MA261 Lesson 52

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Leaderboard

1) $\cancel{\sqrt{\frac{f^2}{x}}}$

$f(x,y,z) = x^2$ $\qquad (z_x = x^2, z_y = 1, z_z = 0)$

0.6

$z_x = 2x \Rightarrow \sqrt{z_x^2 + z_y^2 + z_z^2}$

$= \sqrt{4x^2 + 1}$

$$\int_0^2 \int_0^{\sqrt{15}/2} x\sqrt{4x^2+1}\, dx\, dz$$

let $4x^2 + 1 = t \Rightarrow 8x\, dx = dt$

$dx = \frac{dt}{8x}$

$$\frac{1}{8}\int_0^2 \int_0 \left[\frac{2}{3}\left(4x^2+1\right)^{3/2}\right]_0^{\sqrt{15}/2} dz$$

$$\frac{1}{4}2\left[\frac{1}{6}\left(16^{3/2} - 1^{3/2}\right)\right] = \frac{1}{6}\left[63\right] = \cancel{\frac{31}{3}}\ \frac{21}{2}$$

2)

$$x + y + z = 3$$

$$z = 3 - x - y$$

$$z_x = -1 \quad z_y = -1$$

$$\therefore \sqrt{z_x^2 + z_y^2 + z_z^2} = \sqrt{3}$$

$$\int_0^1 \int_0^1 4\sqrt{3}\,(3 - y - x)\, dy\, dx$$

$$= 4\sqrt{3} \int_0^1 \int_0^1 (3 - y - x)\, dy\, dx$$

$$= 4\sqrt{3} \int_0^1 \left[3y - \frac{y^2}{2} - xy \right]_0^1 dx$$

$$4\sqrt{3} \int_0^1 \left[3 - \frac{1}{2} - x \right] dx$$

$$4\sqrt{3} \int_0^1 \frac{5}{2} - x\, dx = 4\sqrt{3} \left[\frac{5x}{2} - \frac{x^2}{2} \right]_0^1$$

$$= 4\sqrt{3} \left[5/2 - 1/2 \right] = 2 \cdot 4\sqrt{3}$$

$$8\sqrt{3}$$

3) $z^2 = x^2 + y^2 \Rightarrow \langle x^2, y^2, \sqrt{x^2+y^2} \rangle$

$\cancel{f_z = 2x \quad f_y = 2y \quad f_x = 2x}$

$z = \sqrt{x^2+y^2} \Rightarrow f_x = \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}} \quad f_y = \frac{y}{\sqrt{x^2+y^2}} \quad f_z = 1$

$\therefore \sqrt{f_x^2 + f_y^2 + f_z^2} = \sqrt{1 + \frac{y^2+x^2}{x^2+y^2}} = \sqrt{2}$

$= \sqrt{\cancel{2x^2+2y^2}}$

$\int_0^{2\pi}\int_0^{5} \sqrt{2}\, r \, dr\, d\theta = \sqrt{2}\left(\frac{25}{\cancel{2}}\right) \times \cancel{2}\pi$

$= \underline{25\sqrt{2}\pi}$

4) $z = 6x^2$

$f_x = 12x \quad f_y = 0 \quad f_z = 1$

$\int_0^2\int_{-2}^{2} \sqrt{(12x)^2 + 1} \; dx\, dy$

Evaluating $\int \sqrt{144x^2+1}$ can be done using partial

fractions :

$$I = x\sqrt{144x^2+1} - \int \left[\frac{\cancel{288}\,144x}{\cancel{2}\sqrt{144x^2+1}} \cdot x \right] dx$$

$$= x\sqrt{144x^2+1} - \int \frac{144x^2}{\sqrt{144x^2+1}}\, dx$$

$$x\sqrt{144x^2+1} - \int \frac{144x^2+1-1}{\sqrt{144x^2+1}}\, dx$$

$$x\sqrt{144x^2+1} - \int I + \frac{1}{\sqrt{144x^2+1}}\, dx$$

$$\sqrt{a^2+x^2} \to \ln\left| x + \sqrt{a^2+x^2} \right|$$

$$2I = x\sqrt{144x^2+1} + \ln\left| 12x\sqrt{144x^2+1} \right| + C$$

(~~Cost ver~~ answer using symbolab is

5) $z = 2(x^2+y^2)$

$z_x = 4x \quad z_y = 4y \quad z_z = 1$

$$\int_0^{2\pi}\iint \sqrt{z_x^2+z_y^2+z_z^2}$$

$$\sqrt{\cancel{4x^2}\ 16(x^2+y^2)+1}$$

$z = \cancel{8}\,2r^2$

$z = 0 \rightarrow r = 0$

$z = 8 \rightarrow r = 2$

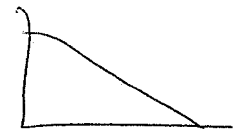
$$\int_0^{2\pi}\int_0^2 r\sqrt{16r^2+1}\,dr\,d\theta$$

$$2\pi\int_0^2 r\sqrt{16r^2+1}\,dr = \frac{\cancel{2}\pi}{\cancel{32}\ \cancel{16}\ 8}\left[\frac{\cancel{2}^{1}}{3}(16r^2+1)^{3/2}\right]$$

$$\frac{\pi}{24}\left[65^{3/2}-1\right]$$

6)

$z_x = -1 \quad z_y = -1 \quad z_z = 1$



$6 - x - y \Rightarrow \cancel{x}$

$$\int_0^6 \int_0^{6-x} \sqrt{3} \cdot 3xy \, dy \, dx$$

$$= 3\sqrt{3} \int_0^6 \left[x \frac{y^2}{2} \right]_0^{6-x} dx$$

$$\frac{3\sqrt{3}}{2} \int_0^6 x(6-x)^2 \, dx$$

$$\frac{3\sqrt{3}}{2} \int_0^6 x(36 - 12x + x^2) \, dx \quad \frac{3\sqrt{3}}{2} \int_0^6 36x - 12x^2 + x^3 \, dx$$

$$\frac{3\sqrt{3}}{2} \left[18x^2 - 4x^3 + \frac{x^4}{4} \right]_0^6$$

$$\frac{3\sqrt{3}}{2} \left(18 \cdot 36 - 4 \cdot 6^3 + \frac{6^4}{4} \right)$$

7)

$z = 8 - 2x - 8y$

$z_x = -2 \quad z_y = -8 \quad z_z = 1$

$\sqrt{64 + 4 + 1} = \sqrt{69}$ $\quad 8y + 2x = 8$

$y = \frac{8-2x}{8} \quad 1 - \frac{x}{4}$

$$\int_0^4 \int_0^{1-x/4} e^{8-2x-8y} \, dy \, dx$$

$$= \frac{-\sqrt{69}}{8} \int_0^4 \left[e^{8-2x-8y} \right]_0^{1-x/4} dx$$

$$\frac{-\sqrt{69}}{8} \int_0^4 \left[e^{8-2x-8\left(1-\frac{x}{4}\right)} - e^{8-2x} \right] dx$$

$$\frac{-\sqrt{69}}{8} \int_0^4 \left[e^{\cancel{8-2x-8+2x}} - e^{8-2x} \right] dx$$

$1 - e^{8-2x}$

$$\frac{-1}{8} \int_0^4 1 - e^{8-2x} \, dx = \frac{-\sqrt{69}}{8} \left[x + \frac{1}{2} e^{8-2x} \right]_0^4 \quad \frac{e^8}{16} - \frac{9}{16}$$

$$\frac{-\sqrt{69}}{8} \left[4 + \frac{1}{2} - \frac{e^8}{2} \right] = \cancel{\frac{e^8}{16}} - \frac{1}{2}$$

8) $z = 7 - 6x - 2y$

$\cancel{t_x} = $

$t_x = -6 \quad t_y = -2 \quad t_z = 1$

$$\sqrt{t_x^2 + t_y^2 + t_z^2} = \sqrt{41}$$

$$\rightarrow \int_{-1}^{1}\int_{-1}^{1} e^{-2}\sqrt{41}\,dy\,dx$$

$$\int_{-1}^{1}\int_{-1}^{1} \sqrt{41}\,dy\,dx$$

$$= \int_{-1}^{1}\int_{-1}^{1} e^{6x+2y-7}\,dy\,dx = \frac{1}{2}\int_{-1}^{1}\left[e^{6x-5} - e^{6x-9}\right]dx$$

$$4\sqrt{41}$$

$$\frac{1}{2}\times\frac{1}{6}\left[e^{6x-5} - e^{6x-9}\right]_{-1}^{1}$$

$$\cancel{\frac{1}{12}\left[e^{1} - 2e^{-15}\right]}$$

$$\frac{1}{48}\left[e - e^{-3} - e^{-11} + e^{-15}\right]$$

$$\frac{1}{\cancel{48\sqrt{41}}}\left[e - e^{15}\right]$$