

Master -  
April 08

E 4.25  
C 4.1  
F 4.23

## Design of Linear Multivariable Control Systems

### Solutions 2008

1. (a) (i) Since  $[A - sI \ B]$  loses rank for  $s = -3$  and  $s = -5$ , they are uncontrollable modes, and since  $[A^T - sI \ C^T]$  loses rank for  $s = 4$  and  $s = -5$ , they are unobservable modes. Since the uncontrollable modes are stable, the realisation is stabilisable, and since one of the unobservable modes is unstable, the realisation is not detectable.
- (ii) Since the modes  $\lambda = -3$  and  $\lambda = -5$  are uncontrollable, they cannot be assigned via state feedback and so they are eigenvalues of  $A - BK$ . Similarly, since  $\lambda = 4$  and  $\lambda = -5$  are unobservable modes, they cannot be assigned via output injection and so they are eigenvalues of  $A - LC$ .
- (iii) By removing the uncontrollable and/or unobservable modes we get the minimal realisation

$$G(s) \stackrel{s}{=} \left[ \begin{array}{c|cc} 1 & 1 & 2 \\ \hline 2 & 1 & 0 \\ 1 & 0 & 1 \end{array} \right] = \left[ \begin{array}{cc} \frac{s+1}{s-1} & \frac{4}{s-1} \\ \frac{1}{s-1} & \frac{s+1}{s-1} \end{array} \right] = \frac{1}{s-1} \begin{bmatrix} s+1 & 4 \\ 1 & s+1 \end{bmatrix}.$$

- (b) (i) The inequality implies that  $A'P + PA < 0$ . Let  $z \neq 0$  be a right eigenvector of  $A$  and let  $\lambda$  be the corresponding eigenvalue. Then multiplying the inequality from the left by  $z'$  and from the right by  $z$  gives  $(\lambda + \bar{\lambda})z'Pz < 0$ . Since  $P > 0$  and  $z \neq 0$  then  $z'Pz > 0$  and it follows that  $\lambda + \bar{\lambda} < 0$  so that  $A$  is stable.
- (ii) Since  $A$  is stable,  $\|H\|_\infty < \gamma$  if and only if, with  $x(0) = 0$ ,

$$J := \int_0^\infty [y'y - \gamma^2 u'u] dt < 0,$$

for all  $u(t)$  such that  $\|u\|_2 < \infty$ . If  $\|u\|_2$  is bounded, then  $\lim_{t \rightarrow \infty} x(t) = 0$ . Now,

$$\int_0^\infty \frac{d}{dt} [x'Px] dt = x(\infty)'Px(\infty) - x(0)'Px(0) = 0.$$

So,

$$\begin{aligned} 0 &= \int_0^\infty \dot{x}'Px + x'P\dot{x} dt = \int_0^\infty [(Ax + Bu)'Px + x'P(Ax + Bu)] dt \\ &= \int_0^\infty [x'(A'P + PA)x + x'PBu + u'B'Px] dt. \end{aligned}$$

Use  $y = Cx$  and add the last expression to  $J$  and using the hint,

$$\begin{aligned} J &= \int_0^\infty [x'(A'P + PA + C'C)x + x'PBu + u'B'Px - \gamma^2 u'u] dt \\ &= \int_0^\infty [x'(A'P + PA + C'C + \gamma^{-2}PBB'P)x - \|(\gamma u - \gamma^{-1}B'Px)\|^2] dt < 0 \end{aligned}$$

from the inequality. It follows that  $\|H\|_\infty < \gamma$ . Comparing with the inequality, it follows that  $\gamma = 2$ .

2. (a) Inject a signal  $d$  in between  $G(s)$  and  $K(s)$  and call the input to  $G(s)$   $u$ . The loop is internally stable if and only if the transfer matrix from  $\begin{bmatrix} d \\ r \end{bmatrix}$  to  $\begin{bmatrix} u \\ e \end{bmatrix}$  is stable (no poles in the closed right half plane). Since

$$\begin{bmatrix} d \\ r \end{bmatrix} = \begin{bmatrix} I & -K \\ -G & I \end{bmatrix} \begin{bmatrix} u \\ e \end{bmatrix} =: T(s) \begin{bmatrix} u \\ e \end{bmatrix}$$

the loop is internally stable if and only if  $T^{-1}(s)$  is stable.

- (b) Since  $G(s)$  is stable, we proceed as follows. Note that

$$\begin{bmatrix} I & -K \\ -G & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ -G & I \end{bmatrix} \begin{bmatrix} I & -K \\ 0 & I - GK \end{bmatrix}.$$

Hence

$$\begin{bmatrix} I & -K \\ -G & I \end{bmatrix}^{-1} = \begin{bmatrix} I & -K \\ 0 & I - GK \end{bmatrix}^{-1} \begin{bmatrix} I & 0 \\ -G & I \end{bmatrix}^{-1} = \begin{bmatrix} I & K(I - GK)^{-1} \\ 0 & (I - GK)^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ G & I \end{bmatrix}.$$

Finally, since  $(I - GK)^{-1} = I + GK(I - GK)^{-1}$ , it follows that if  $G$  is stable, then the loop is internally stable if and only if  $Q := K(I - GK)^{-1}$  is stable. Rearranging terms shows that  $K$  internally stabilising if and only if  $K = Q(I + GQ)^{-1}$  for some stable  $Q$ .

- (c) Since  $K$  is required to be internally stabilising,  $K = Q(I + GQ)^{-1}$  for some stable  $Q$  from Part (b). We search for a stable  $Q$  to satisfy the requirements.

- i Since the transfer matrix from  $r$  to  $e$  is

$$S(s) = (I - G(s)K(s))^{-1} = I + G(s)Q(s)$$

we need

$$\|I + GQ\|_{\infty} < 1.$$

- ii Let the input to  $\Delta$  be  $\epsilon$  while the output from  $\Delta$  be  $\delta$ . Then  $\epsilon = C\delta$  where  $C = (I - GK)^{-1}GK$  which is stable. Now  $C = GK(I - GK)^{-1} = GQ$ . The small gain theorem implies that for  $K$  to stabilise the loop in Figure 2.2 for all  $\Delta$  such that  $\|\Delta\|_{\infty} < 1$ , we must have

$$\|GQ\|_{\infty} < 1.$$

Since  $G(s)$  is minimum phase  $G(s)^{-1}$  is stable and we set  $Q(s) = \alpha G(s)$  and choose  $\alpha$  to satisfy the design specifications. The specification in (i) requires

$$|1 + \alpha| < \frac{1}{2} \Leftrightarrow -\frac{1}{2} < 1 + \alpha < \frac{1}{2} \Leftrightarrow -\frac{3}{2} < \alpha < -\frac{1}{2}.$$

The second specification requires that

$$|\alpha| < 1 \Leftrightarrow -1 < \alpha < 1.$$

Combining these specifications, a family of internally stabilising controllers that achieves the design specifications is given by  $K = Q(I + GQ)^{-1}$  where  $Q = \alpha G(s)^{-1}$  and where  $-1 < \alpha < -\frac{1}{2}$ . That is,  $K = \frac{\alpha}{1+\alpha} G(s)^{-1}$ .

For the last part, since  $KS = Q = \alpha G^{-1}$ , it follows that the smallest achievable  $\|KS\|_{\infty}$  is  $0.5\|G^{-1}\|_{\infty}$ .

3. (a) Let  $V = x^T P x$  and set  $u = Fx$ . Provided that  $P = P^T > 0$  and  $\dot{V} < 0$  along closed-loop trajectories, we can assume  $\lim_{t \rightarrow \infty} x(t) = 0$ . Then

$$\dot{V} = \dot{x}^T P x + x^T P \dot{x} = x^T (A^T P + P A + F^T B^T P + P B F) x.$$

Integrating from 0 to  $\infty$  and using  $x(\infty) = 0$ ,

$$\int_0^\infty x^T (A^T P + P A + F^T B^T P + P B F) x dt = -x_0^T P x_0.$$

- (b) Using the definition of  $J$  and adding the last equation,

$$J = x_0^T P x_0 + \int_0^\infty x^T [A^T P + P A + I + F^T F + F^T B^T P + P B F] x dt.$$

Completing the square using  $(F + B^T P)^T (F + B^T P) = F^T F + F^T B^T P + P B F + P B B^T P$  gives  $J = x_0^T P x_0 + \int_0^\infty \{x^T [A^T P + P A + I - P B B^T P] x + \|(F + B^T P)x\|^2\} dt$ .

Since the last term is always nonnegative, it follows that the minimizing value of  $F$  is given by  $F = -B^T P$ . We can set the term in square brackets to zero provided  $P$  satisfies the Riccati equation,

$$A^T P + P A + I - P B B^T P = 0.$$

It follows that the minimum value of  $J$  is  $x_0^T P x_0$ .

- (c) For closed loop stability we need to prove that  $A_c := A - B B^T P$  is stable. The Riccati equation can be written as  $A_c^T P + P A_c + I + P B B^T P = 0$ . Let  $\lambda \in \mathcal{C}$  be an eigenvalue of  $A_c$  and  $z \neq 0$  be the corresponding eigenvector. Pre- and post-multiplying the Riccati equation by  $z'$  and  $z$  respectively gives  $(\lambda + \bar{\lambda})z' P z + z' z + z' P B B^T P z = 0$ . Since  $P > 0$  and  $z \neq 0$ ,  $z' P z > 0$ ,  $z' z > 0$  and  $z' P B B^T P z \geq 0$ . It follows that  $\lambda + \bar{\lambda} < 0$  and the closed loop is stable.
- (d) By direct evaluation,  $L(j\omega)' L(j\omega) =$

$$I - F(j\omega I - A)^{-1} B - B'(-j\omega I - A')^{-1} F' + B'(-j\omega I - A')^{-1} F' F(j\omega I - A)^{-1} B$$

But  $F' F = A' P + P A + I = -(-j\omega I - A') P - P(j\omega I - A) + I$  from the Riccati equation. So,  $L(j\omega)' L(j\omega)$

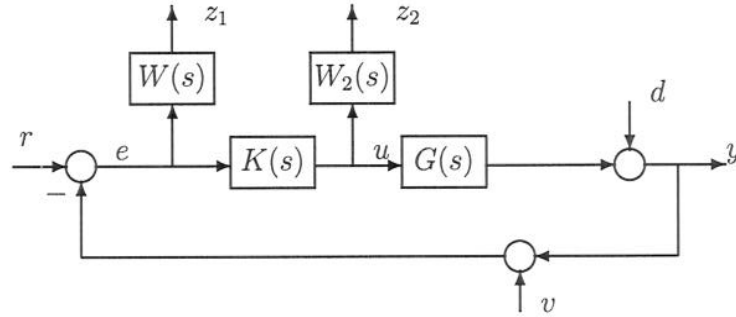
$$\begin{aligned} &= I - F(j\omega I - A)^{-1} B - B'(-j\omega I - A')^{-1} F' \\ &\quad + B'(-j\omega I - A')^{-1} [ -(-j\omega I - A') P - P(j\omega I - A) + I ] (j\omega I - A)^{-1} B \\ &= I - [F + B' P] (j\omega I - A)^{-1} B - B'(-j\omega I - A')^{-1} [F' + P B] \\ &\quad + B'(-j\omega I - A')^{-1} (j\omega I - A)^{-1} B = I + G(j\omega)' G(j\omega) \end{aligned}$$

- (e) Let  $\epsilon$  be the input to  $\Delta$  and  $\delta$  be the output of  $\Delta$ . Then  $\epsilon = \delta + F G \epsilon = (I - F G)^{-1} \delta$ . Using the small gain theorem (since the regulator is stable and the perturbation is assumed stable), the loop is stable if  $\|\Delta (I - F G)^{-1}\|_\infty < 1$ . But Part (d) implies that  $\underline{\sigma}[I - F G(j\omega)] \geq 1 \forall \omega$  which implies  $\|(I - F G)^{-1}\|_\infty \leq 1$ . This shows that the loop will tolerate perturbations  $\Delta$  of size  $\|\Delta\| < 1$  without losing internal stability.

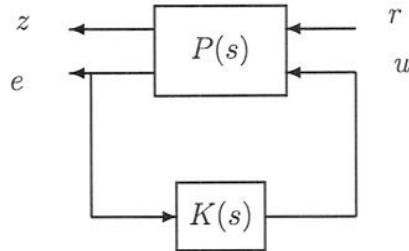
4. (a) It is clear that we require  $K$  to internally stabilize the nominal model.
- (i) Suppose that  $\Delta_1 = 0$  and let the input to  $\Delta_2$  be  $\epsilon_2$  while the output from  $\Delta_2$  be  $\delta_2$ . Then a calculation shows that  $\epsilon_2 = -KS\delta_2$  where  $S = (I + GK)^{-1}$  is the sensitivity which is stable. Using the small gain theorem, to satisfy the first design requirement, it is sufficient that  $\|\Delta_2(j\omega)K(j\omega)S(j\omega)\| < 1, \forall \omega$ . This can be satisfied if  $\|W_2KS\|_\infty < 1$ , where  $W_2 = w_2^{-1}I$ .
  - (ii) An analogous procedure shows that to satisfy the second design requirement, it is sufficient that  $\|\Delta_1(j\omega)S(j\omega)\| < 1, \forall \omega$ . This can be satisfied if  $\|W_1S\|_\infty < 1$ , where  $W_1 = w_1^{-1}I$ .
  - (iii) For the nominal loop,  $y_o = (I + GK)^{-1}GKr$  so that  $(I + GK)y_o = GKr$ . For the loop with  $\Delta_2 = 0$ ,  $y_1 = (I + (I + \Delta_1)^{-1}GK)^{-1}(I + \Delta_1)^{-1}GKr$  so that  $(I + (I + \Delta_1)^{-1}GK)y_1 = (I + \Delta_1)^{-1}GKr$ . Substituting  $(I + GK)y_o = GKr$  and multiplying from the left by  $(I + \Delta_1)$  gives  $(I + \Delta_1 + GK)y_1 = (I + GK)y_o$  and so  $(I + GK)(y_o - y_1) = \Delta_1 y_1$  or  $y_o - y_1 = S\Delta_1 y_1$ . Thus to satisfy the robust tracking requirement, it is sufficient that  $\|\epsilon^{-1}W_1S\|_\infty < 1$ .

We can combine the second and third requirements as  $\|WS\|_\infty < 1$  where  $W = W_1/\min(1, \epsilon)$ . To satisfy all three design requirements, it is sufficient that  $\left\| \begin{bmatrix} WS \\ W_2KS \end{bmatrix} \right\|_\infty < 1$ .

- (b) The design specifications reduce to the requirement that the transfer matrix from  $r$  to  $z = [z_1^T \ z_2^T]^T$  in the following diagram has  $\mathcal{H}_\infty$ -norm less than 1.



The corresponding generalized regulator formulation is to find an internally stabilizing  $K$  such that  $\|\mathcal{F}_l(P, K)\| < 1$ :



$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \left[ \begin{array}{c|c} W & -WG \\ \hline 0 & W_2 \\ \hline I & -G \end{array} \right].$$

5. (a) (i) The  $(1, 1)$  block of the inequality gives the inequality  $A'P + PA < 0$ . Let  $z \neq 0$  be a right eigenvector of  $A$  and let  $\lambda$  be the corresponding eigenvalue. Then multiplying the inequality from the left by  $z'$  and from the right by  $z$  gives  $(\lambda + \bar{\lambda})z'Pz < 0$ . Since  $P > 0$  it follows that  $z'Pz > 0$  and it follows that  $\lambda + \bar{\lambda} < 0$  so that  $A$  is stable.
- (ii) Since  $A$  is stable,  $\|H\|_\infty < \gamma$  if and only if, with  $x(0) = 0$ ,  $J := \int_0^\infty [y'y - \gamma^2 u'u] dt < 0$ , for all  $u(t)$  such that  $\|u\|_2 < \infty$ . If  $\|u\|_2$  is bounded, then  $\lim_{t \rightarrow \infty} x(t) = 0$ . Now,  $\int_0^\infty \frac{d}{dt} [x'Px] dt = x(\infty)'Px(\infty) - x(0)'Px(0) = 0$ . So,

$$0 = \int_0^\infty (\dot{x}'Px + x'P\dot{x}) dt = \int_0^\infty [x'(A'P + PA)x + x'PBu + u'B'Px] dt.$$

Use  $y = Cx + Du$  and add the last expression to  $J$

$$\begin{aligned} J &= \int_0^\infty [x'(A'P + PA + C'C)x + 2x'(PB + C'D)u + u'(D'D - \gamma^2 I)u] dt \\ &= \int_0^\infty \begin{bmatrix} x' & u' \end{bmatrix} \overbrace{\begin{bmatrix} A'P + PA + C'C & PB + C'D \\ B'P + D'C & D'D - \gamma^2 I \end{bmatrix}}^M \begin{bmatrix} x \\ u \end{bmatrix} dt. \end{aligned}$$

It follows that  $J < 0$ , and so  $\|H\|_\infty < \gamma$ , if  $M < 0$ . This proves the result.

- (b) (i) Substituting  $u = Ly$ ,  $y = Cx + w_2$  into the state equation gives

$$\dot{x} = \underbrace{(A + LC)}_{A_c} x + \underbrace{\begin{bmatrix} B & L \end{bmatrix}}_{B_c} w, \quad y = \underbrace{C}_{C_c} x + \underbrace{\begin{bmatrix} 0 & I \end{bmatrix}}_{D_c} w.$$

It follows that  $T_{yw}(s) = D_c + C_c(sI - A_c)^{-1}B_c$ .

- (ii) Using the results of part (a), by replacing  $A, B, C$  and  $D$  by  $A_c, B, C$  and  $D$ , we have that there exists a feasible  $L$  if there exists  $P = P^T > 0$  such that

$$\begin{bmatrix} (A + LC)'P + P(A + LC) + C'C & PB & PL + C' \\ B'P & -\gamma I & 0 \\ L'P + C & 0 & -(\gamma - 1)I \end{bmatrix} < 0.$$

Noting that the only nonlinearity is due to the product  $PL$ , we define  $Z = PL$  and so there exists a feasible  $L$  if there exists  $P = P^T > 0$  and  $Z$  such that

$$\begin{bmatrix} A'P + PA + ZC + C'Z' + C'C & PB & Z + C' \\ B'P & -\gamma I & 0 \\ Z' + C & 0 & -(\gamma - 1)I \end{bmatrix} < 0.$$

6. (a) The generalized regulator formulation is given by

$$\begin{bmatrix} z(s) \\ y(s) \end{bmatrix} = P(s) \begin{bmatrix} w(s) \\ u(s) \end{bmatrix}, \quad u(s) = Fy(s), \quad P(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} \stackrel{s}{=} \left[ \begin{array}{c|c|c} A & B & B \\ \hline C & 0 & 0 \\ \hline 0 & 0 & I \\ \hline I & 0 & 0 \end{array} \right].$$

(b) The requirement  $\|H\|_\infty < \gamma$  is equivalent to  $J := \|z\|_2^2 - \gamma^2 \|w\|_2^2 < 0$ . Let  $V = x^T X x$  and set  $u = Fx$ . Provided that  $X = X^T > 0$  and  $V < 0$  along the closed-loop trajectory, we can assume  $\lim_{t \rightarrow \infty} x(t) = 0$ . Then

$$\dot{V} = \dot{x}^T X x + x^T X \dot{x} = x^T (A^T X + X A + F^T B^T X + X B F) x + x^T X B w + w^T B^T X x.$$

Integrating from 0 to  $\infty$  and using  $x(0) = x(\infty) = 0$ ,

$$0 = \int_0^\infty [x^T (A^T X + X A + F^T B^T X + X B F) x + x^T X B w + w^T B^T X x] dt.$$

Using the definition of  $J$  and adding the last equation,  $J =$

$$\int_0^\infty \{x^T [A^T X + X A + C^T C + F^T F + F^T B^T X + X B F] x - [\beta w^T w - x^T Z^T w - w^T Z x]\} dt$$

where  $Z = F + B^T X$  and  $\beta = \gamma^2 - 1 > 0$  since  $\gamma > 1$  by assumption. Completing the squares by using

$$\begin{aligned} Z^T Z &= F^T F + F^T B^T X + X B F + X B B^T X \\ \|(\sqrt{\beta} w - \sqrt{\beta^{-1}} B^T X x)\|^2 &= \beta w^T w - w^T B^T X x - x^T X B w + \beta^{-1} x^T X B B^T X x, \\ J &= \int_0^\infty \{x^T [A^T X + X A + C^T C - X B B^T X] x + (1 + \beta^{-1}) \|Z x\|^2 - \|\sqrt{\beta} w - \sqrt{\beta^{-1}} Z x\|^2\} dt. \end{aligned}$$

Thus two sufficient conditions for  $J < 0$  are the existence of  $X$  such that

$$A^T X + X A + C^T C - X B B^T X = 0, \quad X = X^T > 0.$$

The feedback gain is obtained by setting  $Z = 0$  so  $F = -B^T X$ . The worst case disturbance is  $w^* = \beta^{-1} Z x = 0$ . The closed-loop with  $u = Fx$  and  $w = w^*$  is  $\dot{x} = [A - B B^T X] x$  and a third condition is  $\text{Re } \lambda_i[A - B B^T X] < 0, \forall i$ . It remains to prove  $\dot{V} < 0$  for  $u = Fx$  and  $w = 0$ . But

$$\dot{V} = x^T (A^T X + X A + F^T B^T X + X B F) x = -x^T (C^T C + X B B^T X) x < 0$$

for all  $x \neq 0$  (since  $(A, B, C)$  is assumed minimal) proving closed-loop stability.

(c) It is clear that our procedure breaks down if  $\gamma \leq 1$  since in that case  $\beta \leq 0$ . Thus the smallest value of  $\gamma$  is 1.