Homework 12

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1. Prove that if $A \subseteq \mathbf{R}$ is closed and bounded, then A is compact. *Hint*: See Problem 8 from Homework 11, and use the above Lemma.

   Let $A \subseteq \mathbf{R}$ closed and bounded.
   By Lemma 1. $\mathbf{R}$ is compact.

   Homework 11.8 states: Suppose A is a closed set. If K is a compact set with $A \subseteq K$, then A is compact.

   Since A is a closed set and $\mathbf{R}$ is a compact set with $A \subseteq \mathbf{R}$, then A is compact.
   Thus we have shown if $A \subseteq \mathbf{R}$ is closed and bounded, then A is compact.

2. Suppose A is sequentially compact. Prove that A is closed and bounded.

   Then by definition 1 every sequence in A has a subsequence that converges to a point in A. That is for every infinite subset of A, the subset has a limit point that is an element of A.

   By homework 5.3 if a set $A \in \mathbf{R}$ has the property that every infinite subset of A has a limit point which is an element of A, then by part (a) the set A must be bounded and by part (b) the set A must be closed.

   Hence, If A is sequentially compact, then A is closed and bounded.

4. Let $K \subseteq \mathbf{R}$ and suppose $f : K \to \mathbf{R}$ is continuous. Prove that if K is compact, then the image $f(K)$ is compact.

Let $(G_i)_{i \in I}$ be an open cover of $f(K)$.

Since $K \subseteq f^{-1}f(K)$. K is contained in the inverse image of the open cover.

$K \subseteq f^{-1}(\bigcup_{i \in I} G_i) = \bigcup_{i \in I} f^{-1}(G_i)$. Since f is a map it is continuous so these sets are still open.

Since K is compact we have an open cover of K, so there exists a finite subcover of G_i that covers K.

Let this cover be $(f^{-1}(K_1), ..., f^{-1}(K_n))$

So $f(K) \subseteq f(\bigcup_{i=1}^{n} f^{-1}(K_i)) \subseteq \bigcup_{i=1}^{n} K_i$, so we have a finite subcover for the open cover $f(K)$.

Since this holds for any open cover of $f(K)$, the space is compact.

10. Suppose $f : (a, b) \to \mathbf{R}$ is differentiable.

   (a) Assume that there is some $c \in (a, b)$ so that $f(c) \geq f(x)$ for all $x \in (a, b)$. Prove that $f'(c) = 0$.

   Let $f : (a, b) \to \mathbf{R}$ be differentiable. Assume that there is some $c \in (a, b)$ so that $f(c) \geq f(x)$ for all $x \in (a, b)$. Let h be some positive or negative number close to zero.

   Then we have $f(c) \geq f(c + h)$
   Then we have $0 \geq f(c + h) - f(c)$

   Then we will divide both sides by h. Since h is positive or negative we get two cases.

   Case 1: Let h be positive.

   Then we have $\frac{f(c+h)-f(c)}{h} \leq 0$.
   Since $h > 0$ we can take the right hand limit of both sides.
   So $\lim_{h^+ \to 0} \frac{f(c+h)-f(c)}{h} \leq \lim_{h^+ \to 0} 0$.

   Since f is differentiable we know $\lim_{h \to 0} f(c)$ exists.
   So $f'(c) = \lim_{h \to 0} \frac{f(c+h)-f(c)}{h} = \lim_{h^+ \to 0} \frac{f(c+h)-f(c)}{h} \leq \lim_{h^+ \to 0} 0 = 0$

   So $f'(c) \leq 0$

   Case 2: Let h be negative.

   Then we have $\frac{f(c+h)-f(c)}{h} \geq 0$.
   Since $h < 0$ we can take the left hand limit of both sides.

   So $f'(c) = \lim_{h \to 0} \frac{f(c+h)-f(c)}{h} = \lim_{h^- \to 0} \frac{f(c+h)-f(c)}{h} \geq \lim_{h^- \to 0} 0 = 0$

   So $f'(c) \geq 0$

   Thus, we have $f'(c) = 0$.

(b) Assume that there is some $c \in (a, b)$ so that $f(c) \leq f(x)$ for all $x \in (a, b)$. Prove that $f'(c) = 0$.

Let $f : (a, b) \to \mathbf{R}$ be differentiable. Assume that there is some $c \in (a, b)$ so that $f(c) \geq f(x)$ for all $x \in (a, b)$. Let h be some positive or negative number close to zero.

Then we have $f(c) \leq f(c + h)$
Then we have $0 \leq f(c + h) - f(c)$

Then we will divide both sides by h. Since h is positive or negative we get two cases.

Case 1: If h is positive then $\frac{f(c+h)-f(c)}{h} \geq 0$
As seen in part (a) case two $f'(c) \geq 0$.

Case 2: If h is negative then $\frac{f(c+h)-f(c)}{h} \leq 0$
As seen in part (a) case one $f'(c) \leq 0$.

Thus, we have $f'(c) = 0$.