Homework 5

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1 (a) Suppose that $A \subseteq B \subseteq \mathbf{R}$. Prove that if B is closed, then $\bar{A} \subseteq B$.

Let A be closed in B. Then $B - A$ is open in B. So there exists a set X open in $\mathbf{R}$ such that $X \cap B = B - A$. Since X is open in $\mathbf{R}$, the set $C = \mathbf{R} - X$ is closed in $\mathbf{R}$. $C \cap B$ is closed since both B and C are closed.
$C \cap B = (\mathbf{R} - X) \cap B = B - X \cap B = B - B + A = A$. So A is closed by an intersection of closed sets.

Since B is a closed subset of $\mathbf{R}$ which contains A. Then B is one of the sets in the intersection of closed sets in $\mathbf{R}$, thus $\bar{A}$ is in the intersection of closed sets that are subsets of B.

So if B is closed, then $\bar{A} \subseteq B$.

(b) For $A \subseteq \mathbf{R}$ show that the closure $\bar{A}$ equals the intersection of all closed sets $B \subseteq \mathbf{R}$ that contain A.

Let $B = \{B_i : A \subseteq B_i, B_i \text{ is closed}\}$
Then let $\bar{A} = \cap B$, thus $\bar{A}$ is closed.

Let $x \in A$. Then for all i, $x \in B_i$.
Then it follows that $x \in \cap B = \bar{A}$.
Thus $A \subseteq \bar{A}$.

Now suppose there exists a closed set C with $A \subseteq C \subseteq \bar{A}$.
But since $C \supseteq A$ implies $C \in B$ so $\bar{A} \subseteq C$ implies $\bar{A} = C$.
Thus $\bar{A}$ is the smallest closed set containing A and it is the closure of A.

2 (a) Let $A_1, ..., A_n \subseteq \mathbf{R}$ be a finite collection of sets, and set $B_n = \cup_{\mathrm{i=1}}^{\mathrm{n}} A_i$. Prove that $\bar{B}_n = \cup_{\mathrm{i=1}}^{\mathrm{n}} \bar{A}_i$.

WTS closure $\overline{\cup_{\mathrm{i=1}}^{\mathrm{n}} A_i} = \cup_{\mathrm{i=1}}^{\mathrm{n}} \bar{A}_i$.

For each $1 \leq k \leq n$.
$A_k \subseteq \cup_{\mathrm{i=1}}^{\mathrm{n}} A_i \subseteq \overline{\cup_{\mathrm{i=1}}^{\mathrm{n}} A_i}$.
$\bar{A}_k \subseteq \overline{\cup_{\mathrm{i=1}}^{\mathrm{n}} A_i}$.
Since this holds for $1 \leq k \leq n$.
$\cup_{\mathrm{i=1}}^{\mathrm{n}} \bar{A}_i \subseteq \overline{\cup_{\mathrm{i=1}}^{\mathrm{n}} A_i}$.

For each $1 \leq k \leq n$
$A_k \subseteq \bar{A}_k \subseteq \cup_{\mathrm{i=1}}^{\mathrm{n}} \bar{A}_i$
Since this holds for $1 \leq k \leq n$.
$\cup_{\mathrm{i=1}}^{\mathrm{n}} A_i \subseteq \cup_{\mathrm{i=1}}^{\mathrm{n}} \bar{A}_i$

Since the closure is the smallest superset of open sets $\overline{\cup_{\mathrm{i=1}}^{\mathrm{n}} A_i} \subseteq \cup_{\mathrm{i=1}}^{\mathrm{n}} \bar{A}_i$.

So $\bar{B}_n = \overline{\cup_{\mathrm{i=1}}^{\mathrm{n}} A_i} = \cup_{\mathrm{i=1}}^{\mathrm{n}} \bar{A}_i$

(b) Let $A_1, A_2, ..., A_n \subseteq \mathbf{R}$ be a finite collection of sets, and set $B = \cup_{\mathrm{i=1}}^{\infty} A_i$. Is the closure $\bar{B}$ necessarily equal to the union $\cup_{\mathrm{i=1}}^{\infty} \bar{A}_i$ of the closures? Either prove these are equal or give a counter-example.

Let $A_i \subseteq \mathbf{R}$ and $A_i = [\frac{1}{i}, 1]$ for all $i > 1$.
Then $\bar{A}_i = A_i$.
$\cup_{\mathrm{i>1}} \bar{A}_i = (0, 1] = \cup_{\mathrm{i>1}} A_i$.
However,
$\overline{\cup_{\mathrm{i>1}} A_i} = [0, 1]$.
So $\overline{\cup_{\mathrm{i>1}} A_i} \neq \cup_{\mathrm{i>1}} \bar{A}_i$

3 Suppose that $E \subseteq \mathbf{R}$ has the property that every infinite subset of E has a limit point which is an element of E.

(a) Prove that such a set E must be bounded.

Suppose E is not bounded.
Then for all $n \in \mathbf{N}$ there exists $x_n \in E$ such that $|x_n| > n$.
So if we construct a set of x_n.
$\{x_1, x_2, ..., x_n\} : x_\infty > \infty$ then
$\cup_{\mathrm{n=1}}^{\infty} \{x_n\}$ is an infinite subset of E.

However it has no limit points which is a contradiction. So E must be bounded.

(b) Now we will prove that such a set E must be closed. Let x be a limit point of E. Construct an infinite subset of E that has x as its sole limit point (and thus that $x \in E$).
(Be careful with part (b). It is not enough to just use that if x is

a limit point, for $\epsilon > 0$, that $(x-\epsilon, x+\epsilon)$ contains infinitely many points from E. What the property gives you at that point is that the $(x-\epsilon, x+\epsilon)$ contains some limit point in E, not that x has to be that limit point. You need to create an infinite subset of E which can *only* have x as its limit point.)

Suppose E is not closed.
Then there exists $x_0 \in E^c$ where x_0 is a limit point of E.
Then for all $\epsilon > 0$ there exists $y \in E \backslash \{x_0\}$ with $y \in (x_0 - \epsilon, x_0 + \epsilon)$.
For all $n \in \mathbf{N}$ there exists $x_n \in E$ such that the difference between x_n and x_0 will be less than $1/n$ or in other words $|x_n - x_0| < 1/n$.
Let $X = \cup_{\mathrm{n=1}}^{\infty}\{x_n\}$ which is an infinite subset of E.
So X has a limit point $x_\infty \in E$.
WTS $x_\infty = x_0$, hence $x_0 \in E$.
Let $n, m \in \mathbf{N}$ since x_∞ is a limit point of X there exists $m \geq n$ such that $|x_m - x_\infty < 1/n$. So $|x_m - x_\infty < 1/m < 1/n$.
$|x_0 - x_\infty| \leq |x_m - x_\infty| + |x_m - x_0| \leq 1/n + 1/n = 2/n$

Since this is true for all $n \in \mathbf{N}$ we conclude $|x_0 - x_\infty| = 0$, so $x_\infty = x_0$.

So $x_0 \in E$ which contradicts $x_0 \in E^c$ thus E is closed and has x_0 as its sole limit point.

4 For each set, find its boundary. Prove one of your answers.

(a) $[-1, 3]$
$\partial = -1, 3$

(b) 7

For all $\epsilon > 0$.
$(7-\epsilon, 7+\epsilon) \cap 7 = 7$.
$(7-\epsilon, 7+\epsilon) \cap 7^c = 7^c$.
So $\partial = 7$

(c) $\mathbf{Q}$
$\partial = \mathbf{R}$

5 Let $A \subseteq \mathbf{R}$ be a set that is bounded above. Prove that $lub(A) \in \partial A$.

Let $u = lub(A)$ and let $\epsilon > 0$.

Then no element of $u + \epsilon$ is in A.
and every element of $u + \epsilon$ is in A^c.
and every element of $u - \epsilon$ is in A.
and no element of $u - \epsilon$ is in A^c.
So then it follows that
$(u - \epsilon, u + \epsilon) \cap A \neq \emptyset$

$(u-\epsilon, u+\epsilon) \cap A^c \neq \emptyset$
So $lub(A) \in \partial A$.

6 Let $A \subseteq \mathbf{R}$, and set $B = A^c$. Prove that $\partial A = \bar{A} \cap \bar{B}$ is the intersection of the closures of A and A^c.

Let $S = ((A^c)^\circ)^c$ so $(A^c)^\circ$ is open and contained in A^c.
Then $((A^c)^\circ)^c$ is closed containing A.
Then $\bar{A} \subseteq S$.

If T is a closed set containing A then T^c is an open subset of A^c.
$T^c \subseteq (A^c)^\circ$ so $T \supseteq S$.
Since T is an arbitrary closed set containing A.
$S \subseteq \bar{A}$ thus $\bar{A} = S$.

Now let $B = A^c$.
Then by using the above;
$((\bar{A^c}))^c = (\bar{B})^c = (((A^c)^\circ)^c)^c = (A^c)^\circ = A^\circ$.
So $A^\circ = ((\bar{A^c}))^c$.
$\partial A = \bar{A} \backslash A^\circ = \bar{A} \cap A^\circ = \bar{A} \cap (((\bar{A^c}))^c)^c = \bar{A^c} = \bar{A} \cap \bar{B}$.
Thus $\partial A = \bar{A} \cap \bar{B}$.