Imperial College
London

M4/5S4

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May-June 2017

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Applied Probability

Date: Friday 26 May 2017
Time: 10:00 - 12:30
Time Allowed: 2.5 Hours
This paper has 5 Questions.
Candidates should use ONE main answer book.
Supplementary books may only be used after the relevant main book(s) are full.
All required additional material will be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers to each question as per the table below.

Raw Mark	Up to 12	13	14	15	16	17	18	19	20
Extra Credit	0	½	1	1 ½	2	2 ½	3	3 ½	4

- Each question carries equal weight.
- Calculators may not be used.

1. (a) Consider a homogeneous Markov chain $(X_n)_{n\in\mathbb{N}_0}$ with state space $E = \{1, 2, 3, 4, 5, 6\}$ and transition matrix given by

$$\mathbf{P} = \begin{pmatrix} \frac{1}{7} & \frac{6}{7} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & \frac{1}{5} & 0 & \frac{4}{5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(i) Draw the transition diagram.

(ii) Specify the communicating classes and determine whether they are transient, null recurrent or positive recurrent. *Please note that you need to justify your answers.*

(iii) Find all stationary distributions.

(b) Consider a homogeneous Markov chain $(X_n)_{n\in\mathbb{N}_0}$ with state space $E = \mathbb{N}$ and transition matrix given by

$$\mathbf{P} = \begin{pmatrix} q & p & 0 & 0 & 0 & \dots \\ q & 0 & p & 0 & 0 & \dots \\ q & 0 & 0 & p & 0 & \dots \\ q & 0 & 0 & 0 & p & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where $0 < p < 1$ and $q = 1 - p$.

(i) Is this Markov chain irreducible?

(ii) Find the stationary distribution(s).

(iii) For each state, determine whether it is transient, null recurrent or positive recurrent. *Please note that you need to justify your answers.*

2. Let $X = (X_n)_{n\in\mathbb{N}_0}$ denote a homogeneous Markov chain with countable state space E and transition matrix $\mathbf{P}$.

(a) Suppose that $i \in E$ is a periodic state with period $d(i) = gcd\{n \in \mathbb{N} : p_{ii}(n) > 0\} > 1$.
Claim: $p_{ii}(d(i)) > 0$.

(i) Is the above claim true or false?

(ii) If it is true, prove it. If it is false, give a counterexample.

(b) Suppose that the transition matrix $\mathbf{P}$ is symmetric, i.e. $p_{ij} = p_{ji}$ for all $i, j \in E$, and that the state space is finite and given by $E = \{1, 2, \ldots, K\}$ for $K \in \mathbb{N}$. Show that the row vector $\boldsymbol{\pi}$ with $\pi_i = \frac{1}{K}$ for $i = 1, \ldots, K$ is a stationary distribution for the Markov chain.

(c) Assume that X is a simple random walk on $E = \mathbb{Z}$.

(i) State the (one-step) transition probabilities.

(ii) Let $n \in \mathbb{N}$. Find the n-step transition probabilities of X.

3. (a) Let X, Y, Z denote i.i.d. random variables with Exp(λ) distribution for $\lambda > 0$. Find

$$\mathbb{P}(X < Y + Z).$$

(b) Suppose that $X = (X_t)_{t\geq 0}$ denotes a compound Poisson process given by

$$X_t = \sum_{i=1}^{N_t} Y_i, \qquad t \geq 0,$$

where $N = (N_t)_{t\geq 0}$ denotes a Poisson process of rate $\lambda > 0$, and $Y_1, Y_2, \ldots$ are i.i.d. with finite first and second moment and are independent of N.
Derive the following quantities:

(i) $\mathbb{E}(X_t)$ for $t \geq 0$.

(ii) $\mathrm{Var}(X_t)$ for $t \geq 0$.

(iii) $\mathrm{Cov}(X_s, X_t)$ for $0 \leq s \leq t$.

4. Consider a linear birth process $N = (N_t)_{t\geq 0}$ with birth rates given by $\lambda_n = n\lambda$, for $n \in \mathbb{N}$. Assume that $N_0 = 1$. Let $p_n(t) = \mathbb{P}(N_t = n)$ for $n \in \mathbb{N}$.

(a) Derive an equation for $p_1'(t)$ in terms of $p_1(t)$.

(b) Derive an equation for $p_n'(t)$ in terms of $p_n(t)$ and $p_{n-1}(t)$ valid for $n \geq 2$.

(c) Show that the system of differential equations derived in (a) and (b) with appropriate initial conditions has solution

$$p_n(t) = (1 - e^{-\lambda t})^{n-1} e^{-\lambda t}, \qquad \text{for } t \geq 0, n \in \mathbb{N}.$$

You might find the hint given below helpful.

(d) Determine whether or not this birth process explodes and justify your answer.

Hint:

Recall that a one-dimensional ordinary differential equation

$$\frac{df(t)}{dt} + \alpha(t) f(t) = g(t), \qquad t \geq 0$$

with continuous functions α, g and initial condition $f(0) = C$ has solution

$$f(t) = \frac{\int_0^t g(u) M(u) du + C}{M(t)},$$

where M is the integrating factor $M(t) = \exp\left\{ \int_0^t \alpha(u) du \right\}$.

5. This question is based on additional reading material on renewal processes.

(a) Define a renewal process.

(b) Give an example of a renewal process which satisfies the Markov property.

(c) Define the renewal function.

(d) Derive the renewal equation for continuous interarrival times. (You need to state clearly any assumptions you make.)

(e) Consider a renewable process with interrival times given by the continuous uniform distribution on the interval $(0, 1)$. Show that the renewal function is given by $U(t) = e^t - 1$, for $0 \leq t \leq 1$. *You might find the hint for Question 4 helpful.*

Imperial College London

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2017

This paper is also taken for the relevant examination for the Associateship.

M3/4/5 S4

Applied Probability (Solutions)

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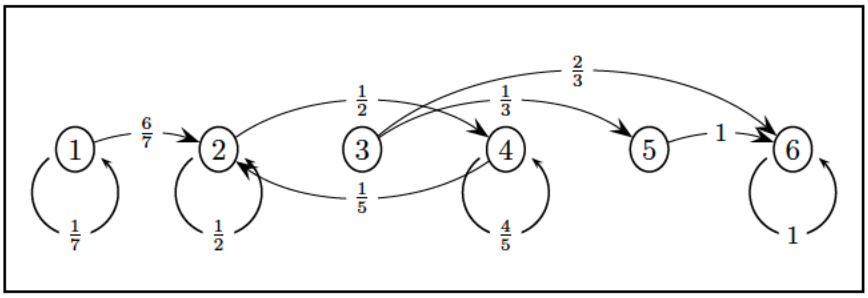
1. (a) (i) The transition diagram is given by $\boxed{\text{meth seen} \Downarrow}$



$\boxed{2}$

(ii) We have a finite state space which can be divided into five communicating classes: The classes $T_1 = \{1\}, T_2 = \{3\}, T_3 = \{5\}$ are not closed and hence transient. $\boxed{1}$

The classes $C_1 = \{2, 4\}, C_2 = \{6\}$ are finite and closed and hence positive recurrent. $\boxed{2}$

(iii) This Markov chain does not have a unique stationary distribution $\boldsymbol{\pi}$ since we have two closed (essential) communicating classes. For the transient states we know from the lectures that $\pi_i = 0$ for $i = 1, 3, 5$. For the positive recurrent $\boxed{1}$ states, we solve $(\pi_2, \pi_4) = (\pi_2, \pi_4) \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{5} & \frac{4}{5} \end{pmatrix}$ and $\pi_6 \cdot 1 = \pi_6$, which leads to $\pi_2 = \frac{2}{5}\pi_4$ and $\pi_6 = \pi_6$. $\boxed{2}$

Altogether, we conclude that the stationary distributions are given by $\boldsymbol{\pi} = (0, \frac{2}{5}\pi_4, 0, \pi_4, 0, \pi_6)$ for all $\pi_4, \pi_6 \geq 0$ with $\frac{7}{5}\pi_4 + \pi_6 = 1$. $\boxed{2}$

$\boxed{\text{meth seen} \Downarrow}$

(b) (i) Yes, this Markov chain is irreducible since all states communicate with each other. $\boxed{1}$

(ii) We need to solve the following system of equations: $\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P}$, where the $\pi_i \geq 0$ for $i \in \mathbb{N}$ and $\sum_{i=1}^{\infty} \pi_i = 1$. We obtain $\boxed{2}$

$$q \underbrace{\sum_{i=1}^{\infty} \pi_i}_{=1} = \pi_1, \qquad \pi_1 p = \pi_2, \qquad \pi_2 p = \pi_3, \qquad \cdots$$

i.e. $\pi_n p = \pi_{n+1}$, for $n \in \mathbb{N}$.

Hence $\pi_1 = q, \pi_2 = pq, \pi_3 = p^2 q, \ldots$. I.e. we found that $\boldsymbol{\pi}$ with $\pi_n = p^{n-1}q$ for $n \in \mathbb{N}$ is the unique stationary distribution of the Markov chain. $\boxed{4}$

(iii) Since the Markov chain is irreducible and has a (unique) stationary distribution, we conclude from a theorem from the lectures that all states in $E = \mathbb{N}$ are positive recurrent. $\boxed{3}$

2. (a) (i) This claim is in general false as the following counterexample will show. $\boxed{\text{meth seen} \Downarrow}$

(ii) Consider a Markov chain with state space $E = \{1, 2, 3, 4\}$ and transition matrix $\boxed{2}$

$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0.5 & 0 & 0.5 & 0 \end{pmatrix}.$$

Then $A_1 := \{n \in \mathbb{N} : p_{11}(n) > 0\} = \{4, 6, 8, 10, \dots\}$ and $d(1) = gcd(A_1) = 2$. However, $p_{11}(2) = 0$. $\boxed{4}$

$\boxed{\text{unseen} \Downarrow}$

(b) Let $\pi_i = \frac{1}{K}$ for $i = 1, \dots, K$ denote the uniform distribution on E. Clearly, the uniform distribution is a *distribution*, so we only need to check that $\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P}$. $\boxed{1}$

For any $i \in \{1, \dots, K\}$, we have

$$(\boldsymbol{\pi}\mathbf{P})_i = \sum_{j=1}^{K} \pi_j p_{ji} \overset{\text{symmetry}}{=} \sum_{j=1}^{K} \pi_j p_{ij} = \sum_{j=1}^{K} \frac{1}{K} p_{ij} = \frac{1}{K} \underbrace{\sum_{j=1}^{K} p_{ij}}_{=1} = \frac{1}{K} = \pi_i,$$

where we used that the sum of the elements in the ith row is equal to one, since $\mathbf{P}$ is a stochastic matrix. $\boxed{4}$

(c) (i) The (one-step) transition probabilities of the simple random walk are given by $\boxed{\text{seen} \Downarrow}$

$$p_{ij} = \begin{cases} p & \text{if } j = i + 1, \\ 1 - p & \text{if } j = i - 1, \\ 0 & \text{otherwise,} \end{cases}$$

for $p \in (0, 1)$. $\boxed{3}$

(ii) We want to find $p_{ij}(n) = \mathbb{P}(X_n = j | X_0 = i)$. In order to get from i to j in n steps, we could go up u times and down d times. Note that we require

$$n = u + d, \qquad i + u - d = j.$$

Solving for u and d we have

$$u = \frac{1}{2}(n - i + j), \qquad d = \frac{1}{2}(n - j + i) \quad \text{for } u, d \geq 0.$$

$\boxed{3}$

There are $\binom{n}{u}$ possibilities of going up u steps, hence, we have

$$p_{ij}(n) = \binom{n}{u} p^u (1 - p)^d = \binom{n}{\frac{1}{2}(n - i + j)} p^{\frac{1}{2}(n-i+j)} (1 - p)^{\frac{1}{2}(n-j+i)},$$

if $n - i + j$ is even and $p_{ij}(n) = 0$ otherwise. $\boxed{3}$

3. (a) We know that a sum of independent exponential variables is Gamma distributed. More precisely, we have $A := Y + Z \sim$ Gamma$(2, \lambda)$, i.e. $\boxed{\text{meth seen} \Downarrow}$

$$f_A(s) = \frac{\lambda^2}{\Gamma(2)} s e^{-\lambda s} = \lambda^2 s e^{-\lambda s}, \quad s > 0.$$

[This result can be derived as follows. Using the continuous version of the law of total probability and the independence of Y and Z, we get for $s > 0$:

$$\begin{aligned}
\mathbb{P}(A \leq s) &= \int_0^\infty \mathbb{P}(Y + Z \leq s | Z = z) f_Z(z) dz = \int_0^s \mathbb{P}(Y \leq s - z) f_Z(z) dz \\
&= \int_0^s (1 - e^{-\lambda(s-z)}) \lambda e^{-\lambda z} dz = \int_0^s \lambda e^{-\lambda z} dz - \int_0^s \lambda e^{-\lambda s} dz \\
&= -e^{-\lambda s} + 1 - \lambda e^{-\lambda s} s, \Rightarrow f_A(s) = \frac{d}{ds} \mathbb{P}(A \leq s) = \lambda^2 s e^{-\lambda s}.
\end{aligned}$$

] $\boxed{2}$

Note that due to the independence of X, Y, Z, we also have that X is independent of A. $\boxed{2}$

Using the results from the lecture notes on conditional distributions, we obtain

$$\begin{aligned}
\mathbb{P}(X < Y + Z) = \mathbb{P}(X < A) &= \int_0^\infty \mathbb{P}(X < A | A = s) f_A(s) ds \\
&= \int_0^\infty \mathbb{P}(X < s | A = s) f_A(s) ds = \int_0^\infty \mathbb{P}(X < s) f_A(s) ds,
\end{aligned}$$

where we used the independence of X and A. Then we plug in the exponential cumulative distribution function for $\mathbb{P}(X < s)$ and the Gamma(2,λ) density for $f_A(s)$ and obtain $\boxed{3}$

$$\begin{aligned}
\mathbb{P}(X < Y + Z) &= \int_0^\infty \mathbb{P}(X < s) f_A(s) ds = \int_0^\infty (1 - \exp(-\lambda s)) \lambda^2 s e^{-\lambda s} ds \\
&= \int_0^\infty \lambda^2 s e^{-\lambda s} ds - \int_0^\infty \lambda^2 s e^{-2\lambda s} ds.
\end{aligned}$$

Using a change of variables in the first integral with $u = \lambda s$ leads to

$$\int_0^\infty \lambda^2 s e^{-\lambda s} ds = \int_0^\infty u^{2-1} e^{-u} du = \Gamma(2) = 1.$$

Similarly, using a change of variables in the second integral with $u = 2\lambda s$ leads to

$$\int_0^\infty \lambda^2 s e^{-2\lambda s} ds = 4^{-1} \int_0^\infty 2\lambda s e^{-2\lambda s} 2\lambda ds = 4^{-1} \int_0^\infty u^{2-1} e^{-u} du = \frac{\Gamma(2)}{4} = 4^{-1}.$$

So we conclude that $\mathbb{P}(X < Y + Z) = 1 - \frac{1}{4} = \frac{3}{4}$. $\boxed{3}$

(b) (i) Note that for $n \in \mathbb{N}_0$ [seen ⇓]

$$\mathbb{E}(X_t|N_t = n) = \mathbb{E}\left(\sum_{i=1}^{n} Y_i|N_t = n\right) = \mathbb{E}\left(\sum_{i=1}^{n} Y_i\right) = n\mathbb{E}(Y_1),$$

where we used the fact that (Y_i) and N are independent and further that the (Y_i) are i.i.d.. Hence, we get $\mathbb{E}(X_t|N_t) = N_t\mathbb{E}(Y_1)$.
Using the properties of the conditional expectation, we get

$$\mathbb{E}(X_t) \quad = \quad \mathbb{E}(\mathbb{E}(X_t|N_t)) = \mathbb{E}(N_t\mathbb{E}(Y_1)) = \mathbb{E}(N_t)\mathbb{E}(Y_1) = \lambda t\mathbb{E}(Y_1).$$

[2]

(ii) For the variance, we use [seen ⇓]

$$\mathrm{Var}(X_t) = \mathrm{Var}\left(\mathbb{E}\left(X_t|\, N_t\right)\right) + \mathbb{E}\left(\mathrm{Var}\left(X_t|\, N_t\right)\right). \tag{1}$$

[1]

Let us compute the conditional variance first:

$$\mathrm{Var}(X_t|N_t = n) = \mathrm{Var}\left(\sum_{i=1}^{n} Y_i|N_t = n\right) = \mathrm{Var}\left(\sum_{i=1}^{n} Y_i\right) = n\mathrm{Var}(Y_1),$$

where we used the fact that (Y_i) and N are independent and further that the (Y_i) are i.i.d.. Hence $\mathrm{Var}(X_t|N_t) = N_t\mathrm{Var}(Y_1)$. Then we have

$$\begin{aligned}\mathbb{E}(\mathrm{Var}(X_t|N_t)) &= \mathbb{E}(N_t\mathrm{Var}(Y_1)) = \mathbb{E}(N_t)\mathrm{Var}(Y_1) = \lambda t\mathrm{Var}(Y_1),\\ \mathrm{Var}(\mathbb{E}(X_t|N_t)) &= \mathrm{Var}(N_t\mathbb{E}(Y_1)) = \mathrm{Var}(N_t)(\mathbb{E}(Y_1))^2 = \lambda t(\mathbb{E}(Y_1))^2.\end{aligned}$$

Using formula (1), we get

$$\mathrm{Var}(X_t) = \lambda t\mathrm{Var}(Y_1) + \lambda t(\mathbb{E}(Y_1))^2 = \lambda t\mathbb{E}(Y_1^2).$$

[3]

(iii) Let $0 \leq s \leq t$. Then $\mathrm{Cov}(X_s, X_t) = \mathbb{E}(X_sX_t) - \mathbb{E}(X_s)\mathbb{E}(X_t)$. [unseen ⇓]
Since N has independent increments, the (Y_i) are i.i.d. and independent of N, X has independent increments, too. Hence

$$\begin{aligned}\mathbb{E}(X_sX_t) &= \mathbb{E}(X_s((X_t - X_s) + X_s)) = \mathbb{E}(X_s)\mathbb{E}(X_t - X_s) + \mathbb{E}(X_s^2)\\ &= \mathbb{E}(X_s)\mathbb{E}(X_t) - (\mathbb{E}(X_s))^2 + \mathbb{E}(X_s^2) = \mathbb{E}(X_s)\mathbb{E}(X_t) + \mathrm{Var}(X_s).\end{aligned}$$

Hence

$$\mathrm{Cov}(X_s, X_t) = \mathrm{Var}(X_s) = \lambda s\mathbb{E}(Y_1^2).$$

[4]

4. (a) Let $\delta > 0$. Then, using the law of total probability, we have $\boxed{\text{meth seen} \Downarrow}$

$$p_1(t+\delta) = \sum_{i=0}^{1} \mathbb{P}(N_{t+\delta} = 1 | N_t = i)\mathbb{P}(N_t = i) = (1 - \lambda\delta)p_1(t) + o(\delta),$$

where we used the single arrival property of a birth process and the initial condition. Then $\boxed{2}$

$$\lim_{\delta \downarrow 0} \frac{p_1(t+\delta) - p_1(t)}{\delta} = p_1'(t) = -\lambda p_1(t).$$

$\boxed{2}$

(b) Let $\delta > 0$ and $n \geq 2$. Then, using the law of total probability, we have $\boxed{\text{meth seen} \Downarrow}$

$$p_n(t+\delta) = \sum_{i=0}^{n} \mathbb{P}(N_{t+\delta} = n | N_t = i)\mathbb{P}(N_t = i) = \lambda(n-1)\delta p_{n-1}(t) + (1 - \lambda n\delta)p_n(t) + o(\delta),$$

where we used the single arrival property of a birth process. Then $\boxed{3}$

$$\lim_{\delta \downarrow 0} \frac{p_n(t+\delta) - p_n(t)}{\delta} = p_n'(t) = \lambda(n-1)p_{n-1}(t) - \lambda n p_n(t).$$

$\boxed{2}$

(c) Here we carry out a proof by induction. We want to prove that $\boxed{\text{seen} \Downarrow}$

$$p_n(t) = (1 - e^{-\lambda t})^{n-1} e^{-\lambda t}, \qquad (2)$$

holds for all $n \geq 1$.

Clearly, by solving the forward equation we get

$$p_1(t) = e^{-\lambda t} = (1 - e^{-\lambda t})^{1-1} e^{-\lambda t}.$$

$\boxed{2}$

Now suppose that (2) holds for an $n \in \mathbb{N}$. We write down the forward equation for $n+1$.

$$p_{n+1}'(t) \quad = \quad -(n+1)\lambda p_{n+1}(t) + n\lambda p_n(t).$$

Using the induction hypothesis, we get

$$p_{n+1}'(t) + (n+1)\lambda p_{n+1}(t) = n\lambda p_n(t) = n\lambda(1 - e^{-\lambda t})^{n-1} e^{-\lambda t}.$$

Using the integrating factor approach (as described in the hint), we get

$$M(x) = \exp\left(\int_0^x \lambda(n+1) ds\right) = \exp(\lambda(n+1)x),$$

where we used the boundary condition. Further,

$$\begin{aligned} p_{n+1}(t) &= \int_0^t n\lambda p_n(u) M(u) du (M(t))^{-1} \\ &= \int_0^t \lambda n(1 - e^{-\lambda u})^{n-1} e^{-\lambda u} e^{\lambda(n+1)u} du e^{-\lambda(n+1)t}. \end{aligned}$$

$\boxed{3}$

Note that an application of the binomial theorem leads to

$$
\begin{aligned}
&\int_0^t \lambda n(1-e^{-\lambda u})^{n-1} e^{-\lambda u} e^{\lambda(n+1)u} du = \int_0^t \lambda n(1-e^{-\lambda u})^{n-1} e^{\lambda n u} du \\
&= \int_0^t \lambda n \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k (e^{-\lambda u})^k e^{\lambda n u} du = \int_0^t \lambda n \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k e^{\lambda(n-k)u} du \\
&= \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k n \int_0^t \lambda e^{\lambda(n-k)u} du = \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k \frac{n}{n-k} \left(e^{\lambda t(n-k)} - 1\right) \\
&= \sum_{k=0}^{n-1} \binom{n}{k} (-1)^k \left(e^{\lambda(n-k)t} - 1\right) = \sum_{k=0}^{n-1} \binom{n}{k} (-1)^k (e^{\lambda t})^{n-k} - \sum_{k=0}^{n-1} \binom{n}{k} (-1)^k \\
&= (e^{\lambda t} - 1)^n - (-1)^n - (-1+1)^n + (-1)^n = (e^{\lambda t} - 1)^n.
\end{aligned}
$$

Hence

$$p_{n+1}(t) = (e^{\lambda t} - 1)^n e^{-\lambda(n+1)t} = (1 - e^{-\lambda t})^n e^{-\lambda t}$$

which concludes the proof. $\boxed{3}$

(d) Using the fact that the harmonic series diverges, we have that $\boxed{\text{meth seen} \Downarrow}$

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \frac{1}{\lambda} \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

From a theorem from the lectures (see below) we can now conclude that the probability that explosion occurs is equal to zero. $\boxed{3}$

Theorem Let N be a birth process started from 1 with rates $\lambda_1, \lambda_2, \ldots$. Let $T_1, T_2, \ldots$ denote the arrival times of the birth process N (i.e. $T_n = \inf\{t \geq 0 : N_t = n\}$). Let the limit of the arrival times be denoted by $T_\infty = \lim_{n\to\infty} T_n$. Then, if $\sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \infty$, then $\mathbb{P}(T_\infty = \infty) = 1$, i.e. the probability that explosion occurs is 0.

5. The following material was unseen in the lectures, but is fully covered in the additional reading material. $\boxed{\text{unseen} \Downarrow}$

(a) Let $X_1, X_2, \ldots$ be independent and identically distributed random variables, called the *interarrival times*. Set

$$S_n = \sum_{i=1}^{n} X_i.$$

Then the renewal process $N = (N_t)_{t\geq 0}$ is defined as

$$N_t = \sum_{n=1}^{\infty} \mathbb{I}_{\{S_n \leq t\}}.$$

Alternatively, one could define it by

$$N_t = \max\{n \in \mathbb{N} : S_n \leq t\}.$$

$\boxed{4}$

(b) A Poisson process is an example of a renewal process with exponential interarrival times. It is in fact the only renewal process which satisfies the Markov property. $\boxed{3}$

(c) The renewal function is defined as

$$U(t) = \mathbb{E}(N_t), \qquad \text{for } t \geq 0,$$

for a renewal process N. $\boxed{2}$

(d) Let N denote a renewal process. Let X_1 denote the first interarrival time and assume that it has a density denoted by f. Then, using the continuous version of the law of total probability, we have

$$U(t) = \mathbb{E}(N_t) = \int_0^{\infty} \mathbb{E}(N_t|X_1 = x) f(x) dx.$$

$\boxed{2}$

Let us focus on the term $\mathbb{E}(N_t|X_1 = x)$. If $X_1 = x$ and $x > t$, then $N_t = 0$ and hence $\mathbb{E}(N_t|X_1 = x) = 0$. $\boxed{1}$

If $X_1 = x$ and $x \leq t$, then there has been one renewal by time x and there can be additional renewals between x and t. From the definition of a renewal process we see that the process probabilistically starts over again after an renewal has occurred. Hence we conclude that $\mathbb{E}(N_t|X_1 = x) = 1 + \mathbb{E}(N_{t-x}) = 1 + U(t - x)$. $\boxed{2}$

Altogether we find that the renewal equation is given by

$$\begin{aligned} U(t) = \mathbb{E}(N_t) &= \int_0^t (1 + \mathbb{E}(N_{t-x})) f(x) dx = \int_0^t f(x) dx + \int_0^t U(t-x) f(x) dx \\ &= F(t) + \int_0^t U(t-x) f(x) dx. \end{aligned}$$

$\boxed{2}$

(e) Using the notation from (d), we have $f(t) = 1$ and $F(t) = t$ for $0 \leq t \leq 1$. Hence for $0 \leq t \leq 1$ we have

$$U(t) = t + \int_0^t U(t-x)dx = t + \int_0^t U(y)dy,$$

where we did a change of variables. Differentiating with respect to t, leads to $\boxed{2}$

$$U'(t) = 1 + U(t).$$

This differential equation can be solved using the hint which leads to $M(t) = \exp(-t)$, $U(0) = 0$, hence

$$U(t) = \int_0^t (-1)e^{-u}du = e^{-t} - 1.$$

$\boxed{2}$