A Kolmogorov Extension Theorem for POVMs*

Roderich Tumulka†

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Abstract

We prove a theorem about positive-operator-valued measures (POVMs) that is an analog of the Kolmogorov extension theorem, a standard theorem of probability theory. According to our theorem, if a sequence of POVMs G_n on $\mathbb{R}^n$ satisfies the consistency (or projectivity) condition $G_{n+1}(A \times \mathbb{R}) = G_n(A)$ then there is a POVM G on the space $\mathbb{R}^{\mathbb{N}}$ of infinite sequences that has G_n as its marginal for the first n entries of the sequence. We also describe an application in quantum theory.

MSC: 81Q99; 46N50. Key words: positive-operator-valued measure (POVM); construction of POVMs; Kolmogorov measure extension theorem; Daniell measure extension theorem; consistent family of measures; projective family of measures.

1 Introduction

A relevant mathematical concept for quantum physics is that of POVM (positive-operator-valued measure). It forms the natural generalization of the concept of observable represented by a self-adjoint operator. One can say that all probability measures that arise in quantum physics are of the form

$$\mathbb{P}(\cdot) = \text{tr}\big(\rho \, G(\cdot)\big), \tag{1}$$

where ρ is a density matrix in the appropriate Hilbert space $\mathscr{H}$ and G is a POVM. We recall the definition of POVM in Section 1.2.

The Kolmogorov extension theorem is a standard theorem of probability and measure theory concerning probability measures [3]. In its simplest version (due to Daniell) [3, Theorem 5.14] it asserts that if for every $n \in \mathbb{N}$, μ_n is a probability measure on $\mathbb{R}^n$ (with its Borel σ-algebra $\mathcal{B}^{\otimes n}$) such that the so-called *consistency (or projectivity) condition*

$$\mu_{n+1}(A \times \mathbb{R}) = \mu_n(A) \quad \forall A \in \mathcal{B}^{\otimes n} \tag{2}$$

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is satisfied then there exists a unique probability measure μ on the space $\mathbb{R}^{\mathbb{N}}$ of all real sequences (with the σ-algebra $\mathcal{B}^{\otimes \mathbb{N}}$ generated by the cylinder sets, i.e., sets depending only on finitely many members of the sequence) such that μ_n is its marginal distribution of the first n components. Note that (2) is a necessary condition for $\mu_n(\cdot)$ being a marginal of some probability measure on $(\mathbb{R}^{\mathbb{N}}, \mathcal{B}^{\otimes \mathbb{N}})$. The more refined versions of the Kolmogorov extension theorem we describe in Section 1.1

In our theorem we replace the probability measures with POVMs. The proof utilizes the Kolmogorov extension theorem, but has to take care of the quadratic dependence of the probabilities on ψ. In the remainder of Section 1, we provide more detail about the Kolmogorov extension theorem and the concept of POVM. In Section 2 we formulate and prove our theorem. In Section 3 we describe an application in quantum physics.

1.1 The Kolmogorov Extension Theorem

Above we formulated the simplest version of the Kolmogorov extension theorem. The statement remains true if $\mathbb{R}$ is replaced by any Borel space. Recall that a Borel space is a measurable space $(M, \mathcal{A})$ that is isomorphic (in the category of measurable spaces) to a measurable subset of the real line. Any Polish space (i.e., complete separable metric space) with its Borel σ-algebra is a Borel space; in particular, $\mathbb{R}$, $\mathbb{R}^n$, separable Hilbert spaces and separable manifolds are all Borel spaces.

The statement also remains true if, instead of M^n and $M^{\mathbb{N}}$, one considers $M_1 \times \cdots \times M_n$ and $\prod_{n \in \mathbb{N}} M_n$, provided every $(M_n, \mathcal{A}_n)$ is a Borel space, with the σ-algebras $\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n$ and $\bigotimes_{n \in \mathbb{N}} \mathcal{A}_n$ (the σ-algebra generated by the cylinder sets).

The statement still remains true if $\mathbb{N}$ is replaced by any (possibly uncountable) set T. The name "Kolmogorov extension theorem" often refers to this version in particular. Then it is to be formulated as follows. Let $\hat{T}$ be the collection of all finite subsets of T. Suppose we are given a family $\{(M_t, \mathcal{A}_t) : t \in T\}$ of Borel spaces and a family $\{\mu_K : K \in \hat{T}\}$ of probability measures μ_K on $(M^K, \mathcal{A}^K) := (\prod_{t \in K} M_t, \bigotimes_{t \in K} \mathcal{A}_t)$ satisfying the consistency condition

$$K \subseteq K' \implies \forall A \in \mathcal{A}^K : \mu_{K'}\Big(A \times \prod_{t \in K' \setminus K} M_t\Big) = \mu_K(A) \,. \tag{3}$$

Then there is a unique probability measure μ on $(M^T, \mathcal{A}^T) := (\prod_{t \in T} M_t, \bigotimes_{t \in T} \mathcal{A}_t)$ (where the σ-algebra is again generated by the cylinder sets, i.e., sets depending only on finitely many t's) such that μ_K is the marginal of μ on M^K.

1.2 The Concept of POVM

Let $\mathscr{H}$ be a complex Hilbert space (not necessarily separable) and $\mathscr{B}(\mathscr{H})$ the space of bounded operators on $\mathscr{H}$. We recall the following (standard) definition.

Definition 1 A **POVM** (positive operator valued measure) on the measurable space $(\Omega, \mathcal{A})$ acting on $\mathscr{H}$ is a mapping $G : \mathcal{A} \to \mathscr{B}(\mathscr{H})$ from a σ-algebra $\mathcal{A}$ on the set Ω such that

(i) $G(\Omega) = I$, the identity operator,

(ii) $G(A) \geq 0$ (i.e., $G(A)$ is a positive operator) for every $A \in \mathcal{A}$, and

(iii) (σ-additivity) for any sequence of pairwise disjoint sets $A_1, A_2, \ldots \in \mathcal{A}$

$$G\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \sum_{i=1}^{\infty} G(A_i) \,, \tag{4}$$

where the sum on the right hand side converges weakly, i.e., $\sum_i \langle \psi | G(A_i) \psi \rangle$ converges for every $\psi \in \mathscr{H}$ to $\langle \psi | G(\cup_i A_i) \psi \rangle$.

If G is a POVM on $(\Omega, \mathcal{A})$ and $\psi \in \mathscr{H}$ with $\|\psi\| = 1$, then $A \mapsto \langle \psi | G(A) \psi \rangle$ is a probability measure on $(\Omega, \mathcal{A})$.

2   Theorem and Proof

Theorem 1 *Let $\mathscr{H}$ be a Hilbert space, T be an arbitrary index set, and for every $t \in T$ let $(M_t, \mathcal{A}_t)$ be a Borel space. Let $\hat{T}$ denote the collection of all finite subsets of T. For every $K \in \hat{T}$ let G_K be a POVM on $(M^K, \mathcal{A}^K) := (\prod_{t \in K} M_t, \bigotimes_{t \in K} \mathcal{A}_t)$ acting on $\mathscr{H}$. If the G_K satisfy the consistency condition*

$$K \subseteq K' \implies \forall A \in \mathcal{A}^K : G_{K'}\Big(A \times \prod_{t \in K' \backslash K} M_t\Big) = G_K(A) \tag{5}$$

then there is a unique POVM G on $(M^T, \mathcal{A}^T) := (\prod_{t \in T} M_t, \bigotimes_{t \in T} \mathcal{A}_t)$ such that G_K is the marginal of G on M^K. Moreover, for every $\psi \in \mathscr{H}$ with $\|\psi\| = 1$ there exists a unique probability measure μ^ψ on $(M^T, \mathcal{A}^T)$ such that for all $K \in \hat{T}$ and all sets $A \in \mathcal{A}^K$,

$$\mu^\psi\Big(A \times \prod_{t \in T \backslash K} M_t\Big) = \langle \psi | G_K(A) \psi \rangle \,, \tag{6}$$

and in fact $\mu^\psi(\cdot) = \langle \psi | G(\cdot) \psi \rangle$.

(To appreciate why the last sentence is not a reformulation of the previous ones but an independent statement, the reader should note that even if the POVM G that extends all the G_K is unique, we must exclude the possibility of a further measure $\tilde{\mu}^\psi$ that is *not* given by a POVM but does extend the $\mu_K^\psi(\cdot) = \langle \psi | G_K(\cdot) \psi \rangle$.)

As a special case of the theorem, we obtain the following statement for $T = \mathbb{N}$ and $(M_t, \mathcal{A}_t) = (M, \mathcal{A})$.

Corollary 1 *Let $(M, \mathcal{A})$ be a Borel space and G_n, for every $n \in \mathbb{N}$, a POVM on $(M^n, \mathcal{A}^{\otimes n})$. If the family $\{G_n\}_{n \in \mathbb{N}}$ satisfies the consistency property*

$$G_{n+1}(A \times M) = G_n(A) \quad \forall A \in \mathcal{A}^{\otimes n} \tag{7}$$

3

then there exists a unique POVM G on $(M^{\mathbb{N}}, \mathcal{A}^{\otimes \mathbb{N}})$ such that for all $n \in \mathbb{N}$ and all sets $A \in \mathcal{A}^{\otimes n}$,

$$G_n(A) = G(A \times M^{\mathbb{N}}). \tag{8}$$

Moreover, for every $\psi \in \mathscr{H}$ with $\|\psi\| = 1$ there exists a unique probability measure μ^{ψ} on $(M^{\mathbb{N}}, \mathcal{A}^{\otimes \mathbb{N}})$ such that for all $n \in \mathbb{N}$ and all sets $A \in \mathcal{A}^{\otimes n}$, $\mu^{\psi}(A \times M^{\mathbb{N}}) = \langle \psi | G_n(A) \psi \rangle$, and in fact $\mu^{\psi}(\cdot) = \langle \psi | G(\cdot) \psi \rangle$.

Proof of Theorem 1. For arbitrary $\psi \in \mathscr{H}$ with $\|\psi\| = 1$, define the probability measure

$$\mu_K^{\psi}(\cdot) = \langle \psi | G_K(\cdot) \psi \rangle \tag{9}$$

on $(M^K, \mathcal{A}^K)$. Because of (5), (3) is fulfilled, so the family $(\mu_K^{\psi})_{K \in \hat{T}}$ of probability measures is consistent, and by the Kolmogorov extension theorem there exists a unique measure μ^{ψ} on $(M^T, \mathcal{A}^T)$ such that for all $K \in \hat{T}$ and all $A_K \in \mathcal{A}^K$,

$$\mu_K^{\psi}(A_K) = \mu^{\psi}\Big(A_K \times \prod_{t \in T \setminus K} M_t \Big). \tag{10}$$

To see that $\mu^{\psi}(\cdot) = \langle \psi | G(\cdot) \psi \rangle$ for some POVM $G(\cdot)$, we first define, for every $A \in \mathcal{A}^T$ and every $\psi \in \mathscr{H}$, the complex number $\tilde{\mu}^{\psi}(A)$ by

$$\tilde{\mu}^{\psi}(A) := \begin{cases} \|\psi\|^2 \, \mu^{\psi/\|\psi\|}(A) & \text{if } \psi \neq 0 \\ 0 & \text{if } \psi = 0 \, , \end{cases} \tag{11}$$

which allows us to use any $\psi \in \mathscr{H}$, also with $\|\psi\| \neq 1$. Furthermore, we define, for every $\psi, \phi \in \mathscr{H}$, the complex number $\mu_{\psi,\phi}(A)$ by "polarization":

$$\mu_{\psi,\phi}(A) := \tilde{\mu}^{\frac{1}{2}\psi + \frac{1}{2}\phi}(A) - \tilde{\mu}^{\frac{1}{2}\psi - \frac{1}{2}\phi}(A) + i\tilde{\mu}^{\frac{1}{2}\psi - \frac{i}{2}\phi}(A) - i\tilde{\mu}^{\frac{1}{2}\psi + \frac{i}{2}\phi}(A) \, , \tag{12}$$

The definitions (11) and (12) are so chosen that if A is a cylinder set,

$$A = A_K \times \prod_{t \in T \setminus K} M_t \, , \tag{13}$$

then

$$\tilde{\mu}^{\psi}(A) = \langle \psi | G_K(A_K) \psi \rangle \quad \text{and} \quad \mu_{\psi,\phi}(A) = \langle \psi | G_K(A_K) \phi \rangle \, . \tag{14}$$

Note that $\mu_{\psi,\phi}$ is a complex measure on $(M^T, \mathcal{A}^T)$ since, by (12), it is a complex linear combination of finite measures. Since the cylinder sets form a $\cap$-stable generator of $\mathcal{A}^T$, two complex measures on $\mathcal{A}^T$ coincide as soon as they agree on the cylinder sets. Therefore, the three equations

$$\mu_{\psi+\psi',\phi}(A) = \mu_{\psi,\phi}(A) + \mu_{\psi',\phi}(A) \quad \forall \psi, \psi', \phi \in \mathscr{H} \, , \tag{15}$$

$$\mu_{z\psi,\phi}(A) = z^* \, \mu_{\psi,\phi}(A) \quad \forall z \in \mathbb{C} \forall \psi, \phi \in \mathscr{H} \, , \tag{16}$$

4

and
$$\mu_{\phi,\psi}(A) = \mu_{\psi,\phi}(A)^* \quad \forall \psi, \phi \in \mathscr{H}, \tag{17}$$

which hold for cylinder sets A because of (14), hold for all $A \in \mathcal{A}^T$. Thus, for every fixed $A \in \mathcal{A}^T$, $(\psi, \phi) \mapsto \mu_{\psi,\phi}(A)$ is a Hermitian sesquilinear form on $\mathscr{H}$. Since

$$\mu_{\psi,\psi}(A) = \tilde{\mu}^\psi(A) = \|\psi\|^2 \, \mu^{\psi/\|\psi\|}(A) \leq \|\psi\|^2, \tag{18}$$

this sesquilinear form is bounded and has, in fact, norm ≤ 1. Therefore, by the Riesz lemma [4, p. 43] there is a bounded operator $G(A)$ such that

$$\mu_{\psi,\phi}(A) = \langle \psi | G(A) \, \phi \rangle. \tag{19}$$

The uniqueness of $G(A)$ follows from the fact that if $G'(A) \neq G(A)$ then there is $\psi \in \mathscr{H}$ with $\|\psi\| = 1$ such that $\langle \psi | G'(A) \, \psi \rangle \neq \langle \psi | G(A) \, \psi \rangle$. We now have that $\tilde{\mu}^\psi(A) = \mu_{\psi,\psi}(A) = \langle \psi | G(A) \, \psi \rangle$.

We check that G is a POVM: $G(A)$ is positive since $\mu_{\psi,\psi}(A) = \tilde{\mu}^\psi(A) \geq 0$ for every ψ; $G(\cdot)$ is σ-additive in the weak sense because $\mu_{\psi,\phi}(\cdot)$ is σ-additive; $G(\cdot)$ restricted to the cylinder sets $A_k \times \prod_{t \in T \setminus K} M_t$ is $G_K(\cdot)$, and thus $G(M^T) = I$.

The uniqueness statement for the extension $G(\cdot)$ of all $G_K(\cdot)$'s follows from the uniqueness of $G(A)$ satisfying (19), together with the fact that for every fixed ψ the measure μ^ψ is unique.

It remains to check that G_K is the appropriate marginal of G. Let A be of the form (13) with $A_K \in \mathcal{A}^K$. For all $\psi \in \mathscr{H} \setminus \{0\}$,

$$\langle \psi | G(A) \, \psi \rangle = \tilde{\mu}^\psi(A) = \|\psi\|^2 \, \mu^{\psi/\|\psi\|}(A) = \|\psi\|^2 \, \mu_K^{\psi/\|\psi\|}(A_K) = \langle \psi | G_K(A_K) \, \psi \rangle. \tag{20}$$

But a positive bounded operator T is uniquely determined by the family of numbers $\langle \psi | T \, \psi \rangle$, hence $G(A) = G_K(A_K)$. $\qquad\square$

To see that Corollary 1 is a special case of Theorem 1, we have to check that, instead of considering *all* finite subsets K of $\mathbb{N}$, it suffices to consider the sets $\{1, \dots, n\}$. Indeed, if the consistency relation (7) holds then we can define

$$G_K(A) = G_n\Big(A \times \prod_{t \in \{1 \dots n\} \setminus K} M_t\Big), \tag{21}$$

where the right hand side does not depend on the choice of n, provided that $n \geq \max K$. Through this definition, we obtain a POVM G_K, and the family $\{G_K\}_{K \in \hat{\mathbb{N}}}$ satisfies the consistency condition (5).

3   Application

The application that has caused me to formulate Theorem 1 and to develop its proof concerns the Ghirardi–Rimini–Weber (GRW) theory of spontaneous wave function collapse [2, 1], a *quantum theory without observers* that has been proposed as a precise

version and explanation of quantum mechanics. In the GRW theory, the collapse of the wave function is an objective physical event governed by a stochastic law, whereas in conventional quantum mechanics the collapse is said to occur whenever an "observer" intervenes. The evolution of the wave function is a stochastic process in Hilbert space, replacing the deterministic unitary Schrödinger evolution. The proof of the rigorous existence of this process [5], and thus the proof that the GRW theory is mathematically well defined, is best done by means of Theorem 1. In more detail, since each collapse in the GRW theory is associated with a space-time point, the wave function process in Hilbert space can be equivalently translated into a point process in space-time, that is, a random sequence of space-time points or a random element of $M^{\mathbb{N}}$, where M is the space-time manifold. Given the stochastic GRW law of wave function evolution, the joint distribution of the first n points can be written down explicitly, and in fact is of the form $\mu_n(\cdot) = \langle \psi | G_n(\cdot)\, \psi \rangle$, where $\psi \in \mathscr{H}$ is the initial state vector with $\|\psi\| = 1$ and G_n is a POVM on M^n acting on $\mathscr{H}$.

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