Imperial College London

[E1.11 (Maths) ISE 2009]

B.ENG. and M.ENG. EXAMINATIONS 2009

MATHEMATICS (INFORMATION SYSTEMS ENGINEERING E1.11)

Date **Wednesday 3rd June 2009** **10.00 am - 1.00 pm**

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

Answer ANY SEVEN questions.

Answers to questions from Section A and Section B should be written in different answer books.

CALCULATORS MAY **NOT** BE USED.

A mathematical formulae sheet is provided.

[Before starting, please make sure that the paper is complete; there should be 7 pages, with a total of 9 questions. Ask the invigilator for a replacement if your copy is faulty.]

SECTION A

1. (i) Find the real and imaginary parts of

$$\frac{2 + 3i}{3 + 2i} .$$

What are the modulus and argument of this number ?

(ii) If x and y are real, find the real and imaginary parts of

$$\sinh (x + i y) .$$

Hence show that

$$| \sinh (x + i y) |^2 = \sinh^2(x) + \sin^2(y) .$$

2. (i) Evaluate the partial sum

$$\sum_{n=1}^{N} \ln\left(\frac{n+2}{n+1}\right) .$$

Evaluate the limit

$$\lim_{n\to\infty} \ln\left(\frac{n+2}{n+1}\right) .$$

State whether the infinite series

$$\sum_{n=1}^{\infty} \ln\left(\frac{n+2}{n+1}\right)$$

is convergent or not.

Is the series

$$\sum_{n=1}^{\infty} (-1)^n \ln\left(\frac{n+2}{n+1}\right)$$

convergent ?

(ii) Explain what is meant by the *radius of convergence* of a power series

$$\sum_{n=0}^{\infty} a_n z^n .$$

Calculate the radii of convergence of the following two power series :

(*a*)
$$\sum_{n=0}^{\infty} \frac{2n+1}{\sqrt{n^2+1}} z^n ,$$

(*b*)
$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} z^n .$$

Do not attempt to sum either series.

PLEASE TURN OVER

3. (i) Evaluate the limits

(a) $$\lim_{n\to 1} \frac{n^2-1}{n^2-3n+2} \, ,$$

(b) $$\lim_{x\to\pi/2} (\sec x - \tan x) \, ,$$

(c) $$\lim_{n\to\infty} \left[n \left((n^3+3n)^{1/3} \; - \; (n^4+n^2)^{1/4} \right) \right].$$

(ii) Using L'Hôpital's rule, evaluate:

$$\lim_{x\to 0} \frac{\exp(x^2)-1}{\sin^2(4x)} \, .$$

4. Evaluate the definite integrals

(i)

$$\int_0^{\frac{\pi}{2}} x^2 \sin(x) \, \mathrm{d}x,$$

(ii)

$$\int_0^\infty \exp(-3x) \, \cos(4x) \, \mathrm{d}x$$

and

$$\int_0^\infty \exp(-3x) \, \sin(4x) \, \mathrm{d}x \, ,$$

(iii)

$$\int_0^\infty \frac{\mathrm{d}x}{(x+1)\,(x^2+4)} \, .$$

PLEASE TURN OVER

5. Solve the ordinary differential equations

(i)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{4x+3y}{x+y} \ ;$$

(ii)

$$\frac{\mathrm{d}y}{\mathrm{d}x} - \tan(x)\, y = 1\,, \quad \text{with} \quad y(0) = 1\ ;$$

(iii)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = \exp(-2x), \quad \text{with} \quad y(0) = 0, \quad \text{and} \quad y'(0) = 0.$$

In each case, find the most general solution possible.

PLEASE TURN OVER

SECTION B

6. (i) Let $u = u(x, y)$, where $x = r\cos\theta$, $y = r\sin\theta$. Show that

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 .$$

(ii) Find the stationary points of the function

$$f(x, y) = (x^2 + 2y)\, e^{x+y}$$

and determine their nature.

7. (i) Find the inverse Laplace transform of the function

$$\frac{e^{-s}}{(s-2)^2} .$$

(ii) Find functions $y(t)$ and $z(t)$ satisfying the simultaneous differential equations

$$\frac{d^2y}{dt^2} - 2y + z = 0 ,$$

$$\frac{d^2z}{dt^2} - 2z + y = 0 ,$$

such that $y(0) = z(0) = 0$, $y'(0) = z'(0) = 1$.

For (i) *you may assume the shift rule*

$$L\,(H_a(t)\, f(t-a)) = e^{-as}\, L\,(f(t)) ,$$

where $H_a(t)$ is the Heaviside function

$$H_a(t) = \begin{cases} 1 , & t > a , \\ 0 , & t \le a . \end{cases}$$

For (ii) *you may assume that*

$$L\left(\frac{d^2 f}{d\,t^2}\right) = -\,f'(0) - s f(0) + s^2\, L\,(f(t)) .$$

PLEASE TURN OVER

8. The function $f(x)$ is defined in the range $-\pi \leq x < \pi$ by

$$f(x) = \begin{cases} 1 + \dfrac{x}{\pi}, & -\pi \leq x \leq 0, \\[2ex] 1 - \dfrac{x}{\pi}, & 0 \leq x < \pi. \end{cases}$$

Sketch the graph of $f(x)$.

Find the Fourier series of $f(x)$ in the range $-\pi \leq x < \pi$.

By substituting a suitable value of x, deduce that

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

9. (i) Find, in terms of K, the determinant of the matrix

$$\begin{pmatrix} 1 & 5 & 3 \\ 5 & 1 & -K \\ 1 & 2 & K \end{pmatrix}.$$

(ii) Let A be the matrix of part (i) with $K = 1$. Find all solutions $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ of the system of linear equations

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

(iii) Let A be as in part (ii). Find all vectors $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ such that the system

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

has no solutions.

END OF PAPER

MATHEMATICS DEPARTMENT

MATHEMATICAL FORMULAE

1. VECTOR ALGEBRA

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1, a_2, a_3)$$

Scalar (dot) product: $\mathbf{a}\,.\,\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

Vector (cross) product:

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Scalar triple product:

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a}\,.\,\mathbf{b}\times\mathbf{c} = \mathbf{b}\,.\,\mathbf{c}\times\mathbf{a} = \mathbf{c}\,.\,\mathbf{a}\times\mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product: $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{c}\,.\,\mathbf{a})\mathbf{b} - (\mathbf{b}\,.\,\mathbf{a})\mathbf{c}$

2. SERIES

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \ldots \quad (\alpha \text{ arbitrary, } |x| < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n\frac{x^{2n}}{(2n)!} + \ldots,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + \ldots,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^n\frac{x^{n+1}}{(n+1)} + \ldots (-1 < x \le 1)$$

3. TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin(a+b) = \sin a\cos b + \cos a\sin b\,;$$

$$\cos(a+b) = \cos a\cos b - \sin a\sin b\,.$$

$$\cos iz = \cosh z\,;\quad \cosh iz = \cos z\,;\quad \sin iz = i\sinh z\,;\quad \sinh iz = i\sin z\,.$$

4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f\,D^n g + \binom{n}{1} Df\,D^{n-1}g + \ldots + \binom{n}{r} D^r f\,D^{n-r}g + \ldots + D^n f\,g\,.$$

(b) Taylor's expansion of $f(x)$ about $x = a$:

$$f(a+h) = f(a) + hf'(a) + h^2 f''(a)/2! + \ldots + h^n f^{(n)}(a)/n! + \epsilon_n(h)\,,$$

where $\epsilon_n(h) = h^{n+1}f^{(n+1)}(a+\theta h)/(n+1)!$, $0 < \theta < 1$.

(c) Taylor's expansion of $f(x, y)$ about (a, b):

$$f(a+h, b+k) = f(a,b) + [hf_x + kf_y]_{a,b} + 1/2!\left[h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}\right]_{a,b} + \ldots$$

(d) Partial differentiation of $f(x, y)$:

i. If $y = y(x)$, then $f = F(x)$, and $\dfrac{dF}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dx}$.

ii. If $x = x(t)$, $y = y(t)$, then $f = F(t)$, and $\dfrac{dF}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$.

iii. If $x = x(u, v)$, $y = y(u, v)$, then $f = F(u, v)$, and

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}, \quad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$

(e) Stationary points of $f(x, y)$ occur where $f_x = 0$, $f_y = 0$ simultaneously.
Let (a, b) be a stationary point: examine $D = [f_{xx}f_{yy} - (f_{xy})^2]_{a,b}$.
If $D > 0$ and $f_{xx}(a, b) < 0$, then (a, b) is a maximum;
If $D > 0$ and $f_{xx}(a, b) > 0$, then (a, b) is a minimum;
If $D < 0$ then (a, b) is a saddle-point.

(f) Differential equations:

i. The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating factor $I(x) = \exp[\int P(x)(dx]$, so that $\dfrac{d}{dx}(Iy) = IQ$.

ii. $P(x, y)dx + Q(x, y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y$.

5. INTEGRAL CALCULUS

(a) An important substitution: $\tan(\theta/2) = t$:
$\sin\theta = 2t/(1+t^2)$, $\cos\theta = (1-t^2)/(1+t^2)$, $d\theta = 2\,dt/(1+t^2)$.

(b) Some indefinite integrals:

$$\int (a^2 - x^2)^{-1/2} dx = \sin^{-1}\left(\frac{x}{a}\right), \quad |x| < a.$$

$$\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1}\left(\frac{x}{a}\right) = \ln\left\{\frac{x}{a} + \left(1 + \frac{x^2}{a^2}\right)^{1/2}\right\}.$$

$$\int (x^2 - a^2)^{-1/2} dx = \cosh^{-1}\left(\frac{x}{a}\right) = \ln\left|\frac{x}{a} + \left(\frac{x^2}{a^2} - 1\right)^{1/2}\right|.$$

$$\int (a^2 + x^2)^{-1} dx = \left(\frac{1}{a}\right)\tan^{-1}\left(\frac{x}{a}\right).$$

6. NUMERICAL METHODS

(a) Approximate solution of an algebraic equation:

If a root of $f(x) = 0$ occurs near $x = a$, take $x_0 = a$ and
$x_{n+1} = x_n - [f(x_n)/f'(x_n)]$, $n = 0, 1, 2 \ldots$

(Newton Raphson method).

(b) Formulae for numerical integration: Write $x_n = x_0 + nh$, $y_n = y(x_n)$.

i. Trapezium rule (1-strip): $\int_{x_0}^{x_1} y(x)dx \approx (h/2)[y_0 + y_1]$.

ii. Simpson's rule (2-strip): $\int_{x_0}^{x_2} y(x)dx \approx (h/3)[y_0 + 4y_1 + y_2]$.

(c) Richardson's extrapolation method: Let $I = \int_a^b f(x)dx$ and let I_1, I_2 be two estimates of I obtained by using Simpson's rule with intervals h and $h/2$. Then, provided h is small enough,

$$I_2 + (I_2 - I_1)/15,$$

is a better estimate of I.

7. LAPLACE TRANSFORMS

Function	Transform	Function	Transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)dt$	$af(t) + bg(t)$	$aF(s) + bG(s)$
df/dt	$sF(s) - f(0)$	d^2f/dt^2	$s^2F(s) - sf(0) - f'(0)$
$e^{at}f(t)$	$F(s-a)$	$tf(t)$	$-dF(s)/ds$
$(\partial/\partial\alpha)f(t,\alpha)$	$(\partial/\partial\alpha)F(s,\alpha)$	$\int_0^t f(t)dt$	$F(s)/s$
$\int_0^t f(u)g(t-u)du$	$F(s)G(s)$		
1	$1/s$	$t^n (n = 1, 2 \ldots)$	$n!/s^{n+1}$, $(s > 0)$
e^{at}	$1/(s-a)$, $(s > a)$	$\sin\omega t$	$\omega/(s^2+\omega^2)$, $(s > 0)$
$\cos\omega t$	$s/(s^2+\omega^2)$, $(s > 0)$	$H(t-T) = \begin{cases} 0, & t < T \\ 1, & t > T \end{cases}$	e^{-sT}/s, $(s, T > 0)$

8. FOURIER SERIES

If $f(x)$ is periodic of period $2L$, then $f(x + 2L) = f(x)$, and

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \text{ where}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad n = 0, 1, 2, \ldots, \text{ and}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1, 2, 3, \ldots$$

Parseval's theorem

$$\frac{1}{L}\int_{-L}^{L} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right).$$

E1.11 ISE1 Maths
Solutions 2009

ISE 1

1

A1. (i) To find the real and imaginary parts of

$$\frac{2+3i}{3+2i},$$

multiply numerator and denominator by the conjugate of the denominator:

4

$$\frac{2+3i}{3+2i} = \frac{(2+3i)(3-2i)}{(3+2i)(3-2i)}$$
$$= \frac{12+5i}{13}.$$

So

$$\Re(\frac{2+3i}{3+2i}) = 12/13, \qquad \Im(\frac{2+3i}{3+2i}) = 5/13.$$

The modulus and argument of this number are respectively

4

$$|\frac{2+3i}{3+2i}| = 1, \qquad \arg(\frac{2+3i}{3+2i}) = \tan^{-1}(5/12).$$

(ii) Using the addition formula for $\sinh$, we have

4

$$\sinh(x+iy) = \sinh(x)\cos(y) + i\cosh(x)\sin(y).$$

Hence the modulus we need is, for real (x, y),

4

$$|\sinh(x+iy)|^2 = \sinh(x)^2\cos^2(y) + \cosh^2(x)\sin(y)^2.$$

Substitute $\cos^2(y) = 1 - \sin^2(y)$, $\cosh^2(x) = 1 + \sinh(x)^2$, getting

$$|\sinh(x+iy)|^2 = \sinh(x)^2(1 - \sin^2(y)) + (1 + \sinh(x)^2)\sin(y)^2 =$$

4

$$= \sinh^2(x) + \sin^2(y).$$

(Total 20)

ISE 1

2

(b) For

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} z^n,$$

4

we have

$$R = \lim_{n\to\infty} \left|\frac{(n!)^2}{(2n)!} \frac{(2(n+1))!}{((n+1)!)^2}\right|$$

$$= \lim_{n\to\infty} \left|\frac{(2n+2)(2n+1)}{(n+1)^2}\right| = 4.$$

(Total 20)

ISE 1

2

A2. (i) The partial sum

$$\sum_{n=1}^{N} \ln\left(\frac{n+2}{n+1}\right).$$

is 3

$$\sum_{n=1}^{N} \ln(n+2) - \ln(n+1)$$
$$= \ln(N+2) - \ln(2).$$

The limit

$$\lim_{n\to\infty} \ln\left(\frac{n+2}{n+1}\right)$$
$$= \lim_{n\to\infty} \ln\left(1+\frac{1}{n+1}\right)$$
$$= \ln(1) = 0.$$

2

This is a necessary, but not sufficient, condition for the series to converge.
The infinite series

$$\sum_{n=1}^{\infty} \ln\left(\frac{n+2}{n+1}\right)$$

is convergent if and only if the sequence of partial sums, here 3

$$S_N = \ln(N+2) - \ln(2),$$

converges - but $\ln(N+2)$ is unbounded - so the series diverges. However, the series

$$\sum_{n=1}^{\infty} (-1)^n \ln\left(\frac{n+2}{n+1}\right)$$

2

does converge, by the alternating series test.

(ii) By the *radius of convergence* of a power series

$$\sum_{n=0}^{\infty} a_n z^n,$$

2

we mean that R such that for $|z| < R$, the series converges, and for $|z| > R$, it diverges. If the limit

$$\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$$

exists, it is equal to the radius of convergence R.

(a) For

$$\sum_{n=0}^{\infty} \frac{2n+1}{\sqrt{n^2+1}} z^n,$$

the radius of convergence is

4

$$R = \lim_{n\to\infty} \left|\frac{2n+1}{\sqrt{n^2+1}} \frac{\sqrt{(n+1)^2+1}}{2(n+1)+1}\right|$$
$$= \lim_{n\to\infty} \left|\frac{2+1/n}{\sqrt{1+1/n^2}} \frac{\sqrt{1+1/(n+1)^2}}{2+1/(n+1)}\right| = 1.$$

ISE 1

3

A3. (a) (i)

$$\lim_{n\to 1}\frac{n^2-1}{n^2-3n+2}$$
$$=\lim_{n\to 1}\frac{(n+1)(n-1)}{(n-2)(n-1)}$$
$$=\lim_{n\to 1}\frac{(n+1)}{(n-2)}=-2.$$

5

(ii)

$$\lim_{x\to\pi/2}(\sec x-\tan x)$$
$$=\lim_{x\to\pi/2}\frac{(\sec^2 x-\tan^2 x)}{(\sec x+\tan x)}$$
$$=\lim_{x\to\pi/2}\frac{1}{(\sec x+\tan x)}=0.$$

5

(iii)

$$\lim_{n\to\infty}[n((n^3+3n)^{1/3}-(n^4+n^2)^{1/4})]$$
$$=\lim_{n\to\infty}[n^2((1+3/n^2)^{1/3}-(1+1/n^2)^{1/4})]$$

5

Using the binomial series, expanding for large n, this becomes

$$\lim_{n\to\infty}\left[n^2\left((1+1/n^2+O(1/n^4))-(1+1/(4n^2)+O(1/n^4))\right)\right]=\frac{3}{4}.$$

(b) Using L'Hopital's rule,

$$\lim_{x\to 0}\frac{\exp(x^2)-1}{\sin^2(4x)}$$
$$=\lim_{x\to 0}\frac{2x\exp(x^2)}{8\sin(4x)\cos(4x)}$$
$$=\lim_{x\to 0}\frac{(4x^2+2)\exp(x^2)}{32\cos^2(4x)-32\sin^2(4x)}=\frac{1}{16}.$$

5

(Total 20)

ISE 1

4

A4. (i) To evaluate

$$\int_0^{\frac{\pi}{2}} x^2 \sin(x)\mathrm{d}x,$$

integrate by parts:

$$\int_0^{\frac{\pi}{2}} x^2 \sin(x)\mathrm{d}x$$

$$= \int_0^{\frac{\pi}{2}} 2x\cos(x)\mathrm{d}x - [x^2\cos(x)]_0^{\frac{\pi}{2}}$$

$$= \int_0^{\frac{\pi}{2}} 2x\cos(x)\mathrm{d}x$$

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integrate by parts again,

$$= -\int_0^{\frac{\pi}{2}} 2\sin(x)\mathrm{d}x + [2x\sin(x)]_0^{\frac{\pi}{2}}$$

$$= [2\cos(x)]_0^{\frac{\pi}{2}} + [2x\sin(x)]_0^{\frac{\pi}{2}} = -2 + \pi.$$

(ii) To evaluate

$$u = \int_0^{\infty} \exp(-3x)\cos(4x)\mathrm{d}x,$$

and

$$v = \int_0^{\infty} \exp(-3x)\sin(4x)\mathrm{d}x,$$

consider

$$u + iv = \int_0^{\infty} \exp(-3x)\exp(4ix)\mathrm{d}x$$

7

$$= \int_0^{\infty} \exp(-(3-4i)x)\mathrm{d}x$$

$$= [-\frac{\exp(-(3-4i)x)}{3-4i}]_0^{\infty} = \frac{1}{3-4i} = \frac{3+4i}{25}.$$

Hence

$$\int_0^{\infty} \exp(-3x)\cos(4x)\mathrm{d}x = \frac{3}{25},$$

$$\int_0^{\infty} \exp(-3x)\sin(4x)\mathrm{d}x = \frac{4}{25}.$$

(iii) To evaluate

$$\int_0^{\infty} \frac{1}{(x+1)(x^2+4)}\mathrm{d}x,$$

expand in partial fractions:

$$\frac{1}{(x+1)(x^2+4)} = \frac{a}{x+1} + \frac{bx+c}{x^2+4},$$

provided

$$1 = a(x^2+4) + (bx+c)(x+1),$$

ISE

4

giving $a+b=0$, $b+c=0$, and $4a+c=1$. Thus $a=1/5$, $b=-1/5$, $c=1/5$, and

$$\int_0^\infty \frac{1}{(x+1)(x^2+4)} = \frac{1}{5}\int_0^\infty \frac{1}{x+1} + \frac{1-x}{x^2+4}\mathrm{d}x$$

$$= \frac{1}{5}\left[\ln(x+1) - \ln(\sqrt{x^2+4}) + \frac{1}{2}\tan^{-1}(\frac{x}{2})\right]_0^\infty$$

$$= \frac{1}{5}\left(0+\ln(2)) + \frac{\pi}{4}\right) = \frac{\ln(2)}{5} + \frac{\pi}{20}.$$

7

(Total 20)

ISE 1

5

A5. (i) To solve the ordinary differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{4x + 3y}{x + y},$$

note that it is homogeneous- set $y = xu(x)$. Then

$$u(x) + x\frac{\mathrm{d}u}{\mathrm{d}x} = -\frac{4 + 3u}{1 + u},$$

or equivalently

$$x\frac{\mathrm{d}u}{\mathrm{d}x} = -\frac{4 + 4u + u^2}{1 + u},$$

which is separable:

$$\int^{u(x)} \frac{(1 + u)\mathrm{d}u}{4 + 4u + u^2} = -\int^{x} \frac{\mathrm{d}x}{x}.$$

$$\int^{u(x)} \frac{((2 + u) - 1)\mathrm{d}u}{(2 + u)^2} = -\int^{x} \frac{\mathrm{d}x}{x}.$$

$$\int^{u(x)} \frac{1\mathrm{d}u}{(2 + u)} - \int^{u(x)} \frac{\mathrm{d}u}{(2 + u)^2} = -\int^{x} \frac{\mathrm{d}x}{x}.$$ 7

Thus

$$\ln(2 + \frac{y(x)}{x}) + \frac{x}{2x + y(x)} = -\ln x + c.$$

(ii)

$$\frac{\mathrm{d}y}{\mathrm{d}x} - \tan(x)y = 1, \quad \text{with} \quad y(0) = 1;$$

Here the integrating factor is seen to be $\cos(x)$:

$$\frac{\mathrm{d}(y\cos(x)}{\mathrm{d}x} = \cos(x), \quad \text{with} \quad y(0) = 1.$$

Integrating from 0 to x

$$y(x)\cos(x) - 1 = \sin(x),$$ 6

so that

$$y(x) = \sec(x) + \tan(x).$$

(iii) The homogeneous equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = 0$$

is linear, with constant coefficients - the auxiliary equation has the root $\lambda = -2$, repeated, so the CF is seen to be

$$y_{CF} = \exp(-2x)(A + Bx).$$

The forcing term $\exp(-2x)$ on the RHS of the given ode

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = \exp(-2x), \text{with} \quad y(0) = 0, \text{and} \quad y'(0) = 0.$$

ISE 1

5

has the same exponential dependence as the CF, so we look for a PI of the form,

$$y_{PI} = \alpha x^2 \exp(-2x).$$

Substituting in, we find $2\alpha = 1$ so the general solution of the given equation is

$$y(x) = \exp(-2x)(A + Bx + \frac{x^2}{2}).$$

The initial condition $y(0) = 0$ gives $A = 0$; then the other condition $y'(0) = 0$ gives $B = 0$. Hence the most general solution satisfying these conditions is

$$y(x) = \exp(-2x)\frac{x^2}{2}.$$

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(Total 20)

Parts

Solution

(a) By chain rule,

$$u_r = \frac{\partial u}{\partial r} = u_x \frac{\partial x}{\partial r} + u_y \frac{\partial y}{\partial r}$$

$$= u_x \cos\theta + u_y \sin\theta$$ 2

$$u_\theta = u_x \frac{\partial x}{\partial \theta} + u_y \frac{\partial y}{\partial \theta}$$

$$= u_x \cdot (-r\sin\theta) + u_y (r\cos\theta)$$

$$= r\left(-u_x \sin\theta + u_y \cos\theta\right).$$ 2

So

$$u_r^2 + \frac{1}{r^2} u_\theta^2 = \left(u_x \cos\theta + u_y \sin\theta\right)^2$$

$$+ \frac{1}{r^2} \cdot r^2 \left(-u_x \sin\theta + u_y \cos\theta\right)^2$$

$$= u_x^2 \cos^2\theta + u_y^2 \sin^2\theta + 2u_x u_y \cos\theta \sin\theta$$

$$+ u_x^2 \sin^2\theta + u_y^2 \cos^2\theta - 2u_x u_y \cos\theta \sin\theta$$

$$= u_x^2 + u_y^2.$$ 4

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~~Question~~ Solution 6, ctd

Marks & seen/unseen

Parts

(b)
$$f_x = \frac{\partial f}{\partial x} = (x^2+2y)e^{x+y} + 2xe^{x+y}$$
$$= (x^2+2x+2y)e^{x+y}$$

$$f_y = (x^2+2y)e^{x+y} + 2e^{x+y}$$
$$= (x^2+2+2y)e^{x+y}.$$

So $f_x = f_y = 0$ when
$$x^2+2x+2y = x^2+2+2y = 0$$

This happens only if $x = 1$, $y = -\frac{3}{2}$.

So only stationary point is $\underline{(1, -\frac{3}{2})}$. 6

At this point

$$A = f_{xx} = (x^2+2x+2y)e^{x+y} + (2x+2)e^{x+y}$$
$$= 4e^{-\frac{1}{2}}$$

$$B = f_{xy} = (x^2+2x+2y)e^{x+y} + 2e^{x+y}$$
$$= 2e^{-\frac{1}{2}}$$

$$C = f_{yy} = (x^2+2+2y)e^{x+y} + 2e^{x+y}$$
$$= 2e^{-\frac{1}{2}}.$$

So $A > 0$ and $AC - B^2 = 8e^{-1} - 4e^{-1} > 0$

So this point is a $\underline{\text{minimum}}$. 6

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Setter's initials: MLL

Checker's initials: [illegible]

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	EXAMINATION QUESTIONS/SOLUTIONS 2008-09	Course ISE1
Question Solution 7.		Marks & seen/unseen
Parts (i)	Using tables provided: $(LT)^{-1}\left(\frac{1}{s^2}\right) = t$ Shift rule $\Rightarrow (LT)^{-1}\frac{1}{(s-2)^2} = e^{2t}.t$ Using 2nd shift rule given in question $(LT)^{-1}\left(\frac{e^{-s}}{(s-2)^2}\right) = H(t-1)(t-1)e^{2(t-1)}$	6
(ii)	Take LT's of both ODEs $-y'(0)-sy(0)+s^2L(y)-2L(y)+L(z)=0$ & $-z'(0)-sz(0)+s^2L(z)-2L(z)+L(y)=0$	4
	but $y(0)=z(0)=0$ & $y'(0)=z'(0)=1$, $\therefore (s^2-2)L(y)+L(z)=1 \quad (1)$ & $L(y)+(s^2-2)L(z)=1 \quad (2)$	4
	Eliminating $L(z)$: $((s^2-2)^2-1)L(y) = s^2-3$ i.e. $(s^2-1)(s^2-3)L(y) = s^2-3$ $\Rightarrow L(y) = \frac{1}{s^2-1} \equiv \frac{1}{2}\left(\frac{1}{s-1}-\frac{1}{s+1}\right)$ Inverting: (using tables) $y = \frac{1}{2}(e^x-e^{-x}) \quad (= \sinh x)$	4
	& then: $z = 2y-y'' = e^x-e^{-x}-\frac{1}{2}(e^x-e^{-x})$ $= \frac{1}{2}(e^x-e^{-x})$	2
		(Total 20)
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~~Question~~ Solution 8

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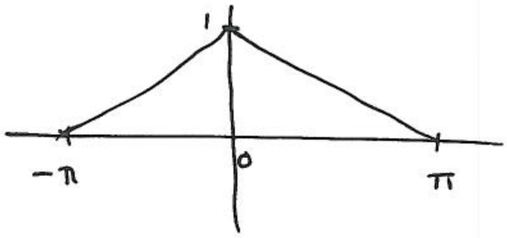
Parts

Sketch:



3

Fourier series: $f(x)$ is even, so this is a cosine series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where

$$a_0 = \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{x}{\pi}\right)dx$$

$$= \frac{2}{\pi}\left[x - \frac{x^2}{2\pi}\right]_0^{\pi} = 1$$

2

and for $n \geq 1$,

$$a_n = \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{x}{\pi}\right)\cos nx \, dx$$

By parts,

$$\frac{\pi a_n}{2} = \left[\left(1-\frac{x}{\pi}\right)\cdot\frac{1}{n}\sin nx\right]_0^{\pi} - \int_0^{\pi} -\frac{1}{\pi}\cdot\frac{1}{n}\sin nx$$

$$= 0 + \frac{1}{\pi n^2}\left[-\cos nx\right]_0^{\pi}$$

$$= \begin{cases} 0, & n \text{ even} \\ \frac{2}{\pi n^2}, & n \text{ odd.} \end{cases}$$

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Parts

So

$$a_n = \begin{cases} 0, & n \text{ even} \\ \dfrac{4}{\pi^2 n^2}, & n \text{ odd}. \end{cases}$$ 9

So Fourier series is

$$\underline{\frac{1}{2} + \frac{4}{\pi^2}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\right)}$$ 2

Putting $x = 0$ (at which f is continuous),

$$1 = f(0) = \frac{1}{2} + \frac{4}{\pi^2}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right)$$

Hence

$$\underline{1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}}.$$ 4

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	EXAMINATION QUESTIONS/SOLUTIONS 2008-09	Course ISE 1
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Parts

(a)
$$\begin{vmatrix} 1 & 5 & 3 \\ 5 & 1 & -k \\ 1 & 2 & k \end{vmatrix} = \begin{vmatrix} 1 & 5 & 3 \\ 0 & -24 & -k-15 \\ 0 & -3 & k-3 \end{vmatrix}$$

$$= -24(k-3) - 3(k+15)$$

$$= -27k + 27 = \underline{27(1-k)}.$$ 4

(b) System is
$$\begin{pmatrix} 1 & 5 & 3 \\ 5 & 1 & -1 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Solve:
$$\left(\begin{array}{ccc|c} 1 & 5 & 3 & 0 \\ 5 & 1 & -1 & 0 \\ 1 & 2 & 1 & 0 \end{array}\right) \longrightarrow \begin{pmatrix} 1 & 5 & 3 & 0 \\ 0 & -24 & -16 & 0 \\ 0 & -3 & -2 & 0 \end{pmatrix}$$

So system reduces to

$$x + 5y + 3z = 0 \quad (1)$$
$$3y + 2z = 0 \quad (2)$$

General soln:

$$\underline{z = 3t,\ y = -2t,\ x = t} \quad (\text{any } t)$$ 8

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(c) System

$$\left(\begin{array}{ccc|c} 1 & 5 & 3 & a \\ 5 & 1 & -1 & b \\ 1 & 2 & 1 & c \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 5 & 3 & a \\ 0 & -24 & -16 & b-5a \\ 0 & -3 & -2 & c-a \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 5 & 3 & a \\ 0 & -3 & -2 & c-a \\ 0 & 0 & 0 & (b-5a)-8(c-a) \end{array}\right)$$

So system has no solutions when

$$b-5a-8(c-a) \neq 0$$

i.e. $\underline{3a+b-8c \neq 0}$ 8

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