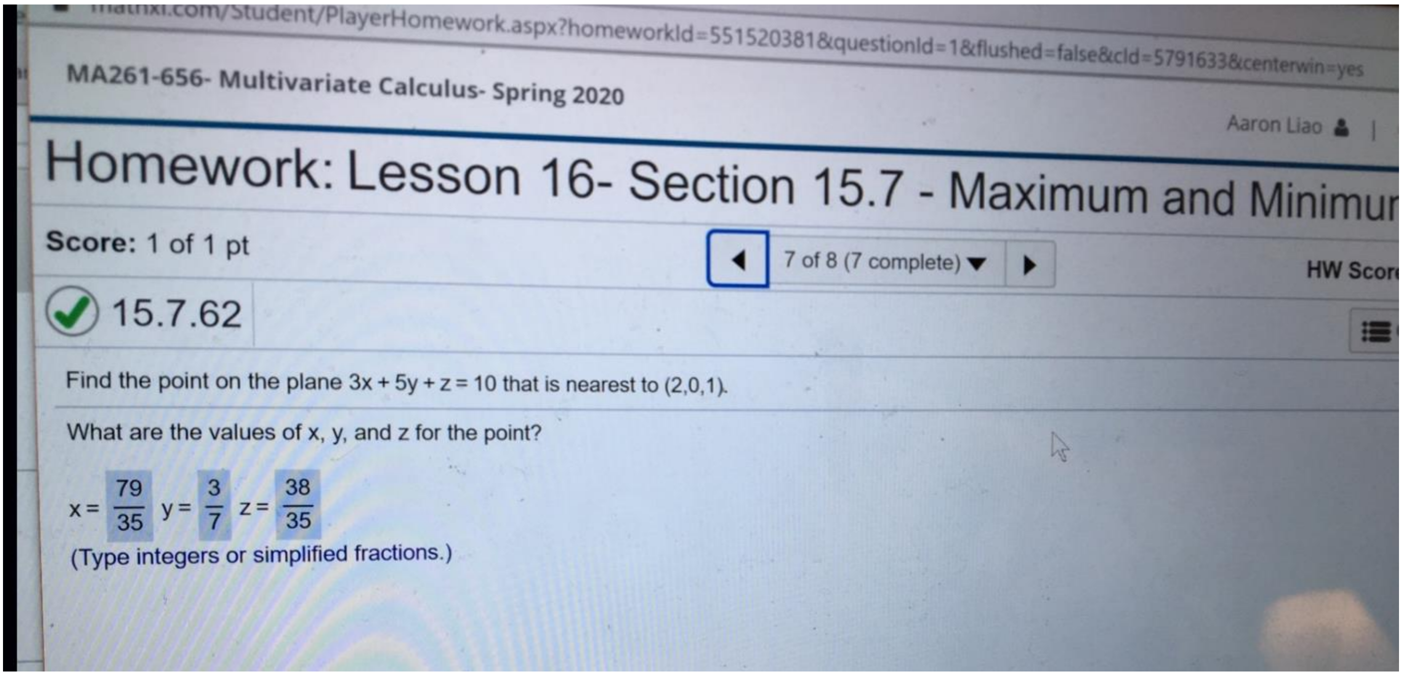
Question



Solution

There are a couple of ways to go about this problem.

The first way is to use Lagrange multipliers. To start with, we want to minimise the distance to a plane. Note that the distance of a point (a, b, c) from (x, y, z) is given by the formula

$$D = \sqrt{(a-x)^2 + (b-y)^2 + (c-z)^2}$$

Hence, in this case, we want to minimise

$$D = \sqrt{(x-2)^2 + y^2 + (z-1)^2}$$

The constraint is the plane: $3x + 5y + z = 10$.

Now we proceed with the Lagrange method: to keep things simple let us take D^2 (to avoid messing with the square roots). Then, $f = (x-2)^2 + y^2 + (z-1)^2$ and $g = 3x + 5y + z - 10$

Note that $f = \lambda g$. Then,

$$(x-2)^2 + y^2 + (z-1)^2 = \lambda(3x + 5y + z - 10)$$

Taking partial derivatives with respect to x,

$$2(x-2) = 3\lambda$$

Doing the same with the other variables,

$$\begin{aligned}&2y = 5\lambda\\&2(z-1) = \lambda\\&(x-2)^2 + y^2 + (z-1)^2 = 3x + 5y + z - 10\end{aligned}$$

We now need to find x, y and z. Then, first find λ. Note that

$$(x-2) = \frac{3}{2}\lambda \rightarrow x = \frac{3}{2}\lambda + 2$$

$$y = \frac{5\lambda}{2}$$
$$(z-1) = \frac{\lambda}{2} \rightarrow z = \frac{\lambda}{2} + 1$$

Then,

$$(x-2)^2 + y^2 + (z-1)^2 = 3x + 5y + z - 10$$
$$\left(\frac{3}{2}\lambda\right)^2 + \frac{25\lambda^2}{4} + \frac{\lambda^2}{4} = 3\left(\frac{3\lambda}{2} + 2\right) + 5\left(\frac{5\lambda}{2}\right) + \frac{\lambda}{2} + 1 - 10$$
$$\frac{35\lambda^2}{4} = \frac{9\lambda}{2} + 6 + 13\lambda - 9$$
$$\frac{35\lambda^2}{4} = \frac{35\lambda}{2} - 3$$
$$\frac{35\lambda^2}{4} = \frac{70\lambda}{4} - \frac{12}{4}$$
$$35\lambda^2 - 70\lambda + 12 = 0$$
$$\lambda = \frac{70 \pm \sqrt{70^2 - 4 * 35 * 12}}{70}$$
$$\lambda = \frac{1}{70}(70 \pm \sqrt{4900 - 1680}$$
$$\lambda = \frac{1}{70}\left(70 \pm \sqrt{3220}\right)$$
$$\lambda = 1 \pm \frac{56.8}{70}$$
$$\lambda = 1.811, 0.19$$

Hence,

$$x = \frac{3}{2}\lambda + 2$$
$$x = \frac{3}{2}(0.189) + 2$$
$$x = \frac{79}{35}$$

Similarly,

$$y = \frac{5\lambda}{2} \rightarrow y = \frac{3}{7}$$
$$z = \frac{0.19}{2} + 1 \rightarrow z = \frac{38}{35}$$

(the values are roughly equivalent to their fractional components)

Another easier alternative is to note that the desired point is always a multiple of the normal vector of the plane. Hence we can say that the desired point will be of the form

$$(2,0,1) + \lambda(3,5,1) = (2 + 3\lambda, 5\lambda, 1 + \lambda)$$

That point is on the plane. Then,

$$3(2 + 3\lambda) + 5(5\lambda) + (1 + \lambda) = 10$$
$$6 + 9\lambda + 25\lambda + 1 + \lambda = 10$$
$$35\lambda = 3$$
$$\lambda = \frac{3}{35}$$

And

$$(x, y, z) = \left(2 + \frac{9}{35}, \frac{3}{7}, 1 + \frac{3}{35}\right)$$