A heat equation approach to temperature prediction of decomposing cadavers

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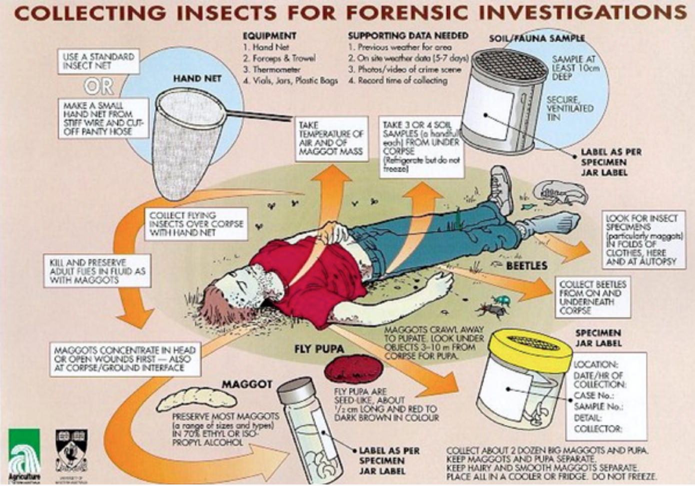
Outline

Introduction

Upon discovery of a crime scene, a critical piece of physical evidence, especially in an outdoor setting, are the insects involved in natural decomposition.



The effects of insect development lasts much longer than chemical changes that occur in the body immediately after death.

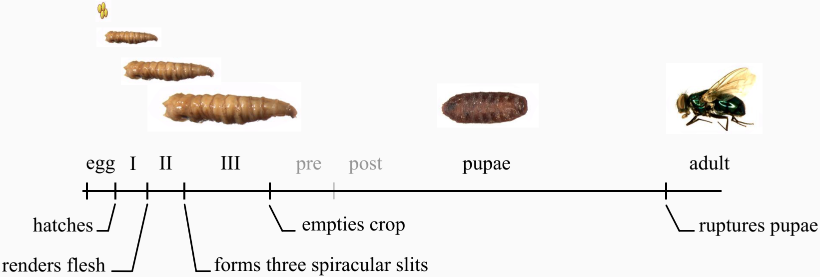


Figure 1: Life cycle of blow fly showing relative time spent in each stage of development.

Progression through an insect's life cycle is temperature dependent, so thermal summation models are often used.

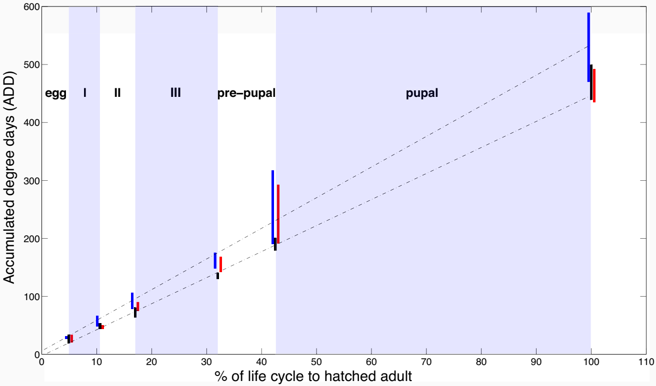


Figure 2: Development data for Calliphoridae at three different temperature profiles as collected in . The ADD range at 15.8 °C (blue), 20.6 °C (black), and 23.3 °C (red) are shown.

To use the thermal summation model historic temperature profiles are required. Current methods use linear regression between the environment and a simulated internal logger.

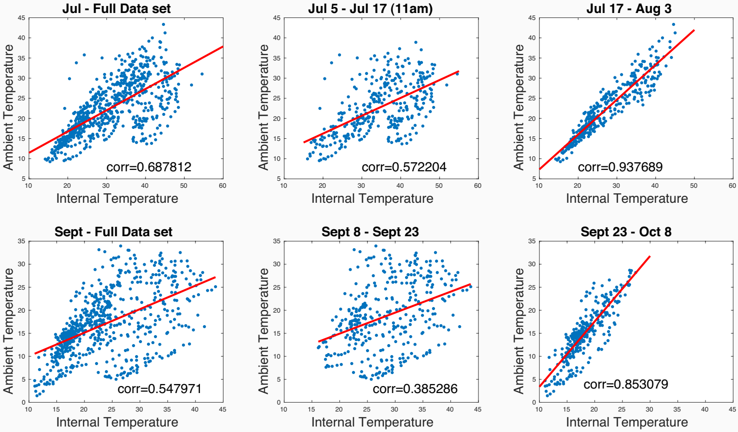


Figure 3: Correlation figures for case study data from July and September 2016 decomposition studies performed with swine cadavers at Ontario Tech.

Model Development

Case study data offers internal logger measurements as well as ambient temperature measurement with nearby external loggers.

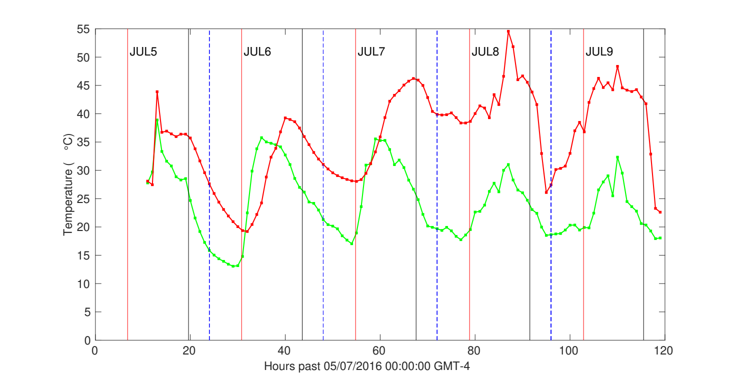


Figure 4: Internal temperature probe (red) and environmental temperatures (green) for the first five days of a swine cadaver decomposition study in July 2016

There are some notable patterns between the two temperature profiles

- There exists a phase shift between the two curves, especially in the first few days;
- The phase shift decreases over time;
- The internal logger doesn't experience the same lows as the ambient temperature, and experiences higher maximums;
- Both temperature profiles have a distinct day/night cycle.

We explore a heat equation approach. Starting with

$$\rho c_p \frac{\partial T}{\partial t} = k\nabla^2 T + Q(\mathbf{x}, t). \qquad (1)$$

We consider the model in cylindrical coordinates to simulate a three dimensional cylindrical vessel, (**??**) becomes

$$\rho c_p \frac{\partial T}{\partial t} = k\left(\frac{\partial^2 T}{\partial^2 t} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial^2 z}\right) + Q(r, z, t). \qquad (2)$$

In terms of boundary conditions we desire that the vessel is insulated on all sides and the bottom, S_2, S_3. The top, S_1, may interact with the environment. This gives way to

$$-k\,\frac{\partial T}{\partial \hat{n}}\bigg|_{S_2,S_3} = 0, \qquad -k\,\frac{\partial T}{\partial \hat{n}}\bigg|_{S_1} = h(T - T_{\mathsf{ext}}(t)) \tag{3}$$

We examine the effects of averaging over the spacial dimensions to obtain an ordinary differential equation which is only time dependent. We rewrite the model where $\bar{\bar{T}}(t) \mapsto \bar{\theta}(\tau)$,

$$\frac{\mathrm{d}\bar{\theta}}{\mathrm{d}\tau} = -\frac{h\Delta T}{Q_0 H}\left(\bar{\theta} - \theta_{\mathsf{ext}}(\tau)\right) + q(\tau), \qquad \bar{\theta}(0) = 0. \tag{4}$$

We run a simulation with a set of parameters to correspond to a bucket of water of depth 30 cm and a significant amount of internal heating. We allow external heating to be sinusoidal. We examine the solution to (**??**) by solving the ODE numerically and with method of lines.

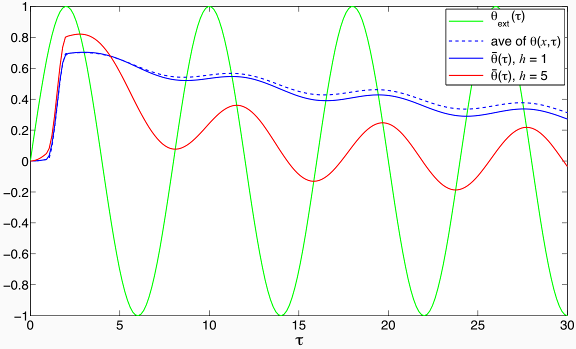


Figure 5: Solution to the ODE with $h = 1$ (blue), $h = 5$ (red), using method of lines (dashed blue) and the external heating (green).

We use the toy model to inform the model for a swine cadaver. We allow for heat flux through the boundary,

$$-k\left.\frac{\partial T}{\partial \hat{n}}\right|_{\partial\Omega} = h(T - T_0) \qquad \mathbf{x} \in \partial\Omega,\ t > 0, \tag{5}$$

and simplify by averaging at every point so that the average temperature is

$$\bar{T}(t) = \frac{1}{|\Omega|}\int_{\Omega} T(\mathbf{x}, t)\,\mathrm{d}\mathbf{x}. \tag{6}$$

We nondimensionalize using

$$\bar{T} = T_{\mathsf{min}} + \Delta T\theta \qquad T_0 = T_{\mathsf{min}} + \Delta T\theta_0 \qquad t = \tau\tilde{t}. \tag{7}$$

The equation for the averaged temperature, as derived analogous to the previous section, becomes

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = c(t)(\theta_0(t) - \theta) + s(t), \tag{8}$$

with

$$c(t) = \frac{h|\partial\Omega|\tau}{\rho c_p|\Omega|}, \quad s(t) = \frac{Q(t)\tau}{\rho c_p \Delta T}, \quad Q(t) = \frac{1}{|\Omega|}\int_\Omega q(\mathbf{x}, t)\,\mathrm{d}\mathbf{x}, \tag{9}$$

Parameter Investigation

We choose the coupling coefficient to be a constant, $c(t) \mapsto c_0$ and the heat flux, $s(t)$, to be piecewise constant with changes at characteristic times, τ_j, corresponding to sunrise and sunset.

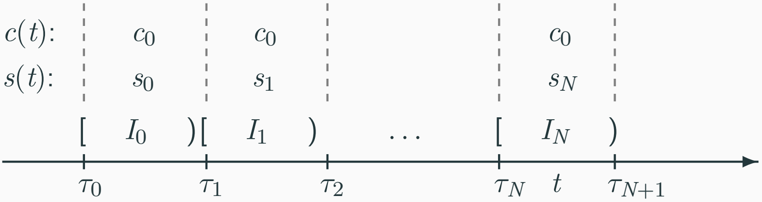


Figure 6: The structure of $c(t)$ and $s(t)$.

Upon rewriting the solution of (**??**) as $\theta = u + v$, to represent two heat sources we obtain a solution in the following form

$$\theta(t; c_0, \mathbf{s}) = \theta(t_0)\mathrm{e}^{-c_0(t-\tau_0)} + c_0 \int_{\tau_0}^{t} \mathrm{e}^{-c_0(t-\eta)}\theta_0(\eta)\,\mathrm{d}\eta \tag{10}$$

$$+ \frac{1}{c_0}\sum_{j=0}^{l-1} s_j \left(\mathrm{e}^{-c_0(t-\tau_{j+1})} - \mathrm{e}^{-c_0(t-\tau_j)}\right) + \frac{s_l}{c_0}\left(1 - \mathrm{e}^{-c_0(t-\tau_l)}\right), \tag{11}$$

for $\tau_l \leq t < \tau_{l+1}$, and $l = 0, 1, \ldots, N$.

Objective Function

With the model representation now having one value for c_0 and $N+1$ values for s, we are able to use ground truth values to perform a one parameter optimization. Measured internal temperatures, $\hat{\theta}$ are used to give the objective function

$$f = \|\hat{\theta} - \theta\|^2_{\ell^2(\mathbb{R}^M)}. \tag{12}$$

Or using a matrix approach

$$\theta = \mathbf{u}(c_0) + B(c_0)\mathbf{s} \tag{13}$$

where $\mathbf{u}(c_0) = (u(t_1; c_0), \ldots, u(t_M; c_0))^\mathsf{T}$, and

$$[B(c_0)]_{ij} = \frac{1}{c_0}\begin{cases} \mathrm{e}^{-c_0(t_i-\tau_j)} - \mathrm{e}^{-c_0(t_i-\tau_{j-1})}, & 1 \le j \le l, \\ 1 - \mathrm{e}^{-c_0(t_i-\tau_l)}, & j = l+1, \\ 0, & l+2 \le j \le N+1. \end{cases} \tag{14}$$

Optimal Parameters

Analysis of the coupling coefficient is performed by examining the percent relative error

$$\text{Percent relative error} = \frac{100}{\|\hat{\theta}\|_{\ell^2(\mathbb{R}^M)}} \left\| \left(\mathbf{I} - B(B^{\mathsf{T}}B)^{-1}B^{\mathsf{T}}\right)(\mathbf{u} - \hat{\theta}) \right\|_{\ell^2(\mathbb{R}^M)} . \tag{15}$$

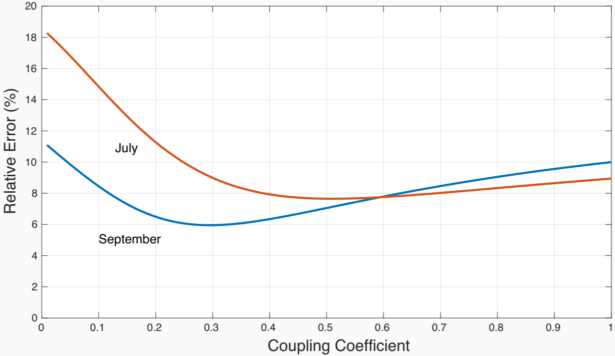


Figure 7: The relative error as a function of c_0 for the duration of each decomposition study for the July ($c_0^* \simeq 0.51$) and the September

We also examine how the error and the coupling coefficient change as a function of time.

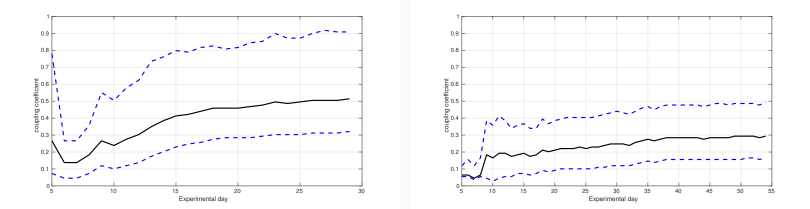


Figure 8: Left: July data set. Right:September data set. Upper and lower 1% errors shown in dashed blue.

Heat flux vs time

After finding the optimal choice of coupling coefficient we are able to extract the optimal piecewise constant heat flux signature.

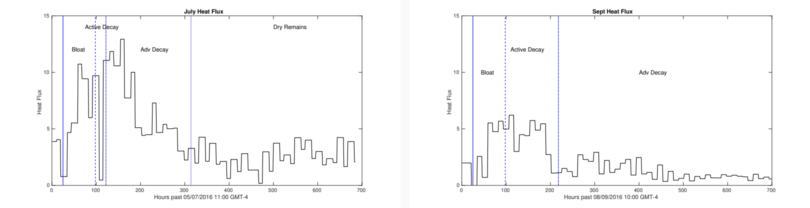


Figure 9: Left: July data set. Right: September data set.defined by Hours past beginning of experiment plotted as the independent axis. Vertical lines indicate progression through observed stages of decomposition.

Heat flux vs ADD

We also examine the heat flux signature as it depends on the ADD.

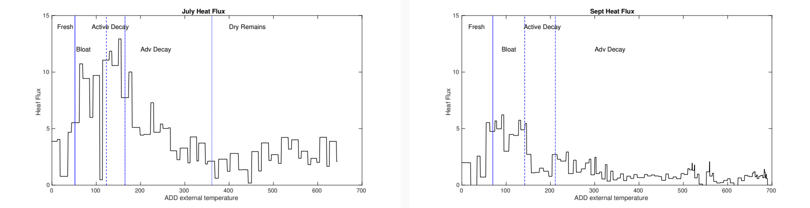


Figure 10: A scaling of the flux $\mathbf{s}^*$ with the abscissa expressed in ADD calculated using the external ambient temperature.

Temperature Prediction

Choice of coupling coefficient

A new swine cadaver decomposition study was performed at Ontario Tech in September 2019. We use the September 2016 data to inform the choice of parameters. We choose $c_0 = 0.29$.

Choice of heat flux signature

We allow the heat flux signature to be a time dependant function with constant background value, and a parabolic arc which encapsulates the increased heat flux as the body progresses through decomposition. The function is,

$$s(t; s_0, s_\infty) = s_\infty + \frac{4(s_0 - s_\infty)}{(b-a)^2}(t-a)(b-t)\chi_{[a,b]}(t), \qquad (16)$$

where a and b are chosen as times that correspond to an ADD of 50 and 220 respectively[1].

[1] These choices of ADD values span the majority of insect activity through active decay.

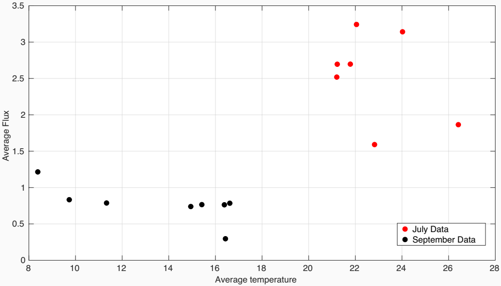
Choice of background and peak

The peak is chosen so that the maximum temperature in bounded below $45°$C. We require

$$s_0^* = \arg\left\{\max_{s_0} \theta\left(t; c_0, s\left(t; s_0, s_\infty\right)\right) = \theta_{\max}\right\} \qquad (17)$$

where $\theta_{\max} = 45$.

The background flux is informed by the behaviour at the tail end of the 2016 studies.



Results

We compare our model against the linear regression model that is built using the last 7 days of the internal pig logger and the ambient logger on the fence,

$$\theta_{\mathsf{reg}} = 0.6510\theta_0 + 6.7724. \tag{18}$$

Based on analysis from 2016 sets, our choice of parameters are $c_0 = 0.29$, $s_\infty = 0.75$ and $s_0 = 5.9$. The resulting relative error is 19.11% for our model and 33.54% for linear regression.

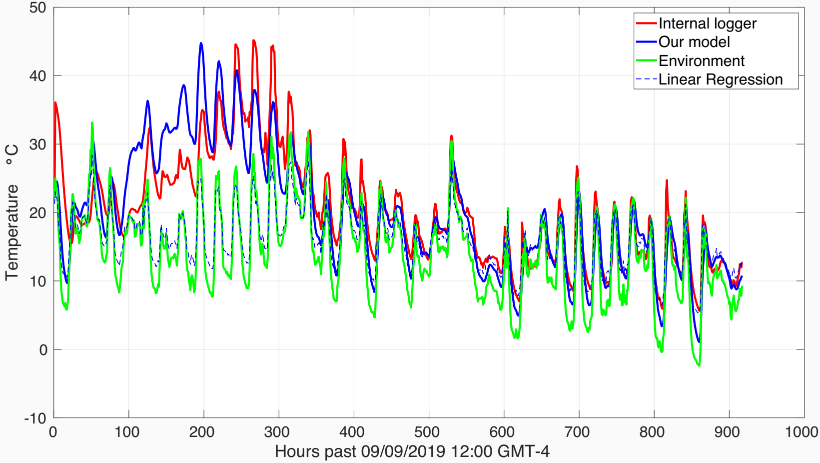


Figure 11: The temperature profiles for the full pig 1 data set commencing at 09/09/2019 12:00:00 GMT-4 (917 hours). Internal cadaver temperature (red), en-vironment (green), predicted cadaver (blue) and linear regression prediction (dahsed blue)

Permutations of parameters

We examine the effects of changing the parameters on the relative error.

Table 1: The effects of changing model parameters on the percent relative error. The nominal values are $c_0 = 0.29$, $s_\infty = 0.75$, and $s_0 = 5.9$.

c_0	% error	s_∞	% error	s_0	% error
0.2	28.61%	0.5	20.23%	4	18.14%
0.3	18.84%	1	18.48%	5	17.80%
0.4	19.59%	2	22.42 %	7	22.29%

Future Work

Further steps can be taken to improve the model's reliability such as

- Test against more data sets to improve parameter prediction
- Test against studies in different locations and times of the year
- Examine the effects of the surrounding environment (shade, wind, tree canopy) on the coupling coefficient
- Find ways to choose start day when information is not available

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