EE1-10B MATHEMATICS II

1. a) Given the function

$$f(t) = e^{at}H(-t)\,,$$

where H is the Heaviside function, obtain $F(\omega)$, the Fourier transform of $f(t)$. State the condition on the constant a which is necessary for the existence of $F(\omega)$. [5]

SOLUTION

$$\begin{aligned}\mathscr{F}\left[f(t)\right] &= \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt = \int_{-\infty}^{0} e^{at}e^{-i\omega t}\,dt\\ &= \left[\frac{1}{a-i\omega}e^{(a-i\omega)t}\right]_{-\infty}^{0} = \frac{1}{a-i\omega} - \lim_{K\to-\infty}\frac{e^{(a-i\omega)K}}{a-i\omega}\\ &= \frac{1}{a-i\omega}\end{aligned}$$

as the limit is zero, provided $a > 0$.

b) Hence, or otherwise, obtain the inverse Fourier Transform of [5]

$$G(\omega) = \frac{1}{4-2i\omega-3i}\,.$$

SOLUTION

Begin by rewriting

$$G(\omega) = \frac{1}{4-2i\left(\omega+\frac{3}{2}\right)} = H\left(\omega+\frac{3}{2}\right)$$

where

$$H(\omega) = \frac{1}{4-2i} = \frac{1}{2}\left(\frac{1}{2-i\omega}\right)\,,$$

and using (a) we have

$$h(t) = \mathscr{F}^{-1}\left[H(\omega)\right] = \frac{1}{2}e^{2t}H(-t)$$

and using the frequency shift from G to H, we obtain

$$g(t) = e^{-\frac{3}{2}it}h(t) = \frac{1}{2}e^{(2-3i/2)t}H(-t)\,.$$

c) Given the plane with equation $\Pi : 2x - 3y + 5z = -4$,

i) Find the minimum distance from the point $P(1, -1, 2)$ to Π; obtain the point on Π nearest to P. [4]

SOLUTION

The intersection of the line through P and normal to the plane will be the desired point, say A; hence the direction vector of the line is the normal vector of the plane, $(2, -3, 5)$. The line has equation

$$\underline{\mathbf{r}} = (x, y, z) = (1, -1, 2) + \lambda(2, -3, 5)$$

and substituting into the equation of the plane:

$$2(1 + 2\lambda) - 3(-1 - 3\lambda) + 5(2 + 5\lambda) = 15 + 38\lambda = -4 \Rightarrow \lambda = -\frac{1}{2},$$

so the interection of line and plane, the point A on Π nearest to P is

$$A : (1, -1, 2) - \frac{1}{2}(2, -3, 5) = \left(0, \frac{1}{2}, -\frac{1}{2}\right)$$

and the minimum distance from P to Π is

$$|\overrightarrow{AP}| = \left|\left(1, -\frac{3}{2}, \frac{5}{2}\right)\right| = \frac{\sqrt{38}}{2}.$$

ii) Another plane has equation $\Phi : x+\alpha y+\beta z=0$. Give all values of α and β that make Π and Φ orthogonal. [3]

SOLUTION For the planes to be orthogonal, the normal vectors need to be orthogonal:

$$(2,-3,5)\cdot(1,\alpha,\beta)=0 \Rightarrow 2-3\alpha+5\beta=0$$

with solutions

$$(\alpha,\beta)=(2/3,0)+t(5/3,1)$$

where t is any real number.

d) Given the vectors $\underline{\mathbf{u}}=(1,2,a)$, $\underline{\mathbf{v}}=(3,-4,b)$ and $\underline{\mathbf{w}}=(-5,6,c)$, find a condition on the scalars a,b,c so that $\underline{\mathbf{u}}\times\underline{\mathbf{v}}\cdot\underline{\mathbf{w}}=0$.

Let this condition be satisfied. The vectors now form what kind of set? What is the determinant of the matrix whose columns are $\underline{\mathbf{u}},\underline{\mathbf{v}},\underline{\mathbf{w}}$? Finally, obtain scalars p,q such that $\underline{\mathbf{u}}=p\underline{\mathbf{v}}+q\underline{\mathbf{w}}$. [7]

SOLUTION

Rewrite as

$$\underline{\mathbf{w}}\cdot(\underline{\mathbf{u}}\times\underline{\mathbf{v}})=\begin{vmatrix}-5 & 6 & c\\ 1 & 2 & a\\ 3 & -4 & b\end{vmatrix}=-5(2b+4a)-6(b-3a)+c(-4-6),$$

so the condition is

$$a+8b+5c=0, \qquad (*)$$

Begin by finding p,q: $\underline{\mathbf{u}}=p\underline{\mathbf{v}}+q\underline{\mathbf{w}} \Rightarrow (1,2,a)=p(3,-4,b)+q(-5,6,c)$ which is a set of three equations:

$$3p-5q=1,\quad -4p+6q=2,\quad \Rightarrow\quad p=-8, q=-5$$

and the third equation:

$$bp+cq=a \Rightarrow -8p-5c=a,$$

is satisfied due to $(*)$.

Thus, the vectors form a linearly dependent set, and a matrix with linearly dependent columns has determinant zero.

Alternatively, from $\underline{\mathbf{w}}\cdot(\underline{\mathbf{u}}\times\underline{\mathbf{v}})=0$ we have that $\underline{\mathbf{w}}$ is orthogonal to $\underline{\mathbf{u}}\times\underline{\mathbf{v}}$, but as this is orthogonal to $\underline{\mathbf{u}}$ and $\underline{\mathbf{v}}$, we deduce that $\underline{\mathbf{w}}$ is in the plane defined by $\underline{\mathbf{u}}$ and $\underline{\mathbf{v}}$ and hence the vectors form a linearly dependent set.

2. a) Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{pmatrix}.$$

i) Calculate A^2 and A^3 and find scalars ϕ and ψ such that [4]

$$A^3 + \phi A^2 + \psi A + I = \underline{\mathbf{0}},$$

where I is the identity matrix.

SOLUTION

$$A^2 = \begin{pmatrix} 1 & 4 & 6 \\ 0 & 8 & 13 \\ -2 & 3 & 6 \end{pmatrix}, \quad A^3 = \begin{pmatrix} -1 & 15 & 25 \\ -5 & 29 & 50 \\ -5 & 10 & 19 \end{pmatrix}.$$

and hence

$A^3 + \phi A^2 + \psi A + I$

$$= \begin{pmatrix} -1+\phi+\psi+1 & 15+4\phi+\psi & 25+6\phi+\psi \\ -5+\psi & 29+8\phi+2\psi+1 & 50+13\phi+3\psi \\ -5-2\phi-\psi & 10+3\phi+\psi & 19+6\phi+2\psi+1 \end{pmatrix}$$

$= \underline{\mathbf{0}}$. The $(2,1)$ entry gives $-5+\psi = 0 \Rightarrow \psi = 5$; the $(1,1)$ entry gives $\phi + \psi = 1 \Rightarrow \phi = -5$. Check with any other entry to confirm these.

ii) Use the result from (i) to find the inverse of A. [4]

SOLUTION

To use (i), we observe that, given $A^3 - 5A^2 + 5A + I = \underline{\mathbf{0}}$ we can multiply on the left:

$$A^{-1}(A^3 - 5A^2 + 5A + I) = A^2 - 5A + 5I + A^{-1} = A^{-1}\underline{\mathbf{0}} = \underline{\mathbf{0}}$$

so that $A^{-1} = -A^2 + 5A - 5I =$

$$= - \begin{pmatrix} 1 & 4 & 6 \\ 0 & 8 & 13 \\ -2 & 3 & 6 \end{pmatrix} + 5 \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{pmatrix} - 5 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 1 & -1 \\ 5 & -3 & 2 \\ -3 & 2 & -1 \end{pmatrix}$$

iii) Confirm your result in (ii) by calculating A^{-1} using Gaussian elimination. [4]

SOLUTION

Set up the augmented matrix and use row operations:

$$(A:I) = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ -1 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \underset{\substack{R_2-R_1 \\ R_3+R_1}}{\sim} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 2 & 3 & 1 & 0 & 1 \end{array}\right)$$

$$\underset{\substack{R_1-R_2 \\ R_3-2R_2}}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & -1 & 3 & -2 & 1 \end{array}\right) \underset{-R_3}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & -3 & 2 & -1 \end{array}\right)$$

$$\underset{\substack{R_1+R_3 \\ R_2-2R_3}}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & -1 \\ 0 & 1 & 0 & 5 & -3 & 2 \\ 0 & 0 & 1 & -3 & 2 & -1 \end{array}\right)$$

confirming the result from (ii).

b) Given a matrix

$$A = \begin{pmatrix} 1 & -4 & 2 \\ -4 & 1 & -2 \\ 2 & -2 & -2 \end{pmatrix}$$

i) Show that $\lambda = -3$ is one of the eigenvalues of A and find the other two. [4]

SOLUTION

Begin with $\det(A - \lambda I) = 0$ and use row/column operations to simplify:

$$\begin{vmatrix} 1-\lambda & -4 & 2 \\ -4 & 1-\lambda & -2 \\ 2 & -2 & -2-\lambda \end{vmatrix} \underset{R_1+R_2}{=} \begin{vmatrix} -3-\lambda & -3-\lambda & 0 \\ -4 & 1-\lambda & -2 \\ 2 & -2 & -2-\lambda \end{vmatrix}$$

$$\underset{C_2-C_1}{=} \begin{vmatrix} -3-\lambda & 0 & 0 \\ -4 & 5-\lambda & -2 \\ 2 & -4 & -2-\lambda \end{vmatrix} = 0$$

showing $\lambda = -3$ is an eigenvalue. Now expand by the first row:

$$-(3+\lambda)[(5-\lambda)(-2-\lambda)+8] = -(3+\lambda)(\lambda^2 - 3\lambda - 18) = 0$$

$\Rightarrow\ -(\lambda+3)^2(\lambda-6) = 0$, so the other eigenvalues are $\lambda = -3$ and $\lambda = 6$.

ii) Find eigenvectors corresponding to the three eigenvalues of A. [4]

SOLUTION

For $\lambda = 6$ we have $(A-6I)\underline{\mathbf{x}} = \underline{\mathbf{0}}$ giving

$$\begin{pmatrix} -5 & -4 & 2 \\ -4 & -5 & -2 \\ 2 & -2 & -8 \end{pmatrix} \underline{\mathbf{x}} = \underline{\mathbf{0}} \Rightarrow \begin{pmatrix} 1 & -1 & -4 \\ 0 & -9 & -18 \\ 0 & -9 & -18 \end{pmatrix} \underline{\mathbf{x}} = \underline{\mathbf{0}} \text{ using Row operations.}$$

Hence $y+2z = 0$ and $x = y+4z$. Choosing $z = 1$ we get the eigenvector $\underline{\mathbf{x}}_1 = (2,-2,1)$

For $\lambda = -3$ we have $(A+3I)\underline{\mathbf{x}} = \underline{\mathbf{0}}$ giving $\begin{pmatrix} 4 & -4 & 2 \\ -4 & 4 & -2 \\ 2 & -2 & 1 \end{pmatrix} \underline{\mathbf{x}} = \underline{\mathbf{0}}$ and all three rows give $2x-2y+z = 0$, two free variables. To ensure linear independence, we choose $y = 2, z = 0$ for one eigenvector and $y = 0, z = 2$ for the other, giving $\underline{\mathbf{x}}_2 = (2,2,0)$ and $\underline{\mathbf{x}}_3 = (-1,0,2)$.

iii) Using projection, or otherwise, find a set of orthonormal eigenvectors for A, and hence obtain the orthogonal diagonalization of A. [5]

SOLUTION

We note that $\underline{\mathbf{x}}_2$ and $\underline{\mathbf{x}}_3$ are both orthogonal to $\underline{\mathbf{x}}_1$, but not to each other. Given that a linear combination of eigenvectors corresponding to the same eigenvalue is also an eigenvector for that eigenvalue, we can use projection to get two orthogonal eigenvectors from $\underline{\mathbf{x}}_1$ and $\underline{\mathbf{x}}_2$.

If $\mu\underline{\mathbf{x}}_2$ is the projection of $\underline{\mathbf{x}}_3$ onto $\underline{\mathbf{x}}_2$, then $\underline{\mathbf{x}}_3 - \mu\underline{\mathbf{x}}_2$ is an eigenvector for $\lambda = -3$ and orthogonal to $\underline{\mathbf{x}}_2$. Hence

$$\mu = \frac{\underline{\mathbf{x}}_3 \cdot \underline{\mathbf{x}}_2}{\underline{\mathbf{x}}_2 \cdot \underline{\mathbf{x}}_2} = \frac{(-1,0,2)\cdot(2,2,0)}{(2,2,0)\cdot(2,2,0)} = -\frac{1}{4}$$

and so

$$\underline{\mathbf{x}}_3 - \mu\underline{\mathbf{x}}_2 = (-1,0,2) + \frac{1}{4}(2,2,0) = \frac{1}{2}(-1,1,4)$$

and we take $\underline{\mathbf{x}}_4 = (-1,1,4)$, for convenience, noting it is orthogonal to $\underline{\mathbf{x}}_1$ and $\underline{\mathbf{x}}_2$, as expected. Hence the set or orthonormal eigenvectors is

$$\hat{\underline{\mathbf{x}}}_1 = \frac{1}{3}(2,-2,1), \quad \hat{\underline{\mathbf{x}}}_2 = \frac{1}{\sqrt{2}}(1,1,0), \quad \hat{\underline{\mathbf{x}}}_4 = \frac{1}{\sqrt{18}}(-1,1,4).$$

Given a symmetric matrix and a set of orthonormal eigenvectors, the orthogonal diagonalization is $A = PDP^T$ where

$$P = \begin{pmatrix} \frac{2}{3} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{18}} \\ -\frac{2}{3} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{18}} \\ \frac{1}{3} & 0 & \frac{4}{\sqrt{18}} \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 6 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

3. a) Find the general solution of the differential equation

$$(3t\cos x - 2x)\frac{dx}{dt} = 4t - 3\sin x\,.$$

Find also the particular solution satisfying the condition $x(1) = 0$. [6]

SOLUTION

Rewrite as

$$(3t\cos x - 2x)\frac{dx}{dt} - 4t + 3\sin x = 0\,.$$

with $P(x,t) = 3t\cos x - 2x$ and $Q(x,t) = -4t + 3\sin x$. Check for exactness:

$$\frac{\partial P}{\partial t} = 3\cos x = \frac{\partial Q}{\partial x}\,,$$

so the ODE is exact in the form $dF = 0$, where F is found by integrating P, Q:

$$\frac{\partial F}{\partial x} = P \Rightarrow F = \int P\,dx = \int 3t\cos x - 2x\,dx = 3t\sin x - x^2 + g(t)$$

where g is an arbitrary function. Similarly

$$\frac{\partial F}{\partial t} = Q \Rightarrow F = \int Q\,dt = \int -4t + 3\sin x\,dt = -2t^2 + 3t\sin x + h(x)$$

where h is another arbitrary function. Equating the two, we have $g(t) = -2t^2$ and $h(x) = -x^2$ so that

$$F(x,t) = 3t\sin x - x^2 - 2t^2$$

with $dF = 0 \Rightarrow F =$constant giving the solution:

$$3t\sin x - x^2 - 2t^2 = C\,.$$

and implementing the initial condition $x(1) = 0$ gives $C = -2$, so the particular solution is

$$3t\sin x - x^2 - 2t^2 = -2\,.$$

b) Given the Bernoulli equation

$$x\frac{dy}{dx} + y = x^2y^2\,,$$

use the substitution $v = y^{-1}$ to obtain a first order linear equation in v, and hence solve for y. [6]

SOLUTION

Following the suggestion, the substitution

$$v = \frac{1}{y} \Rightarrow \frac{dv}{dx} = -\frac{1}{y^2}\frac{dy}{dx}$$

and dividing the original equation by $-xy^2$ we obtain

$$-\frac{1}{y^2}\frac{dy}{dx}-\frac{1}{x}\frac{1}{y}=-x\,,$$

and substituting, we have a linear equation:

$$\frac{dv}{dx}-\frac{1}{x}v=-x$$

with $P(x)=-1/x$ and the integrating factor

$$\mu(x)=e^{-\int\frac{1}{x}\,dx}=\frac{1}{x}\,.$$

Multiplying through by the integrating factor we get

$$\frac{1}{x}\frac{dv}{dx}-\frac{1}{x^2}v=\frac{d}{dx}\left(\frac{1}{x}v\right)=-1\Rightarrow\frac{1}{x}v=-x+C$$

and the solution for v is

$$v=\frac{1}{y}=-x^2+Cx\Rightarrow y=\frac{1}{Cx-x^2}\,.$$

c) Solve the following second order differential equation: [8]

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 26\cos(3x)\,.$$

SOLUTION

The auxiliary equation is $\lambda^2 - 2\lambda + 5 = 0 \Rightarrow \lambda = 1 \pm 2i$ and the complementary function is

$$y_c = e^x(c_1 \cos 2x + c_2 \sin 2x)\,,$$

where $c_{1,2}$ are arbitrary constants. For a particula integral, try

$$\begin{aligned} y_p &= A\cos 3x + B\sin 3x \\ y_p' &= -3A\sin 3x + 3B\cos 3x \\ y_p'' &= -9A\cos 3x - 9B\sin 3x \end{aligned}$$

and substitute into the ODE to get

$$-9A\cos 3x - 9B\sin 3x - 2(-3A\sin 3x + 3B\cos 3x) + 5(A\cos 3x + B\sin 3x) =$$

$$(-9A - 6B + 5A)\cos 3x + (-9B + 6A + 5B)\sin 3x = 26\cos 3x$$

so equating coefficients we have $-4A - 6B = 26\,, \quad 6A - 4B = 0$ so that $A = -2$ and $B = -3$, and the particular integral together with the complementary function gives the general solution

$$y = y_c + y_p = e^x(c_1 \cos 2x + c_2 \sin 2x) - 2\cos 3x - 3\sin 3x\,.$$

d) The height h of a regular cone, with volume V and radius of the circular base r, is found using

$$V = \frac{1}{3}\pi r^2 h\,.$$

Given that the percentage errors in the measurements of r and V are at most 0.5% and 0.2%, respectively, give an estimate for the maximum percentage error in the calculation of h. [5]

SOLUTION

Begin by writing

$$h = \frac{3V}{\pi r^2} \Rightarrow \frac{\partial h}{\partial V} = \frac{3}{\pi r^2}\,, \quad \frac{\partial h}{\partial r} = -\frac{6V}{\pi r^3}\,.$$

The total differential

$$dh = \frac{\partial h}{\partial V}dV + \frac{\partial h}{\partial r}dr$$

is used to estimate small differences $\Delta h\,, \Delta V\,, \Delta r$:

$$\Delta h \approx \frac{\partial h}{\partial V}\Delta V + \frac{\partial h}{\partial r}\Delta r = \frac{3}{\pi r^2}\Delta V - \frac{6V}{\pi r^3}\Delta r$$

and dividing by h on the left, and by $3V/(\pi r^2)$ on the right, we obtain

$$\frac{\Delta h}{h} \approx \frac{\frac{3}{\pi r^2}}{\frac{3V}{\pi r^2}}\Delta V - \frac{\frac{6V}{\pi r^3}}{\frac{3V}{\pi r^2}}\Delta r = \frac{\Delta V}{V} - 2\frac{\Delta r}{r}\,.$$

The relative error could be positive or negative, so we use the triangle inequality to estimate

$$\left|\frac{\Delta h}{h}\right| \leq \left|\frac{\Delta V}{V}\right| + 2\left|\frac{\Delta r}{r}\right| = 0.002 + 2(0.005) = 0.012\,,$$

so the maximum error in calculating h is 1.2%.

4. a) A solution of the second order differential equation

$$\frac{d^2y}{dx^2} - 2xy = 0\,,$$

can be found in the form of a series with the Leibnitz-Maclaurin method. Given the initial conditions $y(0) = 1$ and $y'(0) = 0$, differentiate the ODE n times to obtain the recurrence relation

$$y^{(n+2)}(0) = 2ny^{(n-1)}(0)\,, \quad (n \geq 1)\,,$$

where $y^{(k)}(0)$ is the k^{th} derivative of y, evaluated at zero.

Obtain the first three non-zero terms of the series. [8]

SOLUTION

Using Leibnitz' Theorem we differentiate the ODE n times and get

$$y^{(n+2)} - 2\left[xy^{(n)} + n \cdot 1y^{(n-1)}\right] = 0$$

and setting $x = 0$ we have

$$y^{(n+2)}(0) - 2ny^{(n-1)}(0) = 0 \Rightarrow y^{(n+2)}(0) = 2ny^{(n-1)}(0)$$

as required. The initial conditions give
$y(0) = y^{(0)}(0) = 1$ and $y'(0) = y^{(1)}(0) = 0$.
From the ODE, setting $x = 0$ we obtain $y''(0) = y^{(2)}(0) = 0$. Hence the recurrence gives that
$n = 1 : y^{(3)}(0) = 2(1)y^{(0)}(0) = 2y(0) = 2$
$n = 2 : y^{(4)}(0) = 2(2)y^{(1)}(0) = 0$
$n = 3 : y^{(5)}(0) = 2(3)y^{(2)}(0) = 0$
$n = 4 : y^{(6)}(0) = 2(4)y^{(3)}(0) = 2^4 = 16$
Clearly, we will only get non-zero every third term, so
$n = 5, 6 : y^{(7)}(0) = y^{(8)}(0) = 0$ and
$n = 7 : y^{(9)}(0) = 2(7)y^{(6)}(0) = 7 \cdot 2^5 = 224$
and so on. The Maclaurin series

$$y = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \cdots + \frac{y^{(n)}(0)}{n!}x^n + \cdots$$

gives the solution for this ODE:

$$y = 2 + \frac{16}{6!}x^6 + \frac{7 \cdot 2^5}{9!}x^9 + \cdots = 2 + \frac{1}{45}x^6 + \frac{1}{36 \cdot 45}x^9 + \cdots\,.$$

b) If $u = f(\phi)$ where f is not specified, and $\phi = \dfrac{2x-y}{3xy}$, show that

$$y^2\frac{\partial u}{\partial y} + 2x^2\frac{\partial u}{\partial x} = 0\,,$$

SOLUTION

Begin with the chain rule:

$$\frac{\partial u}{\partial x} = \frac{du}{d\phi}\frac{\partial \phi}{\partial x} = \frac{du}{d\phi}\frac{3y^2}{(3xy)^2}\,,$$

and

$$\frac{\partial u}{\partial y} = \frac{du}{d\phi}\frac{\partial \phi}{\partial y} = \frac{du}{d\phi}\frac{(-6x^2)}{(3xy)^2}\,.$$

Hence

$$y^2\frac{\partial u}{\partial y} + 2x^2\frac{\partial u}{\partial x} = \frac{du}{d\phi}\left[y^2\frac{(-6x^2)}{(3xy)^2} + 2x^2\frac{3y^2}{(3xy)^2}\right] = 0\,,$$

as required.

c) A function of two variables is given as

$$f(x,y) = x(y+1)^2 - x^2 - x\,.$$

i) Find the stationary points of $f(x,y)$ and determine their nature using the Hessian determinant. [7]

SOLUTION

Begin with the partial derivatives, set equal to zero to find the stationary points:

$$\frac{\partial f}{\partial x} = (y+1)^2 - 2x - 1 = 0\,, \quad \frac{\partial f}{\partial y} = 2x(y+1) = 0\,,$$

The second of these gives immediate solutions $x = 0$ or $y = -1$. Substitute into the first equation:

$$x = 0 \Rightarrow (y+1)^2 - 1 = 0 \Rightarrow y+1 = \pm 1 \Rightarrow y = 0, -2$$

giving stationary points at $P_1 : (0,0)$ and $P_2 : (0,-2)$ and

$$y = -1 \Rightarrow -2x - 1 = 0 \Rightarrow x = -\frac{1}{2}$$

giving a third stationary point at $P_3 : (-\frac{1}{2}, -1)$.

To classify the stationary points, first obtain the second partial derivatives

$$f_{xx} = -2, \quad f_{yy} = 2x, \quad f_{xy} = f_{yx} = 2(y+1),$$

Hence the Hessian determinant is

$$D = \begin{vmatrix} -2 & 2(y+1) \\ 2(y+1) & 2x \end{vmatrix} = -4x - 4(y+1)^2,$$

Evaluating the determinant at the stationary points we find:

$P_1 :\ H(0,0) = -4 < 0$ so P_1 is a saddle point;

$P_2 :\ H(0,-2) = -4 < 0$ so P_2 is a saddle point;

$P_3 :\ H(-\frac{1}{2},-1) = 2 > 0$, and $f_{xx}(-\frac{1}{2},-1) = -2 < 0$ so P_3 is a maximum.

ii) Sketch the contours of the surface $z = f(x,y)$. [5]

SOLUTION

First set $f(x,y) = x(y+1)^2 - x^2 - x = x[(y+1)^2 - x - 1] = 0$, so contours with $f = 0$ when $x = 0$ or $x = (y+1)^2 - 1$: the $y-$axis and a horizontal parabola on $[-1,\infty]$. Check that, as expected, the saddle-points are the points of intersection of parabola and $y-$axis.

The maximum at $(-\frac{1}{2},-1)$ is in the area between the two zero contours, so $f > 0$ here, descending to $f = 0$ and then to $f < 0$ in adjacent areas. Crossing the positive $y-$axis from (say) $(-1,2)$ to $(1,2)$ we move from a region with $f < 0$ to $f = 0$ on the axis, and therefore continue with f increasing into a region where $f > 0$. Thus, the two areas in the first quadrant above the parabola and in the second quadrant below the parabola show contours with $f > 0$. This should all appear in the sketch: