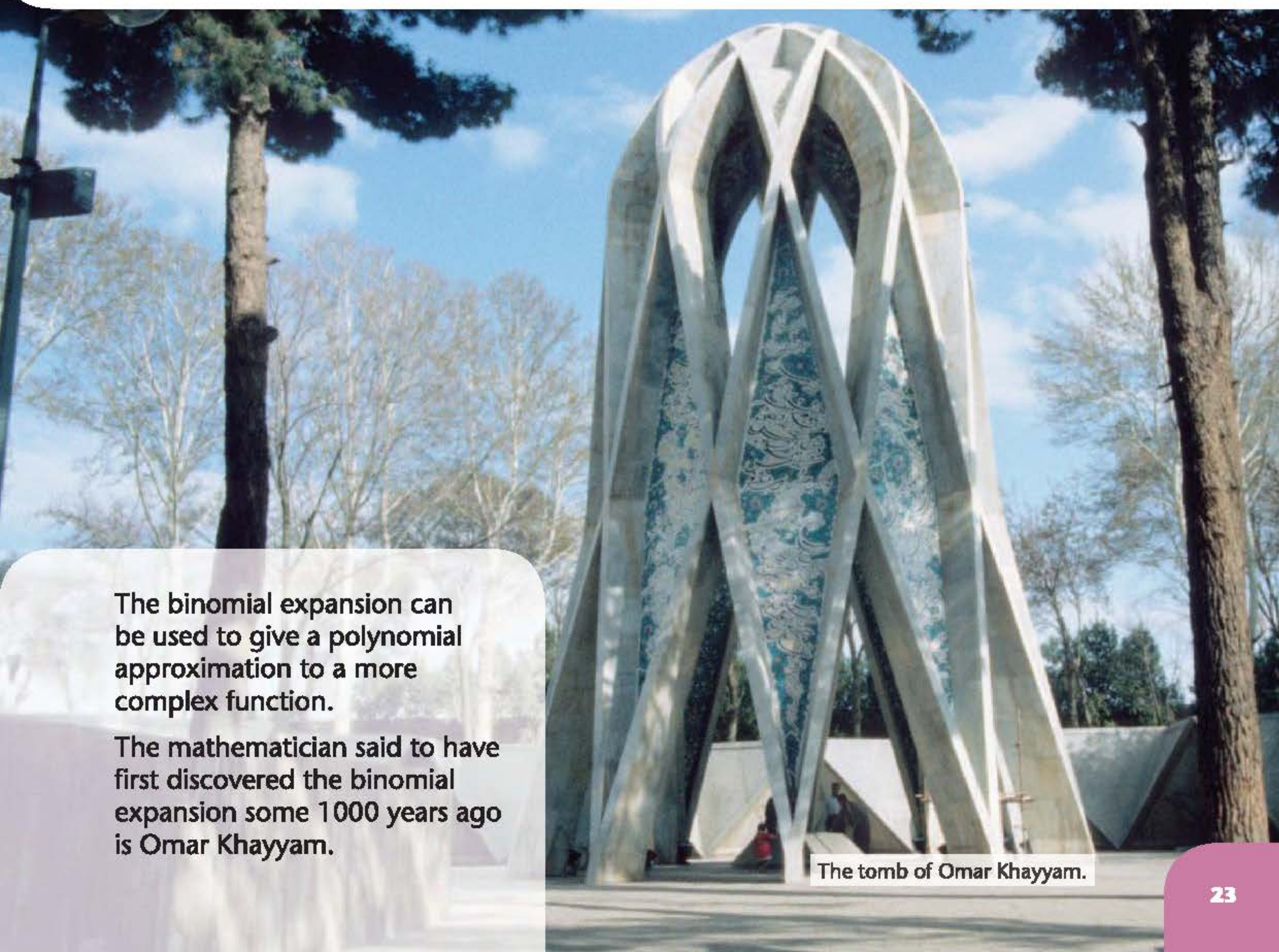
After completing this chapter you should be able to:

- expand $(1 + x)^n$ for any constant $n$
- expand $(a + bx)^n$ for any constants $a$, $b$ and $n$
- determine the range of values of $x$ for which the expansion is valid
- use partial fractions to expand more complex fractional expressions.

# 3 The binomial expansion



The binomial expansion can be used to give a polynomial approximation to a more complex function.

The mathematician said to have first discovered the binomial expansion some 1000 years ago is Omar Khayyam.

The tomb of Omar Khayyam.

**3.1** The binomial expansion is

$$(1+x)^n = 1 + nx + n(n-1)\frac{x^2}{2!} + n(n-1)(n-2)\frac{x^3}{3!} + \ldots + {}^nC_r x^r$$

When $n$ is a positive integer, this expansion is finite and exact. This is not generally the case when $n$ is negative or a fraction.

**Example 1**

Use the binomial expansion to find **a** $(1+x)^4$ **b** $(1-2x)^3$

**a** $(1+x)^4$

$$= 1 + 4x + \frac{4 \times 3x^2}{2!} + \frac{4 \times 3 \times 2x^3}{3!} + \frac{4 \times 3 \times 2 \times 1x^4}{4!} + \frac{4 \times 3 \times 2 \times 1 \times 0x^5}{5!}$$

Replace $n$ by 4 in the formula.

$$= 1 + 4x + \frac{4 \times 3x^2}{2} + \frac{4 \times 3 \times 2x^3}{6} + \frac{4 \times 3 \times 2 \times 1x^4}{24} + \frac{4 \times 3 \times 2 \times 1 \times 0x^5}{120}$$

Simplify coefficients.

$$= 1 + 4x + 6x^2 + 4x^3 + 1x^4 + 0x^5$$

All terms after this will also have zero as a coefficient.

$$= 1 + 4x + 6x^2 + 4x^3 + x^4$$

**b** $(1-2x)^3$

$$= 1 + 3 \times (-2x) + \frac{3 \times 2 \times (-2x)^2}{2!} + \frac{3 \times 2 \times 1 \times (-2x)^3}{3!} + \frac{3 \times 2 \times 1 \times 0 \times (-2x)^4}{4!}$$

Replace $n$ by 3 and $x$ by $-2x$.

$$= 1 - 6x + \frac{3 \times 2 \times 4x^2}{2} + \frac{3 \times 2 \times 1 \times -8x^3}{6} + \frac{3 \times 2 \times 1 \times 0 \times 16x^4}{24}$$

Simplify coefficients.

$$= 1 - 6x + 12x^2 - 8x^3 + 0x^4$$

$$= 1 - 6x + 12x^2 - 8x^3$$

All terms after this will also have zero as a coefficient.

## Example 2

Use the binomial expansion to find the first four terms of **a** $\dfrac{1}{(1+x)}$ **b** $\sqrt{(1-3x)}$

**a** $\dfrac{1}{(1+x)} = (1+x)^{-1}$

Write in index form.

$$= 1 + (-1)(x) + \frac{(-1)(-2)(x)^2}{2!} + \frac{(-1)(-2)(-3)(x)^3}{3!} + \ldots$$

Replace $n$ by $-1$ in the expansion.

$$= 1 - 1x + 1x^2 - 1x^3 + \ldots$$

$$= 1 - x + x^2 - x^3 + \ldots$$

As $n$ is not a positive integer, no coefficient will ever be equal to zero.
The expansion is **infinite**, and convergent when $|x| < 1$.

**b** $\sqrt{(1-3x)} = (1-3x)^{\frac{1}{2}}$

Write in index form.

$$= 1 + (\tfrac{1}{2})(-3x) + \frac{(\frac{1}{2})(\frac{1}{2}-1)(-3x)^2}{2!} + \frac{(\frac{1}{2})(\frac{1}{2}-1)(\frac{1}{2}-2)(-3x)^3}{3!} + \ldots$$

Replace $n$ by $\frac{1}{2}$ and $x$ by $-3x$.

Be careful to write this as $(-3x)^2$, not $-3x^2$.

$$= 1 - \frac{3x}{2} + \frac{(\frac{1}{2})(-\frac{1}{2})9x^2}{2} + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})(-27x^3)}{6} + \ldots$$

Simplify terms.

$$= 1 - \frac{3x}{2} - \frac{9x^2}{8} - \frac{27x^3}{16} + \ldots$$

Because $n$ is not a positive integer, no coefficient will ever be equal to zero.
The expansion is **infinite**, and convergent when $|x| < \frac{1}{3}$ because $|3x| < 1$

## Example 3

Find the binomial expansions of **a** $(1-x)^{\frac{1}{3}}$, **b** $\dfrac{1}{(1+4x)^2}$, up to and including the term in $x^3$.

State the range of values of $x$ for which the expansions are valid.

**a** $(1-x)^{\frac{1}{3}}$

$$= 1 + (\tfrac{1}{3})(-x) + \frac{(\frac{1}{3})(\frac{1}{3}-1)(-x)^2}{2!} + \frac{(\frac{1}{3})(\frac{1}{3}-1)(\frac{1}{3}-2)(-x)^3}{3!} + \ldots$$

Replace $n$ by $\frac{1}{3}$, $x$ by $(-x)$.

Simplify brackets.

$$= 1 + (\tfrac{1}{3})(-x) + \frac{(\frac{1}{3})(-\frac{2}{3})(-x)^2}{2} + \frac{(\frac{1}{3})(-\frac{2}{3})(-\frac{5}{3})(-x)^3}{6} + \ldots$$

Simplify coefficients.

$$= 1 - \tfrac{1}{3}x - \tfrac{1}{9}x^2 - \tfrac{5}{81}x^3 + \ldots$$

Expansion is valid as long as $|-x| < 1$

Terms in expansion are $(-x)$, $(-x)^2$, $(-x)^3$.

$$\Rightarrow |x| < 1$$

**b** $\dfrac{1}{(1+4x)^2}$

$= (1+4x)^{-2}$

Write in index form.

$= 1 + (-2)(4x) + \dfrac{(-2)(-2-1)(4x)^2}{2!} + \dfrac{(-2)(-2-1)(-2-2)(4x)^3}{3!} + \ldots$

Replace $n$ by $-2$, $x$ by '$4x$'.

Simplify brackets.

$= 1 + (-2)(4x) + \dfrac{(-2)(-3)16x^2}{2} + \dfrac{(-2)(-3)(-4)64x^3}{6} + \ldots$

Simplify coefficients.

$= 1 - 8x + 48x^2 - 256x^3 + \ldots$

Terms in expansion are $(4x)$, $(4x)^2$, $(4x)^3$.

Expansion is valid as long as $|4x| < 1$

$\Rightarrow \ |x| < \frac{1}{4}$.

**Example 4**

Find the expansion of $\sqrt{(1-2x)}$ up to and including the term in $x^3$. By substituting in $x = 0.01$, find a suitable decimal approximation to $\sqrt{2}$.

$\sqrt{(1-2x)} = (1-2x)^{\frac{1}{2}}$

Write in index form.

$= 1 + (\tfrac{1}{2})(-2x) + \dfrac{(\frac{1}{2})(\frac{1}{2}-1)(-2x)^2}{2!} + \dfrac{(\frac{1}{2})(\frac{1}{2}-1)(\frac{1}{2}-2)(-2x)^3}{3!} + \ldots$

Replace $n$ by $\frac{1}{2}$, $x$ by $(-2x)$.

Simplify brackets.

$= 1 + (\tfrac{1}{2})(-2x) + \dfrac{(\frac{1}{2})(-\frac{1}{2})(4x^2)}{2} + \dfrac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})(-8x^3)}{6} + \ldots$

Simplify coefficients.

$= 1 - x - \dfrac{x^2}{2} - \dfrac{x^3}{2} + \ldots$

Terms in expansion are $(-2x)$, $(-2x)^2$, $(-2x)^3$.

Expansion is valid if $|2x| < 1$

$\Rightarrow |x| < \frac{1}{2}$.

$$\sqrt{(1-2\times 0.01)} \approx 1 - 0.01 - \frac{(0.01)^2}{2} - \frac{(0.01)^3}{2}$$

Substitute $x = 0.01$ into both sides of expansion. This is valid as $|x| < \frac{1}{2}$.

$$\sqrt{0.98} \approx 1 - 0.01 - 0.00005 - 0.0000005$$

Simplify both sides.
Note that the terms are getting smaller.

$$\sqrt{\frac{98}{100}} \approx 0.9899495$$

Write 0.98 as $\frac{98}{100}$.

$$\sqrt{\frac{49\times 2}{100}} \approx 0.9899495$$

$$\frac{7\sqrt{2}}{10} \approx 0.9899495$$

Use rules of surds.

$\times 10$, $\div 7$

$$\sqrt{2} \approx \frac{0.9899495 \times 10}{7}$$

Simplify.

$$\sqrt{2} \approx 1.414213571$$

## Exercise 3A

**1** Find the binomial expansion of the following up to and including the terms in $x^3$. State the range values of $x$ for which these expansions are valid.

**a** $(1+2x)^3$ **b** $\dfrac{1}{1-x}$ **c** $\sqrt{(1+x)}$ **d** $\dfrac{1}{(1+2x)^3}$

**e** $\sqrt[3]{(1-3x)}$ **f** $(1-10x)^{\frac{3}{2}}$ **g** $\left(1+\dfrac{x}{4}\right)^{-4}$ **h** $\dfrac{1}{(1+2x^2)}$

**2** By first writing $\dfrac{(1+x)}{(1-2x)}$ as $(1+x)(1-2x)^{-1}$ show that the cubic approximation to $\dfrac{(1+x)}{(1-2x)}$ is $1+3x+6x^2+12x^3$. State the range of values of $x$ for which this expansion is valid.

**3** Find the binomial expansion of $\sqrt{(1+3x)}$ in ascending powers of $x$ up to and including the term in $x^3$. By substituting $x = 0.01$ in the expansion, find an approximation to $\sqrt{103}$. By comparing it with the exact value, comment on the accuracy of your approximation.

**4** In the expansion of $(1+ax)^{-\frac{1}{2}}$ the coefficient of $x^2$ is 24. Find possible values of the constant $a$ and the corresponding term in $x^3$.

**5** Show that if $x$ is small, the expression $\sqrt{\left(\dfrac{1+x}{1-x}\right)}$ is approximated by $1+x+\frac{1}{2}x^2$.

**6** Find the first four terms in the expansion of $(1-3x)^{\frac{3}{2}}$. By substituting in a suitable value of $x$, find an approximation to $97^{\frac{3}{2}}$.

**3.2** **You can use the binomial expansion of $(1 + x)^n$ to expand $(a + bx)^n$ for any constants $a$ and $b$ by simply taking out $a$ as a factor.**

### Example 5

Find the first four terms in the binomial expansion of **a** $\sqrt{(4 + x)}$ **b** $\dfrac{1}{(2 + 3x)^2}$.
State the range in values of $x$ for which these expansions are valid.

**a** $\sqrt{(4 + x)} = (4 + x)^{\frac{1}{2}}$

Write in index form.

$= \left[4\left(1 + \dfrac{x}{4}\right)\right]^{\frac{1}{2}}$

Take out a factor of 4.

$= 4^{\frac{1}{2}}\left(1 + \dfrac{x}{4}\right)^{\frac{1}{2}}$

$= 2\left(1 + \dfrac{x}{4}\right)^{\frac{1}{2}}$

Write $4^{\frac{1}{2}}$ as 2.

$$= 2\left[1 + \left(\frac{1}{2}\right)\left(\frac{x}{4}\right) + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2} - 1\right)\left(\frac{x}{4}\right)^2}{2!} + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2} - 1\right)\left(\frac{1}{2} - 2\right)\left(\frac{x}{4}\right)^3}{3!} + \ldots\right]$$

Expand $\left(1 + \dfrac{x}{4}\right)^{\frac{1}{2}}$ using the binomial expansion with $n = \frac{1}{2}$ and $x = \dfrac{x}{4}$.

$$= 2\left[1 + \left(\frac{1}{2}\right)\left(\frac{x}{4}\right) \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{x^2}{16}\right)}{2} + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(\frac{x^3}{64}\right)}{6} + \ldots\right]$$

Simplify coefficients.

$$= 2\left[1 + \frac{x}{8} - \frac{x^2}{128} + \frac{x^3}{1024} + \ldots\right]$$

Multiply by the 2.

$$= 2 + \frac{x}{4} - \frac{x^2}{64} + \frac{x^3}{512} + \ldots$$

Expansion is valid if $\left|\dfrac{x}{4}\right| < 1$

$\Rightarrow \quad |x| < 4.$

Terms in expansion are $\left(\dfrac{x}{4}\right), \left(\dfrac{x}{4}\right)^2, \left(\dfrac{x}{4}\right)^3$.

**b** $\dfrac{1}{(2+3x)^2} = (2+3x)^{-2}$

Write in index form.

$= \left[2\left(1+\dfrac{3x}{2}\right)\right]^{-2}$

Take out a factor of 2.

$= 2^{-2}\left(1+\dfrac{3x}{2}\right)^{-2}$

Write $2^{-2} = \dfrac{1}{2^2} = \dfrac{1}{4}$

$= \dfrac{1}{4}\left(1+\dfrac{3x}{2}\right)^{-2}$

$$= \frac{1}{4}\left[1 + (-2)\left(\frac{3x}{2}\right) + \frac{(-2)(-2-1)\left(\frac{3x}{2}\right)^2}{2!} + \frac{(-2)(-2-1)(-2-2)\left(\frac{3x}{2}\right)^3}{3!} + \ldots\right]$$

Expand $\left(1+\dfrac{3x}{2}\right)^{-2}$ using the binomial expansion with $n = -2$ and $x = \dfrac{3x}{2}$.

$$= \frac{1}{4}\left[1 + (-2)\left(\frac{3x}{2}\right) + \frac{(-2)(-3)\left(\frac{9x^2}{4}\right)}{2} + \frac{(-2)(-3)(-4)\left(\frac{27x^3}{8}\right)}{6} + \ldots\right]$$

Simplify coefficients.

$$= \frac{1}{4}\left[1 - 3x + \frac{27x^2}{4} - \frac{27x^3}{2} + \ldots\right]$$

Multiply by the $\frac{1}{4}$.

$$= \frac{1}{4} - \frac{3}{4}x + \frac{27x^2}{16} - \frac{27x^3}{8} + \ldots$$

Expansion is valid if $\left|\dfrac{3x}{2}\right| < 1$

$\Rightarrow \quad |x| < \dfrac{2}{3}$.

Terms in expansion are $\left(\dfrac{3x}{2}\right), \left(\dfrac{3x}{2}\right)^2, \left(\dfrac{3x}{2}\right)^3$.

## Exercise 3B

**1** Find the binomial expansions of the following in ascending powers of $x$ as far as the term in $x^3$. State the range of values of $x$ for which the expansions are valid.

**a** $\sqrt{(4+2x)}$ **b** $\dfrac{1}{2+x}$ **c** $\dfrac{1}{(4-x)^2}$ **d** $\sqrt{(9+x)}$

**e** $\dfrac{1}{\sqrt{(2+x)}}$ **f** $\dfrac{5}{3+2x}$ **g** $\dfrac{1+x}{2+x}$ **h** $\sqrt{\left(\dfrac{2+x}{1-x}\right)}$

**2** Prove that if $x$ is sufficiently small, $\dfrac{3+2x-x^2}{4-x}$ may be approximated by $\frac{3}{4}+\frac{11}{16}x-\frac{5}{64}x^2$.
What does 'sufficiently small' mean in this question?

**3** Find the first four terms in the expansion of $\sqrt{(4-x)}$. By substituting $x=\frac{1}{9}$ find a fraction that is an approximation to $\sqrt{35}$. By comparing this to the exact value, state the degree of accuracy of your approximation.

**4** The expansion of $(a+bx)^{-2}$ may be approximated by $\frac{1}{4}+\frac{1}{4}x+cx^2$. Find the values of the constants $a$, $b$ and $c$.

## 3.3 You can use partial fractions to simplify the expansions of many more difficult expressions.

### Example 6

**a** Express $\dfrac{4-5x}{(1+x)(2-x)}$ as partial fractions.

**b** Hence show that the cubic approximation of $\dfrac{4-5x}{(1+x)(2-x)}$ is $2-\dfrac{7x}{2}+\dfrac{11}{4}x^2-\dfrac{25}{8}x^3$.

**c** State the range of values of $x$ for which the expansion is valid.

**a** $\dfrac{4-5x}{(1+x)(2-x)} \equiv \dfrac{A}{(1+x)}+\dfrac{B}{(2-x)}$

The denominators must be $(1+x)$ and $(2-x)$.

$\equiv \dfrac{A(2-x)+B(1+x)}{(1+x)(2-x)}$

Add the fractions.

$4-5x \equiv A(2-x)+B(1+x)$

Set the numerators equal.

Substitute $x=2$

$4-10 = A\times 0 + B\times 3$

$-6 = 3B$

$B=-2$

Set $x=2$ to find $B$.

Substitute $x=-1$

$4+5 = A\times 3 + B\times 0$

$9=3A$

$A=3$

Set $x=-1$ to find $A$.

so $\dfrac{4-5x}{(1+x)(2-x)} = \dfrac{3}{1+x}-\dfrac{2}{2-x}$

**b** $\dfrac{4-5x}{(1+x)(2-x)} = \dfrac{3}{(1+x)} - \dfrac{2}{(2-x)}$

$= 3(1+x)^{-1} - 2(2-x)^{-1}$

Write in index form.

The expansion of $3(1+x)^{-1}$

$= 3\left[1 + (-1)(x) + (-1)(-2)\dfrac{(x)^2}{2!}\right.$

$\left. + (-1)(-2)(-3)\dfrac{(x)^3}{3!} + \ldots\right]$

$= 3[1 - x + x^2 - x^3 + \ldots]$

$= 3 - 3x + 3x^2 - 3x^3 + \ldots$

Expand $3(1+x)^{-1}$ using the binomial expansion with $n = -1$.

Take out a factor of 2.

The expansion of $2(2-x)^{-1}$

$= 2\left[2\left(1 - \dfrac{x}{2}\right)\right]^{-1}$

$= 2 \times 2^{-1}\left(1 - \dfrac{x}{2}\right)^{-1}$

$= 1 \times \left[1 + (-1)\left(-\dfrac{x}{2}\right)\right.$

$+ \dfrac{(-1)(-2)\left(-\dfrac{x}{2}\right)^2}{2!}$

$\left. + \dfrac{(-1)(-2)(-3)\left(-\dfrac{x}{2}\right)^3}{3!} + \ldots\right]$

$= 1 \times \left[1 + \dfrac{x}{2} + \dfrac{x^2}{4} + \dfrac{x^3}{8} + \ldots\right]$

$= 1 + \dfrac{x}{2} + \dfrac{x^2}{4} + \dfrac{x^3}{8}$

Expand $\left(1 - \dfrac{x}{2}\right)^{-1}$ using the binomial expansion with $n = -1$ and $x = \dfrac{x}{2}$.

Hence $\dfrac{4-5x}{(1+x)(2-x)}$

$= 3(1+x)^{-1} - 2(2-x)^{-1}$

'Add' both expressions.

$= (3 - 3x + 3x^2 - 3x^3)$

$-\left(1 + \dfrac{x}{2} + \dfrac{x^2}{4} + \dfrac{x^3}{8}\right)$

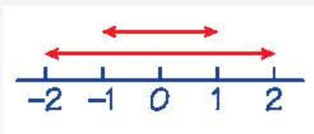
$= 2 - \dfrac{7}{2}x + \dfrac{11}{4}x^2 - \dfrac{25}{8}x^3$

**c** $\dfrac{3}{(1+x)}$ is valid if $|x| < 1$

Terms are $x$, $x^2$, $x^3$.

$\dfrac{2}{(2-x)}$ is valid if $\left|\dfrac{x}{2}\right| < 1 \Rightarrow |x| < 2$

Terms are $\dfrac{x}{2}$, $\left(\dfrac{x}{2}\right)^2$, $\left(\dfrac{x}{2}\right)^3$.




Look for values of $x$ that satisfy both expressions.

Both expressions are valid provided $|x| < 1$.

## Exercise 3C

**1** **a** Express $\dfrac{8x+4}{(1-x)(2+x)}$ as partial fractions.

**b** Hence or otherwise expand $\dfrac{8x+4}{(1-x)(2+x)}$ in ascending powers of $x$ as far as the term in $x^2$.

**c** State the set of values of $x$ for which the expansion is valid.

**2** **a** Express $\dfrac{-2x}{(2+x)^2}$ as a partial fraction.

**b** Hence prove that $\dfrac{-2x}{(2+x)^2}$ can be expressed in the form $0 - \dfrac{1}{2}x + Bx^2 + Cx^3$ where constants $B$ and $C$ are to be determined.

**c** State the set of values of $x$ for which the expansion is valid.

**3** **a** Express $\dfrac{6+7x+5x^2}{(1+x)(1-x)(2+x)}$ as a partial fraction.

**b** Hence or otherwise expand $\dfrac{6+7x+5x^2}{(1+x)(1-x)(2+x)}$ in ascending powers of $x$ as far as the term in $x^3$.

**c** State the set of values of $x$ for which the expansion is valid.

## Mixed exercise 3D

**1** Find binomial expansions of the following in ascending powers of $x$ as far as the term in $x^3$. State the set of values of $x$ for which the expansion is valid.

**a** $(1-4x)^3$ **b** $\sqrt{(16+x)}$ **c** $\dfrac{1}{(1-2x)}$ **d** $\dfrac{4}{2+3x}$

**e** $\dfrac{4}{\sqrt{(4-x)}}$ **f** $\dfrac{1+x}{1+3x}$ **g** $\left(\dfrac{1+x}{1-x}\right)^2$ **h** $\dfrac{x-3}{(1-x)(1-2x)}$

**2** Find the first four terms of the expansion in ascending powers of $x$ of:
$(1-\frac{1}{2}x)^{\frac{1}{2}}$, $|x|<2$
and simplify each coefficient.

*E* (adapted)

**3** Show that if $x$ is sufficiently small then $\dfrac{3}{\sqrt{(4+x)}}$ can be approximated by
$\dfrac{3}{2}-\dfrac{3}{16}x+\dfrac{9}{256}x^2$.

**4** Given that $|x|<4$, find, in ascending powers of $x$ up to and including the term in $x^3$, the series expansion of:

**a** $(4-x)^{\frac{1}{2}}$ **b** $(4-x)^{\frac{1}{2}}(1+2x)$

*E* (adapted)

**5** **a** Find the first four terms of the expansion, in ascending powers of $x$, of
$(2+3x)^{-1}$, $|x|<\frac{2}{3}$

**b** Hence or otherwise, find the first four non-zero terms of the expansion, in ascending powers of $x$, of:
$\dfrac{1+x}{2+3x}$, $|x|<\frac{2}{3}$

*E*

**6** Find, in ascending powers of $x$ up to and including the term in $x^3$, the series expansion of $(4+x)^{-\frac{1}{2}}$, giving your coefficients in their simplest form.

*E*

**7** $f(x)=(1+3x)^{-1}$, $|x|<\frac{1}{3}$.

**a** Expand $f(x)$ in ascending powers of $x$ up to and including the term in $x^3$.

**b** Hence show that, for small $x$:
$$\frac{1+x}{1+3x}\approx 1-2x+6x^2-18x^3.$$

**c** Taking a suitable value for $x$, which should be stated, use the series expansion in part **b** to find an approximate value for $\frac{101}{103}$, giving your answer to 5 decimal places.

*E*

**8** Obtain the first four non-zero terms in the expansion, in ascending powers of $x$, of the function $f(x)$ where $f(x)=\dfrac{1}{\sqrt{(1+3x^2)}}$, $3x^2<1$.

*E*

**9** Give the binomial expansion of $(1+x)^{\frac{1}{2}}$ up to and including the term in $x^3$. By substituting $x=\frac{1}{4}$, find the fraction that is an approximation to $\sqrt{5}$.

**10** When $(1 + ax)^n$ is expanded as a series in ascending powers of $x$, the coefficients of $x$ and $x^2$ are $-6$ and $27$ respectively.

**a** Find the values of $a$ and $n$.

**b** Find the coefficient of $x^3$.

**c** State the values of $x$ for which the expansion is valid.

**E** (adapted)

**11** **a** Express $\dfrac{9x^2 + 26x + 20}{(1 + x)(2 + x)^2}$ as a partial fraction.

**b** Hence or otherwise show that the expansion of $\dfrac{9x^2 + 26x + 20}{(1 + x)(2 + x)^2}$ in ascending powers of $x$ can be approximated to $5 - \dfrac{7x}{2} + Bx^2 + Cx^3$ where $B$ and $C$ are constants to be found.

**c** State the set of values of $x$ for which this expansion is valid.

## Summary of key points

**1** The binomial expansion $(1 + x)^n = 1 + nx + \dfrac{n(n-1)x^2}{2!} + \dfrac{n(n-1)(n-2)x^3}{3!} + \ldots$ can be used to give an exact expression if $n$ is a positive integer, or an approximate expression for any other rational number.

- $(1 + 2x)^3 = 1 + 3(2x) + 3 \times 2\dfrac{(2x)^2}{2!} + 3 \times 2 \times 1 \times \dfrac{(2x)^3}{3!} + 3 \times 2 \times 1 \times 0 \times \dfrac{(2x)^4}{4!}$

  $= 1 + 6x + 12x^2 + 8x^3$ (Expansion is *finite* and *exact*.)

- $\sqrt{(1-x)} = (1-x)^{\frac{1}{2}} = 1 + \left(\dfrac{1}{2}\right)(-x) + \left(\dfrac{1}{2}\right)\left(-\dfrac{1}{2}\right)\dfrac{(-x)^2}{2!} + \left(\dfrac{1}{2}\right)\left(-\dfrac{1}{2}\right)\left(-\dfrac{3}{2}\right)\dfrac{(-x)^3}{3!} + \ldots$

  $= 1 - \dfrac{1}{2}x - \dfrac{1}{8}x^2 - \dfrac{1}{16}x^3 + \ldots$

  (Expansion is *infinite* and *approximate*.)

**2** The expansion $(1 + x)^n = 1 + nx + n(n-1)\dfrac{x^2}{2!} + n(n-1)(n-2)\dfrac{x^3}{3!} + \ldots$, where $n$ is negative or a fraction, is only valid if $|x| < 1$.

**3** You can adapt the binomial expansion to include expressions of the form $(a + bx)^n$ by simply taking out a common factor of $a$:

e.g. $\dfrac{1}{(3 + 4x)} = (3 + 4x)^{-1} = \left[3\left(1 + \dfrac{4x}{3}\right)\right]^{-1}$

$= 3^{-1}\left(1 + \dfrac{4x}{3}\right)^{-1}$

**4** You can use knowledge of partial fractions to expand more difficult expressions, e.g.

$$\frac{7 + x}{(3 - x)(2 + x)} = \frac{2}{(3 - x)} + \frac{1}{(2 + x)}$$

$$= 2(3 - x)^{-1} + (2 + x)^{-1}$$

$$= \frac{2}{3}\left(1 - \frac{x}{3}\right)^{-1} + \frac{1}{2}\left(1 + \frac{x}{2}\right)^{-1}$$