UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 1999

BEng Honours Degree in Computing Part III
MEng Honours Degrees in Computing Part IV
BEng Honours Degree in Information Systems Engineering Part III
MEng Honours Degree in Information Systems Engineering Part III
BSc Honours Degree in Mathematics and Computer Science Part III
MSci Honours Degree in Mathematics and Computer Science Part III
MSci Honours Degree in Mathematics and Computer Science Part IV
MSc Degree in Advanced Computing
MSc Degree in Computing Science
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the
Diploma of Membership of Imperial College
Associateship of the Royal College of Science
Associateship of the City and Guilds of London Institute

PAPER 3.36 / 4.36 / I 3.6

PERFORMANCE ANALYSIS

Monday, April 26th 1999, 10.00 – 12.00

Answer THREE questions

For admin. only:
paper contains 4 questions

1 a i) State the conditions for a continuous time stochastic process $X = \{X_t \mid t \geq 0\}$ to possess the *Markov Property*.

ii) Under what conditions is this process *time-homogeneous*?

iii) Assuming these conditions hold, define its *transition rate matrix*.

iv) Give an example of a *reducible* Markov chain.

v) Let $q_{ij}(t)$ be the probability that the time-homogeneous Markov process X is in state j at time t given that its initial state was i. Prove that

$$q_{ij}(s+t) = \sum_{k \in S} q_{ik}(s)\, q_{kj}(t)$$

vi) State the *steady state theorem* for Markov processes.

b Messages arrive at rate λ to a node in a communication network. Interarrival times are exponential, but each arrival comprises a message of either 1, 2 or 3 packets. The number of packets, N, in one message depends on the number, k, in the previous message according to the probability matrix

$$P(N{=}n \mid k) = \begin{pmatrix} 0.9 & 0.1 & 0.0 \\ 0.4 & 0.5 & 0.1 \\ 0.6 & 0.0 & 0.4 \end{pmatrix}$$

where n and k refer to row and column respectively. Show that the state, X_t at any real time t, defined as the number of packets in the most recent message, follows a Markov process and prove that it has a steady state. Show that on average, the most recent message contains one packet 30/37ths of the time.

The two parts carry, respectively, 60% and 40% of the marks.

2 a i) Prove informally, by a simple charging argument or otherwise, Little's result, namely $L = \lambda W$, explaining carefully the meaning of each of the symbols.

ii) Show that the utilisation at equilibrium of any single server with a buffer in which tasks wait is equal to the ratio of its throughput (or arrival rate) to a suitably defined service rate.

b Outline how a closed Markovian queueing network of M nodes, with a population of K customers, can be analysed by *Mean Value Analysis* of a pseudo-open network.

c In such a closed network, suppose that all of the nodes are *infinite servers* in the sense that each has sufficient servers for every visiting task to be served immediately on arrival. Let the visitation rate of server i be $x_i\, \mu_i$ where μ_i is its service rate ($1 \leq i \leq M$). Use Little's result to show that, at equilibrium, the average number of tasks at node i is proportional to that node's utilisation.

Turn over .....

3 a An M/M/1 queue has constant service rate μ, constant arrival rate λ and first come first served queueing discipline. Show that, in steady state, the queue length is n with probability $(1-\rho)\rho^n$ where $\rho=\lambda/\mu$ and give a necessary and sufficient condition for a steady state to exist.

b Show that the *waiting time probability density function* of the queue is $(\mu-\lambda)\, e^{-(\mu-\lambda)t}$, for example by assuming that the sum of n+1 exponential random variables with parameter μ has probability density function $\mu[(\mu t)^n/n!]\, e^{-\mu t}$. State any other properties you use.

c A similar queue has arrivals from two priority classes at rate λ for each class but with different service rates, μ_1 and μ_2, for classes 1 and 2 respectively. Class 1 arrivals enter the queue ahead of all waiting class 2 tasks.

i) If there is *preemption*, i.e. any class 2 task currently in service is interrupted when a class 1 task arrives and only resumes when there are no class 1 tasks remaining in the queue, show that class 1 tasks have response time probability density function $(\mu_1-\lambda)\, e^{-(\mu_1-\lambda)t}$.

ii) If there is no preemption, i.e. any class 2 task in service is allowed to complete when a class 1 task arrives, show that the *mean class* 1 *queueing time*, Q_1, is given by $Q_1 = (\rho_1/\mu_1 + \rho_2/\mu_2)/(1-\rho_1)$ where $\rho_i = \lambda/\mu_i$ $(i=1,2)$.

Hint: Use Little's result on the class 1 tasks queueing and the fact that the utilisation of the server is ρ_i for class i tasks (i = 1,2).

4 a State *Jackson's Theorem*, first for *open* and then for *closed* queueing networks. Explain how the *visitation rate* may be estimated for a server in a closed network. In what sense is it unique?

b In a closed queueing network of M nodes with population K, node i has constant service rate μ_i and visitation rate $x_i\mu_i$ chosen such that the maximum (over i) of the quantities x_i is equal to one $(1 \le i \le M)$.

i) Define the *normalising constant* $g(M,N)$ for this network's equilibrium state probabilities.

ii) Show that, for $M, N > 0$, $g(M,N) = g(M-1,N) + x_M\, g(M,N-1)$.

iii) Prove that the utilisation of node i is $g(M,N-1)x_i/g(M,N)$ $(1 \le i \le M)$.

iv) By considering the node with the maximum utilisation, or otherwise, prove that $g(M,N-1)/g(M,N) \to 1$ as $N \to \infty$.

v) Hence explain how, if $x_i = 1$ for exactly one server, the network behaves as a pseudo-open network, together with a single bottleneck node, as the population becomes very large.

Reminder: A product-form, M-dimensional probability mass function is proportional to $\prod_{i=1}^{M} x_i^{n_i}$ when the service rates are fixed.

The two parts carry, respectively, 40% and 60% of the marks.

End of paper