EE4-24
EE9-CS1-3

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2013

MSc and EEE/EIE PART IV: MEng and ACGI

DISCRETE-TIME SYSTEMS AND COMPUTER CONTROL

Friday, 17 May 10:00 am

Time allowed: 3:00 hours

There are FOUR questions on this paper.

Answer ALL questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s) :	T. Parisini
	Second Marker(s) :	E.C. Kerrigan

DISCRETE-TIME SYSTEMS AND COMPUTER CONTROL

Information for candidates:

- $\mathscr{Z}\left(\frac{1}{s}\right) = \frac{z}{z-1} = \frac{1}{1-z^{-1}}$

- $\mathscr{Z}\left(\frac{1}{s+a}\right) = \frac{z}{z-e^{-aT}} = \frac{1}{1-z^{-1}e^{-aT}}$

- $\mathscr{Z}\left(\frac{1}{s^2}\right) = T\frac{z}{(z-1)^2} = T\frac{z^{-1}}{(1-z^{-1})^2}$

- $\mathscr{Z}\left(\frac{1}{s^3}\right) = \frac{T^2}{2}\frac{z(z+1)}{(z-1)^3} = \frac{T^2}{2}\frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3}$

- Transfer function of the ZOH: $H_0(s) = \frac{1-e^{-sT}}{s}$

- Definition of the w-plane: $z = \frac{1+\frac{wT}{2}}{1-\frac{wT}{2}}$, $w = \frac{2}{T}\frac{z-1}{z+1}$

- Tustin transformation: $s = \frac{2}{T}\frac{z-1}{z+1}$

- Note that, for a given signal r, or $r(t)$, $R(z)$ denotes its $\mathscr{Z}$-transform.

1. Consider the digital control system in Figure 1.1.

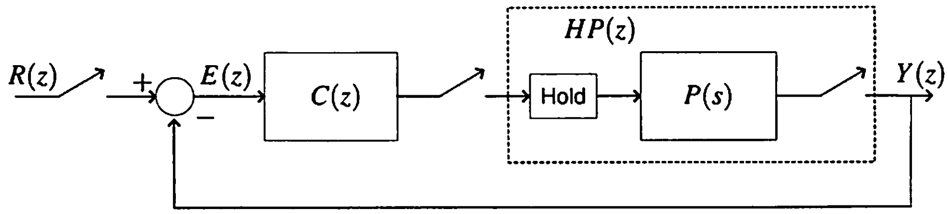

Figure 1.1 Block diagram for Question 1.

Let

$$P(s) = \frac{1}{5s+1}.$$

Assume the hold is a "zero-order-hold" (ZOH in the following) and let the sampling period be $T = 1$.

a) Compute the equivalent discrete-time model $HP(z)$ for the plant interconnected to the hold and the sampler. [4 marks]

b) Consider a continuous-time controller described by the transfer function

$$C(s) = \frac{as+b}{s},$$

with $a > 0$ and $b > 0$ parameters to be selected. Discretize the controller $C(s)$ using the Tustin transformation. Compute explicitly the resulting discrete-time controller. Determine for which values of a and b the controller has a zero inside the unity disk (in this case we say that the controller is minimum phase). [4 marks]

c) Using the results of parts a) and b) compute the closed-loop transfer function from the input $R(z)$ to the output $Y(z)$. [6 marks]

d) Study the stability properties of the discrete-time closed-loop system computed in part c) and discuss if there is a selection of a an b which gives a stable closed-loop system and a minimum phase controller. [6 marks]

2. Consider the digital control system in Figure 2.1.

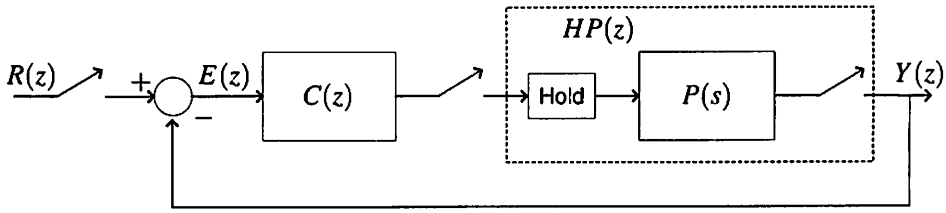

Figure 2.1 Block diagram for Question 2.

Assume the hold is a ZOH and $T > 0$ is the sampling time.

Dahlin's algorithm for the design of a digital control system consists in determining the transfer function $C(z)$ of the controller such that the closed-loop system, from the input $R(z)$ to the output $Y(z)$ is described by the equation

$$\frac{Y(z)}{R(z)} = W_d(z) = \frac{(1-A)z^{-N-1}}{1-Az^{-1}}.$$

with $0 < A < 1$ and $N \geq 0$ and integer.

a) Show that $W_d(z)$ is the discrete-time equivalent model of the continuous-time transfer function

$$W_c(s) = \frac{e^{-hs}}{\lambda s + 1}$$

with $h = NT$ and $\lambda = -T/\log A > 0$. [6 marks]

b) Show that Dahlin's controller, which is a controller that yields the closed-loop transfer function $W_d(z)$ for any open-loop transfer function $HP(z)$, is described by the equation

$$C_D(z) = \frac{(1-A)z^{-N-1}}{1-Az^{-1}-(1-A)z^{-N-1}} \frac{1}{HP(z)}.$$

[8 marks]

c) Let

$$HP(z) = K\frac{z^{-1}}{1-z^{-1}}.$$

i) Assume $K = 1$. Compute Dahlin's controller $C(z)$ for $N = 1$ and $A = 1/2$. [2 marks]

ii) Let $K > 0$. Consider the closed-loop system with the controller designed in part c.i).

Write the characteristic polynomial of the closed-loop system. [2 marks]

Study the stability properties of the closed-loop system as a function of K. [2 marks]

3. The transfer function describing the dynamics of a temperature sensor is given by

$$P(s) = \frac{\tau_m(s)}{\tau(s)} = e^{-10s}\frac{1/10}{s+1/10},$$

where $\tau(s)$ is the actual temperature and $\tau_m(s)$ is the measured temperature.

Assume the actual temperature profile is given (as a function of time) by

$$\tau(t) = \begin{cases} 85°\mathrm{C} & 0 \leq t \leq 10, \\ 70°\mathrm{C} & 10 < t. \end{cases}$$

Assume the temperature is recorded by a computer with a sampling period $T = 10$.

a) Discuss why it is not possible to determine a discrete-time equivalent model for the sensor. [5 marks]

b) Determine the Laplace transform of the signal $\tau(t)$. [5 marks]

c) Determine the Laplace transform of the measured temperature $\tau_m(t)$. [2 marks]

d) Determine the $\mathscr{Z}$-transform of the sampled measured temperature. (Recall that $T = 10$) [8 marks]

4. Consider the transfer function

$$P(s) = \frac{1}{s(s+1)}.$$

Assume it is interconnected to a ZOH and a sampler. Let the sampling period be $T = 1$.

a) Compute the equivalent discrete-time model $HP(z)$ for the plant interconnected to a ZOH and a sampler. [4 marks]

b) Using the definition of the w-plane, determine the transfer function $HP(w)$. [4 marks]

c) Let

$$C(w) = \frac{a}{w + 12.19},$$

with $a > 0$.
Note that there is an approximate cancellation in the product $C(w)HP(w)$.

Let $\tilde{CHP}(w)$ be the transfer function obtained assuming that the approximate cancellation is exact.

In what follows perform all computations using the transfer function $\tilde{CHP}(w)$.

i) Write the characteristic polynomial of the closed-loop system. Study the stability properties of the closed-loop system as a function of $a > 0$. Select a value of a yielding a stable closed-loop system. [8 marks]

ii) Using the value of a determined in part c.i) and the definition of the w-plane, compute the discrete-time controller $C(z)$. Explain why the controller $C(z)$ stabilizes the discrete-time closed-loop system.
(Do not compute the characteristic polynomial of the discrete-time closed-loop system.) [4 marks]

SOLUTIONS: DISCRETE-TIME SYSTEMS AND COMPUTER CONTROL

1. Solution

a) As $T = 1$, to compute the equivalent discrete-time model $HP(z)$ for the plant interconnected to the hold and the sampler we group the ZOH block with $P(s)$, thus having in the Laplace domain

$$H(s)P(s) = (1 - e^{-s})\frac{1}{s(5s+1)}$$

Then

$$\begin{aligned} HP(z) &= \left(1 - z^{-1}\right)\mathscr{Z}\left[\frac{1}{s(5s+1)}\right] = \left(1 - z^{-1}\right)\mathscr{Z}\left[\frac{1}{s} - \frac{1}{s+1/5}\right] \\ &= \left(1 - z^{-1}\right)\left[\frac{1}{1-z^{-1}} - \frac{1}{1-e^{-1/5}z^{-1}}\right] \end{aligned}$$

After some algebra, we finally get

$$HP(z) = \frac{1 - e^{-1/5}}{z - e^{-1/5}}$$

[4 marks]

b) To discretize the controller $C(s) = (as+b)/s$ with the Tustin transformation, we substitute $s = \frac{2}{T}\frac{z-1}{z+1}$ (with $T = 1$), thus obtaining

$$C_D(z) = C(s)|_{s=2\frac{z-1}{z+1}} = \frac{2a\frac{z-1}{z+1} + b}{2\frac{z-1}{z+1}} = \frac{(2a+b)z + b - 2a}{2(z-1)}$$

The zero of $C_D(z)$ is $z_0 = \frac{2a-b}{2a+b}$. It is straightforward to see that $|z_0| < 1, \forall a > 0, b > 0$ thus concluding that any choice of positive a and b yields a minimum-phase discrete-time controller.

[4 marks]

c) According to the samplers locations shown in Fig. 1, it follows that

$$G_{\text{cl}}(z) = \frac{Y(z)}{R(z)} = \frac{C_D(z)HP(z)}{1 + C_D(z)HP(z)}$$

Using the results given in the answers to Question 1a) and 1b), that is

$$HP(z) = \frac{1 - e^{-1/5}}{z - e^{-1/5}}, \quad C_D(z) = \frac{(2a+b)z + b - 2a}{2(z-1)},$$

after some algebraic calculations we finally get

$$G_{\text{cl}}(z) = \frac{(1-e^{-1/5})(2a+b)z+(1-e^{-1/5})(b-2a)}{2e^{1/5}z^2+[(1-e^{-1/5})(2a+b)-2(1+e^{1/5})]z+2+(1-e^{-1/5})(b-2a)}.$$

[6 marks]

d) To study the stability properties of the discrete-time closed-loop system computed in the answer to Question 1c), we have to analyze the location of the closed-loop poles (that is, the roots of the characteristic equation) in the z-plane with respect to the unit-circle. The characteristic equation is:

$$z^2+Az+B=0$$

with

$$A = \tfrac{1}{2}e^{-1/5}\left[(1-e^{-1/5})(2a+b)-2(1+e^{1/5})\right]$$

$$B = \tfrac{1}{2}\left[2+(1-e^{-1/5})(b-2a)\right]$$

By using the bilinear transformation it is easy to see that all roots of the characteristic equation are located strictly inside the unit circle if and only if

$$B > -1-A; \quad B < 1; \quad B > A-1$$

A possible choice is $A = 0$; $B = 0$ from which, after some algebra, we get

$$a = \frac{e^{1/5}}{e^{1/5}-1}+\frac{e^{2/5}}{2(e^{1/5}-1)} \simeq 8.88, \quad b = \frac{e^{2/5}}{e^{1/5}-1} \simeq 6.74$$

which is a feasible choice of a,b to stabilize the system in closed-loop (the controller is minimum-phase for any $a > 0, b > 0$ as reported in the answer to Question 1b)).

[6 marks]

2. Solution

a) To compute the equivalent discrete-time model of $W_c(s)$, we write

$$H(s)W_c(s) = (1-e^{-Ts})\frac{1}{s(\lambda s+1)}e^{-hs}$$

Since $h = NT$, the term e^{-hs} corresponds to a discrete-time delay of N time-steps and hence we have:

$$\begin{aligned} HW_c(z) &= \left(1-z^{-1}\right)\mathscr{Z}\left[\frac{1}{s(\lambda s+1)}\right]z^{-N} = \left(1-z^{-1}\right)\mathscr{Z}\left[\frac{1}{s}-\frac{1}{s+1/\lambda}\right]z^{-N} \\ &= \left(1-z^{-1}\right)\left[\frac{1}{1-z^{-1}}-\frac{1}{1-e^{-T/\lambda}z^{-1}}\right]z^{-N} \\ &= \frac{1-e^{-T/\lambda}}{z-e^{-T/\lambda}}z^{-N} \end{aligned}$$

Since $\lambda = -T/\log A \Longrightarrow \log A = -T/\lambda$, after some algebra, we finally get

$$HW_c(z) = \frac{Y(z)}{R(z)} = W_d(z) = \frac{(1-A)z^{-N-1}}{1-Az^{-1}}.$$

[6 marks]

b) According to the samplers locations shown in Fig. 1, it follows that

$$W_d(z) = \frac{C_D(z)HP(z)}{1+C_D(z)HP(z)}$$

Hence, after some algebra, we obtain

$$C_D(z) = \frac{W_d(z)}{1-W_d(z)}\cdot\frac{1}{HP(z)}$$

and substituting the expression of $W_d(z)$ determined in the answer to Question 2a) it follows that

$$C_D(z) = \frac{\frac{(1-A)z^{-N-1}}{1-Az^{-1}}}{1-\frac{(1-A)z^{-N-1}}{1-Az^{-1}}}\cdot\frac{1}{HP(z)} = \frac{(1-A)z^{-N-1}}{1-Az^{-1}-(1-A)z^{-N-1}}\frac{1}{HP(z)}.$$

[8 marks]

c) i) Since

$$HP(z) = \frac{z^{-1}}{1-z^{-1}}.$$

we have

$$C_D(z) = \frac{\frac{1}{2}z^{-2}}{1-\frac{1}{2}z^{-1}-\frac{1}{2}z^{-2}}\frac{z^{-1}}{1-z^{-1}} = \frac{1}{1+2z}$$

[2 marks]

ii) The closed-loop discrete-time transfer function is

$$W_d(z) = \frac{(1-\frac{1}{2})z^{-2}}{1-\frac{1}{2}z^{-1}} = \frac{\frac{1}{2}}{z(z-\frac{1}{2})}$$

The denominator $z(z-\frac{1}{2})$ is the characteristic polynomial.

[2 marks]

The characteristic polynomial does not depend on K. The roots of the characteristic polynomial are $z_1 = 0$ and $z_2 = \frac{1}{2}$ which are both located strictly inside the unit circle. Hence the closed-loop system is asymptotically stable $\forall K$.

[2 marks]

3. Solution

a) As no sampler has been inserted at the input τ of the sensor, it is not possible to determine a discrete-time equivalent model for the sensor because a transfer function in the z-domain cannot be defined.

[5 marks]

b) The function of time describing the temperature profile at the input to the sensor can be expressed as

$$\tau(t) = 85 \cdot [1(t) - 1(t-10)] + 70 \cdot 1(t-10)$$

where $1(t)$ denotes the continuous-time unit step function. Then

$$\tau(s) = \mathscr{L}[\tau(t)] = \mathscr{L}[85 \cdot [1(t) - 1(t-10)] + 70 \cdot 1(t-10)] = \frac{85 - 15e^{-10s}}{s}$$

[5 marks]

c) Using the expression of $P(s)$, we have:

$$\tau_m(s) = \frac{85 - 15e^{-10s}}{s} \cdot \frac{1/10}{s + 1/10} e^{-10s} = 85 \frac{1/10}{s(s+1/10)} e^{-10s} - 15 \frac{1/10}{s(s+1/10)} e^{-20s}$$

[2 marks]

d) Since $T = 10$, the factors e^{-10s} and e^{-20s} in the s-domain translate in the factors z^{-1} and z^{-2} in the z-domain, respectively. Then:

$$\begin{aligned}
\mathscr{Z}\left[85 \frac{1/10}{s(s+1/10)} e^{-10s}\right] &= \mathscr{Z}\left[85 \left(\frac{1}{s} - \frac{1}{s+1/10}\right) e^{-10s}\right] \\
&= 85 \left(\frac{1}{1-z^{-1}} - \frac{1}{1-\frac{1}{e}z^{-1}}\right) z^{-1} \\
&= \frac{85(1-\frac{1}{e})}{(z-1)(z-\frac{1}{e})}
\end{aligned}$$

Likewise

$$\begin{aligned}
\mathscr{Z}\left[15 \frac{1/10}{s(s+1/10)} e^{-20s}\right] &= \mathscr{Z}\left[15 \left(\frac{1}{s} - \frac{1}{s+1/10}\right) e^{-20s}\right] \\
&= 15 \left(\frac{1}{1-z^{-1}} - \frac{1}{1-\frac{1}{e}z^{-1}}\right) z^{-2} \\
&= \frac{15(1-\frac{1}{e})z^{-1}}{(z-1)(z-\frac{1}{e})}
\end{aligned}$$

and hence, after some algebra, we get

$$\tau_m(z) = 15(1 - 1/e) \frac{\frac{17}{3}z - 1}{z(z-1)(z-\frac{1}{e})}$$

[8 marks]

4. Solution

a) To compute the equivalent discrete-time model $HP(z)$ for the plant interconnected to the hold and the sampler we group the ZOH block with $P(s)$, thus having in the Laplace domain (recall that $T = 1$)

$$H(s)P(s) = (1 - e^{-s})\frac{1}{s^2(s+1)}$$

Then

$$\begin{aligned} HP(z) &= (1 - z^{-1})\,\mathscr{Z}\left[\frac{1}{s^2(s+1)}\right] = (1 - z^{-1})\,\mathscr{Z}\left[\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}\right] \\ &= (1 - z^{-1})\left[\frac{z^{-1}}{(1 - z^{-1})^2} - \frac{1}{1 - z^{-1}} + \frac{1}{1 - \frac{1}{e}z^{-1}}\right] \end{aligned}$$

After some algebra, we finally get

$$HP(z) = \frac{\frac{1}{e}z + 1 - \frac{2}{e}}{(z-1)(z - \frac{1}{e})}$$

[4 marks]

b) According to the definition of the w-plane (recalling again that $T = 1$) we have

$$z = \frac{1 + \frac{1}{2}w}{1 - \frac{1}{2}w}$$

Substituting into the expression of $HP(z)$ gives (after some algebra)

$$\begin{aligned} HP(w) &= HP(z)|_{z=(1+\frac{1}{2}w)/(1-\frac{1}{2}w)} \\ &= \frac{(\frac{1}{4} - \frac{3}{4e})w^2 + (\frac{2}{e} - 1)w + 1 - \frac{1}{e}}{(\frac{1}{2e} + \frac{1}{2})w^2 + (1 - \frac{1}{e})w} \end{aligned}$$

[4 marks]

c) i) The zeros of $HP(w)$ are $z_1 = (2e-2)/(e-3) \simeq -12.1986$ and $z_2 = 2$. Assuming that the approximate cancellation among z_1 and the pole $p = -12.19$ of the controller $C(w)$ is exact, we have:

$$C\tilde{H}P(w) = \frac{a(2-w)}{(\frac{1}{2e} + \frac{1}{2})w^2 + (1 - \frac{1}{e})w}$$

and hence the characteristic polynomial is

$$\begin{aligned} &\left(\frac{1}{2e} + \frac{1}{2}\right)w^2 + \left(1 - \frac{1}{e}\right)w + a(2-w) \\ &= \left(\frac{1}{2e} + \frac{1}{2}\right)w^2 + \left(1 - \frac{1}{e} - a\right)w + 2a \end{aligned}$$

If $a < 1 - 1/e \simeq 0.63$, the roots of the characteristic polynomial have negative real part thus guaranteeing the closed-loop stability.

A possible choice is $a = 1/2$. [8 marks]

ii) The controller $C(z)$ can be computed as follows:

$$\begin{aligned} C(z) &= C(w)|_{w=2(z-1)/(z+1)} = \left.\frac{1/2}{w+12.19}\right|_{w=2(z-1)/(z+1)} \\ &= \frac{z+1}{4z+20.38} \end{aligned}$$

According to the answer to Question 4c i), the choice $a = 1/2$ stabilizes the closed-loop in the w-plane. Owing to the correspondence between the points in the w-plane and the ones in the z-plane, it can be concluded that the discrete-time closed-loop control system is asymptotically stable because, thanks to the controller $C(z)$, the closed-loop poles are located strictly inside the unit circle. [4 marks]