

**B.ENG. AND M.ENG. EXAMINATIONS 2011**

**PART I : MATHEMATICS 2 (ELECTRICAL ENGINEERING)**

**Date Thursday 9th June 2011 10.00 - 12.00**

*DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO*

**Answer Question 1 and THREE of the remaining FIVE questions.**

**Answer Section A and Section B in different answerbooks.**

Question 1 carries twice the marks of each of the other questions.

*A mathematical formulae sheet is provided.*

*[Before starting, please make sure that the paper is complete; there should be 6 pages, with a total of SIX questions. Ask the invigilator for a replacement if your copy is faulty.]*

## SECTION A

1. (i) Determine the union and the intersection of the sets

$$A = \{ n \in \mathbb{N} \mid 0 < n \leq 10 \} ,$$

$$B = \{ n \in \mathbb{N} \mid 0 < n < 10 \text{ and } n \text{ prime} \} .$$

- (ii) Draw the truth table for

$$(p \rightarrow q) \vee p$$

and

$$(p \wedge q) \vee (\bar{p} \wedge \bar{q}) .$$

- (iii) Determine the truth value of the following propositions

$$(a) \quad \forall x \in \mathbb{R} \quad \exists y \in Q \quad \forall z \in \mathbb{R} \quad z > xy ,$$

$$(b) \quad \forall x \in \mathbb{R} \quad \exists y \in Q \quad \exists z \in \mathbb{R} \quad z > xy ,$$

$$(c) \quad \exists x \in \mathbb{R} \quad \forall y \in Q \quad \exists z \in \mathbb{R} \quad z > xy ,$$

$$(d) \quad \forall x \in \mathbb{R} \quad \exists y \in Q \quad \forall z \in \mathbb{R} \quad z^2 > xy .$$

- (iv) Find

$$\int_0^{2\pi} \cos(mx) \cos(nx) dx$$

for positive integers  $m$  and  $n$  .

- (v) Let
- $f(x) = \exp(x)$
- .

$$\text{Determine } \int_0^{2\pi} f(x) \sin(nx) dx \text{ for } n \in \mathbb{N}, n > 0.$$

- (vi) Let
- $f(x, y) = \exp(y \sin(x))$
- .

Determine the partial derivatives  $f_x(x, y)$ ,  $f_y(x, y)$ ,  $f_{xy}(x, y)$  and  $f_{yx}(x, y)$  .

- (vii) Show that

$$x \frac{\partial}{\partial x} u(x, y) + y \frac{\partial}{\partial y} u(x, y) = u(x, y)$$

for  $u(x, y) = x \ln(x/y)$  .

- (viii) Let
- $u(x, y) = \cos(y/x)$
- .

Find  $\frac{d}{dt} u(x(t), y(t))$  in terms of  $t$  for  $x(t) = t$  and  $y(t) = \frac{1}{2} t^2$ , using partial differentiation.

PLEASE TURN OVER

- (ix) Let  $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$  and  $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$   
the identity matrix. Find all  $l$  so that

$$\det(A - lI) = 0 .$$

- (x) Find all vectors  $\mathbf{r}$  such that

$$\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \mathbf{r} = -\mathbf{r} .$$

## SECTION B

2. (i) Given the sets  $S_1 = \{6, 8\}$ ,  $S_2 = \{3, 4\}$ , list all the elements of

- (a)  $S_1 \cup S_2$  ;
- (b)  $S_1 \cap S_2$  ;
- (c)  $S_1 - S_2$  ;
- (d)  $P(S_1)$  .

- (ii) What is meant by a relation  $R$  from a set  $S$  to itself?

What is meant if we say the relation is (i) reflexive, (2) symmetric and (iii) transitive?

Apply these definitions to the relations below and say which are (i) reflexive, (ii) symmetric and (iii) transitive.

I "Is a sibling of" on the set of all people (sibling means brother or sister, with both parents in common).

II "Is the daughter of" on the set of all people.

III "Is the same sex as" on the set of all people.

IV "Is greater than" on the set of all integers.

3. Let

$$f(x) = \begin{cases} x - \frac{\pi}{2} & \text{for } x \in [0, \pi) \\ \frac{3\pi}{2} - x & \text{for } x \in [\pi, 2\pi) \end{cases}$$

define a periodic function with period  $2\pi$ .

(i) Determine the coefficients  $a_n$  for  $n \geq 0$  and  $b_n$  for  $n > 0$  of the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)) .$$

(ii) Express

$$\int_0^{2\pi} f^2(x) dx$$

in terms of  $a_n$  and  $b_n$  and thus determine  $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4}$  .

PLEASE TURN OVER

4. (i) If  $f(x, y) = 2x \tan^{-1} \left( \frac{y}{x} \right) + y \ln(x^2 + y^2)$  find

$$\frac{\partial f}{\partial x} \text{ and } \frac{\partial f}{\partial y}$$

and show that

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = f + 2y.$$

- (ii) The function  $u(x, t)$  satisfies the wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}.$$

New variables  $p$  and  $q$  are defined by

$$p = x - ct \text{ and } q = x + ct.$$

Determine  $\frac{\partial u}{\partial x}$ ,  $\frac{\partial u}{\partial t}$ ,  $\frac{\partial^2 u}{\partial x^2}$  and  $\frac{\partial^2 u}{\partial t^2}$  in terms of derivatives with respect to  $p$  and  $q$  and hence show that the wave equation in the new variables becomes

$$\frac{\partial^2 u}{\partial p \partial q} = 0.$$

Hence find the general solution of the wave equation for  $u$  in terms of  $x$  and  $t$ .

5. You are given

$$A = \begin{pmatrix} -1 & -2 & 0 \\ -1 & 0 & 0 \\ -2 & -2 & 1 \end{pmatrix}.$$

Compute  $A^2$  and  $A^3$ .

Verify  $A^3 - 3A + 2I = 0$ .

Find scalars  $a$  and  $b$  such that

$$A^2 + aA + bI = 0.$$

Hence or otherwise find  $A^{-1}$ .

6. (i) Solve the differential equation

$$(x + 1) \frac{dy}{dx} - x y = e^x$$

subject to the condition  $y = 0$  when  $x = 1$ .

- (ii) Under what circumstances is

$$P(x, y) dx + Q(x, y) dy = 0$$

exact.

Show that

$$(2xy + \frac{1}{3}y^3) dx + (x^2 + xy^2) dy = 0$$

is exact and hence solve the differential equation

$$(2xy + \frac{1}{3}y^3) + (x^2 + xy^2) \frac{dy}{dx} = 0.$$

Note: *Leave the solution in its implicit form.*

END OF PAPER

# M A T H E M A T I C S   D E P A R T M E N T

## MATHEMATICAL   FORMULAE

### 1. VECTOR ALGEBRA

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1, a_2, a_3)$$

Scalar (dot) product:  $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

Vector (cross) product:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Scalar triple product:

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = \mathbf{b} \cdot \mathbf{c} \times \mathbf{a} = \mathbf{c} \cdot \mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product:  $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c} \cdot \mathbf{a})\mathbf{b} - (\mathbf{b} \cdot \mathbf{a})\mathbf{c}$

### 2. SERIES

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \dots \quad (\alpha \text{ arbitrary, } |x| < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots ,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^n \frac{x^{2n}}{(2n)!} + \dots ,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \dots ,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^n \frac{x^{n+1}}{(n+1)} + \dots \quad (-1 < x \leq 1)$$

### 3. TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin(a + b) = \sin a \cos b + \cos a \sin b ;$$

$$\cos(a + b) = \cos a \cos b - \sin a \sin b .$$

$$\cos iz = \cosh z ; \quad \cosh iz = \cos z ; \quad \sin iz = i \sinh z ; \quad \sinh iz = i \sin z .$$

### 4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f D^n g + \binom{n}{1} Df D^{n-1} g + \dots + \binom{n}{r} D^r f D^{n-r} g + \dots + D^n f g .$$

(b) Taylor's expansion of  $f(x)$  about  $x = a$ :

$$f(a + h) = f(a) + hf'(a) + h^2 f''(a)/2! + \dots + h^n f^{(n)}(a)/n! + \epsilon_n(h) ,$$

$$\text{where } \epsilon_n(h) = h^{n+1} f^{(n+1)}(a + \theta h)/(n+1)! , \quad 0 < \theta < 1 .$$

(c) Taylor's expansion of  $f(x, y)$  about  $(a, b)$ :

$$f(a+h, b+k) = f(a, b) + [hf_x + kf_y]_{a,b} + 1/2! [h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}]_{a,b} + \dots$$

(d) Partial differentiation of  $f(x, y)$ :

$$\text{i. If } y = y(x), \text{ then } f = F(x), \text{ and } \frac{dF}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx} .$$

$$\text{ii. If } x = x(t), y = y(t), \text{ then } f = F(t), \text{ and } \frac{dF}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} .$$

iii. If  $x = x(u, v)$ ,  $y = y(u, v)$ , then  $f = F(u, v)$ , and

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u} , \quad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial v} .$$

(e) Stationary points of  $f(x, y)$  occur where  $f_x = 0$ ,  $f_y = 0$  simultaneously.

Let  $(a, b)$  be a stationary point: examine  $D = [f_{xx}f_{yy} - (f_{xy})^2]_{a,b}$ .

If  $D > 0$  and  $f_{xx}(a, b) < 0$ , then  $(a, b)$  is a maximum;

If  $D > 0$  and  $f_{xx}(a, b) > 0$ , then  $(a, b)$  is a minimum;

If  $D < 0$  then  $(a, b)$  is a saddle-point.

(f) Differential equations:

i. The first order linear equation  $dy/dx + P(x)y = Q(x)$  has an integrating factor  $I(x) = \exp[\int P(x)(dx)]$ , so that  $\frac{d}{dx}(Iy) = IQ$ .

ii.  $P(x, y)dx + Q(x, y)dy = 0$  is exact if  $\partial Q/\partial x = \partial P/\partial y$ .



## 5. INTEGRAL CALCULUS

(a) An important substitution:  $\tan(\theta/2) = t$  :

$$\sin \theta = 2t/(1+t^2), \quad \cos \theta = (1-t^2)/(1+t^2), \quad d\theta = 2dt/(1+t^2).$$

(b) Some indefinite integrals:

$$\int (a^2 - x^2)^{-1/2} dx = \sin^{-1} \left( \frac{x}{a} \right), \quad |x| < a.$$

$$\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1} \left( \frac{x}{a} \right) = \ln \left\{ \frac{x}{a} + \left( 1 + \frac{x^2}{a^2} \right)^{1/2} \right\}.$$

$$\int (x^2 - a^2)^{-1/2} dx = \cosh^{-1} \left( \frac{x}{a} \right) = \ln \left| \frac{x}{a} + \left( \frac{x^2}{a^2} - 1 \right)^{1/2} \right|.$$

$$\int (a^2 + x^2)^{-1} dx = \left( \frac{1}{a} \right) \tan^{-1} \left( \frac{x}{a} \right).$$

## 6. NUMERICAL METHODS

(a) Approximate solution of an algebraic equation:

If a root of  $f(x) = 0$  occurs near  $x = a$ , take  $x_0 = a$  and  
 $x_{n+1} = x_n - [f(x_n)/f'(x_n)], \quad n = 0, 1, 2 \dots$

(Newton Raphson method).

(b) Formulae for numerical integration: Write  $x_n = x_0 + nh$ ,  $y_n = y(x_n)$ .

i. Trapezium rule (1-strip):  $\int_{x_0}^{x_1} y(x) dx \approx (h/2) [y_0 + y_1]$ .

ii. Simpson's rule (2-strip):  $\int_{x_0}^{x_2} y(x) dx \approx (h/3) [y_0 + 4y_1 + y_2]$ .

(c) Richardson's extrapolation method: Let  $I = \int_a^b f(x) dx$  and let  $I_1, I_2$  be two

estimates of  $I$  obtained by using Simpson's rule with intervals  $h$  and  $h/2$ .

Then, provided  $h$  is small enough,

$$I_2 + (I_2 - I_1)/15,$$

is a better estimate of  $I$ .

## 7. LAPLACE TRANSFORMS

Function	Transform	Function	Transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t) dt$	$af(t) + bg(t)$	$aF(s) + bG(s)$
$df/dt$	$sF(s) - f(0)$	$d^2 f/dt^2$	$s^2 F(s) - sf(0) - f'(0)$
$e^{at} f(t)$	$F(s - a)$	$tf(t)$	$-dF(s)/ds$
$(\partial/\partial\alpha)f(t, \alpha)$	$(\partial/\partial\alpha)F(s, \alpha)$	$\int_0^t f(t) dt$	$F(s)/s$
$\int_0^t f(u)g(t-u)du$	$F(s)G(s)$		
1	$1/s$	$t^n (n = 1, 2, \dots)$	$n!/s^{n+1}, (s > 0)$
$e^{at}$	$1/(s - a), (s > a)$	$\sin \omega t$	$\omega/(s^2 + \omega^2), (s > 0)$
$\cos \omega t$	$s/(s^2 + \omega^2), (s > 0)$	$H(t - T) = \begin{cases} 0, & t < T \\ 1, & t > T \end{cases}$	$e^{-sT}/s, (s, T > 0)$

## 8. FOURIER SERIES

If  $f(x)$  is periodic of period  $2L$ , then  $f(x + 2L) = f(x)$ , and

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \text{ where}$$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx, \quad n = 0, 1, 2, \dots, \text{ and}$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx, \quad n = 1, 2, 3, \dots$$

Parseval's theorem

$$\frac{1}{L} \int_{-L}^L [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

# E1.14 (Maths 2)

## ~~Questions~~ Solutions

	EXAMINATION <del>QUESTIONS</del> / <u>SOLUTIONS</u> 2010-2011	Course  EE1(2) (1)																									
Question	Long question (Solutions)	Marks & seen/unseen																									
Parts		2																									
i)	$A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ $A \cap B = \{2, 3, 5, 7\}$	2																									
ii)	<table><tr><th><math>p</math></th><th><math>q</math></th><th><math>p \rightarrow q</math></th><th><math>(p \rightarrow q) \cup p</math></th><th><math>(p \wedge q) \cup (\bar{p} \wedge \bar{q})</math></th></tr><tr><td>T</td><td>T</td><td>T</td><td>T</td><td>T</td></tr><tr><td>T</td><td>F</td><td>F</td><td>T</td><td>F</td></tr><tr><td>F</td><td>T</td><td>T</td><td>T</td><td>F</td></tr><tr><td>F</td><td>F</td><td>T</td><td>T</td><td>T</td></tr></table>	$p$	$q$	$p \rightarrow q$	$(p \rightarrow q) \cup p$	$(p \wedge q) \cup (\bar{p} \wedge \bar{q})$	T	T	T	T	T	T	F	F	T	F	F	T	T	T	F	F	F	T	T	T	1 1 1 1
$p$	$q$	$p \rightarrow q$	$(p \rightarrow q) \cup p$	$(p \wedge q) \cup (\bar{p} \wedge \bar{q})$																							
T	T	T	T	T																							
T	F	F	T	F																							
F	T	T	T	F																							
F	F	T	T	T																							
iii)	a) false b) true c) true d) <del>true</del> false	1 1 1 1																									
iv)	Double integration by parts or $\cos(mx) \cos(nx) = \frac{1}{2} (\cos((m+n)x) + \cos((m-n)x))$ Without loss of generality $n \geq 0$ , so integral over $\cos((m+n)x)$ always vanishes: $\int_0^{2\pi} \cos(mx) \cos(nx) dx = \frac{1}{2} \int_0^{2\pi} \cos((m-n)x) dx$ $= \begin{cases} \pi & \text{for } m=n \\ 0 & \text{otherwise} \end{cases}$	1 1 1 1																									
v)	$\int_0^{2\pi} e^x \sinh(nx) dx = [e^x \sin(nx)]_0^{2\pi} - \int_0^{2\pi} e^x n \cos(nx) dx$ $= -[e^x n \cos(nx)]_0^{2\pi} - \int_0^{2\pi} e^x n^2 \sin(nx) dx \Rightarrow$	1 1																									
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# Examination Solutions

## Long questions

2010-2011

EE1(2)

(1)

ix)  $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$

$$A - \lambda \mathbb{I} = \begin{pmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{pmatrix}$$

$$\det(A - \lambda \mathbb{I}) = (1-\lambda)(2-\lambda) - 6$$

$$= \lambda^2 - 3\lambda - 4 = 0$$

Solutions by inspection or otherwise:  $\lambda = -1, 4$

x)  $\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \vec{r} = -\vec{r} \quad \vec{r} = \begin{pmatrix} x \\ y \end{pmatrix}$

$$x + 2y = -x$$

$$3x + 2y = -y$$

$$2x + 2y = 0$$

$$3x + 3y = 0$$

$$x = 1 \Rightarrow y = -1$$

1, 1

Solution: Any multiple of  $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ .



# EXAMINATION QUESTIONS/SOLUTIONS 2010-2011

Course

EEI(2)  
(2)

Question

2b

Marks &  
seen/unseen

Parts

1b.

A relation on a set  $S$  is a subset of  $S \times S$ .

reflexive iff  $(a, a) \in R$  for every  $a \in A$ .

symmetric iff  $(a, b) \in R$  whenever  $(b, a) \in R$ .

transitive iff whenever  $(a, b) \in R$  and  $(b, c) \in R$  then  $(a, c) \in R$

	Reflexive	Symmetric	Transitive
I	X	✓	✓
II	X	X	X
III	✓	✓	✓
IV	X	X	✓

4

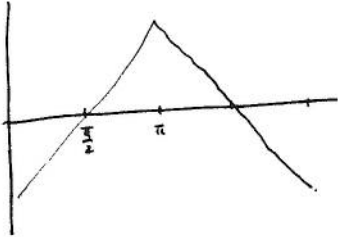
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Page number

	EXAMINATION <del>QUESTIONS</del> /SOLUTIONS 2010-2011	Course  EE1(2) <u>3</u>
Question	Fourier question	Marks & seen/unseen
Parts i)	$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(nx) dx; \quad b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(nx) dx$ <p>Graph of <math>f(x)</math></p>  <p><math>f(x)</math> is even <math>\Rightarrow b_n = 0</math>, and <math>a_0 = 0</math></p> $\pi a_n = \int_0^{\pi} (x - \frac{\pi}{2}) \cos(nx) dx + \int_{\pi}^{2\pi} (\frac{3\pi}{2} - x) \cos(nx) dx$ $= \left[ \frac{1}{n} (x - \frac{\pi}{2}) \sin(nx) \right]_0^{\pi} - \int_0^{\pi} \frac{1}{n} \sin(nx) dx$ $+ \left[ \frac{1}{n} (\frac{3\pi}{2} - x) \sin(nx) \right]_{\pi}^{2\pi} + \int_{\pi}^{2\pi} \frac{1}{n} \sin(nx) dx$ $= \left[ \frac{1}{n} \cos(nx) \right]_0^{\pi} - \left[ \frac{1}{n} \cos(nx) \right]_{\pi}^{2\pi}$ $= \frac{1}{n} \left\{ (1 - (-1)^n) - ((-1)^n - 1) \right\} = \begin{cases} 4/n^2 & \text{for } n \text{ odd} \\ 0 & \text{otherwise} \end{cases}$ <p><math>a_0</math> vanishes, <math>\frac{3\pi}{2} - x = \frac{\pi}{2} - (x - \pi)</math></p> $a_n = -\frac{4}{\pi n^2} \text{ for } n \text{ odd, all other coefficients vanish.}$ <p>ii) Parseval's theorem:</p> $\frac{1}{\pi} \int_0^{2\pi} f^2(x) dx = \sum_{n=1}^{\infty} a_n^2 + b_n^2 + \frac{a_0^2}{2}$ $\frac{1}{\pi} \int_0^{2\pi} f^2(x) dx = \frac{4}{\pi} \int_0^{\pi/2} x^2 dx = \frac{4}{3\pi} \left(\frac{\pi}{2}\right)^3 = \frac{\pi^2}{6} \checkmark$	<p>4</p> <p>Seen Similar (Througout)</p> <p>4</p> <p>4</p> <p>3</p>
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	EXAMINATION <del>QUESTIONS</del> /SOLUTIONS 2010-2011	Course EEL(2) (3)
Question		Marks & seen/unseen
Parts	$\frac{a_0^2}{2} + \sum a_n^2 + b_n^2 = \sum_{m=0}^{\infty} \frac{16}{\pi^2 (2m+1)^4}$ $\Rightarrow \sum_{m=0}^{\infty} \frac{1}{(2m+1)^4} = \frac{\pi^4}{96}$	3  3
	Setter's initials GP	Checker's initials Page number 52

4  
(i)

E E mid paper

EEI(2)

4

$$u = 2x \tan^{-1}\left(\frac{y}{x}\right) + y \ln(x^2 + y^2)$$

$$\frac{\partial u}{\partial x} = 2 \tan^{-1}\left(\frac{y}{x}\right) + \frac{2x}{1 + \left(\frac{y}{x}\right)^2} \left(-\frac{y}{x^2}\right) + \frac{y \cdot 2x}{x^2 + y^2} \quad \leftarrow 4$$

$$= 2 \tan^{-1}\left(\frac{y}{x}\right)$$

$$\frac{\partial u}{\partial y} = \frac{2x}{1 + \left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} + \ln(x^2 + y^2) + \frac{y \cdot 2y}{x^2 + y^2} \quad \leftarrow 4$$

$$= 2 + \ln(x^2 + y^2)$$

$$\therefore x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2x \tan^{-1}\left(\frac{y}{x}\right) + 2y + y \ln(x^2 + y^2) = u + 2y \quad 2$$

(ii)

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} \frac{\partial p}{\partial x} + \frac{\partial u}{\partial q} \frac{\partial q}{\partial x} = \frac{\partial u}{\partial p} + \frac{\partial u}{\partial q} \quad 2$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial p} \frac{\partial p}{\partial t} + \frac{\partial u}{\partial q} \frac{\partial q}{\partial t} = c \left( \frac{\partial u}{\partial q} - \frac{\partial u}{\partial p} \right)$$

$$\frac{\partial^2 u}{\partial x^2} = \left( \frac{\partial}{\partial p} + \frac{\partial}{\partial q} \right) \left( \frac{\partial u}{\partial p} + \frac{\partial u}{\partial q} \right) = \frac{\partial^2 u}{\partial p^2} + 2 \frac{\partial^2 u}{\partial p \partial q} + \frac{\partial^2 u}{\partial q^2} \quad 4$$

$$\frac{\partial^2 u}{\partial t^2} = c \left( \frac{\partial}{\partial q} - \frac{\partial}{\partial p} \right) c \left( \frac{\partial u}{\partial q} - \frac{\partial u}{\partial p} \right) = c^2 \left( \frac{\partial^2 u}{\partial p^2} - 2 \frac{\partial^2 u}{\partial p \partial q} + \frac{\partial^2 u}{\partial q^2} \right)$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} \Leftrightarrow 4 \frac{\partial^2 u}{\partial p \partial q} = 0 \quad 2$$

$$\text{Thus } u = f(p) + g(q) \quad [\text{f and p are arbitrary functions}]$$

$$\text{Thus } u = f(x - ct) + g(x + ct) \quad 2$$

forward travelling wave

backward travelling wave

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Q5

EE and paper

EEI(2) p1/2

$$A^2 = \begin{pmatrix} -1 & -2 & 0 \\ -1 & 0 & 0 \\ -2 & -2 & 1 \end{pmatrix} \begin{pmatrix} -1 & -2 & 0 \\ -1 & 0 & 0 \\ -2 & -2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 2 & 0 \\ 1 & 2 & 0 \\ 2 & 2 & 1 \end{pmatrix}$$

4 ✓

$$A^3 = \begin{pmatrix} -1 & -2 & 0 \\ -1 & 0 & 0 \\ -2 & -2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 2 & 0 \\ 1 & 2 & 0 \\ 2 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -5 & -6 & 0 \\ -3 & -2 & 0 \\ -6 & -6 & 1 \end{pmatrix}$$

4 ✓

$$A^3 - 3A + 2I = \begin{pmatrix} -5 & -6 & 0 \\ -3 & -2 & 0 \\ -6 & -6 & 1 \end{pmatrix} - 3 \begin{pmatrix} -1 & -2 & 0 \\ -1 & 0 & 0 \\ -2 & -2 & 1 \end{pmatrix} + 2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

4 ✓

$$A^2 + aA + bI = \begin{pmatrix} 3 & 2 & 0 \\ 1 & 2 & 0 \\ 2 & 2 & 1 \end{pmatrix} + a \begin{pmatrix} -1 & -2 & 0 \\ -1 & 0 & 0 \\ -2 & -2 & 1 \end{pmatrix} + b \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\left. \begin{array}{l} \text{11 Element in } 3 - a + b = 0 \\ \text{12 Element in } 2 - 2a = 0 \end{array} \right\}$$

$$\Rightarrow a = 1, b = -2$$

4 ✓

Check all other elements are then 0

$$\text{ie } A^2 + A - 2I = 0$$

Multiply by  $A^{-1}$  and manipulate  
to get  $A^{-1} = \frac{1}{2}(A + I)$

$$= \begin{pmatrix} 0 & -1 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -1 & -1 & 1 \end{pmatrix}$$

✓  
4

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Q 6

EE and  
Paper

EE I (2)

6

(i) Divide eqn. by  $(x+1)$  to get standard form;

$$\frac{dy}{dx} = \frac{x}{x+1} \quad y = \frac{e^x}{x+1}$$

Multiply by integrating factor

$$I = \exp\left[-\int \frac{x}{x+1} dx\right] = e^{-x} (x+1) \quad 2$$

$$\text{to get: } e^{-x} (x+1) \frac{dy}{dx} - x e^{-x} y = 1$$

$$\text{i.e. } \frac{d}{dx} [(x+1) e^{-x} y] = 1 \quad 2$$

$$\text{i.e. } (x+1) e^{-x} y = x + C \quad 2$$

Condition implies  $C = 1$ 

$$\text{Thus } y = \frac{x-1}{x+1} e^x \quad 2$$

(ii) Exact if  $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$  2

$$P = 2xy + \frac{1}{3}y^3, \quad Q = x^2 + xy^2 \quad 2$$

$$\frac{\partial P}{\partial y} = 2x + y^2, \quad \frac{\partial Q}{\partial x} = 2x + y^2 \quad 2$$

Equal thus exact 2Solution is  $F(x, y) = C$ 

$$\text{where } \frac{\partial F}{\partial x} = P = 2xy + \frac{1}{3}y^3 \quad \text{--- (1)} \quad 2$$

$$\text{and } \frac{\partial F}{\partial y} = Q = x^2 + xy^2 \quad \text{--- (2)} \quad 2$$

$$\text{(2)} \Rightarrow F = x^2y + \frac{1}{3}xy^3 + f(x) \quad \text{--- (3)} \quad 2$$

Subst. (3) in (1) to get

$$\frac{df}{dx} = 0 \Rightarrow f(x) = \text{const}$$

$$\therefore \text{solution is } x^2y + \frac{1}{3}xy^3 = C \quad 2$$

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