Leaderboard's solution for MA261 Fall '19 Exam 2

Notes:

- Those are just my solutions and thoughts about each question.
- If there's an error or issue, please let me know and I'll try to fix it as soon as possible.
- I did **not** get every question correctly in the exam. For those questions, I'll also explain where I went wrong (as much as possible), as it is likely that others made the same mistake I did.
- I had the **GREEN** version.

Version 1.1
Compiled on 20/11/2019

Changelog

V1.1 (20/11/19) – Updated Q2 analysis as two options were the same

V1.0 (9/11/19) – Initial version

Answer 1

This is a classic Lagrange multipliers problem. To do this, we define f and g as usual:

$$f(x, y) = 8x - 6y$$
$$\lambda(x, y) = (x - 1)^2 + y^2 - 1$$

Then find the partial derivatives as usual:

$$\frac{\delta f}{\delta x} = \frac{\delta \lambda}{\delta x}$$
$$\rightarrow 8 = 2\lambda(x - 1)$$
$$\rightarrow 4 = \lambda(x - 1)$$
$$\frac{\delta f}{\delta y} = \frac{\delta \lambda}{\delta y}$$
$$\rightarrow -6 = 2\lambda y$$
$$-3 = \lambda y$$

Set either variable to one side:

$$-\frac{4}{3} = \frac{x - 1}{y}$$
$$-4y = 3x - 3$$
$$4y = 3 - 3x$$
$$y = \frac{3(1 - x)}{4}$$

Use the constraint to find x and y:

$$(x - 1)^2 + y^2 = 1$$
$$(x - 1)^2 + \frac{9}{16}(x - 1)^2 = 1$$
$$\frac{25}{16}(x - 1)^2 = 1$$
$$(x - 1)^2 = \frac{16}{25}$$
$$(x - 1) = \pm\frac{4}{5}$$
$$x = 1 \pm \frac{4}{5} \rightarrow x = \frac{1}{5}, \frac{9}{5}$$

Find y:

$$y = \frac{3}{4}(1 - x)$$
$$x = \frac{1}{5} => y = \frac{3}{4}\left(1 - \frac{1}{5}\right) \rightarrow y = \frac{3}{4}\left(\frac{4}{5}\right) = \frac{3}{5}$$
$$x = \frac{4}{5} => y = \frac{3}{4}\left(1 - \frac{9}{5}\right) \rightarrow y = \frac{3}{4}\left(-\frac{4}{5}\right) = -\frac{3}{5}$$

Now, which value of y should be picked? Note that the function to maximize is $8x - 6y$, which means that we need a positive x value and a negative y value to get the highest value. Hence, the maximum value occurs when $(x, y) = \left(\frac{9}{5}, -\frac{3}{5}\right)$, and the maximum value is $\left(8 * \frac{9}{5} + 6 * \frac{3}{5}\right) = \frac{90}{5} = 18$.

Notes

If you took $y = \frac{3}{5}$ (+ve instead of -ve), you'll hit a red herring of 10. The *minimum* value (-2) was also a trap, and I *nearly* fell for it before re-reading the question.

Also, as we're dealing with only two variables, Lagrange multipliers isn't needed to work out this problem. It is possible to do this using elementary Calculus 1 knowledge, and that's actually how I worked it out during the exam:

We know that $f(x, y) = 8x - 6y$ is bounded by $(x-1)^2 + y^2 = 1$. Then, we can reduce the function to that of one variable by directly making use of the constraint:

$$f(x) = 8x - 6\left(\sqrt{1-(x-1)^2}\right)$$

Then differentiate:

$$\frac{df}{dx} = 8 - \frac{6\left(-2(x-1)\right)}{2(\sqrt{1-(x-1)^2}}$$
$$\frac{df}{dx} = 8 - \frac{6(1-x)}{\sqrt{1-(x-1)^2}}$$

Setting the derivative to zero,

$$8 = \frac{6(1-x)}{\sqrt{1-(x-1)^2}}$$

And the process continues much the same way as you'll do in the Lagrange method. Which one to use is a matter of taste (but obviously this method fails with three variables).

Answer 2

The region to integrate is purely rectangular, and hence it does not matter which order the integration is performed. So the double integral is

$$\int_0^4 \int_1^2 \frac{x^2 y}{2+x^3} dxdy$$

Integrating over x first,

$$\int_0^4 y \int_1^2 \frac{x^2}{2+x^3} dx\, dy$$

Let $2 + x^3 = t$. Then $3x^2dx = dt \rightarrow dx = 1/3x^2dt$. Then,

$$\int_1^2 \frac{x^2}{2+x^3} dx = \frac{1}{3}\int_1^2 \frac{dt}{t} = \frac{1}{3}[\ln(2+x^3)]_1^2$$
$$= \frac{1}{3}(\ln 10 - \ln 3)$$

Now integrating over y, we have

$$\frac{1}{3}(\ln 10 - \ln 3) \int_0^4 y\, dy$$
$$\frac{1}{3}(\ln 10 - \ln 3) \left[\frac{y^2}{2}\right]_0^4$$
$$= \frac{8}{3}(\ln 10 - \ln 3)$$

Notes

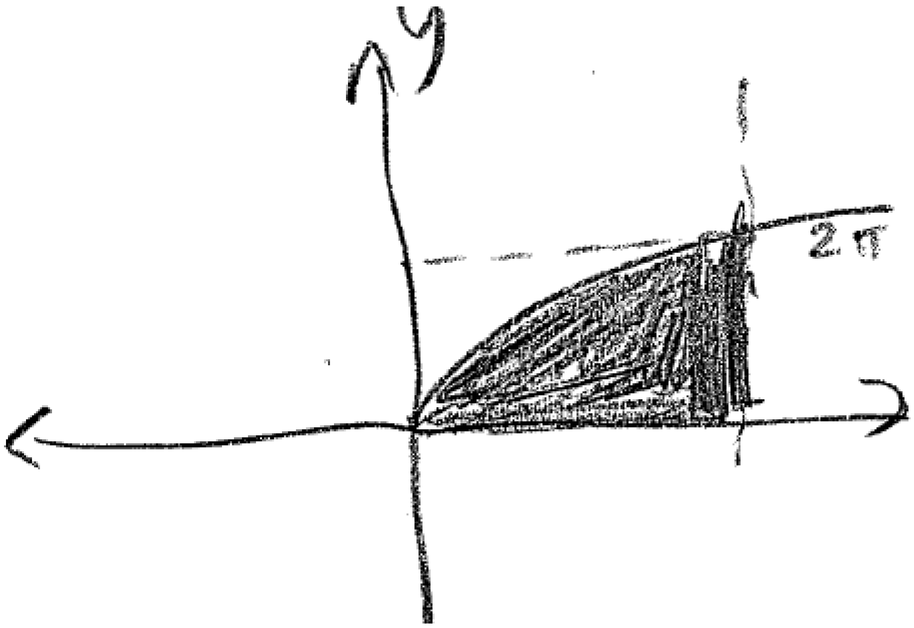
Straightforward question, and there aren't any limits to set up either.

Update: turns out option E and C (for green) were the same. This can be shown by taking logarithmic properties (using the fact that $\ln ab = \ln a + \ln b$ *or* $\ln \frac{a}{b} = \ln a - \ln b$*).*

$$\frac{8}{3}(\ln 10 - \ln 3) = \frac{8}{3}\big(\ln 5 + \ln 2 - (\ln 1.5 + \ln 2)\big) = \frac{8}{3}(\ln 5 - \ln 1.5)$$

Answer 3

It helps to sketch the region of integration.



From there, we get the limits after switching the order of integration:

$$I = \int_0^{2\pi} \int_0^{\sqrt{x}} y \cos x^2 \, dy \, dx$$

Now evaluating I,

$$I = \int_0^{2\pi} \left[\frac{y^2}{2} \cos x^2\right]_0^{\sqrt{x}} dx$$

$$= \frac{1}{2}\int_0^{2\pi} x \cos x^2 \, dx$$

Let $x^2 = t, 2x \, dx = dt \rightarrow dx = \frac{dt}{2x}$. Then,

$$I = \frac{1}{4}\int_0^{2\pi} \cos t \, dt$$

$$I = \frac{1}{4}[\sin x^2]_0^{2\pi}$$

$$I = \frac{1}{4}\sin 4\pi^2$$

Notes

If I was written as $\frac{1}{4}\sin x$ (instead of x^2), you'll get 0, which while a red flag, was also a red herring.

Otherwise, this is a common type of question to come in these exams.

Answer 4

The lower bound for z is 0, and the upper bound for z is the plane $x + y + z = 2$ or $z = 2 - x - y$.

The lower bound for y is the plane $y = x$, and the upper bound for y is when $y = 0$; i.e, $y = 2 - x$.

The lower bound for x is 0, and the upper bound for x is with the plane $y = x$. With $y = 2 - x$, this means that $x = 2 - x \to x = 1$.

Hence, the final integral is

$$\int_0^1 \int_x^{2-x} \int_0^{2-x-y} f(x, y, z)\, dzdydx$$

Notes

I was asked why the region of integration of y is from x to $2 - x$ and not from 0 to $2 - x$. This is because the lower bound of y is the plane $y = x$. Knowing this itself would instantly eliminate three options.

Answer 5

Converting to cylindrical form, we get

$$\int_0^{\frac{\pi}{2}} \int_0^2 \int_r^{\sqrt{2}r} r\, dzdrd\theta$$

Evaluating it,

$$\begin{aligned}
I &= \left(\sqrt{2} - 1\right) \int_0^{\frac{\pi}{2}} \int_0^2 r^2\, drd\theta \\
&= \left(\sqrt{2} - 1\right) \int_0^{\frac{\pi}{2}} \left[\frac{r^3}{3}\right]_0^2 d\theta \\
&= \left(\sqrt{2} - 1\right) \frac{\pi}{2} \left(\frac{8}{3}\right) \\
&= \frac{4\pi}{3} (\sqrt{2} - 1)
\end{aligned}$$

Notes

Reasonably straightforward, and has appeared in past papers. The only thing to note is that it's easy to make simple errors (like forgetting r).

Answer 6

To find c, note the intersection of the two planes $\sqrt{3(x^2 + y^2)}$ and $\sqrt{4 - x^2 - y^2}$:

$$\sqrt{3(x^2+y^2)} = \sqrt{4-x^2-y^2}$$
$$3x^2+3y^2 = 4-x^2-y^2$$
$$4x^2+4y^2 = 4$$
$$x^2+y^2 = 1$$

Hence, the cross-section is a circle of radius 1, and hence $c = 1$.

To find b, note the angle of intersection between the two planes, which is $\varphi = \frac{\pi}{3}$.

$\theta = 2\pi$, as the region covers the whole cross-sectional area.

Hence,

$$\frac{bc}{a} = \frac{1 * \frac{\pi}{3}}{2\pi} = \frac{1}{6}$$

Notes

A medium-difficulty question with tricky options which required good grasp of the concepts.

My mistake

I accidentally forgot to find the angle of intersection (thinking as if it were two spheres), and wrote $\varphi = \pi$, which resulted in my getting the incorrect answer of $1/2$.

Answer 7

We set up the triple integral first:

$$V = \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{12\cos\varphi} r^2 \sin\varphi \, dr\, d\varphi\, d\theta$$

How did we get that? r goes from 0 to the boundary of the surface ($12\cos\varphi$), φ goes from 0 to the angle where $12\cos\varphi = 0$ ($\varphi = \frac{\pi}{2}$). The sphere covers the whole circular cross-sectional area, and hence $\theta = [0,2\pi]$.

Then all that's left is evaluating that integral:

$$V = \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{12\cos\varphi} r^2 \sin\varphi \, dr\, d\varphi\, d\theta$$
$$V = \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \sin\varphi \left[\frac{r^3}{3}\right]_0^{12\cos\varphi} d\varphi\, d\theta$$
$$= \frac{1}{3}\int_0^{2\pi}\int_0^{\frac{\pi}{2}} 1728\cos^3\varphi \sin\varphi \, d\varphi\, d\theta$$
$$= \frac{1728\,(2\pi)}{3}\int_0^{\frac{\pi}{2}} \cos^3\varphi \sin\varphi \, d\varphi$$

Let $\cos\varphi = t$. Then $-\sin\varphi\, d\varphi = dt \rightarrow d\varphi = -\frac{dt}{\sin\varphi}$. Then,

$$V = -576(2\pi)\left[\frac{\cos^4\varphi}{4}\right]_0^{\frac{\pi}{2}}$$
$$V = -576(2\pi)\left(-\frac{1}{4}\right)$$
$$V = 288\pi$$

Notes

While the question is not conceptually hard, it helps if you've seen a similar question before (as the professors gave little indication on how these questions work). The integral to evaluate is also tricky and lengthy.

This is one question where reviewing the homework problems would have helped: a similar question is Q6 in Lesson 23.

My mistake

Very simple: my integral read like this instead –

$$V = \int_0^{2\pi}\int_0^{\pi}\int_0^{12\cos\varphi} r^2 \sin\varphi \, dr \, d\varphi \, d\theta$$

Whoops, wrong bounds for φ! That was a completely careless mistake on my part and evaluating this integral resulted in a value that was not amongst the options.

I wondered whether I had to subtract it from the whole sphere instead (even though that made little sense, I was confused in the exam and wasn't having much time to figure it out).

$$V = \int_0^{2\pi}\int_0^{\pi}\int_{12\cos\varphi}^{12} r^2 \sin\varphi \, dr \, d\varphi \, d\theta$$

This returned the incorrect answer 284π. I'm actually surprised that the two answers were that close.

Answer 8

The first step is to define the plane by which the tetrahedron is bounded. This will be one which has corners $(0,0,0), (1,0,0), (0,2,0)$ and $(0,0,4)$, and hence one plane which satisfies these conditions is $4x + 2y + z = 4$.

With that done, the next step is to define the bounds. z goes from 0 to the plane, y goes from zero to the upper bound of y, which is $4x + 2y = 4 \rightarrow y = 2 - 2x$, and x goes from 0 to 1. Hence, the mass is

$$\int_0^1\int_0^{2-2x}\int_0^{4-2y-4x} \rho(x,y,z) \, dz \, dydx$$

Substituting the value of ρ, and evaluating,

$$M = \int_0^1\int_0^{2-2x}\int_0^{4-2y-4x} 2z \, dz \, dydx$$
$$= \int_0^1\int_0^{2-2x} [z^2]_0^{4-2y-4x} \, dz \, dydx$$

$$= \int_0^1 \int_0^{2-2x} (4 - 2y - 4x)^2 \, dydx$$

Let $4 - 2y - 4x = t$. Then, $-2\,dy = dt \rightarrow dy = -\frac{1}{2}dt$. Then,

$$\int_0^1 \int_0^{2-2x} -\frac{t^2}{2} dtdx$$

$$M = \int_0^1 \left[-\frac{(4 - 2y - 4x)^3}{6} \right]_0^{2-2x} dx$$

$$= -\frac{1}{6}\int_0^1 (4 - 2(2 - 2x) - 4x)^3 - (4 - 4x)^3 \, dx$$

$$= -\frac{1}{6}\int_0^1 -(4 - 4x)^3 \, dx = \frac{1}{6}\int_0^1 (4 - 4x)^3 \, dx$$

Let $4 - 4x = t$. Then $-4\,dx = dt \rightarrow dx = -\frac{1}{4}dt$. Then,

$$M = \frac{-1}{24}\left[\frac{(4 - 4x)^4}{4}\right]_0^1$$

$$M = -\frac{1}{96}[0 - 256]$$

$$M = \frac{256}{96}$$

$$M = \frac{8}{3}$$

Notes

This is a somewhat hard question. Firstly, the intuition of finding the plane bounded by a tetrahedron is not well explained by the professors (at least mine didn't). Those who haven't seen this before (in a past exam paper) will find it harder to get the equation quickly.

And it's awfully lengthy. The integrals aren't complicated per se, but they require tricky algebraic manipulation and it was easy to goof up on one of the steps.

Answer 9

A potential field φ for F has the property that $F =< f, g >= \nabla\varphi$ and that

$$\varphi_{\text{x}} = \sin y$$
$$\varphi_y = x \cos y$$

First integrate φ_{x} with respect to x. This gives $\varphi(x, y) = x \sin y + c(y)$.

To find the arbitrary constant $c(y)$, differentiate $\varphi(x, y)$ with respect to y and equate to g. Then we get $x \cos y + C'(y) = x \cos y \rightarrow C'(y) = 0$. Hence, $C(y) = k$, which is a constant.

Hence the potential function is $x \cos y + k$, and the only option with a constant is $x \cos y + 1$.

Notes

I still stand with my earlier observation that this question should **not** have come up on this exam. It's covered in Section 17.3, but the exam covered only up to the first part of Section 17.2, and it's not that my lecturer gave much of an indication on how this problem should be done… For some reason, the professors disagree with this observation.

However, there is a 'brute-force' way to do this anyway. Simply find the gradient for all the options and only E will match up with the question. But isn't that a bad way of setting a question – wherein the 'normal' way is not in syllabus?

My mistake

I did not conceptually know how to solve this problem properly, having no experience with this type of problem.

Answer 10

We know that $r(t) = [\cos t, \sin t, t]$. Hence, $r'(t) = [-\sin t, \cos t, 1]$ and $|r'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$.

Hence,

$$\int_C f(x,y,z)\,ds = \int_C xy\,ds = \sqrt{2}\int_0^{\frac{\pi}{2}} \cos t \sin t\,dt$$

Evaluating the integral,

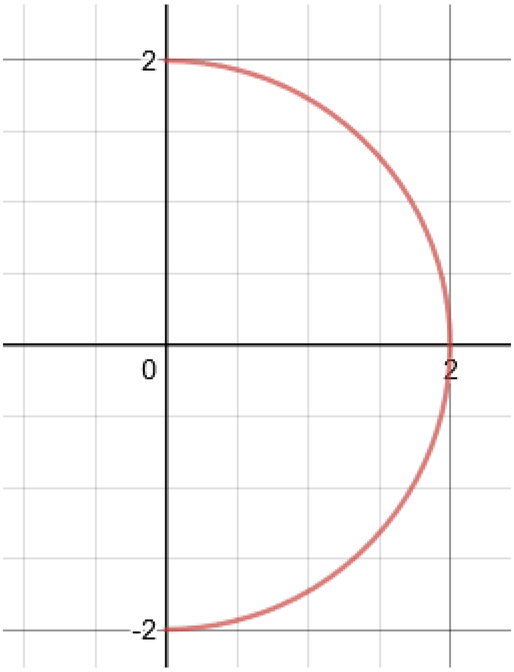
$$\begin{aligned}
&\sqrt{2}\int_0^{\frac{\pi}{2}} \cos t \sin t\,dt \\
&= \frac{\sqrt{2}}{2}\int_0^{\frac{\pi}{2}} \sin 2t\,dt \\
&= \frac{-\sqrt{2}}{4}[\cos 2t]_0^{\frac{\pi}{2}} \\
&= -\frac{\sqrt{2}}{4}(-1-1) \\
&\to \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}
\end{aligned}$$

Notes

Mostly similar to past questions, except that the missing z term in $f(x,y,z) = xy$ might confuse some.

Answer 11

The curve to integrate is the half circle with $x \geq 0$. It looks like this (see right).



The parametric function in this case is $r(t) = \left(\sqrt{4-t^2}, t\right)$, noting that x is positive (and hence $\sqrt{4-t^2}$ is also positive).

With that in mind, we integrate over the curve:

$$r'(t) = \left(\frac{-2t}{2\sqrt{4-t^2}}, 1\right) = \left(-\frac{t}{\sqrt{4-t^2}}, 1\right)$$

$$|r'(t)| = \sqrt{\left(-\frac{t}{\sqrt{4-t^2}}\right)^2 + 1}$$

$$\rightarrow \sqrt{\frac{t^2}{4-t^2} + 1} = \sqrt{\frac{t^2 + 4 - t^2}{4-t^2}} = \frac{2}{\sqrt{4-t^2}}$$

Hence, we have

$$\int_C x\, dS = \int_C f\big(r(t)\big)\, r'(t)\, dt$$

$$\int_{-2}^{2} \sqrt{4-t^2}\left(\frac{2}{\sqrt{4-t^2}}\right) dt$$

(the limits is from -2 to 2 as the parameter t is based on y, which goes over the entire diameter (from (0,2) to (0,-2))

$$= \int_{-2}^{2} 2\, dt = 2[t]_{-2}^{2} = 8$$

Notes

A somewhat tricky question. Firstly, there is nothing to divide! The options were set in such a way that it would be very tempting to simply divide by 2 or even 4 as it was not a full circle. But the best one can parameterize a 'circle' is actually a half circle, as circles are *not* functions! Also, the mere condition of $x \geq 0$ does not mean that we get a quarter-circle, as y isn't restricted.