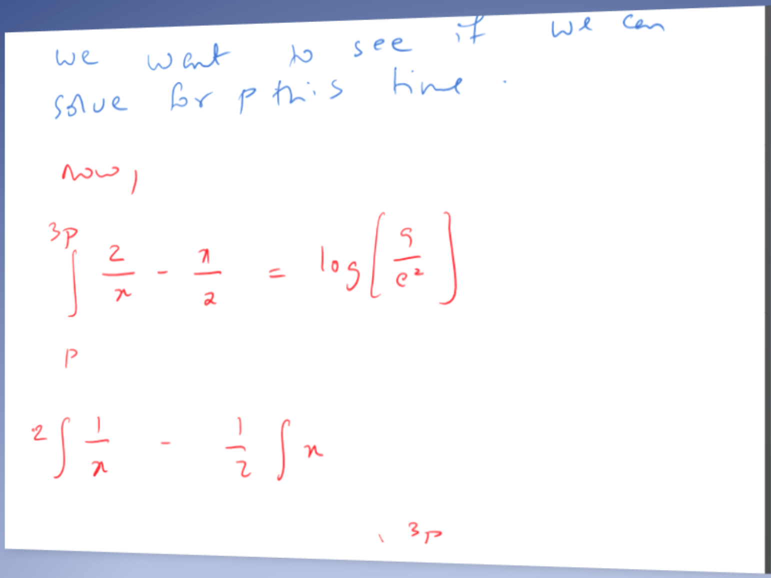


Math Creative Work

V1.1 Update 1, 29-12-2017| By Darsh Manoj

This document contains 5 interesting questions with detailed analysis on (attempting to) getting the answer of each one.

Table of Contents

The questions and related analysis begin on page 5.

Clicking on a section/question in the table of contents will directly take you there.

Changelog

Only a very brief account of changes between major versions and this update are listed here.

V1.0 RTM (link) is the first final version of this document, and contains the initial analysis of all questions (except Q5) up to November 27th 2016, and was submitted for the Spirit of Ramanujan Math Talent Initiative.

V1.1 RTM (link) is the second final version of this document. It adds Q5, and includes all changes and new analysis made up to the creation of this document on November 1st 2017, including a lot of mistakes corrected.

V1.1 Update 1 (this document) is an interim update to V1.1, containing a few minor corrections and new analysis to the oscillatory nature of the Newton-Raphson iteration in Q2 the explanation for the deviation of values in the sin summation in Q5 and graphical testing of regressions in sine products in Q5.

Note that many of the questions and analysis started *before* this document (V1.0 Beta 1, August 2016) was first made, and such analysis have also been included here.

Cover pictures: - The first one comes from a snippet of v1.0 Beta 1(handwritten), while the second one comes from Q4 in a beta version of V1.0. This graph was changed in the final version.

Acknowledgements

While the majority of work & analysis was done on my own, initial questions included, this document would have been difficult to make without the help of some people and tools used. Hence, I'm dividing them separately depending on whether it's a human or not.

Section 1 – Human credits

Natraj Sarma has helped me a lot with scrutinising the questions and checking whether they were appropriate. He also chipped in with some useful analysis of his own, including the Sum-Product relation in Question 1 and helped me to eventually get the correct solution in Question 3. He managed to help me a lot despite his schedule being extremely busy. To this date, we keep in touch via email.

My **parents** also helped me by finding suitable people who could help me with validating my work, as my school teachers were unable to help me despite their best efforts.

The **Spirit of Ramanujan Math Talent Initiative**, as it is what inspired me to put my questions, which were merely internally analysed till then, into this document. This initiative also encouraged me to work on new ideas and keep up my habit on making up math questions.

Section 2 – Non-human credits

1. **Word 2010/16 –** used to create this document.
2. **Excel 2010/16 –** used for statistical analysis
3. **desmos.com** – used for graphical analysis
4. **Ti-Nspire Student's Software** – used for graphical analysis
5. **Visual Studio 2015/17** – used for programming tasks

1 Solve for $\boldsymbol{n}$ if

$$n!\,(n-1)! = \frac{1}{2}(n+1)!$$

1.1 *Abstract*

This is a question that I found when playing with factorials in my recess when I was in Class 10. When analysing that question initially, I found one major difference from the questions that I'd seen before. Using common methods to solve this problem fails miserably, as then we'd have one factorial that we cannot do anything with. When giving the problem to my seniors and teachers, they also ran into the exact same problem, finally informing me that 'it cannot be solved' and 'only trial and error is possible'. Some suggested that I graph it, while that is one 'method' of solving the problem, it is not a particularly good one. After all, you won't be able to have a graphing calculator in your hand all the time… I was looking for a direct algebraic method. I was also curious as to whether we could do some manipulation to get a quadratic like $n^2 - 4n + 3 = 0$ (as then we'd get $n = 1$ or 3). But I found none.

In this case, we first start by taking a look at the 'direct' methods that we would normally use for a problem like this. Then, we detail the unusual method of depriving each term of its factorial. It's weird, but the application of this method brings out both of the correct solutions! Then, we show that this question can be converted to the classic sum = product relation. Finally, we take a graphical look at the problem to find out if there is any 'hidden' solution.

1.2 *Attempting to directly solve*

Let us start by

$$n!\,(n-1)! = \frac{1}{2}(n+1)(n!)$$

or,

$$(n-1)! = \frac{1}{2}(n+1)$$

This simply leads us to

$$2(n-1)! = n+1$$

(1)

We can only solve by trial and error from here. It is worth noting out that trial and error is actually viable, as $(n-1)!$ increases at a much faster rate than $n+1$.

We can try to cancel $(n-1)!$ instead: -

$$n!\,(n-1)! = \frac{1}{2}(n+1)(n)(n-1)!$$
$$n! = \frac{1}{2}(n+1)(n)$$
$$2n! = (n+1)(n)$$

(2)

with the same issue, as above.

We can try to combine the 2 equations and see what we get: -
Taking (2) and making $n+1$ as the subject, we get

$$\frac{2n!}{n} = n + 1$$

(3)

As the RHS of (1) is equal to $n + 1$,
we can write (3) as

$$\frac{2n!}{n} = 2(n-1)!$$

$$\frac{n!}{n} = (n-1)!$$

$$\frac{n(n-1)!}{n} = (n-1)!$$

$$(n-1)! = (n-1)!$$

.. which is an identity. Nothing wrong with it, but unfortunately does not help us at all.

1.3 *The unconventional 'factorial stripping' method*

What makes this question very interesting is that there is an unusual (and mathematically weird) method that can be used to work out the 2 values of n.
I'd call this method as the 'depriving' or the 'factorial stripping' method.

We take the original question,

$$n!\,(n-1)! = \frac{1}{2}(n+1)!$$

and strip off the factorial sign from $n!$, which then makes it n.
It must be noted that when we do this, the domain of n essentially becomes $\{1\,,2\}$ as then we need $n! = n$, for which n can only be 1 or 2.

The question now becomes

$$n(n-1)! = \frac{1}{2}(n+1)(n)(n-1)!$$

(4)

which enables us to cancel like terms, giving us

$$1 = \frac{1}{2}(n+1)$$

$$2 = n + 1$$

$$n = 1$$

Does it work?
When $n = 1$,

$$n!\,(n-1)! = \frac{1}{2}(n+1)!$$

Becomes

$$1! * 0! = \frac{1}{2} * 2!$$

$$1 * 1 = \frac{1}{2} * 2$$
$$1 = 1$$

which is valid!
So, we found one value of n, which is 1.
But how do we find out the other value of n? Amazingly, by depriving the $(n-1)!$ of its factorial!

Again, it must be noted that when we do this, the domain of n essentially becomes $\{2, 3\}$ as then we need $(n-1)! = n-1$, for which n can only be 2 or 3.

So, when we do that, we get

$$n!\,(n-1) = \frac{1}{2}(n+1)!$$

Simplifying the equation,

$$n!\,(n-1) = \frac{1}{2}(n+1)(n!)$$
$$n-1 = \frac{1}{2}(n+1)$$
$$2(n-1) = n+1$$
$$2n-2 = n+1$$
$$n = 3$$

Testing its validity,

$$n!\,(n-1) = \frac{1}{2}(n+1)!$$

when $n = 3$,

$$3! * 2 = \frac{1}{2} * 4!$$
$$6 * 2 = \frac{1}{2} * 24$$
$$12 = 12$$

which is again valid!

But is there a 3rd value of n that we can find by depriving the $(n+1)!$ term of its factorial? Unfortunately not, as then the $n!$ and $(n-1)!$ term won't have anyone to cancel with.

Also, when we stripped the terms of its factorial, $n = 2$ was a *potential* solution in both cases, but in the end, it did not become part of the solution. For some reason, the other term $(1\ and\ 3)$ worked in both cases.

1.4 *Sum and product relation*

Take the question

$$n!\,(n-1)! = \frac{1}{2}(n+1)!$$

and then

$$n! = \frac{1}{2}\frac{(n+1)!}{(n-1)!}$$

$$n! = \frac{1}{2}(n)(n+1)$$

giving us the final relation

$$\prod_{k=1}^{n} k = \sum_{k=1}^{n} k$$

which can be said to be only true when $n = 1$ and $n = 3$, but that is not enough.

Again, it is possible to justify based on the divergence of these functions.

Let

$$f(n) = \prod_{k=1}^{n} k$$

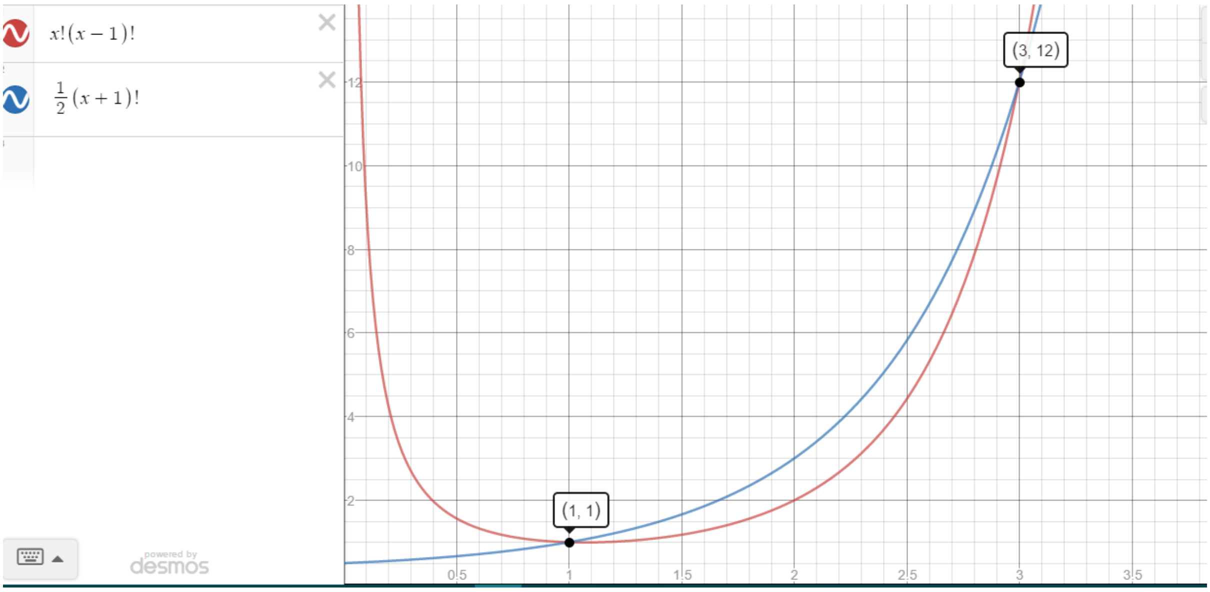
$$g(n) = \sum_{k=1}^{n} k$$

The graphs converge at 2 points and then diverge again. We need to take only the points that correspond to natural numbers. By trial and error (again!), we can say that when the number of terms on both sides is 1 and 3, $f(n) = g(n)$.

However, it is not hard to find out why $f(n) = g(n)$ is not possible when $n > 3$. The rate of increase of $f(n)$ is significantly more than that of $g(n)$. As both diverge to ∞, this is enough to show that they will never meet again. Nevertheless, this is an interesting observation.

1.5 *Graphical verification*

To be sure that n = 1 and n = 3 are the only 2 values that work, we need to graphically verify this.



This serves as proof of the same. Note that the graphs diverge from that point, thus ensuring that there is no other valid 'hidden' solution that could crop up.

But that begs us a very simple question: - How do we *actually* solve for n, without resorting to methods like trial and error?

1.6 *Area between the factorials*

Note that a curious question comes up when you take a look at the graph. How do you find the area under the graph, if you have to?

How would you integrate

$$\int_1^3 x!\,(x-1)!\,dx \textbf{ AND } \int_1^3 \frac{1}{2}(x+1)!\,dx$$

considering that $x!$ is not differentiable, and that it can be safely assumed that the integral of $x!$ would have $x!$? (That is without considering the fact that the first integral would require parts at the very least)

Technically, the area would be

$$\int_1^3 x!\,(x-1)!\,dx - \int_1^3 \frac{1}{2}(x+1)!\,dx$$

As there is no clear way of finding the value of $\int x!\,dx$ (see above), that is not an option. But as the integral we have is definite, we could try to see if the application of definite integral properties could help in this scenario.

Let $I_1 = \int_1^3 x!\,(x-1)!\,dx$ and $I_2 = \int_1^3 \frac{1}{2}(x+1)!\,dx$

We have

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

Applying this property to the integral, we get

$$I_1 = \int_1^3 (2-x)!\,(1-x)!\,dx \ \&\ I_2 = \int_1^3 \frac{1}{2}(3+x)!\,dx$$

and then what?

Unfortunately, this method does not solve the problem we are having (and no, we cannot make out a relation between $(x-1)!$ and $(1-x)!$ as it is embedded in a factorial). The more serious problem is that in I_1 ,instead of the domain being $x \geq 1$,the domain becomes $x \leq 1$, invalidating a correct solution ($x = 3$) immediately!

I was hoping that we could get it in a nice $2I$ form that can be easily integrated…

2 Find value(s) of p for which the integral

$$\int_{p}^{p+2} \frac{2}{x} - \frac{x}{2} dx$$

is equal to $\log\frac{9}{e^2}$, where $\log x$ denotes the logarithm to base e (also known as $\ln x$)

2.1 *Abstract*

I found this question when playing with algebraic expressions of the form

$$\frac{a}{b} + \frac{b}{a} = k$$

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{z}$$ [1]

For fun, I was curious to see what happened if I integrated

$$\int \frac{2}{x} - \frac{x}{2} dx$$

from the limits 1 to 3. I got $\log\frac{9}{e^2}$ as the answer. But I was surprised when I was unable to get the other way around, that is if we are given the answer as $\log\frac{9}{e^2}$ and we needed to integrate the expression from one limit to another which is defined by a relation, and we need to find that limit. I tried a lot to get the answer, but my efforts were thwarted every time by e or $\log x$ which made finding the answer very difficult, despite it looking more like an exponent that could easily be reduced. A combination of an irrational base and variable power on the RHS was the crux of the question. When I tried changing the relationship of the 2 limits (which would still give the same answer) from (see section 2.5)

$$\int_{p}^{p+2} f(x)dx \Longrightarrow \int_{p}^{3p} f(x)dx$$

I got the correct limit as expected. That fuelled my curiosity for this one. Why is it not possible to integrate from the same limits if their relations were different? After all, I could not see how $p + 2$ was so much different from $3p$.

It was even more surprising when I graphed the question. I thought that I would get the solution only when $x = 1$, but I was wrong, as I got two negative values that was correct on verification (only that as it was negative, I had to use $\log|x|$ instead of $\log\ x$).

So, this integral actually has hidden solutions! That made this a very beautiful question in my opinion, as on the outside it looked not much more than a simple application of basic integral properties that we would learn in Class 12, but on the inside, it was much more advanced!

[1] Inspiration often hits you out of the most unusual situations. I took an interest in it after getting a higher-order geometry problem incorrectly in my Class 10 examinations in which we'd had to prove that. This exact expression also formed as the basis for the former Q4 (which was removed later after it was deemed to have a logical error and hence unsuitable for submission).

But it did not end there. Later, when I took another look on this problem, I wondered whether the application of a definite integral property that we would learn in Class 12 could help here (see section 2.3). It looked really promising at first, but it did not work in the end.

I also tried equating[2] the terms. Unfortunately, in both cases (direct and using the property), all the values (but one) of p that I got were incorrect. The only one which was correct was $p = 1$.

I was not interested in iterative-based methods (as I felt that such methods were clumsy), but I was persuaded to try it by my mentor (see *Acknowledgements, pg. 4)*. In Section 2.7, we use the Newton-Raphson iterative method to try to get the final answer using a chain reaction. The main iterative work was done using a custom-made program. It initially showed great promise, but further analysis and testing discovered that it was nothing but a pattern and was unreliable.

2.2 *Attempting to solve directly*

Taking the integral

$$\int_{p}^{p+2} \frac{2}{x} - \frac{x}{2}\,dx$$

and integrating it,

$$2\log|x| - \frac{x^2}{4} + C$$

(We need $|x|$, not just x. This is to allow for the inclusion of negative numbers that may crop up in the solution).
Taking limits and simplifying,

$$\left[2\log|x| - \frac{x^2}{4}\right]_{p}^{p+2}$$

$$2\log|p+2| - 2\log|p| - \frac{(p+2)^2}{4} + \frac{p^2}{4}$$

$$2\log\left|\frac{p+2}{p}\right| - \frac{p^2+4p+4-p^2}{4}$$

$$2\log\left|\frac{p+2}{p}\right| - \frac{4p+4}{4}$$

$$2\log\left|\frac{p+2}{p}\right| - (p+1)$$

Noting that this is equal to $\log\frac{9}{e^2}$,

$$2\log\left|\frac{p+2}{p}\right| - (p+1) = \log\frac{9}{e^2}$$

$$2\log\left|\frac{p+2}{p}\right| - 2\log\left(\frac{3}{e}\right) = p+1$$

[2] which is not mathematically correct like an identity, but is used only as a *potential* option, and should be treated as such. This warning is also repeated in later sections.

$$2\log\left|\frac{p+2}{p}\right| - (2\log 3 - 2\log e) = p+1$$
$$2\log\left|\frac{p+2}{p}\right| - 2\log 3 + 2 = p+1$$
$$2\log\left|\frac{p+2}{p}\right| - 2\log 3 = p-1$$
$$2\log\left|\frac{p+2}{p} * \frac{1}{3}\right| = p-1$$
(the equation third from above is *1, used in section 2.6)
or, which would finally lead us to the simplification by taking antilog on both sides,
$$\left(\frac{p+2}{3p}\right)^2 = e^{p-1}$$
This is as far as our substitution will go.
Note the lack of modulus in the final expression. This happens because of the square term. But it is still needed until then.

Unfortunately, this is precisely where we actually get stuck. It is difficult to get it in the form $e^x = e^y$ or something similar as then we are always bothered with logs that hinder us from going any further.

We could however consider equating the terms like if $a+b=c+d$, we could say that $a=c$ and $b=d$ or $a=d$ and $b=c$ (because of commutativity). Consider
$$2\log\left|\frac{p+2}{p}\right| - 2\log 3 = p-1$$
In which the only actual possibility is to equate
$$2\log\left|\frac{p+2}{p}\right| = -1$$
and
$$-2\log 3 = p$$

Taking the former equation,
$$2\log\left|\frac{p+2}{p}\right| = -1$$
and simplifying,
$$\log\left(\frac{p+2}{p}\right)^2 = -\log e$$
$$\left(\frac{p+2}{p}\right)^2 = -e$$
$$p^2 + 4p + 4 = -ep^2$$
or, giving us an interesting quadratic equation
$$(1+e)p^2 + 4p + 4 = 0$$
But a look into the equation reveals that the solutions are purely complex without actually solving it. As the discriminant is $D = b^2 - 4ac$ and $D = 0$ when $a = 1$, an increase in the value of a (as in this case) will immediately make the value of D negative, which shows that the equation has only complex solutions.

WARNING

The method of 'equating' described here and in a couple of other sections is not actually a mathematically legal method. It is intended solely as 'just' one way of trying to solve the problem, just like the *stripping factorials* method seen in Question 1.

The value of p in the second equation

$$-2\log 3 = p$$

is also incorrect.

2.3 *Using Definite integral property*

Let's see if we can use the fact that

$$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$$

to help us in this integral.
Then the integral becomes

$$\int_p^{p+2} f(x)dx = \int_p^{p+2} f(2p+2-x)dx$$

$$\int_p^{p+2} \frac{2}{x} - \frac{x}{2}dx = \int_p^{p+2} \frac{2}{p+p+2-x} - \frac{p+p+2-x}{2}dx$$

It helps to know that p is a constant for the purposes of integrating, and that we have to find that constant(s).

$$\int_p^{p+2} \frac{2}{2p+2-x} - \frac{2p+2-x}{2}dx$$

$$\int_p^{p+2} \frac{2}{2\left(p+1-\frac{x}{2}\right)}dx - \frac{1}{2}\int_p^{p+2} 2p+2-x\,dx$$

$$\int_p^{p+2} \frac{1}{\left(p+1-\frac{x}{2}\right)}dx - \frac{1}{2}\left[2px+2x-\frac{x^2}{2}\right]_p^{p+2}$$

$$\int_p^{p+2} \frac{1}{\left(p+1-\frac{x}{2}\right)}dx - \frac{1}{2}\left(2p(p+2)-2p^2+2(p+2)-2p-\frac{(p+2)^2}{2}+\frac{p^2}{2}\right)$$

$$\int_p^{p+2} \frac{1}{\left(p+1-\frac{x}{2}\right)}dx - \frac{1}{2}(2p^2+4p-2p^2+2p+4-2p-p-1)$$

$$\int_p^{p+2} \frac{1}{\left(p+1-\frac{x}{2}\right)}dx - \frac{1}{2}(4p+4-p-1)$$

$$\int_p^{p+2} \frac{1}{\left(p+1-\frac{x}{2}\right)}dx - \frac{3}{2}(p+1)$$

The first integral can now be evaluated by taking the substitution

$$p + 1 - \frac{x}{2} = t$$

Then, we get

$$-\frac{1}{2}dx = dt$$

or,

$$dx = -2\,dt$$

Then the integral becomes

$$-2 \int_{p}^{p+2} \frac{1}{t} dt$$

$$[-2 \log t]_{p}^{p+2}$$

$$\left[-2 \log(p + 1 - \frac{x}{2})\right]_{p}^{p+2}$$

$$2 \log\left(p + 1 - \frac{p}{2}\right) - 2 \log\left(p + 1 - \frac{p+2}{2}\right)$$

$$2\left(\log\left(\frac{p}{2} + 1\right) - \log\left(\frac{2p + 2 - p - 2}{2}\right)\right)$$

$$2\left(\log\left(\frac{p}{2} + 1\right) - \log\left(\frac{p}{2}\right)\right)$$

$$2 \log\left(\frac{p+2}{p}\right)$$

Adding the previously integrated term and noting that the value of the final integral is $\log(\frac{9}{e^2})$,

$$2 \log\left(\frac{p+2}{p}\right) - \frac{3}{2}(p + 1) = 2 \log 3 - 2$$

(using log laws on RHS).
At this point, the only thing that comes close to actually solving it, other than pure trial and error, is to equate the terms at this stage.
Taking the first case, we have

$$2 \log\left(\frac{p+2}{p}\right) = 2 \log 3 \Longrightarrow \frac{p+2}{p} = 3$$

$$-\frac{3}{2}(p + 1) = -2 \Longrightarrow 3(p + 1) = 4$$

The first equation gives us $p = 1$, which is correct.
But the second equation gives us $p = \frac{1}{3}$, which is incorrect. This can be verified by direct substitution.
Now only one of the terms is correct, which is weird.
What about equating? Let's see.

We have

$$2 \log\left(\frac{p+2}{p}\right) = -2 \Longrightarrow \log\left(\frac{p+2}{p}\right) = -1$$

$$-\frac{3}{2}(p + 1) = 2 \log 3 \Longrightarrow p + 1 = -\frac{4}{3}\log 3$$

Taking the former equation,

$$\log\left(\frac{p+2}{p}\right) = -1$$

Taking antilog on both sides,

$$\frac{p+2}{p} = -e$$

Simplifying,

$$p + 2 = -ep$$
$$p + ep = -2$$
$$p(1 + e) = -2$$
$$p = \frac{-2}{1+e} \cong -0.538$$

which is anyway incorrect.
and, if

$$p + 1 = -\frac{4}{3}\log 3$$

then,

$$p = -\frac{4}{3}\log 3 - 1 \cong -2.46$$

This is incorrect as well.

2.4 *Using trial and error method*

Taking the simplified form, we obtained before,

$$\left(\frac{p+2}{3p}\right)^2 = e^{p-1}$$

It becomes obvious that $p = 1$ is a solution of the equation.
But as we'll see in section 2.6, there is more than that what meets the eye.

2.5 *The unconventional change of limit trick*

We can also obtain the value by making a nice little change to the limits: -
Instead of

$$\int_{p}^{p+2} \frac{2}{x} - \frac{x}{2}\,dx$$

, we can write it as

$$\int_{p}^{3p} \frac{2}{x} - \frac{x}{2}\,dx$$

It must be noted that p = 1 is the only value that could even work, as if

$$\int_{p}^{p+2} f(x)dx = \int_{p}^{3p} f(x)dx$$

Then $p + 2 = 3p$, or $p = 1$.
We want to see if we can *solve* for it.

Now with

$$\int_{p}^{3p} \frac{2}{x} - \frac{x}{2} dx$$

And integrating normally, we get

$$\left[2\log|x| - \frac{x^2}{4}\right]_p^{3p}$$

Simplifying and equating it to $\log(\frac{9}{e^2})$,

$$2\log|3p| - 2\log|p| - \frac{9p^2}{4} + \frac{p^2}{4} = 2\log 3 - 2$$

$$2\log\left|\frac{3p}{p}\right| - 2p^2 = 2\log 3 - 2$$

$$2p^2 = 2$$

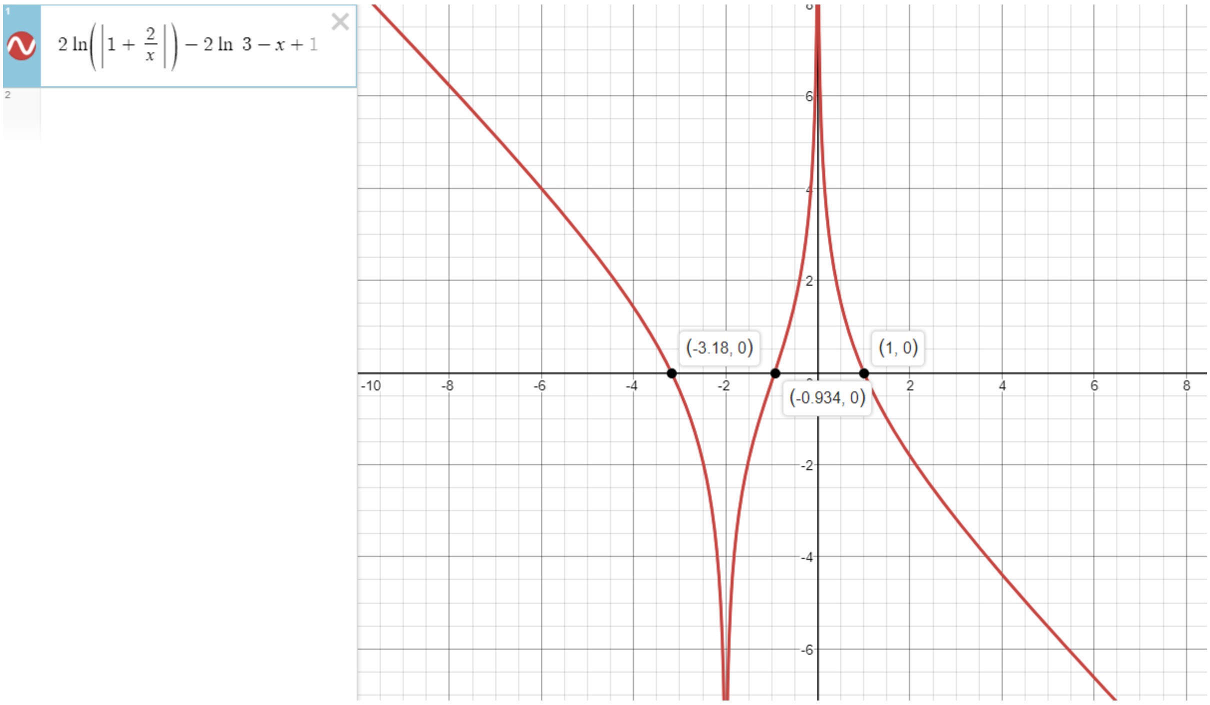
$$p = \pm 1$$

Note that $p = -1$ is incorrect, but $p = 1$ is.
Unfortunately, the question is essentially being modified in this case, ruling the use of this method for anything else. Also, the other values will *never* be obtained here.

2.6 *Graphical analysis*

What happens if we graphically check *1?

We get a beautiful symmetrical graph that reveals 3 solutions, only one of which was found before: -



An attempt to represent the simplified form we obtained before (in 2.2) also gives the same three solutions, but in a much more cunning manner (see box).

One of them is one we already know ($p = 1$) , but the other ones are something new and interesting ($p = -0.934\,, -3.18$).
The graphs diverge in different directions; however, it must be noted that in both graphs that as $x \implies -\infty\,, y \implies 0$. But even that is not very accurate – doing this ended up being a serious blunder! (see box)

Let us now take a look at one of the negative solutions (-0.934) that we discovered only through the graph: -
When verifying the solution,

$$\left(\frac{p+2}{3p}\right)^2 = e^{p-1}$$

On the LHS, $\left(\frac{p+2}{3p}\right)^2$ becomes $\frac{(1.066)^2}{9(-0.934)^2} = 0.1447$
On the RHS, e^{p-1} becomes $e^{-1.934} = 0.144569$
We can thus say that the solutions are the same, as their difference is small enough for this question.

What about the other solution (-3.18)?
On the LHS, $\left(\frac{p+2}{3p}\right)^2$ becomes $\left(\frac{-1.18}{-9.54}\right)^2 = 0.015299$
On the RHS, e^{p-1} becomes $e^{-4.18} = 0.0152985$
.. which are virtually same!

Thus, it can be said that the solutions of p in the integral

$$\int_{p}^{p+2} \frac{2}{x} - \frac{x}{2}\,dx = \log(\frac{9}{e^2})$$

are $1, -3.18$ and -0.934.

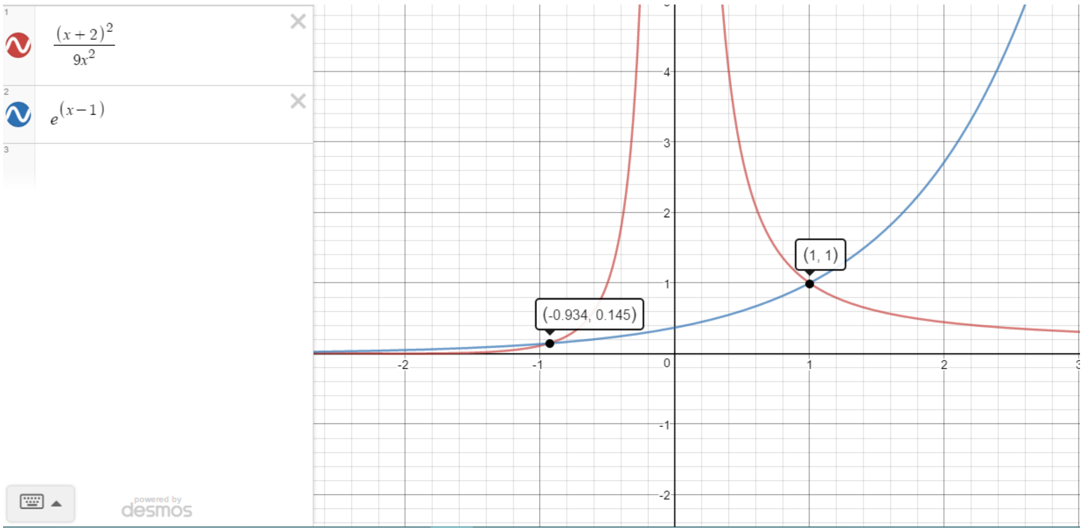
But
how do we *even* solve for p without resorting to nonalgebraic methods?

All versions of this document up to V1.1 Beta 3 contained a serious error as to the number of solutions (and the solutions thus obtained) that would be obtained in the graphical method.

This error was discovered when discussing this question with someone. It turned out that the graph he obtained contained *another* solution that was never considered!

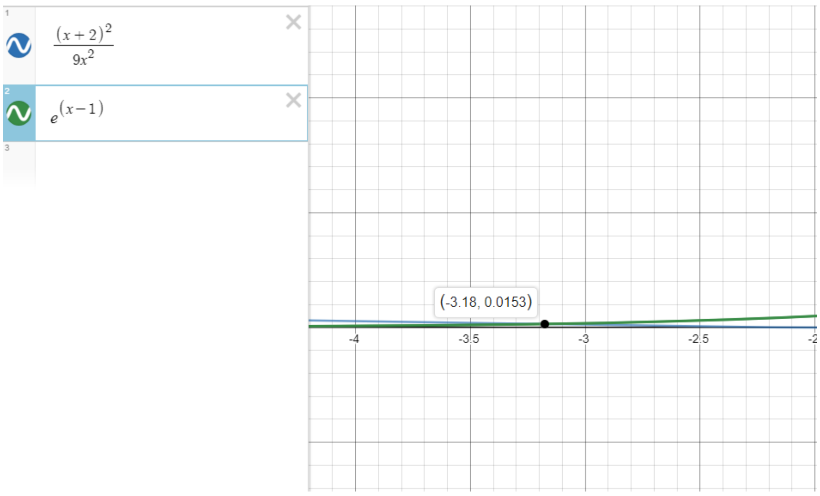
This prompted an investigation, and when I graphed the equation obtained, the 3rd solution was clearly visible! But this was hard to believe – after all, this document was vetted (internally and externally) quite a few times, so it was a rude surprise to find only now (August 2017) that I had missed something quite basic. After a bit of cross-checking, the error was discovered – and it sure was embarrassing! But what was it?

All versions of the document up to V1.1 Beta 3 included this graph, clearly showing the two solutions:



The left part of the diagram was assumed to be limiting at 0, and hence wasn't considered, while the right side of the graph was diverging. This was my logic, and I was right. *Almost*.

I made the error of not vetting the left side of the graph properly enough. When I ran another scan of the two graphs, the missing solution popped up! But, where was it? In a tricky place – and that required a good zoom. I hadn't realised that the graphs converge at that point and then the polynomial graph start to increase slightly again. As I trusted the thumbnail all too much, I hadn't conducted further checks – even dismissing the (correct) solution obtained through the Newton-Raphson method (see section 2.7) as a problem with the method.



2.7 The Newton-Raphson iterative method

This is an iterative-based method that can be used to narrow down the solution of an expression.
I am not a big fan of such methods, as they pose some problems. Iterative methods (for instance this one) only give an approximation (but a very good one, as we'll see). It does not give us the number of solutions that is possible.

Still, we want to give it a try.
We know that the Newton-Raphson method is based on the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

This indicates that we need to find the derivative of x_n.
We now take the simplified form as found in Section 2.2: -

$$\left(\frac{x+2}{3x}\right)^2 = e^{x-1}$$

$$\left(\frac{x+2}{3x}\right)^2 - e^{x-1} = f(x)$$

Simplifying and taking derivative,

$$f(x) = \frac{x^2+4x+4}{9x^2} - e^{x-1}$$
$$f(x) = \frac{1}{9} + \frac{4}{9x} + \frac{4}{9x^2} - e^{x-1}$$
$$f'(x) = 0 - \frac{4}{9(x^2)} - \frac{8}{9(x^3)} - e^{x-1}$$

(by noting that $\frac{d}{dx}(\frac{1}{x}) = -\frac{1}{x^2}$ and $\frac{d}{dx}\left(\frac{1}{x^2}\right) = -\frac{2}{x^3}$)
Now there is no point in trying to find the solution manually due to its complexity. So I built a program to handle this task. (Note that this was done in Visual Studio 2015 in C++)
The code for the same is given below. Note the complex mathematical expression when you try to express it in code!

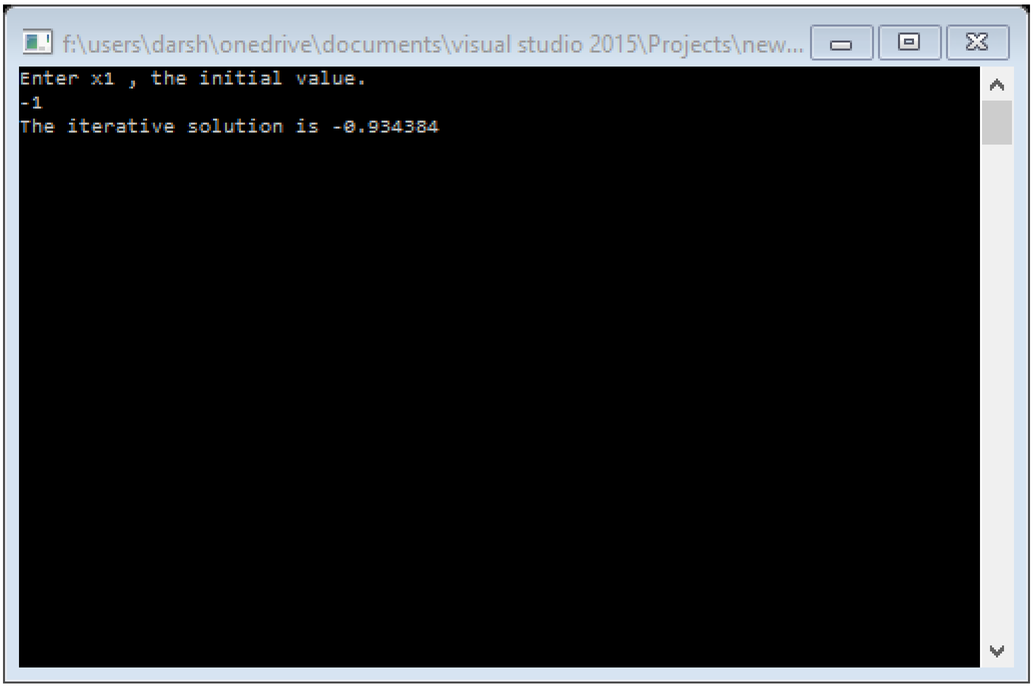
```
#include "stdafx.h"
#include <iostream>
#include <math.h>
#include <conio.h>

void main()
{
	// The Newton-Raphson method uses an iterative process to narrow down the answer until the sign changes.
	// Using the idea xn+1 = xn-(f(x)/f'(x))
	// Here, the function and its derivative are hard-coded.
	// Let the function be the simplified form given in Section 2.2. Its derivative will be calculated.
	float x2, x1;
	// Float's desirable. We want an approximation.
	// As we do not know how many solutions are there, let us ask the user for the initial value.
	std::cout << "Enter x1, the initial value." << '\n';
```

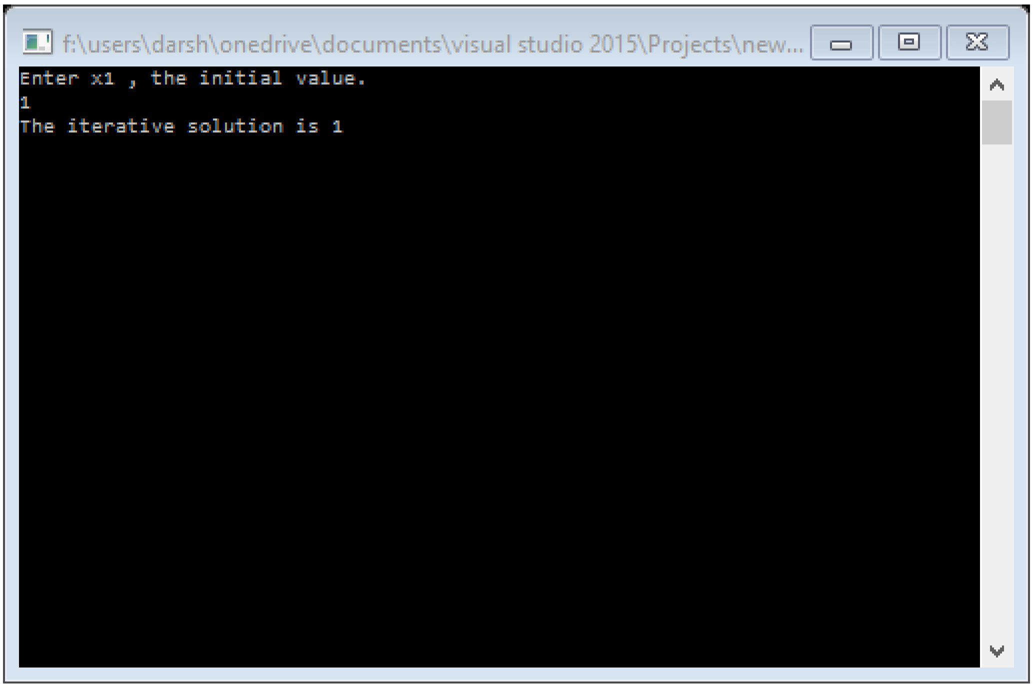
```
	std::cin >> x1;
	x2 = x1 - ((pow((x1+2)/(3*x1),2)- exp(x1 - 1)) / ((-4/(9*(x1*x1)))-
(8/(9*pow(x1,3)))-exp(x1-1)));
	// Now for a looping structure that will iterate the solution until the sign
changes:-
	// Instead of again using that long expression, we can simplify it.
	// Let the expression be k. Then we want k < 0. or x2 = x1-k or k=x1-x2.
	if (x1-x2 < 0)
	{
		do
		{
			x1 = x2;
			x2 = x1 - (((pow((x1 + 2) / (3 * x1), 2)-exp(x1-1)) / ((-4 /
(9 * (x1*x1))) - (8 / (9 * pow(x1, 3))) -exp(x1-1))));
		}
		while (x1-x2 > 0);
	}
	else if (x1-x2 > 0)
	{
		do
		{
			x1 = x2;
			x2 = x1 - ((pow((x1 + 2) / (3 * x1), 2)- exp(x1 - 1)) / ((-4 /
(9 * (x1*x1))) - (8 / (9 * pow(x1, 3))) - exp(x1 - 1)));
		}
		while (x1-x2 < 0);
	}

	// When looping has finished and we have got the answer
	std::cout << "The iterative solution is " << x2;
	_getch();
}
```

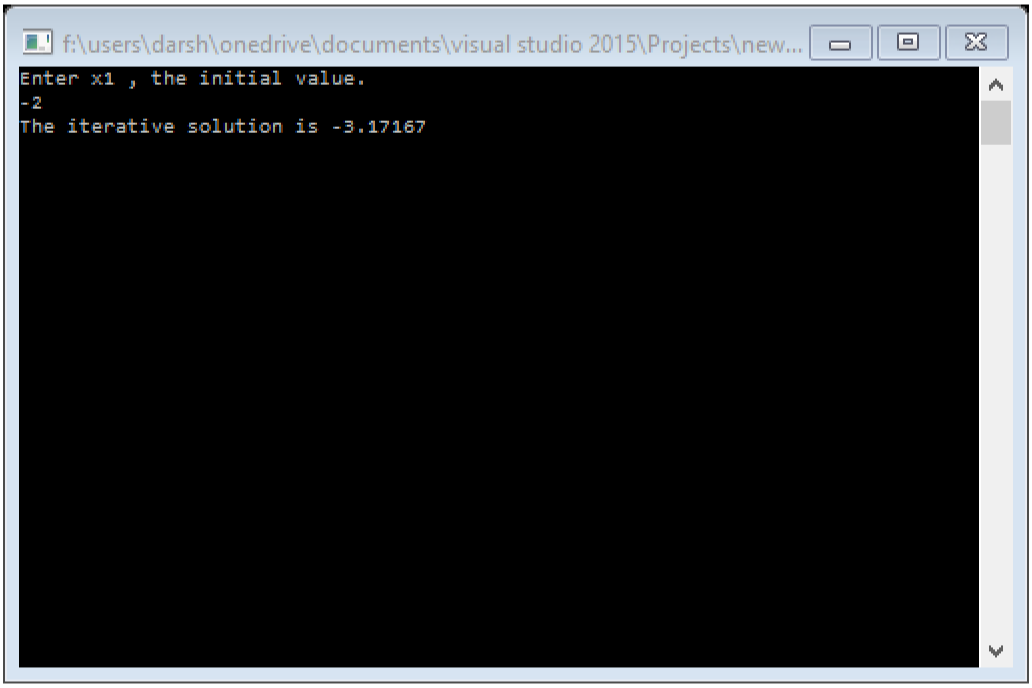
The program takes in x_n , iterates it until the sign changes and displays that value.
Running the program, let $x_n = -1$.
The program gives out the correct iterative answer!



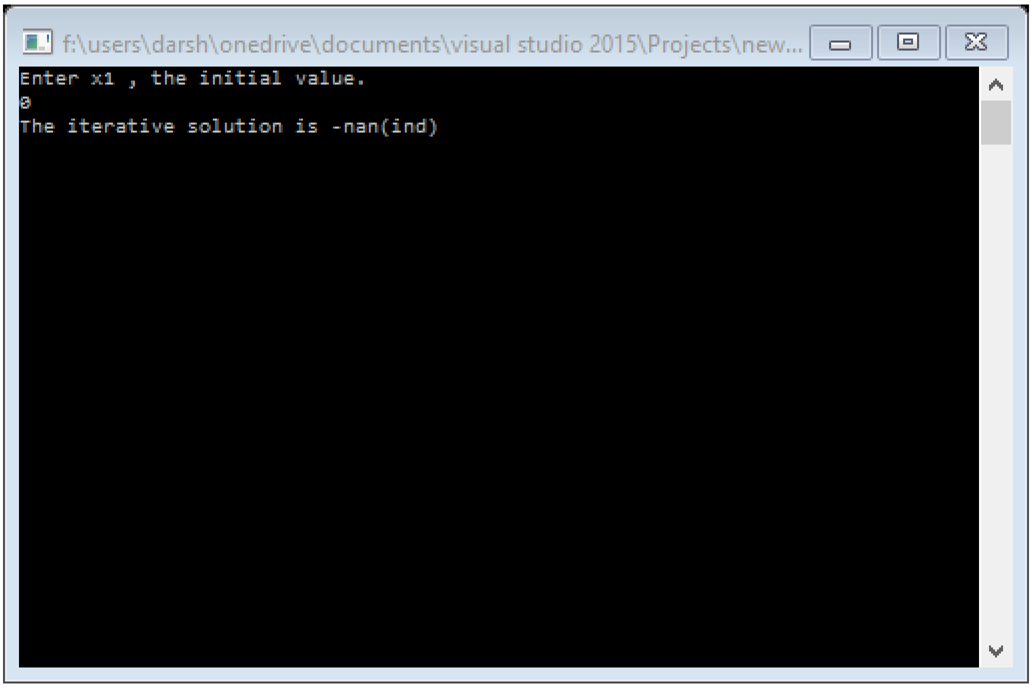
Let us try another value. Why not 1?



That answer is also correct!
Choosing another initial value will allow us to obtain the third iterative solution: -



However, if we take the wrong value of x_n , we get completely incorrect values…



which is just another way of saying that nothing happened, but it should be noted that the design of the code will always give a result like that when the initial value is 0.

f:\users\darsh\onedrive\documents\visual studio 2015\Projects\new...

```
Enter x1 , the initial value.
0.5
The iterative solution is 0.932864
```

Somehow turning out to be almost the negative of one of the solutions.

But the iterative method is designed by principle to locate the closest possible solution. I wondered what values it would give out for different initial iterative values, now that we know that it gives wrong values for some iterative values at least. Which ones? I modified the program to loop the values in this case.

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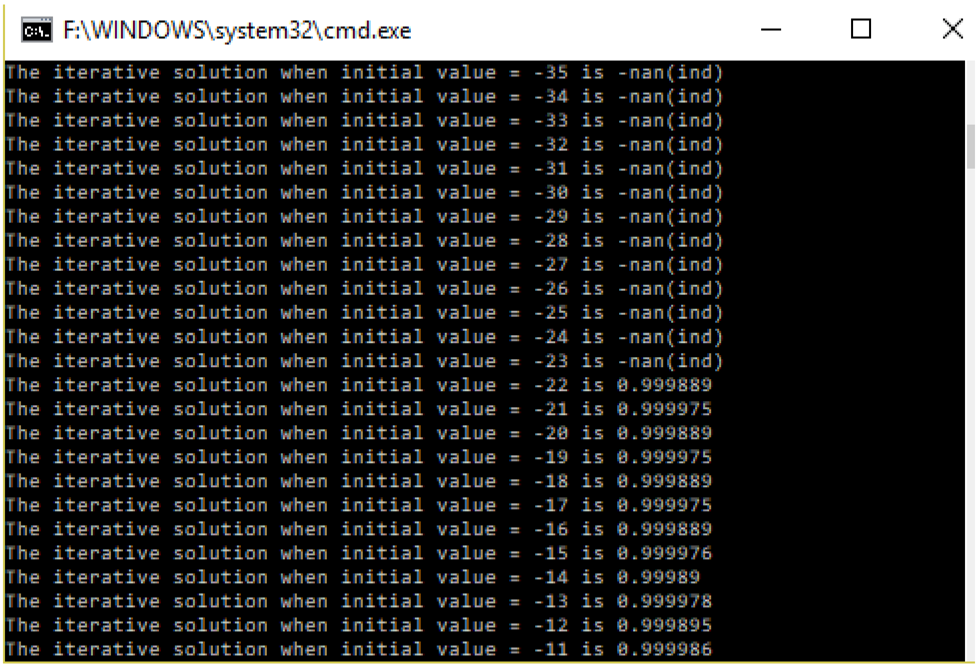
```
The iterative solution when initial value = -10 is 0.99992
The iterative solution when initial value = -9 is 0.999998
The iterative solution when initial value = -8 is 0.999987
The iterative solution when initial value = -7 is 0.99989
The iterative solution when initial value = -6 is -0.934384
The iterative solution when initial value = -5 is -3.17998
The iterative solution when initial value = -4 is -3.17998
The iterative solution when initial value = -3 is -3.17996
The iterative solution when initial value = -2 is -3.17167
The iterative solution when initial value = -1 is -0.934384
The iterative solution when initial value = 0 is -nan(ind)
The iterative solution when initial value = 1 is 1
The iterative solution when initial value = 2 is 0.987606
The iterative solution when initial value = 3 is 1.26111
The iterative solution when initial value = 4 is 2.0658
The iterative solution when initial value = 5 is 3.01886
The iterative solution when initial value = 6 is 4.00588
The iterative solution when initial value = 7 is 5.00192
The iterative solution when initial value = 8 is 6.00065
The iterative solution when initial value = 9 is 7.00022
The iterative solution when initial value = 10 is 8.00008
The iterative solution when initial value = 11 is 9.00003
The iterative solution when initial value = 12 is 10
The iterative solution when initial value = 13 is 11
The iterative solution when initial value = 14 is 12
```

At least within the narrow range of values checked, the negative values show an interesting trend. For the initial few negatives, the iterative values obtained are perfectly correct (from -1 to -6).

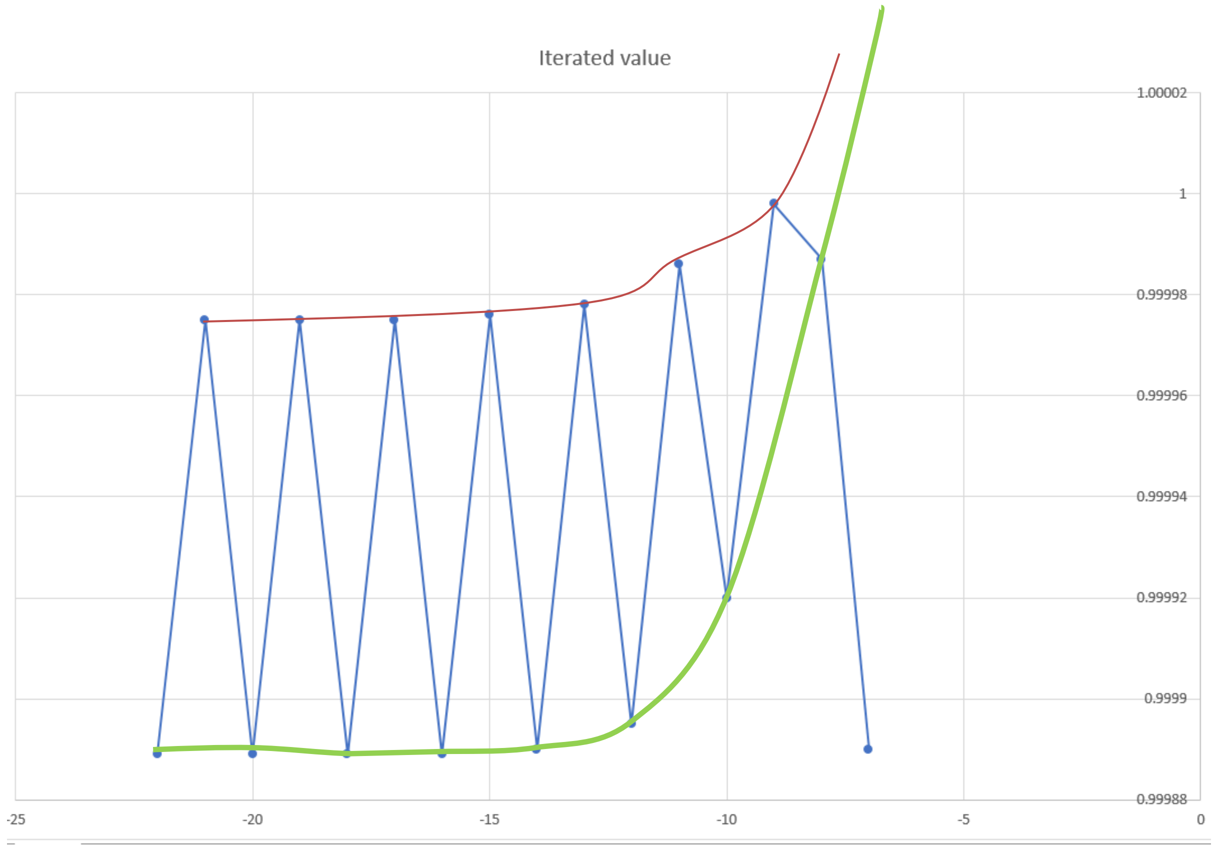
But as you go deeper down, the values oddly start limiting to 1 – which is another correct solution!

Deeper down, the program stops working when the initial value is -23 or less. `-nan(ind)` is an indicator that the main block of code never executed. Unlike when the initial value was equal to 0 (because $\lim_{x \to 0} \frac{f(x)}{f'(x)} = 0$,and then $x_1 - x_2 = 0$ – which satisfied neither of the conditions required for the block of code to execute), clearly if the iterator was limiting to 1 when the initial value was -22, there is no reason for the iterator not to execute when the initial value is -23. It may have to do with the limit of resolution, but I'm not really sure.



```
F:\WINDOWS\system32\cmd.exe
The iterative solution when initial value = -35 is -nan(ind)
The iterative solution when initial value = -34 is -nan(ind)
The iterative solution when initial value = -33 is -nan(ind)
The iterative solution when initial value = -32 is -nan(ind)
The iterative solution when initial value = -31 is -nan(ind)
The iterative solution when initial value = -30 is -nan(ind)
The iterative solution when initial value = -29 is -nan(ind)
The iterative solution when initial value = -28 is -nan(ind)
The iterative solution when initial value = -27 is -nan(ind)
The iterative solution when initial value = -26 is -nan(ind)
The iterative solution when initial value = -25 is -nan(ind)
The iterative solution when initial value = -24 is -nan(ind)
The iterative solution when initial value = -23 is -nan(ind)
The iterative solution when initial value = -22 is 0.999889
The iterative solution when initial value = -21 is 0.999975
The iterative solution when initial value = -20 is 0.999889
The iterative solution when initial value = -19 is 0.999975
The iterative solution when initial value = -18 is 0.999889
The iterative solution when initial value = -17 is 0.999975
The iterative solution when initial value = -16 is 0.999889
The iterative solution when initial value = -15 is 0.999976
The iterative solution when initial value = -14 is 0.99989
The iterative solution when initial value = -13 is 0.999978
The iterative solution when initial value = -12 is 0.999895
The iterative solution when initial value = -11 is 0.999986
```

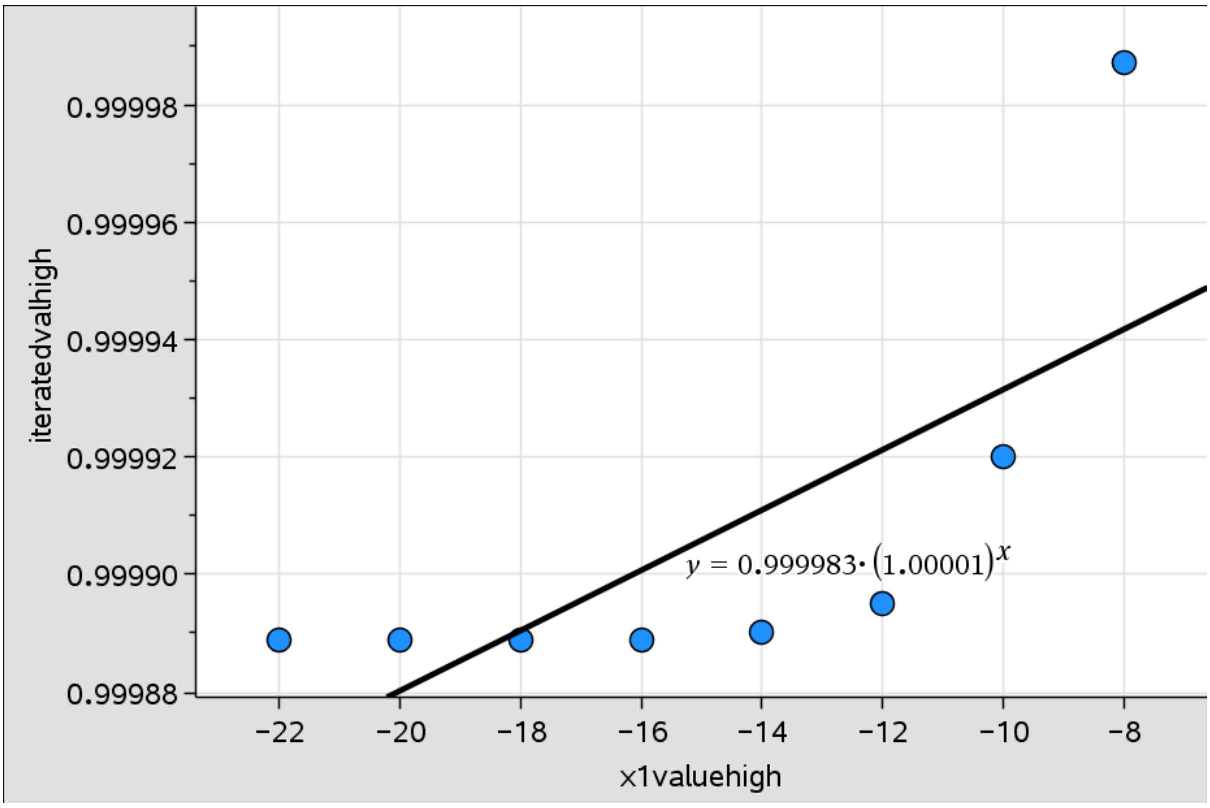
Additionally, the iterative solution actually oscillates a bit between each value. When $x_1 = -11$, for instance, the iterated value is 0.999986. It's slightly more when $x_1 = -12$, and slightly less when $x_1 = -13$. Plotting a scatter graph with the iterated values obtained between $x_1 = -7$ and $x_1 = -22$ shows an interesting trend: -



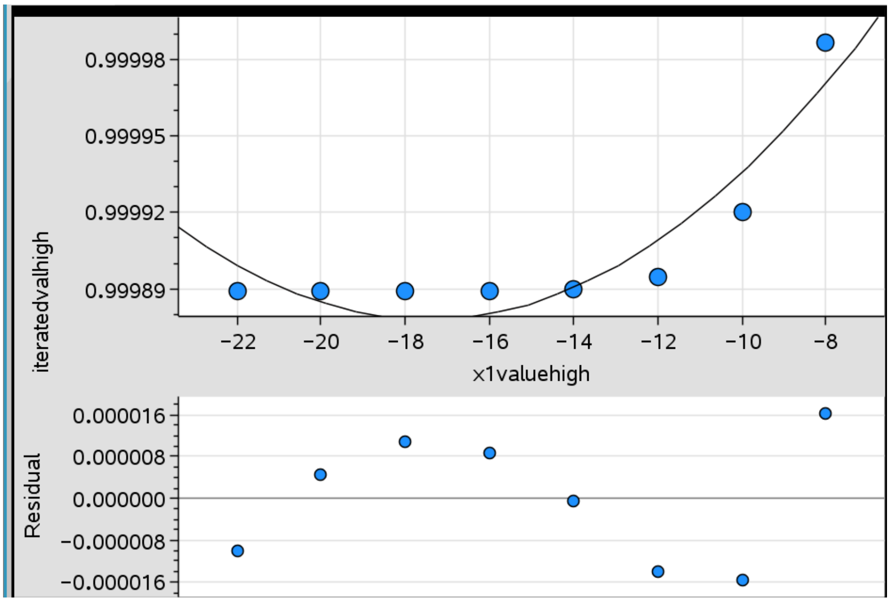
Take a look at the red and green curves carefully, which show the 'highs' and 'lows' of the iteration graph respectively. It should also be noted that the slope of the *lows* drastically increase in comparison with the slope of the *highs*. (All this must be considered in relation with each other; the numerical slope of both curves is still *very* low, and the precision of the graph is higher than usual!)

Both of them follow a roughly exponential graph with relation to each other. Let's try to separate the high and lows and compare based on that.

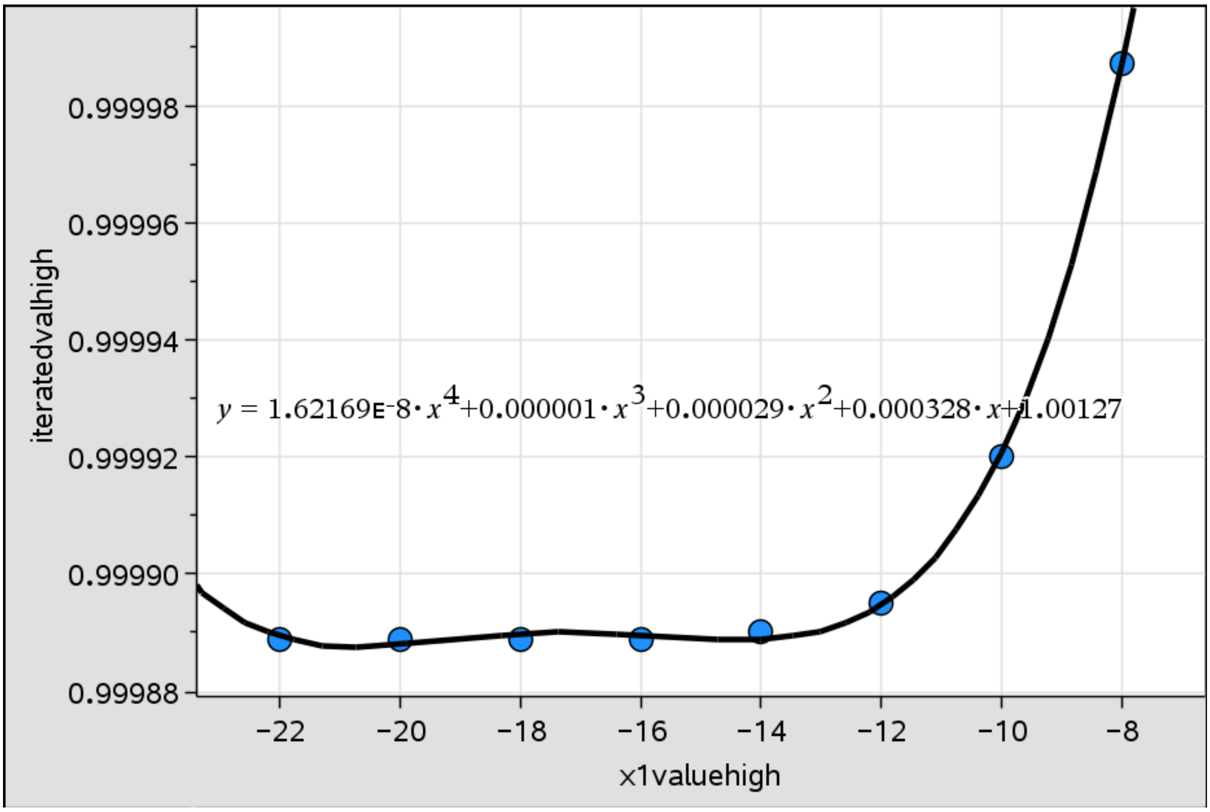
If we separate the *lows* for instance, and graph it, it (still) appears to follow an exponential trend. However, applying an exponential regression backfires. The reason is that δy is very small, so in $y = a.b^x$, $a \sim b \cong 1$, hence making the graph strikingly linear. The same problem will happen if the other one is separated instead.



The polynomial-based graphs also perform poorer than expected, mainly because of the limited sample size and unfavourable characteristics (quadratics don't have $y = 0.99989$ as a horizontal asymptote).

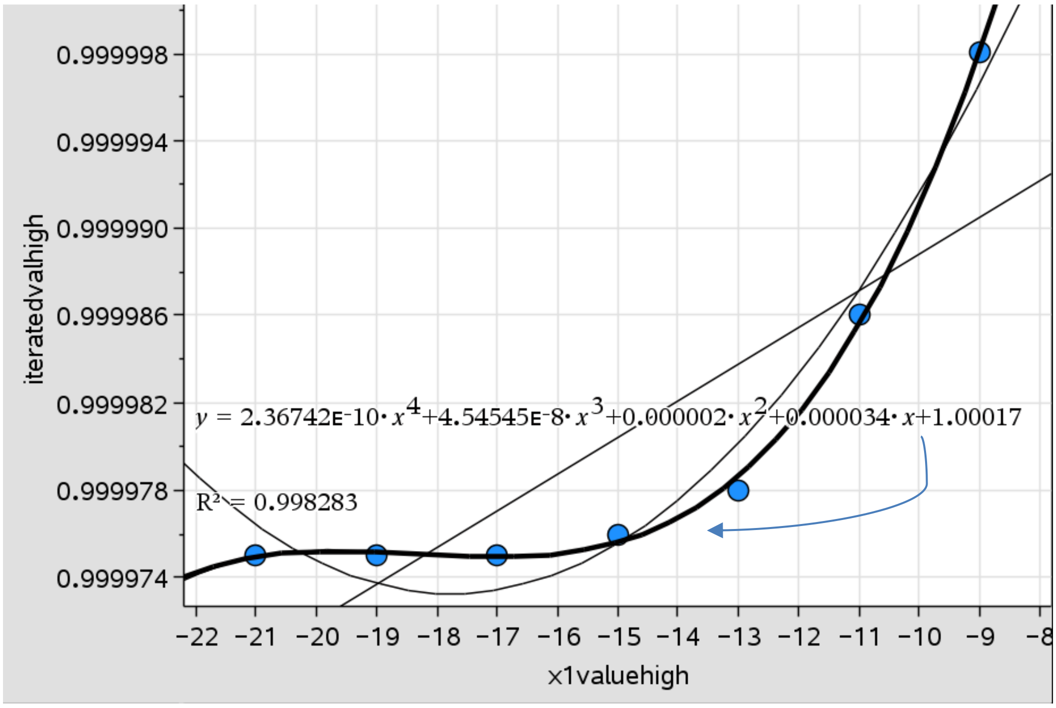


The cubic reduces the error, but only the quartic manages to make it reasonable, though it's still not very desirable because of the presence of very subtle local maximas and minimas, though it can be argued that cannot be removed, because of the limited sample and the fact that the graph appears to stay at 0.99989 as $y \to -\infty$.



What if we graph the *highs* instance? Then the graph roughly follows a similar trend, as shown below, but with a much lower slope. As expected from the analysis of the previous graph, the application of the exponential graph backfires (again).

While the quadratic regression is still undesirable, the cubic regression does quite well this time. The quartic, as from the previous instance, does the best.
The collated graphs, along with the original plotted values, are shown below.



…

But the behaviour of the iterator when the initial value is positive is significantly more predictable, but with incorrect values. As we go up, the iteration values differ from the initial value exactly by 2.

```
F:\WINDOWS\system32\cmd.exe
The iterative solution when initial value = 14 is 12
The iterative solution when initial value = 15 is 13
The iterative solution when initial value = 16 is 14
The iterative solution when initial value = 17 is 15
The iterative solution when initial value = 18 is 16
The iterative solution when initial value = 19 is 17
The iterative solution when initial value = 20 is 18
The iterative solution when initial value = 21 is 19
The iterative solution when initial value = 22 is 20
The iterative solution when initial value = 23 is 21
The iterative solution when initial value = 24 is 22
The iterative solution when initial value = 25 is 23
The iterative solution when initial value = 26 is 24
The iterative solution when initial value = 27 is 25
The iterative solution when initial value = 28 is 26
The iterative solution when initial value = 29 is 27
The iterative solution when initial value = 30 is 28
The iterative solution when initial value = 31 is 29
The iterative solution when initial value = 32 is 30
The iterative solution when initial value = 33 is 31
The iterative solution when initial value = 34 is 32
The iterative solution when initial value = 35 is 33
The iterative solution when initial value = 36 is 34
The iterative solution when initial value = 37 is 35
The iterative solution when initial value = 38 is 36
```

Applying the limit to the derivation shows a fairly interesting result. The steps in the evaluation of the limit are not shown here, but they are fairly obvious – in the limit, all instances of $\frac{a}{bx}$ are zeroed out.

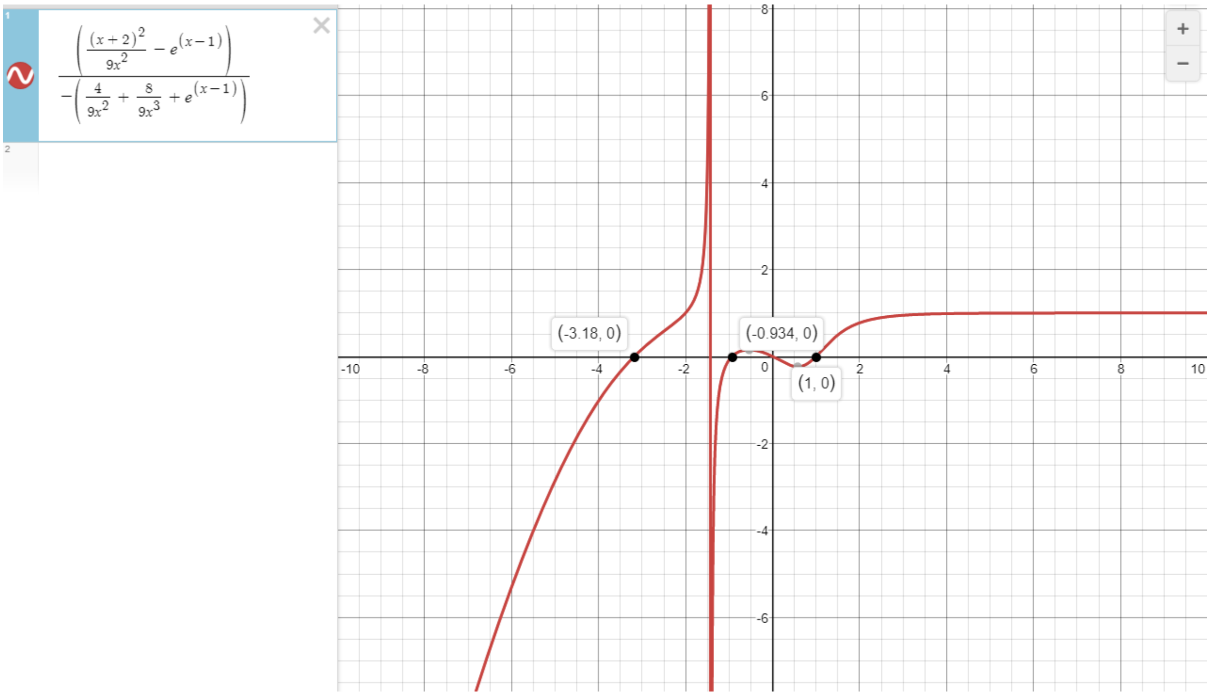
$$\lim_{x\to\infty^+}\frac{f(x)}{f'(x)} = \lim_{x\to\infty^+}\frac{\left(\frac{x+2}{3x}\right)^2 - e^{x-1}}{-\frac{4}{9(x^2)} - \frac{8}{9(x^3)} - e^{x-1}} = 1$$

But

$$\lim_{x\to\infty^-}\frac{f(x)}{f'(x)} = \lim_{x\to\infty^-}\frac{\left(\frac{x+2}{3x}\right)^2 - e^{x-1}}{-\frac{4}{9(x^2)} - \frac{8}{9(x^3)} - e^{x-1}} = -\infty$$

The value of the limit obtained to +∞ can explain the reason behind getting whole numbers in the iteration. It appears from the limit that the iteration gets performed twice – hence the difference by 2. As for -∞, this can be explained by the iteration itself.

A graphical look confirms our suspicions all along – and agrees with it. It must be noted that the three solutions obtained earlier still work in the iterator, interestingly.



While initially I was unfazed by the iteration method, after some analysis (and especially after the blunder), it sure seems like an interesting method, but it is not the best method for the main reason that an iteration method does not exactly allow you to find the value yourself. Additionally, there does seem to be some problems with the method, as can be seen from the incorrect values.

2.8 *Other considered options*

There was a couple of iterative-based method which were taken into consideration. One of them was the Taylor expansion series

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \qquad -\infty < x < \infty$$

which does not really help as it is simply an iteration method. It does not help like this…

$$e^{p-1} = 1 + \frac{p-1}{1!} + \frac{(p-1)^2}{2!} + \frac{(p-1)^3}{3!} + \cdots$$

which could also be simplified as

$$e^{p-1} = p + \frac{(p-1)^2}{2!} + \frac{(p-1)^3}{3!} + \cdots$$

The only other reasonable idea that I could find was to think about the binomial expression

$$(1+x)^n = 1 + \frac{nx}{1!} + \frac{n(n-1)x^2}{2!} + \cdots$$

$$(x+a)^n = \sum_{k=0}^{n} \binom{n}{k} x^k a^{n-k}$$

I wouldn't count on it (especially with the difficulty of getting both as infinite series), but who knows…

3 Given a triangle with sides x, $\frac{1}{x}$ and c, where c is a constant and all the sides are positive reals greater than 0, for a particular value of c, find the minimum and maximum values that x can take.

3.1 *Abstract*

I got this idea when taking a look at the old 7th standard inequality that I learnt and was thinking of an idea that I could put as a higher order thinking skills question for a Class 9 paper that someone had requested me to put. (They learn about this inequality again)

I soon thought of a question I thought was challenging enough: -

Q19 (3 marks) – Given a triangle whose sides are $x + 3$, $3x - 1$ and 3 cm, prove that while x must be greater than $\frac{1}{3}cm$, x must be less than $1\ cm$.

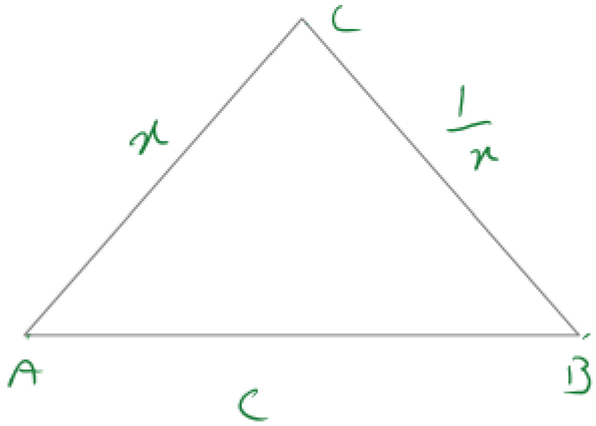
But I discovered an interesting roadblock when working out a solution for this. I discovered that different cases would give out different values of, some of which would completely disrupt the question. I ended up getting $x > 1$ when taking one particular case! In the end that question was abandoned in the paper I gave him in the end.

This prompted me to analyse this question in a much deeper level, and this is what made me come up with this beautiful idea based on the symmetry of the triangle inequality.
Probing this idea further only led me to more and more questions.

The solution is based on the fact that the sum of any 2 sides of a triangle is always greater than the 3rd side. Similarly, the difference of any 2 sides of a triangle is always less than the 3rd side.

We want to use this inequality to solve the problem.
Initial attempts focus on making some use of the 3 equations that are oddly different. After that, we work out the problem by considering 2 separate cases as to which side is the greatest.

3.2 *Initial attempts*



Given the triangle shown above,

We first take $AB + BC > AC$.(Case 1)

This gives us

$$c + \frac{1}{x} > x$$

(1)

Simplifying,

$$\frac{cx + 1}{x} > x$$

$$cx + 1 > x^2$$

$$cx + 1 - x^2 > 0$$

$$x^2 - cx - 1 < 0$$

(2).

Then we take $BC + AC > AB$. This is Case 2.

We should (hopefully) be getting the same equation.

Anyway, we get

$$\frac{1}{x} + x > c$$

(3)

Simplifying,

$$\frac{1 + x^2}{x} > c$$

$$1 + x^2 > cx$$

$$x^2 - cx + 1 > 0$$

(4).

However, it turns out that we are **not** getting the same equation.

What about $AC + AB > BC$?(Case 3)

We then get

$$x + c > \frac{1}{x}$$

(5)

Simplifying,

$$x^2 + cx > 1$$

$$x^2 + cx - 1 > 0$$

(6).

Again, we're getting something else. What's going on? If the equations are different, then how can we find out which one is correct?

It gets no better when we actually find out the solutions to each of the quadratic equations.

Taking the equation on Case 1 (equation 2),

$$x^2 - cx - 1 < 0$$

We get

$$\frac{c \pm \sqrt{c^2 + 4}}{2}$$

Or ,

$$\frac{c + \sqrt{c^2 + 4}}{2} < x < \frac{c - \sqrt{c^2 + 4}}{2}$$

Note that the equation itself is not valid when $c - \sqrt{c^2 + 4}$ is less than 0, which actually covers all values of c in the first place, hence invalidating the whole expression unless we write something like

$$0 < x > \frac{c + \sqrt{c^2 + 4}}{2}$$

(7).

(When $c = 1$, $> \frac{1+\sqrt{5}}{2}$, thus showing that this is always more than the golden ratio!)

Taking the second case (Equation 4), we get

$$x^2 - cx + 1 > 0$$

Or,

$$\frac{c \pm \sqrt{c^2 - 4}}{2}$$

Thus we get

$$x < \frac{c - \sqrt{c^2 - 4}}{2}$$

otherwise

$$x > \frac{c + \sqrt{c^2 - 4}}{2}$$

(8)
However, both equations are invalid when $c < 2$, as then $\sqrt{c^2 - 4}$ is complex.

Hence this equation is itself invalid when $c < 2$.

Now taking the 3rd case (Equation 6),

$$x^2 + cx - 1 > 0$$

Then,

$$\frac{-c \pm \sqrt{c^2 + 4}}{2}$$

Thus we get

$$x < \frac{-c - \sqrt{c^2 + 4}}{2}$$

(9)

or,

$$x > \frac{-c + \sqrt{c^2 + 4}}{2}$$

(10).

But (7) says that

$$x > \frac{c + \sqrt{c^2 + 4}}{2}$$

But because of (10), as $\frac{-c+\sqrt{c^2+4}}{2} < \frac{c+\sqrt{c^2+4}}{2}$, it follows that

$$x > \frac{c+\sqrt{c^2+4}}{2}$$

(11)

This can be disproven with a simple example. When $= 1$, by Equation 11, we have $x > \frac{1+\sqrt{5}}{2}$. This is incorrect, as when $= 1$, the triangle rules are not violated – meaning that something's wrong!

When we take the 'reverse inequality' cases, we get the same equations as obtained before.

3.3 *The 'correct' option*

This time, we also divide this into 3 cases.

Case 1 or 3:- When x *or* $\frac{1}{x}$ is the largest side

If so, then

$$\frac{1}{x} + c > x$$

$$\frac{1}{x} + \frac{cx}{x} > \frac{x^2}{x}$$

$$x^2 - cx - 1 < 0$$

Solutions then would be

$$\frac{c \pm \sqrt{c^2+4}}{2}$$

But $\frac{c-\sqrt{c^2+4}}{2}$ is rejected, because its value is always below 0 for every value of c.

Hence

$$x < \frac{c \pm \sqrt{c^2+4}}{2}$$

But $x > c$(Given).

Hence the solution is then

$$c < x < \frac{c \pm \sqrt{c^2+4}}{2}$$

But what happens if $x < 1$? Then $\frac{1}{x} > 1$ and hence $\frac{1}{x}$ is now the largest side.

For this, we need to take Case 3.

Then, $x + c > \frac{1}{x}$

This would give

$$x^2 + cx - 1 > 0$$

$$\frac{-c \pm \sqrt{c^2 + 4}}{2}$$

In which case,

$$x > \frac{-c - \sqrt{c^2 + 4}}{2}$$

$$x < \frac{-c + \sqrt{c^2 + 4}}{2}$$

Taking modulus on the expressions obtained in Case 3, we get

$$\frac{-c + \sqrt{c^2 + 4}}{2} < x < \frac{c + \sqrt{c^2 + 4}}{2}$$

This represents the maximum and minimum values of x, atleast when $c < 2$.

But what happens when $c \geq 2$? Or when c is the largest side?

<u>Case 2: - When c is the largest side</u>

In this case, $x + \frac{1}{x} > c$

Which would result in

$$x^2 - cx + 1 > 0$$

With the solutions as

$$\frac{c \pm \sqrt{c^2 - 4}}{2}$$

Note that **both** solutions are invalid when $c < 2$.

But then,

$$x > \frac{c - \sqrt{c^2 - 4}}{2}$$

$$x < \frac{c + \sqrt{c^2 - 4}}{2}$$

is the solution.

Hence, the maximum and minimum values of x for a value of c are

$$\min(x) = \begin{cases} \dfrac{-c + \sqrt{c^2 + 4}}{2}, & c < 2\,, c > 2(x > c) \\ \dfrac{c - \sqrt{c^2 - 4}}{2}, & c > 2(c > x) \end{cases}$$

$$\max(x) = \begin{cases} \dfrac{c + \sqrt{c^2 + 4}}{2}, & c < 2\,, c > 2(x > c) \\ \dfrac{c + \sqrt{c^2 - 4}}{2}, & c > 2(c > x) \end{cases}$$

But it would be nice if we could do all this in a ‘one size fits all’ approach…

4 Given that

$$\mathbf{\sin x + \cos x = \tan x}$$

Derive expressions purely in terms of $\mathbf{\sin x}$, $\mathbf{\cos x}$ and $\mathbf{\tan x}$ that can hence be used to find the value of $\boldsymbol{x}$ for which this equation is valid.

Do the same for their reciprocals: - $\operatorname{cosec} x + \sec x = \cot x$ purely in terms of $\operatorname{cosec} x$, $\sec x$ and $\cot x$.

4.1 *Abstract*

The idea comes from how we generally use trigonometric equations for. Mostly, we use it to solve basic trigonometric equations. But it is a very interesting idea to see whether the trigonometric symmetries hold in a different manner.

A student would usually learn trigonometric rations in the following manner: -

$$\sin x , \cos x , \tan x , \operatorname{cosec} x , \sec x , \cot\ x$$

I initially thought that it is impossible for the trigonometric ratios themselves to interact with other ratios, for instance

$$\sin x + \cos x = \tan x (1)$$

$$\operatorname{cosec} x + \sec x = \cot x (2)$$

However, when I graphically tested the first function for fun and was taken by surprise when I actually got *any* value, the adventure began.

We take the equation and then derive expressions in terms of

1 - $\sin x$, $\cos x$ & $\tan x$ for which (1) is valid.

2 - $\operatorname{cosec} x$, $\sec x$ & $\cot\ x$ for which (2) is valid.

This is a multi-part section.

Sections 4.2 to 4.5 will deal with case (1): - this is **Part 1**.

Sections 4.6 to 4.9 will deal with case (2): - this is **Part 2.**

Part 1 starts from the next section.

4.2 *Derivation of the sine expression*

Start with

$$\sin x + \cos x = \tan x$$

And put $\cos x = t$.[3]

Then the equation becomes

$$\sqrt{1-t^2}+t=\frac{\sqrt{1-t^2}}{t}$$

$$\sqrt{1-t^2}-\frac{\sqrt{1-t^2}}{t}=-t$$

$$\sqrt{1-t^2}\left(1-\frac{1}{t}\right)=-t$$

$$\sqrt{1-t^2}=-\frac{t}{1-\frac{1}{t}}$$

$$\sqrt{1-t^2}=-\frac{t}{\frac{t}{t}-\frac{1}{t}}$$

$$\sqrt{1-t^2}+\frac{t^2}{t-1}=0$$

Resubstituting,

$$\sqrt{1-\cos^2 x}+\frac{\cos^2 x}{\cos x-1}=0$$

$$\sin x\,(1-\cos x)=\cos^2 x$$

$$\sin x-\frac{1}{2}\sin 2x=1-\sin^2 x$$

$$\sin x+\sin^2 x-\frac{1}{2}\sin 2x=1$$

(1)

Unfortunately, we can't obtain the quadratic directly by putting $\sin x = u$, as the $\sin 2x$ term blocks that.

But we are persistent to get a quartic with $\sin x = t$.

Take (1).

$$\sin x+\sin^2 x-\frac{1}{2}\sin 2x=1$$

Then, write $\sin 2x$ in terms of sin and cos.

[3] We are not necessarily using the 'quickest', 'fastest' or 'best' method. We're illustrating the method used for derivation when I first did it back in April 2016. Note the relatively convoluted method.

$$\sin x + \sin^2 x - \frac{1}{2}\sin 2x = 1$$

$$\sin^2 x + \sin x - \sin x \cos x = 1$$

Put $\sin x = t$.

$$t^2 + t - t\left(\sqrt{1 - t^2}\right) = 1$$

$$-t\left(\sqrt{1 - t^2}\right) = 1 - t - t^2$$

Squaring both sides and simplifying,

$$t^2(1 - t^2) = (1 - t - t^2)^2$$

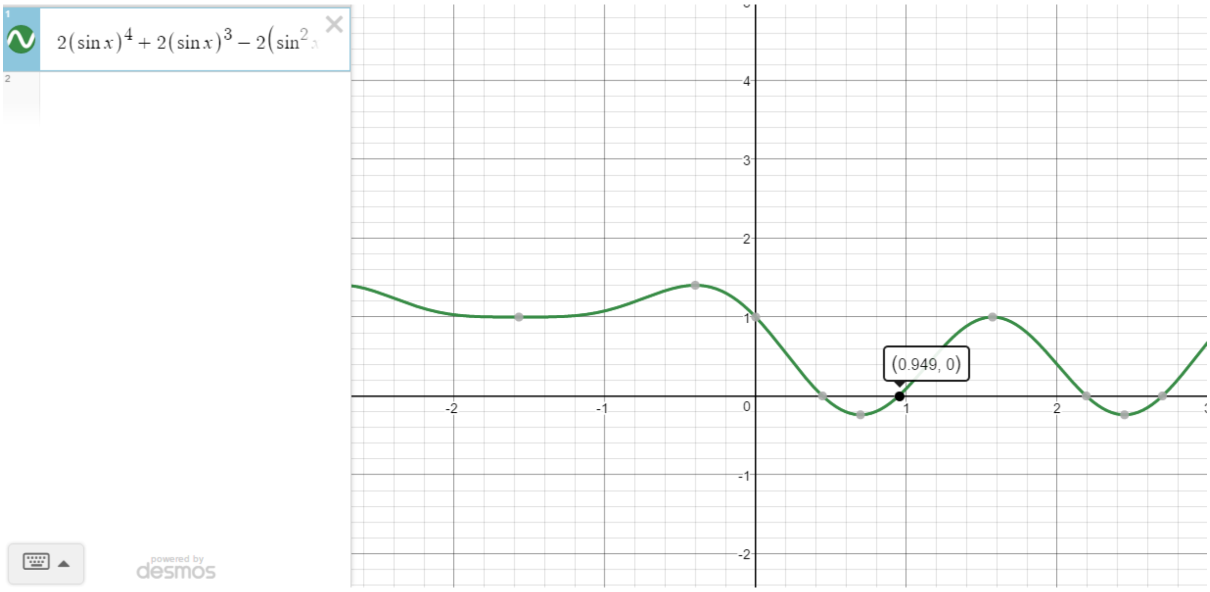
$$t^2 - t^4 = 1 + t^2 + t^4 - 2t + 2t^3 - 2t^2$$

$$2t^4 + 2t^3 - 2t^2 - 2t + 1 = 0$$

Resubstituting,

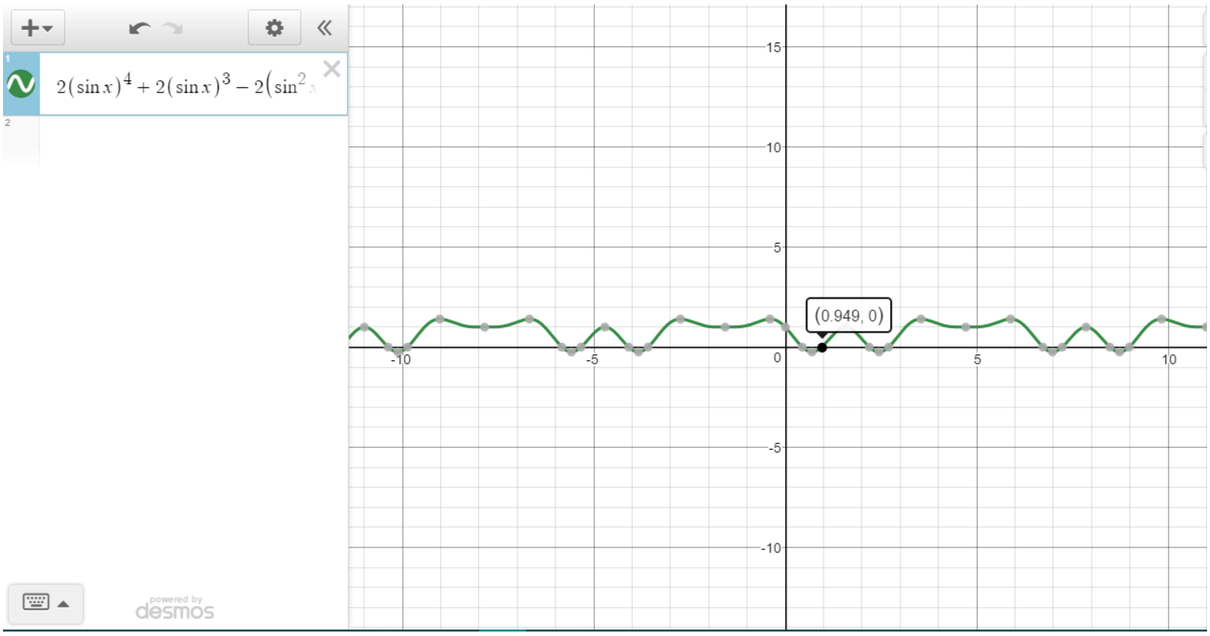
$$2\sin^4 x + 2\sin^3 x - 2\sin^2 x - 2\sin x + 1 - 0$$

A graphical check reveals that we get $x = 0.949$ rad.



Note that we would actually get the same answer had we graphed (1), but everything else will be different.

If we zoom out of the graph, we get a really beautiful ‘cavity curve’ instead!



4.3 *Derivation of the cosine equation*

Again, start with

$$\sin x + \cos x = \tan x$$

And put $\sin x = t$.

Now you get

$$t + \sqrt{1 - t^2} = \frac{t}{\sqrt{1 - t^2}}$$

$$\sqrt{1 - t^2} = \frac{t}{\sqrt{1 - t^2}} - t$$

Squaring both sides,

$$\left(\sqrt{1 - t^2}\right)^2 = \left(\frac{t}{\sqrt{1 - t^2}} - t\right)^2$$

$$1 - t^2 = \frac{t^2}{1 - t^2} - \frac{2t^2}{\sqrt{1 - t^2}} + t^2$$

Resubstituting,

$$1 = \frac{t^2}{1 - \sin^2 x} - \frac{2t^2}{\sqrt{1 - \sin^2 x}} + 2t^2$$

$$1 = \frac{\sin^2 x}{\cos^2 x} - \frac{2\sin^2 x}{\cos x} + 2\sin^2 x$$

$$1 = \sin^2 x\left(\frac{1}{\cos^2 x} - \frac{2}{\cos x} + 2\right)$$

$$1 = (1 - \cos^2 x)(\frac{1}{\cos^2 x} - \frac{2}{\cos x} + 2)$$

Now put $\cos x = t$.

$$1 = (1 - t^2)(\frac{1}{t^2} - \frac{2}{t} + 2)$$

$$t^2 = (1 - t^2)(1 - 2t + 2t^2)$$

Expanding,

$$t^2 = 1 - 2t + 2t^2 - t^2 + 2t^3 - 2t^4$$

$$2t^2 = 1 - 2t + 2t^2 + 2t^3 - 2t^4$$

$$1 - 2t + 2t^3 - 2t^4 = 0$$

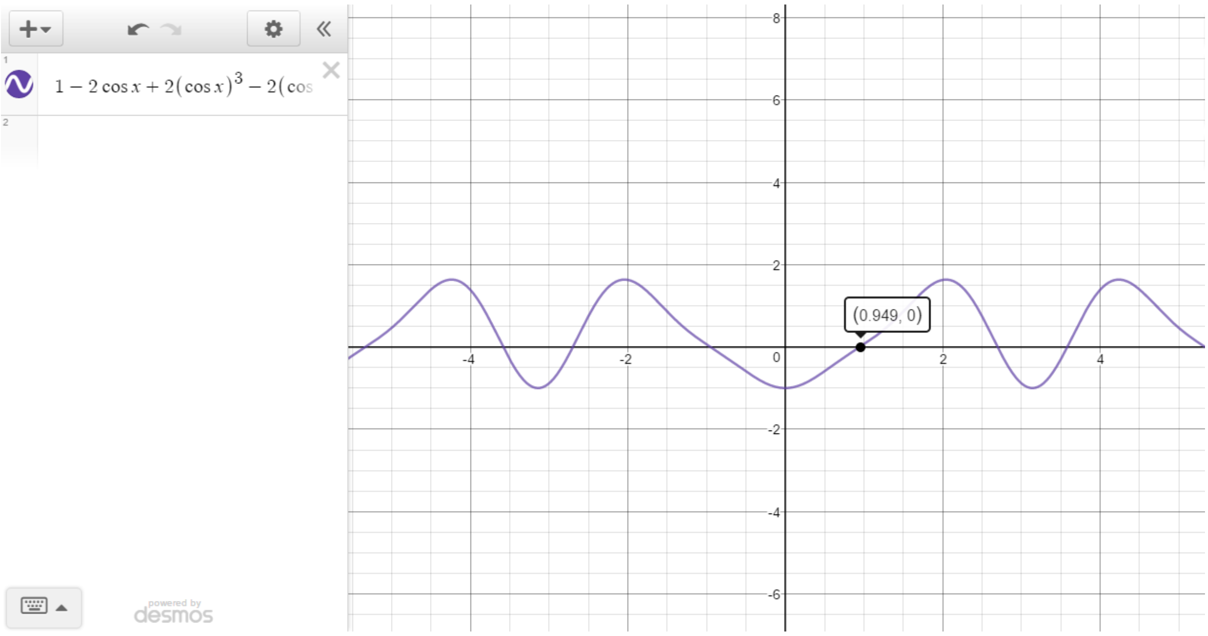
(2)

Replacing back, we get the final equation: -

$$1 - 2\cos x + 2\cos^3 x - 2\cos^4 x = 0$$

(3)

A graphical check reveals the same solution, but with a different curve: -



Note the irregularity of the curve.

4.4 Derivation of the tangent equation

Start with the equation again: -

$$\sin x + \cos x = \tan x$$

Writing expression in terms of $\tan x$,[4]

$$\sqrt{1-\cos^2 x}+\cos x=\tan x$$

$$\sqrt{1-\frac{1}{\sec^2 x}}+\frac{1}{\sec x}=\tan x$$

$$\sqrt{1-\frac{1}{\sec^2 x}}+\frac{1}{\sec x}=\tan x$$

$$\sqrt{1-\frac{1}{1+\tan^2 x}}+\frac{1}{\sqrt{1+\tan^2 x}}=\tan x$$

Put $\tan x=t$.

$$\sqrt{1-\frac{1}{1+t^2}}+\frac{1}{\sqrt{1+t^2}}=t$$

$$\sqrt{\frac{1+t^2-1}{1+t^2}}+\frac{1}{\sqrt{1+t^2}}=t$$

$$\frac{1}{\sqrt{1+t^2}}(t+1)=t$$

Squaring both sides,

$$\frac{1}{1+t^2}(t^2+2t+1)=t^2$$

$$t^2+2t+1=t^2(1+t^2)$$

$$t^2+2t+1=t^2+t^4$$

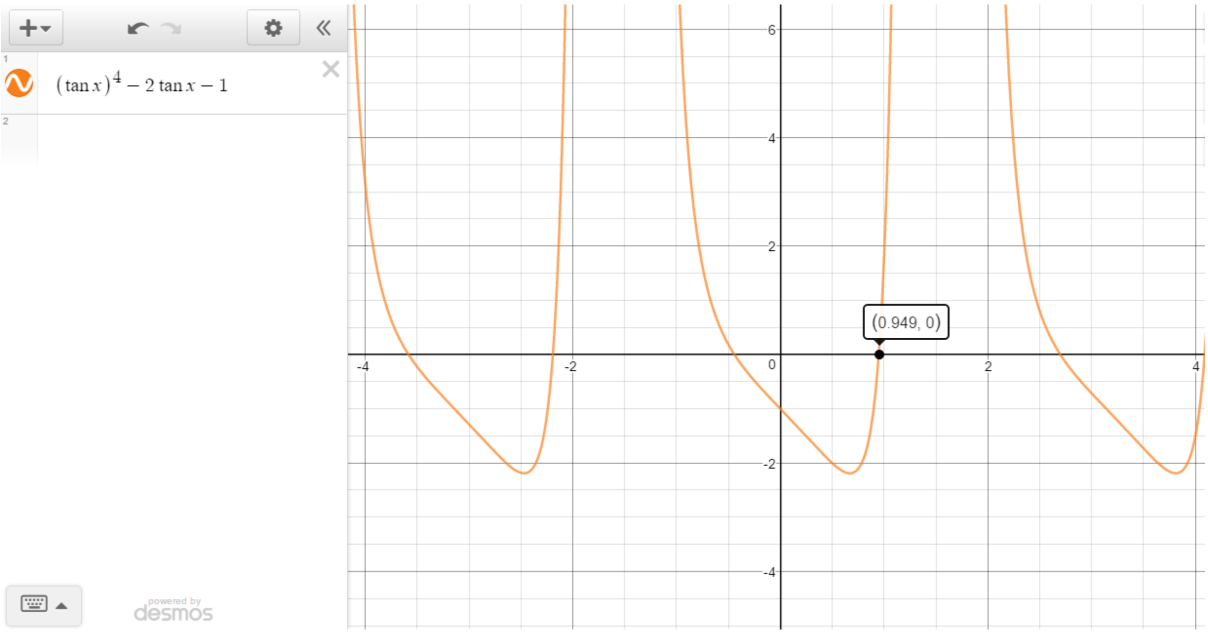
$$t^4-2t-1=0$$

Resubstituting,

$$\tan^4 x-2\tan x-1=0$$

[4] An additional option exists. Square both sides, and write $\sin 2x$ in terms of $\tan x$. Then put $\tan x=t$. We would still get the same expression.

Again, a graphical check reveals the same solution as we found before: -



Note the significant difference of this graph as compared to the other graphs. This is mostly because the range of the tangent function goes to infinity , as compared with the sine and cosine functions , both of which go only up to 1.

<u>Alternative</u>

It should also be possible to get an expression in terms of $\tan\frac{x}{2}$ by using the half angle formulas.

If we start with

$$\sin x + \cos x = \tan x$$

$$\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} + \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}}$$

$$\left\{2\tan\frac{x}{2} + 1 - \tan^2\frac{x}{2}\right)\left(1-\tan^2\frac{x}{2}\right) = 2\tan\frac{x}{2}\left(1+\tan^2\frac{x}{2}\right)$$

It can be noted from here while the sides of the expressions offer some similarity , there is no simplification possible at all.

Let us take $\tan\frac{x}{2} = t$.

This gives us

$$(2t + 1 - t^2)(1 - t^2) = 2t(1 + t^2)$$

It is immediately obvious that there will be something interesting to cancel. Let's go on.

$$2t - 2t^3 + 1 - t^2 - t^2 + t^4 = 2t + 2t^3$$

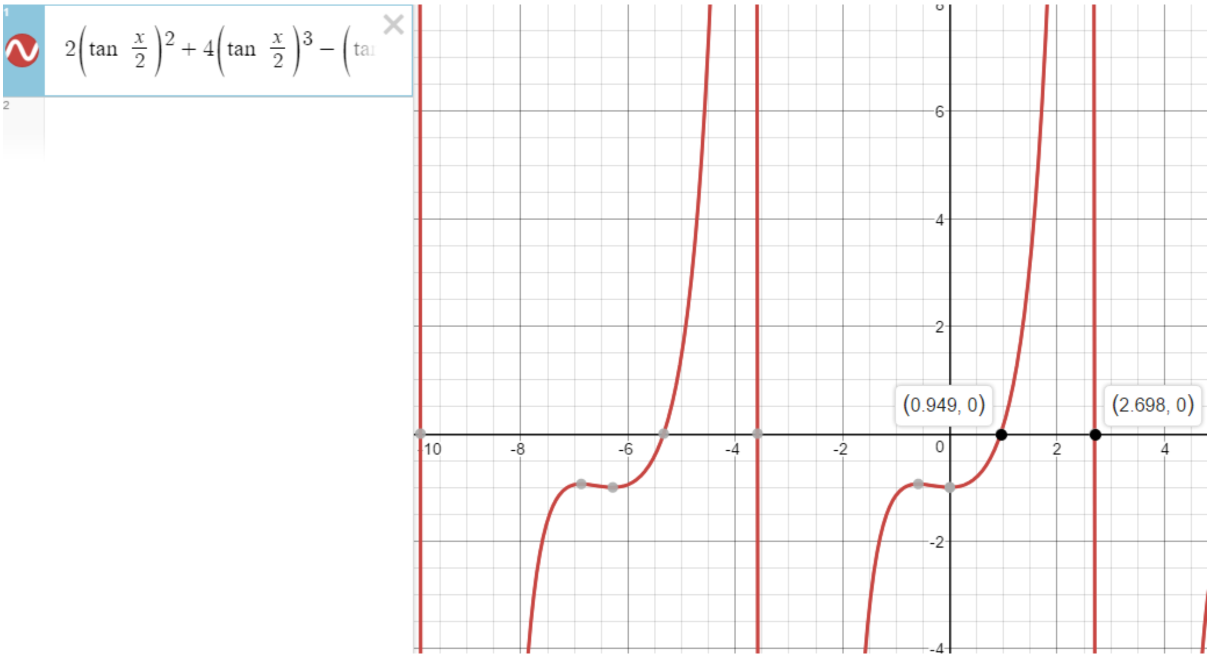
$$-4t^3 + 1 - 2t^2 + t^4 = 0$$

$$2t^2 + 4t^3 - t^4 = 1$$

Unfortunately, the simplification wasn't so nice here. There's no mistake however. Let's resubstitute.

$$2\tan^2\frac{x}{2} + 4\tan^3\frac{x}{2} - \tan^4\frac{x}{2} = 1$$

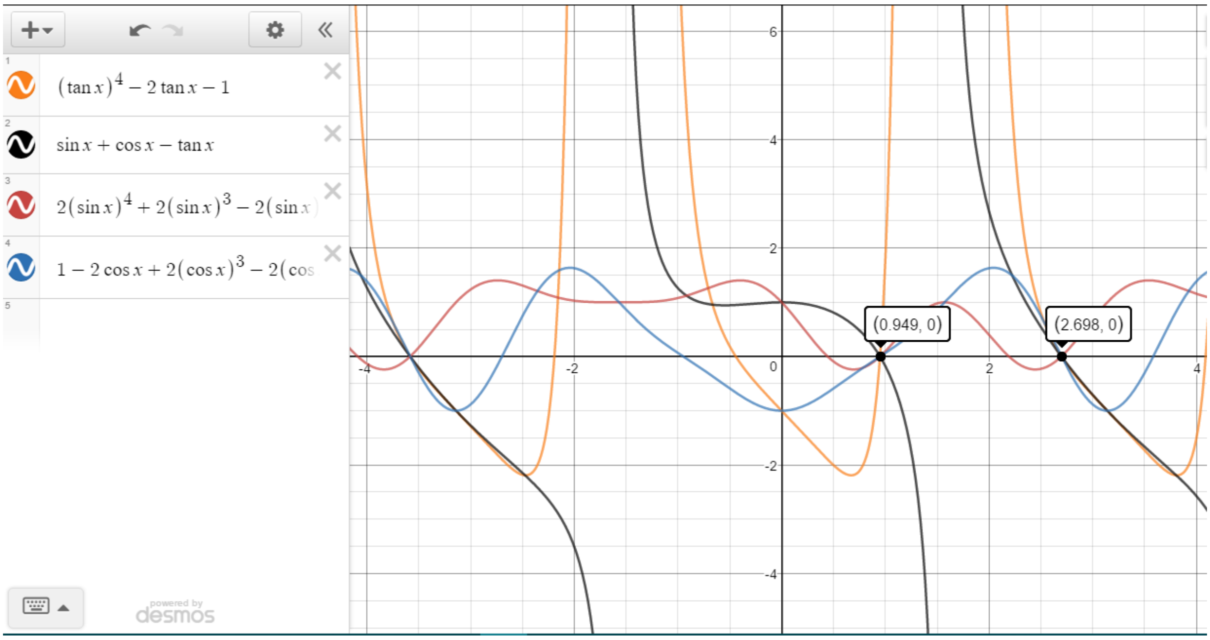
Giving that one a graph ,



The value is correct , and even the graph can be said as remarkably similar.

4.5 *Putting it all together(Part 1)*

As we have seen before, the derivation of the sine, cosine and tangent expressions look completely different at first, but when we check for the solution, one of the solutions is always common, even though the graphs themselves look (and actually are!) completely unrelated to each other. This is more evident if we put all the equations derived, including the original question, into the graph. We get a beautiful series of curves, as shown: -



This serves as a proof that each equation has no connection with the other, that is, we cannot use common trigonometric equations to derive one from another. If that happened, the graphs will coincide with each other at every point.

They also converge again at another point, which is less than even π. Suffice to say, they certainly seem like a mystery!

They also evoke interest with their unusual and irregular graphs.

End of Part 1.

Part 2 starts from the next section.

4.6 *Derivation of the cosecant equation*

Take the equation: -

$$\operatorname{cosec} x + \sec x = \cot\ x$$

and modify the equation appropriately so that it is in terms of $\operatorname{cosec} x$ only: -

$$\operatorname{cosec} x + \frac{1}{\cos x} = \frac{\cos x}{\sin x}$$

$$\operatorname{cosec} x + \frac{1}{\sqrt{1-\sin^2 x}} = \left(\sqrt{1-\sin^2 x}\right).\operatorname{cosec} x$$

$$\operatorname{cosec} x + \frac{1}{\sqrt{1-\left(\frac{1}{\operatorname{cosec}^2 x}\right)}} = \sqrt{1-\left(\frac{1}{\operatorname{cosec}^2 x}\right)}.\operatorname{cosec} x$$

Now put $\operatorname{cosec} x = t$.

$$t + \frac{1}{\sqrt{1 - \left(\frac{1}{t^2}\right)}} = \sqrt{1 - \left(\frac{1}{t^2}\right)}\,.\,t$$

Multiply by $\sqrt{1 - \left(\frac{1}{t^2}\right)}$ everywhere: -

$$t.\sqrt{1 - \left(\frac{1}{t^2}\right)} + 1 = \left(1 - \frac{1}{t^2}\right).\,t$$

Simplify the equation: -

$$t.\sqrt{1 - \left(\frac{1}{t^2}\right)} = \left(1 - \frac{1}{t^2}\right).\,t - 1$$

$$t^2\left(1 - \frac{1}{t^2}\right) = \left[\left(t - \frac{1}{t}\right) - 1\right]^2$$

$$t^2 - 1 = \left(t - \frac{1}{t}\right)^2 - 2\left(t - \frac{1}{t}\right) + 1$$

$$t^2 - 1 = \left(t^2 + \frac{1}{t^2} - 2\right) - 2t + \frac{2}{t} + 1$$

$$t^2 - 1 = t^2 + \frac{1}{t^2} - 1 - 2t + \frac{2}{t}$$

$$\frac{1}{t^2} - 2t + \frac{2}{t} = 0$$

$$1 - 2t^3 + 2t = 0$$

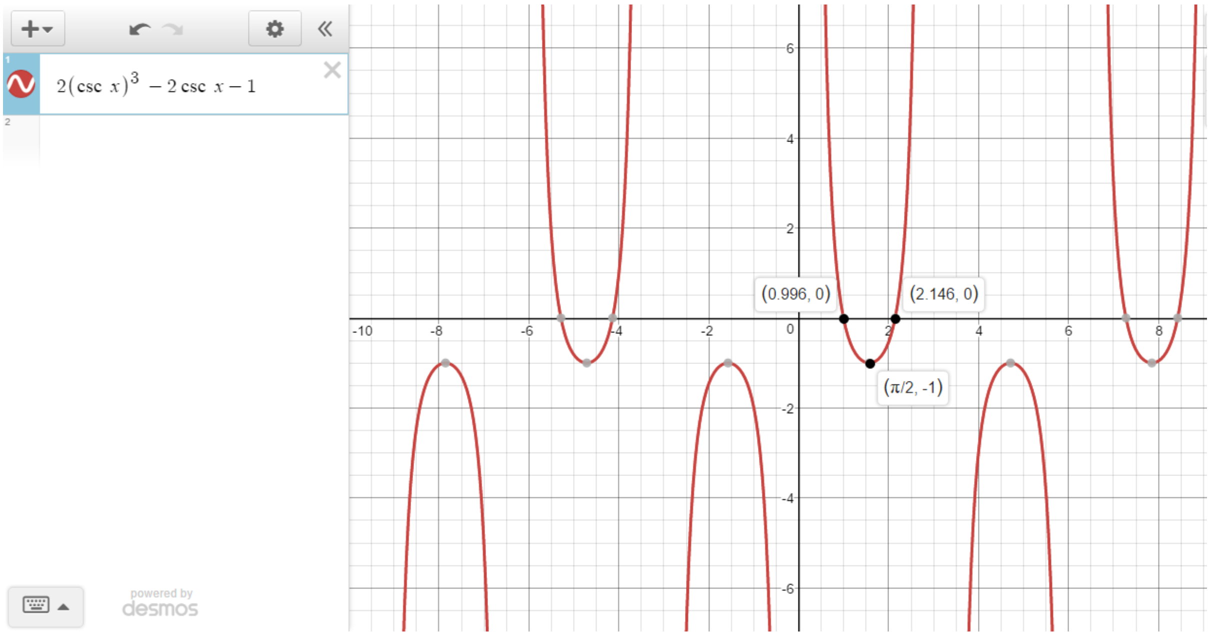
Multiplying by -1 to obtain

$$2t^3 - 2t - 1 = 0$$

Resubstituting,

$$2\operatorname{cosec}^3 x - 2\operatorname{cosec} x - 1 = 0$$

A graphical look reveals some interesting facts: -



Look at the graph. There are 2 distinct solutions in the interval $(0\ ,\pi)$, which is interesting. And look at the asymptote in $(\frac{\pi}{2}, -1)$. These 3 points form what I'd call as the *happy asymptote*. (Note that the central point is the local minima within the differentiable range $(0\ ,\pi)$, while the other points are the local zeros, so you could call it as the zero-minima triplet as well)

When I saw the graph, one thing that immediately came to my mind was to just locate these points and then draw a triangle through them. The symmetry is just so interesting; I'd even claim from it that the triangle drawn is equilateral!

Back to the graph, we see that the 'main' solution is $2.146\ rad$, as we will soon see.

4.7 *Derivation of the secant equation*

Take the same equation as before and write it in terms of $\sec x$ as before: -

$$\operatorname{cosec} x + \sec x = \cot x$$

$$\frac{1}{\sin x} + \sec x = \frac{\cos x}{\sin x}$$

$$\frac{1}{\sqrt{1-\cos^2 x}} + \sec x = \frac{1}{\sec x} . \frac{1}{\sqrt{1-\cos^2 x}}$$

$$\frac{1}{\sqrt{1-\frac{1}{\sec^2 x}}} + \sec x = \frac{1}{\sec x} . \frac{1}{\sqrt{1-\frac{1}{\sec^2 x}}}$$

As before, take $\sec x = t$ (or any other letter in the world[5])

$$\frac{1}{\sqrt{1-\frac{1}{t^2}}} + \text{t} = \frac{1}{t} \cdot \frac{1}{\sqrt{1-\frac{1}{t^2}}}$$

$$\sqrt{1-\frac{1}{t^2}}\left[\frac{1}{\sqrt{1-\frac{1}{t^2}}} + t\right] = \frac{1}{t}$$

$$1 + t\sqrt{1-\frac{1}{t^2}} = \frac{1}{t}$$

$$t\sqrt{1-\frac{1}{t^2}} = \frac{1}{t} - 1$$

$$\sqrt{1-\frac{1}{t^2}} = \frac{\frac{1}{t} - 1}{t}$$

Squaring both sides and simplifying to eliminate the denominator,

$$1 - \frac{1}{t^2} = \frac{(1-t)^2}{t^4}$$

$$\frac{t^2-1}{t^2} = \frac{1-2t+t^2}{t^4}$$

$$t^2(t^2-1) = 1-2t+t^2$$

$$t^4 - t^2 = 1 - 2t + t^2$$

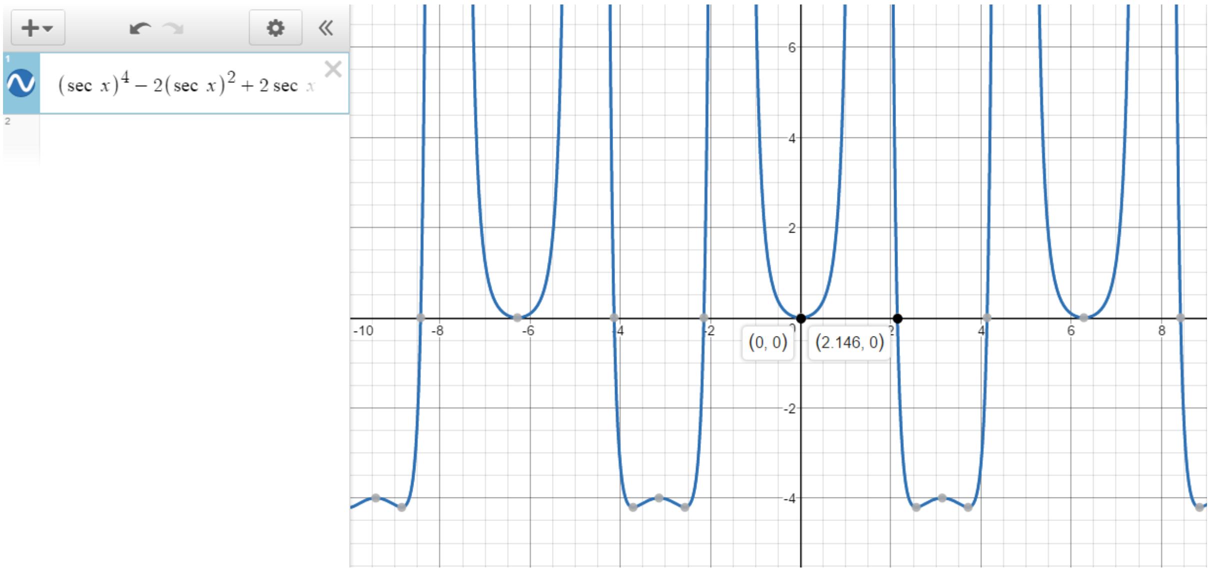
$$t^4 - 2t^2 + 2t - 1 = 0$$

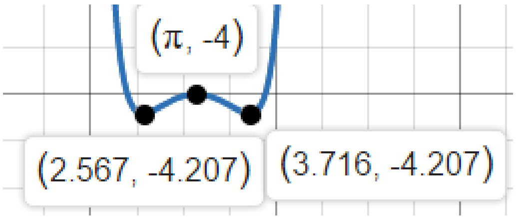
Resubstituting,

$$\sec^4 x - 2\sec^2 x + 2\sec x - 1 = 0$$

Taking a graphical look of the equation,

[5] with the probability of picking my "preferred letter" as $\frac{1}{25}$, because $\sec x \neq x$ generally. It is also worth noting that only English letters are included, otherwise the probability would be far less.





we see that (0,0) and (2.146, 0) are working points (as we'll see later, only the 2nd point is common to all equations). Additionally, the way that the graph moves near π is also a bit unusual. There are 2 points on the graph whose value is minimum, and they both lie on a relatively narrow range.

Is there something else we are missing? Let us find the local minimums and maximums: -

Let

$$f(x) = \sec^4 x - 2\sec^2 x + 2\sec x - 1 = 0$$

Differentiating,

$$f'(x) = 4(\sec^3 x)(\sec x \tan x) - 2\sec x\,(\sec x \tan x) + 2\sec x \tan x$$

$$f'(x) = 2\sec x \tan x\,(2\sec^3 x - \sec x + 1)$$

Equating it to find the main corner ranges to evaluate: -

$$\rightarrow 2\sec x \tan x = 0$$

This is equation (1).

The other equation would be (equation 2): -

$$\rightarrow 2\sec^3 x - \sec x + 1 = 0$$

Take equation 1.

We have

$$\sec x = 0$$

which has no solution.

The other option would then be

$$\tan x = 0$$

In that case, $x = 0$. This is also the solution that you would get if you solve equation (1) directly. Hence it can be inferred that one range is $(-\frac{\pi}{2}, 0)$

Now let us see the possible ranges to evaluate in equation 2.

We have

$$2\sec^3 x - \sec x + 1 = 0$$

It can be quickly seen that when $\sec x = -1$, then the equation is valid. Hence $x = \pi$ is one solution of the equation. We can use this fact to our advantage. As $\sec x = -1$ is one solution of the equation, by Factor Theorem, $\sec x + 1$ is one factor of the equation. Now divide the equation by $\sec x + 1$ to unravel the other factor(s).

$$2\sec^2 x - 2\sec x + 1$$

$$\sec x + 1 \,\big)\, 2\sec^3 x - \sec x + 1$$

$$- 2\sec^3 x \mp 2\sec^2 x$$

$$-2\sec^2 x - \sec x + 1$$

$$\mp 2\sec^2 x \mp 2\sec x$$

$$0 + \sin x + 1$$

$$\mp \sin x \mp 1$$

$$0$$

From that, we see that equation 2 can also be split as

$$(\sec x + 1)(2\sec^2 x - 2\sec x + 1)$$

Or, we have

$$\sec x + 1 = 0 \rightarrow \sec x = -1 \rightarrow x = \pi$$

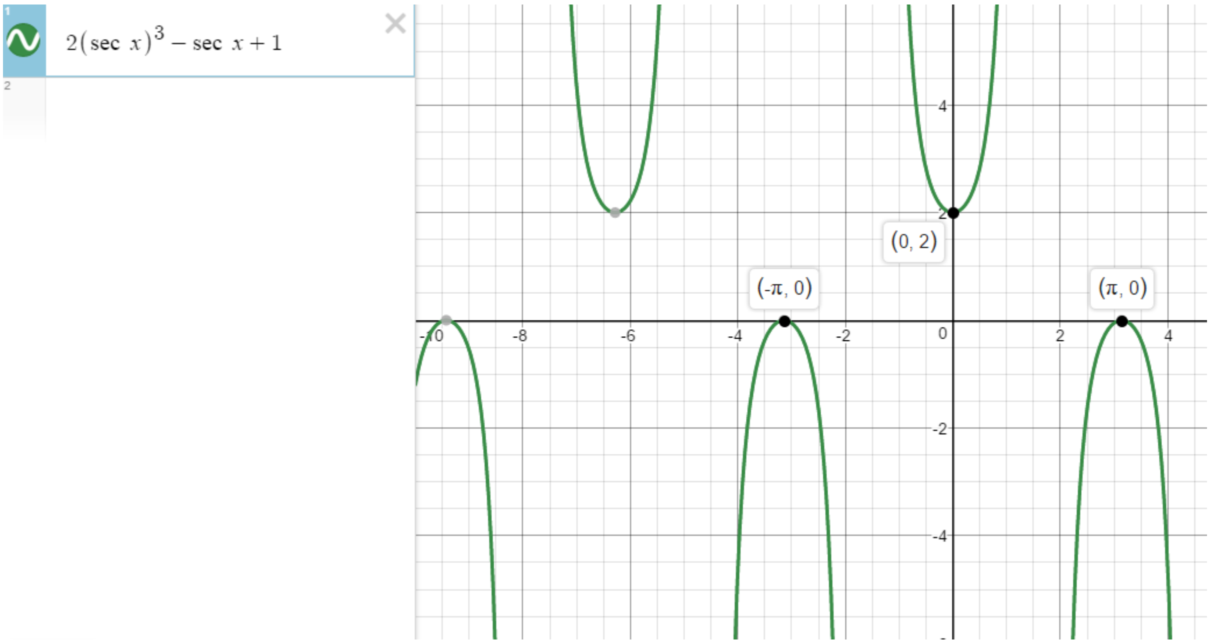
The other equation would then be

$$2\sec^2 x - 2\sec x + 1 = 0$$

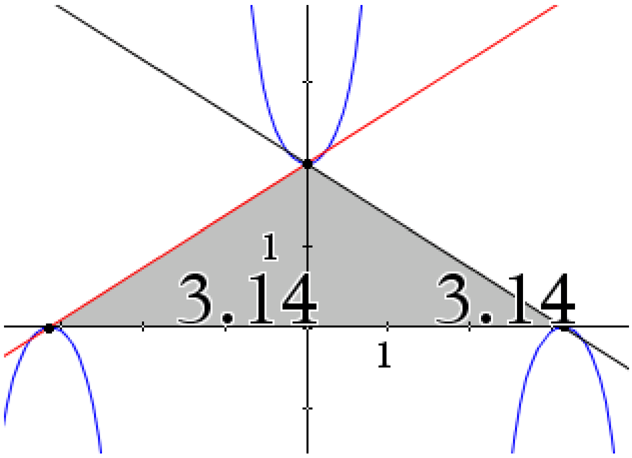
But that equation is sadly a complex quadratic; we cannot find any values of x for which the equation is equal to 0. Again (see section 2.3), we have $D = 0$ when $a = 1$. As $a > 1$, $D < 0$.

Hence another range of the derivative is $(0\,, \pi)$

A graphical check of the original cubic equation confirms our findings about having only 1 zero despite being a cubic: -



(Additionally, I can't help but make a note about the isosceles triangle that will be formed if we take the central points (i.e. when x and y are equal to 0). That looks like a nice little triangle ☺)



Note: *-The intersection point with the line* $y = \frac{2x}{\pi} + 2$ *(red) is with the* x *axis in the graph above, not with* $2\sec^3 x - \sec x + 1 = 0$ *(blue), whose intersection point with the line is slightly different, which is* $(\pm 3.16\,, 0)$ *instead of* $(\pm\pi, 0)$ *shown in the above graph. This similarly applies to the line* $y = 2 - \frac{2x}{\pi}$ *(black). Note though that the difference is purely superficial;* $x = \pi$ *is a solution of the secant cubic.*

Thus, we can say that

$$f'(x) = (2\sec x \tan x)(\sec x - 1)(2\sec^2 x - 2\sec x + 1)$$

We have 2 ranges to evaluate, that is from ($-\frac{\pi}{2}, 0$) and from $(0, \pi)$.

While x is between ($-\frac{\pi}{2}, 0$), $\sec x$ is positive, $\tan x$ is negative, $\sec x - 1$ is negative (as $\sec x < 1$) and $2\sec^2 x - 2\sec x + 1$ is positive (as its roots are complex).

Hence the sign would be $+.-.-.+$ or that the function is **increasing** within that interval.

Within the range $(0\,,\pi)$, $\sec x - 1$ is always negative and $2\sec^2 x - 2\sec x + 1$ is always positive. $\sec x$ & $\tan x$ follow the same sign within the range; within $\left(0, \frac{\pi}{2}\right)$ they are positive, and within $\left(\frac{\pi}{2}, \pi\right)$ they are both negative.

Hence the sign would be $\pm.\pm.-.+$ or that the function is **decreasing** within that interval.

Now we need to find local maximums and minimums. For that we take the 1st derivative test. The points to evaluate are 0 and π.

When $= 0$, $f'(0^-) > 0$. But $f'(0^+) < 0$. Hence it is a point of **maxima.**

When $x = \pi\,, f'(\pi^-) < 0$. However, this time though we have $f'(\pi^+) > 0$. Hence it is a relative **minimum.**

But an important question still remains. A graphical look (see here) shows that we have a couple of minima that appear to be separated by an equal distance from π. We aren't seeing that in our derivative test – what's up?

The derivative test that shows that π is a relative minima cannot be really true in this case as there indeed are values of x whose $f(x) < f(\pi)$. Something isn't really right , as we're not even getting the points in our derivative , let alone dertermining whether it is a maxima or minima!

4.8 *Derivation of the cotangent equation*

As before, start with the equation: -

$$\operatorname{cosec} x + \sec x = \cot x$$

Unlike the approach used in the previous 2 cases, this time we're going to square both sides.

$$(\operatorname{cosec} x + \sec x)^2 = \cot^2 x$$

$$\operatorname{cosec}^2 x + \sec^2 x + 2\operatorname{cosec} x \sec x = \cot^2 x$$

$$1 + \cot^2 x + \sec^2 x + \frac{2}{\sin x \,.\, \cos x} = \cot^2 x$$

$$1 + (1 + \tan^2 x) + 2\operatorname{cosec} x \sec x = 0$$

$$2 + \tan^2 x + 2\left(\frac{\operatorname{cosec} x}{\sec x}\right).\sec^2 x = 0$$

$$2 + \frac{1}{\cot^2 x} + 2\cot x\,(1 + \tan^2 x) = 0$$

$$2 + \frac{1}{\cot^2 x} + 2\cot x + 2\tan x = 0$$

$$2 + \frac{1}{\cot^2 x} + 2\cot x + \frac{2}{\cot x} = 0$$

Now put $\cot x = t$.

$$2 + \frac{1}{t^2} + 2t + \frac{2}{t} = 0$$

$$2t^2 + 1 + 2t^3 + 2t = 0$$

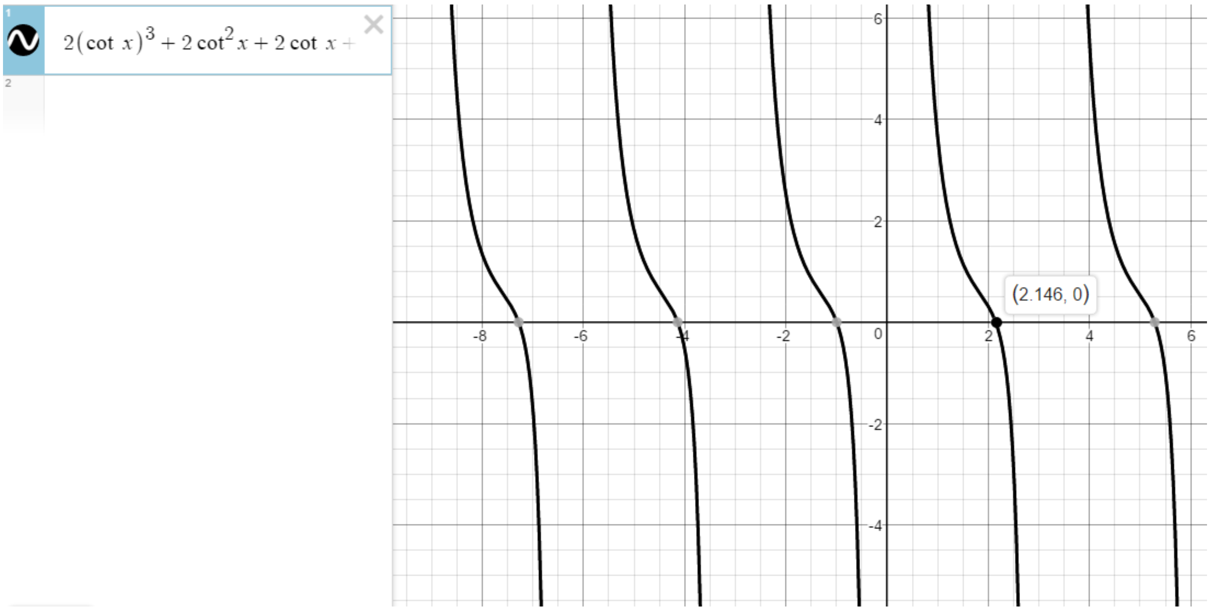
Rearranging powers,

$$2t^3 + 2t^2 + 2t + 1 = 0$$

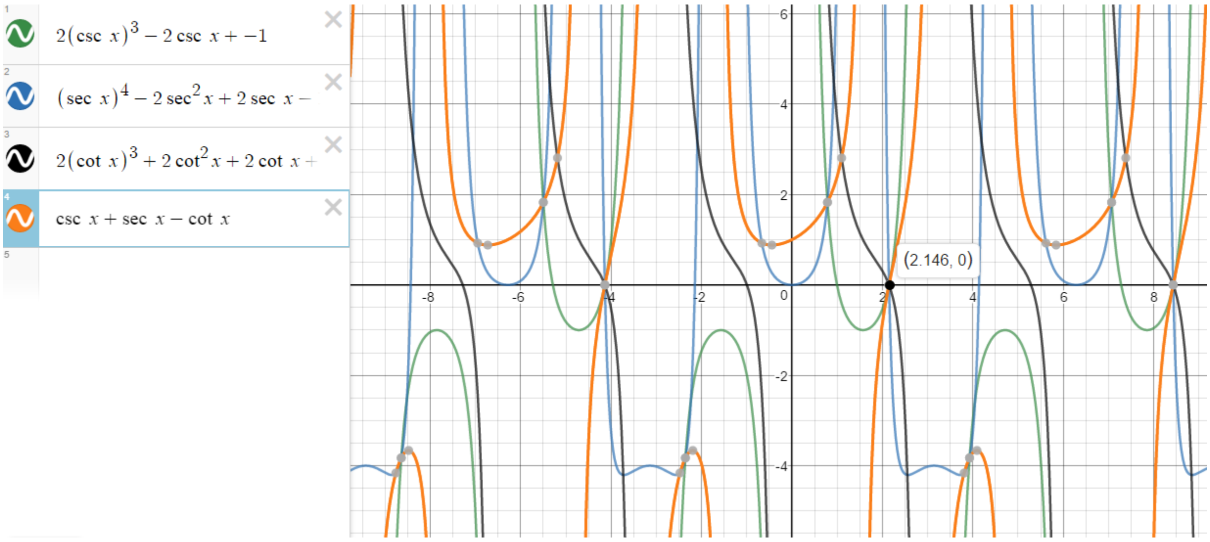
Resubstituting,

$$2\cot^3 x + 2\cot^2 x + 2\cot x + 1 = 0$$

A graphical look shows that the graph is remarkably similar to the typical cotangent graph: -



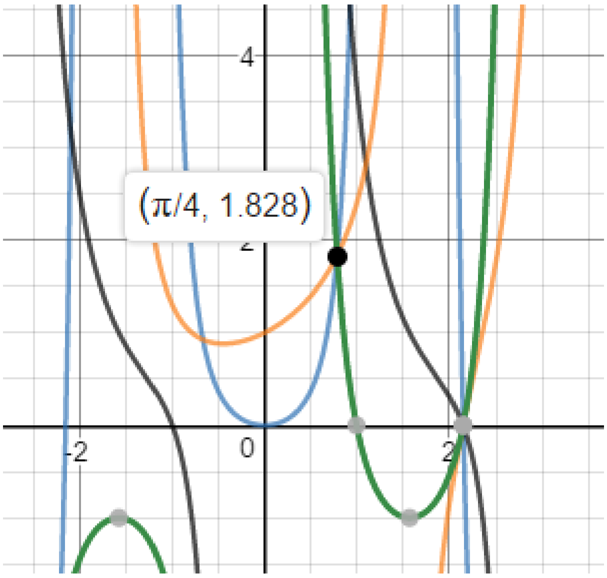
4.9 *Putting it all together(Part 2)*



When you take the equations obtained, including the original equation, together, we get a beautiful series of curves as shown in the graph. As expected, the common solution of $2.146\ rad$ is still valid in this case.

Like the graph obtained in Part 1, the equations are unrelated to each other. However, unlike the more 'even' distribution of lines in Part 1, in this case, they are concentrated in the positive y axis. This can be explained by the fact that while in Part 1, only the tangent had a non-fully real domain; in this case, *every* ratio has a non-fully real domain, which happens because of an eventual $\frac{1}{0}$ at some point of the graph. As in the previous part, the apparent irregularity of the graphs does possess some interest, despite every graph having one thing in common: - there is a point p for each graph when $x \to np(n \in Z)$, $y \to \pm\infty$.

A further look however reveals some interesting parts. Take this point from the combined graphs for instance: -



While the point is not on the line of the cotangent equation, it indeed is in every other equation, including the original equation. Remember that this is not a zero of the equation. It appears that only the 'true' zero of the equation will be intersected by all the equations.

5 Find the value of $\sin 1^\circ + \sin 2^\circ + \sin 3^\circ + \cdots + \sin n^\circ$ to n terms. What happens if you find the product instead?

5.1 *Introduction*

As with virtually all the previous questions, this idea came out during the recess in my school, when I was wondering about finding sum and product of n terms of $\sin x$.

So, I spent some time in the recess and in my Maths period (as I was bored) and soon wrote some basic analysis on a piece of paper and showed it to a Maths teacher in my school. He suggested finding the integral of $\sin x$ from 0 to 90 degrees instead or try for contour integrals(?).

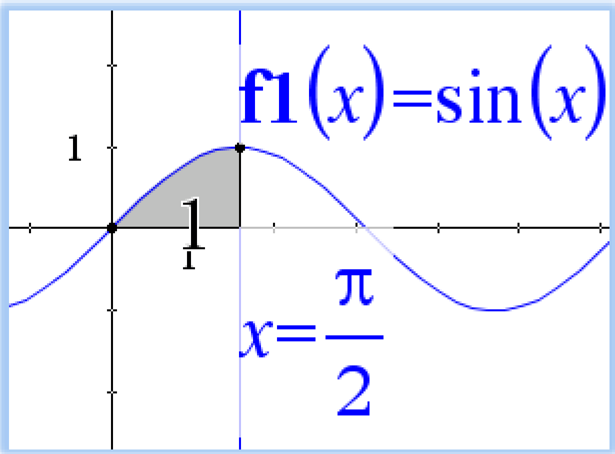
$$\int_0^{90^\circ} \sin x \, dx$$

It's not hard to understand what he had in mind. (but it was something that bore no relation to what I had in mind, as we'll see in Section 5.6)

He meant taking strips of $\sin x$ to indicate each degree, summing it off, using it as an excuse to integrate it instead by taking an infinitesimally small dx, a bit like the *limit of a sum* method which is taught in Class 12.[6]

$$\sum_{x=1}^{90^\circ} \sin x \to \int_0^{90^\circ} \sin x \, dx$$

The above integral would give you 1. But that is not the value of $\sum_{i=1}^{90} \sin i$, as $\sin 90^\circ = 1$ and the other sine terms are non-negative.



However, calling this as a mistake is a little trickier. Note that we are actually taking *integral* strips when summing up, but while integrating, you are actually accounting for the curved nature of $\sin x$. Admittedly though, trying to reduce the interval between each sine terms backfires, as the value of $\sum \sin x$ simply increases. The area under the graph is indeed 1 and it's safe to say that the 'true' vale of the summation is as well. A check with Ti-Nspire will dispel any doubts (see above).

[6] The method *does* have significant relevance with the trapezium (Simpson's) method in which you'd take n strips and approximate the value of that integral. However, the limit of a sum method forces you to take the limit instead, and requires you to sum up all the terms, employing methods like finding sum of a G.P. Take this into account while noticing the 'error' soon after, as the summing method has relevance with Simpson's.

Anyway, in this question, we simply try to summate $\sin x$. In other words,

$$\sum_{x=1°}^{n} \sin x$$

But before that we try to find out what happens if we find the product of $\sin x$ instead.

A brief analysis of it is given in Part 1. Admittedly the reason why Part 1 is so big is partly because of some interesting ideas discovered while writing this section. In fact, the initial plan was to analyse the product sum in Section 5.1 itself!

The observations made in Section 5.6 when we were deriving the expression for the sine summation made it irresistible to do the same for the cosine summation, and make cross-trig comparisons, which were made possible due to the high similarity between the expressions.

We also try out De Moivre's theorem to the product and sum expressions to see how that goes.

This is a multi-part section.

Part 1 deals with the product section of sine. (Section 5.2 to 5.5)

Part 2 deals with the summation of sine (and cosine). (Sections 5.6 to 5.9)

Part 1 starts from the next section.

Please note: - You will see an inconsistency between the use of the degree symbol (°) and without it; some have, some do not, especially in Part 2. While for all purposes the equations in Part 2 work irrespective of whether degree or radians are used (even though degrees are emphasized), for Part 1, only degrees are used; the equations and ideas presented there are incompatible with radians.

5.2 *The product summation: - Introduction & Methodology*

It is easy to find the product of n terms of $\sin x$, as

$$\prod_{x=1°}^{n} \sin x = 0$$

That is, if $n \geq 180^{\circ}$. It is easy to see why, as $\sin 180^{\circ} = 0$. Because of this, it does not matter whether we are finding the product to 181 or 987654321 terms, the answer is 0 itself!

But what happens if $n < 180^{\circ}$? Note that $\sin x$ when x is small or nearly equal to 180 degrees is nearly equal to 0. And there is no value of x when $\sin x > 1$. This means that the value of

$$\prod_{x=1°}^{n} \sin x$$

when $n < 180°$ is nearly equal to 0.

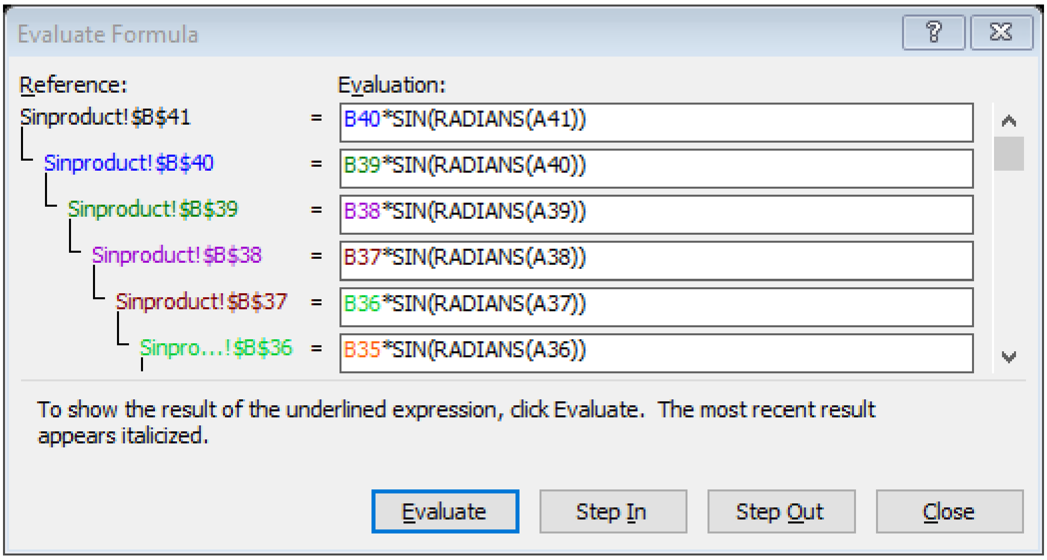
We want to test this. So I found out the value of the product of $\sin x$ from 1 to 179°[7] using Excel 2010.

We use the function

$$\mathrm{B(n) = B(n-1) * sin\left(radians(A(n))\right)}$$

where A is the column containing the degrees and B is the product of sines from 1 to n.

The interesting thing that comes up here is that Excel has to reference every cell in order to find the value, as shown in the screenshot below: -



Indeed, that means 179 clicks through ‘Step In’ to find the end of this chain for the 179th degree!

5.3 *Sine product from 1 to* n°

Degrees	Product of sin x
1	0.017452406
2	0.00060908
3	3.18768E-05
4	2.22361E-06
5	1.93801E-07
6	2.02577E-08
7	2.46879E-09
8	3.43589E-10
9	5.37492E-11

[7] I once used a different method based on a C++ program. However, because of the limited accuracy of taking only 4 digits of π (float), this method ended up causing serious accuracy deviations compared to just using the Excel function. Even though the problem was later fixed by increasing the accuracy of π (double), I decided not to use the method any longer.

10	9.33345E-12
11	1.78091E-12
12	3.70271E-13
13	8.32929E-14
14	2.01504E-14
15	5.2153E-15
16	1.43753E-15
17	4.20294E-16
18	1.29878E-16
19	4.22841E-17
20	1.4462E-17
21	5.18272E-18
22	1.94148E-18
23	7.58597E-19
24	3.08549E-19
25	1.30399E-19
26	5.7163E-20
27	2.59515E-20
28	1.21835E-20
29	5.90666E-21
30	2.95333E-21
31	1.52108E-21
32	8.06049E-22
33	4.39006E-22
34	2.45489E-22
35	1.40807E-22
36	8.2764E-23
37	4.98086E-23
38	3.06653E-23
39	1.92983E-23
40	1.24047E-23
41	8.13821E-24
42	5.44552E-24
43	3.71384E-24
44	2.57985E-24
45	1.82423E-24
46	1.31224E-24
47	9.59712E-25
48	7.13205E-25
49	5.38263E-25
50	4.12333E-25
51	3.20443E-25
52	2.52513E-25
53	2.01665E-25
54	1.63151E-25

55	1.33645E-25
56	1.10797E-25
57	9.29222E-26
58	7.88025E-26
59	6.75469E-26
60	5.84973E-26
61	5.11629E-26
62	4.51742E-26
63	4.02505E-26
64	3.61769E-26
65	3.27874E-26
66	2.99528E-26
67	2.75717E-26
68	2.5564E-26
69	2.38661E-26
70	2.24268E-26
71	2.12049E-26
72	2.01671E-26
73	1.92859E-26
74	1.85388E-26
75	1.79071E-26
76	1.73752E-26
77	1.69298E-26
78	1.65599E-26
79	1.62556E-26
80	1.60087E-26
81	1.58116E-26
82	1.56577E-26
83	1.5541E-26
84	1.54559E-26
85	1.5397E-26
86	1.53595E-26
87	1.53385E-26
88	1.53291E-26
89	1.53268E-26
90	1.53268E-26
91	1.53245E-26
92	1.53151E-26
93	1.52941E-26
94	1.52569E-26
95	1.51988E-26
96	1.51156E-26
97	1.50029E-26
98	1.48569E-26
99	1.4674E-26

100	1.44511E-26
101	1.41855E-26
102	1.38756E-26
103	1.35199E-26
104	1.31183E-26
105	1.26713E-26
106	1.21805E-26
107	1.16482E-26
108	1.10781E-26
109	1.04746E-26
110	9.84289E-27
111	9.18913E-27
112	8.52001E-27
113	7.84271E-27
114	7.16467E-27
115	6.4934E-27
116	5.83623E-27
117	5.20012E-27
118	4.59143E-27
119	4.01576E-27
120	3.47775E-27
121	2.98101E-27
122	2.52804E-27
123	2.12019E-27
124	1.75772E-27
125	1.43984E-27
126	1.16485E-27
127	9.30294E-28
128	7.33082E-28
129	5.69712E-28
130	4.36424E-28
131	3.29374E-28
132	2.44772E-28
133	1.79015E-28
134	1.28773E-28
135	9.10561E-29
136	6.32529E-29
137	4.31384E-29
138	2.88652E-29
139	1.89373E-29
140	1.21726E-29
141	7.66049E-30
142	4.71627E-30
143	2.83832E-30
144	1.66832E-30

145	9.56911E-31
146	5.35098E-31
147	2.91435E-31
148	1.54437E-31
149	7.9541E-32
150	3.97705E-32
151	1.92811E-32
152	9.05194E-33
153	4.10949E-33
154	1.80148E-33
155	7.6134E-34
156	3.09665E-34
157	1.20996E-34
158	4.53258E-35
159	1.62433E-35
160	5.55554E-36
161	1.80871E-36
162	5.58921E-37
163	1.63413E-37
164	4.50426E-38
165	1.16579E-38
166	2.8203E-39
167	6.34429E-40
168	1.31905E-40
169	2.51687E-41
170	4.3705E-42
171	6.83697E-43
172	9.51523E-44
173	1.15961E-44
174	1.21213E-45
175	1.05644E-46
176	7.36934E-48
177	3.85681E-49
178	1.34601E-50
179	2.34911E-52

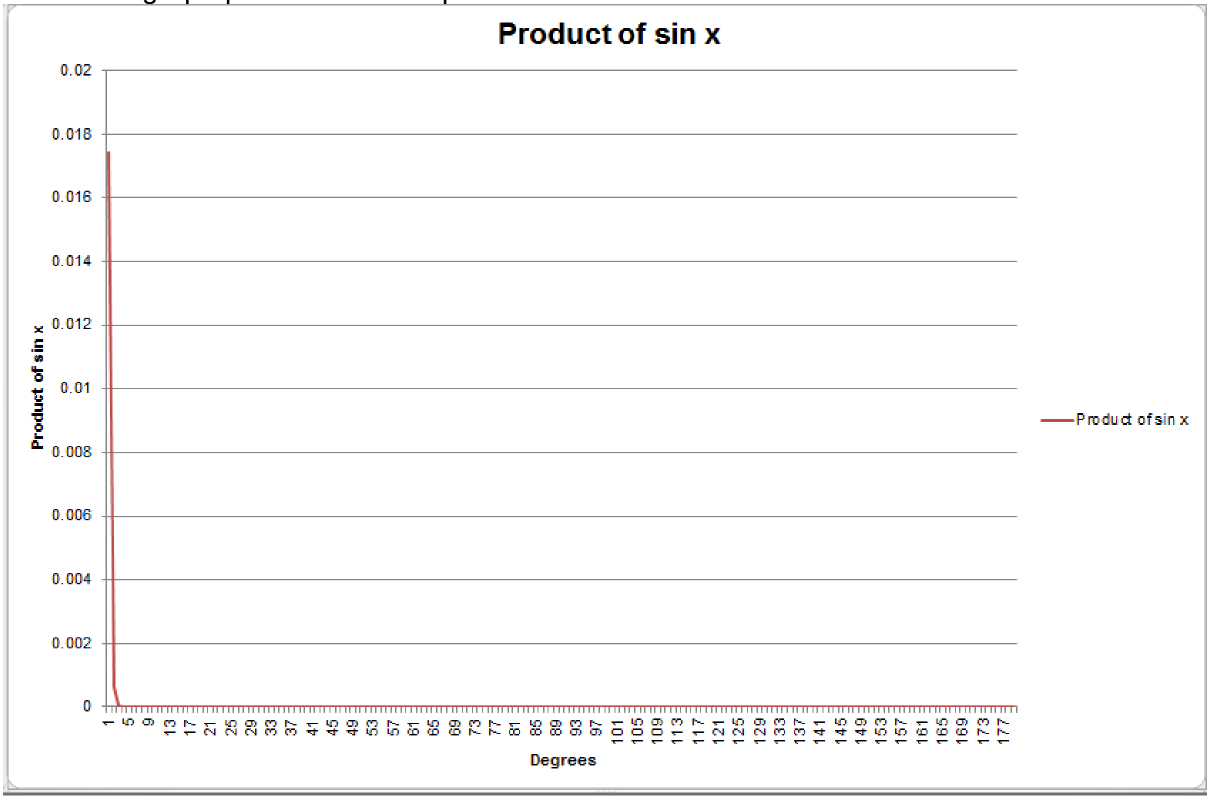
The data supports our reasoning that the value of

$$\prod_{x=1^\circ}^{n} \sin x$$

becomes very low as $n \to 180^\circ$. It must however be noted that the graph of product of sines will abruptly stop at 0.

5.4 *Statistical and graphical analysis*

The linear graph paints a sombre picture: -



… the graph drops so sharply that you wouldn't know whether $\sin 1^{\circ}. \sin 2^{\circ}. \sin 3^{\circ} \ldots \sin 7^{\circ}$ is 0 or not solely from the graph!

What happens if we want to find the area of the graph? Here it gets rather funny and awkward.

The sum of the data is

$$\sum f$$

which is actually

$$\sum_{i=1^{\circ}}^{179^{\circ}} \left[\prod_{x=1^{\circ}}^{n} \sin x \right]$$

What is n? It may seem that we would get a formula, but n is actually the value of n that is currently used in the summation. The formula derived itself is really interesting.

As we are looking to find the area of the graph, we take a small value dx and integrate it.

$$\int_{1^{\circ}}^{179^{\circ}} \prod_{x=1^{\circ}}^{n} \sin x \, dx$$

Then what?
How would you integrate … a product of sines?

What about if we use the traditional method for finding the area of a curve?

$$\int_{a}^{b} y \, dx$$

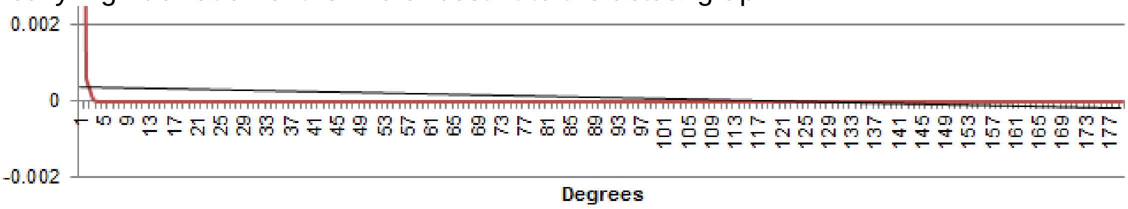
If = $\prod_1^{179} \sin x$, then the area would be

$$\int_1^{179} \prod_{x=1°}^{n} \sin x$$

But again, how do we integrate this?

…

When we try to apply a *linear* line of best fit to the graph, it ends up going negative. Note the really high deviation of the line of best fit to the actual graph.



That is impossible, even if $\sin 180°$ is not 0! This is because as we're dealing with negative values of $\sin x$, the graph should actually jump up and down when $x > 180°$, as then every other integral value of x will change the sign of

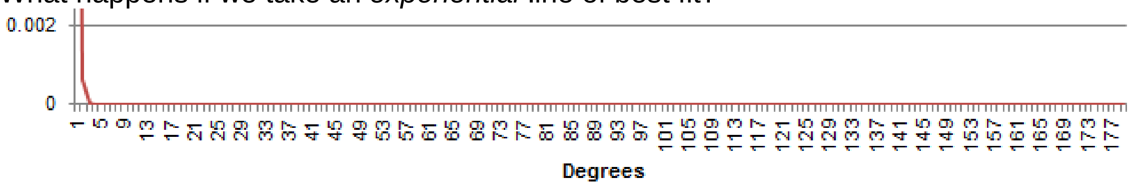
$$\prod_{x=1°}^{n} \sin x$$

In fact, if we were to take

$$\prod_{x=181°}^{n} \sin x$$

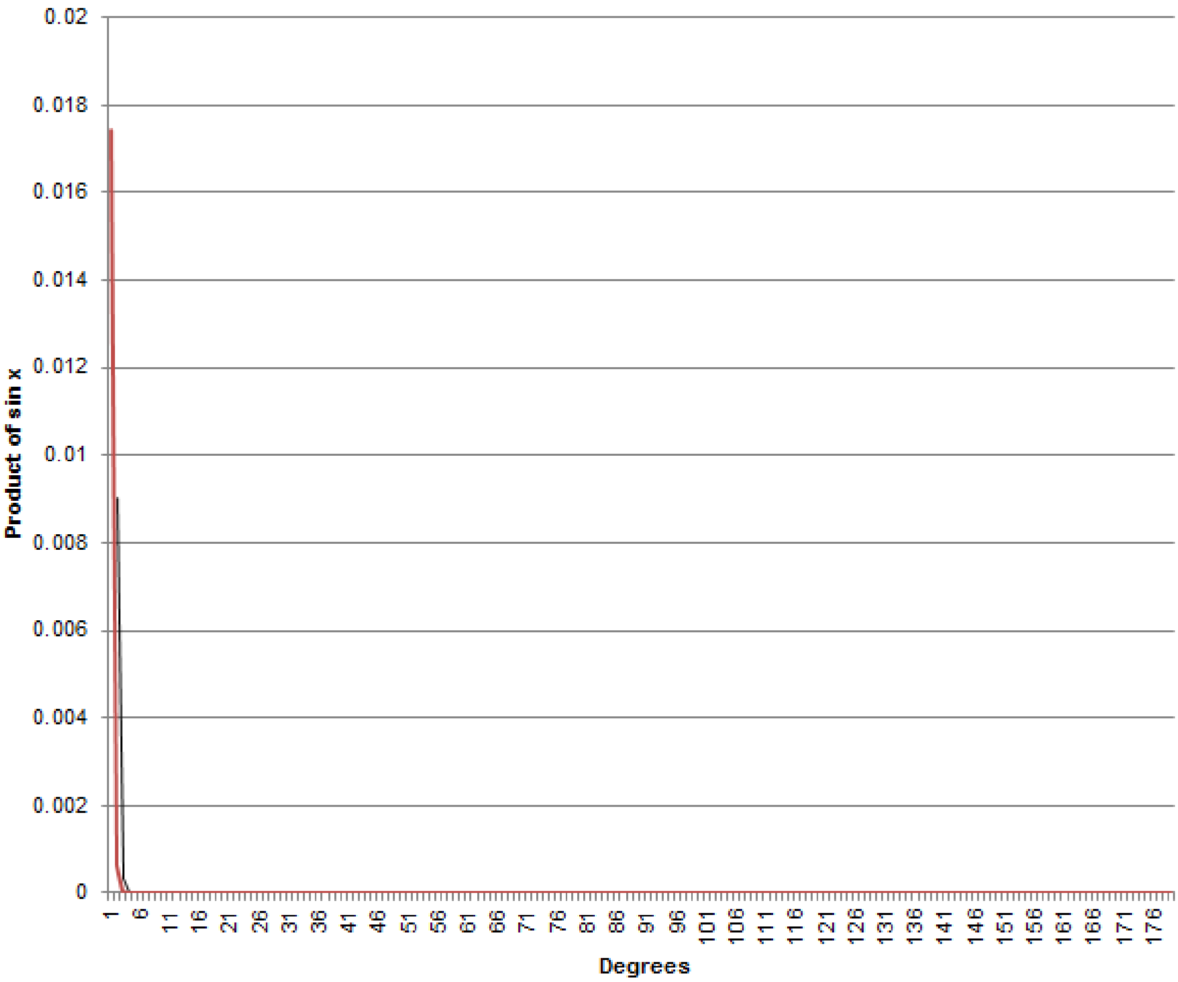
and we know that $n < 360°$, we may think that the graph is going a hard time keeping up as it has to jump between positive and negative values. However, in reality this is not the case. In fact, we're going to get a graph that resembles the graph of damping of sound. Only that it won't be easily visible: – this is because the product is very small anyway.

What happens if we take an *exponential* line of best fit?

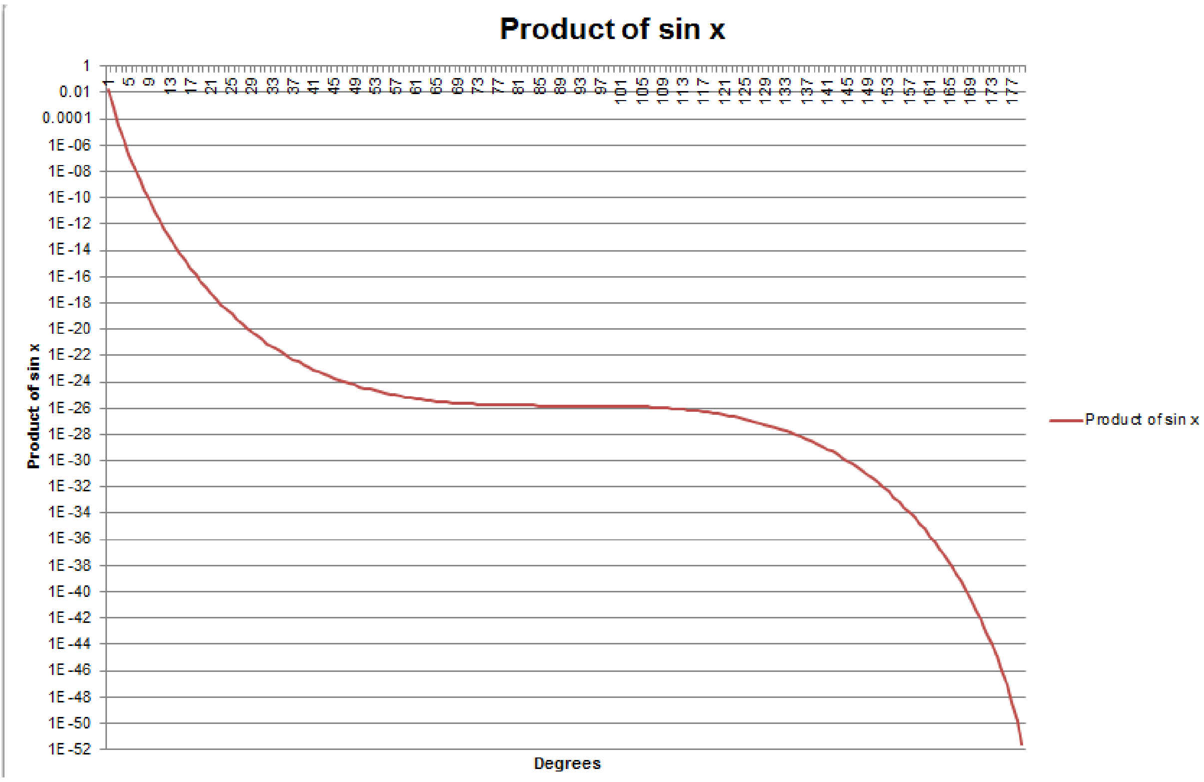


It seems to sync pretty well with the graph, and hence it is not of much benefit to us here.

Taking a *2-point moving average* instead seems to have a noticeable difference for the first couple of sine product values, but then falls back in line with the rest of the graph. This is expected, because most of the visible changes happens during the initial few terms alone.

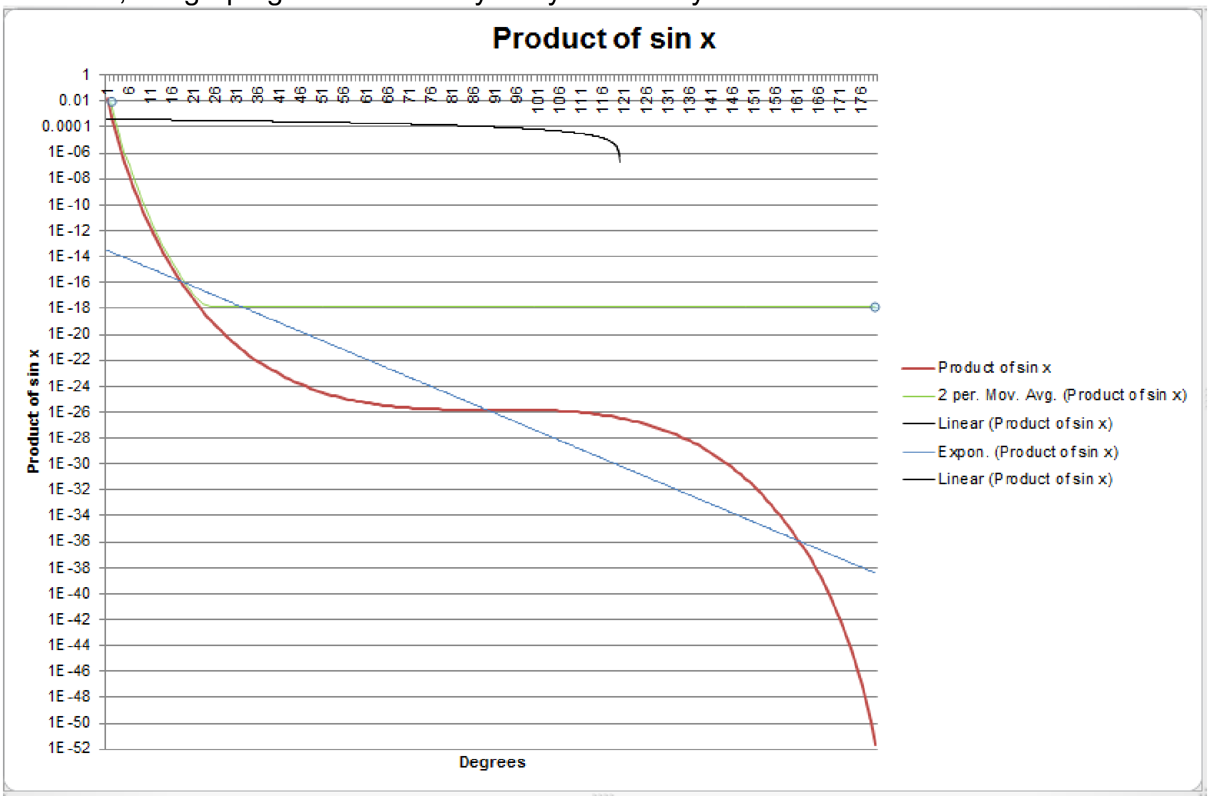


If we actually want to have a graph that will represent the data on offer, then we have to switch to a logarithmic graph. Here we're using base 10. Each 'bar' on the y axis is less than the previous 'bar' by 100 times.



Note how the graph flattens out as we approach 90˚. It diverges again once it's sufficiently away from 90˚. Actually, that's an interesting specimen to find the area of – and it sure makes for a good relaxation chair when one is lazy!

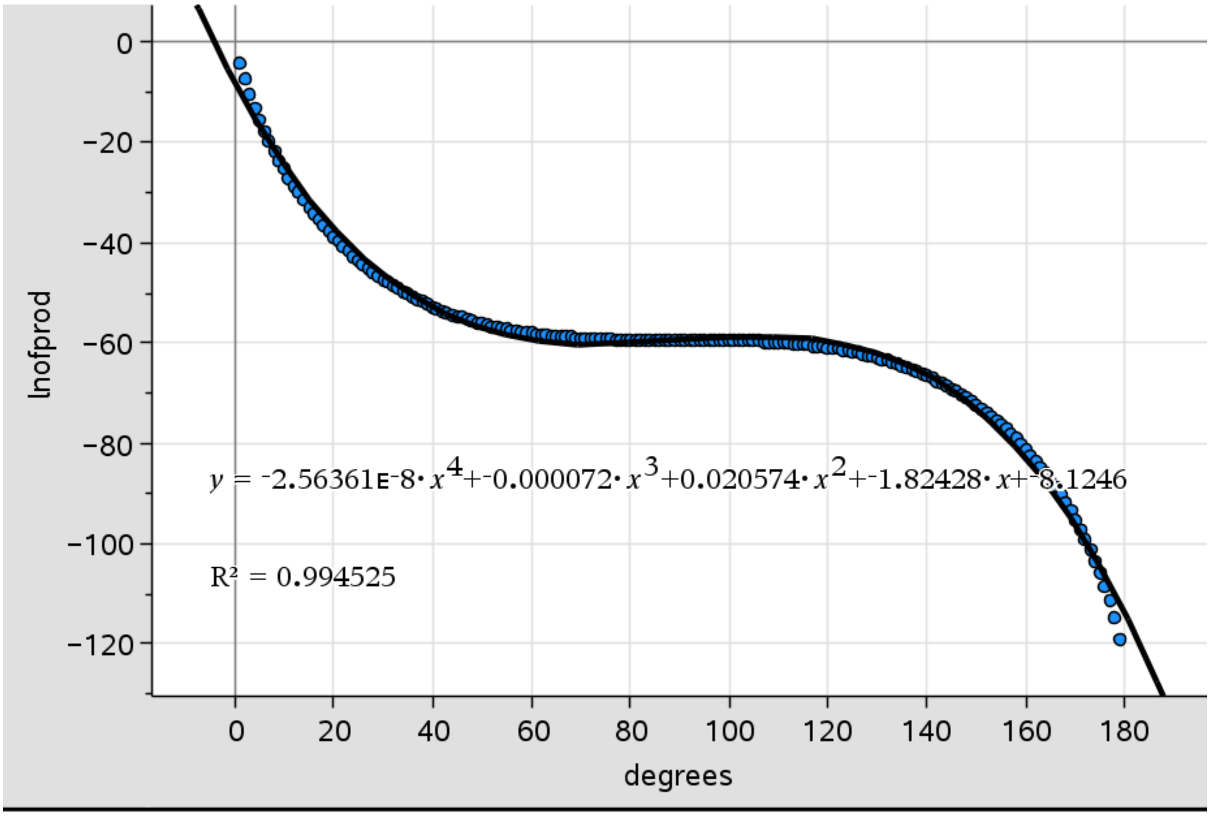
However, the graph goes bonkers if you try to use any 'line of best fit' ideas!



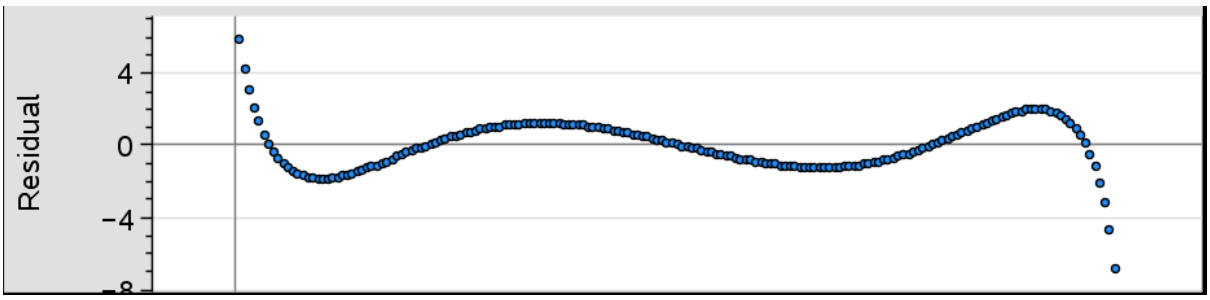
Only the *exponential* trendline manages to keep any pace with the graph, but not only is the starting point incorrect, the graph is highly linear, and hence miserably fails as it has a really high deviation when compared to the actual graph. The linear trendline 'crashes' abruptly right around $\frac{2\pi}{3}$, oddly it appears to curve in a way that would be reminiscent of the *reciprocal* graph ($xy = k$). Anyway, a look at its history says that it would've been doomed from the start. And the linear 2-point moving average works properly *till* 22° and then decides to stay flat (theoretically, that should have worked well).

It seems that the trendlines cannot handle the logarithmic nature of the graph. I would've expected the exponential (and even the 2-point moving average) to sync well with the graph. Even then, I did not expect the linear graph to go *that* bonkers.

However, the quartic regression (not the quadratic or the cubic, mind you!) works nearly perfectly with the graph, only deviating a bit with the first and last few degrees of the graph.



(Note that a different logarithm scale was used, but the graph is still pretty similar.)
This observation is well exemplified in the *residual* graph deviation between the regressional and actual graph) as shown below. It actually forms a beautiful pentadegree polynomial! Note that the graph is actually symmetrical along the 3rd zero of the graph. (the scale on the x axis is the same as before)



When we analyse the statistical values of the sine products, we see that the standard deviation of the data is 0.001305, which is actually pretty high. This is mainly caused by the relatively high product values of the first few terms. In fact, if we were to find the standard deviation of all the sine products except the first 5 terms, then we get a significantly lower standard deviation of $1.54604 * 10^{-9}$. However, the standard deviation increases if the last 6 terms are removed as well, as the standard deviation then is $1.57336 * 10^{-9}$.

For a true statistical comparison, we can take the coefficient of variation instead: -

	Standard deviation (σ)	**Mean (M)**	**Coefficient of variation ($\frac{\sigma}{M} * 100$)**
$\sum \prod_{x=1^\circ}^{179^\circ} \sin x$	0.00130498	0.000101094	1290.859
$\sum \prod_{x=1^\circ}^{179^\circ} \sin x - \sum \prod_{x=1^\circ}^{5^\circ} \sin x$	$1.54604 * 10^{-9}$	$1.32962 * 10^{-10}$	1162.770
$\sum \prod_{x=1^\circ}^{174^\circ} \sin x - \sum \prod_{x=1^\circ}^{5^\circ} \sin x$	$1.56871 * 10^{-9}$	$1.36896 * 10^{-10}$	1145.914

One thing that immediately strikes out is the fact that the CoV of all 3 cases is **extremely high!** Anyway, the data shows that the variability of the last case is slightly less than the second case, which in turn is slightly lesser than the first case.

We now take a look at the interquartile range of the data.
The 25th percentile is $1.09914 * 10^{-28}$ and the 75th percentile is $1.56823 * 10^{-24}$. This gives the interquartile range as $1.56812 * 10^{-24}$, which ends up being really close to the 75th percentile.

A consequence of this is that the variability of the middle 50% is relatively less. This happens because the value of $\sin x$ at these locations is higher than at the extreme points, meaning that the reduction in product is relatively less.

5.5 *De Moivre's Theorem (Part 1)*

It states that[8]

$$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$$

[8] The way the formula is mentioned does mean something. Specifically, that the product of $\sinh x$ won't be too different from the product of $\sin x$, only that the $\frac{1}{2i}$ term is no longer present. Specifically, that $\frac{\prod \sinh x}{(2i)^n}$ is equal to $\prod \sin x$ itself.

‘Producting’,

$$\prod_{\theta=1^\circ}^{179^\circ} \sin\theta = \prod_{\theta=1^\circ}^{179^\circ} \left[\frac{1}{2i}\right]\left(e^{i\theta} - e^{-i\theta}\right)$$

(Note: - We can product to any number of terms as long as it is less than 180˚)

Continuing to product,

$$\left[\frac{1}{2i}\right]\left(e^{i\theta} - \frac{1}{e^{i\theta}}\right)$$
$$\left[\frac{1}{2i}\right]\left(\frac{e^{2i\theta} - 1}{e^{i\theta}}\right)$$

$$\left[\frac{1}{2i}\right]\frac{\left(e^{2i\theta} - 1\right)}{e^{i\theta}}$$

From this, the result when it’s taken to n terms instead ($n < 180°$) become pretty clear: -

$$\prod_{\theta=1^\circ}^{n^\circ} \sin\theta = \left[\frac{1}{2i}\right]^n \frac{\left(e^{2i\theta} - 1\right)^n}{e^{ni\theta}}$$

Unfortunately, we’ve hardly done anything other than rearranging the terms. It’s clear that the application of this method does little in reality.

Additionally, the problem of handling a real valued function of $\sin x$ with complex numbers still persist.

End of Part 1.

Part 2 starts from the next section.

5.6 *The sine summation (Part 2)*

Take

$$\sin 1° + \sin 2° + \sin 3° + \cdots + \sin(n-2)° + \sin(n-1)° + \sin n°$$

Take the first and last, second and second last… and nth and nth last.

$$\sin 1° + \sin n° + \sin 2° + \sin(n-1)° + \cdots + \sin\frac{n}{2}° + \left(\sin\frac{n}{2}° + 1\right)$$

Using the formula

$$\sin\alpha \pm \sin\beta = 2\sin\frac{1}{2}(\alpha \pm \beta)\cos\frac{1}{2}(\alpha \mp \beta)$$

we get

$$\sin 1° + \sin n° = 2\sin\frac{n+1}{2}\cos\frac{1-n}{2}$$

and

$$\sin 2° + \sin(n-1)° = 2\sin\frac{2+n-1}{2}\cos\frac{2-n+1}{2}$$

which is also equal to

$$\sin 2° + \sin(n-1)° = 2\sin\frac{1+n}{2}\cos\frac{3-n}{2}$$

Doing the same for the next pair of terms,

$$\sin 3° + \sin(n-2)° = 2\sin\frac{1+n}{2}\cos\frac{5-n}{2}$$

Eventually summing, the RHS would look like

$$2\sin\left(\frac{n+1}{2}\right)\left[\cos\frac{1-n}{2} + \cos\frac{3-n}{2} + \cdots + \cos\frac{1}{2}\right]$$

(for $\frac{n}{2}$ terms, not n terms. This is because we've already combined the 2 terms into one.)

Note that the last term is actually a constant independent of n. This is because when we generalise the cos term, we get $\cos\frac{((2n-1)-n)}{2}$. Except that in this case, it is $\frac{n}{2}$. Hence the last term would be

$$\cos\left(\frac{2\left(\frac{n}{2}\right)-n-1)}{2}\right) \to \cos\left(-\frac{1}{2}\right) \to \cos\left(\frac{1}{2}\right)$$

Hence at this stage we have

$$\sum_{x=1}^{n}\sin x = 2\sin\frac{(n+1)}{2}\left[\cos\frac{1-n}{2} + \cos\frac{3-n}{2} + \cos\frac{5-n}{2} + \cdots + \cos\frac{3}{2} + \cos\frac{1}{2}\right]$$

But we can actually simplify the cos terms using the $\cos x + \cos y$ formula!
We have

$$\cos\alpha + \cos\beta = 2\cos\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)$$

Taking the first and last term, second and second last term and so on,

$$\cos\frac{1-n}{2} + \cos\frac{1}{2} + \cos\frac{3-n}{2} + \cos\frac{3}{2} + \cdots \cos\frac{\frac{n}{2}-n}{2} + \cos\frac{\frac{n}{2}-n}{2} + 1$$

Using the formula,

$$2\cos\left(\frac{\frac{1-n}{2}+\frac{1}{2}}{2}\right)\cos\left(\frac{\frac{1-n}{2}-\frac{1}{2}}{2}\right) + 2\cos\left(\frac{\frac{3-n}{2}+\frac{3}{2}}{2}\right)\cos\left(\frac{\frac{3-n}{2}-\frac{3}{2}}{2}\right) + \cdots$$

(to $\frac{n}{4}$ terms!)
we get

$$2\cos\frac{2-n}{4}\cos\frac{n}{4} + 2\cos\frac{6-n}{4}\cos\frac{n}{4} + \cdots + 2\cos\frac{1}{2}\cos\frac{n}{4}$$

$$2\cos\frac{n}{4}\left(\cos\frac{2-n}{4} + \cos\frac{6-n}{4} + \cdots + \cos\frac{1}{2}\right)$$

Again, it must be noted that the last term this time as well is a constant. This happens because the generalisation of the cos term in this case would give

$$\cos\frac{(4n-2)-n}{4}$$

This time, n is $\frac{n}{4}$
So now we get (again!)

$$\cos\frac{\left(4\left(\frac{n}{4}\right)-2\right)-n}{4} \to \cos-\frac{1}{2} \to \cos\frac{1}{2}$$

Now we're getting

$$\sum_{x=1}^{n} \sin x = 2\sin\frac{n+1}{2} * 2\cos\frac{n}{4}\left[\cos\frac{2-n}{4} + \cos\frac{6-n}{4} + \cos\frac{10-n}{4} + \cdots + \cos\frac{3}{2} + \cos\frac{1}{2}\right]$$

(to $\frac{n}{4}$ terms)

So it is actually a repetitive process.

When you go for a full generalisation, things get into a pretty interesting state.

Look carefully. Oddly, the first sine term is $\frac{n+1}{2}$, but all the others form a $\frac{n}{2^k}$ pattern, where k is the number of times the process is done and n is the number of terms.

Also take a look at the formation of the first few cosine terms for some values of k:-

$$k = 1 \to \cos\frac{1-n}{2} + \cos\frac{3-n}{2} + \cos\frac{5-n}{2} + \cdots + \cos\frac{2*0+(n-1)*2^1-n}{2^1}$$
$$k = 2 \to \cos\frac{2-n}{4} + \cos\frac{6-n}{4} + \cdots + \cos\frac{2*1+(n-1)*2^2-n}{2^2}$$

Hence, to $\frac{n}{2^k}$ terms, we can say that the final answer is

$$\sum_{x=1}^{n} \sin x = 2^k \sin\frac{n+1}{2}\prod_{i=2}^{k}\cos\frac{n}{2^i}\sum_{i=1}^{\frac{n}{2^k}}\left[\cos\frac{2^{k-1}+(i-1)2^k-n}{2^k}\right]$$

Note that the answer will remain correct whether $k = 1$ or $k = 256$, given that there are actually 2^k number of terms at least.

Note that we've introduced a 3rd variable in the sine summation. This is required, as if there was no 3rd variable, you're summing the same thing over and over!

Proving this statement as-is is a bit dubious. Note that we had not given a long explanation on *how* we managed to obtain the expression given about. It's just that we took a couple of cases and then strung them together to obtain the beautiful final expression. If we generalise it, we'll end up with the same expression.

One option is to prove this by induction. It may seem tempting to prove by induction, as when you're going for the $k+1$ stage, the 2^{k-1} & $k-1$ terms will reduce into a nicer k. But a closer look reveals some difficulties that are usually not encountered when one does a typical proof by induction. For instance, for what are you proving? For n or for k? There are 2 major variables, and neither can be ignored. And how would you proceed? Take *both* n and k and prove their $+1$th case together? Or just do this separately?
That's not it. Remember that the equation is valid no matter what the value of k is.
With that out of the way, let us (try to) work out an inductive proof. To start with, let us prove this for n.

As is the typical method for an inductive proof, we start when $n = 1$. Then we'd have

$$\sum_{x=1}^{1} \sin x = 2^k \sin 1 \prod_{i=2}^{k} \cos\frac{1}{2^i} \sum_{i=1}^{\frac{1}{2^k}} \left[\cos\frac{2^{k-1}+(k-1)2^k-1}{2^k}\right]$$

How would you proceed? The only real way is to substitute a suitable value of k. Let $k = 1$. Then we would have

$$\sum_{x=1}^{1} \sin x = 2 \sin 1 \prod_{i=2}^{1} \cos\frac{1}{2^i} \sum_{i=1}^{\frac{1}{2^1}} \left[\cos\frac{2^{1-1}+(1-1)2^1-1}{2^1}\right]$$

The cosine summation does not get used at all no matter what the value of i is, as since $i > 1$, $\frac{1}{2^i} < 1$ and hence the summation does not proceed. Even if it were to proceed, we would get $\sum \cos 0 = 1$.

In this particular case, the product does not run either, as i starts from 2 but $k < i$. Hence, we are left with

$$\sum_{x=1}^{n} \sin x = 2 \sin 1$$

But the LHS would reduce to $\sin 1$, which is not equal to the RHS. A further thought shows that as the RHS is technically $2^k \sin 1$, which means that you're only going to get multiples of the correct answer, making the *correct* answer *incorrect*!

Does that mean that the statement that we worked out is incorrect? Not necessary! Recall that when the process is worked out k times, the minimum number of terms required is 2^k. But when $k = 1$, $2^1 > 1$, which could help explain why we're getting a faulty answer. What's the fix? This'll only work when $2^k = 1$, or $k = 0$, meaning that the process cannot run at all!

But this only creates a further headache with regards to how we should work out the inductive proof. It is not an option to prove it for $k + 1$ when n is 1. In fact, to have any hope of proving it, $n \geq 2^{k+1}$ is a necessary condition as if $n \geq 2^k < 2^{k+1}$, you will end up running into trouble when you try to prove it for the $k + 1^{\text{th}}$ term. Now that the condition has been clearly established, how do we continue?

As stated above, the first stage has to be showing that $n = 2^k$ is true. But in that case, how would you continue?
Let us do something else instead. We can at least check the validity of the solution. As it is impractical to do this directly, we'll use the power of the computer to assist us in this case instead.

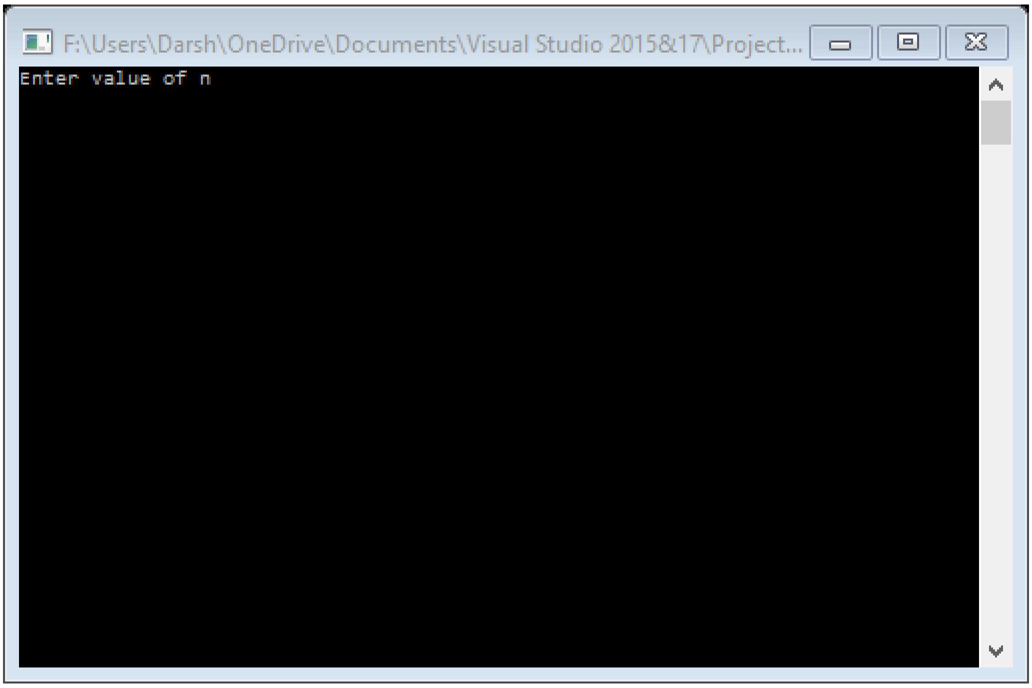
I've coded a program to find the value of the given expression for a particular value of n, looping the program for relevant values of k. This was done in Visual Studio 2017.

The program additionally calculates the actual value of $\sum_{m=1}^{n} \sin m$ to allow us to easily compare the values.

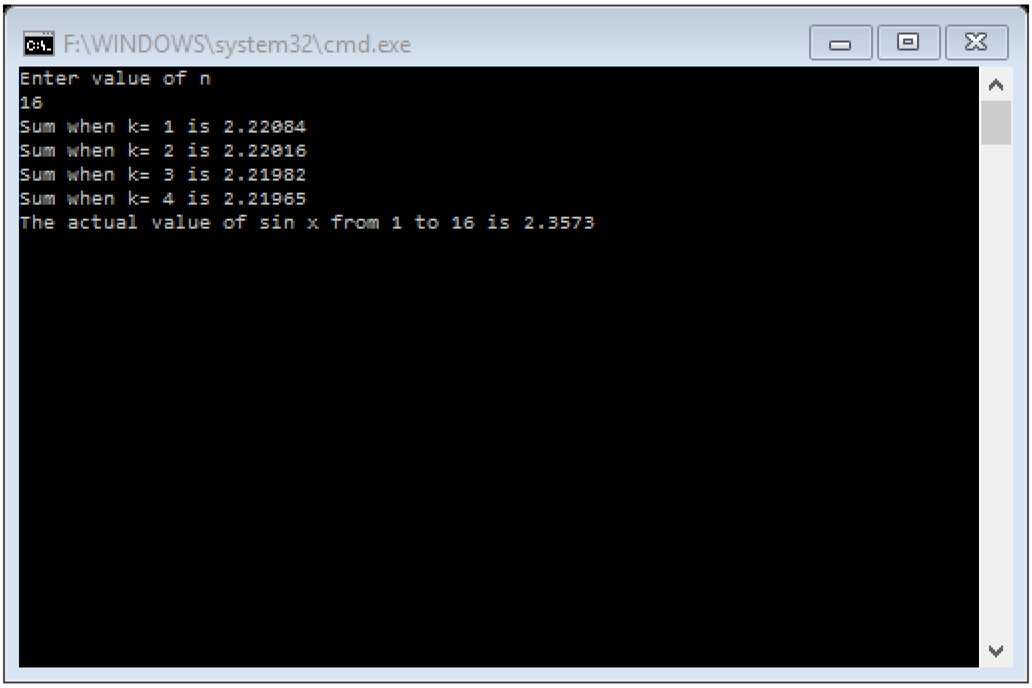
```
// sintest.cpp: main project file.

#include "stdafx.h"
#include <iostream>
#include <math.h>
```

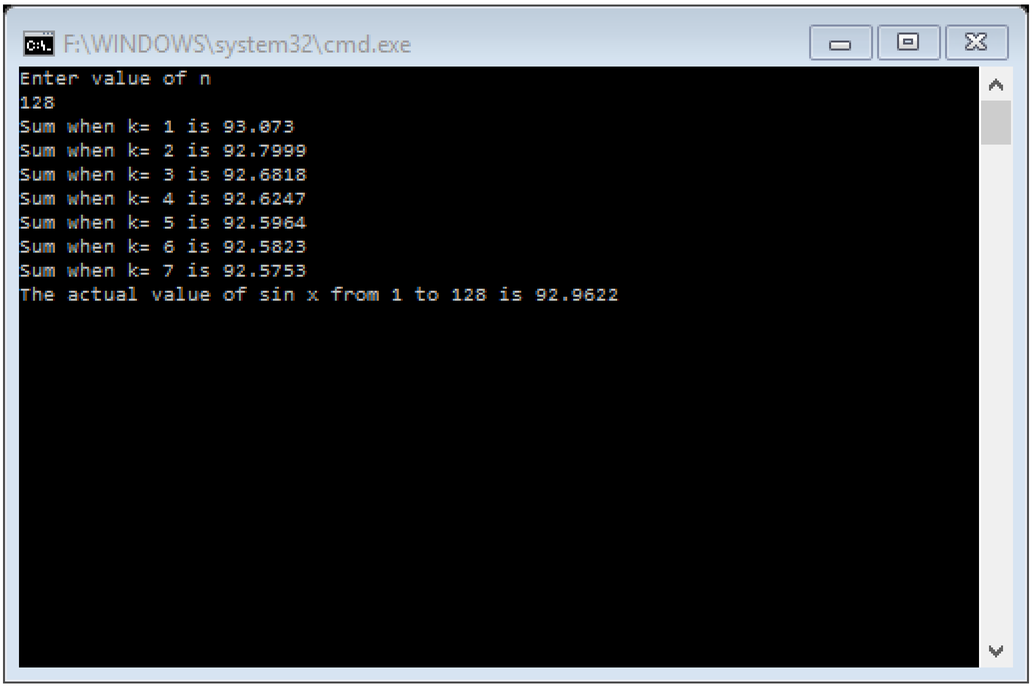
```
#include <conio.h>
using namespace std;
const double pi = 3.14159265358979323846264338327950;
float radical (long deg)
{
	// This function is required to convert degree into radians for C++ to calculate the related trigonometric ratio of.
	double rad = pi / 180 * deg;
	return rad;
}
void sinsummer (long deg)
{
	float sinsuming = 0;
	for (int i = 1; i <= deg; i++)
		sinsuming += sin(radical(i));
	cout << "The actual value of sin x from 1 to " << deg << " is " << sinsuming;
}
void main()
{
	long n;
	cout << "Enter value of n" << '\n';
	cin >> n;
	for (int k = 1; pow(2,k) <= n; k++)
	{
		float sinsum;
		sinsum = pow(2, k)*sin(radical((n + 1) / 2));
		for (int i = 2; i <= k; i++)
			sinsum = sinsum*cos(radical((n / pow(2, i))));
		float q = 0;
		for (int j = 1; j <= n / pow(2, k); j++)
			q = q + cos(radical(((pow(2, k - 1) + (j - 1)*pow(2, k) - n) / pow(2, k))));
		sinsum = sinsum * q;
		cout << "Sum when k= " << k << " is " << sinsum << '\n';
	}
	sinsummer(n);
	_getch();
}
```

This is how the program looks: -



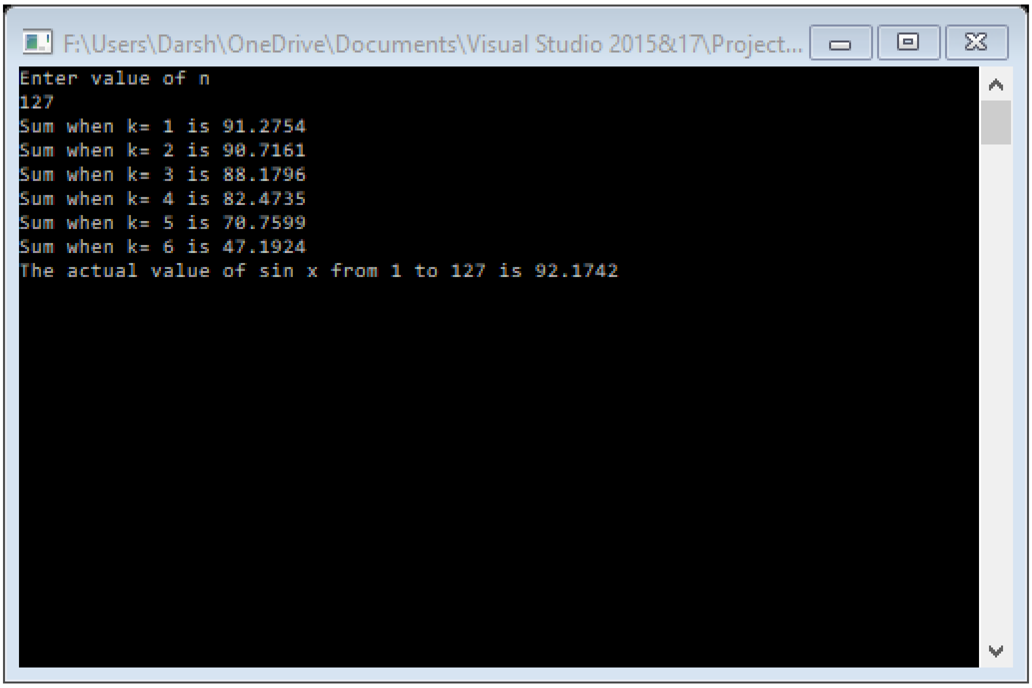
The outputs seem to be fairly accurate when testing with different values of n:-





It wasn't always this accurate. (see page 78)

But the program does show significant deviation for certain values of n, as shown below when n = 127.



Note that the sum obtained, as k increases, decreases. However, when $k = 5$ and $k = 6$, the sum becomes wildly inaccurate.

But why is that? Let's take a look at the formula again: -

$$\sum_{x=1}^{n}\sin x = 2^k \sin\frac{n+1}{2}\prod_{i=2}^{k}\cos\frac{n}{2^i}\sum_{i=1}^{\frac{n}{2^k}}\left[\cos\frac{2^{k-1}+(i-1)2^k-n}{2^k}\right]$$

Note that

$$127 \bmod 2^k = 2^k - 1$$

(as 127 = $2^7 - 1$)

This means that while the formula will slightly overcalculate if the upper limit of the summation was $\frac{n}{2^k} + 1$, the formula will heavily undercalculate if the limit was $\frac{n}{2^k}$. The problem is because in such cases, the value of $\frac{n}{2^{k+1}}$ will be very nearly 1, but not 1. Then the summation for $\frac{n}{2^{k+1}}$ won't run, and hence the underestimation occurs.

The important consideration to determine whether such an underestimation will occur is to find out the value of $n \bmod 2^k$ for respective values of k. This is also why the first few values of k will not show a major deviation if any; the upper value of $n \bmod 2^k$ (which is $(2^k - 1)$) is less. It can hence be proven that the value of the summation formula will decrease with increasing values of k.

It must be emphasized that this does not invalidate the formula, but is an inherent design of the formula.

What happens if, for curiosity, we ask it to find for an obnoxiously large value though? Let's see. The values obtained are imported into Excel for statistical analysis. In this case, we are doing this for $n = 39{,}987{,}651$.

And what does it tell us? For a value this ludicrous, the accuracy isn't that bad (see left picture). The summation values for most values of k do fall within a narrow range whose differences can be easily attributed to range. This also applies to the 'outliers', whose relatively larger differences could easily be due to the exponentially large change of 2^k , negative cosine terms or other such factors.

k	sum
1	35.8044
2	37.2514
3	36.3875
4	37.9884
5	37.665
6	36.1088
7	36.4293
8	36.4289
9	37.1509
10	40.7941
11	37.3877
12	36.9046
13	35.285
14	35.5979
15	36.1864
16	34.8987
17	35.1786
18	34.5138
19	36.0095
20	35.8731
21	35.8179
22	33.9241
23	30.1923
24	30.2061
25	30.2038
Average value	35.607528
Actual sum	36.6406

The average of the values obtained and the actual sum closely correlate with the obtained values as well.

Though it indeed is not invalid to analyse the positive/negative spike in the sums when $k = 10$ and when $k \geq 22$.

Considering the observation made before regarding the decrease in sum for later values of k for certain values of n, I made a quick program to find out the value of $n \bmod 2^k$ for different values of k. The code is not included, but it's not hard to make it anyway.

```
Enter n
39987651

 Values:
When k= 0 remainder = 0
When k= 1 remainder = 1
When k= 2 remainder = 3
When k= 3 remainder = 3
When k= 4 remainder = 3
When k= 5 remainder = 3
When k= 6 remainder = 3
When k= 7 remainder = 67
When k= 8 remainder = 195
When k= 9 remainder = 451
When k= 10 remainder = 451
When k= 11 remainder = 451
When k= 12 remainder = 2499
When k= 13 remainder = 2499
When k= 14 remainder = 10691
When k= 15 remainder = 10691
When k= 16 remainder = 10691
When k= 17 remainder = 10691
When k= 18 remainder = 141763
When k= 19 remainder = 141763
When k= 20 remainder = 141763
When k= 21 remainder = 141763
When k= 22 remainder = 2.23892e+06
When k= 23 remainder = 6.43322e+06
When k= 24 remainder = 6.43322e+06
When k= 25 remainder = 6.43322e+06
```

And the outputs obtained are pretty interesting. They explain why the summation decreases when $k \geq 22$, and also explains why the summation value is the same for $k = 23$ to $k = 25$.

However, this does not explain why the values do not decrease when $k = 18$, nor does it explain why the summation value is greatest for $k = 10$. To which, other factors like negative cosine terms (in some cases) are more reasonable, and this shows that the formula is actually quite reliable.

However, an alternative idea of getting the summation expression exists using the half angle formulas.

If one starts with the original unsummed expression ($\sin x$) and applies half angle formulas, we'd get (remember, we can relate with the previous expression with the help of our beloved k. Note that $k = 1$.)

$$2\sin\frac{x}{2}\cos\frac{x}{2}$$

Repeatedly doing this ($k = 2$)

$$2\left(2\sin\frac{x}{4}\cos\frac{x}{4}\right)\cos\frac{x}{2}$$
$$2^2\sin\frac{x}{2^2}\cos\frac{x}{2^1}\cos\frac{x}{2^2}$$

For $k = 3$.

$$2^3\sin\frac{x}{8}\prod_{i=1}^{3}\cos\frac{x}{2^i}$$

Generalising,

$$\sin x = 2^k\sin\frac{x}{2^k}\prod_{i=1}^{k}\cos\frac{x}{2^i}$$

The high degree of similarity between this equation and the original sine summation expression obtained calls for some interesting comparisons.

For the sake of reference, the original expression is

$$\sum_{x=1}^{n}\sin x = 2^k\sin\frac{n+1}{2}\prod_{i=2}^{k}\cos\frac{n}{2^i}\sum_{i=1}^{\frac{n}{2^k}}\left[\cos\frac{2^{k-1}+(i-1)2^k-n}{2^k}\right]$$

Applying the summation to the half-angle method that we just obtained,

$$\sum_{x=1}^{n}\sin x = \sum_{x=1}^{n}\left\{2^k\sin\frac{x}{2^k}\prod_{i=1}^{k}\cos\frac{x}{2^i}\right\}$$

2^k is a constant, so that one can be taken out.

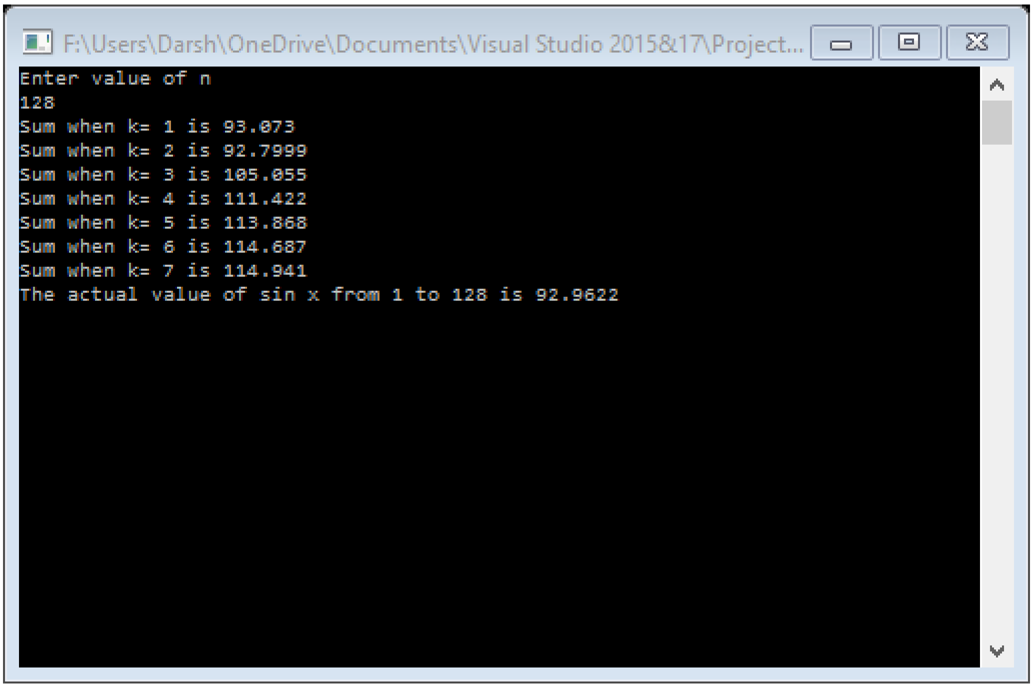
$$\sum_{x=1}^{n}\sin x = 2^k\sum_{x=1}^{n}\left\{\sin\frac{x}{2^k}\prod_{i=1}^{k}\cos\frac{x}{2^i}\right\}$$

This is however a worse expression, because the summator is essentially quite meaningless, just providing a symbol to add everything. The previous expression was better. Additionally, the high dependency on the current sine being added (compared to the previous one, where there was *no* dependency) is another drawback.

Still, I took note of this one because of the apparent similarity. When developing this expression, I thought that I could pull of a full replication of the first expression, so to obtain this one instead is admittedly a bit of a disappointment.

When a letter makes a big difference

During the development of V1.1 Beta 3, when I was testing the validity of the sine function, the outputs coming out of the program were strange and surprising. I went through the code quite a few times, but was still flummoxed when the outputs grew highly inconsistent and inaccurate for higher values of n. To add to this confusion, the outputs only started to be inconsistent when $k \geq 3$. No matter the value of n, $k = 1$ and 2 never had any problems at all.



The above screenshot shows the inconsistency when $n = 128$.
The blunder was discovered only 2 months later, when I was working on the cosine expression. It had suddenly occurred to me that I had then written the sine summation expression as

$$\sum_{x=1}^{n} \sin x = 2^k \sin \frac{n+1}{2} \prod_{i=2}^{k} \cos \frac{n}{2^k} \sum_{i=1}^{\frac{n}{2^k}} \left[\cos \frac{2^{k-1} + (i-1)2^k - n}{2^k} \right]$$

But that means that the cosine multiplier is a constant, which was clearly not what I was looking for. As can be clearly seen from the derivation, the cosine multiplier changes each time. So, all I had to do was change

$$\prod_{i=2}^{k} \cos \frac{n}{2^k}$$

to

$$\prod_{i=2}^{k} \cos \frac{n}{2^i}$$

And reflect the changes in the code, which was just a matter of changing one letter from

```
sinsum = sinsum*cos(radical((n / pow(2, k))));
```

to

```
sinsum = sinsum*cos(radical((n / pow(2, i))));
```

and the problem was solved!

The moral of this is that the power of a letter is far more than it seems. It can do a **lot** of damage if you're not careful.

The intricate expression obtained when finding $\Sigma \sin x$ calls for a natural question: - what would we get with $\Sigma \cos x$? There is something really exciting here: - we should be able to get appreciable expressions that happen to be similarly beautiful to what we obtained in Section 5.6, and hence set the stage for interesting cross-ratio comparisons. This line of thinking is possible because in the sine summation, we noted that there is a term $\sin\left(\frac{n+1}{2}\right)$ that makes its mark, independent of the value of k or n , and the reasoning for this was that the sine expansion

$$\sin\alpha \pm \sin\beta = 2\sin\frac{1}{2}(\alpha\pm\beta)\cos\frac{1}{2}(\alpha\mp\beta)$$

required for the first half, which happened to be a constant, to be a sine and the second half to be cosine terms. However, because of the constant sine term, *every* other term was purely cosine, and to add it up, the cosine summation had only cosines!

So, in theory, when we apply the cosine summation, we should get a remarkably similar expression. Let's start.

Take

$$\sum_{x=1}^{n}\cos x = \cos 1 + \cos 2 + \cdots + \cos n$$

Applying the cosine summation formula

$$\cos\alpha + \cos\beta = 2\cos\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)$$

in the RHS, we get

$$2\cos\frac{1+n}{2}\cos\frac{1-n}{2} + 2\cos\frac{2+(n-1)}{2}\cos\frac{2-(n-1)}{2} + 2\cos\frac{3+(n-2)}{2}\cos\frac{3-(n-2)}{2} + \cdots$$

$$\to 2\cos\left(\frac{1+n}{2}\right)\left(\cos\frac{1-n}{2} + \cos\frac{3-n}{2} + \cdots + \cos\frac{2\left(\frac{n}{2}\right)-1-n}{2}\right)$$

$$\to 2\cos\left(\frac{1+n}{2}\right)\left(\cos\frac{1-n}{2} + \cos\frac{3-n}{2} + \cdots + \cos\left|-\frac{1}{2}\right|\right)$$

$$\sum_{x=1}^{n}\cos x = 2\cos\left(\frac{1+n}{2}\right)\sum_{i=1}^{\frac{n}{2}}\cos\frac{2i-1-n}{2}$$

At this stage, it can be easily argued that the expression is **really** similar to the sine summation. Right now, $k = 1$.

Let us increment k and see what happens.

$$\sum_{x=1}^{n}\cos x = 2\cos\left(\frac{1+n}{2}\right)\left(\cos\frac{1-n}{2} + \cos\frac{3-n}{2} + \cdots + \cos\frac{1}{2}\right)$$

It must be noted that the cosine summation obtained is exactly the same as the one in the sine summation.

Taking the first and last term, second and second last term and so on,

$$\cos\frac{1-n}{2}+\cos\frac{1}{2}+\cos\frac{3-n}{2}+\cos\frac{3}{2}+\ldots+\cos\frac{\frac{n}{2}-n}{2}+\cos\frac{\frac{n}{2}-n}{2}+1$$

Using the formula,

$$2\cos\left(\frac{\frac{1-n}{2}+\frac{1}{2}}{2}\right)\cos\left(\frac{\frac{1-n}{2}-\frac{1}{2}}{2}\right)+2\cos\left(\frac{\frac{3-n}{2}+\frac{3}{2}}{2}\right)\cos\left(\frac{\frac{3-n}{2}-\frac{3}{2}}{2}\right)+\cdots$$

(to $\frac{n}{4}$ terms!)
we get

$$2\cos\frac{2-n}{4}\cos\frac{n}{4}+2\cos\frac{6-n}{4}\cos\frac{n}{4}+\cdots+2\cos\frac{1}{2}\cos\frac{n}{4}$$

$$2\cos\frac{n}{4}\left(\cos\frac{2-n}{4}+\cos\frac{6-n}{4}+\cdots+\cos\frac{1}{2}\right)$$

Giving us

$$\sum_{x=1}^{n}\cos x=4\cos\frac{n}{2}\cos\frac{n}{4}\sum_{i=1}^{\frac{n}{4}}\cos\frac{4i-2-n}{4}$$

The exact copy!

Generalising to the k^{th} time for n terms, we get

$$\sum_{x=1}^{n}\cos x=2^k\prod_{i=1}^{k}\cos\frac{n}{2^i}\sum_{i=1}^{\frac{n}{2^k}}\left[\cos\frac{2^{k-1}+(i-1)2^k-n}{2^k}\right]$$

and it cannot be argued that this marks for some interesting cross-ratio(al) comparisons. This will be seen in the next section.

However, attempting to replicate the ½-angle formula for cosines does not work, mainly because of the power-driven expression of $\cos 2x$.

$$\cos x=2\cos^2\frac{x}{2}-1\longleftrightarrow 2\left(\cos\frac{x}{2}\right)^2-1\longleftrightarrow 2\left(2\cos^2\frac{x}{4}-1\right)^2-1$$

5.8 *Cross-trig comparisons*

Section 1

When $\Sigma\sin x=\Sigma\cos x$ for a particular value of n and k.

In this case, it turns out that due to the heavy similarity of the terms, the simplification becomes drastically simpler, setting the stage for some interesting action: -

$$2^k \sin\frac{n+1}{2}\prod_{i=2}^{k}\cos\frac{n}{2^i}\sum_{i=1}^{\frac{n}{2^k}}\left[\cos\frac{2^{k-1}+(i-1)2^k-n}{2^k}\right]$$
$$= 2^k\prod_{i=1}^{k}\cos\frac{n}{2^i}\sum_{i=1}^{\frac{n}{2^k}}\left[\cos\frac{2^{k-1}+(i-1)2^k-n}{2^k}\right]$$

Cutting off the unnecessary terms,

$$\sin\frac{n+1}{2} = \cos\frac{n}{2}$$

We've got a linear! Simplifying it,

$$\cos\left(\frac{180-n-1}{2}\right) = \cos\frac{n}{2}$$
$$\rightarrow 179 - n = n$$
$$n = 89\frac{1}{2}$$

and that is impossible! Admittedly I thought that there would be a value or two, though this certainly does not seem to be the case. Certainly, the relation $\sin x = \cos(\frac{\pi}{2} - x)$ does not help here. The culprit is the $\frac{1}{2}$ in $\sin\frac{n+1}{2}$ obtained in the sine summation.

One immediate question that crops up is why $n \neq 90°$? After all, if one applies $\sin x = \cos(\frac{\pi}{2} - x)$ to every term in the summing sequence, we would get the same one, right? However, there is a bug in that line of thought. If one takes

$$\sum_{i=1}^{90°}\sin i$$

and applies the formula, then he'll get

$$\sum_{i=1°}^{90°}\cos(90° - i)$$

which is however equal to, when the formula is rewritten in terms of $\cos x$:-

$$\sum_{i=0°}^{89°}\cos i$$

which is certainly not equal to the actual cosine summation, as the limits (and the answer) are different.

<u>Section 2</u>

What about

$$\sum_{x=1}^{n}\sin x + \cos x\,?$$

Due to the similarity between the terms observed, we want to see what type of combined similarity can be observed.

Substituting with the expressions already obtained,

$$\sum_{x=1}^{n} \sin x + \cos x$$

$$= 2^k \sin\frac{n+1}{2}\prod_{i=2}^{k}\cos\frac{n}{2^i}\sum_{i=1}^{\frac{n}{2^k}}\left[\cos\frac{2^{k-1}+(i-1)2^k-n}{2^k}\right]$$

$$+ 2^k\prod_{i=1}^{k}\cos\frac{n}{2^i}\sum_{i=1}^{\frac{n}{2^k}}\left[\cos\frac{2^{k-1}+(i-1)2^k-n}{2^k}\right]$$

Factorising,

$$\sum_{x=1}^{n}\sin x+\cos x = 2^k\sum_{i=1}^{\frac{n}{2^k}}\left[\cos\frac{2^{k-1}+(i-1)2^k-n}{2^k}\right]\left\{\sin\frac{n+1}{2}\prod_{i=2}^{k}\cos\frac{n}{2^i}+\prod_{i=1}^{k}\cos\frac{n}{2^i}\right\}$$

$$\rightarrow \sum_{x=1}^{n}\sin x+\cos x = 2^k\prod_{i=2}^{k}\cos\frac{n}{2^i}\sum_{i=1}^{\frac{n}{2^k}}\left[\cos\frac{2^{k-1}+(i-1)2^k-n}{2^k}\right]\left\{\sin\frac{n+1}{2}+\cos\frac{n}{2}\right\}$$

At this stage, attempting to convert the cosine term into the sine term and then using the $\sin(A+B)$ formula is not the best idea, because n can be greater than 90°. That being said, for those desirous of a purely productial expression, this could be a good idea in a limited-domain scenario.

Section 3

It may seem obvious that

$$\frac{\sum_{i=1}^{n}\sin i}{\sum_{i=1}^{n}\cos i} = \sum_{i=1}^{n}\tan i$$

but unfortunately, we know that

$$\frac{\sum a}{\sum b} \neq \sum\frac{a}{b}$$

However, it can indeed be said that

$$\frac{\sum_{i=1}^{n}\sin i}{\sum_{i=1}^{n}\cos i} = \frac{\sin\frac{n+1}{2}}{\cos\frac{n}{2}}$$

just that the value cannot be 1.

5.9 *De Moivre's Theorem (Part 2)*

What happens if we use De Moivre's theorem instead?
Again, we know that

$$\sin\theta = \frac{1}{2i}\left(e^{i\theta}-e^{-i\theta}\right)$$

Applying summation,

$$\sum_{n=1}^{k} \frac{1}{2i}(e^{ni} - e^{-ni})$$

$$\frac{1}{2i}\left[e^{i} + e^{2i} + \cdots + e^{ni} - (e^{-i} + e^{-2i} + \cdots + e^{-ni})\right]$$

This is a sum of 2 GP's. The first one has e^{i} as the common ratio, while the other term has e^{-i}.

Using the G.P formula (and assuming that $r \geq 1$ – there is no difference if you take $r < 1$ instead),

$$\frac{1}{2i}\left[\frac{e^{i}(e^{in} - 1)}{e^{i} - 1} - \frac{e^{-i}(e^{-in} - 1)}{e^{-i} - 1}\right]$$

Admittedly this looks a bit untidy. The e^{-i} term in particular looks like something that deserves a fix of its own.

Let us see what would happen if we were to write e^{-i} as $\frac{1}{e^{i}}$.

$$\frac{1}{2i}\left[\frac{e^{i}(e^{in} - 1)}{e^{i} - 1} - \frac{\left(\frac{1}{e^{i}}\right)\left(\frac{1}{e^{in}} - 1\right)}{\left(\frac{1}{e^{i}}\right) - 1}\right]$$

$$\frac{1}{2i}\left[\frac{e^{i}(e^{in} - 1)}{e^{i} - 1} - \frac{\left(\frac{1}{e^{in}} - 1\right)}{e^{i}\left(\frac{1}{e^{i}}\right) - 1}\right]$$

$$\frac{1}{2i}\left[\frac{e^{i}(e^{in} - 1)}{e^{i} - 1} - \frac{\left(\frac{1}{e^{in}} - 1\right)}{1 - e^{i}}\right]$$

$$\frac{1}{2i(e^{i} - 1)}\left[\frac{e^{i}(e^{in} - 1)}{1} + \frac{\left(\frac{1}{e^{in}} - 1\right)}{1}\right]$$

$$\frac{1}{2i(e^{i} - 1)}\left[e^{i}(e^{in} - 1) + \frac{1 - e^{in}}{e^{in}}\right]$$

$$\rightarrow \sum_{x=1}^{n} \sin x = \frac{e^{in} - 1}{2i(e^{i} - 1)}\left[e^{i} - \frac{1}{e^{in}}\right]$$

That expression looks much better and is far more elegant.
But there is one obvious disadvantage to the application of de Moivre's theorem. It forces you to use complex numbers. Obviously, one wouldn't expect the sum of sines (which are positive reals) to be imaginary… That is not mentioning the fact that the sum expression obtained cannot be made real at all.

Let's now apply Euler's theorem to the expression:

$$\frac{e^{in}-1}{2i(e^{i}-1)}\left[e^{i}-\frac{1}{e^{in}}\right]=\frac{\cos n+i\sin n-1}{2i\,(\cos 1+i\sin 1-1)}\left[(\cos 1+i\sin 1)-\frac{1}{\cos n+i\sin n-1}\right]$$

Up to this point, we have taken degrees, even if Euler's theorem is based on radians. However, we cannot keep doing it beyond here. First off, Euler's theorem states that

$$e^{i\theta}=\cos\theta+i\sin\theta.$$

But n is not an angle on its own (rather, it is the upper limit of the summation), and while it seems ridiculous that n *could* be taken as an angle in this instance, it must be noted that $\sin n$ is still valid – it does act as an angle.

If the formula were to be compatible with $n,$ and the formula *is* applied with degrees in mind, then we can take some really risky approximations. Let $\cos 1° \cong 1, \sin 1° \cong 0$. Then the denominator in the first term falls apart:

$2i\,(\cos 1+i\sin 1-1)\rightarrow 2i(1+0-1)\rightarrow 2i*0=0$ (obviously incompatible)

OR,

$2i\,(\cos 1+i\sin 1-1)\rightarrow(2i\cos 1-\sin 1-2i)\rightarrow 2i-2i=0$ (same thing)

Even if the denominator was not 0, the value of the first term would tend towards infinity. This is also very risky.

If we take radians instead, the formula will work, but there is still the problem of dealing with complex numbers.

Even through Euler's theorem provided an interesting simplification to the result obtained by De Moivre's theorem, further simplification to the expression is arduous, and its reliance on complex numbers is also a disadvantage.