# Homework 10

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1. Consider the sequence $(a_n)_{n=0}^{\infty}$ defined by $a_n = n^2$.

   (a) For $k \in \mathbf{N}$, define $n_k := 2k$. Give an explicit formula expressing the terms of the subsequence. $(a_{n_k})_{k=1}^{\infty}$.

   $(a_{n_k})_{k=1}^{\infty} = \{(2k)^2\}$

   (b) Consider the sequence $(b_k)_{k=1}^{\infty}$ where $b_k = k^4 + 2k^2 + 1$. Show that $(b_k)_k$ is a subsequence of $(a_n)_n$.

   Let $n_k = k^2 + 1$ where $k \geq 0$.
   Let $a_{n_k} = (k^2 + 1)^2$ be a subsequence of $(a_n)_n$.
   $a_{n_k} = (n_k)^2 = (k^2 + 1)^2 = k^4 + 2k^2 + 1 = b_k$.
   So we have shown that $(b_k)_k$ is a subsequence of $(a_n)_n$.

   (c) Consider the sequence $(c_k)_{k=1}^{\infty}$ where $c_k$ is the kth positive even integer. Show that $(c_k)_k$ is not a subsequence of $(a_n)_n$.

   Let $c_k$ be the 1st positive integer.
   $a_n = n^2$.
   Let $n = 1$ and $n = 2$.
   $a_1 = 1 < 2 < 4 = a_2$.
   So we can see 2 is not in the sequence of $a$ thus it is not a subsequence.

2. Let $(a_n)_n$ be a sequences and $L \in \mathbf{R}$. Proving that $(a_n)_n$ converges to $L$ if and only if every subsequence of $(a_n)_n$ converge to $L$.

   Let $a_n$ be a sequence such that $a_n$ converges to $L$.
   Let $a_{n_k}$ by any subsequence of $a_n$.

   Since $a_n$ converges to $L$, then for some $\epsilon > 0$, there exists a positive integer $m$ such that $|a_n - L| < \epsilon$ for all $n \geq m$.

   Let $S = \{a_1, a_2, ...., a_n\}$ so $|a_n - L| < \epsilon$ for all $a_n \in S$.
   Since $a_{n_k}$ is a subsequence of $a_n$, then $a_{n_k} \in S$ for all $k$.
   So for all $k$, $|a_{n_k} - L| < \epsilon$ for all $k \geq m$.
   $a_{n_k}$ converges to $L$.

   Conversely suppose that every subsequence converges to $L$.
   Since $a_n$ is a subsequence of itself $a_n$ converges to $L$.

   Thus $(a_n)_n$ converges to $L$ if and only if every subsequence of $(a_n)_n$ converge to $L$.

4. Consider the sequence $(a_n)_{n=0}^{\infty}$ defined by $(a_n) = (-1)^n$.

   (a) Identify the accumulation points of $(a_n)_n$.

   $\{-1, 1\}$

   (b) Give an explicit example of a subsequence of $(a_n)_n$ that converges.

   Let $a_{n_k}$ be defined by $n_k = 2k$ for $k \in \mathbf{N}$.
   Then $a_{n_k}$ converges to 1.

   (c) Show that every subsequence of $(a_n)_n$ has a subsequence that converges.

   Since $n$ is natural number it can be even or odd. So we will let $n_k$ be even and $n_m$ be odd subsequences.

   Let $a_{n_k}$ be defined by $n_k = 2k$ for $k \in N$.
   Then $a_{n_k}$ converges to 1.
   Let $a_{n_{k_z}}$ be defined by $k_z = 2z$ for $z \in N$.
   Then $a_{n_{k_z}}$ converges to 1.

   Let $a_{n_m}$ be defined by $n_m = 2m - 1$ for $m \in N$.
   Then $a_{n_m}$ converges to $-1$.
   Let $a_{n_{m_y}}$ be defined by $m_y = 2y - 1$ for $y \in N$.
   Then $a_{n_{m_y}}$ converges to $-1$.

   So we can see every subsequence of $(a_n)_n$ converges to either accumulation point $-1$ or 1, and each subsequence has a subsequence that converges.

   (d) The sequence $(a_n)_n$ diverges (you do not need to prove this). Explain why parts (b) and (c) do not contradiction the results of Problems 2 and 3.

   We have 2 subsequences $a_{n_k}$ and $a_{n_m}$ but they do not converge to the same L. In number (2) and (3) it requires every subsequence to coverge to the same L. And we can see it did not happen in part (c) of this problem, thus it does not contradict problems 2 and 3.

6. Let $(a_n)_n$ be a sequence that converges. Prove that $(a_n)_n$ is a Cauchy sequence.

So $a_n \to L$. Let $\epsilon > 0$, then by definition we can find $N \in \mathbf{N}$ s.t
If $n \geq N$, then $|a_n - L| < \epsilon/2$.

Similarly take $M \in \mathbf{N}$ such if $m \geq M$, then $|a_m - L| < \epsilon/2$

Let $k = max\{N, M\}$.
Then for all $n, m \geq k$

$$|a_n - a_m| \leq |a_n - L| + |L - a_m| < \epsilon/2 + \epsilon/2 = \epsilon$$

Then by definition $(a_n)_n$ is a cauchy sequence.