IMPERIAL COLLEGE LONDON

E4.45
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DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2008

MSc and EEE/ISE PART IV: MEng and ACGI

Corrected Copy

WAVELETS AND APPLICATIONS

Friday, 9 May 10:00 am

Time allowed: 3:00 hours

There are FOUR questions on this paper.

Answer THREE questions.

All questions carry equal marks.

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : P.L. Dragotti

Second Marker(s) : A.G. Constantinides

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Special Information for the Invigilators: NONE

Information for Candidates:

Lipshitz regularity:

The restriction of $f(t)$ to $[a, b]$ is uniformly Lipschitz $\alpha \geq 0$ over $[a, b]$ if there exists a real $K > 0$ such that for all $\nu \in [a, b]$ there exists a polynomial $p_\nu(t)$ of degree $m = \lfloor \alpha \rfloor$ such that

$$\forall t \in (a, b), \quad |f(t) - p_\nu(t)| \leq K|t - \nu|^\alpha.$$

The Questions

1. *The Laplacian Pyramid (LP)* as shown in Figure 1a is frequently used in computer vision. The basic idea of the LP is the following: First, derive a coarse approximation of the original signal by lowpass filtering and downsampling. Based on this coarse version, predict the original (by upsampling and filtering) and then calculate the difference as the predictor error. Transmit $c[n]$ and $d[n]$. The corresponding synthesis system is shown in Figure 1b.

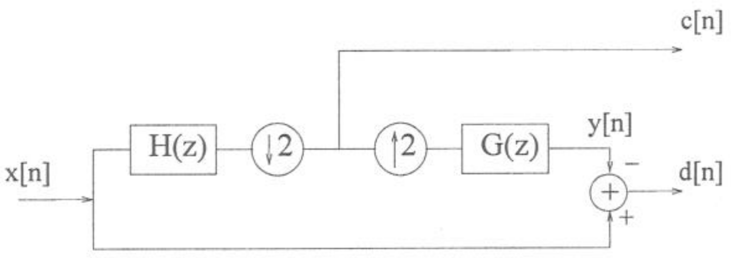

Figure 1a: Decomposition of $x[n]$ using the Laplacian Pyramid.

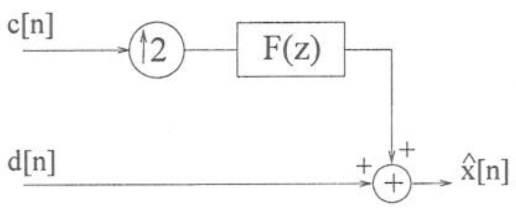

Figure 1b: Reconstruction using the synthesis part of the LP.

(a) Express $\hat{X}(z)$ as a function of $X(z)$ and the filters. Then, derive the perfect reconstruction condition(s) the filters have to satisfy.

[5]

(b) Assume that $G(z)$ is half-band ideal low-pass filter as shown in Figure 1c with $A = \sqrt{2}$, also assume that $H(z) = G(z^{-1})$. Sketch and dimension the Fourier transform of c[n] and d[n] assuming that x[n] has the spectrum shown in Figure 1d.

[5]

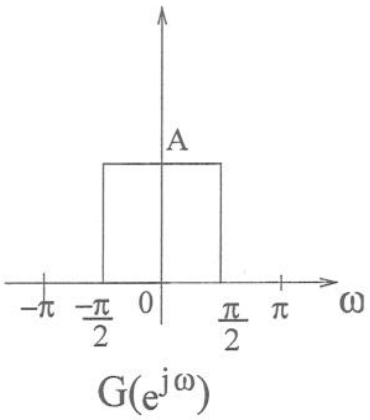

Figure 1c: Lowpass filter.

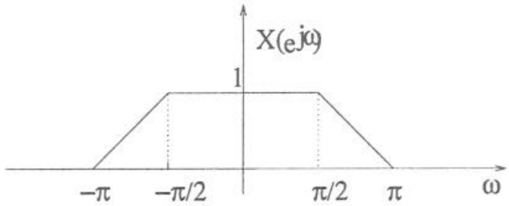

Figure 1d: Fourier transform of $x[n]$

(c) Consider a filter $G(z) = z^{-1} + 2 + z$ and assume $H(z) = G(z)/2$. (This is very similar to the original Laplacian pyramid contruction). Show that the operator P that converts $x[n]$ into $y[n]$ is sub-optimal since it is not idempotent. That is $P^2 \neq P$.

[5]

(d) With $G(z) = z^{-1} + 2 + z$, design a 5-tap symmetric filter $H(z)$ with two zeros at $z = -1$ such that the idempotent contraint is met. That is, design $H(z)$ such that $P^2 = P$.

[5]

2. *Spectral Factorization methods for two-channel filter-banks.* First of all, recall that if a polynomial $P(z)$ is symmetric then if z_k is a root of $P(z)$, so is $1/z_k$. Moreover, when the coefficients of $P(z)$ are real then if z_k is a root of $P(z)$ so is z_k^* where $*$ denotes the complex conjugate. Consider now the two-channel filter bank of Figure 2 and the 10th degree half-band polynomial $P(z) = (1+z)^3(1+z^{-1})^3 Q(z)$, where $Q(z)$ is a symmetric polynomial with real coefficients. Moreover $Q(z)$ has four complex roots in the right half complex plane.

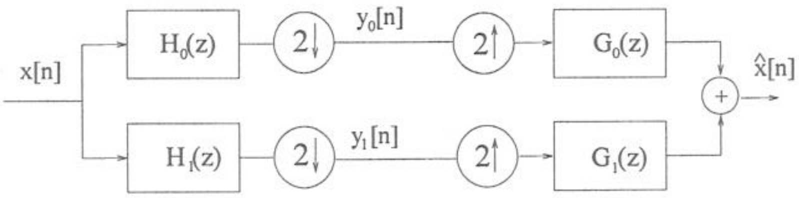

Figure 2: Two-channel filter bank.

(a) Denote with r one of the four complex roots of $Q(z)$ and assume $|r| < 1$. Draw a figure to show the ten roots on the complex plane. Notice that you do not need to compute the actual value of r.

[5]

(b) Without computing r, factorize $P(z)$ in order to have an orthogonal filter bank. Choose $G_0(z)$ to be minimum phase.

[5]

(c) Show that the high-pass branch of the orthogonal filter-bank you have just designed annihilates discrete-time polynomials of maximum degree 2. That is, show that $\sum_k x[n-k]h_1[k] = 0$, for $x[n] = n^l$ and $l = 0, 1, 2$.

[5]

(d) Now factorize $P(z)$ in order to have a biorthogonal filter bank with symmetric filters with real coefficients. There are many different possible factorizations, choose a factorization where both $G_0(z)$ and $H_0(z)$ have at least two zeros at $\omega = \pi$.

[5]

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3. Consider a biorthogonal scaling function $\varphi(t)$ and its dual $\tilde{\varphi}(t)$. The two functions satisfy the following two-scale equations:

$$\varphi(t) = \sqrt{(2)} \sum_n g_0[n] \varphi(2t - n)$$

and

$$\tilde{\varphi}(t) = \sqrt{(2)} \sum_n h_0[n] \tilde{\varphi}(2t - n).$$

(a) Show that the biorthogonality condition $\langle \tilde{\varphi}(t), \varphi(t-n) \rangle = \delta[n]$ implies that $\langle h_0[k+2n], g_0[k] \rangle = \delta[n]$.

[5]

(b) Now assume that $\varphi(t)$ is a linear B-spline given by

$$\varphi(t) = \begin{cases} 1 - |t|, & |t| < 1 \\ 0, & \text{otherwise.} \end{cases}$$

In this case, the z-transform of $g_0[n]$ is

$$G_0(z) = \frac{1}{2\sqrt{2}}(z + 2 + z^{-1}).$$

Using the following half-band filter

$$P(z) = \frac{1}{16}(1+z)^2(1+z^{-1})^2(-z+4-z^{-1}).$$

sketch and dimension the corresponding wavelet

$$\psi(t) = \sqrt{(2)} \sum_n (-1)^{n-1} h_0[1-n] \varphi(2t - n).$$

[5]

(c) The dual of $\psi(t)$ is given by

$$\tilde{\psi}(t) = \sqrt(2) \sum_n (-1)^{n-1} g_0[1-n] \tilde{\varphi}(2t-n).$$

How many vanishing moments has $\tilde{\psi}(t)$? Justify your answer.

[5]

(d) A function $f(t)$ uniformly Lipshitz in $[a,b]$ with Lipshitz coefficients $\alpha = 1.8$ is decomposed using $\psi(t)$:

$$f(t) = \sum_n \sum_m \langle f(t), \tilde{\psi}_{m,n}(t) \rangle \psi_{m,n}(t).$$

Show that the wavelet coefficients in the cone of influence of $t_0 \in [a,b]$ decay as follows: $\langle f(t), \tilde{\psi}_{m,n}(t) \rangle \sim C_1 2^{m(\alpha+1/2)}$ where C_1 is a constant.

[5]

4. Let $\varphi(t)$ and $\psi(t)$ be the Haar scaling and wavelet functions, respectively. Let V_j and W_j be the spaces generated by $\varphi_{j,n}(t) = \sqrt{2^{-j}}\varphi(2^{-j}t - n)$, $n \in \mathbb{Z}$ and $\psi_{j,n}(t) = \sqrt{2^{-j}}\psi(2^{-j}t - n)$, $n \in \mathbb{Z}$, respectively. Consider the function defined on $0 \le t < 1$ given by

$$f(t) = \begin{cases} -1 & 0 \le t < 1/4 \\ 4 & 1/4 \le t < 1/2 \\ 2 & 1/2 \le t < 3/4 \\ -3 & 3/4 \le t < 1. \end{cases}$$

(a) Express $f(t)$ in terms of the basis of V_{-2}. In other words, find the coefficients $c_{-2,n}$, $n \in \mathbb{Z}$ that leads to the decomposition $f(t) = \sum_{n\in\mathbb{Z}} c_{-2,n}\varphi_{-2,n}(t)$.

[5]

(b) Now, decompose $f(t)$ into its component parts W_{-1}, W_0, and V_0. In other words, find the coefficients $c_{0,n}$, $d_{-1,n}$ and $d_{0,n}$, $n \in \mathbb{Z}$ that leads to the following decomposition

$$f(t) = \sum_{n\in\mathbb{Z}} c_{0,n}\varphi_{0,n}(t) + \sum_{j=-1}^{0}\sum_{n\in\mathbb{Z}} d_{j,n}\psi_{j,n}(t).$$

[5]

(c) Sketch and dimension each of the decompositions of part (b).

[5]

(d) Verify the Parseval equality. That is, verify that:

$$\|f(t)\|^2 = \sum_{n} |c_{0,n}|^2 + \sum_{j=-1}^{0}\sum_{n} |d_{j,n}|^2.$$

[5]