

1 Solution

a) Solve the noiseless equation:

$$\ddot{x}_1 = 0$$

$$\text{or } \dot{x}_2 = 0$$

$$\text{So } x_2(t) = x_2(s) \quad (\text{for } t > s)$$

$$x_1(t) = x_1(s) + \int_s^t x_2(r) dr = x_1(s) + x_2(s)(t-s)$$

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$$\text{So } \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 1 & t-s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1(s) \\ x_2(s) \end{pmatrix}$$

which gives the fundamental matrix. The noisy case

is then an application of the variation-of-constants formula.

b) The best predictor conditional mean of  $x(t)$  is  
The mean

$$E x(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} x(0)$$

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The covariance is just that of the integral noise:

$$\begin{aligned} \text{Cov}(x(t)) &= \int_0^t \int_0^t \begin{pmatrix} 1 & t-r \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} (0 \ 1) \begin{pmatrix} 1 & 0 \\ t-r' & 1 \end{pmatrix} \delta(r-r') dr dr' \\ &= \int_0^t \begin{pmatrix} t-r \\ 1 \end{pmatrix} (t-r, 1) dr \\ &= \int_0^t \begin{pmatrix} (t-r)^2 & t-r \\ t-r & 1 \end{pmatrix} dr = \begin{bmatrix} \frac{t^3}{3} & \frac{t^2}{2} \\ \frac{t^2}{2} & t \end{bmatrix} \end{aligned}$$

$$c) E[x(t)|y_1] = E[x(t)] + \text{Cov}(x(t), y_1) (\text{Cov } y_1)^{-1} (y_1 - x_1(0))$$

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$$\text{But } \text{Cov}(x(t), y_1) = \begin{pmatrix} \text{Cov}(x_1) \\ \text{Cov}(x_1, x_2) \end{pmatrix} = \begin{pmatrix} \frac{t^3}{3} \\ \frac{t^2}{2} \end{pmatrix}$$

$$\text{Cov}(y_1) = \frac{t^3}{3} + Q$$

$$\text{So } E[x(t)|y_1] \text{ has the form indicated with } K = \begin{pmatrix} \frac{1}{1 + \frac{3Q}{t^3}} \\ \frac{t}{\frac{2t}{3} + \frac{2Q}{t^2}} \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ as } t \rightarrow \infty.$$

As  $t \rightarrow \infty$ , the estimate 'forgets' the initial condition and relies solely on the single observation.

2 Solution Since all random variables are normal, we have

$$-\frac{1}{2} (x_0 - \hat{x}_{0|k})^T P_{0|k}^{-1} (x_0 - \hat{x}_{0|k}) + \log p(y_k | y_{k-1}, \dots)$$

$$= -\frac{1}{2} (y_k - CA^k x_0)^T Q^{-1} (y_k - CA^k x_0)$$

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$$-\frac{1}{2} (x_0 - \hat{x}_{0|k+1})^T P_{0|k+1}^{-1} (x_0 - \hat{x}_{0|k+1})$$

Equating linear and quadratic terms in  $x_0$  gives

the two expressions for the information filter

6) If  $x_{k+1} = a x_k$ ,  $Q = 1$ ,  $C = 1$

Then  $P_{0|k}^{-1} = P_{0|k-1}^{-1} + a^{2k}$

Further  $P_{0|0}^{-1} \approx 0$ , so

$$P_{0|k}^{-1} = a^2 + a^4 + \dots + a^{2k} = a^2 \left( \frac{1 - a^{2k}}{1 - a^2} \right)$$

If  $a^2 < 1$  this converges to  $\frac{a^2}{1 - a^2}$ .

If  $a = .99$ ,  $a^2 \approx 0.98$ ,  $P_{0|k}^{-1} = \frac{.98}{.02} = 49$ .

So the smallest standard deviation of  $x_0 = \frac{1}{7}$ .

7) If  $a = 1.01$ ,  $a^2 = 1.02$

Then  $P_{0|k}^{-1} = \frac{a^2 (a^{2k} - 1)}{a^2 - 1} = 51 (e^{2k \log a} - 1)$

$$= 51 (e^{(.04)k} - 1)$$

So the conditional standard deviation at  $k$

$$= \left( \frac{1}{51 (e^{.04k} - 1)} \right)^{1/2} \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Solution

= a)

$$E[\hat{A}_N | A, B] = \frac{1}{2N+1} \sum_{k=-N}^N E[A + Bk + W_k | A, B]$$

By the independence of the  $W_k$  and  $A+B$ ,  $E[W_k | A, B] = 0$ .

$$\text{So } E[\hat{A}_N | A, B] = \frac{1}{2N+1} \sum_{k=-N}^N (A + Bk) = A.$$

Similarly

$$E[\hat{B}_N | A, B] = \frac{\sum_{k=1}^N (A + Bk - A + Bk)}{2 \sum_{k=1}^N k^2} = B.$$

b) Being conditionally unbiased the conditional mean of  $\hat{A}_N$

is just  $A$ ; so its variance is

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$$\begin{aligned} E[(\hat{A}_N - A)^2 | A, B] &= \frac{1}{(2N+1)^2} E\left[\left(\sum_{k=-N}^N W_k\right)^2 | A, B\right] \\ &= \frac{1}{(2N+1)^2} (2N+1)Q = \frac{Q}{2N+1} \end{aligned}$$

That of  $\hat{B}_N$  is

$$\begin{aligned} E[(\hat{B}_N - B)^2 | A, B] &= \frac{1}{4\left(\sum_{k=1}^N k^2\right)^2} E\left[\left(\sum_{k=1}^N (kW_k - kW_{-k})\right)^2 | A, B\right] \\ &= \frac{2Q\left(\sum_{k=1}^N k^2\right)}{4\left(\sum_{k=1}^N k^2\right)^2} = \frac{Q}{2\left(\sum_{k=1}^N k^2\right)} \end{aligned}$$

Finally  $\hat{A}_N$  and  $\hat{B}_N$  are uncorrelated since

$$\begin{aligned} E[(\hat{A}_N - A)(\hat{B}_N - B) | A, B] &= \frac{E\left[\left(\sum_{k=-N}^N W_k\right)\left(\sum_{k=1}^N (kW_k - kW_{-k})\right)\right]}{(2N+1) 2 \sum_{k=1}^N k^2} \\ &= \frac{Q \sum_{k=-N}^N k}{2(2N+1) \sum_{k=1}^N k^2} = 0. \end{aligned}$$

Solution  
3 c)

$$E[(\bar{\beta}_N - \beta)^2 | A, B] = E\left[\frac{(w_N - w_{-N})^2}{4N^2}\right]$$
$$= \frac{Q}{2N^2}$$

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So the ratio

$$R_N = \frac{E[(\hat{\beta}_N - \beta)^2 | A, B]}{E[(\bar{\beta}_N - \beta)^2 | A, B]}$$

$$= \frac{2N^2}{Q} \cdot \frac{Q}{2 \sum_1^N k^2} = \frac{N^2}{\sum_1^N k^2}$$

$$\text{But } \sum_1^N k^2 > \frac{N^3}{3} \quad \text{So}$$

$$R_N < \frac{3N^2}{N^3} = \frac{3}{N}$$

which converges to zero as  $N \rightarrow \infty$ .

4 Solution

a) By the 'nesting' property of conditional expectation,

$$\begin{aligned}
 10 \quad E[(x'_k - z_k)^2] &= E[E[(x'_k - \hat{z}_{k|k} + \hat{z}_{k|k} - z_k)^2 | y_{k1}, \dots]] \\
 &= E[(x'_k - \hat{z}_{k|k})^2] + E[E[(\hat{z}_{k|k} - z_k)^2 | y_{k1}, \dots]] \\
 &= E[(x'_k - \hat{z}_{k|k})^2] + E[(\hat{z}_{k|k} - z_k)^2]
 \end{aligned}$$

The second term is unaffected by the controls; so the optimal control law is the same as that for the given cost.

b) The Kalman filter for  $z_{k|k}$  is , where  $a = .99$

$$10 \quad z_{k+1|k+1} = a z_{k|k} + \frac{P_{k+1|k}}{P_{k+1|k} + 1} (y_{k+1} - a \hat{z}_{k|k})$$

If this is time-invariant,  $P_{k+1|k} = P$ , where

$$P = a^2 P + 1 - \frac{a^2 P}{P+1} \quad \text{or} \quad ((a^2 - 1)P + 1)(1 + P) - a^2 P = 0$$

$$\text{or} \quad P^2 - a^2 P - 1 = 0. \quad \text{As } P > 0, \quad P = \frac{a^2}{2} + \frac{1}{2} \sqrt{a^4 + 4}$$

$$\text{If } a = .99, \quad P = \frac{.98}{2} + \frac{1}{2} \sqrt{.96 + 4} = .49 + \sqrt{1.24} \approx 1.6$$

$$\text{The Kalman gain } \frac{P}{P+1} = \frac{1.6}{2.6} = \frac{8}{13} = 0.6$$

The noise term  $y_{k+1} - a \hat{z}_{k|k} = w_{k+1} + z_{k+1} - \hat{z}_{k+1|k}$ ,  
with variance  $1 + P$

$$\begin{aligned}
 \text{The term } \frac{P}{P+1} (y_{k+1} - a \hat{z}_{k|k}) \quad \text{Hence/fore has variance } \frac{P^2}{1+P} \\
 = \frac{(1.6)^2}{2.6} = \frac{8}{13} \times 1.6 = \frac{12.8}{13} \approx 1
 \end{aligned}$$

5 Solution

a) The EKF is based on the approximate model

$$x_{k+1} = f(\hat{x}_{k|k}) + f'(\hat{x}_{k|k})(x_k - \hat{x}_{k|k}) + v_k$$

4 Then  $\hat{x}_{k+1/k} = E[x_{k+1} | y_k, y_{k-1}, \dots] = f(\hat{x}_{k|k})$

b) for the statistical linearization filter

$$\hat{x}_{k+1/k} = E_{k|k}[f(x_k)]$$

$$P_{k+1/k} = E_{k|k}[(x_{k+1} - \hat{x}_{k+1/k})^2]$$

6 
$$= E_{k|k}[R_k P_{k|k}^{-1} P_{k|k} P^{-1} R_k] + Q_s$$

$$= R_k P_{k|k}^{-1} P_{k|k} + Q_s.$$

c) if  $f(x) = x - x^3$

$$\hat{x}_{k+1/k} = E_{k|k}[x_k - x_k^3] = \hat{x}_{k|k} - E_{k|k}[(\hat{x}_{k|k} + (x_k - \hat{x}_{k|k}))^3]$$

$$= \hat{x}_{k|k} - \hat{x}_{k|k}^3 - 3 P_{k|k} \hat{x}_{k|k}.$$

5 
$$= (1 - 3 P_{k|k}) \hat{x}_{k|k} - \hat{x}_{k|k}^3$$

(Note that under the normality assumption,  $E_{k|k}\{(x_k - \hat{x}_{k|k})^3\} = 0$ )

d) The prediction equation for the EKF is

$$\hat{x}_{k+1/k} = \hat{x}_{k|k} - \hat{x}_{k|k}^3.$$

5 if the conditional variance is small (e.g. if  $Q_0$  is small), both filters would behave similarly. However

if  $P_{k|k}$  is large then the statistical linearization filter is likely to be more accurate. The EKF is

likely to produce estimates  $\hat{x}_{k|k}$  that are biased away from the origin.

6 (Solution)

(a) Sufficient conditions for a unique  $S \geq 0$  solving the control ARE are that:  $R$  is positive definite, the pair  $(A, B)$  is stabilizable and  $(Q^{1/2}, A)$  is detectable.

3 A more restrictive set of sufficient conditions is that  $R > 0$ ,  $(A, B)$  is completely controllable and  $(Q^{1/2}, A)$  completely observable.

(b) As  $Q_N = S$ , it follows that from the given Riccati difference equation and the ARE that  $S_{N-1} = S$  and that, similarly,  $S_k = S$  for all  $k = 0, \dots, N$ .

But by the quadratic cost identity, (with  $\text{cov}(x_0) = 0$ )

6

$$\min_u J_{0,N}^u = x^T S_0 x + \sum_{k=0}^{N-1} (0)^T (B^T S_k B + R) (0) + N \text{tr}(S M M^T).$$

(take  $u_k = -F_k x_k$ )

$$= x^T S x + N \text{trace}(S M M^T)$$

(c)  $\lim_{N \rightarrow \infty} \frac{J_{0,N}^u(x)}{N} = \lim_{N \rightarrow \infty} \frac{x^T S x}{N} + \text{trace}(S M M^T)$   
 $= \text{trace}(S M M^T)$

5 It can be shown that this is indeed the minimum value of  $\lim_{N \rightarrow \infty} \left( \frac{J_{0,N}^u}{N} \right)$  for all

"stabilizing"  $\{u_k\}$  for which  $E(x_j^T Q x_j + u_j^T R u_j)$  is bounded in  $j$ .

(d) The separation principle is that the design of the control law and that of the Kalman filter can

6 be treated separately; the 'control' parameters  $B, Q, R$  do not affect the filter;  $H, C, N$  do not affect the optimal control law.