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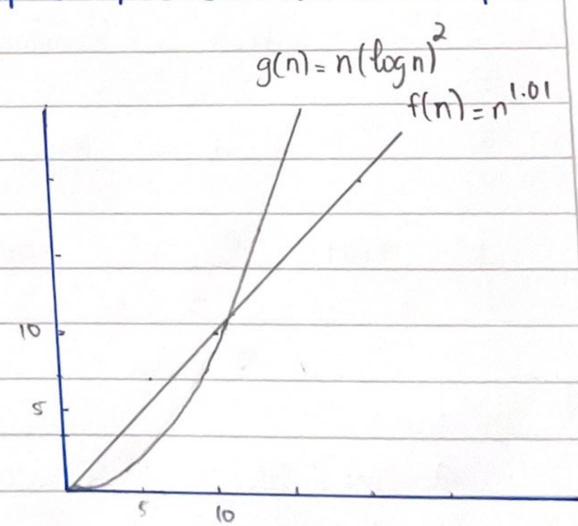
Module 1: **Homework Assignment**

P1. Each element of an array $A[1,\dots,n]$ is a digit $(0,\dots,9)$. The array is ordered: $A[i] \le A[i+1]$ for all i. Consider the problem of finding the sum of array $A[1,\dots,n]$. Can we do it in $O(\log n)$ time?

$\Rightarrow$ Yes we could.

P2. For each of these parts, indicate whether $f = O(g)$, $f = \Omega(g)$ or both (i.e., $f = \Theta(g)$). In each case, give a brief justification for your answer. (Hint: it may help to ~~give~~ plot the functions and obtain an estimate of the relative growth rates. In some cases, it may be also help to express each function as power of 2 and then compare



$f(n) = n^{1.01}$

$g(n) = n(\log n)^2$

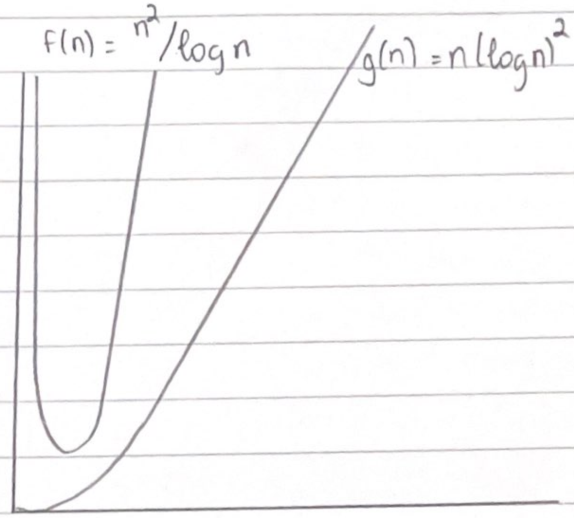
$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \frac{n \cdot n^{0.01}}{n(\log n)^2}$$

$$= \lim_{n\to\infty} \frac{n^{0.01}}{(\log n)^2}$$

$$= \lim_{n\to\infty} \left(\frac{n^{0.005}}{\log n}\right)^2 < 1, \quad n \to \infty$$

$$\boxed{\begin{array}{l} \therefore\ f(n) \le g(n) \\ \therefore\ f = O(g) \end{array}}$$

P2b. $f(n) = \dfrac{n^2}{\log(n)}$; $g(n) = n(\log n)^2$



assuming $\forall\ n \in \mathbb{R}^+$

$$\Rightarrow \lim_{n \to +\infty} \frac{f(n)}{g(n)}$$

$$= \lim_{n \to +\infty} \frac{\frac{n^2}{\log(n)}}{n(\log n)^2}$$

$$= \lim_{n \to +\infty} \frac{n}{(\log n)^3}$$

$$= \lim_{n \to +\infty} \frac{1}{\frac{3(\log x)^2}{x}} \quad \text{(l'hopital rule)}$$

$$= \lim_{n \to +\infty} \frac{x}{3(\log x)^2}$$

$$= \lim_{n \to +\infty} \frac{\cancel{x}\ 1}{\frac{(3)(2)\log x}{x}} \quad \text{(l'hopital rule)}$$

$$= \lim_{n \to +\infty} \frac{x}{6\log x}$$

$$= \lim_{n \to +\infty} \frac{1}{6\left(\frac{1}{x}\right)} \quad \text{(l'hopital rule)}$$

$$= \lim_{n \to +\infty} \left(\frac{1}{6}\right) x$$

$$= +\infty$$

$$\Rightarrow \boxed{f(n) \geqslant g(n) \quad \therefore \quad f(n) \text{ is } \Omega(g(n))}$$

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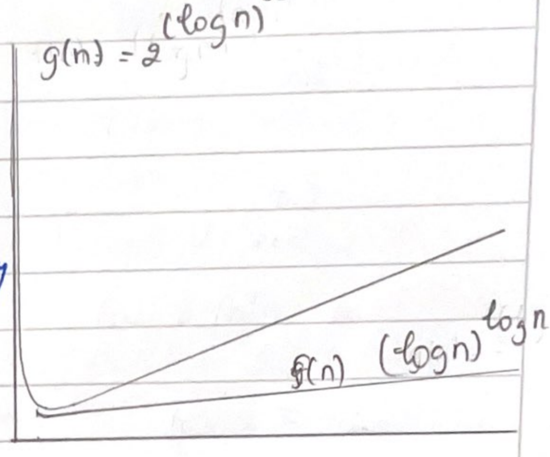
P3a. For each of these parts, indicate whether $f = O(g)$, $f = \Omega(g)$, or both (i.e., $f = \Theta(g)$). In each case, give a brief justification for your answer.

a. $f(n) = (\log n)^{\log n}$; $g(n) = 2^{(\log n)^2}$



$f(n) = (\log n)^{\log n}$ is a logarithmic function with logarithmic power. Hence the growth rate for the function is relatively slow compare to that of $g(n)$.

$g(n) = 2^{(\log n)^2}$ is an exponential growth even though its base power is a logarithmic but it is still fast.

let $n = 1 \Rightarrow (\log 1)^{\log 1} = 1$

$\Rightarrow 2^{(\log n)^2} = 1$

let $n = 100{,}000 \Rightarrow f(n) = (\log 100{,}000)^{\log 100{,}000}$

$= (\log 10^5)^{\log 10^5}$

$= (\log 10^5)^5$

$= (5)^5 = 3125$

$\Rightarrow g(n) = 2^{(\log 100{,}000)^2} = 2^{(5 \log 10)^2} = 2^{25}$

$= 33{,}554{,}432$

$\therefore g(n) \geq f(n)$
$\Rightarrow f(n)$ is $O(g(n))$

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b. $f(n) = \sum_{i=1}^{n} i^K$; $g(n) = n^{K+1}$

$f(n) = \sum_{i=1}^{n} i^K = 1^K + 2^K + 3^K + 4^K + \dots + n^K$

$g(n) = n^{K+1} = n^K + n^K + n^K + \dots + n^K = n.n^K = n^{K+1}$

$\therefore f(n) \leq g(n) \Rightarrow f(n)$ is $O(g(n))$

$f(n) = 1^K + 2^K + 3^K + 4^K + \dots + \left(\frac{n}{2}\right)^K + \dots + n^K$

if we only take the last half of f(n)

$\Rightarrow 1^K + 2^K + 3^K + \dots + \left(\frac{n}{2}\right)^K + \dots + n^K \geq \left(\frac{n}{2}\right)^K + \dots + n^K$

$g(n) = n^K + n^K + n^K + \dots + n^K$

assuming we only take the last half of g(n) so that we could compare it to ~~g(n)~~ f(n)

$\Rightarrow g(n) = \left(\frac{n}{2}\right)^K + \dots + \left(\frac{n}{2}\right)^K = \left(\frac{n}{2}\right)\left(\frac{n}{2}\right)^K = \left(\frac{n}{2}\right)^{K+1}$

$\Rightarrow \left(\frac{n}{2}\right)^K + \dots + n^K \geq \left(\frac{n}{2}\right)\left(\frac{n}{2}\right)^K$

$\left(\frac{n}{2}\right)^K + \dots + n^K \geq \left(\frac{n}{2}\right)^{K+1}$

$\therefore f(n) \geq g(n) \Rightarrow f(n)$ is $\Omega(g(n))$

$\boxed{\text{since } f(n) \leq g(n) \text{ \& } f(n) \geq g(n) \quad \therefore f(n) \text{ is } \Theta(g(n))}$