j\omega t)$

Hence:

$\nabla^2\underline{\mathrm{E}} = -\omega^2\mu_0\varepsilon\underline{\mathrm{E}}$ – time independent vector wave equation

If $\underline{\mathbf{E}} = E_x\, \underline{i}$, then:

$\nabla^2 E_x = -\omega^2\mu_0\varepsilon E_x$ – time independent scalar wave equation

[10]

For plane waves propagating in the z-direction, $\partial E_x/\partial x = \partial E_x/\partial y = 0$. Hence:

$d^2E_x/dz^2 = -\omega^2\mu_0\varepsilon E_x$

If $E_x = E_{x0}\exp(-jkz)$, then $d^2E_x/dz^2 = -jk^2 E_{x0}\exp(-jkz)$

Hence: $-jk^2 E_{x0}\exp(-jkz) = -\omega^2\mu_0\varepsilon E_{x0}\exp(-jkz)$ and $k^2 = \omega^2\mu_0\varepsilon$

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The propagation constant is therefore $k = \omega(\mu_0\varepsilon)^{1/2}$

The phase velocity is $v_{ph} = \omega/k = 1/(\mu_0\varepsilon)^{1/2}$

[4]

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