Homework 2

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1. Suppose $\mathcal{F}$ is a field. Use the field axioms to show the following.

   (a) The additive identity is unique.
   Suppose it is not unique.
   Let $x \in \mathcal{F}$ and let $0, 0^{'} \in \mathcal{F}$ be additive identities such that:
   $0 + x = x = x + 0$ and $0^{'} + x = x = x + 0^{'}$.
   Then $0 + x = x \rightarrow 0 + 0^{'} = 0^{'}$ and $x + 0^{'} = x \rightarrow 0 + 0^{'} = 0$.
   Then $0^{'} = 0 + 0^{'} = 0$ thus $0^{'} = 0$.
   Hence the additive identity is indeed unique.

   (b) For each $x \in \mathcal{F}$, the element y of Axiom $\mathcal{F}2$ is unique.
   Suppose it is not unique.
   Let $x \in \mathcal{F}$ and let $y, y^{'} \in \mathcal{F}$ where $x + y = 0$ and $x + y^{'} = 0$:
   So $x + y = x + y^{'}$, since $0 = 0$.
   Then $y + x + y = y + x + y^{'}$.
   Since $x + y = 0$ and addition is commutative by $\mathcal{F}0 \rightarrow y + x = 0$.
   Then $0 + y = 0 + y^{'}$ and by $\mathcal{F}1$ $y = y^{'}$.
   Hence the element y in $\mathcal{F}2$ is unique, and we can see y is the additive inverse.

   (c) The multiplicative identity is unique.
   Suppose it is not unique.
   Let $x \in \mathcal{F}$ and let $1, 1^{'} \in \mathcal{F}$ be multiplicative identities such that:
   $1 \cdot x = x = x \cdot 1$ and $1^{'} \cdot x = x = x \cdot 1^{'}$.
   Then $1 \cdot x = x \rightarrow 1 \cdot 1^{'} = 1^{'}$ and $x \cdot 1^{'} = x \rightarrow 1 \cdot 1^{'} = 1$.
   Then $1^{'} = 1 \cdot 1^{'} = 1$ thus $1^{'} = 1$.
   Hence the additive identity is indeed unique.

   (d) For each $x \in \mathcal{F} \backslash \{0\}$, the element y of Axiom $\mathcal{F}4$ is unique.
   Suppose it is not unique.
   Let $x \in \mathcal{F}$ and let $y, y^{'} \in \mathcal{F}$ where $x \cdot y = 1$ and $x \cdot y^{'} = 1$:
   So $x \cdot y = x \cdot y^{'}$, since $1 = 1$.
   Then $y \cdot x \cdot y = y \cdot x \cdot y^{'}$.
   Since $x \cdot y = 1$ and multiplication is commutative by $\mathcal{F}0 \rightarrow y \cdot x = 1$.
   Then $1 \cdot y = 1 \cdot y^{'}$ and by $\mathcal{F}1$ $y = y^{'}$
   Hence the element y in $\mathcal{F}4$ is unique, and we can see y is the multiplicative inverse.

2. Suppose $\mathcal{F}$ is a field. Use the field axioms to show the following.

   (a) For all $a, b \in \mathcal{F}$, there exists some $c \in \mathcal{F}$ such t hat $a + c = b$.

   Proof:
   Let $a, b \in \mathcal{F}$.
   Using axiom, $\mathcal{F}2$ since $a \in \mathcal{F}$, there exists $y \in \mathcal{F}$ s.t $a + y = 0$.
   Let $c \in \mathcal{F}$ where $c = y + b$.
   Consider $a + c$.
   $a + c = a + (y + b)$ by substituting c for $y + b$.
   $a + (y + b) = (a + y) + b$ since addition is associative by $\mathcal{F}0$.
   $(a + y) + b = 0 + b$ since $a + y = 0$.
   $0 + b = b + 0$ since addition is associative by $\mathcal{F}0$.
   $b + 0 = b$ by using $\mathcal{F}1$.
   Thus $a + c = b$.
   Hence we have proved for all $a, b \in \mathcal{F}$ there exists $c \in \mathcal{F}$ s.t $a + c = b$.

   (b) For all $a, b \in \mathcal{F}$ with $a \neq 0$, there exists $c \in \mathcal{F}$ s.t $a \cdot c = b$

   Proof:
   Let $a, b \in \mathcal{F}$.
   Using axiom, $\mathcal{F}4$ since $a \in \mathcal{F}$, there exists $y \in \mathcal{F}$ s.t $a \cdot y = 1$.
   Let $c \in \mathcal{F}$ where $c = y \cdot b$.
   Consider $a \cdot c$.
   $a \cdot c = a \cdot (y \cdot b)$ by substituting c for $y \cdot b$.
   $a \cdot (y \cdot b) = (a \cdot y) \cdot b$ since addition is associative by $\mathcal{F}0$.
   $(a \cdot y) \cdot b = 1 \cdot b$ since $a \cdot y = 1$.
   $1 \cdot b = b \cdot 1$ since addition is associative by $\mathcal{F}0$.
   $b \cdot 1 = b$ by using $\mathcal{F}3$.
   Thus $a \cdot c = b$.
   Hence we have proved for all $a, b \in \mathcal{F}$ there exists $c \in \mathcal{F}$ s.t $a \cdot c = b$.

3. Suppose $\mathcal{F}$ is a field and let $a, b, c \in \mathcal{F}$. Use the field axioms to show the following.

   (a) If $a + b = a + c$, then $b = c$.

   Proof:
   Using axiom $\mathcal{F}2$ since $a \in \mathcal{F}$, $\exists\ y \in \mathcal{F}$ s.t $a + y = 0$.
   Since addition is commutative by $\mathcal{F}0$ $y + a = 0$.
   Suppose $a + b = a + c$.
   Then $y + a + b = y + a + c$.
   Then $0 + b = 0 + c$. Also $0 + b = b + 0$ and $0 + c = c + 0$ by $\mathcal{F}0$.
   So $b + 0 = c + 0$.
   Then by $\mathcal{F}1$, $b = c$.
   Hence proved.

   (b) If $a + b = a$, then $b = 0$.

   Proof:
   We showed in problem 1a that the additive identity is unique so if $a + b = a$ then b must be 0.

   (c) If $a + b = 0$, then $b = -a$.

   Proof:
   Let $a \in \mathcal{F}$ and let $y, y^{'} \in \mathcal{F}$ s.t $b = y$ or $b = y^{'}$.
   $a + y = 0$ and $a + y^{'} = 0$:
   So $a + y = a + y^{'}$, since $0 = 0$.
   Then $y + a + y = y + a + y^{'}$.
   Since $a + y = 0$ and addition is commutative by $\mathcal{F}0 \rightarrow y + a = 0$.
   Then $0 + y = 0 + y^{'}$ and by $\mathcal{F}1$ $y = y^{'}$.
   So there can only be one y, or in the case of $a + b = 0$ only one b that this is true. We can define this $b = -a$.
   So we have proved if $a + b = 0$ then $b = -a$.

   (d) $-(-a) = a$.

   Proof:
   We know if $a + b = 0$, then $b = a$.
   We know if $x \in \mathcal{F}$, then $x \cdot 0 = 0$
   $0 = 0$
   $-a \cdot 0 = 0$
   $-a \cdot (1 + (-1)) = 0$
   $(-a) \cdot 1 + (-a) \cdot (-1) = 0$
   $-a + -(-a) = 0$
   $a + (-a) + -(-a) = 0$
   $-(-a) = a$

4. Suppose $\mathcal{F}$ is a field and let $a, b, c \in \mathcal{F}$. Use the field axioms to show the following.

   (a) If $a \neq 0$ and $a \cdot b = a \cdot c$, then b = c.

   Proof:
   Using axiom $\mathcal{F}4$ since $a \in \mathcal{F}$, $\exists\ y \in \mathcal{F}$ s.t $a \cdot y = 1$.
   Since multiplication is commutative by $\mathcal{F}0$ $y \cdot a = 1$.
   Suppose $a \cdot b = a \cdot c$.
   Then $y \cdot a \cdot b = y \cdot a \cdot c$.
   Then $0 \cdot b = 0 \cdot c$. Also $0 \cdot b = b \cdot 0$ and $0 \cdot c = c \cdot 0$ by $\mathcal{F}0$.
   So $b \cdot 0 = c \cdot 0$.
   Then by $\mathcal{F}1$, $b = c$.
   Hence proved.

   (b) If $a \neq 0$ and $a \cdot b = a$ then $b = 1$.

   Proof:
   We showed in 1c the multiplicative identity is unique so is $a \cdot b = a$ then b must equal 1

   (c) If $a \neq 0$ and $a \cdot b = 1$, then $b = 1/a$.

   Proof:
   Let $a \in \mathcal{F}$ and let $y, y^{'} \in \mathcal{F}$ s.t $b = y$ or $b = y^{'}$.
   $a \cdot y = 1$ and $a \cdot y^{'} = 1$:
   So $a \cdot y = a \cdot y^{'}$, since $1 = 1$.
   Then $y \cdot a \cdot y = y \cdot a \cdot y^{'}$.
   Since $a \cdot y = 1$ and addition is commutative by $\mathcal{F}0 \rightarrow y \cdot a = 1$.
   Then $1 \cdot y = 1 \cdot y^{'}$ and by $\mathcal{F}3$ $y = y^{'}$.
   So there can only be one y, or in the case of $a \cdot b = 1$ only one b that this is true. We can define this $b = 1/a$.
   So we have proved if $a \cdot b = 1$ then $b = 1/a$.

   (d) If $a \neq 0$, then $1/(1/a) = a$.

   Proof:
   Proved above if $a \cdot b = 1$, then $b = 1/a$.
   So we know $a \cdot b = 1$ and $a \cdot (1/a) = 1$ and $b \cdot (1/b) = 1$
   $1 = 1$
   $b \cdot (1/b) = a \cdot (1/a)$
   $a \cdot b \cdot (1/b) = a \cdot a \cdot (1/a)$
   $a \cdot b \cdot (1/b) = a$
   $a \cdot (1/a) \cdot (1/(1/a)) = a$
   $(1/(1/a)) = a$
   $1/(1/a) = a$

5. For each set below, determine whether it is bounded above, bounded below, or both. If an upper/lower bound exists, then find one. Otherwise, write DNE for Does Not Exist. You should give answers to all sets, but you only need to justify (prove) your answers for one of the sets (your choice).

   (a) $[0, 1]$ Yes
   $0 \leq 0$ and $0 \leq 1$, thus 0 is a lower bound.
   $0 \leq 1$ and $1 \leq 1$, thus 1 is an upper bound.
   Hence, $[0, 1]$ is indeed a bound.

   (b) $\{\frac{1}{n} : n \in N\}$ Yes

   (c) $\{n + \frac{(-1)^n}{n} : n \in N\}$ No

   (d) $\{r \in Q : r^2 < 4\}$ Yes

   (e) $\{x \in Z : x^3 < 8\}$ No

6. Let $A \subseteq E$ be nonempty and bounded. Suppose a is a lower bound of A and b is an upper bound for A

   (a) Prove that $a \leq b$
   Proof:
   Let $a, b, c \in A$ where c is not a bound.
   Since a is a lower bound, then all elements in A are larger than a.
   Then $a \leq c$.
   Since b is an upper bound, then all elements in A are smaller than b.
   Then $b \geq c$.
   Then we have the inequality $a \leq c \leq b$.
   If $c = a$ or $c = b$, then $a \leq b$.
   If $c < b$ or $a < c$, then $a \leq b$.
   Hence proved.

   (b) Then A would either be a hole or a point.