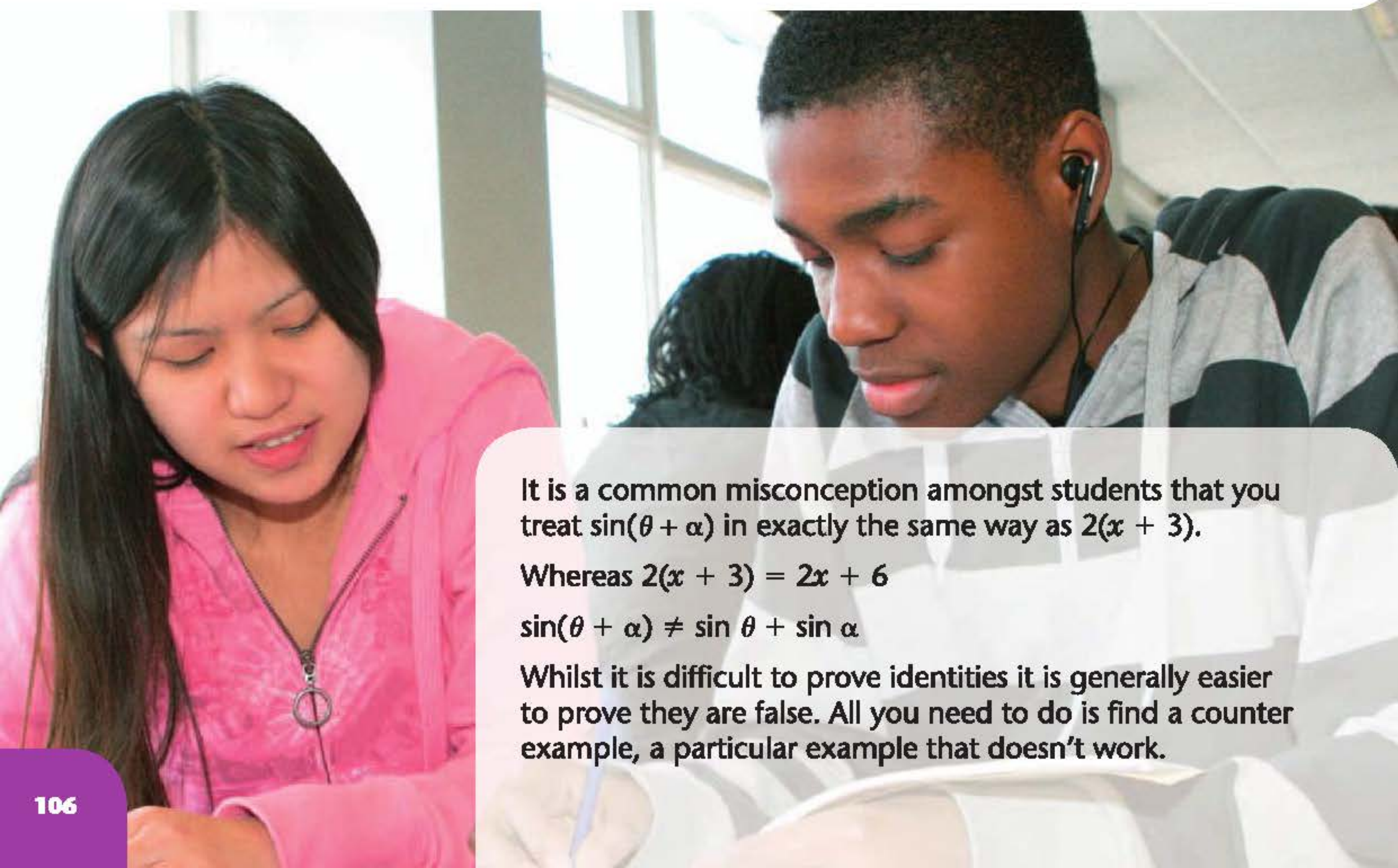
# 7

After completing this chapter you should be able to

1. use the addition formulae
2. use the double angle formulae
3. write expressions of the form $a\cos\theta \pm b\sin\theta$ in the form $R\cos(\theta \pm \alpha)$ and/or $R\sin(\theta \pm \alpha)$
4. use the factor formulae
5. use all of the above to solve equations and prove identities.

# Further trigonometric identities and their applications



It is a common misconception amongst students that you treat $\sin(\theta + \alpha)$ in exactly the same way as $2(x + 3)$.

Whereas $2(x + 3) = 2x + 6$

$\sin(\theta + \alpha) \neq \sin\theta + \sin\alpha$

Whilst it is difficult to prove identities it is generally easier to prove they are false. All you need to do is find a counter example, a particular example that doesn't work.

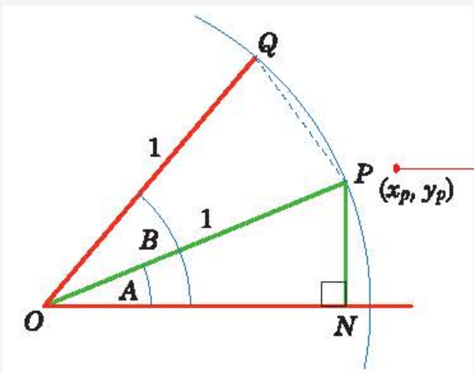
## 7.1 You need to know and be able to use the addition formulae.

- $\sin(A+B) \equiv \sin A \cos B + \cos A \sin B$ $\quad$ $\sin(A-B) \equiv \sin A \cos B - \cos A \sin B$
- $\cos(A+B) \equiv \cos A \cos B - \sin A \sin B$ $\quad$ $\cos(A-B) \equiv \cos A \cos B + \sin A \sin B$
- $\tan(A+B) \equiv \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$ $\quad$ $\tan(A-B) \equiv \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

Although you will not be expected to derive these formulae from first principles, it will help your understanding of them to see how one of them can be derived.

### Example 1

Show that $\cos(A-B) \equiv \cos A \cos B + \sin A \sin B$




Draw a circle, centre the origin and with unit radius. Place points $P$ and $Q$ on the circumference such that $OP$ and $OQ$ make angles $A$ and $B$ with the $x$-axis, as shown.

The coordinates of $P$ are $(\cos A, \sin A)$ and those of $Q$ are $(\cos B, \sin B)$.

In $\triangle PON$ $\cos A = \dfrac{x_P}{1}$, $\sin A = \dfrac{y_P}{1}$, as the radius has length 1 (unit radius).

So

$$PQ^2 = (\cos B - \cos A)^2 + (\sin B - \sin A)^2$$

Use the formula for the distance between two points: $PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$.

$$= (\cos^2 B - 2\cos A \cos B + \cos^2 A) + (\sin^2 B - 2\sin A \sin B + \sin^2 A)$$
$$= (\sin^2 A + \cos^2 A) + (\sin^2 B + \cos^2 B) - 2(\cos A \cos B + \sin A \sin B)$$
$$= 2 - 2(\cos A \cos B + \sin A \sin B)$$

Use $\sin^2 A + \cos^2 A \equiv 1$ and $\sin^2 B + \cos^2 B \equiv 1$.

But $\quad PQ^2 = 1^2 + 1^2 - 2\cos(A-B)$
$$= 2 - 2\cos(A-B)$$

Using the cosine rule in $\triangle POQ$, with $OP = OQ = 1$ and $\angle POQ = (A-B)$.

Comparing the two results for $PQ^2$

$\cos(A-B) \equiv \cos A \cos B + \sin A \sin B$

The other formulae involving sine and cosine can be constructed using the one in the example above.

## Example 2

Use the result in Example 1 to show that:

**a** $\cos(A + B) \equiv \cos A \cos B - \sin A \sin B$

**b** $\sin(A + B) \equiv \sin A \cos B + \cos A \sin B$

**c** $\sin(A - B) \equiv \sin A \cos B - \cos A \sin B$

**a** Replace $B$ by $(-B)$ in

$$\cos(A - B) \equiv \cos A \cos B + \sin A \sin B$$

So $$\cos(A + B) \equiv \cos A \cos(-B) + \sin A \sin(-B)$$

Use the results $\cos(-B) = \cos B$ and $\sin(-B) = -\sin B$
(See Chapter 8 in Book C2.)

$$\therefore \quad \cos(A + B) \equiv \cos A \cos B - \sin A \sin B$$

**b** Replace $A$ by $\left(\frac{\pi}{2} - A\right)$ in

$$\cos(A - B) \equiv \cos A \cos B + \sin A \sin B$$

So $$\cos\left[\left(\frac{\pi}{2} - A\right) - B\right] \equiv \cos\left(\frac{\pi}{2} - A\right)\cos B + \sin\left(\frac{\pi}{2} - A\right)\sin B$$

$$\cos\left[\frac{\pi}{2} - (A + B)\right] \equiv \cos\left(\frac{\pi}{2} - A\right)\cos B + \sin\left(\frac{\pi}{2} - A\right)\sin B$$

Use $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$

and $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$

(See Chapter 8 in Book C2.)

$$\therefore \quad \sin(A + B) \equiv \sin A \cos B + \cos A \sin B$$

**c** Replace $B$ by $(-B)$ in the result in **b**:

$\sin(A - B) \equiv \sin A \cos B + \cos A \sin(-B)$

so $\sin(A - B) \equiv \sin A \cos B - \cos A \sin B$

To find similar expressions for $\tan(A + B)$ and $\tan(A - B)$ you can use the fact that $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ and divide the appropriate results given above.

## Example 3

Show that

**a** $\tan(A + B) \equiv \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

**b** $\tan(A - B) \equiv \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

**a** $\tan(A+B) \equiv \dfrac{\sin(A+B)}{\cos(A+B)}$

$\equiv \dfrac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B}$

Dividing the 'top and bottom' by $\cos A\cos B$ gives

$\tan(A+B)$

$$\equiv \frac{\dfrac{\sin A\cancel{\cos B}}{\cos A\cancel{\cos B}} + \dfrac{\cancel{\cos A}\sin B}{\cancel{\cos A}\cos B}}{\dfrac{\cancel{\cos A}\cancel{\cos B}}{\cancel{\cos A}\cancel{\cos B}} - \dfrac{\sin A\sin B}{\cos A\cos B}}$$

Cancel terms, as shown, and use the result $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

$$\equiv \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

**b** Replace $B$ by $-B$ in the result above:

$$\tan(A-B) \equiv \frac{\tan A + \tan(-B)}{1 - \tan A\tan(-B)}$$

Use the result $\tan(-\theta) = -\tan\theta$. See Chapter 8 in Book C2.

$$\equiv \frac{\tan A - \tan B}{1 + \tan A\tan B}$$

## Example 4

Show, using the formula for $\sin(A-B)$, that $\sin 15° = \dfrac{\sqrt{6}-\sqrt{2}}{4}$

$\sin 15° = \sin(45-30)°$

You know the exact form of sin and cos for many angles, e.g. 30°, 45°, 60°, 90°, 180°..., so write 15° using two of these angles. [You could equally use $\sin(60-45)°$.]

$= \sin 45°\cos 30° - \cos 45°\sin 30°$

$= (\tfrac{1}{2}\sqrt{2})(\tfrac{1}{2}\sqrt{3}) - (\tfrac{1}{2}\sqrt{2})(\tfrac{1}{2})$

$= \tfrac{1}{4}(\sqrt{3}\sqrt{2} - \sqrt{2})$

$= \tfrac{1}{4}(\sqrt{6} - \sqrt{2})$

## Example 5

Given that $\sin A = -\frac{3}{5}$ and $180° < A < 270°$, and that $\cos B = -\frac{12}{13}$ and $B$ is obtuse, find the value of

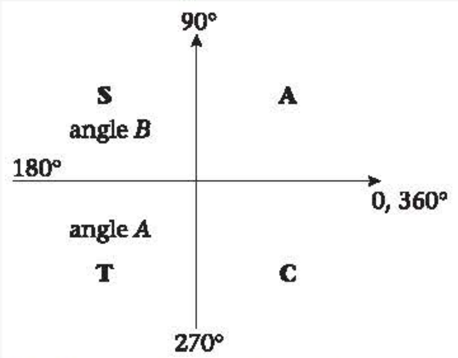
**a** $\cos(A-B)$

**b** $\tan(A+B)$

**a** $\cos(A - B) \equiv \cos A \cos B + \sin A \sin B$

$\cos^2 A \equiv 1 - \sin^2 A = 1 - \left(-\frac{3}{5}\right)^2 = \frac{16}{25}$

So $\cos A = \pm\frac{4}{5}$




but $A$ is in the third quadrant, where $\cos$ is $-$ve,

$\therefore \quad \cos A = -\frac{4}{5}$

$\sin^2 B \equiv 1 - \cos^2 B = 1 - \left(-\frac{12}{13}\right)^2 = \frac{25}{169}$

So $\sin B = \pm\frac{5}{13}$

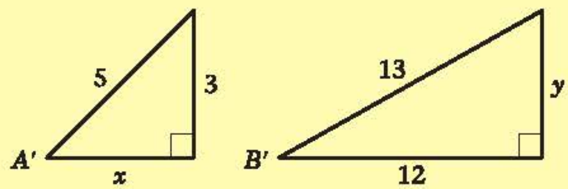
but $B$ is in the 2nd quadrant

$\therefore \quad \sin B = +\frac{5}{13}$

$$\cos(A - B) = \left(-\frac{4}{5}\right)\left(-\frac{12}{13}\right) + \left(-\frac{3}{5}\right)\left(+\frac{5}{13}\right) = \frac{33}{65}$$

You need to find $\cos A$ and $\sin B$.
Take note of the quadrants that $A$ and $B$ are in.

You can also work with the associated acute angles $A'$ and $B'$. Draw right-angled triangles and use Pythagoras' theorem.




$x^2 = 5^2 - 3^2$ so $x = 4$

$y^2 = 13^2 - 12^2$ so $y = 5$

As $A$ is in 3rd quadrant.
$\sin A = -\sin A' = -\frac{3}{5}$
$\cos A = -\cos A' = -\frac{4}{5}$
$\tan A = +\tan A' = +\frac{3}{4}$

As $B$ is in 2nd quadrant.
$\sin B = +\sin B' = +\frac{5}{13}$
$\cos B = -\cos B' = -\frac{12}{13}$
$\tan B = -\tan B' = -\frac{5}{12}$

**b** $\tan(A + B) \equiv \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

$$\text{So } \tan(A + B) = \frac{\frac{3}{4} + \left(-\frac{5}{12}\right)}{1 - \left(\frac{3}{4}\right)\left(-\frac{5}{12}\right)} = \frac{\frac{1}{3}}{\frac{63}{48}} = \frac{1}{3} \times \frac{48}{63} = \frac{16}{63}$$

You can use the above results for $\tan A$ and $\tan B$, or use $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ with the results for $\sin A$, $\cos A$, $\sin B$ and $\cos B$.

## Example 6

Given that $2\sin(x + y) = 3\cos(x - y)$, express $\tan x$ in terms of $\tan y$.

Expanding $\sin(x + y)$ and $\cos(x - y)$ gives

$$2\sin x\cos y + 2\cos x\sin y = 3\cos x\cos y + 3\sin x\sin y$$

$$\text{so } \frac{2\sin x\cos y}{\cos x\cos y} + \frac{2\cos x\sin y}{\cos x\cos y} = \frac{3\cos x\cos y}{\cos x\cos y} + \frac{3\sin x\sin y}{\cos x\cos y}$$

This is similar to the expression seen in deriving $\tan(A + B)$. A good strategy is to divide both sides by $\cos x\cos y$.

$$2\tan x + 2\tan y = 3 + 3\tan x\tan y$$

$$2\tan x - 3\tan x\tan y = 3 - 2\tan y$$

Collect all terms in $\tan x$ on one side.

$$\tan x(2 - 3\tan y) = 3 - 2\tan y$$

Factorise.

$$\text{So} \quad \tan x = \frac{3 - 2\tan y}{2 - 3\tan y}$$

## Exercise 7A

**1** A student makes the mistake of thinking that $\sin(A + B) \equiv \sin A + \sin B$.
Choose non-zero values of $A$ and $B$ to show that this statement is not true for all values of $A$ and $B$.

This is a very common error – don't make the same mistake. One counterexample is sufficient to disprove a statement.

**2** Using the expansion of $\cos(A - B)$ with $A = B = \theta$, show that $\sin^2\theta + \cos^2\theta \equiv 1$.

**3** **a** Use the expansion of $\sin(A - B)$ to show that $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$.

**b** Use the expansion of $\cos(A - B)$ to show that $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$.

**4** Express the following as a single sine, cosine or tangent:

**a** $\sin 15^\circ \cos 20^\circ + \cos 15^\circ \sin 20^\circ$

**b** $\sin 58^\circ \cos 23^\circ - \cos 58^\circ \sin 23^\circ$

**c** $\cos 130^\circ \cos 80^\circ - \sin 130^\circ \sin 80^\circ$

**d** $\dfrac{\tan 76^\circ - \tan 45^\circ}{1 + \tan 76^\circ \tan 45^\circ}$

**e** $\cos 2\theta \cos\theta + \sin 2\theta \sin\theta$

**f** $\cos 4\theta \cos 3\theta - \sin 4\theta \sin 3\theta$

**g** $\sin\frac{1}{2}\theta \cos 2\frac{1}{2}\theta + \cos\frac{1}{2}\theta \sin 2\frac{1}{2}\theta$

**h** $\dfrac{\tan 2\theta + \tan 3\theta}{1 - \tan 2\theta \tan 3\theta}$

**i** $\sin(A + B)\cos B - \cos(A + B)\sin B$

**j** $\cos\left(\dfrac{3x + 2y}{2}\right)\cos\left(\dfrac{3x - 2y}{2}\right) - \sin\left(\dfrac{3x + 2y}{2}\right)\sin\left(\dfrac{3x - 2y}{2}\right)$

**5** Calculate, without using your calculator, the exact value of:

**a** $\sin 30^\circ \cos 60^\circ + \cos 30^\circ \sin 60^\circ$

**b** $\cos 110^\circ \cos 20^\circ + \sin 110^\circ \sin 20^\circ$

**c** $\sin 33^\circ \cos 27^\circ + \cos 33^\circ \sin 27^\circ$

**d** $\cos\frac{\pi}{8}\cos\frac{\pi}{8} - \sin\frac{\pi}{8}\sin\frac{\pi}{8}$

**e** $\sin 60^\circ \cos 15^\circ - \cos 60^\circ \sin 15^\circ$

**f** $\cos 70^\circ(\cos 50^\circ - \tan 70^\circ \sin 50^\circ)$

**g** $\dfrac{\tan 45^\circ + \tan 15^\circ}{1 - \tan 45^\circ \tan 15^\circ}$

**h** $\dfrac{1 - \tan 15^\circ}{1 + \tan 15^\circ}$

**Hint:** $\tan 45^\circ = 1$.

**i** $\dfrac{\tan\left(\frac{7\pi}{12}\right) - \tan\left(\frac{\pi}{3}\right)}{1 + \tan\left(\frac{7\pi}{12}\right)\tan\left(\frac{\pi}{3}\right)}$

**j** $\sqrt{3}\cos 15^\circ - \sin 15^\circ$

**Hint:** Look at **e**.

**6** Triangle $ABC$ is such that $AB = 3$ cm, $BC = 4$ cm, $\angle ABC = 120^\circ$ and $\angle BAC = \theta^\circ$.

**a** Write down, in terms of $\theta$, an expression for $\angle ACB$.

**b** Using the sine rule, or otherwise, show that $\tan\theta^\circ = \dfrac{2\sqrt{3}}{5}$.

**7** Prove the identities

**a** $\sin(A+60°)+\sin(A-60°)\equiv\sin A$

**b** $\dfrac{\cos A}{\sin B}-\dfrac{\sin A}{\cos B}\equiv\dfrac{\cos(A+B)}{\sin B\cos B}$

**c** $\dfrac{\sin(x+y)}{\cos x\cos y}\equiv\tan x+\tan y$

**d** $\dfrac{\cos(x+y)}{\sin x\sin y}+1\equiv\cot x\cot y$

**e** $\cos\left(\theta+\dfrac{\pi}{3}\right)+\sqrt{3}\sin\theta\equiv\sin\left(\theta+\dfrac{\pi}{6}\right)$

**f** $\cot(A+B)\equiv\dfrac{\cot A\cot B-1}{\cot A+\cot B}$

**g** $\sin^2(45+\theta)°+\sin^2(45-\theta)°\equiv1$

**h** $\cos(A+B)\cos(A-B)\equiv\cos^2 A-\sin^2 B$

**8** Given that $\sin A=\frac{4}{5}$ and $\sin B=\frac{1}{2}$, where $A$ and $B$ are both acute angles, calculate the exact values of

**a** $\sin(A+B)$ **b** $\cos(A-B)$ **c** $\sec(A-B)$

**9** Given that $\cos A=-\frac{4}{5}$, and $A$ is an obtuse angle measured in radians, find the exact value of

**a** $\sin A$ **b** $\cos(\pi+A)$ **c** $\sin\left(\dfrac{\pi}{3}+A\right)$ **d** $\tan\left(\dfrac{\pi}{4}+A\right)$

**10** Given that $\sin A=\frac{8}{17}$, where $A$ is acute, and $\cos B=-\frac{4}{5}$, where $B$ is obtuse, calculate the exact value of

**a** $\sin(A-B)$ **b** $\cos(A-B)$ **c** $\cot(A-B)$

**11** Given that $\tan A=\frac{7}{24}$, where $A$ is reflex, and $\sin B=\frac{5}{13}$, where $B$ is obtuse, calculate the exact value of

**a** $\sin(A+B)$ **b** $\tan(A-B)$ **c** $\operatorname{cosec}(A+B)$

**12** Write the following as a single trigonometric function, assuming that $\theta$ is measured in radians:

**a** $\cos^2\theta-\sin^2\theta$ **b** $2\sin4\theta\cos4\theta$ **c** $\dfrac{1+\tan\theta}{1-\tan\theta}$ **d** $\dfrac{1}{\sqrt{2}}(\sin\theta+\cos\theta)$

**13** Solve, in the interval $0°\leqslant\theta<360°$, the following equations. Give answers to the nearest 0.1°.

**a** $3\cos\theta=2\sin(\theta+60°)$

**b** $\sin(\theta+30°)+2\sin\theta=0$

**c** $\cos(\theta+25°)+\sin(\theta+65°)=1$

**d** $\cos\theta=\cos(\theta+60°)$

**e** $\tan(\theta-45°)=6\tan\theta$

**f** $\sin\theta+\cos\theta=1$

**Hint for part f:** Multiply each term by $\dfrac{1}{\sqrt{2}}$

**14** **a** Solve the equation $\cos\theta\cos30°-\sin\theta\sin30°=0.5$, for $0\leqslant\theta\leqslant360°$.

**b** Hence write down, in the same interval, the solutions of $\sqrt{3}\cos\theta-\sin\theta=1$.

**15** **a** Express $\tan(45+30)°$ in terms of $\tan45°$ and $\tan30°$.

**b** Hence show that $\tan75°=2+\sqrt{3}$.

**16** Show that $\sec105°=-\sqrt{2}(1+\sqrt{3})$

**17** Calculate the exact values of

**a** $\cos15°$ **b** $\sin75°$

**c** $\sin(120+45)°$ **d** $\tan165°$

**Hint for part a:** Write 15° as $(45-30)°$

**18** **a** Given that $3\sin(x-y)-\sin(x+y)=0$, show that $\tan x=2\tan y$.

**b** Solve $3\sin(x-45°)-\sin(x+45°)=0$, for $0\leqslant x\leqslant360°$.

**19** Given that $\sin x(\cos y + 2\sin y) = \cos x(2\cos y - \sin y)$, find the value of $\tan(x + y)$.

**20** Given that $\tan(x - y) = 3$, express $\tan y$ in terms of $\tan x$.

**21** In each of the following, calculate the exact value of $\tan x°$.

**a** $\tan(x - 45)° = \frac{1}{4}$

**b** $\sin(x - 60)° = 3\cos(x + 30)°$

**c** $\tan(x - 60)° = 2$

**22** Given that $\tan A° = \frac{1}{5}$ and $\tan B° = \frac{2}{3}$, calculate, without using your calculator, the value of $A + B$,

**a** where $A$ and $B$ are both acute,

**b** where $A$ is reflex and $B$ is acute.

**23** Given that $\cos y = \sin(x + y)$, show that $\tan y = \sec x - \tan x$.

**24** Given that $\cot A = \frac{1}{4}$ and $\cot(A + B) = 2$, find the value of $\cot B$.

**25** Given that $\tan\left(x + \frac{\pi}{3}\right) = \frac{1}{2}$, show that $\tan x = 8 - 5\sqrt{3}$.

## 7.2 You can express sin 2*A*, cos 2*A* and tan 2*A* in terms of angle *A*, using the double angle formulae.

- $\sin 2A \equiv 2\sin A\cos A$
- $\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$
- $\tan 2A \equiv \dfrac{2\tan A}{1 - \tan^2 A}$

You can derive these results from the addition formulae.

### Example 7

Use the expansion of $\sin(A + B)$ to show that $\sin 2A \equiv 2\sin A\cos A$

Using $\sin(A + B) \equiv \sin A\cos B + \cos A\sin B$

$$\sin 2A \equiv \sin A\cos A$$
$$+ \cos A\sin A$$
$$\equiv 2\sin A\cos A$$

Replace $B$ by $A$.

### Example 8

**a** By using an appropriate addition formula show that $\cos 2A \equiv \cos^2 A - \sin^2 A$.

**b** Hence derive the alternative forms $\cos 2A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$.

**a** Using $\cos(A+B)$

$\equiv \cos A \cos B - \sin A \sin B$

So $\cos 2A$

$\equiv \cos A \cos A - \sin A \sin A$

Replace $B$ with $A$.

$\equiv \cos^2 A - \sin^2 A$

**b** $\cos 2A$

$\equiv \cos^2 A - \sin^2 A$

$\equiv \cos^2 A - (1 - \cos^2 A)$

$\equiv 2\cos^2 A - 1$

OR

$\equiv 2(1 - \sin^2 A) - 1$

$\equiv 1 - 2\sin^2 A$

Use $\sin^2 A + \cos^2 A = 1$.

You can express $\cos 2A$ in terms of $\cos^2 A$ and $\sin^2 A$, or $\cos^2 A$ only, or $\sin^2 A$ only.

## Example 9

Express $\tan 2A$ in terms of $\tan A$.

Using $\tan(A+B) \equiv \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

$\tan 2A \equiv \dfrac{2\tan A}{1 - \tan^2 A}$

Replace $B$ by $A$.

## Example 10

Rewrite the following expressions as a single trigonometric function:

**a** $2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}\cos\theta$

**b** $1 + \cos 4\theta$

**a** $2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2} \equiv \sin\theta$

Using $2\sin A\cos A \equiv \sin 2A$ with $A = \dfrac{\theta}{2}$.

So $2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}\cos\theta \equiv \sin\theta\cos\theta$

$\equiv \frac{1}{2}\sin 2\theta$

The double angle formulae allow you to convert an angle into its half angle, and vice versa. In a question it will not always be obvious that the double angle formulae are needed (i.e.you will not always see **2*A*** or **2*θ***).

$\sin 2\theta = 2\sin\theta\cos\theta$.

**b** $\cos 4\theta \equiv 2\cos^2 2\theta - 1$

So $1 + \cos 4\theta \equiv 2\cos^2\theta$

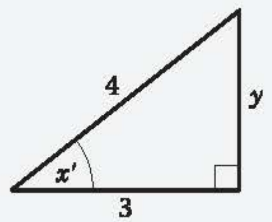
Using $\cos 2A = 2\cos^2 A - 1$ with $A = 2\theta$. Choose this form of $\cos 2A$ because the $-1$ will cancel out the $+1$ in '$1 + \cos 4\theta$'.

## Example 11

Given that $\cos x = \frac{3}{4}$, and that $180° < x < 360°$, find the exact values of

**a** $\sin 2x$ **b** $\tan 2x$

**a**




Draw the right-angled triangle with $\cos x' = \frac{3}{4}$.

Using Pythagoras' theorem

$$y^2 = 4^2 - 3^2 = 7$$

So $y = \sqrt{7}$

$$\therefore \quad \sin x' = \frac{\sqrt{7}}{4} \text{ and } \tan x' = \frac{\sqrt{7}}{3}$$

$$\Rightarrow \quad \sin x = -\frac{\sqrt{7}}{4} \text{ and } \tan x = -\frac{\sqrt{7}}{3}$$

As $\cos x$ is +ve and $x$ is reflex, $x$ must be in the 4th quadrant, so $\sin x = -\sin x'$ and $\tan x = -\tan x'$.

$$\sin 2x = 2\sin x \cos x$$

$$= 2\left(-\frac{\sqrt{7}}{4}\right)\left(\frac{3}{4}\right) = -\frac{3\sqrt{7}}{8}$$

**b** $$\tan 2x = \frac{2\tan x}{1 - \tan^2 x} = \frac{-\frac{2\sqrt{7}}{3}}{1 - \frac{7}{9}}$$

$$= -\frac{2\sqrt{7}}{3} \times \frac{9}{2}$$

$$= -3\sqrt{7}$$

## Exercise 7B

In equations, give answers to 1 decimal place where appropriate.

**1** Write the following expressions as a single trigonometric ratio:

**a** $2\sin 10° \cos 10°$

**b** $1 - 2\sin^2 25°$

**c** $\cos^2 40° - \sin^2 40°$

**d** $\dfrac{2\tan 5°}{1 - \tan^2 5°}$

**e** $\dfrac{1}{2\sin(24\frac{1}{2})° \cos(24\frac{1}{2})°}$

**f** $6\cos^2 30° - 3$

**g** $\dfrac{\sin 8°}{\sec 8°}$

**h** $\cos^2 \dfrac{\pi}{16} - \sin^2 \dfrac{\pi}{16}$

**2** Without using your calculator find the exact values of:

**a** $2\sin(22\tfrac{1}{2})^\circ\cos(22\tfrac{1}{2})^\circ$ **b** $2\cos^2 15^\circ - 1$ **c** $(\sin 75^\circ - \cos 75^\circ)^2$ **d** $\dfrac{2\tan\frac{\pi}{8}}{1-\tan^2\frac{\pi}{8}}$

**3** Write the following in their simplest form, involving only one trigonometric function:

**a** $\cos^2 3\theta - \sin^2 3\theta$ **b** $6\sin 2\theta\cos 2\theta$ **c** $\dfrac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$

**d** $2 - 4\sin^2\frac{\theta}{2}$ **e** $\sqrt{1+\cos 2\theta}$ **f** $\sin^2\theta\cos^2\theta$

**g** $4\sin\theta\cos\theta\cos 2\theta$ **h** $\dfrac{\tan\theta}{\sec^2\theta - 2}$ **i** $\sin^4\theta - 2\sin^2\theta\cos^2\theta + \cos^4\theta$

**4** Given that $\cos x = \frac{1}{4}$, find the exact value of $\cos 2x$.

**5** Find the possible values of $\sin\theta$ when $\cos 2\theta = \frac{23}{25}$.

**6** Given that $\cos x + \sin x = m$ and $\cos x - \sin x = n$, where $m$ and $n$ are constants, write down, in terms of $m$ and $n$, the value of $\cos 2x$.

**7** Given that $\tan\theta = \frac{3}{4}$, and that $\theta$ is acute:

**a** Find the exact value of **i** $\tan 2\theta$ **ii** $\sin 2\theta$ **iii** $\cos 2\theta$

**b** Deduce the value of $\sin 4\theta$.

**8** Given that $\cos A = -\frac{1}{3}$, and that $A$ is obtuse:

**a** Find the exact value of **i** $\cos 2A$ **ii** $\sin A$ **iii** $\operatorname{cosec} 2A$

**b** Show that $\tan 2A = \dfrac{4\sqrt{2}}{7}$.

**9** Given that $\pi < \theta < \dfrac{3\pi}{2}$, find the value of $\tan\dfrac{\theta}{2}$ when $\tan\theta = \frac{3}{4}$.

**10** In $\triangle ABC$, $AB = 4$ cm, $AC = 5$ cm, $\angle ABC = 2\theta$ and $\angle ACB = \theta$. Find the value of $\theta$, giving your answer, in degrees, to 1 decimal place.

**11** In $\triangle PQR$, $PQ = 3$ cm, $PR = 6$ cm, $QR = 5$ cm and $\angle QPR = 2\theta$.

**a** Use the cosine rule to show that $\cos 2\theta = \frac{5}{9}$.

**b** Hence find the exact value of $\sin\theta$.

**12** The line $l$, with equation $y = \frac{3}{4}x$, bisects the angle between the $x$-axis and the line $y = mx$, $m > 0$. Given that the scales on each axis are the same, and that $l$ makes an angle $\theta$ with the $x$-axis,

**a** write down the value of $\tan\theta$.

**b** Show that $m = \frac{24}{7}$.

## 7.3 The double angle formulae allow you to solve more equations and prove more identities.

### Example 12

Prove the identity $\tan 2\theta \equiv \dfrac{2}{\cot\theta - \tan\theta}$

Start on LHS with $\tan 2\theta \equiv \dfrac{2\tan\theta}{1-\tan^2\theta}$

Divide 'top and bottom' by $\tan\theta$.

So $\tan 2\theta \equiv \dfrac{2}{\dfrac{1}{\tan\theta} - \tan\theta}$

$\equiv \dfrac{2}{\cot\theta - \tan\theta}$

There are many starting points here; the more you know the more options you have.

Try starting with $\tan 2\theta \equiv \dfrac{\sin 2\theta}{\cos 2\theta}$ and use the double angle formulae for $\sin 2\theta$ and $\cos 2\theta$ (a bit harder!), or start with the RHS.

### Example 13

By expanding $\sin(2A + A)$ show that $\sin 3A \equiv 3\sin A - 4\sin^3 A$.

$$\sin(2A + A) \equiv \sin 2A\cos A + \cos 2A\sin A$$

So
$$\begin{aligned}\sin 3A &\equiv (2\sin A\cos A)\cos A \\ &\quad + (\cos^2 A - \sin^2 A)\sin A \\ &\equiv 2\sin A\cos^2 A \\ &\quad + \cos^2 A\sin A - \sin^3 A \\ &\equiv 3\sin A\cos^2 A - \sin^3 A \quad * \\ &\equiv 3\sin A(1 - \sin^2 A) - \sin^3 A \\ &\equiv 3\sin A - 4\sin^3 A\end{aligned}$$

$\cos 2A = \cos^2 A - \sin^2 A$ has been used here because the line marked * is a useful result when you need to find the formula for $\tan 3A$.

See Exercise 7C Questions 12 and 13 for similar expansions of $\cos 3A$ and $\tan 3A$.

### Example 14

Given that $x = 3\sin\theta$ and $y = 3 - 4\cos 2\theta$, eliminate $\theta$ and express $y$ in terms of $x$.

The equations can be rewritten as

$$\sin\theta = \frac{x}{3} \qquad \cos 2\theta = \frac{3-y}{4}$$

As $\cos 2\theta = 1 - 2\sin^2\theta$ for all values of $\theta$,

$$\frac{3-y}{4} = 1 - 2\left(\frac{x}{3}\right)^2$$

So
$$\frac{y}{4} = 2\left(\frac{x}{3}\right)^2 - \frac{1}{4}$$

or
$$y = 8\left(\frac{x}{3}\right)^2 - 1$$

Be careful with this manipulation. Many errors occur in the early part of a solution.

This is the relationship: $\theta$ has been eliminated but the solution is not complete. Always make sure that you have answered the question: here you need to write $y = \ldots$ .

## Example 15

Solve $3\cos 2x - \cos x + 2 = 0$ for $0° \leqslant x \leqslant 360°$.

Using a double angle formula for $\cos 2x$

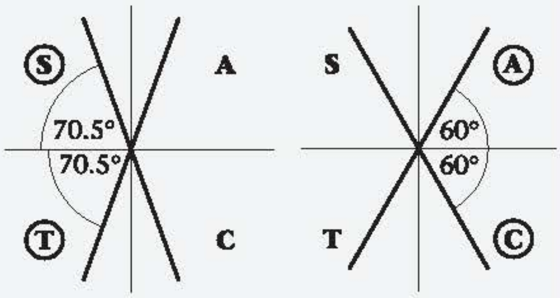
$$3\cos 2x - \cos x + 2 = 0$$

becomes

$$3(2\cos^2 x - 1) - \cos x + 2 = 0$$
$$6\cos^2 x - 3 - \cos x + 2 = 0$$
$$6\cos^2 x - \cos x - 1 = 0$$

So $(3\cos x + 1)(2\cos x - 1) = 0$

Solving: $\cos x = -\frac{1}{3}$ or $\cos x = \frac{1}{2}$




$\cos^{-1}\left(-\frac{1}{3}\right) = 109.5°$ $\quad \cos^{-1}\left(\frac{1}{2}\right) = 60°$

So $x = 60°, 109.5°, 250.5°, 300°$

The term $\cos x$ in the equation dictates the choice you need to make; you need the form of $\cos 2x$ with $\cos x$ only, i.e. $\cos 2x \equiv 2\cos^2 x - 1$.

This quadratic equation factorises $6X^2 - X - 1 = (3X + 1)(2X - 1)$.

Solutions to $\cos x = -\frac{1}{3}$ are in the 2nd and 3rd quadrants. Solutions to $\cos x = \frac{1}{2}$ are in the 1st and 4th quadrants.

Remember that two solutions of $\cos x = k$ are $\cos^{-1} k$ and $360° - \cos^{-1} k$.
In this case they all fall in the required interval.

## Exercise 7C

In equations, give answers to 1 decimal place where appropriate.

**1** Prove the following identities:

**a** $\dfrac{\cos 2A}{\cos A + \sin A} \equiv \cos A - \sin A$

**b** $\dfrac{\sin B}{\sin A} - \dfrac{\cos B}{\cos A} \equiv 2\operatorname{cosec} 2A \sin(B - A)$

**c** $\dfrac{1 - \cos 2\theta}{\sin 2\theta} \equiv \tan\theta$

**d** $\dfrac{\sec^2\theta}{1 - \tan^2\theta} \equiv \sec 2\theta$

**e** $2(\sin^3\theta\cos\theta + \cos^3\theta\sin\theta) \equiv \sin 2\theta$

**f** $\dfrac{\sin 3\theta}{\sin\theta} - \dfrac{\cos 3\theta}{\cos\theta} \equiv 2$

**g** $\operatorname{cosec}\theta - 2\cot 2\theta\cos\theta \equiv 2\sin\theta$

**h** $\dfrac{\sec\theta - 1}{\sec\theta + 1} \equiv \tan^2\dfrac{\theta}{2}$

**i** $\tan\left(\dfrac{\pi}{4} - x\right) \equiv \dfrac{1 - \sin 2x}{\cos 2x}$

**2** **a** Show that $\tan\theta + \cot\theta \equiv 2\operatorname{cosec} 2\theta$.

**b** Hence find the value of $\tan 75° + \cot 75°$.

**3** Solve the following equations, in the interval shown in brackets:

**a** $\sin 2\theta = \sin\theta \quad \{0 \leqslant \theta \leqslant 2\pi\}$

**b** $\cos 2\theta = 1 - \cos\theta \quad \{-180° < \theta \leqslant 180°\}$

**c** $3\cos 2\theta = 2\cos^2\theta \quad \{0 \leqslant \theta < 360°\}$

**d** $\sin 4\theta = \cos 2\theta \quad \{0 \leqslant \theta \leqslant \pi\}$

**e** $2\tan 2y \tan y = 3 \quad \{0 \leqslant y < 360°\}$

**f** $3\cos\theta - \sin\dfrac{\theta}{2} - 1 = 0 \quad \{0 \leqslant \theta < 720°\}$

**g** $\cos^2\theta - \sin 2\theta = \sin^2\theta \quad \{0 \leqslant \theta \leqslant \pi\}$

**h** $2\sin\theta = \sec\theta \quad \{0 \leqslant \theta \leqslant 2\pi\}$

**i** $2\sin 2\theta = 3\tan\theta \quad \{0 \leqslant \theta < 360°\}$

**j** $2\tan\theta = \sqrt{3}(1 - \tan\theta)(1 + \tan\theta) \quad \{0 \leqslant \theta \leqslant 2\pi\}$

**k** $5\sin 2\theta + 4\sin\theta = 0 \quad \{-180° < \theta \leqslant 180°\}$

**l** $\sin^2\theta = 2\sin 2\theta \quad \{-180° < \theta \leqslant 180°\}$

**m** $4\tan\theta = \tan 2\theta \quad \{0 \leqslant \theta < 360°\}$

**4** Given that $p = 2\cos\theta$ and $q = \cos 2\theta$, express $q$ in terms of $p$.

**5** Eliminate $\theta$ from the following pairs of equations:

**a** $x = \cos^2\theta,\ y = 1 - \cos 2\theta$

**b** $x = \tan\theta,\ y = \cot 2\theta$

**c** $x = \sin\theta,\ y = \sin 2\theta$

**d** $x = 3\cos 2\theta + 1,\ y = 2\sin\theta$

**6** **a** Prove that $(\cos 2\theta - \sin 2\theta)^2 \equiv 1 - \sin 4\theta$.

**b** Use the result to solve, for $0 \leqslant \theta < \pi$, the equation $\cos 2\theta - \sin 2\theta = \dfrac{1}{\sqrt{2}}$.
Give your answers in terms of $\pi$.

**7** **a** Show that:

**i** $\sin\theta \equiv \dfrac{2\tan\dfrac{\theta}{2}}{1 + \tan^2\dfrac{\theta}{2}}$ **ii** $\cos\theta \equiv \dfrac{1 - \tan^2\dfrac{\theta}{2}}{1 + \tan^2\dfrac{\theta}{2}}$

**b** By writing the following equations as quadratics in $\tan\dfrac{\theta}{2}$, solve, in the interval $0 \leqslant \theta \leqslant 360°$:

**i** $\sin\theta + 2\cos\theta = 1$ **ii** $3\cos\theta - 4\sin\theta = 2$

**8** **a** Using $\cos 2A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$, show that:

**i** $\cos^2\dfrac{x}{2} \equiv \dfrac{1 + \cos x}{2}$ **ii** $\sin^2\dfrac{x}{2} \equiv \dfrac{1 - \cos x}{2}$

These are known as the **half angle formulae**. (They are useful in integration.)

**b** Given that $\cos\theta = 0.6$, and that $\theta$ is acute, write down the values of:

**i** $\cos\dfrac{\theta}{2}$ **ii** $\sin\dfrac{\theta}{2}$ **iii** $\tan\dfrac{\theta}{2}$

**c** Show that $\cos^4\dfrac{A}{2} \equiv \frac{1}{8}(3 + 4\cos A + \cos 2A)$

**9** **a** Show that $3\cos^2 x - \sin^2 x \equiv 1 + 2\cos 2x$.

**b** Hence sketch, for $-\pi \leqslant x \leqslant \pi$, the graph of $y = 3\cos^2 x - \sin^2 x$, showing the coordinates of points where the curve meets the axes.

**10** **a** Express $2\cos^2\frac{\theta}{2} - 4\sin^2\frac{\theta}{2}$ in the form $a\cos\theta + b$, where $a$ and $b$ are constants.

**b** Hence solve $2\cos^2\frac{\theta}{2} - 4\sin^2\frac{\theta}{2} = -3$, in the interval $0 \leqslant \theta < 360°$.

**11** **a** Use the identity $\sin^2 A + \cos^2 A \equiv 1$ to show that $\sin^4 A + \cos^4 A \equiv \frac{1}{2}(2 - \sin^2 2A)$.

**b** Deduce that $\sin^4 A + \cos^4 A \equiv \frac{1}{4}(3 + \cos 4A)$.

**c** Hence solve $8\sin^4\theta + 8\cos^4\theta = 7$, for $0 < \theta < \pi$.

**12** **a** By expanding $\cos(2A + A)$ show that $\cos 3A \equiv 4\cos^3 A - 3\cos A$.

**b** Hence solve $8\cos^3\theta - 6\cos\theta - 1 = 0$, for $\{0 \leqslant \theta \leqslant 360°\}$.

**13** **a** Show that $\tan 3\theta \equiv \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$.

**b** Given that $\theta$ is acute such that $\cos\theta = \frac{1}{3}$, show that $\tan 3\theta = \dfrac{10\sqrt{2}}{23}$.

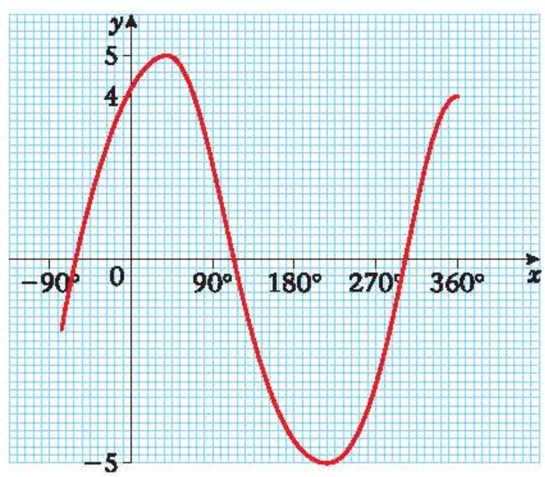
**Hint:** Divide formulae for $\sin 3\theta$ and $\cos 3\theta$. See Example 13 for a useful form of $\sin 3\theta$, and use a similar form for $\cos 3\theta$.

## 7.4 You can write expressions of the form $a\cos\theta + b\sin\theta$, where $a$ and $b$ are constants, as a sine function only or a cosine function only.

If you sketch, or draw on your calculator, the graph of $y = 3\sin x + 4\cos x$ you will see that it has the form of $y = \sin x$ or $y = \cos x$ but stretched vertically and translated horizontally.

If you draw the graph of $y = 5\sin(x + \tan^{-1}\{\frac{4}{3}\})$ or $y = 5\cos(x - \tan^{-1}\{\frac{3}{4}\})$, you will see that they are the same as $y = 3\sin x + 4\cos x$.

Using the addition formulae you can show that all expressions of the form $a\cos\theta + b\sin\theta$ can be expressed in one of the forms
$R\sin(x \pm \alpha)$ where $R > 0$, and $0 < \alpha < 90°$, or
$R\cos(x \pm \beta)$ where $R > 0$, and $0 < \beta < 90°$.




Remember: the graph of $y = a\,\mathrm{f}(x - \alpha)$ is the graph of $y = \mathrm{f}(x)$ stretched vertically by a factor of $a$ and translated horizontally by $\alpha$.

### Example 16

Show that you can express $3\sin x + 4\cos x$ in the form $R\sin(x + \alpha)$, where $R > 0$, $0 < \alpha < 90°$, giving your values of $R$ and $\alpha$ to 1 decimal place where appropriate.

$R\sin(x+\alpha) \equiv R\sin x\cos\alpha + R\cos x\sin\alpha$

Use $\sin(A+B) \equiv \sin A\cos B + \cos A\sin B$, and multiply through by $R$.

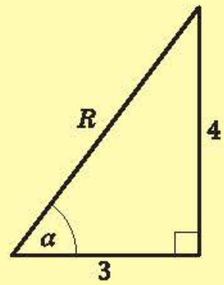
Let $3\sin x + 4\cos x \equiv R\sin x\cos\alpha + R\cos x\sin\alpha$

For this to be true for all values of $x$, the coefficients of $\sin x$ and $\cos x$ on both sides of the identity have to be equal.

So $R\cos\alpha = 3$ and $R\sin\alpha = 4$

Equations of this sort can always be solved, and so $R$ and $\alpha$ can always be found.

You could draw a right-angled triangle with $\cos\alpha = \dfrac{3}{R}$ and $\sin\alpha = \dfrac{4}{R}$




So $\tan\alpha = \frac{4}{3}$ and $R^2 = 3^2 + 4^2 \Rightarrow R = 5$.

Divide the equations to find $\tan\alpha$

$$\frac{R\sin\alpha}{R\cos\alpha} = \tfrac{4}{3} \text{ or } \tan\alpha = \tfrac{4}{3}$$

So $\alpha = 53.1°$ (1 d.p.)

Square and add the equations to find $R^2$:

$$R^2\cos^2\alpha + R^2\sin^2\alpha = 3^2 + 4^2$$

So $R^2(\cos^2\alpha + \sin^2\alpha) = 3^2 + 4^2$

So $R = \sqrt{3^2 + 4^2} = 5$

$3\sin x + 4\cos x \equiv 5\sin(x + 53.1°)$

You could equally have shown that $3\sin x + 4\cos x \equiv 5\cos(x - 36.9°)$ by setting $3\sin x + 4\cos x \equiv R\cos(x-\alpha)$ and solving for $R$ and $\alpha$, as in the example.

## Example 17

**a** Show that you can express $\sin x - \sqrt{3}\cos x$ in the form $R\sin(x-\alpha)$, where $R > 0$, $0 < \alpha < \dfrac{\pi}{2}$.

**b** Hence sketch the graph of $y = \sin x - \sqrt{3}\cos x$.

**a** Set $\sin x - \sqrt{3}\cos x \equiv R\sin(x-\alpha)$

$\sin x - \sqrt{3}\cos x \equiv R\sin x\cos\alpha - R\cos x\sin\alpha$

Expand $\sin(x-\alpha)$ and multiply by $R$.

So $R\cos\alpha = 1$ and $R\sin\alpha = \sqrt{3}$

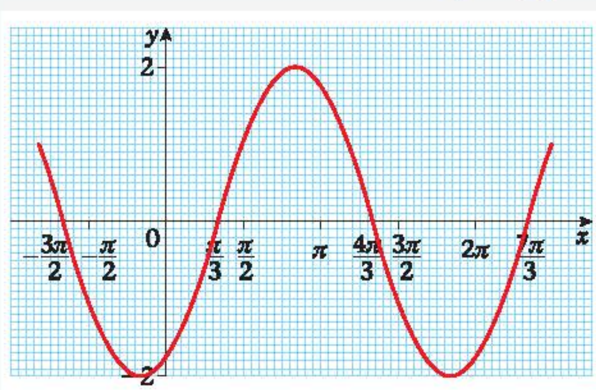
Compare the coefficients of $\sin x$ and $\cos x$ on both sides of the identity.

Dividing, $\tan\alpha = \sqrt{3}$, so $\alpha = \dfrac{\pi}{3}$

Squaring and adding: $R = 2$

So $\sin x - \sqrt{3}\cos x \equiv 2\sin\left(x - \dfrac{\pi}{3}\right)$

**b** $y = \sin x - \sqrt{3}\cos x \equiv 2\sin\left(x - \dfrac{\pi}{3}\right)$




You can sketch $y = 2\sin\left(x - \dfrac{\pi}{3}\right)$ by translating $y = \sin x$ by $\dfrac{\pi}{3}$ to the right and then stretching by a scale factor of 2 in the $y$-direction.

## Example 18

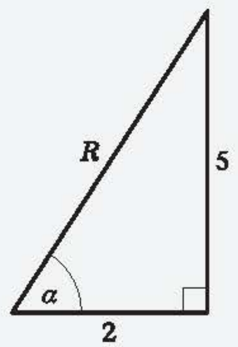
**a** Express $2\cos\theta + 5\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$, $0 < \alpha < 90°$.

**b** Hence solve, for $0 < \theta < 360°$, the equation $2\cos\theta + 5\sin\theta = 3$.

**a** Set $2\cos\theta + 5\sin\theta \equiv R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$

So $R\cos\alpha = 2$ and $R\sin\alpha = 5$

Compare the coefficients of $\sin x$ and $\cos x$ on both sides of the identity.




Draw a right-angled triangle with $\cos\alpha = \frac{2}{R}$ and $\sin\alpha = \frac{5}{R}$

$\therefore$ $\tan\alpha = 2.5$ and $R = \sqrt{2^2 + 5^2} = \sqrt{29}$

So $2\cos\theta + 5\sin\theta \equiv \sqrt{29}\cos(\theta - 68.2°)$

**b** The solutions of $2\cos\theta + 5\sin\theta = 3$ are the same as those of $\sqrt{29}\cos(\theta - 68.2°) = 3$.

Divide the equation by $\sqrt{29}$.

So $\cos(\theta - 68.2°) = \dfrac{3}{\sqrt{29}}$

$\cos^{-1}\left(\dfrac{3}{\sqrt{29}}\right) = 56.1\ldots°$

As $0 < \theta < 360°$, the interval for $(\theta - 68.2°)$ is $-68.2° < (\theta - 68.2°) < 291.8°$.

$\dfrac{3}{\sqrt{29}}$ is +ve, so solutions for $\theta - 68.2$ are in the 1st and 4th quadrants.

So $\theta - 68.2° = -56.1\ldots°, 56.1\ldots°$

$\theta = 12.1°, 124.3°$ (to the nearest 0.1°)

## Example 19

Without using calculus, find the maximum value of $12\cos\theta + 5\sin\theta$, and give the smallest positive value of $\theta$ at which it arises.

Set $12\cos\theta + 5\sin\theta \equiv R\cos(\theta - \alpha)$

The most convenient forms to use here are $R\sin(\theta + \alpha)$ or $R\cos(\theta - \alpha)$ as the sign in the expanded form is the same as that in $12\cos\theta + 5\sin\theta$. If the signs do not match up, it will be more difficult for you.

So $12\cos\theta + 5\sin\theta \equiv R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$

So $R\cos\alpha = 12$ and $R\sin\alpha = 5$

$R = 13$ and $\tan\alpha = \frac{5}{12} \Rightarrow \alpha = 22.6°$

So $12\cos\theta + 5\sin\theta \equiv 13\cos(\theta - 22.6°)$

The maximum value of $13\cos(\theta - 22.6°)$ is 13 and occurs when $\cos(\theta - 22.6°) = 1$;
i.e. when $\theta - 22.6° = \ldots, -360°, 0°, 360°, \ldots$
The smallest positive value of $\theta$, therefore, is 22.6°.

■ **For positive values of $a$ and $b$,**
**$a\sin\theta \pm b\cos\theta$ can be expressed in the form $R\sin(\theta \pm \alpha)$, with $R > 0$ and $0 < \alpha < 90°$ $\left(\text{or } \frac{\pi}{2}\right)$**

**$a\cos\theta \pm b\sin\theta$ can be expressed in the form $R\cos(\theta \mp \alpha)$, with $R > 0$ and $0 < \alpha < 90°$ $\left(\text{or } \frac{\pi}{2}\right)$**

**where $R\cos\alpha = a$ and $R\sin\alpha = b$**

**and $R = \sqrt{a^2 + b^2}$.**

Do not quote these results, but they are useful check points.

**Note:** When solving equations of the form $a\cos\theta + b\sin\theta = c$, use the '$R$ formula', unless $c = 0$, when the equation reduces to $\tan\theta = k$.

## Exercise 7D

Give all angles to the nearest 0.1° and non-exact values of $R$ in surd form.

**1** Given that $5\sin\theta + 12\cos\theta \equiv R\sin(\theta + \alpha)$, find the value of $R$, $R > 0$, and the value of $\tan\alpha$.

**2** Given that $\sqrt{3}\sin\theta + \sqrt{6}\cos\theta \equiv 3\cos(\theta - \alpha)$, where $0 < \alpha < 90°$, find the value of $\alpha$.

**3** Given that $2\sin\theta - \sqrt{5}\cos\theta \equiv -3\cos(\theta + \alpha)$, where $0 < \alpha < 90°$, find the value of $\alpha$.

**4** Show that:

**a** $\cos\theta + \sin\theta \equiv \sqrt{2}\sin\left(\theta + \frac{\pi}{4}\right)$

**b** $\sqrt{3}\sin 2\theta - \cos 2\theta \equiv 2\sin\left(2\theta - \frac{\pi}{6}\right)$

**5** Prove that $\cos 2\theta - \sqrt{3}\sin 2\theta \equiv 2\cos\left(2\theta + \frac{\pi}{3}\right) \equiv -2\sin\left(2\theta - \frac{\pi}{6}\right)$.

**6** Find the value of $R$, where $R > 0$, and the value of $\alpha$, where $0 < \alpha < 90°$, in each of the following cases:

**a** $\sin\theta + 3\cos\theta \equiv R\sin(\theta + \alpha)$

**b** $3\sin\theta - 4\cos\theta \equiv R\sin(\theta - \alpha)$

**c** $2\cos\theta + 7\sin\theta \equiv R\cos(\theta - \alpha)$

**d** $\cos 2\theta - 2\sin 2\theta \equiv R\cos(2\theta + \alpha)$

**7** **a** Show that $\cos\theta - \sqrt{3}\sin\theta$ can be written in the form $R\cos(\theta + \alpha)$, with $R > 0$ and $0 < \alpha < \frac{\pi}{2}$.

**b** Hence sketch the graph of $y = \cos\theta - \sqrt{3}\sin\theta$, $0 < \alpha < 2\pi$, giving the coordinates of points of intersection with the axes.

**8** **a** Show that $3\sin 3\theta - 4\cos 3\theta$ can be written in the form $R\sin(3\theta - \alpha)$, with $R > 0$ and $0 < \alpha < 90°$.

**b** Deduce the minimum value of $3\sin 3\theta - 4\cos 3\theta$ and work out the smallest positive value of $\theta$ at which it occurs.

**9** **a** Show that $\cos 2\theta + \sin 2\theta$ can be written in the form $R\sin(2\theta + \alpha)$, with $R > 0$ and $0 < \alpha < \frac{\pi}{2}$.

**b** Hence solve, in the interval $0 \leqslant \theta < 2\pi$, the equation $\cos 2\theta + \sin 2\theta = 1$, giving your answers as rational multiples of $\pi$.

**10** **a** Express $7\cos\theta - 24\sin\theta$ in the form $R\cos(\theta + \alpha)$, with $R > 0$ and $0 < \alpha < 90°$.

**b** The graph of $y = 7\cos\theta - 24\sin\theta$ meets the $y$-axis at P. State the coordinates of P.

**c** Write down the maximum and minimum values of $7\cos\theta - 24\sin\theta$.

**d** Deduce the number of solutions, in the interval $0 < \theta < 360°$, of the following equations:

**i** $7\cos\theta - 24\sin\theta = 15$ **ii** $7\cos\theta - 24\sin\theta = 26$ **iii** $7\cos\theta - 24\sin\theta = -25$

**11** **a** Express $5\sin^2\theta - 3\cos^2\theta + 6\sin\theta\cos\theta$ in the form $a\sin 2\theta + b\cos 2\theta + c$, where $a$, $b$ and $c$ are constants.

**b** Hence find the maximum and minimum values of $5\sin^2\theta - 3\cos^2\theta + 6\sin\theta\cos\theta$.

**12** Solve the following equations, in the interval given in brackets:

**a** $6\sin x + 8\cos x = 5\sqrt{3}$ [0, 360°] **b** $2\cos 3\theta - 3\sin 3\theta = -1$ [0, 90°]

**c** $8\cos\theta + 15\sin\theta = 10$ [0, 360°] **d** $5\sin\frac{x}{2} - 12\cos\frac{x}{2} = -6.5$ [−360°, 360°]

**13** Solve the following equations, in the interval given in brackets:

**a** $\sin x\cos x = 1 - 2.5\cos 2x$ [0, 360°] **b** $\cot\theta + 2 = \operatorname{cosec}\theta$ [$0 < \theta < 360°$, $\theta \neq 180°$]

**c** $\sin\theta = 2\cos\theta - \sec\theta$ [0, 180°] **d** $\sqrt{2}\cos\left(\theta - \frac{\pi}{4}\right) + (\sqrt{3} - 1)\sin\theta = 2$ [0, 2$\pi$]

**14** Solve, if possible, in the interval $0 < \theta < 360°$, $\theta \neq 180°$, the equation $\dfrac{4 - 2\sqrt{2}\sin\theta}{1 + \cos\theta} = k$ in the case when $k$ is equal to:

**a** 4 **b** 2 **c** 1 **d** 0 **e** −1

**15** A class were asked to solve $3\cos\theta = 2 - \sin\theta$ for $0 \leqslant \theta \leqslant 360°$. One student expressed the equation in the form $R\cos(\theta - \alpha) = 2$, with $R > 0$ and $0 < \alpha < 90°$, and correctly solved the equation.

**a** Find the values of $R$ and $\alpha$ and hence find her solutions.

Another student decided to square both sides of the equation and then form a quadratic equation in $\sin\theta$.

**b** Show that the correct quadratic equation is $10\sin^2\theta - 4\sin\theta - 5 = 0$.

**c** Solve this equation, for $0 \leqslant \theta < 360°$.

**d** Explain why not all of the answers satisfy $3\cos\theta = 2 - \sin\theta$.

## 7.5 You can express sums and differences of sines and cosines as products of sines and/or cosines by using the 'factor formulae'.

- $\sin P + \sin Q \equiv 2\sin\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$
- $\cos P + \cos Q \equiv 2\cos\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$
- $\sin P - \sin Q \equiv 2\cos\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)$
- $\cos P - \cos Q \equiv -2\sin\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)$

These identities are derived from the addition formulae.

## Example 20

Use the formulae for $\sin(A+B)$ and $\sin(A-B)$ to derive the result that

$$\sin P + \sin Q \equiv 2\sin\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$$

$$\sin(A+B) \equiv \sin A\cos B + \cos A\sin B$$

and $\sin(A-B) \equiv \sin A\cos B - \cos A\sin B$

Add the two identities:

$$\sin(A+B) + \sin(A-B) \equiv 2\sin A\cos B$$

Let $A+B=P$ and $A-B=Q$,

then $A = \dfrac{P+Q}{2}$ and $B = \dfrac{P-Q}{2}$

$$\sin P + \sin Q \equiv 2\sin\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$$

The other three factor formulae are proved in a similar manner, by adding or subtracting two appropriate addition formulae.
See Exercise 7E.

This result is useful in integration, e.g.

$$\int 2\sin 4x\cos x\,dx = \int(\sin 5x + \sin 3x)dx.$$

## Example 21

Using the result that $\sin P - \sin Q \equiv 2\cos\left(\dfrac{P+Q}{2}\right)\sin\left(\dfrac{P-Q}{2}\right)$

**a** show that $\sin 105° - \sin 15° = \dfrac{1}{\sqrt{2}}$

**b** solve, for $0 \leqslant \theta \leqslant \pi$, $\sin 4\theta - \sin 3\theta = 0$

**a** $\sin 105° - \sin 15°$

$$= 2\cos\left(\frac{105° + 15°}{2}\right)\sin\left(\frac{105° - 15°}{2}\right)$$

$$= 2\cos 60°\sin 45°$$

$$= 2\left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}}$$

Let $P = 105°$ and $Q = 15°$.

Remember: $\cos 60° = \frac{1}{2}$, $\sin 45° = \dfrac{1}{\sqrt{2}}$.

**b** $\sin 4\theta - \sin 3\theta = 2\cos\left(\frac{7\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)$

Let $P = 4\theta$ and $Q = 3\theta$.

The solutions of $2\cos\left(\frac{7\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) = 0$ are either

$$\cos\left(\frac{7\theta}{2}\right) = 0$$

As $0 \leqslant \theta \leqslant \pi$, the interval for $\frac{7\theta}{2}$ is $0 \leqslant \frac{7\theta}{2} \leqslant \frac{7\pi}{2}$.

so $\quad \frac{7\theta}{2} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$

$\therefore \quad \theta = \frac{\pi}{7}, \frac{3\pi}{7}, \frac{5\pi}{7}, \pi$

or $\quad \sin\left(\frac{\theta}{2}\right) = 0$

The interval for $\frac{\theta}{2}$ is $0 \leqslant \frac{\theta}{2} \leqslant \frac{\pi}{2}$.

so $\quad \frac{\theta}{2} = 0 \quad \therefore \quad \theta = 0$

Answers are $\theta = 0, \frac{\pi}{7}, \frac{3\pi}{7}, \frac{5\pi}{7}, \pi$

## Example 22

Prove that $\dfrac{\sin(x + 2y) + \sin(x + y) + \sin x}{\cos(x + 2y) + \cos(x + y) + \cos x} \equiv \tan(x + y)$.

In the numerator

$\sin(x + 2y) + \sin x$

$\equiv 2\sin\left(\frac{2x + 2y}{2}\right)\cos\left(\frac{2y}{2}\right)$

$\equiv 2\sin(x + y)\cos y$

Use $\sin P + \sin Q \equiv 2\sin\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$ with $P = x + 2y$ and $Q = x$.

So $\sin(x + 2y) + \sin(x + y) + \sin x$

$\equiv \sin(x + y) + 2\sin(x + y)\cos y$

$\equiv \sin(x + y)(1 + 2\cos y)$ ①

Factorise.

Similarly for the denominator

$\cos(x + 2y) + \cos(x + y) + \cos x$

$\equiv \cos(x + y) + 2\cos(x + y)\cos y$

$\equiv \cos(x + y)(1 + 2\cos y)$ ②

Use $\cos P + \cos Q \equiv 2\cos\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$ with $P = x + 2y$ and $Q = x$.

Factorise.

so $\dfrac{\sin(x + 2y) + \sin(x + y) + \sin x}{\cos(x + 2y) + \cos(x + y) + \cos x}$

Use results ① and ②.

$\equiv \dfrac{\sin(x + y)\cancel{(1 + 2\cos y)}}{\cos(x + y)\cancel{(1 + 2\cos y)}}$

Cancel.

$\equiv \tan(x + y)$

## Exercise 7E

**1** **a** Show that $\sin(A+B)+\sin(A-B)\equiv 2\sin A\cos B$.

**b** Deduce that $\sin P+\sin Q\equiv 2\sin\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$.

**c** Use part **a** to express the following as the sum of two sines:
**i** $2\sin 7\theta\cos 2\theta$ **ii** $2\sin 12\theta\cos 5\theta$

**d** Use the result in **b** to solve, in the interval $0\leqslant\theta\leqslant 180°$, $\sin 3\theta+\sin\theta=0$.

**e** Prove that $\dfrac{\sin 7\theta+\sin\theta}{\sin 5\theta+\sin 3\theta}\equiv\dfrac{\cos 3\theta}{\cos\theta}$.

**2** **a** Show that $\sin(A+B)-\sin(A-B)\equiv 2\cos A\sin B$.

**b** Express the following as the difference of two sines:

**i** $2\cos 5x\sin 3x$ **ii** $\cos 2x\sin x$ **iii** $6\cos\frac{3}{2}x\sin\frac{1}{2}x$

**c** Using the result in **a** show that $\sin P-\sin Q\equiv 2\cos\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)$.

**d** Deduce that $\sin 56°-\sin 34°=\sqrt{2}\sin 11°$.

**3** **a** Show that $\cos(A+B)+\cos(A-B)\equiv 2\cos A\cos B$.

**b** Express as a sum of cosines **i** $2\cos\dfrac{5\theta}{2}\cos\dfrac{\theta}{2}$ **ii** $5\cos 2x\cos 3x$

**c** Show that $\cos P+\cos Q\equiv 2\cos\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$.

**d** Prove that $\dfrac{\sin 3\theta-\sin\theta}{\cos 3\theta+\cos\theta}\equiv\tan\theta$.

**4** **a** Show that $\cos(A+B)-\cos(A-B)\equiv -2\sin A\sin B$.

**b** Hence show that $\cos P-\cos Q\equiv -2\sin\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)$.

**c** Deduce that $\cos 2\theta-1\equiv -2\sin^2\theta$.

**d** Solve, in the interval $0\leqslant\theta\leqslant 180°$, $\cos 3\theta+\sin 2\theta-\cos\theta=0$.

**Hint** to Question 4c: What is the value of $\cos 0°$?

**5** Express the following as a sum or difference of sines or cosines:

**a** $2\sin 8x\cos 2x$ **b** $\cos 5x\cos x$ **c** $3\sin x\sin 7x$

**d** $\cos 100°\cos 40°$ **e** $10\cos\dfrac{3x}{2}\sin\dfrac{x}{2}$ **f** $2\sin 30°\cos 10°$

**6** Show, without using a calculator, that $2\sin 82\frac{1}{2}°\cos 37\frac{1}{2}°=\frac{1}{2}(\sqrt{3}+\sqrt{2})$.

**7** Express, in their simplest form, as a product of sines and/or cosines:

**a** $\sin 12x+\sin 8x$ **b** $\cos(x+2y)-\cos(2y-x)$ **c** $(\cos 4x+\cos 2x)\sin x$

**d** $\sin 95°-\sin 5°$ **e** $\cos\dfrac{\pi}{15}+\cos\dfrac{\pi}{12}$ **f** $\sin 150°+\sin 20°$

**8** Using the identity $\cos P+\cos Q\equiv 2\cos\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$, show that

$$\cos\theta+\cos\left(\theta+\frac{2\pi}{3}\right)+\cos\left(\theta+\frac{4\pi}{3}\right)=0.$$

**9** Prove that $\dfrac{\sin 75° + \sin 15°}{\cos 15° - \cos 75°} = \sqrt{3}$.

**10** Solve the following equations:

**a** $\cos 4x = \cos 2x$, for $0 \leqslant x \leqslant 180°$

**b** $\sin 3\theta - \sin \theta = 0$, for $0 \leqslant \theta \leqslant 2\pi$

**c** $\sin(x + 20°) + \sin(x - 10°) = \cos 15°$, for $0 \leqslant x \leqslant 360°$

**d** $\sin 3\theta - \sin \theta = \cos 2\theta$, for $0 \leqslant \theta \leqslant 2\pi$

**11** Prove the identities

**a** $\dfrac{\sin 7\theta - \sin 3\theta}{\sin \theta \cos \theta} \equiv 4 \cos 5\theta$

**b** $\dfrac{\cos 2\theta + \cos 4\theta}{\sin 2\theta - \sin 4\theta} \equiv -\cot \theta$

**c** $\sin^2(x + y) - \sin^2(x - y) \equiv \sin 2x \sin 2y$

**d** $\cos x + 2 \cos 3x + \cos 5x \equiv 4 \cos^2 x \cos 3x$

**12** **a** Prove that $\cos \theta + \sin 2\theta - \cos 3\theta \equiv \sin 2\theta(1 + 2 \sin \theta)$.

**b** Hence solve, for $0 \leqslant \theta \leqslant 2\pi$, $\cos \theta + \sin 2\theta = \cos 3\theta$.

## Mixed exercise 7F

**1** The lines $l_1$ and $l_2$, with equations $y = 2x$ and $3y = x - 1$ respectively, are drawn on the same set of axes. Given that the scales are the same on both axes and that the angles that $l_1$ and $l_2$ make with the positive $x$-axis are $A$ and $B$ respectively,

**a** write down the value of $\tan A$ and the value of $\tan B$;

**b** without using your calculator, work out the acute angle between $l_1$ and $l_2$.

**2** Given that $\sin x = \dfrac{1}{\sqrt{5}}$ where $x$ is acute, and that $\cos(x - y) = \sin y$, show that $\tan y = \dfrac{\sqrt{5} + 1}{2}$.

**3** Using $\tan 2\theta = \dfrac{2 \tan \theta}{1 - \tan^2 \theta}$ with an appropriate value of $\theta$,

**a** show that $\tan \dfrac{\pi}{8} = \sqrt{2} - 1$.

**b** Use the result in **a** to find the exact value of $\tan \dfrac{3\pi}{8}$.

**4** In $\triangle ABC$, $AB = 5$ cm and $AC = 4$ cm, $\angle ABC = (\theta - 30)°$ and $\angle ACB = (\theta + 30)°$. Using the sine rule, show that $\tan \theta = 3\sqrt{3}$.

**5** Two of the angles, $A$ and $B$, in $\triangle ABC$ are such that $\tan A = \frac{3}{4}$, $\tan B = \frac{5}{12}$.

**a** Find the exact value of **i** $\sin(A + B)$ **ii** $\tan 2B$

**b** By writing $C$ as $180° - (A + B)$, show that $\cos C = -\frac{33}{65}$.

**6** Show that

**a** $\sec \theta \operatorname{cosec} \theta \equiv 2 \operatorname{cosec} 2\theta$

**b** $\dfrac{1 - \cos 2x}{1 + \cos 2x} \equiv \sec^2 x - 1$

**c** $\cot \theta - 2 \cot 2\theta \equiv \tan \theta$

**d** $\cos^4 2\theta - \sin^4 2\theta \equiv \cos 4\theta$

**e** $\tan\left(\dfrac{\pi}{4} + x\right) - \tan\left(\dfrac{\pi}{4} - x\right) \equiv 2 \tan 2x$

**f** $\sin(x + y)\sin(x - y) \equiv \cos^2 y - \cos^2 x$

**g** $1 + 2 \cos 2\theta + \cos 4\theta \equiv 4 \cos^2 \theta \cos 2\theta$

**7** The angles $x$ and $y$ are acute angles such that $\sin x = \dfrac{2}{\sqrt{5}}$ and $\cos y = \dfrac{3}{\sqrt{10}}$.

**a** Show that $\cos 2x = -\frac{3}{5}$.

**b** Find the value of $\cos 2y$.

**c** Show without using your calculator, that

**i** $\tan(x + y) = 7$ **ii** $x - y = \dfrac{\pi}{4}$

**8** Given that $\sin x \cos y = \frac{1}{2}$ and $\cos x \sin y = \frac{1}{3}$,

**a** show that $\sin(x + y) = 5 \sin(x - y)$.

Given also that $\tan y = k$, express in terms of $k$:

**b** $\tan x$

**c** $\tan 2x$

**9** Solve the following equations in the interval given in brackets:

**a** $\sqrt{3} \sin 2\theta + 2 \sin^2 \theta = 1 \quad \{0 \leqslant \theta \leqslant \pi\}$

**b** $\sin 3\theta \cos 2\theta = \sin 2\theta \cos 3\theta \quad \{0 \leqslant \theta \leqslant 2\pi\}$

**c** $\sin(\theta + 40°) + \sin(\theta + 50°) = 0 \quad \{0 \leqslant \theta \leqslant 360°\}$

**d** $\sin^2 \dfrac{\theta}{2} = 2 \sin \theta \quad \{0 \leqslant \theta \leqslant 360°\}$

**e** $2 \sin\theta = 1 + 3 \cos\theta \quad \{0 \leqslant \theta \leqslant 360°\}$

**f** $\cos 5\theta = \cos 3\theta \quad \{0 \leqslant \theta \leqslant \pi\}$

**g** $\cos 2\theta = 5 \sin\theta \quad \{-\pi \leqslant \theta \leqslant \pi\}$.

**10** The first three terms of an arithmetic series are $\sqrt{3} \cos\theta$, $\sin(\theta - 30°)$ and $\sin \theta$, where $\theta$ is acute. Find the value of $\theta$.

**11** Solve, for $0 \leqslant \theta \leqslant 360°$, $\cos(\theta + 40°) \cos(\theta - 10°) = 0.5$.

**12** Without using calculus, find the maximum and minimum value of the following expressions. In each case give the smallest positive value of $\theta$ at which each occurs.

**a** $\sin \theta \cos 10° - \cos \theta \sin 10°$

**b** $\cos 30° \cos \theta - \sin 30° \sin \theta$

**c** $\sin \theta + \cos \theta$

**13** **a** Express $\sin x - \sqrt{3} \cos x$ in the form $R \sin(x - \alpha)$, with $R > 0$ and $0 < \alpha < 90°$.

**b** Hence sketch the graph of $y = \sin x - \sqrt{3} \cos x$ $\{-360° \leqslant x \leqslant 360°\}$, giving the coordinates of all points of intersection with the axes.

**14** Given that $7 \cos 2\theta + 24 \sin 2\theta \equiv R \cos(2\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \dfrac{\pi}{2}$, find:

**a** the value of $R$ and the value of $\alpha$, to 2 decimal places

**b** the maximum value of $14 \cos^2 \theta + 48 \sin \theta \cos \theta$

**15** **a** Given that $\alpha$ is acute and $\tan \alpha = \frac{3}{4}$, prove that

$$3 \sin(\theta + \alpha) + 4 \cos(\theta + \alpha) \equiv 5 \cos \theta$$

**b** Given that $\sin x = 0.6$ and $\cos x = -0.8$, evaluate $\cos(x + 270)°$ and $\cos(x + 540)°$. E

**16** **a** Without using a calculator, find the values of:

**i** $\sin 40° \cos 10° - \cos 40° \sin 10°$ **ii** $\frac{1}{\sqrt{2}} \cos 15° - \frac{1}{\sqrt{2}} \sin 15°$ **iii** $\frac{1 - \tan 15°}{1 + \tan 15°}$

**b** Find, to 1 decimal place, the values of $x$, $0 \leqslant x \leqslant 360°$, which satisfy the equation

$$2 \sin x = \cos(x - 60)$$

*E*

**17** **a** Prove, by counter example, that the statement

'$\sec(A + B) \equiv \sec A + \sec B$, for all $A$ and $B$'

is false.

**b** Prove that $\tan \theta + \cot \theta \equiv 2 \operatorname{cosec} 2\theta$, $\theta \neq \frac{n\pi}{2}$, $n \in \mathbb{Z}$.

*E*

**18** Using the formula $\cos(A + B) \equiv \cos A \cos B - \sin A \sin B$:

**a** Show that $\cos(A - B) - \cos(A + B) \equiv 2 \sin A \sin B$.

**b** Hence show that $\cos 2x - \cos 4x \equiv 2 \sin 3x \sin x$.

**c** Find all solutions in the range $0 \leqslant x \leqslant \pi$ of the equation

$$\cos 2x - \cos 4x = \sin x$$

giving all your solutions in multiples of $\pi$ radians.

*E*

**19** **a** Given that $\cos(x + 30°) = 3 \cos(x - 30°)$, prove that $\tan x = -\frac{\sqrt{3}}{2}$.

**b** **i** Prove that $\frac{1 - \cos 2\theta}{\sin 2\theta} = \tan \theta$.

**ii** Verify that $\theta = 180°$ is a solution of the equation $\sin 2\theta = 2 - 2 \cos 2\theta$.

**iii** Using the result in part **i**, or otherwise, find the two other solutions, $0 < \theta < 360°$, of the equation $\sin 2\theta = 2 - 2 \cos 2\theta$.

*E*

**20** **a** Express $1.5 \sin 2x + 2 \cos 2x$ in the form $R \sin(2x + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$, giving your values of $R$ and $\alpha$ to 3 decimal places where appropriate.

**b** Express $3 \sin x \cos x + 4 \cos^2 x$ in the form $a \sin 2x + b \cos 2x + c$, where $a$, $b$ and $c$ are constants to be found.

**c** Hence, using your answer to part **a**, deduce the maximum value of $3 \sin x \cos x + 4 \cos^2 x$.

*E*

## Summary of key points

**1** The addition (or compound angle) formulae are

- $\sin(A + B) \equiv \sin A \cos B + \cos A \sin B$ $\quad\quad$ $\sin(A - B) \equiv \sin A \cos B - \cos A \sin B$
- $\cos(A + B) \equiv \cos A \cos B - \sin A \sin B$ $\quad\quad$ $\cos(A - B) \equiv \cos A \cos B + \sin A \sin B$
- $\tan(A + B) \equiv \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$ $\quad\quad$ $\tan(A - B) \equiv \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

**2** The double angle formulae are

- $\sin 2A \equiv 2 \sin A \cos A$
- $\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$
- $\tan 2A \equiv \dfrac{2 \tan A}{1 - \tan^2 A}$

**3** Expressions of the form $a \sin \theta + b \cos \theta$ can be rewritten in terms of a sine only or a cosine only, as follows:

For positive values of $a$ and $b$,

$a \sin \theta \pm b \cos \theta \equiv R \sin(\theta \pm \alpha)$, with $R > 0$ and $0 < \alpha < 90°$,
$a \cos \theta \pm b \sin \theta \equiv R \cos(\theta \mp \alpha)$, with $R > 0$ and $0 < \alpha < 90°$
where $R \cos \alpha = a$, $R \sin \alpha = b$ and $R = \sqrt{a^2 + b^2}$.

Remember you can always use 'the $R$ formula' to solve equations of the form $a \cos \theta + b \sin \theta = c$, where $a$, $b$ and $c$ are constants, but if $c = 0$, the equation reduces to the form $\tan \theta = k$.

**4** Products of sines and/or cosines can be expressed as the sum or difference of sines or cosines, using the formulae:

$2 \sin A \cos B \equiv \sin(A + B) + \sin(A - B)$ $\quad\quad$ $2 \cos A \cos B \equiv \cos(A + B) + \cos(A - B)$

$2 \cos A \sin B \equiv \sin(A + B) - \sin(A - B)$ $\quad\quad$ $2 \sin A \sin B \equiv -[\cos(A + B) - \cos(A - B)]$

**5** Sums or differences of sines or cosines can be expressed as a product of sines and/or cosines, using 'the factor formulae':

$$\sin P + \sin Q \equiv 2 \sin\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right) \qquad \cos P + \cos Q \equiv 2 \cos\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$$

$$\sin P - \sin Q \equiv 2 \cos\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right) \qquad \cos P - \cos Q \equiv -2 \sin\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)$$