Homework 4

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1. For each of the following sets, determine the interior, closure, whether it is open, and whether it is closed. Prove two of your answers (your choice; you dont need to prove your answers to the other ones).

   (a) $A = (a, b)$
   Interior: (a, b)
   Closure: $[a, b]$
   Open: Yes
   Closed: No

   Proof for interior:
   Given $A = (a, b)$ WTS $A^\circ = (a, b)$.
   Let $x \in A^\circ$.
   Then there exists $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subseteq A$.
   Then $x - \epsilon < x < x + \epsilon$.
   So $x \in (x - \epsilon, x + \epsilon)$.
   Thus $x \in A = (a, b)$.

   For the other inclusion.
   Let $x \in (a, b)$.
   Let $\epsilon = min(b - x, x - a)$.
   Claim 1: $\epsilon > 0$.
   Since $x \in (a, b)$ then $a < x < b$.
   So it follows that $x - a > 0$ and $b - x > 0$.
   So $\epsilon > 0$.
   Claim 2: $(x - \epsilon, x + \epsilon) \subseteq (a, b)$.
   Let $y \in (x - \epsilon, x + \epsilon)$.
   Then it follows $a = x - x + a \leq x - \epsilon < y < x + \epsilon \leq x + b - x = b$.
   So $a < y < b$.
   Then $y \in (a, b)$.

   By the claims it follows that x is an interior point so $x \in A^\circ = (a, b)$.
   Since $A = A^\circ$ it is open.

   Proof for closure:

Given $A = (a, b)$ WTS $\bar{A} = [a, b]$.
$A - \{a\} = (a, b)$.
Let $0 < \epsilon$ and $(a - \epsilon, a + \epsilon)$ be the neighborhood around a.
If $x \in A - \{a\} = (a, b)$ then $a < x < b$. Also observe $a < a + \epsilon$.
If $a + \epsilon < b$ then $a < a + \epsilon < b$.
So an intersection with (a, b) would be $(a, a + \epsilon)$.
If $a + \epsilon \geq b$ then $a < b \leq a + \epsilon$.
So an intersection with (a, b) would be (a, b).
Thus $(a - \epsilon, a + \epsilon) \cap A - \{a\} = \emptyset$.
So a is a limit point.

$A - \{b\} = (a, b)$.
Let $0 < \epsilon$ and $(b - \epsilon, b + \epsilon)$ be the neighborhood around b.
If $x \in A - \{b\} = (a, b)$ then $a < x < b$. Also observe $b - \epsilon < b$.
If $b - \epsilon > a$ then $a < b - \epsilon < b$.
So an intersection with (a, b) would be $(b - \epsilon, b)$.
If $b - \epsilon \leq a$ then $b - \epsilon \leq a < b$.
So an intersection with (a, b) would be (a, b).
Thus $(b - \epsilon, b + \epsilon) \cap A - \{b\} = \emptyset$.
So b is a limit point.

Given $A = (a, b)$ and our limit points $A' = \{a, b\}$, $A \cap A' = \bar{A} = [a, b]$.
Thus $[a, b]$ is the closure of A.
Since $\bar{A} \neq A$, A is not closed.

(b) $B = [a, b]$
Interior: (a, b)
Closure: $[a, b]$
Open: No
Closed: Yes

Proof for interior:
Given $B = (a, b)$ WTS $B^\circ = (a, b)$.
Let $x \in B^\circ$.
Then there exists $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subseteq B$.
Then $x - \epsilon < x < x + \epsilon$.
So $x \in (x - \epsilon, x + \epsilon)$.
Thus $x \in B = (a, b)$.

For the other inclusion.
Let $x \in (a, b)$.
Let $\epsilon = min(b - x, x - a)$.
Claim 1: $\epsilon > 0$.
Since $x \in (a, b)$ then $a < x < b$.
So it follows that $x - a > 0$ and $b - x > 0$.
So $\epsilon > 0$.

Claim 2: $(x-\epsilon, x+\epsilon) \subseteq (a,b)$.
Let $y \in (x-\epsilon, x+\epsilon)$.
Then it follows $a = x - x + a \leq x - \epsilon < y < x + \epsilon \leq x + b - x = b$.
So $a < y < b$.
Then $y \in (a,b)$.

By the claims it follows that x is an interior point so $x \in B^\circ = (a,b)$.
Since $B = B^\circ$ it is open.

Proof for closure:
Given $B = [a,b]$ WTS $\bar{B} = [a,b]$.
$B - \{a\} = (a,b]$.
Let $0 < \epsilon$ and $(a-\epsilon, a+\epsilon)$ be the neighborhood around a.
If $x \in B - \{a\} = (a,b]$ then $a < x \leq b$. Also observe $a < a + \epsilon$.
If $a + \epsilon < b$ then $a < a + \epsilon < b$.
So an intersection with $(a,b]$ would be $(a, a+\epsilon)$.
If $a + \epsilon \geq b$ then $a < b \leq a + \epsilon$.
So an intersection with $(a,b]$ would be $(a,b]$.
Thus $(a-\epsilon, a+\epsilon) \cap B - \{a\} = \emptyset$.
So a is a limit point.

$B - \{b\} = [a,b)$.
Let $0 < \epsilon$ and $(b-\epsilon, b+\epsilon)$ be the neighborhood around b.
If $x \in B - \{b\} = [a,b)$ then $a \leq x < b$. Also observe $b - \epsilon < b$.
If $b - \epsilon > a$ then $a \leq b - \epsilon < b$.
So an intersection with $[a,b)$ would be $(b-\epsilon, b)$.
If $b - \epsilon \leq a$ then $b - \epsilon \leq a < b$.
So an intersection with $[a,b)$ would be (a,b).
Thus $(b-\epsilon, b+\epsilon) \cap B - \{b\} = \emptyset$.
So b is a limit point.

Given $B = [a,b]$ and our limit points $B' = \{a,b\}$, $B \cap B' = \bar{B} = [a,b]$.
Thus $[a,b]$ is the closure of B.
Since $\bar{B} = B$, B is closed.

(c) $C = (a,b]$
Interior: (a,b)
Closure: $[a,b]$
Open: No
Closed: No

(d) $D = \emptyset$
Interior: $\emptyset$
Closure: $\emptyset$
Open: Yes
Closed: Yes

(e) $E = N$

Interior: N
Closure: N
Open: Yes
Closed: Yes

2. (a) $A = Z$
   Interior: $\emptyset$
   Closure: Z
   Open: No
   Closed: Yes

   (b) $B = \{1/n \mid n \in N\}$
   Interior: $\emptyset$
   Closure: $B \cup \{0\}$
   Open: No
   Closed: No

   Proof:
   Let $\epsilon = 1/n - 1/(n+1)$
   Then $(1/n - \epsilon, 1/n + \epsilon) \cap B = \{1/n\}$
   So $1/n$ is not a limit point so each point of B is isolated.
   However every neighborhood centered around 0 will contain points in B. $1/n$ as n gets arbitrarily large approaches 0. But $0 \notin B$ so the closure of B is $B \cup \{0\}$. Since $B \neq B \cup \{0\}$ it is not closed.
   Since $1/n$ is isolated there are no open sets contained in B so the interior is empty.

   (c) $C = Q \cap (0, 1)$
   Interior: N
   Closure: N
   Open: Yes
   Closed: Yes

3. Let $A \subseteq R$.

   (a) Show the interior A° is an open set.

   Proof:
   Suppose $x \in A^\circ$ then there exists an $\epsilon > 0$ such that $(x-\epsilon, x+\epsilon) \subseteq A$.
   $(x - \epsilon, x + \epsilon)$ is open. So for all $y \in (x - \epsilon, x + \epsilon)$ there exists $\epsilon' > 0$ such that $(y - \epsilon', y + \epsilon') \subseteq (x - \epsilon, x + \epsilon) \subseteq A$.
   This means that every point in $(x-\epsilon, x+\epsilon)$ is an interior point of A.
   Thus $(x - \epsilon, x + \epsilon) \subseteq A^\circ$ implies A° is open.

   (b) Show the closure $\bar{A}$ is a closed set.
   A set A is closed if and only if every limit point of A is also a point in A.
   The closure of A is the set $A \cup A'$.
   A' is the set of limit points of A.

The closure of A is denoted $\bar{A} = A \cup A'$.
Since it contains A and its limit points the set $\bar{A}$ is closed.

4. Let $A \subseteq R$. Prove that A is open if and only if the complement A^c is closed.

   Proof:
   Suppose A is open. Since A is open every point in A is an interior point. Let x be a limit point of A^c. Then every neighborhood around x contains a point in A^c so x is not an interior point of A.
   Then $x \in A^c$ and every limit point of A^c is a point in A^c. Hence if A is open, A^c is closed.
   Conversely suppose A^c is closed.
   Let $x \in A$, then $x \notin A^c$. We know x is not a limit point of A^c since it is closed and by definition contains all of its limit points. Thus for some $\epsilon > 0$ there exists some $(x - \epsilon, x + \epsilon)$ where $A^c \cap (x - \epsilon, x + \epsilon) = \emptyset$. That neighborhood around x will be a subset of A. Thus it is an interior point of A. So every point in A is an interior point of A. Then by definition S is open.
   Hence if A^c is closed, A is open.

5. Let $\{A_1, ..., A_n\}$ be a finite family of sets.

   (a) Prove that if A_k is open for each $1 \leq k \leq n$, then the intersection $A := \cap_{\mathrm{k=1}}^{\mathrm{n}} A_k$ is open.

   Proof:
   Let $\{A_1, ..., A_n\}$ be a finite collection of open sets.
   If $a \in \cap_{\mathrm{k=1}}^{\mathrm{n}} A_k$, then a is an element of each of the open sets.
   By the definition of an open set we know for all $1 \leq k \leq n$ there exists some $(a - \epsilon_k, a + \epsilon_k) \subseteq A_k$ for some $\epsilon_k > 0$.
   Let $\epsilon = min\{\epsilon_1, ...\epsilon_n\}$ such that $(a - \epsilon, a + \epsilon) \subseteq (a - \epsilon_k, a + \epsilon_k) \subseteq A_k$ for all k. Hence $(a - \epsilon, a + \epsilon) \subseteq A_k$, thus it is open.

   (b) Prove that if A_k is closed for each $1 \leq k \leq n$, then the union $A := \cup_{\mathrm{k=1}}^{\mathrm{n}} A_k$ is closed.

   Proof:
   We showed in part (a) if A_k is open for all $1 \leq k \leq n$, $A := \cap_{\mathrm{k=1}}^{\mathrm{n}} A_k$ is open.
   $(\cap_{\mathrm{k=1}}^{\mathrm{n}} A_k)^c$ is closed since we showed the complement of the open set is closed.
   And by De Morgans Law
   $(\cap_{\mathrm{k=1}}^{\mathrm{n}} A_k)^c = \cup_{\mathrm{k=1}}^{\mathrm{n}} A_k^c$ is closed.
   A_k^c is closed since A_k is open as show in number 4. So we have shown the finite union of closed sets A_k^c is closed.

6. Let $\{A_i\}_{i \in I}$ be a family of sets indexed by an indexing set I.

(a) Prove that if A_i is open for each $i \in I$, then the union $A =: \cup_{i \in I} A_i$ is open.

Proof:
Let $x \in A$. Then for some $i \in I$, $x \in A_i$. This means $\exists \epsilon > 0$ s.t. $(x - \epsilon, x + \epsilon) \subset A_i$.
So $(x - \epsilon, x + \epsilon) \subset \cup_{i \in I} A_i$, which we'll restate as $(x - \epsilon, x + \epsilon) \subset A$.
Each set is open, so each set A_i contains the interior points of A_i. So by definition of union and open, A is open.

(b) Prove that is A_i is closed for each $i \in I$, then the intersection $B : \cap_{i \in I} A_i$ is closed.

Proof:
We know from Problem 4 that A is closed iff A^c is open. Then by De Morgan's laws $(\cap_{i \in I} A_i)^c = \cup_{i \in I} A_i^c$. From part (a) above, we proved $\cup_{i \in I} A_i$ is open when A_i for each $i \in I$ is open. This means $\cup_{i \in I} A_i^c$ is open, so $(\cap_{i \in I} A_i)^c$ is open. Then $\cap_{i \in I} A_i$ is closed.