DTS AND COMPUTER CONTROL

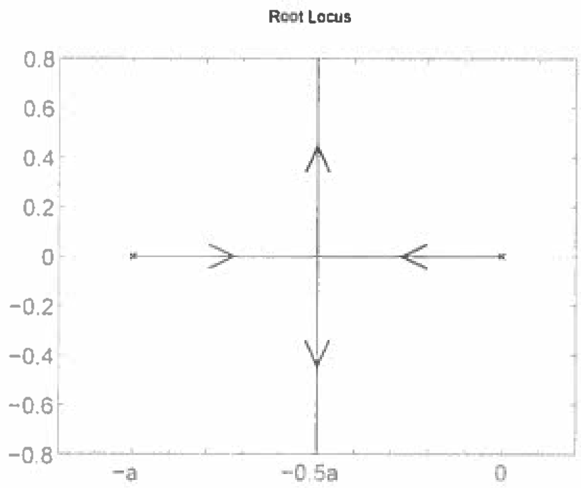


Figure 1.1 Root locus of $C(s)P(s)$.

1. a) Figure 1.1 shows the root locus of $C(s)P(s)$. The break-away point which can be computed with the formula in the formula sheet is $-\frac{1}{2}a$. The closed-loop transfer function is

$$\frac{C(s)P(s)}{1+C(s)P(s)} = \frac{K}{s^2+as+K}.$$

The characteristic polynomial of the closed-loop continuous-time system is

$$s^2+as+K.$$

By Routh test, all roots of the above polynomial are in the left half of the complex plane if $a > 0$ and $K > 0$. [4 marks]

Some students were unable to sketch the root locus. A few students applied the discrete-time Routh test instead of the continuous-time Routh test.

b) The equivalent discrete-time model of the plant is

$$\begin{aligned} HP(z) &= Z\left(\frac{1-e^{-sT}}{s}P(s)\right) = (1-z^{-1})Z\left(\frac{P(s)}{s}\right) = (1-z^{-1})Z\left(\frac{K}{s^2}\right) \\ &= \frac{TKz^{-1}}{1-z^{-1}} = \frac{TK}{z-1}. \end{aligned}$$

[3 marks]

This was an easy question since the Z-transform of $\frac{1}{s^2}$ was in the formula sheet. Yet some students were unable to complete this point.

c) i) To determine $C(z)$ with the impulse response invariance method we compute the z-transform of $C(s)$. As a result

$$C_I(z) = Z(C(s)) = \frac{z}{z-e^{-aT}} = \frac{1}{1-e^{-aT}z^{-1}}.$$

ii) Using the pole-zero correspondence method yields

$$C_{PZ}(z) = k\,\frac{z+1}{z-e^{-aT}} = \frac{1+z^{-1}}{1-e^{-aT}z^{-1}},$$

with $k = \dfrac{1-e^{-aT}}{2a}$ to match the DC gain ($s=0$, $z=1$) of the continuous-time controller. [3 marks]

Some students were not able to complete point i). Typical mistakes include the incorrect computation of the DC gain in point ii) or mapping the zero at infinity to +1 instead of -1.

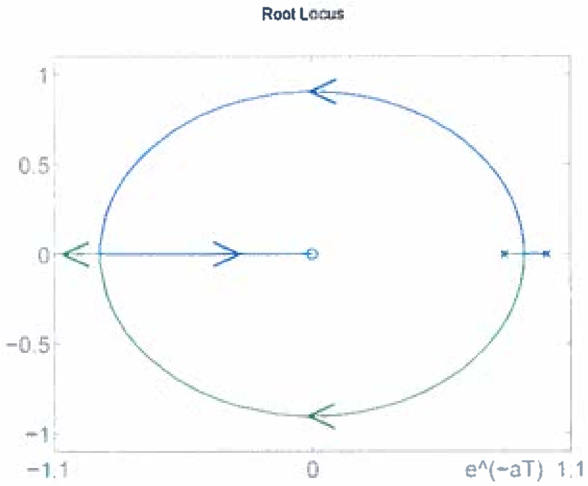


Figure 1.2 Root locus of $C_I(z)P(z)$.

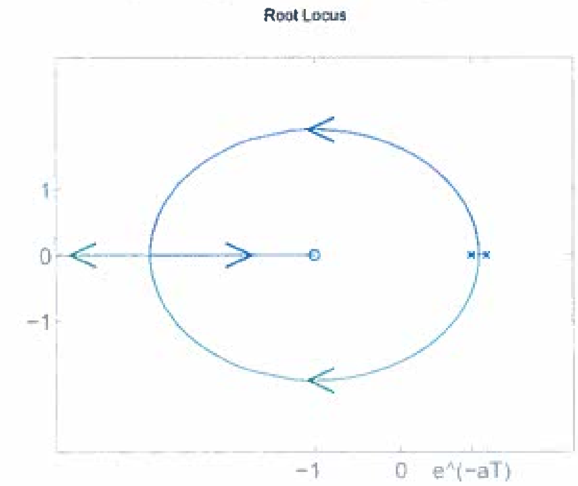


Figure 1.3 Root locus of $C_{PZ}(z)P(z)$.

d) Figure 1.2 shows the root locus of $C_I(z)HP(z)$. The break-away point which can be computed with the formula in the formula sheet is $\sqrt{e^{-aT}}$. The break-in point is $-\sqrt{e^{-aT}}$. Figure 1.3 shows the root locus of $C_{PZ}(z)HP(z)$. The break-away point is $-1+\sqrt{2+2e^{-aT}}$. The break-in point is $-1-\sqrt{2+2e^{-aT}}$.

[4 marks]

Some students were unable to sketch the root loci. A few students did not compute correctly the break-away and break-in points.

e) The characteristic polynomial when we use $C_I(z)$ is

$$z^2 - (1 + e^{-aT} - TK)z + e^{-aT}.$$

To determine the location of the roots of this polynomial we can use the bilinear transformation $z = \dfrac{w+1}{w-1}$, obtaining

$$\left(2 + 2e^{-aT} - TK\right)w^2 + \left(2 - 2e^{-aT}\right)w + TK,$$

and then use the Routh test. A quicker and safer route is to recall that the polynomial

$$z^2 + \alpha z + \beta$$

has all roots in the unit circle if

$$\begin{aligned} &1 + \alpha + \beta > 0, \\ &1 - \beta > 0, \\ &1 - \alpha + \beta > 0. \end{aligned} \tag{1.1}$$

Applying these conditions yields that the closed-loop discrete-time system is asymptotically stable, when we use the controller $C_I(z)$, for any value of K such that

$$0 < K < \frac{2}{T}\left(1 + e^{-aT}\right).$$

The characteristic polynomial when we use $C_{ZP}(z)$ is

$$z^2 - \left(1 + e^{-aT} - \left(\frac{T\left(1 - e^{-aT}\right)}{2a}\right)K\right)z + e^{-aT} + \left(\frac{T\left(1 - e^{-aT}\right)}{2a}\right)K.$$

Applying again the conditions (1.1) yields

$$0 < K < \frac{2}{T}a.$$

Comparing the three ranges of K we observe that the discretization introduces a maximum value of the gain for which the closed-loop system is asymptotically stable. Moreover, in the case of $C_I(z)$, the maximum gain is influenced by a only marginally. If $T \to 0$ then we recover the continuous-time case. In the case of $C_{ZP}(z)$, both T and a greatly influence the maximum gain. If $T \to 0$ or/and $a \to +\infty$ than we recover the continuous-time case. [6 marks]

Typical mistakes include the wrong computation of the stability margins. The computation of the stability margin for C_{ZP} resulted harder. Very few students gave a satisfactory comment regarding the differences between the stability margins.

2. a) Since the sampling time is $T = 3.92699$, the primary strip is enclosed between $-\frac{\omega_s}{2} = -\frac{\pi}{T} = -0.8$ and $\frac{\omega_s}{2} = \frac{\pi}{T} = 0.8$. The s-plane region is mapped into the shaded z-plane region shown in Figure 2.1, where

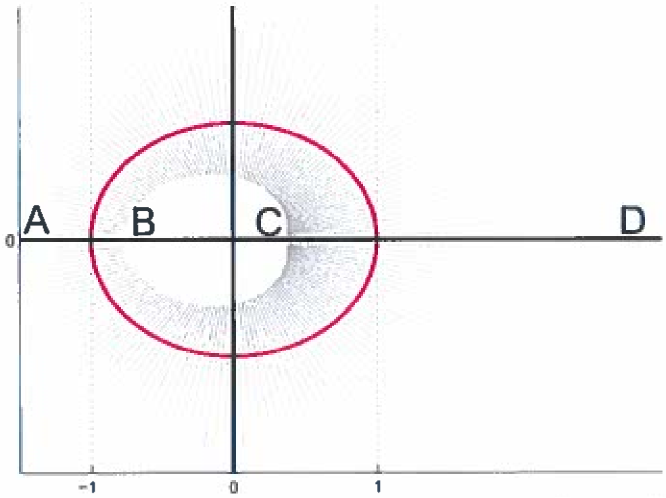


Figure 2.1 Shaded z-plane region.

$$A = e^{(0.2\pm0.8j)T}, \qquad B = e^{(-0.2\pm0.8j)T}, \qquad C = e^{-T}, \qquad A = e^{T}.$$

[4 marks]

Some students were not able to determine the region. A few students did not show that the primary strip is between $\pm 0.8j$ or forgot to multiply the exponent by T when mapping the points in the z-plane.

b) We introduce the auxiliary variables $e_1(t)$ and $e_2(t)$ at the entrance of the first two samplers. At the exit of these two samplers we introduce the variables $e_1^*(t)$ and $e_2^*(t)$. We can now write down the relations between these variables in the Laplace domain, namely

$$\begin{aligned} E_1(s) &= R(s) - H_1(s)Y^*(s), \\ E_2(s) &= C_1(s)E_1^*(s) - H_2(s)Y^*(s), \\ Y(s) &= G(s)C_2(s)E_2^*(s). \end{aligned}$$

We determine the *starred* version of the first and of the second equation, and we replace the result in the third equation

$$\begin{aligned} E_1^*(s) &= R^*(s) - H_1^*(s)Y^*(s), \\ E_2^*(s) &= C_1^*(s)E_1^*(s) - H_2^*(s)Y^*(s), \\ Y(s) &= G(s)C_2(s)\left[C_1^*(s)R^*(s) - C_1^*(s)H_1^*(s)Y^*(s) - H_2^*(s)Y^*(s)\right]. \end{aligned}$$

We now determine the *starred* version of the third equation

$$Y^*(s) = [G(s)C_2(s)]^*\left[C_1^*(s)R^*(s) - C_1^*(s)H_1^*(s)Y^*(s) - H_2^*(s)Y^*(s)\right].$$

From this equation, we solve with respect to $Y^*(s)$ and divide by $R^*(s)$, yielding

$$\frac{Y^*(s)}{R^*(s)} = \frac{C_1^*(s)\,[G(s)C_2(s)]^*}{1 + C_1^*(s)H_1^*(s)\,[G(s)C_2(s)]^* + H_2^*(s)\,[G(s)C_2(s)]^*}.$$

From this last equation we can write directly the pulse transfer function

$$\frac{Y(z)}{R(z)} = \frac{C_1(z)GC_2(z)}{1+C_1(z)H_1(z)GC_2(z)+H_2(z)GC_2(z)}.$$

where $GC_2(z) = Z[G(s)C_2(s)]$. [4 marks]

c) We introduce the auxiliary variables $E(z)$ and $E_{D_1}(z)$ before the blocks $C_1(z)$ and $C_2(z)$ respectively. We can now write down the relations between $D_1(z)$, $E(z)$, $E_{D_1}(z)$ and $Y(z)$, namely

$$\begin{aligned}
Y(z) &= G(z)C_2(z)E_{D_1}(z),\\
E_{D_1}(z) &= D_1(z)+C_1(z)E(z)-H_2(z)Y(z),\\
E(z) &= -H_1(z)Y(z).
\end{aligned}$$

Substituting the third equation in the second and the resulting second equation in the first yields

$$Y(z) = G(z)C_2(z)D_1(z) - G(z)C_2(z)C_1(z)H_1(z)Y(z) - G(z)C_2(z)H_2(z)Y(z).$$

Solving with respect to $Y(z)$ and dividing by $D_1(z)$ yields

$$\frac{Y(z)}{D_1(z)} = \frac{G(z)C_2(z)}{1+G(z)C_2(z)C_1(z)H_1(z)+G(z)C_2(z)H_2(z)}.$$

To minimize the effect of the disturbance on the output the gain of $C_1(z)$ should be selected as large as possible, whereas the gain of $C_2(z)$ should be selected as small as possible. [4 marks]

d) We introduce the auxiliary variables $E_1(z)$, $E_2(z)$ and $E_{D_2}(z)$ before the blocks $C_1(z)$, $C_2(z)$ and $G(z)$, respectively. We can now write down the relations between $D_2(z)$, $E_1(z)$, $E_2(z)$, $E_{D_2}(z)$ and $Y(z)$, namely

$$\begin{aligned}
Y(z) &= G(z)E_{D_2}(z),\\
E_{D_2}(z) &= D_2(z)+C_2(z)E_2(z),\\
E_2(z) &= C_1(z)E_1(z)-H_2(z)Y(z),\\
E_1(z) &= -H_1(z)Y(z).
\end{aligned}$$

Substituting the forth equation in the third, the resulting third in the second and the resulting second in the first yields

$$Y(z) = G(z)D_2(z) - G(z)C_2(z)C_1(z)H_1(z)Y(z) - G(z)C_2(z)H_2(z)Y(z).$$

Solving with respect to $Y(z)$ and dividing by $D_2(z)$ yields

$$\frac{Y(z)}{D_2(z)} = \frac{G(z)}{1+G(z)C_2(z)C_1(z)H_1(z)+G(z)C_2(z)H_2(z)}.$$

To minimize the effect of the disturbance on the output the gains of $C_1(z)$ and $C_2(z)$ should be selected as large as possible. [4 marks]

e) We introduce the auxiliary variables $E_1(z)$ and $E_2(z)$ before the blocks $C_1(z)$ and $C_2(z)$, respectively. We can now write down the relations between $D_3(z)$, $E_1(z)$, $E_2(z)$ and $Y(z)$, namely

$$\begin{aligned}
Y(z) &= G(z)C_2(z)E_2(z),\\
E_2(z) &= C_1(z)E_1(z)-H_2(z)Y(z),\\
E_1(z) &= -H_1(z)(D_3(z)+Y(z)).
\end{aligned}$$

Substituting the third equation in the second and the resulting second in the first yields

$$Y(z) = -G(z)C_2(z)\left[C_1(z)H_1(z)D_3(z) + C_1(z)H_1(z)Y(z) + H_2(z)Y(z)\right].$$

Solving with respect to $Y(z)$ and dividing by $D_3(z)$ yields

$$\frac{Y(z)}{D_3(z)} = \frac{-G(z)C_2(z)C_1(z)H_1(z)}{1 + G(z)C_2(z)C_1(z)H_1(z) + G(z)C_2(z)H_2(z)}.$$

To minimize the effect of the disturbance on the output the gains of $C_1(z)$ and $C_2(z)$ should be selected as small as possible. [4 marks]

The majority of students were able to do most of the questions b), c), d) and e). Typical mistakes include some computational error, or the inability to determine the appropriate gain for C_1 and C_2.

This question has been correctly perceived as the easiest question in the exam.

3. a) The equivalent discrete-time model is

$$HP(z) = \frac{z-1}{z} Z\left(\frac{P(s)}{s}\right) = \frac{z-1}{z} Z\left(\frac{2.5}{s} - \frac{5}{s+1} + \frac{2.5}{s+2}\right)$$
$$= \frac{0.0205+0.0226z}{(z-0.9048)(z-0.8187)}.$$

[4 marks]

Computational mistakes were the most common source of error in this point.

b) The transfer function in the w-plane is (recall that $T = 0.1$)

$$HP(w) = HP(z)\Big|_{z=\left(\frac{1+0.05w}{1-0.05w}\right)} = -0.000622\frac{(w+400.33)(w-20)}{(w+1.9934)(w+0.9992)}.$$

[3 marks]

Computational mistakes were the most common source of error in this point. Not showing the final passages was the most common reason for failing to obtain the full mark.

c) The velocity constant in the w-plane is defined as

$$K_v = \lim_{w\to 0} w\, C_1(w)\, HP(w) = 1.$$

Selecting $r = 0$ yields $K_v = 0$, whereas selecting $r \geq 2$ yields $K_v = \infty$. Selecting $r = 1$ yields

$$K_v = -0.000622\frac{(0+400.33)(0-20)}{(0+1.9934)(0+0.9992)}k = 1$$

that is

$$K_v = 2.5k = 1.$$

Thus $k = 0.4$ and $C_1(w) = \frac{0.4}{w}$. The open-loop transfer function is given by

$$C_1(w)HP(w) = -0.000249\frac{(w+400.33)(w-20)}{w(w+1.9934)(w+0.9992)}.$$

[3 marks]

The majority of students were able to complete this question. Computational mistakes were the most common source of error in this point.

d) The phase margin of the open-loop transfer function is approximatively 30°. We need at least 15° more.

- We start selecting $m = 2$. The phase increase is approximatively 20°. The magnitude decrease is approximatively 3 dB. From the Bode plot, we read that -3 dB is approximatively at $\omega = 0.935$ for which the phase margin is 19°. Since $20+19 = 39 < 45^\circ$, this compensator does not satisfy the phase margin requirement.
- We select $m = 3$. The phase increase is approximatively 30°. The magnitude decrease is approximatively 4.8 dB. From the Bode plot, we read that -4.8 dB is approximatively at $\omega = 1.05$ for which the phase margin is 13°. Since $30+13 = 43 < 45^\circ$, this compensator does not satisfy the phase margin requirement.

- We select $m = 4$. The phase increase is approximatively 37°. The magnitude decrease is approximatively 6 dB. From the Bode plot, we read that -6 dB is approximatively at $\omega = 1.14$ for which the phase margin is 8°. Since $37 + 8 = 45 = 45^\circ$, this is an acceptable solution.
- We select $m = 5$. The phase increase is approximatively 43°. The magnitude decrease is approximatively 7 dB. From the Bode plot, we read that -7 dB is approximatively at $\omega = 1.22$ for which the phase margin is 4°. Since $43 + 4 < 47^\circ$, this is an acceptable solution.

Note that for any value of $m \geq 4$ the phase margin becomes at least 45°. Any of these value of m is acceptable. In the following we select $m = 5$ which gives $\tau = \dfrac{\sqrt{5}}{1.22} = 1.8328$ and the controller

$$C_2(w) = \frac{1 + 1.8328w}{1 + 0.3666w}.$$

The full-page figure in the previous page shows the Bode plot of the uncompensated system (dotted line), of the compensated system with $m = 2$ and $m = 3$ (dashed lines) and of the compensated system with $m = 4$ and $m = 5$ (solid lines). [6 marks]

Some students did not attempt this point. The majority of students that attempted this question was able to obtain the maximum mark. Typical mistakes include not accounting correctly for the change in magnitude due to the compensator.

e) The discrete-time controller is

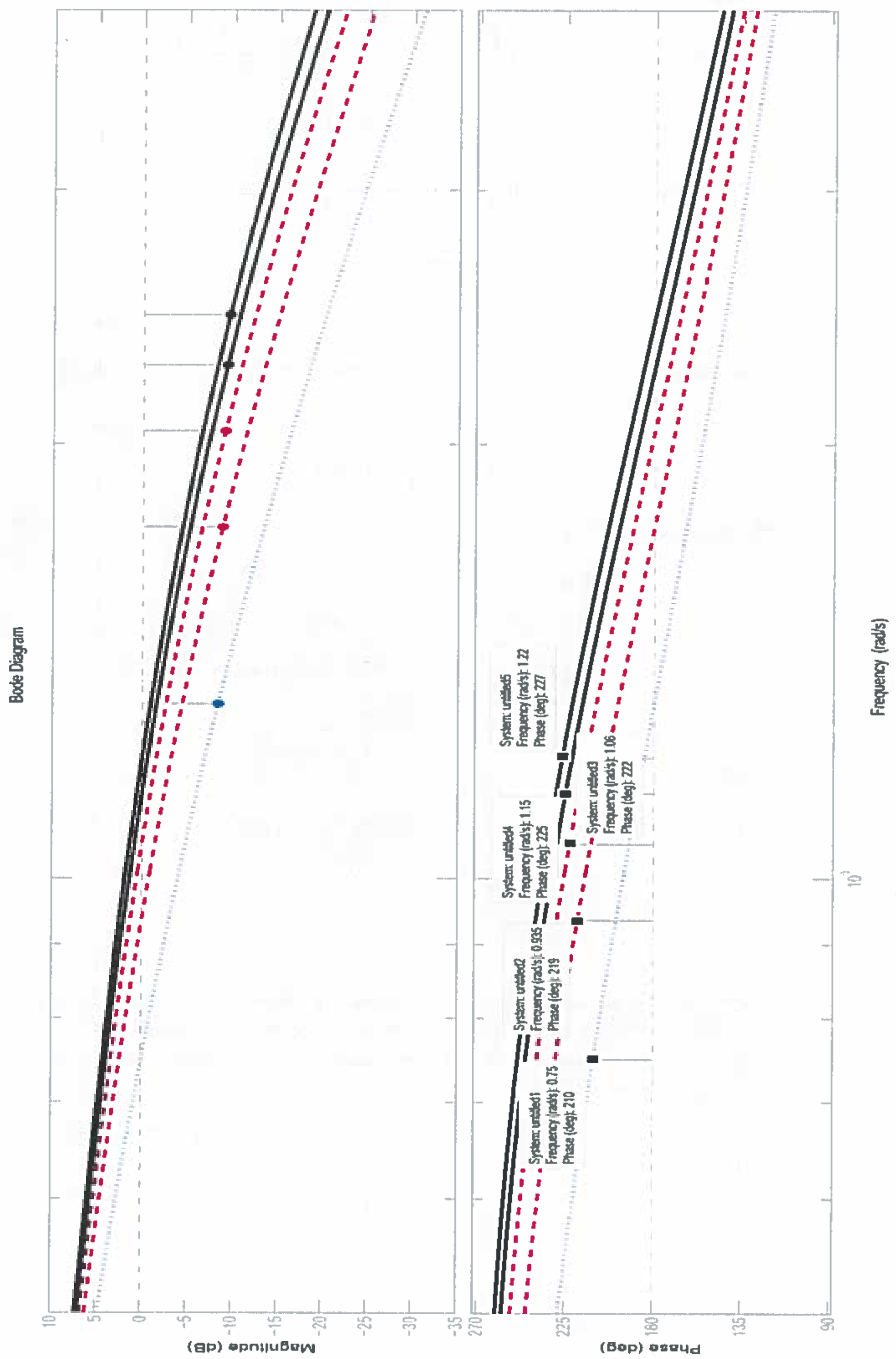
$$C(z) = [C_1(w)C_2(w)]_{w = 20\left(\frac{z-1}{z+1}\right)} = 0.0904\frac{(z - 0.9469)(z+1)}{(z-1)(z-0.75994)}.$$

Note that this is also equivalent to

$$C(z) = [C_1(w)]_{w = 20\left(\frac{z-1}{z+1}\right)} [C_2(w)]_{w = 20\left(\frac{z-1}{z+1}\right)}.$$

[4 marks]

Computational mistakes were the most common source of error in this point.


Bode Diagram
Magnitude (dB)
Phase (deg)
Frequency (rad/s)
System: untitled5
Frequency (rad/s): 1.22
Phase (deg): 227
System: untitled4
Frequency (rad/s): 1.15
Phase (deg): 225
System: untitled3
Frequency (rad/s): 1.06
Phase (deg): 222
System: untitled2
Frequency (rad/s): 0.935
Phase (deg): 219
System: untitled1
Frequency (rad/s): 0.75
Phase (deg): 210

4. a) The equivalent discrete-time model is

$$\begin{aligned}
HP(z) &= (1-z^{-1})Z\left(\frac{P(s)}{s}\right) = (1-z^{-1})Z\left(\frac{e^{-s}}{s(2s+1)}\right) \\
&= (1-z^{-1})z^{-1}Z\left(\frac{1}{s(2s+1)}\right) = (z^{-1}-z^{-2})Z\left(\frac{1}{s}-\frac{1}{s+0.5}\right) \\
&= (z^{-1}-z^{-2})\left(\frac{1}{1-z^{-1}}-\frac{1}{1-e^{-0.5}z^{-1}}\right) = \\
&= \frac{1-e^{-0.5}}{z(z-e^{-0.5})} = \frac{0.3935z^{-2}}{1-0.6065z^{-1}}.
\end{aligned}$$

[4 marks]

Computational mistakes were the most common source of error in this point.

b) Since $\alpha(1-e^{-0.5(3-1)}) = 1$, we have

$$\alpha = \frac{e}{e-1} = 1.5820.$$

The sequence $\tilde{y}(kT)$ is

$$\begin{aligned}
&\tilde{y}(0) = 0 \\
&\tilde{y}(T) = 0 \\
&\tilde{y}(2T) = \frac{e}{e-1}(1-e^{-0.5(2-1)}) = 0.6225 \\
&\tilde{y}(kT) = 1, \qquad k = 3,4,5,\dots
\end{aligned}$$

Hence,

$$\begin{aligned}
C(z) &= 0.6225z^{-2}+z^{-3}+z^{-4}+z^{-5}+\cdots = 0.6225z^{-2}+z^{-3}\frac{1}{1-z^{-1}} \\
&= \frac{0.6225z^{-2}+0.3775z^{-3}}{1-z^{-1}}.
\end{aligned}$$

[6 marks]

The majority of students was able to compute α. Many students were not able to approach the rest of the question correctly. Some of the students that approached the problem correctly did some computational mistakes. Few students earned the full mark.

c) To determine the controller $C(z)$ we compute the closed-loop transfer function between the reference and the output, namely

$$\frac{\tilde{Y}(z)}{R(z)} = \frac{C(z)HP(z)}{1+C(z)HP(z)}.$$

Since $R(z)$, $\tilde{Y}(z)$ and $HP(z)$ are given, it is sufficient to solve this equation with respect to $C(z)$ yielding

$$C(z) = \frac{-\tilde{Y}(z)}{HP(z)(\tilde{Y}(z)-R(z))}.$$

After some computation we obtain

$$C(z) = \frac{1.5820(1+0.6065z^{-1})(1-0.6065z^{-1})}{(1-0.6225z^{-2}-0.3775z^{-3})} = \frac{1.5820(1-0.3678z^{-2})}{(1-0.6225z^{-2}-0.3775z^{-3})}.$$

We observe that the denominator of $C(z)$ can be factorized as

$$1-0.6225z^{-2}-0.3775z^{-3}=(1-z^{-1})(1+z^{-1}+0.3775z^{-2}),$$

which shows that $C(z)$ has a pole at $z=1$. [6 marks]

Few students approached the problem correctly. Very few were able to complete the computation and show that there is a pole in $z=1$. A typical wrong approach consisted in trying to apply some version of the analytical method.

d) Note that $\tilde{Y}(z)=HP(z)U(z)$. Hence,

$$U(z)=\frac{\tilde{Y}(z)}{HP(z)}=\frac{1.5820(1-0.3678z^{-2})}{1-z^{-1}}.$$

By dividing the numerator by the denominator yields

$$U(z)=1.5820+1.5820z^{-1}+z^{-2}+z^{-3}+z^{-4}+\dots$$

which corresponds to the signal $u(0)=1.5820$, $u(T)=1.5820$, $u(kT)=1$ for all $k\geq 2$. The signal $u(t)$ is shown in Figure 4.1.

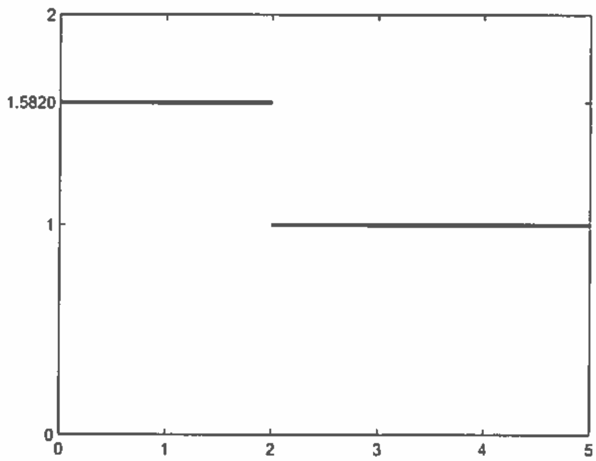


Figure 4.1 Plot of $u(t)$.

[4 marks]

Few students attempted this point and the majority of the ones that attempted the question used an incorrect approach.

This question has been correctly perceived as the hardest question in the exam.