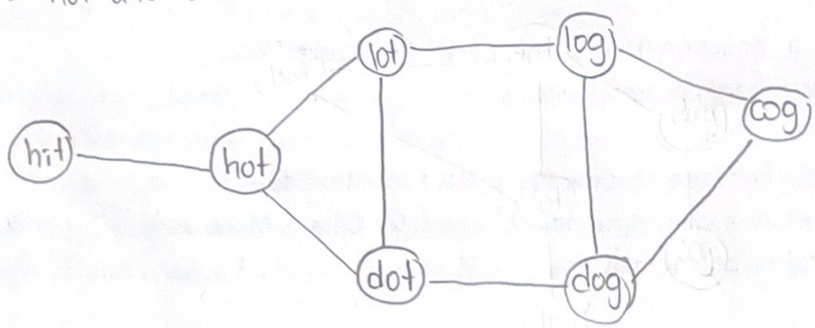
Q1.1. The problem could be formulated as a shortest path problem. To do so, we are going to convert each word into a node. Hence, now we are going to have 7 nodes: hit, cog, hot, dot, dog, lot, and log. If node A could be converted to node B, there will be an edge between node A and B. In our case, cog could be converted to log; hence, there exists an edge between cog and log. Another example would be hot and dot. This could be presented in the graph below.



with the graphical representation above, we can see that the problem is now a shortest path problem. To convert "hit" to "cog", we could traverse through the graph to get the shortest path and that shortest path will provide the fewest number of steps to transform w_1 to w_2.

Q1.2 Show that the graph can be constructed in $O(n^2)$ or $O(Kn^2)$

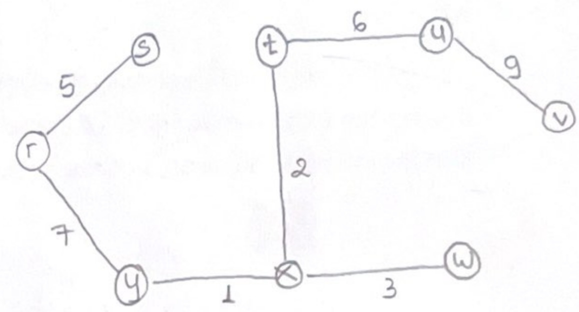
⇒ please see python file: Problem1.2_1.3.py

Q1.3 Get algorithm:

⇒ please see python file: Problem1.2_1.3.py

Q2.1 Give the result of running Kruskal's algorithm

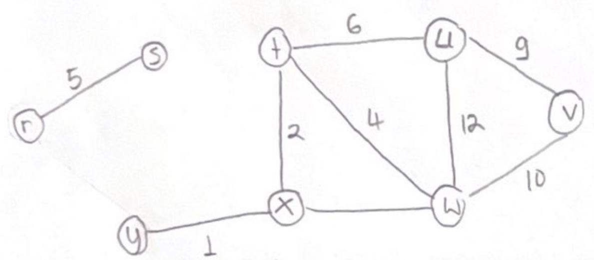
1. First we are going to pick the smallest weight edge of 1: y – x
- x – t with weight of 2
- x – w with weight of 3
- r – s with weight of 5
- t – u with weight of 6
- r – y with weight of 7
- u – v with weight of 9



⇒ Total minimum spanning tree = 1+2+3+5+6+7+9 = 33
with the sequence of (y–x), (x–t), (x–w), (r–s), (t–u), (r–y), (u–v)

Q2.2 exhibit a cut that certifies that the edge ry is in the minimum spanning tree.

- The set of cut are (ry), (ys), (st)



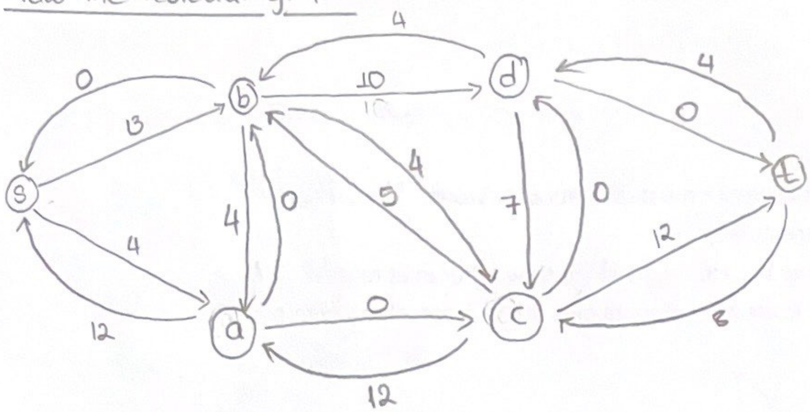
graph after cutting

⇒ S(= rs

⇒ V/S = yxtuwv

⇒ $\boxed{E(S, V/S) = E(rs, yxtuwv)}$

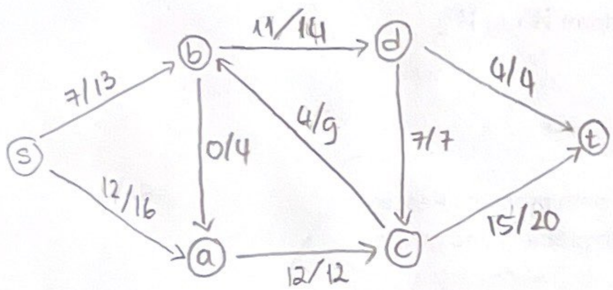
Q3.1 Draw the residual graph



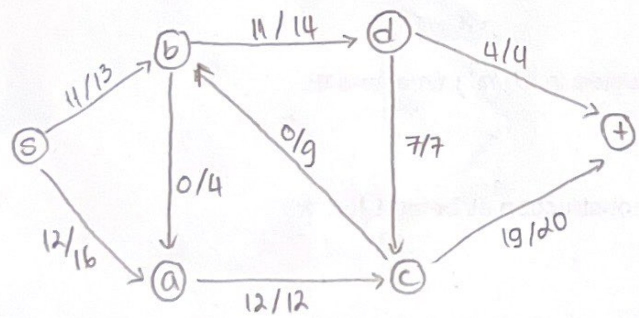
Q3.2 Draw: a) The flow values on the edges & Residual network

a). Flow values on the edges :

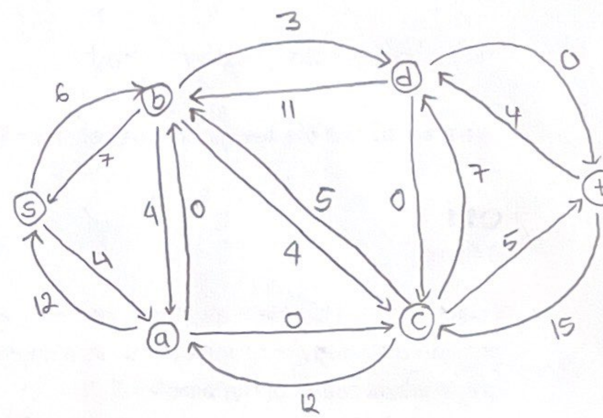
i). s → b → d → c → t : 7 units

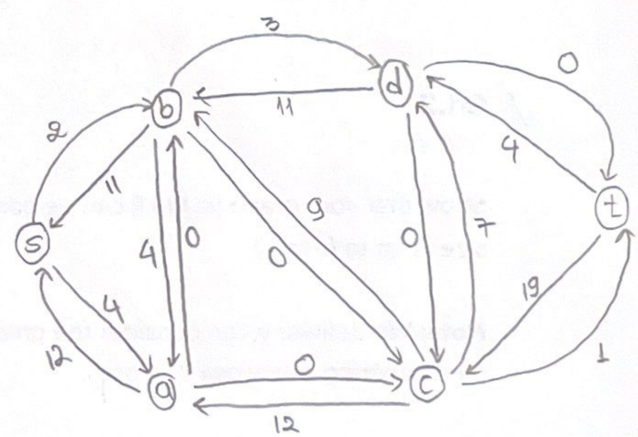


ii). s → b → c → t with 4 units



b). Residual network



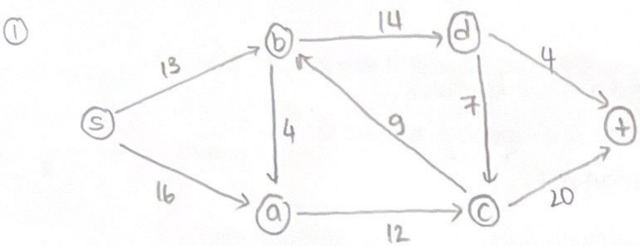


Assuming 3.2 is the continuity of 3.1

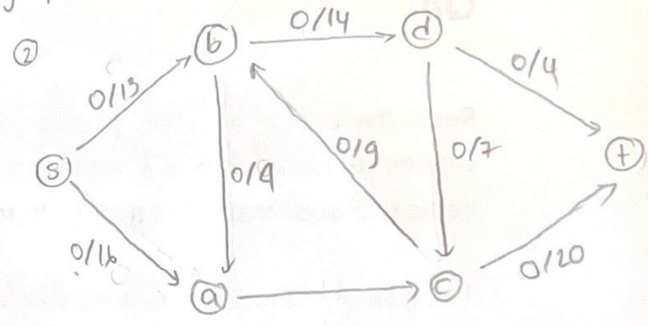
Q3.3 Exhibit a maximum flow with flow values on the edges, state its value on exhibit a cut

We are going to use Ford Fulkerson for our grap.

①

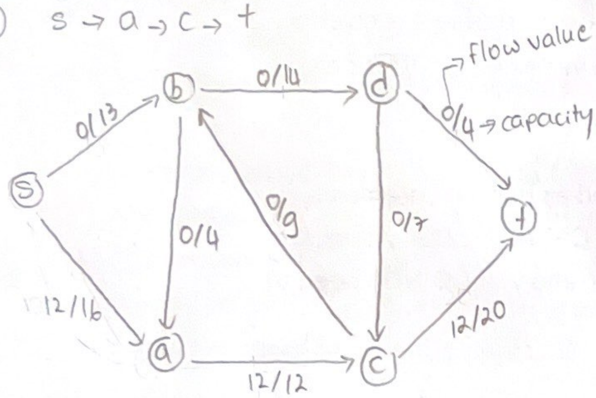


⇒ ②



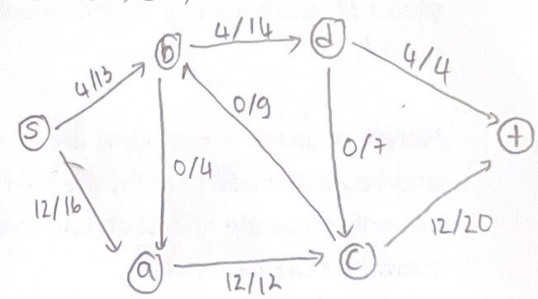
initializing the graph ⇒ Flow value =0

③ $s \to a \to c \to t$



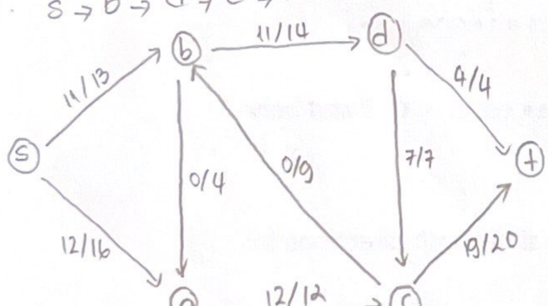
⇒ bottle neck = 12   Flow value = 12

④ $s \to b \to d \to t$



⇒ bottle neck = 4   Flow value = 16

④ $s \to b \to d \to c \to t$



⇒ bottle neck = 7   Flow value = 19+4 or 12+11 = 33

⇒ As for cut, the edges for cuts are (b, d), (a,c), (c,b). Hence, total cut = 11 +12+0 = 33

∴ max flow = min cut = 33

Q4.1 Prove that any subset of vertices S, S is a vertex cover in G if and only if V/s is clique in C.

1). Show that " if S is a vertex cover in G, V/s is a clique in C" by showing for every edge uv in C, at least u or v in VS

- By definition, if a vertex cover S of an undirected graph G(V, E) is a subset of $S \subseteq V$ such that every edge in G has at least one end point in S ($u \in S$ or $v \in S$)
- However, C is a complementary graph of G. Hence, $uv \in C \wedge uv \notin G$. Therefore, we can say for every edge uv in C, at least u or v not in S
- Since S is a vertex cover in G and C is G's complimentary graph, for every uv in C, at least u or v is in V/S.

∴ VS is a clique in C (a)

2). Show that if V/S is a clique in C, then S is a vertex cover in G, by showing for every edge uv in G, at least u or v in S

- By definition, the clique problem is: given an undirected graph G=(V, E) and an integer K, does G have a subset S of K vertices such that for each distinct $u, v \in S$, $\{u, v\} \in E$.
- C is a complementary graph of G. Hence, $uv \in C \wedge uv \notin G$. Thuse, for every edge $uv \in C$, at least one of u or v not in V/S.
- Since V/S is a clique of C, every edge uv in C, at least $u, v \in S$

∴ S is a vertex cover (b)

(a) ∧ (b) indicate that S is a vertex cover in G if and only if VS is a clique in C

Q 4.2 Give reduction from vertex cover to clique.

Assuming we already know clique $\in$ NP and vertex cover $\in$ NP complete $\Rightarrow$ if vertex cover can be reduced to clique $\Rightarrow$ clique is an NP complete.

- To prove reduction from clique problem to Vertex cover, we have to show

1) if there is a solution to vertex cover (G, K), there must be a solution to Clique(H, l) where H is the complement graph $G = (V, \bar{E})$ and $l = V - K$ (this is i in our problem set)

2) if there is a solution to clique (H, l), then there must be a solution to vertex cover(G, K) (this is ii in our problem set).

Proof

①. • Suppose G has a vertex cover $S \Rightarrow |S| \subseteq V$ where $|S| = K$ (vertex cover of size K).

- $\forall u, v \in V$, if $(u, v) \in E$, $u \in S$ or $v \in S$ or both.
- the contrapositive is that $\forall u, v \in V$, if $u \notin S$ and $v \notin S$ then $(u, v) \in H$. This implying that for any pairs of vertices that both are not in S of G, there is an edge between them in H.

$\Rightarrow$ the union of all pairs of vertices that are not in S is equal to $V - S$.

$\therefore$ $V - S$ is a clique in H $\wedge$ $V - S$ has the size of $|V| - K$ (this part, answer problem 4.2.i and the runtime will be explain latter on,

②. suppose H has a clique $V \setminus S \subseteq V$, with $|V \setminus S| = |V| - K$.

- let (u, v) is an edge in E. $\Rightarrow (u, v) \notin E$ if at least one of u or $v \notin V \setminus S$ because every pair of vertices in $V \setminus S$ is connected by an edge of H.

$\therefore$ at least one of u or v is in $V - V \setminus S$

$\Rightarrow$ the edge (u, v) is covered by $V - V \setminus S$.

- Since we arbitrary pick (u, v) from edge E, $\forall$ edge in E is covered by $V - V \setminus S$

$\therefore$ the set $V - V \setminus S$ is a vertex cover of G with size K. (this part answer problem 4.2.ii)

① $\wedge$ ② prove the reduction from clique to vertex cover.

- To generate H, the complementary graph, we need to check all pair of vertices in the original graph, then generate the edge if there is no edge between vertices $\Rightarrow$ the operation can be done in polynomial time. Since vertex cover can be reduced from clique $\Rightarrow$ clique $\in$ NP $\wedge$ vertex cover $\in$ NP complete

$\therefore$ Clique is also NP complete.