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IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2007

MSc and EEE/ISE PART IV: MEng and ACGI

DIGITAL SIGNAL PROCESSING AND DIGITAL FILTERS

Tuesday, 15 May 10:00 am

Time allowed: 3:00 hours

There are FIVE questions on this paper.

Answer THREE questions.

CORRECTED COPY

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : A.G. Constantinides

Second Marker(s) : M.K. Gurcan

The Questions

1

1.1. Define the root moments $\{S_m\}$ of the real polynomial $f(z) = K\prod_{i=1}^{n}\left(1 - r_i z^{-1}\right)$ where m is the degree of the moment, and comment on their dependence on $r_i \quad i = 1,2,\ldots,n$ as $m \to \infty$. [2]

1.2. A Finite Impulse Response transfer function is of the form

$$H(z) = K\prod_{i=1}^{n_1}\left(1 - \alpha_i z^{-1}\right)\prod_{i=1}^{n_2}\left(1 - \beta_i z^{-1}\right),$$

where K is a constant, α_i are the zeros inside the unit circle and β_i are the zeros outside the unit circle.

Set $N_1(z) = \prod_{i=1}^{n_1}\left(1 - \alpha_i z^{-1}\right)$ and $N_2(z) = \prod_{i=1}^{n_2}\left(1 - \beta_i z^{-1}\right)$,

Show that if $H(z)$ is real then the root moments of both $N_1(z)$ and $N_2(z)$ are also real. [2]

1.3. Given the amplitude and phase responses are $A(\theta)$ and $\phi(\theta)$ of $H(z)$ derive the *Fundamental Relationships*

$$\ln(A(\theta)) = \ln(K_1) - \sum_{m=1}^{\infty}\frac{S_m^{N_1} + S_{-m}^{N_2}}{m}\cos(m\theta),$$

$$\phi(\theta) = -n_2\theta + \sum_{m=1}^{\infty}\frac{S_m^{N_1} - S_{-m}^{N_2}}{m}\sin(m\theta)$$

where K_1 is an appropriate real constant, $S_m^{N_1}$ are the root moments of the minimum phase factor and $S_{-m}^{N_2}$ the inverse root moments of the maximum phase factor of $H(z)$. [7]

1.4. Hence show that if the transfer function $H(z)$ is linear phase then it must have zeros located outside the unit circle, and determine their number in relation to the number of zeros located inside the unit circle. [4]

1.5. Determine the *Fundamental Relationships* for the allpass transfer function

$$H(z) = \prod_{i=1}^{m} A_i(z) \text{ where } A_i(z) = \left(\frac{z^{-1} - \alpha_i^*}{1 - \alpha_i z^{-1}}\right).$$ [5]

2.

2.1. Define the normalised group delay $\tau(\theta)$ of a discrete time system of transfer function $H(z)$ and show that if on the unit circle $H(e^{j\theta}) = A(\theta)e^{j\phi(\theta)}$ then we may write $\tau(\theta) = -\operatorname{Im}\left[\frac{d}{d\theta}\left(\ln H(e^{j\theta})\right)\right]$. [2]

2.2. Let the transfer function of a real allpass system of order m that has no real zeros be given by $H(z) = \prod_{i=1}^{m} A_i(z)$ where $A_i(z) = \left(\frac{1-\alpha_i^* z}{z-\alpha_i}\right)$, $\alpha_i = \rho_i e^{j\psi_i}$ and $|\rho_i| < 1$. Show that the phase response of $A_i(z)$ is given by $\arg(A_i(e^{j\theta})) = -\theta - 2\arctan\frac{\rho_i \sin(\theta-\psi_i)}{1-\rho_i\cos(\theta-\psi_i)}$. [2]

2.3. Determine an expression for the overall group delay $\tau(\theta)$ of the real allpass $H(z)$ defined as above. [4]

2.4. Show that for ρ_i as above and for any ψ_i, $\int_0^{2\pi} \frac{d}{d\theta}\frac{\rho_i \sin(\theta-\psi_i)}{1-\rho_i\cos(\theta-\psi_i)}\, d\theta = 0$ [2]

2.5. Hence determine the average group delay $\tau_{av} = \frac{1}{2\pi}\int_0^{2\pi}\tau(\theta)d\theta$ and explain the significance of this result. [5]

2.6. Show that the group delay $\tau(\theta)$ of the real allpass is always positive. [5]

3.

3.1. A real digital filter transfer function $H_N(z)$ is given by

$$H_N(z) = \frac{p_0 + p_1 z^{-1} + \ldots + p_{N-1} z^{-(N-1)} + p_N z^{-N}}{1 + d_1 z^{-1} + \ldots + d_{N-1} z^{-(N-1)} + d_N z^{-N}}.$$

It is proposed to realise this transfer function as in Figure 1 where

$$H_N(z) = Y_1 / X_1.$$

The subsystem S in the figure is linear and is characterised by the relationships $Y_1 = AX_2 + BY_2$ $\quad X_1 = CX_2 + DY_2$. Express $H_N(z)$ as a function of $H_{N-1}(z)$ and A, B, C, D. Determine an expression for $H_{N-1}(z)$ in terms of $H_N(z)$ and the parameters A, B, C, D. [5]

3.2. By examining $\left[H_N(z) - \frac{B}{D}\right]$, or otherwise, determine the condition under which $H_N(z)$ is independent of $H_{N-1}(z)$. [2]

3.3. The parameters of S are chosen so as to make $H_{N-1}(z)$ of degree $(N-1)$. Verify that the following choice satisfies the requirements: $A = p_N z^{-1}$, $B = p_0$, $C = d_N z^{-1}$, $D = 1$. [3]

3.4. Discuss other possible and alternative non-trivial values for these parameters. [3]

3.5. For the given selection above derive the coefficients of $H_{N-1}(z)$ in terms of the coefficients of $H_N(z)$. Explain how such a procedure may be used iteratively to realise a given transfer function, assuming no terms become infinite. For the given selection of parameters as in 3.3 above, produce a minimal component realisable signal flow graph in terms of appropriate adders, multipliers and delays, indicating the first step of the iteration [7]

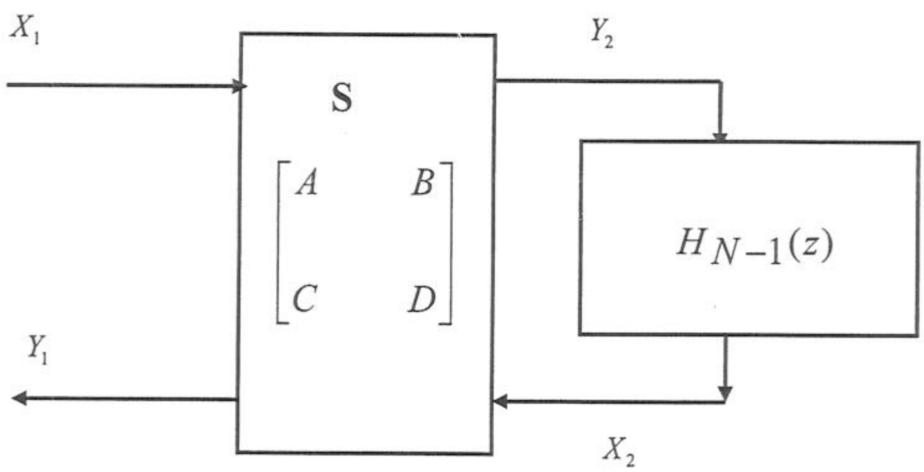


Figure 1

4.

4.1. Consider an ideal linear phase lowpass digital filter transfer function $H(z)$. On the unit circle $z = e^{j\theta}$, the function $H(z)$ takes the values $H(e^{j\theta}) = \begin{cases} e^{-jM\theta} & -\frac{\pi}{M} \le \theta \le \frac{\pi}{M} \\ 0 & elsewhere \end{cases}$

where M is a positive integer. Sketch the amplitude response of $H(e^{j(\theta - \frac{2\pi}{M}r)})$ for $r = 0$, $r = 1$ and $r = 2$. [3]

4.2. Show that the frequency response shown below is allpass and determine its phase response [4]

$$G(e^{j\theta}) = \sum_{r=0}^{M-1} H(e^{j(\theta - \frac{2\pi}{M}r)}).$$

4.3. Let $H(z)$ be expressed as $H(z) = \sum_{r=0}^{M-1} z^{-r} H_r(z^M)$ where $H_r(z)$ are some appropriate subfilter transfer functions. By replacing z by $ze^{-j\frac{2\pi}{M}k}$ in the expression above for $H(z)$ and summing over k, or otherwise, show that the subfilter transfer function $H_0(z^M)$ is given by the expression $H_0(e^{jM\theta}) = \frac{1}{M} \sum_{r=0}^{M-1} H(e^{j(\theta - \frac{2\pi}{M}r)})$. [10]

4.4. What is the amplitude response of $H_0(z^M)$? [3]

5.

5.1. Explain what is meant by terms computational complexity and twiddle factors in the context of evaluating the Discrete Fourier Transform (DFT), and derive the computational complexity of a N-point DFT. [4]

5.2. It is given that $N = N_1 N_2$ with N_1 and N_2 co-prime. It is proposed to carry out on the data array $\{x(n)\}, \quad 0 \le n \le N-1$, the following 1-D to 2-D mapping

$$n = \langle An_1 + Bn_2 \rangle_N \quad \begin{Bmatrix} 0 \le n_1 \le N_1 - 1 \\ 0 \le n_2 \le N_2 - 1 \end{Bmatrix} \quad k = \langle Ck_1 + Dk_2 \rangle_N \quad \begin{Bmatrix} 0 \le k_1 \le N_1 - 1 \\ 0 \le k_2 \le N_2 - 1 \end{Bmatrix} \quad \text{where}$$

$\langle M \rangle_N$ means a reduction of the number M modulo N. Derive the conditions that must prevail on the products AC, BD, AD, and BC in order that all possible twiddle factors in the 2-D DFT computation are eliminated. [10]

5.3. Show that the following set of parameters satisfies these conditions $A = N_2$, $B = N_1$, $C = N_2 \langle N_2^{-1} \rangle_{N_1}$, $D = N_1 \langle N_1^{-1} \rangle_{N_2}$ where $\langle L^{-1} \rangle_P$ denotes the multiplicative inverse of L evaluated modulo P. [3]

5.4. Hence outline the algorithm for the computation of the N-point DFT. [3]

DIGITAL SIGNAL PROCESSING & DIGITAL FILTERS

Question 1 SOLUTIONS MASTER - 16/4/07.

2007

1.1 The root moments S_m are defined as the sum of powers of the roots

$$S_m = \sum_{i=1}^{n} r_i^m$$

$\frac{1}{13}$

If $|r_i| < 1$ $|S_m| \to \infty$ as $m \to \infty$

If $|r_i| > 1$ $|S_m| \to \infty$ as $m \to \infty$ 2

1.2 If $H(z)$ is real then for every complex α_i in the RHS, $\exists$ another factor containing α_i^*. Similarly with β_i.

Hence $$\sum_{i=1}^{n_1} \alpha_i^m = \sum_{i=1}^{n_1/2} \alpha_i^m + (\alpha_i^*)^m \to \text{real}$$

Similarly with β_i 2

1.3 The Fundamental Relationships involve taking logarithms and thus

$$\ln H(z) = \ln K + \sum_{i=1}^{n_1} \ln(1-\alpha_i z^{-1}) + \sum_{i=1}^{n_2} \ln(1-\beta_i z^{-1})$$

The infinite power series involve Taylor expansions but the last term needs to be re-expressed for convergence as $\sum_{i=1}^{n_2} \ln\left[(-\beta_i z^{-1})(1-\frac{1}{\beta_i}z^{-1})\right]$

and hence

$$\ln H(z) = \ln K + \sum_{i=1}^{n_2} \ln(-\beta_i) - n_2 \ln z - \sum_{m=1}^{\infty} \frac{S_m^{N_1}}{m} z^{-m} + \frac{S_{-m}^{N_2}}{m} z^{m}$$

with $z = e^{j\theta}$ and $H(e^{j\theta}) = A(\theta)\exp(j\phi(\theta))$ we have after equating real with real and imaginary with imaginary

$$\ln A(\theta) = \ln K_1 - \sum_{m=1}^{\infty} \frac{S_m^{N_1} + S_{-m}^{N_2}}{m} \cos m\theta$$

2/13

and $\phi(\theta) = -n_2\theta + \sum_{m=1}^{\infty} \frac{S_m^{N_1} - S_{-m}^{N_2}}{m} \cdot \sin m\theta$

where $k_1 = \ln k + \sum_{i=1}^{n_2} \ln(-\beta_i)$ 7

1.4 From the phase expression it is seen that if $S_m^{N_1} = -S_{-m}^{N_2}$ then

$$\phi(\theta) = -n_2\theta$$

ie. the phase is precisely linear. The root moments of N_1 and N_2 must have the above relationship $\forall m$. ie. there are roots located outside $|z|=1$ when there are roots inside $|z|=1$ & there are as many roots in one region as there are in the other, in a reciprocal pairing 4

1.5 If $H(z)$ is allpass then $A(\theta)=1$ since $|A_i(z)|=1$

For any allpass $\ln A_i(z) = \ln(z^{-1}-\alpha_i) - \ln(1-\alpha_i z^{-1})$

or $\ln A_i(z) = \ln\left[(z^{-1})(1-\alpha_i z)\right] - \ln(1-\alpha_i z^{-1})$

$$= -\ln z + \ln(1-\alpha_i z) - \ln(1-\alpha_i z^{-1})$$

$$= -\ln z - \left(\alpha_i z + \frac{\alpha_i^2 z^2}{2} + \frac{\alpha_i^3 z^3}{3} + \cdots\right)$$

$$+ \left(\alpha_i z^{-1} + \frac{\alpha_i^2 z^{-2}}{2} + \frac{\alpha_i^3 z^{-3}}{3} + \cdots\right)$$

$$= -\ln z - \left[\alpha_i(z - z^{-1}) + \frac{\alpha_i^2}{2}(z^2 - z^{-2}) + \frac{\alpha_i^3}{3}(z^3 - z^{-3}) + \cdots\right]$$

On $z = e^{j\theta}$

$\ln A_i(e^{j\theta}) = -j\theta - 2j\left[\alpha_i \cdot \sin\theta + \frac{\alpha_i^2}{2} \cdot \sin 2\theta + \frac{\alpha_i^3}{3} \sin 3\theta + \cdots\right]$

ie as expected completely imaginary

and thus $\phi(\theta) = -\theta - 2\sum_{m=1}^{\infty} \frac{S_m^{A_i}}{m} \cdot \sin m\theta$

where $S_\mu^{A_i} = \alpha_i^{\,\mu}$ and for $H(z)$

$$S_\mu^H = \sum_{i=1}^{m} \alpha_i^{\,\mu}$$, the root moments of the denominator.

5

Question 2.

2.1 The group delay $\tau(\theta)$ is defined as

$$\tau(\theta) = -\frac{d\,\phi(\theta)}{d\theta}$$

where $\phi(\theta)$ is the unwrapped phase response

From $H(e^{j\theta}) = A(\theta)\cdot e^{j\phi(\theta)}$ we have

$$\ln H(e^{j\theta}) = \ln A(\theta) + j\,\phi(\theta)$$

and hence

$$\tau(\theta) = -\operatorname{Im}\frac{d}{d\theta}\ln H(e^{j\theta})$$ 2

2.2

Set $z = e^{j\theta}$ $\alpha_i = \rho_i e^{j\psi_i}$ in $A_i(z)$

so that

$$A_i(e^{j\theta}) = \frac{1 - \rho_i e^{-j\psi_i} e^{+j\theta}}{e^{j\theta} - \rho_i e^{+j\psi_i}}$$

$$= e^{-j\theta} \cdot \frac{1 - \rho_i e^{+j(\theta-\psi_i)}}{1 - \rho_i e^{-j(\theta-\psi_i)}}$$

$$= e^{-j\theta} \cdot \frac{B_i(\theta)\cdot e^{-j\mu_i(\theta)}}{B_i(\theta)\cdot e^{+j\mu_i(\theta)}}$$

where

$$B_i(\theta) = \left|1 - \rho_i e^{+j(\theta-\psi_i)}\right| = \left|1 - \rho_i e^{-j(\theta-\psi_i)}\right|$$

$$\mu_i(\theta) = \tan^{-1}\frac{\rho_i \sin(\theta-\psi_i)}{1-\rho_i\cos(\theta-\psi_i)}$$

Hence

$$\phi_i(\theta) = -\theta - 2\tan^{-1}\frac{\rho_i \sin(\theta-\psi_i)}{1-\rho_i\cos(\theta-\psi_i)}$$ 2

2.3 The group delay associated with $\phi_i(\theta)$ is

$$T_i(\theta) = 1 + 2\cdot\frac{d}{d\theta}\cdot\tan^{-1}\frac{\rho_i s}{1-\rho_i c}$$

where $s = \sin(\theta - \psi_i)$ $\quad c = \cos(\theta - \psi_i)$ 5/13

ie $$\tau_i(\theta) = 1 + 2 \frac{1}{1 + \left(\frac{\rho_i s}{1 - \rho_i c}\right)^2} \cdot \rho_i \frac{[c(1-\rho_i c) - s\rho_i s]}{(1 - \rho_i c)^2}$$

$$= 1 + \frac{2\rho_i(c - \rho_i)}{(1-\rho_i c)^2 + (\rho_i s)^2}$$

Hence
$$\tau(\theta) = \sum_{i=1}^{m} \tau_i(\theta) = m + 2\sum_{i=1}^{m} \frac{\rho_i(c-\rho_i)}{(1-\rho_i c)^2 + (\rho_i s)^2}$$ 4

2.4

Since
$$I = \int_0^{2\pi} \frac{d}{d\theta} \cdot \frac{\rho_i \sin(\theta - \psi_i)}{1 - \rho_i \cos(\theta - \psi_i)} \, d\theta$$

$$= \int_0^{2\pi} d\left[\frac{\rho_i \sin(\theta - \psi_i)}{1 - \rho_i \cos(\theta - \psi_i)}\right]$$

It follows that
$$I = \left.\frac{\rho_i \sin(\theta - \psi_i)}{1 - \rho_i \cos(\theta - \psi_i)}\right|_0^{2\pi} = 0.$$ 2

2.5
$$\tau_{av} = \frac{1}{2\pi} \int_0^{2\pi} \tau(\theta) \cdot d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left(m + 2 \frac{d}{d\theta}\left(\frac{\rho_i s}{1 - \rho_i c}\right)\right) \cdot d\theta$$

and in view of 2.4
$$\tau_{av} = m$$

the average group delay is equal to the number of zeros or degree of the allpass 6/12

5

2.6 From 2.3 we have

$$\tau_i(\theta) = 1 + \frac{2\rho_i(c-\rho_i)}{(1-\rho_i c)^2 + (\rho_i s)^2}$$

$$= \frac{(1-\rho_i c)^2 + (\rho_i s)^2 + 2\rho_i c - 2\rho_i^2}{(1-\rho_i c)^2 + (\rho_i s)^2}$$

$$= \frac{1-\rho_i^2}{(1-\rho_i c)^2 + (\rho_i s)^2}$$

and since $0<\rho_i<1$ it follows that both numerator $[(1-\rho_i^2)]$ and denominator (a sum of squares) are positive

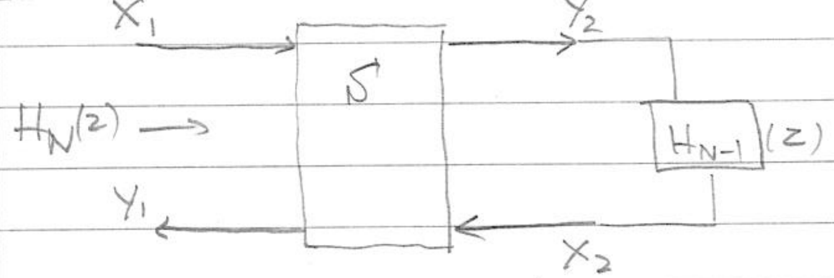
ie $\tau_i(\theta) \geqslant 0$

Hence $\tau(\theta) = \sum_{i=1}^{m} \tau_i(\theta) \geqslant 0$

5

Question 3

3.1



$H_N(z) = Y_1 / X_1 \qquad Y_1 = AX_2 + BY_2$

$\qquad\qquad\qquad\qquad X_1 = CX_2 + DY_2$

Use $\quad X_2 = Y_2 \cdot H_{N-1}$ so that

$$Y_1 = A\,Y_2 H_{N-1} + BY_2$$
$$X_1 = C\,Y_2 H_{N-1} + DY_2$$

and hence

$$\frac{Y_1}{X_1} = \frac{A\,H_{N-1} + B}{C\,H_{N-1} + D}$$

i.e. $$H_N(z) = \frac{A\,H_{N-1}(z) + B}{C\,H_{N-1}(z) + D} \qquad (1)$$ 5

3.2

Examine $\quad H_N(z) - \dfrac{B}{D} = T(z)$ say

$$T(z) = \frac{A\,H_{N-1}(z) + B}{C\,H_{N-1}(z) + D} - \frac{B}{D} = \frac{(AD - BC)\,H_{N-1}(z)}{D\,(C\,H_{N-1}(z) + D)}$$

Thus if $AD - BC = 0$ then $T(z) = 0$ and hence

$$H_N(z) = B / D$$

i.e. independent of $H_{N-1}(z)$. 2

3.3

Write equ(1) (3.1) in terms of $H_{N-1}(z)$ on LHS.

i.e. $\quad H_N \cdot C \cdot H_{N-1} + H_N D = A H_{N-1} + B$

$$\text{or } H_{N-1} = (B - D H_N) / (C H_N - A)$$

8/13

then with the given expression for $H_N(z)$ we have

$$H_{N-1}(z) = \frac{B(1+d_1 z^{-1}+\cdots+d_N z^{-N}) - D(p_0+p_1 z^{-1}+\cdots+p_N z^{-N})}{C(p_0+p_1 z^{-1}+\cdots+p_N z^{-N}) - A(1+d_1 z^{-1}+\cdots+d_N z^{-N})}$$

For $A = p_N z^{-1}$, $B = p_0$, $C = d_N z^{-1}$ $D = 1$ we have

$$H_{N-1}(z) = \frac{0 + (p_0 d_1 - p_1) z^{-1} + \cdots + (p_0 d_N - p_N) z^{-N}}{(d_N - p_N) z^{-1} + (d_N p_1 - p_N d_1) z^{-2} + \cdots + (d_N p_{N-1} - p_N d_{N-1}) z^{-N} + 0}$$

There is a common factor of z^{-1} between the numerator and denominator, which upon cancellation makes $H_{N-1}(z)$ of degree N. 3

3.4.

The selection of $[A, B, C, D]$ must be such that

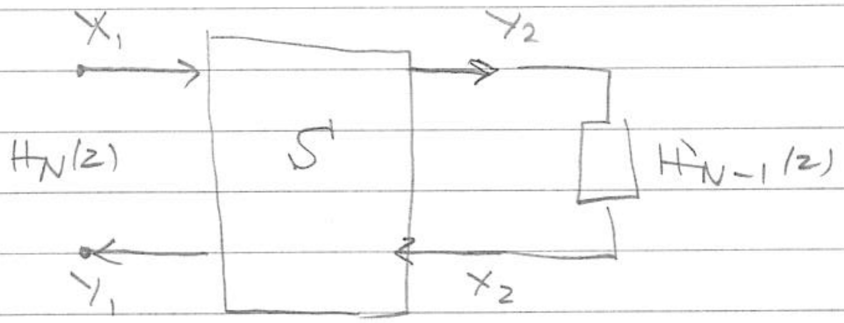
a) $AD - BC \neq 0$ as seen in 3.2. This ensures the dependence of $H_N(z)$ on $H_{N-1}(z)$ and hence the possibility of selecting $H_{N-1}(z)$ appropriately for a given $H_N(z)$.

b) There needs to be a common factor for cancellation as in 3.3. The selection given is not unique, but it is one that makes the common factor very simple ie z^{-1}. Other selections are possible for example by making these parameters second order thereby reducing the degree of $H_{N-1}(z)$ by 2 less than the degree of $H_N(z)$. 3

3.5. The coefficients of $H_{N-1}(z)$ are given already above.

The procedure may be iterated now

with respect to $H_{N-1}(z)$ thereby producing



9/13

$$Y_1 = AX_2 + BY_2 \quad = AX_2 + \frac{B}{D}(X_1 - CX_2)$$

$$Y_2 = \frac{1}{D}(X_1 - CX_2)$$

or $$Y_1 = \left(A - \frac{BC}{D}\right) \cdot X_2 + \frac{B}{D} X_1$$

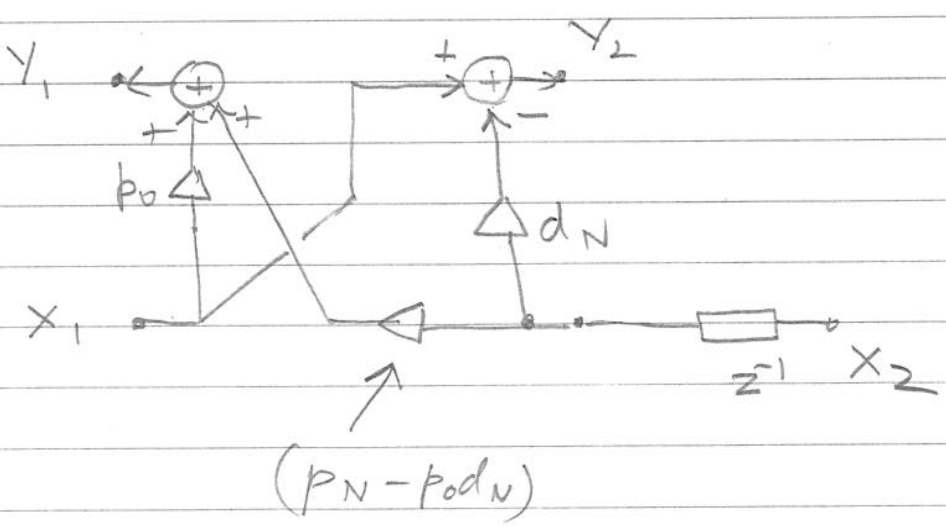
$$Y_2 = \frac{1}{D}(X_1 - CX_2)$$

For the given set

$$Y_1 = \left(p_N z^{-1} - p_0 d_N z^{-1}\right) X_2 + p_0 X_1$$

$$= p_0 X_1 + (p_N - p_0 d_N) \cdot z^{-1} \cdot X_2$$

$$Y_2 = X_1 - d_N z^{-1} \cdot X_2$$



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Question 4

10/13

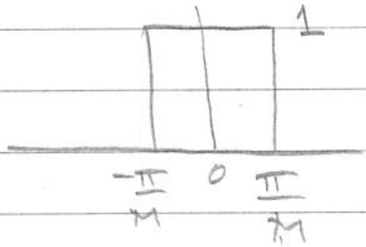
4.1 Since $|H(e^{j\theta})| = 1$ in the range $-\frac{\pi}{M} \le \theta \le \frac{\pi}{M}$ the function

$$|H(e^{j(\theta - \frac{2\pi}{M}\cdot r)})|$$

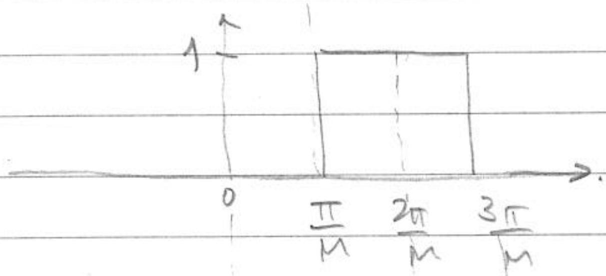
will be unity in the shifted range

$$\left(-\frac{\pi}{M} + \frac{2\pi}{M}r\ ,\ \frac{\pi}{M} + \frac{2\pi}{M}r\right)$$

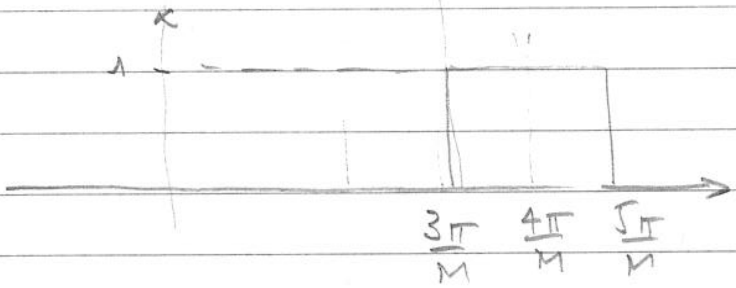
For $r = 0$



For $r = 1$



$r = 2$



3.

4.2 From the given form for $H(z)$ we have

$$G(e^{j\theta}) = \sum_{r=0}^{M-1} e^{-jM(\theta - \frac{2\pi}{M}\cdot r)} = e^{-jM\theta} \cdot \sum_{r=0}^{M-1} e^{-jM\cdot\frac{2\pi}{M}\cdot r}$$

$$= M \cdot e^{-jM\theta}.$$

i.e. $|G(e^{j\theta})| = M$ constant for all frequencies.

Its phase response $\phi(\theta) = M\theta$.

4

4.3

Set $$H(z) = \sum_{r=0}^{M-1} z^{-r} \cdot H_r(z^M)$$ 11/13

Now replace z by $z e^{-j\frac{2\pi}{M}\cdot k}$

$$H(z e^{-j\frac{2\pi}{M}\cdot k}) = \sum_{r=0}^{M-1} z^{-r} \cdot e^{j\frac{2\pi}{M}\cdot kr} \cdot H_r(z^M)$$

Note that $H_r(z^M)$ remain the same.

Now sum over $k = 0, 1, \cdots, M-1$

$$\sum_{k=0}^{M-1} H(z e^{-j\frac{2\pi}{M}k}) = \sum_{r=0}^{M-1} z^{-r} \cdot H_r(z^M) \cdot \sum_{k=0}^{M-1} e^{j\frac{2\pi}{M}\cdot kr}$$

But $$\sum_{k=0}^{M-1} e^{j\frac{2\pi}{M}\cdot kr} = \frac{1 - e^{j\frac{2\pi}{M}\cdot r \cdot M}}{1 - e^{j\frac{2\pi}{M}\cdot r}} = 0$$

for any $r \neq 0$.

For $r = 0$ $$\sum_{k=0}^{M-1} e^{j\frac{2\pi}{M}\cdot kr} = \underbrace{1 + 1 + \cdots + 1}_{M \text{ times}} = M$$

i.e. $$\sum_{k=0}^{M-1} H(z e^{-j\frac{2\pi}{M}\cdot k}) = M \cdot H_0(z^M)$$

or $$H_0(z^M) = \frac{1}{M} \sum_{k=0}^{M-1} H(z e^{-j\frac{2\pi}{M}\cdot k})$$ 10

4.4. Refer to 4.2. It is seen that the expressions are the same and hence $H_0(z^M)$ is allpass

3

Question 5

12/13

5.1 The DFT requires complex multiplications and additions for its evaluation. Thus for

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}\cdot k\cdot n} \qquad k=0,1,\dots,N-1$$

we require N complex multiplications for each value of k. The twiddle factors are the necessary phasors for the implementation of the computational scheme.

For a length N we therefore need N times as many multiplications thereby producing a computational complexity of $O(N^2)$!

4

5.2 It is observed from 5.1 that n and k & nk need only be taken modulo N.

For $n = An_1 + Bn_2$ and $k = Ck_1 + Dk_2$ we have

$$nk = (An_1 + Bn_2)(Ck_1 + Dk_2)$$

$$= \underbrace{ACn_1k_1}_{\text{possible DFT in } n_1} + \underbrace{ADn_1k_2 + BCn_2k_1}_{\text{twiddle factors}} + \underbrace{BDn_2k_2}_{\text{possible DFT in } n_2}$$

$$\left\langle \frac{ACn_1k_1}{N} \right\rangle_N \equiv \frac{n_1k_1}{N_1} \qquad \text{i.e. } \left\langle \frac{AC}{N} \right\rangle_N = N_1^{-1}$$

$$\text{or } AC = N_2$$

$$\langle AD \rangle_N \equiv 0$$

$$\langle BC \rangle \equiv 0$$

and $\left\langle \frac{BD}{N}\, n_2 k_2 \right\rangle_N \equiv \frac{n_2 k_2}{N_2}$ or $\left\langle \frac{BD}{N} \right\rangle_N = N_2^{-1}$

or $BD = N_1$

5.3 From $AC = N_2 \cdot N_2 \langle N_2^{-1} \rangle_{N_1} = N_2 \mod N$ 10

$AD = N_1 N_2 \langle N_1^{-1} \rangle_{N_2} \equiv 0$ —"—

$BC = N_1 N_2 \langle N_2^{-1} \rangle_{N_1} \equiv 0$ —"—

$BD = N_1 \; N_1 \langle N_1^{-1} \rangle_{N_2} \equiv N_1$ —"—

Hence the given values satisfy the required conditions 3

5.4
The algorithm proceeds as follows.
a) The data is sectioned into lengths of N_2 and placed as consecutive rows (columns) in a 2-D array.
b) The 1-D DFT of each row (column) is carried out and placed in the same location.
c) The 1-D DFT of each column (row) is carried out and placed in the same location.
d) The 1-D ~~DFT~~ is read out from the 2-D array according to $k = Ck_1 + Dk_2$ where now the rows and columns are labelled as k_1 and k_2.

3