UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 2002

BEng Honours Degree in Computing Part III
MSc in Computing Science
MSc in Advanced Computing
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the Associateship of the City and Guilds of London Institute

PAPER C336=I3.6

PERFORMANCE ANALYSIS

Wednesday 24 April 2002, 14:00
Duration: 120 minutes

Answer THREE questions

Paper contains 4 questions
Calculators not required

1. a. State the superposition and decomposition theorems for Poisson Processes and prove *either one of them.*

b. In a *Bulk Poisson Process* (BPP) with rate λ and batch size probability mass function $\{p_i \mid i = 1,2,\ldots\}$, batch arrivals occur as a Poisson process with rate λ but at each instant the number of packets that arrive is i with probability p_i. What is

i) The superposition of two BPPs with rates λ, μ and the same batch size probability mass function $\{p_i \mid i = 1,2,\ldots\}$?

ii) The decomposition of a BPP, with rate λ and batch size probability mass function $\{p_i \mid i = 1,2,\ldots\}$ into n streams with independent splitting probabilities $\{q_j \mid j = 1,2,\ldots,n\}$?

c. The superposition of two BPPs, with rates λ, μ and constant positive integer batch sizes a, b respectively, is decomposed with independent splitting probabilities $q_1 = \lambda/(\lambda+\mu)$ and $q_2 = \mu/(\lambda+\mu)$. Show that the resultant processes are BPPs with rates λ, μ and the same batch size probability mass function $p_a = \lambda/(\lambda+\mu)$, $p_b = \mu/(\lambda+\mu)$.

The three parts carry, respectively, 50% *and* 30% *and* 20% *of the marks.*

2. a. i) Give an informal statement, using the notion of *probability flux*, of the Steady State Theorem for continuous time Markov chains (CTMCs).

ii) Give an example of a *discrete time* Markov chain (DTMC) that is periodic with period 3.

iii)Show one way in which this DTMC can be made aperiodic by adding one single-step state transition.

b. Records arrive in a buffer of capacity 2 as a Poisson process with rate λ. The buffer is cleared asynchronously at instants separated by independent exponential random variables with parameter μ.

i) Show that the buffer behaves as a Markov process, draw its state transition diagram, showing the rates of the instantaneous transitions, and explain why it must have a steady state probability distribution (SSPD).

ii) Calculate the SSPD vector, $\pi = (\pi_0, \pi_1, \pi_2)$.

c. The reversed process of a CTMC at equilibrium is defined to be the same equilibrium CTMC running backwards in time, i.e. its state at time t is the state of the original CTMC (here, the buffer) at time $A-t$, for any constant A. (Time is bi-infinite for the equilibrium processes, $-\infty < t < \infty$.)

i) Draw the reversed buffer process's state transition diagram, but *omitting the rates.*

ii) Justify intuitively that an equilibrium Markov process and its reversed process have the same SSPD, for example by considering corresponding sample paths (particular sequences of state transitions and their times).

iii)The transition rate q'_{ij} from state i to state j in the reversed process is defined by $\pi_i q'_{ij} = \pi_j q_{ji}$ where q_{ji} is the rate from state j to state i in the original process and π is the SSPD vector. Determine fully the reversed process of the buffer by calculating the rates in your diagram of part c i).

The three parts carry, respectively, 40%, 25% *and* 35% *of the marks.*

3. a. State Norton's Decomposition Theorem for Markovian queueing networks with constant arrival and service rates.

b. i) Derive the equations of *Mean Value Analysis* (MVA) in closed Jackson queueing networks of M nodes and population K tasks, where node i has service rate μ_i and visitation rate v_i ($1 \le i \le M$).

ii) Use MVA to show that the throughput T through each server in the closed, multiple parallel server network shown in fig. 1 is $k\mu/(m+k-1)$, where m is the number of parallel servers, which are all selected with the same probability $1/m$, μ is the rate of each server and k is the network's population.

c. Use parts a. and b. to decompose the network of fig. 2 into a closed queueing network of two nodes. Node 0 has constant service rate μ_0 and nodes $1,\ldots,m$ have constant service rate μ. Draw the decomposed network and be sure to derive the state-dependent rate function of the aggregated node.

d. Describe your two-node closed network as an equivalent M/M/1 queue with state-dependent arrival and service rates. Hence or otherwise find the equilibrium probability that all k tasks are at the aggregated node.

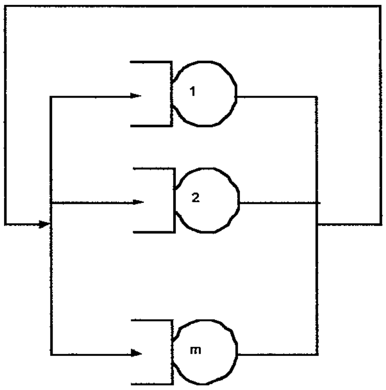


Fig. 1 Parallel server network

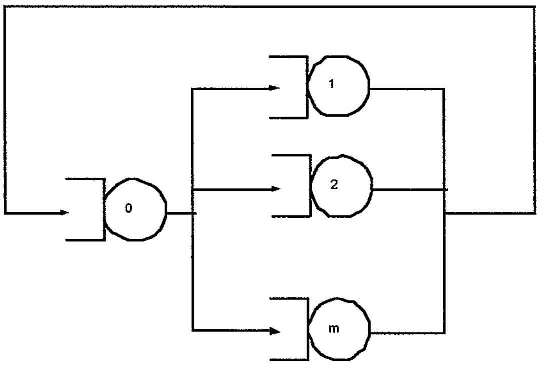
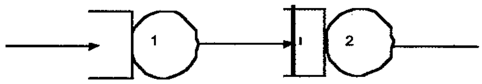


Fig. 2 Network to be decomposed

The four parts carry, respectively, 20%, 50%, 15% *and* 15% *of the marks.*

4. a. State Jackson's theorem for open queueing networks.

b. Consider the open network of two queues shown below in which the second has finite capacity *C*.



Arrivals to queue 1 are Poisson, rate $\lambda<\mu_1$, and service times are exponential random variables with parameters μ_1 and μ_2 at queues 1 and 2 respectively. Arrivals to queue 2 that find it full (i.e. queue length C) are lost.

i) What is the state space of the network?

ii) Why is Jackson's theorem not satisfied?

iii)Show that the balance equations for the network's equilibrium queue length probabilities are, for $n_1 \geq 0$, $0 \leq n_2 \leq C$:

$$\begin{aligned}(\lambda+\varepsilon(n_1)\mu_1+\varepsilon(n_2)\mu_2)p(n_1,n_2) = {} & \varepsilon(n_1)\lambda p(n_1-1,n_2) \\ & +\varepsilon(n_2)\mu_1 p(n_1+1,n_2-1) \\ & +(1-\varepsilon(C-n_2))\mu_1 p(n_1+1,n_2) \\ & +\varepsilon(C-n_2)\mu_2 p(n_1,n_2+1)\end{aligned}$$

where $\varepsilon(n)=0$ if $n=0$ and 1 otherwise.

iv) Verify that a solution is $p(n_1,n_2)=G\left(\lambda/\mu_1\right)^{n_1}\left(\lambda/\mu_2\right)^{n_2}$ for $n_1 \geq 0$, $0 \leq n_2 \leq C$, where G is a normalizing constant that ensures the probabilities sum to 1.

v) By considering infinite rates at queue 2 at appropriate queue lengths, explain how a limiting form of Jackson's theorem can be applied.

The two parts carry, respectively, 25%, and 75% of the marks.

End of paper