Final

Imperial College London

[E1.14 (Maths 2) 2010]

B.ENG. AND M.ENG. EXAMINATIONS 2010

PART I : MATHEMATICS 2 (ELECTRICAL ENGINEERING)

Date Thursday 3rd June 2010 10.00 am - 1.00 pm

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

Answer EIGHT questions.

A mathematical formulae sheet is provided.

[Before starting, please make sure that the paper is complete; there should be 7 pages, with a total of 10 questions. Ask the invigilator for a replacement if your copy is faulty.]

1. Let the function $f : [0, \pi] \to [-1, 1]$ be defined by $f(x) = \sin(x)$.

   (i) Give the definition of domain, co-domain and range of *any* function.

   (ii) Write down the domain, co-domain and range of this particular f.

   (iii) Explain why f does not have a well-defined inverse.

   (iv) Define a new function g that is the same as f but on a restricted domain, in such a way that g has a well-defined inverse. Make the domain as large as possible while maintaining the property that the inverse is well-defined. Write down the definition of g^{-1}, the inverse of g, and specify the domain, co-domain and range of g^{-1}.

2. Let

$$f(x) = \frac{x^2 - 2x + 2}{x - 1} \equiv \frac{(x-1)^2 + 1}{x - 1} .$$

   (i) Show that $f'(x)$ vanishes at two locations, a and b say, (with $a < b$), and find the values of a, b, $f(a)$, and $f(b)$.

   (ii) Determine the point c where $f(x)$ is undefined and show that $a < c < b$.

   (iii) Find the sign of $f'(x)$ in each of the following regions:

   (a) $x < a$;

   (b) $a < x < c$;

   (c) $c < x < b$;

   (d) $x > b$.

   (iv) Are there any values of x for which $f(x) = 0$?
   Explain your answer.

   (v) Determine the limiting behaviour of $f(x)$ as $x \to \pm\infty$.

   (vi) Determine the behaviour of $f(x)$ as $x \to c$ from the left, and $x \to c$ from the right.

   (vii) Classify the stationary points found in part (i).

   (viii) With the aid of the information obtained in parts (i) to (vii), sketch the graph of $f(x)$.

PLEASE TURN OVER

3. (i) Given a function $f(x)$, give the definition of the derivative of f at a point $x = a$.

(ii) Use the above definition to find, from first principles, the first derivative of the function $f(x) = 1/x$.

(iii) Using the product and quotient rules where appropriate, express the derivative of

$$f(x) = \frac{\sin(x^2)\,\sin^2(x)}{1+\sin(x)}$$

in the form

$$f'(x) = \frac{g(x)}{(1+\sin(x))^2}\ ,$$

where $g(x)$ is a function to be determined.

(iv) A circular object has a radius that varies with time. If we know that when its radius is 6, the rate of change of radius is 4, find the rate of change of the *area* when the radius is 6.

4. (i) Give the definition of the integral of $f(x)$ between a and b, expressed as the limit of the sum of n equal-width Riemann rectangles.

(ii) Use the above limit definition to compute, from first principles (i.e. as the limit of a sum)

$$\int_0^b x\,dx\ .$$

(iii) Find all points where the curves

$$f(x) = x^2 \text{ and } g(x) = 2 + \frac{1}{2}x^2$$

intersect, and then sketch both on the same graph.

Use standard integration techniques to compute the area of the region bounded by these two curves.

(*It is not necessary to integrate from first principles.*)

5. (i) Determine if the following series converge or not.

Explain your reasoning in each case.

Where necessary you may assume that $\sum_{n=1}^{\infty} \frac{1}{n}$ *diverges and* $\sum_{n=1}^{\infty} \frac{1}{2^n}$ *converges.*

(a)
$$\sum_{n=1}^{\infty} \frac{n^2}{n!} \; ;$$

(b)
$$\sum_{n=2}^{\infty} \frac{1}{\ln(n)} \; ;$$

(c)
$$\sum_{n=2}^{\infty} \frac{1}{[\ln(n)]^n} \; .$$

(ii) Compute the Maclaurin Series (Taylor series at $x = 0$) for

$$f(x) \;=\; \frac{1}{x-a} \,, \qquad a \neq 0 \; .$$

PLEASE TURN OVER

6. (i) Write $\sin(nx)$ and $\cos(nx)$ in terms of e^{inx} and e^{-inx}.

(ii) Let $f_n(x) = e^{inx}$, where n is an integer.

Show that

(a)
$$\int_{-\pi}^{\pi} f_n(x)\, f_m(x)\, dx \;=\; 0 \;\text{ if }\; n \neq -m \;;$$

(b)
$$\int_{-\pi}^{\pi} |\, f_n(x)\,|^2\, dx \;=\; 2\pi \;.$$

(iii) Using the result of parts (i) and (ii), or otherwise, deduce the values of

(a)
$$\int_{-\pi}^{\pi} \cos(mx)\, \sin(nx)\, dx \;;$$

(b)
$$\int_{-\pi}^{\pi} \cos(mx)\, \cos(nx)\, dx \;;$$

(c)
$$\int_{-\pi}^{\pi} \sin(mx)\, \sin(nx)\, dx \;,$$

where m and n are integers, distinguishing between the cases $n = -m$, $n = m$ and otherwise.

7. (i) Consider the change of variables

$$u = x + y, \quad v = x - y .$$

Express $\dfrac{\partial}{\partial x}$ and $\dfrac{\partial}{\partial y}$ in terms of $\dfrac{\partial}{\partial u}$ and $\dfrac{\partial}{\partial v}$.

Hence show that if Ψ satisfies

$$\frac{\partial^2 \Psi}{\partial x^2} = \frac{\partial^2 \Psi}{\partial y^2}$$

then if Ψ is expressed in terms of u and v

$$\frac{\partial^2 \Psi}{\partial u \, \partial v} = 0 .$$

What is the general solution of this equation?

(ii) The volume of a box of sides x, y, z each is to be calculated approximately by estimating the length of each side. If we want to know the volume with a maximum error of a 1%, what is the maximum (equal) error we can make in measurements of x, y, z?

8. Sketch the graph of the function

$$f(x) = x - 2 + \ln x , \quad x > 0 .$$

Find consecutive integers either side of the root. Use the average value of these integers as an initial guess in Newton's method. Find the root correct to 4 decimal places.

PLEASE TURN OVER

9. (i) Find the solution of

$$\frac{dy}{dx} = \frac{y(y-2)}{x(y-1)}$$

subject to $y = 3$ when $x = 1$.

(ii) Write down the condition that the equation

$$P(x, y) + Q(x, y)\,\frac{dy}{dx} = 0$$

is exact.

Show that

$$y + x\ln x\,\frac{dy}{dx} = 0$$

is not exact.

Show that it can be made exact by multiplying by a suitable function of x.

Hence show that the general solution is

$$y = \frac{C}{\ln x},$$

where C is a constant.

10. The function $f(x)$ is defined by

$$f(x) = x^2, \qquad -\pi \le x \le \pi .$$

Express this function in the real Fourier series form:

$$f(x) = c_0 + 2\,\mathrm{Re}\sum_{n=1}^{\infty} c_n\, e^{inx} .$$

Determine the coefficients c_n, for $n \ge 0$.

By considering the Fourier series at an appropriate value of x, deduce that π can be evaluated using the formula

$$\pi = \sqrt{6\sum_{n=1}^{\infty}\frac{1}{n^2}} .$$

END OF PAPER

MATHEMATICS DEPARTMENT

MATHEMATICAL FORMULAE

1. VECTOR ALGEBRA

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1, a_2, a_3)$$

Scalar (dot) product: $\mathbf{a}\,.\,\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

Vector (cross) product:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Scalar triple product:

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a}\,.\,\mathbf{b} \times \mathbf{c} = \mathbf{b}\,.\,\mathbf{c} \times \mathbf{a} = \mathbf{c}\,.\,\mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product: $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c}\,.\,\mathbf{a})\mathbf{b} - (\mathbf{b}\,.\,\mathbf{a})\mathbf{c}$

2. SERIES

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \ldots \quad (\alpha \text{ arbitrary, } |x| < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n\frac{x^{2n}}{(2n)!} + \ldots,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + \ldots,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^n\frac{x^{n+1}}{(n+1)} + \ldots (-1 < x \leq 1)$$

3. TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin(a+b) = \sin a \cos b + \cos a \sin b\,;$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b\,.$$

$$\cos iz = \cosh z\,; \quad \cosh iz = \cos z\,; \quad \sin iz = i\sinh z\,; \quad \sinh iz = i\sin z\,.$$

4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f\,D^n g + \binom{n}{1} Df\,D^{n-1}g + \ldots + \binom{n}{r} D^r f\,D^{n-r}g + \ldots + D^n f\,g\,.$$

(b) Taylor's expansion of $f(x)$ about $x = a$:

$$f(a+h) = f(a) + hf'(a) + h^2 f''(a)/2! + \ldots + h^n f^{(n)}(a)/n! + \epsilon_n(h)\,,$$

where $\epsilon_n(h) = h^{n+1} f^{(n+1)}(a + \theta h)/(n+1)!, \quad 0 < \theta < 1\,.$

(c) Taylor's expansion of $f(x, y)$ about (a, b):

$$f(a+h, b+k) = f(a,b) + [hf_x + kf_y]_{a,b} + 1/2!\,\left[h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}\right]_{a,b} + \ldots$$

(d) Partial differentiation of $f(x, y)$:

i. If $y = y(x)$, then $f = F(x)$, and $\dfrac{dF}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dx}$.

ii. If $x = x(t)$, $y = y(t)$, then $f = F(t)$, and $\dfrac{dF}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$.

iii. If $x = x(u, v)$, $y = y(u, v)$, then $f = F(u, v)$, and

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}, \quad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$

(e) Stationary points of $f(x, y)$ occur where $f_x = 0$, $f_y = 0$ simultaneously.
Let (a, b) be a stationary point: examine $D = [f_{xx}f_{yy} - (f_{xy})^2]_{a,b}$.
If $D > 0$ and $f_{xx}(a, b) < 0$, then (a, b) is a maximum;
If $D > 0$ and $f_{xx}(a, b) > 0$, then (a, b) is a minimum;
If $D < 0$ then (a, b) is a saddle-point.

(f) Differential equations:

i. The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating factor $I(x) = \exp[\int P(x)(dx)]$, so that $\dfrac{d}{dx}(Iy) = IQ$.

ii. $P(x, y)dx + Q(x, y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y$.

5. INTEGRAL CALCULUS

(a) An important substitution: $\tan(\theta/2) = t$:
$\sin\theta = 2t/(1+t^2)$, $\cos\theta = (1-t^2)/(1+t^2)$, $d\theta = 2\,dt/(1+t^2)$.

(b) Some indefinite integrals:

$$\int (a^2 - x^2)^{-1/2} dx = \sin^{-1}\left(\frac{x}{a}\right), \quad |x| < a .$$

$$\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1}\left(\frac{x}{a}\right) = \ln\left\{\frac{x}{a} + \left(1 + \frac{x^2}{a^2}\right)^{1/2}\right\} .$$

$$\int (x^2 - a^2)^{-1/2} dx = \cosh^{-1}\left(\frac{x}{a}\right) = \ln\left|\frac{x}{a} + \left(\frac{x^2}{a^2} - 1\right)^{1/2}\right| .$$

$$\int (a^2 + x^2)^{-1} dx = \left(\frac{1}{a}\right)\tan^{-1}\left(\frac{x}{a}\right) .$$

6. NUMERICAL METHODS

(a) Approximate solution of an algebraic equation:

If a root of $f(x) = 0$ occurs near $x = a$, take $x_0 = a$ and
$x_{n+1} = x_n - [f(x_n)/f'(x_n)]$, $n = 0, 1, 2 \ldots$

(Newton Raphson method).

(b) Formulae for numerical integration: Write $x_n = x_0 + nh$, $y_n = y(x_n)$.

i. Trapezium rule (1-strip): $\int_{x_0}^{x_1} y(x)dx \approx (h/2)[y_0 + y_1]$.

ii. Simpson's rule (2-strip): $\int_{x_0}^{x_2} y(x)dx \approx (h/3)[y_0 + 4y_1 + y_2]$.

(c) Richardson's extrapolation method: Let $I = \int_a^b f(x)dx$ and let I_1, I_2 be two estimates of I obtained by using Simpson's rule with intervals h and $h/2$. Then, provided h is small enough,

$$I_2 + (I_2 - I_1)/15,$$

is a better estimate of I .

7. LAPLACE TRANSFORMS

Function	Transform	Function	Transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)dt$	$af(t) + bg(t)$	$aF(s) + bG(s)$
df/dt	$sF(s) - f(0)$	d^2f/dt^2	$s^2F(s) - sf(0) - f'(0)$
$e^{at}f(t)$	$F(s-a)$	$tf(t)$	$-dF(s)/ds$
$(\partial/\partial\alpha)f(t,\alpha)$	$(\partial/\partial\alpha)F(s,\alpha)$	$\int_0^t f(t)dt$	$F(s)/s$
$\int_0^t f(u)g(t-u)du$	$F(s)G(s)$		
1	$1/s$	$t^n (n = 1, 2 \ldots)$	$n!/s^{n+1}$, $(s > 0)$
e^{at}	$1/(s-a)$, $(s > a)$	$\sin\omega t$	$\omega/(s^2 + \omega^2)$, $(s > 0)$
$\cos\omega t$	$s/(s^2 + \omega^2)$, $(s > 0)$	$H(t-T) = \begin{cases} 0, & t < T \\ 1, & t > T \end{cases}$	e^{-sT}/s, $(s, T > 0)$

8. FOURIER SERIES

If $f(x)$ is periodic of period $2L$, then $f(x + 2L) = f(x)$, and

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \text{ where}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad n = 0, 1, 2, \ldots, \text{ and}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1, 2, 3, \ldots$$

Parseval's theorem

$$\frac{1}{L}\int_{-L}^{L} [f(x)]^2\, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right) .$$

EXAMINATION QUESTIONS/SOLUTIONS 2009-2010

EE1 - Maths Paper 2 (E1.14)

Course: EE I (2) ①

Question 1

Marks & seen/unseen

Parts

i) Domain: the set X of numbers for which a function is defined. — 2

Range: $\{f(x) : x \in X\}$ — 2

Co-domain: in $f: X \to Y$, Y is the co-domain. It is the set of points containing the range of f; it may be a superset of the range — 2.

ii) Domain of f: $[0, \pi]$ Range: $[0,1]$. — 3

Co-domain: $[-1, 1]$

iii) Because $\exists$ more than one $x \in [0, \pi]$ with $\sin(x) \in [0,1]$. — 4

iv) $g : [0, \frac{\pi}{2}] \to [0,1]$, $g(x) = \sin(x)$ — 2

$g^{-1} : [0,1] \to [0, \frac{\pi}{2}]$, $g^{-1}(x) = \sin^{-1}(x) \equiv \arcsin(x)$ — 2

Domain of g^{-1}: $[0,1]$, Range $= [0, \frac{\pi}{2}]$ — 3

Co-domain: $[0, \frac{\pi}{2}]$

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	EXAMINATION QUESTIONS/SOLUTIONS 2009-2010	Course EE I (2) 2
Question 2		Marks & seen/unseen
Parts		
i)	$$f'(x) = \frac{(x-1)(2x-2) - (x^2-2x+2)}{(x-1)^2}$$ $$= \frac{2x^2-4x+2-x^2+2x-2}{(x-1)^2}$$ $$= \frac{x^2-2x}{(x-1)^2} = \frac{x(x-2)}{(x-1)^2}$$	1
	$\Rightarrow f'(x)$ vanishes at $a=0$, $b=2$. $f(0)=-2$, $f(2)=2$.	2
ii)	$c=1$ since the denominator is zero at $x=c=1$.	1
iii)	a) $x<0 \Rightarrow f'(x)>0$	1
	b) $0<x<1 \Rightarrow f'(x)<0$	1
	c) $1<x<2 \Rightarrow f'(x)<0$	1
	d) $x>2 \Rightarrow f'(x)>0$	1
iv)	The numerator of $f(x)$ is $(x-1)^2+1$, which is always positive, so $f(x)$ never vanishes - it never crosses the x-axis	1
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EXAMINATION QUESTIONS/SOLUTIONS 2009-2010

Course: EE I (2)

2

Question: 2

Marks & seen/unseen

Parts

v) as $x \to \infty$, $f(x) \to \frac{(x-1)^2}{x-1} \to x-1 > 0 \to \infty$ — 1

as $x \to -\infty$, $f(x) \to \frac{(x-1)^2}{x-1} \to x-1 < 0 \to -\infty$ — 1

vi) As $x \to 1$ from the left,

$$f(x) = \frac{(x-1)^2+1}{x-1} \to \frac{0+1}{0_-} \to -\infty$$ — 1

As $x \to 1$ from the right,

$$f(x) = \frac{(x-1)^2+1}{x-1} \to \frac{0+1}{0_+} \to +\infty$$ — 1

vii)

$$f''(x) = \frac{(x-1)^2(2x-2) - x(x-2)\cdot 2(x-1)}{(x-1)^4}$$ — 1

$f''(0) = \frac{-2}{1} < 0$ so $x=0$ is a MAX — 1

$f''(2) = \frac{2-0}{1} > 0$ so $x=2$ is a MIN. — 1

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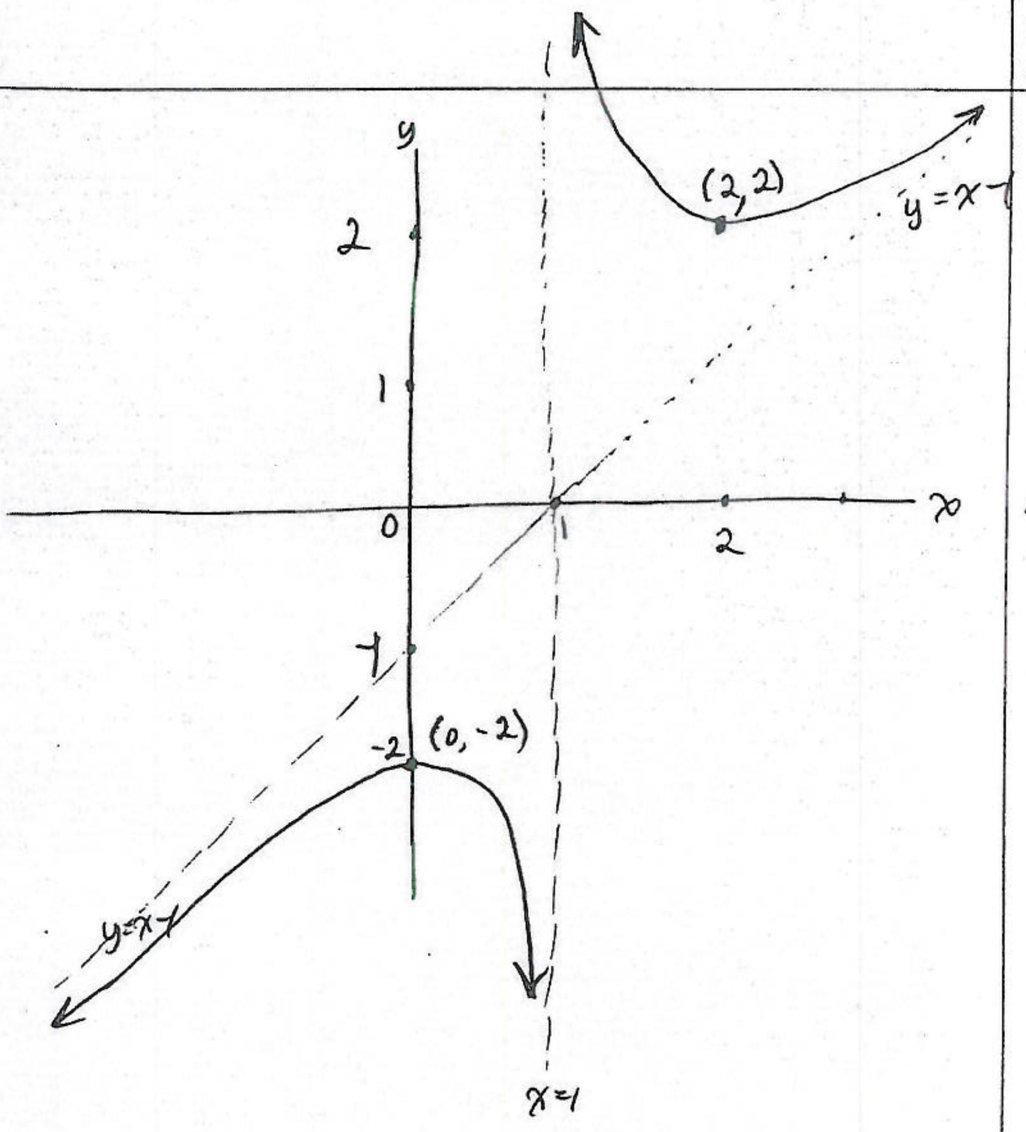
Checker's initials

Page number 2

	EXAMINATION QUESTIONS/SOLUTIONS 2009-2010	Course EE I (2) 2
Question 2		Marks & seen/unseen
Parts viii)		3
	Setter's initials gr. M	Checker's initials	Page number 3

Question 3

Marks & seen/unseen

Parts

i) $$\lim_{\delta \to 0} \frac{f(a+\delta) - f(a)}{\delta}.$$ 2

The derivative at $x = a$ exists iff the above limit exists. 3

ii) $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$ 1

$$= \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$ 1

$$= \lim_{h \to 0} \frac{\left(\frac{x-(x+h)}{x(x+h)}\right)}{h}$$ 1

$$= \lim_{h \to 0} \frac{-h}{h} \cdot \frac{1}{x(x+h)} = -\frac{1}{x^2}$$ 2

$= -x^{-2}$ (this last step is not mandatory)

EXAMINATION QUESTIONS/SOLUTIONS 2009-2010

Course: EE I (2) (3)

Question 3

Marks & seen/unseen

Parts

iii) $f'(x) =$ (by quotient rule) Y

$$\frac{(1+\sin x)\frac{d}{dx}(\sin x^2 \sin^2 x) - \sin x^2 \sin^2 x \cos x}{(1+\sin x)^2}$$ 2

Now apply product rule to $\frac{d}{dx}\sin x^2 \sin^2 x$

$$f'(x) = \frac{(1+\sin x)\left[2x\cos x^2 \cdot \sin^2 x + \sin x^2 \cdot 2\sin x \cos x\right] - Y}{(1+\sin x)^2}$$ 3

iv) $A = \pi r^2 \Rightarrow A'(t) = 2\pi r(t)\, r'(t)$ 2

(by the chain rule).

When $r(t) = 6$, $r'(t) = 4$, so 3

$A'(t) = 2\pi \cdot 6 \cdot 4 = 48\pi$.

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Course EE1 (2) (4)

Question 4

Marks & seen/unseen

Parts

i) $\int_a^b f(x)\,dx = \lim_{n\to\infty} \frac{(b-a)}{n} \sum_{k=0}^{n-1} f\left(a+\frac{k}{n}(b-a)\right)$ 5

(Note that other variations are acceptable, eg. let $\Delta x = \frac{(b-a)}{n}$ and use

$\lim_{\Delta x\to 0} \Delta x \sum_{k=0}^{n-1} f(a+k\Delta x)$, etc.)

ii) $f(x)=x$, so $\int_0^b x\,dx = \lim_{n\to\infty} \frac{(b-a)}{n} \sum_{k=0}^{n-1} f\left(\frac{kb}{n}\right)$ 1

$= \lim_{n\to\infty} \frac{b}{n} \sum_{k=0}^{n-1} \frac{kb}{n} = \lim_{n\to\infty} \frac{b^2}{n^2}\left[\sum_{k=0}^{n-1} k\right]$ 2

$= \lim_{n\to\infty} \frac{b^2}{n^2}\left[\frac{(n-1)n}{2}\right] = \frac{b^2}{2}.$ 2.

iii) They intersect when $f(x)=g(x)$, 1

so $x^2 = 2+\frac{1}{2}x^2 \Rightarrow \frac{1}{2}x^2 = 2$, 1

or $x^2=4 \Rightarrow x=\pm 2$. At 1

those points $f(-2)=g(-2)=4$, 1

$f(2)=g(2)=4$. 1

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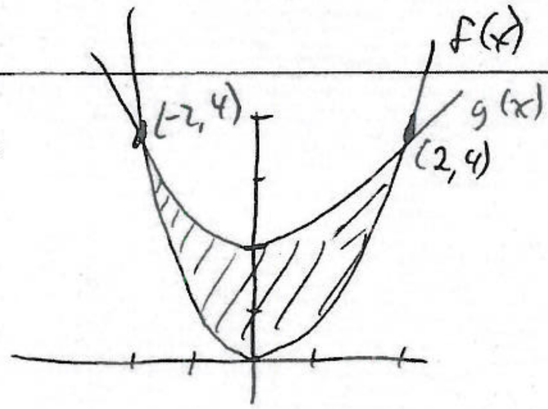
Checker's initials

Question
4

Marks & seen/unseen

Parts

Sketch:



2

The area between the two curves is

$$\int_{-2}^{2} g(x) - f(x)\,dx = \int_{-2}^{2} \frac{x^2}{2} + 2 - x^2\,dx$$

3

$$= \int_{-2}^{2} 2 - \frac{1}{2}x^2\,dx = \left[2x - \frac{1}{6}x^3\right]_{-2}^{2}$$

$$= \left[4 - \frac{8}{6}\right] - \left[-4 + \frac{8}{6}\right] = 8 - \frac{16}{6} = \frac{48-16}{6}$$

$$= \frac{32}{6} = \frac{16}{3}.$$

1

	EXAMINATION QUESTIONS/SOLUTIONS 2009-2010	Course EE I (2) (5)
Question 5		Marks & seen/unseen
Parts		
i) a)	Use ratio test: $$\lim_{n\to\infty} \frac{(n+1)^2/(n+1)!}{n^2/n!} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^2 \frac{1}{n+1} = 0$$ $\Rightarrow$ converges.	2
b)	Divergent by comparison test: $\frac{1}{\ln(n)} > \frac{1}{n}$ and $\sum \frac{1}{n}$ diverges.	2
c)	Note that $(\ln n) > 2$ for $n \gtrsim 9$, so $\frac{1}{(\ln n)^n} < \frac{1}{2^n}$ for $n \geq 9$. Since $\sum \frac{1}{2^n}$ converges, so does $\sum \frac{1}{(\ln n)^n}$, by comparison test.	5
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	EXAMINATION QUESTIONS/SOLUTIONS 2009-2010	Course EE I (2) 5
Question 5	(Series Con't)	Marks & seen/unseen

Parts

(ii) ~~Compute the MacLaurin series (Taylor series at x=0) for the following~~

~~$f(x) = \frac{1}{x-a}$, $a \neq 0$~~

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(0)\frac{x^n}{n!} = f(0) + f'(0)x + f''(0)\frac{x^2}{2} + \cdots$$ 1

Now, $f'(x) = \frac{d}{dx}(x-a)^{-1} = -(x-a)^{-2}$ 1

and $f''(x) = -(-2)(x-a)^{-3} = 2(x-a)^{-3}$ 1

and $f'''(x) = -3\cdot 2(x-a)^{-4}$; $f^{(4)}(x) = 4\cdot 3\cdot 2\cdot (x-a)^{-5}$ 1

So $f(0) = \frac{1}{-a}$, $f'(0) = -(-a)^{-2} = \frac{-1}{a^2}$ 3

$f''(0) = 2(-a)^{-3} = \frac{2}{-a^3}$, $f'''(0) = \frac{-3\cdot 2}{(-a)^4}$ 2

$f^{(4)}(0) = \frac{4\cdot 3\cdot 2}{(-a)^5} = \frac{4\cdot 3\cdot 2}{-a^5}$. ~~So every term is negative~~, and the series is 1

$$f(x) = \frac{-1}{a} - \frac{x}{a^2} - \frac{x^2}{a^3} - \frac{x^3}{a^4} - \cdots$$ 1

$$= -\sum_{n=0}^{\infty} \frac{x^n}{a^{(n+1)}}$$ (last step not mandatory)

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Question 5

Marks & seen/unseen

Parts

(ii) ($\boxed{\text{Alternative}}$ Solution for the Taylor Series) If you are clever, you may note that

$$f(x) = \frac{-1}{a}\left(1-\frac{x}{a}\right), \text{ and } \left(1-\frac{x}{a}\right) \text{ may}$$ 5

be expanded so that

$$f(x) = \frac{-1}{a}\left(1 + \frac{x}{a} + \frac{x^2}{a^2} + \cdots + \frac{x^n}{a^n} + \cdots\right)$$

$$= \frac{-1}{a} - \frac{x}{a^2} - \frac{x^2}{a^3} - \frac{x^3}{a^4} - \cdots \frac{x^n}{a^{n+1}} - \cdots$$

By the uniqueness of the Taylor Series, this must be the Taylor Expansion of $f(x)$ around $x=0$ because it is of the form 4

$$f(x) = \sum_{n=0}^{\infty} a_n x^n,$$ 1

which is exactly the form a Taylor series takes, where $a_n = f^{(n)}(0)/n!$ 1

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Course EE I(2) (6)

Question 6

Marks & seen/unseen

Parts

i) $\sin(nx) = \frac{e^{inx} - e^{-inx}}{2}$, $\cos(nx) = \frac{e^{inx} + e^{-inx}}{2}$ 2.

ii) a) $\int_{-\pi}^{\pi} e^{inx} \cdot e^{imx} dx = \int_{-\pi}^{\pi} e^{i(n+m)x} dx$

$= \frac{1}{i(n+m)} \left[e^{i(n+m)\cdot} - e^{-i(n+m)} \right] = \frac{2i}{i(n+m)} \sin \pi (n+m)$ 3

$= 0$ if $n \neq -m$ $\because \sin k\pi = 0$

b) $\int_{-\pi}^{\pi} |f_n(x)|^2 dx = \int_{-\pi}^{\pi} |e^{inx}|^2 dx$ 3

$= \int_{-\pi}^{\pi} e^{inx} \cdot e^{-inx} dx = \int_{-\pi}^{\pi} 1 \, dx = 2\pi.$

iii) a) $\frac{1}{4}\int_{-\pi}^{\pi} (e^{imx} + e^{-imx})(e^{inx} - e^{-inx}) dx$

$= \frac{1}{4}\int_{-\pi}^{\pi} e^{i(n+m)x} - e^{i(m-n)x} + e^{i(n-m)x} - e^{-i(m+n)x} dx$ 3

$= \frac{1}{4}\int_{-\pi}^{\pi} e^{i(n+m)x} - e^{-i(n+m)x} dx + \frac{1}{4}\int_{\pi}^{\pi} e^{i(n-m)x} - e^{-i(n-m)x} dx$

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Parts

If $n=-m$, this is

$$\underbrace{\frac{1}{4}\int_{-\pi}^{\pi}(1-1)\,dx}_{=\ 0} + \underbrace{\frac{1}{4}\int_{-\pi}^{\pi} e^{2ni x} - e^{-2ni x}\,dx}_{=0 \text{ by Part i) a)}}$$ 3

It is similarly zero if $n=m$.

Otherwise it is still zero, by Part i) a).

Thus, it is always zero

b) $$\frac{1}{4}\int_{-\pi}^{\pi}(e^{imx}+e^{-imx})(e^{inx}+e^{-inx})\,dx$$

$$= \frac{1}{4}\int_{-\pi}^{\pi} e^{i(n+m)x} + e^{i(m-n)x} + e^{i(n-m)x} + e^{-i(n+m)x}\,dx$$

$$= \frac{1}{4}\int_{-\pi}^{\pi} e^{i(m+n)x} + e^{-i(n+m)x}\,dx + \frac{1}{4}\int_{-\pi}^{\pi} e^{i(m-n)x} + e^{-i(m-n)x}\,dx$$ 3

If $m=-n$, this is

$\frac{1}{4}[2\pi + 2\pi] + 0 = \pi.$

If $m=n$, this is $0 + \frac{1}{4}[2\pi + 2\pi] = \pi$

otherwise it is $0+0=0$.

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Course EE I (2)

6

Question 6

Marks & seen/unseen

Parts

c) $\frac{1}{4}\int_{-\pi}^{\pi}(e^{imx}-e^{-imx})(e^{inx}-e^{-inx})dx$

$=\frac{1}{4}\int_{-\pi}^{\pi}e^{i(m+n)x}-e^{i(m-n)x}-e^{i(n-m)x}+e^{-i(n+m)x}dx$ 3

$=\frac{1}{4}\int_{-\pi}^{\pi}e^{i(m+n)x}+e^{-i(m+n)x}dx-\frac{1}{4}\int_{-\pi}^{\pi}e^{i(m-n)x}+e^{-i(m-n)x}dx$

If $n=m$ this is $0-\pi=-\pi$

If $n=-m$ $\pi-0=\pi$

Otherwise $0-0=0$.

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EXAMINATION QUESTIONS/SOLUTIONS 2009-2010

Course EE1(2) 7

Question 7

Solution

Marks & seen/unseen

Parts

(i) Using $\frac{\partial}{\partial x} \to \frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v}$, $\frac{\partial}{\partial y} \to \frac{\partial u}{\partial y}\frac{\partial}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial}{\partial v}$ 2

$\frac{\partial}{\partial x} \to \frac{\partial}{\partial u} + \frac{\partial}{\partial v}$, $\frac{\partial}{\partial y} \to -\frac{\partial}{\partial u} + \frac{\partial}{\partial v}$ 1

$\Psi_{xx} \to \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)^2 \Psi = \Psi_{uu} + \Psi_{vv} + 2\Psi_{uv}$ 2

$\Psi_{yy} \to \left(-\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)^2 \Psi = \Psi_{uu} + \Psi_{vv} - 2\Psi_{uv}$ 2

$\therefore \Psi_{xx} - \Psi_{yy} \to 4\Psi_{uv}$ 1

$\therefore \Psi_{uv} = 0 \Rightarrow \Psi_u = f(u)$

$\Rightarrow \Psi = \int f(u) + g(v)$

$= F(u) + G(v)$ 4

$= F(x+y) + G(x-y)$

where F, G arbitrary functions

(ii) $V = xyz$, $dV = yz\,\delta x + xz\,\delta y + xy\,\delta z$ 2

$\therefore \frac{\delta V}{V} = \frac{\delta x}{x} + \frac{\delta y}{y} + \frac{\delta z}{z}$ 2

$\therefore \left|\frac{\delta V}{V}\right| \leq 3\left|\frac{\delta x}{x}\right|$ 2

so need $\left|\frac{\delta x}{x}\right| \leq \frac{1\%}{3} = .333\%$. 2

Total 20

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Course: EE 1(2) (8)

Solution. Question 8.

Marks & seen/unseen

Parts

~~Sketch the graph of $x-2+\ln x$ for $x>0$, and show~~

use this to sketch $f(x)$ (below)

$f(x) = x-2+\ln x$, $x>0$.

$x \to 0$, $f \to -\infty$.

$x \to \infty$, $f \sim x$

$f'(x) = 1 + 1/x$ so no turning points for $x>0$ ← 2

and just one root.

$f(1) = -1$, $f(2) = \ln 2 > 0$ so root is between $x=1$ and $x=2$. 2

Take initial guess $x = 1.5$ in Newton's method 2

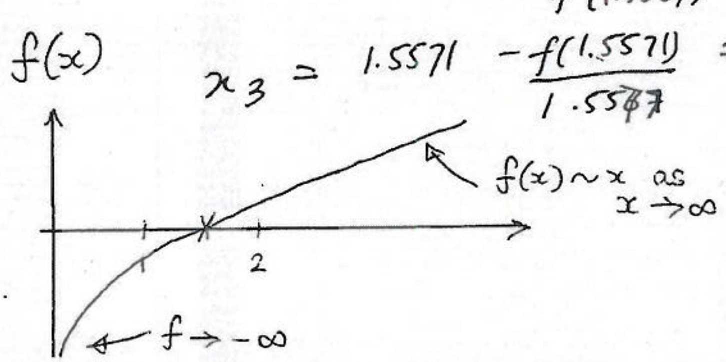
$\dfrac{f'}{f} = \dfrac{1+\frac{1}{x}}{x-2+\ln x}$. $\therefore x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$ 4

$\therefore x_0 = 1.5$,

$x_1 = 1.5 - \dfrac{(-.0945)}{1.6667} = 1.5567$ correct to 4dp 2

$x_2 = 1.5567 - \dfrac{f(1.5567)}{f'(1.5567)} = 1.5571$ correct to 4dp 2

$x_3 = 1.5571 - \dfrac{f(1.5571)}{1.5567} = 1.5571$ correct to 4dp. 2



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Total 20.

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Solution

Question 9

Marks & seen/unseen

Parts

(i)

$$y' = \frac{y(y-2)}{x(y-1)}$$

$$\therefore \int \frac{dy\,(y-1)}{y(y-2)} = \int \frac{dx}{x}$$ 2

but $\dfrac{y-1}{y(y-2)} = \dfrac{A}{y} + \dfrac{B}{y-2}$, 2

inspection gives $A = \frac{1}{2}$, $B = \frac{1}{2}$ 2

Now integrate both sides to give

$$\frac{1}{2}\ln|y| + \frac{1}{2}\ln|y-2| = \ln|x| + C$$ 2

$$y = 3,\ x = 1 \Rightarrow \frac{1}{2}\ln 3 = C$$

$$\therefore \cancel{\ln}\left|\frac{y(y-2)}{\cancel{x^2}}\right| = \cancel{\ln}\, 3x^2$$ 2

ii

$$y + \ln x \cdot x\frac{dy}{dx} = 0$$ In general, condition is $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ ← 2

$$\frac{\partial}{\partial y}(y) = 1 \neq \frac{\partial}{\partial x}(x\ln x) = 1 + \ln x$$ 2

so not exact. Now multiply by $f(x)$. (Total 20)

$$\frac{\partial}{\partial y}(f(x)y) = f(x) = \frac{\partial}{\partial x}(x\ln x\, f) = f(1+\ln x) + x\ln x f'$$ 2

so exact if $f' = -f/x$ $\therefore f = \frac{1}{x}$ 2

$\therefore y/x + \ln x\dfrac{dy}{dx} = 0$ is exact with solution $f(x,y) = C$

$y\ \dfrac{\partial f}{\partial x} = y/x$, $\dfrac{\partial f}{\partial y} = \ln x$. $\therefore f(x,y) = y\ln x = C$. 2

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Course EE I (2) 10

Question 10

Fourier Series

Marks & seen/unseen

Parts

a)

Since $f(x)$ is even, the sine terms will all be zero. 1

$$f(x) = C_0 + 2\,\mathrm{Re} \sum_{n=1}^{\infty} C_n e^{inx}$$

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx$$

2

$$C_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\, dx = \frac{1}{3}\pi^2$$

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{x^2}_{u}\, \underbrace{e^{-inx}}_{v'}\, dx$$ use integration by parts twice

$$= \frac{1}{2\pi}\left[x^2 e^{-inx}\cdot\frac{-1}{in}\right]_{-\pi}^{\pi} - \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{2xi}{n} e^{-inx}\, dx$$ 1

$$= \frac{1}{2\pi}\left[\frac{ix^2}{n} e^{-inx}\right]_{-\pi}^{\pi} - \frac{1}{2\pi}\left(\left[\frac{ix}{n} e^{-inx}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{i}{n} e^{-inx}\, dx\right)\cdot\frac{2i}{n}$$ 2

$$= \frac{i\pi^2}{2n}\left(e^{-in\pi} - e^{in\pi}\right) - \frac{2i}{n}\left[\frac{i\pi}{n}\left(e^{-in\pi} + e^{in\pi}\right) - \left[\frac{1}{2\pi}\frac{i^2}{n^2} e^{-inx}\right]_{-\pi}^{\pi}\right]$$ 2

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Question: 10

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Parts

$$= \frac{\cancel{2} i\pi^2}{n}\left(-i\underbrace{\sin(n\pi)}_{0}\right)$$

$$- \frac{\cancel{2} i}{n\pi}\left(\frac{2i\pi}{n}\cos(n\pi) + \frac{1}{n^2}\underbrace{\left(e^{-in\pi} - e^{in\pi}\right)}_{0}\right) \qquad 2$$

$$= \frac{\cancel{4\pi}}{2n^2}\cos n\pi \qquad 1$$

$$= \frac{\cancel{2\pi}}{n^2}(-1)^n \qquad 1$$

$$\Rightarrow x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx \qquad 1$$

b)

Set ~~an~~ $x = \pi$, therefore 1

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}(-1)^n \qquad 2$$

$$\Rightarrow \frac{2\pi^2}{3} = 4\sum_{n=1}^{\infty}\frac{1}{n^2} \qquad 1$$

$$\Rightarrow \frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2} \quad \Rightarrow \pi = \sqrt{6\sum_{n=1}^{\infty}\frac{1}{n^2}} \qquad 1$$

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