EE4-23
EE9-CS2-1

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2014

MSc and EEE/EIE PART IV: MEng and ACGI

Corrected Copy

STABILITY AND CONTROL OF NON-LINEAR SYSTEMS

Friday, 9 May 10:00 am

Time allowed: 3:00 hours

There are ~~SIX~~ Four questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

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1. Consider the following system of differential equations defined on $\mathbb{R}^2$:

$$\begin{aligned} \dot{x}_1 &= \sin\left(\sqrt{x_1^2+x_2^2}\right) \\ \dot{x}_2 &= -x_2+x_1. \end{aligned}$$

a) Find all equilibria of the system and sketch the nullclines; [3]

b) Discuss existence and unicity of solutions; [2]

c) Identify equilibria where the system is linearizable; [2]

d) Linearize the system around each linearizable equilibrium and discuss the local phase-portrait; [5]

e) Sketch the global phase portrait of the system by taking into account the clues collected in items a), b), c) and d); [4]

f) Prove that the equilibrium $(0,0)$ is not Lyapunov stable (*Hint: use instability criteria based on Lyapunov functions*). [4]

2. Consider the following nonlinear system:

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2^3 \\ \dot{x}_2 &= -x_1 x_2^2 - 2x_2 + d. \end{aligned}$$

with state $x = [x_1, x_2] \in \mathbb{R}^2$ and input $d \in \mathbb{R}$.

a) Show that the system is Input-to-State Stable (*Hint: use a quadratic ISS-Lyapunov function*); [5]

b) Find an upper-bound of class $\mathcal{K}_\infty$ for the gain from d to x; [3]

c) Consider next the following scalar system:

$$\dot{z} = -z(1 + d_1^2 + d_2^2) \tag{2.1}$$

with state $z \in \mathbb{R}$ and input $[d_1, d_2] \in \mathbb{R}^2$. Argue that, regardless of the input d_1, d_2, the corresponding solution is bounded. [2]

d) Show that for all compact sets $D \subset \mathbb{R}^2$ the system (2.1) with input signals taking value in D is Uniformly Globally Asymptotically Stable (UGAS); [4]

e) Consider next the feedback interconnection of the previous systems obtained by letting $d = z$ and $d_1 = x_1$, $d_2 = x_2$. Write its equations and argue that the origin is Globally Attractive. [6]

3. Consider the coupled differential equations given below which model two 1 Kg masses linked by a nonlinear spring and such that the first mass is connected by a linear spring to some fixed position (labeled 0 without loss of generality) while the second one is forced by some external force $u(t)$:

$$\ddot{x}_1(t) = -x_1(t) - k(x_1(t) - x_2(t))$$

$$\ddot{x}_2(t) = k(x_1(t) - x_2(t)) + u(t).$$

In particular, x_1, x_2 and u are scalar functions of time t and $k : \mathbb{R} \to \mathbb{R}$ is a $\mathscr{C}^\infty$ increasing function fulfilling: $k(\Delta)\Delta > 0$ for all $\Delta \neq 0$.

a) Write a state-space description of the system assuming the output y is given as $y(t) = \dot{x}_2(t)$; [3]

b) Show that the system considered is passive from u to y; [3]

c) Show that the static output feedback $u(t) = -y(t)$ globally asymptotically stabilizes the origin of the system (*Hint: use mechanical energy as a Lyapunov function*); [6]

d) Show that the dynamic feedback

$$u(t) = -y(t) - \int_0^t y(\tau)d\tau$$

achieves global convergence of solutions to 0 position and velocities. (*Hint: realize the PI controller as a system of dimension 1; show its passivity; apply the passivity theorem and Lasalle's criterion*). [8]

4. Consider the following 3 dimensional nonlinear affine control system:

$$\begin{aligned}\dot{x}_1 &= -2\text{atan}(x_1)+x_2\\ \dot{x}_2 &= -\text{atan}(x_1)-x_2+x_3\\ \dot{x}_3 &= -\sin(x_3)+ue^{-x_2^2}\end{aligned}$$

with state $x=[x_1,x_2,x_3]'\in\mathbb{R}^3$ and input $u\in\mathbb{R}$.

a) Show that for $u=0$ it admits multiple equilibrium points; [2]

b) Consider the output signal $y=x_1$. Compute the relative degree with respect to such an output selection and specify whether it is globally or locally defined; (justify your answer) [3]

c) Find an input-state linearizing feedback and the associated change of coordinates; [3]

d) Design an auxiliary globally stabilizing controller of the feedback-linearized system by any linear control design method; [2]

e) From now on assume $y=x_3$. Compute the associated relative degree and input-output linearizing feedback. Rewrite the system's equation in *normal form*; [6]

f) Highlight the zero-dynamics and show that they are Input-to-State Stable; argue that this fact can be exploited to design a globally stabilizing controller. [4]