Homework 8

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1. Let $f : X \to Y$ be a function.

   (a) Prove that $f^{-1}(Y \backslash U) = X \backslash f^{-1}(U)$ for every $U \subseteq Y$.

   $f^{-1}(Y \backslash (U)) = \{x \in X \mid f(x) \in Y \backslash U\}$
   $X \backslash f^{-1}(U) = X \backslash \{x \in X \mid f(x) \in U\} = \{x \in X \mid f(x) \in Y \backslash U\}$

   $f^{-1}(Y \backslash U) = \{x \in X \mid f(x) \in Y \backslash U\} = X \backslash f^{-1}(U)$.

   (b) Prove that $f : X \backslash Y$ is continuous if and only if $f-1(C)$ is closed in X for every set C that is closed in Y.

   Let $C \subseteq Y$ be closed. Then $Y \backslash C \subseteq Y$ is open.

   So $f^{-1}(Y \backslash C) \subseteq X$ is open.
   And by part (a) $X \backslash f^{-1}(C) \subseteq X$ is open.
   Then $f^{-1}(C) \subseteq X$ is closed.

   Let $C \subseteq Y$ be open. Then $Y \backslash C \subseteq Y$ is closed.

   So $f^{-1}(Y \backslash C) \subseteq X$ is closed.
   And by part (a) $X \backslash f^{-1}(C) \subseteq X$ is closed.
   Then $f^{-1}(C) \subseteq X$ is open.

   Then $f : X \to Y$.

2. Suppose $D \subseteq R$ is dense, and let $f, g : R \to R$ be continuous functions. Assume that $f(x) = g(x)$ for all $x \in D$. Prove that $f = g$ (that is, prove $f(x) = g(x)$ for all $x \in R$).

   Let $f, g : R \to R$ be continuous s.t $f(x) = g(x)$ for all $x \in D$.

   Since $D \subseteq R$ is dense by the definition of dense $\bar{D} = R$.

   To show that $f(x) = g(x)$ for all $x \in D$ it suffices to prove that $f(x) = g(x)$ for all $x \in D'$ where D' is the limit point of D.

   If $x \in D'$ then there is a sequence $x_n \in D$ s.t $lim_{n\to\infty} = x$.

   Therefore since f and g are continuous $f(x) = lim_{n\to\infty} f(x_n) = lim_{n\to\infty} g(x_n) = g(x)$. So $f(x) = g(x)$.

3. Decide whether each set is connected. If it is disconnected, write it as the union of two nonempty separated sets A and B. Prove at least two of your answers.

   (a) R is connected.

   (b) $(0, 1) \cup (1, 2)$ is disconnected

   Proof:
   Let $A = (0, 1)$ and $B = (1, 2)$
   So $E = A \cup B = (0, 1) \cup (1, 2)$

   $A \cap \bar{B} = (0, 1) \cup [1, 2] = \emptyset$
   $\bar{A} \cap B = [0, 1] \cup (1, 2) = \emptyset$

   So $(0, 1) \cup (1, 2)$
   is disconnected.

   (c) Z is disconnected

   Proof:
   Let $A = (-\infty, .5)$ and $B = (.5, \infty)$.
   $Z = A \cup B = (-\infty, .5) \cup (.5, \infty)$

   $A \cap \bar{B} = (-\infty, .5) \cup [.5, \infty) = \emptyset$
   $\bar{A} \cap B = (-\infty, .5] \cup (.5, \infty) = \emptyset$

   So Z is disconnected.

   (d) $[1, 5)$ is connected

   (e) $[0, 1] \cap Q$ is disconnected

4. Let $E \subseteq R$, and assume there are $x, y \in E$ with $x < y$. Suppose that there exists some $z \in E$ such that $x < z < y$. Prove that E is not connected.

   Let $A = (-\infty, z)$ and $B = (z, \infty)$.
   $E = A \cup B = (-\infty, z) \cup (z, \infty)$

   $A \cap \bar{B} = (-\infty, z) \cup [z, \infty) = \emptyset$
   $\bar{A} \cap B = (-\infty, z] \cup (z, \infty) = \emptyset$

   So E is not connected.

8. Consider the function $h : [0, \infty) \to [0, \infty)$ defined by $h(x) = x^2$. Prove that h is surjective. Hint: Use the Intermediate Value Theorem.

   Definition: let $f : A \to B$ be a function then f is said to be onto if for every $y \in B$, there exists $x \in A$ such that $f(x) = y$.

   Let $h(x) = y$ then $x^2 = y$ then $x = \sqrt{y}$ so $x = \sqrt{y}$
   Since $-\sqrt{y}$ not possible $x \in [0, \infty)$
   Now, $h(x) = h(\sqrt{y)} = (\sqrt{y})^2 = y$
   Then we have shown that for all $x \in [0, \infty)$, $h(x)$ is sent to some y.
   So $h(x)$ is surjective.