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On the Quantum Theory of Sequential Measurements

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The quantum theory of sequential measurements is worked out and is employed to provide an operational analysis of basic measurement theoretical notions such as coexistence, correlations, repeatability, and ideality. The problem of the operational definition of continuous observables is briefly revisited, with a special emphasis on the localization observable. Finally, a brief overview is given of possible applications of the theory to various fields and problems in quantum physics.

1. INTRODUCTION

The aim of the present note is to (re-)investigate the operational content of some important notions of quantum measurement theory. A natural framework for this task is offered by the quantum theory of sequential measurements. We believe that our work fits well with the spirit—though certainly not which the fundamental scope—of Charles Randall's contribution to the foundations of physics; Charles once explained to us his motivations: to understand what we mean operationally when we talk about quantum mechanics. It is an honor to dedicate this modest contribution to his memory.

We shall start with a sketch of the theory of *sequential measurements* within the Hilbert space formulation of quantum mechanics. The instrument of a sequential measurement is determined, and it is observed to

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agree with the sequential composition of the instruments of the measurements constituting the sequential measurement. The problem of interpreting *sequential probabilities as conditional probabilities* will be discussed. With respect to two possible notions of conditional probability, such an interpreation is seen to imply rather strong restrictions on the involved observables and their measurements. The order independence of some of the sequential measurements of a pair of observables guarantees their compatibility or coexistence. The notion of *repeatable* measurements is recalled and some properties of such measurements are reviewed. In particular, repeatable measurements are *preparatory* and of the *first kind.* Moreover, such measurements lead to *strong correlations.* A result of Ozawa is reviewed to show that observables admitting completely positive repeatable measurements are necessarily *discrete.* Some further properties of ideal and repeatable measurements of discrete observables are pointed out. The problem of the operational definition of continuous observables, such as localization, is briefly discussed. The solution is based on *approximate* repeatability. Finally, we shall indicate possible applications of the theory of sequential measurements in various areas of physics, ranging from relativistic quantum theory (localization) over quantum optics (coherent state measurements, quantum nondemolition detectors), signal processing (informational completeness), and continuous observations (Zeno/watchdog effect) to quantum cosmology.

The framework of the paper is the usual Hilbert space formulation of quantum mechanics together with its measurement theory. The notations and terminology shall be fixed next.

1.1. General Framework

Let $\mathscr{H}$ be a complex separable Hilbert space, with the inner product $\langle\cdot|\cdot\rangle$. We let $\mathscr{L}(\mathscr{H})$ denote the set of bounded operators on $\mathscr{H}$, and $\mathscr{L}(\mathscr{H})^+$ the subset of its positive elements. A normalized positive operator-valued (*POV*) measure $A\colon \mathscr{X} \to \mathscr{L}(\mathscr{H})^+$ on a measurable space $(\mathbf{X}, \mathscr{X})$ is defined through the properties: (i) $A(X) \geqslant A(\varnothing)$ for all $X \in \mathscr{X}$; (ii) if (X_i) is a countable collection of disjoint sets in $\mathscr{X}$, then $A(\bigcup X_i) = \sum A(X_i)$, the series converging in the weak operator topology; (iii) $A(\mathbf{X}) = I$, where I is the identity operator on $\mathscr{H}$. The space of trace class operators on $\mathscr{H}$ will be denoted $\mathscr{T}(\mathscr{H})$, and $\mathscr{T}(\mathscr{H})_1^+$ consists of the positive trace one operators on $\mathscr{H}$. The set $\mathscr{T}(\mathscr{H})_1^+$ is convex, and its extremal points are the one-dimensional projection operators $P[\varphi](P[\varphi]\psi = \langle\varphi|\psi\rangle\varphi)$ on $\mathscr{H}$, $\varphi \in \mathscr{H}$.

In the Hilbert space formulation of quantum mechanics the description of a physical system $\mathscr{S}$ is based on some $\mathscr{H}$. Any (generalized) *observable* of $\mathscr{S}$ is represented as (and identified with) a *POV* measure A: $\mathscr{X} \to \mathscr{L}(\mathscr{H})^+$ on a Borel space $(\mathbf{X}, \mathscr{X})$—the value space of A. We recall that a *POV* measure A is a *PV* (i.e., a projection valued) measure if and only if $A(X) = A(X)^2$ for each $X \in \mathscr{X}$. According to the spectral theorem, the usual notion of an observable as a self-adjoint operator in $\mathscr{H}$ refers to the special case of $A: \mathscr{X} \to \mathscr{L}(\mathscr{H})^+$ being a *PV* measure on the real Borel space $(\mathfrak{R}, B(\mathfrak{R}))$. *States* of $\mathscr{S}$ are represented as (and identified with) the elements of $\mathscr{T}(\mathscr{H})_1^+$, and we refer to the unit vectors of $\mathscr{H}$, or to the one-dimensional projections they generate, as the vector states of $\mathscr{S}$. Any pair (A, T) of an observable A and a state T determines a probability measure $A_T: \mathscr{X} \to [0, 1]$, $X \mapsto A_T(X) \doteq \mathrm{tr}[TA(X)]$, where $[0, 1]$ is the unit interval of the real line $\mathfrak{R}$. According to the *minimal interpretation*, $A_T(X)$ is the probability that a measurement of A performed on $\mathscr{S}$ in the state T leads to a result in the value set X. It is another basic fact of the Hilbert space quantum mechanics that the above representations of observables and states are the most general ones compatible with the probability structure of quantum mechanics.

Let $\mathscr{H}_a$ be another complex separable Hilbert space. The Hilbert space tensor product of $\mathscr{H}$ and $\mathscr{H}_a$ is denoted $\mathscr{H} \otimes \mathscr{H}_a$. The sets $\mathscr{L}(\mathscr{H} \otimes \mathscr{H}_a)$, $\mathscr{T}(\mathscr{H} \otimes \mathscr{H}_a)$, etc. are defined as above. The partial trace over $\mathscr{H}_a$, say, is the positive linear map Π_a: $\mathscr{T}(\mathscr{H} \otimes \mathscr{H}_a) \to \mathscr{T}(\mathscr{H})$ defined as follows: $\mathrm{tr}[\Pi_a(W)S] = \mathrm{tr}[WS \otimes I_a]$, where $W \in \mathscr{T}(\mathscr{H} \otimes \mathscr{H}_a)$, $S \in \mathscr{L}(\mathscr{H})$, and I_a is the identity operator on $\mathscr{H}_a$. Similarly, one defines the partial trace over $\mathscr{H}$, and it is denoted Π.

Let $\mathscr{A}$ be another physical system, the description of which is based on $\mathscr{H}_a$. The description of the compound system $\mathscr{S} + \mathscr{A}$ is then based on $\mathscr{H} \otimes \mathscr{H}_a$ along the usual ideas of the theory of compound systems in quantum mechanics. We recall only that any state W of $\mathscr{S} + \mathscr{A}$ uniquely determines the states of the subsystems $\mathscr{S}$ and $\mathscr{A}$ as the partial traces $\Pi_a(W)$ and $\Pi(W)$, respectively.

For more details of the basic Hilbert space formulation of quantum mechanics, see, e.g., Refs. 1 and 2.

1.2. A Sketch of the Measurement Theory

In its usual formulation, the theory of a measurement of an observable A of the object system $\mathscr{S}$ starts with fixing a measuring apparatus $\mathscr{A}$ (with a Hilbert space $\mathscr{H}_a$), its initial state $T_a \in \mathscr{T}(\mathscr{H}_a)$, a pointer observable P_a (i.e., a *POV* measure $P_a: \mathscr{X} \to \mathscr{L}(\mathscr{H}_a)^+$), and a measurement mapping V_a: $\mathscr{T}(\mathscr{H} \otimes \mathscr{H}_a) \to \mathscr{T}(\mathscr{H} \otimes \mathscr{H}_a)$ (a positive linear trace-preserving mapping).

The 4-tuple $\langle \mathscr{H}_a, P_a, T_a, V_a \rangle$ is a *measurement* of A if it satisfies the probability reproducibility condition

$$\begin{aligned} A_T(X) &= P_{a, \Pi(V_a(T \oplus T_a))}(X) \\ &= \operatorname{tr}[V_a(T \otimes T_a) I \otimes P_a(X)] \end{aligned} \tag{1}$$

for all $X \in \mathscr{X}$, $T \in \mathscr{T}(\mathscr{H})_1^+$. The second equality in (1) is due to the properties of the partial trace, and is not to be considered as part of the definition. [In Ref. 3 4-tuples $\langle \mathscr{H}_a, P_a, T_a, V_a \rangle$ satisfying (1) were called premeasurements in order to emphasize that the objectification of the measurement result requires more than just (1).]

Any measurement $\langle \mathscr{H}_a, P_a, T_a, V_a \rangle$ of an observable $A: \mathscr{X} \to \mathscr{L}(\mathscr{H})^+$ determines an *instrument* (a state transformation or operation-valued measure) $\mathscr{I}_{V_a}^A: \mathscr{X} \to \mathscr{L}(\mathscr{T}(\mathscr{H}))$ via the relation

$$\mathscr{I}_{V_a}^A(X) T \doteq \Pi_a(V(T \otimes T_a) I \otimes P_a(X)) \tag{2}$$

for all $X \in \mathscr{X}$, $T \in \mathscr{T}(\mathscr{H})$. The instrument $\mathscr{I}_{V_a}^A$ determines (uniquely) the measured observable via the probability relation

$$\operatorname{tr}[\mathscr{I}_{V_a}^A(X) T] = \operatorname{tr}[T A(X)] \tag{3}$$

for all $X \in \mathscr{X}$, $T \in \mathscr{T}(\mathscr{H})$. Moreover, it gives the (nonnormalized) conditionalized final states $\mathscr{I}_{V_a}^A(X) T$ of S, with the condition that the A-measurement yielded a result in the set X (when S was initially in the state T). In this sense, the instrument $\mathscr{I}_{V_a}^A$ contains all the information on the measurement $\langle \mathscr{H}_a, P_a, T_a, V_a \rangle$ of A which is relevant to S.

Let $\langle \mathscr{H}_a, P_a, T_a, V_a \rangle$ and $\langle \tilde{\mathscr{H}}_a, \tilde{P}_a, \tilde{T}_a, \tilde{V}_a \rangle$ be any two measurements of A. They are *statistically equivalent* if they induce the same instrument, i.e., if $\mathscr{I}_{V_a}^A = \mathscr{I}_{\tilde{V}_a}^A$. A measurement $\langle \mathscr{H}_a, P_a, T_a, V_a \rangle$ of A is a *normal* measurement if the pointer observable P_a is a PV measure on $(\mathfrak{R}, \mathscr{B}(\mathfrak{R}))$ (i.e., a usual observable), the initial state T_a of $\mathscr{A}$ is a vector state (i.e., $T_a = P[\Phi_a]$ for some unit vector $\Phi_a \in \mathscr{H}_a$), and the measurement mapping V_a is unitary (i.e., $V_a(W) = U_a^* W U_a$, $W \in \mathscr{T}(\mathscr{H} \otimes \mathscr{H}_a)_1^+$, for some unitary operator U_a on $\mathscr{H} \otimes \mathscr{H}_a$). If $\langle \mathscr{H}_a, P_a, T_a, V_a \rangle$ is normal, then we use also the notation $\langle \mathscr{H}_a, P_a, \Phi_a, U_a \rangle$ for it, with the understanding that P_a is a PV measure on $(\mathfrak{R}, \mathscr{B}(\mathfrak{R}))$. A measurement $\langle \mathscr{H}_a, P_a, T_a, V_a \rangle$ is *completely positive* if the measurement mapping V_a, or at least $V_a|_{\mathscr{T}(\mathscr{H}) \otimes [T_a]}$, is completely positive. Any completely positive measurement of A is statistically equivalent to some of its normal measurements.[4,5]

2. SEQUENTIAL MEASUREMENTS

2.1. General Formulation

Let $A\colon \mathscr{X} \to \mathscr{L}(\mathscr{H})^+$ and $B\colon \mathscr{Y} \to \mathscr{L}(\mathscr{H})^+$ be any two (generalized) observables of the object system S, and let $\langle \mathscr{H}_a, P_a, T_a, V_a \rangle$ and $\langle \mathscr{H}_b, P_b, T_b, V_b \rangle$ be any of their measurements. These measurements can be combined into a sequential measurement of A and B performing them one after the other, e.g., "first A and then B." We give now a precise formulation of this idea. Let ι_a be the identity transformation on $\mathscr{T}(\mathscr{H}_a)$, and ι_b that of $\mathscr{T}(\mathscr{H}_b)$. Let ι_{ab} be the transformation $\mathscr{T}(\mathscr{H} \otimes \mathscr{H}_a \otimes \mathscr{H}_b) \to \mathscr{T}(\mathscr{H} \otimes \mathscr{H}_b \otimes \mathscr{H}_a)$ induced by the isometric isomorphism $\mathscr{H} \otimes \mathscr{H}_a \otimes \mathscr{H}_b \to \mathscr{H} \otimes \mathscr{H}_b \otimes \mathscr{H}_a$, which switches the positions of $\mathscr{H}_a$ and $\mathscr{H}_b$ in the tensor product $\mathscr{H} \otimes \mathscr{H}_a \otimes \mathscr{H}_b$. Let $\tilde{V}_a = V_a \otimes \iota_b$, and $\tilde{V}_b = \iota_{ab}^{-1} \circ V_b \otimes \iota_a \circ \iota_{ab}$. The partial traces on $\mathscr{T}(\mathscr{H} \otimes \mathscr{H}_a \otimes \mathscr{H}_b)$, like $\tilde{\Pi}_{ab}\colon \mathscr{T}(\mathscr{H} \otimes \mathscr{H}_a \otimes \mathscr{H}_b) \to \mathscr{T}(\mathscr{H})$, are distinguished by the superscript tilde ($\sim$). The 4-tuple $\langle \mathscr{H}_a \otimes \mathscr{H}_b, P_a \otimes P_b, T_a \otimes T_b, \tilde{V}_b \circ \tilde{V}_a \rangle$ is the *sequential AB-measurement* of A and B obtained with performing first the A-measurement $\langle \mathscr{H}_a, P_a, T_a, V_a \rangle$ and then the B-measurement $\langle \mathscr{H}_b, P_b, T_b, V_b \rangle$. A direct computation shows that

$$
\begin{aligned}
\mathscr{I}^{AB}_{V_a, V_b}(X \times Y)T &= \tilde{\Pi}_{ab}(\tilde{V}_b \circ \tilde{V}_a(T \otimes T_a \otimes T_b) I \otimes P_a(X) \otimes P_b(Y)) \\
&= \Pi_b \tilde{\Pi}_a(\tilde{V}_b(V_a(T \otimes T_a) \otimes T_b) I \otimes P_A(X) \otimes P_b(Y)) \\
&= \Pi_b(V_b(\Pi_a(V_a(T \otimes T_a) I \otimes P_a(X)) \otimes T_b) I \otimes P_b(Y)) \\
&= \Pi_b(V_b(\mathscr{I}^A_{V_a}(X) T \otimes T_b) I \otimes P_b(Y)) \\
&= \mathscr{I}^B_{V_b}(Y)(\mathscr{I}^A_{V_a}(X) T) \\
&= \mathscr{I}^B_{V_b}(Y) \circ \mathscr{I}^A_{V_a}(X) T
\end{aligned} \tag{4}
$$

for all $X \in \mathscr{X}$, $Y \in \mathscr{Y}$, and $T \in \mathscr{T}(\mathscr{H})$. Accordingly, the instrument $\mathscr{I}^{AB}_{V_a, V_b}$ of the sequential measurement $\langle \mathscr{H}_a \otimes \mathscr{H}_b, P_a \otimes P_b, T_a \otimes T_b, \tilde{V}_b \circ \tilde{V}_a \rangle$ is exactly the composition of the instruments $\mathscr{I}^A_{V_a}$ and $\mathscr{I}^B_{V_b}$, i.e., $\mathscr{I}^{AB}_{V_a V_b} = \mathscr{I}^B_{V_b} \circ \mathscr{I}^A_{V_a}$.[6]

Consider any two measurements $\langle \mathscr{H}_a, P_a, T_a, V_a \rangle$ and $\langle \mathscr{H}_b, P_b, T_b, V_b \rangle$ of the observables A and B. In general, $\mathscr{I}^{AB}_{V_a V_b} \neq \mathscr{I}^{BA}_{V_b V_a}$ (even when A and B are the same). This means that usually the result of a sequential measurement depends on the order in which the measurements $\langle \mathscr{H}_a, P_a, T_a, V_a \rangle$ and $\langle \mathscr{H}_b, P_b, T_b, V_b \rangle$ are performed on $\mathscr{S}$. If, however, there are measurements of A and B for which the sequential AB-measurement is statistically equivalent with the sequential BA-measurement, i.e., the instruments $\mathscr{I}^{AB}_{V_a V_b}$ and $\mathscr{I}^{BA}_{V_b V_a}$ are the same, then A and B are compatible or coexistent.[7] Clearly, if A and B are discrete PV measures, then also the converse holds true.

2.2. Sequential Probabilities as Conditional Probabilities

We consider next the probabilistic interpretation of sequential measurements. To that end, let $\mathscr{I}^{AB}_{V_aV_b}=\mathscr{I}^{B}_{V_b}\circ\mathscr{I}^{A}_{V_a}$ be the instrument of the sequential AB-measurement of Sec. 2.1. Then, for any $T\in\mathscr{T}(\mathscr{H})^+_1$, the mapping

$$\mathscr{X}\times\mathscr{Y}\to[0,1],\qquad Z\mapsto \operatorname{tr}[\mathscr{I}^{AB}_{V_aV_b}(Z)T] \tag{5}$$

is a probability measure on the product value space $\mathbf{X}\times\mathbf{Y}$ of A and B. In accordance with the minimal interpretation, the number

$$\operatorname{tr}[\mathscr{I}^{AB}_{V_aV_b}(X\times Y)T]=\operatorname{tr}[\mathscr{I}^{B}_{V_b}(Y)\circ\mathscr{I}^{A}_{V_a}(X)T] \tag{6}$$

is the probability that the sequential AB-measurement on the system $\mathscr{S}$ in the state T leads to a result in the product value set $X\times Y$. We call such probabilities *sequential probabilities*. Since

$$\operatorname{tr}[\mathscr{I}^{B}_{V_b}(Y)\circ\mathscr{I}^{A}_{V_a}(X)T]=\operatorname{tr}[(\mathscr{I}^{A}_{V_a}(X)T)\,B(Y)], \tag{7}$$

this number can also be interpreted as the probability that a B-measurement on $\mathscr{S}$ in the (nonnormalized) state $\mathscr{I}^{A}_{V_a}(X)T$ leads to a result in the set Y. But $\mathscr{I}^{A}_{V_a}(X)T$ is the (nonnormalized) state of $\mathscr{S}$ after the A-measurement under the condition that this measurement led to a result in X. Hence it would be natural to try to interpret the sequential probability $\operatorname{tr}[(\mathscr{I}^{A}_{V_a}(X)T)\,B(Y)]$ as the conditional probability: the probability that a B-measurement leads to a result in Y under the condition that the preceding A-measurement on $\mathscr{S}$ in the state T led to a result in X. This intuitive connection between the sequential and conditional probabilities carries, however, with it the problem of defining conditional probabilities in quantum mechanics. We shall now address this question.

In classical probability theory conditional probability can be characterized in different but equivalent ways. In quantum probability theory these different characterizations lead, however, to different notions of conditional probability.[8,9] Thus, to work out the possible connections of sequential probabilities to conditional probabilities, the latter concept, i.e., the intended method of conditioning, has to be specified. In the present context of sequential measurements there are two natural, but inequivalent, ways of conditioning; conditioning with respect to the A-measurement with a particular result X, and conditioning with respect to the A-measurement with fixing the pointer scale but without reading the result. Instead of giving the detailed definitions of the involved conditional probabilities, we refer to their characterizations as obtained in Refs. 8 and 9. In order to apply here directly these results, we assume now that A and B are, in fact, PV measures. This assumption has no bearing on the general conclusions to be drawn.

Consider, again, the sequential probability $\mathrm{tr}[(\mathscr{I}^A_{V_a}(X)T)\,B(Y)]$. A necessary and sufficient condition that this number can always, i.e., for each state T, be interpreted as the conditional probability of a B-measurement to lead to a result in Y given the fact that the A-measurement led to a result in X is that

$$\mathscr{I}^A_{V_a}(X)\,T = A(X)\,TA(X) \tag{8}$$

for each $T \in \mathscr{T}(\mathscr{H})^+_1$.[8] This is a severe restriction on the A-measurement (with the instrument) $\mathscr{I}^A_{V_a}$. Indeed, it may appear that for a given observable A there is no measurement $\langle \mathscr{H}_a, P_a, T_a, V_a \rangle$ such that its instrument $\mathscr{I}^A_{V_a}$ would satisfy the above condition. Then no sequential probability $\mathrm{tr}[(\mathscr{I}^A_{V_a}(X)T)\,B(Y)]$ would admit the above interpretation as conditional probability. As an example, the usual position Q of an object is such an observable (cf. Sec. 3.3). In addition, for each observable A there are measurements which do not fulfill the above condition. [This can be seen, e.g., with choosing properly an instrument of the type (29) of Sec. 3.3.] Thus, in general, the sequential probabilities $\mathrm{tr}[(\mathscr{I}^A_{V_a}(X)T)\,B(Y)]$ do not admit the above type of interpretation as conditional probabilities. Clearly, there are also important cases where such an interpretation can be given. Indeed, for a fixed $X \in \mathscr{X}$, one may consider, instead of the AB-measurement, e.g., an $\chi_X(A)B$-measurement with the first measurement being the von Neumann–Lüders measurement. Here χ_X is the characteristic function of the set X so that $\chi_X(A)$ is the PV measure with the range $\{O, A(X), I-A(X), I\}$. Then $\mathrm{tr}[(\mathscr{I}^{\chi_X(A)}_{vNL}(\{1\})T)\,B(Y) = \mathrm{tr}[A(X)\,TA(X)\,B(Y)]$ for each Y and T.

As the second example, consider the sequential probability $\mathrm{tr}[(\mathscr{I}^A_{V_a}(\mathbf{X})T)\,B(Y)]$, and let (X_i) be a partition of $\mathbf{X}$ into disjoint value sets $X_i \in \mathscr{X}$. Then $\mathscr{I}^A_{V_a}(\mathbf{X})T = \sum \mathscr{I}^A_{V_a}(X_i)T$ is the state of $\mathscr{S}$ after the A-measurement with fixing the pointer scale (i.e., the partition of $\mathbf{X}$) but without reading the result X_i. The sequential probability $\mathrm{tr}[(\mathscr{I}^A_{V_a}(\mathbf{X})T)\,B(Y)]$, with the partition $\mathbf{X} = \bigcup X_i$, admits an interpretation as conditional probability exactly when

$$\mathscr{I}^A_{V_a}(\mathbf{X})\,T\left(= \mathscr{I}^A_{V_a}\left(\bigcup X_i\right)T = \sum \mathscr{I}^A_{V_a}(X_i)\,T\right) = \sum A(X_i)\,TA(X_i) \tag{9}$$

for all T.[9] Again, it is obvious that the conditional probability interpretation in the present sense is a strong restriction on the A-measurement. Indeed, any A-measurement which admits such an interpretation is of the first kind (cf. Sec. 3.2.2). Clearly, if A is discrete, then the sequential probabilities $\mathrm{tr}[(\mathscr{I}^{A}_{V_a}(\mathbf{X})T(B(Y)]$ admit the conditional probability interpretation with respect to the partition of $\mathbf{X}$ into the eigenvalues of A when the A-measurement $\langle \mathscr{H}_a, P_a, T_a, V_a \rangle$ is the von Neumann–Lüders measurement. [To see this, see Eq. (18).]

3. REPEATABLE MEASUREMENTS

3.1. Definition

The notion of sequential measurement leads to a natural formulation of the notion of repeatable measurement. Consider an observable A, and let $\langle \mathscr{H}_a, P_a, T_a, V_a \rangle$ be any of its measurements. If we apply twice this measurement on $\mathscr{S}$ it may, or may not, happen that new results are obtained. The measurement $\langle \mathscr{H}_a, P_a, T_a, V_a \rangle$ is *repearable* if its repetition does not lead—from the probabilistic point of view—to a new result, i.e , if

$$\begin{aligned} \mathrm{tr}[\mathscr{I}^{AA}_{V_aV_a}(X\times Y)T] &= \mathrm{tr}[\mathscr{I}^{A}_{V_a}(Y)\circ\mathscr{I}^{A}_{V_a}(X)T] \\ &= \mathrm{tr}[\mathscr{I}^{A}_{V_a}(Y\cap X)T] \end{aligned} \tag{10}$$

for all $X, Y\in\mathscr{X}$, and for all $T\in\mathscr{T}(\mathscr{H})^+_1$. Sometimes condition (10) is taken to characterize *weak repeatability* of a measurement $\langle \mathscr{H}_a, P_a, T_a, V_a \rangle$ to distinguish it from the *strong repeatability* condition

$$\mathscr{I}^{AA}_{V_aV_a}(X\times Y)T=\mathscr{I}^{A}_{V_a}(Y\cap X)T, \qquad X, Y\in\mathscr{X},\ T\in\mathscr{T}(\mathscr{H})^+_1 \tag{11}$$

of a measurement $\langle \mathscr{H}_a, P_a, T_a, V_a \rangle$. Thus $\langle \mathscr{H}_a, P_a, T_a, V_a \rangle$ is strongly repeatable if the sequential AA-measurement obtained with performing this measurement twice on $\mathscr{S}$ is statistically equivalent to $\langle \mathscr{H}_a, P_a, T_a, V_a \rangle$. Here we shall mainly consider the weaker, purely probabilistic condition (10), and we refer to it as the repeatability condition. We observe that (10) is equivalent to

$$\mathrm{tr}[\mathscr{I}^{A}_{V_a}(X)\circ\mathscr{I}^{A}_{V_a}(X)T]=\mathrm{tr}[\mathscr{I}^{A}_{V_a}(X)T], \qquad X\in\mathscr{X},\ T\in\mathscr{T}(\mathscr{H})^+_1 \tag{12}$$

Indeed, if (12) holds, then $\mathrm{tr}[\mathscr{I}^{A}_{V_a}(X)\circ\mathscr{I}^{A}_{V_a}(Z)T]=0$ for any $T\in\mathscr{T}(\mathscr{H})^+_1$, whenever $X\cap Z=\varnothing$. This suffices to show that (12) implies (10).

We say that an observable A admits a repeatable measurement if there is an A-measurement $\langle \mathscr{H}_a, P_a, T_a, V_a \rangle$ which is repeatable.

3.2. Some Properties

We shall now point out some of the important properties of repeatable measurements. Such measurements turn out to be preparatory, and of the first kind, and they produce strong correlations between the measured observable and the pointer observable. The problem of determining the general form of such measurements is also reviewed.

3.2.1. Preparatory. A measurement of an observable is preparatory if after the measurement the measured observable has with certainty (i.e., with probability equal to 1) the measured value. Consider a measurement $\langle \mathscr{H}_a, P_a, T_a, V_a \rangle$ of A. Let T^X_{S, V_a} be the normalized conditional final state of $\mathscr{S}$, with the condition that the measurement yielded a result in X. (Note that this presupposes that $\mathrm{tr}[TA(X)] = \mathrm{tr}[\mathscr{I}^A_{V_a}(X) T] \neq 0$, which guarantees that $\mathscr{I}^A_{V_a}(X) T$ can be normalized.) Thus, in the present formulation, the measurement $\langle \mathscr{H}_a, P_a, T_a, V_a \rangle$ of A is *preparatory* if

$$\mathrm{tr}[T^X_{S, V_a} A(X)] = 1 \tag{13}$$

for any $X \in \mathscr{X}$, and for all $T \in \mathscr{T}(\mathscr{H})^+_1$ for which $\mathrm{tr}[TA(X)] \neq 0$. It is now immediate to observe that a repeatable A-measurement $\langle \mathscr{H}_a, P_a, T_a, V_a \rangle$ is also preparatory.

3.2.2. First Kind. A measurement is of the first kind if the probability to obtain a particular result is the same before and after the measurement. In the present formulation, an A-measurement $\langle \mathscr{H}_a, P_a, T_a, V_a \rangle$ is of the *first kind* if

$$\mathrm{tr}[TA(X)] = \mathrm{tr}[\Pi_a(V_a(T \otimes T_a))\, A(X)] \tag{14}$$

for all $X \in \mathscr{X}$, $T \in \mathscr{T}(\mathscr{H})^+_1$. If $\langle \mathscr{H}_a, P_a, T_a, V_a \rangle$ is a repeatable A-measurement, then it is also a first kind measurements since $\mathrm{tr}[\mathscr{I}^A_{V_a}(\mathfrak{R})\, TA(X)] = \mathrm{tr}[\mathscr{I}^A_{V_a}(X) \circ \mathscr{I}^A_{V_a}(\mathfrak{R}) T] = \mathrm{tr}[\mathscr{I}^A_{V_a}(X) T] = \mathrm{tr}[TA(X)]$ for all $X \in \mathscr{X}$, $T \in \mathscr{T}(\mathscr{H})^+_1$.

3.2.3. Strong Correlations. Let $\langle \mathscr{H}_a, P_a, T_a, V_a \rangle$ be a measurement of A, and assume that P_a is a PV measure on $(\mathfrak{R}, \mathscr{B}(\mathfrak{R}))$. The measurement $\langle \mathscr{H}_a, P_a, T_a, V_a \rangle$ produces *strong correlations between the observables A and P_a* if

$$\rho(A, P_a, V_a(T \otimes T_a)) = 1 \tag{15}$$

for each $T \in \mathscr{T}(\mathscr{H})^+_1$ [for which $\mathrm{Var}(A, \Pi_a(V(T \otimes T_a)) \neq 0 \neq \mathrm{Var}(P_a, \Pi(V(T \otimes T_a)))]$. If $\langle \mathscr{H}, P_a, T_a, V_a \rangle$ is repeatable, and thus of the first kind, then $A_T = P_{a,\Pi(V(T\otimes T_a))} = A_{\Pi_a(V(T\otimes T_a))}$. This implies that (15) holds, i.e., $\langle \mathscr{H}_a, P_a, T_a, V_a \rangle$ produces strong correlations between the measured observable A and the pointer observable P_a.[3,10]

The is another (stronger) type of correlations a measurement may produce, namely correlations between the values of the measured observable and the pointer observable. In fact, this is the type of correlation required in determining the value of the measured observable from that of the pointer reading. A measurement $\langle \mathscr{H}_a, P_a, T_a, V_a \rangle$ produces *strong correlations between the values* of A and P_a if

$$\rho(A(X), P_a(X), V_a(T\otimes T_a)) = 1 \tag{16}$$

for all value sets $X \in \mathscr{X}$ and for all states $T \in \mathscr{T}(\mathscr{H})_1^+$ (for which the condition is meaningful). A general study of this condition can, again, be given. Here we note only that if A is a PV measure, then the strong-value correlation condition (16) is equivalent with the first kind condition (14) showing that a repeatable measurement (of a usual observable) has also the strong value correlation property.[3,10]

3.2.4. Discrete. An observable A is *discrete* if there is a countable subset $\mathbf{X}_0$ of the value space $\mathbf{X}$ such that $A(\mathbf{X}\backslash\mathbf{X}_0) = O$. It is an old issue in quantum measurement theory, dating from the pioneering work of von Neumann,[11] whether observables admitting repeatable measurements are necessarily discrete.[2] According to a theorem of Ozawa, if A admits a complete positive repeatable measurement, then A is discrete.[4] This result resolves the issue to a large extent. We recall that completely positive measurements of an observable A are exactly those which are statistically equivalent with some of its normal measurements. As normal measurements refer, in particular, to unitary measurements, complete positivity is certainly a very important measurement theoretical concept. We return to this in Sec. 3.2.6 where a characteristic property of a completely positive measurement is given. For the sake of notational simplicity we shall hereon describe discrete observables as POV measures on the subsets of positive integers $\mathbf{N}$.

3.2.5. Ideal. A measurement of an observable is ideal if it does not change the state of the system whenever a measurement result can be predicted with certainty. This notion can be formulated in a meaningful way only for discrete observables admitting probabilities equal to one. Thus assuming a discrete observable A (on $\mathbf{N}$) admitting probabilities equal to one, an A-measurement $\langle \mathscr{H}_a, P_a, T_a, V_a \rangle$ is *ideal* if

$$\mathrm{tr}[\mathscr{I}^A_{V_a}(\{n\})T] = 1 \text{ implies } \mathscr{I}^A_{V_a}(\{n\})T = T \text{ for any } T \in \mathscr{T}(\mathscr{H})_1^+, n \in \mathbf{N} \tag{17}$$

Note that if the assumption that the observable A admits probabilities equal to one is dropped, the notion could be empty: there are (generalized) observables A such that no $A(X)$ $(\neq I)$ possesses eigenvalue 1. Then any instrument of such an A would trivially satisfy (17). We note that ideal, (weakly) repeatable measurements are strongly repeatable. In general, ideality of a measurement is not a consequence of, and does not imply, repeatability.

3.2.6. Connections. It is rewarding to investigate the implications of some of the above properties of measurements on the structure of the resulting instruments. Before doing that, we recall that if A is a discrete PV measure, then its von Neumann–Lüders measurement, with the associate instrument

$$\mathscr{I}^A_{vNL}(X)T = \sum_{n \in X} A(\{n\})\, TA(\{n\}), \qquad X \subset \mathbf{N},\ T \in \mathscr{T}(\mathscr{H})^+_1 \tag{18}$$

has all the above discussed properties, and, in fact, somewhat more. It is an old problem to try to characterize the von Neumann–Lüders measurement (i.e., measurements with the above type of instrument), and to characterize observables admitting such measurements.

We recall first that if A is a discrete PV measure, then (18) can be characterized as an ideal, strong-value correlation measurement, or, equivalently, as an ideal, first-kind measurement. For a proof of this well-known results, see, e.g., Ref. 3. As was pointed out above, the discreteness assumption can be dropped if the measurement is completely positive. Hence, if A is a PV measure, then (18) can be characterized as an ideal, repeatable, completely positive measurement. If one would stick to the idea that PV measures represent properly—and exhaustively—the observable concept, then one could argue that these results are complete. However, if one follows quantum measurement theory, then one immediately observes that the observable concept arrived at there is rather a POV measure. Then, clearly, the above results are not enough and one has to look for more. For that, we recall next a result of Davies[2] which states that a *generalized* discrete observable admits an ideal, repeatable, nondegenerate (see below) instrument if and only if this observable is a PV measure. In that case, the relevant instrument is exactly the von Neumann–Lüders instrument (18). An instrument $\mathscr{I}$ on $\mathbf{N}$ is *nondegenerate* if it fulfills the following condition:

$$\text{if } B \in \mathscr{L}(\mathscr{H})^+,\ \mathrm{tr}[B\mathscr{I}_n T] = 0 \text{ for all } T \in \mathscr{T}(\mathscr{H})^+_1,\ n \in \mathbf{N}, \text{ then } B = O \tag{19}$$

Again, the discreteness assumption can be dropped here by assuming that the measurement is completely positive. But since the nondegeneracy assumption appears as a technical condition which is difficult to interpret physically, it is worth trying to exploit further the ideality and repeatability conditions together with the complete positivity assumption. Before doing that, we shall, however, give an example which shows that ideality, repeatability, and complete positivity of a measurement are not enough to fix the form of the associated instrument to be the von Neumann–Lüders one.

Example. Let $\mathscr{H}$ be a 3-dimensional Hilbert space with an orthonormal basis $\{\varphi, \psi, \xi\}$. Define a *POV* measure A and an associated instrument $\mathscr{I}^A$ on the two-point value space $\{1, 2\}$ as follows:

$$A(\{1\}) = P[\varphi] + \frac{1}{2} P[\xi], \qquad A(\{2\}) = P[\psi] + \frac{1}{2} P[\xi]$$

$$\mathscr{I}^A(\{1\})T = B_1 T B_1^* + B_2 T B_2^*, \qquad B_1 = P[\varphi], \qquad B_2 = \frac{1}{\sqrt{2}} |\varphi\rangle\langle\xi|$$

$$\mathscr{I}^A(\{2\})T = B_1' T B_1'^* + B_2' T B_2'^*, \qquad B_1' = P[\psi], \qquad B_2' = \frac{1}{\sqrt{2}} |\psi\rangle\langle\xi|$$

One verifies quickly that $\mathscr{I}^A$ is completely positive [see Eq. (20) below], ideal, and repeatable although A is not a *PV* measure. Clearly, 0 and 1 are also eigenvalues of $A(\{1\})$ and $A(\{2\})$.

We shall now return to study the consequences of ideality and repeatability under the assumption of complete positivity. This gives us also an opportunity to correct an error in Theorem 1 of Ref. 12.

We note first that, if an observable A admits a repeatable measurement, then the spectrum of any $A(X)$, $X \in \mathscr{X}$, (except O and I) contains 0 and 1 as eigenvalues. This follows immediately, e.g., from the fact that repeatable measurements are also preparatory (so that then obviously $A(X)\varphi = \varphi$ and $A(X)\psi = 0$ for $\varphi \in \operatorname{ran}(T^X)$ and $\psi \in \operatorname{ran}(T^X)^\perp$).

Next we shall concentrate on simple observables which are defined on the two-point set $\{1, 2\}$, like in the above Example. For short, let $A(\{1\}) = E$ and $A(\{2\}) = E'$ so that $E' = I - E$. Any instrument of such an A is generated by two operations, as well: $\mathscr{I}^A(\{1\}) = \Phi$ and $\mathscr{I}^A(\{2\}) = \Phi'$, with $\operatorname{tr}[\Phi T] = \operatorname{tr}[TE]$ and $\operatorname{tr}[\Phi' E'] = \operatorname{tr}[TE']$ for all $T \in \mathscr{T}(\mathscr{H})_1^+$. The results obtained below for simple observables will extend in an obvious way to general observables.

An operation Φ is *completely positive* exactly when it has the following representation:

$$\Phi T = \sum_{k \in \mathbf{N}} B_k T B_k^*, \qquad T \in \mathscr{T}(\mathscr{H}) \tag{20}$$

for some countable set $\{B_k\} \subset \mathscr{L}(\mathscr{H})$.[2,13] If the range of an instrument consists solely of completely positive operations, then the instrument is completely positive. Such instruments arise, indeed, from completely positive measurements.

Consider now a simple observable A with the range $\{O, E, E', I\}$, and let $\mathscr{I}^A$ be an associated instrument generated by the operations Φ and Φ'. Assume that $\mathscr{I}^A$ is completely positive, with the representations $\Phi T = \sum_{k \in \mathbf{N}} B_k T B_k^*$ and $\Phi' T = \sum_{k \in \mathbf{N}} B'_k T B'^*_k$. Assume further that 0 and 1 are eigenvalues of both E and E'. Let P and P' be the spectral projections of E and E' associated with the eigenvalue 1 (cf. the above example). Then the following assertions are true:

$$\mathscr{I}^A \text{ is ideal iff } B_k P = \beta_k P \text{ (for all } k), \sum_k |\beta_k|^2 = 1$$

$$\text{and } B'_k P = \beta'_k P' \text{ (for all } k), \sum_k |\beta'_k|^2 = 1 \tag{21}$$

$$\mathscr{I}^A \text{ is repeatable iff } PB_k = B_k,\ P'B'_k = B'_k \text{ (for all } k) \tag{22}$$

$$A \text{ is a } PV \text{ measure iff } B_k P = B_k,\ B'_k P' = B'_k \text{ (for all } k) \tag{23}$$

This result then shows that the von Neumann–Lüders instrument $\mathscr{I}^A_{vNL}$ is the only completely positive, ideal instrument associated with a PV measure A. We note next that under the assumption of complete positivity, ideality, and repeatability the nondegeneracy assumption yields exactly that $E = P$ and $E' = P' = P^\perp$. This shows then in a very straightforward way that only PV measures admit ideal, repeatable, nondegenerate, completely positive instruments, and that, indeed, the nondegeneracy assumption is unavoidable in characterizing the von Neumann–Lüders measurements.

3.3. Continuous Observables, Localization Observable

An observable $A\colon \mathscr{X} \to \mathscr{L}(\mathscr{H})^+$ is called *continuous* if A, as a *measure*, is atom-free. An element $X \in \mathscr{X}$ is an *atom* of the Boolean lattice $\mathscr{X}$ with respect to the finite (POV) measure A if $A(X) \neq O$ and $X' \in \mathscr{X}$, $X' \subset X$ implies $A(X') = O$ or $A(X') = A(X)$.

As a consequence of the above-cited theorem of Ozawa, continuous observables do not admit repeatable, completely positive measurements. This fact causes some difficulties in understanding the operational definition of continuous observables, among them position, momentum, and energy—observables which are of utmost importance for the concept of particle in quantum physics.

As an example one may consider the localization observable $A = E^Q$, the spectral measure of the position operator Q. The spectrum of Q is absolutely continuous, and so is the spectral measure E^Q. One might try to define "localization" by means of a slit (represented by an interval $I \in \mathscr{X} = \mathscr{B}(\mathfrak{R})$) in a diaphragm, the action of which on the particle should be describable by the familiar state transformation (the Lüders operation):

$$\mathscr{I}^Q(I)\,T = E^Q(I)\,TE^Q(I) \tag{24}$$

Such an operation determines uniquely the spectral projection $E^Q(I)$, which, by varying over all intervals, determine the position Q. But, as we shall see below, there is no instrument $\mathscr{I}^Q$ having the property (24) for any of the intervals of $\mathfrak{R}$.

In the measurement theory of generalized observables A the so-called *generalized Lüders operations* are important:

$$\mathscr{I}^A_L(X)\,T = A(X)^{1/2}\,TA(X)^{1/2} \tag{25}$$

The existence of such an operation in the range of an instrument $\mathscr{I}^A$ has interesting implications on the observable A. Indeed, consider a continuous observable A on $(\mathbf{X}, \mathscr{X})$, and assume that there is an instrument $\mathscr{I}^A$ of A such that $\mathscr{I}^A(X_0)$ has the form (25) for some value set $X_0 \in \mathscr{X}$. Then A and $\mathscr{I}^A$ have the following form:

$$A(X) = \lambda(X \cap X_0)\,A(X_0) + A(X \cap X_0'), \qquad X \in \mathscr{X} \tag{26}$$

$$\mathscr{I}^A(X) = \lambda(X \cap X_0)\,\mathscr{I}^A(X_0) + \mathscr{I}^A(X \cap X_0'), \qquad X \in \mathscr{X} \tag{27}$$

where λ is a (continuous) normalized measure vanishing on $X \cap X_0'$, $X_0' = \mathbf{X} \backslash X_0$. The proof of this result is given in the Appendix.

As a corollary, we note that if A is a continuous PV measure, then A admits no instrument containing in its range a Lüders operation. When combined with the results of Sec. 3.2.6, one obtains that any PV measure A on $(\mathfrak{R}, \mathscr{B}(\mathfrak{R}))$ which admits a completely positive instrument generated (as a measure) by ideal (and therefore Lüders) operations is necessarily discrete.

Returning to the localization observable Q, the preceding considerations show that its operational definition as a *continuous* observable in terms of ideal and repeatable measurements in impossible. This means that the standard way of defining particles by means of systems of imprimitivity does not seem to have an operational foundation in quantum mechanics.

We shall now sketch possible remedies. Though there exist investigations[6,14] dealing with the problem on a rather general level, we shall, for the sake of simplicity, only deal here with the position observable. The usual way out of the dilemma consists in introducing discretized versions of position. This was already suggested by von Neumann.[11] But this approach destroys the translational covariance characteristic of the localization concept. Another approach proposed by Davies and Lewis[6] is to relax strict repeatability into a weaker δ-repeatability. An instrument $\mathscr{I} = \mathscr{I}^Q$, associated to Q, is *δ-repeatable* if for all states $T \in \mathscr{T}(\mathscr{H})_1^+$ and for all $X \in \mathscr{B}(\mathfrak{R})$ the following holds:

$$\operatorname{tr}[\mathscr{I}(X_\delta)\,\mathscr{I}(X)\,T] = \operatorname{tr}[\mathscr{I}(X)\,T] \tag{28}$$

where $X_\delta = \{x \in \mathfrak{R}\colon |x - x'| \leqslant \delta \text{ for all } x' \in X\}$ is the closed δ-neighborhood of X. The following instrument $\mathscr{I}$ is then δ-repeatable. Let $\{X_i\}$ be a countable partition of $\mathfrak{R}$ into disjoint Borel intervals, and let $\{T_i\}$ be a collection of vector states localized on X_i ($\operatorname{tr}[T_i E^Q(X_i)] = 1$). The formula

$$\mathscr{I}^Q(X)\,T \doteq \sum \operatorname{tr}[TQ(X \cap X_i)]\,T_i \tag{29}$$

(with $X \in \mathscr{X}$, $T \in \mathscr{T}(\mathscr{H})_1^+$) defines a completely positive, δ-repeatable instrument of Q. Due to its complete positivity, there is a normal measurement of Q, which defines this instrument.[4] Clearly, the above partition also gives rise to a discretized position $f(Q)$, say, which admits a standard measurement with the associated instrument being the von Neumann–Lüders instrument. However, the covariance is still not satisfied (for the instrument). But the following yields even a *covariant*, δ-repeatable, completely positive instrument for Q. Let T_0 be a vector state localized on $[-\delta, \delta]$. Define $T_x = U_x T_0 U_x^*$, where U_x is the unitary representation of the translation group on $\mathfrak{R}$. Thus T_x is localized on $[x - \delta,\ x + \delta]$, and $\operatorname{tr}[T_x E^Q(X_\delta)] = 1$ whenever $x \in X$. This guarantees that the instrument

$$\mathscr{I}(X)\,T = \int_X T_x \operatorname{tr}[TE^Q(dx)] \tag{30}$$

satisfies (28). One may still object that this construction appears rather artificial, and indeed, it seems difficult to imagine a realistic measurement leading to such an instrument. In our opinion the most natural way consists in yet another version of approximate repeatability: an operation Φ is called *ε-repeatable* $(0 < \varepsilon < 1/2)$ if $\operatorname{tr}[\Phi^2 T] \geqslant (1 - \varepsilon)\operatorname{tr}[\Phi T]$ for all states T. Then an instrument $\mathscr{I}$ will be called *(ε, δ)-repeatable* if the following condition is satisfied for all states T and for all value sets X:

$$\operatorname{tr}[\mathscr{I}(X_\delta)\,\mathscr{I}(X)\,T] \geqslant (1 - \varepsilon)\operatorname{tr}[\mathscr{I}(X)\,T] \tag{31}$$

This concept turns out to be the only realizable one if one considers genuine generalized observables which do not allow probabilities equal to one. It has been shown that even phase space can be given an operational meaning in quantum mechanics on the basis of (ε, δ)-repeatability.[14] Furthermore, also the concept of ideality can be relaxed into *approximate* ideality, which is a property of, say, the generalized Lüders operation (25). In this way it is possible to modify all relevant measurement theoretical notions so as to make them applicable to an operational understanding of generalized observables. For a more detailed exposition of these questions, the reader is referred to Ref. 14.

4. SUMMARY AND OUTLOOK

The theory of sequential measurements outlined in this paper allows one to discuss systematically the significance of the well-known measurement theoretical notions to the operational definition of observables in quantum mechanics. In particular, we have revisited the problems arising in the case of continuous observables. The considerations given at the end of the preceding section suggests also that it is possible to achieve at least an approximate constitution of physical objects within quantum mechanics.

The present investigation belongs to the field of *generalized quantum mechanics*, which is based on the most general notion of an observable compatible with the probabilistic structure of quantum mechanics. In a recent survey paper an overview of this rather new field has been given.[15] Therefore we shall now only sketch a few potential applications of the theory.

First, we mention the problem of localization of relativistic quantum systems. It turns out that there exists a consistent solution obtained by generalizing the usual system of imprimitivity by allowing there a generalized observable concept. Then localization is represented by means of a genuine *POV* measure, and the above ideas of approximate repeatability immediately apply to it.

Next, quantum phase space observables play an important role in quantum optics and signal processing. In these fields they allow for a consistent description of coherent state determination, they open the possibility to certain quantum nondemolition measurements, and they provide a means for informationally complete signal analysis. Indeed, the only way to understand, e.g., the nondemolition condition is, again, in terms of (ε, δ)-repeatability.[14]

As a final example we mention that effects and *POV* measures have very recently raised the interest of researchers in quantum cosmology. It seems possible to understand the Gell-Mann—Hartle notion of *approximate probability for alternative time histories* in terms of sequential measurements of generalized observables.[16]

APPENDIX

Assume that A, $\mathscr{I}^A$, and X_0 are as specified in the context of Eqs. (26) and (27). Since A and $\mathscr{I}^A$ are measures, it is enough to show that for all $X \subset X_0$, $X \in \mathscr{X}$,

$$A(X) = \lambda(X)\, A(X_0) \tag{A1}$$

$$\mathscr{I}^A(X)\, T = \lambda(X)\, \mathscr{I}^A(X_0)\, T \tag{A2}$$

for all $T \in \mathscr{T}(\mathscr{H})_1^+$. By assumption,

$$\mathscr{I}^A(X_0)\, T = A(X_0)^{1/2}\, TA(X_0)^{1/2} \qquad \text{for all} \quad T \in \mathscr{T}(\mathscr{H})_1^+ \tag{A3}$$

Denote $\Phi_0 = \mathscr{I}^A(X_0)$, $\Phi = \mathscr{I}^A(X)$, $\Phi' = \mathscr{I}^A(X_0 \backslash X)$, and $B_0 = A(X_0)^{1/2}$. From $\Phi_0 = \Phi + \Phi'$ we conclude that, together with Φ_0, Φ and Φ' are also pure operations. According to a theorem of Davies,[(2)] any pure operation Φ is one of the following three forms:

$$\Phi T = BTB^*, \qquad B \text{ bounded, linear} \tag{A4}$$

$$\Phi T = BT^*B^*, \qquad B \text{ bounded, antilinear} \tag{A5}$$

$$\Phi T = \mathrm{tr}[CT]\, P[\gamma], \qquad C \text{ bounded, positive, } \gamma \in \mathscr{H},\ \langle \gamma \mid \gamma \rangle = 1 \tag{A6}$$

Thus we have to deal with three cases.

Case 1. $\Phi T = BTB^*$, B linear, bounded. It follows that for any $\varphi \in \mathscr{H}$, $B\varphi = \alpha(\varphi)\, B_0 \varphi$. For $B_0 \varphi = 0$, $\alpha(\varphi)$ remains undetermined and can be chosen freely. Assume $B_0 \varphi \neq 0$. We shall exploit the linearity of B and B_0. Then it follows that $\alpha(c\varphi) = \alpha(\varphi)$. From $B(\varphi + \psi) = B(\varphi) + B(\psi)$, we obtain

$$0 = \{\alpha(\varphi + \psi) - \alpha(\varphi)\}\, B_0 \varphi + \{\alpha(\varphi + \psi) - \alpha(\psi)\}\, B_0 \psi \tag{A7}$$

Assume $\{B_0 \varphi, B_0 \psi\}$ is linearly independent. Then $\alpha(\varphi) = \alpha(\varphi + \psi) = \alpha(\psi)$. Next, assume that $\{B_0 \varphi, B_0 \psi\}$ is not linearly independent.

(a) If there is a $\chi \in \mathscr{H}$ such that both sets $\{B_0 \varphi, B_0 \chi\}$ and $\{B_0 \psi, B_0 \chi\}$ are linearly independent, then it follows as above that $\alpha(\varphi) = \alpha(\chi) = \alpha(\psi)$. So, in this situation (irrespective of whether $\{B_0 \varphi, B_0 \psi\}$ is linearly independent or not), the function α may be chosen constant on $\mathscr{H}$. Then $B = l(X) B_0$, and (A1) and (A2) follow.

(b) If there is no χ such that both sets $\{B_0 \varphi, B_0 \chi\}$ and $\{B_0 \psi, B_0 \chi\}$ are linearly independent, then, in fact, $B_0 \chi = \beta(\chi) B_0 \varphi = \beta(\chi) \varphi_0$ for all $\chi \in \mathscr{H}$. This implies (by Riesz' Theorem) that $B_0 = \beta_0 P[\varphi_0]$. Furthermore,

$B\varphi = \alpha(\varphi)\, B_0 \varphi = \alpha(\varphi)\, \beta_0 P[\varphi_0]\, \varphi$, so that by linearity of B, α can be chosen constant and $B = \beta(X) B_0$. This again yields (A1) and (A2).

We note that, together with $\Phi = \mathscr{I}^A(X) = \lambda(X)\, \mathscr{I}^A(X_0)$, Φ' is also of the same form: $\Phi' = \mathscr{I}^A(X_0 \backslash X) = (1 - \lambda(X))\, \mathscr{I}^A(X_0) = \lambda(X_0 \backslash X)\, \mathscr{I}^A(X_0)$.

Case 2. $\Phi T = \mathrm{tr}[CT]\, P[\gamma]$, $C = A(X)$. Then for all $\varphi \in \mathscr{H}$, $\Phi_0 P[\varphi] = \mathscr{I}^A(X_0)\, P[\varphi] = \alpha(\varphi)\, P[\gamma]$ (since Φ_0 is a pure operation). Now one proceeds as in part (b) of the previous case to obtain (A1) and (A2).

Case 3. $\Phi T = BT^*B^*$, B bounded, antilinear. The same line of argument as in Case 1 leads to the relation $B\varphi = \alpha(\alpha)\, B_0 \varphi$ for all $\varphi \in \mathscr{H}$.

(a) If $B_0 = \beta_0 P[\varphi_0]$, then it follows from the antilinearity of B that $B\varphi = \beta(\varphi, \varphi_0)\varphi_0$ and therefore $\Phi T = |\beta|^2\, \mathrm{tr}[P[\varphi_0]\, T^*]\, P[\varphi_0]$. Now Φ' also must be one of $\Phi' T = |\beta'|^2\, \mathrm{tr}[P[\varphi_0]\, T]\, P[\varphi_0]$ or $\Phi' T = |\beta'|^2\, \mathrm{tr}[P[\varphi_0]\, T^*]\, P[\varphi_0]$. In either case, however, $\Phi + \Phi' \neq \Phi_0$. Thus Case 3 cannot occur in this situation.

(b) If B_0 is not of the form $\beta_0 P[\varphi_0]$, then one can proceed as in (a) of Case 1 to infer from the additivity of B that $\alpha(\varphi)$ can be chosen constant. Again, this in contradiction to the antilinearity of B, so that Case 3 is finally excluded.

REFERENCES

1. E. Beltrametti and G. Cassinelli, *The Logic of Quantum Mechanics* (Addision-Wesley, Reading, Massacusetts, 1981).
2. E. B. Davies, *Quantum Theory of Open Systems* (Academic Press, New York, 1976).
3. E. Beltrametti, G. Cassinelli, and P. Lahti, "Unitary measurements of discrete quantities in quantum mechanics," *J. Math. Phys.* **31**, 91–98 (1990).
4. M. Ozawa, "Quantum measuring processes of continuous observables," *J. Math. Phys.* **25**, 79–87 (1984).
5. P. Busch and P. Lahti, "Completely positive mappings in quantum dynamics and measurement theory," *Found. Phys.* **20** (1990), to appear.
6. E. B. Davies and J. T. Lewis, "An operational approach to quantum probability," *Commun. Math. Phys.* **17**, 239–260 (1970).
7. P. Lahti and S. Bugajski, "Fundamental principles of quantum theory, part 2," *Int. J. Theor. Phys.* **24**, 1051–1080 (1985).
8. G. Cassinelli and N. Zanghi, "Conditional probabilities in quantum mechanics I," *Nuovo Cimento* **73B**, 237–245 (1983).
9. G. Cassinelli and N. Zanghi, "Conditional probabilities in quantum mechanics II," *Nuovo Cimento* **79B**, 141–154 (1984).
10. G. Cassinelli and P. Lahti, "Strong correlation measurements in quantum mechanics," manuscript submitted.
11. J. von Neumann, *Mathematische Grundlagen der Quantenmechanik* (Springer, Berlin, 1932; English translation, Princeton University Press, Princeton, New Jersey, 1955).

12. P. Busch, "Unsharp reality and the question of quantum systems," in *Symposium on the Foundations of Modern Physics 1987*, P. Lahti and P. Mittelsttaedt, eds. (World Scientific, Singapore, 1988), pp. 105–125.
13. K. Kraus, *States, Effects, and Operations* (Springer-Verlag, Berlin, 1983).
14. P. Busch and P. Lahti, "Some remarks on unsharp quantum measurements, quantum nondemolition, and all that," *Ann. Phys. (Leipzig)* **47** (1990), in press.
15. P. Busch, M. Grabowski, and P. Lahti, "Some remarks on effects, operations, and unsharp quantum measurements," *Found. Phys. Lett.* **2**, 331–345 (1989).
16. J. B. Hartle, private communication to one of us (P.B.), 1989.