Information for Students

Fourier Transform Pairs

Pair Number	$x(t)$	$X(f)$
1.	$\Pi\left(\frac{t}{\tau}\right)$	$\tau \operatorname{sinc} \tau f$
2.	$2W \operatorname{sinc} 2Wt$	$\Pi\left(\frac{f}{2W}\right)$
3.	$\Lambda\left(\frac{t}{\tau}\right)$	$\tau \operatorname{sinc}^2 \tau f$
4.	$\exp(-\alpha t)u(t),\ \alpha > 0$	$\frac{1}{\alpha + j2\pi f}$
5.	$t \exp(-\alpha t)u(t),\ \alpha > 0$	$\frac{1}{(\alpha + j2\pi f)^2}$
6.	$\exp(-\alpha\lvert t\rvert),\ \alpha > 0$	$\frac{2\alpha}{\alpha^2 + (2\pi f)^2}$
7.	$e^{-\pi(t/\tau)^2}$	$\tau e^{-\pi(f\tau)^2}$
8.	$\delta(t)$	1
9.	1	$\delta(f)$
10.	$\delta(t - t_0)$	$\exp(-j2\pi f t_0)$
11.	$\exp(j2\pi f_0 t)$	$\delta(f - f_0)$
12.	$\cos 2\pi f_0 t$	$\frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)$
13.	$\sin 2\pi f_0 t$	$\frac{1}{2j}\delta(f - f_0) - \frac{1}{2j}\delta(f + f_0)$
14.	$u(t)$	$(j2\pi f)^{-1} + \frac{1}{2}\delta(f)$
15.	$\operatorname{sgn} t$	$(j\pi f)^{-1}$
16.	$\frac{1}{\pi t}$	$-j \operatorname{sgn}(f)$
17.	$\hat{x}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(\lambda)}{t - \lambda}\, d\lambda$	$-j \operatorname{sgn}(f)X(f)$
18.	$\sum_{m=-\infty}^{\infty} \delta(t - mT_s)$	$f_s \sum_{m=-\infty}^{\infty} \delta(f - mf_s)$, $f_s = T_s^{-1}$

Useful Relations and Formulas

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

$$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

$$\tan(\frac{\pi}{2} - x) = \cot x$$

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x} = \frac{2\cot x}{\cot^2 x - 1} = \frac{2}{\cot x - \tan x}$$

$$\tan\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}} = \frac{\sin x}{1+\cos x} = \frac{1-\cos x}{\sin x}$$

$$\sin x \cos y = \frac{1}{2}[\sin(x+y) + \sin(x-y)]$$

$$\cos x \sin y = \frac{1}{2}[\sin(x+y) - \sin(x-y)]$$

$$\sin x \sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$$

$$\cos x \cos y = \frac{1}{2}[\cos(x+y) + \cos(x-y)]$$

$$\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$$

$$\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$$

$$\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$$

$$\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$$

Differentiation Rule of Leibnitz

Let $F(z) = \int_{a(z)}^{b(z)} f(x,z)dx$. Then we have

$$\frac{dF(z)}{dz} = \frac{db(z)}{dz}f(b(z),z) - \frac{da(z)}{dz}f(a(z),z) + \int_{a(z)}^{b(z)} \frac{\partial f(x,z)}{\partial z}dx$$

Fourier Transform Theorems[a]

Name of Theorem		
1. Superposition (a_1 and a_2 arbitrary constants)	$a_1x_1(t) + a_2x_2(t)$	$a_1X_1(f) + a_2X_2(f)$
2. Time delay	$x(t - t_0)$	$X(f)e^{-j2\pi f t_0}$
3a. Scale change	$x(at)$	$\lvert a\rvert^{-1}X\left(\frac{f}{a}\right)$
b. Time reversal	$x(-t)$	$X(-f) = X*(f)$
4. Duality	$X(t)$	$x(-f)$
5a. Frequency translation	$x(t)e^{j\omega_0 t}$	$X(f - f_0)$
b. Modulation	$x(t)\cos\omega_0 t$	$\frac{1}{2}X(f - f_0) + \frac{1}{2}X(f + f_0)$
6. Differentiation	$\frac{d^n x(t)}{dt^n}$	$(j2\pi f)^n X(f)$
7. Integration	$\int_{-\infty}^{t} x(t')\,dt'$	$(j2\pi f)^{-1}X(f) + \frac{1}{2}X(0)\delta(f)$
8. Convolution	$\int_{-\infty}^{\infty} x_1(t - t')x_2(t')\,dt'$ $= \int_{-\infty}^{\infty} x_1(t')x_2(t - t')\,dt'$	$X_1(f)X_2(f)$
9. Multiplication	$x_1(t)x_2(t)$	$\int_{-\infty}^{\infty} X_1(f - f')X_2(f')\,df'$ $= \int_{-\infty}^{\infty} X_1(f')X_2(f - f')\,df'$

Q Function

$$Q(x) = \frac{1}{\sqrt{2\pi}}\int_x^{\infty} e^{-t^2/2}dt$$

Joint Gaussian density

The joint probability density function (pdf) of two correlated Gaussian random variables X and Y is given by

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}e^{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_X)^2}{\sigma_X^2} - \frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} + \frac{(y-\mu_Y)^2}{\sigma_Y^2}\right]}.$$

where $\mu_X = E[X]$, $\mu_Y = E[Y]$ are the mean values, σ_X and σ_Y are the standard deviation of X and Y, respectively, and ρ is the correlation coefficient defined as

$$\rho = \frac{\mathrm{Cov}(X,Y)}{\sigma_X\sigma_Y}.$$

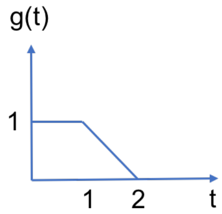
1. a) i) Why are digital signals more immune to channel noise compared to analogue signals? [2]

Very few students mentioned the benefits of coding in digital transmission.

ii) What is the minimum sampling rate such that the function $f(t) = \text{sinc}^2(3t)$ can be reconstructed exactly from its samples? [2]

mostly answered correctly.

iii) Let $g(t)$ be the following pulse shape:



Some students could not recall that the matched filter is the optimal one, and how it can be obtained.

Assume that the receiver receives $y(t) = a \cdot g(t) + w(t)$, where $w(t)$ is white Gaussian noise, and a is an unknown constant that the receiver wishes to detect. $y(t)$ is passed through a filter and then sampled at $t = 2$. Assuming that $g(t)$ is known at the receiver, what is the best filter that minimizes the effect of noise? [3]

iv) Consider a QPSK system, where the transmitted signal is denoted by

$$x(t) = A_c \cos(2\pi f_c t + \phi), \qquad \phi \in \left\{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\right\}.$$

Draw the diagram of a coherent QPSK receiver, and explain the function of each component of the receiver. [5]

Was answered satisfactorily by a surprisingly small number of students, although this was done in the class.

b) Let $X(t)$ and $Y(t)$ be two random processes. State whether each of the following statements are true or false, and discuss your answer:

i) If $X(t)$ is strict sense stationary (SSS), it is also wide sense stationary (WSS). [2]

ii) If $X(t)$ is WSS, it is also SSS. [2]

iii) If $X(t)$ is a white process, then it is Gaussian. [2]

iv) If $X(t)$ and $Y(t)$ are WSS, so is $X(t) + Y(t)$. [3]

This question was mostly answered correctly, apart from the last one, which is a bit tricky as the students tend to assume the two processes are independent.

c) Assume that $X(t)$ is a zero-mean WSS random process with power spectral density $S_X(f) = \Pi(\frac{f}{3000})$, where $\Pi(x)$ is the rectangular function defined as follows:

$$\Pi(x) \triangleq \begin{cases} 1 & \text{if } |x| < 1/2, \\ 1/2 & \text{if } |x| = 1/2, \\ 0 & \text{otherwise .} \end{cases}$$

i) What is the maximum sampling rate that will lead to uncorrelated samples? This was confused with the Nyquist sampling rate by many of the students. [4]

ii) If the sampling rate is 1KHz, and each sample is quantized by a 10-bit quantizer, how much storage is needed to store a 5 second time-frame of signal $X(t)$? Almost all answered correctly. [2]

d) Assume that $X(t)$ is a real zero-mean WSS Gaussian random process. $X(t)$ is passed through a linear time invariant (LTI) system, and the output is denoted by $Y(t)$. Let $h(t) = \text{sinc}(t)$ denote the impulse response of the LTI system.

i) Which of the following three statements are true?

1. $Y(t)$ is Gaussian.
2. $S_Y(f)$ is bandlimited.
3. $Y(t)$ is WSS, but not necessarily SSS.

[6]

ii) Find $\Pr\{Y(1) \geq 0\}$. [3]

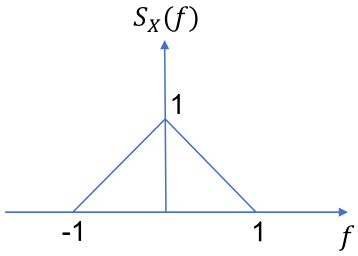
iii) If $\Pr\{Y(1) + Y(2) \leq 2\} = 0.3$, find

$$\Pr\{Y(-1) + Y(-2) > 2\}.$$

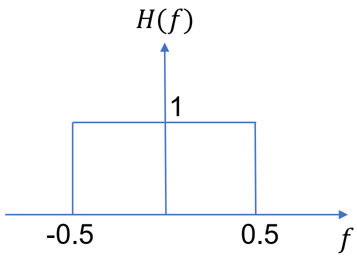
[4]

The first two parts were answered correctly about half of the students. I guess there was almost no correct answers for the last bit. It seems that nobody thought about exploiting the strict sense stationarity to argue that Y(1) and Y(-2) have the same joint probability distribution as Y(1) and Y(2).

2. a) A WSS Gaussian random process $X(t)$ has the following power spectral density (PSD):



Assume that $\mathbb{E}[X(t)] = \frac{1}{2}$ for all t, and $X(t)$ is passed through a linear time invariant (LTI) filter with the following frequency response, and the output is denoted by $Y(t)$.



i) Find $\mathbb{E}[Y(t)]$. [3]

ii) Find the PSD of $Y(t)$, i.e., $S_Y(f)$. [4]

iii) Find $\Pr\{Y(0) \geq 2\}$. [5]

iv) Find $\Pr\{Y(1)+Y(2)+Y(3) \geq 3/2\}$. [3]

While most students got the first two parts correctly, very few could answer the last two. Again, the students seem not to realize that $Y(0)$ is a Gaussian random variable whose mean is found in part i), so all that remains is to find its variance to obtain its pdf. In the last part we don't even need the variance since $Y(1)+Y(2)+Y(3)$ is a Gaussian r.v. with mean 1.5.

b) A binary message source generates bit "0" with probability p_0 and "1" with probability p_1. These bits are transmitted over a binary digital communication system. Bit "0" is transmitted with a pulse of amplitude -1, and bit "1" is transmitted with a pulse of amplitude 1. The noise in the channel is zero-mean additive white Gaussian with variance 0.5. The receiver uses a matched filter followed by threshold detection.

i) Determine the optimum detection threshold if $p_1 = 0.5$. [3]

ii) Determine the optimum detection threshold if $p_1 = 0.2$. What is the corresponding probability of error? [6]

iii) Assume that the receiver sets the optimum threshold as derived in question ii). However; the message source generates bits with $p_1 = 0.7$. What is the probability of error? How does this compare with the probability of error you found above? [6]

There were many correct or almost correct answers. Most students understood the derivation of the error probabilities in this setting.

3. a) Let $m(t) = \sqrt{6}\cos(4\pi t)$ denote a baseband message signal. The signal undergoes standard amplitude modulation (AM). Assume that the additive noise is Gaussian and white, with the autocorrelation function $R_N(\tau) = \frac{1}{2}\delta(\tau)$.

i) Draw the diagram of a coherent AM detector, and explain the function of each component. [5]

ii) Calculate the signal to noise ratio (SNR) at the receiver output. [6]

iii) How does the performance of the above system compare with that of a baseband system with the same transmitted power? [4]

Unfortunately the performance for this problem was disappointing. Although this was meant as a straightforward analog communications problem, it seems that most students did not have a good understanding of the receiver structure and SNR analysis.

b) Assume that a memoryless source outputs symbols A, B and C with the corresponding probabilities 0.5, 0.3 and 0.2, respectively.

i) Design a Huffman code for compressing this source output. [3]

ii) What is the average codeword length of this code? [3]

iii) What is Shannon's theoretical bound on the minimum average codeword length for this source? [3]

iv) Assume that we want to compress two-letter words independently generated by this source, i.e., AA, AB, AC, ... Design a Huffman code for compressing these two-letter words. What is the average codeword length per symbol for this code? [6]

This was the question with a very high success rate. It seems that most students understood the structure of Huffman coding and the meaning of the Shannon bound.

