MA 26100 Exam 1 Sums

Vectors AND Equation of Lines and Planes:

Spring 2015, E1

1. Find $\cos\theta$ where θ is the angle between $\vec{u} \times \vec{v}$ and $\vec{w}$, where $\vec{u} = \vec{i} + 2\vec{j} + \vec{k}$, $\vec{v} = -\vec{i} + \vec{j} + 2\vec{k}$, and $\vec{w} = -2\vec{i} - \vec{j} + 2\vec{k}$.

A. $4/(3\sqrt{26})$

B. $2/\sqrt{27}$

C. $4/(3\sqrt{3})$

D. $3/(\sqrt{27})$

$\boxed{\text{E. } 1/\sqrt{27}}$

$U \times V = \begin{vmatrix} i & j & k \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{vmatrix} = \langle 3, -3, 3 \rangle \qquad |U \times V| = \sqrt{3 \cdot 9} = 3\sqrt{3}$

$|w| = \sqrt{4+1+4} = 3$

$\therefore \cos\theta = \dfrac{\langle 3,-3,3\rangle \cdot \langle -2,-1,2\rangle}{|U \times V|\,|w|} = \dfrac{3}{9\sqrt{3}} = \dfrac{1}{3\sqrt{3}} = \dfrac{1}{\sqrt{27}}$

Spring 2006

1. The area of the triangle with vertices $\overset{A}{(2,0,0)}, \overset{B}{(0,4,0)}, \overset{C}{(0,0,6)}$ is

A. 12

$\boxed{\text{B. } 14}$

C. 16

D. 18

E. 20

$\vec{AB} = \langle -2, 4, 0 \rangle$

$\vec{AC} = \langle -2, 0, 6 \rangle$

$\vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 4 & 0 \\ -2 & 0 & 6 \end{vmatrix} = \langle 24, 12, 8 \rangle$

$\text{Ar of } \Delta = \dfrac{|\vec{AB} \times \vec{AC}|}{2} = \dfrac{\sqrt{24^2 + 12^2 + 8^2}}{2} = \dfrac{\sqrt{784}}{2}$

$= \dfrac{28}{2} = 14$

Find the line of intersection of the plane given by 3x+6y−5z=−3 and the plane given by −2x+7y−z=24

$\vec{n}_1 = \langle 3, 6, -5 \rangle$

$\vec{n}_2 = \langle -2, 7, -1 \rangle$

$\vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} i & j & k \\ 3 & 6 & -5 \\ -2 & 7 & -1 \end{vmatrix}$

$= \langle 29, 13, 33 \rangle$

Assume both plane intersect xy plane, $\therefore z = 0$

$\Rightarrow 3x + 6y = -3$

$\Rightarrow -2x + 7y = 24$

$\Rightarrow x = -5, y = 2$ (Solving Simultaneously)

$\therefore$ Common Point: $(-5, 2, 0)$ $\quad \therefore \underline{\ell(t) = \langle -5, 2, 0 \rangle + t\langle 29, 13, 33 \rangle}$

Fall 2015, E1

PROBLEM 1: The angle between the planes given by the equations

$$x + y = 2 \text{ and } x + y + \sqrt{2}z = \sqrt{6}$$

is

A. $\frac{\pi}{2}$

$\boxed{\text{B. } \frac{\pi}{4}}$

C. $\frac{\pi}{6}$

D. π

E. $\frac{\pi}{3}$

$\vec{n}_1 = \langle 1, 1, 0 \rangle \qquad \vec{n}_2 = \langle 1, 1, \sqrt{2} \rangle$

$|\vec{n}_1| = \sqrt{2} \qquad |\vec{n}_2| = \sqrt{4} = 2$

$\vec{n}_1 \cdot \vec{n}_2 = 1 + 1 + 0 = 2$

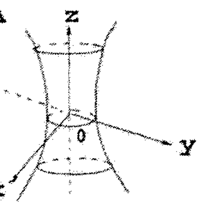
$\therefore \cos\theta = \dfrac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|} = \dfrac{2}{2\sqrt{2}} = \dfrac{1}{\sqrt{2}}$

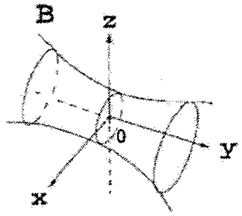
$\therefore \theta = \dfrac{\pi}{4}$

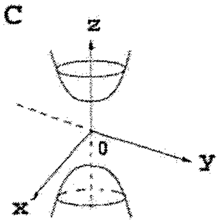
Quadric Surfaces AND Function Of Several Variables (Domain and Contour Maps):

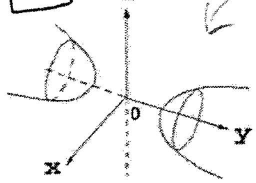
Fall 2007:

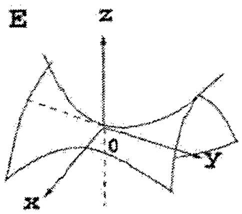
3. The graph of the surface $x^2 - \frac{y^2}{4} + \frac{z^2}{9} = -1$ looks most like :

A


B


C


$\boxed{\text{D}}$


E


$$x^2 - \frac{y^2}{4} + \frac{z^2}{9} = -1$$

$$\Rightarrow -x^2 + \frac{y^2}{4} - \frac{z^2}{9} = 1$$

2-ve signs, All terms squared } Hyperboloid of 2sheets

Oriented along positive variable: y

Fall 2007

9. Find the domain of

$$f(x, y) = \ln\left(\frac{x}{y+2}\right)$$

A. $y \neq -2,\ x > 0$

$\boxed{\text{B. } y > -2,\ x > 0 \text{ or } y < -2,\ x < 0}$

C. $y > -2,\ x > 0$

D. $y > 0,\ x > 0$

E. $y > 0,\ x > 0$ or $y < -2,\ x < 0$

$\frac{x}{y+2} > 0$

Case I:

$x > 0$ and $y + 2 > 0$

$\Rightarrow y > -2$

Case II:

$x < 0$ and $y + 2 < 0$

$\Rightarrow y < -2$

Spring 2006

8. The level curves of $f(x, y) = x^2 - 2x + 4y^2$ include:

k

< 0

$x^2 - 2x + 4y^2 = -1$

$\Rightarrow (x-1)^2 + 4y^2 = 0$

$\Rightarrow (x-1)^2 = -4y^2$

No Solution

0

$x^2 - 2x + 4y^2 = 0$

$\Rightarrow x^2 - 2x + 1 + 4y^2 = 1$

$\Rightarrow (x-1)^2 + 4y^2 = 1$

ellipse

> 0

$x^2 - 2x + 4y^2 = 1$

$\Rightarrow (x-1)^2 + 4y^2 = 2$

ellipse

A. ellipses

B. hyperbolas

C. parabolas

D. two lines

E. both B) and D)

Spring 2001

(7) 3. Which of the following surfaces represents the graph of $f(x, y) = 4x^2 + y^2 - 4$?

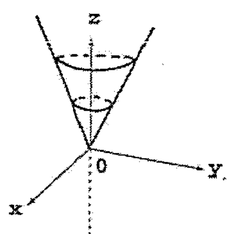
$z = x^2 + y^2 - 4$

$\Rightarrow (z + 4) = x^2 + y^2$

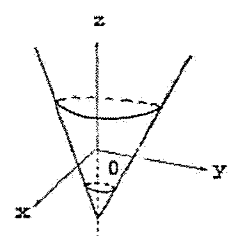
means Surface moved down 4 units

$\left\{ \begin{array}{l} z - \text{linear} \\ x, y - \text{squared and positive} \end{array} \right\}$ Elliptical paraboloid

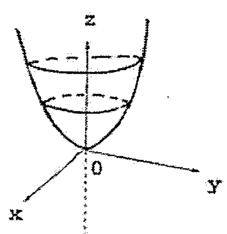
A



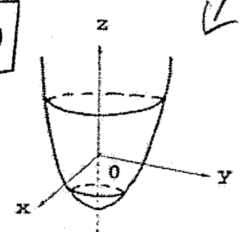
B



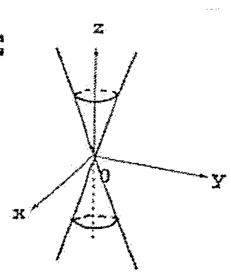
C



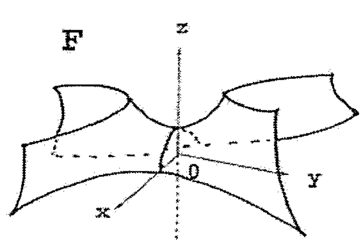
$\boxed{D}$



E



F



Spring 2008:

11. The level curve $f(x,y)=2$ of the function $f(x,y)=x^2-y^2+8x-7$ is

A. a parabola

B. $\boxed{\text{a hyperbola}}$

C. two lines

D. an ellipse but not a circle

E. a circle

$x^2-y^2+8x-7=2$

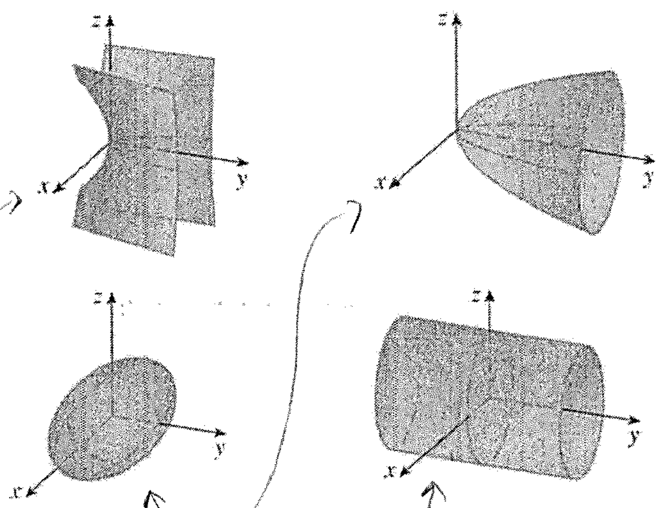
$(x^2+8x+16)-y^2=2+7+16$

$(x+4)^2-y^2=25$

$\therefore$ Hyherbola

Fall 2016

3. Which of the following equations produces a surface that is NOT shown here?



A. $y=x^2-z^2$

B. $y=2x^2+z^2$

C. $x^2+4y^2+9z^2=1$

D. $x^2+2z^2=1$

$\boxed{\text{E. } -x^2+y^2-z^2=1}$ → Hyperboloid of 2 sheet

Derivative and Integration of Vector Functions AND Motion in Space Vel. And Acc.:

Fall 2017

12. Find $f'(1)$, where $f(t) = \mathbf{u}(t)\cdot\mathbf{v}(t)$, $\mathbf{u}(1) = \langle 1,1,1\rangle$, $\mathbf{u}'(1) = \langle 1,2,3\rangle$, and $\mathbf{v}(t) = \langle t, t^2, t^3\rangle$. $v(1)=\langle 1,1,1\rangle$

$v'(t)=\langle 1,2t,3t^2\rangle$

$v'(1)=\langle 1,2,3\rangle$

$f'(t)=u'(t)\cdot v(t)+u(t)\cdot v'(t)$

$\therefore f'(1)=u'(1)\cdot v(1)+u(1)\cdot v'(1)$

$=\langle 1,2,3\rangle\cdot\langle 1,1,1\rangle+\langle 1,1,1\rangle\cdot\langle 1,2,3\rangle$

$=1+2+3+1+2+3$

$=12$

A. 6

B. 14

C. 28

$\boxed{\text{D. } 12}$

E. 24

Spring 1998

5. Find the speed $\|v(2)\|$ where $\vec{a}(t) = -10\vec{k}$ and $v(1) = \vec{i} - \vec{j} + \vec{k}$.

$\vec{a}(t)=\langle 0,0,-10\rangle$

$\vec{v}(t)=\langle 0,0,-10t\rangle+\vec{c}$

$\vec{v}(1)=\langle 1,-1,1\rangle=\langle 0,0,-10\rangle+\vec{c}$

$\Rightarrow \vec{c}=\langle 1,-1,11\rangle$

$\therefore \vec{v}(t)=\langle 1,-1,-10t+11\rangle$

$\vec{v}(2)=\langle 1,-1,-9\rangle$

$|\vec{v}(2)|=\sqrt{1^2+1+9^2}=\sqrt{83}$

A. $\sqrt{84}$

$\boxed{\text{B. } \sqrt{83}}$

C. $\sqrt{82}$

D. 9

E. $\sqrt{80}$

Spring 2000

5) An object has acceleration $\overrightarrow{a(t)} = e^t\vec{i}+2\vec{k}$, initial velocity $\overrightarrow{v(0)} = \vec{i}$, and initial position $\overrightarrow{r(0)} = 2\vec{j}$. Find the position vector of the object at time $t = 1$.

$\vec{a}(t)=\langle e^t,0,2\rangle$

$\vec{v}(t)=\langle e^t,0,2t\rangle+\vec{c}$

$\vec{v}(0)=\langle 1,0,0\rangle=\langle 1,0,0\rangle+\vec{c} \Rightarrow \vec{c}=\langle 0,0,0\rangle$

$\therefore \vec{v}(t)=\langle e^t,0,2t\rangle$

$\vec{r}(t)=\langle e^t,0,t^2\rangle+\vec{d}$

$\vec{r}(0)=\langle 0,2,0\rangle=\langle 1,0,0\rangle+\vec{d} \Rightarrow \vec{d}=\langle -1,2,0\rangle$

$\therefore \vec{r}(t)=\langle e^t-1,2,t^2\rangle$

$\vec{r}(1)=\langle e-1,2,1\rangle$

A. $(e-1)\vec{i}-2\vec{j}+\vec{k}$

$\boxed{\text{B. } (e-1)\vec{i}+2\vec{j}+\vec{k}}$

C. $e\vec{i}-2\vec{j}$

D. $e\vec{i}+2\vec{j}+\vec{k}$

E. $e\vec{i}+2\vec{j}-\vec{k}$

Fall 2001

9. A particle has acceleration $\vec{a}(t) = 6t\vec{j} + 2\vec{k}$. The initial position is $\vec{r}(0) = \vec{j}$ and the initial velocity is $\vec{v}(0) = \vec{i} - \vec{j}$. The distance from the position of the particle at time $t = 1$ to the point $(2, 2, 3)$ is

$\vec{a}(t)=\langle 0,6t,2\rangle$

$\vec{v}(t)=\langle 0,3t^2,2t\rangle+\vec{c}$

$\vec{v}(0)=\langle 1,-1,0\rangle=\langle 0,0,0\rangle+\vec{c} \Rightarrow \vec{c}=\langle 1,-1,0\rangle$

$\therefore \vec{v}(t)=\langle 1,3t^2-1,2t\rangle$

$\vec{r}(t)=\langle t,t^3-t,t^2\rangle+\vec{d}$

$\vec{r}(0)=\langle 0,1,0\rangle=\langle 0,0,0\rangle+\vec{d} \Rightarrow \vec{d}=\langle 0,1,0\rangle$

$\therefore \vec{r}(t)=\langle t,t^3-t+1,t^2\rangle$

$\therefore \vec{r}(1)=\langle 1,1,1\rangle$

$\therefore d=\sqrt{(2-1)^2+(2-1)^2+(3-1)^2}=\sqrt{1+1+4}=\sqrt{6}$

A. 3

B. $\sqrt{7}$

$\boxed{\text{C. } \sqrt{6}}$

D. 4

E. 2

~~Fall 2018~~

FALL 2018

5. A particle has acceleration $\vec{a}(t) = < 0, 2t, \sqrt{2} >$ with an initial velocity of $< 1, 0, 0 >$ at $t = 0$. Find the distance traveled for $0 \le t \le 3$. $\longrightarrow$ distance = Arc Length

$\vec{a}(t) = \langle 0, 2t, \sqrt{2} \rangle$

$\vec{v}(t) = \langle 0, t^2, \sqrt{2}t \rangle + \vec{c}$

$\vec{v}(0) = \langle 1, 0, 0 \rangle = \langle 0, 0, 0 \rangle + \vec{c} \implies \vec{c} = \langle 1, 0, 0 \rangle$

$\therefore \vec{v}(t) = \langle 1, t^2 + \sqrt{2}t \rangle = r'(t)$

$\therefore |r'(t)| = \sqrt{1^2 + (t^2)^2 + (\sqrt{2}t)^2}$

$= \sqrt{1 + t^4 + 2t^2} = \sqrt{(t^2+1)^2} = t^2 + 1$

$\therefore d = \int_0^3 |r'(t)| dt = \int_0^3 t^2 + 1 \, dt$

$= \left[\frac{t^3}{3} + t\right]_0^3 = 9 + 3 - 0 = 12$

A. 3

$\boxed{\text{B. } 12}$

C. $2\cosh(3) - 1$

D. $4\sinh(3)$

E. $\frac{3\pi}{2}$

Arc Length and Curvature:

Fall 1998

4. Compute the length of the curve

$$\vec{r}(t) = \frac{t^3}{3}\vec{i} + t^2\vec{j} + \vec{k}, \quad 0 \le t \le \sqrt{5}$$

$\vec{r}'(t) = \langle t^2, 2t, 0 \rangle$

$|r'(t)| = \sqrt{t^4 + 4t^2} = \sqrt{t^2(t^2+4)} = t\sqrt{t^2+4}$

$L = \int_0^{\sqrt{5}} |r'(t)| dt = \int_0^{\sqrt{5}} t\sqrt{t^2+4} \, dt$

let $u = t^2 + 4$
$\Rightarrow du = 2t\,dt$
$\Rightarrow \frac{du}{2} = t\,dt$

$\Rightarrow \int_{t=0}^{\sqrt{5}} \sqrt{u} \cdot \frac{1}{2} du$

$= \frac{1}{2}\left[\frac{u^{3/2}}{3/2}\right]_{t=0}^{\sqrt{5}}$

$\Rightarrow \frac{1}{2} \cdot \frac{2}{3}\left[(t^2+4)^{3/2}\right]_0^{\sqrt{5}} = \frac{1}{3}(9^{3/2} - 4^{3/2}) = \frac{1}{3}(27 - 8) = \frac{19}{3}$

$\boxed{\text{A. } \frac{19}{3}}$

B. $\frac{5^{3/2} - 8}{3}$

C. 3

D. $5^{3/2}$

E. $\frac{5^{3/2} - 1}{3}$

Fall 1998

5. Find the unit tangent vector to the curve $\vec{r}(t) = 2t^2\vec{i} + t^3\vec{j} + \vec{k}$ at the point $(2, 1, 1)$.

$\vec{r}(t) = \langle 2t^2, t^3, 1 \rangle = \langle 2, 1, 1 \rangle$

$\therefore t^3 = 1 \Rightarrow \boxed{t = 1}$

$r'(t) = \langle 4t, 3t^2, 0 \rangle$

$\vec{r}'(1) = \langle 4, 3, 0 \rangle$

$|r'(1)| = \sqrt{16 + 9 + 0^2} = 5$

$\therefore \vec{T}(1) = \frac{\vec{r}'(1)}{|r'(1)|} = \frac{\langle 4, 3, 0 \rangle}{5}$

A. $\frac{4}{5}\vec{i} - \frac{3}{5}j$

B. $3\vec{i} + 4j$

$\boxed{\text{C. } \frac{4}{5}i + \frac{3}{5}j}$

D. $4\vec{i} + 3\vec{j}$

E. $\frac{2}{5}\vec{i} + \frac{3}{5}j + \frac{1}{5}\vec{k}$

Fall 2018:

4. If $\vec{r}(t) = < 1, 5t^2, 4t >$, find $\kappa(0)$ (i.e., the curvature at t=0).

$\vec{r}\,'(t) = \langle 0, 10t, 4 \rangle$ $\quad |\vec{r}\,'(t)| = \sqrt{100t^2+16}$

$\vec{r}\,''(t) = \langle 0, 10, 0 \rangle$ $\quad |\vec{r}\,'(0)| = \sqrt{0+16} = 4$

$$\vec{r}\,'(t) \times \vec{r}\,''(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 10t & 4 \\ 0 & 10 & 0 \end{vmatrix} = \langle 40, 0, 0 \rangle$$

$\therefore |\vec{r}\,'(t) \times \vec{r}\,''(t)| = \sqrt{40^2} = 40$

$$K(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$$

$$\therefore K(0) = \frac{40}{4^3} = \frac{5}{8}$$

A. 0

B. $\frac{5}{4}$

$\boxed{\text{C. } \frac{5}{8}}$

D. 1

E. $-\frac{5}{4}$

Fall 2001

10. The curvature of the curve defined by the intersection of the cylinder $x^2 + y^2 = 1$ with the plane $y + z = 2$ at $(0, 1, 1)$ is

Parameterize cylinder: $x = \cos t$, $y = \sin t$

$\therefore z = 2 - y = 2 - \sin t$

$\therefore \vec{r}(t) = \langle \cos t, \sin t, 2 - \sin(t) \rangle = \langle 0, 1, 1 \rangle \Rightarrow t = \pi/2$

$\vec{r}\,'(t) = \langle -\sin t, \cos t, -\cos t \rangle$ $\quad \vec{r}\,'(\pi/2) = \langle -1, 0, 0 \rangle$

$\vec{r}\,''(t) = \langle -\cos t, -\sin t, \sin t \rangle$ $\quad \vec{r}\,''(\pi/2) = \langle 0, -1, 1 \rangle$

$\vec{r}\,'(\pi/2) \times \vec{r}\,''(\pi/2) = \langle 0, 1, 1 \rangle$

$$\therefore K(\pi/2) = \frac{|\vec{r}\,'(\pi/2) \times \vec{r}\,''(\pi/2)|}{|\vec{r}\,'(\pi/2)|^3} = \frac{\sqrt{0+1+1}}{\sqrt{1+0+0}} = \sqrt{2}$$

A. 1

B. $\frac{1}{2}$

$\boxed{\text{C. } \sqrt{2}}$

D. $\frac{\sqrt{2}}{2}$

E. 2

Partial Derivatives (Includes ALL Implicit Differentiation) AND Chain Rule:

Fall 2001:

7. Let $f(x, y) = e^{xy} \sin(x^2)$. Then $\frac{\partial^2 f}{\partial x \partial y}(\sqrt{\pi}, 0) =$

$$\frac{\partial f}{\partial y} = x e^{xy} \sin(x^2)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = e^{xy} \sin(x^2) + xy e^{xy} \sin(x^2) + 2x^2 e^{xy} \cos(x^2)$$

$$\therefore \frac{\partial^2 f}{\partial x \partial y}(\sqrt{\pi}, 0) = e^0 \cdot \overbrace{\sin(\pi)}^{0} + 0 + 2\pi e^0 \cdot \cos(\pi) = -2\pi$$

$\boxed{\text{A. } -2\pi}$

B. $-2\sqrt{\pi}$

C. 0

D. π

E. $\sqrt{2\pi}$

Spring 1998

9. Suppose that $\sqrt{x^3y + x^2y^2} = 10$ implicitly defines y as a function of x. Find $\dfrac{dy}{dx}$.

$\Rightarrow x^3y + x^2y^2 = 100$

$\Rightarrow F(x,y) = x^3y + x^2y^2 - 100 = 0$

$\therefore \dfrac{dy}{dx} = \dfrac{-F_x}{F_y} = \dfrac{-(3x^2y + 2xy^2)}{x^3 + 2x^2y}$

A. $\dfrac{3x^2y + 2xy^2}{2\sqrt{x^3y + x^2y^2}}$

B. $\boxed{-\dfrac{3x^2y + 2xy^2}{x^3 + 2x^2y}}$

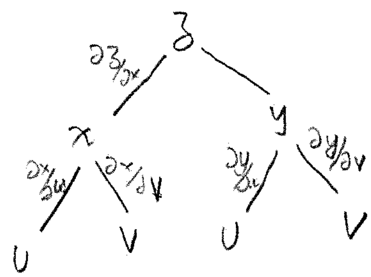
C. $\dfrac{\sqrt{x^3y + x^2y^2} - 10}{x^3 - 3xy^2}$

D. $\dfrac{x^3y + 2x^2y}{3x^2y + 2xy^2}$

E. $\dfrac{x^2y + 2xy^2 - 10}{\sqrt{x^3y + x^2y^2}}$

Spring 1998

8. Let $z = \sqrt{x^2 + y^2}$, $x = uv$, $y = u^2 - v^2$. Find $\dfrac{z_u}{z_v}$.



$z_u = \dfrac{\partial z}{\partial u} = \dfrac{\partial z}{\partial x} \cdot \dfrac{\partial x}{\partial u} + \dfrac{\partial z}{\partial y} \cdot \dfrac{\partial y}{\partial u}$

$= \left(\dfrac{x}{\sqrt{x^2+y^2}}\right) \cdot v + \left(\dfrac{y}{\sqrt{x^2+y^2}}\right) \cdot (2u)$

$z_v = \dfrac{\partial z}{\partial v} = \dfrac{\partial z}{\partial x} \cdot \dfrac{\partial x}{\partial v} + \dfrac{\partial z}{\partial y} \cdot \dfrac{\partial y}{\partial v} = \left(\dfrac{x}{\sqrt{x^2+y^2}}\right) \cdot u + \left(\dfrac{y}{\sqrt{x^2+y^2}}\right) \cdot (-2v)$

$\therefore \dfrac{z_u}{z_v} = \dfrac{xv + 2uy}{xu - 2vy} = \dfrac{uv \cdot v - 2u(v^2 - u^2)}{uv \cdot u - 2v(v^2 - u^2)} = \dfrac{2u^3 - uv^2}{2v^3 - u^2v}$

A. $\dfrac{xv + 2yu}{xu + 2yv}$

B. $\dfrac{u^2v + 2v^3}{\sqrt{x^2 + y^2}}$

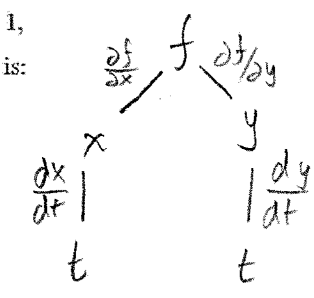
C. $\dfrac{2v^3 + u^2v}{2u^3 + uv^2}$

D. $\dfrac{uv}{u^2 - v^2}$

E. $\boxed{\dfrac{2u^3 - uv^2}{2v^3 - u^2v}}$

Spring 2000

(10) 6) Let $f(x,y) = \ln(x^2 + y^2)$ with $x = g(t)$ and $y = h(t)$. Assuming that $g(0) = 1$, $h(0) = 3$, $g'(0) = 2$, and $h'(0) = 4$, the value of $\dfrac{d}{dt}(f(g(t),\ h(t))$ when $t = 0$ is:



$\dfrac{df}{dt} = \dfrac{\partial f}{\partial x} \cdot \dfrac{dx}{dt} + \dfrac{\partial f}{\partial y} \cdot \dfrac{dy}{dt}$

$= \dfrac{2x}{x^2 + y^2} \cdot g'(t) + \dfrac{2y}{x^2+y^2} \cdot h'(t)$

$= \dfrac{2g(t)}{g^2(t) + h^2(t)} \cdot g'(t) + \dfrac{2y(t)}{g^2(t) + h^2(t)} \cdot h'(t)$

$\therefore \dfrac{df}{dt}(t=0) = \dfrac{2 \cdot 1}{1+9} \cdot 2 + \dfrac{2 \cdot 3}{1+9} \cdot 4 = \dfrac{28}{10} = \dfrac{14}{5}$

A. $\frac{1}{5}$

B. $\frac{2}{5}$

C. $\frac{3}{5}$

D. $\frac{7}{5}$

E. $\boxed{\frac{14}{5}}$

Fall 1999

7. Find a value of a for which the function $z = 4\cos(x + ay)$ satisfies
$$\frac{\partial^2 z}{\partial y^2} = 9\,\frac{\partial^2 z}{\partial x^2}.$$

$$\frac{\partial z}{\partial x} = -4\sin(x+ay) \qquad \frac{\partial z}{\partial x^2} = -4\cos(x+ay)$$

$$\frac{\partial z}{\partial y} = -4a\sin(x+ay) \qquad \frac{\partial z}{\partial x^2} = -4a^2\cos(x+ay)$$

$$\therefore\ \cancel{-}4a^2\cancel{\cos(x+ay)} = 9\cdot(\cancel{-}4\cancel{\cos(x+ay)})$$

$$\Rightarrow a^2 = 9$$

$$a = \pm 3$$

A. $a = 2$

B. $a = 0$

C. $a = \frac{1}{2}$

D. $a = 1$

$\boxed{\text{E. } a = 3}$

Tangent Plane & Linear Approximation:

Fall 1999

12. A right circular cylinder has a radius and altitude that vary with time. At a certain instant the altitude is increasing at 0.5 ft/sec and the radius is decreasing at 0.2 ft/sec. How fast is the volume changing if at this time the radius is 20 feet and the altitude is 60 feet.

$$V = \pi r^2 h \qquad dh = +0.5,\ dr = -0.2\ ;\ r = 20,\ h = 60$$

$$dV = \frac{\partial V}{\partial r}\cdot dr + \frac{\partial V}{\partial h}\cdot dh = (2\pi r)\,dr + (\pi r^2)\,dh$$

$$= (2\pi\cdot 20)\cdot(-0.2) + (\pi\cdot 20^2)(0.5)$$

$$= -480\pi + 200\pi = \boxed{-280\pi\ ft^3/s}$$

Fall 2006

2. The approximate change of $z = \sqrt{1 + x + y^2}$ as (x, y) changes from $(2, 1)$ to $(1.9, 1.2)$ is

$$z_x = \frac{1}{2\sqrt{1+x+y^2}}\bigg|_{(2,1)} = \frac{1}{2(1+2+1)} = \frac{1}{4}$$

$$z_y = \frac{\cancel{2}y}{\cancel{2}\sqrt{1+x+y^2}}\bigg|_{(2,1)} = \frac{1}{2}.$$

$$z(x,y) - z(2,1) = \frac{1}{4}\cdot(1.9-2) + \frac{1}{2}(1.2-1)$$

$$= -\frac{1}{4}\cdot\frac{1}{10} + \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{10}$$

$$= \frac{3}{40}$$

A. $\frac{1}{10}$

B. $\frac{1}{\sqrt{10}}$

$\boxed{\text{C. } \frac{3}{40}}$

D. $-\frac{1}{40}$

E. $-\frac{1}{20}$

Fall 2006

5. Find the equation of the tangent plane to $z = e^{xy}$ at the point $(1, 1, e)$

$f_x = ye^{xy}\big|_{(1,1)} = e$

$f_y = xe^{xy}\big|_{(1,1)} = e$

$z(1,1) = e$

$\therefore z - e = e(x-1) + e(y-1)$

$\Rightarrow z = ex + ey - e - \cancel{e} + \cancel{e}$

A. $z = ex + ey + 1$

B. $z = x+y+e-2$

C. $z = ex + ey + e$

D. $\boxed{z = ex + ey - e}$

E. $z = x + y + 1$

Fall 2015

PROBLEM 7: Consider the function

$$f(x, y, z) = xyz$$

which of the following is true. $\nabla f = (yz, xz, xy)$

× (1) $df = xdx + ydy + zdz$ X $df = yzdx + xzdy + xydz$

× (2) Its linear approximation is the tangent plane. X Tangent space as $w = f(x,y,z)$

✓ (3) If $\Delta x = \Delta y = \Delta z = 0.2$ then the error estimated by using differentials at $(1, 2, 1)$ is 1 $\Delta f = yzdx + xzdy + xydz = 2\cdot 0.2 + 2\cdot 0.2 + 1\cdot 0.2 = 1$ ✓

✓ (4) Its gradient is $\langle yz, xz, xy\rangle$ ✓

✓ (5) Its linear approximation at $(1, 1, 1)$ is $L(x, y, z) = x + y + z - 2$. ✓ $f(1,1,1) = 1$

$L(x,y,z) = 1 + 1\cdot(x-1) + 1\cdot(y-1) + 1\cdot(z-1)$

$= x + y + z - 2$

A. 1,2,3 are true.

B. $\boxed{\text{3,4,5 are true.}}$

C. all are true.

D. none is true.

E. only 4 is true.

Directional Derivatives:

Fall1999

8. Find the maximal directional derivative of

$f(x, y, z) = e^x + e^y + e^{2z}$

at $(1, 1, -1)$. $\nabla f = \langle e^x, e^y, 2e^{2z}\rangle\big|_{(1,1,-1)} = \langle e, e, 2e^{-1}\rangle$

$D_u f_{max} = |\nabla f(1,1,-1)| = \sqrt{e^2 + e^2 + \frac{4}{e^2}} = \sqrt{2e^2 + \frac{4}{e^2}}$

A. $e\sqrt{3 - 2e}$

B. $\boxed{\sqrt{2e^2 + 4e^{-4}}}$

C. $\frac{1}{e}\sqrt{2 - 4e^{-3}}$

D. $\sqrt{2e^2 + e^{-4}}$

E. $\sqrt{e^2 + 2e^{-4}}$

Spring 1998

10. The directional derivative at $(-1,1)$ of $f(x,y) = e^{-\frac{x^2}{2}-\frac{y^3}{3}}$ in the direction of $\vec{a} = -2\vec{i}+\vec{j}$ is

$$\nabla f = \langle (-x)\cdot e^{\frac{-x^2}{2}-\frac{y^3}{3}}, -y^2 \cdot e^{\frac{-x^2}{2}-\frac{y^3}{3}} \rangle$$

$$\nabla f(-1,1) = \langle e^{-5/6}, -e^{-5/6} \rangle$$

$$\vec{a} = \langle -2, 1 \rangle, \quad |\vec{a}| = \sqrt{4+1} = \sqrt{5}$$

$$\hat{a} = \langle \frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \rangle$$

$$\therefore D_{\vec{u}} f(-1,1) = \frac{1}{\sqrt{5}} (-2\cdot e^{-5/6} - 1\cdot e^{-5/6})$$

$$= \frac{-3}{\sqrt{5}} e^{-5/6}$$

A. $\frac{1}{5}e^{-\frac{5}{6}}$

B. $-\frac{1}{\sqrt{5}}\vec{i} - \frac{3}{\sqrt{5}}\vec{j}$

$\boxed{\text{C. } -\frac{3}{\sqrt{5}}e^{-\frac{5}{6}}}$

D. $2e^{\frac{7}{6}}$

E. $-\frac{2}{5}\vec{i} + \frac{1}{\sqrt{5}}e^{-\frac{5}{6}}\vec{j}$

Spring 2009

7. Find all the points on the circle $x^2+y^2=1$ at which the direction of fastest change of the function $f(x,y) = x^2+y^2-10x-8y$ is parallel to $\vec{i}+\vec{j}$.

$$\nabla f = \langle 2x-10, 2y-8 \rangle \qquad \vec{u} = \langle k, k \rangle$$

$$\therefore 2x-10 = k \qquad 2y-8 = k$$

$$\Rightarrow 2x-10 = 2y-8$$

$$\Rightarrow 2x-2y-2 = 0$$

$$\Rightarrow x-y-1 = 0$$

$$\Rightarrow y = x-1 \qquad y=-1,\ y=0$$

$$\therefore x^2+y^2=1 \Rightarrow x^2+(x-1)^2 = 1$$

$$\Rightarrow x^2+x^2+\cancel{1}-2x = \cancel{1}$$

$$\Rightarrow x(2x-2) = 0 \Rightarrow x=0 \text{ or } x=1$$

$\boxed{\text{A. } (1,0) \text{ and } (0,-1)}$

B. $(1,0)$ and $(-1,0)$

C. $(0,1)$ and $(-1,0)$

D. $(-1,0)$ and $(0,-1)$

E. $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$

$\therefore (1,0)$ and $(0,-1)$

Fall 2014

8. **(8 points)** For which direction $\mathbf{u}$ will the directional derivative of $f(x,y) = xy^{-2}$ at the point $(2,1)$ have the value 0?

$$\nabla f = \langle \frac{1}{y^2}, \frac{-2x}{y^3} \rangle \Big|_{(2,1)} = \langle 1, -4 \rangle$$

$$\therefore D_{\vec{u}} f(2,1) = \nabla f \cdot u = 0$$

$$= \langle 1, -4 \rangle \cdot \langle u_1, u_2 \rangle = 0$$

Test all answers, True only for E

A. $\mathbf{u} = \langle 1, -4 \rangle$

B. $\mathbf{u} = \langle 1, 4 \rangle$

C. $\mathbf{u} = \langle \frac{1}{\sqrt{17}}, \frac{4}{\sqrt{17}} \rangle$

D. $\mathbf{u} = \langle \frac{-1}{\sqrt{17}}, \frac{4}{\sqrt{17}} \rangle$

$\boxed{\text{E. } \mathbf{u} = \langle \frac{4}{\sqrt{17}}, \frac{1}{\sqrt{17}} \rangle}$

Local/Absolute Max and Min:

Fall 2013

$f_x = 6x^2 + 6y = 0$ $\quad 6y^2+6y=0$, $y(y+1)=0$, $y=0$ or $y=-1$

$f_y = 6x + 6y = 0 \Rightarrow x = -y$ $\quad \therefore x=0$ or $x=1$

$cc = \{(0,0),(1,-1)\}$

11. **(10 points)** Let $f(x, y) = 2x^3 + 6xy + 3y^2$. Which of the following is true?

A. $(1, -1)$ corresponds to a local maximum and $(0, 0)$ to a saddle point.

$\boxed{\text{B.}}$ $(1, -1)$ corresponds to a local minimum and $(0, 0)$ to a saddle point.

C. $(1, -1)$ corresponds to a saddle point and $(0, 0)$ to a local minimum.

D. both $(1, -1)$ and $(0, 0)$ correspond to saddle points.

E. $(1, -1)$ corresponds to a local minimum and $(0, 0)$ is not a critical point.

	$(0,0)$	$(1,-1)$
$f_{xx}=12x$	0	$12>0$
$f_{yy}=6$	6	6
$f_{xy}=6$	6	6
$D=72x-36$	$-36<0$	$36>0$
	Saddle	local Min

Spring 2000

(12) 2) Find the minimum and maximum values of the function $f(x, y) = x^2 - 4x + y^2 - 2y$ on the disk $x^2 + y^2 \le 20$:

$4-8+1-2$

A. 0 and 32

B. 0 and 40

C. -5 and 32

$\boxed{\text{D. } -5 \text{ and } 40}$

E. 32 and 40

$f_x = 2x - 4 = 0$, $x = 2$ $\qquad f_y = 2y - 2 = 0$, $y = 1$ $\qquad (2, +1)$, $f(2,+1) = \boxed{-5}$

$x = \sqrt{20-y^2}$

$\therefore g(y) = 20 - y^2 + y^2 - 4\sqrt{20-y^2} - 2y$

$g'(y) = \dfrac{-4}{2\sqrt{20-y^2}} \cdot (-2y) - 2 = 0$

$\Rightarrow \dfrac{4y}{\sqrt{20-y^2}} = 2 \Rightarrow \dfrac{16y^2}{20-y^2} = 4$

$\Rightarrow 20y^2 = 80 \rightarrow y^2 = 4 \Rightarrow y = \pm 2$

$\hookrightarrow x = \pm 4, \mp 4$

$(4,-2)(4,2)(-4,2),(-4,-2)$

$f(4,-2) = \boxed{8}$

$f(4,2) = \boxed{0}$

$f(-4,2) = \boxed{32}$

$f(-4,-2) = \boxed{40}$

Fall 2018:

1. The shortest distance from $(1, 1, 0)$ to the plane $x + y + z = 1$ is

$d = \sqrt{(x-1)^2 + (y-1)^2 + z^2}$ $\qquad z = 1 - x - y$

$f(x,y) = d^2 = (x-1)^2 + (y-1)^2 + z^2$

$= (x-1)^2 + (y-1)^2 + (1-x-y)^2$

A. $\sqrt{3}/2$

B. $\sqrt{3}$

C. $\sqrt{3}/4$

$\boxed{\text{D. } \sqrt{3}/3}$

E. 3

$f_x = 2(x-1) - 2(1-x-y) = 0$ $\qquad f_y = 2(y-1) - 2(1-x-y) = 0$

$\Rightarrow 2x + y - 2 = 0$ (1) $\qquad 2y + x - 2 = 0$ (2)

(1) $-$ 2(2) $\Rightarrow$ $2x + y - 2 = 0$

$(-)$ $2x + 4y - 4 = 0$

$-3(y) - 6 = 0$

$\Rightarrow y = 2/3$

from (2) $x = 2 - 2y = 2 - \frac{4}{3} = \frac{2}{3}$

$\therefore z = 1 - x - y = 1 - \frac{2}{3} - \frac{2}{3} = -\frac{1}{3}$

$\therefore d = \sqrt{\left(\frac{2}{3}-1\right)^2 + \left(\frac{2}{3}-1\right)^2 + \left(\frac{-1}{3}\right)^2} = \sqrt{\frac{1}{9} + \frac{1}{9} + \frac{1}{9}} = \sqrt{\frac{3}{9}} = \frac{\sqrt{3}}{3}$

.d ALL critical points for the following function, you do not need the classify them.

$$f = 3y^3 - x^2y^2 + 8y^2 + 4x^2 - 20y + 4^{5^{6^7}}$$

You may need to use determinant formula =] $(16^2 - 4(9)(-20) = 976)$

$f_x = -2xy^2 + 8x = 0$

$\Rightarrow x(-2y^2+8) = 0$

$x = 0$ or $y = \pm 2$

$f_y = 9y^2 - 2x^2y + 16y - 20$

$y = -2 \quad x = \pm 2$

$y = 2 \quad x = \pm\sqrt{12}$

$x = 0 \quad y = \dfrac{-16 \pm \sqrt{976}}{8}$

$\therefore CC = \left\{ \left(0, \dfrac{-16 \pm \sqrt{976}}{8}\right), (\pm 2, -2), (\pm\sqrt{12}, 2) \right\}$

Limits of multivariate functions:

Spring 2000

(10) 2) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{(x^2+y^2)} \cdot (y+2)$ is equal to:

A. 0

B. 1

$\boxed{\text{C. } 2}$

D. 4

E. Does not exist.

$\Rightarrow \displaystyle \lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{(x^2+y^2)} \cdot \lim_{(x,y)\to(0,0)} y+2$

let $z = x^2+y^2$ $\qquad$ 2

$\Rightarrow \displaystyle \lim_{z\to 0} \frac{\sin z}{z} \left[\frac{0}{0}\right] \to \lim_{z\to 0} \frac{\cos z}{1} = 1 \to 1 \cdot 2 = \boxed{2}$

Fall 1999

6. If $f(x,y) = \dfrac{3x^2+yx}{x^2+y^2}$, $(x,y) \neq (0,0)$, let ℓ be the limit of $f(x,y)$ as $(x,y) \to (0,0)$ along the y-axis, and let m be the limit of $f(x,y)$ as $(x,y) \to (0,0)$ along the line $y = x$. Then

$x = 0$

$\displaystyle \lim_{(0,y)\to(0,0)} \frac{0}{y^2} = 0$

$y = x$

$\displaystyle \lim_{(x,x)\to 0,0} \frac{3x^2+x^2}{x^2+x^2} = 2$

A. $\ell = 3, \quad m = 2$

$\boxed{\text{B. } \ell = 0, \quad m = 2}$

C. $\ell = 0, \quad m = \frac{3}{2}$

D. $\ell = 3, \quad m = 3$

E. $\ell = \frac{1}{2}, \quad m = \frac{1}{2}$

Spring 2014

1. Evaluate the limit if it exists

$$\lim_{(x,y)\to(0,0)} \frac{x^4 - y^4 + xy}{x^2 + y^2}$$

Along $y=0$

$$\lim_{(x,0)\to(0,0)} \frac{x^4}{x^2} = \lim_{x\to 0} \frac{4x^3}{2x} = \lim_{x\to 0} \frac{12x^2}{2} = 0$$

Along $y=x$

$$\lim_{(x,x)\to(0,0)} \frac{x^4 - \cancel{x^4} + x^2}{x^2+x^2} = \lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2}$$

$\neq$ } DNE

A. −1
B. 0
C. 1
D. 2
E. The limit does not exist

Fall 2007

8. Evaluate

$$\lim_{(x,y)\to(2,0)} x^2 e^{-y^2}$$

$$\lim_{(x,y)\to 2,0} \frac{x^2}{e^{y^2}} = \frac{4}{e^0} = 4$$

A. −4

B. 4

C. 0

D. e^{-4}

E. does not exist

Fall 2018

8. If

$$\lim_{(x,y)\to(0,0)} \frac{x^4 - 3a(x^2+y^2) - y^4}{x^2+y^2} = 12,$$

then the number a must be equal to

$$\frac{x^4 - 3a(x^2+y^2) - y^4}{x^2+y^2} = \frac{x^4-y^4}{x^2+y^2} - \frac{3a(x^2+y^2)}{x^2+y^2} = (x^2-y^2) - 3a$$

$$\therefore \lim_{(x,y)\to 0,0} (x^2-y^2) - 3a = -3a = 12$$

$$\Rightarrow a = -4$$

A. 4

B. 6

C. 12

D. −4

E. 3