EEE-U8 2016 Solutions Maths 2A

marks

Q1 a) i) $u = \sinh x \cos y + 2\cosh x \sin y$

First $u_x = \cosh x \cos y + 2\sinh x \sin y$

$u_{xx} = \sinh x \cos y + 2\cosh x \sin y = u$ (2)

And $u_y = -\sinh x \sin y + 2\cosh x \cos y$

$u_{yy} = -\sinh x \cos y - 2\cosh x \sin y = -u$

$\Rightarrow u_{xx} + u_{yy} = u - u = 0$, so satisfies Laplace's eqn. (2)

ii) C-R: $u_x = v_y \Rightarrow v = \int u_x \, dy$ *

$u_y = -v_x \Rightarrow v = -\int u_y \, dx$ **

} must do both

* $\rightarrow v = \int \cosh x \cos y + 2\sinh x \sin y \, dy$

$= \cosh x \sin y - 2\sinh x \cos y + f(x)$

** $\rightarrow v = -\int -\sinh x \sin y + 2\cosh x \cos y \, dx$

$= \cosh x \sin y - 2\sinh x \cos y + g(y)$

} arbitrary fns (3)

Compare both solns $\Rightarrow f = g =$ Constant.

Hence $v = \cosh x \sin y - 2\sinh x \cos y + C \; (\in \mathbb{R})$ (1)

iii) $w = u + iv$, simplify!

Use $\cosh x = \cos(ix)$, $\sinh x = -i\sin(ix)$ to get

$w = -i\sin(ix)\cos y + 2\cos(ix)\sin y + i\left[\cos(ix)\sin y - 2(-i\sin(ix))\cos y\right] + iC$ (2)

$= (2+i)\cos(ix)\sin(y) - (2+i)\sin(ix)\cos y + iC$

$= (2+i)\sin(y - ix) + iC$

$= (2+i)\sin(-iz) + iC$

Hence $C_1 = 2+i$, $C_2 = -i$, $C_3 = iC$ (3)

(with $C \in \mathbb{R}$)

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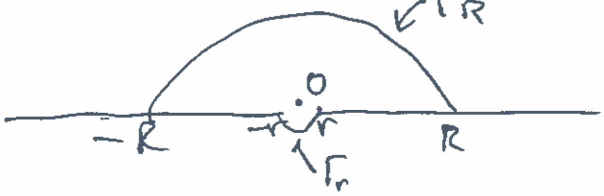
b) i) Residue at $z=0$:

$\lim_{z\to 0} zF(z) = \lim_{z\to 0} \frac{e^{iz}}{z^2+9} = \frac{1}{9}$ (2)

Residue at $z = 3i$:

$\lim_{z\to 3i} (z-3i)\frac{e^{iz}}{z(z^2+9)} = \lim_{z\to 3i} \frac{e^{iz}}{z(z+3i)} = \frac{e^{3i^2}}{3i(6i)} = -\frac{e^{-3}}{18}$ (2)

, Draw Γ:



Γ_R, Γ_r : arcs of semicircles

$\Rightarrow \Gamma = \Gamma_R \cup [-R,-r] \cup \Gamma_r \cup (r,R]$

ii) $I = \int_{\Gamma_r} f(z)\,dz$ Let $z = re^{i\theta}$, where $\theta = \cancel{\pi} \ldots 2\pi$ (anticlockwise)

$dz = ire^{i\theta} = iz\,d\theta$

Substitute $\Rightarrow$

$$I = \int_{\pi}^{2\pi} \frac{e^{i(re^{i\theta})}\, i \cancel{z}\, d\theta}{\cancel{z}(r^2e^{i2\theta}+9)} \qquad (3)$$

$$\lim_{r\to 0} I = \frac{i}{9}\int_{\pi}^{2\pi} 1\,d\theta = \frac{i\pi}{9} \quad \text{(as } re^{i\theta},\ r^2e^{i2\theta} \to 0 \text{ as } r \to 0) \qquad (2)$$

iii) $\lim_{R\to\infty} \int_{\Gamma_R} f(z)\,dz = 0$ as conditions hold for Jordan's lemma; $\lim_{R\to\infty} \int_{\Gamma_R} e^{imz} F(z)\,dz$

i) Only singularities are Poles

ii) $m = 1 > 0$

iii) $|F(z)| = \left|\frac{1}{z(z^2+9)}\right| \to 0$ fast enough, as $R \to \infty$ (3)

iv) Residue Theorem $\Rightarrow$

$$2\pi i\left(\frac{1}{9} - \frac{e^{-3}}{18}\right) = \oint_{\Gamma} f(z)\,dz = \int_{\Gamma_r} f(z)\,dz + \int_{\Gamma_R} f(z)\,dz + \int_{-R}^{-r} f(x)\,dx + \int_{r}^{R} f(x)\,dx$$

limit as $r \to 0$, $R \to \infty$ $\Rightarrow$

$$2\pi i\left(\frac{1}{9} - \frac{e^{-3}}{18}\right) = \frac{i\pi}{9} + 0 + \int_{-\infty}^{+\infty} \frac{e^{ix}}{x(x^2+9)}\,dx \qquad (3)$$

$$\Rightarrow \quad i\left(\frac{\pi}{9} - \frac{\pi e^{-3}}{9}\right) = \underbrace{\int_{-\infty}^{+\infty} \frac{\cos x}{x(x^2+9)}\,dx}_{=0} + i\int_{-\infty}^{\infty} \frac{\sin x}{x(x^2+9)}\,dx$$

as integrand odd

$$\Rightarrow \quad \int_{-\infty}^{+\infty} \frac{\sin x}{x(x^2+9)}\,dx = \frac{\pi}{9}\left(1 - e^{-3}\right) \qquad (2)$$

2) a) $\mathcal{L}[H(t-a)] = \int_0^\infty H(t-a)e^{-st}\,dt$ — Marks

$$= \int_a^\infty e^{-st}\,dt \qquad (2)$$

$$\left.\frac{e^{-st}}{-s}\right]_a^\infty = \frac{e^{-as}}{s}$$

$$\left(\lim_{t\to\infty}\left(\frac{e^{-st}}{-s}\right) = 0, \text{ provided } R(s) > 0\right) \qquad (2)$$

b) $f(t) = 3(H(t) - H(t-2))$

$$\mathcal{L}[f(t)] = \frac{3}{s} - \frac{3e^{-2s}}{s} \qquad (3)$$

c) $$* \; \frac{1}{s(s^2+4s+3)} = \frac{1}{s(s+3)(s+1)} = \frac{A}{s} + \frac{B}{s+3} + \frac{C}{s+1}$$

$\Rightarrow 1 = A(s+3)(s+1) + Bs(s+1) + Cs(s+3)$

Let $s = 0 \Rightarrow 1 = 3A \Rightarrow A = \frac{1}{3}$

$s = -1 \Rightarrow 1 = -C(2) \Rightarrow C = -\frac{1}{2} \Rightarrow$ (3)

$s = -3 \Rightarrow 1 = 6B \Rightarrow B = \frac{1}{6}$

$$\mathcal{L}^{-1}[*] = \frac{1}{3}\mathcal{L}^{-1}\left[\frac{1}{s}\right] + \frac{1}{6}\mathcal{L}^{-1}\left[\frac{1}{s+3}\right] - \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s+1}\right]$$

$$= \frac{1}{3} + \frac{1}{6}e^{-3t} - \frac{1}{2}e^{-t} \qquad (2)$$

Marks

d) $\mathcal{L}[\ddot{x} + 4\dot{x} + 3x] = \mathcal{L}\{f(t)\} \Rightarrow$

$s^2 F(s) - x(0)s - x'(0) + 4(sF(s) - x(0)) + 3F(s) = \frac{3}{s} - \frac{3e^{-2s}}{s}$ (2)

ICs: $x(0) = 1, \; x'(0) = 0 \Rightarrow$

$$(s^2+4s+3)\,F(s) = \frac{3}{s} + s + 4 - \frac{3e^{-2s}}{s}$$

$$= \frac{s^2+4s+3}{s} - \frac{3e^{-2s}}{s}$$

$$\Rightarrow F(s) = \frac{1}{s} - \frac{3e^{-2s}}{s(s^2+4s+3)}$$ (2)

$$x(t) = \mathcal{L}^{-1}[F(s)] = 1 - 3\,\mathcal{L}^{-1}\left[\frac{e^{-2s}}{s(s^2+4s+3)}\right]$$

$$= 1 - 3H(t-2)\,\mathcal{L}^{-1}\left[\frac{1}{s(s^2+4s+3)}\right]_{t\to t-2}$$ (second shift)

$$= 1 - 3H(t-2)\left[\frac{1}{3} + \frac{1}{6}e^{-3t} - \frac{1}{2}e^{-t}\right]_{t\to t-2}$$ (from (c))

$$= 1 + H(t-2)\left[-1 - \frac{1}{2}e^{-3(t-2)} + \frac{3}{2}e^{-(t-2)}\right]$$ (2)

$$= \begin{cases} 1, & 0 \le t < 2 \\ -\frac{1}{2}e^{-3(t-2)} + \frac{3}{2}e^{-(t-2)}, & t \ge 2 \end{cases}$$ (2)

as required

Q1 a) seen similar; partly unseen in (cii)

b) Seen similar; partly unseen in (ii), (iv)

Q2 a) Bookwork

b, c) seen similar

d) seen similar, last step unseen