Leaderboard’s solution for MA261 Exam 1

Notes:

- Those are just my solutions and thoughts about each question.
- If there’s an error or issue, please let me know and I’ll try to fix it as soon as possible.

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Answer 1

Solution

Make use of the *alternative curvature formula*:

$$\kappa = \frac{r'(t) \times r''(t)}{r'(t)^3}$$

$$r'(t) = < 2\cos t\,, 0, -2\sin t >$$
$$r''(t) =< -2\sin t\,, 0, -2\cos t >$$

Then, applying cross product,

$$r'(t) \times r''(t) = \begin{vmatrix} i & j & k \\ 2\cos t & 0 & -2\sin t \\ -2\sin t & 0 & -2\cos t \end{vmatrix}$$
$$= |j(-4\cos^2 t - 4\sin^2 t)| \to 4$$

and

$$|r'(t)| = \sqrt{4\cos^2 t + 4\sin^2 t} = 2$$

Hence,

$$\kappa = \frac{r'(t) \times r''(t)}{r'(t)^3} = \frac{4}{2^3} = \frac{1}{2}$$

Notes

There is no need to substitute the value for t; as we can see, the curvature remains the same.

Other than that, this is a straightforward question.

Answer 2

Solution

Use the formula to find the arc-length:

$$L = \int_a^b \sqrt{|< x'(t), y'(t), z'(t) >|}\, dt$$

Hence, in this case, we have

$$L = \int_0^{2\pi} \sqrt{|< \cos t - (\cos t - t\sin t), -\sin t + (\sin t + t\cos t) >|}\, dt$$
$$L = \int_0^{2\pi} \sqrt{|< t\sin t\,, t\cos t >|}$$
$$L = \int_0^{2\pi} t\, dt \to L = \left[\frac{t^2}{2}\right]_0^{2\pi} \to L = 2\pi^2$$

Notes

Another simple and straightforward question.

Answer 3

Solution

Due to the simplicity of this problem, it suffices to work purely in the $x - y$ coordinate frame.

We have

$$a = < 0, -g >$$
$$v = < 0,19.6 - gt >$$

The time taken for the ball to reach its highest point is when the vertical velocity vector is zero. Then, solving for t, we get $19.6 - gt \rightarrow t = 2$ sec.

But what about the distance? We integrate the velocity vector:

$$s = < 0,19.6t - \frac{gt^2}{2} >$$

We already found $t = 2$. Hence substitute it in the displacement vector to get the final answer:

$$19.6t - \frac{gt^2}{2} = 39.2 - 19.6 = 19.6\,m$$

Notes

It is possible to solve this purely using high school physics knowledge. Either the projectile motion formula (with $\theta = 90°$)

$$H = \frac{u^2 \sin^2 \theta}{2g}$$

Or even

$$v^2 - u^2 = 2gS$$
$$\rightarrow S = \frac{u^2}{2g}$$

Both (or alternatively the traditional Calculus 3 method described above) will work.

Answer 4

Solution

Find the velocity and acceleration vectors by differentiating $\boldsymbol{r}(t)$:

$$\boldsymbol{v}(t) = \boldsymbol{r}'(t) = < -\sin t\,, t, -\cos t >$$
$$\boldsymbol{a}(t) = \boldsymbol{r}''(t) = < -\cos t\,, 1, \sin t >$$

Then find the angle between the two vectors:

$$\cos\theta = \frac{v(t).\,a(t)}{|v(t)||a(t)|}$$
$$\cos\theta = \frac{\sin t \cos t + t - \cos t \sin t}{\sqrt{\sin^2 t + t^2 + \cos^2 t}\ \sqrt{\cos^2 t + 1 + \sin^2 t}}$$
$$\cos\theta = \frac{t}{\sqrt{1 + t^2}\sqrt{2}}$$

At $t = 1$,

$$\cos\theta = \frac{1}{\sqrt{2}\sqrt{2}}$$
$$\theta = \cos^{-1}\left(\frac{1}{2}\right)$$
$$\theta = \frac{\pi}{3}$$

Notes

This question looked confusing from my eyes (how can “angle between velocity and acceleration” make sense?) but isn’t hard once the idea is understood (all that they want is the angle; couldn’t that just be written as two arbitrary vectors instead?).

Answer 5

Solution

Complete the square:

$$\begin{aligned}
&2x^2 + 3z^2 = 4x + 2y^2 \\
&2x^2 - 4x - 2y^2 = -3z^2 \\
&2(x^2 - 2x) - 2y^2 = -3z^2 \\
&2((x-1)^2 - 1) - 2y^2 = -3z^2 \\
&2(x-1)^2 - 2 - 2y^2 = 3z^2 \\
&2(x-1)^2 - 2y^2 + 3z^2 = 2 \\
&(x-1)^2 - y^2 + \frac{3}{2}z^2 = 1
\end{aligned}$$

This is of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, which is a hyperboloid of one sheet.

Notes

The hint was unusual and made it obvious what to do.

And, the most common mistake I would expect is to mix up one sheet with a hyperboloid of *two* sheet. Note that the signs are flipped in that case!

Answer 6

Solution

Find the derivatives:

$$f_x'(x,y) = \frac{2x}{x^2 + y^4 + 2}$$
$$f_{xy}'(x,y) = \frac{\delta f}{\delta y}\left(\frac{\delta f}{\delta x} f(x,y)\right) = -\frac{2x(4y^3)}{(x^2 + y^4 + 2)^2}$$
$$f_{xy}'(2,1) = -\frac{2 * 2 * 4}{(4 + 1 + 2)^2} = -\frac{16}{49}$$

Notes

If you did $f_{yx}'(x,y)$ instead, you’ll get the same result (Fubini’s theorem holds here).

Otherwise, it’s a straightforward question.

Answer 7

Solution

First note that the direction of the steepest *ascent* is the gradient of the function. Hence

$$\nabla f = <\frac{\delta f}{\delta x}, \frac{\delta f}{\delta y}>$$
$$\nabla f = < y(e^{xy} + xye^{xy}), x(e^{xy} + xye^{xy}) >$$

Evaluating ∇f at the indicated point, we get $< 3(e^6 + 6e^6), 2(e^6 + 6e^6) >$. Hence, $< 3,2 >$ is a vector in the *direction* of the steepest *ascent*.

Then what about descent? It's in the opposite direction to that of ascent, so hence the vector becomes $< -3, -2 >$.

Notes

This question was a bit trickier for many, partially because of the red herrings possible. It was possible to forget that the question asked for *descent* rather than *ascent*; alternatively, quite a few seemed to pick $< 2,3 >$ thinking that switching the values would change the direction. This isn't correct; $< 2,3 >$ is a completely different direction.

Answer 8

Solution

This question can be solved with intuition. The function $d(x, y) = \sqrt{(x-2)^2 + (y-2)^2 + 4}$ is strictly increasing on both x and y due to the square. This means that the maximum value automatically becomes ∞ on $\mathbb{R}^2$, and the minimum value of $d(x, y)$ is when $(x-2)^2 + (y-2)^2$ is minimum.

Now when the domain is brought into the picture, the maximum value immediately comes into the picture: that's just $(-2, -1)$ as that's when $(x-2)^2 + (y-2)^2$ is maximised. Hence, $d(x, y) = \sqrt{9 + 16 + 4} = \sqrt{29}$, and that is the global maximum value.

Similarly the global minimum value can be found by taking the value closest to the actual minimum (2,2) of $d(x, y)$ that is still valid in the domain. This occurs (again) at the boundary of $d(x, y)$ (i.e, at (2,1)), which is $\sqrt{0 + 1 + 4} = \sqrt{5}$.

Hence, we have an absolute maximum value of $\sqrt{29}$ and an absolute minimum value of $\sqrt{5}$.

Notes

The structure of the function means that we do not have to go through the (relatively) tedious process of checking the partial derivatives and finding the Hessian matrix; in fact, if you were to do *only* that, you may get the incorrect coordinate (2,2) (which is not even valid).

Answer 9

Solution

For this one, our only option is to take the partial derivates as usual:

$$f'_x = y^4 - 1 - x$$

$$f'_y = 4xy^3$$

Finding the critical points:

$4xy^3 = 0$ implies that $(0,0)$ is a critical point.

Also, $y^4 - 1 = x \to 4(y^4 - 1)y^3 = 0 \to y = \pm 1$. At those points, $x = 0$ as well.

Hence the critical points are $(0,0), (0,-1), (0,1)$.

Now we classify them. First find f_{xx}' and f_{yy}':

$f_{xx}' = -1$ and $f_{yy}' = 12xy^2$. Then take the Hessian:

$$H = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} \to H = \begin{vmatrix} -1 & 4y^3 \\ 4y^3 & 12xy^2 \end{vmatrix}$$

But for all the critical points, $x = 0$. Hence H simplifies to

$$H = \begin{vmatrix} -1 & 4y^3 \\ 4y^3 & 0 \end{vmatrix}$$
$$H = -16y^6$$

This means that we have one indeterminate point (when $y = 0, H = 0$) and two saddle points (when $y = \pm 1$).

Notes

The options in this question had a flaw in it (admitted by the question-setter) – option (d) was also possible. The indeterminate point turned out to be a maximum, and this was caught by a student of the setter (can also be verified though a graph). Hence both options were accepted as correct.

Answer 10

Solution

Take cross-product of the normals:

$$\begin{vmatrix} i & j & k \\ 1 & -1 & 2 \\ 3 & 0 & 2 \end{vmatrix} = -2i + 4j + 3k$$

Now, the equation of the plane through the point (x_0, y_0, z_0) and perpendicular to (a, b, c) is given by

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Substituting, we get

$$-2(x - 0) + 4(y - 1) + 3(z - 2) = 0$$
$$-2x + 4y - 4 + 3z - 6 = 0$$
$$-2x + 4y + 3z = 10$$

Notes

Similar to past and homework questions, straightforward.

Answer 11

Solution

Using the fact that $\lim_{x\to 0}\frac{\sin(x)}{x} = 1$, it follows that (for the two-variable case)

$$\lim_{(x,y)\to(0,0)}\frac{\sin(x+y)}{x+y} = 1$$

So, for this question, divide the numerator and the denominator by $4x^2 + 8y^2$:

$$\lim_{(x,y)\to(0,0)}\frac{\frac{\sin(4x^2+8y^2)}{4x^2+8y^2}}{\frac{x^2+2y^2}{4x^2+8y^2}}$$

$$= \lim_{(x,y)\to(0,0)}\frac{1}{\frac{x^2+2y^2}{4(x^2+2y^2)}}$$

$$\to \frac{1}{\frac{1}{4}} = 4$$

Notes

If one did not recall the above formula, the question could still be solved. The idea of applying L'Hopital's comes up, but this only works in the single-variable case. So in that case, it's necessary to convert to polar coordinates ($x = r\cos\theta\,, y = r\sin\theta$) and then continue.