COMPGV19: Tutorial 4

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Exercise 1

Implement the linear preconditioned Conjugate Gradient method

```
clear all, close all;
```

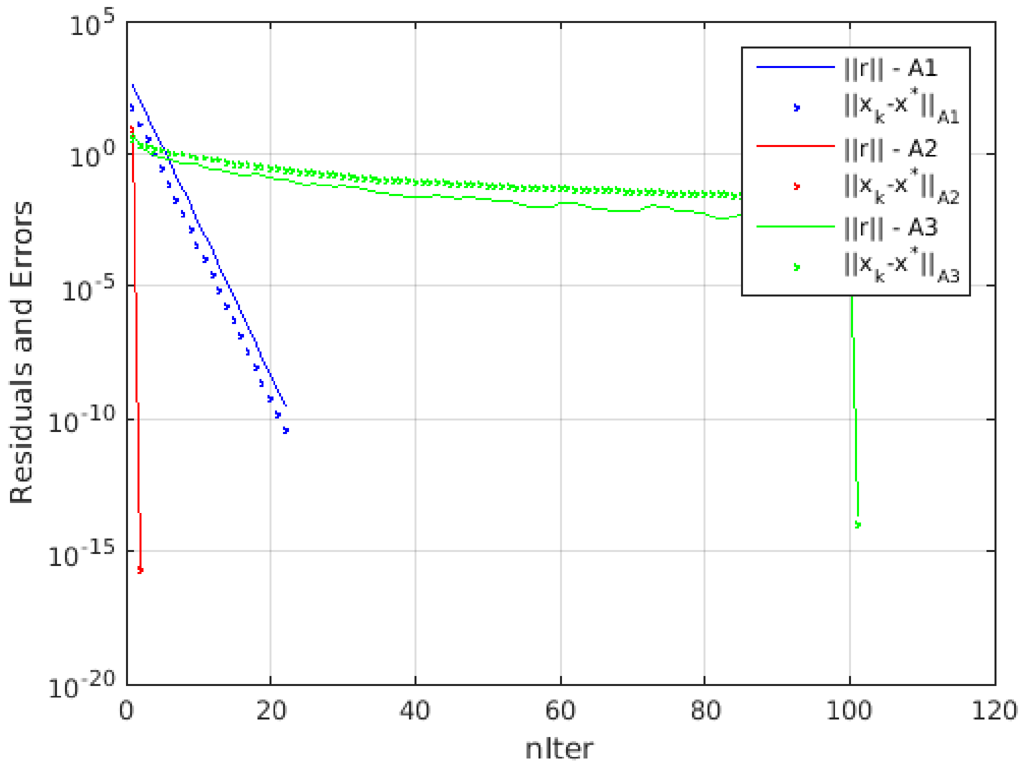
Initialisation

```
n = 1e2;
tol = 1e-12;
maxIter = 1e3;
% Initial point
x0 = zeros(n, 1);

% Matrix definition - Use sparse matrices
b = ones(n, 1);
A1 = spdiags((1:n)', 0, n, n);
A2 = spdiags([ones(n-1, 1); 100], 0, n, n);
A3 = spdiags([-ones(n-1, 1); 0], -1, n, n) + spdiags(2*ones(n, 1), 0,
 n, n) + spdiags([0; -ones(n-1, 1)], 1, n, n);
% A4 = hilb(n);

% Define xtrue
xtrue = zeros(n,1);
xtrue(floor(n/4):floor(n/3)) = 1;
xtrue(floor(n/3)+1:floor(n/2)) = -2;
xtrue(floor(n/2)+1:floor(3/4*n)) = 1/2;

% Alternative xtrue
%  [V, Lambda] = eig(full(A3));
%  k = 5;
%  xtrue = V(:,1:5)*randn(5,1);
```

SOLVE - Without preconditioning

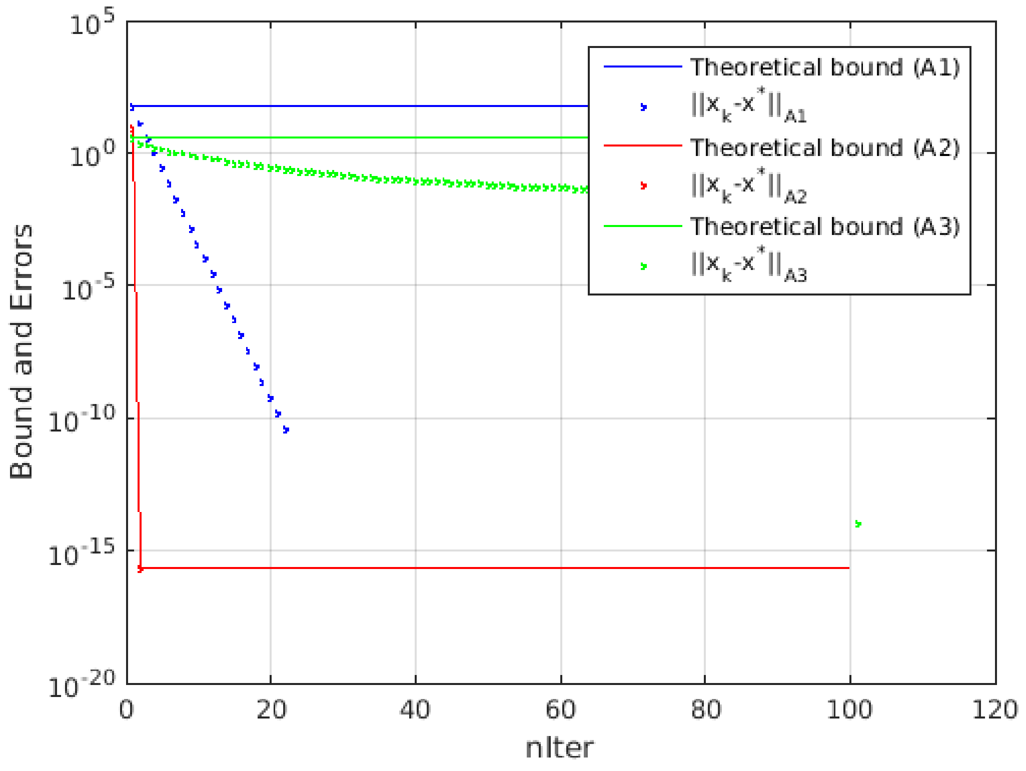
```
% Identity operator
M = @(y) y;
```

```
% Linear preconditioned Conjugate gradient backtracking line search
[xMin1, nIter1, resV1, infoCG1] = conjugateGradient(A1, A1*xtrue, tol,
 maxIter, M, x0, xtrue);
[xMin2, nIter2, resV2, infoCG2] = conjugateGradient(A2, A2*xtrue, tol,
 maxIter, M, x0, xtrue);
[xMin3, nIter3, resV3, infoCG3] = conjugateGradient(A3, A3*xtrue, tol,
 maxIter, M, x0, xtrue);

% Plot Residuals and Errors
figure;
semilogy(resV1, 'b');
hold on;
semilogy(infoCG1.errA, '*b', 'MarkerSize', 2);
semilogy(resV2 + eps, 'r'); % add eps for log plot
semilogy(infoCG2.errA + eps, '*r', 'MarkerSize', 2);
semilogy(resV3, 'g');
semilogy(infoCG3.errA, '*g', 'MarkerSize', 2);
xlabel('nIter');
ylabel('Residuals and Errors');
legend('||r|| - A1', '||x_k-x^*||_{A1}', '||r|| - A2', '||x_k-x^*||
_{A2}', '||r|| - A3', '||x_k-x^*||_{A3}');
grid on;
saveas(gcf, '../figs/01_01_ResidualsErrors', 'png');
```



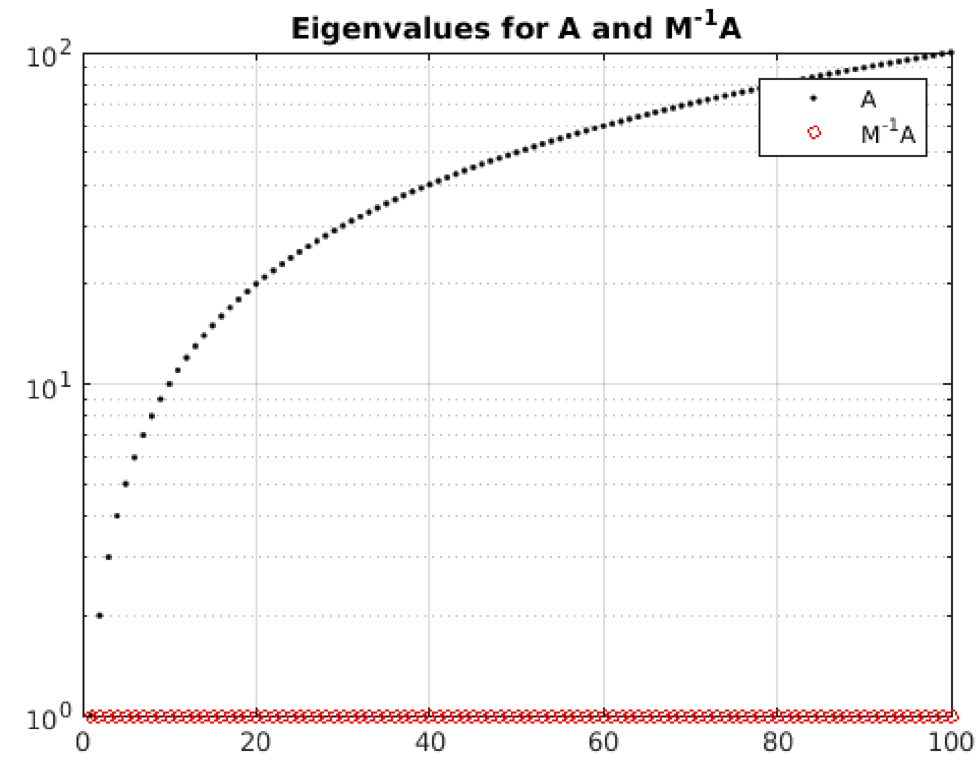
```
% Plot Residuals and Errors
lambda1 = sort(eig(A1), 'descend');
```

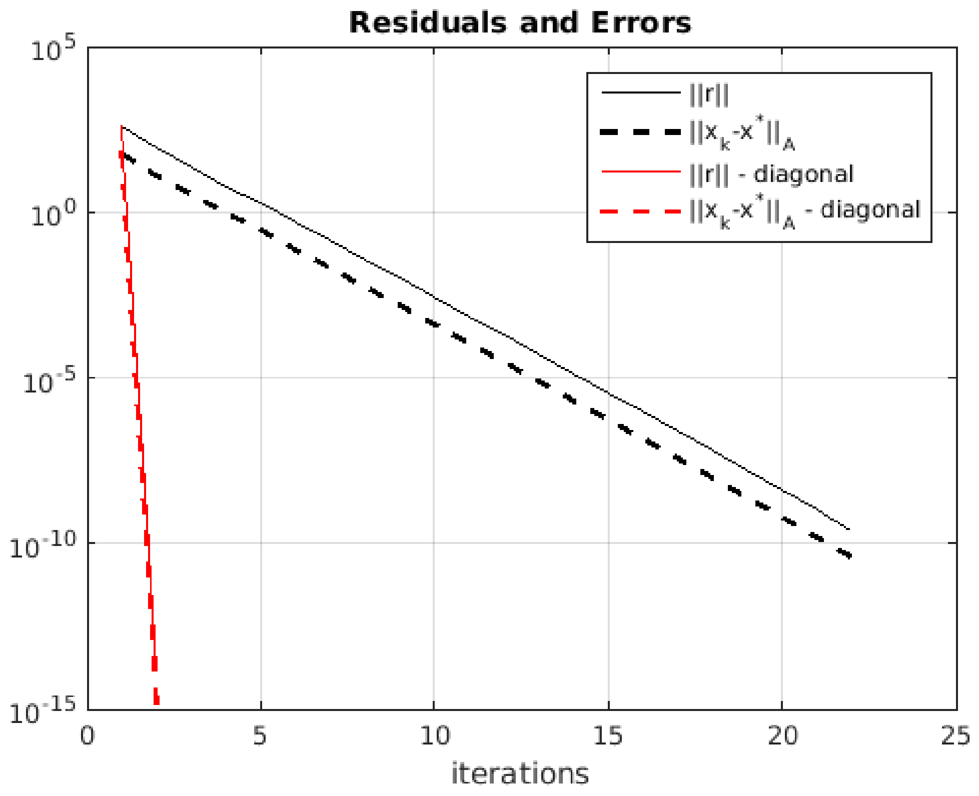
```
lambda2 = sort(eig(A2), 'descend');
lambda3 = sort(eig(A3), 'descend');
bound1 = abs((lambda1 - lambda1(end))./(lambda1 +
 lambda1(end))).*sqrt(xtrue'*A1*xtrue);
bound2 = abs((lambda2 - lambda2(end))./(lambda2 +
 lambda2(end))).*sqrt(xtrue'*A2*xtrue);
bound3 = abs((lambda3 - lambda3(end))./(lambda3 +
 lambda3(end))).*sqrt(xtrue'*A3*xtrue);
figure;
semilogy(bound1, 'b');
hold on;
semilogy(infoCG1.errA, '*b', 'MarkerSize', 2);
semilogy(bound2 + eps, 'r'); % add eps for log plot
semilogy(infoCG2.errA + eps, '*r', 'MarkerSize', 2);
semilogy(bound3, 'g');
semilogy(infoCG3.errA, '*g', 'MarkerSize', 2);
xlabel('nIter');
ylabel('Bound and Errors');
legend('Theoretical bound (A1)', '||x_k-x^*||_{A1}', 'Theoretical
 bound (A2)', '||x_k-x^*||_{A2}', 'Theoretical bound (A3)', '||x_k-
x^*||_{A3}');
grid on;
saveas(gcf, '../figs/01_02_ErrorTheoretical', 'png');
```



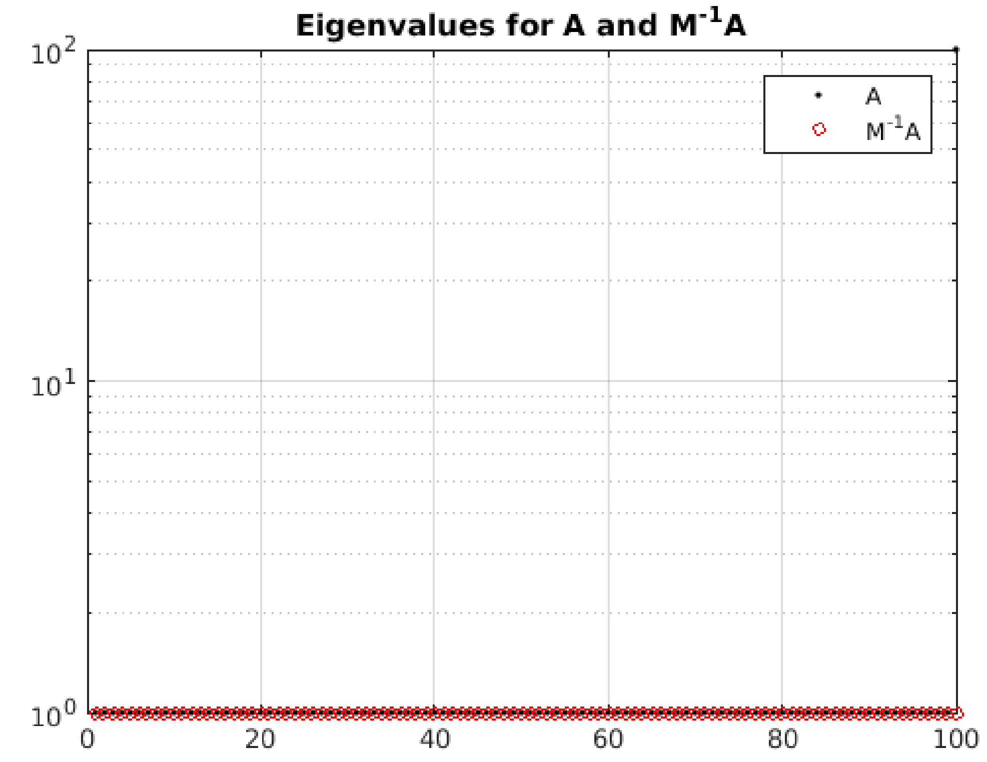
SOLVE - With preconditioning

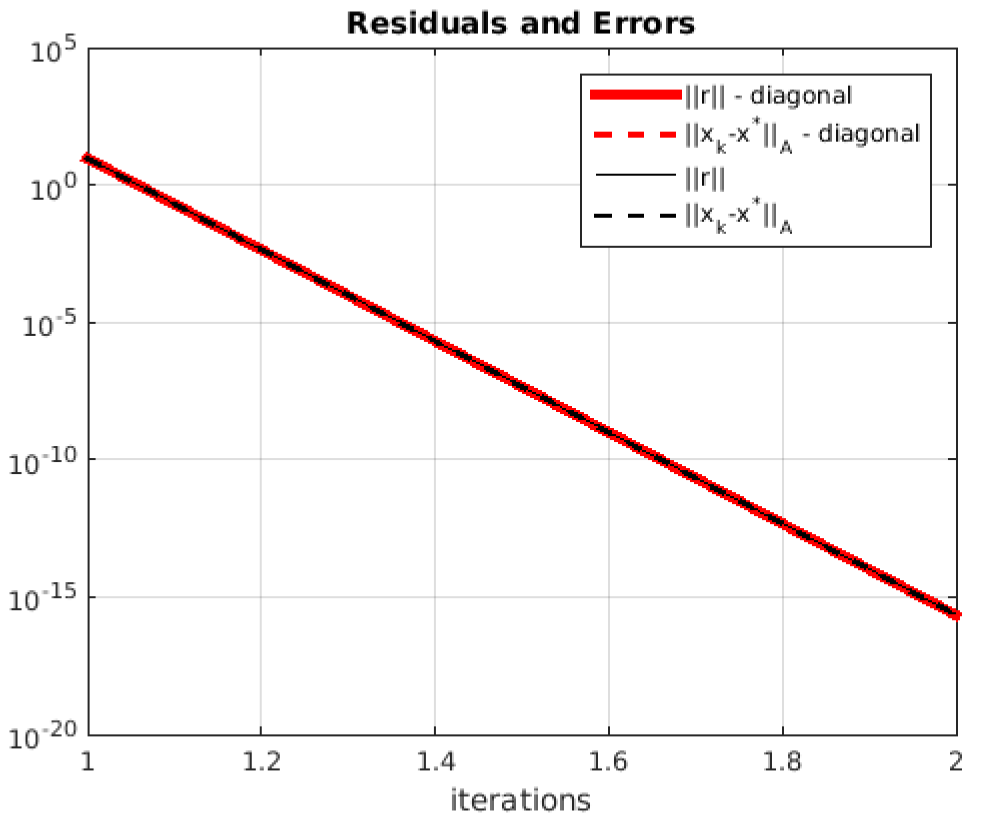
```
%========================================================================
% A1
%===========================================================================
A = A1;

% Diagonal preconditioner is equivalent to Cholesky for diagonal
 matrices
M = @(y) diag(1./diag(full(A)))*y;

% Eigenvalue distribution of A and M^(-1)*A
figure;
semilogy(eig(full(A)), 'k.');
hold on;
semilogy(real(eig(M(full(A)))), 'ro', 'MarkerSize', 5);
legend('A',  'M^{-1}A')
title('Eigenvalues for A and M^{-1}A');
grid on;
saveas(gcf, '../figs/01_03_EigenvaluesA1', 'png');

% Solution with CG
[xMin_d, nIter_d, resV_d, infoCG_d] = conjugateGradient(A, A*xtrue,
 tol, maxIter, M, x0, xtrue);

% Plot residuals and errors
figure;
semilogy(resV1, 'k');
hold on;
semilogy(infoCG1.errA, 'k--', 'LineWidth', 2);
semilogy(resV_d, 'r');
semilogy(infoCG_d.errA, 'r--', 'LineWidth', 2);
grid on;
xlabel('iterations');
legend('||r||', '||x_k-x^*||_A', '||r|| - diagonal', '||x_k-x^*||_A -
 diagonal');
title('Residuals and Errors');
saveas(gcf, '../figs/01_04_ResidualsPreconditioningA1', 'png');
```


Eigenvalues for A and $M^{-1}A$
A
$M^{-1}A$
10^2
10^1
10^0
0
20
40
60
80
100


Residuals and Errors
||r||
$||x_k-x^*||_A$
||r|| - diagonal
$||x_k-x^*||_A$ - diagonal
10^5
10^0
10^{-5}
10^{-10}
10^{-15}
0
5
10
15
20
25
iterations

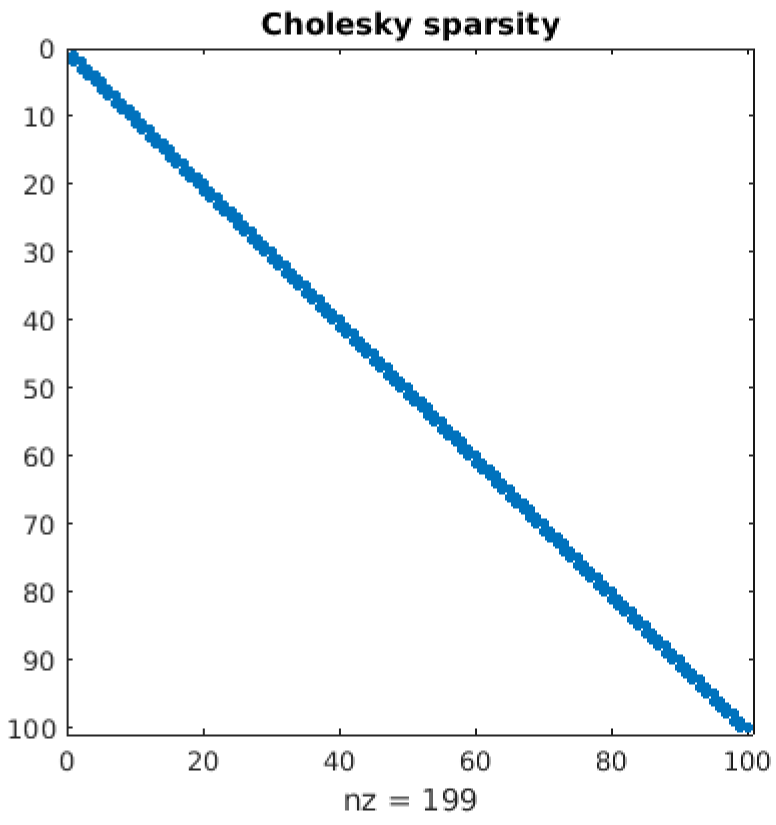
```
%==================================================================
% A2
%==========================================================================
A = A2;

% Diagonal preconditioner is equivalent to Cholesky for diagonal
 matrices
M = @(y) diag(1./diag(full(A)))*y;

% Eigenvalue distribution of A and M^(-1)*A
figure;
semilogy(eig(full(A)), 'k.');
hold on;
semilogy(real(eig(M(full(A)))), 'ro', 'MarkerSize', 5)
legend('A',  'M^{-1}A')
title('Eigenvalues for A and M^{-1}A');
grid on;
saveas(gcf, '../figs/01_05_EigenvaluesA2', 'png');

% Solution with CG
[xMin_d, nIter_d, resV_d, infoCG_d] = conjugateGradient(A, A*xtrue,
 tol, maxIter, M, x0, xtrue);

% Plot residuals and errors
figure;
semilogy(resV_d + eps, 'r', 'LineWidth', 4);
hold on;
semilogy(infoCG_d.errA + eps, 'r--', 'LineWidth', 2);
semilogy(resV2 + eps, 'k');
semilogy(infoCG2.errA + eps, 'k--', 'LineWidth', 2);
grid on;
xlabel('iterations');
legend('||r|| - diagonal', '||x_k-x^*||_A - diagonal', '||r||', '||
x_k-x^*||_A');
title('Residuals and Errors');
saveas(gcf, '../figs/01_06_ResidualsPreconditioningA2', 'png');
```

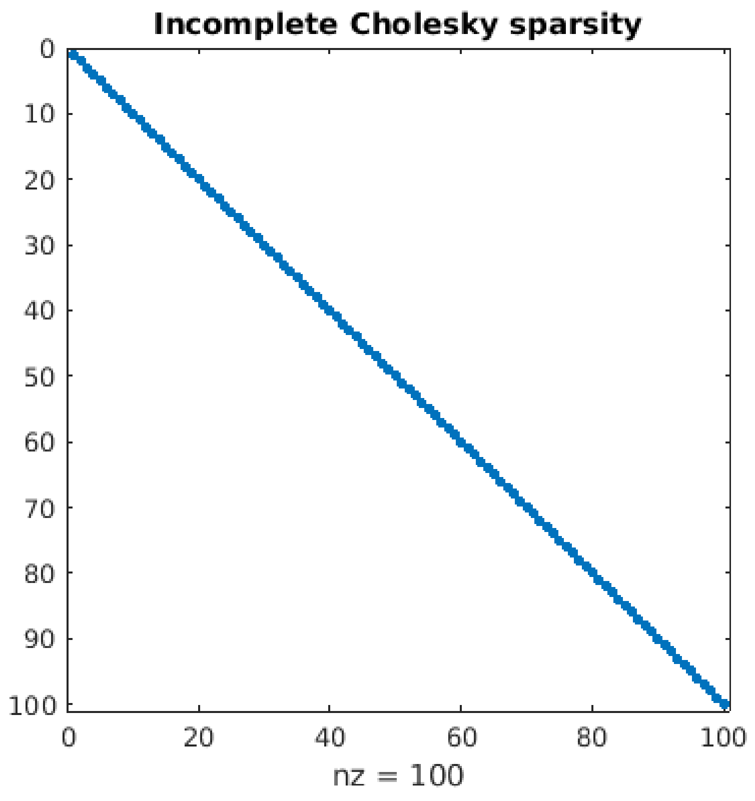

Eigenvalues for A and $M^{-1}A$
A
$M^{-1}A$
10^2
10^1
10^0
0
20
40
60
80
100

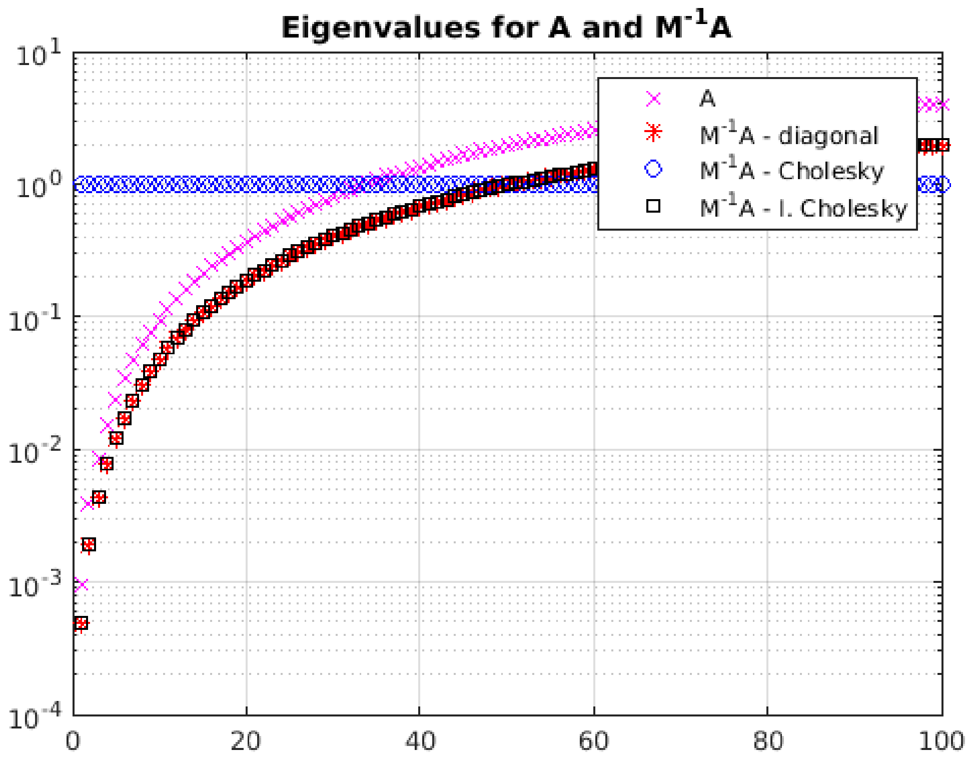

Residuals and Errors
||r|| - diagonal
$||x_k-x^*||_A$ - diagonal
||r||
$||x_k-x^*||_A$
10^5
10^0
10^{-5}
10^{-10}
10^{-15}
10^{-20}
1
1.2
1.4
1.6
1.8
2
iterations

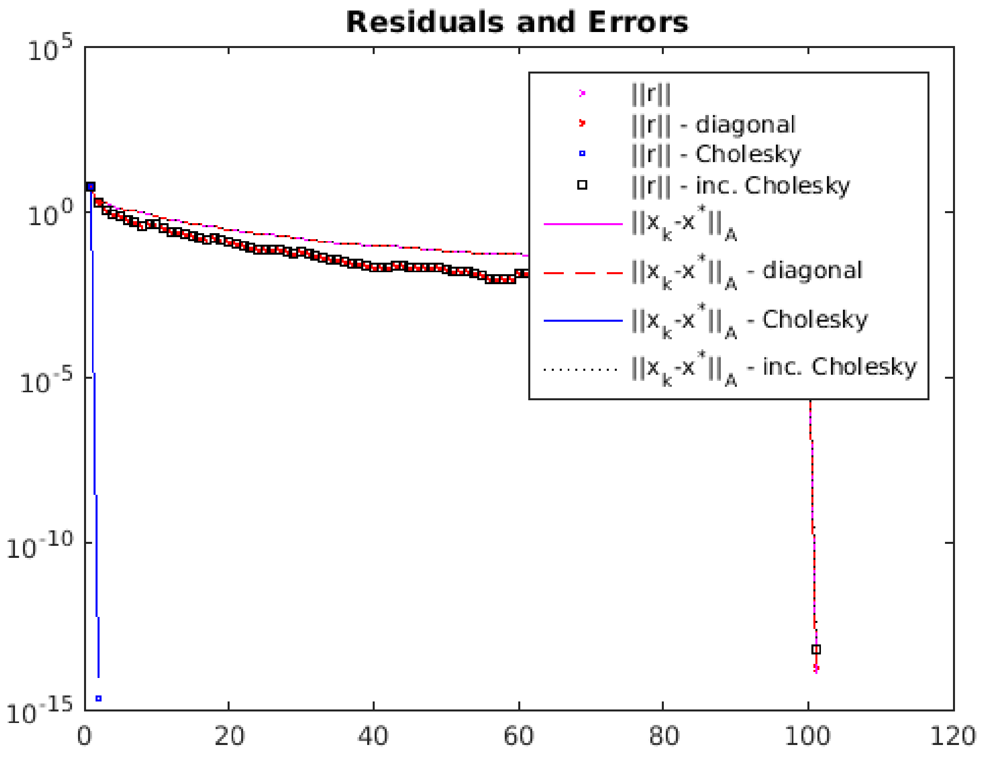
```
%==========================================================================
% A3
%==========================================================================
A = A3;

% Diagonal
M = @(y) diag(1./diag(full(A)))*y;
% Cholesky
L_c = chol(A, 'lower'); % M = L*L' -> M*x = y -> (L*L')*x = y -> x =
 L'\(L\y);
M_c = @(y) L_c'\(L_c\y);
% Incomplete Cholesky
icholOpts.type = 'ict';
icholOpts.droptol = 4e-1;
icholOpts.michol = 'off'; % this tridiagonal matrix is to simple for
 modification
L_i = ichol(A,  icholOpts);
M_i = @(y) L_i'\(L_i\y);

% Sparsity pattern
figure;
spy(L_c);
title('Cholesky sparsity');
saveas(gcf, '../figs/01_07_SparsityCholeski', 'png');

figure;
spy(L_i);
title('Incomplete Cholesky sparsity');
saveas(gcf, '../figs/01_08_SparsityIncompleteChol', 'png');

% Eigenvalue distribution
figure;
semilogy(eig(full(A)), 'mx');
hold on;
semilogy(real(eig(M(full(A)))), 'r*') % eigenvalues of A and M^(-1)*A
semilogy(real(eig(M_c(full(A)))), 'bo') % eigenvalues of A and
 M^(-1)*A
semilogy(real(eig(M_i(full(A)))), 'ks') % eigenvalues of A and
 M^(-1)*A
legend('A',  'M^{-1}A - diagonal', 'M^{-1}A - Cholesky', 'M^{-1}A - I.
 Cholesky');
title('Eigenvalues for A and M^{-1}A');
grid on;
saveas(gcf, '../figs/01_09_EigenvaluesA3', 'png');

% Solution with CG
[xMin_c, nIter_c, resV_c, infoCG_c] = conjugateGradient(A, A*xtrue,
 tol, maxIter, M_c, x0, xtrue);
[xMin_i, nIter_i, resV_i, infoCG_i] = conjugateGradient(A, A*xtrue,
 tol, maxIter, M_i, x0, xtrue);
[xMin_d, nIter_d, resV_d, infoCG_d] = conjugateGradient(A, A*xtrue,
 tol, maxIter, M, x0, xtrue);

figure;
```

```
grid on;
semilogy(resV3,'xm','MarkerSize', 2); hold on;
semilogy(resV_d,'*r','MarkerSize', 2);
semilogy(resV_c,'ob', 'MarkerSize', 2);
semilogy(resV_i,'sk', 'MarkerSize', 4);

semilogy(infoCG3.errA, 'm-','MarkerSize', 2);
semilogy(infoCG_d.errA, 'r--','MarkerSize', 2);
semilogy(infoCG_c.errA, 'b-', 'MarkerSize', 2);
semilogy(infoCG_i.errA, 'k:', 'MarkerSize', 2);
legend('||r||', '||r|| - diagonal', '||r|| - Cholesky', '||r|| - inc.
 Cholesky', '||x_k-x^*||_A ', '||x_k-x^*||_A - diagonal', '||x_k-x^*||
_A - Cholesky', '||x_k-x^*||_A - inc. Cholesky');
title('Residuals and Errors');
saveas(gcf, '../figs/01_10_ResidualsPreconditioningA3', 'png');
```




Incomplete Cholesky sparsity
0
10
20
30
40
50
60
70
80
90
100
nz = 100


Eigenvalues for A and M^{-1}A
A
M^{-1}A - diagonal
M^{-1}A - Cholesky
M^{-1}A - I. Cholesky
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
0
20
40
60
80
100



============================ Subfunctions ============================

conjugateGradient.m

Conjugate gradient method.

```
function [xMin, nIter, resV, info] = conjugateGradient(A, b, tol,
 maxIter, M, x0, xtrue)
% CONJUGATE GRADIENT Solves the implicitely symmetric preconditioned
 CG
%function [x, flag, relres, iter, resvec, xs, V] = mcg(A, b, tol,
 maxit, M, x0)
% INPUTS
% A: symmetric matrix
% b: specifies the vector to solve Ax = b
% tol: tolerance for the residual
% maxIter: maximum number of iterations
% M: linear preconditioner matrix
% x0: initial iterate
%
% OUTPUTS
% xMin: solution of the system
% nIter: number of iterations taken
% resV: vector of residuals
% info: structure with information about the iteration
%   - xs: iterate history
%   Notice that the left preconditioned NE (with M inner product) and
%   the right preconditioned NE (with M^-1 inner product) produce the
 same algorithm,
%   hence only one version.
%
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if ~strcmp(class(A), 'function_handle')
  afun = @(x) A*x;
else
  afun = A;
end
if ~strcmp(class(M), 'function_handle')
  mfun = @(x) M\x;
else
  mfun = M;
end

n = length(b);

if nargin < 6
    x0 = zeros(n,1);
end

%Initialize:
x = x0;              % Intial x
r0 = b - afun(x);    % residual: b - Ax
z0 = mfun(r0);       % M^(-1)r0
p0 = z0;             % conjugate gradient
rnorm = norm(r0);    % residual norm |r|
bnorm = rnorm;

alpha = zeros(maxIter,1);
beta = zeros(maxIter,1);
resV = zeros(maxIter+1,1);
resV(1) = rnorm;
xs = zeros(n, maxIter+1);
V = zeros(n, maxIter+1);
xs(:,1) = x;
V(:,1) = z0;
j = 1;
while j <= maxIter && rnorm/bnorm >= tol %|r|/|b| > tol

    %Preconditioned Conjugate Gradient step
    Ap0 = afun(p0);
    alpha(j) = (r0'*z0) / (Ap0'*p0);
    x = x + alpha(j)*p0;
    r1 = r0 - alpha(j)*Ap0;
    z1 = mfun(r1);
    beta(j) = (r1'*z1) / (r0'*z0);
    p1 = z1 + beta(j)*p0;

    %Compute norm
    rnorm = norm(r1);
    resV(j+1) = rnorm;
    xs(:,j+1) = x;

    %Compute the Lanczos vectors : v = scalar * r
    V(:,j+1) = z1;

    %Next step
    p0 = p1;
    r0 = r1;
    z0 = z1;
    j = j+1;
end

%Assign output
if rnorm/bnorm > tol,
  %If MCG iterated maxIter without convergence, return the solution
 with the minimal residual
    flag = 1;
    [~, jmin] = min(resV);
    x = xs(:,jmin);
    rnorm = resV(jmin);
    nIter = jmin-1;
else
    flag = 0;
    nIter = j-1;
end

relres = rnorm/norm(b);
resV = resV(1:(min(j,maxIter+1)));
info.xs = xs(:,1:(min(j,maxIter+1)));
V = V(:,1:(min(j,maxIter+1)));
xMin = x;

if nargin > 6
  for k = 1:min(j,maxIter+1)
    info.errA(k) = sqrt((xs(:,k)-xtrue)'*afun(xs(:,k)-xtrue));
  end
end
```

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