Homework 6

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1. Prove the following.

   (a) Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = x$. Then $lim_{x \to x_0} f(x) = x_0$ for every $x_0 \in \mathbf{R}$.

   Proof: Assume $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = x$.
   Let $\epsilon > 0$.
   Set $\delta = \epsilon$.
   Then $\delta > 0$.
   Let $x \in \mathbf{R}$ with $0 < |x - x_0| < \delta$.
   Then $|f(x) - x_0| = |x - x_0| < \delta = \epsilon$.
   So $|f(x) - x_0| < \epsilon$.
   So $lim_{x \to x_0} f(x) = x_0 = L$ for all $x_0 \in \mathbf{R}$.

   (b) Fix $c \in \mathbf{R}$ and let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = c$. Then $lim_{x \to x_0} f(x) = c$ for every $x_0 \in \mathbf{R}$.

   Proof:
   Let $\epsilon > 0$.
   Set $\delta = |x - x_0| + \epsilon$.
   Then $\delta > 0$ because $|x - x_0| > 0$ and $\epsilon > 0$.
   Let $x \in \mathbf{R}$ with $0 < |x - x_0| < \delta$.
   Then $|f(x) - c| = |c - c| = 0 < \epsilon$.
   So $|f(x) - c| < \epsilon$.
   So $lim_{x \to x_0} f(x) = c = L$.

2. (a) Suppose $a, b, n \in \mathbf{R}$ and $n > 0$. Prove that $|a - b| < n$ if and only if $a \in (b - n, b + n)$.

Proof:
If $|a - b| < n$.
Then $-n < a - b < n$.
Then $b - n < a < n + b$.
So $a \in (b - n, b + n)$.

If $a \in (b - n, b + n)$.
Then $b - n < a < b + n$.
Then $-n < a - b < n$.
So $|a - b| < n$.
Hence, we have proven that $|a-b| < n$ if and only if $a \in (b-n, b+n)$.

(b) Explain why $lim_{x\to x_0} f(x) = L$ if and only if $\forall\epsilon > 0$, $\exists\delta > 0$ such that if $x \in (x_0 - \delta, x_0 + \delta) \cap X\backslash\{x_0\}$, then $f(x) \in (L - \epsilon, L + \epsilon)$.

Proof:
If $lim_{x\to x_0} f(x) = L$ exists then by definition $\forall\epsilon > 0, \exists\delta > 0$ s.t if $x \in X$ satisfies $0 < |x - x_0| < \delta$ then $|f(x) - L| < \epsilon$.

So in the case of $0 < |x - x_0| < \delta$, we have
$-\delta < x - x_0 < \delta$
so $x_0 - \delta < x < x_0 + \delta$
so $x \in (x_0 - \delta, x_0 + \delta)$.
However since x only approaches x_0 then $x \in (x_0 - \delta, x_0 + \delta) \cap X\backslash\{x_0\}$ where x_0 will not be in the interval.

Then comes the case of $|f(x) - L| < \epsilon$. Then we have $-\epsilon < f(x) - L < \epsilon$. Then $L - \epsilon < f(x) < L + \epsilon$. Then $f(x) \in (L - \epsilon, L + \epsilon)$.
So we have $\forall\epsilon > 0$, $\exists\delta > 0$ such that if $x \in (x_0 - \delta, x_0 + \delta) \cap X\backslash\{x_0\}$, then $f(x) \in (L - \epsilon, L + \epsilon)$.

In the other direction:
If we have $\forall\epsilon > 0$, $\exists\delta > 0$ such that if $x \in (x_0 - \delta, x_0 + \delta) \cap X\backslash\{x_0\}$, then $f(x) \in (L - \epsilon, L + \epsilon)$.

In the case of $f(x) \in (L - \epsilon, L + \epsilon)$.Then we have $L - \epsilon < f(x) < L + \epsilon$, then $-\epsilon < f(x) - L < \epsilon$, then $|f(x) - L| < \epsilon$.

In the case of $x \in (x_0 - \delta, x_0 + \delta) \cap X\backslash\{x_0\}$. We have $x \in (x_0 - \delta, x_0 + \delta)$ where at all values of $\delta > 0$ $x_0 \notin (x_0 - \delta, x_0 + \delta)$.
So $x_0 - \delta < x < x_0 + \delta$.
Then $-\delta < x - x_0 < \delta$.
So it follows that $0 < |x - x_0| < \delta$ since $|x - x_0| \neq 0$ since $x \neq x_0$.

Then we have satisfied the conditions for the existence of $lim_{x\to x_0} f(x) = L$.

Thus $lim_{x\to x_0} f(x) = L$ if and only if $\forall\epsilon > 0$, $\exists\delta > 0$ such that if $x \in (x_0 - \delta, x_0 + \delta) \cap X\backslash\{x_0\}$, then $f(x) \in (L - \epsilon, L + \epsilon)$.

(c) Use the definition of the absolute value to prove that $|a+b| \leq |a|+|b|$ for all $a, b \in \mathbf{R}$ (this is called the triangle inequality).

We have 4 possible cases without loss of generality:
$a = 0$
$a > 0, b > 0$
$a < 0, b < 0$
$a > 0, b < 0$

Case 1: Suppose $a = 0$ then we have,
$|a + b| = |b|$
$|a| + |b| = |b|$
so $|a + b| = |a| + |b|$

Case 2: Suppose $a > 0, b > 0$ then we have,
$|a + b| = a + b$
$|a| + |b| = a + b$
so $|a + b| = |a| + |b|$

Case 3: Suppose $a < 0, b < 0$ then we have,
$|a| = -a$ and $|b| = -b$ then $|a| + |b| = (-a) + (-b)$
$|a + b| = -(a + b) = (-a) + (-b)$
so $|a + b| = |a| + |b|$

Case 4: Suppose $a > 0, b < 0$. Then we have 3 cases to consider here.
$a + b = 0$
$a + b > 0$
$a + b < 0$

Suppose $a + b = 0$ then we have,
$a = -b$
$|a + b| = 0$
$|a| + |b| = a + (-b) = a + a > 0$ because $a > 0$
So $|a + b| < |a| + |b|$.

Suppose $a + b > 0$ then we have,
$a > -b > 0$ since $b < 0$ then $-b > 0$
$|a + b| = a + b$
$|a| + |b| = a + (-b)$
$b < (-b)$ so $a + b < a + (-b)$
So $|a + b| < |a| + |b|$.

Suppose $a + b < 0$ then we have,
$a < (-b)$ so $b < 0 < a < (-b)$
$|a + b| = -(a + b) = (-a) + (-b)$

$|a| + |b| = a + (-b)$
$(-a) < a$ since $a > 0$ then $-a < 0$
$(-a) + (-b) < a + (-b)$
So $|a + b| < |a| + |b|$

Thus we have covered all possible cases and have proved the triangle inequality using the definition of the absolute value.

3. Suppose f and g are functions with the same domain X. Let $x_0 \in X'$, and assume $\lim_{x \to x_0} f(x) = L$ and $\lim_{x \to x_0} g(x) = M$. Prove that $\lim_{x \to x_0} f(x) + g(x) = L + M$.

   Proof:
   Let $\epsilon > 0$
   Since $\lim_{x \to x_0} f(x) = L$ there exists a $\delta_1 > 0$ such that if $0 < |x - x_0| < \delta_1$ then $|f(x) - L| < \epsilon/2$.
   Since $\lim_{x \to x_0} g(x) = M$ there exists a $\delta_2 > 0$ such that if $0 < |x - x_0| < \delta_2$ then $|g(x) - M| < \epsilon/2$.
   Set $\delta = min\{\delta_1, \delta_2\}$ and suppose $0 < |x - x_0| < \delta$. Then $0 < |x - x_0| < \delta_1$ so $|f(x) - L| < \epsilon/2$ and $0 < |x - x_0| < \delta_2$ so $|g(x) - M| < \epsilon/2$.
   $|(f(x) + g(x)) - (L + M)| = |(f(x) - L) + (g(x) - M)|$
   Using the Archimedean property:
   $|(f(x) - L) + (g(x) - M)| \leq |f(x) - L| + |g(x) - M| < \epsilon/2 + \epsilon/2 = \epsilon$
   So $|(f(x) + g(x)) - (L + M)| < \epsilon$. Thus we have proved $\lim_{x \to x_0} f(x) + g(x) = L + M$.

4 Suppose $f : X \to Y$ is a function and $\lim_{x \to x_0} f(x) = L$. For $c \in \mathbf{R}$, prove $\lim_{x \to x_0} cf(x) = cL$.

   Proof:
   If $c = 0$ then $\lim_{x \to x_0} cf(x) = cL$ will become $\lim_{x \to x_0} 0 = 0$ which is true.
   If $c \neq 0$
   Let $\epsilon > 0$
   Since $\lim_{x \to x_0} f(x) = L$, there exists $\delta > 0$ such that $0 < |x - x_0| < \delta$ then $|f(x) - L| < \epsilon/|c|$.
   Then $|cf(x) - cL| = |c||f(x) - L| < \epsilon$
   So $|cf(x) - cL| < \epsilon$
   So we have proved $\lim_{x \to x_0} cf(x) = cL$ exists.