M3/M4/M5
P6

Imperial College
London

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2014

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Probability Theory

Date: Wednesday, 28 May 2014. Time: 2.00pm – 4.00pm. Time allowed: 2 hours.

This paper has FOUR questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the main book is full.

Statistical tables will not be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers to each question as per the table below.

Raw mark	up to 12	13	14	15	16	17	18	19	20
Extra credit	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4

- Each question carries equal weight.
- Calculators may not be used.

Q1

Q1.

(1.i) (1.i) Define the distribution function of a random variable explaining carefully all notions involved.

(1.ii) (1.ii) Which of the following is a distribution function and which is not? Justify you answer.

(1.ii.a) (1.ii.a)

$$F(x) = \begin{cases} 0, \; if \; x \in (-\infty, -1); \\ \frac{1}{3}, \; if \; x \in [-1, +\infty); \end{cases}$$

(1.ii.b) (1.ii.b)

$$F(x) = \frac{1}{2} \int_{-\infty}^{x} e^{-|s|} ds;$$

(1.ii.c) (1.ii.c)

$$F(x) = 1 - e^{-|x|};$$

(1.iii) (1.iii) For a real valued random X on a probability space $\{\Omega, \Sigma, \mu\}$ prove or disprove that

$$\lim_{n \to \infty} \mu(\{X \leq x + \frac{1}{n}\}) = \mu(\{X \leq x\})$$

Q2

Q2.

(2.i) (2.i) Define independent random variables on a given probability space.

(2.ii) (2.ii) Which of the following are independent random variables ?

(2.ii.a) (2.ii.a)

$\cos(nx)$, $n \in \mathbb{N}$, in $([0, 2\pi], \Sigma_{Leb}, \frac{1}{2\pi}\lambda)$

(2.ii.b) (2.ii.b)

$\sin(2^n \pi_n)$, $n \in \mathbb{N}$, in $(([0,1), \Sigma_{Leb} \cap [0,1))^{\mathbb{N}}, \widetilde{\lambda})$ where $[0,1)^{\mathbb{N}} \ni \omega \longmapsto \pi_n(\omega) \equiv \omega_n \in [0,1)$ and $\widetilde{\lambda}$ denotes the restriction of the Lebesgue measure to the σ-algebra $\Sigma_{Leb} \cap [0,1)$.

(2.ii.c) (2.ii.c) $x^n \chi_{[n,n+1)}$, $n \in \mathbb{N}$, in $(\mathbb{R}, \Sigma_{Leb}, \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \lambda)$.

(2.iii) (2.iii) Prove or disprove the following statement.

$$\forall \varepsilon > 0, \qquad \lim_{n \to \infty} \lambda \left\{ \left| \frac{1}{n} \sum_{k=1}^{n} \cos(kx) \right| > \varepsilon \right\} = 0$$

Q3

Q3.

(3.i) (3.i) State the Kolmogorov's 0-1 law explaining carefully all notions involved.

(3.ii) (3.ii) Which of the following is true. Justify your answer. In $((\Omega_0 \equiv \{0,1\}, 2^{\Omega_0})^{\mathbb{N}}, \mu \equiv \nu_0^{\otimes\mathbb{N}})$ with $\nu_0(\{0\}) = \frac{1}{2}$ and $\omega \equiv (\omega_i)_{i\in\mathbb{N}} \in \Omega_0^{\mathbb{N}}$,

(3.ii.a) (3.ii.a) $\mu\{\lim_{n\to\infty} \frac{1}{\log(1+n)} \sum_{j=1}^{n} \frac{1}{j}\omega_j = \frac{1}{2}\} = \frac{1}{5}$

(3.ii.b) (3.ii.b) $\mu\{\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n}(1-2\omega_j) = 0\} = 0$

(3.iii) (3.iii) Prove or disprove that in a sequence of tossing a fair coin, series of n consecutive heads of length $L(n) \to \infty$ as $n \to \infty$ occur infinitely often.

Q4

Q4.

(4.i) (4.i) Define the characteristic function of a random variable.

Which of the following is a characteristic function of a random variable. Justify your answer.

(4.ii.a) (4.ii.a) $\frac{1}{2}e^{-\frac{t^2}{2}} + \frac{1}{2}e^{-|t|}$

(4.ii.b) (4.ii.b) $\frac{3}{2}e^{-\frac{t^2}{2}} \cdot e^{-|t|}$

(4.ii.c) (4.ii.c) $\frac{1}{1+t^2} \cdot e^{-|t|}$

(4.ii) (4.ii) State the Levy continuity theorem.

(4.iii) (4.iii) Using characteristic function prove or disprove the convergence in probability for $s_n \equiv \frac{1}{n}\sum_{k=1}^{n} X_k$ where X_k, $k \in \mathbb{N}$, are i.i.d. with finite first moment.

P6
SOLUTIONS

Sln 1.

(1.i) Let (Ω, Σ, μ) be a probability space, i.e. a nonempty set Ω, a family Σ of subset of Ω containing this subset, closed with respect to taking the complement and a countable union of sets, and a probability measure $\mu : \Sigma \to [0,1]$ such that $\mu(\Omega) = 1$ and for any family of disjoint sets $A_n \in \Sigma$ one has the following σ-additivity property

$$\mu(\bigcup_{n\in\mathbb{N}} A_n) = \sum_{n\in\mathbb{N}} \mu(A_n).$$

A function $X : \Omega \to \mathbb{R}$ is called a (real valued) random variable iff the preimage $X^{-1}((-\infty, x])$ belongs to Σ, for any $x \in \mathbb{R}$. A distribution function F_X of the random variable X on the probability space (Ω, Σ, μ) is defined by

$$F_X(z) \equiv \mu(\{X \leq z\})$$

for $z \in \mathbb{R}$.
(Remark: In the literature on can find a definition of the distribution function with $<$ replacing $\leq$.)
(1.ii)
(1.ii.a) It follows from the definition of the distribution function that $\lim_{z\to\infty} F(z) = 1$, so

$$F(x) = \begin{cases} 0, \; if\, x \in (-\infty, -1); \\ \frac{1}{3}, \; if\, x \in [-1, +\infty); \end{cases}$$

is not a distribution function.
(1.ii.b)

$$F(x) = \frac{1}{2}\int_{-\infty}^{x} e^{-|s|} ds;$$

This is a distribution function of a random variable x on real line with a probability measure $\frac{1}{2}e^{-|s|}\lambda(ds)$, where λ denotes the Lebesgue measure.
(1.ii.c) It follows from the definition of the distribution function that $\lim_{z\to-\infty} F(z) = 0$, so

$$F(x) = 1 - e^{-|x|};$$

cannot be a distribution function.

(1.iii) For a real valued random X on a probability space $\{\Omega, \Sigma, \mu\}$ prove or disprove that

$$\lim_{n\to\infty} \mu(\{X \leq x + \frac{1}{n}\}) = \mu(\{X \leq x\})$$

We note that

$$\{X \leq x\} = \cap_{n\in\mathbb{N}}\{X \leq x + \frac{1}{n}\} = \Omega \setminus (\cup_{n\in\mathbb{N}} A_n)$$

where in the last step, we denoted $A_n \equiv \{X \leq x + \frac{1}{n}\}^c \equiv \Omega \setminus \{X \leq x + \frac{1}{n}\}$, and we used De'Morgan laws. Hence

$$\mu(\{X \leq x\}) = \mu\left(\Omega \setminus (\cup_{n\in\mathbb{N}} A_n)\right) = 1 - \mu\left(\cup_{n\in\mathbb{N}} A_n\right)$$

Next we note that, because $\{X \leq x + \frac{1}{n+1}\} \subseteq \{X \leq x + \frac{1}{n}\}$, we have $A_n \subseteq A_{n+1}$, and hence (by monotone convergence theorem),

$$\mu\left(\cup_{n\in\mathbb{N}} A_n\right) = \lim_{n\to\infty} \mu\left(A_n\right)$$

This implies the desired property.

Sln 2.

(2.i) Real valued random variables X and Y defined on a probability space $\{\Omega, \Sigma, \mu\}$ are called independent iff the σ-algebras $X^{-1}(\mathcal{B})$ and $Y^{-1}(\mathcal{B})$ are independent, where $\mathcal{B}$ denotes the Borel σ-algebra in $\mathbb{R}$. Two σ-algebras $\Sigma_1, \Sigma_2 \subset \Sigma$ are independent iff $\forall A \in \Sigma_1\, and\, B \in \Sigma_2$ one has $\mu(A \cap B) = \mu(A)\mu(B)$.

(2.ii)

(2.ii.a)

The random variables $\cos(nx)$ in $([0, 2\pi], \Sigma_{Leb}, \frac{1}{2\pi}\lambda)$ are not independent. For example

$$\int_0^{2\pi} \cos^2(x)\cos(2x)\frac{1}{2\pi}\lambda \neq \int_0^{2\pi} \cos^2(x)\frac{1}{2\pi}\lambda \cdot \int_0^{2\pi} \cos(2x)\frac{1}{2\pi}\lambda$$

(2.ii.b) Since $\pi_n(\omega) \equiv \omega_n$ are independent, (by construction of the product measure), so also $\sin(2^n\pi_n)$ in $(([0,1), \Sigma_{Leb} \cap [0,1))^{\mathbb{N}}, \widetilde{\lambda})$ are independent.

(2.ii.c) The product of any (measurable) functions of two random variables $x^n\chi_{[n,n+1]}$, $n \in \mathbb{N}$, is equal to zero a.e. and so the integral of the product is zero. But the product of integrals can be different from zero. That is these variables are not independent.

(2.iii) By Markov inequality we have

$$\frac{1}{2\pi}\lambda\left\{\left|\frac{1}{n}\sum_{k=1}^{n}\cos(kx)|\right| > \varepsilon\right\} \leq \varepsilon^{-2}\frac{1}{n^2}\int_0^{2\pi}\left|\frac{1}{n}\sum_{k=1}^{n}\cos(kx)\right|^2 \frac{1}{2\pi}d\lambda$$

$$= \varepsilon^{-2}\frac{1}{n^2}\cdot\frac{n}{2} = \varepsilon^{-2}\frac{1}{2n} \to_{n\to\infty} 0$$

where we used the fact that $\int_0^{2\pi}\cos(kx)\cos(mx)\frac{1}{2\pi}d\lambda = \frac{1}{2}\delta_{km}$.

Sln 3.

(3.i) Let X_n, $n \in \mathbb{N}$, be a sequence of independent random variables on a probability space (Ω, Σ, μ). A σ-algebra at infinity is defined as follows

$$\Sigma_\infty \equiv \bigcap_{n \in \mathbb{N}} \Sigma_{>n}$$

where $\Sigma_{>n}$ denotes a smallest σ-algebra generated by the random variables X_k, $k \geq n$.

Theorem: (Kolmogorov's 0-1 law)
Σ_∞ consists of events which have either probability 0 or 1.

(3.ii)

(3.ii.a) $\mu\{\lim_{n\to\infty} \frac{1}{\log(1+n)} \sum_{j=1}^{n} \frac{1}{j}\omega_j = \frac{1}{2}\} = \frac{1}{5}$
Since, the limit $\lim_{n\to\infty} \frac{1}{\log(1+n)} \sum_{j=1}^{n} \frac{1}{j}\omega_j$ is the same as $\lim_{n\to\infty} \frac{1}{\log(1+n)} \sum_{j=N}^{n} \frac{1}{j}\omega_j$, for any fixed $N \in \mathbb{N}$, so the event belongs to Σ_∞. Thus the statement in question cannot be true, as according to Kolmogorov's 0-1 law it can only have a probability either 0 or 1.

(3.ii.b) In the setup of the question the SLLN is true and one has

$$\mu\{\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} (1 - 2\omega_j) = 0\} = 1.$$

Thus the statement is not true.

(3.iii) The answer depends on how quickly the function $L(n)$ grows with $n \in \mathbb{N}$. If the growth is fast enough, so that we have

$$\sum_{n=1}^{\infty} 2^{-L(n)} < \infty$$

then by the 1^{st} Borel-Cantelli lemma, the probability that series of n consecutive heads of length $L(n)$ will occur infinitely often is equal to zero. On the other hand if the above sum diverges, (then $L(n)$ grows in a sublinear way), taking into the account that events

$$A_n \equiv \{\omega \in \Omega_0^{\mathbb{N}} : \omega_i = head,\ i = 2n+1, .., 2n + L(n)\}$$

are independent (for large n), by the 2^{nd} Borel-Cantelli lemma, the probability that series of n consecutive heads of length $L(n)$ will occur infinitely often is equal to one.

Sln 4.

(4.i) Let X be a random variable on a probability space (Ω, Σ, μ). A function

$$\mathbb{R} \ni t \longmapsto \varphi_X(t) \equiv \mu(e^{itX})$$

is called a characteristic function (c.f.) of the random variable X.

A function $\varphi(t)$ is a characteristic function of a random variable iff at 0 it equals 1, it is continuous at zero and is positive definite.

(4.ii.a) $\frac{1}{2}e^{-\frac{t^2}{2}} + \frac{1}{2}e^{-|t|}$
is a characteristic function as a convex linear combination of Gaussian and Cauchy characteristic functions

(4.ii.b) $\frac{3}{2}e^{-\frac{t^2}{2}} \cdot e^{-|t|}$
This function is not equal to 1 at zero, so is not a c.f..

(4.ii.c) $\frac{1}{1+t^2} \cdot e^{-|t|}$
This function is defined as a product of two characteristic functions and therefore satisfies all conditions of c.f..

(4.ii)

Levy continuity theorem:

Let F_n, $n \in \mathbb{N}$, and F be distribution functions with c.f.s φ_n, $n \in \mathbb{N}$, and φ, respectively. If $F_n \to F$ at any continuity point of F, then $\varphi_n \to \varphi$ uniformly on any finite interval.
Conversely, if φ_n, $n \in \mathbb{N}$, are c.f.s corresponding to distribution functions F_n, $n \in \mathbb{N}$, and if $\varphi_n \to \varphi$ pointwise everywhere, with some φ which is continuous at zero, then φ is a c.f. of some distribution function F and $F_n \to F$ at any continuity point of F.

(4.iii) We note that for i.i.d. random variables X_k, $k \in \mathbb{N}$ one has $\varphi_{s_n}(t) = \varphi_{X_1}(\frac{t}{n})^n$. Since by our assumption X_k's are integrable, so the corresponding characteristic function is continuously differentiable. Using this, the fact that a c.f. is equal to one at zero, we can use continuity to show that $\log \varphi_{X_1}(\frac{t}{n})$ is well defined for sufficiently large n on any finite interval of t. Hence expanding log to the first order and again using continuous differentiability of c.f., one concludes that the sequence $\varphi_{s_n}(t)$ converges to a characteristic function $e^{it\mu X_1}$. Hence convergence in distribution follows via Levy continuity theorem. Since for any $\varepsilon > 0$

$$\mu(\{|s_n - \mu X_1| > \varepsilon\}) = \mu(\{s_n - \mu X_1 > \varepsilon\}) + \mu(\{s_n - \mu X_1 < -\varepsilon\}) = 1 - F_n(\varepsilon) + F_n(-\varepsilon)$$

with $F_n(x)$ denoting distribution function of $s_n - \mu X_1$, the convergence in probability follows.