Imperial College London

[E2.9 (Maths 4) 2013]

B.ENG. AND M.ENG. EXAMINATIONS 2013

PART II Paper 4 : MATHEMATICS (ELECTRICAL ENGINEERING)

Date Friday 31st May 2013 2.00 - 4.00 pm

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.

Answer FOUR questions.

Please answer questions from Section A and Section B in separate answer-books.

A mathematical formulae sheet is provided.

Statistical datasheets are provided.

[*Before starting, please make sure that the paper is complete; there should be 5 pages, with a total of 6 questions. Ask the invigilator for a replacement if your copy is faulty.*]

SECTION A

1. (i) Evaluate the path integral in the plane:

$$I = \oint_C (x^3\,dy - y^3\,dx)\,,$$

where C is the closed curve

$$x = \cos(\theta), \qquad y = \sin(\theta), \qquad 0 < \theta < 2\pi,$$

taken in the anticlockwise direction.

(ii) State Green's Theorem, and hence show that the above path integral may be written as an area integral. Write this area integral down explicitly, and evaluate it directly, using polar coordinates.

2. (i) Write down the necessary conditions on the vector-valued function $\boldsymbol{F}(x, y, z)$, such that

$$\boldsymbol{F} = \nabla\phi,$$

for some potential $\phi(x, y, z)$. Write this condition down in vector form.

(ii) Write down the condition for a line integral in three dimensions

$$\int_P^Q (F_1\boldsymbol{i} + F_2\boldsymbol{j} + F_3\boldsymbol{k}).\,d\boldsymbol{r}$$

to be independent of the path from P to Q.

(iii) Verify that the integral

$$\int_{(0,0,0)}^{(1,1,2)} [(y + z^2)\,dx + (x + z)\,dy + (2xz + y)\,dz]$$

is path-independent, find the corresponding potential, and hence evaluate the integral between these end points.

(iv) Write down an example of a vector field $\boldsymbol{G}(x, y, z)$ such that the integral

$$\int_P^Q (G_1\boldsymbol{i} + G_2\boldsymbol{j} + G_3\boldsymbol{k}).\,d\boldsymbol{r}$$

does depend on the path from P to Q, and show for your example that the condition for path independence is not satisfied.

PLEASE TURN OVER

SECTION B

3. (a) In the game of 'craps', a player rolls two fair six-sided dice, and the sum of their scores, X, is recorded. If X is 7 or 11, the player wins the game immediately. If X is 2, 3, or 12, the player loses the game immediately.

   (i) Find the probability mass function (PMF) of X.

   (ii) What is the probability that the player wins on the first throw?

   (iii) Suppose that the game ended on the first throw, but we do not know the result of this throw. What is the probability that the player lost?

   If X is 4, 5, 6, 8, 9, or 10, the player continues to roll the dice until either the sum of 7 is scored (in which case the player loses), or the original sum X is scored again (in which case the player wins).

   (iv) Suppose that the result of the first throw is $X = 4$. Show that the probability the player goes on to win the game is $1/3$.

   (b) A system has 'strength' $X \sim \text{Exponential}\,(k\lambda)$, and 'stress' $Y \sim \text{Exponential}\,(\lambda)$ is placed upon it, where $k,\, \lambda > 0$. X and Y are independent, and the system fails if $Y > X$.

   (i) Write down $f_{X,Y}(x, y)$, the joint probability density function (PDF) of X and Y.

   (ii) By integrating $f_{X,Y}(x, y)$ in the appropriate region, find the probability that the system fails. What happens when $k = 1$?

4. Let $X_1, \ldots, X_n$ be a random sample from $\text{Uniform}\,(0, \theta)$. Define $Y = \max_i X_i$ to be the largest observation in this sample.

   (a) Prove that the cumulative distribution function (CDF) of Y is $F_Y(y) = (F_{X_1}(y))^n$, and use this result to find $f_Y(y)$, $\text{E}(Y)$, and $\text{Var}(Y)$.

   (b) Show that the maximum-likelihood estimator of θ is $\hat{\theta}_{ML} = Y$, and prove that it is biased.

   (c) Use $\hat{\theta}_{ML}$ to construct an unbiased estimator of θ. Of the maximum-likelihood estimator and the unbiased estimator, which has smaller mean squared error (MSE)?

5. (a) The lifespan, T, of an electronic component has hazard function

$$h_T(t) \;=\; c\,(t+1)^{-1}\,, \quad t > 0,$$

where $c > 1$ is a parameter. Find:

(i) the cumulative hazard function, $H_T(t)$.

(ii) the reliability function, $R_T(t)$.

(iii) the mean time to failure, $\mathrm{E}(T)$.

Another electronic component has lifespan S and hazard function

$$h_S(t) \;=\; \sqrt{t}\,, \quad t > 0.$$

(iv) Which of the two components is more likely to still be functioning at time $t = 1$? (*Hint:* $\ln 2 > 2/3$)

(b) Consider the time series model

$$y_t \;=\; 0.5\,y_{t-2} \,+\, \varepsilon_t\,,$$

where $\{\varepsilon_t\}$ is a white noise process with a mean of zero and variance σ_ε^2.

(i) Write the model equation in the form

$$y_t \;=\; \beta(B)^{-1}\varepsilon_t\,,$$

where $\beta(B)$ is a polynomial in B. What are the roots of this polynomial? Is the time series (weakly) stationary?

(ii) By considering the power series expansion of $\beta(B)^{-1}$, show that $\gamma_0 = 2\,\sigma_\varepsilon^2$ and $\gamma_1 = 0$.

PLEASE TURN OVER

6. Consider the time series model

$$y_t = u_t + \frac{1}{2}\, u_{t-1}\,,$$

where $\{u_t\}$ is a white noise process with mean zero and variance σ_u^2.

(a) Define 'white noise'.

(b) Find $\mathrm{E}(y_t)$ and $\mathrm{Cov}(y_t, y_{t+s})$ for $s = 0, 1, 2, \ldots$ Is $\{y_t\}$ (weakly) stationary?

(c) Find the spectrum $f(\omega)$ of this time series.

For parts (d) and (e), you may refer to the results given below the question.

(d) Put the time series $\{y_t\}$ in state-space form.
Hint: Define the state at time t as $(u_t, u_{t-1})^T$.

(e) What are suitable initial values a_1, P_1 for the Kalman filter? Perform the first iteration of the filter to obtain a_2.

<u>State-space form</u>

$$\begin{aligned} y_t &= Z_t \alpha_t + \varepsilon_t \\ \alpha_{t+1} &= T_t \alpha_t + R_t \eta_t\,, \quad t = 1, 2, 3, \ldots, \end{aligned}$$

where $\mathrm{E}(\varepsilon_t) = \mathrm{E}(\eta_t) = 0$, $\mathrm{Var}(\varepsilon_t) = h_t$, $\mathrm{Var}(\eta_t) = Q_t$, and the processes $\{\varepsilon_t\}$, $\{\eta_t\}$ are mutually and serially independent.

<u>Kalman filter recursions</u>

$$\begin{aligned} v_t &= y_t - Z_t a_t \\ F_t &= Z_t P_t Z_t^T + h_t \\ K_t &= T_t P_t Z_t^T F_t^{-1} \\ a_{t+1} &= T_t a_t + K_t v_t \\ P_{t+1} &= T_t P_t (T_t - K_t Z_t)^T + R_t Q_t R_t^T\,, \end{aligned}$$

for $t = 1, 2, 3, \ldots$. Recursions are initialised with $a_1 = \mathrm{E}(\alpha_1)$ and $P_1 = \mathrm{Var}(\alpha_1)$.

END OF PAPER

A1. (i) The given path integral in the plane is

$$I = \oint_C (x^3 \mathrm{d}y - y^3 \mathrm{d}x),$$

where C is the closed curve

$$x = \cos(\theta), \qquad y = \sin(\theta), \qquad 0 < \theta < 2\pi.$$

Using $\mathrm{d}x = (-\sin(\theta)\mathrm{d}\theta$, $\mathrm{d}y = (\cos(\theta)\mathrm{d}\theta$, this becomes

$$\int_0^{2\pi} \cos^3(\theta)(\cos(\theta)\mathrm{d}\theta) - \sin^3(\theta)(-\sin(\theta)\mathrm{d}\theta) =$$

$$\int_0^{2\pi} [\cos^4(\theta) + \sin^4(\theta)]\mathrm{d}\theta =$$

(4 marks)

$$= \int_0^{2\pi} \frac{1}{4}[(1+\cos(2\theta))^2 + (1-\cos(2\theta))^2]\mathrm{d}\theta =$$

$$\frac{1}{4}\int_0^{2\pi} [2 + 2\cos^2(2\theta)]\mathrm{d}\theta = 2\pi\frac{3}{4} = \frac{3\pi}{2}.$$

(5 marks)

(ii) Green's Theorem states that

$$\oint_{\partial\Omega} P(x,y)\mathrm{d}x + Q(x,y)\mathrm{d}y = \int\int_{\Omega} [Q_x - P_y]\mathrm{d}x\mathrm{d}y,$$

where Ω is a simply connected region of the plane, and $\partial\Omega$ is its boundary.

(3 marks)

For the given example, Ω is the unit disc, and $\partial\Omega$ is the unit circle. We have $P = -y^3$, $Q = x^3$, so that $Q_x - P_y = 3(x^2 + y^2)$. The double integral is thus

$$\int\int_{\Omega} [3[x^2 + y^2]\mathrm{d}x\mathrm{d}y =$$

$$\int_0^{2\pi} \left[\int_0^r 3r^2 r\mathrm{d}r\right] \mathrm{d}\theta$$

on changing to polar coordinates.

(4 marks)

Evaluating this, we get

$$\int_0^{2\pi} \left[\int_0^r 3r^2 r\mathrm{d}r\right] \mathrm{d}\theta = 2\pi[\frac{3r^4}{4}]_0^1 = \frac{3\pi}{2},$$

as before.

(4 marks)

A2. (i) If

$$\mathbf{F} = \nabla\phi,$$

then

$$F_1 = \frac{\partial\phi}{\partial x},$$

$$F_2 = \frac{\partial\phi}{\partial y},$$

$$F_3 = \frac{\partial\phi}{\partial z}.$$

It follows, by cross-differentiating, that

$$\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y},$$

$$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z},$$

$$\frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x},$$

that is,

$$\nabla \wedge \mathbf{F} = \mathrm{curl}\mathbf{F} = 0.$$

(5 marks)

(ii) If the above condition holds, then

NOT ASKED

$$\int_P^Q (F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}).\mathrm{d}\mathbf{r} = \int_P^Q \nabla\phi.\mathrm{d}\mathbf{r} = \phi(Q) - \phi(P).$$

The converse also holds - if the integral is path-independent, then $\nabla \wedge \mathbf{F} = 0$.
(~~5~~ 4 marks)

(iii) We may verify directly that for $\mathbf{F} = ((y+z^2),(x+z),(2xz+y))$, $\nabla\wedge\mathbf{F} = 0$, proving the path-independence of the integral. Explicitly,

$$1 = \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y} = 1,$$

$$1 = \frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z} = 1,$$

$$2z = \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x} = 2z,$$

(4 marks for this or equivalent)

This being satisfied, we may set

$$y + z^2 = \frac{\partial\phi}{\partial x},$$

$$x + z = \frac{\partial\phi}{\partial y},$$

$$2xz + y = \frac{\partial \phi}{\partial z},$$

and we find

$$\phi = xy + xz^2 + yz,$$

up to an arbitrary constant.

The integral is thus

$$\int_{(0,0,0)}^{(1,1,2)} [(y + z^2)\mathrm{d}x + (x + z)\mathrm{d}y + (2xz + y)\mathrm{d}z] = [xy + xz^2 + yz]_{(0,0,0)}^{(1,1,2)} = 7.$$

(4 marks)

(iv) The vector field $\mathbf{G} = (y, 0, 0)$ is not curl-free,

$$\nabla \wedge \mathbf{G} = -\mathbf{k} \neq 0,$$

so the integral

$$\int_P^Q y\mathrm{d}x$$

is not path-independent. (3 marks)

EXAMINATION QUESTIONS/SOLUTIONS 2012-13

Course: ELEC ENG 2

Question 3

TOPIC Statistics

Marks & seen/unseen

Parts

(a) i.

x	2	3	4	5	6	7	8	9	10	11	12
$f_x(x)$	$1/36$	$2/36$	$3/36$	$4/36$	$5/36$	$6/36$	$5/36$	$4/36$	$3/36$	$2/36$	$1/36$

2 seen

ii. Define events

W: 'Player wins'

E_i: 'Game ends on i^{th} throw'

$$P(W \cap E_1) = P(X=7) + P(X=11)$$
$$= 6/36 + 2/36 = 2/9$$

3 seen similar

iii. $$P(W|E_1) = \frac{P(W \cap E_1)}{P(E_1)}$$

$$= \frac{P(X \in \{7, 11\})}{P(X \in \{2, 3, 7, 11, 12\})}$$

$$= \frac{2/9}{\frac{1}{36} + \frac{2}{36} + \frac{6}{36} + \frac{2}{36} + \frac{1}{36}} = 2/3$$

so $P(\bar{W}|E_1) = 1 - 2/3 = 1/3$

4 seen similar

iv. Let X be the result of the first throw.

Then $P(W|X=4) = \sum_{j=2}^{\infty} P(W \cap E_j | X=4)$,

as the game cannot end on 1st throw if $X=4$.

Setter's initials: MM

Checker's initials: ATJ

Parts

Winning on the j^{th} throw (given $X=4$) requires rolling something other than 4 or 7 on the preceding $j-2$ throws, followed by a 4 on the j^{th} throw. Thus:

$$P(W|X=4)=\sum_{j=2}^{\infty}\left(1-\tfrac{3}{36}-\tfrac{6}{36}\right)^{j-2}\tfrac{3}{36}$$

$$=\sum_{j=2}^{\infty}\left(\tfrac{3}{4}\right)^{j-2}\tfrac{1}{12}=\frac{1/12}{1-3/4}=1/3.$$

5 unseen

(b) i. By independence,

$$f_{X,Y}(x,y)=f_X(x)\,f_Y(y)$$

$$=\lambda k e^{-\lambda k x}\,\lambda e^{-\lambda y}$$

$$=\lambda^2 k e^{-\lambda(kx+y)},\quad x,y>0$$

2 seen similar

ii.

$$P(Y>X)=\int_{-\infty}^{\infty}\int_{x}^{\infty} f_{X,Y}(x,y)\,dy\,dx$$

$$=\int_0^{\infty}\int_x^{\infty}\lambda k e^{-\lambda k x}\lambda e^{-\lambda y}\,dy\,dx$$

$$=\int_0^{\infty}\left[-\lambda k e^{-\lambda k x}e^{-\lambda y}\right]_{y=x}^{y\to\infty}dx$$

$$=\int_0^{\infty}\lambda k e^{-\lambda(k+1)x}\,dx$$

$$=\left[-\frac{\lambda k}{\lambda(k+1)}e^{-\lambda(k+1)x}\right]_0^{\infty}=\frac{k}{k+1}$$

4 unseen

This is $1/2$ when $k=1$, as X,Y are iid.

Question 4

TOPIC

Marks & seen/unseen

Parts

(a) $F_Y(y) = P(Y \le y) = P(\max_i X_i \le y)$

$= P(X_1 \le y, X_2 \le y, \dots, X_n \le y)$

$= \prod_{i=1}^{n} P(X_i \le y)$ (by independence)

$= \prod_{i=1}^{n} F_{X_i}(y) = \left(F_{X_1}(y)\right)^n$ (identically distr.)

Hence: $F_Y(y) = \left(\frac{y}{\theta}\right)^n$, for $y \in [0, \theta]$

$\Rightarrow f_Y(y) = \frac{d}{dy} F_Y(y) = \begin{cases} \frac{n y^{n-1}}{\theta^n}, & y \in [0,\theta] \\ 0, & o/w \end{cases}$

7 unseen

$\mathbb{E}(Y) = \int_{-\infty}^{\infty} y f_Y(y)\, dy = \int_0^\theta \frac{n y^n}{\theta^n}\, dy$

$= \left[\frac{n y^{n+1}}{(n+1)\theta^n}\right]_0^\theta = \frac{n}{n+1}\theta$

$\mathbb{E}(Y^2) = \int_{-\infty}^{\infty} y^2 f_Y(y)\, dy = \int_0^\theta \frac{n y^{n+1}}{\theta^n}\, dy$

$= \frac{n}{n+2}\theta^2$

Parts

(a) cont.

So $Var(Y) = \mathbb{E}(Y^2) - \mathbb{E}(Y)^2$

$$= \frac{n}{n+2}\theta^2 - \left(\frac{n}{n+1}\theta\right)^2$$

$$= \frac{n^2+2n+1-n^2-2n}{(n+2)(n+1)^2} n\theta^2$$

$$= \frac{n\theta^2}{(n+2)(n+1)^2}$$

(b) $L(\theta;\underline{x}) = \prod_{i=1}^{n} f_{X_i}(x_i;\theta)$

$$= \prod_{i=1}^{n} \frac{1}{\theta} \quad , \quad \theta \geq x_i,\ i=1,\dots,n$$

$$= \theta^{-n} \quad , \quad \theta \geq y$$

The likelihood is zero for $\theta < Y$ and decreasing in the interval $[Y, \infty)$, so $\hat{\Theta}_{ML} = Y$.

3 seen

$\mathbb{E}(\hat{\Theta}_{ML}) = \mathbb{E}(Y) = \frac{n}{n+1}\theta$, so it is biased

EXAMINATION QUESTIONS/SOLUTIONS 2012-13

Course: ELEC ENG 2

Question 4

TOPIC

Marks & seen/unseen

Parts

(c) $\hat{\theta}_U = \frac{n+1}{n}\hat{\theta}_{ML}$ is an unbiased estimator of θ.

$$MSE_\theta(\hat{\theta}_U) = \cancel{\left(Bias_\theta(\hat{\theta}_U)\right)^2}^{\to 0} + Var(\hat{\theta}_U)$$

$$= \left(\frac{n+1}{n}\right)^2 Var(Y)$$

$$= \frac{\cancel{(n+1)^2}}{n^2} \cdot \frac{n}{(n+2)\cancel{(n+1)^2}}\theta^2$$

$$= \frac{\theta^2}{n(n+2)}$$

10

unseen

$$MSE_\theta(\hat{\theta}_{ML}) = \left(Bias_\theta(\hat{\theta}_{ML})\right)^2 + Var(\hat{\theta}_{ML})$$

$$= \left(\frac{n}{n+1}\theta - \theta\right)^2 + \frac{n}{(n+2)(n+1)^2}\theta^2$$

$$= \frac{1}{(n+1)^2}\theta^2 + \frac{n}{(n+2)(n+1)^2}\theta^2$$

$$= \frac{n+2+n}{(n+2)(n+1)^2}\theta^2 = \frac{2\cancel{(n+1)}}{(n+2)(n+1)^{\cancel{2}}}\theta^2$$

Setter's initials: MM

Checker's initials: AW

Parts

(c) cont.

so: $\underset{\theta}{MSE}(\hat{\theta}_{ML}) \geqslant \underset{\theta}{MSE}(\hat{\theta}_{U})$

$\Leftrightarrow \frac{2}{(n+2)(n+1)}\theta^2 \geqslant \frac{1}{n(\cancel{n+2})}\theta^2$

$\Leftrightarrow \quad 2n \geqslant n+1$

$\Leftrightarrow \quad n \geqslant 1$

That is, the unbiased estimator has MSE smaller than or equal to that of the MLE (with strict inequality when we have more than one observation)

Question 5

TOPIC

Marks & seen/unseen

Parts

(a) all unseen

i. $H_T(t) = \int_0^t h_T(u)\,du = \int_0^t c(u+1)^{-1}\,du$ 2

$= \left[c\log(u+1)\right]_0^t = c\log(t+1)$

ii. $R_T(t) = e^{-H_T(t)} = e^{-c\log(t+1)}$ 1

$= (t+1)^{-c}$

iii. Easier to use the formula:

$E(T) = \int_0^\infty R_T(t)\,dt$ 3

$= \int_0^\infty (t+1)^{-c}\,dt = \left[\frac{(t+1)^{1-c}}{1-c}\right]_0^\infty$

$= (c-1)^{-1}$.

iv. $H_S(t) = \int_0^t h_S(u)\,du = \int_0^t \sqrt{u}\,du$

$= \left[\frac{u^{3/2}}{3/2}\right]_0^t = \frac{2}{3}t^{3/2}$

At $t=1$, $H_S(1) = \frac{2}{3} < \log 2 < c\log 2 = H_T(1)$ 5

so $R_S(1) > R_T(1)$, and we conclude that the second component is likelier to still be functioning.

	EXAMINATION QUESTIONS/SOLUTIONS 2012-13	Course ELEC ENG 2
Question 5	TOPIC	Marks & seen/unseen

Parts

(b) i.

$$y_t = 0.5 y_{t-2} + \varepsilon_t$$

$$\Leftrightarrow y_t - 0.5 y_{t-2} = \varepsilon_t$$

$$\Leftrightarrow (1 - 0.5 B^2) y_t = \varepsilon_t$$

$$\Leftrightarrow y_t = \underbrace{(1 - 0.5 B^2)^{-1}}_{b(B)} \varepsilon_t$$

The roots of the polynomial are $\pm\sqrt{2}$, both outside the unit circle, so the AR(2) process is stationary.

4 seen similar

ii.

$$y_t = \sum_{j=0}^{\infty} (0.5 \cdot B^2)^j \varepsilon_t$$

$$= \sum_{j=0}^{\infty} 0.5^j \varepsilon_{t-2j}$$

$$\Rightarrow \gamma_0 = Var(y_t) = \sum_{j=0}^{\infty} 0.5^j \sigma_\varepsilon^2$$

$$= \frac{\sigma_\varepsilon^2}{1-0.5} = 2\sigma_\varepsilon^2$$

5 seen similar

$$\gamma_1 = Cov(y_t, y_{t+1}) = Cov\left(\sum_{j=0}^{\infty} 0.5^j \varepsilon_{t-2j}, \sum_{j=0}^{\infty} 0.5^j \varepsilon_{t+1-2j}\right)$$

$= 0$, because the two sums have no common terms.

Setter's initials MM | Checker's initials AN |

	EXAMINATION QUESTIONS/SOLUTIONS 2012-13	Course ELEC ENG 2
Question 6	TOPIC	Marks & seen/unseen

Parts

~~state~~

(a) Bookwork: $\{\varepsilon_t\}$ is white noise if $\mathbb{E}(\varepsilon_t)=0$ for all t, $Var(\varepsilon_t)=\sigma_\varepsilon^2$ for all t, and $Cov(\varepsilon_t,\varepsilon_s)=0$, for $t\neq s$ — 1 seen

(b) $$\mathbb{E}(y_t)=\mathbb{E}(u_t)+\frac{1}{2}\mathbb{E}(u_{t-1})=0$$

$$Var(y_t)=Var(u_t)+\left(\frac{1}{2}\right)^2 Var(u_{t-1})=\frac{5}{4}\sigma_u^2$$

$$Cov(y_t,y_{t+1})=Cov\left(u_t+\frac{1}{2}u_{t-1},\ u_{t+1}+\frac{1}{2}u_t\right)$$

$$=\frac{1}{2}Cov(u_t,u_t)=\frac{1}{2}\sigma_u^2$$

5 seen similar

$$Cov(y_t,y_{t+s})=0 \quad \text{for } s=2,3,\ldots$$

Thus $\{y_t\}$ is stationary

(c) $$f(\omega)=\gamma_0+2\sum_{k=1}^{\infty}\gamma_k\cos(\omega k)$$

$$=\frac{5}{4}\sigma^2+2\,\frac{1}{2}\sigma^2\cos(\omega)$$

$$=\sigma^2\left(\frac{5}{4}+\cos\omega\right)$$

3 seen similar

Setter's initials MM	Checker's initials ATU	Page number 9

Parts

(d)

$$y_t = \underbrace{\begin{pmatrix}1 & 1/2\end{pmatrix}}_{Z_t} \underbrace{\begin{pmatrix}u_t \\ u_{t-1}\end{pmatrix}}_{\alpha_t}, \quad (h_t = 0)$$

$$\underbrace{\begin{pmatrix}u_{t+1} \\ u_t\end{pmatrix}}_{\alpha_{t+1}} = \underbrace{\begin{pmatrix}0 & 0 \\ 1 & 0\end{pmatrix}}_{T_t} \underbrace{\begin{pmatrix}u_t \\ u_{t-1}\end{pmatrix}}_{\alpha_t} + \underbrace{\begin{pmatrix}1 \\ 0\end{pmatrix}}_{R_t} \underbrace{u_{t+1}}_{\eta_t},$$

$(Q_t = \sigma_u^2)$

4 seen

(e) Initialise KF with

$$a_1 = \mathbb{E}(\alpha_1) = \begin{pmatrix}0 \\ 0\end{pmatrix}, \quad P_1 = Var(\alpha_1) = \begin{pmatrix}\sigma_u^2 & 0 \\ 0 & \sigma_u^2\end{pmatrix} = \sigma_u^2 I_2$$

$$V_1 = y_1 - \cancel{Z_1 a_1} = y_1$$

$$F_1 = Z_1 P_1 Z_1^T + \cancel{h_1} = \begin{pmatrix}1 & 1/2\end{pmatrix} \sigma_u^2 I_2 \begin{pmatrix}1 \\ 1/2\end{pmatrix} = \frac{5}{4}\sigma_u^2$$

$$K_1 = T_1 P_1 Z_1^T F_1^{-1} = \begin{pmatrix}0 & 0 \\ 1 & 0\end{pmatrix} \cancel{\sigma_u^2} I_2 \begin{pmatrix}1 \\ 1/2\end{pmatrix} \left(\frac{5}{4}\cancel{\sigma_u^2}\right)^{-1}$$

$$= \begin{pmatrix}0 \\ 4/5\end{pmatrix}$$

7 seen similar

$$\therefore a_2 = \cancel{T_1 a_1} + K_1 V_1 = \begin{pmatrix}0 \\ \frac{4}{5} y_1\end{pmatrix}$$