Homework 9

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1. Each of the following sequences converges. Identify the limit of each one. Prove one of your answers.

   (a) $(1/n^2)_{n=1}^{\infty}$ converges to 0.

   proof
   Let $\epsilon > 0$.
   Take $N \in \mathbf{N}$ such that $\frac{1}{N^2} < \epsilon$.
   For all $n > N$ we have,
   $|\frac{1}{n^2} - 0| = |\frac{1}{n^2}| = \frac{1}{n^2} < \frac{1}{N^2} < \epsilon$
   then $\lim_{n\to\infty} a_n = 0$.

   (b) $(2^{1/n})_{n=1}^{\infty}$ converges to 1.

   (c) $(\sqrt{n^2+1} - n)_{n=1}^{\infty}$ converges to 0.

3. Assume $(a_n)_n$ and $(b_n)_n$ are sequences that converge to a and b, respectively. Prove two of the following:

   (a) $\lim_{n\to\infty} a_n + b_n = a + b$.

   proof
   The problem states we have two known limits.
   For all ϵ_1, there exists N_1 s.t if $n > N_1$, then $0 \leq |a_n - a| \leq \epsilon_1$.
   For all ϵ_2, there exists N_2 s.t if $n > N_2$, then $0 \leq |b_n - b| \leq \epsilon_2$.

   Suppose $\epsilon_1 = \epsilon_2 = \epsilon/2$, then there exists a N_1 and N_2.

   Choose $N = \min\{N_1, N_2\}$ s.t for $n > N_1$ and $n > N_2$ we have $n > N$.

   This implies that $|a_n - a| \leq \epsilon/2$ and $|b_n - b| \leq \epsilon/2$

   So $|a_n - a| + |b_n - b| < \epsilon$
   Then by triangle inequality
   $|(a_n + b_n) - (a + b)| = |a_n - a + b_n - b| \leq |a_n - a| + |b_n - b| < \epsilon$

   Thus $\lim_{n\to\infty} a_n + b_n$ exists and it is equal to $a + b$.

   (b) $\lim_{n\to\infty} a_n b_n = ab$

   proof
   The problem states we have two known limits.
   For all ϵ_1, there exists N_1 s.t if $n > N_1$, then $0 \leq |a_n - a| \leq \epsilon_1$.
   For all ϵ_2, there exists N_2 s.t if $n > N_2$, then $0 \leq |b_n - b| \leq \epsilon_2$.

   Suppose we have,
   $\epsilon_1 = min(\frac{\epsilon}{3(b+1)}, (\frac{\epsilon}{3})(\frac{1}{2}))$
   $\epsilon_2 = min(\frac{\epsilon}{3(a+1)}, (\frac{\epsilon}{3})(\frac{1}{2}))$

   Then there exists a corresponding N_1 and N_2.

   Choose $N = max\{N_1, N_2\}$ so $n > N_1$ and $n > N_2$ then $n > N$.

   Then,
   $|a_n b_n - ab| = |(a_n - a)b + (b_n - b)a + (a_n - a)(b_n - b)| \leq$
   $|a_n - a||b| + |b_n - b||a| + |a_n - a||b_n - b| \leq \frac{\epsilon b}{3(b+1)} + \frac{\epsilon a}{3(a+1)} + \frac{\epsilon}{3} \leq \epsilon.$

   Thus the $\lim_{n\to\infty} a_n b_n$ exists and is equal to ab.

6. (a) Prove that if a sequence $(a_n)_{n=n_0}^{\infty}$ converges, then it is bounded.

proof
Let a_n be a sequence which converges to l.
Let let $\epsilon = 1$ so $\epsilon > 0$.

$|a_n - l| < \epsilon$
$-\epsilon < a_n - l < \epsilon$
$l - 1 < a_n < l + 1$

$l \leq |l|$
$l + 1 \leq |l| + 1$

$-l \leq |l|$
$-|l| \leq l$
$-|l| - 1 \leq l - 1$

$-|l| - 1 \leq l - 1 < a_n < l + 1 \leq |l| + 1$
$-|l| - 1 < a_n < |l| + 1$
$|a_n| \leq |l| + 1$

Let $M = max\{|a_1|, |a_2|, ..., |a_{N-1}|, |l| + 1\}$
Then given any a_n, if $n < N$, $|a_n| \leq M$
If $n \geq N$, then $|a_n| < |l| + 1 \leq M$.
Hence if the sequence converges to some arbitrary l then it is bounded by some M so it is bounded.

(b) Give a counterexample showing that the converse of the statement in (a) is false.

proof

Consider the sequence $\{a_n\}$, $a_n = (-1)^n$.
So it is bounded in $[-1, 1]$ since $a_n = \{-1, 1\}$ for all $a_n \in \{a_n\}$

Suppose $\{a_n\}$ converges.
Let $\epsilon = 1$
If $n > N$ is odd then $|-1-L| = |-(1+L)| = |L+1| < 1$ so $-2 < L < 0$.
If $n > N$ is even then $|1-L| = |-(-1+L)| = |L-1| < 1$ so $0 < L < 2$.

Since $(-2, 0)$ and $(0, 2)$ are disjoint sets then we have a contradiction since L cannot be in both.
Thus $\{a_n\}$ does not converge.
So we have proved the converse of (a) does not necessarily hold.

7. Suppose that $\{a_n\}$ is monotone. Prove that $\{a_n\}$ converges if and only if it is bounded. (This is called the Monotone Convergence Theorem.)

   Suppose $\{a_n\}$ is monotone and bounded.
   W.L.O.G we will assume that the sequence is increasing.
   The sequence has a set of points $\{a_n : n \in \mathbf{N}\}$ which is bounded by the supposition.

   As (a_n) is increasing, let our limit be $s = sup\{a_n : n \in \mathbf{N}\}$.

   Because s is the supremum of $\{a_n : n \in \mathbf{N}\}$, the number $s - \epsilon$ is not an upper bound, so there is a number a_N in the sequence where $s - \epsilon < a_N$.

   Since (a_n) is increasing, we know that $a_N \leq a_n$ for all $n \geq N$.

   Since s is a supremum we can obtain $s - \epsilon < a_N \leq a_n \leq s < s + \epsilon$ for all $n \geq N$.

   So $|a_n - s| < \epsilon$ for all $n \geq N$. So we have found that the sequence converges.

   For the other direction suppose $\{a_n\}$ is monotone and convergent.

   We proved in problem 6 if a sequence $(a_n)_{n=n_0}^{\infty}$ converges, then it is bounded.
   So $\{a_n\}$ is bounded.

   Thus we have proved that given $\{a_n\}$ is monotone, $\{a_n\}$ converges if and only if it is bounded.