Exploring Quarto and Latex

Leah A. Camonayan

4.1. ANTIDIFFERENTIATION AND INDEFINITE INTEGRALS

1 4.1.2 Integration by Substitution

1.1 Theorem 4.1.11 (Substitution Rule)

Theorem 1.1. *If $u = g(x)$ is a differentiable function whose range is an interval I and f is continues on I, then*

$$\int f(g(x)) \cdot g'(x)dx = \int f(u)du.$$

Example 1.1. Example 4.1.12.

1. $\int (1-4x)^{\frac{1}{2}} dx$

   If we let $u = 1-4x$, then $du = -4dx$. We multiply the integrand by $\frac{-4}{-4}$.Thus,

   $$\int (1-4x)^{1/2} dx = \int (1-4x)^{1/2} \cdot \frac{-4}{-4} dx = \int u^{1/2} \left(-\frac{du}{4}\right) = -\frac{1}{4} \int u^{1/2} du = -\frac{1}{4} \cdot \frac{2u^{3/2}}{3} + C.$$

   We put the final answer in terms of x by substituting $u = 1-4x$. Therefore,

   $$\int (1-4x)^{1/2} dx = \frac{(1-4x)^{3/2}}{6} + C.$$

2. $\int x^2 \left(x^3-1\right)^{10} dx$

   Let $u = x^3 - 1$. Then $du = 3x^2dx$ or $\frac{du}{3} = x^2dx$. By substitution,

   $$\int x^2 \left(x^3-1\right)^{10} dx = \frac{du}{3} = \frac{1}{3} \int u^{10} du = \frac{u^{11}}{33} + C = \frac{\left(x^3-1\right)^{11}}{33} + C$$

3. $\int \frac{x}{\left(x^2+1\right)^3} dx$

Let $u=x^2+1$. Then $du=2x dx$, or $\frac{du}{2}=x dx$. By substitution,

$$\frac{x}{\left(x^2+1\right)^3} dx=\frac{1}{2} \int u^{-3} du=\frac{1}{2} \cdot \frac{u^{-2}}{-2}+C=-\frac{1}{4\left(x^2+1\right)^2}+C .$$

4. $\int \cos^4 x \sin x dx$ Let $u=\cos x$. Then $du=-\sin x dx$, or $-du=\sin x dx$. By substitution,

$$\int \cos^4 x \sin x dx=-\int u^4 du=-\frac{u^5}{5}+C=-\frac{\cos^5 x}{5}+C .$$

5. $\int x \sec^3\left(x^2\right) \tan\left(x^2\right) dx$

Let $u=\sec\left(x^2\right)$. Then $du=\sec\left(x^2\right) \tan\left(x^2\right) \cdot 2x dx$, or $\frac{du}{2}=\sec\left(x^2\right) \tan\left(x^2\right) \cdot x dx$. By substitution,

$$\begin{aligned}
\int x \sec^3\left(x^2\right) \tan\left(x^2\right) dx & =\int \sec^2\left(x^2\right) \sec\left(x^2\right) \tan\left(x^2\right) \cdot x dx \\
& =\int u^2 du=\frac{1}{2} \cdot \frac{u^3}{3}+C \\
& =\frac{\sec^3\left(x^2\right)}{6}+C .
\end{aligned}$$

6. $\int \frac{\tan \frac{1}{s}+\tan \frac{1}{s} \sin \frac{1}{s}}{s^2 \cos \frac{1}{s}} ds$
Let $u=\frac{1}{s}$.Then $du=-\frac{1}{s} ds$ or $-du=\frac{ds}{s}$. By substitutuion,

$$\begin{aligned}
\int \frac{\tan \frac{1}{s}+\tan \frac{1}{s} \sin \frac{1}{s}}{s^2 \cos \frac{1}{s}} ds & =-\int \frac{\tan u+\tan u \sin u}{\cos u} du \\
& =-\int\left(\sec u \tan u+\tan^2 u\right) du \\
& =-\int\left(\sec u \tan u+\sec^2 u-1\right) du \\
& =-(\sec u+\tan u-u)+C \\
& =-\sec \frac{1}{s}-\tan \frac{1}{s}+\frac{1}{s}+C
\end{aligned}$$

7. $\int t \sqrt{t-1} dt$
Let $u=t-1$.\$ Then $u=dt$. Also, $t=u+1$. By substitution,

$$\begin{aligned}
\int t \sqrt{t-1} dt=\int(u+1) u^{1/2} du=\int\left(u^{3/2}+u^{1/2}\right) du=\frac{2 u^{5/2}}{5}+\frac{2 u^{3/2}}{3}+C \\
=\frac{2(t-1)^{5/2}}{5}+\frac{2(t-1)^{3/2}}{3}+C .
\end{aligned}$$

8. $\int \frac{t^3}{\sqrt{t^2+3}} dt$
Let ($u=t^2+3$. Then $du=2tdt$, or $\frac{du}{2}=tdt$. Also, \$ t^{2}=u-3.\$ By substitution,

$$
\begin{aligned}
\int \frac{t^3}{\sqrt{t^2+3}} dt=\int \frac{t^2 \cdot t}{\sqrt{t^2+3}} dt & =\int u^{-1/2}(u-3) \frac{du}{2} \\
& =\frac{1}{2} \int\left(u^{1/2}-3 u^{-1/2}\right) du=\frac{1}{2}\left(\frac{2 u^{3/2}}{3}-6 u^{1/2}\right)+C \\
& =\frac{\left(t^2+3\right)^{3/2}}{3}-3\left(t^2+3\right)^{1/2}+C
\end{aligned}
$$

9. $\int \sqrt{4+\sqrt{x}} dx$ Let $u=4+\sqrt{x}$. Then $du=\frac{1}{2\sqrt{x}} dx$ or $2du=\frac{dx}{\sqrt{x}}$. By substitution,

$$
\begin{aligned}
\int \sqrt{4+\sqrt{x}} dx & =\int \sqrt{4+\sqrt{x}} \cdot \frac{\sqrt{x}}{\sqrt{x}} dx \\
& =\int \sqrt{4+\sqrt{x}} \cdot \sqrt{x} \cdot \frac{dx}{\sqrt{x}} \quad(\sqrt{x}=u-4) \\
& =\int u^{1/2} \cdot(u-4) \cdot 2 du \\
& =\int\left(2 u^{3/2}-8 u^{1/2}\right) du \\
& =\frac{2 \cdot 2 u^{5/2}}{5}-\frac{2 \cdot 8 u^{3/2}}{3}+C \\
& =\frac{4(4+\sqrt{x})^{5/2}}{5}-\frac{16(4+\sqrt{x})^{3/2}}{3}+C .
\end{aligned}
$$

4.1.3 Particular Antiderivatives

Now suppose that given a function $f(x)$, we wish to find a particular antiderivatives $F(x)$ of $f(x)$ that satisfies a given condition. Such a condition is called an initial or boundary condition.