# Chapter 4
## Dilation Theorems

We saw our first example of a dilation theorem in Chapter 1. Sz.-Nagy's dilation theorem (Theorem 1.1) showed that every contraction operator on a Hilbert space $\mathcal{H}$ was the compression of a unitary operator on a Hilbert space $\mathcal{K}$ that contains $\mathcal{H}$. In this chapter we focus on dilation theorems that characterize various classes of maps into $B(\mathcal{H})$ as compressions to $\mathcal{H}$ of "nicer" maps into $B(\mathcal{K})$, where $\mathcal{K}$ is a Hilbert space containing $\mathcal{H}$. One of the most general dilation theorems of this type is Stinespring's theorem, which characterizes completely positive maps from $C^*$-algebras into $B(\mathcal{H})$ in terms of $*$-homomorphisms into $B(\mathcal{K})$ for some other Hilbert space $\mathcal{K}$.

**Theorem 4.1 (Stinespring's dilation theorem).** *Let $\mathcal{A}$ be a unital $C^*$-algebra, and let $\phi\colon \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a completely positive map. Then there exists a Hilbert space $\mathcal{K}$, a unital $*$-homomorphism $\pi\colon \mathcal{A} \to B(\mathcal{K})$, and a bounded operator $V\colon \mathcal{H} \to \mathcal{K}$ with $\|\phi(1)\| = \|V\|^2$ such that*

$$\phi(a) = V^*\pi(a)V.$$

*Proof.* Consider the algebraic tensor product $\mathcal{A} \otimes \mathcal{H}$, and define a symmetric bilinear function $\langle , \rangle$ on this space by setting

$$\langle a \otimes x, b \otimes y\rangle = \langle \phi(b^*a)x, y\rangle_{\mathcal{H}}$$

and extending linearly, where $\langle , \rangle_{\mathcal{H}}$ is the inner product on $\mathcal{H}$.

The fact that $\phi$ is completely positive ensures that $\langle , \rangle$ is positive semidefinite, since

$$\left\langle \sum_{j=1}^{n} a_j \otimes x_j, \sum_{i=1}^{n} a_i \otimes x_i \right\rangle = \left\langle \phi_n((a_i^* a_j)) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right\rangle_{\mathcal{H}^{(n)}} \geq 0,$$

where $\langle , \rangle_{\mathcal{H}^{(n)}}$ denotes the inner product on the direct sum $\mathcal{H}^{(n)}$ of $n$ copies of $\mathcal{H}$, given by

$$\left\langle \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \right\rangle_{\mathcal{H}^{(n)}} = \langle x_1, y_1 \rangle_{\mathcal{H}} + \cdots + \langle x_n, y_n \rangle_{\mathcal{H}}.$$

Positive semidefinite bilinear forms satisfy the Cauchy–Schwarz inequality,

$$|\langle u, v \rangle|^2 \leq \langle u, u \rangle \cdot \langle v, v \rangle.$$

Thus, we have that

$$\{u \in \mathcal{A} \otimes \mathcal{H} \mid \langle u, u \rangle = 0\} = \{u \in \mathcal{A} \otimes \mathcal{H} \mid \langle u, v \rangle = 0 \text{ for all } v \in \mathcal{A} \otimes \mathcal{H}\}$$

is a subspace, $\mathcal{N}$, of $\mathcal{A} \otimes \mathcal{H}$. The induced bilinear form on the quotient space $\mathcal{A} \otimes \mathcal{H}/\mathcal{N}$ defined by

$$\langle u + \mathcal{N}, v + \mathcal{N} \rangle = \langle u, v \rangle$$

will be an inner product. We let $\mathcal{K}$ denote the Hilbert space that is the completion of the inner product space $\mathcal{A} \otimes \mathcal{H}/\mathcal{N}$.

If $a \in \mathcal{A}$, define a linear map $\pi(a)$: $\mathcal{A} \otimes \mathcal{H} \to \mathcal{A} \otimes \mathcal{H}$ by

$$\pi(a) \left( \sum a_i \otimes x_i \right) = \sum (aa_i) \otimes x_i.$$

A matrix factorization shows that the following inequality in $M_n(\mathcal{A})^+$ is satisfied:

$$(a_i^* a^* a a_j) \leq \|a^* a\| \cdot (a_i^* a_j),$$

and consequently,

$$\begin{aligned} &\left\langle \pi(a) \left( \sum a_j \otimes x_j \right), \pi(a) \left( \sum a_i \otimes x_i \right) \right\rangle \\ &\quad = \sum_{i,j} \langle \phi(a_i^* a^* a a_j) x_j, x_i \rangle_{\mathcal{H}} \leq \|a^* a\| \cdot \sum_{i,j} \langle \phi(a_i^* a_j) x_j, x_i \rangle_{\mathcal{H}} \\ &\quad = \|a\|^2 \cdot \left\langle \sum a_j \otimes x_j, \sum a_i \otimes x_i \right\rangle. \end{aligned}$$

Thus, $\pi(a)$ leaves $\mathcal{N}$ invariant and consequently induces a quotient linear transformation on $\mathcal{A} \otimes \mathcal{H}/\mathcal{N}$, which we still denote by $\pi(a)$. The above inequality also shows that $\pi(a)$ is bounded with $\|\pi(a)\| \leq \|a\|$. Thus, $\pi(a)$ extends to a bounded linear operator on $\mathcal{K}$, which we still denote by $\pi(a)$.

It is straightforward to verify that the map $\pi\colon \mathcal{A} \to B(\mathcal{K})$ is a unital $*$-homomorphism.

Now define $V : \mathcal{H} \to \mathcal{K}$ via

$$V(x) = 1 \otimes x + \mathcal{N}.$$

Then $V$ is bounded, since

$$\|Vx\|^2 = \langle 1 \otimes x, 1 \otimes x \rangle = \langle \phi(1)x, x \rangle_{\mathcal{H}} \leq \|\phi(1)\| \cdot \|x\|^2.$$

Indeed, it is clear that $\|V\|^2 = \sup\{\langle \phi(1)x, x\rangle_{\mathcal{H}} : \|x\| \leq 1\} = \|\phi(1)\|$.

To complete the proof, we only need observe that

$$\langle V^*\pi(a)Vx, y\rangle_{\mathcal{H}} = \langle \pi(a)1 \otimes x, 1 \otimes y\rangle_{\mathcal{K}} = \langle \phi(a)x, y\rangle_{\mathcal{H}}$$

for all $x$ and $y$, and so $V^*\pi(a)V = \phi(a)$. □

There are several remarks to be made. First, any map of the form $\phi(a) = V^*\pi(a)V$ is easily seen to be completely positive. Thus, Stinespring's dilation theorem characterizes the completely positive maps from any $C^*$-algebra into the algebra of bounded operators on any Hilbert space.

Second, note that if $\phi$ is unital, then $V$ is an isometry. In this case we may identify $\mathcal{H}$ with the subspace $V\mathcal{H}$ of $\mathcal{K}$. With this identification, $V^*$ becomes the projection of $\mathcal{K}$ onto $\mathcal{H}$, $P_{\mathcal{H}}$. Thus, we see that

$$\phi(a) = P_{\mathcal{H}}\pi(a)|_{\mathcal{H}}.$$

If $T$ is in $B(\mathcal{K})$, then the operator $P_{\mathcal{H}}T|_{\mathcal{H}}$ in $B(\mathcal{H})$ is called the *compression of $T$ to $\mathcal{H}$*. If we decompose $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^{\perp}$ and, using this decomposition, regard $T$ as a $2 \times 2$ operator matrix, then the compression of $T$ to $\mathcal{H}$ is just the (1,1) entry of this operator matrix for $T$. Thus, when $\phi(1) = 1$, Stinespring's theorem shows that every completely positive map into $B(\mathcal{H})$ is the compression to $\mathcal{H}$ of a *-homomorphism into a Hilbert space that contains $\mathcal{H}$.

The third point to be made is that Stinespring's theorem is really the natural generalization of the Gelfand–Naimark–Segal (GNS) representation of states. Indeed, if $\mathcal{H} = \mathbb{C}$ is one-dimensional so that $B(\mathbb{C}) = \mathbb{C}$, then an isometry $V : \mathbb{C} \to \mathcal{K}$ is determined by $V1 = x$ and we have

$$\phi(a) = \phi(a)1 \cdot 1 = V^*\pi(a)V1 \cdot 1 = \langle \pi(a)V1, V1\rangle_{\mathcal{K}} = \langle \pi(a)x, x\rangle.$$

In fact, rereading the above proof with $\mathcal{H} = \mathbb{C}$ and $\mathcal{A} \otimes \mathbb{C} = \mathcal{A}$, the reader will find a proof of the GNS representation of states.

Finally, we note that if $\mathcal{H}$ and $\mathcal{A}$ are separable, then the space $\mathcal{K}$ constructed above will be separable as well. Similarly, if $\mathcal{H}$ and $\mathcal{A}$ are finite-dimensional, then $\mathcal{K}$ is finite-dimensional.

We now turn our attention to considering the uniqueness of the Stinespring representation. We shall call a triple $(\pi, V, \mathcal{K})$ as obtained in Stinespring's theorem a *Stinespring representation for* $\phi$. Given a Stinespring representation $(\pi, V, \mathcal{K})$, let $\mathcal{K}_1$ be the closed linear span of $\pi(\mathcal{A})V\mathcal{H}$. It is easily verified that $\mathcal{K}_1$ reduces $\pi(\mathcal{A})$ so that the restriction of $\pi$ to $\mathcal{K}_1$ defines a $*$-homomorphism, $\pi_1\colon \mathcal{A} \to B(\mathcal{K}_1)$.

Clearly, $V\mathcal{H} \subseteq \mathcal{K}_1$, so we have that $\phi(a) = V^*\pi_1(a)V$, i.e., that $(\pi_1, V, \mathcal{K}_1)$ is also a Stinespring representation. It enjoys one additional property, namely, that $\mathcal{K}_1$ is the closed linear span of $\pi_1(\mathcal{A})V\mathcal{H}$. Whenever the space of the representation enjoys this additional property, we call the triple a *minimal Stinespring representation*. The following result summarizes the importance of this minimality condition.

**Proposition 4.2.** *Let $\mathcal{A}$ be a $C^*$-algebra, let $\phi\colon \mathcal{A} \to B(\mathcal{H})$ be completely positive, and let*

$$(\pi_i, V_i, \mathcal{K}_i), \qquad i = 1, 2,$$

*be two minimal Stinespring representations for $\phi$. Then there exists a unitary $U\colon \mathcal{K}_1 \to \mathcal{K}_2$ satisfying $UV_1 = V_2$ and $U\pi_1 U^* = \pi_2$.*

*Proof.* If $U$ exists, then necessarily,

$$U\left(\sum_i \pi_1(a_i)V_1 h_i\right) = \sum_i \pi_2(a_i)V_2 h_i.$$

Thus, it will be sufficient to verify that the above formula yields a well-defined isometry from $\mathcal{K}_1$ to $\mathcal{K}_2$, since by the minimality condition, $U$ will have dense range and hence be onto.

To this end, note that

$$\left\|\sum_i \pi_1(a_i)V_1 h_i\right\|^2 = \sum_{i,j} \langle V_1^*\pi_1(a_i^* a_j)V_1 h_j, h_i\rangle$$
$$= \sum_{i,j} \langle \phi(a_i^* a_j)h_j, h_i\rangle = \left\|\sum_i \pi_2(a_i)V_2 h_i\right\|^2,$$

so $U$ is isometric and consequently well defined, which is all that we needed to show. □

We now show how some other dilation theorems can be deduced from Stinespring's result.

The following is a slightly refined version of Theorem 1.1. It is interesting to compare the proofs.

**Theorem 4.3 (Sz.-Nagy's dilation theorem).** *Let $T \in B(\mathcal{H})$ with $\|T\| \le 1$. Then there exists a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ as a subspace and a unitary $U$ on $\mathcal{K}$ with the property that $\mathcal{K}$ is the smallest closed reducing subspace for $U$ containing $\mathcal{H}$ such that*

$$T^n = P_{\mathcal{H}} U^n|_{\mathcal{H}} \quad \text{for all nonnegative integers} \quad n.$$

*Moreover, if $(U', \mathcal{K}')$ is another pair with the above properties, then there is a unitary $V: \mathcal{K} \to \mathcal{K}'$ such that $Vh = h$ for $h \in \mathcal{H}$ and $VUV^* = U'$.*

*Proof.* By Theorem 2.6 and Exercise 2.2, the map $\phi(p + \bar{q}) = p(T) + q(T)^*$, where $p$ and $q$ are polynomials, extends to a positive map of $C(\mathbb{T})$ into $B(\mathcal{H})$. This map is completely positive by Theorem 3.11.

Let $(\pi, V, \mathcal{K})$ be a minimal Stinespring representation of $\phi$, and recall that since $\phi(1) = 1$, we may identify $V\mathcal{H}$ with $\mathcal{H}$. Setting $\pi(z) = U$, where $z$ is the coordinate function, we have that $U$ is unitary and that

$$T^n = \phi(z^n) = P_{\mathcal{H}} \pi(z^n)|_{\mathcal{H}} = P_{\mathcal{H}} U^n|_{\mathcal{H}}.$$

The minimality condition on $(\pi, V, \mathcal{K})$ is equivalent to requiring that the span of

$$\{U^n \mathcal{H}: n = 0, \pm 1, \pm 2, \dots\}$$

be dense in $\mathcal{K}$, which is equivalent to the requirement that there be no closed reducing subspace for $U$ containing $\mathcal{H}$ other than $\mathcal{K}$ itself.

The final statement of the theorem is a consequence of the uniqueness of a minimal Stinespring representation up to unitary equivalence, Proposition 4.2. □

The techniques used to prove Theorem 4.3 can be used to prove a far more general result. Let $X \subseteq \mathbb{C}$ be a compact set, and let $\mathcal{R}(X)$ be the algebra of rational functions on $X$. An operator $T$ is said to have a *normal $\partial X$-dilation* if there is a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ as a subspace and a normal operator $N$ on $\mathcal{K}$ with $\sigma(N) \subseteq \partial X$ such that

$$r(T) = P_{\mathcal{H}} r(N)|_{\mathcal{H}}$$

for all $r$ in $\mathcal{R}(X)$. We shall call $N$ a *minimal normal $\partial X$-dilation of $T$*, provided that $\mathcal{K}$ is the smallest reducing subspace for $N$ that contains $\mathcal{H}$.

Clearly, when $T$ has a normal $\partial X$ dilation $N$,

$$\|r(T)\| \le \|r(N)\| \le \sup\{|r(z)|: z \in \partial X\},$$

and so a necessary condition for $T$ to have a normal $\partial X$-dilation is that $X$ is a spectral set for $T$. It is a long-standing problem to determine if this condition is also sufficient [231]. That is, if $X$ is a spectral set for $T$, then does it necessarily follow that $T$ has a normal $\partial X$-dilation?

No compact subsets of $\mathbb{C}$ are known where this fails to be true. Yet the collection of sets $X$ for which this statement is known to always hold is somewhat limited. It has been verified that this condition is sufficient for $X$ an annulus [1], but the answer to this question is still unknown even when $X$ is a nice region with two holes and $T$ is a finite matrix. On the other hand, if $T$ is restricted to be a $2 \times 2$ matrix, then this statement is known to be true [63] for every set $X$.

If $\mathcal{S} = \mathcal{R}(X) + \overline{\mathcal{R}(X)}$ is dense in $C(\partial X)$, then $\mathcal{R}(X)$ is called a *Dirichlet algebra on* $\partial X$. For example, if $\mathbb{C}/X$ has only finitely many components and the interior of $X$ is simply connected, then $\mathcal{R}(X)$ is a Dirichlet algebra on $\partial X$. See [68] for this and further topological conditions on $X$ that ensure that $\mathcal{R}(X)$ is a Dirichlet algebra on $\partial X$.

**Theorem 4.4 (Berger–Foias–Lebow).** *Let $\mathcal{R}(X)$ be a Dirichlet algebra on $\partial X$. If $X$ is a spectral set for $T$, then $T$ has a minimal normal $\partial X$-dilation. Moreover, any two minimal normal $\partial X$-dilations for $T$ are unitarily equivalent, via a unitary which leaves $\mathcal{H}$ invariant.*

*Proof.* Let $\rho: \mathcal{R}(X) \to B(\mathcal{H})$ be the unital contraction defined by $\rho(r) = r(T)$, so that $\tilde{\rho}: \mathcal{S} \to B(\mathcal{H})$ is positive, where $\mathcal{S} = \mathcal{R}(X) + \overline{\mathcal{R}(X)}$. Since $\mathcal{S}$ is dense in $C(\partial X)$ and positive maps are bounded, $\tilde{\rho}$ extends to a positive map $\phi$ on $C(\partial X)$. But by Theorem 3.11, $\phi$ is completely positive. The remainder of the proof proceeds as in Theorem 4.3. □

When $\mathcal{R}(X)$ is not a Dirichlet algebra, minimal normal $\partial X$-dilations of an operator need not be unitarily equivalent [165].

To state the next dilation theorem, we need to introduce some notation. Let $X$ be a compact Hausdorff space, and let $\mathcal{B}$ be the $\sigma$-algebra of Borel sets on $X$. A *$B(\mathcal{H})$-valued measure on $X$* is a map $E: \mathcal{B} \to B(\mathcal{H})$ that is weakly countably additive, that is, if $\{B_i\}$ is a countable collection of disjoint Borel sets with union $B$, then

$$\langle E(B)x, y\rangle = \sum_i \langle E(B_i)x, y\rangle$$

for all $x, y$ in $\mathcal{H}$. The measure is *bounded* provided that

$$\sup\{\|E(B)\|: B \in \mathcal{B}\} < \infty,$$

and we let $\|E\|$ denote this supremum. The measure is *regular* provided that for all $x$, $y$ in $\mathcal{H}$, the complex measure given by

$$\mu_{x,y}(B) = \langle E(B)x, y\rangle \tag{$*$}$$

is regular.

Given a regular bounded $B(\mathcal{H})$-valued measure $E$, one obtains a bounded, linear map

$$\phi_E \colon C(X) \to B(\mathcal{H})$$

via

$$\langle \phi_E(f)x, y\rangle = \int f \, d\mu_{x,y}. \tag{$**$}$$

Conversely, given a bounded, linear map $\phi \colon C(X) \to B(\mathcal{H})$, if one defines regular Borel measures $\{\mu_{x,y}\}$ for each $x$ and $y$ in $\mathcal{H}$ by the above formula $(**)$, then for each Borel set $B$, there exists a unique, bounded operator $E(B)$, defined by the formula $(*)$, and the map $B \to E(B)$ defines a bounded, regular $B(\mathcal{H})$-valued measure. Thus, we see that there is a one-to-one correspondence between the bounded, linear maps of $C(X)$ into $B(\mathcal{H})$ and the regular bounded $B(\mathcal{H})$-valued measures. Such measures are called

(i) *spectral* if $E(M \cap N) = E(M) \cdot E(N)$,
(ii) *positive* if $E(M) \geq 0$,
(iii) *self-adjoint* if $E(M)^* = E(M)$,

for all Borel sets $M$ and $N$.

Note that if $E$ is spectral and self-adjoint, then $E(M)$ must be an orthogonal projection and hence $E$ is also positive.

The following proposition, whose proof we leave to the reader (Exercise 4.10), summarizes the relationships between the above properties of measures and the properties of their corresponding linear maps.

**Proposition 4.5.** *Let $E$ be a regular bounded $B(\mathcal{H})$-valued measure, and let $\phi \colon C(X) \to \mathcal{B}(\mathcal{H})$ be the corresponding linear map. Then:*

*(i) $\phi$ is a homomorphism if and only if $E$ is spectral;*
*(ii) $\phi$ is positive if and only if $E$ is positive;*
*(iii) $\phi$ is self-adjoint if and only if $E$ is self-adjoint;*
*(iv) $\phi$ is a $*$-homomorphism if and only if $E$ is self-adjoint and spectral.*

The correspondence between these measures and linear maps leads to a dilation result for these measures.

**Theorem 4.6 (Naimark).** *Let $E$ be a regular, positive, $B(\mathcal{H})$-valued measure on $X$. Then there exists a Hilbert space $\mathcal{K}$, a bounded linear operator $V: \mathcal{H} \to \mathcal{K}$, and a regular, self-adjoint, spectral, $B(\mathcal{K})$-valued measure $F$ on $X$, such that*

$$E(B) = V^* F(B) V.$$

*Proof.* Let $\phi: C(X) \to B(\mathcal{H})$ be the positive, linear map corresponding to $E$. Then $\phi$ is completely positive by Theorem 3.11, and so we may apply Stinespring's theorem to obtain a $*$-homomorphism $\pi: C(X) \to B(\mathcal{K})$ and a bounded, linear operator $V: \mathcal{H} \to \mathcal{K}$ such that $\phi(f) = V^*\pi(f)V$ for all $f$ in $C(X)$. If we let $F$ be the $B(\mathcal{K})$-valued measure corresponding to $\pi$, then it is easy to verify that $F$ has the desired properties. □

As another application of Stinespring's theorem, we shall give a characterization of the completely positive maps between two matrix algebras. In contrast, there is an entire panoply of classes of positive maps between matrix algebras and much that is not known about the relationships between these various classes (see, for example [46] and [244]).

We begin by remarking that if $\pi: M_n \to B(\mathcal{K})$ is a $*$-homomorphism, then up to unitary equivalence, $\mathcal{K}$ decomposes as an orthogonal, direct sum of $n$-dimensional subspaces such that $\pi$ is the identity representation on each of the subspaces (Exercise 4.11).

Now let $\phi: M_n \to M_k = B(\mathbb{C}^k)$, and let $(\pi, V, \mathcal{K})$ be a minimal Stinespring representation for $\phi$. By the construction of the space $\mathcal{K}$ given in Theorem 4.1, we see that $\dim(\mathcal{K}) \le \dim(M_n \otimes \mathbb{C}^k) = n^2 k$.

Thus, up to unitary equivalence, we may decompose $\mathcal{K}$ into the direct sum of fewer than $nk$ subspaces, each of dimension $n$, such that $\pi: M_n \to B(\mathcal{K})$ is the identity representation on each one. So let us write $\mathcal{K} = \sum_{i=1}^{\ell} \oplus C_i^n$, $\ell \le nk$, and let $P_i$ denote the projection of $\mathcal{K}$ onto $\mathbb{C}_i^n$. We have that for any $A$ in $M_n$,

$$P_i \pi(A)|_{\mathbb{C}_i^n} = A.$$

Now, if we let $V_i: \mathbb{C}^k \to \mathbb{C}_i^n$ be defined by $V_i = P_i V$, then

$$\phi(A) = V^*\pi(A)V = \sum_{i,j=1}^{\ell} V_i^*\pi(A)V_j = \sum_{i=1}^{\ell} V_i^* A V_i,$$

after identifying each $\mathbb{C}_i^n$ with $\mathbb{C}^n$.

We summarize these calculations in the following:

**Proposition 4.7 (Choi).** *Let $\phi: M_n \to M_k$ be completely positive. Then there exist fewer than nk linear maps, $V_i: \mathbb{C}^k \to \mathbb{C}^n$, such that $\phi(A) = \sum_i V_i^* A V_i$ for all $A$ in $M_n$.*

There is another dilation theorem due to Naimark whose proof is closely related to the proof of Stinespring's theorem. Let $G$ be a group and let $\phi\colon G \to B(\mathcal{H})$. We call $\phi$ *completely positive definite* if for every finite set of elements $g_1, \ldots, g_n$ of $G$, the operator matrix $(\phi(g_i^{-1} g_j))$ is positive.

If $G$ is a topological group, then we call a map $\phi\colon G \to B(\mathcal{H})$ *weakly continuous* provided $\langle \phi(g_\lambda)x, y\rangle \to \langle \phi(g)x, y\rangle$ for every pair of vectors $x, y$ in $\mathcal{H}$, and every net $\{g_\lambda\}$ in $G$ converging to $g$. Similarly, $\phi$ is called *strongly continuous* provided $\|\phi(g_\lambda)x - \phi(g)x\| \to 0$, and $*$-*strongly continuous* provided $\|\phi(g_\lambda)x - \phi(g)x\| \to 0$ and $\|\phi(g_\lambda)^* x - \phi(g)x\| \to 0$.

**Theorem 4.8 (Naimark).** *Let $G$ be a topological group, and let $\phi\colon G \to B(\mathcal{H})$ be weakly continuous and completely positive definite. Then there exists a Hilbert space $\mathcal{K}$, a bounded operator $V\colon \mathcal{H} \to \mathcal{K}$, and a $*$-strongly continuous unitary representation $\rho\colon G \to B(\mathcal{H})$ such that*

$$\phi(g) = V^* \rho(g) V.$$

*Consequently, $\phi$ is automatically $*$-strongly continuous.*

*Proof.* Consider the vector space $C_0(G, \mathcal{H})$ of finitely supported functions from $G$ to $\mathcal{H}$ and define a bilinear function on this space by

$$\langle f_1, f_2\rangle = \sum_{g,g'} \langle \phi(g^{-1}g') f_1(g'), f_2(g)\rangle_{\mathcal{H}}.$$

As in the proof of Stinespring's theorem, we have that $\langle f, f\rangle \geq 0$ and that the set $\mathcal{N} = \{f \mid \langle f, f\rangle = 0\}$ is a subspace of $C_0(G, \mathcal{H})$. We let $\mathcal{K}$ be the completion of $C_0(G, \mathcal{H})/\mathcal{N}$ with respect to the induced inner product.

For $h$ in $\mathcal{H}$, define $Vh$ by

$$(Vh)(g) = \begin{cases} h & \text{if } g = e, \\ 0 & \text{if } g \neq e, \end{cases}$$

where $e$ denotes the identity of $G$, and let $\rho\colon G \to B(\mathcal{K})$ be left translation, that is,

$$(\rho(g) f)(g') = f(g^{-1} g').$$

It is straightforward to check that $V$ is bounded and linear, that $\rho$ is a unitary representation, and that $\phi(g) = V^* \rho(g) V$.

Now we show that $\rho$ is weakly continuous. Let $\{g_\lambda\}$ be a net in $G$ that converges to $g_0$. Since $\rho$ is a unitary representation, it will suffice to show that $\rho(g_\lambda)$ converges weakly to $\rho(g_0)$ on a dense subspace.

Let $f_1, f_2$ be in $C_0(G, \mathcal{H})$. Then

$$\begin{aligned}\langle \rho(g_\lambda) f_1, f_2 \rangle &= \sum_{g,g'} \langle \phi(g^{-1}g') f_1\big(g_\lambda^{-1}g'\big), f_2(g) \rangle_{\mathcal{H}} \\ &= \sum_{g,g''} \langle \phi(g^{-1}g_\lambda g'') f_1(g''), f_2(g) \rangle_{\mathcal{H}},\end{aligned}$$

which, since both sums involved are finite, converges to

$$\sum_{g,g''} \langle \phi(g^{-1}g_0 g'') f_1(g''), f_2(g) \rangle_{\mathcal{H}} = \langle \rho(g_0) f_1, f_2 \rangle.$$

Thus, we see that $\rho(g_\lambda)$ converges weakly to $\rho(g_0)$.

Now it is easily checked that if a net of unitaries converges in the weak operator topology to another unitary, then it converges $*$-strongly. Thus, $\rho$ is $*$-strongly continuous. The $*$-strong continuity of $\phi$ now follows from the representation and the $*$-strong continuity of $\rho$. $\square$

It is useful to consider the special case of the above theorem when $G = \mathbb{Z}$, the group of integers. Setting $\phi(n) = A_n$, we see that $\phi$ is completely positive definite if and only if $(A_{n_i - n_j})_{i,j=1}^k$ is positive for every choice of finitely many integers $n_1, \ldots, n_k$ with $k$ arbitrary. Taking $n_i = i$, we see that this implies the formal positivity of $(A_{i-j})_{i,j=0}^\infty = T$. A little careful reflection shows that an arbitrary choice of $n_1, \ldots, n_k$ is simply restricting $T$ to the entries represented by the set $\{n_1, \ldots, n_k\}$ and then permuting these entries.

Thus, a completely positive definite function on $\mathbb{Z}$ is just an operator-valued trigonometric moment sequence (Exercise 3.15). Naimark's dilation theorem now tells us that there exists a unitary $U = \rho(1)$ and $V$ such $A_n = V^* U^n V$ for all $n$.

As with Stinespring's representation, there is a minimality condition that guarantees the uniqueness of this representation up to unitary equivalence (Exercise 4.12).

A map $\phi\colon G \to B(\mathcal{H})$ will be called *positive definite* if for every choice of $n$ elements $g_1, \ldots, g_n$ of $G$, and scalars, $\alpha_1, \ldots, \alpha_n$, the operator

$$\sum_{i,j} \bar{\alpha}_i \alpha_j \phi\big(g_i^{-1} g_j\big)$$

is positive. We remark that this is equivalent to requiring that for every $x$ in $\mathcal{H}$, the map $\phi_x\colon G \to \mathbb{C}$, given by $\phi_x(g) = \langle \phi(g)x, x \rangle$, be completely positive definite.

We caution the reader that our terminology is not standard. What we have chosen to call completely positive definite is usually called simply positive definite, and the concept that we have introduced above and called positive definite is usually not introduced at all. Our rationale for this slight deviation in notation will be clear in the section "Group $C^*$-Algebras" at the end of this chapter, where we will show a correspondence between the two classes of maps on $G$ that are defined above and maps on a certain $C^*$-algebra associated with $G$, $C^*(G)$. Not surprisingly, this correspondence will carry (completely) positive definite maps on $G$ to (completely) positive maps on $C^*(G)$.

We begin by describing this correspondence in one case of particular interest. Let $\mathbb{Z}^n$ be the Cartesian product of $n$ copies of the integers, and let $\mathbb{T}^n$ be the Cartesian product of $n$ copies of the circle. Let $J = (j_1, \ldots, j_n)$ be in $\mathbb{Z}^n$, and let $z_j$ be the $j$th coordinate function on $\mathbb{T}^n$. We set $z^J = z_1^{j_1} \cdots z_n^{j_n}$. Note that there is a one-to-one correspondence between unitary representations $\rho\colon \mathbb{Z}^n \to B(\mathcal{H})$ and $*$-homomorphisms $\pi\colon C(\mathbb{T}^n) \to B(\mathcal{H})$, given by $\pi(z_j) = \rho(e_j)$, where $e_j$ is the $n$-tuple that is 1 in the $j$th entry and 0 in the remaining entries.

**Proposition 4.9.** *Let $\phi\colon \mathbb{Z}^n \to B(\mathcal{H})$ be (completely) positive definite. Then there is a uniquely defined, (completely) positive map $\psi\colon C(\mathbb{T}^n) \to B(\mathcal{H})$, given by $\psi(z^J) = \phi(J)$. Conversely, if the (completely) positive map $\psi$ is given, then the above formula defines a (completely) positive definite function $\phi$ on $\mathbb{Z}^n$.*

*Proof.* First, we consider the case where $\phi$ is completely positive definite. Let $(\rho, V, \mathcal{K})$ be the Naimark dilation of $\phi$, so that $\phi(J) = V^*\rho(J)V$, and let $\pi\colon C(\mathbb{T}^n) \to B(\mathcal{H})$ be the $*$-homomorphism associated with $\rho$. If we set $\psi(f) = V^*\pi(f)V$, then we obtain a map $\psi\colon C(\mathbb{T}^n) \to B(\mathcal{H})$, which is completely positive. Moreover, $\psi$ satisfies $\psi(z^J) = V^*\pi(z^J)V = V^*\rho(J)V = \phi(J)$.

The proof of the converse in the completely positive case is identical.

Now, suppose that $\phi$ is only positive definite. If we fix $x$ in $\mathcal{H}$ and set $\phi_x(J) = \langle \phi(J)x, x\rangle$, then $\phi_x$ is a completely positive definite function on $\mathbb{Z}^n$. Thus, by the above there is a positive map $\psi_x\colon C(\mathbb{T}^n) \to \mathbb{C}$ with $\psi_x(z^J) = \phi_x(J)$. For fixed $f$ in $C(\mathbb{T}^n)$, the function $x \to \psi_x(f)$ as $x$ varies over $\mathcal{H}$ is a bounded, quadratic function (see Exercise 4.18), and hence there exists a bounded operator $\psi(f)$ such that $\langle \psi(f)x, x\rangle = \psi_x(f)$. This defines a linear map $\psi\colon C(\mathbb{T}^n) \to B(\mathcal{H})$, which is easily seen to be positive.

The converse in the positive case is similar. □

**Corollary 4.10.** *For $\mathbb{Z}^n$, the sets of positive definite and completely positive definite operator-valued functions coincide.*

*Proof.* Clearly, every completely positive definite map is positive definite. Now let $\phi\colon \mathbb{Z}^n \to B(\mathcal{H})$ be positive definite. Then $\psi\colon C(\mathbb{T}^n) \to B(\mathcal{H})$, given $\psi(z^J) = \phi(J)$ is a positive linear map. By Theorem 3.11, $\psi$ is completely positive and hence $\phi$ is completely positive definite. □

Again it is useful to understand what this corollary says in the case of $Z$. Setting $\phi(n) = A_n$, we have seen that $\phi$ is completely positive definite if and only if $(A_{i-j})_{i,j=0}^{\infty}$ is formally positive. But $\phi$ is positive definite if and only if $\phi_x$ is positive definite for all $x$, which is if and only if $(\langle A_{i-j}x, x\rangle)_{i,j=0}^{\infty}$ is formally positive for all $x$. Thus, for Toeplitz operator matrices, formal positivity of $(A_{i-j})$ is equivalent to formal positivity of $(\langle A_{i-j}x, x\rangle)$ for every vector $x$.

Corollary 4.10 is a generalization of this observation to multivariable Toeplitz operators. We now wish to discuss the analogue for more general groups.

## Group $C^*$-Algebras

The above results are part of a more general duality. Let $G$ be a locally compact group, and let $dg$ be a (left) Haar measure on $G$. The space $L^1(G)$ of integrable functions, on $G$ can be made into a $*$-algebra by defining

$$f_1 \times f_2(g') = \int f_1(g) f_2(g^{-1}g')\, dg,$$

and a $*$-operation by

$$f^*(g) = \Delta(g)^{-1}\overline{f(g^{-1})},$$

where $\Delta(\cdot)$ is the modular function. It is then possible to endow $L^1(G)$ with a norm such that the completion of $L^1(G)$ is a $C^*$-algebra, denoted by $C^*(G)$ (see [73] or [173]).

There is a one-to-one correspondence between weakly continuous, unitary representations $\rho\colon G \to B(\mathcal{H})$, and $*$-homomorphisms $\pi\colon C^*(G) \to B(\mathcal{H})$, given by

$$\pi(f) = \int f(g)\rho(g)\, dg$$

when $f$ is in $L^1(G)$. See [173] for a development of this theory.

In a similar fashion, there are one-to-one correspondences between the weakly continuous, (completely) positive definite, operator-valued functions on $G$ and the (completely) positive maps on $C^*(G)$, given by

$$\psi(f) = \int f(g)\phi(g)\, dg$$

for $f$ in $L^1(G)$. The proof that the above formula defines a one-to-one correspondence between these two classes of maps is similar to the proof of Proposition 4.9 and is left as an exercise (Exercise 4.16).

Proposition 4.9 follows from the above correspondence and the fact [173] that $C^*(Z^n) = C(\mathbb{T}^n)$.

Paralleling Corollary 4.10, we see that when $G$ is commutative the positive definite and completely positive definite functions correspond.

## Notes

Naimark's two dilation theorems ([150] and [151]) preceded Stinespring's dilation theorem [221]. Stinespring [221] defined completely positive maps on $C^*$-algebras, proved that positive maps on $C(X)$ were completely positive (Theorem 3.11), and then proved Theorem 4.1 as a generalization of Naimark's dilation theorem for positive, operator-valued measures.

Arveson [6] realized the important role that the theory of completely positive maps can play in the study of normal $\partial X$-dilations and gave the proofs of Theorems 4.3 and 4.4 that are presented here.

Other proofs of Sz.-Nagy's dilation theorem have used the theory of positive definite functions on $Z$ or the "geometric" technique that we presented in Chapter 1, where the unitary and the space that it acts on are explicitly constructed ([228] and [210]). Two beautiful results of the geometric dilation techniques are Ando's dilation theorem for commuting pairs of contractions [5] and the Sz.-Nagy–Foias commutant lifting theorem [231]. We will present these topics in Chapter 5.

The correspondence between bounded, regular, operator-valued measures on a compact Hausdorff space $X$ and bounded, linear maps on $C(X)$ is discussed in Hadwin [109].

Proposition 4.7 was proved by Choi [43], who also developed the theory of multiplicative domains [42] (Exercise 4.2).

See Bunce [37] for more on Korovkin-type theorems (Exercise 4.9).

Exercise 4.16 is from Berger [15].

The material distinguishing positive definite and completely positive definite functions on groups seems to be new.

## Exercises

4.1 Use Stinespring's representation theorem to prove that $\|V\|^2 = \|\phi\|_{\text{cb}}$ when $\phi$ is completely positive. Also, use the representation theorem to prove that $\phi(a)^*\phi(a) \le \|\phi(1)\|^2\phi(a^*a)$.

4.2 (Multiplicative domains) In this exercise, we present an alternative proof of Theorem 3.18. Let $\mathcal{A}$ be a $C^*$-algebra with unit, and let $\phi\colon \mathcal{A} \to B(\mathcal{H})$ be completely positive, $\phi(1) = 1$, with minimal Stinespring representation $(\pi, V, \mathcal{K})$.

   (i) Prove that $\phi(a)^*\phi(a) = \phi(a^*a)$ if and only if $V\mathcal{H}$ is an invariant subspace for $\pi(a)$.

(ii) Use this to give another proof that $\{a \in \mathcal{A}: \phi(a)^*\phi(a) = \phi(a^*a)\} = \{a \in \mathcal{A}: \phi(ba) = \phi(b)\phi(a) \text{ for all } b \in \mathcal{A}\}$. Recall that this set is the *right multiplicative domain* of $\phi$.

(iii) Similarly, show that $\phi(a)^*\phi(a) = \phi(a^*a)$ and $\phi(a)\phi(a)^* = \phi(aa^*)$ if and only if $V\mathcal{H}$ is a reducing subspace for $\pi(a)$. Deduce that the set of all such elements is a $C^*$-subalgebra of $\mathcal{A}$. Recall that this subalgebra is the *multiplicative domain of* $\phi$.

4.3 (Bimodule Maps) Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be $C^*$-algebras with unit, and suppose that $\mathcal{C}$ is contained in both $\mathcal{A}$ and $\mathcal{B}$, with $1_{\mathcal{C}} = 1_{\mathcal{A}}$ and $1_{\mathcal{C}} = 1_{\mathcal{B}}$. A linear map $\phi: \mathcal{A} \to \mathcal{B}$ is called a $\mathcal{C}$-*bimodule* map if $\phi(c_1 a c_2) = c_1\phi(a)c_2$ for all $c_1, c_2$ in $\mathcal{C}$. Let $\phi: \mathcal{A} \to \mathcal{B}$ be completely positive.

(i) If $\phi(1) = 1$, then prove that $\phi$ is a $\mathcal{C}$-bimodule map if and only if $\phi(c) = c$ for all $c$ in $\mathcal{C}$.

(ii) Prove, in general, that $\phi$ is a $\mathcal{C}$-bimodule map if and only if $\phi(c) = c \cdot \phi(1)$ for all $c$ in $\mathcal{C}$. Moreover, in this case $\phi(1)$ commutes with $\mathcal{C}$.

4.4 Let $\mathcal{D}_n$ be the $C^*$-subalgebra of diagonal matrices in $M_n$. Prove that a linear map $\phi: M_n \to M_n$ is a $\mathcal{D}_n$-bimodule map if and only if $\phi$ is the Schur product map $S_T$ for some matrix $T$.

4.5 Let $\phi: \mathcal{A} \to \mathcal{A}$ be a completely positive projection with $\phi(1) = 1$.

(i) Show that if $\phi(a) = a$, then $\phi(ax) = \phi(a\phi(x))$ for every $x$.

(ii) Show that $\mathcal{B} = \{a: \phi(a) = a\}$ is a $C^*$-algebra, in the product $a \circ b = \phi(ab)$, but that in general, $\mathcal{B}$ is distinct from the multiplicative domain of $\phi$.

(iii) Show that $\phi$ is a $(\mathcal{B}, \circ)$-bimodule map.

4.6 Let $\phi: G \to B(\mathcal{H})$ be completely positive definite. Prove that $\phi$ is weakly continuous if and only if $\phi$ is strongly continuous.

4.7 Let $\phi: G \to M_n$ be continuous. Prove that $\phi$ is completely positive definite if and only if there exists a Hilbert space $\mathcal{H}$ and continuous functions $x_i: G \to \mathcal{H}$, $i = 1, \ldots, n$, such that $\phi(g^{-1}g') = (\langle x_j(g'), x_i(g)\rangle)$.

4.8 A semigroup $G$ with an involution $g \to g^*$ satisfying $(g_1g_2)^* = g_2^*g_1^*$, $1^* = 1$ is called a $*$-*semigroup*. A function $\phi: G \to B(\mathcal{H})$ is called *completely positive definite* if $(\phi(g_i^*g_j))$ is positive for every set of finitely many elements $g_1, \ldots, g_n$ of $G$, and *bounded* if $(\phi(g_i^*g^*gg_j)) \le M_g(\phi(g_i^*g_j))$ where $M_g$ is a constant depending only on $g$. Show that every group is a $*$-semigroup if we set $g^* = g^{-1}$. Prove a version of Naimark's dilation theorem for $*$-semigroups.

4.9 Let $\mathcal{A}$ be a $C^*$-algebra with unit, and let $\phi_n: \mathcal{A} \to B(\mathcal{H})$ be a sequence of completely positive maps such that $\phi_n(1) \to 1$ in the weak operator topology.

(i) Prove that $\{a\colon\ \phi_n(a)^*\phi_n(a) - \phi_n(a^*a) \to 0$ and $\phi_n(a)\phi_n(a)^* - \phi_n(aa^*) \to 0$ in the weak operator topology$\}$ is a $C^*$-subalgebra of $\mathcal{A}$.

(ii) (Korovkin) Prove that if $\phi_n\colon C([0, 1]) \to C([0, 1])$ is a sequence of positive maps such that $\phi_n(1)$, $\phi_n(t)$, and $\phi_n(t^2)$ converge in norm to $1$, $t$, and $t^2$, respectively, then $\phi_n(f)$ converges in norm to $f$, for all $f$.

4.10 Prove Proposition 4.5.

4.11 Let $\pi\colon M_n \to B(\mathcal{K})$ be a unital $*$-homomorphism. Prove that up to unitary equivalence, $\mathcal{K} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$, with $\mathcal{H}_i = \mathcal{H}$, $i = 1, \ldots, n$, such that $\pi(E_{i,j})$ is the identity map from $\mathcal{H}_j$ to $\mathcal{H}_i$. Show that up to unitary equivalence, $\pi$ is the direct sum of $\dim(\mathcal{H})$ copies of the identity map.

4.12 Give a minimality condition for the Naimark representation of completely positive definite functions on a group, and prove uniqueness of minimal representations up to unitary equivalence.

4.13 Let $t \to A(t)$, $t \geq 0$, be a weakly continuous semigroup of contraction operators, $A(0) = I$. For $t < 0$, set $A(t) = A(-t)^*$. Prove that this extended map is completely positive definite in $\mathbb{R}$. What does Naimark's dilation theorem imply? (Hint: Recall the proof of Theorem 2.6.)

4.14 (Trigonometric moments) Let $\{A_n\}_{n=-\infty}^{+\infty}$ be a sequence of bounded linear operators on a Hilbert space $\mathcal{H}$. Prove that $\{A_n\}_{n=-\infty}^{+\infty}$ is an operator-valued trigonometric moment sequence if and only if there exists a Hilbert space $\mathcal{K}$, a unitary operator $U$ on $\mathcal{K}$, and a bounded linear operator $V\colon \mathcal{H} \to \mathcal{K}$ such that $A_n = V^*U^nV$ for all $n$.

4.15 (Hausdorff moments) Let $\{A_n\}_{n=0}^{+\infty}$ be a sequence of bounded linear operators on a Hilbert space $\mathcal{H}$. Prove that $\{A_n\}_{n=0}^{+\infty}$ is an operator-valued Hausdorff moment sequence if and only if there exists a Hilbert space $\mathcal{K}$, a positive contraction $P$ on $\mathcal{K}$, and a bounded linear operator $V\colon \mathcal{H} \to \mathcal{K}$ such that $A_n = V^*P^nV$ for all $n \geq 0$.

4.16 Verify the claims of the subsection on group $C^*$-algebras.

4.17 (Berger) Let $T$ be an operator on a Hilbert space $\mathcal{H}$. Prove that $w(T) \leq 1$ if and only if there exists a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a unitary operator $U$ on $\mathcal{K}$ such that $T^n = 2P_{\mathcal{H}}U^n|_{\mathcal{H}}$ for all $n \geq 1$.

4.18 Let $\mathcal{H}$ be a Hilbert space, and let $\gamma\colon \mathcal{H} \to \mathbb{C}$ be a function. We call $\gamma$ *quadratic* provided that $\gamma(\lambda x) = |\lambda|^2\gamma(x)$ and $\gamma(x + y) + \gamma(x - y) = 2(\gamma(x) + \gamma(y))$ for all $\lambda$ in $\mathbb{C}$ and for every $x$ and $y$ in $\mathcal{H}$. If, in addition, there exists a constant $M$ such that $|\gamma(x)| \leq M\|x\|^2$ for all $x$ in $\mathcal{H}$, then we call $\gamma$ *bounded*. Prove that $\gamma$ is a bounded, quadratic function on $\mathcal{H}$ if and only if there exists $T$ in $B(\mathcal{H})$ such that $\gamma(x) = \langle Tx, x\rangle$, and that $T$ is unique.