IMPERIAL COLLEGE LONDON

B.Eng and M.Eng Examinations 2016 - 2017

Part 1

BE1-HMATH1 Mathematics I

Monday, 22 May 2017 10:00-11:30
(duration: 90 minutes)

All three questions are compulsory.

Please answer each question in separate answer book.

A list of formulae is provided separately.

Each question is worth 100 marks.
Marks for questions and parts of questions are shown next to the question. The marks for questions (and parts thereof) are indicative, and they may be slightly moderated at the discretion of the Examiner.

Question 1 This question has two parts.

a) **i)** Show that, if $-1 \leq x \leq 1$, then

$$x^2 = \frac{1}{3} + 4\sum_{n=1}^{\infty}(-1)^n \frac{\cos(n\pi x)}{n^2\pi^2}.$$

35 marks

ii) and that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \cong 1.65.$$

15 marks

b) **i)** Let $w = x + iy$ be a complex number, show that

$$2\ \mathrm{Im}\left[\cosh^2 w\right] = (\sin 2y)(\sinh 2x).$$

20 marks

ii) Find all the solutions to the complex equation

$$z\bar{z} = \mathrm{Im}\left[z^2\right]\ .$$

10 marks

iii) Find all the solutions to the complex equation

$$x^8 + x^5 - 2x^2 = 0\ .$$

10 marks

iv) Draw the solutions from ii) and iii) on an Argand diagram.

10 marks

The two parts carry equal marks.

Question 2 This question has two parts.

a) **i)** Use the series expansion of an appropriate function to estimate $\sqrt[3]{1.3}$ using 3 terms ($n = 0, 1, 2$).

25 marks

ii) Estimate the error in using only the first 3 terms of the expansion.

15 marks

b) **i)** Solve the following differential equation:

$$\frac{dy}{dx} = y - x$$

with the condition $y(0) = 1/2$.

30 marks

ii) Sketch the solution $y(x)$. Identify clearly the y-intercept (you may ignore the x-intercepts), local maxima/minima, intervals in which $y(x)$ is increasing and decreasing, intervals in which $y(x)$ is concave up and down, and asymptotes.

30 marks

The two parts carry, respectively, 40%, and 60% of the marks.

Question 3 **a)** Determine whether the set of three vectors $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is linearly independent or dependent.

10 marks

b) Determine whether the set of three vectors $\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$ is linearly independent or dependent.

10 marks

c) Is it possible to express the vector $\begin{pmatrix} 2 \\ 6 \\ -4 \end{pmatrix}$ as a linear combination of $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$?

20 marks

d) Which vectors in **R**^{3} can be expressed as a linear combination of $\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$? Provide both the algebraic and geometric answers.

20 marks

e) Consider three spheres of radius 2. The first sphere is centered at $A = (1, 0, -1)$, the second sphere is centered at $B = (0, 1, 1)$, and the third sphere is centered at $C = (1, 1, 0)$. Find a plane which touches all of the three spheres from the same side.

40 marks

The five parts carry, respectively, 10%, 10%, 20%, 20%, and 40% of the marks.

Question 1 This question has two parts.

a) **i)** Show that, if $-1 \leq x \leq 1$, then

$$x^2 = \frac{1}{3} + 4\sum_{n=1}^{\infty}(-1)^n \frac{\cos(n\pi x)}{n^2\pi^2}.$$

35 marks

Marks: $\overline{35}$

ii) and that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \cong 1.65.$$

15 marks

Marks: $\overline{15}$

b) **i)** Let $w = x + iy$ be a complex number, show that

$$2\ \mathrm{Im}\left[\cosh^2 w\right] = (\sin 2y)(\sinh 2x).$$

20 marks

$2\mathrm{Im}\left[\cosh^2 w\right] = \mathrm{Im}\left[\frac{e^{2w}+e^{-2w}+2}{2}\right] = \mathrm{Im}\left[\frac{e^{2x}\sin 2y - e^{-2x}\sin 2y}{2}\right] = (\sin 2y)(\sinh 2x)$.

Marks: $\overline{20}$

ii) Find all the solutions to the complex equation

$$z\bar{z} = \mathrm{Im}\left[z^2\right] .$$

10 marks

Let $z = x + iy$, then the equation says that $x^2 + y^2 = 2xy$ which leads to the equation $(x - y)^2 = 0$, i.e., the line $y = x$.

Marks: $\overline{10}$

iii) Find all the solutions to the complex equation

$$x^8 + x^5 - 2x^2 = 0 \ .$$

10 marks

$x^8 + x^5 - 2x^2 = x^2(x^6 + x^3 - 2) = x^2(x^3 + 2)(x^3 - 1) = 0$. *So the roots are* $\{0, 1, e^{i2\pi/3}, e^{i4\pi/3}, 2^{1/3}e^{i\pi/3}, 2^{1/3}e^{i\pi}, 2^{1/3}e^{i5\pi/3}\}$.

Marks:

$\overline{10}$

iv) Draw the solutions from ii) and iii) on an Argand diagram.

10 marks

ii) The line $y = x$. *iii) The 7 points given by the cylindrical coordinates in iii).*

Marks:

$\overline{10}$

The two parts carry equal marks.

Question 2 This question has two parts.

a) **i)** Use the series expansion of an appropriate function to estimate $\sqrt[3]{1.3}$ using 3 terms ($n = 0, 1, 2$).

25 marks

Marks: $\overline{25}$

ii) Estimate the error in using only the first 3 terms of the expansion.

15 marks

Marks: $\overline{15}$

b) **i)** Solve the following differential equation:

$$\frac{dy}{dx} = y - x$$

with the condition $y(0) = 1/2$.

30 marks

The equation is linear, so we first find the integrating factor, which is e^x, and the solution is $1 + x + Ce^x$ and $C = -1/2$ given the initial condition.

Marks: $\overline{30}$

ii) Sketch the solution $y(x)$. Identify clearly the y-intercept (you may ignore the x-intercepts), local maxima/minima, intervals in which $y(x)$ is increasing and decreasing, intervals in which $y(x)$ is concave up and down, and asymptotes.

30 marks

y-intercept is 1/2.. Local maximum at $x = \ln 2$. y is decreasing in $[\ln 2, \infty)$ and increasing in $(-\infty, \ln 2)$, and is always concave down. There are no asymptotes.

Marks: $\overline{30}$

The two parts carry, respectively, 40%, and 60% of the marks.

Question 3 a) Determine whether the set of three vectors $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is linearly independent or dependent.

10 marks

Independent because $\det \begin{bmatrix} 1 & 1 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 1 \end{bmatrix} \neq 0.$

Marks: $\overline{\mathbf{10}}$

b) Determine whether the set of three vectors $\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$ is linearly independent or dependent.

10 marks

Dependent because $\det \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix} = 0.$

Marks: $\overline{\mathbf{10}}$

c) Is it possible to express the vector $\begin{pmatrix} 2 \\ 6 \\ -4 \end{pmatrix}$ as a linear combination of $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$?

20 marks

No. $\begin{pmatrix} 2 \\ 6 \\ -4 \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ b \\ -a+b \end{pmatrix}$ *does not have a solution, since* $a = 2$ *and* $b = 6$ *result in* $-a + b = 4 \neq -4$.

Marks: $\overline{\mathbf{20}}$

d) Which vectors in $\mathbf{R}^3$ can be expressed as a linear combination of

$\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$? Provide both the algebraic and geometric answers.

20 marks

The vectors $a \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a+c \\ b+c \\ -a+b \end{pmatrix}$*, which lie on the plane* $x - y + z = 0$*.*

Marks:

$\overline{\mathbf{20}}$

e) Consider three spheres of radius 2. The first sphere is centered at $A = (1, 0, -1)$, the second sphere is centered at $B = (0, 1, 1)$, and the third sphere is centered at $C = (1, 1, 0)$. Find a plane which touches all of the three spheres from the same side.

40 marks

The plane $x - y + z = 0$ *goes through the points* A, B *and* C*. The plane which is parallel to* $x - y + z = 0$ *with the distance* 2 *goes through the point* $(1, 0, -1) \pm \frac{2}{\sqrt{3}}(1, -1, 1)$*, and is described by*
$x - y + z = 1 - 1 \pm \frac{2}{\sqrt{3}}(1 + 1 + 1) = \pm 2\sqrt{3}$*.*

Marks:

$\overline{\mathbf{40}}$

The five parts carry, respectively, 10%, 10%, 20%, 20%, and 40% of the marks.