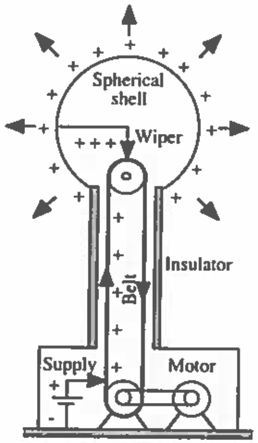
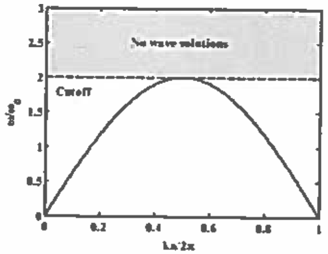
Electromagnetic Fields 2018 – Solutions

1. a) The Van de Graaf generator is a simple example of Gauss's law Flux out = Charge enclosed. Friction contacts are used to transfer charge from a DC supply to a moving belt, and then onto a spherical shell. The charge stored on the shell then creates a large electric flux density D_r, whose value at radius r can be found by applying Gauss' law over a spherical surface of radius r as $D_r = Q/4\pi r^2$. The corresponding electric field is $D_r = Q/4\pi\varepsilon_0 r^2$. [4]



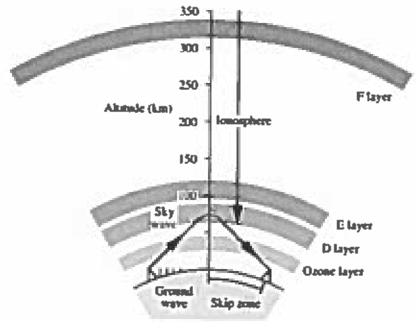
[4]

b) A dispersion diagram (otherwise known as a ω-k diagram) is a plot of angular frequency (ω) against propagation constant (k) for a medium that supports waves. It shows the frequency range(s) within which propagating waves can exist, and provides a simple method of determining the phase velocity $v_{ph} = \omega/k$ and the group velocity $v_g = d\omega/dk$. A simple example is given by a low-pass ladder network, which has the dispersion relation $\omega = 2\omega_0 \sin(ka/2)$, and the dispersion diagram shown below. There are no propagating waves above the cutoff frequency $\omega = 2\omega_0$. [4]



[4]

c) The 'ionosphere' is a set of concentric layers of ionized gas in the upper atmosphere, created by continual bombardment with energetic particles from outer space and first discovered by Edward Appleton in 1924. The layers reflect low-frequency waves, providing a propagation pathway around the curved surface of the earth and forming the basis of the early trans-Atlantic radio communication pioneered by Marconi in 1901. However, the layers are transparent to high frequency waves, allowing over the horizon communication via a geostationary satellite. [4]



[4]

d) Stokes' theorem $\oint_L \mathbf{F} \cdot d\mathbf{L} = \iint_A (\nabla \times \mathbf{F}) \cdot d\underline{a}$ is one of the two integral theorems used to transform Maxwell's equations from integral form into a more easily soluble differential form (the other is Gauss' theorem). [2]

For example, the integral version of Faraday's law is $\oint_L \underline{E} \cdot d\underline{L} = -\iint_A \partial\underline{B}/\partial t \cdot d\underline{a}$, a relation between a line integral and a surface integral. Applying Stokes' Theorem to the LHS we get:

$\oint_L \underline{E} \cdot d\underline{L} = \iint_A (\nabla \times \mathbf{E}) \cdot d\underline{a}$

Consequently, Faraday's law can be rewritten as:

$\iint_A (\nabla \times \underline{E}) \cdot d\underline{a} = -\iint_A \partial\underline{B}/\partial t \cdot d\underline{a}$

This equation is still an integral equation, but now relates two surface integrals. Since the integration range A is undefined, the integrands themselves must be equal, and we obtain:

$\nabla \times \underline{E} = -\partial\underline{B}/\partial t$

[4]

This is the desired differential form of Faraday's law. Stokes' theorem may also be used to transform the integral version of Ampere's law into differential form.

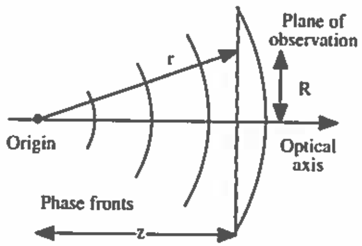
[2]

e) A paraxial wave is an approximation to a spherical wave that is valid for distances close to the optical axis. A suitable expression may be derived as follows. The general expression for the electric field of a spherical wave emanating from the origin is $E(r) = E_0/r \exp(-jk_0 r)$, where E_0 is the wave amplitude, k_0 is the propagation constant and r is radial distance.

[2]

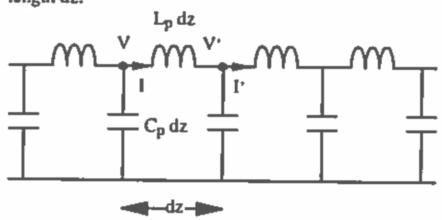
For the (x, y) plane a distance z from the origin, $r^2 = z^2 + R^2$, where $R^2 = x^2 + y^2$
Hence, $r = \sqrt{(z^2 + R^2)}$. If we now write $r = z\sqrt{(1 + R^2/z^2)}$, and R/z is small, we can use the binomial approximation to obtain $r \approx z(1 + R^2/2z^2) \approx z + R^2/2z$
The phase term in the expression for a spherical wave may then be approximated as:
$\exp(-jk_0 r) \approx \exp(-jk_0 z) \exp(-jk_0 R^2/2z)$
The amplitude term varies more slowly, and consequently may be approximated as $E_0/r \approx E_0/z$
Combining these expressions together, we obtain:
$E(R, z) \approx E_0/z \exp(-jk_0 z) \exp(-jk_0 R^2/2z)$
Or $E(R, z) \approx A(z) \exp(-jk_0 R^2/2z)$, where A(z) is constant over the plane of observation. The phase front of a paraxial wave therefore varies parabolically.

[4]



[2]

2. The figure below shows part of a ladder model of a transmission line. The circuit has series inductance L_p and parallel capacitance C_p per-unit-length, and has been divided into sections of length dz.



a) Using Kirchhoff's law at angular frequency ω, we can write the following relations between the nodal voltages and series currents:
$V' = V - j\omega L_p dz I$
$I' = I - j\omega C_p dz V'$

[2]

Now, a continuous system would have:
$V' = V + dV/dz\ dz$
$I' = I + dI/dz\ dz$

[2]

Comparison of the two sets of equations implies that:
$dV/dz = -j\omega L_p I$ (1)
$dI/dz = -j\omega C_p V'$ (2a)

[2]

However if V' and V are sufficiently close we can approximate (2a) as:
$dI/dz = -j\omega C_p V$ (2b)

[2]

b) Differentiating (1) we have:
$d^2V/dz^2 = -j\omega L_p dI/dz$
Substituting using (2b) we then obtain:
$d^2V/dz^2 = (-j\omega L_p)(-j\omega C_p)V = -\omega^2 L_p C_p V$ (3)

[2]

Now, wave solutions travelling in the +z direction have the form $V = V_0 \exp(-jkz)$. Differentiating twice and substituting into (3) we obtain:
$(-jk)(-jk)\ V_0 \exp(-jkz) = -\omega^2 L_p C_p\ V_0 \exp(-jkz)$, or $k^2 = \omega^2 L_p C_p$
Hence the propagation constant is $k = \omega\sqrt{(L_p C_p)}$

[2]

The phase velocity is $v_{ph} = \omega/k$
Substituting the result for k found above, we obtain $v_{ph} = 1/\sqrt{(L_p C_p)}$

[2]

Now from (1), $I = (-1/j\omega L_p)\ dV/dz$
Differentiating the travelling wave solution for voltage and substituting we obtain:

$I = (-1/j\omega L_p)(-jk)\, V_0 \exp(-jkz) = (k/\omega L_p)\, V_0 \exp(-jkz)$
Hence, if I is written in the form $I = I_0 \exp(-jkz)$, we must have $I_0 = (k/\omega L_p)\, V_0$

[2]

Substituting for k we obtain $I_0 = \{\omega\sqrt{(L_pC_p)}/\omega L_p\}\, V_0 = \sqrt{(C_p/L_p)}\, V_0$
The characteristic impedance $Z_0 = V_0/I_0$ is then $Z_0 = \sqrt{(L_p/C_p)}$

[2]

Wave solutions travelling in the -z direction have the form $V = V_0 \exp(+jkz)$ and $I = I_0 \exp(+jkz)$.
Differentiating and substituting into (3) we obtain:
$-k^2 V_0 \exp(+jkz) = -\omega^2 L_p C_p V_0 \exp(+jkz)$
Comparing with the previous result, we can see that the propagation constant (and consequently the phase velocity) must be unchanged.

[2]

However, differentiating the travelling wave solution for voltage and substituting we obtain
$I = (-1/j\omega L_p)(+jk)\, V_0 \exp(+jkz) = -(k/\omega L_p)\, V_0 \exp(+jkz)$
Hence, if I is written in the form $I = I_0 \exp(+jkz)$, we must have $I_0 = -(k/\omega L_p)\, V_0$
Consequently, there is now a sign change, so $I_0 = -V_0/Z_0$

[2]

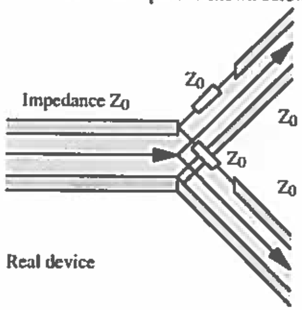
c) If $v_{ph} = 1/\sqrt{(L_pC_p)}$ and $Z_0 = \sqrt{(L_p/C_p)}$, then $Z_0/v_{ph} = L_p$
Consequently, to construct a transmission line with $Z_0 = 50\ \Omega$ and $v_{ph} = c/1.5$ m/s, we require:
$L_p = 50/(2 \times 10^8) = 2.5 \times 10^{-7}$ H/m $= 0.25\ \mu$H/m

[2]

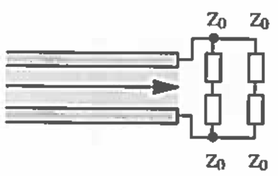
C_p can then be found from $C_p = 1/(L_p v_{ph}^2)$, as:
$C_p = 1/\{2.5 \times 10^{-7} \times (2 \times 10^8)^2\} = 1 \times 10^{-10}$ F/m $= 100$ pF/m

[2]

If the line above is to be connected to two other lines to obtain a reflection-less 1 x 2 splitter, the two lines must present a combined impedance of 50 Ω. Since they are in parallel, each must present an impedance of 100 Ω. If they are similar to the original line, and each have a characteristic impedance of 50 Ω, this value can be obtained by inserting an additional 50 Ω series resistor in each path as shown below.



Real device



Equivalent

[4]

3a) For x-polarized plane waves travelling in the z-direction in a dielectric medium, the electric field must satisfy the equation
$d^2E_x/dz^2 = -\omega^2\mu_0\varepsilon E_x$
where ω is the angular frequency, μ_0 is the permeability and ε is the permittivity.
Assuming a z-going solution in the form $E_x = E_0 \exp(-jkz)$, where k is the propagation constant, we obtain:
$(-jk)(-jk)\, E_0 \exp(-jkz) = -\omega^2\mu_0\varepsilon\, E_0 \exp(-jkz)$
Consequently, $k^2 = \omega^2\mu_0\varepsilon$

[2]

Assuming that the permittivity can be written in the form $\varepsilon = \varepsilon' - j\varepsilon''$ we obtain:
$k^2 = \omega^2\mu_0(\varepsilon' - j\varepsilon'') = \omega^2\mu_0\varepsilon'(1 - j\varepsilon''/\varepsilon')$
Consequently,
$k = \omega\sqrt{(\mu_0\varepsilon')}\sqrt{(1 - j\varepsilon''/\varepsilon')}$
If $\varepsilon'' << \varepsilon'$ we can use a binomial approximation for the second square root to obtain:
$k \approx \omega\sqrt{(\mu_0\varepsilon')}(1 - j\varepsilon''/2\varepsilon')$

[2]

If we now write $k = k' - jk''$ then the real and imaginary parts can be found separately as:
$k' = \omega\sqrt{(\mu_0\varepsilon')}$ and $k'' = k'\varepsilon''/2\varepsilon'$

[2]

If k is complex, the wave solution $E_x = E_0 \exp(-jkz)$ becomes:
$E_x = E_0 \exp\{-j(k' - jk'')z\} = E_0 \exp(-jk'z) \exp(-k''z)$
Consequently the wave decays exponentially as it propagates.

[2]

b) Assuming now that the medium is a metal, the dielectric constant can be written in terms of the conductivity σ as $\varepsilon = \sigma/j\omega$. We now obtain:
$k^2 = \omega^2\mu_0(\sigma/j\omega)$
Now $1/j = -j$, and $\sqrt{(-j)} = (1 - j)/\sqrt{2}$.

[2]

Consequently, the propagation constant is now:
$k = (1 - j)\sqrt{(\omega\mu_0\sigma/2)}$
Assuming that this can be written as $k = k' - jk''$ we obtain:
$k' = k'' = \sqrt{(\omega\mu_0\sigma/2)} = \sqrt{(\pi f\mu_0\sigma)}$

[2]

If the wave decays as before, its amplitude will reach 1/e of its initial value when $z = 1/k''$, the skin depth δ. Assuming that $\mu_0 = 4\pi \times 10^{-7}$ H/m, $\sigma = 5.96 \times 10^7$ S/m and f = 100 MHz we get:
$\delta = 1/\sqrt{(\pi \times 10^8 \times 4\pi \times 10^{-7} \times 5.96 \times 10^7)}$ m $= 6.52 \times 10^{-6}$ m, or 6.52 μm.

[2]

c) If a radio transmitter has a power output of P and an isotropic antenna, the power density at radius R is $S = P/4\pi R^2$
If $P = 10^3$ W and $r = 10^3$ m, then $S = 10^3/\{4\pi \times (10^3)^2\} = 7.96 \times 10^{-5}$ W/m^2

[2]

The antenna gain G is related to the efficiency η and directivity by $G = \eta D$
If η = 0.5 and D = 100, then the gain of the new antenna is G = 0.5 x 100 = 50.

The peak power density with the new antenna is then is $S' = GS$.
If $G = 50$ and $S = 7.96 \times 10^{-5}$ W/m^2, $S' = 3.98 \times 10^{-3}$ W/m^2.

[4]

d) At a frequency f, the wavelength is $\lambda = c/f$, where $c = 3 \times 10^8$ m/s is the velocity of light.
Assuming that $f = 100 \times 10^6$ Hz, $\lambda = 3 \times 10^8/10^8 = 3$ m.
The effective area A_e is related to the directivity by $A_e = \lambda^2 D/4\pi$.
Assuming that $D = 100$, the effective area is $A_e = 3^2 \times 100/4\pi = 71.62$ m^2.

[2]

Using this antenna, the peak received power is $P_R = \eta S' A_e$
In this case, $P_R = 0.5 \times 3.98 \times 10^{-3} \times 71.62$ W $= 0.143$ W

[4]

The received power falls off as $1/R^2$. If the minimum detectable power is $P_{Rmin} = 10$ μW, and the received power is P_R at range R, the maximum link length is $R_{max} = \sqrt{(P_R/P_{Rmin})} \times R$.
In this case, $R_{max} = \sqrt{(0.143/10^{-5})} \times 10^3$ m $= 119.6$ km

[2]