University of Minho
School of Engineering

Quantum Algorithms and Applications

Leander Reascos

Quantum Computing School @ Yachay

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Factoring

What are the prime factors of

$$p \cdot q = 21606371083676 3057$$

Factoring

$$p = 225865261$$

$$q = 956604437$$

Shor's Algorithm

Shor's Algorithm

- Shor's algorithm is a quantum algorithm for integer factorization.
- It was invented in 1994 by Peter Shor.
- It solves the following problem in polynomial time:

$$a^r \equiv 1 \pmod{N} \tag{1}$$

- Where N is a composite number and $2 \leq a < N$ is a random integer.

Shor's Algorithm

Find r such that $a^r \equiv 1 \pmod{N}$

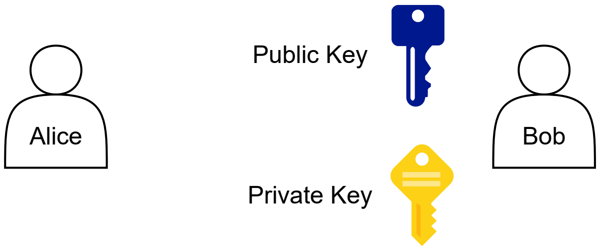
$$\frac{N}{a^r - 1} = \frac{N}{\left(a^{\frac{r}{2}} - 1\right)\left(a^{\frac{r}{2}} + 1\right)} \tag{2}$$

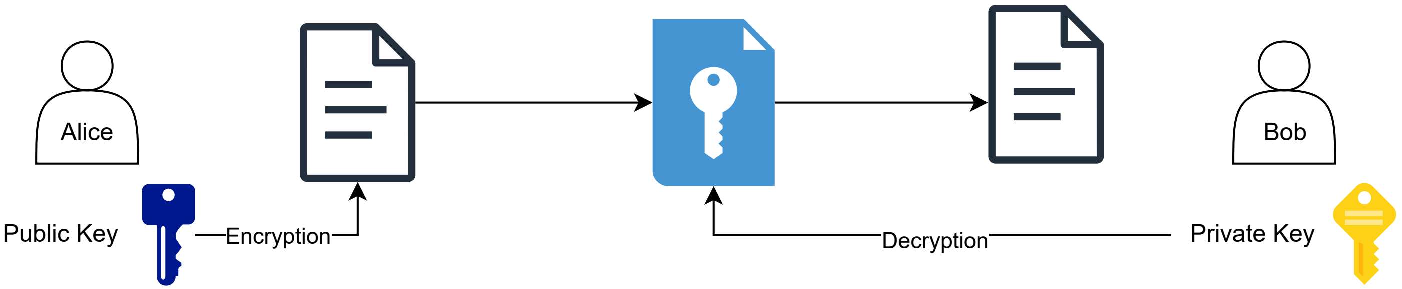
Implications

- Shor's algorithm can be used to break RSA encryption.
- It can also be used to solve the discrete logarithm problem.
- It can be used to solve the hidden subgroup problem.

RSA Protocol

RSA encryption is based on the difficulty of factoring large integers (2048 bits - 600 decimal digits).





NISQ Era

John Preskill coined the term Noisy Intermediate-Scale Quantum (NISQ) in 2017.

- NISQ computers are quantum computers with 50-100 qubits.
- They are noisy and have a short coherence time.
- They are not powerful enough to run Shor's algorithm.
- They can be used to run variational quantum algorithms.

Grover's Algorithm

Grover's Algorithm

- Grovers algorithm is a quantum search algorithm.
- It was invented in 1996 by Lov Grover.
- It solves the following problem in polynomial time:

$$f(x) = 1 \tag{3}$$

- Where f is a function that takes n-bit strings as input and returns a single bit.

Grover Operator

Let's define the Grover operator G as follows:

$$G \equiv U_d U_f \tag{4}$$

where,

$$U_f \left|x\right\rangle = (-1)^{f(x)} \left|x\right\rangle \quad \text{Oracle} \tag{5}$$

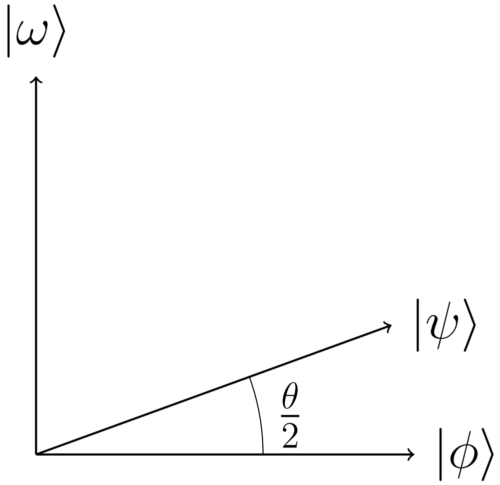
U_d Diffusion Operator

$$U_d = 2 \left|\psi\right\rangle \left\langle\psi\right| - I \tag{6}$$

where,

$$\left|\psi\right\rangle = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} \left|x\right\rangle \tag{7}$$

Geometry of Grover's Algorithm

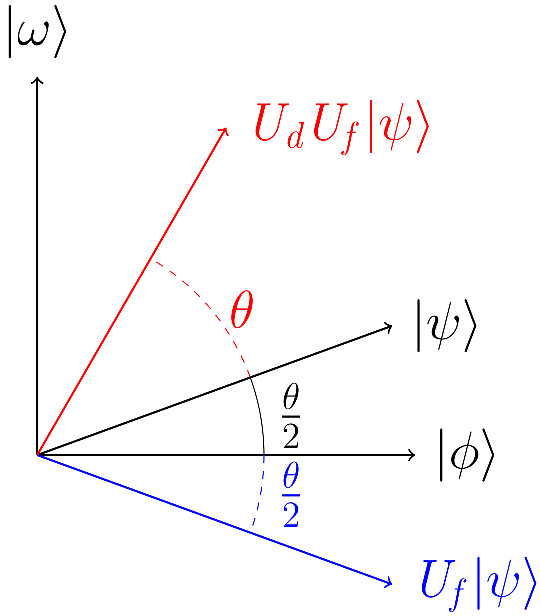


$$|\psi\rangle = \frac{\sqrt{N-1}}{\sqrt{N}}|\phi\rangle + \frac{1}{\sqrt{N}}|\omega\rangle \tag{8}$$

$$|\phi\rangle = \frac{1}{\sqrt{N-1}}\sum_{x\neq\omega}|x\rangle \tag{9}$$

$$|\psi\rangle = \cos\frac{\theta}{2}|\phi\rangle + \sin\frac{\theta}{2}|\omega\rangle \tag{10}$$

Geometry of Grover's Algorithm



$$U_f |\psi\rangle = \cos\frac{\theta}{2}|\phi\rangle - \sin\frac{\theta}{2}|\omega\rangle$$

$$U_d U_f |\psi\rangle = \cos\left(\frac{\theta}{2}+\theta\right)|\phi\rangle + \sin\left(\frac{\theta}{2}+\theta\right)|\omega\rangle$$

Grover Iterator

The Grover iterator G is defined as follows:

$$G = U_d U_f \tag{11}$$

Each iteration of G increases the angle $\frac{\theta}{2}$ by θ.

$$G^k |\psi\rangle = \cos\left(\frac{\theta}{2} + k\theta\right)|\phi\rangle + \sin\left(\frac{\theta}{2} + k\theta\right)|\omega\rangle \tag{12}$$

The number of iterations required to find the solution is $k \leq \frac{\pi}{4}\sqrt{N}$.

Implications

- Cryptography
- Golberg's Conjecture
- Amplitude Amplification

Complexity Theory

Complexity Theory

- **P**: Problems that can be solved in polynomial time.
- **NP**: Problems that can be verified in polynomial time.
- **BQP**: Problems that can be solved in polynomial time by a quantum computer.

Note: The Godel's incompleteness theorem states that there are problems that cannot be solved by any computer.

Cryptography

Cryptography

- **Cryptography** is the study of techniques for secure communication in the presence of third parties.
- **Classical Cryptography** is based on the difficulty of solving mathematical problems.
- **Quantum Cryptography** is based on the laws of quantum mechanics.

Quantum Cryptography

- Quantum cryptography is the science of exploiting quantum mechanical properties to perform cryptographic tasks.
- The best known example of quantum cryptography is **quantum key distribution** which offers an information-theoretically secure solution to the key exchange problem.

Post-Quantum Cryptography

- **Post-quantum cryptography** refers to cryptographic algorithms (usually public-key algorithms) that are thought to be secure against an attack by a quantum computer. (Dilithium)
- **Quantum-resistant** algorithms are designed to be secure against both quantum and classical computers.

Quantum Internet

- The **quantum internet** is a network that will let quantum devices exchange some information within an environment that harnesses quantum mechanics' weird properties.
- The quantum internet will be used to distribute quantum keys.

Discussion

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