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ISE3.17

IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE
UNIVERSITY OF LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2001

EEE/ISE PART III/IV: M.Eng., B.Eng. and ACGI

ADVANCED SIGNAL PROCESSING

Wednesday, 16 May 10:00 am

There are FIVE questions on this paper.

Answer ONE question from Section A, and TWO from Section B.

Use the same answer book for each section.

Time allowed: 3:00 hours

Corrected Copy

Examiners: Ward,D.B. and Constantinides,A.G.

Special instructions for invigilators:	One main answer book only is needed on each desk (not one each for Sections A and B).
Information for candidates:	Write your answers for Sections A and B in the same answer book.

Section A

1.

(a) Explain what is meant by the following terms used to describe a parameter estimator

(i) unbiased (ii) minimum variance (iii) efficient

[5 marks]

(b) A linear model is given by

$$\underline{x} = H\underline{\theta} + \underline{w}$$

where $\underline{x}$ is an $N \times 1$ column vector of data observations, H is a known $N \times p$ observation matrix, with $N > p$ and full column rank, $\underline{\theta}$ is an unknown $p \times 1$ parameter vector and $\underline{w}$ is an $N \times 1$ vector with multivariate normal distribution $N(\underline{0}, \sigma^2 \underline{I})$.

(i) Determine the minimum variance unbiased (MVU) estimator $\hat{\underline{\theta}}$, given the observation $\underline{x}$. **[10 marks]**

(ii) Calculate the covariance matrix of $\hat{\underline{\theta}}$. **[2 marks]**

(c) The observations from a seismic sensor are assumed to satisfy

$$x[n] = \sum_{i=1}^{p} A_i (r_i)^n + w[n] \quad n = 0, 1, \ldots, N-1$$

where w[n] is zero mean white normally distributed noise with variance σ^2.

(i) Show how these can be put in the form of the linear model in (b). **[3 marks]**

(ii) Evaluate the MVU estimator of the amplitudes, A_i, when $p = 2$, $r_1 = 1$, $r_2 = -1$, and N is even.

[5 marks]

2.

(a) Define the power spectral density of a wide sense stationary discrete time random signal and comment upon its properties. **[3 marks]**

(b) A linear shift-invariant discrete time system with z-domain transfer function

$$H(z) = \frac{1 - \frac{1}{2} z^{-1}}{1 - \frac{1}{3} z^{-1}}$$

is excited by zero mean exponentially correlated noise x[n], which has autocorrelation sequence

$$r_{xx}(k) = \left(\frac{1}{2}\right)^{|k|}.$$

Denoting the output of the system by y[n], which is calculated from y[n] = x[n] * h[n], where * represents discrete time convolution, then

(i) Calculate the z-domain power spectrum, $P_{yy}(z)$, of y[n]. **[3 marks]**

(ii) Evaluate the autocorrelation sequence, $r_{yy}(k)$, of y[n]. **[3 marks]**

(iii) Determine the cross-correlation sequence, $r_{xy}(k)$, between x[n] and y[n]. **[10 marks]**

(iv) Evaluate the z-domain cross-power spectral density, $P_{xy}(z)$. **[2 marks]**

(c) Design a whitening filter for y[n]. **[4 marks]**

Section B

3.

(a) Summarize the difference between the problems of filtering, smoothing and prediction. **[5 marks]**

(b) The output of a forward prediction error filter at discrete time N is given by

$$e[N]=x[N]+\hat{x}[N]=x[N]+\sum_{k=1}^{N}a[k]x[N-k].$$

(i) Formulate , in matrix form, the parameter vector $\underline{a}_{opt}=[a_{opt}[1],a_{opt}[2],\ldots,a_{opt}[N]]^T$ which minimizes the mean square error $E\{e[N]^2\}$. The discrete time random input signal x[n] is zero mean and wide sense stationary. **[8 marks]**

(ii) Evaluate the corresponding minimum mean squared error. **[2 marks]**

(c) Given that the autocorrelation function of x[n] is

$$r_{xx}[k]=\frac{\sigma^2}{1-\rho^2}(-\rho)^{|k|} \quad \forall k, \ \ \rho\in(-1,1)$$

find the optimal in the minimum mean square error sense parameters and the corresponding mean square error of the forward prediction error filter for N = 1 and 2, and comment upon the results. **[5 marks]**

(d) Show how the elements of the optimal parameter vector for a forward predictor are related to those of a backward prediction error filter with output error of the form

$$e_b[N]=x[0]+\hat{x}[N]=x[0]+\sum_{k=1}^{N}a[k]x[k].$$

[5 marks]

4.

(a) Explain the difference between the term estimate and estimator of an unknown parameter θ. **[3 marks]**

(b) Given that the joint probability density function of the measurement vector $\underline{x} = [x[0], x[1], \ldots, x[N-1]]^T$, parameterised by the unknown scalar parameter θ, i.e. $p(\underline{x};\theta)$, satisfies the regularity condition

$$E\left[\frac{\partial \ln p(\underline{x};\theta)}{\partial \theta}\right] = 0 \quad \forall \theta$$

and the Cauchy-Schwartz inequality

$$\left[\int w(\underline{x})g(\underline{x})h(\underline{x})d\underline{x}\right]^2 \leq \int w(\underline{x})g^2(\underline{x})d\underline{x} \int w(\underline{x})h^2(\underline{x})d\underline{x}$$

where g(.) and h(.) are arbitrary functions, and $w(\underline{x}) \geq 0$ for all $\underline{x}$,

prove that the variance of any unbiased estimator of θ, i.e. $\hat{\theta}$, must satisfy

$$\mathrm{var}(\hat{\theta}) \geq \frac{1}{-E\left[\frac{\partial^2 \ln p(\underline{x};\theta)}{\partial \theta^2}\right]}.$$

[13 marks]

(c) Determine the Cramer Rao Lower Bound for an unbiased estimator of the phase, ϕ, within the model $x[n] = A\cos(2\pi f_0 n + \phi) + w[n]$, $n = 0, 1, \ldots, N-1$, where $w[n]$ is zero mean white Gaussian noise with variance σ^2. The amplitude A and frequency f_o are assumed to be fixed.

[7 marks]

(d) Comment upon the existence of an efficient estimator of the phase in the model in (c).

[2 marks]

5.

(a) Describe the difference between the linear and nonlinear least squares problems.
[3 marks]

(b) Find the least squares estimator and the corresponding minimum least squares error for the parameter A in the signal model

$$s[n] = \begin{cases} A & 0 \le n \le M-1 \\ -A & M \le n \le N-1 \end{cases}$$

given the observation $x[n] = s[n] + w[n]$ for $n = 0,1,\ldots N-1$. **[10 marks]**

(c) Calculate the probability density function of the estimator in (b) if $w[n]$ is zero mean white Gaussian noise with variance σ^2. **[5 marks]**

(d) Formulate a least squares estimator for θ given the observations

$$x[n] = \exp(\theta) + w[n] \quad n = 0,1,\ldots,N-1$$

[7 marks]

1 a) (i) Unbiased - $E\{\hat{\theta}\} = \theta$; $\theta \in [a, b]$ (Entire parameter range)
$\hat{\theta}$ - Estimator, θ - True value

(ii) Minimum mean square error: $\min E\{(\hat{\theta}-\theta)^2\}$; generally leads to an estimator which would $\hat{\theta}$ be a function of the true value - impractical. Hence, constrain estimator to be unbiased, then MSE = VAR, and therefore minimum variance = minimum mean square error.

(iii) Efficient - The minimum variance solution attains the CRLB for $\theta \in [a, b]$ - no other unbiased estimator has a lower error variance. (5)

b) (i) $$p(\underline{x}; \underline{\theta}) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left(\frac{-1}{2\sigma^2}(\underline{x}-H\underline{\theta})^T(\underline{x}-H\underline{\theta})\right)$$

$$\frac{\partial \ln p(\underline{x};\underline{\theta})}{\partial \underline{\theta}} = \frac{\partial}{\partial \underline{\theta}}\left[-\ln(2\pi\sigma^2)^{N/2} - \frac{1}{2\sigma^2}(\underline{x}-H\underline{\theta})^T(\underline{x}-H\underline{\theta})\right] = \frac{1}{\sigma^2}\left[H^T\underline{x} - H^T H \underline{\theta}\right]$$

$$I(\underline{\theta}) \Rightarrow \frac{\partial}{\partial \underline{\theta}}\left(\frac{\partial \ln p(\underline{x},\underline{\theta})}{\partial \underline{\theta}}\right)^T = \frac{-H^T H}{\sigma^2} \Rightarrow I(\underline{\theta}) = \frac{H^T H}{\sigma^2}$$

Therefore, using CRLB theory

$$\frac{\partial \ln p(\underline{x};\underline{\theta})}{\partial \underline{\theta}} = \frac{H^T H}{\sigma^2}\left[\underbrace{(H^T H)^{-1} H^T \underline{x}}_{g(\underline{x})} - \underline{\theta}\right] \Rightarrow \hat{\theta}_{MVU} = \underbrace{(H^T H)^{-1}}_{\text{Invertible, since } \underline{H} \text{ has full column rank}} H^T \underline{x}$$ (10)

(ii) $$C_{\hat{\theta}} = I^{-1}(\underline{\theta}) = \sigma^2 (H^T H)^{-1}$$ (2)

c) (i)
$$\underbrace{\begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}}_{\underline{x}} = \underbrace{\begin{bmatrix} (r_1)^0 & (r_2)^0 & \cdots & (r_p)^0 \\ (r_1)^1 & (r_2)^1 & \cdots & (r_p)^1 \\ \vdots & & & \\ (r_1)^{N-1} & (r_2)^{N-1} & \cdots & (r_p)^{N-1} \end{bmatrix}}_{H} \underbrace{\begin{bmatrix} A_1 \\ \vdots \\ A_p \end{bmatrix}}_{\underline{\theta}} + \underbrace{\begin{bmatrix} w[0] \\ w[1] \\ \vdots \\ w[N-1] \end{bmatrix}}_{\underline{w}}$$ $\underline{w} \sim \mathcal{N}(\underline{0}, \sigma^2 \underline{I})$ (3)

(ii) $$\hat{\underline{\theta}} = \begin{bmatrix} \hat{A}_1 \\ \hat{A}_2 \end{bmatrix} = (H^T H)^{-1} H^T \underline{x} \quad \text{where} \quad H = \begin{bmatrix} 1 & -1 \\ \vdots & \vdots \\ 1 & -1 \end{bmatrix} \Rightarrow H^T H = \begin{bmatrix} N & 0 \\ 0 & N \end{bmatrix}$$

$$\hat{\underline{\theta}} = \frac{1}{N}\begin{bmatrix} \sum_{i=0}^{N-1} x[i] \\ \sum_{i=0}^{N-1} (-1)^i x[i] \end{bmatrix}$$ (5)

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2) (a) $P(f) = \sum_{k=-\infty}^{\infty} r_{xx}[k] e^{-j2\pi fk} \qquad f \in [-\frac{1}{2}, \frac{1}{2})$ (3)

Properties – Real, even and non negative

(b) (i) $$\underbrace{\frac{\frac{3}{4}}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{2}z)}}_{\text{Driving, correlated process}} \cdot \frac{(1-\frac{1}{2}z^{-1})(1-\frac{1}{2}z)}{(1-\frac{1}{3}z^{-1})(1-\frac{1}{3}z)} = \frac{\frac{3}{4}}{(1-\frac{1}{3}z^{-1})(1-\frac{1}{3}z)}$$ (3)

$$P_{xx}(z) \quad \cdot \quad |H(z)|^2 \quad = \quad P_{yy}(z)$$

[First order AR process]

(ii) $r_{yy}(k) = \frac{\frac{3}{4}}{1-\frac{1}{9}}\left[\frac{1}{3}\right]^{|k|}$ (3)

(iii) $r_{xy}(k) = E\{x[n]y[n+k]\} = h[k] * r_{xx}[k]$

$h[k] = \frac{1}{3}^k u[k] - \frac{3}{2}\left(\frac{1}{3}\right)^k u[k-1]$

$r_{xx}(k) = \left(\frac{1}{2}\right)^{|k|}$

$\frac{1}{3}^k u(k) * \frac{1}{2}^{|k|}$

For $k \le 0$

$r_{xy}(k) = \sum_{l=-\infty}^{k} \left(\frac{1}{3}\right)^{(k-l)} \frac{1}{2}^{-l} = \frac{1}{3}^k \frac{1}{6}^{-k} \frac{1}{1-\frac{1}{6}} = \frac{6}{5} 2^k$

For $k > 0$

$r_{xy}(k) = \sum_{l=-\infty}^{0} \left(\frac{1}{3}\right)^{k-l} \frac{1}{2}^{-l} + \sum_{l=1}^{k} \frac{1}{3}^{k-l} \frac{1}{2}^{l}$

$= \frac{1}{3}^{(k-1)}\left[\left(\frac{3}{2}\right)^k - \frac{3}{5}\right]$

$\frac{1}{3}^k u(k-1) * \frac{1}{2}^{|k|}$

For $k \le 0$

$r_{xy}(k) = \sum_{l=-\infty}^{k-1} \left(\frac{1}{3}\right)^{k-l} \frac{1}{2}^{-l}$

$= \frac{1}{5} 2^k$

For $k \ge 1$

$r_{xy}(k) = \frac{1}{3}^k\left(\frac{6}{5}\right) + \frac{1}{3}^{(k-1)}\left(\frac{3}{2}^k - 1\right)\frac{1}{2}^k$

(10)

Combining: $r_{xy}(k) = \frac{9}{10} 2^k u[-k-1] + \frac{9}{10}\left(\frac{1}{3}\right)^k u[k]$ (2)

(iv) $P_{xy}(z) = \frac{\frac{9}{10}}{1-2z^{-1}} + \frac{\frac{9}{10}}{1-\frac{1}{3}z^{-1}} = \frac{\frac{3}{10}[6-7z^{-1}]}{(1-2z^{-1})(1-\frac{1}{3}z^{-1})}$ (4)

(c) $H_W(z) = \sqrt{\frac{4}{3}}\left(1 - \frac{1}{3}z^{-1}\right)$

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3) a) Filtering – Given $x[m] = \underset{\text{Signal}}{s[m]} + \underset{\text{Noise}}{w[m]}$ $\quad m = 0, 1, \ldots, n$

filter the signal from the noise using present and past data only

Smoothing – As filtering, except that future data is also used to estimate the signal – generally block-based.

Prediction – To estimate data outside of the observation interval – L-step forward prediction uses $\{x[0], x[1], \ldots, x[N-1]\}$ to predict $x[N-1+L]$ (5)

b) (i) $J \triangleq E\{e[N]^2\} = E\{(x[N] + \sum_{k=1}^{N} a[k]x[N-k])^2\}$

$$\frac{\partial J}{\partial a[m]} \quad m=1,2,\ldots,N \quad = 2E\{(x[N] + \sum_{k=1}^{N} a[k]x[N-k])x[N-m]\} = 0$$

For solution.

i.e. $r_{xx}[-m] = -\sum_{k=1}^{N} a_{opt}[k]\, r_{xx}[k-m]$

In matrix form

$$\underbrace{\begin{bmatrix} r_{xx}[-1] \\ r_{xx}[-2] \\ \vdots \\ r_{xx}[-N] \end{bmatrix}}_{\underline{r}_{xx}} = - \underbrace{\begin{bmatrix} r_{xx}[0] & r_{xx}[1] & \cdots & r_{xx}[N-1] \\ r_{xx}[-1] & r_{xx}[0] & \cdots & r_{xx}[N-2] \\ \vdots & \vdots & \ddots & \\ r_{xx}[1-N] & r_{xx}[2-N] & \cdots & r_{xx}[0] \end{bmatrix}}_{R_{xx}} \underbrace{\begin{bmatrix} a[1] \\ a[2] \\ \vdots \\ a[N] \end{bmatrix}}_{\underline{a}_{opt}}$$

$\underline{a}_{opt} = -R_{xx}^{-1}\underline{r}_{xx}$ (8)

(ii) $J_{min} = r_{xx}[0] + \underline{a}_{opt}^T \underline{r}_{xx}$ (2)

c) $r_{xx}[0] = \sigma^2/1-\rho^2$, $r_{xx}[1] = -\sigma^2\rho/1-\rho^2$, $r_{xx}[2] = \sigma^2\rho^2/1-\rho^2$

N=1 $\quad r_{xx}[-1] = -r_{xx}[0]a[1] \Rightarrow a[1] = \rho \qquad J_{min}^{(1)} = r_{xx}[0] + \rho r_{xx}[-1]$
$= r_{xx}[0]\left(1 - \left(\frac{r_{xx}[-1]}{r_{xx}[0]}\right)^2\right)$

N=2 $\quad \begin{bmatrix} r_{xx}[-1] \\ r_{xx}[-2] \end{bmatrix} = -\begin{bmatrix} r_{xx}[0] & r_{xx}[1] \\ r_{xx}[1] & r_{xx}[0] \end{bmatrix}\begin{bmatrix} a'[1] \\ a'[2] \end{bmatrix} \Rightarrow \begin{bmatrix} a'[1] \\ a'[2] \end{bmatrix} = \frac{-\rho}{1-\rho^2}\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\begin{bmatrix} -1 \\ \rho \end{bmatrix} = \begin{bmatrix} \rho \\ 0 \end{bmatrix}$

$\therefore$ $a[1] = a'[1]$, $J_{MIN}^{(2)} = J_{MIN}^{(1)}$ – $r_{xx}[k]$ is ACF for an AR(1) process (5)

d) $J_b \triangleq E\{e_b^2[n]\}$ as in b), $E\{(x[0] + \sum_{k=1}^{N} a[k]x[k])x[m]\} = 0 \quad m = 1, \ldots, N$

$r_{xx}[m] + \sum_{k=1}^{N} a[k]r_{xx}[m-k] = 0$

Due to even symmetry in $r_{xx}[k]$, a consequence of the WSS property, elements are identical. (5)

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4) a) Estimate is the value of the parameter for a given observation of the data.

Estimator is a rule that assigns a value to the parameter for any observation of the data. (3)

b) Given a scalar parameter $\alpha = g(\theta)$, and only estimators which are unbiased $E\{\hat{\alpha}\} = \alpha = g(\theta)$,

thus

$$\int \hat{\alpha}\, p(\underline{x};\theta)\, d\underline{x} = g(\theta)$$

Differentiating w.r.t. θ and assuming regularity conditions are satisfied

$$\int \hat{\alpha} \frac{\partial p(\underline{x};\theta)}{\partial \theta} d\underline{x} = \frac{\partial g(\theta)}{\partial \theta}$$

or

$$\int \hat{\alpha} \frac{\partial \ln p(\underline{x};\theta)}{\partial \theta} p(\underline{x};\theta)\, d\underline{x} = \frac{\partial g(\theta)}{\partial \theta}$$

Adding zero to both sides

$$\int \underbrace{(\hat{\alpha}-\alpha)}_{g(\underline{x})} \underbrace{\frac{\partial \ln p(\underline{x};\theta)}{\partial \theta}}_{h(\underline{x})} \underbrace{p(\underline{x};\theta)}_{w(\underline{x})} d\underline{x} = \frac{\partial g(\theta)}{\partial \theta}$$

Thus

$$\left(\frac{\partial g(\theta)}{\partial \theta}\right)^2 \leq \int (\hat{\alpha}-\alpha)^2 p(\underline{x};\theta)\, d\underline{x} \int \left(\frac{\partial \ln p(\underline{x};\theta)}{\partial \theta}\right)^2 p(\underline{x},\theta)\, d\underline{x}$$

or

$$\mathrm{var}(\hat{\alpha}) \geq \frac{\left(\frac{\partial g(\theta)}{\partial \theta}\right)^2}{E\left[\left(\frac{\partial \ln p(\underline{x};\theta)}{\partial \theta}\right)^2\right]}$$

Identity

$$E\left[\left(\frac{\partial \ln p(\underline{x};\theta)}{\partial \theta}\right)^2\right] = -E\left[\frac{\partial^2 \ln p(\underline{x};\theta)}{\partial \theta^2}\right]$$

and if $\alpha = g(\theta) = \theta$ $\quad \frac{\partial g(\theta)}{\partial \theta} = 1$

hence

$$\mathrm{var}(\hat{\theta}) \geq \frac{1}{-E\left\{\frac{\partial^2 \ln p(\underline{x};\theta)}{\partial \theta^2}\right\}}$$

(13)

c) $$p(\underline{x};\phi) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left\{ -\frac{1}{2\sigma^2} \sum_{n=0}^{N-1} [x[n] - A\cos(2\pi f_0 n + \phi)]^2 \right\}$$

$$\frac{\partial}{\partial\phi} \ln p(\underline{x};\phi) = -\frac{A}{\sigma^2} \sum_{n=0}^{N-1} \left[x[n] \sin(2\pi f_0 n + \phi) - \frac{A}{2} \sin(4\pi f_0 n + 2\phi) \right]$$

and

$$\frac{\partial^2 \ln p(\underline{x};\phi)}{\partial\phi^2} = -\frac{A}{\sigma^2} \sum_{n=0}^{N-1} \left[x[n] \cos(2\pi f_0 n + \phi) - A\cos(4\pi f_0 n + 2\phi) \right]$$

$$-E\left[\frac{\partial^2 \ln p(\underline{x};\phi)}{\partial\phi^2} \right] = \frac{A^2}{\sigma^2} \sum_{n=0}^{N-1} \left[\frac{1}{2} + \frac{1}{2}\cos(4\pi f_0 n + 2\phi) - \cos(4\pi f_0 n + 2\phi) \right]$$

$$\approx \frac{NA^2}{2\sigma^2} \quad \text{since } \frac{1}{N} \sum_{n=0}^{N-1} \cos(4\pi f_0 n + 2\phi) \approx 0$$

for $f_0 \not\approx 0/\tfrac{1}{2}$

$$\therefore \operatorname{var}(\hat{\phi}) \geq \frac{2\sigma^2}{NA^2} \quad \text{— C.R.L.B}$$

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d) A phase estimator does not exist which is unbiased and attains the CRLB (with equality) – no efficient estimator

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5 (a) Least squares problem

$$\min_{\theta} J(\theta) = \sum_{n=0}^{N-1} (\underset{\uparrow\ \text{Observation}}{x[n]} - \overset{\text{Model}\ \downarrow}{s[n]})^2$$

- If the model is linear in θ, linear least squares and easy to solve analytically, e.g. θn, $\theta \sin(2\pi f n)$
- If the model is non linear in θ, non linear least squares and difficult to solve analytically, save for special case when substitution is possible, e.g. $\exp(\theta n)$, $\sin(2\pi \theta n)$

(3)

(b) $$J(A) = \sum_{n=0}^{M-1} (x[n]-A)^2 + \sum_{n=M}^{N-1} (x[n]+A)^2$$

$$\frac{\partial J(A)}{\partial A} = -2\sum_{n=0}^{M-1}(x[n]-A) + 2\sum_{n=M}^{N-1}(x[n]+A) = 0 \quad - \ (*)$$

$$\Rightarrow -2\sum_{n=0}^{M-1} x[n] + 2MA + 2\sum_{n=M}^{N-1} x[n] + 2(N-M)A = 0$$

$$\Rightarrow \hat{A} = \frac{1}{N}\left(\sum_{n=0}^{M-1} x[n] - \sum_{n=M}^{N-1} x[n]\right)$$

$$J_{MIN} = \sum_{n=0}^{M-1}(x[n]-\hat{A})(x[n]-\hat{A}) + \sum_{n=M}^{N-1}(x[n]+\hat{A})(x[n]+\hat{A})$$

$$= \sum_{n=0}^{M-1} x[n](x[n]-\hat{A}) + \sum_{n=M}^{N-1} x[n](x[n]+\hat{A}) \quad \text{using } (*)$$

$$= \sum_{n=0}^{N-1} x^2[n] - \hat{A}\left(\sum_{n=0}^{M-1} x[n] - \sum_{n=M}^{N-1} x[n]\right)$$

$$= \sum_{n=0}^{N-1} x^2[n] - N\hat{A}^2$$

(10)

(c) $$E[\hat{A}] = \frac{1}{N}\left[MA - (N-M)A\right] = \frac{2MA - NA}{N} = A'$$

$$VAR(\hat{A}) = \frac{1}{N^2}\left[var\left(\sum_{n=0}^{M-1} x[n]\right) + var\left(\sum_{n=M}^{N-1} x[n]\right)\right] = \frac{1}{N^2}\left[M\sigma^2 + (N-M)\sigma^2\right] = \frac{\sigma^2}{N}$$

$$\Rightarrow \hat{A} \sim \mathcal{N}\left(A', \frac{\sigma^2}{N}\right)$$ - follows from $\hat{A}$ being a linear function of the $x[n]$'s

(5)

d) Quick method – $\alpha = e^{\theta} \Rightarrow \hat{\alpha} = \bar{x}$ (sample mean)
and hence $\hat{\theta} = \ln(\bar{x})$

Iteratively – Newton Rapson

$$\theta_{k+1} = \theta_k + \left(\underline{H}^T(\theta_k)\,\underline{H}(\theta_k) - \sum_{n=0}^{N-1} G_n(\theta_k)(x[n] - e^{\theta_k})\right)^{-1} \underline{H}^T(\theta_k)(\underline{x} - e^{\theta_k}\underline{1})$$

$$[H(\theta)]_i = \frac{\partial s[i]}{\partial \theta} = e^{\theta} \quad i = 0, 1, \ldots, N-1$$

$$[G_n(\theta)]_{ii} = \frac{\partial^2 s[n]}{\partial \theta^2} = e^{\theta}$$

$$\Rightarrow H(\theta) = e^{\theta}\underline{1} \qquad G_n(\theta) = e^{\theta}$$

$$\theta_{k+1} = \theta_k + \left(N e^{2\theta_k} - \sum_{n=0}^{N-1} e^{\theta_k}(x[n] - e^{\theta_k})\right)^{-1} e^{\theta_k}\underline{1}^T(\underline{x} - e^{\theta_k}\underline{1})$$

$$= \theta_k + \frac{e^{\theta_k}(N\bar{x} - N e^{\theta_k})}{N e^{2\theta_k} - N\bar{x}e^{\theta_k} + N e^{2\theta_k}}$$

$$= \theta_k + \frac{\bar{x} - e^{\theta_k}}{2e^{\theta_k} - \bar{x}}$$ 7

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Jonathon Chambers
20-01-01