Homework 7

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1. Define $f : [0, \infty) \to R$ by $f(x) = \sqrt{x}$. Prove that f is continuous.

   Unless both x, x_0 are zero we have:
   $\sqrt{x} - \sqrt{y} = (\sqrt{x} - \sqrt{y})\frac{(\sqrt{x}+\sqrt{y})}{(\sqrt{x}+\sqrt{y})} = \frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$

   We have two cases where $x > 0$ and $x = 0$.

   Case 1: $x > 0$
   Let $\delta = x\sqrt{\epsilon}$ s.t $|x - x_0| < \delta$ then
   $|\sqrt{x} - \sqrt{x_0}| = \frac{|x-x_0|}{|\sqrt{x}-\sqrt{x_0}|} \leq \frac{|x-x_0|}{\sqrt{x}} < \frac{\epsilon\sqrt{x}}{\sqrt{x}} = \epsilon$

   So $|\sqrt{x} - \sqrt{x_0}| < \epsilon$

   Case 2: $x = 0$
   Let $\delta = \epsilon^2$ s.t $|x - x_0| < \delta$ so $x_0 > 0$ then
   $|\sqrt{x} - \sqrt{x_0}| = \frac{|x-x_0|}{\sqrt{x}+\sqrt{x_0}} \leq \frac{|0-x_0|}{\sqrt{0}+\sqrt{x_0}} = \sqrt{x_0} < \sqrt{\epsilon^2} = \epsilon$

   So for all $x_0 \in [0, \infty)$, x_0 is a limit point of $[0, \infty)$ so f is continuous.

2. Define $f : R \to R$ by $f(x) = x$ if $x \in \mathbf{Q}$ and $f(x) = 0$ if $x \notin \mathbf{Q}$. Prove that f is continuous at 0. (This function f is called the modified Dirichlet function.)

   Define $f : \mathbf{R} \to \mathbf{R}$
   $f(x) = x$ if $x \in \mathbf{Q}$
   $f(x) = 0$ if $x \notin \mathbf{Q}$

   Let $\epsilon > 0$
   Set $\delta = \epsilon$ s.t $|x - 0| = |x| < \delta$
   if $x \in \mathbf{Q}$ then $|f(x) - f(0)| = |x - 0| = |x| < \delta = \epsilon$
   if $x \notin \mathbf{Q}$ then $|f(x) - f(0)| = |0 - 0| = |0| < \epsilon$

   So for all $\epsilon > 0$ there exists $\delta > 0$ s.t if $|x - 0| < \delta$, then $|f(x) - 0| < \epsilon$. Thus f is continuous at 0.

3. Define $f : R \to R$ by $f(x) = 1$ if $x \in \mathbf{Q}$ and $f(x) = 0$ if $x \notin \mathbf{Q}$. Prove that f is not continuous at any $x_0 \in \mathbf{R}$. (This function f is called the Dirichlet function.)

   Suppose f is continous at x_0.
   Set $\epsilon = 1$.
   Then there exists $\delta > 0$ s.t for all $x \in \mathbf{R}$ s.t $|x - x_0| < \delta$ so $|f(x) - f(x_0)| < 1$

   If $x_0 \in \mathbf{Q}$ then $f(x_0) = 1$.
   By density of irrationals, pick $x \in (x_0 - \delta, x + \delta)$ so $f(x) = 0$ then $|f(x) - f(x_0)| < 1$ so $|0 - 1| < 1$ so $1 < 1$.

   If $x_0 \in \mathbf{Q}$ then $f(x_0) = 0$
   By density of rationals, pick $x \in (x_0 - \delta, x + \delta)$ so $f(x) = 0$ then $|f(x) - f(x_0)| < 1$ so $|1 - 0| < 1$ so $1 < 1$.

   So in both cases we have a contradiction so x_0 is not continuous at any point for all $x_0 \in \mathbf{R}$.

7. Let $X \subseteq R$ and $C \subseteq X$. Prove that C is closed in X if and only if $X \backslash C$ is open in X

   Assume that C is closed in X. Then there exists a closed set D so that $C = X \cap D$.

   Then we have:
   $X \backslash C = X \cap C^c = X \cap (X \cap D)^c = X \cap (X^c \cup D^c) = (X \cap X^c) \cup (X \cap D^c) = \emptyset \cup (X \cap D^c) = X \cap D^c$.

   Since D is closed D^c is open. So $X \backslash C$ is open in X. So if C is closed in X then $X \backslash C$ is open in X

   For the other way we assume that $X \backslash C$ is open in X then there exists an open set V s.t $X \backslash C = X \cap V$. So $X \backslash (X \backslash C)$ is closed.

   Then we have:
   $X \backslash (X \backslash C) = X \backslash (X \cap C^c) = X \cap (X \cap C^c)^c = X \cap (X^c \cup C) = (X \cap X^c) \cup (X \cap C) = \emptyset \cup C = C$.

   So C is closed. So if $X \backslash C$ is open in X then C is closed in X.

   Thus, C is closed in X if and only if $X \backslash C$ is open in X

9. Assume that $f : X \to Y$ has the property that $f^1(U)$ is open in X for all sets $U \subseteq Y$ that are open in Y . Prove that f is continuous.

   Let $U \subseteq Y$ be open in Y. Then for all $u_0 \in U$, there exists an $\epsilon > 0$ such that $(u_0 - \epsilon, u_0 + \epsilon) \subseteq Y$.

   Then by assumption $f^{-1}(U)$ is open in X. For all $f^{-1}(u_0) = x \in f^{-1}(U)$ there exists $\delta > 0$ s.t $(u_0 - \epsilon, u_0 + \epsilon) \subseteq X$.

   Then x is not a limit point of X so f is continuous at x.