Quantum measuring processes of continuous observables

Masanao Ozawa
Department of Information Sciences, Tokyo Institute of Technology, Oh-Okayama, Meguro-ku, Tokyo 152, Japan

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The purpose of this paper is to provide a basis of theory of measurements of continuous observables. We generalize von Neumann's description of measuring processes of discrete quantum observables in terms of interaction between the measured system and the apparatus to continuous observables, and show how every such measuring process determines the state change caused by the measurement. We establish a one-to-one correspondence between completely positive instruments in the sense of Davies and Lewis and the state changes determined by the measuring processes. We also prove that there are no weakly repeatable completely positive instruments of nondiscrete observables in the standard formulation of quantum mechanics, so that there are no measuring processes of nondiscrete observables whose state changes satisfy the repeatability hypothesis. A proof of the Wigner–Araki–Yanase theorem on the nonexistence of repeatable measurements of observables not commuting conserved quantities is given in our framework. We also discuss the implication of these results for the recent results due to Srinivas and due to Mercer on measurements of continuous observables.

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1. INTRODUCTION

In the last decade, some attempts were developed to construct a satisfactory theory of the quantum mechanical measurement of an observable with continuous spectrum.[1–9] However, we have found no satisfactory solution of the fundamental problem to determine the state changes caused by measurements of continuous observables. In spite of these difficulties in continuous spectrum, the theory for discrete spectrum has a conventionally accepted solution since the pioneering work of von Neumann.[10]

Let $A = \Sigma_i \lambda_i P_i$ be an observable with simple discrete spectrum $\lambda_1, \lambda_2, \cdots$. Then von Neumann[10] showed the following:

(1) By the repeatability hypothesis, the state change $\rho \rightarrow \rho'$ caused by the measurement of A is determined as $\rho' = \Sigma_i P_i \rho P_i$.

(2) The above state change $\rho \rightarrow \rho'$ is compatible with the Hamiltonian formalism in the description of the measuring process in terms of the time evolution of the composite system of the observed system and the measuring apparatus.

In the present paper, we shall show the following:

(1) The description of measuring processes has a satisfactory generalization to continuous observables.

(2) Every measuring process determines a state change caused by the measurement.

(3) There are no measuring processes of a nondiscrete observable whose state changes satisfy the repeatability hypothesis.

In order to clarify the present situation, we shall review some developments on the problem so far. In the early stage, Umegaki and Nakamura[11] showed that the state change $\rho \rightarrow \rho' = \Sigma_i P_i \rho P_i$ is just an example of Umegaki's noncommutative conditional expectations[12] onto the von Neumann algebra generated by A, and they conjectured that the state change caused by the measurement of a continuous observable would also be such a noncommutative conditional expectation. However, it is shown by Areveson[13] that such conditional expectations do not exist for continuous observables. In view of these results, Davies and Lewis[1] established the mathematical concept of instruments which enables us to treat statistical correlations of outcomes of successive measurements, and formulate the repeatability hypothesis for continuous observables. They conjectured the nonexistence of repeatable instruments for continuous observables and proposed the more flexible approach to measurements of continuous observables abandoning repeatability hypothesis. Recently, Srinivas[8] generalized the concept of instruments and showed the existence of such generalized instruments for continuous observables which satisfy the repeatability hypothesis. He proposed a generalized collapse postulate which determines such repeatable generalized instruments to describe the state changes caused by measurements of continuous observables. More recently, Mercer[9] considered a wider class of state transformations than conditional expectations and proposed the state change should be described by such a transformation with the locality introduced by him. It is a remarkable fact that these attempts are concerned only with the first half of von Neumann's work cited above. An operator theoretical analysis on von Neumann's second result was done by Kraus.[14] He established the complete positivity of state changes caused by the general measuring processes, but his result is concerned only with the yes–no measurements.

In this paper, we shall show that the state changes determined by measuring processes naturally correspond to completely positive instruments and vice versa. We prove Davies and Lewis's conjecture for completely positive instruments, i.e., completely positive instruments cannot be weakly repeatable unless the corresponding observable is discrete. These results show that Srinivas's generalized col-

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lapse postulate cannot be compatible for continuous observables with the Hamiltonian description of measuring processes. We shall also show that if they can be realized by some measuring processes, Mercer's local transition maps correspond to repeatable measurements, and hence they cannot exist for continuous observables.

The nonexistence of repeatable measuring processes of continuous observables suggests that we should investigate the approximately repeatable measuring processes as models of measurements in quantum mechanics. Moreover, this direction of investigation is appropriate not only for continuous observables. Indeed, even in measurements of discrete observables, it is known that the repeatable measurement is impossible unless observed quantity commutes with conserved quantity under some conservation law (see Refs. 15 and 16, also Sec. 8). The author believes that, in future investigations on really existing approximately repeatable measurements, our framework of measuring processes will provide a nice basis. However, we shall discuss these problems elsewhere.

In Sec. 2, we give some preliminaries on semiobservables and conditional expectations. Our concept of observed quantities allows the nonorthogonal resolutions of identity, called semiobservables. In Sec. 3, we generalize von Neumann's measuring processes to continuous observables and show that every measuring process determines the state change caused by the measurement. In Sec. 4, we provide a dilation theorem and a decomposition theorem of completely positive instruments which are useful in the later sections. In Sec. 5, we shall establish the one-to-one correspondence between measuring processes and completely positive instruments. If the observed quantity is a usual one, the obtained correspondence is reduced to very simple form by the decomposition theorem, that is, measuring processes are determined by their transition $\rho \to \rho'$. In Sec. 6, we study the repeatability hypothesis and prove the nonexistence of weakly repeatable completely positive instruments for nondiscrete observables in the standard formulation of quantum mechanics. In Sec. 7, we study the local transition maps and prove the nonexistence of local transition maps corresponding to measuring processes of nondiscrete observables. In Sec. 8, we shall give a proof of the Wigner–Araki–Yanase theorem in our framework, which states the nonexistence of repeatable measuring processes of the observables which do not commute with the conserved quantity. In Sec. 9, we shall give a characterization of the measuring processes discussed in the conventional measurement theory among our general measuring processes.

2. OBSERVABLES AND CONDITIONAL EXPECTATIONS

Let $\mathscr{H}$ be a Hilbert space. Denote by $\mathscr{L}(\mathscr{H})$ the algebra of bounded operators on $\mathscr{H}$ and by $\mathscr{T}(\mathscr{H})$ the space of trace class operators on $\mathscr{H}$. A *state* ρ on $\mathscr{H}$ is a positive trace one operator on $\mathscr{H}$. Denote by $\Sigma(\mathscr{H})$ the space of all states on $\mathscr{H}$. Let $(\Omega,\mathscr{B})$ be a Borel space. A *semiobservable* X on $\mathscr{H}$ with value space $(\Omega,\mathscr{B})$ is a positive operator valued measure $X: \mathscr{B} \to \mathscr{L}(\mathscr{H})$ such that $X(\Omega) = 1$. An *observable* X is a semiobservable which is projection valued. Denote by $\mathscr{B}(\mathbf{R}^n)$ the Borel σ-field of $\mathbf{R}^n$. By the spectral theory, we shall identify an observable X on $\mathscr{H}$ with value space $(\mathbf{R}^n, \mathscr{B}(R^n))$ and the corresponding mutually commutable family $\{x_1,\dots,x_n\}$ of self-adjoint operators on $\mathscr{H}$ such that

$$x_i = \int_{\mathbf{R}} \lambda X(\mathbf{R}\times\cdots\times \underset{i}{d\lambda} \times\cdots\times\mathbf{R}). \tag{2.1}$$

An observable X with value space $(\mathbf{R}, \mathscr{B}(R))$ is called *bounded* if $x = \int_{\mathbf{R}} \lambda\, X(d\lambda)$ is bounded. Let X be a semiobservable on $\mathscr{H}$ with value space $(\Omega,\mathscr{B})$. If the system is in the state ρ at the instant before a measurement of X, then the probability distribution $\mathrm{Prob}(X \in B;\rho)$ of the outcomes of this measurement is given by

$$\mathrm{Prob}(X\in B;\rho) = \mathrm{Tr}[\rho X(B)], \tag{2.2}$$

for any B in $\mathscr{B}$. For a semiobservable X, we shall denote by $X(\mathscr{B})$ the range of X, i.e., $X(\mathscr{B}) = \{X(B); B \in \mathscr{B}\}$. A *conditional expectation* T on $\mathscr{L}(\mathscr{H})$ onto a von Neumann algebra $\mathscr{M}$ on $\mathscr{H}$ is a normal completely positive linear map T on $\mathscr{L}(\mathscr{H})$ with range $\mathscr{M}$ such that $T(axb) = aT(x)b$ for all a,b in $\mathscr{M}$, x in $\mathscr{L}(\mathscr{H})$. It is known[17] that an ultraweakly continuous linear map T on $\mathscr{L}(\mathscr{H})$ is a conditional expectation if and only if it is a projection of norm 1 onto $\mathscr{M}$.

Let $\mathscr{K}$ be another Hilbert space. Let σ be a state on $\mathscr{K}$. Then the formula

$$\mathrm{Tr}[\rho E_\sigma(x)] = \mathrm{Tr}[(\rho\otimes\sigma)x], \tag{2.3}$$

where $x\in\mathscr{L}(\mathscr{H}\otimes\mathscr{K})$ and $\rho\in\mathscr{T}(\mathscr{H})$, defines a normal completely positive linear map $E_\sigma: \mathscr{L}(\mathscr{H}\otimes\mathscr{K}) \mapsto \mathscr{L}(\mathscr{H})$ such that $E_\sigma(a\otimes 1) = a$ for any a in $\mathscr{L}(\mathscr{H})$. Thus the formula $x \to E_\sigma(x)\otimes 1$, for x in $\mathscr{L}(\mathscr{H}\otimes\mathscr{K})$, defines a conditional expectation on $\mathscr{L}(\mathscr{H}\otimes\mathscr{K})$ onto $\mathscr{L}(\mathscr{H})\otimes \mathbf{C}1$. It is easily seen that the map E_σ is the adjoint of the map $\rho\to\rho\otimes\sigma$ from $\mathscr{T}(\mathscr{H})$ into $\mathscr{T}(\mathscr{H}\otimes\mathscr{K})$. The formula

$$\mathrm{Tr}[E_{\mathscr{K}}(\phi)a] = \mathrm{Tr}[\phi(a\otimes 1)], \tag{2.4}$$

where $\phi\in\mathscr{T}(\mathscr{H}\otimes\mathscr{K})$ and $a\in\mathscr{L}(\mathscr{H})$, defines a completely positive linear map $E_{\mathscr{K}}: \mathscr{T}(\mathscr{H}\otimes\mathscr{K}) \mapsto \mathscr{T}(\mathscr{H})$, which is called the *partial trace* over $\mathscr{K}$. The partial trace $E_{\mathscr{K}}$ also satisfies that for any ξ,η in $\mathscr{H}$, and any orthogonal basis $\{\psi_i\}$, we have

$$(E_{\mathscr{K}}(\rho)\xi,\eta) = \sum_i (\rho(\xi\otimes\psi_i),\eta\otimes\psi_i), \tag{2.5}$$

for any ρ in $\mathscr{T}(\mathscr{H}\otimes\mathscr{K})$. It is easily seen that the adjoint of $E_{\mathscr{K}}$ is the map $a \to a\otimes 1$ from $\mathscr{L}(\mathscr{H})$ into $\mathscr{L}(\mathscr{H}\otimes\mathscr{K})$.

The following lemmas can be verified by easy computations.

Lemma 2.1: Let $\rho\in\mathscr{T}(\mathscr{H})$, $\sigma\in\mathscr{T}(\mathscr{H}\otimes\mathscr{K})$, and $b\in\mathscr{L}(\mathscr{K})$. If we have $\mathrm{Tr}[a\rho] = \mathrm{Tr}[(a\otimes b)\sigma]$ for any $a\in\mathscr{L}(\mathscr{H})$, then we have

$$\rho = E_{\mathscr{K}}[(1\otimes b)\sigma]. \tag{2.6}$$

Lemma 2.2: Let $T: \mathscr{T}(\mathscr{H}) \mapsto \mathscr{T}(\mathscr{H})$ be a bounded linear map, and let $U\in\mathscr{L}(\mathscr{H}\otimes\mathscr{K})$, $b\in\mathscr{L}(\mathscr{K})$, and $\sigma\in\mathscr{T}(\mathscr{K})$. Then

$$T(\rho) = E_{\mathscr{K}}[U(\rho\otimes\sigma)U^*(1\otimes b)], \tag{2.7}$$

for any ρ in $\mathscr{T}(\mathscr{H})$ if and only if

$$T^*(a) = E_\sigma[U^*(a\otimes b)U], \tag{2.8}$$

for any a in $\mathscr{L}(\mathscr{H})$.

Lemma 2.3: Let $\sigma = \Sigma_i \lambda_i |\xi_i\rangle\langle\xi_i|$ be the spectral decomposition of σ in $\Sigma(\mathscr{K})$. Then

$$E_\sigma[A] = \sum_i \lambda_i E_{|\xi_i\rangle\langle\xi_i|}[A], \tag{2.9}$$

for any A in $\mathscr{L}(\mathscr{H} \otimes \mathscr{K})$, where the sum is convergent in the weak operator topology.

3. MEASURING PROCESSES

In order to determine the possible transformations of states associated with the measurement of an observable, we shall consider the description of the measuring process in terms of the interaction between the observed system and the apparatus, which is a generalization of von Neumann's description of the measuring process for an observable with discrete spectrum (Ref. 10, Chap. IV). Our mathematical formulation of the measuring process is as follows.

Definition 3.1: Let $\mathscr{H}$ be a Hilbert space and X be a semiobservable on $\mathscr{H}$ with value space $(\Omega,\mathscr{B})$. A *measuring process* M of X is a 4-tuple $M = \langle \mathscr{K},\widetilde{X},\sigma,U\rangle$ consisting of a Hilbert space $\mathscr{K}$, an observable $\widetilde{X}$ on $\mathscr{K}$ with value space $(\Omega,\mathscr{B})$, a state σ on $\mathscr{K}$, and a unitary operator U on $\mathscr{H}\otimes\mathscr{K}$ satisfying the relation

$$X(B) = E_\sigma[U^*(1\otimes\widetilde{X}(B))U] \tag{3.1}$$

for any B in $\mathscr{B}$.

Now we shall explain the physical interpretation of the measuring process $M = \langle \mathscr{K},\widetilde{X},\sigma,U\rangle$ of a semiobservable X of a Hilbert space $\mathscr{H}$ with value space $(\Omega,\mathscr{B})$. The Hilbert space $\mathscr{H}$ and $\mathscr{K}$ describe, respectively, the measured system I and the apparatus II. The semiobservable X is to be measured by this measuring process. The observable $\widetilde{X}$ is to show the value of X on a scale in the apparatus which is actually measured by the observer, i.e., $\widetilde{X}$ is the position of the pointer on this scale. The state σ is the initially prepared state of the apparatus. The measurement is carried out by the interaction between the observed system and the apparatus during a finite time interval from time 0 to t. The unitary operator U describes the time evolution of the composite system, i.e.,

$$U = \exp[-it(H_{\rm I}\otimes 1 + 1\otimes H_{\rm II} + H_{\rm int})], \tag{3.2}$$

where $H_{\rm I}$ and $H_{\rm II}$ are Hamiltonians of the observed system I and the apparatus II, respectively, and $H_{\rm int}$ represents the interaction. Suppose that at the instant before the interaction the measured system is in the (unknown) state ρ. Then the composite system is in the state $\rho\otimes\sigma$ at time 0 and by the interaction it is in the state $U(\rho\otimes\sigma)U^*$ at time t. Thus the probability distribution $\mathrm{Prob}(X\in B;\rho)$ of the outcomes of this measurement must coincide with the probability distribution $\mathrm{Prob}(\widetilde{X}\in B;t)$ of the observable $\widetilde{X}$ at time t. Since $\mathrm{Prob}(X\in B;\rho) = \mathrm{Tr}[\rho X(B)]$ and $\mathrm{Prob}(\widetilde{X}\in B;t) = \mathrm{Tr}[U(\rho\otimes\sigma)U^*\widetilde{X}(B)]$, we should impose the requirement

$$\mathrm{Tr}[\rho X(B)] = \mathrm{Tr}[U(\rho\otimes\sigma)U^*\widetilde{X}(B)] \tag{3.3}$$

for any B in $\mathscr{B}$, ρ in $\Sigma(\mathscr{H})$. It is easy to see that the requirement (3.3) is equivalent to the requirement (3.1) in Definition 3.1.

We shall now show that the measuring process $M = \langle \mathscr{K},\widetilde{X},\sigma,U\rangle$ determines a unique state change caused by this measurement. Suppose that a measuring process $M = \langle \mathscr{K},\widetilde{X},\sigma,U\rangle$ of X is carried out in the initial state ρ of $\mathscr{H}$. Let $B\in\mathscr{B}$. Denote by ρ^B the state, at the instant after the measurement, of the subensemble of the measured system in which the outcomes of the measurement lie in B. In order to determine the state ρ^B, suppose that the observer were to measure the simultaneously measurable observables A in I and $\widetilde{X}$ in II, where A is an arbitrary bounded observable with value space $(\mathbf{R},\mathscr{B}(\mathbf{R}))$. Then we have the joint probability distribution of their values:

$$\mathrm{Prob}(A\in d\lambda,\widetilde{X}\in d\omega) = \mathrm{Tr}[U(\rho\otimes\sigma)U^*(A(d\lambda)\otimes\widetilde{X}(d\omega))]. \tag{3.4}$$

Thus, if $\mathrm{Prob}(\widetilde{X}\in B)\neq 0$, we have also the conditional probability distribution of A conditioned by the value of $\widetilde{X}$ lying in B,

$$\begin{aligned}\mathrm{Prob}(A\in d\lambda\,|\widetilde{X}\in B) &= \mathrm{Prob}(A\in d\lambda,\widetilde{X}\in B)/\mathrm{Prob}(\widetilde{X}\in B)\\ &= \mathrm{Tr}[U(\rho\otimes\sigma)U^*(A(d\lambda)\otimes\widetilde{X}(B))]/\mathrm{Tr}[\rho X(B)],\end{aligned} \tag{3.5}$$

and the conditional expectation $\mathrm{Ex}(A\,|\widetilde{X}\in B)$ of A conditioned by the value of $\widetilde{X}$ lying in B,

$$\begin{aligned}\mathrm{Ex}(A\,|\widetilde{X}\in B) &= \int_{\mathbf{R}} \lambda\,\mathrm{Prob}(A\in d\lambda\,|\widetilde{X}\in B)\\ &= \mathrm{Tr}[U(\rho\otimes\sigma)U^*(a\otimes\widetilde{X}(B))]/\mathrm{Tr}[\rho X(B)],\end{aligned} \tag{3.6}$$

where $a = \int_{\mathbf{R}}\lambda A(d\lambda)$. On the other hand, by the probabilistic interpretation of the state ρ^B, the state ρ^B must satisfy the relation

$$\mathrm{Prob}(A\in d\lambda\,|\widetilde{X}\in B) = \mathrm{Tr}[\rho^B A(d\lambda)] \tag{3.7}$$

or, equivalently,

$$\mathrm{Ex}(A\,|\widetilde{X}\in B) = \mathrm{Tr}[\rho^B a]. \tag{3.8}$$

By the arbitrariness of A, we can determine the state ρ^B uniquely by Eqs. (3.6) and (3.8). That is, by Lemma 2.1, we have

$$\rho^B = \{1/\mathrm{Tr}[\rho X(B)]\}E_{\mathscr{K}}[U(\rho\otimes\sigma)U^*(1\otimes\widetilde{X}(B))], \tag{3.9}$$

where $E_{\mathscr{K}}: \mathscr{T}(\mathscr{H}\otimes\mathscr{K})\mapsto\mathscr{T}(\mathscr{H})$ is the partial trace over $\mathscr{K}$. In particular, we have

$$\rho^\Omega = E_{\mathscr{K}}[U(\rho\otimes\sigma)U^*]. \tag{3.10}$$

Therefore, we have determined the state change $\rho\rightarrow\rho^B$ caused by the measuring process $M = \langle \mathscr{K},\widetilde{X},\sigma,U\rangle$ of the semiobservable X on $\mathscr{H}$ with value space $(\Omega,\mathscr{B})$.

Let $M = \langle \mathscr{K},\widetilde{X},\sigma,U\rangle$ be a measuring process of a semiobservable X. For any a in $\mathscr{L}(\mathscr{H})$, $\mathrm{Ex}^M(a|B;\rho)$ will denote the conditional expectation of the outcome of a measurement of a at that instant after the measuring process M under the condition that the measuring process M of X has been carried out in the initial state ρ on $\mathscr{H}$ and its outcome lies in $B\in\mathscr{B}$. Then from the above discussions, we have

$$\begin{aligned}\mathrm{Ex}^M(a|B;\rho) &= \mathrm{Tr}[\rho^B a]\\ &= \{1/\mathrm{Tr}[\rho X(B)]\}\\ &\quad\times\mathrm{Tr}[U(\rho\otimes\sigma)U^*(a\otimes\widetilde{X}(B))].\end{aligned} \tag{3.11}$$

Conclusion: Every measuring process $M = \langle \mathscr{K}, \tilde{X}, \sigma, U \rangle$ of a semiobservable X determines a state change $\rho \to \rho^B$ caused by the measurement, where ρ^B is the state, at the instant after the measurement, of the subensemble of the measured system in which outcomes of the measurement in the initial state ρ lies in $B \in \mathscr{B}$.

4. COMPLETELY POSITIVE INSTRUMENTS

From the investigations of von Neumann's repeated measurements, Davies and Lewis[1] introduced a mathematical notion of instruments which represents statistical correlations of outcomes of successive measurements. For the theory of instruments, called operational quantum probability theory, we refer the reader to Refs. 1 and 4. In the present section, we shall provide some general results on instruments imposed complete positivity.

Our setting for operational quantum probability theory consists of a von Neumann algebra $\mathscr{M}$ on a Hilbert space $\mathscr{H}$ and a Borel space $(\Omega, \mathscr{B})$. A *state* ρ of $\mathscr{M}$ is a normal state on $\mathscr{M}$. Denote by $\mathscr{M}_*$ the predual of $\mathscr{M}$ and by $\Sigma(\mathscr{M})$ the space of all normal states on $\mathscr{M}$. A *semiobservable* X in $\mathscr{M}$ is a semiobservable on $\mathscr{H}$ whose range is contained in $\mathscr{M}$. A *subtransition map* T on $\mathscr{M}$ is a normal completely positive linear map $T: \mathscr{M} \to \mathscr{M}$ such that $0 \leqslant T(1) \leqslant 1$. A *transition map* T is a subtransition map such that $T(1) = 1$. We define the right action of a subtransition map T on $\mathscr{M}_*$ by the duality

$$\langle \rho, Ta \rangle = \langle \rho T, a \rangle, \tag{4.1}$$

for all a in $\mathscr{M}$, ρ in $\mathscr{M}_*$. A *CP instrument* $\mathscr{I}$ on $\mathscr{M}$ with value space $(\Omega, \mathscr{B})$ is a subtransition map valued measure on $(\Omega, \mathscr{B})$ such that (i) for each countable family $\{B_i\}$ of pairwise disjoint sets in $\mathscr{B}$,

$$\left\langle \rho, \mathscr{I}\Big(\bigcup_i B_i\Big)a \right\rangle = \sum_i \langle \rho, \mathscr{I}(B_i)a \rangle, \tag{4.2}$$

for all a in $\mathscr{M}$, ρ in $\mathscr{M}_*$ and that (ii) $\mathscr{I}(\Omega)1 = 1$. The condition (i) is equivalent to countable additivity of the right action in the strong operator topology on $\mathscr{L}(\mathscr{M}_*, \mathscr{M}_*)$. In what follows we shall also use the notation $\mathscr{I}(\cdot,\cdot)$ for a CP instrument $\mathscr{I}$ in such a way $\mathscr{I}(B,a) = \mathscr{I}(B)a$ for all B in $\mathscr{B}$, a in $\mathscr{M}$. By the same argument as in Ref. 1, Theorem 1, we can prove the following.

Proposition 4.1: For every CP instrument $\mathscr{I}$ on $\mathscr{M}$ with value space $(\Omega, \mathscr{B})$ there is a unique semiobservable X in $\mathscr{M}$ with value space $(\Omega, \mathscr{B})$ such that $X(B) = \mathscr{I}(B,1)$ for all B in $\mathscr{B}$. Every semiobservable is determined in such a way by at least one CP instrument.

Let $\mathscr{I}$ be a CP instrument. We say that a semiobservable X is the *associate semiobservable* of $\mathscr{I}$, if $X(B) = \mathscr{I}(B,1)$ for any B in $\mathscr{B}$ and that a transition map T is the *associate map* of $\mathscr{I}$ if $T(a) = \mathscr{I}(\Omega, a)$ for any a in $\mathscr{M}$. Let X be a semiobservable. A CP instrument $\mathscr{I}$ is called *X-compatible* if X is the associate semiobservable of $\mathscr{I}$. A transition map T is called *X-compatible* if the range of T is contained in $X(\mathscr{B})'$.

The following proposition is very useful in dealing with CP instruments which is a modification of the Stinespring theorem on completely positive maps.[18]

Proposition 4.2: For any CP instrument $\mathscr{I}$ of $\mathscr{M}$ with value space $(\Omega, \mathscr{B})$ there is a Hilbert space $\mathscr{H}_0$, a spectral measure $E: \mathscr{B} \to \mathscr{L}(\mathscr{H}_0)$, a nondegenerate normal*-representation $\pi: \mathscr{M} \to \mathscr{L}(\mathscr{H}_0)$ and a linear isometry $V: \mathscr{H} \to \mathscr{H}_0$ satisfying

$$\mathscr{I}(B,a) = V^* E(B) \pi(a) V, \tag{4.3}$$

$$E(B)\pi(a) = \pi(a) E(B), \tag{4.4}$$

for any B in $\mathscr{B}$ and a in $\mathscr{M}$.

Proof: Denote by $B(\Omega)$ the space of all bounded $\mathscr{B}$-measurable functions on Ω. Consider the algebraic tensor product $B(\Omega) \otimes \mathscr{M} \otimes \mathscr{H}$. We define a sesquilinear form $(\cdot,\cdot)$ on $B(\Omega) \otimes \mathscr{M} \otimes \mathscr{H}$ as follows:

$$(\xi, \eta) = \sum_i \int_\Omega g_j(\omega)^* f_i(\omega) (\mathscr{I}(d\omega, b_j^* a_i)\xi_i, \eta_j),$$

for $\xi = \Sigma_i f_i \otimes a_i \otimes \xi_i$, $\eta = \Sigma_j g_j \otimes b_j \otimes \eta_j$ in $B(\Omega) \otimes \mathscr{M} \otimes \mathscr{H}$. Then we can prove that $(\xi, \xi) \geqslant 0$ by just a similar way as the proof of Ref. 18, Theorem 4, and thus $\xi \to \|\xi\| = (\xi,\xi)^{1/2}$ is a seminorm. Define actions π of $\mathscr{M}$ and E of $\mathscr{B}$ on $B(\Omega) \otimes \mathscr{M} \otimes \mathscr{H}$ as follows:

$$\pi(x)\xi = \sum_i f_i \otimes x a_i \otimes \xi_i,$$

$$E(B)\xi = \sum_i \chi_B f_i \otimes a_i \otimes \xi_i,$$

for x in $\mathscr{M}$, B in $\mathscr{B}$, and $\xi = \Sigma_i f_i \otimes a_i \otimes \xi_i$. Then we have that $\|\pi(x)\xi\| \leqslant \|x\| \, \|\xi\|$ and $\|E(B)\xi\| \leqslant \|\xi\|$. Thus the both actions are well defined also on the $\|\cdot\|$-completion $\mathscr{H}_0$ of the quotient space $B(\Omega) \otimes \mathscr{M} \otimes \mathscr{H} / \mathscr{N}$, where $\mathscr{N} = \{\xi \mid \|\xi\| = 0\}$. Define a map $V: \mathscr{H} \to \mathscr{H}_0$ as $V\phi = (1 \otimes 1 \otimes \phi) + \mathscr{N}$, for any ϕ in $\mathscr{H}$. Then the assertions can be checked in a routine manner (Ref. 18 and Ref. 19, p. 194). QED

A CP instrument $\mathscr{I}$ is called *decomposable* if it is of the form $\mathscr{I}(B,a) = X(B)T(a)$ for all B in $\mathscr{B}$, a in $\mathscr{M}$, where X is the associate semiobservable of $\mathscr{I}$ and T is the associate map of $\mathscr{I}$.

Proposition 4.3: A CP instrument $\mathscr{I}$ is decomposable if its associate semiobservable X is projection-valued or if its associate map T is homomorphic [i.e., $T(a^*a) = T(a)^*T(a)$ for all a in $\mathscr{M}$].

Proof: First suppose that T is homomorphic. We can suppose that $\mathscr{I}$ is of the form $\mathscr{I}(B,a) = V^*E(B)\pi(a)V$ as in Proposition 4.2. Since $T(a) = V^*\pi(a)V$ and $V^*V = 1$, we have

$$(\pi(a)V - VT(a))^*(\pi(a)V - VT(a)) = T(a^*a) - T(a)^*T(a) = 0.$$

Thus $\pi(a)V = VT(a)$ for all a in $\mathscr{M}$, and hence we obtain that $\mathscr{I}(B,a) = V^*E(B)\pi(a)V = V^*E(B)VT(a) = X(B)T(a)$ for any B in $\mathscr{B}$, a in $\mathscr{M}$. The proof for the case that X is projection-valued is similar. QED

Proposition 4.4: Let X be an observable in $\mathscr{M}$ with value space $(\Omega, \mathscr{B})$. Then there is a one-to-one correspondence between X-compatible CP instruments $\mathscr{I}$ on $\mathscr{M}$ and X-compatible transition maps T on $\mathscr{M}$, which is given by $\mathscr{I}(B,a) = X(B)T(a)$ for any B in $\mathscr{B}$, a in $\mathscr{M}$.

Proof: If a CP instrument $\mathscr{I}$ is decomposable, then its associate map T is X-compatible, since $X(B)T(a) = (X(B)T(a))^* = T(a)X(B)$ for any $a \geqslant 0$ in $\mathscr{M}$, B in $\mathscr{B}$.

Conversely, if T is an X-compatible transition map then it is easy to check that the relation $\mathscr{I}(B,a) = X(B)T(a)$, where $a \in \mathscr{M}$ and $B \in \mathscr{B}$, defines an X-compatible CP instrument. Thus the assertion follows immediately from Proposition 4.3. QED

5. CLASSIFICATION OF MEASURING PROCESSES

Let $\mathscr{H}$ be a Hilbert space and X be a semiobservable on $\mathscr{H}$ with value space $(\Omega,\mathscr{B})$. We say that two measuring processes M_1 and M_2 of X are *statistically equivalent* if

$$\mathrm{Ex}^{M_1}(a|B;\rho) = \mathrm{Ex}^{M_2}(a|B;\rho), \tag{5.1}$$

for any a in $\mathscr{L}(\mathscr{H})$, B in $\mathscr{B}$, ρ in $\Sigma(\mathscr{H})$. Since every two statistically equivalent measuring processes give the same state change, it is desirable to classify these equivalence classes by more tractable mathematical objects concerned only with the observed system. In this section, we shall carry out such classification.

Let $M = \langle \mathscr{K},\tilde{X},\sigma,U\rangle$ be a measuring process of X. Consider the following relation:

$$\mathscr{I}(B)a = E_\sigma[U^*(a\otimes\tilde{X}(B))U], \tag{5.2}$$

for any B in $\mathscr{B}$, a in $\mathscr{L}(\mathscr{H})$. Then it is not hard to check that Eq. (5.2) defines an X-compatible CP instrument $\mathscr{I}$ on $\mathscr{L}(\mathscr{H})$. By Lemma 2.2, Eq. (5.2) is equivalent to

$$\rho\mathscr{I}(B) = E_{\mathscr{K}}[U(\rho\otimes\sigma)U^*(1\otimes\tilde{X}(B))], \tag{5.3}$$

for all B in $\mathscr{B}$, ρ in $\mathscr{T}(\mathscr{H})$. By Eqs. (3.1) and (3.9), we have

$$X(B) = \mathscr{I}(B,1), \tag{5.4}$$

$$\rho^B = (1/\mathrm{Tr}[\rho\mathscr{I}(B)])\rho\mathscr{I}(B),$$
$$\text{whenever}\quad \mathrm{Tr}[\rho X(B)]\neq 0, \tag{5.5}$$

for all ρ in $\Sigma(\mathscr{H})$, B in $\mathscr{B}$. Thus the CP instrument $\mathscr{I}$ defined by Eq. (5.2) retains the all statistical data of the measuring process M, that is, the probability distribution of outcomes of the measurement and the state change caused by the measurement. The following theorem shows that every CP instrument on $\mathscr{L}(\mathscr{H})$ arises in this way.

Theorem 5.1: Let X be a semiobservable on $\mathscr{H}$ with value space $(\Omega,\mathscr{B})$. Then there is a one-to-one correspondence between statistical equivalence classes of measuring processes M of X and X-compatible CP instruments $\mathscr{I}$ on $\mathscr{L}(\mathscr{H})$, which is given by the relation

$$\mathrm{Tr}[\rho\mathscr{I}(B)]\mathrm{Ex}^M(a|B;\rho) = \mathrm{Tr}[\rho\mathscr{I}(B)a], \tag{5.6}$$

for all B in $\mathscr{B}$, ρ in $\Sigma(\mathscr{H})$, a in $\mathscr{L}(\mathscr{H})$.

Proof: Let $M = \langle\mathscr{K},\tilde{X},\sigma,U\rangle$ be a measuring process of X. Then it is easy to see that the CP instrument $\mathscr{I}$ defined by Eq. (5.2) is a unique CP instrument which satisfies Eq. (5.6). It follows that the statistically equivalent measuring processes determine the same CP instrument by Eq. (5.2). Now it suffices to construct a measuring process of X which determines by Eq. (5.2) a given X-compatible CP instrument. Let $\mathscr{I}$ be an X-compatible CP instrument on $\mathscr{L}(\mathscr{H})$ with value space $(\Omega,\mathscr{B})$. Let $\mathscr{H}_0$, E, π, and V be such as obtained in Proposition 4.2 for the CP instrument $\mathscr{I}$. Since every nondegenerate normal $*$-representation of $\mathscr{L}(\mathscr{H})$ is unitarily equivalent to the multiple of the identity representation (Ref. 4, Lemma 9.2.2), there is a Hilbert space $\mathscr{H}_1$ such that $\mathscr{H}_0 = \mathscr{H}\otimes\mathscr{H}_1$ and that $\pi(a) = a\otimes 1$ for any a in $\mathscr{L}(\mathscr{H})$.

Then by Eq. (4.3) and by the commutation theorem of von Neumann algebras, for any B in $\mathscr{B}$ there is a projection $E_1(B)$ in $\mathscr{L}(\mathscr{H}_1)$ such that $E(B) = 1\otimes E_1(B)$. Obviously, the correspondence $E_1: B\to E_1(B)$ is a projection-valued measure from $\mathscr{B}$ to $\mathscr{L}(\mathscr{H}_1)$. By Eq. (4.3), we have

$$\mathscr{I}(B,a) = V^*(a\otimes E_1(B))V,$$

for any B in $\mathscr{B}$, a in $\mathscr{L}(\mathscr{H})$. Let η_0 be a unit vector in $\mathscr{H}_0$ and η_1 be a unit vector in $\mathscr{H}_1$. Define an isometry V_0 on $\mathscr{H}\otimes[\eta_1]\otimes[\eta_0]$ into $\mathscr{H}\otimes\mathscr{H}_1\otimes\mathscr{H}_0$ by the relation

$$V_0(\xi\otimes\eta_1\otimes\eta_0) = V\xi\otimes\eta_0,$$

for any ξ in $\mathscr{H}$. Then, since $\dim(\mathscr{H}_0) = \dim(\mathscr{H}\otimes\mathscr{H}_1)$, by the usual computations of cardinal numbers, it is easy to show that

$$\dim(\mathscr{H}\otimes\mathscr{H}_1\otimes\mathscr{H}_0 - \mathscr{H}\otimes[\eta_1]\otimes[\eta_0])$$
$$= \dim(\mathscr{H}\otimes\mathscr{H}_1\otimes\mathscr{H}_0 - V_0(\mathscr{H}\otimes[\eta_1]\otimes[\eta_0])).$$

It follows that there is a unitary operator U on $\mathscr{H}\otimes\mathscr{H}_1\otimes\mathscr{H}_0$ which is an extension of V_0. Now let $\mathscr{K}$, σ, and $\tilde{X}$ be such that

$$\mathscr{K} = \mathscr{H}_1\otimes\mathscr{H}_0,\quad \sigma = |\eta_1\otimes\eta_0\rangle\langle\eta_1\otimes\eta_0|,$$
$$\text{and}\quad \tilde{X}(B) = E_1(B)\otimes 1\quad\text{on }\mathscr{H}_1\otimes\mathscr{H}_0,$$

for any B in $\mathscr{B}$. Then we shall claim that $\langle\mathscr{K},\tilde{X},\sigma,U\rangle$ is a measuring process which determines the CP instrument $\mathscr{I}$ by Eqs. (5.2). For any a in $\mathscr{L}(\mathscr{H})$, ξ in $\mathscr{H}$, and B in $\mathscr{B}$, we have that

$$\begin{aligned}
(\mathscr{I}(B,a)\xi,\xi) &= (V^*(a\otimes E_1(B))V\xi,\xi)\\
&= ((a\otimes E_1(B))V\xi,V\xi)\\
&= ((a\otimes E_1(B))V\xi\otimes\eta_0,V\xi\otimes\eta_0)\\
&= ((a\otimes E_1(B)\otimes 1)U(\xi\otimes\eta_1\otimes\eta_0),U(\xi\otimes\eta_1\otimes\eta_0))\\
&= (U^*(a\otimes\tilde{X}(B))U(\xi\otimes\eta_1\otimes\eta_0),\xi\otimes\eta_1\otimes\eta_0)\\
&= \mathrm{Tr}[U^*(a\otimes\tilde{X}(B))U(|\xi\rangle\langle\xi|\otimes\sigma)]\\
&= \mathrm{Tr}[|\xi\rangle\langle\xi|E_\sigma[U^*(a\otimes\tilde{X}(B))U]]\\
&= (E_\sigma[U^*(a\otimes\tilde{X}(B))U]\xi,\xi).
\end{aligned}$$

It follows that

$$\mathscr{I}(B,a) = E_\sigma[U^*(a\otimes\tilde{X}(B))U],$$

for any a in $\mathscr{L}(\mathscr{H})$ and B in $\mathscr{B}$. Therefore, $\langle\mathscr{K},\tilde{X},\sigma,U\rangle$ is a measuring process of X which determines $\mathscr{I}$ by Eq. (5.2). QED

We say that a measuring process M is a *realization* of a CP instrument $\mathscr{I}$ if M and $\mathscr{I}$ satisfies Eq. (5.6). The above theorem asserts that every CP instrument has its realization. In the conventional theory of quantum mechanics, it is always assumed that the Hilbert space is separable and the value space is a standard Borel space, i.e., a Borel space which is Borel isomorphic to a separable complete metric space.[20] Thus it is desirable that the realization is also with a separable Hilbert space in such circumstances. We say that realization $M = \langle\mathscr{K},\tilde{X},\sigma,U\rangle$ of a CP instrument $\mathscr{I}$ is *separable* if the Hilbert space $\mathscr{K}$ is separable.

Corollary 5.2: Let $\mathscr{I}$ be a CP instrument on $\mathscr{L}(\mathscr{H})$ with value space $(\Omega,\mathscr{B})$. If $\mathscr{H}$ is separable and $(\Omega,\mathscr{B})$ is a standard Borel space, then there is a separable realization of $\mathscr{I}$.

Proof (the notations are the same as in the proof of Theorem 5.1): It is easy to see that we can assume that $\mathscr{H}_0$ in Proposition 4.2 is spanned by $\{E(B)\pi(a)V\xi;\ B\in\mathscr{B},\ a\in\mathscr{L}(\mathscr{H}),\text{ and } \xi\in\mathscr{H}\}$. Since $\mathscr{H}$ is separable, there is a countable family $\{a_n\}$ of a_n in $\mathscr{L}(\mathscr{H})$ which is dense in $\mathscr{L}(\mathscr{H})$ in the strong operator topology. Let $\{B_n\}$ be a countable generator of $\mathscr{B}$ and $\{\xi_n\}$ be a countable dense subset of $\mathscr{H}$. Then it is easy to see that the countable family $\{E(B_i)\times\pi(a_j)V\xi_k;\ i,j,k=1,2,\cdots\}$ spans $\mathscr{H}_0$, so that $\mathscr{H}_0$ is separable. Since $\mathscr{H}\otimes\mathscr{K}=\mathscr{H}_0\otimes\mathscr{H}_0$, $\mathscr{K}$ is separable. QED

We say that a measuring process $M=\langle\mathscr{K},\widetilde{X},\sigma,U\rangle$ is *pure* if σ is pure state, i.e., there is a unit vector ξ in $\mathscr{K}$ such that $\sigma=|\xi\rangle\langle\xi|$. In the conventional argument of quantum measurement, the assumption that the prepared state of the apparatus is pure has been justified in some contexts. The following is one of such justification from a most general point of view.

Corollary 5.3: Every measuring process is statistically equivalent to a pure measuring process.

Proof: The assertion is immediate from the construction of the measuring process in the proof of Theorem 5.1. QED

Let $M=\langle\mathscr{K},\widetilde{X},|\eta\rangle\langle\eta|,U\rangle$ be a pure measuring process. Define an isometry $V:\mathscr{H}\to\mathscr{H}\otimes\mathscr{K}$ by $V\xi=U(\xi\otimes\eta)$ for all ξ in $\mathscr{H}$. Let $\mathscr{I}$ be the corresponding CP instrument. Then it is easy to see that

$$\mathscr{I}(B,a)=E_\sigma[U^*(a\otimes\widetilde{X}(B))U]=V^*(a\otimes\widetilde{X}(B))V,$$

for all a in $\mathscr{L}(\mathscr{H})$, B in $\mathscr{B}$.

The following result justifies our postulate, which is tacit in Eq. (2.2), that *semiobservables can be measured.*

Corollary 5.4: For any semiobservable X, there is a measuring process of X.

Proof: By proposition 4.1, for any semiobservable X, there is an X-compatible CP instrument $\mathscr{I}$. Then any realization of $\mathscr{I}$ obtained by Theorem 5.1 is a measuring process of X. QED

Consider the case that X is an observable. In this case the classification of measuring processes is surprisingly simpler, that is, the measuring processes of X are determined by their total state changes $\rho\to\rho^{\Omega}$.

Theorem 5.5: Let X be an observable on $\mathscr{H}$ with value space $(\Omega,\mathscr{B})$. Then there is a one-to-one correspondence between statistical equivalence classes of measuring processes M of X and X-compatible transition maps T on $\mathscr{L}(\mathscr{H})$, which is given by the relation

$$\mathrm{Tr}[\rho X(B)]\mathrm{Ex}^M(a|B;\rho)=\mathrm{Tr}[\rho X(B)T(a)],\qquad(5.7)$$

for any a in $\mathscr{L}(\mathscr{H})$, ρ in $\Sigma(\mathscr{H})$, B in $\mathscr{B}$.

Proof: The assertion follows immediately from Proposition 4.4 and Theorem 5.1. QED

6. REPEATABILITY

Consider von Neumann's repeatability hypothesis (Ref. 10, pp. 214, 335):

(M) If the physical quantity is measured twice in succession in a system, then we get the same value each time.

Let $M=\langle\mathscr{K},\widetilde{X},\sigma,U\rangle$ be a measuring process of a semiobservable X. If X is discrete, then it is easy to see that (M) is equivalent to

$$(\mathrm{M}')\quad \mathrm{Ex}^M(X(\{\lambda\})|\{\mu\};\rho)=\delta_{\lambda,\mu},$$

for all ρ in $\Sigma(\mathscr{H})$ and all λ,μ in Ω, whenever $\mathrm{Tr}[\rho X(\{\mu\})]\neq0$. We say that a measuring process M of X is *weakly repeatable* if

$$(\mathrm{R})\quad \mathrm{Ex}^M(X(C)|B;\rho)=\mathrm{Tr}[\rho X(B\cap C)]/\mathrm{Tr}[\rho X(B)],$$

for any ρ in $\Sigma(\mathscr{H})$, B, C in $\mathscr{B}$, whenever $\mathrm{Tr}[\rho X(B)]\neq0$. Then it is easy to see that if X is discrete the condition (M′) and (R) are equivalent. The condition (R) appeared first in Ref. 1 for instruments. We say that a CP instrument $\mathscr{I}$ is *weakly repeatable* if $\mathscr{I}(B)X(C)=X(B\cap C)$ for all B,C in $\mathscr{B}$, where X is the associate semiobservable of $\mathscr{I}$. It is easily seen that a measuring process M is weakly repeatable if and only if the corresponding CP instrument $\mathscr{I}$ is weakly repeatable. In Ref. 1, p. 247, it is conjectured that the existence of repeatable instruments for continuous observables is doubtful even in the case of standard quantum theory. In the present section, we shall prove this conjecture, that is, we shall prove that there is at least one X-compatible weakly repeatable CP instrument on $\mathscr{L}(\mathscr{H})$ if and only if X is discrete.

Let $\mathscr{M}$ be a von Neumann algebra on $\mathscr{H}$ and $(\Omega,\mathscr{B})$ be a Borel space. Let $\mathscr{I}$ be a weakly repeatable CP instrument on $\mathscr{M}$ with value space $(\Omega,\mathscr{B})$, X its associate semiobservable, and T its associate map. We can assume that $\mathscr{I}$ is of the form $\mathscr{I}(B,a)=V^*E(B)\pi(a)V$ for any B in $\mathscr{B}$, a in $\mathscr{M}$, as in Proposition 4.2.

Lemma 6.1: For any B,C in $\mathscr{B}$, a in $\mathscr{M}$, we have
(1) $T(X(B)^2)=X(B)$,
(2) $\mathscr{I}(B\cap C,a)=\mathscr{I}(C,aX(B))=\mathscr{I}(C,X(B)a)$,
(3) $\mathscr{I}(B,a)=T(aX(B))=T(X(B)a)$.

Proof: Since $\mathscr{I}(B,X(B))=X(B)$ by the weak repeatability of $\mathscr{I}$, a routine computation leads that

$$\begin{aligned}&(\pi(X(B))V-E(B)V)^*(\pi(X(B))V\\&\quad-E(B)V)=T(X(B)^2)-X(B),\end{aligned}\qquad(6.1)$$

for any B in $\mathscr{B}$. Thus we have $T(X(B)^2)\geqslant X(B)$. On the other hand, we have $X(B)^2\leqslant X(B)$, since $0\leqslant X(B)\leqslant1$. By weak repeatability, $T(X(B))=X(B)$, so that $X(B)=T(X(B))\geqslant T(X(B)^2)$. Thus we have the relation (1). It follows that the left-hand side of Eq. (6.1) is 0, so that we have $\pi(X(B))V=E(B)V$ and $V^*\pi(X(B))=V^*E(B)$. Thus for any B,C in $\mathscr{B}$, a in $\mathscr{M}$, we have $\mathscr{I}(B\cap C,a)=V^*E(B\cap C)\pi(a)V=V^*E(B)E(C)\pi(a)V=V^*\pi(X(B))E(C)\pi(a)V=V^*E(C)\pi(X(B)a)V=\mathscr{I}(C,X(B)a)$. By the analogous way we can show that $\mathscr{I}(B\cap C,a)=\mathscr{I}(C,aX(B))$. Thus we obtain the relation (2). The relation (3) is obtained by putting $C=\Omega$ in (2). QED

Let p be the least projection in $X(\mathscr{B})''$ such that $T(p)=1$.

Lemma 6.2: For any x in $\mathscr{M}$, $T(x)=T(xp)=T(px)=T(pxp)$.

Proof: For any ξ,η in $\mathscr{H}$, we have

$$\begin{aligned}|(T(x-px)\xi,\eta)|&=|(V^*\pi(1-p)\pi(x)V\xi,\eta)|\\&=|(\pi(x)V\xi,\pi(1-p)V\eta)|\leqslant\|\pi(x)V\xi\|\ \|\pi(1-p)V\eta\|\\&=\|\pi(x)V\xi\|\ \|(V^*\pi(1-p)V\eta,\eta)\|^{1/2}\\&=\|\pi(x)V\xi\|(T(1-p)\eta,\eta)^{1/2}=0.\end{aligned}$$

Thus we have $T(x)=T(px)$. The rest of the assertions are immediate. QED

Lemma 6.3: For every x in $X(\mathscr{B})''$ with $x \geqslant 0$, if $T(x) = 0$, then $pxp = 0$.

Proof: Let e be the range projection of x. Since e is a limit of polynomials of x not containing the constant term in the strong operator topology, we have $T(e) = 0$. Thus $1 - e \geqslant p$ so that $ep = pe = 0$. It follows that $pxp = 0$. QED

Define a positive operator valued measure P: $\mathscr{B} \rightarrow X(\mathscr{B})''$ by the relation $P(B) = pX(B)p$ for all B in $\mathscr{B}$.

Lemma 6.4: P is a projection valued measure such that $P(B) = pX(B) = X(B)p$ for any B in $\mathscr{B}$.

Proof: By Lemma 6.2, we have $T(P(B)) = T(pX(B)p) = T(X(B))$. By Lemmas 6.1 and 6.2, we have $T(P(B)^2) = T(pX(B)pX(B)p) = T(X(B)pX(B)) = \mathscr{I}(B, pX(B)) = \mathscr{I}(B \cap B, p) = \mathscr{I}(B, p) = T(pX(B)) = T(X(B))$. It follows that $T(P(B) - P(B)^2) = 0$. Since $P(B) - P(B)^2$ belongs to $X(\mathscr{B})''$, we have $P(B)^2 = P(B)$ by Lemma 6.3. Thus P is a projection-valued measure. We have $T((P(B) - X(B)p)^*(P(B) - X(B)p)) = 0$, by the routine computations. Thus, by Lemma 6.3, $P(B) = X(B)p$, since $P(B) - X(B)p$ is in $X(\mathscr{B})''$. By the positivity, we have $P(B) = pX(B)$. QED

Theorem 6.5: For any weakly repeatable CP instrument $\mathscr{I}$ on $\mathscr{M}$ with value space $(\Omega, \mathscr{B})$, there is a projection-valued measure P: $\mathscr{B} \rightarrow X(\mathscr{B})''$ such that

$$\mathscr{I}(B,a) = T(aP(B)) = T(P(B)a)$$

and that

$$P(B) = P(\Omega)X(B) = X(B)P(\Omega),$$

for any B in $\mathscr{B}$, a in $\mathscr{L}(\mathscr{H})$.

Proof: The assertion follows immediately from Lemmas 6.1 and 6.4. QED

We suppose for the rest of this section that the value space $(\Omega, \mathscr{B})$ is a standard Borel space and that the Hilbert space $\mathscr{H}$ is separable. We say that a positive operator valued measure P is *discrete* if there is a countable set $\Omega_0 \subseteq \Omega$ such that $P(\Omega \setminus \Omega_0) = 0$ and that a CP instrument is *discrete* if the associate semiobservable is discrete.

Theorem 6.6: Let $(\Omega, \mathscr{B})$ be a standard Borel space, and let $\mathscr{H}$ be a separable Hilbert space. Then every weakly repeatable CP instrument $\mathscr{I}$ on $\mathscr{L}(\mathscr{H})$ with value space $(\Omega, \mathscr{B})$ is discrete.

Proof: Let P be a projection-valued measure obtained in Theorem 6.5. By the relation $X(B) = \mathscr{I}(B,1) = T(P(B))$ for every B in $\mathscr{B}$, we have only to show that P is discrete. By Ref. 4, Lemma 4.4.1, there is a countable set B_0 such that $B \rightarrow P(B \cap B_0)$ is a discrete projection-valued measure with values in $\mathscr{L}(P(B_0)\mathscr{H})$ and $B \rightarrow P(B \setminus B_0)$ is a continuous projection-valued measure with values in $\mathscr{L}(P(\Omega \setminus B_0)\mathscr{H})$. Let Q be such that $Q = P(\Omega \setminus B_0)$. Then it suffices to prove that $Q = 0$. Let T be the associate map of $\mathscr{I}$ and T_0 be such that $T_0(a) = QT(a)Q$ for all a in $\mathscr{L}(Q\mathscr{H})$. Then $T_0(Q) = QT(Q)Q = QT(X(\Omega \setminus B_0))Q = QX(\Omega \setminus B_0)Q = Q$, and hence T_0 is a transition map on $\mathscr{L}(Q\mathscr{H})$. Thus there is a trace-preserving linear map S: $\mathscr{T}(Q\mathscr{H}) \rightarrow \mathscr{T}(Q\mathscr{H})$ such that $S^* = T_0$. For any a in $\mathscr{L}(Q\mathscr{H})$, B in $\mathscr{B}$, ρ in $\mathscr{T}(Q\mathscr{H})$, we have

$$\begin{aligned}\mathrm{Tr}[aP(B \setminus B_0)S(\rho)] &= \mathrm{Tr}[T_0(aP(B \setminus B_0))\rho] \\ &= \mathrm{Tr}[QT(aP(B \setminus B_0))Q\rho] = \mathrm{Tr}[QT(P(B \setminus B_0)a)Q\rho] \\ &= \mathrm{Tr}[T_0(P(B \setminus B_0)a)\rho] = \mathrm{Tr}[P(B \setminus B_0)aS(\rho)].\end{aligned}$$

It follows that $P(B \setminus B_0)S(\rho) = S(\rho)P(B \setminus B_0)$ for any B in $\mathscr{B}$, ρ in $\mathscr{T}(Q\mathscr{H})$. Since $B \rightarrow P(B \setminus B_0)$ is a continuous projection-valued measure, we can conclude that $S = 0$ (see, Ref. 4, Theorem 4.3.3), and hence $Q = T_0(Q) = 0$. QED

7. LOCALITY

Let $\mathscr{H}$ be a Hilbert space and $\mathscr{M}$ a von Neumann algebra on $\mathscr{H}$. Let X be an observable in $\mathscr{M}$ with value space $(\Omega, \mathscr{B})$. A transition map T on $\mathscr{M}$ is called *X-local* if $T(X(B)) = X(B)$ for any B in $\mathscr{B}$. It is easy to see that T is X-local if and only if $Tx = x$ for any x in $X(\mathscr{B})''$.

Let $\{x_1, \dots, x_n\}$ be a mutually commutable family of self-adjoint operators on $\mathscr{H}$ corresponding to a family of simultaneously measurable observables of a quantum system. Suppose that X is the joint spectral measure of $\{x_1, \dots, x_n\}$ on $\mathscr{H}$ with value space $(\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n))$. Recently, Mercer[9] proposed that the total state change $\rho \rightarrow \rho'$ caused by a simultaneous measurement of $x_1, \dots, x_n$ should be described by an X-local transition map T on $\mathscr{L}(\mathscr{H})$ in such a way $\rho' = \rho T$ (see Ref. 9, p. 244). However, we should notice that the X-locality is not sufficient for describing state transformations caused by measurements. In fact, the identity transformation on $\mathscr{L}(\mathscr{H})$ is obviously an X-local transition map for any observable X, in spite of the fact that we cannot measure any nontrivial quantum observable unchanging every state of the system. Thus we have to impose some further requirements for eliminating such physically irrelevant X-local transition maps in order to describe a state change caused by the measurement of X. A moderate one of such requirements seems the existence of a measuring process for observables $x_1, \dots, x_n$, whose state change is the given X-local transition map. The following result is an easy consequence of the results obtained in the previous sections, but shows that such requirement cannot be fulfilled unless all observables $x_1, \dots, x_n$ are discrete.

Proposition 7.1: Let $\mathscr{M}$ be a von Neumann algebra on a Hilbert space $\mathscr{H}$ and X an observable in $\mathscr{M}$ with value space $(\Omega, \mathscr{B})$. There is a one-to-one correspondence between X-compatible X-local transition maps T on $\mathscr{M}$ and X-compatible weakly repeatable CP-instruments $\mathscr{I}$ on $\mathscr{M}$, which is given by

$$\mathscr{I}(B,a) = X(B)T(a), \tag{7.1}$$

for any B in $\mathscr{B}$, a in $\mathscr{M}$.

Proof: It is known in the proof of Ref. 1, Theorem 7, that a decomposable CP instrument $\mathscr{I}$ is weakly repeatable if and only if

$$T(X(B)) = X(B) \quad \text{and} \quad X(B \cap C) = X(B)X(C),$$

for any B, C in $\mathscr{B}$. Since every X-compatible CP instrument is decomposable, the assertion follows immediately from Proposition 4.4 QED

Theorem 7.2: Let X be an observable on a separable Hilbert space $\mathscr{H}$ whose value space is a standard Borel space and T be an X-local transition map on $\mathscr{L}(\mathscr{H})$. If there is a measuring process $M = \langle \mathscr{K}, \tilde{X}, \sigma, U \rangle$ of X such that $\rho^{\Omega} = \rho T$ for any ρ in $\Sigma(\mathscr{H})$ [see Eq. (3.10)], then X is discrete.

Proof: It is obvious that T is the associate map of the CP instrument $\mathscr{I}$ determined by the measuring process M.

Thus, by Proposition 7.1, the CP instrument $\mathscr{I}$ is weakly repeatable and hence by Theorem 6.6 the corresponding observable X is discrete. QED

8. THE WIGNER–ARAKI–YANASE THEOREM

It was pointed out by Wigner[15] that the presence of a conservation law puts a limitation of the measurement of an operator which does not commute with the observed quantity. A proof of the above assertion was given by Araki and Yanase[16] in the conventional framework of measurement theory. In this section, we shall give another proof in our framework and under somewhat general assumptions. Our assertion is the following.

Theorem 8.1: Let X be an observable on a Hilbert space $\mathscr{H}$ with value space $(\Omega,\mathscr{B})$. Let $M = \langle \mathscr{K},\widetilde{X},\sigma,U\rangle$ be a weakly repeatable measuring process of X. Suppose that there is L_1 in $\mathscr{L}(\mathscr{H})$ and L_2 in $\mathscr{L}(\mathscr{K})$ such that $[U,L] = 0$, where $L = L_1 \otimes 1 + 1 \otimes L_2$. Then $L_1 \in X(\mathscr{B})'$.

For the proof we use the following.

Lemma 8.2. Let $M = \langle \mathscr{K},\widetilde{X},\sigma,U\rangle$ be a measuring process of an observable X on $\mathscr{H}$, and $\sigma = \Sigma_i \lambda_i |\eta_i\rangle\langle\eta_i|$ be the spectral decomposition of σ. Then for any i, $M_i = \langle \mathscr{K},\widetilde{X}, |\eta_i\rangle\langle\eta_i|,U\rangle$ is a pure measuring process of X such that

$$E_\sigma[U^*AU] = \sum_i \lambda_i E_{|\eta_i\rangle\langle\eta_i|}[U^*AU], \tag{8.1}$$

for any A in $\mathscr{L}(\mathscr{H}\otimes\mathscr{K})$. If M is weakly repeatable, then M_i is also weakly repeatable for every i.

Proof: The formula (8.1) is obtained from Lemma 2.3. Let $B \in \mathscr{B}$. Then

$$X(B) = E_\sigma[U^*(1\otimes\widetilde{X}(B))U]$$
$$= \sum_i \lambda_i E_{|\eta_i\rangle\langle\eta_i|}[U^*(1\otimes\widetilde{X}(B))U].$$

Since any projection is an extreme point of the positive part of the unit sphere of $\mathscr{L}(\mathscr{H})$, we have that

$$X(B) = E_{|\eta_i\rangle\langle\eta_i|}[U^*(1\otimes\widetilde{X}(B))U],$$

for any i. Thus M_i is a measuring process of X. Since M_i is weakly repeatable if $X(B) = E_{|\eta_i\rangle\langle\eta_i|}[U^*(X(B)\otimes 1)U]$ for any B in $\mathscr{B}$ by Proposition 7.1, the assertion for the weak repeatability follows from the same reasoning. QED

Proof of Theorem 8.1: By Theorem 5.5, there is an X compatible transition map T such that $E_\sigma[U^*(a\otimes\widetilde{X}(B))U] = X(B)T(a)$ for any B in $\mathscr{B}$, a in $\mathscr{L}(\mathscr{H})$. Then we have

$$T(L_1) + E_\sigma[U^*(1\otimes L_2)U]$$
$$= E_\sigma[U^*(L_1\otimes 1 + 1\otimes L_2)U]$$
$$= E_\sigma[L_1\otimes 1 + 1\otimes L_2]$$
$$= L_1 + [\mathrm{Tr}(\sigma L_2)]1.$$

Since T is X-compatible, $T(L_1)\in X(\mathscr{B})'$. Thus we have only to show that $E_\sigma[U^*(1\otimes L_2)U]\in X(\mathscr{B})'$. By Lemma 8.2, we can assume without any loss of generality that there is a unit vector η in $\mathscr{K}$ such that $\sigma = |\eta\rangle\langle\eta|$, so that there is an isometry V: $\mathscr{H}\to\mathscr{H}\otimes\mathscr{K}$ such that $E_\sigma[U^*AU] = V^*AV$ for all A in $\mathscr{L}(\mathscr{H}\otimes\mathscr{K})$, where $V\xi = U(\xi\otimes\eta)$ for any ξ in $\mathscr{H}$. Let $B\in\mathscr{B}$. Since the CP instrument $\mathscr{I}$ such that $\mathscr{I}(B,a) = V^*(a\otimes\widetilde{X}(B))V$ is weakly repeatable, we have $V^*(X(B)\otimes 1)V = \mathscr{I}(\Omega,X(B)) = X(B)$. Thus by the simple computations we have

$$((X(B)\otimes 1)V - VX(B))^*((X(B)\otimes 1)V - VX(B)) = 0,$$

and hence $(X(B)\otimes 1)V = VX(B)$ and $V^*(X(B)\otimes 1) = X(B)V^*$. It follows that

$$V^*(1\otimes L_2)VX(B) = V^*(X(B)\otimes L_2)V = X(B)V^*(1\otimes L_2)V.$$

Thus we conclude that $E_\sigma[U^*(1\otimes L_2)U]\in X(\mathscr{B})'$. QED

9. CONVENTIONAL MEASURING PROCESSES

In the conventional theory of quantum measurement, the only class of measuring processes studied at all seriously is as follows. Let $\mathscr{H}$ be a separable Hilbert space and X be a discrete observable on $\mathscr{H}$ with value space $(\mathbb{R},\mathscr{B}(\mathbb{R}))$. Let $\{\xi_{ij}\}$ be a complete orthonormal set of eigenvectors of X where the eigenvalue of ξ_{ij} is λ_i. Let $\mathscr{K}$ be another separable Hilbert space with complete orthonormal vectors $\{\eta_i\}$. Let η be a unit vector of $\mathscr{K}$ and U be a unitary operator on $\mathscr{H}\otimes\mathscr{K}$ satisfying

$$U(\xi_{ij}\otimes\eta) = \xi_{ij}\otimes\eta_i \tag{9.1}$$

for any i, j. Then $M = \langle \mathscr{K},\widetilde{X},|\eta\rangle\langle\eta|,U\rangle$ is a measuring process of X, where $\widetilde{X} = \Sigma_i\lambda_i|\eta_i\rangle\langle\eta_i|$. In the sequel, we call this measuring process a *conventional* measuring process of X. The total state change corresponding to M is of the form

$$\rho\to\rho' = \sum_i P_i\rho P_i, \tag{9.2}$$

where $P_i = X(\{\lambda_i\})$, i.e., $P_i = \Sigma_j|\xi_{ij}\rangle\langle\xi_{ij}|$. In fact, for $\rho = \Sigma_{ijkl}\,\mu_{ijkl}|\xi_{ij}\rangle\langle\xi_{kl}|$ in $\Sigma(\mathscr{H})$, we have

$$\rho' = E_{\mathscr{K}}[U(\rho\otimes|\eta\rangle\langle\eta|)U^*]$$
$$= \sum_{ijkl}\mu_{ijkl}E_{\mathscr{K}}[|\xi_{ij}\otimes\eta_i\rangle\langle\xi_{kl}\otimes\eta_k|]$$
$$= \sum_{ijkl}\mu_{ijkl}(\eta_i,\eta_k)|\xi_{ij}\rangle\langle\xi_{kl}|$$
$$= \sum_i P_i\rho P_i$$

[see Eq. (3.10)]. Conversely, every state change given by Eq. (9.2) is realized as the above measuring process M as shown by von Neumann (see Ref. 10, p. 442). By Eq. (9.2) the corresponding CP instrument $\mathscr{I}$ is of the form

$$\mathscr{I}(B,a) = \sum_{\lambda_i\in B} P_i a P_i, \tag{9.3}$$

for any B in $\mathscr{B}(\mathbb{R})$, a in $\mathscr{L}(\mathscr{H})$, and the corresponding transition map T is a conditional expectation onto $X(\mathscr{B}(\mathbb{R}))'$.

In the present section, we shall give a characterization of the above conventional measuring processes up to statistical equivalence. A similar problem is considered in Refs. 1 and 21 in different methods.

Definition 9.1: Let X be a semiobservable on $\mathscr{H}$ with value space $(\mathbb{R},\mathscr{B}(\mathbb{R}))$. A measuring process M of X is called *standard* if it satisfies the following three conditions.

(WR) (Weak repeatability) M is weakly repeatable.

(MD) (Minimal disturbance condition) The set $\{\rho\in\Sigma(\mathscr{H});\rho^{\mathrm{R}}\neq\rho\}$ is minimal in the set inclusion among all measuring process of X.

(ND) (Nondegeneracy condition) For any B in $\mathscr{B}(\mathbb{R})$ with $X(B)\neq 0$, there is some ρ in $\Sigma(\mathscr{H})$ such that $\mathrm{Tr}[\rho^{\mathrm{R}}X(B)]\neq 0$.

Let M be a measuring process of X. Denote by $F(M)$ the set of all nondisturbed states, i.e., $F(M)=\{\rho\in\sigma(\mathscr{H});\rho^{\mathrm{R}}=\rho\}$. Obviously, M satisfies (MD) if and only if for any measuring process M' of X, $F(M)\subseteq F(M')$ implies $F(M')\subseteq F(M)$.

Proposition 9.2: Let M be a conventional measuring process of a discrete observable X. Then M is standard.

Proof: It is well known that M is weakly repeatable. The condition (ND) is easy to check. Thus we shall prove that M satisfies the condition (MD). Let M' be a measuring process of X such that $F(M)\subseteq F(M')$. Denote by T and S the transition maps corresponding to M and M', respectively. Let $\rho\in\mathscr{T}(\mathscr{H})$ be such that $\rho S=\rho$. Then it suffices to show that $\rho T=\rho$. Since T is a conditional expectation onto $X(\mathscr{B}(\mathbb{R}))'$ and by the X-compatibility of S the range of S is contained in $X(\mathscr{B}(\mathbb{R}))'$, we have $TS=S$. Since $T^2=T$, we have $(\rho T)T=\rho T$ so that $\rho T\in F(M)$. Thus by the assumption that $F(M)\subseteq F(M')$, $\rho T\in F(M')$. It follows that $\rho T=\rho TS=\rho S=\rho$. This concludes the proof. QED

Theorem 9.3: Let $\mathscr{H}$ be a separable Hilbert space and X be a semiobservable on $\mathscr{H}$ with value space $(\mathbf{R},\mathscr{B}(\mathbf{R}))$. Let M be a standard measuring process of X. Then X is a discrete observable, and M is statistically equivalent to a conventional measuring process of X.

Proof: Let $\mathscr{I}$ be the CP instrument corresponding to a standard measuring process M of X. Since $\mathscr{I}$ is weakly repeatable, by Theorem 6.6, X is discrete and, by Theorem 6.5, there is an orthogonal family $\{P_\lambda;\lambda\in R\}$ of projections in $X(\mathscr{B}(R))''$ such that

$$\mathscr{I}(B,a)=T\Big(\sum_{\lambda\in B}P_\lambda aP_\lambda\Big), \tag{9.4}$$

for all B in $\mathscr{B}$, a in $\mathscr{L}(\mathscr{H})$. Let Q be a projection in $X(\mathscr{B}(\mathbb{R}))''$ such that $Q=1-\Sigma_{\lambda\in R}P_\lambda$. Then we have $T(Q)=0$. It follows from the condition (ND) that $Q=0$ so that $\Sigma_{\lambda\in R}P_\lambda=1$. Thus by Lemma 6.4 we have $X(B)=\Sigma_{\lambda\in B}P_\lambda$ for any B in $\mathscr{B}(\mathbb{R})$. It follows that X is an observable. Let M' be a conventional measuring process of X and $\mathscr{I}'$ be the corresponding CP instrument. Then

$$\mathscr{I}'(B,a)=\sum_{\lambda\in B}P_\lambda aP_\lambda, \tag{9.5}$$

for any B in $\mathscr{B}$, a in $\mathscr{L}(\mathscr{H})$. Denote by T and S the corresponding transition maps of M and M', respectively. Since T is X-compatible, we have $T(a)=\Sigma_{\lambda\in R}P_\lambda T(a)=\Sigma_{\lambda\in R}P_\lambda T(a)P_\lambda=ST(a)$, for any a in $\mathscr{L}(\mathscr{H})$. On the other hand, by Eq. (9.4) we have $T=TS$. It follows that $T=ST=TS$. For any ρ in $\mathscr{T}(\mathscr{H})$, if $\rho T=\rho$, then $\rho S=\rho TS=\rho T=\rho$ and hence $F(M)\subseteq F(M')$. Thus by the condition (MD), $F(M')=F(M)$. Let $\rho\in\mathscr{T}(\mathscr{H})$. Then since $S^2=S,\rho S\in F(M')$, so that $\rho ST=\rho S$. It follows that $S=ST$. Thus we have $T=S$. Therefore, by Theorem 5.5, M is statistically equivalent to a conventional measuring process M' of X. QED

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