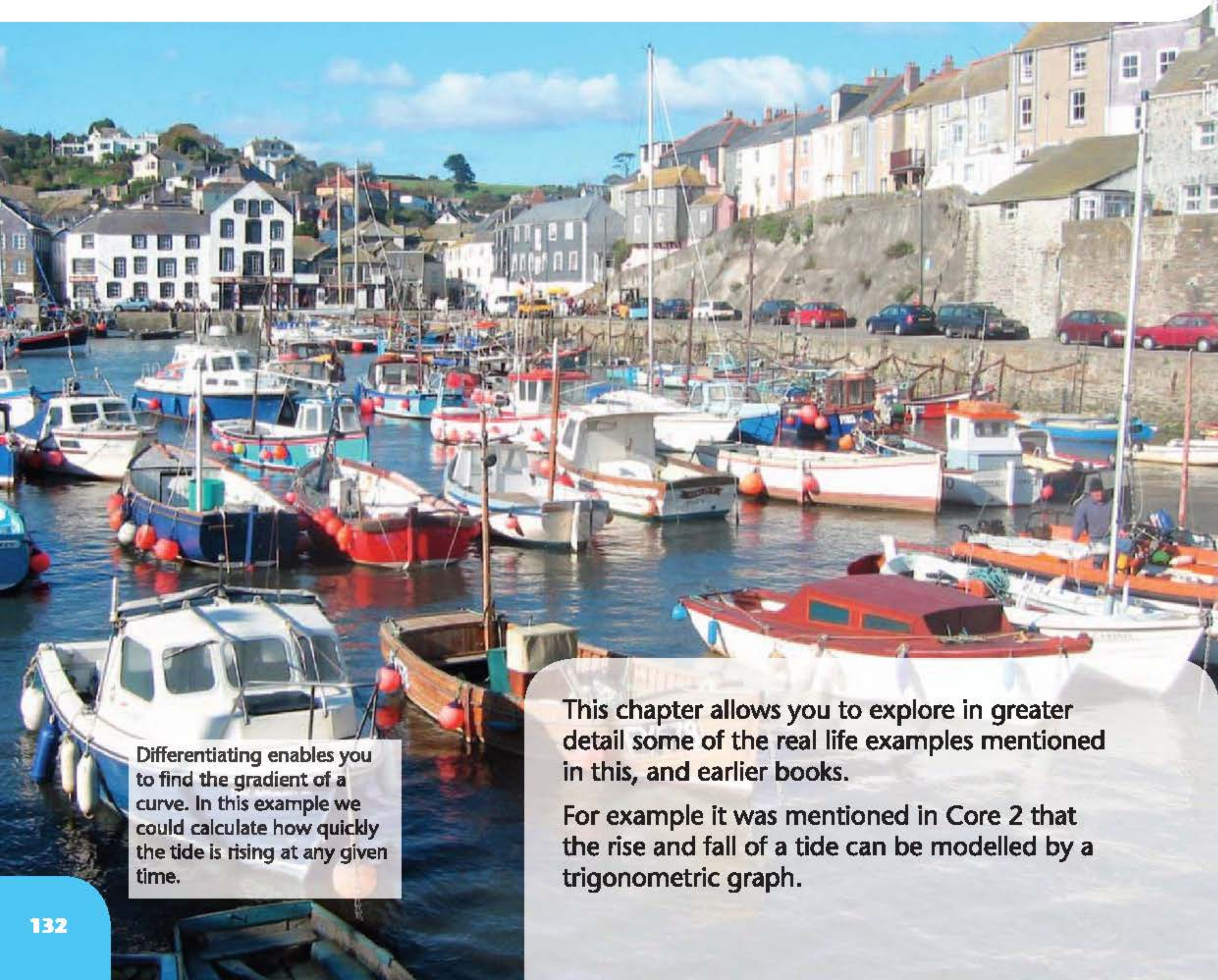
8 Differentiation

After completing this chapter you should be able to

1. differentiate a composite function using the chain rule
2. differentiate functions that are multiplied together by using the product rule
3. differentiate rational functions using the quotient rule
4. differentiate variations on the functions of e^x and $\ln(x)$
5. differentiate variations on the functions $\sin x$, $\cos x$ and $\tan x$.



Differentiating enables you to find the gradient of a curve. In this example we could calculate how quickly the tide is rising at any given time.

This chapter allows you to explore in greater detail some of the real life examples mentioned in this, and earlier books.

For example it was mentioned in Core 2 that the rise and fall of a tide can be modelled by a trigonometric graph.

8.1 You need to be able to differentiate a function of a function, using the chain rule.

■ **The chain rule enables you to differentiate a function of a function. In general,**

- **if $y = [f(x)]^n$ then $\dfrac{dy}{dx} = n[f(x)]^{n-1}f'(x)$**
- **if $y = f[g(x)]$ then $\dfrac{dy}{dx} = f'[g(x)]g'(x)$**

You should learn these results.

Example 1

Given that $y = (3x^4 + x)^5$ find $\dfrac{dy}{dx}$ using the chain rule.

Here $f(x) = 3x^4 + x$
So $f'(x) = 12x^3 + 1$

Using the chain rule

$$\frac{dy}{dx} = 5(3x^4 + x)^4(12x^3 + 1)$$
$$= 5(12x^3 + 1)(3x^4 + x)^4$$

This uses the chain rule with $n = 5$.

Example 2

Given that $y = \sqrt{5x^2 + 1}$ find the value of $\dfrac{dy}{dx}$ at (4, 9).

Let $f(x) = 5x^2 + 1$
Then $f'(x) = 10x$

Using the chain rule

$$\frac{dy}{dx} = \tfrac{1}{2}(5x^2 + 1)^{-\frac{1}{2}}(10x)$$
$$= 5x(5x^2 + 1)^{-\frac{1}{2}}$$

Required value is $2\frac{2}{9}$.

This time $n = \frac{1}{2}$ and $\frac{1}{2}(10x)$ is simplified to $5x$.

Substitute $x = 4$ to give the required value.

■ **Another form of the chain rule is**

- $$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

where y is a function of u, and u is a function of x.

Example 3

Given that $y = (x^2 - 7x)^4$ find $\frac{dy}{dx}$, using the chain rule.

Let $u = x^2 - 7x$, then $y = u^4$

$\therefore \quad \frac{du}{dx} = 2x - 7 \quad$ and $\quad \frac{dy}{du} = 4u^3$

Then, using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= 4u^3 \times (2x - 7)$$
$$= 4(2x - 7)(x^2 - 7x)^3$$

When the substitution is not given in a question you should put the bracket equal to u.

Use the chain rule to find $\frac{dy}{dx}$.

Ensure that you give your answer in terms of x, with no u terms present.

Example 4

Given that $y = \frac{1}{\sqrt{6x - 3}}$ find the value of $\frac{dy}{dx}$ at $(2, \frac{1}{3})$.

Let $u = 6x - 3$, then $y = u^{-\frac{1}{2}}$

$\therefore \quad \frac{du}{dx} = 6 \quad$ and $\quad \frac{dy}{du} = -\frac{1}{2}u^{-\frac{3}{2}}$

Then, as $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$$\frac{dy}{dx} = -\frac{1}{2}u^{-\frac{3}{2}} \times 6$$
$$= -3(6x - 3)^{-\frac{3}{2}}$$

Required value is $-\frac{1}{9}$.

Put u equal to the expression in the bracket.

Substitute $x = 2$ to give the required value.

■ **Also a particular case of the chain rule is the result**

- $$\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)}$$

This arises since $\frac{dy}{dx} \times \frac{dx}{dy} = \frac{dy}{dy} = 1$

Then you make $\frac{dy}{dx}$ the subject of the formula.

Example 5

Find the value of $\frac{dy}{dx}$ at the point (2, 1) on the curve with equation $y^3 + y = x$.

$$\frac{dx}{dy} = 3y^2 + 1$$

Start with $x = y^3 + y$ and differentiate with respect to y.

$$\therefore \quad \frac{dy}{dx} = \frac{1}{3y^2 + 1}$$

Use $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$

$$= \tfrac{1}{4}$$

Substitute $y = 1$.

Exercise 8A

1 Differentiate:

a $(1 + 2x)^4$ b $(3 - 2x^2)^{-5}$ c $(3 + 4x)^{\frac{1}{2}}$ d $(6x + x^2)^7$

e $\frac{1}{3 + 2x}$ f $\sqrt{7 - x}$ g $4(2 + 8x)^4$ h $3(8 - x)^{-6}$

2 Given that $y = \frac{1}{(4x + 1)^2}$ find the value of $\frac{dy}{dx}$ at $(\frac{1}{4}, \frac{1}{4})$.

3 Given that $y = (5 - 2x)^3$ find the value of $\frac{dy}{dx}$ at (1, 27).

4 Find the value of $\frac{dy}{dx}$ at the point (8, 2) on the curve with equation $3y^2 - 2y = x$.

5 Find the value of $\frac{dy}{dx}$ at the point $(2\frac{1}{2}, 4)$ on the curve with equation $y^{\frac{1}{2}} + y^{-\frac{1}{2}} = x$.

8.2 You need to differentiate functions that are multiplied together, by using the product rule.

■ **To differentiate the product of two functions, differentiate the first function and leave the second function alone, then differentiate the second one and leave the first function alone, then add all this together.**

- **If $y = uv$ then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$,**

where u and v are both functions of x. This is called the product rule.

Here is a proof of this rule:

Let $y = uv$ where u and v are two functions of x. Suppose that a small increment δx in the variable x results in a small change δu in u and a small change δv in v, which in turn results in a small change δy in the variable y.

Then $\quad y + \delta y = (u + \delta u)(v + \delta v) \qquad (1)$

But $\quad y = uv \qquad (2)$

Subtract ① − ②

$$\therefore \quad \delta y = (u + \delta u)(v + \delta v) - uv$$
$$= uv + u\delta v + v\delta u + \delta u\delta v - uv$$
$$= u\delta v + v\delta u + \delta u\delta v$$
$$\therefore \quad \frac{\delta y}{\delta x} = u\frac{\delta v}{\delta x} + v\frac{\delta u}{\delta x} + \frac{\delta u}{\delta x}\delta v$$

You will *not* need to prove this result in an examination.

As $\delta x \to 0$ then $\frac{\delta y}{\delta x} \to \frac{dy}{dx}, \frac{\delta u}{\delta x} \to \frac{du}{dx}$ and $\frac{\delta v}{\delta x} \to \frac{dv}{dx}$.

Also $\delta v \to 0$ and thus $\frac{\delta u}{\delta x}\delta v \to 0$.

$$\therefore \quad \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

You should learn this result, and learn where it is appropriate to use it.

Example 6

Given that $f(x) = x^2\sqrt{3x - 1}$, find $f'(x)$.

Recognise that this is a product of two functions.

Let $u = x^2$ and $v = \sqrt{3x - 1} = (3x - 1)^{\frac{1}{2}}$

Then $\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = 3 \times \frac{1}{2}(3x - 1)^{-\frac{1}{2}}$

The second function is a function of a function and requires the chain rule.

Using $\frac{dy}{dx} = u\frac{dv}{dx} + u\frac{du}{dx}$

$$f'(x) = x^2 \times \tfrac{3}{2}(3x - 1)^{-\frac{1}{2}} + \sqrt{3x - 1} \times 2x$$
$$= \frac{3x^2 + 12x^2 - 4x}{2\sqrt{3x - 1}}$$
$$= \frac{15x^2 - 4x}{2\sqrt{3x - 1}}$$
$$= \frac{x(15x - 4)}{2\sqrt{3x - 1}}$$

Collect terms to simplify, and factorise to give the final answer.

Exercise 8B

1 Differentiate:

a $x(1 + 3x)^5$ **b** $2x(1 + 3x^2)^3$ **c** $x^3(2x + 6)^4$ **d** $3x^2(5x - 1)^{-1}$

2 **a** Find the value of $\frac{dy}{dx}$ at the point (1, 8) on the curve with equation $y = x^2(3x - 1)^3$.

b Find the value of $\frac{dy}{dx}$ at the point (4, 36) on the curve with equation $y = 3x(2x + 1)^{\frac{1}{2}}$.

c Find the value of $\frac{dy}{dx}$ at the point $(2, \frac{1}{5})$ on the curve with equation $y = (x - 1)(2x + 1)^{-1}$.

3 Find the points where the gradient is zero on the curve with equation $y = (x - 2)^2(2x + 3)$.

8.3 You need to be able to differentiate rational functions using the quotient rule.

■ **A rational function has the form $\frac{u(x)}{v(x)}$, where $u(x)$ and $v(x)$ are functions.**

- **If $y = \frac{u(x)}{v(x)}$ then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$**

This is called the quotient rule.

Example 7

Given that $y = \frac{x}{2x+5}$ find $\frac{dy}{dx}$.

Let $u = x$ and $v = 2x + 5$

$\frac{du}{dx} = 1$ and $\frac{dv}{dx} = 2$

Using $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$$\frac{dy}{dx} = \frac{(2x+5) \times 1 - x \times 2}{(2x+5)^2}$$

$$= \frac{5}{(2x+5)^2}$$

Recognise that y is a quotient and use the quotient rule.

Simplify the numerator of the fraction.

Example 8

By expressing $y = \frac{u}{v}$ as $y = uv^{-1}$, prove the quotient rule.

You can use the product rule to give

$$\frac{dy}{dx} = u\frac{d}{dx}(v^{-1}) + v^{-1}\frac{du}{dx}$$

$$= u\left(-v^{-2}\frac{dv}{dx}\right) + v^{-1}\frac{du}{dx}$$

$$= -\frac{u}{v^2}\frac{dv}{dx} + \frac{1}{v}\frac{du}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Use the chain rule to differentiate v^{-1}.

Then use a common denominator v^2.

You should learn this result, and learn where it is appropriate to use it.

Exercise 8C

1 Differentiate:

a $\dfrac{5x}{x+1}$ **b** $\dfrac{2x}{3x-2}$ **c** $\dfrac{x+3}{2x+1}$ **d** $\dfrac{3x^2}{(2x-1)^2}$ **e** $\dfrac{6x}{(5x+3)^{\frac{1}{2}}}$

2 Find the value of $\dfrac{dy}{dx}$ at the point $(1, \frac{1}{4})$ on the curve with equation $y = \dfrac{x}{3x+1}$.

3 Find the value of $\dfrac{dy}{dx}$ at the point $(12, 3)$ on the curve with equation $y = \dfrac{x+3}{(2x+1)^{\frac{1}{2}}}$.

8.4 You need to be able to differentiate the exponential function.

In Chapter 3 you met the exponential function e^x. This is a special function because it is the only function for which $f(x) = f'(x)$.

You should learn this result.

■ **If $y = e^x$ then $\dfrac{dy}{dx} = e^x$**

You can prove this result from first principles by the method introduced in Book C1.

Use the definition $f'(x) = \lim\limits_{\delta x \to 0}\left[\dfrac{f(x+\delta x) - f(x)}{\delta x}\right]$

If $f(x) = e^x$ then $f'(x) = \lim\limits_{\delta x \to 0}\left[\dfrac{e^{x+\delta x} - e^x}{\delta x}\right]$

$$= \lim_{\delta x \to 0}\left[\frac{e^x(e^{\delta x} - 1)}{\delta x}\right]$$

$$= e^x \lim_{\delta x \to 0}\left[\frac{(e^{\delta x} - 1)}{\delta x}\right]$$

The table below shows values for $\left[\dfrac{(e^{\delta x} - 1)}{\delta x}\right]$ for $e = 2.718\,282$ for progressively smaller values of δx.

	$\delta x = 0.1$	$\delta x = 0.01$	$\delta x = 0.001$	$\delta x = 0.000\,1$	$\delta x = 0.000\,01$
$\left[\dfrac{(e^{\delta x} - 1)}{\delta x}\right]$	1.051 709 25	1.005 016 772	1.000 500 23	1.000 050 06	1.000 005 063

From this table you can see that $\left[\dfrac{(e^{\delta x} - 1)}{\delta x}\right]$ approaches a limiting value of 1 as $\delta x \to 0$. This means that if $f(x) = e^x$, then $f'(x) = 1 \times e^x$. e^x is called the exponential function, where $e = 2.718\,282$ to 6 d.p. and has the property that if $y = e^x$ then $\dfrac{dy}{dx} = e^x$ also.

This result can be used together with the chain rule and the product and quotient rules to enable you to differentiate a wide range of functions. In particular:

■ **$y = e^{f(x)}$ then $\dfrac{dy}{dx} = f'(x)e^{f(x)}$**

Example 9

Differentiate **a** $5e^x$ **b** e^{2x+3} **c** xe^{x^2} **d** $\dfrac{e^{2x+3}}{x}$

a Let $y = 5e^x$

Then $\dfrac{dy}{dx} = 5e^x$

Differentiate e^x to give e^x and multiply the result by 5.

b Let $y = e^{2x+3}$, then $y = e^t$ where $t = 2x + 3$.

$\therefore \dfrac{dy}{dt} = e^t$ and $\dfrac{dt}{dx} = 2$

$\dfrac{dy}{dx} = 2e^t$

$= 2e^{2x+3}$

Use the chain rule to differentiate this function of a function.

Remember to give the answer in terms of x.

c Let $y = xe^{x^2}$

Let $u = x$ and $v = e^{x^2}$

Then $\dfrac{du}{dx} = 1$ and $\dfrac{dv}{dx} = 2xe^{x^2}$

$\therefore \dfrac{dy}{dx} = x(2xe^{x^2}) + e^{x^2}$

$= e^{x^2}(2x^2 + 1)$

Use the product rule and then use the chain rule to differentiate e^{x^2}.

Simplify the answer by factorising.

d Let $y = \dfrac{e^{2x+3}}{x}$

then $\dfrac{dy}{dx} = \dfrac{x \times 2e^{2x+3} - e^{2x+3}}{x^2}$

$= \dfrac{(2x-1)e^{2x+3}}{x^2}$

Use the quotient rule, together with the chain rule to differentiate e^{2x+3}.

Exercise 8D

1 Differentiate:

a e^{2x} **b** e^{-6x} **c** e^{x+3} **d** $4e^{3x^2}$ **e** $9e^{3-x}$

f xe^{2x} **g** $(x^2+3)e^{-x}$ **h** $(3x-5)e^{x^2}$ **i** $2x^4e^{1+x}$ **j** $(9x-1)e^{3x}$

k $\dfrac{x}{e^{2x}}$ **l** $\dfrac{e^{x^3}}{x}$ **m** $\dfrac{e^x}{x+1}$ **n** $\dfrac{e^{-2x}}{\sqrt{x+1}}$

2 Find the value of $\dfrac{dy}{dx}$ at the point $\left(1, \dfrac{1}{e}\right)$ on the curve with equation $y = xe^{-x}$.

3 Find the value of $\dfrac{dy}{dx}$ at the point $(0, 3)$ on the curve with equation $y = (2x+3)e^{2x}$.

4 Find the equation of the tangent to the curve $y = xe^{2x}$ at the point $(\frac{1}{2}, \frac{1}{2}e)$.

5 Find the equation of the tangent to the curve $y = \dfrac{e^{\frac{x}{3}}}{x}$ at the point $(3, \frac{1}{3}e)$.

6 Find the coordinates of the turning points on the curve $y = x^2e^{-x}$, and determine whether these points are maximum or minimum points.

7 Given that $y = \dfrac{e^{3x}}{x}$, find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$, simplifying your answers.
Use these answers to find the coordinates of the turning point on the curve with equation $y = \dfrac{e^{3x}}{x}$, $x > 0$, and determine the nature of this turning point.

8.5 You need to be able to differentiate the logarithmic function.

In Chapter 3 you were introduced to the logarithmic function, $\ln x$, which was defined as the inverse of the exponential function e^x.

You are now going to use the derivative of e^x to find the derivative of $\ln x$.

Let $y = \ln x$

Then $x = e^y$

You can make x the subject of the formula using the inverse function exp.

So $\dfrac{dx}{dy} = e^y$

But $\dfrac{dy}{dx} = \dfrac{1}{\frac{dx}{dy}}$

You can now use the result $\dfrac{dy}{dx} = \dfrac{1}{\frac{dx}{dy}}$, which was quoted at the end of Section 8.1, as a special case of the chain rule.

$\therefore \quad \dfrac{dy}{dx} = \dfrac{1}{e^y}$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{1}{x}$

■ **So if $y = \ln x$ then $\dfrac{dy}{dx} = \dfrac{1}{x}$**

You should learn this result.

This result can also be used together with the chain rule and the product and quotient rules to enable you to differentiate a wide range of functions.

In particular

■ **If $y = \ln[f(x)]$ then $\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$**

You put $f(x) = u$, so $y = \ln u$. Then, using the chain rule, $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$, so $\dfrac{dy}{dx} = \dfrac{1}{u} \times f'(x) = \dfrac{f'(x)}{f(x)}$

Example 10

Differentiate **a** $5\ln x$ **b** $\ln(6x - 1)$ **c** $x^3 \ln x$ **d** $\dfrac{\ln 5x}{x}$ **e** $2x + e^x \ln x$

a $y = 5\ln x$

$\therefore \quad \dfrac{dy}{dx} = 5 \times \dfrac{1}{x}$

$= \dfrac{5}{x}$

If $y = af(x)$ then $\dfrac{dy}{dx} = af'(x)$.

b $y = \ln(6x - 1)$

$$\therefore \quad \frac{dy}{dx} = 6 \times \frac{1}{6x-1}$$
$$= \frac{6}{6x-1}$$

This is a function of a function. Use the chain rule.
Let $u = 6x - 1$, so $y = \ln u$.
$\frac{du}{dx} = 6$ and $\frac{dy}{du} = \frac{1}{u}$ $\therefore$ $\frac{dy}{dx} = 6 \times \frac{1}{u}$.

c $y = x^3 \ln x$

$$\therefore \quad \frac{dy}{dx} = x^3 \times \frac{1}{x} + 3x^2 \times \ln x$$
$$= x^2 + 3x^2 \ln x$$

Use the product rule here, with $u = x^3$ and $v = \ln x$.

d $y = \frac{\ln 5x}{x}$

$$\therefore \quad \frac{dy}{dx} = \frac{x\left(\frac{5}{5x}\right) - 1 \times \ln 5x}{x^2}$$
$$= \frac{1 - \ln 5x}{x^2}$$

Use the quotient rule, and use the chain rule to differentiate the ln $5x$ term.

e $y = 2x + e^x \ln x$

$$\therefore \quad \frac{dy}{dx} = 2 + \left(e^x \times \frac{1}{x} + e^x \ln x\right)$$
$$= 2 + \frac{e^x}{x}(1 + x \ln x)$$

Use the product rule to differentiate the second term.

Exercise 8E

1 Find the function $f'(x)$ where $f(x)$ is

a $\ln(x + 1)$	**b** $\ln 2x$	**c** $\ln 3x$	**d** $\ln(5x - 4)$
e $3 \ln x$	**f** $4 \ln 2x$	**g** $5 \ln(x + 4)$	**h** $x \ln x$
i $\frac{\ln x}{x+1}$	**j** $\ln(x^2 - 5)$	**k** $(3 + x) \ln x$	**l** $e^x \ln x$

8.6 You need to be able to differentiate trigonometric functions. You can use the formula for $f'(x)$ to differentiate $\sin x$.

Earlier you were reminded that if you wish to differentiate a function then you must use the definition introduced in Book C1. That is,

$$f'(x) = \lim_{\delta x \to 0}\left[\frac{f(x + \delta x) - f(x)}{\delta x}\right]$$

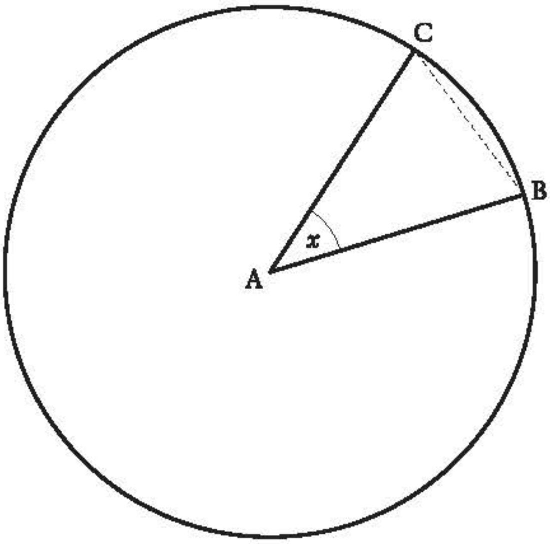
Let $f(x) = \sin x$.

$$\text{Then } f'(x) = \lim_{\delta x \to 0}\left[\frac{\sin(x + \delta x) - \sin(x)}{\delta x}\right]$$
$$= \lim_{\delta x \to 0}\left[\frac{\sin x \cos \delta x + \cos x \sin \delta x - \sin x}{\delta x}\right] \quad *$$

Now use the compound angle formula to expand $\sin(A + B)$.

As with many of these limiting values the numerator and the denominator of this fraction both approach zero, and so you need to investigate the behaviour of $\sin x$ and $\cos x$ for small values of x.

Consider first a circle with radius r, with radii AB and AC such that angle BAC is x, where x is measured in radians.



You use the formula introduced in the radians section:
area of sector $= \frac{1}{2}r^2\theta$

The area of sector ABC is $\frac{1}{2}r^2x$ and the area of triangle ABC is $\frac{1}{2}r^2 \sin x$. As x becomes small the area of the triangle becomes close to the area of the sector. Thus $\frac{1}{2}r^2 \sin x \approx \frac{1}{2}r^2 x \Rightarrow \sin x \approx x$, where x is small and is measured in radians.
Also $\cos x \approx 1$ for small values of x.
So in equation $*$ on page 141, replace $\cos \delta x$ by 1 and replace $\sin \delta x$ by δx, since δx is small.

You use area of triangle $= \frac{1}{2}ab \sin C$ with $a = b = r$ and $C = x$.

Then $f'(x) = \lim\limits_{\delta x \to 0}\left[\dfrac{\sin x + \cos x \times \delta x - \sin x}{\delta x}\right]$

$= \cos x$

- **So if $y = \sin x$ then $\dfrac{dy}{dx} = \cos x$**

This formula applies where x is measured in radians.

- **And, by the chain rule, if $y = \sin f(x)$ then $\dfrac{dy}{dx} = f'(x) \cos f(x)$**

Learn these two key points.

Example 11

Differentiate **a** $y = \sin 3x$ **b** $y = \sin\frac{2}{3}x$ **c** $y = \sin^2 x$

a $y = \sin 3x$

$\dfrac{dy}{dx} = 3\cos 3x$

Use the chain rule with $f(x) = 3x$, so $f'(x) = 3$.

b $y = \sin\frac{2}{3}x$

This time put $f(x) = \frac{2}{3}x$.

$\dfrac{dy}{dx} = \frac{2}{3}\cos\frac{2}{3}x$

c $y = \sin^2 x = (\sin x)^2$

$\dfrac{dy}{dx} = 2(\sin x)^1 \cos x$

Use the chain rule, with $u = \sin x$, so $\dfrac{du}{dx} = \cos x$ and $\dfrac{dy}{du} = 2u$.

$= 2\sin x \cos x$

Exercise 8F

1 Differentiate:

a $y = \sin 5x$ **b** $y = 2\sin\frac{1}{2}x$ **c** $y = 3\sin^2 x$ **d** $y = \sin(2x + 1)$

e $y = \sin 8x$ **f** $y = 6\sin\frac{2}{3}x$ **g** $y = \sin^3 x$ **h** $y = \sin^5 x$

8.7 You can use the result obtained for the derivative of $\sin x$ to differentiate $\cos x$.

Let $y = \cos x$

Then $y = \sin\left(\frac{\pi}{2} - x\right)$

This uses the expansion of $\sin(A - B)$ together with $\sin\frac{\pi}{2} = 1$ and $\cos\frac{\pi}{2} = 0$.

Using the result that $\frac{dy}{dx} = f'(x)\cos f(x)$ for $y = \sin f(x)$ and

$f(x) = \frac{\pi}{2} - x$

$$\frac{dy}{dx} = -\cos\left(\frac{\pi}{2} - x\right)$$
$$= -\sin x$$

- **So if $y = \cos x$ then $\frac{dy}{dx} = -\sin x$**

Remember x is measured in radians.

- **Also, by the chain rule, if $y = \cos f(x)$ then $\frac{dy}{dx} = -f'(x)\sin f(x)$**

Learn these two key points.

Example 12

Differentiate **a** $y = \cos(4x - 3)$ **b** $y = \cos x°$ (x degrees) **c** $y = \cos^3 x$

a $y = \cos(4x - 3)$

$\frac{dy}{dx} = -4\sin(4x - 3)$

Put $f(x) = 4x - 3$, and use the chain rule.

b $y = \cos\left(\frac{\pi x}{180}\right)$

When x is given in degrees you need to change the angle into radians before differentiating.

$\frac{dy}{dx} = -\frac{\pi}{180}\sin\left(\frac{\pi x}{180}\right)$

$= -\frac{\pi}{180}\sin x°$

c $y = \cos^3 x = (\cos x)^3$

$\frac{dy}{dx} = 3(\cos x)^2(-\sin x)$

Use the chain rule with $u = \cos x$. Ensure that you have no u terms in the answer.

$= -3\cos^2 x \sin x$

Exercise 8G

1 Differentiate:

a $y = 2\cos x$ **b** $y = \cos^5 x$ **c** $y = 6\cos\frac{5}{6}x$ **d** $y = 4\cos(3x + 2)$

e $y = \cos 4x$ **f** $y = 3\cos^2 x$ **g** $y = 4\cos\frac{1}{2}x$ **h** $y = 3\cos 2x$

8.8 You can use the quotient rule, together with the results obtained for the derivatives of sin x and cos x, to differentiate tan x.

Let $y = \tan x$.

Then $y = \dfrac{\sin x}{\cos x}$, which is a quotient.

This is the definition for $\tan x$.

Use the quotient rule

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

with $u = \sin x$ and $v = \cos x$.

Then
$$\frac{dy}{dx} = \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x}$$
$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$
$$= \frac{1}{\cos^2 x}$$
$$= \sec^2 x$$

Using the result that $\sin^2 x + \cos^2 x = 1$.

As $\sec x = \dfrac{1}{\cos x}$

- **So if x is measured in radians**
  - **$y = \tan x$ implies that $\dfrac{dy}{dx} = \sec^2 x$**

You should learn these key points.

- **Also by the chain rule, if $y = \tan f(x)$ then $\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$**

Example 13

Differentiate **a** $y = x\tan 2x$ **b** $y = \tan^4 x$

a $y = x\tan 2x$

$$\frac{dy}{dx} = x2\sec^2 2x + \tan 2x$$
$$= 2x\sec^2 2x + \tan 2x$$

This is a product.
Use $u = x$ and $v = \tan 2x$, together with the product formula.

b $y = \tan^4 x = (\tan x)^4$

$$\frac{dy}{dx} = 4(\tan x)^3(\sec^2 x)$$
$$= 4\tan^3 x\sec^2 x$$

Use the chain rule with $u = \tan x$.

Exercise 8H

1 Differentiate:

a $y = \tan 3x$ **b** $y = 4\tan^3 x$ **c** $y = \tan(x - 1)$ **d** $y = x^2 \tan \frac{1}{2}x + \tan(x - \frac{1}{2})$

8.9 The remaining trigonometric functions can be differentiated using the chain rule, together with the results obtained so far for $\sin x$, $\cos x$ and $\tan x$.

Let $y = \operatorname{cosec} x$.

Then $y = \dfrac{1}{\sin x} = (\sin x)^{-1}$

So $\dfrac{dy}{dx} = -(\sin x)^{-2}(\cos x)$

$= -\dfrac{\cos x}{\sin^2 x}$

This equals $-\dfrac{1}{\sin x} \times \dfrac{\cos x}{\sin x}$

$= -\operatorname{cosec} x \cot x$

- **$y = \operatorname{cosec} x$ implies that $\dfrac{dy}{dx} = -\operatorname{cosec} x \cot x$**
- **Also by the chain rule, if $y = \operatorname{cosec} f(x)$ then $\dfrac{dy}{dx} = -f'(x) \operatorname{cosec} f(x) \cot f(x)$**

Let $y = \sec x$.

Then $y = \dfrac{1}{\cos x} = (\cos x)^{-1}$

So $\dfrac{dy}{dx} = -(\cos x)^{-2}(-\sin x)$

$= \dfrac{\sin x}{\cos^2 x}$

Note that $\dfrac{\sin x}{\cos^2 x} = \dfrac{1}{\cos x} \times \dfrac{\sin x}{\cos x}$

$= \sec x \tan x$

- **$y = \sec x$ implies that $\dfrac{dy}{dx} = \sec x \tan x$**
- **Also by the chain rule, if $y = \sec f(x)$ then $\dfrac{dy}{dx} = f'(x) \sec f(x) \tan f(x)$**

Let $y = \cot x$.

Then $y = \dfrac{1}{\tan x} = (\tan x)^{-1}$

So $\dfrac{dy}{dx} = -(\tan x)^{-2}(\sec^2 x)$

$= \dfrac{\sec^2 x}{\tan^2 x}$

$\dfrac{\sec^2 x}{\tan^2 x} = \dfrac{1}{\cos^2 x} \times \dfrac{\cos^2 x}{\sin^2 x} = \dfrac{1}{\sin^2 x}$

$= -\operatorname{cosec}^2 x$

- **$y = \cot x$ implies that $\dfrac{dy}{dx} = -\operatorname{cosec}^2 x$**
- **Also by the chain rule, if $y = \cot f(x)$ then $\dfrac{dy}{dx} = -f'(x) \operatorname{cosec}^2 f(x)$**

Collecting all these results together,

- $y = \sin x \Rightarrow \dfrac{dy}{dx} = \cos x$
- $y = \cos x \Rightarrow \dfrac{dy}{dx} = -\sin x$
- $y = \tan x \Rightarrow \dfrac{dy}{dx} = \sec^2 x$
- $y = \operatorname{cosec} x \Rightarrow \dfrac{dy}{dx} = -\operatorname{cosec} x \cot x$
- $y = \sec x \Rightarrow \dfrac{dy}{dx} = \sec x \tan x$
- $y = \cot x \Rightarrow \dfrac{dy}{dx} = -\operatorname{cosec}^2 x$

These results, obtained so far, should all be learned. They can be used together with the chain rule and the product and quotient rules to enable you to differentiate a wide range of functions.

Example 14

Differentiate **a** $y = \dfrac{\operatorname{cosec} 2x}{x^2}$ **b** $y = \sec^3 x$

a

$y = \dfrac{\operatorname{cosec} 2x}{x^2}$

So $\dfrac{dy}{dx}$

$= \dfrac{x^2(-2 \operatorname{cosec} 2x \cot 2x) - \operatorname{cosec} 2x \times 2x}{x^4}$

$= \dfrac{-2 \operatorname{cosec} 2x(x \cot 2x + 1)}{x^3}$

Use the quotient rule with $u = \operatorname{cosec} 2x$ and $v = x^2$.

b $y = \sec^3 x = (\sec x)^3$

$\dfrac{dy}{dx} = 3(\sec x)^2 (\sec x \tan x)$

$= 3 \sec^3 x \tan x$

Use the chain rule with $u = \sec x$.

Exercise 8I

1 Differentiate

a $\cot 4x$ **b** $\sec 5x$ **c** $\operatorname{cosec} 4x$ **d** $\sec^2 3x$

e $x \cot 3x$ **f** $\dfrac{\sec^2 x}{x}$ **g** $\operatorname{cosec}^3 2x$ **h** $\cot^2(2x - 1)$

8.10 **You are now able to differentiate functions that are formed from a combination of trigonometric, exponential, logarithmic and polynomial functions.**

Example 15

Differentiate **a** $y = e^x \sin x$ **b** $y = \dfrac{\ln x}{\sin x}$

a $y = e^x \sin x$

$\dfrac{dy}{dx} = e^x \cos x + e^x \sin x$

Use the product rule with $u = e^x$ and $v = \sin x$.

b $y = \dfrac{\ln x}{\sin x}$

$\dfrac{dy}{dx} = \dfrac{\sin x \times \frac{1}{x} - \ln x \times \cos x}{\sin^2 x}$

$= \dfrac{\sin x - x \cos x \ln x}{x \sin^2 x}$

Use the quotient rule with $u = \ln x$ and $v = \sin x$.

Exercise 8J

1 Find the function $f'(x)$ where $f(x)$ is

a $\sin 3x$ **b** $\cos 4x$ **c** $\tan 5x$ **d** $\sec 7x$

e $\operatorname{cosec} 2x$ **f** $\cot 3x$ **g** $\sin \dfrac{2x}{5}$ **h** $\cos \dfrac{3x}{7}$

i $\tan \dfrac{2x}{5}$ **j** $\operatorname{cosec} \dfrac{x}{2}$ **k** $\cot \frac{1}{3}x$ **l** $\sec \dfrac{3x}{2}$

2 Find the function $f'(x)$ where $f(x)$ is

a $\sin^2 x$ **b** $\cos^3 x$ **c** $\tan^4 x$ **d** $(\sec x)^{\frac{1}{2}}$

e $\sqrt{\cot x}$ **f** $\operatorname{cosec}^2 x$ **g** $\sin^3 x$ **h** $\cos^4 x$

i $\tan^2 x$ **j** $\sec^3 x$ **k** $\cot^3 x$ **l** $\operatorname{cosec}^4 x$

3 Find the function $f'(x)$ where $f(x)$ is

a $x \cos x$ **b** $x^2 \sec 3x$ **c** $\dfrac{\tan 2x}{x}$ **d** $\sin^3 x \cos x$

e $\dfrac{x^2}{\tan x}$ **f** $\dfrac{1 + \sin x}{\cos x}$ **g** $e^{2x} \cos x$ **h** $e^x \sec 3x$

i $\dfrac{\sin 3x}{e^x}$ **j** $e^x \sin^2 x$ **k** $\dfrac{\ln x}{\tan x}$ **l** $\dfrac{e^{\sin x}}{\cos x}$

Mixed exercise 8K

1 Differentiate with respect to x:

a $\ln x^2$ **b** $x^2 \sin 3x$ E

2 Given that

$$f(x) \equiv 3 - \frac{x^2}{4} + \ln\frac{x}{2}, \quad x > 0$$

find $f'(x)$. E

3 Given that $2y = x - \sin x \cos x$, show that $\frac{dy}{dx} = \sin^2 x$. E

4 Differentiate, with respect to x,

a $\frac{\sin x}{x}, \quad x > 0$ **b** $\ln\frac{1}{x^2+9}$ E

5 Use the derivatives of $\sin x$ and $\cos x$ to prove that the derivative of $\tan x$ is $\sec^2 x$. E

6 $f(x) = \frac{x}{x^2+2}, \quad x \in \mathbb{R}$

Find the set of values of x for which $f'(x) < 0$. E

7 The function f is defined for positive real values of x by

$$f(x) = 12 \ln x - x^{\frac{3}{2}}$$

Write down the set of values of x for which $f(x)$ is an increasing function of x. E

8 Given that $y = \cos 2x + \sin x$, $0 < x < 2\pi$, and x is in radians, find, to 2 decimal places, the values of x for which $\frac{dy}{dx} = 0$. E

9 The maximum point on the curve with equation $y = x\sqrt{\sin x}$, $0 < x < \pi$, is the point A. Show that the x-coordinate of point A satisfies the equation $2 \tan x + x = 0$. E

10 $f(x) = e^{0.5x} - x^2, \quad x \in \mathbb{R}$

a Find $f'(x)$.

b By evaluating $f'(6)$ and $f'(7)$, show that the curve with equation $y = f(x)$ has a stationary point at $x = p$, where $6 < p < 7$. E

11 $f(x) \equiv e^{2x} \sin 2x, \quad 0 \leqslant x \leqslant \pi$

a Use calculus to find the coordinates of the turning points on the graph of $y = f(x)$.

b Show that $f''(x) = 8e^{2x} \cos 2x$.

c Hence, or otherwise, determine which turning point is a maximum and which is a minimum. E

12 The curve **C** has equation $y = 2e^x + 3x^2 + 2$. The point A with coordinates (0, 4) lies on **C**. Find the equation of the tangent to **C** at A. E

13 The curve **C** has equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = 3\ln x + \frac{1}{x},\ x > 0$$

The point P is a stationary point on **C**.

a Calculate the x-coordinate of P.

The point Q on **C** has x-coordinate 1.

b Find an equation for the normal to **C** at Q. *E*

14 Differentiate $e^{2x}\cos x$ with respect to x.
The curve **C** has equation $y = e^{2x}\cos x$.

a Show that the turning points on **C** occur when $\tan x = 2$.

b Find an equation of the tangent to **C** at the point where $x = 0$. *E*

15 Given that $x = y^2 \ln y,\ y > 0$,

a find $\dfrac{dx}{dy}$

b use your answer to part **a** to find in terms of e, the value of $\dfrac{dy}{dx}$ at $y = e$. *E*

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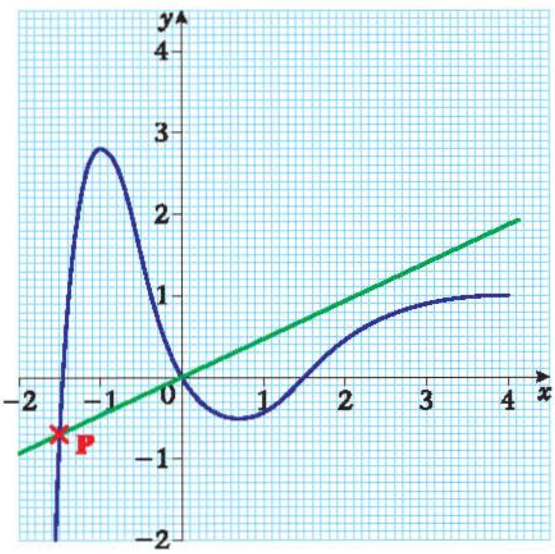


The figure shows part of the curve **C** with equation $y = \mathrm{f}(x)$, where $\mathrm{f}(x) = (x^3 - 2x)e^{-x}$

a Find $\mathrm{f}'(x)$.

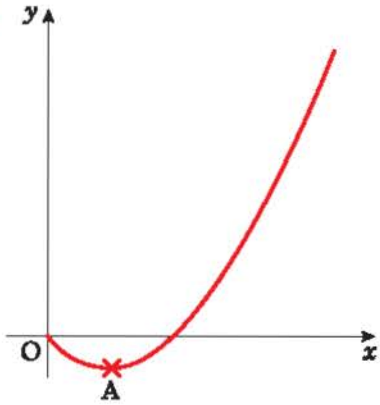
The normal to **C** at the origin O intersects **C** at a point **P**, as shown in the figure.

b Show that the x-coordinate of **P** is the solution of the equation

$$2x^2 = e^x + 4.$$

E

17



The diagram shows part of the curve with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = x(1 + x)\ln x \quad \{x > 0\}$$

The point A is the minimum point of the curve.

a Find $\mathrm{f}'(x)$.

b Hence show that the x-coordinate of A is the solution of the equation $x = \mathrm{g}(x)$, where

$$\mathrm{g}(x) = \mathrm{e}^{-\frac{1+x}{1+2x}}$$

E

Summary of key points

You should learn all of these results.

1 You can use the chain rule to differentiate a function of a function:

- if $y = [\mathrm{f}(x)]^n$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = n[\mathrm{f}(x)]^{n-1}\mathrm{f}'(x)$
- if $y = \mathrm{f}[\mathrm{g}(x)]$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}'[\mathrm{g}(x)]\mathrm{g}'(x)$

2 Another form of the **chain rule** states that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}$ where y is a function of u, and u is a function of x.

3 A particular case of the chain rule is the result $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)}$

4 You can use the product rule when two functions $u(x)$ and $v(x)$ are multiplied together.

- If $y = uv$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$

5 You can use the quotient rule when one function $u(x)$ is divided by another function $v(x)$, to form a rational function.

- If $y = \frac{u}{v}$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$

6 If $y = \mathrm{e}^x$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^x$ also and if $y = \mathrm{e}^{\mathrm{f}(x)}$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}'(x)\mathrm{e}^{\mathrm{f}(x)}$

7 If $y = \ln x$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x}$ and if $y = \ln[\mathrm{f}(x)]$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{f}'(x)}{\mathrm{f}(x)}$

8 If $y = \sin x$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = \cos x$.

9 If $y = \cos x$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = -\sin x$.

10 If $y = \tan x$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = \sec^2 x$.

11 If $y = \operatorname{cosec} x$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = -\operatorname{cosec} x \cot x$.

12 If $y = \sec x$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = \sec x \tan x$.

13 If $y = \cot x$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = -\operatorname{cosec}^2 x$.

The chain rule can be used with each of these functions to obtain further results. (See the examples in the section.)