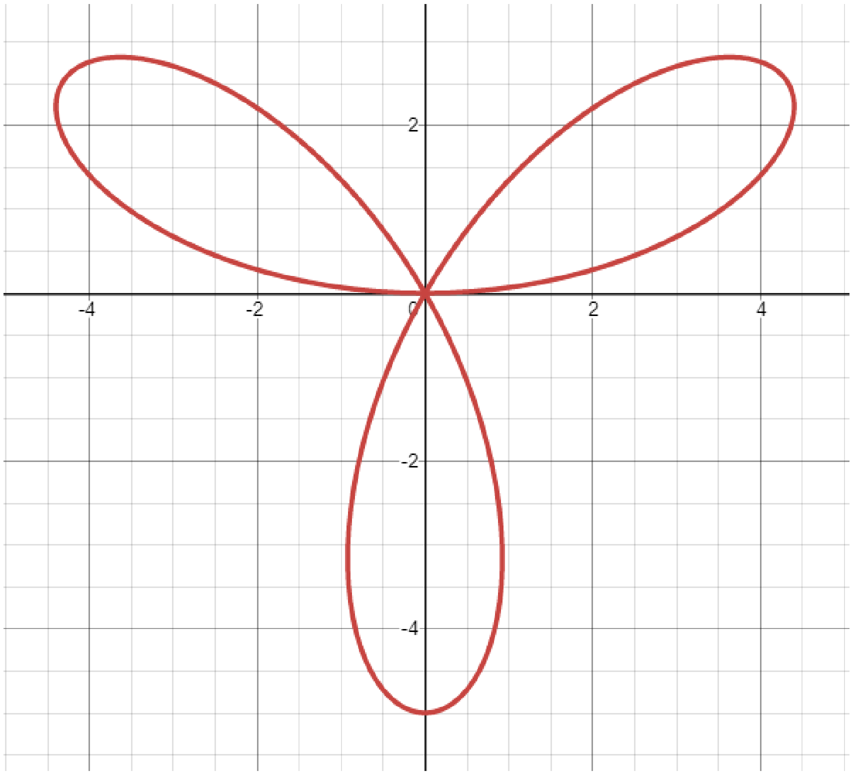
Question

31. The area of one leaf of the three-leaved rose bounded by the graph of $r = 5\sin 3\theta$ is

A. $\frac{5\pi}{6}$ B. $\frac{25\pi}{12}$ C. $\frac{25\pi}{6}$ D. $\frac{5\pi}{3}$ E. $\frac{25\pi}{3}$.

Solution

It helps to think about the graph of such a polar surface:



Note that the curve $r = 5\sin 3\theta$ has extremums at $\theta = 0$ and $\theta = \frac{\pi}{3}$ (for the first leaf).

So, the upper bound for r is the curve, and it goes from 0 to $\frac{\pi}{3}$. Hence, the integral we need to compute is

$$A = \int_0^{\frac{\pi}{3}} \int_0^{5\sin 3\theta} r\, dr\, d\theta$$

Evaluating it,

$$A = \int_0^{\frac{\pi}{3}} \frac{25}{2} \sin^2 3\theta\, d\theta$$

We know that $\cos 2\theta = 1 - 2\sin^2\theta$, or $\sin^2\theta = \frac{1-\cos 2\theta}{2}$. Hence, the surface area of **one** leaf is

$$A = \int_0^{\frac{\pi}{3}} \frac{25}{4}(1 - \cos 6\theta)\, d\theta$$

$$A = \frac{25}{4}\left[\theta - \frac{1}{6}\sin 6\theta\right]_0^{\frac{\pi}{3}}$$
$$A = \frac{25}{4}\left[\frac{\pi}{3}\right]$$
$$A = \frac{25\pi}{12}$$

Hint: This can also be solved using Calculus 2 concepts. Note that the area between two polar curves is (where α and β are the angles)

$$A = \frac{1}{2}\int_{\alpha}^{\beta} |f(\theta)|^2 d\theta$$

In this case, we'll get

$$A = \int_0^{\frac{\pi}{3}} \frac{1}{2}(5\sin 3\theta)^2\, d\theta$$

The result will still be the same.