Imperial College London

BSc/MSci EXAMINATION June 2012

This paper is also taken for the relevant Examination for the Associateship

STRUCTURE OF MATTER, VIBRATIONS AND WAVES, AND QUANTUM PHYSICS

For 1st-Year Physics Students

Monday, 11th June 2012: 14:00 to 16:00

Answer ALL questions from Section A, ONE question from Section B, ONE question from Section C, and ONE question from Section D.
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the 6 answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in 6 answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

SECTION A

1. (i) The atomic mass of sodium is 23. A hamburger is reported to contain 5mg of sodium per 100g. Assuming that a mouthful is about 10g, estimate the total number of sodium atoms consumed with every bite. [2 marks]

(ii) Determine the gauge pressure P, at a depth h, below the surface of an incompressible liquid of density ρ. [3 marks]

(iii) The mean free path of a molecule in a gas is given by

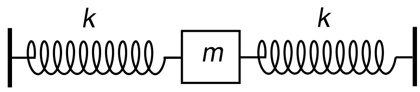
$$\lambda = \frac{1}{4\sqrt{2}\pi n a^2}$$

where n is the number density of the molecules (assumed to be spheres of radius a). Calculate the mean free path of helium atoms at a pressure of 10 Pa and a temperature of 200 K, assuming that helium can be treated as an ideal gas at this pressure, and that the atoms are spheres of radius 0.5×10^{-10} m.

[3 marks]

[Total 8 marks]

2. (i) Calculate the frequency of the oscillations of the mass in the system shown for $m = 0.1$ kg and $k = 20$ N/m:



[2 marks]

(ii) Two loudspeakers separated by a distance d, are emitting sound waves of wavelength λ (where $\lambda \ll d$) in antiphase. Sketch the sound intensity as a function of angle far away from the speakers. [3 marks]

(iii) A longitudinal wave is given by

$$\psi(z, t) = (0.4\,\text{mm})\sin[(2.0\,\text{m}^{-1})z + (5.0\,\text{s}^{-1})t]$$

Identify the frequency f, wavelength λ, amplitude A, phase velocity v and propagation direction of the wave and the direction of motion of the medium.

[3 marks]

[Total 8 marks]

3. (i) X-rays may be produced by shining a beam of electrons of energy $E = 60\,\text{keV}$ at a solid. Explain why the frequencies of the emitted X-ray photons show a material-independent cutoff. Work out the corresponding cut-off wavelength.

[3 marks]

(ii) The Compton scattering formula is

$$\lambda' - \lambda = \frac{h}{mc}(1 - \cos\theta).$$

Draw a diagram of a Compton scattering event and use it to explain the meanings of the symbols λ, λ', m and θ. [3 marks]

(iii) Find the value of the real positive constant A that normalises the wave function

$$\phi(x) = \begin{cases} A(1+i)e^{-x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

[3 marks]

[Total 9 marks]

SECTION B

4. (i) With the aid of a diagram, show that the work done, W, on a gas in a closed system, that undergoes a quasi-static isothermal volume change from V_0 to V_1 can be expressed as

$$W = -\int_{V_0}^{V_1} P dV$$

where P is the pressure. [4 marks]

(ii) Determine W for the case of an ideal gas and explain why $W > 0$ for a gas compression. [3 marks]

(iii) The van der Waals equation of state is given by

$$(P + a(N^2/V^2))(V - bN) = Nk_B T.$$

where T is the absolute temperature, N the number of gas molecules, and k_B the Boltzmann constant. Without proof, briefly explain the origins of the constants a and b. [2 marks]

(iv) Use the equation above to show that the work done on a van der Waals gas, W_{vdW}, during a volume change of the type described in part (i) is

$$W_{vdW} = -Nk_B T \ln\left(\frac{V_1 - bN}{V_0 - bN}\right) - aN^2\left(\frac{1}{V_1} - \frac{1}{V_0}\right).$$

and show that your answer agrees with the ideal-gas case when you set $a = b = 0$. [5 marks]

(v) For ethane gas (C_2H_6), $a = 2.77 \times 10^{-48}$ Jm^3 and $b = 1.06 \times 10^{-28}$ m^3. By using van der Waals equation of state, calculate the work done when 1 mole of ethane expands from 2×10^{-3} m^3 to 6×10^{-3} m^3 at a constant temperature of 300 K. Provide quantitative arguments to explain whether or not it is justified to treat ethane as an ideal gas under such conditions.

[6 marks]

(vi) The internal energy of a van der Waals gas with n_d degrees of freedom is given by

$$U = \frac{n_d}{2} Nk_B T - a\frac{N^2}{V} \qquad \text{[DO NOT PROVE]} .$$

By considering the change in internal energy, ΔU, during an isothermal volume change, derive an expression for Q, the heat change during this process and calculate Q for the change considered in part (v). Does heat flow *into*, or *out of* the gas? [5 marks]

[Total 25 marks]

5. (i) The Maxwell-Boltzmann distribution for the speed, v, of particles in a gas at absolute temperature T is given by;

$$f(v) = \left[4\pi\left(\frac{m}{2\pi k_B T}\right)^{3/2}\right] v^2 e^{-mv^2/2k_B T}$$

where m is the particle mass, and k_B the Boltzmann constant. Explain how this function implies that only a small fraction of particles will possess very high or very low speeds. What is the significance of the term in square brackets?

[3 marks]

(ii) Sketch the shape of $f(v)$ as a function of (v) and indicate on the graph the most probable speed v_{mp}, and determine the value of the distribution function at this speed. [4 marks]

(iii) On the same sketch, indicate what would happen to the distribution if the temperature was increased to $2T$. State quantitatively what happens to v_{mp}, and the value of $f(v_{mp})$ in this case? [4 marks]

(iv) Find the mean speed $\bar{v}$, of the distribution for a temperature T. Hence calculate the mean speed of N_2 in air at 25^o C (the atomic mass of nitrogen is 14.0). You may use the standard integral without proof:

$$\int_0^{+\infty} x^3 e^{-\alpha x^2} dx = \frac{1}{2\alpha^2}$$

[6 marks]

(v) Assume that the internal energy of an ideal gas is all translational kinetic energy. By considering the equipartition theorem, show that the average speed, $< v >$, of the gas particles can be written as

$$< v > = \sqrt{(3RT/M_r)}$$

where M_r is the molar mass, and R the gas constant. [3 marks]

(vi) The escape velocity, v_e, from a planet of mass M is given by $(2GM/r)^{1/2}$ where r is the distance from the centre of mass and G, the gravitational constant. Theory suggests that a planet will retain its atmosphere when the escape velocity required is at least 6 times greater than the average velocity of the particles.

Use the information above and data below, to provide quantitative arguments that explain why at room temperature, N_2 is abundant in the Earth's lower atmosphere whereas H_2 is not.

[Mass of Earth = 6×10^{24} kg, Radius of Earth = 6.4×10^3 km] [5 marks]

[Total 25 marks]

SECTION C

6. (i) Explain what are meant by normal and anomalous dispersion. How are non-dispersive waves defined? [2 marks]

The equation of motion for transverse waves on a rigid bar is

$$\frac{\partial^2\psi}{\partial t^2} = -\alpha\frac{\partial^4\psi}{\partial z^4}$$

where α is a constant that depends on the properties of the bar.

(ii) Show that the bar can support harmonic waves of the form $e^{i(\omega t-kz)}$ provided $\omega^2 = \alpha k^4$ [3 marks]

(iii) Sketch the dispersion relation indicating on your graph the phase and group velocities for a given value of k. What type of dispersion is this? What are the physical interpretations of the phase and group velocities? [5 marks]

(iv) A uniform steel rail is struck from the side by a hammer. Describe how the resulting transverse wave pulse changes as it propagates along the rail. The oscillations are monitored 100 m further along the rail. At a time 0.2 s after the rail was struck, the instantaneous frequency of these oscillations is 500 Hz. Calculate α for the rail.

[6 marks]

A bar of length L is fixed at both ends such that the displacement at the ends is zero, but the ends are free to pivot. The ends of the rail are at $z = 0$ and $z = L$.

(v) Sketch the first three harmonic standing wave modes of the bar. Write down an expression for the wavenumber of the n^{th} harmonic mode and hence show that the frequencies of the harmonic standing wave modes are given by

$$f_n = \frac{\pi\alpha^{1/2}n^2}{2L^2}$$

for $n = 1, 2, 3, ...$ [6 marks]

(vi) The bar is 0.2000 m long and has a fundamental frequency of 800.0 Hz. A second slightly longer, but otherwise identical bar is placed next to the first. When the two bars are struck simultaneously the volume of the resulting sound is modulated at a frequency of 70.00 Hz. What is this phenomenon? Calculate the length of the second bar (give your answer to 4 s.f.). [3 marks]

[Total 25 marks]

7. An LCR circuit is driven by an oscillating voltage source $V_0 \cos \omega t$.

The equation of motion for the charge q on the capacitor is

$$\frac{d^2q}{dt^2} + \gamma \frac{dq}{dt} + \omega_0^2 q = \frac{V_0}{L} \cos \omega t$$

where $\gamma = R/L$ and $\omega_0^2 = 1/(LC)$.

(i) Explain how this system is equivalent to a mechanical mass-spring system stating the mechanical equivalences for the quantities L, C and R. [2 marks]

(ii) Show that the steady state solution is $q = A \exp[i(\omega t + \phi)]$ where

$$A \exp(i\phi) = \frac{V_0/L}{\omega_0^2 - \omega^2 + i\gamma\omega}$$

[2 marks]

(iii) The voltage across the capacitor ($V = q/C$) is monitored with an oscilloscope. The amplitude of the source is set to 1V while its frequency is adjusted until the oscillations lag behind the drive voltage by exactly $\pi/2$. This frequency is found to be 3.5 kHz at which point the amplitude of the voltage oscillations across the capacitor is 17V. Use these values to calculate ω_0 and γ for the system. What is the Q-value of the system? [5 marks]

(iv) Write down an expression for the current $I = \dot{q}$ and hence show that the time-averaged power dissipated by the resistor is

$$P_{\mathrm{av}} = \frac{1}{2} \frac{\gamma\omega^2 V_0^2/L}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}$$

[5 marks]

(v) Sketch the time-average power absorption of the oscillator as a function of ω (for $Q \gg 1$), marking on your graph the behaviour for $\omega \ll \omega_0$, for $\omega \gg \omega_0$ and in the vicinity of the resonance. Calculate the maximum time-average power absorption for $V_0 = 1$V if the inductance is 100mH. What happens as the resistance is reduced to zero? [7 marks]

(vi) Calculate the time for the amplitude of the oscillations to settle to within 1% of the steady state amplitude after the voltage source is first switched on. [You may use the fact that the motion of a damped free oscillator is of the form $x = A_0 \exp(-\gamma t/2) \cos(\omega_D t + \phi)$, where ω_D is the damped frequency.] [4 marks]

[Total 25 marks]

SECTION D

8. (i) The position-momentum version of Heisenberg's uncertainy principle states that $\Delta x \Delta p_x \geq \hbar/2$. Assuming that you have been given many identical quantum systems, each containing a single particle in the same initial quantum state, and that you are able to measure the position or momentum of the particle in every system, describe how you would check this inequality. [7 marks]

(ii) Write down the dispersion relation for a free non-relativistic quantum mechanical particle of mass m. Derive expressions for the phase and group velocities of the particle. What do the phase and group velocities represent? [6 marks]

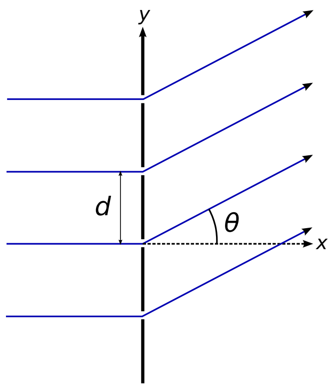
(iii) A free non-relativistic quantum mechanical particle is represented by a wave packet of root-mean-square (rms) length Δx, mean momentum p_x, and rms momentum uncertainty Δp_x.

(a) Express the energy uncertainty ΔE in terms of Δp_x, p_x and m. You may assume that $\Delta p_x \ll \langle p_x \rangle$. [Note: if the uncertainty in a real variable η is $\Delta\eta$, where $\Delta\eta \ll \eta$, the uncertainty in the value of any smooth function $f(\eta)$ is given by $\Delta f \approx |df/d\eta|\,\Delta\eta$.] [2 marks]

(b) Use the appropriate velocity from part (ii) to relate Δx to the time Δt required for the wave packet to pass a fixed point. [2 marks]

(c) Hence show how to transform the position-momentum form of the uncertainty principle into the energy-time form. [2 marks]

A particle wave of wavelength λ moving in the x direction strikes a diffraction grating at normal incidence.



The first non-central maximum in the interference pattern occurs when

$$\sin\theta = \frac{\lambda}{d} = \frac{h}{p_x d}, \qquad \text{[DO NOT PROVE]}$$

where d is the slit spacing and p_x is the momentum of the incident particle.

(iv) Explain why the diffraction pattern appears less sharp if the length Δx of the incident wave packet is small. Assuming minimum uncertainty, derive an expression for the smearing, $\Delta\theta$, in terms of Δx, p_x, d, and θ. (You may assume that $\Delta\theta$ is small, but not that θ is small.) [6 marks]

[Total 25 marks]

9. (i) Write down the time-independent Schrödinger equation satisfied by the energy eigenfunctions $\phi(x)$ of a particle of mass m moving in a simple harmonic potential well of the form

$$V(x) = \frac{1}{2}sx^2,$$

where s is the spring constant. [4 marks]

(ii) The ground-state energy eigenfunction $\phi_0(x)$ is of the form $e^{-\alpha x^2}$. By substituting this trial solution into the time-independent Schrödinger equation and equating coefficients of like powers of x, obtain an expression for the constant α and show that the ground-state energy eigenvalue is

$$E_0 = \frac{1}{2}\hbar\omega_{\text{cl}},$$

where ω_{cl} is the classical angular frequency of the oscillator. [8 marks]

(iii) If an atom of carbon (atomic weight 12) is displaced from its equilibrium position in a diamond, it feels a harmonic restoring force of spring constant $s = 585\,\text{Jm}^{-2}$. Work out the total zero-point vibrational energy of a diamond of mass 1 g. Note that every atom is able to vibrate in *all three* Cartesian directions. [6 marks]

(iv) The vibrational frequency ω_{cl} increases as a diamond is squashed. This effect is usually described in terms of the Gruneisen parameter,

$$\gamma = -\frac{V}{\omega_{\text{cl}}}\frac{d\omega_{\text{cl}}}{dV},$$

which is equal to 0.97 for diamond and is approximately independent of the volume V. Given that the density of diamond is $3530\,\text{kg m}^{-3}$, evaluate the zero-point contribution to the pressure $P = -dE/dV$ and compare your result to atmospheric pressure ($\approx 10^5\,\text{Nm}^{-2}$). [7 marks]

[Total 25 marks]